On matrix Painlevé-4 equations.
Part 1: Painlevé–Kovalevskaya test

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Abstract

Using the Painlevé–Kovalevskaya test, we find several new matrix generalizations of the Painlevé-4 equation. Some limiting transitions reduces them to known matrix Painlevé -2 equations.

Keywords: Matrix ODEs, Painlevé–Kovalevskaya test, Painlevé equations

1 Introduction

In the paper [2], it was shown that the second Painlevé equation admits at least three non-abelian generalizations that satisfy the matrix Painlevé–Kovalevskaya test and possess isomonodromic Lax pairs. Certainly, a similar variety should be expected for other Painlevé equations. Although the literature on non-abelian generalizations is quite rich, the question remains as to how many kinds of them there are and, in particular, how non-abelian constants can be included into the answer.

In this paper, we consider matrix generalizations of the Painlevé equation $P_4$

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \gamma)y + \frac{\delta}{y}. \tag{1}$$

Here $'$ means the derivative with respect to the variable $z$.

It is known that equation (1) is equivalent to the system

$$\begin{cases}
u' &= -u^2 + 2uv - 2zu + c_1, \\
u' &= -v^2 + 2uv + 2zv + c_2, \tag{2}
\end{cases}$$

where $y = u$, $\gamma = 1 + \frac{1}{2}c_1 - c_2$, $\delta = -\frac{1}{2}c_1^2$. Using the matrix Painlevé test, we look for integrable matrix generalizations of system (2) of the form

$$\begin{cases}
u' &= -u^2 + 2uv + \alpha(uv - vu) - 2zu + b_1u + ub_2 + b_3v + vb_4 + b_5 \\
u' &= -v^2 + 2vu + \beta(vu - uv) + 2zv + c_1v + vc_2 + c_3u + uc_4 + c_5, \tag{3}
\end{cases}$$

where the coefficients $\alpha$, $\beta$ are scalar and others are constant matrices.
Remark 1. System (3) is invariant under transformations of the form
\[ b_1 \mapsto b_1 - 2\gamma_1 I, \quad b_2 \mapsto b_2 - 2\gamma_2 I, \quad c_1 \mapsto c_1 + 2\gamma_1 I, \quad c_2 \mapsto c_2 + 2\gamma_2 I, \quad z \mapsto z - \gamma_1 - \gamma_2. \] \quad (4)

Transformations
\[ u \mapsto v, \quad v \mapsto u \] \quad (5)
and
\[ u \mapsto u^T, \quad v \mapsto v^T, \] \quad (6)
where \( T \) means the matrix transposition, map a system (3) to another system of the same form but change the coefficients.

One more class of transformations we use in this paper is given by the formula
\[ u \mapsto e^{zK}(u + Q_1)e^{-zK}, \quad v \mapsto e^{zK}(v + Q_2)e^{-zK}, \] \quad (7)
where \( K \) and \( Q_i \) are constant matrices. In general, a transformation (7) takes a system of the form (3) outside of the class such systems. But in very particular cases transformations (7) can be applied.

The principal homogeneous part of system (2) has the form
\[ \begin{align*}
   u' &= -u^2 + 2uv, \\
   v' &= -v^2 + 2uv.
\end{align*} \] \quad (8)
This system is integrable in any acknowledged sense. In particular, it has an infinite sequence of polynomial infinitesimal symmetries of the form
\[ \begin{align*}
   u_\tau &= I^N(-u^2 + 2uv), \\
   v_\tau &= I^N(-v^2 + 2uv),
\end{align*} \] \quad (9)
where
\[ I = u(u - v) \]
is an integral of motion for system (8). In addition, system (8) satisfies the Painlevé-Kovalevskaya test. Namely it has formal Laurent solutions of the following three types:

1: \quad \begin{align*}
   u &= -\frac{1}{z - z_0} + O(1), \\
   v &= -\frac{1}{z - z_0} + O(1);
\end{align*}

2: \quad \begin{align*}
   u &= \frac{1}{z - z_0} + O(1), \\
   v &= O(1),
\end{align*}

3: \quad \begin{align*}
   u &= O(1), \\
   v &= \frac{1}{z - z_0} + O(1)
\end{align*}
that contains, apart from \( z_0 \), another arbitrary constant. For example, a solution of type 1 is
\[ u = -\frac{1}{z - z_0} + \tau(z - z_0)^2 - \frac{3}{4}\tau^2(z - z_0)^5 + \cdots, \quad v = -\frac{1}{z - z_0} - \tau(z - z_0)^2 + \frac{3}{4}\tau^2(z - z_0)^5 + \cdots. \]

Homogeneous non-abelian generalizations of system (8) of the form
\[ \begin{align*}
   u' &= -u^2 + 2uv + \alpha(uv - vu), \\
   v' &= -v^2 + 2vu + \beta(vu - uv),
\end{align*} \] \quad (10)
where \( \alpha, \beta \in \mathbb{C} \), were studied in [7].

The transformations \( u \leftrightarrow v \) and \( u \mapsto u^T, v \mapsto v^T \) preserve the class of such systems changing the parameters in the following way: \( \alpha \leftrightarrow \beta \) and \( \alpha \mapsto -\alpha - 2, \beta \mapsto -\beta - 2. \)

In [7], the following statement was proved:
Theorem 1. Up to transformations (5) and (6), there are only the following pairs of parameters $\alpha, \beta$:

1. $\alpha = -1, \beta = -1$,
2. $\alpha = 0, \beta = -1$,
3. $\alpha = 0, \beta = -2$,
4. $\alpha = 0, \beta = 0$,
5. $\alpha = 0, \beta = -3$,

under which system (10) has matrix polynomial symmetries that, in the case of matrices of size 1, coincide with (9).

The full set of acceptable parameter values is represented by dots in the following figure:

Figure 1: The points highlighted in red are sets of parameters from Theorem 1. Admissible transformations correspond to the reflection in the line $\beta = \alpha$ and in the point $(-1, -1)$. Using these symmetries, all the points highlighted in blue can be obtained from the red ones.

Let us denote by $\Sigma$ the set of thirteen integer points shown in Figure 1.

In Section 2, we study Painlevé properties of matrix systems (10). It turns out that under some assumptions (see Theorem 5), they satisfy the matrix Painlevé–Kovalevskaya test [3] only for points $(\alpha, \beta)$ that are shown in Figure 1.

For homogeneous systems described in Section 2, we find linear terms that can be added to it so that the resulting system (3) still satisfies the matrix Painlevé–Kovalevskaya test.

Let us formulate the main result of Section 3

Theorem 2. Any system (3) that satisfies the Painlevé–Kovalevskaya test can be reduced by transformations from Remark 1 to one of the following:

$$
\begin{align*}
\begin{cases}
u' & = -u^2 + uv + vu - 2zu + hu + \gamma_1 I, \\
v' & = -v^2 + vu + uv + 2zv - vh + \gamma_2 I
\end{cases}
\end{align*}
$$
Here $\gamma_1, \gamma_2 \in \mathbb{C}$, $h$ is arbitrary matrix and two constant matrices $h_1$, $h_2$ are connected by relation $[h_2, h_1] = -2h_1$.

The parameter $\alpha$ and the coefficients $b_3, b_4$ are equal to zero in systems $P_2^4 - P_5^4$. Therefore the unknown variable $v$ can be eliminated and as a result we arrive at a matrix $P_4^4$ equation of the form

$$y'' = \frac{1}{2} (y' + k_1) y^{-1} (y' + k_2) + \frac{3}{2} y^3 + \kappa (y', y) + 4z y^2 + yk_3y + k_4y + yk_5 + 2z^2y$$

for $y = u$. The coefficients $k_i$ are expressed in terms of the corresponding system (3) by the formulas

$$\kappa = \beta + \frac{3}{2}, \quad k_1 = -k_2 = b_5, \quad k_3 = 2c_3,$$
$$k_4 = -2 - \left( \beta + \frac{3}{2} \right) b_5, \quad k_5 = 2c_3 + \left( \beta + \frac{1}{2} \right) b_5.$$

System $P_1^4$ is equivalent to an obvious matrix generalization of the dressing chain with $N = 3$ investigated in [8].

In order to express the variable $v$ from the first equation of this system, one needs to invert the operator $P_u^u : P_u^u(v) = uv + vu$ and, therefore, $u''$ is not a rational function in the variables $u$ and $u'$.

In Section 4 we find limiting transitions from the matrix Painlevé -4 systems to the matrix Painlevé -2 equations found in [2].

## 2 Matrix Painlevé–Kovalevskaya test for homogeneous systems (10)

Let us find out when system (10) possesses a formal solution of the form

$$u = \frac{p}{z - z_0} + a_0 + a_1(z - z_0) + \cdots, \quad v = \frac{q}{z - z_0} + b_0 + b_1(z - z_0) + \cdots$$

where $p, q, a_j, b_j \in \text{Mat}_n(\mathbb{C})$, $z_0 \in \mathbb{C}$, containing the maximum possible number $2n^2$ of arbitrary constants. We will call such a solution maximal.
Substituting the series (12) into the system (10), we obtain the following recurrence relations for their coefficients:

\[ -p^2 + 2pq + \alpha [p, q] + p = 0, \quad -q^2 + 2qp + \beta [q, p] + q = 0, \]

\[ -pa_k - a_k p + 2 (pb_k + a_k q) + \alpha ([p, b_k] + [a_k, q]) = \gamma (a_k - b_k + 1), \]

\[ -qb_k - b_k q + 2 (qa_k + b_k p) + \beta ([q, a_k] + [b_k, p]) = \gamma (b_k - a_k - 1), \]

where \( f_\gamma (X, Y) \) is defined by the formula

\[ f_\gamma (X, Y) = \sum_{i=0}^{k} \left( \frac{1}{2} (X_i X_{k-i} + X_{k-i} X_i) - 2 X_i Y_{k-i} - \gamma [X_i, Y_{k-i}] \right). \]

The relations (14) can be written as

\[ (\mathcal{L} - k I) \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} f_\alpha (a_{k-1}, b_{k-1}) \\ f_\beta (b_{k-1}, a_{k-1}) \end{pmatrix}, \quad k \in \mathbb{Z}_{\geq 0}, \]

where the operator \( \mathcal{L} : \text{Mat}_n (\mathbb{C}) \oplus \text{Mat}_n (\mathbb{C}) \to \text{Mat}_n (\mathbb{C}) \oplus \text{Mat}_n (\mathbb{C}) \) acts according to the rule

\[ \mathcal{L} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -pX - Xp + 2 (pY + Xq) + \alpha ([p, Y] + [X, q]) \\ -qY - Yq + 2 (qX + Yp) + \beta ([q, X] + [Y, p]) \end{pmatrix}. \]

2.1 System of matrix quadratic equations (13)

Let us consider the system (13) for the residues \( p \) and \( q \). Commuting each of the equations of system (13) with \( p \) and \( q \), we get 4 linear equations with constant coefficients with respect to the unknowns \( p[p, q], [p, q], p[p, q], q[p, q], [p, q], [p, q] \). The determinant of the matrix consisting of the coefficients at the first four unknowns is equal to \( 4 \Delta \), where

\[ \Delta = \alpha^2 + \beta^2 + \alpha \beta + 3(\alpha + \beta + 1). \]

If \( \Delta = 0 \), then it follows from the system that \( [p, q] = 0 \).

In the case \( \Delta \neq 0 \) solving the system, we get

\[ p[p, q] = \mu_1 [p, q], \quad [p, q]p = \mu_2 [p, q], \quad q[p, q] = \mu_3 [p, q], \quad [p, q]q = \mu_4 [p, q], \]

where

\[ \mu_1 = -\frac{\alpha (3 + \alpha + 2 \beta)}{2 \Delta}, \quad \mu_2 = -\frac{(2 + \alpha) (3 + \alpha + 2 \beta)}{2 \Delta}, \]

\[ \mu_3 = -\frac{\beta (3 + \beta + 2 \alpha)}{2 \Delta}, \quad \mu_4 = -\frac{(2 + \beta) (3 + \beta + 2 \alpha)}{2 \Delta}. \]

Relations (18) imply \( pq[p, q] = qp[p, q] = \mu_1 \mu_3 [p, q] \) and hence \( [p, q]^2 = 0 \).

Remark 2. From (18) it follows that the vector space spanned by \( p, q, [p, q] \) is a Lie algebra whose square is one-dimensional.

Proposition 1. Let \( (p, q) \) be a solution of system (13) such that \( [p, q] = 0 \). Then the matrices \( p \) and \( q \) are simultaneously diagonalizable.
Proof. Denote \( P_1 \overset{def}{=} -p^2 + 2pq + p = 0, P_2 \overset{def}{=} -q^2 + 2pq + q = 0 \). It is easy to check that

\[
0 = P_1 \left( q - \frac{3}{2}p - \frac{3}{2} \right) + 2pP_2 = \frac{3}{2} (p^3 - p).
\]

From this equation for \( p \) it follows that the Jordan form of \( p \) is diagonal. Similarly, the matrix \( q \) is diagonalizable. If two matrices are diagonalizable and commute, then they are diagonalizable simultaneously. Indeed, \( q \) acts in a diagonalizable way on the eigenspaces of the matrix \( p \). Choosing a bases in these eigenspaces consisting of the eigenvectors of the matrix \( q \), we give rise to a basis in which both matrices are diagonal. \( \square \)

Suppose that the matrices \( p \) and \( q \) are diagonal (see Proposition 1). Then from (13) it follows that one may set

\[
p = \text{diag} (-I_{k_1}, I_{k_2}, 0_{k_3}, 0_{k_4}), \quad q = \text{diag} (-I_{k_1}, 0_{k_2}, I_{k_3}, 0_{k_4}),
\]

where \( k_1 + k_2 + k_3 + k_4 = n \).

Consider now the case \([p, q] \neq 0\). Denote by \( \Sigma_0 \) the collection of all integer points \((\alpha, \beta)\), shown in Figure 1, with the exception of the point \((-1, -1)\).

**Proposition 2.** Suppose that there exists a solution \( p, q \) of system (13) such that \([p, q] \neq 0\), then \((\alpha, \beta) \in \Sigma_0\).

**Proof.** Multiplying each of equations (13) by the commutator \([p, q]\) from the left and from the right and using relations (18), we derive that the equations

\[
-\mu_1^2 + 2\mu_1\mu_3 + \mu_1 = 0, \quad -\mu_3^2 + 2\mu_1\mu_3 + \mu_3 = 0,
\]

\[
-\mu_2^2 + 2\mu_2\mu_4 + \mu_2 = 0, \quad -\mu_4^2 + 2\mu_2\mu_4 + \mu_4 = 0
\]

must be satisfied. Substituting the values (19) for \( \mu_i \) and solving the resulting system with respect to \( \alpha, \beta \), we arrive at the condition \((\alpha, \beta) \in \Sigma_0\). \( \square \)

**Proposition 3.** Let \( n = 2 \). Then for each point \((\alpha, \beta) \in \Sigma_0\) there exists a unique up to a conjugation solution of system (13) such that \([p, q] \neq 0\). The solution can be normalized by the conditions \(^1\)

\[
p = \begin{pmatrix} \mu_1 & X \\ 0 & \mu_2 \end{pmatrix}, \quad q = \begin{pmatrix} \mu_3 & Y \\ 0 & \mu_4 \end{pmatrix}, \quad [p, q] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

where \( \mu_i \) are defined by formulas (19). The pairs \((X, Y)\) can be chosen as it follows:

\[
\begin{array}{ccl}
(1, -2) : & X = -1, & Y = 0; \\
(0, 0) : & X = -1, & Y = 0; \\
(0, -1) : & X = 0, & Y = 1;
\end{array}
\]

\[
\begin{array}{ccl}
(0, -2) : & X = -1, & Y = 0; \\
(0, -3) : & X = 0, & Y = -1; \\
(-1, 0) : & X = -1, & Y = 0;
\end{array}
\]

\[
\begin{array}{ccl}
(-1, -2) : & X = 1, & Y = 0; \\
(-2, 1) : & X = 0, & Y = 1; \\
(-2, 0) : & X = 1, & Y = 0;
\end{array}
\]

\[
\begin{array}{ccl}
(-2, -1) : & X = 0, & Y = -1; \\
(-2, -2) : & X = 1, & Y = 0;
\end{array}
\]

\[
\begin{array}{ccl}
(-3, 0) : & X = 1, & Y = 0.
\end{array}
\]

**Remark 3.** It is easy to verify that if in the solutions described in Proposition 3 we replace the numbers with the corresponding scalar matrices of size \( m \times m \), then the resulting \( 2m \times 2m \)-matrices define solutions of system (13) for \( n = 2m \).

\(^1\)Such a solution is unique up to a conjugation by means of a upper-triangular matrix with units on the diagonal.
Let us describe all solutions of system \((13)\), for which \([p,q] \neq 0\). Since \([p,q]^2 = 0\), it is possible to reduce the commutator to the block form

\[
K \overset{\text{def}}{=} [p,q] = \begin{pmatrix} m & m & k \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} m , \quad k + 2m = n \quad \tag{21}
\]

by a conjugation. It follows from \((18)\) that \(p\) and \(q\) have the following structure:

\[
p = \begin{pmatrix} m & m & k \\ \mu_1 \mathbb{I} & p_{12} & p_{13} \\ 0 & \mu_2 \mathbb{I} & 0 \\ 0 & p_{32} & p_{33} \end{pmatrix} m , \quad q = \begin{pmatrix} m & m & k \\ \mu_3 \mathbb{I} & q_{12} & q_{13} \\ 0 & \mu_4 \mathbb{I} & 0 \\ 0 & q_{32} & q_{33} \end{pmatrix} m .
\]

The normalization \((21)\) and equations \((13)\) provide the fact that the blocks \(p_{33}, q_{33}\) commute and satisfy system \((13)\). Conjugating \(p\) and \(q\) with a matrix of the form \(\text{diag}(I_m, I_m, T_k)\), we reduce \(p_{33}, q_{33}\) to a diagonal form (see Proposition \(1\) and formula \((20)\)). As a result, we obtain

\[
p = \begin{pmatrix} m & m & k_1 & k_2 & k_3 & k_4 \\ \mu_1 \mathbb{I} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & \mu_2 \mathbb{I} & 0 & 0 & 0 & 0 \\ 0 & p_{32} & -\mathbb{I} & 0 & 0 & 0 \\ 0 & p_{42} & 0 & \mathbb{I} & 0 & 0 \\ 0 & p_{52} & 0 & 0 & 0 & 0 \end{pmatrix} m , \quad q = \begin{pmatrix} m & m & k_1 & k_2 & k_3 & k_4 \\ \mu_3 \mathbb{I} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} \\ 0 & \mu_4 \mathbb{I} & 0 & 0 & 0 & 0 \\ 0 & q_{32} & -\mathbb{I} & 0 & 0 & 0 \\ 0 & q_{42} & 0 & \mathbb{I} & 0 & 0 \\ 0 & q_{52} & 0 & 0 & \mathbb{I} & 0 \end{pmatrix} m .
\]

\[
\tag{22}
\]

**Theorem 3.** For each point \((\alpha, \beta) \in \Sigma_0\) and any non-negative integers \(m, k_1, k_2, k_3, k_4\) there exists a unique up to a conjugation solution of system \((13)\). It can be reduced to the form \((22)\), where \(p_{1,i} = q_{1,i} = p_{i,2} = q_{i,2} = 0\) for \(i = 3, 4, 5, 6\), and \(p_{1,2} = X \mathbb{I}, \quad q_{1,2} = Y \mathbb{I}\) (the numbers \(X\) and \(Y\) are given in Proposition \(3\)).

**Proof.** We outline the proof for \(\alpha = 0, \beta = -3\). In other cases, the proof is similar. The formula \((19)\) gives \(\mu_1 = 0, \mu_2 = 1, \mu_3 = \mu_4 = 0\). Relations \((13), (21)\) are equivalent to the
The dimension of the stabilizer for the pair of matrices is equal to the orbit $O$ of arbitrary parameters that can be contained in the residues. Since it can be calculated as the difference between $p_{33} = \text{diag}(-I_{k_1}, I_{k_2}, 0_{k_3}, 0_{k_4})$, and $q_{33} = \text{diag}(-I_{k_1}, 0_{k_2}, I_{k_3}, 0_{k_4})$, we have: $k_1 + k_2 + k_3 + k_4 = k$.

\section*{2.1.1 Maximal solutions with commuting residues}

Let us consider first the maximal formal solutions of (12) under the condition $[p, q] = 0$. In this case, the canonical form of the residues is given by the formula (20). Since the system (10) admits transformations $u \mapsto Su^{-1}$, $v \mapsto S\nu S^{-1}$, where $S$ is an arbitrary non-degenerate matrix, the group $GL_n$ acts on the residues $p, q$. The maximum number of arbitrary parameters that can be contained in the residues is equal to the dimension of the orbit $O$ of this action. Since it can be calculated as the difference between $n^2$ and the dimension of the stabilizer for the pair of matrices $p$ and $q$, we have: \[\dim O = n^2 - (k_1^2 + k_2^2 + k_3^2 + k_4^2).\]

Other arbitrary parameters can appear in the solutions of the system (16) when $k$ belongs to the spectrum of the operator $L$. For any maximal solution (12) the total number of parameters must be equal to $2n^2 - 1$. One more arbitrary parameter is $z_0$. 

\begin{equation}
\begin{pmatrix}
m & m & k_1 & k_2 & k_3 & k_4 \\
0 & p_{12} & q_{13} & p_{14} & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 2q_{32} & -I & 0 & 0 & 0 \\
0 & 0 & 0 & -I & 0 & 0 \\
0 & -q_{52} & 0 & 0 & 0 & 0 \\
p_{62} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
m \begin{pmatrix}
m & m & k_1 & k_2 & k_3 & k_4 \\
0 & -I - q_{13} q_{32} + q_{15} q_{52} & q_{13} & 0 & 0 & 0 \\
0 & 0 & -q_{32} & -I & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & -p_{62} & 0 \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & &
\end{pmatrix}
m
\end{equation}

After conjugation $p \mapsto gpg^{-1}$, $q \mapsto gpg^{-1}$, where

\begin{equation}
\begin{pmatrix}
m & m & k_1 & k_2 & k_3 & k_4 \\
I & -p_{12} - 2q_{13} q_{32} - q_{15} q_{52} & q_{13} & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & -q_{32} & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & q_{62} & 0 & 0 & 0 & 0 \\
0 & -p_{62} & 0 & 0 & 0 & I \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & &
\end{pmatrix}
m
\end{equation}

we obtain

\begin{equation}
p = \begin{pmatrix}
m & m & k \\
0 & I & 0 \\
0 & 0 & p_{33}
\end{pmatrix} m, \quad q = \begin{pmatrix}
m & m & k \\
0 & -I & 0 \\
0 & 0 & q_{33}
\end{pmatrix} m,
\end{equation}

\[p_{33} = \text{diag}(-I_{k_1}, I_{k_2}, 0_{k_3}, 0_{k_4}), \quad q_{33} = \text{diag}(-I_{k_1}, 0_{k_2}, I_{k_3}, 0_{k_4}), \quad k_1 + k_2 + k_3 + k_4 = k.\]
Proposition 4. Suppose that the matrices \( p \) and \( q \) have the form (20). Then the spectrum of the operator (17) belongs to the set

\[
\{ \lambda = -2, \quad \lambda_2 = 2, \quad \lambda_3 = -1, \quad \lambda_4 = 0, \quad \lambda_5 = -\alpha, \quad \lambda_6 = -\beta, \quad \lambda_7 = \alpha + 2, \quad \lambda_8 = \beta + 2, \\
\lambda_9 = 4 + \alpha + 2\beta, \quad \lambda_{10} = 4 + 2\alpha + \beta, \quad \lambda_{11} = 3 + \alpha + \beta, \quad \lambda_{12} = -2 - \alpha - 2\beta, \\
\lambda_{13} = -2 - 2\alpha - \beta, \quad \lambda_{14} = 1 + \alpha - \beta, \quad \lambda_{15} = 1 - \alpha + \beta, \quad \lambda_{16} = -1 - \alpha - \beta \}
\]

The dimensions of the corresponding eigenspaces are given by the formulas

\[
d_1 = d_2 = k_1^2 + k_2^2 + k_3^2, \quad d_3 = 2 (k_1 k_2 + k_1 k_3 + k_2 k_3 + k_1 k_4 + k_2 k_4 + k_3 k_4), \quad d_4 = 2k_4^2 \\
d_5 = d_7 = k_3 k_4, \quad d_6 = d_8 = k_2 k_4, \quad d_9 = d_{12} = k_1 k_2, \quad d_{10} = d_{13} = k_1 k_3, \\
d_{11} = d_{16} = k_1 k_4, \quad d_{14} = d_{15} = k_2 k_3.
\]

Proof. Let us represent \( X \) and \( Y \) as block 4 \( \times \) 4-matrices:

\[
X = \begin{pmatrix}
k_1 & k_2 & k_3 & k_4 \\
k_1 x_{11} & k_2 x_{21} & k_3 x_{31} & k_4 x_{41} \\
k_1 x_{12} & k_2 x_{22} & k_3 x_{32} & k_4 x_{42} \\
k_1 x_{13} & k_2 x_{23} & k_3 x_{33} & k_4 x_{43} \\
k_1 x_{14} & k_2 x_{24} & k_3 x_{34} & k_4 x_{44}
\end{pmatrix}, \quad Y = \begin{pmatrix}
k_1 & k_2 & k_3 & k_4 \\
k_1 y_{11} & k_2 y_{21} & k_3 y_{31} & k_4 y_{41} \\
k_1 y_{12} & k_2 y_{22} & k_3 y_{32} & k_4 y_{42} \\
k_1 y_{13} & k_2 y_{23} & k_3 y_{33} & k_4 y_{43} \\
k_1 y_{14} & k_2 y_{24} & k_3 y_{34} & k_4 y_{44}
\end{pmatrix}, \quad (25)
\]

where the sizes of the blocks are prescribed by formula (20). Then the system

\[
(\mathcal{L} - \lambda \mathbb{I}) \begin{pmatrix} X \\ Y \end{pmatrix} = 0
\]

is splitted into 16 independent subsystems with respect to pairs of blocks \( x_{ij}, y_{ij} \). For example, for the blocks \( x_{31}, y_{31} \) we have

\[
\left\{ \begin{array}{l}
(-1 - 2\alpha - \lambda) x_{31} + \alpha y_{31} = 0, \\
(2 + 2\beta) x_{31} + (-2 - \beta - \lambda) y_{31} = 0.
\end{array} \right.
\]

Equating the determinant of this system to zero, we find that \( \lambda = -1 \) and \( \lambda = -2 - 2\alpha - \beta \) belong to the spectrum of \( \mathcal{L} \). The dimension of the eigenspace corresponding to the second of the values of \( \lambda \) is determined by the size of the corresponding block and is equal to \( k_1 k_3 \). The eigenvalue \( \lambda_3 = -1 \) occurs in other blocks as well. The sum \( d_3 \) of the numbers of elements in these blocks is given in Proposition 4.

The blocks that form the eigenspaces are called resonance.

Lemma 4. Suppose that a solution (12) is maximal. Then \( k_1^2 + k_2^2 + k_3^2 = 1 \) and all eigenvalues of the operator \( \mathcal{L} \) except \( \lambda_3 = -1 \) and \( \lambda_4 = -2 \) are non-negative integers.

Proof. If all eigenvalues except \( \lambda_3 = -1 \) and \( \lambda_4 = -2 \) are non-negative integers, then the possible number of arbitrary constants is equal to

\[
2n^2 - d_1 - d_3 + \dim \mathcal{O} = 2n^2 - k_1^2 - k_2^2 - k_3^2.
\]

Thus, as in the scalar case, there are three types of possible maximal solutions:
• 1: \( k_1 = 1, \quad k_2 = k_3 = 0, \quad k_4 = n - 1 \);
• 2: \( k_2 = 1, \quad k_1 = k_3 = 0, \quad k_4 = n - 1 \);
• 3: \( k_3 = 1, \quad k_1 = k_2 = 0, \quad k_4 = n - 1 \).

In each of these cases, \( \dim \mathcal{O} = 2n - 2 \). Thus, for fixed \( p \) and \( q \), the coefficients of a maximum series must contain \( 2n^2 - 2n + 1 \) arbitrary constants.

**Proposition 5.** i) A maximal solution of type 1 for system (10) exists iff \( \alpha + \beta \) is an integer satisfying the inequalities \(-1 \geq \alpha + \beta \geq -3\). ii) A maximal solution of type 2 exists iff \( \beta \) is an integer and \( 0 \geq \beta \geq -2 \). iii) A maximal solution of type 3 exists iff \( \alpha \) is an integer and \( 0 \geq \alpha \geq -2 \).

**Proof.** In the case 1, the spectrum of the operator \( \mathcal{L} \) is the set \( \lambda_1, \ldots, \lambda_4 \), \( \lambda_{11} = 3 + \alpha + \beta \), and \( \lambda_{16} = -1 - \alpha - \beta \). The conditions of Lemma 4 hold if \( \lambda_{11}, \lambda_{16} \in \mathbb{Z}_{\geq 0} \). This means that \( \alpha + \beta \) is an integer satisfying the inequalities \(-1 \geq \alpha + \beta \geq -3 \). Similarly, we show the necessity of the conditions of the proposition for solutions of the types 2 and 3.

For series of type 1, there exist three cases \( \alpha + \beta = -1, -2, -3 \) with different sets of resonance blocks. To prove the sufficiency in the case 1, we have to verify that in all these cases the right hand sides of relations (14) are equal zero in the position of resonance blocks. This can be done by a direct calculation based on formula (15). The series of types 2 and 3 are considered in a similar way.

**Remark 4.** Points from the set \( \Sigma \) (see Figure 1) are distinguished by the requirement that system (10) has more than one maximal solution with commuting residues. For seven of these points, there are three different maximal solutions of types 1-3, and for the rest (marked in Figure 1 with the dots surrounded by an orange rim), there are two. As will be shown below, the latter have one more maximal solution with non-commuting residues.

**2.1.2 Maximal solutions with non-commuting \( p \) and \( q \)**

If \( (\alpha, \beta) \in \Sigma_0 \) there exist residues more general than (20) (see Theorem 3). Consider the case \( \alpha = 0, \beta = -3 \) as an example. One can check that the eigenvalues for the operator (17) are \( \lambda = -2, \ldots, 4 \). The dimensions \( d_\lambda \) of the corresponding eigenspaces are given by:

\[
\begin{align*}
d_{-2} &= m^2 + m(k_1 + 2k_2 + k_3) + k_1^2 + k_2^2 + k_3^2 + k_1k_2 + k_2k_3, \\
d_{-1} &= 3m^2 + m(4k_1 + 3k_2 + 4k_3 + 3k_4) + 2k_1k_2 + 2k_1k_3 + 2k_2k_3 + 2k_1k_4 + 3k_2k_4 + 2k_3k_4, \\
d_0 &= 2m^2 + m(k_1 + k_3 + 4k_4) + 2k_1^2 + k_1k_4 + k_3k_4, \quad d_1 = 2k_1k_3, \\
d_2 &= m^2 + m(k_1 + 2k_2 + k_3) + k_1^2 + k_2^2 + k_3^2 + k_1k_4 + k_3k_4, \\
d_3 &= m^2 + m(k_2 + k_4) + k_2k_4, \quad d_4 = m(k_1 + k_3) + k_1k_2 + k_2k_3.
\end{align*}
\]

The dimension of the stabilizer of the pair of residues \((p, q)\) is equal to

\[d = m^2 + m(k_2 + k_4) + k_1^2 + k_2^2 + k_3^2 + k_4^2.\]

Since \( \dim \mathcal{O} = n^2 - d \), the possible number of arbitrary constants \( M \) in the formal solution (12) is given by

\[M = 2n^2 - d_{-2} - d_{-1} + \dim \mathcal{O} = 2n^2 - 2m^2 - m(k_1 + 2k_2 + 3k_4) - k_1^2 - k_2^2 - k_3^2 - k_2(k_1 + k_3 + k_4). \quad (26)\]
A similar calculation shows that for $\alpha = \beta = 0$ the number $M$ of parameters is defined by the formula

$$M = 2n^2 - m^2 - m(2k_1 + k_2 + k_3) - k_1^2 - k_2^2 - k_3^2 - k_1(k_2 + k_3 + k_4);$$  \hspace{1cm} (27)

for $\alpha = 0$, $\beta = -1$ we obtain

$$M = 2n^2 - 2m^2 - 2m(k_1 + k_3) - k_1^2 - k_2^2 - k_3^2 - k_1k_3;$$  \hspace{1cm} (28)

and in the case $\alpha = 0$, $\beta = -2$ we have

$$M = 2n^2 - 2m^2 - 2m(k_2 + k_3) - k_1^2 - k_2^2 - k_3^2 - k_2k_3.$$  \hspace{1cm} (29)

For the remaining points of $\Sigma_0$ the number $M$ can be found by means of the discrete symmetries (5) and (6).

The maximal solutions for which $m = 0$, are described in the previous section. Let $m > 0$. In the cases $\alpha = 0$, $\beta = -3$ and $\alpha = \beta = 0$ from formulas (26), (27) it follows that $M$ may be equal $2n^2 - 1$ only if $m = 1$, $k_1 = k_2 = k_3 = 0$, $k_4 = n - 2$.

To prove that for $m = 1$ the formal solution (12) is indeed maximal in these two cases, we represent the matrix coefficients of the formal series in the block form

$$x_k = \begin{pmatrix} 1 & 1 & n - 2 \\ x_{k,11} & x_{k,12} & x_{k,13} \\ x_{k,21} & x_{k,22} & x_{k,23} \\ x_{k,31} & x_{k,32} & x_{k,33} \end{pmatrix} 1, \quad y_k = \begin{pmatrix} 1 & 1 & n - 2 \\ y_{k,11} & y_{k,12} & y_{k,13} \\ y_{k,21} & y_{k,22} & y_{k,23} \\ y_{k,31} & y_{k,32} & y_{k,33} \end{pmatrix} 1.$$  \hspace{1cm} (28)

The resonances in these cases are $k = 0, 1, 2, 3$. Considering the recursion relations (14) for these $k$, we find that the arbitrary blocks are $x_{0,11}$, $x_{0,12}$, $x_{0,13}$, $x_{0,31}$, $x_{0,32}$, $x_{0,33}$, $y_{0,11}$, $y_{0,12}$, $y_{0,13}$, $y_{0,31}$, $y_{0,32}$, $y_{0,33}$. The sum of dimensions of these blocks is equal to $2n^2 - 3n + 2$.

From the formulas (28) and (29), it follows that in the cases of $\alpha = 0$, $\beta = -1$ and $\alpha = 0$, $\beta = -2$, there are no maximal solutions with $m > 0$.

Putting together the results of sections 2.1.1 and 2.1.2, we arrive at the following statement:

**Theorem 5.** System (10) has three different maximal solutions (12) (cf. with the scalar case) iff $(\alpha, \beta) \in \Sigma$. For the points that are vertices of the star in Figure 1, two of these solutions have commuting residues, and for the third one the residues do not commute. For the remaining seven points, the residues commute with each other for all maximal solutions.

### 3 Inhomogeneous integrable systems of $P_4$ type

In the scalar case, the procedure we use is as follows. Suppose we want to find linear integrable deformations of a homogeneous system (8) of the form

$$ \begin{cases} u' = -u^2 + 2uv - 2zu + b_1u + b_2v + c_1, \\ v' = -v^2 + 2uv + 2zv + b_3v + b_4u + c_2, \end{cases} \hspace{1cm} b_i, c_i \in \mathbb{C},$$  \hspace{1cm} (30)

satisfying the Painlevé–Kovalevskaya test and to “regain” system (2). If desired, unknown linear polynomials in $u, v$ could be used in this ansatz as coefficients of $z$. This somewhat complicates the calculations and leads, in addition to system (8), to several systems in which the explicit dependence on $z$ can be eliminated by a changing of variables.
In order for us to be able to add the dimension 2, (3.1 Linear deformation of system) arbitrary blocks are this, we subdivide the matrices involved in the recurrence relations into the solutions of the type 1, 2, or 3, containing the maximum possible number of arbitrary constants.

When restoring linear terms, the following principle works efficiently:

- All maximal solutions of a homogeneous system have to allow prolongation to solutions of the inhomogeneous system while preserving the property of their maximality.

In particular, in the scalar example discussed above, we need to require the existence of maximal deformations of all three solutions, and then the answer will only be system (2).

Below we use this procedure to find integrable linear deformations (3) of homogeneous matrix systems (10), corresponding to the pairs \((\alpha, \beta)\), shown by the red dots in Figure 1. For the remaining points, the answer can be obtained using transformations (5) and (6).

For diagonal \(p\) and \(q\) a maximal solution has to contain \(2n^2 - 2n + 1\) arbitrary constants. In order for us to be able to add the dimension \(2n - 2\) of the orbit \(O\) to this number, such a maximal solution must exist for arbitrary matrices \(p, q \in O\). Of course, we can reduce these matrices to a diagonal form by a conjugation, but all matrix coefficients \(b_1, c_1\) will also be conjugated. Therefore, if we obtain some condition for these coefficients, it must also be satisfied when replacing \(b_1 \mapsto Sb_1S^{-1}, c_1 \mapsto Sc_1S^{-1}\), where \(S\) is an arbitrary non-degenerate matrix. In other words, the conditions are supposed to be \(GL_n\)-invariant.

Remark 5. If we discovered that for some matrix \(A\) the condition \(a_{i,j} = 0\) is satisfied for \(i \neq j\), then \(GL_n\)-invariance implies that \(A = \gamma I, \gamma \in \mathbb{C}\).

### 3.1 Linear deformation of system (10) with \(\alpha = \beta = -1\)

In this case system (10) possesses maximal solution of each of the three types 1, 2, 3. The arbitrary blocks are \(x_{0,2}, y_{0,4}, x_{1,2}, x_{1,4}, x_{2,2}, y_{2,2}\), where \(\mathbf{j}\) means the type of the series.

When substituting series of the form (12) into the system (3), recurrent relations of the form (16) appear, in which the operator \(L\) in the left part is the same as in the homogeneous case, and the right hand sides

\[
\begin{align*}
   f_1(x_k, y_k) &= f_\alpha(x_{k-1}, y_{k-1}) + 2z_0x_{k-1} + 2x_{k-2} (1 - \delta_{0,k}) \\
   & \quad - b_1x_{k-1} - X_{k-1}b_2 - b_3y_{k-1} - y_{k-1}b_4 - b_5\delta_{1,k}, \\
   f_2(x_k, y_k) &= f_\beta(y_{k-1}, x_{k-1}) - 2z_0y_{k-1} - 2y_{k-2} (1 - \delta_{0,k}) \\
   & \quad - c_1y_{k-1} - Y_{k-1}c_2 - c_3x_{k-1} - x_{k-1}c_4 - c_5\delta_{1,k},
\end{align*}
\]

(31)

where \(\delta\) is the Kronecker delta and \(X_{-1} = p, Y_{-1} = q\), now depend not only on the previous coefficients of the series, but also on \(b_1, c_1\).

According to the requirement of integrability of deformation (3), the same blocks as for system (10) must remain arbitrary. This means that in the recurrence relations, the corresponding blocks in the right hand sides are to be zero. We call this equality the resonance condition in such a block.

Consider the matrix equations (16) with right hand sides (31) at \(k = 0, 1, 2\). To do this, we subdivide the matrices involved in the recurrence relations into \(2 \times 2\)-blocks of the
corresponding sizes:

\[
x_k = \begin{pmatrix} x_{k,ij} & x_{k,j4} \\ x_{k,4j} & x_{k,44} \end{pmatrix} \begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix}, \quad y_k = \begin{pmatrix} y_{k,ij} & y_{k,j4} \\ y_{k,4j} & y_{k,44} \end{pmatrix} \begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix},
\]

\[
b_i = \begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix}, \quad c_i = \begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix},
\]

where \( x_k, y_k \) are matrix coefficients of formal solution (12) and \( b_i, c_i, i = 1, \ldots, 5 \) are matrix coefficients of the deformed system (3).

**Resonance conditions at** \( k = 0 \)

When defining the matrices \( x_0, y_0 \) for series of all three types, the solvability conditions do not arise, the blocks \( x_{0,44}, y_{0,44} \) are arbitrary, and the remaining blocks are uniquely defined.

**Resonance conditions at** \( k = 1 \)

For the series of type 2, two resonance conditions arise:

\[
2z_0c_{4,24} - (c_{3,24} - c_{4,24}) x_{0,44} - (b_{2,24} + c_{1,24}) y_{0,44}
+ \frac{1}{2} (-b_{2,24}c_{3,22} + c_{2,24}c_{3,22} + b_{2,24}c_{4,22} + c_{2,24}c_{4,22} + 2c_{2,24}c_{4,24} + 2c_{2,44}c_{4,24}
+ 2c_{3,22}c_{4,24} - 2b_{2,24}c_{4,44} + 2c_{2,44}c_{4,44} - 2c_{3,24}) = 0,
\]

\[
2z_0c_{3,42} + (c_{3,42} - c_{4,42}) x_{0,44} - (b_{1,42} + c_{2,42}) y_{0,44}
+ \frac{1}{2} (b_{1,42}c_{3,22} + c_{1,42}c_{3,22} + 2c_{1,44}c_{3,4} + 2b_{1,42}c_{3,4} - 2b_{1,42}c_{3,44} + 2c_{3,42}c_{3,44}
- b_{1,42}c_{4,22} + c_{1,42}c_{4,22} + 2c_{3,42}c_{4,22} - 2c_{3,42}) = 0.
\]

Due to the arbitrariness of the point \( z_0 \) and the blocks \( x_{0,44}, y_{0,44} \), these relations are equivalent to a system of eight equations. From the six simplest equations and the \( GL_n \)-invariance of the conditions (see Remark 5), it follows that

\[
c_1 = -b_2 + \gamma_1 I, \quad c_2 = -b_1 + \gamma_2 I, \quad c_3 = \gamma_3 I, \quad c_4 = (\gamma_3 + \gamma_4) I, \quad \gamma_i \in \mathbb{C}. \quad (32)
\]

Taking into account relations (32), we find that the remaining two equations are equivalent to the relation

\[
c_5 = -\left( \gamma_3 + \frac{1}{2} \gamma_4 \right) (b_1 + b_2) + \gamma_5 I, \quad \gamma_5 \in \mathbb{C}. \quad (33)
\]

If the matrices \( b_i, c_i \) satisfy the conditions (32) – (33), then for \( k = 1 \) a formal solution of type 2 has two arbitrary blocks \( x_{1,24} \) and \( x_{1,42} \).

For a solution of the type 3, two solvability conditions also arise:

\[
-2z_0b_{3,34} - (b_{3,34} + c_{3,34}) x_{0,44} - (b_{3,34} - c_{2,34}) y_{0,44}
+ \frac{1}{2} (b_{2,34}b_{3,33} + b_{2,34}b_{4,33} + 2b_{1,33}b_{3,34} + 2b_{2,44}b_{4,34} + 2b_{3,33}b_{4,34} + 2b_{3,34}b_{4,44}
- b_{3,33}c_{2,34} + b_{4,33}c_{2,34} - 2b_{4,44}c_{2,34} - 2b_{5,34}) = 0,
\]
Substituting them into the resonance conditions at arbitrary blocks respectively. The corresponding system of four equations has two solutions:

\[-2z_0b_{3,43} - (b_{2,43} + c_{1,43})x_{0,44} + (b_{3,43} - b_{4,43})y_{0,44} + \frac{1}{2}(b_{1,43}b_{3,33} + 2b_{1,44}b_{3,43} + 2b_{2,33}b_{3,43} + 2b_{3,43}b_{4,44} + b_{1,43}b_{4,33} + 2b_{3,43}b_{4,33} + b_{3,33}c_{1,43} - 2b_{3,44}c_{1,43} - b_{4,33}c_{1,43} - 2b_{5,43}) = 0.\]

Reasoning as above, we can check that the six simplest equations derived from the conditions (34) – (35) have a $GL_n$-invariant solution of the form

\[c_1 = -b_2 + \delta_1 \mathbb{I}, \quad c_2 = -b_1 + \delta_2 \mathbb{I}, \quad c_3 = \delta_3 \mathbb{I}, \quad c_4 = (\delta_3 + \delta_4) \mathbb{I}, \quad \delta_i \in \mathbb{C}.\] (36)

It follows from the two remaining equations that

\[b_5 = \left(\delta_3 + \frac{1}{2}\delta_4\right)(b_1 + b_2) + \delta_5 \mathbb{I}, \quad \delta_5 \in \mathbb{C}.\] (37)

If the conditions (36) – (37) for the matrices $b_i, c_i$ hold, then a series of type 3 has two arbitrary blocks $x_{1,34}$ and $x_{1,43}$.

Taking together, the conditions (32) – (33) and (36) – (37) are equivalent to the relations

\[c_1 = -b_2 + \gamma_1 \mathbb{I}, \quad c_2 = -b_1 + \gamma_2 \mathbb{I}, \quad c_3 = \gamma_3 \mathbb{I}, \quad c_4 = (\gamma_3 + \gamma_4) \mathbb{I},\]

\[c_5 = -\left(\gamma_3 + \frac{1}{2}\gamma_4\right)(b_1 + b_2) + \gamma_5 \mathbb{I};\]

\[b_3 = \delta_3 \mathbb{I}, \quad b_4 = (\delta_3 + \delta_4) \mathbb{I}, \quad b_5 = \left(\delta_3 + \frac{1}{2}\delta_4\right)(b_1 + b_2) + \delta_5 \mathbb{I}.\]

Substituting them into the resonance conditions at $k = 1$ for a solution of type 1, we get that it has two arbitrary blocks $x_{1,41}$ and $x_{1,14}$ iff

\[[b_1, b_2] = \frac{1}{2}(\gamma_1 + \gamma_2)(b_1 + b_2) + \varepsilon \mathbb{I}, \quad \varepsilon \in \mathbb{C}.\] (38)

**Resonance conditions at $k = 2$**

Each of the series of type 2 and 3 has only one resonance solvability condition. They are given by

\[(\gamma_1 + \gamma_2)(2\gamma_3 + \gamma_4)z_0 + \frac{1}{4}(8\gamma_3 - 4\gamma_1^2\gamma_3 - 8\gamma_1\gamma_2\gamma_3 - 4\gamma_2^2\gamma_3 - 4\gamma_1\gamma_3^2 + 4\gamma_2\gamma_3^2 + 4\gamma_1\gamma_2\gamma_4 - 4\gamma_2^2\gamma_4 - 4\gamma_1\gamma_3\gamma_4 - 4\gamma_2\gamma_3\gamma_4 - \gamma_1\gamma_4^2 - \gamma_2\gamma_4^2 + 4\gamma_1\gamma_5 + 4\gamma_2\gamma_5) = 0,\]

\[(\gamma_1 + \gamma_2)(2\delta_3 + \delta_4)z_0 + \frac{1}{4}(-8\delta_3 - 4\delta_4 - 4\delta_3^2\gamma_1 - 4\delta_3\delta_4\gamma_1 - \delta_4^2\gamma_1 + 4\delta_5\gamma_1 - 4\delta_3^2\gamma_2 - 4\delta_3\delta_4\gamma_2 - \delta_4^2\gamma_2 + 4\delta_5\gamma_2) = 0,\]

respectively. The corresponding system of four equations has two solutions:

\[\gamma_2 = -\gamma_1, \quad \gamma_4 = -2\gamma_3, \quad \delta_4 = -2\delta_3;\] (39)

and

\[\gamma_5 = \delta_5 = 0, \quad \gamma_4 = -2\gamma_3, \quad \delta_4 = -2\delta_3.\] (40)
If relations (39) are satisfied, then the series of type 1, 2, 3 have an arbitrary block \(x_{2,ij}\), i.e. they are maximal. In system (38), one have to put \(\varepsilon = 0\), since the commutator of two matrices is a traceless matrix.

The system corresponding to solution (39) has the form
\[
\begin{align*}
\begin{cases} 
 u' = -u^2 + uv + vu - 2zu + b_1 u + wb_2 + \gamma_1 \mathbb{I}, \\
 v' = -v^2 + vu + uv + 2zv - b_2 v - vh_1 + \gamma_2 \mathbb{I},
\end{cases}
\end{align*}
\]
\(\begin{bmatrix} b_1, b_2 \end{bmatrix} = 0, \quad \gamma_1, \gamma_2 \in \mathbb{C}. \quad (41)\)

In the case of solution (40), the requirement for the existence of a maximal solution of type 1 leads to the relation
\[
\frac{1}{4}(\gamma_1 + \gamma_2)(-12 + 12\gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2) + (\gamma_1 + \gamma_2)^2 z_0 = 0.
\]
This implies that \(\gamma_1 = -\gamma_2\) and we arrive at a particular case of the system (41).

**Remark 6.** This system can be rewritten in the spirit of the paper [6] using the noncommutative independent variable \(\bar{z}\):
\[
\begin{align*}
\begin{cases} 
 u' = -u^2 + uv + vu + (k - 2)u\bar{z} - ku\bar{z} + \gamma_1 \mathbb{I}, \\
 v' = -v^2 + vu + uv + k\bar{z}v - (k - 2)v\bar{z} + \gamma_2 \mathbb{I}.
\end{cases}
\end{align*}
\]
Applying the following transformation of the form (7):
\[
u \mapsto e^{zb_2} u e^{-zb_2}, \quad v \mapsto e^{zb_2} v e^{-zb_2},
\]
and renaming \(h = b_1 + b_2\), we reduce system (41) to \(P_4\).

### 3.2 Case \(\alpha = 0, \beta = -3\)

In contrast to the previous case, the homogeneous system (10) has only two maximal formal solutions of types 1 and 3 with the resonance blocks \(x_{0,14}, x_{0,44}, y_{0,44}, x_{2,11}, y_{2,41} \) and \(x_{0,34}, x_{0,44}, y_{0,44}, x_{2,33}, x_{2,43}\) respectively. We require that the same holds true for the deformation (3).

Calculations similar to those described above lead to the system
\[
\begin{align*}
\begin{cases} 
 u' = -u^2 + 2uv - 2zu - [u, h_1] - 3h_2 u + h_2 v + h_4, \\
 v' = -v^2 + 3uv - vu + 2zv - [v, h_1] + 3vh_2 + h_3 u - 3uh_2 + (2h_4 + \frac{1}{2}h_3 h_2 + \gamma \mathbb{I}),
\end{cases}
\end{align*}
\]
\(\begin{bmatrix} h_1, h_2 \end{bmatrix} = [h_1, h_3] = [h_2, h_3] = 0, \quad [h_4, h_2 - h_3] = -2(h_2 - h_3), \quad [h_4, 2h_1 - 5h_2] = -2h_2, \quad \gamma \in \mathbb{C}. \quad (42)\)

System (42) can be simplified by a proper transformation of the form (7). It follows from the commutator relations that
\[
e^{-\frac{1}{2}z(2h_1 - 5h_2)} h_4 e^{\frac{1}{2}z(2h_1 - 5h_2)} = h_4 - \frac{1}{2} z[2h_1 - 5h_2, h_4] = h_4 - zh_2.
\]
Using this formula, one can verify that the mapping
\[
u \mapsto e^{-\frac{1}{2}z(2h_1 - 5h_2)} \left(u + \frac{1}{2}h_2\right) e^{\frac{1}{2}z(2h_1 - 5h_2)}, \quad v \mapsto e^{-\frac{1}{2}z(2h_1 - 5h_2)} (v - h_2) e^{\frac{1}{2}z(2h_1 - 5h_2)}.
\]
reduces system (42) to \(P_4\).
3.3 Case $\alpha = \beta = 0$

As in the previous case, the homogeneous system (10) with $\alpha = \beta = 0$ has only two maximal series of types 2 and 3 with the resonance blocks $x_{0,24}$, $x_{0,44}$, $y_{0,44}$, $x_{2,22}$, $y_{2,42}$ and $x_{0,34}$, $x_{0,44}$, $y_{0,44}$, $x_{2,33}$, $x_{2,43}$ respectively. From the requirement of the existence of the same resonance blocks in the formal solution (12) for the deformation (3), it follows that the system has the form

$$
\begin{align*}
&u' = -u^2 + 2uv - 2zu + [u, h_1] + h_2v + h_4, \\
&v' = -v^2 + 2vu + 2zv + [v, h_1] + h_3u + (-h_4 + \frac{1}{2}h_2h_3 + \gamma I),
\end{align*}
$$

(43)

and a renaming the coefficients, system (43) is reduced to the canonical form $P^4_3$.

3.4 Case $\alpha = 0$, $\beta = -2$

Similar to the case $\alpha = \beta = -1$, the homogeneous system (10) has three maximal formal solutions. The resonant blocks are $x_{0,44}$, $y_{0,44}$, $x_{1,14}$ in a series of type 1, $y_{0,42}$, $x_{0,44}$, $y_{0,44}$, $x_{2,22}$, $x_{2,24}$ in a series of type 2, and $x_{0,34}$, $x_{0,44}$, $x_{2,33}$, $x_{2,43}$ in a series of type 3. We require that the same holds true for the deformation (3).

After a renaming the coefficients the result can be written as

$$
\begin{align*}
&u' = -u^2 + 2uv - 2zu + [u, h_1] + 2uh_2 - h_2v + (h_3 + \gamma I), \\
&v' = -v^2 + 2uv + 2zv + [v, h_1] - 2h_2v + uh_2 + h_3,
\end{align*}
$$

(44)

and the subsequent renaming $h = h_3 + \frac{3}{2}h_2^2$.

3.5 Case $\alpha = 0$, $\beta = -1$

In this case, homogeneous system (10) has three maximal formal solutions with resonance blocks $y_{0,41}$, $x_{0,44}$, $y_{0,44}$, $x_{2,11}$, $x_{2,14}$ in a series of type 1, $x_{0,44}$, $y_{0,44}$, $x_{1,24}$, $y_{1,42}$, $x_{2,22}$ in a series of type 2, and $x_{0,34}$, $x_{0,44}$, $y_{0,44}$, $x_{2,33}$, $x_{2,43}$ in a series of type 3. By requiring the same blocks to be resonance in the formal solution (12) for the deformation (3), we obtain the system

$$
\begin{align*}
&u' = -u^2 + 2uv - 2zu + h_1u + uh_2 - (h_1 + h_2)v + h_3, \\
&v' = -v^2 + vu + uv + 2zv - vh_1 - h_2v + \gamma I,
\end{align*}
$$

(45)

and the subsequent renaming $h = h_3 - \frac{1}{2}h_2$. 

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System (45) coincides with the example from the paper [2] if \( h_1 = -h_2 = \delta I \).
To obtain the canonical form \( P_4^1 \), one can use the following transformation of the form (7):
\[
\begin{align*}
    u &\mapsto e^{-\frac{1}{2}z(h_1-h_2)}\left(u - \frac{1}{2}(h_1 + h_2)\right)e^{\frac{1}{2}z(h_1-h_2)}, \\
v &\mapsto e^{-\frac{1}{2}z(h_1-h_2)}ve^{\frac{1}{2}z(h_1-h_2)}.
\end{align*}
\]

**Remark 7.** In order to get system (41), we had to use the requirement of the existence of three maximal solutions of types \(1\)-\(3\). One can verify that to find any of systems (42)–(45), it is sufficient to require that there exist only two maximal solutions of types \( (1, 2) \), \( (1, 3) \), or \( (2, 3) \). In other words, the inhomogeneous systems corresponding to the points from \( \Sigma_0 \) are distinguished by the existence of two maximal solutions. It turns out that for each of these systems there exists a third maximal solution (12) with non-commuting residues \( p \) and \( q \) (see Section 2.1.2).

### 4 Degeneracies to Painlevé-2 equations

In this section, we will use the notation from [2], where the matrix Painlevé-2 equations were found.

Consider the scalar system (2), in which the shift \( z \mapsto z - \frac{1}{2}b_1 \) is made, and perform the following transformation:
\[
\begin{align*}
    z &= \frac{1}{4} \varepsilon^{-3} - \varepsilon x, \\
u(z) &= -\frac{1}{4} \varepsilon^{-3} - \varepsilon^{-1} f(x), \\
v(z) &= -2 \varepsilon g(x), \\
c_1 &= -\frac{1}{16} \varepsilon^{-6}, \\
c_2 &= 2 \theta, \quad \theta \in \mathbb{C}.
\end{align*}
\]

As a result, the system will look like
\[
\begin{align*}
f' &= 2 \varepsilon^2 (2fg - xf) - f^2 + g - \frac{1}{2} x, \\
g' &= 2 \varepsilon^2 (g^2 - xg + 2fg + \theta).
\end{align*}
\]
Passing to the limit \( \varepsilon \to 0 \), we obtain a system, which is equivalent to the Painlevé-2 equation
\[
y'' = 2y^3 + xy + \left( \theta - \frac{1}{2} \right)
\]
with respect to \( y(x) = f(x) \).

Let us demonstrate that all matrix Painlevé-2 equations found in [2] can be obtained from systems of Section 3 by a similar limiting transition.

The matrix system (3) as a result of the same as in the scalar case substitution (46), supplemented by the shift \( b_5 \mapsto b_5 + \frac{1}{16} \varepsilon^{-6} I \), takes the form
\[
\begin{align*}
f' &= 2 \varepsilon^3 (-b_3g - gb_4) + 2 \varepsilon^2 \left( 2fg + \alpha [f, g] - xf + \frac{1}{2} b_5 \right) + \varepsilon (-b_1f - fb_2) \\
    &\quad + \left( -f^2 + g - \frac{1}{2} x I \right) + \frac{1}{4} \varepsilon^{-1} (-b_1 - b_2), \\
g' &= 2 \varepsilon^2 (g^2 + xg) + \varepsilon (-c_1g - gc_2) + \left( 2fg + \beta [g, f] + \frac{1}{2} c_5 \right) \\
    &\quad + \frac{1}{2} \varepsilon^{-1} (-c_3f - fc_4) + \frac{1}{8} \varepsilon^{-3} (-c_3 - c_4).
\end{align*}
\]
Let us consider the limits of this system in particular cases.
4.1 Case $\alpha = \beta = -1$

For system (41) corresponding to this case, we have

\[ b_1 = h_1, \quad b_2 = h_2, \quad b_3 = b_4 = 0, \quad b_5 = \gamma_1, \]
\[ c_1 = -h_2, \quad c_2 = -h_1, \quad c_3 = c_4 = 0, \quad c_5 = \gamma_2, \quad \gamma_i \in \mathbb{C}, \]

where the matrices $h_1, h_2$ commute: $[h_1, h_2] = 0$. Let $h_1 = h_2 = 2\varepsilon b$, where $b \in \text{Mat}_n(\mathbb{C})$. Then the limiting system (47) has the form

\[
\begin{cases}
  f' = -f^2 + g - \frac{1}{2}xI - b, \\
g' = gf + fg + \gamma.
\end{cases}
\]

The latter system is equivalent to the equation $P_2^0$ from [2] with respect to $y(x) = f(x)$.

4.2 Case $\alpha = 0, \beta = -2$

The coefficients of the deformed system (44) are given by

\[ b_1 = -h_1, \quad b_2 = h_1 + 2h_2, \quad b_3 = -h_2, \quad b_4 = 0, \quad b_5 = h_3 + \gamma, \]
\[ c_1 = -h_1 - 2h_2, \quad c_2 = h_1, \quad c_3 = 0, \quad c_4 = h_2, \quad c_5 = h_3, \]

and satisfy the following commutation relations:

\[ [h_1, h_2] = 0, \quad [h_3, 2h_1 + h_2] = -2h_2. \]

Let $a = \frac{1}{2}h_3, h_1 = 0$ and $h_2 \mapsto \varepsilon^{-4}h_2$. Then after taking the limit $\varepsilon \to 0$ in (47) we get the system

\[
\begin{cases}
  f' = -f^2 + g - \frac{1}{2}xI, \\
g' = 2fg + a,
\end{cases}
\]

which is equivalent to the equation $P_2^1$ from [2] with respect to $y(x) = f(x)$.

The system (48) is also equivalent to the equation

\[ w'' = \frac{1}{2}(w' + a)w^{-1}(w' - a) + 2w^2 - xw \]

for $w(x) = g(x)$, which coincides with the Painlevé-34 equation in the scalar case.

4.3 Case $\alpha = 0, \beta = -3$

In this case for system (42) we have

\[ b_1 = -3h_2 + h_1, \quad b_2 = -h_1, \quad b_3 = h_2, \quad b_4 = 0, \quad b_5 = h_4, \]
\[ c_1 = h_1, \quad c_2 = 3h_2 - h_1, \quad c_3 = h_3, \quad c_4 = -3h_2, \quad c_5 = 2h_4 + \frac{1}{2}h_3h_2 + \gamma, \]

where the matrix coefficients are connected by the relations

\[ [h_1, h_2] = [h_1, h_3] = [h_2, h_3] = 0, \quad [h_4, h_2 - h_3] = -2(h_2 - h_3), \quad [h_4, 2h_1 - 5h_2] = -2h_2. \]
Let $a = h_4 + \frac{1}{2} \gamma I$ and $h_1 = h_3 = 3h_2$, $h_2 = -\frac{4}{3} \varepsilon b$, with $b \in \text{Mat}_n(\mathbb{C})$. Then, passing to the limit $\varepsilon \to 0$ in (47), we obtain the system

\[
\begin{align*}
  f' &= -f^2 + g - \frac{1}{2} x I - b, \\
  g' &= 3f g - g f - 2[f, b] + a,
\end{align*}
\]

which is equivalent to the equation $P^2_2$ from [2] with respect to $y(x) = f(x)$.

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