CONVERGENCE TO THE CRITICAL ATTRACTOR AT INFINITE AND TANGENT BIFURCATION POINTS.

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Abstract

The dynamics of the convergence to the critical attractor for the logistic map is investigated. At the border of chaos, when the Liapunov exponent is zero, the use of the non-extensive statistical mechanics formalism allows to define a weak sensitivity or insensitivity to initial conditions. Using this formalism we analyse how a set of initial conditions spread all over the phase space converges to the critical attractor in the case of infinite bifurcation and tangent bifurcation points. We show that the phenomena is governed in both cases by a power-law regime but the critical exponents depend on the type of bifurcation and may also depend on the numerical experiment set-up. Differences and similarities between the two cases are also discussed.
1 Introduction

Nonlinear dissipative maps are widely used in the description of a variety of systems exhibiting complex behaviour, given their suitability for numerical simulations. Sensitivity to initial conditions is easy to investigate in association to the dynamics onto the fractal attractor. Physical properties like entropies and relaxation phenomena are easily computed and generalised to more sophisticated systems. The logistic map is the most famous and used among the nonlinear maps. Nevertheless it still presents some aspect of the dynamics not well understood or exploited. This is the case for special points of the map, like the border or onset of chaos, constituted by the infinite bifurcation points or the tangent bifurcation points, often indicated as critical points, where the Lyapunov exponent vanishes and sensitivity to initial conditions is not exponential but polynomial in time. A vast literature [C. Tsallis et al., 1997; F. Baldovin et al., 2002; U. Tirnakli et al., 1999; M. Coraddu et al., 2004] has recently tried to investigate these critical points especially in association to entropic or relaxation properties. The main feature of the logistic map at these points is a power-law sensitivity or insensitivity to initial conditions [P. Grassberger and M. Scheunert, 1981; C. Tsallis, 1988; F.A.B.F. De Moura et al., 2000], with consequent peculiar behaviour for the time evolution and the convergence to the
attractor of an ensemble of points. Recently a power-law sensitivity to initial conditions has been rewritten [C. Tsallis et al., 1997; M.L. Lyra and C. Tsallis 1998] in the form
\[ \xi(t) = \left[ 1 + (1 - q)\lambda_q t \right]^{1/(1-q)}, \]
where \( \lambda_q \) defines the time scale after which the power-law behaviour begins. When used to investigate the expansion (weak sensitivity, namely \( 1/(1-q) \geq 0 \)) or contraction (weak insensitivity, conversely \( 1/(1-q) \leq 0 \)) of a compact ensemble of points or grouped initial conditions this relation is connected to the evolution of the entropy of such a system. To be more clear let consider a partition of the phase space of the logistic map in \( W_{box} \) cells, where we use the parameterisation
\[ x_{n+1} = 1 - ax_n^2 \]
so that the phase space is \(-1 \leq x \leq +1\). One of the standard ways to define the entropy of a system is the Boltzmann-Gibbs (BG) formulation
\[ S_{BG} = - \sum_{i=1}^{W_{box}} p_i \log(p_i) \]
where \( p_i \) defines the probability of occupation of the \( i \)-th cell as the ratio of points inside the cell with respect to the total number of points in the phase space. When the parameter \( a \) assumes values such that the map behaves chaotically, the Liapunov exponent is positive and there is *strong sensitivity* to initial conditions.
If one groups a set of points close together they will expand exponentially in time (at least for small time scales) and correspondingly the BG entropy will increase. It has been demonstrated [A.N. Kolmogorov, 1958; Ya. G. Sinai, 1959] that the rate of increase of the BG entropy, known as Kolmogorov-Sinai (KS) entropy and defined as

\[ S_{KS} = \lim_{t \to \infty} \frac{S(t) - S(0)}{t} \]  

equals the Liapunov exponent under certain conditions, giving the well known Pesin equality \( \lambda = S_{KS} \) [Ya. Pesin, 1977], so that the entropy increases linearly with time. Conversely, if the map behaves periodically, the Liapunov exponent is negative and the BG entropy will decrease.

When the parameter \( a \) is set in a critical point the Liapunov exponent vanishes and a set of different features appears. Both at the onset of chaos and at tangent bifurcation points new definitions of the sensitivity and of the entropies are suitable to describe the weak sensitive/insensitive behaviour. The term weak sensitivity (insensitivity) designs a separation that increases (decreases) not exponentially (the Liapunov exponent is zero) but polynomially in time.

Two different numerical experiments present a notable interest with regard to the way the points evolve in time approaching the attractor.

On the one extreme, it is interesting to choose (randomly or not), all the
initial conditions in a single cell, so that the initial entropy is zero [E. Borges et al., 2002; M. Coraddu et al., 2004]. The number of cells $W(t)$ occupied at time $t$ then grows as:

$$W(t) = [1 + (1 - q)K_q t]^{1/(1-q)} \tag{5}$$

where $K_q$ is the generalised KS entropy, defined as the rate of increase of the non-extensive generalised Tsallis entropy

$$S_q^T = (1 - \sum p_i^q)/(q - 1), \tag{6}$$

appropriate to describe the behaviour of the logistic map at critical points [C. Tsallis 1988].

On the other extreme one can choose to spread the initial conditions all over the whole phase space, occupying most of (or all, if the set of initial points is at least equal to the number of cells) the partition cells [F.A.B.F. De Moura et al., 2000]. The fact that the parameter $a$ has been set in a critical point allows the system to exhibit peculiar behaviour. The main interest of this paper consists in the numerical analysis of the time evolution of a set of initial conditions spread in the whole phase space and the way they converge to the critical attractor.

We analyse the case of weak insensitivity and make a comparison with the weak sensitivity case. Due to the criticality of the system, the temporal evolution may depend on the particular choice of the initial conditions. In our case,
with initial conditions spread over the phase space, one expects a shrink toward the attractor in both cases, namely $a$ corresponding to the infinite bifurcation point ($a_{\infty} = 1.401155189$) and to the tangent bifurcation point ($a_{tg} = 1.75$).

2 The Convergence to the Critical Attractor

We analyse the temporal evolution of an ensemble of $N$ initial conditions randomly chosen and uniformly distributed in the whole phase space. Following the work done in [F.A.B.F. de Moura et al.] we partitioned the phase space in $W_{box}$ equal cells and examined the number of occupied cells $W(t)$ at time $t$ for different ratios $r = N/W_{box}$. Physically $r$ is an index giving the sampling ratio of the phase space. We used $W_{box}$ ranging from 5000 to 128000 cells for the partition and ratios $r$ varying from $r = 0.1$ to $r = 10000$, for both cases $a = a_{\infty} = 1.401155189$ and $a = a_{tg} = 1.75$. When the convergence to the critical attractor is analysed using the non-extensive Tsallis entropy [G.F.J. Ananos et al., 2004 and references therein] such that the entropy $S^T_q$ evolves in time at constant rate $K_q$, with

$$K_q = \lim_{N \to \infty} \frac{[S_q(t) - S_q(0)]}{t}$$

In the case of relaxation onto the attractor, the number of occupied cells decreases in time giving a negative $K_q$. In the simplest case of equiprobability, so
that each cell contains the same number of points, the relation (6) gives

\[ S^T_q = \frac{W(t)^{1-q} - 1}{1 - q} \]  

and the number of occupied cells is expected to present a power-law decay

\[ W(t) = [W(0)^{1-q} + (1 - q)K_q t]^{1/(1-q)} \]  

with asymptotic exponent \( \alpha = 1/(1-q) \leq 0. \)

In order to exploit most of the features of the convergence to the critical attractor we set up a series of different numerical experiments varying \( W_{box} \) at constant \( r \) and vice-versa, for the two cases \( a = a_\infty \) and \( a = a_{tg}. \) The numerical experiments consist in selecting, using a uniform random distribution, at time zero, \( N \) initial points in the phase space of the logistic map \( x \in [-1,1], \) that has been partitioned in \( W_{box} \) boxes. As each of the \( N \) points is iterated in time according to the logistic map \( W(t) \) boxes, among all the \( W_{box} \), will be not empty at time \( t. \) The number \( W(t) \) generally will decrease iterating the map and is the quantity used in the following to characterize the different regimes of convergence to the attractor of the \( N \) initial points.

As a first case we analyse the infinite bifurcation point. Here the expected power-law behaviour (Eq. (9)) [P. Grassberger and M. Scheunert, 1981] sets up for \( a = a_\infty \) even at low \( r \) values \((r = 0.1); increasing \( r \) a superimposed log-log oscillation appears clearly and becomes more evident at higher \( r \) values (Fig. 1).
In these cases there is a transient behaviour, preceding the power-law regime, which is longer for larger $r$.

It is important to note that, in a logarithmic-to-logarithmic scale, the transient time $\tau_{\text{crossover}}$ after which the convergence starts is translated by almost the same amount when $r$ increases of a factor 10. This feature seems to persist up to the values of $r = 10000$, reached in our experiments. Higher values for the sampling ratio are in order to be used to verify the power-law regime in the limit $r \to \infty$.

The dependence of $\tau_{\text{crossover}}$ on the sampling ratio $r$ is clearly evidenced in Fig. 1. Previous works analysed values of sampling ratio up to $r = 10$, obtaining as a result a crossover time $\tau_{\text{crossover}}$ independent on $r$ and related to the Eq. (??) by $\tau_{\text{crossover}} \sim \frac{1}{W(0)}$. Analysing sampling ratios up to $r = 10000$, for many different $W_{\text{box}}$, this picture breaks down revealing a net and regular variation of $\tau_{\text{crossover}}$ with $r$.

In our analysis the power-law exponent seems to be not independent on $r$. The relation

$$W(t) = t^{-\mu} P\left(\frac{\ln(t)}{\ln(\lambda)}\right)$$ (10)

holds in the hypothesis of log-log oscillations [D. Sornette et al., 1998; C. Tsallis et al., 1997], where $P()$ is a period one function and $\lambda$ is a scaling factor between
two oscillation periods. If the exponent $\mu$ is constant a plot of $W(t)/t^{-\mu}$ should show a periodic function in log(t) with constant average. Figure 2 shows that the log-periodic oscillations exist but the function $P()$ can be globally increasing or decreasing in average, depending on the value of $r$. This trend can be explained if the exponent $\mu$ is not perfectly constant but slightly varying with $r$ (at least for the considered values; it may be possible that increasing $r$ a limit value for $\mu$ shows up. This value should probably be less than -0.71, around -0.8). In the figure we used the value $\mu = -0.71$ according to [F.A.B.F. De Moura et al., 2000]. However to the same sampling ratio $r$ corresponds the same exponent irrespectively of $W_{box}$.

In the case of the tangent bifurcation point the regime of convergence to the critical attractor also presents a power-law behaviour, but with different exponents (Fig. 3). Fitting the curves we found $\mu = -1.55$ for $r = 1000$ and $\mu = -1.45$ for $r = 10000$ (Fig. 4a). This difference is not surprising, since the exponent is related to the structure of the chaotic attractor [F.A.B.F. De Moura et al., 2000]. Nevertheless a power-law behaviour was not obvious at the tangent bifurcation point because the critical attractor has only three fixed point and does not posses a multifractal structure. To our knowledge this is the first study of such a power-law behaviour during the convergence to the attractor
of a statistical ensemble of initial conditions in a critical point with not infinite bifurcations or multifractal structure. We found (Fig. 3), as in the case of the infinite bifurcation point (Fig. 1), small irregular oscillations perturbing the power-law behaviour (to not be confused with the log-log oscillations) due to the finite size partition of the phase space. The larger is the number of cells $W_{\text{box}}$ (the finer is the partition) the smaller are the oscillations around the power-law behaviour. These oscillations are irregular and grid (and/or $r$) depending, and are almost completely reduced when $W_{\text{box}}$ is big enough.

No log-log oscillations are observed in this case (Fig. 4b), reflecting the different structure of the attractor.

Again, increasing $r$, in the log-to-log scale, the transient time $\tau_{\text{crossover}}$ after which the power-law regime of convergence begins is translated by approximatively the same amount when $r$ is increased of a factor 10 (Fig. 3). The ratio $W(t)/W(0)$ before $\tau_{\text{crossover}}$ is determined in both cases by the portion of phase space occupied by the attractor at the edge of chaos.

We repeated the experiments with a different kind of initial set-up, choosing all the initial $N$ points concentrated in a single cell, for $a = a_{\infty}$ and $a = a_{\text{tg}}$, using as a test case a grid of 64000 cells. At the infinite bifurcation point we found that, after a transient time, the system reaches a power-law regime with a
different exponent with respect to the experiment with all the initial conditions spread in $x \in [-1,1]$ (Fig. 5a), so that the exponent is sensitive to the system set-up, namely to its past history. This behaviour has already been observed in others numerical experiments at the infinite bifurcation point [F. Baldovin and A. Robledo, 2002b; U. Tirnakli et al., 1999].

A main difference is observed in the case of the tangent bifurcation. We repeated the same experiment for $a = a_{tg}$ with all the $N$ initial conditions arranged in a single cell and let them expand entropically until the process of convergence to the attractor begins. After a transient time we found exactly the same power-law index as in the case of initial conditions uniformly distributed over the phase space (Fig. 5b). In this case the system seems to be not sensitive to the system set-up or to its past history.

3 Conclusions

In this paper we numerically studied how a statistical ensemble of initial conditions converges to the critical attractor using the logistic map. We compared the behaviour in two selected critical points: $a_{\infty}$ and $a_{tg}$. The main results can be summarised in the following points:

- for $a = a_{\infty}$ a power-law regime with superimposed log-log periodic oscillations sets up after different crossover times, depending on the sampling ratio $r$ and
there is no strict numerical evidence for the power-law exponent to be a constant independent on the sampling ratio, at least for \( r \leq 100 \).

- for \( a = a_{tg} \) a power-law convergence to the critical attractor, with different exponent, appears after a transient time even if the attractor has not fractal structure. The crossover time depends on the sampling ratio \( r \) in the same fashion as in the \( a_{\infty} \) case and is partition independent;

- the choice of an ensemble of initial conditions concentrated on a single cell presents the same power-law exponent in the \( a_{tg} \) case, whilst a different regime of convergence to the critical attractor appears in the case \( a = a_{\infty} \).
References.

Ananos G.F.J. and Constantino Tsallis [2004] “Ensemble Averages and Nonextensivity at the Edge of Chaos of One-Dimensional Maps”, Phys. Rev. Lett. 93, 020601.

Baldovin F., A. Robledo [2002], “Universal renormalization-group dynamics at the onset of chaos in logistic maps and nonextensive statistical mechanics”, Phys. Rev. E66, 045104.

Baldovin F., A. Robledo [2002], “Sensitivity to initial conditions at bifurcations in one-dimensional nonlinear maps: Rigorous nonextensive solutions”, Europhys. Lett. 60, 518-524.

Borghes E., C. Tsallis, G.F.J. Ananos, P.M.C. de Oliveira [2002] “Nonequilibrium Probabilistic Dynamics of the Logistic Map at the Edge of Chaos”, Phys. Rev. Lett. 89, 254103.

Coraddu M., F. Meloni, G. Mezzorani and R. Tonelli [2004], “Weak insensitivity to initial conditions at the edge of chaos in the logistic map”, Physica A340, 234-239.

De Moura F.A.B.F., U. Tirnakli and M.L. Lyra [2000], “Convergence to the critical attractor of dissipative maps: Log-periodic oscillations, fractality, and nonextensivity”, Phys. Rev. E62, 6361-6365.
Grassberger P. and Scheunert M. [1981], “Some More Universal Scaling Laws for Critical Mappings”, J. Stat. Phys. 26, 697.

Kolmogorov A.N. [1958], “A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces”, Dokl. Akad. Nauk. SSSR 119, 861; Ya.G. Sinai [1959], “On the Concept of Entropy of a Dynamical System”, ibid. 124, 768.

Lyra M.L. and C. Tsallis [1998], “Nonextensivity and Multifractality in Low-Dimensional Dissipative Systems”, Phys. Rev. Lett. 80, 53-56.

Pesin Ya. [1977], “Characteristic Lyapunov exponents and smooth ergodic theory”, Russ. Math. Surveys 32, 55-114.

Sornette D. et al. [1996], “Complex Fractal Dimensions Describe the Hierarchical Structure of Diffusion-Limited-Aggregate Clusters”, Phys. Rev. Lett. 76, 251-254.

Tirnakli U., C. Tsallis, and M.L. Lyra [1999], “Circular-like maps: sensitivity to the initial conditions, multifractality and nonextensity”, Eur. Phys. J. B11, 309-315.

Tsallis C., A.R. Plastino, and W.-M. Zheng [1997], “Power-law sensitivity to initial conditions-new entropic representation”, Chaos, Solitons Fractals 8, 885-891.
Tsallis C. [1988], “Possible Generalization of Boltzmann-Gibbs Statistics”,
J. Stat. Phys. 52, 479-487.

Tsallis C. et al. [1997], “Specific heat anomalies associated with Cantor-set energy spectra” Phys. Rev. E56, R4922-R4925.
We report for \( a = a_\infty \), in a log-log scale, the time evolution of the ratio between the occupied boxes and the available boxes. In frame (a) we show for different partitions \( W_{\text{box}} \) of the interval [-1,1] the superimposed curves corresponding to the same \( r \). We used the values \( W_{\text{box}} = [20, 64, 128] \times 10^3 \) for \( r = 0.1 \), \( W_{\text{box}} = [5, 10, 20, 64, 128] \times 10^3 \) for \( r = 10 \) and \( W_{\text{box}} = [10, 20, 128] \times 10^3 \) for \( r = 1000 \). It is evident the occurrence of a power-law regime. Superimposed log-log oscillations appear for higher sampling ratios. Crossover times are also indicated. To the same sampling ratio \( r \) corresponds the same power-law exponent for the various grids until saturation is reached at different times. In frame (b) we show in sequence the values \( r = 0.1, 1, 10, 100, 1000 \) and 10000 for the two grids \( W_{\text{box}} = 10000 \) (red) and \( W_{\text{box}} = 128000 \) (black). The indicated crossover times show a net dependence on the sampling ratio \( r \).
We show the ratio $W(t)/t^{-0.71}$ versus log(t) for the infinite bifurcation point. In frame (a) we used $W_{\text{box}} = [10 \text{ (black)}, 20 \text{ (blue)}, 128 \text{ (red)}] \times 10^3$ for different sampling ratios $r$. Frame (b) shows different sampling ratios calculated using $W_{\text{box}} = 20000 \text{ (red)}$ and $W_{\text{box}} = 128000 \text{ (black)}$. Higher sampling ratios exhibit a decreasing mean value of $W(t)/t^{-0.71}$, while the lower ones exhibit an increasing mean value, suggesting a slight dependence of the power-law exponent on $r$. 
Figure 3

We report for $a = a_{tg}$, in a log-log scale, the time evolution of the ratio of occupied boxes to the available boxes. In frame (a) we show for different partitions of the interval $[-1,1]$ the superimposed curves corresponding to the same $r$. We used the values $W_{box} = [20, 64, 128] \times 10^3$ for $r = 0.1$, $W_{box} = [5, 10, 20, 64, 128] \times 10^3$ for $r = 10$ and $W_{box} = [10, 20, 128] \times 10^3$ for $r = 1000$, as in the $a_{\infty}$ case. It is evident the occurrence of a power-law regime well defined for higher sampling ratios with not superimposed log-log oscillations. Crossover times are indicated. Again, to the same sampling ratios $r$ corresponds the same power-law exponent for the various grids. In frame (b) we show in sequence the values $r = 0.1, 1, 10, 100, 1000$ and $10000$ for the two grids $W_{box} = 10000$ (red) and $W_{box} = 128000$ (black). The indicated crossover times show again a net dependence on the sampling ratio $r$. The power-law regime is well pronounced at higher $r$ values and extends in time over different orders of magnitude. Saturation effects appear for lower grids.
Figure 4

Frame (a) evidences the power-law regimes for the tangent bifurcation point, with \( W_{\text{box}} = [10 \text{ (blue)}, 128 \text{ (black)}] \times 10^3 \) and \( r = 1000 \), and with \( W_{\text{box}} = 10000 \) and \( r = 10000 \) (blue). Red lines are fits to the curves, with exponents \( \mu = -1.55 \) for \( r = 1000 \) and \( \mu = -1.45 \) for \( r = 10000 \). Different exponents describe the power-law regime for different sampling ratios. The power-law persists long in time over different orders of magnitude. Frame (b) shows the evolution of \( W(t)/t^{-1.55} \) versus \( \log(t) \) for various \( r \) and grids \( W_{\text{box}} = 128000 \) (black) and \( W_{\text{box}} = 10000 \) (other colours). The exponent \( \mu = -1.55 \) is appropriate only for the case \( r = 1000 \) where \( W(t)/t^{-1.55} \) shows a constant mean value.
Figure 5

We compare the different behaviours for $a = a_\infty$ (frame (a)) and $a = a_{tg}$ (frame (b)), when all the $N$ initial conditions are randomly selected inside a single bin of the phase space partition. We used in both cases $W_{\text{box}} = 64000$. Frame (a) illustrates the two different power-law coefficients obtained for $a = a_\infty$ when the initial conditions are uniformly spread all over the phase space or are constrained inside a single bin. In the frame we show four different selections of the initial bin in different colours. The four curves overlap in the regime of convergence to the attractor, oscillating at high frequency among two states every temporal step and giving rise to the broad coloured curves. In frame (b) the two different selections of initial conditions exhibit the same power-law exponent.
Figure 1: We report for $a = a_{\infty}$, in a log-log scale, the time evolution of the ratio between the occupied boxes and the available boxes. In frame (a) we show for different partitions $W_{\text{box}}$ of the interval $[-1,1]$ the superimposed curves corresponding to the same $r$. We used $W_{\text{box}} = [20, 64, 128] \times 10^{3}$ for $r = 0.1$, $W_{\text{box}} = [5, 10, 20, 64, 128] \times 10^{3}$ for $r = 10$ and $W_{\text{box}} = [10, 20, 128] \times 10^{3}$ for $r = 1000$. It is evident the occurrence of a power-law regime. Superimposed log-log oscillations appear for higher sampling ratios. Crossover times are also indicated. To the same sampling ratio $r$ corresponds the same power-law exponent for the various grids until saturation is reached at different times. In frame (b) we show in sequence the values $r = 0.1, 1, 10, 100, 1000$ and 10000 for the two grids $W_{\text{box}} = 10000$ (red) and $W_{\text{box}} = 128000$ (black). The indicated crossover times show a net dependence on the sampling ratio.
Figure 2: We show the ratio $W(t)/t^{-0.71}$ versus $\log(t)$ for the infinite bifurcation point. In frame (a) we used $W_{\text{box}} = [10 \text{ (black)}, 20 \text{ (blue)}, 128 \text{ (red)}] \times 10^3$ for different sampling ratios $r$. Frame (b) shows different sampling ratios calculated using $W_{\text{box}} = 20000 \text{ (red)}$ and $W_{\text{box}} = 28000 \text{ (black)}$. Higher sampling ratios exhibit a decreasing mean value of $W(t)/t^{-0.71}$, while the lower ones exhibit an increasing mean value, suggesting a slight dependence of the power-law exponent on $r$. 
Figure 3: We report for $a = a_{tg}$, in a log-log scale, the time evolution of the ratio of occupied boxes to the available boxes. In frame (a) we show for different partitions of the interval $[-1,1]$ the superimposed curves corresponding to the same $r$. We used the values $W_{box} = [20, 64, 128] \times 10^3$ for $r = 0.1$, $W_{box} = [5, 10, 20, 64, 128] \times 10^3$ for $r = 10$ and $W_{box} = [10, 20, 128] \times 10^3$ for $r = 1000$, as in the $a_\infty$ case. It is evident the occurrence of a power-law regime well defined for higher sampling ratios with not superimposed log-log oscillations. Crossover times are indicated. Again, to the same sampling ratios $r$ corresponds the same power-law exponent for the various grids. In frame (b) we show in sequence the values $r = 0.1, 1, 10, 100, 1000$ and 10000 for the two grids $W_{box} = 10000$ (red) and $W_{box} = 128000$ (black). The indicated crossover times show again a net dependence on the sampling ratio $r$. The power-law regime is well pronounced at higher $r$ values and extends in time over different orders of magnitude. Saturation effects appear for lower grids.
Figure 4: Frame (a) evidences the power-law regimes for the tangent bifurcation point, with $W_{box} = [10 \text{ (blue), } 128 \text{ (black)}] \times 10^3$ and $r = 1000$, and with $W_{box} = 10000$ and $r = 10000 \text{ (blue)}$. Red lines are fits to the curves, with exponents $\mu = -1.55$ for $r = 1000$ and $\mu = -1.45$ for $2^{14} = 10000$. Different exponents describe the power-law regime for different sampling ratios. The power-law persists long in time over different orders of magnitude. Frame (b) shows the evolution of $W(t)/t^{-1.55}$ versus log(t) for various $r$ and grids $W_{box} = 128000 \text{ (black)}$ and $W_{box} = 10000 \text{ (other colours)}$. The exponent $\mu = -1.55$ is appropriate only for the case $r = 1000$ where $W(t)/t^{-1.55}$ shows a constant mean value.
Figure 5: We compare the different behaviours for $a = a_\infty$ (frame (a)) and $a = a_{tg}$ (frame (b)), when all the $N$ initial conditions are randomly selected inside a single bin of the phase space partition. We used in both cases $W_{box} = 64000$. Frame (a) illustrates the two different power-law coefficients obtained for $a = a_\infty$ when the initial conditions are uniformly spread all over the phase space or are constrained inside a single bin. In the frame we show four different selections of the initial bin in different colours. The four curves overlap in the regime of convergence to the attractor, oscillating at high frequency among two states every temporal step and giving rise to the broad coloured curves. In frame (b) the two different selections of initial conditions exhibit the same power-law exponent.