Light Spin-One Particles Imply Gauge Invariance

C.P. Burgess\textsuperscript{a} and David London\textsuperscript{b}

\textsuperscript{a} Physics Department, McGill University
3600 University St., Montréal, Québec, CANADA, H3A 2T8.

\textsuperscript{b} Laboratoire de Physique Nucléaire, Université de Montréal
C.P. 6128, Montréal, Québec, CANADA, H3C 3J7.

Abstract

Recently, calculations which consider the implications of anomalous trilinear gauge-boson couplings, both at tree-level and in loop-induced processes, have been criticized on the grounds that the lagrangians employed are not $SU_L(2) \times U_Y(1)$ gauge invariant. We prove that, in fact, the general Lorentz-invariant and $U(1)_{em}$ invariant but not $SU_L(2) \times U_Y(1)$ invariant action is equivalent to the general lagrangian in which $SU_L(2) \times U_Y(1)$ appears but is nonlinearly realized. We demonstrate this equivalence in an explicit calculation, and show how it is reconciled with loop calculations in which the different formulations can (superficially) appear to give different answers. In this sense any effective theory containing light spin-one particles is seen to be automatically gauge invariant.

1. Introduction

Perhaps the biggest pothole in the otherwise reasonably well-maintained surface that
is high-energy theory is our ignorance of the origin of particle masses. This ignorance is patched over in the standard model through the introduction of the Higgs couplings, but a better understanding is expected once shorter distance scales have been probed. One way in which the physics underlying the Higgs sector might make itself known in accelerator collisions is through the deviations from standard-model predictions it can produce in the couplings of particles to gauge-boson probes. The couplings of the massive $W^\pm$ and $Z^0$ bosons themselves are particularly interesting in this regard since they directly involve the symmetry-breaking physics through their longitudinal modes.

This type of reasoning has led to considerable effort in outlining the potential form that the anomalous couplings of these particles to the photon and the $Z^0$ might take, since these are the probes that are currently the most cleanly available in collider experiments. Since the experimental success of the standard model up to and beyond the $Z^0$ mass can be interpreted as saying that the energy scale appropriate to any new physics must be large, the analysis of potential anomalous couplings has focused on the lowest electromagnetic and electroweak moments of fermions [1], [2], [3], [4] and gauge bosons [5], [6], [7], [8], [9] that would dominate interactions at low energies. The natural theoretical framework for this type of analysis is an effective-lagrangian approach [10] in which the influence of any at-present-unknown new heavy particles is parameterized through the effective nonrenormalizable interactions that they generate among the lighter particles.

Essentially only two ingredients are required to specify such a low-energy effective lagrangian: the low-energy particle content and the symmetries that their interactions preserve. Although by and large there is agreement on the low-energy particle content, there are currently two main choices that are made concerning the symmetries that should be required of the low-energy lagrangian. One school [1], [3], [6], [7], [11] imposes only the very minimal conditions of Lorentz-invariance, $SO(3,1)$, and electromagnetic gauge invariance, $U_{em}(1)$. The alternative procedure [2], [12] is to require invariance with respect to the full electroweak gauge group, $SU_L(2) \times U_Y(1)$, but with all but the unbroken $U_{em}(1)$ subgroup being nonlinearly realized.† In this second framework the unknown symmetry-breaking sector is assumed at low energies to contain only the three Nambu-Goldstone bosons which are eaten by the massive $W^\pm$ and $Z^0$ particles. The transformation properties of all fields are then determined by general arguments [13] that were developed within the framework of chiral perturbation theory many years ago.

The principal goal of this article is to demonstrate the equivalence of these two schemes. We show that each may be obtained from the other via field redefinitions. This

† A third choice [4], [8], [9] is to linearly realize $SU_L(2) \times U_Y(1)$-invariance by explicitly including the standard-model Higgs doublet in the low-energy theory. We do not pursue this option further here.
demonstration is given in section (2) below. A practical implication of this equivalence is to permit the application of renormalizable gauges to loop calculations in what is nominally not a gauge-invariant theory. At a conceptual level it illustrates that spontaneously broken gauge invariance is automatic for any effective theory containing light spin-one particles.

Recently, de Rújula and coworkers have criticized most analyses involving anomalous trilinear gauge-boson couplings, saying that the lagrangians used are not gauge invariant [9]. Although many of their conclusions are basically correct, the equivalence theorem established in this paper shows that the supposed non-gauge invariance of such effective lagrangians is actually a red herring. Incorrect conclusions that are based on the non-gauge invariant effective lagrangian really arise from other abuses of the effective-lagrangian formalism. We pursue these related issues in a separate publication [14].

Of course, the claimed equivalence only makes sense within the domain of applicability of both formulations of the effective theory. For both approaches this is necessarily restricted to energy scales that are not too large compared to the spin-one boson masses, $M$. For a weak spin-one coupling, $g$, the maximum applicable scale may be estimated to be $\simeq 4\pi M/g$ in order of magnitude. At higher energies pathologies such as the failure of perturbative unitarity may be expected, indicating a breakdown of the low-energy approximation and the appearance of some sort of ‘new physics’.

In order to bring out some of the peripheral issues which can confuse this equivalence we compute the one-loop-induced weak fermion dipole moment that would be generated by a particular (anapole) anomalous moment in the $WWZ$ interaction. We show that although the equivalence is manifest within a gauge-invariant regularization—such as dimensional regularization—it is hidden when a cutoff is used (as is frequently done in the literature). In this case the induced weak dipole moment can be quadratically or logarithmically divergent depending on the gauge, or even on the field variables that are employed.

Although some of these points are undoubtedly known to the effective-lagrangian cognicenti, it is evident that they have not percolated out into the wider community which is now finding applications for these techniques. (This is particularly clear in criticisms [9] of the ‘non-gauge invariant’ formulation discussed above.) For this reason we feel that a re-examination of these issues is appropriate here.

2. The Equivalence Result

There are two natural ways to incorporate spontaneously broken gauge symmetries within a low-energy effective lagrangian:
• **No Gauge Invariance:** In the first formulation massive spin-one bosons are represented by vector fields and the lagrangian is only required to be Lorentz invariant. Only invariance with respect to unbroken gauge symmetries is imposed and all broken gauge symmetries are simply ignored.

• **Nonlinearly-Realized Gauge Invariance:** The alternative is the second approach in which both Lorentz and gauge symmetries are built in from the beginning. Spontaneous symmetry breaking is incorporated by coupling all fields to a symmetry-breaking sector. All that is assumed about this sector is that its only light degrees of freedom are the appropriate set of Nambu-Goldstone bosons that are required on general grounds by Goldstone’s theorem. These are, of course, ultimately ‘eaten’ by the gauge bosons via the Higgs mechanism once the nonlinearly-realized action of the broken-symmetry transformations amongst the Nambu-Goldstone bosons is ‘gauged’.

We demonstrate in this section a precise form for the equivalence of these two formulations for the low-energy lagrangian. Although the arguments can be made quite generally, we restrict ourselves here to establishing this equivalence for two specific cases: a simplified toy model involving a single massive spin-one particle, as well as the realistic case appropriate to the couplings of the electroweak gauge bosons, $W^\pm, Z^0$ and the photon, $\gamma$.

2.1) A Toy Example

In order to describe the argument within its simplest context, consider first the coupling of a single massive spin-one particle, $V_\mu$, coupled to various forms of spinless or spin-half matter, $\psi$. We first state the two alternative forms for the effective lagrangian and then demonstrate their equivalence.

• **No Gauge Invariance:** The lagrangian in the first formulation then takes the form:

$$\mathcal{L}_1 = \mathcal{L}_1(V_\mu, \psi),$$

in which $\mathcal{L}_1$ is *a priori* an arbitrary local Lorentz-invariant function of the fields $V_\mu$, $\psi$ and their spacetime derivatives. Since $\psi$ and $V_\mu$ are independent degrees of freedom the quantum theory could be defined in this case by a functional integral of the form:

$$Z_1 = \int [d\psi] [dV_\mu] \exp \left[ i \int d^4x \mathcal{L}_1(V_\mu, \psi) \right].$$

(2)
Nonlinearly Realized Gauge Invariance: The alternative formulation is to consider a $U(1)$ gauge theory with matter fields, $\chi_i$, carrying $U(1)$ charges $q_i$. The gauge symmetry transformations acting on these fields and on the gauge potential, $A_\mu$, are the usual ones:

$$\chi_i \rightarrow e^{i q_i \omega} \chi_i; \quad g A_\mu \rightarrow g A_\mu + \partial_\mu \omega.$$  \hspace{1cm} (3)

$g$ here is the gauge coupling constant.

Symmetry breaking is incorporated by coupling these matter and gauge fields in a completely general way to a single Nambu-Goldstone boson, $\varphi$, for a spontaneously broken $U(1)$. The action of the $U(1)$ on the Nambu-Goldstone bosons may always be chosen to take a standard form [13], which becomes in this case

$$\varphi \rightarrow \varphi + f \omega.$$  \hspace{1cm} (4)

$f$ here is the Nambu-Goldstone boson’s decay constant which is of the order of the scale at which the $U(1)$ symmetry is spontaneously broken. It is related to the mass of the gauge boson by the relation $M = gf$.

The most general gauge-invariant low-energy lagrangian may then be written in the following form:

$$\mathcal{L}_2 = \mathcal{L}_2(D_\mu \varphi, \chi'),$$  \hspace{1cm} (5)

in which the redefined field is $\chi'_i \equiv e^{-i q_i \varphi / f} \chi_i$ and the gauge-covariant derivative for $\varphi$ is given by $D_\mu \varphi \equiv \partial_\mu \varphi - g f A_\mu$. Notice that all of the dependence on $A_\mu$ in $\mathcal{L}_2$ arises through this gauge-covariant derivative. For example, the gauge field strength is given by $g f F_{\mu \nu} = \partial_\mu D_\nu \varphi - \partial_\nu D_\mu \varphi$.

The corresponding functional integral defining the quantum theory then has the standard form:

$$Z_2 = \int [d\chi'_i] [dA_\mu] [d\varphi] \exp \left[ i \int d^4 x \mathcal{L}_2(D_\mu \varphi, \chi') \right] \delta[G] \text{ Det} \left( \frac{\delta G}{\delta \omega} \right),$$  \hspace{1cm} (6)

in which the second-to-last term is the functional delta function, $\delta[G]$, which enforces the gauge condition $G = 0$, and the last term is the associated Fadeev-Popov-DeWitt—or ghost—functional determinant.

It is crucial for the remainder of the argument that both $\chi'_i$ and $D_\mu \varphi$ are invariant—as opposed to being covariant—with respect to gauge transformations. As a result even if
the lagrangian, $L_2$, is only required to be Lorentz invariant it becomes automatically also
gauge invariant.

• **Equivalence:** Now comes the main point. The two lagrangians, $L_1$ and $L_2$, are identical
to one another. There is a one-to-one correspondence between the terms in each given by the
replacement $\psi \leftrightarrow \chi_i'$ and $D_\mu \varphi \leftrightarrow -gf V_\mu$. This is only possible because both $L_1$ and
$L_2$ are constrained only by Lorentz invariance and so any interaction which is allowed for
one is equally allowed for the other.

More formally, the functional integral of eq. (2) may be obtained from that of eq.
(6) by simply choosing unitary gauge, defined by the condition $G \equiv \varphi(x)$, and using the
functional delta function to perform the integration over $\varphi$. The ghost ‘operator’ is in this
case $\delta G(x)/\delta \omega(x') = f \delta^4(x - x')$ and so the ghost determinant contributes just a trivial
field-independent normalization factor.

The integration over the ‘extra’ Nambu-Goldstone degree of freedom of the gauge-
invariant theory is thereby seen to be precisely compensated by the freedom to choose a
gauge.

2.2) Applications to the Electroweak Bosons

The argument as applied to a more complicated symmetry-breaking pattern, such
as appears in the electroweak interactions, has essentially the same logic although the
technical details are slightly more intricate.

• **No Gauge Invariance:** We take for the purposes of illustration the degrees of freedom in
the low-energy effective lagrangian for the electroweak interactions of leptons and quarks.
These are: the massless photon, $A_\mu$, the massive weak vector bosons, $W_\mu$ and $Z_\mu$, and
the usual fermions, $\psi$. Although other particles such as gluons may also be very simply
included we do not do so here for simplicity of notation. The general lagrangian for these
fields may be written:

$$L_1 = L_1(A_\mu, W_\mu, Z_\mu, \psi),$$

in which $L_1$ is a general local and Lorentz-invariant function whose form is constrained
only by the requirement of invariance with respect to the unbroken electromagnetic gauge
transformations, $U_{\text{em}}(1)$. All derivatives are taken to be the $U_{\text{em}}(1)$ gauge-covariant deriva-
tive, $D_\mu$, which for fermions takes the form $D_\mu \psi = \partial_\mu \psi - ieQA_\mu \psi$. $Q$ here denotes the
diagonal matrix of fermion electric charges.
The quantum theory is given in terms of a functional integral of the form

$$Z_1 = \int [dW_\mu] [dW^{*}_\mu] [dZ_\mu] [dA_\mu] [d\ell_i] \exp \left[ i \int d^4x \, \mathcal{L}_1 \right] \delta [G_{em}] \, \mathrm{Det} \left( \frac{\delta G_{em}}{\delta \omega_{em}} \right). \quad (8)$$

We next outline the nonlinear realization of $SU_L(2) \times U_Y(1)$.

- **Nonlinearly Realized Gauge Invariance:** The first step is to briefly review the formulation for realizing the symmetry-breaking pattern $SU_L(2) \times U_Y(1) \rightarrow U_{em}(1)$ nonlinearly [13].

Consider, therefore, a collection of matter fields, $\psi$, on which $SU_L(2) \times U_Y(1)$ is represented (usually reducibly) by the matrices $G = \exp[i\omega_2 T_a + i\omega_1 Y]$. We choose here a slightly unconventional normalization for the generator $s T_a$ and $Y$, viz $\text{tr}[T_a T_b] = \frac{1}{2}\delta_{ab}$, $\text{tr}[T_a Y] = 0$ and $\text{tr}[Y^2] = \frac{1}{2}$. Finally define the matrix-valued scalar field containing the Nambu-Goldstone bosons by $\xi(x) = \exp[iX_a \varphi^a(x)/f]$, in which the three $X_a$’s represent the spontaneously broken generators $X_1 = T_1$, $X_2 = T_2$ and $X_3 = T_3 - Y$. $X_3$ here is chosen to be orthogonal to the unbroken generator of $U_{em}(1): Q = T_3 + Y$.

The action of the gauge group $SU_L(2) \times U_Y(1)$ on $\xi$ and $\psi$ may be written in the standard form [13]:

$$\psi \rightarrow G \psi \quad \text{and} \quad \xi \rightarrow \xi', \quad \text{where} \quad G \xi = \xi' H^\dagger. \quad (9)$$

Here $H = \exp[iQ u(\xi, \xi', G)]$ and $u = u(\xi, \xi', G)$ is implicitly defined by the condition that $\xi'$ on the right-hand-side of eq. (9) involves only the broken generators.

As was the case for the toy example, for the purposes of constructing the lagrangian it is convenient to define new matter fields, $\psi'$, according to $\psi' \equiv \xi^\dagger \psi$ since this has the $SU_L(2) \times U_Y(1)$ transformation rule:

$$\psi' \rightarrow \xi'^\dagger \, G \, \psi$$
$$= H \, \psi'. \quad (10)$$

Notice that even for global $U_Y(1)$ rotations, for which $\omega_1$ is constant, $u(\xi, \xi', G)$ is spacetime dependent because of its dependence on the scalar field $\xi(x)$.

The next step is the construction of the general locally $SU_L(2) \times U_Y(1)$ invariant effective lagrangian. To this end consider the auxiliary quantity $\mathcal{D}_\mu(\xi)$ which may be defined in terms of $\xi$ and the $SU_L(2) \times U_Y(1)$ gauge potentials $W_\mu = g_2 W_\mu^a T_a + g_1 B_\mu Y$ by

$$\mathcal{D}_\mu(\xi) \equiv \xi^\dagger \partial_\mu \xi - i\xi^\dagger W_\mu \xi. \quad (11)$$
In terms of this quantity it is possible to construct fields which transform in a simple way with respect to $SU_L(2) \times U_Y(1)$. Together with their transformation rules these are,

$$e A_\mu \equiv i \text{tr}[QD_\mu(\xi)], \quad e A_\mu \rightarrow eA_\mu + \partial_\mu u;$$

$$\sqrt{g_1^2 + g_2^2} Z_\mu \equiv 2i \text{tr}[(T_3 - Y)D_\mu(\xi)], \quad Z_\mu \rightarrow Z_\mu;$$

$$g_2 W_\mu^\pm \equiv i\sqrt{2} \text{tr}[T_\pm D_\mu(\xi)], \quad W_\mu^\pm \rightarrow e^{\pm iuQ} W_\mu^\pm. \quad (14)$$

$T_\pm$ is defined as usual to be $T_1 \pm iT_2$. The first of these fields, $A_\mu(\xi)$, transforms in such a way as to permit the construction of a covariant derivative for the local transformations as realized on $\psi'$:

$$D_\mu \psi' \equiv (\partial_\mu - ieA_\mu Q) \psi'. \quad (15)$$

The main point to be appreciated here is that all of the fields $\psi'$, $D_\mu \psi'$, $A_\mu(\xi)$, $Z_\mu(\xi)$ and $W_\mu^\pm(\xi)$ transform purely electromagnetically under arbitrary $SU_L(2) \times U_Y(1)$ transformations. This ensures that once the lagrangian is constructed to be invariant under the unbroken group, $U_{em}(1)$, it is automatically invariant with respect to the full nonlinearly-realized group $SU_L(2) \times U_Y(1)$.

With these transformation rules the most general $SU_L(2) \times U_Y(1)$-invariant lagrangian becomes

$$L_2 = L_2(A_\mu, W_\mu, Z_\mu, \psi') \quad (16)$$

with $L_2$ restricted only by the unbroken $U_{em}(1)$ gauge invariance. The functional integral which defines the quantum theory may then be written

$$Z_2 = \int [dW_\mu] [d\xi] [d\psi'] \exp \left[i \int d^4x \ L_2 \right] \delta [G_a] \ \text{Det} \left( \frac{\delta G_a}{\delta \omega^a} \right). \quad (17)$$

Four gauge conditions, $G_a = 0$, $a = 1, \ldots 4$, are required—one for each generator of $SU_L(2) \times U_Y(1)$.

**Equivalence:** The demonstration of the equivalence between eqs. (8) and (17) proceeds along lines that are similar to those used in the abelian toy example presented previously. As was the case in this earlier example, the equivalence works term-by-term in the lagrangian. The correspondence between the field variables is

$$A_\mu \leftrightarrow A_\mu, \quad Z_\mu \leftrightarrow Z_\mu, \quad W_\mu^\pm \leftrightarrow W_\mu^\pm, \quad \psi' \leftrightarrow \psi. \quad (18)$$
The equivalence is explicit in unitary gauge, which is defined in this case by the condition $\varphi^a(x) \equiv 0$, or equivalently $\xi(x) \equiv 1$, throughout spacetime. As is seen from the transformation rules of eq. (9) this condition does not completely fix the gauge. It is preserved by the unbroken electromagnetic transformations which satisfy $G = H = e^{i\omega_{em}}$. In this gauge the relations for $Z_\mu$, $W_\mu$ and $\psi$ indicated in eqs. (18) above simply become equalities.

More formally, using the unitary gauge-condition to perform the functional integral over $\xi$ in eq. (17), gives the result

$$Z_2 = \int [dW_\mu] [d\psi] \exp \left[ i \int d^4x \mathcal{L}_2 \right] \delta [G_{em}] \operatorname{Det} \left( \frac{\delta G_{em}}{\delta \omega_{em}} \right) \operatorname{Det} \left( \frac{\delta \varphi^a}{\delta \omega^b} \right) \bigg|_{\varphi=0}. \quad (19)$$

Since $\mathcal{L}_2(\xi = 1) = \mathcal{L}_1$ this clearly agrees with eq. (8) apart from the final Fadeev-Popov-DeWitt ghost determinant that is associated with the choice of unitary gauge

$$\delta \varphi^a(x)/\delta \omega^b(x') \equiv \Delta^a_b(x) \delta^4(x - x'). \quad (20)$$

The final point is that the identity $\operatorname{Det} \equiv \exp \text{Tr Log}$ may be used to rewrite this determinant as the exponential of a local, Lorentz- and $U_{em}(1)$-invariant function. As such it may be considered as a shift in the parameters appearing in the original lagrangian, $\mathcal{L}_2$. Furthermore, since its contribution to $\mathcal{L}_2$ is proportional to $\delta^4(x = 0)$ its coefficients are ultraviolet divergent and so their contribution may be absorbed into the renormalizations that are anyhow required in defining the functional integral of eq. (19). At a practical level, the Fadeev-Popov determinant does not in any case arise until at least two-loop order.

### 3. An Illustrative Calculation

In order to illustrate explicitly the equivalence of the two formulations, we will compute the $CP$-violating ‘weak dipole moment’ [1] (which we denote by $Z_{dm}$) of the $\tau$ lepton,

$$\mathcal{L}_{zdm} = -iz \, \tau \gamma_5 \sigma^{\mu\nu} \tau \partial_\mu Z_\nu, \quad (21)$$

that is induced at one loop by an anomalous $WWZ$ vertex. We consider for these purposes the following $CP$-violating anomalous anapole coupling such as appears in the non-gauge
invariant formulation of Hagiwara et.al. ref. [5]:

$$\mathcal{L}_a = -a W^*_\mu W_\nu \left( \partial^\mu Z^\nu + \partial^\nu Z^\mu \right).$$  \hspace{1cm} \text{(22)}$$

We may translate this effective interaction into a form in which the gauge invariance is nonlinearly realized using the general correspondence of the previous section. The result is to simply make the substitutions of eqs. (18) in eq. (22).

In order to illustrate the equivalence of these two formulations we next compute the Zdm using the anapole vertex as derived from interaction (22) before and after making the substitution (18).

3.1) Unitary Gauge Calculation

In the non-gauge-invariant formulation the anapole vertex of Fig. 1 is represented by the following Feynman rule

$$a \left( k^\beta g^{\mu\alpha} + k^\alpha g^{\mu\beta} \right),$$ \hspace{1cm} \text{(23)}$$
and the gauge bosons propagate with the usual massive vector-boson propagator

$$G^{\mu\nu}_U(k) = -i \frac{P^{\mu\nu}(k)}{k^2 - M_W^2},$$

with

$$P^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{M_W^2}.$$  \hspace{1cm} \text{(24)}$$

The expression for $z$ may then be read from the amplitude (see Fig. 2)

$$\mathcal{T}^\mu = -\frac{ag_w^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{D} \left( k^\beta g^{\mu\alpha} + k^\alpha g^{\mu\beta} \right) P_\alpha^\rho(q + p_2) P_{\beta\sigma}(q - p_1) \overline{u}_{\tau}(p_2) \gamma^\rho \gamma^\sigma v_{\tau}(p_1),$$ \hspace{1cm} \text{(25)}$$

$D$ here represents the denominators of the propagators that appear in the graph

$$D = (q^2 - m_{W^\pm}^2) [(q + p_2)^2 - M_W^2] [(q - p_1)^2 - M_W^2].$$  \hspace{1cm} \text{(26)}$$

\[ \dagger \] The coefficient ‘$a$’ in this equation corresponds to $g_4^Z$ of ref. [5].
Since this amplitude diverges we regularize the integral by working in \( n \neq 4 \) dimensions. We will return to the issue of regularization later in this section. The divergent part may be explicitly evaluated to be

\[
\mathcal{T}^\mu = -\frac{ag_w^2}{384\pi^2} \frac{m_\tau (m_\tau^2 - m_{\nu_e}^2)}{M_W^4} T_\tau (p_2) \sigma^{\mu\nu} k_\nu \gamma_5 v_\tau (p_1) \left( \frac{2}{4 - n} \right),
\]

and may be absorbed by renormalizing the coefficient \( z \) of the \( Z \)dm operator of eq. (21). This determines how these operators mix due to renormalization. In the minimal subtraction scheme we therefore find:

\[
z(\mu) = z(\mu') + \frac{g_w^2}{384\pi^2} \frac{m_\tau (m_\tau^2 - m_{\nu_e}^2)}{M_W^4} a(\mu') \log \left( \frac{\mu^2}{\mu'^2} \right). \quad (28)
\]

3.2) Renormalizable-Gauge Calculation

The same calculation may be performed in a general gauge using the Feynman rules appropriate to the effective lagrangian with nonlinearly-realized gauge invariance. The principal difference here is that there are now four diagrams – that of Fig. 2, and those in which one or both of the \( W^\pm \)'s is replaced by the corresponding would-be-Goldstone boson (WBGB), \( \phi^\pm \).

In the standard family of covariant renormalizable gauges parameterized by the variable \( \alpha \) the \( \phi^\pm \)-scalar and \( W^\pm \)-boson propagators are respectively given by

\[
G_{(\alpha)}(k) = \frac{i}{k^2 - \alpha M_W^2}
\]

and

\[
G_{(\alpha)}^{\mu\nu}(k) = -i \frac{1}{k^2 - M^2} \left[ g^{\mu\nu} + (\alpha - 1) \frac{k^\mu k^\nu}{k^2 - \alpha M_W^2} \right] = G_U^{\mu\nu}(k) - \frac{k^\mu k^\nu}{M_W^2} G_{(\alpha)}(k). \quad (30)
\]

As is clear from the expansion of \( W_\mu(\xi) \) in terms of powers of fields:

\[
W_\mu^\pm = g_2 \left[ W_\mu^\pm + \frac{1}{M_W} \partial_\mu \phi^\pm + \cdots \right], \quad (31)
\]

the Feynman rule for the emission of a WBGB, \( w \), of four-momentum \( k^\mu \) from the anapole vertex is found by simply contracting the result for the emission of the corresponding
gauge particle—\textit{i.e.} that of eq. (23)—by $k^\mu/M_W$. The same is true for the emission of a WBGB by a fermion line. As may be easily verified these are precisely the vertices that are required to preserve the $\alpha$-independence of tree level amplitudes.

From these Feynman rules it is immediately clear that the sum of the four graphs that contribute in the renormalizable gauges precisely corresponds to the four terms that would be obtained by substituting eq. (30) into the unitary-gauge result of eq. (25). This demonstrates the equivalence of the induced $Z_{dm}$ as computed with the non-gauge-invariant and the nonlinearly-realized gauge-invariant formulations.

Notice that this equivalence has relied on the WBGB’s having derivative couplings to fermions as well as to the anapole vertex. Such couplings are an automatic consequence of the replacement (18) in the nonlinearly-realized effective lagrangian. They differ superficially from those that appear in the standard model, however, where the WBGB’s couple to fermions \textit{via} renormalizable Yukawa couplings. This difference is irrelevant because one set of couplings may be changed into the other by performing an appropriate field redefinition, which cannot alter any scattering amplitudes. It is in fact straightforward to check that use of these Yukawa couplings in the previous calculation does not at all alter our conclusions.

### 3.3) Related Red Herrings

This equivalence as outlined appears to be so simple as to be almost trivial. It is therefore worth outlining some circumstances which can act, and have acted in the literature, to obscure this conclusion.

The main obstacle to understanding this equivalence is the widespread use of cutoffs to regularize the divergent integrals that arise in loop-level effective-lagrangian applications. For the present purposes an uncritical use of cutoffs can cause confusion in two distinct ways. At a purely technical level they can hide the transformation properties of the theory under field redefinitions in general, and gauge transformations in particular, and so can give the impression of obtaining differing results in different gauges. Cutoffs also introduce a more conceptual difficulty once an attempt is made to associate a physical interpretation with the cutoff-dependence of a given amplitude. We speak briefly to each of these issues in the following paragraphs.

At the technical level, it is notoriously easy to inadvertently break gauge-invariance with a cutoff regularization. One way to see this is to implement the cutoff in the effective theory by adding higher-derivative kinetic terms to the lagrangian. This has the effect of multiplying each propagator by a form factor which separately implements the cutoff
on each internal line of any graph and ensures, for example, that the cutoff result is independent of extraneous issues such as how momentum is routed through the graph. Considered this way, however, it is clear that higher-derivative terms cannot be gauge invariant unless the derivatives used are gauge covariant. Gauge covariant derivatives necessarily imply additional cutoff-dependent interaction terms, however, whose effects are easily missed if cutoffs are simply applied \textit{a posteriori} to loop integrals.

A related issue concerns the behaviour of cutoff-regulated amplitudes under field redefinitions. For instance, in the example considered above it is superficially possible to change the divergent behaviour of the result simply by performing a field redefinition. This may be seen by comparing the result of evaluating the given graph using two kinds of WBGB–fermion couplings: on the one hand using the derivative WBGB–fermion couplings which come from the general substitution (18), and on the other hand using the standard-model Yukawa-type couplings between these particles.

In order to see these difficulties explicitly consider using the following form factor regularization in the one-loop-generated $Z$dm

\begin{equation}
\frac{-\Lambda^2}{q^2 - \Lambda^2} \frac{-\Lambda^2}{(q + p_2)^2 - \Lambda^2} \frac{-\Lambda^2}{(q - p_1)^2 - \Lambda^2}.
\end{equation}

Using this regularization together with the derivatively-coupled fermion–WBGB vertex one finds the following quadratic divergence

\begin{equation}
\mathcal{T}^\mu = \frac{-a g_w^2}{2304 \pi^2} \frac{\Lambda^2}{M_W^4} m_{\tau} \bar{\pi}_\tau (p_2) \sigma^{\mu \nu} k_\nu \gamma_5 v_\tau (p_1).
\end{equation}

This result holds for both the unitary-gauge and the $\alpha$-gauge calculations.

Performing the same calculation using Yukawa-type WBGB–fermion vertices in $\alpha$-gauge gives instead only linear and logarithmic divergences. These arise only from the graph in which both vector bosons in Fig. 2 are replaced by WBGB’s. The result from this graph is

\begin{equation}
\mathcal{T}^\mu = \frac{-a g_w^2}{2M_W^4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{D} [2q^\mu q \cdot k - q \cdot k (p_1 - p_2)^\mu] \bar{\pi}_\tau (p_2) \left[ g (m_{\tau}^2 \gamma_R + m_{\nu}^2 \gamma_L) - m_{\tau} m_{\nu}^2 \right] v_\tau (p_1).
\end{equation}

Regularizing using eq. (32) as before, we find

\begin{equation}
\mathcal{T}^\mu = \frac{-a g_w^2}{384 \pi^2} \frac{m_{\tau} (m_{\tau}^2 - m_{\nu}^2)}{M_W^4} \ln \frac{\Lambda^2}{M_W^2} \bar{\pi}_\tau (p_2) \sigma^{\mu \nu} k_\nu \gamma_5 v_\tau (p_1),
\end{equation}

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which is only logarithmically divergent, as advertised.

The problem here is that these two kinds of Feynman rules for the fermion–WBGB vertex may be obtained from one another by performing a WBGB-dependent nonlinear field redefinition on the fermion fields. The answer would be unchanged if the higher-derivative term which implements the cutoff were also transformed, since this transformation would introduce new cutoff-dependent fermion–WBGB interactions. Of course, this is not what was compared between eqs. (33) and (35).

There are two lessons to be learned from this example. The first is that it is very simple to miss contributions when performing field redefinitions on cutoff-regulated quantities. More important, however, is the realization that the cutoff dependence of an amplitude in an effective theory is not necessarily simply related to its dependence on the heavy mass scales that appear within whatever short-distance physics generates that effective lagrangian. Since cutoffs are frequently used to estimate the scale of new physics which might be probed in proposed experiments, we will deal with this issue in more depth in a separate publication [14]. It suffices here to remark that the connection between cutoffs and the scale of new physics is completely unrelated to how gauge-invariance is realized in the effective lagrangian. Furthermore, we repeat that superficial gauge-variance of cutoff-regulated results can usually be traced to the non-invariance of the regularization – and not to the lagrangian itself.

4. Conclusions

Effective lagrangians are the natural way to parameterize the effects of the new physics that is ultimately responsible for the breaking of the electroweak gauge group. If one does not wish to explicitly include a Higgs scalar in the low-energy theory, there are two principal candidates for such an effective lagrangian – one which requires only $U_{\text{em}}(1)$ gauge invariance, but not $SU_L(2) \times U_Y(1)$ gauge invariance, and one which imposes the full $SU_L(2) \times U_Y(1)$ gauge invariance, nonlinearly realized. We have demonstrated the equivalence of these two lagrangians.

The same arguments as are used here may be similarly used to prove this equivalence for more general symmetry-breaking patterns $G \rightarrow H$. This shows that any effective theory containing light spin-one particles automatically has a (spontaneously broken) gauge invariance. Alternatively, one can say that at low energies there is little to choose between a spontaneously-broken gauge invariance and no gauge invariance at all. It also shows that criticisms of effective lagrangians based on the absence of gauge invariance are actually red herrings. Problems with these lagrangians tend to arise for other reasons, such as the careless use of cutoffs to regularize loop diagrams.
At a practical level this equivalence has the advantage that it allows the use of the techniques of renormalizable gauges for calculations in what is nominally not a gauge-invariant theory. This is useful when powercounting arguments are being used in that all propagators explicitly vary like $1/p^2$ for large four-momenta. As a simple example, this equivalence provides an extremely easy way to see why QED remains renormalizable even after it is supplemented by a photon mass term while a nonabelian gauge theory like the standard model does not. The difference may be most easily seen in the version of these theories in which the WBGB’s are explicit. It arises because although it is possible to construct an invariant power-counting renormalizable lagrangian for a $U(1)$ WBGB – simply its kinetic term $-\frac{1}{2}D_\mu \varphi D^{\mu} \varphi$ – such a term is not possible for a nonabelian symmetry group. This is because the kinetic terms are in this case not by themselves invariant with respect to the nonlinearly-realized symmetries.

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Figure Captions

• Figure 1: The Feynman rule for the CP-violating anomalous gauge-boson vertex discussed in the text. All momenta are outgoing.

• Figure 2: The Feynman graph through which the anomalous gauge-boson vertex contributes to fermion weak dipole moments.
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