Notes on squeezed states in $x$-space representation

Alexander N. Korotkov
Department of Electrical and Computer Engineering,
University of California, Riverside, California 92521
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In these notes, we discuss squeezed states using the elementary quantum language based on one-dimensional Schrödinger equation. No operators are used. The language of quantum optics is mentioned only for a hint to solve a differential equation. Sections II and III (squeezed vacuum and pure squeezed states) can be used in a standard undergraduate course on quantum mechanics after discussion of a harmonic oscillator. Section IV discusses density matrix of mixed squeezed states in $x$-representation. These notes present explicit (therefore somewhat lengthy) step-by-step derivations. These notes are not intended for publication as a journal paper.

I. INTRODUCTION

In quantum optics [1–4], squeezed states are usually introduced via the squeezing operator and discussed in the phase space, with the formalism based on creation and annihilation operators. Here we discuss one-dimensional squeezed states in the $x$-representation, using the formalism based only on elementary quantum mechanics [5, 6].

II. SQUEEZED VACUUM

Let us consider a harmonic oscillator with mass $m$ and frequency $\omega$, so the potential energy is

$$V(x) = \frac{1}{2} m \omega^2 x^2.$$ (1)

The ground state of this oscillator is [5, 6]

$$\psi_{\text{gr}}(x,t) = \frac{e^{-i\omega t/2}}{\sqrt{2\pi \sigma_{\text{gr}}^2}} \exp\left(-\frac{x^2}{4\sigma_{\text{gr}}^2}\right), \quad \sigma_{\text{gr}}^2 = \frac{\hbar}{2m\omega},$$ (2)

where $\sigma_{\text{gr}}^2$ is the $x$-space variance of the ground state probability distribution $|\psi_{\text{gr}}(t)|^2$.

Suppose we have prepared an initial state $\psi(x,0) = \exp(-x^2/4D)/\text{Norm}$. What is the further evolution $\psi(x,t)$ of this state? If $D = \sigma_{\text{gr}}^2$, then we have prepared the ground state and therefore it will not evolve (except for the trivial time-factor $e^{-i\omega t/2}$ due to the ground state energy $\hbar \omega/2$). If $D \neq \sigma_{\text{gr}}^2$, then this state is called a squeezed vacuum (especially if $D < \sigma_{\text{gr}}^2$). Our goal in this section is to find the explicit time-dependence $\psi(x,t)$ for the squeezed vacuum in $x$-space.

In quantum optics language, the time-dependence of the squeezed vacuum corresponds to the rotation of quadrature axes with frequency $\omega$. From the picture of an ellipse representing the squeezed vacuum in optics, we know that $\psi(x,t)$ should be periodic with frequency $2\omega$ (except for an accumulating phase). And we also know that a Gaussian state should remain Gaussian with rotation of axes. Also, since $\psi(x,0)$ is symmetric (even), this symmetry should remain in time.

Therefore, the squeezed vacuum should be

$$\psi(x,t) = \frac{1}{\sqrt{2\pi \sigma_{\text{gr}}^2}} \exp\left(-\frac{x^2(1 + iB(t))}{4\sigma_{\text{gr}}^2}\right) e^{-i\phi(t)},$$ (3)

where $A(t)$, $B(t)$, and $\phi(t)$ are real dimensionless functions of time, with $A(t)$ and $B(t)$ being $2\omega$-periodic (note that this state is already normalized). In particular, for the ground state $A(t) = 1$, $B(t) = 0$, and $\phi(t) = \omega t/2$. Let us find $A(t)$, $B(t)$, and $\phi(t)$ for the squeezed vacuum.

To use the Schrödinger equation [5, 6],

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi,$$ (4)

we need to calculate $\partial \psi/\partial t$ and $\partial^2 \psi/\partial x^2$. This can be easily done for the wavefunction $\psi$,

$$\frac{\partial \psi}{\partial t} = \left[ -\frac{\dot{A}}{4A} + \frac{x^2(1 + iB)A}{4\sigma_{\text{gr}}^2 A^2} - \frac{i\dot{B}}{2\sigma_{\text{gr}}^2 A} - i\dot{\phi} \right] \psi,$$ (5)

$$\frac{\partial^2 \psi}{\partial x^2} = \left[ -\frac{(x(1 + iB))^2}{2\sigma_{\text{gr}}^2 A} - \frac{1 + iB}{2\sigma_{\text{gr}}^2 A} \right] \psi.$$ (6)

Substituting these derivatives into the Schrödinger equation, we obtain

$$i\hbar \left[ -\frac{\dot{A}}{4A} + \frac{x^2(1 + iB)A}{4\sigma_{\text{gr}}^2 A^2} - \frac{i\dot{B}}{2\sigma_{\text{gr}}^2 A} - i\dot{\phi} \right] =$$

$$= -\frac{\hbar^2}{2m} \left[ \frac{x^2(1 + iB)^2}{4\sigma_{\text{gr}}^2 A^2} - \frac{1 + iB}{2\sigma_{\text{gr}}^2 A} \right] + \frac{1}{2} m \omega^2 x^2,$$ (7)

which is the same as

$$-\frac{\dot{A}}{A} + \frac{x^2(1 + iB)}{\sigma_{\text{gr}}^2 A^2} - \frac{i\dot{B}}{\sigma_{\text{gr}}^2 A} - 4i\dot{\phi} =$$

$$= \frac{i\hbar}{2m} \left[ \frac{x^2(1 + iB)^2}{\sigma_{\text{gr}}^2 A^2} - 2 \frac{1 + iB}{\sigma_{\text{gr}}^2 A} \right] - \frac{2i}{\hbar} m \omega^2 x^2.$$ (8)

Since this relation is valid for any $x$, it gives us two equations: for $x^2$-terms and for $x^0$-terms. Let us start with the equation for $x^2$-component (multiplied by $\sigma_{\text{gr}}^2$),

$$\frac{(1 + iB)A}{A^2} - \frac{i\dot{B}}{2m\sigma_{\text{gr}}^2 A} - \frac{2i}{\hbar} m \omega^2 \sigma_{\text{gr}}^2 =$$ (9)
Using $\sigma_{gr}^2 = h/(2m\omega)$ and multiplying all terms by $A^2$, we can rewrite Eq. (9) as
\[(1 + iB)\dot{A} - iA\dot{B} = i\omega(1 + iB)^2 - i\omega A^2.\] (10)
Since $A$ and $B$ are real, this equation gives us two equations: for the real and imaginary components,
\[\dot{A} = -2\omega B,\] (11)
\[BA - A\dot{B} = \omega(1 - B^2 - A^2).\] (12)

These two equations are sufficient to find $A(t)$ and $B(t)$. We will discuss the solution a little later; before that, let us consider the $x^0$-component of Eq. (5), which is
\[-\frac{\dot{A}}{A} - 4i\dot{\varphi} = -\frac{i\hbar}{m} \frac{1 + iB}{\sigma_{gr}^2 A} = \frac{2\omega}{A} (-i + B).\] (13)
The real part of this equation gives $\dot{A} = -2\omega B$, which is the same as Eq. (11). The imaginary part gives
\[\dot{\varphi} = \frac{\omega}{2A},\] (14)
from which we can find $\varphi(t)$ if we know $A(t)$.

Thus, we only have to solve Eqs. (11) and (12). Since we know that both $A(t)$ and $B(t)$ should be periodic with frequency $2\omega$, it is quite simple to guess the solution:
\[A(t) = A_0 + \Delta A \cos(2\omega t + \phi_{sq}),\] (15)
\[B(t) = \Delta A \sin(2\omega t + \phi_{sq}),\] (16)
where $\phi_{sq}$ is the initial phase for oscillations of squeezing and $A_0 > \Delta A \geq 0$. It is easy to see that Eq. (11) is satisfied for any $\Delta A$, $A_0$, and $\phi_{sq}$, while to satisfy Eq. (12) we need the condition
\[(A_0 + \Delta A)(A_0 - \Delta A) = 1,\] (17)
which is a familiar condition for the product of minimum and maximum quadrature variances of a squeezed state. Note that for the initial state discussed at the beginning, we need $\phi_{sq} = n\pi$ with an integer $n$. Also note that the squeezed vacuum is a minimum-uncertainty state, $\sigma_x \sigma_p = \hbar/2$, only when $2\omega t + \phi_{sq} = n\pi$ with integer $n$, i.e., four times per oscillator period.

Thus, we have found the evolution of the squeezed vacuum up to the overall phase $\varphi(t)$, which is not quite important and can be found via Eq. (14). It is interesting that there is an explicit solution,
\[\varphi(t) = \frac{\omega}{2} \int^t dt' \frac{A_0 + \Delta A \cos(2\omega t' + \phi_{sq})}{(A_0 - \Delta A) \tan(\omega t' + \phi_{sq})/2} + \text{const},\] (18)
where each discontinuity of the tangent should be compensated by an increase of the constant by $\pi/2$. Note that in deriving Eq. (18) we have used Eq. (17). It is easy to see that for each $\Delta t = \pi/\omega$ (half-period), the phase increases by $\Delta\varphi = \pi/2$, so that $\Delta\varphi/\Delta t = \omega/2$, which is the same value as for the ground state.

### III. Pure Squeezed States

Now let us generalize the $x$-space evolution of the squeezed vacuum to the $x$-space evolution of an arbitrary (pure) squeezed state. Since we know that for a squeezed state the evolution of the center is decoupled from squeezing, it is natural to guess the following wavefunction for a squeezed state:
\[\psi(x, t) = \frac{1}{\sqrt{2\pi \sigma_{gr}^2 A}} \exp \left[ -\frac{(x - x_c)^2}{4\sigma_{gr}^2 A} \right] \times \exp(i\phi_c x/\hbar) e^{-i\varphi},\] (20)
where $A(t)$ and $B(t)$ are the same functions of time as for the squeezed vacuum [given by Eqs. (15)–(17)], while the overall phase $\varphi(t)$ is now different, and the state center evolves classically,
\[x_c = X_{amp} \cos(\omega t + \phi_c),\] (21)
\[p_c = m\dot{x}_c = -m\omega X_{amp} \sin(\omega t + \phi_c),\] (22)
where $X_{amp}$ is the amplitude of classical oscillations of the center and the initial phase $\phi_c$ is not related to the initial phase $\phi_{sq}$ of squeezing oscillations. Let us show that the time-dependence of an arbitrary squeezed state given by Eqs. (20) and (21) and Eqs. (15)–(17) satisfy the Schrödinger equation.

The derivatives of $\psi(x, t)$ given by Eq. (20) are
\[\frac{\partial \psi}{\partial t} = \left[ -\frac{\dot{A}}{4A} + \frac{(x - x_c)^2(1 + iB)}{4\sigma_{gr}^2 A^2} \frac{\dot{\varphi}}{4\sigma_{gr}^2 A} - i\frac{(x - x_c)^2}{4\sigma_{gr}^2 A} \right] \psi,\] (23)
\[\frac{\partial^2 \psi}{\partial x^2} = \left[ \frac{(x - x_c)^2(1 + iB)^2}{4\sigma_{gr}^2 A^2} + \frac{i p_c}{\hbar} \frac{-1 + iB}{2\sigma_{gr}^2 A^2} \right] \psi,\] (24)
so the Schrödinger equation is [instead of Eq. (5)]
\[- \frac{\dot{A}}{A} + \frac{(x - x_c)^2(1 + iB)}{\sigma_{gr}^2 A^2} \frac{\dot{\varphi}}{\sigma_{gr}^2 A} - i\frac{(x - x_c)^2}{\sigma_{gr}^2 A} - 4i\dot{\varphi} + 2(x - x_c) \dot{x}_c (1 + iB) + i p_c \frac{-1 + iB}{\hbar} \frac{-1 + iB}{\hbar} \frac{1 + iB}{\hbar} = i\hbar \left[ \frac{(x - x_c)^2(1 + iB)^2}{\sigma_{gr}^2 A^2} - \frac{2}{\sigma_{gr}^2 A^2} - 4 \frac{p_c^2}{(x - x_c) (1 + iB)} - 4i \frac{\varphi}{\hbar} \frac{-1 + iB}{\hbar} \frac{-1 + iB}{\hbar} \frac{1 + iB}{\hbar} \right] \psi.\] (25)

This equation now gives three equations: for $x^2$, $x^3$, and $x^0$ components. It is easy to check that the $x^2$-component still gives Eq. (9). Therefore, Eqs. (11)–(12) for $A(t)$ and $B(t)$ and correspondingly their solutions, Eqs. (15)–(17), are still valid for an arbitrary squeezed state. The $x^1$-component of Eq. (25) multiplied by $\sigma_{gr}^2$
we obtain
\[ -\frac{2x_c(1+iB)A}{A^2} + \frac{2i\dot{x}_cB}{A} + \frac{2\dot{x}_c(1+iB)}{A} + 4i\frac{\dot{p}_c}{\hbar} \frac{\sigma^2_{gr}}{2m} = \frac{i\hbar}{2m} -2x_c(1+iB)^2 \frac{\sigma^2_{gr}}{A^2} + \frac{2p_c}{m} 1+iB A, \quad (26) \]

Note that several terms here are the same as the terms in Eq. (15) multiplied by \(-2x_c\). Subtracting Eq. (15) multiplied by \(-2x_c\) from Eq. (26) and using \(\sigma^2_{gr} = \hbar/2m\), we obtain
\[ \frac{2\dot{x}_c(1+iB)}{A} + 2i\frac{\dot{p}_c}{m} = \frac{2p_c}{m} 1+iB A - 2i\dot{x}_c^2, \quad (27) \]

which is obviously satisfied for \(x_c\) and \(p_c\) given by Eqs. (21) and (22). Thus, the \(x^1\)-component of Eq. (26) is satisfied. Finally, the \(x^2\)-component gives
\[ \frac{-\dot{A}}{A} + \frac{x^2_c(1+iB)A}{\sigma^2_{gr}A^2} - \frac{iA\dot{x}_c}{\sigma^2_{gr}A} - \frac{2i\dot{x}_c(1+iB)}{\sigma^2_{gr}A} + 4i\frac{\dot{p}_c}{\hbar} \frac{\sigma^2_{gr}}{A^2} = \frac{i\hbar}{2m} \left[ x^2_c(1+iB)^2 - 2 \frac{1+iB}{\sigma^2_{gr}A} - 4 \frac{\sigma^2_{gr}}{\hbar^2} + 4i \frac{\dot{p}_c}{\hbar} \frac{x_c(1+iB)}{\sigma^2_{gr}A} \right]. \quad (28) \]

The real part of this equation is
\[ \frac{-\dot{A}}{A} + \frac{x^2_cA}{\sigma^2_{gr}A^2} - \frac{2x_c\dot{x}_c}{\sigma^2_{gr}A} = \frac{\hbar B}{m\sigma^2_{gr}A} - \frac{\hbar x^2_cB}{m\sigma^2_{gr}A^2} - \frac{2p_c x_c}{m\sigma^2_{gr}A}. \quad (29) \]

The first terms on both sides are equal to each other because \(A = -2\omega B\) [see Eq. (11)]. The second terms are equal because of the same reason. The third terms are equal because \(\dot{x}_c = p_c/m\). So, Eq. (29) is satisfied. Finally, the imaginary part of Eq. (28) gives (after some algebra)
\[ \dot{\varphi} = \frac{\omega}{2A} + \frac{x^2_c A}{\sigma^2_{gr}A^2} - \frac{x^2_c B}{4\sigma^2_{gr}A} - \frac{\omega x^2_c (1-B^2)}{4\sigma^2_{gr}A} + \frac{p_c^2}{2\hbar m}, \quad (30) \]

It is simple to solve this equation by adding the explicit solution [19] for \(x_c = X_{\text{amp}} = 0\) and the integral of the last term in Eq. (30).

Thus, we have shown that the general (pure) squeezed state in \(x\)-space has the form given by Eqs. (20), (22), (15)–(17), and (30).

IV. MIXED SQUEEZED (GAUSSIAN) STATES

Using the wavefunction (20), let us construct the density matrix
\[ \rho(x,x') = \psi(x) \psi^*(x') = \frac{1}{\sqrt{2\pi\sigma^2_{x}}} \exp \left[ \frac{-(x-x')^2}{2\sigma^2_{x}} \right] \]
\[ \times \exp \left[ \frac{-iB(x-x')^2}{2\sigma^2_{x}} \right] \exp \left[ \frac{-iB x^2}{2\sigma^2_{x}} \right] \exp \left[ i\frac{p_c(x-x')}{\hbar} \right]. \quad (31) \]

Now let us probabilistically mix the states with \(x_c\) and \(p_c\) having the Gaussian probability distribution
\[ p(x_c,p_c) = \frac{1}{\sqrt{2\pi \sigma^2_{x}}} \exp \left[ \frac{- (x_c-x_c')^2}{2\sigma^2_{x}} \right] \]
\[ \times \frac{1}{\sqrt{2\pi (m\omega \sigma^2_{x})^2}} \exp \left[ \frac{- (p_c-p_c')^2}{2(m\omega \sigma^2_{x})^2} \right], \quad (32) \]

so that the additional \(x_c\)-spread has variance \(\sigma^2_{x} \) and the spread of \(p_c/m\) has the same variance \(\sigma^2_{p} \), while the averaged centers \(\bar{x}_c\) and \(\bar{p}_c\) evolve according to Eqs. (21) and (22), i.e.,
\[ \dot{\bar{x}}_c = \bar{p}_c/m, \quad \dot{\bar{p}}_c = -m\omega^2 \bar{x}_c. \quad (33) \]

It is easy to see that the evolution of \(\bar{x}_c\) and \(\bar{p}_c\) does not change the probability distribution for \(x_c\) and \(p_c\), which evolve according to Eqs. (21) and (22). The easiest way to see this fact is to consider an evolution on the plane of \(x_c\) and \(p_c/m\), so that Eqs. (21) and (22) as well as (33) describe a rotation with frequency \(\omega \) along a circle, while the Gaussian distribution (32) is isotropic. Therefore, if we average the density matrix (31) over \(x_c\) and \(p_c\) with the distribution (32), the result automatically satisfies the Schrödinger equation.

For averaging the density matrix (31) over \(x_c\), the following formula is useful:
\[ \int_{-\infty}^{\infty} \exp \left[ \frac{- (x+y)^2 + a(x+y)}{2\sigma^2_{x}} \right] \exp \left[ \frac{-y^2}{2\sigma^2_{y}} \right] dy = \sqrt{2\pi \sigma^2_{y}} \exp \left[ \frac{a^2}{2\sigma^2_{y}} \right] \]
\[ \exp \left[ \frac{-\sigma^2_{x}+a^2\sigma^2_{y}}{2\sigma^2_{x}+\sigma^2_{y}} \right] \exp \left[ \frac{a^2\sigma^2_{y}}{8\sigma^2_{x} \left( \sigma^2_{x} + \sigma^2_{y} \right)} \right]. \quad (34) \]

where \(a\) can be complex. Similarly, for averaging over \(p_c\) we can use
\[ \int_{-\infty}^{\infty} e^{a(x+y)} \exp \left[ \frac{-y^2}{2\sigma^2_{y}} \right] dy = e^{ax} \exp \left[ \frac{a^2\sigma^2_{y}}{2} \right]. \quad (35) \]

Therefore, averaging of the density matrix (31) over the
distribution \( \sigma \) gives
\[
\rho(x, x') = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left[ -\frac{(x + x' - x_c)^2}{2\sigma^2} \right]
\times \exp \left[ -iB \frac{(x + x' - x_c)(x - x')}{2\sigma^2} \right]
\times \exp \left[ \frac{i\bar{p}_c(x - x')}{\hbar} \right],
\]
where \( \sigma^2 = \sigma_a^2 A + \sigma_p^2 \). Using Eqs. (15)–(17), the last exponential factor can be expressed as
\[
\exp \left[ -\frac{(x - x')^2}{2\sigma^2} \right] \frac{\sigma^4}{\sigma_a^2} \frac{\sigma^4}{\sigma_p^2} + 2A_0 \sigma_a^2 \sigma_p^2.
\]
We see that if we introduce \( \tilde{A} = A + \sigma_a^2/A \) and \( \tilde{A}_0 = A_0 + \sigma_a^2/A \), then \( \sigma^2 = \sigma_a^2 \tilde{A} \) and the second fraction in Eq. (37) can be rewritten as \( 1 + (\sigma_a^2/\sigma_p^2) + 2A_0 \sigma_a^2 = P = (\tilde{A}_0 - \Delta A). \)

Therefore, removing tilde signs and overbars (\( \tilde{A} \to A \), \( \tilde{A}_0 \to A_0 \), \( x_c \to x_c \), \( \bar{p}_c \to p_c \)), we can rewrite the averaged density matrix as
\[
\rho(x, x') = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left[ -\frac{(x + x' - x_c)^2}{2\sigma^2} \right]
\times \exp \left[ -P \frac{(x - x')^2}{2\sigma^2} \right] \exp \left[ -iB \frac{(x + x' - x_c)(x - x')}{2\sigma^2} \right]
\times \exp \left[ \frac{i\bar{p}_c(x - x')}{\hbar} \right],
\]
which differs from Eq. (31) for the pure state by only the factor \( P \) in the second exponent. This is the general form of a Gaussian state of an oscillator. In Eq. (38), the time-dependence of the center \( x_c(t), p_c \) is classical and is given by Eqs. (21) and (22); the center oscillates with frequency \( \omega \). The parameters \( A(t) \) and \( B(t) \) are given by Eqs. (15) and (16), they oscillate with frequency \( 2\omega \). The parameter \( P \) is the product of maximum and minimum dimensionless variances,
\[
P = A_{\text{max}} A_{\text{min}} = (A_0 + \Delta A)(A_0 - \Delta A) \geq 1,
\]
so that \( P = 1 \) corresponds to a pure state, while \( P > 1 \) corresponds to a mixed Gaussian state.

V. SUMMARY

In the \( x \)-space representation, the squeezed vacuum wavefunction is given by Eq. (3) with dimensionless variance \( A(t) \) and phase parameter \( B(t) \) given by Eqs. (15)–(17), and the overall phase \( \varphi(t) \) given by Eq. (19). For an oscillator with frequency \( \omega \), the shape of the wavefunction oscillates with frequency \( 2\omega \).

The wavefunction of a pure squeezed state is given by Eq. (20) with the same parameters \( A(t) \) and \( B(t) \) as for the squeezed vacuum, while the center position \( x_c(t) \) and momentum \( p_c(t) \) are given by Eqs. (21) and (22). The center oscillates with frequency \( \omega \), while the variance oscillates with frequency \( 2\omega \).

The density matrix of a mixed (Gaussian) squeezed state is given by Eq. (38) with the same equations for \( A(t), B(t), x_c(t), \) and \( p_c(t) \), but with an additional parameter \( P \) given by Eq. (39), which replaces Eq. (17).

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