Majorana representation of adiabatic and superadiabatic processes in three-level systems

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We show that stimulated Raman adiabatic passage (STIRAP) and its superadiabatic version (saSTIRAP) have a natural geometric two-star representation on the Majorana sphere. In the case of STIRAP, we find that the evolution is confined to a vertical plane. A faster evolution can be achieved in the saSTIRAP protocol, which employs a counterdiabatic Hamiltonian to nullify the non-adiabatic excitations. We observe how, under realistic experimental parameters, the counterdiabatic term corrects the trajectory of the Majorana stars toward the dark state. We also introduce a spin-1 average vector and present its evolution during the two processes. We show that the Majorana representation can be used as a sensitive tool for the detection of process errors due to ac Stark shifts and non-adiabatic transitions.

I. INTRODUCTION

Geometric representations play a key role in modern quantum information science. As far as the dynamics of quantum states is concerned, they may be used as a probe to look into various processes and develop intuitive ideas. For a spin-1/2, the mapping of states to the two-dimensional Bloch sphere is a well-known result dating back to the early work of Bloch and Rabi [1], a representation which is ubiquitously used nowadays for visualizing the states of qubits [2]. Clearly, for higher-dimensional Hilbert spaces this is a non-trivial task [3]. Majorana’s key insight from 1932 [4] was to represent these states as a “constellation” of several points (Majorana stars). According to this idea, a pure state of a particle with spin ‘s’ is represented by ‘2s’ points on a unit sphere [4].

This concept has been proved useful in various experimental contexts, such as for representing states of polarization of light [5], for characterizing the symmetry of the order parameter in spinor Bose-Einstein condensates [6, 7], and for the decomposition of quantum gates used in nuclear magnetic resonance into experimentally implementable pulses [8]. In theoretical quantum information, the Majorana representation has enabled the geometric study of symmetric multi-qubit states [9, 10], the construction of geometrically mutually unbiased bases and symmetric informationally complete positive operator valued measures [11], the study of Berry phases [12, 13], and the classification of high-dimensional entanglement [10, 14, 15]. It has also inspired the search for alternative geometric representations of quantum states [16–21].

Our goal in this paper is to obtain and study the Majorana representation of single-qutrit dynamics for the stimulated Raman adiabatic passage (STIRAP) and for its superadiabatic version (saSTIRAP). STIRAP is a well-known protocol [22, 23], widely used to perform certain non-trivial quantum operations such as laser-induced population exchange between the energy levels of atoms and molecules. In circuit quantum electrodynamics, the protocol for STIRAP pulses was first benchmarked in Ref. [24], where it was experimentally realized using the first three energy levels of a transmon, and demonstrating population exchange between the energy levels using microwave fields. With the emergence of more and more controllable multilevel systems, and stimulated by research into quantum technologies, the applications of STIRAP are likely to expand [25].

However, adiabaticity requires ideally infinitely long operation times, and therefore for realistic finite-time experimental conditions one expects a trade-off between time and fidelity. It is possible to accelerate STIRAP without loss of fidelity by the use of an additional drive (referred to as counterdiabatic) which exactly cancels the non-adiabatic excitations [26–29]. This superadiabatic drive, also known as transitionless driving or assisted adiabatic passage has recently been experimentally implemented in superconducting transmons [30]. The superadiabatic STIRAP is a specific form of quantum control from the larger class of shortcuts to adiabaticity [31], which have found a wide variety of applications, for example in electron transfer in quantum dots [32], state transfer in nitrogen vacancy centers [33], for the design of fast and error-resistant single-qubit [34] and two-qubit [35] gates, and for the quantum simulation of spin systems [36]. Here we show that the dynamics induced by the counterdiabatic correction applied to the STIRAP protocol has a very intuitive picture when represented geometrically, as it brings the Majorana stars closer to the path of the dark state.

The paper is organized as follows. We begin with a brief introduction to the Majorana representation for a qutrit, see Section [1]. Section [II] briefly describes STIRAP, followed by the corresponding dark-state dynamics on the Majorana sphere in Section [IV]. Further, the superadiabatic protocol is discussed in Section [V] and the corresponding dynamics is analysed in detail. Results from various simulations with experimentally feasible pa-
parameters are presented as well, followed by concluding remarks in Section VI.

II. MAJORANA REPRESENTATION

In 1932 Ettore Majorana introduced a representation of states with any angular momentum, as a starting point of his solution to the problem of atoms in a magnetic field [4]. Consider the standard basis $\{ | jm \rangle \}$ of angular momentum states, where $j$ and $m$ are the quantum numbers of the total angular momentum and its $z$-axis component. Specifically, $J_+ | j m \rangle = \hbar m | j m \rangle$ and $J_- | j m \rangle = \hbar j | j m \rangle$. The Majorana polynomial appears naturally when the spin-coherent representation of angular momentum states is considered. Writing $\mathbf{J} = J_x \mathbf{e}_x + J_y \mathbf{e}_y + J_z \mathbf{e}_z$, where $\mathbf{e}_{x,y,z}$ are the unit vectors along the axes, we notice that the average of the angular momentum operator in the state $| j j \rangle$ equals the total angular momentum $\hbar j$, $( j j | \mathbf{J} | j j \rangle = \hbar j \mathbf{e}_z$. We can obtain a vector with the same property but oriented along an arbitrary direction $\mathbf{n}$ by an appropriate rotation of $| j j \rangle$,

$$| j, \mathbf{n} \rangle = e^{-i \varphi J_z / \hbar} e^{-i \theta J_y / \hbar} | j j \rangle,$$

(1)

which define a spin-coherent state. Here the angles $\theta$ and $\varphi$ define the spherical parametrization of $\mathbf{n}$ as $\mathbf{n} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$. By using Wigner’s rotation matrices, we can expand $| j, \mathbf{n} \rangle$ in the $| jm \rangle$ basis,

$$| j, \mathbf{n} \rangle = \sum_{m=-j}^{j} \sqrt{(2j)! / (j + m)!(j - m)!} \left( \frac{\cos \theta}{2} \right)^{j+m} \left( \frac{\sin \theta}{2} \right)^{j-m} \times e^{-im\varphi} | jm \rangle$$

(2)

Now, given a general state

$$| \Psi \rangle = \sum_{m=-j}^{j} c_m | jm \rangle,$$

(3)

where $c_m$ are complex amplitude probabilities normalized as $\sum_{m=-j}^{j} | c_m |^2 = 1$, the complex amplitude probability for transitions to the spin-coherent state oriented along $-\mathbf{n}$ can be obtained as

$$\langle j, -\mathbf{n} | \Psi \rangle = \sqrt{(2j)!} (-1)^j \frac{\sqrt{2j} \hbar^2}{\sqrt{2 \pi}} P_{| \Psi \rangle} (\xi)$$

(4)

where we define the Majorana polynomial using the same notations as in the original paper [4],

$$P_{| \Psi \rangle} (\xi) = \sum_{r=0}^{2j} a_r \xi^{2j-r},$$

(5)

in the complex variable $\xi = \tan \frac{\theta}{2} e^{i \varphi}$. The relation between the coefficients of the polynomial and the complex amplitudes of the state is

$$a_r = \frac{(-1)^r}{\sqrt{(2j-r)! r!}} c_{j-r}.$$

(6)

Applying the fundamental theorem of algebra, it follows that $P_{| \Psi \rangle} (\xi)$ has $2j$ roots, which correspond to points in the complex $xOy$ plane.

Next, we notice that if we represent these roots by the angles $\theta$ and $\varphi$ as above, the roots can be represented by an inverse stereographic projection on the Riemann sphere, with respect to the South Pole as the reference. It is straightforward to prove geometrically that a line that connects the South Pole with a point $\zeta = \tan \frac{\theta}{2} e^{i \varphi}$ in the $xOy$ plane will intersect the unit sphere at a point with spherical coordinates $(\theta, \varphi)$. This unique configuration of $2j$ Majorana stars is invariant under rotations and achieves a geometrical representation of $| \Psi \rangle$ called Majorana constellation. For instance, $(\theta = \pi/2, \varphi = 0)$ is a Majorana star on the South Pole and a point at infinity in the complex plane, while $(\theta = 0, \varphi = 0)$ is a Majorana star placed at the North Pole and corresponds to the center of the complex plane. The Majorana representation of the state $| jm \rangle$ consists of $j + m$ stars at the North Pole and $j - m$ stars at the South Pole. Also, for a general state of a spin-$1/2$, $| \cos(\theta/2)|1/2, 1/2\rangle + |\sin(\theta/2)e^{i\varphi}|1/2, -1/2\rangle$ one can verify immediately that we recover the standard Bloch representation, namely that the Majorana star is a point on the Bloch sphere, with spherical coordinates $(\theta, \varphi)$.

Now we can introduce the geometrical picture of a qutrit ($s = 1$), which is a three level quantum system - thus its Majorana geometrical representation consists of two Majorana stars. An arbitrary state of a qutrit in the computational basis $\{ | 0 \rangle, | 1 \rangle, | 2 \rangle \}$ is given by

$$| \Psi \rangle = C_0 | 0 \rangle + C_1 | 1 \rangle + C_2 | 2 \rangle,$$

(7)

where $| 0 \rangle$, $| 1 \rangle$, and $| 2 \rangle$ is the computational bases for a single-qutrit with complex coefficients $C_0$, $C_1$, and $C_2$ respectively. The second-degree Majorana polynomial $P_{| \Psi \rangle} (\xi) = a_0 \xi^2 + a_1 \xi + a_2$ associated with the state of a qutrit is obtained by identifying the $j = 1$ basis $| jm \rangle$ ($m = -1, 0, 1$) with the computational basis, $|1m\rangle = | 1, m \rangle$, and the coefficients are obtained from Eq. (3) as $a_2 = C_2/\sqrt{2}$, $a_1 = -C_1$, and $a_0 = C_0/\sqrt{2}$. For the state $| 0 \rangle$ both Majorana points lie on the North Pole, $| 2 \rangle$ has both points lying on the South Pole, while $| 1 \rangle$ is represented by one point on the North Pole and another point on the South Pole. We define the distance between two Majorana points as $\eta = \cos^{-1}(\mathbf{S}_1 \cdot \mathbf{S}_2) = \cos^{-1}[\sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) + \cos \theta_1 \cos \theta_2]$, where the Majorana stars $\mathbf{S}_k$ have respective spherical coordinates $(\theta_k, \varphi_k), k \in \{1, 2\}$. This distance can be interpreted as a direct measure of entanglement between two qubits in the symmetrized-state representation of the qutrit [14] [15]. Explicitly, any spin-$1$ state - and, by the identification above, any qutrit state Eq. (7) can be represented as a symmetric combination of two spin-$1/2$ (qubit) states $| \Psi \rangle = \cos(\theta_k/2)|1/2, 1/2\rangle + \sin(\theta_k/2)e^{i\varphi}|1/2, -1/2\rangle$ (with $k = 1, 2$), as

$$| \Psi \rangle = \frac{1}{\sqrt{2[1 + \cos^2(\eta/2)]}} \sum_{\sigma} | \psi_{\sigma(1)} \rangle \otimes | \psi_{\sigma(2)} \rangle$$

(8)
where $\sigma$ are permutations of the indices $k$. Using the definitions above, the Majorana stars of this state coincides with the two points with spherical coordinates $(\theta_k, \varphi_k)$, representing the two qubits in the standard Bloch representation.

It follows immediately that the concurrence between the two qubits is measured by the angle between the Majorana stars,

$$C_{\Psi} = \frac{\sin^2(\eta/2)}{1 + \cos^2(\eta/2)}, \tag{9}$$

which is obtained by applying Wooters’ formula [37].

Majorana representation is complete and unique up to a global phase. Given a qutrit state $|\Psi\rangle$, one can always find a pair of points representing the state on the Majorana sphere. Alternatively, corresponding to any arbitrary pair of points on the Majorana sphere, one can construct a unique $|\Psi\rangle$ in the three-dimensional Hilbert space of a qutrit.

To obtain a complete representation of the operators acting on the qutrit Hilbert space we employ the set of eight Gell-Mann matrices (also called lambda matrices), which form a representation of the generators of the Lie algebra associated with the group SU(3), where

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These elements $T_i = \Lambda_i/2$, along with the $3 \times 3$ identity matrix $(I_{3 \times 3})$, generate various unitary transformations of a qutrit.

Further, we consider the $x, y, z$-components $J_x, J_y, J_z$ of the angular momentum operator, given by $J_x = (\Lambda_1 + \Lambda_6)/\sqrt{2}$, $J_y = (\Lambda_2 + \Lambda_7)/\sqrt{2}$, and $J_z = (\Lambda_3 + \sqrt{3}\Lambda_8)/2$. The average of the spin-1 vector $\mathbf{J}$ expressed in terms of the expectation values $\langle J_k \rangle = \langle \Psi | J_k | \Psi \rangle$ of $J_k$ ($k \in \{x, y, z\}$) in state $|\Psi\rangle$ is given by

$$\mathbf{J} = \langle J_x \rangle \mathbf{e}_x + \langle J_y \rangle \mathbf{e}_y + \langle J_z \rangle \mathbf{e}_z, \tag{11}$$

which is a vector on the Majorana sphere [8] with magnitude

$$|\mathbf{J}| = \frac{2 \cos(\eta/2)}{1 + \cos^2(\eta/2)}, \tag{12}$$

and direction given by the bisector of angle $\eta$. Interestingly, $\eta$ is the distance between the Majorana stars, and thus it follows that $|\mathbf{J}|$ is manifestly invariant under rotations generated by $J_x, J_y, J_z$.

We now show that there is a deeper connection between the average of the angular and the entanglement properties in the symmetrized state representation of spin 1. Starting from Eq. (8), we take the partial trace of the total density matrix $|\Psi\rangle\langle\Psi|$ over one of the spin-1/2. The resulting reduced density matrix $\rho$ does not depend on which spin-1/2 we traced out, and it reads

$$\rho = \frac{I_2}{2} + \frac{1}{2} r \sigma,$$ \tag{13}

where

$$r = \frac{1}{1 + \cos^2(\eta/2)}(S_1 + S_2). \tag{14}$$

We see that the vector $\mathbf{r}$ which parametrizes the reduced density matrix can be obtained, up to a normalization factor, by the vectorial addition of the Majorana vectors. We can now calculate immediately the average of the total angular momentum. Since this is a sum of the angular momenta $\sigma/2$ of each of the two qubits, we get

$$\langle \mathbf{J} \rangle = Tr(r \sigma) = \mathbf{r}. \tag{15}$$

Thus, the average vector angular momentum is given by the vector $\mathbf{r}$ which parametrizes the reduced density matrix. The length of this vector can be obtained from Eq. (14)

$$|\mathbf{r}| = \frac{2 \cos(\eta/2)}{1 + \cos^2(\eta/2)}, \tag{16}$$

and clearly coincides with the result Eq. (12). The purity of the reduced state $1 - Tr(\rho^2) = 1 - 2|\mathbf{r}|^2$ can be used to calculate the concurrence $C_{\Psi} = \sqrt{2[1 - Tr(\rho^2)]}$, with the result $C_{\Psi} = \sin^2(\eta/2)/(1 + \cos^2(\eta/2))$, identical with the one calculated above in Eq. (9).

Next, we closely follow the dynamics of the Majorana stars and its corresponding spin-1 average vector, as our single qutrit pure state undergoes unitary evolution generated by STIRAP and superadiabatic(sa)-STIRAP Hamiltonians.

### III. STIRAP

The Stimulated Raman Adiabatic Passage (STIRAP) is a fundamental quantum mechanical process for transferring population between the single-qutrit ground state $|0\rangle$ and the second excited state $|2\rangle$ without populating the intermediate state $|1\rangle$. This is achieved via a counterintuitive sequence [22, 23], whereby the driving between levels $|1\rangle$ and $|2\rangle$ commences prior to that of $|0\rangle$ and $|1\rangle$. For example, consider a transmon driven by two microwave fields with time-dependent amplitudes $\Omega_{12}(t)$ and $\Omega_{01}(t)$, phases $\phi_{01}$ and $\phi_{12}$, and driving frequencies $\omega_{01}^{(T)}$ and $\omega_{12}^{(T)}$, coupling the levels $|0\rangle - |1\rangle$ and $|1\rangle - |2\rangle$ respectively [35]. These microwave fields may be slightly detuned
from the respective transmon transition frequencies \(\omega_{01}\) and \(\omega_{12}\) by \(\delta_0\) and \(\delta_{12}\) respectively. Then, under the rotating wave approximation, the effective Hamiltonian governing STIRAP is given by \(21\):

\[
\begin{align*}
    H_0 &= \frac{\hbar}{2} \Omega_{01}(t) \left[ \cos(\phi_0)A_1 + \sin(\phi_0)A_2 \right] \\
    &+ \frac{\hbar}{2} \Omega_{12}(t) \left[ \cos(\phi_1)A_6 + \sin(\phi_1)A_7 \right] \\
    &- \frac{\hbar}{2} \left[ \delta_0 A_3 + \frac{\delta_{01} + 2\delta_{12}}{\sqrt{3}} A_8 - \frac{2\delta_{01} + \delta_{12}}{3} I_1 \right].
\end{align*}
\]

(17)

The time-varying amplitudes of the microwave drives are chosen of Gaussian form in time-domain, with the same standard deviation \(\sigma\) and therefore the same width. They are separated in time by an amount \(t_s\), which, when negative, leads to the counterintuitive sequence. The corresponding Rabi couplings read

\[
\Omega_{01}(t) = \Omega_{01} e^{-t^2/2\sigma^2} \quad \text{and} \quad \Omega_{12}(t) = \Omega_{12} e^{-(t-t_s)^2/2\sigma^2}.
\]

An adiabatic evolution is ideally infinitely slow and would require the system to be in an eigenstate of the instantaneous Hamiltonian at all times. At the two-photon resonance condition (i.e. \(\delta_{01} = -\delta_{12}\)), the eigenvectors of \(H_0\) are given by

\[
\begin{align*}
    |+\rangle &= \sin \Phi \sin \Theta |0\rangle + \sin \Phi \cos \Theta |2\rangle + \cos \Phi |1\rangle, \\
    |\rangle &= \cos \Phi \sin \Theta |0\rangle + \cos \Phi \cos \Theta |2\rangle - \sin \Phi |1\rangle, \\
    |D\rangle &= \cos \Theta |0\rangle - \sin \Theta |2\rangle,
\end{align*}
\]

(18)

with eigenvalues \(\omega_{\pm} = \delta_{01} \pm \sqrt{\delta_{01}^2 + \Omega_{01}^2 + \Omega_{12}^2}\) and \(\omega_D = 0\) respectively. Here the angles \(\Theta\) and \(\Phi\) are given by

\[
\begin{align*}
    \tan \Theta &= \frac{\Omega_{01}(t)}{\Omega_{12}(t)}, \\
    \tan \Phi &= \frac{\sqrt{(\Omega_{01}^2 + \Omega_{12}^2)/2}}{\sqrt{(\Omega_{01}^2 + \Omega_{12}^2 + \delta_{12}^2)/2} + \delta_0/\sqrt{2}}.
\end{align*}
\]

(20)

A convenient choice for the state to be followed adiabatically is the dark state \(|D\rangle\), which does not contain any component from the intermediate level \(|1\rangle\). Interestingly, \(|D\rangle\) is the same as the single-qutrit canonical state \(|S\rangle\), up to a change of variables \(\Theta \rightarrow \Theta - \pi/2\). The canonical state is a single-parameter state which spans the entire qutrit Hilbert space under SO(3) rotations generated by \(J_x, J_y,\) and \(J_z\).

These unitary operations rotate the Majorana stars like a rigid body in three-dimensional real space \(\mathbb{R}^3\); thus, any qutrit state can be parametrized with one parameter from the canonical state and three parameters from the three rotations.

Under adiabatic evolution, as a consequence of the adiabatic theorem \(20\), the system remains in the state \(|D\rangle\) at all times. In order to transfer the population from \(|0\rangle\) to \(|2\rangle\), one can initialize the system in a state corresponding to \(\Theta = 0\) at \(t = t_i\), which eventually transforms into \(\Theta = \pi/2\) at \(t = t_f\), with total duration of the sequence being \(T = t_f - t_i\), with \(t_i = -110\) ns and \(t_f = 80\) ns as shown in Fig. 1. The Gaussian pulse profiles \(\Omega_{01}(t)\) and \(\Omega_{12}(t)\) are shown in Fig. 1(a) for \(t_s = -30\) ns, \(\sigma = 20\) ns, \(\Omega_{01} = \Omega_{12} = 2\pi \times 25.5\) MHz. The corresponding variation of \(\Theta\) from 0 to \(\pi/2\) is clearly reflected from Fig. 1(b), and the probabilities of occupation of the energy levels in terms of the squares of the absolute values of the coefficients, \(p_0(t) = \cos^2 \Theta(t)\) and \(p_2(t) = \sin^2 \Theta(t)\) are plotted in Fig. 1(c). The rate of change of the populations is

\[
\dot{p}_0(t) = -\dot{p}_2(t) = -\dot{\Theta}(t) \sin [2\Theta(t)],
\]

(21)

which tends to zero as the mixing angle approaches its extreme values \(\Theta = 0\) and \(\pi/2\). The rate of change of populations attains its maximum value at \(t = T/2 = (t_f - t_i)/2\). A geometrical picture of this dynamics may be obtained on the Majorana sphere, which is discussed further in Section IV.

IV. MAJORANA REPRESENTATION OF THE DARK STATE \(|D\rangle\)

Recalling that the dark state is \(|D\rangle = \cos \Theta |0\rangle - \sin \Theta |2\rangle\), the Majorana representation of \(|D\rangle\) consists of two stars \(S_1(\theta, \pi)\) and \(S_2(\theta, 0)\), where \(\theta = \pi - 2\tan^{-1} \sqrt{\cot \Theta}\). These points lie in the \(xz\)-plane with Cartesian co-ordinates \((\pm x_D, y_D, z_D)\), such that

\[
\begin{align*}
    x_D &= \frac{\sqrt{2} \sin 2\Theta}{\cos \Theta + \sin \Theta}, \\
    y_D &= 0, \\
    z_D &= \frac{\cos \Theta - \sin \Theta}{\cos \Theta + \sin \Theta},
\end{align*}
\]

(22)
where $\Theta \in [0, \pi/2]$.

FIG. 2. A cross-section of the Majorana sphere in the plane $y = 0$, which shows the Majorana representation of the dark state $|D\rangle$ and its dynamics under STIRAP Hamiltonian. (a) Majorana stars are labeled by $S_1(x_1, y_1, z_1)$ and $S_2(x_2, y_2, z_2)$ with red circle and blue square respectively, such that $-x_1 = x_2 = x_D$, $y_1 = y_2 = 0$, and $z_1 = z_2 = z_D$. (b) Trajectory of both the points on the Majorana sphere as $\Theta$ varies from 0 to $\pi/2$. (c) Variation in real time of the co-ordinates of Majorana points $S_1(x_1, y_D, z_D)$ and $S_2(x_2, y_D, z_D)$ is plotted as $\Theta$ changes from 0 to $\pi/2$ as in Fig. 1(b).

Majorana stars representing $|D\rangle$ lie on the great circle in the plane $y = 0$, with the same latitude $\theta$. The angle $\eta$ formed by the vectors $S_1$ and $S_2$ is given by $\eta = 2\theta$. As $\Theta$ varies from $0 \leq \Theta \leq \pi/2$, $\eta$ assumes values from 0 to $\pi$ and then 0 again. Thus the two points $S_1$ and $S_2$ lie all along the great circle in the plane $y = 0$ as shown in Fig. 2(a). This readily leads to the fact that the dynamics of $|D\rangle$ under the STIRAP Hamiltonian is confined to the plane $y = 0$ on the Majorana sphere i.e. the longitudes remain unchanged while sweeping the latitudes $\theta_1 = \theta_2 = \theta$ from 0 to $\pi$, see Fig. 2(b). Under STIRAP, both stars representing $|D\rangle$ on the Majorana sphere move symmetrically with varying $\Theta$, and any deviation from this behaviour signals a loss of adiabaticity. The trajectories of these Majorana stars in the co-ordinate space under the action of STIRAP are shown in Fig. 2(c). The trajectories are plotted in time with the same scales as described in Section III.

The dark state can be also represented using symmetrized spin-1/2 states, as described in Section II, in the form $|D\rangle = \cos \Theta |1/2, 1/2\rangle |1/2, 1/2\rangle - \sin \Theta |1/2, -1/2\rangle |1/2, -1/2\rangle$. Thus, the resulting symmetrized dark state is manifestly in the Schmidt form $|\eta\rangle$. The concurrence is obtained as $C_{|D\rangle} = \sin 2\Theta$ and the $r$-parameter of the reduced density matrix is $r = \cos 2\Theta$.

The Majorana representation of the spin-1 average vector in the dark state $|D\rangle$ is obtained using the bisector of angle $\eta$ as shown with $OO'$ in Fig. 2(a), such that (see also Eq.(12))

$$J = \frac{2OO'}{1 + |OO'|^2}. \quad (23)$$

Following Eq (11), for the dark state $|D\rangle$: $\langle J_+ \rangle = 0$, $\langle J_0 \rangle = 0$, and $\langle J_- \rangle = \cos 2\Theta$.

The magnitude of the spin-1 average vector is given by $J = |J| = \cos 2\Theta$, which varies with time as

$$\dot{J} = 2\dot{\Theta} \sin 2\Theta. \quad (24)$$

This is twice the rate of change of populations $2\dot{p}_2(t)$ from Eq.(21), which puts in evidence that the Majorana geometrical picture reflects the rate of evolution of a quantum state with time. Indeed, the co-ordinates of the Majorana stars vary slowly with time near the starting and end points, and change significantly faster in between, as expected for an adiabatic evolution.

V. SUPERADIABATIC(SA)-STIRAP

It is interesting to observe on the Majorana sphere the dynamics of the quantum state under the combined effect of STIRAP Hamiltonian and a counter-diabatic drive. When acting together, this realizes the so-called
superadiabatic(sa)-STIRAP. The counterdiabatic part achieves the cancellation of spurious non-adiabatic excitations, leading to the expected final state with high precision at finite times. The counterdiabatic term in case of a three-level quantum system assumes a very simple form \[36\],

\[ H_{cd} = -\frac{\hbar}{2} \Omega_{2\beta} (t) [\cos(\phi_{2\beta})A_4 + \sin(\phi_{2\beta})A_5], \tag{25} \]

where the Rabi coupling is \( \Omega_{2\beta}(t) = 2\Theta(t) \) and \( \phi_{2\beta} \) is the phase of the \(|0\rangle - |2\rangle \) drive, such that \( \phi_{1} + \phi_{12} + \phi_{2\beta} = \pi/2 \). For \( \phi_{2\beta} = \pi/2 \), the dynamics under \( H_{cd} \) is the unitary \( \exp \left[ i\hbar A_5 \int_0^t \Theta(t) dt \right] \) generated by \( A_5 \), under which both stars move together on the Majorana sphere. This is experimentally realized by a two-photon resonance pulse \[36\], which simultaneously drives the \(|0\rangle - |1\rangle \) and \(|1\rangle - |2\rangle \) transitions with detunings \( \pm \Delta \) respectively, where \( \Delta = (\omega_1 - \omega_2)/2 \). The Hamiltonian in the doublyrotating frame driven by tones with frequencies \( \omega_1 \) and \( \omega_2 \) is given by

\[ H_{2\beta} = \frac{\hbar}{2} \Omega_{2\beta} [\cos(\phi_{2\beta} - \Delta t)A_1 - \sin(\phi_{2\beta} - \Delta t)A_2] \]
\[ + \frac{\hbar}{\sqrt{2}} \Omega_{2\beta} [\cos(\phi_{2\beta} + \Delta t)A_6 - \sin(\phi_{2\beta} + \Delta t)A_7], \tag{26} \]

with the amplitude of the drive \( |\Omega_{2\beta}| = \sqrt{2\Delta} \Omega_{2\beta} \) and phase \( \phi_{2\beta} = -\left( \phi_{2\beta} + \pi \right)/2 \).

A. Majorana trajectories under \( H_0 \) and \( H_{2\beta} \)

We simulate the dynamics of a qutrit with transition frequencies \( \omega_{01}/2\pi = 5.27 \) GHz and \( \omega_{12}/2\pi = 4.82 \) GHz. This may be considered as a three-level systems with unequally spaced energy levels with energy level spacings \( \omega_{01} \) and \( \omega_{12} \) and anharmonicity \( \omega_{01} - \omega_{12} \). This system, initialized in the state \(|0\rangle \), is driven resonantly by the STIRAP Hamiltonian with \( \sigma = 35 \) ns, \( \Omega_{01}/2\pi = \Omega_{12}/2\pi = 45 \) MHz, \( t_s/\sigma = -1.2 \), and it is evolved from \( t_i = -182 \) ns to \( t_f = 140 \) ns in 1800 time steps. The driving frequencies of the Gaussian pulses are taken to be resonant to the respective qutrit transition frequencies. We calculate the dynamics of a qutrit and plot the corresponding trajectories on the Majorana sphere as shown in Fig. 3(a). We see that STIRAP alone is not working perfectly, with cusps appearing along the trajectory, and as a result the final state misses the South Pole. These are due to non-adiabatic transitions, and we see that the Majorana representation is very sensitive to these errors. For example, the final state in this figure has populations \( p_0(t_f) = 0.010 \), \( p_1(t_f) = 0.003 \), and \( p_2(t_f) = 0.987 \) on the states \(|0\rangle, |1\rangle, \) and \(|2\rangle \) respectively, yet the stars appear clearly distinct and separated from the South Pole. This sensitivity is a general feature of the Majorana representation, which will be visible in all the following figures.

Next, with the same parameters for the Gaussian pulses, we simulate the two-photon resonant drive \( H_{2\beta} \) at different values of the detunings \( \Delta \). From Eq. (26) we see that the constant part of the phase, \( \phi_{2\beta} \), shifts the overall plane of the trajectory, while time-dependent effects arise from \( \Delta t \). This produces a rapidly oscillating phase of the drive, yielding a wiggling trajectory on the Majorana sphere as shown in Fig. 3(b)-(d). The frequency of this wiggling is \( \Delta/2\pi \), which, as expected, is same as the frequency of the rotating frame. Thus, for a system with larger (smaller) anharmonicity \( (\omega_1 - \omega_2) \), or for a two-photon resonance pulse which is more (less) detuned from the respective \(|0\rangle - |1\rangle \) and \(|1\rangle - |2\rangle \) transitions, the rate of wiggling is higher (lower), as shown in Fig. 3(b),(c),(d) for detunings \( \Delta/2 \), \( \Delta \) and \( 2\Delta \). Interestingly, we observe wiggles also in the black curve representing the bisector. Unlike the case of Majorana sphere trajectories (trajectories traversed by the Majorana stars on the sphere), the rate of wiggling in this case relates to the speed of the single-qutrit evolution, which in a given time interval is directly proportional to the pitch of the wiggling black line. Thus, we can observe visually that the evolution is faster in the middle while slower close to the initial and the target states. Further, the amplitude of the wiggles in the trajectory is more pronounced in the middle of the drive, when the first excited state \(|1\rangle \) gets populated. The amplitudes of the wiggles are largest in Fig. 3(b), where the detuning of the two-photon drive is relatively small and the system posses lesser anharmonicity, which makes it more likely for the first excited state to get populated. A reverse situation may be seen in Fig. 3(d). Thus, this demonstrates visually the advantage of using a system with large anharmonicity.

B. Dynamical phase corrections

The two-photon pulse is also responsible for producing ac-Stark shifts of the energy levels. These can be compensated by using a dynamical phase corrections \[30\] \[34\] \[40\]. The correction is applied to the phases of the drives, such that \( \phi_{nk}(t) = \phi_{nk} + \int_0^t \epsilon_{nk}(t') dt'/\hbar \), where \( n, k = 0, 1, 2 \) are the labels of energy levels and \( \epsilon_{nk}(t) \) is the ac Stark shift resulting from the \( n-k \) drive at a given time \( t \). As shown in Ref. \[30\] \[34\], respective ac Starks shifts are given by \( \epsilon_{01}(t) = h|\Omega_{2\beta}|^2/\Delta \), \( \epsilon_{12}(t) = -5h|\Omega_{2\beta}|^2/4\Delta \), and \( \epsilon_{02}(t) = -h|\Omega_{2\beta}|^2/4\Delta \).

The corresponding dynamic phase corrections are thus given by

\[ \phi_{01}(t) = \phi_{01} + 2\sqrt{2}h\Theta(t), \]
\[ \phi_{12}(t) = \phi_{12} - \frac{5h}{\sqrt{2}} \Theta(t), \]
\[ \phi_{02}(t) = \phi_{02} - \frac{h}{\sqrt{2}} \Theta(t). \tag{27} \]
The trajectories on the Majorana sphere of a qutrit driven by the two-photon resonant pulse $H_{2\text{ph}}$ and corrected for the ac Stark shifts are shown in Fig. 4(b). We simulate the Majorana trajectory with same parameters as before, as specified in the STIRAP Hamiltonian, $H_{\text{STIRAP}} = H_0 + H_{2\text{ph}}$, with $\Theta = \pi/2$. The state is evolved from $t_i = -182$ ns to $t_f - t_i = 140$ ns with $T = t_f - t_i$ in $1800$ time steps. Driving frequencies of the Gaussian pulses are the same as those of the respective qubit transition frequencies.

C. Majorana trajectory under sa-STIRAP

We observe the single-qutrit dynamics under the sa-STIRAP Hamiltonian,

$$H_{\text{sa-STIRAP}} = H_0 + H_{2\text{ph}},$$

with the qutrit initialized in the dark state $|D\rangle$ with $\Theta = 0$. $H_{\text{sa-STIRAP}}$ preserves the robustness of the STIRAP Hamiltonian ($H_0$), while concurrently has improved performance due to super adiabatic part $H_{2\text{ph}}$, and precisely returns the expected final state $|D\rangle$ in an experimentally feasible time. The corresponding trajectory of the qutrit on the Majorana sphere is presented in Fig. 5. The simulation is performed with same set of parameters as before, as specified in the

FIG. 4. (a) Majorana trajectories (in red and blue colors) resulting from the two-photon drive with a constant phase $\phi_{2\text{ph}}$ as in Eq. 26. The dynamically phase-corrected Majorana trajectories, with a time-dependent phase $\phi_{2\text{ph}}(t)$, is shown in (b). Plots in (c) and (d) present the corresponding variation of distance between the Majorana stars and the evolution of occupation probabilities with time. The dashed lines are uncorrected values, while the continuous lines include the ac Stark shift correction. Parameters: qubit transition frequencies, $\omega_{12}/2\pi = 5.27$ GHz, $\omega_{2}/2\pi = 4.82$ GHz, $\sigma = 35$ ns, $\Omega_0/2\pi = \Omega_{12}/2\pi = 45$ MHz, $t_0/\sigma = -1.2$, and the state is evolved from $t_i = -182$ ns to $t_f = 140$ ns with $T = t_f - t_i$ in 1800 time steps. Driving frequencies of the Gaussian pulses are taken to be same as that of respective qubit transition frequencies.

FIG. 5. (a) Majorana trajectories in sa-STIRAP, and (b)-(e) their evolution at different moments of time. Parameters: qutrit transition frequencies, $\omega_{12}/2\pi = 5.27$ GHz, $\omega_{2}/2\pi = 4.82$ GHz, $\sigma = 35$ ns, $\Omega_0/2\pi = \Omega_{12}/2\pi = 45$ MHz, $t_0/\sigma = -1.2$, state is evolved from $t_i = -182$ ns to $t_f - t_i = 140$ ns in 1800 time steps. The driving frequencies of the Gaussian pulses are the same as that of the respective qubit transition frequencies.
VI. CONCLUSIONS

We have investigated the Majorana representation of the evolution under STIRAP and superadiabatic STIRAP. We have shown that the dark states are represented by two stars on the circle defined by the intersection of the $xOz$ plane with the Majorana sphere. We have also represented a introduction of the spin-1 average vector, which evolves along the $Oz$ axis, and we have shown how its rate of change in the three-dimensional space is a measure of a state change of the qutrit in the Hilbert space. The representation puts clearly in evidence the role of the counterdiabatic drive in saSTIRAP that corrects the deviations of the trajectory from the adiabatic path, and as such it offers a sensitive visual diagnosis tool for errors caused by non-adiabaticity and ac Stark shifts. We have done an in-depth analysis of the Majorana trajectories resulting from STIRAP and saSTIRAP with and without the dynamically corrected phases used to compensate for the errors in gates based on shortcuts to adiabaticity.
for the ac Stark shifts. We have also analyzed the effectiveness of the STIRAP and saSTIRAP processes via the distance between the Majorana stars.

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