Infinite Energy Dyon Solutions

D. Singleton

Department of Physics, Virginia Commonwealth University, 1020 West Main St., Box 842000, Richmond, VA 23284-2000

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Abstract

Three dyon solutions to the SU(2) Yang-Mills-Higgs system are presented. These solutions are obtained from the BPS dyon solutions by allowing the gauge fields to be complex or by letting the free parameter of the BPS solution become imaginary. In all cases however the physically measurable quantities connected with these new solutions are entirely real. Although the new solutions are mathematically simple variations of the BPS solution, they have one or more physically distinct characteristics.

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I. THE DYON SOLUTIONS

The system studied in this paper is an SU(2) gauge theory coupled to a scalar field in the triplet representation. The scalar field is taken to have no mass or self interaction. The Lagrangian for this system is

\[ \mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + \frac{1}{2} (D_\mu \phi_a) (D^\mu \phi^a) \]  

(1)

where \( G^a_{\mu\nu} \) is the field tensor of the SU(2) gauge fields \( (W^a_\mu) \) and \( D_\mu \) is the covariant derivative of the scalar field. The equations of motion for this system are simplified through the use of a generalized Wu-Yang ansatz [1] which was used by Witten [2] to study multi-instanton solutions

\[
W^a_i = \epsilon_{aij} \frac{r_j}{gr^2} [1 - K(r)] + \left( \frac{r_i r_a}{r^2} - \delta_{ia} \right) \frac{G(r)}{gr} \\
W^a_0 = \frac{r_a}{gr^2} J(r) \\
\phi^a = \frac{r_a}{gr^2} H(r)
\]  

(2)

where \( K(r), G(r), J(r), \) and \( H(r) \) are the ansatz functions to be determined by the equations of motion. In terms of this ansatz the field equations from the Lagrangian in Eq. (1) reduce to the following set of coupled, non-linear equations.

\[
r^2 K'' = K(K^2 + G^2 + H^2 - J^2 - 1) \\
r^2 G'' = G(K^2 + G^2 + H^2 - J^2 - 1) \\
r^2 J'' = 2J(K^2 + G^2) \\
r^2 H'' = 2H(K^2 + G^2)
\]  

(3)

where the primes denote differentiation with respect to \( r \). The solution to these equations discovered by Prasad and Sommerfield [3] and independently by Bogomolnyi [4] is

\[
K(r) = \cos(\theta) Cr \text{csch}(Cr) \\
G(r) = \sin(\theta) Cr \text{csch}(Cr)
\]
\[ J(r) = \sinh(\gamma)[1 - Cr \coth(Cr)] \]
\[ H(r) = \cosh(\gamma)[1 - Cr \coth(Cr)] \] (4)

where \( C, \theta \) and \( \gamma \) are arbitrary constants. One of the nice properties of this solution is that the energy in its fields is finite (as compared to the field energy of a classical point charged particle, which has a divergent energy from the singularity at the origin). In terms of the ansatz functions one can use standard Lagrangian techniques to obtain the energy density of the fields
\[
T^{00} = \frac{1}{g^2} \left( K'^2 + G'^2 + \frac{(K^2 + G^2 - 1)^2}{2r^2} + \frac{J^2(K^2 + G^2)}{r^2} + \frac{(rJ' - J)^2}{2r^2} + \frac{H^2(K^2 + G^2)}{r^2} + \frac{(rH' - H)^2}{2r^2} \right) \]
(5)

For the solution in Eq. (4) this gives
\[
T^{00} = \frac{2cosh^2(\gamma)}{g^2} \left[ C^2 \text{csch}^2(Cr) \left( 1 - Cr \coth(Cr) \right) \right] \]
(6)

This energy density can be integrated over all space to yield the total field energy of the BPS solution
\[
E = \int T^{00} d^3x = \frac{4\pi C}{g^2} \cosh^2(\gamma) \]
(7)

To investigate the electromagnetic properties of this solution ’t Hooft defined a generalized, gauge invariant, electromagnetic field strength tensor \[ F_{\mu\nu} = \partial_{\mu}(\hat{\phi}^a W_\nu^a) - \partial_{\nu}(\hat{\phi}^a W_\mu^a) - \frac{1}{g} \epsilon^{abc} \hat{\phi}^a(\partial_{\mu}\hat{\phi}^b)(\partial_{\nu}\hat{\phi}^c) \]
(8)

where \( \hat{\phi}^a = \phi^a(\phi^b\phi^b)^{-1/2} \). This generalized U(1) field strength tensor reduces to the usual expression for the field strength tensor if one performs a gauge transformation to the “Abelian” gauge where the scalar field only points in one direction in isospin space (i.e. \( \phi^a = \delta^{3a}v \)) \[ \footnote{ } \]

Thus the electric and magnetic fields of the BPS solution become
\[
E_i = F_{i0} = \frac{r_i}{gr} \frac{d}{dr} \left( \frac{J(r)}{r} \right) = \frac{\sinh(\gamma)r_i}{gr^3} \left( C^2 r^2 \text{csch}^2(Cr) - 1 \right) \]
\[
B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{r_i}{gr^3} \]
(9)
The magnetic field is that due to a point monopole of strength $-4\pi/g$ located at the origin. The electric field is that due to some extended charge configuration of total charge $Q = -4\pi\sinh(\gamma)/g$. The charge density which gives rise to the electric field can be calculated using

$$\rho(r) = \nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \mathcal{E}_r) = \frac{2C^2 \sinh(\gamma) \text{csch}^2(Cr)}{gr} (1 - Cr \coth(Cr))$$

(10)

Since the BPS solution has finite field energy, this has led to its interpretation as a magnetically and electrically charged particle, whose mass is given by Eq. (7). The ansatz functions $K(r)$, $G(r)$, $J(r)$, and $H(r)$ (and therefore the gauge and scalar fields) are real. If complex gauge fields and/or infinite energy configurations are allowed there are several more solutions which can be found for Eq. (3). First by looking at the complementary hyperbolic functions one finds the following complex solution

$$K(r) = i\cos(\theta)Cr \text{ sech}(Cr)$$

$$G(r) = i\sin(\theta)Cr \text{ sech}(Cr)$$

$$J(r) = \sinh(\gamma)[1 - Cr \tanh(Cr)]$$

$$H(r) = \cosh(\gamma)[1 - Cr \tanh(Cr)]$$

(11)

Since the ansatz functions $K(r)$ and $G(r)$ are now imaginary, the space components of the gauge fields will be complex. Despite this one finds that all the above listed quantities (i.e the field energy density, the electric field, the magnetic field, and the charge density) associated with this complex solution are completely real. Although this solution was found by making some simple changes we will see that it has some physical features which are distinct from the BPS solution (e.g. the electric charge density and the energy density). Inserting the ansatz functions of Eq. (11) into Eq. (5) we find that the field energy density is

$$T^{00} = \frac{2\cosh^2(\gamma)}{g^2} \left[ -C^2 \text{sech}^2(Cr) \left(1 - Cr \tanh(Cr)\right)^2 + \frac{(C^2 r^2 \text{sech}^2(Cr) + 1)^2}{2r^2} \right]$$

(12)
The energy density is real even though the space components of the gauge fields are complex. However, the total energy in this field configuration is infinite due to the singularity in the energy density at \( r = 0 \). Thus the above solution is more like a Wu-Yang monopole or a charged point particle, as opposed to a BPS dyon which has non-singular fields and finite energy. Using Eq. (9) we find that the electric and magnetic fields associated with the solution in Eq. (11) are

\[
\mathcal{E}_i = \frac{-\sinh(\gamma) r_i}{gr^3} (C^2 r^2 \text{sech}^2(Cr) + 1) \\
\mathcal{B}_i = -\frac{r_i}{gr^3}
\]

(13)

The complex solution has the same \(-4\pi/g\) magnetic charge as the BPS solution. The electric field is that of some finite distribution of electric charge of total charge \(-4\pi \sinh \gamma/g\). This is the same as the total electric charge carried by the BPS solution. By using Eq. (10) we find that the electric charge density for the complex solution is given by

\[
\rho(r) = \frac{-2C^2 \sinh(\gamma) \text{sech}^2(Cr)}{gr} \left(1 - Cr \tanh(Cr)\right)
\]

(14)

This charge density is real, has a singularity at the origin, and falls off exponentially for large \( r \). Even though the space components of the gauge fields are complex all the physical quantities calculated from it are real. The main difference between this solution and the BPS solution is the infinite field energy of the complex solution.

To obtain the next solution we apply the transformation, \( C \rightarrow iC \) to the BPS solution of Eq. (4). This changes the hyperbolic functions into their trigonometric counterparts, and yields the following solution to Eq. (3)

\[
K(r) = \cos(\theta)Cr \csc(Cr) \\
G(r) = \sin(\theta)Cr \csc(Cr) \\
J(r) = \sinh(\gamma)[1 - Cr \cot(Cr)] \\
H(r) = \cosh(\gamma)[1 - Cr \cot(Cr)]
\]

(15)

This solution is completely real as opposed to the hyperbolic solution in Eq. (11). Even though this solution was obtained from the original BPS solution via a trivial transformation,
it has very different physical properties. Most obviously the ansatz functions, and therefore the gauge and scalar fields, become singular when $Cr = n\pi$ where $n = 1, 2, 3, 4...$. Thus this solution exhibits a series of concentric spherical surfaces on which its fields become singular. These singularities also show up in the energy density of this solution. Inserting the ansatz functions of Eq. (13) in Eq. (5) we find that the energy density of this solution is

$$T^{00} = \frac{2\cosh^2(\gamma)}{g^2} \left[ C^2 \csc^2(Cr) \left( 1 - Cr \cot(Cr) \right)^2 + \frac{\left( C^2 r^2 \csc^2(Cr) - 1 \right)^2}{2r^2} \right]$$

The energy density becomes singular on the same spherical surfaces as the gauge and scalar fields. These spherical shells on which the energy density becomes infinite cause the total field energy of this solution $(E = \int T^{00}d^3x)$ to diverge. The electric and magnetic fields of this solution are obtained using Eq.(11)

$$\mathcal{E}_i = \frac{-\sinh(\gamma) r_i}{gr^3}(C^2 r^2 \csc^2(Cr) - 1)$$
$$\mathcal{B}_i = -\frac{r_i}{gr^3}$$

The magnetic field is exactly the same as that of the BPS solution or the solution of Eq. (11). The electric field is very unusual for this solution. First, because of the $C^2 r^2 \csc^2(Cr)$ term, the electric field does not fall off for large $r$, but has a periodic behaviour. Second, the electric field becomes singular on the spherical shells given by $Cr = n\pi$. One could take the electric charge of this solution as located on these singular surfaces. Finally, the electric charge of this solution is infinite, as is indicated by the electric field or by directly looking at the charge density

$$\rho(r) = \frac{2C^2 \sinh(\gamma) \csc^2(Cr)}{gr} \left( 1 - Cr \cot(Cr) \right)$$

The infinite electric charge of this solution is its worst feature. However, for the special case of this solution where $\gamma = 0$, one finds that the solution carries no electric charge, but only a magnetic charge. Even in this case the energy density still becomes singular on the concentric spherical surfaces. Though this solution was obtained from the original
BPS solution by a trivial transformation, its physical characteristics are different. Both the BPS solution and the solution from Eq. (11) had finite magnetic and electric charges for any finite value of $\gamma$. Thus both solutions could be viewed as dyonic particles (The BPS solution had the additional nice feature that its total field energy was finite.). The solution given in Eq. (13), while having the same magnetic charge as the other two solutions, has an infinite electric charge in the general case when $\gamma \neq 0$. Although this solution is a dyon in the sense that it carries both magnetic and electric charge it is probably not correct to view it as a particle-like solution. At this point it is unclear how one should view this solution. The spherical singular surfaces of this solution are similar to that of the Schwarzschild-like solution presented in Ref. [7]. However, the solution in Ref. [7] only possessed one spherical surface on which the fields and energy density diverged, and it carried no net electric charge. When the solution of Ref. [7] was treated as a background field in which a test particle was placed it was found that the spherical singularity acted as an impenetrable barrier which would trap the test particle either in the interior or the exterior of the sphere [8]. Similar results have been found for other singular solutions [9] [10]. Treating the present solution as a background field would trap the test particle between any two of the concentric spherical singularities.

Finally one can obtain a third solution to Eq. (3) by applying the transformation $C \rightarrow iC$ to the complex solution in Eq. (11). This yields

$$K(r) = -\cos(\theta)Cr \sec(Cr)$$
$$G(r) = -\sin(\theta)Cr \sec(Cr)$$
$$J(r) = \sinh(\gamma)[1 + Cr \tan(Cr)]$$
$$H(r) = \cosh(\gamma)[1 + Cr \tan(Cr)] \quad (19)$$

This solution is basically the same as the one obtained by taking $C \rightarrow iC$ for the BPS solution. Most of the comments concerning the solution in Eq. (15) apply here as well. The singularities in the fields and energy density are now located at $r = 0$ and on the spherical surfaces $Cr = n\pi/2$ where $n = 1, 3, 5, 7,...$. The solution also has infinite total field energy,
and infinite electric charge, unless $\gamma = 0$. As with all the other solutions it possesses a magnetic charge of $-4\pi/g$.

Many of the physical characteristics of the solutions were substantially different in each case. However the magnetic field and magnetic charge of all the solutions were the same. This comes about since the magnetic charge of each solution is a topological charge which carries the same value for each field configuration. To investigate this we look at the topological current, $k_\mu$, of Ref. \[6\]

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\rho\sigma\tau} \epsilon_{abc} \partial^\rho \hat{\phi}^a \partial^\sigma \hat{\phi}^b \partial^\tau \hat{\phi}^c$$

(20)

The topological charge of this field configuration is then

$$q = \int k_0 d^3x = \frac{1}{8\pi} \int (\epsilon_{ijk} \epsilon_{abc} \partial^i \hat{\phi}^a \partial^j \hat{\phi}^b \partial^k \hat{\phi}^c) d^3x$$

(21)

For all the solutions one finds that $\hat{\phi}^a = r^a/r$ which is the same regardless of the ansatz function $H(r)$. In all cases we find that the topological charge is $q = 1$. Since the magnetic charge is identified with this topological charge via ‘t Hooft’s generalized electromagnetic tensor, the magnetic charge of each solution is the same.

II. DISCUSSION AND CONCLUSION

In this paper we have presented three new, exact solutions for the SU(2) Yang-Mills-Higgs system, where the scalar field has vanishing mass and self coupling. All three solutions were connected to the well known BPS solution either by allowing the gauge fields to be complex or by allowing the free parameter in the BPS solution to be imaginary. In the first solution given in Eq. \[11\] we replaced the hyperbolic functions of the BPS solution with their complements and let the ansatz functions, $K(r)$ and $G(r)$, be imaginary, thus making the space components of the gauge fields complex. Despite having complex gauge fields, this solution gave real results for all the physical quantities calculated from it (i.e. the field
energy density, the electric and magentic fields, and the topological index of the solution were all real). The magnetic and electric fields of this solution indicated that it was a dyon carrying a magnetic charge of $-4\pi/g$ and an electric charge of $-4\pi \sinh(\gamma)/g$, with the magnetic charge being concentrated at the origin while the electric charge was spread out in an exponentially decaying distribution around the origin. The only drawback of this solution is that, unlike the BPS solution, it has a divergent field energy due to the singularity in the field energy density at the origin. Thus, although this solution looks mathematically similar to the BPS solution, physically it may be more correct to think of it as a dyonic version of the Wu-Yang monopole, which also has divergent field energy due to a singularity at $r = 0$.

The two other new solutions presented here, were derived from the BPS solution and the complementary solution of Eq. (11) by allowing the free parameter $C$ of these solutions to become imaginary (i.e. $C \to iC$). This resulted in solutions with completely real gauge and scalar fields, and with the hyperbolic functions replaced by their trigonometric counterparts. In both cases the solutions carried the same magnetic charge as the previous hyperbolic solutions. Now however the gauge and scalar fields developed singularities on an infinite series of concentric spherical shells (the solution in Eq. (19) also had a singularity at $r = 0$). In addition, unless $\gamma = 0$, both solutions carry an infinite electric charge, which can be thought of as concentrated on the spherical shells. Thus, although in the general case these solutions do carry both magnetic and electric charge, it is difficult to see how they could be viewed as particle-like, and the name dyon is probably inappropriate (even in the case of the complex solution of Eq. (11), which had infinite field energy, the fact that it had localized charges, and a localized divergent energy density makes a particle-like interpretation reasonable). The spherically, singular surfaces of the trigonometric solutions are somewhat similar to the spherically singular surface of the Schwarzchild-like solution of Ref. [7]. The Schwarzchild-like solution, however, had only one spherically singular surface rather than an infinite series of concentric surfaces, and it carried no electric charge rather than the infinite electric charge carried by the trigonometric solutions in the general $\gamma \neq 0$ case. At present it is not clear what interpretation can be given to these trigonometric
solutions or what physical role if any they may play. One could argue that the singularities in the trigonometric solutions and to a lesser extent the singularity in the solution of Eq. (11) might indicate that they are not physically interesting. However this is not necessarily the case as can be seen by the example of the meron solution which is singular, and yet is thought to play a role in some non-Abelian gauge theories. One might consider using the trigonometric solutions to solve the field equations in some finite region around the origin, and patching them together with one of the other solutions (either the BPS solution or the Schwarzschild-like solution) which are better behaved as $r \to \infty$. This would put some conditions on the fields and their derivatives at the point where the solutions are joined, thus possibly fixing some of the arbitrary constants ($e.g. C$) of the solutions. In one sense the trigonometric solutions, despite their singularities, are interesting solutions since they have entirely different physical characteristics from any previously given solutions to the SU(2) Yang-Mills-Higgs system.

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