Conformally Lifshitz solutions from Horava–Lifshitz Gravity

Mohsen Alishahiha\textsuperscript{1} and Hossein Yavartanoo\textsuperscript{2,3}

\textsuperscript{1} School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran
\textsuperscript{2} Department of Physics, Kyung-Hee University, Seoul 130-701, Korea
\textsuperscript{3} State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

E-mail: alishah@ipm.ir and yavar@itp.ac.cn

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Abstract
We show that the IR action of the healthy non-projectable Hořava–Lifshitz gravity and its small modification exhibit Lifshitz and hyperscaling violating solutions, respectively, though it does not have the corresponding static black hole solutions. The model may also have an AdS\textsubscript{2} \( \times \mathbb{R}^{d} \) vacuum solution.

Keywords: Horava–Lifshitz gravity, AdS/CFT, black hole

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1. Introduction

The Lifshitz fixed point is a critical point where space and time scale differently. For the spatially isotropic scale invariant case, the corresponding fixed point is characterized by a dynamical exponent, \( z \), as follows

\[
t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad i = 1, \cdots, d.
\]

(1.1)

In the context of AdS/CFT correspondence [1], the gravity dual of the Lifshitz fixed point is provided by the following metric [2]\textsuperscript{4}

\[
dx^2 = -r^{2c} \, dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d} dx_i^2.
\]

(1.2)

This metric, whose isometry group contains the generators of spatial rotations and translations, time translations and dilations, is known as Lifshitz geometry. Note that the isometry group of the Lifshitz geometry does not mix time and space coordinates.

\textsuperscript{4} See also [3] for an earlier work on a geometry with the Lifshitz scaling.
More generally one may consider a metric which is conformally Lifshitz. The corresponding geometry may be written as follows [4]

$$ds^2_{d+2} = r^{-2\alpha} \left( -r^{2\alpha} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d} dx_i^2 \right),$$

(1.3)

where the constant $\theta$ is the hyperscaling violation exponent and the metric is known as hyperscaling violating geometry. The reason for this name is as follows. Since with a non-zero $\theta$ the distance is not invariant under the scaling, using AdS/CFT correspondence, it indicates violations of hyperscaling in the dual field theory. More precisely, while in $(d + 1)$-dimensional theories without hyperscaling (dual to background (1.2)) the entropy scales as $T^{d-\theta}/z$ with temperature. In the present case (dual to background (1.3)) it scales as $T^{(d-\theta)/2}$ [5, 6].

It is obvious that the above metrics are not solutions of a pure cosmological Einstein gravity, simply because in pure Einstein gravity there is nothing to produce an anisotropy in space-time. In fact to obtain such solutions one needs to couple Einstein gravity to other fields. In the minimal case the extra field could be a massive gauge field [7] which can, indeed, produce an anisotropy in the space-time leading to Lifshitz geometry (see also [8]). More naturally the hyperscaling violating metrics may be found in the Einstein–Maxwell–Dilaton theory (see for example [9, 10]).

It is important to note that so far the non-relativistic solutions (1.2) and (1.3) have mostly been studied in the context of relativistic theories where the corresponding actions are invariant under the full space-time diffeomorphism. On the other hand, taking into account that conformally Lifshitz geometries are not invariant under the full Lorentz group, one may naturally pose a question whether the above solutions may be found from a non-relativistic action such as that of Hořava–Lifshitz (HL) gravity [11, 12].

Actually this question has been recently addressed in [13] where the authors have shown that the Lifshitz metric can be obtained from HL gravity. Indeed in a bottom-up approach using the following ADM decomposition of the metric

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a - N^a dt)(dx^b - N^b dt).$$

(1.4)

The authors of [13] considered an action as follows

$$S = \frac{1}{2\kappa^2} \int dt dr d^dx \sqrt{gN} \left( K_{ab} K^{ab} - \lambda K^2 + \beta (R - 2\Lambda) + \frac{\alpha^2}{2} \nabla_a N \nabla^a N \right).$$

(1.5)

where $K_{ab} = \frac{1}{N} (\partial_t g_{ab} - \nabla_a N_b - \nabla_b N_a)$ is the extrinsic curvature of the foliation, $K = g^{ab} K_{ab}$ and $R$ is the scalar curvature of the metric $g_{ab}$. In the context of HL gravity the above action consists of the most relevant terms at low energies. Moreover the gauge symmetries of the action are the foliation preserving diffeomorphisms which could naturally contains the Lifshitz group. This is, indeed, the IR action of the healthy non-projectable model introduced in [14, 15].

In the present paper we would like to further explore a possibility of having non-relativistic vacuum solutions in the action (1.5). In particular we shall show that with a small modification, namely assuming an $r$-dependent $\Lambda$, the model could support hyperscaling violating solutions. Alternatively the hyperscaling violation solutions may be obtained by adding a minimally non-relativistic scalar field with a wrong sign in the potential. Interestingly enough, even though the system is unchanged, the model also supports an AdS$_2 \times \mathbb{R}^d$ solution which might be thought of as an emergent IR fixed point$^5$. This might be expected because an $AdS_2 \times \mathbb{R}^d$ may be thought of as a Lifshitz geometry with $z \to \infty$ dynamical exponent.

More interestingly, one observes that although the model admits Lifshitz and hyperscaling violating solutions, the corresponding black holes metrics are not solutions of the model. It seems that setting $\alpha \neq 0$ prevents the model from having black hole solutions.

$^5$ Typically an AdS$_2$ factor appears at the near horizon of the extremal black holes where the backgrounds are charged.
Note that for non-zero $\alpha$, AdS$_{d+2}$ is not a solution of the equations of motion, though setting $\alpha = 0$ the equations of motion admits an AdS$_{d+2}$ vacuum solution. It is, however, important to mention that even though with $\alpha = 0$ we get an AdS solution, the gravitational model is not the standard GR. This is due to the fact that the model does not have general temporal diffeomorphism. Note also that for $\alpha = 0$ the model has AdS black hole solutions.

The paper is organized as follows. In the next section we will obtain a Lifshitz solution for the action (1.5), where we show that the model does not admit the corresponding black hole solutions. Then assuming an $r$-dependent $\Lambda$ (or by coupling the gravity to a scalar field) we find hyperscaling violating solutions. In section 3 we rewrite the model in a covariant form where we shall argue that within the context of HL gravity the Lifshitz fixed point might be treated as a CFT deformed by the time component of a vector operator. The last section is devoted to conclusions.

2. Non-relativistic solutions from HL gravity

2.1. Lifshitz geometry

In this section we would like to study a general solution of HL gravity which is asymptotically Lifshitz geometry. To begin with we consider the following ansatz

$$d^2s = e^{2f(r)}dr^2 + \frac{dr^2}{e^{2h(r)}} + e^{2l(r)}dx^2.$$  \hspace{1cm} (2.1)

Of course with $\alpha = 0$, even though we have still two more free parameters, one would not expect to get a Lifshitz solution from the action (1.5), while when $\alpha$ is non-zero this minimal model has a chance to have such a vacuum.

For the ansatz (2.1) one has

$$N = e^f(r), \quad g_{r r} = \frac{1}{e^{2h(r)}}, \quad g_{r a} = e^{2l(r)}, \quad N_a = 0, \quad g_{ij} = 0, \quad \text{for } i \neq j.$$  \hspace{1cm} (2.2)

Thus

$$\sqrt{g} = e^{l(r) - h(r)}, \quad K_{ab} = K = 0.$$  \hspace{1cm} (2.3)

Note also that

$$R = -\frac{d}{2}e^{2h(r)[2l''(r) + (d + 1)l'^2(r) + 2l'(r)h'(r)].}$$  \hspace{1cm} (2.4)

Plugging this ansatz into the action (1.5) and after integration by parts one arrives at

$$S = \frac{d\beta v}{2\kappa^2} \int dr \ e^{d+l+f} \left[ (d - 1)l'^2 + 2l'f' + \delta f'^2 - 2\frac{\Lambda}{d} e^{-2h} \right] \equiv \frac{d\beta v}{2\kappa^2} \int dr \ e^{d+l+f} \mathcal{L}_0.$$  \hspace{1cm} (2.5)

where $\nu = \int dt \ d^d x$ and $\delta = \frac{\alpha^2}{3d}$. To find the functions $f, l$ and $h$, one may write down the equations of motion of the above one-dimensional action. Furthermore, the equations of motion are supplemented by a zero energy constraint.

The corresponding equations of motion of $l$ and $f$ are

$$[e^{dl+hf} (\delta f' + l')]' = \frac{1}{2} e^{dl+hf} \mathcal{L}_0, \quad [e^{dl+hf} (f' + (d - 1)l')]' = \frac{d}{2} e^{dl+hf} \mathcal{L}_0.$$  \hspace{1cm} (2.6)

On the other hand since the derivative of $h$ does not appear in the action (2.5), its equation of motion gives us the following constraint

$$\mathcal{L}_0 = -4\frac{\Lambda}{d} e^{-2h},$$  \hspace{1cm} (2.7)

which is, indeed, the zero energy constraint.
By making use of the zero energy constraint (2.7), the equations of motion (2.6) may be simplified as follows

\[ [e^{\delta l} + h f (d' + l')]' = -2 \frac{A}{d} e^{\delta l - h f}, \quad [e^{\delta l} + h f (d' + (d - 1)l')]' = -2 \Lambda e^{\delta l - h f}. \] (2.8)

One may further simplify the above equations to get

\[ (1 - d \delta) f' - l' = c e^{-d \delta - h f} \] (2.9)
where \( c \) is a constant of integration.

Of course, in general, it is not an easy task to solve these equations. Nevertheless, inspired by the known solutions in the literature, one may guess an ansatz with free parameters and try to fix the parameters by plugging it in the above equations. To proceed we will consider the following ansatz

\[ f = z \ln r + \frac{1}{4} \ln \xi (r), \quad h = \ln r + \frac{1}{2} \ln \xi (r), \quad l = \ln r. \] (2.10)

Plugging this ansatz into the equation (2.8) and (2.9) one observes that there could be a non-trivial solution if we take

\[ \delta = \frac{z - 1}{dz}, \quad \Lambda = -\frac{(d + z - 1)(d + z)}{2}, \] (2.11)
by which the solution reads

\[ ds^2 = -r^2 \xi (r) dr^2 + \frac{dr^2}{r^2 \xi (r)} + r^2 d\xi^2, \quad \xi (r) = 1 - \left( \frac{r_0}{r} \right)^{d+z}, \] (2.12)
which looks like a black brane solution in an asymptotically Lifshitz geometry. It is, however, important to note that the resultant metric must also satisfy the zero energy condition. Plugging the above metric into (2.7) one observes that it is a solution if \( r_0 = 0 \) which is a Lifshitz solution.

In a more general setup, one may also seek for a black hole solution in Lifshitz background in this theory. To solve the equations we can always take \( l(r) = \ln r \). A combination of two equations in (2.8) gives an algebraic equation on \( h(r) \) and substituting its solutions into the other equations gives second rank differential equations on \( f(r) \),

\[ (d \delta - d - 1) r^2 f'' + (d^2 \delta + 2 - d - \delta) r f' + (2d \delta + \delta - 2) r^2 f'^2 + (d \delta^2 - \delta) r^3 f^3 + d - 1 = 0. \] (2.13)

Assuming the metric has a horizon at \( r = r_0 \), we can take \( e^{2f} = (r - r_0)^m g(r) \), where \( m \) is a positive real number and \( g(r) \) does not have any zero or pole for \( r > r_0 \) and \( g = r^{2m} \) at large \( r \). It is straightforward to check that for generic values of \( d, \alpha \) and \( \beta \), equation (2.13) does not admit such a solution. We conclude that the model does not admit a black hole solution.

Although the general solution of equation (2.13) is cumbersome, however there are some special cases where the general solution has a rather simple form. For example if we take \( \delta = \frac{1}{d-1} \) we get

\[ f = C - (d - 1) \ln r, \quad \Lambda = 0, \] (2.14)
where \( C \) is a constant. If we take \( \delta = \frac{1}{2} \) we get the following solution

\[ e^{2f} = \frac{4d}{r^{2d}(c_1 r^{\sqrt{2}} + c_2 r^{-\sqrt{2}})^2}, \quad e^{2h} = \frac{\Lambda r^2 (c_1 r^{\sqrt{2}} + c_2 r^{-\sqrt{2}})^2}{2d c_1 c_2}. \] (2.15)

In next section we will add more stuffs in the model to see if the model could have a black hole solution.
2.2. Hyperscaling violating metric

In this subsection we would like to investigate a possibility of having a conformally Lifshitz solution in HL gravity. It is clear that an action in the form of (1.5) cannot exhibit a conformally Lifshitz solution. This is due to the fact that although the action has an anisotropic nature which is needed for Lifshitz geometry, it does not have enough free parameters to support the degree of freedom needed to have a hyperscaling violation solution. Nevertheless one may have a simple modification of the action (1.5) which could support a hyperscaling violation metric as its vacuum solution.

Actually one could still work with the same action as (1.5) with an assumption that the cosmological constant $\Lambda_1$ is not, indeed, a constant. In other words we may assume an $r$-dependent $\Lambda_1$. In this case, essentially, the equations of motion are the same as that in the previous subsection, except one should replace $\Lambda_1 \rightarrow \Lambda_1(r)$. With this assumption we will proceed with the following ansatz

$$f = \left(z - \frac{\theta}{d}\right) \ln r + \frac{1}{2} \ln \xi(r), \quad h = \left(1 + \frac{\theta}{d}\right) \ln r + \frac{1}{2} \ln \xi(r), \quad l = \left(1 - \frac{\theta}{d}\right) \ln r.$$  \hspace{1cm} (2.16)

Plugging this ansatz in the equation (2.9) one finds

$$\xi(r) = 1 - \left(\frac{r_0}{r}\right)^{d+z-\theta}, \quad \delta = \frac{z - 1}{dz - \theta}. \hspace{1cm} (2.17)$$

On the other hand using the equation (2.8) one gets

$$\Lambda_1(r) = -\frac{(d+z-1-\theta)(d+z-\theta)}{2} r^{2z} \hspace{1cm} (2.18)$$

Although this metric looks like a black brane in an asymptotically hyperscaling violating geometry [9], again the zero energy condition forces us to set $r_0 = 0$. Therefore we just get a zero temperature solution. It is clear that for $\theta = 0$ the above solution reduces to (2.12) in the previous subsection.

Alternatively the hyperscaling violating geometry may be obtained from HL gravity if one couples the action (1.5) to a non-relativistic matter. In the minimal model the corresponding matter field could be a real scalar field whose action consists of a quadratic kinetic term with the right symmetries and a potential. In general the action for a scalar field may be written as [17]

$$S_{Sc} = \frac{1}{2\kappa^2} \int d^dx \sqrt{|g|} \left(\frac{1}{N^2} (\Phi - N a_\alpha_a \Phi)^2 - V(\partial_\alpha \Phi, \Phi)\right). \hspace{1cm} (2.19)$$

Therefore adding this scalar action with the gravity sector (1.5) would provide a model in which we will seek a hyperscaling violation solution. To find such a solution we will consider the following potential for the scalar field

$$V(\partial_\alpha \Phi, \Phi) = -\frac{1}{2} \delta g^{ab} \partial_a \Phi \partial_b \Phi - V_0 e^{\gamma \Phi} \hspace{1cm} (2.20)$$

Note that the first term, comparing with the conventional relativistic scalar field, has a ‘wrong sign’. For a static ansatz for the scalar field and ansatz (2.1) for the metric, the total action is reduced to the following one-dimensional action

$$S = \frac{d\beta v}{2\kappa^2} \int dr \ e^{d\beta + h + f} \left[(d-1)l'f'' + 2l' f' + f^2 + \frac{1}{d\beta} \Phi^2 + \frac{V_0}{d\beta} e^{\gamma \Phi - 2\kappa} \right]$$

$$\equiv \frac{d\beta v}{2\kappa^2} \int dr \ e^{d\beta + h + f} L. \hspace{1cm} (2.21)$$

6 Note that in this case we can absorb the cosmological constant in the definition of potential of the scalar field.
Therefore the corresponding equations of motion of $l$ and $f$ are

$$[e^{dl+h/f}(\delta f'+l')]' = \frac{1}{2} e^{dl+h/f} \mathcal{L}, \quad [e^{dl+h/f} (f' + (d - 1)l')]' = \frac{d}{2} e^{dl+h/f} \mathcal{L}. \quad (2.22)$$

On the other hand the equation of motion for $h$ gives us the following constraint

$$\mathcal{L} = 2V_0 e^{d\Phi - 2h}, \quad (2.23)$$

while for the scalar field one finds

$$[e^{dl+h/f} \Phi']' = \frac{V_0 \gamma}{2} e^{d\Phi - h + y \Phi}. \quad (2.24)$$

Then it is easy to check that the ansatz

$$f = \left(1 - \frac{\theta}{d}\right) \ln r, \quad h = \left(1 + \frac{\theta}{d}\right) \ln r, \quad l = \left(1 - \frac{\theta}{d}\right) \ln r, \quad \Phi = \phi_1 \ln r \quad (2.25)$$

solves the equations of motion if one takes

$$\delta = \frac{z-1}{dz-\theta}, \quad V_0 = (d+z-\theta-1)(d+z-\theta), \quad \gamma^2 = \frac{4\theta}{d(d+z-\theta-1)\beta} \quad (2.26)$$

and then

$$\phi_1 = \frac{2\theta}{d} \ln r. \quad (2.27)$$

Therefore the solution is

$$ds^2 = r^{-2\delta} \left(-r^2 \frac{dr^2}{r^2} + r^2 \sum_{i=1}^d dx_i^2\right), \quad \Phi = \frac{2\theta}{d} \ln r. \quad (2.28)$$

It is worth noting that, at least when the matter field is given by a minimally coupled non-relativistic scalar field, the hyperscaling violating solution does exist if the potential term of the scalar field has a wrong sign. Of course this is consistent with the fact that the hyperscaling violating metric is a solution of pure gravity (the action (1.5)) with the $r$-dependent $\Lambda$. This means that the matter field should mimic the nature of a cosmological constant. Then it is natural to examine the stability of the solution, though we will postpone it for a further study. On the other hand if we had worked with the plus sign in front of $(\partial_\phi \Phi)^2$ term in the potential, the value for $\gamma$ would have been imaginary, leading to an imaginary solution.

Moreover the running of the scalar field might suggest that the solution is not IR complete and therefore one cannot trust the solution for small $r$. In fact the situation is similar to the relativistic case studied in [18] where it was shown that by taking into account the quantum corrections, the magnetically charged hyperscaling violating solution could be completed at IR by an emergent $\text{AdS}_2 \times \mathbb{R}^d$ solution.

Although the aim of the present paper is not to explore a possible IR completion of the geometry, it is important to note that with a non-zero $\alpha$ the model admits another interesting solution as follows

$$\Phi = \Phi_0, \quad \delta = \frac{1}{d}, \quad V_0 = -1, \quad \gamma = 0. \quad (2.29)$$

Therefore the corresponding metric is

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + \sum_{i=1}^d dx_i^2, \quad (2.30)$$

which is an $\text{AdS}_2 \times \mathbb{R}^d$ solution. It is very interesting that HL gravity with non-zero $\alpha$ admits an $\text{AdS}_2 \times \mathbb{R}^d$ solution, though it may have $\text{AdS}_{d+2}$ only if $\alpha = 0$. Of course it can be understood from the fact that the AdS vacuum is an isotropic solution.

Having found an $\text{AdS}_2 \times \mathbb{R}^d$ solution, one may wonder that this solution can be thought of as an emergent IR fixed point which could complete the geometry even though we are dealing with the HL gravity.
3. General model

As we have seen in the previous section, as long as \( \alpha \) is non-zero, the metric (2.12) is a solution of the equations of motion for arbitrary \( \lambda \) and \( \beta \). Therefore for \( \alpha \neq 0 \) one may set \( \lambda = \beta = 1 \), so that the first terms of the action (1.5) reduce to the standard Einstein–Hilbert action. More precisely setting

\[
G_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_b & g_{ab} \end{pmatrix}
\]

the action (1.5) may be recast into the following form

\[
S = \frac{1}{2\kappa^2} \int dt \, dr \, d^d x \, \sqrt{-G} \left( R - \frac{\alpha^2}{2} \nabla_a N \nabla^a N \right),
\]

where \( R \) is the scalar curvature of the metric \( G_{\mu\nu} \).

It is important to note that when we set \( \alpha = 0 \), although the action gets the same form as the Einstein–Hilbert one, the model does not reduce to standard GR theory due to the missing of temporal diffeomorphism. Nevertheless as it is evident from the equations (2.11) and (2.12) when \( \alpha \) is zero the natural vacuum of the model is an AdS geometry. While for non-zero \( \alpha \) the natural solution is the one with an anisotropic direction such as the Lifshitz geometry.

If we think of the model given by the action (3.2) as a gravity which provides a gravitational description for a non-relativistic field theory, then having the \( \alpha \)-term might be thought of as perturbing the dual theory by an operator which induces an RG flow from an isotropic CFT vacuum into a vacuum with an anisotropy along the time direction.

To explore this point better, following [14] we utilize the St"uckelberg formalism by which the general covariance may be restored by introducing a scalar field \( \varphi \), known as the khronon.

In this formalism the surfaces of foliation are defined by \( \varphi = \text{constant} \). Then it is useful to define a unit normal vector, \( u_\mu \),

\[
\mu = \frac{\partial \varphi}{\sqrt{G_{\mu\nu} \partial \varphi \partial \varphi}},
\]

by which the action (3.2) may be generalized to

\[
S = \frac{1}{2\kappa^2} \int dt \, dr \, d^d x \, \sqrt{-G} \left( R - \frac{\alpha^2}{2} a_\mu a^\mu \right),
\]

where \( a_\mu = u^\nu \partial \nu u_\mu \).

Then it is possible to compensate the non-invariance of HL gravity under temporal diffeomorphisms by a suitable transformation of \( \varphi \). Note that the HL gravity can be obtained by a foliation defined by \( \varphi = t \) which is known as the 'unitary gauge'. In other words, the action may be thought of as a deviation of Einstein gravity.

Deforming the action by the \( \alpha \) term in the gravity side would correspond to perturbing the dual CFT. Of course, in general, it is difficult to make this statement precise. Namely it is not clear how to identify the corresponding dual operator in the field theory side. Heuristically, one may treat \( a_\mu \) as a vector field in the bulk by which the action is deformed. Then there would be a vector operator in the dual theory which perturbs the corresponding CFT. In particular if we consider the time component of the dual operator to be non-zero then in the bulk gravity we have to deform the theory by \( a_t \) which in turn is equivalent to \( \varphi = t \) foliation. In this case the end point of the RG flow will be a Lifshitz fixed point. It is an easy exercise to see this procedure in the covariant form. Indeed from the ansatz (2.1) one has

\[
R = -2\alpha^2 \left[ d(r^2 + f^2 + f^2) + d' f' + d' h' + f' h' \right], \quad \sqrt{-G} = e^{d + b + f}. \quad (3.5)
\]

7 It is, however, important to note that setting \( \alpha = 0 \), will remove the kinetic term of the scalar field from the action and therefore the limit is degenerate [14]. Nevertheless we will consider the \( \alpha \) term as a sufficiently small perturbation.
Plugging these expressions in the above action and performing an integration by parts and setting $\varphi = t$ one arrives at the same one-dimensional action of that in the previous section whose natural vacuum is a Lifshitz metric.

More generally we may perturb the dual CFT by a spatial component of the dual operator, for example along $x_1$ direction; $O_1$. Then the gravity will be deformed by $a_{x_1}$. A natural way to get the desired field is to foliate the space-time by $\varphi = x_1$. More explicitly one has

$$\text{d}s^2 = -A^2 \text{d}t^2 + g_{mn} (\text{d}x^m - B^m \text{d}t) (\text{d}x^n - B^n \text{d}t), \quad m, n = t, 2, \ldots, d. \quad (3.6)$$

Using a proper ansatz and plugging it into the covariant form of the action (3.4), one can show that the resultant metric may has the following anisotropic scaling

$$r \rightarrow \lambda^{-1} r, \quad x_1 \rightarrow \lambda x_1, \quad (t, x_i) \rightarrow \lambda (t, x_i), \quad \text{for } i = 2, \ldots, d, \quad (3.7)$$

with $z = 2/(2 - \alpha^2)$. More generally one finds

$$\text{d}s^2 = -r^2 \text{d}t^2 + r^{2z} \text{d}x_1^2 + \frac{\text{d}r^2}{r^2} + r^2 \sum_{i=2}^{d} \text{d}x_i^2. \quad (3.8)$$

Therefore one may conclude that holographically the Lifshitz fixed point may be considered as an AdS solution perturbed by the time component of the vector operator $O_\mu. \quad (8)$

Other components of the operator $O$ perturb the theory to other vacua which have anisotropy in spatial directions.

4. Conclusions

In this paper we have shown that the minimal model defined by the action (1.5) admits Lifshitz and hyperscaling violating vacua. An interesting observation we have made is that the resultant solution (the equations (2.12) and (2.11)) does not depend on the parameter $\lambda$. Moreover the parameter $\beta$ can be absorbed by a redefinition of $\alpha$, though the solution does crucially depend on $\alpha$. Therefore for $\alpha \neq 0$ by making use of the St"uckelberg formalism one may rewrite the action as a covariant form.

Using the covariant form of the model we have shown that the Lifshitz fixed point may be thought of as CFT perturbed by the time component of a vector operator.

We have also shown that the HL gravity given by the action (1.5) has an IR emergent fixed point where the background develops an AdS$_2$ geometry. In other words we find a vacuum with AdS$_2 \times \mathbb{R}^d$ geometry in the model. This is an interesting observation in the sense that we arrive at an AdS$_2$ geometry even though the background metric is not charged.

We have also shown that with a small modification, namely assuming an $r$-dependent $\Lambda$ the theory could also support vacua with hyperscaling violating geometry.

Finally we note that although the model has a Lifshitz solution the corresponding black hole geometry is not a solution of the model. It is surprising to note that if one sets $\alpha = 0$, both AdS and AdS black hole solutions exist in the theory. In other words having non-zero $\alpha$ prevents the model from having a black hole solution. It would be interesting to explore this point better.

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8 In the context of AdS/CFT correspondence a similar idea—rather more precise—has been explored by Skenderis et al [19].
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