Abstract

We study the geometry of the canonical connection on a quasi-Kähler manifold with Norden metric. We consider the cases when the canonical connection has Kähler curvature tensor and parallel torsion, and derive conditions for an isotropic-Kähler manifold. We give the relation between the canonical connection, the $B$-connection, and the connection with totally skew symmetric torsion on quasi-Kähler manifolds with Norden metric.

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Key words. Norden metric, almost complex manifold, quasi-Kähler manifold, indefinite metric, canonical connection, parallel torsion.

Introduction

A fundamental fact about almost complex manifolds with Hermitian metric is that the action of the almost complex structure on the tangent space is isometry. There is another kind of metric, called a Norden metric, or a $B$-metric, on an almost complex manifold, such that the action of the almost complex structure is anti-isometry with respect to the metric. Such a manifold in [8] is called a generalized $B$-manifold, while in the present paper we adopt the more widely used term of almost complex manifold with Norden metric [2].

It is known [11, 12, 7] that on an almost Hermitian manifold there exists a unique linear connection $\nabla'$ with a torsion $T$ such that $\nabla' J = \nabla' g = 0$ and $T(x, Jy) = T(Jx, y)$, for all vector fields $x, y$ on $M$. The group of the conformal transformations on the metric $g$ generates the conformal group of transformations of $\nabla'$. An analogous problem for almost complex manifolds with Norden metric is treated in [4], where a canonical connection on such a manifold is obtained and proved unique.

In the present paper we consider this canonical connection on an almost complex manifold with Norden metric in the case when it is a quasi-Kähler manifold. We establish that the manifold is isotropic-Kähler manifold with Norden metric iff the scalar curvatures of the canonical connection and the Levi-Civita connection are equal. We consider the cases of the canonical connection with Kähler curvature tensor and parallel torsion. We give the relation between the canonical connection, the $B$-connection ([4]), and the connection with totally skew symmetric torsion ([5]) on quasi-Kähler manifolds with Norden metric. The $B$-connection is an analogue of the first canonical Lichnerowicz connection in the Hermitian geometry [13, 18, 6]. The connection with a totally skew-symmetric torsion is known as a Bismut connection or a $KT$-connection. It is
applied in mathematics as well as in theoretical physics. For instance, the local index theorem for non-Kähler Hermitian manifolds is proved in [1] using this connection. Moreover, this connection is applied in string theory [17].

1 Preliminaries

Let $(M, J, g)$ be a $2n$-dimensional almost complex manifold with Norden metric, i.e.

\[ J^2 x = -x, \quad g(Jx, Jy) = -g(x, y), \quad (1.1) \]

for all differentiable vector fields $x, y$ on $M$. The associated metric $\tilde{g}$ of $g$ on $M$, given by $\tilde{g}(x, y) = g(x, Jy)$, is a Norden metric, too. The signature of both metrics is necessarily $(n, n)$.

Further, $x, y, z, w$ will stand for arbitrary differentiable vector fields on $M$ (or vectors in the tangent space of $M$ at an arbitrary point $p \in M$).

The Levi-Civita connection of $g$ is denoted by $\nabla$. The tensor field $F$ of type $(0, 3)$ on $M$ is defined by

\[ F(x, y, z) = g((\nabla_x J)y, z). \quad (1.2) \]

It has the following properties [8]:

\[ F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz). \quad (1.3) \]

In [2], the considered manifolds are classified into eight classes with respect to $F$. The class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric belongs to any of the other seven classes. It is determined by the condition $F(x, y, z) = 0$, which is equivalent to $\nabla J = 0$.

The condition

\[ \mathcal{S}_{x,y,z} F(x, y, z) = 0, \quad (1.4) \]

where $\mathcal{S}_{x,y,z}$ is the cyclic sum over $x, y, z$, characterizes the class $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric. This is the only class of manifolds with non-integrable almost complex structure $J$, i.e. manifolds with non-zero Nijenhuis tensor $N$, where

\[ N(x, y) = (\nabla_x J) y - (\nabla_y J) Jx + (\nabla_{Jx} J)y - (\nabla_{Jy} J)x. \quad (1.5) \]

The associated tensor $N^*$ of $N$ is defined by

\[ N^*(x, y) = (\nabla_x J) Jy + (\nabla_y J) Jx + (\nabla_{Jx} J)y + (\nabla_{Jy} J)x. \quad (1.6) \]

The condition (1.4) is equivalent (8) to the condition

\[ N^*(x, y) = 0 \quad (1.7) \]

and (1.6) to the condition

\[ \mathcal{S}_{x,y,z} F(Jx, y, z) = 0. \quad (1.8) \]
Let \( \{ e_i \} \) \((i = 1, 2, \ldots, 2n)\) be an arbitrary basis of the tangent space of \( M \) at a point \( p \in M \). The components of the inverse matrix of \( g \), with respect to this basis, are denoted by \( g^{ij} \).

Following [5], the square norm \( \| \nabla J \|_2^2 \) of \( \nabla J \) is defined in [16] by

\[
\| \nabla J \|_2^2 = g^{ij} g^{ks} g((\nabla e_i J) e_k, (\nabla e_j J) e_s).
\]

(1.9)

There, the manifold with \( \| \nabla J \|_2 = 0 \) is called an isotropic-Kähler manifold with Norden metric. It is clear that every Kähler manifold with Norden metric is isotropic-Kähler, but the inverse implication is not always true.

For quasi-Kähler manifolds with Norden metric the following equality is valid [16]

\[
\| \nabla J \|_2^2 = -2g^{ij} g^{ks} g((\nabla e_i J) e_k, (\nabla e_j J) e_s).
\]

(1.10)

Let \( R \) be the curvature tensor of \( \nabla \), i.e. \( R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z \) and the corresponding \((0, 4)\)-tensor is determined by \( R(x, y, z, w) = g(R(x, y)z, w) \). The Ricci tensor \( \rho \) and the scalar curvature \( \tau \) with respect to \( \nabla \) are defined by

\[
\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j).
\]

A tensor \( L \) of type \((0, 4)\) with the properties

\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),
\]

(1.11)

\[
\sum_{x, y, z} L(x, y, z, w) = 0 \quad \text{(the first Bianchi identity)}
\]

(1.12)

is called a curvature-like tensor. Moreover, if the curvature-like tensor \( L \) has the property

\[
L(x, y, Jz, Jw) = -L(x, y, z, w),
\]

(1.13)

we call it a Kähler tensor [3].

2 A natural connection on an almost complex manifold with Norden metric

Let \( \nabla' \) be a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) and a torsion tensor \( T \), i.e.

\[
\nabla'_x y = \nabla x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].
\]

The corresponding \((0, 3)\)-tensors are defined by

\[
Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z).
\]

(2.1)

The symmetry of the Levi-Civita connection implies

\[
T(x, y) = Q(x, y) - Q(y, x),
\]

(2.2)
\[ T(x, y) = -T(y, x). \] (2.3)

A partial decomposition of the space \( \mathcal{T} \) of the torsion (0,3)-tensors \( T \) (i.e., \( T(x, y, z) = -T(y, x, z) \)) is valid on an almost complex manifold with Norden metric \((M, J, g)\): \( \mathcal{T} = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \), where \( T_i \) (i = 1, 2, 3, 4) are invariant orthogonal subspaces \([4]\). For the projection operators \( p_i \) of \( \mathcal{T} \) in \( T_i \) is established:

\[
\begin{align*}
4p_1(x, y, z) &= T(x, y, z) - T(Jx, Jy, z) - T(Jx, y, Jz) - T(x, Jy, Jz), \\
4p_2(x, y, z) &= T(x, y, z) - T(Jx, Jy, z) + T(Jx, y, Jz) + T(x, Jy, Jz), \\
8p_3(x, y, z) &= 2T(x, y, z) - T(y, z, x) - T(z, x, y) - T(Jy, z, Jx) \\
&- T(y, Jx, Jz) + 2T(Jx, y, z) - T(Jy, z, Jx) \\
&- T(Jz, Jx, y) + T(y, Jz, Jx) + T(Jz, x, Jy), \\
8p_4(x, y, z) &= 2T(x, y, z) + T(y, z, x) + T(z, x, y) + T(Jy, z, Jx) \\
&+ T(y, Jx, Jz) + 2T(Jx, y, z) + T(Jy, Jz, x) \\
&+ T(Jz, Jx, y) - T(y, Jz, Jx) - T(Jz, x, Jy).
\end{align*}
\] (2.4)

A linear connection \( \nabla' \) on an almost complex manifold with Norden metric \((M, J, g)\) is called a natural connection if \( \nabla' J = \nabla' g = 0 \). The last conditions are equivalent to \( \nabla' g = \nabla' \tilde{g} = 0 \). If \( \nabla' \) is a linear connection with a tensor \( Q \) of the transformation \( \nabla \to \nabla' \) on an almost complex manifold with Norden metric, then it is a natural connection iff the following conditions are valid:

\[
\begin{align*}
F(x, y, z) &= Q(x, y, Jz) - Q(x, Jy, z), \\
Q(x, y, z) &= -Q(x, z, y). \\
\end{align*}
\] (2.5, 2.6)

Let \( \Phi \) be the (0,3)-tensor determined by

\[
\Phi(x, y, z) = g \left( \nabla_x y - \nabla_x y - z \right),
\] (2.7)

where \( \nabla \) is the Levi-Civita connection of the associated metric \( \tilde{g} \).

**Theorem 2.1** \([4]\). *A linear connection with the torsion tensor \( T \) on an almost complex manifold with Norden metric \((M, J, g)\) is natural iff*

\[
\begin{align*}
4p_2(x, y, z) &= g(N(x, y), z) = 2 \left\{ \Phi(z, Jx, Jy) - \Phi(z, x, y) \right\}, \\
4p_3(x, y, z) &= -\Phi(x, y, z) + \Phi(y, z, x) + \Phi(x, Jy, Jz) \\
&\quad + \Phi(y, Jz, Jx) - 2\Phi(z, Jx, Jy).
\end{align*}
\] (2.8, 2.9)

For an arbitrary almost complex manifold with Norden metric \((M, J, g)\) the following equality is valid \([3]\)

\[
\Phi(x, y, z) = \frac{1}{2} \left\{ F(Jz, x, y) - F(x, y, Jz) - F(y, Jz, x) \right\}.
\] (2.10)
Applying (1.4) to (2.10) we obtain that \((M, J, g)\) is a quasi-Kähler manifold with Norden metric iff
\[ \Phi(x, y, z) = F(Jz, x, y). \] (2.11)

**Theorem 2.2.** For a natural connection with a torsion tensor \(T\) on a quasi-Kähler manifold with Norden metric \((M, J, g)\), which is non-Kählerian, the following properties are valid
\[ p_2 \neq 0, \quad p_3 = 0. \] (2.12)

**Proof.** If we suppose \(p_2 = 0\) then (2.8) implies \(N = 0\). Because of the last condition and (1.7) the manifold \((M, J, g)\) becomes Kählerian, which is a contradiction. Therefore, \(p_2 \neq 0\) is valid. From equalities (2.11), (2.9), (1.3), (1.4) we get \(p_3 = 0\). \(\square\)

### 3 A canonical connection on a quasi-Kähler manifold with Norden metric

**Definition 3.1 ([4]).** A natural connection with torsion tensor \(T\) on an almost complex manifold with Norden metric \((M, J, g)\) is called a canonical connection if
\[ T(x, y, z) + T(y, z, x) - T(Jx, y, Jz) - T(y, Jz, Jx) = 0. \] (3.1)

In [4] it is shown that (3.1) is equivalent to the condition
\[ p_1 = p_4 = 0, \] (3.2)
i. e. to the condition \(T \in T_2 \oplus T_4\). The same paper shows that on every almost complex manifold with Norden metric \((M, J, g)\) there exists an unique canonical connection \(\nabla'\), and it is determined by
\[ g(\nabla'_x y, z) = g(\nabla_y z, x) + \frac{1}{4} \{ \Phi(x, y, z) - \Phi(z, x, y) - \Phi(Jz, x, Jy) \}. \] (3.3)

In [4], the conditions for the torsion tensor \(T\) of the canonical connection are used to obtain new characteristics for the eight classes of almost complex manifolds with Norden metric. The class \(W_0\) of the Kähler manifold with Norden metric is characterized by the condition \(T(x, y, z) = 0\). The class \(W_3\) of the quasi-Kähler manifold with Norden metric is characterized by the condition
\[ T(Jx, y, z) = -T(x, y, Jz). \] (3.4)

The equalities (2.1), (2.3) and (3.4) for the torsion tensor \(T\) of the canonical connection on a quasi-Kähler manifold with Norden metric imply the properties:
\[ T(x, y, z) = -T(y, x, z), \quad T(Jx, y, z) = T(x, Jy, z) = -T(x, y, Jz). \] (3.5)

From Theorem 2.2 and condition (3.2) we obtain immediately the following
Theorem 3.1. For the torsion tensor $T$ of the canonical connection on a quasi-Kähler manifold with Norden metric the equality $T = p_2$ is valid, i.e. $T \in T_2$.

Equalities (2.11) and (3.3) imply the following

**Proposition 3.2.** The canonical connection $\nabla'$ on a quasi-Kähler manifold with Norden metric $(M, J, g)$ is determined by

$$\nabla'_x y = \nabla_x y + \frac{1}{4} \left\{ (\nabla_y J) Jx - (\nabla_J y) Jx + 2 (\nabla_x J) Jy \right\}. \quad (3.6)$$

Let $\nabla'$ be the canonical connection on a quasi-Kähler manifold with Norden metric $(M, J, g)$. According to (3.6), for the tensor $Q$ of the transformation $\nabla \rightarrow \nabla'$ we have

$$Q(x, y) = \frac{1}{4} \left\{ (\nabla_y J) Jx - (\nabla_J y) Jx + 2 (\nabla_x J) Jy \right\}. \quad (3.7)$$

Then

$$T(x, y) = \frac{1}{2} \left\{ (\nabla_x J) y + (\nabla_J y) y \right\}. \quad (3.8)$$

Substituting $y \leftrightarrow z$ into the above, according to (1.3), we get

$$T(x, z, y) = \frac{1}{2} \left\{ -F(x, Jy, z) + F(Jx, y, z) \right\}. \quad (3.9)$$

Subtracting this from (3.8) and replacing $y$ with $Jy$ in the result, we have

$$F(x, y, z) = T(x, z, Jy) - T(x, Jy, z). \quad (3.10)$$

The equalities (3.7), (2.1) and (1.2) imply

$$Q(x, y, z) = \frac{1}{4} \left\{ F(y, Jx, z) - F(Jy, x, z) + 2F(x, Jy, z) \right\}. \quad (3.11)$$

Hence, because of (1.3) and (1.4), we conclude that

$$Q(x, y, z) = -Q(y, x, z) + F(Jz, x, y). \quad (3.12)$$

Theorem 3.3. Let $\tau'$ and $\tau$ be the scalar curvatures for the canonical connection $\nabla'$ and the Levi-Civita connection $\nabla$, respectively, on a quasi-Kähler manifold with Norden metric. Then

$$\tau' = \tau - \frac{1}{4} \|\nabla J\|^2. \quad (3.13)$$
Proof. According to (1.1) and (1.3), for an almost complex manifold with Norden metric we have \( g^{ij} F(Jz, e_i, e_j) = 0 \). Then, from (3.11), after contraction by \( x = e_i, y = e_j \), we obtain

\[
g^{ij} Q(e_i, e_j, z) = 0. \tag{3.13}
\]

Because of \( \nabla g_{ij} = 0 \) (for the Levi-Civita connection \( \nabla \)) and (3.13), we get

\[
g^{ij} (\nabla x Q)(e_i, e_j, z) = 0. \tag{3.14}
\]

It is known that for the curvature tensors \( R' \) and \( R \) of \( \nabla' \) and \( \nabla \), respectively, the following is valid:

\[
R'(x, y, z, w) = R(x, y, z, w) + (\nabla x Q)(y, z, w) - (\nabla y Q)(x, z, w) + g(Q(x, w), Q(y, z)) - g(Q(y, w), Q(x, z)). \tag{3.15}
\]

Then from (2.6) and (2.1) it follows that

\[
R'(x, y, z, w) = R(x, y, z, w) + (\nabla x Q)(y, z, w) - (\nabla y Q)(x, z, w) - g(Q(x, w), Q(y, z)) + g(Q(y, w), Q(x, z)),
\]

for an almost complex manifold with Norden metric \((M, J, g)\).

Using a contraction by \( x = e_i, \; w = e_j \) in (3.15) and combining (2.6), (3.13) and (3.14), we find that the Ricci tensors \( \rho' \) and \( \rho \) for \( \nabla' \) and \( \nabla \) satisfy

\[
\rho'(y, z) = \rho(y, z) + g^{ij} (\nabla e_i Q)(y, z, e_j) + g^{ij} g(Q(y, e_j), Q(e_i, z)). \tag{3.16}
\]

Similarly, after a contraction by \( y = e_k, \; z = e_s \) in (3.16) and according to (3.14), we obtain

\[
\tau' = \tau + g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)), \tag{3.17}
\]

for the scalar curvatures \( \tau' \) and \( \tau \) for \( \nabla' \) and \( \nabla \). The equalities (3.17) and (3.7) imply

\[
g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)) = \frac{1}{16} g^{ij} g^{ks} g(P_{jk}, P_{si}) \tag{3.18}
\]

for a quasi-Kähler manifold with Norden metric \((M, J, g)\), where

\[
P_{jk} = (\nabla e_j J) e_k - (\nabla J e_j) e_k + 2 (\nabla e_k J) e_j.
\]

From (3.18), (1.1), (1.9) and (1.10) we get

\[
g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)) = -\frac{1}{4} \|\nabla J\|^2.
\]

The last equality and (3.17) imply (3.12). \( \Box \)

**Corollary 3.4.** A quasi-Kähler manifold with Norden metric is isotropic-Kählerian if and only if the scalar curvatures for the canonical connection and the Levi-Civita connection are equal.
4 A canonical connection with Kähler curvature tensor on a quasi-Kähler manifold with Norden metric

The curvature tensor \( R' \) of a natural connection \( \nabla' \) on an almost complex manifold with Norden metric \((M, J, g)\) satisfies property (1.11), according to (3.15). Since \( \nabla' J = 0 \), the property (1.13) is also valid. Therefore, \( R' \) is Kählerian iff the first Bianchi identity (1.12) is satisfied. On the other hand, it is known ([10]) that for every linear connection \( \nabla' \) with a torsion \( T \) and a curvature tensor \( R' \) the following equality (the first Bianchi identity) is valid

\[
S_{x,y,z} R'(x, y)z = S_{x,y,z} \{(\nabla'_x T)(y, z) + T(T(x, y), z)\}.
\]

Since we have \( \nabla' g = 0 \), the last equality implies

\[
S_{x,y,z} R'(x, y, z, w) = S_{x,y,z} \{(\nabla'_x T)(y, z, w) + T(T(x, y), z, w)\}.
\]

Thus, \( R' \) satisfies (1.12) iff

\[
S_{x,y,z} \{(\nabla'_x T)(y, z, w) + T(T(x, y), z, w)\} = 0. \hspace{1cm} (4.1)
\]

This leads to the following

**Lemma 4.1.** The curvature tensor for the natural connection \( \nabla' \) with a torsion \( T \) on an almost complex manifold with Norden metric is Kählerian iff (4.1) is valid.

We substitute \( Jz \) for \( z \) and \( Jw \) for \( w \) in (4.1). Hence, according to (3.5), we obtain

\[
(\nabla'_x T)(y, z, w) - (\nabla'_y T)(z, x, w) + (\nabla'_y z) (x, y, Jw) + T(T(x, y), z, w) + T(T(y, Jz), x, w) + T(T(Jz, x), y, Jw) = 0.
\]

We subtract the last equality from (4.1), and substitute \( Jx \) for \( x \) and \( Jw \) for \( w \) in the result. Then, using (3.5), we get

\[
(\nabla'_x T)(x, y, z) + (\nabla'_y z) (x, Jy, w) + 2T(T(y, z), x, w) + 2T(T(z, Jx), y, Jw) = 0. \hspace{1cm} (4.2)
\]

We substitute \( Jy, Jz \) for \( y, z \), respectively, and we apply (3.5). We subtract the obtained equality from (4.1), and we reapply (3.5). This leads to

\[
T(T(z, x), y, w) = 0.
\]

Hence, (3.9) and (1.3) imply

\[
F(Jy, w, T(z, x)) = T(y, w, T(z, x)),
\]

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and from (1.2) and (2.1) we obtain
\[ g(T(x, z), T(y, w) - (\nabla J y) w) = 0. \] (4.3)

Since, according to (3.8) and (1.2), we have
\[ T(y, w) = \frac{1}{2} \{ (\nabla y J) J w + (\nabla J y) J w \}, \]
the following equality is valid
\[ T(y, w) - (\nabla J y) w = \frac{1}{2} \{ (\nabla y J) J w - (\nabla J y) J w \}. \]

Thus, using (4.3), we arrive at the following

**Theorem 4.2.** Let \((M, J, g)\) be a quasi-Kähler manifold with Norden metric, whose canonical connection has a Kähler curvature tensor. Then the following equality is valid
\[ g((\nabla_x J) J z + (\nabla J x) J z, (\nabla J y) J w - (\nabla y J) J w) = 0. \]

5 **A canonical connection with parallel torsion on a quasi-Kähler manifold with Norden metric**

In this section we consider a canonical connection \(\nabla'\) with parallel torsion \(T\) (i.e. \(\nabla'T = 0\)) on a quasi-Kähler manifold with Norden metric \((M, J, g)\).

According to the Hayden theorem ([9])
\[ Q(x, y, z) = \frac{1}{2} \{ T(x, y, z) - T(y, z, x) + T(z, x, y) \}. \] (5.1)

Combining this with (2.2), (2.5), (3.10), leads to the following

**Proposition 5.1.** Let \(\nabla'\) be a natural connection on an almost complex manifold with Norden metric \((M, J, g)\). Then the tensors \(T, Q\) and \(F\) are parallel or non-parallel at the same time with respect to \(\nabla'\).

Let \(\nabla'\) be a natural connection with parallel torsion on an almost complex manifold with Norden metric \((M, J, g)\). According to Proposition 5.1 we have \(\nabla'Q = 0\). Then, having in mind the formula for the covariant derivative of \(Q\), we obtain
\[ xQ(y, z, w) - Q(\nabla_x y, z, w) - Q(y, \nabla_x z, w) - Q(y, z, \nabla_x w) = 0. \] (5.2)

Since \(Q\) is the tensor of the deformation \(\nabla \rightarrow \nabla'\), applying the formula for the covariant derivative of \(Q\) with respect to \(\nabla\) and equalities (2.1) and (2.2), we obtain the following
Lemma 5.2. Let $R'$ be the curvature tensor for a natural connection $\nabla'$ with a parallel torsion $T$ on an almost complex manifold with Norden metric $(M, J, g)$. Then the following equality is valid

$$R'(x, y, z, w) = R(x, y, z, w) + Q(T(x, y), z, w) + g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)).$$

(5.3)

Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric whose canonical connection $\nabla'$ has a parallel torsion $T$. Then, according to (2.6), (3.11) and (1.2), we have

$$Q(T(x, y), z, w) = g(Q(z, w), T(x, y)) + g((\nabla_J w) z, T(x, y)).$$

The last equality and Lemma 5.2 imply

Theorem 5.3. Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric whose canonical connection $\nabla'$ has a parallel torsion $T$. Then for the curvature tensor $R'$ of $\nabla'$ we obtain

$$R'(x, y, z, w) = R(x, y, z, w) + g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) + g(Q(z, w), T(x, y)) + g((\nabla_J w) z, T(x, y)).$$

(5.4)

Because of (3.13) we have $g^{ij} Q(e_i, e_j) = 0$. Then, from (5.4) via a contraction by $x = e_i$, $w = e_j$, we get

$$\rho'(y, z) = \rho(y, z) - g^{ij} g(Q(e_i, z), Q(y, e_j)) + g^{ij} g(Q(z, e_j), T(e_i, y)) + g^{ij} g((\nabla_{Je_j} J) z, T(e_i, y)), \quad (5.5)$$

where $\rho'$ and $\rho$ are the Ricci tensors for $\nabla'$ and $\nabla$, respectively.

Combining (2.11), (3.11), (3.5), (1.2), (5.2) and (1.2), we obtain

$$g(Q(z, e_j), T(e_i, y)) = g(Q(e_j, z), Q(y, e_i)) - g(Q(e_j, z), Q(e_i, y)) - g((\nabla_{Je_j} J) z, T(e_i, y)) - g((\nabla_J z) e_j, e_i, T(e_i, y)).$$

(5.6)

We get the following equality from (5.5) and (5.6):

$$\rho'(y, z) = \rho(y, z) - g^{ij} g(Q(e_j, z), Q(e_i, y)) - g^{ij} g((\nabla_{Je_j} J) e_j, T(e_i, y)).$$

(5.7)

A contraction by $y = e_k$, $z = e_s$ leads to

$$\tau' = \tau - g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) - g^{ij} g^{ks} g((\nabla_{Je_j} J) e_j, T(e_i, e_k)), \quad (5.8)$$

where $\tau'$ and $\tau$ are the respective scalar curvatures for $\nabla'$ and $\nabla$.

Using (5.6), (1.10) and (1.9), we get

$$g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) = -\frac{1}{2} \|\nabla J\|^2. \quad (5.9)$$
From (1.10) and 2T(e_i, e_j) = (∇_{e_i}J)e_k + (∇_{e_j}J)e_k we have
\[ g^{ij}g^{ks}g((\nabla_{e_i}J)e_j, T(e_i, e_k)) = \frac{1}{4} \|\nabla J\|^2. \] (5.10)

Then, (5.8), (5.9) and (5.10) imply
\[ \tau' = \tau + \frac{1}{4} \|\nabla J\|^2. \]

From the last equality and (3.12) we obtain the following

Theorem 5.4. Let \((M, J, g)\) be a quasi-Kähler manifold with Norden metric whose canonical connection \(\nabla'\) has a parallel torsion \(T\). Then \((M, J, g)\) is isotropic-Kählerian.

6 A relation between the canonical connection, the \(B\)-connection, and the \(KT\)-connection on a quasi-Kähler manifold with Norden metric

Let \((M, J, g)\) be a quasi-Kähler manifold with Norden metric. Let us consider the following connections on \((M, J, g)\): the canonical connection \(\nabla^C\), the \(B\)-connection \(\nabla^B\) ([14]) and the connection \(\nabla^{KT}\) with a totally skew-symmetric torsion ([15]).

Let
\[ \nabla^B = \nabla + Q^B, \quad \nabla^{KT} = \nabla + Q^{KT}, \quad \nabla^C = \nabla + Q^C. \]

Then, according to [14] [15] and (3.10), we have
\[ Q^B(x, y, z) = \frac{1}{2} F(x, Jy, z), \]
\[ Q^{KT}(x, y, z) = -\frac{1}{4} \sum_{x,y,z} F(x, y, Jz), \]
\[ Q^C(x, y, z) = \frac{1}{4} \left\{ F(Jz, x, y) - F(x, y, Jz) - F(Jy, x, z) \right\}. \]

After that, using (1.3) and (1.4), we obtain
\[ Q^B = \frac{1}{2} \left( Q^{KT} + Q^C \right). \]

This leads to

Proposition 6.1. The \(B\)-connection on a quasi-Kähler manifold with Norden metric is the mean connection for the canonical connection and the \(KT\)-connection.
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