Simple views on symmetries and dualities in the theory of elasticity

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Microscopic symmetries impose strong constraints on the elasticity of a crystalline solid. In addition to the usual spatial symmetries captured by the tensorial character of the elastic tensor, hidden non-spatial symmetries can occur microscopically in special classes of mechanical structures. Examples of such non-spatial symmetries occur in families of mechanical metamaterials where a duality transformation relates pairs of different configurations. We show on general grounds how the existence of non-spatial symmetries further constrains the elastic tensor, reducing the number of independent moduli. In systems exhibiting a duality transformation, the resulting constraints on the number of moduli are particularly stringent at the self-dual point but persist even away from it, in a way reminiscent of critical phenomena.

Classical elasticity describes how rigid objects respond to deformations [1–4]. New facets of this time-honored subject continue to emerge in often unexpected guises and contexts. Recent examples range from quantum elasticity [5–7] and fractons [8–10], non-orientable elasticity [11], the odd elasticity of active solids [12] and topological elasticity [13–16] inspired by static mechanical topological metamaterials [17–23].

The very existence of rigid objects would seem rather mysterious if we were not so used to them in daily life: it is a consequence of the spontaneous breaking of translational invariance that occurs when a fluid condenses into a solid [24]. This spontaneously broken symmetry guarantees the existence of excitations with arbitrarily low energies called Nambu-Goldstone modes [25–28]. In mechanics, the Goldstone modes are familiar objects: phonons of arbitrarily large wavelength [29, 30]. Elasticity can be viewed as the effective field theory of such Goldstone modes: a continuum description that ignores irrelevant microscopic details and instead focuses on the behavior at large scales relevant to our direct interactions with elastic bodies.

The coarse-graining procedure that goes from a microscopic description to a continuum elastic theory should discard irrelevant details, but must crucially preserve symmetries [31, 32]. The spatial symmetries of a crystal can be gathered in a space group, containing all spatial transformations that leave the crystal invariant [33, 34]. These space groups put strong constraints on the elasticity of the corresponding crystal, e.g., on the number of independent moduli [2, 35, 36]. For instance, the elasticity of a two-dimensional crystal with triangular symmetry is isotropic (i.e., it is the same for all orientations) and, as a consequence, can display at most two independent elastic moduli.

In addition to spatial symmetries, additional non-spatial symmetries can occur microscopically. Recent studies revealed that such additional symmetries can emerge, for instance, in families of mechanical metamaterials where a duality transformation relates pairs of distinct configurations that surprisingly exhibit identical spectra of vibrational modes [37]. In self-dual systems (mapped onto themselves by the duality) the duality transformation becomes an additional symmetry distinct from spatial ones.

In this Letter, we seek to determine the consequences of these additional constraints on the linear elasticity of a material. More precisely, we consider the following question: how do microscopic symmetries affect the coarse-grained tensor of elastic moduli? Formally, we will determine the relation between the elastic tensors $\tilde{c}_{ijkl}$ and $\bar{c}_{ijkl}$ of the systems described by the momentum-space force-constant matrices $S(q)$ and $\tilde{S}(q) = U(q) S(u \cdot q) U(q)^{-1}$ (see next section for precise definitions). For standard spatial symmetries, the answer is simply contained in the fact that $c_{ijkl}$ must transforms as a tensor. Our analysis goes beyond this simple case and allows to analyze the effect of additional hidden (non-spatial) symmetries of the force-constant matrix, that can result in even stronger constraints. In addition, it applies to the case of dualities whereby the force-constant matrices of two different systems are related to each other by a nontrivial transformation.

We apply our general formulas to the example of twisted Kagome lattices (see Fig. 1), a family of two-dimensional crystals exhibiting a duality with a self-dual point where a non-spatial symmetry emerges [37]. When all point group symmetries are lifted, six independent elastic moduli are expected in the continuum description of such systems. However, the self-dual twisted Kagome lattices have isotropic elasticity with only one elastic modulus, despite not having any microscopic symmetry beyond Bravais lattice translations. Most strikingly, the elastic tensor is also constrained away from the self-dual point, reducing the number of elastic moduli to three. Our theory explains these counter-intuitive properties and casts them in a general formalism applicable beyond this concrete example.

Linear Elasticity. – Linear elasticity describes the relation between the stress tensor $\sigma_{ij}$ and the displacement (or strain) tensor $\epsilon_{kl}$, respectively representing the long-wavelength forces and deformations in a solid.
More precisely, the displacement tensor is $\epsilon = J - \text{Id}$ where $J_{\mu\nu} = \partial X^{\nu}/\partial x^{\mu}$ is the Jacobian of the transformation $X(x)$ giving the position after deformation of the point originally at $x$, while the stress tensor is defined such that its divergence is the surface force $f_{\mu} = \partial_{\nu} S_{\mu\nu}$ acting on an infinitesimal patch of material continuum. We choose to work with the nonsymmetric tensors to encompass recent extensions of elasticity where the antisymmetric components are relevant [12]. Hooke’s law in continuum form

$$\sigma_{ij} = c_{ijkl} \epsilon_{kl} \quad (1)$$

linearly relates $\sigma_{ij}$ and $\epsilon_{kl}$ through the elastic tensor $c_{ijkl}$, whose entries are the elastic moduli of the solid.

Spatial symmetries put strong constraints on the material properties of a crystal such as its elastic tensor $c_{ijkl}$. This is because the elastic tensor $c_{ijkl}$ unsurprisingly transforms as a tensor under a spatial transformation $T \in O(d)$:

$$c_{ijkl} \mapsto \hat{c}_{ijkl} = T_{ii'}T_{jj'}T_{kk'}T_{ll'}c_{i'j'k'l'}. \quad (2)$$

Hence, there is only a certain number of entries in $c_{ijkl}$ (i.e., of elastic moduli) that can be independent of each other, and those are prescribed by the symmetry of the material (we refer the reader to the SI and references therein for a short summary). Yet, nothing guarantees that all of these moduli must be independent, especially when additional constraints not originating from purely spatial symmetries exist.

Microscopically, we describe the elastic material as a set of massive particles arranged on a d-dimensional crystal and ruled by Newton equations $M \ddot{\mathbf{u}} = F$, where $\mathbf{u} = x - x_{\text{eq}}$ are the displacements of the masses with respect to their equilibrium positions $x_{\text{eq}}$, and $M$ is a mass matrix describing the inertia of the particles. The forces $F$ between the particles are given in the harmonic approximation by $F = -S\mathbf{u}$ where the force-constant matrix $S$ is essentially the matrix of second derivatives of the potential in the absence of pre-stress [38]. Hooke’s law (1) is the macroscopic version of this equation. Hence, the elastic tensor $c_{ijkl}$ can in principle be computed explicitly from the force-constant matrix $S$, see Ref. [39] (also Refs. [40–42]).

Here, we specialize to the case of a crystal, where particles are arranged in a spatially periodic fashion. Hence, we can use Bloch theorem to block-diagonalize Newton equations and to write $M \ddot{\mathbf{u}}(q) = F(q) = -S(q)\mathbf{u}(q)$ where $q$ is the quasi-momentum vector. Elasticity describes the long-wavelength modes $q \rightarrow 0$ projected to the kernel of $S(0)$ with the constraint that the projection of the force $F(q)$ on the orthogonal complement of the kernel (i.e. fast modes with a finite frequency at $q = 0$) are assumed to relax and are integrated out, in agreement with the the zero temperature limit of finite-temperature elasticity [41].

The elastic tensor can then be obtained from the momentum-space force-constant matrix $S(q)$ near zero momentum as [12, 38, 39]

$$\frac{c_{ijkl}}{\rho} = \left[ \frac{\partial^2 S}{\partial q_i \partial q_k} - \frac{\partial S}{\partial q_i} \left[ S^{-1} \right] \frac{\partial S}{\partial q_k} \right]_{j\ell} \quad (3)$$

where $\rho$ is the density. This expression is taken at momentum $q = 0$ and the inverse $S^{-1}$ is computed in the orthogonal complement of the kernel of the matrix. Technically, $j, \ell$ label a basis of the kernel of $S(0)$ but they can be chosen to correspond to space dimensions.

It is convenient to write the stress and strain as vectors and the elastic tensor as the matrix [12]

$$K^{ab} = \frac{1}{4} \sum_{ijkl} \tau^{a}_{ij} c_{ijkl} \tau^{b}_{kl}. \quad (4)$$

where the matrices $\tau^a = \tau^{a\nu}$ are a suitable orthonormal basis of the vector space $M_d(\mathbb{R})$ (with scalar product $\langle M, N \rangle = \text{tr}(M^{\mathsf{T}} N)/2$). The elastic tensor $c_{ijkl}$ and the elastic matrix $K^{ab}$ contain exactly the same information, only ordered in different ways. This decomposition inspired by the study of anomalous hydrodynamics [43] is described in Ref. [12]. For instance in $d = 2$ the basis matrices can be chosen as $(\tilde{\tau}^0, \tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3) = (\sigma^0, -i\sigma^2, \sigma^3, \sigma^1)$ where $\sigma^a$ are Pauli matrices. In this case, the four components of the deformation (stress) vector correspond to compression, rotation, and two linearly independent shear strains (stresses) [see SI for a visual representation].

Symmetries and dualities and their effect on the elastic tensor. — We now consider a situation where a momentum-space force-constant matrix $\tilde{S}(q)$ is related to another force-constant matrix $S(q)$ by a relation of the form

$$\tilde{S}(q) = (USU^{-1})(u \cdot q) \quad (5)$$

where $U$ is unitary and $u$ is orthogonal. This relation can be seen as a symmetry if $\tilde{S} = S$, or as a duality relation between two distinct systems if $\tilde{S} \neq S$. The two force-constant matrices $S(q)$ and $\tilde{S}(q)$ define two elastic tensors $\tilde{c}_{ijkl}$ and $\tilde{c}_{ijkl}$ (equivalently, two elastic matrices $K^{ab}$ and $\tilde{K}^{ab}$) through equation (3). We now proceed to determine the relation between $\tilde{c}_{ijkl}$ and $\tilde{c}_{ijkl}$ imposed by Eq. (5). Using equations (3) and (5), one obtains by a direct calculation (see SI)

$$\tilde{c}_{imjn} = u_{i1} [U_{\mathcal{Z}Z}]_{m\nu} u_{j'j} [U_{\mathcal{Z}Z}^{-1}]_{\nu'n'} c_{i'm'j'n'}. \quad (6)$$

where $U_{\mathcal{Z}Z}$ is the projection of $U(0)$ on the kernel of $S(0)$ (and restricted to this subspace). Notably, this relation only requires the knowledge of $U(0)$ (and not of any derivative with respect to momentum). In terms of the elastic matrix defined in Eq. (4), this relation can be cast in the more compact form

$$\tilde{K} = VKV^\dagger \quad (7)$$
Below the critical angle. (b) At the critical angle. (c) Above the critical angle. Inset: definition of the twisting angle. Twisted Kagome lattices.

\( V^{ab} = \frac{1}{2} \text{tr} \left[ \tau^a U_{ZZ} [\tau^b]^T u \right]. \) (8)

The standard result Eq. (2) is recovered from Eq. (6) when \( U_{ZZ} = u^T \equiv T \in O(d) \). However, this particular case does not exhaust Eq. (6) as the relation (5) is not necessarily a spatial symmetry, i.e., an element of the space group of the crystal. As such, the projection \( U_{ZZ} \) of \( U(0) \) on the kernel of \( S(0) \) is not necessarily the representation of a spatial symmetry, and it is not necessarily related to \( u^T \). In the next section, we shall present a concrete example of such a situation.

Twisted Kagome lattices. – We now consider the example of a family of mechanical structures called twisted Kagome lattices [15, 17, 44–47]. These are two-dimensional periodic structures composed of three particles per unit cell on a triangular lattice, with each particle connected to four neighbors, as represented in Fig. 1. We consider a situation where inequivalent bonds (i.e., those not related by Bravais lattice translations) have different spring stiffnesses \( k_i, i = 1, 2, 3 \) (see figure 1). This family is parametrized by a simple geometric parameter: the twisting angle \( \theta \) between two connected triangles, see the inset of Fig. 1. It was shown in Ref. [37] that a duality relates the dynamical matrices of the structures with \( \theta \) and \( \theta^* = 2\theta_c - \theta \) (with \( \theta_c = \pi/4 \)) through a relation of the form [37]

\( \mathcal{U}(k)D(\theta^*, -k)\mathcal{U}^{-1}(k) = D(\theta, k) \) (9)

where \( \mathcal{U}(k) = \text{diag}(i\sigma_y, i\sigma_y e^{-ikx_2}, i\sigma_y e^{ikx_1}) \). In this expression, the matrices \( i\sigma_y \) act on the displacements \((x, y)\) of each of the three masses in the unit cell of the crystal, \( \sigma_i \) are Pauli matrices, and \( a_i = [\cos((i-1)\pi/3), \sin((i-1)\pi/3)]^T \) are primitive vectors of the triangular Bravais lattice. The duality (9) typically relates different systems, with different twisting angles, such as the mechanical networks represented in Fig. 1 (a) and (c). However, there is a particular self-dual angle \( \theta_c = \pi/4 \) such that \( \theta^*_c = \theta_c \) (see Fig. 1), where the duality relation becomes an additional non-spatial symmetry of the dynamical matrix.

When Eq. (9) is applied to the Kagome lattices, one finds that \( U_{ZZ} = i\sigma_y \) and \( u = -I_d \). Upon substituting these results in Eq. (8), we obtain

\( V = \sigma_3 \otimes i\sigma_2 \) (10)

where \( \otimes \) is the Kronecker product and \( \sigma_i \) are Pauli matrices. It is instructive to write the most general form of the elastic matrix for a standard material (i.e., energy and angular momentum are conserved and solid-body rotations do not change the elastic energy). In this situation, \( K^{ab}_0 = 0 = K^{\text{ob}} \) and \( K^{ab} = K^{ba} \) (see Ref. [12] and SI for details), so we have

\[ K = \begin{pmatrix} K^{00} & K^{02} & K^{03} \\ 0 & 0 & 0 \\ K^{02} & K^{22} & K^{23} \\ K^{03} & K^{23} & K^{33} \end{pmatrix} \] (11)

The elastic matrices \( K(\theta) \) and \( K(\theta^* ) \) of two twisted Kagome lattices must indeed have the form (11). Following the preceding analysis, the duality relation (9) implies an additional set of constraints

\[ VK(\theta)V^\dagger = K(\theta^* ) \] (12)
with the transformation matrix $V$ defined in Eq. (10). As a consequence, we find that

$$K(\theta) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & K^{22}(\theta) & K^{23}(\theta) \\
0 & 0 & K^{23}(\theta) & K^{33}(\theta)
\end{pmatrix}$$

(13)

with

$$K^{22}(\theta) = K^{33}(\theta^*)$$

(14a)

$$K^{33}(\theta) = K^{22}(\theta^*)$$

(14b)

$$K^{23}(\theta) = -K^{23}(\theta^*).$$

(14c)

In particular, the constraint $V K(\theta_c) V^\dagger = K(\theta_c)$ at the critical angle $\theta_c = \theta_c^*$ leads to $K^{22}(\theta_c) = K^{33}(\theta_c)$ while $K^{23}(\theta_c) = 0$.

Hence, the duality relation (12) implies two striking consequences. First, twisted Kagome lattices have only shear moduli: the coefficients $K^{00}$, $K^{02}$ and $K^{03}$ always vanish [see Eq. (13)]. Crucially, the duality constrains the elastic moduli everywhere along the duality line (not only at the self-dual point). Physically, the lack of bulk moduli is related to the existence of a Guest-Hutchinson mechanism [15, 17, 44–46], see in particular Ref. [46]. Second, a stronger constraint occurs at the self-dual point where the elastic tensor becomes isotropic and characterized by a single shear modulus, despite no change in symmetry in the lattice. The occurrence of an isotropic elastic tensor holds even when all point group symmetries are lifted (i.e. the space group is $p1$). A direct computation of the elastic tensor from the dynamical matrix shown in Fig. 2, using either Eq. (3) or the real-space equivalent [39] confirms all our results [37].

Conclusions. – We have shown how hidden non-spatial symmetries (originating, for instance, from dualities) strongly constrain the elastic moduli of a solid. Our results suggest a general mechanism limited to elasticity by which microscopic dualities and non-spatial symmetries impose constraints on generalized rigidities and response functions. These subtle effects are not captured by an analysis based on the spatial symmetry (i.e., the point group or space group) of the underlying structure. They are therefore likely to be overlooked in a symmetry analysis performed from the point of view of the coarse-grained theory.

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FIG. 2. Elastic constants for an anisotropic Kagome lattices. The elastic moduli $K^{22}$, $K^{33}$ and $K^{23} = K^{32}$ computed from the microscopic description of Kagome lattices according to Eq. (3) are plotted as a function of the twisting angle $\theta$ for a generic situation where all inequivalent springs in the unit cell have different stiffnesses (see figure 1) [57]. The duality (represented by black arrows) exchanges $K^{22}$ and $K^{33}$ as well as $K^{23}$ and $−K^{23}$. We have set $k_1 = k_0, k_1 = 2k_0, k_3 = 3k_0.$
In general different (this number is greater or equal to the one we compute). One could also rotate a particular sample in order to reduce as much as possible the number of moduli, such as in Ref. [2, § 10] (this number is smaller or equal to the one we compute). Note that unlike the choice of a particular symmetry axis, that can in principle be done independently, this choice has nothing to do with symmetry: it is a choice of coordinate system along principal axes.

We now briefly describe how the standard formula (15) is obtained, and refer the reader to e.g. [50, § VIII.4.1.2] for more details (the same results can be obtained by direct inspection of the effects of group operations on the elastic tensor as explained e.g. in [35]; see also [51] for good summary of the method and its extension to quasicrystals). The main idea in counting the number of independent coefficients is that the elastic tensor should be invariant under symmetries, and hence should transform according to the identity representation of the symmetry group (see e.g. [33, 52–54] for references on group theory). First, we have to determine how the elastic tensor transforms under spatial transformations. This question can be rephrased as follows: under what representation $\Gamma$ of $O(d)$ (and hence of the point groups of interest) does the elastic tensor transform? We can then decompose $\Gamma$ into irreducible representations of the point group $G$; this decomposition looks like $\Gamma = n_1\Gamma_1 \oplus n_2\Gamma_2 \oplus \cdots \oplus n_N\Gamma_N$, where $\Gamma_i$ are the irreducible representations of $G$. In general, such a decomposition means that there are $n_1$ basis tensors $T^{\alpha}_i$ (with $\alpha = 1, \ldots, n_1$) transforming under $\Gamma_1$, etc., such that

$$c_{ijk\ell} = \sum_{k=1}^N \sum_{\alpha=1}^{n_k} c^{\alpha}_k [T^{\alpha}_k]_{ijk\ell}$$

where $c^{\alpha}_k$ are the coefficients in the decomposition. The elastic tensor must be invariant under the symmetry group $G$. This means that in this decomposition, only the part transforming along the identity representation $\Gamma_1$ (i.e., not transforming at all, as they are invariant) can stay. Hence, there are $n_1$ independent elastic moduli. To obtain this number explicitly, it is enough to know the character of the representation $\Gamma_i$, i.e. the trace of the representation applied to each element of the group. The numbers $n_k(\chi)$ can be computed as

$$n_k(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi_k(g)$$

where $|G|$ is the number of elements in the group. The character of the identity representation is $\chi_1(g) = 1$ for all $g \in G$, which gives the equation (15).

This number depends on the character $\chi$ of the representation $\Gamma$, which depends on our choices of constraints. For instance, if we do not impose any symmetrization...
TABLE I. Possible constraints on the elastic tensor. Indices or groups of indices in parentheses are symmetrized. For instance, $2\epsilon_{ij(kl)} = \epsilon_{ijkl} + \epsilon_{klij}$, while $2\epsilon_{ij(kl)} = \epsilon_{ijkl} + \epsilon_{klji}$, etc.

| point group | $c_{ijkl}$ | $c_{ij(kl)}$ | $c_{ij(lk)}$ | $c_{ij(kl)}$ | $c_{ijkl}$ |
|-------------|------------|--------------|--------------|--------------|------------|
| 1           | $C_1$      | 16           | 12           | 9            | 10         | 6          |
| 2           | $C_2$      | 16           | 12           | 9            | 10         | 6          |
| m           | $C_m$      | 8            | 6            | 5            | 6          | 4          |
| 2mm         | $C_{2v}$   | 8            | 6            | 5            | 6          | 4          |
| 4           | $D_4$      | 8            | 6            | 5            | 6          | 4          |
| 4mm         | $C_{4v}$   | 4            | 3            | 3            | 4          | 3          |
| 3           | $D_3$      | 6            | 4            | 3            | 4          | 2          |
| 3m          | $C_{3v}$   | 3            | 2            | 2            | 3          | 2          |
| 6           | $D_6$      | 6            | 4            | 3            | 4          | 2          |
| 6mm         | $C_{6v}$   | 3            | 2            | 2            | 3          | 2          |

TABLE II. Number of elastic moduli in 2D. The number of $c_{ijkl}$ is identical to the number of $\epsilon_{ijkl}$. The number of moduli was computed with GAP [48] using the crystallographic database package CrystCat [49]. The corresponding point groups are labeled with the conventions of Ref. [34] in Hermann-Mauguin notation (first column), as well as in Schoenflies notation (second column).

| point group | $c_{ijkl}$ | $c_{ij(kl)}$ | $c_{ij(lk)}$ | $c_{ij(kl)}$ | $c_{ijkl}$ |
|-------------|------------|--------------|--------------|--------------|------------|
| 1           | $C_1$      | 16           | 12           | 9            | 10         | 6          |
| 1           | $C_1$      | 16           | 12           | 9            | 10         | 6          |
| m           | $C_m$      | 8            | 6            | 5            | 6          | 4          |
| 2           | $C_2$      | 16           | 12           | 9            | 10         | 6          |
| m           | $C_m$      | 8            | 6            | 5            | 6          | 4          |
| 2           | $C_2$      | 4            | 3            | 3            | 4          | 3          |
| 3           | $D_3$      | 6            | 4            | 3            | 4          | 2          |
| 6           | $D_6$      | 6            | 4            | 3            | 4          | 2          |
| 6mm         | $C_{6v}$   | 3            | 2            | 2            | 3          | 2          |

TABLE III. Number of elastic moduli in 3D. The number of $c_{ijkl}$ is identical to the number of $\epsilon_{ijkl}$. The number of moduli was computed with GAP [48] using the crystallographic database package CrystCat [49]. The corresponding point groups are labeled with the conventions of Ref. [34] in Hermann-Mauguin notation (first column), as well as in Schoenflies notation (second column).

| point group | $c_{ijkl}$ | $c_{ij(kl)}$ | $c_{ij(lk)}$ | $c_{ij(kl)}$ | $c_{ijkl}$ |
|-------------|------------|--------------|--------------|--------------|------------|
| 1           | $C_1$      | 81           | 54           | 36           | 45         | 21         |
| 1           | $C_1$      | 81           | 54           | 36           | 45         | 21         |
| m           | $C_m$      | 8            | 6            | 5            | 6          | 4          |
| 2           | $C_2$      | 41           | 28           | 20           | 25         | 13         |
| m           | $C_m$      | 41           | 28           | 20           | 25         | 13         |
| 2           | $C_2$      | 21           | 15           | 12           | 15         | 9          |
| 3           | $D_3$      | 21           | 15           | 12           | 15         | 9          |
| 4           | $S_4$      | 21           | 14           | 10           | 13         | 7          |
| 4           | $S_4$      | 21           | 14           | 10           | 13         | 7          |
| 4/m         | $C_{4h}$   | 21           | 14           | 10           | 13         | 7          |
| 4/m         | $C_{4h}$   | 21           | 14           | 10           | 13         | 7          |
| 4           | $S_4$      | 21           | 14           | 10           | 13         | 7          |
| 4           | $S_4$      | 21           | 14           | 10           | 13         | 7          |
| 6           | $C_6$      | 19           | 12           | 8            | 11         | 5          |
| 6           | $C_6$      | 19           | 12           | 8            | 11         | 5          |
| 2           | $D_2$      | 10           | 7            | 6            | 8          | 5          |
| 2/m         | $D_{2h}$   | 10           | 7            | 6            | 8          | 5          |
| 6           | $C_6$      | 10           | 7            | 6            | 8          | 5          |
| 6           | $C_6$      | 10           | 7            | 6            | 8          | 5          |
| 3           | $T_3$      | 7            | 5            | 4            | 5          | 3          |
| 3           | $T_3$      | 7            | 5            | 4            | 5          | 3          |
| 3/m         | $T_3$      | 7            | 5            | 4            | 5          | 3          |
| 3/m         | $T_3$      | 7            | 5            | 4            | 5          | 3          |
| 4            | $T_4$      | 4            | 3            | 3            | 4          | 3          |
| 4            | $T_4$      | 4            | 3            | 3            | 4          | 3          |
| m3           | $O_h$      | 4            | 3            | 3            | 4          | 3          |
| m3           | $O_h$      | 4            | 3            | 3            | 4          | 3          |

where the matrices $\tau^{\alpha} = \tau^{\alpha}$ (seen as vectors) form a suitable orthonormal basis of $M_d(\mathbb{R})$ (seen as a vector space endowed with a scalar product such as $\langle M, N \rangle = \text{tr}(M^T N)/2$, as we will assume in the following). Although any basis can formally be chosen, it is convenient to choose basis matrices from symmetry, see Ref. [12].

To recover the matrix $K^{ab}$ from the elastic tensor, we
write
\[
\frac{1}{4} \epsilon_{ij}^c \epsilon_{kl}^d c_{ijkl} = \sum_{a,b} K^{ab}_{ijkl} \epsilon_{ij}^a \epsilon_{kl}^b \epsilon_{ij}^c \epsilon_{kl}^d
\] (23)

where we recognize \( \tau_\alpha^a, \tau_\alpha^c = \text{tr}(\tau^\alpha \tau^\alpha) = 2 \delta^{ac} \) (similarly, we obtain \( 2 \delta^{bd} \)). Hence,
\[
K^{ab} = \frac{1}{4} \sum_{ijkl} \epsilon_{ij}^a \epsilon_{kl}^b c_{ijkl}.
\] (24)

The duality relation (9) implies that for all \( \theta \),
\[
VK(\theta)V^{-1} = K(\theta^*)
\] (28)

where both \( K(\theta) \) and \( K(\theta^*) \) are constrained to be of the form (25).

Hence, we find that we always have
\[
K^{00}(\theta) = K^{02}(\theta) = K^{03}(\theta) = 0.
\] (29)

Besides, the remaining coefficients at \( \theta \) and \( \theta^* \) are related through
\[
K^{22}(\theta) = K^{33}(\theta^*)
\] (30a)
\[
K^{33}(\theta) = K^{22}(\theta^*)
\] (30b)
\[
K^{23}(\theta) = -K^{23}(\theta^*).
\] (30c)

Hence, at the self-dual point, \( K^{22}(\theta_c) = K^{33}(\theta_c) \) and \( K^{23}(\theta_c) = 0 \).

Duality between elastic matrices in twisted Kagome lattices

Consider the most general elastic matrix satisfying the constraints of standard elasticity (see figure 3) in two dimensions,
\[
K = \begin{pmatrix}
K^{00} & 0 & K^{02} & K^{03} \\
0 & 0 & 0 & 0 \\
K^{02} & 0 & K^{22} & K^{23} \\
K^{03} & 0 & K^{23} & K^{33}
\end{pmatrix}.
\] (25)

With the matrix
\[
V = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\] (26)

we have
\[
V^T K V = \frac{1}{2} \left[ \text{tr}(T^2) + \text{tr}(T^2) \right] \text{tr}(T^2)
\]
\[
c_{ijkl} = \frac{1}{2} \left[ \text{tr}(T^2) + \text{tr}(T^2) \right] \text{tr}(T^2)
\]
\[
c_{ijkl} = \frac{1}{4} \left[ \text{tr}(T^2) + \text{tr}(T^2) \right] \text{tr}(T^2)
\]
\[
c_{ijkl} = \frac{1}{2} \left[ \text{tr}(T^2) + \text{tr}(T^2) \right] \text{tr}(T^2)
\]
\[
c_{ijkl} = \frac{1}{4} \left[ \text{tr}(T^2) + \text{tr}(T^2) \right] \text{tr}(T^2)
\]

The duality relation (9) implies that for all \( \theta \),
\[
VK(\theta)V^{-1} = K(\theta^*)
\] (28)

where both \( K(\theta) \) and \( K(\theta^*) \) are constrained to be of the form (25).

Hence, we find that we always have
\[
K^{00}(\theta) = K^{02}(\theta) = K^{03}(\theta) = 0.
\] (29)

Besides, the remaining coefficients at \( \theta \) and \( \theta^* \) are related through
\[
K^{22}(\theta) = K^{33}(\theta^*)
\] (30a)
\[
K^{33}(\theta) = K^{22}(\theta^*)
\] (30b)
\[
K^{23}(\theta) = -K^{23}(\theta^*).
\] (30c)

Hence, at the self-dual point, \( K^{22}(\theta_c) = K^{33}(\theta_c) \) and \( K^{23}(\theta_c) = 0 \).

Determination of the elastic tensor from the microscopic description

In this Appendix, we review the coarse-graining of the microscopic equations of motion summarized in the force constant matrix to the elastic tensor. We refer the reader to Refs. [12, 15, 38–42, 55] for more details.

We first diagonalize the momentum-space force constant matrix \( S(0) \) at \( q = 0 \). The corresponding basis of orthogonal eigenvectors are ordered by increasing eigenvalues, and \( S(q) \) is written in the block-diagonal form
\[
S(q) = \begin{pmatrix}
S_{ZZ}(q) & S_{ZF}(q) \\
S_{FZ}(q) & S_{FF}(q)
\end{pmatrix}
\] (31)

where the block \( Z \) corresponds to the kernel of \( S(0) \) i.e. to zero-frequency modes, while \( F \) corresponds to the remaining modes with finite frequency. By definition, \( S(0) \) is block-diagonal and \( S_{ZZ}(0) = 0 \) while \( S_{ZZ}(0) \equiv S_{ZZ} \) is invertible.

We further assume that \( S_{ZZ}(q) = S_{ZZ}^{\mu\nu} q_\mu q_\nu + O(q^2) \) i.e. that (i) there is no linear term in the series expansion of \( S_{ZZ}(q) \) near \( q = 0 \) and (ii) the second-order term is non-vanishing. This is not necessarily true: for instance, the system may be pre-stressed, or there may be lines of zero-frequency modes in the Brillouin zone (see e.g. Ref. [15]).

At lowest order in each block, the force-constant matrix then reads
\[
S(q) = \begin{pmatrix}
S_{ZZ}^{\mu\nu} q_\mu q_\nu & S_{ZF}^{\mu\nu} \\
S_{FZ}^{\mu\nu} & S_{FF}^{\mu\nu}
\end{pmatrix}
\] (32)

where
\[
S_{ZZ}^{\mu\nu} = \frac{\partial^2 S_{ZZ}}{\partial q_\mu \partial q_\nu} \bigg|_{q=0} \quad S_{ZF}^{\mu\nu} = \frac{\partial S_{ZF}(FZ)}{\partial q_\mu} \bigg|_{q=0}
\] (33)
and $S_{FF} = S_{FF}(q = 0)$.

Let $u_{Z,n}$ with $n = 1, \ldots, d$ be a basis of the nullspace of $S(0)$. These can usually be chosen as the solid-body motion of all particles in the unit cell in a given space direction (by construction), and labeled with spatial directions. The displacement tensor (here written in momentum space) is related to the displacement field $u(q)$ by $\epsilon_{\mu\nu} = i (u_{Z,\mu} q_\nu u(q))$, i.e. it corresponds to the gradients of the projection on the nullspace of $S(0)$ of the displacements.

Similarly, the force (density) $f$ acting on the elastic body is identified to the projection $F_Z$ on the elastic degrees of freedom of the force $F = -Su$, so that $f_{\mu}(q) = \rho F_{Z,\mu}(q) = i q_\mu \sigma_{\mu\nu}$. We note that a general definition of the stress tensor requires some caution, especially when forces with long range (with respect to the microscopic scales in the lattice) are present (here, we avoided those issues by assuming a [possibly effective] description in terms of pairwise harmonic interactions).

First, a distinction has to be made between body forces and surface forces: we refer the reader to Refs. [56, 57] for discussions. Besides, the uniqueness of the stress tensor is a controversial issue (at first sight, it is uniquely defined only up to divergence free terms, but some additional assumptions appear to make it unique), and we refer the reader to Refs. [58–61] for more details.

The projection on the finite frequency part of the displacement is called non-affine displacement and is determined by assuming that the corresponding (non-elastic) projection of the force vanishes, $F_{F,\mu} = 0$, see Refs. [15, 38–42, 55]. In other words, we integrate out the irrelevant degrees of freedom at high frequency by solving for $u_F$ such that $F_{F,\mu} = 0$ and replacing in the equations. Physically, this is because the non-affine forces $F_{F,\mu}$ relax due to thermal fluctuations: the additional term (called non-affine term) in the elastic tensor that accounts for this relaxation is the zero temperature limit of the term accounting for fluctuations in finite-temperature elasticity [41].

Hence, we have

$$F = \begin{pmatrix} F_Z \\ 0 \end{pmatrix} = -\begin{pmatrix} q_\mu \\ 0 \end{pmatrix} \begin{pmatrix} S_{wZ}^{\mu} \\ S_{FZ}^{\mu} \end{pmatrix} \begin{pmatrix} q_\nu \\ 0 \end{pmatrix} \begin{pmatrix} u_Z \\ u_F \end{pmatrix}$$

(34)

where we have factorized the force-constant matrix. Hence,

$$\begin{pmatrix} q_\mu^{-1} F_Z \\ 0 \end{pmatrix} = -\begin{pmatrix} S_{Zw}^{\mu} \\ S_{Fw}^{\mu} \end{pmatrix} \begin{pmatrix} q_\nu u_Z \\ u_F \end{pmatrix}$$

(35)

Note that multiplication by $q_\mu^{-1}$ corresponds to integration (effectively, we want to find $q$ given $f$ and the relation $f = \text{Div}(q)$), so integration constants may in general appear [12]. We shall assume that such constants vanish.

Solving for $u_F$ and replacing then yields

$$q_\mu^{-1} F_Z = -[S_{ZZ}^{\mu} - S_{ZF}^{\mu} S_{FZ}^{-1} q_\nu] u_Z$$

(36)
where we recognize the deformation tensor and the stress tensor, so that
\[ \sigma_{\mu \alpha} = \rho [\Sigma_{\mu \nu}^{\alpha \beta} - S_{\nu \xi \mu \zeta}^{\mu \beta} \xi \eta \zeta |_{\alpha \beta} \epsilon_{\nu \xi \zeta} ]. \]  
(37)

The elastic tensor is then
\[ c_{\mu \alpha \nu \beta} = \rho [\Sigma_{\mu \nu}^{\alpha \beta} ] \alpha \beta. \]  
(38)

where
\[ \Sigma_{\mu \nu}^{\alpha \beta} = S_{\mu \nu}^{\alpha \beta} - S_{\nu \xi \mu \zeta}^{\mu \beta} \xi \eta \zeta. \]  
(39)

Symmetries of the elastic tensor from the symmetries of the force constant matrix

We consider a force constant matrix \( S(q) \), giving rise to an elastic tensor \( c_{ijkl} \). Let us consider the new force constant matrices (a) \( \tilde{S}(q) = U(q)S(q)U(q)^{-1} \) and (b) \( \tilde{S}(q) = S(u \cdot q) \) (we shall then combine the results) and determine the corresponding elastic tensors \( \tilde{c}_{ijkl} \) in terms of the initial elastic tensor.

Consider first the case (a) where
\[ \tilde{S}(q) = U(q)S(q)U(q)^{-1}. \]  
(40)

For spatial symmetries, it is usually possible to assume that the symmetry operator does not depend on the momentum. However, this is not the case for the duality operator considered in the main text. Hence, we must consider cases where the unitary matrix \( U(q) \) depends explicitly on \( q \). However, we shall see that only the projection \( U_{\nu \xi} \) to the kernel of \( S(0) \) taken at \( q = 0 \) appears in the transformation of the elastic tensor, as if only \( U(0) \) was considered. We first write the symmetry operator at the lowest non-trivial order in each block
\[ U(q) = \begin{pmatrix} U_{\nu \xi} & U_{\nu \eta} q_{\nu} U_{\xi \eta} \\ U_{\nu \eta} q_{\nu} U_{\xi \eta} & U_{\xi \eta} \end{pmatrix} \]  
(41)

and
\[ U^{-1}(q) = \begin{pmatrix} U_{\nu \xi}^\dagger & U_{\nu \eta}^\dagger q_{\nu} U_{\xi \eta}^\dagger \\ U_{\nu \eta}^\dagger q_{\nu} U_{\xi \eta}^\dagger & U_{\xi \eta}^\dagger \end{pmatrix}. \]  
(42)

At lowest order in each block,
\[ U(q)S(q)U^{-1}(q) \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} q_{\nu} \tilde{S}_{\xi \eta \mu \zeta}^{\alpha \beta} q_{\nu} \]  
(43)


\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} \]  
(44a)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} \]  
(44b)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} \]  
(44c)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} \]  
(44d)

We also must have \( \tilde{U}_{\nu \xi}^\dagger = [U_{\nu \xi}]^\dagger \) so that \( [\tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta}]^\dagger = \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} \). Combining the preceding relations into
\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} + U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi} \]  
(45)

yields, after simplification
\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = U_{\nu \xi} S_{\nu \xi \mu \zeta}^{\alpha \beta} U_{\nu \xi}. \]  
(46)

Consider now the case (b) where
\[ \tilde{S}(q) = S(u \cdot q) \]  
(47)

where \( u \in O(d) \) acts canonically on \( q \in \mathbb{R}^d \), namely so that \( (u \cdot q)_\mu = u_\mu q_\nu \). A direct computation shows that the blocks indeed transform like tensors, namely
\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = S_{\nu \xi \mu \zeta}^{\alpha \beta} u_\mu u_\nu \]  
(48a)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = S_{\nu \xi \mu \zeta}^{\alpha \beta} u_\mu u_\nu \]  
(48b)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = S_{\nu \xi \mu \zeta}^{\alpha \beta} u_\mu u_\nu \]  
(48c)

\[ \tilde{S}_{\nu \xi \mu \zeta}^{\alpha \beta} = S_{\nu \xi \mu \zeta}^{\alpha \beta} u_\mu u_\nu \]  
(48d)

As a consequence,
\[ \Sigma_{\mu \nu}^{\alpha \beta} = \Sigma_{\mu \nu}^{\alpha \beta} u_\mu u_\nu \]  
(49)

Finally, consider the combination of cases (a) and (b),
\[ \tilde{S}(q) = (USU^{-1})(u \cdot q). \]  
(50)

By combining the previous results, we obtain
\[ c_{ijmn} = [U_{\nu \xi}]_{ij} c_{\alpha \beta \gamma \delta} [U_{\nu \xi}]_{\gamma \delta} u_{\alpha m} u_{\beta n}. \]  
(51)

Finally, let us write the constraint in terms of the elastic matrix. We want to compute
\[ \tilde{K}^{ab} = \frac{1}{4} \sum_{ijkl} \tilde{c}_{ij}^{ab} \tilde{c}_{ijkl} \]  
(52)

in terms of
\[ K^{ab} = \frac{1}{4} \sum_{ijkl} c_{ij}^{ab} c_{ijkl} \]  
(53)

Using \( c_{i'm'j'n'} = c_{i'm'}^{i'j'} c_{r'd}^{i'j'} \) we get
\[ \tilde{K}^{ab} = V^{ac} K^{cd} W^{db} \]  
(54)

with
\[ V^{ac} = \frac{1}{2} \tilde{c}_{im}^{ac} [U_{\nu \xi}]_{mn} \tilde{c}_{i'm'}^{i'j'} u_{i'j'} \]  
(55)

\[ W^{db} = \frac{1}{2} u_{j'j} \tilde{c}_{i'm'}^{i'j'} [U_{\nu \xi}]_{n'n'} \tilde{c}_{r'd}^{i'j'} \]  
(56)

Provided that \( u \in O(d) \) and that \( \tau \) matrices are real, \( W^{db} = \bar{V}^{bd} \), and we can write
\[ \tilde{K} = VKV^\dagger \]  
(57)

where (as defined above)
\[ V^{ac} = \frac{1}{2} \text{Tr} [\tau^{ac} U_{\nu \xi} [\tau^c]^T U]. \]  
(58)