A REMARK ON A TRACE PALEY–WIENER THEOREM

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Abstract. In this paper we prove a version of a trace Paley–Wiener theorem for tempered representations of a reductive $p$–adic group. This is applied to complete certain investigation of Shahidi on the proof that a Plancherel measure is invariant of a $L$–packet of discrete series.

1. Introduction

Let $G$ be a reductive $p$–adic group. Let $Rep(G)$ be the category of smooth admissible complex representations of $G$ of finite length, and let $R(G)$ be the corresponding Grothendieck group. We write $Ψ(G)$ (resp., $Ψ^u(G)$) for the group of (resp., unitary) unramified characters of $G$. $Ψ(G)$ has a structure of algebraic variety (a complex tours), and algebra of regular functions $C[Ψ(G)]$ is generated by evaluations with elements of $G$. The subgroup $Ψ^u(G)$ is Zariski dense in $Ψ(G)$. We say that a complex function is regular on $Ψ^u(G)$ if it is a restriction of a regular function on $Ψ(G)$. We observe that the restriction map from $C[Ψ(G)]$ into functions on $Ψ^u(G)$ is injective since $Ψ^u(G)$ is Zariski dense in $Ψ(G)$.

We fix a minimal parabolic subgroup $P_0$, its Levi decomposition $P_0 = M_0U_0$, and, as usual related to these choices, we fix a set of standard parabolic subgroups $P = MU$, where $M_0 ⊂ M$, $P = MP_0$. Since the standard parabolic subgroup is determined by the choice of Levi subgroup, the normalized parabolic induction $Ind_M^G(σ)$, where $σ$ is a smooth representation of $M$, we write as usual $i_{GM}(σ)$.

In [3], Bernstein, Deligne, and Kazhdan proved a trace Paley–Wiener theorem for category $Rep(G)$. We consider a full subcategory $Rep_t(G)$ of $Rep(G)$ consisting of representations having all irreducible subquotients tempered. Let $R_t(G)$ be the corresponding Grothendieck group. We write $R_t^i(G)$ for the subgroup generated by $i_{GM}(σ)$, where $M$ ranges over standard Levi subgroups of $G$ (including $G$), and $σ$ ranges over a set of square–integrable irreducible representations of $M$. We warn the reader that this notion is not an analogue of the notion of structly induced modules from (3, 3.1).

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We prove the following version of a trace Paley–Wiener theorem:

**Theorem 1.1.** Let \( f : R_\ell(G) \to \mathbb{C} \) be the \( \mathbb{Z} \)-linear form such that there exists an open compact subgroup \( K \subset G \) which dominates \( f \) (i.e., \( f \) is non-zero only on those irreducible tempered representations which have a non-trivial space of \( K \)-invariant vectors), and, for each standard maximal Levi subgroup \( M \) and a square-integrable modulo center representation \( \sigma \) of \( M \), the function \( \psi \mapsto f(i_{GM}(\psi \sigma)) \) is regular on \( \Psi_u(M) \), and for any other proper standard Levi subgroup \( N \) we have \( f(i_{GM}(\psi \sigma)) = 0 \) for all \( \psi \in \Psi_u(N) \). Then, there exists \( F \in C_c^\infty(G) \) such that \( f(\pi) = \text{tr}(\pi(F)) \), for all \( \pi \in R_i^\ell(G) \).

An effective version of this theorem is given by Proposition 3-1. This constructs correct function needed in the proof of ([9], Proposition 9.3 2)) in the case when \( M \) (see notation there is a Levi of a maximal parabolic subgroup. We remark that since Plancherel factors are multiplicative, it is enough to prove ([9], Proposition 9.3 2)) for maximal Levi subgroup.

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2. **Proof of Theorem**

This theorem is proved by reduction to the main result of [3]. Let \( M \) be a standard Levi subgroup then we write \( \Psi(M)^r \) the group of all unramified characters \( \psi \) such that \( \psi(G) \subset \mathbb{R}_{>0} \).

For a standard Levi subgroup \( M \), a tempered representation \( \pi \) of \( M \), and a “positive character”, with respect to \( P \), \( \psi \in \Psi(M)^r \), the module \( i_{GM}(\psi \pi) \) is called a standard module; it has a unique (Langlands quotient) \( L(i_{GM}(\psi \pi)) \). The condition of positivity is empty if \( M = G \). By the Langlands classification, every irreducible representation can be expressed in the form \( L(i_{GM}(\psi \pi)) \) for unique such datum \( (M, \pi, \psi) \). The following standard result will be used in the proof:

**Lemma 2-1.** The standard modules of \( G \) form a \( \mathbb{Z} \)-basis of \( R(G) \).

**Proof.** As in ([8], Proposition 1).

In analogy with ([3], 2.1), we make the following definitions.

Let \( \sigma \in \text{Irr}(M) \) where \( M \) is a standard Levi subgroup of \( G \). We define the usual affine variety attached to \( \sigma \)

\[ \text{Irr}(M) \supset D(\sigma) = \Psi(M)\sigma = \Psi(M)/\text{Stab}_{\Psi(M)}(\sigma), \]

\[ \text{Irr}(M) \supset D(\sigma) = \Psi(M)\sigma = \Psi(M)/\text{Stab}_{\Psi(M)}(\sigma), \]
where $\text{Stab}_{\Psi(M)}(\sigma)$ is a finite group consisting of all $\psi \in \Psi(M)$ such that $\psi \sigma \simeq \sigma$.

If $A$ is a maximal split torus in the centre of $M$, then the restriction map $\Psi(M) \to \Psi(A)$ is surjective, and the kernel is a finite group. Therefore, by considering the restriction to $A$ we find that

$$\text{Stab}_{\Psi^u(M)}(\sigma) = \text{Stab}_{\Psi(M)}(\sigma).$$

So, we may consider

$$D^u(\sigma) \overset{\text{def}}{=} \Psi^u(M)/\text{Stab}_{\Psi^u(M)}(\sigma) \subset D(\sigma).$$

It is easy to see that $D^u(\sigma)$ is Zariski dense in $D(\sigma)$.

The action of the Weyl group

$$W(M) = N_G(M)/M$$
on $\Psi(M)$ is algebraic. Moreover, $w \in W(M)$ transforms $\text{Stab}_{\Psi(M)}(\sigma)$ onto $\text{Stab}_{\Psi(M)}(w(\sigma))$. So that it maps $D(\sigma)$ (resp., $D^u(\sigma)$) onto $D(w(\sigma))$ (resp., $D^u(w(\sigma))$).

Put $D = D(\sigma)$ and $D^u = D^u(\sigma)$. As usual, we consider the group $W(D)$ of all $w \in W(M)$ such that $w(D) = D$. Explicitly, $W(D)$ consists of all $w \in W(M)$ such that there exists $\psi_w \in \Psi(G)$ such that

$$(2-2) \quad w(\sigma) \simeq \psi_w \sigma.$$

The character $\psi_w$ is determined uniquely modulo $\text{Stab}_{\Psi(M)}(\sigma)$.

The resulting orbit space

$$D/W(D)$$
is affine variety with algebra of regular functions given by

$$\mathbb{C}[D]^{W(D)}.$$ Explicitly, this variety is isomorphic to the orbit space of

$$\Psi(G)/\text{Stab}_{\Psi(M)}(\sigma)$$
under the action of $W(D)$ given by (see (2-2))

$$(2-3) \quad w.\psi \text{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi) \text{Stab}_{\Psi(M)}(\sigma).$$

This explicit realization will be important in the next section where we perform explicit constructions.

One can construct a regular function $D/W(D)$ in the following way:

**Lemma 2-4.** Let $f \in C^\infty_c(G)$. Then $\psi \mapsto tr(i_{GM}(\psi \sigma))$ belongs to $\mathbb{C}[D]^{W(D)}$. 
Proof. It is standard that this function is regular on $D$. We show that it is $W(D)$–invariant. Let $w \in W(D)$. By (3, Lemma 5.3 (iii)), we have

$$tr(i_{GM}(\psi \sigma)) = tr(i_{GM}(w(\psi \sigma)))$$

which completes the proof. \hfill $\square$

The above explicit description shows that analogously defined group $W(D_u)$ is a subgroup of $W(D)$. In fact, those groups are equal if $\sigma$ has unitary central character. In this case $D_u/W(D)$ is a subset of $D/W(D)$. It is Zariski dense in $D/W(D)$.

Now, we begin the preparation for the proof of the theorem. The following lemma is a fundamental result of Harish–Chandra:

**Lemma 2-5.** Assume that $M$ and $N$ are standard Levi subgroups of $G$, and $\sigma$ and $\tau$ are square–integrable modulo center representations of $M$ and $N$, respectively. Then, $i_{GM}(\sigma)$ and $i_{GN}(\tau)$ have a common irreducible sub-representation if and only if there exists $w \in G$ such that $N = wMw^{-1}$ and $\tau \simeq w(\sigma)$, where $w(\sigma)$ is defined by $w(\sigma)(n) = \sigma(w^{-1}nw)$, $n \in N$. Moreover, if there exists $w \in G$ such that $N = wMw^{-1}$, then $i_{GM}(\sigma)$ and $i_{GM}(w(\sigma))$ are isomorphic, and in particular equal in $R_t(G)$. 

Proof. [10]. \hfill $\square$

For irreducible $\pi \in R_t(G)$, there exists a standard Levi subgroup $M$ and a square–integrable modulo center $\sigma$ of $M$ such that $\pi \hookrightarrow i_{GM}(\sigma)$, the pair is $(M,\sigma)$ is unique up to a conjugation by Lemma 2-5. We call the equivalence class $[M,\sigma]$ under conjugation of the pair $(M,\sigma)$ the $t$–infinitesimal character of $\pi$. The set of equivalence of such pairs we denote by $\Theta_t(G)$.

For a pair $(M,\sigma)$, we define a natural map $\Psi^u(M) \longrightarrow \Theta_t(G)$ given by

$$\psi \longmapsto [M,\psi \sigma].$$

The image is called a connected component of $\Theta_t(G)$. We denote it by $\Theta_t(M,\sigma)$. This map induces a bijection which enable us to identify

$$\Theta_t(M,\sigma) = D^u(\sigma)/W(D(\sigma))$$

which gives an embedding

$$\Theta_t(M,\sigma) \subset D(\sigma)/W(D(\sigma))$$

realizing $\Theta_t(M,\sigma)$ as a Zariski dense subset of affine variety $D(\sigma)/W(D(\sigma))$.

As in (3, 2.1), we can decompose

$$R_t(G) = \bigoplus_\theta R_t(G)(\theta),$$

where $\theta$ ranges over connected components of $\Theta_t(G)$. Here

$$R_t(G)(\theta)$$
is generated with all tempered representations which \( t \)-infinitesimal characters belonging to \( \theta \).

In analogy with [3], we make the following definition. We say that a \( \mathbb{Z} \)-linear form \( f : \mathbb{R}_t(G) \to \mathbb{C} \) is \textit{good} if there exists an open compact subgroup \( K \subset G \) which dominates \( f \) (i.e., \( f \) is non-zero only on those irreducible tempered representations which have a non-trivial space of \( K \)-invariant vectors), and, for each standard Levi subgroup \( M \) and a square-integrable modulo center representation \( \sigma \) of \( M \), the function \( \psi \mapsto f(i_{GM}(\psi \sigma)) \) is regular on \( \Psi^u(M, \sigma) \) (consequently, using ([3], Lemma 5.3 (iii)), as in the proof of Lemma 2.4 it is regular function on \( \Theta_t(M, \sigma) \)).

The support of a \( \mathbb{Z} \)-linear form \( f : \mathbb{R}_t(G) \to \mathbb{C} \) is the set of all connected components \( \theta \) of \( \Theta_t(G) \) such that \( f \neq 0 \) on \( \mathbb{R}_t(G)(\theta) \).

**Lemma 2.6.** Every good functional is supported only on finitely many connected components.

**Proof.** Obvious. \( \square \)

Lemma 2.6 reduces the proof of the theorem to the case where \( f \) is supported on only one orbit. We consider this case and even more in the next section.

### 3. Proof of the Theorem; main part

In this section we fix a standard maximal Levi subgroup \( M \), a square-integrable representation \( \sigma \in \text{Irr}(M) \), and a regular function \( a \in \mathbb{C}[\Psi(M)] \) which is invariant under right-translations under \( Stab_{\Psi(M)}(\sigma) \) and \( W(D) \)-invariant under the action given by (2.3) where \( D = \Psi(M) \sigma \). This notation is explained in the previous section. We prove the following proposition:

**Proposition 3.1.** Under above assumptions, there exists \( F \in C_c^\infty(G) \) such that

\[
tr(\pi(F)) = \begin{cases} 
a(\psi) & \text{for } \pi = i_{GM}(\psi \sigma), \ \psi \in \Psi(M), \\0 & \text{for } \pi = i_{GN}(\psi \tau), \ \psi \in \Psi(N),\end{cases}
\]

for any standard proper Levi subgroup \( N \) and a square-integrable representation \( \tau \) such that \( \Theta_t(N, \tau) \neq \Theta_t(M, \sigma) \).

The proof of Proposition 3.1 is a generalization of ([8], 4.2, Proposition 1) where the proof of existence of pseudo-coefficients for semisimple \( G \) is given based also on [3].

We remark that \( \Psi^u(G) \) acts on \( \Psi^u(M) \) in a usual way:

\[
\psi \mapsto \chi|_M \psi, \ \chi \in \Psi^u(G), \ \psi \in \Psi^u(M).
\]
For $\psi \in \Psi^u(M)$, the stabilizer

$$\text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$$

is the group of all $\chi \in \Psi^u(G)$ such that

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

We remind the reader that for all $\chi \in \Psi^u(G)$ we have

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma).$$

**Lemma 3-2.** Assume that $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$. Then, for each irreducible constituent $\pi$ of $i_{GM}(\psi\sigma)$, the multiplicity of $\chi\pi$ in $\chi i_{GM}(\psi\sigma)$ is same as that of $\pi$ in $i_{GM}(\psi\sigma)$.

**Proof.** Obvious. $\square$

**Lemma 3-3.** Assume that for $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$ there exists an irreducible constituent $\pi$ of $i_{GM}(\psi\sigma)$ such that $\chi\pi$ is an irreducible constituent of $i_{GM}(\psi\sigma)$. Then, $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$.

**Proof.** First, $\chi\pi$ is a common constituent of $i_{GM}(\psi\sigma)$ and $i_{GM}(\chi|_M\psi\sigma)$. So, by Lemma [2-5] there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma = w(\psi\sigma).$$

Then, again by Lemma [2-5] we obtain

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

$\square$

**Lemma 3-4.** Let $\psi \in \Psi^u(M)$. Then, we have the following:

(i) If $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, then $a(\chi|_M\psi) = a(\psi)$.
(ii) For each $\eta \in \Psi(G)$ and $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we have

$$a(\chi|_M\eta|_M\psi) = a(\eta|_M\psi).$$

**Proof.** We prove (i). Since $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we obtain

$$i_{GM}(\chi|_M\psi\sigma) \simeq \chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

So, by Lemma [2-5] there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma = w(\psi\sigma).$$

This implies

$$w \in W(D),$$

and

$$\chi|_M\psi\sigma = \psi_w w(\psi)\sigma.$$
Consequently (see (2-3))

\[ \chi|_M \psi Stab_{\Psi(M)}(\sigma) = \psi_w w(\psi) Stab_{\Psi(M)}(\sigma) = w. \psi Stab_{\Psi(M)}(\sigma). \]

This implies \( a(\chi|_M \psi) = a(\psi) \). This proves (i).

To prove (ii), we may assume that \( \eta \) is unitary. Then, we obviously have

\[ Stab_{\Psi^u(G)}(i_{GM}(\eta|_M \psi \sigma)) = Stab_{\Psi^u(G)}(i_{GM}(\psi \sigma)). \]

Now, the claim follows from (i).

□

Now, we define a \( \mathbb{Z} \)-linear form \( f : R(G) \rightarrow \mathbb{C} \) in several steps.

1. For each \( \Psi^u(G) \)-orbit \( \mathcal{O} \) in \( \Psi^u(M) \), we fix a representative \( \psi_\mathcal{O} \in \mathcal{O} \) and an irreducible constituent \( \pi_\mathcal{O} \) in \( i_{GM}(\psi_\mathcal{O} \sigma) \). By Lemma 3-2 we have

\[ (3-5) \quad Stab_{\Psi^u(G)}(\pi_\mathcal{O}) \subset Stab_{\Psi^u(G)}(i_{GM}(\psi_\mathcal{O} \sigma)). \]

The quotient is finite and if \( \chi \) ranges over representatives of the quotient, then \( \chi \pi_\mathcal{O} \) ranges over the set of all mutually non-equivalent irreducible sub-representations in \( i_{GM}(\psi_\mathcal{O} \sigma) \) which are \( \Psi^u(G) \)-equivalent to \( \pi_\mathcal{O} \). Any of those representations, have the same multiplicity in \( i_{GM}(\psi_\mathcal{O} \sigma) \). Let \( m_\mathcal{O} \) be the sum of their multiplicities. We define:

\[ f(\chi \pi_\mathcal{O}) = \frac{a(\psi_\mathcal{O})}{m_\mathcal{O}}, \quad \chi \in Stab_{\Psi^u(G)}(i_{GM}(\psi_\mathcal{O} \sigma)). \]

Lemma 3-4 shows that this is well-defined.

2. For each \( \chi \in \Psi^u(G) \), we obviously have

\[ Stab_{\Psi^u(G)}(\chi \pi_\mathcal{O}) = Stab_{\Psi^u(G)}(\pi_\mathcal{O}) \]

and

\[ Stab_{\Psi^u(G)}(i_{GM}(\chi|_M \psi_\mathcal{O} \sigma)) = Stab_{\Psi^u(G)}(i_{GM}(\psi_\mathcal{O} \sigma)). \]

By, Lemma 3-2 and these remarks, the sum of multiplicities of \( \Psi^u(G) \)-equivalent representations of \( \pi_\mathcal{O} \) which belong to \( i_{GM}(\chi|_M \psi_\mathcal{O} \sigma) \) is again \( m_\mathcal{O} \). We let

\[ f(\chi \pi_\mathcal{O}) = \frac{a(\chi|_M \psi_\mathcal{O})}{m_\mathcal{O}}, \quad \chi \in \Psi^u(G). \]

3. For any other tempered irreducible representation \( \pi \) of \( G \) we let

\[ f(\pi) = 0. \]

4. For any quasi-tempered irreducible representation \( \pi \) of \( G \), we write \( \pi = \chi \pi^u \), where \( \chi \) is real unramified character of \( G \) and \( \pi^u \) is tempered. We let

\[ f(\pi) = 0. \]
if $\pi^u$ is not in $\Psi^u(G)\piO$ for any orbit $O$. But if $\pi^u \in \Psi^u(G)\piO$ for some $O$, then we can write $\pi^u = \psi\piO$, for some $\psi \in \Psi^u(G)$ uniquely determined modulo $\text{Stab}_{\Psi^u(G)}(\piO)$. We let

$$f(\pi) = \frac{a(\chi|_{M}\psi|M\psiO)}{mO}.$$ 

Using (3-5) and Lemma 3-4 we see that this is well-defined.

**Lemma 3-6.** $f$ is supported on $\Theta_t(M, \sigma)$. Moreover, we have the following:

$$f(\pi) = \begin{cases} a(\psi) & \text{for } \pi = i_{GM}(\psi\sigma), \; \psi \in \Psi^u(M), \\ 0 & \text{for } \pi = i_{GN}(\psi\tau), \; \psi \in \Psi^u(N), \end{cases}$$

for any standard Levi $N$ and a square–integrable representation $\tau$ such that $\Theta_t(N, \tau) \neq \Theta_t(M, \sigma)$.

6. Finally, we define $f$ on non–tempered Langlands quotients (see Lemma 2-1). let $f$ to be equal to zero on all standard modules (induced from proper parabolic subgroups) except at those of the form

$$i_{GM}(\chi\psi\sigma),$$

where $\psi \in \Psi^u(M)$, $\chi \in \Psi(M)^r$ “positive character”, with respect to $P$, where $P$ is standard parabolic subgroup with Levi $M$. In this case we let

$$f(i_{GM}(\chi\psi\sigma)) = a(\chi\psi).$$

It is also possible that $\chi \in \Psi(M)^r$ “positive character”, with respect to $P$. Then, there exists standard maximal parabolic subgroup $Q$ with standard Levi and $w \in G$ such that $N = wMw^{-1}$. Now, in $R(G)$, we have

$$i_{GM}(\chi\psi\sigma) = i_{GN}(w(\chi)w(\psi)w(\sigma)),$$

and $w(\chi)$ “positive character”, with respect to $P$. On the standard module $i_{GN}(w(\chi)w(\psi)w(\sigma))$ we let

$$f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi).$$

The third case that $\chi \in \Psi(M)^r$ is in neither chamber. Then it is in $\Psi(G)^r$, and

$$i_{GM}(\chi\psi\sigma) = \chi i_{GM}(\psi\sigma)$$

One can check the assumptions of (Theorem 1.2, [3]) as in [8]. This completes the proof of Proposition 3-1. Having completed the proof of Proposition 3-1, Theorem 1-1 is proved.
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