Universal Elementary Constituents of Potential Transformations

Shifting Waves

(qualitative Theory)

V. M. Chabanov, B. N. Zakhariev

Laboratory of the Theoretical Physics, Joint Institute for Nuclear Research Dubna, 141980, Russia; e-mail: zakharev@thsun1.jinr.dubna.su ; URL: http://thsun1.jinr.ru/~zakharev/

S. A. Sofianos

Department of Physics, University of South Africa, Pretoria, South Africa

Abstract

It is shown that the potential perturbation that shifts a chosen standing wave in space is a block of potential barrier and well for every wave bump between neighbouring knots. The algorithms shifting the range of the primary localization of a chosen bound state in a potential well of finite width are as well applicable to the scattering functions if states of the continuous spectrum are considered as bound states normalized to unity but distributed on an infinite interval with vanishing density. The potential perturbations of the same type on the half-axis concentrate the scattering wave at the origin, thus creating a bound state embedded into the continuous spectrum (zero width resonance). It is an improved version of paper: Annalen der Physik, 6 (1997) 136.

PACS: 24.10.-i, 03.65-w
I. INTRODUCTION

In the recent years it became clear [2,13,15] what qualitative transformations of any initial potential $V(x)$ are required to change an energy level or to shift the region of main concentration of any bound state in configurational space (i.e. the distribution of the corresponding standing wave over the $x$-axis, cf. for example Fig.1a,b). This means in effect the extraction of physical essence from the mathematical apparatus of the quantum inverse problem (Gelfand-Levitan-Marchenko formalism) [1–3], Darboux transformation, factorization method and supersymmetry theory [4–8]. It was done via visualization of analytical formulae corresponding to exactly solvable models. The simple and universal rules of qualitative prediction of the transformed potentials and resulting systems systems (intuitive quantum design [2,13,15]) go beyond the scope of these models.

It is also possible to control the features of states of the continuous spectrum (reflection, transparency, resonance parameters [13]). In the present paper it is shown that an additional potential $\Delta V(x)$ shifting the $n$–th bound state over $x$ axis to the right (left) can be considered as a sum of $n$ elementary universal blocks consisting of a potential barrier and a well. We show below that perturbations constructed out of these blocks can ”gather” also the scattering states rolled out on the half-axis confining them into bound states embedded in the continuous spectrum (BSEC) at a fixed energy; these states can be considered as resonance states with vanishing width.

We discuss below the qualitative aspects of the theory in Sec. II, give the corresponding formalism in Sec. III and summarize our conclusions in Sec. IV.

II. QUALITATIVE CONSIDERATIONS

2
A. Shifting of standing waves

We have found that the perturbation of a potential shifting the ground state localisation in space (see Fig. 1c,b for the case of initial infinite rectangular well) consisting of a well and a barrier can be considered as an universal elementary block for shifting the localisation of arbitrary bound states over $x$ axis without actually shifting any energy level (here the influences of barriers and wells are mutually compensated). To shift, for instance, the second bound state to the left we need two such blocks (Fig. 1f). The n-th bound state can be shifted with n blocks. Every block corresponds to one bump in the initial bound state standing wave. The boundaries of each block coincide with the relevant neighboring knots of the wave function. The amplitudes of the potential oscillations are decreasing together with the transformation of the standing wave which is concentrating towards the origin. The same situation holds for shifting waves to the right, but with mirror reflection of potentials and wave functions: barrier-well block for each wave bump instead of wave-barrier blocks.

Now we can see that potentials confining a BSEC are also constructed of an infinite number of these universal blocks with slowly decreasing amplitudes as $\sim 1/r$ (see Fig. 2). Neuman and Wigner [9] were the first who obtained formulae for BSEC. Further work on BSEC was undertaken by a number of other authors (see [11-13] and references therein and experimental paper [15]). We conclude that the construction of BSEC can be considered as ”gathering” scattering states and concentrating them in the vicinity of the origin.

B. Scattering functions as ”bound” states

Let us consider the most simple case of a continuous spectrum wave function for a free particle. The function describing free motion on the half-axis $0 < r < \infty$ at fixed energy $E_b = k_b^2$ is given by $\sin(k_b r)/k_b$. Usually the scattering functions are normalized to a $\delta$-function. However, we shall show that one can treat them as bound states normalized to
unity.

If an infinite unpenetrable potential wall would be placed at the point of the first knot \( r = \pi/k_b \) of this sine–function, then the first half–wave of this sine will coincide with the bound ground state \( \psi(E_b, x) \) formed by an infinite potential well. In order to normalize the obtained function to unity one simply multiplies the sine by \( \sqrt{2k_b/\pi} \). Let us now move the potential wall to the point \( r = 2\pi/k_b \) of the second knot of the free wave. Now on the finite segment of double length the sinusoid \( \sin(k_b r) \) will coincide exactly with the wave function of the second bound state in a more wide well. The required normalization is then given by \( \sqrt{k_b/\pi} \). If we continue to expand the well we will get, at the \( n \)th step, the \( n \)th bound state of the rectangular well coinciding with a chosen sinusoid and the corresponding normalization factor will be \( \sqrt{2k_b/(n\pi)} \). In the limit \( n \to \infty \) we will obtain a ”bound” state \( \psi(E_b, x) \) of an infinitely wide rectangular well with ”vanishing” normalization factor while normalized to unity,

\[
\int_0^\infty \psi^2(E_b, x)dx = 1.
\] (1)

This bound state is distributed over the infinite interval with vanishing density. The situation is somewhat analogous to the infinitely narrow but normalized to unity peak of a \( \delta \)–function

\[
\int_{-\infty}^{\infty} \delta(x)dx = 1.
\] (2)

Using inverse problem or SUSY algorithms [2,8,10,11] 1991, each of the bound states considered above could be moved to the left wall of the infinitely deep well using blocks of transformations of the potential described above. The derivative of a normalized wave function at the origin can be done arbitrarily large. Fig. 2a exhibits the functions of BSEC by consecutive growth of \( \psi'(E_b, 0) \) and corresponding perturbations of the potential. For a higher concentration of the function near the origin perturbations \( V(x) \) with increasing amplitudes of oscillations are needed, see Fig. 2b.
This view allows to bridge the differences in the notions of bound and continuum states.

An interesting fact is that the knots of the BSEC do not move by increasing $\psi'(E_b, 0)$, a phenomenon which must have a clear qualitative explanation.

The blocks of potentials producing BSEC’s can be arbitrarily rearranged without losing the gathering property. Then the wave functions on the corresponding intervals must also be rearranged. We must only change their amplitudes for smooth continuation through the junction points (block boundaries). Some of the blocks can also be repeated; an interesting limiting case is the periodic potential on the half-axis with the bound state at the boundary of the conduction band. Some blocks can be missed, i.e. substituted by intervals with zero potential. The tail of the gathering potential can be deleted from some point. Then BSEC will transform into a quasi-bound (resonance) state with finite width which must be smaller the further the tail is cut. The system can be continued to the whole axis using mirror reflections of blocks and corresponding solutions to create BSEC decreasing in both opposite directions $x \to \pm \infty$.

Scattering wave functions at energies neighboring to those of a BSEC become asymptotically ($x \to \infty$) free waves without phase shift. This effect can be explained by total reciprocal compensation of the influence of intervals where the wave is gathered and intervals from which the wave is forced out (see beatings in wave function shown in Fig. 3). The period of such beatings increases without bound when approaching the energy and in the limit there $k_b^2$ of the BSEC, $\lim_{r \to k_b} \psi(r, k \to k_b) = \sin(k_b r)/k_b$. So, in the limit there is left only one beating on the full half-axis that is what presents BSEC (Fig. 2).

C. Background Formulae

It is the purpose of this paper to clarify the general qualitative peculiarities of evolution of potentials and wave functions by transformations shifting waves at fixed energies which where not mentioned before. We thus give here only the final formulae only (without any
derivation which is given, for example, in [2,14]). The transformation of the potential caused by the change of the spectral weight parameter from $c_{0,\nu}$ to $c_{\nu}$ ($c_{\nu} = \psi''(E_{0,\nu}, 0)$ where $\psi(E_{0,\nu}, x)$ is the normalized wave function at the chosen energy) is given by

$$V(r) = V_0(r) - 4\delta c^2_{\nu} \varphi_0'(E_{0,\nu}, r) \varphi_0(E_{0,\nu}, r) p^{-1}(r) + 2(\delta c^2_{\nu})^2 \varphi_0^4(E_{0,\nu}, r) p^{-2}(r),$$  \(3\)

where

$$\delta c^2_{\nu} = c^2_{\nu} - c^2_{0,\nu}; \quad c^{-2}_{\nu} = \int_{0}^{\infty} \phi^2(E_{0,\nu}, r) dr \quad (4)$$

and

$$p(r) = 1 + \delta c^2_{\nu} \int_{0}^{r} \varphi_0^2(E_{0,\nu}, t) dt. \quad (5)$$

The corresponding perturbed regular solution fulfilling the boundary conditions $\varphi(E, 0) = 0$ and $\varphi'(E, 0) = 1$ at an arbitrary $E \neq E_{0,\nu}$, is given by

$$\varphi(E, r) = \varphi_0(E, r) - \delta c^2_{\nu} \varphi_0(E_{0,\nu}, r) p^{-1}(r) \int_{0}^{r} \varphi_0(E_{0,\nu}, t) \varphi_0(E, t) dt. \quad (6)$$

The regular solution at $E_{0,\nu}$ is

$$\varphi(E_{0,\nu}, r) = \varphi_0(E_{0,\nu}, r) p^{-1}(r). \quad (7)$$

The BSEC for initial repulsive Coulomb potential is shown in Fig.5. Shifting of bound states in attractive Coulomb potentials was considered in [17]. The case of a creation BSEC on a slope of the initial linear potential had been considered in the paper [12]. For this case the potential perturbations confining the bound state grow without limit by an amplitude which moves away from the direction of the turning point of the barrier.

It is interesting to consider the multichannel systems and find there also some universal blocks of potential transformations. It is easy to generalize one-channel case to the multichannel one with equal channels. In the general case it is an open problem.
III. CONCLUSION

We have shown that the additional potential $\Delta V(x)$ which shifts $n$–th bound state consists of $n$ elementary blocks (barrier + well), one of which is shown in Fig. 1c, shifting the ground state.

We have managed to extract the physical essence from proper formulas, reveal universal "atoms” of potential perturbations, performing such transformations (quantum "design"). It is remarkable that then distinct islands of quantum intuition began gradually merging into united qualitative theory of spectral, scattering and decay control. The BSEC’s are explained as scattering states normalized to unity and gathered to the origin. The zero scattering phase shift for solutions with energies near to BSEC are simply explained as result a total reciprocal compensation of gathering and pulling under the potential oscillations.

It is of interest to find some universal building blocks in multi–channel systems. In this case the different and coincident thresholds or closed channels could be considered. The first results in this direction were already achieved [13,16].

ACKNOWLEDGMENTS

The authors gratefully acknowledge financial support from the Russian Funds of Fundamental Research (RFFR), Soros Fund, the Foundation of Research and Development, and the University of South Africa.
FIGURE CAPTIONS

Fig. 1: Increase in the spectral weight parameter (the derivative $\Psi'(x = 0)$) of a chosen bound state related to initial rectangular potential well with conservation of the normalization of the wave function leads to the concentration of distribution of probability for given state in a near vicinity of the origin. b) and d) – the transformation of the chosen ground and the second states c) and f) – corresponding perturbations $\Delta V_{1,2}(x)$ of the flat bottom of the initial well; a) and e) – examples of a reaction of all states except for the chosen one: second and the first states $\Psi_{2,1}$ which undergo the recoil in opposite direction in comparison with the chosen one. The thick arrows ==¿ and ¡== show the direction of the average localization shifts of the states. In the limit $\Psi'(x = 0) \to \infty$ there occurs ”pressing” of a chosen state into the wall at the origin.

Fig. 2: The potentials b) performing a confinement of the bound states in continuous spectrum at the fixed energy $E_b$ and the proper wave functions a) for different values the weight spectral factors ”c” (values of derivatives at the origin). With growth of ”c” BSEC are more and more pressed to the origin.

Fig. 3: Scattering function related to energy $E_b + \varepsilon$ that is close to BSEC energy $E_b$. The correlation of their oscillations with potential ones is violated, the properties of the potential to gather wave function remain only on finite sections of the half-line $r$ which leads to ”gathering” them but on finite parts of co-ordinate axis. Out of these sections the oscillations of the functions and the potential appear to be opposite in phase and there occurs ejecting the wave function to nearest regions of wave gathering. This develops in beatings dying at infinity with decreasing of potential. The asymptotic of such functions coincides with unperturbed solutions (with zero phase shifts) due to mutual compensation of ”concentrations” and ”displacements”.

Fig. 4

(a) A free motion wave function $\sin(k_br)$ at fixed energy in continuous spectrum $E_b = k_b^2$. 

8
(b) An infinite unpenetrable potential wall at the point of the first knot of \( \sin(k_br) \) creates infinitely deep potential well in which scattering half-wave coincides with the wave function of bound state.

(c) When shifting the wall in \( n \)th knot, \( \sin(k_br) \) is already \( n \)th bound state of the new well and strictly coincides with the free scattering solution inside the well. In the limit \( n \to \infty \) the scattering function can be considered as a bound state, distributed on the whole half-axis \( 0 < r < \infty \), only if one needs to normalize it by unity, vanishing normalization factor \( \sim 1/\sqrt{n} \) should be inserted.

Fig. 5: Transformation of initial repulsive coulomb potential to create BSEC.
REFERENCES

[1] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. NY, Springer, 1982.

[2] B. N. Zakhariev and A. A. Suzko, *Direct and Inverse Problems*, Heidelberg, Springer-Verlag, 1990.

[3] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* 2nd ed., Heidelberg, Springer, 19

[4] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep., 251, No 5 & 6, 268 (1995), (and Refs therein).

[5] A. A. Andrianov, M. V. Ioffe, and V. P. Spiridonov, Phys. Lett. A 174, 273 (1993); V. P. Berezovoy, A. I. Pashnev, Z. Phys. C 51, 525 (1991).

[6] D. Baye, J.Phys. A28, 5079 (1995) and references therein.

[7] C. V. Sukumar, J. Phys. A 18, 2917; 2937 (1985), A 21, L455 (1988).

[8] J. Pöschel and E. Trubowitz E., *Inverse Spectral Theory*, New York, Academic, 1987.

[9] J. Neuman and E. Wigner, Phys. Z. 1929, 30, p.465.

[10] A. A. Stahlhofen, Phys. Rev. A51, 934 (1995)

[11] F. H. Stillinger and D. R. Herrik, Phys. Rev. A 11, 446 (1975); N. Meyer-Vernet, Am. J. Phys.,50, 353 (1982); J. Papademos, U. Sukhatme, and A. Pagnamenta, Phys. Rev. A 48, 3525 (1992).

[12] F. Calogero and A. Degasperis, Lett. Nuovo Cim. 23, 143 (1978).

[13] B. N. Zakhariev, V. M. Chabanov, Ann. Phys. 285, 1, 2000; Inverse Problems, 13, R47, 1997; Phys. Lett. A255, 123, 1999; Phys. Part.& Nucl., 30, 111, 1999. Fiz. Elem. Chastits At. Yadra 25, 1561, 1994; [Sov.J.Part.& Nucl. 25, 662 (1994)];

[14] B. N. Zakhariev et al, 1992, Fiz. Elem. Chastits At.Yadra 23 1387 [Sov.J.Part.& Nucl.
23, 603 (1992)]; 1990, Fiz. Elem. Chastits At.Yadra 21 914 [Sov.J. Part.& Nucl.21 384 (1990)];

[15] B. N. Zakhariev, Lessons on Quantum Intuition, Dubna, JINR, 1996; New ABC of Quantum Mechanics, Izhevsk, UdSU, 1998, in Proc.Conf. Inverse Problems: Proceedings of the International conference, Montpellier, 1986 (Acad. Press, New York) 141; 1990, JINR Rapid Commun. 6[45].

[16] T. Stroh, B. N. Zakhariev, Physica Scripta 55, 9, 1997.

[17] D.J. Fernandez, Lett. Math. Phys. 8, 337, 1984.
Fig. 1
Fig. 3
Fig. 4
Fig. 5