A NOTE ON THE RICCI FLOW ON NONCOMPACT
MANIFOLDS

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Abstract. Let \((M^3, g_0)\) be a complete noncompact Riemannian 3-manifold with nonnegative Ricci curvature and with injectivity radius bounded away from zero. Suppose that the scalar curvature \(R(x) \to 0\) as \(x \to \infty\). Then the Ricci flow with initial data \((M^3, g_0)\) has a long time solution on \(M^3 \times [0, \infty)\). This extends a recent result of Ma and Zhu. We also have a higher dimensional version, and we reprove a Kähler analogy due to Chau, Tam and Yu.

1. Introduction

In a recent paper [MZ] Ma and Zhu announced the following result: Let \((M^3, g_0)\) be a complete noncompact Riemannian 3-manifold with nonnegative sectional curvature such that \(|Rm|(x) \to 0\) as \(x \to \infty\). Then the Ricci flow with initial data \((M^3, g_0)\) has a long time solution on \(M^3 \times [0, \infty)\).

In this short note we’ll improve Ma and Zhu’s result by showing the following

**Theorem 1** Let \((M^3, g_0)\) be a complete noncompact Riemannian 3-manifold with nonnegative Ricci curvature and with injectivity radius bounded away from zero. Further assume that the scalar curvature \(R(x) \to 0\) as \(x \to \infty\). Then the Ricci flow with initial data \((M^3, g_0)\) has a long time solution on \(M^3 \times [0, \infty)\).

Similarly one has

**Theorem 2** Let \((M^n, g_0)\) be a complete noncompact Riemannian \(n\)-manifold with nonnegative curvature operator and with injectivity radius bounded away from zero. Further assume that the scalar curvature \(R(x) \to 0\) as \(x \to \infty\). Then the Ricci flow with initial data \((M^n, g_0)\) has a long time solution on \(M^n \times [0, \infty)\).

(The case \(n = 3\) is due to Ma and Zhu ([MZ, Theorem 1.1]), where the injectivity radius condition is not assumed, but actually this condition does not follow from the nonnegativity of curvature operator.)

We also reprove a Kähler analogy due to Chau, Tam and Yu.

**Theorem 3** ([CTY,Theorem 1.2]) Let \((M^n, g_0)\) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and with injectivity radius bounded away from zero. Further assume that \(|Rm|(x) \to 0\) as
\[ x \to \infty. \] Then the Kähler-Ricci flow with initial data \((M^n, g_0)\) has a long time solution on \(M^n \times [0, \infty)\).

(Actually the condition \(|Rm|(x) \to 0\) as \(x \to \infty\) may be replaced by \(R(x) \to 0\) as \(x \to \infty\).)

Both theorems are consequences of Perelman’s no local collapsing theorem, pseudolocality theorem (actually its generalization due to Chau, Tam and Yu [CTY]), and his theorem on asymptotic volume ratio on ancient solution to Ricci flow (or its Kähler analogy due to Ni [N]), as shown in the next section.

2. Proof of Theorems

Proof of Theorem 1. First note that since \((M^3, g_0)\) has nonnegative Ricci curvature and the scalar curvature \(R(x) \to 0\) as \(x \to \infty\), we have \(|Rm|(x) \to 0\) as \(x \to \infty\), so the sectional curvature of \((M^3, g_0)\) is bounded, then we have the short time existence of a solution \((M^3, g(t))\) to the Ricci flow equation \(\frac{\partial g}{\partial t} = -2R_{ij}\) with initial data \((M^3, g_0)\) by Shi([S2]). Also note that the Ricci curvature of \(g(t)\) remains nonnegative along the Ricci flow again by Shi ([S1]).

Let the maximal time interval of existence of \((M^3, g(t))\) be \([0, T)\). We’ll prove by contradiction that \(T = \infty\). Suppose \(T < \infty\). Then since \((M^3, g_0)\) has bounded sectional curvature and has injectivity radius bounded away from zero, by the virtue of Perelman’s work [P], see Kleiner-Lott [KL,Theorem 26.2], \((M^3, g(t))\) is non-collapsed (in the sense of [KL,Definition 26.1]).

Since \((M^3, g_0)\) has injectivity radius bounded away from zero, and satisfies \(|Rm|(x) \to 0\) as \(x \to \infty\), and \(T < \infty\), by a theorem of Chau, Tam and Yu([CTY, Theorem 1.1]) which extends Perelman’s pseudolocality theorem in [P]) we know that there exists some compact set \(S \subset M^3\) such that \(|Rm|(x, t)\) is uniformly bounded on \((M^3 \setminus S) \times [0, T)\). But the sectional curvature of \((M^3, g(t))\) must blow up at time \(T\), hence there exist \(t_n \to T\), \(p_n \in S\) such that \(Q_n := |Rm|(p_n, t_n) = \sup_{x, t \in M^3, t_n \leq t} |Rm|(x, t) \to \infty\) as \(n \to \infty\). By Hamilton’s compactness theorem for Ricci flow ([H]), the rescaled solutions \((M^3, Q_n g(t_n + Q_n^{-1} t), p_n)\) sub-converge to a complete non-flat ancient solution \((M^\infty, g^\infty(t), g)\) to Ricci flow, which has (bounded) nonnegative sectional curvature by the Hamilton-Ivey curvature pinching estimate. Then \((M^\infty, g^\infty(t))\) has zero asymptotic volume ratio, again by Perelman [P].

So given any \(\varepsilon > 0\), there exists \(r > 0\) such that

\[
\frac{Vol_{g^\infty(0)}(q, r)}{(r^3)^{\frac{3}{2}}} < \varepsilon. \tag{*}
\]

Then, when \(n\) is sufficiently large we have

\[
\frac{Vol_{g(t_n)}(p_n, Q_n^{-\frac{1}{2}} r)}{(Q_n^{-\frac{1}{2}} r)^3} < \varepsilon.
\]

Since \(|Ric|(x, t)\) is uniformly bounded on \((M^3 \setminus S) \times [0, T)\), the volume of any compact region in \(M^3 \setminus S\) decays with a controllable speed. So there exist a positive constant \(\delta\) and a compact region \(\Omega \subset M^3 \setminus S\) such that Vol\(_{g(t)}\Omega \geq \delta, t \in [0, T)\).

Choose \(r_0\) sufficiently large such that \(\Omega \subset B_{g_0}(p, r_0)\) for any \(p \in S\). Then since the Ricci curvature of \(g(t)\) remains nonnegative, we know that the distance function of \((M^3, g(t))\) is decreasing, so we have \(\Omega \subset B_{g(t)}(p, r_0)\), for any \(t \in [0, T)\), and any \(p \in S\).

Choose \(n\) sufficiently large such that \(Q_n^{-\frac{1}{2}} r < r_0\). Then by Bishop-Gromov theorem we have
\[
\frac{\text{Vol} B_{\varepsilon}(t_0)(p_0, Q_0, r_0)}{(Q_0 - \varepsilon r_0)^3} \geq \frac{\text{Vol} B_{\varepsilon}(t_0)(p_0, r_0)}{r_0^3} \geq \frac{\delta}{r_0^3}.
\]

But if we choose \(\varepsilon < \delta/r_0^3\), then the above inequality contradicts the inequality (*), and we are done.

**Remark** The above proof is a modification of that of Ma and Zhu ([MZ]). It seems to me that the arguments in [MZ] may have some gaps. In their arguments in [MZ], Ma and Zhu invoke a theorem of Hamilton (see Theorem 2.3 in [MZ]) to imply the existence of \(S\) (where it is denoted by \(K\)), but it seems that Hamilton’s theorem is not sufficient for this implication.

The prove of Theorem 2 is similar, and is omitted. (When the condition in Theorem 2 on nonnegativity of curvature operator is replaced by positivity of curvature operator, then the injectivity radius condition may be removed, and the same result holds, since in this case one has the Gromoll-Meyer injectivity radius estimate. Actually in this case the proof is simpler, since one does not need Perelman’s no local collapsing theorem.)

To prove Theorem 3, one only needs to replace Perelman’s theorem on asymptotic volume ratio by Ni’s([N,Theorem 2]), which says that a non-flat ancient solution to Kähler-Ricci flow with bounded nonnegative holomorphic sectional curvature must have zero asymptotic volume ratio. Then after minor modifications the arguments in the proof of Theorem 1 apply to this case.

**Reference**

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