EXTENSION THEOREM OF WHITNEY TYPE FOR $S(\mathbb{R}^d_+)$ BY THE USE OF THE KERNEL THEOREM

SMILJANA JAKŠIĆ AND BOJAN PRANGOSKI

Abstract. We study the expansions of the elements in $S(\mathbb{R}^d_+)$ and $S'(\mathbb{R}^d_+)$ with respect to the Laguerre orthonormal basis, extending the result of M. Guilmont-Teissier [4] in the case $d = 1$. As a consequence, we obtain the Schwartz kernel theorem for $S(\mathbb{R}^d_+)$ and $S'(\mathbb{R}^d_+)$ and the extension theorem of Whitney type for $S(\mathbb{R}^d_+)$.}

1. Introduction

We denote by $\mathbb{R}^d_+$ the set $(0, \infty)^d$ and by $\overline{\mathbb{R}^d_+}$ its closure, i.e. $[0, \infty)^d$. We will consider the space $S(\overline{\mathbb{R}^d_+})$ which consists of all $f \in C^\infty(\overline{\mathbb{R}^d_+})$ such that all derivatives $D^p f$, $p \in \mathbb{N}^d_0$, extend to continuous functions on $\mathbb{R}^d_+$ and

$$\sup_{x \in \mathbb{R}^d_+} x^k |D^p f(x)| < \infty, \forall k, p \in \mathbb{N}^d_0.$$  

With this system of seminorms $S(\mathbb{R}^d_+)$ becomes an $(F)$-space.

The results concerning the extension of a smooth function or a function of class $C^k$ out of some region and various reformulation of such problems are called extension theorems of Whitney type. One can see Whitney [11], Seeley [8] and Hörmander [3, Theorem 2.3.6, p. 48]. Here we deal with a problem of extension of a function from $S(\mathbb{R}^d_+)$ onto $S(\mathbb{R}^d)$. Theorem 4.3 is the main result of the paper. For the purpose of this theorem we prove the Schwartz kernel theorem for $S(\mathbb{R}^d_+)$ and $S'(\mathbb{R}^d_+)$, Theorem 4.2.

Recall, for $n = 0, 1, 2...$ the functions

$$L_n(x) = \frac{e^x}{n!} \left(\frac{d}{dx}\right)^n (e^{-x} x^n), \quad x > 0$$

are the Laguerre polynomials and $L_n(x) = L_n(x)e^{-\frac{x}{2}}$ are the Laguerre functions; $\{L_n(x), n = 0, 1, ...\}$ is an orthonormal basis for $L^2(0, \infty)$ ([10] p.108).

The problem of expanding the elements of $S'(\mathbb{R}_+)$ with respect to the Laguerre orthonormal basis has been treated by Guilmont-Teissier in [4] and Duran in [1]:

If $T \in S'(\mathbb{R}_+)$ and $a_n = \langle T, L_n(x) \rangle$ then $T = \sum_{n=0}^{\infty} a_n L_n(x)$ and $\{a_n\}_{n=0}^{\infty}$ decreases slowly. Conversely, if $\{a_n\}_{n=0}^{\infty}$ decreases slowly, then there exists $T \in S'(\mathbb{R}_+)$ such that $T = \sum_{n=0}^{\infty} a_n L_n(x)$.

The works [7], [12] and [13] contain expansions of the same kind as in [4] and [1].
The novelty of this paper is the extension of the results of [4] for the \( d \) dimensional case. This leads to the Schwartz kernel theorem (Theorem 4.2) which states that there is one-to-one correspondence between elements from \( S'(\mathbb{R}^{m+n}_+) \) in two sets of variables \( x \) and \( y \) and the continuous linear mappings of \((S(\mathbb{R}_+^n))_y\) into \((S'(\mathbb{R}_+^m))_x\). As a consequence of Theorem 4.3 we explain the convolution in \( S'(\mathbb{R}_+^d) \) in the last remark.

The plan of the paper is as follows. We recall in section 3 some properties of the Laguerre series and we prove the convergence of the Laguerre series in \( S(\mathbb{R}_+^d) \) and the continuous linear mappings of \((S(\mathbb{R}_+^d))_y\) into \((S'(\mathbb{R}_+^m))_x\). In section 4 we state the Schwartz's kernel theorem for \( S(\mathbb{R}_+^d) \) and we prove the extension theorem of Whitney type for \( S(\mathbb{R}_+^d) \).

2. Notation

We use the standard multi-index notation. Given \( \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d \), we write \( |\alpha| = \sum_{i=1}^d \alpha_i \), \( x^\alpha = (x_1, ..., x_d)^{\alpha_1, ..., \alpha_d} = \prod_{i=1}^d x_i^{\alpha_i} \), \( D^\alpha = \prod_{i=1}^d \frac{\partial^\alpha}{\partial x_i^{\alpha_i}} \) for the partial derivative and \( X^\alpha f(x) = x^\alpha f(x) \) for the multiplication operator. For \( x \in \mathbb{R}^d \), \(|x| \) stands for the standard Euclidean norm in \( \mathbb{R}^d \).

Let \( s \) be the space of rapidly decreasing sequences

\[
\{a_n\}_{n \in \mathbb{N}_0^d} \in s \iff \sum_{n \in \mathbb{N}_0^d} |a_n|^2 n^{2k} < \infty, \quad \forall k \in \mathbb{N}.
\]

Then \( s' \) stands for the strong dual of \( s \), the space of slowly increasing sequences

\[
\{a_n\}_{n \in \mathbb{N}_0^d} \in s' \iff \sum_{n \in \mathbb{N}_0^d} |a_n|^2 n^{-2k} < \infty, \quad \exists k \in \mathbb{N}.
\]

3. Laguerre series

The \( d \)-dimensional Laguerre functions

\[
\mathcal{L}_n(x) = \mathcal{L}_{n_1}(x_1) \cdots \mathcal{L}_{n_d}(x_d) = \prod_{i=1}^d \mathcal{L}_{n_i}(x_i)
\]

form an orthonormal basis for \( L^2(\mathbb{R}^d_+) \) and are the eigenfunctions of the Laguerre operator \( E = (D_1(x_1D_1) - \frac{x_1}{4}) \cdots (D_d(x_dD_d) - \frac{x_d}{4}) \), \( E : S(\mathbb{R}_+^d) \to S(\mathbb{R}_+^d) \)

\[
\mathcal{L}_n(x) \to E(\mathcal{L}_n(x)) = \prod_{i=1}^d -n_i + \frac{1}{2} \mathcal{L}_n(x).
\]

Notice that \( E \) is a self-adjoint operator, i.e.

\[
\langle Ef, g \rangle = \langle f, Eg \rangle, \quad f, g \in \text{dom}(E) = \{ f \in L^2(\mathbb{R}_+^d); \, Ef \in L^2(\mathbb{R}_+^d) \}.
\]

For \( f \in S(\mathbb{R}_+^d) \) we define the \( n \)-th Laguerre coefficient by \( a_n = \int_{\mathbb{R}_+^d} f(x) \mathcal{L}_n(x) dx \). The Laguerre series of the function \( f \in S(\mathbb{R}_+^d) \) is \( \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n(x) \).

In [4], p. 547 the following bound on the one-dimensional Laguerre functions is obtained:
\[ |x^k \left(\frac{d}{dx}\right)^p \mathcal{L}_n(x)| \leq C_{p,k}(n+1)^{p+k}, \ x \geq 0, \ n, p, k \geq 0. \]

Finding the bound on the \( d \)-dimensional Laguerre functions involves not complicated calculation. Hence:

\[ |x^k D^p \mathcal{L}_n(x)| \leq C_{p,k} \prod_{i=1}^{d} (n_i + 1)^{p_i+k_i}, \ x \in \mathbb{R}_+^d, \ n, p, k \in \mathbb{N}_0^d. \quad (1) \]

### 3.1. Convergence of the Laguerre series in \( \mathcal{S}(\mathbb{R}_+^d) \).

**Theorem 3.1.** For \( f \in \mathcal{S}(\mathbb{R}_+^d) \) let \( a_n(f) = \int_{\mathbb{R}_+^d} f(x) \mathcal{L}_n(x) \, dx \). Then \( f = \sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{L}_n \) and the series converges absolutely in \( \mathcal{S}(\mathbb{R}_+^d) \). Moreover the mapping \( \iota : \mathcal{S}(\mathbb{R}_+^d) \to \mathcal{S} \), \( \iota(f) = \{a_n(f)\}_{n \in \mathbb{N}_0^d} \) is a topological isomorphism.

**Proof.** For \( f \in \mathcal{S}(\mathbb{R}_+^d) \)

\[ a_n(Ef) = \langle Ef, \mathcal{L}_n \rangle = \langle f, E(\mathcal{L}_n) \rangle = a_n(f)(-1)^d \prod_{i=1}^{d} \left(n_i + \frac{1}{2}\right). \]

Moreover,

\[ a_n(E^p f) = a_n(f) \prod_{i=1}^{d} (-1)^{p_i}(n_i + \frac{1}{2})^{p_i} \]

for any \( p \in \mathbb{N}_0^d \). As \( E^p f \in \mathcal{S}(\mathbb{R}_+^d) \subset L^2(\mathbb{R}_+^d) \), we have

\[ \sum_{n \in \mathbb{N}_0^d} |a_n(f)|^2 \prod_{i=1}^{d} \left(n_i + \frac{1}{2}\right)^{2p_i} < \infty, \text{ for every } p \in \mathbb{N}_0^d, \]

i.e. \( \{a_n(f)\}_{n \in \mathbb{N}_0^d} \in s. \) Clearly \( f = \sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{L}_n \) as elements of \( L^2(\mathbb{R}_+^d) \). By (1), we obtain

\[ \sum_{n \in \mathbb{N}_0^d} |x^k D^p(a_n(f) \mathcal{L}_n(x))| \leq C_{p,k} \sum_{n \in \mathbb{N}_0^d} |a_n(f)| \prod_{i=1}^{d} (n_i + 1)^{p_i+k_i} < \infty \quad (2) \]

which yields the absolute convergence of the series in \( \mathcal{S}(\mathbb{R}_+^d) \).

To prove that \( \iota \) is topological isomorphism, first observe that by the above consideration it is well defined and it is clearly an injection. Let \( \{a_n\}_{n \in \mathbb{N}_0^d} \in s. \) Define \( f = \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n \in L^2(\mathbb{R}_+^d). \) Now (2) proves that this series converges in \( \mathcal{S}(\mathbb{R}_+^d) \), hence \( f \in \mathcal{S}(\mathbb{R}_+^d). \) Thus \( \iota \) is bijective. Observe that, (2) proves that \( \iota^{-1} \) is continuous. Since \( \mathcal{S}(\mathbb{R}_+^d) \) and \( s \) are \( (F) \)-spaces, the open mapping theorem proves that \( \iota \) is topological isomorphism. \[ \square \]
3.2. Convergence of the Laguerre series in $S'(\mathbb{R}^d_+)$.

**Theorem 3.2.** For $T \in S'(\mathbb{R}^d_+)$, let $b_n(T) = \langle T, \mathcal{L}_n \rangle$. Then $T = \sum_{n \in \mathbb{N}_0} b_n(T) \mathcal{L}_n$ and \{b_n(T)\}_{n \in \mathbb{N}_0} \in s'$. The series converges absolutely in $S'(\mathbb{R}^d_+)$. Conversely, if \{b_n\}_{n \in \mathbb{N}_0} \in s'$, then there exists $T \in S'(\mathbb{R}^d_+)$ such that $T = \sum_{n \in \mathbb{N}_0} b_n \mathcal{L}_n$. As a consequence, $S'(\mathbb{R}^d_+)$ is topologically isomorphic to $s'$.

**Proof.** Let \{b_n\}_{n \in \mathbb{N}_0} \in s'$. There exists $k \in \mathbb{N}$ such that $\sum_{n \in \mathbb{N}_0} |b_n|^2(|n| + 1)^{-2k} < \infty$. For a bounded subset $B$ of $S(\mathbb{R}^d_+)$, Theorem 3.1 implies that there exists $C > 0$ such that
\[
\sum_{n \in \mathbb{N}_0^d} |a_n(f)|^2(|n| + 1)^{2k} \leq C, \forall f \in B,
\]
where we denote \{a_n(f)\}_{n \in \mathbb{N}_0} = \iota(f)$. Observe that for arbitrary $q \in \mathbb{N}$ we have
\[
\sum_{|n| \leq q} \sup_{f \in B} |\langle b_n \mathcal{L}_n, f \rangle| \leq \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} |\langle b_n \mathcal{L}_n, a_m(f) \mathcal{L}_m \rangle| = \sup_{f \in B} \sum_{n \in \mathbb{N}_0} |b_n| |a_n(f)| \leq C',
\]
i.e.
\[
\sum_{n \in \mathbb{N}_0} \sup_{f \in B} |\langle b_n \mathcal{L}_n, f \rangle| < \infty,
\]
hence $\sum_{n \in \mathbb{N}_0} b_n \mathcal{L}_n$ converges absolutely in $S'(\mathbb{R}^d_+)$. Let $T \in S'(\mathbb{R}^d_+)$. Theorem 3.1 implies that $'\iota : s' \to S'(\mathbb{R}^d_+)$ is an isomorphism (‘$\iota$ denotes the transpose of $\iota$). Now, one easily verifies that $(\iota)^{-1}T = \{b_n\}_{n \in \mathbb{N}_0^d}$, where $b_n(T) = \langle T, \mathcal{L}_n \rangle$. Observe that for $f \in S(\mathbb{R}^d_+)$
\[
\langle T, f \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) \langle T, \mathcal{L}_n \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) b_n(T) = \left\langle \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n, f \right\rangle,
\]
i.e. $T = \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n$. \hfill $\square$

4. Kernel theorem

The completions of the tensor product are denoted by $\hat{\otimes}_\epsilon$ and $\hat{\otimes}_\pi$ with respect to $\epsilon$ and $\pi$ topologies. If they are equal we drop the subindex.

**Proposition 4.1.** The spaces $S(\mathbb{R}^d_+)$ and $S'(\mathbb{R}^d_+)$ are nuclear.

**Proof.** Since $s$ is nuclear Theorem 3.1 implies that $S(\mathbb{R}^d_+)$ is also nuclear. Now $S'(\mathbb{R}^d_+)$ is nuclear as the strong dual of a nuclear $(F)$-space. \hfill $\square$

**Theorem 4.2.** The following canonical isomorphisms hold:
\[
S(\mathbb{R}^m_+) \hat{\otimes} S(\mathbb{R}^n_+) \cong S(\mathbb{R}^{m+n}_+), \quad S'(\mathbb{R}^m_+) \hat{\otimes} S'(\mathbb{R}^n_+) \cong S'(\mathbb{R}^{m+n}_+).
\]
implies that the underlying spaces are \((\mathbb{R}^m_+) \otimes \mathbb{R}^n_+\). Clearly \(\mathbb{R}^m_+\) and \(\mathbb{R}^n_+\) are \((F)\)-spaces. By the open mapping theorem, the restriction mapping \(f \mapsto f \otimes g\) of \(\mathbb{R}^m_+ \times \mathbb{R}^n_+\) into \(\mathbb{R}^m+n_+\) is separately continuous it follows that it is continuous \((\mathbb{S}(\mathbb{R}^m_+))\) and \(\mathbb{S}(\mathbb{R}^+)^n_+\) are \((F)\)-spaces. The continuity of this bilinear mapping proves that the inclusion \(\mathbb{S}(\mathbb{R}^m_+) \otimes \mathbb{S}(\mathbb{R}^n_+) \rightarrow \mathbb{S}(\mathbb{R}^m+n_+)\) is continuous, hence the topology \(\pi\) is stronger than the induced one from \(\mathbb{S}(\mathbb{R}^m+n_+)\) onto \(\mathbb{S}(\mathbb{R}^m_+) \otimes \mathbb{S}(\mathbb{R}^n_+)\).

Step 2: Let \(A'\) and \(B'\) be equicontinuous subsets of \(\mathbb{S}'(\mathbb{R}^m_+)\) and \(\mathbb{S}'(\mathbb{R}^n_+)\), respectively. There exist \(C > 0\) and \(j, l \in \mathbb{N}\) such that such that

\[
\sup_{T \in A'} |\langle T, \varphi \rangle| \leq C \|\varphi\|_{j,l} \quad \text{and} \quad \sup_{F \in B'} |\langle F, \psi \rangle| \leq C \|\psi\|_{j,l},
\]

where

\[
\|f\|_{j,l} = \sup_{|k| \leq j, \chi(x,y) \in \mathbb{R}^m_+} \sup_{|p| \leq l} |x^k y^l f(x)| < \infty.
\]

(3)

For all \(T \in A'\) and \(F \in B'\) we have

\[
|\langle T_x \otimes F_y, \chi(x,y) \rangle| = |\langle F_y, \langle T_x, \chi(x,y) \rangle \rangle| \leq C \sup_{|k| \leq j, \chi(x,y) \in \mathbb{R}^m_+} \sup_{|p| \leq l} |y^k \langle T_x, D_y^p \chi(x,y) \rangle| \leq C^2 \sup_{|k| \leq j} \sup_{|p| \leq l} |x^k y^l D_y^p \chi(x,y)|
\]

\[
\leq C^2 \|\chi(x,y)\|(k',k) \langle p',p \rangle, \forall \chi \in \mathbb{S}(\mathbb{R}^m_+) \otimes \mathbb{S}(\mathbb{R}^n_+).
\]

It follows that the \(\epsilon\) topology on \(\mathbb{S}(\mathbb{R}^m) \otimes \mathbb{S}(\mathbb{R}^n)_+\) is weaker than the induced one from \(\mathbb{S}(\mathbb{R}^m+n_+)\).

As a consequence of this theorem we have the following important

**Theorem 4.3.** The restriction mapping \(f \mapsto f|_{\mathbb{R}^d_+}\), \(\mathbb{S}(\mathbb{R}^d) \rightarrow \mathbb{S}(\mathbb{R}^d_+)\) is a topological homomorphism onto.

The space \(\mathbb{S}(\mathbb{R}^d_+)\) is topologically isomorphic to the quotient space \(\mathbb{S}(\mathbb{R}^d)/N\), where \(N = \{f \in \mathbb{S}(\mathbb{R}^d) | \sup_{\mathbb{R}^d} f = \mathbb{R}^d \setminus \mathbb{R}^d_+\}\). Consequently, \(\mathbb{S}(\mathbb{R}^d_+)\) can be identified with the closed subspace of \(\mathbb{S}(\mathbb{R}^d)\) which consists of all tempered distributions with support in \(\mathbb{R}^d_+\).

**Proof.** Obviously, the restriction mapping \(f \mapsto f|_{\mathbb{R}^d_+}\), \(\mathbb{S}(\mathbb{R}^d) \rightarrow \mathbb{S}(\mathbb{R}^d_+)\) is continuous. We prove its surjectivity by induction on \(d\). For clarity, denote the \(d\)-dimensional restriction by \(R_d\). For \(d = 1\), the surjectivity of \(R_1\) is proved in [11, p. 168]. Assume that \(R_d\) is surjective. By the open mapping theorem, \(R_d\) and \(R_1\) are topological homomorphisms onto since all the underlying spaces are \((F)\)-spaces. By the above theorem \(R_d \otimes \mathbb{R}^d_+\) is continuous mapping from \(\mathbb{S}(\mathbb{R}^d)\) to \(\mathbb{S}(\mathbb{R}^d_+)\) \((\mathbb{S}(\mathbb{R}^d) \otimes \mathbb{S}(\mathbb{R}) \cong \mathbb{S}(\mathbb{R}^d_+)\) by the Schwartz kernel theorem). Clearly \(R_d \otimes \mathbb{R}^d_+ R_1 = R_{d+1}\). As \(\mathbb{S}(\mathbb{R}^d_+)\) and \(\mathbb{S}(\mathbb{R}^d_+)\) are \((F)\)-spaces \([3\text{, Theorem 7, p. 189}]\) implies that \(R_{d+1}\) is also surjective.
The surjectivity of the restriction mapping together with the open mapping theorem implies that it is homomorphism. Clearly $N$ is closed subspace of $\mathcal{S}(\mathbb{R}^d)$ and $\ker R_d = N$. Thus $R_d$ induces natural topological isomorphism between $\mathcal{S}(\mathbb{R}^d)/N$ and $\mathcal{S}(\mathbb{R}^d_+)$. Hence $(\mathcal{S}(\mathbb{R}^d)/N)'$ is topologically isomorphic to $\mathcal{S}'(\mathbb{R}^d_+)$ (the index $b$ stands for the strong dual topology). Since $\mathcal{S}(\mathbb{R}^d)$ is an $(FS)$-space, [5, Theorem A.6.5, p. 255] implies that $(\mathcal{S}(\mathbb{R}^d)/N)'_b$ is topologically isomorphic to the closed subspace 

$$N^\perp = \{ T \in \mathcal{S}'(\mathbb{R}^d) | \langle T, f \rangle = 0, \forall f \in N \}$$

of $\mathcal{S}'(\mathbb{R}^d)$ which is exactly the subspace of all tempered distributions with support in $\mathbb{R}^d_+$.

Given $f, g \in \mathcal{S}'(\mathbb{R}^d_+)$, Theorem 4.3 implies that we can consider them as elements of $\mathcal{S}'(\mathbb{R}^d)$ with support in $\mathbb{R}^d_+$. Now, one easily verifies that for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(f(x) \otimes g(y))\varphi(x + y) \in \mathcal{D}_{L,1}(\mathbb{R}^{2d}),$$

hence the $\mathcal{S}'$-convolution of $f$ and $g$ exists (see [9, p. 26]). Also, if $\supp \varphi \cap \mathbb{R}^d_+ = \emptyset$, then $(f(x) \otimes g(y))\varphi(x + y) = 0$, hence $\supp f \ast g \subseteq \mathbb{R}^d_+$, i.e. $f \ast g \in \mathcal{S}'(\mathbb{R}^d_+)$. Thus

$$\langle f \ast g, \varphi \rangle = \langle f(x) \otimes g(y), \varphi(x + y) \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

(observe that the function $\varphi^A(x, y) = \varphi(x + y)$ is an element of $\mathcal{S}(\mathbb{R}^{2d}_+))$.

**Remark 4.4.** ([1, Remark 3.7 for $d=1$]) Let us show that $\mathcal{S}'(\mathbb{R}^d_+)$ is a convolution algebra. Given $f, g \in \mathcal{S}'(\mathbb{R}^d_+)$, we compute the $n$-th Laguerre coefficient of $f \ast g$ if $a_n = \langle f, L_n \rangle$ and $b_n = \langle g, L_n \rangle$ then

$$\langle f \ast g, L_n(t) \rangle = \langle f(x) \otimes g(y), L_n(x + y) \rangle.$$  

Now, $L_n^1(x + y) = \sum_{k=0}^n L_{n-k}(x)L_k(y)$ and $L_n(t) = L_n^1(t) - L_{n-1}^1(t)$ (see [2, p. 192]), where $L_n^1(x) = \sum_{k=0}^n \binom{n}{k}(-x)^k/k!$. In order to simplify the proof, we consider the case $d = 2$. Then

\[
\langle f \ast g, L_n(t) \rangle = \langle f(x) \otimes g(y), \prod_{i=1}^2 (L_{n_i}^1(x_i + y_i) - L_{n_i-1}^1(x_i + y_i) ) \rangle \\
= \langle f(x) \otimes g(y), \prod_{i=1}^2 \left( \sum_{k_i=0}^{n_i} L_{n_i-k_i}(x_i) L_k(y_i) - \sum_{k_i=0}^{n_i-1} L_{n_i-k_i-1}(x_i) L_k(y_i) \right) \rangle \\
= \langle f(x) \otimes g(y), \sum_{k \leq (n_1,n_2)} L_{(n_1,n_2)-k}(x) L_k(y) - \sum_{k \leq (n_1-1,n_2)} L_{(n_1-1,n_2)-k}(x) L_k(y) \\
- \sum_{k \leq (n_1,n_2-1)} L_{(n_1,n_2-1)-k}(x) L_k(y) + \sum_{k \leq (n_1-1,n_2-1)} L_{(n_1-1,n_2-1)-k}(x) L_k(y) \rangle 
\]
\[ \begin{align*}
= & \sum_{k \leq (n_1, n_2)} a_{(n_1, n_2) - k} b_k - \sum_{k \leq (n_1 - 1, n_2)} a_{(n_1 - 1, n_2) - k} b_k \\
- & \sum_{k \leq (n_1, n_2 - 1)} a_{(n_1, n_2 - 1) - k} b_k + \sum_{k \leq (n_1 - 1, n_2 - 1)} a_{(n_1 - 1, n_2 - 1) - k} b_k,
\end{align*} \]

where \( a_n \) or \( b_n \) equals zero if some component of the subindex \( n \) is less than zero. It is easy to verify that if \( (a_n)_{n \in \mathbb{N}^2} \in s' \) and \( (b_n)_{n \in \mathbb{N}^2} \in s' \) then \( \langle f \ast g, L_n(t) \rangle \in s' \).

Acknowledgement. The paper was supported by the projects Modelling and harmonic analysis methods and PDEs with singularities, No. 174024 financed by the Ministry of Science, Republic of Serbia.

REFERENCES

[1] A. J. Duran, Laguerre expansions of Tempered Distributions and Generalized Functions, Journal of Mathematical Analysis and Applications 150 (1990), 166-180.
[2] A. Erdelyi, Higher Transcendentals Function, Vol. 2, McGraw-Hill, New York, 1953.
[3] L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis, Springer-Verlag, 1990.
[4] M. Guilleminot-Teissier, Développements des distributions en séries de fonctions orthogonales. Série de Legendre et de Laguerre, Annali della Scuola Normale Superiore di Pisa (3) 25 (1971), 519-573.
[5] M. Morimoto, An introduction to Sato’s hyperfunctions. Vol. 129. American Mathematical Soc., 1993.
[6] G. Köthe, Topological vector spaces II, Vol.II. Springer-Verlag, New York Inc., 1979.
[7] S. Pilipovic, On the Laguerre expansions of generalized functions, C. R. Math. Rep. Acad. Sci. Canada 11 (1989), no. 1, 23-27
[8] R. T. Seeley, Extension of C8 functions defined in a half space, Proc. Amer. Math. Soc. 15 (1964), 625-626.
[9] R. Shiraishi, On the definition of convolutions for distributions, J. Sci. Hiroshima Univ. Ser. A 23 (1959), 19-32.
[10] G. Szego, Orthogonal polynomials, Am. Math. Soc. Colloquium, 1959.
[11] H. Whitney, Analytic extensions of functions defined in closed sets, Transactions of the American Mathematical Society 36 (1934), 63-89.
[12] A. I. Zayed, Laguerre series as boundary values. SIAM J. Math. Anal. 13 (1982), no. 2, 263-279
[13] A. H. Zemanian, Generalized Integral Transformations, Intersci. , New York, 1968.

FACULTY OF FORESTRY, BELGRADE UNIVERSITY, KNEZA VIŠESLAVA 1, BELGRADE, SERBIA, TEL.: +381-11-305398, FAX: +381-11-3053988
E-mail address: smiljana.jaksic@sf.bg.ac.rs

FACULTY OF MECHANICAL ENGINEERING, UNIVERSITY SS. CYRIL AND METHODIUS, KARPOS II BB, 1000 SKOPJE, MACEDONIA
E-mail address: bprangoski@yahoo.com