RIESZ TRANSFORM ON MANIFOLDS WITH QUADRATIC CURVATURE DECAY.

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ABSTRACT. We investigate the $L^p$-boundness of the Riesz transform on Riemannian manifolds whose Ricci curvature has quadratic decay. Two criteria for the $L^p$-unboundness of the Riesz transform are given. We recover known results about manifolds that are Euclidean or conical at infinity.

RÉSUMÉ : On étudie la continuité de la transformée de Riesz sur les espaces $L^p$ pour des variétés dont la courbure de Ricci décroît quadratiquement. Nous donnons aussi deux critères géométriques impliquant la non continuité de la transformée de Riesz. Notre méthode nous permet de retrouver les résultats connus pour les variétés euclidiennes ou coniques à l’infini.

1. INTRODUCTION

Let $(M^n, g)$ be a complete Riemannian manifold with infinite volume, and let $\Delta$ be its associated Laplacian. The Green formula:

$$\forall f \in C_0^\infty(M), \int_M |df|^2_g d\text{vol}_g = \langle \Delta f, f \rangle_{L^2} = \int_M |\Delta^{1/2} f|^2 d\text{vol}_g,$$

implies that the Riesz transform

$$R := d\Delta^{-1/2} : L^2(M) \to L^2(T^*M)$$

is a bounded operator. It is well known [38] that on an Euclidean space, the Riesz transform has a bounded extension $R : L^p(\mathbb{R}^n) \to L^p(T^*\mathbb{R}^n)$ for every $p \in (1, +\infty)$. In general, it is of interest to figure out the range of $p$ for which the Riesz transform extends to a bounded operator $R : L^p(M) \to L^p(T^*M)$ [39].

In this article, we are going to use recent results of A. Grigor’yan and L. Saloff-Coste [20] and V. Minerbe [33] together with some idea of P.

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Auscher, T. Coulhon, X-T. Duong and S. Hoffmann [1] to study this ques-
tion on a Riemannian manifold whose Ricci curvature satisfies a quadratic
decay lower bound
\[
(QD) \quad \text{Ricci} \geq -\frac{\kappa^2}{r^2(x)}g,
\]
where for a fixed point \(o \in M\) : \(r(x) := d(o, x)\).

A Riemannian manifold \((M^n, g)\) that outside a compact set \((M, g)\) is
isometric to the warped product
\[
([1, \infty) \times \Sigma, (dr)^2 + r^{2\gamma}h)
\]
where \((\Sigma, h)\) is a compact manifold with non negative Ricci curvature and
\(\gamma \in (0, 1)\) satisfies the quadratic decay condition \((QD)\). And our analysis
will show that :

- If \(\Sigma\) is connected then the the Riesz transform is bounded on \(L^p\) for
every \(p \in (1, +\infty)\)
- If \(\Sigma\) is not connected then the Riesz transform is bounded on \(L^p\) if
  and only if \(1 < p \leq 2\) or \(1 < p < (n - 1)\gamma + 1\).

According to [20, 33], under this condition \((QD)\) we have a good un-
derstanding of the behavior of the heat kernel \(\{h_t(x, y)\}\) i.e. the Schwarz
kernel of the heat operator \(e^{-t\Delta}\) :
\[
e^{-t\Delta} f(x) = \int_M h_t(x, y) f(y) dy.
\]

This will be the case when \((M, g)\) satisfies the following two conditions:
- The volume comparison : for some constant \(C\) and for any \(R \geq 1\)
  and any \(x \in \partial B(o, R)\) :
  \[
  (VC) \quad \text{vol } B(o, R) \leq C \text{ vol } B(x, R/2)
  \]
- The RCE (Relatively Connected to an End) condition : there is a
  constant \(\theta \in (0, 1)\) such that for any \(R \geq 1\) and any \(x \in \partial B(o, R)\)
  can be connected through a path \(c\) : \([0, 1] \rightarrow B(o, R) \setminus B(o, \theta R)\)
  with the foot of a geodesic ray : \(\gamma\) : \([0, +\infty) \rightarrow M \setminus B(o, R)\) i.e.
  \(c(0) = x, c(1) = \gamma(0)\) and such that the lengh of \(c\) is not too long :
  \[
  \mathcal{L}(c) \leq \frac{R}{\theta}.
  \]

These two conditions imply that the heat kernel satisfies the upper bound:
for all \(t > 0, x, y \in M\):
\[
(DUE) \quad h_t(x, y) \leq \frac{C}{\text{vol } B(x, \sqrt{t})},
\]
and that \((M^n, g)\) is doubling: there is a constant \(\vartheta\) such that for any \(x \in M\) and radius \(R > 0\):

\[
(D) \quad \text{vol } B(x, 2R) \leq \vartheta \text{ vol } B(x, R)
\]

Our first result is the following

**Theorem A.** Let \((M^n, g)\) be a complete Riemannian manifold whose Ricci curvature satisfies the quadratic decay control (QD). Assume also that \((M, g)\) satisfies the hypothesis (VC & RCE) or (DUE & D). If for some positive constants \(c\) and \(\nu > 2\), balls anchored at \(o\) satisfy the reverse doubling hypothesis:

\[
\forall R \geq r \geq 1 : \quad c \left(\frac{r}{R}\right)^\nu \text{ vol } B(o, r) \leq \text{ vol } B(o, R);
\]

then:

- The Riesz transform is bounded on \(L^p\) for any \(p \in (1, \nu)\).
- if \((M, g)\) is \(\beta\)-hyperbolic and if the \(\beta\)-capacity of anchored balls satisfy:

\[
\frac{\text{vol } B(o, R)}{R^\beta} \leq C \text{ cap}_{\beta}(B(o, R))
\]

then the Riesz transform is bounded on \(L^p\) for any \(p \in (1, \beta)\).

Recall that a complete Riemannian manifold \((M, g)\) is said to be \(\beta\)-hyperbolic if the \(\beta\)-capacity of some/any bounded open subset \(O \subset M\) is positive, where the \(\beta\)-capacity is defined by:

\[
\text{cap}_{\beta} O := \inf \left\{ \int_M |d\varphi|^\beta \text{ dvol}_g, \varphi \in C^\infty_0(M) \text{ and } \varphi \geq 1 \text{ on } O \right\},
\]

Our proof is based on estimates of the Schwarz kernel of the Riesz transform. Outside the diagonal of \(M \times M\), this kernel is smooth and given by:

\[
R(x, y) = \int_0^\infty \nabla_x h_t(x, y) \frac{dt}{\sqrt{\pi t}}.
\]

Following the philosophy of Pseudo-differential operator on open manifold ([32]) and that we already used in [6], we separate our analysis in two parts: the part closer to the diagonal \(\{(d(x, y) \leq \kappa r(x)\}\) where we can use the result of [1], and the off diagonal part, where we get the estimate:

\[
|R(x, y)| \leq \frac{d(x, y)}{r(x)} \frac{C}{\text{vol } B(o, d(x, y))}.
\]

When \((M, g)\) is a manifold with Euclidean ends, our estimates correspond to the one obtain in [6] and are sharp. The control on the Ricci curvature will be used to get the gradient estimate for the heat kernel ([29]). We will not use the Kato inequality to control the behavior Laplacian on 1-form by
a Schrödinger operator. In particular, we do not get Gaussian estimates on the heat kernel of the Hodge Laplacian on 1-forms. It is known that such a Gaussian upper bound implies a boundness result for the Riesz transform ([35]). And in our setting, it is possible to get such upper bound with some sub-critical assumptions on the Ricci curvature ([13], [15], [17]) and it has recently been shown that this sub-critical assumptions yields results for the Riesz transform on 1 and 2-forms ([31]).

We’ll improve an earlier result of [6] and show that if \( M \) has two ends then there are restriction on the range of \( p \) where the Riesz transform is \( L^p \) bounded:

**Theorem B.** On a non-2-parabolic and \( p > 2 \)-parabolic manifold with at least two ends, the Riesz transform can not be bounded simultaneously on \( L^p \) and on \( L^{p-\frac{1}{2}} \).

Recall that a non-parabolic manifold (or equivalently a 2-hyperbolic) manifold is a manifold carrying a positive Green kernel. According to [10], the conditions (DUE) and (D) imply that the Riesz transform is bounded on \( L^q \) for any \( q \in (1,2] \) hence in our setting the above criterium is mainly a criterium for the unboundness of the Riesz transform on \( L^p \). Moreover, this criterium implies that the gluing result of B. Devyver is optimal. In [16], B. Devyver shows that if we consider two manifolds \( M_1, M_2 \) where the Riesz transform is bounded on \( L^p \), such that they satisfy a Sobolev inequality, a lower bound on the Ricci curvature and if the connected sum \( M_1 \# M_2 \) is \( p \)-hyperbolic then the Riesz transform is \( L^p \)-bounded on \( M_1 \# M_2 \).

When \( M \) has only one end, we can improve the range of \( p \) where the Riesz transform is bounded on \( L^p \). Indeed in this case, the manifold \((M,g)\) satisfies the RCA condition introduced by A. Grigor’yan and L. Saloff-Coste and we get a scale Poincaré inequality : for any ball \( B \subset M \) and any function \( f \in C_c(2B) \) we have

\[
\| f - f_B \|_{L^2(B)} \leq C r(B) \| df \|_{L^2(B)}.
\]

Where we have use the following notation :

- if \( B \subset M \) is a ball we note \( r(B) \) its radius and for \( \theta > 0 \), \( \theta B \) is the ball with the same center and radius \( \theta r(B) \);
- if \( \mathcal{O} \subset M \) and \( f \in L^1(\mathcal{O}) \) we note \( f_{\mathcal{O}} \) the mean of \( f \) on \( \mathcal{O} \)

\[
f_{\mathcal{O}} = \frac{1}{\text{vol}\mathcal{O}} \int_{\mathcal{O}} f.
\]

In particular, we get some \( \varepsilon \)-Hölder regularity ([34, theorem 4.1], [35]) on the heat kernel :

\[
|h_t(x,y) - h_t(z,y)| \leq C \left( \frac{d(x,z)}{\sqrt{t}} \right)^\varepsilon e^{\frac{-d^2(x,y)}{4ct}} \frac{1}{\text{vol}\, B(x, \sqrt{t})}
\]
for all $t > 0$, $x, y, z \in M$ such that $d(x, z) \leq \sqrt{t}$.

In [1], the authors provide an analytic criteria for the $L^p$-boundness of the Riesz transform in terms of a control of $L^p$ norm of the gradient of the heat kernel. Using this result, it is possible to show that in the setting of the Theorem A, the Riesz transform is bounded on $L^p$ when $(1 - \varepsilon)p < \nu$. But we will obtain our result under a slightly weaker hypothesis.

**Definition 1.1.** Let $\alpha \in [0, 1]$. A complete Riemannian manifold $(M^n, g)$ is said to satisfy the scale $\alpha$-Hölder Elliptic estimate if there is a constant $C$ such that for any ball $B \subset M$ and any harmonic function $h$ defined on $3B$, we have for all $x, y \in B$:

$$|h(x) - h(y)| \leq C \left( \frac{d(x, y)}{r(B)} \right)\alpha \sup_{z \in 2B} |h(z)|.$$

The above $\varepsilon$-Hölder regularity on the heat kernel implies a scale $\varepsilon$-Hölder Elliptic estimate. A result of W. Hebisch and L. Saloff-Coste shows that under the hypothesis (D) and (DUE) the $\alpha$-Hölder Elliptic estimate implies some $\varepsilon$-Hölder regularity on the heat kernel where the exponent $\varepsilon$ that can be expressed in term of $\alpha$ and the other geometrical constants ([26]). We don’t know in which setting both are equivalent, with the same exponent. It should be remark that in general the Hölder regularity on the heat kernel is stronger that the Hölder Elliptic estimate ([4] [14]).

**Theorem C.** Let $(M^n, g)$ be a complete Riemannian manifold whose Ricci curvature satisfies the quadratic decay control (QD) and the reverse doubling hypothesis for some exponent $\nu > 2$. Assume also that $(M, g)$ satisfies the hypothesis (VC) and (RCA) and the scale $\alpha$-Hölder Elliptic estimate. Then the Riesz transform is bounded on $L^p$ for any $p$ such that $(1 - \alpha)p < \nu$.

Our proof is based on a result of P. Auscher and T. Coulhon, in [2] they prove that on a manifold that is doubling and that satisfies the scale $L^1$-Poincaré inequality, the Riesz transform is bounded on $L^p$ provide some $L^{p > p}$-Reverse Hölder inequality holds for the gradient of harmonic functions (see also [37]). Although it was not noticed by the authors, this result and the Cheng-Yau’s gradient estimate ([21]) provided another proof of D. Bakry’s famous result [3]: on a manifold with non negative Ricci curvature the Riesz transform is bounded on $L^p$, for all $p \in (1, \infty)$.

It is easy to see that if $(M, g)$ carries a non constant sublinear harmonic function $h$ :

$$h(x) = O \left( r^\beta (x) \right),$$

then $(M, g)$ can not satisfy the $\alpha$-Hölder Elliptic estimate for any $\alpha > \beta$. We’ll show that the existence of such a sublinear harmonic function yields some restrictions on the range of $p$ where the Riesz transform is on $L^p$-bounded:
Proposition D. Let \((M^n, g)\) be a complete Riemannian manifold whose Ricci curvature satisfies the quadratic decay control (QD) and the reverse doubling hypothesis for some exponent \(\nu > 2\). Assume also that \((M, g)\) satisfies the hypothesis (VC) and (RCE). If the volume growth of \((M, g)\) is control by \(R^\mu\):
\[
\forall R \geq 1 : \text{vol } B(o, R) \leq CR^\mu,
\]
and if \((M, g)\) carries a non constant sublinear harmonic function \(h\):
\[
h(x) = O \left( r^\beta(x) \right),
\]
then the Riesz transform can not be bounded on \(L^p\) for any \(p > 2\) such that \(p \geq \mu/(1 - \beta)\).

Applied to manifolds with conical ends, our analysis provides another proof of results of H-Q. Li, C. Guillarmou and A. Hassell and of A. Hassell and P. Lin ([28, 21, 22, 25]).

With H-J. Hein [7], we have obtained as a corollary of the Theorem C:

Corollary E. Let \((M^n, g)\) be a complete Riemannian manifold that satisfy the condition (VC) whose Ricci curvature satisfies the quadratic decay control (QD). Assume that the diameter of geodesic sphere growth slowly
\[
\text{diam } \partial B(o, R) = \sup_{x,y \in \partial B(o,R)} d(x, y) = o(R)
\]
then the Riesz transform is bounded on \(L^p\) for every \(p \in (1, +\infty)\).

Other results and extensions are possible:

According to the perturbation result of T. Coulhon and N. Dungey ([12]), if \((M, g)\) is a complete Riemannian manifold satisfying the hypothesis of the Theorem A or Theorem C and a non collapsing hypothesis:
\[
\inf_{x \in M} \text{vol } B(x, 1) > 0
\]
then the same conclusion holds for any other metric \(\tilde{g}\) that satisfies for some \(\varepsilon > 0\):
\[
\tilde{g} - g = O(r^{-\varepsilon}(x)).
\]

The situation is much more involved when we deals with the Riesz transform associated to Schrödinger operator \(R_V = d(\Delta + V)^{-\frac{1}{2}}\) ([17, 21, 22, 25]).

Recently, C. Guillarmou and D. Sher have investigated the Riesz transform on differential forms on manifold with conical ends ([23]) ; it is tempting to analyze what can be done on a manifold whose curvature tensor decay quadratically, however in order to used our analysis, we ’ll need Gaussian upper bound on the heat kernel on forms and such an estimate already provided some boundness result for the Riesz transform according to a general principle discovered by A. Sikora ([36]).
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2. Analysis on manifolds with a quadratic decay of the Ricci curvature

2.1. Setting: In this section, we consider a complete Riemannian manifold $(M^n, g)$ such that for a fixed point $o \in M$, the Ricci curvature satisfies:

$$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g_x,$$

where we have defined $r(x) = d(o, x)$. We'll review geometric conditions that insures that $(M, g)$ satisfies the so-called relative Faber-Krahn inequalities: there are uniform positive constants $C, \mu$ such that for any $x \in M$ and $R > 0$ and any open domain $\Omega \subset B(x, R)$:

$$\lambda^D_{\Omega} \geq \frac{C}{R^2} \left( \frac{\text{vol} \Omega}{\text{vol} B(x, R)} \right)^{-\frac{2}{\mu}}.$$

We are note by $\lambda^D_{\Omega}$ the first non zero eigenvalue of the Dirichlet Laplacian on $\Omega$:

$$\lambda^D_{\Omega} = \inf_{\varphi \in C^\infty_0(\Omega)} \frac{\int_{\Omega} |d \varphi|^2}{\int_{\Omega} |\varphi|^2}.$$

Our discussion is based on the results of A. Grigor’yan and L. Saloff-Coste [20] and V. Minerbe [33].

It is well known [18] that the relative Faber-Krahn inequalities are equivalent to the conjunction of the following two properties:

- $(M, g)$ is doubling: there is a constant $\vartheta$ such that for any $x \in M$ and radius $R > 0$:
  $$\text{vol} B(x, 2R) \leq \vartheta \text{ vol} B(x, R)$$

- The heat kernel $h_t(x, y)$ satisfies the uniform upper bound: for all $t > 0$ and all $x, y \in M$,
  $$h_t(x, y) \leq \frac{C}{\text{vol} B(x, \sqrt{t})}.$$

2.2. Remote balls. A ball $B(x, \rho) \subset M$ is called remote if its center $x$ and radius $\rho$ satisfy

$$\rho \leq \frac{r(x)}{2}.$$
The Bishop-Gromov comparison theorem implies that all remote ball satisfy the doubling condition: If $B$ is a remote ball and if $\theta \in (0, 1)$ then
\[
\theta^n \text{vol}(B) \leq C(n, \kappa) \text{vol}(\theta B).
\]
The quadratic lower bound implies that remote balls satisfy the Poincaré inequality and a Relative Faber-Krahn inequality:

**Lemma 2.1.** If $B \subset M$ is a remote ball, then for all $\varphi \in C^1(B)$:
\[
\|\varphi - \varphi_B\|_{L^1(B)}^2 \leq B(n, \kappa) r(B) \|d\varphi\|_{L^1(B)},
\]
and for all domain $\Omega \subset B$
\[
\lambda^D_1(\Omega) \geq \frac{C(n, \kappa)}{r(B)^2} \left( \frac{\text{vol} \Omega}{\text{vol} B} \right)^{-\frac{2}{n}}.
\]

A ball centered at $o$ will be called **anchored**.

### 2.3. The Doubling condition.
This condition is insured by the volume comparison (VC) assumption: there is a constant $C$ such that for any $x \in M$:

\[ (VC) \quad \text{vol} B(o, r(x)) \leq C \text{vol} B(x, r(x)/2) \]

We recall that the doubling condition implies that the volume of balls varies slowly with the center of the balls: for any $\gamma \geq 1$ there is a constant $C_\gamma$ such that if $d(x, y) \leq \gamma R$ and $\gamma^{-1}R \leq r \leq \gamma R$ then:
\[
C_\gamma^{-1} \leq \frac{\text{vol} B(x, R)}{\text{vol} B(y, r)} \leq C_\gamma.
\]
In particular if $R \geq r(x)/\gamma$ then
\[ (2) \quad C_\gamma^{-1} \text{vol} B(o, R) \leq \text{vol} B(x, R) \leq C_\gamma \text{vol} B(o, R). \]

### 2.4. Geometry of annulus.

#### 2.4.1. Number of ends.
We remark that the doubling condition implies that $(M, g)$ has a finite of ends: i.e. there is a integer $N$ such that for any $R$, the complement of $B(o, R)$ has at most $N$ unbounded connected components. Indeed if $\mathcal{O} \subset M \setminus B(o, R)$ is an unbounded connected component, there is a point $x_{\mathcal{O}} \in \mathcal{O} \cap \partial B(o, 2R)$, we have in the inclusions $B(x_{\mathcal{O}}, R) \subset \mathcal{O}$ and $B(x_{\mathcal{O}}, R) \subset B(o, 3R)$, hence we get
\[
\sum_{\mathcal{O}} \text{vol} B(x_{\mathcal{O}}, R) \leq \text{vol} B(o, 3R)
\]
However using the doubling condition we get:
\[
\text{vol} B(o, 3R) \leq \text{vol} B(x_{\mathcal{O}}, 5R) \leq \theta^3 \text{vol} B(x_{\mathcal{O}}, R).
\]
Hence $M \setminus B(o, R)$ has at most $\theta^3$ unbounded connected components.
A slight variation of this argument shows that for any $\lambda > 1$, the annulus $A_{\lambda,R} = B(o, \lambda R) \setminus B(o, R)$ has at most $N(\lambda, \theta)$ connected components, however some these connected components will not necessarily intersect an unbounded connected component of $M \setminus B(o, R)$.

2.4.2. The RCE condition. The following condition is a natural extension of the RCA (Relatively Connected Annuli) condition introduced by A. Grigor’yan and L. Saloff-Coste in the setting where the manifold has several ends:

**Definition 2.2.** We say that a complete Riemannian manifold $(M, g)$ with a finite number of ends satisfies the Relatively Connected to an End (RCE) condition if there is a constant $\theta \in (0, 1)$ such that for any point $x$ with $r(x) \geq 1$ there is a continuous path $c: [0, \infty) \to M$ such that:

- the length of $c$ is bounded by $L(c) \leq r(x)/\theta$,
- $c(0) = x$,
- $c([0, 1]) \subset B(o, \theta^{-1}r(x)) \setminus B(o, \theta r(x))$.

for all $\tau \geq 1$:

$$r(c(\tau)) \geq \theta \tau.$$  

The RCE condition says that any point can be connected to an end of $M$ by a path that stays at bounded distance away from the origin. It is easy to see that if $M$ has only one end, the RCE condition is just the RCA condition of A. Grigor’yan and L. Saloff-Coste.

Eventually we remark that the doubling condition implies a reverse doubling condition: there is a constant $\delta > 0$, depending only on the doubling constant $\vartheta$ such that for all $x \in M$ and all $R > r$

$$\delta \left( \frac{R}{r} \right)^{\delta} \leq \frac{\text{vol } B(x, R)}{\text{vol } B(x, r)}.$$  

2.5. Relative Faber-Krahn inequality. The results of A. Grigor’yan and L. Saloff-Coste and of V. Minerbe imply:

**Theorem 2.3.** Let $(M^n, g)$ be a complete Riemannian manifold whose Ricci curvature satisfies:

$$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g.$$  

If we assume moreover that $(M^n, g)$ satisfies the volume comparison condition (VC) and the RCE condition then $(M, g)$ satisfies the relative Faber-Krahn inequality: for some $\mu > 0$ and $C > 0$ and for any ball $B \subset M$ and any domain $\Omega \subset B$

$$\lambda_1^D(\Omega) \geq C \frac{\text{vol } \Omega}{r(B)^2} \left( \frac{\text{vol } \Omega}{\text{vol } B} \right)^{-\frac{2}{n}}.$$
When $M$ has only one end, then $(M, g)$ satisfies the scale $L^1$ Poincaré inequality: for any ball $B$ and any function $f \in C^\infty(2B)$ then

$$\|f - f_B\|_{L^1(B)} \leq Cr(B) \|df\|_{L^1(2B)}.$$ 

The second assertion is one of the main results of [20, thm 5.2]—a priori the article deals with the scale $L^2$ Poincaré inequality but the argument carries over the case of any $L^p$ Poincaré inequality. Strictly speaking, the first assertion cannot be found in the paper of V. Minerbe; however, a quick glimpse on the argumentation shows that the limitation on the exponent $\nu > 1$ in the reverse doubling condition,

$$\forall R > r : \text{vol } B(o, R) \geq \varepsilon \left( \frac{R}{r} \right)^{\nu} \text{vol } B(o, r)$$

is made only to ensure the RCA condition. And under the assumptions of the Theorem [2,3] we get that for any $p \geq n$ there is a constant $C$ such that the weighted Sobolev inequality holds

$$\forall f \in C^\infty_0(M) : \left( \int_M |f(x)|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \leq C \int_M \frac{r(x)^2}{(\text{vol } B(o, r(x)))^{\frac{2}{p}}} |df|^2(x) dx.$$ 

In our case, it is enough to verify that the Relative Faber Krahn inequality holds for anchored balls i.e. balls centered at $o$. But the doubling condition yields a constant $\mu \geq n$ such that

$$\forall R > r : \text{vol } B(o, R) \leq C \left( \frac{R}{r} \right)^{\mu} \text{vol } B(o, r).$$

In particular using the above Sobolev inequality for $p = \mu$, we get for any function $f \in C^\infty_0(B(o, R))$:

$$\left( \int_{B(o, R)} |f(x)|^{\frac{2\mu}{\mu-2}} dx \right)^{\frac{\mu-2}{\mu}} \leq C \int_{B(o, R)} \frac{r(x)^2}{(\text{vol } B(o, r(x)))^{\frac{2}{\mu}}} |df|^2(x) dx$$

$$\leq C \left( \frac{R^2}{(\text{vol } B(o, R))^{\frac{2}{\mu}}} \right)^{\frac{2}{\mu}} \int_{B(o, R)} |df|^2(x) dx .$$

Then with the Hölder inequality, we get for any domain $\Omega \subset B(o, R)$:

$$1 \leq C \frac{R^2}{(\text{vol } B(o, R))^{\frac{2}{\mu}}} (\text{vol } \Omega)^{\frac{2}{\mu}} \lambda^D_1(\Omega).$$

**Remark 2.4.** In fact a remarkable result of V. Minerbe [33, Prop. 2.8] (see also [27, Proposition 4.5] for an earlier result) shows that the RCA condition is insured by an anchored Poincaré inequality and a reverse doubling condition:
Theorem 2.5. Assume that \((M, g)\) is a complete Riemannian manifold that is doubling and such that balls \(B = B(o, R)\) centered at \(o\) satisfy the Poincaré inequalities:
\[
\forall f \in C^\infty(2B) : \|f - f_B\|_{L^p(B)} \leq CR\|df\|_{L^p(2B)}
\]
and that for positive constants \(C\) and \(\nu > p\), we have the Reverse doubling condition:
\[
\forall R > r : \text{vol } B(o, R) \geq C \left( \frac{R}{r} \right)^\nu \text{vol } B(o, r)
\]
then \((M, g)\) satisfies the RCA condition.

3. Estimates on the Riesz Kernel

3.1. Assumptions. In the section, we assume that \((M^n, g)\) is a complete Riemannian manifold with a based point \(o \in M\) satisfying the following conditions:

i) A quadratic decay control on the negative part of the Ricci curvature
\[
\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g.
\]

ii) There are positive constants \(\mu\) and \(C\), such that we have the Relative Faber-Krahn inequality: for any ball \(B \subset M\) and any domain \(\Omega \subset B\)
\[
\lambda^p(\Omega) \geq \frac{C}{r(B)^2} \left( \frac{\text{vol } \Omega}{\text{vol } B} \right)^{-\frac{2}{n}}.
\]

iii) for some positive constants \(c\) and \(\nu > 2\), we have the reverse doubling condition for anchored balls:
\[
\forall R > r : \text{vol } B(o, R) \geq c \left( \frac{R}{r} \right)^\nu \text{vol } B(o, r).
\]

Remark 3.1. The limitation \(\nu > 2\) is not essential, we can handle the case where \(\nu > 1\) but the estimate on the Riesz kernel are more complicated and the conclusion of the main theorem are interesting only when \(\nu > 2\). Indeed the Relative Faber-Krahn hypothesis implies that the Riesz transform is bounded on all \(L^p\), \(p \in (1, 2]\) (10).

3.2. Li and Yau inequality. When \(B \subset M\) is a remote ball, then on \(\frac{3}{2}B\) the Ricci curvature is bounded from below by \(-16\kappa^2 r(B)^{-2}\), so that according to P. Li and S-T. Yau’s Harnack inequality [29]: there is a constant \(c(n, \kappa)\) such that for any positive solution of the heat equation \(u: [0, 2T] \times \frac{3}{2}B \to \mathbb{R}^*_+\), we have on \([T, 2T] \times B\):
\[
\frac{|\nabla u|^2}{u^2} - 2 \frac{\partial u}{u \partial t} \leq c(n, \kappa) \left( \frac{1}{T} + \frac{1}{r^2(B)} \right).
\]
3.3. **Spatial derivative of the heat kernel.** In our setting according to A. Grigor’yan [18, 19], the heat kernel satisfies the Gaussian upper bound: for all \( t > 0 \) and \( x, y \in M \):

\[
h_t(x, y) \leq \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}};
\]

\[
\left| \frac{\partial}{\partial t} h_t(x, y) \right| \leq \frac{C}{\sqrt{t} \text{ vol } B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}}.
\]

Let \( t > 0 \) and \( x, y \in M \) and let \( u(s, z) = h_s(z, y) \). On the parabolic ball \([t/2, t] \times B(x, \sqrt{t})\), the function \( u \) satisfies

\[
u(s, z) + \sqrt{t} \left| \frac{\partial}{\partial s} u(s, z) \right| \leq \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}}.
\]

Then the Li and Yau’s results yields the following estimate on the gradient of the heat kernel:

\[
|\nabla_x h_t(x, y)| \leq \left( \frac{1}{\sqrt{t}} + \frac{1}{r(x)} \right) \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}}
\]

**3.4. Estimate on the Riesz kernel.** Recall that the Riesz transform is the operator

\[
R = d\Delta^{-\frac{1}{2}} : L^2(M) \rightarrow L^2(T^*M),
\]

its Schwarz kernel is smooth on \( M \times M \setminus \text{Diag} \) and is given by:

\[
R(x, y) = \int_0^{+\infty} \nabla_x h_t(x, y) \frac{dt}{\sqrt{\pi t}},
\]

if \( x \neq y \in M \) then \( R(x, y) \in T^*_x M \). We are going to estimate \(|R(x, y)|\) in three different regimes; let \( \kappa \geq 4 \):

i) First regime: \( d(x, y) \geq \frac{1}{\kappa} r(x) \) and \( \frac{1}{\kappa} r(x) \leq r(y) \leq \kappa r(x) \),

ii) The short to long range regime: \( r(x) \geq \kappa r(y) \),

iii) The long to short range regime: \( r(y) \geq \kappa r(x) \).

**3.4.1. First regime.** In this regime, we have \( r(x) \simeq r(y) \simeq d(x, y) \) hence:

\[
|R(x, y)| \leq C \left[ \int_0^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{\text{vol } B(x, \sqrt{t})} dt + \int_{r^2(x)}^{+\infty} \frac{e^{-\frac{r^2(x)}{ct}}}{r(x) \text{ vol } B(x, \sqrt{t}) \sqrt{t}} dt \right],
\]

Using the doubling assumption, the first integral is bounded by

\[
C \frac{r(x)^\mu}{\text{vol } B(x, r(x))} \int_0^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{t^{\frac{\mu}{2} + 1}} dt \leq C \frac{C}{\text{vol } B(x, r(x))} \leq \frac{C}{\text{vol } B(\omega, r(x))}
\]
For the second integral, we use the fact that if $\sqrt{t} \geq r(x)$ then

$$\text{vol } B(x, \sqrt{t}) \geq C \text{ vol } B(o, \sqrt{t}) \geq C \left( \frac{\sqrt{t}}{r(x)} \right)^\nu \text{ vol } B(o, r(x))$$

and because $\nu > 2$, we obtain

$$\int_{r^2(x)}^{+\infty} \frac{e^{-\frac{r^2(x)}{ct}}}{r(x) \text{ vol } B(x, \sqrt{t}) \sqrt{t}} \, dt \leq C \frac{r^{-\nu-1}(x)}{\text{ vol } B(x, r(x))} \int_{r^2(x)}^{+\infty} \frac{e^{-\frac{r^2(x)}{ct}}}{t^{\frac{\nu+1}{2}}} \, dt \leq \frac{C}{\text{ vol } B(o, r(x))}.$$  

### 3.4.2. The short to long range regime.

In this regime, we have $d(x, y) \simeq r(x)$, and the same estimate is valid.

### 3.4.3. The long to short range regime.

In this regime we have $r(y) \simeq d(x, y)$ hence we have

$$|R(x, y)| \leq C \int_{0}^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{\text{ vol } B(x, \sqrt{t}) \sqrt{t}} \, dt \leq C \frac{r(y)^\mu}{\text{ vol } B(o, r(y))} \int_{0}^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{t^{\frac{\nu+1}{2}}} \, dt \leq \frac{C}{\text{ vol } B(o, r(y))}.$$  

Using the same technics, we get

$$\int_{0}^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{\text{ vol } B(x, \sqrt{t}) t} \, dt \leq C \frac{r(y)^\mu}{\text{ vol } B(o, r(y))} \int_{0}^{r^2(x)} \frac{e^{-\frac{r^2(x)}{ct}}}{t^{\frac{\nu+1}{2}}} \, dt \leq \frac{C}{\text{ vol } B(o, r(y))}.$$  

and

$$\int_{r^2(x)}^{+\infty} \frac{e^{-\frac{r^2(x)}{ct}}}{r(x) \text{ vol } B(x, \sqrt{t}) \sqrt{t}} \, dt \leq C \frac{r(y)}{r(x) \text{ vol } B(o, r(y))}.$$  

similarly, using the reverse doubling hypothesis, we get :

$$\int_{r^2(y)}^{+\infty} \frac{e^{-\frac{r^2(y)}{ct}}}{r(x) \text{ vol } B(x, \sqrt{t}) \sqrt{t}} \, dt \leq C \frac{r(y)}{r(x) \text{ vol } B(o, r(y))}.$$  

As a conclusion, we have obtained :

**Lemma 3.2.** There is a positive constant $C$ such that for all $x, y \in M$ satisfying $d(x, y) \geq \frac{1}{k}r(x)$ and $\frac{1}{k}r(x) \leq r(y) \leq kr(y)$ or when $r(x) \geq kr(y)$

$$|R(x, y)| \leq C \frac{r(y)}{r(x) \text{ vol } B(o, r(y))},$$

and when when $r(y) \geq kr(x)$ :

$$|R(x, y)| \leq C \frac{r(y)}{\text{ vol } B(o, r(y))}.$$
3.5. The Green kernel. Because we have assumed that \( \nu > 2 \), the Riemannian manifold \((M, g)\) is non parabolic and the minimal Green kernel is finite and we have:

\[
G(x, y) = \int_0^{+\infty} h_t(x, y) dt.
\]

It is classical that the relative Faber-Krahn inequality yields the following estimate of the Green kernel:

\[
G(x, y) \leq C \int_{d(x,y)}^{+\infty} \frac{t}{\text{vol } B(x,t)} dt.
\]

Using the reverse doubling hypothesis, we get for \( r(y) \geq \kappa r(x) \):

\[
G(x, y) \leq C \frac{r^2(y)}{\text{vol } B(o, r(y))}.
\]

Recall that we say that \((M^n, g)\) satisfies the scale \( \alpha \)-Hölder Elliptic estimate if there is a constant \( C \) if for any ball \( B \subset M \) and any harmonic function \( h \) defined on \( 3B \), we have for all \( x, y \in B \):

\[
|h(x) - h(y)| \leq C \left( \frac{d(x, y)}{r(B)} \right)^\alpha \sup_{z \in 2B} |h(z)|.
\]

Lemma 3.3. Assume moreover that \((M^n, g)\) satisfies the scale \( \alpha \)-Hölder Elliptic estimate, then there is a constant \( C \) such that when \( r(y) \geq \kappa r(x) \) then

\[
|\nabla x G(x, y)| \leq C \left( \frac{r(y)}{r(x)} \right)^{1-\alpha} \frac{r(y)}{\text{vol } B(o, r(y))}.
\]

Proof. Indeed we apply the above \( \alpha \)-Hölder regularity estimate for the function \( f(z) = G(z, y) - G(x, y) \). And we consider the ball \( B = B(o, (1+\delta)r(x)) \) If \( r(y) \geq (3+3\delta)r(x) \), we have

\[
\sup_{z \in 2B} |h(z)| \leq C \frac{r^2(y)}{\text{vol } B(o, r(y))}.
\]

Hence on the ball \( B(x, \delta r(x)) \) we obtain

\[
|h(z)| \leq C \left( \frac{r(x)}{r(y)} \right)^\alpha \frac{r^2(y)}{\text{vol } B(o, r(y))}.
\]

Hence by Cheng and Yau’s gradient estimate \([9]\), we get:

\[
|\nabla h|(x) = |\nabla x G(x, y)| \leq \frac{C}{r(x)} \left( \frac{r(x)}{r(y)} \right)^\alpha \frac{r^2(y)}{\text{vol } B(o, r(y))}.
\]

\( \Box \)
Remark 3.4. In order to obtain this improvement, we need only that the scale \( \alpha \)-Hölder Elliptic estimate hold for anchored balls; however, an argument (see Lemma 5.2) shows that with the doubling condition, the scale \( \alpha \)-Hölder Elliptic estimate hold for all balls provided its hold for any remote and anchored balls. In our setting, the quadratic decay control of the negative part of the Ricci curvature implying a Lipschitz Elliptic estimate for harmonic function on remote balls; hence the scale \( \alpha \)-Hölder Elliptic estimate hold for any ball if and only if it holds for anchored balls.

4. Boundness of the Riesz Transform

4.1. In this section, we’ll prove Theorem A, hence we consider a complete Riemannian manifold \((M, g)\) satisfying the hypothesis of this theorem.

If \( f \in \mathcal{C}_0^\infty(M) \), we decompose \( R(f) \) in three parts:

\[
R(f) = R_d(f) + R_0(f) + R_1(f)
\]

where the Schwarz kernel of \( R_0 \) and \( R_1 \) are smooth and given by the restriction of the Schwarz kernel of \( R \) restricted to the sets

\[
\Omega_0 := \{ (x, y) \in M \times M, d(x, y) \geq \kappa^{-1} r(x) \text{ and } \kappa r(x) \geq r(y) \}
\]

\[
\Omega_1 := \{ (x, y) \in M \times M, \kappa r(x) \leq r(y) \}
\]

That is to say for \( \alpha \in \mathcal{C}_0^\infty(T^* M) \) and \( f \in \mathcal{C}_0^\infty(M) \):

\[
\langle \alpha, R_0(f) \rangle_{L^2} = \int_{\Omega_0} \langle \alpha(x), R(x, y) \rangle_g f(y) dy dx.
\]

and similarly for \( R_1 \). The Schwarz kernel of the Riesz transform has a singularity along the diagonal of \( M \times M \) and \( R_d \) is the restriction the kernel of the Riesz transform to the following neighborhood of the diagonal:

\[
\mathcal{V}(Diag) := \{ (x, y) \in M \times M, d(x, y) \leq \kappa^{-1} r(x) \}
\]

4.2. The short to long range part. This part is now relatively easy to handle:

Proposition 4.1. The operator \( R_0 \) is bounded \( L^\infty(M) \rightarrow L^\infty(T^* M) \) and \( L^1 \rightarrow L^1_w : \) that is to say for any \( f \in L^1 \) we have

\[
\text{vol}\{ x \in M, |R_0(f)(x)| > t \} \leq \frac{C}{t} \| f \|_{L^1}.
\]

In particular by interpolation, \( R_0 : L^p(M) \rightarrow L^p(T^* M) \) is bounded for any \( p \in (1, +\infty) \).
Proof. As a matter of fact, our previous analysis (Lemma 3.2) implies that if \( f \in C_0^\infty(M) \) then

\[
|R_0(f)(x)| \leq \frac{C}{\text{vol} B(o, r(x))} \int_{B(o, kr(x))} |f(y)|dy
\]

hence the boundness \( L^\infty(M) \to L^\infty(T^*M) \) is a direct consequence of the doubling property. Moreover this also implies that

\[
\{ x \in M, \ |R_0(f)(x)| > t \} \subset B(o, \rho)
\]

where \( \rho \) satisfies

\[
\text{vol} B(o, \rho) < C \frac{t \|f\|_{L^1}}{t}
\]

And the boundness \( L^1 \to L^1_w \) follows immediately.

4.3. **The diagonal part.** In this part, we are going to use an idea from [10, section 4] and a result from [1, section 4]:

**Proposition 4.2.** The operator \( R_d \) is bounded on \( L^p \) for every \( p \in (2, +\infty) \).

**Proof.** We build a cover of \( M \) by remote balls. By induction of \( N \in \mathbb{N} \):

- \( B_{0,1} = B(o, 1) \).
- We cover \( B(o, 2^N) \setminus \bigcup_{i<N, j} B_{i,j} \) by a collection of balls \( B_{N,1}, \ldots, B_{N,k_N} \) of radius \( 2^{N-10} \) that are centered on \( B(o, 2^N) \setminus B(o, 2^{N-1}) \) and such that the balls \( \frac{1}{2} B_{N,1}, \ldots, \frac{1}{2} B_{N,k_N} \) are disjoints and included in \( B(o, 2^N) \setminus \bigcup_{i<N, j} B_{i,j} \).

At each stage \( N \), the number of balls is bounded independently:

\[
k_N \leq m(\emptyset).
\]

We obtain in this way a cover

\[
M = \bigcup_{A \in A} B_A
\]

where \( A \subset \mathbb{N} \) by balls \( B_A = B(x, r_A) \). Note that we have by construction: \( 2^{-10} r(x) \leq r_A \leq 2^{-9} r(x) \). Moreover this cover has a finite multiplicity: there is a constant \( p \) such that for any \( x \in M \):

\[
\text{card} \{ \alpha \in A, x \in B_A \} \leq p.
\]

Let \( \chi_{\alpha} \) be a partition of unity subordinate to this covering. If \( \kappa \) is chosen large enough (\( \kappa \geq 2^{10} \)), then we have

\[
|R_d(f)(x)| \leq \sum_{\alpha} |1_{4B_A}(x) R(\chi_{\alpha} f)(x)|.
\]

Let \( R_\alpha = 1_{4B_A} R_{\chi_{\alpha}} \), we decompose

\[
R_\alpha = R_{\alpha,0} + R_{\alpha,1}.
\]
where
\[ R_{\alpha,0}(f)(x) = 1_{4B_\alpha}(x) \int_0^{r_\alpha} \nabla_x e^{-\tau \Delta} (\chi_\alpha f)(x) \frac{d\tau}{\sqrt{\pi \tau}} \]
and
\[ R_{\alpha,1}(f)(x) = 1_{4B_\alpha}(x) \int_{r_\alpha^2}^{\infty} \nabla_x e^{-\tau \Delta} (\chi_\alpha f)(x) \frac{d\tau}{\sqrt{\pi \tau}} \]

Because the covering \( M = \bigcup_{\alpha \in A} B_\alpha \) has finite multiplicity and because \((M, g)\) is doubling, we only need to prove that there is a uniform constant \( C \) such that for all \( \alpha \)

\[ \| R_{\alpha,0} \|_{L^p \to L^p} \leq C, \quad \| R_{\alpha,1} \|_{L^p \to L^p} \leq C. \]

Let’s us explain how the arguments of [1, subsection 3.2] and [1, section 4] together with the estimates on the gradient of the heat kernel : for all \( x, y \in M \) and all \( t \in (0, Ar^2(x)) \)

\[ |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t \text{ vol}(B(x, \sqrt{t})}} e^{-\frac{d^2(x,y)}{ct}}. \]

yield that there is a constant \( C \) independent of \( \alpha \) such that :

\[ \| R_{\alpha,0} \|_{L^p \to L^p} \leq C. \]

We will apply [1, Theorem 2.4]. The setting is the following :

- \((M, g)\) is a complete Riemannian manifold,
- \( T : L^2(M) \to L^2(M) \) is a bounded sublinear operator,
- \( \{ A_r \}_{r>0} \) is a family of bounded operator on \( L^2 \) :
  \[ \sup_{r>0} \| A_r \|_{L^2 \to L^2} < \infty. \]
- \( U \subset \Omega \subset M \) are two open subset ssuch that \( \Omega \) satisfies the relative doubling condition : there is a constant \( \tilde{\theta} \) such that for all ball \( B \subset M \) :
  \[ \text{vol}(2B \cap \Omega) \leq \tilde{\theta} \text{ vol}(B \cap \Omega). \]
- \( S : L^p(U) \to L^p(\Omega) \) is a bounded operator for all \( p > 2 \).

The assumption are
i) For all \( p > 2 \), the sublinear operator \( \mathcal{M}^\# \) defined by
\[ \mathcal{M}^\#(f)(x) = \sup_{B, B \cap \Omega \ni x} \frac{1}{\text{vol}(\Omega \cap B)} \int_{B \cap \Omega} |T(\text{Id} - A_r(B))(f)|^2 \]
is bounded on \( L^p \).

ii) For all \( f \in L^p(U) \) and all ball \( B \subset M \) and \( x, y \in \Omega \cap B \)
\[ |TA_r(B)(f)|^2(y) \leq C \mathcal{M}_\Omega \left( |T(f)|^2 + |S(f)|^2 \right)(x). \]
Where we have note $\mathcal{M}_\Omega$ the maximal operator related to $\Omega$:

$$\mathcal{M}_\Omega(f)(x) = \sup_{B,B \cap \Omega \ni x} \frac{1}{\operatorname{vol}(\Omega \cap B)} \int_{B \cap \Omega} |f|$$

The conclusion is that operator

$$T: L^p(U) \to L^p(\Omega)$$

is bounded and there is an upper bound on operator norm of $T: L^p \to L^p$ that depends only on the constants involved in the setting and the hypothesis.

We will use the following fact:

**Lemma 4.3.** If $(M, g)$ is doubling then

$$\int_{M \setminus B(x,r)} e^{-\frac{d^2(x,y)}{ct}} |f(y)| dy \leq C \operatorname{vol} B(x, \sqrt{t}) e^{-\frac{r^2}{2ct}} M(f)(x)$$

where

$$M(f)(x) = \sup_{B,B \ni x} \frac{1}{\operatorname{vol}(B)} \int_B |f|$$

is the maximal operator associated to $(M, g)$.

**Lemma 4.4.** If $B(x,r) \subset M$ is a remote ball and if $M$ satisfies the (QD) condition, then for all $f \in C^1(B(x,r))$:

$$|f(x) - f_{B(x,r)}| \leq C r \sup_{0<s\leq r} \frac{1}{\operatorname{vol}(B(x,s))} \int_{B(x,s)} |df|.$$

Using the first lemma, the gradient estimate (3) and the argumentation of [1, subsection 3.2] we easily get:

$$\mathcal{M}^\#(f)(x) \leq C \sqrt{\mathcal{M}_\Omega(|f|^2)(x)}.$$

Whereas using also the second lemma, we also get that if $f \in C^1(0,M)$ and if $B$ is a remote ball of radius $r$ and if $x, y \in B$ then

$$\left| \nabla e^{-r^2\Delta} f \right|
\leq C \sup_{0<s\leq r} \frac{1}{\operatorname{vol}(B(x,s))} \int_{B(x,s)} |df| + \frac{C}{r} \mathcal{M}(|f|)(x).$$

We will use [1, Theorem 2.4] with $U = B_\alpha$, $\Omega = 4B_\alpha$, $A_r = e^{-r^2\Delta}$ and

$$T(f)(x) = \left| \int_0^{r^2_\alpha} \left( de^{-\tau\Delta} f \right)(x) \frac{d\tau}{\sqrt{\pi \tau}} \right|$$

Let $v \in L^p(U)$, if we apply the last inequality (4) to

$$f = S(v) = \int_0^{r^2_\alpha} e^{-\tau\Delta} v \frac{d\tau}{\sqrt{\pi \tau}},$$
then we get that for all ball $B$ and all $x, y \in B \cap \Omega$:

$$|TA_{r(B)}v|(y) \leq C \sup_{0 < s \leq r_\alpha} \frac{1}{\text{vol}(B(x, s))} \int_{B(x, s)} |T(v)| + \frac{C}{r_\alpha} \mathcal{M}(S(v))(x).$$

The estimate of the gradient of the heat kernel implies that

$$\int_{B(x, s)} |T(v)| \leq \int_{B(x, s) \cap \Omega} |T(v)| + C \text{vol}(B(x, s) \setminus \Omega) \frac{1}{\text{vol} U} \int_U |v| ;$$

and similarly for $\int_{B(x, s)} |S(v)|$. Hence we get

$$|TA_{r(B)}v|(y) \leq C \mathcal{M}_\Omega(T(v))(x) + \frac{C}{r_\alpha} \mathcal{M}_\Omega(S(v))(x) + c \mathcal{M}_\Omega(v)(x)$$

The fact that $\|S\|_{L^p \to L^p} \leq C r_\alpha$ yields the uniform estimate

$$\|T\|_{L^p(U) \to L^p(\Omega)} \leq C.$$

Concerning the $L^p$ boundness of $T_{\alpha,1}$, we use the fact that for all $y \in B_\alpha$, $x \in 4B_\alpha$ and $t \geq r_\alpha^2$ then

$$|\nabla_x h_t(x, y)| \leq \frac{C}{r_\alpha \text{vol} B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}}.$$

So that the Schwarz kernel of $T_{\alpha,1}$ is bounded by

$$C 1_{4B_\alpha}(x) 1_{B_\alpha}(y) \int_{r_\alpha}^{\infty} \frac{e^{-\frac{d^2(x, y)}{ct}}}{r_\alpha \text{vol} B(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}} \frac{dt}{\sqrt{t}}.$$

And using the slow variation of the volume of balls, we get that for all $x \in 4B_\alpha$ and all $t \geq r_\alpha^2$, $\text{vol} B(x, \sqrt{t}) \simeq \text{vol} B(o, \sqrt{t})$, so that using the reverse doubling condition we obtain:

$$|T_{\alpha,1}(f)(x)| \leq \frac{C 1_{4B_\alpha}(x)}{\text{vol} B_\alpha} \int_{B_\alpha} |f|(y)dy.$$

Hence $T_{\alpha,1}$ is bounded on $L^1$ and on $L^\infty$ with an operator norm bounded independently of $\alpha$.

4.4. The long to short range part. This is the most significant part. In order to study the $L^p$ boundness of this part, we only need to find conditions under which the operator

$$T(f)(x) := \frac{1}{r(x)} \int_{M \setminus B(o, kr(x))} \frac{r(y)}{\text{vol} B(o, r(y))} |f(y)| dy$$

is bounded $L^p \to L^p_w$. 

When \( f \in L^p(M) \), we have:

\[
|T(f)(x)| \leq M_p(x) \|f\|_{L^p}
\]

where

\[
M_p(x) = \frac{1}{r(x)} \left( \int_{M \setminus B(o, kr(x))} \left( \frac{r(y)}{\text{vol} B(o, r(y))} \right)^\frac{p}{p-1} dy \right)^{1-\frac{1}{p}}.
\]

If we introduce the Riemann-Stieljes measure associated to the non decreasing function \( V(r) = \text{vol} B(o, r) \), we get (by integrating by parts)

\[
\int_{M \setminus B(o,R)} \left( \frac{r(y)}{\text{vol} B(o, r(y))} \right)^\frac{p}{p-1} dy = \int_{R} \left( \frac{r}{\text{vol} B(o, r)} \right)^\frac{p}{p-1} dV(r) = (p-1) \frac{R^p}{V(R)^{\frac{1}{p-1}}} + p \int_{R} \frac{r^\frac{1}{p}}{V(r)^{\frac{1}{p-1}}} dr.
\]

provided

- \( \lim_{R \to \infty} \frac{R^\frac{p}{p-1}}{V(R)^{\frac{1}{p-1}}} = 0 \) and
- \( \int_{1}^{\infty} \left( \frac{r}{V(r)} \right)^\frac{1}{p-1} dr < \infty \).

The second condition implies the first one and in our setting, the second condition is equivalent to the \( p \)-hyperbolicity of the manifold \((M, g)\) (\cite{[27]}).

Recall that if \( \mathcal{O} \subset M \) then its \( p \)-capacity is defined by:

\[
\text{cap}_p \mathcal{O} := \inf \left\{ \int_M |d\varphi|^p d\text{vol}_g \mid \varphi \in C^\infty_0(M) \text{ and } \varphi \geq 1 \text{ on } \mathcal{O} \right\},
\]

and that \((M, g)\) is said to be \( p \)-hyperbolic if the \( p \)-capacity of bounded open subset is positive. Using the argument of the proof of the Proposition 4.1, we obtain as before that

\[
T : L^p \to L^p_w
\]

is bounded provided that for some constant \( C \) independent of \( R \), we have:

\[
\int_{R}^{\infty} \left( \frac{r}{V(r)} \right)^\frac{1}{p-1} dr \leq C \left( \frac{r^p}{V(r)} \right)^\frac{1}{p-1}.
\]

Using the reverse doubling condition, we get that this condition is satisfied when \( p < \nu \), this prove the first assertion of the Theorem \[A\].

According to \[11\], in our setting the \( p \) capacity of a anchored ball can be estimate:

\[
\text{cap}_p (B(o, R)) \leq C \left( \int_{R}^{\infty} \frac{r^\frac{p-1}{p}}{V(r)^{\frac{1}{p-1}}} dr \right)^{1-p}.
\]
Hence, when the $p$-capacity of anchored ball satisfy the uniform estimate:
\[
cap_p(B(o, R)) \geq c \frac{\vol B(o, R)}{R^p}
\]
then the condition (5) is satisfied. This finishes the proof of the second assertion of Theorem A.

5. PASSING THE CRITICAL EXPONENT

We will improve our Theorem A and prove Theorem C when $(M, g)$ has only one end and satisfies the scale $\alpha -$ Hölder Elliptic regularity estimate. In this case, the scale $L^1$ Poincaré inequalities hold: for any ball $B \subset M$ and any $f \in C^\infty(2B)$:
\[
\|f - f_B\|_{L^1(B)} \leq Cr(B) \leq \|df\|_{L^1(2B)}.
\]
These Poincaré inequalities and the doubling condition implies ([2]) that for all $q \in (1, 2]$, the reverse Riesz transform is bounded in $L^q$: there is a constant $C$ such that for any $f \in C^\infty_0(M)$:
\[
\|\sqrt{\Delta}f\|_{L^q} \leq C\|df\|_{L^q}.
\]
Hence when $p > 2$, the Riesz transform is bounded on $L^p$ as soon as the Hodge projector $\Pi = d\Delta^{-1}d^*: L^2(T^*M) \to L^2(T^*M)$ has a bounded extension to $L^p$ (cf. [2, lemma 0.1]).

Now the proof of the implication $1) \Rightarrow 2)$ of the [2, theorem 2.1] (see also [37]) shows that the Hodge projector has a bounded extension on $L^p$ provided for some $\tilde{p} > p$, we have a $L^{\tilde{p}}$-reverse Hölder inequality for the gradient of harmonic functions.

**Definition 5.1.** A complete Riemannian manifold $(M, g)$ is said to satisfy the $L^p$-reverse Hölder inequality if for some constants $C > 0$, $\tilde{\alpha} > \alpha > 1$ and for any ball $B \subset M$ and any harmonic function $h$ defined in $\tilde{\alpha}B$, one has the reverse Hölder inequality:
\[
\left( \frac{1}{\vol B} \int_B |dh|^p \right)^{\frac{1}{p}} \leq C \left( \frac{1}{\vol(\tilde{\alpha}B)} \int_{\tilde{\alpha}B} |dh|^2 \right)^{\frac{1}{2}}.
\]

In our case, the quadratic decay control of the negative part of the Ricci curvature implies a Lipschitz/$L^\infty$-reverse Hölder inequality for remote balls. The following lemma shows that in our setting will get the $L^p$-reverse Hölder inequality provide it hold for anchored balls.

**Lemma 5.2.** Let $(M, g)$ be a complete Riemannian manifold that satisfies the doubling condition. The $L^p$-reverse Hölder inequality hold provide it holds for remote and anchored balls.
Proof. Assume that the $L^p$-reverse Hölder inequality holds for remote and anchored balls with parameters $\bar{\alpha} > \alpha > 1$.

Let $B(x, r)$ a ball that is not anchored nor remote, i.e. $x \neq o$ and $r \geq \frac{r(x)}{2}$. Let $\lambda \geq 1$ a real parameter.

i) Assume that $r \geq \lambda r(x)$ and let $B' = B(o, (1+\lambda^{-1})r)$ we get $B(x, r) \subset B'$ and $\alpha B' \subset \beta B(x, r)$ provided $\beta = (1 + \lambda^{-1})\alpha + \lambda^{-1}$. Defined now $\bar{\beta} = (1 + \lambda^{-1})\bar{\alpha} + \lambda^{-1}$ and $\bar{\alpha} = (1 + \lambda^{-1})\bar{\alpha} + \lambda^{-1}$. The six balls $B(x, r), \beta B(x, r), \bar{\beta} B(x, r), B', \bar{\alpha} B', \bar{\alpha} B'$ have a comparable volume then the inclusion $B(x, r) \subset B'$, $\bar{\alpha} B' \subset \bar{\beta} B(x, r)$ and $\alpha B' \subset \beta B(x, r)$ together with $L^p$-reverse Hölder inequality for the ball $B'$ yields that if $h$ is a harmonic function defined on $\bar{\beta} B(x, r)$, then

$$\left( \frac{1}{\text{vol} B(x, r)} \int_{B(x, r)} |dh|^p \right)^\frac{1}{p} \leq C \left( \frac{1}{\text{vol}(\beta B(x, r))} \int_{\beta B(x, r)} |dh|^2 \right)^\frac{1}{2}.$$ 

ii) Assume now that $\frac{r(x)}{2} \leq r \leq \lambda r(x)$ : and let $h$ be a harmonic function defined on $B(x, 4r)$. Then $B(o, (4 - \lambda^{-1})r) \subset B(x, 4r)$. We consider a minimal covering of $B(o, (4 - \lambda^{-1})r) \setminus B(o, 4\delta r)$ by balls of radius $\delta r$:

$$B(o, (4 - \lambda^{-1})r) \setminus B(o, 4\delta r) = \bigcup_{\alpha \in A} B_\alpha$$

All the balls $B_\alpha$ are remote and for some constant $N$ depending only on $\delta$ and on the doubling constant $\delta$

$$\text{card } A \leq N.$$ 

Moreover all the balls $B_\alpha, B(o, 4\delta r)$ have a comparable volume. We choose $\delta$ so that

$$8\delta < 1 \text{ and } (1 + \bar{\alpha})\delta < 1$$

Let $\mathcal{B}_\alpha$ be the collection of the balls $B(o, 4\delta r), B_\alpha \in A$ and let

$$\mathcal{B} = \{ B \in \mathcal{B}_\alpha, B \cap B(x, r) \neq \emptyset \}.$$ 

Then we get that if $B \in \mathcal{B}$ then $\bar{\alpha} B \subset B(x, 2r)$ and also $\bar{\alpha} B \subset B(x, 4r)$, moreover
\[
\int_{B(x,r)} |dh|^p \leq \sum_{B \in \mathcal{B}} \int_{B} |dh|^p \\
\leq C \sum_{B \in \mathcal{B}} (\text{vol } B)^{1-\frac{2}{p}} \left( \int_{\alpha B} |dh|^2 \right)^{\frac{p}{2}} \\
\leq C (\text{card } A + 1)(\text{vol } B(x,r))^{1-\frac{2}{p}} \left( \int_{2B(x,r)} |dh|^2 \right)^{\frac{p}{2}}.
\]

Hence the result.

\[\square\]

In order to prove the Theorem C. We only need to prove that if \((M, g)\) satisfies the hypothesis of this theorem in particular the scale \(\alpha\)-Hölder Elliptic regularity estimate then for all \(p\) such that \((1 - \alpha)p < \nu\), \((M, g)\) satisfies the \(L^p\)-reverse Hölder inequality for anchored balls.

With J.Cheeger and T.Colding [8], we can build a smooth function \(\chi\) such that
- \(\chi = 0\) on \(M \setminus B(o, 3R/4)\)
- \(\chi = 1\) on \(B(o, R/2)\)
- \(R |d\chi| + R^2 |\Delta \chi| \leq C\)

Indeed, according to [8, Theorem 6.33], if \(B\) is a remote ball then there is a smooth function \(\chi_B\) with compact support in \(B\) such that \(\chi_B = 1\) on \(\frac{1}{2}B\) and such that
\[
r(B)|d\chi_B| + r^2(B)|\Delta \chi_B| \leq C(n, \kappa).
\]
Using the doubling hypothesis, we can cover \(\partial B(o, R/2)\) by at most \(N\) balls of radius \(R/8\) and centered in \(\partial B(o, R/2)\):
\[
\partial B(o, R/2) \subset \bigcup_i B_i
\]
Introduce \(\varphi = \sum_i \chi_{2B_i}\) on \(\partial B(o, R/2)\) we have \(\varphi \geq 1\) and \(\varphi\) has compact support in \(B(o, \frac{3}{4}R)\). Moreover there is a constant (independent of \(R\)) such that : \(R |d\varphi| + R^2 |\Delta \varphi| \leq C\). We let \(\chi = u(\varphi)\) where \(u : [0, \infty] \rightarrow [0, 1]\) is a smooth function such that \(u = 1\) on \([1, \infty)\) and \(u = 0\) on \([0, 1/2]\).

We'll use that annuli satisfies a scale Poincaré inequality (according to [20]): There is some constant \(\kappa > 1\) and \(C > 0\) such that if for any \(R > 1\) if we let for \(A_R := B(o, R) \setminus B(o, R/2)\) and \(A_R^* := B(o, \kappa R) \setminus B(o, R/(2\kappa))\) then
\[
\forall f \in C^\infty(A_R^*) : \|f - f_{A_R}\|_{L^2(A_R)} \leq CR\|df\|_{L^2(A_R^*)}
\]
Let $h$ be a harmonic function on $B(o, R/2\kappa)$. Using the Green kernel we have for $x \in B(o, R/(2\kappa))$ and any constant $c$:

$$h(x) - c = \int_{A_R} G(x, y) \Delta(\chi(h - c))(y) dy$$

Using the fact that $h$ is harmonic, we know that

$$\Delta(\chi(h - c)) = (\Delta \chi)(h - c) - 2\langle d \chi, dh \rangle g.$$ 

With Cauchy-Schwarz inequality, we obtain

$$\|dh\|^2(x) \leq C \int_{A_R} |\nabla x G(x, y)|^2 dy \times \int_{A_R} \left[R^{-4}|h(y) - c|^2 + R^{-2}|dh(y)|^2\right] dy.$$ 

We choose

$$c = \frac{1}{\text{vol}(A_R)} \int_{A_R} h(y) dy = h_{A_R}$$

and we use the Poincaré inequality for the annulus and get

$$\int_{A_R} |h(y) - c|^2 dy \leq CR^2 \int_{A_R} |dh(y)|^2 dy.$$ 

Eventually, we obtain

$$|dh|^2(x) \leq CR^{-2} \int_{A_R} |\nabla x G(x, y)|^2 dy \|dh\|^2_{L^2(B(o, R/2\kappa))}.$$ 

Using our estimates of the gradient of the Green kernel we get for $x \in B(o, R/(2\kappa))$ and $y \in B(o, R) \setminus B(o, R/2)$:

$$|\nabla x G(x, y)| \leq C \left(\frac{R}{r(x)}\right)^{1-\alpha} \frac{R}{\text{vol} B(o, R)}.$$ 

And we get that

$$\left(\frac{1}{\text{vol} B(o, R/(2\kappa))} \int_{B(o,R/(2\kappa))} |dh|^p \right)^{\frac{1}{p}} \leq C(R) \left(\frac{1}{\text{vol} B(o, \kappa R)} \int_{B(o,\kappa R)} |dh|^2 \right)^{\frac{1}{2}},$$

where

$$C^p(R) = \frac{1}{\text{vol} B(o, R/(2\kappa))} \int_{B(o,R/(2\kappa))} \left(\frac{R}{r(x)}\right)^{p(1-\alpha)} d\text{vol}(x).$$

Now it is easy to show that the Reverse doubling assumption yields an uniform bound on $C(R)$ as soon as

$$p(1 - \alpha) < \nu.$$
5.1. **On the scale Hölder elliptic regularity estimate.** In our setting, the scale $\alpha$-Hölder Elliptic regularity estimate is equivalent to a monotonicity result for the $L^2$ norm of the gradient of harmonic function.

**Proposition 5.3.** Let $(M, g)$ be a complete Riemannian manifold whose Ricci curvature satisfies

$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g$.

Assume moreover that $(M, g)$ is doubling and satisfies the RCA condition. Let $\alpha \in (0, 1]$. Then $(M, g)$ satisfies the $\alpha$-Hölder Elliptic regularity estimate if and only if there is some constants $\kappa > 1$, $C > 0$ such that for any harmonic function $h$ on $B(o, \kappa R)$ and any $1 \leq r \leq R$

$$\frac{r^{2-2\alpha}}{\text{vol} B(o, r)} \int_{B(o,r)} |dh|^2 \leq C \frac{R^{2-2\alpha}}{\text{vol} B(o, R)} \int_{B(o,R)} |dh|^2.$$ 

**Proof.** If $f$ is a continuous function on a subset $O \subset M$, we define its oscillation by:

$$\text{Osc}_O f = \sup_{x \in O} f(x) - \inf_{y \in O} f(y).$$

Let $R \geq 1$ and $h$ a harmonic function defined on $B(o, 2\kappa R)$. If we let again $A_R := B(o, R) \setminus B(o, R/2)$ and $A^*_R := B(o, \kappa R) \setminus B(o, R/(2\kappa))$. We have the Poincaré inequality

$$\|h - h_{A_R}\|_{L^2(A_R)}^2 \leq CR^2\|dh\|_{L^2(A_R)}^2.$$ 

But according to [20], we have a Harnack inequality on annuli, so that

$$\text{(Osc}_{A_R} h)^2 \leq \frac{C}{\text{vol} A_R} \|h - h_{A_R}\|_{L^2(A_R)}^2 \leq \frac{R^2}{\text{vol} B(o, \kappa R)}\|dh\|_{L^2(B(o,\kappa R))}^2.$$ 

Using the function $\chi$ defined by

$$\chi(x) = \begin{cases} 
1 & \text{on } B(o, R/2) \\
2 - \frac{2r(x)}{R} & \text{on } B(o, R) \setminus B(o, R/2) \\
0 & \text{on } M \setminus B(o, R),
\end{cases}$$

and we get

$$\int_{B(o,R/2)} |dh|^2 \leq \int_{B(o,R)} |d(\chi(h - h_{A_R}))|^2$$

$$= \int_{A_R} (h - h_{A_R})^2 |d\chi|^2 = \frac{4}{R^2} \int_{A_R} (h - h_{A_R})^2.$$
In particular we get
\begin{equation}
\frac{1}{\text{vol } B(o, R/2)} \int_{B(o, R/2)} |dh|^2 \leq \frac{C}{R^2} (\text{Osc}_{A_h} h)^2.
\end{equation}

But in our case, the $\alpha$-Hölder Elliptic regularity estimate is equivalent to a monotonicity inequality for $\rho \mapsto \rho^{-\alpha} \text{Osc}_{A_h} h$. The result is then a consequence of the inequalities (6) and (7). □

6. Non Boundedness of the Riesz Transform

In this section, we give two criteria that implies that the Riesz transform is not bounded on $L^p$.

6.1. Parabolicity and the Riesz transform. Our argument is a slight improvement of earlier results [6]. The starting point is to understanding the $L^p$ closure of the space of differential of smooth compactly support function; the following lemma is a $L^p$ adaptation of an idea of [5].

**Lemma 6.1.** Let $(M^n, g)$ be a complete Riemannian manifold and let $f \in W^{1,p}_{\text{loc}}$ such that $df \in L^p$. Assume moreover that $M(r) = \int_{B(o, r)} |f|^p$ satisfies
\[
\int_1^\infty \left( \frac{r}{M(r)} \right)^{\frac{1}{p-1}} \, dr = \infty,
\]
then there is a sequence $(\chi_\ell)$ of smooth function with compact support such that
\[
\lim_{\ell \to \infty} \|df - d(\chi_\ell f)\|_{L^p} = 0
\]
in particular :
\[
df \in dC_{0}^\infty(M)^{L^p}.
\]

**Proof.** Let $r < R$, we define a function $\chi_{r, R}$ by letting $\chi_{r, R} = 1$ on $B(o, r)$, $\chi_{r, R} = 0$ outside $B(o, R)$ and if $x \in B(o, R) \setminus B(o, r)$ then
\[
\chi_{r, R}(x) = \xi_{r, R}(r(x)) = \varepsilon(r, R) \int_{r(x)}^{R} \left( \frac{s}{M(s)} \right)^{\frac{1}{p-1}} \, ds,
\]
where
\[
\varepsilon(r, R) = \left( \int_{r}^{R} \left( \frac{s}{M(s)} \right)^{\frac{1}{p-1}} \, ds \right)^{-1}.
\]
Let $f$ be a function satisfying the hypothesis of the lemma, then we get :
\[
\|df - d(\chi_{r, R} f)\|_{L^p}^p \leq C \left( \int_{M \setminus B(o, r)} |df|^p + \int_{B(o, R) \setminus B(o, r)} |f|^p |d\chi_{r, R}|^p \right).
\]
But if we introduce the Riemann-Stieljes measure associated to the non decreasing function \( s \mapsto M(s) \) we have
\[
\int_{B(o,R) \setminus B(o,r)} |f|^p d\chi \, R \, |p| = \int_{r}^{R} |\xi_{r,R}(s)|^p dM(s)
\]
\[
= \varepsilon(r, R)^p \int_{r}^{R} s^{\frac{p}{p-1}} \frac{1}{M(s)^{\frac{1}{p-1}}} dM(s)
\]
\[
= \varepsilon(r, R)^p \left( r^{\frac{p}{p-1}} \frac{1}{M(r)^{\frac{1}{p-1}}} + \int_{r}^{R} \frac{p \, s^{\frac{1}{p-1}}}{M(s)^{\frac{1}{p-1}}} ds \right)
\]
\[
= (p - 1) \varepsilon(r, R)^p R^{\frac{p}{p-1}} \frac{1}{M(r)^{\frac{1}{p-1}}} + p \varepsilon(r, R)^{p-1}.
\]

With the hypothesis \( \int_{1}^{\infty} \left( \frac{r}{M(r)} \right)^{\frac{1}{p-1}} dr = \infty \) it is possible to find two increasing and divergent sequences of \( r_\ell < R_\ell \) such that
\[
\lim_{\ell \to \infty} \varepsilon(r_\ell, R_\ell) = 0
\]
and
\[
\lim_{\ell \to \infty} \varepsilon(r_\ell, R_\ell)^p R_\ell^{\frac{p}{p-1}} \frac{1}{M(r_\ell)^{\frac{1}{p-1}}} = 0.
\]
Hence the results.

Assume now that \((M, g)\) is non parabolic and that the Riesz transform is bounded on \( L^p \) and on \( L^{\frac{p}{p-1}} \), then the Hodge projector
\[
\Pi = d\Delta^{-1} d^* = (d\Delta^{-\frac{1}{2}})(\Delta^{-\frac{1}{2}} d^*) = (d\Delta^{-\frac{1}{2}})(d\Delta^{-\frac{1}{2}})^*
\]
extends from \( L^2 \cap L^p \) to a bounded operator on \( L^p(T^*M) \). Hence \( \Pi(\mathcal{C}_0^\infty(T^*M)) \) is dense in \( \Pi(L^p(T^*M)) \). Also by definition \( \Pi \) is the identity on \( d\mathcal{C}_0^\infty(M) \) hence
\[
\overline{d\mathcal{C}_0^\infty(M)}^{L^p} \subset \Pi(L^p(T^*M)).
\]
If \( \alpha \in \mathcal{C}_0^\infty(T^*M) \) then \( \Pi(\alpha) = df \) where
\[
f(x) = \int_{M} G(x, y)d^*\alpha(y)dy
\]
and \( G(x, y) \) is the Green kernel of \((M, g)\). Because \( d^*\alpha \) has compact support, the growth of \( r \mapsto \int_{B(o,r)} f^p \) is controlled by
\[
\mathcal{G}_p(r) := \int_{B(o,r)} G(x, o)^p dx.
\]
Hence a direct consequence of the last lemma is the following proposition:
Proposition 6.2. Assume that \((M, g)\) is a complete Riemannian manifold such that

- \((M, g)\) is non parabolic and its Green kernel satisfies:
  \[
  \int_1^{\infty} \left( \frac{r}{G_p(r)} \right)^{\frac{1}{p-1}} \, dr = \infty.
  \]
  
- The Riesz transform is bounded on \(L^p\) and on \(L^{\frac{p}{p-1}}\)

then

\[
\Pi(L^p(T^*M)) = dC_0^\infty(M)^{L^p}.
\]

Recall that \((M, g)\) is said to be \(p\)-parabolic if we can find a sequence of smooth function with compact support \((\chi_k)\) such that

\[0 \leq \chi_k \leq 1 \quad \lim_{k \to \infty} \|d\chi_k\|_{L^p} = 0 \quad \text{and} \quad \chi_k \to 1 \text{ uniformly on compact set.}\]

A consequence of the definition is that on a \(p\)-parabolic manifold, any bounded function with \(L^p\) gradient has its gradient in the \(L^p\)-closure of \(dC_0^\infty(M)\). Now if \((M, g)\) is non-parabolic (i.e. non 2-parabolic or 2-hyperbolic) then its Green kernel \(G(x, y)\) is bounded outside its pole: that is to say if \(r > 0\) is small enough then \(x \in M \setminus B(y, r) \mapsto G(x, y)\) is positive and bounded by \(\max_{x \in \partial B(y, r)} G(x, y)\). In particular we get

Proposition 6.3. Assume that \((M, g)\) is a complete Riemannian manifold such that

- \((M, g)\) is non 2-parabolic and \(p\)-parabolic for some \(p > 2\).
- The Riesz transform is bounded on \(L^p\) and on \(L^{\frac{p}{p-1}}\)

then

\[
\Pi(L^p(T^*M)) = dC_0^\infty(M)^{L^p}.
\]

This two results should be compared with the one of [6, lemma 7.1]. Then we can show the following adaptation of [6, corollary 7.5]:

Theorem 6.4. Under the hypothesis of the Proposition \[6,3\] (\(M, g\)) has only one end.

Proof. If \(M\) has at most two ends, we can find a compact set \(K \subset M\) such that \(M \setminus K = \mathcal{O}_- \cup \mathcal{O}_+\) with \(\mathcal{O}_- \cap \mathcal{O}_+ = \emptyset\) and such that both \(\mathcal{O}_\pm\) are unbounded. And we can build a smooth function \(\varphi\) such that \(\varphi = \pm 1\) on \(\mathcal{O}_\pm\). Then \(\Delta \varphi \in C_0^\infty(M)\), \(d\varphi \in C_0^\infty(T^*M)\), We can defined \(h: M \to [-1, 1]\) by:

\[
h(x) = \varphi(x) - \int_M G(x, y) \Delta \varphi(y) \, dy,
\]

\(h\) is a harmonic function and by construction

\[
dh = d\varphi - \Pi(d\varphi).
\]
Then $h$ being bounded and $(M, g)$ is assumed to be $p$-parabolic we get that:

$$\text{d}h \in \text{d}C_\infty^0(M)^{L^p}$$

So that $\Pi(\text{d}h) = \text{d}h$; but on $L^2$, we have by construction $\Pi(\text{d}h) = 0$, hence the contradiction. □

**Corollary 6.5.** On a non-parabolic and $p > 2$-parabolic manifold with at least two ends, the Riesz transform can not be bounded simultaneously on $L^p$ and on $L^{\frac{p}{p-1}}$.

6.2. **Sublinear harmonic function and the Riesz transform.** Using the same kind of idea, we are going to give another criterium for the unbound-edness of the Riesz transform on $L^p$ that is based on the existence of sub-linear harmonic function.

**Proposition 6.6.** Let $(M, g)$ be a complete Riemannian manifold whose Ricci curvature satisfies

$$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g.$$ 

Assume moreover that $(M, g)$ is doubling and satisfies the RCE condition and that the volume of anchored balls satisfy for some $\mu \geq \delta > 2$ and positive constant $c$:

$$c \left( \frac{R}{r} \right)^\delta \text{vol } B(o, r) \leq \text{vol } B(o, R) \leq c R^\mu.$$ 

If $(M, g)$ carries a non constant harmonic function $h$ with sublinear growth: for some $\alpha \in (0, 1)$

$$h(x) = O(r^\alpha(x)),$$

then the Riesz transform is not bounded on $L^p$ for any $p > 2$ and $p \geq \frac{\mu}{1 - \alpha}$.

A quick inspection of the proof below shows that we only need that $\delta > \frac{\mu}{p-1}$.

**Proof.** By contraposition, we assume that the Riesz transform is bounded on $L^p$ with $p(1 - \alpha) \geq \mu$ and consider a harmonic function $h$ such that

$$h(x) = O(r^\alpha(x))$$

and we will show that necessary $h$ is constant. We remark that our conditions implies a Relative Faber-Krahn inequality and then by [10] the Riesz transform is bounded on $L^{\frac{\mu}{1 - \alpha}}$. The quadratic decay of the negative part of the Ricci curvature together with the Yau’s gradient estimates implies that

$$\text{d}h(x) = O \left( r^{\alpha-1}(x) \right),$$
hence the condition
\[ \text{vol } B(o, r) = O(r^\nu) \]
implies that \( dh \in L^p \) and we also have \( M(r) = \int_{B(o,r)} h^p \leq Cr^{\nu+p\alpha} \leq C r^p \), so that
\[ \left( \int_1^\infty \frac{r}{M(r)} \right)^{\frac{1}{p-1}} dr = \infty, \]
and with the Lemma [6.1], we get that:
\[ dh \in dC_0^\infty(M)^{L^p}. \]
The volume growth assumptions implies that \((M, g)\) is \(p\)-parabolic then the Proposition [6.2] implies that
\[ \Pi(dh) = dh. \]
Let \( \alpha \in C_0^\infty(T^*M) \) then the Hodge projector being bounded on \( L^p \) and \( L^{\frac{p}{2-1}} \), we get
\[ \langle dh, \alpha \rangle = \langle \Pi(dh), \alpha \rangle = \langle dh, \Pi(\alpha) \rangle \]
But
\[ \Pi(\alpha) = df \]
where \( f \) is given by
\[ f(x) = \int_M G(x, y) d^* \alpha(y) dy. \]
There is a \( R > 0 \) so that \( f \) is harmonic outside \( B(o, R) \) and using the Green kernel estimate, we know that \( f \) tends to zero at infinity. We are going to estimate the decay of \( f \). Let \( q = p/(p-1) \). Because \( \Pi \) is bounded on \( L^q \), we have \( df \in L^q \).
Let \( x \in M \setminus B(o, 2R) \), \( f \) is harmonic on the remote ball \( B(x, r(x)/2) \) and \( u = |df| \) satisfies the Elliptic inequality:
\[ \Delta u \leq \frac{C}{r(x)^2} u \text{ on } B(x, r(x)/2). \]
The lower bound on the Ricci curvature implies that
\[ u^q(x) \leq \frac{C}{\text{vol } B(x, r(x)/2)} \int_{B(x, r(x)/2)} u^q = \frac{o(1)}{\text{vol } B(o, r(x))}. \]

hence:
\[ |df(x)| \leq \frac{o(1)}{(\text{vol } B(o, r(x)))^{\frac{1}{q}}}. \]
Using the RCE condition, we can integrate this inequality along a path starting from $x$ and escaping to infinity and get
\[
|f(x)| \leq \frac{o(r(x))}{(\text{vol } B(o, r(x)))^{\frac{1}{q}}} + o(1) \int_{r(x)}^{\infty} \frac{1}{(\text{vol } B(o, s))^{\frac{1}{q}}} ds.
\]
Using the reverse doubling condition for the anchored ball we get that
\[
|f(x)| \leq C \frac{o(r(x))}{(\text{vol } B(o, r(x)))^{\frac{1}{q}}}
\]
Now if $\chi_k$ is the function defined by
\[
\chi_k(x) = \begin{cases} 
1 & \text{on } B(o, k) \\
2 - \frac{r(x)}{k} & \text{on } B(o, 2k) \setminus B(o, k) \\
0 & \text{on } M \setminus B(o, 2k)
\end{cases}
\]
Because $dh \in L^p$ and $df \in L^q$, we have
\[
\langle dh, \alpha \rangle = \lim_{k \to \infty} \langle dh, \chi_k df \rangle = \lim_{k \to \infty} \langle dh, d(\chi_k f) \rangle - \langle dh, d\chi_k \rangle
\]
However $h$ is harmonic hence
\[
\langle dh, d(\chi_k f) \rangle = 0,
\]
moreover
\[
|\langle dfh, d\chi_k \rangle| \leq C \frac{\text{vol } B(o, 2k) o(k)}{(\text{vol } B(o, k))^{\frac{1}{q}}} k^{(\alpha - 1)} k^{-1}
\]
\[
\leq o(1)
\]
Our hypothesis implies that this quantity tends to zero when $k$ tends to infinity. Eventually we obtain that for all $\alpha \in C_0^\infty (T^*M) : \langle dh, \alpha \rangle = 0$ hence $dh = 0$. \qed

7. Examples

In this section, we describe three series of examples where our results apply.

7.1. Manifolds with conical ends. These manifolds $(M^n, g)$ are isometric outside a compact set to a truncated cone:
\[
C_R(\Sigma) := \{(r, \infty) \times \Sigma, (dr)^2 + r^2 h\}
\]
where $(\Sigma, h)$ is compact Riemannian manifold. Its easy to check that the hypothesis of Theorem [A] are satisfied for $\nu = n$, hence when $\Sigma$ has several connected component then the Riesz transform is bounded on $L^p$ for all $p \in (1, n)$ but unbounded on $L^p$ when $p \geq \max \{2, n\}$. Where as when
Σ is connected then an application of the Theorem gives that the Riesz transform is bounded on \( L^{p>1} \) if and only if:

\[(1 - \alpha)p < n\]

where

- the first non zero eigenvalue of the Laplacian is \( \alpha(n - 2 - \alpha) \) when the first non zero eigenvalue of the Laplacian on \((\Sigma, h)\) is smaller than \( n - 1 \).
- \( \alpha = 1 \) if the first non zero eigenvalue of the Laplacian on \((\Sigma, h)\) is greater than \( n - 1 \).

In this case, the \( \alpha \)-Hölder elliptic estimate is obtained with the Proposition and a separation of variables argument to analyze harmonic functions on the cone.

Hence we recover earlier results of H-Q. Li, C. Guillarmou and A. Hassell and of A. Hassell and P. Lin (\cite{28, 21, 22, 25}).

7.2. Models manifolds. We consider a Riemannian manifold \((M^n, g)\) that is isometric outside a compact set to the warped product:

\[C_R(\Sigma) := ((R, \infty) \times \Sigma, (dr)^2 + f^2(r)h)\]

where

\[f(r) = e^{u(\ln(r))}\]

for some function \( u : [1, \infty) \to \mathbb{R} \) with bounded second derivative and

\[\alpha \leq u' \leq 1.\]

When \((\Sigma, h)\) is a compact manifold with non negative Ricci curvature then \((M, g)\) satisfies the quadratic decay control on the negative part of the Ricci curvature, the condition (VC) and (RCE). Moreover anchored balls satisfies the \( \nu \)-Reverse doubling condition for the exponent \( \nu = (n - 1)\bar{\alpha} + 1 \). But also when \( u \) satisfies the asymptotic condition:

\[u'(\theta) \leq \bar{\alpha} - \psi(\theta) \text{ for some } \psi \in L^1\]

then the exponent in the Reverse doubling is improve to \( \nu = (n - 1)\bar{\alpha} + 1 \). As a consequence we get:

**Proposition 7.1.** Assume that the function \( u \) satisfies the condition (8), then the Riesz transform is bounded on \( L^p \) when \( p < (n - 1)\bar{\alpha} + 1 \) and is not bounded on \( L^p \) if \( \Sigma \) is non connected and if

\[
\int_1^\infty \frac{1}{f^{\frac{n+1}{p-1}}(r)} dr = \infty.
\]
7.3. **Examples with infinite topological type.** Our last example is inspired by a construction of J.Lott and Z.Shen [30]. Assume that $(P^n, g)$ is a compact Riemannian manifold with boundary $\partial P = \Sigma_- \cup \Sigma_-$ and that $\Sigma_{\pm}$ have collared neighborhood $U_{\pm}$ that are diffeomorphic $f: U_- \to U_+$ with 

$$f^*g = 4g.$$ 

For $\ell \in \mathbb{N}$, we define $2^\ell P$ to be the rescaled Riemannian manifold

$$(P, 4^\ell g).$$

Using the map $f$ we can glue all the $2^\ell P$ and get a Riemannian manifold $(X, g)$ with boundary $\partial X = \Sigma_-$. If there is a compact manifold with boundary diffeomorphic to $\Sigma_-$ then we can form $X_0 = X \cup K$. We can also form $X_1 = X \#_{\Sigma_-} (-X)$ the double of $X$. These manifolds have quadratic curvature decay and Euclidean growth. Hence

**Proposition 7.2.** 

- For $X_0$ and $X_1$, the Riesz transform is bounded on $L^p$ for every $p \in (1, n)$.
- If $p \geq n > 2$, on $X_1$, the Riesz transform is not bounded on $L^p$.
- If $P$ is connected, then there is some $\varepsilon \in (0, 1]$ such that on $X_0$, the Riesz transform is $L^p$-bounded for every $p \in (1, \frac{n}{1-\varepsilon})$.

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