Homotopy Operators and One-Loop Vacuum Energy at the Tachyon Vacuum

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We construct homotopy operators for the BRST operator in the theory around identity-based solutions, which are believed to represent the tachyon vacuum in the cubic bosonic open string field theory. Using the homotopy operators, we find that the one-loop vacuum energy at the tachyon vacuum is independent of moduli such as interbrane distances, which are included in the BRST operator. We also revisit the cohomology problem, which was solved earlier without the homotopy operators.

Subject Index: 126, 128

§1. Introduction

The cubic bosonic open string field theory (SFT) has classical solutions describing the tachyon vacuum. In the theory expanded around the tachyon vacuum solution, the BRST cohomology vanishes from a physical Hilbert space and the annihilation of D-branes can be described using the solution. However, there are apparently different results for the cohomology at the tachyon vacuum. For the solution constructed using wedge states,1) the cohomology completely vanishes at all ghost numbers.2) On the other hand, it was proved for the identity-based solution3) that there exists a nonempty BRST cohomology at unphysical ghost numbers.4) Also in numerical calculation, while the cohomology was proposed to be trivial,5) it was reported from a different analysis that the cohomology exists at nonstandard ghost numbers.6),7)

The vanishing cohomology for the wedge-like solution is proved using a homotopy operator.2) The anticommutator between the homotopy operator and BRST operator at the solution is equal to unity. Then, all the BRST closed states turn out to be BRST exact. In the case of the identity-based solution, the BRST operator at the solution is represented as a well-defined operator acting on a Fock space.3) By a similarity transformation and a level shift operation in the ghost sector, the cohomology can be obtained4) from the known results for the ordinary BRST operator.8)–10)

In this work, we will construct homotopy operators for the identity-based solutions. We will find that the cohomologically nontrivial part is given by acting the BRST operator to states, which are obtained by acting the homotopy operator, living outside a single Fock space. Accordingly, although the resulting cohomology vanishes, there is no contradiction with the earlier result since it was solved only in a single Fock space.
Given the homotopy operator, we cannot find nonzero scattering amplitudes with on-shell external lines. Then, it is natural to ask what happens to a one-loop vacuum amplitude without external lines at the tachyon vacuum. At the perturbative vacuum, the one-loop vacuum amplitude of an open string depends on interbrane distances and it can be interpreted as an amplitude for the exchange of a closed string between D-branes. Since D-branes disappear at the tachyon vacuum, we can speculate that the one-loop vacuum amplitude becomes independent of interbrane distances.

To calculate the vacuum amplitude, gauge fixing is needed and it seems a non-trivial problem at the tachyon vacuum. However, the Siegel gauge works well for the theory around the identity-based solution. By using the Siegel gauge level expansion, the unstable perturbative vacuum solution was found with high precision (up to the truncation level \((26, 78)^{11})\) and the one-loop vacuum amplitude was investigated numerically.\(^{12}\) Here, we will use the homotopy operator to analyze the vacuum amplitude in the Siegel gauge and demonstrate the independence of interbrane distances at the tachyon vacuum.

This paper is organized as follows. In §2, we briefly review the identity-based solution characterized by some function. Then, we will explicitly construct the homotopy operator for a class of the identity-based solution. In §3, we will evaluate the variation of the one-loop vacuum energy with respect to moduli such as interbrane distances and we will discuss the cohomology in the theory around the identity-based solution. We will also comment on homotopy operators for other solutions. Finally, we will give concluding remarks in §4.

§2. Homotopy operators for the identity-based solution

2.1. Identity-based solution

We consider the bosonic open string field theory with a midpoint interaction. The action is given by

\[
S[\Psi] = -\frac{1}{g^2} \int \left( \frac{1}{2} \Psi * Q_B \Psi + \frac{1}{3} \Psi * \Psi * \Psi \right),
\]  

(2.1)

where the operator \(Q_B\) is the Kato-Ogawa BRST charge, which is defined on perturbative vacuum. The equation of motion is derived from the variation of the action as

\[
Q_B \Psi + \Psi * \Psi = 0.
\]  

(2.2)

We can construct an exact classical solution of the equation of motion (2.2) using half-string operators and the identity string field \(I):^{3)}

\[
\Psi_0 = Q_L (e^h - 1) I - C_L ((\partial h)^2 e^h) I,
\]  

(2.3)

where \(Q_L(f)\) and \(C_L(f)\) are integrations of the BRST current \(j_B(z)\) and the ghost \(c(z)\) with a function \(f(z)\) along a half-unit disc:

\[
Q_L(f) = \int_{C_{\text{left}}} \frac{dz}{2\pi i} f(z) j_B(z), \quad C_L(f) = \int_{C_{\text{left}}} \frac{dz}{2\pi i} f(z) c(z).
\]  

(2.4)
We can find that the equation of motion holds for the function \( h(z) \) such that \( h(-1/z) = h(z) \) and \( h(\pm i) = 0 \). Expanding the string field as \( \Psi = \Psi_0 + \Phi \) and subtracting \( S[\Psi_0] \), we obtain the action for the fluctuation \( \Phi \) around the solution \( \Psi_0 \) as

\[
S'[\Phi] = -\frac{1}{g^2} \int \left( \frac{1}{2} \Phi \ast Q' \Phi + \frac{1}{3} \Phi \ast \Phi \ast \Phi \right). \tag{2.5}
\]

The operator \( Q' \) in the quadratic term is given by

\[
Q' = Q(e^h) - C((\partial h)^2 e^h), \tag{2.6}
\]

where the operators \( Q(f) \) and \( C(f) \) are defined as integrations along a unit circle:

\[
Q(f) = \oint \frac{dz}{2\pi i} f(z) j_B(z), \quad C(f) = \oint \frac{dz}{2\pi i} f(z) c(z). \tag{2.7}
\]

The classical solution (2.3) includes an arbitrary function, which is changed by gauge transformations. Most of the solutions are regarded as a trivial pure gauge transformation from the trivial solution, \( \Psi_0 = 0 \), but nontrivial solutions can be obtained at the boundary of some function spaces. For example, we consider the classical solution constructed using the function

\[
h_a(z) = \log \left( 1 + \frac{a}{2} (z + z^{-1})^2 \right). \tag{2.8}
\]

This function includes one parameter \( a \), which is larger than or equal to \(-1/2\). This range of the parameter \( a \) is determined with the reality condition for the classical solution. The solution for \( h_a(z) \) corresponds to a trivial pure gauge for \( a > -1/2 \), but we find that it becomes nontrivial at the boundary \( a = -1/2 \). In the case of \( a = -1/2 \), the operator (2.6) has no cohomology in the ghost number one sector. In addition, the theory based on the action (2.5) has an unstable vacuum solution corresponding to the perturbative string vacuum. Consequently, we conclude that the nontrivial solution at \( a = -1/2 \) is the tachyon vacuum solution.

The function (2.8) for the solution (2.3) can be generalized as

\[
h_a^l(z) = \log \left( 1 - \frac{a}{2} (-1)^l (z^l - (-1)^l z^{-l})^2 \right), \tag{2.9}
\]

where \( l \) is a positive integer and \( a \geq -1/2 \). This includes the function (2.8) as the \( l = 1 \) case. For all \( l \), the classical solution is expected to be the tachyon vacuum solution at \( a = -1/2 \). For the function \( h_a^{l=1/2}(z) \), the BRST operator (2.6) at the tachyon vacuum can be written as

\[
Q_l = Q(F) + C(G), \tag{2.10}
\]

\[
F(z) = \frac{(-1)^l}{4} \left( z^l + (-1)^l z^{-l} \right)^2, \quad G(z) = -(-1)^l z^{-2} \left( z^l - (-1)^l z^{-l} \right)^2. \tag{2.11}
\]
2.2. Homotopy operators

The operator product expansion (OPE) of the BRST current with the antighost \( b(z) \) is

\[
j_B(z)b(z') \sim \frac{3}{(z-z')^3} + \frac{1}{(z-z')^2} j_{gh}(z') + \frac{1}{z-z'} T(z'),
\]

(2.12)

where \( j_{gh}(z) \) is the ghost number current and \( T(z) \) is the total energy momentum tensor. From this OPE, we can derive the anticommutation relation of \( Q(f) \) and \( b(z) \) as

\[
\{ Q(f), b(z) \} = \frac{3}{2} \partial^2 f(z) + \partial f(z) j_{gh}(z) + f(z) T(z).
\]

(2.13)

Similarly, from the OPE of \( c(z) \) and \( b(z) \), we obtain the anticommutation relation,

\[
\{ C(f), b(z) \} = f(z).
\]

(2.14)

Using (2.13) and (2.14), we can calculate the anticommutation relation of the BRST operator (2.10) with \( b(z) \). The important point is that the function \( F(z) \) in (2.11) has second-order zeros at the points,

\[
z_k = \begin{cases} 
e^{i\frac{k-\frac{1}{2}}{l}\pi} \quad \text{for odd } l, \\ e^{i\frac{2k}{2l}\pi} \quad \text{for even } l, \end{cases} \quad (k = 1, 2, \ldots, 2l)
\]

(2.15)

which are solutions to the equation: \( z^{2l} + (-1)^l = 0 \). Therefore, the anticommutator becomes a c-number:

\[
\{ Q_l, b(z_k) \} = \frac{3}{2} \partial^2 F(z_k) + G(z_k) = z_k^{-2} l^2.
\]

(2.16)

It should be noted that the function \( e^{H_k} \) has only first-order zeros for \( a > -1/2 \), and therefore, the above anticommutator depends on the ghost number current \( j_{gh} \) for trivial pure gauge solutions.

Alternatively, we can explicitly compute the anticommutation relation (2.16) in terms of the oscillator expression. The operator \( Q_l \) is expanded as

\[
Q_l = \frac{1}{2} Q_B + \frac{(-1)^l}{4} (Q_{2l} + Q_{-2l}) + 2 l^2 c_0 - (-1)^l l^2 (c_{2l} + c_{-2l}),
\]

(2.17)

where we have expanded the BRST current and the ghost as \( j_B(z) = \sum_n Q_n z^{-n-1} \) and \( c(z) = \sum_n c_n z^{-n+1} \). Using the oscillator expressions of \( b(z) = \sum_n b_n z^{-n-2} \), \( T(z) = \sum_n L_n z^{-n-2} \) and \( j_{gh}(z) = \sum_n q_n z^{-n-1} \), we find the anticommutation relation of \( Q_m \) and \( b_n \) from (2.12),

\[
\{ Q_m, b_n \} = L_{m+n} + m q_{m+n} + \frac{3}{2} m(m-1) \delta_{m+n,0}.
\]

(2.18)
Using (2.17) and (2.18), we can calculate the left-hand side of (2.16) as

\[
\{Q_l, b(z_k)\} = \frac{1}{2} \sum_{n=\infty}^{\infty} L_n(z_k)^{-n-2}
+ \frac{(-1)^l}{4} \left( \sum_{n=\infty}^{\infty} (L_{n+2l} + 2lq_{n+2l})(z_k)^{-n-2} + \frac{3}{2} 2l(2l+1)(z_k)^{-2l-2} \right)
+ \frac{(-1)^l}{4} \left( \sum_{n=\infty}^{\infty} (L_{n-2l} - 2lq_{n-2l})(z_k)^{-n-2} + \frac{3}{2} 2l(2l-1)(z_k)^{2l-2} \right)
+ 2l^2(z_k)^{-2} - (-1)^l l^2 ((z_k)^{-2l-2} + (z_k)^{2l-2})
= z_k^{-2l^2}, \quad (2.19)
\]

where we have used \((z_k)^{2l} = -(-1)^l\). Thus, Eq. (2.16) can be derived from the mode expansion without any divergence.

The anticommutation relation (2.16) implies that we can define a homotopy operator \(\hat{A}\) corresponding to the BRST operator \(Q_l\) at the solution (2.3) with the function (2.9) at \(a = -1/2\):

\[
\hat{A} = \sum_{k=1}^{2l} a_k l^{-2} z_k^2 b(z_k), \quad \sum_{k=1}^{2l} a_k = 1, \quad (2.20)
\]

which satisfies the relations

\[
\{Q_l, \hat{A}\} = 1, \quad \hat{A}^2 = 0. \quad (2.21)
\]

If we choose the coefficients \(a_k\) as

\[
a_k = a_{l-k+2}, \quad (k = 1, 2, \cdots, l+1); \quad a_k = a_{3l-k+2}, \quad (k = l+2, l+3, \cdots, 2l)
\]

(2.22)

for odd \(l\),

\[
a_k = a_{l-k+1}, \quad (k = 1, 2, \cdots, l); \quad a_k = a_{3l-k+1}, \quad (k = l+1, l+2, \cdots, 2l)
\]

(2.23)

for even \(l\) and \(a_k \in \mathbb{R} (k = 1, 2, \cdots, 2l)\), the operator \(\hat{A}\) is BPZ even and Hermitian. (The conditions (2.22) and (2.23) imply that the coefficients in (2.20) corresponding to each pair of \(z_k\), which are symmetric points with respect to the imaginary axis, are equal.) In this case, \(\hat{A}\) is explicitly expressed in terms of oscillators as

\[
\hat{A} = l^{-2} \sum_{m=-\infty}^{\infty} \sum_{k=1}^{2l} a_k \left( b_{2m} \cos \frac{2m(k-1)\pi}{l} - ib_{2m-1} \sin \frac{(2m-1)(k-1)\pi}{l} \right)
\]

(2.24)
for odd $l$ and

$$
\hat{A} = l^{-2} \sum_{m=\infty}^{\infty} \sum_{k=1}^{2l} a_k \left( b_{2m} \cos \frac{2m(2k-1)\pi}{2l} - ib_{2m-1} \sin \frac{(2m-1)(2k-1)\pi}{2l} \right)
$$

(2.25)

for even $l$.

Note that the above homotopy operator $\hat{A}$ can be rewritten as

$$
\hat{A}\Phi = \frac{1}{2} \left( A \Phi + (-1)^{|\Phi|} \Phi A \right),
$$

(2.26)

using the homotopy state $A \equiv \hat{A} I$. In order to obtain this expression, we have used (2.22), (2.23) and

$$
z^{-4} b(-1/z) \Phi_1 \Phi_2 = (-1)^{|\Phi_1|} \Phi_1 b(z) \Phi_2,
$$

(2.27)

$$
z^{-4} b(-1/z) I = b(z) I.
$$

(2.28)

Using $Q_{l} I = 0$ and (2.21), we have

$$
Q_{l} A = I, \quad A \star A = 0
$$

(2.29)

for the homotopy state $A$.

§3. One-loop vacuum energy and cohomology

3.1. One-loop vacuum energy at the tachyon vacuum

We consider a string field theory at the tachyon vacuum, in which the BRST operator is given by $Q_{l}$. We impose the Siegel gauge condition, $b_{0} \Phi = 0$. Then, the one-loop vacuum energy is given by the integration over the moduli $t$ of the partition function:

$$
Z(t) = \text{Tr} \left[ (-1)^{N_{FP}} e^{-tL'} b_{0} c_{0} \right].
$$

(3.1)

Here, $L' = \{Q_{l}, b_{0}\}$ is the Siegel gauge inverse propagator$^{*}$) and $N_{FP}$ is the operator counting ghost number: $N_{FP} = c_{0} b_{0} + \sum_{n \geq 1} (c_{-n} b_{n} - b_{-n} c_{n})$. The trace $\text{Tr}$ is defined by the sum over all the Fock space states and the projection operator $b_{0} c_{0}$ is inserted into the trace to restrict to the Siegel gauge subspace.

Suppose that there are multiple separated D-branes at the perturbative vacuum. Then, the BRST operator $Q_{B}$ depends on interbrane distances through the zero mode of string coordinates in the nondiagonal sector of a string field with the Chan-Paton

$^{*}$) There are several works about the Siegel gauge theory around the identity-based solution. In Ref. 15), it was suggested that purely closed string amplitudes could be derived from open string fields by using the kinetic operator $L'$. It was found in Ref. 16) that all the scattering amplitudes vanish. The vacuum structure in the theory with $L'$ was numerically evaluated up to level 26,13,14) and the resulting structure agrees with that of the tachyon vacuum.
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indices.\textsuperscript{a}) In this case, $\Psi_0$ (2.3) with the function $h = h_{\alpha=-1/2}^a$ is a solution by including the identity matrix in the identity state $I$ and it is expected to represent the tachyon vacuum. The partition function (3.1) includes the trace over the Chan-Paton indices and apparently depends on the brane distance parameters through $L'$ or the BRST operator at the solution $Q_l$. However, the one-loop vacuum energy is expected not to include the distance parameters, since the D-branes no longer exist at the tachyon vacuum.

Let us show that (3.1) is indeed independent of the interbrane distances. Under an infinitesimal change of modulus such as the D-brane positions, the BRST operator changes to $Q_l' = Q_l + \delta Q_l$. The variation of $L'$ is given by that of the BRST charge:

$$\delta L' = \{\delta Q_l, b_0\}. \quad (3.2)$$

The key ingredient of the proof is the existence of the homotopy operator $\hat{A}$. Since the homotopy operator (2.20) is defined only by the antighost, $\hat{A}$ and $b_0$ anticommute with each other:

$$\{\hat{A}, b_0\} = 0. \quad (3.3)$$

From the variation of (2.21), we find

$$\{\delta Q_l, \hat{A}\} = 0. \quad (3.4)$$

Using the Jacobi identity, we have

$$[L', \hat{A}] = [(Q_l, b_0), \hat{A}] = -[\{b_0, \hat{A}\}, Q_l] - [\{\hat{A}, Q_l\}, b_0] = 0. \quad (3.5)$$

Now, we are ready to evaluate the change of the partition function:

$$\delta Z(t) = -t \int_0^1 d\alpha \text{Tr} \left( (-1)^{N_{FP}} e^{-atL'} \{\delta Q_l, b_0\} e^{-(1-\alpha)tL'b_0c_0} \right). \quad (3.6)$$

Using the commutation relations $[L', b_0] = 0$ and the cyclic invariance of the trace, we can rewrite the integrand in (3.6) as

$$\text{Tr} \left( (-1)^{N_{FP}} e^{-atL'} b_0 \delta Q_l e^{-(1-\alpha)tL'b_0c_0} \right)$$

$$= -\text{Tr} \left( (-1)^{N_{FP}} e^{-atL'} \delta Q_l e^{-(1-\alpha)tL'b_0c_0} b_0 \right)$$

$$= -\text{Tr} \left( (-1)^{N_{FP}} e^{-atL'} \delta Q_l e^{-(1-\alpha)tL'} b_0 \right). \quad (3.7)$$

In this equation, we insert $\{Q_l, \hat{A}\}(=1)$ between $e^{-(1-\alpha)tL'}$ and $b_0$:

$$= -\text{Tr} \left( (-1)^{N_{FP}} e^{-atL'} \delta Q_l e^{-(1-\alpha)tL'} \{Q_l, \hat{A}\} b_0 \right)$$

$$= -\text{Tr} \left[ (-1)^{N_{FP}} e^{-atL'} \delta Q_l e^{-(1-\alpha)tL'} Q_l \hat{A} b_0 \right]$$

$$-\text{Tr} \left[ (-1)^{N_{FP}} e^{-atL'} \delta Q_l e^{-(1-\alpha)tL'} \hat{A} Q_l b_0 \right]. \quad (3.8)$$

\textsuperscript{a}) To introduce Wilson lines in the theory on D25 branes, we have only to replace momentum zero modes of string coordinates as $p^m \rightarrow p^m + (\theta_i - \theta_j)/\pi R$, where $i$ and $j$ are Chan-Paton indices.\textsuperscript{17} Therefore, in the T-dual picture, we can describe separated multiple D-branes only by changing zero modes of string coordinates in the SFT action of coincident multiple D-branes.\textsuperscript{17}
In the second term, we move \( \hat{A} \) to the left using the (anti) commutation relations, (3.4) and (3.5). Then, we have

\[
\begin{align*}
\delta Z(t) &= -\text{Tr} \left[ (-1)^{\mathcal{N}_{FP}} e^{-\alpha t L'} \delta Q_l e^{-(1-\alpha)t L'} Q_l b_0 \hat{A} b_0 \right] \\
&\quad -\text{Tr} \left[ (-1)^{\mathcal{N}_{FP}} e^{-\alpha t L'} \delta Q_l e^{-(1-\alpha)t L'} Q_l b_0 \hat{A} \right],
\end{align*}
\]

using cyclic invariance of the trace. These two terms cancel each other, thanks to (3.3). Thus, we finally obtain

\[
\delta Z(t) = 0,
\]

and we conclude that the one-loop vacuum energy is independent of interbrane distances.

We note that the above proof of \( \delta Z(t) = 0 \) does not depend on the details of the variation of \( Q_l \) and the calculation is rather formal. If we restrict the space in the definition of the trace, it is necessary to treat the relation \( \{ Q_l, \hat{A} \} = 1 \) carefully. We will discuss the related issue in the next subsection.

3.2. Homotopy operator versus cohomology

Once the homotopy operator \( \hat{A} \) exists at the tachyon vacuum, a BRST invariant state \( \psi \) such that \( Q_l \psi = 0 \) is a BRST exact state, namely,

\[
Q_l \psi = 0 \quad \Leftrightarrow \quad \psi = Q_l(\hat{A} \psi),
\]

because of the commutation relation \( \{ Q_l, \hat{A} \} = 1 \) (2.21).

On the other hand, the cohomology of the BRST operator \( Q_l \) was derived earlier in Ref. 4) by referring to the results for \( Q_B \):

\[
Q_l \psi = 0 \quad \Leftrightarrow \quad |\psi\rangle = |P\rangle \otimes U_l b_{-2l+1} \cdots b_{-2} |0\rangle + |P'\rangle \otimes U_l b_{-2l+2} \cdots b_{-2} |0\rangle + Q_l |\phi\rangle,
\]

where \( |P\rangle \) and \( |P'\rangle \) are positive-norm states in the matter sector such as DDF states, and the operator \( U_l \) is given by

\[
U_l = \exp \left( -2 \sum_{n=1}^{\infty} \frac{(-1)^{n(l+1)}}{n} q_{-2nl} \right).
\]

According to the result of (3.12), the cohomology exists in the Hilbert space of the ghost numbers \(-2l+1\) and \(-2l+2\). This result is apparently incompatible with vanishing cohomology in all the ghost number sectors, which can be read off from (3.11).

In order to resolve the discrepancy between (3.11) and (3.12), we investigate the cohomologically nontrivial states in (3.12)

\[
|\varphi\rangle = |P\rangle \otimes U_l b_{-2l} b_{-2l+1} \cdots b_{-2} |0\rangle + |P'\rangle \otimes U_l b_{-2l+1} b_{-2l+2} \cdots b_{-2} |0\rangle.
\]

According to the proposition (3.11), we can represent the state \( |\varphi\rangle \) as a BRST exact state:

\[
|\varphi\rangle = Q_l(\hat{A} |\varphi\rangle).
\]
Here, let us rewrite the state $\hat{A}|\varphi\rangle$ using a “normal ordered” expression, namely moving $\hat{A}$, which includes the positive frequency modes, to the right of $U_l$. Using the commutation relation $[q_m, b_n] = -b_{m+n}$, we have

$$b(z) U_l = \exp\left(-2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{-2nl}\right) U_l b(z),$$

for $U_l$ given in (3.13). By taking a limit, $z \to z_k$ with $(z_k)^{2l} = (-1)^l$,

$$b(z_k) U_l b_{-m} \cdots b_{-2}|0\rangle = \exp\left(-2 \sum_{n=1}^{\infty} \frac{1}{n}\right) U_l b(z_k)b_{-m} \cdots b_{-2}|0\rangle = 0$$

is obtained. Because the homotopy operator $\hat{A}$ (2.20) is a linear combination of $b(z_k)$, this implies that the state $\hat{A}|\varphi\rangle$ becomes zero in the Fock space expression. Namely, all the coefficients of $\hat{A}|\varphi\rangle$ vanish when it is expanded in terms of the conventional oscillators, $b_{-n}, c_{-m}$, on the conformal vacuum $|0\rangle$. However, the relation (3.15) implies that the state $\hat{A}|\varphi\rangle$ is not truly zero in some enlarged space. The conclusion is that the state $|\varphi\rangle$ is an exact state obtained by applying $Q_l$ to the state outside a single Fock space,* which is beyond the scope of the proposition (3.12).

3.3. **Comments on homotopy operators for other solutions**

Firstly, we comment on another type of identity-based solution,\(^{18}\) which is given by a function $h_a^{(4)}$ with a parameter $a \geq -1/2$:

$$h_a^{(4)}(z) = \log \left(1 + 2a - \frac{a}{8}(z - z^{-1})^4\right)$$

for $h$ in (2.3). At the boundary of the parameter, i.e., $a = -1/2$, the solution is expected to represent a nontrivial solution. For the function $h_{-1/2}^{(4)}$, the BRST operator around the solution is

$$Q^{(4)} = Q(F_4) + C(G_4)$$

$$= \frac{3}{8}Q_B - \frac{1}{4}(Q_2 + Q_{-2}) + \frac{1}{16}(Q_4 + Q_{-4}) + 2c_0 - c_4 - c_{-4},$$

$$F_4(z) = \frac{1}{16}(z - z^{-1})^4, \quad G_4(z) = -z^{-2}(z^2 - z^{-2})^2.$$

In this case, the function $F_4(z)$ has fourth-order zeros at $z = \pm 1$. Using the anti-commutation relations, (2.13) and (2.14), we have

$$\{Q^{(4)}, b(\pm 1)\} = 0, \quad \{Q^{(4)}, \partial b(\pm 1)\} = 0,$$

\(^*\) In this paper, we regard a space in which any state can be expressed by a linear combination of Fock bases with finite coefficients as a “single Fock space”, where a single Fock vacuum and a single set of creation-annihilation operators are fixed. (In the present case, they correspond to $c_1|0\rangle$ and $\{b_n, c_m\}$ in the ghost sector, respectively. Conversely, we bear in mind the Bogoliubov transformation, for example, as outside of a single Fock space.) Although we should define the space as a completion with respect to appropriate norm mathematically, we leave it as a future problem. Intuitively, we may be able to interpret that $\hat{A}|\varphi\rangle$ weakly converges to zero but not in a strong sense. We can find a similar situation with respect to the “phantom term” $\psi_\infty$ in Ref. 1).
\{Q^{(4)}, \partial^2 b(\pm 1)\} = \frac{3}{2} \partial^4 F_4(\pm 1) + \partial^2 G_4(\pm 1) = 4. \tag{3.22}

Therefore, we obtain a homotopy operator \( \hat{A}^{(4)} \):
\[
\hat{A}^{(4)} = \frac{1}{8}(\partial^2 b(1) + \partial^2 b(-1)) + \frac{5}{8}(\partial b(1) - \partial b(-1)) + \frac{1}{2}(b(1) + b(-1))
\]
\[=
\sum_{n=-\infty}^{\infty} n^2 b_{2n}\tag{3.23}
\]
which is BPZ even and Hermitian and satisfies
\[
\{Q^{(4)}, \hat{A}^{(4)}\} = 1, \quad (\hat{A}^{(4)})^2 = 0. \tag{3.24}
\]

The first anticommutation relation can also be obtained using mode expansion from (3.19), (3.23), and (2.18). Noting the relation \( \{\hat{A}^{(4)}, b_0\} = 0 \), we can trace the same computation as \S 3.1 to demonstrate \( \delta Z(t) = 0 \) (3.10).

In general, we can construct classical solutions with higher-order zeros.\(^{18}\) If the function corresponding to the solution has an \( n \)-th order zero at \( z = z_0 \), the anticommutator between \( \partial^{k-2} b(z) \) and the BRST operator at the solution becomes a nonzero c-number at \( z = z_0 \) for \( k = n \) and it vanishes for \( k < n \). Using these anti-commutation relations, the homotopy operator can be obtained as a linear combination of \( \partial^{k-2} b(z) \) (\( k = 2, \cdots, n \)). Therefore, we expect that the classical solution with higher-order zeros corresponds to the tachyon vacuum.

The cohomology of \( Q^{(4)} \) was derived in Ref. 18) and the result is
\[
Q^{(4)} |\psi\rangle = 0 \quad \Leftrightarrow \quad |\psi\rangle = |P\rangle \otimes U_{(4)} b_{-4} b_{-3} b_{-2} |0\rangle + |P'\rangle \otimes U_{(4)} b_{-3} b_{-2} |0\rangle + Q^{(4)} |\phi\rangle, \tag{3.25}
\]
in a similar way to (3.12), where
\[
U_{(4)} = \exp \left(-4 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n}\right). \tag{3.26}
\]
Namely, in the Fock space expression, a state of the form
\[
|\varphi_{(4)}\rangle = |P\rangle \otimes U_{(4)} b_{-4} b_{-3} b_{-2} |0\rangle + |P'\rangle \otimes U_{(4)} b_{-3} b_{-2} |0\rangle \tag{3.27}
\]
represents a nontrivial state of \( Q^{(4)} \) cohomology. However, from (3.24), it could be rewritten as
\[
|\varphi_{(4)}\rangle = Q^{(4)} (\hat{A}^{(4)} |\varphi_{(4)}\rangle). \tag{3.28}
\]
This apparent inconsistency can be resolved as in \S 3.2. Noting the relations
\[
b(z) U_{(4)} = \exp \left(-4 \sum_{n=1}^{\infty} \frac{z^{-2n}}{n}\right) U_{(4)} b(z) = (1 - z^{-2})^4 U_{(4)} b(z), \tag{3.29}
\]
\[
\partial b(z) U_{(4)} = (1 - z^{-2})^4 U_{(4)} \partial b(z) + 8 z^{-3} (1 - z^{-2})^3 U_{(4)} b(z), \tag{3.30}
\]
\[
\partial^2 b(z) U_{(4)} = (1 - z^{-2})^4 U_{(4)} \partial^2 b(z) + 16 z^{-3} (1 - z^{-2})^3 U_{(4)} \partial b(z)
\]
\[+ 24 z^{-4} (1 - 3 z^{-2}) (1 - z^{-2})^2 U_{(4)} b(z), \tag{3.31}
\]
and taking a limit $z \to \pm 1$, we have
\begin{align*}
    b(\pm 1) U(4) b_{-m} \cdots b_{-2} |0\rangle &= 0, \\
    \partial b(\pm 1) U(4) b_{-m} \cdots b_{-2} |0\rangle &= 0, \\
    \partial^2 b(\pm 1) U(4) b_{-m} \cdots b_{-2} |0\rangle &= 0,
\end{align*}
by reexpressing the left-hand sides as normal ordered forms. Because the homotopy operator $\hat{A}(4)$ is given as a linear combination of $b(\pm 1), \partial b(\pm 1)$ and $\partial^2 b(\pm 1)$ as in (3.23), the above equations imply that $\hat{A}(4) |\varphi(4)\rangle$ is zero in the Fock space expression. Hence, it is necessary to use the appropriate expression beyond a single Fock space to conclude that $|\varphi(4)\rangle$ is $Q^{(4)}$ exact in the sense of (3.28).

Next, we briefly mention the case of solutions constructed using the $K BC$ subalgebra. The Schnabl solution\textsuperscript{1)} and Erler-Schnabl solution\textsuperscript{19)} are in this category and they are considered to represent the tachyon vacuum. The homotopy states, which satisfy (2.29), for the BRST operator around these solutions were obtained in Refs. 2) and 19). These states can be rewritten as the homotopy operators such as (2.21) through the definition given by (2.26).\textsuperscript{2)} It turns out that both of them do not anticommute with $b_0$. Hence, we cannot apply the same procedure as in §3.1 for these solutions.*

However, for a solution $\Psi = \sqrt{1 - \beta K} \beta^{-1} c \sqrt{1 - \beta K}$, which is a real form of an identity-based solution: $\beta^{-1} c - cK$, in terms of the $KBC$ subalgebra, the homotopy state is $A = \beta B$ as is given in Ref. 25). Therefore, the homotopy operator $\hat{A}$ such as (2.21), which is a linear combination of $b_n$, satisfies (3.3) and, therefore, the same computation in §3.1 can be applicable to prove $\delta Z(t) = 0$ (3.10).

\section*{§4. Concluding remarks}

In this work, we have constructed a homotopy operator $\hat{A}$ for the BRST operator $Q_i$ in the theory around a type of identity-based solution associated with particular functions $h_{a,-l/2}^l$ ($l = 1, 2, 3, \cdots$).\textsuperscript{4)} Using the operator $\hat{A}$, we have demonstrated that the one-loop vacuum energy at the solution is independent of moduli such as interbrane distances. These results are consistent with the interpretation that the solution represents the tachyon vacuum. We have also found a homotopy operator for another type of identity-based solution whose associated function has higher order zeros.\textsuperscript{18)} We can apply the same procedure to prove $\delta Z(t) = 0$ (3.10) for this solution and a particular type of solution in the $KBC$ subalgebra,\textsuperscript{25)} which is a real form of an identity-based solution.

We have also revisited the cohomology problem for the identity-based solutions.\textsuperscript{4),18)} Using the obtained homotopy operator, one can conclude that there is no cohomology at all the ghost number sectors. The nontrivial cohomology part of (3.12) (or (3.25)) cannot be regarded as a BRST exact state within a single Fock space. This is not the first appearance of such a state in SFT. In the bosonic closed light-cone SFT, a classical solution associated with the dilaton vacuum expectation

\footnote{In the theory around the regularized identity-based solution,\textsuperscript{20)-22)} we cannot apply the procedure for the same reason. Although we have used the Siegel gauge condition for the evaluation of the one-loop vacuum energy, other gauge conditions\textsuperscript{23),24)} might be useful for these solutions.}
value was constructed and then it was impossible to realize it within a single Fock space. Then, it was remarked that the space of string fields should be much larger than a single Fock space. Also, in the study of target space duality, it was emphasized that classical solutions in SFT must live outside the Hilbert space of the original background. More recently, a tachyon vacuum solution based on wedge-like states includes a so-called phantom state. The phantom state is effectively zero in a Fock space, but it is indispensable to derive the vacuum energy correctly. Once again, for the cohomology of $Q_l$, we are forced to incorporate the state outside a single Hilbert space.

In the discussion so far, the anticommutation relation $\{Q_l, \hat{A}\} = 1$ is respected on any state. However, the relation (3.17) might imply that $\varphi = (Q_l \hat{A} + \hat{A} Q_l)\varphi$ is not equal to $Q_l(\hat{A}\varphi) + \hat{A}(Q_l\varphi)$, which gives zero for (3.14), if one interprets Eq. (3.17) as it stands. Namely, the associativity of multiplication of the operators may be broken on the states of the form (3.14) because multiple infinite summations of oscillators are included in the expression and we have interchanged the order of limits naively. In order to avoid this ambiguity, some regularizations should be introduced and/or the space of states should be restricted appropriately. Together with the issue stated in the previous paragraph, a mathematically more rigorous treatment of the space of states in SFT is desired in future developments. Although one can find some investigations in this direction in the context of the $K_{BC}$ subalgebra in Ref. 25), for example, a wider class of string fields should be incorporated to resolve the delicate problems mentioned above.

Finally, we comment on the result for the one-loop vacuum energy at the tachyon vacuum from the viewpoint of the BRST quartet mechanism. In the present case, the BRST charge is the operator $Q_l$ and any states are classified into the irreducible representations of the algebra of $Q_l$ and the FP ghost charge. The existence of the homotopy operator for $Q_l$ shows that there are no BRST singlet states. Therefore, we might be able to interpret that the vanishing result of the trace (3.6) is closely related to the norm cancellation among quartet states. In contrast, if the anticommutation relation $\{Q_l, \hat{A}\} = 1$ is broken on the nontrivial part of (3.12), the state (3.14) seems to propagate in the trace as a BRST singlet. However, the state cannot form a singlet pair with a nonzero inner product. In fact, its dual state in the trace should have the ghost number $2l + 2$ or $2l + 1$ and such a state does not belong to the BRST singlet representation as seen in (3.12). Therefore, the result $\delta Z(t) = 0$ in §3.1 seems to be plausible also from the above speculation.

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References

1) M. Schnabl, Adv. Theor. Math. Phys. 10 (2006), 433.
2) I. Ellwood and M. Schnabl, J. High Energy Phys. 02 (2007), 096.
3) T. Takahashi and S. Tanimoto, J. High Energy Phys. 03 (2002), 033.
4) I. Kishimoto and T. Takahashi, Prog. Theor. Phys. 108 (2002), 591.
5) I. Ellwood, B. Feng, Y. H. He and N. Moeller, J. High Energy Phys. 07 (2001), 016.
6) S. Giusto and C. Imbimbo, Nucl. Phys. B 677 (2004), 52.
7) C. Imbimbo, Nucl. Phys. B 770 (2007), 155.
8) M. Kato and K. Ogawa, Nucl. Phys. B 212 (1983), 443.
9) M. Henneaux, Phys. Lett. B 177 (1986), 35.
10) I. B. Frenkel, H. Garland and G. J. Zuckerman, Proc. Natl. Acad. Sci. USA 83 (1986), 8442.
11) I. Kishimoto, Prog. Theor. Phys. Suppl. No. 188 (2011), 155.
12) T. Takahashi, Prog. Theor. Phys. Suppl. No. 188 (2011), 163.
13) I. Kishimoto and T. Takahashi, Theor. Math. Phys. 163 (2010), 717.
14) I. Kishimoto and T. Takahashi, Prog. Theor. Phys. 122 (2009), 385.
15) N. Drukker, J. High Energy Phys. 08 (2003), 017.
16) T. Takahashi and S. Zeze, Prog. Theor. Phys. 110 (2003), 159.
17) J. Polchinski, *String Theory I, An Introduction to the Bosonic String* (Cambridge University Press, 1998).
18) Y. Igarashi, K. Itoh, F. Katsumata, T. Takahashi and S. Zeze, Prog. Theor. Phys. 114 (2005), 695.
19) T. Erler and M. Schnabl, J. High Energy Phys. 10 (2009), 066.
20) E. A. Arroyo, J. of Phys. A 43 (2010), 445403.
21) S. Zeze, J. High Energy Phys. 10 (2010), 070.
22) E. A. Arroyo, J. High Energy Phys. 11 (2010), 135.
23) M. Kiermaier and B. Zwiebach, J. High Energy Phys. 07 (2008), 063.
24) M. Asano and M. Kato, Nucl. Phys. B 807 (2009), 348.
25) M. Schnabl, arXiv:1004.4858.
26) T. Yoneya, Phys. Lett. B 197 (1987), 76.
27) T. Kugo and B. Zwiebach, Prog. Theor. Phys. 87 (1992), 801.
28) T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. No. 66 (1979), 1.
29) H. Hata and T. Kugo, Phys. Rev. D 21 (1980), 3333.