Higher-rank isomonodromic deformations and $W$-algebras

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Received: 26 February 2018 / Revised: 1 May 2019 / Accepted: 4 September 2019 / Published online: 14 September 2019 © Springer Nature B.V. 2019

Abstract
We construct the general solution of a class of Fuchsian systems of rank $N$ as well as the associated isomonodromic tau functions in terms of semi-degenerate conformal blocks of $W_N$-algebra with central charge $c = N - 1$. The simplest example is given by the tau function of the Fuji–Suzuki–Tsuda system, expressed as a Fourier transform of the 4-point conformal block with respect to intermediate weight. Along the way, we generalize the result of Bowcock and Watts on the minimal set of matrix elements of vertex operators of the $W_N$-algebra for generic central charge and prove several properties of semi-degenerate vertex operators and conformal blocks for $c = N - 1$.

Keywords $W$-algebras · Conformal field theory · Isomonodromic deformations

Mathematics Subject Classification 81R10 · 81T40 · 34M56

1 Introduction

The theory of monodromy preserving deformations has recently gained new insights from its connections to the two-dimensional conformal field theory. The most relevant
for the present work is the solution of the inverse monodromy problem for rank 2 Fuchsian systems and associated isomonodromic tau function with Fourier transforms of \( c = 1 \) Virasoro conformal blocks \([37]\). The simplest instance of this correspondence expresses the tau function of the Painlevé VI equation in terms of 4-point conformal blocks \([30]\); see also \([10]\) for a different proof of this statement.

The last relation has been extended to a number of confluent limits including Painlevé V, IV and III \([31,51]\), as well as to the \( q \)-difference \([11,42]\) and non-commutative \([9]\) setting. On the other hand, the case of Fuchsian systems of higher rank \( N > 2 \) is as yet only superficially explored. A natural guess is that their fundamental solutions and tau functions are related to conformal blocks of \( W_N \)-algebras \([32]\) with integer central charge \( c = N - 1 \) \([30]\). However, trying to make this claim well-defined and constructive one faces a number of obstacles.

The \( W_N = W(\mathfrak{sl}_N) \) algebras appeared in \([24,25,48,61]\) as extensions of the Virasoro algebra including chiral currents with higher spins. One of the problems appearing in their investigation is that, for general \( N \), they do not have known explicit and convenient definition in terms of generators and relations. An additional complication is that \( W_N \)-algebras are not Lie algebras for \( N > 2 \). Even though in the \( W_3 \) case it is possible to proceed in the study of the algebra and its representations (see for example \([14,15]\)), for \( W_N \)-algebras with \( N > 3 \) the direct approach becomes quite involved. The first definition of \( W_N \)-algebras for \( N > 3 \) was given in terms of bosonic fields by means of the quantum Miura transformation \([24,48]\). Subsequently, new approaches appeared, the most productive of them being the quantum Drinfeld–Sokolov reduction \([26]\). It gives rise to the most general definition of \( W \)-algebras associating them with pairs formed by a simple Lie algebra and a nilpotent element therein \([44]\). The recent progress in the representation theory of \( W \)-algebras \([2,21]\) was also made in this framework. The present paper uses a bosonic realization of \( W_N \)-algebras coming from the quantum Miura transformation, but we also need some results from \([2]\).

A more significant problem is that, in contrast to the Virasoro \( (N = 2) \) case, the local \( W \)-invariance does not fix vertex operators uniquely: their descendant matrix elements cannot be reduced to the primary one. This produces an infinite number of free parameters (even in the 3-point conformal blocks!) which have to be determined or constrained by other means such as crossing symmetry. For \( N = 3 \), a characterization of the minimal set of independent matrix elements was given in \([16]\). One may also adopt the point of view that such matrix elements can be fixed arbitrarily. However, the resulting vertex operators may be plagued by divergencies in the multi-point conformal blocks. Even their compositions with the fundamental degenerate vertex operator\(^1\) may give rise to a divergent series. A satisfactory definition of the vertex operator should produce a convergent expansion for the appropriate 4-point conformal block with consistent global analytic (monodromy) properties with respect to the position of the degenerate field.

In the case \( c = N - 1 \), where the CFT/isomonodromy correspondence is expected to hold true, a definition of the general vertex operator for the \( W_N \)-algebra was proposed in \([32]\) by employing the isomonodromic tau functions and the corresponding 3-

\(^1\) Vertex operator corresponding to the highest weight vector of an irreducible representation of \( W_N \)-algebras associated with one of the \( N \)-dimensional fundamental representations of \( \mathfrak{sl}_N \).
point Fuchsian systems. The elements of the basis of vertex operators are labeled in this approach by a finite number of moduli parameterizing the monodromy data [19,33,34,58]. For generic central charge, analogous definition is not available so far, but it is expected to be consistent with an action of the algebra of Verlinde loop operators on the space of 3-point conformal blocks, see recent work [20].

There is a special class of vertex operators of $W_N$-algebras relevant to the construction in this paper. It consists of the so-called semi-degenerate vertex operators corresponding to 3-point conformal blocks with one field having highest weight labeled by $ah_1$, where $a \in \mathbb{C}$ and $h_1$ is the highest weight of the first fundamental representation of $sl_N$. It is expected that arbitrary 3-point conformal blocks involving one semi-degenerate field (primary or its descendant) are uniquely determined by the primary ones as in the Virasoro case (for $N = 3$, this statement [45] can be deduced from the results of [16] quoted above). The multi-point conformal blocks of $W_N$-algebras which appear in the AGT-type relation [1,23,49,60] to Nekrasov instanton partition functions [53] are precisely those obtained by compositions of semi-degenerate vertex operators.

On the differential equations side, the tau function of a Fuchsian system with $n$ regular singular points can be written [18,33] as a Fredholm determinant whose integral kernel is expressed in terms of fundamental solutions of 3-point Fuchsian systems which arise upon decomposition of the $n$-punctured Riemann sphere into pairs of pants. While for $N = 2$ these auxiliary systems can be explicitly solved in terms of Gauss hypergeometric functions, the construction of 3-point solutions (series or integral representations, connection formulas, etc) for $N \geq 3$ is a major open problem.

An important exception is given by the 3-point Fuchsian systems with one of the singular points having special spectral type $(N-1,1)$. Their solutions can be expressed in terms of the Clausen–Thomae hypergeometric functions $N F_{N-1}$. For this reason, the constructions of [18,33] can be made completely explicit for Fuchsian systems with 2 generic singular points and the remaining $n - 2$ singularities of the above special type. In this paper, such systems are called semi-degenerate; they are the closest higher-rank relatives of the $N = 2$ ones. The specialization of the Schlesinger isomonodromic deformation equations to semi-degenerate 4-point Fuchsian system (i.e. higher-rank semi-degenerate analog of Painlevé VI) is known as the Fuji–Suzuki–Tsuda system. It was discovered in [29,55] as a similarity reduction of Drinfeld–Sokolov hierarchies, and related to deformations of Fuchsian equations in [59], where semi-degenerate Fuchsian systems with arbitrary $n \geq 4$ also appear.

The aim of this paper is to extend the results of [37] to higher-rank case by relating fundamental solutions and tau functions of semi-degenerate Fuchsian systems to Fourier transformed semi-degenerate conformal blocks of the $W_N$-algebras with $c = N - 1$. In particular, we claim that, under suitable normalization of vertex operators,

$$
\tau (z) = \sum_{w_1, \ldots, w_{n-3} \in \mathbb{N}} e^{(\beta_1, w_1)+\ldots+(\beta_{n-3}, w_{n-3})} \prod_{n=-2}^{n-3} \frac{z_{n-2} z_{n-3} \sigma_{n-3}}{-\theta_{n-3} + w_{n-3} \theta_{n-3}} \sigma_{n-2} a_{n-3}^{1} \cdots \sigma_{0} a_{1}^{n-1} a_{1}^{a_1} \cdots a_{n-3}^{a_{n-3}} z_{1}^{a_1} z_{2}^{a_2} \cdots z_{n-3}^{a_{n-3}} \sigma_{n-3}^{1} \cdots \sigma_{1}^{a_1} z_{0}^{a_0} \theta_{0}^{0}
$$

(1)
where $\tau (z)$ is the isomonodromic tau function depending on the positions $z = \{z_1, \ldots, z_{n-2}\}$ of the special punctures, the generic punctures are located at $z_0 = 0$ and $z_{n-1} = \infty$, and the trivalent graph on the right denotes appropriate $n$-point conformal block. The parameters $\theta_0, -\theta_{n-1} \in \mathbb{C}^N$, $a_1, \ldots, a_{n-2} \in \mathbb{C}$ assigned to external edges describe local monodromy exponents of the Fuchsian system. The labels $\sigma_1, \ldots, \sigma_{n-3} \in \mathbb{C}^N$ of the internal edges as well as Fourier momenta $\beta_1, \ldots, \beta_{n-3} \in \mathbb{C}^N$ are explicitly related to the remaining moduli of semi-degenerate monodromy; the components of all vectors in $\mathbb{C}^N$ sum up to zero. The summation in (1) is carried over the root lattice $\mathcal{R}$ of $\mathfrak{sl}_N$. Analogous statement for the fundamental solution involves extra degenerate insertions.

In the way of establishing the correspondence, we proved several statements about conformal blocks of the $W_N$-algebras which, to the best of our knowledge, remained so far at the level of folklore for $N > 3$. For arbitrary central charge, we developed a reduction procedure of matrix elements of vertex operators to a minimal set, thereby generalizing the results of [16]. For $c = N - 1$, we proved that the descendant 3-point functions involving semi-degenerate field are uniquely expressed in terms of the 3-point function of primaries. For the fundamental degenerate field, we found restrictions (fusion rules) to be satisfied to allow for non-vanishing 3-point functions. They are then used to prove the well-known hypergeometric formulas [22] for the 4-point conformal blocks with one semi-degenerate and one degenerate field using the rigidity property of the associated Fuchsian system.

We end this introduction by mentioning a few more relevant papers. A survey of recent results in the representation theory of $W$-algebras may be found in [3]. The properties of semi-degenerate $W_N$-conformal blocks describing correlation functions of the Toda CFT and their gauge theory counterparts have been studied, for instance, in [17,36]. An interesting direction which is in a sense close to ours is the construction of integral representation of the 4-point conformal blocks of $W_3$-algebra involving one semi-degenerate field of higher level (whose matrix elements cannot be reduced to primary ones only) and one fundamental degenerate field [5]. This construction is based on the middle convolution from the Katz theory of rigid systems. Connections between Fuchsian systems and $W$-algebras were also studied in [6], with further links to topological recursion suggested in [7].

The paper is organized as follows. In Sect. 2, we introduce semi-degenerate Fuchsian systems and provide an explicit parameterization of their monodromy by suitable coordinates (Proposition 1). Section 3 is devoted to $W_N$-algebras and their representations. The minimal set of matrix elements is described by Theorem 2, after which we proceed to the proof of uniqueness of semi-degenerate vertex operator (Proposition 2) and fusion rules for completely degenerate fields. Section 4 computes the operator-valued monodromy of semi-degenerate conformal blocks with respect to positions of additional degenerate fields. Diagonalizing this monodromy by Fourier transform, in Sect. 5 we obtain the fundamental solution (Theorems 4 and 5) and the tau function (Proposition 7) of the semi-degenerate Fuchsian systems in terms of $W_N$-algebra conformal blocks. Some technical results are relegated to appendices.
2 Semi-degenerate Fuchsian systems

2.1 Generalities

We are interested in the analysis of Fuchsian systems of rank $N$ having $n$ regular singular points $z := \{z_0, \ldots, z_{n-2}, z_{n-1} \equiv \infty\}$ on the Riemann sphere $\mathbb{CP}^1$:

$$\frac{d\Phi(y)}{dy} = \Phi(y) A(y), \quad A(y) = \sum_{k=0}^{n-2} \frac{A_k}{y - z_k}. \quad (2)$$

Here the residues $A_0, \ldots, A_{n-2} \in \text{Mat}_{N \times N}(\mathbb{C})$ and $\Phi(y)$ is the fundamental $N \times N$ matrix solution which may be normalized as $\Phi(y_0) = I$, with $y_0 \in \mathbb{CP}^1 \setminus z$. It will be assumed that the matrices $A_0, \ldots, A_{n-2}$ and $A_{n-1} := -\sum_{k=0}^{n-2} A_k$ are diagonalizable and non-resonant, i.e. the pairwise differences of the eigenvalues of each $A_k$ are not nonzero integers.

The solution $\Phi(y)$ is a multivalued function on $\mathbb{CP}^1 \setminus z$. Its monodromy realizes a representation of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus z, y_0)$ in $GL(N, \mathbb{C})$. This group is generated by the paths $\xi_0, \ldots, \xi_{n-1}$ around the points $z_0, \ldots, z_{n-1}$ on $\mathbb{CP}^1$ indicated in Fig. 1, which satisfy one relation $\xi_0 \cdots \xi_{n-1} = 1$. In what follows, their orientations will be referred to as positive. Denoting by $M_k$ the monodromy of $\Phi(y)$ along the loop $\xi_k$, we similarly have $M_0 \cdots M_{n-1} = I$.

The fundamental matrix $\Phi(y)$ is uniquely fixed by the following properties:

(a) $\Phi(y)$ is holomorphic and invertible on the universal cover of $\mathbb{CP}^1 \setminus z$ and has constant monodromy under analytic continuation.
(b) $\Phi(y)$ satisfies the normalization condition $\Phi(y_0) = I$.
(c) In sufficiently small neighborhoods of $z_k, k = 0, \ldots, n-1$, the behavior of $\Phi(y)$ is

$$\Phi(y \to z_k) = C_k (z_k - y)^{\Theta_k} G_k(y), \quad (3)$$

Fig. 1 Basis of loops $\xi_0, \ldots, \xi_{n-1}$ in $\pi_1(\mathbb{CP}^1 \setminus z, y_0)$
where \( G_k(y) \) is holomorphic and invertible in the vicinity \( z_k \); \( C_k \) is a non-degenerate constant matrix; \( \Theta_k \) is a diagonal matrix conjugate to \( A_k \). (The asymptotics at \( z_{n-1} = \infty \) should be rewritten in terms of a suitable local parameter). Note that \( M_k = C_k e^{2\pi i \Theta_k} C_k^{-1} \).

**Definition 1** The Riemann–Hilbert problem associated with the Fuchsian system (2) is the problem of reconstruction of \( \Phi(y) \) satisfying the conditions (a)–(c) for a given monodromy data: \( M_k \in GL(N, \mathbb{C}), \ k = 0, \ldots, n-1 \), subject to the relation \( M_0 \ldots M_{n-1} = I \).

Instead of normalizing the fundamental solution \( \Phi(y) \) by the condition (b), we could also use another normalization which combines (b) and (c): namely, one may fix the connection matrix \( C_l = I \) at one of the singular points.

Different choices of the normalization point \( y_0 \) lead to an overall conjugation of all monodromies. We identify the corresponding monodromy data and consider the space

\[
\mathcal{M} = Hom \left( \pi_1 \left( \mathbb{C} P^1 \setminus z, y_0 \right), GL(N, \mathbb{C}) \right) / GL(N, \mathbb{C}).
\]

(4)

It is often convenient to work with the slice \( \mathcal{M}_\Theta \subset \mathcal{M} \) corresponding to fixed local monodromy exponents \( \Theta = \{ \Theta_0, \ldots, \Theta_{n-1} \} \).

Besides \( \Phi(y) \), we will be also interested in the isomonodromic tau function \( \tau(z) \) of Jimbo-Miwa-Ueno [41]. It is defined by the following 1-form

\[
dz \log \tau(z) := \frac{1}{2} \sum_{k=0}^{n-1} \text{res}_{y = z_k} \text{Tr} A_2^2(y) \, dz_k,
\]

(5)

which is closed provided the monodromy of (2) is kept constant. The tau function \( \tau(z) \equiv \tau(z | \Theta, m) \) with \( m \in \mathcal{M}_\Theta \) is a generating function of the Hamiltonians which govern the isomonodromic evolution of \( A_0, \ldots, A_{n-2} \) with respect to times \( z \).

### 2.2 Semi-degenerate monodromy

Let \( \mathcal{Y} \) be the set of partitions \( \lambda = [\lambda_1, \ldots, \lambda_\ell], \lambda_1 \geq \ldots \geq \lambda_\ell > 0 \), and \( \mathcal{Y}_k \) be the set of all partitions of \( k \in \mathbb{Z}_{\geq 0} \). One can decompose the space of Fuchsian systems (2) according to their spectral type \( s = (s^{(0)}, \ldots, s^{(n-1)}) \in \mathcal{Y}_N \), where the partition \( s^{(i)} \vdash N \) encodes the multiplicities of the eigenvalues of \( \Theta_i \) or \( A_i \). Thus, for example, \( \ell(s^{(i)}) \) is the number of distinct eigenvalues of \( \Theta_i \) and \( s_{ij}^{(i)} \) is the multiplicity of its most degenerate eigenvalue.

The dimension of the space \( \mathcal{M}_\Theta \) of monodromy data for irreducible systems of spectral type \( s \) coincides with the number of accessory parameters and is known to be given by

\[
\dim \mathcal{M}_\Theta = (n - 2) N^2 + 2 - \sum_{i=0}^{n-1} \sum_{j=1}^{\ell_i} \left( s_{ij}^{(i)} \right)^2.
\]

(6)
Generic Fuchsian systems have spectral type \( s_{\text{gen}} = \left( \left( 1^N \right), \ldots, \left( 1^N \right) \right) \). It then follows from the last formula that

\[
\dim \mathcal{M}_{\Theta, \text{gen}} = 2 \left( n - 3 \right) \left( N - 1 \right) + \left( n - 2 \right) \left( N - 1 \right) \left( N - 2 \right).
\]  

(7)

This expression has a geometric interpretation. The \( n \)-punctured Riemann sphere can be decomposed into \( n - 2 \) pairs of pants (3-punctured spheres) by \( n - 3 \) closed curves. To each of these curves, one may assign \( 2 \left( N - 1 \right) \) monodromy parameters which play the role of Fenchel–Nielsen-type coordinates (lengths and twists) and give the 1st term in (7). The 2nd term comes from \( \left( N - 1 \right) \left( N - 2 \right) \) parameters associated with each pair of pants with fixed conjugacy classes of local monodromies at 3 boundaries. The presence of such parameters is the principal new feature of the higher rank \( N \geq 3 \).

We are interested in the Fuchsian systems with 2 generic \( \left( 1^N \right) \)-punctures at \( z_0 \) and \( z_{n-1} \), and \( n - 2 \) singular points of spectral type \( \left( N - 1, 1 \right) \) at \( z_1, \ldots, z_{n-2} \). The systems of this type will be called semi-degenerate. The dimension of the relevant space of monodromy data is readily computed to be

\[
\dim \mathcal{M}_{\Theta, \text{sd}} = 2 \left( n - 3 \right) \left( N - 1 \right).
\]  

(8)

For \( n = 3 \), this dimension vanishes, meaning that the Fuchsian system with 2 generic punctures and one puncture of type \( \left( N - 1, 1 \right) \) is rigid. The conjugacy class of monodromy is then completely determined by the local exponents \( \Theta \), i.e. the pants carry no internal moduli. For \( n \geq 4 \), there exist decompositions of the \( n \)-punctured sphere into such semi-degenerate pants, which explains the difference between (7) and (8).

Our next task is to provide an explicit parameterization of semi-degenerate monodromy. The construction of solution of the corresponding Riemann–Hilbert problem and the associated isomonodromic tau function constitutes the main goal of the present work.

**Assumption 1** The monodromy matrices \( M_k \in SL \left( N, \mathbb{C} \right) \), \( k = 0, \ldots, n - 1 \) satisfying the cyclic condition \( M_0 \ldots M_{n-1} = \mathbb{I} \) are assumed to be diagonalizable, i.e. \( M_k \sim \exp(2\pi i \Theta_k) \), where \( \Theta_k = \text{diag} \theta_k \) with \( \theta_k = (\theta_k^{(1)}, \ldots, \theta_k^{(N)}) \in \mathbb{C}^N \) are traceless diagonal matrices. For \( k = 1, \ldots, n - 2 \), these matrices are fixed to be

\[
\theta_k = a_k \left( \frac{N-1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N} \right), \quad a_k \in \mathbb{C}.
\]  

(9)

It is further assumed that the products \( M_{[k]} := M_0 \cdots M_k \) with \( k = 0, \ldots, n - 2 \) are also diagonalizable and their eigenvalues \( \text{Spec} M_{[k]} \) are pairwise distinct:

\[
M_{[k]} \sim \exp \left( 2\pi i \mathcal{G}_k \right), \quad \mathcal{G}_k = \text{diag} \sigma_k, \quad \sigma_k = (\sigma_k^{(1)}, \ldots, \sigma_k^{(N)}),
\]  

(10)

where \( \text{Tr} \mathcal{G}_k = 0 \). Note that \( M_{[0]} = M_0, M_{[n-2]} = M_{n-1}^{-1} \), so that we can identify \( \mathcal{G}_0 = \Theta_0, \mathcal{G}_{n-2} = -\Theta_{n-1}, \sigma_0 = \theta_0, \sigma_{n-2} = -\theta_{n-1} \).

For \( n = 3 \), the semi-degenerate monodromy is described by the following result, see e.g. [12].
Lemma 1 (Rigidity Lemma) If $M_A, M_B \in GL(N, \mathbb{C})$ are diagonalizable with non-intersecting sets of eigenvalues $\text{Spec } M_A = \{\alpha_1, \ldots, \alpha_N\}$, $\text{Spec } M_B = \{\beta_1, \ldots, \beta_N\}$ and $M_B^{-1}M_A$ is a reflection (a rank 1 perturbation of the identity matrix), then there exists a unique (up to overall rescaling) basis in which

$$M_A = \begin{pmatrix}
0 & 0 & 0 & \ldots & (-1)^{N+1}e_N(A) \\
1 & 0 & 0 & \ldots & (-1)^N e_{N-1}(A) \\
0 & 1 & 0 & \ldots & (-1)^{N-1}e_{N-2}(A) \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & e_1(A)
\end{pmatrix},$$

$$M_B = \begin{pmatrix}
0 & 0 & 0 & \ldots & (-1)^{N+1}e_N(B) \\
1 & 0 & 0 & \ldots & (-1)^N e_{N-1}(B) \\
0 & 1 & 0 & \ldots & (-1)^{N-1}e_{N-2}(B) \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & e_1(B)
\end{pmatrix},$$

where $e_k(A)$ and $e_k(B)$ denote the $k$th elementary symmetric polynomials in the eigenvalues of $M_A$ and $M_B$, respectively. In this case, $\text{Spec } M_B^{-1}M_A = \{\prod_{k=1}^N \alpha_k \beta_k^{-1}, 1, \ldots, 1\}$.

The matrices $W_A$ and $W_B$ defined by $(W_A)_{kl} = \alpha_k^{l-1}$ and $(W_B)_{kl} = \beta_k^{l-1}$ diagonalize, respectively, $M_A$ and $M_B$:

$$W_A M_A W_A^{-1} = D_A, \quad W_B M_B W_B^{-1} = D_B,$$

where $D_A = \text{diag}(\alpha_1, \ldots, \alpha_N)$ and $D_B = \text{diag}(\beta_1, \ldots, \beta_N)$. The matrix $W_B W_A^{-1}$ is a transition matrix between the eigenbases of $M_A$ and $M_B$. Its matrix elements are

$$(W_B W_A^{-1})_{kl} = \prod_{s(\neq l)} \frac{\beta_k - \alpha_s}{\alpha_l - \alpha_s}. \quad (11)$$

Note that the general form of a matrix which relates a basis where $M_A$ is diagonal to another basis where $M_B$ is diagonal is given by $R_B W_B W_A^{-1} R_A^{-1}$, where $R_A$ and $R_B$ are non-degenerate diagonal matrices.

Now one may use Lemma 1 to parameterize recursively the monodromy matrices $M_k$ of semi-degenerate Fuchsian systems. To this end, observe that it suffices to parameterize instead a related set of matrices $M_{[k]} = M_0 \ldots M_k$ with $k = 0, \ldots, n-2$. Indeed, we have $M_0 = M_{[0]}$, $M_{n-1} = M_{[n-2]}^{-1}$ and $M_k = M_{[k-1]}^{-1}M_{[k]}$ for $k = 1, \ldots, n-2$.

Proposition 1 Let $M_k \in SL(N, \mathbb{C})$, $k = 0, \ldots, n-1$ be the monodromy matrices of a semi-degenerate Fuchsian system satisfying genericity conditions of Assumption 1.
They can be parameterized as follows:

\[ M_{[k]} = W_{[k]}^{-1} \exp (2\pi i \Theta_k) W_{[k]}, \]  
\[ W_{[k]} = R_k W_{k+1} R_{k+1} \ldots R_{n-3} W_{n-2} R_{n-2}, \]

where \( R_k = \text{diag} (r_k) \) are diagonal matrices from \( SL(N, \mathbb{C}) \) and \( \Theta_k \) are parameters.

Proof The idea is to use Lemma 1 successively for the pairs of matrices

\[ M_A = e^{2\pi i a_n N} M_{[k]}, \quad M_B = M_{[k-1]}, \quad k = n - 2, \ldots, 1, \]

where the factor \( e^{2\pi i a_n N} \) ensures that the eigenvalue of \( M_B^{-1} M_A = e^{2\pi i a_n N} M_k \) with multiplicity \( N - 1 \) is equal to 1. We start the parameterization from the pair

\[ M_A = e^{2\pi i a_{n-2}/N} M_{[n-2]}, \quad M_B = M_{[n-3]}, \]

assuming that \( M_A \) is diagonal: \( M_A = e^{2\pi i a_{n-2}/N} \exp (2\pi i \Theta_{n-2}) \), where \( \Theta_{n-2} = -\Theta_{n-1} \). Then

\[ M_B = R_{n-2}^{-1} W_{n-2}^{-1} R_{n-3}^{-1} \exp (2\pi i \Theta_{n-3}) R_{n-3} W_{n-2} R_{n-2}, \]

where, as follows from (11), \( W_{n-2} \) is given by (14) and \( R_k = \text{diag} (r_k) \) are arbitrary diagonal matrices from \( SL(N, \mathbb{C}) \). Continuing the recursive procedure, we get the parameterization (12) for all \( M_{[k]} \). The matrix \( W_{[k]} \) defined by (13) is a transition matrix between the eigenbases of \( M_{[n-2]} = M_{[n-1]}^{-1} \) and \( M_{[k]} \), which is obtained as a composition of elementary transition matrices at each step. \( \square \)

Observe that the diagonal matrix \( R_k \) cancels out in (12); however, we keep it for later use. Since \( R_{n-2} \) only produces an overall conjugation of all monodromies, it does not enter into the parameterization of \( \mathcal{M}_{\Theta, s, \delta} \). Thus the semi-degenerate monodromy is parameterized by \( \sigma_k, r_k \) with \( k = 1, \ldots, n - 3 \), which of course agrees with the dimension (8).

2.3 Three-point case

This subsection gives an explicit solution of the semi-degenerate Fuchsian system (2) with 3 singular points \( z_0 = 0, z_1 = 1, z_2 = \infty \), and the connection \( A (y) \) having traceless residues \( A_0, A_1, A_\infty = -A_0 - A_1 \) at these poles. We suppose that \( A_0, A_1, A_\infty \) are diagonalizable to \( \Theta_0, \Theta_1, \Theta_\infty \). Moreover, it is convenient to choose the gauge so that \( A_\infty \) is diagonal. Thus

\[ A_0 = G_0^{-1} \Theta_0 G_0, \quad A_1 = G_1^{-1} \Theta_1 G_1, \quad A_\infty = \Theta_\infty, \]

where \( G_0, G_1 \) are diagonalizable to \( \Theta_0, \Theta_1 \).
where $G_0, G_1 \in GL(N, \mathbb{C})$ and
\[
\begin{align*}
\Theta_0 &= \text{diag}\left(\theta_0^{(1)}, \ldots, \theta_0^{(N)}\right), \quad \Theta_\infty = \text{diag}\left(\theta_\infty^{(1)}, \ldots, \theta_\infty^{(N)}\right), \quad \text{Tr } \Theta_0 = \text{Tr } \Theta_\infty = 0, \\
\Theta_1 &= a \cdot \text{diag}\left(\frac{N-1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N}\right).
\end{align*}
\]
(16)

Recall that while $\Theta_1$ has an eigenvalue of multiplicity $N - 1$, all eigenvalues of $\Theta_0$ and $\Theta_\infty$ are distinct. Such data correspond to a rigid local system and the matrix elements of $A_1$ can be derived from an additive variant of Lemma 1:
\[
(A_1)_{jm} = -\frac{r_j}{r_m} \cdot \frac{\prod_{k}(\theta_\infty^{(j)} - a/N + \theta_0^{(k)})}{\prod_{k(\neq m)}(\theta_\infty^{(m)} - \theta_\infty^{(k)})} - \delta_{jm} \frac{a}{N},
\]
(17)

where $r_1, \ldots, r_N$ are arbitrary nonzero parameters. They appear due to the possibility of overall conjugation of $A_0, A_1, A_\infty$ by the diagonal matrix $R = \text{diag}(r_1, \ldots, r_N)$, preserving the diagonal form of $A_\infty$.

**Theorem 1** The solution of the Fuchsian system
\[
\frac{d\Phi(y)}{dy} = \Phi(y) A(y), \quad A(y) = \frac{A_0}{y} + \frac{A_1}{y - 1},
\]
with $A_1$ fixed by (17) and $A_0 = -A_1 - \Theta_\infty$, which has the asymptotics $\Phi(y) = y^{-\Theta_\infty} (1 + O\left(y^{-1}\right))$ as $y \to \infty$ is
\[
\begin{align*}
\Phi_{jm}(y) &= N_{jm} y^{-\theta_\infty^{(j)} - 1 + \delta_{jm}} \left(1 - \frac{1}{y}\right)^{-a/N} \\
&\quad \times _{N F_{N-1}} \left(\begin{array}{c}
1 - \delta_{jm} - a/N + \theta_0^{(k)} + \theta_\infty^{(j)} \\
1 + \theta_\infty^{(j)} - \theta_\infty^{(k)} + \delta_{mk} - \delta_{jm}
\end{array}\right)_{k=1,N; k \neq j} \left(1\right)_{y},
\end{align*}
\]
(19)

where
\[
N_{jm} = \begin{cases} 
\frac{(A_1)_{jm}}{\theta_\infty^{(m)} - \theta_\infty^{(j)} - 1}, & j \neq m, \\
1, & j = m,
\end{cases}
\]

and $_{N F_{N-1}} (\ldots | x)$ denotes the generalized hypergeometric function.

Let us comment on the computation of the coefficients $N_{jm}$. They can be derived from the expansion of $\Phi(y)$ near infinity. Indeed, as $y \to \infty$, one has
\[
A(y) = -\frac{A_\infty}{y} + \frac{A_1}{y^2} + O\left(y^{-3}\right).
\]

\footnote{One can derive the additive variant of Lemma 1 from the original multiplicative one by taking the limit $M_\nu = I + \epsilon A_\nu$ for $\epsilon \to 0$.}
The solution $\Phi(y)$ can be found iteratively using the expansion

$$\Phi(y) = y^{-A_\infty} \left( \mathbb{I} + H_1 y^{-1} + H_2 y^{-2} + O(y^{-3}) \right).$$  \hspace{1cm} (20)

Substituting this expression into the Fuchsian system yields $H_1 = [H_1, A_\infty] - A_1$. Under the non-resonance condition for $A_\infty$, it follows that

$$(H_1)_{jm} = \frac{(A_1)_{jm}}{\theta^{(m)}_\infty - \theta^{(j)}_\infty - 1}.$$ 

Comparing (19) and (20) as $y \to \infty$, one finds that the coefficients of the off-diagonal leading terms and diagonal next-to-leading terms are given by the matrix elements of $H_1$. In particular, $N_{jm} = (H_1)_{jm}$ for $j \neq m$.

### 3 $W_N$-algebras and their representations

It is important for us that $W_N$-algebras with central charge $c = N - 1$ have a bosonic realization related to the bosonic realization of $\hat{\mathfrak{sl}}_N$ at level 1. Since this value of $c$ is exactly what we need for applications to isomonodromy, the bosonic realization of $W_N$-algebras will be used as their basic definition. Thanks to the boson-fermion correspondence, in this case there also exists a fermionic realization of $W_N$-algebras. In this section, we will use the results of [2], where the definition of $W$-algebras is given in the quantum Drinfeld–Sokolov formalism. The equivalence of the latter definition with the definition in terms of bosonic fields can be found in [3, Sections 5.8, 5.9], [4].

#### 3.1 Definition of $W_N$-algebras with $c = N - 1$

We are going to define the $W_N$-algebras, $W_N = W(\mathfrak{sl}_N)$, with $c = N - 1$, as abstract operator algebras starting from their realizations in terms of $N$ free bosonic fields $\phi_k(z)$, $k = 1, \ldots, N$, subject to one relation $\sum_{k=1}^{N} \phi_k(z) = 0$. The currents $J_k(z) = i \partial \phi_k(z)$ have the operator product expansions (OPE) of the form

$$J_k(z) J_l(z') = \frac{\delta_{kl} - \frac{1}{N}}{(z - z')^2} + \text{regular}. \hspace{1cm} (21)$$

The modes $a_p^{(k)}$ of the currents $J_k(z)$ are defined by

$$J_k(z) = \sum_{p \in \mathbb{Z}} \frac{a_p^{(k)}}{z^{p+1}}.$$

$$

\begin{align*}
\text{Springer}
\end{align*}$$
These currents define $W_N$-algebra currents $W^{(2)} (z), \ldots, W^{(N)} (z)$ as sums of normal-ordered monomials:

$$W^{(j)} (z) = \sum_{1 \leq i_1 < \cdots < i_j \leq N} : J_{i_1} (z) \cdots J_{i_j} (z) : , \quad j = 2, \ldots, N. \quad (23)$$

For example,

$$W^{(2)} (z) = \sum_{k < l} : J_k (z) J_l (z) : = -\frac{1}{2} \sum_{k=1}^N : J_k (z)^2 : = -T (z),$$

where $T (z)$ is the holomorphic component of the energy-momentum tensor. The OPEs of currents (23) between themselves can be rewritten in terms of products of currents from the same set and their derivatives giving an abstract definition of $W_N$-algebras. From the OPE of the energy-momentum tensor $T (z)$ with itself we get the central charge $c = N - 1$. The modes $W_p^{(j)}$ of the currents $W^{(j)} (z)$ are defined by

$$W^{(j)} (z) = \sum_{p \in \mathbb{Z}} \left. W^{(j)}_p \right|_{z^p + j} .$$

We will not need the explicit form of all OPEs of the currents $W^{(j)} (z)$ and the explicit formulas for the commutation relations of their modes $W_p^{(j)}$. However, it will be important for us that the commutation relations respect two structures on the $W_N$-algebra: $\mathbb{Z}$-gradation with respect to the adjoint action of $L_0 = -W^{(2)}_0$

$$\deg_{L_0} W_p^{(j)} = -p, \quad (24)$$

and quasi-commutativity with respect to the $W$-filtration [2] defined by

$$\deg_W W_p^{(j)} = j - 1. \quad (25)$$

Namely, we will need the relations

$$\deg_{L_0} [W_p^{(j_1)}, W_p^{(j_2)}] = -(p_1 + p_2), \quad (26)$$

$$\deg_W [W_p^{(j_1)}, W_p^{(j_2)}] < (j_1 - 1) + (j_2 - 1). \quad (27)$$

The latter inequality means that the modes of $W_N$-algebra currents commute up to elements of smaller degree with respect to $\deg_W$ (quasi-commutativity which is commutativity of the corresponding graded algebra).
3.2 Vertex operators

Given $\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{C}^N$ with $\sum_{k=1}^N \theta_k = 0$, one may introduce the exponential vertex operator

$$V_\theta(z) = : e^{i(\theta, \phi(z))} :$$

It has the following OPEs with the currents $J_k (z)$:

$$J_k (z) V_\theta(z') = \frac{\theta_k V_\theta(z')}{z - z'} + \text{regular},$$

which in turn imply that

$$W^{(j)} (z) V_\theta(z') = \sum_{k=0}^{\infty} \frac{\left( W^{(j)} - k V_\theta \right)(z')}{(z' - z)^{j-k}} , \quad (28)$$

where

$$(W^{(j)} V_\theta)(z) = e_j(\theta) V_\theta(z) . \quad (29)$$

and $e_j(\theta)$ is the $j$th elementary symmetric polynomial in the variables $\theta$. The relations (28), (29) define the primary field $V_\theta(z)$ of the $W_N$-algebra. Due to the state-field correspondence, to every such primary field we can associate the highest weight vector $|\theta\rangle$ of a Verma module $M_\theta$ of the $W_N$-algebra.

The Verma module $M_\theta$ is induced from the one-dimensional module with the basis element $|\theta\rangle$ of the subalgebra of $W_N$-algebra generated by $\{W^{(j)}\}$:

$$W^{(j)}_0 |\theta\rangle = e_j(\theta) |\theta\rangle , \quad W^{(j)}_{k>0} |\theta\rangle = 0 . \quad (30)$$

The $W_N$-algebra admits a Poincaré–Birkhoff–Witt basis [2, Proposition 4.12.1 and Section 5.1]. This means that there is a linear basis in the Verma module $M_\theta$ consisting of the elements $W_\lambda |\theta\rangle$, where $\lambda = (\lambda^{(2)}, \ldots, \lambda^{(N)}) \in \mathbb{Z}^{N-1}$ is an $(N-1)$-tuple of partitions and

$$W_\lambda |\theta\rangle = W^{(N)}_{-\lambda^{(N)}} \cdots W^{(2)}_{-\lambda^{(2)}} |\theta\rangle , \quad W^{(s)}_{-\lambda^{(s)}} := W^{(s)}_{-\lambda^{(s)}_1} \cdots W^{(s)}_{-\lambda^{(s)}_{\ell_s}} . \quad (31)$$

Similarly one can define the action of $W_N$-algebra on the space of fields which are local with respect to the currents $W^{(j)} (z)$ (but not necessary pairwise local) using operator product expansion (see, for example, the definition in [27, Section 3.3])

$$W^{(j)} (z) V(z') = \sum_{k=-\infty}^{\infty} \frac{(W^{(j)} - k V)(z')}{(z' - z)^{j-k}} . \quad (32)$$
This action defines the structure of Verma modules (or their quotients) with highest weight vectors being primaries \( V_\theta (z) \). We are interested in 3-point functions, the matrix elements of descendents of the vertex operator \( V_\theta (z) \):

\[
\langle \theta'' | W_{\lambda''}^\dagger (W_\lambda V_\theta) (z) W_{\lambda'} | \theta' \rangle, \quad \lambda, \lambda', \lambda'' \in \mathbb{Y}^{N-1}, \tag{33}
\]

where \( \dagger \) is an anti-linear involutive anti-automorphism of the \( W_N \)-algebra uniquely defined by \( (W_{k}^{(j)})^\dagger = W_{-k}^{(j)} \), and \( \langle \theta | \rangle \) satisfies the conditions analogous to (30):

\[
\langle \theta | W_0^{(j)} = \langle \theta | e_j (\theta) \rangle, \quad \langle \theta | W_{k<0}^{(j)} = 0. \tag{34}
\]

It is well known that in the case of the Virasoro algebra (i.e. for \( N = 2 \)) thanks to the Ward identities all matrix elements (33) can be expressed in terms of the matrix element of \( V_\theta (z) \) between the highest weight vectors, \( \langle \theta'' | V_\theta (z) | \theta' \rangle \) (3-point function of primaries). In order to simplify these matrix elements as much as possible in the \( N \geq 3 \) case, we will use the identities [13,27,43]

\[
[W_p^{(j)}, V (z)] = \sum_{k=1-j}^{\infty} z^{p-k} \binom{p+j-1}{k+j-1} (W_k^{(j)} V) (z), \tag{35}
\]

\[
(W_p^{(j)} V) (z) = \sum_{k=0}^{\infty} (-z)^k \binom{j+p-1}{k} W_{p-k}^{(j)} V (z) - \sum_{k=0}^{\infty} (-z)^{p+j-k-1} \binom{j+p-1}{k} V (z) W_{1-j+k}^{(j)}, \tag{36}
\]

valid for any descendant \( V (z) \) of the primary vertex operator \( V_\theta (z) \) and any \( p \in \mathbb{Z} \) (and, in fact, for any central charge \( c \)).

The following theorem naturally generalizes the corresponding result for the \( W_3 \)-algebra [16,45] and is valid for any \( c \). (Although the \( W_N \) algebra was defined above only for \( c = N - 1 \), the argument only uses the fact that the algebra is generated by \( N - 1 \) currents \( W^{(j)} (z) \), the existence of the PBW basis and the definition (28), (29) of primary field.)

**Theorem 2** General matrix elements of the vertex operators of the \( W_N \)-algebra can be reduced to the following linear combinations:

\[
\langle \theta'' | W_{\lambda''}^\dagger (W_\lambda V_\theta) (z) W_{\lambda'} | \theta' \rangle = \sum_{\mu} A_{\mu} (z) \langle \theta'' | V_\theta (z) W_{-\mu^{(N)}}^{(N)} \ldots W_{-\mu^1(j)}^{(3)} | \theta' \rangle, \tag{37}
\]

where the coefficients \( A_{\mu} (z) \) are labeled by \( \mu = (\mu^{(3)}, \ldots, \mu^{(N)}) \in \mathbb{Y}^{N-2} \), and the corresponding partitions are restricted so that \( \mu^{(1)}_1 \leq j - 2 \) for \( j = 3, \ldots, N \).

**Proof** First let us move all \( W_p^{(j)} \) with \( p > 0 \) in (33) to the right of the vertex operator with the help of (35). After this procedure, we obtain matrix elements of the form (33) but without \( W_{\lambda''}^\dagger \).
At the next step, use the identities (36) to reduce the matrix elements of descendant operators \( V(z) = (W_\lambda V_\theta)(z) \) to those of primary vertex operator \( V_\theta(z) \). Note that for \( p < 0 \) the matrix elements corresponding to the first sum on the right of (36) vanish due to (34). Therefore after the 2nd step we come to linear combinations of matrix elements of type

\[
\{\theta''\} V_\theta(z) W_\lambda |\theta'\rangle,
\]

with so far unrestricted vectors (31).

Finally, let us change the basis (31) in the Verma module \( M_\theta \). We will use new generators of the \( W_N \)-algebra (see [16] for the \( W_3 \) case):

\[
w_p^{(j)}(z) = \sum_{k=0}^{j} (-1)^{k} \binom{j}{k} W_{p-k}^{(j)}, \quad \tilde{w}_0^{(j)}(z) = \sum_{k=1}^{j} (-1)^{k} \binom{j}{k} W_{-k}^{(j)}.
\]

They satisfy the relations (for all \( p \in \mathbb{Z} \))

\[
[V_\theta(z), w_p^{(j)}(z)] = 0, \quad [V_\theta(z), \tilde{w}_0^{(j)}(z)] = (W_0^{(j)} V_\theta)(z),
\]

which imply the following action formulas:

\[
\{\theta''\} V_\theta(z) w_p^{(j)}(z) = 0, \quad p < 0,
\]

\[
\{\theta''\} V_\theta(z) w_0^{(j)}(z) = e_j(\theta'') \{\theta''\} V_\theta(z), \quad \{\theta''\} V_\theta(z) \tilde{w}_0^{(j)}(z) = e_j(\theta) \{\theta''\} V_\theta(z).
\]

The new PBW basis in the Verma module is labeled by \( (\lambda, \mu, k, \tilde{k}) \), where

\[
\lambda = \left(\lambda^{(2)}, \ldots, \lambda^{(N)}\right) \in \mathbb{Y}^{N-1}, \quad k = (k_2, \ldots, k_N) \in \mathbb{Z}_{\geq 0}^{N-1},
\]

\[
\tilde{k} = (\tilde{k}_2, \ldots, \tilde{k}_N) \in \mathbb{Z}_{\geq 0}^{N-1},
\]

\[
\mu = \left(\mu^{(3)}, \ldots, \mu^{(N)}\right) \in \mathbb{Y}^{N-2}, \quad \text{with } \mu_{1}^{(j)} \leq j - 2 \text{ for } j = 3, \ldots, N,
\]

and is given by the vectors

\[
W_{(\lambda, \mu, k, \tilde{k})} |\theta\rangle = w_{-\lambda^{(N)}}^{(N)} \cdots w_{-\lambda^{(2)}}^{(2)} w_{0}^{(N)} \tilde{k}_N \tilde{w}_0^{(N)} \tilde{w}_0^{(2)} \tilde{k}_2 \cdots \left( w_0^{(2)} \right)^{k_2} \left( \tilde{w}_0^{(2)} \right)^{\tilde{k}_2} W_{-\mu^{(N)}}^{(N)} \cdots W_{-\mu^{(3)}}^{(3)} |\theta\rangle,
\]

where we omitted the argument \( z \) in all \( w_p^{(j)}(z), \tilde{w}_0^{(j)}(z) \) to lighten the notation. Thanks to (39), (40), the matrix elements \( \{\theta''\} V_\theta(z) W_{(\lambda, \mu, k, \tilde{k})} |\theta\rangle \) can be expressed in terms of the matrix elements

\[
\{\theta''\} V_\theta(z) W_{(\emptyset, \mu, 0, 0)} |\theta\rangle = \{\theta''\} V_\theta(z) W_{-\mu^{(N)}}^{(N)} \cdots W_{-\mu^{(3)}}^{(3)} |\theta\rangle
\]
labeled by tuples of partitions $\mu \in \mathcal{Y}_{N-2}$ which satisfy the above restrictions $\mu^{(j)}_1 \leq j - 2$. Note that $\mu^{(j)}$ may be equivalently represented by $j - 2$ non-increasing non-negative integers. For example, for $N = 4$, the minimal set of matrix elements can be chosen as

$$\langle \theta'' \mid V_\theta (z) \left( W^{(4)}_{-2} \right)^{l_2} \left( W^{(4)}_{-1} \right)^{l_1-l_2} \left( W^{(3)}_{-1} \right)^{l} \mid \theta' \rangle, \quad l_1 \geq l_2 \geq 0, \quad l \geq 0.$$  

For generic weights $\theta, \theta', \theta''$, the matrix elements (41) cannot be related by means of the Ward identities. However, if one of these weights is of semi-degenerate type (to be discussed in the next subsection), then all these matrix elements can be expressed in terms of $\langle \theta'' \mid V_\theta (z) \mid \theta' \rangle$, just as in the case of the Virasoro algebra.

### 3.3 Semi-degenerate representations

We will need special reducible Verma modules with $\theta = a h_1$, where $a$ is a complex number and $h_s, s = 1, \ldots, N$, are the weights of the first fundamental representation of $\mathfrak{sl}_N$ with the components

$$h_s^{(k)} = \delta_{sk} - 1/N, \quad k = 1, \ldots, N. \quad (42)$$

The irreducible quotient with the highest weight $\theta = a h_1$ is called semi-degenerate representation. We have $N - 2$ relations

$$\left[ W^{(r)}_{-1} - \left( \frac{N - 2}{r - 2} \right) \left( - \frac{a}{N} \right)^{r-2} W^{(2)}_{-1} \right] |a h_1\rangle = 0, \quad r = 3, \ldots, N,$$

corresponding to singular vectors on the first level of the $L_0$-gradation in the Verma module. All the relations needed for derivation on the $p$th level are given by

$$\left[ W^{(r)}_{-p} + (-1)^{r+p} \sum_{s=2}^{p+1} \left( \frac{N - s}{r - s} \right) \left( \frac{r - s - 1}{p - s + 1} \right) \left( \frac{a}{N} \right)^{r-s} W^{(s)}_{-p} \right] |a h_1\rangle = 0, \quad 2 \leq p + 1 < r \leq N, \quad (43)$$

and correspond to factoring out different proper submodules in the Verma module. The derivation of these relations is given in “Appendix A.”

The following proposition shows how the relations (43) can be used for further reduction of the matrix elements appearing in (37).

**Proposition 2** Matrix elements of the semi-degenerate vertex operator $V_{a h_1} (z)$ and its descendants can be expressed in terms of the primary matrix element

$$\langle \theta' \mid V_{a h_1} (z) \mid \theta \rangle = \mathcal{N} \left( \theta', a h_1, \theta \right) z^{\Delta_{\theta'} - \Delta_{a h_1} - \Delta_{\theta}}, \quad (44)$$

where $\Delta_{\theta} = -e_2 (\theta) = \theta^2 / 2$.  

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**Proof** Theorem 2 allows us to start the reduction procedure from the matrix elements of the form

\[
\{ \theta' \mid V_{ah_1}(z) W^{(j)}_{-p} \mathcal{W} \mid \theta \}, \quad 1 \leq p \leq j - 2,
\]

where \( \mathcal{W} \) is a product of the generators of the \( W_N \)-algebra. We will reduce such matrix elements to \( \{ \theta' \mid V_{ah_1}(z) \tilde{\mathcal{W}} \mid \theta \} \) with \( \tilde{\mathcal{W}} \) having \( \deg \mathcal{W} \tilde{\mathcal{W}} < j - 1 + \deg \mathcal{W} \mathcal{W} \), cf (25), (27). Since \( \deg \mathcal{W} W^{(j)}_{-p} = j - 1 \), it then suffices to use induction in \( \deg \mathcal{W} \mathcal{W} \).

The identity (35) can be rewritten for \( V(z) = V_{\theta} (z) \) and any \( p \in \mathbb{Z} \) as

\[
\left[ W^{(j)}_{-p}, V_{\theta}(z) \right] = z^{-p} \sum_{k=0}^{j-1} z^k \binom{j - p - 1}{j - k - 1} \left( W^{(j)}_{-k} V_{\theta} \right)(z).
\]

This commutation relation allows to transform (45) into a linear combination of matrix elements

\[
\{ \theta' \mid (W^{(j)}_{-p} V_{ah_1}) (z) \mathcal{W} \mid \theta \}
\]

with \( 1 \leq p \leq j - 1 \). Moreover, we can exclude \( \{ \theta' \mid (W^{(j)}_{-1-p} V_{ah_1})(z) \mathcal{W} \mathcal{W} \mid \theta \} \) from this set of matrix elements using the relation (46) with \( p = 0 \), which produces one more linear combination of matrix elements (47). It can be found since \( \{ \theta' \mid W^{(j)}_{0} \mathcal{W} \mid \theta \} \) may be computed independently using (30), (34) and the fact that

\[
W^{(j)}_{0} \mathcal{W} = \mathcal{W} W^{(j)}_{0} + \mathcal{W}, \quad \deg \mathcal{W} \mathcal{W} < \deg \mathcal{W} (W^{(j)}_{0} \mathcal{W}),
\]

by the induction assumption. Thus the problem is reduced to finding matrix elements (47) for \( 1 \leq p \leq j - 2 \).

The next step is to use (43). The identity (36) then produces matrix elements of the form \( \{ \theta' \mid V_{ah_1}(z) \mathcal{W} \mathcal{W} \mid \theta \} \) for \( \mathcal{W} \) with \( \deg \mathcal{W} \mathcal{W} < j - 1 + \deg \mathcal{W} \mathcal{W} \). These elements are known by the induction assumption.

The starting matrix element of the induction procedure is \( \{ \theta' \mid V_{ah_1}(z) \mid \theta \} \). It can be calculated, up to a normalization factor, from the relation \( \{ \theta' \mid (L_{-1} V_{ah_1})(z) \mid \theta \} = \partial_z \{ \theta' \mid V_{ah_1}(z) \mid \theta \} \), with \( L_p = -W^{(2)}_{p} \). This yields (44). \( \square \)

### 3.4 Degenerate representations

We will need even more special irreducible representation with \( a = 1 \), i.e. \( \theta = h_1 \). It corresponds to the first fundamental representation of \( \mathfrak{sl}_N \). Another important representation has \( \theta = -h_N \) and corresponds to the last fundamental representation of \( \mathfrak{sl}_N \). Irreducible representations with highest weights \( \theta = h_1 \) and \( \theta = -h_N \) are called (completely) degenerate representations.

One may expect that all the normalization coefficients \( \mathcal{N} (\theta', a h_1, \theta) \) (structure constants) are nonzero for generic \( a, \theta, \theta' \). However, in the case of degenerate vertex operator with \( a = 1 \) there are additional restrictions on the possible values of \( \theta' \) to
have non-vanishing $\mathcal{N} (\theta', ah_1, \theta)$ due to additional singular vectors in the Verma module $M_{h_1}$.

**Proposition 3** The fusion rule for $V_{h_1} (z)$ is

$$V_{h_1} (z) | \theta \rangle = \sum_{s=1}^{N} \mathcal{N} (\theta + h_s, h_1, \theta) z^{\Delta_{\theta + h_s} - \Delta_{h_1} - \Delta_{\theta}} [| \theta + h_s \rangle + O (z)].$$

Similarly, the fusion rule for $V_{-h_N} (z)$ is

$$V_{-h_N} (z) | \theta \rangle = \sum_{s=1}^{N} \mathcal{N} (\theta - h_s, h_1, \theta) z^{\Delta_{\theta - h_s} - \Delta_{h_1} - \Delta_{\theta}} [| \theta - h_s \rangle + O (z)].$$

**Proof** The derivation of these fusion rules is given in “Appendix B.”

In what follows, we will use the projector $\mathcal{P}_{\theta}$ to the irreducible module with the highest weight $\theta$ and a special notation for the degenerate vertex operators restricted to a particular fusion channel:

$$\psi_s (y) = \mathcal{P}_{\theta + h_s} V_{h_1} (y), \quad \bar{\psi}_s (y) = \mathcal{P}_{\theta - h_s} V_{-h_N} (y).$$

Sometimes we will use shorthand notations $\psi_s (y)$ and $\bar{\psi}_s (y)$ if a particular $\theta$ is understood. Explicitly, the fusion rules for them are

$$\psi_s (y) | \theta \rangle = \mathcal{N} (\theta + h_s, h_1, \theta) y^{\Delta_{\theta + h_s} - \Delta_{h_1} - \Delta_{\theta}} [| \theta + h_s \rangle + O (y)],$$

$$\bar{\psi}_s (y) | \theta \rangle = \mathcal{N} (\theta - h_s, h_1, \theta) y^{\Delta_{\theta - h_s} - \Delta_{h_1} - \Delta_{\theta}} [| \theta - h_s \rangle + O (y)].$$

The singular parts of their OPEs become

$$\psi_s (z) \bar{\psi}_{s'} (w) \sim \frac{\delta_{s,s'}}{(z - w)^{(N-1)/N}}, \quad \psi_s (z) \psi_{s'} (w) \sim 0, \quad \bar{\psi}_s (z) \bar{\psi}_{s'} (w) \sim 0,$$

provided we choose normalizations so that $\mathcal{N} (\theta, h_1, \theta + h_s) = \mathcal{N}^{-1} (\theta + h_s, h_1, \theta)$.

## 4 Conformal blocks of $W$-algebras and their properties

### 4.1 Hypergeometric conformal blocks

Below we will need special conformal blocks which can be expressed in terms of hypergeometric functions, and their properties. In this subsection, the non-vanishing 3-point functions of semi-degenerate vertex operators are normalized by $\mathcal{N} (\theta', ah_1, \theta) = 1$, but later this normalization will be changed to a more convenient one. In what follows, we will use a shorthand notation $V_a (z) := V_{ah_1} (z)$ for the semi-degenerate vertex operators.
Theorem 3 The following conformal blocks have hypergeometric expressions:

\[
\langle \theta_\infty | V_a (z) \psi_s (y) | \theta_0 \rangle = z^{\Delta_1 - \Delta_2 - \Delta_3} (y/z)^{\Delta_1 - 1} \langle y/z \rangle^{a/N} \mathcal{G}_s (y/z),
\]

where, recalling the notation \( \Delta_\theta = \theta^2 / 2 \) for conformal weights, we have

\[
\begin{align*}
\Delta_1 &= \Delta_{\theta_0}, \quad \Delta_2 = \Delta_{h_1} = \frac{N-1}{2N}, \quad \Delta_3 = \Delta_{a h_1} = a^2 \cdot \frac{N-1}{2N}, \quad \Delta_4 = \Delta_{\theta_\infty}, \\
\Delta_{1,s} &= \Delta_{\theta_0 + h_1}, \quad \Delta_{4,s} = \Delta_{\theta_\infty - h_1}, \\
\mathcal{G}_s (x) &= N F_{N-1} \left( \{(N - a - 1) / N + \theta_0^{(s)} - \theta_{\infty}^{(k)} \}_{k=1,N}, \{ 1 + \theta_0^{(s)} - \theta_{\infty}^{(k)} \}_{k=1,N; k \neq s} \right) x, \\
\mathcal{G}_s' (x) &= N F_{N-1} \left( \{(N - a - 1) / N + \theta_0^{(k)} - \theta_{\infty}^{(s)} \}_{k=1,N}, \{ 1 + \theta_{\infty}^{(k)} - \theta_{\infty}^{(s)} \}_{k=1,N; k \neq s} \right) x.
\end{align*}
\]

Proof In this proof, we use the standard assumption on the analytic properties of conformal blocks in the left-hand sides of (54) and (55): namely, the only singularities in \( y \) are branching points at \( 0, z, \infty \) with exponents given by the Proposition 2. The idea of the proof is to use rigidity of the 3-point Fuchsian system (18). We claim that the solution of the system can be given in terms of 4-point conformal blocks with degenerate fields \( \psi_j (y) \) and a semi-degenerate field \( V_a (1) = V_{ah_1} (1) \):

\[
\tilde{\Phi}_{jm} (y) = K_{jm} \langle -\theta_\infty + h_m | \psi_j (y) V_a (1) | \theta_0 \rangle.
\]

The constants \( K_{jm} \) will be fixed later. As \( y \to \infty \), the right-hand side behaves as

\[
K_{jm} y^{-\theta_{\infty}^{(j)} + \delta_{jm} - 1} \left[ 1 + O \left( y^{-1} \right) \right],
\]

which follows from the fusion rules (51). The leading asymptotics of the matrix elements of \( \tilde{\Phi} (y) \) as \( y \to \infty \) can also be rewritten as

\[
\tilde{\Phi}_{jm} (y) = y^{-\theta_{\infty}^{(j)}} \left[ \delta_{jm} K_{jj} + O \left( y^{-1} \right) \right].
\]

Each column \( m \) of the matrix \( \tilde{\Phi} (y) \) constitutes a basis in the \( N \)-dimensional space \( \mathcal{V}_m \) of conformal blocks (due to the Proposition 3) with fixed external weights labeled by \( \theta_0, \theta_1 = ah_1, h_1, -\theta_\infty + h_m \). This space is invariant under analytic continuation in \( y \) around \( z_0 = 0, z_1 = 1 \) and \( z_2 = \infty \) giving the monodromies \( M_0^{(m)}, M_1^{(m)} \) and \( M_\infty^{(m)} \), respectively. The spectra of these monodromy matrices are independent of \( m \) and can be found from the conformal dimensions of fields using (51). Namely, all these bases are associated with different channels corresponding to the fusion of the degenerate field \( \psi_j (y) \) with a generic primary at \( z_2 = \infty \).
There are two more bases in each of $\mathcal{V}_m$ associated with the channels corresponding to the fusion of the degenerate field $\psi_j(y)$ with the fields at $z_0 = 0$ and $z_1 = 1$, respectively. The leading terms (up to constant prefactors) of the basis conformal blocks for each $\mathcal{V}_m$ at $z_0 = 0$ are $y^{\theta_0} = (y^{\theta_0(1)}, \ldots, y^{\theta_0(N)})$. Similarly, the leading terms of the basis conformal blocks for each $\mathcal{V}_m$ at $z_1 = 1$ are $(y - 1)^{\theta_1} = ((y - 1)^{\theta_1(1)}, \ldots, (y - 1)^{\theta_1(N)})$. These two bases are distinguished by the property of having diagonal monodromies exp $2\pi i \theta_0$ and exp $2\pi i \theta_1$ (see (16) for the definition of $\Theta_0$ and $\Theta_1$) under analytic continuation around $z_0$ and $z_1$, respectively. In the initial basis, which is distinguished by the property of having diagonal monodromy around $z_2 = \infty$, we have

$$M_0^{(m)} = C_0^{(m)} e^{2\pi i \theta_0 (C_0^{(m)})^{-1}}, \quad M_1^{(m)} = e^{2\pi i \theta_1}, \quad M_\infty^{(m)} = C_1^{(m)} e^{2\pi i \theta_1 (C_1^{(m)})^{-1}}.$$  

It follows from Lemma 1 that the triples $(M_0^{(m)}, M_1^{(m)}, M_\infty^{(m)})$ of monodromy matrices for different $m$ are related by an overall similarity transformation. Since $M_\infty^{(m)}$ are already diagonal for all $m$ and have simple spectrum, the remaining freedom is given by the conjugation by diagonal matrices. Such type of similarity transformations corresponds to choosing the coefficients $K_{jm}$, $j = 1, \ldots, N$, for each $m$. Fix these coefficients so that all triples $(M_0^{(m)}, M_1^{(m)}, M_\infty^{(m)})$ coincide with the triple $(M_0, M_1, M_\infty)$ of monodromy matrices of the actual fundamental solution $\Phi(y)$ given by (19). Then the elements of the matrix $\Phi(y)^{-1} \tilde{\Phi}(y)$ are given by single-valued meromorphic functions with the only possible poles at 0, 1 or $\infty$. However, $\Phi(y)$ and $\tilde{\Phi}(y)$ have the same local monodromy exponents, and hence this matrix is in fact constant. From the normalization of $\Phi(y)$, it follows that the constant matrix is diagonal and may be chosen as the identity matrix, in which case $K_{jj} = 1$ and $\Phi(y) = \tilde{\Phi}(y)$.

The diagonal elements of the latter relation give the hypergeometric representation (55) with $z = 1$; arbitrary $z$ may be obtained by a scale transformation of conformal blocks. The proof of (54) is completely analogous. □

Conformal blocks (54) are naturally defined as convergent series for $|y| < |z|$. Their analytic continuation in $y$ to the region with $|y| > |z|$ can be compared to conformal blocks (55), given by convergent series in the latter domain. The relation between the two sets of conformal blocks may be expressed with the help of well-known $N \times N$ connection formulas, see e.g. [54]:

$$\langle \theta_\infty \mid V_a(z) \psi_j(y, \xi_z) \mid \theta_0 \rangle = \sum_{j=1}^N e^{-i \pi ((N+1)/2 - \theta_\infty^{(j)})} F_{ij}^{[\infty, 0]} (\theta_\infty, a, \theta_0) \langle \theta_\infty \mid \psi_j(y) V_a(z) \mid \theta_0 \rangle.$$  \hspace{1cm} (56)

3 Since the spectrum of $\Theta_1$ is degenerate of spectral type $(N - 1, 1)$, actually there is an ambiguity in the choice of the basis in the space of conformal blocks: taking their differences it is possible to choose another basis in which the leading terms are multiplied by $(y - 1)^k$, $k \in \mathbb{Z}_{>0}$. Springer
Here \( y, \xi_z \) stands for the analytic continuation along a contour \( \xi_z \) going around \( z \) in the positive direction and \( \tilde{F}^{[\infty,0]} \) is the fusion matrix “\( 0 \to \infty \)” with the elements

\[
\tilde{F}_{lj}^{[\infty,0]}(\theta_\infty, a, \theta_0) = \frac{\prod_{p(l)} \Gamma((1 + \theta_0^{(l)} - \theta_0^{(p)}) \cdot \prod_{p(\neq j)} \Gamma(\theta_\infty^{(j)} - \theta_\infty^{(p)})}{\prod_{p(l)} \Gamma((1 + a)/N + \theta_\infty^{(j)} - \theta_\infty^{(p)}) \cdot \prod_{p(\neq j)} \Gamma((N - 1 - a)/N + \theta_0^{(l)} - \theta_\infty^{(p)})},
\]

where \( \Gamma(x) \) is the gamma function.

In fact, the fusion matrix in (56) will not change if instead of the vectors \( |\theta_0\rangle \) and \( |\theta_\infty\rangle \) we use any of their descendants, obtained by the action of the creation operators \( W_k^{(j)}, k < 0 \). This is due to the fact that all such conformal blocks can be obtained by an action of differential operators in \( y \) and \( z \), which commutes (intertwines) with the crossing symmetry transformation (56). Combining the vertex operators \( \psi_l(y) \), \( l = 1, \ldots, n \), into a column matrix \( \Psi(y) \), one may therefore rewrite (56) as an operator relation

\[
\mathcal{P}_{\theta_\infty} V_a(z) \Psi(y, \xi_z) \mathcal{P}_{\theta_0} = B^{-1}(\theta_0) \tilde{F}^{[\infty,0]}(\theta_\infty, a, \theta_0) B'(\theta_\infty) \cdot \mathcal{P}_{\theta_\infty} \Psi(y) V_a(z) \mathcal{P}_{\theta_0},
\]

where the braiding matrices \( B \) and \( B' \) are diagonal and their nonzero elements are given by

\[
B_{ll}(\theta_0) = \exp i\pi \theta_0^{(l)}, \quad B'_{jj}(\theta_\infty) = \exp i\pi \left(\theta_\infty^{(j)} - \frac{N-1}{N}\right).
\]

The notations for transformations \( F \) and \( B \) are motivated by the Moore–Seiberg formalism [50], where they correspond to fusion and braiding moves.

Analytic continuation of \( \psi_l(y) |\theta_0\rangle \) in \( y \) around \( 0 \) in the positive direction leads to multiplication by the diagonal braiding matrix \( B^2(\theta_0) \). Indeed,

\[
B_{ll}^2(\theta_0) = \exp \{2\pi i (\Delta_{\theta_0 + h_l} - \Delta_{\theta_0} - \Delta_{h_l})\} = \exp 2\pi i \theta_0^{(l)}.
\]

Taking into account that the braiding matrix is the same for descendants of \( |\theta_0\rangle \), we write the braiding relation as

\[
\Psi(y e^{2\pi i}) \mathcal{P}_{\theta_0} = B^2(\theta_0) \cdot \Psi(y) \mathcal{P}_{\theta_0}.
\]

Similarly, the analytic continuation of \( \langle \theta_\infty | \psi_j(y) \rangle \) in \( y \) around \( \infty \) in the negative direction leads to multiplication by the diagonal braiding matrix \( B'^2(\theta_\infty) \), whose non-vanishing elements are

\[
B_{jj}'^2(\theta_\infty) = \exp \{2\pi i (\Delta_{\theta_\infty} - \Delta_{\theta_\infty - h_j} - \Delta_{h_j})\} = \exp 2\pi i \theta_\infty^{(j)} - \frac{N-1}{N}.
\]
This leads to
\[ P_{\theta_{\infty}} \Psi \left( y e^{2\pi i} \right) = B'_{\infty} (\theta_{\infty}) \cdot P_{\theta_{\infty}} \Psi (y). \] (60)

Finally, let us introduce the following formal transformations which are a consequence of the definition (50):
\[ \psi_s (y) P_{\sigma} = P_{\sigma + h_s} \psi_s (y) = \nabla_{\sigma, s} P_{\sigma} \psi_s (y), \]
where \( \nabla_{\sigma, s} \) is the shift operator defined by \( \nabla_{\sigma, s} F (\sigma) = F (\sigma + h_s) \) for any function \( F \) depending on \( \sigma \). We combine the shifts \( \nabla_{\sigma, s}, s = 1, \ldots, N \), into the diagonal matrix \( \nabla_{\sigma} = \text{diag} (\nabla_{\sigma, 1}, \ldots, \nabla_{\sigma, N}) \), to write compactly
\[ \Psi (y) P_{\sigma} = \nabla_{\sigma} P_{\sigma} \Psi (y). \] (61)

Note that there is a useful relation between the two types of braiding,
\[ B (\sigma) \nabla_{\sigma} = \nabla_{\sigma} B' (\sigma). \] (62)

### 4.2 Normalization of vertex operators and properties of fusion matrices

Let us change the normalization of the vertex operators in (44): instead of \( \mathcal{N} (\sigma', a h_1, \sigma) = 1 \), we will use
\[ \mathcal{N} (\sigma', a h_1, \sigma) = \prod_{l, j} \frac{G(1 - a/N + \sigma^{(l)} - \sigma^{(j)})}{G(1 + \sigma^{(p)} - \sigma^{(m)}) G(1 - \sigma^{(p)} + \sigma^{(m)})}, \] (63)
or
\[ \mathcal{N} (\sigma', a h_1, \sigma) = \prod_{p < m} \frac{G(1 + a/N - \sigma^{(l)} + \sigma^{(j)})}{G(1 + \sigma^{(p)} - \sigma^{(m)}) G(1 - \sigma^{(p)} + \sigma^{(m)})}, \] (64)
where \( G (x) \) is the Barnes \( G \)-function. Both expressions can be employed for the generic semi-degenerate vertex operators (we will use the first one), but for the degenerate field \( \psi_s (z) \) we can use only the second expression since the first one becomes singular:
\[ \langle \sigma + h_s | \psi_s (1) | \sigma \rangle = \mathcal{N} (\sigma + h_s, h_1, \sigma). \] (65)

In order to preserve the operator product expansion (53) between \( \psi_s (z) \) and \( \bar{\psi}_s (w) \), we choose the following normalization for the degenerate field \( \bar{\psi}_s (w) \):
\[ \langle \sigma | \bar{\psi}_s (1) | \sigma + h_s \rangle = \mathcal{N}^{-1} (\sigma + h_s, h_1, \sigma). \] (66)
The braiding matrices $B$ Therefore the new fusion matrix $F (\sigma', a, \sigma) \equiv F^{[\infty,0]} (\sigma', a, \sigma)$ is given by

$$F_{ij} (\sigma', a, \sigma) = N_{ij} (\sigma', a, \sigma) F^{[\infty,0]} (\sigma', a, \sigma) = \prod_{p(\neq i)} \pi((a + 1)/N + \sigma^{(j)} - \sigma^{(p)}) \sin \pi((a + 1)/N + \sigma^{(j)} - \sigma^{(p)}) \sin \pi(\sigma^{(p)} - \sigma^{(l)})$$

(67)

Its trigonometric form is the crucial advantage of the modified normalization of vertex operators. Note that the inverse fusion matrix admits a simple expression:

$$F^{-1} (\sigma', a, \sigma) = F (-\sigma, a, -\sigma').$$

(68)

The braiding matrices $B (\sigma), B' (\sigma)$ defined by (58) are not affected by the change of normalization. Therefore, the fusion relation (57) transforms into

$$P_{\sigma} V_{\alpha} (z) \Psi (y, \xi z) P_{\sigma} = B^{-1} (\sigma) F (\sigma', a, \sigma) B' (\sigma') : P_{\sigma} \Psi (y) V_{\alpha} (z) P_{\sigma}.$$ 

(69)

In what follows, we will also need additional easily verifiable properties of $F (\sigma', a, \sigma)$.

**Proposition 4** The fusion matrix $F (\sigma', a, \sigma)$ satisfies the following shift transformations:

$$F (\sigma', a, \sigma \pm h_m) = D_m^{N-1} F (\sigma', a \pm 1, \sigma),$$

$$F (\sigma' \pm h_m, a, \sigma) = F (\sigma', a \mp 1, \sigma) D_m^{N-1},$$

(70)

where $D_m$ is a diagonal matrix with matrix elements $(D_m)_{ij} = (-1)^{\delta_{im}} \delta_{ij}$ and the weight vectors $h_m$ are given by (42). Compositions of the transformations (70) give the following identities:

$$F (\sigma', a, \sigma + h_m - h_s) = D_m^{N-1} D_s^{N-1} F (\sigma', a, \sigma),$$

$$F (\sigma' + h_m - h_s, a, \sigma) = F (\sigma', a, \sigma) D_m^{N-1} D_s^{N-1}.$$ 

(71)

The main result of the paper is an explicit solution of the Riemann–Hilbert problem for semi-degenerate Fuchsian systems in terms of $W_N$-conformal blocks. We will find the monodromy matrices of the proposed solution and compare them with the
monodromy matrices (12) parameterized by Proposition 1 with the help of Lemma 1. For such comparison, the following proposition will be useful.

**Proposition 5** The matrix $F(\sigma', a, \sigma)$ given by (67) and the matrices $W_k$ given by (14) are related by

$$F(\sigma_k, a_k - 1, \sigma_{k-1}) = X_k W_k Y_k^{-1},$$

where the diagonal matrices $X_k$ and $Y_k$ are defined by their diagonal elements $x_k^{(l)}$ and $y_k^{(j)}$, $l, j = 1, \ldots, N$, respectively:

$$
\begin{align*}
(x_k^{(l)})^{-1} &= e^{i\pi(N-1)\sigma_k^{(l)}} \prod_{p=1}^{N} \sin \pi(\hat{\sigma}_k^{(p)} - \sigma_k^{(l)}) \prod_{p(\neq l)} \sin \pi(\sigma_{k-1}^{(p)} - \sigma_k^{(l)}), \\
(y_k^{(j)})^{-1} &= e^{i\pi(N-1)\sigma_k^{(j)}} \prod_{p=1}^{N} \sin \pi(\hat{\sigma}_k^{(j)} - \sigma_k^{(p)}) \prod_{p(\neq j)} \sin \pi(\sigma_{k-1}^{(p)} - \sigma_k^{(j)}),
\end{align*}
$$

and we use the notation $\hat{\sigma}_k^{(p)} = \sigma_k^{(p)} + a_k/N$.

### 4.3 Monodromy of conformal blocks

This subsection is devoted to the computation of monodromy of multi-point conformal blocks using the transformation properties (69), (59)–(62).

Consider the following column $\tilde{F}_m(\sigma | y)$, $\sigma = (\sigma_1, \ldots, \sigma_{n-3})$ of conformal blocks:

$$
\tilde{F}_m(\sigma | y) = C_m (-\theta_{n-1} + h_m | \Psi(y) V_{a_n-2} (z_{n-2}) \mathcal{P}_{\sigma_{n-3}} V_{a_{n-3}-2} (z_{n-3}) \mathcal{P}_{\sigma_{n-4}} \cdots \mathcal{P}_{\sigma_1} V_{a_1} (z_1) | \theta_0),
$$

where $C_m$ is the diagonal matrix with matrix elements $(C_m)_{jj} = (-1)^{N\delta_{jm}}$. The advantage of incorporating $C_m$ into the definition of $\tilde{F}_m(\sigma | y)$ is that its monodromy becomes independent of $m$. Denote by $\tilde{F}_m(\sigma | y)$ the analytic continuation of $\tilde{F}_m(\sigma | y)$ in $y$ along the path $\xi_{[k]} := \xi_0 \cdots \xi_k$ encircling $z_0(= 0), z_1, \ldots, z_k$. We have

$$
\tilde{F}_m(\sigma | y, \xi_{[k]}) = \hat{M}_{[k]} \tilde{F}_m(\sigma | y),
$$

where the operator-valued monodromy matrix $\hat{M}_{[k]}$ is

$$
\begin{align*}
\hat{M}_{[k]} &= \hat{V}_{[k]} B^2(\sigma_k)(\hat{V}_{[k]})^{-1}, \\
\hat{V}_{[k]} &= C_m T_{n-2} \mathcal{V}_{\sigma_{n-3}} T_{n-3} \mathcal{V}_{\sigma_{n-4}} T_{n-4} \cdots \mathcal{V}_{\sigma_{k+1}} T_{k+1}, \\
T_i &= B'(\sigma_i) F^{-1}(\sigma_i, a_i, \sigma_{i-1}) B^{-1}(\sigma_{i-1}),
\end{align*}
$$
and $\sigma_{n-2} := -\theta_{n-1} + h_m$. Although it may seem that $\hat{V}'_{[k]}$ depends on $m$, thanks to the identity
\[
C_m B' (-\theta_{n-1} + h_m) F^{-1} (-\theta_{n-1} + h_m, a_{n-2}, \sigma_{n-3})
= -B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}),
\]
this dependence actually disappears. One may also use the identity $B^{-1} (\sigma_l) \nabla_{\sigma_l} B' (\sigma_l) = \nabla_{\sigma_l}$ (cf 62) to simplify the expression (75) for the operator-valued monodromy matrix $\hat{M}_{[k]}$:
\[
\begin{align*}
\hat{M}_{[k]} &= \hat{V}_{[k]} B^2 (\sigma_k) \hat{V}_{[k]}^{-1}, \\
\hat{V}_{[k]} &= B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) \nabla_{\sigma_{n-3}} F^{-1} (\sigma_{n-3}, a_{n-3}, \sigma_{n-4}) \\
&\quad \times \nabla_{\sigma_{n-4}} F^{-1} (\sigma_{n-4}, a_{n-4}, \sigma_{n-5}) \nabla_{\sigma_{n-5}} \cdots \nabla_{\sigma_{k+1}} F^{-1} (\sigma_{k+1}, a_{k+1}, \sigma_k).
\end{align*}
\]

Recall that there is a simple formula (68) for $F^{-1}$.

The operator-valued monodromy matrix $\hat{M}_{[k]}$ has periodic dependence on $\sigma = (\sigma_1, \ldots, \sigma_{n-3})$. Namely, $\hat{M}_{[k]}$ is invariant under the shifts of any $\sigma_l, l = 1, \ldots, n - 3$, by the vectors $w_l$ of root lattice $\mathfrak{R}$ of $\mathfrak{sl}_N$ embedded into $\mathbb{C}^N$, i.e. $w_l \in \mathbb{Z}^N$, $\sum_{s=1}^N w_l^{(s)} = 0$. This claim is equivalent to the following proposition.

**Proposition 6** The operator-valued monodromy matrix $\hat{M}_{[k]}$ has the following property:

$$
\nabla_{\sigma_l, m} \nabla_{\sigma_l, s} \hat{M}_{[k]} = \hat{M}_{[k]} \nabla_{\sigma_l, m} \nabla_{\sigma_l, s}^{-1}.
$$

**Proof** For $l < k$, $\hat{M}_{[k]}$ does not depend on $\sigma_l$. Therefore it is sufficient to prove the commutativity for the cases with $l \geq k$.

For $l = k$, let us extract explicitly the part of $\hat{M}_{[k]}$ given by (77) depending on $\sigma_k$:

$$
\hat{M}_{[k]} = \cdots F^{-1} (\sigma_{k+1}, a_{k+1}, \sigma_k) B^2 (\sigma_k) F (\sigma_{k+1}, a_{k+1}, \sigma_k) \cdots.
$$

Using the identities (71) and $B^2 (\sigma_k) = \exp (2\pi i \text{ diag } \sigma_k)$, we then obtain

$$
\nabla_{\sigma_k, m} \nabla_{\sigma_k, s} \hat{M}_{[k]} = \cdots F^{-1} (\sigma_{k+1}, a_{k+1}, \sigma_k + h_m - h_s) B^2 (\sigma_k + h_m - h_s) \\
\times F (\sigma_{k+1}, a_{k+1}, \sigma_k + h_m - h_s) \cdots \nabla_{\sigma_k, m} \nabla_{\sigma_k, s}^{-1} = \hat{M}_{[k]} \nabla_{\sigma_k, m} \nabla_{\sigma_k, s}^{-1}.
$$

For $l > k$, it suffices to show the commutativity property for $\hat{V}_{[k]}$:

$$
\nabla_{\sigma_l, m} \nabla_{\sigma_l, s} \hat{V}_{[k]} = \hat{V}_{[k]} \nabla_{\sigma_l, m} \nabla_{\sigma_l, s}^{-1}.
$$

Indeed, the part of the operator-valued matrix $\hat{V}_{[k]}$ which depends on $\sigma_l$ is given by

$$
\hat{V}_{[k]} = \cdots F^{-1} (\sigma_{l+1}, a_{l+1}, \sigma_l) \nabla_{\sigma_l} F^{-1} (\sigma_l, a_l, \sigma_{l-1}) \cdots.
$$
The property (78) easily follows from (71) and the diagonal form of \( \nabla_{\sigma_i} \). □

Although \( \mathcal{F}_m (\sigma \mid y) \) is not invariant with respect to the shifts of \( \sigma_i \) by root vectors, the above proposition suggests to consider Fourier transform of \( \mathcal{F}_m (\sigma \mid y) \) with respect to such shifts. It will be shown in the next section that this Fourier transform gives the solution of the Riemann–Hilbert problem we are interested in.

5 Solution of semi-degenerate Fuchsian system and isomonodromic tau function

As above, let \( \mathcal{R} \) denote the root lattice of \( sl_N \) embedded into \( \mathbb{C}^N \), i.e. \( w \in \mathcal{R} \) if and only if \( w = \bigoplus_{k=1}^{N-1} \mathbb{Z} (h_k - h_{k+1}) \), where \( h_k \) are the weights of the first fundamental representation of \( sl_N \), cf (42). Equivalently, \( w \in \mathcal{R} \) if and only if \( w \in \mathbb{Z}^N \) and the sum of its components vanishes.

**Theorem 4** Let \( \Phi (y) \) be the solution of the semi-degenerate Fuchsian system (2) having the asymptotics \( \Phi (y) = y^{-\Theta_{n-1}} [I + \mathcal{O} (y^{-1})] \) as \( y \to \infty \) and monodromies \( \{ M_k \}_{k=0, \ldots, n-1} \), described by Proposition 1 with parameters \( \theta_0, \Theta_{n-1}; \{ r_k, a_k \}_{k=1, \ldots, n-2}, \{ \sigma_i \}_{i=1, \ldots, n-3} \). The matrix elements of \( \Phi (y) \) can be written in terms of conformal blocks of the \( W_N \)-algebra:

\[
\Phi_{jm}(y) = \frac{(-1)^{N(\delta_{jm}-1)} \langle -\theta_{n-1} + h_m \mid \psi_j(y) \mid \Theta^D_{jm} \rangle}{\mathcal{N}_m \langle -\theta_{n-1} \mid \Theta^D \rangle}, \tag{79}
\]

where

\[
\Theta_{jm} (\sigma_1, \ldots, \sigma_{n-3}) := \mathcal{P}_{\theta_{n-1}-h_j+h_m} V_{a_{n-2}} (z_{n-2}) \mathcal{P}_{\sigma_{n-3}} V_{a_{n-3}} (z_{n-3}) \mathcal{P}_{\sigma_{n-4}} \cdots \mathcal{P}_{\sigma_1} V_{a_1} (z_1) \mid \theta_0 \rangle,
\]

\[
\Theta^D_{jm} := \sum_{w_1, \ldots, w_{n-3} \in \mathcal{R}} e^{(\beta_1, w_1) + \cdots + (\beta_{n-3}, w_{n-3})} \langle \Theta_{jm} (\sigma_1 + w_1, \ldots, \sigma_{n-3} + w_{n-3}) \mid \Theta^D \rangle,
\]

\[
\Theta^D := \Theta^D_{11} = \cdots = \Theta^D_{NN}, \tag{80}
\]

and \( \mathcal{R} \) is the root lattice of \( sl_N \). The relation between the monodromy parameters \( r_k \) and conjugate Fourier momenta \( \beta_k \) is given by the formulas (73), (74) of Proposition 5 and

\[
R_k = Y_{k-1}^{-1} H_k X_{k+1}, \quad 1 \leq k \leq n-3, \quad R_{n-2} = Y_{n-2}^{-1} B (\theta_{n-1}),
\]

\[
R_k = \text{diag} \, r_k, \quad H_k = \text{diag} (e^{\beta_1 k}, \ldots, e^{\beta_{n-3} k}). \tag{81}
\]

The normalization coefficients \( \mathcal{N}_m \) follow from the normalization (65) of \( \psi_m (y) \):

\[
\mathcal{N}_m = \mathcal{N} (-\theta_{n-1} + h_m, h_1, -\theta_{n-1}).
\]
We have to show that proposed matrix $\Phi(y)$ solves the initial Riemann–Hilbert problem with the monodromy matrices specified above. The monodromy matrix $M_{n-1}$ around $z = \infty$ in the positive direction on the Riemann sphere is diagonal and has matrix elements

$$(M_{n-1})_{jj} = (B'/2 (-\theta_{n-1} + h_m))_{jj} = \exp \left\{ 2\pi i \left( \theta_{n-1}^{(j)} - h_m^{(j)} + \frac{N-1}{N} \right) \right\} = e^{2\pi i \theta_{n-1}^{(j)}}.$$  

The next monodromy matrix is still numerical,

$$\hat{M}_{[n-3]} = B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) B^2 (\sigma_{n-3}) F (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) B^{-1} (-\theta_{n-1}).$$

The next one, however, has operator-valued entries:

$$\hat{M}_{[n-4]} = B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) \nabla_{\sigma_{n-3}} F^{-1} (\sigma_{n-3}, a_{n-3}, \sigma_{n-4})$$

$$\times B^2 (\sigma_{n-4}) F (\sigma_{n-3}, a_{n-3}, \sigma_{n-4}) \nabla_{\sigma_{n-3}}^{-1}$$

$$F (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) B^{-1} (-\theta_{n-1}).$$

This operator expression is invariant with respect to the shifts of intermediate weights $\sigma_{n-3}$ and $\sigma_{n-4}$ by root vectors $h_m - h_1$ of $\mathfrak{sl}_N$ (see Proposition 6), and therefore it is the same for all conformal blocks appearing in (80).

Let us move the components of $\nabla_{\sigma}$ to the right to act on conformal blocks. We need a few identities which follow from (70) and (71) written in components,

$$\nabla_{\sigma,l} F_{lj} (-\sigma', a, -\sigma) = (-1)^{N-1} F_{lj} (-\sigma', a - 1, -\sigma) \nabla_{\sigma,l},$$

$$\nabla_{\sigma,m} F_{jl} (\sigma, a, \sigma') = (-1)^{(N-1)\delta_{ml}} F_{jl} (\sigma, a - 1, \sigma') \nabla_{\sigma,m},$$

$$\nabla_{\sigma,m} \nabla_{\sigma,l}^{-1} F_{lp} (\sigma'', a, \sigma) = (-1)^{(N-1)\delta_{ml}-1} F_{lp} (\sigma'', a, \sigma) \nabla_{\sigma,m} \nabla_{\sigma,l}^{-1},$$

to obtain

$$\hat{M}_{[n-4]} = B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) \tilde{\nabla}_{\sigma_{n-3}}$$

$$\times B^2 (\sigma_{n-4}) F (\sigma_{n-3}, a_{n-3} - 1, \sigma_{n-4}) \tilde{\nabla}_{\sigma_{n-3}}^{-1}$$

$$F (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) B^{-1} (-\theta_{n-1}),$$

where the components of $\tilde{\nabla}_{\sigma_{n-3}}$ act directly on conformal blocks. Therefore, the action on the Fourier transformed conformal blocks diagonalizes producing a numerical monodromy matrix,

$$M_{[n-4]} = B (-\theta_{n-1}) F^{-1} (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) H_{n-3}^{-1} F^{-1} (\sigma_{n-3}, a_{n-3} - 1, \sigma_{n-4})$$

$$\times B^2 (\sigma_{n-4}) F (\sigma_{n-3}, a_{n-3} - 1, \sigma_{n-4}) H_{n-3}$$

$$F (-\theta_{n-1}, a_{n-2} - 1, \sigma_{n-3}) B^{-1} (-\theta_{n-1}).$$
where $H_k$ is the diagonal matrix with the elements $(H_k)_{mj} = e^{\delta_{kj}} \delta_{mj}$.

Analogous procedure may be applied to other monodromy matrices. We note that their structure as products of elementary building blocks reproduces the structure of the corresponding products (12), (13) in Proposition 1. The exact relation (81) between the parameters is obtained using (72).

There is a related theorem for an alternative normalization of the fundamental solution.

**Theorem 5** The solution of the semi-degenerate Fuchsian system with the same monodromies, normalized as $\Phi(y_0) = I$, can be written in terms of conformal blocks as

$$
\Phi_{jm}(y) = (-1)^{N(\delta_{jm}-1)} (y_0 - y)^{(N-1)/N} \begin{pmatrix} -\theta_{n-1} |\bar{\psi}_m(y_0) \psi_j(y) \rangle_{\Theta^D} \\ -\theta_{n-1} \langle \Theta^D \end{pmatrix},
$$

where $|\Theta^D_{jm}\rangle$ and $|\Theta^D\rangle$ are given by the same formulas (80). The normalizations of $\psi_j(y)$ and $\bar{\psi}_m(y_0)$ are fixed by (65) and (66).

The proof of Theorem 5 goes along the same lines as the proof of Theorem 4. Note that the solution (79) can be obtained from the solution (82) by fusion of $(-\theta_{n-1} |\bar{\psi}_m(y_0)$ in the limit $y_0 \to \infty$. The sign factors $(-1)^{N(\delta_{jm}-1)}$ in (79) and (82) have their origin from the sign factors in the formulas of Proposition 4 and can be removed by an appropriate change of normalization of $\psi_j(y)$ and $\bar{\psi}_m(y_0)$.

By the arguments given in [30], one obtains the isomonodromic tau function (5) as the Fourier transform of the $W_N$-conformal blocks which appears in the denominators of the fundamental solutions (79) and (82):

**Proposition 7** The isomonodromic tau function of Jimbo-Miwa-Ueno for semi-degenerate Fuchsian systems is given by

$$
\tau(z) = \langle -\theta_{n-1} | \Theta^D \rangle,
$$

where $|\Theta^D\rangle$ is given by (80).

Let us exemplify the above results with the simplest example of $n = 4$, i.e. two generic punctures and two punctures of spectral type $(N-1,1)$. The former are located at $z_0 = 0$ and $z_3 = \infty$. It can be also assumed that $z_2 = 1$, so that the only remaining time parameter is $z_1 \equiv t$. The Fuchsian system (2) then reduces to

$$
\frac{d\Phi(y)}{dy} = \Phi(y) \left( \frac{A_0}{y} + \frac{A_1}{y-t} + \frac{A_1}{y-1} \right).
$$

It should be stressed that the matrices $A_0, A_1, A_1$ are traceless (this assumption involves no loss of generality but is crucial from the CFT perspective). Their respective diagonalizations are

$$
\text{diag } \theta_0, \text{ diag } \left( \frac{N-1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N} \right) a_t, \text{ diag } \left( \frac{N-1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N} \right) a_1,
$$

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with \( a_t, a_1 \in \mathbb{C} \). We also have \( A_\infty = -A_0 - A_t - A_1 = \text{diag} \theta_\infty \). The monodromy preserving deformations for (84) are described by the equations

\[
\frac{dA_0}{dt} = [A_0, A_t] t, \quad \frac{dA_1}{dt} = [A_1, A_t] t - 1.
\] (85)

equivalent to the polynomial Hamiltonian Fuji–Suzuki–Tsuda system [59]. The corresponding tau function is defined, up to arbitrary nonzero constant factor, by

\[
\frac{d \ln \tau_{\text{FST}}(t)}{dt} = \text{Tr} A_0 A_t + \frac{\text{Tr} A_1 A_t}{t - 1}.
\] (86)

Proposition 7 states that \( \tau_{\text{FST}}(t) \) is nothing but the Fourier transform of the 4-point semi-degenerate conformal block,

\[
\tau_{\text{FST}}(t) = \text{const} \cdot \sum_{w \in \mathfrak{R}} e^{(\beta, w)} \frac{1}{t} \frac{t}{a_1} \frac{a_t}{\sigma + w} 0 \quad \infty -\theta_\infty \theta_0
\] (87)

where the vertices represent appropriate chiral vertex operators in (80). The parameters \( \sigma = (\sigma(1), \ldots, \sigma(N)) \in \mathbb{C}^N \) with \( \sum_{s=1}^N \sigma(s) = 0 \) are related to diagonalized composite monodromy of \( \Phi(y) \) around 0 and \( t: M_0 M_t \sim e^{2\pi i \sigma} \).

There is a freedom in the choice of \( \sigma \), which can be shifted by any vector in \( \mathfrak{R} \). Obviously, such a shift does not affect the result (87). From the point of view of asymptotic analysis of \( \tau_{\text{FST}}(t) \) as \( t \to 0 \), it is convenient to choose \( \sigma \) so that \( |\Re \sigma(i) - \Re \sigma(j)| \leq 1 \) for \( i, j = 1, \ldots, N \). Indeed,

\[
\tau_{\text{FST}}(t) \simeq t^{\frac{1}{2}} \left( \sigma_0^2 - \sigma_0 a_1^2 \right) \sum_{w \in \mathfrak{R}} \mathcal{C}(\sigma, w) e^{(\beta, w)} \left[ \frac{w^2}{2} + (\sigma, w) \right] \left[ 1 + O(t) \right],
\] (88)

as \( t \to 0 \), where

\[
\mathcal{C}(\sigma, w) = \mathcal{N}(\sigma + w, a_1 h_1, \sigma) \mathcal{N}(\sigma + w, a_1 h_1, \theta_0) \mathcal{N}(\sigma, a_1 h_1, \theta_0).
\] (89)

The normalization coefficients \( \mathcal{N}(\sigma', a h_1, \sigma) \) are given by (63), which implies that the structure constants \( \mathcal{C}(\sigma, w) \) can be expressed in terms of gamma functions. In the generic case of strict inequality \( |\Re \sigma(i) - \Re \sigma(j)| < 1 \), the leading asymptotic contribution to (88) is determined by \( w = 0 \), and the subleading terms correspond to
the roots, \( w = h_i - h_j \). It follows that

\[
\tau_{\text{FST}}(t) \simeq t^{\frac{1}{2}}(\theta^2 - \theta_0^2 - \sigma^2 h_i^2) \left[ 1 + \sum_{i \neq j} e^\theta (\sigma, h_i - h_j) e^{\beta(i) - \beta(j)} t^{1 + \sigma(i) - \sigma(j)} + \mathcal{O}(t) \right],
\]

(90)

\[
\mathcal{O}(t) = O(t) + \sum_{i,j,k,l=1} O(t^2 + \Re(\sigma(i) - \sigma(j) + \sigma(k) - \sigma(l))).
\]

(91)

The asymptotics (90) is a specialization to the semi-degenerate case of the Proposition 3.9 of [39], which provides a higher-rank analogue of the Jimbo’s asymptotic formula [40] for Painlevé VI.

6 Discussion

We conclude with some open research directions. As already mentioned in the introduction, the fundamental solutions of 3-point Fuchsian systems allow to construct Fredholm determinant representation of the \( n \)-point tau function [18,33]. The principal minor expansion of the determinant gives a series representation for \( \tau(z) \), which in the semi-degenerate case may be expected to coincide with Nekrasov formulas [53] for instanton partition functions of linear quiver gauge theories in the self-dual \( \Omega \)-background. After the first version of the present work has appeared on arXiv, this proposal was proved in [35], which produces a new (direct) proof of the AGT-W relation [1,23,49,60] for \( c = N - 1 \).

Let us note that the combinatorics of the tau function expansions is the same in the generic and semi-degenerate case. Any progress on the 3-point solutions would provide new information on more general vertex operators and conformal blocks for \( W_N \)-algebras.

It might be possible to adapt the technique developed in [39] to compute the connection coefficient for the tau function of the Fuji–Suzuki–Tsuda system (relative normalization of the asymptotics as \( t \to 0 \) and as \( t \to \infty \)). The CFT counterpart of this quantity is the fusion kernel relating the \( s \)- and \( u \)-channel semi-degenerate \( c = N - 1 \) conformal blocks (see [38] for \( N = 2 \) case):

\[
\int d\sigma' \quad F\left[ a_2, a_1, \frac{\sigma'}{\sigma} \right] = \int \Sigma \sigma' \quad F\left[ a_2, a_1, \frac{\sigma'}{\sigma} \right] \quad \int \Sigma \sigma' \quad F\left[ a_2, a_1, \frac{\sigma'}{\sigma} \right]
\]

Besides the explicit evaluation of \( F[...] \), it would be interesting to clarify the relation of this quantity to symplectic geometry of the moduli space of semi-degenerate monodromy data.

Monodromy preserving deformations and Fuchsian systems are also related to quasi-classical (\( c \to \infty \)) limit of CFT [47,52,56,57]. One may wonder whether a
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direct connection between the quasi-classical and $c = N - 1$ $W_N$-conformal blocks can be established. Quantitative aspects of this relation for $N = 2$ have been the subject of recent study [46].

**Acknowledgements** We would like to thank M. Bershtein, A. Marshakov, R. Santachiara and G. Watts for useful discussions and comments. The present work was supported by the NASU-CNRS PICS Project "Isomonodromic deformations and conformal field theory." The work of P.G. was partially supported the Russian Academic Excellence Project '5-100' and by the RSF Grant No. 16-11-10160. In particular, formulas of Section 4 have been obtained using support of Russian Science Foundation. P.G. is a Young Russian Mathematics award winner and would like to thank its sponsors and jury. The work of N.I. was partially supported by NAS of Ukraine (Project No. 0117U000238). N.I. thanks Max Planck Institute for Mathematics (Bonn), where a part of this research was done, for hospitality and excellent working conditions.

**A Singular vectors of semi-degenerate Verma modules**

For $c = N - 1$, we are going to use a free-boson realization of the $W_N$-algebra to find the singular vectors of the semi-degenerate Verma modules. It will be helpful to extend $W_N = W(\mathfrak{sl}_N)$ to $W(\mathfrak{gl}_N)$ by introducing one more free-boson field $J(z)$ with the OPE

$$J(z) J(w) = \frac{1/N}{(z - w)^2} + \text{regular},$$

and regular OPEs with the currents $J_k(z)$ entering the definition of $W_N$: $J_k(z) J(w) = \text{regular}$. Introduce the currents $\tilde{J}_k(z) = J_k(z) + J(z)$ which have the following OPEs:

$$\tilde{J}_k(z) \tilde{J}_l(w) = \frac{\delta_{kl}}{(z - w)^2} + \text{regular},$$

and define the generators of the $W(\mathfrak{gl}_N)$-algebra:

$$\tilde{W}^{(s)}(z) = \sum_{1 \leq i_1 < \ldots < i_s \leq N} : \tilde{J}_{i_1}(z) \cdots \tilde{J}_{i_s}(z) :, \quad s = 1, \ldots, N. \quad (92)$$

We will use the mode expansion

$$\tilde{J}_k(z) = \sum_{p \in \mathbb{Z}} \tilde{a}^{(k)}_p z^{p+1}, \quad (93)$$

with modes acting on the bosonic Fock space $\mathcal{F}_\theta$ generated from the vacuum state $|\theta\rangle$, $\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{C}^N$:

$$\tilde{a}^{(k)}_0 |\theta\rangle = \theta_k |\theta\rangle, \quad \tilde{a}^{(k)}_p |\theta\rangle = 0, \quad k = 1, \ldots, N, \quad p > 0. \quad (94)$$

The semi-degenerate module $\tilde{L}_\theta$, $\theta = (\alpha, 0, \ldots, 0)$, of the $W(\mathfrak{gl}_N)$-algebra is a $W(\mathfrak{gl}_N)$-submodule of $\mathcal{F}_\theta$ generated by $|\theta\rangle$. In what follows, only such $\theta$ will be
used. For the action of modes of \( \tilde{W}^{(s)}(z) \) defined by

\[
\tilde{W}^{(s)}(z) = \sum_{p \in \mathbb{Z}} \tilde{W}^{(s)}_{p} z^{p+s},
\]

using (92), (93) and (94), we have the following relations in \( W(gl_{N}) \)-module \( \tilde{L}_{\theta} \):

\[
\tilde{W}^{(s)}_{-p} |\theta\rangle = 0, \quad 0 \leq p \leq s - 2.
\] (95)

We would like to find the implication of these relations for \( W(sl_{N}) \)-submodule \( L_{\theta} \) of module \( \tilde{L}_{\theta} \) generated by \( |\theta\rangle \). We have a relation which can be obtained from the expansion of (92) with the use of (23):

\[
\tilde{W}^{(s)}(z) = \sum_{r=0}^{s} \binom{N-r}{N-s} J^{s-r}(z) :W^{(r)}(z):.
\] (96)

In order to find the relations for the elements of \( W_{N} = W(sl_{N}) \) acting on the vector \( |\theta\rangle \) in the module \( \tilde{L}_{\theta} \), let us use the relation (96) acting on \( |\theta\rangle \) modulo vectors from the \( W_{N} \)-submodule \( L_{\theta}' \) generated by \( a_{-p_{1}} \cdots a_{-p_{l}}|\theta\rangle \), \( l > 0, \ p_{1}, \ldots, p_{l} > 0 \). We have, for \( p > 0 \),

\[
\tilde{W}^{(s)}_{-p} |\theta\rangle = \sum_{r=0}^{s} \binom{N-r}{N-s} (a/N)^{s-r} W^{(r)}_{-p} |\theta\rangle \mod L_{\theta}'.
\] (97)

These relations are linear and can be inverted:

\[
W^{(r)}_{-p} |\theta\rangle = \sum_{s=0}^{r} \binom{N-s}{N-r} (-a/N)^{r-s} \tilde{W}^{(s)}_{-p} |\theta\rangle \mod L_{\theta}'.
\] (98)

Using (95), rewrite (98) as

\[
W^{(r)}_{-p} |\theta\rangle = \sum_{s=2}^{p+1} \binom{N-s}{N-r} (-a/N)^{r-s} \tilde{W}^{(s)}_{-p} |\theta\rangle \mod L_{\theta}'
= \sum_{s=2}^{p+1} \binom{N-s}{N-r} (-a/N)^{r-s} \sum_{t=2}^{s} \binom{N-t}{N-s} (a/N)^{s-t} W^{(t)}_{-p} |\theta\rangle \mod L_{\theta}'.
\] (99)

Since both hand sides of the relation are written in terms of elements of \( W(sl_{N}) \) subalgebra, it can be considered as an exact relation in \( W(sl_{N}) \)-submodule \( L_{\theta} \). After
summation over $s$ and changing summation index $t$ to $s$, we finally get

$$\left(W^{(r)}_{-p} + (-1)^{r+p} \sum_{s=2}^{p+1} \binom{N-s}{r-s} \binom{r-s-1}{p-s+1} \left(\frac{a}{N}\right)^{r-s} W^{(s)}_{-p}\right) |\theta\rangle = 0,$$

$$2 \leq p + 1 < r \leq N.$$

(100)

B Null vectors and fusion rules for completely degenerate fields

This appendix uses a free-fermionic realization of the extension of $W_N = W(sl_N)$ to $W(gl_N)$. It is convenient since the fermionic fields realize completely degenerate fields for $W(gl_N)$ and their properties can be studied easily.

The algebra of $N$-component free-fermionic fields is generated by $\psi^+_\alpha(z), \psi^-_\alpha(w)$, $\alpha = 1, \ldots, N$, with the standard singular part of the OPEs:

$$\psi^+_\alpha(z) \psi^-_\beta(w) \sim \delta_{\alpha, \beta} \frac{z-w}{z-w}, \quad \psi^+_\alpha(z) \psi^+_\beta(w) \sim 0, \quad \psi^-_\alpha(z) \psi^-_\beta(w) \sim 0. \quad (101)$$

The $W(gl_N)$-algebra is a subalgebra of the free-fermionic algebra generated by the fields

$$\hat{W}^{(k)}(z) = \sum_{\alpha=1}^{N} \partial^{k-1} \psi^+_\alpha(z) \psi^-_\alpha(z) \circ, \quad k = 1, \ldots, N,$$  

(102)

where $\circ \circ \circ$ denotes fermionic normal ordering (moving positive fermionic modes to the right). It is convenient to extend this definition to all integer $k > 0$. Using the bosonization formulas $\psi^\pm_\alpha(z) \equiv \exp(\pm i \phi_\alpha(z)) \circ, \alpha = 1, \ldots, N$, we get a free-boson realization of $\hat{W}^{(k)}(z)$ as differential polynomials in bosonic currents $\tilde{J}_\alpha(z) = i \partial \phi_\alpha(z)$:

$$\hat{W}^{(k)}(z) = \frac{1}{k} \sum_{\alpha=1}^{N} \partial^k e^{i \phi_\alpha(z)} \cdot e^{-i \phi_\alpha(z)} \circ. \quad (103)$$

This free-boson realization of the currents $\hat{W}^{(k)}(z)$ does not coincide with the currents $\tilde{W}^{(k)}(z)$ given by the formula (92) of "Appendix A", but they generate the same algebra (a proof of this fact for fermionic realization (102) of generators of $W(gl_N)$ can be found in e.g. [8, the argument after Eq. (2.10))]. A similar fermionic realization of $W(gl_N)$ was used in [28].

The OPE

$$\hat{W}^{(k)}(z) \psi^+_\alpha(w) \sim \frac{\partial^{k-1} \psi^+_\alpha(w)}{z-w}.$$

(104)
\[ \hat{W}^{(k)}_{-k} | \psi_{\alpha}^+ \rangle = L^{k-1}_{-1} | \psi_{\alpha}^+ \rangle, \quad \hat{W}^{(k)}_m | \psi_{\alpha}^+ \rangle = 0, \quad m > 1 - k, \quad (105) \]

which gives us the list of convenient null vectors for the degenerate fields \( \psi_{\alpha}^+ (z) \). (They can also be rewritten in terms of standard generators.) Note that the OPEs (104) are the same for all \( \alpha \) and, in fact, give the OPEs of the \( W_N \)-algebra currents with the completely degenerate fields.

To find the fusion rules for \( \psi_{\alpha}^+ (z) \), consider the conformal block

\[ \Omega^{(k)}_{\alpha} (z) = \langle \theta_\infty | \hat{W}^{(k)} (z) \psi_{\alpha}^+ (1) | \theta_0 \rangle. \quad (106) \]

Since

\[ \langle \theta_\infty | \psi_{\alpha}^+ (t) | \theta_0 \rangle = t^{\Delta_\infty - \Delta_{\theta_0} - \Delta (\psi_{\alpha})} = t^{\frac{1}{2} (\theta_\infty^2 - \theta_0^2 - 1)}, \quad (107) \]

where \( \Delta (\psi_{\alpha}) = 1/2 \) is the conformal dimension of the field \( \psi_{\alpha}^+ (t) \), we have

\[ \langle \theta_\infty | \partial^{k-1} \psi_{\alpha}^+ (1) | \theta_0 \rangle = [\frac{\theta_\infty^2 - \theta_0^2 - 1}{2}]_{k-1} =: A_k, \quad (108) \]

where \( [x]_k = x (x - 1) \cdots (x - k + 1) \) denotes the falling factorial. This fixes the singular part of \( \Omega^{(k)}_{\alpha} (z) \) near \( z = 1 \) because of (104):

\[ \Omega^{(k)}_{\alpha} (z) = \frac{A_k}{z - 1} + O (1) \quad \text{as} \quad z \to 1. \quad (109) \]

In order to compute the asymptotics of \( \Omega^{(k)}_{\alpha} (z) \) near \( z = 0 \) and \( z = \infty \), one may use the mode expansion

\[ \hat{W}^{(k)} (z) = \sum_{n \in \mathbb{Z}} \hat{W}^{(k)}_n \frac{z^{-n+k}}{w_n}, \]

considered for \( |z| < 1 \) and \( |z| > 1 \), respectively. It gives the asymptotics

\[ \Omega^{(k)}_{\alpha} (z) = \frac{w_k}{z^k} + \frac{c_{k-1}}{z^{k-1}} + \cdots + \frac{c_1}{z} + O (1) \quad \text{as} \quad z \to 0, \quad (110) \]

\[ \Omega^{(k)}_{\alpha} (z) = \frac{w'_k}{z^k} + O \left( \frac{1}{z^k} \right) \quad \text{as} \quad z \to \infty, \quad (111) \]

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where \( w_k \) and \( w'_k \) are the eigenvalues \( w_k = \frac{1}{k} \sum_\alpha [\theta_0^{(\alpha)}]_k \) and \( w'_k = \frac{1}{k} \sum_\alpha [\theta_\infty^{(\alpha)}]_k \) of \( \hat{W}_0^{(k)} \) acting on the two vacua.\(^4\) Now from the asymptotics (109)–(111) of \( \Omega_\alpha^{(k)}(z) \) near 0, 1 and \( \infty \) follows an exact formula

\[
\Omega_\alpha^{(k)}(z) = \frac{w_k}{z^k} + \frac{c_{k-1}}{z^{k-1}} + \ldots + \frac{c_1}{z} + \frac{A_k}{z - 1},
\]

where

\[
c_1 = \ldots = c_{k-1} = -A_k, \quad w_k + A_k = w'_k.
\]

The equations \( w_k + A_k = w'_k \), rewritten explicitly as

\[
\sum_{\alpha=1}^N [\theta_\infty^{(\alpha)}]_k - k \left[ \frac{\theta_\infty^2 - \theta_0^2}{2} \right]_{k-1} - \sum_{\alpha=1}^N [\theta_0^{(\alpha)}]_k = 0,
\]

give restrictions on the possible values of \( \theta_\infty \) in terms of \( \theta_0 \) (fusion rules). Indeed, using the identity \([x + 1]_k - [x]_k = k [x]_{k-1}\), rewrite (114) as

\[
\sum_{\alpha=0}^N [x_\alpha]_k = \sum_{\alpha=0}^N [y_\alpha]_k,
\]

where \( k = 1, 2, \ldots \) and

\[
x_\alpha = \theta_\infty^{(\alpha)}, \quad y_\alpha = \theta_0^{(\alpha)}, \quad \alpha > 0,
\]

\[
x_0 = \frac{1}{2} \left( \theta_\infty^2 - \theta_0^2 - 1 \right), \quad y_0 = \frac{1}{2} \left( \theta_\infty^2 - \theta_0^2 + 1 \right).
\]

The equations (115) require the coincidence of all symmetric polynomials in \( N + 1 \) variables on two sets: \( X = \{x_0, x_1, \ldots, x_N\} \) and \( Y = \{y_0, y_1, \ldots, y_N\} \). This is

\(^4\) To compute \( w_k \) explicitly, it is convenient to introduce the generating function

\[
\hat{W}(z, w) = \sum_{k=1}^{\infty} \frac{(z - w)^k}{(k - 1)!} \hat{W}_0^{(k)}(w) = \sum_{k, \alpha} \frac{(z - w)^{k-1}}{(k - 1)!} \alpha^{k-1} \psi^+_{\alpha}(w) \psi^-_{\alpha}(w) \frac{\phi_{\alpha}(z)}{\phi_{\alpha}(w)}
\]

\[
= \sum_{\alpha} \psi^+_{\alpha}(z) \psi^-_{\alpha}(w) \frac{\phi_{\alpha}(z)}{\phi_{\alpha}(w)} - \frac{N}{z - w}.
\]

In the bosonized picture, this generating function becomes \( \hat{W}(z, w) = \sum_{\alpha} e^{i \phi_{\alpha}(z)} e^{-i \phi_{\alpha}(w)} = - \frac{N}{z - w} \). Now, computing the expectation value of the product of this expression and two exponential fields, one obtains

\[
\langle \hat{W}(z, w) : e^{-i(\theta, \phi(\infty))} \rangle = \sum_{\alpha} \frac{\theta_{\alpha} w^{-\theta_{\alpha}} - 1}{z - w} = \sum_{k=1}^{\infty} \frac{(z - w)^{k-1}}{(k - 1)!} w^{-k} \frac{[\theta^{(\alpha)}]_k}{k}.
\]
possible only if these two sets coincide. Since $x_0 \neq y_0$, it means that there exist $\alpha', \alpha'' > 0$ such that

$$\theta_{\infty}^{(\alpha'')} = \frac{1}{2} (\theta_{\infty}^2 - \theta_0^2 + 1), \quad \theta_0^{(\alpha')} = \frac{1}{2} (\theta_{\infty}^2 - \theta_0^2 - 1).$$  \hspace{1cm} (117)

and the sets formed by all the other $\theta$’s coincide. We immediately deduce from these equations that

$$\theta_{\infty}^{(\alpha'')} = \theta_0^{(\alpha')} + 1.$$  \hspace{1cm} (118)

Since the variables in the sets $X$ and $Y$ are not independent, one also has to check consistency of the obtained solution; indeed,

$$\theta_0^{(\alpha')} = \frac{1}{2} ((\theta_0^{(\alpha')} + 1)^2 - (\theta_0^{(\alpha')})^2 - 1).$$  \hspace{1cm} (119)

Finally, let us recall that the $W(gl_N)$-modules generated from $\theta$ and $\theta'$ are isomorphic if the components of $\theta'$ are obtained from those of $\theta$ by a permutation. From this point of view, the fusion rules (118) can be rewritten as $N$ possible channels (labeled by $\alpha = 1, \ldots, N$) of changing $\theta_0$ to obtain $\theta_{\infty}$:

$$\theta_{\infty}^{(\alpha)} = \theta_0^{(\alpha)} + 1, \quad \theta_{\infty}^{(\beta)} = \theta_0^{(\beta)}, \quad \beta \neq \alpha.$$  \hspace{1cm} (120)

Moreover, each of these fusion channels is realized by the fusion with $\psi_\alpha^+ (z), \alpha = 1, \ldots, N$. This claim becomes clear in the bosonized picture, where

$$\psi_\alpha^+ (z) = : e^{i \phi_\alpha (z)} : , \quad |\theta_0 \rangle = : e^{i (\theta_0 - \phi (0))} : |0 \rangle.$$  \hspace{1cm} (121)

Returning to $W_N = W(sl_N)$, we introduce the bosonic field $\phi_\alpha (z) = N^{-1} \sum_{\alpha=1}^N \phi_\alpha (z)$ and correct the fermionic fields by $\exp i \phi (z)$ to obtain the fields

$$\psi_\alpha (z) = : e^{-i \phi (z)} \psi_\alpha^+ (z) :, \quad \bar{\psi}_\alpha (z) = : e^{i \phi (z)} \psi_\alpha^- (z) :.$$  \hspace{1cm} (122)

They have the following fusion rules:

$$\langle \theta_{\infty} | \psi_\alpha (z) | \theta_0 \rangle \neq 0 \quad \text{if and only if} \quad \theta_{\infty} = \theta_0 + h_\alpha,$$

$$\langle \theta_{\infty} | \bar{\psi}_\alpha (z) | \theta_0 \rangle \neq 0 \quad \text{if and only if} \quad \theta_{\infty} = \theta_0 - h_\alpha,$$  \hspace{1cm} (123, 124)

where $h_\alpha, \alpha = 1, \ldots, N$, are the weights of the first fundamental representations of $sl_N$, with components $h_\alpha^{(s)} = \delta_{\alpha,s} - 1/N$.

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