Resolution of a conjecture on variance functions for one-parameter natural exponential family

Xiongzhi Chen*

Abstract

One-parameter natural exponential family (NEF) plays fundamental roles in probability and statistics. A conjecture of Bar-Lev, Bshouty and Enis states that a polynomial with a simple root at 0 and a complex root with positive imaginary part is the variance function of some NEF with mean domain \((0, \infty)\) if and only if the real part of the complex root is not positive. We prove the conjecture in this note.

Keywords: Natural exponential family; polynomial variance functions.

MSC 2010 subject classifications: Primary 60E05; Secondary 62E10.

1 Introduction

One-parameter natural exponential family (NEF) is an important family of probability measures and has been widely used in statistical modelling (McCullagh and Nelder, 1989; Letac, 1992). Let \(\beta\) be a positive Radon measure on \(\mathbb{R}\) that is not concentrated on one point. Suppose the interior \(\Theta\) of

\[\tilde{\Theta} = \left\{ \theta \in \mathbb{R} : g(\theta) = \int e^{\theta x} \beta(dx) < \infty \right\}\]

is not empty. Then \(\mathcal{F} = \{F_\theta : \theta \in \Theta\}\) with

\[F_\theta(dx) = \exp[\theta x - \log g(\theta)] \beta(dx)\]

forms a NEF with respect to the basis \(\beta\). Without loss of generality, we can assume \(0 \in \Theta\), so that \(\beta(\mathbb{R}) = 1\).

For each NEF \(\mathcal{F}\), the mapping \(\mu(\theta) = \frac{d}{d\theta} \log g(\theta)\) defines the mean function \(\mu : \Theta \to U\) with \(U = \mu(\Theta)\), and \(U\) is called the “mean domain”. Let \(\theta = \theta(\mu)\) be the inverse function of \(\mu\). Define the function \(v\) on \(U\) as

\[v(\mu) = \int (x - \mu)^2 F_{\theta(\mu)}(dx)\]

for \(\mu \in U\). Then the pair \((v, U)\) is called the variance function (VF) of \(\mathcal{F}\). Note that \(v\) is a positive, real analytic function on \(U\) and that \(v\) characterizes the NEF \(\mathcal{F}\).

---

*Center for Statistics and Machine Learning, and Lewis-Sigler Institute for Integrative Genomics, Princeton University, Princeton, NJ 08544. Email: xiongzhi@princeton.edu
Let $G_D$ be the set of positive, real analytic functions on a domain $D \subseteq \mathbb{R}$. In order to use an $f \in G_D$ to model the variance-mean relationship for some data and assume that the associated random errors follow a NEF, it is crucial to ensure that $(f, D)$ is indeed the VF of some NEF. Therefore, identifying which $f \in G_D$ are VFs of NEFs is of importance. For few of these results, see, e.g., Bar-Lev et al. (1991); Letac (1992); Letac (2016).

The authors of a ground breaking paper Bar-Lev et al. (1992) raised the following:

**Conjecture.** Let
\[ v(u) = a_0 u (u - u_1)^n (u - \bar{u}_1)^n, \]
where $a_0 > 0$, $\Im(u_1) > 0$, and $\bar{u}_1$ is the complex conjugate of $u_1$. Then $(v, (0, \infty))$ is a variance function for all $n \in \mathbb{N}$ if and only if $\Re(u_1) \leq 0$.

Let $\mathbb{R}^+ = (0, \infty)$. When $\Re(u_1) \leq 0$ the polynomial $v$ has non-negative coefficients. By Bar-Lev (1987) and Proposition 4.4 of Letac and Mora (1990), $(v, \mathbb{R}^+)$ is a VF of an infinitely divisible NEF concentrated on the set $\mathbb{N}$ of non-negative integers. This justifies the sufficiency in the conjecture, and we are generating probability distributions on $\mathbb{N}$ using $v$. Thus, it is left to prove the necessity. Bar-Lev et al. (1992) derived a powerful result, their Theorem 1, for this purpose. Specifically, this theorem requires to determine the sign of $\Re(\tau)$ of the residue $\tau = -2\pi i \text{Res}(1/v, u_1)$, where $i = \sqrt{-1}$. Bar-Lev et al. (1992) directly computed $\tau$ to determine the sign of $\Re(\tau)$ and only verified the necessity for $n \in \{1, 2, 3\}$. However, when $n$ is large computing $\tau$ becomes intractable since $1/v$ is the reciprocal of a polynomial $v$ of degree $2n + 1$. This is revealed by the already complicated expression for $\tau$ when $n = 3$ in their Theorem 6. So a general, abstract approach is needed to determine the sign of $\Re(\tau)$ in order to prove the necessity for all $n$.

In this note, we prove the necessity for all $n \in \mathbb{N}$ using a different strategy that exploits the algebraic and analytic properties of $v$. The positive answer to the conjecture enlarges the family of polynomials that can be the VFs of NEFs and the variance-mean relationships that can be used for statistical modelling using NEFs.

## 2 Proof of necessity

Let $\mu_0 = \mu(0)$ and $\theta_0 = \lim_{u \to +\infty} \int_0^u d\mu_0/v(t)$. Then $0 < \theta_0 < \infty$ when $(v, \mathbb{R}^+)$ is a VF and $\Theta = (-\infty, \theta_0)$. Let $d = \frac{2\pi}{v'(0)}$, $\Theta + 2^{-1}id = \{z \in \mathbb{C} : z = \theta + 2^{-1}id, \theta \in \Theta\}$ and $S_0^+ = \{z \in \mathbb{C} : \Re(z) \in \Theta \text{ and } 0 < \Im(z) < 2^{-1}d\}$.

Our proof relies on a partial result of Theorem 1 of Bar-Lev et al. (1992) rephrased in our notations as:
Figure 1: Contour $\Gamma$ in the proof of Theorem 1, where the arrow indicates orientation.

**Proposition 1.** Let $v$ be a polynomial of degree greater than 2 such that it has a simple zero at the origin, only one zero with positive imaginary part and its conjugate, and no other zeros. Then, if $(v, R^+)$ is a VF of some NEF, it is necessary that $\theta_0 + \tau \not\in S_0^+$ and $\theta_0 + \tau \not\in \Theta + 2^{-1}id$.

Instead of computing $\tau$ to determine the sign of $\Re(\tau)$, our strategy is to show $\Im(\tau) = 2^{-1}d$ and directly establish the correspondence such that $\Re(\tau) < 0$ whenever $\Re(u_1) > 0$ for all $n \in \mathbb{N}$.

From this, an application of Proposition 1 gives $\Re(\tau) \geq 0$ when $(v, R^+)$ is a VF, which by (2) forces $\Re(u_1) \leq 0$ and yields the necessity. We now proceed to the proof.

**Theorem 1.** The necessity is true for all $n \in \mathbb{N}$.

**Proof.** Without loss of generality, we can assume $a_0 = 1$. Pick $\rho$ and $R$ such that $5^{-1} |u_1| > \rho > 0$ and $R > 2 |u_1|$. Define the contour $\Gamma = \bigcup_{i=1}^4 \Gamma_i$ depicted in Figure 1, where $\Gamma_1 = \{Re^{i\omega} : 0 \leq \omega \leq \pi\}$, $\Gamma_2 = \{pe^{i\omega} : \pi \geq \omega \geq 0\}$, $\Gamma_3 = \{x : -R \leq x \leq -\rho\}$, and $\Gamma_4 = \{x : \rho \leq x \leq R\}$. Then $\tau = -\int_{\Gamma} \frac{d\zeta}{v(\zeta)}$. However, $\int_{\Gamma_2} \frac{d\zeta}{v(\zeta)} = -\frac{i\pi}{v'(0)}$, $\lim_{R \to \infty} \left| \int_{\Gamma_1} \frac{d\zeta}{v(\zeta)} \right| \leq \lim_{R \to \infty} \frac{\pi}{6R^{n+1}} = 0$, and $L_R = \int_{\Gamma_3 \cup \Gamma_4} \frac{d\zeta}{v(\zeta)}$ is real. So, $\Im(\tau) = \frac{\pi}{v'(0)} = 2^{-1}d$ and

$$\Re(\tau) = -\lim_{R \to \infty} L_R = -\lim_{R \to \infty} \int_{\rho}^{R} \alpha(\zeta) \, d\zeta,$$

where

$$\alpha(z) = \frac{1}{z} \left( \frac{1}{|z-u_1|^{2n}} - \frac{1}{|z+u_1|^{2n}} \right) = \frac{\vartheta(-z) - \vartheta(z)}{z \vartheta(-z) \vartheta(z)},$$

and

$$\vartheta(z) = \left( z^2 - 2z \Re(u_1) + |u_1|^2 \right)^n.$$

It suffices now to show that $\Re(u_1) > 0$ is untenable. Given $\Re(u_1) > 0$, then $\alpha(z) > 0$ for
$z \geq \rho$ and $\inf_{z \in [\rho, \rho_1]} \alpha(z) > 0$ for any finite $\rho_1 \geq \rho$. So, $\Re(u_1) > 0$ implies $\Re(\tau) < 0$, i.e., (2) holds. Let $\theta_1 = \theta_0 + \tau$. Then $\theta_1 = \theta_0 + \Re(\tau) + 2^{-1}id$. Since $(v,\mathbb{R}^+)$ is a VF, Proposition 1 forces $\theta_1 \notin \Theta + 2^{-1}id$ and $\Re(\tau) \geq 0$, a contradiction. The proof is complete. 

Acknowledgements

This research is funded by the Office of Naval Research grant N00014-12-1-0764. I am very grateful to the Associate Editor and two anonymous referees for valuable suggestions and comments. I would like to thank John D. Storey, Daoud Bshouty, Gérard Letac and Persi Diaconis for support, comments, discussions and encouragements.

References

Bar-Lev, S. K. (1987). Contribution to discussion of paper by B. Jorgensen: Exponential dispersion models, J. R. Statist. Soc. Ser. B 49: 153–154.

Bar-Lev, S. K., Bshouty, D. and Enis, P. (1991). Variance functions with meromorphic means, Ann. Probab. 19(3): 1349–1366.

Bar-Lev, S. K., Bshouty, D. and Enis, P. (1992). On polynomial variance functions, Probab. Theory Relat. Fields 94(1): 69–82.

Letac, G. (1992). Lectures on natural exponential families and their variance functions, Monografías de matemática, 50, IMPA, Rio de Janeiro.

Letac, G. (2016). Associated natural exponential families and elliptic functions, in M. Podolskij, R. Stelzer, S. Thorbjørnsen and A. E. D. Veraart (eds), The Fascination of Probability, Statistics and their Applications, Springer International Publishing, pp. 53–83.

Letac, G. and Mora, M. (1990). Natural real exponential families with cubic variance functions, Ann. Statist. 18(1): 1–37.

McCullagh, P. and Nelder, J. A. (1989). Generalized linear models, 2nd edn, Chapman and Hall, New York.
Reduction functions for the variance function of one-parameter natural exponential family

Xiongzhı Chen

Abstract

In this note, we consider the problem of constructing an unbiased estimator of the variance of a random variable $\xi$ from a single realization. We show that, when $\xi$ has parametric distributions that form an infinitely divisible (i.d.) natural exponential family (NEF) whose induced measure is absolutely continuous with respect to the basis measure of the NEF, there exists a deterministic function $h$, called “reduction function”, such that $h(\xi)$ is an unbiased estimator of the variance function of $\xi$. In particular, such an $h$ exists when $\xi$ follows an i.d. NEF concentrated on the set of nonnegative integers, a NEF with cubic variance function, or a NEF with power variance function. The utility of the reduction function is illustrated by an application to estimating the latent linear space in high-dimensional data.

Keywords: Natural exponential family, reduction function, variance function.

MSC 2010 subject classifications: Primary 60E07; Secondary 62E10.

1 Introduction

An important task in statistical inference is to construct of a simple, unbiased estimator of the variance of a random variable (rv). Usually such an estimator is obtained from multiple realizations of the rv. In the paper, we explore the possibility of obtaining such an estimator from a transformation of a single realization of the rv. One motivation behind our study is the need to adjust for the variances in order to estimate the latent space that generates the means of the observations; see Theorem 2 of Chen and Storey (2015). From a theoretical perspective, our study is essentially classifying probability measures with special variance-mean relationship (VMR).

Let $E$ and $V$ be the mean and variance operators. Stated formally, we attempt to answer the following the question: for which rv $\xi$ with finite variance does there exist a function $\varphi$ such that $E[\varphi(\xi)] = V[\xi]$? When $\varphi$ exists, it is called a “reduction function (RF)”. It can be perceived that the existence of $\varphi$ depends crucially on the VMR of $\xi$. For example, when $\xi$ is a Poisson rv $\varphi$ exists and is the identity. However, when $\xi$ has a different distribution, $\varphi$ may not exist at all. Unfortunately, a complete answer to the question seems to be out of reach. So, we focus on the case where $\xi$ has a parametric distribution in the form of some natural exponential family.

*Center for Statistics and Machine Learning, and Lewis-Sigler Institute for Integrative Genomics, Princeton University, Princeton, NJ 08544. Email: xiongzh@princeton.edu
(NEF) \( \mathcal{F} \). This allows partial but useful results on the existence of \( \varphi \) due to the richness and wide usage of NEFs in probability and statistics (McCullagh and Nelder, 1989; Letac, 1992).

We show that \( \varphi \) always exists if the NEF \( \mathcal{F} \) is infinitely divisible (i.d.) and an associated convolution of measures is absolutely continuous (a.c.) with respect to (wrt) the basis measure of \( \mathcal{F} \); see Proposition 2. Our results imply that \( \varphi \) exists if \( \xi \) follows a NEF with cubic variance function (NEF-CVF, Letac and Mora, 1990), a NEF with polynomial variance function (NEF-pVF, Bar-Lev et al., 1992), or a NEF with power variance function (NEF-PVF, Bar-Lev and Enis, 1986). These NEFs cover many distributions used in statistical modelling.

In its most general form, the existence of \( \varphi \) for a NEF \( \mathcal{F} \) with basis measure \( \beta \) and maximal open parameter space \( \Theta \) is equivalent to the existence of \( \varphi \) satisfying the integro-differential equation

\[
\int \varphi(x) e^{\theta x} \beta(dx) = \left[ \frac{d^2}{d\theta^2} \log \int e^{\theta x} \beta(dx) \right] \int e^{\theta x} \beta(dx), \forall \theta \in \Theta.
\]

However, this is a very challenging problem whose complete answer is left for future research.

2 Preliminaries on NEFs

We review the definition of a NEF and some properties of its mean and variance functions, which can be found in Letac (1992). Let \( \beta \) be a positive Radon measure on \( \mathbb{R} \) that is not concentrated on one point. Let \( L(\theta) = \int e^{\theta x} \beta(dx) \) for \( \theta \in \mathbb{R} \) be its Laplace transform and \( \Theta \) be the maximal open set containing \( \theta \) such that \( L(\theta) < \infty \). Suppose \( \Theta \) is not empty. Then

\[
\mathcal{F} = \{ F_\theta : F_\theta(dx) = \exp[\theta x - \log L(\theta)] \beta(dx), \theta \in \Theta \}
\]

forms a NEF with respect to the basis \( \beta \). Without loss of generality (WLOG), we can assume \( 0 \in \Theta \), so that \( \beta(\mathbb{R}) = 1 \).

Let \( \kappa(\theta) = \log L(\theta) \) be the cumulant function of \( \beta \). For the NEF \( \mathcal{F} \), the mapping

\[
\mu(\theta) = \int x F_\theta(dx) = \kappa'(\theta)
\]

defines the mean function \( \mu : \Theta \to U \) with \( U = \mu(\Theta) \), and \( U \) is called the "mean domain". Further, the mapping

\[
V(\theta) = \int (x - \mu(\theta))^2 F_\theta(dx) = \kappa''(\theta)
\]

defines the variance function (VF). Let \( \theta = \theta(\mu) \) be the inverse function of \( \mu \). Then \( V \) can be parametrized by \( \mu \) as

\[
V(\mu) = \int (x - \mu)^2 F_{\theta(\mu)}(dx) \text{ for } \mu \in U,
\]

and the pair \((V, U)\) is called the VF of \( \mathcal{F} \).
3 Reduction functions for some infinitely divisible NEFs

Let \( \xi_\theta \) with \( \theta \in \Theta \) have distribution \( F_\theta \in F \), then its mean is \( \mu (\theta) \) and variance \( V (\theta) \) as defined by (1) and (2). We will identify some \( F \) for which there exists a function \( \varphi \) independent of \( \theta \) such that

\[
\mathbb{E} [\varphi (\xi_\theta)] = V (\xi_\theta), \quad \forall \theta \in \Theta.
\]  

(3)

Note that (3) is equivalent to

\[
\int \varphi (x) e^{\theta x} \beta (dx) = \int e^{\theta x} \alpha (dx), \quad \forall \theta \in \Theta.
\]  

(4)

Let \( \delta_y \) be the Dirac mass at \( y \in \mathbb{R} \). We provide simple sufficient conditions on the existence of \( \varphi \).

**Proposition 2.** \( F \) is an i.d. NEF iff \( \kappa'' (\theta) = \int e^{\theta x} \rho (dx) \) where

\[
\rho (dx) = \sigma^2 \delta_0 (dx) + x^2 \nu (dx)
\]  

(5)

and the Lévy triple \( (\epsilon, \sigma^2, \nu) \) is given in (7). Therefore, if \( F \) is i.d., then setting \( \alpha = \beta \ast \rho \) gives the equivalent to (4) as

\[
\int \varphi (x) e^{\theta x} \beta (dx) = \int e^{\theta x} \alpha (dx).
\]  

(6)

Further, if \( \alpha \ll \beta \), then \( \varphi \) exits as the Radon-Nikodym derivative \( \varphi = \frac{d\alpha}{d\beta} \).

**Proof.** Since \( 0 \in \Theta \), then \( \beta \in F \). By Proposition 4.1 of Letac (1992), \( \beta \) is i.d iff \( F \) is. By Theorem 6.2 of Letac (1992), \( \beta \) is i.d. iff there exist some constants \( \epsilon \) and \( \sigma^2 \geq 0 \) and Lévy measure \( \nu \) such that

\[
\kappa (\theta) = \epsilon \theta + 2^{-1} \sigma^2 \theta^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{\theta x} - 1 - \theta \tau (x)) \nu (dx)
\]  

(7)

for a centering function \( \tau : \mathbb{R} \to \mathbb{R} \) such that \( x^{-2} (\tau (x) - x) \) is bounded as \( x \to 0 \). Again, by Theorem 6.2 of Letac (1992), \( \beta \) is i.d. iff \( \kappa'' (\theta) = \int e^{\theta x} \rho (dx) \) with \( \rho \) given by (5). Obviously, \( \int_{|x| \geq 1} x^{-2} \rho (dx) < \infty \) since \( \nu \) is a Lévy measure. Therefore, \( \kappa'' (\theta) g (\theta) = \int e^{\theta x} \alpha (dx) \), and (4) becomes (6). When \( \alpha \ll \beta \), we see that \( \varphi = \frac{d\alpha}{d\beta} \) by Radon-Nikodym theorem (e.g., in Halmos, 1950). This completes the proof.

We call \( \rho \) the measure induced by the i.d. NEF \( F \). By Proposition 2, to find \( \varphi \) we first need to check if a NEF is i.d., and if so, we then check if \( \alpha \ll \beta \). We will focus on i.d. NEFs whose VFs are simple but flexible for statistical applications rather than exhausting all i.d. NEFs. Bar-Lev (1987) and Chen (2015) showed that a NEF-pVF with \( V (u) = \sum_{k=1}^n a_k u^k \) and mean domain \( \mathbb{R}_+ := (0, \infty) \) is i.d. when all \( a_k \)'s are nonnegative. In particular, such NEFs include six
members of the NEF-CVF that are concentrated on the set \( \mathbb{N} \) of nonnegative integers; see Table 2 in Letac and Mora (1990).

We now show the existence and formula for the RF \( \varphi \) for i.d. NEFs that are concentrated on \( \mathbb{N} \).

**Corollary 1.** Suppose \( \mathcal{F} \) is i.d. and concentrated on \( \mathbb{N} \). Then

\[
c_n = \left. \frac{d^n}{dz^n} \kappa'(\log z) \right|_{z=0}
\]

is well defined for \( n \geq 1 \), and the measure \( \alpha \) on \( \mathbb{N} \) with

\[
\alpha(n) = \sum_{k=0}^{n} \beta(n-k) (k+2)(k+1) c_{k+2}
\]

is well-defined. Further, \( \varphi \) exists and \( \varphi(n) = \alpha(n) / \beta(n) \) for \( n \in \mathbb{N} \).

**Proof.** Since \( \beta \) is concentrated on \( \mathbb{N} \), we can write \( \beta(dx) = \sum_{n=0}^{\infty} \beta_n \delta_n(dx) \) with \( \beta_n = \beta(\{n\}) \) for each \( n \in \mathbb{N} \). By the criterion on page 290 of Feller (1971), \( \beta \) is i.d. and concentrated on \( \mathbb{N} \) iff there exists a non-negative sequence \( \{c_n\}_{n \geq 1} \) such that \( \sum_{n=1}^{\infty} c_n z^n \) has radius of convergence \( R \geq 1 \) and that \( \kappa(\theta) = \sum_{n=0}^{\infty} c_n e^{n\theta} \) with \( c_0 = -\sum_{n=1}^{\infty} c_n \) for \( \theta \in \Theta = (-\infty, \log R) \). So, the sequence \( \{c_n\}_{n \geq 1} \) is given by (8) and

\[
\kappa''(\theta) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} e^{n\theta}, \: \forall \theta \in \Theta.
\]

Therefore,

\[
\rho(dx) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} \delta_n(dx)
\]

satisfies \( \kappa''(\theta) = \int e^{\theta x} \rho(dx) \). On the other hand,

\[
L(\theta) = \exp(\kappa(\theta)) = \sum_{n=0}^{\infty} \beta_n e^{n\theta}, \: \forall \theta \in \Theta
\]

and \( \beta_0 > 0 \). The measure \( \alpha \) with \( \alpha_n \) in (9) is well defined. Let \( A = \{n \in \mathbb{N} : c_n > 0\} \) and \( S = \{n \in \mathbb{N} : \beta_n > 0\} \). Then \( \{n \in \mathbb{N} : \alpha_n > 0\} \subseteq S \) and \( \alpha \ll \beta \). Thus, \( \varphi : \mathbb{N} \to \mathbb{R}_+ \) such that \( \varphi(n) = \alpha_n / \beta_n \) is the RF. This completes the proof.

**Corollary 1** provides the generic formula to obtain the RF \( \varphi \) for i.d. NEFs concentrated on \( \mathbb{N} \). However, it is not always easy to obtain the sequences \( \{c_n\}_{n \geq 1} \) and \( \{\beta_n\}_{n \geq 1} \) in the series expansions of \( \kappa(\theta) \) and \( L(\theta) \) for such a NEF in order to get \( \varphi \); see Examples 4 to 6 on NEF-CVFs in Section 3.1. In the next two subsections, we will provide the RF \( \varphi \) for NEF-CVF and NEF-PVF.
\[ \text{Table 1: The reduction function } \varphi \text{ such that } \mathbb{E}[\varphi(\xi)] = \mathbb{V}[\xi] \text{ when the parametric distribution of } \xi \text{ forms } \mathcal{F} = \{ F_\theta : \theta \in \Theta \}, \text{ a NEF-QVF in Morris (1982). Here } \logit(x) = \log(x/(1-x)) \text{ for } x \in (0,1), m \in \mathbb{N} \text{ and it is bigger than } 1 \text{ for Binomial distributions, and } \text{“GHS” stands for “generalized hyperbolic secant distribution”. Note that GHS is called “hyperbolic cosine” by Letac and Mora (1990).}

3.1 RFs for NEFs with cubic variance functions

**Example 1.** NEF-QVF. Recall that a NEF-QVF in Morris (1982) has VF \( V(u) = a_0 + a_1 u + a_2 u^2 \) for some \( a_i \in \mathbb{R} \) for \( i = 0, 1, 2 \). For such a NEF, \( \varphi \) can be obtained by directly solving (4) without using Corollary 1. In fact, (3) holds with

\[ \varphi(t) = (1 + a_2)^{-1} (a_0 + a_1 t + a_2 t^2) \]

when \( a_2 \neq -1 \). Note that \( a_2 = -1 \) corresponds to a Bernoulli distribution for which \( \varphi \) does not exist. Table 1 gives \( \varphi \) for NEF-QVF and is reproduced from Chen and Storey (2015).

For a NEF-CVF with \( \text{deg} V = 3 \), directly solving (4) for \( \varphi \) is no longer preferred. However, when \( \text{deg} V = 3 \), the corresponding NEF is i.d. with mean domain \( U = (0, \infty) \). So, we will use Proposition 2 or Corollary 1 to derive \( \varphi \). The following three NEF-CVFs are concentrated on \( \mathbb{N} \) and generated by analytic functions; see Proposition 4.3 of Letac and Mora (1990).

**Example 2.** Inverse Gaussian. Its VF is \( V(u) = u^3 \) and

\[ \beta(x) = (2\pi)^{-1/2} x^{-3/2} \exp(-1/(2x)) 1_{(0,\infty)}(x) dx. \]

So, \( \kappa(\theta) = -\sqrt{-2\theta} \) with \( \theta < 0 \), and \( \rho(dx) = (2\pi)^{-1/2} x^{1/2} 1_{(0,\infty)}(x) dx. \) Take \( E \in \mathcal{B}(\mathbb{R}_+) \), then \( \beta(E) = 0 \) if the Lebesgue measure of \( E \) is 0, i.e., \( \chi(E) = 0. \) Since \( \chi \) is translation invariant, \( \int_{-\infty}^\infty 1_{E}(x+y) \beta(dx) = 0 \) if \( \beta(E) = 0 \) for any \( y > 0 \). This implies \( \alpha = \beta \ast \rho \) satisfies \( \alpha \ll \beta. \) The RF \( \varphi \) can be found by computing the inverse Laplace transform \( L_\kappa \) of \( L(z) \kappa''(z) \) for \( z \) such that \( \Re(z) = \theta \in \Theta. \) Details are given below. Since

\[ L(z) \kappa''(z) = (-2z)^{-3/2} \exp\left( -\sqrt{-2z} \right), \]

we see

\[ |L(z)\kappa''(z)| \leq (2|z|)^{-3/2} \sim |\Im(z)|^{-3} \quad \text{as } |\Im(z)| \to \infty \]
uniformly for $\theta$ in any compact subset of $(-\infty, 0)$. So, by Theorem 19a of Widder (1946) $\hat{L}_\kappa$ is well-defined. In fact,

$$\hat{L}_\kappa(x) = (2\pi)^{-1} \left( \sqrt{2\pi} \exp \left( -2^{-1}x^{-1} \right) - \pi x^{-1/2} + \pi \Phi \left( 2^{-1/2}x^{-1/2} \right) \right),$$

where $\Phi(x) = 2\pi^{-1/2} \int_{0}^{x} \exp(-t^2) \, dt$. Therefore, $\varphi(x) = \frac{\hat{L}_\kappa(x)}{f(x)} \mathbf{1}_{(0, \infty)}$ is the RF.

**Example 3.** Strict Arcsine. Its VF is $V(u) = u(1 + u^2)$ and $\beta(dx) = \sum_{n=0}^{\infty} p_n(1) \frac{\delta_n(dx)}{n!}$, where

$$p_{2n}(t) = \prod_{k=0}^{n-1} (t^2 + 4k^2) \quad \text{and} \quad p_{2n+1}(t) = t \prod_{k=0}^{n-1} \left( t^2 + (2k + 1)^2 \right). \quad (12)$$

The generating function of $\beta$ is $f(z) = \exp(\arcsin z)$ for $|z| \leq 1$ and $\kappa(\theta) = \arcsin e^\theta$ with $\theta < 0$. By Kokonendji and Seshadri (1994), $\rho(dx) = \sum_{n=1}^{\infty} \delta_{2n+1}(dx)$.

**Example 4.** Ressel family. Its VF is $V(u) = u^2(1 + \mu)$ and $\beta(dx) = \frac{x^e e^{-x}}{\Gamma(x + 2)} \mathbf{1}_{(0, \infty)} dx$. By Lemma 4.1 of Kokonendji and Seshadri (1994), $\rho(dx) = \sum_{n=0}^{\infty} \frac{(n+2)!}{2^{n+1}} \beta^{*n+2}$, where $\beta^{*m}$ means the convolution of $\beta$ by itself $m$ times.

The next three NEF-CVFs with $\deg V = 3$ are concentrated on $\mathbb{N}$ and generated by a composition of two analytic functions; see Theorem 4.5 of Letac and Mora (1990). To compute $\rho$, we reformulate Theorem 4.5 of Letac and Mora (1990) and Lemma 4.2 of Kokonendji and Seshadri (1994) as follows.

**Proposition 3.** Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ with radius of convergence $R(g) \geq 1$ such that $g_n \geq 0$ and $g_0 g_1 \neq 0$. Then there exists some $0 < R(h) < R(g)$ such that $h(w) - wg(h(w)) = 0$ implicitly defines an analytic function $h$ for $|w| < R(h)$. Let

$$\beta_n = \left. \frac{1}{(n + 1)!} \left( \frac{d}{dz} \right)^n (g(z))^n \right|_{z=0}.$$ 

If $\mu_n \geq 0$ for all $n \in \mathbb{N}$, then the measure $\beta(dx) = \sum_{n=0}^{\infty} \beta_n \delta_n(dx)$ is well-defined and

$$g(h(w)) = \frac{h(w)}{w} = \sum_{n=0}^{\infty} \beta_n w^n.$$

Further, setting $w = e^\theta$ gives

$$\kappa''(\theta) = \left( 1 - \frac{h'g}{g} \right)^{-3} \frac{h(gg' + hgg'' - hg'')}{g^2}, \quad (13)$$

where the derivative is taken wrt to $\theta$. 

6
Since $\kappa''(\theta) = \int e^{\theta x} \beta(dx)$, taking $h(e^{\theta})$ in (13) as the argument gives the generating function $f_\rho$ for $\rho$ as

$$f_\rho (z) = H_\rho (h(z)) = \sum_{n=0}^{\infty} w^n \rho \{n\}$$

and

$$H_\rho (x) = x \left(1 - \frac{xg'(x)}{g(x)}\right)^{-3} \left(\frac{g'(x)}{g(x)} + \frac{xg''(x)}{g(x)} - x \left(\frac{g'(x)}{g(x)}\right)^2\right).$$ (14)

Note that (14) is also given in Kokonendji and Seshadri (1994). However, computing $f_\rho^{(n)} (0)$ from (14) in order to get $\rho \{n\}$ can be quite tedious.

Proposition 3 provides $\rho$ and hence $\varphi$ for the following:

**Example 5.** Abel family. Its VF is $V(u) = u(1+u)^2$ and $\beta(dx) = \sum_{n=0}^{\infty} \beta_n \delta_n(dx)$ with $\beta_n = (1+n)^{n-1}/n!$. $\beta$ is induced by $g(h) = e^{h}$ and $h(w) = \sum_{n=0}^{\infty} \beta_n w^{n+1}$.

**Example 6.** Takács family. Its VF is $V(u) = u(1+u)(1+2u)$ and $\beta(dx) = \sum_{n=0}^{\infty} \beta_n \delta_n(dx)$ with $\beta_n = \frac{(2n)!}{n!(n+1)!}$. $\beta$ is induced by $g(h) = (1-h)^{-1}$ with $h(w) = \sum_{n=0}^{\infty} \beta_n w^{n+1}$.

**Example 7.** Large Arcsine. Its VF is $V(u) = u(1+2u+2u^2)$ and $\beta(dx) = \sum_{n=0}^{\infty} \beta_n \delta_n(dx)$ with $\beta_n = \frac{p_n(1+n)}{(n+1)!}$, where $p_n(t)$ is defined in (12). $\beta$ is induced by $g(h) = \exp(\arcsin h)$ with $h(w) = \sum_{n=0}^{\infty} \beta_n w^{n+1}$.

### 3.2 RFs for NEFs with power variance functions

Let $D = \{0\} \cup [1, \infty)$. Bar-Lev and Enis (1986) showed that for a $V(u) = au^r$ for $a > 0$ and $r \in \mathbb{R}$ to be a VF of some NEF, it is necessary that $r \in D$. They further showed that NEF-PVF are i.d. It should be pointed out that NEFs with PVFs include the Tweedie family of Tweedie (1984). It is easy to see that $U = \mathbb{R}$ iff $r = 0$ and that $U = \mathbb{R}_+$ when $r \in D \setminus \{0\}$. For more information on NEF-PVF, we refer the reader to Bar-Lev and Enis (1986). Since NEF-PVF with $r \in \mathbb{N}$ corresponds to a NEF-pVF discussed in Corollary 1 and Section 3.1, we will only show the existence of $\varphi$ when $r \in D \setminus \mathbb{N}$.

**Corollary 2.** Suppose $\mathcal{F}$ is a NEF-PVF with VF $V(u) = au^r$ for $r \in D \setminus \mathbb{N}$. Then $\alpha \ll \beta$ and $\varphi = \frac{da}{d\theta}$.

**Proof.** WLOG, assume $a = 1$. Then $\theta = (1-r)^{-1} u^{1-r}$, $\theta < 0$ and $u = ((1-r) \theta)^{\frac{1}{1-r}}$. Let $\gamma = \frac{2-r}{1-r}$. Set $C_{0,r} = (2-r)^{-1} (1-r)^{\gamma}$ and $C_{2,r} = (1-r)^{\gamma-2}$.

**Case 1:** $r \in (1,2)$, i.e., $\gamma < 0$. Then $\kappa(\theta) = (2-r)^{-1} u^{2-r} = C_{0,r} \theta^\gamma$ and $\kappa''(\theta) = C_{2,r} \theta^{\gamma-2}$. So,

$$L(\theta) = \exp(C_{0,r} \theta^\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{0,r}^n \theta^{n\gamma}$$
and

\[ L (\theta) \kappa'' (\theta) = C_{2,r} \theta^{r-2} \sum_{n=0}^{\infty} \frac{\theta^{n+2}}{n!} C_{0,r}^n \sum_{n=0}^{\infty} \frac{1}{n!} C_{2,r}^n \theta^{n+2} \kappa'' \theta^{n+2} \]  

Since \( \Gamma (t) = \int_0^\infty e^{-x} x^{t-1} dx \) for \( \Re (t) > 0 \), we see \((-\theta)^{-t} = \frac{1}{\Gamma (t)} \int_0^\infty x^{t-1} e^{\theta x} dx \). Therefore, setting

\[ q (x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n! \Gamma (-n+1)} \]

and

\[ \beta (dx) = \delta_0 (dx) + q (x) 1_{(0,\infty)} (x) dx \]

gives \( L (\theta) \kappa'' (\theta) = \int_0^\infty e^{\theta x} \beta (dx) \). Similarly, setting \( \alpha (dx) = \tilde{q} (x) dx \) with

\[ \tilde{q} (x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n! \Gamma (-n+1)} \]

gives \( L (\theta) \kappa'' (\theta) = \int_0^\infty e^{\theta x} \alpha (dx) \). Further, \( \alpha = \beta \ast \rho \) satisfies \( \alpha \ll \beta \) and \( \varphi = \frac{d\alpha}{d\beta} \).

It is easily seen that when \( \gamma = 1 \), i.e., \( r = 3/2 \),

\[ \frac{d^3}{dx^3} \frac{x^{-n+1}}{\Gamma (-n+1)} = \frac{x^{-n+1}}{\Gamma (-n+1)} \]

which implies \( q (x) = (-1)^{\gamma-2} C_{2,r} \tilde{q} (x) \) and \( \alpha \ll \beta \). So, \( h = \frac{d\alpha}{d\beta} \) when \( r = 3/2 \). Explicitly, \( C_{0,3/2} = -4, C_{2,3/2} = -8 \).

\[ q (x) = \delta_0 (dx) + \sum_{n=1}^{\infty} \frac{4^n x^{n+1}}{n! (n+1)!} \]

and \( \alpha (dx) = \sum_{n=0}^{\infty} \frac{4^n x^{n+2}}{(n+2)!} 1_{(0,\infty)} (x) \). Clearly, \( \alpha \ll \beta \), we have \( \varphi = \frac{d\alpha}{d\beta} \).

**Case 2:** \( r > 2 \), i.e., \( 0 < \gamma < 1 \). By the proof of Theorem 4.1 of Bar-Lev and Enis (1986), we can do the following. Let \( G_{\gamma} \) be a stable distribution with characteristic component \( \gamma \) and let \( \tilde{G}_\gamma (x) = G_{\gamma} (x/a) \) for \( x \in (0,\infty) \), where \( a = \gamma^{-1/\gamma} (1-\gamma)^{1/\gamma-1} \). Then \( \tilde{G}_\gamma \) has Laplace transform \( L (\theta) = \exp (-\langle \hat{a} \theta \rangle^\gamma) \) for \( \theta \leq 0 \), \( \kappa (\theta) = C_{0,r} (-\theta)^\gamma \) and \( \kappa'' (\theta) = C_{2,r} (-\theta)^{-2} \). In fact, we can set

\[ q (x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sin (n \pi \gamma) \Gamma (1+\gamma n)}{n!} \]

and \( \beta (dx) = \tilde{G}_\gamma (dx) = (1/a) q (x/a) dx \); see Section 5 of Bar-Lev and Enis (1986). Clearly,

\[ \rho (dx) = C_{2,r} \frac{x^{1-\gamma}}{\Gamma (2-\gamma)} \]

and \( \kappa'' (\theta) = \int e^{\theta x} \rho (dx) \). So, \( \alpha = \beta \ast \rho \) satisfies \( \alpha \ll \beta \) and \( \varphi = \frac{d\alpha}{d\beta} \).
4 An application to estimating latent linear space

We provide an application of the RF $\varphi$ to estimating the latent space in the means of high-dimensional data. For $k \gg n$ the data matrix $Y = (y_{ij}) \in \mathbb{R}^{k \times n}$ contains $n$ independent measurements of $k$ objects under study. Suppose the means $\mathbb{E}[y_{ij}]$ are subject to a latent structure such that

$$\mathbb{E}[Y | M] = (\mathbb{E}[y_{ij} | M]) = \Phi M$$

for an unknown $M \in \mathbb{R}^{r \times n}$ of rank $1 \leq r \leq n$ and unknown $\Phi \in \mathbb{R}^{k \times r}$ of rank $r$. In (15), $M$ can represent a design matrix or $n$ realizations of $r$ factors or $r$ latent variables. Then all $\mathbb{E}[y_{ij}]$ lie in the row space $S_M$ of $M$, and capturing $S_M$ is essential to understanding the systematic variations in the data $Y$. By Theorem 2 of Chen and Storey (2015), $S_M$ can be consistently estimated by the space spanned by the $r$ leading eigenvectors of

$$G_k = k^{-1} Y^T Y - D_k$$

as $k \to \infty$ with $n$ fixed, where $D_k = \text{diag}\{\sigma_{i,k}, \ldots, \sigma_{n,k}\}$ and $\sigma_{i,k} = k^{-1} \sum_{j=1}^{k} \mathbb{V}[y_{ij}]$. Note that $\sigma_{i,k}$ is the average of the variances of $y_{ij}$’s in the $i$th column of $Y$ and that $S_M$ cannot be consistently estimated from $k^{-1} Y^T Y$ when $D_k$ is not a multiple of the identity matrix $I_n$. Therefore, we need to consistently estimate $D_k$, i.e., to estimate $\sigma_{i,k}$ for $1 \leq i \leq n$.

If $y_{ij}$ follows a parametric distribution $F_{\theta_{ij}}$ for $\theta_{ij} \in \Theta$ and $\mathcal{F} = \{F_{\theta}: \theta \in \Theta\}$ forms an i.d. NEF with RF $\varphi$, then

$$\mathbb{E}[\varphi(y_{ij})] = \mathbb{V}[y_{ij}]$$

and $\hat{\sigma}_{i,k} = k^{-1} \sum_{j=1}^{k} \varphi(y_{ij})$ satisfies $\mathbb{E}[\hat{\sigma}_{i,k}] = \sigma_{i,k}$. In other words, the RF $\varphi$ induces an unbiased estimator $\hat{\sigma}_{i,k}$ of $\sigma_{i,k}$, for which $\hat{D}_k = \text{diag}\{\hat{\sigma}_1,k, \ldots, \hat{\sigma}_n,k\}$ is an unbiased estimator of $D_k$. Under a moment condition, it can be shown that $\lim_{k \to \infty} |\hat{\sigma}_{i,k} - \sigma_{i,k}| = 0$ for $1 \leq i \leq n$ and $\lim_{k \to \infty} \|\hat{D}_k - D_k\| = 0$ hold almost surely; see Lemma 8 in Chen and Storey (2015). In view of this, $\varphi$ induces an unbiased estimator of $D_k$ and enables consistent estimation of the latent space $S_M$ when $y_{ij}$’s follow a NEF we have identified.

Acknowledgements

This research is funded by the Office of Naval Research grant N00014-12-1-0764. I would like to thank John D. Storey for support, Gérard Letac for guidance that has led to much improved results, Peter Jones for a discussion on the maximal zero free domain of a Laplace transform, and Wayne Smith for a discussion on the inverse of analytic functions.
References

Bar-Lev, S. K. (1987). Contribution to discussion of paper by B. Jorgensen: Exponential dispersion models. *J. R. Statist. Soc. Ser. B* 49, 153–154.

Bar-Lev, S. K., D. Bshouty, and P. Enis (1992). On polynomial variance functions. *Probab. Theory Relat. Fields* 94(1), 69–82.

Bar-Lev, S. K. and P. Enis (1986). Reproducibility and natural exponential families with power variance functions. *Ann. Statist.* 14(4), 1507–1522.

Chen, X. (2015). Natural exponential families with reduction functions and resolution of a conjecture. [http://arxiv.org/abs/1510.03966](http://arxiv.org/abs/1510.03966).

Chen, X. and J. D. Storey (2015). Consistent estimation of low-dimensional latent structure in high-dimensional data. [http://arxiv.org/abs/1510.03497](http://arxiv.org/abs/1510.03497).

Feller, W. (1971). *An Introduction to Probability Theory and its Applications*, Volume I. Wiley, New York, NY.

Halmos, P. R. (1950). *Measure Theory*. Springer Verlag.

Kokonendji, C. C. and V. Seshadri (1994). The lindsay transform of natural exponential families. *Canad. J. Statist.* 22(2), 259–272.

Letac, G. (1992). *Lectures on natural exponential families and their variance functions*. Monografías de matemática, 50, IMPA, Rio de Janeiro.

Letac, G. and M. Mora (1990). Natural real exponential families with cubic variance functions. *Ann. Statist.* 18(1), 1–37.

McCullagh, P. and J. A. Nelder (1989). *Generalized linear models* (2nd ed.). New York: Chapman and Hall.

Morris, C. N. (1982). Natural exponential families with quadratic variance functions. *Ann. Statist.* 10(1), 65–80.

Tweedie, M. C. K. (1984). An index which distinguishes between some important exponential families. In *Statistics: Applications and New Directions*, Calcutta: Indian Statistical Institute, pp. 579–604.

Widder, D. V. (1946). *The Laplace Transform*. Princeton University Press.