Semiclassical Quantization of Giant Gravitons

Peter Ouyang
Joseph Henry Laboratories, Princeton University,
Princeton, New Jersey 08544, USA

Abstract

We study excited spherical branes ("giant gravitons") in $AdS \times S$ spacetimes with background flux. For large excitation, these branes may be treated semiclassically. We compute their spectra using Bohr-Sommerfeld quantization and use the AdS/CFT correspondence to relate them to anomalous dimensions in the dual field theory at strong coupling, expressed as a series expansion in powers of $1/N$. These effects resemble those due to $k$-body forces between quarks in Hartree-Fock models of baryons at large $N$. For branes expanded in AdS, we argue that the anomalous dimensions are due to loop corrections to the effective action.

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1 Introduction

The AdS/CFT correspondence \cite{1, 2, 3} relates weakly coupled string theories in backgrounds with flux and curvature to strongly coupled field theories. This duality is well-understood in the flat string background is nearly flat, but our understanding is more rudimentary in highly curved backgrounds; in the former case the massive string modes decouple and we can work with a low-energy effective theory, whereas in the latter case string effects are often non-negligible.

One approach to studying nonzero curvature relies on semiclassical methods \cite{4}. Highly excited strings can probe length scales on the order of the radius of curvature of the ambient spacetime, which normally makes calculations difficult. However, in the limit of high excitation semiclassical methods are reliable and, at least for some states of high symmetry, it turns out that we can extract some physical insight without working with the full quantum string theory. The authors of \cite{4} worked primarily with “folded” closed strings, but also remarked on the existence of pulsating string solutions, which \cite{5} subsequently studied using Bohr-Sommerfeld techniques.

In this paper, we present some investigations on the effects of finite curvature for excited spinning spherical brane probes. Unlike the case of F-strings studied in \cite{4, 5}, these curvature effects introduce $1/N$ corrections rather than $g_s N$ corrections to operator dimensions in the field theory. Brane probes also differ crucially from the previously studied strings in that they couple non-trivially to the background RR flux filling the spacetime, as observed by \cite{6, 7, 8, 9, 10}. Interaction with the background flux causes the branes to expand to finite size; these expanded branes are supersymmetric, and are known in the literature as “giant gravitons.” Some small fluctuations about these BPS states have been studied in earlier work \cite{11, 12, 13, 14}; we will compute the dimensions of the operators corresponding to these states when the fluctuations are not small. In Section 2, we present the case of spherical D3-branes in $AdS_5 \times S^5$. We briefly remark on the case of $AdS_3 \times S^3$ in Section 3, for which our techniques do not apply. In Section 4, we give the analogous results for the M-theory solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$, whose field theory duals are poorly understood. Finally, we comment on the range of validity and physical interpretation of our results in Section 5.

2 D3-Branes in $AdS_5 \times S^5$

Let us begin by studying oscillating branes in $AdS_5 \times S^5$. Our convention for the full ten-dimensional metric is

$$ds^2 = L^2 \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_3^2 \right)$$

(1)
where we have chosen global coordinates for the $AdS_5$. The RR four-form potential is

$$C_4 = \frac{L^4}{g_s} \left( \sinh^4 \rho dt \wedge \Omega_3 - \sin^4 \theta d\psi \wedge \Omega_3' \right)$$  \hspace{1cm} (2)

and all the other supergravity fields vanish. The AdS radius is given by $L^4 = 4\pi g_s N\alpha'^2$.

### 2.1 Oscillating D3-Branes in $AdS_5$

A simple way to embed an oscillating D3-brane in this geometry is to wrap the brane on the 3-sphere, and to let the radial position be a function of coordinate time on the brane, $\rho = \rho(\tau)$. If we also introduce angular momentum on the $S^5$ by making $\psi$ a function of $\tau$, then the action, consisting of the Dirac-Born-Infeld and Wess-Zumino terms, is

$$S = -T_3 L^4 \int \Omega_3 \int d\tau \left( \sinh^3 \rho \sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - \dot{\psi}^2} - \sigma \dot{t} \sinh^4 \rho \right).$$  \hspace{1cm} (3)

In this form, AdS coordinate time is $t = t(\tau)$. We also introduce $\sigma = \pm 1$, with the upper sign corresponding to a D3-brane and the lower to an anti-D3. Notice that the coefficient of the action is simply

$$T_3 L^4 \int \Omega_3 = N.$$  \hspace{1cm} (4)

For the semiclassical calculation we perform, the relevant quantity is really the action in $\hbar$ units: $S/\hbar \sim N/\hbar$. Thus, from the standpoint of the D-brane action (we ignore fluctuations of supergravity fields in the bulk of AdS), the semiclassical expansion is a series in powers of $\hbar/N$. We will set $\hbar = 1$ in the rest of this paper and in this section take the equivalent $N \gg 1$ limit.

Now let us find the classical equations of motion from this action. The Lagrangian does not depend on $t$, so the quantity

$$\frac{\partial L}{\partial \dot{t}} = -N \left( \frac{i \sinh^3 \rho \cosh^2 \rho}{\sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - \dot{\psi}^2}} - \sigma \sinh^4 \rho \right) \equiv -E$$  \hspace{1cm} (5)

is conserved. In terms of the integration constant $E$, we have

$$(-\dot{\rho}^2 - \dot{\psi}^2 + i^2 \cosh^2 \rho)(E/N + \sigma \sinh^4 \rho)^2 = i^2 \cosh^4 \rho \sinh^6 \rho.$$  \hspace{1cm} (6)

A convenient gauge choice is

$$i^2 \cosh^4 \rho = (E/N + \sigma \sinh^4 \rho)^2.$$  \hspace{1cm} (7)
From the \( \psi \) equation of motion, and using the gauge choice (7), we identify the conserved angular momentum

\[
\frac{\partial L}{\partial \dot{\psi}} = N \frac{\dot{\psi} \sinh^3 \rho}{\sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - \dot{\psi}^2}} = N \dot{\psi} \equiv J
\]  

which allows us to write the once-integrated equation of motion for \( \rho \) as

\[
\dot{\rho}^2 = \frac{(E/N + \sigma \sinh^4 \rho)^2}{\cosh^2 \rho} - \sinh^6 \rho - \left(\frac{J}{N}\right)^2.
\]  

Note that this equation of motion corresponds to one-dimensional motion of a particle in a potential

\[
V(\rho) = \frac{1}{2} \frac{E}{\cosh^2 \rho} \left(2\sigma \frac{E}{N} \sinh^4 \rho - \sinh^6 \rho - \left(\frac{E}{N}\right)^2 \sinh^2 \rho\right)
\]  

with energy \( \frac{1}{2} \left(\frac{E - J}{N}\right)^2 \). It is instructive to consider the unintegrated equation of motion for \( \rho \), which we can obtain by differentiating equation (8) with respect to \( \tau \):

\[
\ddot{\rho} = -\frac{\sinh \rho \cosh^3 \rho}{\cosh^3 \rho} (E/N + \sigma \sinh^4 \rho)^2 + 2\sigma \frac{\sinh^3 \rho \cosh \rho}{\cosh^3 \rho} (E/N + \sigma \sinh^4 \rho) - 3 \sinh^5 \rho \cosh \rho.
\]  

The first and last terms in equation (11) correspond to forces directed radially inward, while the middle term gives an outward force for \( \sigma = 1 \) (as it should, for a D3-brane) and an inward force for \( \sigma = -1 \) (for an anti-brane.)

When \( E = J \), there are special stable states. The equation of motion becomes

\[
\dot{\rho}^2 = -\frac{\sinh^2 \rho}{\cosh^2 \rho} \left(\frac{E}{N} - \sigma \sinh^2 \rho\right)^2
\]  

so that both \( \dot{\rho} \) and \( \ddot{\rho} \) vanish at \( \rho = 0 \) and, for \( \sigma = +1 \), at \( \rho = \sinh^{-1} \sqrt{\frac{E}{N}} \). These static spherical branes are the BPS "dual giant gravitons" discovered in [9, 10]. Whether or not the brane at \( \rho = 0 \) is a physically sensible state is not obvious – the brane becomes highly curved and we should not trust the DBI action. We will only consider the expanded branes, and will always take

\[
E \gtrsim N
\]  

so that the size of the D3-brane remains finite as we take \( N \gg 1 \). For the remainder of this subsection, \( \sigma = +1 \).

The frequency of small oscillations about the expanded BPS brane state is relatively simple to calculate. This computation was previously carried out in [11, 12]; we
will reproduce their result with a slightly different method. Making the convenient change of variables $y = \sinh \rho$, equation (9) becomes

$$y^2 = \left(\frac{E}{N}\right)^2 - \left(\frac{J}{N}\right)^2 (y^2 + 1) + 2\frac{E}{N}y^4 - y^6$$

(14)

with static brane solutions at $y = 0$ and $y^2 = E/N$. To find the frequency of oscillations about the latter solution, we set $J = E$ and treat the left hand side of (14) as an inverted potential $-2U(y)$. The frequency of small oscillations is then

$$\Omega^2 = U''(y = \sqrt{E/N}) = 4 \left(\frac{E}{N}\right)^2$$

(15)

This agrees with the result of [11] for spherical branes, provided that we convert our result to static gauge ($t = \tau$).

Now let us analyze these excited giant gravitons using semiclassical methods. The basic approach is to use Bohr-Sommerfeld quantization, as was done for oscillating strings by [5]. The momentum conjugate to $\rho$ is

$$P_\rho = N \frac{\sinh^3 \rho \dot{\rho}}{\sqrt{\dot{\rho}^2 \cosh^2 \rho - \dot{\rho}^2 - \psi^2}} = N \dot{\rho}$$

(16)

$$= N \dot{\rho}$$

(17)

One period consists of the expansion of the brane from a minimum radius $\rho_1$ to its maximum radius $\rho_2$ and the subsequent contraction. The Bohr-Sommerfeld condition, which will be a good approximation for $n \gg 1$, is

$$2\pi n = \oint P_\rho d\rho = 2 \int_{\rho_1}^{\rho_2} N \dot{\rho} d\rho$$

(18)

$$= 2N \int_{\rho_1}^{\rho_2} \frac{d\rho}{\cosh \rho} \sqrt{\left(\frac{E}{N}\right)^2 - \left(\frac{J}{N}\right)^2 + 2\frac{E}{N} \sinh^4 \rho - \sinh^6 \rho - \left(\frac{J}{N}\right)^2 \sinh^2 \rho}.$$  

(19)

This integral can be expressed in terms of highly unenlightening elliptic integrals. We can learn much more in the case where the giant gravitons are not too strongly excited:

$$E - J \ll J, N.$$  

(20)

Now let us define $x = \sinh^2 \rho$. Inside the square root, we find a cubic polynomial. For the trajectories of interest, there are three real roots, which we call $a, b_1, b_2$. The integral above becomes

$$N \int_{b_1}^{b_2} \frac{dx}{x^{3/2}(1+x)} \sqrt{(x-a)(x-b_1)(b_2-x)}$$

(21)
and for small $\Delta \equiv (E - J)/N$, we can expand the roots as

$$
a \simeq 2\Delta \omega + \frac{8 + \omega}{\omega^2} \Delta^2 \quad (22)
$$

$$
b_1 \simeq \omega - \sqrt{2(1 + \omega)} \Delta^{1/2} + \frac{\omega - 1}{\omega} \Delta - \frac{5 + 3\omega + \omega^2}{2^{3/2}(1 + \omega)^{1/2}\omega^2} \Delta^{3/2} - \frac{8 + \omega}{2\omega^3} \Delta^2 \quad (23)
$$

$$
b_2 \simeq \omega + \sqrt{2(1 + \omega)} \Delta^{1/2} + \frac{\omega - 1}{\omega} \Delta + \frac{5 + 3\omega + \omega^2}{2^{3/2}(1 + \omega)^{1/2}\omega^2} \Delta^{3/2} - \frac{8 + \omega}{2\omega^3} \Delta^2 \quad (24)
$$

where we have defined $\omega = J/N$. To evaluate the integral perturbatively in $\Delta$, we make a change of variables:

$$
x \equiv \left(\frac{b_2 - b_1}{2}\right) \xi + \left(\frac{b_2 + b_1}{2}\right) \quad (25)
$$

Then the action integral becomes

$$
2\pi n = N \left(\frac{b_2 - b_1}{2}\right)^2 \int_{-1}^{1} d\xi \sqrt{1 - \xi^2} \left[\frac{1}{1 + \omega} + \Delta \left(\frac{2\xi^2}{(1 + \omega)^2} - \frac{1 + \omega^{-2}}{(1 + \omega)^2}\right)\right] \quad (26)
$$

Evaluating the integrals, the quantization condition is, at quadratic order in $\Delta$,

$$
2n = N \left(\Delta + \frac{3}{2\omega^2} \Delta^2\right) \quad (27)
$$

or, solving for $E - J$,

$$
E - J = 2n - \frac{6n^2N}{J^2}. \quad (28)
$$

In the AdS/CFT correspondence, we identify the energy $E$ in global coordinates as the operator dimension in the field theory and $J$ as the R-charge, so we have computed the spectrum of dimensions of operators corresponding to these excited branes. We will comment further on this result in section 5.

Our analysis breaks down if the D3-brane becomes too highly excited. Consider the one-dimensional effective potential in terms of the coordinate $\rho$, as determined from Eq. (9), which we show in Figure 1. For small $E - J$, the oscillations are small and we can consider the solutions which are always at large radius. However, if $E - J$ is too large, then the amplitude of oscillations can be large enough to excite the D3-brane over the potential barrier in Figure 1, and the brane will collapse to small radius. As mentioned earlier, at small $\rho$ the brane is highly curved and we can no longer rely on the DBI action.

For large energy, there is a substantial regime of validity of our calculation. In this limit, it is simple to determine the minimum R-charge $J$ compatible with stable
oscillations – we simply take the zeros of $V'$, computed from (10) and ask whether $\dot{\rho}^2$ is positive, taking $J \ll E$. The result is that if

$$\frac{J}{N} > \sqrt{2} \left( \frac{E}{N} \right)^{3/4}$$

then the oscillations are constrained to large radius.

### 2.2 Oscillating D3-Branes on $S^5$

The original giant gravitons of [8] were D3-branes wrapped on an $S^3$ in the $S^5$. They also move along a circle in the $S^5$ and are thus coupled to the background RR five-form flux. The $S^3$ on which these 3-branes are wrapped is contractible, but the RR-flux exerts a repulsive force on the brane, stabilizing it at finite size on the $S^5$. We will now study excitations of these branes.

Using the spacetime geometry of (11), we take the embedding $t = t(\tau), \psi = \psi(\tau)$, and $\theta = \text{constant}$. The D3-brane action becomes

$$S = -N \int d\tau \left( \sin^3 \theta \sqrt{\dot{t}^2 - \cos^2 \theta \dot{\psi}^2 - \dot{\theta}^2 + \sigma \sin^4 \theta \dot{\psi}} \right)$$

Figure 1: $V(\rho)$, for $E/N = 10$
We choose the gauge

\[ i^2 - \cos^2 \theta \dot{\psi}^2 - \dot{\theta}^2 = \sin^6 \theta \]  

(31)

and define the conserved quantities

\[ E = -\frac{\partial L}{\partial \dot{t}} = N\sqrt{\dot{\theta}^2 + \cos^2 \theta \dot{\psi}^2 + \sin^2 \theta} = N\dot{t}, \]  

(32)

\[ J = N\omega = \frac{\partial L}{\partial \dot{\psi}} = N\left(\dot{\psi} \cos^2 \theta - \sigma \sin^4 \theta\right) \]  

(33)

in terms of which we may write the momentum conjugate to \( \dot{\theta} \),

\[ P_\theta = N\dot{\theta} = N\sqrt{\left(\frac{E}{N}\right)^2 - \left(\frac{\omega + \sigma \sin^4 \theta}{\cos^2 \theta}\right)^2 - \sin^6 \theta}. \]  

(34)

The static “giant graviton” solutions exist for \( \sigma = -1 \), \( \omega = E/N \), and \( \omega \leq 1 \).

The Bohr-Sommerfeld quantization condition may be analyzed in much the same way as in the previous subsection. Though the expressions differ slightly in intermediate steps, the final result is the same:

\[ E - J \simeq 2n - \frac{6n^2 N}{J^2}. \]  

(35)

3 Remark on Long Strings in \( AdS_3 \)

The oscillating string solutions have already been thoroughly studied in the case of \( AdS_3 \times S^3 \) [6, 7, 15], so we will only discuss them briefly here. The action for a string coupled to the background flux is

\[ S = -\eta \int d\tau \left( \sinh \rho \sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - \dot{\psi}^2 - \sigma \sinh^2 \rho} \right) \]  

(36)

where \( \eta = 2\pi TL^2 \). We again identify the conserved energy as \( E = -\frac{\partial L}{\partial \dot{t}} \) and the angular velocity \( \omega = \dot{\psi} \), and choose the gauge

\[ i^2 \cosh^4 \rho = (E/\eta + \sigma \sinh^2 \rho)^2. \]  

(37)

Then we find that \( P_\rho = \eta \dot{\rho} \) and that

\[ \dot{\rho}^2 = \frac{(E/\eta + \sigma \sinh^2 \rho)^2}{\cosh^2 \rho} - \sinh^2 \rho - \omega^2 \]  

(38)

\[ = \frac{1}{\cosh^2 \rho} \left( (E/\eta)^2 - \omega^2 - (\omega^2 - 2\sigma E/\eta + 1) \sinh^2 \rho \right). \]  

(39)
Notice that if $\omega = 0$ and
\[
\frac{2\sigma E}{\eta} \geq 1
\] (40)
then there is no value of $\rho$ for which $\dot{\rho}$ vanishes. These solutions, for which there is no turning point of the motion, are the so-called “long strings” in $AdS_3$. For nonzero $\omega$, there are no giant graviton solutions; from Eq. (39) we can see that there is at most one point at which $\dot{\rho} = 0$. Thus for large energy the radial motion cannot be periodic, and the Bohr-Sommerfeld analysis of Section 2 is inapplicable.

4 Oscillating Branes in $AdS_4 \times S^7$ and $AdS_7 \times S^4$

Calculations similar to those in Section 2 can be done for AdS spaces of dimensions other than 5. In this section we examine the cases of special interest, $AdS_4 \times S^7$ and $AdS_7 \times S^4$, which are solutions of M-theory. The calculations are very similar to those for $AdS_5 \times S^5$, so we will list our results first; the details follow for interested readers. For M2-branes oscillating in $AdS_4$,
\[
E - \frac{J}{2} = n - 12\frac{Nn^2}{J^3}, \quad J \sim N^{1/2}.
\] (41)
For M5-branes in $S^7$,
\[
E - \frac{J}{2} = 2n - \frac{15n^2 N^{1/2}}{2J^{3/2}}, \quad J \sim N.
\] (42)
For M5-branes oscillating in $AdS_7$,
\[
E - 2J = 4n - \frac{15Nn^2}{2J^{3/2}}, \quad J \sim N^2.
\] (43)
For M2-branes in $S^4$,
\[
E - 2J = 2n - \frac{6N^2n^2}{J^3}, \quad J \sim N.
\] (44)

At $n = 0$, these calculations reproduce the BPS bounds for the conformal field theories on the boundaries of $AdS_4$ and $AdS_7$, while at nonzero $n$ the leading terms are consistent with a semiclassical calculation in field theory. We will comment further on these results in Section 5.
4.1 Oscillating Membranes in AdS4

The eleven-dimensional metric is
\[ ds^2 = L^2 \left( -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2 \right) + 4L^2 \left( \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_5^2 \right) \]
(45)
where we have chosen global coordinates for the AdS4. The three-form potential of M-theory is
\[ A_3 = L^3 \sinh^3 \rho dt \wedge \Omega_2 \] \quad (46)

The action for an M2-brane wrapped on the S2 and spinning in the S7 is
\[ S = -T_{M2} L^3 Vol(S^2) \int d\tau \left( \sinh^2 \rho \sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\psi}^2 - \sigma \sinh^3 \rho} \right). \] \quad (47)

Recall that the tension of an M2-brane is \( T_{M2} = (2\pi)^{-2}l_p^{-3} \) and that the AdS4 radius for the AdS4 x S7 solution of M-theory is given by \( L_{AdS4} = \left( \frac{\pi^2 N^2}{2} \right)^{1/6} l_p \). The coefficient of the action is therefore \( \sqrt{N^2} \). The conserved energy is
\[ \frac{\partial L}{\partial \dot{t}} = -\sqrt{\frac{N}{2}} \left( \frac{i \sinh^2 \rho \cosh^2 \rho}{\sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\psi}^2}} - \sigma \sinh^3 \rho \right) \equiv -E \] \quad (48)
and the conserved angular momentum is
\[ \frac{\partial L}{\partial \dot{\psi}} = \sqrt{\frac{N}{2}} \left( \frac{4\dot{\psi} \sinh^2 \rho}{\sqrt{i^2 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\psi}^2}} \right) \equiv J. \] \quad (49)

We define \( c = E \sqrt{\frac{2}{N}} \) and \( 4\omega = J \sqrt{\frac{2}{N}} \). With the gauge choice
\[ i^2 \cosh^4 \rho = \left( c + \sigma \sinh^3 \rho \right)^2 \] \quad (50)
we obtain \( \dot{\psi} = \omega \), and the equation of motion for the radial position \( \rho \) is
\[ \dot{\rho}^2 = \frac{(c + \sigma \sinh^3 \rho)^2}{\cosh^2 \rho} - \sinh^4 \rho - 4\omega^2 \] \quad (51)

When \( c = 2\omega \) and \( \sigma = +1 \) there are special static solutions. The momentum conjugate to \( \rho \) is \( P_\rho = \sqrt{\frac{N}{2}} \dot{\rho} \). By a similar analysis as in the previous section, we find that to quadratic order in \( \Delta = c - 2\omega \), is that
\[ n = \sqrt{\frac{N}{2}} \left( \Delta + \frac{3}{8\omega^3} \Delta^2 \right). \] \quad (52)

As a condition on allowed energies, we find
\[ E - \frac{J}{2} = n - 12 \frac{Nn^2}{J^3} \] \quad (53)
4.2 Oscillating M5-branes in $AdS_4 \times S^7$

In the $AdS_4 \times S^7$ solution, we can also embed an M5-brane by wrapping it on a 5-sphere in the $S^7$. The M5 also spins in the $\psi$-direction, as usual, and expands to an angle $\theta$. The relevant six-form potential of M-theory is (using the requirement $F_7 = \ast F_4$)

$$A_6 = -(2L)^6 \sin^6 \theta d\psi \wedge \Omega_5$$

As in the previous subsection, the $AdS_4$ radius $L$ is given by

$$L_{AdS_4} = \left(\frac{\pi^2 N}{2}\right)^{1/6} l_p.$$ 

The M5 action is

$$S = -32T_{M5} L^6 \text{Vol}(S^5) \int d\tau \left( \sin^5 \theta \sqrt{i^2 \cosh^2 \rho - 4\dot{\theta}^2 - 4 \cos^2 \theta \dot{\psi}^2} - 2\sigma \dot{\psi} \sin^6 \theta \right).$$

Because the tension of an M5-brane is $T_{M5} = (2\pi)^{-5/2} l_p^{-6}$, we find that the coefficient in front of the action integral is $\frac{N}{2}$. The conserved quantities $E$ and $J$ are given by

$$\frac{\partial L}{\partial \dot{t}} = -\frac{N}{2} \left( \frac{\dot{t} \sin^5 \theta}{\sqrt{i^2 - 4\dot{\theta}^2 - 4 \cos^2 \theta \dot{\psi}^2}} \right) \equiv -E$$

and

$$\frac{\partial L}{\partial \dot{\psi}} = 2N \frac{\dot{\psi} \cos^2 \theta \sin^5 \theta}{\sqrt{i^2 - 4\dot{\theta}^2 - 4 \cos^2 \theta \dot{\psi}^2}} - N \sigma \sin^6 \theta \equiv J.$$ 

The most convenient gauge choice is

$$\sqrt{i^2 - 4\dot{\theta}^2 - 4 \cos^2 \theta \dot{\psi}^2} = \sin^5 \theta$$

so that $E = \frac{N}{2} i$ and $J = 2N \dot{\psi} \cos^2 \theta$. Moreover, the momentum conjugate to $\theta$ is

$$P_{\theta} = 2N \dot{\theta}$$

$$= \frac{N}{\cos \theta} \sqrt{\left(\frac{2E}{N}\right)^2 - \left(\frac{J}{N}\right)^2 - \left(\frac{2E}{N}\right)^2 \sin^2 \theta - \frac{2J}{N} \sigma \sin^6 \theta - \sin^{10} \theta}$$

We see that the giant graviton solutions will have $\sigma = -1$. Computing the Bohr-Sommerfeld integral as before,

$$2\pi n = \oint P_{\theta} d\theta$$

where now we work perturbatively in $(2E - J)/N$, we obtain the spectrum

$$E - \frac{J}{2} = 2n - \frac{15n^2 N^{1/2}}{2J^{3/2}}.$$
4.3 Oscillating M5-branes in $AdS_7 \times S^4$

The eleven-dimensional metric is

$$ds^2 = L^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2\right) + \frac{L^2}{4} \left(\cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_5^2\right),$$

where we have chosen global coordinates for the $AdS_7$. The six-form potential of M-theory is

$$A_6 = -L^6 \sinh^6 \rho dt \wedge \Omega_5.$$  

The action for an M5-brane wrapped on the $S^2$ and spinning in the $S^7$ is

$$S = -T_{M5} L^6 Vol(S^5) \int d\tau \left(\sinh^5 \rho \sqrt{\dot{\rho}^2 \cosh^2 \rho - \dot{\rho}^2 - \frac{1}{4} \dot{\psi}^2 + \sigma \sinh^6 \rho}\right).$$

Recall that the tension of an M5-brane is $T_{M5} = (2\pi)^{-5/2} l_p^{-6}$ and that the $AdS_7$ radius for the $AdS_7 \times S^4$ solution of M-theory is given by $L_{AdS_7} = 2 (\pi N)^{1/3} l_p$. The coefficient of the action is therefore $2N^2$. The conserved energy is

$$\frac{\partial L}{\partial \dot{t}} = -2N^2 \left(\frac{\dot{\psi} \sinh^5 \rho \cosh^2 \rho}{\sqrt{\dot{\rho}^2 \cosh^2 \rho - \dot{\rho}^2 - \frac{1}{4} \dot{\psi}^2}} + \sigma \sinh^6 \rho\right) \equiv -E,$$

and the conserved angular momentum is

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{N^2}{2} \left(\frac{\dot{\psi} \sinh^5 \rho}{\sqrt{\dot{\rho}^2 \cosh^2 \rho - \dot{\rho}^2 - \frac{1}{4} \dot{\psi}^2}}\right) \equiv J.$$  

We define $c = E/2N^2$ and $\omega = 2J/N^2$. With the gauge choice

$$i^2 \cosh^4 \rho = \left(c - \sigma \sinh^6 \rho\right)^2$$

we obtain $\dot{\psi} = \omega$, and the equation of motion for the radial position $\rho$ is

$$\dot{\rho}^2 = \left(c - \sigma \sinh^6 \rho\right)^2 \cosh^2 \rho - \sinh^2 \rho - \frac{\omega^2}{4}.$$

Here, we take $c \simeq \omega/2$ and $\sigma = -1$ to study oscillations about the special static solutions. The momentum conjugate to $\rho$ is $P_\rho = 2N^2 \dot{\rho}$. Computing the integrals as before, we obtain

$$E - 2J = 4n - \frac{15Nn^2}{2J^{3/2}}.$$
4.4 Oscillating M2 branes in $AdS_7 \times S^4$

In the eleven-dimensional geometry of the previous subsection, we may also study giant gravitons which are M2-branes wrapped on a two-sphere embedded in the $S^4$. The 3-form potential is

$$A_3 = \left( \frac{L}{2} \right)^3 \sin^3 \theta d\psi \wedge \Omega_2$$  \hspace{1cm} (71)

The action for an M2-brane wrapped on the $S^2$ and spinning in the $S^4$ (along an angular coordinate $\psi$) is

$$S = -\frac{1}{8} T_{M2} L^3 Vol(S^2) \int d\tau \left( 2 \sin^2 \theta \sqrt{i^2 - \frac{\dot{\theta}^2}{4} - \frac{\dot{\psi}^2 \cos^2 \theta}{4} - \sigma \dot{\psi} \sin^3 \theta} \right).$$  \hspace{1cm} (72)

The tension of an M2-brane is $T_{M2} = (2\pi)^{-2} l_p^{-3}$ and the $AdS_7$ radius is given by $L = 2 (\pi N)^{1/3} l_p$, so the coefficient of the action is just $N$. In the gauge

$$\sqrt{i^2 - \frac{\dot{\theta}^2}{4} - \frac{\dot{\psi}^2 \cos^2 \theta}{4}} = \sin^2 \theta$$  \hspace{1cm} (73)

we have

$$\frac{\partial L}{\partial \psi} = J = \frac{N}{2} \dot{\psi} \cos^2 \theta - \sigma N \sin^3 \theta$$  \hspace{1cm} (74)

$$-\frac{\partial L}{\partial t} = E = 2N \dot{t}$$  \hspace{1cm} (75)

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = \frac{N}{2} \dot{\theta}.$$  \hspace{1cm} (76)

Now, we take $\sigma = -1$ and work perturbatively in $\frac{E}{2N} - \frac{J^3}{N}$. Evaluating the Bohr-Sommerfeld integral in this case yields

$$E - 2J = 2n - \frac{6N^2 n^2}{J^3}.$$  \hspace{1cm} (77)

5 Comments and Interpretation

Our results are easiest to interpret for the oscillating branes on $S^5$. For these states, there is a conjecture that the dual field theory operators are subdeterminants \[16\] of the form

$$Z_{i_1}^{j_1} \cdots Z_{i_J}^{j_J} \epsilon^{i_1 \cdots i_J} \epsilon^{j_1 \cdots j_J} n^{i_N} \epsilon^{j_1 \cdots j_J}$$  \hspace{1cm} (78)
where the summed indices are $SU(N)$ gauge indices. Subsequent work supporting this conjecture includes \[7, 13\]. Following \[13\], non-BPS excitations of this giant graviton should correspond to insertions of fields into the operator (78). These insertions ought to involve scalars $\phi$ and possibly derivatives $\nabla_\mu$ (fermions and gauge fields are associated with scalar and two-form potentials in the string background.) For fixed R-charge, any $\phi$ must actually appear in the combination $\phi\bar{\phi}$, and Poincaré symmetry of the background requires the derivative to be contracted into something – perhaps $\nabla^2$. Then from the usual dimension counting in a four-dimensional field theory, the quantity $E - J$ must change in units of two.

The interesting physics arises from the corrections to the leading behavior, which we can interpret as arising from many-body forces on the field theory side. Consider replacing some of the $Z_j^i$ fields in the operator (78) by $(Z\phi\bar{\phi})^j_i$ (we might also imagine taking two $Z$ fields and replacing one by $Z\phi$ and the other by $Z\bar{\phi}$.) Excitations of this form have been conjectured to correspond to open string states \[19, 21, 14\], which can change the size and shape of the D3-brane. This insertion of “impurities” into the subdeterminant operator is reminiscent of a Hartree-Fock analysis of baryons \[20\], in which we excite individual quarks. The correction term, which we found to be of order $n^2/N$, then corresponds to two-body interactions between impurities (the two-body force is of order $1/N$, and there are roughly $n^2$ ways to pick two impurities.) The minus sign indicates that the force between excited quarks is attractive. Note that if we had computed the spectrum to higher orders in $\Delta$ in Section 2, we would have obtained terms of order $n^k/N^{k-1}$. Assuming that the semiclassical expansion remains valid for these terms, we can thus continue the expansions performed in this paper to obtain leading contributions to the $k$-body forces for any $k$.

The results for M-branes which have expanded on the sphere are more difficult to interpret because we lack a Lagrangian description of the dual field theories. However, the corrections in the semiclassical expansion again have the form $n^k/N^{k-1}$, suggesting that the picture of having $k$-body forces between quarks extends to the M-brane case.

Similar results have appeared previously \[21, 14\] in the context of the plane-wave approximation of $AdS_5 \times S^5$ \[22\]. We have performed our calculations in a different regime – we take $J \sim N$, as opposed to the standard plane-wave limit, for which $J^2/N$ is held fixed. Note that the corrections to the giant graviton spectrum computed by \[21, 14\], of the form \( g_s^2 N n^2 / J^2 \), are also of order $n^2/N$ in the appropriate limit.

The AdS giants, for which our semiclassical considerations are most reliable, are harder to analyze as operators on the field theory side. Some of the difficulties are even apparent in the AdS dual. For example, it costs infinite energy to take these states to large $\rho$, and they are not localized along the directions tangent to the boundary, so it is not clear that there is a simple local operator description in the dual field theory.
A proposal for the operators corresponding to the AdS giants has been given in \[17\]; a subsequent paper \[14\] studied these operators in the plane wave limit.

An alternative approach, in the spirit of the analysis of monopoles and solitons \[23\], is to identify a classical field configuration which solves the equations of motion and then to consider quantum fluctuations about this solution. The authors of \[10\] (whose argument we follow) found the relevant field configuration for an \(m\)-dimensional field theory from the action for the scalars

\[
S = -\frac{1}{2g_{YM}^2} \int d^m x \left[ (\partial \phi_1)^2 + (\partial \phi_2)^2 + \frac{(m-2)^2}{4L_{AdS}^2} (\phi_1^2 + \phi_2^2) \right].
\]

The fields \(\phi_{1,2}\) are the scalars corresponding to the plane of rotation of the expanded brane. Because the geometry of the branes is of the form \(R \times S^{m-1}\), there is a mass term for the \(\phi\) fields, whose form is fixed by conformal invariance. Integrating over the directions of the \(S^{m-1}\), the action becomes

\[
S = \frac{L^3 \Omega_{m-1}}{2g_{YM}^2} \int dt \left[ \dot{\phi}_1^2 + \dot{\phi}_2^2 - \frac{(m-2)^2}{4L_{AdS}^2} (\phi_1^2 + \phi_2^2) \right].
\]

The action is simply that of a two-dimensional harmonic oscillator, so the dynamics are exactly solvable. We now set \(L = 1\) and make the change of variables

\[
\phi_1 = \sqrt{\frac{g_{YM}^2}{\Omega_{m-1}}} \eta \cos \theta, \quad \phi_2 = \sqrt{\frac{g_{YM}^2}{\Omega_{m-1}}} \eta \sin \theta
\]

in terms of which the action is

\[
S = \frac{N}{2} \int dt \left( \dot{\eta}^2 + \dot{\theta}^2 - \frac{(m-2)^2}{4\eta^2} \right).
\]

The Hamiltonian and angular momentum \(J = \frac{dS}{d\theta}\) are conserved. The energy is minimized at

\[
\eta_0^2 = \frac{2J}{N(m-2)}
\]

and the spectrum is given by

\[
E - \frac{m-2}{2} J = (m-2)n.
\]

This calculation reproduces the leading terms in \(28,41,43\).

The effective action \(80\) is accurate for perturbations about a BPS configuration, but may receive large corrections for non-BPS backgrounds, which are our main interest. It is natural to suspect that these corrections are responsible for the subleading
terms in the brane spectra computed earlier. To see how these terms arise in field theory, let us take \(d = 4\) for specificity, and note that we can regard the brane configuration with a single AdS giant graviton in the background of \(N\) static D3-branes as a theory with \(SU(N + 1)\) gauge symmetry spontaneously broken to \(SU(N) \times U(1)\). The modes not in the light \(SU(N)\) or \(U(1)\) are Higgsed by the vacuum expectation values of the \(\phi\) fields. When we integrate out these massive modes there is a \(\phi^4\) coupling induced at one-loop order, which will scale as \(1/N\). We note that such a term, proportional to \(\frac{E^2-L^2}{E}\), was identified on the gravity side in [7], and corresponded precisely to a non-cancellation of brane tension and RR flux forces.

By considering a few simple one-loop Feynman diagrams, we can schematically reproduce the \(Nn^2/J^2\) dependence of the anomalous dimension in (28) in weakly coupled field theory. A direct comparison with the gravity result, which is valid at strong coupling in the field theory, is of course impossible, so we will not worry about the precise numerical factor or factors of \(g_{YM}^2 N\). Also, we will measure all dimensionful quantities in units of \(L_{AdS}\). The relevant Feynman diagrams, shown in Figure 2, have four external scalar legs; for definiteness we can consider the \(U(1)\) component of the field \(\eta\), which is described in terms of the \(\phi\) fields in (81). The internal lines are given by the Higgsed fields, which carry a single \(SU(N)\) index and have masses proportional to \(\eta\). In the supersymmetric case, we have a constant \(\eta = \eta_0 = \sqrt{J/N}\), and therefore constant masses. In this case, the divergent part of the one-loop amplitude cancels, leaving a finite piece of order \(1/N\). In the semiclassical limit \(n \gg 1\), this correction turns out to be small compared to the non-supersymmetric contribution, and so we will ignore it. In the non-supersymmetric case, when the D3-branes are oscillating, there is an additional finite piece which we cannot ignore. The most straightforward way to compute this correction is to write \(\eta = \eta_0 + \eta_1(t)\) and treat \(\eta_1\) as a field with a Yukawa-type coupling to the massive fields. Thus, for example, the Higgsed scalars have a coupling of the form

\[
\frac{1}{2} \eta^2 \phi^2 \sim \frac{1}{2} \eta_0^2 \phi^2 + \eta_0 \eta_1(t) \phi^2
\]  

and the associated vertex appears in Figure 3. Now we allow the \(\eta\) field to fluctu-
Figure 3: Coupling of $\eta_1$ to the scalars. There are analogous vertices for the vector particles and fermions.

Figure 4: A typical loop diagram with two $\eta_1$ insertions.

ate quantum-mechanically around the classical solution and consider the four-scalar vertex. Working in momentum space, we hold the four fluctuating scalar legs at zero momentum. The loop diagrams with one insertion of $\eta_1$ do not contribute because of momentum conservation, so the leading effect of the time dependence arises from two $\eta_1$ insertions; a typical Feynman diagram is shown in Figure 4. Allowing these insertions to carry momentum $p$, a typical loop integral will have the form

$$\frac{1}{N} \int d^4k \frac{n_0^2 n_1(p)n_1(-p)}{(k^2 + n_0^2)((k + p)^2 + n_0^2)^2} = \frac{1}{N} \int d^4k n_0^2 n_1(p)n_1(-p) \left[ \frac{1}{(k^2 + n_0^2)^4} - \frac{2p^2}{(k^2 + n_0^2)^5} + \frac{12(p \cdot k)^2}{(k^2 + n_0^2)^6} \right] \quad (86)$$

$$\sim \frac{n_1(p)n_1(-p)p^2}{N n_0^2}. \quad (87)$$

We have made the approximation that $p^2 \ll n_0^2$ in the top line and retained the overall factor of $1/N$. In the second line, we have discarded the zero-order term in the $p^2$ expansion, which must cancel amongst the one-loop diagrams. Integrating over $p$ and converting back to position space, we obtain the time-average $\langle \eta_1^2 \rangle \sim n$ in the numerator, or

$$\frac{\langle \eta_1^2 \rangle}{N n_0^2} \sim \frac{Nn}{J^2}. \quad (88)$$

Up to numerical and $g_{YM}^2 N$ factors, this is the one-loop effective coupling for $\eta^4$, and
including it in a semiclassical computation gives a correction of order $Nn^2/J^2$ to the spectrum.

An important issue is that in our analysis we have ignored effects due to backreaction of the probe D3-branes on the string background. These effects are of order $1/N$, and may change our results. One effect of the backreaction, for example, is that the probe brane should change the location of the center of mass of the system by an amount of order $1/N$ – essentially, there ought to be a reduced mass shift of the spectrum. Still, there should be a regime in which our results are robust. Backreaction effects like the reduced mass shift should give a correction to the spectrum of order $n/N$. On the field theory side, for sphere giants, this might correspond to a $1/N$ correction to the bare masses of the $n$ impurities, or a combinatoric correction ($\frac{n(n-1)}{2}$ rather than $n^2/2$ ways having pairwise interactions, for example); for AdS giants there are subleading finite contributions to the effective action after integrating out massive fields. Then, in the limit that $n \gg 1$ but $n \ll N$, this backreaction effect is small compared to the $n^2/N$ contributions we computed above. Clearly, a more detailed computation of the backreaction in supergravity is necessary to confirm or deny these handwaving arguments. Such a computation might not be hopelessly ambitious, as it should eventually boil down to a one-dimensional mechanics problem.

We have ignored various other effects in our calculation, in particular the effects of massive string modes and of closed string emission. Notice that we have written the metric (1) in such a way that all the factors of $g_sN$ factor out. In the subsequent analysis, no factors of $g_sN$ could appear. However, we considered only supergravity fields and not massive string modes, which would have introduced $\alpha'/L^2 \sim 1/(g_sN)^{1/2}$ corrections. Thus to ensure that we can ignore these effects, we must also take $1/(g_sN)^{1/2} \ll n/N$. As for closed string emission, at leading order in $1/N$ these effects cannot change the spectrum (though they do allow our non-BPS states to decay.) Radiative corrections to the spectrum due to closed strings (which roughly correspond to non-planar diagrams in the field theory) only appear at order $1/N^2$, and should not affect our results at order $1/N$. To summarize, we expect our semiclassical calculations for D3-branes in $AdS_5 \times S^5$ to be sensible if

$$\frac{1}{N}, \frac{1}{(g_sN)^{1/2}} \ll \frac{n}{N} \ll 1. \quad \text{(89)}$$

Though we have seen that semiclassical methods can give useful information on both sides of the gauge/gravity duality, much remains to be done. On the gravity side, we need to understand backreaction and stringy effects better, and on the field

1We note, however, that in the case of M5-branes in $AdS_7$, the semiclassical expansion proceeds in powers of $1/N^2$, so $1/N$ backreaction effects could actually be dominant.
theory side, we need more detailed computations of corrections to the effective action. We also hope that the semiclassical results for M-branes may give some hints about the structure of the dual gauge theories on the boundaries of $AdS_4$ and $AdS_7$.

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