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Nel’s category theory based
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Did Newton know category theory?

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Abstract

In a series of publications in the early 1990s, L D Nel set up a study of
non-normable topological vector spaces based on methods in category
theory. One of the important results showed that the classical oper-
ations of derivative and integral in Calculus can in fact be obtained
by a rather simple construction in categories. Here we present this
result in a concise form. It is important to note that the respective
differentiation does not lead to any so called generalized derivatives,
for instance, in the sense of distributions, hyperfunctions, etc., but it
simply corresponds to the classical one in Calculus.
Based on that categorial construction, Nel set up an infinite dimen-
sional calculus which can be applied to functions defined on non-
convex domains with empty interior, a situation of great importance
in the solution of partial differential equations.

1. The Setup

The presentation follows mainly Nel [1], where further details as well
as the proofs can be found. For convenience, a few basic concepts needed from Category Theory are recalled in short in the Appendix.

Let \( I \subseteq \mathbb{R} \) be any compact interval and let \( E \) be a real Banach space. We consider the Banach space structure on the set \( \mathcal{C}(I, E) \) of all continuous functions \( f : I \rightarrow E \), induced by the \( \mathcal{L}^\infty \) norm \( || \cdot ||_\infty \).

Our first aim is to define a derivative for functions \( f \) in a suitable subset of \( \mathcal{C}(I, E) \), and do so by means of a simple construction in categories. Among such functions \( f \) for which a derivative can be defined by categorial means are those called paths in \( E \), and they are characterized as follows.

Let us consider in \( \mathcal{C}(I, E) \) the subset

\[
Path \mathcal{C}(I, E)
\]

of paths in \( E \), given by all \( f \in \mathcal{C}(I, E) \), for which there exists \( h_f \in \mathcal{C}(I \times I, E) \), such that

\[
f(y) - f(x) = (y - x)h_f(x, y), \quad x, y \in I
\]

We give several useful properties of such paths. For that purpose, similar with \( \mathcal{C}(I, E) \), we first define a Banach space structure on the set \( \mathcal{C}(I \times I, E) \) of all continuous functions \( h : I \times I \rightarrow E \).

Let us now consider in \( \mathcal{C}(I \times I, E) \) the following closed subspace

\[
ad \mathcal{C}(I \times I, E)
\]

given by all \( h \in \mathcal{C}(I \times I, E) \), such that

\[
(y-x)h(x, y) + (z-y)h(y, z) + (x-z)h(z, x) = 0, \quad x, y, z \in I
\]

It is easy to see that, with the notation in (1.2), we have
∀ \( f \in \text{path} \, C(I, E) \):

(1.5)

\[ (*) \quad h_f \in \text{ad} \, C(I \times I, E) \]

\[ (**) \quad h_f \text{ is unique} \]

Furthermore, if \( f \in \text{path} \, C(I, E) \), then the classical derivative of Calculus, namely, \( f'(x) \) exists for every \( x \in I \), and

(1.6)

\[ f'(x) = h_f(x, x) = \lim_{y \to x, \, z \to x, \, y \neq z} \frac{(f(z) - f(y))/(z - y)}{y - x}, \quad x \in I \]

This property, however, will not be used in the categorial definition of the derivative given in the sequel, and instead, it will follow from that definition.

**Remark**

1) Regardless of (1.6), in the purely algebraic condition (1.4) which defines the elements in \( \text{ad} \, C(I \times I, E) \), there is no involvement of any kind of operation of limit, derivation, or even merely of a division with a quantity which may eventually vanish. And as we shall see in the sequel, that purely algebraic condition (1.4) is sufficient in order to define and effectively compute both the derivative and the integral by using a simple diagonal construction in a suitable category.

2) However, the idea in the rather unusual definition (1.3), (1.4) is, in view of (1.2), (1.6), quite straightforward. Namely, for every function \( f : I \to E \), we can define the function

(1.7)

\[ h : I \times I \to E \]

by the expression

(1.8)

\[ h(x, y) = \begin{cases} 
(f(y) - f(x))/(y - x) & \text{if } x, \, y \in I, \, x \neq y \\
\text{an arbitrary element } a_x \in E & \text{if } x = y \in I 
\end{cases} \]

and then, clearly, \( h \) satisfies (1.2) and (1.4).
The issue, however, is whether \( h \) is continuous on \( I \times I \), more precisely, whether in (1.8) one can choose \( a_x \in E \), with \( x \in I \), so that \( h \) becomes continuous. And obviously, this can be done, if and only if \( h \in ad \mathcal{C}(I \times I, E) \), in which case \( f \) is a path and \( h = h_f \), thus (1.6) holds.

\[ \square \]

Let us note two further useful properties of the functions in \( ad \mathcal{C}(I \times I, E) \).

First, it is easy to see that every \( h \in ad \mathcal{C}(I \times I, E) \) is symmetric.

Second, given two compact intervals \( I = [a, b], \ J = [b, c] \subset \mathbb{R} \) together with functions \( h \in ad \mathcal{C}(I \times I, E), \ k \in ad \mathcal{C}(J \times J, E) \). If \( h(b, b) = k(b, b) \), then there exists a unique function \( l \in ad \mathcal{C}((I \cup J) \times (I \cup J), E) \), such that \( l_{I \times I} = h \) and \( l_{J \times J} = k \).

### 2. A Few Related Categories

Let us denote by \( \text{Ban} \) the category of Banach spaces and their continuous linear mappings.

For every compact interval \( I \subset \mathbb{R} \), we obtain two covariant functors

\[
\mathcal{C}(I, -) : \text{Ban} \rightarrow \text{Ban}
\]

\[
(2.1)
\]

\[
\mathcal{C}(I \times I, -) : \text{Ban} \rightarrow \text{Ban}
\]

acting as follows. Given two Banach spaces \( E \) and \( F \) and a continuous linear mapping \( u : E \rightarrow F \), then

\[
\mathcal{C}(I, u) : \mathcal{C}(I, E) \ni f \mapsto u \circ f \in \mathcal{C}(I, F)
\]

\[
(2.2)
\]

\[
\mathcal{C}(I \times I, u) : \mathcal{C}(I \times I, E) \ni h \mapsto u \circ h \in \mathcal{C}(I \times I, F)
\]

Consequently, in view of (1.1), we can define the covariant functor
(2.3) \( \text{ad} \mathcal{C}(I \times I, -) : \text{Ban} \rightarrow \text{Ban} \)

as the restriction of the functor \( \mathcal{C}(I \times I, -) : \text{Ban} \rightarrow \text{Ban} \), namely, for \( u : E \rightarrow F \) as above, we have

(2.4) \( \text{ad} \mathcal{C}(I \times I, u) : \text{ad} \mathcal{C}(I \times I, E) \ni h \mapsto u \circ h \in \text{ad} \mathcal{C}(I \times I, F) \)

Finally, let us define the diagonal evaluation mapping

(2.5) \( \text{ed}_{I, E} : \text{ad} \mathcal{C}(I \times I, E) \rightarrow \mathcal{C}(I, E) \)

by

(2.6) \( (\text{ed}_{I, E}(h))(x) = h(x, x) \)

for \( h \in \text{ad} \mathcal{C}(I \times I, E) \) and \( x \in I \).

The basic result is given in

**Theorem on Diagonal Evaluation (L D Nel)**

For every compact interval \( I \subset \mathbb{R} \), the mapping \( \text{ed}_{I, -} \) is a natural isomorphism and isometry between the functors \( \text{ad} \mathcal{C}(I \times I, -) \) and \( \mathcal{C}(I, -) \).

### 3. The Derivative

Given a compact interval \( I \subset \mathbb{R} \) and a Banach space \( E \), the derivative \( D \) is defined as the mapping

(3.1) \( D : \text{Path} \mathcal{C}(I, E) \rightarrow \mathcal{C}(I, E) \)

where for \( f \in \text{Path} \mathcal{C}(I, E) \) we have, see (1.5)

(3.2) \( Df = \text{ed}_{I, E}(hf) \)

Here the important point to note is that, as mentioned at pct. 1 in the
Remark above, this definition of derivative does not refer in any way to limits, derivation, or even merely to division with a quantity which may eventually vanish. Yet, as seen next, this concept of derivative recovers the usual derivative in Calculus for functions $f : I \rightarrow E$, see Nel [1] for further details.

Indeed, in view of (2.6) and (1.6), we have

\[(Df)(x) = h_f(x, x) = f'(x), \quad x \in I\]

Therefore it follows that for every path $f \in \text{Path} \ C(I, E)$ and $x \in I$, we have

\[(Df)(x) = \lim_{y \to x, z \to x, y \neq z} \frac{f(z) - f(y)}{z - y}\]

which is the classical definition of derivative in Calculus.

Here we give a simple example which shows that the derivative in (3.1), (3.2) does not lead to any concept of derivation is a generalized sense, like for instance, distributional. Instead, it corresponds to the classical derivative in Calculus.

Let $I = [-1, 1], E = \mathbb{R}$, and let us take the function $f \in C(I, E)$ defined by $f = x^+, \text{that is, } f(x) = 0, \forall x \in [-1, 0], \text{while } f(x) = x, \forall x \in [0, 1]$.

Then $f \in C(I, E)$, however $f$ is not a path, therefore, the derivation in (3.1) - (3.3) cannot be applied to it.

Indeed, assume that $f$ is a path. Then (1.4) gives $h_f \in \text{ad} \ C(I \times I, E)$ such that $f(y) - f(x) = (y - x)h_f(x, y)$, for $x, y \in I$. Consequently (3.4), (3.5) imply that

$$\lim_{x \to 0, x \neq 0} \frac{f(x)}{x} = h_f(0, 0)$$

which is absurd, since the limit in the left hand term does not exist.

It follows that the that categorial definition of derivative in (3.1), (3.2)
rather corresponds to the usual derivative of $C^1$-smooth functions.

Let us further illustrate the above categorial concept of derivative in (3.1), (3.2) in the case of some simple functions.

A function $f \in C(I, E)$ is called **affine**, if and only if, for suitable $c, d \in E$, we have

\[(3.5) \quad f(x) = c + xd, \quad x \in I\]

We denote by $\text{Aff}(I, E)$ the subset of all affine functions $f \in C(I, E)$.

A function $f \in C(I, E)$ is called **polygonal**, if and only if it is piece-wise affine on $I$. We denote by $\text{Poly}(I, E)$ the subset of all polygonal functions $f \in C(I, E)$.

It is well known, Dieudonne, that the set of polygonal functions $\text{Poly}(I, E)$ is **dense** in $C(I, E)$.

Now for every affine function $f \in \text{Aff}(I, E)$ let us define the corresponding **averaging function**

\[(3.6) \quad \text{av}_f \in C(I \times I, E)\]

by

\[(3.7) \quad \text{av}_f(x, y) = c + ((x + y)/2)d, \quad x, y \in I\]

Then obviously

\[(3.8) \quad \text{av}_f \in ad C(I \times I, E), \quad ed_{I, E}(\text{av}_f) = f\]

It follows therefore, Nel [1, p.53], that one can in a piece-wise manner extend the averaging function in (3.6), (3.7) to all polygonal functions $f \in \text{Poly}(I, E)$, and obtain relations similar to those in (3.8).

Moreover, since the polygonal functions are dense in $C(I, E)$, it follows that one can further extend the averaging function in (3.6), (3.7) to
the whole of $\mathcal{C}(I, E)$, with the preservation of the relations in (3.8).

4. The Integral

Based on the categorial concept of derivative in (3.1), (3.2), we can define as well a notion of *integral*. Indeed, such an integral is given by the mapping

$$\int : I \times I \times \mathcal{C}(I, E) \ni (a, b, f) \mapsto \int_a^b f \overset{\text{def}}{=} (b - a) \text{av}_f(a, b) \in E$$

Now, the following two properties result:

For every path $f \in \text{Path} (I, E)$ and every $a, b \in I$, we have the classical Newton-Leibniz formula of Calculus

$$\int_a^b f' = f(b) - f(a)$$

where the derivative $f'$ is given by the categorial concept in (3.1), (3.2).

Conversely, for every given $f \in \mathcal{C}(I, E)$ and $a \in I$, let us define $F \in \mathcal{C}(I, E)$ by

$$F(x) = \int_a^x f, \quad x \in I$$

Then $F$ is a path in $\mathcal{C}(I, E)$, that is, $F \in \text{Path} (I, E)$, and with the categorial concept of derivative in (3.1), (3.2), we have

$$F' = f$$

Appendix

One way to define a *category* $\mathcal{A}$ is as being a directed graph which has the following properties:

C1) The vertices are called *objects*, while the directed edges are called *morphisms*, and given two objects $A$ and $B$, the totality of morphisms
$A \xrightarrow{f} B$ is a set denoted by $\text{Hom}_A(A, B)$.

C2) For every object $A$ there exists a morphism $A \xrightarrow{id_A} A$ with property C3.2) below.

C3) For every two morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, there exists a well defined morphism $A \xrightarrow{g \circ f} C$, such that :

C3.1) the operation $\circ$ is associative

C3.2) for every morphism $A \xrightarrow{f} B$, we have $f \circ id_A = f = id_B \circ f$

A functor $F$ acts between two categories $\mathcal{A}$ and $\mathcal{X}$, namely

\[ \mathcal{A} \xrightarrow{F} \mathcal{X} \]

as follows :

That is, $F$ takes objects $A$ in category $\mathcal{A}$ into objects $X = F(A)$ in category $\mathcal{X}$.

Further, $F$ takes morphisms $A \xrightarrow{f} B$ in category $\mathcal{A}$ into morphisms $X = F(A) \xrightarrow{\xi = F(f)} Y = F(B)$ in category $\mathcal{X}$.  

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The above actions of the functor $F$ have the properties:

For every object $A$ in category $\mathcal{A}$, we have in category $\mathcal{X}$ the relation

F1) $F(id_A) = id_{F(A)}$

Also, for every two morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in category $\mathcal{A}$, we have in category $\mathcal{X}$ the relation

F2a) $F(g \circ f) = F(g) \circ F(f)$

in which case the functor $F$ is called *covariant*.

If on the other hand, we have in category $\mathcal{X}$ the relation

F2b) $F(g \circ f) = F(f) \circ F(g)$

then the functor $F$ is called *contravariant*.

A *natural transformation* $\nu$ acts between two functors $F$ and $G$ which, on their turn, act between the same two categories $\mathcal{A}$ and $\mathcal{X}$, namely

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\nu} & \mathcal{A} \\
\downarrow F & & \downarrow G \\
\mathcal{X} & & \mathcal{X}
\end{array}
\]

And the action of $\nu$ is as follows:

N1) To every object $A$ in category $\mathcal{A}$ corresponds in category $\mathcal{X}$ a morphism
N2) Every morphisms $A \xrightarrow{f} B$ in category $\mathcal{A}$ generates the commutative diagram in category $\mathcal{X}$, given by
We recall that $\nu_A$ and $\nu_B$ above are morphisms in the category $\mathcal{X}$.

A natural transformation $\nu$ between categories $\mathcal{A}$ and $\mathcal{X}$ is called a *natural isomorphism*, if and only if for every object $A$ in category $\mathcal{A}$, the morphism

$$
\begin{array}{c}
F(A) \\
\downarrow \nu_A \\
G(A)
\end{array}
$$

is an isomorphism in category $\mathcal{X}$, that is, there exists a morphism $\mu_A : G(A) \rightarrow F(A)$ in category $\mathcal{X}$, such that

$$
\mu_A \circ \nu_A = id_{F(A)}, \quad \nu_A \circ \mu_A = id_{G(A)}
$$

References

[1] Dieudonné J : Foundations of Modern Analysis. Acad. Press, New York, 1969

[2] Herrlich H, Strecker G E : Category Theory. Heldermann, Berlin, 1973

[3] Nel L D [1] : Differential calculus founded on an isomorphism. Applied Categorical Structures, Vol. 1, 1993, 51-57

[4] Nel L D [2] : Foundational isomorphism in continuous differentiation theory. Applied Categorical Structures, 1977, Vol. 5, 1-9

[5] Nel L D [3] : Infinite dimensional calculus allowing for nonconvex domains with empty interior. Mh. Math., Vol. 110, 1990, 145-166

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[6] Nel L D [4] : Newton’s method and Frobenius-Dieudonné theorem in nonnormable spaces. Applied Categorical Structures, Vol. 5, 1997, 205-216

[7] Nel L D [5] : Nonlinear existence theorems in nonnormable analysis. In (Eds. Herrlich H, Porst H-E) Category Theory at Work, Heldermann, Berlin, 1991, 343-365

[8] Nel L D [6] : Effective categories of complete separated spaces. In Recent Developments General Topology and its Applications. International Conference in Memory of Felix Hausdorff. Mathematical Research 67, Berlin 1992, 242-251

[9] Nel L D [7] : Categorical Differential Calculus for Infinite Dimensional Spaces. Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol. 29-4, 1988, 257-286

[10] Nel L D [8] : Introduction to Categorical Methods. Parts 1 and 2. Carleton-Ottawa Mathematical Lecture Note Series, No. 11, August 1991

[11] Nel L D [9] : Introduction to Categorical Methods. Part 3. Carleton-Ottawa Mathematical Lecture Note Series, No. 12, March 1992

[12] Monadi A, Nel L D [10] : Holomorphy in Convergence Spaces. Applied Categorical Structures, Vol. 1, 1993, 233-245

[13] Monadi A, Nel L D [11] : Holomorphic Maps on Domains with Empty Interior. Quaestiones Mathematicae, Vol. 19, 1996, 409-415