A Free Field Representation of the $Osp(2|2)$ current algebra at level $k = -2$, and Dirac Fermions in a random $SU(2)$ gauge potential

Andreas W.W. Ludwig

Department of Physics, University of California, Santa Barbara, CA 93106, and,
Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam (The Netherlands)

(October 28, 2018)

The $Osp(2|2)$ current algebra at level $k = -2$ is known to describe the IR fixed point of 2D Dirac fermions, subject to a random $SU(2)$ gauge potential. We show that this theory has a simple free-field representation in terms of a compact, and a non-compact free scalar field, as well as a free fermionic ghost, at $c = -2$. The fermionic twist fields are crucial for the construction.

The logarithmic current-algebra primary field with vanishing scaling dimension, transforming in the 0 sector, can be represented in terms of a compact, and a non-compact free scalar field, as well as a free fermionic ghost, at $c = -2$.

I. INTRODUCTION

The current algebra based on the two-dimensional $Osp(2|2)$ Wess-Zumino-Witten model at level $k = -2$, is known to provide a supersymmetric description of the the infrared fixed point, existing in the problem of two species of Dirac fermions in two spatial dimensions, subject to a random, static and short-ranged $SU(2)$-gauge potential. The latter problem has been studied extensively by various authors, including the work by Mudry, Chamon, and Wen, and by Caux, Taniguchi and Tsvelik.

A discussion of the $Osp(2|2)$ current algebra at general level $k$ was given by Maassarani and Serban. (The central charge is $c = 0$.) Here we show that at level $k = -2$, relevant for the random gauge model, special simplifications occur: at this level the current algebra and its primary fields can be represented in terms of a compact and a non-compact free scalar field (each at central charge $c = 1$), as well as a free fermionic ghost at central charge $c = -2$. In order to obtain the representations of the $Osp(2|2)_{-2}$ current algebra, it is crucial to include the twist fields of the fermionic ghost sector. As pointed out by Gurarie, this $c = -2$ sector contains a zero-dimensional logarithmic operator, reflecting a Jordan-block structure of the Virasoro generators. In our free field representation it is this same operator, which gives rise to the zero conformal weight $Osp(2|2)_{-2}$ current algebra primary field transforming in an indecomposable representation.

II. MOTIVATION

The conformal weights of the $Osp(2|2)_{-2}$ current algebra, corresponding to Kac-Moody primary fields transforming under $Osp(2|2)$ in the (’typical’) representation labeled by $[b, q]$ are given by

$$\Delta = \frac{q^2 - b^2}{2},$$

where $q = 0, 1/2, 1, \ldots$ and we consider $b$ real. We wish to exhibit a free field representation of $Osp(2|2)_{-2}$ in terms of (i): a compact free bosonic $\varphi$, obtained by bosonization of the $SU(2)_1$ sub-current algebra, (ii): a non-compact bosonic $\varphi'$, and (iii): a system of free fermionic ghosts $\chi \equiv \chi^\dagger \equiv \chi_+ \equiv \chi_{s}$ at central charge $c = -2$, normalized such that

$$< \chi_a(z, \bar{z}) \chi_b(w, \bar{w}) > = e_{ab} \ln(z - w)(\bar{z} - \bar{w})$$

($e_{ab} = -e^{ab} =$ antisymmetric, $e_{-} = 1$. ) [From now on we will consider only the left-moving (holomorphic) sectors, denoting $\varphi_L(z)$ by $\varphi$, $\varphi'_L(z)$ by $\varphi'(z)$ etc.] The total central charge $c = 1 + 1 - 2$ adds up to zero. In the $c = -2$ sector it will be crucial to also consider the fermionic twist fields $\mu$ and $\sigma_a$, $a = \pm$, of conformal weights $-1/8$ and $+3/8$, respectively. To motivate the possibility of describing $Osp(2|2)_{-2}$ within the tensor product of the three free theories above, consider the 4-dimensional representation $[b, q] = [0, 1/2]$ of conformal weight $\Delta = 1/8$. The representation decomposes under the bosonic subalgebra into an $SU(2)$ doublet with charge $b = 0$, and two $SU(2)$ singlets with charge $b = \pm 1/2$. We claim that the doublet may be represented by

$$\psi = e^{\frac{i}{\sqrt{2}}} \varphi \varphi', \quad (\alpha = \pm)$$

(conformal weights : $\Delta = 1/4 - 1/8 = 1/8$),

(2.1)

where $\alpha/2 = \pm 1/2$ is the z-quantum number of $SU(2)$, and the singlets by

$$\sigma_a = e^{\frac{i}{\sqrt{2}}} \varphi \varphi', \quad (a = \pm)$$

(conformal weights : $\Delta = -1/4 + 3/8 = 1/8$),

(2.2)

where $a/2 = \pm 1/2$ measures the charge quantum number $b$. Note that the conformal weights add up as required.

* Conventional normalization: $< \varphi(z, \bar{z}) \varphi(0) > = -\ln(z\bar{z})$.

† We choose $i\varphi'$ real with conventional correlator: $< i\varphi'(z, \bar{z}) i\varphi'(0) > = -\ln(z\bar{z})$. 


III. CURRENT ALGEBRA

In order to establish the free field representation, we first show that the eight $Osp(2|2)_{-2}$ currents can be entirely expressed in terms of the free fields. The four bosonic currents consist of three $SU(2)_1$ currents $Q_{++}, Q_{--}, Q_z$, together with the charged current $B$, measuring the quantum number $b$. These can be written as

$$Q_{++} = e^{+\frac{i}{\sqrt{2}}\partial \phi}, \quad Q_{--} = e^{-\frac{i}{\sqrt{2}}\partial \phi}, \quad Q_z = \frac{1}{\sqrt{2}} i \partial \phi, \quad B = -\frac{1}{\sqrt{2}} i \partial \phi'$$

(3.1)

The fermionic currents consist of two global $SU(2)$ doublets, one $(\alpha)\) with charge $b = +1/2$, and one $(\alpha)\) with charge $b = -1/2$. We claim that these can be written as

$$V_\alpha = e^{\frac{i}{\sqrt{2}}(\alpha \phi + \phi')} \partial \chi$$

$$W_\alpha = e^{\frac{i}{\sqrt{2}}(\alpha \phi - \phi')} \partial \chi$$

(3.2)

It is easily checked that the bosonized expressions defined above satisfy the OPE’s of the $osp(2|2)_{-2}$ current algebra (listed in the Appendix), hence proving the claim.

IV. FOUR-DIMENSIONAL REPRESENTATION ($\Delta = 1/8$)

Next we establish that the OPE’s of the currents (as defined in the previous paragraph) with the four fields defined in Eq. (3.1) are indeed those required for the corresponding Kac-Moody primary field $[b, q] = [0, 1/2]$. The OPE’s with the bosonic currents (3.1) are obviously correct. In considering the OPE’s with the fermionic currents (3.2), the twist fields of the $c = -2$ fermionic ghost sector are crucial. A useful tool in handling those is an algebra (listed in the Appendix), hence proving the claim.

$$\partial \chi_{\pm}(z) \mu(w) \sim (1/\sqrt{2}) \frac{1}{(z-w)^{3/2}} \mu(w)$$

(4.2)

when we adopt the following normalizations of the 2pt. functions

$$\sigma_\alpha(z) \sigma_b(0) = \frac{\epsilon_{ab}}{z^{3/4}}, \quad < \mu(z) \mu(0) > = \frac{1}{z^{-1/4}}$$

(4.3)

More generally ($n \geq 0$),

$$\partial \chi_{\pm}(z) \sigma_{+n}(w) \sim \tilde{C}_n (z-w)^{n-1/2} \sigma_{+(n+1)}(w),$$

$$\partial \chi_{\pm}(z) \sigma_{-n}(w) \sim C_n (z-w)^{n-1/2} \sigma_{-(n+1)}(w)$$

where $\tilde{C}_n, C_n$ are constants. Here $\sigma_{\pm n}$ is the $c = -2$ Kac degenerate field of conformal weight $\Delta_{n+1/2} = 2(\frac{n}{2})^2 - \frac{1}{4}$, which is a highest weight under the global isospin $SU(2)$ symmetry, with quantum numbers $j_3 = \pm j = \pm n/2$.

Using Eq. (3.2) one easily verifies the OPE of the eight $Osp(2|2)_{-2}$ currents, collectively denoted by $J_A$ (where $A$ is the adjoint index), with the four-dimensional KM primary $\phi^A$:

$$J_A(z) \phi^A(0) = \frac{1}{z} (t_A)^B \phi_B + ...$$

(4.4)

The representation matrices $(t_A)^B$ for the generators $V_\pm, W_\pm$ in the 4-dimensional representation (index $a, b, ...$) under consideration are e.g. given in Ref. [4] in the basis $(\phi_1, \phi_2, \phi_3, \phi_4) = (v_+, w_-, w_+, v_-)$. Since the Knizhnik-Zamolodchikov (KZ) equation is a consequence of the OPE (4.3), the four point functions of the bosonized expressions given in (4.2) need to satisfy the KZ equation given in Ref. [3].

It is interesting to consider the OPE of two fields in the four-dimensional representation. Group-theoretically, the tensor product (16 states) decomposes into an 8-dimensional adjoint and another 8-dimensional, but indecomposable representation. This can be studied by looking at the correlator of four fields $v_a$ and $w_a$, representing the components of the conformal block of $Osp(2|2)_{-2}$ primary fields in the representation $[b, q] = [0, 1/2]$. We now focus on this correlator.

A. Four-point function (conformal blocks), for $\Delta = 1/8$

Consider first the correlator of four $v_a$, which may be decomposed into the two $SU(2)$ invariant tensors

$$(1^2)_{\epsilon_{a_1,a_2,a_3,a_4}} = \epsilon_{a_1,a_2} \epsilon_{a_3,a_4}, \quad (1^4)_{\epsilon_{a_1,a_2,a_3,a_4}} = \epsilon_{a_4,a_1} \epsilon_{a_2,a_3}$$

§ We also see this explicitly below.
yielding

\[< v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) v_{\alpha_3}(z_3) v_{\alpha_4}(z_4) >=\]

\[= [(\mathbf{I}^2) G_2 + (\mathbf{I}^1) G_1] < \mu \mu \mu \mu >\]

Here \(G_1\) are correlators of four free boson exponentials.

Evaluation of \(< v_+ v_+ v_+ v_- >\) and \(< v_- v_+ v_+ >\) gives

\[G_2 = \left[\frac{1}{z_{13} z_{24}} \left(1 - \xi \right)\right]^{1/2}, \quad G_1 = \left[\frac{1}{z_{13} z_{24}} \left(\frac{\xi}{(1 - \xi)}\right)\right]^{1/2}\]

where

\[\xi = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad (z_{ij} = z_i - z_j)\]

is a crossratio. The two-dimensional space of conformal blocks of the twist \(\mu\) is described in terms of hypergeometric functions

\[< \mu(z_1) \mu(z_2) \mu(z_3) \mu(z_4) >=\]

\[= [z_{13} z_{24} \xi (1 - \xi)]^{1/4} f(\xi),\]

where \(f(\xi)\) is an arbitrary linear combination of the two functions

\[f^{(0)}(\xi) = F[1/2, 1/2; 1; \xi] = 1 + \xi/4 + ...\]

and

\[f^{(1)}(\xi) = f^{(0)}(1 - \xi) = f^{(0)}(\xi) \ln(\xi) + h(\xi)\]

(4.5)

\([h(\xi)\) is a function, regular at \(\xi = 0\). Note that \(f^{(1)}(\xi)\) has a logarithmic singularity at \(\xi = 0\). The conformal block in question can therefore be written in the following simple form

\[< v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) v_{\alpha_3}(z_3) v_{\alpha_4}(z_4) >=\]

\[= \frac{1}{[z_{13} z_{24}]^{1/4}} \left\{\left(\frac{1}{\xi (1 - \xi)}\right)^{1/4} (\mathbf{I}^2 (1 - \xi) + \mathbf{I}^1 \xi) f(\xi)\right\}\]

(4.6)

**B. Knizhnik-Zamolodchikov equation of the current algebra, for \(\Delta = 1/8\)**

It is instructive to make connection with the description of the same conformal block in terms of current algebra. In general, the \(Osp(2|2)_{-2}\) conformal block

\[\text{is described in terms of hypergeometric functions where} f(\xi) \text{is an arbitrary linear combination of the two functions}

\[f^{(0)}(\xi) = F[1/2, 1/2; 1; \xi] = 1 + \xi/4 + ...\]

and

\[f^{(1)}(\xi) = f^{(0)}(1 - \xi) = f^{(0)}(\xi) \ln(\xi) + h(\xi)\]

(4.5)

\([h(\xi)\) is a function, regular at \(\xi = 0\). Note that \(f^{(1)}(\xi)\) has a logarithmic singularity at \(\xi = 0\). The conformal block in question can therefore be written in the following simple form

\[< v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) v_{\alpha_3}(z_3) v_{\alpha_4}(z_4) >=\]

\[= \frac{1}{[z_{13} z_{24}]^{1/4}} \left\{\left(\frac{1}{\xi (1 - \xi)}\right)^{1/4} (\mathbf{I}^2 (1 - \xi) + \mathbf{I}^1 \xi) f(\xi)\right\}\]

(4.6)

\[\text{As discussed in the Appendix, this condition is consistent with the } Osp(2|2) \text{ Knizhnik-Zamolodchikov (KZ)}

\[\text{equation. The KZ-equation implies the following two additional statements (see Appendix). First, when parametrizing}

\[F_1(z) = \frac{1}{z} F_1(z)\]

(4.7)

\[f(z) \text{ satisfies the defining second order differential equation for the elliptic function } F[1/2, 1/2; 1; z]. \text{ Second, the}

\[KZ \text{ equation also implies}

\[F_3(z) = (\frac{\alpha}{z})^{3/4} (z - 1)^{-3/4} \frac{1}{z} f(z) + 2 \frac{d}{dz} f(z)\]

(4.9)

\[\text{to be distinguished from the two } SU(2) \text{ tensors above (in boldface).} \]
Using standard identities for elliptic functions, one finds that this expression is again related to the solution of a hypergeometric equation:

\[ F_3(z) = \left( \frac{4}{\alpha} \right) z^{-1/4}(z - 1)^{-1/4} g(z) \]

where

\[ g^{(0)}(z) = F[-1/2, 1/2; 1, z], \quad g^{(1)}(z) = g^{(0)}(1 - z) \]

Note that the same expression for \( F_3 \) is obtained by using the bosonized expressions in Eq. (2.3). Hence one can see directly that the bosonized expressions satisfy the relevant KZ equation, as expected from the OPE’s.

**C. Single-valued Combinations**

Since the only non-trivial conformal blocks arise from the \( c = -2 \) sector, the problem of combining holomorphic and antiholomorphic conformal blocks is resolved by forming off-diagonal combinations of the two conformal blocks, in the same way as at \( c = -2 \), in Ref. [3].

**V. EIGHT-DIMENSIONAL INDECOMPOSABLE REPRESENTATION AND LOGARITHMS (\( \Delta = 0 \))**

As mentioned above, the OPE of two 4-dimensional representations (\( \Delta = 1/8 \)), decomposes group theoretically into an adjoint and an indecomposable representation. Since the adjoint channel does not contain an \( Osp(2|2) \) singlet, the singlet will necessarily be a component of the 8-dimensional indecomposable, of conformal weight \( \Delta = 0 \). The latter results from the logarithmic conformal block of the \( c = -2 \) twist fields of Eq. (4.3) above. In order to see this explicitly, consider the \( SU(2) \)-singlet channel of (4.3):

\[
\frac{\epsilon_{\alpha_1\alpha_2\epsilon_{\alpha_3\alpha_4}}}{4} < v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) v_{\alpha_3}(z_3) v_{\alpha_4}(z_4) > = \\
= \frac{1}{[z_{12} z_{34}]^{1/4}} \{ C_0 f^{(0)}(\xi) + C_1 f^{(1)}(\xi) + O(\xi) \} = \\
= \frac{1}{[z_{12} z_{34}]^{1/4}} \{ C_0 + C_1 [\ln z_{12} + \ln z_{34} - 2 \ln z_{24}] + \ldots \}
\]

Hence the OPE in the singlet channel is

\[
\frac{\epsilon_{\alpha_1\alpha_2}}{2} v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) \sim \frac{1}{(z_{12})^{1/4}} \{[\ln z_{12}] \phi_s(z_2) + \phi_t(z_2) + \ldots \} \quad (5.1)
\]

with the two point functions

\[
< \phi_s(z_2) \phi_s(z_4) > = 0, \\
< \phi_s(z_2) \phi_t(z_4) > = C_1, \\
< \phi_t(z_2) \phi_t(z_4) > = -2C_1 \log z_{24} + C_0 \quad (5.2)
\]

Here \( \phi_s, \phi_t \) are the ‘bottom-’ and ‘top-’ component of the indecomposable multiplet, respectively, (of zero charge \( B \), and zero spin) of Ref. [3]. The constant \( C_0 \) arises from the freedom to redefine \( \phi_t \) by addition of \( \phi_s \) with an arbitrary coefficient. This freedom gives rise to the presence of the conformal block \( f^{(0)} \), which is free of logarithms (in this channel).

Note that \( \phi_t \) plays here the role of the identity operator, since

\[
< \phi_s > = 0, \quad < \phi_t > = \text{const.}
\]

The vanishing of the first expectation value follows from supersymmetry — similarly for the first correlator in Eq. (5.2). The fact that \( \phi_t \) has an expectation value ensures that the OPE of Eq. (5.1) implies a ‘usual’ nonvanishing two point function

\[
< v_{\alpha_1}(z_1) v_{\alpha_2}(z_2) > \sim \epsilon_{\alpha_1\alpha_2} (z_{12})^{1/4}
\]

Similarly, in the spin-triplet channel (which lies in the \( Osp(2|2) \)-adjoint) the current-algebra descendant of the indecomposable weight-zero operator appears. Relevant OPE’s can be obtained in an analogous fashion, by considering for example the \( \xi \rightarrow 0 \) limit of the correlator \( < v_+ v^- v^- > \) [from Eq. (4.3)].

**VI. OTHER REPRESENTATIONS**

General representations of the \( Osp(2|2) \) current algebra are labeled by \( \{b, q\} \). One expects that representations with \( q \geq 1 \) cannot occur at level \( k = -2 \), because they would contain inadmissible representations of \( SU(2)_1 \) (that is, primaries of this current algebra with spin \( q \geq 1 \)).

**VII. STRESS TENSOR MULTIPLET**

The total stress tensor of the \( c = 0 \) theory is the sum of the three free field stress tensors

\[
T^\varphi = \frac{1}{4} (\partial \varphi)^2, \quad T^{\varphi'} = \frac{1}{4} (\partial \varphi')^2, \\
T^\chi = \frac{1}{2} \epsilon^{ab} (\partial \chi_a)(\partial \chi_b) = (\partial \chi^a)(\partial \chi)
\]
of central charge \( c = +1, +1, -2 \) respectively. Clearly, the total stress tensor is that of the \( gl(1|1) \) current-subalgebra, generated by \( W_+, V_-, Q_3, B \).

It was pointed out recently by Gurarie that in theories with (at least) a global \( gl(1|1) \) symmetry the stress-tensor transforms in the (indecomposable) adjoint representation, with the ‘top’-component denoted by \( t \). In the present case, one finds

\[
t = T^c - (T^c' + T^x) \equiv T^f - T^b
\]

Clearly, the so-defined stress tensors \( T^f \) and \( T^b \) commute, and have central charges \( c = +1 \) and \( c = -1 \), respectively. The anomaly parameter of Ref. [12] is therefore \( b = 1 \). The OPE of the stress tensor \( T \) and its ‘companion’ \( t \) is not that of the logarithmic extension noted recently in the 2D percolation problem [4].

\[
Q_3(z)Q_3(0) \sim \frac{1}{2} \frac{\partial}{\partial z}, \quad B(z)B(0) \sim \frac{-1}{2} \frac{\partial}{\partial z}
\]

\[
Q_{++}(z)Q_{--}(0) \sim \frac{1}{z} V_+(0), \quad Q_{++}(z)W_-(0) \sim \frac{1}{z} W_+(0)
\]

\[
Q_{--}(z)V_+(0) \sim \frac{1}{z} V_-(0), \quad Q_{--}(z)W_+(0) \sim \frac{1}{z} W_-(-0)
\]

\[
Q_3(z)\bar{P}_\pm(0) \sim \pm \frac{1}{2} \bar{P}_\pm(0), \quad Q^3(z)\bar{P}_\pm(0) \sim \pm \frac{1}{2} \bar{P}_\pm(0)
\]

\[
B(z)\bar{P}_\pm(0) \sim \frac{1}{2} \bar{P}_\pm(0), \quad B(z)\bar{P}_\mp(0) \sim \frac{1}{2} \bar{P}_\mp(0)
\]

\[
W_+(z)V_+(0) \sim \frac{1}{z} Q_{++}(0), \quad W_+(z)V_-(0) \sim \frac{1}{z} Q_{--}(0)
\]

\[
V_+(z)W_-(0) \sim -\frac{1}{2} + \frac{B(0) - Q_3(0)}{z}
\]

\[
V_-(z)W_+(0) \sim \frac{1}{2} + \frac{B(0) - Q_3(0)}{z}
\]

It may be useful to write OPE’s involving \( SU(2) \) tensor products in more compact form:

\[
Q_A(z)Q_B(0) \sim \frac{d_{AB}}{z^2} + \frac{f_{AB}C}{z} Q_C(0)
\]

\[
Q_A(z)V_\alpha(0) \sim \frac{(\tau_A)^\beta}{z} V_\beta(0)
\]

\[
Q_A(z)W_\alpha(0) \sim \frac{(\tau_A)^\beta}{z} W_\beta(0)
\]

\[
V_\alpha(z)W_\beta(0) \sim \frac{\epsilon_{\alpha\beta}}{z^2} + \frac{\epsilon_{\alpha\beta} B(0) + (\tau_A)_{\alpha\beta} Q_A(0)}{z}
\]

Here, \( A,B,C,... \in \{++, --, 3\} \) is a spin-1 (adjoint) index, \( d_{AB} \) and \( f_{AB}C \) are the metric and structure constants, respectively. \( \alpha, \beta \) are spin-1/2 indices, and \((\tau_A)^\alpha_\beta\) is the representation matrix in the spin-1/2 representation (Pauli-matrices). Spinor indices are lowered and raised with \( \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} \) and adjoint indices with \( d_{AB}, d^{AB} \).
APPENDIX B: KNIZHNIK-ZAMOLODCHIKOV EQUATION

In this Appendix we briefly discuss the $Osp(2|2)$ Knizhnik-Zamolodchikov (KZ) equation(s) at level $k = -2$, and their consequences. These equations are

$$4 \frac{d}{dz} F_1(z) = \frac{-F_1}{z(z-1)} + \frac{-\alpha^2 F_3}{z-1},$$

$$4 \frac{d}{dz} F_2(z) = \frac{1}{z(z-1)} [2F_1 + (-2z + 3)F_2 + \frac{\alpha}{2} F_3],$$

$$4 \frac{d}{dz} (\frac{\alpha}{2} F_3) = \frac{1}{z(z-1)} [2(-F_1 + (z-2)F_2) - \frac{\alpha}{2} F_3],$$

(B1)

where $\alpha$ is a parameter, denoted by $\frac{1}{\epsilon^4}$ in Ref.[6].

By adding the first two KZ equations one derives for $H(z) = F_1 + zF_2$ a simple equation which, upon integration, yields

$$H(z) = \text{Const.} \ z^{1/4}(z-1)^{1/4}$$

(B2)

Furthermore, combining the first and the third KZ equations, and using (B2) one finds that the function $f(z)$ defined in (4.8) satisfies the differential equation:

$$[z(z-1) \frac{d^2 f}{dz^2} + (2z-1) \frac{df}{dz} + \frac{1}{4} f] =$$

$$= -\text{Const.} \ (z-2)z^{-3/2}(z-1)^{-1/2}$$

(B3)

Moreover, the first KZ equation yields directly the desired relationship (4.9). For $\text{Const.} = 0$ the solutions of (B3) are the functions recorded in (4.5).

---

* Permanent address.
1 A.W.W. Ludwig (unpublished).
2 D. Bernard, A. LeClair, cond-mat/0003073.
3 C. Mudry, C. Chamon, X.-G. Wen, Nucl. Phys. B466 (1966) 383.
4 J.-S. Caux, N. Taniguchi, A. M. Tsvelik, Nucl.Phys. B525 (1998) 671-696; (and references therein).
5 M. Scheunert, W. Nahm, R. Rittenberg, J. Math. Phys 18 (1977) 155.
6 Z. Maassarani, D. Serban, Nucl. Phys. B489 (1997) 603.
7 D. Friedan, E. Martinec, S. Shenker, Nucl. Phys. B271 (1986) 93.
8 V. Gurarie, Nucl. Phys. B410 (1993) 535.
9 For recent work on $c = -2$, see e.g. the following, and reference therein: H. G. Kausch, hep-th/9510149.
10 V. Gurarie, A.W.W. Ludwig, N. Read, Phys. Rev. Lett. 82 (1999), 4524.
11 I. Gruzberg, A.W.W. Ludwig, Nucl. Phys. B546 (1999) 765.
12 V. Gurarie, Nucl. Phys. B546 (1999) 765.
13 For example: I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press 1980, New York).
14 V. Gurarie, A.W.W. Ludwig, cond-mat/9911392.