On Optimal Harvesting Problems in Random Environments

Qingshuo Song∗ Richard H. Stockbridge† Chao Zhu‡

September 30, 2014

Abstract

This paper investigates the optimal harvesting strategy for a single species living in random environments whose growth is given by a regime-switching diffusion. Harvesting acts as a (stochastic) control on the size of the population. The objective is to find a harvesting strategy which maximizes the expected total discounted income from harvesting up to the time of extinction of the species; the income rate is allowed to be state- and environment-dependent. This is a singular stochastic control problem with both the extinction time and the optimal harvesting policy depending on the initial condition. One aspect of receiving payments up to the random time of extinction is that small changes in the initial population size may significantly alter the extinction time when using the same harvesting policy. Consequently, one no longer obtains continuity of the value function using standard arguments for either regular or singular control problems having a fixed time horizon. This paper introduces a new sufficient condition under which the continuity of the value function for the regime-switching model is established. Further, it is shown that the value function is a viscosity solution of a coupled system of quasi-variational inequalities. The paper also establishes a verification theorem and, based on this theorem, an optimal harvesting strategy is constructed under certain conditions on the model. Two examples are analyzed in detail.

Key Words. Regime-switching diffusion, singular stochastic control, quasi-variational inequality, viscosity solution, verification theorem.

AMS subject classification. 93E20, 60J60.

∗Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon Tong, Hong Kong, song.qingshuo@cityu.edu.hk. The research of this author was supported in part by the Research Grants Council of Hong Kong No. CityU 100310.

†Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, stockbri@uwm.edu. The research of this author was supported in part by the U.S. National Security Agency under Grant Agreement Number H98230-09-1-0002.

‡Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, zhu@uwm.edu.
1 Introduction

One of the most important yet difficult problems in modern natural resources management is the establishment of ecologically, environmentally and economically reasonable wildlife management and harvesting policies. There are many occurrences where myopic unconstrained harvesting has led to local and/or global extinctions. Lande et al. [17] documents many such examples. Real ecological communities are random by nature. As a result of developments in stochastic analysis and stochastic control techniques, there has been a resurgent interest in determining the optimal harvesting strategies in the presence of stochastic fluctuations (see, e.g., [2, 4, 19, 29]). Unfortunately, most of the current research on harvesting problems, including the aforementioned references, are primarily focused on a single species in a static environment. The paper by Lungu and Øksendal [18] makes a first step in the analysis of the harvesting problem for interacting populations but does not consider changes in the environment.

As noted in [6, 13, 31], the variations in the external environment (for example, weather or anthropogenic) can have important effects on the dynamics of the populations of the ecosystem. In addition to the random fluctuations of the populations (usually modeled by white noise; see, e.g., [3]), certain biological parameters such as the growth rates and the carrying capacities often demonstrate abrupt changes due to environmental noise. Moreover, the qualitative changes of those parameters form an essential part of the dynamics of the ecosystem. For example, Medina-Reyna [22] demonstrates that the mean growth rates of white shrimp (Litopenaeus vannamei) in the Mar Muerto Lagoon, Southern Mexico are significantly different in various salinity levels. Similar observations were made in [26] for reproduction performance of crossbred goats in a derived Guinea savanna zone. For another example, the carrying capacities often vary according to the changes in nutrition, water supply, living spaces, and/or food resources (see [30] for many such examples). In the mathematical community, people are paying more attention to the modeling and analysis of population dynamics subject to both white and colored noises; see, e.g., [7, 20, 38] and references therein.

Naturally, one expects that the optimal harvesting strategies may vary according to the changes of the environment. Despite the increasing interests in the mathematical modeling and analysis of population dynamics, to our best knowledge, there are relatively few results in the literature that address harvesting strategies in random environments. This paper addresses this hole in the literature by examining optimal harvesting problems of a species in random environments.

Suppose there is a single species whose growth is subject to the usual fluctuations as well as the abrupt changes of the random environments. Harvesting strategies are introduced to derive financial benefit as well as to control the growth of the population. The goal is to find a harvesting strategy which maximizes the expected total discounted income from harvesting, up to the time when the population falls to a given threshold (e.g., extinction). Harvesting may occur instantaneously so results in a singular stochastic control problem in the sense that the optimal harvesting strategy may not be absolutely continuous with respect
to the Lebesgue measure of time. In other words, in contrast to the regular stochastic control problems, in which the displacement of the state due to control is differentiable in time, the harvesting problem considered in this work allows the displacement to be discontinuous. This paper establishes a verification theorem and, based on the theorem, explicitly constructs an \( \varepsilon \)-optimal harvesting strategy. Both the extinction time and harvesting policy may depend on the initial conditions. As a result, continuity of the value function can not be obtained using the standard arguments for regular or singular stochastic control problems in a fixed time horizon. This paper provides a sufficient condition under which the continuity of the value function is guaranteed. It is further shown that the value function is a viscosity solution of a coupled system of quasi-variational inequalities (2.1).

The novelty of this work arises in two distinct ways. The modeling of random environments through the use of a continuous-time finite-state Markov chain introduces coupling of the value function for each environment in the quasi-variational inequalities for the verification theorem (Theorem 2.1). An \( \varepsilon \)-optimal harvesting policy (Theorem 2.4) is determined under certain conditions that involves quickly harvesting very small amounts until the species becomes extinct in a very small time interval. We should also remark that the proof of Theorem 2.4 is very technical and non-trivial. In addition to the subtle analysis in dealing with the controlled process \( \hat{X} \), the presence of environmental switching adds much difficulties in the proof. The introduction of different regimes necessarily implies that the optimal harvesting strategy will depend on the current environment (Example 3.1). The determination of an environment-dependent optimal policy is non-trivial as one must overcome some significant technical challenges.

This paper’s second contribution comes from identifying a new sufficiency condition for the continuity of the value function as a function of the initial state. The fact that payment is received only until the random time of extinction and this time strongly depends on the harvesting policy adopted means that a small decrease in initial population size may result in a significant decrease of the extinction time. Continuity is therefore not a direct extension of standard results for a fixed time horizon. Theorem 4.4 establishes a sufficient condition under which the value function can be proven to be continuous for this criterion involving the hitting time of the population at 0 (or any quasi-extinction level from which the population will not rebound). Once the value function is shown to be continuous, it is then proven to be a viscosity solution of the quasi-variational inequalities (Theorem 4.9); even this analysis is technically challenging due to the existence of multiple environments.

Note that this work is expressed entirely in terms of harvesting of a single species in random environments, but as in Miller and Voltaire [23], the harvesting problem is a paradigm that has many additional economic applications.

Besides the optimal harvesting problems considered in this paper and [2, 18, 19], singular stochastic control has found applications in many other areas. For example, singular stochastic control problems naturally arise in monotone follower problems [14], optimal dividend distribution schemes [5, 27], portfolio selection management with transaction cost [21, 25], diffusion control of many-server queues [33], and heavy traffic modeling and control problems [34]. We refer the reader to [8, 16, 36] for more such examples. See also [11, 12] for
a general singular stochastic control problem for a multidimensional Itô diffusion on a fixed
time horizon, in which the existence of the optimal control and the characterization of the
value function as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation are
established. Most, if not all, of the existing literature on singular stochastic controls consider
with Itô (jump) diffusions.

Regular control and optimal stopping problems for regime switching diffusions have be-
come more popular recently (see, for example, [9, 10, 32, 37] and references therein). Less is
known for singular control of regime switching diffusions. Moreover, a common assumption is
that the marginal yield from exerting the singular control is constant. Two exceptions are in
[1, 19], where the marginal yields depend on state and time, respectively. The assumption of
constant marginal yield seems rather restrictive since in the real world, the unit price usually
depends on the current state of the system. This paper considers state-and-regime-dependent
marginal yields from harvesting; that is, the unit price depends on both the current state of
the population size of the species and the regime of the environment. This additional feature
of the model is not merely an extension of the traditional models, but in fact, introduces
many interesting mathematical problems for the analysis. More specifically, there may not
exist admissible optimal harvesting strategy under this setting; see Theorem 2.4, Remark
3.2 and Example 3.3 for more details. Nevertheless, using detailed and careful analysis of
the sample path properties of the controlled process, we constructs an explicit admissible
ε-optimal harvesting strategy.

The rest of the paper is organized as follows. A precise formulation of the problem is
presented in Section 1.1. Then a verification theorem is proven in Section 2 and is used to
explicitly construct an ε-optimal harvesting strategy under additional conditions. Two
examples are given in Section 3 to illustrate these results. Section 4 derives the continuity
of the value function $V$ and characterizes it as a viscosity solution of a coupled system of
quasi-variational inequalities (2.1). Section 5 contains concluding remarks.

A few words about notation is needed. A function from $[0, \infty)$ to some Polish space $E$
is càdlàg if it is right continuous and has left limits in $E$. When $E = \mathbb{R}$ and $\xi$ is càdlàg,
then $\Delta \xi(t) = \xi(t) - \xi(t-) \text{ for } t > 0$ and the convention $\Delta \xi(0) = \xi(0)$ is used. As usual,
sup $\emptyset = -\infty$ and inf $\emptyset = +\infty$. For any $a, b \in \mathbb{R}$, $a^+ = \max \{a, 0\}$ and $a \wedge b = \min \{a, b\}$. If
$B$ is a set, $I_B$ denotes the indicator function of $B$.

1.1 Formulation

Suppose a certain species, whose population size at time $t$ is denoted by $X(t)$, lives in
random environments. As alluded in Section 1 in addition to the random fluctuations of the
population, we also assume that the growth of the species is subject to abrupt changes of
the environment. For simplicity, we assume that the switching among different environments
is memoryless and the waiting time for the next switch is exponentially distributed. In
fact, this phenomenon is also frequently observed in the nature; see the aforementioned
references. Thus we can model the random environments and other random factors in the
ecological system by a continuous-time Markov chain $\{\alpha(t), t \geq 0\}$ with a finite state space

4
Let the continuous time Markov chain $\alpha(\cdot)$ be generated by $Q = (q_{ij})$, that is,

$$P \{ \alpha(t + \Delta t) = j | \alpha(t) = i, \alpha(s), s \leq t \} = \begin{cases} q_{ij} \Delta t + o(\Delta t), & \text{if } j \neq i \\ 1 + q_{ii} \Delta t + o(\Delta t), & \text{if } j = i, \end{cases} \quad (1.1)$$

where $q_{ij} \geq 0$ for $i, j = 1, \ldots, m$ with $j \neq i$ and $q_{ii} = -\sum_{j \neq i} q_{ij} < 0$ for each $i = 1, \ldots, m$.

In light of the above discussion, in an effort to capture the salient feature that continuous dynamics and discrete events coexist in the ecosystem, we model the evolution of $X(t)$ in the absence of harvesting by the stochastic differential equation

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t), \quad X(0) = x, \alpha(0) = \alpha, \quad (1.2)$$

where $w(\cdot)$ is a 1-dimensional standard Brownian motion which provides the random fluctuations in the population’s size, and $b$ and $\sigma$ are real-valued functions. Further, we assume that the Brownian motion $w(\cdot)$ and the Markov chain $\alpha(\cdot)$ are independent, a standard assumption in the literature.

Assume throughout the paper that $b$ and $\sigma$ satisfy the usual Lipschitz condition and the linear growth condition. That is, there exists some $\kappa_0 > 0$ such that for any $x, y \in \mathbb{R}$ and each $\alpha \in \mathcal{M}$, we have

$$\begin{align*}
|b(x, \alpha) - b(y, \alpha)| + |\sigma(x, \alpha) - \sigma(y, \alpha)| & \leq \kappa_0 |x - y|, \\
|b(x, \alpha)| + |\sigma(x, \alpha)| & \leq \kappa_0 (1 + |x|).
\end{align*} \quad (1.3)$$

Consequently, the solution $X^{x,\alpha}(\cdot)$ of (1.2) exists and is unique in the strong sense (see [35] for details). Moreover, the solution $X^{x,\alpha}(\cdot)$ will not explode in finite time with probability 1 or it is regular in the sense of Khasminskii [15]. We refer the reader to [35] for related results on the regularity of regime switching diffusions.

If the species is subject to harvesting, and if $Z(t)$ denotes the total amount harvested from the species up to time $t$, then $\hat{X}(\cdot)$, the population size of the harvested population, satisfies

$$d\hat{X}(t) = b(\hat{X}(t), \alpha(t))dt + \sigma(\hat{X}(t), \alpha(t))dw(t) - dZ(t), \quad (1.4)$$

with initial conditions

$$\hat{X}(0-) = x \in \mathbb{R}_+ , \quad \alpha(0) = \alpha \in \mathcal{M}. \quad (1.5)$$

Note that $\hat{X}(0)$ may not equal to $\hat{X}(0-)$ due to an instantaneous harvest $Z(0)$ at time 0. Throughout the paper we use the convention that $\hat{Z}(0-) = 0$. The jump size of $Z$ at time $t \geq 0$ is denoted by $\Delta Z(t) := Z(t) - Z(t-)$, and $Z^c(t) := Z(t) - \sum_{0 \leq s \leq t} \Delta Z(s)$ denotes the continuous part of $Z$. Also note that $\Delta X(t) := X(t) - X(t-) = -\Delta Z(t)$ for any $t \geq 0$.

Denote the solution to (1.4) with initial condition specified by (1.5) by $\hat{X}^{x,\alpha}(\cdot)$ if necessary.

We say that $Z$ is an admissible harvesting strategy if

(i) $Z(t)$ is nonnegative for any $t \geq 0$ and nondecreasing with respect to $t$,

(ii) $\hat{X}(t) \geq 0$, for any $t \leq \tau$, where $\tau$ is the extinction time defined in (1.6) below,

\[ \hat{X}(\tau) = 0. \]
(iii) \( Z(t) \) is càdlàg and adapted to \( \mathcal{F}_t := \sigma \{ w(s), \alpha(s), 0 \leq s \leq t \} \), and

(iv) \( J(x, \alpha, Z) < \infty \) for any \( x > 0 \) and \( \alpha \in M \), where \( J \) is the functional defined in (1.7) below.

Let \( \mathcal{A} \) denote the collection of all admissible harvesting strategies.

Let \( f(\cdot, \cdot) : \mathbb{R}_+ \times M \mapsto \mathbb{R}_+ \) represent the instantaneous marginal yields accrued from exerting the harvesting strategy \( Z \). Assume \( f \) is continuous and non-increasing with respect to \( x \). Thus \( f(x, \alpha) \geq f(y, \alpha) \) for each \( \alpha \in M \) whenever \( x \leq y \). Moreover, we assume \( 0 < f(0, \alpha) < \infty \) for each \( \alpha \in M \). Let \( S = (0, \infty) \), which may be regarded as the survival set of the species. Denote the extinction time by

\[
\tau := \tau^{x,\alpha} = \inf \left\{ t \geq 0, \hat{X}^{x,\alpha}(t) \notin S \right\}.
\]  

(1.6)

Then for a fixed harvesting process \( Z \in \mathcal{A} \), the expected total discounted value from harvesting is

\[
J(x, \alpha, Z) := \mathbb{E}_{x,\alpha} \int_0^\tau e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) = \mathbb{E} \int_0^\tau e^{-rs} f(\hat{X}^{x,\alpha}(s-), \alpha(s-))dZ(s),
\]

(1.7)

where \( r \geq 0 \) is the discounting factor and \( \mathbb{E}_{x,\alpha} \) denotes the expectation with respect to the probability law when the process (1.4) starts with initial condition \( (x, \alpha) \) as specified in (1.5).

The goal is to maximize the expected total discounted value from harvesting and find an optimal harvesting strategy \( Z^* \):

\[
V(x, \alpha) = J(x, \alpha, Z^*) := \sup_{Z \in \mathcal{A}} J(x, \alpha, Z).
\]

(1.8)

The dynamic programming principle takes the form (see [8, 28]):

\[
V(x, \alpha) = \sup_{Z \in \mathcal{A}} \mathbb{E}_{x,\alpha} \left[ \int_0^{\tau \wedge \eta} e^{-rs} f(\hat{X}^{x,\alpha}(s-), \alpha(s-))dZ(s) + e^{-r(\tau \wedge \eta)} V(\hat{X}^{x,\alpha}(\tau \wedge \eta), \alpha(\tau \wedge \eta)) \right]
\]

(1.9)

for every \( (x, \alpha) \in S \times M \) and any stopping time \( \eta \).

For later convenience, we introduce the generator of the paired process \((X^{x,\alpha}, \alpha)\), in which \( X^{x,\alpha} \) satisfies (1.2). For any \( h(\cdot, \alpha) \in C^2, \alpha \in M \), we define

\[
\mathcal{L}h(x, \alpha) = b(x, \alpha) h'(x, \alpha) + \frac{1}{2} \sigma^2(x, \alpha) h''(x, \alpha) + \sum_{j \in M} q_{\alpha j} [h(x, j) - h(x, \alpha)],
\]

where \( h' \) and \( h'' \) denote the first and second order derivatives of \( h \) with respect to \( x \), respectively.
2 Verification Theorem and an $\varepsilon$-Optimal Policy

This section establishes a verification theorem whose proof utilizes the generalized Itô formula, the monotonicity of $f$, and the regularity of the process $\tilde{X}^{x,\alpha}$. We further construct an $\varepsilon$-optimal harvesting strategy explicitly in Corollary 2.4 based on the verification theorem and the imposition of additional conditions.

**Theorem 2.1.** Suppose there exists a function $\phi : S \times \mathcal{M} \to \mathbb{R}_+$ such that $\phi(\cdot, \alpha) \in C^2(S)$ for each $\alpha \in \mathcal{M}$ and that $\phi$ solves the following coupled system of quasi-variational inequalities:

$$\max \{ (\mathcal{L} - r)\phi(x, \alpha), f(x, \alpha) - \phi'(x, \alpha) \} = 0, \quad (x, \alpha) \in S \times \mathcal{M},$$

where $(\mathcal{L} - r)\phi(x, \alpha) = \mathcal{L}\phi(x, \alpha) - r\phi(x, \alpha)$.

(a) Then $\phi(x, \alpha) \geq V(x, \alpha)$ for every $(x, \alpha) \in S \times \mathcal{M}$.

(b) Define the continuation region

$$\mathcal{C} = \{(x, \alpha) \in S \times \mathcal{M} : f(x, \alpha) - \phi'(x, \alpha) < 0 \}.$$

Assume there exists a harvesting strategy $\tilde{Z} \in \mathcal{A}$ and corresponding process $\tilde{X}$ satisfying (1.4) such that,

$$\tilde{X}(t, \alpha(t)) \in \mathcal{C} \text{ for Lebesgue almost all } 0 \leq t \leq \tau,$$

$$(\tilde{X}(s), \alpha(s)) \in A \text{ for every } s \leq t,$$

$$\lim_{N \to \infty} \mathbb{E}_{x,\alpha} \left[ e^{-r(\tau \land N \land \beta_N)} \phi(\tilde{X}(\tau \land N \land \beta_N), \alpha(\tau \land N \land \beta_N)) \right] = 0,$$

and if $\tilde{X}(s) \neq \tilde{X}(s^-)$, then

$$\phi(\tilde{X}(s), \alpha(s^-)) - \phi(\tilde{X}(s), \alpha(s)) = -f(\tilde{X}(s^-), \alpha(s^-)) \Delta \tilde{Z}(s),$$

where $\beta_N := \inf \{ t \geq 0 : \tilde{X}(t) \geq N \}$. Then $\phi(x, \alpha) = V(x, \alpha)$ for every $(x, \alpha) \in S \times \mathcal{M}$ and $\tilde{Z}$ is an optimal harvesting strategy.

**Proof.** (a) Fix some $(x, \alpha) \in S \times \mathcal{M}$ and $Z \in \mathcal{A}$ and let $\hat{X}$ denote the corresponding solution to (1.4). Choose $N$ sufficiently large so that $|x| < N$ and define $\beta_N := \inf \{ t \geq 0 : |\hat{X}(t)| \geq N \}$. By virtue of [35, Section 2.3],

$$\beta_N \to \infty \text{ a.s. as } N \to \infty.$$ 

Write $T_N := N \land \beta_N \land \tau$. Then Itô’s formula leads to

$$\mathbb{E}_{x,\alpha}[e^{-rT_N}\phi(\hat{X}(T_N), \alpha(T_N))] - \phi(x, \alpha)$$

$$= \mathbb{E}_{x,\alpha} \int_0^{T_N} e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))ds - \mathbb{E}_{x,\alpha} \int_0^{T_N} e^{-rs}\phi'(\hat{X}(s), \alpha(s))d\tilde{Z}^c(s)$$

$$+ \mathbb{E}_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} \left[ \phi(\hat{X}(s), \alpha(s^-)) - \phi(\hat{X}(s), \alpha(s)) \right].$$
It follows from (2.1) that
\[
E_{x,\alpha}[e^{-rT_N} \phi(\hat{X}(T_N), \alpha(T_N))] - \phi(x, \alpha) \\
\leq - E_{x,\alpha} \int_0^{T_N} e^{-rs} \phi' (\hat{X}(s), \alpha(s)) dZ^c(s) + E_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} \Delta \phi (\hat{X}(s), \alpha(s-)),
\]
where \( \Delta \phi (\hat{X}(s), \alpha(s-)) = \phi(\hat{X}(s), \alpha(s-)) - \phi(\hat{X}(s-), \alpha(s-)) \). Apply the mean value theorem to \( \Delta \phi (\hat{X}(s), \alpha(s-)) \) and we obtain
\[
\Delta \phi (\hat{X}(s), \alpha(s-)) = \phi'(\xi(s), \alpha(s-)) \Delta \hat{X}(s) = -\phi'(\xi(s), \alpha(s-)) \Delta Z(s),
\]
where \( \xi(s) = \theta(s) \hat{X}(s) + (1 - \theta(s)) \hat{X}(s) \) for some \( \theta(s) \in (0, 1) \). Note that \( \hat{X}(s) \leq \xi(s) \leq \hat{X}(s-). \) Thus it follows that
\[
\phi(x, \alpha) \geq E_{x,\alpha}[e^{-rT_N} \phi(\hat{X}(T_N), \alpha(T_N))] + E_{x,\alpha} \int_0^{T_N} e^{-rs} \phi' (\hat{X}(s), \alpha(s)) dZ^c(s) \\
+ E_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} \phi' (\xi(s), \alpha(s-)) \Delta Z(s).
\]
Using (2.1) again and noting that \( \phi \) is nonnegative and that \( f(\cdot, \alpha) \) is nonincreasing for each \( \alpha \in \mathcal{M} \), it follows that
\[
\phi(x, \alpha) \geq E_{x,\alpha} \int_0^{T_N} e^{-rs} f(\hat{X}(s), \alpha(s)) dZ^c(s) + E_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} f(\xi(s), \alpha(s-)) \Delta Z(s) \\
\geq E_{x,\alpha} \int_0^{T_N} e^{-rs} f(\hat{X}(s), \alpha(s)) dZ^c(s) + E_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} f(\hat{X}(s-), \alpha(s-)) \Delta Z(s) \\
= E_{x,\alpha} \int_0^{T_N} e^{-rs} f(\hat{X}(s-), \alpha(s-)) dZ(s).
\]
Now letting \( N \to \infty \), it follows from (2.6) and the bounded convergence theorem that
\[
\phi(x, \alpha) \geq E_{x,\alpha} \int_0^{\tau} e^{-rs} f(\hat{X}(s-), \alpha(s-)) dZ(s).
\]
Finally, taking supremum over all \( Z \in \mathcal{A} \), we obtain \( \phi(x, \alpha) \geq V(x, \alpha) \), as desired.

(b) Let \( \tilde{Z} \in \mathcal{A} \) satisfy (2.2)–(2.5). Define \( \beta_N \) and \( T_N \) as before with \( \tilde{X} \) replacing \( \hat{X} \). As in part (a), we have from Itô’s formula that
\[
E_{x,\alpha}[e^{-rT_N} \phi(\tilde{X}(T_N), \alpha(T_N))] - \phi(x, \alpha) \\
= E_{x,\alpha} \int_0^{T_N} e^{-rs} (\mathcal{L} - r) \phi(\tilde{X}(s), \alpha(s)) ds - E_{x,\alpha} \int_0^{T_N} e^{-rs} \phi'(\tilde{X}(s), \alpha(s)) d\tilde{Z}^c(s) \\
+ E_{x,\alpha} \sum_{0 \leq s \leq T_N} e^{-rs} \left[ \phi(\tilde{X}(s), \alpha(s-)) - \phi(\tilde{X}(s-), \alpha(s-)) \right].
\]
By (2.2), \((\mathcal{L} - r)\phi(\tilde{X}(s), \alpha(s)) = 0\) for almost all \(s \in [0, \tau]\). This, together with (2.3) and (2.5), implies that 
\[
\phi(x, \alpha) = \mathbb{E}_{x, \alpha}[e^{-rT_N}\phi(\tilde{X}(T_N), \alpha(T_N))] + \mathbb{E}_{x, \alpha}\int_0^{T_N} e^{-rs} f(\tilde{X}(s -), \alpha(s -))d\tilde{Z}(s).
\]

Letting \(N \to \infty\) and using (2.4) and (2.6), we obtain
\[
\phi(x, \alpha) = \mathbb{E}_{x, \alpha}\int_0^\tau e^{-rs} f(\tilde{X}(s -), \alpha(s -))d\tilde{Z}(s).
\]

This shows that \(\phi(x, \alpha) = V(x, \alpha)\) for every \((x, \alpha) \in S \times \mathcal{M}\) and \(\tilde{Z}\) is an optimal harvesting strategy. \(\square\)

**Remark 2.2.** The conditions of Theorem 2.1 can be weakened. In fact, by virtue of [24, Appendix D], we need only to assume that (i) \(\phi(\cdot, \alpha) \in C^1(S) \cap C^2(S - D)\) for each \(\alpha \in \mathcal{M}\), where \(D\) is countable set of points, and (ii) \(\phi''(x+) < \infty, \phi''(x-) < \infty\) for all \(x \in D\). Under these conditions, there exist sequences \(\{\phi_j(\cdot, \alpha)\}^\infty_{j=1}, \alpha \in \mathcal{M}\) such that \(\phi_j(\cdot, \alpha) \in C^2(S)\) for each \(\alpha \in \mathcal{M}\). Moreover, the following are satisfied:

(a) for each \(\alpha \in \mathcal{M}\), \(\lim_{j \to \infty} \phi_j(\cdot, \alpha) \to \phi(\cdot, \alpha)\) uniformly on compact subsets of \(S\),

(b) \(\lim_{j \to \infty} (\mathcal{L} - r)\phi_j(x, \alpha) \to \phi(x, \alpha)\) uniformly on compact subsets of \(S - D, \alpha \in \mathcal{M}\), and

(c) \(\{(\mathcal{L} - r)\phi_j\}^\infty_{j=1}\) is locally bounded on \(S \times \mathcal{M}\).

Then, we can first work with the sequence \(\phi_j\) exactly the same way as in the proof of Theorem 2.1. Next, using (a), (b), and (c), we can pass to the limit as \(j \to \infty\) to obtain the same conclusions. The reader is referred to [24] for details.

By virtue of Theorem 2.1(a), any sufficiently smooth solution to (2.1) is an upper bound for the value function \(V\). Further, the additional conditions in Theorem 2.1(b) will help us to find an optimal harvesting strategy. In practice, it is, however, usually very hard to find an explicit solution to (2.1). In particular, with the presence of regime switching, (2.1) is a coupled system of quasi-variational inequalities, a closed form solution is virtually impossible except in some special cases (see Examples 3.1 and 3.3). Nevertheless, some results about the value function can be derived; Proposition 2.3 below gives an upper bound for \(V\) when \(f(\cdot, \alpha)\) is smooth for each \(\alpha\). Furthermore, under additional assumptions, we explicitly construct an \(\varepsilon\)-optimal harvesting strategy in Theorem 2.4.

For any \(x > 0\) and \(\alpha \in \mathcal{M}\), define
\[
g(x, \alpha) = \int_0^x f(y, \alpha)dy. \tag{2.7}
\]

Then it follows that \(g\) is nonnegative and \(g'(x, \alpha) = f(x, \alpha)\). Moreover, if \(f(\cdot, \alpha) \in C^1(S)\), then \(g''(x, \alpha) = f'(x, \alpha) \leq 0\) because \(f(\cdot, \alpha)\) is nonincreasing for each \(\alpha \in \mathcal{M}\). This shows that \(g(\cdot, \alpha)\) is concave for each \(\alpha \in \mathcal{M}\).
Proposition 2.3. Assume that \( f(\cdot, \alpha) \in C^1(S) \) and \( f(\cdot, \alpha) \) is non-increasing for each \( \alpha \in \mathcal{M} \). Then we have

\[
V(x, \alpha) \leq g(x, \alpha) + \sup_{Z \in A} E_{x, \alpha} \int_0^T e^{-rs} (\mathcal{L} - r) g(\hat{X}(s), \alpha(s)) ds. \tag{2.8}
\]

Proof. Fix some \((x, \alpha) \in S \times \mathcal{M}\) and \(Z \in A\) and let \(\hat{X}\) denote the corresponding solution to (1.3). Let \(T_N\) be as in the proof of Theorem 2.1. Apply Itô’s formula using \(g\) to obtain

\[
E_{x, \alpha} \left[ e^{-rT_N} g(\hat{X}(T_N), \alpha(T_N)) \right] - g(x, \alpha) \]

\[
= E_{x, \alpha} \int_0^{T_N} e^{-rs} (\mathcal{L} - r) g(\hat{X}(s), \alpha(s)) ds - E_{x, \alpha} \int_0^{T_N} e^{-rs} g'(\hat{X}(s), \alpha(s)) dZ(s) \]

\[
+ E_{x, \alpha} \sum_{0 \leq s \leq T_N} e^{-rs} \left[ g(\hat{X}(s), \alpha(s)) - g(\hat{X}(s), \alpha(s)) - g'(\hat{X}(s), \alpha(s)) \Delta \hat{X}(s) \right].
\]

Since \(g(\cdot, \alpha)\) is concave for each \(\alpha \in \mathcal{M}\), it follows that

\[
g(\hat{X}(s), \alpha(s)) \leq g(\hat{X}(s), \alpha(s)) + g'(\hat{X}(s), \alpha(s)) (\hat{X}(s) - \hat{X}(s)).
\]

Thus we have

\[
g(x, \alpha) \geq E_{x, \alpha} \int_0^{T_N} e^{-rs} f(\hat{X}(s), \alpha(s)) dZ(s) - E_{x, \alpha} \int_0^{T_N} e^{-rs} (\mathcal{L} - r) g(\hat{X}(s), \alpha(s)) ds.
\]

Now letting \(N \to \infty\) and using the same argument as in the proof of Theorem 2.1, we obtain

\[
g(x, \alpha) \geq E_{x, \alpha} \int_0^{T} e^{-rs} f(\hat{X}(s), \alpha(s)) dZ(s) - E_{x, \alpha} \int_0^{T} e^{-rs} (\mathcal{L} - r) g(\hat{X}(s), \alpha(s)) ds,
\]

from which (2.8) follows by taking supremum over \(Z \in A\). \(\square\)

Theorem 2.4. Assume, in addition to the conditions of Proposition 2.3, \((\mathcal{L} - r) g(x, \alpha) \leq 0\) for all \((x, \alpha) \in S \times \mathcal{M}\).

(i) Suppose that there exists a constant \(L > 0\) such that

\[
|f(x, \alpha) - f(y, \alpha)| \leq L |x - y|, \text{ for all } x, y \in S \text{ and } \alpha \in \mathcal{M}. \tag{2.9}
\]

Then for any \(\varepsilon > 0\), there exists a harvesting strategy \(Z^\varepsilon \in A\) under which

\[
g(x, \alpha) - \varepsilon \leq J(x, \alpha, Z^\varepsilon) \leq V(x, \alpha) \leq g(x, \alpha). \tag{2.10}
\]

The harvesting strategy \(Z^\varepsilon\) is a “chattering policy” that instantaneously harvests a sufficiently small amount many times in a sufficiently small interval of time until the species becomes extinct.

(ii) In particular, if \(f(x, \alpha) \equiv f(\alpha)\) for all \(x \in S\) and each \(\alpha \in \mathcal{M}\), then

\[
V(x, \alpha) = g(x, \alpha) = f(\alpha)x, \tag{2.11}
\]

and the optimal harvesting strategy is to drive the process instantaneously to extinction.
Proof. Fix some \((x, \alpha) \in S \times M\). By Proposition 2.3 and the condition \((L - r)g(x, \alpha) \leq 0\), we have

\[
V(x, \alpha) \leq g(x, \alpha).
\] (2.12)

The rest of proof is divided into two parts. Part 1 is devoted to the proof of (2.10) while the second part establishes (2.11).

Part 1. Since \(f(\cdot, \alpha)\) is continuous, for any \(\varepsilon > 0\), there exists an \(N \in \mathbb{N}\) such that

\[
R(f, \alpha) := \sum_{i=0}^{n-1} f(x_i, \alpha) \delta > \int_{0}^{x} f(y, \alpha) dy - \varepsilon/3 = g(x, \alpha) - \varepsilon/3, \quad \text{for any } n \geq N,
\]

where \(\delta = x/n\) and \(x_i = x - i\delta\), \(i = 0, 1, \ldots, n - 1\). Note that we also have \(R(f, \alpha) \leq g(x, \alpha)\) since \(f(\cdot, \alpha)\) is non-increasing. Thus it follows that

\[
|R(f, \alpha) - g(x, \alpha)| < \varepsilon/3.
\] (2.13)

Let \(\varsigma = n^{-5}\) and \(t_i = i\varsigma/n\), \(i = 0, 1, \ldots, n\). We construct a harvesting strategy \(Z = Z^\varepsilon\) which increases only on the set \(\{t_i : i = 0, \ldots, n\}\); denote the corresponding harvested process by \(\hat{X}\). Note that \(\hat{X}(t_0) = x_0 = x\). Define \(\Delta Z(t_0) = Z(t_0) = \delta\), observe \(\hat{X}(t_0) = x_1\) and it therefore follows that

\[
\hat{X}(t_1) = X(t_0) + \int_{t_0}^{t_1} b(\hat{X}(s), \alpha(s)) ds + \int_{t_0}^{t_1} \sigma(\hat{X}(s), \alpha(s))dw(s).
\]

At time \(t = t_1\), define \(\Delta Z(t_1) = (\hat{X}(t_1) - x_2)^+\) so that \(\hat{X}(t_1) \leq x_2\) and allow the process \(\hat{X}\) to diffuse until time \(t = t_2\). In general, for \(i = 1, \ldots, n - 1\), we define

\[
\Delta Z(t_i) = \left(\hat{X}(t_i) - x_{i+1}\right)^+
\]

so that

\[
\hat{X}(t_i) = \hat{X}(t_{i-}) - \Delta Z(t_i),
\]

and

\[
\hat{X}(t_{i+1}) = \hat{X}(t_i) + \int_{t_i}^{t_{i+1}} b(\hat{X}(s), \alpha(s)) ds + \int_{t_i}^{t_{i+1}} \sigma(\hat{X}(s), \alpha(s))dw(s).
\]

Note that \(\hat{X}(t_i) = x_{i+1}\) if \(\Delta Z(t_i) > 0\). The expected total discounted income from the harvesting strategy \(Z\) is

\[
J(x, \alpha, Z) = E_{x, \alpha} \sum_{i=0}^{n-1} e^{-rt_i} f(\hat{X}(t_i), \alpha(t_{i-})) \Delta Z(t_i).
\]
Next we want to show that $|J(x, \alpha, Z) - R(f, \alpha)| < \varepsilon/3$. In fact, we have

$$
|J(x, \alpha, Z) - R(f, \alpha)| \leq \sum_{i=0}^{n-1} \mathbb{E}_{x,\alpha} \left| e^{-rt_i} f(\hat{X}(t_i^-), \alpha(t_i^-)) \Delta Z(t_i) - f(x_i, \alpha) \delta \right|
$$

$$
\leq \sum_{i=0}^{n-1} \left[ \mathbb{E}_{x,\alpha} \left| f(\hat{X}(t_i^-), \alpha(t_i^-)) - f(x_i, \alpha) \right| \delta \\
+ \mathbb{E}_{x,\alpha} \left| f(\hat{X}(t_i^-), \alpha(t_i^-)) \Delta Z(t_i) - \delta \right|
+ \mathbb{E}_{x,\alpha} \left| e^{-rt_i} - 1 \right| f(\hat{X}(t_i^-), \alpha(t_i^-)) \Delta Z(t_i) \right]
$$

$$
=: \sum_{i=1}^{n} (A_i + B_i + C_i).
$$

In the following, we analyze the terms $A_i$, $B_i$, and $C_i$ separately. To this end, for any $i = 0, 1, \ldots, n - 1$, we apply [35, Proposition 2.3] to obtain

$$
\mathbb{E} \left| \int_{t_i}^{t_{i+1}} b(\hat{X}(s), \alpha(s)) ds + \int_{t_i}^{t_{i+1}} \sigma(\hat{X}(s), \alpha(s)) dw(s) \right|^2 \leq K(t_{i+1} - t_i) = Kt_1,
$$

$$
\mathbb{E} |\Delta Z(t_i)| = \mathbb{E} |(\hat{X}(t_i^-) - x_{i+1})^+| \leq K,
$$

where $K$ is a generic positive constant depending only on $x$, $m$, and the constant $\kappa_0$ in (1.3). Also, in the sequel, the exact value of $K$ may change in different appearances. Then it follows from the Tchebychev inequality that

$$
P \{\Delta Z(t_1) = 0\} = P \left\{\hat{X}(t_1^-) \leq x_2\right\}
$$

$$
= P \left\{ \int_{t_0}^{t_1} b(\hat{X}(s), \alpha(s)) ds + \int_{t_0}^{t_1} \sigma(\hat{X}(s), \alpha(s)) dw(s) \leq -\delta \right\}
$$

$$
\leq P \left\{ \int_{t_0}^{t_1} b(\hat{X}(s), \alpha(s)) ds + \int_{t_0}^{t_1} \sigma(\hat{X}(s), \alpha(s)) dw(s) \geq \delta \right\}
$$

$$
\leq \frac{Kt_1}{\delta^2}.
$$

Note that $\hat{X}(t_1) = x_2$ if $\Delta Z(t_1) > 0$. Thus we have

$$
P \left\{ \hat{X}(t_1) \neq x_2 \right\} \leq P \{\Delta Z(t_1) = 0\} \leq \frac{Kt_1}{\delta^2}.
$$

(2.18)

Using the same arguments as those in (2.17) and (2.18), we have

$$
P \{\Delta Z(t_2) = 0\} = P \left\{\Delta Z(t_2) = 0, \hat{X}(t_1) = x_2\right\} + P \left\{\Delta Z(t_2) = 0, \hat{X}(t_1) \neq x_2\right\}
$$

$$
\leq \frac{Kt_1}{\delta^2} + \frac{Kt_1}{\delta^2} = \frac{Kt_2}{\delta^2},
$$

and

$$
P \left\{ \hat{X}(t_2) \neq x_3 \right\} \leq P \{\Delta Z(t_2) = 0\} \leq \frac{Kt_2}{\delta^2}.
$$

(2.20)
Continuing in this manner, it follows that for any $i = 1, 2, \ldots, n - 1$,

$$
P \{ \Delta Z(t_i) = 0 \} \leq \frac{K t_i}{\delta^2}, \quad \text{and} \quad P \{ \hat{X}(t_i) \neq x_{i+1} \} \leq \frac{K t_i}{\delta^2}.
$$

(2.21) (2.22)

Using the conditions that $f$ is Lipschitz continuous and uniformly bounded, we compute

$$
A_i \leq E \left| f(\hat{X}(t_i), \alpha) - f(x_i, \alpha) \right| \delta + E \left| f(\hat{X}(t_i), \alpha(t_i)) - f(\hat{X}(t_i), \alpha) \right| \delta
$$

$$
\leq LE \left| \hat{X}(t_i) - x_i \right| \delta + K P \{ \alpha(t_i) \neq \alpha \} \delta
$$

$$
\leq LE \left| \hat{X}(t_i) - x_i \right| \delta + K t_i \delta,
$$

where in the last inequality, we used (1.1). But using (2.22), (2.15), and [35 Proposition 2.3], we obtain

$$
E \left| \hat{X}(t_i) - x_i \right| \leq E \left| \hat{X}(t_{i-1}) - x_i \right| + E \left| \int_{t_{i-1}}^{t_i} b(\hat{X}(s), \alpha(s)) ds + \int_{t_{i-1}}^{t_i} \sigma(\hat{X}(s), \alpha(s)) dw(s) \right|
$$

$$
\leq E^{1/2} \left| \hat{X}(t_{i-1}) - x_i \right|^{2} + K \sqrt{t_i} \left| I_{\{ \hat{X}(t_{i-1}) \neq x_i \}} \right|
$$

$$
\leq K \sqrt{t_i} \delta + K t_i \delta.
$$

Thus it follows that

$$
A_i \leq K \delta \left( \sqrt{t_i} + t_i + t_1 \right) = K \left( \sqrt{t_i} + t_i \delta + t_1 \delta \right).
$$

(2.23)

Next we estimate $B_i$. Since $f$ is uniformly bounded, it follows that

$$
B_i \leq K E \left| \Delta Z(t_i) - \delta \right|
$$

$$
= K E \left| (\Delta Z(t_i) - \delta) I_{\{ \Delta Z(t_{i-1}) = 0 \}} + K E \left| (\Delta Z(t_i) - \delta) I_{\{ \Delta Z(t_{i-1}) \neq 0 \}} I_{\{ \hat{X}(t_{i-1}) = x_i \}} \right|
$$

$$
+ K E \left| (\Delta Z(t_i) - \delta) I_{\{ \Delta Z(t_{i-1}) \neq 0 \}} I_{\{ \hat{X}(t_{i-1}) \neq x_i \}} \right|
$$

$$
:= B_{i1} + B_{i2} + B_{i3}.
$$

Note that (2.21) implies that $B_{i1} \leq \delta \frac{K t_i}{\delta^2} = \frac{K t_i}{\delta}$. Using the definition of $\Delta Z(t_i)$ and (2.15), we have

$$
B_{i2} = K E \left| (\hat{X}(t_i) - x_{i+1} - \delta) I_{\{ \Delta Z(t_{i-1}) \neq 0 \}} I_{\{ \hat{X}(t_{i-1}) = x_i \}} \right|
$$

$$
= K E \left| \int_{t_{i-1}}^{t_i} b(\hat{X}(s), \alpha(s)) ds + \int_{t_{i-1}}^{t_i} \sigma(\hat{X}(s), \alpha(s)) dw(s) \right| I_{\{ \Delta Z(t_{i-1}) \neq 0 \}} I_{\{ \hat{X}(t_{i-1}) = x_i \}}
$$

$$
\leq K t_i.
$$

Concerning the term $B_{i3}$, we use the Cauchy-Schwartz inequality, (2.16), and (2.22):

$$
B_{i3} \leq K E^{1/2} \left| (\Delta Z(t_i) - \delta) I_{\{ \Delta Z(t_{i-1}) \neq 0 \}} \right| E^{1/2} \left| I_{\{ \hat{X}(t_{i-1}) \neq x_i \}} \right|^2
$$

$$
\leq K \frac{\sqrt{t_i}}{\delta} \leq K \frac{\sqrt{t_i}}{\delta}.
$$

13
Putting these estimates together, we obtain
\[ B_i \leq K \frac{t_i}{\delta} + K t_1 + K \sqrt{t_i} \leq K(t_1 + \frac{\sqrt{t_i}}{\delta}). \quad (2.24) \]

For the term \( C_i \), we again use the uniform boundedness of \( f \) and (2.16) to obtain
\[ C_i = \mathbb{E} \left| (e^{-rt_i} - 1) f(\hat{X}(t_i -), \alpha(t_i -)) \Delta Z(t_i) \right| \leq K(1 - e^{-rt_i}) = Krt_i + o(t_i) \leq Kt_i. \quad (2.25) \]

Now using the estimates (2.23), (2.24), and (2.25) in (2.14), and noting \( \delta = xn^{-1}, t_i = it_1, \) and \( t_1 = n^{-6}, \)
\[
|J(x, \alpha, Z) - R(f, \alpha)| \leq \sum_{i=1}^{n} \left( K(\sqrt{t_i} + t_i\delta + t_1\delta) + K(t_1 + \frac{\sqrt{t_i}}{\delta}) + Kt_i \right) \\
\leq K \left( t_1 \sum_{i=1}^{n} (i + 1) + \sqrt{t_1}\frac{\delta + 1}{\delta} \sum_{i=1}^{n} \sqrt{t} \right) \\
\leq K \left( n^2 t_1 + \sqrt{t_1}\frac{\delta + 1}{\delta} n^{3/2} \right) \\
\leq Kn^2 n^{-6} + n^{-3} n^{3/2} \leq Kn^{-1/2}.
\]

Finally we choose \( n \) sufficiently large so that (2.13) holds and \( |J(x, \alpha, Z) - R(f, \alpha)| < \varepsilon / 3. \) Then it follows that
\[
|J(x, \alpha, Z) - g(x, \alpha)| \leq |J(x, \alpha, Z) - R(f, \alpha)| + |R(f, \alpha) - g(x, \alpha)| < \varepsilon / 3 + \varepsilon / 3 < \varepsilon.
\]
Now (2.10) follows in view of (2.12).

**Part 2.** If \( f(x, \alpha) \equiv f(\alpha) \) for all \( x \in S \) and each \( \alpha \in \mathcal{M} \), then we choose \( Z \) to be the harvesting policy which drives the process \( \hat{X} \) instantaneously from state \( x \) to state 0. It follows that \( \tau = 0 \) and
\[
J(x, \alpha, Z) = f(\alpha)x = g(x, \alpha) = V(x, \alpha).
\]

This finishes the proof. \( \Box \)

**Remark 2.5.** In [1, Corollary 1], it was commented that “if the convenience yield from holding reserves is non-positive at all states then the optimal policy is to deplete the reserves at an infinitely fast rate but only in small proportions at a time (a form of a ‘chattering policy’).” While the intuition in [1] is correct, the optimal policy in [1] is not admissible in our context, because it is not well defined at time 0. In Theorem 2.4, we explicitly constructed an admissible harvesting policy, under which the expected total discounted income from harvesting is \( \varepsilon \)-optimal.

### 3 Examples

We provide two examples to demonstrate our results in the previous section. They reveal that in the setting of regime switching, it is much harder to obtain the value functions and
the \((\varepsilon)-\) optimal harvesting strategies due to the coupling in the system of quasi variational inequalities.

**Example 3.1.** We assume the growth of a certain species (or a certain risky investment) is governed by a regime switching geometric Brownian motion

\[
dX(t) = \mu(\alpha(t))X(t)dt + \sigma(\alpha(t))X(t)dw(t),
\]

and the harvested process is given by

\[
d\hat{X}(t) = \mu(\alpha(t))\hat{X}(t)dt + \sigma(\alpha(t))\hat{X}(t)dw(t) - dZ(t),
\]

where \(Z(t)\) denotes the total amount of harvest (or dividends) up to time \(t\), \(w\) is a standard Brownian motion, \(\alpha\) is a continuous time Markov chain with state space \(M = \{1, \ldots, m\}\), and for each \(\alpha \in M\), \(\mu_\alpha = \mu(\alpha)\) and \(\sigma_\alpha = \sigma(\alpha)\) are constants. Our objective is to maximize the expected discounted income from harvest and find an optimal harvesting policy, i.e., we want to find

\[
V(x, \alpha) = J(x, \alpha, Z^*) = \sup_{Z \in A} \mathbb{E}_{x, \alpha} \int_0^\tau e^{-rs}dZ(s),
\]

where \(r > 0\) is the discount factor. Note that \(f \equiv 1\) in this example.

First consider the case when \(m = 1\); that is, there is only a static environment so no regime switching occurs. It is clear that if \(\mu > r\), then \(V(x, 1) = \infty\). On the other hand, if \(\mu \leq r\), then \(V(x, 1) = x\) and the optimal harvesting policy is to drive the process instantaneously to extinction \((\tau = 0\ a.s.)\). We refer the reader to [1] or [2] for details.

Now let \(m = 2\) and assume that the continuous time Markov chain \(\alpha\) is generated by

\[
Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},
\]

where \(\lambda_1 > 0\) and \(\lambda_2 > 0\). Without loss of generality, we further assume that \(\mu_1 \leq \mu_2\).

Case 1: \(\mu_1 \leq r\) and \(\mu_2 \leq r\). In this case, we have \(g(x, 1) = g(x, 2) = x\) and

\[
(\mathcal{L} - r)g(x, i) = \mu_i x - (\lambda_i + r) x + \lambda_i x = (\mu_i - r) x \leq 0, \quad x > 0, \quad i = 1, 2.
\]

Then Theorem 2.4 implies that \(V(x, 1) = V(x, 2) = x\) and that the optimal policy is to drive the process instantaneously to 0.

Case 2: \(\mu_1 < r < \mu_2 \leq \xi\), where

\[
\xi = \frac{r\lambda_1 + (r - \mu_1)(r + \lambda_2)}{r + \lambda_1 - \mu_1}.
\]

Note that \(\xi > r\). In this case, it can be shown that the unique solution to the system of coupled quasi variational inequalities

\[
\max \{ (\mathcal{L} - r)\phi(x, \alpha), 1 - \phi'(x, \alpha) \} = 0, \quad x > 0, \quad \alpha = 1, 2,
\]

is

\[
\phi(x, 1) = x, \quad \phi(x, 2) = \frac{\lambda_2}{\lambda_2 + r - \mu_2} x.
\]
Therefore Theorem 2.1 implies that $\phi(x, \alpha) \geq V(x, \alpha)$, $\alpha = 1, 2$.

Next we show that there is a harvesting strategy $Z^*$ under which $J(x, \alpha, Z^*) = \phi(x, \alpha)$ for all $x > 0$ and $\alpha = 1, 2$. To this end, we denote the harvesting region by $\mathcal{H} := (0, \infty) \times \{1\}$ and the continuation region by $\mathcal{C} := (0, \infty) \times \{2\}$. Let the harvesting policy $Z^*$ be such that it drives the process instantaneously to the origin once the Markov chain enters state 1 or the process enters the harvesting region. Consequently, the extinction time is

$$
\tau = \tau^{x, \alpha} = \inf \{t \geq 0 : \alpha(t) = 1\} = \inf \left\{t \geq 0 : (\hat{X}^{x, \alpha}(t), \alpha(t)) \in \mathcal{H} \right\}.
$$

One can verify that this harvesting policy and the corresponding harvested process satisfy all conditions in Theorem 2.1, part (b). In fact, it is obvious that $J(x, 1, Z^*) = x = \phi(x, 1)$. Next we consider $J(x, 2, Z^*)$. Note that $\tau = \tau^{x, 2}$ has exponential distribution with parameter $\lambda_2$ and that

$$
\hat{X}(t) = \hat{X}^{x, 2}(t) = x \exp \left\{(\mu_2 - \frac{1}{2}\sigma_2^2)t + \sigma_2 w(t)\right\}, \text{ for all } t \in [0, \tau].
$$

Therefore it follows that

$$
J(x, 2, Z^*) = \mathbb{E}_{x, 2} \int_0^\tau e^{-rs}dZ^*(s) = \mathbb{E}_{x, 2}[e^{-\tau \hat{X}(\tau)}]
= \int_0^\infty e^{-\tau x} \exp \left\{(\mu_2 - \frac{1}{2}\sigma_2^2)t + \frac{1}{2}\sigma_2^2 t\right\} \lambda_2 e^{-\lambda_2 t} dt
= \frac{\lambda_2}{\lambda_2 + r - \mu_2} x = \phi(x, 2).
$$

Hence $V(x, \alpha) = \phi(x, \alpha)$ for all $(x, \alpha) \in (0, \infty) \times \{1, 2\}$ and $Z^*$ is an optimal harvesting strategy.

Case 3: $\mu_1 < r < \xi < \mu_2$, where $\xi$ is defined in (3.4). We claim that $V(x, 1) = V(x, 2) = \infty$ for any $x > 0$. This is quite interesting. It indicates that even though $\mu_1 < r$, we still have $V(x, 1) = \infty$ thanks to the switching component $\alpha$. In the sequel, we demonstrate that there exists an admissible harvesting policy $Z$ under which $J(x, 1, Z)$ and $J(x, 2, Z)$ can be arbitrarily large and hence the claim follows.

Fix some $M > 0$ and define $\eta := \inf \{t \geq 0 : (X(t), \alpha(t)) = (M, 2)\}$. Note that the function $u(x, i) = \mathbb{E}_{x, i}[e^{-r\eta}]$, $0 < x < M$, $i = 1, 2$, solves the differential equation $(\mathcal{L} - r)u(x, i) = 0$. That is, $u$ is a solution to the coupled system of differential equations

$$
\begin{align*}
\frac{1}{2}\sigma_1^2 x^2 u''(x, 1) + \mu_1 xu'(x, 1) - (r + \lambda_1)u(x, 1) + \lambda_1 u(x, 2) &= 0, \\
\frac{1}{2}\sigma_2^2 x^2 u''(x, 2) + \mu_2 xu'(x, 2) - (r + \lambda_2)u(x, 2) + \lambda_2 u(x, 1) &= 0.
\end{align*}
$$

(3.6)

The characteristic equation of (3.6) is $h(x) = g_1(x)g_2(x) - \lambda_1 \lambda_2 = 0$, where $g_i(x) = \frac{1}{2}\sigma_i^2 x(x - 1) + \mu_i x - r - \lambda_i$, $i = 1, 2$. As argued in [9], $h(x)$ has four real roots $\beta_1 > \beta_2 > 0 > \beta_3 > \beta_4$. Moreover, the condition $\mu_1 < r < \xi < \mu_2$ implies that $1 > \beta_2 > 0$. Therefore $u(x, i)$, a
solution of (3.6), can be written as

\[ u(x, i) = E_{x,i}[e^{-r\eta}] = \sum_{j=1}^{4} C_j^{x, \beta_j} \quad 0 < x < M, \quad i = 1, 2 \]

for some constants \( C_j^{x, \beta_j}, \ i = 1, 2 \) and \( j = 1, 2, 3, 4 \). As noted in [9], \( C_j^{x, \beta_j} = l_j C_j^{1, \beta_j} \), where \( l_j = -\frac{\lambda_j}{g_2(\beta_j)} = -\frac{g_1(\beta_j)}{\lambda_j} \). But as

\[ X(t) = x \exp \left\{ \int_0^t [\mu(\alpha(s)) - \frac{1}{2}\sigma^2(\alpha(s))]ds + \int_0^t \sigma(\alpha(s))dw(s) \right\}, \]

it follows that \( \eta \to \infty \) a.s. as \( x \to 0 \). Thus for \( i = 1, 2 \), we have \( E_{x,i}[e^{-r\eta}] \to 0 \) as \( x \to 0 \) and hence

\[ E_{x,1}[e^{-r\eta}] = C_1 x^{\beta_1} + C_2 x^{\beta_2}, \quad E_{x,2}[e^{-r\eta}] = l_1 C_1 x^{\beta_1} + l_2 C_2 x^{\beta_2}, \]

where \( C_1 \) and \( C_2 \) are constants, and

\[ l_1 = -\frac{\lambda_2}{g_2(\beta_1)} = -\frac{g_1(\beta_1)}{\lambda_1} < 0, \quad l_2 = -\frac{\lambda_2}{g_2(\beta_2)} = -\frac{g_1(\beta_2)}{\lambda_1} > 0. \]

Now the boundary conditions yield

\[ E_{M,1}[e^{-r\eta}] = c = C_1 M^{\beta_1} + C_2 M^{\beta_2}, \quad E_{M,2}[e^{-r\eta}] = 1 = l_1 C_1 M^{\beta_1} + l_2 C_2 M^{\beta_2}, \]

where \( 0 < c \leq 1 \). Solve the above equations for \( c \) and \( l_1 \) and we obtain

\[ C_1 = \frac{l_2 c - 1}{(l_2 - l_1) M^{\beta_1}}, \quad C_2 = \frac{1 - l_1 c}{(l_2 - l_1) M^{\beta_2}}. \]

Notice that \( C_2 > 0 \). Consequently, we can write for \( x \in (0, M) \) that

\[ E_{x,1}[e^{-r\eta}] = \frac{l_2 c - 1}{(l_2 - l_1) M^{\beta_1}} x^{\beta_1} + \frac{1 - l_1 c}{(l_2 - l_1) M^{\beta_2}} x^{\beta_2}, \]

and

\[ E_{x,2}[e^{-r\eta}] = l_1 \frac{l_2 c - 1}{(l_2 - l_1) M^{\beta_1}} x^{\beta_1} + l_2 \frac{1 - l_1 c}{(l_2 - l_1) M^{\beta_2}} x^{\beta_2}. \]

Now we choose \( Z(t) = MI_{[M,\infty)\times\{2\}}(X(t), \alpha(t)), \ t \geq 0 \). Also, let

\[ \tilde{\eta} := \inf \{ t \geq 0 : (X(t), \alpha(t)) \in [M, \infty) \times \{2\} \}. \]

Then we have \( \tilde{\eta} \leq \eta \leq \tau \). Therefore the fact \( 1 > \beta_2 > 0 \) leads to

\[ J(x, 1, Z) = E_{x,1} \int_0^\tau e^{-r\eta}dZ(s) \geq ME_{x,1}[e^{-r\tilde{\eta}}] \geq ME_{x,1}[e^{-r\eta}] \]

\[ = \frac{l_2 c - 1}{(l_2 - l_1)} x^{\beta_1} M^{1-\beta_1} + \frac{1 - l_1 c}{(l_2 - l_1)} x^{\beta_2} M^{1-\beta_2} \]

\[ \to \infty, \quad \text{as } M \to \infty. \]
Similar calculation shows that
\[ J(x, 2, Z) \geq \frac{l_1 (l_2 c - 1)}{(l_2 - l_1)} x^{\beta_1} M^{1 - \beta_1} + \frac{l_2 (1 - l_1 c)}{(l_2 - l_1)} x^{\beta_2} M^{1 - \beta_2} \rightarrow \infty, \quad \text{as} \ M \rightarrow \infty. \]
The claim that \( V(x, 1) = V(x, 2) = \infty \) thus follows.

Case 4: \( \mu_1 \geq r \) and \( \mu_2 > r \). As in Case 3, we can show that the characteristic equation of (3.6) \( h(x) = g_1(x)g_2(x) - \lambda_1 \lambda_2 = 0 \) has a solution \( 0 < \beta_2 < 1 \) and hence similar arguments as in Case 3 reveals that \( V(x, 1) = V(x, 2) = \infty \).

Remark 3.2. In Theorem 2.1, part (b), condition (2.3) suggests that the optimal harvesting strategy \( \tilde{Z} \) will harvest only in the harvesting region
\[ \mathcal{H} = \mathbb{R}_+ \times \mathcal{M} - \mathcal{C} = \{(x, \alpha) \in S \times \mathcal{M} : \phi'(x, \alpha) = f(x, \alpha) \}, \]
which, in turn, implies that \( \phi(x, \alpha) = \int^x f(y, \alpha) dy \) for \( (x, \alpha) \in \mathcal{H} \). Further, if we do harvest at time \( s \) (so \( \tilde{X}(s) \neq \tilde{X}(s^-) \)), the amount of harvest \( \Delta \tilde{Z}(s) \) must satisfy condition (2.5). In other words, if we denote \( \tilde{X}(s^-) = x, \alpha(s^-) = \alpha, \) and \( \Delta \tilde{Z}(s) = \tilde{X}(s) - \tilde{X}(s) = \delta x > 0 \), then we must have
\[ -f(x, \alpha) \delta x = \phi(x - \delta x, \alpha) - \phi(x, \alpha) = -\int_{x-\delta x}^x f(y, \alpha) dy. \quad (3.7) \]
However, if \( f(x, \alpha) \) is strictly decreasing with respect to \( x \) for \( (x, \alpha) \in \mathcal{H} \), (3.7) can never be satisfied. In other words, there is no admissible optimal harvesting strategy at all. Then a natural question arises: Can we find an admissible \( \varepsilon \)-optimal harvesting policy? In the following example, the answer to this question is positive.

Example 3.3. As in Example 3.1, let the harvested process be given by (3.2) and the random environments be modeled by a two-state continuous time Markov chain \( \alpha \) whose generator is \( Q \). Our objective is to maximize the expected total discounted income from harvest
\[ V(x, \alpha) = \max_{Z \in \mathcal{A}} \mathbb{E}_{x, \alpha} \int_0^\tau e^{-rs} (1 + \tilde{X}(s^-))^{-\gamma} dZ(s), \quad (3.8) \]
where \( 0 < \gamma < 1 \), and \( r, \tau, \) and \( Z \) are as in Example 3.1

Assume that \( \mu_1 > r \) and \( \mu_2 > r \). As a result, the positive roots
\[ p_i = \frac{1}{2} - \frac{\mu_i}{\sigma_i^2} + \sqrt{\left( \frac{1}{2} - \frac{\mu_i}{\sigma_i^2} \right)^2 + \frac{2r}{\sigma_i^2}} \]
of the equations \( \frac{1}{2} \sigma_i^2 x(x-1) + \mu_i x - r = 0 \) satisfy \( 0 < p_i < 1, i = 1, 2 \). Suppose that
\[ p_1 = p_2 = p. \quad (3.9) \]
Note that there are many nontrivial examples (in the sense that \( \mu_1 \neq \mu_2 \) and \( \sigma_1 \neq \sigma_2 \)) where condition (3.9) is satisfied. For example, if \( \mu_1 = 1, \sigma_1^2 = 2, \mu_2 = 2 - \sqrt{7}, \) and \( \sigma_2^2 = 4 \), where \( 0 < r < 1 \), then \( p_1 = p_2 = p = \sqrt{7} \). Under condition (3.9), we compute
\[ h(p) = g_1(p)g_2(p) - \lambda_1 \lambda_2 = (-\lambda_1)(-\lambda_2) - \lambda_1 \lambda_2 = 0, \]
where \( h, g_1, \) and \( g_2 \) are defined in Example 3.1. Consequently, it follows that \((L - r)x^p = 0.\)

Assume \( 1 - p < \gamma < 1.\) Detailed calculations reveal that

\[
\phi(x, 1) = \phi(x, 2) = \begin{cases} 
\frac{(1 + b)^{-\gamma}}{p^p - 1} x^p, & \text{if } x \in (0, b), \\
\frac{(1 + x)^{1 - \gamma} - (1 + b)^{1 - \gamma}}{1 - \gamma} + \frac{b(1 + b)^{-\gamma}}{p}, & \text{if } x \in [b, +\infty),
\end{cases}
\]

(3.10)
solves the quasi-variational inequality

\[
\max \left\{ (L - r)\phi(x, \alpha), (1 + x)^{-\gamma} - \phi'(x, \alpha) \right\} = 0, \quad x > 0, \quad \alpha = 1, 2,
\]

where \( b = \frac{1 - p}{p + \gamma - 1} > 0.\) Therefore by virtue of Theorem 2.1, \( V(x, \alpha) \leq \phi(x, \alpha) \) for all \( x > 0 \) and \( \alpha = 1, 2.\)

Next, we construct an \( \varepsilon \)-optimal harvesting strategy \( Z^\varepsilon.\) To this end, we denote the continuation region by \( C := (0, b) \times \{1, 2\} \) and the harvesting region by \( H := [b, \infty) \times \{1, 2\}.\) Let \((x, \alpha) \in C \) and \( L_b \) be the local time process for the process \( \hat{X} \) at the point \( b.\) Then for the function \((x, \alpha) \mapsto x^p,\) we have

\[
E_{x, \alpha}[e^{-rT_N} \hat{X}(T_N)^p] - x^p = E_{x, \alpha} \int_0^{T_N} e^{-rs}(L - r)\hat{X}(s)^p ds - E_{x, \alpha} \int_0^{T_N} e^{-rs} p\hat{X}(s)^{p-1}dL_b(s),
\]

where \( T_N = \tau \wedge \delta \) as in the proof of Theorem 2.1. Thus it follows that

\[
pb^{p-1} E_{x, \alpha} \int_0^{T_N} e^{-rs}dL_b(s) = x^p - E_{x, \alpha}[e^{-rT_N} \hat{X}(T_N)^p].
\]

Using (2.10), we can readily verify that \( E_{x, \alpha}[e^{-rT_N} \hat{X}(T_N)^p] \to 0 \) as \( N \to \infty.\) Hence by letting \( N \to \infty, \) we obtain

\[
E_{x, \alpha} \int_0^{\tau} e^{-rs}dL_b(s) = \frac{x^p}{pb^{p-1}}.
\]

(3.11)

Now let \( Z = L_b.\) Then

\[
J(x, \alpha, Z) = E_{x, \alpha} \int_0^{\tau} e^{-rs}(1 + \hat{X}(s))^{-\gamma}dZ(s) = E_{x, \alpha} \int_0^{\tau} e^{-rs}(1 + \hat{X}(s))^{-\gamma}dL_b(s)
\]

\[
= (1 + b)^{-\gamma} E_{x, \alpha} \int_0^{\tau} e^{-rs}dL_b(s) = \frac{(1 + b)^{-\gamma}}{pb^{p-1}} x^p = \phi(x, \alpha).
\]

Next, let \((x, \alpha) \in H.\) If \( x = b,\) define \( Z^1 \in A \) such that \( Z^1(0) = \Delta Z^1(0) = \varepsilon > 0 \) and \( Z^1(t) = L_b(t) \) for \( t > 0.\) Then

\[
J(x, \alpha, Z^1) = (1 + b)^{-\gamma} \varrho + J(b - \varrho, \alpha, L_b)
\]

\[
= (1 + b)^{-\gamma} \varrho + \frac{(1 + b)^{-\gamma}}{pb^{p-1}} (b - \varrho)^p
\]

\[
> \frac{b(1 + b)^{-\gamma}}{p} - \varepsilon/3 = \phi(x, \alpha) - \varepsilon/3
\]

(3.12)
for $\varrho$ sufficiently small.

If $x > b$. Then as in the proof of Theorem 2.4 for $n$ sufficiently large, let $\delta = \frac{x-b}{n}$, $\varsigma = n^{-5}$, and $t_i = i\varsigma/n$ for $i = 0, 1, \ldots, n - 1$. Let $Z^2 \in A$ increase only at times $t_i$, $i = 0, 1, \ldots, n - 1$. More specifically, $Z^2(t_0) = \Delta Z^2(t_0) = \delta$ and for $i = 1, 2, \ldots, n - 1$,

$$\Delta Z(t_i) = \left(\hat{X}(t_{i-1}) - (x - (i + 1)\delta)\right)^+.$$

Denote $\theta := \inf \left\{t \geq 0 : \hat{X}(t) \leq b\right\}$. Note that $\theta \leq t_{n-1} < n^{-5}$ and $\hat{X}(\theta) = b$ a.s. Define

$$Z^\varepsilon(t) = Z^2(t)I_{\{t < \theta\}} + Z^1(t)I_{\{t \geq \theta\}}. \quad (3.13)$$

It is easy to verify that for any $x, y > 0$, we have

$$\left|(1 + x)^{-\gamma} - (1 + y)^{-\gamma}\right| \leq \left|x - y\right|,$$

and hence $f$ is Lipschitz continuous. As a result, similar arguments as those in the proof of Theorem 2.4 reveal that

$$E_{x,\alpha} \int_0^\theta e^{-rs}(1 + \hat{X}(s-))^{-\gamma}dZ^2(s) > \int_b^x (1 + y)^{-\gamma}dy - \frac{\varepsilon}{2} \quad (3.14)$$

by choosing $n$ sufficiently large. Then for $n$ sufficiently large, we have from (3.12) and (3.14) that

$$J(x, \alpha, Z^\varepsilon) = E_{x,\alpha} \int_0^\tau e^{-rs}(1 + \hat{X}(s-))^{-\gamma}dZ^\varepsilon(s)$$

$$= E_{x,\alpha} \int_0^\theta e^{-rs}(1 + \hat{X}(s-))^{-\gamma}dZ^2(s) + E_{x,\alpha} \int_\theta^\tau e^{-rs}(1 + \hat{X}(s-))^{-\gamma}dZ^1(s)$$

$$\geq E_{x,\alpha} \int_0^\theta e^{-rs}(1 + \hat{X}(s-))^{-\gamma}dZ^2(s) + E_{x,\alpha}[e^{-r\theta}(J(b, 1, Z^1) \wedge J(b, 2, Z^1))]$$

$$> \int_b^x (1 + y)^{-\gamma}dy - \varepsilon/2 + e^{-rn_B}(\frac{b(1 + b)^{-\gamma}}{p} - \varepsilon/3)$$

$$> \frac{(1 + x)^{1-\gamma} - (1 + b)^{1-\gamma}}{1-\gamma} + \frac{b(1 + b)^{-\gamma}}{p} - \varepsilon = \phi(x, \alpha) - \varepsilon,$$

This shows that the harvesting policy $Z^\varepsilon$ is $\varepsilon$-optimal.

## 4 Properties of the Value Function

Theorem 2.4 gives sufficient conditions for a function $\phi$ to coincide with the value function. In particular, the function $\phi$ must satisfy the system of quasi-variational inequalities (2.1). It is natural to ask whether the converse is true: “Does the value function $V$ defined in (1.8) always satisfy (2.1)?” In general, the answer is no since the solution $V$ is not necessarily smooth enough. An alternative definition for a solution to the quasi-variational inequalities (2.1) is that of a viscosity solution (see Fleming and Soner [3]).

Therefore this section is devoted to the properties of the value function $V$. We present sufficient conditions under which the value function $V$ is continuous. Also, we show that $V$ is a viscosity solution to the coupled system of quasi variational inequalities (2.1).
4.1 Continuity

As we indicated in Section 1 in the definition of \( J \) in (1.7), both the extinction time \( \tau \) and the harvesting strategy \( Z \) may depend on the initial condition \( \hat{X}(0-) = x \). Consequently the standard arguments for continuity of \( V \) using Lipschitz continuity and Gronwall’s inequality (as in [36, Proposition 4.3.1]) or Dini’s theorem (as in [8, Lemma V.2.1]) do not apply here. In the sequel, we establish the continuity of \( V \) by first studying some elementary properties of \( V \).

The following lemma is immediate, which asserts that the value function \( V(x, \alpha) \) is non-decreasing with respect to the \( x \) variable.

**Lemma 4.1.** For each \( \alpha \in \mathcal{M} \) and any \( 0 < y \leq x \), we have

\[
V(x, \alpha) \geq f(x, \alpha)(x - y) + V(y, \alpha). \tag{4.1}
\]

**Proof.** Fix some \( \alpha \in \mathcal{M} \). If \( y \leq x \), then for any harvesting strategy \( Z \in \mathcal{A} \), we define \( \tilde{Z} \) to be the harvesting strategy such that \( \tilde{Z}(t) = Z(t) + (x - y) \) for any \( t \geq 0 \). It is obvious that \( \tilde{Z} \in \mathcal{A} \). Note also that \( \Delta \tilde{Z}(t) = \Delta Z(t) \) for all \( t > 0 \). Let \( \hat{X} \) denote the process satisfying (1.4) with initial condition \( \hat{X}(0-) = y \) and harvesting strategy \( Z \). Similarly, \( \tilde{X} \) denotes the process satisfying (1.4) with initial condition \( \hat{X}(0-) = x \) and harvesting strategy \( \tilde{Z} \). Then we have \( \hat{X}(t) = \tilde{X}(t) \) for all \( t > 0 \). Consequently, it follows that

\[
J(x, \alpha, \tilde{Z}) = f(x, \alpha)(x - y) + J(y, \alpha, Z).
\]

Since \( V(x, \alpha) \geq J(x, \alpha, \tilde{Z}) \), we have

\[
V(x, \alpha) \geq f(x, \alpha)(x - y) + J(y, \alpha, Z),
\]

from which (4.1) follows by taking supremum over \( Z \in \mathcal{A} \). \( \square \)

**Lemma 4.2.** For each \( \alpha \in \mathcal{M} \) and any \( 0 < y \leq x \), we have

\[
V(x, \alpha) \leq V(y, \alpha) + \max \{V(x - y, j) : j = 1, \ldots, m\}. \tag{4.2}
\]

**Proof.** Let \( x > y > 0 \). We consider a harvested process \( \hat{X} \) with initial conditions \( \hat{X}(0-) = x \), \( \alpha(0) = \alpha \) and harvesting strategy \( Z \in \mathcal{A} \). Define \( \theta := \inf\{t \geq 0 : \hat{X}(t) \leq x - y\} \) and \( \tau := \inf\{t \geq 0 : \hat{X}(t) \notin S\} \). Then \( \tau \geq \theta \).

Case 1: \( \theta = \infty \). Note that \( \tau = \infty \). Let \( y \in (0, x) \) and \( \tilde{X} \) be another harvested process with initial conditions \( \tilde{X}(0-) = y \), \( \alpha(0) = \alpha \) and harvesting strategy \( \tilde{Z} \in \mathcal{A} \), where we choose \( \tilde{Z}(t) = \hat{Z}(t) \) for all \( t \geq 0 \). Consequently, it follows that \( 0 \leq \tilde{X}(t) = \hat{X}(t) - (x - y) \leq \hat{X}(t) \) for all \( t \geq 0 \). Using the assumption that \( f(\cdot, \alpha) \) is nonincreasing for each \( \alpha \in \mathcal{M} \), we have

\[
J(x, \alpha, Z) = \mathbb{E} \int_0^\tau e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) \\
\leq \mathbb{E} \int_0^\tau e^{-rs} f(\tilde{X}(s-), \alpha(s-))d\tilde{Z}(s) \\
= J(y, \alpha, \tilde{Z}) \leq V(y, \alpha). \tag{4.3}
\]
Case 2: $\theta < \infty$. Then we can write

\[
J(x, \alpha, Z) = \mathbb{E}\left[\int_{[0,\theta]} e^{-rs}f(\tilde{X}(s), \alpha(s-))dZ(s)\right] \\
= \mathbb{E}\left[\int_{[0,\theta]} e^{-rs}f(\tilde{X}(s), \alpha(s-))dZ(s) + e^{-r\theta}f(\tilde{X}(\theta-), \alpha(\theta-))\Delta Z(\theta) \right. \\
+ \left. \int_{[\theta, \tau]} e^{-rs}f(\tilde{X}(s), \alpha(s-))dZ(s) \right].
\]

As in Case 1, we consider $y \in (0, x)$ and another harvested process $\tilde{X}$ with initial conditions $\tilde{X}(0-) = y, \alpha(0) = \alpha$ and harvesting strategy $\tilde{Z} \in \mathcal{A}$, where we choose $\tilde{Z}(t) = Z(t)$ for all $0 \leq t < \theta$ and $\tilde{Z}(\theta) = Z(\theta-) + \tilde{X}(\theta-)$. As a result, $\tilde{X}(t) - \tilde{X}(t) = x - y$ for all $0 \leq t < \theta$ and $\tilde{X}(\theta) = 0$. Then it follows from the monotonicity of $f$ that

\[
J(x, \alpha, Z) \leq \mathbb{E}\left[\int_{[0,\theta]} e^{-rs}f(\tilde{X}(s), \alpha(s-))d\tilde{Z}(s) + e^{-r\theta}f(\tilde{X}(\theta-), \alpha(\theta-))\Delta\tilde{Z}(\theta) \right. \\
+ \left. \int_{[\theta, \tau]} e^{-rs}f(\tilde{X}(s), \alpha(s-))d\tilde{Z}(s) \right] \leq J(y, \alpha, \tilde{Z}) + \mathbb{E}\int_{[\theta, \tau]} e^{-rs}f(\tilde{X}(s), \alpha(s-))d\tilde{Z}(s).
\]

Note that $\tilde{X}(\theta) \leq x - y$. Therefore by virtue of Lemma 4.1 we have

\[
\mathbb{E}\int_{[\theta, \tau]} e^{-rs}f(\tilde{X}(s), \alpha(s-))d\tilde{Z}(s) = \mathbb{E}\left[\mathbb{E}\left[\int_{[\theta, \tau]} e^{-rs}f(\tilde{X}(s), \alpha(s-))d\tilde{Z}(s)|\mathcal{F}_\theta]\right]\right] \\
\leq \mathbb{E}[e^{-r\theta}V(\tilde{X}(\theta), \alpha(\theta))] \leq \max \{V(x - y, j), j = 1, \ldots, m\}.
\]

Hence it follows that

\[
J(x, \alpha, Z) \leq J(y, \alpha, \tilde{Z}) + \max \{V(x - y, j), j = 1, \ldots, m\} \\
\leq V(y, \alpha) + \max \{V(x - y, j), j = 1, \ldots, m\}. \tag{4.4}
\]

Combining cases 1 and 2, we conclude that for any $Z \in \mathcal{A}$, we have

\[
J(x, \alpha, Z) \leq V(y, \alpha) + \max \{V(x - y, j), j = 1, \ldots, m\}.
\]

Now (4.2) follows by taking supremum over $Z \in \mathcal{A}$. \qed

For $h \geq 0$, we denote

\[
\zeta_h := \inf \{t \geq 0 : X^{z, \alpha}(t) = h\}, \quad \text{where } X^{z, \alpha}(t) \text{ is the solution to (1.2).} \tag{4.5}
\]

22
Lemma 4.3. Suppose that for any $t > 0$ and $h > 0$, we have

$$
\mathbb{P}_{x,\alpha}\left\{ \max_{0 \leq s \leq \zeta_0 \wedge t} X(s) < h \right\} \to 1, \text{ as } x \downarrow 0, \text{ for each } \alpha \in \mathcal{M}. \tag{4.6}
$$

If $V(x_0, \alpha) < \infty$ for some $x_0 > 0$ and every $\alpha \in \mathcal{M}$, then

$$
\lim_{x \downarrow 0} V(x, \alpha) = 0, \text{ for each } \alpha \in \mathcal{M}. \tag{4.7}
$$

Proof. It follows from Lemmas 4.1 and 4.2 that $V(x, \alpha) < \infty$ for all $(x, \alpha) \in S \times \mathcal{M}$. Fix some $(x, \alpha) \in \mathbb{R}^+ \times \mathcal{M}$ and $Z \in \mathcal{A}$. For any $h > 0$ and $t > 0$, let $\eta : = \tau \wedge \zeta_h \wedge t$, where $\tau$ is the extinction time defined in (1.6) and $\zeta_h$ in (4.5). Then $\eta \leq \tau$ and $\eta < \tau$ if and only if $\tau > \zeta_h \wedge t$. Note that (4.6) implies that

$$
\mathbb{P}_{x,\alpha}\{ \zeta_0 < \zeta_h \wedge t \} \to 1, \text{ as } x \downarrow 0, \text{ for each } \alpha \in \mathcal{M}. \tag{4.8}
$$

Hence for any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$
\mathbb{P}_{x,\alpha}\{ \zeta_0 < \zeta_h \wedge t \} > 1 - \varepsilon, \text{ for any } 0 \leq x < \delta \text{ and each } \alpha \in \mathcal{M}. \tag{4.9}
$$

On the other hand, by virtue of [35, Proposition 2.3], we have

$$
\mathbb{E}_{x,\alpha}\left[ \sup_{0 \leq s \leq \zeta_h \wedge t} X(s) \right] < \varepsilon, \text{ for any } 0 \leq x < \delta \text{ and each } \alpha \in \mathcal{M}. \tag{4.10}
$$

Now we compute

$$
J(x, \alpha, Z) = \mathbb{E}_{x,\alpha}\int_0^\tau e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) = \mathbb{E}_{x,\alpha}\int_0^\eta e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) + \mathbb{E}_{x,\alpha}\left[ I_{\{\tau \geq \eta\}} \int_\eta^\tau e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) \right] := A + B.
$$

Since $f(\cdot, \alpha)$ is nonincreasing and that $\hat{X}(s-) \geq 0$ for all $0 \leq s \leq \eta$, it follows that

$$
f(\hat{X}(s-), \alpha(s-)) \leq f(0, \alpha(s-)) \leq \max_{i \in \mathcal{M}} f(0, i) = K,
$$

where in the above and hereafter, $K$ is a generic positive constant not depending on $x$ or $t$ whose exact value may change in different appearances. Hence it follows that $A \leq K \mathbb{E}_{x,\alpha}[Z(\eta)]$. Further, since $Z$ is an admissible harvesting strategy, we have

$$
Z(\eta) \leq X(\eta) \leq \sup_{0 \leq s \leq \zeta_0 \wedge t} X(s).
$$
Note in the above, we used the fact that $\eta = \tau \wedge \zeta_h \wedge t \leq \tau \wedge t \leq \zeta_0 \wedge t$. Thus we have from (4.10) that $A \leq E_{x,\alpha}[Z(\eta)] < K\varepsilon$.

On the other hand,

$$B = E_{x,\alpha} \left[ I_{\{\tau > \eta\}} \int_\eta^\tau e^{-rs} f(\hat{X}(s), \alpha(s)) dZ(s) \right]$$

$$= E_{x,\alpha} \left[ I_{\{\tau > \eta\}} E \left[ \int_\eta^\tau e^{-rs} f(\hat{X}(s), \alpha(s)) dZ(s) | \mathcal{F}_\eta \right] \right]$$

$$\leq E_{x,\alpha} \left[ I_{\{\tau > \eta\}} e^{-e\eta} V(X(\eta), \alpha(\eta)) \right].$$

By the definition of $\eta$, we have $\hat{X}(\eta) \leq X(\eta) \leq h$. Thus Lemma 4.1 leads to

$$V(X(\eta), \alpha(\eta)) \leq V(h, \alpha(\eta)) \leq \max_{i \in \mathcal{M}} V(h, i) < \infty.$$ 

Therefore it follows from the above computations and (4.9) that

$$J(x, \alpha, Z) \leq K\varepsilon + \max_{i \in \mathcal{M}} V(h, i) \mathbb{P}_{x,\alpha}\{\tau > \eta\} < K\varepsilon.$$

Taking supremum over $Z \in \mathcal{A}$, we obtain $V(x, \alpha) < K\varepsilon$, for all $0 \leq x < \delta$ and each $\alpha \in \mathcal{M}$. Therefore (4.7) follows and this completes the proof of the lemma. □

By virtue of Lemmas 4.1, 4.2, and 4.3, we have the following theorem, which presents a sufficient condition for continuity of the value function.

**Theorem 4.4.** Let the conditions of Lemma 4.3 be satisfied. Then the value function $V$ defined in (1.8) is continuous with respect to the variable $x$.

**Remark 4.5.** Note that (4.6) is the crucial assumption in Theorem 4.4, it also plays a key role of the proof of Lemma 4.3. One may wonder under what condition(s), is (4.6) valid?

**Example 4.6.** If the unharvested process is given by

$$dX(t) = b(\alpha(t))dt + \sigma(\alpha(t))dw(t),$$

where for each $\alpha \in \mathcal{M}$, $b(\alpha) \in \mathbb{R}$ and $\sigma(\alpha) > 0$, then using the same argument as that of [3], Lemma 1], we obtain (4.6).

Next we present a sufficient condition for (4.6).

**Proposition 4.7.** If there exists a function $W : S \times \mathcal{M} \mapsto \mathbb{R}^+$ satisfying

(i) for each $\alpha \in \mathcal{M}$, $W(\cdot, \alpha)$ is continuous on $[0, \infty)$ and vanishes only at $x = 0$,

(ii) $\mathcal{L}W(x, \alpha) \leq 0$ for all $(x, \alpha) \in S \times \mathcal{M}$.

Then for any $t > 0$ and $h > 0$, we have

$$\mathbb{P}_{x,\alpha}\left\{ \max_{0 \leq s \leq \zeta_0 \wedge t} X(s) < h \right\} \rightarrow 1, \text{ as } x \downarrow 0, \text{ for each } \alpha \in \mathcal{M}.$$
Proof. Fix some \( t > 0 \) and \( h > 0 \). Let \((x, \alpha) \in S \times \mathcal{M}\) with \( x < h \). Denote \( W_h := \inf \{ W(y, j) : y \in S, y \geq h, j \in \mathcal{M}\} \). Then assumption (i) implies that \( W_h > 0 \). By virtue of Itô’s formula and assumption (ii), we have
\[
E_{x,\alpha} W(X(t \wedge \zeta_0 \wedge \zeta_h), \alpha(t \wedge \zeta_0 \wedge \zeta_h)) = W(x, \alpha) + E_{x,\alpha} \int_0^{t \wedge \zeta_0 \wedge \zeta_h} LW(X(s), \alpha(s)) ds \leq W(x, \alpha).
\]
Now since \( W \) is nonnegative, it follows that
\[
W(x, \alpha) \geq E_{x,\alpha} \left[ W(X(\zeta_h), \alpha(\zeta_h))I_{\{\zeta_h < \zeta_0 \wedge t\}} \right] \geq W_h P_{x,\alpha} \{ \zeta_h < \zeta_0 \wedge t \}.
\]
Thus we have
\[
P_{x,\alpha} \{ \zeta_h < \zeta_0 \wedge t \} \leq \frac{W(x, \alpha)}{W_h}.
\]
This, together with assumption (i), leads to
\[
P_{x,\alpha} \{ \zeta_0 \wedge t < \zeta_h \} \geq 1 - \frac{W(x, \alpha)}{W_h} \to 1, \quad \text{as } x \downarrow 0.
\]
Note that \( \zeta_0 \wedge t < \zeta_h \) if and only if \( \max_{0 \leq s \leq \zeta_0 \wedge t} X(s) < h \). Therefore the desired assertion follows. \( \square \)

### 4.2 Viscosity Solution

In this subsection, we aim to characterize the value function as a viscosity solution of the coupled system of quasi-variational inequalities (2.1). Let’s recall the notion of viscosity solution.

**Definition 4.8.** A function \( u \) is said to be a **viscosity subsolution** of (2.1), if for any \((x_0, \alpha_0) \in S \times \mathcal{M}\) and function \( \varphi(\cdot, \alpha) \in C^2(S) \) satisfying \( \varphi(x_0, \alpha_0) = u(x_0, \alpha_0) \) and \( \varphi(x, \alpha) \leq u(x, \alpha) \) for all \( x \) in a neighborhood of \( x_0 \) and each \( \alpha \in \mathcal{M} \), we have
\[
\max \{(\mathcal{L} - r)\varphi(x_0, \alpha_0), f(x_0, \alpha_0) - \varphi'(x_0, \alpha_0)\} \leq 0.
\]

Similarly, a function \( u \) is said to be a **viscosity supersolution** of (2.1), if for any \((x_0, \alpha_0) \in S \times \mathcal{M}\) and function \( \varphi(\cdot, \alpha) \in C^2(S) \) satisfying \( \varphi(x_0, \alpha_0) = u(x_0, \alpha_0) \) and \( \varphi(x, \alpha) \geq u(x, \alpha) \) for all \( x \) in a neighborhood of \( x_0 \) and each \( \alpha \in \mathcal{M} \), we have
\[
\max \{(\mathcal{L} - r)\varphi(x_0, \alpha_0), f(x_0, \alpha_0) - \varphi'(x_0, \alpha_0)\} \geq 0.
\]

The function \( u \) is said to be a **viscosity solution** of (2.1), if it is both a viscosity subsolution and a viscosity supersolution.

**Theorem 4.9.** Assume the conditions of Theorem 4.4. Then the value function \( V \) is a viscosity solution of the coupled system of quasi-variational inequalities (2.1) with boundary condition (4.7).
Proof. The proof is motivated by [5] and [8, Theorem VIII 5.1]. We divide the proof into two parts. The first part shows that \( V \) is a viscosity subsolution of (2.1), while the second part establishes that \( V \) is viscosity supersolution of (2.1).

Step 1. We show that \( V \) is a viscosity subsolution of (2.1). That is, for any \((x_0, \alpha_0) \in S \times M\) and any \(C^2\) function \(\phi(\cdot, \cdot)\) satisfying \(\phi(x_0, \alpha_0) = V(x_0, \alpha_0)\) and that \(\phi(x, \alpha) \leq V(x, \alpha)\) for all \(x\) in a neighborhood of \(x_0\) and each \(\alpha \in M\), we have

\[
\max \{(\mathcal{L} - r)\phi(x_0, \alpha_0), f(x_0, \alpha_0) - \phi'(x_0, \alpha_0)\} \leq 0. \tag{4.12}
\]

Let \(B_\varepsilon(x_0) := \{x \in \mathbb{R} : |x - x_0| < \varepsilon\}\), where \(\varepsilon > 0\) is sufficiently small so that (i) \(\overline{B}_\varepsilon(x_0) \subset S\) and (ii) \(\phi(x, \alpha) \leq V(x, \alpha)\) for all \((x, \alpha) \in \overline{B}_\varepsilon(x_0) \times M\), where \(\overline{B}_\varepsilon(x_0) = \{x \in \mathbb{R} : |x - x_0| \leq \varepsilon\}\) denotes the closure of \(B_\varepsilon(x_0)\). Choose \(Z \in \mathcal{A}\) such that \(Z(0-) = 0\) and \(Z(t) = \eta\) for all \(t \geq 0\), where \(0 \leq \eta < \varepsilon\). Let \(\hat{X}(\cdot) = \hat{X}_{x_0,\alpha_0}(\cdot)\) be the corresponding harvested process with initial condition \((x_0, \alpha_0)\) and harvesting strategy \(Z(\cdot)\). Put

\[
\theta := \inf \left\{ t \geq 0 : \hat{X}(t) \notin B_\varepsilon(x_0) \right\}.
\]

Note that the chosen harvesting strategy \(Z\) guarantees that \(\hat{X}(\cdot)\) has at most one jump at \(t = 0\) and remains continuous on \([0, \theta]\). This, together with the choice of \(\varepsilon\), implies that \(\theta \leq \tau\) and that \(\hat{X}(t) \in \overline{B}_\varepsilon(x_0)\) for all \(0 \leq t \leq \theta\). By virtue of the dynamic programming principle [11,9], for any \(h > 0\), we have

\[
\phi(x_0, \alpha_0) = V(x_0, \alpha_0)
\geq \mathbb{E} \left[ \int_0^{\theta \wedge h} e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) + e^{-r(\theta \wedge h)} V(\hat{X}(\theta \wedge h), \alpha(\theta \wedge h)) \right]
\geq \mathbb{E} \left[ \int_0^{\theta \wedge h} e^{-rs} f(\hat{X}(s-), \alpha(s-))dZ(s) + e^{-r(\theta \wedge h)} \phi(\hat{X}(\theta \wedge h), \alpha(\theta \wedge h)) \right]. \tag{4.13}
\]

Applying the generalized Itô formula to the process \(e^{-rs} \phi(\hat{X}(s), \alpha(s))\), we obtain

\[
e^{-r(\theta \wedge h)} \phi(\hat{X}(\theta \wedge h), \alpha(\theta \wedge h)) - \phi(x_0, \alpha_0)
= \int_0^{\theta \wedge h} e^{-rs} (\mathcal{L} - r) \phi(\hat{X}(s), \alpha(s))ds + \int_0^{\theta \wedge h} e^{-rs} \phi'(\hat{X}(s), \alpha(s))\sigma(\hat{X}(s), \alpha(s))dw(s)
- \int_0^{\theta \wedge h} e^{-rs} \phi'(\hat{X}(s), \alpha(s))dZ^c(s) + \sum_{0 \leq s \leq \theta \wedge h} e^{-rs} \left[ \phi(\hat{X}(s), \alpha(s-)) - \phi(\hat{X}(s-), \alpha(s-)) \right].
\]

Since \(\phi \in C^2\), \(\sigma\) is continuous, and \(\hat{X}(s) \in \overline{B}_\varepsilon(x_0)\) for all \(0 \leq s \leq \theta\), it follows that

\[
\mathbb{E} \int_0^{\theta \wedge h} e^{-rs} \phi'(\hat{X}(s), \alpha(s))\sigma(\hat{X}(s), \alpha(s))dw(s) = 0.
\]
Consequently, we have
\[
\phi(x_0, \alpha_0) = E e^{-r(\theta \wedge h)} \phi(\hat{X}(\theta \wedge h), \alpha(\theta \wedge h)) - E \int_0^{\theta \wedge h} e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))ds
\]
\[
+ E \int_0^{\theta \wedge h} e^{-rs} \phi'(\hat{X}(s), \alpha(s))dZ^c(s)
\]
\[
- E \sum_{0 \leq s \leq \theta \wedge h} e^{-rs} \left[ \phi(\hat{X}(s), \alpha(s)) - \phi(\hat{X}(s), \alpha(s)) \right].
\]

A combination of (4.13) and (4.14) leads to
\[
0 \geq E \int_0^{\theta \wedge h} e^{-rs} f(\hat{X}(s), \alpha(s))dZ(s) + E \int_0^{\theta \wedge h} e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))ds
\]
\[
- E \int_0^{\theta \wedge h} e^{-rs} \phi'(\hat{X}(s), \alpha(s))dZ^c(s)
\]
\[
+ E \sum_{0 \leq s \leq \theta \wedge h} e^{-rs} \left[ \phi(\hat{X}(s), \alpha(s)) - \phi(\hat{X}(s), \alpha(s)) \right].
\]

Now let \( \eta = 0 \), i.e., \( Z(t) \equiv 0 \) for any \( t \geq 0 \). Then (4.15) can be rewritten as
\[
0 \geq E \int_0^{\theta \wedge h} e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))ds
\]
\[
= E \int_0^h e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))I_{\{s \leq \theta\}}ds.
\]

Note that \( e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))I_{\{s \leq \theta\}} \) is bounded for all \( 0 \leq s \leq h \) by the definition of \( \theta \).

Hence there exists some \( 0 \leq \xi^h \leq h \) such that
\[
0 \geq E \int_0^h e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))I_{\{s \leq \theta\}}ds \geq hE[e^{-r\xi^h}(\mathcal{L} - r)\phi(\hat{X}(\xi^h), \alpha(\xi^h))I_{\{\xi^h \leq \theta\}}].
\]

Note that as \( h \downarrow 0 \), \( \xi^h \downarrow 0 \). This implies that \( (\hat{X}(\xi^h), \alpha(\xi^h)) \rightarrow (x_0, \alpha_0) \) a.s. due to the choice of \( Z \equiv 0 \). With the continuity of \( (\mathcal{L} - r)\phi \), we conclude that
\[
e^{-r\xi^h}(\mathcal{L} - r)\phi(\hat{X}(\xi^h), \alpha(\xi^h))I_{\{\xi^h \leq \theta\}} \rightarrow (\mathcal{L} - r)\phi(x_0, \alpha_0), \text{ a.s. as } h \downarrow 0.
\]

Therefore it follows from the bounded convergence theorem that
\[
(\mathcal{L} - r)\phi(x_0, \alpha_0) \leq 0.
\]

On the other hand, if we choose \( 0 < \eta < \varepsilon \), then (4.15) reduces to
\[
E \int_0^{\theta \wedge h} e^{-rs}(\mathcal{L} - r)\phi(\hat{X}(s), \alpha(s))ds + f(x_0, \alpha_0)\eta + \phi(x_0 - \eta, \alpha_0) - \phi(x_0, \alpha_0) \leq 0.
\]

Now sending \( h \downarrow 0 \), we have
\[
f(x_0, \alpha_0)\eta + \phi(x_0 - \eta, \alpha_0) - \phi(x_0, \alpha_0) \leq 0.
\]
Finally, divide the above inequality by $\eta$ and let $\eta \to 0$, it follows that
\[ f(x_0, \alpha_0) - \phi'(x_0, \alpha_0) \leq 0. \tag{4.17} \]
Now (4.12) follows from a combination of (4.16) and (4.17).

Step 2. We need to show that $V$ is also a viscosity supersolution of (2.1). That is, for any $(x_0, \alpha_0) \in S \times \mathcal{M}$ and any $\varphi \in C^2$ such that $\varphi(x_0, \alpha_0) = V(x_0, \alpha_0)$ and that $\varphi(x_0, \alpha_0) \geq V(x_0, \alpha_0)$ for $x$ in a neighborhood of $x_0$ and each $\alpha \in \mathcal{M}$, we have
\[ \max \left\{ (L - r)\varphi(x_0, \alpha_0), f(x_0, \alpha_0) - \varphi'(x_0, \alpha_0) \right\} \geq 0. \tag{4.18} \]
Suppose on the contrary that (4.18) was wrong, then there would exist some $(x_0, \alpha_0) \in S \times \mathcal{M}$, a $\varphi \in C^2$, and a constant $A > 0$ such that
\[ \max \left\{ (L - r)\varphi(x_0, \alpha_0), f(x_0, \alpha_0) - \varphi'(x_0, \alpha_0) \right\} \leq -2A < 0. \tag{4.19} \]

In what follows, we will derive a contradiction to (4.19). This is achieved in several steps. First we use the generalized Itô formula and (4.20) to obtain (4.21). Next, detailed analysis using the monotonicity of the functions $V$ and $f$ leads to (4.21). Then we claim in (4.27) that the last term in (4.26) is bounded below by a positive constant, from which, with the aid of dynamic programming (1.9), we obtain a contradiction to (4.19). The final step of the proof is devoted to the proof of (4.27).

Fix some $Z \in \mathcal{A}$ and let $\hat{X}(\cdot) = \hat{X}_{x_0,\alpha_0}(\cdot)$ be the corresponding harvested process. Define $B_\varepsilon(x_0)$ as in Step 1, where $\varepsilon > 0$ is small enough so that (i) $\overline{B_\varepsilon}(x_0) \subset S$, (ii) $\varphi(x, \alpha) \geq V(x, \alpha)$ for all $(x, \alpha) \in \overline{B_\varepsilon}(x_0) \times \mathcal{M}$, and (iii)
\[ \max \left\{ (L - r)\varphi(x, \alpha), f(x, \alpha) - \varphi'(x, \alpha) \right\} \leq -A < 0, \quad (x, \alpha) \in \overline{B_\varepsilon}(x_0) \times \mathcal{M}. \tag{4.20} \]

Let $\theta := \inf \left\{ t \geq 0 : \hat{X}(t) \notin B_\varepsilon(x_0) \right\}$. Then $\theta \leq \tau$. It follows from the generalized Itô formula that
\begin{align*}
E e^{-r\theta} &\varphi(\hat{X}(\theta^-), \alpha(\theta^-)) - \varphi(x_0, \alpha_0) \\
&= E \int_0^{\theta^-} e^{-rs}(L - r)\varphi(\hat{X}(s), \alpha(s))ds - E \int_0^{\theta^-} e^{-rs} \varphi'(\hat{X}(s), \alpha(s))dZ^c(s) \\
&\quad + E \sum_{0 \leq s < \theta} e^{-rs} \left[ \varphi(\hat{X}(s), \alpha(s-)) - \varphi(\hat{X}(s^-), \alpha(s-)) \right].
\end{align*}
Note that
\begin{align*}
\varphi(\hat{X}(s), \alpha(s-)) - \varphi(\hat{X}(s^-), \alpha(s-)) \\
&= (\hat{X}(s) - \hat{X}(s^-))\varphi'(\hat{X}(s^-) + z(\hat{X}(s) - \hat{X}(s^-)), \alpha(s-)) \\
&= -\Delta Z(s)\varphi'(\hat{X}(s^-) + z(\hat{X}(s) - \hat{X}(s^-)), \alpha(s))
\end{align*}
for some $z \in [0, 1]$. But by virtue of (4.20), for all $0 \leq s < \theta$, we have
\[-\varphi'(\hat{X}(s^-) + z(\hat{X}(s) - \hat{X}(s^-)), \alpha(s-)) \leq -f(\hat{X}(s^-) + z(\hat{X}(s) - \hat{X}(s^-)), \alpha(s-)) - A.
\]
Further, since \( \hat{X}(s) \leq \hat{X}(s-)+z(\hat{X}(s)-\hat{X}(s-)) \) and \( f(\cdot, \alpha) \) is non-increasing, we have
\[
-f(\hat{X}(s-)) + z(\hat{X}(s)-\hat{X}(s-)), \alpha(s-)) \leq -f(\hat{X}(s-), \alpha(s-)).
\]
Hence it follows from (4.20) that
\[
Ee^{-r\theta}\varphi(\hat{X}(\theta-), \alpha(\theta-)) - \varphi(x_0, \alpha_0) \\
\leq E \int_0^{\theta-} e^{-rs}(-A)ds + E \int_0^{\theta-} e^{-rs}(-f(\hat{X}(s), \alpha(s)) - A)\Delta Z(s) \\
+ E \sum_{0 \leq s < \theta} e^{-rs}(-f(\hat{X}(s), \alpha(s)) - A)\Delta Z(s) \\
= -E \int_0^{\theta-} e^{-rs}f(\hat{X}(s), \alpha(s))dZ(s) - AE \int_0^{\theta-} e^{-rs}ds - AE \int_0^{\theta-} e^{-rs}dZ(s).
\]
Therefore
\[
\varphi(x_0, \alpha_0) \geq Ee^{-r\theta}\varphi(\hat{X}(\theta-), \alpha(\theta-)) + E \int_0^{\theta-} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) \\
+ AE \left[ \int_0^{\theta} e^{-rs}ds + \int_0^{\theta-} e^{-rs}dZ(s) \right].
\]
(4.21)
Note that \( \hat{X}(\theta) \leq \hat{X}(\theta-) \) and \( \hat{X}(\theta-) \in B_{\varepsilon}(x_0). \) Thus there exists some \( \lambda \in [0, 1] \) such that
\[
x_\lambda := \hat{X}(\theta-)+\lambda(\hat{X}(\theta)-\hat{X}(\theta-)) = \hat{X}(\theta)-\lambda\Delta Z(\theta) \in \partial B_{\varepsilon}(x_0).
\]
Moreover, \( \hat{X}(\theta) \leq x_\lambda \leq \hat{X}(\theta-). \) Note that
\[
\varphi(\hat{X}(\theta-), \alpha(\theta-)) - \varphi(x_\lambda, \alpha(\theta-)) = (\hat{X}(\theta-)-x_\lambda)\varphi'(\hat{X}(\theta-)+z(\hat{X}(\theta)-x_\lambda), \alpha(\theta-)) \\
= \lambda\Delta Z(\theta)\varphi'(\hat{X}(\theta-)+z(\hat{X}(\theta)-x_\lambda), \alpha(\theta-)).
\]
But (4.20) and the monotonicity of \( f(\cdot, \alpha) \) imply that
\[
\varphi'(\hat{X}(\theta-)+z(\hat{X}(\theta)-x_\lambda), \alpha(\theta-)) \geq f(\hat{X}(\theta-)+z(\hat{X}(\theta)-x_\lambda), \alpha(\theta-)) + A \\
\geq f(\hat{X}(\theta-), \alpha(\theta-)) + A.
\]
This, together with the fact that \( \Delta Z(\theta) \geq 0 \), leads to
\[
\varphi(\hat{X}(\theta-), \alpha(\theta-)) - \varphi(x_\lambda, \alpha(\theta-)) \geq \lambda\Delta Z(\theta) \left[ f(\hat{X}(\theta-), \alpha(\theta-)) + A \right].
\]
(4.22)
Combining (4.21) and (4.22), we obtain
\[
V(x_0, \alpha_0) = \varphi(x_0, \alpha_0) \\
\geq E \int_0^{\theta-} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) + E e^{-r\theta}\varphi(x_\lambda, \alpha(\theta-)) \\
+ AE \left[ \int_0^{\theta} e^{-rs}ds + \int_0^{\theta-} e^{-rs}dZ(s) \right] + \lambda E e^{-r\theta} \Delta Z(\theta) \left[ f(\hat{X}(\theta-), \alpha(\theta-)) + A \right].
\]
(4.23)
Note that \( x_\lambda \in \overline{B}_x(x_0) \). Therefore we have \( \varphi(x_\lambda, \alpha(\theta-)) \geq V(x_\lambda, \alpha(\theta-)) \). On the other hand, since \( \hat{X}(\theta) \leq x_\lambda \), it follows from (4.1) that

\[
V(x_\lambda, \alpha(\theta-)) \geq V(\hat{X}(\theta), \alpha(\theta-)) + [x_\lambda - \hat{X}(\theta)]f(x_\lambda, \alpha(\theta-)) \\
\geq V(\hat{X}(\theta), \alpha(\theta-)) + (1 - \lambda)\Delta Z(\theta)f(\hat{X}(\theta-), \alpha(\theta-)).
\]

(4.24)

Note that

\[
\mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta-)) = \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta)).
\]

(4.25)

In fact, by virtue of [35, Theorem 2.12], \( \alpha(\cdot) \) is continuous in mean square. Hence it follows that

\[
\left| \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta-)) - \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta)) \right|
\]

\[
\leq \left| \mathbf{E}e^{-r\theta}[V(\hat{X}(\theta), \alpha(\theta)) - V(\hat{X}(\theta), \alpha(\theta-))]I_{\{\alpha(\theta) \neq \alpha(\theta-)\}} \right|
\]

\[
\leq KP \{\alpha(\theta) \neq \alpha(\theta-)\} = 0,
\]

where \( K \) is some positive constant. Therefore (4.25) follows. Now put (4.24) and (4.25) into (4.23) and we obtain

\[
V(x_0, \alpha_0) \geq \mathbf{E} \int_0^{\theta-} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) + \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta))
\]

\[
+ A\mathbf{E} \left[ \int_0^\theta e^{-rs}ds + \int_0^{\theta-} e^{-rs}dZ(s) \right] + (1 - \lambda)\mathbf{E}e^{-r\theta}\Delta Z(\theta)f(\hat{X}(\theta-), \alpha(\theta-))
\]

\[
+ \lambda\mathbf{E}e^{-r\theta}\Delta Z(\theta) \left[ f(\hat{X}(\theta-), \alpha(\theta-)) + A \right]
\]

\[
= \mathbf{E} \int_0^{\theta} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) + \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta))
\]

\[
+ A\mathbf{E} \left[ \int_0^\theta e^{-rs}ds + \int_0^{\theta-} e^{-rs}dZ(s) + \lambda e^{-r\theta}\Delta Z(\theta) \right].
\]

(4.26)

We now claim that for some constant \( \kappa > 0 \), we have

\[
\mathbf{E} \left[ \int_0^\theta e^{-rs}ds + \int_0^{\theta-} e^{-rs}dZ(s) + \lambda e^{-r\theta}\Delta Z(\theta) \right] \geq \kappa.
\]

(4.27)

Assume (4.27) for the moment. Then (4.26) can be rewritten as

\[
V(x_0, \alpha_0) \geq \mathbf{E} \int_0^{\theta} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) + \mathbf{E}e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta)) + A\kappa.
\]

(4.28)

Taking supremum over \( Z \in \mathcal{A} \), it follows that

\[
V(x_0, \alpha_0) \geq \sup_{Z \in \mathcal{A}} \mathbf{E} \left[ \int_0^{\theta} e^{-rs}f(\hat{X}(s-), \alpha(s-))dZ(s) + e^{-r\theta}V(\hat{X}(\theta), \alpha(\theta)) \right] + A\kappa.
\]

(4.29)
But in view of the dynamic programming principle (1.9), (4.29) can be rewritten as

\[ V(x_0, \alpha_0) \geq V(x_0, \alpha_0) + A\kappa > V(x_0, \alpha_0). \]

This is a contradiction. So we must have (4.18) and hence (4.27). Now it remains to show (4.27). To this end, we consider the function \( \tilde{W}(x, \alpha) := |x - x_0|^2 - \varepsilon^2 \) for \((x, \alpha) \in B_\varepsilon(x_0) \times \mathcal{M}\). Then it follows that

\[
(\mathcal{L} - r)\tilde{W}(x, \alpha) = 2(x - x_0)b(x, \alpha) + \frac{1}{2}2\sigma^2(x, \alpha) - r(|x - x_0|^2 - \varepsilon^2).
\]

Since \( \tilde{W}, b, \) and \( \sigma \) are continuous, and \( \mathcal{M} \) is a finite set, it is obvious that

\[
| (\mathcal{L} - r)\tilde{W}(x, \alpha) | \leq K < \infty
\]

for some positive constant \( K \). Now let \( K_0 := \frac{1}{2\varepsilon + K} \) and define \( W(x, \alpha) = K_0\tilde{W}(x, \alpha) \) for \((x, \alpha) \in B_\varepsilon(x_0) \times \mathcal{M}\). Then it follows immediately that

\[
|(\mathcal{L} - r)W(x, \alpha)| < 1, \quad (x, \alpha) \in B_\varepsilon(x_0) \times \mathcal{M}. \tag{4.30}
\]

Moreover, we have

\[
W'(x, \alpha) = 2K_0(x - x_0) \geq -1. \tag{4.31}
\]

Now apply the generalized Itô formula to \( e^{-rs}W(\hat{X}(s), \alpha(s)) \),

\[
\mathbb{E}[e^{-r\theta}W(\hat{X}(\theta-), \alpha(\theta-))] - W(x_0, \alpha_0)
\]

\[
= \mathbb{E} \int_0^\theta e^{-rs}(\mathcal{L} - r)W(\hat{X}(s), \alpha(s)) \, ds - \mathbb{E} \int_0^\theta e^{-rs}W'(\hat{X}(s), \alpha(s)) \, ds \tag{4.32}
\]

\[
+ \mathbb{E} \sum_{0 \leq s < \theta} e^{-rs}[W(\hat{X}(s), \alpha(s-)) - W(\hat{X}(s-), \alpha(s-))].
\]

But by virtue of (4.31), we have

\[
W(\hat{X}(s), \alpha(s-)) - W(\hat{X}(s-), \alpha(s-))
\]

\[
= W'(\hat{X}(s-)) + z(\hat{X}(s) - \hat{X}(s-)), \alpha(s))(\hat{X}(s) - \hat{X}(s-))
\]

\[
= W'(\hat{X}(s-)) + z(\hat{X}(s) - \hat{X}(s-)), \alpha(s))\Delta Z(s)
\]

\[
\leq -\Delta Z(s). \tag{4.33}
\]

Hence it follows from (4.30) – (4.33) that

\[
\mathbb{E}[e^{-r\theta}W(\hat{X}(\theta-), \alpha(\theta-))] - W(x_0, \alpha_0)
\]

\[
\leq \mathbb{E} \int_0^\theta e^{-rs} \, ds + \mathbb{E} \int_0^\theta e^{-rs} \, ds \tag{4.34}
\]

\[
+ \mathbb{E} \sum_{0 \leq s < \theta} e^{-rs} \Delta Z(s)
\]

\[
= \mathbb{E} \int_0^\theta e^{-rs} \, ds + \mathbb{E} \int_0^\theta e^{-rs} \, ds.
\]
Also, recall that \( X(\theta) \leq x_\lambda \leq \hat{X}(\theta^-) \). It follows from (4.31) that

\[
W(\hat{X}(\theta^-), \alpha(\theta^-)) - W(x_\lambda, \alpha(\theta^-)) = W'(x_\lambda + z(\hat{X}(\theta^-) - x_\lambda), \alpha(\theta^-)) \left[ \hat{X}(\theta^-) - x_\lambda \right] \\
= \lambda W'(x_\lambda + z(\hat{X}(\theta^-) - x_\lambda), \alpha(\theta^-)) \Delta Z(\theta) \\
\geq -\lambda \Delta Z(\theta).
\]

Combining (4.34) and (4.35), we have

\[
E \int_0^\theta e^{-rs} ds + E \int_0^{\theta-} e^{-rs} dZ(s) + \lambda E e^{-r\theta} \Delta Z(\theta) \geq E e^{-r\theta} W(x_\lambda, \alpha(\theta^-)) - W(x_0, \alpha_0).
\]

But \( x_\lambda \in \partial B_\varepsilon(x_0) \). Consequently, \( W(x_\lambda, \alpha(\theta^-)) = 0 \). Also, it is immediate that \( W(x_0, \alpha_0) = -K_0 \varepsilon^2 \). Hence it follows that

\[
E \int_0^\theta e^{-rs} ds + E \int_0^{\theta-} e^{-rs} dZ(s) + \lambda E e^{-r\theta} \Delta Z(\theta) \geq K_0 \varepsilon^2 = \kappa > 0.
\]

This establishes (4.27) and hence finishes the proof of the theorem. \( \square \)

5 Conclusions and Remarks

In this work, we considered the optimal harvesting problem for a single species living in random environments. We first established a verification theorem, based on which we explicitly constructed an \( \varepsilon \)-optimal harvesting strategy under additional conditions. Next we obtained the continuity of the value function and further characterized it as a viscosity solution of the coupled system of quasi-variational inequalities (2.1).

In examples 3.1 (cases 1 and 2) and 3.3, thanks to the special structures of the harvesting and continuation regions, we were able to obtain the value functions and \( (\varepsilon-) \)-optimal harvesting policies. It will be very interesting to investigate whether \( (\varepsilon-) \)-optimal harvesting policies exist in more general settings.

The next logical step is to consider optimal harvesting strategy for multiple but finite number of interacting species in random environments. For virtually all ecosystems, the species often interact with each other and a small change of one population may have significant effects on other populations. Therefore to apply the mathematical findings in real worlds, one must consider the interactions among the species in the ecosystem. It seems that some results of this paper can be extended to multiple interacting species. For example, a verification theorem like Theorem 2.1 can be established using almost the same argument. But one may no longer get closed-form value functions and optimal controls by solving the corresponding quasi-variational inequalities. Also, we may be able to show that the value function is a viscosity solution of the quasi-variational inequalities but it is not immediate to identify condition(s) under which the value function is continuous.
In view of [11, Section 4], within the same framework and the same optimality criterion considered in this paper, we may consider a more general problem where the controlled state process is given by

\[ d\hat{X}(t) = b(\hat{X}(t), \alpha(t))dt + \sigma(\hat{X}(t), \alpha(t))dw(t) - \gamma(\hat{X}(t^-), \alpha(t^-))dZ(t), \]

where \( \hat{X}, Z \in \mathbb{R}^n \), \( b, \sigma, \gamma \) are suitable functions with appropriate dimensions, and \( w \) is an \( n \)-dimensional standard Brownian motion. Since every process of bounded variation can be written as a difference of two nondecreasing processes, the control space can be enlarged by allowing the singular control \( Z \) to be an adapted process with bounded variation.

A number of other questions deserve further investigations. In particular, in many practical situations, it is virtually impossible to obtain the explicit form of the value function and an optimal control by solving (2.1). Therefore a viable alternative is to employ numerical approximations. The controlled Markov chain approximation method developed in [16] seems promising. We may also consider relaxed control, under which we may achieve the optimal value with an optimal control. Another problem of great interests is to consider the case when the random environment or the Markov chain \( \alpha \) is unobservable.

References

[1] L.H.R. Alvarez. Singular stochastic control in the presence of a state-dependent yield structure. Stochastic Process. Appl., 86:323–343, 2000.

[2] L.H.R. Alvarez and L.A. Shepp. Optimal harvesting of stochastically fluctuating populations. J. Math. Biol., 37:155–177, 1998.

[3] L. Arnold, W. Horsthemke, and J.W. Stucki. The influence of external real and white noise on the Lotka-Volterra model. Biomedical J., 21:451–471, 1979.

[4] C.A. Brauman. Variable effort harvesting models in random environments: generalization to density-dependent noise intensities. Mathematical Biosciences, 177 & 178:229–245, 2002.

[5] T. Choulli, M. Taksar, and X.Y. Zhou. A diffusion model for optimal dividend distribution for a company with constraints on risk control. SIAM J. Control Optim., 41(6):1946–1979, 2003.

[6] J.E. Cohen, T. Luczak, C.M. Newman, and Z.-M. Zhou. Stochastic structure and nonlinear dynamics of food webs: qualitative stability in a Lotka-Volterra cascade model. Proc. R. Soc. Lond. B, 240:607–627, 1990.

[7] N.H. Du and V.H. Sam. Dynamics of a stochastic Lotka-Volterra model perturbed by white noise. J. Math. Anal. Appl., 324:82–97, 2006.

[8] W.H. Fleming and H.M. Soner. Controlled Markov Processes and Viscosity Solutions, volume 25 of Stochastic Modelling and Applied Probability. Springer-Verlag, New York, NY, second edition, 2006.

33
[9] X. Guo and Q. Zhang. Closed-form solutions for perpetual american put options with regime switching. *SIAM J. Appl. Math.*, 64:2034–2049, 2004.

[10] X. Guo and Q. Zhang. Optimal selling rules in a regime switching model. *IEEE Transactions on Automatic Control*, 50:1450–1455, 2005.

[11] U. Haussmann and W. Suo. Singular optimal stochastic controls I: Existence. *SIAM J. Control Optim.*, 33(3):916–936, 1995.

[12] U. Haussmann and W. Suo. Singular optimal stochastic controls II: Dynamic programming. *SIAM J. Control Optim.*, 33(3):937–959, 1995.

[13] C. Jeffries. Stability of predation ecosystem models. *Ecology*, 57:1321–1325, 1976.

[14] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, New York, second edition, 1991.

[15] R.Z. Khasminskii. *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands, 1980.

[16] H.J. Kushner and P. Dupuis. *Numerical Methods for Stochastic Control Problems in Continuous Time*, volume 24 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2001.

[17] R. Lande, S. Engen, and B. Sæther. Optimal harvesting of fluctuating populations with a risk of extinction. *The American Naturalist*, 145:728–745, 1995.

[18] E.M. Lungu and B. Øksendal. Optimal harvesting from a population in a stochastic crowded environment. *Math. Biosci.*, 145:47–75, 1997.

[19] E.M. Lungu and B. Øksendal. Optimal harvesting from interacting populations in a stochastic environment. *Bernoulli*, 7:527–539, 2001.

[20] Q. Luo and X. Mao. Stochastic population dynamics under regime switching. *J. Math. Anal. Appl.*, 334:69–84, 2007.

[21] J. Ma, Q.S. Song, J. Xu, and J. Zhang. Impulse control and optimal portfolio selection with general transaction cost. 2008. preprint.

[22] C.E. Medina-Reyna. Growth and emigration of white shrimp, Litopenaeus vannamei, in the Mar Muerto Lagoon, Southern Mexico. *Naga, The ICLARM Quarterly*, 24:30–34, 2001.

[23] R.A. Miller and K. Voltaire. A stochastic analysis of the three paradigm. *J. Econ. Dyn. Control*, 6:371–386, 1983.

[24] B. Øksendal. *Stochastic differential equations, An introduction with applications*. Springer-Verlag, Berlin, 6th edition, 2003.

[25] B. Øksendal and Agnès Sulem. Optimal consumption and portfolio with both fixed and proportional transaction costs. *SIAM J. Control Optim.*, 40:1765–1790, 2002.

[26] M.O. Otuma and I.I. Osakwe. Assessment of the reproductive performance and post weaning growth of crossbred goat in derived Guinea Savanna Zone. *Research Journal of Animal Sciences*, 2:87–91, 2008.
[27] J. Paulsen. Optimal dividend payouts for diffusions with solvency constraints. *Finance Stoch.*, 7:457–473, 2003.

[28] H. Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2009.

[29] D. Ryan and F.B. Hanson. Optimal harvesting of a logistic population in an environment with stochastic jumps. *J. Math. Biol.*, 24:259–277, 1986.

[30] N.F. Sayre. The genesis, history, and limits of carrying capacity. *Annals of the Association of American Geographers*, 98:120–134, 2008.

[31] M. Slatkin. The dynamics of a population in a Markovian environment. *Ecology*, 59:249–256, 1978.

[32] M. Taksar and X. Zeng. On maximizing CRRA utility in regime switching markets with random endowment. *SIAM J. Control Optim.*, 48:2984–3002, 2009/10.

[33] A. Weerasinghe and A. Mandelbaum. *Abandonment vs. blocking in many-server queues: asymptotic optimility in the QED regime*, 2009. preprint.

[34] L.M. Wein. Optimal control of a two station brownian network. *Math. Oper. Res.*, 15:215–242, 1990.

[35] G. Yin and C. Zhu. *Hybrid Switching Diffusions: Properties and Applications*, volume 63 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2010.

[36] J. Yong and X.Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*, volume 43 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, New York, 1999.

[37] X.Y. Zhou and G. Yin. Markowitzs mean variance portfolio selection with regime switching: A continuous-time model. *SIAM J. Control Optim.*, 42:1466–1482, 2003.

[38] C. Zhu and G. Yin. On competitive Lotka-Volterra model in random environments. *J. Math. Anal. Appl.*, 357:154–170, 2009.