Research Article

Investigation of Extended $k$-Hypergeometric Functions and Associated Fractional Integrals

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Hypergeometric functions have many applications in various areas of mathematical analysis, probability theory, physics, and engineering. Very recently, Hidan et al. (Math. Probl. Eng., ID 5535962, 2021) introduced the $(p, k)$-extended hypergeometric functions and their various applications. In this line of research, we present an expansion of the $k$-Gauss hypergeometric functions and investigate its several properties, including, its convergence properties, derivative formulas, integral representations, contiguous function relations, differential equations, and fractional integral operators. Furthermore, the current results contain several of the familiar special functions as particular cases, and this extension may enrich the theory of special functions.

1. Introduction

Special functions are important tools in solving certain problems arising from many different research areas in mathematical physics, astronomy, chemistry, applied statistics, and engineering (see, e.g., [1–3]). Hypergeometric functions are among most important special functions mainly because they have a lot of applications in a variety of research branches such as (for example) quantum mechanics, electromagnetic field theory, probability theory, analytic number theory, and data analysis (see, e.g., [1, 2, 4–6]). Also, a number of elementary functions and special polynomials are expressed in terms of hypergeometric functions. Accordingly, a number of various extensions of hypergeometric functions have been introduced and investigated.

Throughout this research, $\mathbb{N}$: $= \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ denotes the set of negative integers, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$, $\mathbb{R}^+$ denotes the set of positive real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

Traditionally, the hypergeometric function known as Gauss function is defined by

$$W(v) = F(\delta_1, \delta_2, \delta_3; v) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n} \frac{v^n}{n!}, \quad v \in \mathbb{C}, \quad (1)$$

which is absolutely and uniformly convergent if $|v| < 1$, divergent when $|v| > 1$, and absolutely convergent when $|v| = 1$, if $\text{Re}(\delta_3 - \delta_1 - \delta_2) > 0$, where $\delta_1, \delta_2,$ and $\delta_3$ are complex parameters with $\delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and
is the Pochhammer symbol (or the shifted factorial) and \( \Gamma(\cdot) \) is gamma function. The function in (1) satisfies the following differential equation:
\[
v(1-v)Wu + [\delta_3 - (\delta_1 + \delta_2 + 1)v]Wv - \delta_1 \delta_2 W = 0.
\] (3)

Nowadays, numerous investigations, for example, in recent works of Srivastava et al. [7, 8], Jana et al. [9, 10], Goswami et al. [11, 12], Fuli et al. [13], and Abdalla and Bakhet [14, 15] to introduce extensions and generalizations of the hypergeometric functions, defined by Euler-type integrals, are associated with properties and applications.

In particular, Diaz and Pariguan [16] introduced the \( k \)-analogue of gamma, beta, and hypergeometric functions and proved a number of their properties. Since that period, many different results concerning the \( k \)-hypergeometric function and related functions have been considered by many researchers, for instance, Agarwal et al. [17], Mubeen et al. [18–20], Rahman et al. [21], Chinch et al. [22], Korkmaz-Duzgun and Erkus-Duman [23], Nisar et al. [24], Li and Dong [25], Yilmaz et al. [26], Hidan et al. [27], and Yilmazer and Ali [28].

Motivated by some of these aforesaid studies of the \( k \)-hypergeometric functions and related functions, we introduce the \( (p,k) \)-extended Gauss and Kummer hypergeometric functions and their properties. Relevant connections of some of the discussed results here with those presented in earlier references are outlined.

The manuscript is organized as follows. In Section 2, we list some basic definitions and terminologies that are needed in the paper. In Section 3, we introduce the \( (p,k) \)-extended Gauss and Kummer (or confluent) hypergeometric functions and discuss their regions of convergence. In Section 4, we obtain integral and differentiation formulas of the \( (p,k) \)-extended Gauss and Kummer hypergeometric functions. In addition, contiguous function relations and differential equations connecting these functions are established in Section 5. Compositions of the \( k \)-Riemann–Liouville fractional integral operators of these functions are presented in Section 6. Finally, we point out outlook and observations in Section 7.

## 2. Preliminaries

In this section, we give some basic definitions and terminologies which are used further in this manuscript.

**Definition 1** (see [16, 26]). For \( k \in \mathbb{R}^+ \), the \( k \)-gamma function \( \Gamma_k(u) \) is defined by
\[
\Gamma_k(u) = \int_0^\infty y^{u-1}e^{-(y/k)}dy,
\] (4)

where \( u \in \mathbb{C}\setminus k\mathbb{Z}^- \). We note that \( \Gamma_k(u) \rightarrow \Gamma(u) \), for \( k \rightarrow 1 \), where \( \Gamma(u) \) is the classical Euler's gamma function and \( (u)_{m,k} \) is the \( k \)-Pochhammer symbol given in the form
\[
(u)_{m,k} = \Gamma_k(u + m - 1)/\Gamma_k(u),
\] (5)

where \( u \in \mathbb{C}\setminus k\mathbb{Z}^- \). The relation between \( \Gamma_k(u) \) and gamma function \( \Gamma(u) \) follows easily that
\[
\Gamma_k(u) = k(u/k)^{-1}\Gamma_k\left(\frac{u}{k}\right).
\] (6)

**Definition 2** (see [16, 26]). For \( u, v \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \), the \( k \)-beta function \( B_k(u,v) \) is defined by
\[
B_k(u,v) = \frac{1}{k} \int_0^1 y^{(u/k)-1}(1-y)^{v(k)-1}dy = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)},
\] (7)

where \( \text{Re}(u) > 0 \) and \( \text{Re}(v) > 0 \).

Clearly, the case \( k = 1 \) in (7) reduces to the known beta function \( B(u,v) \), and the relation between the \( k \)-beta function \( B_k(u,v) \) and the original beta function \( B(u,v) \) is
\[
B_k(u,v) = \frac{1}{k} B\left(\frac{u}{k}, \frac{v}{k}\right).
\] (8)

**Definition 3** (see [16, 26, 28]) Let \( k \in \mathbb{R}^+ \) and \( s_1, s_2, \eta \in \mathbb{C} \) and \( s_1 \in \mathbb{C}\setminus \mathbb{Z}^- \); then, \( k \)-Gauss hypergeometric function is defined in
\[
U(\eta) = \sum_{m=0}^\infty \frac{(s_1)_{m,k}(s_2)_{m,k}}{(s_3)_{m,k}} \eta^m / m!, \quad |\eta| < \frac{1}{k},
\] (9)

where \((s_1)_{m,k}\) is the \( k \)-Pochhammer symbol defined in (5). Obviously, if \( k = 1 \), equation (9) is reduced to (1).

**Proposition 1** (see [16, 26]) For any \( \delta \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \), the following identity holds
\[
\gamma_k\left(\frac{\delta}{-\eta}\right) = \sum_{n=0}^\infty (\delta)_{n,k} \eta^n / n!, \quad |\eta| < \frac{1}{k},
\] (10)

The \( k \)-hypergeometric differential equation of second order is defined in [18, 25, 26, 28] by
\[ k\eta(1-k\eta)U^{n} + [s_{3} - (s_{1} + s_{2} + k)\eta]U^{n-1} - s_{1}s_{2}U = 0. \]  

(11)

Particular choices of the parameters \(s_{1}, s_{2}, s_{3}\), and \(k\) in the linearly independent solutions of the differential equation (11) yield more than 24 special cases. Also, the \(k\)-hypergeometric function can be given an integral representation in the following result [20, 26]:

**Theorem 1.** Assume that \(\eta, s_{1}, s_{2}, s_{3} \in \mathbb{C}\) such that \(\text{Re}(s_{3}) > \text{Re}(s_{2}) > 0\) and \(k \in \mathbb{R}^{+}\); then, the integral formula of the \(k\)-hypergeometric function is given by

\[
\int_{0}^{1} y^{(s_{2}/k)-1} (1 - y)^{(s_{1}-s_{2})/k-1} (1 - k\eta y)^{-s_{1}/k} dy. \tag{12}
\]

(2) Setting \(k = 1\), we obtain a \(p\)-extension of the Gauss and Kummer hypergeometric functions in the following forms, respectively (see Chapter 3 in [29]):

\[
\mathcal{W}(p; \xi) = 2\mathcal{B}_{1}^{(p,1)} \left[ \frac{\xi_{1}, \xi_{2}}{\xi_{3}} ; \xi \right] \tag{16}
\]

\[
\mathcal{Y}(p; \xi) = \mathcal{B}_{1}^{(p,1)} \left[ \frac{\xi_{1}}{\xi_{3}} ; \xi \right] \tag{17}
\]

(3) Taking \(k = 1\) and \(p = 1\) in (11), we produce the standard Gauss hypergeometric function in (1).

(4) When \(k = 1\) and \(p = 1\), (12) yields the following special case (see, e.g., [1, 2]).

The following theorem shows the convergence property of series (14).

**Theorem 2.** For all \(k \in \mathbb{R}^{+}\) and \(p > 1\), the \((p, k)\)-extended Gauss hypergeometric function \(\mathcal{W}(p, k; \xi)\) given by (14) is an entire function.

**Proof.** For this proof, we relabel and write (14) as

\[
\mathcal{B}_{1}^{(p,k)} \left[ \frac{\xi_{1}, \xi_{2}}{\xi_{3}} ; \xi \right] = \sum_{m=0}^{\infty} U_{m}(\xi), \tag{18}
\]

where

\[
U_{m}(\xi) = \frac{(\xi_{1})_{m,k}(\xi_{2})_{m,k}}{(\xi_{3})_{m,k}} \cdot \xi^{m} (pm)^{m}. \tag{19}
\]

By using the ratio test and according to the identity \((\xi)_{n+1,k} = (\xi + mk)(\xi)_{m,k}\), we see that
Thus, the power series (14) is convergent for all $|\xi| < \infty$, under the hypothesis $p > 1$, $k \in \mathbb{R}^+$, and $\zeta \in \mathbb{C} \setminus \mathbb{Z}_0$. Thus, it yields our desired result.

The following result can be verified in a similar way.

**Theorem 3.** For all $k \in \mathbb{R}^+$ and $p > 1$, the $(p,k)$-extended Kummer hypergeometric function $\mathcal{Y}(p,k; \xi)$ given by (15) is an entire function.

**Corollary 1.** For all $p > 1$, the power series (16) and (17) are an entire function.

**Remark 2.** For $p = 1$ in Theorems 2 and 3, we get the convergence property of the $k$-Gauss hypergeometric function $\mathcal{M}(1,k; \xi)$ and the $k$-Kummer hypergeometric function $\mathcal{Y}(1,k; \xi)$, provided that $k \in \mathbb{R}^+$ and $\zeta \in \mathbb{C} \setminus \mathbb{Z}_0$ (see [16]).

\[
\lim_{m \to \infty} \frac{U_{m+1}(\xi)}{U_m(\xi)} = \lim_{m \to \infty} \left( \frac{\zeta_1_{m+1,k} \zeta_2_{m+1,k}^{m+1}}{(p(m+1))! \zeta_1_{m,k} \zeta_2_{m,k}^m \Gamma_k(\zeta_1/2) \Gamma_k(\zeta_2/2)} \right)
\]

\[
= \lim_{m \to \infty} \frac{(\zeta_1 + mk)(\zeta_2 + mk)(pm)!\xi^n}{(\zeta_1 + mk)(\zeta_2 + mk)(pm + p - 1)\cdots(pm + 1)(pm)!}\]

\[
= \lim_{m \to \infty} \frac{m^2((\zeta_1/m) + k)((\zeta_2/m) + k)}{m^{p + 1}((\zeta_1/m) + k)(p + (p/m))(p + ((p - 1)/m))\cdots(p + (1/m))}\]

\[
= 0.
\]

**Remark 3.** For $p = 1$ in Corollary 1, we obtain the convergence property of the usual Gauss and Kummer hypergeometric series (see [1, 2]).

### 4. Integral Representations and Derivative Formulae

#### 4.1. Integral Representations

In the following, we establish the following theorems in terms of the $k$-integral representations of the $(p,k)$-extended Gauss and Kummer hypergeometric functions.

**Theorem 4.** The following integral representation for $\mathcal{F}_{1}^{(p,k)}$ in (14) holds true:

\[
\mathcal{F}_{1}^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \\ \end{array} ; \xi \right] = \frac{\Gamma_k(\zeta_1)}{\Gamma_k(\zeta_2) \Gamma_k(\zeta_3 - \zeta_2)} \int_0^1 t^{(\zeta_1/k - 1)}(1 - t)^{((\zeta_1/2) - k)}t^{(\zeta_1 - \zeta_2)} dt,
\]

where $\zeta_1, \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0$, $\Re(\zeta_1) > \Re(\zeta_2) > 0$, $\Re(\zeta_1) > 0, k \in \mathbb{R}^+$, and $p > 1$.

**Proof.** Considering the following elementary identity involving the $k$-Beta function $B_k(u,v)$,

\[
\left( \frac{\zeta_1^{m,k}}{\zeta_2^{m,k}} \right) = \frac{\Gamma_k(\zeta_1 + mk)}{\Gamma_k(\zeta_2) \Gamma_k(\zeta_3 - \zeta_2)} \frac{\Gamma_k(\zeta_1)}{\Gamma_k(\zeta_3 - \zeta_2)}
\]

\[
= \frac{\Gamma_k(\zeta_1)}{\Gamma_k(\zeta_2) \Gamma_k(\zeta_3 - \zeta_2)} \frac{1}{k} \int_0^1 t^{(\zeta_1/k) - m - 1}(1 - t)^{((\zeta_1 - \zeta_2)/k - 1)} dt, \quad (\Re(\zeta_1) > \Re(\zeta_2)),
\]

in (14), and using relation (17), we get the required integral formula (21).

**Theorem 5.** The following integral representation for $\mathcal{F}_{1}^{(p,k)}$ in (14) holds true:
\[ 2\mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \zeta \right] = \frac{1}{\Gamma(\zeta)} \int_0^\infty t^{\zeta-1} e^{-t} \mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_2 \\ \zeta_3 \end{array} ; t \right] dt, \]

(23)

where \( \zeta, \zeta_1, \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \Re(\zeta_1) > 0, p > 1, \) and \( k \in \mathbb{R}^+ \) when \( p = 1. \)

**Proof.** Inserting the \( k \)-Pochhammer symbol \((\alpha)_n\) from (5) in definition (14) by its integral form given by (4) and from relation (15), we thus obtain the desired result (23).

By virtue of the same theorems, we give the following theorem:

**Theorem 6.** The following integral representations for \( \mathcal{B}_1^{(p,k)} \) in (15) hold true:

\[ \mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \zeta \right] = \frac{1}{\Gamma(\zeta_2)} \int_0^\infty t^{\zeta_2-1} e^{-t} \mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_2 \\ \zeta_3 \end{array} ; \zeta \right] dt, \]

(24)

**Remark 5.** The substitution \( p = 1 \) in (21)–(25) leads to the integral formulas of the \( k \)-analogue of Gauss and Kummer hypergeometric functions (see [20, 24, 26]).

Also, the special cases of (3.6)–(3.9) when \( k = 1 \) and \( p = 1 \) are seen to yield the classical integral representations of the Gauss and Kummer hypergeometric functions (see, e.g., [1, 2, 6]).

**4.2. Derivative Formulae**

**Theorem 7.** The following derivative formulas hold true:

\[ \frac{d^n}{d\zeta^n} \left\{ 2\mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \zeta \right] \right\} = \left( \frac{(\zeta_1)_{nk}(\zeta_2)_{nk}}{p^{\zeta_3} n_k} \right) 2\mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 + nk, \zeta_2 + nk \\ \zeta_3 + nk \end{array} ; \zeta \right] \]

(26)

and

\[ \frac{d^n}{d\zeta^n} \left\{ \mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 \\ \zeta_3 \end{array} ; \zeta \right] \right\} = \frac{1}{p} \left( \frac{\zeta_1 \zeta_2}{\zeta_3} \right) 2\mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 + k, \zeta_2 + k \\ \zeta_3 + k \end{array} ; \zeta \right]. \]

(27)

where \( k \in \mathbb{R}^+, p > 1, \) and \( n \in \mathbb{N}_0. \)

**Proof.** Result (26) is obviously valid in the trivial case when \( n = 0. \) For \( n = 1, \) by the power series representation (14) of \( 2\mathcal{B}_1^{(p,k)}, \) we see from (26) that

\[ \frac{d}{d\zeta} \left\{ 2\mathcal{B}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \zeta \right] \right\} = \sum_{m=1}^\infty \left( \frac{(\zeta_1)_{mk}(\zeta_2)_{mk}}{\Gamma(\zeta_3+pm-1)} \right) \zeta_3^{m-1}. \]

(28)

Replacing the \( k \)-Pochhammer symbols \((\zeta_1 + k)_{mk}\) by relation (5), we arrive at

Proof.
where \( k \in \mathbb{R}^+, \ p > 1, \) and \( n \in \mathbb{N}_0. \)

**Proof.** By using series (14) in (30) and differentiating term by term under the sign of summation, we observe that

\[
\frac{d^n}{d\xi^n} \left[ \xi^{\zeta_1-1} \Phi_2^{(p,k)} \left( \zeta_1, \zeta_2; \xi \right) \right] = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \frac{d^n}{d\xi^n} \left[ \xi^{km\zeta_1-1} \right] \]  

\[
= \Gamma_k(\zeta_3) \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3 + mk - n)(pm)!} \xi^{km},
\]

which, in view of series (14), yields the coveted formula (30).

Similarly, we can derive the derivative formula (31). \( \square \)

**Remark 6.** The special cases of (30) and (31) when \( p = 1 \) are easily seen to reduce to the known derivative formulas of the \( k \)-Gauss and Kummer hypergeometric functions (see [18]).

**Remark 7.** If we take \( p = 1 \) and \( k = 1 \) in the abovementioned theorems, we obtain the corresponding results for the classical hypergeometric functions \( _2\Phi_1^{(1,1)} \) and \( _1\Phi_1^{(1,1)} \) (cf. [6]).

### 5. Contiguous Function Relations and Differential Equations

The \( k \)-analogue of theta operator \( k\Theta \), as given in [18, 19, 25], takes the form \( k\Theta = k\xi (d/d\xi) \). This operator has the particularly pleasant property that \( k\Theta \xi^m = km\xi^m \), which makes it handy to be used on power series. In this section, relying on Definition 1, we present some results concerning contiguous function relations and differential equations for the \((p,k)\)-extended Gauss hypergeometric function \( _2\Phi_1^{(p,k)} \) and \((p,k)\)-extended Kummer hypergeometric function \( _1\Phi_1^{(p,k)} \).

To realize that, we increase or decrease one or more of the parameters of the \((p,k)\)-extended Gauss hypergeometric function:

\[
W = W(p,k;\xi) = \Phi_2^{(p,k)} \left[ \zeta_1, \zeta_2; \xi \right], \quad k \in \mathbb{R}^+, \ p > 1.
\]

(33)

By \( \pm k \), then the resultant function is said to be contiguous to \( W(p,k;\xi) \). For simplicity, we use the following notations:

\[
W(p,k;\zeta_1 \pm k) = \Phi_2^{(p,k)} \left[ \zeta_1 \pm k, \zeta_2; \xi \right].
\]

(34)

Now, we consider

\[
W(p,k;\zeta_1 +) = \sum_{m=0}^{\infty} \frac{(\zeta_1 + k)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \frac{\xi^m}{(pm)!}.
\]

(35)

where \( a_1(a_1 + k)_{nk} = (a_1 + nk)(a_1)_{nk} \) and \( U_{m,k}(\xi) \) is defined in (19).

Similarly, we can write \( W(p,k;\zeta_1 -) \) as

\[
W(p,k;\zeta_1 -) = \sum_{m=0}^{\infty} \frac{(\zeta_1 - k)_{m,k}}{(\zeta_3)_{m,k}} U_{m,k}(\xi),
\]

(36)

where \((\zeta_1 (m - 1)k)(\zeta_1 - k)_{m,k} = (\zeta_1 - k)(\zeta_1)_{m,k} \). Similarly, for \( W(p,k;\zeta_2 \pm) \), and \( W(p,k;\zeta_3 \pm) \).

By the help of differential operator \( k\Theta = k\xi (d/d\xi) \), we get the following relations:

\[
(k\Theta + \zeta_1)W(p,k;\xi) = \zeta_1 W(p,k;\zeta_1 +),
\]

(37)

\[
(k\Theta + \zeta_2)W(p,k;\xi) = \zeta_2 W(p,k;\zeta_2 +),
\]

and \( (k\Theta + \zeta_3 - k)W(p,k;\xi) = (\zeta_3 - k)W(p,k;\zeta_3 -) \).

From the above relations, we can easily obtain the following results:
\[ (\zeta_1 - \zeta_2) \mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1) - \zeta_2 \mathcal{W}(p, k; \zeta_2), \quad (38) \]
\[ (\zeta_1 - \zeta_2) \mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1 + \zeta_2) - \zeta_2 \mathcal{W}(p, k; \zeta_1), \quad (39) \]
\[ (\zeta_1 - \zeta_3 + k) \mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1) - (\zeta_3 - k) \mathcal{W}(p, k; \zeta_1 + 2k), \quad (40) \]
\[ (\zeta_2 - \zeta_3 + k) \mathcal{W}(p, k; \xi) = \zeta_2 \mathcal{W}(p, k; \zeta_2) - (\zeta_3 - k) \mathcal{W}(p, k; \zeta_2 + 2k). \quad (41) \]

**Remark 8.** Other contiguous function relations for the \(k\)-Gauss hypergeometric function may be derived from the relations in (38) to (41) and the same manner, and other results can also be obtained.

**Remark 9.** We can easily obtain many known results in [19, 22] by setting the parameters in our main findings. Therefore, the obtained results here extend to those results.

**Remark 10.** It is easy to see that, in (38) to (41), if we take \(k = 1\), we get hypergeometric contiguous function relations (see [6]). Furthermore, the operator \(k\theta = k \xi (\partial / \partial \xi)\), which is used in the derivation of the contiguous function relations, is also used in deriving the differential equations satisfied by \(\mathcal{W}(p, k; \xi)\) and \(\mathcal{Y}(p, k; \xi)\) as follows:

\[
\left[ \Theta \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \ldots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi)
= \sum_{m=1}^{\infty} \frac{m(m-1/p)(m-2/p)\ldots(m-(p-1)/p)(\zeta_1)_{m,k}(\zeta_2)_{m,k} \xi^m}{(m!)^m (\zeta_3)_{m,k} (\zeta_3 + m)_{m,k}}
= \frac{\xi}{p^p} \sum_{m=0}^{\infty} \frac{(\zeta_1 + m)(\zeta_2 + m) \xi^m}{(\zeta_3 + m)_{m,k}} \mathcal{U}_{m,k}(\xi).
\]

Using the following identity \((a)_{n+1,k} = (a)_{nk}(a + nk)\), we find that

\[
\left[ \Theta \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \ldots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi) = \frac{\xi}{p^p} \sum_{m=0}^{\infty} \frac{(\zeta_1 + m)(\zeta_2 + m) \xi^m \mathcal{U}_{m,k}(\xi)}{(\zeta_3 + m)}
= \frac{\xi}{p^p} \left[ k\mathcal{W}(p, k; \xi) + (\zeta_1 + \zeta_2 - \zeta_3) \mathcal{W}(p, k; \xi) + \frac{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)}{\xi_3} \mathcal{W}(p, k; \zeta_3) \right].
\]

Thus, we get the following differential equation:

\[
\left[ \Theta \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \ldots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi)
- \frac{\xi}{p^p} \left[ k\mathcal{W}(p, k; \xi) + (\zeta_1 + \zeta_2 - \zeta_3) \mathcal{W}(p, k; \xi) + \frac{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)}{\xi_3} \mathcal{W}(p, k; \zeta_3) \right] = 0.
\]
Similarly, we can derive the following result of the 
\((p, k)\)-extended Kummer hypergeometric function \(Y(p, k; \xi)\):

\[
\left[ \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \cdots \left( \Theta - \frac{p-1}{p} \right) \right] Y(p, k; \xi) = \frac{\xi(\zeta_1 - \zeta_3)}{p^p} Y(p, k; \zeta) = 0. \tag{45}
\]

In addition, we consider

\[
\left[ \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \cdots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi) = \frac{1}{p^p} \sum_{m=0}^{\infty} \frac{(km - \zeta_3 - k)(\zeta_1 m)(\zeta_2 m)}{(pm - p)! (\zeta_3 m)} \xi^m. \tag{46}
\]

Replacing \(m\) by \(m+1\) and according to the identity

\[
(\zeta_1)_{m+1,k} = (\zeta_1 + mk)(\zeta_1)_{m,k},
\]

we have

\[
\left[ \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \cdots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi) = \frac{\xi}{p^p} (\zeta_1 + mk)(\zeta_2 + mk) \mathcal{W}(p, k; \xi). \tag{47}
\]

We thus get the following differential equation:

\[
\left[ \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \cdots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{W}(p, k; \xi) = 0. \tag{48}
\]

**Remark 11.** For \(p = 1\) in (48), it obviously reduces to the usual differential equation of the \(k\)-Gauss hypergeometric function in (11) (see, [18]).

A similar procedure yields differential equation of the 
\((p, k)\)-extended Gauss hypergeometric function \(W(p, k, \xi)\),

\[
\left[ \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \cdots \left( \Theta - \frac{p-1}{p} \right) \right] \mathcal{Y}(p, k; \xi) = 0. \tag{49}
\]

**Remark 12.** The special cases of (48) and (49) when \(k = 1\) and \(p = 1\) are seen to yield the classical differential equations of Gauss and Kummer hypergeometric functions (see, for details, [6]).
6. The k-Fractional Integral Operators

Nowadays, computations of images of the k-analogues of special functions under operators of k-fractional calculus have found significant importance and applications by many references (for instance, see, [20, 21, 30–34]).

The k-Riemann–Liouville fractional integral operators of the \((p,k)\)-extended Gauss and Kummer hypergeometric functions.

\[
(\Gamma_k^p f(\tau))(x) = \frac{1}{k!} \int_0^x f(\tau)(x - \tau)^{(\nu/k)-1} \, d\tau, \quad \nu, \rho > 0, \, k \in \mathbb{R}^+.
\]

In current section, we consider compositions of the k-Riemann–Liouville fractional integral operators of the \((p,k)\)-extended Gauss and Kummer hypergeometric functions.

\[
\frac{1}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^x \frac{(x - \rho)^{(n/k)-1} (\rho - \varrho) (\zeta_1/k) - 1}{\varrho}\left[\begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} : (\rho - \varrho) \right] d\rho = \frac{k(x - \varrho) ((n\zeta_1) - 1)}{\Gamma^k(n + \zeta_3)} \left[\begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} : (x - \varrho) \right].
\]

**Theorem 9.** Assume that \(\rho, k \in \mathbb{R}^+, \, p \in \mathbb{N}, \, \zeta_1, \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0, \, \text{Re}(\zeta_1) > 0, \, \text{Re}(\zeta_2) > 0.\) Then, for \(n > 0, x > \rho,\) the following formula holds true:

**Proof.** By virtue of relations (14) and (50) and putting \(\omega = ((\rho - \varrho)/(x - \varrho)),\) we arrive at

\[
\frac{1}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^x \frac{(x - \rho)^{(n/k)-1} (\rho - \varrho) (\zeta_1/k) - 1}{\varrho}\left[\begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} : (\rho - \varrho) \right] d\rho
\]

\[
= \frac{1}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^x \frac{[(x - \rho) - (\rho - \varrho)]^{(n/k)-1} (\rho - \varrho) (\zeta_1/k) - 1}{\varrho}\left[\begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} : -(\rho - \varrho) \right] d\rho
\]

\[
= \frac{(x - \rho)^{(n\zeta_1) - 1}}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^1 (1 - \omega)^{(n/k)-1} \omega (\zeta_1/k - 1) \frac{(\zeta_1, \zeta_2)}{\varrho^k}\left[\begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} : (x - \varrho) \right] d\omega
\]

Changing order of summation and integration and applying relation (7), we obtain

\[
\frac{(x - \rho)^{(n\zeta_1) - 1}}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^1 (1 - \omega)^{(n/k)-1} \omega (\zeta_1/k - 1) \sum_{m=0}^{\infty} \frac{(\zeta_1, \zeta_2, \rho)_{\omega,\zeta_3}}{\varrho^{(m/k)}(\rho, \zeta_3)} \frac{(x - \varrho)^m}{(pm)!} d\omega
\]

\[
= \frac{(x - \rho)^{(n\zeta_1) - 1}}{\Gamma^k(n)\Gamma^k(\zeta_3)} \sum_{m=0}^{\infty} \frac{(\zeta_1, \zeta_2, \rho)_{\omega,\zeta_3}}{\varrho^{(m/k)}(\rho, \zeta_3)} \frac{(x - \varrho)^m}{(pm)!} \frac{1}{\Gamma^k(n)\Gamma^k(\zeta_3 + mk)}.
\]
We thus get the required formula (51).

\[
\frac{1}{\Gamma^k(n)\Gamma^k(\zeta_3)} \int_0^x (x-\rho)^{(m\zeta_3)-1} (\rho-\zeta_1)^{(\zeta_1/\zeta_3)-1} \, \Phi^{(p,k)}(\rho) \, d\rho = \Psi^{(p,k)}(x)
\]

Proof. The proof here would run in parallel with that of Theorem 9. The details are omitted.

Theorem 11. Let \( k \in \mathbb{R}^+, \rho \in \mathbb{N}, (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}, \zeta_3 \in \mathbb{C} \backslash \mathbb{Z}_0, \) \( \Re(\zeta_1) > 0, \Re(\zeta_2) > 0. \) Then, for \( n > 0, x > 0, \rho < \infty, \) the following formula holds true:

\[
\int_0^x (x-\rho)^{(\zeta_1/\zeta_3)-1} \, \Phi^{(p,k)}(\rho) \, d\rho = \Psi^{(p,k)}(x)
\]

Then, the proof would flow along the lines of that of Theorem 2.4 in [31]. The details are omitted.

Remark 13. The present study in the above theorems is assumed to be extensions of the results in [31].

7. Concluding Remarks

Recently, many studies and extensions of the well-known special functions have been considered by various researchers.

In this paper, we obtained a new extension of the Gauss and Kummer hypergeometric functions, so-called \((p,k)\)-extended Gauss and Kummer hypergeometric functions. Also, we gave some of their main properties, namely, the convergence properties, integral representations, differential formulas, contiguous function relations, differential equations, and fractional integral operators.

We have spotted that, by setting \( p = 1, \) the various outcomes presented in this article will reduce to the corresponding outcomes derived earlier in [16, 18, 19, 24, 25, 31]. Furthermore, if we let \( k = 1, \) then we obtain several interesting new outcomes for the \( p \)-extended Gauss and Kummer hypergeometric functions. Finally, we have spotted that if \( p = 1 \) and \( k = 1, \) then we obtain some known results for the usual Gauss and Kummer hypergeometric functions defined and established in [1, 2, 6]. Additional research and application on this topic is now under preparation and will be presented in forthcoming articles.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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