On the synchrotron emission in kinetic simulations of runaway electrons in magnetic confinement fusion plasmas

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Abstract

Developing avoidance or mitigation strategies of runaway electrons (REs) in magnetic confinement fusion (MCF) plasmas is of crucial importance for the safe operation of ITER. In order to develop these strategies, an accurate diagnostic capability that allows good estimates of the RE distribution function in these plasmas is needed. Synchrotron radiation (SR) of RE in MCF, besides of being one of the main damping mechanisms for RE in the high energy relativistic regime, is routinely used in current MCF experiments to infer the parameters of RE energy and pitch angle distribution functions. In the present paper we address the long standing question about what are the relationships between different REs distribution functions and their corresponding synchrotron emission: full-orbit effects, information of the spectral and angular distribution of SR of each electron, and basic geometric optics of a camera. We study the spatial distribution of the SR on the poloidal plane, and the statistical properties of the expected value of the synchrotron spectra of REs. We observe a strong dependence of the synchrotron emission measured by the camera on the pitch angle distribution of runaways, namely we find that crescent shapes of the spatial distribution of the SR as measured by the camera relate to RE distributions with small pitch angles, while ellipse shapes relate to distributions of runaways with larger the pitch angles. A weak dependence of the synchrotron emission measured by the camera with the RE energy, value of the $q$-profile at the edge, and the chosen range of wavelengths is observed. Furthermore, we find that oversimplifying the angular dependence of the SR changes the shape of the synchrotron spectra, and overestimates its amplitude by approximately 20 times for avalanching runaways and by approximately 60 times for mono-energetic distributions of runaways

Keywords: runaway electrons, synchrotron radiation, kinetic simulations, synthetic camera diagnostic, full-orbit effects

(Some figures may appear in colour only in the online journal)

1. Introduction

Runaway electrons (REs), thermal electrons accelerated to relativistic energies during the rapid termination of a magnetic confinement fusion (MCF) plasma, pose a threat to ITER if they are not avoided or mitigated before they hit the wall, causing damage to plasma facing components [1, 2]. Various strategies to avoid or mitigate RE in MCF plasmas have been proposed, e.g. using resonant magnetic perturbations to...
deconfin RE before they reach high energies [3], or using either massive gas injection (MGI) or shattered pellet injection of high Z impurities to slow down RE through collisional drag and by enhancing synchrotron radiation (SR) losses of RE through pitch angle scattering driven by collisions [4–6]. An accurate diagnostic capability that allows good estimates of the RE parameters in MCF plasmas is needed to gain a better understanding of the underlying physics of the RE dynamics, as well as to guide the development of the avoidance and mitigation strategies.

SR of RE in MCF plasmas is important for two reasons: SR is one of the main damping mechanisms for RE in the high energy relativistic regime, limiting the maximum energy that RE can reach during a disruption [7, 8], and substantially reducing the RE rate for weak (near critical) E fields [9]. On the other hand, SR is routinely measured in current MCF experiments to infer the RE energy and pitch angle distribution function. The latter is done by interpreting the measured SR spectra and geometric features of the radiation spatial patterns seen by the visible and infrared cameras [10–16]. Most recently, SR was used to infer the characteristic energy and pitch angle of RE in the DIII-D tokamak when MGI was used as the mitigation mechanism [17, 18]. In these plasmas the RE parameters were obtained by fitting the measured SR spectra with the single-particle spectrum, that is, the spectrum of a single electron calculated with a single energy and pitch angle, and using characteristic parameters of the plasma as measured at the magnetic axis. In [19] the authors showed that using single-particle spectra overestimates the SR by orders of magnitude and this can be misleading when inferring the RE parameters, and in general one should iteratively use the SR spectrum of a guess distribution function for the RE until the best fit to the experimental data is found. This overestimation, that depending on the wavelength can reach several orders of magnitude, is due to assuming that all runaways emit as much SR as the most strongly emitting particle in the actual runaway distribution function. Later, the study of [20] went a step further by solving the Fokker–Planck equation to obtain the RE distribution function for a given set of plasma parameters, and then calculating the corresponding SR spectrum. Again, the authors found that the single-particle spectra can be misleading when used to infer the RE parameters of more realistic RE distribution functions. Importantly, the above studies did not include any information of the electrons’ orbits, thus ignoring confinement and collisionless pitch angle scattering that can substantially modify the SR spectra [21].

In the present paper we address the long standing question about what are the relationships between different REs distribution functions and their corresponding synchrotron emission simultaneously including: full-orbit effects, information of the spectral and angular distribution of SR of each electron, and basic geometric optics of a camera. We follow the full-orbit dynamics of ensembles of REs in the magnetic field model in [21] using the new Kinetic Orbit Runaway electrons Code (KORC) to generate synthetic data to calculate different aspects of their SR. First, we use mono-energy and mono-pitch angle distributions of runaways as initial conditions in our simulations to study the spatial distribution of the SR on the poloidal plane, and the statistical properties of the expected value of the synchrotron spectra of REs. Then, we find relations between the REs’ parameters and both the spatial distribution of the synchrotron emission and the synchrotron spectra as observed by a camera placed at the outer midplane plasma. Finally, we use these results to interpret the synchrotron emission for an avalanche RE distribution function. In our simulations we observe a strong dependence of the spatial distribution of the radiation on the pitch angle distribution of the runaways. Also, we find that the synchrotron spectra is very sensitive to over-simplifications of the angular distribution of the SR, dramatically changing its shape and amplitude.

The rest of the paper is organized as follows: in section 2 we present a brief summary of the theory of the SR used throughout the paper. In section 3 we describe the parameters used in our simulations. In section 4 we present the study of the relationship between various distribution functions of REs and their synchrotron emission on the poloidal plane and as measured by a camera. Finally, in section 5 we summarize our results and discuss their implications in the interpretation of experimental data. Details on the synthetic camera diagnostic are provided in the appendix.

2. Synchrotron radiation theory

In our full-orbit simulations of REs we calculate the total radiated power, the SR spectra, and the spectral and angular distribution of the radiation. The total instantaneous synchrotron radiated power of a relativistic electron moving in an arbitrary orbit with velocity \( \mathbf{v} \) is given by:

\[
P_T = \frac{e^2}{6\pi\epsilon_0 c^3} \gamma^4 \nu^4 \kappa^2,
\]

where \( \kappa \) is the instantaneous curvature of the electron orbit, \( \gamma = 1/\sqrt{1-v^2/c^2} \) is the relativistic factor, \( e \) is the magnitude of the electron charge, \( c \) is the speed of light, and \( \epsilon_0 \) is the vacuum permittivity. For a relativistic electron moving in an electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) field, the instantaneous curvature \( \kappa \) is given by:

\[
\kappa = \frac{e}{\gamma m_e c^3} |\mathbf{v} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B})|.
\]

In the case when \( \mathbf{E} \ll \mathbf{v} \times \mathbf{B} \), the curvature can be approximated as \( \kappa \approx e B \sin \theta / \gamma m_e c \), where \( \theta \) is the pitch angle of the electron, that is, the angle between the vectors \( \mathbf{v} \) and \( \mathbf{B} \).

The spectral distribution of the SR of relativistic electrons is given by [22]:

\[
P_\lambda(\lambda) = \frac{1}{\sqrt{3}} \frac{c e^3}{\epsilon_0 \lambda^3} \frac{m_e c^2}{\mathcal{E}} \int_{\lambda/\lambda_c}^\infty K_{5/3}(\eta) d\eta.
\]

Here, \( \mathcal{E} = \gamma m_e c^2 \) is the relativistic electron’s energy, \( K_{5/3}(\eta) \) is the modified Bessel function of the second kind of order 5/3, and \( \lambda_c \) is the wavelength at which the electron is radiating. The critical wavelength \( \lambda_c = 4\pi/(3\kappa c^3) \) is the wavelength
characterizing $P_R(\lambda)$, dividing the spectra into two parts of equal radiated power, that is, half the total power is radiated at wavelengths $\lambda > \lambda_c$, and the rest is radiated at wavelengths $\lambda < \lambda_c$. We should note that equation (3) is completely general and can be used for calculating the synchrotron spectrum of a relativistic electron moving in an arbitrary orbit with radius of curvature $1/\kappa$. In [24] an approximate expression for the spectral distribution of the SR of REs with small pitch angle in tokamaks was derived, and used in [13, 17, 19].

The most detailed level of description for the SR emitted by a relativistic electron is given by its spectral and full angular distribution, which in the case when the angle between the direction of emission and motion is small is given by:

$$P_R(\lambda, \psi, \chi) = \frac{ce^2}{\sqrt{3} \epsilon_0 \hbar \gamma^4}(1 + \gamma^2 \psi^2)^2$$

$$\times \left\{ \frac{\gamma^2 \psi^2}{1 + \gamma^2 \psi^2} K_{1/3}(\xi) \cos \left[ \frac{3}{2}(z + \frac{1}{3} \xi^2) \right] \right\}$$

$$- \frac{1}{2} K_{1/3}(\xi)(1 + z^2) \cos \left[ \frac{3}{2}(z + \frac{1}{3} \xi^2) \right]$$

$$+ K_{2/3}(\xi)z \sin \left[ \frac{3}{2}(z + \frac{1}{3} \xi^2) \right],$$

where $\xi = 2\pi(1/\gamma^2 + \psi^2)^{1/2}/3\lambda\kappa$, $z = \gamma\chi/\sqrt{1 + \gamma^2 \psi^2}$. $\psi$ is the angle between the direction of emission \(\hat{n}\) and the instantaneous orbital plane containing the tangent and normal vectors, that is, $\psi$ is the complementary angle to the angle between \(\hat{n}\) and the binormal vector defined below; $\chi$ is the angle between the projection of \(\hat{n}\) on the instantaneous orbital plane and the instantaneous direction of motion $\hat{v}$, respectively. The unit vector defining the direction of emission $\hat{n}$ points from the electron’s position towards where an observer measuring the radiation is. It is worth mentioning that equation (4) is obtained when going from equation (II.31) to equation (II.32) of [22]. In equation (4) it is assumed that the SR is emitted mainly along $\hat{v}$, that is, small $\psi$ and $\chi$, $K_{1/3}$ and $K_{2/3}$ are the modified Bessel functions of second kind of order $1/3$ and $2/3$, respectively. The instantaneous orbital plane of the electron is uniquely determined by the tangent vector \(\hat{T}\), which corresponds to the electron unit velocity $\hat{v} = \gamma/\gamma$, the normal vector $\hat{N}$, and the binormal vector $\hat{B} = \hat{T} \times \hat{N}$, which is perpendicular to the instantaneous orbital plane. For a relativistic electron moving in an arbitrary electric and magnetic field

$$\hat{B} = \frac{v \times \hat{v}}{|v \times \hat{v}|} = \frac{v \times (E + v \times B)}{|v \times (E + v \times B)|}$$

$P_R(\lambda, \psi, \chi)$ in equation (4) decreases exponentially as a function of $\psi$ through the function $\xi$, this due to $K_{1/3}(\xi)$ and $K_{2/3}(\xi)$. On the other hand equation (4) shows oscillations as a function of $\chi$ through the function $z$, and can become negative for large values of $\chi$ or $\psi$. In order to make an efficient search on the $\psi\chi$-plane where $P_R(\lambda, \psi, \chi)$ is positive and thus physically meaningful, we restrict our search to a rectangular domain containing the region of validity defined by the values [25]:

$$\psi_c = \left( \frac{3\kappa \lambda}{4\pi} \right)^{1/3},$$

and $\chi_c$, which is a solution of the equation:

$$\frac{\gamma^3}{3} \chi_c^3 + \gamma \chi_c - \frac{\pi}{3\xi} = 0.$$

Figure 1(a) shows an example of $P_R(\lambda, \psi, \chi)$ in the domain defined by $\psi_c$ and $\chi_c$ for a relativistic electron with energy $\mathcal{E} = 30$ MeV and pitch angle $\theta_0 = 10^\circ$ at the high-field side (HFS). From this figure we observe that the SR is emitted within an ellipse in the $\psi$ and $\chi$ plane with major and minor radii bounded by $\psi_c$ and $\chi_c$. This means that the radiation is emitted within an elliptic cone with its axis along $\hat{v}$.

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Previous studies where synchrotron emission has been used to diagnose REs [10, 11, 17, 26] have simplified the synchrotron angular distribution to either a $\delta$ function in space, that is, $P_R(\lambda) = P_R(\lambda) \cdot \delta(\psi) \cdot \delta(\chi)$, or to a circular cone with ‘natural aperture’ $\alpha = 1/\gamma$, that is, $P_R^{\text{ci}}(\lambda) = P_R(\lambda)/\Omega_n$, for $\psi^2 + \chi^2 \leq \alpha^2$, and $P_R^{\text{ci}}(\lambda) = 0$ otherwise. In the former case,
all the radiation $P_R(\lambda)$ of equation (3) is emitted along the velocity of the particle, while in the latter case $P_R(\lambda)$ is allowed to ‘spread’ uniformly within the solid angle subtended by $\alpha$, that is, $\Omega_\alpha = 2\pi (1 - \cos(\alpha))$. Throughout this paper, we will refer to $P_R^{ls}(\lambda)$ as the simplified model for the angular distribution of the SR. In figure 1(b) we show the corresponding $P_R^{ls}(\lambda)$ for the same values of figure 1(a).

3. Simulations setup

In order to study the relationship between the RE distribution functions and different aspects of their SR we have used KORC. This code is a parallel Fortran 95 code that follows large ensembles of REs in the full 6D phase space. KORC efficiently exploits the shared memory computational systems by using the hybrid open MP + MPI paradigm for parallelization, showing nearly ideal weak and strong scaling. KORC incorporates the Landau–Lifshitz formulation of the radiation reaction force [21], and Coulomb collisions of RE with the thermal plasma using the model of [3].

In all the simulations reported in this paper we have used the analytical field of [21], which in toroidal coordinates is given by:

$$B(r, \vartheta) = \frac{1}{1 + \eta \cos \vartheta} [B_0 \hat{\varsigma} + B_1(r) \hat{\vartheta}],$$

where $\eta = r / R_0$ is the aspect ratio, $B_1(r) = \eta B_0 / q(r)$ is the poloidal magnetic field. The safety factor is

$$q(r) = q_0 \left[ 1 + \frac{r^2}{\varepsilon^2} \right].$$

The constant $\varepsilon$ is obtained from the values of $q_0$ and $q(r)$ at the plasma edge $r = r_{edge}$. The coordinates $(r, \vartheta, \varsigma)$ are defined as $x = (R_0 + r \cos \vartheta) \sin \varsigma$, $y = (R_0 + r \cos \vartheta) \cos \varsigma$, and $z = r \sin \vartheta$, where $(x, y, z)$ are the Cartesian coordinates. In these coordinates, $r$ denotes the minor radius, $\vartheta$ the poloidal angle, and $\varsigma$ the toroidal angle. Note that in this right-handed toroidal coordinate system, the toroidal angle $\varsigma$ rotates clockwise, that is, it is anti-parallel to the azimuthal angle, $\phi = \pi / 2 - \varsigma$, of the standard cylindrical coordinate system. The magnitude of the magnetic field at the magnetic axis, located at $R_0 = 1.5$ m, is $B_0 = 2.1$ T. The safety factor $q_0 = 1$ at the magnetic axis and $q_{edge} = 3$ at the plasma edge, which is located at $r_{edge} = 0.5$ m. These are characteristic values of DIII-D plasmas. Note however that the direction of the poloidal magnetic field in equation (8) is opposite to that reported in [17] for DIII-D. As a result, in our simulations the REs drift inwards whereas in the studies in [17, 27] the REs drift outward.

Throughout the paper we use different distribution functions for the REs in the energy and pitch-angle space $f_{RE}(E, \theta)$. We use $5 \times 10^6$ computational particles uniformly distributed in a torus as the spatial distribution of the different $f_{RE}(E, \theta)$, see for example figures 2(c) or 12(a). The location of the runaway beam is specified in each section, and the initial radius of the beam is fixed at $r = 0.2$ m for all the simulations reported in this paper. The simulation time $t_{sim} \sim 10 \mu s$ is set so that the less energetic RE considered in our simulations undergo 30 poloidal turns. Because this time is much smaller than both the collisional time $\tau_{coll} = 4\pi \varepsilon_0 m_e^2 c^3 / (n_e e^4 \log \Lambda) \sim 10$ ms and the characteristic time for radiation losses $\tau_R = 6\pi \varepsilon_0 (m_e c^2) / (e^4 B^2) \sim 1$ s, we have turned off the radiation reaction force and collisions in KORC, so the energy is conserved in these simulations. For the cases of interest, extending the simulations beyond $t_{sim} \sim 10 \mu s$ will not lead to any modification.

4. Full-orbit effects on synchrotron emission of various RE distribution functions

In this section we study the collisionless pitch angle dispersion effects on the SR spectra emitted by various RE distribution functions. Previous studies using a full-orbit description of RE in toroidal plasmas [21, 27, 28] have shown that due to the variation of the magnetic field seen by runaways along their orbits, they experience collisionless pitch angle dispersion, even in the case where collisions or SR losses are not included. Because the SR of each electron strongly depends on its pitch angle, it is expected that the resulting synchrotron emission of different ensembles of REs will show non-trivial changes with respect to the results inferred from distributions that do not take into account collisionless pitch angle dispersion effects. The aim of this sections is to study these changes in detail.

4.1. Synchrotron emission of mono-energetic and mono-pitch angle RE distributions on the poloidal plane

We start our study of the collisionless pitch angle dispersion effects on SR emission by using mono-energetic and mono-pitch angle RE distributions as the initial condition of KORC simulations. The kinetic energies (i.e. not including the rest mass energy $m_e c^2$) of the simulated runaways are $E_0 = 10$ and $30$ MeV, and initial pitch angles of $\theta_0 = 5^\circ$, $10^\circ$, $15^\circ$, and $20^\circ$. This means that our initial distributions functions are delta functions in the energy and pitch angle $f_{RE}(E, \theta, t = 0) = \delta(\theta - \theta_0) \delta(E - E_0)$. The initial radial position of the runaway beam is $R = 1.475$ m and $R = 1.43$ m for RE with $E = 10$ MeV and $30$ MeV, respectively. In our simulations we evolve the REs by $t \sim 10 \mu s$, which is enough for reaching a steady state distribution function.

In figure 2(a) we show the spatial distribution of the total synchrotron radiated power $P_T$ of equation (1) for the ensemble of REs with $E = 30$ MeV and $\theta_0 = 10^\circ$. The intensity of the radiation is higher at the HFS and lower at the low-field side, and the spatial distribution shows an up–down symmetry. Figure 2(b) shows the spatial distribution of the radiated power $P_R(\lambda)$ of equation (3) integrated over $\lambda \in (100, 10,000)$ nm. This range of wavelengths encompasses the visible and a part of the infrared portions of the electromagnetic spectrum, usually used in experimental studies. The same qualitative features of $P_T$ are observed. Figure 2(c) shows the spatial distribution of runaways on the poloidal plane of the simulation of panels (a) and (b). For producing these figures we computed the histograms of each
we computed the histograms of each quantity using a grid of 75 bins. These same features of radiated power are observed in all the other simulations of initially mono-energetic and mono-pitch angle distributions of runaway electrons. For producing these figures we computed the histograms of each quantity using a grid of 75 × 75 bins.

quantity using a grid of 75 × 75 bins. These same features of \( P_T \) and the integrated SR power are observed in all the other simulations of initially mono-energetic and mono-pitch angle distributions of REs.

In figure 3 we show the comparison between the expected value of the SR spectra for different full orbit

\[
P_{\lambda}(\lambda) = \int \int P_{\lambda}(E, \theta) P_{\lambda}(\lambda, E, \theta) dE d\theta,
\]

and the so-called single-particle spectrum, namely, the synchrotron spectrum of equation (3) computed using the initial values for the energy and pitch angle of the runaways, and characteristic values for the magnetic field (taken at the magnetic axis). In this figure we only show the simulations with \( E_0 = 30 \text{ MeV} \) and \( \theta_0 = 5^\circ, 10^\circ, \) and \( 20^\circ \). The other simulations show similar results. Among the differences between \( P_{\lambda}(\lambda) \) and the corresponding single-particle spectra we observe that the maximum of \( P_{\lambda}(\lambda) \) tends to move towards smaller wavelengths, and its magnitude is larger in all cases. These changes in the shape of \( P_{\lambda}(\lambda) \) are particularly important because the REs’ parameters are usually inferred by fitting the experimentally measured synchrotron spectrum with the single-particle spectrum. In [19] the authors used pre-computed distribution functions for the runaways to show that \( P_{\lambda}(\lambda) \) can be very different from what is called the single-particle spectrum. This was also shown in [20] for distribution functions of runaways obtained from solving the Fokker-Plank equation with radiation losses and collisions in 0-D simulations, that is, not including spatial information. In our simulations any departure of \( P_{\lambda}(\lambda) \) from the single-particle spectra results from allowing the magnetic field to have a spatial dependence, which in turn translated into collisionless pitch angle dispersion. In figure 4(b) we show the full orbit, steady state distribution functions. We observe that as the value of the relative dispersion of the pitch angle \( \sigma_\theta/\mu_\theta \) (figure 4(a)) increases, the departure of \( P_{\lambda}(\lambda) \) from the single-particle spectra becomes larger; we measure this departure using the relative difference between the integrated power of the two spectra in the range of wavelengths \( \lambda \in (100, 10\,000) \text{ nm} \), this is shown on the right axis of figure 4(a) as \( \Delta P_{\lambda} \). Here \( \mu_\theta \)
and $\sigma_0$ are the mean and standard deviation of the full orbit $f_{\text{RE}}(\mathcal{E}, \theta)$.

### 4.2. Synchrotron emission of mono-energetic and mono-pitch angle RE distributions as measured by a camera

We now go a step further and calculate the spatial distribution and spectra of the SR as measured by a camera placed at the outer midplane plasma. In this calculation each pixel of the camera measures the SR integrated along the corresponding line of sight. To the best of our knowledge this calculation is the first of its kind, including the exact full-orbit dynamics of REs in toroidal magnetic fields and the basic geometric optics of a camera. In the appendix we describe in detail the setup of the camera in the simulations. For this calculation we have used the full orbit information of each electron in our simulations and two models for the angular distribution, namely, the full angular distribution $P_R(\lambda, \psi, \chi)$ in equation (4) and the simplified model for the angular distribution $P^{\Omega}_{R}(\lambda) = P_R(\lambda)/\Omega_k$. In figure 5 we show the spatial distribution of the integrated synchrotron emission of simulations with $E_0 = 30$ MeV and $\theta_0 = 5^\circ, 10^\circ,$ and $20^\circ$ calculated with $P_R(\lambda, \psi, \chi)$. We have integrated the radiation over the range of wavelengths $\lambda \in (100, 10 000)$ nm. No significant difference is observed if a visible or infrared filter is used for the SR. Using $P^{\Omega}_{R}(\lambda)$ for calculating the spatial distribution of the synchrotron emission yields to qualitatively similar results, showing the same spatial features, but having an intensity order of magnitude larger. Contrary to the spatial distribution of the synchrotron emission on the poloidal plane (see figure 2), the spatial distribution of the synchrotron emission seen by the camera shows a variety of different non-symmetric shapes, they transition from a crescent shape to an ellipse shape as the mean pitch angle increases. For distributions of REs with $E_0 < 30$ MeV and with pitch angles in the range $\theta_0 < 20^\circ$ we always observe crescent shapes. In addition to the different shapes of the radiation seen by the camera, we observe a shift of the bright regions towards the HFS as we increase the pitch angle, despite the actual spatial distribution of the runaways remain fairly symmetric and localized around the magnetic axis, see figure 2(c). This shift of the bright regions towards the HFS strongly depends on the pitch angle of the electrons, becoming larger as we increase the pitch angle; its dependence on energy is observed to be rather weak, increasing as we increase the energy only for $\theta_0 \geq 20^\circ$.

Finally, we calculate the SR spectra of the simulated distributions of runaways as measured by the camera. In this case we regard the camera as one big spectrometer, merging the information of all the pixels of the camera. This calculation can be done using only one or a small subset of pixels of the camera if needed. In figure 6 we show the spectra of simulated REs with $E_0 = 30$ MeV and various pitch angles. We calculate the spectra using both the full angular distribution $P_R(\lambda, \psi, \chi)$, and the simplified angular distribution $P^{\Omega}_{R}(\lambda)$. The spectra calculated using the full angular distribution $P_R(\lambda, \psi, \chi)$ shows the same features than the spectra of figure 3, namely, the amplitude of the spectra becomes larger and the maximum of the spectra shifts towards smaller wavelengths as the pitch angle increases. The differences between the spectra of $P_R(\lambda, \psi, \chi)$ and $P^{\Omega}_{R}(\lambda)$ are in their magnitude, being approximately sixty times larger when calculated using $P^{\Omega}_{R}(\lambda)$ than when using $P_R(\lambda, \psi, \chi)$, and in their shape, having the maximum of the spectra shifted towards larger wavelengths when using $P^{\Omega}_{R}(\lambda)$; these large differences may result in underestimating the RE density and pitch angles of the REs if $P^{\Omega}_{R}(\lambda)$ is used to interpret the experimental measurements. We have explored the case when the ‘natural aperture’ $\alpha$ of the cone defining the emission region of $P^{\Omega}_{R}(\lambda)$ becomes smaller than $1/\gamma$. In this case, the spatial distribution and the shape of the spectra of the synchrotron emission measured by the camera remains practically unchanged, but the amplitude of the synchrotron spectra becomes even larger than in the case where $\alpha = 1/\gamma$.
\[ E_{\parallel} = E_{\perp}/E_{\parallel}, \quad E_{\parallel} \text{ is the parallel electric field normalized to the critical electric field } E_{\parallel} = m_{e}c/(e\tau_{\text{coll}}), \text{ and } \gamma_{e} = \sqrt{3(Z_{\text{eff}} + 1)}/\pi \log \Lambda. \]

We use equation (11) as the initial condition of our simulations with \( n_{e} = 3.9 \times 10^{20} \text{ m}^{-3} \), which results in \( \tau_{\text{coll}} \sim 10 \text{ ms} \), \( E_{\parallel} = 0.15 \text{ V m}^{-1} \) and we consider \( Z_{\text{eff}} = 1 \) and \( Z_{\text{eff}} = 10 \) for simulating an hydrogenic plasma and a plasma with high concentration of impurities, respectively. We choose \( E_{\parallel} = 0.74 \text{ V m}^{-1} \) motivated by the use of the same value in section 4.B of [19], where DIII-D parameters of shot number 146 704 at \( t = 2290 \text{ ms} \) were used. Larger (smaller) values of \( E_{\parallel} \) result in narrower (wider) pitch-angle distributions and longer (shorter) tails of the energy distribution of avalanching runaways. As expected, different values of \( E_{\parallel} \) leading to different avalanche distributions modify the corresponding synchrotron emission. The initial radial position of the runaway beam is \( R = 1.37 \text{ m} \). In figure 7(a) we show the filled contours of \( f_{\text{RE}}(E, \theta) \) using equation (11) with \( Z_{\text{eff}} = 1 \); using \( Z_{\text{eff}} = 10 \) results in a wider distribution in pitch angle space at low energies \( \mathcal{E} \sim 10 \text{ MeV} \). Here, \( \mathcal{E} = c \sqrt{\frac{p^{2} + m_{e}^{2}c^{2}}{2m_{e}}} \text{ and } \theta = \arccos \gamma_{e}. \) We observe only small fluctuations for the difference between the analytical and the initial condition of our simulations, that is, \( \sqrt{(f_{\text{RE}} - f_{\text{sim}})^2} \sim 0.01 \), where \( f_{\text{sim}} \) is the sampled distribution function used as the initial condition of our simulations. We sample \( f_{\text{RE}}(p, \eta) \) using the Metropolis-Hastings algorithm. As shown in figure 7(b), the collisionless orbit-induced dynamics slightly modifies the distribution in the time scale of the computation. We infer a linear relation between the energy of the bulk of the distribution and the pitch angle given by \( \mathcal{E} \approx 10 \times \theta. \)

As for the case of the mono-energy and mono-pitch angle distributions, we first calculate the spatial distribution of the total and the integrated synchrotron radiated power for the avalanche distribution function. This is shown in figure 8 for
4.4. Synchrotron emission of avalanching RE as measured by a camera

Next, we compute the spatial distribution and the spectra of the SR as measured by a camera placed a the outer midplane plasma. For this calculations the parameters of the camera are the same as in section 4.2 and in the appendix. In figure 10 we show the spatial distribution of the integrated SR calculated using the full angular distribution $P^\text{full}_{R}(\lambda)$. We have integrated the radiation over the range of wavelengths $\lambda \in (100, 10000)$ nm. No significant difference is observed if a visible or infrared filter is used for the SR. Using the simplified angular distribution $P^{\text{eff}}_{R}(\lambda)$ results in similar features of the spatial distribution of the radiation. Consistent with the results of section 4.2, we observe the transition from a crescent to an ellipse shape for the spatial distribution of the radiation as we increase $Z_{\text{eff}}$, as we are effectively increasing the pitch angle of the bulk of the runaway distribution function.

The crescent shape of the spatial distribution of the SR observed in figures 10(a) and 5(a) results from the contribution of REs with small pitch angle that follow the winding of the magnetic field lines. In figure 11 we show the contribution of different toroidal sectors of the runaway beam to figure 10(a); as it can be seen, the larger contribution to the crescent shape of the SR spatial distribution comes from the toroidal sector with $\varphi \in (40^\circ, 70^\circ)$, where $\varphi$ is the toroidal angle as defined in figure 12(c). As the pitch angle of the runaways increases, their velocity vector is not longer pointing along the magnetic field lines, resulting in shapes similar to figures 10(b) and 5(c).

Finally, we calculate the spectra of the SR as measured by the camera, these are shown in figures 9(c)–(d). As for the simulations of section 4.2, we observe large differences between the spectra calculated using the two different angular distributions for the radiation, namely, the magnitude of the spectra calculated using $P^{\text{full}}_{R}(\lambda)$ is approximately twenty times larger than when using $P_{R}(\lambda, \psi, \chi)$, also the maximum of the spectra are shifted towards larger wavelengths in the case when $P^{\text{eff}}_{R}(\lambda)$ is used. As discussed before, this may result in underestimating the RE density and pitch angles of the REs if $P^{\text{eff}}_{R}(\lambda)$ is used to interpret the experimental measurements.

5. Discussion and conclusions

In this paper we have addressed the long standing question about what are the relationships between different REs distribution functions and their corresponding synchrotron emission including: full-orbit effects, information of the spectral and angular distribution of SR of each electron, and the basic geometric optics of a camera. We performed kinetic simulations of the full-orbit dynamics of different ensembles of REs to study in detail various aspects of their synchrotron emission.

In sections 4.1 and 4.2, we used mono-energetic and mono-pitch angle distribution functions as the initial conditions of the simulations. For these simulations we calculated the spatial distribution on the poloidal plane of the total and the integrated synchrotron radiated power, which show bright regions of radiation at the HFS and up–down symmetry. Then we compared the synchrotron spectra of the full orbit distributions of runaways with the so-called single-particle spectra, showing that full orbit effects and in particular collisionless pitch angle dispersion effects cause the former to depart from the single-particle spectra. These effects become more evident as the relative dispersion of the pitch angle $\sigma_\theta/\mu_\theta$ increases, see figures 3 and 4.
Then, we calculated the spatial distribution and spectra of the SR as measured by a camera placed at the outer midplane plasma. To the best of our knowledge this calculation is the first of its kind, including the exact full-orbit dynamics of REs in toroidal magnetic fields and the basic geometric optics of a camera. We used two models for the angular distribution of the SR, namely, the full spectral and angular distribution of equation (4), and a simplified model where the radiation is emitted isotropically within a circular cone with ‘natural aperture’ \( \alpha = 1/\gamma \). Using either model for the angular distribution we observed a rich variety of non-symmetric shapes for the spatial distribution of the radiation that strongly depend on the pitch angle distribution of the runaways, and weakly depend on the runaways energy distribution, value of the \( q \)-profile at the plasma edge, and the chosen range of wavelengths. We noticed a transition from a crescent shape to an ellipse shape as the mean pitch angle increases, see figure 5. On the other hand, we found that the magnitude of the synchrotron spectra measured by the camera is overestimated by approximately a factor of 60 when the angular distribution is oversimplified, and the shape is affected too, moving to larger wavelengths when we use the simplified angular distribution \( P_R^{\alpha} (\lambda) \), see figure 6. This may result in underestimating the RE density and pitch angles of the REs if \( P_R^{\alpha} (\lambda) \) is used to interpret the experimental measurements.

In sections 4.3 and 4.4 we repeated the analysis of previous sections for an avalanche RE distribution function. We studied the case of a hydrogenic plasma (\( Z_{\text{eff}} = 1 \)) and a plasma with a high content of impurities (\( Z_{\text{eff}} = 10 \)). We find that collisionless pitch angle dispersion modifies the initial distribution function (see figure 7), so that there exist a deviation of the pitch angle of the bulk distribution as function of the runaways’ energy, that is, \( \mathcal{E} \approx 10 \times \theta \). In this case we also observed a complex structure of the spatial distribution of the SR on the poloidal plane with a non-trivial relation to the spatial density of REs, see figure 8. As in the simulations of section 4.1, the synchrotron spectra of the full orbit
avalanche distributions depart from the analytical approximation, showing larger departures for the case of $Z_{\text{eff}} = 1$, see figures 9(a) and (b). On the other hand, the spatial distribution of the synchrotron emission measured by the camera in our simulations showed a transition from a crescent shape to an ellipse shape as we increased $Z_{\text{eff}}$, this due to the effective increase of the pitch angle of the bulk distribution as $Z_{\text{eff}}$ becomes larger, see figure 5. We expect that in longer time scales, especially in plasmas containing high-Z impurities, the collisionless pitch-angle dispersion will be modified by collisions. This is a problem that we plan to address in a future publication. Regarding the synchrotron spectra measured by the camera, similarly in the simulations of section 4.2, we found that its amplitude is overestimated by approximately a factor of 20 when $P^b_R(\lambda)$ is used, and its maximum is shifted to larger wavelengths with respect to the spectra of equation (4), see figures 9(c) and (d).

In addition, we investigated the effect of changing the value of the $q$-profile at the plasma edge on the synchrotron emission. Consistent with the results of [21], we observe smaller radial shifts of the spatial distribution of runaways as $q(\rho_{\text{edge}})$ decreases. When reducing the value from $q(\rho_{\text{edge}}) = 3$, the value used in all the above results, to $q(\rho_{\text{edge}}) = 2$ we observe a smaller radial shift towards the HFS of the spatial distribution of the SR measured by the camera, and a drop of the amplitude of the synchrotron spectra of up 5%. The main features of the spatial distribution of the synchrotron emission remain practically unchanged when changing $q(\rho_{\text{edge}})$, also a small shift of up to 100 nm of the maximum of the spectra is observed for $q(\rho_{\text{edge}}) = 2$ with respect to the cases with $q(\rho_{\text{edge}}) = 3$. In [31] a study was presented on the dependence of the synchrotron emission on the shape of the $q$-profile. This type of more detailed study will be the subject of a future publication.

These results shed some light into the relationship between a given runaway distribution function and its corresponding synchrotron emission in magnetic confinement plasmas. This might help to find better ways to interpret experimental measurements of SR to obtain better estimates of the RE parameters, and so help to both formulate better theoretical descriptions of the runaways in these plasmas, and to improve the mechanisms for avoiding and/or mitigating REs.

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Appendix. Setup of the synthetic camera diagnostic

The camera diagnostic in KORC consists of an array of pixels on a rectangular detector placed at $R_{sc}= (R_{sc}, Z_{sc})$, where in a cylindrical coordinate system with origin at the center of the tokamak, $R_{sc}$ is the cylindrical radial position of the camera, and $Z_{sc}$ the corresponding camera position along the $z$-axis. For simplicity we assume that the camera is placed along the $x$-axis of a Cartesian coordinate system, and that the radial camera position $R_{sc}$ defines the outer wall at the midplane plasma, too. The horizontal and vertical size of the camera detector determine the optics of the camera, that is, the horizontal and vertical angles of view of the camera. We assume
that the camera has a single lens located at $R_{sc}$, so that each pixel has a single line of sight that connects the center of each pixel to the center of the lens, and then extends into the plasma.

In figure 12(a) we show the setup of the synthetic camera placed at $R_{sc} = 2.4$ and $Z_{sc} = 0$. The size of the detector is $40 \text{ cm} \times 40 \text{ cm}$, and the pixel array is made of $75 \times 75$ pixels. The blue lines show the horizontal angle of view, while the green line shows the main line of sight, that is, the line of sight joining the center of the detector and the lens. With this setup the main line of sight of the camera starts outside the plasma and goes all the way through the plasma. In figure 12(c) we show the top view of the camera setup. In this figure the dotted lines show some lines of sight of different pixels of the camera. Another parameter of the camera is its focal length $f$, which is the distance between the lens and the center of the camera detector and is chosen to be $f = 50 \text{ cm}$. Finally, the incline of the camera $\theta$, which is the angle between the main line of sight (green line) and the solid horizontal red line in figure 12(c), can be used to aim the camera. We choose $\theta = 55^\circ$ for all the simulations in this work. In this way, the size of the detector, the focal length of the camera detector, the size of the detector, and the focal length of the camera determine the camera’s field of view (see figure 12(b)).

The frequency at which the camera can take snapshots is equal to or lower than the inverse of the time step used in a KORC simulation, with an exposure time that depends on how many snapshots are used to produce the final picture of the SR. Therefore each pixel of the camera measures the line integrated synchrotron emission over the whole exposure time.

On the other hand, in axisymmetric plasmas the electromagnetic fields and particles’ variables are independent of the cylindrical azimuthal angle $\phi$ (or the toroidal angle $\zeta$ in toroidal coordinates). Thus, any rigid rotation of the electron’s variables by an arbitrary angle in the azimuthal direction is a possible realization of an electron in the plasma. The above implies that an electron can be detected by more than one pixel of the camera. In the camera set-up of figure 12 the azimuthal angle $\phi$ is measured anticlockwise. A potential complication is that for every snapshot taken by the camera the radiation spectra would have to be calculated for each electron of the simulation—in a typical KORC simulation we simultaneously follow hundreds of thousands ($\sim 10^6$) of REs. It can be seen that for an array of $100 \times 100$ pixels the number of computations involved is larger than $10^5$, increasing quadratically with the number of pixels of the camera detector. This computation can become computationally costly if it is not done in an efficient way. In order to reduce the number of computations involved, we pre-select those runaways that are more likely to be seen by the camera.

The pre-selection of the electrons is done as follows: for each electron with velocity $v_i$ and position $R_i$, we extend $v_i$ and calculate $R_i^\alpha = (R_{sc}, Z_{sc})$, the point at which $v_i$ intersects the outer wall. Here, the outer wall is modeled as an infinitely long cylindrical shell with inner radius $R_{sc}$. Note that $v_i$ is a vector with origin at the electron position $R_i$. Then, we measure the angle between the electron’s velocity and the vector $R_i^\alpha - R_i$, which is given by $\cos \zeta_i = v_i \cdot (R_i^\alpha - R_i)/|v_i||R_i^\alpha - R_i|$. In the simplest approximation for the angular distribution of the SR, the radiation is emitted within a circular cone with its axis along $v_i$, and aperture $\alpha = 1/\gamma$, where $\gamma$ is the relativistic gamma factor of the particle. See section 2 for details. Only electrons with $\zeta_i \leq \alpha$ are kept for the calculation of the camera snapshot. Next, we iterate over each pixel of the camera detector and calculate the contribution of each electron to the line integrated emission measured by that pixel. The process for computing the radiation emitted by the $i$th electron and measured by each pixel is a two-step process: The first step is to find the columns of pixels that detect the $i$th electron. We note that the pixels in the same column of the camera detector share the same line of sight when the camera setup is seen from the top, see figure 12(c). We say that the $j$th electron is detected by the $j$th column of pixels when the circle with radius $R_j = \sqrt{x_j^2 + y_j^2}$ defined by the position of
Here \( x_i = (x, y, z_i) \) is the position of the \( i \)th electron. For the \( i \)th electron seen by the \( j \)th column of pixels we calculate the angle \( \varphi_{ij} \), which is the angle between the camera position and the position at which the circle with radius \( R_i \) intersects the \( j \)th line of sight. This angle is measured anticlockwise from the solid red line of figure 12(a). In the second step we identify the row of pixels that detect the \( i \)th electron. This is done by identifying the row of pixels that the unitary vector \( \hat{n}_i \), the direction of emission of the \( i \)th electron, hits when it extends from the electrons’ position to the plane of the camera detector. Here \( \hat{n}_i = \hat{T}_y \hat{r}_i - \hat{R}_{xz}, \hat{T}_z \) are rigid rotations along the \( z \)-axis by an angle \( \varphi_i \) and \( \phi_0 \) is the azimuthal angle defined by the position of the particle \( x_i \). Once that we have identified which pixels detect which electrons we compute their contribution to the measured synchrotron emission using either model for the angular distribution of the SR of section 2.

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References

[1] Hender T et al 2007 Nucl. Fusion 47 S128–202
[2] Hollmann E et al 2011 J. Nucl. Mater. 415 S27–34
[3] Papp G, Drevlak M, Fülöp T and Helander P 2011 Nucl. Fusion 51 043004
[4] Lehnen M et al 2011 Nucl. Fusion 51 123010
[5] Hollmann E M, Parks P B, Commaux N, Eidietis N W, Moyer R A, Shiraki D, Austin M E, Lasnier C J, Paz-Soldan C and Rudakov D L 2015 Phys. Plasmas 22 056108
[6] Commaux N et al 2016 Nucl. Fusion 56 046007
[7] Martin-Solis J R, Alvarez J D, Sanchez R and Esposito B 1998 Nucl. Fusion 38 2370
[8] Andersson F, Helander P and Eriksson L G 2001 Phys. Plasmas 8 5221–9
[9] Stahl A, Hirvijoki E, Decker J, Embréus O and Fülöp T 2015 Plasma Phys. Control. Fusion 59 (2017) 124001 L Carbajal and D del-Castillo-Negrete
[10] Finken K H, Winters J G, Rusbildt D, Corbett W J, Dippel K H, Goebel D M and Moyer R A 1990 Nucl. Fusion 30 859–70
[11] Jaspers R 1995 Relativistic runaway electrons tokamak PhD Thesis (https://pure.tue.nl/ws/files/1475618/431410.pdf)
[12] Jaspers R, Lopes Cardozo N J and Finken K H 1999 Plasma Phys. Control. Fusion 41 377–98
[13] Jaspers R, Lopes Cardozo N J, Donnè A J H, Widdershoven H L M and Finken K H 2001 Rev. Sci. Instrum. 72 466–70
[14] Kudiyakov T, Finken K H, Jakubowski M W, Lehen M, Xu Y, Schweer B, Toncian T, Wassenhove G V and Willi O 2008 Nucl. Fusion 48 122002
[15] De Angelis R and Di Matteo L 2010 Nucl.Instrum. Methods Phys. Res. A 623 815–7
[16] Shi Y, Fu J, Li J, Yang Y, Wang F, Li Y, Zhang W, Wan B and Chen Z 2010 Rev. Sci. Instrum. 81 033506
[17] Yu J H, Hollmann E M, Commaux N, Eidietis N W, Humphreys D A, James A N, Jernigan T C and Moyer R A 2013 Phys. Plasmas 20 042113
[18] Hollmann E et al 2013 Nucl. Fusion 53 083004
[19] Stahl A, Landreman M, Papp G, Hollmann E and Fülöp T 2013 Phys. Plasmas 20 093302
[20] Landreman M, Stahl A and Fülöp T 2014 Comput. Phys. Commun. 185 847–55
[21] Carbajal L, del-Castillo-Negrete D, Spong D, Seal S and Baylor I 2017 Phys. Plasmas 24 042512
[22] Schwinger J 1949 Phys. Rev. 75 1912–25
[23] Balerna A and Mobilio S 2015 Synchrotron radiation: basics, methods and applications Synchrotron Radiation ed S Mobilio et al (Berlin: Springer) pp 3–28
[24] Pankratov I M 1999 Plasma Phys. Rep. 25 145–8
[25] Jackson J 2007 Classical Electrodynamics (New York: Wiley)
[26] Wongrach K, Finken K H, Abdullaev S S, Koslowski R, Willi O, Zeng L and The TEXTOR Team 2014 Nucl. Fusion 54 43011
[27] Liu J, Wang Y and Qin H 2016 Nucl. Fusion 56 064002
[28] Wang Y, Qin H and Liu J 2016 Phys. Plasmas 23 062505
[29] Rosenbluth M N and Putvinski S V 1997 Nucl. Fusion 1 1355–62
[30] Fülöp T, Pokol G, Helander P and Lisak M 2006 Phys. Plasmas 13 062506
[31] Zhou R J, Pankratov I M, Hu L Q, Xu M and Yang J H 2014 Phys. Plasmas 21 063302