On the fundamental group of II$_1$ factors and equivalence relations arising from group actions

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Dedicated to Alain Connes at the occasion of his 60th birthday.

Abstract

Given a countable group $G$, we consider the sets $S_{\text{factor}}(G)$, $S_{\text{eqrel}}(G)$, of subgroups $\mathcal{F} \subset \mathbb{R}_+$ for which there exists a free ergodic probability measure preserving action $G \curvearrowright X$ such that the fundamental group of the associated II$_1$ factor $L^\infty(X) \rtimes G$, respectively orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$, equals $\mathcal{F}$. We prove that if $G = \Gamma^\infty * \mathbb{Z}$, with $\Gamma \neq 1$, then $S_{\text{factor}}(G)$ and $S_{\text{eqrel}}(G)$ contain $\mathbb{R}_+$ itself, all of its countable subgroups, as well as uncountable subgroups whose log can have any Hausdorff dimension $\alpha \in (0, 1)$. We deduce that there exist II$_1$ factors of the form $M = L^\infty(X) \rtimes \mathbb{F}_\infty$ such that the fundamental group of $M$ is $\mathbb{R}_+$, but $M \otimes \mathbb{B}(\ell^2(\mathbb{N}))$ admits no continuous trace scaling action of $\mathbb{R}_+$. We then prove that if $G = \Gamma * \Lambda$, with $\Gamma, \Lambda$ finitely generated ICC groups, one of which has property (T), then $S_{\text{factor}}(G) = S_{\text{eqrel}}(G) = \{\{1\}\}$.

1 Introduction

Some of the most intriguing phenomena concerning group measure space II$_1$ factors $M = L^\infty(X) \rtimes G$ and orbit equivalence relations $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$, arising from free ergodic probability measure preserving (p.m.p.) actions $G \curvearrowright X$ of countable groups $G$ on probability spaces $(X, \mu)$, pertain to their fundamental group $\mathcal{F}(M)$, $\mathcal{F}(\mathcal{R})$ ([18]). Although much progress has been made in understanding and calculating these invariants, many natural questions on how the group $G$ may affect the behavior of $\mathcal{F}(M)$, $\mathcal{F}(\mathcal{R})$, remain open.

A first indication that certain properties of $G$ can impact the invariants independently of the way it acts, appeared in Connes’ ground breaking work on the classification and the structure of von Neumann factors, from the 1970’s. Thus, a side effect of the uniqueness of the amenable II$_1$ factor [5] and of the amenable II$_1$ equivalence relation [7], is that $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}) = \mathbb{R}_+$ whenever the group $G$ is amenable. On the other hand, arguments from Connes’ rigidity paper [4] were used to show that if $G$ is infinite conjugacy class (ICC) and has the property (T) of Kazhdan, then $\mathcal{F}(M), \mathcal{F}(\mathcal{R})$ are countable for any free ergodic p.m.p. action of $G$ ([23], [15]).

Then in the late 1990’s, Gaboriau discovered that certain groups $G$, such as the free groups with finitely many generators, $F_n$, $2 \leq n < \infty$, give rise to orbit equivalence relations $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ with $\mathcal{F}(\mathcal{R}) = \{1\}$, for any free ergodic p.m.p. action $F \curvearrowright X$ [13]. Moreover, many factors of the form $L^\infty(X) \rtimes F_n$ were shown to have trivial fundamental group as well (cf. [27], [22]) and it is strongly believed that, in fact, this holds true for all $F_n \curvearrowright X$.

In turn, a completely new type of phenomena emerged in the case $G = \mathbb{F}_\infty$, where it was shown that there exist free ergodic p.m.p. actions $\mathbb{F}_\infty \curvearrowright X$ with the fundamental group of the associated II$_1$

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factors and equivalence relations, \( \mathcal{F}(M), \mathcal{F}(\mathcal{R}) \), ranging over a “large” family of subgroups \( \mathcal{F} \subset \mathbb{R}_+ \), containing \( \mathbb{R}_+ \) itself, all its countable subgroups, as well as “many” uncountable subgroups \( \neq \mathbb{R}_+ \) [29]. In fact, it was conjectured in [29] that any group \( \mathcal{F} \) that can be realized as a fundamental group of a \( \Pi_1 \) factor or equivalence relation, can also be realised as \( \mathcal{F}(L^\infty(X) \times \mathbb{F}_\infty), \mathcal{F}(\mathcal{R}(\mathbb{F}_\infty \cap X)) \), for some free ergodic p.m.p. action \( \mathbb{F}_\infty \cap X \).

Related to all these phenomena, we introduced in [29] the set \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \), of subgroups \( \mathcal{F} \subset \mathbb{R}_+ \) for which there exists a free ergodic m.p. action \( G \rhd X \) such that \( \mathcal{F}(L^\infty(X) \times G) = \mathcal{F} \), respectively \( \mathcal{F}(\mathcal{R}(G \rhd X)) = \mathcal{F} \). Using this notation, the result in [29] shows, more precisely, that \( S_{\text{factor}}(\mathbb{F}_\infty) \cap S_{\text{eqrel}}(\mathbb{F}_\infty) \) contains the set \( S_{\text{centr}} \) of all subgroups \( \mathcal{F} \subset \mathbb{R}_+ \) for which there exists a free ergodic action of an amenable group \( \Lambda \) on an infinite measure space \((Y, \mu)\), such that the set of scalars \( t > 0 \) that can appear as scaling constants of non-singular automorphisms \( \theta \) of \((Y, \mu)\) commuting with \( \Lambda \rhd Y \), equals \( \mathcal{F} \). In turn, \( S_{\text{centr}} \) is shown to contain \( \mathbb{R}_+ \), all its countable subgroups and uncountable subgroups \( \mathcal{F} \subset \mathbb{R}_+ \) with the Hausdorff dimension of \( \log(\mathcal{F}) \subset \mathbb{R} \) ranging over all the interval \((0, 1)\) ([29]). While an abstract characterization of \( S_{\text{factor}}(\mathbb{F}_\infty) \), \( S_{\text{eqrel}}(\mathbb{F}_\infty) \) remains elusive, it was noticed in [29] that subgroups in either set, as in fact subgroups in \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \) for any \( G \), must be Borel sets and Polishable.

Our purpose in this paper is to estimate (or even completely calculate) the invariants \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \) for other classes of groups \( G \). We target two types of results: on the one hand, detecting classes of groups \( G \) for which \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \) are “large”, containing for instance the set \( S_{\text{centr}} \) defined above (like in the case case \( G = \mathbb{F}_\infty \)); on the other hand, detecting classes of groups \( G \) for which \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \) contain only “small” subgroups of \( \mathbb{R}_+ \) (e.g. countable, or just \{1\}).

Thus, our first result enlarges considerably the class of groups \( G \) for which we can show that the set \( S_{\text{centr}} \) is contained in both \( S_{\text{factor}}(G) \) and \( S_{\text{eqrel}}(G) \).

**Theorem 1.1.** Let \( \Gamma \) be a non-trivial group, \( \Sigma \) an infinite amenable group and denote \( G = \Gamma^\ast \Sigma \). Then,

\[
S_{\text{centr}} \subset S_{\text{factor}}(G) \quad \text{and} \quad S_{\text{centr}} \subset S_{\text{eqrel}}(G).
\]

Moreover, there exist free ergodic p.m.p. actions \( G \rhd (X, \mu) \) such that the \( \Pi_1 \) factor \( M = L^\infty(X) \rtimes G \) has fundamental group \( \mathcal{F}(M) = \mathbb{R}_+ \), but the \( \Pi_\infty \) factor \( M \overline{\otimes} B(\ell^2(\mathbb{N})) \) admits no trace scaling action of \( \mathbb{R}_+ \).

In Section 6, we will show that if the full group of an equivalence relation \( \mathcal{R} \) on a probability space \((X, \mu)\) contains a property (T) group acting ergodically on \( X \), then \( \mathcal{F}(\mathcal{R}) \) follows countable. Thus, if a group \( \Gamma \) appearing in Theorem 1.1 contains an infinite subgroup \( \Lambda \) with the property (T) and if \( G = \Gamma^\ast \Sigma \rhd X \) is a free ergodic p.m.p. action such that \( \mathcal{R}_G \) has fundamental group equal to an uncountable group in \( S_{\text{centr}} \), then the restriction of \( G \rhd X \) to \( \Lambda \) cannot be ergodic.

Note that the last part of Theorem 1.1 provides group measure space \( \Pi_1 \) factors \( M = L^\infty(X) \rtimes G \) which do have fundamental group equal to \( \mathbb{R}_+ \) yet cannot appear in the continuous decomposition of a type \( \Pi_1 \) factor. The problem of whether such \( \Pi_1 \) factors exist was posed over the years by several people, including Connes, Takesaki, and more recently Shlyakhtenko. The fact that there are even factors of the form \( L^\infty(X) \rtimes \mathbb{F}_\infty \) satisfying this property (by simply taking \( \Gamma = \Sigma = \mathbb{Z} \) in 1.1) should be contrasted with the fact that the \( \Pi_\infty \) factor associated with \( L(\mathbb{F}_\infty) \) does admit a trace scaling action of \( \mathbb{R}_+ \), by [32].

Note that all groups of the form \( G = \Gamma^\ast \Sigma \), covered by the above theorem, have infinite first \( \ell^2 \)-Betti number, \( \beta_1^{(2)}(G) = \infty \), and in fact \( \beta_n^{(2)}(G) = \infty, 0, \forall n \geq 2 \). On the other hand, by Gaboriau’s scaling formula for \( \ell^2 \)-Betti numbers [13], any free ergodic p.m.p. action of a group \( G \)
with \( \rho_n^{(2)}(G) \neq 0, \infty \), for some \( n \), gives rise to an equivalence relation \( \mathcal{R}_G \) with trivial fundamental group, \( \mathcal{F}(\mathcal{R}_G) = \{1\} \). In other words, \( S_{\text{eqrel}}(G) = \{\{1\}\} \). While it is still an open question whether the corresponding \( \Pi_1 \) factors \( M = L^\infty(X) \rtimes G \) satisfy \( \mathcal{F}(M) = \{1\} \) as well (i.e. \( S_{\text{factor}}(G) = \{\{1\}\} \)), our next result provides a large class of groups \( G \) for which this is indeed the case.

**Theorem 1.2.** Let \( \Gamma \) and \( \Lambda \) be infinite, finitely generated groups. Assume that \( \Gamma \) is ICC and that one of the following conditions holds.

a) \( \Gamma = \Gamma_1 \times \Gamma_2 \), with \( \Gamma_1 \) non-trivial and \( \Gamma_2 \) non-amenable,

b) \( \Gamma \) admits a non virtually abelian, normal subgroup \( \Gamma_1 \) with the relative property (T).

Then, \( S_{\text{factor}}(\Gamma \star \Lambda) = S_{\text{eqrel}}(\Gamma \star \Lambda) = \{\{1\}\} \).

When viewed from the perspective of Connes’ discrete decomposition of type III factors with \( 0 < \lambda < 1 \) ([6]) and respectively Connes-Takesaki continuous decomposition of type \( \Pi_1 \) factors ([10]), the above result provides a large class of groups \( G \) with the property that no \( \Pi_1 \) factor \( M \) arising from an arbitrary free ergodic p.m.p. action of \( G \) can appear in the decomposition of a type III factor (i.e., as Connes puts it, no such \( M \) can appear as the “shadow” of a type III factor).

While \( \Pi_1 \) factors \( M = L^\infty(X) \rtimes \Gamma \) arising from free ergodic p.m.p. actions \( \Gamma \ltimes X \) of ICC property (T) groups always have countable fundamental group (cf. [4], [23], [15]), it was not known whether there exist cases when \( \mathcal{F}(M) \neq \{1\} \). Our next result gives the first such examples. It also provides the first “concrete” examples of free ergodic p.m.p. actions \( \Gamma \ltimes X \) with the associated \( \Pi_1 \) factors \( M \) having fundamental group \( \neq \{1\}, \mathbb{R}_+ \). Indeed, the actions in Theorem 1.1 above and in [29] are shown to exist by using a Baire-category argument, at some point, while in 1.3 below they are specific \( G \)-actions, obtained as diagonal products of Bernoulli and profinite actions.

**Theorem 1.3.** Let \( \mathcal{F} \subset \mathbb{Q}_+ \) be a subgroup generated by a subset of the prime numbers. Let \( G = \mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z}) \) with \( n \geq 3 \). Then, \( G \) admits a free ergodic p.m.p. action \( G \ltimes (X, \mu) \) such that the fundamental group of \( L^\infty(X) \rtimes G \) and of \( \mathcal{R}(G \ltimes X) \) equals \( \mathcal{F} \).

We in fact believe that any subgroup of \( \mathbb{Q}_+ \) can be realized as the fundamental group of a factor or equivalence relation arising from a free ergodic p.m.p. action of \( \mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z}) \), \( n \geq 3 \). The question of whether there exist free ergodic p.m.p. actions of an ICC property (T) group \( G \ltimes X \) such that \( \mathcal{F}(\mathcal{R}_G) \) or \( \mathcal{F}(L^\infty(X) \rtimes G) \) contain irrational numbers, remains open. In fact, it is not even known whether the union of all the fundamental groups of \( \Pi_1 \) factors and equivalence relations arising from free ergodic p.m.p. actions of a fixed ICC property (T) group \( G \) is necessarily countable or not.

Finally, noticing that for a large number of groups \( G \) it is known that \( \{1\} \in S_{\text{factor}}(G) \) (see e.g. [25], [27], [28]), we conjecture that this is in fact the case for all non-amenable groups \( G \). If true, this would also show that the only possibilities for \( S_{\text{factor}}(G), S_{\text{eqrel}}(G) \) to be single point sets are \( S_{\text{factor}}(G) = S_{\text{eqrel}}(G) = \{\mathbb{R}_+\} \), \( S_{\text{factor}}(G) = S_{\text{eqrel}}(G) = \{\{1\}\} \), the first situation corresponding to \( G \) being amenable. This would provide a new, interesting facet of the dichotomy amenable/non-amenable for groups.
2 Preliminaries

The fundamental group $\mathcal{F}(M)$ of a $\text{II}_1$ factor $M$, introduced in [18], is defined as the following subgroup of $\mathbb{R}_+$.

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid p, q \text{ are non-zero projections in } M \text{ such that } pMp \cong qMq\}.$$

We call $\text{II}_1$ equivalence relation on a standard probability space $(X, \mu)$ every ergodic probability measure preserving (p.m.p.) measurable equivalence relation with countable equivalence classes.

The fundamental group $\mathcal{F}(\mathcal{R})$ of a $\text{II}_1$ equivalence relation $\mathcal{R}$ is defined as

$$\mathcal{F}(\mathcal{R}) = \{\mu(Y)/\mu(Z) \mid \mathcal{R}|_Y \cong \mathcal{R}|_Z\}.$$

Whenever $\Gamma \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, we denote by $\mathcal{R}(\Gamma \curvearrowright X)$ the associated $\text{II}_1$ orbit equivalence (OE) relation and by $L^\infty(X) \rtimes \Gamma$ the associated group measure space $\text{II}_1$ factor [18].

**Definition 2.1.** A free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is called rigid if the corresponding inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ is rigid in the sense of [27, Proposition 4.1].

Some sets of subgroups of $\mathbb{R}$ and ergodic measures

Given a countable group $\Gamma$, we are interested in

$$S_{\text{factor}}(\Gamma) := \{\mathcal{F} \subset \mathbb{R}_+ \mid \text{there exists a free ergodic p.m.p. action } \Gamma \curvearrowright (X, \mu) \text{ such that } \mathcal{F}(L^\infty(X) \rtimes \Gamma) = \mathcal{F}\},$$

$$S_{\text{eqrel}}(\Gamma) := \{\mathcal{F} \subset \mathbb{R}_+ \mid \text{there exists a free ergodic p.m.p. action } \Gamma \curvearrowright (X, \mu) \text{ such that } \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \mathcal{F}\}.$$

In [29, Theorem 5.3 and formula (2.2)], we have shown that both $S_{\text{factor}}(\mathcal{F}_\infty)$ and $S_{\text{eqrel}}(\mathcal{F}_\infty)$ contain $S_{\text{centr}}$ defined as

$$S_{\text{centr}} := \{\mathcal{F} \subset \mathbb{R}_+ \mid \text{there exists } \Lambda \curvearrowright (Y, \eta) \text{ free ergodic m.p. action, with } \Lambda \text{ amenable and } \text{mod(Centr}_\Lambda(Y)) = \mathcal{F}\}.$$

Following [1, Section 4], we call ergodic measure on $\mathbb{R}$ any $\sigma$-finite measure $\nu$ on the Borel sets of $\mathbb{R}$ having the following properties, where we denote $\lambda_x(y) = x + y$.

- For all $x \in \mathbb{R}$, either $\nu \circ \lambda_x = \nu$ or $\nu \circ \lambda_x \perp \nu$.
- There exists a countable subgroup $Q \subset \mathbb{R}$ such that $\nu \circ \lambda_x = \nu$ for all $x \in Q$ and such that every $Q$-invariant Borel function on $\mathbb{R}$ is $\nu$-almost everywhere constant.

For every ergodic measure $\nu$ on $\mathbb{R}$, one defines

$$H_\nu := \{x \in \mathbb{R} \mid \nu \circ \lambda_x = \nu\}.$$

As shown in [1], the groups $H_\nu$ can have arbitrary Hausdorff dimension and all $\exp(H_\nu)$ belong to $S_{\text{centr}}$. We refer to [29, Section 2 and proof of Theorem 5.3] for a detailed exposition.
Intertwining by bimodules and the notation $A \prec_M B$

In Sections 3 and 4, we use the method of intertwining by bimodules, introduced by the first author in [27]. Let $(M, \tau)$ be a von Neumann algebra with faithful normal tracial state $\tau$. We use the notation $M^n = M_n(\mathbb{C}) \otimes M$. When $A, B \subset M^n$ are possibly non-unital embeddings, we write $A \prec_B \subset_M$ if there exists a non-zero partial isometry $v \in \mathcal{I}_A(M_{n,m}(\mathbb{C}) \otimes M)$ and $a$, possibly non-unital, normal $^*$-homomorphism $\rho: A \to B^m$ satisfying $av = \rho(a)$ for all $a \in A$. Several equivalent formulations of this property can be given, see [24, Theorem 2.1] (see also [35, Theorem C.3]).

Suppose that $A$ and $B$ are Cartan subalgebras of the II$_1$ factor $M$. Let $A_0 \subset A$ be a von Neumann subalgebra such that $A_0 \cap M = A$. By [27, Theorem A.1], $A_0 \prec_B \subset_M$ if and only if there exists a unitary $u \in M$ such that $uAu^* = B$.

3 Groups $G$ for which $S_{\text{factor}}(G)$ contains uncountable groups

The following theorem, whose proof is given at the end of the section, provides a large family of groups $G$ such that $S_{\text{factor}}(G)$ and $S_{\text{eqrel}}(G)$ is large, in the sense that both contain $S_{\text{centr}}$. Moreover, we prove that $G$ admits free ergodic p.m.p. actions $G \acts (X, \mu)$ such that the II$_1$ factor $M := \mathcal{L}_\infty(X) \rtimes G$ has fundamental group $\mathbb{R}_+$, but nevertheless, the II$_\infty$ factor $M \mathcal{O} B(\ell^2(\mathbb{N}))$ admits no strongly continuous trace scaling action of $\mathbb{R}_+$.

The groups $G$ involved are infinite free product groups and should be opposed to the groups $G$ treated in Theorem 4.1, for which $S_{\text{factor}}(G)$ is trivial (cf. Remark 4.2).

**Theorem 3.1.** Let $\Gamma$ be a non-trivial group, $\Sigma$ an infinite amenable group and denote $G = \Gamma^{*\infty} \ast \Sigma$. Then,

$S_{\text{centr}} \subset S_{\text{factor}}(G)$  \quad and \quad $S_{\text{centr}} \subset S_{\text{eqrel}}(G)$.

Moreover, there exist free ergodic p.m.p. actions $G \acts (X, \mu)$ such that the II$_1$ factor $M = \mathcal{L}_\infty(X) \rtimes G$ has fundamental group $\mathcal{F}(M) = \mathbb{R}_+$, but the II$_\infty$ factor $M \mathcal{O} B(\ell^2(\mathbb{N}))$ admits no trace scaling action of $\mathbb{R}_+$.

In the course of the proof of Theorem 3.1, we will also obtain the following result.

**Theorem 3.2.** There exist II$_1$ factors $M_1$ and $M_2$ such that $\mathcal{F}(M_1) \neq \mathbb{R}_+ \neq \mathcal{F}(M_2)$, but nevertheless $\mathcal{F}(M_1 \mathcal{O} M_2) = \mathbb{R}_+$.

Let $G$ be a countable group with subgroup $\Gamma$. Suppose that $G \acts (X, \mu)$ is a free p.m.p. action such that the restriction to $\Gamma$ is ergodic. Slightly changing notations compared to [29, Section 2], denote by $\text{Emb}(\Gamma, G)$ the set of non-singular partial automorphisms $\phi$ of $(X, \mu)$ satisfying $\phi(g \cdot x) \in G \cdot \phi(x)$ for all $g \in \Gamma$ and almost all $x \in X$ with $x, g \cdot x \in D(\phi)$. Denote by $\lvert [G] \rvert$ the full pseudo-group of the OE relation $\mathcal{R}(G \acts X)$, i.e. the set of a partial automorphisms $\phi$ of $(X, \mu)$ satisfying $\phi(x) \in G \cdot x$ for almost all $x \in D(\phi)$.

The following lemma generalizes [29, Theorem 4.1].

**Lemma 3.3.** Let $\Gamma$ be an infinite group, $\Lambda$ an arbitrary group, both acting freely and p.m.p. on $(X, \mu)$. There exists a free p.m.p. action $\Gamma^{*\infty} \ast \Lambda \acts (X, \mu)$ with the following properties.

- The restriction of $\alpha$ to $\Gamma^{*\infty}$ is ergodic and rigid (in the sense of Definition 2.1).
• $\text{Emb}(\Gamma^* \infty, \Gamma^* * \Lambda) = [[\Gamma^* * \Lambda]]$.

• The restriction of $\alpha$ to any of the copies of $\Gamma$, resp. to $\Lambda$, is conjugate with the originally given action.

Proof. Denote the given actions by $\Gamma \overset{\beta}{\curvearrowright} (X, \mu)$ and $\Lambda \overset{\rho}{\curvearrowright} (X, \mu)$. We introduce the following notations:

$$\Gamma^* \infty = \bigstar_{n=-1}^{\infty} G_n \quad \text{with all } G_n \cong \Gamma,$$

$$\Gamma_n := \bigstar_{k=-1}^{n} G_k,$$

$$\Gamma_E := G_{-1} * G_0 * \bigstar_{n \in E} G_n \quad \text{whenever } E \subset \mathbb{N}.$$

By [14, Theorem 1.2], take a free ergodic p.m.p. action $\Gamma_0 \overset{\alpha_0}{\curvearrowright} (X, \mu)$ such that $\alpha_0$ is a rigid action and such that the restrictions of $\alpha_0$ to $G_{-1}$ and $G_0$ are conjugate with the action $\beta$. By [34, Category Lemma] and [17, Lemma A.1], extend $\alpha_0$ to a free action of $\Gamma_0 * \Lambda$ on $(X, \mu)$, still denoted by $\alpha_0$, whose restriction to $\Lambda$ is conjugate with the action $\rho$.

Extend the action $\alpha_0$ inductively to free actions $\Gamma_n * \Lambda \overset{\alpha_n}{\curvearrowright} (X, \mu)$ following the procedure in [29, Section 3] and such that the restriction of $\alpha_n$ to $G_k \subset \Gamma_n$ is conjugate with $\beta$ for all $k \leq n$. We end up with the free action $\Gamma^* \infty * \Lambda \overset{\alpha}{\curvearrowright} (X, \mu)$. For every infinite subset $E \subset \mathbb{N}$, we denote by $\alpha_E$ the restriction of $\alpha_n$ to $\Gamma_E * \Lambda$. Following the proof of [29, Theorem 4.1], there exists an infinite subset $E \subset \mathbb{N}$ such that $\text{Emb}(\Gamma_E, \Gamma_E * \Lambda) = [[\Gamma_E * \Lambda]]$. Since $\Gamma_E \cong \Gamma^* \infty$, the lemma is proved.

Remark 3.4. Using the methods of [14, Section 2.3], Lemma 3.3 can be shown for $\Gamma_1 * \Gamma_2$ instead of the infinite free product $\Gamma^* \infty$, for arbitrary infinite groups $\Gamma_1, \Gamma_2$ with given free p.m.p. actions on $(X, \mu)$. Such a generalization does not provide a refinement for Theorem 3.1 though, since the proof of Theorem 3.1 involves taking once more an infinite free product.

For the formulation of the following theorem, recall that the automorphism group $\text{Aut}(N)$ of a von Neumann algebra with separable predual is a Polish group under the topology making the maps $\text{Aut}(N) \rightarrow N_* : \alpha \mapsto \omega \circ \alpha$ continuous for all $\omega \in N_*$. Similarly, the group $\text{Aut}(Y, \eta)$ of non-singular isomorphisms of $(Y, \eta)$ (up to equality almost everywhere) is a Polish group and $\text{Cenr}_{\text{Aut}}(\Gamma_2)$ is a closed subgroup whenever $\Gamma_2 \curvearrowright (Y, \eta)$ is a non-singular action.

**Theorem 3.5.** Let $\Gamma_1 * \Gamma_2 \overset{\alpha}{\curvearrowright} (X, \mu)$ be a free p.m.p. action. Let $\Gamma_2 \overset{\rho}{\curvearrowright} (Y, \eta)$ be a free ergodic action preserving the infinite standard measure $\eta$. Consider the action $\Gamma_1 * \Gamma_2 \overset{\alpha}{\curvearrowright} X \times Y$ given by

$$g \cdot (x, y) = (g \cdot x, y) \quad \forall g \in \Gamma_1, \quad h \cdot (x, y) = (h \cdot x, h \cdot y) \quad \forall h \in \Gamma_2.$$  \hspace{1cm} (3.1)

Make the following assumptions.

• The restriction of $\alpha$ to $\Gamma_1$ is ergodic and rigid.

• We have $\text{Emb}(\Gamma_1, \Gamma_1 * \Gamma_2) = [[\Gamma_1 * \Gamma_2]]$.

• $\Gamma_2$ is amenable.

Then, the following holds.
1. The map

\[ \Theta : \text{Centr}_{\text{Aut}Y}(\Gamma_2) \to \text{Aut}(\mathcal{R}(\Gamma_1 \ast \Gamma_2 \curvearrowright X \times Y)) : \Delta \mapsto \Theta_\Delta \text{ where } \Theta_\Delta(x, y) = (x, \Delta(y)) \]

induces an onto group isomorphism between \text{Centr}_{\text{Aut}Y}(\Gamma_2) and \text{Out}(\mathcal{R}(\Gamma_1 \ast \Gamma_2 \curvearrowright X \times Y)).

2. Define the II_\infty factor \( N := \text{L}^\infty(X \times Y) \rtimes (\Gamma_1 \ast \Gamma_2). \) Denote for every \( \Delta \in \text{Centr}_{\text{Aut}Y}(\Gamma_2), \) by \( \theta_\Delta \) the corresponding automorphism of \( N. \)

(a) The group \( \text{Aut}(N) \) is generated by the three subgroups \( \{ \theta_\Delta \mid \Delta \in \text{Centr}_{\text{Aut}Y}(\Gamma_2) \} \), the inner automorphism group \( \text{Inn}(N) = \{ \text{Ad} u \mid u \in \mathcal{U}(N) \} \) and the group of automorphisms^{(5)} \( H := \{ \theta \in \text{Aut}(N) \mid \theta(a) = a \text{ for all } a \in \text{L}^\infty(X \times Y) \}. \)

(b) The subgroup \( \text{Inn}(N) \cdot H \) of \( \text{Aut}(N) \) is closed and normal in \( \text{Aut}(N) \) and the map

\[ \text{Centr}_{\text{Aut}Y}(\Gamma_2) \to \frac{\text{Aut}(N)}{\text{Inn}(N) \cdot H} : \Delta \mapsto \theta_\Delta \]

is an isomorphism and homeomorphism of Polish groups.

Proof. The proof of point 1 is identical to [29, Lemma 5.1]. It remains to prove point 2.

Write \( A = \text{L}^\infty(X) \) and \( B = \text{L}^\infty(Y). \) We first prove that every automorphism of \( N \) preserves the Cartan subalgebra \( A \otimes B \) up to unitary conjugacy. Together with point 1, this implies 2(a). So, let \( \theta \) be an automorphism of \( N := (A \otimes B) \rtimes (\Gamma_1 \ast \Gamma_2). \) Take a projection \( p \in A \otimes B \) of finite trace and put \( q = \theta(p). \) After unitary conjugacy, we may assume that \( q \in A \otimes B. \) By [27, Theorem A.1], it is sufficient to prove that \( \theta(Ap) \preceq (A \otimes B)q. \)

Since \( \theta(Ap) \subset qNq \) is rigid, [17, Theorem 5.1] implies that

\[ \theta(Ap) \preceq q((A \otimes B) \rtimes \Gamma_i)q \text{ for some } i = 1, 2. \]

Since \( \theta(Ap) \) is quasi-regular in \( qNq, \) [17, Theorem 1.1] implies that \( \theta(Ap) \preceq (A \otimes B)q. \)

We finally prove 2(b). Observe that \( \mathcal{U}(N) \) is a Polish group in a natural way and that the map \( \mathcal{U}(N) \to \text{Aut}(N) : u \mapsto \text{Ad} u \) is a continuous group morphism. Define \( H \) as in the formulation of the theorem and note that \( H \) is a closed subgroup of \( \text{Aut}(N). \) We form the semi-direct product Polish group \( \mathcal{U}(N) \rtimes H \) in such a way that \( \pi : \mathcal{U}(N) \rtimes H \to \text{Aut}(N) : \pi(u, \theta) = (\text{Ad} u) \circ \theta \) is a group morphism. Note that \( \pi \) is continuous and denote \( K := (\mathcal{U}(N) \rtimes H)/\text{Ker} \pi. \) Again, \( K \) is a Polish group. We form the semi-direct product Polish group \( K \rtimes \text{Centr}_{\text{Aut}Y}(\Gamma_2) \) in such a way that

\[ \rho : K \rtimes \text{Centr}_{\text{Aut}Y}(\Gamma_2) \to \text{Aut}(N) : \rho(k, \Delta) = \pi(k)\theta_\Delta \]

is a group morphism. Then, \( \rho \) is a continuous and injective group morphism between Polish groups. Moreover, by 2(a), \( \rho \) is onto. So, \( \rho \) is a homeomorphism. Hence, \( \text{Inn}(N) \cdot H = \rho(K) \) is closed and normal in \( \text{Aut}(N) \) and the map \( \Delta \mapsto \theta_\Delta \) provides an isomorphism and homeomorphism between \( \text{Centr}_{\text{Aut}Y}(\Gamma_2) \) and \( \text{Aut}(N)/(\text{Inn}(N) \cdot H). \)

---

^{(5)} \text{Note that } H \text{ is isomorphic to the group of } S^1\text{-valued 1-cocycles for the action } \Gamma_1 \ast \Gamma_2 \curvearrowright X \times Y. \
Lemma 3.6. Let \( \Gamma \ast \Lambda \curvearrowright (X, \mu) \) be a free p.m.p. action with the restriction to \( \Gamma \) being ergodic. Let \( \Lambda \curvearrowright (Y, \eta) \) be a free ergodic action preserving the infinite standard measure \( \eta \). Assume that \( \Lambda \) is amenable. Consider \( \Gamma \ast \Lambda \curvearrowright X \times Y \) as in (3.1). Let \( Z \subseteq Y \) be a subset of finite measure and define the II\(_1\) equivalence relation \( \mathcal{R} \) as the restriction of \( \mathcal{R}(\Gamma \ast \Lambda \curvearrowright X \times Y) \) to \( Z \).

Whenever \( \Sigma \) is an infinite amenable group, there exists a free ergodic p.m.p. action \( \Gamma \ast \Sigma \curvearrowright X \times Z \) such that \( \mathcal{R} = \mathcal{R}(\Gamma \ast \Sigma \curvearrowright X \times Z) \).

Proof. Denote by \( \mathcal{R}_1 \) the equivalence relation given by the restriction of \( \mathcal{R}(\Lambda \curvearrowright X \times Y) \) to \( X \times Z \). Note that \( \mathcal{R}_1 \) need not be ergodic. Since \( \Lambda \) is amenable and almost every equivalence class of \( \mathcal{R}_1 \) is infinite, the results in [7] and [20] allow to take a free p.m.p. action \( \Sigma \curvearrowright X \times Z \) whose OE relation is precisely \( \mathcal{R}_1 \).

Since the action of \( \Lambda \) on \( (Y, \eta) \) is ergodic, take \( \phi_n \in [[\Lambda]] \) with \( \text{dom}(\phi_n) = X \times Z \) and \( \text{range}(\phi_n) = X \times Z_n \), where \( Z_n, n \in \mathbb{N} \), forms a partition of \( Y \) (up to measure zero). Since the action of \( \Gamma \) leaves every \( X \times Z_n \) globally invariant, we can view \( \phi_n^{-1}\Gamma_\phi_n \) as a group of automorphisms of \( X \times Z \). It is now an exercise to check that \( \mathcal{R} \) is freely generated by the OE relations of \( \phi_n^{-1}\Gamma_\phi_n, n \in \mathbb{N} \), together with \( \Sigma \curvearrowright X \times Z \). This provides us with the required free action of \( \Gamma \ast \Sigma \curvearrowright X \times Z \). \( \square \)

The following is the final ingredient in the proof of Theorem 3.1.

Lemma 3.7. There exist ergodic measures \( \nu, \nu' \) on \( \mathbb{R} \) such that \( H_\nu \neq \mathbb{R} \neq H_{\nu'} \) and \( H_\nu + H_{\nu'} = \mathbb{R} \).

Proof. As explained in [29, Section 2], an ergodic measure \( \nu \) on \( \mathbb{R} \) can be associated to any pair \( (a_n), (b_n) \) of sequences in \( \mathbb{N} \) satisfying \( \sum_{n=1}^{\infty} b_n^{-1} < \infty \) and \( b_n < a_n/2 \) for all \( n \), in such a way that

\[
H_\nu = \left\{ x \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{a_n}{b_n} \| a_1 \cdots a_{n-1} x \| < \infty \right\}
\]

where \( \| x \| \) denotes the distance of \( x \in \mathbb{R} \) to \( \mathbb{Z} \subseteq \mathbb{R} \). Take \( a_n = 2^{2n+2} \), \( b_n = 2^{2n} \) and associate with it the ergodic measure \( \nu \). Take \( a'_n = 2^{2n+1} \), \( b'_n = 2^{2n-1} \) and associate with it the ergodic measure \( \nu' \). First of all,

\[
\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \notin H_\nu \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b'_n}{a'_1 \cdots a'_n} \notin H_{\nu'}
\]

proving that \( H_\nu \neq \mathbb{R} \neq H_{\nu'} \).

Let now \( x \in \mathbb{R} \) and write

\[
x = x_0 + \sum_{n=1}^{\infty} \frac{x_n}{a_1 \cdots a_n} \quad \text{with} \quad x_n \in \{0, \ldots, a_n - 1\}.
\]

Write for every \( n \in \mathbb{N} \), \( x_n = y_n + \sqrt{a_n}z_n \) with \( y_n, z_n \in \{0, \ldots, \sqrt{a_n}\} \). Define

\[
y = x_0 + \sum_{n=1}^{\infty} \frac{y_n}{a_1 \cdots a_n} \quad \text{and} \quad z = \sum_{n=1}^{\infty} \frac{2z_n}{a'_1 \cdots a'_n}.
\]

One checks that \( y \in H_\nu \), \( z \in H_{\nu'} \) and \( x = y + z \). So, \( H_\nu + H_{\nu'} = \mathbb{R} \). \( \square \)
Proof of Theorem 3.1. Since \( \Gamma^{*\infty} = (\Gamma \ast \Gamma)^{*\infty} \), we may assume that \( \Gamma \) is an infinite group. Let \( \Sigma, \Lambda \) be infinite amenable groups and \( \Lambda \triangleleft (Y, \eta) \) a free ergodic action preserving the infinite standard measure \( \eta \). Set \( G := \Gamma^{*\infty} \ast \Sigma \). We prove the existence of a free ergodic p.m.p. action of \( G \triangleleft \Sigma \) such that the associated \( \Pi_1 \) factor \( M := L^\infty(Z) \times G \) and equivalence relation \( R := R(G \triangleleft \Sigma) \) have the following properties.

1. The fundamental group of \( M \) and the fundamental group of \( R \) equal \( \text{mod}(\text{Centr}_{\text{Aut}} Y(\Lambda)) \).

2. The \( \Pi_\infty \) factor \( M \otimes B(\ell^2(\mathbb{N})) \) admits a strongly continuous trace-scaling action of \( \mathbb{R}_+ \) if and only if the group morphism
   \[
   \text{mod} : \text{Centr}_{\text{Aut}} Y(\Lambda) \rightarrow \mathbb{R}_+
   \] is onto and splits continuously.

Choose any free p.m.p. actions \( \Gamma \triangleleft (X, \mu) \) and \( \Lambda \triangleleft (X, \mu) \). Take a free p.m.p. action \( \Gamma^{*\infty} \ast \Lambda \triangleleft (X, \mu) \) satisfying the conclusions of Lemma 3.3. Define \( \Gamma^{*\infty} \ast \Lambda \triangleleft X \times Y \) by (3.1), with \( \Gamma_1 = \Gamma^{*\infty} \) and \( \Gamma_2 = \Lambda \). Define the \( \Pi_1 \) equivalence relation \( R \) by restricting \( R(\Gamma^{*\infty} \ast \Lambda \triangleleft X \times Y) \) to a subset \( Z \) of finite measure. By Lemma 3.6, we can take a free ergodic p.m.p. action \( G \triangleleft Z \) whose OE relation equals \( R \). By point 1 of Theorem 3.5,
   \[
   \mathcal{F}(R) = \text{mod}(\text{Centr}_{\text{Aut}} Y(\Lambda)) .
   \]
Put \( M := L^\infty(Z) \times G \) and note that \( M \otimes B(\ell^2(\mathbb{N})) \cong L^\infty(X \times Y) \times (\Gamma^{*\infty} \ast \Lambda) \). By point 2(a) of Theorem 3.5, also
   \[
   \mathcal{F}(M) = \text{mod}(\text{Aut}(N)) = \text{mod}(\text{Centr}_{\text{Aut}} Y(\Lambda)) .
   \]
If the group morphism (3.2) splits continuously, it is clear that \( N \) admits a strongly continuous trace scaling action. The converse follows from point 2(b) of Theorem 3.5.

In order to conclude the proof of Theorem 3.1, we have to construct an action \( \Lambda \triangleleft (Y, \eta) \) such that \( \text{mod}(\text{Centr}_{\text{Aut}} Y(\Lambda)) = \mathbb{R}_+ \), but the morphism (3.2) does not split continuously. By Lemma 3.7, we can take ergodic measures \( \nu_1, \nu_2 \) on \( \mathbb{R} \) such that \( H_{\nu_1} \neq \mathbb{R} \neq H_{\nu_2} \), while \( H_{\nu_1} + H_{\nu_2} = \mathbb{R} \). By formula (2.2) in [29], we can take amenable groups \( \Lambda_1, \Lambda_2 \) and free ergodic infinite measure preserving actions \( \Lambda_i \triangleleft (Y_i, \eta_i) \) such that \( \text{mod}(\text{Centr}_{\text{Aut}} Y_i(\Lambda_i)) = \exp(H_{\nu_i}) \). Since the homomorphism mod is continuous, we equip \( \exp(H_{\nu_i}) \) with the (Polish) quotient topology. In this way, the \( H_{\nu_i} \) become Polish groups and the embedding \( H_{\nu_i} \hookrightarrow \mathbb{R} \) continuous.

We prove that
   \[
   \text{mod} : \text{Centr}_{\text{Aut}}(Y_1 \times Y_2)(\Lambda_1 \times \Lambda_2) \rightarrow \mathbb{R}_+
   \]
admits no continuous splitting. Assume that it does. Since the left-hand side of the previous formula equals \( \text{Centr}_{\text{Aut}} Y_1(\Lambda_1) \times \text{Centr}_{\text{Aut}} Y_2(\Lambda_2) \), the homomorphism
   \[
   H_{\nu_1} \times H_{\nu_2} \rightarrow \mathbb{R} : (x, y) \mapsto x + y
   \]
admits a continuous splitting. We then find continuous homomorphisms \( \theta_i : \mathbb{R} \rightarrow H_{\nu_i} \) such that \( x = \theta_1(x) + \theta_2(x) \) for all \( x \in \mathbb{R} \). Since the embedding \( H_{\nu_i} \hookrightarrow \mathbb{R} \) is continuous, there exist \( \lambda_i \in \mathbb{R} \) such that \( \theta_i(x) = \lambda_i x \) for all \( x \in \mathbb{R} \). But, \( H_{\nu_i} \neq \mathbb{R} \), forcing \( \lambda_i = 0 \) for \( i = 1, 2 \), a contradiction.

Proof of Theorem 3.2. By Lemma 3.7, we can take ergodic measures \( \nu_1, \nu_2 \) on \( \mathbb{R} \) such that \( H_{\nu_1} \neq \mathbb{R} \neq H_{\nu_2} \), while \( H_{\nu_1} + H_{\nu_2} = \mathbb{R} \). By formula (2.2) in [29], \( \exp(H_{\nu_i}) \in S_{\text{centr}} \), so that by Theorem 3.1, we can take \( \Pi_1 \) factors \( M_i \) with \( \mathcal{F}(M_i) = \exp(H_{\nu_i}) \). Then, the fundamental group of \( M_1 \otimes M_2 \) contains \( \exp(H_{\nu_1} + H_{\nu_2}) \) and hence equals \( \mathbb{R}_+ \).
4 Groups $G$ for which $S_{\text{factor}}(G)$ is trivial

Combining results from [3, 13, 17], we prove that for the following groups $G$, $S_{\text{factor}}(G)$ is trivial.

**Theorem 4.1.** Let $\Gamma$ and $\Lambda$ be infinite, finitely generated groups. Assume that $\Gamma$ is ICC and that one of the following conditions holds.

a) $\Gamma = \Gamma_1 \times \Gamma_2$ is a non-trivial direct product with $\Gamma_2$ being non-amenable,

b) $\Gamma$ admits a non virtually abelian, normal subgroup $\Gamma_1$ with the relative property $(T)$.

Then, $S_{\text{factor}}(\Gamma \ast \Lambda) = S_{\text{eqrel}}(\Gamma \ast \Lambda) = \{\{1\}\}$.

**Remark 4.2.** Observe that Theorem 4.1 implies that, in general, Theorem 3.1 is false if we only take a finite free product.

**Proof of Theorem 4.1.** Let $\Gamma \ast \Lambda \actson (X, \mu)$ be a free ergodic p.m.p. action. Note first that by [13, Propriétés 1.5], we have $0 < \beta^{(3)}(\Gamma \ast \Lambda) < \infty$. Hence, by [13, Corollaire 5.7], the fundamental group of the OE relation $\mathcal{R}(\Gamma \ast \Lambda \actson X)$ is trivial.

Write $A = L^\infty(X)$, $M_1 = A \rtimes \Gamma$, $M_2 = A \rtimes \Lambda$. Finally, set $M = A \rtimes (\Gamma \ast \Lambda) = M_1 \ast_A M_2$. Suppose that $p \in A$ is a projection and $\theta : M \to pMp$ a $*$-isomorphism. It remains to prove that $\theta(A)$ and $pA$ are unitarily conjugate, since this implies that $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}(\Gamma \ast \Lambda \actson X))$.

Under assumption a), we invoke [3, Theorem 4.2] and under assumption b), we invoke [17, Theorem 5.1] and conclude in both cases that $\theta(L(\Gamma_1)) \prec M_i$ for some $i = 1, 2$. Take a projection $q \in M_i$, a non-zero partial isometry $v \in p(M_{1,n}(C) \otimes M)q$ and a unital $*$-homomorphism $\rho : L(\Gamma_1) \to qM_i q$ satisfying $\theta(a)v = v\rho(a)$ for all $a \in L(\Gamma_1)$. In both cases a) and b), the group $\Gamma_1$ is not virtually abelian. Hence, $\rho(L(\Gamma_1)) \not\prec A$. By [17, Theorem 1.1], the normalizer of $\rho(L(\Gamma_1))$ inside $qM_i q$ is contained in $qM_i q$. Since $v^*v$ commutes with $\rho(L(\Gamma_1))$, we may first of all assume that $q = v^*v$. Next, it follows that $v^*\theta(L(\Gamma))v \subset qM_i q$. Hence, $\theta(L(\Gamma)) \not\prec M_i$.

Repeating the previous paragraph, we may assume that $\rho : L(\Gamma) \to qM_i q$, $\theta(a)v = v\rho(a)$ for all $a \in L(\Gamma)$ and $v^*v = q$. Since $\Gamma$ is an ICC group, we get that

$$M \cap L(\Gamma)' = M_1 \cap L(\Gamma)' = A^\Gamma.$$  

So, $vv^* \in \theta(A^\Gamma)$. It follows that $v^*\theta(A)v$ is a Cartan subalgebra of $qM^n q$. Moreover, for all $g \in \Gamma$, the unitary $v^*\theta(u_g)v = \rho(u_g)$ belongs to $qM^n q$ and normalizes $v^*\theta(A)v$. Then, [17, Theorem 1.8] implies that there exists $w \in qM^n$ such that $ww^* = q$, $w^*w \in A^n$ and $w^*v^*\theta(A)vw = w^*wA^n$. It follows that $\theta(A)$ and $pA$ are unitarily conjugate. \hfill $\square$

5 $S_{\text{factor}}(\mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z}))$ is non-trivial, for all $n \geq 3$

When $\Gamma$ is an ICC property (T) group, all groups in $S_{\text{factor}}(\Gamma)$ are countable (cf. [15, Proof of Theorem 1.7], or [23, Theorem 4.5.1]). Nevertheless, $S_{\text{factor}}(\Gamma)$ can be non-trivial, as shown by the next theorem, in which we show that if $\Gamma = \mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z})$, $n \geq 3$, then $S_{\text{factor}}(\Gamma)$ contains “many” subgroups of $ \mathbb{Q}_+$. It is unclear though whether there exists a free ergodic p.m.p. action $\Gamma \actson (X, \mu)$ of an ICC property (T) group $\Gamma$ such that $\mathcal{F}(\mathcal{R}(\Gamma \actson X)) \not\subset \mathbb{Q}_+$. 

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Theorem 5.1. Let $F \subset \mathbb{Q}_+$ be a subgroup generated by a subset of the prime numbers. Let $\Gamma = \mathbb{Z}^n \times \text{SL}(n, \mathbb{Z})$ with $n \geq 3$. Then, $\Gamma$ admits a “concrete” free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ such that both the fundamental group of $L^\infty(X) \times \Gamma$ and of $R(\Gamma \curvearrowright X)$ equal $F$.

We prove Theorem 5.1 as a consequence of the following more general result.

Theorem 5.2. Let $\Gamma$ be a group having a normal, non virtually abelian subgroup $\Sigma$ with the relative property (T) and with $\Gamma/\Sigma$ being finitely generated. Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a decreasing sequence of finite index subgroups such that the action $\Gamma \curvearrowright (X, \mu) := \lim_{\Gamma \rightarrow \Gamma_0} \Gamma/\Gamma_n$ is essentially free. Consider the diagonal product action $\Gamma \curvearrowright X \times [0,1]^\Gamma$ of $\Gamma \curvearrowright X$ and the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$.

Then, the fundamental groups of the associated $II_1$ factor and $II_1$ equivalence relation are both equal to

$$\left\{ \left[ \frac{\Gamma : \Lambda_1}{\Gamma : \Lambda_2} \right] \mid \Gamma_n \subset \Lambda_1 \cap \Lambda_2 \text{ for large enough } n, \text{ and } \right.$$

$$\Lambda_1 \curvearrowright \lim_{\Gamma \rightarrow \Gamma_0} \Lambda_1/\Gamma_n \text{ is conjugate with } \Lambda_2 \curvearrowright \lim_{\Gamma \rightarrow \Gamma_0} \Lambda_2/\Gamma_n \right\} \quad (5.1)$$

Conjugacy of two profinite actions can be expressed in purely group-theoretic terms, see e.g. [16, Proposition 1.8].

Before proving Theorem 5.2, we introduce some terminology and an auxiliary result. Recall that a 1-cocycle $\omega : \Gamma \times X \rightarrow \Lambda$ for an action $\Gamma \curvearrowright (X, \mu)$ with values in a countable group $\Lambda$, is a measurable map satisfying

$$\omega(gh,x) = \omega(g,h \cdot x) \omega(h,x) \quad \text{for all } g,h \in \Gamma \text{ and almost all } x \in X.$$ 

The 1-cocycles $\omega, \omega' : \Gamma \times X \rightarrow \Lambda$ are called cohomologous if there exists a measurable map $\varphi : X \rightarrow \Lambda$ satisfying $\omega'(g,x) = \varphi(g \cdot x) \omega(g,x) \varphi(x)^{-1}$ almost everywhere. We identify homomorphisms from $\Gamma$ to $\Lambda$ with 1-cocyles $\omega$ that are independent of the $x$-variable.

Definition 5.3. Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. We say that a 1-cocycle $\omega : \Gamma \times X \rightarrow \Lambda$ virtually untwists if there exists

- a finite index subgroup $\Gamma_0 < \Gamma$ and a quotient map $\pi : \Gamma \rightarrow \Gamma/\Gamma_0$ satisfying $\pi(g \cdot x) = g\pi(x)$ almost everywhere,
- a 1-cocycle $\omega' : \Gamma \times \Gamma/\Gamma_0 \rightarrow \Lambda$ for the action $\Gamma \curvearrowright \Gamma/\Gamma_0$,

such that $\omega$ is cohomologous with the 1 cocycle $(g,x) \mapsto \omega'(g,\pi(x))$.

We call $\Gamma \curvearrowright (X, \mu)$ virtually cocycle superrigid (with countable target groups) if every 1-cocycle with values in a countable group $\Lambda$, virtually untwists.

A stable orbit equivalence between free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X', \mu')$ is a map $\Delta : X \rightarrow X'$ satisfying the following properties.

- For almost every $x \in X$, we have $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.
- There exists a partition $X = \bigsqcup_n X_n$ of $X$ into measurable subsets $X_n \subset X$ and there exist measurable subsets $X'_n \subset X'$ such that for every $n \in \mathbb{N}$, the restriction of $\Delta$ to $X_n$ is a non-singular isomorphism between $X_n$ and $X'_n$. 

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By ergodicity, every of these non-singular isomorphisms $\Delta|_{X_n} : X_n \rightarrow X'_n$ is measure scaling, with the scaling being independent of $n$. This scaling factor is called the compression constant of the stable OE $\Delta$ and denoted by $c(\Delta)$.

The Zimmer 1-cocycle $\omega : \Gamma \times X \rightarrow \Lambda$ associated with the stable OE $\Delta$ is defined by

$$\Delta(g \cdot x) = \omega(g, x) : \Delta(x) \quad \text{almost everywhere.}$$

Two stable OEs $\Delta_1, \Delta_2 : X \rightarrow X'$ are called similar if $\Delta_1(x) \in \Lambda \cdot \Delta_2(x)$ for almost all $x \in X$. Note that similar stable OEs give rise to cohomologous 1-cocycles.

Whenever $X_0 \subset X$ and $X'_0 \subset X'$ are non-negligible measurable subsets and $\Delta_0 : X_0 \rightarrow X'_0$ is a non-singular isomorphism satisfying $\Delta_0(\Gamma \cdot x \cap X_0) = \Lambda \cdot \Delta_0(x) \cap X'_0$ for almost all $x \in X_0$, ergodicity allows to choose a measurable map $p : X \rightarrow X_0$ with $p(x) \in \Gamma \cdot x$ for almost all $x \in X$ and then, $\Delta := \Delta_0 \circ p$ defines a stable OE. Another choice of $p$ gives rise to a similar stable OE. It follows that

$$\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \{c(\Delta) \mid \text{\Delta is a stable OE between} \ \Gamma \curvearrowright X \ \text{and} \ \Gamma \curvearrowright X \}.$$ 

Let $\Gamma \curvearrowright (X, \mu)$. We say that the action $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$ if $X_1$ is a non-negligible measurable subset of $X$ and $\Gamma_1 < \Gamma$ is a finite index subgroup such that $g \cdot X_1 = X_1$ for all $g \in \Gamma_1$ and $\mu(g \cdot X_1 \cap X_1) = 0$ if $g \in \Gamma - \Gamma_1$. Obviously, in this situation $\Gamma \curvearrowright X$ is stably orbit equivalent with $\Gamma_1 \curvearrowright X_1$ with compression constant $[\Gamma : \Gamma_1]^{-1}$.

The following provides one more instance of a general principle going back to [36, Proposition 4.2.11]. For other versions of this, see [26, Proposition 5.11] and [35, Lemma 4.7].

**Proposition 5.4.** Let $\Delta : X \rightarrow X'$ be a stable OE between the free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X', \mu')$. If the associated Zimmer 1-cocycle virtually untwists (see Definition 5.3), there exist finite index subgroups $\Gamma_1 < \Gamma$, $\Lambda_1 < \Lambda$, non-negligible measurable subsets $X_1 \subset X$, $X'_1 \subset X'$ and a finite normal subgroup $H < \Gamma_1$ such that

1. $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$,
2. $\Lambda \curvearrowright Y$ is induced from $\Lambda_1 \curvearrowright Y_1$,
3. the actions $\Gamma_1/H \curvearrowright X_1/H$ and $\Lambda_1 \curvearrowright Y_1$ are conjugate,

and such that the stable OE $\Delta$ is similar to the composition of the canonical stable OEs given by 1, 3 and 2. In particular, the compression constant of $\Delta$ equals

$$c(\Delta) = \frac{[\Lambda : \Lambda_1]}{[\Gamma : \Gamma_1]|H|}.$$ 

**Proof.** Let $\Delta(g \cdot x) = \omega(g, x) : \Delta(x)$ almost everywhere. By our assumption, take a finite index subgroup $\Gamma_1$, a quotient map $\pi : X \rightarrow \Gamma/\Gamma_1$ and a 1-cocycle $\omega' : \Gamma \times \Gamma/\Gamma_1 \rightarrow \Lambda$ such that $\pi(g \cdot x) = g\pi(x)$ almost everywhere and such that $\omega$ is cohomologous with the 1-cocycle $(g, x) \mapsto \omega'(g, \pi(x))$. Define $X_1 = \pi^{-1}(e\Gamma_1)$. By construction, $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$.

Denote by $\Delta_1$ the restriction of $\Delta$ to $X_1$. Then, $\Delta_1$ is a stable OE between $\Gamma_1 \curvearrowright X_1$ and $\Lambda \curvearrowright Y$. By construction, the 1-cocycle associated with $\Delta_1$ is cohomologous to a homomorphism from $\Gamma_1$ to $\Lambda$. The conclusion of the proposition now follows from [35, Lemma 4.7].
In order to show the equality of the fundamental groups of the $\Pi_1$ factor and the $\Pi_1$ equivalence relation associated with the $\Gamma$-actions defined in Theorem 5.2, we need the following result about automatic preservation of Cartan subalgebras.

**Proposition 5.5.** Let $\Gamma$ be a countable group having a normal, non virtually abelian subgroup $\Sigma$ with the relative property (T). Let $\Gamma \actson (X, \mu)$ be a free ergodic p.m.p. action and assume that this action admits a free and profinite quotient: there exists a free profinite p.m.p. action $\Gamma \actson (X_1, \mu_1)$ and a quotient map $\pi : X \to X_1$ satisfying $\pi(g \cdot x) = g \cdot \pi(x)$ almost everywhere.

Let $(Y_0, \eta_0)$ be a non-trivial standard probability space and set $(Y, \eta) = (Y_0, \eta_0)^\Gamma$. Consider the diagonal action $\Gamma \actson (X \times Y, (\mu \times \eta))$. Set $M = L^\infty(X \times Y) \rtimes \Gamma$.

Then, every isomorphism $\theta : M \to pMp$ preserves, up to unitary conjugacy, the natural Cartan subalgebras of $M$, $pMp$.

**Proof.** Set $A = L^\infty(X)$ and $B = L^\infty(Y)$. Let $\theta : M \to pMp$ be an isomorphism. Denote $A_1 = L^\infty(X_1)$ and view $A_1$ as a globally $\Gamma$-invariant von Neumann subalgebra of $A$.

Almost literally repeating [24, Theorem 4.1] (see also [35, Lemma 6.1]), we find that $\theta(L(\Sigma)) \subset A \rtimes \Gamma$. Take a projection $q \in (A \rtimes \Gamma)^n$, a non-zero partial isometry $v \in p(M_{1,n}(\mathbb{C}) \otimes M)q$ and a unital $^*$-homomorphism $\alpha : L(\Sigma) \to q(A \rtimes \Gamma)^n q$ such that $\theta(a)v = v\alpha(a)$ for all $a \in L(\Sigma)$.

Since $\Sigma$ is normal in $\Gamma$ and since $\Sigma \actson X_1$ is profinite, the quasi-normalizer of $L(\Sigma)$ inside $M$ contains $A_1 \rtimes \Gamma$. Since $\Sigma$ is non virtually abelian, $L(\Sigma)$ cannot be embedded in an amplification of $A$. So, by [35, Proposition D.5], $v^*\theta(A_1 \rtimes \Gamma)v \subset (A \rtimes \Gamma)^n$. It follows in particular that $\theta(A_1) \subset A \rtimes \Gamma$.

We claim that in fact $\theta(A_1) \subset A_1$. Indeed, if this would not be the case, applying once more [35, Proposition D.5] (and using the regularity of $A_1 \subset (A \rtimes B) \rtimes \Gamma$) would yield $M \subset A \rtimes \Gamma$, a contradiction. This proves the claim.

Since $\Gamma \actson X_1$ is free, we have $A_1^I \cap M = A \rtimes B$. Hence, the proposition follows from [27, Theorem A.1].

We are now ready to prove Theorem 5.2.

**Proof of Theorem 5.2.** Put $(X, \mu) = \varprojlim \Gamma / \Gamma_n$ as in the formulation of the theorem. We assume $\Gamma \actson X$ to be (essentially) free. Let $Y = [0, 1]^\Gamma$ and denote by $\eta$ the infinite product of the Lebesgue measure on $[0, 1]$. We consider the diagonal action $\Gamma \actson X \times Y$.

By Proposition 5.5, we have

$$\mathcal{F}(L^\infty(X \times Y) \rtimes \Gamma) = \mathcal{F}(\mathcal{R}(\Gamma \actson X \times Y))$$.

Whenever $A_1 < \Gamma$ is a subgroup containing $\Gamma_n$ for large enough $n$, the action $\Gamma \actson (X, \mu)$ is induced from the action $A_1 \actson X_1 := \varprojlim A_1 / \Gamma_n$ and hence, $\Gamma \actson X \times Y$ is induced from $\Gamma_1 \actson X_1 \times Y$.

Since $\Gamma_1 \actson [0, 1]^\Gamma$ and $\Gamma_1 \actson [0, 1]^\Gamma_1$ are isomorphic actions, it follows that the set defined by (5.1) is part of the fundamental group $\mathcal{F}(\mathcal{R}(\Gamma \actson X \times Y))$.

A combination of [26, Theorem 0.1] and [16, Theorem B] yields that the diagonal action $\Gamma \actson X \times Y$ is virtually cocycle superrigid in the sense of Definition 5.3. So, we can apply Proposition 5.4.

Let $\Delta : X \times Y \to X \times Y$ be a stable OE between $\Gamma \actson X \times Y$ and itself. We have to prove that $c(\Delta)$ belongs to the set defined in (5.1). Proposition 5.4 provides us with finite index subgroups
$G_1, G_2$ of $\Gamma$, non-negligible measurable subsets $Z_1, Z_2 \subset X \times Y$ and a finite normal subgroup $H$ of $G_1$ such that $\Gamma \curvearrowright X \times Y$ is induced from $G_i \curvearrowright Z_i$ and such that $G_1/H \curvearrowright Z_1/H$ is conjugate with $G_2 \curvearrowright Z_2$, say through the isomorphism $\Delta : Z_1/H \rightarrow Z_2$ and the group isomorphism $\delta : G_1 \rightarrow G_2$. Finally, $c(\Delta) = \frac{|\Gamma : G_2|}{|\Gamma : G_1||H|}$.

Since the Bernoulli action $G_i \curvearrowright Y$ is mixing, we have $Z_i = X_i \times Y$, with $\Gamma \curvearrowright X$ being induced from $G_i \curvearrowright X_i$. Moreover, still because the Bernoulli action is mixing, $\Delta(x, y) = (\Delta_0(x), \ldots)$, where $\Delta_0 : X_1/H \rightarrow X_2$ is an isomorphism conjugating the actions $G_1/H \curvearrowright X_1/H$ and $G_2 \curvearrowright X_2$ through the group isomorphism $\delta : G_1/H \rightarrow G_2$.

Denote by $\pi_n : X \rightarrow \Gamma/\Gamma_n$ the natural quotient map. By [16, Lemma 4.1], we find $k \in \mathbb{N}$ and $g \in \Gamma$ such that $g\Gamma_k g^{-1} \subset G_1$ and $X_1 = \pi_k^{-1}(G_1g\Gamma_k)$. Moreover, since $\Gamma \curvearrowright X$ is free, we can take $k$ large enough and assume that $H \cap \Gamma_k = \{e\}$. Replacing $G_1$ by $g^{-1}G_1g$ and $X_1$ by $g^{-1} \cdot X_1$, we may assume that $g = e$. Note that $G_1/H \curvearrowright X_1/H = \lim\lim G_1/(\Gamma_nH)$ is induced from $(\Gamma_kH)/H \curvearrowright \lim\lim(\Gamma_kH)/(\Gamma_nH)$ and that the latter is conjugate with $\Gamma_k \curvearrowright \lim\lim\Gamma_k/\Gamma_n$, because $\Gamma_k \cap H = \{e\}$.

It follows that $\Gamma \curvearrowright X$ is induced from $\delta((\Gamma_kH)/H) \sim \Delta_0(\lim(\Gamma_kH)/(\Gamma_nH))$. Applying as above [16, Lemma 4.11], we find $h \in \Gamma$ such that, after replacing $\delta$ by $g \mapsto h\delta(g)h^{-1}$ and $\Delta_0$ by $x \mapsto h \cdot \Delta_0(x)$, we have $\Gamma_n \subset \Lambda_2 := \delta((\Gamma_kH)/H))$ for $n$ large enough and

$$\Delta_0(\lim(\Gamma_kH)/(\Gamma_nH)) = \lim\lim \Lambda_2/\Gamma_n.$$

Denoting $\Lambda_1 = \Gamma_k$, we have constructed finite index subgroups $\Lambda_1, \Lambda_2 \subset \Gamma$ such that $\Gamma_n \subset \Lambda_1 \cap \Lambda_2$ for $n$ large enough and such that the actions $\Lambda_i \curvearrowright \lim\lim \Lambda_i/\Gamma_n$ are conjugate for $i = 1, 2$. Tracing back the construction, we also have

$$c(\Delta) = \frac{[\Gamma : \Lambda_2]}{[\Gamma : \Lambda_1]}$$

concluding the proof of the theorem.

**Proof of Theorem 5.1.** Let $\mathcal{F}$ be a subgroup of $\mathbb{Q}_+$ generated by a non-empty subset $\mathcal{P}$ of the prime numbers. The case $\mathcal{F} = \{1\}$ will be discussed at the end of the proof. Denote by $R$ the subring of $\mathbb{Q}$ generated by $\mathcal{P}^{-1}$. Note that $R^* = \mathcal{F} \cup (-\mathcal{F})$. Set $G = R^* \times \text{GL}(n, R)$ and $\Gamma = \mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z})$.

Let $G = \{g_1, g_2, \cdots\}$ and define the finite index subgroups $\Gamma_k < \Gamma$ as $\Gamma_k = \Gamma \cap \bigcap_{i=1}^{k} g_i\Gamma_{g_i}^{-1}$. Define the profinite action $\Gamma \curvearrowright (X, \mu) := \lim\lim \Gamma/\Gamma_k$.

We first argue why $\Gamma \curvearrowright (X, \mu)$ is essentially free. Let $p \in \mathcal{P}$ and take $k_1 < k_2 < k_3 < \cdots$ such that $(0, p^1) \in \{g_1, \ldots, g_k\}$. One checks that $\Gamma_{k_i} \subset G_i := p^i \mathbb{Z}^n \rtimes \text{SL}(n, \mathbb{Z})$ and hence, it suffices to prove freeness of $\Gamma \curvearrowright \lim\lim \Gamma/\Gamma_i$. The latter has been shown in [16, discussion before Corollary 5.8].

Consider the diagonal action $\Gamma \curvearrowright X \times [0, 1]^\Gamma$. Denote by $\mathcal{F}$ the set defined in (5.1). By Theorem 5.2, we have to prove that $\mathcal{F} = R_+^\Gamma$. It is more convenient to write $X = \lim\lim \Gamma/\Gamma_F$, where $F$ runs through the finite subsets of $G$ and $\Gamma_F := \Gamma \cap \bigcap_{g \in F} g\Gamma g^{-1}$. Whenever $g \in G$, the action $\Gamma \curvearrowright X$ is induced from

$$\Gamma \cap g\Gamma g^{-1} \sim \lim\lim \Gamma \cap g\Gamma g^{-1}$$

and is induced from

$$g^{-1}\Gamma \cap g \sim \lim\lim g^{-1}\Gamma \cap g.$$

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Both actions are conjugate by construction. If \( g = (x, A) \), one checks that

\[
\frac{[\Gamma : \Gamma \cap g\Gamma g^{-1}]}{[\Gamma : g^{-1}\Gamma g \cap \Gamma]} = |\det A|.
\]

It follows that \( R^*_+ \subset F \).

Conversely, we claim that whenever \( \Lambda_1, \Lambda_2 < \Gamma \) are isomorphic finite index subgroups of \( \Gamma \) such that \( \Gamma_k \subset \Lambda_1 \cap \Lambda_2 \) for some \( k \), then \([\Gamma : \Lambda_1]/[\Gamma : \Lambda_2] \in R^*_+\). Once this claim is proven, we get the required equality \( F = R^*_+ \). Denoting by \( \Gamma \rtimes \mathbb{R} \) the required equality \( F = R^*_+ \). Let \( \delta : \Lambda_1 \to \Lambda_2 \) be an isomorphism and \( \Gamma_k \subset \Lambda_1 \cap \Lambda_2 \). We have \( \delta(\Lambda_1 \cap \mathbb{Z}^n) = \Lambda_2 \cap \mathbb{Z}^n \). An elementary argument for this fact can be given by repeating the beginning of the proof of [30, Proposition 7.1]. Denoting by \( \pi : \Gamma \to \text{SL}(n, \mathbb{Z}) \) the quotient map, \( \pi(\Lambda_1) \) and \( \pi(\Lambda_2) \) are isomorphic finite index subgroups of \( \text{SL}(n, \mathbb{Z}) \). Using [16, Lemma 5.2], it follows that \([\text{SL}(n, \mathbb{Z}) : \pi(\Lambda_1)] = [\text{SL}(n, \mathbb{Z}) : \pi(\Lambda_2)]\). Hence, we get

\[
\frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} = \frac{[\text{SL}(n, \mathbb{Z}) : \pi(\Lambda_1)]}{[\text{SL}(n, \mathbb{Z}) : \pi(\Lambda_2)]} = \frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_1]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_2]}.
\]

Being finite index subgroups of \( \mathbb{Z}^n \), we have \( \Lambda_1 \cap \mathbb{Z}^n = B_i \mathbb{Z}^n \) for some \( B_i \in \text{M}_n(\mathbb{Z}) \) with \( \det B_i \neq 0 \), \( i = 1, 2 \). It follows that there exists \( A \in \text{GL}(n, \mathbb{Q}) \) such that \( \delta(x, 1) = (Ax, 1) \) for all \((x, 1) \in \Lambda_1 \cap \mathbb{Z}^n\). Hence,

\[
\frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_1]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_2]} = \frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap A^{-1}\mathbb{Z}^n]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap AZ^n]}.
\]

Since \( \Gamma_k \subset \Lambda_1 \cap \Lambda_2 \subset \mathbb{Z}^n \cap AZ^n \cap A^{-1}\mathbb{Z}^n \), we find \( \alpha \in R^* \cap (\mathbb{N} - \{0\}) \) such that \( \alpha \mathbb{Z}^n \subset \mathbb{Z}^n \cap AZ^n \cap A^{-1}\mathbb{Z}^n \) for \( i = 1, 2 \). We conclude that \( A \in \text{GL}(n, \mathbb{R}) \) and finally,

\[
\frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} = |\det A|^{-1} \in R^*_+.
\]

To conclude the proof of the theorem, we need to construct a free ergodic p.m.p. action \( \Gamma \rtimes (X, \mu) \) such that the associated \( \text{II}_1 \) factor has trivial fundamental group. By [25, Corollary 0.2], the Bernoulli action \( \Gamma \rtimes [0, 1]^F \) has this property. Other examples can be given as follows. Let \( p_1, p_2, \ldots \) be an enumeration of the prime numbers and set \( \Gamma_k = p_1 \cdots p_k \mathbb{Z}^n \times \text{SL}(n, \mathbb{Z}) \). By Theorem 5.2 and [16, Corollary 5.8], the diagonal product of the Bernoulli action \( \Gamma \rtimes [0, 1]^\Gamma \) and the profinite action \( \Gamma \rtimes \lim \Gamma/\Gamma_k \), provides a crossed product \( \text{II}_1 \) factor with trivial fundamental group.

\section{Property (T) and countability of the fundamental group}

In his celebrated “rigidity paper” [4], Connes showed that \( \text{II}_1 \) factors arising from ICC groups with the property (T) of Kazhdan have countable fundamental group. Using the same ideas, it was later shown that for a separable \( \text{II}_1 \) factor \( M \) to have countable \( \mathcal{F}(M) \), it is in fact sufficient that \( M \) contains a subfactor with the property (T) in the sense of [9] and having trivial relative commutant, \( N' \cap M = \mathbb{C} \) (cf. Theorem 4.6.1 in [23]; see also [19] for a more general statement). It was also shown that if \( M \) is a separable \( \text{II}_1 \) factor, then the family of subfactors \( N_i \subset M, i \in I \), having property (T) and trivial relative commutant, is countable modulo conjugacy by unitaries in \( M \) (cf. Theorem 4.5.1 in [23]; see also [21] for a related result). In this section, we prove some analogous results for \( \text{II}_1 \) equivalence relations.

In particular, these results show that given any Kazhdan group \( \Gamma \), \( S_{\text{eqrel}}(\Gamma) \) can only contain countable subgroups of \( \mathbb{R}_+ \), and if in addition \( \Gamma \) is ICC then the same holds true for \( S_{\text{factor}}(\Gamma) \). More
generally, Part 1 of Theorem 6.4 below shows that this is still the case if the center of $\Gamma$ “virtually coincides” with its FC radical (as defined before 6.4). However, if one drops this assumption on $\Gamma$, then the situation seems to become quite complicated. Thus, Part 2 of Theorem 6.4 shows that if a property (T) group $\Gamma$ is residually finite and has non-virtually abelian FC radical, then $\Gamma$ admits free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ such that $L^{\infty}(X) \rtimes \Gamma$ is McDuff and hence, its fundamental group is equal to $\mathbb{R}_+$. 

We first need some notations. Thus, if $R$ group is equal to $R$. Note that, since space $(X, \mu \rtimes G)$ free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ of $\Gamma$ is II$_1$ (as defined before 6.4). However, if one drops this assumption on $\Gamma$, then the situation seems to become quite complicated. Thus, Part 2 of Theorem 6.4 shows that if a property (T) group $\Gamma$ is residually finite and has non-virtually abelian FC radical, then $\Gamma$ admits free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ such that $L^{\infty}(X) \rtimes \Gamma$ is McDuff and hence, its fundamental group is equal to $\mathbb{R}_+$.

We first need some notations. Thus, if $R$ is a II$_1$ equivalence relation on the standard probability space $(X, \mu)$, then we denote by $[R]$ the full group of the equivalence relation $R$, consisting of all non-singular isomorphisms $\phi : X \to X$ satisfying $(x, \phi(x)) \in R$ for almost all $x \in X$. The full pseudogroup $R$ is denoted by $[[R]]$ and consists of all non-singular partial automorphisms $\phi$ between measurable subsets $D(\phi), R(\phi) \subset X$, satisfying $(x, \phi(x)) \in R$ for almost all $x \in D(\phi)$. Note that, since $R$ is II$_1$, every $\phi \in [[R]]$ is measure preserving. If $\Gamma \subset [[R]]$ is a subgroup, we denote by $s(\Gamma) \subset X$ its support, i.e. the subset $Y \subset X$ with the property that $R(g) = D(g) = Y$, $\forall g \in \Gamma$. Two such subgroups $\Gamma, \Lambda \subset [[R]]$ are conjugate by an element in $[[R]]$ if there exists $\phi \in [[R]]$ such that $D(\phi) = s(\Gamma), R(\phi) = s(\Lambda)$ and $\phi \Gamma = \Lambda \phi$.

**Theorem 6.1.** Let $R$ be a II$_1$ equivalence relation on the probability space $(X, \mu)$.

1. If $[R]$ contains a property (T) group $\Gamma$ implementing an ergodic action on $(X, \mu)$, then $\mathcal{F}(R)$ is countable. More generally, if $[R]$ contains a countable group $\Gamma$ having a subgroup $H \subset \Gamma$ with the relative property (T) implementing an ergodic action on $(X, \mu)$, then $\mathcal{F}(R)$ is countable.

2. Let $T$ be the set of property (T) subgroups $\Gamma \subset [[R]]$ acting ergodically on $s(\Gamma)$. Then $T$ is countable, modulo conjugacy by elements in $[[R]]$.

Note that the ergodicity assumption of the action of $\Gamma$ on $(X, \mu)$ in Part 1 of the above statement is crucial. Indeed, Theorem 3.1 provides examples of free ergodic p.m.p. actions $G \curvearrowright (X, \mu)$ such that $R(G \rtimes X)$ has uncountable fundamental group, but nevertheless $G$ contains a subgroup having property (T) (which, a fortiori, acts non-ergodically on $(X, \mu)$). In turn, the existence of a property (T) subgroup of $[R]$ acting freely and ergodically on $X$, does not insure that the II$_1$ factor $L(R)$ has countable fundamental group. Indeed, by [8] there exist free ergodic p.m.p. actions of groups of the form $G = \Gamma \rtimes \Sigma$, with $\Gamma$ having property (T) and acting by Bernoulli shifts (thus ergodically), such that $M = L^{\infty}(X) \rtimes G$ splits off the hyperfinite II$_1$ factor, and thus $\mathcal{F}(M) = \mathbb{R}_+$. In fact, as pointed out in [26], more than being countable, the fundamental group of $R(G \rtimes X)$ is trivial.

Note also that in the case $R$ comes from a free ergodic action of a property (T) group, $\Gamma \curvearrowright (X, \mu)$, Part 1 of Theorem 6.1 was already shown in [15, Corollary 1.8], in the case $\Gamma$ is ICC, and in [16, Theorem 5.9], in the general case. We will use the above result in [31], to prove that the II$_1$ equivalence $R$ obtained by restricting the II$_\infty$ equivalence relation implemented by $SL(n, \mathbb{Z}) \rtimes \mathbb{R}^n$, $n \geq 4$, to a subset of measure 1, has property (T) in the sense of Zimmer, yet cannot be implemented by an action (even non-free) of a property (T) group because $\mathcal{F}(R) = \mathbb{R}_+$.

We will prove Theorem 6.1 by contradiction, using the property (T) of the subgroups and a “separability” argument, in the spirit of [23]. For more on this strategy of proofs, which grew out of Connes’ rigidity paper [4], we send the interested reader to Section 4 in [28]. As a result of this argument, we obtain two copies $\Gamma_1, \Gamma_2 \subset [[R]]$ of the same property (T) group, which are uniformly close one to the other. This in turn gives rise to a non-zero intertwiner $\phi \in [[R]]$ between $\Gamma_1, \Gamma_2$. But if the $\Gamma_i$-actions are assumed ergodic, this forces $\mu(s(\Gamma_1)) = \mu(s(\Gamma_2))$ and the conjugacy of $\Gamma_1, \Gamma_2$. 

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The existence of an intertwiner between uniformly close subgroups in \([{|R|}}\) is the subject of the next lemma. Recall that \([{|R|}}\) has a natural metric space structure, inherited from the Hilbert-norm \(\| \cdot \|_2\) of the underlying II_1 factor \(L(R)\) associated with \(R\), by viewing every \(\phi \in [|R|]\) as a partial isometry in \(L(R)\) and using the \(\| \cdot \|_2\)-norm on the latter. The metric can be concretely written as

\[
d(\phi, \psi)^2 = \mu(D(\phi) \triangle D(\psi)) + 2\mu(\{x \in D(\phi) \cap D(\psi) \mid \phi(x) \neq \psi(x)\}) ,
\]

where \(\triangle\) denotes the symmetric difference of two sets. We will also need the natural \(\sigma\)-finite measure \(\mu^{(1)}\) on \(\mathcal{R} \subset X \times Y\), defined by the formula

\[
\mu^{(1)}(\mathcal{U}) = \int_X \# \{y \mid (x, y) \in \mathcal{U}\} \, d\mu(x) = \int_X \# \{x \mid (x, y) \in \mathcal{U}\} \, d\mu(y)
\]

for all measurable subsets \(\mathcal{U} \subset \mathcal{R}\).

**Lemma 6.2.** Let \(\mathcal{R}\) be a II_1 equivalence relation on the standard probability space \((X, \mu)\). Suppose that \(\Gamma\) is a countable group, \(X_0, Y_0 \subset X\) and let

\[
\alpha : \Gamma \to [\mathcal{R}|_{X_0}] \quad \text{and} \quad \beta : \Gamma \to [\mathcal{R}|_{Y_0}]
\]

be group morphisms satisfying \(d(\alpha_g, \beta_g) \leq 1/5\) for all \(g \in \Gamma\) and \(\mu(X_0), \mu(Y_0) \geq 3/4\). Then, there exist non-negligible measurable subsets \(X_1 \subset X_0, Y_1 \subset Y_0\) and \(\phi \in [|R|]\) with \(D(\phi) = X_1, R(\phi) = Y_1\) such that

\[
X_1 \quad \text{is globally} \quad (\alpha_g)_{g \in \Gamma}\text{-invariant} , \quad Y_1 \quad \text{is globally} \quad (\beta_g)_{g \in \Gamma}\text{-invariant}, \quad \text{and} \quad \phi(\alpha_g(x)) = \beta_g(\phi(x)) \quad \text{for almost all} \quad x \in D(\phi).
\]

**Proof.** Denote by \(\text{Tr}\) the normal faithful semi-finite trace on \(L^\infty(\mathcal{R})\) given by integration along \(\mu^{(1)}\). Let \(p \in L^\infty(\mathcal{R})\) be the projection onto \(\mathcal{R} \cap X_0 \times Y_0\) and \(e \leq p\) the projection onto \(\{(z, z) \mid z \in X_0 \cap Y_0\}\). Set \(B = L^\infty(\mathcal{R})p\). The group \(\Gamma\) acts by automorphisms \(\rho_g\) of \(B\) given by

\[
(\rho_g F)(x, y) = F(\alpha_g^{-1}(x), \beta_g^{-1}(y)) \quad \text{for all almost} \quad (x, y) \in \mathcal{R} \cap X \times Y .
\]

Since \(\|\rho_g(e) - e\|_{2, \text{Tr}}^2 = 2\mu(\{z \in X_0 \cap Y_0 \mid \alpha_g^{-1}(z) \neq \beta_g^{-1}(z)\})\), we get

\[
\|\rho_g(e) - e\|_{2, \text{Tr}} \leq \frac{1}{5} \quad \text{for all} \quad g \in \Gamma .
\]

Define \(a \in B^+\) as the unique element of minimal \(\| \cdot \|_{2, \text{Tr}}\) in the weakly closed convex hull \(\text{conv}\{\rho_g(e) \mid g \in \Gamma\}\). It follows that \(\|a - e\|_{2, \text{Tr}} \leq 1/5\) and that \(\rho_g(a) = a\) for all \(g \in \Gamma\). Note that \(0 \leq a \leq 1\). Defining \(f\) as the spectral projection \(f = \chi_{[1/2, 1]}(a)\), we find that \(\|f - e\|_{2, \text{Tr}} \leq 2/5\) and \(\rho_g(f) = f\) for all \(g \in \Gamma\). We write \(f = \chi_W\), where \(W \subset \mathcal{R} \cap X_0 \times Y_0\) is globally \((\rho_g)_{g \in \Gamma}\)-invariant and satisfies

\[
\mu^{(1)}(W \triangle \{(z, z) \mid z \in X_0 \cap Y_0\}) \leq \frac{4}{25} . \tag{6.1}
\]

Denote \(z_W := \{y \in Y_0 \mid (x, y) \in W\}\) and \(W_y := \{x \in X_0 \mid (x, y) \in W\}\). Define

\[
W_0 := \{(x, y) \in W \mid z_W\text{ and } W_y\text{ are singletons}\} .
\]

Then, \(W_0\) is still globally \((\rho_g)_{g \in \Gamma}\)-invariant. Since \(\mu(X_0), \mu(Y_0) \geq 3/4\) and \(\mu(X_0 \triangle Y_0) \leq 1/25\), we have \(\mu(X_0 \cap Y_0) \geq 3/4 - 1/25\). By (6.1), the set of \(x \in X_0\) such that \(z_W\) is a singleton, has measure at least \(3/4 - 1/25 - 4/25\). The same holds for the set of \(y \in Y_0\) such that \(W_y\) is a singleton. So, \(W_0\) has measure at least \(1/10\). By construction, \(W_0\) is the graph of a partial automorphism \(\phi \in [|R|]\) satisfying all the conclusions of the lemma. \(\square\)
Proof of Theorem 6.1. Let us first prove Part 1 of the theorem. By the relative property (T) of $H \subset \Gamma$, there exist $F \subset \Gamma$ and $0 < \varepsilon < 1/4$ such that whenever $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ a unit vector satisfying $\| \pi(g)\xi_0 - \xi_0 \| \leq \varepsilon$ for all $g \in F$, then $\| \pi(h)\xi_0 - \xi_0 \| \leq 1/8$ for all $h \in H$.

Choose for every $t \in (0, 1)$ a measurable subset $Y_t \subset X$ with $\mu(Y_t) = t$ and such that $Y_s \subset Y_t$ if $s \leq t$. Assume that the fundamental group of $\mathcal{R}$ is uncountable. For every $t \in \mathcal{F}(\mathcal{R}) \cap (3/4, 1)$, choose an isomorphism $\Delta_t : X \to Y_t$ between $\mathcal{R}$ and $\mathcal{R}|_{Y_t}$. Note that $\Delta_t$ scales the measure $\mu$ by $t$.

Define $\alpha_t^g = \Delta_t \circ \alpha_g \circ \Delta_t^{-1}$. Note that $\alpha_t^g \in \{[\mathcal{R}]\}$ with $D(\alpha_t^g) = R(\alpha_t^g) = Y_t$. Since $\mathcal{F}(\mathcal{R}) \cap (3/4, 1)$ is uncountable, separability of the metric space $([\mathcal{R}], d)$ yields $s, t \in \mathcal{F}(\mathcal{R}) \cap (3/4, 1)$ with $s < t$ and $d(\alpha_s^g, \alpha_t^g) \leq \varepsilon/2$ for all $g \in F$.

Define the Hilbert space $\mathcal{H} = L^2(\mathcal{R} \cap Y_s \times Y_t, \mu^{(1)})$ and the unitary representation

$$\pi : \Gamma \to \mathcal{U}(\mathcal{H}) : (\pi(g))((x,y) = \xi(\alpha_{t-1}^g(x), \alpha_{t-1}^g(y)).$$

Set $\Delta_s := \{(y, y) \mid y \in Y_s\}$ and $\xi_0 := s^{-1/2} \chi_{\Delta_s}$. Then, $\xi_0$ is a unit vector in $\mathcal{H}$ and, for all $g \in F$,

$$\| \pi(g)\xi_0 - \xi_0 \| = 2s^{-1} \mu\{(y \in Y_s \mid \alpha_s^g(y) \neq \alpha_t^g(y))\} \leq s^{-1} d(\alpha_s^g, \alpha_t^g)^2 \leq \varepsilon^2.$$

It follows that $\| \pi(h)\xi_0 - \xi_0 \| \leq 1/8$ for all $h \in H$. So, for all $h \in H$, we have

$$2\mu\{(y \in Y_s \mid \alpha_s^h(y) \neq \alpha_t^h(y))\} \leq s \leq \frac{1}{64}.$$ Since also, given $g \in F$,

$$\mu(Y_t \setminus Y_s) \leq d(\alpha_s^g, \alpha_t^g)^2 \leq \frac{\varepsilon^2}{4} \leq \frac{1}{64},$$

it follows that

$$d(\alpha_s^g, \alpha_t^h)^2 \leq \mu(Y_t \setminus Y_s) + \frac{1}{64} < \frac{1}{25}$$

for all $h \in H$. Since $(\alpha_h)_{h \in H}$ implements an ergodic action on $(X, \mu)$, the same holds for $(\alpha_s^h)_{h \in H}$, $(\alpha_t^h)_{h \in H}$ and so, Lemma 6.2 provides an element $\phi \in \{[\mathcal{R}]\}$ with $D(\phi) = Y_s$ and $R(\phi) = Y_t$. Since $\phi$ is a measure preserving isomorphism between $Y_s$ and $Y_t$ and $\mu(Y_s) = s < t = \mu(Y_t)$, we reached a contradiction.

To prove Part 2 of Theorem 6.1, assume by contradiction that there exist uncountably many subgroups $\{\Gamma_i \mid i \in I\}$ in $\{[\mathcal{R}]\}$ which have property (T) and are non-conjugate in $\{[\mathcal{R}]\}$. We continue to use the measurable subsets $Y_t \subset X$ with $\mu(Y_t) = t$ and $Y_s \subset Y_t$ whenever $s \leq t$. By the ergodicity of $\mathcal{R}$, we may assume that for every $i \in I$, the support of $\Gamma_i$ is one of the $Y_s$.

By Shalom’s theorem [33, Theorem 6.7], every property (T) group is the quotient of a finitely presented property (T) group. Since there are only countably many finitely presented groups, we may assume that all $\Gamma_i$’s are representations $\alpha_i$ of the same property (T) group $\Gamma$. Finally, we may assume that there exists $t \in (0, 1)$ such that $\mu(s(\Gamma_i)) \in (3t/4, t)$ for all $i \in I$. So, replacing $\mathcal{R}$ by $\mathcal{R}|_{Y_t}$, we may assume that $\mu(s(\Gamma_i)) \in (3/4, 1)$ for all $i \in I$.

By the property (T) of $\Gamma$, there exist $F \subset \Gamma$ and $0 < \varepsilon < 1/4$ such that whenever $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ a unit vector satisfying $\| \pi(g)\xi_0 - \xi_0 \| \leq \varepsilon$ for all $g \in F$, then $\| \pi(g)\xi_0 - \xi_0 \| \leq 1/8$ for all $g \in \Gamma$.

Now, by the separability of $([\mathcal{R}], d)$, there exist $i \neq j$ such that $d(\alpha_i(g), \alpha_j(g)) \leq \varepsilon/2$, $\forall g \in F$.

Let $Y_i \subset X$, $Y_j \subset X$ be the support of $\Gamma_i$ resp. $\Gamma_j$ and assume $Y_i \subset Y_j$. We define $\mathcal{H}$, $\xi_0 \in \mathcal{H}$,
π : Γ → U(H) as before, but replacing α_δ by α_i(g), α_δ^t by α_j(g), Y_δ by Y_i and Y_t by Y_j. The same estimates then show that ξ_0 is a unit vector satisfying ∥π_δ(ξ_0) − ξ_0∥ ≤ ε, ∀g ∈ F. Thus, ∥π(g)ξ_0 − ξ_0∥ ≤ 1/8 for all g ∈ Γ. As before, this translates into d(α_i(g), α_j(g)) ≤ 1/4, ∀g ∈ Γ. By Lemma 6.2, this implies Γ_i, Γ_j are conjugate by an element in [[R]], contradicting our initial assumption. □

Part 2 of Theorem 6.1 readily implies that the functor Γ → RΓ, from free ergodic p.m.p. actions of property (T) groups with morphisms given by conjugacy, to the associated equivalence relations with morphisms given by orbital isomorphism, is “countable to one”. In other words, there are at most countably many non-conjugate free ergodic p.m.p. actions in each OE class of a free ergodic p.m.p. action of a property (T) group. In fact, even more is true: any free ergodic p.m.p. action of a property (T) group follows “orbit equivalent superrigid, modulo countable classes”, in a sense made precise below.

**Corollary 6.3.** Let Γ ∼ X be a free ergodic p.m.p. action of a property (T) group. Let Λ_i ∼ X_i, i ∈ I, be a family of free ergodic p.m.p. actions such that RΓ ∼ R^i_Λ, for some i > 0. Then the family I is countable, modulo conjugacy of actions.

**Proof.** We may assume that t_i ≥ 1/c for all i ∈ I and some c > 0. Setting R = (R_Γ)^c, we can view all Λ_i as subgroups of [[R]], with the action of Λ_i on s(Λ_i) ⊂ X being conjugate to Λ_i ∼ X_i. By [12, Corollary 1.4], all Λ_i have property (T). So, by Part 2 of Theorem 6.1, the family I is countable modulo conjugacy of actions. □

When Γ is an ICC property (T) group, all groups in S_{factor}(Γ) are countable (cf. [15, Proof of Theorem 1.7], or [23, Theorem 4.5.1]). The next theorem generalizes this result to Kazhdan groups Γ with the property that the center Z(Γ) has finite index in the FC-radical Γ_f of Γ, defined by

Γ_f := {g ∈ Γ | g has a finite conjugacy class }.

As far as we know, the only examples of Kazhdan groups Γ with infinite FC-radical Γ_f that exist in the literature are such that Z(Γ) has finite index in Γ_f (see e.g. [2, Example 1.7.13] and [11, Definition 2.4]). Concerning the remaining case, we prove in the second part of the theorem below that if Γ is a residually finite property (T) group such that Γ_f is not virtually abelian (i.e., Z(Γ_f) < Γ_f has infinite index), then Γ admits a free ergodic p.m.p. action on (X, µ) with L^∞(X) × Γ being McDuff and hence, R+ ∈ S_{factor}(Γ). While we were unable to show whether Kazhdan groups with these properties exist or not, after discussing this problem with several specialists, it was indicated to us by Mark Sapir and Denis Osin that such groups probably do exist, but their actual construction may require a substantial effort.

**Theorem 6.4.** Let Γ be a property (T) group.

1. If Z(Γ) has finite index in the FC-radical Γ_f, then S_{factor}(Γ) only contains countable groups.

2. If Γ is residually finite and [Γ_f, Z(Γ_f)] = ∞, then Γ admits a free ergodic profinite p.m.p. action on (X, µ) such that L^∞(X) × Γ is McDuff.

**Proof.** Whenever H ⊂ Γ, denote by C_G(H) the centralizer of H inside Γ.

Assume first that Z(Γ) has finite index in Γ_f. Let Γ ∼ (X, µ) be free ergodic p.m.p. Write A := L^∞(X) and M := A × Γ. Define Λ := C_G(Γ_f). Since Z(Γ) has finite index in Γ_f, it follows
that Λ has finite index in Γ. A fortiori, the subgroup Λ_1 := Λ \cdot Γ_f has finite index in Γ. Also, the subalgebra A^Λ of Λ-invariant functions in A, is finite dimensional and globally Λ_1-invariant. Consider the subalgebra B := A^Λ \times Λ_1 of M. Since L(Λ_1) \subset B has finite index, it follows that B has property (T). On the other hand, M \cap B' \subset M \cap L(Λ)' and it is straightforward to check that M \cap L(Λ)' \subset A^Λ \times Γ_f. So, we get M \cap B' \subset B. By [19, Theorem A.1], it follows that F(M) is countable.

Suppose from now on that Γ is residually finite and [Γ_f, Z(Γ_f)] = ∞. Let Γ_f = \{h_1, h_2, \ldots\} be an enumeration. Let H_n \triangleleft Γ be a decreasing sequence of normal, finite index subgroups with \bigcap_n H_n = \{e\}. Define
\[ Γ_n := H_n \cap \bigcap_{s \in Γ/\Gamma_n} s C_Γ(h_1, \ldots, h_n) s^{-1}. \]

By construction, Γ_n is a decreasing sequence of normal, finite index subgroups with \bigcap_n Γ_n = \{e\} and such that for all h \in Γ_f, we have Γ_n \subset C_Γ(h) for all n large enough.

Denote (X, μ) = \lim(Γ/Γ_n, counting probability measure). Consider the natural free, ergodic, profinite, p.m.p. action Γ ⋊ (X, μ). Put A = L^∞(X) and M := A ⋊ Γ. For every s ∈ Γ and n ∈ \mathbb{N}, denote by \chi_{s\Gamma_n} the function equal to 1 on sΓ_n and zero elsewhere and interpret \chi_{s\Gamma_n} as a projection in A.

For every h ∈ Γ_f, define the unitary \upsilon_h ∈ M \cap L(Γ)' by
\[ \upsilon_h := \sum_{s \in Γ/Γ_n} \chi_{s\Gamma_n} u_{sh^{-1}s^{-1}} \text{ for } n \text{ large enough, meaning } Γ_n \subset C_Γ(h). \]

It is straightforward to check that Γ_f → U(M \cap L(Γ)'): h ↦ \upsilon_h is a group morphism and that τ(\upsilon_h) = 0 whenever h \neq e.

Claim. If for all n ∈ \mathbb{N}, we have h_n ∈ Γ_f ∩ Γ_n with h_n \neq e, then (\upsilon_{h_n}) is a central sequence in M with τ(\upsilon_{h_n}) = 0 for all n. For all n, we have \upsilon_{h_n} ∈ L(Γ)'. So, to prove the claim, it suffices to take k ∈ \mathbb{N}, g ∈ Γ and prove that
\[ \lim_n \|[\chi_{g\Gamma_k}, \upsilon_{h_n}]\|_2 = 0. \]

But, by construction, \chi_{g\Gamma_k} and \upsilon_{h_n} commute when n ≥ k.

Since Z(Γ_f) < Γ_f has infinite index and since Γ_f has finite conjugacy classes, it follows that Γ_f has no finite index abelian subgroups. So, for every n, the finite index subgroup Γ_f \cap Γ_n of Γ_f is non-abelian. Therefore, we can choose h_n, h'_n ∈ Γ_f ∩ Γ_n such that h_n h'_n h_n^{-1} h'_n^{-1} ≠ e. By the claim above, \upsilon_{h_n} and \upsilon_{h'_n} are central sequences. By construction, τ(\upsilon_{h_n} \upsilon_{h'_n} \upsilon_{h_n}^{*} \upsilon_{h'_n}^{*}) = 0 for all n. So, M is McDuff.

\[\square\]

References

[1] J. Aaronson, The intrinsic normalising constants of transformations preserving infinite measures. J. Analyse Math. 49 (1987), 239-270.

[2] B. Bekka, P. de la Harpe and A. Valette, Kazhdan’s property (T). New Mathematical Monographs 11. Cambridge University Press, Cambridge, 2008.

[3] I. Chifan and C. Houdayer, Prime factors and amalgamated free products. Preprint. arXiv:0805.1566
[4] A. Connes, A factor of type II\textsubscript{1} with countable fundamental group. J. Operator Theory \textbf{4} (1980), 151-153.
[5] A. Connes, Classification of injective factors. Ann. Math. \textbf{104} (1976), 73-115.
[6] A. Connes, Une classification des facteurs de type III, Ann. Ec. Norm. Sup. \textbf{6} (1973), 133-252.
[7] A. Connes, J. Feldman & B. Weiss, An amenable equivalence relation is generated by a single transformation. Ergodic Theory Dynam. Systems \textbf{1} (1981), 431-450.
[8] A. Connes and V.F.R. Jones, A II\textsubscript{1} factor with two non-conjugate Cartan subalgebras, Bull. Amer. Math. Soc. \textbf{6} (1982), 211-212.
[9] A. Connes and V.F.R. Jones, Property (T) for von Neumann algebras, Bull. London Math. Soc. \textbf{17} (1985), 57-62.
[10] A. Connes and M. Takesaki, The flow of weights of a factor of type III. Tohoku Math. J. \textbf{29} (1977), 473-575.
[11] Y. de Cornulier, Finitely presentable, non-Hopfian groups with Kazhdan’s property (T) and infinite outer automorphism group. Proc. Amer. Math. Soc. \textbf{135} (2007), 951-959.
[12] A. Furman, Gromov’s measure equivalence and rigidity of higher rank lattices. Ann. of Math. \textbf{150} (1999), 1059–1081.
[13] D. Gaboriau, Invariants l\textsuperscript{2} de relations d’équivalence et de groupes. Publ. Math. Inst. Hautes Études Sci. \textbf{95} (2002), 93-150.
[14] D. Gaboriau, Relative property (T) actions and trivial outer automorphism groups. Preprint. arXiv:0804.0358
[15] S.L. Gefter and V.Ya. Golodets, Fundamental groups for ergodic actions and actions with unit fundamental groups. Publ. Res. Inst. Math. Sci. \textbf{24} (1988), 821-847.
[16] A. Ioana, Cocycle superrigidity for profinite actions of property (T) groups. Preprint. arXiv:0805.2998
[17] A. Ioana, J. Peterson and S. Popa, Amalgamated free products of w-rigid factors and calculation of their symmetry groups. Acta Math. \textbf{200} (2008), 85-153.
[18] F. Murray, J. von Neumann, Rings of operators IV, Ann. Math. \textbf{44} (1943), 716-808.
[19] R. Nicoara, S. Popa and R. Sasyk, On II\textsubscript{1} factors arising from 2-cocycles of w-rigid groups. J. Funct. Analysis \textbf{242} (2007), 230-246.
[20] D.S. Ornstein and B. Weiss, Ergodic theory of amenable group actions. Bull. Amer. Math. Soc. (N.S.) \textbf{2} (1980), 161-164.
[21] N. Ozawa, There is no separable universal II\textsubscript{1}-factor, Proc. Amer. Math. Soc. \textbf{132} (2004), 487-490.
[22] N. Ozawa and S. Popa, On a class of II\textsubscript{1} factors with at most one Cartan subalgebra, preprint math.OA/0706.3623, to appear in Ann. Math.
[23] S. Popa, Correspondences, INCREST preprint No 56, 1986 (unpublished). http://www.math.ucla.edu/~popa/preprints.html
[24] S. Popa, Strong rigidity of II\textsubscript{1} factors arising from malleable actions of w-rigid groups, I. Invent. Math. \textbf{165} (2006), 369-408.
[25] S. Popa, Strong rigidity of II\textsubscript{1} factors arising from malleable actions of w-rigid groups, II. Invent. Math. \textbf{165} (2006), 409-452.
[26] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups. Invent. Math. \textbf{170} (2007), 243-295.
[27] S. Popa, On a class of type II\textsubscript{1} factors with Betti numbers invariants. Ann. of Math. \textbf{163} (2006), 809-899.
[28] S. Popa, Deformation and rigidity for group actions and von Neumann algebras, in “Proceedings of the International Congress of Mathematicians” (Madrid 2006), Volume I, EMS Publishing House, Zurich 2006/2007, pp. 445-479.
[29] S. Popa and S. Vaes, Actions of $\mathbb{F}_\infty$ whose II\textsubscript{1} factors and orbit equivalence relations have prescribed fundamental group. Preprint. arXiv:0803.3351
[30] S. Popa and S. Vaes, Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups. Adv. Math. \textbf{217} (2008), 833–872.
[31] S. Popa and S. Vaes, Cocycle and orbit superrigidity for lattices in $\text{SL}(n, \mathbb{R})$ acting on homogeneous spaces. Preprint. arXiv:0810.3630

[32] F. Radulescu, A one parameter group of automorphisms of $L(F_{\infty})$ scaling the trace. Comptes Rendus Acad. Sci. Paris 314 (1992), 1027-1032.

[33] Y. Shalom, Rigidity of commensurators and irreducible lattices. Invent Math. 141 (2000), 1-54.

[34] A. Törnquist, Orbit equivalence and actions of $F_n$. J. Symbolic Logic 71 (2006), 265-282.

[35] S. Vaes, Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa). Séminaire Bourbaki, exp. no. 961. Astérisque 311 (2007), 237-294.

[36] R.J. Zimmer, Ergodic theory and semisimple groups. Birkhäuser, Boston, 1984.