A SHORT PROOF FOR HOPF BIFURCATION IN GURTIN-MACCAMY’S POPULATION DYNAMICS MODEL

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Abstract. In this paper, we provide a short proof for the Hopf bifurcation theorem in the Gurtin-MacCamy’s population dynamics model. Here we use the Crandall and Rabinowitz’s approach, based on the implicit function theorem. Compared with previous methods, here we require the age-specific birth rate to be slightly smoother (roughly of bounded variation), but we have a huge gain for the length of the proof.

1. Age-Structured Models

We consider the existence of periodic solutions for the following equation,

\[
\begin{cases}
(\partial_t + \partial_a)u(t, a) = -mu(t, a), & a \in (0, +\infty), \\
u(t, 0) = f(\nu, \int_0^\infty b(a)u(t, a)\,da), \\
u(0, \cdot) = \psi \in L^1_+(\mathbb{R}),
\end{cases}
\]

where \(m > 0\) and \(b \in L^\infty_+(0, \infty)\) are mortality rate and fertility rate of the population respectively, the nonlinear function \(f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})\) describes the birth limitations for the population while \(\nu \in \mathbb{R}\) is regarded as a bifurcation parameter. Here \(u(t, a)\) denotes the density of a population at time \(t\) with age \(a\). This equation is referred as the Gurtin-MacCamy’s age-structured equation and was introduced in its nonlinear form by Gurtin and MacCamy in \([8]\) to study temporal evolution of biological populations.

The existence of nontrivial periodic solution induced by Hopf bifurcation has been observed in various specific age-structured models (Cushing \([3, 4]\), Prüss \([16]\), Swart \([17]\), Kostava and Li \([11]\), Bertoni \([1]\), Magal and Ruan \([14]\)). In this paper we shall use the implicit function theorem to establish the Hopf bifurcation theorem that is used to obtain the existence of nontrivial periodic solutions of the age-structured model \((1.1)\), that is, a nontrivial periodic solution bifurcated from the equilibrium of \((1.1)\) when the bifurcation parameter \(\nu\) takes some critical values.

For two dimensional ordinary differential equations Crandall and Rabinowitz \([2]\) requires less regularity \((C^2\text{-right hand side})\) than the standard result by Hale and Kocack \([9]\) (which requires \(C^3\text{-right hand side}\)) and Hassard, Kazarinoff and Wan \([10]\) (which requires \(C^4\text{-right hand side}\)). Here we assume that the function \(f\) is only \(C^2\) which corresponds to the regularity imposed by Crandall and Rabinowitz \([2]\) for their result applied to ordinary differential equations. Such a regularity assumption has been mentioned already in Liu, Magal and Ruan \([12, \text{see Remark 2.5}]\). In the context of partial differential equations Crandall and Rabinowitz \([2]\) original theorem only applies to parabolic PDE since their proof strongly uses the fact that the semigroup is generated by a sectorial operator \(A : D(A) \subset X \rightarrow X\) on a Banach space \(X\) and verifies that the map

Key words and phrases. Age structure; Population dynamics; Hopf bifurcation.

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\( t \rightarrow e^{At}x \) (with \( x \in X \)) is of class \( C^1 \) on \((0, \infty)\). Here we are working with hyperbolic operator, therefore such a property is not satisfied. Nevertheless by exploiting the special structure of the system, and imposing some extra regularity for the birth function \( a \rightarrow b(a) \) (i.e. \( b \) has bounded variation on \([0, \infty)\)) we are still able to apply Crandall and Rabinowitz’s ideas.

Note that the previous result by Magal and Ruan [14] and Liu, Magal and Ruan [12] only assume that the birth function \( b \) belongs to \( L^\infty(0, \infty) \). Here the fact that \( a \rightarrow b(a) \) has bounded variation on each \([0, \infty)\), means that \( a \rightarrow b(a) \) has a finite number of discontinuity point on each bounded interval on \([0, \infty)\) and that \( b(a) \) is continuous in between two successive discontinuity points. Such an assumption is sufficient for most practical examples. Finally this paper is closely related to the work of Cushing [3] in which he considered an equation with age and delay at birth. In [3], the function \( a \rightarrow b(a) \) is assumed to be of class \( C^1 \) which is stronger than our bounded variation assumption.

This paper is organized as follows. In Section 2, we give the well-posedness result of (1.1). In Section 3, we provide the assumptions for our Hopf bifurcation theorem, while Section 4 is devoted to state and prove the Hopf bifurcation theorem.

2. WELL-POSEDNESS

Set

\[ X = \mathbb{R} \times L^1((0, \infty), \mathbb{R}) \text{ and } X_0 = \{0\} \times L^1((0, \infty), \mathbb{R}). \]

Assume that \( X \) is endowed with the product norm

\[ \|x\| = |\alpha| + \|\psi\|_{L^1((0, \infty), \mathbb{R})}, \quad \forall x = \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \in X. \]

Consider the linear operator \( A : D(A) \subset X \rightarrow X \) given by

\[ A \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} -\psi(0) \\ -\psi' - m\psi \end{pmatrix} \]

with

\[ D(A) = \{0\} \times W^{1,1}((0, \infty), \mathbb{R}). \]

Recall that \( X_0 \) is the closure of \( D(A) \) in \( X \). In addition, note that \( A_0 \), the part of \( A \) in \( X_0 \), generates a \( C_0 \)-semigroup of bounded linear operators, denoted by \( \{T_{A_0}(t)\}_{t \geq 0} \) and explicitly given by

\[ T_{A_0}(t) \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{T}_{A_0}(t)\psi \end{pmatrix}, \]

wherein we have set, for all \( t \geq 0 \) and \( \psi \in L^1((0, \infty), \mathbb{R}) \),

\[ \hat{T}_{A_0}(t)(\psi)(a) = \begin{cases} e^{-mt}\psi(a - t), & \text{if } a \geq t, \\ 0, & \text{if } a \leq t. \end{cases} \]

Moreover \( A \) generates an integrated semigroup of \( X \), denoted by \( \{S_A(t)\}_{t \geq 0} \) and defined, for \( t \geq 0 \) by

\[ S_A(t) \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \left( L(t)\alpha + \int_0^t \hat{T}_{A_0}(s)\psi ds \right), \quad \forall \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \in X, \]

where

\[ L(t)(\alpha)(a) = \begin{cases} 0, & \text{if } a \geq t, \\ e^{-ma}\alpha, & \text{if } a \leq t. \end{cases} \]
Define the map $H : \mathbb{R} \times X_0 \to X$ by

$$H\left(\nu, \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right) = \left( f\left(\nu, \int_0^\infty b(a)\psi(a)da\right), 0 \right).$$

Then by identifying $u(t)$ with $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, problem (1.1) can be rewritten as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H(\nu, v(t)), \quad t \geq 0, \quad v(0) = y \in X_0.$$

Since $f$ is Lipschitz continuous on bounded sets, the general results proved in Magal and Ruan [13, see section 5] or [15, Chapter 5] ensure that the Cauchy problem (2.1) generated a maximal semiflow (with eventually some blowup), denoted below by $\{U_{\nu}(t)\}_{t \geq 0}$.

3. Assumptions

Regularity of the birth function: Let us recall some definition and properties of the so-called bounded variation functions. Let $F : [0, \infty) \to \mathbb{R}$ be some function. For each $a > 0$ define

$$T_F(a) := \sup \left\{ \sum_{j=1}^n |F(a_j) - F(a_{j-1})| : n \in \mathbb{N}, 0 = a_0 < \cdots < a_n = a \right\} \in [0, \infty],$$

where the supremum is taken over all finite strictly increasing sequences in $[0, a]$.

**Definition 3.1.** A function $F : [0, \infty) \to \mathbb{R}$ is said to be of bounded variation on $[0, \infty)$ if

$$\sup_{a > 0} T_F(a) < \infty.$$

In that case the function $a \to T_F(a)$ is bounded and increasing on $[0, \infty)$.

Let $F : [0, \infty) \to \mathbb{R}$ be a right continuous function of bounded variation on $[0, \infty)$, then $a \to T_F(a)$ is also right continuous and according to Folland [7, Theorem 3.29], there exists a unique Borel measure $\mu_F$ such that

$$\mu_F([a, b]) = F(b) - F(a),$$

for all $a, b \in [0, \infty)$, with $a < b$.

Furthermore, its total variation $|\mu_F|$ is the positive and bounded Borel measure associated to the right continuous and increasing function $a \to T_F(a)$.

Next let us recall the integration by parts formula proved by Folland [7, Theorem 3.36] as well. If $G : [0, \infty) \to \mathbb{R}$ is of class $C^1$ then for all $0 \leq a < b$, one has

$$\int_a^b G(s)\mu_F(ds) = [G(b)F(b-) - G(a)F(a+)] - \int_a^b G'(s)F(s)ds.$$

We now make a set of assumptions on the fertility rate $b$.

**Assumption 3.2.** Assume that the function $\chi(a) := b(a)e^{-ma}$ satisfies the two following properties

(i) Assume that $\chi \in L^1(0, \infty)$ with

$$\int_0^\infty \chi(a)da = 1 \iff \int_0^\infty b(a)e^{-ma}da = 1.$$

(ii) We assume that the function $\tau : a \to a\chi(a)$ is right continuous and of bounded variation on $[0, \infty)$. We let $\mu_\tau$ be the unique Borel measure associated to $\tau$. 
Remark 3.3. Note that when \( b \in L^\infty(0, \infty) \), for each integer \( n \in \mathbb{N} \) one has
\[
a \to a^n \chi(a) = a^n b(a) e^{-ma} \in L^1((0, \infty), \mathbb{R}).
\]

**Equilibria:** Recall that \( \left( \begin{array}{c} 0 \\ \bar{u} \end{array} \right) \in D(A) \) is an equilibrium of the semiflow \( \{U_\nu(t)\}_{t \geq 0} \) if and only if
\[
\left( \begin{array}{c} 0 \\ \bar{u} \end{array} \right) \in D(A) \quad \text{and} \quad A \left( \begin{array}{c} 0 \\ \bar{u} \end{array} \right) + H \left( \nu, \left( \begin{array}{c} 0 \\ \bar{u} \end{array} \right) \right) = 0.
\]
As a consequence a positive equilibrium is given by
\[
\bar{u}(a) = \bar{w}_\nu e^{-ma}, \ a \geq 0
\]
where \( \bar{w}_\nu > 0 \) becomes a solution of the following equation
\[
\bar{w}_\nu = f(\nu, \bar{w}_\nu).
\]

Our next assumption is concerned with the existence of such equilibrium point and its regularity with respect to the parameter \( \nu \).

**Assumption 3.4.** We assume that there exists an open interval \( I \) such that for each \( \nu \in I \) there exists a constant solution \( \bar{w}_\nu \in \mathbb{R} \) of the equation \( \bar{w}_\nu = f(\nu, \bar{w}_\nu) \).

We assume further that the map \( \nu \to \bar{w}_\nu \) is continuously differentiable on the interval \( I \). In the sequel we set
\[
\bar{v}_\nu = \left( \begin{array}{c} 0 \\ \bar{w}_\nu \end{array} \right) \quad \text{with} \quad \bar{u}_\nu(a) = \bar{w}_\nu e^{-ma}, \ \forall \nu \in I.
\]

In the following we will use the notation \( \mathcal{L}(Y, Z) \) to denote the space of the linear bounded operators from \( Y \) to \( Z \) where \( Y \) and \( Z \) are two Banach spaces. Define for \( \nu \in I \) the linear operator \( B_\nu : D(B_\nu) \subset X \to X \) as follows,
\[
D(B_\nu) = D(A) \quad \text{and} \quad B_\nu x = Ax + \partial_v H(\nu, \bar{v}_\nu)x, \ \forall x \in D(B_\nu),
\]
wherein \( \partial_v \) corresponds to the partial derivative of \( H(\nu, v) \) with respect to \( v \). The bounded linear operator \( \partial_v H(\nu, \bar{v}_\nu) \in \mathcal{L}(X_0, X) \) is defined by
\[
\partial_v H(\nu, \bar{v}_\nu) \left( \begin{array}{c} 0 \\ \varphi \end{array} \right) = \left( \partial_w f(\nu, \bar{w}_\nu) \int_0^{\infty} b(l) \varphi(l) dl \right) \left( \begin{array}{c} 0 \\ \varphi \end{array} \right) \in X_0.
\]
Herein \( \partial_w \) denotes the partial derivative of \( f = f(\nu, w) \) with respect to \( w \).

By using the result of Ducrot, Liu and Magal \([6]\), the essential growth rate of the semigroup generated by \( (B_\nu)_0 \), the part of \( B_\nu \) in the closure of its domain, satisfies
\[
\omega_{0,\text{ess}}((B_\nu)_0) \leq -m < 0.
\]
The following result follows from \([15, \text{Theorem 4.3.27, Lemma 4.4.2, Theorem 4.4.3-(ii)}]\) to which we refer the reader for a proof and more details.

**Lemma 3.5.** The spectrum of \( B_\nu \) in the half plane
\[
\Omega := \{ \lambda \in \mathbb{C} : \text{Re} \lambda > -m \}
\]
contains only isolated eigenvalues which are poles of the resolvent of \( B_\nu \).
Recall that the characteristic function, describing the spectrum of \( B_\nu \) in \( \Omega \), is obtained by computing the resolvent of \( B_\nu \) as presented in Liu, Magal and Ruan \([12]\). We define the characteristic function for \( \nu \in I \) and \( \lambda \in \Omega \) as follows
\[
\Delta(\nu, \lambda) := 1 - \partial_w f (\nu, \overline{w}_\nu) \int_0^\infty b(l)e^{-(\nu + \lambda)l}dl.
\]

Recall that the resolvent set \( \rho(A) \) of \( A \) contains \( \Omega \) and for each \( \lambda \in \Omega \) the resolvent of \( A \) is defined by the following formula
\[
(\lambda I - A)^{-1} \begin{pmatrix}
\alpha \\
\psi
\end{pmatrix} = \begin{pmatrix}
0 \\
\varphi
\end{pmatrix}
\]
\[\Leftrightarrow \varphi(a) = e^{-(\lambda + m)a} + \int_0^a e^{-(\lambda + m)(a-s)}\psi(s)ds.
\]
We now recall some result already presented (in a more general framework in the Section 5.2 in \([12]\])

**Lemma 3.6.** For each \( \nu \in I \) the resolvent set \( \rho(B_\nu) \) of \( B_\nu \) satisfies
\[\lambda \in \rho(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu, \lambda) \neq 0,
\]
or equivalently the spectrum \( \sigma(B_\nu) := \mathbb{C} \setminus \rho(B_\nu) \) of \( B_\nu \) satisfies
\[\lambda \in \sigma(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu, \lambda) = 0.
\]
Moreover one has
\[
(\lambda I - B_\nu)^{-1} \begin{pmatrix}
\alpha \\
\psi
\end{pmatrix} = \begin{pmatrix}
0 \\
\varphi
\end{pmatrix}
\]
\[\Leftrightarrow \varphi(a) = e^{-(\lambda + m)a} + \int_0^a e^{-(\lambda + m)(a-s)}\psi(s)ds,
\]
where
\[\alpha_1 := \Delta(\nu, \lambda)^{-1} \left[ \alpha - \partial_w f (\nu, \overline{w}_\nu) \int_0^\infty b(a) \int_0^a e^{-(\lambda + m)(a-s)}\psi(s)dsda \right].
\]

**Proof.** For each \( \lambda \in \Omega \) the linear operator \( \lambda I - B_\nu \) is invertible if and only if the linear operator is invertible
\[I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1}
\]
In that case we have
\[\left(\lambda I - B_\nu\right)^{-1} = (\lambda I - A)^{-1} \left[ I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1} \right]^{-1}.
\]

**Computation of the inverse of** \( I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1} :** We have
\[
\begin{pmatrix}
\hat{\alpha} \\
\hat{\varphi}
\end{pmatrix} = \left[ I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1} \right] \begin{pmatrix}
\alpha \\
\varphi
\end{pmatrix}
\]
\[\Leftrightarrow \begin{pmatrix}
\hat{\alpha} \\
\hat{\varphi}
\end{pmatrix} = \left[ \alpha - \partial_w f (\nu, \overline{w}_\nu) \int_0^\infty b(a) \left[ e^{-(\lambda + m)a} + \int_0^a e^{-(\lambda + m)(a-s)}\varphi(s)ds \right] da \right] \begin{pmatrix}
\alpha \\
\varphi
\end{pmatrix}.
\]
Therefore \( I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1} \) is invertible (for \( \lambda \in \Omega \)) if and only if \( \Delta(\nu, \lambda) \neq 0 \) and one has
\[
\begin{pmatrix}
\alpha \\
\varphi
\end{pmatrix} = \left[ I - \partial_v H(\nu, \overline{v}_\nu) (\lambda I - A)^{-1} \right]^{-1} \begin{pmatrix}
\hat{\alpha} \\
\hat{\varphi}
\end{pmatrix}
\]
\[\Leftrightarrow \begin{pmatrix}
\alpha \\
\varphi
\end{pmatrix} = \left[ \Delta(\nu, \lambda)^{-1} \left[ \alpha - \partial_w f (\nu, \overline{w}_\nu) \int_0^\infty b(a) \int_0^a e^{-(\lambda + m)(a-s)}\varphi(s)ds da \right] \right] \begin{pmatrix}
\hat{\alpha} \\
\hat{\varphi}
\end{pmatrix}.
\]
The result follows. □

**Assumption 3.7.** There exists \( \nu_0 \in I \) and \( \omega_0 > 0 \) such that the following properties are satisfied

(i) \[ \Delta(\nu_0, \omega_0) = 0. \]

(ii) Crandall and Rabinowitz’s condition
\[ \Delta(\nu_0, k \omega_0 i) \neq 0, \forall k \in \mathbb{N} \text{ and } k \neq 1. \]

(iii) Simplicity of the eigenvalue \( \omega_0 i \)
\[ \partial_\lambda \Delta(\nu_0, \omega_0 i) \neq 0, \]
which is also equivalent to
\[ \partial_\nu f(\nu_0, \nu_0) \int_0^\infty b(l) e^{-(m+i\omega_0)l} dl \neq 0. \]

(iv) Transversality condition
\[ \text{Re} \left( \frac{\partial_\nu \Delta(\nu_0, \omega_0 i) \times \partial_\lambda \Delta(\nu_0, \omega_0 i)}{} \right) \neq 0. \]

We can observe that by combining (i) and (iii) and by using the implicit function theorem there exists a branch \( \lambda : (\nu_0 - \eta, \nu_0 + \eta) \subset I \to \mathbb{C} \) with some \( \eta > 0 \) small enough such that for each \( \nu \in (\nu_0 - \eta, \nu_0 + \eta) \), \( \lambda(\nu) = \alpha(\nu) + i\omega(\nu) \) and \( \lambda(\nu) = \alpha(\nu) - i\omega(\nu) \) satisfying solution of
\[ \Delta(\nu, \lambda(\nu)) = 0 \]
and
\[ \lambda(\nu_0) = i\omega_0. \]

Moreover by combining (iii)-(iv) we deduce that the transversality condition is satisfied. Namely, one has
\[ \text{Re} \frac{d\lambda(\nu_0)}{d\nu} \neq 0. \]

Moreover by using the property (iii) in Lemma 5.8 in [12] we also deduce that \( i\omega_0 \) is a simple eigenvalue of \( B_{\nu_0} \) since
\[ \lim_{\lambda \to \lambda_0} \frac{\Delta(\nu_0, \lambda)}{\lambda - \lambda_0} \neq 0 \Leftrightarrow \partial_\lambda \Delta(\nu_0, \lambda_0) \neq 0. \ (\text{with } \lambda_0 = \lambda(\nu_0)). \]

The condition (ii) avoids to assume that the purely imaginary spectrum is reduced to a single pair of purely imaginary eigenvalues. Such a condition has been introduced in the Crandall and Rabinowitz’s proof [2].

4. **Main Result**

In this section we state the main theorem of this paper. The following result is inspired by Crandall and Rabinowitz [2].

**Theorem 4.1.** Let Assumptions 3.2, 3.4 and 3.7 be satisfied. Then there exist a constant \( \delta > 0 \) and two \( C^1 \) maps, \( s \to \nu(s) \) from \( (-\delta, \delta) \) to \( \mathbb{R} \) and \( s \to \omega(s) \) from \( (-\delta, \delta) \) to \( \mathbb{R} \) such that for each \( s \in (-\delta, \delta) \) there exists a \( 2\pi/\omega(s) \)-periodic solution \( u(s) \) of class of \( C^1 \) which is a solution of (1.1) with the parameter \( \nu = \nu(s) \). Moreover, the branch of periodic orbit is bifurcating from \( \nu_0 \) at \( \nu = \nu_0 \), that is to say that
\[ \nu(0) = \nu_0, \quad \omega(0) = \omega_0. \]
Proof. (Proof of Theorem 4.1) Up to time rescaling we can assume, without loss of
generality, that \( \omega_0 = 1 \). Observe that Assumption 3.7-(i) implies that (1.1) linearized
about \( u = \bar{u}_\nu \) for \( \nu = \nu_0 \) has nontrivial \( 2\pi \)-periodic solutions. We now seek nontrivial
\( 2\pi/\omega \)-periodic solutions of (1.1) with \( \omega \) close to 1 and \((u, \nu) \) close to \((\bar{u}_\nu, \nu_0)\).

Solving (1.1) along the characteristic line \( t - a = \text{constant} \), one obtains
\[
[u](t,a) = u(t-a,0)e^{-ma}, \quad t \in \mathbb{R}, \quad a > 0.
\]
Thus \( v = v(t) \) given by
\[
v(t) = u(t,0),
\]
satisfies the renewal equation
\[
v(t) = f\left( \nu, \int_0^\infty b(a)v(t-a)e^{-ma}da \right), \quad t \in \mathbb{R}.
\]
Setting
\[
w(t) = v\left( \frac{t}{\omega} \right),
\]
yields the following equation for the \( 2\pi \)-periodic function \( w = w(t) \)
\[
w(t) = v\left( \frac{t}{\omega} \right) = f\left( \nu, \int_0^\infty b(a)v(t/\omega-a)e^{-ma}da \right)
= f\left( \nu, \int_0^\infty b(a)w(t-\omega a)e^{-ma}da \right), \quad t \in \mathbb{R}.
\]
Recalling the definition of \( \chi \) in Assumption 3.2 and using the change of the variable
\( l = \omega a \) in the integral lead to the following equation for \( w = w(t) \)
\[
(4.1) \quad w(t) = f\left( \nu, \int_0^\infty \frac{1}{\omega} \chi\left( \frac{l}{\omega} \right)w(t-l)dl \right), \quad t \in \mathbb{R}.
\]
Now the existence of nontrivial \( 2\pi/\omega \)-periodic solution of (1.1) becomes equivalent to
the one of nontrivial \( 2\pi \)-periodic solution of (4.1). Next we shall apply the implicit
function theorem to investigate the existence of nontrivial \( 2\pi \)-periodic solution of (4.1)
for \( \nu \) close to \( \nu_0 \).

Let \( C_{2\pi}(\mathbb{R}) \) be the Banach space of the continuous \( 2\pi \)-periodic functions. Define the
map \( F : \mathbb{R}^2 \times C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R}) \) by
\[
F(\omega, \nu, x)(t) = x(t) - f\left( \nu, \int_0^\infty \frac{1}{\omega} \chi\left( \frac{l}{\omega} \right)x(t-l)dl \right), \quad \forall (\omega, \nu, x) \in \mathbb{R}^2 \times C_{2\pi}(\mathbb{R}).
\]
We now aim at investigating the zeros of the equation
\[
F(\omega, \nu, x) = 0,
\]
for \((\omega, \nu, x) \) close to \((1, \nu_0, \bar{w}_{\nu_0})\) using the implicit function theorem. To do so, we need
first to verify the smoothness of
\[
\int_0^\infty \frac{1}{\omega} \chi\left( \frac{l}{\omega} \right)x(t-l)dl = \int_0^\infty \chi(l)x(t-\omega l)dl,
\]
with respect to \( \omega \).

The first main step of this proof is the following lemma.

Lemma 4.2. The map \( G : \mathbb{R} \times C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R}) \) defined by
\[
G(\omega, x)(t) = \int_0^\infty \chi(l)x(t-\omega l)dl, \quad t \in \mathbb{R},
\]

is continuously differentiable with respect to \( \omega \in \mathbb{R} \) and its partial derivative with respect to \( \omega \), denoted by \( \partial_\omega G \), is given

\[
\partial_\omega G(\omega, x)(\cdot) = \int_0^\infty x(\cdot - \omega l)\mu_\tau(dl), \quad \forall (\omega, x) \in \mathbb{R} \times C_{2\pi}(\mathbb{R}).
\]

Herein \( \tau \) and \( \mu_\tau \) are defined in Assumption 3.2-(ii).

Proof. First observe that using Fubini theorem, for any \( x \in C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R}) \) and \( \omega \in \mathbb{R} \), we have, for any \( t \in \mathbb{R} \),

\[
\int_0^\omega \int_0^\infty -x'(t - \sigma l) \chi(l) dl d\sigma = \int_0^\infty \int_0^\omega -lx'(t - \sigma l) d\sigma \chi(l) dl = \int_0^\infty x(t - \sigma l)|_{\sigma=0} \chi(l) dl = \int_0^\omega \chi(l)x(t - \omega l)dl - \int_0^t \int_0^\infty \chi(l)dl,
\]

so that, since \( \int_0^\infty \chi(l)dl = 1 \) (see Assumption 3.2-(i)), we get

\[
(4.2) \quad \int_0^\omega \int_0^\infty -x'(t - \sigma l)l \chi(l) dl d\sigma = \int_0^\infty \chi(l)x(t - \omega l)dl - x(t),
\]

that is for all \( (\omega, x) \in \mathbb{R} \times \left(C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})\right) \) and \( t \in \mathbb{R} \)

\[
G(\omega, x)(t) = x(t) - \int_0^\omega \int_0^\infty x'(t - \sigma l)\tau(l) dl d\sigma.
\]

We deduce that for all \( x \in C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R}) \) the map \( \omega \mapsto G(\omega, x) = \int_0^\infty \chi(l)x(\cdot - \omega l)dl \) is of class \( C^1 \) and

\[
(4.3) \quad \frac{d}{d\omega} \int_0^\infty \chi(l)x(t - \omega l)dl = -\int_0^\omega x'(t - \omega l)l \chi(l)dl.
\]

Moreover by using again the formula (4.2) we get

\[
\int_\omega^{\omega+\varepsilon} \int_0^\infty -x'(t - \sigma l)l \chi(l) dl d\sigma = \int_0^{\infty} \chi(l)x(t - (\omega + \varepsilon) l)dl - \int_0^\infty \chi(l)x(t - \omega l)dl.
\]

hence

\[
(4.4) \quad \int_\omega^{\omega+\varepsilon} \frac{d}{d\sigma} \int_0^\infty \chi(l)x(t - \sigma l) dl d\sigma = \int_0^\infty \chi(l)x(t - (\omega + \varepsilon) l)dl - \int_0^\infty \chi(l)x(t - \omega l)dl.
\]

By using the integration by parts formula (3.1) and (4.3), we obtain for all \( (\omega, x) \in \mathbb{R} \times \left(C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})\right) \)

\[
\partial_\omega G(\omega, x)(t) = \frac{d}{d\omega} \int_0^\infty \chi(l)x(t - \omega l)dl = - \int_0^\omega x'(t - \omega l)l \chi(l)dl = \int_0^\omega x(t - \omega l)\mu_\tau(dl).
\]

Then using Assumption 3.2-(ii) we infer from the above equality that

\[
\|\partial_\omega G(\omega, x)\|_{C_{2\pi}(\mathbb{R})} \leq \|x\|_{C_{2\pi}(\mathbb{R})} |\mu_\tau|((0, \infty)) < \infty, \quad \forall (\omega, x) \in \mathbb{R} \times C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R}),
\]

wherein \( |\mu_\tau|((0, \infty)) \) is nothing but the variation of \( \tau(a) \) on \((0, \infty)\). That is supremum over all the subdivision \( 0 = t_0 < t_1 < t_2 < \ldots < t_n = M \) of

\[
|\mu_\tau|((0, M]) = \sup_{0 = t_0 < t_1 < t_2 < \ldots < t_n = M} \sum_{i=0}^n |\tau(t_{i+1}) - \tau(t_i)|,
\]

and

\[
|\mu_\tau|((0, \infty)) = \lim_{{M \to +\infty}} |\mu_\tau|((0, M]).
\]
We now define \( L_\omega \in \mathcal{L}(C_{2\pi}(\mathbb{R})) \) by
\[
L_\omega(x)(t) := \int_0^\infty x(t - \omega l)\mu_\tau(dl).
\]
Now the result follows by using the (4.4) and the density of \( C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R}) \) into \( C_{2\pi}(\mathbb{R}) \), since it implies that
\[
\int_\omega^{\omega + \varepsilon} L_\sigma(x)(t) d\sigma = \int_0^\infty \chi(l)x(t - (\omega + \varepsilon)l) dl - \int_0^\infty \chi(l)x(t - \omega l) dl.
\]
It remains to prove that the map \((\omega, x) \to \partial_\omega G(\omega, x) = \int_0^\infty x'(t - \omega l)\mu_\tau(dl)\) is continuous from \( \mathbb{R} \times C_{2\pi}(\mathbb{R}) \) into \( C_{2\pi}(\mathbb{R}) \). To see this fix \((\omega_1, x_1)\) in \( \mathbb{R} \times C_{2\pi}(\mathbb{R}) \) and observe that for all \((\omega, x) \in \mathbb{R} \times C_{2\pi}(\mathbb{R})\) one has:
\[
\partial_\omega G(\omega, x) - \partial_\omega G(\omega_1, x_1) = \int_0^\infty [x'(t - \omega l) - x_1'(t - \omega l)]\mu_\tau(dl) = J_1 + J_2,
\]
wherein we have set
\[
J_1 := \int_0^\infty [x'(t - \omega l) - x_1'(t - \omega l)]\mu_\tau(dl), J_2 := \int_0^\infty [x_1'(t - \omega l) - x_1'(t - \omega l)]\mu_\tau(dl).
\]
We first observe that
\[
||J_1||_{C_{2\pi}(\mathbb{R})} \leq ||x - x_1||_{C_{2\pi}(\mathbb{R})}||\mu_\tau||((0, \infty)) \to 0 \text{ uniformly for } \omega \in \mathbb{R} \text{ as } ||x - x_1||_{C_{2\pi}(\mathbb{R})} \to 0.
\]
On the other hand fix \( \varepsilon > 0 \) and since one has
\[
||\mu_\tau||((M, \infty)) := ||\mu_\tau||((0, \infty)) - ||\mu_\tau||((0, M)) \to 0 \text{ as } M \to \infty,
\]
choose \( M > 0 \) large enough so that \( 2||x_1||_{C_{2\pi}(\mathbb{R})}||\mu_\tau||((M, \infty)) \leq \varepsilon \). With such a choice we split \( J_2 \) as follows \( J_2 = I_1 + I_2 \) with
\[
I_1 := \int_0^M [x_1'(t - \omega l) - x_1'(t - \omega l)]\mu_\tau(dl), I_2 := \int_M^\infty [x_1'(t - \omega l) - x_1'(t - \omega l)]\mu_\tau(dl).
\]
Hence we get
\[
||I_2||_{C_{2\pi}(\mathbb{R})} \leq 2||x_1||_{C_{2\pi}(\mathbb{R})}||\mu_\tau||((M, \infty)) \leq \varepsilon,
\]
and, since \( x_1 \) is continuous and \( 2\pi \)-periodic, it is uniformly continuous on \( \mathbb{R} \). Thus for all \( \omega \) is sufficiently close to \( \omega_1 \), we have
\[
||I_1||_{C_{2\pi}(\mathbb{R})} \leq \sup_{0 \leq l \leq M} |x_1(t - \omega l) - x_1(t - \omega l)|\mu_\tau((0, M)) \leq \varepsilon.
\]
As a consequence \( \partial_\omega G \) is continuous and the proof is completed. \( \Box \)

**Computation of the derivatives of \( F \):** One can calculate the following derivatives directly,
\[
\partial_x F(\omega, \nu, \varphi)(x)(t) = x(t) - \partial_\omega f(\nu, \varphi) \int_0^\infty \chi(l)x(t - \omega l) dl,
\]
\[
\partial_\nu \partial_x F(\omega, \nu, \varphi)(x)(t) = - \left[ \partial_\nu \partial_\omega f(\nu, \varphi) + \partial^2_\nu f(\nu, \varphi) \partial_\omega \varphi \right] \int_0^\infty \chi(l)x(t - \omega l) dl,
\]
and by Lemma 4.2
\[
\partial_\omega \partial_x F(\omega, \nu, \varphi)(x)(t) = - \partial_\nu f(\nu, \varphi) \int_0^\infty x(t - \omega l)\mu_\tau(dl).
\]

**State space decomposition:** Note that by Assumption 3.7-(i) we have
\[
1 = \partial_\omega f(\nu_0, \varphi_0) \int_0^\infty \chi(l)e^{\pm il} dl,
\]
which re-writes as
\begin{equation}
1 = \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l) \cos ld\l and 0 = \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l) \sin ld\l
\end{equation}
and thus the kernel \(N(\partial_x F(1, \nu_0, \overline{w}_{\nu_0}))\) contains the space \(X_1\) given by
\[X_1 := \text{span}\{\cos, \sin\} \subset C_{2\pi}(\mathbb{R}).\]

Now define the closed space
\[X_2 := \left\{ z \in C_{2\pi}(\mathbb{R}) : \int_0^{2\pi} z(t) \cos(t) dt = \int_0^{2\pi} z(t) \sin(t) dt = 0 \right\}.
\]
This space turns out to be a complement of \(X_1\) as stated in the next lemma.

**Lemma 4.3.** We have the following state space decomposition
\[C_{2\pi}(\mathbb{R}) = X_1 \oplus X_2.\]

**Proof.** This property is directly inherited from the decomposition of \(L^2((0, 2\pi), \mathbb{R})\) as
\[L^2((0, 2\pi), \mathbb{R}) = X_1 \oplus X_1^\perp,
\]
with
\[X_1^\perp = \left\{ z \in L^2((0, 2\pi), \mathbb{R}) : \int_0^{2\pi} z(t) e^{it} dt = 0 \right\},
\]
Now if \(z \in C_{2\pi}(\mathbb{R})\) then \(z \in L^2((0, 2\pi), \mathbb{R})\) and the above \(L^2((0, 2\pi), \mathbb{R})\)–decomposition ensures that there exist unique \(z_1 \in X_1\) and \(z_2 \in X_1^\perp\) such that
\[z = z_1 + z_2.
\]
Now since \(z_1 = c_1 \cos + c_2 \sin\) (for some constants \(c_1\) and \(c_2\)) this ensures that \(z_2 = z - z_1\) is also continuous and \(z_2 \in X_2\). The state space decomposition follows. \(\Box\)

Now let us define the map \(h : \mathbb{R}^3 \times X_2 \to C_{2\pi}(\mathbb{R})\) by
\[h(s, \omega, \nu, z) = \begin{cases} s^{-1} F(\omega, \nu, \overline{w}_{\nu} + s(u_1 + z)), & \text{if } s \neq 0, \\ \partial_x F(\omega, \nu, \overline{w}_{\nu})(u_1 + z), & \text{if } s = 0, \end{cases}\]
where
\[u_1(t) = \cos(t), \forall t \in \mathbb{R}.
\]
Now let us observe that since \(f = f(\nu, x)\) is of class \(C^2\), \(h\) is of class \(C^1\). One also has \(h(0, 1, \nu_0, 0) = 0\) while the derivative with respect to \((\omega, \nu, z)\) is given, for all \((\tilde{\omega}, \tilde{\nu}, \tilde{z}) \in \mathbb{R} \times \mathbb{R} \times C_{2\pi}(\mathbb{R})\), by
\[D_{(\omega, \nu, z)} h(0, 1, \nu_0, 0)(\tilde{\omega}, \tilde{\nu}, \tilde{z}) = \partial_x F(1, \nu_0, \overline{w}_{\nu_0}) \tilde{z} + \tilde{\nu} \partial_\nu \partial_x F(1, \nu_0, \overline{w}_{\nu_0}) u_1 + \tilde{\omega} \partial_\omega \partial_x F(1, \nu_0, \overline{w}_{\nu_0}) u_1,
\]
hence by using (4.5)-(4.7) we obtain
\[D_{(\omega, \nu, z)} h(0, 1, \nu_0, 0)(\tilde{\omega}, \tilde{\nu}, \tilde{z}) = \tilde{z}(t) - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l) \tilde{z}(t-l) dl - \tilde{\nu} \left[ \partial_{\nu} \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_{\nu}^2 f(\nu_0, \overline{w}_{\nu_0}) \partial_\nu \overline{w}_{\nu_0} \right] \int_0^\infty \chi(l) u_1(t-l) dl - \tilde{\omega} \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty u_1(t-l) \mu(t) dl.
\]
The second main step of the proof is the following lemma.

**Lemma 4.4.** The bounded linear operator
\[D_{(\omega, \nu, z)} h(0, 1, \nu_0, 0) \in \mathcal{L}(\mathbb{R}^2 \times X_2, C_{2\pi}(\mathbb{R}))
\]
is invertible.
Proof. Let us first define the linear bounded operator $K : X_2 \to C_{2\pi}(\mathbb{R})$ by

$$K\phi := \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l) \phi(\cdot - l) dl.$$ 

Step 1: Let us prove that $K(X_2) \subset X_2$. Indeed, by using Fubini’s theorem, for all $\phi \in C_{2\pi}(\mathbb{R})$ one has

$$\int_0^{2\pi} \int_0^\infty \chi(l) \phi(t - l) dt dl e^{it} dt = \int_0^\infty \int_0^{2\pi} \phi(t - l) e^{i(t - l)} dl e^{it} \chi(l) dl$$

and since $t \to \phi(t) e^{it}$ is $2\pi$-periodic we deduce that

$$\int_0^{2\pi} \phi(t - l) e^{i(t - l)} dt = 0, \forall l \geq 0.$$ 

This completes the first step.

Step 2: Let us now prove that $N(I - K) = \{0\}$ whenever $I - K \in \mathcal{L}(X_2)$. In order to compute the kernel of $I - K$ in $X_2$, consider $g \in N(I - K)$, that is $g \in X_2$ such that

$$g(t) - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l) g(t - l) dl = 0, \forall t \in \mathbb{R}.$$ 

Multiplying the above equality by $e^{-int}$, for some $n \in \mathbb{Z}$ and integrating between 0 and $2\pi$ we obtain

$$[1 - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \tilde{\chi}(n)] \hat{g}(n) = 0,$$

wherein we have set

$$\hat{g}(n) := \int_0^{2\pi} g(l) e^{-int} dl,$$

and

$$\tilde{\chi}(n) := \int_0^\infty \chi(l) e^{-int} dl.$$ 

Now for $n \neq \pm 1$ we deduce by Assumption 3.7(ii) that

$$\hat{g}(n) = [1 - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \tilde{\chi}(n)]^{-1} 0 = 0.$$ 

Since $g \in X_2$, it follows that $g = 0$ and $N(I - K) = \{0\}$.

Step 3: Let us prove that $I - K \in \mathcal{L}(X_2)$ is invertible. Next note that it follows from the continuity of the translation in $L^1$ that $K$ is a compact operator. Thus one has $R(I - K) = X_2$ by Fredholm Alternative (see [15, Lemma 4.3.17]), where $R(I - K)$ denotes the range of $I - K$. Hence we have that $I - K$ is invertible and that the inverse is continuous by bounded inverse theorem.

Step 4: Let us prove that $D_{(\omega, \nu, z)} h(0, 1, \nu_0, 0)$ is invertible. To prove this, let $y \in C_{2\pi}(\mathbb{R})$ be given. Set $L := D_{(\omega, \nu, z)} h(0, 1, \nu_0, 0)$ and let us consider the equation

$$(\tilde{\omega}, \tilde{\nu}, \tilde{z}) \in \mathbb{R} \times \mathbb{R} \times X_2, \quad L(\tilde{\omega}, \tilde{\nu}, \tilde{z}) = y.$$ 

Define the projectors $P_1 : C_{2\pi}(\mathbb{R}) \to X_1$ and $P_2 : C_{2\pi}(\mathbb{R}) \to X_2$ associated to the state space decomposition of Lemma 4.3 and set $y_1 := P_1 y$ and $y_2 := P_2 y$. Next projecting (4.9) along $P_1$ and $P_2$ yields the following system, for all $t \in \mathbb{R},$

$$y_1(t) = -\tilde{\nu} \left[ \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_\nu f(\nu_0, \overline{w}_{\nu_0}) \partial_\nu \overline{w}_{\nu_0} \right] \int_0^\infty \chi(l) u_1(t - l) dl$$

$$-\tilde{\omega} \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty u_1(t - l) \mu_\omega(dl), \quad 0 \leq \omega \leq \tilde{\omega}, \quad 0 \leq \nu \leq \tilde{\nu},$$

$$(4.10)$$
and
\[ y_2(t) = \hat{z}_2(t) - \partial_w f(\nu_0, w_{\nu_0}) \int_0^\infty \chi(t)\hat{z}_2(t - t)dl. \]

Observe that \( I - K \) is invertible in \( X_2 \), thus \( \hat{z}_2 \) can be solved by
\[ \hat{z}_2 = (I - K)^{-1}y_2. \]

Let us focus on the resolution of (4.10). To that aim, recall that \( u_1 = \cos(\cdot) \) and (4.8) so that (4.10) rewrites
\[ y_1(t) = -\dot{\nu} \left[ \partial_w \partial_w f(\nu_0, w_{\nu_0}) + \partial_w^2 f(\nu_0, w_{\nu_0}) \partial_w w_{\nu_0} \right] \int_0^\infty \chi(t) \cos t \left[ \int_0^\infty \cos \mu_\tau (dl) \cos t + \int_0^\infty \sin \mu_\tau (dl) \sin t \right], \]

Furthermore by applying the integration by parts formula (3.1), we obtain
\[ \int_0^\infty \cos \mu_\tau (dl) = \lim_{M \to \infty} \int_0^M \cos \mu_\tau (dl) \]
\[ = \lim_{M \to \infty} \left\{ [\cos M\tau(M) - \cos 0\tau(0)] + \int_0^M \tau(t) \sin t dl \right\}, \]
hence
\[ \int_0^\infty \cos(t) \mu_\tau (dl) = \int_0^\infty l\chi(l) \sin(t)dl. \]

Similarly, one can obtain
\[ \int_0^\infty \sin(t) \mu_\tau (dl) = -\int_0^\infty \chi(l) \cos(t)dl. \]

On the other hand, since \( y_1 \in X_1 \), there exist two constants \( c_1, c_2 \in \mathbb{R} \) such that \( y_1 = c_1 \cos(t) + c_2 \sin(t) \), while the \( y_1 \)-equation can be rewritten as the following system, for all \( t \in \mathbb{R} \),
\[ c_1 \cos t + c_2 \sin t \]
\[ = -\dot{\nu} \left[ \partial_w \partial_w f(\nu_0, w_{\nu_0}) + \partial_w^2 f(\nu_0, w_{\nu_0}) \partial_w w_{\nu_0} \right] \int_0^\infty \chi(t) \cos t \left[ \int_0^\infty l\chi(l) \sin(t)dl - \int_0^\infty l\chi(l) \cos(t)dl \right], \]
and identifying the coefficients of \( \cos(t) \) and \( \sin(t) \) we end up with the resolution of the following two-dimensional linear system
\[ \left( \begin{array}{c} \int_0^\infty \chi(t) \cos t \int_0^\infty l\chi(l) \sin(t)dl \\ \int_0^\infty \chi(t) \cos t \int_0^\infty l\chi(l) \cos(t)dl \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) = \left( \begin{array}{c} \dot{\nu} \\ 0 \end{array} \right), \]

To solve this linear system, let us show that our assumptions for the characteristic equation, namely Assumption 3.7, ensures that the determinant of the above matrix is non-zero, that reads as
\[ (4.11) - \partial_w f(\nu_0, w_{\nu_0}) \left[ \partial_w \partial_w f(\nu_0, w_{\nu_0}) + \partial_w^2 f(\nu_0, w_{\nu_0}) \partial_w w_{\nu_0} \right] \int_0^\infty \chi(t) \cos t \int_0^\infty t\chi(l) \cos(t)dl \neq 0. \]

To check this property, recalling (4.8), it is sufficient to check that
\[ \left[ \partial_w \partial_w f(\nu_0, w_{\nu_0}) + \partial_w^2 f(\nu_0, w_{\nu_0}) \partial_w w_{\nu_0} \right] \int_0^\infty t\chi(l) \cos(t)dl \neq 0. \]

Next set for \( \theta \in \mathbb{R} \),
\[ \hat{x}(\theta) = \int_0^\infty \chi(t)e^{-i\theta t}dt, \]
and observe that the above condition (or equivalently (4.11)) is equivalent to
\[ (4.12) \quad \left[ \partial_w \partial_w f(\nu_0, w_{\nu_0}) + \partial_w^2 f(\nu_0, w_{\nu_0}) \partial_w w_{\nu_0} \right] \text{Im} \hat{x}'(1) \neq 0 \]
On the other hand, by differentiating the characteristic equation (3.2) with respect to $\nu$ at $\nu = \nu_0$ (and recalling that $\omega_0 = 1$), we have

$$\frac{d\lambda(\nu_0)}{d\nu} = \frac{\partial_\nu f(\nu_0, \bar{w}_{\nu_0})}{\partial_\nu f(\nu_0, \bar{w}_{\nu_0})} \int_0^\infty \chi(l)e^{-il}dl = \left[ \partial_\nu \partial_\nu f(\nu_0, \bar{w}_{\nu_0}) + \partial_\nu^2 f(\nu_0, \bar{w}_{\nu_0}) \partial_\nu \bar{w}_{\nu_0} \right] \hat{\chi}(1),$$

which implies

$$\frac{d\lambda(\nu_0)}{d\nu} = \frac{\partial_\nu \partial_\nu f(\nu_0, \bar{w}_{\nu_0}) + \partial_\nu^2 f(\nu_0, \bar{w}_{\nu_0})}{\partial_\nu f(\nu_0, \bar{w}_{\nu_0})} \int_0^\infty \chi(l)e^{-il}dl.$$

Thus the transversality condition, that is Assumption 3.7-(iv), becomes

$$\text{Re} \left\{ \frac{d\lambda(\nu_0)}{d\nu} \right\} = \text{Re} \left\{ \left[ \partial_\nu \partial_\nu f(\nu_0, \bar{w}_{\nu_0}) + \partial_\nu^2 f(\nu_0, \bar{w}_{\nu_0}) \partial_\nu \bar{w}_{\nu_0} \right] \hat{\chi}(1) \partial_\nu f(\nu_0, \bar{w}_{\nu_0}) \hat{\chi}(1) \right\} = 0,$$

therefore (4.12) holds true and we can find a unique $(\tilde{\nu}, \tilde{\omega}) \in \mathbb{R}^2$ solving the above two-dimensional linear system. This completes the proof of the lemma.

Last part of the proof of Theorem 4.1: To conclude the proof of the Hopf bifurcation Theorem 4.1, we apply the implicit function theorem (see Deimling [5, Theorem 15.2]) to the function $h : \mathbb{R}^3 \times X_2 \rightarrow C_{2\pi}(\mathbb{R})$ and we deduce that there exists a $C^1$-mapping $(\omega, \nu, z) : (-\delta, \delta) \rightarrow \mathbb{R}^2 \times X_2$, for some $\delta > 0$ small enough, such that

$$h(s, \omega(s), \nu(s), z(s)) = 0, \quad \forall s \in (-\delta, \delta).$$

By the definition of $h$, this is equivalent to say that

$$F(\omega(s), \nu(s), \bar{w}_{\nu(s)} + s(u_1 + z(s))) = 0,$$

when $s \neq 0$ with $(\omega(0), \nu(0), z(0)) = (1, \nu_0, \bar{w}_{\nu_0})$. We see that $(\omega(s), \nu(s), \bar{w}_{\nu(s)} + s(u_1 + z(s)))$ is the desired curve of solutions of $F = 0$. Thus the proof is complete.

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