ON A REVERSE HÖLDER INEQUALITY FOR SCHRÖDINGER OPERATORS

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Abstract. We obtain a reverse Hölder inequality for the eigenfunctions of the Schrödinger operator with slowly decaying potentials. The class of potentials includes singular potentials which decay like $|x|^{-\alpha}$ with $0 < \alpha < 2$, in particular the Coulomb potential.

1. Introduction

In this paper we are concerned with a reverse Hölder inequality for the eigenfunctions of the Schrödinger operator $-\Delta + V(x)$ in $L^2(\mathbb{R}^n)$. More generally, we consider second-order elliptic operators of the form

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x)u_{x_i}) + V(x)u$$

where $a_{ij}(x)$ is a measurable and real-valued function, and the matrix $(a_{ij}(x))_{n \times n}$ is uniformly elliptic. Namely, there exists a positive constant $\Lambda$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq \Lambda |\xi|^2$$

(1.1)

for $x, \xi \in \mathbb{R}^n$. Particularly when $a_{ij} = \delta_{ij}$ (Kronecker delta function), the operator $L$ becomes equivalent to the classical Schrödinger operator. In this regard, we shall call a real-valued function $V(x)$ the potential.

Reverse Hölder inequalities for solutions to the following Dirichlet boundary problem have been studied for a long time:

$$\begin{cases}
Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + V(x)u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(1.2)

where $\Omega$ is a bounded region in $\mathbb{R}^n$. When $n = 2$, Payne and Rayner showed that if $\lambda$ is the first eigenvalue and $u$ is the corresponding eigenfunction of the problem

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with $a_{ij} = \delta_{ij}$ and $V(x) \equiv 0$,\n
\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

then the following reverse Schwarz inequality holds:

\[
\|u\|_{L^2(\Omega)} \leq \sqrt{\frac{\lambda}{4\pi}} \|u\|_{L^1(\Omega)}.
\]

This result was extended to higher dimensions by Kohler-Jobin [7] (see also [9]). In the general setting (1.2), the reverse Hölder inequalities,

\[
\|u\|_{L^q(\Omega)} \leq C_{p,q,\lambda,n} \|u\|_{L^p(\Omega)}, \quad q \geq p > 0,
\]

were obtained later by Talenti [10] for $q = 2$ and $p = 1$, and by Chiti [3] for all $q \geq p > 0$, but with a nonnegative potential $V \geq 0$ and with symmetric coefficients $a_{ij} = a_{ji}$.  

Our aim in this paper is to remove these restrictions. Namely, we obtain a reverse Hölder inequality for solutions of (1.2) where $V$ is allowed to be negative and we do not need to assume the symmetry, $a_{ij} = a_{ji}$.  

For the purpose, we use a completely different approach based on a combination between the Fefferman-Phong inequality and the classical Moser’s iteration technique. Compared with the approach, the authors of the previous results [8, 7, 3] mentioned above are mainly interested on isoperimetric inequalities by which they obtain explicit constants in their reverse Hölder inequalities and could characterize equality which is not the main issue in the present work.

Before stating our results, we introduce the Morrey-Campanato class $\mathcal{L}^{\alpha,r}$ of potentials $V$, which is defined for $\alpha > 0$ and $1 \leq r \leq n/\alpha$ by

\[
V \in \mathcal{L}^{\alpha,r} \iff \sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{\alpha-n/r} \left( \int_{B_{\rho}(x)} |V(y)|^r dy \right)^{1/r} < \infty,
\]

where $B_{\rho}(x)$ is the ball centered at $x$ with radius $\rho$. In particular, $\mathcal{L}^{\alpha,n/\alpha} = L^{n/\alpha}$ and $1/|x|^\alpha \in L^{n/\alpha,\infty} \subset \mathcal{L}^{\alpha,r}$ if $1 \leq r < n/\alpha$. Let us next make precise what we mean by a weak solution of the problem (1.2). We say that a function $u \in H^1_0(\Omega)$ is a weak solution if

\[
\int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) u_{x_i} \phi_{x_j} + \int_{\Omega} V(x) u(x) \phi(x) dx = \int_{\Omega} \lambda u(x) \phi(x) dx \tag{1.3}
\]

for every $\phi \in H^1_0(\Omega)$. Our result is then the following theorem.

**Theorem 1.1.** Let $n \geq 3$. Assume that $u \in H^1_0(\Omega)$ is a weak solution of the problem (1.2) with $\lambda \in \mathbb{R}$ and $V \in \mathcal{L}^{\alpha,r}$ for $\alpha < 2$ and $r > 2/\alpha$. Then we have

\[
\|u\|_{L^q(\Omega)} \leq C C_{\alpha,r} \max\{p, 2\}^{\frac{n}{p(n-\alpha)}} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{p(n-\alpha)}} \|u\|_{L^p(\Omega)} \tag{1.4}
\]
for all \( q \geq p > 0 \). Here, \( C \) is a constant depending on \( \Lambda, \lambda, p, q, n \) and \( \Omega \), and

\[
C_\alpha = 1 + \alpha \frac{\alpha}{2 - \alpha} \left( \frac{2C_n}{\Lambda} \| V \|_{L^{\alpha,r}} \right)^{2/(2-\alpha)}
\]

with a constant \( C_n \) depending on \( n \) and arising from the Fefferman-Phong inequality (2.6).

Remark 1.2. The class \( L^{\alpha,r}, \alpha < 2 \), of potentials in the theorem includes the positive homogeneous potentials \( a|x|^{-\alpha} \) with \( a > 0 \) and \( 0 < \alpha < 2 \) in three and higher dimensions, in particular the Coulomb potential.

In Section 2, we prove Theorem 1.1. Compared with the previous results [7, 3] based on rearrangements of functions, our approach works also for negative potentials and for non-symmetric coefficients \( a_{ij} \). We use a completely different approach based on a combination between the Fefferman-Phong inequality and the classical Moser’s iteration technique. By making use of a two-dimensional analogue of the Fefferman-Phong inequality, we obtain a reverse Hölder inequality for \( n = 2 \) as well. See Section 3 for details.

Throughout this paper, we denote \( A \lesssim B \) to mean \( A \leq CB \) with unspecified constant \( C > 0 \) which may be different at each occurrence.

2. Proof of Theorem 1.1

In this section we prove the reverse Hölder inequality (1.1). Since a complex-valued solution \( u \) satisfies (1.3) for every complex \( \phi \in H^1_0(\Omega) \), one can easily see that real and imaginary parts of the solution also satisfy (1.3) for every real \( \phi \in H^1_0(\Omega) \). On the other hand, once we prove the inequality for the real and imaginary parts, we get the same inequality for \( u \). Indeed, using the inequality \( (a + b)^s \leq C(a^s + b^s) \) for \( a, b > 0 \) and \( s > 0 \), one can see

\[
\| u \|^q_{L^q(\Omega)} = \int_{\Omega} (|\text{Re} u + i\text{Im} u|^2)^{q/2} dx
\]

\[
= \int_{\Omega} (|\text{Re} u|^2 + |\text{Im} u|^2)^{q/2} dx
\]

\[
\leq C \int_{\Omega} |\text{Re} u|^q + |\text{Im} u|^q dx
\]

\[
\leq C(\|\text{Re} u\|_{L^q(\Omega)}^q + \|\text{Im} u\|_{L^q(\Omega)}^q)
\]

\[
\leq C(\|\text{Re} u\|_{L^p(\Omega)}^q + \|\text{Im} u\|_{L^p(\Omega)}^q)
\]

\[
\leq C\| u \|^q_{L^p(\Omega)}.
\]

Hence we may assume that the solution \( u \) is a real-valued function.
Now we decompose \( u \) into two parts, \( f = \max\{u, 0\} \) and \( g = -\min\{u, 0\} \). Then it is enough to prove that (1.4) holds for \( f \) and \( g \). Indeed,
\[
\|u\|_{L^\gamma(\Omega)} = \|f - g\|_{L^\gamma(\Omega)} \\
\leq \|f\|_{L^\gamma(\Omega)} + \|g\|_{L^\gamma(\Omega)} \\
\leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}) \\
\leq C\|u\|_{L^p(\Omega)}.
\]
We only consider \( f \) because the proof for \( g \) follows obviously from the same argument. To prove (1.4) for \( f \), we now divide cases into two parts, \( p \geq 2 \) and \( p < 2 \).

2.1. The case \( p \geq 2 \). In this case we will show that for all \( \tau \geq 2 \)
\[
\|f\|_{L^{\tau}(\Omega)} \lesssim C^{1/\tau}_{\alpha} \tau^{2(2-\alpha)/\alpha} \|f\|_{L^\tau(\Omega)} \tag{2.1}
\]
with \( \omega = n/(n-2) \). Beginning with \( \tau = p \), we then iterate as \( \tau = p, p\omega, p\omega^2, \cdots \), to obtain (1.4). Indeed, first put
\[
\tau_i = p\omega^i
\]
for \( i = 0, 1, 2, \cdots \). Since \( \tau_i = \tau_{i-1}\omega \), we then get
\[
\|f\|_{L^{\tau_i}(\Omega)} \lesssim C^{1/\tau_i}_{\alpha} \tau_{i-1}^{2(2-\alpha)/\alpha} \|f\|_{L^{\tau_{i-1}}(\Omega)} \\
= C^{\omega_{i-1}}_{\alpha} (p\omega^i)^{2/(2-\alpha) - i} \|f\|_{L^{\tau_{i-1}}(\Omega)}
\]
for \( i = 1, 2, \cdots \), which implies by iteration that
\[
\|f\|_{L^{\tau}(\Omega)} \lesssim (C_\alpha p^2/(2-\alpha))^{\sum_{i=1}^{\infty} (p\omega^{k-1})^{-i}} (\omega^{2/(2-\alpha)} \sum_{i=1}^{\infty} (p\omega^{k})^{-i})^{-1} \|f\|_{L^p(\Omega)}.
\]
Since \( \omega = n/(n-2) > 1 \), by letting \( i \to \infty \), this implies
\[
\|f\|_{L^{\tau}(\Omega)} \lesssim \|f\|_{L^\infty(\Omega)} \lesssim C^{\omega}_{\alpha} p^{\omega/(\alpha-2)} n \|f\|_{L^p(\Omega)}
\]
as desired.

It remains to show (2.1). Since \( f \in H^1_0(\Omega) \) is a positive part of the weak solution \( u \), it follows that
\[
\int_\Omega \sum_{i,j=1}^n a_{ij}(x)f_x \phi_{x_j} \, dx + \int_\Omega V(x)f(x)\phi(x) \, dx = \int_\Omega \lambda f(x)\phi(x) \, dx \tag{2.2}
\]
for every real \( \phi \in H^1_0(\Omega) \) supported on \( \{x \in \mathbb{R}^n : u(x) > 0\} \). For \( l > 0 \) and \( m > 0 \), we set \( \tilde{f} = f + l \), and let
\[
\tilde{f}_m = \begin{cases} 
  l + m & \text{if } f \geq m, \\
  \hat{f} & \text{if } f < m.
\end{cases}
\]
We now consider the following test function
\[
\phi = \tilde{f}_m^\beta \tilde{f} - l^{\beta+1} \in H^1_0(\Omega)
\]
for \( \beta \geq 0 \). We then compute
\[
\phi_{x_{j}} = \beta f^{\beta} (f_{m})_{x_{j}} + f^{\beta}_{m} f_{x_{j}}
\]
using the fact that
\[
(f_{m})_{x_{j}} = 0 \text{ in } \{ x : f(x) \geq m \} \quad \text{and} \quad f_{m} = f \text{ in } \{ x : f(x) < m \}.
\]  

Substituting \( \phi \) into (2.2) and using (2.3) together with the trivial fact \( f_{x_{i}} = f_{x_{i}} \), the first term on the left-hand side of (2.2) is written as
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) f_{x_{i}} \phi_{x_{j}} dx
\]
\[
= \beta \int_{\Omega} f^{\beta}_{m} \sum_{i,j=1}^{n} a_{ij}(x) (f_{m})_{x_{i}} (f_{m})_{x_{i}} dx + \int_{\Omega} \tilde{f}^{\beta}_{m} \sum_{i,j=1}^{n} a_{ij}(x) \tilde{f}_{x_{i}} \tilde{f}_{x_{i}} dx.
\]

Then it follows from the ellipticity (1.1) that
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) f_{x_{i}} \phi_{x_{j}} dx \geq \Lambda \beta \int_{\Omega} f^{\beta}_{m} |\nabla f_{m}|^{2} dx + \Lambda \int_{\Omega} \tilde{f}^{\beta}_{m} |\nabla \tilde{f}|^{2} dx.
\]  

Combining (2.2) and (2.4), we conclude that
\[
\Lambda \beta \int_{\Omega} f^{\beta}_{m} |\nabla f_{m}|^{2} dx + \Lambda \int_{\Omega} \tilde{f}^{\beta}_{m} |\nabla \tilde{f}|^{2} dx
\]
\[
\leq \int_{\Omega} -V f (f^{\beta}_{m} \tilde{f} - l^{\beta+1}) dx + \int_{\Omega} \lambda f (f^{\beta}_{m} \tilde{f} - l^{\beta+1}) dx.
\]

Note here that
\[
|\nabla (f^{\beta/2} f)|^{2} \leq 2(1 + \beta) \left( \beta f^{\beta}_{m} |\nabla f_{m}|^{2} + \tilde{f}^{\beta}_{m} |\nabla \tilde{f}|^{2} \right)
\]
which follows by a direct computation together with (2.3). We therefore get
\[
\int_{\Omega} |\nabla (f^{\beta/2} f)|^{2} dx
\]
\[
\leq \frac{2(1 + \beta)}{\Lambda} \int_{\Omega} -V f (f^{\beta}_{m} \tilde{f} - l^{\beta+1}) dx + \frac{2 \lambda (1 + \beta)}{\Lambda} \int_{\Omega} f (f^{\beta}_{m} \tilde{f} - l^{\beta+1}) dx
\]
\[
\leq \frac{2(1 + \beta)}{\Lambda} \int_{\Omega} |V f^{\beta}_{m} \tilde{f}|^{2} dx + \frac{2 \lambda (1 + \beta)}{\Lambda} \int_{\Omega} \tilde{f}^{\beta}_{m} \tilde{f}^{2} dx. \]  

To control the term involving the potential in (2.5), we now use the so-called Fefferman-Phong inequality (6),
\[
\int_{\mathbb{R}^{n}} |g|^{2} v(x) dx \leq C_{n} \| v \|_{L^{2, r}} \int_{\mathbb{R}^{n}} |\nabla g|^{2} dx,
\]  

where \( C_{n} \) is a constant depending on \( n \), and \( 1 < r \leq n/2 \). (It is not valid for \( r = 1 \) as remarked in [4].) Applying this inequality along with Hölder’s inequality, the first
integral on the right-hand side of (2.9) is bounded as
\[
\int_{\Omega} |V| \tilde{f}_{m}^\beta \tilde{f} dx \leq \left( \int_{\Omega} |V|^{\frac{\alpha}{2}} \tilde{f}_{m}^\beta \tilde{f}^2 dx \right)^{\frac{\alpha}{2}} \left( \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f}^2 dx \right)^{\frac{2-\alpha}{2}}
\leq C_n \|V\|_{L^\infty} \left( \int_{\Omega} |\nabla (\tilde{f}_{m}^\beta \tilde{f})|^2 dx \right)^{\frac{\alpha}{2}} \left( \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f}^2 dx \right)^{\frac{2-\alpha}{2}}
\]
for all \(1 < \tilde{r} \leq n/2\). We note here that \(\|V\|_{L^{\infty}} \approx \|V\|_{L^{\alpha, 2\tilde{r}/\alpha}}\) and apply Young’s inequality
\[
abla \leq \frac{\alpha}{2} (\varepsilon a)^{2/\alpha} + \frac{2-\alpha}{2} (\varepsilon^{-1}\beta)^{2/(2-\alpha)}
\]
with \(0 < \alpha < 2\) and \(\varepsilon > 0\) to obtain
\[
\int_{\Omega} |V| \tilde{f}_{m}^\beta \tilde{f} dx \leq C_n \|V\|_{L^{\alpha, 2\tilde{r}/\alpha}} \left( \frac{\alpha}{2} \varepsilon^{2/\alpha} \int_{\Omega} |\nabla (\tilde{f}_{m}^\beta \tilde{f})|^2 dx + \frac{2-\alpha}{2} \varepsilon^{-2/(2-\alpha)} \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f}^2 dx \right).
\] (2.8)
By setting \(r = 2\tilde{r}/\alpha\) (since \(\tilde{r} > 1\), setting \(r = 2\tilde{r}/\alpha\) determines the condition \(r > 2/\alpha\)
in the theorem) and taking \(\varepsilon^{2/\alpha} = c(1 + \beta)^{-1}\) with \(c = \frac{\Lambda}{2\alpha C_n \|V\|_{L^{\alpha, r}}}\) so that
\[
C_n \|V\|_{L^{\alpha, r}} \frac{\alpha}{2} \varepsilon^{2/\alpha} \frac{2(1 + \beta)}{\Lambda} = 1/2,
\]
the gradient term in (2.8) can be absorbed into the left-hand side of (2.9), as follows:
\[
\int_{\Omega} |\nabla (\tilde{f}_{m}^\beta \tilde{f})|^2 dx \leq e^{-\frac{\alpha}{2}} \left( \frac{2-\alpha}{\alpha} \right) (\beta + 1)^{\frac{2-\alpha}{\alpha}} \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f}^2 dx
+ \frac{2|\Lambda|(1 + \beta)}{\Lambda} \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f}^2 dx.
\] (2.9)
Finally, applying the Gagliardo-Nirenberg-Sobolev inequality (15) to the left-hand side of (2.9), we see
\[
\left( \int_{\Omega} \tilde{f}_{m}^\beta \tilde{f} dx \right)^{1/\omega} \leq \int_{\Omega} |\nabla (\tilde{f}_{m}^\beta \tilde{f})|^2 dx
\]
with \(\omega = n/(n - 2)\). Using the fact that \(\tilde{f}_{m} \leq \tilde{f}\) and setting \(\beta + 2 = \tau\), we therefore get
\[
\left( \int_{\Omega} \tilde{f}_{m}^\tau dx \right)^{1/\omega} \leq e^{-\frac{\alpha}{2\alpha}} \left( \frac{2-\alpha}{\alpha} \right) (\tau - 1)^{\frac{2-\alpha}{\alpha}} \int_{\Omega} \tilde{f}^\tau dx
+ \frac{2|\Lambda|(\tau - 1)}{\Lambda} \int_{\Omega} \tilde{f}^\tau dx,
\]
which implies the desired estimate
\[
\left( \int_{\Omega} f^\tau dx \right)^{1/\omega} \leq \left( 1 + \alpha^{\frac{2}{2\alpha}} \right) \left( \frac{2C_n \|V\|_{L^{\alpha, r}}}{\Lambda} \right)^{\frac{\alpha}{2}} \frac{2}{\tau} \int_{\Omega} f^\tau dx
\]
by letting \(m \to \infty\) and \(l \to 0\).
2.2. The case \( p < 2 \). From the case \( p = 2 \), we have \( \|f\|_{L^\infty(\Omega)} < \infty \) and

\[
\|f\|_{L^\infty(\Omega)} \lesssim C^p_n 2^{\frac{n}{p(2-p)}} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{p(2-p)}} \|f\|_{L^p(\Omega)}
\]

\[
\leq C^p_n 2^{\frac{n}{p(2-p)}} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{p(2-p)}} \|f\|_{L^\infty(\Omega)}^{(2-p)/2} |f|_{L^p(\Omega)}^{p/2}
\]

\[
\leq \frac{1}{2} \|f\|_{L^\infty(\Omega)} + C^p_n \frac{p}{2} \left( \frac{1}{2-p} \right)^{1-\frac{2}{p}} 2^{\frac{n}{p(2-p)}} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{p(2-p)}} \|f\|_{L^p(\Omega)}. \tag{2.10}
\]

For the third inequality, we used here Young’s inequality,

\[
ab \leq \left( 1 - \frac{p}{2} \right) (\epsilon a)^{\frac{p}{2}} + \frac{p}{2} (\epsilon^{-1} b)^{\frac{2}{p}}
\]

with \( \epsilon = \left( \frac{1}{2-p} \right) (2-p)/2) \),

\[
a = \|f\|_{L^\infty(\Omega)}^{(2-p)/2} \quad \text{and} \quad b = C^p_n \frac{2}{2} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{2(2-p)}} \|f\|_{L^p(\Omega)}^{p/2}.
\]

Byabsorbingthe firstterm on the right-hand side of (2.10) into the left-hand side, we conclude that

\[
\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^\infty(\Omega)} \lesssim C^p_n \frac{2}{2} \left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{2(2-p)}} \|f\|_{L^p(\Omega)}
\]

as desired.

3. Concluding remarks

Finally we obtain a reverse Hölder inequality for \( n = 2 \) by making use of a two-dimensional analogue (see Theorem 3.1) of the Fefferman-Phong inequality (2.6).

We first recall the function space \( \mathcal{M} \log L \) introduced in [1], which is defined by

\[
V \in \mathcal{M} \log L \iff \sup_{0 < t < |\Omega|} \log \left( \frac{|\Omega|}{t} \right) \int_0^t V^*(s) ds < \infty.
\]

For \( t > 0 \), \( V^*(t) = \inf \{ s > 0 : d_V(s) \leq t \} \) is the decreasing rearrangement of \( V \) with \( d_V(s) = |\{ x \in \Omega : |V(x)| > s \} | \) for \( s > 0 \). Note that \( L^p(\Omega) \subset L \log L \subset \mathcal{M} \log L \) (see [1] [2]).

A two-dimensional version of Theorem [3,3] is now stated in the following theorem. Indeed, the constant \( C^\sharp_n \max\{p, 2\} \frac{2}{2-n}\left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{2(2-p)}} \) in (1.4) boils down to the one in (3.1) if \( n \to 2 \).

**Theorem 3.1.** Let \( n = 2 \). Assume that \( u \in H^s_0(\Omega) \) is a weak solution of the problem (1.2) with \( \lambda \in \mathbb{R} \) and \( V^\sigma \in \mathcal{M} \log L \) for some \( \sigma > 1 \). Then we have

\[
\|u\|_{L^q(\Omega)} \leq C \max\{p, 2\} \frac{2}{2-n}\left( \frac{n}{n-2} \right)^{\frac{n(n-2)}{2(2-p)}} \|u\|_{L^p(\Omega)} \tag{3.1}
\]

for all \( q \geq p > 0 \). Here, \( C \) is a constant depending on \( \Lambda, \lambda, p, q \) and \( \Omega \), and

\[
C_\sigma = 1 + \left( \frac{2}{\sigma} \right)^{\frac{1}{p-1}} \left( \frac{2||V^\sigma\|_{\mathcal{M} \log L}}{\Lambda} \right)^{\frac{2}{p-1}}.
\]
Proof. We briefly sketch the proof since it is an obvious modification of the one for Theorem 1.1. Indeed, it is enough to show the following inequality corresponding to (2.1): for all \( \tau \geq 2 \)
\[ \|f\|_{L^\infty(\Omega)} \lesssim C_{\sigma}^{1/\tau} \tau^{\sigma/\tau} \|f\|_{L^n(\Omega)} \] (3.2)
with \( \omega = (2 - \varepsilon)/\varepsilon \) for all arbitrarily small \( \varepsilon > 0 \). By iteration as before, we then get
\[ \|f\|_{L^p(\Omega)} \lesssim \|f\|_{L^n(\Omega)} \lesssim C_{\sigma}^{2p-\varepsilon \sigma/(2-\varepsilon)} \|f\|_{L^n(\Omega)}. \]
Letting \( \varepsilon \to 0 \) implies (3.1) for \( p \geq 2 \). The case \( p < 2 \) follows from the case \( p = 2 \) in the same way as before.

It remains to show (3.2). To control the term involving the potential in (2.5), we use the following inequality (11)
\[ \int |g|^2 v(x) \, dx \leq \frac{\|v\|_{M\log L}}{\pi} \int |\nabla g|^2 \, dx \] (3.3)
instead of the Fefferman-Phong inequality (2.6). Using (3.3) along with Hölder’s inequality, the first integral on the right side of (3.5) is indeed bounded as
\[ \int V |\tilde{f}_m|^2 \, dx \leq \left( \int V |\tilde{f}_m|^2 \, dx \right)^{\frac{1}{2}} \left( \int \tilde{f}_m^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq |||V|||_{M\log L} \left( \int V (\tilde{f}_m^2)^2 \, dx \right)^{\frac{1}{2}} \left( \int \tilde{f}_m^2 \, dx \right)^{\frac{1}{2}} \]
for \( \sigma > 1 \). By using Young’s inequality (2.7) with \( \alpha = 2/\sigma \), we have
\[ \int V |\tilde{f}_m|^2 \, dx \]
\[ \leq |||V|||_{M\log L} \left( \frac{1}{\sigma} \int \nabla (\tilde{f}_m^2) \, dx \right)^{\frac{1}{2}} \left( \frac{\sigma - 1}{\sigma} \right)^{\frac{1}{2}} \left( \int \tilde{f}_m^2 \, dx \right)^{\frac{1}{2}}. \] (3.4)

By taking \( \varepsilon^\sigma = c(1 + \beta)^{-1} \) with \( c = \frac{\sigma A}{\pi \|V\|^2 \|M\log L\}} \) so that
\[ |||V|||_{M\log L} \frac{1}{\sigma} \varepsilon^\sigma \frac{2(1 + \beta)}{\Lambda} = \frac{1}{2}, \]
the gradient term in (3.4) can be absorbed into the left-hand side of (2.5), as follows:
\[ \int \nabla (f_m^2/\tilde{f}) \, dx \leq c^{-\frac{\sigma}{2(\sigma-1)}} (\sigma - 1)(\beta + 1)^{-\frac{\sigma}{2(\sigma-1)}} \int \tilde{f}_m^2 \, dx \]
\[ + \frac{2\Lambda(1 + \beta)}{\Lambda} \int \tilde{f}_m^2 \, dx. \] (3.5)

Finally, we apply the Gagliardo-Nirenberg-Sobolev inequality (5) after using Hölder’s inequality in the left side of (3.5) to obtain
\[ \int \nabla (f_m^2/\tilde{f}) \, dx \gtrsim \left( \int \nabla (f_m^2/\tilde{f}) \, dx \right)^{2(2-\varepsilon)} \gtrsim \left( \int \tilde{f}_m^2 \, dx \right)^{1/\omega} \] (3.6)
with $\omega = (2 - \varepsilon)/\varepsilon$ for all arbitrarily small $\varepsilon > 0$. Using the fact that $\tilde{f}_m \leq \tilde{f}$ and setting $\beta + 2 = \tau$, we therefore get
\[
\left( \int_{\Omega} \tilde{f}^{\omega} \tau \, dx \right)^{1/\omega} \lesssim \left( 1 + \left( \frac{2}{\beta} \right)^{\frac{1}{\tau-1}} \left( \frac{2 \|V|^{\sigma} \|M \log L}{\Lambda} \right)^{\frac{1}{\tau-1}} \right)^{\frac{1}{\tau-1}} \int_{\Omega} \tilde{f}^{\tau} \, dx,
\]
which implies the desired estimate (3.2) by letting $m \to \infty$ and $l \to 0$. □

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