From the asymmetric simple exclusion processes to the stationary measures of the KPZ fixed point on an interval

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Abstract. Barraquand and Le Doussal [2] introduced a family of stationary measures for the (conjectural) KPZ fixed point on an interval with Neumann boundary conditions, and predicted that they arise as scaling limit of stationary measures of all models in the KPZ universality class on an interval. In this paper, we show that the stationary measures for KPZ fixed point on an interval arise as the scaling limits of the height increment processes for the open asymmetric simple exclusion process in the steady state, with parameters changing appropriately as the size of the system tends to infinity.

MSC2020 subject classifications: Primary 60K35, 60F05; secondary 82C22

Keywords: asymmetric simple exclusion process, scaling limit, KPZ fixed point on an interval

1. Introduction and main result

1.1. KPZ fixed point on an interval

The asymmetric simple exclusion process (ASEP) in one dimension is one of the most widely investigated models for open non-equilibrium systems in the physics literature and serves as a basic model in the Kardar–Parisi–Zhang (KPZ) universality class. In particular, investigations on the connection between the ASEP and the so-called KPZ fixed point, the conjectural [15] limiting space-time random field for the KPZ universality class which was rigorously defined on the real line by Matetski, Quastel and Remenik [25], have been among the most active areas in mathematical physics in recent decades. While most activities focus on the KPZ equation and KPZ fixed point on the real line (see e.g. [12, 17, 27, 30, 29, 31, 32] and more references therein), recent progress has been made regarding the models defined on an interval instead of the real line, with appropriate boundary conditions. The investigations of an open ASEP on an interval (in a weakly asymmetric regime) turned out to be an effective tool to study the open KPZ equation ([16, 26]), which then lead Corwin and Knizel [14] to the construction and characterization of the stationary measures of the KPZ on [0, 1] (see also [2] and [5]). We refer to the references therein and to the review [13] for more background on the topic.

The KPZ fixed point on an interval has not yet been rigorously defined. However, building on the work of Corwin and Knizel [14], Barraquand and Le Doussal [2] determined the large scale limit of the stationary measures of the KPZ equation under the appropriate rescaling, and postulated that this limit should correspond to the stationary measures of the (conjectural) KPZ fixed point on [0, 1] that are expected to arise as the scaling limit of stationary measures of all models in the KPZ class on an interval. The postulated stationary measures of the KPZ fixed point on [0, 1] depend on two boundary parameters \( a, c \), and can be represented as the laws of the processes

\[
\begin{align*}
\{ \tilde{H}(x) \}_{x \in [0,1]} = \{ \tilde{B}_x + \tilde{X}_x \}_{x \in [0,1]},
\end{align*}
\]

where \( \tilde{B} \) is the standard Brownian motion multiplied by \( 1/\sqrt{2} \) and \( \tilde{X} \) is an independent stochastic process with continuous trajectories. The law \( P_{\tilde{X}} \) of \( \tilde{X} \) is absolutely continuous with respect to the law \( P_{\tilde{B}} \) of process \( \tilde{B} \) with the Radon–Nikodym derivative

\[
\frac{dP_{\tilde{X}}}{dP_{\tilde{B}}} (\beta) \propto e^{\beta \min_x (\beta - \beta_1) + a \min_x (\beta_x - \beta_1)}, \quad \beta = (\beta_x)_{x \in [0,1]} \in C([0,1]),
\]

where

\[
\beta_1(x) = \begin{cases} 
1, & x < 1, \\
0, & x = 1.
\end{cases}
\]
The finite dimensional distributions for $\tilde{X}$ and their Laplace transforms were given in [2, Supplementary material, formulas (53) and (55)], see also [4]. (We note change of notation here: the boundary parameters in [2] are $\bar{v} = a/2$ and $\bar{u} = c/2$.)

The recent advances [2, 14] leading to the process (1.1) can be interpreted as a double-limit procedure: one first takes the limit of the increments of the height function of an ASEP in a weakly asymmetric regime as the size of the system tends to infinity (leading to stationary measure for open KPZ equation), and then scales both the boundary parameters and the magnitudes appropriately to obtain process $H$.

The contribution of this paper is a limit theorem for the open ASEP that leads to the aforementioned process (1.1) as a single limit when the boundary parameters change. The single-limit procedure seems to be of a different nature than the double-limit procedure described above. In particular the parameter $q$ is fixed. It is remarkable that $q$ does not appear in the limit process, and moreover the appearance of the process $H$ is actually a surprise to us, and we do not have a simple explanation on why the two procedures lead to the same limit process. More specifically, in Theorem 1.5 we show that under appropriate scaling the spatial height increment process converges in finite-dimensional distributions to the sum of two independent processes

$$\frac{1}{\sqrt{2}} \left\{ B_x + \eta_x^{(a,c)} \right\}_{x \in [0,1]}$$

where $B$ is the standard Brownian motion, and process $\eta_x^{(a,c)}$ is introduced in Section 1.3. When the sum $a + c$ is finite and non-negative, convergence is in Skorokhod’s space $D[0, 1]$ of càdlàg functions and process (1.3) has the same law as process (1.1), see Remark 1.6. Our proof relies on a different representation of the limiting process, not on the Radon–Nikodym derivative (1.2). Our limit theorem has actually a larger family of processes in the limit, allowing that $a$ and/or $c$ are infinite.

1.2. Open ASEP with changing parameters

The asymmetric simple exclusion process is an irreducible continuous time Markov process on the finite state space $\{0, 1\}^n$ with parameters

$$\alpha > 0, \quad \beta > 0, \quad \gamma \geq 0, \quad \delta \geq 0, \quad \text{and} \quad 0 \leq q < 1.$$  \hfill (1.4)

Informally, the process models the evolution of the particles located at sites $1, \ldots, n$ that can jump to the neighbor cell to the right with rate $1$ and to the left with rate $q$, if the target site is unoccupied. Furthermore, particles arrive at site $1$ from the left reservoir (respectively, at site $n$ from the right reservoir), if empty, at rate $\alpha$ (respectively, $\delta$), and exit the system into the right reservoir at site $n$ (respectively, exit the system into the left reservoir at site $1$), if occupied, at rate $\beta$ (respectively, $\gamma$). The transition rates are summarized in Figure 1. Since $q < 1$, particles move in an asymmetric way, with higher rate to the right than to the left; in the special case $q = 0$, particles move only to the right and the model is known as the totally asymmetric simple exclusion process.

We let $\tau_1(t), \ldots, \tau_n(t)$ denote the occupations of the sites: $\tau_j(t) = 1$ if the $j$-th location is occupied by a particle at time $t \geq 0$, and $\tau_j(t) = 0$ otherwise. The height function is defined for $x \in [0, 1]$ and $t \geq 0$ as

$$h_n(x, t) = h_n(0, t) + \sum_{j=1}^{\lfloor nx \rfloor} (2\tau_j(t) - 1), \quad h_n(0, t) = -2N_n(t),$$

with $N_n(t)$ denoting the net flow of particles into the system from the left reservoir up to time $t$, i.e, total number of particles that arrived at site $1$ from the left reservoir up to time $t$ minus the number of particles that have exited from site $1$ into the left reservoir up to time $t$. This height function is piece-wise constant in variable $x$, which is more convenient for our approach than the continuous height function obtained by the piece-wise linear interpolation between the jumps.

We denote by $\mu_n$ the stationary distribution of the ASEP as a Markov process on $\{0, 1\}^n$. Under $\mu_n$, the process is also referred to be in the steady state in the physics literature. Following the common notation in the physics literature, we will denote the expectation with respect to $\mu_n$ by $\langle \cdot \rangle_n$.

Started in the steady state, i.e., with $\mu_n$ as the initial distribution of $(\tau_1(0), \ldots, \tau_n(0))$, the law of the height increment process $\{h_n(x, t) - h_n(0, t)\}_{x \in [0,1]}$ does not change with time $t \geq 0$. We can therefore omit the dependence on $t$ and consider a single instance $(\tau_1, \ldots, \tau_n) \in \{0, 1\}^n$ as a random variable with the law $\mu_n$. Our main object of interest is then the stationary measure for the open ASEP height function process (see [14]),

$$h_n(x) := \sum_{j=1}^{\lfloor nx \rfloor} (2\tau_j - 1), \quad x \in [0, 1].$$  \hfill (1.5)
We shall consider the case when the parameters $\alpha, \beta, \gamma, \delta$ vary with $n$, while we shall keep $q \in [0,1)$ fixed. Consequently, $\mu_n$ denotes the stationary distribution of the ASEP with varying parameters $\alpha_n, \beta_n, \gamma_n, \delta_n$ and $q \in [0,1)$ fixed. As in [11, 7], it is convenient to reparametrize the ASEP using
\[
A_n = \kappa_+(\beta_n, \delta_n), \quad B_n = \kappa_-(\beta_n, \delta_n), \quad C_n = \kappa_+(\alpha_n, \gamma_n), \quad D_n = \kappa_-(\alpha_n, \gamma_n).
\]
(1.6)

with
\[
\kappa_\pm(u,v) := \frac{1}{2u} \left( 1 - q - u + v \pm \sqrt{(1-q-u+v)^2 + 4uv} \right).
\]

In particular $A_n, C_n \geq 0$, and $B_n, D_n \in (-1,0]$ as explained in [7].

We shall specify how the parameters $\alpha_n, \beta_n, \gamma_n, \delta_n$ of the ASEP vary by specifying how the parameters $A_n, B_n, C_n, D_n$ vary. We will be interested in convergence to the "triple point" in the phase diagram, $A_n \to 1$ and $C_n \to 1$. This point lies at the intersection of three regions of the phase diagram for the open ASEP, where the high density, low density and the maximal current regimes meet, see Fig. 2. For parameters $B_n, D_n$ we shall consider more general limits, with controlled rates of convergence only when $B_n \to -1, D_n \to -1$. The key assumption on the rates of convergence is as follows.

**Assumption 1.1.** In addition to the assumptions that $A_n, C_n \geq 0, B_n, D_n \in (-1,0]$ for all $n \geq 1$, which are implied by (1.6), we assume that $A_n C_n < 1$, and

\[
\begin{align*}
\lim_{n \to \infty} A_n &= 1 \text{ with } \lim_{n \to \infty} \sqrt{n}(1-A_n) = a \in (-\infty, \infty], \\
\lim_{n \to \infty} C_n &= 1 \text{ with } \lim_{n \to \infty} \sqrt{n}(1-C_n) = c \in (-\infty, \infty],
\end{align*}
\]

(1.7) (1.8)

where $a + c \geq 0$. We also assume that

\[
\lim_{n \to \infty} B_n = B \in [-1,0] \quad \text{and} \quad \lim_{n \to \infty} D_n = D \in [-1,0],
\]

(1.9)

and

\[
\lim_{n \to \infty} \frac{1}{n} \log(1-B_n D_n) = 0.
\]

(1.10)

We use the convention that $a + c > 0$ includes the cases $a = \infty$ and/or $c = \infty$.

We remark that (1.10) holds if (1.9) holds with $BD < 1$, or if $BD = 1$ but the speed of convergence in one of the limits, say to $B = -1$, is restricted by a condition such as $n^\theta(1+B_n) \to \infty$ for some $\theta > 0$.

1.3. Main result

Let
\[
q_t(x,y) := \frac{1}{\sqrt{2\pi t}} \left[ \exp \left( -\frac{1}{2t}(x-y)^2 \right) - \exp \left( -\frac{1}{2t}(x+y)^2 \right) \right], \quad x, y, t > 0,
\]

(1.11)

denote the transition kernel of the Brownian motion killed at hitting zero. Introduce also
\[
\ell_x(y) := \frac{y}{\sqrt{2\pi x^3}} \exp \left( -\frac{y^2}{2x} \right), \quad x, y > 0.
\]

(1.12)
We first recall two classical stochastic processes. The Brownian excursion, denoted by $B^\text{ex}$, is the process with $B^\text{ex}_0 = B^\text{ex}_1 = 0$ and with finite-dimensional density at time points $0 < x_1 < \ldots < x_{d-1} < 1$ of the form

$$\sqrt{8\pi} \ell_1(y_1) \ell_{1-x_{d-1}}(y_{d-1}) \prod_{k=2}^{d-1} q_{x_k-x_{k-1}}(y_{k-1}, y_k), \quad y_1, \ldots, y_{d-1} > 0.$$  

(1.13)

The Brownian meander, denoted by $B^\text{me}$, is the process with $B^\text{me}_0 = 0$ and with finite-dimensional density at time points $0 < x_1 < \ldots < x_{d-1} < x_d = 1$ of the form

$$\sqrt{2\pi} \ell_1(y_1) \prod_{k=2}^{d} q_{x_k-x_{k-1}}(y_{k-1}, y_k), \quad y_1, \ldots, y_d > 0.$$  

(1.14)

Next, we introduce an auxiliary Markov process $\tilde{\eta}^{(a,c)} = \left\{ \tilde{\eta}^{(a,c)}_x \right\}_{x \in [0,1]}$ parameterized by $a, c$ with $a + c \geq 0$.

**Definition 1.2.**

(i) The process $\tilde{\eta}^{(\infty,\infty)}$ is the Brownian excursion $B^\text{ex}$.

(ii) For $a \in \mathbb{R}$, the process $\tilde{\eta}^{(a,\infty)}$ is the process with finite-dimensional density at time points $0 < x_1 < \ldots < x_{d-1} < x_d = 1$ of the form

$$\frac{1}{\mathcal{C}_a,\infty} \ell_1(y_1) e^{-ay_d/\sqrt{2}} \prod_{k=2}^{d} q_{x_k-x_{k-1}}(y_{k-1}, y_k), \quad y_1, \ldots, y_d > 0,$$  

(1.15)

where the normalization constant $\mathcal{C}_a,\infty$ is given in (1.17) below. We set $\tilde{\eta}^{(0,\infty)} = 0$. In particular, $\tilde{\eta}^{(0,\infty)}$ is the Brownian meander.

(iii) For $c \in \mathbb{R}$, the process $\tilde{\eta}^{(\infty,c)}$ is defined as the process with finite-dimensional density at time points $0 = x_0 < x_1 < \ldots < x_{d-1} < 1$ of the form

$$\frac{1}{\mathcal{C}_\infty,c} \ell_{1-x_{d-1}}(y_{d-1}) e^{-cy_0/\sqrt{2}} \prod_{k=1}^{d-1} q_{x_k-x_{k-1}}(y_{k-1}, y_k), \quad y_0, \ldots, y_{d-1} > 0,$$  

where the normalization constant $\mathcal{C}_\infty,c = \mathcal{C}_c,\infty$ is given in (1.17) below. We set $\tilde{\eta}^{(\infty,\infty)} = 0$. 

**Fig 2.** Phase diagram for the open ASEP with maximal current (MC), low density (LD), high density (HD) regions, and with shaded fan region $AC < 1$. Our parameters converge from within the shaded fan region to the triple point (1,1).
(iv) For \( a + c \in (0, \infty) \), the process \( \tilde{\eta}^{(a,c)} \) is defined as the process with finite-dimensional density at time points 0 = \( x_0 < x_1 < \cdots < x_d = 1 \) of the form

\[
\tilde{p}^{(a,c)}_{x_0, \ldots, x_d}(y_0, \ldots, y_d) := \frac{1}{\mathcal{C}_{a,c}} e^{-(c y_0 + a y_d)/\sqrt{2}} \prod_{k=1}^{d} q_{x_k}^*(y_{k-1}, y_k), \quad y_0, \ldots, y_d > 0, \tag{1.16}
\]

with the expression of \( \mathcal{C}_{a,c} \) in (1.17) below. (We note that for \( a + c \in (0, \infty) \), process \( \frac{1}{\sqrt{2}} \tilde{\eta}^{(a,c)} \) appeared in [4, Theorem 2.1].)

**Remark 1.3.** The normalization constant \( \mathcal{C}_{a,c} \) is

\[
\mathcal{C}_{a,c} = \begin{cases} 
\int_{(0,\infty)^2} e^{-\frac{a x + c y}{\sqrt{2}}} q_1(x, y) \, dx \, dy, & \text{if } a + c > 0, \ a, c \in \mathbb{R}, \\
\int_{(0,\infty)} \phi_1(y) e^{-x^2/2} \, dy, & \text{if } a \in \mathbb{R}, \ c = \infty,
\end{cases}
\]

which gives

\[
\mathcal{C}_{a,c} = \begin{cases} 
\sqrt{2} \frac{a H(a/2) - c H(c/2)}{a^2 - c^2}, & \text{if } a \neq c, a + c > 0, \\
\frac{2 + a^2}{2\sqrt{2} a} H(a/2) - \frac{1}{\sqrt{2} \pi}, & \text{if } a = c > 0, \\
\frac{1}{\sqrt{2} \pi} \frac{a H(a/2)}{2\sqrt{2}}, & \text{if } a \in \mathbb{R}, \ c = \infty, \\
\frac{1}{\sqrt{2} \pi} \frac{c H(c/2)}{2\sqrt{2}}, & \text{if } a = \infty, \ c \in \mathbb{R},
\end{cases} \tag{1.17}
\]

where for \( x \in \mathbb{R} \),

\[
H(x) = e^{x^2} \text{erfc}(x) \quad \text{with} \quad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt. \tag{1.18}
\]

The integrals for the normalization constant can be computed with symbolic software, see also Lemma A.2.

Since \( q_1(x, y) = q_1(y, x) \), from the form of the joint densities (1.13), (1.14), (1.15) and (1.16) we see that for all \( a + c > 0 \), including the cases when one or both parameters are \( \infty \), we have

\[
\left\{ \tilde{\eta}^{(a,c)} \right\}_{x \in [0,1]} \overset{d}{=} \left\{ \eta^{(a,c)}_{x} \right\}_{x \in [0,1]}.
\tag{1.19}
\]

We can now introduce the limit stochastic processes \( \eta^{(a,c)} = \left\{ \eta^{(a,c)}_{x} \right\}_{x \in [0,1]} \) that shall arise in the scaling limit of the stationary measure height function process \( h_n \) in addition to a Brownian motion component. For \( a + c > 0 \), we define the process \( \eta^{(a,c)} \) as

\[
\eta^{(a,c)}_{x} := \tilde{\eta}^{(a,c)} - \tilde{\eta}^{(c,a)}_{0}, \ x \in [0,1]. \tag{1.20}
\]

For \( a + c = 0 \), we define the process \( \eta^{(a,-a)} \), \( a \in \mathbb{R} \), as

\[
\eta^{(a,-a)}_{x} := \mathbb{B}_{x} - \frac{a}{\sqrt{2} \pi}, \ x \in [0,1]. \tag{1.21}
\]

From definition (1.21) for the case \( a + c = 0 \) and from definition (1.20) combined with (1.19) for the case \( a + c > 0 \), we get the following.

**Remark 1.4.** For all \( a + c \geq 0 \), we have

\[
\left\{ \eta^{(a,c)}_{x} \right\}_{x \in [0,1]} \overset{d}{=} \left\{ \eta^{(c,a)}_{1-x} - \eta^{(c,a)}_{1} \right\}_{x \in [0,1]}.
\tag{1.22}
\]
In Theorem 1.5, which is our main result, we establish convergence in $D[0, 1]$ for fluctuations of the height function of the open ASEP in the steady state to the processes predicted in [2], including a slightly broader class of limits under convergence of the finite-dimensional distributions.

**Theorem 1.5.** Under Assumption 1.1 and under the stationary distribution $\mu_n$ with $h_n$ as in (1.5), if $a, c$ are finite we have

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0, 1]} \xrightarrow{\text{f.d.d.}} \frac{1}{\sqrt{2}} \left\{B_x + \eta^{(a,c)}_x\right\}_{x \in [0, 1]}$$

as processes in the space $D[0, 1]$ of càdlàg functions with the Skorokhod metric and the limit process has continuous trajectories. Here $B$ is a standard Brownian motion, $\eta^{(a,c)}$ is introduced above, and the two processes are independent.

When $n$ is allowed as a value for $a$ and/or $c$, we still have convergence of the finite-dimensional distributions,

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0, 1]} \xrightarrow{\text{f.d.d.}} \frac{1}{\sqrt{2}} \left\{B_x + \eta^{(a,c)}_x\right\}_{x \in [0, 1]}$$

(1.23)

We note that cases $a = \infty$ or $c = \infty$ in (1.23) include Brownian excursion, Brownian meander, and its reversal as well as some slightly more general processes where one of the parameters is infinite while the other one takes arbitrary real values. Brownian excursion and Brownian meander appeared also in a related context for ASEP (but with fixed parameters) in [11] and in the non-rigorous discussion of the phase diagram and formula (3) in [2].

**Remark 1.6.** When $a + c \geq 0$ are finite, process $\tilde{X}$ from [2] has the same law as process $\frac{1}{\sqrt{2}} \eta^{(a,c)}$. Indeed, for $a + c = 0$, this holds because both of these processes are just the Brownian motion $\tilde{B} = B/\sqrt{2}$, with the same drift $-ax/2$. For $a + c > 0$, this can be seen by comparing formula (49) for the heat kernel, formula (53) for the joint probability function, and formula (57) for the normalizing constant from the supplementary material in [2] with the expressions (1.11), (1.16), (1.17) respectively. The Radon–Nikodym representation (1.2) for process $\frac{1}{\sqrt{2}} \eta^{(a,c)}$ is also discussed in [4].

**1.4. Organization of the paper**

In Section 2 we review Askey–Wilson processes, their relation to the matrix product ansatz for the open ASEP, the particle-hole duality, and a coupling technique which we use in Section 3 to prove Theorem 1.5 for $a + c = 0$.

In Section 4 we prove Theorem 1.5 in the case $a + c > 0$. The proof consists of two main steps. First, in Theorem 4.2 we compute the limit of the Laplace transform

$$\left\langle \exp \left( -\frac{1}{\sqrt{n}} \sum_{k=1}^{d} c_k h_n(x_k) \right) \right\rangle_n, \quad 0 < x_1 < \cdots < x_d = 1,$$

as $n \to \infty$. Second, in Section 4.2 we identify the limit as the Laplace transform of the process $(B + \eta^{(a,c)})/\sqrt{2}$. In the proof, we work under the additional assumption that $A_n \geq C_n$ (thus $a \leq c$), and the result for $A_n < C_n$ follows by the particle-hole duality.

At the heart of our computation of the limit Laplace transform, we rely on the matrix ansatz of Derrida, Evans, Hakim and Pasquier [20] and its representation via Askey–Wilson Markov processes, which are reviewed in Sections 2.1 and 2.2. Our approach is the same as the one used in recent developments on limit theorems for the height increment process of the open ASEP with fixed parameters [11] and for open weakly asymmetric simple exclusion process [14]. The analysis here is more involved than the one in [11], but much less so than the one in [14] where all the parameters depend on the size of the system.

**2. Preliminaries**

**2.1. Askey–Wilson processes**

The Askey–Wilson polynomials were introduced by Askey and Wilson [1]. The polynomials satisfy a three step recursion, so when Favard’s theorem applies the polynomials are orthogonal with respect to a probability measure on the real line, which we shall call the Askey–Wilson measure. The Askey–Wilson processes are a family of Markov processes based on these Askey–Wilson measures. The material of this section is mainly based on [11], which in turn is based mostly on [1, 6]. The Askey–Wilson probability measure $\nu(dy; a, b, c, d, q)$ depends on five parameters $a, b, c, d, q$ and it is invariant
The expression for $p$ and the corresponding probabilities are

\[
ac, ad, bc, bd, qac, qad, qbc, qbd, abcd, qabcd \notin [1, \infty).
\] (2.1)

Conditions that allow some of the above products to be in $[1, \infty)$ are listed in [6, Lemma 3.1]. In this paper we will encounter the degenerate Askey–Wilson measure corresponding to [6, Lemma 3.1(iii)] with $N = 0$.

In general, the Askey–Wilson measure is of mixed type

\[
\nu(dy; a, b, c, d, q) = f(y; a, b, c, d, q)dy + \sum_{z \in F(a, b, c, d, q)} p(z)\delta_z(dy),
\]

with the absolutely continuous part supported on $[-1, 1]$ and with the discrete part supported on a finite or empty set $F$. For certain choices of parameters, the measure can be only discrete or only absolutely continuous. The absolutely continuous part is

\[
f(y; a, b, c, d, q) = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty \sqrt{1 - y^2}} \left| \frac{(e^{2i\theta_y}; q)_\infty}{(ae^{i\theta_y}, be^{i\theta_y}, ce^{i\theta_y}, de^{i\theta_y}; q)_\infty} \right|^2,
\] (2.2)

where $y = \cos \theta_y$ (with the convention that $f(y; a, b, c, d, q) = 0$ when $|y| > 1$). Here and below, for complex $\alpha, \alpha_1, \ldots, \alpha_k$, $n \in \mathbb{N} \cup \{\infty\}$ and $|q| < 1$ we use the $q$-Pochhammer symbol

\[
(\alpha; q)_n = \prod_{j=0}^{n-1} (1 - \alpha q^j), \quad (\alpha_1, \ldots, \alpha_k; q)_n = \prod_{j=1}^{k} (\alpha_j; q)_n.
\]

It will be convenient to use the identity $(\alpha; q)_\infty = (1 - a)(qa; q)_\infty$ to separate the first factors in (2.2) from the remaining infinite products, so we write

\[
f(y; a, b, c, d, q) = \frac{2}{\pi} J(y; a, b, c, d) R(y; a, b, c, d, q)
\]

with

\[
J(y; a, b, c, d) = \frac{(1 - ab)(1 - ac)(1 - ad)(1 - bc)(1 - bd)(1 - cd)\sqrt{1 - y^2}}{(1 - abcd)(1 - ae^{i\theta_y})(1 - be^{i\theta_y})(1 - ce^{i\theta_y})(1 - de^{i\theta_y})^2},
\] (2.3)

\[
R(y; a, b, c, d, q) = \frac{(q, qab, qac, qad, qbc, qbd, qcd; q)_\infty}{(qabcd; q)_\infty} \left| \frac{(qe^{2i\theta_y}; q)_\infty}{(qae^{i\theta_y}, qbe^{i\theta_y}, qce^{i\theta_y}, qde^{i\theta_y}; q)_\infty} \right|^2.
\] (2.4)

The set $F = F(a, b, c, d, q)$ of atoms of $\nu(dy; a, b, c, d, q)$ is non-empty if there is a real parameter $\alpha \in \{a, b, c, d\}$ with $|\alpha| > 1$. Each such parameter generates atoms. For example, if $|\alpha| > 1$ then it generates the atoms

\[
y_j = \frac{1}{2} \left( \alpha q^j + \frac{1}{\alpha q^j} \right) \text{ for } j = 0, 1, \ldots \text{ such that } |\alpha q^j| \geq 1,
\]

and the corresponding probabilities are

\[
p(y_0; a, b, c, d, q) = \frac{(a^{-2}, bc, bd, cd; q)_\infty}{(b/a, c/a, d/a, abcd; q)_\infty},
\]

\[
p(y_j; a, b, c, d, q) = p(y_0; a, b, c, d, q) \left( \frac{a^2, ab, ac, ad; q)_j}{(q, qa/b, qa/c, qa/d; q)_j} \right) \left( \frac{q}{abcd} \right)^j, j \geq 1.
\]

The expression for $p(y_j; a, b, c, d, q)$ given here only applies for $a, b, c, d \neq 0$, and takes a different form otherwise. We shall however only need $p(y_0; a, b, c, d, q)$ in this paper (with $a = 0$ defined as the limit $a \to 0$).
The Askey–Wilson process is a time-inhomogeneous Markov process introduced in [6], based on the Askey–Wilson measures. It is then explained in [7] how each ASEP with parameters $\alpha, \beta > 0, \gamma, \delta \geq 0, q \in [0, 1)$ is associated to an Askey–Wilson process $Y$, the parameters of which are denoted by $A, B, C, D, q$, with $A, B, C, D$ given in (1.6).

As we already noted, (1.6) implies $A, C \geq 0$ and $-1 < B,D \leq 0$. Throughout the paper we shall assume $AC < 1$, which ensures that condition (2.1) holds for the marginal laws (but not for the transition probabilities). Then, the Askey–Wilson process with parameters $(A, B, C, D, q)$ is introduced as the Markov process with marginal distribution

$$
P(Y_t \in dy) = \nu \left(dy; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}, q\right), \quad 0 < t < \infty,
$$

and the transition probabilities

$$
P(Y_t \in dz \mid Y_s = y) = \nu \left(dz; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}(y + \sqrt{y^2 - 1}), \sqrt{s/t}(y - \sqrt{y^2 - 1})\right),
$$

for $0 < s < t, y, z > 0$. When $|y| < 1$, expression $y \pm \sqrt{y^2 - 1}$ is understood as $e^{\pm i\theta_y}$ with $\cos \theta_y = y$. It was shown in [6] that the above marginal laws and transition probabilities satisfy the Chapman-Kolmogorov equations and hence determine a Markov process indexed by $t \in [0, \infty)$.

More explicit expressions for the law of $Y$ will appear below when needed in the proofs.

### 2.2. Matrix ansatz for open ASEP and Askey–Wilson processes

Recall that $\langle \cdot \rangle_n$ denotes the expectation with respect to the invariant measure $\mu_n$ of the open ASEP. [20] introduced the celebrated matrix product ansatz that provides an explicit expression for the joint generating function. Formally, for any $t_1, \ldots, t_n > 0$, from [20] one can write

$$
\left\langle \prod_{j=1}^{n} t_j^{y_j} \right\rangle_n = \frac{\langle W | (E + t_1 D) \times \cdots \times (E + t_n D) | V \rangle}{\langle W | (E + D)^n | V \rangle},
$$

for a pair of infinite matrices $D, E$, a row vector $\langle W \rangle$ and a column vector $| V \rangle$, satisfying

$$
DE - qED = D + E,
$$

$$
\langle W | (\alpha E - \gamma D) \rangle = \langle W \rangle,
$$

$$
\langle \beta D - \delta E | V \rangle = | V \rangle.
$$

See [18, 19] for reviews of the literature. See also [8], in particular Appendix C there, for a discussion of the case where the matrix approach fails. However, for our purpose, we shall apply an alternative expression developed recently in [7, Theorem 1], summarized in the following theorem.

**Theorem 2.1.** Consider the parametrization $A, B, C, D$ in (1.6) for an open ASEP with parameters $\alpha, \beta > 0, \gamma, \delta \geq 0$ and $q \in [0, 1)$. Suppose that $AC < 1$.

If $0 < t_1 \leq t_2 \leq \cdots \leq t_n$, then the joint generating function of the stationary distribution $\mu_n$ of the ASEP is

$$
\left\langle \prod_{j=1}^{n} t_j^{y_j} \right\rangle_n = \frac{\mathbb{E} \left[ \prod_{j=1}^{n} (1 + t_j + 2\sqrt{t_j} Y_{t_j}) \right]}{2^n \mathbb{E}(1 + Y_1)^n},
$$

where $\{Y_t\}_{t \geq 0}$ is the Askey–Wilson process with parameters $(A, B, C, D, q)$.

### 2.3. Particle-hole duality

The asymptotics for the height function in the low density phase $A_n \leq C_n$ is an immediate consequence of the asymptotics for the high density phase $A_n \geq C_n$, by the particle-hole duality which we now explain.

Consider an open ASEP of size $n$ with parameters $(\alpha, \beta, \gamma, \delta, q)$ and stationary law that is convenient here to denote by $\mu_n^{\alpha,\beta,\gamma,\delta}$. (Since $q$ is fixed throughout the paper, we suppress the dependence on $q$.) Instead of thinking of particles jumping around, we can view the particles as background and allow the holes to jump around. In this way, equivalently a hole jumps to the unoccupied left and right sites with rates 1 and $q$, respectively, and disappears at site 1 with rate $\alpha$ and at site $n$ with rate $\delta$, and enters site $n$ if unoccupied with rate $\beta$ and site 1 if unoccupied with rate $\gamma$. The holes...
form the open ASEP with parameters \((\alpha, \beta, \gamma, \delta, q)\), if we relabel the sites \(\{1, \ldots, n\}\) as \(\{n, \ldots, 1\}\) by \(j \mapsto N - j + 1\). Consequently, the particles occupations \(\tau_1, \ldots, \tau_n\), represented by the black disks in Fig. 1, are related to the holes occupations \(\varepsilon_1, \ldots, \varepsilon_n\), represented by the white disks in Fig. 1, by

\[
\varepsilon_j = 1 - \tau_{n-j+1}, \quad 1 \leq j \leq n, \tag{2.8}
\]

and the stationary law of \((\varepsilon_1, \ldots, \varepsilon_n)\) is \(\mu_n^{\beta, \alpha, \delta, \gamma}\). Introduce

\[
\hat{\mu}_n(x) = \sum_{j=1}^{\lfloor nx \rfloor} (2\varepsilon_j - 1).
\]

The above argument shows that \(\{\hat{\mu}_n(x)\}_{x \in [0,1]}\) with respect to \(\mu_n^{\beta, \alpha, \delta, \gamma}\) has the same law as \(\{h_n(x)\}_{x \in [0,1]}\) (defined in (1.5)) with respect to \(\mu_n^{\alpha, \beta, \gamma, \delta}\). This allows us to switch the roles of the pairs of parameters \((A, B)\) and \((C, D)\) in (1.6) which will simplify the proof of Theorem 1.5.

**Proposition 2.2.** If Theorem 1.5 holds under an additional assumption that \(A_n \geq C_n\) for all \(n\), then it holds also without this additional assumption.

**Proof.** Suppose that a sequence \((A_n, B_n, C_n, D_n)\) satisfies Assumption 1.1. Consider sets of indexes \(\mathbb{N}_+ = \{n \in \mathbb{N} : A_n \geq C_n\}\) and \(\mathbb{N}_- = \mathbb{N} \setminus \mathbb{N}_+\), at least one of which must be infinite.

If \(\mathbb{N}_+\) is infinite, we can extend \((A_n, B_n, C_n, D_n)_{n \in \mathbb{N}_+}\) to a sequence \((\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n)_{n \in \mathbb{N}}\) such that \(\tilde{A}_n \geq \tilde{C}_n\) for all \(n\). (For example, if \(\mathbb{N}_+ = \{n_1, n_2, \ldots\}\), with \(n_0 = 0\) we can take \(\tilde{A}_n = A_{n_k}\) for \(n \in [n_k - 1, n_k]\).) So our assumption implies that the limit (1.23) holds over the sub-sequence \(\mathbb{N}_+\).

It remains to prove that if \(\mathbb{N}_-\) is an infinite set, then we have convergence in (1.23) over \(\mathbb{N}_-\) to the same limit. With some abuse of notation, let us re-parameterize the stationary measure of the open ASEP by the parameters \(A_n, B_n, C_n, D_n\), writing \(\mu_n^{\alpha_n, \beta_n, C_n, D_n}\) instead of \(\mu_n^{\alpha, \beta, \gamma, \delta}\). The stationary measure for the hole occupations is then \(\mu_n^{\beta_n, \alpha_n, \delta_n, \gamma_n}\), which, using (1.6), we write as \(\mu_n^{C_n, D_n, A_n, B_n}\). For \(n \in \mathbb{N}_-\), we have \(C_n > A_n\), so our assumption implies that Theorem 1.5 holds for \(\hat{h}_n\), with the limit taken over \(\mathbb{N}_-\). We get

\[
\frac{1}{\sqrt{n}} \left\{ \hat{h}_n(x) \right\}_{x \in [0,1]} \xrightarrow{fdd} \sqrt{2} \left\{ \mathbb{B}_x + \eta_2^{(a,c)} \right\}_{x \in [0,1]}.
\]

Observe that (2.8) gives

\[
h_n(x) - \left( \hat{h}_n(1-x) - \hat{h}_n(1) \right) = (1 - 2\tau_{\lfloor nx \rfloor}) \mathbb{1}_{nx \in [nx]}.
\]

Since the difference (2.9) is uniformly bounded, the finite-dimensional distributions of \(n^{-1/2}\{h_n(x)\}_{x \in [0,1]}\) have the same limit as the finite-dimensional distributions of \(n^{-1/2}\{\hat{h}_n(1-x) - \hat{h}_n(1)\}_{x \in [0,1]}\), and we arrive at

\[
\frac{1}{\sqrt{n}} \left\{ h_n(x) \right\}_{x \in [0,1]} \xrightarrow{fdd} \sqrt{2} \left\{ \mathbb{B}_{x} - \mathbb{B}_1 + \eta_1^{(e,a)} - \eta_1^{(e,a)} \right\}_{x \in [0,1]} = \frac{1}{\sqrt{2}} \left\{ \mathbb{B}_x + \eta_2^{(a,c)} \right\}_{x \in [0,1]},
\]

where in the last equality we used \(\{\mathbb{B}_{1-x} - \mathbb{B}_1\}_{x \in [0,1]} = \{\mathbb{B}_x\}_{x \in [0,1]}\) and (1.22) from Remark 1.4. Since \(\mathbb{N} = \mathbb{N}_+ \cup \mathbb{N}_-\), this ends the proof.

### 2.4. Coupling and tightness in \(D[0,1]\)

As in [14], we will deduce tightness by coupling a realization of the simple exclusion processes in the steady state with two sequences of \(\{0, 1\}\) valued random variables that have products of Bernoulli measures as the marginal laws.

As in Section 2.3, we denote by \(\mu_n^{A_n, B_n, C_n, D_n}\) the stationary distribution of the ASEP on \(\{1, \ldots, n\}\) with parameters \((q, A_n, B_n, C_n, D_n)\), where \(q\) remains fixed.

Since we are interested only in the case \(A_n \to 1\) and \(C_n \to 1\), without loss of generality we assume that parameters \(A_n, C_n\) are not zero for all \(n\).

Let \((\tau_1, \ldots, \tau_n)\) be a vector with the stationary law \(\mu_n^{A_n, B_n, C_n, D_n}\). Define \(\tilde{A}_n = 1/C_n\) and \(\tilde{C}_n = 1/A_n\). Let \((\tilde{\tau}_1, \ldots, \tilde{\tau}_n)\) be a vector with the stationary law \(\mu_n^{A_n, B_n, \tilde{C}_n, D_n}\) (just one parameter changed), and let \((\hat{\tau}_1, \ldots, \hat{\tau}_n)\) be a vector with the stationary law \(\mu_n^{A_n, \tilde{B}_n, C_n, D_n}\).

From [14, Lemma 5.1] we then deduce the following result. (A similar coupling for a pair of ASEP\(s\) is constructed in [22, Lemma 2.1].)
Proposition 2.3. (i) Random variables \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are independent Bernoulli random variables with \( \mathbb{P}(\tilde{\tau}_j = 1) = \frac{1}{1+\tilde{A}_n} \), and random variables \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are independent Bernoulli random variables with \( \mathbb{P}(\tilde{\tau}_j = 1) = \frac{A_n}{1+\tilde{A}_n} \).

(ii) The three vectors can be defined together on a single probability space in such a way that

\[
\tilde{\tau}_j \leq \rho_j \leq \tilde{\tau}_j, \quad j = 1, 2, \ldots, n. \tag{2.10}
\]

Proof. Since \( \tilde{A}_n C_n = 1 \) it is clear that random variables \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are Bernoulli with \( \mathbb{P}(\tilde{\tau}_j = 1) = \frac{\tilde{A}_n}{1+\tilde{A}_n} = \frac{1}{1+C_n} \), see [11, Remark 2.4]. Similarly, random variables \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are Bernoulli with \( \mathbb{P}(\tilde{\tau}_j = 1) = \frac{A_n}{1+\tilde{A}_n} \). This proves statement (i).

To construct the coupling of the three vectors, recall that ASEP parameters for \( \tau_1, \ldots, \tau_n \) are given by the inverse of formula (1.6), which gives

\[
\begin{align*}
\alpha_n &= \frac{1-q}{(1+C_n)(1+D_n)}, & \beta_n &= \frac{1-q}{(1+A_n)(1+B_n)}, \\
\gamma_n &= \frac{-(1-q)C_n D_n}{(1+C_n)(1+D_n)}, & \delta_n &= \frac{-(1-q)A_n B_n}{(1+A_n)(1+B_n)}. 
\end{align*}
\]

Recall that under our assumption we have \( A_n, C_n > 0, 1+B_n, 1+D_n > 0 \), and \( -B_n, -D_n \geq 0 \) and also \( 1-q > 0 \). Since \( \tilde{A}_n = 1/C_n > A_n \), therefore, the ASEP parameters for \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are \( \tilde{\alpha}_n = \alpha_n, \tilde{\beta}_n \leq \beta_n, \tilde{\gamma}_n = \gamma_n, \tilde{\delta}_n \geq \delta_n \). Similarly, since \( \tilde{C}_n > C_n \) the ASEP parameters for \( \tilde{\tau}_1, \ldots, \tilde{\tau}_n \) are \( \tilde{\alpha}_n \leq \alpha_n, \tilde{\beta}_n = \beta_n, \tilde{\gamma}_n \geq \gamma_n, \tilde{\delta}_n = \delta_n \). Invoking [14, Lemma 5.1], we get (2.10).

For reader's convenience we describe the construction from [14, Lemma 5.1], adapted to our setting and notation.

Sketch of proof of Lemma 5.1 in [14] specialized to the case (2.10). Consider a three-species ASEP, with three types of particles: the high-priority red, mid-priority blue, low-priority gray particles and with the empty slots that can be interpreted as an additional type of particle with the lowest priority. At any time \( t \geq 0 \), a site can be occupied by at most one particle. The time evolution of the three-species ASEP may start from any initial state, which for concreteness we take to be an empty state and is a Markov process with the \( 4^n \)-element state space \( \{\text{red, blue, gray, empty}\}^n \).

We first define the transition rates "away from the boundaries". (This is of course a somewhat informal description, see [22, Lemma 2.1] for a more precise approach.)

1. A red particle may jump to the right from site \( j = 1, \ldots, n-1 \) or to the left at rate \( q \) from site \( j = 2, \ldots, n \), provided that the target site does not have a red particle. When a red particle moves to an empty site or a site occupied by a blue, or gray particle, the particles swap their locations.

2. A blue particle may jump to the right at rate 1 and to the left at rate \( q \), provided that the target site does not have a red or a blue particle. When a blue particle moves to the empty site or a site occupied by a gray particle, the particles swap the locations.

3. A gray particle may jump to the right at rate 1 or to the left at rate \( q \), provided that the target site is empty.

The evolution at the end points is described by the "mutation rates", where a particle, including an empty site, can "mutate" into one of the other species. The mutation rates are specified in the two tables, one for each of the end-point locations. The tables list the transition (mutation) rates from a color specified in the left column into one of the colors listed in the first row. For example, at the left endpoint, transitions gray \( \rightarrow \) blue occur at rate \( \alpha_n - \tilde{\alpha}_n \).

| Left endpoint transition rates: | Right endpoint transition rates: |
|-------------------------------|-------------------------------|
|                             | empty | gray | blue | red | empty | gray | blue | red |
| empty                        | 0     | 0    | \( \alpha_n - \tilde{\alpha}_n \) | \( \alpha_n \) | empty | 0    | \( \delta_n - \tilde{\delta}_n \) | 0    | \( \delta_n \) |
| gray                         | \( \gamma_n \) | 0    | \( \alpha_n - \tilde{\alpha}_n \) | \( \tilde{\alpha}_n \) | gray | \( \tilde{\beta}_n \) | 0    | 0    | \( \delta_n \) |
| blue                         | \( \gamma_n \) | 0    | 0    | \( \tilde{\alpha}_n \) | blue | \( \tilde{\beta}_n \) | \( \beta_n - \tilde{\beta}_n \) | 0    | \( \delta_n \) |
| red                          | \( \gamma_n \) | 0    | \( \tilde{\gamma}_n - \gamma_n \) | 0    | red | \( \tilde{\beta}_n \) | \( \beta_n - \tilde{\beta}_n \) | 0    | 0    |

The above multi-species ASEP evolution defines the following three vectors in \( \{0, 1\}^n \):
From the proof of tightness in the Donsker’s theorem we know that for any centering as in (1.4):

- \( \bar{\tau}_j(t) = 1 \) when at time \( t \) the \( j \)-th location is occupied by a red particle
- \( \bar{\tau}_j(t) = 1 \) when at time \( t \) the \( j \)-th location is occupied either by a red or a blue particle
- \( \bar{\tau}_j(t) = 1 \) when at time \( t \) the \( j \)-th location is non-empty.

It is clear that

\[ \bar{\tau}_j(t) \leq \tau_j(t) \leq \hat{\tau}_j(t), \quad j = 1, 2, \ldots, n. \]

It is also clear that the evolution of vector \( (\tau_1(t), \ldots, \tau_n(t)) \) alone (marginal law) is an ASEP with parameters \( (q, A_n, B_n, C_n, D_n) \) and similarly, the other two vectors are the ASEP with parameters \( (q, A_n, B_n, \tilde{C}_n, D_n) \) and \( (q, A_n, B_n, C_n, D_n) \) respectively. To conclude the construction, we define the joint law of \( (\tau_1, \ldots, \tau_n, \tilde{\tau}_1, \ldots, \tilde{\tau}_n, \hat{\tau}_1, \ldots, \hat{\tau}_n) \) as the stationary law of the Markov process, i.e., as the weak limit

\[ (\tau_1(t), \ldots, \tau_n(t), \tilde{\tau}_1(t), \ldots, \tilde{\tau}_n(t), \hat{\tau}_1(t), \ldots, \hat{\tau}_n(t)) \Rightarrow (\tau_1, \ldots, \tau_n, \tilde{\tau}_1, \ldots, \tilde{\tau}_n, \hat{\tau}_1, \ldots, \hat{\tau}_n) \]

as \( t \to \infty \). The limit defines the appropriate joint law such that (2.10) holds, and the marginal laws are the stationary laws of the corresponding ASEP. (The first ASEP consists of only red particles. The second ASEP consists of red or blue particles. The third ASEP is color-blind.)

A useful technical consequence of Proposition 2.3 is tightness.

**Proposition 2.4.** If Assumption 1.1 holds with finite \( a, c \), then the laws of

\[
\frac{1}{\sqrt{n}} \{ h_n(x) \}_{x \in [0,1]} \]

are tight in \( D[0,1] \) and their subsequential limits have continuous trajectories.

**Proof.** In the context of Proposition 2.3, in addition to \( h_n \) defined in (1.5), consider the height function processes for the two new ASEP introduced there. It will be convenient to consider the “high density centering” and the "low density" centering as in [11, (1.4)):

\[
\tilde{h}_n^H(x) := \sum_{j=1}^{[nx]} \left( 2\bar{\tau}_j - \frac{2A_n}{1 + A_n} \right), \quad x \in [0,1].
\]

\[
\hat{h}_n^H(x) := \sum_{j=1}^{[nx]} \left( 2\hat{\tau}_j - \frac{2}{1 + C_n} \right), \quad x \in [0,1].
\]

Also introduce two deterministic functions:

\[
\tilde{\varepsilon}_n(x) := [nx] \frac{A_n - 1}{1 + A_n}, \quad \text{and} \quad \hat{\varepsilon}_n(x) := [nx] \frac{1 - C_n}{1 + C_n}.
\]

Denoting \( \Delta_{x_0,x_1}(f) := f(x_1) - f(x_0) \), from Proposition 2.3 we get

\[
\Delta_{x_0,x_1}(\tilde{h}_n^H) + \Delta_{x_0,x_1}(\tilde{\varepsilon}_n) \leq \Delta_{x_0,x_1}(h_n) \leq \Delta_{x_0,x_1}(\hat{h}_n^H) + \Delta_{x_0,x_1}(\hat{\varepsilon}_n), \quad 0 \leq x_0 < x_1 \leq 1.
\]

Since \( \Delta_{x_0,x_1}(f) = -\Delta_{x_1,x_0}(f) \), (2.14) implies that for any \( x_0, x_1 \in [0,1] \) we have

\[
|\Delta_{x_0,x_1}(h_n)| \leq |\Delta_{x_0,x_1}(\tilde{h}_n^H)| + |\Delta_{x_0,x_1}(\hat{h}_n^H)| + |\Delta_{x_0,x_1}(\tilde{\varepsilon}_n)| + |\Delta_{x_0,x_1}(\hat{\varepsilon}_n)|.
\]

From the proof of tightness in the Donsker’s theorem we know that for any \( \varepsilon, \eta > 0 \) there exists \( \delta \in (0,1) \) and \( n_0 \) such that

\[
P \left( \sup_{|x_1 - x_0| < \delta} \left\{ |\Delta_{x_0,x_1}(\tilde{h}_n^H)| + |\Delta_{x_0,x_1}(\hat{h}_n^H)| \right\} \geq \varepsilon \sqrt{n} \right) \leq \eta, \quad n \geq n_0.
\]
Furthermore, uniformly in $x \in [0, 1]$, we have

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \tilde{\varepsilon}_n(x) = \lim_{n \to \infty} \frac{|nx|}{n} \sqrt{n(1 - C_n)} = x \frac{a}{2} \quad (2.16)$$

and

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \tilde{\varepsilon}_n(x) = \lim_{n \to \infty} \frac{|nx|}{n} \sqrt{n(A_n - 1)} = -x \frac{a}{2}, \quad (2.17)$$

so $\sup_{|x_0 - x_1| < \delta} (|\Delta_{x_1, x_1}(\tilde{\varepsilon}_n)| + |\Delta_{x, x_1}(\tilde{\varepsilon}_n)|) \leq \delta(|a| + |c|) \sqrt{n}$ for large $n$. Therefore, (2.15) implies that for any $\varepsilon, \eta > 0$ there exists $\delta \in (0, 1)$ and $n_0$ such that

$$P \left( \sup_{|x_1 - x_0| < \delta} |\Delta_{x_0, x_1}(h_n)| \geq \varepsilon \sqrt{n} \right) \leq \eta, \quad n \geq n_0.$$ 

Since $h_n(0) = 0$, this shows, see [3, Theorem 15.5], that the sequence of processes $\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0, 1]}$ is tight in $D[0, 1]$ and the laws of subsequential weak limits are supported on $C[0, 1]$. \hfill \Box

3. Proof of Theorem 1.5 for $a + c = 0$

The case of Theorem 1.5 for $a + c = 0$ is a quick application of coupling from Proposition 2.3 combined with tightness from Proposition 2.4.

**Proof of Theorem 1.5 for $a + c = 0$.** Recalling (2.11), (2.12) and (2.13), consider

$$\tilde{h}_n(x) := \sum_{j=1}^{[nx]} (2\tilde{r}_j - 1) = \tilde{h}_n^h(x) + \tilde{\varepsilon}_n(x), \quad x \in [0, 1],$$

and

$$\hat{h}_n(x) := \sum_{j=1}^{[nx]} (2\hat{r}_j - 1) = \hat{h}_n^w(x) + \hat{\varepsilon}_n(x), \quad x \in [0, 1].$$

Since $a + c = 0$, from Donsker’s theorem and (2.16), (2.17) we see that both processes have the same limit

$$\frac{1}{\sqrt{n}} \tilde{h}_n(x) \xrightarrow{f.d.d.} \left\{ R_x - \frac{a}{2} x \right\}_{x \in [0, 1]}, \quad \frac{1}{\sqrt{n}} \hat{h}_n(x) \xrightarrow{f.d.d.} \left\{ R_x - \frac{a}{2} x \right\}_{x \in [0, 1]} \quad (3.1)$$

(In fact, convergence is in $D[0, 1]$.) From Proposition 2.3 it is clear that for $0 \leq x \leq 1$ we have

$$\tilde{h}_n(x) \leq h_n(x) \leq \hat{h}_n(x). \quad (3.2)$$

In view of (3.1), we see that

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0, 1]} \xrightarrow{f.d.d.} \left\{ R_x - \frac{a}{2} x \right\}_{x \in [0, 1]}$$

and since $a, c$ are finite, by Proposition 2.4 convergence is in $D[0, 1]$.

\hfill \Box

4. Proof of Theorem 1.5 for $a + c > 0$

In view of Proposition 2.2, it is enough to prove Theorem 1.5, under an additional assumption $A_n \geq C_n$, which implies $a \leq c$. Note that this excludes the case $a = \infty, c \in \mathbb{R}$ but includes the cases $a = c = \infty$ and $a \in \mathbb{R}, c = \infty$. For convenience, we restate Assumption 1.1 with these additional constraints on $a, c$ stated explicitly.
Assumption 4.1. We assume that \( A_n, C_n \geq 0, A_n C_n < 1, A_n \geq C_n, B_n, D_n \in (-1,0) \) for all \( n \geq 1 \). Moreover, we assume that

\[
\lim_{n \to \infty} C_n = 1 \quad \text{and} \quad \lim_{n \to \infty} \sqrt{n} (1 - C_n) = c \in (0, \infty),
\]

and

\[
\lim_{n \to \infty} A_n = 1 \quad \text{and} \quad \lim_{n \to \infty} \sqrt{n} (1 - A_n) = a \in (-c, c),
\]

(so \( a + c > 0 \)). We also assume that (1.9) and (1.10) hold.

For \( d \in \mathbb{N} \), and \( x = (x_1, \ldots, x_d) \) with \( x_0 := 0 < x_1 < \cdots < x_d = 1 \), we introduce the Laplace transform \( \varphi_{x,n} \) by the formula

\[
\varphi_{x,n}(c) := \left( \exp \left( -\sum_{k=1}^{d} c_k h_n(x_k) \right) \right)_n, \quad c = (c_1, \ldots, c_d) \in \mathbb{R}^d.
\]

Below we shall compute the limit of \( \varphi_{x,n}(c/\sqrt{n}) \) as \( n \to \infty \). By [11, Appendix], this shall determine the convergence of the finite-dimensional distributions, even though the coefficients \( c = (c_1, \ldots, c_d) \) will be restricted to an open subset of \( \mathbb{R}^d \) determined by inequalities (4.3).

The limit is expressed in terms of the process, called the 1/2-stable Biane process in [9], which can also be defined as a square of the radial part of a 3-dimensional Cauchy process [24, Corollary 1]. This is a time-homogeneous Markov process taking values in \((0, \infty)\) with transition probability density function

\[
p_t(u, v) = \frac{2t^{\sqrt{v}}}{\pi [t^4 + 2t^2(u + v) + (u - v)^2]} 1_{\{u, v > 0\}}, \quad t > 0.
\]

Theorem 4.2. Under Assumption 4.1, for all \( c = (c_1, \ldots, c_d) \in \mathbb{R}^d \) such that

\[
c_1, \ldots, c_{d-1} > 0 \quad \text{and} \quad -a < c_d < c_1 + \cdots + c_d < c,
\]

we have

\[
\lim_{n \to \infty} \varphi_{x,n} \left( \frac{c}{\sqrt{n}} \right) = \mathbb{E} \exp \left( -\sum_{k=1}^{d} \frac{c_k}{\sqrt{2}} \mathbb{E}_{x_k} \right) \cdot \Psi_{x}^{(a,c)}(c),
\]

with \( \Psi_{x}^{(a,c)} \) which has three different forms as follows:

\[
\Psi_{x}^{(a,c)}(c) = \begin{cases} 
\frac{\sqrt{2}}{\pi \mathfrak{c}_{a,c}} \int_{\mathbb{R}^+} \frac{\sqrt{u_1}}{(c - s_1)^2 + u_1((a + c_d)^2 + u_d)} L_{x,c}(u) \, du & a, c \in \mathbb{R}, \\
\frac{1}{2\sqrt{2} \pi \mathfrak{c}_{a,c}} \int_{\mathbb{R}^+} \frac{\sqrt{u_1}}{(a + c_d)^2 + u_d} L_{x,c}(u) \, du & a \in \mathbb{R}, c = \infty, \\
\frac{1}{4\sqrt{\pi}} \int_{\mathbb{R}^+} \sqrt{u_1} L_{x,c}(u) \, du & a = \infty, c = \infty,
\end{cases}
\]

where \( s_1 = c_1 + \cdots + c_d \), constant \( \mathfrak{c}_{a,c} \) is defined in (1.17), and

\[
L_{x,c}(u) = \exp \left( -\frac{1}{4} \sum_{k=1}^{d} (x_k - x_{k-1}) u_k \right) \prod_{k=1}^{d-1} \mathbb{P}_{c_k}(u_k, u_{k+1}).
\]

The proof of Theorem 4.2 consists of several steps and is presented in Section 4.1.

The second step in the proof of Theorem 1.5 for \( a + c > 0 \) is to express \( \Psi_{x}^{(a,c)} \) as the Laplace transform of a stochastic process. We will show the following.
Proposition 4.3. Fix \( a \leq c \). For \( d \in \mathbb{N}, x = (x_1, \ldots, x_d) \) with \( x_0 := 0 < x_1 < \cdots < x_d = 1 \), and \( c = (c_1, \ldots, c_d) \in \mathbb{R}^d \) such that (4.3) holds, we have

\[
\Psi_{x}^{(a, c)}(c) = \mathbb{E}\exp\left(-\frac{1}{\sqrt{2}} \sum_{k=1}^{d} c_k \eta_{x_k}^{(a, c)}\right),
\]

where \( \eta_{x_k}^{(a, c)} \) is defined in Section 1.3.

We shall deduce Proposition 4.3 from a more general Proposition 4.10 in Section 4.2. Assuming these two results, we can now prove Theorem 1.5 for \( a + c > 0 \).

Proof of Theorem 1.5 for \( a + c > 0 \). By Proposition 2.2 we can replace Assumption 1.1 by Assumption 4.1 and apply Theorem 4.2.

By Theorem 4.2, the Laplace transform (4.1) converges to the product of two Laplace transforms. In view of Proposition 4.3, we recognize the Laplace transform of the desired limit. To conclude the proof, we invoke the fact that convergence of Laplace transforms of probability measures to a Laplace transform of a probability measure on an open set of arguments \( c \) implies weak convergence of measures, [11, Theorem A.1]. This proves convergence of finite dimensional distributions.

When \( a, c \) are finite, convergence in \( D[0, 1] \) follows from Proposition 2.4.

It remains to prove Theorem 4.2 and Proposition 4.3.

4.1. Proof of Theorem 4.2

We start by expressing the Laplace transform

\[
\varphi_{x,n}(c) = \exp\left(-\sum_{k=1}^{d} c_k \sum_{j=\lfloor nx_k \rfloor+1}^{\lfloor nx_k \rfloor} (2\tau_j - 1) (c_k + \cdots + c_d)\right),
\]

in terms of the Askey–Wilson process \( (Y_t^{(n)})_{t \geq 0} \) with parameters \( A_n, B_n, C_n, D_n, q \), which has marginal laws (2.5) and transition probabilities (2.6). Note that the range of the legitimate time index of the Askey–Wilson process \( Y^{(n)} \) depends on the parameters, and under Assumption 4.1, the domain \([0, \infty)\) follows from [6, Eq. (1.21)].

Write

\[
s_k := c_k + \cdots + c_d, \quad k = 1, \ldots, d, \quad \text{and} \quad n_k := \lfloor nx_k \rfloor, \quad k = 0, 1, \ldots, d.
\]

(Recall that \( 0 = x_0 < x_1 < x_2 < \cdots < x_d = 1 \).

By (2.7) we have,

\[
\varphi_{x,n}(c) = \exp\left(\sum_{k=1}^{d} s_k (n_k - n_{k-1})\right) \prod_{k=1}^{d} \prod_{j=n_{k-1}+1}^{n_k} (e^{-2s_k \tau_j}) \mathbb{E} \left[ \prod_{k=1}^{d} (1 + e^{-2s_k} + 2e^{-s_k} Y^{(n)}_{e^{-2s_k} x_k})^{n_k-n_{k-1}} \right] / 2^n \mathbb{E}(1 + Y^{(n)}_1)^n
\]

Then,

\[
\varphi_{x,n}(c) = \mathbb{E} \left[ \prod_{k=1}^{d} \left( \cosh(\sqrt{n} s_k) + Y^{(n)}_{e^{-2s_k} \sqrt{n}} \right)^{n_k-n_{k-1}} \right] / \mathbb{E}(1 + Y^{(n)}_1)^n
\]

We shall consider separately the numerator and the denominator.
4.1.1. Asymptotics of the numerator in (4.7)

To analyze the numerator we first denote

\[ t_{k,n} = e^{-2sk / \sqrt{n}}, \quad k = 1, \ldots, d. \]

It follows from (4.3) that for \( n \) large enough,

\[ A_n \sqrt{t_{k,n}} < 1, \quad B_n \sqrt{t_{k,n}} < 1, \quad C_n \sqrt{t_{k,n}} < 1, \]

and hence they do not create atoms for the marginal law (2.5) at time \( t = t_{k,n} \).

However \( D_n \) can create atoms, or more precisely \( -D_n / \sqrt{t_{k,n}} > 1 \) is possible if \( D = -1 \).

In this case, the atom of the marginal law of \( Y_{t_{k,n}}^{(n)} \) is created by \( D_n \) with value

\[ \frac{1}{2} \left( D_n / \sqrt{t_{k,n}} + \frac{1}{D_n / \sqrt{t_{k,n}}} \right) = \frac{(1 + D_n / \sqrt{t_{k,n}})^2}{2D_n / \sqrt{t_{k,n}}} - 1 < -1. \]

Consequently, since \( D_n / \sqrt{t_{k,n}} \rightarrow -1 \) when \( D = -1 \), for a fixed \( \delta \in (0, 1) \) and large enough \( n \), we have

\[ Y_{t_{k,n}}^{(n)} \in (-1 - \delta, 1]. \] (4.9)

In view of the first two inequalities in (4.8), we also do not have atoms in the transition probabilities (2.6) from \( s = t_{k,n} \) to \( t = t_{k+1,n} \), when starting at \( y \in [-1, 1] \).

Using (4.9), for large enough \( n \) we write

\[ \mathbb{E} \left[ \prod_{k=1}^{d} \left( \cosh(s_k / \sqrt{n}) + Y_{t_{k,n}}^{(n)} e^{-2sk / \sqrt{n}} \right)^{n_k - n_{k-1}} \right] = I_{n,1} + I_{n,2} \] (4.10)

with

\[ I_{n,1} := \mathbb{E} \left[ 1_{Y_{t_{1,n}}^{(n)} \in (-1 - \delta, 0)} \prod_{k=1}^{d} \left( \cosh(s_k / \sqrt{n}) + Y_{t_{k,n}}^{(n)} e^{-2sk / \sqrt{n}} \right)^{n_k - n_{k-1}} \right], \]

\[ I_{n,2} := \mathbb{E} \left[ 1_{Y_{t_{1,n}}^{(n)} \in [0,1]} \prod_{k=1}^{d} \left( \cosh(s_k / \sqrt{n}) + Y_{t_{k,n}}^{(n)} e^{-2sk / \sqrt{n}} \right)^{n_k - n_{k-1}} \right]. \]

We first analyze \( I_{n,2} \), which is a multivariate integral with \( Y_{t_{1,n}}^{(n)} \in [0,1] \). We introduce the process

\[ \widetilde{Y}_{t}^{(n)} := 2n \left( 1 - Y_{t}^{(n)} e^{2t / \sqrt{n}} \right), \quad t \in (-\infty, \infty). \]

Since \( Y_{t_{1,n}}^{(n)} \in [0,1] \), therefore \( Y_{t_{k,n}}^{(n)} \in [-1,1], \ k = 2, \ldots, d \). Hence, we have \( \widetilde{Y}_{t_{1,n}}^{(n)} \in [0,2n] \) and \( \widetilde{Y}_{t_{k,n}}^{(n)} \in [0,4n], \ k = 2, \ldots, d \).

We shall use the notations \( \pi_t^{(n)} (\pi_i^{(n)} \) respectively), for the absolutely continuous parts of the marginal laws of \( Y_t^{(n)} \) (\( Y_i^{(n)} \) respectively) which are supported on \([0, 2n]\) ([\(-1, 1\)] respectively).

Recalling (2.2) we write

\[ \pi_t^{(n)}(y) = \frac{1}{2n} \pi_t^{(n)} (1 - \frac{y}{2n}) \quad \text{with} \quad \pi_i^{(n)}(y) = f(y; A_n \sqrt{t}, B_n \sqrt{t}, C_n / \sqrt{t}, D_n / \sqrt{t}, q). \]

Using this density (which may be sub-probabilistic) and

\[ G_{x,c,n}(u) := 2^{-n} \prod_{k=1}^{d} \left( 1_{[0,4n]}(u_k) \left( \cosh \left( \frac{S_k}{\sqrt{n}} \right) + 1 - \frac{u_k}{2n} \right)^{n_k - n_{k-1}} \right) \]

\[ = \prod_{k=1}^{d} \left( 1_{[0,4n]}(u_k) \left( 1 + \sinh^2 \left( \frac{S_k}{2\sqrt{n}} \right) - \frac{u_k}{4n} \right)^{n_k - n_{k-1}} \right), \quad (4.11) \] (4.12)
we can write
\[ \frac{I_{n,2}}{2^n} = \int_{[0,2n]} 1_{[0,2n]}(u) E \left( G_{x,c,n}(u, \tilde{Y}_n^{(1)}, \ldots, \tilde{Y}_n^{(s)}) \big| \tilde{Y}_n^{(s)} = u \right) \tilde{\pi}_n^{(s)}(u) \, du. \] (4.13)

In the next two lemmas we address the asymptotics of the density \( \tilde{\pi}_n^{(s)} \) and of the transition probability density for the process \( \tilde{Y}_n^{(s)} \).

**Lemma 4.4.** Under Assumption 4.1, for every \(-c < t < a\) we have the following.

(i) Pointwise convergence: for every \( u \geq 0 \),

\[
\lim_{n \to \infty} \frac{\tilde{\pi}_n^{(s)}(u)}{\Pi_n} = \begin{cases} \frac{\sqrt{n}}{((a-t)^2 + u)((c+t)^2 + u)}, & \text{if } a, c \in \mathbb{R}, \\ \frac{\sqrt{n}}{(a-t)^2 + u}, & \text{if } a \in \mathbb{R}, c = \infty, \\ \sqrt{n}, & \text{if } a = c = \infty, \end{cases}
\] (4.14)

with

\[
\Pi_n = \frac{1 - \frac{B_n D_n}{1 - A_n B_n C_n D_n}}{\pi} \begin{cases} a + c, & \text{if } a, c \in \mathbb{R}, \\ \frac{1}{\sqrt{n}(1 - C_n)}, & \text{if } a \in \mathbb{R}, c = \infty, \\ \frac{1 - A_n C_n}{n^{3/2}(1 - A_n)^2(1 - C_n)^2}, & \text{if } a = c = \infty. \end{cases}
\] (4.15)

(ii) Uniform bound: There are constants \( N, K > 0 \) (possibly depending on \( t \)) such that for all \( n > N \) and all \( u \geq 0 \),

\[
1_{u \in [0,2n]} \frac{\tilde{\pi}_n^{(s)}(u)}{\Pi_n} \leq K \sqrt{u}. \] (4.16)

**Remark 4.5.** The exact asymptotic of \( \Pi_n \) is not needed for the proof of our main theorem, because of a cancellation later in the calculations. However we need the following consequence of Assumption 4.1

\[
\limsup_{n \to \infty} \frac{1}{n} \log(1/\Pi_n) \leq 0. \] (4.17)

Note that unless \( BD = 1 \), we have \( \frac{1 - B_n D_n}{1 - A_n B_n C_n D_n} \sim 1 \), which further simplifies expression (4.15).

**Proof of Lemma 4.4.** Consider

\[ x_n = 1 - \frac{u}{2n}, \quad t_n = e^{2t/\sqrt{n}}. \]

(Strictly speaking, \( t_n \) is a function of \( t \).) Since \( (\alpha; q)_{\infty} = (1 - \alpha)(\alpha q; q)_{\infty} \), from [11, Eq. (4.4)],

\[
\tilde{\pi}_n^{(s)}(u) = \frac{1}{2^n} \pi_n(x_n; A_n, B_n, C_n, D_n, q) = \frac{1}{\pi n} J_n(t, u) R_n(t, u) \] (4.18)

with

\[
J_n(t, u) := J \left( 1 - \frac{u}{2n}; A_n \sqrt{t_n}, B_n \sqrt{t_n}, C_n \sqrt{t_n}, D_n \sqrt{t_n} \right),
\]

\[
R_n(t, u) := R \left( 1 - \frac{u}{2n}; A_n \sqrt{t_n}, B_n \sqrt{t_n}, C_n \sqrt{t_n}, D_n \sqrt{t_n}, q \right),
\]
where $J$ and $R$ are given by (2.3) and (2.4). Applying (B.1) to each factor in

$$R_n(t,u) = \frac{\left| (qe^{i\theta u}; q)_\infty \right|^2}{(q A_n B_n C_n D_n; q)_\infty} \cdot \frac{(q A_n B_n t_n, q A_n C_n, q A_n D_n, q B_n C_n, q B_n D_n, q C_n D_n/t_n; q)_\infty}{\left| (q B_n \sqrt{t_n e^{i\theta u}}, q D_n e^{i\theta u}/\sqrt{t_n}, q A_n \sqrt{t_n e^{i\theta u}}, q C_n e^{i\theta u}/\sqrt{t_n}; q) \right|_\infty^2},$$

we see that

$$R_n(t,u) \to 1.$$

The explicit formula for $J_n(t,u)$ is

$$J_n(t,u) = \frac{\sqrt{1-x_n^2}}{\left| (1-A_n \sqrt{t_n e^{i\theta u}}) (1-C_n e^{i\theta u}/\sqrt{t_n}) \right|_\infty^2} \cdot \frac{(1-A_n B_n t_n)(1-A_n C_n)(1-A_n D_n)(1-B_n C_n)(1-B_n D_n)(1-C_n D_n/t_n)}{(1-A_n B_n C_n D_n)(1-B_n \sqrt{t_n e^{i\theta u}})(1-D_n e^{i\theta u}/\sqrt{t_n})},$$

Since

$$\frac{(1-A_n B_n t_n)(1-B_n C_n)(1-A_n D_n)(1-C_n D_n/t_n)}{(1-B_n \sqrt{t_n e^{i\theta u}})(1-D_n e^{i\theta u}/\sqrt{t_n})} \sim \frac{(1-B)(1-B)(1-D)(1-D)}{(1-B)^2(1-D)^2} = 1,$$

these factors do not contribute to the asymptotics. Noting that $\sqrt{1-x_n^2} \sim \sqrt{u/n}$, we get

$$J_n(t,u) \sim \frac{\sqrt{u}}{\sqrt{n}} \cdot \frac{1-B_n D_n}{1-A_n B_n C_n D_n} \cdot \frac{1-A_n C_n}{\left| (1-A_n \sqrt{t_n e^{i\theta u}}) (1-C_n e^{i\theta u}/\sqrt{t_n}) \right|_\infty^2}.$$ 

We observe that

$$1 - A_n C_n \sim \begin{cases} \frac{a+c}{\sqrt{n}}, & \text{if } a, c \in \mathbb{R}, \\ 1 - C_n, & \text{if } a \in \mathbb{R}, c = \infty, \end{cases}$$

(keeping the factor $1 - A_n C_n$ unchanged when $a = c = \infty$) and

$$\left| 1 - A_n \sqrt{t_n e^{i\theta u}} \right|^2 = (1 - A_n \sqrt{t_n})^2 + 2 A_n \sqrt{t_n} (1 - x_n) = (1 - A_n \sqrt{t_n})^2 + A_n \sqrt{t_n} u/n.$$  \hspace{1cm} (4.21)

Thus,

$$\left| 1 - A_n \sqrt{t_n e^{i\theta u}} \right|^2 \sim \begin{cases} \frac{(a-t)^2 + u}{n}, & \text{if } a \in \mathbb{R}, \\ (1-A_n)^2, & \text{if } a = \infty. \end{cases}$$

Similarly

$$\left| 1 - C_n \sqrt{t_n e^{i\theta u}} \right|^2 \sim \begin{cases} \frac{(c+t)^2 + u}{n}, & \text{if } c \in \mathbb{R}, \\ (1-C_n)^2, & \text{if } c = \infty. \end{cases}$$
Thus

\[ J_n(t, u) \sim \frac{1 - B_n D_n}{1 - A_n B_n C_n D_n} \left\{ \begin{array}{ll}
\frac{\sqrt{u}}{(a - t)^2 + u} & \text{if } a, c \in \mathbb{R}, \\
\frac{\sqrt{u}}{(1 - C_n) ((a - t)^2 + u)} & \text{if } a \in \mathbb{R}, c = \infty, \\
\left(\frac{1 - A_n C_n}{(1 - A_n)^2 (1 - C_n)^2}\right) \cdot \sqrt{u} & \text{if } a = c = \infty.
\end{array} \right. \tag{4.22} \]

Combining (4.18), (4.19) and (4.22) yields the desired asymptotic equivalence (4.14) for \( \bar{\pi}_t^{(n)} \).

Next we prove part (ii). From (B.2) with \( \alpha_n \), replaced by \( q \alpha_n \), and elementary calculations, like

\[ \sup_{u \in [0, 4n]} |1 - q^k A_n \sqrt{t_n} e^{i\theta_n}|^2 = -(1 - q^k A_n \sqrt{t_n})^2, \]

compare (4.21), we see that the sequence \( \sup_{u \in [0, 4n]} R_n(t, u) \) converges as \( n \to \infty \). Thus there exists \( N > 0 \) and a constant \( K > 0 \) (here and in the remaining part of this proof \( K \) stands for a generic constant which may change from line to line, may depend on \( t \), but is free of \( n \) and \( u \)) such that for all \( u \in [0, 4n] \) and all \( n > N \) we have

\[ 0 \leq R_n(t, u) \leq K. \tag{4.23} \]

The bound for \( J_n(t, u) \) is slightly more delicate and introduces an additional constraint on \( u \). Notice that for all \( u \in [0, 2n] \) (so \( x_n \geq 0 \)),

\[ |1 - B_n \sqrt{t_n} e^{i\theta_n}|^2 = 1 - 2B_n \sqrt{t_n} x_n + B_n^2 t_n \geq 1. \]

Similarly,

\[ |1 - D_n e^{i\theta_n}/\sqrt{t_n}|^2 \geq 1 \quad \text{for } u \in [0, 2n]. \]

Note also that \( \sqrt{1 - x_n^2} \leq \sqrt{u/n} \). Thus, recalling (4.21) and its analogue for \( C_n \), we conclude that for \( N \) large enough and \( n > N \) there exists \( K \) such that

\[ 0 \leq J_n(t, u) \leq K \frac{\sqrt{u}(1 - B_n D_n)(1 - A_n C_n)}{\sqrt{u}(1 - A_n B_n C_n D_n)(1 - A_n \sqrt{t_n}/\sqrt{C_n})(1 - C_n/\sqrt{C_n})^2}, \quad u \in [0, 2n]. \tag{4.24} \]

Moreover, due to (4.20), (4.21) and its analogue for \( C_n \), we can increase \( N \) to ensure that for all \( n > N \) we have

\[ 1 - A_n C_n \leq \left\{ \begin{array}{ll}
\frac{2(a + c)}{\sqrt{n}} & \text{if } a, c < \infty, \\
2(1 - C_n) & \text{if } a < \infty, c = \infty,
\end{array} \right. \tag{4.25} \]

\[ |1 - A_n \sqrt{t_n}|^2 \geq \left\{ \begin{array}{ll}
\frac{(a - t)^2}{2n} & \text{if } a < \infty, \\
\frac{1}{2}(1 - A_n)^2 & \text{if } a = \infty,
\end{array} \right. \tag{4.26} \]

and

\[ |1 - C_n/\sqrt{t_n}|^2 \geq \left\{ \begin{array}{ll}
\frac{(c + t)^2}{2n} & \text{if } c < \infty, \\
\frac{1}{2}(1 - C_n)^2 & \text{if } c = \infty.
\end{array} \right. \tag{4.27} \]
Inserting (4.25), (4.26) and (4.27) into (4.24) we thus obtain

\[
1_{u\in[0,2n]}J_n(t, u) \leq K \sqrt{u} \frac{1 - B_nD_n}{1 - A_nB_nC_nD_n} \left\{ \begin{array}{ll}
\frac{(a - t)^2(c + t)^2}{n} & \text{if } a, c < \infty, \\
\frac{\sqrt{n}}{(a - t)^2(1 - C_n)} & \text{if } a < \infty, c = \infty, \\
\frac{1 - A_nC_n}{\sqrt{n}(1 - A_n)^2(1 - C_n)^2} & \text{if } a = c = \infty.
\end{array} \right.
\]

(4.28)

Recalling (4.15) we see that (4.28) yields \( J_n(t, u) \leq K \Pi_n n \sqrt{u} \) for all \( u \in [0, 2n] \), which, in view of (4.23) and (4.18), proves (4.16).

Next we address the asymptotics of the transition densities of \( \tilde{Y}^{(n)} \). We shall use the notations \( \tilde{P}_{s,t,u}^{(n)} \) and \( P_{s,t,u}^{(n)} \), for the density of the transition probabilities of \( \tilde{Y}^{(n)} \) and \( Y^{(n)} \), respectively. These densities are supported on the intervals \([0, 4n]\) and \([-1, 1]\) respectively.

**Lemma 4.6.** Under Assumption 4.1 we have, for all \( s, t \) such that \( s < t < a \),

\[
\lim_{n \to \infty} \tilde{P}_{s,t,u}^{(n)}(u, v) = \begin{cases} 
p_{s-t}(u, v) \frac{(a - s)^2 + u}{(a - t)^2 + v}, & \text{if } a < \infty, \\
p_{s-t}(u, v), & \text{if } a = \infty,
\end{cases}
\]

(4.29)

and \( p_s(x, y) \) is the transition probability density function of the \( 1/2 \)-stable Biane process as in (4.2).

**Proof.** Consider \( x_n = 1 - u/(2n) \), \( y_n = 1 - v/(2n) \). Recall \( s_n < t_n = e^{2t/\sqrt{n}} \), and hence \( A_n \sqrt{s_n} < A_n \sqrt{t_n} \leq 1 \) eventually by assumption as \( s < t < a \). In this case definition (2.6) gives the following transition probability density of the Askey–Wilson process:

\[
\tilde{P}_{s,t,u}^{(n)}(u, v) = \frac{1}{2n} P_{s,t,u}^{(n)}(x_n, y_n) = \frac{1}{\tau^2} J_n(s, t, u, v) R_n(s, t, u, v),
\]

(4.30)

where

\[
J_n(s, t, u, v) = J \left( y_n; A_\sqrt{t_n}, B_\sqrt{t_n}, \sqrt{\frac{s_n}{t_n}} e^{i\theta x_n}, \sqrt{\frac{t_n}{s_n}} e^{-i\theta x_n} \right),
\]

\[
R_n(s, t, u, v) = R \left( y_n; A_\sqrt{t_n}, B_\sqrt{t_n}, \sqrt{\frac{s_n}{t_n}} e^{i\theta x_n}, \sqrt{\frac{t_n}{s_n}} e^{-i\theta x_n}, q \right),
\]

with functions \( J \) and \( R \) introduced in (2.3) and (2.4).

Similarly as in the proof of Lemma 4.4, by (B.1) we see that \( R_n(s, t, u, v) \to 1 \).

Next we consider

\[
\begin{align*}
J_n(s, t, u) &= 1 - A_n B_n t_n \\
&\quad \frac{1}{1 - A_n B_n s_n} \cdot \frac{|1 - B_n \sqrt{s_n e^{i\theta x_n}}|^2}{|1 - B_n \sqrt{t_n e^{i\theta y_n}}|^2} \\
&\quad \times \frac{|1 - A_n \sqrt{s_n e^{i\theta x_n}}|^2}{|1 - A_n \sqrt{t_n e^{i\theta y_n}}|^2} \cdot \frac{\sqrt{1 - \rho_n^2(1 - s_n/t_n)}}{(1 - \sqrt{s_n/t_n} e^{i(\theta x_n + \theta y_n)})(1 - \sqrt{s_n/t_n} e^{(-\theta x_n + \theta y_n)})}.
\end{align*}
\]

Note that

\[
\lim_{n \to \infty} \frac{1 - A_n B_n t_n}{1 - A_n B_n s_n} \cdot \frac{|1 - B_n \sqrt{s_n e^{i\theta x_n}}|^2}{|1 - B_n \sqrt{t_n e^{i\theta y_n}}|^2} = 1,
\]
and by (4.21), we have

\[
\lim_{n \to \infty} \left| \frac{1 - A_n \sqrt{s_n e^{i\theta_{x_n}}}}{1 - A_n \sqrt{t_n e^{i\theta_{y_n}}}} \right|^2 = \frac{(a - s)^2 + u}{(a - t)^2 + v},
\]

where the ratio is understood as 1 if \( a = \infty \).

Moreover, as in [11, P. 2185], we have

\[
|(1 - \sqrt{s_n/t_n} e^{(i\theta_{x_n} + \theta_{y_n})})|^2 = 1 + t_n/s_n - 2\sqrt{t_n/s_n} \left( x_n y_n + \sqrt{1 - x_n^2} \sqrt{1 - y_n^2} \right)
\]

\[
\sim \frac{1}{n} \left[ (t - s)^2 + (\sqrt{u} \pm \sqrt{v})^2 \right].
\]

Since \( \sqrt{1 - y_n^2} (1 - s_n/t_n) \sim 2(t - s)\sqrt{v}/n \), we get

\[
J_n(s, t, u) \sim 2n \frac{(t - s)\sqrt{v}}{((t - s)^2 + (\sqrt{u} + \sqrt{v})^2)((t - s)^2 + (\sqrt{u} - \sqrt{v})^2)} \frac{(a - s)^2 + u}{(a - t)^2 + v}.
\]

Recalling (4.30) and (4.2), the desired limit (4.29) now follows. \( \square \)

To determine the asymptotic of \( I_{n, 2}/2^n \) in (4.13), we need the following lemma about the limiting properties of function \( G_{x, c, n} \) defined in (4.11).

**Lemma 4.7.** If Assumption 4.1 and (4.3) hold, then

\[
\lim_{n \to \infty} \frac{1}{\Pi_n} \int_0^{2n} E \left( G_{x, c, n}(u, \bar{y}^{(n)}_{x_2}, \ldots, \bar{y}^{(n)}_{x_d}) \mid \bar{y}^{(n)}_{x_1} = u \right) \pi^{(n)}_{x_1}(u) du = \exp \left( \frac{1}{4} \sum_{k=1}^d s_k^2 (x_k - x_{k-1}) \right) \Psi^{(a, c)}(u),
\]

(4.31)

where \( \Psi^{(a, c)}(u) \) is defined in (4.5) and (4.6).

**Proof.** Since \( \sinh^2 x = x^2 + O(x^4) \) as \( x \to 0 \), we have

\[
\left( 1 + \sinh \left( \frac{s}{2\sqrt{n}} \right) - \frac{u}{4n} \right)^n \to e^{(s^2 - u)/4}.
\]

Since \( (n_k - n_{k-1})/n \to x_k - x_{k-1} \), from (4.12) we get

\[
\lim_{n \to \infty} G_{x, c, n}(u) = \exp \left( \frac{1}{4} \sum_{k=1}^d (s_k^2 - u_k)(x_k - x_{k-1}) \right) =: G_{x, c}(u)
\]

for \( u \in [0, \infty)^d \).

Adapting [11, proof of Proposition 4.7, (4.24)] we also have

\[
G_{x, c, n}(u) \leq K \prod_{k=1}^d \exp \left( -\frac{n_k - n_{k-1}}{4n} u_k \right) \text{ for all } u \in \mathbb{R}_+^d,
\]

(4.32)

where \( K \) is a constant independent of \( n \). We will use the following fact which follows from [3, Theorem 5.5].

**Lemma 4.8.** Suppose a sequence \( \mu^{(n)} \) of probability measures on \( \mathbb{R}_+^d \) converges weakly, \( \mu^{(n)} \Rightarrow \mu \). Let \( G_n \) be a sequence of uniformly bounded measurable functions, \( G_n : \mathbb{R}_+^d \to [-K, K] \) for some \( K > 0 \), such that for all \( u_n \to u \in \mathbb{R}_+^d \) we have

\[
\lim_{n \to \infty} G_n(u_n) = G(u).
\]

Then we have

\[
\lim_{n \to \infty} \int G_n(u) \mu^{(n)}(du) = \int G(u) \mu(du).
\]

(4.33)
From (4.32) we see that functions \( G_{x,c,n} \) are uniformly bounded in \( u \in \mathbb{R}^d_+ \). We also note that
\[
\lim_{n \to \infty} G_{x,c,n}(u_n) = G_{x,c}(u), \quad \text{if } u_n \to u \in \mathbb{R}^d_+ \text{ as } n \to \infty,
\]
see [11, (4.25)].

We now apply (4.33) to a sequence of probability measures
\[
\mu_{u_1}^{(n)}(du_2, \ldots, du_d) = \prod_{k=1}^{d-1} P_{-s_k, -s_{k+1}}(u_{k}, u_{k+1}) du_2 \ldots du_d
\]
with fixed \( u_1 > 0 \), which by Scheffé’s Lemma converges to the measure \( \mu \) constructed from the transition densities (4.29). Recall that we have established the convergence of transition densities \( p_{s,t}^{(n)} \) in Lemma 4.6.

It then follows that for all \( u_1 > 0 \) we have
\[
G_n(u_1) := \mathbb{E}\left(G_{x,c,n}(u_1, \tilde{Y}^{(n)}_{-s_2}, \ldots, \tilde{Y}^{(n)}_{-s_d}) \mid \tilde{Y}^{(n)}_{-s_1} = u_1\right) \to G(u_1) := \begin{cases} 
\int_{\mathbb{R}^d_+} G_{x,c}(u_1, u_2, \ldots, u_d) \frac{(a + s_1)^2 + u_1}{(a + s_d)^2 + u_d} \prod_{k=1}^{d-1} P_{s_k, -s_{k+1}}(u_{k}, u_{k+1}) du_2 \ldots du_d, & a < \infty, \\
\int_{\mathbb{R}^d_+} G_{x,c}(u_1, u_2, \ldots, u_d) \prod_{k=1}^{d-1} P_{s_k, -s_{k+1}}(u_{k}, u_{k+1}) du_2 \ldots du_d, & a = \infty,
\end{cases}
\]
as \( n \to \infty \). To prove (4.31) we analyze
\[
\lim_{n \to \infty} \frac{1}{\Pi_n} \int_{\mathbb{R}_+} 1_{u \in [0,2n]} G_n(u) \tilde{p}_{-s_1}^{(n)}(u) \, du.
\]
In view of bounds (4.16) and (4.32), we can apply the dominated convergence theorem. From pointwise convergence of \( u \to 1_{u \in [0,2n]} G_n(u) \), and \( \tilde{p}_{-s_1}^{(n)}/\Pi_n \), see (4.34) and (4.14), we get
\[
\lim_{n \to \infty} \frac{1}{\Pi_n} \int_{\mathbb{R}_+} 1_{u \in [0,2n]} G_n(u) \tilde{p}_{-s_1}^{(n)}(u) \, du = \begin{cases} 
\int_{\mathbb{R}_+} G(u) \frac{\sqrt{u}}{(a + s_1)^2 + u((c - s_1)^2 + u)} \, du, & \text{if } a, c \in \mathbb{R}, \\
\int_{\mathbb{R}_+} G(u) \frac{\sqrt{u}}{(a + s_1)^2 + u} \, du, & \text{if } a \in \mathbb{R}, c = \infty, \\
\int_{\mathbb{R}_+} G(u) \sqrt{u} \, du, & \text{if } a = c = \infty.
\end{cases}
\]

Recalling the definition (4.34) of \( G(u) \), this ends the proof. \( \square \)

Next we analyze the integral \( I_{n,1} \) introduced in (4.10). We observe that for a fixed \( \delta \in (0,1) \) and large enough \( n \) we have \( \cosh(s_k/\sqrt{n}) \leq 1 + \frac{s_k^2}{n} < 1 + \delta \). So with \( Y^{(n)}_{t_1,n} \in (-1 - \delta, 0) \),
\[
-\delta = 1 - (1 + \delta) < \cosh(s_1/\sqrt{n}) + Y^{(n)}_{t_1,n} < 1 + s_1^2/n < 1 + \delta.
\]
Consequently, recalling that \( \cosh(s_k/\sqrt{n}) \leq 1 + s_k^2/n \leq 1 + s_1^2/n \) if \( s_1^2/n < 1 \),
\[
|I_{n,1}| \leq (2 + \frac{s_1^2}{n})^{n-n_1} (1 + \delta)^{n_1} \leq e^{s_1^2/2} 2^n (\frac{1 + \delta}{2})^{n_1}. \tag{4.35}
\]
Referring to (4.17), we see that
\[
\lim_{n \to \infty} \frac{I_{n,1}}{2^n \Pi_n} = 0.
\]
Combining the asymptotics of \(I_{n,1}\) and \(I_{n,2}\) we see that

\[
\lim_{n \to \infty} \frac{\mathbb{E} \left[ \prod_{k=1}^{d} \left( \cosh \left( \sqrt{n} \right) \right) + Y_{r(n)} \right] 2^n \Pi_n}{2^n} \]

is given by the right hand side of (4.31).

4.1.2. Asymptotics of the denominator in (4.7)

Lemma 4.9. Under the Assumption 4.1, we have

\[
\lim_{n \to \infty} \frac{1}{2^n} \mathbb{E} \left( 1 + Y_{1}^{(n)} \right)^n = \begin{cases} 
\frac{\pi}{\sqrt{2}} c_{a,c}, & \text{if } a, c \in \mathbb{R}, \\
2\sqrt{2} \pi c_{a,\infty}, & \text{if } a \in \mathbb{R}, c = \infty, \\
4\sqrt{\pi}, & \text{if } a = c = \infty,
\end{cases}
\]

where \(c_{a,c}\) is defined in (1.17)

Proof. Since \(c > 0\) and \(B_n, D_n \in (-1,0]\), the atoms for \(Y_{1}^{(n)}\) may only arise from \(A_n\) if \(a < 0\).

We first consider the case \(a \geq 0, c > 0\). The proof is similar as in [11, Lemma 4.5]. Since \(Y_{1}^{(n)}\) has a density supported on \([-1,1]\), we have

\[
0 \leq \mathbb{E} \left( 1 + Y_{1}^{(n)} \right)^n - \mathbb{E} \left[ 1_{Y_{1}^{(n)} \in [0,1]} \left( 1 + Y_{1}^{(n)} \right)^n \right] = \int_{-1}^{0} (1 + u)^n \pi_{1}^{(n)}(u) du \leq 1.
\]

Therefore by (4.17) and (4.15) we get

\[
\Pi_n^{-1} 2^{-n} \mathbb{E} \left( 1 + Y_{1}^{(n)} \right)^n \sim \Pi_n^{-1} 2^{-n} \mathbb{E} \left[ 1_{Y_{1}^{(n)} \in [0,1]} \left( 1 + Y_{1}^{(n)} \right)^n \right] = \Pi_n^{-1} \mathbb{E} \left[ 1_{Y_{0}^{(n)} \in [0,2n]} \left( 1 - \frac{\sqrt{\gamma}}{4n} \right)^n \right]
\]

\[
\sim \begin{cases} 
\int_{R_+} \sqrt{\pi} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du, & 0 \leq a < \infty, 0 < c < \infty, \\
\int_{R_+} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du, & 0 \leq a < \infty, c = \infty, \\
\int_{R_+} \sqrt{\pi} e^{-u/4} du, & a = c = \infty.
\end{cases}
\]

see (4.31) and (4.5) with \(d = 1\) and \(s_1 = 0\), compare (4.11).

Recall (1.18). To evaluate the integrals, we use the identity

\[
\int_{0}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du = \pi \left( \frac{2\sqrt{\pi}}{\sqrt{\pi}} - |a| H(|a|/2) \right),
\]

which is valid for all real \(a\). (This is [21, Section 4.2 formula (22) pg 136] evaluated at \(p = 1/4\).)

For \(a \geq 0, c > 0, a \neq c\), we can apply (4.38) directly. We get

\[
\int_{R_+} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du = \int_{R_+} \left( \frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \right) \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du = \pi \frac{c H(c/2) - a H(a/2)}{c^2 - a^2},
\]

which proves (4.37) for this case. To prove (4.37) for \(a \geq 0, c = \infty\) and \(a = c = \infty\), we apply the identity (4.38) and \(\int_{0}^{\infty} e^{-u/4} \sqrt{u} du = 4\sqrt{\pi} \) respectively.

For \(a = c > 0\), we use instead the identity

\[
\int_{R_+} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{\pi}} e^{-u/4} du = \pi \frac{\pi}{2a} \left( \frac{1}{\sqrt{\pi}} \right) H(a/2) - \frac{a}{\sqrt{\pi}}.
\]
which can be obtained by differentiating (4.38) with respect to \( a \), and we get (4.37). This completes the proof for the case \( a \geq 0, c > 0 \).

Next, we consider \( a < 0 < c \) and \( a + c > 0 \). The law of \( Y_1^{(n)} \) now has both atomic and continuous part, and for the continuous part, restricted to \([-1, 1]\), we use (4.38) for \( a < 0 \). Namely, we get

\[
2^{-n} E \left( \left(1 + Y_1^{(n)} \right)^n 1 \{ Y_1^{(n)} \in [-1, 1] \} \right) \sim \pi \Pi_n \times \begin{cases} \frac{c H(\frac{c}{2}) + a H(\frac{-a}{2})}{c^2 - a^2} & a < 0, c < \infty, \\ \left( \frac{2}{\sqrt{n}} + a H(\frac{-a}{2}) \right) & a < 0, c = \infty. \end{cases}
\] (4.39)

It remains to compute the atomic part. Again, for \( n \) large enough there is a single atom at

\[ y_n := \frac{1}{2} \left( A_n + \frac{1}{A_n} \right), \]

and its probability is

\[ p_n = \frac{(A_n^{-2}, B_n C_n, B_n D_n, C_n D_n; q)_{\infty}}{(B_n/A_n, C_n/A_n, D_n/A_n, A_n B_n C_n D_n; q)_{\infty}} \sim \frac{(1 - A_n^{-2})}{(1 - C_n/A_n)} \cdot \frac{1 - B_n D_n}{1 - A_n B_n C_n D_n} \]

\[ \sim \frac{2(A_n - 1)}{A_n - C_n} \cdot \frac{1 - B_n D_n}{1 - A_n B_n C_n D_n}, \]

see (B.1).

We note that

\[ 2^{-n} (1 + y_n)^n = \frac{1}{2^n} \left( A_n + \frac{1}{A_n} + 2 \right)^n = \left( 1 + \frac{1}{4A_n} (A_n - 1)^2 \right)^n \sim e^{a^2/4}. \]

We also note that

\[ \frac{(1 - A_n^{-2})}{(1 - C_n/A_n)} \sim \frac{2(A_n - 1)}{A_n - C_n} \sim \begin{cases} \frac{-2a}{c - a} & a < 0, 0 < c < \infty, \\ \frac{-2a}{\sqrt{n}(1 - C_n)} & a < 0, c = \infty. \end{cases} \]

It follows that

\[ 2^{-n} E \left( \left(1 + Y_1^{(n)} \right)^n 1 \{ Y_1^{(n)} \not\in [-1, 1] \} \right) \sim \frac{1 - B_n D_n}{1 - A_n B_n C_n D_n} e^{a^2/4} \times \begin{cases} \frac{-2a}{c - a} & a < 0, 0 < c < \infty, \\ \frac{-2a}{\sqrt{n}(1 - C_n)} & a < 0, c = \infty. \end{cases} \]

In view of (4.15) we obtain

\[ 2^{-n} E \left( \left(1 + Y_1^{(n)} \right)^n 1 \{ Y_1^{(n)} \not\in [-1, 1] \} \right) \sim \pi \Pi_n e^{a^2/4} \times \begin{cases} \frac{-2a}{c^2 - a^2} & a < 0, 0 < c < \infty, \\ -2a & a < 0, c = \infty. \end{cases} \] (4.40)

Combining (4.39) and (4.40), we have

\[ 2^{-n} E \left( 1 + Y_1^{(n)} \right)^n \sim \pi \Pi_n \times \begin{cases} \frac{c H(\frac{c}{2}) - a H(\frac{a}{2})}{c^2 - a^2} & a < 0, 0 < c < \infty, \\ \left( \frac{2}{\sqrt{n}} - a H(\frac{a}{2}) \right) & a < 0, c = \infty, \end{cases} \]

where we used the identity \( H(x) + H(-x) = 2 \exp(x^2). \) □
Proof of Theorem 4.2. It remains to put all pieces together. To prove (4.4), we refer to (4.7). The normalizing constant in $\Psi_{x,c}^\beta(c)$ arises from the asymptotics of the denominator in (4.7) given in Lemma 4.9. The remaining part of (4.4) is a consequence of the asymptotics of the numerator in (4.7) which is given in (4.36) and (4.31). In particular, the first factor on the right hand side of (4.31) gives the fluctuation part corresponding to the Brownian motion:

$$\exp \left( \frac{1}{4} \sum_{k=1}^{d} s_k^2 (x_k - x_{k-1}) \right) = \mathbb{E} \exp \left( - \sum_{k=1}^{d} \frac{c_k}{\sqrt{2}} \xi_{x_k} \right).$$

\[\square\]

4.2. A dual representation for the Laplace transform

Informally, a dual representation for the Laplace transform expresses the Laplace transform of the finite-dimensional distributions of a process as a Laplace transform of another process by exchanging the roles of the arguments of the Laplace transform and the time variables for the two processes. We shall establish a general dual representation identity that applies to the three limit Laplace transforms in Theorem 4.2, see (4.5). Recall that $p_{t-x}(x,y)$ is the transition probability density function (4.2) of the 1/2-stable Biane process, and $q_t(x,y)$ is the transition sub-probability density function (1.11) of the Brownian motion killed at hitting zero.

Proposition 4.10. Let $f, g$ be two measurable functions on $\mathbb{R}_+$. With $c_1, \cdots, c_{d-1} > 0$ and $0 = x_0 < x_1 < \cdots < x_d$,

$$\int_{\mathbb{R}_+^d} e^{-\frac{1}{2} \sum_{k=1}^{d} (x_k - x_{k-1})^2 u_k^2} f(u_1) \left( \prod_{k=1}^{d-1} p_{c_k}(u_k, u_{k+1}) \right) g(u_d) du = \frac{8}{\pi} \int_{\mathbb{R}_+^{d-1}} e^{-\frac{1}{2} \sum_{k=1}^{d-1} c_k x_k} \tilde{f}(z_1) \left( \prod_{k=2}^{d-1} q_{x_k-x_{k-1}}(z_k-1, z_k) \right) \tilde{g}(z_{d-1}) dz,$$

where

$$\tilde{f}(z) := \int_{\mathbb{R}_+} f(2u^2) \sin(uz)e^{-x_1 u^2/2} du,$$

$$\tilde{g}(z) := \int_{\mathbb{R}_+} g(2u^2)u \sin(uz)e^{-(x_d-x_{d-1})u^2/2} du,$$

provided that the functions under the multiple integrals in (4.41) are absolutely integrable.

The proof follows the same idea as in [10], where the cases $a = 0, c = \infty$ (Brownian excursion) and $a = c = \infty$ (Brownian excursion) have been established therein. The novelty is that now new expressions of $f$ and $g$ show up in applications and they lead to new dual representations of the Laplace transforms in Proposition 4.3. Another dual representation formula can be found in [5].

Proof. With a change of variables $u_k \mapsto u_k^2$, the left-hand side of (4.41) becomes

$$2^d \int_{\mathbb{R}_+^d} e^{-\frac{1}{2} \sum_{k=1}^{d} (x_k - x_{k-1})^2 u_k^2} u_1 f(u_1^2) \left( \prod_{k=1}^{d-1} u_{k+1} p_{c_k}(u_k^2, u_{k+1}^2) \right) g(u_d^2) du.$$

(4.42)

Recall that [10, after Eq. (2.2)]

$$yp_i(x^2, y^2) = \frac{1}{\pi x} \int_{\mathbb{R}_+} e^{-tx} \sin(xz) \sin(zy) dz,$$

and

$$\int_{\mathbb{R}_+} e^{-tx^2/2} \sin(xy_1) \sin(xy_2) dx = \frac{\pi}{2} q_t(y_1, y_2).$$
Then, (4.42) becomes
\[
\frac{2^d}{\pi^{d-1}} \int_{\mathbb{R}_+^d} u_1 f(u_1^2) \left( \prod_{k=1}^{d-1} u_k \right) e^{-c_k z_k^2} \sin(u_k z_k) \sin(u_{k+1} z_k) dz_k \right) g(u_d^2) e^{-\frac{d}{2} \sum_{k=1}^{d} (x_k - x_{k-1}) u_k^2} du_d
\]
\[
= \frac{4}{\pi} \int_{\mathbb{R}_+^d} e^{-\frac{d}{2} \sum_{k=1}^{d} c_k z_k^2} \left( \int f(u_1^2) e^{-x_1 u_1^2/4} du_1 \prod_{k=2}^{d-1} \frac{1}{\pi} \int_{\mathbb{R}_+} e^{-(x_k - x_{k-1}) u_k^2/4} \sin(u_k z_k) \sin(u_{k+1} z_k) du_k \right)
\]
\[
\times \int_{\mathbb{R}_+} u_d e^{-(x_d - x_{d-1}) u_d^2/4} \sin(u_d z_{d-1}) g(u_d^2) du_d \right) dz.
\]
One can then show that the above is the same as
\[
\frac{4}{\pi} \int_{\mathbb{R}_+^d} e^{-\frac{d}{2} \sum_{k=1}^{d} c_k z_k^2} \left( \sqrt{2}f(\sqrt{2}z_1) \prod_{k=2}^{d-1} q(x_k - x_{k-1}) (\sqrt{2}\hat{g}(\sqrt{2}z_{d-1})) \right) dz.
\]
The desired result now follows from \( q_{x/2}(z, z') = \sqrt{2}q_z(\sqrt{2}z, \sqrt{2}z') \) and changes of variables \( \sqrt{2}z_k \mapsto z_k, k = 1, \ldots, d - 1 \).

**Proof of Proposition 4.3.** We consider separately the three cases in (4.5), starting with the easiest.

(i) Case \( a = \infty, c = \infty \): the goal is to show
\[
\Psi_{x, \infty}(c) = \mathbb{E} \exp \left( -\frac{1}{\sqrt{2}} \sum_{k=1}^{d-1} c_k \eta_{x_k}^{c_k} \right).
\]
We apply Proposition 4.10 with \( f(u) = \sqrt{u}, g(u) = 1, x_d = 1 \). Then by straightforward calculation we get,
\[
\hat{f}(z) = \sqrt{2} \int_{\mathbb{R}_+} u e^{-x_1 u^2/2} \sin(u z) du = \sqrt{2} \pi \ell_{x_1}(z),
\]
\[
\hat{g}(z) = \int_{\mathbb{R}_+} u e^{-(1-x_{d-1}) u^2/2} \sin(u z) du = \pi \ell_{1-x_{d-1}}(z),
\]
see (1.12). Looking at the right hand side of (4.41) we then recognize
\[
\sqrt{8\pi} \ell_{x_1}(z_1) \prod_{k=2}^{d-1} q_{x_k-x_{k-1}}(z_{k-1}, z_k) \ell_{1-x_{d-1}}(z_{d-1}), \quad (z_1, \ldots, z_{d-1}) \in \mathbb{R}_+^{d-1},
\]
as the joint probability density of a Brownian excursion at times \( x_1, \ldots, x_{d-1} \), see (1.13). The desired identity now follows.

(ii) Case \( a \in \mathbb{R}, c = \infty \): the goal is to show
\[
\Psi_{x, \infty}(c) = \mathbb{E} \exp \left( -\frac{1}{\sqrt{2}} \sum_{k=1}^{d} c_k \eta_{x_k}^{a, \infty} \right).
\]
This time, consider \( f(u) = \sqrt{u} \) as before (so \( \hat{f}(z) \) is in (4.43)), but
\[
g(u) = \frac{1}{\alpha^2 + u}, \quad \alpha = a + s_d = a + c_d \geq 0.
\]
Then,
\[
\hat{g}(z) = \int_{\mathbb{R}_+} e^{-(1-x_{d-1}) u^2/2} \frac{u}{\alpha^2 + u} \sin(u z) du
\]
\[
= \frac{1}{4} \int_0^{\infty} e^{-(1-x_{d-1}) u^2/2} \sin(\sqrt{u} z) du = \frac{1}{4} F \left( z \left| \frac{\alpha}{\sqrt{2}}, \frac{1-x_{d-1}}{2} \right. \right).
\]
Lemma A.1. For all $c \in \mathbb{R}$, the desired result now follows. In view of (1.16) the desired Laplace transform, we invoke (1.15). Therefore,}

\[
\hat{g}(z_{d-1}) = \frac{\pi}{4} \int_{0}^{\infty} q_{1-x_{d-1}}(z_{d-1}, z_d) e^{-\alpha z_d/\sqrt{2}} dz_d. 
\] (4.47)

Now, (4.41) becomes

\[
\Psi^{(a, \infty)}(c) = \frac{8}{\pi^{2} \sqrt{8c} a^{\infty}} \int_{\mathbb{R}^{d+}} \hat{f}(z_1) \prod_{k=2}^{d-1} q_{x_k-x_{k-1}}(z_{k-1}, z_k) \hat{g}(z_{d-1}) e^{-\gamma_k \sum_{k=1}^{d-1} c_k z_k} dz 
\]

\[
= \frac{1}{c a} \int_{\mathbb{R}^{d+}} e^{x_1(z_1)} \prod_{k=2}^{d} q_{x_k-x_{k-1}}(z_{k-1}, z_k) e^{-\gamma_k \sum_{k=1}^{d} c_k z_k} e^{-\alpha z_d/\sqrt{2}} dz. 
\]

To recognize the above as the desired Laplace transform, we invoke (1.15).

(iii) Case $a, c \in \mathbb{R}, a + c > 0$: the goal is to show

\[
\Psi^{(a,c)}(c) = \exp \left( -\frac{1}{\sqrt{2}} \sum_{k=1}^{d} c_k \left( \hat{\Psi}_{x_k}^{(a,c)} - \hat{\Psi}_{x_k}^{(a,c)}(0) \right) \right) . 
\]

We apply Proposition 4.10 with $x_d = 1$. This time, consider $f(u) = \sqrt{u}/(\gamma^2 + u)$, where $\gamma = c - s_1 = c_1 - \cdots - c_d \geq 0$, and we take $g(u)$ as in (4.45). We have, by the same calculation for $\hat{g}(z)$ as before, see (4.46),

\[
\hat{f}(z) = \int_{\mathbb{R}^{+}} \frac{\sqrt{2 u}}{\gamma^2 + 2u^2} \sin(uz) e^{-x_1 u^2/2} du = \frac{1}{2 \sqrt{2}} F \left( z \left| \frac{x_1}{\sqrt{2}}, \frac{x}{2} \right. \right) = \frac{\pi}{2 \sqrt{2}} \int_{\mathbb{R}^{+}} q_{x_1}(z_0, z_1) e^{-\gamma_z \sqrt{2}} dz_0. 
\] (4.48)

By (4.47) and (4.48), the duality (4.41) gives

\[
\Psi^{(a,c)}(c) = \frac{1}{c a} \int_{\mathbb{R}^{d+}} e^{-x_0/\sqrt{2} - a z_d/\sqrt{2}} \prod_{k=1}^{d} q_{x_k-x_{k-1}}(z_{k-1}, z_k) e^{-\gamma_k \sum_{k=1}^{d} c_k (z_{k-1} - z_0)} dz. 
\]

In view of (1.16) the desired result now follows. \hfill \Box

Acknowledgement

The authors thank Ivan Corwin and Alisa Knizel for sharing an early version of [14]. We thank Guillaume Barraquand for information about Ref. [2] and the discussion of its contents. We also thank Alexey Kuznetsov for several discussions that inspired Proposition 4.10.

WEB’s research was partially supported by Simons Foundation/SFARI Award Number: 703475, US. YW’s research was partially supported by Army Research Office, US (W911NF-20-1-0139). JW’s research was partially supported by grant IDUB no. 1820/366/201/2021, Poland.

Appendix A: Auxiliary integrals

Recall $q_t$ as in (1.11). Recall $H(x) := e^{x^2} \text{erfc}(x)$ in (1.18).

Lemma A.1. For all $c \in \mathbb{R}$,

\[
\int_{0}^{\infty} q_t(x, y) e^{-\eta y} dy = \frac{1}{2} e^{-x^2/(2t)} \left( H \left( \frac{ct - x}{\sqrt{2t}} \right) - H \left( \frac{ct + x}{\sqrt{2t}} \right) \right).
\]
Proof. Write
\[\int_0^\infty q_t(x,y)e^{-cy}dy = \int_0^\infty \frac{1}{\sqrt{2\pi t}} \left( \exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right) \right)e^{-cy}dy.\]
First we compute
\[\int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)} e^{-cy}dy = e^{(c-ct)^2/(2t)} e^{-x^2/(2t)} \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-(y-(x-ct))^2/(2t)}dy\]
\[= \frac{1}{2} H\left(\frac{ct-x}{\sqrt{2t}}\right)e^{-x^2/(2t)}.\]
The second term is obtained by replacing \(x\) by \(-x\). \qed

Lemma A.2. For \(a, c \in \mathbb{R}\) and \(a + c > 0\),
\[\int_0^\infty \int_0^\infty q_t(x,y)e^{-ax-cy}dxdy = \begin{cases} aH(a \sqrt{t}/2) - cH(c \sqrt{t}/2), & \text{if } a \neq c, \\ \frac{1}{a^2 - c^2} & \text{if } a = c. \end{cases}\]

Proof. Assume \(a \neq c\) first. We first compute
\[\frac{1}{\sqrt{2\pi t}} \int_0^\infty \int_0^\infty e^{-(x+y)^2/(2t) - ax-cy}dxdy = \frac{1}{2(a-c)} \left(H(c \sqrt{t}/2) - H(a \sqrt{t}/2)\right). \quad (A.1)\]
By change of variables \(x = \theta u, y = (1-\theta)u\), the double integral becomes
\[\int_0^\infty u e^{-u^2/(2t) - cu}du = \frac{1}{a-c} \left(\int_0^\infty e^{-u^2/(2t) - cu}du - \int_0^\infty e^{-u^2/(2t) - au}du\right).\]
Since
\[\frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-u^2/(2t) - cu}du = \frac{e^{t/2}}{\sqrt{2\pi t}} \int_0^\infty e^{-(u+ct)^2/(2t)}du = \frac{1}{2} H\left(\frac{ct}{\sqrt{2t}}\right), \quad (A.2)\]
it follows that (A.1) holds. Next, we show
\[\frac{1}{\sqrt{2\pi t}} \int_0^\infty \int_0^\infty e^{-(x-y)^2/(2t) - ax-cy}dxdy = \frac{1}{2(a+c)} \left(H(c \sqrt{t}/2) + H(a \sqrt{t}/2)\right). \quad (A.3)\]
This time, first consider the region \(U = \{(x, y) \in \mathbb{R}^2_+, x > y\}\), and for this region consider \(x = u\theta, y = u(\theta - 1)\). Then
\[\int_U e^{-(x-y)^2/(2t) - ax-cy}dxdy = \int_0^\infty u e^{-u^2/(2t) + cu}du \int_0^\infty e^{-(a+c)u}du = \frac{1}{a+c} \int_0^\infty e^{-u^2/(2t) - au}du.\]
A similar calculation holds for the region \(\{(x, y) \in \mathbb{R}^2_+, x < y\}\). Then, combining with (A.2) we obtain (A.3). The desired result follows from (A.1) and (A.3).

Next, assume \(a = c\). Then (A.1) becomes
\[\frac{1}{\sqrt{2\pi t}} \int_0^\infty u e^{-u^2/(2t) - au}du = \frac{e^{t/2}}{\sqrt{2\pi t}} \int_0^\infty u e^{-(u+at)^2/(2t)}du \]
\[= \frac{e^{t/2}}{\sqrt{2\pi t}} \left(\int_0^\infty u e^{-u^2/(2t)}du - at \int_0^\infty e^{-(u+at)^2/(2t)}du\right) \]
\[= \frac{t}{\sqrt{2\pi t}} - a e^{t/2} \left(\frac{1}{2} H(a \sqrt{t}/2)\right).\]
This time, by the above and (A.3), we have the desired result for $a = c$.

Appendix B: Asymptotics of Pochhammer symbols

**Lemma B.1.** Fix $0 \leq q < 1$. For complex numbers $\alpha_n \in \mathbb{C}$ such that $\alpha_n \to \alpha$ with $|q\alpha| < 1$ we have

$$(\alpha_n; q)_\infty = z_n(1 - \alpha_n)(\alpha q; q)_\infty \text{ for some complex sequence } z_n \to 1.$$  

(B.1)

Furthermore, if $|\alpha_n| \leq 1$ then

$$(q; q)_\infty |1 - \alpha_n| \leq |(\alpha_n; q)_\infty| \leq (-q; q)_\infty |1 - \alpha_n|.$$  

(B.2)

**Proof.** It is well known [23] that function

$$z \mapsto (z; q)_\infty = \sum_{k=0}^{\infty} (-z)^k q^{(k-1)/2} / (q; q)_k$$

is analytic if $|z| < 1$. Therefore,

$$(\alpha_n; q)_\infty = (1 - \alpha_n)(\alpha_n q; q)_\infty = (1 - \alpha_n)(\alpha q; q)_\infty z_n,$$

where

$$z_n = \frac{(\alpha_n q; q)_\infty}{(\alpha q; q)_\infty} \to 1 \text{ as } n \to \infty.$$

For the second part of the proof, since $|\alpha_n| \leq 1$ and $q \geq 0$ we have

$$|(\alpha_n; q)_\infty| = |1 - \alpha_n||1 - \alpha_nq|\cdots|1 - \alpha_nq^k| \cdots \geq |1 - \alpha_n|(1 - |\alpha_nq|)\cdots(1 - |\alpha_nq^k|)\cdots$$

$$\geq |1 - \alpha_n|(1 - q)\cdots(1 - q^k)\cdots = |1 - \alpha_n|(q; q)_\infty.$$

Similarly, $|(\alpha_n; q)_\infty| \leq |1 - \alpha_n|(1 + q)\cdots(1 + q^k)\cdots = |1 - \alpha_n|(-q; q)_\infty$. □

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