ENHANCED KOSZUL PROPERTIES IN GALOIS COHOMOLOGY

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To David Eisenbud, with admiration and gratitude.

Abstract. We prove that, under a well known conjecture in the finitely generated case, Galois cohomology satisfies several surprisingly strong versions of Koszul properties. We point out several unconditional results which follow from our work. We show how these enhanced versions are preserved under certain natural operations on algebras, generalising several results that were previously established only in the commutative case.

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1. Motivation. An important outstanding problem in Galois theory is to distinguish absolute Galois groups of fields among all profinite groups. The first significant results in this direction have been obtained by E. Artin and O. Schreier in 1927 in [AS27a, AS27b], where they show that any finite subgroup of an absolute Galois group has order \( \leq 2 \). In 2011, M. Rost and V. Voevodsky completed the proof of Bloch-Kato conjecture (cf. [SJ06, HW09, Voe10, Voe11]), which describes the Galois cohomology of a field in terms of generators and relations, provided the field contains enough roots of unity. Recently, there has been a considerable new activity related to the vanishing of higher Massey products in Galois cohomology and to defining relations of Galois groups (see [HW15], [MT16], [EM17], [GMTW18], [Mat18], [GM], [EQ], [MRT]). Nevertheless, a complete classification of absolute Galois groups seems to be an overly difficult goal at the moment. One can ask the simpler question of classifying all maximal pro-\( p \) quotients of absolute Galois groups for a given prime number \( p \), but this turns out to be a difficult open problem as well.

In our paper we approach the quest of understanding Galois cohomology by resolving its relations, through basic methods of homological algebra that date back to D. Hilbert [Hil90]. In other words, we consider relations between relations, relations between relations between relations, and so on. Following Hilbert, we call these higher order relations syzygies. In [Pri70], S.B. Priddy singled out a class of graded algebras over fields with remarkable syzygies, which he named Koszul algebras. There is a fascinating dichotomy involving them: namely, Koszul property is very restrictive, yet Koszul algebras surprisingly appear in lots of exciting branches of representation theory, algebraic geometry, combinatorics, computer algebra.

Koszul algebras are quadratic, that is, they are generated by degree 1 elements and relations are generated by degree 2 elements. A standard reference on quadratic algebras is [PP05]. On the other side, the important consequence of Bloch-Kato conjecture is that Galois cohomology with \( \mathbb{F}_p \) coefficients is a quadratic algebra, provided the ground field contains a primitive \( p \)th root of unity. L. Positselski in his PhD thesis [Pos98] (cf. also [Pos05]) and Positselski and A. Vishik in [PV95] therefore started considering the idea that the mod-\( p \) Milnor K-theory and the Galois cohomology of a field as above may be Koszul algebras.

Priddy also showed in [Pri70] that a quadratic algebra is Koszul if and only if a certain specific and easy to understand complex is acyclic. In this case, that complex is actually a resolution of the relations of the algebra, thereby providing an immediate solution to our initial question. Taking into account that all relations in Galois cohomology are consequences of the simple equations \( a + b = 1 \) for \( a, b \) in the given ground field, we see that when Galois cohomology is indeed Koszul, it reveals itself to be as clear, tidy and harmonious like a Mozart’s symphony.

Formally, Positselski conjectured that the Galois cohomology with \( \mathbb{F}_p \) coefficients and the mod-\( p \) Milnor K-theory of all fields containing a primitive \( p \)th root of unity are Koszul, and proved the assertion for local and global fields, in 2014 in [Pos14]. On the other hand, from 2000 onwards, various stronger versions of Koszulity were proposed and investigated in commutative algebra, mostly in relation to algebraic geometry (cf. [Con00], [HR00], [CRV01], [CTV01], [CDNR13], [EHH15]), and
extended to the noncommutative setting by D.I. Piontkovski˘ı in [Pio05]. A standard reference on commutative algebra is [Eis95].

In general, deciding whether a given quadratic algebra is Koszul is already difficult, because of lack of any combinatorial criteria and screening tests (the failure of the natural candidate criteria is well expressed in [Pos95], [Roo95] and [Pio01]). In addition, some of these enhanced versions of Koszulity are very restrictive even among Koszul algebras, hence one expects algebras with these properties to be extremely rare. However, paradoxically, proving or disproving these enhanced versions may be easier than deciding about original Koszulity. This is indeed true in our case. Surprisingly, in this paper we show that, if a certain well-known conjecture in Galois theory is true (see Section 3), then nearly all of these enhanced forms of Koszulity hold for the Galois cohomology of those maximal pro-$p$ quotients of absolute Galois groups that are finitely generated.

1.2. Notation and results. Throughout the paper, $p$ is a prime number, $F$ is a field, $G_F = \text{Gal}(\overline{F}/F)$ is the absolute Galois group of $F$ and $G_F(p) = \text{Gal}(F(p)/F)$ is its maximal pro-$p$ quotient. Unless otherwise stated, we always assume the following.

**Standing Hypothesis 1.**

1. We work in the category of profinite groups, therefore all groups are understood to be topological groups. In particular, all homomorphisms are tacitly assumed to be continuous and all subgroups to be closed. Moreover, a group $G$ is said to be finitely generated if it is *topologically* finitely generated, that is, there are finitely many elements of $G$ such that the smallest (closed) subgroup of $G$ containing all these elements is $G$ itself.

2. The field $F$ contains a primitive $p^{th}$ root of unity.

3. The group $G_F(p)$ is finitely generated.

Further, we denote $H^\bullet(G, \mathbb{F}_p)$ the group cohomology with continuous cochains of a profinite group $G$ with coefficients in $\mathbb{F}_p$ (see [NSW08, Chapter 1], [Ser02]). Milnor’s paper [Mil70], introduces a quadratic algebra which is now called Milnor K-theory $K^\bullet(F)$. This is the quotient of the tensor algebra on the multiplicative group $F\times$ by the two-sided ideal generated by the tensors $a \otimes (1-a)$, $a \in F \setminus \{0,1\}$. The important special case of Bloch-Kato conjecture claims that there is a natural graded isomorphism, the norm-residue map,

$$h_\bullet : \frac{K^\bullet(F)}{pK^\bullet(F)} \to H^\bullet(G_F, \mathbb{F}_p).$$

The proof of this conjecture was completed by Rost and Voevodsky in 2011 (see [SJ06] [HW09] [Voe10] [Voe11]). Two relevant corollaries are that $H^\bullet(G_F, \mathbb{F}_p)$ is a quadratic algebra, and that the inflation map

$$\text{inf} : H^\bullet(G_F(p), \mathbb{F}_p) \to H^\bullet(G_F, \mathbb{F}_p)$$

is an isomorphism.

For the algebras $H^\bullet(G_F, \mathbb{F}_p)$, or equivalently $H^\bullet(G_F(p), \mathbb{F}_p)$, we investigate some properties related to Koszulity: the original Koszulity, the existence of Koszul flags, the existence of Koszul filtrations, universal Koszulity, strong Koszulity, PBW property. The definitions of these properties are given in Section 2 along with the relevant bibliography. The known relations between them are summarised in the following diagram, in which all implications are known to be not invertible.
Interestingly, the cohomology algebras in (6.1) and (6.3) provide examples of universally Koszul algebras that are not strongly Koszul (cf. Propositions 43 and 46 and Propositions 44 and 47). We are not aware of any previous example of this behaviour in the literature.

**Theorem A.** Suppose that $G_F(p)$ is either a free or a Demushkin pro-$p$ group. Then $H^\ast(G_F(p), F_p)$ is universally Koszul and strongly Koszul.

This is proved in Subsections 4.1, 4.2, 5.1, 5.2.

Free and Demushkin pro-$p$ groups are the building blocks of all known examples of finitely generated groups of type $G_F(p)$. Thus, it has been conjectured that all finitely generated maximal pro-$p$ quotients of absolute Galois groups of fields satisfying Hypothesis I are obtained assembling free and Demushkin pro-$p$ groups via free products and certain semidirect products. This is the Elementary Type Conjecture for pro-$p$ groups: see Section 3 for details. On the level of cohomology, the operation corresponding to a free product of pro-$p$ groups is the direct sum of algebras and the operation corresponding to the aforementioned semidirect product is the twisted extension of algebras, which often reduces to a skew-symmetric tensor product with an exterior algebra (Definition 21 and Remark 22).

**Theorem B.** The direct sum of any two universally Koszul algebras (respectively, strongly Koszul algebras, algebras with a Koszul filtration) is universally Koszul (respectively, is strongly Koszul, has a Koszul filtration).

This is proved in Propositions 28, 37 and 48.

**Theorem C.** Any twisted extension of a universally Koszul algebra (respectively, an algebra with a “good” Koszul filtration) is universally Koszul (respectively, has a “good” Koszul filtration).

This is proved in Proposition 29 and in Proposition 49. By ”good” Koszul filtration we mean a Koszul filtration which satisfies the condition labeled by $♥$ in Proposition 49.

The analogous result for strong Koszulity does not hold in general: the cohomology algebra of a rigid field of level 0 or 2 is a twisted extension of the strongly Koszul algebra $\mathbb{F}_2(t)$ or $\mathbb{F}_2(t \mid t^2)$ respectively, yet we prove the following (cf. Propositions 44 and 47).

**Proposition D.** Let $F$ be a 2-rigid field of level 0 or 2, such that $G_F(2)$ has a minimal set of generators with at least 3 elements. Then $H^\ast(G_F(2), \mathbb{F}_2)$ is not strongly Koszul.
Note that this negative result does not break the general harmony of our picture. In fact, strong Koszulity imposes the too strict condition of working only within a specific set of generators. This restriction was natural in the setting of the paper [HHR00] in which strong Koszulity was introduced, but it is not always so in our context.

The aforementioned examples of universally Koszul algebras that are not strongly Koszul arise as the cohomology algebras of the previous proposition.

Moreover, strong Koszulity is preserved by twisted extensions whenever they reduce to skew-symmetric tensor products.

**Theorem E.** The skew-symmetric tensor product of a strongly Koszul algebra with a finitely generated exterior algebra is strongly Koszul.

This is Proposition 32, the analogous statement for universal Koszul algebras and for algebras with Koszul filtrations is a corollary of Theorem C.

All together, these results prove that, under the Elementary Type Conjecture for pro-$p$ groups, the algebra $H^\bullet(G_F(p), \mathbb{F}_p)$ is universally Koszul for any finitely generated $G_F(p)$, and hence in particular it is Koszul. This provides a new proof of the main result of [MPQT]. But the different perspective adopted here adds deep insights on the homological behaviour of modules over Galois cohomology. In fact, an algebra is Koszul if its augmentation ideal, or equivalently its ground field, has a remarkably good resolution (a linear resolution, see Definition 3). As pointed out by Piontkovski˘ı in [Pio05], algebras that satisfy enhanced versions of Koszulity have more modules with linear resolutions. The Galois cohomology of a pro-$p$ group as above enjoys this property at its maximum: an algebra is universally Koszul if every ideal generated by elements of degree 1, or equivalently every cyclic module, has a linear resolution.

There are specific situations in which some form of Elementary Type Conjecture is known to be true. In any of these situations, we thus get unconditional results. Some significant examples are included in Section 8 and summarised in the next theorem.

**Theorem F.** If $F$ has characteristic $\text{char } F \neq 2$ and either

1. $|F^\times/(F^\times)^2| \leq 32$,
2. $F$ has at most 4 quaternion algebras,
3. $F$ is Pythagorean and formally real, or more generally $E \subseteq F \subseteq E(2)$, with $E$ a Pythagorean and formally real field and $F/E$ a finite extension,

then $H^\bullet(G_F(2), \mathbb{F}_2)$ is universally Koszul.

Bloch-Kato Conjecture and its proof are marvelous advances in Galois theory, however, it seems to us that they are not the end of the story and that much stronger properties hold for Galois cohomology. We believe that universal Koszulity is the right strengthening of Bloch-Kato, independently of the Elementary Type Conjecture:

**Conjecture 2.** In the standing hypothesis that $F$ contains a primitive $p^{th}$ root of unity and $G_F(p)$ is finitely generated, then $H^\bullet(G_F(p), \mathbb{F}_p)$ is universally Koszul.

It may be even possible to extend most of our results to infinitely generated maximal pro-$p$ quotients of absolute Galois groups (in fact, in [Pos14] Positselski
has already successfully proved PBW property of the Galois cohomology of general local and global fields). This possible extension, along with the previous conjecture, are research in progress.

We made a special effort to write this paper in such a way that mathematicians from various branches like Galois theory, commutative algebra, algebraic geometry may find it accessible.

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2. Koszulity and relatives

In this section, $K$ is an arbitrary field.

2.1. Original Koszul property. An associative, unital $K$-algebra $A$ is graded if it decomposes as the direct sum of $K$-vector spaces $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that, for all $i, j \in \mathbb{Z}$, $A_i A_j \subseteq A_{i+j}$. $A_i$ is the homogeneous component of degree $i$. A graded $K$-algebra $A$ is connected if $A_i = 0$ for all $i < 0$ and $A_0 = K \cdot 1_A \cong K$; generated in degree 1 if all its (algebra) generators are in $A_1$; locally finite-dimensional if $\dim_K A_i < \infty$ for all $i \in \mathbb{Z}$ (another common terminology for this is finite-type). A module $M$ over a graded algebra $A$ is graded if it decomposes as the direct sum of $K$-vector spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that, for all $i, j \in \mathbb{Z}$, $A_i M_j \subseteq M_{i+j}$. A graded $A$-module $M$ is connected if there is an $n \in \mathbb{Z}$ such that $M_i = 0$ for all $i < n$; generated in degree $n$ if all its (module) generators are in $M_n$; locally finite-dimensional or finite type if $\dim_K M_i < \infty$ for all $i \in \mathbb{Z}$.

For a vector space $V$ over $K$, let $T(V)$ denote the tensor $K$-algebra generated by $V$, that is, $T(V) = \bigoplus_{n \geq 0} V^\otimes n$, with $V^\otimes 0 = K$.

It is a graded algebra with algebra multiplication given by the tensor product. Whenever possible, in writing the elements of a tensor algebra we will omit tensor signs, and simply use juxtaposition. In particular, if $V$ is finite-dimensional with a basis $\{x_1, \ldots, x_n\}$, then $T(V)$ will be identified with the algebra $K\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials in the variables $x_1, \ldots, x_n$, graded by polynomial degree.

Ignoring the grading, tensor algebras are the free objects in the category of (associative, unital) algebras (cf. [LYT² §1.1.3]), so any algebra is a quotient of a tensor algebra. When an algebra is presented as a quotient of a tensor algebra, we will simply identify an element of the former with each of its representatives in the latter. If the aforementioned quotient is over an ideal generated by homogeneous elements, then the new algebra inherits a well defined grading from the tensor algebra.

For any graded algebra $A$, there exists a unique morphism of graded algebras $\pi : T(A_1) \to A$ that is the identity in degree 1. Clearly, $A$ being generated in degree 1 is equivalent to $\pi$ being surjective. The algebra $A$ is quadratic if it is generated in degree 1 and $\text{Ker}\pi = (\text{Ker}\pi_2) = T(A_1)\text{Ker}\pi_2 T(A_1)$. In this case, we
call $V = A_1$ the \textit{space of generators} and $R = \text{Ker} \pi_2$ the \textit{space of relators}, and we use the notation $A = Q(V, R)$.

Henceforth we assume graded algebras to be locally finite-dimensional, connected and generated in degree 1, unless otherwise specified. In particular, every such algebra $A$ is equipped with the \textit{augmentation map} $\varepsilon : A \to K$ that is the projection onto $A_0$. Its kernel is the \textit{augmentation ideal} $A_+ = \bigoplus_{n \geq 1} A_n$. The augmentation map gives $K$ a canonical structure of graded $A$-module concentrated in degree 0, via the action $a \cdot k = \varepsilon(a)k (a \in A, k \in K)$. In what follows, ground fields of algebras will be always understood to be equipped with this module structure.

A graded module $M$ over a quadratic algebra $A$ is generated in degree $n$ if and only if the natural map \( \varpi : A \otimes M_n \to M \) induced by the module structure is surjective. $M$ is \textit{quadratic} if, for some $n \in \mathbb{Z}$, $M$ is generated in degree $n$ and $\text{Ker} \varpi$ is generated, as an $A$-submodule, by $\text{Ker} \varpi \cap (A_1 \otimes M_n)$. Informally, this means that $M$ has a presentation with degree $n$ generators and degree $n + 1$ relators.

We adhere to the convention that differentials of complexes of graded modules have degree 0.

\textbf{Definition 3.} Let $A$ be a graded $K$-algebra and $M$ a graded $A$-module concentrated in degree $n$. A resolution $(P_\bullet, d_\bullet)$ of $M$ by free graded $A$-modules is said to be \textit{linear} if each $P_i$ is generated in degree $i + n$: $P_i = A \cdot (P_i)_{i+n}$.

\textbf{Remark 4.} This terminology is due to the fact that, if the modules $M_i$ are finitely generated, the entries of the matrix representing each $d_i$ in suitable bases are all in $A_1$.

\textbf{Definition 5.} A quadratic $K$-algebra $A$ is \textit{Koszul} if $K$, as trivial graded $A$-module concentrated in degree 0, has a linear resolution.

\textbf{Definition 6.} A graded module $M$ generated in degree $n$ over a quadratic algebra $A$ is \textit{Koszul} if it has a linear resolution.

\textbf{Remark 7.} Our definition differs from the original one in \cite[Section 2.1]{PP05} in that, following \cite{Pio05}, we allow modules generated in arbitrary degree to be Koszul. This difference is not serious though. In fact, for any graded module $M$ and any integer $z$ we can define the \textit{shifted module} $M(z)$, that is the same module with shifted grading: $M(z)_i = M_{z+i}$. A module is Koszul according to our definition if and only if its appropriate shifted module is Koszul according to \cite{PP05}. A free resolution of $K$ can be contracted to a free resolution of $A_+$ and, conversely, a free resolution of $A_+$ can be extended to a free resolution of $K$, so the following statements are equivalent:

1. $A$ is Koszul;
2. $K$ is a Koszul module;
3. $A_+$ is a Koszul module.

Since in our assumptions the ground field of an algebra, as a graded module, is concentrated in degree 0, Definition 3 is an extension of Definition 5. Moreover, in the case of locally finite-dimensional objects, the existence of linear resolutions can be equivalently expressed in homological terms: a graded module $M$ generated in degree $n$ over a quadratic $K$-algebra $A$ is Koszul if and only if $H_i(M)_j := \text{Tor}^A_{i,j}(M, K) = 0$ for all $i \geq 0$ and $j \neq i + n$. From this, it follows that a Koszul module is necessarily quadratic.
2.2. Enhanced forms of Koszulity. Koszul property is difficult to check by the definition. It turns out that there are properties that imply Koszulity and are usually easier to check.

**Definition 8.** A quadratic algebra \( A = Q(V, R) \) is PBW if the two-sided ideal \((R)\) has a quadratic Gröbner basis.

Then there is a family of conditions based on “divide and conquer” strategies. We first recall a concept they rely on.

**Definition 9.** Let \( A \) be a \( K \)-algebra and \( I, J \subseteq A \) two ideals. The colon ideal \( I : J \) is \( \{ a \in A \mid aJ \subseteq I \} \). If \( J = (x) \) is a principal ideal, we also use the notation \( I : x \).

**Remark 10.** If \( I \) and \( J \) are homogeneous ideals of a graded algebra \( A \), then the colon ideal \( I : J \) is also homogeneous.

**Definition 11.** Let \( A \) be a quadratic algebra and let \( X = \{ u_1, \ldots, u_d \} \) be a minimal system of homogeneous generators of the augmentation ideal \( A_+ \). A graded \( A \)-module \( M \) is linear if it admits a system of homogeneous generators \( \{ g_1, \ldots, g_m \} \), all of the same degree, such that, for each \( j = 1, \ldots, m \), the colon ideal of \( A (Ag_{i_1} + \cdots + Ag_{i_k-1}) : u_{i_r} \) is generated by a subset of \( X \). Such a system is called a set of linear generators of \( M \).

**Definition 12.** A quadratic algebra \( A \) is called strongly Koszul if its augmentation ideal \( A_+ \) admits a minimal system of homogeneous generators \( X = \{ u_1, \ldots, u_d \} \) such that for every subset \( Y = \{ u_{i_1}, \ldots, u_{i_k} \} \subseteq X \) and for every \( r = 1, \ldots, k - 1 \) the colon ideal \( (u_{i_1}, \ldots, u_{i_{r-1}}) : u_{i_r} = \{ a \in A \mid au_{i_r} \in (u_{i_1}, \ldots, u_{i_{r-1}}) \} \) is generated by a subset of \( X \).

Our definition of strongly Koszul is taken from [CDNR13] and it differs from the original one of [HHR00]. In [HHR00], the system of generators is assumed to be totally ordered and the property that \( (u_{i_1}, \ldots, u_{i_{r-1}}) : u_{i_r} \) is generated by a subset of \( X \) is only required for \( u_{i_1} < \cdots < u_{i_{r-1}} < u_{i_r} \). Since in Galois cohomology there is no natural total order on generators, we prefer the given version.

**Remark 13.** Keeping the same notation as in the previous definitions, \( A \) is strongly Koszul if and only if all ideals \( (u_{i_1}, \ldots, u_{i_{r-1}}) \) are linear \( A \)-modules.

In [HHR00], it is also proved that, over a strongly Koszul commutative \( K \)-algebra with a minimal system of homogeneous generators \( \{ u_1, \ldots, u_d \} \), any ideal of the form \( (u_{i_1}, \ldots, u_{i_k}) \) has a linear resolution, and in particular \( A \) is Koszul. In the general (that is, possibly noncommutative) setting, it is immediate that strong Koszulity implies the existence of a Koszul filtration, defined below.

**Definition 14.** A collection \( \mathcal{F} \) of ideals of a quadratic algebra \( A \) is a Koszul filtration if

1. each ideal \( I \in \mathcal{F} \) is generated by elements of \( A \);
2. the zero ideal and the augmentation ideal \( A_+ \) belong to \( \mathcal{F} \);
(3) for each nonzero ideal \( I \in \mathcal{F} \), there exist an ideal \( J \in \mathcal{F} \), \( J \neq I \), and an element \( x \in A_1 \) such that \( I = J + (x) \) and the colon ideal
\[
J : I = \{a \in A \mid xa \in J\}
\]
lies in \( \mathcal{F} \).

The concept of Koszul filtration was first introduced for commutative algebras in [CTV01] as a more flexible version of strong Koszulity. In fact, the definition encodes precisely the properties of strong Koszulity needed to imply the Koszulity of the relevant ideals, but the formulation is coordinate-free, allowing more choice in the collection of ideals. The definition in the general setting appeared in [Pio05], as well as the proof that each ideal of a Koszul filtration is a Koszul module.

In particular, out of a Koszul filtration we can choose ideals with any number of generators in the range from 1 to the minimal number of generators of the algebra. This leads to the following definition, also appeared in [Pio05].

**Definition 15.** Let \( A \) be a quadratic algebra with \( d \)-dimensional space of generators. A sequence of ideals \((I_0, \ldots, I_d)\) of \( A \) is a **Koszul flag** if

1. \((0) = I_0 < I_1 < \cdots < I_d = A_+;\)
2. for all \( k \) there is \( x_k \in A_1 \) such that \( I_{k+1} = I_k + (x_{k+1});\)
3. for all \( k \), \( I_k \) has a linear free graded resolution.

Algebras satisfying one of the preceding conditions have sufficiently many Koszul cyclic modules. At the extreme, one may ask the algebra to have a Koszul filtration as big as possible, so that all ideals generated in degree 1, or equivalently all cyclic modules, are Koszul. In the commutative setting, this property has been introduced in [Con00] under the name of **universal Koszulity**. We extend the definition to the general setting.

**Definition 16.** A quadratic algebra \( A \) is called **universally Koszul** if every ideal generated in degree 1 has a linear resolution.

As in the commutative setting in [Con00], we define
\[
\mathcal{L}(A) = \{I \trianglelefteq A \mid I = AI_1\}
\]
to be the set of all ideals of \( A \) generated in degree 1 and we have the following characterisation of universally Koszul algebras, which is proved exactly as in the commutative setting ([Con00, Proposition 1.4]).

**Proposition 17.** Let \( A \) be a quadratic \( K \)-algebra. The following conditions are equivalent:

1. \( A \) is universally Koszul;
2. for every \( I \in \mathcal{L}(A) \) one has \( \text{Tor}^2_{A_j}(A/I, K) = 0 \) for every \( j > 2;\)
3. for every \( I \in \mathcal{L}(A) \) and every \( x \in A_1 \setminus I \) one has \( I : (x) \in \mathcal{L}(A);\)
4. \( \mathcal{L}(A) \) is a Koszul filtration;

In our treatment of enhanced Koszulity, we will often deal with extensions of algebras, and in particular with ideals of the smaller algebra that have to be extended to the bigger one or vice-versa. The two processes are defined below.

**Definition 18.** Let \( A \subseteq B \) be two \( K \)-algebras. Given an ideal \( I \) of \( A \), we define its **extension** to \( B \) to be the intersection \( I^e \) of all ideals of \( B \) that contain \( I \). In other words, \( I^e \) is the ideal of \( B \) generated by \( I \). Given an ideal \( J \) of \( B \), we define its
contraction to $A$ to be the ideal $J^c = J \cap A$ of $A$. In case $A_1, A_2$ are two subalgebras of $B$, we will use the notation $J^c_{A_1}$ or $J^c_{A_2}$ to differentiate between the contractions of $J$ to either subalgebra.

In the same spirit, when dealing with colon ideals, when we need to specify the algebra in which they are defined, we add the name of that algebra as a subscript to the colon sign.

3. Elementary type pro-$p$ groups

Recall that $G_F(p)$ is the maximal pro-$p$ quotient of the absolute Galois group of $F$. The class of all finitely generated pro-$p$ groups of the form $G_F(p)$, for some field $F$ as in Hypothesis [1] is far from being well understood. A conjectural description of such groups has been proposed in [JW89] for $p = 2$ and in [Efr95] for $p$ odd. It is called Elementary Type Conjecture and it was motivated by the homonymous conjecture for Witt rings of quadratic forms (see Conjecture [60]).

In principle, it is possible to formulate a unified Elementary Type Conjecture that encompasses both the case of $p = 2$ and the case of odd $p$. However, for the purpose of this paper it is sufficient (and convenient) to state a separate weaker conjecture for $p = 2$.

We start with the case of odd $p$.

Definition 19. A cyclotomic pair is a couple $(G, \chi)$ made of a pro-$p$ group $G$ and a continuous homomorphism $\chi$ from $G$ to the multiplicative group of units of $\mathbb{Z}_p$. We call $\chi$ a cyclotomic character. The vocabulary of group properties is extended to cyclotomic pairs, so that for example a finitely generated cyclotomic pair is a cyclotomic pair $(G, \chi)$ such that $G$ is finitely generated as a pro-$p$ group.

This terminology is justified in view of the situation when $G = G_F(p)$ for a field $F$ as in Hypothesis [1] The action of $G_F(p)$ on the group $\mu_{p^\infty}$ of the roots of unity of order a power of $p$ lying in $F_{sep}$ induces a homomorphism $\chi_p : G \to \text{Aut}(\mu_{p^\infty})$. This homomorphism is known in the literature as the $(p$-adic) cyclotomic character.

A Demushkin group is a finitely generated pro-$p$ group $G$ such that $H^2(G, \mathbb{F}_p)$ is a 1-dimensional $\mathbb{F}_p$-vector space and the cup product induces a non-degenerate pairing

$$H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p).$$

In this situation, we identify $H^2(G, \mathbb{F}_p)$ with $\mathbb{F}_p$. The only finite Demushkin group is $C_2$. The classification of Demushkin groups was achieved by J. Labute ([Lab67]), building on previous contributions by S.P. Demushkin ([Dem61] and [Dem63]) and J.-P. Serre ([Ser95]).

Definition 20. For $p$ odd, the class $\mathcal{E}_p$ of elementary type cyclotomic pairs is the smallest class of cyclotomic pairs such that

(a) any pair $(S, \chi)$, with $S$ a finitely generated free pro-$p$ group and $\chi$ an arbitrary cyclotomic character, is in $\mathcal{E}_p$; the pair $(1, 1)$ consisting of the trivial group and the trivial character is not excluded;

(b) any pair $(G, \chi)$, with $G$ a Demushkin group and $\chi$ its unique cyclotomic character;

(c) if $(G_1, \chi_1), (G_2, \chi_2) \in \mathcal{E}_p$, then also the free product $(G_1 *_p G_2, \chi_1 *_p \chi_2)$ is in $\mathcal{E}_p$.
(d) if \((G, \chi) \in \mathcal{E}_p\), then for any positive integer \(m\) also the cyclotomic semi-direct product \((\mathbb{Z}_p^m \rtimes G, \chi \circ \pi)\) is in \(\mathcal{E}_p\). An elementary type pro-\(p\) group (ET group for short) is a group \(G\) appearing in a pair in \(\mathcal{E}_p\). The Elementary Type Conjecture for pro-\(p\) groups states that all finitely generated pro-\(p\) groups of the form \(G_F(p)\), for \(p\) odd and some field \(F\) containing a primitive \(p^{th}\) root of unity, are ET groups. A more detailed exposition can be found in [MPQT].

There is a parallel inductive description of Galois cohomology algebras of ET groups in terms of two basic operations.

**Definition 21.** Let \(A = Q(V_A, R_A)\) and \(B = Q(V_B, R_B)\) be two quadratic algebras over a field \(K\).

1. The **direct sum** of \(A\) and \(B\) is the quadratic algebra
   \[
   A \oplus B = \frac{T(A_1 \oplus B_1)}{(R)}, \quad \text{with} \quad R = R_A \oplus R_B \oplus (A_1 \otimes B_1) \oplus (B_1 \otimes A_1).
   \]
2. The **skew-symmetric tensor product** of \(A\) and \(B\) is the quadratic algebra
   \[
   A \odot B = \frac{T(A_1 \oplus B_1)}{(R)}, \quad \text{with} \quad R = R_A \oplus R_B \oplus (ab + ba \mid a \in A_1, b \in B_1).
   \]
3. Suppose that \(A\) has a distinguished element \(t\) such that \(t + t = 0\) and let \(X_J = \{x_j \mid j \in J\}\) be a set of distinct symbols not in \(A\). The twisted extension of \(A\) by \(X_J\) is the quadratic \(K\)-algebra \(A(t \mid X_J)\) with space of generators \(\text{span}_K \{t, a_1, \ldots, a_d, x_j \mid j \in J\}\) and space of relators \(\text{span}_K (R \cup \{x_i x_j + x_j x_i + t x_i x_j + t x_j x_i, x_j a_k + a_k x_j, x_j^2 - t x_i \mid i, j \in J, k = 1, \ldots, d\})\). When \(X_J = \{x_1, \ldots, x_m\}\), we will use the notation \(A(t \mid x_1, \ldots, x_m)\).

**Remark 22.** Note that the condition \(t + t = 0\) in the definition of twisted extension forces \(t = 0\) whenever the characteristic of the ground field is different from 2. If the characteristic is 2, then \(t\) may or may not be 0. A twisted extension \(A(0 \mid X_J)\) with \(t = 0\) is the same as the skew-symmetric tensor product of \(A\) with the exterior algebra \(\Lambda(X_J)\). Moreover, \(A(0 \mid x_1, \ldots, x_m)\) admits an additional \(\oplus_{i=1}^m (\mathbb{Z}/2\mathbb{Z})\)-grading, characterized by

\[
A(0 \mid x_1, \ldots, x_m)(\varepsilon_1 + 2\mathbb{Z}, \ldots, \varepsilon_m + 2\mathbb{Z}) = \text{span}_K\{a x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m} \mid a \in A\}.
\]

In particular, \(A(0 \mid x)\) has the additional \(\mathbb{Z}/2\mathbb{Z}\)-grading
\[
A(0 \mid x)_{0 + 2\mathbb{Z}} = A, A(0 \mid x)_{1 + 2\mathbb{Z}} = Ax.
\]
Independently of whether \(t = 0\) or not, elements of a twisted extension \(A(t \mid x)\) can be written in the normal form \(p + q x\) for suitable \(p, q \in A\). By the shape of the defining relations of \(A(t \mid x)\), a monomial in \((x)\) can be written as a word with only one \(x\), but it is not possible to write it as a word without \(x\). As a consequence, the coefficients \(p\) and \(q\) above are uniquely determined: if \(p + q x = p' + q' x\), then \(p - p' = (q' - q)x\); since the left hand side belongs to \(A\), \(q = q'\) and then also \(p = p'\).

**Theorem 23** ([NSW08], Theorem 4.1.1). Let \(G_1\) and \(G_2\) be ET pro-\(p\) groups. Then
\[
H^*_p(G_1 \ast_p G_2, \mathbb{F}_p) \cong H^*_p(G_1, \mathbb{F}_p) \cap H^*_p(G_2, \mathbb{F}_p).
\]

**Theorem 24** ([MPQT], Section 5.4). Let \(G_0\) be an ET pro-\(p\) group. Then
\[
H^*_p(\mathbb{Z}_p^m \rtimes G, \mathbb{F}_p) \cong H^*_p(G, \mathbb{F}_p)(0 \mid x_1, \ldots, x_m).
\]
We now turn our attention to the case $p = 2$. An elementary type (ET) pro-$2$ group is a finitely generated pro-$2$ group $G$ such that $H^*(G, \mathbb{F}_2)$ can be obtained via a finite sequence of direct sums and twisted extensions of cohomology algebras with $\mathbb{F}_2$ coefficients of free pro-$2$ groups and cohomology algebras with $\mathbb{F}_2$ coefficients of maximal pro-$2$ quotients of absolute Galois groups of local fields (including $\mathbb{R}$). The Elementary Type Conjecture for pro-$2$ groups claims that all finitely generated pro-$2$ groups of the form $G_F(2)$, for some field $F$ containing a primitive $2^{\text{nd}}$ root of unity, are ET groups.

4. Universal Koszulity of ET groups

4.1. Free pro-$p$ groups.

**Proposition 25.** Suppose that $G$ is a finitely generated free pro-$p$ group. Then $H = H^*(G, \mathbb{F}_p)$ is universally Koszul.

**Proof.** Since the cohomology is concentrated in degrees 0 and 1, the product of any two elements of positive degree is 0. Hence, $I : x = H_+$ for all $I \subseteq H$, $I \neq H_+$ and for all $x \in H_1 \setminus I$. The result follows from condition 3 of Proposition 17. □

4.2. Demushkin groups.

**Proposition 26.** Suppose that $G$ is an infinite Demushkin pro-$p$ group. Then $H = H^*(G, \mathbb{F}_p)$ is universally Koszul.

**Proof.** Let $I \in \mathcal{L}(H) \setminus \{H_+\}$ and let $x \in H_1 \setminus I$. Let us first address the case $I = (0)$. The ideal $(0) : x$ is made of all the solutions of the equation $ax = 0$ in the variable $a \in H$. This equation has a nonzero solution in $H_1$ for any $x$, as can clearly be seen, for example, writing $a$ and $x$ as linear combinations of an alternating basis of $H_1$. So $(0) : x \neq (0)$, whence $H_1((0) : x) \supseteq H_2$ by nondegeneracy of the cup product. For a general ideal $I$, we have $(0) : x \subseteq I : x$, so again $H_1(I : x) \supseteq H_2$.

But in general if $J$ is an ideal of a graded algebra $A$ generated in degree 1 such that $A_1J \supseteq A_2$, then $J = AJ_1$. In fact, using the hypothesis that $A$ is generated in degree 1, for all $n \in \mathbb{N}$

$$A_{n+1} = A_{n-1}A_2 \subseteq A_{n-1}A_1J = A_nJ.$$ 

Now, if $a$ is a homogeneous element of $J_{n+1}$ for some $n \in \mathbb{N}$, then $a \in A_{n+1} \subseteq A_nJ$ and so, taking the degree of $a$ into account, $a \in A_nJ_1$.

This shows that condition 3 of Proposition 17 is satisfied. □

**Proposition 27.** The cohomology $H = H^*(C_2, \mathbb{F}_2)$ of the group of order 2 is universally Koszul.

**Proof.** In this case $H = \mathbb{F}_2[t]$ is a free algebra on one generator. Therefore, for all $I \in \mathcal{L}(H) \setminus \{H_+\}$ and all $x \in H_1 \setminus I$, $I : x = (0)$. The result follows from condition 3 of Proposition 17. □

4.3. Direct sum.

**Proposition 28.** Let $A$ and $B$ be universally Koszul algebras over an arbitrary field $K$. Then the direct sum $C = A \cap B$ is universally Koszul.
Proof. The proof is almost verbatim the same as in the commutative case \cite[Lemma 1.6(3)]{Con00}. Let \( I \in \mathcal{L}(C) \setminus \{C_+\} \) and let \( x \in C_1 = A_1 \oplus B_1 \setminus I \). Write \( I = (a_1 + b_1, \ldots, a_k + b_k) \) and \( x = a + b \), with \( a_i, a \in A_1 \) and \( b_i, b \in B_1 \). Set \( J_A = (a_1, \ldots, a_k) \), \( J_B = (b_1, \ldots, b_k) \) and \( J = J^e_A + J^e_B = J_A + J_B \). Then \( I \subseteq J \). Moreover, if \( c \in J_i \) for \( i \geq 2 \), then \( c \in I \), because \( A_4 \) annihilates \( B_+ \) and conversely. We claim that \( I : (x) = (J : (x)) \cap C_+ \). In fact, on one side, if \( x \in C \) is such that \( cx \in I \), then \( cx \in J \), so \( c \in J : (x) \), and \( c \in C_+ \) since by assumption \( x \notin I \). On the other side, \( c \in J : (x) \) and \( c \in C_+ \) imply \( cx \in \oplus_{i \geq 2} J_i = \oplus_{i \geq 2} I_i \), so \( c \in I : (x) \).

As a consequence, if \( x \in J \), then \( I : (x) = C_+ \in \mathcal{L}(C) \). Otherwise, \( I : (x) = J : (x) \) and there are three cases to consider.

1. If \( a \notin J_A \) and \( b \notin J_B \), then \( J : (x) = (J_A :_{A} a)^e + (J_B :_{B} b)^e \). Indeed, let \( c \in J : x \). Using Remark \cite{10} we may assume that \( c \) is homogeneous, and \( c \in C_+ \) since \( x \notin J \). Then \( cx \in J_A + J_B \). Writing \( c = c_A + c_B \) with \( c_A \in A_+ \) and \( c_B \in B_+ \), we get that \( cx = (c_A + c_B)a + (c_A + c_B)b = c_Aa + c_Bb \in J_A + J_B \). Hence, \( c_Aa \in J_A \) and \( c_Bb \in J_B \). Therefore, \( c_Aa \in J_A :_{A} a \) and \( c_Bb \in J_B :_{B} b \), whence \( c \in (J_A :_{A} a)^e + (J_B :_{B} b)^e \). Conversely, if \( c = c_A + c_B \in (J_A :_{A} a)^e + (J_B :_{B} b)^e \), then \( c_Aa \in J_A \) and \( c_Bb \in B_+ \), as \( a \notin J_A \) and \( b \notin J_B \). Hence, \( cx = c_Aa + c_Bb \) with \( c_Aa = ca \in J_A \) and \( c_Bb = cb \in J_B \). Therefore, \( cx \in J_A + J_B \), or equivalently \( c \in J : x \).

2. If \( a \in J_A \) and \( b \notin J_B \), then \( J : x = (A_+)^e + (J_B :_{B} b)^e \). In fact, if \( c \in J : x \), we may again assume that \( c \) is homogeneous, by Remark \cite{10} and that \( c \in C_+ \), because \( x \notin J \). Then, with the same notation as before, \( cx = c_Aa + c_Bb \in J_A + J_B \). Since \( a \in J_A \), for all \( c_A \in A_+ \), \( c_Aa \in J_A \), hence \( c_Bb \in (J_A + J_B) \cap B = J_B \). Summing up, \( c \in (A_+)^e + (J_B :_{B} b)^e \). The reverse inclusion is obvious.

3. If \( a \notin J_A \) and \( b \in J_B \), then \( J : x = (J_A :_{A} a)^e + (B_+)^e \). The proof is analogous to the previous case.

In all cases, since \( J_A :_{A} a \in \mathcal{L}(A) \supseteq A_+ \) and \( J_B :_{B} b \in \mathcal{L}(B) \supseteq B_+ \), we deduce that \( J : x \notin \mathcal{L}(C) \). The statement now follows from condition 3 of Proposition \cite{17}.

4.4. Twisted extension.

Proposition 29. Let \( A = Q(V, R) \) be a universally Koszul algebra over an arbitrary field \( K \). Then the twisted extension \( B := A(t \mid x_1, \ldots, x_m) \) is universally Koszul.

Proof. Since \( A(t \mid x_1, \ldots, x_m) = A(t \mid x_1)(t \mid x_2) \ldots (t \mid x_m) \), by induction it is enough to prove the claim for

\[ B = A(t \mid x) = \frac{T(V \oplus \text{span}_K \{x\})}{(R \cup \{x^2 - tx, xa + ax \mid a \in V\})}. \]

The collection \( \mathcal{L}(B) \) obviously satisfies the first two conditions in the definition of Koszul filtration, so we focus on the third. Let \( I \in \mathcal{L}(B) \setminus \{(0)\} \).

(1) Suppose that \( I \) has a complete set of generators belonging to \( A \), that is, \( I = I_A^e \) for some \( I_A \in \mathcal{L}(A) \). Then by the universal Koszulity of \( A \) there exist \( J_A \in \mathcal{L}(A) \setminus \{I_A\} \) and \( a \in A_1 \) such that \( I_A = J_A + (a) \) and \( J_A :_{A} a \in \mathcal{L}(A) \). Set \( J = J_A^e \).

First, we claim that \( J \cap A = J_A \). Indeed, let \( \{a_i\} \) be a set of generators of \( J_A \). The same set also generates \( J \) as an ideal of \( B \). Let \( p \in J \cap A \); then \( p = \sum_i (p_i + q_i x)a_i = \sum_i p_i a_i - (\sum_i q_i a_i) x \); since \( p \in A \), it follows from
the uniqueness of the normal form of elements in $B$ that $p \sum p_i \tilde{a}_i$, whence $p \in J_A$. The reverse inclusion is obvious.

Then we claim that $J \neq I = J + (a)$. The equality is a direct consequence of $I_A = J_A + (a)$. As regards the inequality, suppose by contradiction that $J = I$; then $a \in J$; but since $a \in A$, by the previous claim $a \in J_A$; as a consequence, $I_A = J_A + (a) = J_A$, in contradiction with the construction of $J_A$.

Finally, we claim that $J : B a = (J_A : A a)^c$. In fact, if $J : B a$ contains an arbitrary element in normal form $p + qx \in B$, $p, q \in A$ (recall Remark 22), then $pa + qx = pa − qax \in J$. Writing $pa − qax = \sum_i (p_i + q_i x) \tilde{a}_i$, where the $\tilde{a}_i$‘s are the generators of $J$, and factoring $x$ out in the right hand side, by uniqueness of normal form $pa = \sum_i p_i \tilde{a}_i \in J \cap A$ and $qa = \sum_i q_i \tilde{a}_i \in J \cap A$ separately. This means that $p, q \in J_A : A a$, whence $p + qx \in (J_A : A a)^c$. For the reverse inclusion, the generators of $J_A : A a$ belong to $J : B a$ by construction. This implies that $J : B a \in \mathcal{L}(B)$.

(2) Suppose now that $I$ does not have a set of generators all included in $A$. Using suitable K-linear combinations of generators, which of course preserve degrees, we can reduce to the case $I = I_A^c + (x + l)$, where $I_A \in \mathcal{L}(A)$ and $l \in A \setminus I$. We first address the case $l = 0$, that is, $I = I_A^c + (x)$. Set $J = I_A^c$. Then $J \neq I = J + (x)$ and $J : B x = J + (−t + x)$. In fact, if $J : B x$ contains an arbitrary element $p + qx \in B$ in normal form as before, then $px + qx^2 = px + qtx \in J$. Writing $px + qtx = \sum_i (p_i + q_i x) \tilde{a}_i$, where the $\tilde{a}_i$‘s are the generators of $J$, by uniqueness of normal form $p + qt = \sum_i q_i \tilde{a}_i \in J$. But then $p + qx = p + qt + q(-t + x) \in J + (−t + x)$. For the reverse inclusion, the generators of $J + (−t + x)$ belong to $J : B x$ thanks to the defining relation $x^2 − tx$ of $B$. Again, it follows that $J : B x \in \mathcal{L}(B)$.

Now, if $l$ is arbitrary, we consider the automorphism of graded algebra $\hat{\tau} : B \to B$ defined as $\hat{\tau}(p + qx) = p + q(x − l)$. This automorphism sends $J$ to $J^c + (x)$, so the general case reduces to the previously proved special case.

The statement now follows from condition 3 of Proposition 17. 

5. Strong Koszulity of ET groups

5.1. Free pro-p groups.

Proposition 30. Suppose that $G$ is a free pro-$p$ group on $n$ generators. Then $H = H^*(G, F_p)$ is strongly Koszul.

Proof. In this case, the cohomology is concentrated in degrees 0 and 1, that is, the space of relators of the cohomology is the span of all quadratic monomials. The result then follows from \cite{EH12} Corollary 6.6. 

5.2. Demushkin groups. Let $d$ denote the cardinality of a minimal set of generators of a Demushkin group $G$.

Proposition 31. Suppose that $G$ is a Demushkin group with invariant $q \neq 2$. Then $H = H^*(G, F_p)$ is strongly Koszul.

\footnote{The symbol $\hat{\tau}$ was chosen and strongly defended by the second author, despite the doubts of the third author}
Remark 32. These Demushkin groups are those studied by Demushkin in [Dem61] and [Dem63].

Proof. In this case, $d$ is necessarily even, say $d = 2k$. Moreover, there exists a symplectic basis $X = \{a_1, \ldots, a_{2k}\}$ of $H^1(G, \mathbb{F}_p)$ such that
\[
\begin{align*}
    a_1 \cup a_2 &= a_3 \cup a_4 = \cdots = a_{2k-1} \cup a_{2k} = 1, \\
    a_2 \cup a_1 &= a_4 \cup a_3 = \cdots = a_{2k} \cup a_{2k-1} = -1, \\
    a_i \cup a_j &= 0 \text{ for all other } i, j
\end{align*}
\]
(cf. [Lab67] Proposition 4).

Now there are only two cases to consider in order to verify the condition of Definition 12.

1. If the subset $Y$ is a singleton $\{a_i\}$, then the only quotient ideal involved is
\[
\langle 0 \rangle: a_i = \begin{cases} 
\{\{a_j \mid j \neq i-1\}\} & \text{if } i \text{ odd} \\
\{\{a_j \mid j \neq i+1\}\} & \text{if } i \text{ even}
\end{cases}
\]

2. If the subset $Y$ contains at least 2 elements, then, besides a quotient ideal of the preceding type, the quotient ideals of the form $(a_{i_1}, \ldots, a_{i_{\ell-1}}) : a_{i\ell}$ for $j > 1$ are involved. But each of these ideals is just the whole $(X)$.

\[\square\]

Proposition 33. Suppose that $G$ is a Demushkin pro-$2$ group on $d = 2k + 1$ generators, $k \in \mathbb{N} \setminus \{0\}$. Then $H = H^\bullet(G, \mathbb{F}_2)$ is strongly Koszul.

Remark 34. These Demushkin groups are those studied by Serre in [Ser95].

Proof. There is a basis $X = \{a_1, \ldots, a_{2k+1}\}$ of $H^1(G, \mathbb{F}_2)$ such that
\[
\begin{align*}
    a_1 \cup a_1 &= a_2 \cup a_3 = a_4 \cup a_5 = \cdots = a_{2k} \cup a_{2k+1} = 1, \\
    a_3 \cup a_2 &= a_5 \cup a_4 = \cdots = a_{2k+1} \cup a_{2k} = 1, \\
    a_i \cup a_j &= 0 \text{ for all other } i, j
\end{align*}
\]
(cf. [Lab67] Proposition 4).

1. If the subset $Y$ is a singleton $\{a_i\}$, then the only quotient ideal involved is
\[
\langle 0 \rangle: a_i = \begin{cases} 
\{\{a_j \mid j \neq 1\}\} & i = 1 \\
\{\{a_j \mid j \neq i+1\}\} & i \text{ even} \\
\{\{a_j \mid j \neq i-1\}\} & i \geq 3 \text{ odd}
\end{cases}
\]

2. If the subset $Y$ contains at least 2 elements, then we argue exactly as in Proposition 31.

\[\square\]

Proposition 35. Suppose that $G$ is a Demushkin pro-$2$ group on $d = 2k$ generators, $k \in \mathbb{N} \setminus \{0\}$, with invariant $q = 2$. Then $H = H^\bullet(G, \mathbb{F}_2)$ is strongly Koszul.

Remark 36. These Demushkin groups are those studied by Labute in [Lab67].

Proof. There is a basis $X = \{a_1, \ldots, a_{2k}\}$ of $H^1(G, \mathbb{F}_2)$ such that
\[
\begin{align*}
    a_1 \cup a_1 &= a_1 \cup a_2 = a_3 \cup a_4 = \cdots = a_{2k-1} \cup a_{2k} = 1, \\
    a_2 \cup a_1 &= a_4 \cup a_3 = \cdots = a_{2k} \cup a_{2k-1} = 1, \\
    a_i \cup a_j &= 0 \text{ for all other } i, j
\end{align*}
\]
the cup product is given by

\[ b_1 = a_1, \quad b_2 = a_2 + a_1, \quad b_i = a_i \ (i = 3, \ldots, 2k), \]

the cup product is given by

\[
\begin{align*}
b_1 \cup b_1 &= b_2 \cup b_2 = b_3 \cup b_4 = \cdots = b_{2k-1} \cup b_{2k} = 1, \\
b_4 \cup b_3 &= \cdots = b_{2k} \cup b_{2k-1} = 1, \\
a_i \cup a_j &= 0 \text{ for all other } i, j.
\end{align*}
\]

This provides the following description of the relevant quotient ideals.

1. If the subset \( Y \) is a singleton \( \{b_i\} \), then the only quotient ideal involved is

\[
(0): b_i = \begin{cases} 
\{\{b_j \mid j \neq 1\}\} & i = 1 \\
\{\{b_j \mid j \neq 2\}\} & j = 2 \\
\{\{b_j \mid j \neq i + 1\}\} & i > 1 \text{ odd} \\
\{\{b_j \mid j \neq i - 1\}\} & i > 2 \text{ even.}
\end{cases}
\]

2. If the subset \( Y \) contains at least 2 elements, then we argue exactly as in Proposition 31.

\[ \square \]

5.3. Direct sum.

**Proposition 37.** Let \( A \) and \( B \) be strongly Koszul algebras over an arbitrary field \( K \) with respect to the minimal systems of homogeneous generators \( X_A = \{a_1, \ldots, a_c\} \) and \( X_B = \{b_1, \ldots, b_d\} \) respectively. Then the direct sum \( A \oplus B \) is strongly Koszul with respect to the minimal system of generators \( \{a_1, \ldots, a_c, b_1, \ldots, b_d\} \).

**Proof.** The two key ideas are that every element \( x \in A \oplus B \) has a unique decomposition \( x = x_A + x_B \), with \( x_A \in A \) and \( x_B \in B \), and that elements of \( A \) annihilate \( x_B \) and conversely.

1. An ideal of \( A \cap B \) of the shape \( I = (a_{i_1}, \ldots, a_{i_{k-1}}) : a_{i_k} \) coincides with \((J_A)^r + (J_B)^r\). By hypothesis \( J_A \) is generated, as an ideal of \( A \), by a subset \( Y_A \) of \( X_A \), so \( I = (Y_A \cup Y_B) \). An analogous argument works for an ideal of \( A \cap B \) of the shape \( (b_{i_1}, \ldots, b_{i_{k-1}}) : b_{i_k} \).

2. An ideal of \( A \cap B \) of the shape \( I = (a_{i_1}, \ldots, a_{i_{k-1}}, b_{j_1}, \ldots, b_{j_{k-1}}) : b_{j_k} \) coincides with \((A_A)^r + ((b_{j_1}, \ldots, b_{j_{k-1}}) : b_{j_k})_{B}^r\). By hypothesis, \((b_{j_1}, \ldots, b_{j_{k-1}}) : b_{j_k})_{B}^r \) is generated, as an ideal of \( B \), by a subset \( Y_B \) of \( X_B \), so \( I = (X_A \cup Y_B) \). An analogous argument works for an ideal of \( A \cap B \) of the shape \( I = (a_{i_1}, \ldots, a_{i_{k-1}}, b_{j_1}, \ldots, b_{j_k}) : a_{i_k} \).

\[ \square \]

5.4. Twisted extension.

**Proposition 38.** Let \( A = Q(V, R) \) be a strongly Koszul \( K \)-algebra over an arbitrary field \( K \) with respect to the minimal system of homogeneous generators \( X_A = \{a_1, \ldots, a_d\} \) and let \( X_J = \{x_1, \ldots, x_m\} \). Then the twisted extension \( B := A(0 \mid X_J) = A \otimes^{-1} \lambda(x_1, \ldots, x_m) \) is strongly Koszul with respect to the minimal system of homogeneous generators \( \{a_1, \ldots, a_d, x_1, \ldots, x_m\} \).
Proof. Since \( A(0 \mid X_f) = A(0 \mid x_1)(0 \mid x_2)\ldots(0 \mid x_m) \), by induction it is enough to prove the statement for the algebra

\[
B = A(0 \mid x) = \frac{T(\mathop{\text{span}}_K (X_A \cup \{x\}))}{(R \cup \{x^2, xa_i + a_ix \mid a_i \in X_A\})}.
\]

The only relation between \( x \) and any element of \( A_+ \) is skew-commutativity. As a consequence,

1. If, in \( A \), \( (a_{i_1}, \ldots, a_{i_k}) : a_{i_{k+1}} = (a_{j_1}, \ldots, a_{j_r}) \), then in \( B \)

\[
(a_{i_1}, \ldots, a_{i_k}) : a_{i_{k+1}} = (a_{j_1}, \ldots, a_{j_r}),
\]

\[
(a_{i_1}, \ldots, a_{i_k}, x) : a_{i_{k+1}} = (a_{j_1}, \ldots, a_{j_r}, x).
\]

2. \( (a_{i_1}, \ldots, a_{i_k}) : x = (a_{i_1}, \ldots, a_{i_k}, x) \).

In fact, any element \( b \in B \) can be expressed as a sum \( p + qx \) with \( p, q \in A \), and the summands \( p \) and \( qx \) are exactly the homogeneous components of \( b \) with respect to the additional grading \((3.1)\). Since the left hand side colon ideals are homogeneous with respect to the additional grading \((3.1)\), an element \( b \) belongs to one of these ideals if and only if both \( p \) and \( qx \) belong to that ideal. \( \square \)

6. Exceptions to Strong Koszulity

Strong Koszulity may not be preserved by twisted extensions in characteristic 2, when \( t \neq 0 \). In this section we present two notable families of examples.

**Definition 39** (Lam05). The level (Stufe) of a field \( F \), denoted \( s(F) \), is the smallest positive integer \( k \) such that \(-1\) can be written as a sum of \( k \) squares in \( F \), provided such integer exists, and 0 otherwise.

**Remark 40.** If \( \text{char } F = 2 \), then \( s(F) = 1 \), since \(-1\) is a square. If \( \text{char } F \neq 2 \), then \( s(F) = 0 \) if and only if the field is formally real. Otherwise, \( s(F) \) is a power of 2 (see Lam05).

The level of a field \( F \) is related to the value set of some quadratic forms over it. For \( a \in F^x \), \( [a] \) denotes the image of \( a \) in \( F^x/(F^x)^2 \). The value set of a quadratic form \( \phi \) over \( F \) is defined as \( D_F\phi = \{ [a] \in F^x/(F^x)^2 \mid a \text{ is represented by } \phi \} \).

**Definition 41.** A field \( F \) is 2-rigid if, for all \( a \in F \) such that \([a] \neq 1\) and \([a] \neq [-1]\), the value set of the quadratic form \( \langle 1, a \rangle = X^2 + aY^2 \) is included in \( \{ [1], [a] \} \subseteq F^x/(F^x)^2 \).

By [War78, Theorem 1.9, Proposition 1.1], the level of a 2-rigid field \( F \) is either 0, 1 or 2. The case \( s(F) = 1 \) corresponds to the condition that \( F \) contains a square root of \(-1\). 2-rigid fields of level 0 are called superpythagorean.

**6.1. Superpythagorean fields.** Let \( F \) be a superpythagorean field such that \( \dim_{F_2} F^x/(F^x)^2 = d < \infty \) and let \( \{[-1], [a_2], \ldots, [a_d]\} \) be a fixed \( F_2 \)-basis of \( F^x/(F^x)^2 \). Then a presentation of \( H = H^*(G_F(2), F_2) \) is

\[
H = F_2(t, a_2, \ldots, a_n \mid a_\alpha a_t = a_\alpha t a_t, \alpha_i a_i = ta_i),
\]

where \( t = \ell([-1]), \alpha_i = \ell([a_i]) \) and cup products are omitted (cf. Wad83). For all \( n \geq 1 \), the set

\[
B_n = \{ t^{n-r} a_{i_1} \cdots a_{i_r} \mid 0 \leq r \leq d, \quad 0 < i_1 < \cdots < i_r \}
\]

forms a \( F_2 \)-basis of \( H^n(G_F(2), F_2) \) (cf. EL72 Theorem 5.13(2)) and Voc03 Corollary 7.5).
Proposition 42. If $F$ is a superpythagorean field with $\dim_{\mathbb{F}_2} F^*/(F^*)^2 < \infty$, then $H = H^*(G_F(2), \mathbb{F}_2)$ is PBW.

Proof. We keep the previous notation and we apply the Rewriting Method of [LV12, Section 4.1], to which we also refer for the terminology. Consider the degree-lexicographic order on the monomials of $T(t, \alpha_2, \ldots, \alpha_d)$ induced by the total order $t < \alpha_2 < \cdots < \alpha_d$. Then a normalized basis for the space of relations of $H$ is
\[ \{ \alpha_j \alpha_i - \alpha_i \alpha_j \mid 2 \leq i < j \leq d \} \cup \{ \alpha_i t - t \alpha_i \mid 2 \leq i \leq d \} \cup \{ \alpha_i \alpha_i - t \alpha_i \mid 2 \leq i \leq d \}. \]

The corresponding critical monomials are
1. $\alpha_i \alpha_i \alpha_i \ (2 \leq i \leq d)$;
2. $\alpha_j \alpha_j \alpha_i \ (2 \leq i < j \leq d)$;
3. $\alpha_j \alpha_i \alpha_i \ (2 \leq i < j \leq d)$;
4. $\alpha_i \alpha_i t \ (2 \leq i \leq d)$;
5. $\alpha_k \alpha_j \alpha_i \ (2 \leq i < j < k \leq d)$;
6. $\alpha_j \alpha_i t \ (2 \leq i < j \leq d)$.

The graphs of the critical monomials are

- **Type (1):**
  \[
  \alpha_i \alpha_i \alpha_i \quad \xrightarrow{\alpha_i t \alpha_i} \quad \alpha_i t \alpha_i \quad \xrightarrow{\alpha_i \alpha_i \alpha_i} \quad t \alpha_i \alpha_i \quad \xrightarrow{t \alpha_i \alpha_i} \quad t t \alpha_i.
  \]

- **Type (2):**
  \[
  \alpha_j \alpha_j \alpha_i \quad \xrightarrow{t \alpha_j \alpha_i} \quad \alpha_j \alpha_i \alpha_i \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad \alpha_j t \alpha_i \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad \alpha_i t \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad t \alpha_i \alpha_j.
  \]

- **Type (3):**
  \[
  \alpha_j \alpha_i \alpha_i \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad \alpha_i \alpha_i \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad \alpha_i \alpha_i \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad t \alpha_i \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad t \alpha_i \alpha_j.
  \]

- **Type (4):**
  \[
  \alpha_i \alpha_i t \quad \xrightarrow{\alpha_i \alpha_i t} \quad \alpha_i \alpha_i \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad t \alpha_i \alpha_j \quad \xrightarrow{\alpha_i \alpha_i \alpha_j} \quad t \alpha_i \alpha_j.
  \]

- **Type (5):**
  \[
  \alpha_i \alpha_k \alpha_i \quad \xrightarrow{\alpha_i \alpha_k \alpha_i} \quad \alpha_k \alpha_i \alpha_i \quad \xrightarrow{\alpha_k \alpha_i \alpha_i} \quad \alpha_i \alpha_k \alpha_i \quad \xrightarrow{\alpha_i \alpha_k \alpha_i} \quad \alpha_i \alpha_j \alpha_k \quad \xrightarrow{\alpha_i \alpha_j \alpha_k} \quad \alpha_i \alpha_j \alpha_k.
  \]

- **Type (6):**
  \[
  \alpha_j \alpha_i t \quad \xrightarrow{\alpha_j \alpha_i t} \quad \alpha_j \alpha_i \alpha_i \quad \xrightarrow{\alpha_j \alpha_i \alpha_i} \quad \alpha_i t \alpha_i \quad \xrightarrow{\alpha_i t \alpha_i} \quad \alpha_j t \alpha_i \quad \xrightarrow{\alpha_j t \alpha_i} \quad \alpha_j t \alpha_i.
  \]

Since all critical monomials are confluent, $H$ is PBW. $\square$
Proposition 43. If $F$ is a superpythagorean field with $\dim_{\mathbb{F}_2} F^x/(F^x)^2 < \infty$, then $H = H^\bullet(G_F(2), \mathbb{F}_2)$ is universally Koszul.

Proof. Since $H = \mathbb{F}_2[t] | t \alpha_1, \ldots, \alpha_d)$ is a twisted extension of the universally Koszul algebra $\mathbb{F}_2[t]$, the statement follows from $[29]$. □

Proposition 44. If $F$ is a superpythagorean field with $3 \leq \dim_{\mathbb{F}_2} F^x/(F^x)^2 < \infty$, then $H = H^\bullet(G_F(2), \mathbb{F}_2)$ is not strongly Koszul.

Proof. We first note that, by the shape of the defining relations of $H$, for any $i = 1, \ldots, d$,

\[(6.2) \quad (0): \alpha_i = (t + \alpha_i).
\]

Since the system of generators $\{t, \alpha_1, \ldots, \alpha_d\}$ of $H$ is arbitrary, a similar equality holds in complete generality. Explicitly, for all $a \in F^x$ such that $[a] \neq [1]$ and $[a] \neq [-1]$, we have that

\[(0): \ell([a]) = (\ell([-a])).
\]

Since $\dim_{\mathbb{F}_2} F^x/(F^x)^2 \geq 3$, any basis of $H^1(G_F(2), \mathbb{F}_2)$ contains at least two elements $\ell([a]), \ell([b])$ such that the elements $\ell([-1]), \ell([a]), \ell([b]), \ell([-a]), \ell([-b])$ are all distinct. Now suppose that $H$ be strongly Koszul with respect to a minimal system of homogeneous generators $X = \{a_1, \ldots, a_n\}$ such that $\ell([a]), \ell([b])$. Applying the previous equality to (0): $\ell([a])$ and (0): $\ell([b])$ forces

\[X \supseteq \{\ell([a]), \ell([b]), \ell([-a]), \ell([-b])\}.
\]

But then the system $X$ is not minimal, as it satisfies the relation

\[\ell([a]) + \ell([-a]) + \ell([b]) + \ell([-b]) = \ell([-1]) + \ell([-1]) = 0 \in H^1(G_F(2), \mathbb{F}_2),
\]
a contradiction. □

6.2. 2-Rigid fields of level 2. Let $F$ be a 2-rigid field of level 2 such that $\dim_{\mathbb{F}_2} F^x/(F^x)^2 = d < \infty$ and let $\{[-1], [a_2], \ldots, [a_d]\}$ be a fixed $\mathbb{F}_2$-basis of $F^x/(F^x)^2$. Then a presentation of $H = H^\bullet(G_F(2), \mathbb{F}_2)$ is

\[(6.3) \quad H = \mathbb{F}_2[t, \alpha_2, \ldots, \alpha_d | \alpha_j \alpha_i = \alpha_i \alpha_j, \alpha_i t = t \alpha_i, \alpha_i \alpha_i = t \alpha_i, tt = 0],
\]

where $t = \ell([-1]), \alpha_i = \ell([a_i])$ and cup products are omitted (cf. [Wad83]). This graded algebra is concentrated in degrees 0 to $d$. For all $n \leq d$, the set

\[B_n = \{t^\delta \alpha_{i_1} \cdots \alpha_{i_{n-\delta}} | \delta = 0, 1, \quad i_1 < \cdots < i_{n-\delta}\}
\]

forms a $\mathbb{F}_2$-basis of $H^n(G_F(2), \mathbb{F}_2)$.

We use the previous notation throughout this subsection, unless otherwise specified.

Proposition 45. If $F$ is a 2-rigid field of level 2 with $\dim_{\mathbb{F}_2} F^x/(F^x)^2 < \infty$, then $H = H^\bullet(G_F(2), \mathbb{F}_2)$ is PBW.

Proof. Consider the degree-lexicographic order on the monomials of $T(t, \alpha_2, \ldots, \alpha_d)$ induced by the total order $t < \alpha_2 < \cdots < \alpha_d$. Then a normalized basis for the space of relations of $H$ in the sense of [LV12, Section 4.1] is

\[\{\alpha_j \alpha_i - \alpha_i \alpha_j, \quad \alpha_i t - t \alpha_i, \quad \alpha_i \alpha_i - t \alpha_i | 2 \leq i < j \leq d\} \cup \{tt\}.
\]

The six types of monomials introduced in the proof of Proposition [12] are again critical. In addition, there are two new types:

\[(1') \quad ttt;
\]
The graphs of critical monomials of types (2), (3), (5) and (6) are exactly the same as in Proposition 42. The graphs of types (1) and (4) shall be modified as follows:

**Type (1):**

\[ \alpha_i \alpha_i \alpha_i \]

\[ t \alpha_i \alpha_i \]

\[ tt \alpha_i \quad 0 \]

**Type (4):**

\[ \alpha_i \alpha_i t \]

\[ t \alpha_i t \]

\[ tt \alpha_i \quad 0 \]

Finally, the graphs of the new critical monomials are

**Type (1’):**

\[ ttt \quad 0 \]

**Type (3’):**

\[ \alpha_i tt \]

\[ t \alpha_i t \]

\[ tt \alpha_i \quad 0 \]

Since all critical monomials are confluent, \( H \) is PBW.

**Proposition 46.** If \( F \) is a 2-rigid field of level 2 with \( \dim_{\mathbb{F}_2} F^x/(F^x)^2 < \infty \), then \( H = H^\bullet(G_F(2), \mathbb{F}_2) \) is universally Koszul.

**Proof.** Since \( H = \mathbb{F}_2[t \mid t^2](t \mid \alpha_1, \ldots, \alpha_d) \) is a twisted extension of the universally Koszul algebra \( \mathbb{F}_2[t \mid t^2] \), the statement follows from [29] □

**Proposition 47.** If \( F \) is a 2-rigid field of level 2 with \( 3 \leq \dim_{\mathbb{F}_2} F^x/(F^x)^2 < \infty \), then \( H = H^\bullet(G_F(2), \mathbb{F}_2) \) is not strongly Koszul.

**Proof.** Identical to that of Proposition 44, observing that Equation (6.2) holds in this situation as well. □

7. Koszul filtrations and ET operations

It is natural to investigate the compatibility of Koszul filtrations with direct sums and twisted extensions. In view of the sharper results in Section 4, the content of this section is not needed for our applications to Galois groups, but it seems to be interesting in its own right.

In this section algebras are defined over an arbitrary field.

**Proposition 48.** Let \( A \) and \( B \) be algebras with respective Koszul filtrations \( \mathcal{F} \) and \( \mathcal{G} \). Then the direct sum \( C = A \sqcap B \) has the Koszul filtration \( \mathcal{H} = \mathcal{F} \sqcap \mathcal{G} = \{I_A^e + I_B^e \mid I_A \in \mathcal{F}, I_B \in \mathcal{G}\} \).
Proposition 49. Let $t \in A$ satisfy $t + t = 0$ and that $F$ has the property\footnote{The tag $\triangledown$ represents the authors’ love for this property.} \( (\triangledown) \)

\[
J \in F \Rightarrow J + (t) \in F.
\]

Then any twisted extension $C = A(t \mid x_1,\ldots,x_n)$ has the Koszul filtration

\[
\mathcal{H} = \{I^e + (Y) \mid I \in F, Y \subseteq \{x_1,\ldots,x_m, t - x_1,\ldots,t - x_m\}\}.
\]

Proof. We first prove the result for the case $C = A(t \mid x), \mathcal{H} = \{I^e + (Y) \mid I \in F, Y \subseteq \{x, t - x\}\}$. Conditions (1) and (2) of Definition \ref{defn:koszul} are clearly satisfied. As regards Condition (3), any $c \in C$ can be written in normal form as $c = p + qx$ for some $p, q \in A$ (Remark \ref{rem:koszul}). We now address several cases separately.

(1) $J = I_A^e$ for $I_A \in F \setminus \{(0)\}$.

By hypothesis, there are $J_A = (a_1,\ldots,a_r)_A \in F \setminus \{I_A\}$ and $a \in A_1$ such that $J_A = J_A + (a)_A$ and $J_A :_A a \in F$. We then set $J = J_A^e \leq C$, so that $J \neq I = J + (a)_C$, and we claim that $J :_C a = (J_A :_A a)^e \in H$. In fact, all the generators of $J_A :_A a$ belong to $J :_C a$ by construction. For the reverse inclusion, take any $c = p + qx \in J :_C a$. Then $ca = pa - qax = \sum_{i=1}^r (p_i + q_i a_i)$, for suitable $p_i, q_i \in A$. By uniqueness of the normal form,

\[
\begin{align*}
pa &= \sum_{i=1}^r p_i a_i \in J, \\
qa &= \sum_{i=1}^r q_i a_i \in J.
\end{align*}
\]

But then $p, q \in (J :_C a) \cap A = J_A :_A a \leq A$, hence $p + qx \in (J_A :_A a)^e \leq C$.

(2) $J = I_A^e + (x)_C$ for $I_A = (a_1,\ldots,a_d)_A \in F, d \geq 0$ and all $a_i \in A_1$.

We set $J = I_A^e = (a_1,\ldots,a_d)_C$, so that $J \not= I = J + (x)_C$, and we claim that $J :_C x = I_A^e + (t - x)_C \in H$. In fact, all the generators of the right hand side ideal belong to $J :_C x$ by construction, using the defining relations of $C$. For the reverse inclusion, take any $c = p + qx \in J :_C x$
x. Then \( cx = px + qx^2 = px + qtx = (p + qt)x \) belongs to \( J \), that is, 
\( (p + qt)x = \sum_{i=1}^d (p_i + q_i)x_i \) for suitable \( p_i, q_i \in A \). Since \( p + qt \in A \), by 
uniqueness of the normal form, \( p + qt = -\sum_{i=1}^d q_i a_i \in I_A \subseteq I_A^e \). Finally, 
\( p + qx = p + qt - q(t - x) \in I_A^e + (t - x)_C \).

(3) \( I = I_A^e + (t - x)_C \) for \( I_A \in \mathcal{F} \).

This case is completely analogous to the previous one, interchanging the 
roles of \( x \) and \( t - x \).

(4) \( I = I_A^e + (x, t - x)_C \) for \( I_A \in \mathcal{F} \).

We can write \( I = I_A^e + (t)_C + (x)_C \). Thanks to property \( \bigvee \), \( I_A + (t)_A \in \mathcal{F} \),

so this case is brought back to case 2.

The proof of the particular case \( C = A(t \mid x) \) is complete.

Note that \( \mathcal{H} \) inherits Property \( \bigvee \) from \( \mathcal{F} \). Therefore, since 
\( A(t \mid x_1, \ldots, x_n) = A(t \mid x_1)(t \mid x_2) \ldots (t \mid x_m) \),

the general result follows by induction. \( \square \)

**Corollary 50.** Let \( A \) be a quadratic algebra with a Koszul filtration \( \mathcal{F} \). Then any 
twisted extension \( C = A(0 \mid x_1, \ldots, x_n) \) has the Koszul filtration \( \mathcal{H} = \{ I^e + (Y) \mid I \in \mathcal{F}, Y \subseteq \{ x_1, \ldots, x_n \} \} \).

**Proof.** If \( t = 0 \), \( \bigvee \) is automatically satisfied. \( \square \)

## 8. Unconditional results

### 8.1. Abstract Witt rings

Let \( F \) be a field of char \( F \neq 2 \), or equivalently a 
field in which \(-1\) is a primitive second root of 1. In his celebrated paper [Mil70],
J. Milnor observed the existence of a deep connection between three central arith-
metic objects: the associated graded ring of the Witt ring \( W(F) \) of quadratic forms
over \( F \), the Galois cohomology \( H^\bullet(G_F, \mathbb{F}_2) \) of the absolute Galois group of \( F \), and
the reduced Milnor K-theory \( K_\bullet(F)/2K_\bullet(F) \). This connection takes the shape of homomorphisms of graded algebras

\[
\begin{align*}
\xymatrix{
K_\bullet(F)/2K_\bullet(F) 
\ar[r]^-{\eta} & H^\bullet(G_F, \mathbb{F}_2), 
\ar[l]_-{\nu} \ar[r]^-{c} & \text{gr} W(F) 
}
\end{align*}
\]

which Milnor proved to be isomorphisms in some special circumstances. Milnor conjectures claim that the above maps are isomorphisms of graded algebras for all fields of characteristic different from 2. Milnor conjectures have recently been 
proved in full generality: the map \( \eta \) was shown to be always an isomorphism by
Voevodsky in [Voe03], while the map \( c \) was shown to be always well defined and 
an isomorphism by D. Orlov, Vishik and Voevodsky in [OVV07]. The precise connection 
between Galois groups and Witt rings is established in [MS96]. Analogous results for odd \( p \) and the graded Witt ring replaced by Galois cohomology are treated in [EM11] and [CEM12].

There have been several attempts to encode the abstract ring-theoretic properties
of Witt rings of quadratic forms into a set of axioms. The aim was to define a
unified class of rings that includes traditional Witt rings of quadratic forms and
at the same time is flexible enough to also describe other families of Witt rings.
Remark 52. The Witt ring of quadratic forms over a field $F$ with $G$ contains $-1$, it has exponent 2, and it generates $R$ as an additive group; (the augmentation ideal) satisfies

\begin{align*}
\text{(AP1)}: &\quad \text{If } a, b, c \in G, \text{ then } a + b = 0; \\
\text{(WC)}: &\quad \text{If } a_1 + \cdots + a_n = b_1 + \cdots + b_n, \text{ with } n \geq 3 \text{ and all } a_i, b_i \in G, \text{ then there exist } a, b, c, \ldots, c_n \in G \text{ such that } a_2 + \cdots + a_n = a + c_3 \cdots + c_n \text{ and } a_1 + a = b_1 + b.
\end{align*}

A morphism of abstract Witt rings $(R_1, G_1) \to (R_2, G_2)$ is a ring homomorphism $\alpha : R_1 \to R_2$ such that $\alpha(G_1) \subseteq \alpha(G_2)$.

Remark 53. The Witt ring of quadratic forms over a field $F$ can be viewed as an abstract Witt ring $W = (W(F), G)$ with $G = F^\times/(F^\times)^2$, and $I_W$ is the ideal of $W(F)$ of even-dimensional quadratic forms. Axioms (AP1) and (AP2) are the first two instances of the infinite family of Arason-Pfister properties

\[ AP(k) : \text{ if } a_1 + \cdots + a_n \in I_W^k \text{ and } n < 2^k, \text{ then } a_1 + \cdots + a_n = 0. \]

Axiom (WC) is a consequence of Witt chain lemma and (Dickson-)Witt Cancellation Theorem \cite{Dic07, Wit37}.

Observe that, for any abstract Witt ring $W = (R, G), W/I_W = \mathbb{Z}/2\mathbb{Z}$, as it is for traditional Witt rings of quadratic forms.

In the category of abstract Witt rings one can perform two basic operations: the direct product and the group ring construction.

Definition 53. Let $W_1 = (R_1, G_1)$ and $W_2 = (R_2, G_2)$ be abstract Witt rings. The \textit{direct product} of $W_1$ and $W_2$ is the abstract Witt ring $W_1 \times W_2 = (R, G)$ with $G = (G_1, G_2)$ as a group and $R$ being the subring of the ring-theoretic direct product $R_1 \times R_2$ that is additively generated by $G$.

Definition 54. The \textit{group ring} of the abstract Witt ring $W = (R, G)$ over the group $C_2 = \{1, x\}$ of order 2 is $W[x] = (R[C_2], G \times C_2)$.

To each abstract Witt ring we associate a graded object defined by subsequent powers of its augmentation ideal.

Definition 55. Let $W = (R, G)$ be an abstract Witt ring with augmentation ideal $I_W$. The associated graded abstract Witt ring is

\[ \text{gr } W = \oplus_{i=0}^{\infty} I_W^i/I_W^{i+1}, \]

where by convention $I_W^0 = R$. If $r \in I_W$, the corresponding element $r = r + I_W^{i+1} \in I_W^i/I_W^{i+1}$ of $\text{gr } W$ is called the \textit{initial form} of $r$.

The product of $R$ induces a well defined product on $\text{gr } W$ in the usual way: for $r \in I_W^i$ and $s \in I_W^j$,

\[ r \cdot s = rs + I_W^{i+j+1} \in I_W^{i+j}/I_W^{i+j+1}. \]
Proposition 56. Let $W_1 = (R_1, G_1)$, $W_2 = (R_2, G_2)$ be abstract Witt rings. Then $\text{gr}(W_1 \oplus W_2) = \text{gr} W_1 \cap \text{gr} W_2$.

Proof. The direct product is defined in such a way that, for $i \geq 1$, $I_{W_i \oplus W_2} = I_{W_i} \oplus I_{W_2}$. Now, for any $i \geq 1$ the maps

$$
\Phi_i : I_{W_1}/I_{W_1}^{i+1} \oplus I_{W_2}/I_{W_2}^{i+1} \to (I_{W_1} \oplus I_{W_2})/(I_{W_1}^{i+1} \oplus I_{W_2}^{i+1})
$$

$$
\Phi_i(a + I_{W_1}^{i+1}, b + I_{W_2}^{i+1}) = a + b + (I_{W_1}^{i+1} \oplus I_{W_2}^{i+1})
$$

are well defined isomorphisms of $\mathbb{F}_2$-vector spaces. We also define $\Phi_0 = \text{id}_{\mathbb{F}_2}$. These maps are also compatible with the product, in the sense that the diagram

$$
\begin{array}{ccc}
\frac{I_{W_1}}{I_{W_1}^{i+1}} \oplus \frac{I_{W_2}}{I_{W_2}^{i+1}} & \xrightarrow{\Phi_i} & \frac{I_{W_1}}{I_{W_1}^{i+1}} \oplus \frac{I_{W_2}}{I_{W_2}^{i+1}} \\
\frac{I_{W_1}}{I_{W_1}^{i+1}} \oplus \frac{I_{W_2}}{I_{W_2}^{i+1}} & \xrightarrow{\Phi_j} & \frac{I_{W_1}}{I_{W_1}^{i+1}} \oplus \frac{I_{W_2}}{I_{W_2}^{i+1}} \\
\end{array}
$$

where $\Phi_i \cdot \Phi_j = \Phi_{i+j}$ commutes for any $i, j \geq 1$ and a similar compatibility relation holds for $\Phi_0$. Hence $\Phi = \bigoplus_{i=0}^{\infty} \Phi_i : \text{gr} W_1 \cap \text{gr} W_2 \to \text{gr} (W_1 \oplus W_2)$ is an isomorphism of graded algebras.

\[ \square \]

Proposition 57. Let $W = (R, G)$ be an abstract Witt ring. Then $\text{gr} W[x] = (\text{gr} W)(t \mid y)$, where $t = (1 + 1) \in I_{W[x]}/I_{W[x]}^2$ and $y = (1 + x) \in I_{W[x]}/I_{W[x]}^2$.

Proof. We begin noticing that, since $1 + x \in I_{W[x]}$, $I_{W[x]}^2 = I_{W} \oplus (1 + x)I_{W}$. Now, for any $i \geq 1$ the maps

$$
\Phi_i : I_{W}/I_{W}^{i+1} \oplus yI_{W}^{-1}/I_{W} \to (I_{W} \oplus (1 + x)I_{W}^{-1})/(I_{W}^{i+1} \oplus (1 + x)I_{W})
$$

$$
\Phi_i(a + I_{W}^{i+1}, y(b + I_{W})) = a + (1 + x)b + (I_{W}^{i+1} \oplus (1 + x)I_{W})
$$

are well defined isomorphisms of $\mathbb{F}_2$-vector spaces. We also define $\Phi_0 = \text{id}_{\mathbb{F}_2}$.

Moreover, since $x$ has order 2 by construction (see Definition 54), $(1 + x)(1 + x) = (1 + 1)(1 + 1) = 1$ in $W[x]$, that is, $y^2 = ty$ in $\text{gr} W[x]$. As a consequence, the maps $\Phi_i$ are also compatible with the product, in the sense that the diagram

$$
\begin{array}{ccc}
\frac{I_{W}}{I_{W}^{i+1}} \oplus \frac{yI_{W}^{-1}}{I_{W}} & \xrightarrow{\Phi_i} & \frac{I_{W}}{I_{W}^{i+1}} \oplus \frac{yI_{W}^{-1}}{I_{W}} \\
\frac{I_{W}}{I_{W}^{i+1}} \oplus (1 + x)I_{W} & \xrightarrow{\Phi_j} & \frac{I_{W}}{I_{W}^{i+1}} \oplus (1 + x)I_{W} \\
\end{array}
$$

where $\Phi_i \cdot \Phi_j = \Phi_{i+j}$ commutes for any $i, j \geq 1$ and a similar compatibility relation holds for $\Phi_0$. Hence $\Phi = \bigoplus_{i=0}^{\infty} \Phi_i : (\text{gr} W)(t \mid y) \to \text{gr} W[x]$ is an isomorphism of graded algebras.

\[ \square \]

Definition 58. An abstract Witt ring $(R, G)$ is realisable if there is a field $F$ of char $F \neq 2$ such that $R \cong W(F)$ as rings and $G \cong F^\times/(F^\times)^2$ as groups. An abstract Witt ring $(R, G)$ is finitely generated if $|G| < \infty$.
The basic examples of realisable abstract Witt rings are the trivial ring $W(\mathbb{C})$ and the rings $W(F)$, where $F$ is a local field, that is, $F$ is either $\mathbb{R}$ or a finite extension of $\mathbb{Q}_p$ for some prime $p$. In addition, the direct product of two realisable abstract Witt rings is realisable (cf. [Kul79, Kul85]). On the Galois-theoretic level, the direct product of realisable Witt rings corresponds to the free pro-$p$ product of pro-$p$ groups (cf. also [MPQT, Section 4.2]). Finally, if $W$ is realisable as the Witt ring of quadratic forms over $F$, then also the group ring $W[ x ]$ is realisable as the Witt ring of quadratic forms over the field of formal power series $F((X))$. On the Galois-theoretic level, the group ring construction of a realisable Witt ring corresponds to the cyclotomic semidirect product of a pro-$p$ group with $\mathbb{Z}_p$ (cf. also [MPQT, Section 4.2]).

**Definition 59.** An abstract Witt ring is said to be of *elementary type* if it is obtained by applying finitely many direct products and group ring constructions, starting with the basic Witt rings $W(\mathbb{C})$ and $W($local field$)$.

**Conjecture 60** (Elementary Type Conjectures for Witt rings).

- **Weak ETC:** Every realisable finitely generated abstract Witt ring is of elementary type.
- **Strong ETC:** Every finitely generated abstract Witt ring is of elementary type (and hence realisable).

Up to now, only very partial results in the direction of Elementary Type Conjectures for Witt rings are known. Notably,

**Theorem 61** ([CMS82, Section 5]). **Strong ETC holds for abstract Witt rings** $(R, G)$ **with** $|G| \leq 32$.

From this and our results in Section 6 we obtain immediately the following unconditional result.

**Theorem 62.** If $F$ is a field of char $F \neq 2$ and $|F^\times/(F^\times)^2| \leq 32$, then the algebra $H^\bullet(G_F(2), \mathbb{F}_2)$ has a Koszul filtration, and in particular it is Koszul.

Similarly, in [Cor82] Witt rings of fields which contain at most four quaternion algebras are classified. In particular, they satisfy the Strong ETC, so we obtain the following.

**Theorem 63.** If $F$ is a field of char $F \neq 2$ and containing at most 4 quaternion algebras, then the algebra $H^\bullet(G_F(2), \mathbb{F}_2)$ has a Koszul filtration, and in particular it is Koszul.

By Merkurjev’s Theorem [Mer81], the number of quaternion algebras over a field $F$, of char $F \neq 2$, coincides with the cardinality $|H^2(G_F(2), \mathbb{F}_2)|$. Hence, the hypothesis of Theorem 63 can be equivalently expressed by saying that char $F \neq 2$ and dim$\mathbb{F}_2 H^2(G_F(2), \mathbb{F}_2) \leq 2$. But this dimension is the minimal number of generating relations of $G_F(2)$ (cf. [Ser02, Section I.4.3]). Thus, Theorem 63 can be reformulated as follows.

**Theorem 64.** If $F$ is a field of char $F \neq 2$ and $G_F(2)$ has a presentation with finitely many generators and at most 2 generating relations, then $H^\bullet(G_F(2), \mathbb{F}_2)$ has a Koszul filtration, and in particular it is Koszul.

This generalises the main result of [Qua] in the case $p = 2$. 

8.2. Pythagorean formally real fields. A field $F$ of char $F \neq 2$ is Pythagorean if $F^2 + F^2 = F^2$ and formally real if $-1$ is not a sum of squares in $F$. A form of Elementary Type Conjecture has been proved for the family $\mathcal{PFR}$ of Pythagorean formally real fields with finitely many square classes (cf. [Min86], [Jac81]). Namely, $\mathcal{PFR}$ contains Euclidean fields, that are fields $F$ such that $(F^*)^2 + (F^*)^2 = (F^*)^2$ and $F^* = (F^*)^2 \cup -(F^*)^2$ (cf. [Bec74]) and

**Theorem 65.** The family of maximal pro-$2$ quotients of absolute Galois groups of fields in $\mathcal{PFR}$ is described inductively as follows.

1. For any Euclidean field $E \in \mathcal{PFR}$ (for instance, $E = \mathbb{R}$), $G_E(2) = \mathbb{Z}/2\mathbb{Z}$.

2. For any two fields $F_1, F_2 \in \mathcal{PFR}$ there exists $F \in \mathcal{PFR}$ such that $G_{F_1}(2) *_2 G_{F_2}(2) \cong G_F(2)$.

3. For any field $F_0 \in \mathcal{PFR}$ and any finite product $Z = \prod_{i=1}^m \mathbb{Z}_2$ there exists $F \in \mathcal{PFR}$ such that $Z \rtimes G_{F_0}(2) \cong G_F(2)$, where the action of $G_{F_0}(2)$ on $Z$ is given by

$$\sigma^{-1}z\sigma = z^{-1} \quad \text{for any } \sigma \in G_{F_0}(2) \backslash \{1\}, \sigma^2 = 1 \text{ and } z \in Z$$

Moreover, each maximal pro-$2$ quotient of the absolute Galois group of a field in $\mathcal{PFR}$ can be obtained inductively from Galois groups of Euclidean fields applying a finite sequence of the two operations described in (2) and (3).

For any two fields $F_1, F_2 \in \mathcal{PFR}$, $H^\bullet(G_{F_1}(2) *_2 G_{F_2}(2), \mathbb{F}_2) \cong H^\bullet(G_{F_1}(2), \mathbb{F}_2) \cap H^\bullet(G_{F_2}(2), \mathbb{F}_2)$ (cf. Theorem 28). Also, for any field $F_0 \in \mathcal{PFR}$ and any finite product $Z = \prod_{i=1}^m \mathbb{Z}_2$, $H^\bullet(Z \rtimes G_{F_0}(2), \mathbb{F}_2) = H^\bullet(G_{F_0}(2), \mathbb{F}_2)(t \mid x_1, \ldots, x_m)$, with $t$ corresponding to the square class of $-1$ in Bloch-Kato isomorphism. Since $H^\bullet(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2[t]$ is universally Koszul, using Propositions 28 and 29 we get the following.

**Theorem 66.** Let $F$ be a Pythagorean formally real field. If $|F^*/(F^*)^2| < \infty$, then the algebra $H^\bullet(G_F(2), \mathbb{F}_2)$ is universally Koszul.

**Remark 67.** To any locally finite-dimensional graded $K$-algebra $A$ we can associate the Hilbert series

$$h_A(z) = \sum_{n \in \mathbb{N}} (\dim_K A_n) z^n.$$

If in addition $A$ is Koszul, then we can attach to it a sequence of Betti numbers $\beta_n$, which measure the number of free summands in the component of degree $n$ of the Koszul complex of $A$ (see [PP03] Sections 2.2, 2.3 or [EH12] Section 6.1). The formal power series

$$p_A(z) = \sum_{n \in \mathbb{N}} \beta_n z^n$$

is called the Poincaré series of $A$. The two series are related by the formula

$$(8.1) \quad h_A(z)p_A(-z) = 1.$$

In [Min93], for any Pythagorean formally real field $F$ with finitely many square classes, the Hilbert series of the algebra $H^\bullet(G_F(2), \mathbb{F}_2)$ is introduced under the name of Poincaré series of $G_F(2)$. These series are then effectively determined from the structure of the space of orderings of $F$ via a simple recursive formula ([Min93] Theorems 2 and 10]). Using Theorem 65 and Equation (8.1) above, these results in [Min93] can be extended immediately to derive an effective formula for the Poincaré series of the algebras $H^\bullet(G_F(2), \mathbb{F}_2)$. 
In [JW91, Corollary 2.2], B. Jacob and R. Ware prove that if $F$ is a field of char $F \neq 2$, with $|F^\times/(F^\times)^2| < \infty$ and with elementary type Witt ring of quadratic forms, then for any $a \in F$ the Witt ring of quadratic forms of $F[\sqrt{a}]$ is of elementary type too.

From basic Galois theory and the structure of $2$-groups, we get that if $F$ is a field of char $F \neq 2$ and $K/F$ is any finite (not necessarily Galois) $2$-extension such that $K \subseteq F(2)$, then there is a finite tower of degree 2 extensions $F \subset F_1 \subset \cdots \subset F_n \subset K$. Therefore, by induction, we obtain a strengthening of Theorem 66.

**Theorem 68.** Let $F$ be Pythagorean formally real with finitely many square classes and let $K/F$ be a finite $2$-extension such that $K \subseteq F(2)$. Then $H^\bullet(G_K(2), \mathbb{F}_2)$ is universally Koszul.

**References**

[AS27a] E. Artin and O. Schreier, *Algebraische Konstruktion reeller Körper*, Abh. Math. Sem. Univ. Hamburg 5 (1927), no. 1, 85–99. Reprinted in: Artin, E., *Collected Papers* (S. Lang and J. Tate, ed.), Springer, 1965, 258–272.

[AS27b] ———, *Eine Kennzeichnung der reell abgeschlossenen Körper*, Abh. Math. Sem. Univ. Hamburg 5 (1927), no. 1, 225–231. Reprinted in: Artin, E., *Collected Papers* (S. Lang and J. Tate, ed.), Springer, 1965, 289–295.

[Bec74] E. Becker, *Euklidische Körper und euklidische Hüllen von Körpern*, J. Reine Angew. Math. 268/269 (1974), 41–52. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II.

[CDNR13] A. Conca, E. De Negri and M.E. Rossi, *Koszul algebras and regularity*, Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of His 65th Birthday (I. Peeva, ed.), Springer, 2013, 285–315.

[CEM12] S.K. Chebolu, I. Efrat and J. Mináč, *Quotients of absolute Galois groups which determine the entire Galois cohomology*, Math. Ann. 352 (2012), no. 1, 205–221.

[CM82] A.B. Carson and M. Marshall, *Decomposition of Witt rings*, Canad. J. Math. 34 (1982), no. 6, 1276–1302.

[Cou00] A. Conca, *Universally Koszul algebras*, Math. Ann. 317 (2000), no. 2, 329–346.

[Cor82] C.M. Cordes, *Quadratic forms over fields with four quaternion algebras*, Acta Math. 41 (1982), no. 1, 55–70.

[CRV01] A. Conca, M.E. Rossi and G. Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. 129 (2001), no. 1, 95–121.

[CTV01] A. Conca, N.V. Trung and G. Valla, *Koszul property for points in projective spaces*, Math. Scand. 89 (2001), no. 2, 201–216.

[Dem61] S.P. Demuskin, *The group of a maximal p-extension of a local field*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 329–346.

[Dem63] ———, *On 2-extensions of a local field*, Sibirsk. Mat. Z. 4 (1963), 951–955.

[Dic07] L.E. Dickson, *On quadratic forms in a general field*, Bull. Amer. Math. Soc. 14 (1907), no. 3, 108–115.

[Efr95] I. Efrat, *Orderings, valuations, and free products of Galois groups*, Sem. Structure Algébriques Ordonnées, Univ. Paris VII (1995).

[EHI12] V. Ene and J. Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, vol. 130, American Mathematical Society, Providence, RI, 2012.

[EHH15] V. Ene, J. Herzog and T. Hibi, *Linear flags and Koszul filtrations*, Kyoto J. Math. 55 (2015), no. 3, 517–530.

[Eis95] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[EL72] R. Elman and T.Y. Lam, *Quadratic forms over formally real fields and Pythagorean fields*, Amer. J. Math. 94 (1972), 1153–1194.

[EM11] I. Efrat and J. Mináč, *On the descending central sequence of absolute Galois groups*, Amer. J. Math. 133 (2011), no. 6, 1503–1532.

[EM17] I. Efrat and E. Matzri, *Triple Massey products and absolute Galois groups*, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 12, 3629–3640.
I. Efrat and C. Quadrelli, *The Kummerian property and maximal pro-$p$ Galois groups*, arXiv preprint arXiv:1707.07018.

P. Guillot and J. Mináč, *Extensions of unipotent groups, Massey products and Galois cohomology*, arXiv preprint arXiv:1711.07711.

P. Guillot, J. Mináč and A. Topaz, *Four-fold Massey products in Galois cohomology. With an appendix by O. Wittenberg*, Compos. Math. 154 (2018), no. 9, 1921–1959.

J. Herzog, T. Hibi and G. Restuccia, *Strongly Koszul algebras*, Math. Scand. 86 (2000), no. 2, 161–178.

D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. 36 (1890), no. 4, 473–534.

C. Haesemeyer and C. Weibel, *Norm varieties and the chain lemma (after Markus Rost)*, Algebraic topology, Abel Symp., vol. 4, Springer, 2009, 95 –130.

C. Haesemeyer and C. Weibel, *Splitting varieties for triple Massey products*, J. Pure Appl. Algebra 219 (2015), no. 5, 1304–1319.

B. Jacob, *On the structure of Pythagorean fields*, J. Algebra 68 (1981), no. 2, 247–267.

B. Jacob and R. Ware, *A recursive description of the maximal pro-$2$ Galois group via Witt rings*, Math. Z. 200 (1991), no. 2, 193–208.

M. Kula, *Fields with prescribed quadratic form schemes*, Math. Z. 167 (1979), no. 3, 201–212.

J.P. Labute, *Classification of Demushkin groups*, Canad. J. Math. 19 (1967), 106–132.

T.Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.

D. Orlov, A. Vishik and V. Voevodsky, *An exact sequence for $K_{2}$ with applications to quadratic forms*, Ann. of Math. (2) 165 (2007), no. 1, 1–13.
D.I. Piontkovskii, *On Hilbert series of Koszul algebras*, Funct. Anal. Appl. **35** (2001), no. 2, 133–137.

D.I. Piontkovskii, *Koszul algebras and their ideals*, Funct. Anal. Appl. **39** (2005), no. 2, 120–130.

L. Positselski, *The correspondence between Hilbert series of quadratically dual algebras does not imply their having the Koszul property*, Funct. Anal. Appl. **29** (1995), no. 3, 213–217.

L. Positselski, *Koszul property and Bogomolov’s conjecture*, Ph.D. thesis, Harvard University, 1998, Available at https://conf.math.illinois.edu/K-theory/0296/.

L. Positselski, *Koszul property and Bogomolov’s conjecture*, Int. Math. Res. Not. (2005), no. 31, 1901–1936.

L. Positselski and A. Vishik, *Koszul duality and Galois cohomology*, Math. Res. Lett. **2** (1995), no. 6, 771–781.

C. Quadrelli, *One relator maximal pro-p Galois groups and Koszul algebras*, arXiv preprint arXiv:1601.04480.

J.-E. Roos, *On the characterisation of Koszul algebras. Four counterexamples*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 1, 15–20.

J.-P. Serre, *Structure de certains pro-p-groupes (d’après Demushkin)*, Séminaire Bourbaki, Vol. 8, Soc. Math. France, Paris, 1995, Exp. No. 252, 145–155.

J.-P. Serre, *Galois cohomology*, English ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

A. Suslin and S. Joukhovitski, *Norm varieties*, J. Pure Appl. Algebra **206** (2006), no. 1-2, 245–276.

V. Voevodsky, *Motivic cohomology with Z/2-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104.

V. Voevodsky, *Motivic Eilenberg-Maclane spaces*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 112, 1–99.

V. Voevodsky, *On motivic cohomology with Z/I-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438.

A.R. Wadsworth, *p-Henselian fields: K-theory, Galois cohomology, and graded Witt rings*, Pacific J. Math. **105** (1983), no. 2, 473–496.

R. Ware, *When are Witt rings group rings? II*, Pacific J. Math. **76** (1978), no. 2, 541–564.

E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. **176** (1937), 31–44.