Electrically gauged $\mathcal{N} = 4$ supergravities in $D = 4$ with $\mathcal{N} = 2$ vacua

Christoph Horst$^{a,b}$, Jan Louis$^{a,b}$ and Paul Smyth$^c$

$^a$II. Institut für Theoretische Physik der Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany

$^b$Zentrum für Mathematische Physik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg

$^c$Institut de Théorie des Phénomènes Physiques, EPFL, CH-1015 Lausanne, Switzerland

christoph.horst@desy.de, jan.louis@desy.de
paul.smyth@epfl.ch

ABSTRACT

We study $\mathcal{N} = 2$ vacua in spontaneously broken $\mathcal{N} = 4$ electrically gauged supergravities in four space-time dimensions. We argue that the classification of all such solutions amounts to solving a system of purely algebraic equations. We then explicitly construct a special class of consistent $\mathcal{N} = 2$ solutions and study their properties. In particular we find that the spectrum assembles in $\mathcal{N} = 2$ massless or BPS supermultiplets. We show that (modulo $U(1)$ factors) arbitrary unbroken gauge groups can be realized provided that the number of $\mathcal{N} = 4$ vector multiplets is large enough. Below the scale of partial supersymmetry breaking we calculate the relevant terms of the low-energy effective action and argue that the special Kähler manifold for vector multiplets is completely determined, up to its dimension, and lies in the unique series of special Kähler product manifolds.

December 2012
1 Introduction

The issue of spontaneous partial breaking in theories with extended supersymmetry has long been studied [1–3]. The case of spontaneous $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking in Minkowski vacua is of particular interest due to its phenomenological relevance and the early no-go theorems of [1,2]. In $\mathcal{N} = 2$ globally supersymmetric theories the no-go theorems could be evaded in the presence of electric and magnetic Fayet-Iliopoulos terms that are not aligned [4,5]. In supergravity the no-go theorem was circumvented in simple examples by formulating the problem in a symplectic frame in which no prepotential exists for the special geometry of the vector multiplets [6–8]. Recently, a systematic analysis in $\mathcal{N} = 2$ supergravity with general matter content was carried out [9–11] using the embedding tensor formalism [12].

Spontaneous partial supersymmetry breaking in $\mathcal{N} = 4$ gauged supergravity has been much less studied since the original examples were found [13,14]. Motivated by the fact that the original $\mathcal{N} = 4$ supergravities did not have vacua with non-zero cosmological constant $\Lambda$, more general deformations were introduced via a set of $SU(1,1)$ phases associated to the angles between the semi-simple factors of the gauge group [15,16], now known as de Roo-Wagemans angles. In the embedding tensor language, non-trivial de Roo-Wagemans angles correspond to particular non-vanishing embedding tensor components which imply the simultaneous appearance of electric and magnetic gaugings [17,18]. These additional deformations were seen to allow for vacua with non-zero $\Lambda$ which can spontaneously break supersymmetry to all $\mathcal{N} < 4$ [13]. The problem of partially breaking $\mathcal{N} = 4$ supersymmetry in Minkowski vacua was then studied [13], where it was found that one could break to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry, but not $\mathcal{N} = 3$.

More recently, examples of vacua with supersymmetry spontaneously broken to $\mathcal{N} < 4$ have been found and their relation to string theory compactifications have been studied in some detail (see, for example, [19–23] and references therein), but a systematic analysis of the problem has yet to be carried out. The purpose of this paper is to initiate such an analysis in $\mathcal{N} = 4$ gauged supergravity by solving the supersymmetry conditions for the charges and gaugings that allow for a specified amount of preserved supersymmetry. As a first step, we shall focus on the specific case of spontaneous $\mathcal{N} = 4 \to \mathcal{N} = 2$ breaking with only electric gaugings.

In ungauged $\mathcal{N} = 4$ supergravity with $n$ Abelian vector multiplets the scalar field space is fixed to be the homogeneous space [16,24,25]

$$M = \frac{SL(2)/SO(2) \times SO(6,n)/SO(6)\times SO(n)}{O},$$

(1.1)

where the first factor is spanned by the two scalars in the $\mathcal{N} = 4$ gravitational multiplet (the dilaton and axion), while the second factor is spanned by the $6n$ scalars of the vector multiplets. No scalar potential is allowed and thus all values of the scalar fields correspond to degenerate $\mathcal{N} = 4$ backgrounds.

This situation changes if one considers gauged $\mathcal{N} = 4$ supergravities [16,17]. For simplicity, we confine our interest to $\mathcal{N} = 4$ supergravity coupled to $n$ vector multiplets transforming in the adjoint representation of an electric gauge group $G_{\mathcal{N}=4}$. This induces additional couplings and, in particular, a scalar potential $V$ which is characterized

\footnote{That is, we do not consider the situation where (some of) the vector fields carry charges under dual magnetic gauge fields.}
by the structure constants of $G_{N=4}$. In this case the analysis of possible backgrounds and
the amount of supersymmetry they preserve becomes non-trivial. The order parameters
of supersymmetry breaking are the scalar parts of the fermionic supersymmetry trans-
formations, which generically depend on the scalar fields and the structure constants $f_{MNP}$.

Spontaneous $\mathcal{N} = 4 \to \mathcal{N} = 2$ supersymmetry breaking occurs at points in the $\mathcal{N} = 4$
field space where the supersymmetry transformations of two supercharges vanish (or are
proportional to the square root of the cosmological constant) while the remaining two
are non-zero. This will impose a set of conditions on the structure constants $f_{MNP}$, which
must also satisfy a complicated set of constraints (termed quadratic constraints
in the following) such that the theory itself is gauge invariant and supersymmetric. The
supersymmetry conditions are significantly simplified by using the symmetries of the
theory and the fact that $M$ is a homogeneous space and therefore we can always choose
to perform our analysis at the origin of field space [26]. We shall see that this allows us
to find a purely algebraic reformulation of the problem, part of which can be discussed in
terms of the representation theory of a solvable Lie algebra. We find that all maximally
symmetric vacua of the electrically gauged theory with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry
preserved are necessarily Minkowski and that $\mathcal{N} = 3$ vacua do not exist, as was already
observed in [14]. We then turn to solving the quadratic constraints, which prove too
complicated to solve in complete generality. In order to progress, we impose an additional
condition on the $f_{MNP}$, which holds automatically when the number of vector multiplet $n$
is less or equal than six. It corresponds to a particular choice of gauging which minimizes
the mixing between the gaugini and the gravitini in the Lagrangian. Indeed, we shall
see that in this case one can arrange for only one $\mathcal{N} = 4$ vector multiplet to contribute
to the gravity/Goldstini sector. For this class of gaugings we give the explicit solutions
of the quadratic constraints and the unbroken gauge groups when $n \leq 6$. Moreover, for
arbitrary $n$ we give solutions with an additional set of gaugings (and couplings) turned
off. In the appendix we show that if any other solution were to exist, then it would
necessarily require the number of vector multiplet to be $n > 6$.

Well below the scale of the partial supersymmetry breaking $m_{3/2}$ one can derive
a low-energy effective theory by integrating out the two heavy gravitini together with
all other fields which gain a mass of order $m_{3/2}$. This effective theory is an $\mathcal{N} = 2$
supergravity which only contains light (with respect to $m_{3/2}$) $\mathcal{N} = 2$ multiplets. We
observe that all fields come in complete $\mathcal{N} = 2$ supermultiplets with appropriate mass
degeneracies. Furthermore, the two heavy gravitini which gain a mass $m_{3/2}$ via the
super-Higgs mechanism have to be in a single $\mathcal{N} = 2$, spin-$\frac{3}{2}$ BPS multiplet.

In the scalar sector we find that one of the $\mathcal{N} = 2$ vector multiplets contains the
dilaton/axion of the original $\mathcal{N} = 4$ gravitational multiplet and that its field space
$SL(2)/SO(2)$ descends unchanged to the effective $\mathcal{N} = 2$ theory. In particular no mixing
with the other scalar fields occurs in the kinetic terms. As the scalars in $\mathcal{N} = 2$ vector
multiplets must span a special Kähler manifold, we can use this observation to conclude
that the $\mathcal{N} = 2$ scalar field space lies in the unique series of special Kähler product
manifolds. This follows by a theorem of [27], where it was shown that the only special
Kähler manifolds which split into a direct product are the manifolds

$$M_{\text{SK}} = \frac{SL(2)/SO(2) \times SO(2,k)/SO(2) \times SO(k)}{SO(2)}.$$  (1.2)
Since for the case at hand the first factor of $M$ coincides with the first factor of $M_{SK}$, we can conclude that the $\mathcal{N} = 2$ vector multiplet field space is given by (1.2).

The rest of this paper is organized as follows. In Section 2 we review the main properties of electrically gauged $\mathcal{N} = 4$ supergravities. In Section 3 we formulate the conditions for supersymmetry preserving vacua, focussing on the case of spontaneous $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ breaking. We then present the solution of the $\mathcal{N} = 2$ vacuum conditions for a particular subclass of possible gaugings, leaving the derivations and the discussion of the general case to the appendices. In Section 4 we investigate the structure of the mass terms and their consistency with the unbroken $\mathcal{N} = 2$ supersymmetry. We then discuss the possible unbroken gauge groups and comment on the geometry of the scalar manifold of the low energy effective $\mathcal{N} = 2$ theory. Our conventions and further technical details are gathered in the appendices.

2 Electrically gauged $\mathcal{N} = 4$ supergravities in $D = 4$

Let us briefly recall some properties of $\mathcal{N} = 4$ gauged supergravity in four dimensions. The generic spectrum consists of the gravity multiplet together with $n$ vector multiplets. The graviton multiplet contains the graviton $g_{\mu\nu}$, four gravitini $\psi^i_{\mu}$, $(i = 1, \ldots, 4)$, six vectors $A^m_{\mu}$, $(m = 1, \ldots, 6)$, four spin-1/2 fermions $\chi^i$ and two scalars. We label the vector multiplets with the index $a = 1, \ldots, n$ and each contains a vector $A^a_{\mu}$, 4 spin-1/2 fermions $\lambda^{ai}$ and 6 scalars. In this paper we only consider theories where the above fields carry charges with respect to the electric gauge bosons.

The bosonic Lagrangian for this class of theories is given by [18]

$$e^{-1} L_{\text{bos.}} = \frac{1}{2} R - \frac{1}{4} \text{Im}(\tau) M_{MN} H^M_{\mu\nu} H^{\mu\nu N} + \frac{1}{8} \text{Re}(\tau) \eta_{MN} \epsilon^{\mu\rho\lambda} H^M_{\mu\nu} H^{\rho\lambda N} + \frac{1}{16} (D_\mu M_{MN})(D^\mu M^{MN}) - \frac{1}{4 \text{Im}(\tau)} (\partial_\mu \tau)(\partial^\mu \tau^*) - V ,$$

where $R$ is the Ricci-scalar of the spacetime metric $g_{\mu\nu}$ and $e = \sqrt{|\det g|}$. The field strengths of the vectors are defined by

$$H^M_{\mu\nu} = 2 \partial_{[\mu} A^M_{\nu]} - f_{NP}^M A^N_{[\mu} A^P_{\nu]} ,$$

where the index $M = (m, a) = 1, \ldots, 6 + n$ labels all the vector fields $A^M_{\mu} = (A^m_{\mu}, A^a_{\mu})$. In (2.1), the matrix $M = (M_{MN}) = \mathcal{V} \mathcal{V}^T$ with $\mathcal{V} \in SO(6, n)$ describes a (left) coset of $SO(6, n)/SO(6) \times SO(n)$ which is the target manifold of the scalars of the vector multiplets. Similarly, $\tau \in \mathbb{C}$ with $\text{Im} \tau > 0$ parametrizes $SL(2)/SO(2)$ which is the target manifold for the two scalars of the gravity multiplet (see Appendix A.1 for further details).

The gauge covariant derivative acting on the vector multiplet scalars is defined as

$$D_\mu M_{MN} = \partial_\mu M_{MN} + 2 A^P_{\mu} f_{P(M} M_{N)Q} ,$$

More generally, one could also allow for charges with respect to dual magnetic gauge bosons. Such magnetically gauged theories can be described by means of the embedding tensor formalism [12, 18]. Here, we choose a symplectic frame such that the $A^m_{\mu}, A^a_{\mu}$ are the electric gauge bosons and restrict ourselves to electric gaugings only.
where $f_{MNP}$ are the real deformation parameters of the theory (with $f_{MNP} = 0$ in the ungauged theory). Supersymmetry and closure of the gauge Lie algebra require the $f_{MNP}$ to satisfy the following linear and quadratic constraints \cite{18,24}

$$f_{MNP} = f_{[MNP]}, \quad f_{R[MN} f_{PQ]}^R = 0 ,$$

where the indices are raised and lowered with the $SO(6, n)$ invariant metric

$$\eta = (\eta_{MN}) = (\eta^{MN}) = \text{diag}( -1, \ldots, -1, 1, \ldots, 1) .$$

In the formalism used in \cite{18} the $f_{MNP}$ are specific components of the embedding tensor, which is a spurionic matrix of charges. For purely electric gaugings the $f_{MN}^P$ are the structure constants of the gauge Lie algebra and the quadratic constraint in \cite{24} is the Jacobi identity. Note, however, that not all gauge algebras can occur since the $f_{MNP} = f_{MN}^L \eta_{LP}$ have to be completely antisymmetric. Here, the occurrence of the $SO(6, n)$ invariant metric $\eta_{MN}$ puts constraints on the possible Lie algebras that can be gauged \cite{18,24}. In the following we will not initially specify the gauge group, but rather carry out the analysis for arbitrary $f_{MNP}$. Later, when we discuss a restricted class of solutions for vacua with $\mathcal{N} = 2$ supersymmetry, we shall also be able to determine the possible gauge groups. Finally, the scalar potential is given by

$$V = \frac{1}{16 \text{Im} t} f_{MNP} f_{QRS} \left[ \frac{1}{3} M^{MQ} M^{NR} M^{PS} + \left( \frac{2}{3} \eta^{MQ} - M^{MQ} \right) \eta^{NR} \eta^{PS} \right] .$$

For our analysis we also need the fermionic bilinear couplings, which for the gravitini are \cite{18} \cite{4}

$$e^{-1} \mathcal{L}_{3/2} = \frac{2}{3} A_1^{ij} (\psi_{ij})^* \bar{\sigma}^{\mu \nu} \epsilon (\psi_i^j)^* + \text{h.c.} + \frac{1}{3} A_2^{ij} (\psi_{ij})^* \sigma^{\mu \nu} \epsilon (\lambda^j)^* + \text{h.c.} ,$$

while the bilinear couplings of the spin-1/2 fermions read

$$e^{-1} \mathcal{L}_{1/2} = - A_{2a}^j \chi^j (\lambda^a)^* + \text{h.c.} + \frac{1}{3} A_2^{ij} (\lambda^a)^* \epsilon (\lambda^j)^* + A_{ab}^{ij} (\lambda^a)^* \epsilon (\lambda^b)^* + \text{h.c.} .$$

The scalar shift matrices $A$ appearing in \cite{2.7} and \cite{2.8} depend on the vielbein $\mathcal{V}$ for $SO(6, n)$ and $(\mathcal{V}_a) = (\mathcal{V}_-, \mathcal{V}_+)$ for $SL(2)$ which are defined in Appendix [A.1]. They are given by

$$A_1^{ij} = (\mathcal{V}_-)^* f_{MN}^{NP} \mathcal{V}^M_{[kl]} \mathcal{V}^N_{[ik]} \mathcal{V}_P^{[jl]} ,$$

$$A_2^{ij} = \mathcal{V}_- f_{MN}^{NP} \mathcal{V}^M_{[kl]} \mathcal{V}^N_{[ik]} \mathcal{V}_P^{[jl]} ,$$

$$A_{2a}^j = \mathcal{V}_- f_{MN}^{NP} \mathcal{V}^M_a \mathcal{V}^N_{[ik]} \mathcal{V}_P^{[jk]} ,$$

$$A_{ab}^{ij} = \mathcal{V}_- f_{MN}^{NP} \mathcal{V}^M_a \mathcal{V}^N_b \mathcal{V}_P^{[ij]} ,$$

\footnote{In contrast, for a semisimple Lie algebras with structure constants $f_{abc}$ the Killing form $\kappa_{ab}$ is non-degenerate and can therefore be used to raise/lower indices. Then $f_{abc} = f_{ab}^d \kappa_{cd}$ would be automatically completely antisymmetric.}

\footnote{In Appendix [B] we will give our spinor conventions and relate the Weyl spinors used here to Dirac spinors which are used frequently in the literature. Also note that in \cite{2.7} we removed factors of $i$ in the mixed terms of gravitini and spin-1/2 fermions given in [B].}
where we again use a double index notation $M = (m, a)$ with $m = 1, \ldots, 6$, $a = 1, \ldots, n$. Indices $i, j, k, \ldots$ run from 1 to 4 and will turn out to be $SU(4)$ indices. More precisely, objects with upper/lower indices transform under the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$, respectively, and complex conjugation interchanges upper and lower indices, e.g. $(\psi_\mu^i)^* \text{ transforms as } a \bar{\mathbf{4}}$. Note that supersymmetry relates the $A$-matrices in (2.7) to the scalar potential via the generalized Ward identity [18]

$$\frac{1}{3} A_1^{ik} (A_1^{jk})^* - \frac{1}{9} A_2^{ik} (A_2^{jk})^* - \frac{1}{2} A_{2a_j}^k (A_{2a_i})^* = -\frac{1}{4} \delta^i_j V ,$$

with $V$ given in (2.6).

The full Lagrangian $L$ (2.1) + fermionic terms) is gauge invariant under local gauge transformations of a gauge group which satisfies (2.4). In addition, $L$ has a global $G = SO(6, n)$ symmetry under which the vectors and matter scalars (i.e. the scalars in the vector multiplets) transform in the fundamental representation of $SO(6, n)$ provided that the $f_{MNP}$ transform as a completely antisymmetric rank 3 tensor with respect to $SO(6, n)$. It is in this sense that capitalized indices $M, N, \ldots$ are referred to as $SO(6, n)$ indices. Furthermore, $L$ is also invariant under the local (i.e. spacetime dependent) symmetry $H = SU(4) \times SO(n)$ acting non-trivially on fermionic fields and matter scalars. Indices $a, b, \ldots = 1, \ldots, n$ and $m, n, \ldots$ are indices with respect to $SO(6, n)$.

Indices $a, b, \ldots = 1, \ldots, n$ and $m, n, \ldots$ are indices with respect to $SO(6, n)$ and $SO(6) \sim SU(4) \subset H$. In addition to $H$, there is another local $U(1)$ symmetry acting both on fermions and on the vielbein $V_\alpha$ of $SL(2)/SO(2)$ by multiplication with phase factors. The representations of the fields with respect to the two groups $G$ and $H$, as well as the additional $U(1)$ symmetry are summarized in Table 2.1.

| field   | $G = SO(6, n)$ | $H = SU(4) \times SO(n)$ | $U(1)$ charges |
|---------|----------------|---------------------------|----------------|
| $g_{\mu\nu}$ | 1              | (1, 1)                    | 0              |
| $\psi^i_\mu$ | 1              | (4, 1)                    | $-1/2$         |
| $A^M_\mu$    | $\Box$         | (1, 1)                    | 0              |
| $\chi^i$     | 1              | (4, 1)                    | 3/2            |
| $\lambda^a_i$| 1              | (4, n)                    | 1/2            |
| $V = V_{SO(6,n)}$ | $V \rightarrow gV$ | $V \rightarrow V h(x)$ | 0              |
| $V_\alpha$   | 1              | (1, 1)                    | 1              |

Table 2.1: $G$ and $H$ representations of the fields. Here $g \in SO(6, n)$ and $h(x) \in SO(6) \times SO(n)$, i.e. in particular, matter scalar representatives $V$ are charged with respect to $SO(6) \sim SU(4) \subset H$.

Since we are interested in vacua with a reduced number of supercharges we need to identify the order parameters of this spontaneous supersymmetry breaking. In a maximally symmetric background they are the scalar parts of the fermionic supersymmetry transformations which depend on the $A$-matrices and are given by [18]

$$\delta \psi^i_\mu = 2D_\mu \epsilon^i + \frac{2}{3} A_1^{ij} \bar{\sigma}_\mu \epsilon (e^j)^*,$$

$$\delta \chi^i = \frac{3}{2} i A_1^{ij} \epsilon (e^j)^*,$$

$$\delta \lambda^a_i = 2i A_2^{aj} \epsilon^j.$$

$^5$Here the fields are understood to be background configurations.
Here the supersymmetry parameter $\epsilon^i$ is a Weyl spinor that forms the right-handed spinor part of a Dirac spinor. It can be decomposed into a product of a spacetime independent (complex) $SU(4)$ vector $q^i$ and a Killing spinor $\eta$ of the spacetime according to $\epsilon^i = q^i \eta$. The covariant derivative is then given by

$$D_\mu \epsilon^i = D_\mu q^i \eta = -q^i \sqrt{-\frac{1}{12} V} \, \bar{\sigma}_\mu \epsilon \eta^* .$$

(2.12)

For supersymmetric vacua the background value of the scalar potential $V$ is either zero (Minkowski) or negative (anti-de Sitter).

3 Supersymmetric vacua and partial supersymmetry breaking

3.1 Preliminaries

Let us first recall that an $\mathcal{N} = 4$ supersymmetric background is defined by the conditions

$$\delta_\epsilon \psi^i = 0 , \quad \delta_\epsilon \chi^i = 0 , \quad \delta_\epsilon \lambda^{ai} = 0$$

(3.1)

for all free indices $i, a$ and for all supersymmetry parameters $\epsilon^i$. Using the decomposition $\epsilon^i = q^i \eta$ introduced in the previous section this translates into

$$A_1^{ij} q_j = \sqrt{-\frac{3}{4} V} q^i , \quad q_j A_2^{ij} = 0 , \quad A_{2ai}^i q^i = 0 , \quad \forall i, a .$$

(3.2)

Using that in electrically gauged theories the symmetric matrices $(A_1^{ij})$ and $(A_2^{ij})$ differ only by an overall phase factor, see (2.9), we can immediately conclude that in this class of theories an $\mathcal{N} = 4$ background has

$$(A_1^{ij}) = (A_2^{ij}) = (A_{2ai}^i) = 0 , \forall a \quad \text{and} \quad V = 0 .$$

(3.3)

We also observe that by the same token (3.2) implies that in electrically gauged theories any background with at least one preserved supersymmetry is necessarily Minkowski, i.e. $V = 0$ [14]. We will not investigate $\mathcal{N} = 4$ backgrounds any further here, but instead shift our attention to backgrounds with less supersymmetry. Examples of vacua of $\mathcal{N} = 4$ gauged supergravity with various amounts of preserved supersymmetry have been discussed, for example, in [21, 22] and references therein.

Our goal here is to classify the solution of (3.2) which preserve only two out of the four supercharges in a maximally-symmetric background. Ordinarily, one should first pick a particular $\mathcal{N} = 4$ supergravity theory, i.e. a specific gauging, and then look for solutions of the Killing spinor equations. Rather than take that approach, we shall follow [9] and first specify the vacuum and the amount of preserved supersymmetry, and then use the Killing spinors equations to solve for the embedding tensor components, i.e. the gaugings, that give rise to this vacuum. In this way we are solving the Killing spinor equations to

\textsuperscript{6}This result is implicit in [14], in that the $SU(1,1)$ phases are set equal there. This is equivalent to saying that there are only electric gaugings [18].
find the theory, rather than the vacuum. To this end, one requires that (3.2) hold for any preserved supersymmetry associated to a supercharge $\epsilon^i = q^i \eta$, while for spontaneously broken supersymmetries (3.2) shall not be satisfied. Such a system of equations and inequalities at an arbitrary critical point of the scalar manifold is best solved (for the $f_{MNP}$) by using the symmetry to go, without loss of generality, to the origin of the matter scalar manifold, cf. [26], and, secondly, (by using the residual symmetry) to diagonalize the gravitini mass matrix at such a critical point.

3.1.1 Going to the origin of the matter scalar manifold

We assume that a given consistent electrically gauged theory has a stable scalar vacuum, i.e. a critical point of the scalar potential at some point in the scalar manifold:

$$(\mathcal{V}_{SL(2)}, \mathcal{V}_{SO(6,n)}) \in SL(2) \times SO(6, n).$$

Using the $G = SO(6, n)$ symmetry that, in particular, acts on scalar fields according to Table 2.1 we can transform $\mathcal{V}_{SO(6,n)}$ to the unit matrix $\mathbb{1}_{6+n} \in \text{Mat}_{6+n,6+n}$ and, hence, obtain a theory given in terms of redefined fields, new components $f_{MNP}$ and a critical point

$$(\mathcal{V}_{SL(2)}, \mathbb{1}_{6+n}) \in SL(2) \times SO(6, n).$$

This is of help because the shift-matrices in (3.2) evaluated at (3.5) end up being disentangled with respect to certain components of $f_{MNP}$, as we will see below. The residual symmetry in a theory with vacuum (3.5) is a combination of $SO(6) \times SO(n) \subset G$ and global $H$ symmetries such that their compositions leave $\mathbb{1}_{6+n}$ invariant. In contrast, $SL(2)$ is not part of the global symmetry of the Lagrangian and, thus, cannot be used to also transform $\mathcal{V}_{SL(2)}$ to the origin $\mathbb{1}_2 \in SL(2)$ without loss of generality. Being an on-shell symmetry that maps the system of equations of motion and Bianchi identities onto another such system, general $SL(2)$ transformations would lead to non-electrically gauged theories which (for simplicity) we do not want to consider here. Using the additional local symmetry $U(1) \sim SO(2)$ of the Lagrangian which acts as in Table 2.1 both on gravity scalar representatives $\mathcal{V}_\alpha$ and on fermions, we can bring the gravity vielbein to a form such that $\mathcal{V}_- = 1/\sqrt{\text{Im} \tau} > 0$ without loss of generality (see Appendix A.1 for details). This comes at the cost of redefining the fermion fields but simplifies the $A$-matrices in (2.9), in that $\mathcal{V}_- > 0$ becomes an overall scaling factor. As a result, the components of the $A$-matrices at the critical point (3.3) can be expressed as

$$A_1^{ij} = A_2^{ij} = \frac{1}{8} \mathcal{V}_- ([G_m]_{ik})^*[G_n]_{kt}([G_p]_{lj})^* f_{mnp},$$

$$A_{2ai}^j = -\frac{1}{4} \mathcal{V}_- [G_m]_{ik}([G_n]_{kj})^* f_{amn},$$

$$A_{ab}^{ij} = -\frac{1}{2} \mathcal{V}_- [G_m]^ij f_{abm},$$

where $G$ are the 't Hooft matrices, which we review in Appendix A.1. Note that at a critical point $(1_2, 1_{6+n})$ one has $\mathcal{V}_- = 1.$

7 General magnetic gaugings are described in terms of embedding tensors $f_{\alpha MNP} = f_{\alpha[MNP]}$ and $\xi_{\alpha M}$ where the index $\alpha$ is a vector index with respect to $SL(2)$ [17]. In our symplectic frame purely electric gaugings are given by $f_{-MNP} = 0$ and $\xi_{\alpha M} = 0$: $f_{MNP} \equiv f_{+MNP}$.
From the generalized Ward identity (2.10) or the explicit form of the scalar potential given in (2.6) one finds that the scalar potential scales with a factor \((V-)^2\). As a consequence, the Killing spinor equations (3.2) do not depend on \(V\), i.e. the analysis of partial supersymmetry breaking does not depend on the critical point \(V_{SL(2)} \in SL(2)\) in the gravity scalar manifold. However, we observe that a generic \(V_{SL(2)}\) leads to an overall scaling of all mass terms. Note that it is only upon canonically normalizing the gauge kinetic terms that the mass terms for the vector bosons also scale appropriately. Also recall that for any value of \(V_{SL(2)}\) (or \(\tau\)) the background potential vanishes and, hence, \(\tau\) is a flat complex direction of the potential.

### 3.1.2 Gravitino masses

For all supergravity theories unbroken supersymmetries are in one-to-one correspondence with massless gravitini [3]. Therefore, it will be instructive to first consider mass terms for the gravitini,

\[
e^{-1}L_{m/2} = \frac{2}{3} A^{ij}_1 (\psi^i)_{\mu}^* \sigma^{\mu\nu} \epsilon (\psi^j_\nu)^* + \text{h.c.}
\]

(3.7)

An arbitrary symmetric complex matrix \((A^{ij}_1)\) can be diagonalized by means of an \(SU(4)\) transformation. This is a consequence of the Autonne decomposition [28]: One can always find an \(S \in SU(4)\) such that

\[
S(A^{ij}_1)S^T = \text{diag}(|a_1|, |a_2|, |a_3|, |a_4|),
\]

(3.8)

with \(|a_1| \leq \ldots \leq |a_4|\). Note, however, that diagonalizing a non-diagonal matrix \((A^{ij}_1)\) at the origin transforms also the matrices \((A^{ij}_2)\) and \((A^{2ai}_2)\), and affects the vacuum by an \(SO(6) \subset H\) rotation moving it away from the critical point (3.5). Of course, the scalar vacuum always remains in the same coset of \(G/H\). We now think of such an \(H\) transformation as acting globally and apply its inverse as a \(G\) transformation on the vacuum, the \(f_{MNP}\), and the vector bosons. In doing so, one returns to the origin of \(SO(6,n)\) and at the same time has a diagonal gravitino mass matrix. Moreover, one now knows the \(A\)-matrices in terms of the transformed \(f_{MNP}\). We therefore may assume that, without loss of generality, \((A^{ij}_1)\) is of the form (3.8) and the \(A\)-matrices are explicitly given as in Appendix [C]. Inspecting (3.7) we see that the gravitini mass parameters are given by \(2/3 \cdot |a_1|, \ldots, 2/3 \cdot |a_4|\).

According to the Killing spinor equations (3.2) (with \(V = 0\) for electric gaugings) one requires for any unbroken supersymmetry labelled by \(q_i\) a zero diagonal entry of \((A^{ij}_1) = (A^{ij}_2)\). In contrast, for a broken supersymmetry direction \(q_i\) it is necessary that the diagonal entries be positive. Furthermore, for each unbroken \(q_i\) one needs a zero row in matrices \((A_{2ai}^{2j})\) for all \(a\). It is apparent from the explicit form of the shift matrices given in (C.1), (C.2) and (C.3) that the \(f_{MNP}\) can be chosen in such a way that the Killing spinor equations (and their inequalities) are fulfilled at the critical point (3.5) for any number of preserved supersymmetries. Note that this system of equations and inequalities is linear in the \(f_{MNP}\) and, hence, can easily be solved. On the other hand, consistency of the gauged supergravities requires solving the quadratic equations given in (2.4) which we discuss in Section 3.2.3. Note that it is by means of (2.10) that the first equation of (3.2) already implies the other two\(^8\) which means that in principle we need

---

\(^8\)This is always true for extended supergravity theories, see [3].
not demand zero rows in \((A_{2ai})^j\) since this will follow from a solution of the quadratic constraints. However, solving the constraints is difficult and introducing zero rows in the \((A_{2ai})^j\) is a useful measure to simplify computations.

### 3.2 \(\mathcal{N} = 2\) vacuum

Let us now turn to our main problem, which is to study spontaneous breaking of \(\mathcal{N} = 4\) to \(\mathcal{N} = 2\) supersymmetry. For unbroken \(\mathcal{N} = 2\) supersymmetry, one generically has

\[
(A_1^{ij}) = (A_2^{ij}) = \text{diag}(0,0,\mu_1,\mu_2), \quad A_{2a1}^i = A_{2a2}^i = 0, \quad \forall i,a .
\]

(3.9)

Recall that the vacuum is necessarily Minkowski which implies that the first two eigenvalues of \((A_1^{ij})\) are zero. Before we solve (3.9) let us study the decomposition of \(\mathcal{N} = 4\) multiplets in terms of \(\mathcal{N} = 2\) multiplets. This is of interest as partial supersymmetry breaking requires massive gravitini to be embedded into massive supermultiplets of the preserved supersymmetry [29].

#### 3.2.1 Multiplets of \(\mathcal{N} = 4, 2\) and their relations

In this section we review the decomposition of the two massless \(\mathcal{N} = 4\) multiplets into \(\mathcal{N} = 2\) multiplets. Let us denote a multiplet of \(\mathcal{N}\)-extended supersymmetry with mass \(m\) and highest spin/helicity \(s\) in Minkowski space by \(M_{\mathcal{N},s,m}\). Using this terminology the \(\mathcal{N} = 4\) gravitational multiplet and the massless vector multiplet together with their component spectrum read

\[
\mathcal{N} = 4 \text{ gravitational multiplet: } M_{4,2,0} = \left( [2], 4[\frac{3}{2}], 6[1], 4[\frac{1}{2}], 2[0] \right),
\]

\[
\mathcal{N} = 4 \text{ vector multiplet: } M_{4,1,0} = \left( [1], 4[\frac{1}{2}], 6[0] \right).
\]

(3.10)

where \([s]\) denotes the spin/helicity of the component and the number in front is its multiplicity. The massless \(\mathcal{N} = 2\) multiplets are

\[
\mathcal{N} = 2 \text{ gravitational multiplet: } M_{2,2,0} = \left( [2], 2[\frac{3}{2}], [1] \right),
\]

\[
\mathcal{N} = 2 \text{ gravitino multiplet: } M_{2,3/2,0} = \left( [\frac{3}{2}], 2[1], [\frac{1}{2}] \right),
\]

\[
\mathcal{N} = 2 \text{ vector multiplet: } M_{2,1,0} = \left( [1], 2[\frac{3}{2}], 2[0] \right),
\]

\[
\mathcal{N} = 2 \text{ hypermultiplet: } M_{2,1/2,0} = \left( 2[\frac{1}{2}], 4[0] \right).
\]

(3.11)
while the massive $\mathcal{N} = 2$ multiplets read

\[
\begin{align*}
\mathcal{N} = 2 \text{ massive gravitino multiplet:} & \quad M_{2,3/2,m \neq 0} = \left( [\frac{3}{2}], 4[1], 6[\frac{1}{2}], 4[0] \right), \\
\mathcal{N} = 2 \text{ BPS gravitino multiplet:} & \quad M_{2,3/2,\text{BPS}} = \left( 2[\frac{3}{2}], 4[1], 2[\frac{1}{2}] \right), \\
\mathcal{N} = 2 \text{ massive vector multiplet:} & \quad M_{2,1,m \neq 0} = \left( [1], 4[\frac{3}{2}], 5[0] \right), \\
\mathcal{N} = 2 \text{ BPS vector multiplet:} & \quad M_{2,1,\text{BPS}} = \left( [1], 2[\frac{1}{2}], 1[0] \right), \\
\mathcal{N} = 2 \text{ BPS hypermultiplet:} & \quad M_{2,1/2,\text{BPS}} = \left( 2[\frac{1}{2}], 4[0] \right).
\end{align*}
\]

Note that there are two distinct $\mathcal{N} = 2$ massive gravitino multiplets, the BPS gravitino multiplet $M_{2,3/2,\text{BPS}}$ and the long massive gravitino multiplet $M_{2,3/2,m \neq 0}$. They differ in that only the BPS gravitino multiplet transforms under a central charge of the supersymmetry algebra in precisely the way that leads to multiplet shortening. BPS gravitini can only occur in pairs as each of them carries a non-vanishing BPS charge which by itself would not be CPT-invariant. This implies that $\mathcal{N} = 4$ cannot be broken to $\mathcal{N} = 3$ with a BPS gravitino multiplet.

The branching rules of the two $\mathcal{N} = 4$ multiplets in terms of massless $\mathcal{N} = 2$ multiplets are as follows

\[
M_{4,2,0} = M_{2,2,0} + 2M_{2,3/2,0} + M_{2,1,0} , \quad M_{4,1,0} = M_{2,1,0} + M_{2,1/2,0} ,
\]

from which we see that in breaking $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ the gravity multiplet gives rise to a vector multiplet containing the dilaton and axion in the $\mathcal{N} = 2$ spectrum.

As all degrees of freedom must be embedded into complete $\mathcal{N} = 2$ multiplets, the two heavy gravitini must lie in massive $\mathcal{N} = 2$ supermultiplets. We thus have two options regarding the type of the gravitino multiplet(s). For the situation where the heavy $\mathcal{N} = 2$ gravitini are in non-BPS multiplets one has

\[
M_{4,2,0} + nM_{4,1,0} \rightarrow M_{2,2,0} + 2M_{2,3/2,m \neq 0} + n'_v M_{2,1,m \neq 0} + n_v M_{2,1,\cdot} + n_h M_{2,1/2,\cdot} ,
\]

where $n'_v$ counts long massive vector multiplets, $n_v$ counts BPS vector multiplets and massless vector multiplets (as they have the same field content) and $n_h$ counts BPS or massless hypermultiplets (as they also have the same field content). We use $\cdot$ to denote either massless or BPS multiplets. Inserting the spectrum \((3.10)-(3.12)\) one finds the consistency conditions

\[
n_v = n - 3 - n'_v , \quad n_h = n - 2 - n'_v .
\]

Thus in this case there have to be at least three $\mathcal{N} = 4$ vector multiplets in the spectrum, i.e. $n \geq 3$. In this minimal case with also $n'_v = 0$ there are, apart from the $\mathcal{N} = 2$ gravitational multiplet and the two heavy gravitino multiplets, one massive or massless hypermultiplet after the symmetry breaking.

In case that the heavy $\mathcal{N} = 2$ gravitini are contained in a BPS multiplet one has

\[
M_{4,2,0} + nM_{4,1,0} \rightarrow M_{2,2,0} + M_{2,3/2,BPS} + n'_v M_{2,1,m \neq 0} + n_v M_{2,1,\cdot} + n_h M_{2,1/2,\cdot} ,
\]
with the consistency conditions
\[ n_v = n + 1 - n'_v , \quad n_h = n - 1 - n'_v , \] (3.17)
and thus there has to be at least one \( \mathcal{N} = 4 \) vector multiplet in the spectrum, i.e. \( n \geq 1 \). In this minimal case with \( n'_v = 0 \), one finds after the symmetry breaking the \( \mathcal{N} = 2 \) gravitational multiplet, the BPS gravitino multiplet, and two massless/BPS vector multiplets. Note that according to equations (3.15) and (3.17) the case with two long massive gravitino multiplets \( M_{2,3/2,m \neq 0} \), relative to the BPS case, yields one fewer hypermultiplet and four fewer vector multiplets in the spectrum.

### 3.2.2 The linear conditions

In this section we first solve the linear \( \mathcal{N} = 2 \) conditions (3.9) and then embark on solving the quadratic constraints (2.4). While the linear equations can easily be solved, it is hard to find the general solution for the quadratic constraints.

Let us first focus on the zero entries in \( A_1 (= A_2) \). Using the explicit form given in Appendix C one easily finds that only four of the \( f_{mnp} \) can be non-zero and they depend on only two parameters which we denote by \( c \) and \( d \). More precisely one finds
\[ f_{234} = f_{456} =: c , \quad f_{126} = f_{135} =: d , \] (3.18)
while all other \( f_{mnp} \) vanish. Moreover, \( A_{33}^1 \) and \( A_{44}^1 \) which are related to the gravitini mass parameters \( \mu_1 \) and \( \mu_2 \) introduced in (3.9) also depend on \( c \) and \( d \) via
\[ A_{33}^1 = -\frac{3}{2} \mathcal{V}_-(c + d) = \mu_1 > 0 , \quad A_{44}^1 = -\frac{3}{2} \mathcal{V}_-(c - d) = \mu_2 \geq \mu_1 , \] (3.19)
where as pointed out before \( \mu_2 \geq \mu_1 \) is chosen without loss of generality. Let us now turn to the last set of equations in (3.9) and solve the system of linear equations for the shift matrices \( (A_{2a}^j) \). Using (C.2) and (C.3) the potentially non-trivial components of \( f_{amn} \) turn out to be
\[ f_{a25} = -f_{a36} =: e_a , \quad f_{a23} = f_{a56} =: f_a , \quad f_{a26} = f_{a35} =: g_a , \] (3.20)
while \( f_{a1n} = f_{a4n} = 0 \) for all \( a \) and \( n \). Thus, for any \( a \), the matrix \( A_{2a}^j \) is non-trivial only in its lower right block and given by
\[ (A_{2a}^j) = \begin{pmatrix} 0 & 0 \\ 0 & Z_a \end{pmatrix} , \quad Z_a = f_a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i \begin{pmatrix} -e_a & g_a \\ g_a & e_a \end{pmatrix} . \] (3.21)
This concludes our analysis of the linear equations arising from the Killing spinor equations (3.9). Let us now turn to the quadratic constraints.

### 3.2.3 Partial solution of the quadratic conditions

In order to ensure that a given choice of gauging is consistent with supersymmetry and gauge invariance of the Lagrangian, we need to impose the quadratic constraints (2.4) \( f_{R[MN} f_{PQ]}^R = 0 \) \cite{12,17,18}. However, in practice it is difficult to solve these constraints...
in general and we will have to make much use of their symmetry properties. For instance, (2.4) are \(SO(6,n)\) tensor equations and it will prove crucial to exploit all the symmetries.

Let us first look at the component \((M,N,P,Q) = (m,n,p,q) = (1,2,4,5)\) of the quadratic constraints (2.4) and insert (3.18) to arrive at
\[
c \cdot d = 0.
\] (3.22)

Since \(c = 0\) is inconsistent with the gauge choice of (3.19), we need to have \(d = 0\) and \(c < 0\). This in turn implies a first result, namely that the two heavy gravitini have to be degenerate in mass
\[
m_{3/2} := \frac{2}{3} A_{11}^{33} = \frac{2}{3} A_1^{44} = -c \mathcal{V}_-,
\] (3.23)
as one expects when some fraction of supersymmetry is preserved in a Minkowski background. Let us also note that (3.22) immediately implies that in electrically gauged theories one can never break \(N = 4\) to \(N = 3\) since \(A_{11}^{33} = 0, A_1^{44} \neq 0\) requires \(c = -d \neq 0\), as was first shown in [14].

In order to proceed, it is necessary to make some simplifying assumptions. By inspection, one finds that for \(g_{a} = 0\) the equations simplify considerably and therefore some of them can be solved. On the other hand, the \(g_{a} \neq 0\) case is much more involved and solutions — should they exist — would have to be more sophisticated, as we point out in Appendix D.1.2. In what follows we will therefore assume that \(g_{a} = 0\), which also implies \(e_{a} = 0\) due to the quadratic constraint for \((M,N,P,Q) = (b,n,p,q) = (b,2,4,6)\). This choice corresponds to turning-off certain components of the \(A\)-matrices and minimizes the coupling between gravitini and gaugini in the Lagrangian (2.7). Indeed, we shall see later that with this choice it is only the “first” \(N = 4\) vector multiplet that contributes to the gravity/Goldstini sector. The fact that it is the components \(g_{a} = f_{a26} = f_{a35} = 0\) and \(e_{a} = f_{a25} = -f_{a36} = 0\) that allow for this simplification is due to our particular \(SU(4)\) gauge choice for which gravitini remain massless (3.9), suitably translated into \(SO(6)\) indices using the 't Hooft matrices (see (A.1)).

Let us now consider the quadratic constraint \((M,N,P,Q) = (m,n,p,q) = (2,3,5,6)\). Inserting (3.18) and (3.20) we find
\[
\sum_{a} f_{a}^{2} = c^{2} > 0,
\] (3.24)
i.e. at least one \(f_{a}\) must be different from zero. This implies (via (3.21)) that \((A_{2a}^{j})\) has non-zero entries and from (2.7) and (2.8) we see that additional fermionic couplings have to be non-zero and related to the gravitino mass. As we will see in Section 4.1 (3.24) is necessary for the super-Higgs mechanism and the appropriate couplings of the Goldstone fermions to the gravitinos. In order to simplify the analysis we use an \(SO(n)\) transformation that leaves the origin invariant and choose \(f_{a} = c \delta_{a7}\) which obviously solves (3.24). The quadratic constraints \((M,N,P,Q) = (b,n,p,q)\) then imply
\[
f_{7bm} = 0, \quad \forall b,m.
\] (3.25)

In Appendix D.2 we list the remaining non-trivial quadratic constraints. A subset of them, (D.67a) - (D.67u), can be written in terms of the antisymmetric real \((n-1) \times (n-1)\) matrices
\[
G_{m} = (f_{bkm}) \quad \text{and} \quad G_{7} = (f_{bcka}) \quad \text{with} \ b, c = 8, \ldots, 6+n.
\] (3.26)
which satisfy
\[
\begin{align*}
[G_2, H_+] &= -2c G_3, & [G_3, H_+] &= +2c G_2, & [G_2, G_3] &= c H_-, \\
[G_5, H_+] &= -2c G_6, & [G_6, H_+] &= +2c G_5, & [G_5, G_6] &= c H_-, \\
\end{align*}
\tag{3.27}
\]
where \( H_{\pm} = G_4 \pm G_7 \) and with the remaining commutators all vanishing. (3.27) defines a Lie bracket on the 7-dimensional real vector space spanned by abstract elements \( \{G_1, G_2, G_3, G_5, G_6, H_+, H_-\} \) and it can be checked that the Jacobi identities are satisfied.

Note that \( G_1 \) commutes with all other elements and thus we have a real 7-dimensional Lie algebra \( \mathfrak{g} \) which decomposes into a sum of two ideals,
\[
\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{s},
\tag{3.28}
\]
spanned by \( G_1 \) and \( \{G_2, G_3, G_5, G_6, H_+, H_-\} \), respectively. It can be further checked that \( \mathfrak{s} \) is a solvable Lie algebra of dimension 6.\footnote{Recall that a Lie algebra \( \mathfrak{g} \) is solvable if and only if the (upper) derived series of Lie algebras \( (\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \ldots) \) terminates after finitely many steps.} The problem of finding solutions to the quadratic constraints \( (D.67a) - (D.67u) \) is now equivalent to finding antisymmetric finite-dimensional representations of \( \mathfrak{g} \). One obvious class of solutions is given by
\[
G_2 = G_3 = G_5 = G_6 = H_- = 0
\tag{3.29}
\]
and an arbitrary, antisymmetric \( H_+ \) that commutes with \( G_1 \). In this case one has \( G_4 = G_7 \). In Appendix \( \text{D.2.1} \) we will prove that no other solution exists. Our proof is based on Lie’s theorem concerning complex representations of complex solvable Lie algebras.

The remaining equations \( (D.67a) - (D.68c) \) to be solved now simplify to
\[
\begin{align*}
f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{c} \tilde{1}} - f_{\tilde{a} \tilde{b} \tilde{1}} f_{\tilde{a} \tilde{c} \tilde{4}} &= 0 \tag{3.30a} \\
f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{1}} + f_{\tilde{a} \tilde{b} \tilde{1}} f_{\tilde{a} \tilde{c} \tilde{d}} - f_{\tilde{a} \tilde{b} \tilde{d}} f_{\tilde{a} \tilde{c} \tilde{1}} &= 0 \tag{3.30b} \\
f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{4}} + f_{\tilde{a} \tilde{b} \tilde{4}} f_{\tilde{a} \tilde{c} \tilde{d}} - f_{\tilde{a} \tilde{b} \tilde{d}} f_{\tilde{a} \tilde{c} \tilde{4}} &= 0 \tag{3.30c} \\
f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{a}} + f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{a}} - f_{\tilde{a} \tilde{b} \tilde{d}} f_{\tilde{a} \tilde{c} \tilde{c}} &= f_{\tilde{1} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{1}} + f_{\tilde{1} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{1}} - f_{\tilde{1} \tilde{b} \tilde{d}} f_{\tilde{1} \tilde{c} \tilde{c}}. \tag{3.30d}
\end{align*}
\]
Note that the gravitino mass parameter \( c \) has disappeared from the equations. Unfortunately, it is still hard to solve these equations in generality for any given integer \( n \).

Let us first consider \( G_1 = G_4 = 0 \). In this case the only remaining non-trivial equation is
\[
f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{a}} + f_{\tilde{a} \tilde{b} \tilde{c}} f_{\tilde{a} \tilde{d} \tilde{a}} - f_{\tilde{a} \tilde{b} \tilde{d}} f_{\tilde{a} \tilde{c} \tilde{c}} = 0, \tag{3.31}
\]
which tallies with the Jacobi identity in the adjoint representation of the compact form of a reductive Lie algebra of rank \( (n - 1) \) when expressed in an appropriate basis. Based on the classification of simple Lie algebras, solutions to (3.31) are well-understood. As we will see in Section 4.2, when exponentiated this gives rise to a compact reductive Lie group that leaves invariant the vacuum of the theory and, hence, corresponds to the unbroken gauge group.

Now we turn to the case of non-trivial \( G_1 \) and \( G_4 \). In Appendix \( \text{D.2.2} \) we will solve (3.30a), which in matrix notation reads
\[
[G_1, G_4] = 0. \tag{3.32}
\]
Here we will only explain the result. The solution of this \( SO(n-1) \) tensor equation could be given in terms of \( SO(n-1) \) representatives of an orbit of solutions. However, as it is also an \( O(n-1) \) tensor equation, it is more convenient to give its solution in terms of \( O(n-1) \) representatives, up to an additional simple reflection, so as to obtain this gauge by a \( SO(n-1) \) rotation. Regardless of this subtlety our gauge choice proves useful in the following analysis. One finds that the most general solution consists of simultaneously block-diagonal \( G_1 \) and \( G_4 \) with blocks that square to a matrix proportional to the identity of the block. The explicit form of \( G_1 \) and \( G_4 \) in our gauge is given as follows: First of all, we have

\[
G_1 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix},
\]

(3.33)

where \( D = \text{diag}(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots) \) is a diagonal matrix with ordered positive eigenvalues \( x_1 > x_2 > \ldots > 0 \) and \( \varepsilon \) is the antisymmetric \( 2 \times 2 \) matrix with \( \varepsilon_{12} = 1 \); the zeros in (3.33) denote zero matrices of appropriate dimensions. Then, we have

\[
G_4 = \begin{pmatrix} A & 0 \\ 0 & (D' \otimes \varepsilon) \oplus 0 \end{pmatrix},
\]

(3.34)

where \( A \) is an antisymmetric matrix (of the same matrix dimensions as \( D \otimes \varepsilon \)) satisfying

\[
[D \otimes \varepsilon, A] = 0,
\]

(3.35)

and \( D' \) is another invertible diagonal matrix. Furthermore, we show in Appendix D.2.2 that both \( D \otimes \varepsilon \) and \( A \) are block-diagonal. Furthermore, as a result, we list the four different types of blocks that can appear in Table 3.1.

| \( G_1 \) block | \( G_4 \) block |
|----------------|-----------------|
| \( x_i \, \mathbb{1} \otimes \varepsilon \) | \( 0 \cdot \mathbb{1} \otimes \mathbb{1}_2 \) |
| \( x_i \, \begin{pmatrix} 1 & 0 \\ 0 & 1' \end{pmatrix} \otimes \varepsilon \) | \( \begin{pmatrix} y_{ij} & \frac{1}{2} \\ \frac{1}{2} & y_{ij} \end{pmatrix} \otimes \varepsilon \) |
| \( x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon) \) | \( \begin{pmatrix} \cos \phi_{ijk} & \frac{1}{2} \\ \frac{1}{2} & \cos \phi_{ijk} \end{pmatrix} \otimes \varepsilon + \sin \phi_{ijk} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \) |
| \( x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon) \) | \( D^{(ij0)} \otimes (\varepsilon \otimes \mathbb{1}_2) \) |

Table 3.1: The four different types of blocks appearing in the solution of \( [D \otimes \varepsilon, A] = 0 \). The label \( i \) refers to blocks in \( G_1 \) with eigenvalues \( -x_i^2 \neq 0 \) of \( G_1 \). Similarly, the label \( j \) is associated to subblocks in \( G_4 \) with eigenvalues \( -y_{ij}^2 \neq 0 \) of \( G_4 \). Moreover, \( D^{(ij0)} \) is a diagonal matrix with eigenvalues \( \pm y_{ij} \) and \( \phi_{ijk} \in (0, \pi/2) \). Finally, \( k \) labels different possible angles \( \phi_{ijk} \).

We will now solve the tensor equation given in (3.30b). For a given \( G_1 \), these equations are linear in \( f_{\tilde{a}0} \) and can easily be solved for the latter in the gauge (D.72). Before we state the result, we introduce some index notation in that we distinguish \( SO(n-1) \) indices \( \tilde{a}, \tilde{b}, \ldots \) depending on whether or not they correspond to non-zero or zero blocks in \( G_1 \): Components of non-zero \( 2 \times 2 \) blocks shall have subindices, e.g. \( \tilde{a}_1 = 1, 2 \), indicating the block they belong to. On the other hand, components associated to the zero block in \( G_1 \) shall be denoted by \( \tilde{a}_0 \). Furthermore, we introduce matrices

\[
G^{(x_1)}_{\tilde{a}_0} = (G_{\tilde{a}_0} \tilde{b}_1 \tilde{c}_2) = f_{\tilde{a}_0 \tilde{b}_1 \tilde{c}_2},
\]

(3.36)
where \( \tilde{b}_1, \tilde{c}_2 \) run over all indices associated to blocks with \( x_1 \) in \( G_1 \). The solution of (3.30b) is given in terms of three classes of potentially non-trivial components \( f_{\tilde{a} \tilde{b} \tilde{c}} \). First,

\[
f_{\tilde{a} \tilde{b} \tilde{c}} \in \mathbb{R} ,
\]

(3.37)
can be arbitrary; then one finds

\[
G_{\tilde{a}}^{(x_1)} = S(x_1) \otimes \varepsilon + A(x_1) \otimes 1_2 ,
\]

(3.38)
for a symmetric matrix \( S^{(x_1)} \) and an antisymmetric \( A^{(x_1)} \); finally components \( f_{\tilde{a}_1 \tilde{b}_2 \tilde{c}_3} \) are given in terms of two real numbers,

\[
\begin{align*}
f_{21223} &= f_{11213} , \\
f_{11223} &= -f_{11213} , \\
f_{22123} &= f_{11213} , \\
f_{21123} &= -f_{21223} , \\
f_{11213} &= f_{21221} , \\
f_{111213} &= f_{21221} ,
\end{align*}
\]

(3.39)
for \( x_1 = x_2 + x_3 \) (\( x_1 \geq x_2 \geq x_3 \)).

Unfortunately we are unable to solve equations (3.30c) and (3.30d) in full generality. We will therefore proceed by discussing certain special solutions of them (still in the case \( g_a = 0 \)).

### 3.3 Special solutions

We will discuss two special classes of solutions to the equations given in (3.30a) to (3.30d). First we will give all solutions in the case of \( n \leq 6 \), and secondly we construct special but physically non-trivial solutions that work for any \( n \in \mathbb{N} \).

#### 3.3.1 Solutions for \( n \leq 6 \)

In Appendix D.1.2 we show that for \( n \leq 6 \) consistency requires \( g_a = 0 \). As a consequence, the equations to be solved are precisely the ones in (3.30a) to (3.30d). As in (3.33), we will bring \( G_1 \) to the following gauge

\[
G_1 = \left[ \left( \begin{array}{c} m_1 \\ 0 \\ m_2 \end{array} \right) \otimes \varepsilon \right] \oplus 0 \in \text{Mat}_{5,5}
\]

(3.40)
for \( n = 6 \) with \( m_1, m_2 \in \mathbb{R} \), or to obvious truncations of (3.40) to matrices in \( \text{Mat}_{n-1,n-1} \) for \( n \leq 5 \). As discussed in Appendix D.2.2 we distinguish between the following two cases: Given that matrices \( (G_1)^2 \) and \( (G_4)^2 \) have four nonzero degenerate eigenvalues each (which can only happen for \( n \geq 5 \)), \( G_4 \) can be written as

\[
G_4 = \pm n_1 \left[ \cos \phi \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \otimes \varepsilon + \sin \phi \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right) \otimes 1_2 \right] \oplus 0
\]

(3.41)
for \( n = 6 \) or its obvious truncation in the case of \( n = 5 \), while otherwise we can write

\[
G_4 = \left[ \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \otimes \varepsilon \right] \oplus 0
\]

(3.42)

for \( n = 6 \) or truncations thereof for \( n \leq 5 \). Here, we introduced \( n_1, n_2 \in \mathbb{R} \) and \( \phi \in [0, \pi/2] \). Note that the dimension of the matrices \( G_1 \) and \( G_4 \) being smaller than 6 does not allow for non-trivial deformation components of the kind given in (3.39). However, in general we will find components as in (3.37) that, as we will see, correspond to the structure constants of the unbroken gauge Lie algebra, as well as components as in (3.38) that in some cases for \( n = 6 \) are required to be non-trivial.

We state the result for \( n = 5 \) in terms of representatives of \( SO(n - 1) \) orbits in Table 3.2. In anticipation of phenomenological aspects to be discussed in Section 4 we also list some physical properties for the consistent solutions. Note that for \( n \leq 4 \) consistency is trivially given. Furthermore, in the case of \( n = 5 \) one cannot have \( m_1, m_2 \neq 0 \) which excludes solutions of the type (3.41).

| \( n \) | non-trivial components | \( \mathcal{N} = 2 \) multiplets | unbroken gauge group |
|-------|----------------------|------------------|-----------------|
| 1     | no \( G_1, G_4 \)    | \( 2 \times M_{2,1,0} \) | \( U(1)^3 \) |
| 2     | \( G_1 = G_4 = 0 \)  | \( 3 \times M_{2,1,0}, \) | \( U(1)^{3+1} \) |
|       |                      | \( 1 \times M_{2,1/2,BPS} \) of mass \( |c| \) |                 |
| 3     | \( G_1 = G_4 = 0 \)  | \( 4 \times M_{2,1,0}, \) | \( U(1)^{3+2} \) |
|       | \( m_1 \neq 0 \lor n_1 \neq 0 \) | \( 2 \times M_{2,1/2,BPS} \) of mass \( |c| \) | \( U(1)^3 \) |
|       |                      | \( 2 \times M_{2,1,BPS} \) of mass \( m_1^2 + n_1^2 \), |                 |
|       |                      | \( 1 \times M_{2,1/2,} \) of mass \( m_1^2 + |c| - n_1 - n_1^2 \), |                 |
|       |                      | \( 1 \times M_{2,1/2,BPS} \) of mass \( m_1^2 + |c| + n_1^2 \) |                 |
| 4     | \( G_1 = G_4 = 0, g_{123} \neq 0 \) | \( g_{123} \neq 0 \), | \( U(1)^4 \times SU(2) \) |
|       | \( G_1 = G_4 = 0, g_{123} = 0 \) | \( g_{123} = 0 \), | \( U(1)^{3+3} \) |
|       | \( m_1 \in \mathbb{R}, n_1 \neq 0, g_{123} \in \mathbb{R} \) | \( m_1 \neq 0 \lor n_1 \neq 0 \), | \( U(1)^{3+1} \) |
| 5     | \( G_1 = G_4 = 0, g_{123} \neq 0 \) | \( g_{123} \neq 0 \), | \( U(1)^{3+4} \times SU(2) \) |
|       | \( G_1 = G_4 = 0, g_{123} = 0 \) | \( g_{123} = 0 \), | \( U(1)^3 \) |
|       | \( G_1 = 0, n_1, n_2 \neq 0 \) | \( m_1 \neq 0 \lor n_1 \neq 0, m_2, n_2 \neq 0 \) | \( U(1)^3 \) |
|       | \( m_1 \neq 0, m_2 = 0, n_2 \neq 0 \) and \( g_{123} \in \mathbb{R} \) | \( m_1 \neq 0, m_2 = 0, n_2 \neq 0 \) |                 |
|       | \( m_1 \neq 0, m_2 = 0, n_2 \neq 0 \) | \( m_1 \neq 0, m_2 = 0, n_2 \neq 0 \) |                 |

Table 3.2: Consistent electric gaugings with \( \mathcal{N} = 2 \) vacuum for \( n \leq 5 \). Explanations are given in Section 3.3.1. We also always have the \( \mathcal{N} = 2 \) gravity multiplet \( M_{2,2,0} \) and the \( \mathcal{N} = 2 \) BPS gravitino multiplet \( M_{2,3/2,BPS} \) of mass \( |c| \). For brevity for \( n \geq 4 \) we do not list the \( \mathcal{N} = 2 \) spectrum (the \( \ldots \)). Note that here for convenience we set \( \psi_\pm = 1 \).

The result for \( n = 6 \) is given in terms of \( SO(5) \) gauge representatives in Table 3.3.\textsuperscript{10} We observe that consistent solutions may still have non-trivial deformation spaces.

\textsuperscript{10} There exist also solutions that are obtained from the ones given in Table 3.3 by a reflection \( T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_3 \in O(5) \).
| $G_1$ | $G_4$ | solutions: non-trivial $f_{\alpha \beta \epsilon}$, etc. | unbr. g. group |
|-------|-------|---------------------------------|----------------|
| $m_1, m_2 = 0$ | $n_1, n_2 = 0$ | $f_{123} = 0$ | $U(1)^{3+9}$ $U(1)^{3+2} \times SU(2)$ |
| $m_1, m_2 = 0$ | $n_1 \neq 0, n_2 = 0$ | $f_{123} \in \mathbb{R}$ $f_{345} \neq 0$ | $U(1)^{3+3}$ $U(1)^{3} \times SU(2)$ |
| $m_1, m_2 = 0$ | $0 \neq n_1^2 \neq n_2^2 \neq 0$ | $f_{123} \in \mathbb{R}$ | $U(1)^{3+1}$ |
| $m_1, m_2 = 0$ | $0 \neq n_1^2 = n_2^2 \neq 0$ | $G_5 = \begin{pmatrix} f_{345}^2 & f_{345} \ 0 & f_{345} \end{pmatrix} \otimes \varepsilon + \begin{pmatrix} 0 & f_{245} \ -f_{245} & 0 \end{pmatrix} \otimes \mathbb{1}_2$ | $U(1)^{3+1}$ |
| $m_1 \neq 0, m_2 = 0$ | $n_1 \in \mathbb{R}, n_2 = 0$ | $f_{123} \in \mathbb{R}$ $f_{345} \neq 0$ | $U(1)^{3+3}$ $U(1)^{3} \times SU(2)$ |
| $m_1 \neq 0, m_2 = 0$ | $n_1 \in \mathbb{R}, n_2 \neq 0$ | $f_{123} \in \mathbb{R}$ | $U(1)^{3+1}$ |
| $0 \neq m_1^2 \neq m_2^2 \neq 0$ | $n_1, n_2 \in \mathbb{R}$ | $G_5 = \begin{pmatrix} f_{345} \ 0 & 0 \ m_{12} & f_{345} \end{pmatrix} \otimes \varepsilon$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $n_1, n_2 = 0$ | $G_5 = \begin{pmatrix} f_{125} \ 0 & 0 \ 0 & f_{345} \end{pmatrix}$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $n_1 \neq 0, n_2 = 0$ | $G_5 = \begin{pmatrix} f_{125} \ 0 & 0 \ 0 & f_{345} \end{pmatrix}$ with $m_2^2 = f_{125} f_{345} - f_{245}^2$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $0 \neq n_1^2 \neq n_2^2 \neq 0$ | $G_5 = \begin{pmatrix} f_{125} \ 0 & 0 \ 0 & f_{345} \end{pmatrix}$ with $m_2^2 = f_{125} f_{345} - f_{245}^2$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $0 \neq n_1^2 = n_2^2 \neq 0$ | $G_5 = \begin{pmatrix} f_{125} \ 0 & 0 \ 0 & f_{345} \end{pmatrix}$ with $m_2^2 = f_{125} f_{345} - f_{245}^2$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $0 \neq n_1^2 = n_2^2 \neq 0$ | $G_5 = \begin{pmatrix} f_{345} \ 0 & 0 \ f_{345} & f_{345} \end{pmatrix}$ with $m_2^2 = f_{345}^2 - f_{245}^2$ | $U(1)^{3+1}$ |
| $m_1 = m_2 \neq 0$ | $0 \neq n_1^2 = n_2^2 \neq 0$ | $G_5 = \begin{pmatrix} f_{345} \ 0 & 0 \ f_{345} & f_{345} \end{pmatrix}$ with $m_2^2 = f_{345}^2 - f_{245}^2 + 2 \cot \phi f_{245} f_{345}$ | $U(1)^{3+1}$ |

Table 3.3: Consistent electric gaugings with $\mathcal{N} = 2$ vacuum for $n = 6$. Explanations are given in Section 3.3.1

### 3.3.2 Special solutions with $g_a = 0$ and $G_1 = 0$ for arbitrary $n \in \mathbb{N}$

A class of special solutions with $g_a = 0$ for arbitrary $n$ is obtained by setting $G_1 = 0$ which drastically simplifies the equations (3.30a) to (3.30d). Similarly to the discussion for general $G_1$ in Section 3.2.3, we can write $G_4$ as

$$G_4 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix},$$

(3.43)

where $D = diag(y_1, \ldots, y_1, y_2, \ldots, y_2, \ldots)$ is a diagonal matrix with ordered positive eigenvalues $y_1 > y_2 > \ldots > 0$. In doing so, the full solution to equation (3.30d) is...
analogous to the one given in terms of the (a priori) non-trivial components in (3.37), (3.38), and (3.39). For general such components, it is still hard to solve the last equations (3.30d). However, an interesting class of solutions is obtained after setting all but \( f_{\tilde{a} \tilde{b} \tilde{c}} \) to zero since then (3.30d) is just the Jacobi identity (3.31) for the gauge Lie algebra with structure constants \( f_{\tilde{a} \tilde{b} \tilde{c}} \in \mathbb{R} \). As stated above many non-trivial solutions to these equations are known, each of which corresponds to a compact reductive group \( G \). As we will see in Section 4.2 in those cases the unbroken gauge group that leaves the vacuum invariant is

\[
U(1)^3 \times G_{N=2}.
\]

Finally, anticipating the discussion of mass terms, we list the \( N=2 \) spectrum for such solutions in Table 3.4.

| block in \( G_4 \) | mass | \( N=2 \) multiplets |
|----------------------|------|---------------------|
| \( \theta_\kappa \)   | 0    | \( k \times M_{2,1,0} \) |
| \( |c| \)              |      | \( k \times M_{2,1/2,BPS} \) |
| \( y_i \otimes \varepsilon \) | \( y_i \) | \( 2 \times M_{2,1,BPS} \) |
| \( |c| - y_i \)         |      | \( 1 \times M_{2,1/2,BPS} \) |
| \( |c| + y_i \)         |      | \( 1 \times M_{2,1/2,BPS} \) |

Table 3.4: \( N=2 \) multiplets in the matter sector for the solutions in Section 3.3.2. In the gravity sector one has the \( N=2 \) gravity multiplet \( M_{2,2,0} \), the \( N=2 \) BPS gravitino multiplet \( M_{2,3/2,BPS} \) of mass \( |c| \), and two more \( N=2 \) vector multiplets \( M_{2,1,0} \). The consistency condition given in (3.17) is fulfilled with \( n'_v = 0 \), i.e. no non-BPS massive vector multiplets. Furthermore, note that for blocks with \( y_i = |c| \) one obtains massless hypermultiplets. This is of interest because together with massless vector multiplets these give rise to a non-trivial geometry of the scalar manifold in the effective \( N=2 \) theory.

## 4 Aspects of the \( N=2 \) low-energy effective theory

In an \( N=2 \) vacuum of \( N=4 \) supergravity the low-energy effective theory should be consistent with \( N=2 \) supersymmetry. In particular, we will show that the various fields can be consistently embedded into complete \( N=2 \) multiplets. We will then discuss the unbroken gauge group and, finally, we will comment on the effective Lagrangian below the scale of partial supersymmetry breaking. Bearing in mind that we have not yet fully solved the quadratic constraint equations, we will start generally but then restrict ourselves to the solutions with \( g_a = 0 \).

### 4.1 Mass terms in the gauged theory

The fermionic mass terms of the theory emerge from the fermion bilinears given in equations (2.7) and (2.8) after evaluating the \( A \)-matrices at the critical point (3.5). By construction, the gravitini mass matrix is diagonal and its two non-zero eigenvalues are given by (3.23). Masses for vector bosons arise from the gauge-covariant derivative acting
on the scalar fields. At the same time, the mixed couplings of vector bosons and scalar fields single out the pseudo-Goldstone fields that provide the longitudinal polarization of massive vector bosons. In the case of electric gaugings the scalars in the gravity multiplet are neutral \((D_\mu M_{\alpha\beta} = \partial_\mu M_{\alpha\beta})\) and thus the pseudo-Goldstone fields can only arise from the scalars of the vector multiplets. Using (2.3) together with all the information about the \(f_{MNP}\) obtained in the previous section, the gauged kinetic term of those scalars yields

\[
\frac{1}{16} (D_\mu M_{MN}) (D^\mu M^{MN}) = \frac{1}{16} (\partial_\mu M_{MN}) (\partial^\mu M^{MN}) \\
- \frac{g^2}{2} \sum_{a=1}^{n} \left( e_a^2 + f_a^2 + g_a^2 \right) \sum_{m \in \{2,3,5,6\}} A_m^a A^\mu_m \\
- \frac{g^2}{2} \sum_{b,c=1}^{n} O_{bc} A^b_\mu A^{\mu c} + \ldots ,
\]

(4.1)

where we introduced a symmetric and positive semi-definite matrix \((O_{ab}) \in \text{Mat}_{n,n}\) with components

\[
O_{bc} \equiv \sum_{a=1}^{n} \sum_{m=1}^{6} f_{abm} f_{acm}.
\]

(4.2)

The \(\ldots\) in (4.1) denote couplings of vectors and Goldstone bosons. Note that in (4.1) the terms mixing \(A^\mu_m\) and \(A^\mu_b\) are absent due to the quadratic constraints \((b, m, n, p)\) for \(m, n, p \in \{2, 3, 5, 6\}\).

Before reading off the masses of the vector bosons one has to canonically normalize their kinetic terms in (2.1). To this end, we redefine \(A'^\mu M = \sqrt{\text{Im} \tau} A^\mu M\), for a given background value \(\tau\), which amounts to scaling all mass terms in (3.1) by a factor of \(1/\text{Im} \tau\) as required by supersymmetry, cf. Section 3.1.1. It is then apparent that only four gauge bosons \((A^\mu_2, A^\mu_3, A^\mu_5, A^\mu_6)\) of the gravity multiplet become heavy and, due to (3.24), (D.11), their masses are degenerate and equal to the gravitino mass (3.23):

\[
m^2_{A^\mu_2, A^\mu_3, A^\mu_5, A^\mu_6} = \mathcal{V}^2 \sum_{a=1}^{n} \left( e_a^2 + f_a^2 + g_a^2 \right) = c^2 \mathcal{V}^2 = (m_{3/2})^2 .
\]

(4.3)

Thus, an \(\mathcal{N} = 2\) vacuum with two non-BPS gravitino multiplets would require at least four vector multiplets (i.e. \(n \geq 4\)), as in this case eight massive vector bosons are contained in the two gravitino multiplets (3.12). Eventually, the symmetric mass matrix \((O_{ab})\) will be diagonalized by means of an \(SO(n)\) transformation and being positive semi-definite it will give rise to well-defined mass terms. Note that for the solutions discussed in Section 3.3 we always have \(g_a = e_a = 0\) and \(G_2 = G_3 = G_5 = G_6 = 0\) and the above expressions are much simpler.

In order to analyze the potential (2.6) in a neighborhood of the origin of the scalar manifold, we employ the following chart

\[
\mathbb{R}^{6n} \supset U \quad \rightarrow \quad W \subset SO(6,n)/SO(6) \times SO(n), \\
\phi^{ma} \quad \mapsto \quad \exp \left( \sum_{m,a} \phi^{ma} [t_{ma}] \right) \equiv \mathcal{V}(\phi^{ma}) \equiv \mathcal{V} ,
\]

(4.4)
where \( [t_{ ma}]_M^N = \delta_{ [m}^N \eta_{ a] M} \) are the non-compact generators of the coset space associated to the vector multiplets. We can then express the scalar kinetic term as

\[
\frac{1}{16} (\partial_\mu M_{MN}) (\partial^\mu M^{MN}) = -\frac{1}{2} (\partial_\mu \phi^m) (\partial^\mu \phi^m) + O((\partial\phi)^2 \phi^2). \tag{4.5}
\]

As this kinetic term is canonically normalized, we can identify the coordinates \( \phi^m \) with the scalar degrees of freedom. Geometrically, these can be interpreted as fluctuations in \( SO(6, n)/[SO(6) \times SO(n)] \) around the critical point (3.5). It turns out that in the case of electric gaugings the two scalars of the gravity multiplet remain massless. Therefore, in an infinitesimal neighborhood of the origin where higher-order interactions are negligible, the scalar manifold of the gravity multiplet remains unaffected and thus can be ignored in what follows. Up to cubic terms, one finds:

\[
\mathcal{L}_{\text{pot}} = -\frac{V^2}{2} \left[ \sum_c (e_c^2 + f_c^2 + g_c^2) \sum_{m \in \{2,3,5,6\}} (\phi^m)^2 + \sum_{b,c} O_{bc} \sum_{m=1}^6 \phi^m \phi^m \right.
\]

\[
\left. + \sum_{a,b} \sum_{l,k=1}^6 \left( \sum_c f_{abc} f_{lbc} + \sum_{m=1}^6 f_{alm} f_{lmc} + \sum_c f_{akc} f_{lbc} + \sum_{m=1}^6 f_{akm} f_{lbm} \right) \phi^a \phi^b \right] + O(\phi^3). \tag{4.6}
\]

Note that the absence of linear terms in (4.6) is a necessary condition for metastability. Furthermore, the fact that the cosmological constant vanishes is due to the quadratic constraint (D.11), as we have seen earlier.

Now that we know all mass terms we can check the super-Higgs mechanism that is required by partial supersymmetry breaking. First, we will consider the gravity/Goldstini sector, and secondly, we will discuss the matter sector. As a result, we will also show that the vacuum solutions are metastable, as required by the preserved \( \mathcal{N} = 2 \) supersymmetry. We will restrict ourselves to the case \( g_a = 0 \), which as we have seen in Section 3.2.3 implies \( e_a = 0 \) and \( G_2 = G_3 = G_5 = G_6 = 0 \). For such solutions the potential simplifies to

\[
\mathcal{L}_{\text{pot}} = -\frac{V^2}{2} \left[ c^2 \sum_{m \in \{2,3,5,6\}} \phi^m \phi^m + \sum_{m \in \{2,3,5,6\}} O_{\bar{a} \bar{b}} \phi^m \phi^m + 4c f_{\bar{a} \bar{b} 4} (\phi^{2 \bar{a}} \phi^{3 \bar{b}} + \phi^{5 \bar{a}} \phi^{6 \bar{b}}) \right.
\]

\[
\left. + f_{\bar{c} \bar{a} 4} f_{\bar{d} \bar{b} 4} \phi^{1 \bar{a}} \phi^{1 \bar{b}} + f_{\bar{c} \bar{a} 1} f_{\bar{d} \bar{b} 1} \phi^{4 \bar{a}} \phi^{4 \bar{b}} - 2f_{1 \bar{a} \bar{c}} f_{4 \bar{b} \bar{e}} \phi^{4 \bar{a}} \phi^{1 \bar{b}} \right] + O(\phi^3), \tag{4.7}
\]

where as before we denote the potentially non-trivial embedding tensor components by \( f_{\bar{a} \bar{b} m} \) for \( SO(n-1) \) indices \( \bar{a}, \bar{b}, \) etc.

### 4.1.1 Gravity/Goldstini sector

In the gauge where \( f_a = c \delta_{a7} \) it is only the “first” \( \mathcal{N} = 4 \) vector multiplet that contributes to the gravity/Goldstini sector. After canonically diagonalizing the kinetic terms of the
fermions by means of the field redefinition \( \chi^i = \frac{1}{\sqrt{2}} \lambda^i \) we find that the fermionic mass terms in this sector read\(^{11}\)

\[
c[\psi_\mu^3 \epsilon^{\sigma \mu} \psi_\sigma^3 + \frac{1}{2} \sqrt{2} \bar{\eta}^{(3)} \sigma^\mu \psi_\mu^3 \\
+ \psi_\mu^4 \epsilon^{\sigma \mu} \psi_\sigma^4 + \frac{1}{2} \sqrt{2} \bar{\eta}^{(4)} \sigma^\mu \psi_\mu^4 \\
- \sqrt{2} \chi^3 (\lambda^{74})* - \frac{1}{2} (\lambda^{74})* \epsilon (\lambda^{74})* \\
+ \sqrt{2} \chi^4 (\lambda^{73})* - \frac{1}{2} (\lambda^{74})* \epsilon (\lambda^{74})* \] + h.c. ,
\]

where the would-be Goldstino combinations eaten by the massive gravitini are

\[
\bar{\eta}^{(3)} = \bar{\eta}^{(3)A} = \epsilon \bar{A} \bar{B} \bar{\chi}^3_B + \sqrt{2}(\lambda^{74A})* , \quad \bar{A}, \bar{B} = 1, 2 \\
\bar{\eta}^{(4)} = \bar{\eta}^{(4)A} = \epsilon \bar{A} \bar{B} \bar{\chi}^4_B - \sqrt{2}(\lambda^{73A})* .
\]

The mass terms for the spin-1/2 fermions \( \chi_1^i, \chi_2^j, \lambda^j, \lambda_i^2 \) are absent in (4.8) and thus these fermions are massless. As in [30], mixed terms involving both a gravitino and a spin-1/2 fermion can be removed by means of the following gravitino shifts

\[
\bar{\eta}^{(3)} = \bar{\eta}^{(3A)} = \epsilon \bar{A} \bar{B} \bar{\chi}^3_B + \sqrt{2}(\lambda^{74A})* , \\
\bar{\eta}^{(4)} = \bar{\eta}^{(4A)} = \epsilon \bar{A} \bar{B} \bar{\chi}^4_B - \sqrt{2}(\lambda^{73A})* .
\]

yielding additional contributions to the mass matrix of the spin-1/2 fermions. As a result, their mass terms read

\[
\begin{align*}
\frac{c}{2} \left[ \left( (\lambda^{74A})* , \epsilon \bar{A} \bar{B} \bar{\chi}^3_B \right) \epsilon_{AC} M(-) \left( (\lambda^{74C})* \right) \\
+ \left( (\lambda^{73A})* , \epsilon \bar{A} \bar{B} \bar{\chi}^4_B \right) \epsilon_{AC} M(+) \left( (\lambda^{73C})* \right) \right] + h.c. ,
\end{align*}
\]

where the mass matrices \( M(\pm) \) are given by

\[
M(\pm) = \frac{1}{3} \left( \begin{array}{cc} 1 & \pm \sqrt{2} \\
\pm \sqrt{2} & 2 \end{array} \right) ,
\]

and both have eigenvalues 0 and 1. In fact, the two zero eigenvalues give rise to two massless helicity-1/2 fermions to be identified as the would-be Goldstini associated to the broken supersymmetry. On the other hand, one finds two spin-1/2 fermions of mass \( |c| \) that together with the two massive gravitini fit into the \( \mathcal{N} = 2 \) BPS gravitino multiplet.

As to the bosons in this sector, (4.11) shows that the only massive vectors are \( A_\mu^2, A_\mu^3, A_\mu^5, A_\mu^6 \) while the massless ones are \( A_\mu^1, A_\mu^4, A_\mu^7 \). The four massive vectors belong to the \( \mathcal{N} = 2 \) BPS gravitino multiplet as we shall show in (4.13). Finally, all eight scalars of this sector are massless, as can be seen from (4.17), four of which are to be interpreted as the would-be Goldstone bosons. In an infinitesimal neighborhood around the critical point these fluctuations are described by \( \phi_1^7, \phi_3^7, \phi_5^7, \phi_6^7 \).

---

\(^{11}\)From now on we will drop the overall scaling factor of \( \sqrt{2} \).
To conclude, we have shown that the fields in the massive BPS gravitino multiplet all have the same mass, consistent with $\mathcal{N} = 2$ supersymmetry. Furthermore, in the gravity/Goldstini sector the $\mathcal{N} = 2$ gravity multiplet and the massive $\mathcal{N} = 2$ BPS gravitino multiplet are accompanied by two massless $\mathcal{N} = 2$ vector multiplets, which are the remnants of the minimal $\mathcal{N} = 4$ multiplets required for spontaneous partial supersymmetry breaking to $\mathcal{N} = 2$.

![Table 4.1: Gravity/Goldstini sector of the $\mathcal{N} = 2$ spectrum.](image)

**4.1.2 Matter sector**

The mass squared matrix for vector bosons $A^{i\tilde{a}}$ defined in (4.2) now reads

$$O = -G_1^2 - G_4^2,$$

which according to the discussion in Section 3.2.3 is already diagonal. For each block in $G_1$ and $G_4$ with degenerate eigenvalues

$$(G_1^{(ij)})^2 = -x^2 \ 1_t, \quad (G_4^{(ij)})^2 = -y^2 \ 1_t,$$

where $x, y \in \mathbb{R}$, one finds $l$ vectors of mass squared $x^2 + y^2$.

Using the explicit expression given for the A-matrices in (C.4) the mass terms (2.8) for the fermions $\lambda^{1\tilde{a}}, \lambda^{2\tilde{a}}$ are given by

$$\frac{1}{2} ((\lambda^{\tilde{a}1})^*, (\lambda^{\tilde{a}2})^*) \epsilon U \left( (\lambda^{\tilde{a}1})^* \right) + \text{h.c.},$$

with

$$U = \begin{pmatrix} 0 & iG_1 + G_4 \\ -iG_1 - G_4 & 0 \end{pmatrix}.$$

Thus, their mass squared matrix

$$UU^\dagger = \begin{pmatrix} -G_1^2 - G_4^2 & 0 \\ 0 & -G_1^2 - G_4^2 \end{pmatrix},$$

is also diagonal by virtue of the quadratic constraints (D.71). Similarly, the mass terms for $\lambda^{3\tilde{a}}, \lambda^{4\tilde{a}}$ in (2.8) are given by

$$\frac{1}{2} ((\lambda^{\tilde{a}3})^*, (\lambda^{\tilde{a}4})^*) \epsilon V \left( (\lambda^{\tilde{a}3})^* \right) + \text{h.c.},$$

where

$$V = \begin{pmatrix} -c & -iG_1 + G_4 \\ iG_1 - G_4 & -c \end{pmatrix}.$$
The corresponding mass squared matrix reads
\[ VV^\dagger = \begin{pmatrix} c^2 - G_1^2 - G_4^2 & -2cG_4 \\ 2cG_4 & c^2 - G_1^2 - G_4^2 \end{pmatrix}. \] (4.20)
As in (4.14), it can be shown that for each block in \( G_1 \) and \( G_4 \) the eigenvalues are
\[ x^2 + (|c| \pm |y|)^2, \] (4.21)
with degeneracy \( l \) each.

We can read off the mass terms for the scalar fields \( \phi^{1\tilde{a}}, \phi^{4\tilde{a}} \) directly from (4.7),
\[-\frac{1}{2} \left( \phi^{1\tilde{a}}, \phi^{4\tilde{a}} \right) Z \begin{pmatrix} \phi^{1\tilde{b}} \\ \phi^{4\tilde{b}} \end{pmatrix}, \] (4.22)
where
\[ Z = \begin{pmatrix} -G_4^2 & G_4G_1 \\ G_1G_4 & -G_1^2 \end{pmatrix}. \] (4.23)
Obviously, for the trivial block in \( G_1 \) and \( G_4 \) with \( x, y = 0 \) one obtains (2\( l \)) massless scalars. On the other hand, for each block with \( x \neq 0 \) or \( y \neq 0, l = 2l' \) has to be even and the eigenvalues of \( Z \) turn out to have (2\( l' \))-fold degenerate eigenvalues
\[ 0, \quad (x^2 + y^2). \] (4.24)
The zero eigenvalue set precisely corresponds to the would-be Goldstone modes eaten by the (2\( l' \)) vector bosons that become massive. Finally, the mass terms for the remaining scalars \( \phi^{2\tilde{a}}, \phi^{3\tilde{a}}, \phi^{5\tilde{a}}, \phi^{6\tilde{a}} \) turn out to be
\[-\frac{1}{2} \left( \phi^{2\tilde{a}}, \phi^{3\tilde{a}} \right) \begin{pmatrix} c^2 - G_1^2 - G_4^2 & -2cG_4 \\ 2cG_4 & c^2 - G_1^2 - G_4^2 \end{pmatrix} \begin{pmatrix} \phi^{3\tilde{b}} \\ \phi^{2\tilde{b}} \end{pmatrix}, \]
\[-\frac{1}{2} \left( \phi^{6\tilde{a}}, \phi^{5\tilde{a}} \right) \begin{pmatrix} c^2 - G_1^2 - G_4^2 & -2cG_4 \\ 2cG_4 & c^2 - G_1^2 - G_4^2 \end{pmatrix} \begin{pmatrix} \phi^{6\tilde{b}} \\ \phi^{5\tilde{b}} \end{pmatrix}, \] (4.25)
where the mass squared matrices are precisely \( VV^\dagger \). As a result, one has (2\( l' \)) scalars for each mass in (4.21). It is then clear that all masses-squared are positive and therefore metastability is guaranteed, as required for a supersymmetric theory with Minkowski background. Furthermore, one finds that all degrees of freedom in the matter sector fit into complete \( \mathcal{N} = 2 \) supermultiplets. The resulting \( \mathcal{N} = 2 \) spectrum is summarized in Table 4.2. Note that blocks in \( G_1 \) and \( G_4 \) with \( x = 0 \) and \( |y| = |c| \) give rise to massless \( \mathcal{N} = 2 \) hypermultiplets.

### 4.1.3 BPS multiplets

So far in the discussion of mass terms we have only shown that all fields fit into complete \( \mathcal{N} = 2 \) multiplets. In particular, according to our assignments in Tables 4.1 and 4.2 all massive fields lie in BPS representations. In the generic case where the masses of the various \( \mathcal{N} = 2 \) superfields are all different, the above assignments are obviously correct.
Table 4.2: Matter sector of the $\mathcal{N} = 2$ spectrum. The matrices $G_1, G_4 \in \text{Mat}_{n-1,n-1}$ are simultaneously block-diagonal with non-trivial blocks of the type given in Table (3.1) or zero blocks.

However, in the case of mass degeneracies between various short $\mathcal{N} = 2$ superfields one should exclude the case where short multiplets combine in order to form long multiplets. In fact, in what follows we will show that in the case of $g_a = 0$ all massive fields have to be in BPS representations and that no long $\mathcal{N} = 2$ multiplet can occur in this super-Higgs mechanism. To this end we will study the crucial parts of the supersymmetry transformations of the bosonic fields that we take from [24]. It suffices to analyze the supersymmetry transformations of the massive bosons.

We first consider the massive vectors $A_{\mu}^{2}, A_{\mu}^{3}, A_{\mu}^{5}, A_{\mu}^{6}$ in the gravity/Goldstini sector. Evaluating their supersymmetry transformations at the origin (3.5) of $SO(6, n)$ one finds

$$\delta_{\epsilon} A_{\mu}^{m} \sim [G_{m}]_{ij} (\epsilon^i \epsilon^j \phi_{\mu}^{\gamma} + \epsilon^i \epsilon^j \phi_{\mu}^{\chi}) + \text{h.c.}$$  \hspace{1cm} (4.26)

for $m = 2, 3, 5, 6$. Moreover, as in (2.11), $\epsilon^i = q^{i} \eta$ contains the $SU(4)$ vector $q^{i}$ and $[G_{m}]_{ij}$ denote the 't Hooft matrices given in (3.9). In our gauge, cf. (3.9), the unbroken supersymmetry directions are given by linear combinations of $q^{1}$ and $q^{2}$ (or $\epsilon^1$ and $\epsilon^2$). As a result, for $m = 2, 3, 5, 6$ the massive vectors $A_{\mu}^{m}$ transform into the fermions $\psi_{\mu}^{3}, \psi_{\mu}^{4}, \chi^{3}, \chi^{4}$. While massive scalars are not present in the gravity/Goldstini sector, we will now inspect the transformations of the four Goldstone bosons that provide the longitudinal polarization of the massive vector bosons. In an infinitesimal neighborhood of the origin these fluctuations are described by the scalars $\phi^{27}, \phi^{37}, \phi^{57}, \phi^{67}$. Using the explicit chart (4.4) of $SO(6, n)$ one finds

$$\delta_{\epsilon} \mathcal{V}_{\mu}^{a} = \delta_{\epsilon} \phi^{ma} + \mathcal{O}(\phi \delta \phi),$$  \hspace{1cm} (4.27)

which when evaluated at the origin can again be expressed in terms of the 't Hooft matrices as

$$\delta_{\epsilon} \phi^{ma} \sim [G_{m}]_{ij} \epsilon^i \epsilon^j \chi^{a} + \text{h.c.}. \hspace{1cm} (4.28)$$

In particular, we find that the Goldstone bosons $\phi^{27}, \phi^{37}, \phi^{57}, \phi^{67}$ transform under $\mathcal{N} = 2$ into fermions $\lambda^{73}, \lambda^{74}$. As a result, the massive bosons of the gravity/Goldstini sector transform into the massive fermions of the same sector. Note that the gravitino shifts in (4.10) also only involves the aforementioned fermions.

Next, we will analyze the supersymmetry transformations of the bosonic fields in the matter sector. The supersymmetry transformations of the massive vectors $A_{\mu}^{8}$ evaluated

\footnote{While our proof is somewhat indirect, it does not require the supersymmetry transformations of the fermions which are not fully given in [24].}
at the origin are given by \( \delta \epsilon A_{\mu}^\hat{a} \sim \epsilon \bar{\epsilon} \tilde{\epsilon} \epsilon (\lambda^{\hat{a}i})^* + \text{h.c.} \) \hspace{1cm} (4.29)

As a consequence, restricting the transformations to \( N = 2 \) one finds that each massive vector boson \( A_{\mu}^\hat{a} \) rotates into the gaugini \( \lambda^{\hat{a}1} \) and \( \lambda^{\hat{a}2} \) but not into \( \lambda^{\hat{a}3} \) and \( \lambda^{\hat{a}4} \). Furthermore, as we discussed below \( \text{(4.22)} \), the associated Goldstone bosons are accompanied by massive scalars. Infinitesimally, all of them are described by linear combinations of the scalar fields \( \phi^{\hat{1}a} \) and \( \phi^{\hat{4}a} \).

In particular, this also shows that neither the would-be Goldstone combinations nor the massive scalars in \( \text{(4.22)} \) transform into \( \lambda^{\hat{a}3} \) and \( \lambda^{\hat{a}4} \). Furthermore, it is worth mentioning that neither \( A_{\mu}^\hat{a} \) nor the massive scalars in \( \text{(4.22)} \) transform into the spin-1/2 fermions in the gravity/Goldstini sector given in \( \text{(4.11)} \), let alone into the massless gravitini. Finally, the only remaining potentially massive bosons are the scalars \( \phi^{\hat{2}a}, \phi^{\hat{3}a}, \phi^{\hat{5}a}, \phi^{\hat{6}a} \) in \( \text{(4.25)} \).

As can again be seen from \( \text{(4.28)} \), they only transform into fermions \( \lambda^{\hat{a}3}, \lambda^{\hat{a}4} \) and never into \( \lambda^{\hat{a}1}, \lambda^{\hat{a}2} \), let alone into fermions of the gravity/Goldstini sector.

We can now conclude that all massive \( N = 2 \) supermultiplets have to be BPS multiplets. The argument goes as follows: We found that the massive fields in the gravity/Goldstini sector and the massive fields in the matter sector are not related by supersymmetry transformations acting on the bosonic fields. This implies that the massive fields in the gravity/Goldstini sector have to lie in a BPS gravitino multiplet as massive long gravitino multiplets can never be decomposed into two non-trivial sets of bosons and fermions such that within each set the bosons only mix into the fermions, respectively. This follows from the construction of supermultiplets as representations of the Clifford algebra. Furthermore, by the same token, the remaining massive vector bosons have to be in \( N = 2 \) BPS vector multiplets.

4.2 Unbroken gauge group

We shall now investigate the unbroken gauge group at the \( N = 2 \) critical point, i.e. the group that leaves the scalar vacuum configuration for consistent electric gaugings with \( g_a = 0 \) invariant. First, we note that the critical point in \( SL(2)/SO(2) \) is not affected by gauge transformations. However, on the scalar matter fields a generic gauge transformations parametrized by a gauge parameter \( \theta^P \) acts as

\[
M_{MN} \rightarrow M_{MN} + 2 \theta^P f_{PM}Q M_{NQ},
\]

and, in particular, the coset representative of the origin of \( SO(6, n)/[SO(6) \times SO(n)] \) transforms as

\[
1_{MN} \rightarrow 1_{MN} + 2 \theta^P (f_{PM}N + f_{PN}M).
\]

In demanding invariance of the origin under \( \text{(4.31)} \), the gauge parameters are restricted to the ones with \( \theta^m = 0 \) for \( m = 2, 3, 5, 6 \), and \( \theta^{\hat{a}} = 0 \) for each massive vector boson

\[^{13}\text{As in Section 4.2 indices } \hat{a}, \hat{b}, \ldots \text{ denote } SO(n - 1) \text{ indices } \hat{a}, \hat{b}, \ldots \text{ associated to massive vector bosons, i.e. to non-trivial blocks in either } G_1 \text{ or } G_4.\]
\( A_\mu^\alpha \), the latter of which requires a non-zero block in \( G_1 \) or \( G_4 \). Gauge transformations of vector fields read \( [12, 18] \)

\[
\delta A_\mu^M = \partial_\mu \theta^M + X_{PQ}^M A_\mu^P \theta^Q,
\]

where one has

\[
X_{MN}^P = - f_{MN}^P.
\]

Using our knowledge of certain embedding tensor components in the case of \( g_a = 0 \) one can compute the gauge transformation for the massless vector bosons, which in this section we will denote as \( A_\mu^\nu \) so as to distinguish them from massive vectors \( A_\mu^\alpha \). While we dropped the \( \sim \) above indices, \( \bar{a} \) and \( \hat{a} \) are still understood as \( SO(n-1) \) indices. One finds

\[
\delta A_\mu^1 = \partial_\mu \theta^1,
\]

\[
\delta A_\mu^4 = \partial_\mu \theta^4,
\]

\[
\delta A_\mu^7 = \partial_\mu \theta^7,
\]

\[
\delta A_\mu^\bar{a} = \partial_\mu \theta^{\bar{a}} - f_{\bar{a}bc} A_\mu^b \theta^c.
\]

Note that in the last line of (4.34) we made use of \( f_{\bar{a}bc} = 0 \), which we learned from the quadratic constraints \((\tilde{b}, \tilde{c}, \tilde{d}, 1)\) and \((\tilde{b}, \tilde{c}, \tilde{d}, 4)\). The transformations (4.34) imply that we can interpret the three fields \( A_\mu^1, A_\mu^4, A_\mu^7 \) as the vector bosons of a gauge group \( U(1)^3 \). On the other hand, the embedding tensor components \( f_{\bar{a}bc} \) amount to the structure constants of the gauge Lie algebra associated to the massless vector bosons \( A_\mu^\bar{a} \). In fact, as already pointed out in the simple case of (3.31), the quadratic constraints for \((\tilde{b}, \tilde{c}, \tilde{d}, \tilde{e})\) are simply the Jacobi identity

\[
f_{\bar{a}bc} f_{\bar{d}e\bar{a}} + f_{\bar{a}bc} f_{\bar{d}e\bar{a}} - f_{\bar{a}bd} f_{\bar{ace}} = 0,
\]

that gives rise to a gauge Lie group \( G_{N=2} \). Its dimension equals the number of massless vector bosons \( \leq n-1 \). If \( n \) is sufficiently large, any compact reductive Lie group can be chosen in order to satisfy (4.35). As a result, the full unbroken gauge symmetry is

\[
U(1)^3 \times G_{N=2}.
\]

On the other hand, it is important to note that there is an additional set of constraints on the components \( f_{\bar{a}bc} \) coming from the quadratic equations for \((\tilde{b}, \tilde{c}, \tilde{d}, \tilde{e})\):

\[
f_{\bar{a}bc} f_{\bar{d}e\bar{a}} + f_{\bar{a}bc} f_{\bar{d}e\bar{a}} - f_{\bar{a}bd} f_{\bar{ace}} = 0.
\]

As we have seen in Section 3.3 it is not always possible to set all \( f_{\bar{a}bc} \) (i.e. the components given in (3.38)) to zero such that (4.37) is trivially satisfied. However, we have already shown in Section 3.3.2 that consistent examples exist for any given compact reductive Lie group \( G_{\bar{N}=2} \).

4.3 Scalar manifold in the effective theory

Below the scale of supersymmetry breaking \( m_{3/2} \) we may integrate out heavy particles and, in doing so, arrive at an \( \mathcal{N} = 2 \) supersymmetric effective action. We are particularly interested in the geometry of the scalar manifold of this effective action. As before, we
will consider the case of electric gaugings with \( g_a = 0 \). In the limit where momenta \( p \ll m_{3/2} \) can be neglected, the equations of motion for the massive vectors are purely algebraic and can be solved for the massive vector bosons since their mass terms are automatically diagonal, as we discussed in Section 4.1.2. One finds

\[
A^a_\mu = -\frac{1}{2} \sum_{m=\{2,3,5,6\}} \left( \partial_\mu M_{m\bar{a}} \right) f_{7mn},
\]

\[
A^b_\mu = -\frac{1}{2m_{(b)}^2} \sum_{m=\{1,4\}} \left( \partial_\mu M_{m\bar{a}} \right) f_{\bar{a}bm}.
\]  

(4.38)

for each \( n \in \{2,3,5,6\} \) and massive vectors with index \( \hat{b} \). When inserted back into the Lagrangian and using our knowledge about certain embedding tensor components, the scalar kinetic term yields\(^1\)

\[
\mathcal{L}_{\text{eff}} = \frac{1}{16} \left[ 2 \sum_{m=\{2,3,5,6\}} \left( \partial_\mu M_{m\bar{a}} \right) \left( \partial^\mu M^{\bar{m}\bar{a}} \right) + 2 \sum_{m=\{1,4\}} \left( \partial_\mu M_{m\bar{a}} \right) \left( \partial^\mu M^{m\bar{7}} \right) + 2 \sum_{m=\{1,4\}} \left( \partial_\mu M_{m\bar{a}} \right) \left( \partial^\mu M^{\bar{m}\bar{7}} \right) + \sum_{m=\{1,4\}} \left( \partial_\mu M_{m\bar{a}} \right) \left( \partial^\mu M^{m\bar{7}} \right) + \left( \partial_\mu M_{mn} \right) \left( \partial^\mu M^{mn} \right) + \left( \partial_\mu M_{ab} \right) \left( \partial^\mu M^{ab} \right) \right].
\]  

(4.39)

Using the chart \([1,4]\) one finds

\[
-\frac{1}{2} \sum_{m=\{2,3,5,6\}} \left( \partial_\mu \phi^{m\bar{a}} \right) \left( \partial^\mu \phi^{m\bar{a}} \right) - \frac{1}{2} \sum_{m=\{1,4\}} \left( \partial_\mu \phi^{m\bar{7}} \right) \left( \partial^\mu \phi^{m\bar{7}} \right) - \frac{1}{2} \sum_{m=\{1,4\}} \left( \partial_\mu \phi^{m\bar{a}} \right) \left( \partial^\mu \phi^{m\bar{a}} \right) - \frac{1}{2} \left( \partial_\mu \phi^{\bar{1}\bar{a}}, \partial_\mu \phi^{\bar{4}\bar{a}} \right) (\left( O^{\text{massive}} \right)^{-1} Z^{\text{massive}})_{\hat{a}b} \left( \partial^{\mu} \phi^{\bar{1}\bar{b}} \right) + O \left( \left( \partial \phi \right)^2 \phi^2 \right),
\]  

(4.40)

where \( O^{\text{massive}} \) is the truncation of \([1,13]\) to an invertible matrix obtained after deleting all its zero rows and columns, and similarly, \( Z^{\text{massive}} \) is the analogous truncation of the mass matrix \( Z \) defined in \([1,23]\). Note that kinetic terms for the Goldstone modes \( \phi^{m\bar{a}} \) for \( m = 2, 3, 5, 6 \) are absent in \([1,40]\) as these scalars have been eaten by the massive vector bosons \( A^{m\bar{a}} \) for \( m = 2, 3, 5, 6 \). Moreover, the same diagonalization scheme of Section 4.1.2 also diagonalizes the kinetic terms of the scalars \( \phi^{\bar{1}\bar{a}} \) and \( \phi^{\bar{4}\bar{a}} \) associated to massive vectors with indices \( \hat{a} \). As before, the zero eigenvalues of \( Z^{\text{massive}} \) ensure that the kinetic terms of the Goldstone modes in the matter sector vanish (again the Goldstone modes are eaten by the vector bosons \( A^{\mu\bar{a}} \) that acquire mass). On the other hand, its nonzero eigenvalues are such that the remaining kinetic terms are canonically normalized, which justifies the mass assignment in Section 4.1.2.

\(^1\)Repeated indices are summed over their full index range unless otherwise specified by explicit summation symbols.
Let us now summarize the dynamical degrees of freedom in an infinitesimal neighborhood of the origin. The scalars $\phi^{m \tilde{a}}$ for $m = 2, 3, 5, 6$ lie in light (with respect to $m_{3/2}$) $\mathcal{N} = 2$ (BPS) hypermultiplets, while $\phi^{17}$ and $\phi^{47}$ and the two scalars of $SL(2)/SO(2)$ lie in the two massless $\mathcal{N} = 2$ multiplets that descend from the gravity/Goldstini sector. The scalars $\phi^{1\tilde{a}}, \phi^{4\tilde{a}}$ form $\mathcal{N} = 2$ massless vector multiplets, while the non-Goldstone modes of the $\phi^{1\tilde{a}}, \phi^{4\tilde{a}}$ belong to $\mathcal{N} = 2$ BPS vector multiplets. Note, however, that in the effective theory below the scale of partial supersymmetry breaking $m_{3/2}$, all scalars (and their supersymmetry partners) with masses larger than $m_{3/2}$ should also be integrated out.

As the scalars of $SL(2)/SO(2)$, described by $\tau$, are moduli that lie in a massless $\mathcal{N} = 2$ vector multiplet, the $SL(2)/SO(2)$ factor of the $\mathcal{N} = 4$ scalar manifold descends without change to the scalar field space of the massless $\mathcal{N} = 2$ vector multiplets in the low-energy theory. If the number of these vector multiplets is $(k + 1)$, we conjecture that the vector multiplet field space of the $\mathcal{N} = 2$ low-energy theory is the following product of coset spaces,

$$SL(2)/SO(2) \times SO(2,k)/SO(2) \times SO(k),$$

which is known to be the only series of special Kähler product manifolds including a factor of $SL(2)/SO(2)$ \[27\]. Moreover, since we only analyze the potential to quadratic order, we can only infer that the moduli space is a submanifold of $(4.41)$. To see this explicitly, one should reconstruct the metric of the scalar manifold order by order (due to the power expansion of the exponential map in $(4.4)$). As we saw in Section 3.3.2, it is also possible to have light or massless hypermultiplets, in which case $\mathcal{N} = 2$ supersymmetry requires the field space to be quaternionic Kähler. However, a complete analysis of the scalar geometry is beyond the scope of this paper.

5 Conclusion

We have studied $\mathcal{N} = 2$ vacua of gauged $\mathcal{N} = 4$ supergravity theories focusing on the class of theories with only electric gaugings i.e. vanishing de Roo-Wagemans angles. We reviewed the early result that in such an electrically gauged $\mathcal{N} = 4$ theory, vacua which preserve $\mathcal{N} = 1, 2$ or 4 are necessarily Minkowski and that $\mathcal{N} = 3$ vacua do not exist. Following the observation in \[26\], we discussed in detail how the homogeneity of the scalar manifold and the symmetry of the Lagrangian allows one to carry out the analysis of the gravitino mass matrices and supersymmetry conditions at the origin, which leads to significant simplifications when studying supersymmetry breaking.

In order to construct explicit solutions with spontaneous partial supersymmetry breaking, we then focused on $\mathcal{N} = 2$ vacua. We discussed the possible branching rules for $\mathcal{N} = 4$ supermultiplets, showing that it was possible to have an $\mathcal{N} = 2$ spectrum with one short massive BPS gravitino multiplet or two long massive gravitino multiplets. We then constructed the solutions to the linear conditions that follow from the Killing spinor equations for an $\mathcal{N} = 2$ vacuum, given in terms of a set of embedding tensor components (charges). Consistency of the corresponding gaugings with supersymmetry and gauge invariance required that this set of embedding tensor components satisfy the quadratic constraints (2.4).
We believe that it is difficult to solve the quadratic constraints in general (as argued to some extent in Appendix D.1.2) and so we focussed on the case where a subset of the embedding tensor components vanish \((g_a = 0)\), which holds automatically when the number of \(N = 4\) vector multiplets \(n\) is less or equal than six. In the appendix we showed that if a solution with \(g_a \neq 0\) were to exist, then it would necessarily require the number of vector multiplet \(n\) to be greater than 6. Setting \(g_a = 0\) corresponds to minimizing the couplings between the gaugini and the gravitini in the \(N = 4\) Lagrangian, and therefore heuristically should make it easier to guarantee supersymmetry and gauge invariance. We showed that when \(g_a = 0\) one can arrange for only one \(N = 4\) vector multiplet to contribute to the Goldstini. For the class of gaugings with \(g_a = 0\) and \(n \leq 6\) we gave the solutions of the quadratic constraints and the unbroken gauge groups. We also found solutions for \(n > 6\) with an additional set of gaugings (and couplings) turned off.

We then analyzed the mass terms and showed that all fields assembled in \(N = 2\) multiplets with appropriate mass degeneracies. In particular, all massive \(N = 2\) multiplets (including the gravitino multiplet) have to be BPS. We further showed that vacua exist with unbroken gauge group

\[
U(1)^3 \times G_{N=2},
\]

where \(G_{N=2}\) can be any compact reductive Lie group if the number \(n\) of \(N = 4\) vector multiplets is sufficiently large.

Finally, we computed the effective \(N = 2\) action which is valid below the scale of supersymmetry breaking. We found that the complex scalar \(\tau\) of the \(N = 4\) gravity multiplet cannot contribute to the super-Higgs mechanism i.e. it is not charged under the \(N = 4\) gauge group. This implies that the \(SL(2)/SO(2)\) factor parametrized by \(\tau\) in the \(N = 4\) moduli space descends directly to an \(SL(2)/SO(2)\) factor in the \(N = 2\) moduli space. For vacua with additional \((k + 1)\) massless \(N = 2\) vector multiplets we therefore conjectured that the vector multiplet moduli space has to be

\[
SL(2)/SO(2) \times SO(2,k)/SO(2) \times SO(k)
\]

as this series is the only possible special Kähler manifolds that are product manifolds [27]. We also found that it is possible to have massless hypermultiplets. In this case \(N = 2\) requires a field space which is quaternionic Kähler. We leave a complete analysis of the scalar geometry for future work.

## 6 Acknowledgements

We would like to thank Vicente Cortés, Diederik Roest, Henning Samtleben, Claudio Scrucca, Wolfgang Soergel, Hagen Triendl and Owen Vaughan for useful discussions. This work was partly supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 “Particles, Strings and the Early Universe”. The research of P.S. is supported by the Swiss National Science Foundation under the grant PP00P2-135164.
Appendix

A Conventions

The spacetime metric $g_{\mu\nu}$ used in this paper has signature $(-,+,+,+)$ and the totally antisymmetric tensor $e^{\mu\nu\rho\lambda}$ is defined with $e^{0123} = e^{-1}, e_{0123} = -e = -\sqrt{|\det g|}$.

We use the following indices:

| indices | group |
|---------|-------|
| $\alpha, \beta, \gamma, \ldots \in \{-, +\}$ | $SL(2)$ |
| $M, N, P, \ldots \in \{1, \ldots, 6 + n\}$ | $SO(6, n)$ |
| $m, n, p, \ldots \in \{1, \ldots, 6\}$ | $SO(6)$ |
| $i, j, k, \ldots \in \{1, \ldots, 4\}$ | $SU(4)$ |
| $a, b, c, \ldots \in \{1, \ldots, n\}$ | $SO(n)$ |
| $\tilde{a}, \tilde{b}, \tilde{c}, \ldots \in \{1, \ldots, n - 1\}$ | $SO(n - 1) \subset SO(n)$ |

All indices other than the ones of $SU(4)$ transform under the fundamental representation of the given groups. In the case of $SU(4)$ upper/lower indices transform under the $4$ ($\bar{4}$), respectively. Upon complex conjugation such upper and lower indices are interchanged, e.g. $(X_i^j)^* = X^{ij}$.

In addition, for the $g_a = 0$ solutions discussed in Sections 4.2 and 4.3 we use $SO(n - 1) \subset SO(n)$ indices, $\tilde{a} \ldots$ and $\hat{a} \ldots$ which are associated to massless vectors $A^{\mu \tilde{a}}$ and massive vectors $A^{\mu \hat{a}}$, respectively.

A.1 Coset space representatives

The coset space $SO(6,n)/SO(6) \times SO(n)$ is represented by a matrix $\mathcal{V} = (\mathcal{V}_M^N) \in SO(6, n)$. Raising/lowering $SO(6, n)$ indices is defined via the $SO(6, n)$ invariant metric

$$\eta = (\eta_{MN}) = (\eta^M_N) = \text{diag}(-\ldots-, +\ldots+),$$

and $\mathcal{V}^{-1T} = (\mathcal{V}_M^N)$. $\mathcal{V}$ transforms as

$$\mathcal{V} \rightarrow g \mathcal{V} h(x),$$

which in terms of indices reads

$$\mathcal{V}_M^N \rightarrow g_M^P \mathcal{V}_P^Q h(x)_Q^N,$$

$$\mathcal{V}_M^N \rightarrow g_M^P \mathcal{V}_P^Q h(x)_Q^N,$$

where $g = (g_M^P) \in SO(6, n)$ and a spacetime dependent $h(x) = (h(x)_Q^N) \in SO(6) \times SO(n)$ and $g_M^P$ and $h(x)_Q^N$ are obtained from the former via lowering/raising indices. It is apparent that global $SO(6, n)$ acts only on the first index of $\mathcal{V}_M^N$ and $\mathcal{V}_M^N$ while
local $SO(6) \times SO(n)$ acts only on the second. The bosonic part of the Lagrangian can be conveniently expressed in terms of a symmetric positive definite matrix

$$M = (M_{MN}) := \mathcal{V}\mathcal{V}^T,$$

which transforms as a tensor of $SO(6, n)$, i.e.

$$M_{MN} \rightarrow g_M^Q g_N^R M_{QR},$$

and is manifestly invariant under local $SO(6) \times SO(n)$ transformations. One also has $M^{-1} = (M^{MN})$ transforming as

$$M^{MN} \rightarrow g^M Q g^N R M^{QR}.$$ 

In describing the supergravity theory index calculus seems to be indispensable because $SO(6, n)$ indices associated to $SO(6) \times SO(n)$ need to be decomposed into those of $SO(6)$ and $SO(n)$, of which the $SO(6)$ indices are to be transferred to indices of the universal cover $SU(4)$ in order to describe the coupling of scalar representatives to fermions. The relation between these indices is due to the fact that in terms of representations of their common complex Lie algebra one has $(4 \otimes 4)_{\text{antisymmetric}} \simeq 6$. As in the Appendix of [26], we therefore associate to every vector index $m$ of $SO(6)$ a pair of anti-symmetric $SU(4)$ indices $[ij]$ in the following way

$$\phi_{ij} = \frac{1}{2} \sum_{m=1}^{6} \phi_m [G_m]_{ij}, \quad \phi^{ij} = -\frac{1}{2} \sum_{m=1}^{6} \phi_m [G_m]^{ij},$$

where $\phi_m$ shall be a generic $SO(6)$ vector and the $G$'s are the 't Hooft matrices

$$[G_1]_{ij} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad [G_2]_{ij} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$[G_3]_{ij} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad [G_4]_{ij} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$[G_5]_{ij} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad [G_6]_{ij} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (A.7)$$

Furthermore, for every $m = 1, \ldots, 6$ one defines

$$[G_m]^{ij} = -\frac{1}{2} \epsilon^{ijkl} [G_m]_{kl} = -([G_m]_{ij})^*, \quad (A.8)$$

so as to obtain $(\phi_{ij})^* = \phi^{ij}$.

At the origin of $SO(6, n)$, cf. [3.5], one finds $\mathcal{V} = \mathcal{V}^{-1T} = 1$ which in components reads

$$\mathcal{V}_m^n = \mathcal{V}_m^m = \delta_m^n, \quad \mathcal{V}_m^a = \mathcal{V}_m^m = 0,$$

$$\mathcal{V}_a^b = \mathcal{V}_a^b = \delta_a^b, \quad \mathcal{V}_a^m = \mathcal{V}_a^m = 0. \quad (A.9)$$
In terms of $SU(4)$ indices one now has
\[ \mathcal{V}_M^{ij} = \begin{cases} \frac{1}{2} [G_m]^{ij}, & \text{if } M = m, \\ 0, & \text{if } M = a. \end{cases} \] (A.10)

As to $SL(2)/SO(2)$, a generic representative would be $\mathcal{V} = (\mathcal{V}_\alpha^\beta) \in SL(2)$. Raising/lowering indices is defined via the antisymmetric matrix $\epsilon = (\epsilon_{\alpha\beta}) = (\epsilon^{\alpha\beta})$ with $\epsilon^{12} = 1$ in such a way that
\[ (\mathcal{V}^\alpha_\beta) = (\epsilon^{\alpha\gamma} \mathcal{V}^\gamma_\delta \epsilon_{\delta\beta}) = \epsilon \mathcal{V} = -\mathcal{V}^{-1T}. \] (A.11)

As before, transformations in terms of indices are
\[ \mathcal{V} = (\mathcal{V}_\alpha^\beta) \rightarrow g \mathcal{V} h(x) = (g_\alpha^\gamma \mathcal{V}^\gamma_\delta h(x)_\delta^\beta), \]
\[ -\mathcal{V}^{-1T} = (\mathcal{V}^\alpha_\beta) \rightarrow (g^\alpha_\gamma \mathcal{V}^\gamma_\delta h(x)_\delta^\beta), \] (A.12)
and the bosonic Lagrangian can be written in terms of the symmetric positive definite matrix
\[ M := \mathcal{V} \mathcal{V}^T = (M_{\alpha\beta}), \] (A.13)
that can be expressed in terms of $\tau \in \mathbb{C}$ with $\text{Im} \tau > 0$ as
\[ M_{\alpha\beta} = \frac{1}{\text{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \text{Re}(\tau) \\ \text{Re}(\tau) & 1 \end{pmatrix}. \] (A.14)
Its inverse is $M^{-1} = (M^{\alpha\beta})$ and transforms accordingly. The fermionic sector of the supergravity theory requires a different representation of cosets, namely, in terms of $(\mathcal{V}_\alpha) \in \mathbb{C}^2$ such that
\[ M_{\alpha\beta} = \text{Re}(\mathcal{V}_\alpha \mathcal{V}_\beta^*) \] (A.15)
For (A.14) one can always find appropriate $\mathcal{V}_\alpha$. Letting them transform as vectors under global $SL(2) = SL(2, \mathbb{R})$ gives the right transformation for $M_{\alpha\beta}$. For a given $\tau$ as above, $\mathcal{V}_\alpha$ is unique up to local $U(1)$ transformations
\[ \mathcal{V}_\alpha \rightarrow e^{i\phi(x)} \mathcal{V}_\alpha \] (A.16)
for arbitrary $\phi(x) \in \mathbb{R}$ (and up to a sign ambiguity\textsuperscript{15}). As fermions also transform under this $U(1)$, they couple to coset representatives $\mathcal{V}_\alpha$. At the origin $\mathcal{V} = 1$ and thus in an appropriate gauge one finds $(\mathcal{V}_\alpha) = (1, i)^T$.

**B Weyl & Dirac spinor conventions**

While we find it more convenient to work with Weyl spinors, the fermionic terms in the literature\textsuperscript{[18, 24]} are given in terms of Dirac spinors. Based on the conventions given in\textsuperscript{[18]} we express Dirac spinors in terms of Weyl spinors. In what follows we will first summarize their conventions and then express fermionic terms using Weyl spinors.

\textsuperscript{15}Fixing the gauge such that $\mathbb{R} \ni V_1 > 0$, one finds a sign ambiguity in the imaginary part of $V_2$ as is apparent from $M_{\alpha\beta} = (\text{Re} V_\alpha)(\text{Re} V_\beta) + (\text{Im} V_\alpha)(\text{Im} V_\beta)$. 32
The metric \((\eta_{\mu\nu})\) has signature \((-,+,+,+).\) The \(\gamma\)-matrices \(\Gamma_\mu\) satisfying
\[
\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}
\] (B.1)
are (chirally) represented by
\[
\Gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma_\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix},
\] (B.2)
where
\[
\sigma_\mu = (1, \bar{\sigma}) = \bar{\sigma}^\mu, \quad \sigma^\mu = \eta^{\mu\nu} \sigma_\nu = (-1, \bar{\sigma}) = \bar{\sigma}_\mu,
\] and \(\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\) is built from the usual \(\sigma\)-matrices. One then has
\[
\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (B.3)
and
\[
(\Gamma^\mu)^\dagger = \eta^{\mu\nu}\Gamma_\nu = \Gamma_0\Gamma_\mu\Gamma_0, \quad (\Gamma^\mu)^\dagger = (\eta^{\mu\nu}\Gamma_\nu)^\dagger = \Gamma_0\Gamma^\mu\Gamma_0,
\] (B.4)
\[
\Gamma^0 = -\Gamma_0, \quad (\Gamma^{\mu\nu})^\dagger = \frac{1}{2}[^{\mu\nu}][\Gamma^\mu, \Gamma^\nu]^\dagger = -\Gamma_0\Gamma^{\mu\nu}\Gamma_0.
\]
In particular,
\[
\Gamma^{\mu\nu} = 2 \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix},
\] (B.5)
where
\[
\sigma^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu), \quad \bar{\sigma}^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu).
\] (B.6)
Using the charge conjugation matrix
\[
B = i\Gamma_5\Gamma_2 = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \text{ with } \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\] (B.7)
one defines for a generic Dirac spinor \(\phi^i\) transforming in the \(4\) of \(SU(4)\)
\[
\phi_i = B(\phi^i)^* ,
\] (B.8)
which transforms again as Dirac spinor, but now in the complex conjugate representation \(\bar{4}\) of \(SU(4)\). For a chiral spinor with \(\Gamma_5\phi^i = \pm\phi^i\), one finds \(\Gamma_5\phi_i = \mp\phi_i\), i.e. charge conjugation also flips the chirality of chiral spinors. Furthermore, one defines
\[
\bar{\phi}_i = (\phi^i)^\dagger\Gamma_0, \quad \bar{\phi}_i = (\phi_i)^* B.
\] (B.9)
The fermionic spectrum of \(\mathcal{N} = 4\) supergravity in \(D = 4\) with a gravity multiplet and \(n\) vector multiplets consists of Dirac spinors \(\psi^i_\mu, \lambda^{ai}, \chi^i\) that have the following chirality:
\[
\psi^i_\mu = \begin{pmatrix} (\psi^i_\mu)^A \\ 0 \end{pmatrix}, \quad \Gamma_5\psi^i_\mu = \psi^i_\mu,
\]
\[
\lambda^{ai} = \begin{pmatrix} (\lambda^{ai})^A \\ 0 \end{pmatrix}, \quad \Gamma_5\lambda^{ai} = \lambda^{ai},
\]
\[
\chi^i = \begin{pmatrix} 0 \\ (\chi^i)^A \end{pmatrix}, \quad \Gamma_5\chi^i = -\chi^i.
\] (B.10)
Note that we have not introduced new symbols for Weyl spinors but the latter are recognized in the van der Waerden notation by undotted $(A, \ldots)$ and dotted indices $(\dot{A}, \ldots)$ transforming with respect to the two different $SU(2)$ groups of the Lorentz group. We can now express all the fermionic mass terms in terms of Weyl spinors

\[
\bar{\psi}_i \Gamma^{\mu \nu} \psi_{\nu j} + \text{h.c.} = 2 (\psi^i_\mu)^* \sigma^{\mu \nu} \epsilon (\psi^j_\nu)^* - 2 (\psi^i_\mu) \epsilon \sigma^{\mu \nu} (\psi^j_\nu)^*,
\]

\[
\bar{\psi}_i \Gamma^\mu \chi_j + \text{h.c.} = - (\psi^i_\mu)^* \sigma^\mu \epsilon (\chi^j_\nu)^* + (\chi^j_\nu) \epsilon \sigma^\mu (\psi^i_\mu)^*,
\]

\[
\bar{\psi}_i \Gamma^\mu \lambda_j^\alpha + \text{h.c.} = - (\psi^i_\mu)^* \sigma^\mu \epsilon (\lambda^j_\alpha)^* - (\lambda^j_\alpha) \epsilon \sigma^\mu (\psi^i_\mu)^*,
\]

\[
\bar{\chi}^i \lambda_j^\alpha + \text{h.c.} = (\chi^i)^* (\lambda^j_\alpha)^* + (\lambda^j_\alpha) (\chi^i)^*,
\]

where on the right hand side we suppressed all dotted/undotted spinor indices. Note that we have not introduced new symbols for Weyl spinors but the latter are recognized in the van der Waerden notation by undotted $(A, \ldots)$ and dotted indices $(\dot{A}, \ldots)$ transforming with respect to the two different $SU(2)$ groups of the Lorentz group.

\[\text{C } A\text{-matrices at the origin}\]

Here we state the results for the $A$-matrices in (2.9) evaluated at the origin $(1_2, 1_{6+n})$\[\text{[16]}\]

For $A^i_1 = A^i_2$ the result is:

\[
A^{11}_1 = \frac{3}{4} \left[ (-f_{456} + f_{234} - f_{135} + f_{126}) + i(f_{123} - f_{156} + f_{246} - f_{345}) \right],
\]

\[
A^{22}_1 = \frac{3}{4} \left[ (-f_{456} + f_{234} + f_{135} - f_{126}) + i(f_{123} - f_{156} - f_{246} + f_{345}) \right],
\]

\[
A^{33}_1 = \frac{3}{4} \left[ (-f_{456} - f_{234} - f_{135} - f_{126}) + i(f_{123} + f_{156} + f_{246} + f_{345}) \right],
\]

\[
A^{44}_1 = \frac{3}{4} \left[ (-f_{456} - f_{234} + f_{135} + f_{126}) + i(f_{123} + f_{156} - f_{246} - f_{345}) \right],
\]

\[
A^{12}_1 = \frac{3}{4} \left[ (-f_{125} - f_{136}) + i(-f_{346} - f_{245}) \right],
\]

\[
A^{24}_1 = \frac{3}{4} \left[ (f_{125} - f_{136}) + i(f_{346} - f_{245}) \right],
\]

\[
A^{13}_1 = \frac{3}{4} \left[ (f_{124} - f_{236}) + i(-f_{356} + f_{145}) \right],
\]

\[
A^{24}_4 = \frac{3}{4} \left[ (f_{124} + f_{236}) + i(-f_{356} - f_{145}) \right],
\]

\[
A^{14}_4 = \frac{3}{4} \left[ (f_{134} + f_{235}) + i(f_{256} + f_{146}) \right],
\]

\[
A^{23}_1 = \frac{3}{4} \left[ (-f_{134} + f_{235}) + i(-f_{256} + f_{146}) \right].
\]

\[\text{[C.1]}\]

\[\text{[16]}\]The result for a critical point \[\text{[B.1]}\] is obtained by multiplying all components by $V_-$. \[34\]
The components of the symmetric matrix \( (A_{ij}^{1}) \) depend on 20 real parameters \( f_{mnp} \). It is apparent that any symmetric complex \( 4 \times 4 \) matrix can be written in this form. As to \( (A_{2a}^{j}) \) for all \( a = 1, \ldots, n \), the components of \( (A_{2a}^{j}) \) read:

\[
A_{2a1}^{1} = \frac{1}{2}i(f_{a14} + f_{a25} + f_{a36}),
\]

\[
A_{2a2}^{2} = \frac{1}{2}i(f_{a14} - f_{a25} - f_{a36}),
\]

\[
A_{2a3}^{3} = \frac{1}{2}i(-f_{a14} + f_{a25} - f_{a36}),
\]

\[
A_{2a4}^{4} = \frac{1}{2}i(-f_{a14} - f_{a25} + f_{a36}),
\]

\[
A_{2a1}^{2} = -\frac{1}{2}[(f_{a23} - f_{a56}) + i(f_{a26} - f_{a35})],
\]

\[
A_{2a3}^{4} = -\frac{1}{2}[-(f_{a23} - f_{a56}) + i(-f_{a26} - f_{a35})],
\]

\[
A_{2a1}^{3} = -\frac{1}{2}[-(f_{a13} + f_{a46}) + i(-f_{a16} + f_{a34})],
\]

\[
A_{2a2}^{4} = -\frac{1}{2}[-(f_{a13} - f_{a46}) + i(f_{a16} + f_{a34})],
\]

\[
A_{2a1}^{4} = -\frac{1}{2}[(f_{a12} - f_{a45}) + i(f_{a15} - f_{a24})],
\]

\[
A_{2a2}^{3} = -\frac{1}{2}[-(f_{a12} - f_{a45}) + i(f_{a15} + f_{a24})].
\]

Moreover,

\[
A_{2a2}^{1} = \frac{1}{2}[-(f_{a23} - f_{a56}) + i(f_{a26} - f_{a35})],
\]

etc. where the real part is always multiplied by an extra minus sign. We conclude that \( A_{1} = A_{2} \) depends only on \( f_{mnp} \) while matrices \( A_{2a} \) are built from \( f_{amn} \). Note that at the origin \( f_{abm} \) and \( f_{abc} \) do not appear in the fermion shift matrices (and therefore also not in the Killing spinor equations).

Finally, we give an explicit result for the antisymmetric \( A \)-matrices \( (A_{ab}^{ij}) \) for all \( a, b \). At the origin of the scalar manifold they are entirely given in terms of components \( f_{abm} \):

\[
(A_{ab}^{ij}) = \frac{1}{2} \begin{pmatrix}
0 & if_{ab1} + f_{ab4} & if_{ab2} + f_{ab5} & if_{ab3} + f_{ab6} \\
-\ast & 0 & -if_{ab3} + f_{ab6} & if_{ab2} - f_{ab5} \\
-\ast & -\ast & 0 & -if_{ab1} + f_{ab4} \\
-\ast & -\ast & -\ast & 0
\end{pmatrix}
\]

for all \( a, b \).
D Partial solution of the quadratic constraints

D.1 Discussing constraint equations for \( g_a \neq 0 \)

The quadratic constraints for electric gaugings in the case of \( g_a \neq 0 \) are hard to solve. In fact, so far we have not found any example of a consistent solution with \( g_a \neq 0 \). Here we will discuss the following two aspects: First, we will show that an electrically gauged \( \mathcal{N} = 4 \) theory with \( \mathcal{N} = 2 \) vacuum requires \( f_a \neq 0 \); secondly, we will give some details on a lengthy but elementary calculation that shows that \( g_a \neq 0 \) solutions, if at all, exist only in \( n > 6 \). These two aspects illustrate that \( g_a \neq 0 \) consistent solutions would have to be rather sophisticated. As in Section 3.2.3 we label the quadratic constraints given in (2.4) by the quadruple \((M,N,P,Q)\) of \( SO(6,n) \)-indices.

D.1.1 \( \mathcal{N} = 2 \) vacua require \( f_a \neq 0 \)

We will prove this claim by contradiction; we therefore assume \( f_a = 0 \). The constraint equations to be used in this proof are

\[
\begin{align*}
(2,3,5,6) & \quad e^2 + \bar{g}^2 = e^2 \neq 0, & (D.1) \\
(b,2,4,5) & \quad F_4 \bar{e} = 2c \bar{g}, & (D.2) \\
(b,2,4,6) & \quad F_4 \bar{g} = -2c \bar{e}, & (D.3) \\
(b,2,3,5) & \quad F_2 \bar{g} = F_3 \bar{e}, & (D.4) \\
(b,2,3,6) & \quad F_3 \bar{g} = -F_2 \bar{e}, & (D.5) \\
(b,c,2,3) & \quad ([G_2,G_3])_{bc} = c(F_4)_{bc} + 2(e_c g_b - e_b g_c), & (D.6)
\end{align*}
\]

where for better legibility we use a matrix notation with \( SO(n) \) vectors \( \bar{e}, \bar{g} \) and matrices \((F_m)_{ab} = f_{mab} \). It is obvious from (D.1), (D.2), (D.3) that both \( \bar{e} \) and \( \bar{g} \) must be nonzero because an \( \mathcal{N} = 2 \) vacuum requires \( c \neq 0 \). Thus, without loss of generality, using first an \( SO(n) \) transformation and subsequently a transformation of the residual \( SO(n-1) \) symmetry,\(^{17}\) one can write

\[
\bar{e} = \begin{pmatrix} e \\ 0 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} g' \\ g \end{pmatrix},
\]

with \( e \neq 0 \), \( g, g' \in \mathbb{R} \). Then equations (D.2), (D.3) show that \( g' = 0, g = \sigma e \) with \( \sigma = \pm 1 \) and

\[
F_4 = \begin{pmatrix} 0 & -2c\sigma & 0 \\ 2c\sigma & 0 & 0 \\ 0 & 0 & F_4 \end{pmatrix},
\]

where \( F_4 \in \text{Mat}_{n-2,n-2} \). Furthermore, (D.4) and (D.5) imply

\[
F_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \bar{v} & \bar{w} & F_2 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \sigma \bar{w} & -\sigma \bar{v} & F_3 \end{pmatrix},
\]

\(^{17}\)We assume that \( n \) is large enough.
with $\vec{v}, \vec{w} \in \text{Mat}_{n-2,1}$ and antisymmetric matrices $\tilde{F}_2, \tilde{F}_3 \in \text{Mat}_{n-2,n-2}$. As a consequence, (D.1) and (D.6) yield
\[
\sigma[F_2, F_3]_{78} = -3c^2 = \vec{v}^2 + \vec{w}^2 \geq 0,
\]
which contradicts $c \neq 0$. Hence, $\vec{f}$ cannot vanish in consistent solutions with $N = 2$ vacuum. This ends the proof.

### D.1.2 $g_a \neq 0$ solutions do not exist in $n \leq 6$

First we will concentrate on the subset of non-trivial quadratic constraints in (2.4) that can easily be solved:

\[
\begin{align*}
(2, 3, 5, 6) & \quad \vec{e}^2 + \vec{f}^2 + \vec{g}^2 = c^2 \neq 0, \quad \text{(D.11)} \\
(b, 1, 2, 3) & \quad F_1 \vec{f} = 0, \quad \text{(D.12)} \\
(b, 1, 2, 5) & \quad F_1 \vec{e} = 0, \quad \text{(D.13)} \\
(b, 1, 2, 6) & \quad F_1 \vec{g} = 0, \quad \text{(D.14)} \\
(b, 2, 3, 4) & \quad F_4 \vec{f} = 0, \quad \text{(D.15)} \\
(b, 2, 4, 5) & \quad F_4 \vec{e} = 2c \vec{g}, \quad \text{(D.16)} \\
(b, 2, 4, 6) & \quad F_4 \vec{g} = -2c \vec{e}, \quad \text{(D.17)} \\
(b, 2, 3, 5) & \quad F_3 \vec{e} - F_5 \vec{f} - F_2 \vec{g} = 0, \quad \text{(D.18)} \\
(b, 2, 3, 6) & \quad F_2 \vec{e} - F_6 \vec{f} + F_3 \vec{g} = 0, \quad \text{(D.19)} \\
(b, 2, 5, 6) & \quad F_6 \vec{e} + F_2 \vec{f} - F_5 \vec{g} = 0, \quad \text{(D.20)} \\
(b, 3, 5, 6) & \quad F_5 \vec{e} + F_3 \vec{f} + F_6 \vec{g} = 0 . \quad \text{(D.21)}
\end{align*}
\]

Here we use the same matrix notation as in Section D.1.1. Having shown that $\vec{f} = 0$ is impossible, without loss of generality we write it as
\[
\vec{f} = \begin{pmatrix} f \\ 0 \end{pmatrix},
\]
with $f \neq 0$ and due to (D.12) and (D.15) find
\[
F_1 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad F_4 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},
\]
with certain matrices $* \in \text{Mat}_{n-1,n-1}$. Unlike in Section 3.2.3 here we consider the case where $\vec{g} \neq 0$. Analogously to the discussion in Section D.1.1, one can, without loss of generality and using (D.16) and (D.17), write
\[
\vec{g} = \begin{pmatrix} 0 \\ \sigma e \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e} = \begin{pmatrix} 0 \\ 0 \\ e \\ 0 \end{pmatrix},
\]
\[37\]
with \( e \neq 0 \) and \( \sigma = \pm 1 \) to find

\[
F_1 = 0_{3,3} \oplus \tilde{F}_1, \quad F_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\sigma c \\ 0 & -2\sigma c & 0 \end{pmatrix},
\]

with matrices \( \tilde{F}_1, \tilde{F}_4 \in \text{Mat}_{n-3,n-3} \). Furthermore, equations (D.18) to (D.21) are solved by

\[
F_2 = \begin{pmatrix} \tilde{a} & \tilde{b} & 0_{3,3} \\ \tilde{c} & \tilde{d} & -\sigma \tilde{d} + f/\epsilon \tilde{c} \end{pmatrix} F_2, \quad F_3 = \begin{pmatrix} \tilde{c} & \tilde{d} & 0_{3,3} \\ \tilde{c} & \tilde{d} & \sigma \tilde{b} + f/\epsilon \tilde{a} \end{pmatrix} F_3, \\
F_5 = \begin{pmatrix} \tilde{a} & \tilde{b} & 0_{3,3} \\ \tilde{c} & \tilde{d} & -\sigma \tilde{d} - f/\epsilon \tilde{c} \end{pmatrix} F_5, \quad F_6 = \begin{pmatrix} \tilde{c} & \tilde{d} & 0_{3,3} \\ \tilde{c} & \tilde{d} & \sigma \tilde{b} - f/\epsilon \tilde{a} \end{pmatrix} F_6,
\]

with \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a}', \tilde{b}', \tilde{c}', \tilde{d}' \in \text{Mat}_{1,n-3} \) and antisymmetric \( \tilde{F}_2, \tilde{F}_3, \tilde{F}_5, \tilde{F}_6 \in \text{Mat}_{n-3,n-3} \).

There remain a large number of non-trivial quadratic constraints which we do not know how to fully solve. Here, we list only those that are useful in our argument:

\[
\begin{align*}
(b, c, 1, m) & \quad [F_1, F_m] = 0, \\
(b, c, 2, 4) & \quad [F_2, F_4] = -c F_3, \\
(b, c, 3, 4) & \quad [F_3, F_4] = c F_2, \\
(b, c, 4, 5) & \quad [F_5, F_4] = -c F_6, \\
(b, c, 4, 6) & \quad [F_6, F_4] = c F_5, \\
(b, c, 2, 3) & \quad ([F_2, F_3])_{bc} = c(F_4)_{bc} - f(F_7)_{bc} + 2(e_c g_b - e_b g_c), \\
(b, c, 5, 6) & \quad ([F_5, F_6])_{bc} = c(F_4)_{bc} - f(F_7)_{bc} + 2(e_c g_b - e_b g_c), \\
(b, c, 2, 6) & \quad ([F_2, F_6])_{bc} = -g(F_8)_{bc} + 2(e_b f_c - e_c f_b), \\
(b, c, 3, 5) & \quad ([F_3, F_5])_{bc} = -g(F_8)_{bc} + 2(e_b f_c - e_c f_b), \\
(b, c, 2, 5) & \quad ([F_2, F_5])_{bc} = -e(F_9)_{bc} - 2(f_c g_b - f_b g_c), \\
(b, c, 3, 6) & \quad ([F_3, F_6])_{bc} = e(F_9)_{bc} + 2(f_c g_b - f_b g_c).
\end{align*}
\]

Here, \( (F_7)_{ab} = f_{rab} \), etc. for the first three \( SO(n) \) indices denoted by 7, 8, 9. Using (D.25) and (D.26), equations (D.28) to (D.31) are equivalent to:

\[
\begin{align*}
\tilde{F}_4 \tilde{a} &= c \tilde{c}, \\
\tilde{F}_4 \tilde{b} &= 3c \tilde{d} - 2\sigma e \tilde{c}, \\
\tilde{F}_4 \tilde{c} &= -c \tilde{a}, \\
\tilde{F}_4 \tilde{d} &= -3c \tilde{b} - 2\sigma e \tilde{a}, \\
\tilde{F}_4 \tilde{a}' &= c \tilde{c}', \\
\tilde{F}_4 \tilde{b}' &= 3c \tilde{d}' - 2\sigma e \tilde{c}'.
\end{align*}
\]
While tedious, it is possible to find the general solution to equations (D.38) to (D.45). Rather than discussing this in detail we will content ourselves with showing that consistent solutions require as a necessary condition that \( \vec{a}, \vec{b}, \) etc. be at least nonzero column vectors of dimension 4. This then immediately allows us to prove the claim of this section (obviously due to (D.26)). To this end, we solve equations (D.32) to (D.37) for \( F_7, F_8, F_9 \), respectively, and invoke the antisymmetry of \( f_{abc} \). This gives rise to another set of quadratic constraints. The ones of interest for this argument are

\[
\vec{F}_4 \vec{b} = 3c \vec{d}' + 2c\sigma \vec{c}, \quad (D.43)
\]
\[
\vec{F}_4 \vec{c} = -c \vec{a}', \quad (D.44)
\]
\[
\vec{F}_4 \vec{d} = -3c \vec{b} + 2c\sigma \vec{\bar{a}}, \quad (D.45)
\]
\[
[\vec{F}_2, \vec{F}_4] = -c \vec{F}_3, \quad (D.46)
\]
\[
[\vec{F}_3, \vec{F}_4] = c \vec{F}_2, \quad (D.47)
\]
\[
[\vec{F}_5, \vec{F}_4] = -c \vec{F}_6, \quad (D.48)
\]
\[
[\vec{F}_6, \vec{F}_4] = c \vec{F}_5. \quad (D.49)
\]

\[
\vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{c}, \quad (D.50)
\]
\[
\vec{a}' \cdot \vec{d}' = \vec{b}' \cdot \vec{c}', \quad (D.51)
\]
\[
\bar{a} \cdot \bar{d} = \bar{b} \cdot \bar{c}, \quad (D.52)
\]
\[
\bar{c} \cdot \bar{b} = \bar{d} \cdot \bar{a}, \quad (D.53)
\]
\[
\sigma(\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d}) = \frac{\sigma}{\epsilon}(\vec{c} \cdot \vec{c} - \vec{a} \cdot \vec{a}'), \quad (D.54)
\]
\[
\sigma(\vec{a}' \cdot \vec{b} + \vec{c}' \cdot \vec{d}') = -\frac{\sigma}{\epsilon}(\vec{c} \cdot \vec{c}' - \vec{a} \cdot \vec{a}), \quad (D.55)
\]
\[
\sigma(\vec{b} \cdot \vec{b} + \vec{d} \cdot \vec{d}) = \frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d}), \quad (D.56)
\]
\[
\sigma(\vec{b} \cdot \vec{b} + \vec{d} \cdot \vec{d}) = -\frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{d} + \vec{c} \cdot \vec{b}), \quad (D.57)
\]
\[
\sigma(\vec{d} \cdot \vec{a}' - \vec{a} \cdot \vec{d}) = \frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{c} + \vec{a}' \cdot \vec{c}'), \quad (D.58)
\]
\[
\sigma(\vec{b} \cdot \vec{c}' - \vec{c} \cdot \vec{b}) = -\frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{c} + \vec{a}' \cdot \vec{c}), \quad (D.59)
\]
\[
\sigma(\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}) = \frac{\sigma}{\epsilon}(\vec{b} \cdot \vec{c} + \vec{b}' \cdot \vec{c}), \quad (D.60)
\]
\[
\sigma(\vec{b} \cdot \vec{d}' - \vec{d} \cdot \vec{b}) = -\frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{d} + \vec{a}' \cdot \vec{d}'), \quad (D.61)
\]
\[
\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}' = \vec{d} \cdot \vec{c} - \vec{c} \cdot \vec{d}', \quad (D.62)
\]
\[
\sigma(\vec{b}^2 + \vec{d}^2) + \frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{a}' - \vec{a} \cdot \vec{d}') = \sigma(\vec{b}^2 + \vec{d}^2) - \frac{\sigma}{\epsilon}(\vec{a} \cdot \vec{b} - \vec{c} \cdot \vec{d}'), \quad (D.63)
\]
\[
\sigma(\vec{a} \cdot \vec{b} + \vec{d} \cdot \vec{c}) - \frac{\sigma}{\epsilon}(\vec{a}^2 + \vec{c}^2) = -\sigma(\vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{d}) - \frac{\sigma}{\epsilon}(\vec{a}^2 + \vec{c}^2), \quad (D.64)
\]
\[
\sigma e(6e^2 + \vec{b}^2 + \vec{d}^2) = f \left( -\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}' + \frac{\sigma}{\epsilon} \sigma(\vec{a}^2 + \vec{c}^2) \right), \quad (D.65)
\]
\[ \sigma e(6\epsilon^2 + \vec{b}^2 + \vec{d}^2) = f(\vec{a} \cdot \vec{b} + \vec{d} \cdot \vec{c} - 2\vec{b} \cdot \vec{a}) , \] (D.66)

where for the last two equations we also used (D.11) and (D.24). Those two equations imply that not all \( \vec{a}, \vec{b}, \ldots \) can vanish because by assumption \( \epsilon \neq 0 \). Furthermore, one finds that solutions satisfying (D.38) to (D.45) subject to the additional constraints (D.50) to (D.66) necessarily require nonzero column vectors \( \vec{a}, \vec{b}, \ldots \) of dimension at least 4. Since \( \vec{a}, \vec{b}, \ldots \in \text{Mat}_{1,n-3} \) we conclude that \( g_a \neq 0 \) solutions do not exist in \( n \leq 6 \).

### D.2 Discussing constraint equations for \( g_a = 0 \)

Here we list the quadratic constraint equations that are not trivially satisfied, c.f. Section 3.2.3. In what follows the quadruple \( (M, N, P, Q) \) in the first column refers to the free indices in (2.4):

\begin{align*}
(b, c, 1, 2) & \quad f_{a b 2} f_{a c 1} - f_{a b 1} f_{a c 2} = 0 , \; \text{(D.67a)} \\
(b, c, 1, 3) & \quad f_{a b 3} f_{a c 1} - f_{a b 1} f_{a c 3} = 0 , \; \text{(D.67b)} \\
(b, c, 1, 4) & \quad f_{a b 4} f_{a c 1} - f_{a b 1} f_{a c 4} = 0 , \; \text{(D.67c)} \\
(b, c, 1, 5) & \quad f_{a b 5} f_{a c 1} - f_{a b 1} f_{a c 5} = 0 , \; \text{(D.67d)} \\
(b, c, 1, 6) & \quad f_{a b 6} f_{a c 1} - f_{a b 1} f_{a c 6} = 0 , \; \text{(D.67e)} \\
(b, c, 2, 5) & \quad f_{a b 5} f_{a c 2} - f_{a b 2} f_{a c 5} = 0 , \; \text{(D.67f)} \\
(b, c, 2, 6) & \quad f_{a b 6} f_{a c 2} - f_{a b 2} f_{a c 6} = 0 , \; \text{(D.67g)} \\
(b, c, 3, 5) & \quad f_{a b 5} f_{a c 3} - f_{a b 3} f_{a c 5} = 0 , \; \text{(D.67h)} \\
(b, c, 3, 6) & \quad f_{a b 6} f_{a c 3} - f_{a b 3} f_{a c 6} = 0 , \; \text{(D.67i)} \\
(b, c, 3, 4) & \quad f_{a b 4} f_{a c 3} - f_{a b 3} f_{a c 4} = c f_{b c 2} , \; \text{(D.67j)} \\
(b, c, 2, 4) & \quad f_{a b 4} f_{a c 2} - f_{a b 2} f_{a c 4} = -c f_{b c 3} , \; \text{(D.67k)} \\
(b, c, 4, 5) & \quad f_{a b 5} f_{a c 4} - f_{a b 4} f_{a c 5} = c f_{b c 6} , \; \text{(D.67l)} \\
(b, c, 4, 6) & \quad f_{a b 6} f_{a c 4} - f_{a b 4} f_{a c 6} = -c f_{b c 5} , \; \text{(D.67m)} \\
(b, c, 2, 3) & \quad f_{a b 3} f_{a c 2} - f_{a b 2} f_{a c 3} = c (f_{b c 4} - f_{7 b c}) , \; \text{(D.67n)} \\
(b, c, 5, 6) & \quad f_{a b 6} f_{a c 5} - f_{a b 5} f_{a c 6} = c (f_{b c 4} - f_{7 b c}) , \; \text{(D.67o)} \\
(b, c, 7, 1) & \quad f_{a b 1} f_{a c 7} - f_{a b 7} f_{a c 1} = 0 , \; \text{(D.67p)} \\
(b, c, 7, 4) & \quad f_{a b 4} f_{a c 7} - f_{a b 7} f_{a c 4} = 0 , \; \text{(D.67q)} \\
(b, c, 7, 2) & \quad f_{a b 2} f_{a c 7} - f_{a b 7} f_{a c 2} = c f_{b c 3} , \; \text{(D.67r)} \\
(b, c, 7, 3) & \quad f_{a b 3} f_{a c 7} - f_{a b 7} f_{a c 3} = -c f_{b c 2} , \; \text{(D.67s)} \\
(b, c, 7, 5) & \quad f_{a b 5} f_{a c 7} - f_{a b 7} f_{a c 5} = c f_{b c 6} , \; \text{(D.67t)}
\end{align*}
\[(b, \tilde{c}, 7, 6)\]
\[f_{\tilde{a}b} f_{\tilde{a}c} - f_{\tilde{a}b} f_{\tilde{a}c} = -c f_{b \tilde{c}}, \quad (D.67u)\]

\[(b, \tilde{c}, \tilde{d}, m)\]
\[0 = f_{\tilde{a}b} f_{\tilde{a}d} + f_{\tilde{a}m} f_{\tilde{a}c} - f_{\tilde{a}b} f_{\tilde{a}c} = f_{r \tilde{b}} f_{d \tilde{c}} + f_{r \tilde{b}} f_{c \tilde{d}} - f_{r \tilde{b}} f_{c \tilde{d}}. \quad (D.68b)\]

(D.2.1) The most general solution to equations (3.27)

Here we will prove the claim that the most general solution of equations (3.27) is given by (3.29) and an arbitrary, antisymmetric \(A\) with a \(P\) and an arbitrary, antisymmetric \(A\). Hence, nilpotent.

**Theorem:** We shall prove the following theorem:

\[\text{Theorem:} \quad \text{The most general solution to system (D.69) consists of solutions with} \]
\[G_2 = G_3 = H_0 = 0, \quad H_+ = -H_+^T \text{ arbitrary.} \quad (D.70)\]

Our proof requires two elementary lemmata about matrices and a corollary of Lie’s theorem concerning finite-dimensional representations of complex, solvable Lie algebras.

**Lemma:** An antisymmetric matrix \(A \in \text{Mat}(\mathbb{R}, m \times m)\) is nilpotent if and only if \(A = 0\).

**Proof:** Being antisymmetric \(A\) can be brought to diagonal form \(PAP^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_m)\) with \(P \in \text{GL}(\mathbb{C}, m \times m)\) and \(\lambda_i \in i\mathbb{R}\). As \(PA^n P^{-1} = (PAP^{-1})^n\) for all \(n \in \mathbb{N}\), nilpotency is basis-independent. It is then obvious that,
\[(PAP^{-1})^n = \text{diag}(\lambda_1^n, \ldots, \lambda_m^n),\]

is nilpotent iff \(\lambda_i = 0 \forall i\) which implies \(A = 0\). The converse is trivial.

**Lemma:** Given matrices \(A_1, \ldots, A_k \in \text{Mat}(\mathbb{C}, m \times m)\) for \(k \in \mathbb{N}\). For simultaneously triangularizable matrices \(A_1, \ldots, A_k\) the commutator \([A_i, A_j]\) is nilpotent for all \(i, j = 1, \ldots, k\).

**Proof:** The commutator of two upper triangular matrices is strictly upper triangular and, hence, nilpotent.

**Corollary of Lie’s theorem\(^{18}\):** Let \(\mathfrak{g}\) be a complex, solvable Lie algebra and \((V, \rho)\) a finite-dimensional representation of \(\mathfrak{g}\). Then there exists a basis of \(V\) such that all

\(^{18}\)Given a complex, solvable Lie algebra, then all its finite-dimensional irreducible representations are one-dimensional.
elements of \( \mathfrak{g} \) are represented as upper triangular matrices.

Proof: Lecture script by W. Soergel [31].

In order to be able to apply this corollary we need to complexify our real Lie algebra (D.69).

Lemma: Given a real Lie algebra \( \mathfrak{g} \) and a finite-dimensional real representation \((V, \rho)\) of \( \mathfrak{g} \). Then one finds a finite-dimensional representation \((V_\mathbb{C}, \rho_\mathbb{C})\) of the complexified Lie algebra \( \mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \) (with \( \mathbb{C} \)-linear extension of the Lie bracket) defined by \( V_\mathbb{C} := V \otimes_{\mathbb{R}} \mathbb{C} \) and

\[
\rho_\mathbb{C}(X + iY) := \rho(X) + i\rho(Y),
\]

for all \( X, Y \in \mathfrak{g} \).

Proof: \( \mathbb{C} \)-linearity of \( \rho_\mathbb{C} \) is obvious and so is the proof of

\[
\rho_\mathbb{C}([X + iY, U + iV]) = [\rho_\mathbb{C}(X + iY), \rho_\mathbb{C}(U + iV)]
\]

for all \( X, Y, U, V \in \mathfrak{g} \). As a result, \((V_\mathbb{C}, \rho_\mathbb{C})\) is a finite-dimensional representation of the complex Lie algebra \( \mathfrak{g}_\mathbb{C} \).

Now we can prove the theorem:

Proof of the theorem: Assume that there exists a solution of (D.69) with an antisymmetric \( G_2 \neq 0 \in \text{Mat}(\mathbb{R}, m \times m) \). Any such solution would be a finite-dimensional real representation \((\mathbb{R}^m, \rho)\) of our real solvable Lie algebra \( \mathfrak{s}' \). In this proof such a solution will be denoted by \( \rho(G_2), \rho(G_3), \rho(H_-), \rho(H_+) \) with \( \rho(G_2) \neq 0 \) by assumption, while \( G_2, G_3, H_-, H_+ \in \mathfrak{s}' \) shall refer to the abstract elements of the Lie algebra. We denote the induced representation of the complexified Lie algebra \( \mathfrak{s}'_\mathbb{C} \) as \((\mathbb{C}^m, \rho_\mathbb{C})\). Since also \( \mathfrak{s}'_\mathbb{C} \) is solvable, we apply the corollary and find that \( \rho_\mathbb{C}(G_2), \rho_\mathbb{C}(G_3), \rho_\mathbb{C}(H_-), \rho_\mathbb{C}(H_+) \in \text{Mat}(\mathbb{C}, m \times m) \) are simultaneously triangularizable. Then, according to the second lemma we find that, in particular \((p = 1)\),

\[
[\rho_\mathbb{C}(G_3), \rho_\mathbb{C}(H_+)] = 2c \rho_\mathbb{C}(G_2)
\]

is nilpotent. As \( c \neq 0 \) one finds \( \rho_\mathbb{C}(G_2) = \rho(G_2) \) is nilpotent. However, being antisymmetric \( \rho(G_2) \) must be zero by the first lemma which is in contradiction with \( \rho(G_2) \neq 0 \).

We therefore conclude that \( \rho(G_2) = 0 \) which, by means of the Lie algebra (D.69), immediately implies \( \rho(G_3) = \rho(H_-) = 0 \). As a result, solutions (D.70) are already the most general solutions to (D.69). This ends the proof.

D.2.2 Solving \([G_1, G_4] = 0\)

We will now solve (3.30a), which in matrix notation reads

\[
[G_1, G_4] = 0.
\]  

(D.71)

It is by means of an \( O(n-1) \) transformation that, without loss of generality, any \( G_1 \) can be written in block-diagonal form as

\[
G_1 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix},
\]  

(D.72)
where \( D = \text{diag}(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots) \) is a diagonal matrix with ordered positive eigenvalues \( x_1 > x_2 > \ldots > 0 \) and \( \varepsilon \) is the antisymmetric \( 2 \times 2 \) matrix with \( \varepsilon_{12} = 1 \); the zeros in (D.72) denote zero matrices of appropriate dimensions. Note that, in general, this gauge can only be obtained by also using reflections (in addition to rotations). While strictly speaking we are only allowed to use \( \text{SO}(n-1) \subset G \) rotations, the quadratic constraints (3.30a) - (3.30d) are also \( O(n-1) \) tensor equations. We may therefore also use reflections to arrive, as an intermediate step, at the gauge (D.72) — which simplifies the subsequent analysis — as long as, in the end, we return to only using rotations, in that we apply another reflection that flips two directions but preserves the block structure (e.g. \( x_i \rightarrow -x_i \) for one \( 2 \times 2 \) block). Since \( D \otimes \varepsilon \) is invertible, (D.71) implies (also using another gauge choice for the lower right block)

\[
G_4 = \begin{pmatrix} A & 0 \\ 0 & (D' \otimes \varepsilon) \oplus 0 \end{pmatrix},
\]

(D.73)

where \( A \) is an antisymmetric matrix (of the same matrix dimensions as \( D \otimes \varepsilon \)) satisfying

\[
[D \otimes \varepsilon, A] = 0
\]

and \( D' \) is another invertible diagonal matrix. In order to solve (D.74) we note that any even-dimensional antisymmetric \( A \) can be written as

\[
A = S \otimes \varepsilon + A_1 \otimes 1 + A_2 \otimes \sigma_1 + A_3 \otimes \sigma_3,
\]

(D.75)

where \( S \) is symmetric, \( A_1, A_2, A_3 \) are antisymmetric, and \( \sigma_1, \sigma_3 \) are the usual Pauli matrices. Now (D.74) implies

\[
[D, S] = 0, \quad [D, A_i] = 0, \quad \{D, A_2\} = 0, \quad \{D, A_3\} = 0,
\]

(D.76)

which in the reflection gauge (D.72) implies \( A_2 = A_3 = 0 \) and \( S_{ij} = (A_1)_{ij} = 0 \) for all \( i, j \) with \( x_i \neq x_j \). As a result, we obtain

\[
A = S \otimes \varepsilon + A_1 \otimes 1,
\]

(D.77)

where now \( S \) and \( A_1 \) are block-diagonal with blocks associated to degenerate \( x_i \) in \( D \). We will now refine the block-structure in \( G_4 \). To this end, we will use the residual symmetry of the blocks in \( G_1 \) and \( G_4 \) to bring each \( G_4 \) block associated to some \( x_i \) to the form

\[
(i \text{th block in } G_4) = (\text{diag}(y_{i1}, \ldots, y_{i1}, y_{i2}, \ldots, y_{i2}, \ldots) \otimes \varepsilon) \oplus 0,
\]

(D.78)

with \( y_{i1} > y_{i2} > \ldots > 0 \). While this, of course, temporarily spoils the gauge (D.72), it is by means of (D.71) that we find, using the same argument as before, that the ith block in \( G_1 \) has a subblock structure with blocks associated to degenerate \( y_{ij} \) or zero in the ith \( G_4 \) block. Now we apply symmetries that respect these subblocks to bring \( G_1 \) back to our gauge (D.72) and at the same time maintain the subblock structure in \( G_4 \). Then, repeating the argument that lead to (D.77), we know that the subblock associated to \( x_i \) in \( G_1 \) and \( y_{ij} \) in \( G_4 \) is given by

\[
((i, j) \text{ block in } G_4) = S^{(ij)} \otimes \varepsilon + A_1^{(ij)} \otimes 1,
\]

(D.79)

19\{\ldots\} denotes the anticommutator.
where
\[
(S^{(ij)} \otimes \varepsilon + A_{1}^{(ij)} \otimes 1)^2 = -(y_{ij})^2 1 \otimes 1.
\] (D.80)

The \((i, j)\) block in \(G_1\) is \(x_i 1 \otimes \varepsilon\) and is thus invariant under orthogonal transformations that only act on the first tensor product factor. Such transformations can be used to bring \(S^{(ij)}\) to diagonal form
\[
D^{(ij)} = \text{diag}(d_{ij1}, \ldots, d_{ij1}, -d_{ij1}, \ldots, -d_{ij1}, \ldots) \oplus 0,
\] (D.81)
where \(d_{ijk} > 0\) and the dimensions of positive and negative eigenvalues can in general be different. In doing so, (D.80) gives rise to the following system of equations
\[
(A_1^{(ij)})^2 + (y_{ij})^2 = (D^{(ij)})^2, \quad \{D^{(ij)}, A_1^{(ij)}\} = 0.
\] (D.82)

The second equation gives
\[
(A_1^{(ij)})_{kl} = 0 \quad \lor \quad (D^{(ij)})_{kk} = -(D^{(ij)})_{ll}
\] (D.83)
and, hence, the \(D^{(ij)}\) and \(A_1^{(ij)}\) have the following block-diagonal form
\[
D^{(ij)} = \begin{pmatrix}
d_{ij1} & 0 & \cdots & 0 \\
0 & -d_{ij1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_1^{(ij)} = \begin{pmatrix}
0 & F^{(ij1)} & \cdots & 0 \\
-F^{(ij1)}T & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F^{(ij0)}
\end{pmatrix}
\] (D.84)
where \(F^{(ijk)}\) are rectangular matrices and \(F^{(ij0)}\) are antisymmetric square matrices subject to the following conditions (from (D.82)):
\[
d_{ijk}^2 + \begin{pmatrix}
F^{(ijk)}F^{(ijk)T} \\
0
\end{pmatrix} = y_{ij}^2,
\] (D.85)
\[
(F^{(ij0)})^2 = -y_{ij}^2.
\]

Without loss of generality we can use the residual symmetry to bring each \(F^{(ij0)}\) into diagonal form
\[
D^{(ij0)} \otimes \varepsilon,
\] (D.86)
where the eigenvalues of \(D^{(ij0)}\) must be \(\pm y_{ij}\) in order to satisfy (D.85). In particular, \(F^{(ij0)}\) must have even dimension. As to the \(F^{(ijk)}\), (D.85) implies that
\[
F^{(ijk)}F^{(ijk)T} = \xi_{ijk} 1, \quad F^{(ijk)T}F^{(ijk)} = \xi_{ijk} 1',
\] (D.87a, D.87b)
for some non-negative number \(\xi_{ijk}\). In the case where \(\xi_{ijk} = 0\) one finds \(F^{(ijk)} = 0\), and (D.85) implies \(d_{ijk} = y_{ij}\). On the other hand, for \(\xi_{ijk} > 0\), (D.87a), (D.87b), respectively, shows that the rows/columns of \(1/\sqrt{\xi_{ijk}}F^{(ijk)}\) are orthonormal which, however, is only possible if \(F^{(ijk)}\) is a square matrix. In this case, \(1/\sqrt{\xi_{ijk}}F^{(ijk)}\) is an orthogonal matrix that without loss of generality can be orthogonally transformed to the unit element: In fact, the \((i, j, k)\) block in \(D^{(ij)}\)
\[
\begin{pmatrix}
d_{ijk} 1 & 0 \\
0 & -d_{ijk} 1
\end{pmatrix}
\] (D.88)
is invariant under an orthogonal transformation
\[
\begin{pmatrix}
T & 0 \\
0 & S
\end{pmatrix}
\] (D.89)
that at the same time acts on the \((i, j, k)\) block in \(A^{(ij)}\) as
\[
\begin{pmatrix}
0 & F^{(ijk)} \\
-F^{(ijk)T} & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
T^T & 0 \\
0 & S^T
\end{pmatrix} \begin{pmatrix}
0 & F^{(ijk)} \\
-F^{(ijk)T} & 0
\end{pmatrix} \begin{pmatrix}
T & 0 \\
0 & S
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
-(T^TF^{(ijk)}S)^T & T^TF^{(ijk)}S
\end{pmatrix}.
\] (D.90)
Choosing \(S = 1, T = 1/\sqrt{\xi_{ijk}}\) one obtains
\[
F^{(ijk)} = \sqrt{\xi_{ijk}} 1.
\] (D.91)
The condition (D.85) finally reads
\[
d^2_{ijk} + \xi_{ijk} = y^2_{ij}
\] (D.92)
and, hence,
\[
d_{ijk} = |y_{ij}| \cos \phi_{ijk}, \quad \sqrt{\xi_{ijk}} = |y_{ij}| \sin \phi_{ijk}
\] (D.93)
for some angle \(\phi_{ijk} \in (0, \pi/2)\). To conclude, we have the following block types,
\[
G^{(ijk)}_1 = x_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \varepsilon,
\]
\[
G^{(ijk)}_4 = |y_{ij}| \left( \cos \phi_{ijk} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \varepsilon + \sin \phi_{ijk} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_2 \right)
\] (D.94)
for \(\phi_{ijk} \in (0, \pi/2)\), while blocks with \(F^{(ijk)} = 0\) read
\[
G^{(ijk)}_1 = x_i \begin{pmatrix} 1 & 0 \\ 0 & 1' \end{pmatrix} \otimes \varepsilon,
\]
\[
G^{(ijk)}_4 = |y_{ij}| \begin{pmatrix} 1 & 0 \\ 0 & -1' \end{pmatrix} \otimes \varepsilon.
\] (D.95)
Finally, zero blocks in \(D^{(ij)}\) give rise to the following blocks:
\[
G^{(ij0)}_1 = x_i(1 \otimes 1_2) \otimes \varepsilon,
\]
\[
G^{(ij0)}_4 = (D^{(ij0)} \otimes \varepsilon) \otimes 1_2.
\] (D.96)
Using appropriate orthogonal transformations it is possible to write (D.94) as
\[
G^{(ijk)}_1 = x_i 1 \otimes (1_2 \otimes \varepsilon),
\]
\[
G^{(ijk)}_4 = |y_{ij}| 1 \otimes \left( \cos \phi_{ijk} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \varepsilon + \sin \phi_{ijk} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_2 \right),
\] (D.97)
and, similarly, we transform (D.96) to
\[
G_{1}^{(ij0)} = x_i \, 1 \otimes (1_2 \otimes \varepsilon),
\]
\[
G_{4}^{(ij0)} = D^{(ij0)} \otimes (\varepsilon \otimes 1_2).
\] (D.98)

Note that both (D.97) and (D.98) are block-diagonal matrices with non-trivial $4 \times 4$ blocks. From these blocks and using (D.73) we can construct the full solution of (D.71) for the gauge choice outlined above. As mentioned already, in the end one may have to apply another reflection so that this gauge can be obtained from generic matrices $G_1$ and $G_4$ only by rotations, rather than reflections.

\[^{20}\text{Note that (D.98) yields (D.97) for } \phi_{ijk} = \pi/2 \text{ provided that } D^{(ij0)} \text{ has only positive eigenvalues. But the latter need not be the case in general.}\]
References

[1] E. Witten, *Dynamical Breaking of Supersymmetry*, Nucl.Phys. B188 (1981) 513.

[2] S. Cecotti, L. Girardello and M. Porrati, *Two into one won’t go*, Phys.Lett. B145 (1984) 61.

[3] S. Cecotti, L. Girardello and M. Porrati, *Constraints on partial superhiggs*, Nucl.Phys. B268 (1986) 295–316.

[4] J. Bagger and A. Galperin, *Matter couplings in partially broken extended supersymmetry*, Phys. Lett. B336 (1994) 25–31 [hep-th/9406217].

[5] I. Antoniadis, H. Partouche and T. Taylor, *Spontaneous breaking of N=2 global supersymmetry*, Phys.Lett. B372 (1996) 83–87 [hep-th/9512006].

[6] S. Ferrara, L. Girardello and M. Porrati, *Minimal Higgs branch for the breaking of half of the supersymmetries in N=2 supergravity*, Phys.Lett. B366 (1996) 155–159 [hep-th/9510074].

[7] S. Ferrara, L. Girardello and M. Porrati, *Spontaneous breaking of N=2 to N=1 in rigid and local supersymmetric theories*, Phys.Lett. B376 (1996) 275–281 [hep-th/9512180].

[8] P. Fre, L. Girardello, I. Pesando and M. Trigiante, *Spontaneous N=2 → N=1 local supersymmetry breaking with surviving compact gauge group*, Nucl.Phys. B493 (1997) 231–248 [hep-th/9607032].

[9] J. Louis, P. Smyth and H. Triendl, *Spontaneous N=2 to N=1 supersymmetry breaking in supergravity and type II string theory*, JHEP 1002 (2010) 103 [0911.5077].

[10] J. Louis, P. Smyth and H. Triendl, *The N=1 low-energy effective action of spontaneously broken N=2 supergravities*, JHEP 1010 (2010) 017 [1008.1214].

[11] V. Cortés, J. Louis, P. Smyth and H. Triendl, *On certain Kähler quotients of quaternionic Kähler manifolds*, 36 pages.

[12] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, JHEP 0509 (2005) 016 [hep-th/0507289].

[13] M. de Roo and P. Wagemans, *Partial supersymmetry breaking in N=4 supergravity*, Phys.Lett. B177 (1986) 352.

[14] P. Wagemans, *Breaking of N=4 supergravity to N=1, N=2 at Λ = 0*, Phys.Lett. B206 (1988) 241.

[15] S. J. Gates and B. Zwiebach, *Searching for all N=4 supergravities with global SO(4)*, Nucl.Phys. B238 (1984) 99.

[16] M. de Roo and P. Wagemans, *Gauge matter coupling in N=4 supergravity*, Nucl.Phys. B262 (1985) 644.
[17] J. Schon and M. Weidner, *Gauged N=4 supergravities*, JHEP **0605** (2006) 034 [hep-th/0602024].

[18] M. Weidner, *Gauged supergravities in various spacetime dimensions*, Fortsch.Phys. **55** (2007) 843–945 [hep-th/0702084].

[19] V. Tsokur and Y. Zinovev, *Spontaneous supersymmetry breaking in N=4 supergravity with matter*, Phys.Atom.Nucl. **59** (1996) 2192–2197 [hep-th/9411104].

[20] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Lledo, *Super Higgs effect in extended supergravity*, Nucl.Phys. **B640** (2002) 46–62 [hep-th/0202116].

[21] G. Dall’Agata, G. Villadoro and F. Zwirner, *Type-IIA flux compactifications and N=4 gauged supergravities*, JHEP **0908** (2009) 018 [0906.0370].

[22] G. Dibitetto, A. Guarino and D. Roest, *Charting the landscape of N=4 flux compactifications*, JHEP **1103** (2011) 137 [1102.0239].

[23] D. Cassani and P. Koerber, *Tri-Sasakian consistent reduction*, JHEP **1201** (2012) 086 [1110.5327].

[24] E. Bergshoeff, I. Koh and E. Sezgin, *Coupling of Yang-Mills to N=4, D=4 supergravity*, Phys.Lett. **B155** (1985) 71.

[25] M. de Roo, *Gauged N=4 matter couplings*, Phys.Lett. **B156** (1985) 331.

[26] A. Borghese and D. Roest, *Metastable supersymmetry breaking in extended supergravity*, JHEP **1105** (2011) 102 [1012.3735].

[27] S. Ferrara and A. Van Proeyen, *A theorem on N=2 special Kähler product manifolds*, Class.Quant.Grav. **6** (1989) L243.

[28] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge Univ. Press, Cambridge, 1985. Corollary 4.4.4.

[29] S. Ferrara and P. van Nieuwenhuizen, *Noether coupling of massive gravitinos to N=1 supergravity*, Phys. Lett. **B127** (1983) 70.

[30] J. Wess and J. Bagger, *Supersymmetry and Supergravity*. Princeton Univ. Press, 1991. p.136.

[31] W. Soergel, *Lie-Algebren*. http://home.mathematik.uni-freiburg.de/soergel/Skripten/LIE.pdf, Korollar 1.3.26.