Heavy tailed time series with extremal independence

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Abstract

We consider heavy tailed time series whose finite-dimensional distributions are extremally independent in the sense that extremely large values cannot be observed consecutively. This calls for methods beyond the classical multivariate extreme value theory which is convenient only for extremally dependent multivariate distributions. We use the Conditional Extreme Value approach to study the effect of an extreme value at time zero on the future of the time series. In formal terms, we study the limiting conditional distribution of future observations given an extreme value at time zero. To this purpose, we introduce conditional scaling functions and conditional scaling exponents. We compute these quantities for a variety of models, including Markov chains, exponential autoregressive models, stochastic volatility models with heavy tailed innovations or volatilities.

Keywords: Multivariate regular variation, extremal independence, conditional scaling exponent, Markov chains, stochastic volatility models.

1 Introduction

Let \( \{X_t, t \in \mathbb{Z}\} \) be a strictly stationary time series. We say that \( \{X_t\} \) is regularly varying if all its finite dimensional distributions are regularly varying, i.e. for each \( h \geq 0 \), there exists a nonzero Radon measure \( \nu_h \) on \( [-\infty, \infty]^{h+1} \setminus \{0\} \), called the exponent measure, which puts zero mass at infinity, and a scaling function \( c \) such that, as \( s \to \infty \),

\[
P\left( \frac{(X_0, \ldots, X_h)}{c(s)} \in \cdot \right) \rightharpoonup \nu_h, \tag{1.1}
\]

where \( \rightharpoonup \) means vague convergence, to be understood here on the space \( [-\infty, \infty]^{h+1} \setminus \{0\} \). Recall that a sequence of measures \( \nu_n \) defined on a complete separable metric space \( E \) (endowed with its Borel \( \sigma \)-field) is said to converge vaguely to a measure \( \nu \) if \( \nu_n(f) \to \nu(f) \) for all continuous functions with compact support, or equivalently \( \nu_n(K) \to \nu(K) \) for all compact sets \( K \) with \( \nu(\partial K) = 0 \). See [Res87] for more details. This assumption implies that the function \( c \) is regularly varying with index \( 1/\alpha \) for some \( \alpha > 0 \), the measure \( \nu_h \) is homogeneous of degree \( -\alpha \) and the marginal distribution of \( X_0 \) is heavy tailed with positive tail index \( \alpha \). To avoid trivialities, we will only consider distributions that are not totally skewed to the left, that is we assume that \( \lim_{x \to \infty} \frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(|X_0| > x)} > 0 \). In that case, a possible choice for the scaling function in (1.1) is \( c(s) = F_0^{-1}(1 - 1/s) \), where \( F_0 \) is the distribution function of \( X_0 \), and we can rewrite (1.1) as

\[
\frac{\mathbb{P}\left( \frac{(X_0, \ldots, X_h)}{c(s)} \in \cdot \right)}{\mathbb{P}(X_0 > x)} \rightharpoonup \nu_h, \tag{1.2}
\]
on \( [-\infty, \infty]^{h+1} \setminus \{0\} \), as \( x \to \infty \).

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If \( h \geq 1 \), there exist two fundamentally different cases: either the exponent measure is concentrated on the axes or it is not. The former case is referred to as extremal independence and the latter as extremal dependence. In other words, extremal independence means that no two components can be extremely large at the same time, and extremal dependence means that two components can be simultaneously extremely large. The suitably renormalized componentwise maxima of an i.i.d. sequence of extremally independent random vectors converge to a max-stable distribution with independent marginals. See e.g. [Res02] or [Res07]. This definition is rather weak and it must be noted that if two components of the vector are extremally dependent then the whole vector is extremally dependent; e.g. the vector \((X, X, Y)\), if multivariate regularly varying (MRV), is extremally dependent even if \(X\) and \(Y\) are independent. For most time series models, the distribution of \((X_0, \ldots, X_h)\) is either extremally dependent or extremally independent for all \(h\).

In a time series context, we may want to assess the influence of an extreme event at time zero on future observations. If the finite dimensional distributions of the time series model under consideration are extremally independent or more generally if the vector \((X_0, X_m, \ldots, X_h)\) is extremally independent for some \(m \geq 1\), then, for any set \(A\) which is bounded away from zero in \(\mathbb{R}^{h-m+2}\),

\[
\lim_{x \to \infty} \frac{\mathbb{P}(X_0 > xu_0, (X_m, \ldots, X_h) \in xA)}{\mathbb{P}(X_0 > x)} = 0. \tag{1.3}
\]

Thus in case of extremal independence the exponent measure \(\nu_h\) provides no information on (most) extreme events occurring after an extreme event at time 0. In concrete terms, if the series \(\{X_t\}\) represent financial losses, extremal independence means that an extreme loss at time zero will be followed by another extreme loss of at least the same magnitude with an extremely small probability. This is good news, but it is still of great importance to know how likely a one million euro loss is to be followed by a smaller loss of, say, a hundred thousand euros, which might be disastrous after the previous loss. A moderate flood can still be devastating after a major one. Since the exponent measure provides no information, other tools must be used to quantify the influence of an extreme event at time zero on future events.

In order to obtain a non-degenerate limit in (1.3) and a finer analysis of the sequence of extreme values, it is necessary to change the normalization in (1.2), and possibly the space on which we will assume that vague convergence holds. One idea is to find a sequence of normalizations \(b_j(x)\), \(j \geq 1\) such that for each \(h \geq 1\), the conditional distribution of \((X_0/x, X_1/b_1(x), \ldots, X_h(x)/b_h(x))\) given \(X_0 > x\) has a non-degenerate limit. Finding a limiting conditional distribution of a random vector given one extreme component is a very old problem. It was recently rigorously investigated for bivariate distributions using the concept of regular variation on cones by [HR07] and [DR11]. See the references in these papers for the earlier literature. If such a limiting distribution exists, the vector \((X_0, \ldots, X_h)\) is said to satisfy the “Conditional Extreme Value” (hereafter CEV) assumption. This expression was introduced in this context by [DR11]\(^3\). It must be noted that if they exist, limiting conditional distributions given an extreme component need not be extreme value distributions and the variables \(X_1, \ldots, X_h\) need not be in the domain of attraction of an extreme value distribution. Considering i.i.d. random variables, it is easily seen that any distribution can arise as a limiting distribution. Another important feature of the CEV approach is that it is applicable to extremely dependent regularly varying multivariate distributions as well; in that case, the limiting distribution is entirely determined by the exponent measure. This is important in view of statistical inference, since the same methodology can be applied to both types of distributions.

It is the goal of this paper to apply the CEV approach to the finite dimensional distributions of stationary (and some non-stationary) time series, both extremally dependent and independent, though with a main focus on extremally independent time series.

\(^3\)Note that this expression is also used in the extreme value literature to refer to the standard extreme value theory in presence of a covariate.
In Section 2, we will state Assumption 1 which expresses the CEV condition in the correct vague convergence framework introduced by [HR07] and give several applications. One important problem with extremally independent random variables is the tail behavior of their product. The CEV condition alone does not guarantee that the product is regularly varying. In Section 2.1, we will strengthen it and obtain a result on the tail behavior of products. In Section 2.2, we will further strengthen it to obtain the convergence of conditional moments. This convergence can be applied to study risk measures such as the conditional tail expectation; this will be discussed in Section 2.3. In Section 2.4 we extend the tail process, introduced by [BS09] in the extremally dependent case, to the extremally independent case and we give a representation of the limiting conditional distributions in terms of this tail process. In Section 2.5 we will compare the CEV approach to Hidden Regular Variation.

In the following Sections 3, 4, and 5 we will study several models of regularly varying time series which satisfy Assumption 1. In Section 3, we give a general result for extremally independent Markov chains. In Section 4, we study a non-Markovian exponential linear process and in Section 5, we will finally consider stochastic volatility models with light tailed or heavy tailed volatilities. It should be pointed out that the models studied in Sections 4 and 5 allow for some form of long memory. This is of practical importance since it is one of the so-called stylized facts of financial time series (log-returns) that volatility may exhibit long memory.

Section 6 contains the proof of our main result on Markov chains and Section 7 discusses some directions of further research, the most important one being statistical inference.

2 Limiting conditional distributions and conditional scaling exponents

We now introduce the main Assumption of this paper. It is stated in terms of regular variation on space smaller than \([-\infty, \infty]^{h+1} \setminus \{0\}.

Assumption 1. There exist scaling functions \(b_j, j \geq 1\) and Radon measures \(\mu_h, h \geq 1\), on \((0, \infty] \times [-\infty, +\infty]^h\), \(h \geq 1\), such that

\[
\frac{1}{\mathbb{P}(X_0 > x)} \mathbb{P}\left( \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \in \cdot \right) \xrightarrow{\text{v}} \mu_h,
\]

on \((0, \infty] \times [-\infty, +\infty]^h\) and for all \(y_0 > 0\),

a. the measure \(\mu_h([y_0, \infty] \times \cdot)\) on \(\mathbb{R}^h\) is not concentrated on a line through infinity;

b. the measure \(\mu_h([y_0, \infty] \times \cdot)\) on \(\mathbb{R}^h\) is not concentrated on a hyperplane;

c. the measure \(\mu_h(\cdot \times \mathbb{R}^h)\) on \((0, \infty]\) is not concentrated at infinity.

Notice that vague convergence here must hold on a different space than in (1.1). This is of importance since the compact sets of \([-\infty, \infty]^{h+1} \setminus \{0\}\) and \((0, \infty] \times [-\infty, +\infty]^h\) differ. For instance, if \(h = 1\), \([0, \infty] \times [1, \infty]\) is compact in \([-\infty, \infty]^2 \setminus \{0\}\) but not in \((0, \infty] \times [-\infty, +\infty]\). More generally, a subset \(K\) of \([-\infty, \infty]^{h+1} \setminus \{0\}\) is relatively compact if there exists \(\epsilon > 0\) such that \(x \in K\) implies that at least one component of \(x\) is greater than \(\epsilon\); a subset \(L\) of \((0, \infty] \times [-\infty, \infty]^h\) is relatively compact if there exists \(\epsilon > 0\) such that \(x \in L\) implies that the first component of \(x\) is greater than \(\epsilon\).

For \(h = 1\), Assumption 1 is Condition (5) in [HR07]. We extend it here to a multidimensional framework and to different scaling functions \(b_j, j \geq 1\). This is a fundamental necessity in the time series context. As already mentioned in the introduction, Assumption 1 does not require stationarity of the time series \(\{X_t\}\) and is compatible with both extremal dependence and independence.
We now make some comments on the conditions in Assumption 1.

- Assumption 1a implies that the scaling functions $b_j$ are not too small.
- Assumption 1b implies that the scaling functions $b_j$ are not too large. For instance, if $X_0$ and $X_1$ are independent, choosing $b_1(x) = x$ would yield a measure concentrated on $(0, \infty) \times \{0\}$.
- Assumption 1c implies that the marginal distribution of $X_0$ is heavy tailed with positive tail index. Hereafter, we let $\alpha$ denote the tail index.
- By construction, $\mu_h([1, \infty) \times \mathbb{R}^h) = 1$, i.e. the measure $\mu_h$ restricted to $[1, \infty) \times \mathbb{R}^h$ is a probability measure. Thus we can define the multivariate distribution functions $\Psi_h$ on $[1, \infty) \times \mathbb{R}^h$ by

$$
\Psi_h(y) = \mu_h \left( [1, y_0] \times \prod_{j=1}^h [-\infty, y_j] \right),
$$

where $y = (y_0, y_1, \ldots, y_h) \in [1, \infty) \times \mathbb{R}^h$. For all continuity points $y$ of $\Psi_h$, we obtain

$$
\Psi_h(y) = \lim_{x \to \infty} \mathbb{P} \left( \frac{X_0}{x} \leq y_0, \frac{X_1}{b_1(x)} \leq y_1, \ldots, \frac{X_h}{b_h(x)} \leq y_h \mid X_0 > x \right). \quad (2.3)
$$

The most important consequence of Assumption 1, is that the functions $b_j$, $j \geq 1$ are regularly varying and that the limiting measure $\mu_h$ has some homogeneity property. We state these properties as a lemma whose proof is a standard application of the Convergence to Type Theorem. See [HR07, Proposition 1].

**Lemma 2.1.** If Assumption 1 holds, then there exists $\kappa_j \in \mathbb{R}$ such that

$$
\lim_{t \to \infty} \frac{b_j(ty)}{b_j(t)} = y^{\kappa_j}
$$

and for all $y_0 > 0$ and $(y_1, \ldots, y_h) \in \mathbb{R}^h$,

$$
\mu_h \left( (ty_0, \infty) \times \prod_{i=1}^h [-\infty, t^{\kappa_i} y_i] \right) = t^{-\alpha} \mu_h \left( (y_0, \infty) \times \prod_{i=1}^h [-\infty, y_i] \right). \quad (2.4)
$$

To put emphasis on the regular variation of the functions $b_j$, we introduce the following definition.

**Definition 1** (Conditional scaling exponent). *Under Assumption 1, for $h \geq 1$, we call the index $\kappa_h$ of regular variation of the functions $b_h$ the (lag $h$) conditional scaling exponent.*

The exponents $\kappa_h$, $h \geq 1$ reflect the influence of an extreme event at time zero on future lags. Even though we expect this influence to decrease with the lag in the case of extremal independence, these exponents are not necessarily monotone decreasing. See Sections 4 and 5.3.

Considering only the bivariate distribution of $(X_0, X_h)$, we have the following properties.

- If $(X_0, X_h)$ is multivariate regularly varying in the sense of (1.1) and Assumption 1 holds, then $\kappa_h \leq 1$. If $(X_0, X_h)$ is extremally dependent then $\kappa_h = 1$. If $b_h(x) = o(x)$, which holds in particular if $\kappa_h < 1$, then $(X_0, X_h)$ is extremally independent. Negative values of $\kappa_h$ are allowed. This means that extremely large values are typically followed by extremely small (absolute) values.
- Condition (1.1) and extremal independence do not imply that Assumption 1 holds, i.e. the existence of limiting conditional distributions. See Sections 2.5 and 2.6.
- In most of the examples investigated in the next sections, it will hold that $0 \leq \kappa_h < 1$ for all $h$. However, there are natural examples where the scaling exponent is larger than 1. See Section 3.1.
2.1 Tail of products

One application of Assumption 1 is to obtain the tail of products of regularly varying random variables. If a pair \((X_0, X_h)\) is jointly regularly varying with tail index \(\alpha\) and is extremally dependent, then it is well known that the tail of the product \(X_0X_h\) is \(\alpha/2\); see e.g. [Res07, Proposition 7.6]. In the case of extremal independence, many different tail behaviors of the product are possible. Under Assumption 1 and an additional technical condition, we can obtain the tail index of \(X_0X_h\). The next result generalizes [MRR02, Theorem 3.1] who consider only the case \(\kappa_0 = 0\); see also [SM11]. As before, we denote \(y = (y_0, \ldots, y_h)\).

**Proposition 2.2.** Let Assumption 1 hold and assume moreover that
\[
\int_{[0,\infty)^{h+1}} 1_{\{y_0y_h > 1\}} \mu_h(dy) < \infty ,
\]
and there exists \(\delta > 0\) such that
\[
\lim_{\epsilon \to 0} \limsup_{x \to \infty} \frac{\mathbb{E} \left\{ \left( \frac{X_0 1_{\{X_0 \leq \epsilon \}} X_h}{\beta_h(x)} \right)^{\delta} \right\}}{\mathbb{P}(X_0 > x)} = 0 .
\]
Then
\[
\lim_{x \to \infty} \frac{\mathbb{P}(X_0X_h > xb_h(x)u)}{\mathbb{P}(X_0 > x)} = u^{-\alpha/(1+\kappa_h)} \int_{[0,\infty)^{h+1}} 1_{\{y_0y_h > 1\}} \mu_h(dy) .
\]
Thus, the right tail index of the product \(X_0X_h\) is \(\alpha/(1+\kappa_h)\).

**Proof.** Fix some \(\epsilon > 0\). Then, by vague convergence,
\[
\lim_{x \to \infty} \frac{\mathbb{P}(X_0 1_{\{X_0 \leq \epsilon \}} X_h > xb_h(x)u)}{\mathbb{P}(X_0 > x)} = \int_{(\epsilon,\infty] \times [0,\infty]^h} 1_{\{y_0y_h > u\}} \mu_h(dy) .
\]
and by Markov’s inequality,
\[
\frac{\mathbb{P}(X_0 1_{\{X_0 \leq \epsilon \}} X_h > xb_h(x)u)}{\mathbb{P}(X_0 > x)} \leq \mathbb{E} \left\{ \left( \frac{X_0 1_{\{X_0 \leq \epsilon \}} X_h}{\beta_h(x)} \right)^{\delta} / u^\delta \mathbb{P}(X_0 > x) \right\} .
\]
Conditions (2.5) and (2.6) ensure that
\[
\lim_{x \to \infty} \frac{\mathbb{P}(X_0X_h > xb_h(x)u)}{\mathbb{P}(X_0 > x)} = \int_{[0,\infty)^{h+1}} 1_{\{y_0y_h > u\}} \mu_h(dy) .
\]
This yields (2.7) by the homogeneity property (2.4) and the change of variable \(y_0 = u^{1/(1+\kappa_h)}x_0\), \(y_i = u^{\kappa_i/(1+\kappa_h)}x_i\), \(1 \leq i \leq h\). \(\square\)

2.2 Convergence of moments

The following lemma states that under suitable moment assumptions, the convergence (2.3) can be extended to unbounded functionals.

**Lemma 2.3.** Let Assumption 1 hold. Assume moreover that there exists \(x_0 > 0\) and \(q_0, \ldots, q_h > 0\) such that
\[
\sup_{x \geq x_0} \mathbb{E} \left[ \frac{X_0}{x} \prod_{i=1}^h \frac{X_i}{b_i(x)} | X_0 > x \right]^{q_i} < \infty .
\]
Let \( g \) be a continuous function defined on \([1, \infty) \times \mathbb{R}^h\) such that

\[
|g(x_0, \ldots, x_h)| \leq C \prod_{i=0}^{h} (|x_i| \vee 1)^{q_i},
\]

for some \( q_i < q_0 \), \( 0 \leq i \leq h \) and a positive constant \( C \). Then

\[
\lim_{x \to \infty} \mathbb{E} \left[ g \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \mid X_0 > x \right] = \int_{\mathbb{R}^h} g(y) \mu_h(dy). \tag{2.10}
\]

**Proof.** Let \( \mu_{h,x} \) be the measure defined on \((0, \infty) \times \mathbb{R}^h\) by

\[
\mu_{h,x}(E) = \frac{1}{\mathbb{P}(X_0 > x)} \mathbb{P} \left( \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \in E \right). \tag{2.11}
\]

Then, we have

\[
\mathbb{E} \left[ g \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \mid X_0 > x \right] = \int_{\mathbb{R}^h} g(y) \mu_{h,x}(dy). \]

Note that \( \mu_{h,x} \) is a probability measure on \([1, \infty) \times \mathbb{R}^h\) which converges weakly to \( \mu_h \). Let \( Y_{h,x} \) be a sequence of random variables with distribution \( \mu_{h,x} \). Then \( Y_{h,x} \) converges weakly to a random variable \( Y_h \) with distribution \( \mu_h \). Therefore, the convergence (2.10) holds for all bounded and continuous function \( g \). If \( g \) is unbounded and satisfies (2.9), then (2.8) ensures the uniform integrability of the sequence \( g(Y_{h,x}) \) and thus \( \lim_{x \to \infty} \mathbb{E}[g(Y_{h,x})] = \mathbb{E}[g(Y_h)] \). \( \square \)

**Remark 2.4.** Condition (2.8) ensures the uniform integrability needed to obtain the convergence of the expectation in (2.10). In the case of extremal dependence, it is necessary that \( q_0 + \cdots + q_h \leq \alpha \) for (2.8) to hold. In the case of extremal independence, this is in general neither sufficient nor necessary. For each model, the range of the admissible exponents \( q_i \) must then be given.

### 2.3 Conditional Tail Expectation

Assumption 1 and Lemma 2.3 can be applied to study certain risk measures. In a time series context, we may be interested in the limiting behavior as \( x \to \infty \) of the Conditional Tail Expectation (CTE), defined by

\[
\text{CTE}_h(x) = \mathbb{E}[X_h \mid X_0 > x].
\]

This quantity is related to the expected shortfall (ES), defined by

\[
\text{ES}_h(u) = \mathbb{E}[X_h \mid X_0 > \text{VAR}_{X_0}(u)],
\]

where \( \text{VAR}_{X_0}(u) \) is the Value-at-Risk associated with the random variable \( X_0 \), at the level \( u \). Note that the expected shortfall (originally defined with \( h = 0 \)) is a coherent risk measure in the sense of [ADEH99]. The previous quantities could be zero. In a risk measure context, one might rather be interested in \( \text{CTE}_h^+(x) = \mathbb{E}[(X_h)_{+} \mid X_0 > x] \) where \( (X_h)_{+} \) represent the future losses in absolute values.

If for some \( h \geq 0 \) the vector \((X_0, X_h)\) is extremally dependent and if \( \alpha > 1 \), then \( \text{CTE}_h^+(x) \) will grow linearly with \( x \), i.e. \( \lim_{x \to \infty} x^{-1} \text{CTE}_h^+(x) > 0 \). For a large class of regularly varying sequences (e.g. stationary solutions of stochastic recurrence equations), all the bivariate marginal distributions of the pairs \((X_0, X_h)\) are extremally dependent. This means that a large value of \( X_0 \) yields the same order of magnitude of the \( \text{CTE}_h^+ \) for all lags \( h \). This may not seem reasonable for
many real data sets, e.g. for high frequency financial data. In the case of extremal independence, under Assumption 1, if there exists \( \epsilon > 0 \) such that
\[
\sup_{x \geq x_0} \frac{\mathbb{E}[|b_h^{-1}(x)X_h|^{1+\epsilon} \mathbb{1}_{\{X_0 > x\}}]}{\mathbb{P}(X_0 > x)} < \infty ,
\] (2.12)
then \( \lim_{x \to \infty} x^{-1} \text{CTE}_h^+(x) = 0 \). Again, this does not mean that the CTE is uninformative, but that a smaller normalization is needed in order to obtain a non trivial limit. Lemma 2.3 implies that we can define
\[
m_h = \int_0^\infty \int_{\mathbb{R}^h} (y_h)_+ \Psi_h(dy_0,\ldots,dy_h) ,
\] (2.13)
and we have \( \text{CTE}_h^+(x) \sim b_h(x)m_h \).

2.4 The tail process

In [BS09] the authors define the tail process as the distributional limit of the sequence \( X_0/x, X_1/x, \ldots, X_h/x, \ldots \) conditionally on \( X_0 > x \). In the case of extremal independence, this \( X_i/x \) converges weakly to 0 for all \( t > 0 \). Our approach suggests the following definition which includes the ordinary tail process.

**Definition 2.** Assume that Assumption 1 holds. The tail process \( \{Y_t\} \) is the distributional limit of the sequence
\[
\frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)}, \ldots
\]
conditionally on \( X_0 > x \).

Note that the distribution of \( (Y_0,\ldots,Y_h) \) is \( \Psi_h \). We now give a representation of the tail process. Define the measure \( G_h \) on \( \mathbb{R}^h \) by
\[
G_h(y_1,\ldots,y_h) = \int_0^\infty \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_h} \mu_h(du_0,u_0^{\kappa_1}du_1,\ldots,u_0^{\kappa_h}du_h) .
\]
Then, using the homogeneity property (2.4), we obtain
\[
\int_{y_0}^\infty \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_h} \mu_h(du_0,u_0^{\kappa_1}du_1,\ldots,u_0^{\kappa_h}du_h) = y_0^{-\alpha}G_h(y_1,\ldots,y_h) .
\]
Let \( (J_1,\ldots,J_h) \) be a random vector with distribution \( G_h \). Assume that \( (Y_0,\ldots,Y_h) \) and \( (J_1,\ldots,J_h) \) are defined on the same probability space and such that \( Y_0 \) and \( (J_1,\ldots,J_h) \) are independent. Note that \( (J_1,\ldots,J_h) \) need not be independent. Then the previous identity yields that
\[
(Y_0,\ldots,Y_h) \overset{d}{=} (Y_0^{\kappa_1},J_1,\ldots,Y_0^{\kappa_h}J_h) .
\]
We can use the tail process to interpret the moment condition (2.5) in Proposition 2.2 which can now be expressed as
\[
\mathbb{E}[J_h^{\alpha/(1+\kappa_h)}] < \infty .
\] (2.14)
The limit distribution of \( X_0X_h \) given \( X_0 > x \) is thus the distribution of \( Y_0^{1+\kappa_h}J_h \). The independence of \( J_h \) and \( Y_0 \) and Condition (2.6) imply that the tail of \( Y_0^{1+\kappa_h}J_h \) is \( \alpha/(1+\kappa_h) \) and that the tail index of \( X_0X_h \) is the same as that of \( Y_0^{1+\kappa_h}J_h \).
2.5 Comparison with Hidden Regular Variation

A different way to quantify the joint extremal behavior of extremally independent distributions is Hidden Regular Variation (HRV), introduced in [Res02]. To simplify the notation, we will discuss hidden regular variation for non negative random variables only. Let $C^2_{h+1}$ be the subset of $[0,\infty]^{h+1}$ comprised of vectors with at least two positive components (denoted $\mathbb{E}^0$ in [Res02]); that is, $C^2_{h+1}$ is $[0,\infty]^2$ with the axes removed. For $h = 1$, $C^2_2 = (0, \infty)^2$; for $h = 2$,

$$C^2_3 = \left( (0, \infty)^2 \times [0, \infty) \right) \cup \left( (0, \infty] \times (0, \infty) \right) \cup \left( (0, \infty] \times (0, \infty]^2 \right).$$

A vector $(X_0, \ldots, X_h)$ satisfying (1.1) is said to have Hidden Regular Variation if there exists a scaling function $d$ such that $\lim_{s\to\infty} c(s)/d(s) = \infty$ and the sequence of measures $s\mathbb{P}(d^{-1}(s)(X_0, \ldots, X_h) \in \cdot)$ converges vaguely to a non zero Radon measure on $C^2_{h+1}$. Under suitable non degeneracy conditions, the function $d$ must then be regularly varying with some index $\beta \geq \alpha$. HRV implies extremal independence because of the condition $c(s)/d(s) \to \infty$ but HRV does not imply the existence of a limiting conditional distribution. See e.g. [HR07, Section 6] or [DR11, Example 4]. Conversely, it is conjectured but not proved in [DR11] that extremal independence and existence of a conditional limit law implies HRV.

Let us now highlight the differences between these two concepts. The fundamental theoretical difference between HRV and CEV lies in the space where vague convergence holds. This difference entails the following one: HRV only deals with joint exceedances of two components of the vector, and in dimension greater than two, the hidden exponent measure may be concentrated on hyperplanes; this is the case for instance of a vector with three i.i.d. regularly varying components. The CEV Assumption 1 prevents such a degeneracy. For example, if $X, Y, Z$ are i.i.d. non negative regularly varying random variables with tail index $\alpha$, then $(X, Y, Z)$ has HRV with $\beta = 2\alpha$ but, for $u, v, w > 0$,

$$\lim_{x \to \infty} \frac{\mathbb{P}(X > xu, Y > xv, Z > xw)}{\mathbb{P}(X > x)} = 0,$$
$$\lim_{x \to \infty} \frac{\mathbb{P}(X > xu, Y > v, Z > w)}{\mathbb{P}(X > x)} = u^{-\alpha} \mathbb{P}(Y > v) \mathbb{P}(Z > w).$$

Therefore HRV is uninformative for such exceedances but CEV yields a non degenerate limit.

The CEV assumption is also more flexible than HRV since it can also accommodate extremely dependent vectors (ruled out by HRV) with extremally independent subvectors. For example, we already seen that if $X$ and $Y$ are i.i.d. non negative regularly varying random variables with tail index $\alpha$, then $(X, X, Y)$ is extremally dependent; however, for $u, u', v > 0$,

$$\lim_{x \to \infty} \frac{\mathbb{P}(X > xu, X > xu', Y > xv)}{\mathbb{P}(X > x)} = 0,$$
$$\lim_{x \to \infty} \frac{\mathbb{P}(X > xu, X > xu', Y > v)}{\mathbb{P}(X > x)} = (u \vee u')^{-\alpha} \mathbb{P}(Y > v).$$

The first limit is obtained by the standard multivariate regular variation property and is zero even though the vector is extremally dependent, and the second one shows that the CEV assumption is fulfilled.

In conclusion, we can say that the practical purposes of HRV and CEV are different: HRV gives an approximation of the probability of exceedances of pairs of components of an extremely independent vector over a “not too extreme” level, whereas CEV quantifies the influence of an extreme component (an extreme event at time zero in a time series context) on all other components (the future observations), be they extremely independent or dependent. There is no implication or exclusion between HRV and CEV and one approach cannot be deemed superior to the other. In higher dimensions, the CEV approach seems more flexible.
2.6 A counter example

We now give an example, where the conditional laws do not exist. Consider a stationary standard Gaussian process \( \{ \xi_t, t \in \mathbb{N} \} \) and define \( X_t = e^{c \xi_t} \), with \( c < 1/2 \). Assume moreover that \( |\text{cov}(\xi_0, \xi_n)| < 1 \) for all \( n \geq 1 \). This is not a stringent assumption since a sufficient condition is that the process \( \{ \xi_t \} \) has a spectral density \( f \) such that \( \int \pi f(t) \, dt = 1 \). In that case, extremal independence holds for the bivariate distributions, but a non trivial limiting conditional distribution \( e^{c \xi} \) given \( e^{c \xi} > x \) does not exist. See [HR07, section 2.4].

3 Extremally independent Markov chains

The extremal properties of Markov chains have received considerable attention recently; see [JS13], [RZ13] and the references therein. The aforementioned papers deal with extremal dependence. In this section, we will extend some results of [RZ13] to the present context which allows for extremal independence. Since the distribution of a Markov chain is entirely determined by its initial distribution and the transition kernel, denoted by \( \Pi \), it is natural in this context to replace Assumption 1 by the following one, which is similar to [RZ13, Assumption 2.5]. For simplicity, we assume that the state space is \([0, \infty)\).

**Assumption 2.** There exist a function \( b \), regularly varying at infinity with index \( \kappa \geq 0 \) and a distribution function \( G \) on \([0, \infty)\), not concentrated on one point such that

\[
\lim_{x \to \infty} \Pi(x, b(x) A) = G(A) \tag{3.1}
\]

for all Borel sets \( A \subset [0, \infty) \) such that \( G(\partial A) = 0 \).

This means that the transition kernel is asymptotically homogeneous. It also means that conditionally on \( X_0 = x \), the distribution of \( X_1/b(x) \) converges weakly to the distribution \( G \).

The main result of this section states that Assumption 2 implies Assumption 1. Define \( b_0(x) = x \), \( b_1(x) = b(x) \) and for \( h \geq 1 \), \( b_h = b_{h-1} \circ b \).

**Theorem 3.1.** Let \( \{ X_t \} \) be a Markov chain whose transition kernel satisfies Assumption 2 and with initial distribution having right tail index \( \alpha > 0 \). Assume moreover that \( G(\{0\}) = 0 \). Then Assumption 1 holds and the limiting conditional distribution of

\[
\left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)}, \ldots \right)
\]

given \( X_0 > x \) when \( x \to \infty \) is the distribution of the exponential AR(1) process \( \{ Y_t, t \geq 0 \} \) defined by \( Y_t = Y_{t-1} W_t \) where \( \{ W_t \} \) is an i.i.d. sequence with distribution \( G \), independent of the standard Pareto random variable \( Y_0 \) with tail index \( \alpha \).

The proof is in Section 6. For a Markov chain, the tail process is called the tail chain. With the normalization used here, we obtain a new type of tail chain which is an exponential AR(1) process. In the case of extremal dependence, the usual tail chain is an exponential random walk. This corresponds to the case \( \kappa_j = 1 \) for all \( j \).

Since a Markov chain \( \{ X_t \} \) can always be expressed as \( X_{t+1} = \Phi(X_t, \epsilon_{t+1}) \), where \( \epsilon_t, t \geq 1 \) is an i.i.d. sequence (the innovations), independent of \( X_0 \), Condition (3.1) is equivalent to the weak convergence of \( b^{-1}(x) \Phi(x, \epsilon_0) \) to the distribution \( G \) in (3.1). This is the framework considered in [JS13] under the additional assumption that \( b_j(x) = x \) for all \( j \).

If \( G(\{0\}) > 0 \), then Theorem 3.1 may no longer be true. However, without this condition, it can be seen from the proof that the convergence still holds provided \( X_1, \ldots, X_{h-1} \) are separated from
zero, i.e.

\[
\lim_{x \to \infty} \mathbb{P}\left( \frac{X_0}{x} \leq y_0, \epsilon \leq \frac{X_1}{b_1(x)} \leq y_1, \ldots, \epsilon \leq \frac{X_{h-1}}{b_{h-1}(x)} \leq y_{h-1}, \frac{X_h}{b_h(x)} \leq y_h \mid X_0 > x \right) = \mathbb{P}(Y_0 \leq y_0, \epsilon \leq Y_1 \leq y_1, \ldots, \epsilon \leq Y_{h-1} \leq y_{h-1}, Y_h \leq y_h), \tag{3.2}
\]

with \(\{Y_j\}\) is the tail process as described in Theorem 3.1.

In the extremally dependent case (where \(b_j(x) = x\) for all \(x\)), the convergence still holds under an additional regularity condition, see [RZ13, Proposition 5.1]. It would be possible to generalize this condition to the extremally independent context, but as in the extremally dependent case, this condition would not be necessary for the convergence to hold. Therefore we do not pursue in this direction. Note finally that if \(G(\{0\}) > 0\) then the tail process is identically zero after a geometric time with mean \(1/G(\{0\})\).

We now give examples of Markov chains satisfying condition (3.1).

### 3.1 Exponential AR(1)

Let the time series \(\{V_t\}\) be defined by \(V_t = e^{\xi_t}\) with

\[
\xi_t = \phi \xi_{t-1} + \epsilon_t, \tag{3.3}
\]

where \(0 \leq \phi < 1\) and \(\{\epsilon_t, t \in \mathbb{Z}\}\) is an i.i.d. sequence such that \(\mathbb{E}[\epsilon_0] = 0\) and

\[
\mathbb{P}(e^{\alpha \epsilon_0} > x) = x^{-\alpha} \ell(x), \tag{3.4}
\]

for some \(\alpha > 0\) and a slowly varying function \(\ell\). In other words, \(e^{\epsilon_0}\) has a regularly varying right tail with index \(\alpha\). [MR13, Section 3] studied the regular variation and proved the extremal independence of this model. Let \(\xi_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}\) be the stationary solution of the AR(1) equation (3.3). Condition (3.4) implies that for \(\phi \in [0, 1)\) and \(j \geq 1\),

\[
\mathbb{E}[e^{\alpha \phi^j \epsilon_0}] < \infty.
\]

Thus, applying Breiman’s lemma, we have

\[
\mathbb{P}(V_t > x) = \mathbb{P}(e^{\xi_t} > x) = \mathbb{P}(e^{\alpha \epsilon_0} \sum_{j=1}^{\infty} \phi^j \epsilon_{t-j} > x) \\
\sim \mathbb{P}(e^{\alpha \epsilon_0} > x) \mathbb{E}\left[e^{\alpha \sum_{j=1}^{\infty} \phi^j \epsilon_{t-j}}\right] = \mathbb{P}(e^{\alpha \epsilon_0} > x) \prod_{j=1}^{\infty} \mathbb{E}\left[e^{\alpha \phi^j \epsilon_{t-j}}\right].
\]

That is, \(V_t\) has a regularly varying right tail and is tail equivalent to \(e^{\epsilon_0}\). The Exponential AR(1) satisfies the equation

\[
V_{t+1} = V_t^\phi e^{\epsilon_{t+1}}. \tag{3.5}
\]

This corresponds to the functional representation \(\Phi(x, \epsilon) = x^\phi \epsilon\). We have

\[
\Pi(x, A) = \mathbb{P}(e^{\epsilon_0} \in x^{-\phi} A),
\]

and thus, with \(G\) the distribution of \(e^{\epsilon_0}\), we have

\[
\Pi(x, x^{\phi} A) = G(A).
\]

Since \(G(\{0\}) = 0\), Theorem 3.1 is applicable. The tail chain is a non stationary exponential AR(1) process \(\{Y_t\}\) defined by \(Y_t = Y_{t-1}^\phi e^{\epsilon_t}\) and \(Y_0\) is a standard Pareto random variable.
For \((y_0, \ldots, y_h) \in [1, \infty) \times \mathbb{R}^h\), we have
\[
\lim_{x \to \infty} \mathbb{P}(V_0 \leq x y_0, V_1 \leq x^{\phi} y_1, \ldots, V_h \leq x^{\phi_h} y_h \mid V_0 > x) = \int_1^y \mathbb{P}(e^{x_0} \leq x^{-\phi} y_1, \ldots, e^{x_h} \leq x^{\phi_h} y_h) \alpha v^{-\alpha-1} dv ,
\]
where for \(h \geq 1, \xi_{0,h} = \sum_{j=0}^{h-1} \phi^j \epsilon_h - j\). The limiting conditional distribution of \(V_h\) given \(V_0 > x\) is thus
\[
\lim_{x \to \infty} \mathbb{P}(V_h \leq x^{\phi_h} y \mid V_0 > x) = \int_1^\infty \mathbb{P}(e^{x_0} \leq v^{-\phi} y) \alpha v^{-\alpha-1} dv .
\]
The conditional scaling exponent is \(\kappa_h = \phi_h\). We also note that since \(\kappa_h \in (0, 1)\), this distribution is tail equivalent to the distribution of \(e^{\epsilon_0}\), i.e. we have as \(y \to \infty\),
\[
\mathbb{P}(V_h > x^{\phi_h} y \mid V_0 > x) = \int_1^\infty \mathbb{P}(e^{x_0} > v^{-\kappa_h} y) \alpha v^{-\alpha-1} dv \\
\sim \mathbb{P}(e^{x_0} > y) \int_1^\infty v^{-\kappa_h} \alpha v^{-\alpha-1} dv \\
= \frac{\mathbb{P}(e^{x_0} > y)}{1 - \kappa_h} \sim \frac{\mathbb{P}(e^{x_0} > y)}{(1 - \kappa_h) \mathbb{E}[e^{\alpha \kappa_h}]} .
\]
If \(\alpha > 1\), we can apply Lemma 2.3 and we obtain
\[
\lim_{x \to \infty} \mathbb{E} \left[ \frac{V_h}{x^{\kappa_h}} \mid V_0 > x \right] = \frac{\alpha \mathbb{E}[e^{x_0}]}{\alpha - \kappa_h} = \frac{\alpha \mathbb{E}[V_0]}{\alpha - \kappa_h} \mathbb{E}[V_0^{\alpha \kappa_h}] .
\]

**Tail of \(V_0 V_h\).** The recursion (3.3) yields \(V_0 V_h = V_0^{1+\phi_h} e^{\sum_{j=0}^{h-1} \phi^j \epsilon_h - j}\) and the series in the exponential is independent of \(V_0\). Also, \(\mathbb{E}[e^{(1+\phi_h) \sum_{j=0}^{h-1} \phi^j \epsilon_h - j}] < \infty\) and by Breiman’s Lemma, we obtain directly that the tail index of \(V_0 V_h\) is \(\alpha/(1 + \phi_h)\). We can also check that Condition (2.6) holds. Fix \(\delta < \alpha\) such that \(\delta (1 + \phi_h) > \alpha\) and define \(b_h(x) = x^{\phi_h}\). Then, for some constant \(C > 0\),
\[
\mathbb{E} \left[ \left( \frac{V_0}{x} 1_{\{V_0 \leq \varepsilon x\}} \frac{V_h}{b_h(x)} \right)^\delta \right] = \mathbb{E} \left[ \frac{V_0^{\delta (1 + \phi_h)}}{x^{\delta (1 + \phi_h)}} 1_{\{V_0 \leq \varepsilon x\}} \right] \mathbb{E} \left[ e^{\delta \sum_{j=0}^{h-1} \phi^j \epsilon_h - j} \right] \leq C e^{\delta (1 + \phi_h)} \mathbb{E}[e^{\alpha \epsilon_0}] .
\]
This yields
\[
\limsup_{x \to \infty} \frac{1}{\mathbb{P}(e^{\epsilon_0} > x)} \mathbb{E} \left[ \left( \frac{V_0}{x} 1_{\{V_0 \leq \varepsilon x\}} \frac{V_h}{b_h(x)} \right)^\delta \right] \leq C e^{\delta (1 + \phi_h) - \alpha} .
\]
By the choice of \(\delta\), this yields the negligibility condition (2.6).

**Convergence of moments.** Using the same decomposition as above, we obtain that the moment condition (2.8) holds if, for \(i = 0, \ldots, h\),
\[
\sum_{j=0}^i \phi^j q_{h-j} < \alpha .
\]
This implies in particular that \(q_j < \alpha\) for all \(j = 0, \ldots, h\).
Explosive case. Consider now the case $\phi > 1$. If the exponential AR(1) model is defined by the recurrence equation (3.5): $V_{t+1} = V_t e^{\phi \epsilon_{t+1}}$ with $\{\epsilon_t, t \geq 1\}$ independent of $V_0$, then the limit (3.6) still holds, but a stationary measure for this Markov chain does not exist.

On the other hand, the stationary (non-causal and non-Markovian) solution of Equation (3.3) is given by $\xi_t = \sum_{j=1}^{\infty} (-\phi)^j \epsilon_{t-j}$. Then the sequence $\{e^{\theta_j}\}$ is stationary and regularly varying but the tail index is now $\alpha \phi$ and no conditional limiting distribution exist.

Finally, it is obvious that the time-reversed chain has an invariant measure and the limiting conditional distribution $\mathbb{P}(V_0 < x^{1/\phi} y_1 \mid V_1 > x)$ exists.

3.2 Switching exponential AR(1)

Let $\{U_t\}$ be an i.i.d. sequence with uniform marginal distribution on $[0, 1]$, and let $\{R_t\}$ be an i.i.d. sequence with marginal distribution $F_R$ concentrated on $[0, \infty)$, independent of the sequence $\{U_t\}$. Let $\phi > 0$, $k : [0, \infty) \to [0, 1]$ be a measurable function and define a Markov chain $\{X_t\}$ by $X_0$ and

$$X_{t+1} = R_{t+1}(X_t^\phi I_{\{k(X_t) \leq U_{t+1}\}} + I_{\{k(X_t) > U_{t+1}\}}).$$

This is a multiplicative version of the Stochastic Unit Root process; see [GR06]. The transition kernel $\Pi$ of the chain is defined by

$$\Pi(x, A) = F_R(x^{-\phi} A)(1 - k(x)) + F_R(A)k(x).$$

If $\lim_{x \to \infty} k(x) = \eta$, then Condition (3.1) holds with $G$ defined by

$$G(A) = \lim_{x \to \infty} \Pi(x, x^\phi A) = F_R(A)(1 - \eta) + \eta \delta_0(A),$$

where $\delta_0$ is the Dirac mass at 0. If $\eta = 0$, then we can apply Theorem 3.1. The conditional scaling exponent at lag 1 is $\kappa_1 = \phi$. If $\eta > 0$, then $G(\{0\}) > 0$ and Theorem 3.1 is not applicable. However, in the simple case $k(x) = \eta$, it is readily checked that the conclusion of Theorem 3.1 nevertheless holds, i.e. if the distribution of $X_0$ has a right tail index $\alpha > 0$, then, conditionally on $X_0 > x$, $(x^{-1}X_0, x^{-\phi}X_1, \ldots, x^{-\phi}X_h)$ converges to the tail process $(Y_0, Y_1, \ldots, Y_h)$ where $Y_j = Y_{j-1}^{-\phi}W_j$, $j \geq 1$, $Y_0$ has a standard Pareto distribution with tail index $\alpha$ and $W_1, \ldots, W_h$ are i.i.d. with distribution $G$.

Let us now briefly discuss the existence of a stationary distribution for this Markov chain. We apply the Foster-Lyapunov criterion. See [MT09]. Define $V(x) = \log(x)$ and $c = \mathbb{E}[\log(R_0)]$. Then we have $\Pi V(x) = \phi(1-k(x))V(x) + c$. Assume that $R_0$ has an absolutely continuous distribution with a positive density around 0 so that the chain is irreducible. If $\phi \in (0, 1)$, then $\Pi V(x) \leq \phi V(x) + c$, and thus there exists a unique invariant distribution and the chain is geometrically ergodic.

4 Exponential linear process

The result for the Exponential AR(1) model can be extended to a (non-Markovian) exponential linear process $e^{\epsilon t}$ with possible long memory.

Lemma 4.1. Define $\xi_t = \sum_{j=0}^{\infty} \phi_j \epsilon_{t-j}$, where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables such that $\mathbb{E}[\epsilon_0] = 0$ and such that (3.4) holds, $\phi_0 = 1$, $\sum_{j=1}^{\infty} \phi_j^2 < \infty$ and $0 \leq \phi_j < 1$ for all $j \geq 1$.

The sequence of measures defined on $(0, \infty) \times [0, \infty)^h$ by

$$\frac{1}{\mathbb{P}(e^{\xi_0} > x)} \mathbb{P}\left(\frac{e^{\xi_0}}{x}, \frac{e^{\xi_1}}{b_1(x)}, \ldots, \frac{e^{\xi_h}}{b_h(x)} \right) \in \cdot$$
Equivalently, we obtain the following limiting conditional distribution. For \( y_0 \geq 1 \) and \((y_0, y_1, \ldots, y_h) \in (0, \infty) \times \mathbb{R}^h\),

\[
\lim_{x \to \infty} \mathbb{P}(e^{\xi_0} > x y_0, e^{\xi_1} \leq x^{\phi_1} y_1, \ldots, e^{\xi_h} \leq x^{\phi_h} y_h | e^{\xi_0} > x) = \frac{1}{E[e^{\alpha \xi_0}]} \int_0^\infty \mathbb{P} \left( e^{\xi_0} > v y_0, e^{\xi_1} \leq v^{\phi_1} y_1, \ldots, e^{\xi_h} \leq v^{\phi_h} y_h \right) a v^{-\alpha-1} dv. \tag{4.1}
\]

Thus, the lag \( h \) conditional scaling index is \( \phi_h \). If the coefficients \( \phi_j \) are decreasing, i.e. \( \phi_j > \phi_{j+1} \) for all \( j \geq 0 \), then the index of hidden regular variation of \((e^{\xi_0}, e^{\xi_1})\) is \( \alpha(2 - \phi) \). Otherwise, it is the solution of an infinite dimensional optimization problem and may take any value. See [JD13] for more details.

**Proof of Lemma 4.1.** To avoid trivialities, we assume that \( \phi_j \neq 0 \) for at least one index \( j \geq 1 \). By definition, we have \( \xi_k = \phi_k \epsilon_0 + \xi_k^\star \) for all \( k \geq 0 \). Note that \( \epsilon_0 \) is independent of \( \xi_k^\star \), \( k \geq 0 \). Denote the distribution of \( e^{\epsilon_0} \) by \( F \), and define the measure \( \sigma_x \) by \( \sigma_x (dv) = F(x dv) / F(x) \). The measure \( \sigma_x \) converges vaguely on \((0, \infty)\) to the measure with density \( \alpha v^{-\alpha-1} dv \); see [Res07, Theorem 3.6]. Then, for \((y_0, \ldots, y_h) \in (0, \infty) \times [0, \infty]^h\),

\[
\mathbb{P}(e^{\xi_0} > x y_0, e^{\xi_1} \leq x^{\phi_1} y_1, \ldots, e^{\xi_h} \leq x^{\phi_h} y_h) = \int_{u=0}^{\infty} \mathbb{P}(e^{\xi_0} > (x/u) y_0, e^{\xi_1} \leq (x/u)^{\phi_1} y_1, \ldots, e^{\xi_h} \leq (x/u)^{\phi_h} y_h) F_{e}(du) = \tilde{F}_{e}(x) \int_{u=0}^{\infty} \mathbb{P}(e^{\xi_0} > v^{-1} y_0, e^{\xi_1} \leq v^{-\phi_1} y_1, \ldots, e^{\xi_h} \leq v^{-\phi_h} y_h) \sigma_x (dv).}
\]

In order to prove the convergence of the integral, we must split it into two parts. Define the function \( K_h \) on \((0, \infty)^2 \times \mathbb{R}^h\) by

\[
K_h(v, y_0, \ldots, y_h) = \mathbb{P}(e^{\xi_0} > v^{-1} y_0, e^{\xi_1} \leq v^{-\phi_1} y_1, \ldots, e^{\xi_h} \leq v^{-\phi_h} y_h).
\]

The function \( K_h \) is uniformly bounded (by one), thus for \( c > 0 \), we have,

\[
\lim_{x \to \infty} \int_c^{\infty} K_h(v, y_0, y_1, \ldots, y_h) \sigma_x (dv) = \int_c^{\infty} K_h(v, y_0, y_1, \ldots, y_h) a v^{-\alpha-1} dv.
\]

Let \( \phi^* = \sup_{j \geq 1} \phi_j \). By assumption, \( 0 < \phi^* < 1 \). Thus \( e^{\xi_0} \) has the tail index \( \alpha / \phi^* > \alpha \). Moreover, since \( \epsilon_0 \) is independent of \( \xi_0^\star \), by Markov’s inequality, we have, for \( \alpha < q < \alpha / \phi^* \),

\[
\int_0^{c} K_h(v, y_0, y_1, \ldots, y_h) \sigma_x (dv) \leq \mathbb{P}(e^{\epsilon_0} e^{\xi_0} > x y_0, e^{\epsilon_0} < c x) \frac{F_{e}(x)}{E[e^{\xi_0}]} \frac{\mathbb{E}[e^{\xi_0}] \mathbb{E}[e^{\epsilon_0 \xi_0} 1(e^{\epsilon_0} < c x)]}{(x y_0)^q F_{e}(x)}.
\]

We obtain that

\[
\limsup_{x \to \infty} \int_0^{c} K_h(v, y_0, y_1, \ldots, y_h) \sigma_x (dv) = O(e^{q^\alpha})
\]
We may now conclude that
\[
\lim_{c \to 0} \lim_{x \to \infty} \int_0^c K_h(v, y_0, y_1, \ldots, y_h) \sigma_x(v) \, dv = 0.
\]

Since \( \lim_{x \to \infty} P(e^{\xi_0} > x) / P(e^{\xi_0} > x) = E[e^{\alpha \xi_0}] \), we finally obtain
\[
\lim_{x \to \infty} P(e^{\xi_0} > x, y_0, e^{\xi_1} \leq x^{\phi_1} y_1, \ldots, e^{\xi_h} \leq x^{\phi_h} y_h \mid e^{\xi_0} > x)
= \frac{1}{E[e^{\alpha \xi_0}]} \int_0^\infty K_h(v, y_0, y_1, \ldots, y_h) v^{-\alpha-1} \, dv,
\]
which is exactly (4.1).

The AR(1) process is a particular case of a linear process with \( \phi_j = \phi^j \) for all \( j \geq 0 \), so (4.1) and (3.6) must coincide in this case. To check this, recalling the notation of Section 3.1, note that for the exponential AR(1) we have \( \xi_0^* = \xi_{0,k} + \phi^k \xi_0^* \) and \( \xi_0^* \) is independent of \( \xi_{0,k} \) for each \( k \geq 1 \). Denoting by \( F_\ast \) the distribution of \( e^{\xi_0} \) and considering for clarity only the case \( h = 1 \), we have
\[
\frac{1}{E[e^{\alpha \xi_0^*}]} \int_0^\infty P(e^{\xi_0^*} > v^{-1} y_0, e^{\xi_1} \leq v^{-\phi} y_1) v^{-\alpha-1} \, dv
\]
\[
= \frac{1}{E[e^{\alpha \xi_0^*}]} \int_0^\infty \int_{s=v^{-1} y_0}^\infty P(e^{\xi_0^*} \leq (sv)^{-\phi} y_1) F_\ast(ds) \, v^{-\alpha-1} \, dv
\]
\[
= \frac{1}{E[e^{\alpha \xi_0^*}]} \int_0^\infty \int_{s=v^{-1} y_0}^\infty P(e^{\xi_0^*} \leq (sv)^{-\phi} y_1) v^{-\alpha-1} \, dv F_\ast(ds)
\]
\[
= \frac{1}{E[e^{\alpha \xi_0^*}]} \int_0^\infty s^{-\alpha} F_\ast(ds) \int_{y_0}^\infty P(e^{\xi_0^*} \leq u^{-\phi} y_1) u^{-\alpha-1} \, du
\]
\[
= \int_{y_0}^\infty P(e^{\xi_0^*} \leq u^{-\phi} y_1) u^{-\alpha-1} \, du.
\]
That is, with \( h = 1 \), Equation (4.1) reduces to (3.6).

## 5 Stochastic volatility models

### 5.1 Stochastic volatility process with heavy tailed volatility

Assume now as in [MR13] that \( X_t = V_t Z_t = e^{\xi_t} Z_t \), where \( \{\xi_t, t \in \mathbb{Z}\} \) is the AR(1) process considered in Section 3.1 and \( \{Z_t, t \in \mathbb{Z}\} \) is a sequence of i.i.d. random variables such that \( E[|Z_0|^q] < \infty \) for some \( q > \alpha \), independent of the sequence \( \{\xi_t, t \in \mathbb{Z}\} \). Breiman’s lemma yields, for \( x \to \infty \),
\[
P(X_0 > x) \sim E[(Z_0)^q] P(e^{\xi_0} > x), \tag{5.1}
\]
\[
P(X_0 < -x) \sim E[(Z_0)^q] P(e^{\xi_0} > x). \tag{5.2}
\]
Let $F_0$ be the distribution function of $X_0$ and $\nu_\delta$ is the measure defined by $\nu_\delta(du) = F_0(xdu)/F_0(x)$. Then, conditioning on $X_0$ and the sequence $\{Z_t\}$, we have, for $y_0 \geq 1$ and $(y_1,\ldots,y_n) \in \mathbb{R}^h$,

$$
\mathbb{P}(X_0 > x, y_0, X_1 \leq x^{\phi_1}y_1, \ldots, X_h \leq x^{\phi_h}y_h \mid X_0 > x) = \frac{1}{\mathbb{P}(X_0 > x)} \int_0^1 \mathbb{P}(Z_0 > (x/\xi)y_0, Z_1 e^{\delta_0,1} \leq (x/\xi)^{\phi_1}y_1, \ldots, Z_h e^{\delta_0,h} \leq (x/\xi)^{\phi_h}y_h) F_0(du) \mathbb{P}(V_0 > x) \int_0^1 \mathbb{P}(Z_0 > v^{-1}y_0, Z_1 e^{\delta_0,1} \leq v^{-\phi_1}y_1, \ldots, Z_h e^{\delta_0,h} \leq v^{-\phi_h}y_h) \nu_\delta(dv) .
$$

By arguments similar to those in the proof of (4.1), we obtain

$$
\lim_{x \to \infty} \mathbb{P}(X_0 > x, y_0, X_1 \leq x^{\phi_1}y_1, \ldots, X_h \leq x^{\phi_h}y_h \mid X_0 > x) = \frac{1}{\mathbb{E}(Z_0)^+} \int_0^\infty \mathbb{P}(Z_0 > v^{-1}y_0, Z_1 e^{\delta_0,1} \leq v^{-\phi_1}y_1, \ldots, Z_h e^{\delta_0,h} \leq v^{-\phi_h}y_h) \alpha v^{\alpha-1} dv .
$$

The conditional scaling exponent is thus $\kappa_h = \phi^h$ and the limiting conditional distribution of $x^{-\delta_h}X_h$ given $X_0 > x$ is

$$
\lim_{x \to \infty} \mathbb{P}(X_h \leq x^{\phi_h}y \mid X_0 > x) = \int_0^\infty \mathbb{P}(Z_0 > v^{-1}, Z_1 e^{\delta_0,1} \leq v^{-\phi_1}y_1, \ldots, Z_h e^{\delta_0,h} \leq v^{-\phi_h}y_h) \alpha v^{\alpha-1} dv .
$$

If $\alpha > 1$, we can apply Lemma 2.3 to obtain

$$
\lim_{x \to \infty} \mathbb{E} \left[ \frac{(X_h)^+}{x^{\phi_h}} \mid X_0 > x \right] = \frac{\alpha \mathbb{E}(Z_0)^{\alpha-\kappa_h} \mathbb{E}(Z_0) \mathbb{E}(X_0)}{(\alpha - \kappa_h) \mathbb{E}(Z_0)^+ \mathbb{E}(X_0^{\phi_h})} .
$$

**Tail of $X_0X_h$.** Using similar computation as in Section 3.1 we have

$$
\mathbb{E} \left[ \frac{X_0 - x \mathbb{1}{X_0 \leq \xi}}{x^{\phi_h}} \right] = \mathbb{E} \left[\frac{V_0^{(1 + \phi_h)}Z_0}{x^{(1 + \phi_h)}} \mathbb{1}{V_0 \leq \xi} \right] \mathbb{E} \left[ e^{\delta_0 \sum_{j=1}^{h-1} \phi^j} \right] \mathbb{E}[Z_0^\delta] .
$$

Moreover, if $F_Z$ is the distribution function of $Z_0$, then

$$
\frac{1}{\mathbb{P}(V_0 > x)} \mathbb{E} \left[\frac{V_0^{(1 + \phi_h)}Z_0}{x^{(1 + \phi_h)}} \mathbb{1}{V_0 \leq \xi} \right] = \frac{1}{\mathbb{P}(V_0 > x)} \int \mathbb{E} \left[\frac{V_0^{(1 + \phi_h)}u}{x^{(1 + \phi_h)}} \mathbb{1}{V_0 \leq \xi/u} \right] F_Z(du) \leq e^{\delta(1 + \phi_h)} \int u^{-\delta(1 + \phi_h)} \frac{\mathbb{P}(V_0 > x/u)}{\mathbb{P}(V_0 > x)} u F_Z(du) \leq C e^{\delta(1 + \phi_h) - \alpha} \int u^{\alpha + 1 - \delta(1 + \phi_h)} F_Z(du) .
$$

If $\delta$ is chosen such that $0 < \alpha < \delta(1 + \phi_h) < \alpha + 1$, then the condition (2.6) holds.

### 5.2 Stochastic volatility process with heavy tailed innovation

Assume that $X_t = \sigma_t Z_t$, where $\{Z_t, t \in \mathbb{Z}\}$ is an i.i.d. sequence with regularly varying marginal distribution with tail index $\alpha$, $\sigma_t$ is non negative, $\mathbb{E}[\sigma_t^q] < \infty$ for some $q > \alpha$ and $\{\sigma_t\}$ and $\{Z_t\}$ are independent. Then, by Breiman’s lemma, $X_t$ is regularly varying, and has extremal independence:

$$
\mathbb{P}(X_0 > x) \sim \mathbb{E}[\sigma_0^\alpha] F_Z(x) ,
$$
and
\[ P(X_0 > x, X_h > x) = o(\tilde{F}_Z(x)), \]
where \( F_Z \) is the distribution function of \( Z_0 \). For any integer \( h > 0 \), we have
\[ \lim_{x \to \infty} P \left( \frac{X_0}{x} > y_0, X_1 \leq y_1, \ldots, X_h \leq y_h \mid X_0 > x \right) = \frac{E[\sigma_0^\alpha F_Z(y_h/\sigma_h)] \cdots F_Z(y_h/\sigma_h)}{y_0^\alpha E[\sigma_0^\alpha]} . \] (5.3)
In particular,
\[ \lim_{x \to \infty} P(X_h \leq y_h \mid X_0 > x) = \frac{E[\sigma_0^\alpha F_Z(y_h/\sigma_h)]}{E[\sigma_0^\alpha]} . \]
The conditional scaling exponent \( \kappa_h \) is 0 at all lags \( h \geq 1 \). Note also that
\[ \lim_{x \to \infty} P(X_h > y_h \mid X_0 > x) = \frac{E[\sigma_0^\alpha F_Z(y_h/\sigma_h)]}{E[\sigma_0^\alpha]} \sim \tilde{F}_Z(y_h) \frac{E[\sigma_0^\alpha \sigma_h^\alpha]}{E[\sigma_0^\alpha]} \sim \tilde{F}_X(y_h) \frac{E[\sigma_h^\alpha \sigma_h^\alpha]}{E[\sigma_0^\alpha] E[\sigma_0^\alpha]} , \]
as \( y_h \to \infty \). Hence, the limiting conditional distribution is tail equivalent to the unconditional distribution.

For more details on the extremal behavior of this model we refer to [DM01] and [KS13]. In particular, in the latter paper the conditional model and its extensions to different conditioning events is considered (cf. the discussion in Section 7), together with relevant statistical inference.

If \( \alpha > 1 \), then Lemma 2.3 applies and we obtain
\[ \lim_{x \to \infty} E[(X_h)_+ \mid X_0 > x] = \frac{E[(Z_0)_+ \mid E[\sigma_h \sigma_0^\alpha]]}{E[\sigma_0^\alpha]} . \]

### 5.3 Stochastic volatility process with heavy tailed innovation and leverage

We now consider a stochastic volatility process \( X_t = \sigma_t Z_t \) and assume that the volatility \( \sigma_t \) has the form
\[ \sigma_t = \sigma(\xi_t), \]
where \( \sigma \) is a positive function, \( \xi_t = \sum_{j=1}^{\infty} c_j \eta_{t-j}, \sum_{j=1}^{\infty} c_j^2 < \infty \) and \( \{Z_t, \eta_t\} \) is an i.i.d. sequence, but for each \( t, Z_t \) and \( \eta_t \) may be dependent. This implies that the volatility \( \sigma_t \) is independent of the innovation \( Z_t \) for each \( t \), but \( \sigma_t \) may be dependent of \( \{Z_j, j < t\} \). This allows for some leverage: today's value impacts future volatility. We still assume that the distribution of \( Z_0 \) is regularly varying with index \( \alpha \). For each \( t, Z_t \) and \( \sigma_t \) are independent, thus, if \( E[\sigma_t^q] < \infty \) for some \( q > \alpha \), Breiman's lemma applies and we obtain
\[ P(X_0 > x) \sim E[\sigma_0^\alpha] \tilde{F}_Z(x) . \]

Consider now the probability of joint exceedances. Since \( \sigma_h \) and \( Z_0 \) may be dependent, we have,
\[ P(X_0 > x, X_h > x) = P(Z_0 \sigma_0 > x, Z_h \sigma_h > x) \]
\[ = E \left[ \tilde{F}_Z(x/\sigma_h) 1_{\{Z_0 \sigma_0 > x\}} \right] \]
\[ \sim \tilde{F}_Z(x) E \left[ \sigma_h^\alpha 1_{\{Z_0 \sigma_0 > x\}} \right] = o(\tilde{F}_Z(x)) . \]
(For the last part, a bounded convergence argument is used.) Thus there is still extremal independence, as in the previous model with no leverage, but the rate of decay of the joint exceedances probability is affected by the dependence between \( \sigma_h \) and \( Z_0 \).
Under additional assumptions, we can obtain the limiting conditional distributions. Assume that 
\[ \eta_j = \log(\|Z_j\|) - \mathbb{E}[\log(\|Z_j\|)], \sigma(x) = e^x \text{ and } 0 < c_j < 1 \text{ for all } j \geq 1. \] 
Define \( \tilde{\sigma}_j = \exp\{\sum_{i=1}^{j-1} c_i \eta_{j-i} - c_j \mathbb{E}[\log(\|Z_0\|)] + \sum_{i=j+1}^\infty c_i \eta_{j-i}\}. \) Then, \( X_j = \tilde{\sigma}_j Z_0 \) and by the same type of arguments as previously, we obtain, for \((y_0, \ldots, y_h) \in [1, \infty) \times \mathbb{R}^h,\)

\[
\lim_{x \to \infty} \mathbb{P}(X_0 > xy_0, X_1 \leq x^{c_1} y_1, \ldots, X_h \leq x^{c_h} y_h \mid X_0 > x) = \int_0^\infty \mathbb{P}(\sigma_0 > y_0 u^{-1}, Z_1 \tilde{\sigma}_1 \leq y_1 u^{-c_1}, \ldots, Z_h \tilde{\sigma}_h \leq y_h u^{-c_h}) \alpha u^{-\alpha-1} du.
\]

The conditional scaling exponent depends on \( h \): \( \kappa_h = c_h. \) The marginal limiting distributions are also tail equivalent to the distribution of \( X_0 \).

If \( \alpha > 1 \), Lemma 2.3 applies again and we obtain

\[
\lim_{x \to \infty} \mathbb{E}\left[ \frac{(X_h)_+}{x^{\kappa_h}} \mid X_0 > x \right] = \frac{\alpha \mathbb{E}(Z_h)_+}{(\alpha - \kappa_h)\mathbb{E} \sigma_0^{\alpha - \kappa_h}} = \frac{\alpha \mathbb{E}(Z_h)_+}{(\alpha - \kappa_h)\mathbb{E}\|Z_0\|^\kappa_h} = \frac{\alpha \mathbb{E}(Z_h)_+}{(\alpha - \kappa_h)\mathbb{E}\sigma_0^{\alpha - \kappa_h}}.
\]

6 Proof of Theorem 3.1

The following result is related to [Bil68, Theorem 5.5, page 34] and is sometimes referred to as the second continuous mapping theorem. See also [RZ13, Lemma 8.4].

**Theorem 6.1.** Let \((E, d)\) be a complete locally compact separable metric space. Let \( \mu_n \) be a sequence of probability measures which converge weakly to a probability measure \( \mu \) on \( E \).

(i) If \( \varphi_n \) is a uniformly bounded sequence of continuous functions which converge uniformly on compact sets of \( E \) to a function \( \varphi \), then \( \varphi \) is continuous and bounded on \( E \) and \( \lim_{n \to \infty} \mu_n(\varphi_n) = \mu(\varphi) \).

(ii) Let \( F \) be a topological space. If \( g_n \) is a sequence of uniformly bounded, continuous functions on \( F \times E \) which converge uniformly on compact sets of \( F \times E \) to a function \( g \), then \( g \) is continuous and bounded on \( F \times E \) and the sequence of functions \( \int_E g_n(u, v) \mu_n(du) \) converge uniformly on compact sets of \( F \) to \( \int_E g(u, v) \mu_\mu(du) \)

**Proof.** We start by proving (i). Let \( C \) be such that \( \sup_{n \geq 1} \|\varphi_n\|_\infty \leq C \) and \( \|\varphi\|_\infty \leq C \). Fix some \( \epsilon > 0 \) and let \( K \) be a compact set such that \( \mu(\partial(K^c)) = 0 \) and \( \mu(K^c) \leq \epsilon/(2C) \). Let \( K_\epsilon = \{x \in E \mid d(x, K) \leq \epsilon\} \) and let \( \psi \) be a continuous function such that \( 0 \leq \psi(x) \leq 1 \) for all \( x \in E \), \( \psi(x) = 1 \) if \( x \in K_\epsilon \) and \( \psi(x) = 0 \) if \( x \notin K_\epsilon \).

\[
\mu_n(\varphi_n) - \mu(\varphi) = \mu_n(\varphi_n) - \mu_n(\varphi) + \mu_n(\varphi) - \mu(\varphi).
\]

By weak convergence, \( \lim_{n \to \infty} \mu_n(\varphi) = \mu(\varphi) \), so we only need to consider \( \mu_n(\varphi_n) - \mu_n(\varphi) \). Using the function \( \psi \) defined above, we have

\[
|\mu_n(\varphi_n) - \mu_n(\varphi)| \leq |\mu_n(\varphi_n \psi) - \mu_n(\varphi \psi)| + |\mu_n((1 - \psi)\varphi_n) - \mu_n(\varphi(1 - \psi))| \\
\leq \mu_n(|\varphi \psi - \varphi_n \psi|) + 2C \mu_n(1 - \psi).
\]

Since \( \varphi_n \) converges to \( \varphi \) uniformly on compact sets and the function \( 1 - \psi \) is bounded and continuous, we obtain

\[
\limsup_{n \to \infty} |\mu_n(\varphi_n) - \mu_n(\varphi)| \leq 2C \mu(1 - \psi) \leq 2C \mu(K^c) \leq \epsilon.
\]

Since \( \epsilon \) is arbitrary, the proof of (i) is concluded.
We now prove (ii). Define \( L_n(u) = \int_E g_n(u,v)\mu_n( dv), \) \( \bar{L}_n(u) = \int_E g(u,v)\mu_n( dv) \) and \( L(u) = \int_E g(u,v)\mu( dv). \) Since \( g \) is bounded and continuous, the first part of the theorem implies that \( L_n \) converges uniformly to \( L \) on compact sets of \( F \). We now prove that \( L_n - \bar{L}_n \) converges to zero uniformly on compact sets of \( F \). Fix \( \epsilon > 0 \) and let \( K_x \) be as above. Since \( g_n \) and \( g \) are uniformly bounded, there exists \( C > 0 \) such that

\[
|L_n(u) - \bar{L}_n(u)| \leq \sup_{v \in K_x} |g_n(u,v) - g(u,v)| + 2C\epsilon.
\]

For any compact set \( S \) of \( F \), \( g_n \) converges uniformly on \( S \times K_x \) to \( g \), thus

\[
\limsup_{n \to \infty} \sup_{u \in S} |L_n(u) - \bar{L}_n(u)| \leq 2C\epsilon.
\]

Since \( \epsilon \) is arbitrary, this proves that \( L_n - \bar{L}_n \) converges to 0 uniformly on compact sets of \( F \). \( \square \)

We finally need the following lemma. Let \( \Pi \) and \( G \) be as in Assumption 2 and define the kernels \( \Pi_x \) and \( G_1 \) by

\[
\Pi_x f(u) = \int_0^\infty f(v)\Pi(xu, b(x) dv),
\]

\[
G_1 f(u) = \int_0^\infty f(u^{-}v)G( dv) = \int_0^\infty f(v)G(u^{-}dv).
\]

**Lemma 6.2.** Let \( f, f_x, x > 0 \), be uniformly bounded, continuous functions on \([0, \infty)\). Assume that

(i) either \( f_x \) converges uniformly on compact sets of \([0, \infty)\) to \( f \);

(ii) or \( f_x \) converges uniformly on compact sets of \((0, \infty)\) to \( f \) and \( G(\{0\}) = 0 \).

Then \( \Pi_x f_x \) converges uniformly on compact sets of \((0, \infty)\) to \( G_1 f \).

**Proof.** Fix some positive real numbers \( 0 < a_0 < a_1 \). Since \( b \) is regularly varying at infinity with positive index, without loss of generality, we can assume that \( b \) is increasing and positive on \((a_0, \infty)\). Then, the ratio \( b(xu)/b(x) \) is uniformly bounded on \([a_0, a_1]\), i.e.

\[
0 < \sup_{x \geq 1} \sup_{a_0 \leq u \leq a_1} \frac{b(xu)}{b(x)} < \infty.
\]

(6.1)

Fix some \( \epsilon > 0 \). Then, there exists \( A_x \) such that

\[
\limsup_{x \to \infty} \sup_{a_0 \leq u \leq a_1} \Pi(xu, (b(x)A_x, \infty)) \leq \epsilon, \quad \sup_{a_0 \leq u \leq a_1} G((u^\kappa A_x, \infty)) \leq \epsilon.
\]

(6.2)

Moreover, if \( G(\{0\}) = 0 \), then there also exists \( \eta > 0 \) such that

\[
\limsup_{x \to \infty} \sup_{a_0 \leq u \leq a_1} \Pi(xu, [0, b(x)\eta]) \leq \epsilon, \quad \sup_{a_0 \leq u \leq a_1} G([0, u^\kappa \eta]) \leq \epsilon.
\]

(6.3)

Let now \( f_x \) and \( f \) be as in the statement of the lemma. Then, by the uniform boundedness assumption and by (6.2), there exists \( C > 0 \) such that, for \( u \in [a_0, a_1] \),

\[
|\Pi_x f_x(u) - \Pi_x f(u)| \leq \int_0^{A_x} |f_x(v) - f(v)|\Pi(xu, b(x) dv) + C\epsilon.
\]

In case (i), it is assumed that \( f_x \) converges uniformly on the compact sets of \([0, \infty)\), thus the previous bound yields

\[
\limsup_{x \to \infty} \sup_{a_0 \leq u \leq a_1} |\Pi_x f_x(u) - \Pi_x f(u)| \leq C\epsilon.
\]
Since ε is arbitrary, this yields

\[ \limsup_{x \to \infty} \sup_{a_0 \leq u \leq a_1} |\Pi_x f_x(u) - \Pi_x f(u)| = 0 . \quad (6.4) \]

In case (ii), we must further decompose the integral into \( f^A_\eta = f^\eta_\eta + f^A_\eta \) and use the bound (6.3) to obtain (6.4).

We now prove that \( \Pi_x f \) converges uniformly on compact sets of \( (0, \infty) \) to \( G_1 f \). Define the function \( f_t \) on \( (0, \infty) \times [0, \infty) \) by \( f_t(u, v) = f(vb(t)/b(t/u)) \). By the uniform convergence for regularly varying functions, \( f_t \) converges uniformly on compact sets of \( (0, \infty) \times [0, \infty) \) to \( f(u^\alpha v) \). Define \( F_t(u) = \int f_t(u, v) \Pi(t, b(t)dv) \). By item (ii) of Theorem 6.1, \( F_t \) converges to \( G_1 f \) uniformly on compact sets of \( (0, \infty) \). Note that by change of variables we can write

\[ \Pi_x f(u) = \int f(v) \Pi(xu, b(x)dv) = \int f \left( \frac{vb(xu)}{b(x)} \right) \Pi(xu, b(xu)dv) \]

\[ = \int f_x(u, v) \Pi(xu, b(xu)dv) = F_x(u) , \]

so \( \Pi_x f \) converges uniformly on compact sets of \( (0, \infty) \) to \( G_1 \).

Proof of Theorem 3.1. We must prove that for all \( h \geq 1 \),

\[ \lim_{x \to \infty} \mathbb{P} \left( \frac{X_0}{x} \leq y_0, \frac{X_1}{b_1(x)} \leq y_1, \ldots, \frac{X_h}{b_h(x)} \leq y_h \mid X_0 > x \right) = \mathbb{P}(Y \leq y_0, Y_1 \leq y_1, \ldots, Y_h \leq y_h) . \quad (6.5) \]

The proof is by induction on \( h \). We start by proving (6.5) in the case \( h = 1 \). Recall that \( F_0 \) is the distribution of \( X_0 \) and define the measure \( \nu_x \) by \( \nu_x(du) = F_0(xdu)/F_0(x) \). Let \( f \) be a bounded continuous function on \( [0, \infty) \). Then

\[ \mathbb{E} \left[ f \left( \frac{X_1}{b(x)} \right) \mid X_0 > x \right] = \int_{u=1}^\infty \int_{v=0}^\infty f(v) \Pi(xu, b(x)dv) \nu_x(du) = \int_{u=1}^\infty \Pi_x f(u) \nu_x(du) , \]

and thus we must prove that

\[ \lim_{x \to \infty} \int_1^\infty \Pi_x f(u) \nu_x(du) = \int_1^\infty G_1 f(u) \alpha u^{-\alpha-1} du . \quad (6.6) \]

We know that the measure \( \nu_x \) converges vaguely on \( (0, \infty) \) to the Pareto measure \( \alpha u^{-\alpha-1} du \). By Lemma 6.2 (applied with \( f_x = f \), thus not requiring the assumption \( G(\{0\}) = 0 \)), \( \Pi_x f \) converges to \( G_1 f \) uniformly on compact sets of \( (0, \infty) \). Thus, applying Theorem 6.1(i) we obtain (6.6).

Consider now the higher dimensional distributions. Define the transition kernel \( \Pi_{x, h} \) on \( [0, \infty) \times [0, \infty)^h \) by

\[ \Pi_{x, h}(u_0, du) = \prod_{i=1}^h \Pi(b_{i-1}(x)u_{i-1}, b_i(x)du_i) , \]

with the convention \( b_0(x) = x \). For \( f \) bounded and continuous on \( [0, \infty)^h \), define

\[ \Pi_{x, h} f(u_0) = \int_{u \in [0, \infty)^h} f(u) \Pi_{x, h}(u_0, du) . \]
Then,
\[
\mathbb{E} \left[ f \left( \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \mid X_0 > x \right] = \int_{u_0=1}^{\infty} \Pi_{x,h} f(u_0)\nu_x(du_0) .
\]

Define also the kernel \( G_h \) on \((0, \infty) \times [0, \infty)^h \) by
\[
G_h f(u_0) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(u_1, \ldots, u_h) \prod_{i=1}^{h} G(u_i^{-\kappa}du_i) .
\]

for any bounded continuous function \( f \). What we must prove is that \( \Pi_{x,h} f \) converges uniformly on compact sets of \((0, \infty) \) to \( G_h f \) for any bounded continuous function \( f \) on \([0, \infty)^h \). By Theorem 6.1(i), this will yield the required result. For \( h = 1 \) this is what we have just proved. Assume now that for \( h \geq 1 \), and any bounded continuous function \( f \) on \([0, \infty)^h \), \( \Pi_{x,h} f \) converges uniformly on compact sets of \((0, \infty) \) to \( G_h f \). Without loss of generality, we can assume that the function \( f \) is of the form \( f(u_1, \ldots, u_{h+1}) = f_1(u_1)f_2(u_2, \ldots, u_{h+1}) \), where \( f_1 \) and \( f_2 \) are continuous and bounded on \([0, \infty) \) and \([0, \infty)^h \), respectively. Then, recalling that \( b_h = b_{h-1} \circ b \),
\[
\Pi_{x,h+1} f(u_0) = \int_{0}^{\infty} f_1(u_1)\Pi_{b(x),h} f_2(u_1)\Pi(xu_0, b(x)du_1) = \Pi_x(f_1\Pi_{x,h} f_2)(u_0) \quad (6.7)
\]

By the induction assumption, the sequence functions \( f_1\Pi_{x,h} f_2 \) converges uniformly on compact sets of \((0, \infty) \) to the continuous and bounded function \( f_1 G_h f_2 \). Thus, by Lemma 6.2, \( \Pi_{x,h+1} f \) converges to \( G_1(f_1G_h f_2) = G_{h+1}f_1 f_2 = G_h f \) uniformly on the compact sets of \((0, \infty) \). \( \square \)

### 7 Concluding remarks

In this paper, we have put the concept of conditional extreme values in the context of univariate time series. We have introduced the conditional scaling exponent \( \kappa_h \) of a time series \( \{X_t\} \) at lag \( h \). If the time series is stationary and its finite dimensional marginal distributions are regularly varying, then \( \kappa_h \in [0, 1] \) and a value \( \kappa_h < 1 \) implies the extremal independence of the bivariate distribution \((X_0, X_h)\). We have given conditions for Markov chains and other time series models commonly used in financial econometrics. This work is part of an ongoing project on extremally independent time series. We now briefly discuss several possible future lines of research.

**Vector valued time series.** Consider a \( d \)-dimensional vector valued times series \( \{X_t, t \in \mathbb{Z}\} \) such that for each \( h \geq 0 \), the \((h+1)d\)-dimensional vector \((X_0, \ldots, X_h)\) is regularly varying with index \(-\alpha\). For a relatively compact Borel set \( C \subseteq \mathbb{R}^{h+1} \setminus \{0\} \) (possibly with further regularity conditions), we may be interested in the limiting distribution of \((X_1, \ldots, X_h)\) given that \( X_0 \in xC \), where \( xC = \{xy, y \in C\} \) and \( x \) is large. In the case of extremal dependence, the exponent measure of the vector \((X_0, \ldots, X_h)\) provides the necessary information. In the case of extremal independence, it is useless, and we must investigate the existence of scaling functions \( b_1, \ldots, b_h \) such that the conditional distribution of
\[
\begin{pmatrix} X_1 \\ b_1(x) \end{pmatrix}, \ldots, \begin{pmatrix} X_h \\ b_h(x) \end{pmatrix}
\]
given \( X_0 \in xC \) converges to a proper probability distribution. The choice of the set \( C \) is determined by the problem considered. It could be the complement of the unit ball for some norm \( \| \cdot \| \) on \( \mathbb{R}^d \), if the event of interest is that \( \|X_0\| \) is large, or a half-space such as \( C = \{y \in \mathbb{R}^d \mid a_1y_1 + \cdots + a_ky_k > 1\} \), if the event of interest is that a certain linear combination (a portfolio) is large.
Different conditioning events. We can also consider univariate time series and various conditioning events such as \( \{y_0 > x, \ldots, y_k > x\}\) \((k+1)\) successive values are large), or \( \{\max\{y_0, \ldots, y_k\} > x\}\) \((at least one large value among the first \(k + 1\))\), or any combination of such events. Again, in the case of extremal dependence the appropriate scaling is given by the multivariate regular variation property and the entire information is given by the exponent measure. In the case of extremal independence different scaling functions must be used for different lags and the limiting distributions are not given by the exponent measure.

Statistical procedures. The next step is obviously to provide valid statistical procedures to estimate the conditional scaling exponents, the scaling functions, the conditional limiting distributions and other quantities of interests such as the CTE. As usual in extreme value theory, these quantities cannot be estimated empirically since they are relevant only in a domain where few observations are available. Therefore extrapolation outside the range of available data is needed and semiparametric estimators must be defined. For instance, one can estimate \( m_h \) (defined in (2.13)) and \( \kappa_h \) and then to estimate \( \text{CTE}_h^\text{SP}(x) \) for \( x \) outside the range of the data by

\[
\hat{\text{CTE}}_h^\text{SP}(x) = x^{\hat{\kappa}_h} \hat{m}_h.
\]

Nonparametric estimators of the conditional limiting distributions and of the scaling functions, as well as semiparametric estimators of the conditional scaling exponents are the subject of our future research.

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