Fisher information, Wehrl entropy, and Landau Diamagnetism

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Abstract

Using information theoretic quantities like the Wehrl entropy and Fisher’s information measure we study the thermodynamics of the problem leading to Landau’s diamagnetism, namely, a free spinless electron in a uniform magnetic field. It is shown that such a problem can be “translated” into that of the thermal harmonic oscillator. We discover a new Fisher-uncertainty relation, derived via the Cramer-Rao inequality, that involves phase space localization and energy fluctuations.

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INTRODUCTION

The last years have witnessed a great deal of activity revolving around physical applications of Fisher’s information measure (FIM) $I$ (as a rather small sample, see for instance, [1, 2, 3, 4, 5]). Frieden and Soffer [1] have shown that Fisher’s information measure provides one with a powerful variational principle, the extreme physical information one, that yields most of the canonical Lagrangians of theoretical physics [1, 2]. Additionally, $I$ has been shown to provide an interesting characterization of the “arrow of time”, alternative to the one associated with Boltzmann’s entropy [6, 7].

For our present purposes, the point to emphasize is that equilibrium thermodynamics can be entirely based upon Fisher’s measure (via a kind of “Fisher-MaxEnt”), that exhibits definite advantages over conventional text-book treatments [8]. Evaluating $I$ for a given system is tantamount to possessing complete thermodynamic information about it [8].

Unravelling the multiple FIM facets and their links to physics should be of general interest to a vast audience. Our subject here is the thermodynamics of Landau’s diamagnetism. We show, using FIM that, at temperature $T$, the pertinent physics reduces to that of a thermal harmonic oscillator whose frequency is the cyclotron one of the magnetic problem. In doing so, a new Fisher-uncertainty relation involving phase space localization and energy fluctuations is discovered.

Wehrl entropy and Husimi distribution

Quantum-mechanical phase-space distributions expressed in terms of the celebrated coherent states $|z\rangle$ of the harmonic oscillator, have been proved to be useful in different contexts [9, 10, 11]. Particular reference is to be made to the illuminating work of Andersen and Halliwell [12], who discuss, among other things, the concepts of Husimi distributions and Wehrl entropy. Coherent states are eigenstates of a general annihilation operator $a$, appropriate for the problem at hand, i.e., $a|z\rangle = z|z\rangle$ [9, 10, 11]. In the special case of the harmonic oscillator, for instance, one has

$$\mathcal{H}_o = \hbar \omega [a^+a + 1/2] = i(2\hbar \omega m)^{-1/2}p + (m\omega/2\hbar)^{1/2}x$$

$$z = (m\omega/2\hbar)^{1/2}x + i(2\hbar \omega m)^{-1/2}p. \quad (1)$$

Coherent states are often employed together with the concept of Wehrl entropy $W$ [12].
a special instance of Shannon’s logarithmic information measure that constitutes a powerful tool in statistical physics. $W$ is defined as

$$W = -\int \frac{dx \, dp}{2\pi \hbar} \mu(x, p) \ln \mu(x, p),$$  (2)

where $\mu(x, p) = \langle z | \rho | z \rangle$ is the “semi-classical” phase-space distribution function associated to the density matrix $\rho$ of the system [9, 10, 11]. The distribution $\mu(x, p)$ is normalized in the fashion $\int (dx \, dp/2\pi \hbar) \mu(x, p) = 1$, and is often referred to as the Husimi distribution [15]. The distribution $\mu(x, p)$ is a Wigner function smeared over an $\hbar$ sized region of phase space [12]. The smearing renders $\mu(x, p)$ a positive function, even is the Wigner distribution does not have such a character. The semi-classical Husimi probability distribution refers to a special type of probability: that for simultaneous but approximate location of position and momentum in phase space [12].

The usual treatment of equilibrium in statistical mechanics makes use of the celebrated Gibbs’ canonical distribution, whose associated, “thermal” density matrix is given by

$$\rho = Z^{-1} e^{-\beta H},$$

with $Z = Tr(e^{-\beta H})$ the partition function, $\beta = 1/kT$ the inverse temperature ($T$), and $k$ the Boltzmann constant. Our present Husimi functions will be constructed with such a $\rho$. In order to conveniently write down an expression for $W$ one considers, for the pertinent Hamiltonian $H$, its eigenstates $|n\rangle$ and eigen-energies $E_n$, because one can always write

$$\mu(x, p) = \langle z | \rho | z \rangle = \frac{1}{Z} \sum_n e^{-\beta E_n} |\langle z | n \rangle|^2.$$  (3)

A useful route to $W$ starts then with (3) and continues with (2).

**Electron without spin in a uniform magnetic field**

Consider the kinetic momentum

$$\vec{\pi} = \vec{p} + \frac{e}{c} \vec{A},$$  (4)

of a particle of charge $e$, mass $m_r$, and linear momentum $\vec{p}$, subject to the action of a vector potential $\vec{A}$. These are the essential ingredients of the well-known Landau model for diamagnetism: a spinless electron in a magnetic field of intensity $H$ (we follow the presentation of Feldman et al. [16]). The Hamiltonian is

$$H = \frac{\vec{\pi} \cdot \vec{\pi}}{2m_r},$$  (5)
and the magnetic field is $\vec{H} = \vec{\nabla} \times \vec{A}$. The vector potential is chosen in the symmetric gauge as $\vec{A} = (-Hy/2, Hx/2, 0)$, which corresponds to a uniform magnetic field along the $z-$direction. One also needs the step operators $\pi_{\pm} = px \pm ipy \pm i\hbar/2l^2 (x \pm iy)$ (6)

Motion along the $z-$axis is free (16). For the transverse motion, the Hamiltonian specializes to $H_t = \pi_+ \pi_- + \frac{1}{2} \hbar \Omega$. (7)

Two important quantities characterize the problem, namely, $\Omega = eH/m_r c$, the cyclotron frequency and the length $l = (\hbar c/eH)^{1/2}$ (17). The pertinent eigenstates $|N, m\rangle$ are determined by two quantum numbers: $N$ (associated to the energy) and $m$ (to the $z-$projection of the angular momentum). As a consequence, they are simultaneously eigenstates of both $H_t$ and the angular momentum operator $L_z$, so that

$$H_t|N, m\rangle = \left( N + \frac{1}{2} \right) \hbar \Omega |N, m\rangle = E_N|N, m\rangle$$

and

$$L_z|N, m\rangle = m\hbar|N, m\rangle.$$ (9)

Notice that the eigenvalues of $L_z$ are not bounded by below ($m$ takes the values $-\infty, \ldots, -1, 0, 1, \ldots, N$) (16). This agrees with the fact that the energies $(N + 1/2)\hbar \Omega$ are infinitely degenerate (17). Moreover, $L_z$ is not an independent constant of the motion (17).

We face a bi-dimensional phase-space problem. The corresponding four phase-space variables can conveniently be called $x, y, p_x, \text{and} p_y$, since $\pi_z$ is a constant of the motion (17) and the motion along the $z-$axis is that of a free particle. The pertinent coherent states $|\alpha, \xi\rangle$ are defined as the simultaneous eigenstates of the two commuting non-Hermitian operators which annihilate the ground state (16)

$$\pi_-|N = 0, m = 0\rangle = 0$$

$$X_+|N = 0, m = 0\rangle = 0,$$ (10)

with

$$X_{\pm} = x - \frac{\pi_y}{m_r \Omega} \pm i \left( y + \frac{\pi_x}{m_r \Omega} \right),$$ (11)
that are called orbit-center coordinate operators that step only the angular momentum \( m \) and not the energy \[^{16}\]. We have then

\[
\pi_-|\alpha, \xi\rangle = \frac{\hbar \alpha}{l^2} |\alpha, \xi\rangle
\]

\[
X_+|\alpha, \xi\rangle = \xi |\alpha, \xi\rangle,
\]

where the above defined quantity \( l \) represents the classical radius of the ground-state’s Landau orbit. Evaluating now \( \langle \alpha, \xi | \pi_+ \pi_- | \alpha, \xi \rangle \) we immediately find the modulus squared of eigenvalue \( \alpha \) as given by \[^{16}\]

\[
|\alpha|^2 = \frac{l^4}{\hbar^2} \left\{ \left( p_x - \frac{\hbar y}{2l^2} \right)^2 + \left( p_y + \frac{\hbar x}{2l^2} \right)^2 \right\}.
\]

The terms within the brackets (divided by \( 2m_r \)) yield the classical energy \( \mathcal{E}_{mag} \) of an electron in a uniform magnetic field. As noted in \[^{16}\], the modulus of both \( \alpha \) and \( \xi \) has dimensions of length.

After expanding the states \( |\alpha, \xi\rangle \) in the complete set of energy eigenfunctions \( |N, m\rangle \) given above, and conveniently using Eqs. (3.4) and (3.6) of \[^{16}\], we immediately obtain

\[
|\langle N, m | \alpha, \xi \rangle|^2 = \frac{|\alpha|^{2N} |\xi|^{2(N-m)}}{(2l^2)^N N! (2l^2)^{N-m} (N-m)!} e^{-|\alpha|^2+|\xi|^2/2l^2}.
\]

Our coherent states \( |\alpha, \xi\rangle \) satisfy the closure relation \[^{16}\]

\[
\int \frac{d^2\alpha d^2\xi}{4\pi^2l^4} |\alpha, \xi\rangle \langle \alpha, \xi| = 1,
\]

as expected.

**HUSIMI DISTRIBUTION**

We begin at this point our present endeavor, i.e., introducing thermodynamics into the model of the preceding Section, by calculating the appropriate Husimi distribution \[^{3}\] that our model requires. Such distribution adopts the appearance

\[
\mu(x, p_x; y, p_y) = \frac{1}{Z} \sum_{N=0}^{\infty} \sum_{m=-\infty}^{N} e^{-\beta\mathcal{E}_N} |\langle N, m | \alpha, \xi \rangle|^2.
\]

Using \[^{15}\] one can rewrite the above expression in the fashion

\[
\mu(x, p_x; y, p_y) = \frac{e^{-\beta\Omega/2}}{Z} e^{-|\alpha|^2+|\xi|^2/2l^2} \sum_{N=0}^{\infty} \frac{|\alpha|^{2N} |\xi|^{2N} e^{-\beta\Omega N}}{(2l^2)^{2N} N!} \sum_{m=-\infty}^{N} \left( \frac{2l^2}{|\xi|^2} \right)^m \frac{1}{(N-m)!},
\]

as expected.
and pass to the evaluation of the sum
\[
\sum_{m=-\infty}^{N} \left( \frac{2l^2}{|\xi|^2} \right)^m \frac{1}{(N-m)!} = \left( \frac{|\xi|^2}{2l^2} \right)^{-N} e^{\frac{|\xi|^2}{2l^2}}. \tag{19}
\]

This last result is now replaced into (18) so as to arrive at
\[
\mu(x, p_x; y, p_y) = \frac{e^{-\beta h \Omega/2}}{Z} e^{-|\alpha|^2/2l^2} \sum_{N=0}^{\infty} \left[ \frac{|\alpha|^2}{2l^2} e^{-\beta h \Omega/2} \right]^N \frac{1}{N!}, \tag{20}
\]
which immediately leads to the desired Husimi result we were looking for (our first new result), namely,
\[
\mu(x, p_x; y, p_y) = \frac{e^{-\beta h \Omega/2}}{Z} e^{-(1-e^{-\beta h \Omega})|\alpha|^2/2l^2}. \tag{21}
\]

Feldman \textit{et al.} have given the pertinent partition function \(Z\) that we need here, for a particle in a cylindrical geometry (length \(L\) and radius \(R\)), oriented along the magnetic field. One has \(Z_{\text{perp}} Z_{\text{parall}}\), where \(Z_{\text{parall}}\) is the usual partition function for one-dimensional free motion \(Z_{\text{parall}} = (L/h)(2\pi m_r kT)^{1/2}\) \textbf{[16]}. \(Z\) has the form \textbf{[16]}
\[
Z = V \frac{(2\pi m_r kT)^{1/2} m_r \Omega}{h} \frac{1}{4\pi^2 \sinh(\beta h \Omega/2)} \tag{22}
\]
Using it we can easily recast \(\mu(x, p_x; y, p_y)\) as
\[
\mu(x, p_x; y, p_y) = \frac{4\pi^2 h^2}{V m_r \Omega (2\pi m_r kT)^{1/2}} (1 - e^{-\beta h \Omega}) e^{-(1-e^{-\beta h \Omega})|\alpha|^2/2l^2}. \tag{23}
\]
This last expression is not yet normalized (the pertinent normalization integral equals \(2\pi h/(L \sqrt{2\pi m_r kT})\), with \(L\) the length of the sample). This can be remedied by scaling the above Husimi distribution. We proceed in two steps. First we define
\[
\varphi(x, p_x; y, p_y) = \frac{V m_r \Omega (2\pi m_r kT)^{1/2}}{4\pi^2 h^2} \mu(x, p_x; y, p_y) \tag{24}
\]
and write
\[
\varphi(x, p_x; y, p_y) = (1 - e^{-\beta h \Omega}) e^{-(1-e^{-\beta h \Omega})|\alpha|^2/2l^2}. \tag{25}
\]
Although this is not yet normalized, it is dimensionless. Now the corresponding normalization integral yields \(Am_r \Omega/(2\pi h)\). Finally, the normalized distribution is, of course,
\[
\phi(x, p_x; y, p_y) = \frac{2\pi h}{Am_r \Omega} (1 - e^{-\beta h \Omega}) e^{-(1-e^{-\beta h \Omega})|\alpha|^2/2l^2}. \tag{26}
\]
Obviously, we write now the Wehrl entropy in terms of the distribution function \(\phi(x, p_x; y, p_y)\) and get
\[
W = - \int \frac{d^2 x d^2 p}{4\pi^2 l^4} \phi(x, p_x; y, p_y) \ln \phi(x, p_x; y, p_y), \tag{27}
\]
so that, after replacing (26) into $W$ we find

$$W = 1 - \ln(1 - e^{-\beta\Omega}) - \ln \left( \frac{2\pi l^2}{A} \right),$$

(28)

where we have used the following result given in [16]

$$\int d^2\alpha d^2\xi e^{-(1-e^{-\beta\Omega})|\alpha|^2/2l^2} = \frac{A\mu}{2\pi\hbar} \frac{1}{1 - e^{-\beta\Omega}}. $$

(29)

$W$ depends on the sample’s dimensions via the third term in (28). The effect of the magnetic field is reflected via $\Omega$. The important point is the following: the present Wehrl measure is, save for the above mentioned (constant) third term, identical to that of an harmonic oscillator of frequency $\Omega$ at the temperature $T$ [18]. This constitutes our second original (present) contribution. It is to be pointed out that this result confirms an hypothesis made 10 years ago in [12], whose authors conjectured that the form (28) found for the harmonic oscillator could be of a rather general character.

**FISHER’S INFORMATION MEASURE**

R. A. Fisher advanced, already in the twenties, a quite interesting information measure (for a detailed study see [1, 2]). Consider a $\theta - z$ “scenario” in which we deal with a system specified by a physical parameter $\theta$, while $z$ is a stochastic variable ($z \in \mathbb{R}^M$) and $f_\theta(z)$ the probability density for $z$ (that depends also on $\theta$). One makes a measurement of $z$ and has to best infer $\theta$ from this measurement, calling the resulting estimate $\tilde{\theta} = \tilde{\theta}(z)$. The question is how well $\theta$ can be determined. Estimation theory [2] states that the best possible estimator $\tilde{\theta}(z)$, after a very large number of $z$-samples is examined, suffers a mean-square error $\epsilon^2$ from $\theta$ that obeys a relationship involving Fisher’s $I$, namely, $I\epsilon^2 = 1$, where the Fisher information measure $I$ is of the form

$$I(\theta) = \int dz f_\theta(z) \left( \frac{\partial \ln f_\theta(z)}{\partial \theta} \right)^2. $$

(30)

This “best” estimator is the so-called efficient estimator. Any other estimator exhibits a larger mean-square error. The only caveat to the above result is that all estimators be unbiased, i.e., satisfy $\langle \tilde{\theta}(z) \rangle = \theta$. Fisher’s information measure has a lower bound: no matter what parameter of the system one chooses to measure, $I$ has to be larger or equal
than the inverse of the mean-square error associated with the concomitant experiment. This result,

$$I \epsilon^2 \geq 1,$$

(31)
is referred to as the Cramer–Rao bound [2]. The uncertainty principle can be regarded as a special instance of (31) [2]. One often speaks of “generalized” uncertainty relations.

A particular $I$-case is of great importance: that of translation families [2, 3], i.e., distribution functions (DF) whose form does not change under $\theta$-displacements. These DF are shift-invariant (à la Mach, no absolute origin for $\theta$), and for them Fisher’s information measure adopts the somewhat simpler appearance [2]

$$I = \int dz f(z) \left\{ \frac{\partial \ln f(z)}{\partial z} \right\}^2 .$$

(32)

Fisher’s measure is additive [2]. Here we deal with the issue of estimating localization in a thermal scenario that revolves around a four dimensional phase-space, i.e., $z \equiv (z_1, z_2, z_3, z_4)$ is a 4-dimensional vector. Such an estimation task leads, as shown in [8], to the thermodynamics of the problem. Our Fisher measure acquires the appearance [18],

$$I = \sum_i I_i = \sum_i \int dz_i f(z_1, z_2, z_3, z_4) \left\{ \frac{\partial \ln f(z_i)}{\partial z_i} \right\}^2 .$$

(33)

PRESENT APPLICATION

Since $\ln \phi = \ln \left(2\pi \hbar/A m_r \Omega \right) + \ln(1 - e^{-\beta \hbar \Omega}) - (1 - e^{-\beta \hbar \Omega})|\alpha|^2 / 2l^2$, the above result (14) allows for the immediate finding

$$\frac{\partial \ln \phi}{\partial x} = \frac{1 - e^{-\beta \hbar \Omega}}{2\hbar} \left( p_y + \frac{\hbar x}{2l^2} \right) ,$$

(34)

$$\frac{\partial \ln \phi}{\partial y} = \frac{1 - e^{-\beta \hbar \Omega}}{2\hbar} \left( p_x - \frac{\hbar y}{2l^2} \right) ,$$

(35)

$$\frac{\partial \ln \phi}{\partial p_x} = \frac{l^2 (1 - e^{-\beta \hbar \Omega})}{\hbar^2} \left( p_x - \frac{\hbar y}{2l^2} \right) ,$$

(36)

and

$$\frac{\partial \ln \phi}{\partial p_y} = \frac{l^2 (1 - e^{-\beta \hbar \Omega})}{\hbar^2} \left( p_y + \frac{\hbar x}{2l^2} \right) .$$

(37)

With the above expressions we can now recast (14) in the fashion

$$|\alpha|^2 = \frac{2l^4}{(1 - e^{-\beta \hbar \Omega})^2} A,$$

(38)
where
\[ A = \left( \frac{\partial \ln \phi}{\partial x} \right)^2 + \left( \frac{\partial \ln \phi}{\partial y} \right)^2 + \frac{\hbar^2}{4l^4} \left[ \left( \frac{\partial \ln \phi}{\partial p_x} \right)^2 + \left( \frac{\partial \ln \phi}{\partial p_y} \right)^2 \right]. \] (39)

We are now in a position to write down the Fisher measure by following the prescription (33) and then write
\[ I = \int \frac{d^2\alpha d^2\xi}{4\pi^2 l^4} \phi(x, p_x; y, p_y) I^2 A, \] (40)
which, after a little algebra, turns out to be
\[ I = (1 - e^{-\beta \Omega})^2 \int \frac{d^2\alpha d^2\xi}{4\pi^2 l^4} |\alpha|^2 \phi(x, p_x; y, p_y). \] (41)

The integration is performed by appropriately using the pertinent derivatives of (29). We finally obtain
\[ I = 1 - e^{-\beta \Omega}. \] (42)

A glance at [18] tells us that the above is just the Fisher measure for the harmonic oscillator, which constitutes our third original result. We can finally compare the information (42) with the Wehrl measure (28), concluding that
\[ W = 1 - \ln I - \ln \left( \frac{2\pi l^2}{A} \right), \] (43)
i.e., they are complementary informational quantities [18]. As a matter of fact, we establish here one of the few existing direct Shannon-Fisher links.

For didactic reasons it is now convenient to focus attention on the quantity \(|\alpha|^2 = 2m_r (l^4/\hbar^2) \mathcal{E}_{mag}\), the “natural variable” of our scenario, go back to Eq. (41), and notice that the integral is just \(\langle |\alpha|^2 \rangle\), i.e., proportional to the semi-classical mean magnetic energy \(\langle \mathcal{E}_{mag} \rangle\) (see the comment that follows Eq. (14)). In other words, estimating localization in phase space is for the present problem equivalent to evaluating the average energy of our electron. It is pertinent to ask now about \(|\alpha|-\)fluctuations. A quick calculation yields
\[ \langle |\alpha|^2 \rangle = \frac{\pi l^2}{2I}, \] (44)
and
\[ (\Delta \langle |\alpha|^2 \rangle)^2 = \langle |\alpha|^2 \rangle - \langle |\alpha|^2 \rangle^2 = \frac{4 - \pi}{2} \frac{l^2}{I}. \] (45)
Out phase space localization problem becomes intimately linked to these fluctuations. The ensuing \((\Delta \langle |\alpha|\rangle)^2 I\)-product, i.e., the \(|\alpha|\)-Cramer-Rao bound \((31)\) (generalized uncertainty principle \([2]\)) is

\[
(\Delta \langle |\alpha|\rangle)^2 I = \frac{4 - \pi}{2} l^2 = \frac{4 - \pi}{2} \frac{c}{eH} \hbar,
\]

and we observe: i) as an equal sign is obtained, the estimation is optimal in the sense that the lower bound of the inequality \((31)\) is always obtained \([2]\), ii) the associated uncertainty is independent of the temperature, and iii) as we increase localization-quality \((I)\) increases, the size of \(|\alpha|\)-fluctuations, reasonably enough, decreases. A control-parameter, namely, the magnetic field intensity \(H\), is available. The larger the intensity, the better the overall quality. Nature imposes the ultimate control, however, as given by \(\hbar\).

The difference between \((4 - \pi)/2\) and \(1/2\) (of the order of 0.36) is due to the semi-classical character of our treatment.

We look now for a Cramer-Rao inequality that directly involves the energy \(E_{mag}\). Things will drastically change because to get the energy from \(|\alpha|^2\) one must divide by \(l^4\), which in turn will reverse the \(H\)-role. We immediately find

\[
\langle E_{mag}\rangle = \frac{\hbar \Omega}{I},
\]

and

\[
\langle E_{mag}^2 \rangle = 2 \frac{\hbar^2 \Omega^2}{I^2},
\]

so that for the energy-fluctuation \(\Delta^2 E_{mag} = \langle E_{mag}^2 \rangle - \langle E_{mag}\rangle^2\) one finds

\[
\Delta E_{mag} I = \hbar \Omega = \hbar \frac{eH}{m_e c},
\]

which, once again, is independent of \(T\). The effect of \(H\) is clearly different now, as anticipated. It is a simple matter to verify that \((49)\) also gives a localization-energy fluctuations Cramer-Rao uncertainty for the harmonic oscillator. The smaller the energy fluctuations, the better the localization estimation via \(I\).

CONCLUSIONS

A semi-classical information theory undertaking was tackled here: i) trying to estimate phase-space location via Fisher information and ii) evaluating the semi-classical Wehrl en-
tropy, for the celebrated Landau’s diamagnetism problem. Evaluating the Fisher measure $I$ appropriate for the problem yields its thermodynamics $^8$. As a summary:

- Using the coherent states discussed in Ref. $^1$ we have explicitly given the form of the Husimi distribution function for a spinless electron in a uniform magnetic field (Cf. Eq. (21)).

- We have discovered that the Wehrl entropy for Landau’s diamagnetism is, save for a constant term that depends on the size of the sample, that of a thermal harmonic oscillator whose frequency is the cyclotron one.

- For the corresponding Fisher measure the above similitude becomes identity. The thermo-statistics of the two problems is thus the same at the semi-classical level.

- We confirmed a conjecture made in $^1$; in the sense that the form $^2$ could be of a rather general character.

- An uncertainty relation linking phase space localization with energy fluctuations has been discovered (Cf. Eq. (49)).

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