THE URYSOHN UNIVERSAL METRIC SPACE IS HOMEOMORPHIC TO A HILBERT SPACE

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Abstract. The Urysohn universal metric space $U$ is characterized up to isometry by the following properties: (1) $U$ is complete and separable; (2) $U$ contains an isometric copy of every separable metric space; (3) every isometry between two finite subsets of $U$ can be extended to an isometry of $U$ onto itself. We show that $U$ is homeomorphic to the Hilbert space $l_2$ (or to the countable power of the real line).

1. Introduction

The Urysohn universal metric space $U$ is characterized up to isometry by the following properties: (1) $U$ is complete and separable; (2) $U$ contains an isometric copy of every separable metric space; (3) every isometry between two finite subsets of $U$ can be extended to an isometry of $U$ onto itself. (An isometry is a distance-preserving bijection; an isometric embedding is a distance-preserving injection.) The aim of the present paper is to show that the Urysohn space $U$ is homeomorphic to a Hilbert space (equivalently, to the countable power of the real line). This answers a question raised by Bogatyi, Pestov and Vershik.

There is another characterization of $U$. Let us say that a metric space $M$ is injective with respect to finite spaces, or finitely injective for short, if for every finite metric space $L$, every subspace $K \subset L$ and every isometric embedding $f : K \to M$ there exists an isometric embedding $\bar{f} : L \to M$ which extends $f$. Define compactly injective metric spaces similarly, considering compact (rather than finite) spaces $K$ and $L$. If a metric space $M$ contains an isometric copy of every finite metric space and satisfies the condition (3) above, then $M$ is finitely injective. Indeed, given finite metric spaces $K \subset L$ and an isometric embedding $f : K \to M$, we find an isometric embedding $g : L \to M$ and extend the isometry $gf^{-1} : f(K) \to g(K)$ to an isometry $h$ of $M$ onto itself. Then $h^{-1}g : L \to M$ is an isometric embedding of $L$ which extends $f$. Conversely, let $M$ be a finitely injective metric space. Then every countable metric space admits an isometric embedding into $M$ (use induction). If $M$ is also complete, it follows that $M$ contains an isometric copy of every separable metric space. Assume now that $M$ is also separable, and let $f : K \to L$ be an isometry between two finite subsets of $M$. Enumerating points of a dense countable subset of $M$ and using the back-and-forth method we can extend $f$ to an isometry between two dense countable subsets of $M$ and then to an isometry of $M$ onto itself. The same argument shows

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that any two complete separable finitely injective metric spaces are isometric. Thus the Urysohn space $U$ is the unique (up to isometry) metric space which is complete, separable and finitely injective.

The existence of $U$ was proved by Urysohn [8, 9]. An easier construction was found some 50 years later by Katetov [2], who also gave an example of a non-complete separable metric space satisfying the conditions (2) and (3) above, thus answering a question of Urysohn. Katetov’s construction was used in [10, 11, 12] to prove that the topological group $\text{Is}(U)$ of all isometries of $U$ is universal, in the sense that it contains an isomorphic copy of every topological group with a countable base. A deep result concerning the group $G = \text{Is}(U)$ was established by V.Pestov: the group $G$ is extremely amenable, i.e., every compact space with a continuous action of $G$ has a $G$-fixed point [6, 3].

A.M.Vershik showed that the space $U$ can be obtained as the completion of a countable metric space equipped with a metric which is either “random” or generic in the sense of Baire category [13, 14, 15].

Bogaty˘ı [1] proved that any isometry between two compact subsets of $U$ can be extended to an isometry of $U$ onto itself. It follows by the same argument that we used above for finitely injective spaces that $U$ is compactly injective (and is the unique complete separable compactly injective metric space). Using this, we deduce our Main Theorem from Toruńczyk’s Criterion [4, Theorem 5.2.12]: a complete separable metric space $M$ is homeomorphic to the Hilbert space $l_2$ if and only if $M$ is AR (= absolute retract) and has the discrete approximation property (this notion is defined below). Recall that all infinite-dimensional separable Banach spaces are homeomorphic to each other and to the countable power of the real line.

Given an open cover $\mathcal{U}$ of a space $X$, two points $x, y \in X$ are said to be $\mathcal{U}$-close if there exists $U \in \mathcal{U}$ such that $x, y \in U$. A family of subsets of a space $X$ is discrete if every point in $X$ has a neighbourhood which meets at most one member of the family. A metric space $M$ has the discrete approximation property if for every sequence $K_1, K_2, \ldots$ of compact subspaces of $M$ and every open cover $\mathcal{U}$ of $M$ there exists a sequence of maps $f_i : K_i \to M$ such that for every $i$ and every $x \in K_i$ the points $x$ and $f_i(x)$ are $\mathcal{U}$-close and the sequence $(f_i(K_i))$ is discrete. Equivalently [7], a metric space $(M, d)$ has the discrete approximation property if and only if for every sequence $K_1, K_2, \ldots$ of compact subspaces of $M$ and every continuous function $h$ on $M$ with values $> 0$ there exists a sequence of maps $f_i : K_i \to M$ such that $d(x, f_i(x)) \leq h(x)$ for every $i$ and every $x \in K_i$, and the sequence $(f_i(K_i))$ is discrete.

Let us reformulate Toruńczyk’s Criterion in the form that is convenient for our purposes. We say that a topological space $X$ is homotopically trivial if $X$ has trivial homotopy groups, that is, every map of the $n$-sphere $S^n = \partial B^{n+1}$ to $X$ admits an extension over the $(n + 1)$-ball $B^{n+1}$ ($n = 0, 1, \ldots$). (The term weakly homotopically trivial might be more appropriate.) Every contractible space is homotopically trivial; the converse in general is not true. The empty space is homotopically trivial. If a metric space $M$ has a base $\mathcal{B}$ such that for every non-empty finite subfamily $\mathcal{F} \subset \mathcal{B}$ the intersection $\cap \mathcal{F}$ is homotopically trivial, then $M$ is ANR [4, Theorem 5.2.12]. A metric space is AR if and only if it is homotopically trivial and ANR [4, Theorem 5.2.15]. Thus Toruńczyk’s Criterion can be reformulated as follows:
Theorem 1.1 (Toruńczyk’s Criterion). A complete separable metric space \( M \) is homeomorphic to a Hilbert space if and only if the following conditions hold:

(i) there is a base \( \mathcal{B} \) for \( M \) such that \( U, V \in \mathcal{B} \) implies \( U \cap V \in \mathcal{B} \), and every \( U \in \mathcal{B} \) is homotopically trivial;

(ii) \( M \) is homotopically trivial;

(iii) \( M \) has the discrete approximation property.

In the next section we show that the space \( U \) satisfies the conditions of this criterion.

2. Proof of the main theorem

Theorem 2.1 (Main Theorem). The Urysohn universal space \( U \) is homeomorphic to a Hilbert space.

Proof. We check the three conditions of Toruńczyk’s criterion.

(a) Let \( \mathcal{B} \) be the base for \( U \) consisting of all open balls \( O(a, r) = \{ x \in U : d(x, a) < r \} \) and their finite intersections. We claim that every member \( V = \bigcap_{i=1}^{k} O(a_i, r_i) \) of \( \mathcal{B} \) is homotopically trivial. Let a map \( f : S^n \to V \) be given. We must construct an extension \( \bar{f} : B^{n+1} \to V \).

Every metric space admits an isometric embedding into a normed linear space. Thus we may consider \( U \) as a subspace of a Banach space \( B \). Let \( V' = \bigcap_{i=1}^{k} O'(a_i, r_i) \), where \( O'(a, r) \) is the open ball centered at \( a \) of radius \( r \) in the space \( B \). Then \( V = V' \cap U \).

(b) Let \( \bar{f} \) be a continuous function \( \bar{f} : \bigcap_{i=1}^{k} O(a_i, r_i) \to G \) such that \( \bar{f} \) is contained in \( V' \), since for every \( x \in B^{n+1} \) and \( i = 1, \ldots, k \) we have \( d(f(x), a_i) = d(h(g(x)), h(a_i)) = d(g(x), a_i) < r_i \) (note that \( h(a_i) = a_i \), since \( a_i \in K \) and \( h \) fixes all points in \( K \)).

(c) The space \( U \) is homotopically trivial. The proof is the same as above but easier, since we do not have to care about points \( a_1, \ldots, a_k \).

We prove that \( U \) has the discrete approximation property. Let \( K_1, \ldots, K_n, \ldots \) be a sequence of non-empty compact subsets of \( U \), and let \( h \) be a continuous function on \( U \) with values \( > 0 \). We must construct a discrete sequence \( (L_n) \) of compact subsets of \( U \) and a sequence of maps \( f_n : K_n \to L_n \) such that \( d(f_n(x), x) \leq h(x) \) for every \( n \geq 1 \) and \( x \in K_n \).

We’ll need the notion of union of two metric spaces with a subspace amalgamated. Suppose that \( M_1, M_2, A \) are metric spaces, \( A \neq \emptyset \), and isometric embeddings \( f_i : A \to M_i, i = 1, 2 \), are given. The union \( M \) of \( M_1 \) and \( M_2 \) with the subspace \( A \) amalgamated is characterized by the following properties: there exist isometric embeddings \( h_i : M_i \to M \) such that \( M = h_1(M_1) \cup h_2(M_2) \), \( h_1f_1 = h_2f_2 \), and for every \( x \in M_1 \setminus f_1(A) \), \( y \in M_2 \setminus f_2(A) \)

\[
d(h_1(x), h_2(y)) = \inf\{d_1(x, f_1(z)) + d_2(f_2(z), y) : z \in A\},
\]

where \( d, d_1, d_2 \) are the metrics on \( M \), \( M_1 \), \( M_2 \), respectively. It is easy to see that such a space \( M \) exists and in the obvious sense is unique.
Let $N_i \subset K_i \times \mathbb{R}$ be the union of $K_i \times \{0\}$ and the graph of the restriction of $h$ on $K_i$. Equip $K_i \times \mathbb{R}$ with the metric $\rho$ defined by

$$\rho((x,t), (y,s)) = d(x,y) + |s-t|,$$

and consider the induced metric on $N_i$.

We now construct a sequence $(L_n)$ of compact subsets of $U$ by induction. Suppose the sets $L_i$ have been defined for $i < n$. We define $L_n$. Consider two compact metric spaces: $K_n \cup \bigcup_{i<n} L_i$ and $N_n$. Since $K_n$ lies in the first space and has a natural embedding into the second one (we mean the embedding $x \mapsto (x,0)$), we can construct their union with the subspace $N_n$ amalgamated. Write this union as $P = \bigcup_{i<n} L_i \cup K_n \cup \Gamma_n$, where $\Gamma_n = \{(x,h(x)) : x \in K_n\}$ is the graph of $h \upharpoonright K_n$. Since $U$ is compactly injective, there exists an isometric embedding $\phi : P \to U$ which is identity on each $L_i$ ($i < n$) and on $K_n$. Let $L_n = \phi(\Gamma_n)$. Let $f_n : K_n \to L_n$ be the composition of the map $x \mapsto (x,h(x))$ from $K_n$ onto $\Gamma_n$ and $\phi$. For every $x \in K_n$ the distance from $(x,0)$ to $(x,h(x))$ in $N_n$ is equal to $h(x)$, hence the distance from $x$ to $(x,h(x))$ in $P$ and the distance from $x$ to $f_n(x)$ in $U$ also are equal to $h(x)$. Thus $f_n$ moves every $x \in K_n$ by $h(x)$.

Note that for every $x \in K_n$ and $y \in \bigcup_{i<n} L_i$ the distance from $f_n(x)$ to $y$ is $\geq h(x)$. Indeed, by our construction this distance is equal to the distance from $(x,h(x)) \in \Gamma_n$ to $y$ in $P$ and thus also to

$$\inf\{d(y,z) + \rho((z,0), (x,h(x))) : z \in K_n\} = \inf\{d(y,z) + d(z,x) + h(x) : z \in K_n\} = d(y,x) + h(x) \geq h(x).$$

To conclude the proof, we must show that the sequence $(L_n)$ is discrete. Assume the contrary. Since the sequence $(L_n)$ is disjoint, there exists an infinite set $A$ of positive integers and points $y_i \in L_i$ ($i \in A$) such that the sequence $\{y_i : i \in A\}$ converges to some $p \in U$. Write $y_i = f_i(x_i)$, where $x_i \in K_i$. The distance from $y_n$ to $\{y_i : i < n, i \in A\} \subset \bigcup_{i<n} L_i$ tends to zero as $n \in A$ tends to infinity. On the other hand, we saw in the preceding paragraph that this distance is $\geq h(x_n)$. Therefore the sequence $\{h(x_n) : n \in A\}$ tends to zero. Since $d(x_n,y_n) = d(x_n,f_n(x_n)) = h(x_n) \to 0$ and $y_n \to p$, it follows that $x_n \to p$. But this contradicts the continuity of $h$ at $p$: we have $h(p) > 0$, $x_n \to p$ and $h(x_n) \to 0$.

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