Dynamical structures specific to strong gravitational field: Quantum formalism

Andrzej Góźdź, 1,* Włodzimierz Piechocki, 2,† and Grzegorz Plewa 2,‡

1 Institute of Physics, Maria Curie-Skłodowska University,  
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland  
2 Department of Fundamental Research,  
National Centre for Nuclear Research,  
Hoża 69, 00-681 Warszawa, Poland  
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Abstract

The dynamics of the general Bianchi IX spacetime, near the gravitational singularity, underlies the Belinskii, Khalatnikov and Lifshitz scenario. Asymptotically, near the singularity, the oscillations of the directional scale factors (defining the spacetime metric) are freezed so the evolution of the Bianchi IX model is devoid of chaotic features. However, it includes special structures, that we call the wiggles, which change their properties in the asymptotic regime. We propose the formalism, based on an affine quantization scheme, to examine the fate of the wiggles at the quantum level. We present the way of comparing the classical wiggles with their quantum counterparts. We expect that our work may contribute towards understanding some quantum aspects of the BKL scenario.

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*andrzej.gozdz@umcs.lublin.pl  
†wladzimierz.piechocki@ncbj.gov.pl  
‡grzegorz.plewa@ncbj.gov.pl
I. INTRODUCTION

The Belinskii, Khalatnikov and Lifshitz (BKL) scenario is thought to describe a generic solution to the Einstein equations near either spacelike \([1, 2]\) or timelike \([3, 4]\) gravitational singularities. This scenario was obtained by the generalization of the dynamics of the general Bianchi IX model.
It is well known that the dynamics of the vacuum Bianchi IX includes the oscillations of the dynamical scale factors \([1]\), which underly the chaotic dynamics of that evolution (see, e.g., \([5]\)). If one makes the analysis of the Bianchi IX dynamics corresponding to the general case, when the 3-metric of space cannot be diagonalized globally once for all moments of time\(^1\), one can find that asymptotically near the singularity the oscillations are freezed so the evolution is devoid of the chaotic behaviour \([6–8]\). However, there are other interesting structures, which we call the wiggles, specific to this dynamics.

The existence of physical structures in the evolution of spacetime is of fundamental importance since if they exist, they can be treated as seeds of real structures in the Universe like, e.g., inhomogeneities of the CMB spectrum. The latter is believed to be the origin of large scale structures, like galaxies and clusters of galaxies, visible presently on the sky. On the other hand, if quantization does not erase these structures, they can be used to get insight into quantum fluctuations underlying quantum gravity expected to play an important role near the cosmological and astrophysical singularities. This paper is devoted to the construction of the formalism to be used in the examination of the dynamical gravitational structures at the quantum level.

Our paper is organized as follows: In Sec. II we recall (to make our paper self-consistent) the Hamiltonian formulation of our gravitational system. Section III is devoted to the construction of the quantum formalism. It is based on using the affine coherent states ascribed to the physical phase space and the resolution of the unity in the carrier space of the unitary representation of the affine group. The quantum dynamics is defined in Sec. IV. Section V concerns the probability measure defined on the physical phase space. An outline of how the formalism might be applied is presented in Sec. VI.

II. CLASSICAL DYNAMICS

The asymptotic form (near the singularity) of the dynamical equations of the general (nondiagonal) Bianchi IX model is the following \([6, 7]\)

\[
\frac{d^2 \ln a}{dT^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dT^2} = a^2 - b/a + c/b, \quad \frac{d^2 \ln c}{dT^2} = a^2 - c/b, \tag{1}
\]

where \(a, b, c\) are functions of an evolution parameter \(T\), and are interpreted as the directional scale factors of considered anisotropic universe. The solution to (1) should satisfy the dynamical constraint

\[
\frac{d \ln a}{dT} \frac{d \ln b}{dT} + \frac{d \ln a}{dT} \frac{d \ln c}{dT} + \frac{d \ln b}{dT} \frac{d \ln c}{dT} = a^2 + b/a + c/b. \tag{2}
\]

\(^1\) In the vacuum case the 3-metric can be globally diagonalized.
Eqs. (1) and (2) define a nonlinear coupled system of differential equations. We recommend Sec. 6 of Ref. [7] for the derivation of these equations from the exact dynamics.

Recently, we have derived the reduced phase space formulation of the classical dynamics corresponding to the above dynamics [9]. The two form \( \Omega \) defining the Hamiltonian formulation is given by

\[
\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + dt \wedge dH,
\]

(3)

where the variables \((q_1, q_2, p_1, p_2)\) parameterise the physical phase space, \( H = H(q_1, q_2, p_1, p_2, t) \) is the Hamiltonian generating the classical dynamics, and \( t \) is an evolution parameter specific to the Hamiltonian formulation. There is no explicit relation between the evolution parameters \( T \) and \( t \). The variables \( q_1 \) and \( q_2 \) are related to \( a \) and \( b \) via \( q_1 = \ln a \) and \( q_2 = \ln b \), and the variable \( c \) is expressed in terms of other variables to implement the constraint (2) into the Hamiltonian scheme.

The Hamiltonian reads [9]

\[
H(q_1, q_2, p_1, p_2; t) = q_2 + \ln \left[-e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(p_1^2 + p_2^2 + t^2) + \frac{1}{2}(p_1p_2 + p_1t + p_2t)\right],
\]

(4)

and Hamilton’s equations are:

\[
\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1} = \frac{p_2 - p_1 + t}{2F},
\]

(5)

\[
\frac{dq_2}{dt} = \frac{\partial H}{\partial p_2} = \frac{p_1 - p_2 + t}{2F},
\]

(6)

\[
\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1} = 2e^{2q_1} - e^{q_2-q_1}
\]

(7)

\[
\frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2} = -1 + \frac{e^{q_2-q_1}}{F},
\]

(8)

where

\[
F(q_1, q_2, p_1, p_2, t) := -e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(p_1^2 + p_2^2 + t^2) + \frac{1}{2}(p_1p_2 + p_1t + p_2t) > 0. \quad (9)
\]

The constraint in the r.h.s. of (9) is not of dynamical origin; it results from the restriction of the dynamics to the physical phase space parameterized by real (not complex) variables. Equations (5)-(8) define a coupled system of nonlinear ordinary differential equations. The solution defines the physical phase space of our gravitational system. The regularized version of the Hamiltonian (4) (to be used in calculations) is presented in App. A.
Applying the simple algebraic identity \((A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2BC + 2AC\) to Eq. (9) gives:

\[
F(q_1, q_2, p_1, p_2, t) = -e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(p_1 - p_2 + t)^2 + p_1 t, \tag{10}
\]

\[
F(q_1, q_2, p_1, p_2, t) = -e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(-p_1 + p_2 + t)^2 + p_2 t, \tag{11}
\]

\[
F(q_1, q_2, p_1, p_2, t) = -e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(p_1 + p_2 - t)^2 + p_1 p_2. \tag{12}
\]

Combining (9) and (10) we get

\[p_1 > \frac{1}{t} \left[ e^{2q_1} + e^{q_2-q_1} + \frac{1}{4}(p_1 - p_2 + t)^2 \right], \tag{13}\]

whereas (9) and (11) give

\[p_2 > \frac{1}{t} \left[ e^{2q_1} + e^{q_2-q_1} + \frac{1}{4}(-p_1 + p_2 + t)^2 \right]. \tag{14}\]

Making use of (9) and (12) leads to

\[p_1 p_2 > e^{2q_1} + e^{q_2-q_1} + \frac{1}{4}(p_1 + p_2 - t)^2. \tag{15}\]

It is clear that the signs of both r.h.s. of Eq. (13) and Eq. (14) depend only on the sign of \(t\). It results from (13)–(15) that the signs of both \(p_1\) and \(p_2\) are the same for any value of \(t \neq 0\). The two sectors, \(t > 0\) and \(t < 0\), are dynamically independent. In what follows we consider the case \(t > 0\), which implies \(p_1 > 0\) and \(p_2 > 0\).

The range of the variables \(q_1\) and \(q_2\) results from the physical interpretation ascribed to them \([9]\). Since \(0 < a < +\infty\) and \(0 < b < +\infty\), we have \((q_1, q_2) \in \mathbb{R}^2\). Thus, the physical phase space \(\Pi\) consists of the two half planes:

\[\Pi = \Pi_1 \times \Pi_2 := \{(q_1, p_1) \in \mathbb{R} \times \mathbb{R}_+\} \times \{(q_2, p_2) \in \mathbb{R} \times \mathbb{R}_+\}, \tag{16}\]

where \(\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}\).

### III. QUANTIZATION

Suppose we have reduced phase space Hamiltonian formulation of classical dynamics of a gravitational system. It means dynamical constraints have been resolved and the Hamiltonian is a generator of the dynamics. By quantization we mean
(roughly speaking) a mapping of such Hamiltonian formulation into a quantum system described in terms of quantum observables (including Hamiltonian) represented by a set of self-adjoint operators acting in a Hilbert space. The construction of the Hilbert space may make use some mathematical properties of phase space like, e.g., symplectic structure, geometry or topology. The quantum Hamiltonian is used to define the Schrödinger equation. In what follows we make specific the above procedure by using the affine coherent states approach. Since the variables parameterizing our phase space \( \Pi \) are dimensionless, it is convenient to assume \( \hbar = 1 \) (unless otherwise specified) at the quantum level.

A. Affine coherent states

The Hilbert space \( \mathcal{H} \) of the entire system consists of the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) corresponding to the phase spaces \( \Pi_1 \) and \( \Pi_2 \), respectively. In the sequel the construction of \( \mathcal{H}_1 \) is followed by merging of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \).

As both half-planes \( \Pi_1 \) and \( \Pi_2 \) have the same mathematical structure, the corresponding Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are identical so we first consider only one of them. In what follows we present the formalism for \( \Pi_1 \) and \( \mathcal{H}_1 \) to be extended later to the entire system.

1. Affine coherent states for half-plane

The phase space \( \Pi_1 \) may be identified with the affine group \( G_1 \equiv \text{Aff}(\mathbb{R}) \) by defining the multiplication law as follows

\[
(q', p') \cdot (q, p) = (p'q + q'p', p'),
\]

with the unity \((0, 1)\) and the inverse

\[
(q', p')^{-1} = \left( -\frac{q'}{p'}, \frac{1}{p'} \right).
\]

The affine group has two, nontrivial, inequivalent irreducible unitary representations \([10]\) and \([11, 12]\). Both are realized in the Hilbert space \( \mathcal{H}_1 = L^2(\mathbb{R}_+, d\nu(x)) \), where \( d\nu(x) = dx/x \) is the invariant measure\(^2\) on the multiplicative group \((\mathbb{R}_+, \cdot)\). In what

\(^2\) The general notion of invariant measure \( dm(x) \) on the set \( X \) in respect to the transformation \( h : X \rightarrow X \) can be approximately defined as follows: for every function \( f : X \rightarrow \mathbb{C} \) the integral defined by this measure fulfills the invariance condition:

\[
\int_X dm(x)f(h(x)) = \int_X dm(x)f(x).
\]

This property is often written as: \( dm(h(x)) = dm(x) \).
follows we choose a different representation defined by the following action:

$$U(q, p)\psi(x) = e^{iqx}\psi(px), \quad (19)$$

where $|\psi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$. Eq. (19) defines the representation as we have

$$U(q', p')[U(q, p)\psi(x)] = U(q', p')\psi(px) = e^{i(p'q + q'x)}\psi(p'px),$$

and on the other hand

$$[U(q', p')U(q, p)]\psi(x) = U(p'q + q', p'p)\psi(x) = e^{i(p'q + q')x}\psi(p'px).$$

This action is unitary in respect to the scalar product in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\int_0^\infty d\nu(x)[U(q, p)f_2(x)]^*U(q, p)f_1(x) = \int_0^\infty d\nu(x)[e^{iqx}f_2(px)]^*[e^{iqx}f_1(px)]$$

$$= \int_0^\infty d\nu(x)f_2(px)^*f_1(px) = \int_0^\infty d\nu(x)f_2(x)^*f_1(x). \quad (20)$$

The last equality results from the invariance of the measure $d\nu(px) = d\nu(x)$.

The affine group is not the unimodular group. The left and right invariant measures are given by

$$d\mu_L(q, p) = dq \frac{dp}{p^2} \quad \text{and} \quad d\mu_R(q, p) = dq \frac{dp}{p}, \quad (21)$$

respectively.

The left and right shifts of any group $G$ are defined differently by different authors. Here we adopt the definition from [14]:

$$L^L_h f(g) = f(h^{-1}g) \quad \text{and} \quad L^R_h f(g) = f(gh^{-1}) \quad (22)$$

for a function $f : G \to \mathbb{C}$ and all $g \in G$.

For simplicity of notation, let us define integrals over the affine group $G_1 = \text{Aff}(\mathbb{R})$ as:

$$\int_{G_1} d\mu_L(q, p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \int_0^{+\infty} \frac{dp}{p^2} \quad \text{and} \quad \int_{G_1} d\mu_R(q, p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \int_0^{+\infty} \frac{dp}{p}. \quad (23)$$

\footnote{We use Dirac’s notation whenever we wish to deal with abstract vector, instead of functional representation of the vector.}
In many formulae it is useful to use shorter notation for points in the phase space \( \xi \equiv (q, p) \) and identify them with elements of the affine group. In this case the product (17) is denoted as \( \xi' \cdot \xi \). Depending on needs we will use both notations.

Fixing the normalized vector \( \left| \Phi \right\rangle \in L^2(\mathbb{R}_+, d\nu(x)) \), called the *fiducial* vector, one can define a continuous family of affine coherent states \( |q, p\rangle \in L^2(\mathbb{R}_+, d\nu(x)) \) as follows

\[
|q, p\rangle = U(q, p)\left| \Phi \right\rangle.
\]  

As we have two invariant measures, one can define two operators which potentially can lead to the unity in the space \( L^2(\mathbb{R}_+, d\nu(x)) \):

\[
B_L = \int_{G_1} d\mu_L(q, p)|q, p\rangle\langle q, p| \quad \text{and} \quad B_R = \int_{G_1} d\mu_R(q, p)|q, p\rangle\langle q, p|.
\]  

Let us check which one is invariant under the action \( U(q, p) \) of the affine group:

\[
U(q', p')B_LU(q, p)^\dagger = \int_{G_1} d\mu_L(q, p)|p'q + q', p'p\rangle\langle p'q + q', p'p|.
\]  

One needs to replace the variables under integral:

\[
\tilde{q} = p'q + q' \quad \text{and} \quad \tilde{p} = p'p
\]

\[
q = \frac{1}{p'}(\tilde{q} - q') \quad \text{and} \quad p = \frac{\tilde{p}}{p'}.
\]  

Calculating the Jacobian \( \frac{\partial(q,p)}{\partial(\tilde{q}, \tilde{p})} = \frac{1}{(pq)^2} \) one gets

\[
d\mu_L(q, p) = \frac{1}{p^2(p')^2} d\tilde{q}d\tilde{p} = \frac{1}{p^4} d\tilde{q}d\tilde{p} = d\mu_L(\tilde{q}, \tilde{p}).
\]  

The last result proves that

\[
U(q', p')B_LU(q, p)^\dagger = \int_{G_1} d\mu_L(\tilde{q}, \tilde{p})|\tilde{q}, \tilde{p}\rangle\langle \tilde{q}, \tilde{p}| = B_L.
\]  

This also means that \( B_R \) is not invariant under the action \( U(q, p) \).

The *irreducibility* of the representation, used to define the coherent states (24), enables making use of Schur’s lemma \([15]\), which leads to the resolution of the unity in \( L^2(\mathbb{R}_+, d\nu(x)) \):

\[
\int_{G_1} d\mu_L(q, p)|q, p\rangle\langle q, p| = A_\Phi \mathbb{I},
\]  

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where the constant $A_\Phi$ can be calculated using any arbitrary, normalized vector $|f\rangle \in L^2(\mathbb{R}_+, d\nu(x))$:

$$A_\Phi = \int_{\mathcal{G}_1} d\mu_L(q,p) \langle f|q,p \rangle \langle q,p|f \rangle.$$  \hfill (32)

This formula can be calculated by making use of invariance of the measure:

$$A_\Phi = \int_{\mathcal{G}_1} d\mu_L(q,p)$$

$$\int_0^\infty d\nu(x') \int_0^\infty d\nu(x) (f(x')^* e^{i q x'} \Phi(p x')) (e^{-i q x} \Phi(p x)^* f(x))$$

$$= \int_0^\infty \frac{dx'}{x'} \int_0^\infty \frac{dx}{x} \int_0^\infty \frac{dp}{p^2} \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq e^{i q (x'-x)} \right] f(x')^* f(x) \Phi(p x') \Phi(p x)^*$$

$$= \int_0^\infty \frac{dx}{x} |f(x)|^2 \int_0^\infty \frac{dp}{p^2} |\Phi(p)|^2$$

$$= \left( \int_0^\infty \frac{dx}{x} |f(x)|^2 \right) \left( \int_0^\infty \frac{dp}{p^2} |\Phi(p)|^2 \right) = \int_0^\infty \frac{dp}{p^2} |\Phi(p)|^2$$  \hfill (33)

because $\langle f|f \rangle = 1$. Thus, the normalization constant is dependent on the fiducial vector.

2. Structure of the fiducial vector

The problem which influences the structure of quantum state space is a possible degeneration of the space due to specific structure of the fiducial vector. In the case of quantum states the vectors which differ by a phase factor represent the same quantum state. Thus, let us consider the states satisfying the above condition for physically equivalent state vectors [16]:

$$U(\tilde{q}, \tilde{p}) \Phi(x) = e^{i\beta(\tilde{q}, \tilde{p})} \Phi(x), \quad \text{where} \quad \beta(\tilde{q}, \tilde{p}) \in \mathbb{R}.$$  \hfill (34)

The phase space points $\tilde{\xi} = (\tilde{q}, \tilde{p})$ treated as elements of the affine group Aff($\mathbb{R}$) forms its subgroup $\mathcal{G}_\Phi$. The left-hand side of Eq. (34) can be rewritten as:

$$e^{i\tilde{q}x} \Phi(\tilde{p}) = e^{i\beta(\tilde{q}, \tilde{p})} \Phi(x).$$  \hfill (35)

If the generalized stationary group $\mathcal{G}_\Phi$ of the fiducial vector $\Phi$ is a nontrivial group, then the phase space points $(q', p')$ and $(\tilde{q}, \tilde{p}) \cdot (q', p') = (\tilde{p} q', \tilde{p} p')$ are represented by
the same state vector $U(q', p')\Phi(x)$, for all transformations $(\tilde{q}, \tilde{p}) \in G_\Phi$. This is due to the equality

$$U(q', p')\Phi(x) = U((\tilde{q}, \tilde{p}) \cdot (q', p'))\Phi(x).$$

(36)

In this case, to have a unique relation between phase space and the quantum states, the phase space has to be restricted to the quotient structure $\text{Aff}(\mathbb{R})/G_\Phi$. From the physical point of view, in most cases, this is an undesired property.

How to construct the fiducial vector to have $G_\Phi = \{ e_G \}$, where $e_G$ is the unit element in this group?

It is seen that Eq. (35) cannot be fulfilled for $\tilde{q} \neq 0$, independently of chosen fiducial vector. This suggests that the generalized stationary group $G_\Phi$ is parameterized only by the momenta $(0, \tilde{p})$, i.e. it has to be a subgroup of the multiplicative group of positive real numbers, $G_\Phi \subseteq (\mathbb{R}_+, \cdot)$.

On the other hand, Eq. (35) implies that $|\Phi(\tilde{p}x)| = |\Phi(x)|$ for all $(0, \tilde{p}) \in G_\Phi$. In addition, for the fiducial vectors $\Phi(x) = |\Phi(x)|e^{i\gamma(x)}$ the phases of these complex functions are bounded by $0 \leq \gamma(x) < 2\pi$. Due to Eq. (35) the phases $\gamma(x)$ and $\beta(0, \tilde{p})$ have to fulfil the following condition $\gamma(\tilde{p}x) - \gamma(x) = \beta(0, \tilde{p})$. One of the solutions to this equation is the logarithmic function $\gamma(x) = \ln(x)$.

In what follows, to have the unique representation of the phase space as a group manifold of the affine group, we require the generalized stationary group to be the group consisted only of the unit element. This can be achieved by the appropriate choice of the fiducial vector.

The unit operator (31) depends explicitly on the fiducial vector

$$\mathbb{I}[\Phi] = \frac{1}{A_\Phi} \int_{G_1} d\mu_L(\xi) U(\xi) |\Phi\rangle \langle U(\xi)\rangle^\dagger,$$

(37)

This suggests that the most natural transformation of vectors from the representation given by the fiducial vector $|\Phi\rangle$ to the representation given by another fiducial vector $|\Phi'\rangle$ can be constructed as the product of two unit operators $\mathbb{I}[\Phi']\mathbb{I}[\Phi]$.

Let us consider an arbitrary vector $|\Psi\rangle$ from the space $L^2(\mathbb{R}_+, d\nu(x))$ and its representation in the space spanned with a help of the fiducial vector $|\Phi\rangle$:

$$|\Psi\rangle = \mathbb{I}[\Phi]|\Psi\rangle = \frac{1}{A_\Phi} \int_{G_1} d\mu_L(\xi) U(\xi) |\Phi\rangle \langle U(\xi)\rangle^\dagger |\Psi\rangle$$

(38)

The same vector can be represented in terms of another fiducial vector $|\Phi'\rangle$:

$$|\Psi\rangle = \mathbb{I}[\Phi']|\Psi\rangle = \frac{1}{A_{\Phi'}} \int_{G_0} d\mu_L(\xi) U(\xi) |\Phi'\rangle \langle U(\xi)\rangle^\dagger |\Psi\rangle$$

(39)
However, one can transform the vector (38) into the vector (39) using the product of two unit operators:

$$|\Psi\rangle = \frac{1}{A_{\Phi} A_{\Phi'}} \int_{G_1} d\mu_L(q, p) \int_{G_1} d\mu_L(q', p') U(\xi)|\Phi\rangle$$

$$\langle \Phi'|U(\xi^{-1} \cdot \xi')|\Phi\rangle \langle \Phi|U(\xi')|\Psi\rangle$$

(40)

In this way, we see that a choice of the fiducial vector is formally irrelevant. However, as we will see later a relation between the classical model and its quantum realization depends on this choice. The sets of affine coherent states generated from different fiducial vectors may be not unitarily equivalent, but lead in each case to acceptable affine representations of the Hilbert space [13].

3. Phase space and quantum state spaces

The quantization procedure requires understanding the relations among the classical phase space and quantum states space. We have three spaces to be considered:

- The phase space $\Pi$, which consists of two half-planes $\Pi_1$ and $\Pi_2$ defined by (16). It is the background for the classical dynamics\(^4\).

- The carrier spaces $\mathcal{H}_1 := L^2(\mathbb{R}_+, d\nu(x))$ of the unitary representation $U(q, p)$, with the scalar product defined as

$$\langle \psi_2|\psi_1 \rangle = \int_0^{\infty} d\nu(x) \psi_2^*(x) \psi_1(x).$$

(41)

- The space of square integrable functions on the affine group $\mathcal{K}_G = L^2(\text{Aff}(\mathbb{R}), d\mu_L(q, p))$. The scalar product is defined as follows

$$\langle \psi_{G2}|\psi_{G1} \rangle_G = \frac{1}{A_{\phi}} \int_{\text{Aff}} d\mu_L(q, p) \psi_{G2}^*(q, p) \psi_{G1}(q, p),$$

(42)

where $\psi_G(q, p) := \langle q, p|\psi \rangle = \langle \Phi|U(q, p)^\dagger|\psi\rangle$ with $|\psi\rangle \in \mathcal{H}_1$. The Hilbert space $\mathcal{K}_G$ is defined to be the completion in the norm induced by (42) of the span of the $\psi_G$ functions.

\(^4\) For simplicity we consider here only one half-plane, but the results can be easily extended to $\Pi$.  

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We show below that the spaces $\mathcal{H}_1$ and $\mathcal{K}_G$ are unitary isomorphic. First, one needs to check that the functions $\psi_G \in \mathcal{K}_G$ are square integrable function belonging to $L^2(\text{Aff}(\mathbb{R}), d\mu_L(q,p))$. Using the decomposition of unity we get

$$\frac{1}{A_\Phi} \int_{G_1} d\mu_L(q,p) |\langle q,p | \psi \rangle|^2 < \langle \psi | \psi \rangle_{\mathcal{H}_1} < \infty. \quad (43)$$

The definition of the space $\mathcal{K}_G$ shows that for every $\psi_1, \psi_2 \in \mathcal{H}_1$ we have the corresponding functions $\psi_{G1}, \psi_{G2}$ for which the scalar products are equal (unitarity of the transformation between both spaces): $\langle \psi_2 | \psi_1 \rangle_{\mathcal{H}_1} = \langle \psi_{G1} | \psi_{G2} \rangle_{\mathcal{K}_G}$.

Let us now denote by $|e_n\rangle$ the orthonormal basis in $\mathcal{H}_1$ (see App. C). The corresponding functions $e_{Gn}(q,p) = \langle q,p | e_n \rangle$ furnish the orthonormal set:

$$\langle e_{Gn}|e_{Gm} \rangle_{\mathcal{K}_G} = \frac{1}{A_\Phi} \int_{G_1} d\mu_L(q,p) e_{Gn}^*(q,p) e_{Gm}(q,p)$$

$$= \frac{1}{A_\Phi} \int_{G_1} d\mu_L(q,p) \langle e_n|q,p \rangle \langle q,p | e_m \rangle = \langle e_n|e_m \rangle = \delta_{nm}. \quad (44)$$

It is obvious that the vectors $|e_{Gn}\rangle$ define the orthonormal basis in the space $\mathcal{K}_G$. For every vector $|\psi\rangle \in \mathcal{H}_1$

$$|\psi\rangle = \sum_n \langle e_n|\psi \rangle |e_n\rangle. \quad (45)$$

Closing both sides of the above equation with $\langle q,p |$ gives the unique decomposition of the vector $\psi_G(q,p) \equiv \langle q,p | \psi \rangle \in \mathcal{K}_G$ in the basis $|e_{Gn}\rangle_G$:

$$\psi_G(q,p) \equiv \langle q,p | \psi \rangle = \sum_n \langle e_n|\psi \rangle \langle q,p | e_n \rangle = \sum_n \langle e_n|\psi \rangle |e_{Gn}\rangle_G. \quad (46)$$

Note that the vector $|\psi\rangle \in \mathcal{H}_1$ and the vector $|\psi_G\rangle \in \mathcal{K}_G$ have the same expansion coefficients in the corresponding bases. This define the unitary isomorphism between both spaces. It means that we can work either with the quanutm state space represented by the space $\mathcal{H}_1$ or $\mathcal{K}_G$.

4. Affine coherent states for the entire system

The phase space $\Pi$ of our classical system has the structure of the Cartesian product of two partial phase spaces $\Pi_1$ and $\Pi_2$: $\Pi = \Pi_1 \times \Pi_2$. The partial phase spaces $\Pi_l$, where $l = 1, 2$, are identified with the corresponding affine groups which we denote by $G_l = \text{Aff}_l(\mathbb{R})$. The simple product of both affine groups $G_\Pi = (G_1 = \text{Aff}_1(\mathbb{R})) \times (G_2 = \text{Aff}_2(\mathbb{R}))$ can be identified with the whole phase space $\Pi$:

$$(\xi_1, \xi_2) \rightarrow |\xi_1, \xi_2\rangle = U(\xi_1, \xi_2)|\Phi\rangle := U_1(\xi_1) \otimes U_2(\xi_2)|\Phi\rangle, \quad (47)$$
where \( \xi_l = (q_l, p_l) \), \( l = 1, 2 \), the fiducial vector \( |\Phi\rangle \) belongs to the simple product of two Hilbert spaces \( (\mathcal{H}_1 = L^2(\mathbb{R}_+, d\nu(x_1))) \times (\mathcal{H}_2 = L^2(\mathbb{R}_+, d\nu(x_2))) = L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\nu(x_1, x_2)) \), and where the measure \( d\nu(x_1, x_2) = d\nu(x_1)d\nu(x_2) \). The scalar product in \( \mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\nu(x_1, x_2)) \) reads

\[
\langle \psi_2 | \psi_1 \rangle = \int_0^\infty d\nu(x_1) \int_0^\infty d\nu(x_2) \psi_1(x_1, x_2)^* \psi_2(x_1, x_2).
\] (48)

The fiducial vector \( \Phi(x_1, x_2) \) is constructed as a product of two fiducial vectors \( \Phi(x_1, x_2) = \Phi_1(x_1)\Phi_2(x_2) \) generating the appropriate quantum partners for the phase spaces \( \Pi_1 \) and \( \Pi_2 \). The fiducial vector of this type does not add any correlations between both partial phase spaces. A nonseparable form of \( \Pi \) spaces and where the measure \( d\nu(x_1, x_2) \) might lead to reducible representation of \( G_\Pi \) in which case Schur’s lemma could not be applied to get the resolution of unity in \( \mathcal{H} \).

Let us denote by \( \hat{\mathbb{I}}_{12} \) the linear extension of the tensor product \( \hat{\mathbb{I}}_1 \otimes \hat{\mathbb{I}}_2 \), where the unit operators in \( \mathcal{H}_k \) are expressed in terms of the appropriate coherent states

\[
\hat{\mathbb{I}}_k = \frac{1}{A_{\Phi_k}} \int_{G_\Phi} d\mu_L(\xi_k) U_k(\xi_k) |\Phi_k\rangle \langle \Phi_k| U_k(\xi_k)^\dagger, \quad k = 1, 2.
\] (49)

Let us consider the orthonormal basis \( \{e_n^{(1)}(x_1) \otimes e_n^{(2)}(x_2)\} \) in the Hilbert space \( \mathcal{H} \) and an arbitrary vector \( \Psi(x_1, x_2) = \sum_{nm} a_{nm} e_n^{(1)}(x_1) \otimes e_m^{(2)}(x_2) \) belonging to this space (where the basis \( e_n(x) \) is defined in App. C). Acting on this vector with the operator \( \hat{\mathbb{I}}_{12} \) one gets:

\[
\hat{\mathbb{I}}_{12}\Psi(x_1, x_2) = \sum_{nm} a_{nm} (\hat{\mathbb{I}}_1 e_n^{(1)}(x_1)) \otimes (\hat{\mathbb{I}}_2 e_m^{(2)}(x_2)) = \Psi(x_1, x_2).
\] (50)

The operator \( \hat{\mathbb{I}}_{12} \) is identical with the unit operator \( \hat{\mathbb{I}} \) on the space \( \mathcal{H} \).

The explicit form of the action of the group \( G_\Pi \) on the vector \( \Psi(x_1, x_2) \) reads:

\[
U(q_1, p_1, q_2, p_2)\Psi(x_1, x_2) = \sum_{nm} a_{nm} \{ U_1(q_1, p_1) e_n^{(1)}(x_1) \} \otimes \{ U_2(q_2, p_2) e_m^{(2)}(x_2) \}
\]

\[
= \sum_{nm} a_{nm} \{ e^{iq_1x_1} e_n^{(1)}(p_1 x_1) \} \otimes \{ e^{iq_2x_2} e_m^{(2)}(p_2 x_2) \} = e^{iq_1x_1} e^{iq_2x_2} \sum_{nm} a_{nm} e_n^{(1)}(p_1 x_1) \otimes e_m^{(2)}(p_2 x_2)
\]

\[
= e^{iq_1x_1} e^{iq_2x_2} \Psi(p_1 x_1, p_2 x_2).
\] (51)
B. Quantum observables

Making use of the resolution of the identity (31), we define the quantization of a classical observable \( f \) on a half-plane as follows

\[
\mathcal{F} \ni f \rightarrow \hat{f} := \frac{1}{A_{\Phi}} \int_{G_1} d\mu_L(q,p) |q,p\rangle f(q,p) \langle q,p| \in \mathcal{A},
\]

where \( \mathcal{F} \) is a vector space of real continuous functions on a phase space, and \( \mathcal{A} \) is a vector space of operators (quantum observables) acting in the Hilbert space \( \mathcal{H}_1 = L^2(\mathbb{R}_+, d\nu(x)) \). It is clear that (52) defines a linear mapping and the observable \( \hat{f} \) is a symmetric (Hermitian) operator. Let us evaluate the norm of the operator \( \hat{f} \):

\[
\| \hat{f} \| \leq \frac{1}{A_{\Phi}} \int_{G_1} d\mu_L(q,p) |f(q,p)| \| |q,p\rangle \langle q,p| \| \leq \frac{1}{A_{\Phi}} \int_{G_1} d\mu_L(q,p) |f(q,p)|.
\]

This implies that, if the classical function \( f \) belongs to the space of integrable functions \( L^1(\text{Aff}(\mathbb{R}), d\mu_L(q,p)) \), the operator \( \hat{f} \) is bounded so it is a self-adjoint operator. Otherwise, it is defined on a dense subspace of \( L^2(\mathbb{R}_+, d\nu(x)) \), and its possible self-adjointness becomes an open problem as symmetricity does not assure self-adjointness, and further examination is required [19]. The quantization (52) can be applied to any type of observables including non-polynomial ones, which is of primary importance for us due to the functional form of the Hamiltonian (4).

It is not difficult to show that the mapping (52) is covariant in the sense that one has

\[
U(\xi_0) \hat{f} U^*(\xi_0) = \frac{1}{A_{\Phi}} \int_{G_1} d\mu_L(\xi)|\xi\rangle f(\xi^{-1}_0 \cdot \xi) \langle \xi| = \mathcal{L}_{\xi_0} L_{\xi_0} \hat{f},
\]

where \( \mathcal{L}_{\xi_0} f(\xi) = f(\xi^{-1}_0 \cdot \xi) \) is the left shift operation (22) and \( \xi^{-1}_0 \cdot \xi = (q_0, p_0)^{-1} \cdot (q, p) = (\frac{q-q_0}{p_0}, \frac{p-p_0}{p_0}) \).

The mapping (52) extended to the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\nu(x_1, x_2)) \) of the entire system and applied to an observable \( \hat{f} \) reads

\[
\hat{f}(t) = \frac{1}{A_{\Phi_1} A_{\Phi_2}} \int_{G_{11}} d\mu_L(\xi_1, \xi_2) |\xi_1, \xi_2\rangle f(\xi_1, \xi_2) \langle \xi_1, \xi_2|,
\]

where \( d\mu_L(\xi_1, \xi_2) := d\mu_L(q_1, p_1) d\mu_L(q_2, p_2) \).

IV. QUANTUM DYNAMICS

The mapping (55) applied to the classical Hamiltonian (4) reads

\[
\hat{H}(t) = \frac{1}{A_{\Phi_1} A_{\Phi_2}} \int_{G_{11}} d\mu_L(\xi_1, \xi_2) |\xi_1, \xi_2\rangle H(\xi_1, \xi_2, t) \langle \xi_1, \xi_2|,
\]
where $t$ is an evolution parameter of the classical level.

Suppose that $\hat{H}$ is bounded on $\mathcal{H}$ so it is self-adjoint on $\mathcal{H}$. Therefore, we can define the quantum evolution using the Schrödinger equation as follows

$$i\frac{\partial}{\partial \tau}\ket{\psi(\tau)} = \hat{H}(t)\ket{\psi(\tau)} ,$$

where $\ket{\psi} \in \mathcal{H}$, and where $\tau$ is an evolution parameter at the quantum level.

In general, the parameters $t$ and $\tau$ are quite different. To get the consistency between the classical and quantum levels we postulate that $t = \tau$, which defines the time variable at both levels.

V. THE PROBABILITY MEASURE

Let us define the following probability measure on the phase space:

$$\Pi \supset \Omega \rightarrow \hat{M}(\Omega) := \frac{1}{A_{\Phi_1} A_{\Phi_2}} \int_{\Omega} d\mu_L(\xi_1) d\mu_L(\xi_2)|\xi_1, \xi_2\rangle \langle \xi_1, \xi_2|,$$

where $|\Phi_1\rangle \in \mathcal{H}_1$ and $|\Phi_2\rangle \in \mathcal{H}_2$ are the fiducial vectors. This probability measure represents the observable: “the system is in the region $\Omega$ of the phase space $\Pi$”.

In what follows we take $\Phi_1(x) = \Phi_2(x) = \Phi(x)$ because the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are identical, and any two fiducial vectors can be linked by the unitary map of the form (19).

The measure (58) is of the POV type [20] as it is

- positive, $\hat{M}(\Omega) \geq 0$, $\forall \Omega \subset \Pi$;
- additive, $\hat{M}(\bigcup_k \Omega_k) = \sum_k \hat{M}(\Omega_k)$ if $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$;
- normalized, $\hat{M}(\Pi) = I$.

The volume $V(\Omega)$ of $\Omega$ is defined as

$$V(\Omega) = \int_{\Omega} d\mu_L(q_1, p_1) d\mu_L(q_2, p_2).$$

One needs to remember the factor $1/(2\pi)^2$ hidden in this integral, see 23.

Now, suppose that $\Omega$ is a small neighbourhood of $(\xi_1, \xi_2) := (q_1, p_1, q_2, p_2)$. We define the probability density measure operator $\hat{M}(\xi_1, \xi_2)$, which determines the probability that the quantum system is at the state $|\xi_1, \xi_2\rangle$ of the phase space $\Pi$. It is defined as follows

$$\hat{M}(\xi) = \lim_{V(\Omega) \rightarrow 0} \frac{\hat{M}(\Omega)}{V(\Omega)} \text{ where } (\xi_1, \xi_2) \in \Omega.$$
The r.h.s. of (60) is well defined due to the mean value theorem applied to the measure (58). This probability density operator can be rewritten as

$$\hat{M}(\xi_1, \xi_2) = \frac{1}{A_\Phi^2} |\xi_1, \xi_2\rangle \langle \xi_1, \xi_2|.$$  \hspace{1cm} (61)

Let $f : \Pi \to \mathbb{R}$ be a classical observable and $|\psi\rangle$ a quantum state of our system. The expectation value of the corresponding quantum operator $\hat{f}$ is given by the standard expression

$$\langle \hat{f} \rangle = \langle \psi | \hat{f} | \psi \rangle = \frac{1}{A_\Phi^2} \int_{G_{\Pi}} d\mu_L(\xi_1, \xi_2) \langle \psi | \xi_1, \xi_2 \rangle f(\xi_1, \xi_2) \langle \xi_1, \xi_2 | \psi \rangle,$$  \hspace{1cm} (62)

where $d\mu_L(\xi_1, \xi_2) := d\mu_L(q_1, p_1)d\mu_L(q_2, p_2)$.

Let $\Pi = \bigcup_k \Omega_k$, with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. The probability density that $(\xi_{1k}, \xi_{2k}) \in \Omega_k$ is given by

$$p((\xi_{1k}, \xi_{2k}) \in \Omega_k) := \langle \psi | \hat{M}(\Omega_k) | \psi \rangle.$$  \hspace{1cm} (63)

The consistency between (62) and (63) occurs if one has

$$\langle \hat{f} \rangle \approx \sum_k f(\xi_{1k}, \xi_{2k}) p((\xi_{1k}, \xi_{2k}) \in \Omega_k).$$  \hspace{1cm} (64)

One can show that (64) is satisfied due to the mean value theorem applied to the measure (58). In the limit $V(\Omega_k) \to 0$, such that $\Pi = \cup_k \Omega_k$ Eq. (64) is exact.

**VI. PROSPECTS**

Recently, special structures have been found in the evolution of gravitational systems. They are named spikes [21] and wiggles [22].

To compare the classical and quantum dynamics of the wiggles, one needs to compare the classical solutions $\zeta(t) = (\zeta_1(t), \zeta_2(t), \zeta_3(t), \zeta_4(t)) := (q_1(t), p_1(t), q_2(t), p_2(t))$ of the Hamilton equations (5)–(8) with the average values of the corresponding quantum operators:

$$\hat{\zeta}_k = \frac{1}{A_\Phi^2} \int_{G_{\Pi}} d\mu_L(\zeta) |\zeta\rangle \langle \zeta_k| \hat{M}(\zeta).$$  \hspace{1cm} (65)

The expectation value is given by

$$\langle \hat{\zeta}_k \rangle_{\psi(t)} = \langle \psi(t) | \hat{\zeta}_k | \psi(t) \rangle = \frac{1}{A_\Phi^2} \int_{G_{\Pi}} d\mu_L(\zeta) \langle \psi(t) | \zeta\rangle \zeta_k \langle \zeta | \psi(t) \rangle.$$  \hspace{1cm} (66)
This average is calculated in the state $|ψ(t)⟩$ evolving according to the Schrödinger equation (57) with the initial condition corresponding to the initial condition used in solution of the Hamilton equations. If the classical initial condition is given by $ζ_k(t_0) = ζ_k^{(0)}$, then the corresponding initial condition for the Schrödinger equation is defined to be

$$\langle ψ(t_0)|\hat{ζ}_k|ψ(t_0)⟩ = ζ_k^{(0)}. \quad (67)$$

In such a case the classical trajectories $ζ_k(t)$ and the quantum “trajectories” $⟨\hat{ζ}_k⟩ψ(t)$ are treated on the same footing.

The wiggles are parametric curves, $ζ : \mathbb{R} \supset [t_1, t_2] \rightarrow Π \subset \mathbb{R}^4$, in the physical phase space $Π$. The observables invariant with respect to reparametrization of a curve can be chosen to be [23]:

- the length $s[ζ]$ of a curve (global observable),
- the generalized curvatures $\{χ_1(t), χ_2(t), χ_3(t)\}$ (local observables) of a curve defined by the Frenet vectors.

The curvatures of these curves depend on the location of the initial conditions in the physical phase space $Π$ specifying the dynamics, and on the stage of the evolution of the system.

Both the length and the curvatures can be determined for the quantum “curves” $⟨\hat{ζ}_k⟩ψ(t)$ as well. Thus, the comparison of the observables for the classical and quantum curves will enable finding the influence of the quantization on the classical wiggles.

Does quantization suppress these structures, leave them almost unchanged, or turns them into some quantum structures? Our next paper will be devoted to the examination of this issue.

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Appendix A: Regularized Hamiltonian

The Hamiltonian (4) is restricted to the region of $Π$ corresponding to $F > 0$. We extend it to the whole phase space by introducing an additional functional coefficient which vanishes for $F \leq 0$. It has the form

$$H_{new} := \theta \left[ p_1 p_2 - e^{2q_1} - e^{q_2-q_1} - \frac{1}{4}(p_1 + p_2 - t)^2 \right] H(q_1, q_2; p_1, p_2; t), \quad (A1)$$
where \( H(q_1, q_2; p_1, p_2; t) \) is the original Hamiltonian (4), while \( \theta \) stands for the Heaviside step function. However, so defined new Hamiltonian is not differentiable at \( F = 0 \). To get a better behaviour we adopt the following “smooth” version of \( \theta \):

\[
\theta_\epsilon(x) := \begin{cases} 
0 : x \leq 0, \\
\frac{1}{2}(1 - \cos(\frac{\pi x}{\epsilon})) : 0 < x \leq \epsilon, \\
1 : x > \epsilon.
\end{cases}
\tag{A2}
\]

Here, \( \epsilon \) is a small parameter \( 0 < \epsilon \ll 1 \). Using (A2) we rewrite (A1):

\[
H_\epsilon(q_1, q_2; p_1, p_2; t) := \theta_\epsilon \left[ p_1 p_2 - e^{2q_1} - e^{q_2 - q_1} - \frac{1}{4}(p_1 + p_2 - t)^2 \right] H(q_1, q_2; p_1, p_2; t).
\tag{A3}
\]

The Hamiltonian \( H_\epsilon \) is well defined and differentiable in the whole phase space. Indeed, for any \( (q_1, q_2; p_1, p_2; t) \) corresponding to \( F > 0 \) one finds \( H_\epsilon(q_1, q_2; p_1, p_2; t) \approx H(q_1, q_2; p_1, p_2; t) \). The smaller \( \epsilon \) is, the better approximation we get. Also the function (A2) guarantees that the Hamiltonian (A3) vanishes smoothly in the limit \( F \to 0 \). Taking the limit \( \epsilon \to 0^+ \) restores to the original expression (4).

**Appendix B: Alternative affine coherent states for half-plane**

The phase space \( \Pi_1 \) may be identified with the affine group \( \text{Aff}(\mathbb{R}) \) by defining the multiplication law as follows

\[
(q', p') \cdot (q, p) = \left( \frac{q}{p'} + q', p' p \right),
\tag{B1}
\]

with the unity \((0, 1)\) and the inverse

\[
(q', p')^{-1} = (-q' p', \frac{1}{p'}).
\tag{B2}
\]

The affine group has two, nontrivial, inequivalent irreducible unitary representations [10] and [11, 12]. Both are realized in the Hilbert space \( L^2(\mathbb{R}_+, d\nu(x)) \), where \( d\nu(x) = dx/x \) is the invariant measure on the multiplicative group \( (\mathbb{R}_+, \cdot) \). In what follows we choose the one defined by

\[
U(q, p)\psi(x) = e^{iqx} \psi(x/p),
\tag{B3}
\]

where \( |\psi\rangle \in L^2(\mathbb{R}_+, d\nu(x)) \).
For simplicity of notation, let us define integrals over the affine group \( \text{Aff}(\mathbb{R}) \) as follows:

\[
\int_{\text{Aff}(\mathbb{R})} d\mu_L(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \int_0^\infty dp \frac{dp}{p^2}, 
\]  \( \text{(B4)} \)

\[
\int_{\text{Aff}(\mathbb{R})} d\mu_R(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \int_0^\infty dp \frac{dp}{p}, 
\]  \( \text{(B5)} \)

\[
\int_{\text{Aff}(\mathbb{R})} d\mu_U(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \int_0^\infty dp \rho(q,p). 
\]  \( \text{(B6)} \)

The last one is intended to be used as invariant measure in respect with the action \( U(q,p) \).

Fixing the normalized vector \( |\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x)) \), called the fiducial vector, one can define a continuous family of affine coherent states \( |q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x)) \) as follows

\[
|q,p\rangle = U(q,p)|\phi\rangle. 
\]  \( \text{(B7)} \)

As we have three measures, one can define three operators which potentially can leads to the unity in the space \( L^2(\mathbb{R}_+, d\nu(x)) \):

\[
B_L = \int_{\text{Aff}(\mathbb{R})} d\mu_L(q,p) |q,p\rangle \langle q,p|, 
\]  \( \text{(B8)} \)

\[
B_R = \int_{\text{Aff}(\mathbb{R})} d\mu_R(q,p) |q,p\rangle \langle q,p|, 
\]  \( \text{(B9)} \)

\[
B_U = \int_{\text{Aff}(\mathbb{R})} d\mu_U(q,p) \rho(q,p) |q,p\rangle \langle q,p|. 
\]  \( \text{(B10)} \)

Let us check which one is invariant under the action \( U(q,p) \) of the affine group:

\[
U(q',p')B_U U(q',p')^\dagger = \int_{-\infty}^{+\infty} dq \int_0^\infty dp \rho(q,p) |q/p' + q',p' p\rangle \langle q/p' + q',p' p| 
\]  \( \text{(B11)} \)

One needs to replace the variables under the integral:

\[
\tilde{q} = q/p' + q' \quad \text{and} \quad \tilde{p} = p' p, 
\]  \( \text{(B12)} \)

\[
q = p'(\tilde{q} - q') \quad \text{and} \quad p = \frac{\tilde{p}}{p'}. 
\]  \( \text{(B13)} \)

Calculating the Jacobian \( \frac{\partial(q,p)}{\partial(\tilde{q},\tilde{p})} = 1 \) one gets:

\[
d\mu_U(q,p) = \rho(q,p) d\tilde{q} d\tilde{p} = \rho(p'(\tilde{q} - q'), \frac{\tilde{p}}{p'}) d\tilde{q} d\tilde{p} 
\]  \( \text{(B14)} \)
This implies, the transformed weight should be equal to the initial one, \( \rho(p'(\tilde{q} - q'), \tilde{p}) = \rho(q, p) \) for every \((q', p')\). The simplest solution is \( \rho(q, p) = \text{const} \), so we get \( d\mu_U(q, p) = dq dp \).

It also implies that the operators \( B_L \) and \( B_R \) do not commute with the affine group. The action \((B3)\) is not compatible neither with the left invariant, nor with right invariant measures on the affine group.

The irreducibility of the representation, used to define the coherent states \((B7)\), enables making use of Schur’s lemma \([15]\), which leads to the resolution of the unity in \( L^2(\mathbb{R}^+, d\nu(x)) \):

\[
\int_{\text{Aff}(\mathbb{R})} d\mu_U(q, p) |q, p\rangle\langle q, p| = A_\phi \mathbb{I},
\]

where the constant \( A_\phi \) can be calculated using any arbitrary, normalized vector \( |f\rangle \in L^2(\mathbb{R}^+, d\nu(x)) \):

\[
A_\phi = \int_{\text{Aff}(\mathbb{R})} d\mu_U(q, p) \langle f|q, p\rangle\langle q, p|f\rangle .
\]

This formula can be calculated directly:

\[
A_\phi = \int_{\text{Aff}(\mathbb{R})} d\mu_U(q, p)
\int_0^\infty d\nu(x') \int_0^\infty d\nu(x) (f(x')^{*} e^{iqx'} \phi(x'/p))(e^{-iqx} \phi(x/p)^{*} f(x))
\]

\[
= \int_0^\infty \frac{dx'}{x'} \int_0^\infty \frac{dx}{x} \int_0^\infty dp \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq e^{iq(x'-x)} \right] f(x')^{*} f(x) \phi(x'/p)\phi(x/p)^{*}
\]

\[
= \int_0^\infty \frac{dx}{x^2} |f(x)|^2 \int_0^\infty dp |\phi(x/p)|^2
\]

\[
= \left( \int_0^\infty \frac{dx}{x} |f(x)|^2 \right) \left( \int_0^\infty \frac{dp}{p^2} |\phi(p)|^2 \right) = \int_0^\infty \frac{dp}{p^2} |\phi(p)|^2
\]

(B17)

if \( \langle f|f\rangle = 1 \).

In the derivation of \((B17)\) we have used the equations:

\[
\langle x|x'\rangle = x \delta(x - x'), \quad \int_0^\infty \frac{dx}{x} |x\rangle\langle x| = \mathbb{I}, \quad \int_0^\infty \frac{dx}{x} \delta(x - x') f(x) = f(x').
\]

(B18)

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Appendix C: Orthonormal basis of the carrier space

The basis of the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$ is known to be [17]

$$e^{(\alpha)}_n(x) = \sqrt{\frac{n!}{(n+\alpha)!}} e^{-x/2} x^{(1+\alpha)/2} L^{(\alpha)}_n(x), \quad (C1)$$

where $L^{(\alpha)}_n(x)$ is the Laguerre polynomial, $\alpha > -1$, and $(n+\alpha)! = \Gamma(n+\alpha+1)$. One can verify that $\int_0^\infty e^{(\alpha)}_n(x)e^{(\alpha)}_m(x)d\nu(x) = \delta_{nm}$ so that $e^{(\alpha)}_n(x)$ is an orthonormal basis.

[1] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, “Oscillatory approach to a singular point in the relativistic cosmology”, Adv. Phys. 19, 525 (1970).
[2] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, “A general solution of the Einstein equations with a time singularity”, Adv. Phys. 31, 639 (1982).
[3] S. L. Parnovsky, “Gravitation fields near the naked singularities of the general type”, Physica A 104, 210 (1980).
[4] S. L. Parnovsky, “A general solution of gravitational equations near their singularities”, Class. Quant. Grav. 7, 571 (1990).
[5] N. J. Cornish and J. J. Levin, “The Mixmaster universe is chaotic,” Phys. Rev. Lett. 78, 998 (1997).
[6] V. A. Belinskii, I. M. Khalatnikov, and M. P. Ryan, “The oscillatory regime near the singularity in Bianchi-type IX universes”, Preprint 469 (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as Secs. 1 and 2 in M. P. Ryan, Ann. Phys. 70, 301 (1971).
[7] V. A. Belinski, “On the cosmological singularity”, Int. J. Mod. Phys. D 23, 1430016 (2014).
[8] E. Czuchry and W. Piechocki, “Asymptotic Bianchi IX model: diagonal and general cases,” arXiv:1409.2206 [gr-qc].
[9] E. Czuchry and W. Piechocki, “Bianchi IX model: Reducing phase space,” Phys. Rev. D 87, 084021 (2013).
[10] I. M. Gel’fand and M. A. Naïmark, “Unitary representations of the group of linear transformations of the straight line”, Dokl. Akad. Nauk. SSSR 55, 567 (1947).
[11] E. W. Aslaksen and J. R. Klauder, “Unitary Representations of the Affine Group”, J. Math. Phys. 9, 206 (1968).
[12] E. W. Aslaksen and J. R. Klauder, “Continuous Representation Theory Using Unitary Affine Group”, J. Math. Phys. 10, 2267 (1969). III A 1, B
[13] J. R. Klauder, private communication. III A 2
[14] J. Q. Chen, J. Ping and F. Wang, Group Representation Theory for Physicists (World Scientific, 2002). III A 1
[15] A. O. Barut and R. Rączka, Theory of group representations and applications (PWN, Warszawa, 1977). III A 1, B
[16] A. Perelomov, Generalized coherent states and their applications (Springer-Verlag, Berlin, 1986). III A 2
[17] J. P. Gazeau and R. Murenzi, “Covariant affine integral quantization(s)”, arXiv:1512.08274 [quant-ph]. C
[18] H. Bergeron and J. P. Gazeau, “Integral quantizations with two basic examples,” Annals Phys. 344, 43 (2014). III B
[19] M. Reed and B. Simon, Methods of Modern Mathematical Physics (San Diego, Academic Press, 1980), Vols I and II. III B
[20] P. Busch, P.J. Lahti and P. Mittelstaedt, The Quantum Theory of Measurement (Springer, 1966), sec. edition. V
[21] E. Czuchry, D. Garfinkle, J. R. Klauder and W. Piechocki, “Do spikes persist in a quantum treatment of spacetime singularities?,” Phys. Rev. D 95, 024014 (2017). VI
[22] W. Piechocki and G. Plewa, “Structures arising in the asymptotic dynamics of the Bianchi IX model,” arXiv:1611.05262 [gr-qc]. VI
[23] W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds (American Mathematical Society, 2002), sec. edition. VI
