A NOTE ON GENERALIZED EQUIVARIANT HOMOTOPY GROUPS

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ABSTRACT. In this paper, we generalize the equivariant homotopy groups or equivalently the Rhodes groups. We establish a short exact sequence relating the generalized Rhodes groups and the generalized Fox homotopy groups and we introduce $\Gamma$-Rhodes groups, where $\Gamma$ admits a certain co-grouplike structure. Evaluation subgroups of $\Gamma$-Rhodes groups are discussed.

INTRODUCTION

In 1966, F. Rhodes [8] introduced the fundamental group of a transformation group $(X, G)$ for a topological space on which a group $G$ acts. This group, denoted by $\sigma_1(X, x_0, G)$, is the equivariant analog of the classical fundamental group $\pi_1(X, x_0)$. Rhodes showed that $\sigma_1(X, x_0, G)$ is a group extension of $\pi_1(X, x_0)$ with quotient $G$. Thus, $\sigma_1(X, x_0, G)$ incorporates the $G$-action as well as the action of $\pi_1(X, x_0)$ on the universal cover $\tilde{X}$ of the space $X$. This group has been used in [10] to study the Nielsen fixed point theory for equivariant maps. In 1969, F. Rhodes [9] extended $\sigma_1(X, x_0, G)$ to $\sigma_n(X, x_0, G)$, which is the equivariant higher homotopy group of $(X, G)$. Like $\sigma_1(X, x_0, G)$, $\sigma_n(X, x_0, G)$ is an extension of the Fox torus homotopy group $\tau_n(X, x_0)$ but not of the classical homotopy group $\pi_n(X, x_0)$ by $G$. The Fox torus homotopy groups were first introduced by R. Fox [2] in 1948 in order to give a geometric interpretation of the classical Whitehead product. Recently, a modern treatment of $\tau_n(X, x_0)$ and of $\sigma_n(X, x_0, G)$ has been given in [4] and in [5], respectively. In [5], we further investigated the relationships between the Gottlieb groups of a space and of its orbit space, analogous to the similar study in [3]. Further properties...
of the Fox torus homotopy groups, their generalizations, and Jacobi identities were studied in [6]. It is therefore natural to generalize \( \sigma_n(X, x_0, G) \) to more general constructions with respect to general spaces and to co-grouplike spaces \( \Gamma \) other than the 1-sphere \( S^1 \).

The main objective of this paper is to generalize \( \sigma_n(X, x_0, G) \) of a \( G \)-space \( X \) with respect to a space \( W \) and also with respect to a pair \((W, \Gamma)\), where \( W \) is a space and \( \Gamma \) satisfies a suitable notion of the classical co-grouplike space. W e prove in section 1 that the Rhodes exact sequence of [9] can be generalized to \( \sigma^W_W(X, x_0, G) := \{ [f; g] | f : (\hat{\Sigma}W, v_1, v_2) \to (X, x_0, gx_0) \} \), the \( W \)-Rhodes group, with the generalized Fox torus homotopy group \( \tau_W(X, x_0) \) as the kernel. In section 2, we further extend the construction of Rhodes groups to \( \sigma^\Gamma_W(X, x_0, G) := \{ [f; g] | f : (\Gamma(W), \bar{\gamma}_1, \bar{\gamma}_2) \to (X, x_0, gx_0) \} \), the \( W \)-\( \Gamma \)-Rhodes groups, where \( \Gamma \) admits a co-grouplike structure with two basepoints. Under such assumptions, we obtain a \( W \)-\( \Gamma \)-generalization of the Rhodes exact sequence [9]. In the last section, we generalize the notion of the Gottlieb (evaluation) subgroup to that of a \( W \)-\( \Gamma \)-Rhodes group and we establish a short exact sequence generalizing [5, Theorem 2.2]. Throughout, \( G \) denotes a group acting on a compactly generated Hausdorff path-connected space \( X \) with a basepoint \( x_0 \). The associated pair \((X, G)\) is called in the literature a transformation group.

1. Generalized Rhodes groups

For \( n \geq 1 \), F. Rhodes [9] defined higher homotopy groups \( \sigma_n(X, x_0, G) \) of a pair \((X, G)\) which is an extension of \( \tau_n(X, x_0) \) by \( G \) so that

\[
1 \to \tau_n(X, x_0) \to \sigma_n(X, x_0, G) \to G \to 1
\]

(1.1)

is exact. Here, \( \tau_n(X, x_0) \) denotes the \( n \)-th torus homotopy group of \( X \) introduced by R. Fox [2]. The group \( \tau_n = \tau_n(X, x_0) \) is defined to be the fundamental group of the function space \( X^{T^{n-1}} \) and is uniquely determined by the groups \( \tau_1, \tau_2, \ldots, \tau_{n-1} \) and the Whitehead products, where \( T^{n-1} \) is the \((n - 1)\)-dimensional torus. The group \( \tau_n \) is non-abelian in general.

Now we recall the construction of \( \sigma_n(X, x_0, G) \) presented in [9]. Suppose that \( X \) is a \( G \)-space with a basepoint \( x_0 \in X \) and let \( C_n = I \times T^{n-1} \). We say that a map \( f : C_n \to X \) is of order \( g \in G \) provided \( f(0, t_2, \ldots, t_n) = x_0 \) and \( f(1, t_2, \ldots, t_n) = g(x_0) \) for \( (t_2, \ldots, t_n) \in T^{n-1} \). Two maps \( f_0, f_1 : C^n \to X \) of order \( g \) are said to be homotopic if there exists a continuous map \( F : C^n \times I \to X \) such that:

- \( F(t, t_2, \ldots, t_n, 0) = f_0(t, t_2, \ldots, t_n) \);
• \( F(t, t_2, \ldots, t_n, 1) = f_1(t, t_2, \ldots, t_n); \)
• \( F(0, t_2, \ldots, t_n, s) = x_0; \)
• \( F(1, t_2, \ldots, t_n, s) = gx_0 \) for all \((t_2, \ldots, t_n) \in \mathbb{T}^{n-1}\) and \(s, t \in I.\)

Denote by \([f; g]\) the homotopy class of a map \(f : C_n \to X\) of order \(g\) and by \(\sigma_n(X, x_0, G)\) the set of all such homotopy classes. We define an operation \(*\) on the set \(\sigma_n(X, x_0, G)\) by

\[
[f' ; g'] * [f ; g] := [f' + g'f ; g'g].
\]

This operation makes \(\sigma_n(X, x_0, G)\) a group.

We have generalized in [4] the Fox torus homotopy groups. In this section, we give a similar generalization of Rhodes groups. In a special case, we obtain an extension group of the Abe group considered in [1].

Let \(X\) be a path-connected space with a basepoint \(x_0\). For any space \(W\), we let

\[
\sigma_W(X, x_0, G) := \{[f ; g] | f : (\hat{\Sigma}W, v_1, v_2) \to (X, x_0, gx_0)\}
\]

where \([f ; g]\) denotes the homotopy class of the map \(f\) of order \(g \in G\), \(v_1\) and \(v_2\) are the vertices of the cones \(C^+W\) and \(C^-W\), respectively and \(\hat{\Sigma}W = C^+W \cup C^-W\). Under the operation \([f_1 ; g_1] * [f_2 ; g_2] := [f_1 + g_1f_2 ; g_1g_2]\), \(\sigma_W\) is a group called a \(W\)-Rhodes group.

Write \(C(W, X)\) for the mapping space of all continuous maps from \(W\) to \(X\) with the compact-open topology. We point out that \(\sigma_W(X, x_0, G) = \sigma_1(C(W, X), \bar{x}_0, G)\) provided \(W\) is a locally-compact space, where \((gf)(x) = gf(x)\) for \(f \in C(W, X)\), \(g \in G\) and \(\bar{x}_0\) denotes the constant map from \(C(W, X)\) determined by the point \(x_0 \in X\).

The canonical projection \(\sigma_W(X, x_0, G) \to G\) given by \([f ; g] \mapsto g\) for \([f ; g] \in \sigma_W(X, x_0, G)\) has the kernel \(\{[f ; 1] | f : (\hat{\Sigma}W, v_1, v_2) \to (X, x_0, x_0)\}\). It is easy to see that this kernel is isomorphic to the generalized Fox torus group \([\Sigma(W \sqcup *), X] = \tau_W(X, x_0)\) defined in [4]. Therefore, we get the following result.

**Theorem 1.1.** The following sequence

\[
1 \to \tau_W(X, x_0) \to \sigma_W(X, x_0, G) \to G \to 1
\]

is exact.

**Remark 1.1.** When \(W = \mathbb{T}^{n-1}\), the \((n-1)\)-dimensional torus, \(\sigma_W\) coincides with the \(n\)-th Rhodes group \(\sigma_n\) and (1.2) reduces to (1.1). When \(W = \mathbb{S}^{n-1}\), the \((n-1)\)-sphere, \(\tau_W\) becomes
\( \kappa_n \), the \( n \)-th Abe group (see [2] or [4]). Thus, by Theorem 1.1 we have the following exact sequence

\[
1 \to \pi_n(X, x_0) \rtimes \pi_1(X, x_0) \cong \kappa_n(X, x_0) \to \sigma_{\Sigma-1}(X, x_0, G) \to G \to 1.
\]

One can also generalize the split exact sequence for Rhodes groups from [9] as follows.

**Theorem 1.2.** Let \( W \) be a space with a basepoint \( w_0 \). Then, for any space \( V \), the following sequence

\[
1 \to [(V \times W)/V, \Omega X] \to \sigma_{V \times W}(X, x_0, G) \overset{\sim}{\longrightarrow} \sigma_V(X, x_0, G) \to 1
\]

is split exact.

**Proof.** By [4, Theorem 3.1], we have the following split exact sequence

\[
1 \to [(V \times W)/V, \Omega X] \to \tau_{V \times W}(X) \overset{\sim}{\longrightarrow} \tau_V(X) \to 1.
\]

Given \( [F; g] \in \sigma_{V \times W}(X, x_0, G) \), where \( F : \tilde{\Sigma}(V \times W) \to X \), let \( f : \tilde{\Sigma}V \to X \) be the composite map of \( \tilde{\Sigma}V \approx \tilde{\Sigma}(V \times \{w_0\}) \to \tilde{\Sigma}(V \times W) \) with \( F \). This map gives rise to a homomorphism \( \sigma_{V \times W}(X, x_0, G) \to \sigma_V(X, x_0, G) \). Likewise, using the projection \( V \times W \to V \), one obtains a section \( \sigma_V(X, x_0, G) \to \sigma_{V \times W}(X, x_0, G) \). We have the following commutative diagram

\[
1 \longrightarrow \tau_{V \times W}(X, x_0) \longrightarrow \sigma_{V \times W}(X, x_0, G) \longrightarrow G \longrightarrow 1
\]

\[
1 \longrightarrow \tau_V(X, x_0) \longrightarrow \sigma_V(X, x_0, G) \longrightarrow G \longrightarrow 1,
\]

where the first two vertical homomorphisms have sections. Combining with (1.5), the assertion follows.

As an immediate corollary of Theorem 1.2 we have the following:

**Corollary 1.3.** The following sequence

\[
1 \to [W; \Omega X] \to \sigma_W(X, x_0, G) \overset{\sim}{\longrightarrow} \sigma_1(X, x_0, G) \to 1
\]

is split exact.

**Proof.** The result follows from Theorem 1.2 by letting \( V \) be a point.
Remark 1.2. For any space $W$, Corollary 1.3 asserts that $\sigma_1(X, x_0, G)$ acts on $[\Sigma W, X] = [W, \Omega X]$ according to the splitting. Furthermore, when $W = S^{n-1}$, this corollary gives an alternate description of the action of $\sigma_1$ on $\pi_n(X)$ as described in [5 Remark 1.4]. In this case, $\sigma_W(X, x_0, G) = \sigma_{S^{n-1}}(X, x_0, G)$ is the extension group of the $n$-th Aba group $\kappa_n(X, x_0)$ [1] as in (1.3). Thus, one can either embed $\sigma_1$ in $\sigma_n$ as in [5 Remark 1.4] or in $\sigma_{S^{n-1}}(X, x_0, G)$.

Unlike the reduced suspension $\Sigma$ which has the loop functor $\Omega$ as its right adjoint, the un-reduced suspension $\Sigma$ does not admit a right adjoint. Nevertheless, one can describe the adjoint property for the $W$-Rhodes groups as follows. Recall that a typical element in $\sigma_W(X, x_0, G)$ is a homotopy class $[f; g]$ where $f : (\Sigma W, v_1, v_2) \to (X, x_0, gx_0)$. Thus, $\sigma_W$ is a subset of $[\Sigma W, X]_0 \times G$, where $[\Sigma W, X]_0$ denotes the homotopy classes of maps $f : \Sigma W \to X$ such that $f(v_1) = x_0$ and $f(v_2)$ is independent of the homotopy class of $f$. Then, $\sigma_W$ is also a subset of $[W, \mathcal{P}_{x_0}]^* \times G$, where $[W, \mathcal{P}_{x_0}]^*$ denotes the set of homotopy classes of unpointed maps from $W$ to the space $\mathcal{P}_{x_0}$ of paths originating from $x_0$. In the special case when $G = \{1\}$, $\sigma_W = [\Sigma (W \cup *) , X] = \sigma^*_W = [W, \Omega Y]^* = [W \cup *, \Omega X]$.

2. Generalized $W$-$\Gamma$-Rhodes groups

In the definition of the generalized Rhodes group $\sigma_W(X, x_0, G)$, the two cone points from the un-reduced suspension $\Sigma W = C^+W \cup C^-W$ play an important role. Therefore in replacing $S^1$ with arbitrary co-grouplike space, we require that the space has two distinct basepoints.

Let $\Gamma$ be a space and $\gamma_1, \gamma_2 \in \Gamma$ satisfying the following conditions:

(I) there exists a map $\nu : (\Gamma, \gamma_1, \gamma_2) \to (\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2))$ such that $\text{proj}_i \circ \nu \simeq \text{id}$ as maps of triples for each $i = 1, 2$, where $\text{proj}_1, \text{proj}_2 : (\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2)) \to (\Gamma, (\gamma_1, \gamma_2))$ are the canonical projections;

(II) there exists a map $\eta : \Gamma \to \Gamma$ such that:

(a) $\eta(\gamma_1) = \gamma_2, \eta(\gamma_2) = \gamma_1$;

(b) $\nabla \circ (\text{id} \vee \overline{\eta}) \circ \nu$ is homotopic to the constant map at $\gamma_1$, where

$$\overline{\text{id} \vee \overline{\eta}} : \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2) \to \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_2), (\gamma_2, \gamma_1)$$

with $\overline{\text{id}}(\gamma, \gamma_1) = (\gamma, \gamma_2)$, $\overline{\eta}(\gamma_2, \gamma) = (\gamma_2, \eta(\gamma))$ for $\gamma \in \Gamma$ and $\nabla : (\Gamma \times \{\gamma_2\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_2), (\gamma_2, \gamma_1)) \to (\Gamma, (\gamma_1, \gamma_2))$ is the folding map;
We can easily show:

\[ \tilde{id}(\gamma_2, \gamma) = (\gamma_1, \gamma), \text{ and } \tilde{\eta}(\gamma, \gamma_1) = ((\eta(\gamma), (\gamma_1)) \text{ for } \gamma \in \Gamma; \]

(III) Moreover, we have co-associativity so that the following diagram

\[
\begin{array}{ccc}
(\Gamma, \gamma_1, \gamma_2) & \xrightarrow{\nu} & (\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2, \gamma_2)) \\
\downarrow \nu & & \downarrow \text{id} \circ \nu \circ \tilde{\nu} \\
(\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (v_2, v_2)) & \xrightarrow{\tilde{\nu}(\gamma_2, \gamma)} & (\Gamma \times \{\gamma_1\} \cup \{v_2\} \times \Gamma \times (\gamma_1, \gamma_1), (v_2, v_2))
\end{array}
\]

is commutative up to homotopy, where \( \gamma_1^* = (\gamma_1, (\gamma_1, \gamma_1)), \gamma_2^* = (\gamma_2, (\gamma_2, \gamma_2)), \) and the maps \( \tilde{\nu}(\gamma_2, \gamma) = ((\gamma_2, \gamma_2, \gamma)), \tilde{\nu}(\gamma, \gamma_1) = (\nu(\gamma), (\gamma_1)) \) for \( \gamma \in \Gamma. \)

Now, we generalize the notion of a co-grouplike space presented e.g., in [7]. A co-grouplike space with two basepoints \( \Gamma = (\Gamma, \gamma_1, \gamma_2; \nu, \eta) \) consists of a topological space \( \Gamma \) together with basepoints \( \gamma_1, \gamma_2 \) and maps \( \nu, \eta \) satisfying conditions (I) - (III). For any space \( W, \) the smash product is given by

\[ \Gamma(W) := W \times \Gamma / \{(w, \gamma_1) \sim (w', \gamma_1), (w, \gamma_2) \sim (w', \gamma_2)\} \]

for any \( w, w' \in W. \)

For instance, if \( \Gamma = ([0,1], 0, 1; \nu, \eta) \) with \( \nu(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 - 2t, & \text{if } \frac{1}{2} < t \leq 1 \end{cases} \) and \( \eta(t) = 1 - t \)

for \( t \in [0,1] \) then \( \Gamma(W) = \hat{\Sigma}W, \) the un-reduced suspension of \( W. \)

**Remark 2.1.** Note that if \( \gamma_1 = \gamma_2, \) we obtain the usual co-grouplike structure and \( \Gamma_0 := \Gamma / \sim \) given by identifying the basepoints \( \gamma_1 \) and \( \gamma_2 \) is a co-grouplike space as well.

Next, we define the \( W-\Gamma \)-Rhodes groups.

Let \( \Gamma \) be a co-grouplike space with two basepoints, \( (X, G) \) a \( G \)-space and \( W \) a space. The \( W-\Gamma \)-Rhodes group of \( X \) with respect to \( W \) is defined to be

\[ \sigma^\Gamma_W(X, x_0, G) = \{ [f; g] : (\Gamma(W), \gamma_1, \gamma_2) \to (X, x_0, gx_0) \}. \]

Write \( \tau^\Gamma_W(X, x_0) \) for the \( \Gamma_0 \)-\( W \)-Fox group considered in [6].

We can easily show:
Proposition 2.1. Let \( \pi : \sigma^W_0(X, x_0, G) \rightarrow G \) be the projection sending \([f; g] \mapsto g\). By identifying the two basepoints of \( \Gamma(W) \), the quotient space \( \Gamma(W)/\sim \) is canonically homeomorphic to \( \Gamma_0 \wedge (W \cup \{\ast\}) \). Furthermore,

\[
\text{Ker} \, \pi \cong [\Gamma(W)/\sim, X] = \tau^\Gamma_W(X, x_0).
\]

Then, we obtain a general \( \Gamma \)-Rhodes exact sequence, generalizing (1.2).

Theorem 2.2. The following sequence

\[
1 \rightarrow \tau^\Gamma_0(X, x_0) \rightarrow \sigma^\Gamma_0(X, x_0, G) \xrightarrow{\pi} G \rightarrow 1
\]

is exact.

We now derive the following generalized split exact sequence for the \( W \)-\( \Gamma \)-Rhodes groups.

Corollary 2.3. Let \( W \) be a space with a basepoint \( w_0 \) and \( \Gamma \) be a co-grouplike space with two basepoints. The following sequence

\[
(2.1) \quad 1 \rightarrow [\Gamma_0 \wedge ((V \times W)/V), \Omega X] \rightarrow \sigma^\Gamma_{V \times W}(X, x_0, G) \xrightarrow{\tau^\Gamma_{V \times W}(X, x_0, G) \rightarrow} 1
\]

is split exact.

Proof. From Theorem 2.2, we have the following two short exact sequences:

\[
1 \rightarrow \tau^\Gamma_0(X, x_0) \rightarrow \sigma^\Gamma_{V \times W}(X, x_0, G) \xrightarrow{\pi} G \rightarrow 1
\]

and

\[
1 \rightarrow \tau^\Gamma_0(X, x_0) \rightarrow \sigma^\Gamma(V, x_0, G) \xrightarrow{\pi} G \rightarrow 1.
\]

Moreover, the following split exact sequence was shown in [6, Theorem 4.1]:

\[
1 \rightarrow [\Gamma_0 \wedge ((V \times W)/V), \Omega X] \rightarrow \tau^\Gamma_{V \times W}(X, x_0, G) \xrightarrow{\tau^\Gamma_{V \times W}(X, x_0, G) \rightarrow} 1.
\]

A straightforward diagram chasing argument involving these short exact sequences yields the desired split exact sequence. \( \Box \)
3. Evaluation subgroups of $W$-$\Gamma$-Rhodes groups

We end this note by extending a result concerning the evaluation subgroups of the Rhodes groups and the Fox torus homotopy groups obtained in [5, Theorem 2.2].

Given a $G$-space $X$, the function space $X^X$ is also a $G$-space where the action is pointwise, that is, $(gf)(x) = gf(x)$ for $f \in X^X$, $g \in G$ and $x \in X$. Let $\Gamma$ be a co-group like space with two basepoints and $W$ be a space.

The *evaluation subgroup* of the $W$-$\Gamma$-Rhodes group of $X$ is defined by

$$G\sigma^\Gamma_W(X, x_0, G) := \text{Im}(ev_* : \sigma^\Gamma_W(X^X, \text{id}_X, G) \to \sigma^\Gamma_W(X, x_0, G)).$$

Similarly, the *evaluation subgroup* of $\tau^\Gamma_0 W(X, x_0)$ is defined by

$$G\tau^\Gamma_0 W(X, x_0) := \text{Im}(ev_* : \tau^\Gamma_0 W(X^X, \text{id}_X) \to \tau^\Gamma_0 W(X, x_0)).$$

It is straightforward to see that the proof of [5, Theorem 2.2] is also valid in the setting of $W$-$\Gamma$-Rhodes groups. Therefore, we have the following generalization.

**Theorem 3.1.** Let $G_0$ be the subgroup of $G$ consisting of elements $g$ considered as homeomorphisms of $X$ which are freely homotopic to the identity map $\text{id}_X$. Then the following sequence

$$1 \to G\tau^\Gamma_0 W(X, x_0) \to G\sigma^\Gamma_W(X, x_0, G) \to G_0 \to 1.$$ 

is exact.

**References**

[1] Abe, M., Über die stetigen Abbildungen der $n$-Sphäre in einen metrischen Raum, *Japanese J. Math.* **16** (1940), 169-176.

[2] Fox, R., Homotopy groups and torus homotopy groups, *Ann. of Math.* **49** (1948), 471-510.

[3] Golasiński, M. and Gonçalves, D., Postnikov towers and Gottlieb groups of orbit spaces, *Pacific J. Math.* **197** (2001), no. 2, 291–300.

[4] Golasiński, M., Gonçalves, D. and Wong, P., Generalizations of Fox homotopy groups, Whitehead products, and Gottlieb groups, *Ukrain. Math. Zh.* **57** (2005), no. 3, 320–328 (translated in Ukrainian Mat. J. **57** (2005) no. 3, 382-393).

[5] Golasiński, M., Gonçalves, D. and Wong, P., Equivariant evaluation subgroups and Rhodes groups, *Cah. Topol. Géom. Différ. Catég.* **48** (2007), 55–69.

[6] Golasiński, M., Gonçalves, D. and Wong, P., On Fox spaces and Jacobi identities, *Math. J. Okayama Univ.*, to appear.
[7] Oda, N. and T. Shimizu, A Γ-Whitehead product for track groups and its dual, *Quaest. Math.* **23** (2000), 113-128.

[8] Rhodes, F., On the fundamental group of a transformation group, *Proc. London Math. Soc.* **16** (1966), 635–650.

[9] Rhodes, F., Homotopy groups of transformation groups, *Canad. J. Math.* **21** (1969), 1123–1136.

[10] Wong, P., Equivariant Nielsen numbers, *Pacific J. Math.* **159** (1993), 153–175.

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