Nondispersive solutions to the mass critical half-wave equation in two dimensions

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ABSTRACT

We consider the half-wave equation with mass critical in two dimensions

\[
\begin{aligned}
    i u_t &= Du - |u|u, \\
    u(t_0, x) &= u_0(x),
\end{aligned}
\]

First, we prove the existence of a family of traveling solitary waves. We then show the existence of finite-time blowup solutions with ground state mass \( ||u_0||_2 = ||Q||_2 \), where \( Q \) is the ground state solution of equation \( DQ + Q = Q^2 \).

1. Introduction

In this paper, we consider the half-wave equation in two dimensions

\[
\begin{aligned}
    i \partial_t u &= Du - |u|u, \\
    u(t_0, x) &= u_0(x), \quad u : I \times \mathbb{R}^2 \to \mathbb{C}.
\end{aligned}
\]

Here, \( I \subset \mathbb{R} \) is an interval containing the initial time \( t_0 \in \mathbb{R} \), and

\[
(\mathcal{D} f)(\xi) = |\xi| \hat{f}(\xi)
\]
denotes the first-order nonlocal fractional derivative. Let us mention that nonlinear half-wave equation have recently attracted some attentions in the area of dispersive nonlinear PDE. The evolution problems like (1.1) arise in various physical settings, which include equations range from turbulence phenomena \cite{1, 2}, wave propagation \cite{3}, continuum limits of lattice systems \cite{4} and models for gravitational collapse in astrophysics \cite{5–7}. We also refer to \cite{8–11} and the references therein for the background of the fractional Schrödinger model in mathematics, numerics and physics.

Let us review some basic properties of this equation. The Cauchy problem (1.1) is an infinite-dimensional Hamiltonian system, which has the following two conservation laws:
Mass \( M(u) = \int_{\mathbb{R}^2} |u(t,x)|^2 dx = M(u_0) \), \hspace{1cm} (1.2)

Energy \( E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \ddot{u}(t,x)Du(t,x)dx - \frac{1}{3} \int_{\mathbb{R}^2} |u(t,x)|^3 dx = E(u_0) \). \hspace{1cm} (1.3)

The equation (1.1) also has the following symmetries:

- **Phase:** if \( u(t,x) \) is a solution, then for all \( \theta \in \mathbb{R} \), \( u(t,x)e^{i\theta} \) is also a solution.
- **Translation:** if \( u(t,x) \) is a solution, then for all \( t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^2, u(x-x_0,t-t_0) \) is also a solution.
- **Scaling:** if \( u(t,x) \) is a solution, then for all \( k > 0 \)
  \[
  u_k(t,x) = \frac{1}{k} u\left(\frac{t}{k}, \frac{x}{k}\right)
  \] is also a solution.

The Cauchy problem (1.1) is \( L^2 \)-critical since the \( L^2 \)-norm is invariant under the scaling rule (1.4):

\[
||u_\lambda||_2 = ||u||_2, \text{ for all } \lambda > 0.
\]

The local existence in the energy space \( H^1_{rad}(\mathbb{R}^2) \) for the Cauchy problem (1.1) is known from [12]. Furthermore, Hidano and Wang [13] have improved this result, establishing a local existence result in the space \( H^s_{rad}(\mathbb{R}^2) \), where \( s \in (\frac{1}{2}, 1) \) or in the space \( H^s(\mathbb{R}^2) \), where \( s \in (\frac{3}{4}, 1) \).

On the other hand, from [14], it is known that the mass critical one dimensional half wave equation is local well-posed in the Sobolev space \( H^s(\mathbb{R}) \) with \( s \geq \frac{1}{2} \) and in particular in the energy space \( H^{1/2}(\mathbb{R}) \). Moreover, if \( u(t) \in H^1(\mathbb{R}) \) is the unique solution with its maximal time of existence \( T \in (t_0, \infty) \), then we have

\[
T < +\infty \text{ implies } \lim_{t \to T^-} ||u(t)||_{H^{1/2}} = +\infty.
\] (1.5)

A classical criterion of global-in-time existence for \( H^{1/2}(\mathbb{R}) \) initial data is derived by using the Gagliardo-Nirenberg inequality

\[
||u||_{H^1}^4 \leq C_{opt} ||Du||_{L^2}^2 ||u||_{L^2}^2, \text{ for } u \in H^{1/2}(\mathbb{R}),
\] (1.6)

where \( C_{opt} = \frac{2}{||Q||_2^2} \) is the best Gagliardo-Nirenberg constant and \( Q \) is the unique ground state solution to

\[
DQ + Q = Q^3, Q(x) > 0, Q(x) \in H^{1/2}(\mathbb{R}).
\] (1.7)

Note that the existence of solution to this equation follows from standard variational techniques, while the uniqueness of \( Q \) follows from the result of Frank, Lenzmann and Silvestre in [6, 15]. A combination of the mass and energy conservation and the blowup criterion (1.5) implies that initial data \( u_0 \in H^{1/2}(\mathbb{R}) \) with

\[
||u_0||_2 < ||Q||_2
\]
generate global-in-time solution.
In this paper, we study the following two nondispersive phenomena connected with the focusing 2D half—wave equation (1.1).

1. **Traveling solitary waves of the form.**

\[ u(t, x) = e^{i\mu} Q_v(x - vt) \]

with some \( \mu \in \mathbb{R} \) and traveling velocity \( v \in \mathbb{R}^2 \). Bellazzini, Georgiev, Lenzmann and Visciglia [16] proved that traveling solitary waves for speed \( |v| > 1 \) do not exist and small data scattering failed in any space dimension. We also refer to [10, 14, 17–20] and the references therein for the traveling solitary waves of the fractional Schrödinger operator, square root Klein-Gordon operator \( \sqrt{-\Delta + m^2} \) and other nonlinearities. In what follows, let \( Q \in H^{1/2}(\mathbb{R}^2) \) be the unique ground state solution of (1.7). We can obtain the existence of traveling solitary waves by using a variational approach and adapting the proof in [10]. For the half-wave equation (1.1), we have the following result.

**Theorem 1.1.** For any \( v \in \mathbb{R}^2 \) with \( 0 < |v| < 1 \), there exists a profile \( Q_v \in H^{1/2}(\mathbb{R}^2) \) such that

- \( u(t, x) = e^{i\mu} Q_v(x - vt) \) is a traveling solitary waves solution to (1.1);
- if \( v = e \lambda \) with \( |e| = 1 \) and \( \lambda > 0 \), then the mass \( ||Q_v||_2 \) is strictly decreasing with respect to \( \lambda \), and for any \( \lambda \in (0, 1) \), the profile \( Q_v \) has strictly subcritical mass;

\[ ||Q_v||_2 < ||Q||_2. \] (1.8)

- the following limits hold:

\[ \begin{align*}
&\int ||Q_v||_2 \to ||Q||_2 \quad \text{as} \quad |v| = \lambda \to 0, \\
&\int ||Q_v||_2 \to 0 \quad \text{as} \quad |v| = \lambda \to 1.
\end{align*} \]

2) **Blowup solutions in mass critical case.** There is no general criterion for blowup solutions in \( \mathbb{R}^2 \) for \( L^2 \)-critical and \( L^2 \)-supercritical half-wave equation. This is still an open problem (see [21] for more details).

For the classical \( L^2 \)-critical nonlinear Schrödinger equation, we have the Variance-Virial Laws, which can be expressed as

\[ \frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^2} |x|^2 |u(t)|^2 dx \right) = 2 \frac{d}{dt} \left( \Re \int u \cdot \nabla u dx \right) = 8E(u_0). \]

Unlike the \( L^2 \)-critical NLS, for the \( L^2 \)-critical half-wave equation, we only have

\[ \frac{d}{dt} \left( \Re \int u \cdot \nabla u dx \right) = 2E(u_0). \]

However, it seems difficult to represent the term \( \Re \int u \cdot \nabla u dx \) as the derivative of some nonnegative one. Possible analogue of the variance for half-wave equation was suggested in [21].
\[ V(u(t)) := \int \bar{u}(t) x \cdot (-\Delta)^{1/2} u(t) dx = \|(-\Delta)^{1/2} u(t)\|_2^2. \]

However, the identity
\[ \frac{d}{dt} V(u(t)) = 8 \Re \left( \int \bar{u}(t) x \cdot \nabla u(t) dx \right) \]
is true only if \( u(t) \) is a solution of free half-wave equation \( i\partial_t u = \sqrt{-\Delta} u \). This observation shows the difficulty to use viral type identity and prove a blow-up result in the mass critical case.

Another difficulty arises, when one tries to construct a minimal blow up solution, following the approach for NLS. This difficulty is connected with the lack of pseudo-conformal symmetry, that is an essential advantage of NLS. However, Krieger, Lenzmann and Raphaël [14] constructed a minimal mass blow-up solutions to the mass critical Half-wave equation in one dimension and they obtained that the blowup speed is
\[ \|D^{1/2} u(t)\|_2 \sim \frac{C(u_0)}{|t|} \quad \text{as} \quad t \to 0^- . \]

But unlike the mass critical NLS [22], the uniqueness for this minimal mass blow-up solution is still not known. Our next result shows that there is a solution with ground state mass that blows up in finite time.

**Theorem 1.2.** For all \( E_0 \in \mathbb{R}^+ \) and \( \delta \in (0, \frac{1}{2}) \), there exists \( t^* < 0 \) independent of \( E_0 \) and a solution \( u \in C^0([t^*, 0]; H^{1/2+\delta} (\mathbb{R}^2)) \) of equation (1.1) with
\[ ||u||_2 = ||Q||_2, \quad E(u) = E_0, \]
which blow up at time \( T=0 \). More precisely, it holds that
\[ u(t,x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right) e^{\gamma t} \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^2) \quad \text{as} \quad t \to 0^- , \]
where
\[ \lambda(t) = \lambda^* t^2 + O(t^5), \quad \gamma(t) = \frac{1}{\lambda^* |t|} + O(t), \]
with some constant \( \lambda^* > 0 \), and the blowup speed is given by:
\[ ||D^{1/2} u(t)||_2 \sim \frac{C(u_0)}{|t|} \quad \text{as} \quad t \to 0^- , \]
where \( C(u_0) > 0 \) is constant depending only on the initial data \( u_0 \).

Let us make some comments on the proof of Theorem 1.2.

1. We aim to construct an exact solution of the form
\[ u(t,x) = \frac{1}{\lambda(t)} [Q + \varepsilon] \left( t, \frac{x}{\lambda(t)} \right) e^{\gamma t} = \tilde{Q} + \tilde{\varepsilon} . \]
In case of higher dimensions \( N \geq 2 \) the mass critical exponent \( 1 + \frac{2}{N} \) becomes smaller, the modulation estimates, the refine energy estimates and the bootstrap
argument become more complicated. Another difficulty is to obtain a backward 
monotonicity mixed energy/virial estimate
\[
\frac{d}{dt} \left\{ \frac{1}{\lambda} \left[ \int \left( |D^1 \xi|^2 + |\xi|^2 \right) dx + a \Im \left( \int_{|y| \leq 1} y \cdot \nabla \xi \xi \right) \right] \right\} \geq 0 + \text{lower order terms.}
\]

2. One can control the \(| \cdot | \) norm from above by \(H^2(\mathbb{R})\)–norm in one dimensional case and this is a crucial tool to evaluate some terms in the refine energy estimates and bootstrap argument. In higher dimensional case \(N \geq 2\), such a control is impossible, so we need new tools.

3. In this work the profile of the blow up of the solution is given by
\[
u(t, x) = \frac{1}{\lambda(t)} v \left( s, \frac{x}{\lambda(t)} \right) e^{i \nu(t)} ds, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}. \tag{1.11}
\]
where \(\nu(s, y)\) is a function radial in \(y\). This restriction is important to avoid the difficulties explained above and also simplify the construction of approximate solution. However, this choice gives a restriction on the other conservation quantity—the moment
\[
P(u) = \Re \int_{\mathbb{R}^2} -i \nabla u(t, x) \bar{u}(t, x) dx,
\]
that vanishes for the case of radial \(u\).

This paper is organized as follows: in Section 2, we prove the Theorem 1.1; in Section 3, we construct the high order approximation \(Q_P\) solution of the renormalized equation; in Section 4, we decompose the solution and estimate the modulation parameters; in Section 5, we establish a refine energy/virial type estimate; in Section 6, we apply the energy estimate to establish a bootstrap argument that will be needed in the construction of solutions that blow up and have ground state critical mass; in Section 7, we prove the Theorem 1.2; The Section 8 is Appendix.

**Notations**

- \((f, g) = \int \hat{f} \hat{g}\) as the inner product on \(L^2(\mathbb{R}^2)\).
- \(|| \cdot ||_{L^p}\) denotes the \(L^p(\mathbb{R}^2)\) norm for \(p \geq 1\).
- \(\hat{f}\) denotes the Fourier transform of function \(f\).
- We shall use \(X \lesssim Y\) to denote that \(X \leq CY\) holds, where the constant \(C > 0\) may change from line to line, but \(C\) is allowed to depend on universally fixed quantities only.
- Likewise, we use \(X \sim Y\) to denote that both \(X \approx Y\) and \(Y \approx X\) hold.

**2. Proof of Theorem 1.1**

In this section we prove Theorem 1.1, which establishes the existence and properties of traveling solitary waves for (1.1).

Let \(\nu \in \mathbb{R}^2\) with \(|\nu| < 1\) be given. By making the ansatz \(u(t, x) = e^{it} Q_\nu(x - \nu t)\) for equation (1.11), we find that the profile \(Q_\nu \in H^{1/2}(\mathbb{R}^2)\) has to satisfy
\[
DQ_v + Q_v + i(v \cdot \nabla)Q_v = |Q_v|Q_v. \tag{2.1}
\]

Following an idea in [10], we obtain nontrivial solutions \(Q_v \in H^{1/2}(\mathbb{R}^2)\) as optimizers for the interpolation inequality

\[
\int |u|^3 \leq C_v \left( \int \bar{u} Du + \bar{u} (iv \cdot \nabla u) \right) \left( \int |u|^2 \right)^{1/2}. \tag{2.2}
\]

Here \(C_v > 0\) denotes the optimal constant given by Weinstein functional

\[
\frac{1}{C_v} = \inf_{u \in H^{1/2}(\mathbb{R}^2) \setminus \{0\}} \left( \frac{\int \bar{u} Du + \bar{u} (iv \cdot \nabla u) \left( \int |u|^2 \right)^{1/2}}{\int |u|^3} \right). \tag{2.3}
\]

By Sobolev inequalities, we see that the infimum on the right is strictly positive (and hence \(C_v < +\infty\)). Furthermore, the fact that this infimum is, indeed, attained can be deduced from the concentration-compactness arguments, which is our case follow from a direct adaption of the proof given in [10]. In particular, optimizers \(Q_v \in H^{1/2}(\mathbb{R}^2)\) for (2.2) exist, and after a suitable rescaling \(Q_v(x) \mapsto aQ(bx)\) with \(a, b > 0\) they are found to satisfy equation (2.1). Following the terminology introduced in [10], we refer to optimizers such as \(Q_v\) that solve Equation (2.1) as boosted ground states (with velocity \(v\)) in what follows. In particular, the unboosted ground state \(Q_v = 0\) is the unique (modulo symmetries) radial ground state solving (1.7) above. Finally, we observe that

\[
C_v = \frac{3}{2} ||Q_v||_2^{-1}, \tag{2.4}
\]

which follows from the fact that \(Q_v\) is an optimizer (2.2) and satisfy equation (2.1). In particular, the relation (2.4) shows that two different boosted ground states \(Q_v\) and \(\tilde{Q}_v\) with the same velocity \(v\) must satisfy \(||Q_v||_2 = ||\tilde{Q}_v||_2\).

We may reformulate (2.4) as follows. Let the energy functional

\[
E_v(u) = \frac{1}{2} \int \bar{u} Du + \frac{1}{2} \int \bar{u} (iv \cdot \nabla u) - \frac{1}{3} \int |u|^3, \tag{2.5}
\]

then by the standard Pohozaev identity

\[
E_v(Q_v) = 0. \tag{2.6}
\]

Using (2.4) and the sharp Gagliardo-Nirenberg interpolation inequality:

\[
E_v(u) \geq \frac{1}{2} \left( \int \bar{u} Du + \bar{u} (iv \cdot \nabla u) \right) \left( 1 - \frac{||u||_2}{||Q_v||_2} \right). \tag{2.7}
\]

From the previous paragraph we know that boosted ground states \(Q_v\) satisfying equation (2.1) exist. Now we prove the behavior of the boosted ground states.

**Step 1** Sign of the momentum. Let \(0 \leq |v| < 1\). We claim:

\[
v \cdot \int \bar{Q}_v(i\nabla Q_v) \leq 0. \tag{2.8}
\]

Indeed, assume on the contrary that \(v \cdot \int \bar{Q}_v(i\nabla Q_v) > 0\) holds. We define the reflected function \(\tilde{Q}_v(x) := Q_v(-x)\). Note that \(\int|\tilde{Q}_v|^2 = \int|Q_v|^2\) and \(v \cdot \int \bar{Q}_v(i\nabla Q_v) < 0\). Since the remaining terms in \(E_v(u)\) are invariant with respect to space reflections, we
find that $E_\nu(\tilde{Q}_v) < E_\nu(Q_x) = 0$. But $||\tilde{Q}_v||_2 = ||Q_v||_2$ implies $E_\nu(\tilde{Q}_v) \geq 0$ from (2.7), a contradiction. We conclude that (2.8) holds. In particular, by a suitable (possibly improper) rotation in $\mathbb{R}^2$, we can henceforth assume that

$$\nu = |\nu|e_1 = (|\nu|, 0) \in \mathbb{R}^2$$

points in (positive) $x_1$—direction. We can see

$$\int Q_v(i\tilde{\xi}_i Q_v) \leq 0 \text{ for } 0 < |\nu| < 1. \quad (2.9)$$

For the case $\nu = 0$, we recall that the fact from [15] that (after translation and shift by a complex constant phase) the functions $Q_{\nu=0}(x) = Q(|x|)$ are radial. Hence, in this special case, we have

$$\int Q_{\nu=0}(i\tilde{\xi}_i Q_{\nu=0}) = 0 \quad (2.10)$$

**Step 2** The mass is non-increasing. We claim the monotonicity:

$$||Q_{\nu_1}||_2 \leq ||Q_{\nu_2}||_2 \text{ for } 0 \leq |\nu_1| < |\nu_2| < 1, \quad (2.11)$$

where $\nu_j = |\nu_j|e_1 = (|\nu_j|, 0) \in \mathbb{R}^2$, $j = 1, 2$.

Note that this implies, in particular, the subcritical mass property:

$$||Q_{\nu_1}||_2 < ||Q||_2 \text{ for } 0 < |\nu| < 1.$$

Indeed, let $Q_{\nu_1}$ and $Q_{\nu_2}$ be two boosted ground states satisfying (2.1) with $\nu = \nu_1$ and $\nu = \nu_2$, respectively. Since $E_{\nu_1}(Q_{\nu_1}) = 0$ by (2.6), we find using (2.9), if $|\nu_1| > 0$ and (2.8) if $|\nu_1| = 0$, that

$$E_{\nu_2}(Q_{\nu_1}) = E_{\nu_1}(Q_{\nu_1}) + (\nu_2 - \nu_1) \cdot \int Q_{\nu_1}(i\tilde{\xi}_i Q_{\nu_1}),$$

since $\nu_2 - \nu_1 = (|\nu_2| - \nu_1)e_1 = (|\nu_2| - |\nu_1|)e_1, 0)$ and $(\nu_2 - \nu_1) \cdot \int Q_{\nu_1}(i\tilde{\xi}_i Q_{\nu_1}) = (|\nu_2| - \nu_1)e_1 \int Q_{\nu_1}(i\tilde{\xi}_i Q_{\nu_1}) \leq 0$. Hence $E_{\nu_2}(Q_{\nu_1}) \leq 0$, which together with (2.7) implies $||Q_{\nu_1}||_2 \geq ||Q_{\nu_2}||_2$. In the case of equality, $||Q_{\nu_1}||_2 = ||Q_{\nu_2}||_2$, $Q_{\nu_1}$ attains the minimization problem (2.3) with $\nu_2$. In particular, the function $Q_{\nu_1}$ satisfies the equation

$$DQ_{\nu_1} + \lambda Q_{\nu_1} + \nu_2 \cdot \nabla Q_{\nu_1} - |Q_{\nu_1}|Q_{\nu_1} = 0$$

with the Lagrange multiplier $\lambda \in \mathbb{R}$. On the other hand, by assumption, the boosted ground state $Q_{\nu_1}$ also satisfies equation (2.1) with $\nu = \nu_1$. By subtracting the equations satisfied by $Q_{\nu_1}$, we obtain that

$$(\lambda - 1)Q_{\nu_1} + (\nu_2 - \nu_1) \cdot \nabla Q_{\nu_1} = 0.$$

Since $\nu_1 \neq \nu_2$ by assumption and $Q_{\nu_1} \rightarrow 0$ as $|x| \rightarrow \infty$, we deduce from this equation that $Q_{\nu_1} \equiv 0$ holds, which is absurd.

**Step 3** Limits. We claim:

$$\begin{cases} ||Q_{\nu_1}||_2 \rightarrow ||Q||_2 \text{ as } |\nu| \rightarrow 0, \\ ||Q_{\nu_1}||_2 \rightarrow 0 \text{ as } |\nu| \rightarrow 1. \end{cases}$$

To show this, we argue as follows. From $|\xi| - \nu \cdot \xi \geq (1 - |\nu|)|\xi|$ for $\xi \in \mathbb{R}^2$ and Plancherel’s identity, we deduce that $C_{\nu} \leq (1 - |\nu|)^{-1}C_{\nu=0}$ for the optimal constants in
(2.2). From this simple bound and rescaling (2.4) and the monotonicity (2.11), we deduce that the bounds

\[
\sqrt{1 - |v|} ||Q||_2 \leq ||Q_v||_2 \leq ||Q||_2.
\]

Hence it follows that \( ||Q_v||_2 \to ||Q||_2 \) as \( v \to 0 \).

It remain to show \( ||Q_v||_2 \to 0 \) as \( |v| \to 1 \). To prove this, from [16], we know that for \( |v| < 1 \), we have the estimate

\[
C_v \sim (1 - |v|)^{-1} \quad \text{and} \quad ||Q_v||_2 \sim (1 - |v|)^2.
\]

Hence, we can easily obtain our result.

3. Approximate blowup profile

This section is devoted to the construction of the approximate blowup profile. For a sufficiently regular function \( f: \mathbb{R}^2 \to \mathbb{C} \), we define the generator of \( L^2 \) scaling given by

\[
\Lambda f := f + x \cdot \nabla f.
\]

Note that the operator \( \Lambda \) is skew-adjoint on \( L^2(\mathbb{R}^2) \), that is, we have

\[
(\Lambda f, g) = -(f, \Lambda g).
\]

We write \( \Lambda^k f \), with \( k \in \mathbb{N} \), for the iterates of \( \Lambda \) with the convention that \( \Lambda^0 f \equiv f \).

In some parts of this paper, it will be convenient to identify any complex-valued function \( f: \mathbb{R}^2 \to \mathbb{C} \) with the function \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) by setting

\[
f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \Re f \\ \Im f \end{bmatrix}.
\]

We also define

\[
f \cdot g = f_1 g_1 + f_2 g_2.
\]

Corresponding, we will identify the multiplication by \( i \) in \( \mathbb{C} \) with the multiplication by the real \( 2 \times 2 \)-matrix defined as

\[
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

We start with a general observation: If \( u = u(t,x) \) solves (1.1), then we define the function \( v = v(s,y) \) by setting

\[
u(t,x) = \frac{1}{\lambda(t)} v\left(s, \frac{x}{\lambda(t)}\right) e^{\Gamma(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}.
\]

(3.1)

It is easy to check that \( v = v(s,y) \) with \( y = \frac{x}{\lambda} \) satisfies

\[
iv - Dv - v + |v|^2 = i\frac{\lambda_y}{\lambda} \Lambda v + \tilde{\gamma} v,
\]

(3.2)

where we set \( \tilde{\gamma} = \gamma - 1 \). Here the operators \( D \) is understood as \( D = D_y \). Following the slow modulated ansatz strategy developed in [14, 22, 23], we freeze the modulation
\[ -\frac{\lambda_s}{\lambda} = a. \]  

(3.3)

And we look for an approximate solution of the form

\[ v(s, y) = Q_{\mathcal{P}(s)}(y), \mathcal{P}(s) = a(s), \]  

(3.4)

where

\[ Q_{\mathcal{P}}(y) = Q(y) + \sum_{k \geq 1} a^k R_k(y), \]

where \( \mathcal{P} = a \in \mathbb{R}. \)

We shall define ODE for \( a(s) \) of type

\[ a_s = P_1(a), \]

where \( P_1 \) are appropriate polynomials in \( a. \)

Using the heuristic asymptotic expansions

\[ \lambda(t) \sim t^2. \]

from \( \frac{ds}{dt} = -\frac{1}{\lambda(t)} \) we see that \( s = s_0 - 1/t \) goes to \( \infty \) as \( t \to 0 \) and \( t = 1/(s_0 - s) \sim -1/s \) as \( s \to +\infty. \) Moreover, the modulation relations (3.3) show that

\[ a(s) = -\frac{\lambda_s}{\lambda} \sim \frac{1}{s}. \]

These asymptotic expansions suggests to define \( a(s) \) so that

\[ a_s = -\frac{a^2}{2}. \]  

(3.5)

Moreover the asymptotic expansions for \( a(s) \) show that we can consider \( \mathcal{P} = a \) close to the origin with norm

\[ ||\mathcal{P}||^2 \sim a^2. \]

The terms \( R_k(y) \) is decomposed in real and imaginary parts as follows

\[ R_k(y) = T_k(y) + iS_k(y). \]

We adjust the modulation equation for \( a(s) \) to ensure the solvability of the obtained system, and a specific algebra leads to the laws to leading order:

\[ a_s = -\frac{a^2}{2}. \]

From (3.4) we have

\[ \partial_s v = -\frac{a^2}{2} \partial_a Q_{\mathcal{P}}. \]

Therefore, our purpose is to construct a high order approximation \( Q(y, a) = Q_{\mathcal{P}} \) that is solution to
where $\mathcal{P} = a$ is close to 0 and $\Phi_P$ is some small term of order $O(||\mathcal{P}||^2) = O(a^2)$.

We have the following result about an approximate blowup profile $Q_\mathcal{P}$, parameterized by $\mathcal{P} = a$, around the ground state $Q = [Q,0]^\top$.

**Lemma 3.1.** (Approximate Blowup Profile) Let $\mathcal{P} = a$. There exists a smooth function $Q_\mathcal{P} = Q_\mathcal{P}(x)$ of the form

$$Q_\mathcal{P} = Q + aR_1 + a^2R_2 + a^3R_3 + a^4R_4$$

that satisfies the equation

$$-i\frac{a^2}{2}\partial_a Q_\mathcal{P} - DQ_\mathcal{P} - Q_\mathcal{P} + ia\Lambda Q_\mathcal{P} + |Q_\mathcal{P}|Q_\mathcal{P} = -\Phi_\mathcal{P},$$

(3.6)

Here the functions $\{R_k\}_{0 \leq k \leq 4}$ satisfy the following regularity and decay bounds:

$$||R_k||_{H^m} + ||\Lambda R_k||_{H^m} + ||\Lambda^2 R_k||_{H^m} \leq 1, \text{ for } m \in \{0,1\},$$

(3.7)

$$|R_k| + |\Lambda R_k| + |\Lambda^2 R_k| \leq \langle x \rangle^{-3}, \text{ for } x \in \mathbb{R}^2.$$  

(3.8)

Moreover, the term on the right-hand side of (3.6) satisfies

$$||\Phi_\mathcal{P}||_{H^m} \leq O(a^5), |\nabla \Phi_\mathcal{P}| \leq O(a^5)\langle x \rangle^{-3},$$

(3.9)

for $m \in \{0,1\}$ and $x \in \mathbb{R}^2$.

**Proof.** We recall that the definition of the linear operator

$$L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}$$

acting on $L^2(\mathbb{R}^2,\mathbb{R}^2)$, where $L_+$ and $L_-$ denote the unbounded operators acting on $L^2(\mathbb{R}^2,\mathbb{R}^2)$ given by

$$L_+ = D + 1 - 2Q, \quad L_- = D + 1 - Q.$$

From [10] we have the key property that the kernel of $L$ is given by

$$\ker L = \text{span}\left\{\begin{bmatrix} \partial_{x_1}Q, \partial_{x_2}Q \\ (0,0) \end{bmatrix}, \begin{bmatrix} 0 \\ Q \end{bmatrix}\right\}.$$  

Note that the bounded inverse $L^{-1} = \text{diag}\{L_+^{-1}, L_-^{-1}\}$ exists on the orthogonal complement $\ker L^{-1} = \{(\partial_{x_1}Q, \partial_{x_2}Q)\} \perp \{Q\} \perp$.

**Step 1** Determining the functions $R_k$. We discuss our ansatz for $Q_\mathcal{P}$ to solve (3.6) order by order. The proof of the regularity and decay bounds for the functions $R_k$ will be given further below.

**Order $O(1)$**: Clearly, we have that

$$DQ + Q - |Q|Q = 0.$$  

Since $Q = [Q,0]^\top$, with $Q = Q(|x|) > 0$ being the ground state solution.
Order $O(a)$: We note that
\[ |Q_\mu|Q_\mu = Q^2 + 2aQ^3(R_1) + aQ^3(R_1) + O(a^3). \]

Hence, we obtain the equation
\[ LR_1 = J\Lambda Q. \]

Note that $J\Lambda Q = [0, \Lambda Q]^T$ satisfies $J\Lambda Q \perp \ker L$ due to the fact that $(\Lambda Q, Q) = 0$, which can be easily seen by using the $L^2$-criticality. Hence we can find a unique solution $R_1 \perp \ker L$ to the equation above. In what follows, we denote
\[ R_1 = L^{-1}J\Lambda Q = \begin{bmatrix} 0 \\ L^{-1}\Lambda Q \end{bmatrix}. \]

Order $O(a^2)$:
\[
|Q + aR_1 + a^2R_2|(Q + aR_1 + a^2R_2) \\
= Q \left( 1 + \frac{a^2(R_2 + \bar{R}_2)}{Q} + \frac{a^2|R_1|^2}{Q^2} \right)^{1/2} (Q + aR_1 + a^2R_2) \\
= QQ + 2a^2Q^3(R_2) + a^2Q^3(R_2) + \frac{a^2}{2}|R_1|^2Q^{-1}Q + O(a^3).
\]

We find the equation
\[ LR_2 = -\frac{1}{2}/R_1 + J\Lambda R_1 + \frac{1}{2}|R_1|^2Q^{-1}Q. \]

Since $R_1 = [0, S_1]^T$ with $L^2S_1 = \Lambda Q$, the solvability condition for $R_2$ reduces to
\[ \frac{1}{2}(\nabla Q, S_1) - (\nabla Q, \Lambda S_1) + \frac{1}{2}(\nabla Q, S_1^2) = 0. \]

However, this is true, since $S_1$ and $Q$ are the radial functions. Thus there exists a unique solution $R_2 \perp \ker L$, which is given by
\[ R_2 = \begin{bmatrix} L^{-1} \left( \frac{1}{2}S_1 + \Lambda S_1 - \frac{1}{2}|S_1|^2 \right) \\ 0 \end{bmatrix}. \]

Order $O(a^3)$:
\[
|Q + aR_1 + a^2R_2 + a^3R_3|(Q + aR_1 + a^2R_2 + a^3R_3) \\
= Q \left( 1 + \frac{a(R_1 + \bar{R}_1)}{Q} + \frac{a^2(R_2 + \bar{R}_2)}{Q} + \frac{a^3(R_3 + \bar{R}_3)}{Q} + \frac{a^3(R_1 \cdot \bar{R}_2 + \bar{R}_1 \cdot R_2)}{Q^2} \right)^{1/2} (Q + aR_1 + a^2R_2 + a^3R_3) \\
= Q \left( 1 + \frac{2a^2R_2}{Q} + \frac{a^3(R_3 + \bar{R}_3)}{Q} + \frac{a^2|R_1|^2}{Q^2} \right)^{1/2} (Q + aR_1 + a^2R_2 + a^3R_3) \\
= QQ + 2a^3Q^3(R_3) + a^3Q^3(R_3) + a^3Q^3(R_2)R_1 + \frac{a^3}{2}Q^{-1}|R_1|^2R_1 + O(a^4). \]
We notice that $\mathbf{R}_1 \cdot \mathbf{R}_2 = 0$ and we obtain the equation
\[
L\mathbf{R}_3 = -J\mathbf{R}_2 + J\Lambda \mathbf{R}_2 + \mathcal{R}(\mathbf{R}_2)\mathbf{R}_1 + \frac{1}{2} Q^{-1}|\mathbf{R}_1|^2\mathbf{R}_1.
\] (3.10)

Note that the right side is of the form $[0,f]^\top$ with some nontrivial $f$. Hence the solvability condition for $\mathbf{R}_3$ is equivalent to
\[
-(Q, T_2) + (Q, \Lambda T_2) + (Q, T_2 S_1) + \frac{1}{2} (Q, Q^{-1}S_1^2 S_1) = 0,
\] (3.11)
where the functions $S_1$ and $T_2$ satisfy
\[
L_- S_1 = \Lambda Q, \quad L_+ T_2 = \frac{1}{2} S_1 - \Lambda S_1 + \frac{1}{2} |S_1|^2.
\] (3.12)

To see that (3.11) holds, we first note that

Left-hand side of (3.11)
\[
= -(Q, T_2) - (\Lambda Q, T_2) + (Q, T_2 S_1) + \frac{1}{2} (Q, Q^{-1}S_1^2 S_1)
= -(Q, T_2) - (L_- S_1, T_2) + (Q, T_2 S_1) + \frac{1}{2} (Q, Q^{-1}S_1^2 S_1)
= -(Q, T_2) - (L_+ S_1, T_2) + \frac{1}{2} (Q, Q^{-1}S_1^2 S_1)
= -(Q, T_2) - \frac{1}{2} (S_1, S_1) + (S_1, \Lambda S_1) - \frac{1}{2} (S_1, S_1) + \frac{1}{2} (Q, Q^{-1}S_1^2 S_1)
= -(Q, T_2) - \frac{1}{2} (S_1, S_1),
\]

where in the last step we used that $(S_1, \Lambda S_1) = 0$, since $\Lambda^* = -\Lambda$. Thus it remains to show that
\[
-(Q, T_2) = \frac{1}{2} (S_1, S_1).
\] (3.13)

Indeed, by using $L_+ \Lambda Q = -Q$ and the equations for $T_2$ and $S_1$ above, we deduce
\[
-(Q, T_2) = \left( \Lambda Q, \frac{1}{2} S_1 - \Lambda S_1 + \frac{1}{2} |S_1|^2 \right)
= \frac{1}{2} (L_- S_1, S_1) - (L_- S_1, \Lambda S_1) + \frac{1}{2} (\Lambda Q, S_1^2)
= \frac{1}{2} (S_1, D S_1) + \frac{1}{2} (S_1, S_1) - \frac{1}{2} (S_1, Q S_1)
= -(L_- S_1, \Lambda S_1) + \frac{1}{2} (\Lambda Q, S_1^2).
\] (3.14)

Next, we apply the commutator formula $(L_- f, \Lambda f) = \frac{1}{2} (f, [L_-, \Lambda] f)$, which shows that
\[
(L_- S_1, \Lambda S_1) = \frac{1}{2} (S_1, [L_- , \Lambda] S_1) = \frac{1}{2} (S_1, [D, \Lambda] S_1) - \frac{1}{2} (S_1, [Q, \Lambda] S_1)
= \frac{1}{2} (S_1, D S_1) + \frac{1}{2} (S_1, (x \cdot \nabla Q) S_1),
\]
where we use the commutator \([D, \mathbf{x} \cdot \nabla] = D\) and \([Q, \Lambda] = -\mathbf{x} \cdot \nabla Q\) holds. Furthermore, we have the pointwise identity
\[ -\mathbf{x} \cdot \nabla Q + \Lambda Q = Q. \]

If now we insert the above two equalities into (3.14), we obtain the desired relation (3.13), and thus the solvability condition (3.11) holds. Note that \(R_3 = [0, S_3]^T\) with some radial function \(S_3\).

**Order \(O(a^4)\):**
\[
|Q + aR_1 + a^2R_2 + a^3R_3 + a^4R_4| = Q\left(1 + \frac{a(R_1 + \bar{R}_1)}{Q} + \frac{a^2(R_2 + \bar{R}_2)}{Q} + \frac{a^3(R_3 + \bar{R}_3)}{Q} + \frac{a^4(R_4 + \bar{R}_4)}{Q} + \frac{a^4(R_1 \cdot R_3 + R_1 \cdot \bar{R}_3)}{Q^2} + \frac{a^3(R_1 \cdot R_2 + R_1 \cdot \bar{R}_2)}{Q^2} + \frac{a^2|R_1|^2}{Q^2} + \frac{a^4|R_2|^2}{Q^2}\right)^{1/2}
\]
\[
(Q + aR_1 + a^2R_2 + a^3R_3 + a^4R_4)
\]
\[
= Q\left(1 + 2\left(\frac{2a^2|\mathbf{R}_2|}{Q} + \frac{a^4(R_4 + \bar{R}_4)}{Q} + \frac{2a^4(R_1 \cdot \bar{R}_3)}{Q} + \frac{2a^4|R_1|^2}{Q^2} + \frac{2a^4|R_2|^2}{Q^2}\right) - \frac{1}{8}\left(\frac{2a^2|\mathbf{R}_2|}{Q} + \frac{a^2|R_1|^2}{Q^2}\right)^2\right)\left(Q + aR_1 + a^2R_2 + a^3R_3 + a^4R_4\right)
\]
\[
= QQ + 2a^4Q\mathbf{R}_4 + a^4Q\mathbf{R}_4 + a^4(R_1 \cdot \bar{R}_3)Q^{-1}Q + \frac{a^4}{2}|R_2|^2Q^{-1}Q + a^4|R_2|^2Q^{-1}Q - \frac{a^4}{2}|R_2|^2Q^{-1}Q - \frac{a^4}{8}|R_1|^4Q - \frac{1}{2}a^4|R_2|^2Q^2Q + O(a^5)
\]
\[
= QQ + 2a^4Q\mathbf{R}_4 + a^4Q\mathbf{R}_4 + a^4(R_1 \cdot \bar{R}_3)Q^{-1}Q + a^4|R_2|^2Q^{-1}Q - \frac{a^4}{8}|R_1|^4Q - \frac{1}{2}a^4|R_2|^2Q^2Q + O(a^5).
\]

We obtain the equation
\[
LR_4 = -\frac{3}{2}fR_3 + f\Lambda R_3 + +a^4(R_1 \cdot \bar{R}_3)Q^{-1}Q + |R_2|^2Q^{-1}Q (3.15)
\]
where we use the fact that \(R_1 \cdot Q = R_3 \cdot Q = 0\). Moreover, we easily see that

Right-hand side of \((3.15) \perp \ker L\),

since the right-hand side of (3.15) is radial and \((F, \nabla Q) = 0\) for any radial function \(F \in L^2(\mathbb{R}^2)\). Hence there is a unique solution \(R_4 \perp \ker L\) of equation (3.15), and we have that \(R_4 = [T_4, 0]^T\) holds with some radial function \(T_4\).

**Step 2** Regularity and decay bounds. Let \(m \in \{0, 1\}\) be given. First, we recall that \(|Q|_{H^m} \leq 1\) and \(|Q(x)| \leq |x|^{-3}\) holds. Since, moreover, \(L_+ \Lambda Q = -Q\) and \((\Lambda Q, Q) = 0\), we can apply Lemma A.1 to conclude that
Next, by applying $\Lambda$ to the equation $L_+ \Lambda Q = -Q$ and using that $[L_+, \Lambda] = D + 2x \cdot \nabla Q$, we deduce that

$$L_+ \{-\Lambda^2 Q + \Lambda Q + \alpha Q\} = -(Q^2 + x \cdot \nabla Q) \Lambda Q - (1 - 2\alpha)Q^2.$$

for $\alpha \in \mathbb{R}$.

Using the previous bounds for $Q$ and $\Lambda Q$ (and hence for $x \cdot \nabla Q$) as well, we can apply Lemma A.1 again to obtain the bounds

$$||\Lambda^2 Q||_{H^m} \leq 1, |\Lambda^2 Q(x)| \leq \langle x \rangle^{-3}.$$

Having these bounds for $Q = [Q, 0]^T, \Lambda Q = [\Lambda Q, 0]^T$, and $\Lambda^2 Q = [\Lambda^2 Q, 0]^T$ at hand, we can now prove the claimed bound (3.7) and (3.8) by iterating the equations satisfied by the functions $\{R_k\}_{0 \leq k \leq 4}$ above. For instance, recall that $R_1 = [0, S_1]^T$ with $L_- S_1 = \Lambda Q$ and hence $\Lambda L_- S_1 = \Lambda^2 Q$. Then, by using the commutator $[L_+, \Lambda]$ and the previous estimates for $\{Q, \Lambda Q, \Lambda^2 Q\}$, we derive that

$$||\Lambda^k S_1||_{H^m} \leq 1, |\Lambda^k S_1(x)| \leq \langle x \rangle^{-3}, \text{ for } k = 0, 1, 2 \text{ and } m \in \{0, 1\}.$$

Using this and proceeding in the above, we deduce that (3.7) and (3.8) holds.

Finally, we mention that the bounds (3.9) for the error term $\Phi_P$ follow from expanding $|Q_P|Q_P$ and using the regularity and decay bounds for the functions $\{R_k\}$. We omit the straightforward details. The proof of Lemma 3.1 is now complete. \qed

Remark 1. Note that $L_- > 0$ on $Q^\perp$ and we have $S_1 \perp Q$.

Remark 2. The proof of Lemma 3.1 will actually show that the functions $\{R_k\}$ have the following symmetry structure

$$R_1 = \begin{bmatrix} 0 \\ \text{symmetry} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \text{symmetry} \\ 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 \\ \text{symmetry} \end{bmatrix}, \quad R_4 = \begin{bmatrix} \text{symmetry} \\ 0 \end{bmatrix}.$$

These symmetry properties will be of essential use in the sequel.

We now turn to some key properties of the approximate blowup profile $Q_P$ constructed in Lemma 3.1.

Lemma 3.2. The mass and the energy of $Q_P$ satisfy

$$\int |Q_P|^2 = \int Q^2 + O(a^4),$$

$$E(Q_P) = e_1 a^2 + O(a^4).$$

Here $e_1 > 0$ is the positive constant given by

$$e_1 = \frac{1}{2} (L_- S_1, S_1) > 0,$$

where $S_1$ satisfy $L_- S_1 = \Lambda Q.$
Proof. From the proof of Lemma 3.1, we recall that the facts that \( R_1 = [0, S_1] \) is the radial symmetry function. Hence we have \( \int Q \cdot R_1 = 0 \). Next, we recall that \( R_2 = [T_2, 0] \) satisfies \( (Q', S_1) + 2(Q, T_2) = 0 \), shown in (3.13) above. In summary, we see that

\[
\int |Q_p|^2 = \int Q^2 + O(a^4).
\]

To treat the expansion of the energy, we first recall that \( E(Q) = 0 \) and \( DQ + Q - Q^2 = 0 \) and \( E'(Q) = -Q \). Moreover, since we have \( (Q, S_1) = 0 \), we obtain

\[
E(Q_p) = \frac{1}{2}(Q_p, DQ) - \frac{1}{3}(Q_p, |Q_p|Q_p) = \frac{1}{2}(Q, DQ) - \frac{1}{3}(Q, Q^2) + a^2\left\{ \frac{1}{2}(S_1, DS_1) + (T_2, DQ) \right\}
- (T_2, Q^2) - \frac{1}{2}(S_1^2, Q) + O(a^4)
= a^2\left\{ \frac{1}{2}(S_1, DS_1) + (T_2, DQ) \right\}
+ a^2\left\{ -(T_2, Q^2) - \frac{1}{2}(S_1^2, Q) \right\} + O(a^4).
\]

Using \( DQ + Q - Q^2 = 0 \) and (3.14), we obtain that

\[
E(Q_p) = a^2\left\{ \frac{1}{2}(S_1, DS_1) - (T_2, Q) - \frac{1}{2}(S_1^2, Q) \right\} + O(a^4)
= a^2\left\{ \frac{1}{2}(S_1, DS_1) + \frac{1}{2}(L_-S_1, S_1) - (L_-S_1, AS_1) + \frac{1}{2}(\Lambda Q, S_1^2) \right\}
- \frac{1}{2}(S_1^2, Q) + O(a^4)
= a^2\left( \frac{L_-S_1, S_1}{2} + a^2\left( 1 - \frac{1}{2} \right) (S_1^2, Q) + O(a^4) \right)
= a^2\left( \frac{L_-S_1, S_1}{2} + O(a^4) \right)
= a^2e_1 + O(a^4).
\]

The proof of Lemma 3.2 is now complete. □

4. Modulation estimates

We start with a general observation: If \( u = u(t, x) \) solves (1.1), then we define the function \( v = v(s, y) \) by setting

\[
u(t, x) = \frac{1}{\lambda(t)} v\left( s, \frac{x}{\lambda(t)} \right) e^{iy(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)},
\]

here \( s = s(t) \). It is easy to check that \( v = v(s, y) \) with \( y = \frac{x}{\lambda(t)} \) satisfies
\[
i\partial_s v - Dv - v + |v|v = i\frac{\lambda_s}{\lambda} \Lambda v + \tilde{\gamma}_s v,
\]
where we set \(\tilde{\gamma}_s = \gamma_s - 1\). Here the operators \(D\) is understood as \(D = D_r\).

### 4.1. Geometrical decomposition and modulation equations

Let \(u\) be a solution of (1.1) on some time interval \([t_0, t_1]\) with \(t_1 < 0\). Assume that \(u(t)\) admits a geometrical decomposition of the form

\[
u(t, x) = \frac{1}{\lambda(t)} \left[ Q_{P(t)} + \epsilon \right] \left( s, \frac{x}{\lambda(t)} \right) e^{i\varphi(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)},
\]

with \(P(t) = a(t)\), and we impose the uniform smallness bound

\[
a^2(t) + ||\epsilon||_{L^2}^2 \ll 1.
\]

Furthermore, we assume that \(u(t)\) has almost critical mass in the sense that

\[
\int |u(t)|^2 - \int Q^2 \lesssim \lambda^2(t), \quad \forall t \in [t_0, t_1].
\]

To fix the modulation parameters \(\{a(t), \lambda(t), \gamma(t)\}\) uniquely, we impose the following orthogonality conditions on \(\epsilon = \epsilon_1 + i\epsilon_2\) as follows:

\[
\begin{align*}
(\epsilon_2, \Lambda Q_{1P}) - (\epsilon_1, \Lambda Q_{2P}) &= 0, \\
(\epsilon_2, \partial_a Q_{1P}) - (\epsilon_1, \partial_a Q_{2P}) &= 0, \\
(\epsilon_2, \rho_1) + (\epsilon_1, \rho_2) &= 0,
\end{align*}
\]

the function \(\rho = \rho_1 + i\rho_2\) is defined by

\[
L_+ \rho_1 = S_1, \quad L_- \rho_2 = aS_1\rho_1 + a\Lambda \rho_1 - 2aT_2,
\]

where \(S_1, T_2\) are the functions introduced in the proof of Lemma 3.1. Note that \(L_{-1}^2\) exists on \(L^2_{rad}(\mathbb{R}^2)\) and thus \(\rho_1\) is well-defined. Moreover, it is easy to see that the right-hand side in the equation for \(\rho_2\) is orthogonality to \(Q\). Indeed

\[
(Q, S_1\rho_1 + \Lambda \rho_1 - 2T_2) = (QS_1, \rho_1) - (\Lambda Q, \rho_1) - 2(Q, T_2) = (QS_1, \rho_1) - (S_1, L_- \rho_1) + (S_1, S_1) = -(S_1, L_+ \rho_1) + (S_1, S_1) = 0,
\]

using that \((S_1, S_1) = -2(T_2, Q)\), see (3.13), and the definition of \(\rho_1\). Hence \(\rho_2\) is well-defined.

In the conditions (4.4), we use the notation

\[
Q_P = Q_{1P} + iQ_{2P},
\]

which (in terms of the vector notation used in Section 3) means that

\[
Q_P = \begin{bmatrix} Q_{1P} \\ Q_{2P} \end{bmatrix}.
\]

We refer to Appendix B.1 for some standard arguments, which show that the orthogonality condition (4.4) imply that the modulation parameters \(\{a(t), \lambda(t), \gamma(t)\}\) are
Lemma 4.1. For $t \in [t_0, t_1]$ with $t_1 < 0$, it holds that
\[ a^2 + \|\epsilon\|_{H^{1/2}}^2 \leq \lambda |E_0| + O(\lambda^2 + a^4). \]

Here $E_0 = E(u_0)$ denote the conserved energy of $u = u(t, x)$.

Proof. By the conservation of $L^2$-mass and Lemma 3.2, we find that
\[ \int |u|^2 = \int |Q_p + \epsilon|^2 = \int |Q|^2 + 2\Re(Q, Q_p) + \int |\epsilon|^2 + O(a^4). \]
By assumption (4.3), this implies
\[
2\Re(\epsilon, Q_P) + \int |\epsilon|^2 = \mathcal{O}(\lambda^2 + a^4).
\] (4.10)

Next, we recall that \( v = Q_P + \epsilon \) and the assumed form of \( u = u(t,x) \). Hence, by energy conservation and scaling, we obtain
\[
E(v) = \lambda E(u_0).
\] (4.11)

On the other hand, from Lemma 3.2 and by expanding the energy functional
\[
E(v) = E(Q_P + \epsilon)
\]
\[
= \frac{1}{2} (Q_P + \epsilon, D(Q_P + \epsilon)) - \frac{1}{3} (Q_P + \epsilon, |Q_P + \epsilon|(Q_P + \epsilon))
\]
\[
= \frac{1}{2} \{(Q_P, DQ_P) + (\epsilon, D\epsilon)\} + \Re(\epsilon, DQ_P)
\]
\[
- \frac{1}{3} \{(Q_P, |Q_P + \epsilon|Q_P) + 2(\epsilon, |Q_P + \epsilon|Q_P) + (\epsilon, |Q_P + \epsilon|\epsilon)\}
\]
\[
= E(Q_P) + \frac{1}{2} (\epsilon, D\epsilon) + \Re(\epsilon, DQ_P - |Q_P|Q_P)
\]
\[
- \frac{1}{2} \int |Q_P|\epsilon^2 + \frac{1}{8} \int |Q_P|^{-1}(2Q_1\epsilon_1 + 2Q_2\epsilon_2)^2
\]
\[
+ \mathcal{O}(||\epsilon||_{H^{1/2}}^3 + a^2 ||\epsilon||_{H^{1/2}}^2).
\]

Combining the above equality and (4.10), (4.11) we find that
\[
\lambda E_0 = a^2 \epsilon_1 + \Re(\epsilon, DQ_P + Q_P - |Q_P|Q_P) + \frac{1}{2} \{(M_+\epsilon_1, \epsilon_1) + (M_-\epsilon_2, \epsilon_2)\}
\]
\[
+ \mathcal{O}(||\epsilon||_{H^{1/2}}^3 + a^2 ||\epsilon||_{H^{1/2}}^2 + a^4),
\]
where \( \epsilon_1 = \frac{1}{2}(L_-S_1, S_1) > 0 \). In the previous equation, we note that the linear term in \( \epsilon = \epsilon_1 + i\epsilon_2 \) satisfies
\[
\Re(\epsilon, DQ_P + Q_P - |Q_P|Q_P) = \mathfrak{R}(\epsilon, a^2 \partial_a Q_P - a\Lambda Q_P) + \mathcal{O}(a^4)
\]
\[
= \mathcal{O}(a^4),
\]
thanks to the orthogonality condition (4.4). Next, we observe that quadratic form \( M = (M_+, M_-) \) is a small deformation of the quadratic form given by the linearization \( L = (L_+, L_-) \) around \( Q \). Hence, we deduce
\[
a^2 \epsilon_1 + \frac{1}{2} \{(L_+\epsilon_1, \epsilon_1) + (L_-\epsilon_2, \epsilon_2)\}
\]
\[
= \lambda E_0 + \mathcal{O}(||\epsilon||_{H^{1/2}}^3 + a^4) + o(||\epsilon||_{H^{1/2}}^2).
\] (4.12)

Next, we recall from Lemma C.4 the coercivity estimate
\[
(L_+\epsilon_1, \epsilon_1) + (L_-\epsilon_2, \epsilon_2)
\]
\[
\geq c_0||\epsilon||_{H^{1/2}}^2 - \frac{1}{c_0} \{(\epsilon_1, Q)^2 + (\epsilon_1, S_1)^2 + (\epsilon_2, p_1)^2\},
\] (4.13)
with some universal constant \( c_0 > 0 \). Note that the orthogonality condition (4.4) imply that

\[
(e_1, S_1)^2 = \mathcal{O}(P\|e\|_2^2), (e_2, \rho_1)^2 = \mathcal{O}(P\|e\|_2^2).
\]

Furthermore, from the relation (4.10) we deduce that

\[
|(e_1, Q)|^2 = o(||e||_2^2) + \mathcal{O}(\lambda^2 + a^4).
\]

Combining these bounds with (4.13) and the universal smallness assumption for \( P \) and \( ||e||_{H^{1/2}} \), we obtain that

\[
(L_+ e_1, e_1) + (L_- e_2, e_2) \geq \frac{c_0}{2} ||e||_{H^{1/2}}^2 + \mathcal{O}(a^4).
\]

Inserting this bound into (4.12) and recall that \( e_1 = \frac{1}{2} (L_- S_1, S_1) > 0 \) holds, we have

\[
a^2 + ||e||^2_{H^{1/2}} \leq \lambda E_0 + \mathcal{O}(\lambda^2 + a^4).
\]

(4.14)

We complete the proof of this lemma.

\[\square\]

### 4.2. Modulation estimates

We continue with estimating the modulation parameters. To this end, we define the vector-valued function

\[
\text{Mod}(t) := \left( a_s + \frac{1}{2} a^2, \tilde{\gamma}_s, \frac{\lambda_s}{\lambda} \right).
\]

(4.15)

We have the following result.

**Lemma 4.2.** For \( t \in [t_0, t_1] \) with \( t_1 < 0 \), we have the bound

\[
|\text{Mod}(t)| \leq \lambda^2 + a^4 + a^2 ||e||_2 + ||e||_2^2 + ||e||_{H^{1/2}}^3.
\]

(4.16)

Furthermore, we have the improved bound

\[
\left| \frac{\lambda_s}{\lambda} + a \right| \leq a^5 + a^2 ||e||_2 + ||e||_2^2 + ||e||_{H^{1/2}}^3.
\]

**Proof.** We divide the proof into the following steps, where we also make use of the estimate (B.1)-(B.3), which are shown in Lemma B.1. Now, we recall that

\[
\Lambda Q_{1P} = \Lambda Q + \mathcal{O}(P^2), \quad \Lambda Q_{2P} = a\Lambda S_1 + \mathcal{O}(P^2),
\]

\[
\partial_a Q_{1P} = 2a T_2 + \mathcal{O}(P^2), \quad \partial_a Q_{2P} = S_1 + \mathcal{O}(P^2).
\]

**Step 1 Law for \( a \).** We multiply both sides of the equation (4.6) and (4.7) by \(-\Lambda Q_{2P}\) and \(\Lambda Q_{1P}\), respectively. Adding this and using (B.1) yields, after some calculation (also using the condition (4.4)), we have
We deduce that
\[
\left( a_s + \frac{1}{2} a^2 \right) \left( \partial_a Q_{1P} - \Lambda Q_{2P} \right) + (\partial_a Q_{2P}, \Lambda Q_{1P}) = \left( \partial_a e_1, -\Lambda Q_{2P} \right) + (\partial_a e_2, \Lambda Q_{1P}) - \Re(\epsilon, Q_P)
\]
\[
= \left( \frac{\bar{\lambda}_s}{\lambda} + a \right) \left[ (\Lambda Q_{1P} + \Lambda e_1, -\Lambda Q_{2P} + (\Lambda Q_{2P} + \Lambda e_2, \Lambda Q_{1P}) \right]
\]
\[
+ \tilde{\gamma}_1 [(Q_{2P} + e_2, -\Lambda Q_{2P}) - (Q_{1P} + e_1, \Lambda Q_{1P})] + (R_2(\epsilon), \Lambda Q_{2P}) + (R_1(\epsilon), \Lambda Q_{1P})
\]
\[
- (\Im(\Phi_P), \Lambda Q_{2P}) + (\Re(\Phi_P), \Lambda Q_{1P}) + O(P^2 ||\epsilon||_2).
\]

Hence, we obtain that
\[
- \left( a_s + \frac{1}{2} a^2 \right) \left[ (L_S, S_1) + O(P^2) \right]
\]
\[
= \Re(\epsilon, Q_P) + (R_2(\epsilon), \Lambda Q_{2P}) + (R_1(\epsilon), \Lambda Q_{1P}) - (\Im(\Phi_P), \Lambda Q_{2P}) + (\Re(\Phi_P), \Lambda Q_{1P})
\]
\[
+ O\left( (P^2 + \mod(t))(||\epsilon||_2 + P^2) \right).
\]

Next, from the Lemma 3.2 the constants
\[
e_1 = \frac{1}{2} (L_S, S_1) > 0.
\]

and
\[
2\Re(\epsilon, Q_P) = - \int |\epsilon|^2 + \left( \int |u|^2 - \int |Q|^2 \right) + O(a^4).
\]

We deduce that
\[
- \left( a_s + \frac{1}{2} a^2 \right) \left[ 2\epsilon_1 + O(P^2) \right]
\]
\[
= - \int |\epsilon|^2 + (R_2(\epsilon), \Lambda Q_{2P}) + (R_1(\epsilon), \Lambda Q_{1P})
\]
\[
+ O\left( (P^2 + \mod(t))(||\epsilon||_2 + P^2) + ||u||^2 - ||Q||^2 + a^4 \right).
\]

**Step 2 Law for \( \lambda \).** We multiply both sides of the equation (4.6) and (4.7) by \(-\partial_a Q_{2P}\) and \(\partial_a Q_{1P}\), respectively. After some calculation (also using the condition (4.4)), we have
\[
\left( a_s + \frac{1}{2} a^2 \right) \left[ (\partial_a Q_{1P} - \partial_a Q_{2P}) + (\partial_a Q_{2P}, \partial_a Q_{1P}) \right] + [(\partial_a e_1, -\partial_a Q_{2P}) + (\partial_a e_2, \partial_a Q_{1P})]
\]
\[
= \left( \frac{\bar{\lambda}_s}{\lambda} + a \right) \left[ (\Lambda Q_{1P} + \Lambda e_1, -\partial_a Q_{2P} + (\Lambda Q_{2P} + \Lambda e_2, \partial_a Q_{1P}) \right]
\]
\[
+ \tilde{\gamma}_1 [(Q_{2P} + e_2, -\partial_a Q_{2P}) - (Q_{1P} + e_1, \partial_a Q_{1P})] + (R_2(\epsilon), \partial_a Q_{2P})
\]
\[
+ (R_1(\epsilon), \partial_a Q_{1P}) - (\Im(\Phi_P), \partial_a Q_{2P})
\]
\[
+ (\Re(\Phi_P), \partial_a Q_{1P}) + O(P^2 ||\epsilon||_2).
\]
Hence, we deduce
\[
0 = \left(\frac{\lambda_s}{\lambda} + a\right) \left([(\Lambda Q, -S_1) + \mathcal{O}(\mathcal{P}^2)] + (R_2(e), \partial_a Q_{2P}) + (R_1(e), \partial_a Q_{1P}) + \mathcal{O}\left((\mathcal{P}^2 + |\text{mod}(t)|)(||e||_2 + \mathcal{P}^2)\right)\right).
\]

Therefore, we obtain
\[
\left(\frac{\lambda_s}{\lambda} + a\right) [2e_1 + \mathcal{O}(\mathcal{P}^2)] = (R_2(e), \partial_a Q_{2P}) + (R_1(e), \partial_a Q_{1P}) + \mathcal{O}\left((\mathcal{P}^2 + |\text{mod}(t)|)(||e||_2 + \mathcal{P}^2) + a^5\right).
\]

Furthermore, by this estimate, we deduce the improved bound for $|\frac{\lambda_s}{\lambda} + a| > 0$.

**Step 3 Law for $\tilde{\gamma}_s$.** We multiply both sides of the equation (4.6) and (4.7) by $-P_2$ and $P_1$, respectively. Adding this and using (B.3) yields, after some calculation, we have
\[
\left(a_s + \frac{1}{2}a^2\right) [(\partial_a Q_{1P} - \partial_a Q_{2P}, P_1) + [(\partial_a e_1, \partial_a P_1) + (\partial_a e_2, P_1)]
\]
\[
= \left(\frac{\lambda_s}{\lambda} + a\right) [(\Lambda Q, P_2) + (a\Lambda, P_1) + \mathcal{O}(\mathcal{P}^2)]
\]
\[
+ (R_2(e), P_2) + (R_1(e), P_1) - (\mathcal{S}(\Phi_P), P_2)
\]
\[
+ (\mathcal{R}(\Phi_P, P_1) + \mathcal{O}(\mathcal{P}^2||e||_2).
\]

Hence, we have
\[
\tilde{\gamma}_s[(Q, P_1) + \mathcal{O}(\mathcal{P}^2)]
\]
\[
= -\left(a_s + \frac{1}{2}a^2\right) [(S_1, P_1) + \mathcal{O}(\mathcal{P}^2)] + \left(\frac{\lambda_s}{\lambda} + a\right) [(\Lambda Q, P_2) + (a\Lambda S_1, P_1) + \mathcal{O}(\mathcal{P}^2)]
\]
\[
+ (R_2(e), P_2) + (R_1(e), P_1) - (\mathcal{S}(\Phi_P), P_2)
\]
\[
+ (\mathcal{R}(\Phi_P, P_1) + \mathcal{O}(\mathcal{P}^2||e||_2)
\]
\[
= -\left(a_s + \frac{1}{2}a^2\right) [(S_1, P_1) + \mathcal{O}(\mathcal{P}^2)] + \left(\frac{\lambda_s}{\lambda} + a\right) \mathcal{O}(\mathcal{P})
\]
\[
+ (R_2(e), P_2) + (R_1(e), P_1) + \mathcal{O}\left((\mathcal{P}^2 + |\text{mod}(t)|)(||e||_2 + \mathcal{P}^2) + a^5\right).
\]

Here we used the definition of $\rho$.

**Step 4 Conclusion.** We collect the previous equation and estimate the nonlinear terms in $\epsilon$ by Sobolev inequalities. This gives us
\[
(A + B)\text{mod}(t) = \mathcal{O}\left((\mathcal{P}^2 + |\text{mod}(t)|)(||e||_2^1 + ||e||_2^2 + ||e||_2^3 + ||e||_2^{3/2})
\]
\[
+ ||\epsilon||_2^2 - ||Q||_2^2 + a^4\right).
\]

Here $A = \mathcal{O}(1)$ is invertible $3 \times 3$-matrix, and $B = \mathcal{O}(\mathcal{P})$ is some $3 \times 3$-matrix that is polynomial in $\mathcal{P} = a$. For $|\mathcal{P}| \ll 1$, we can thus invert $A + B$ by Taylor expansion and derive the estimate for $\text{mod}(t)$ stated in this lemma.

\[\square\]


5. Refined energy bounds

In this section, we establish a refined energy estimate, which will be a key ingredient in the compactness argument to construct ground state mass blowup solutions.

Let $u = u(t, x)$ be a solution (1.1) on the time interval $[t_0, 0)$ and suppose that $w$ is an approximate solution to (1.1) such that

$$i\tilde{Q}_t - D\tilde{Q} + |\tilde{Q}|\tilde{Q} = \psi,$$

with the priori bounds

$$||\tilde{Q}||_2 \leq 1, ||D^{\frac{1}{2}}\tilde{Q}||_2 \leq \lambda^{-\frac{1}{2}}, ||\nabla\tilde{Q}||_2 \leq \lambda^{-1}. \tag{5.2}$$

We decompose $u = \tilde{Q} + \tilde{e}$, and hence $\tilde{e}$ satisfies

$$i\tilde{e}_t - D\tilde{e} + (|u|u - |\tilde{Q}|\tilde{Q}) = -\psi, \tag{5.3}$$

where we assume the priori estimate

$$||D\frac{1}{2}\tilde{e}||_2 \leq \lambda^\frac{1}{2}, ||\tilde{e}||_2 \leq \lambda, \tag{5.4}$$

as well as

$$|\lambda_t + a| \leq \lambda^2, a \leq \lambda^\frac{1}{2}, |a_t| \leq 1. \tag{5.5}$$

Next, Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth and radial function with the following properties

$$\phi'(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 3 - e^{-|x|} & \text{for } x \geq 2, \end{cases} \tag{5.6}$$

and the convexity condition

$$\phi''(x) \geq 0 \text{ for } x \geq 0. \tag{5.7}$$

Furthermore, we denote

$$F(u) = \frac{1}{3}|u|^3, \ f(u) = |u|u, F'(u) \cdot h = \Re(f(u)\bar{h}).$$

Let $A > 0$ be a large constant and define the quantity

$$I_A(u) := \frac{1}{2}\int |D\frac{1}{2}\tilde{e}|^2 + \frac{1}{2}\int \frac{\tilde{e}^2}{\lambda} - \int [F(u) - F(\tilde{Q}) - F'(\tilde{Q}) \cdot \tilde{e}]
+ \frac{a}{2} \Im \left( \int A \nabla \phi \left( \frac{x}{A} \right) \cdot \nabla \tilde{e} \tilde{e} \right). \tag{5.8}$$

Our strategy will be to use the preceding functional to bootstrap control over $||\tilde{e}||_{H^2}$.
Lemma 5.1. (Localized energy estimate) Let $J_A$ be as above. Then we have
\[
\frac{dJ_A}{dt} = \mathcal{J} \left( \psi, D\tilde{e} + \frac{1}{\lambda} \tilde{e} - f'(\tilde{Q})\tilde{e} \right) - \frac{1}{\lambda} \left( \tilde{e}, f'(\tilde{Q})\tilde{e} \right) - \Re \left( \partial_t \tilde{Q}, (f(u) - f(\tilde{Q})) \cdot \tilde{e} \right)
+ \frac{a}{2\lambda} \int_0^{+\infty} \left| \tilde{e} \right|^2 - 2a \int_0^{+\infty} \left[ \Delta \phi \left( \frac{x}{A\lambda} \right) \right] \left| \nabla \tilde{e} \right|^2 dx ds
+ \frac{a}{2A^2\lambda} \int_0^{+\infty} \sqrt{s} \int_{\mathbb{R}^2} \Delta^2 \phi \left( \frac{x}{A\lambda} \right) \left| \tilde{e} \right|^2 dx ds
+ \mathcal{J} \left( \int \left( iA \nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \psi + i \frac{a}{2\lambda} \Delta \phi \left( \frac{x}{A\lambda} \right) \psi \right) \tilde{e} \right)
+ aR \left( \int A \nabla \phi \left( \frac{x}{A\lambda} \right) \left( \frac{3}{4} |\tilde{Q}|^{-1} |\tilde{e}|^2 \tilde{Q} + \frac{1}{4} |\tilde{Q}|^{-1} \tilde{e} \cdot \tilde{Q} \right) \cdot \nabla \tilde{Q} \right)
+ O \left( \lambda^2 ||\psi||_2 + ||\tilde{e}||^2_{H^3/2} + \lambda^2 ||\tilde{e}||^2_{H^3/2} \right),
\]
(5.9)
where $\tilde{e}_s := \sqrt{\frac{2}{\pi}} \frac{1}{\lambda s^3} \tilde{e}$ with $s > 0$.

Proof. Step 1: (Estimating the energy part). Using (5.3), a computation
\[
\frac{d}{dt} \left\{ \frac{1}{2} \int |D\tilde{e}|^2 + \frac{1}{2} \int \left| \tilde{e} \right|^2 - \int \left[ F(u) - F(\tilde{Q}) - F'(\tilde{Q}) \cdot \tilde{e} \right] \right\}
= \Re \left( \partial_t \tilde{Q}, D\tilde{e} + \frac{1}{\lambda} \tilde{e} - (f(u) - f(\tilde{Q})) \right) - \frac{\lambda_4}{2\lambda^2} \int \left| \tilde{e} \right|^2
- \Re \left( \partial_t \tilde{Q}, (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{e}) \right)
= -\mathcal{J} \left( \psi, D\tilde{e} + \frac{1}{\lambda} \tilde{e} - (f(u) - f(\tilde{Q})) \right) - \frac{\lambda_4}{2\lambda^2} \int \left| \tilde{e} \right|^2 - \Re \left( \partial_t \tilde{Q}, (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{e}) \right)
- \mathcal{J} \left( D\tilde{e} - (f(u) - f(\tilde{Q})), D\tilde{e} + \frac{1}{\lambda} \tilde{e} - (f(u) - f(\tilde{Q})) \right)
= -\mathcal{J} \left( \psi, D\tilde{e} + \frac{1}{\lambda} \tilde{e} - (f(u) - f(\tilde{Q})) \right) - \frac{\lambda_4}{2\lambda^2} \int \left| \tilde{e} \right|^2 + \mathcal{J} \left( f(u) - f(\tilde{Q}), \frac{1}{\lambda} \tilde{e} \right)
- \Re \left( \partial_t \tilde{Q}, (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{e}) \right)
= -\mathcal{J} \left( \psi, D\tilde{e} + \frac{1}{\lambda} \tilde{e} - f'(\tilde{Q})\tilde{e} \right) - \frac{1}{\lambda} \left( \tilde{e}, f'(\tilde{Q})\tilde{e} \right) - \frac{\lambda_4}{2\lambda^2} \int \left| \tilde{e} \right|^2
+ \mathcal{J} \left( \psi - \frac{1}{\lambda} \tilde{e}, f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{e} \right) - \Re \left( \partial_t \tilde{Q}, (f(u) - f(\tilde{Q}) - f'(\tilde{Q}) \cdot \tilde{e}) \right),
\]
(5.10)
where we denote
\[
f'(\tilde{Q})\tilde{e} = \frac{3}{2} |\tilde{Q}| \tilde{e} + \frac{1}{2} |\tilde{Q}|^{-1} \tilde{Q} \cdot \tilde{e}.
\]
From (5.5) we obtain that
\[
-\frac{\lambda_t}{2\lambda^2} \int |\dot{\mathbf{e}}|^2 = \frac{a}{2\lambda} \int \frac{|\dot{\mathbf{e}}|^2}{\lambda} - \frac{1}{2\lambda^2} (\lambda_t + a) ||\mathbf{e}||_2^2
\]
\[
= \frac{a}{2\lambda} \int \frac{|\dot{\mathbf{e}}|^2}{\lambda} + \mathcal{O}(||\mathbf{e}||_{H^{1/2}}^2).
\] (5.11)

Next, we estimate
\[
\mathcal{J} \left( \psi - \frac{1}{\lambda} \mathbf{e}, f(u) - f(\bar{Q}) - f'(\bar{Q}) \cdot \mathbf{e} \right)
\]
\[
\leq \left( ||\psi||_2 + \lambda^{-1} ||\mathbf{e}||_2 \right) ||f(u) - f(\bar{Q}) - f'(\bar{Q}) \cdot \mathbf{e}||_2
\]
\[
\leq \left( ||\psi||_2 + \lambda^{-1} ||\mathbf{e}||_2 \right) ||\mathbf{e}||^2_4
\]
\[
\leq \left( ||\psi||_2 + \lambda^{-1} ||\mathbf{e}||_2 \right) ||\mathbf{e}||_{H^{1/2}}^2
\]
\[
\leq \lambda^2 ||\psi||_2 + ||\mathbf{e}||_{H^{1/2}}^2
\] (5.12)

where we used the Hölder inequality and Sobolev inequality together with the assumed a-priori estimate (5.2) and (5.4). Here we also used the following estimate
\[
|g(u + v) - g(u) - g'(u) \cdot v| \leq |v|^p,
\] (5.13)

for \(1 < p \leq 2\) and \(g(u) = |u|^{p-1} u\).

We now insert (5.11) and (5.12) into (5.10). Combined with the assumed a priori bounds on \(\mathbf{e}\), we conclude
\[
\frac{d}{dt} \left\{ \frac{1}{2} \int |D^2 \mathbf{e}|^2 + \frac{1}{2} \int \frac{|\dot{\mathbf{e}}|^2}{\lambda} - \int [F(u) - F(\bar{Q}) - F'(\bar{Q}) \cdot \mathbf{e}] \right\}
\]
\[
= -3 \left( \psi, D \mathbf{e} + \frac{1}{\lambda} \dot{\mathbf{e}} - f'(\bar{Q}) \cdot \mathbf{e} \right) - \frac{1}{\lambda} (\dot{\mathbf{e}}, f'(\bar{Q}) \cdot \mathbf{e}) - \mathcal{R} (\partial_t \bar{Q}, (f(u) - f(\bar{Q}) - f'(\bar{Q}) \cdot \mathbf{e})
\]
\[
+ \frac{a}{2\lambda} \int \frac{|\dot{\mathbf{e}}|^2}{\lambda} + \mathcal{O}(\lambda^2 ||\psi||_2 + ||\mathbf{e}||_{H^{1/2}}^2).
\] (5.14)

**Step 2:** Estimating the localized virial part. We set
\[
\nabla \bar{\phi}(t, x) = aA \nabla \phi \left( \frac{x}{A\lambda} \right).
\]

Then we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( a^3 \left( \int A \nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \mathbf{e} \mathbf{e} \right) \right)
\]
\[
= \frac{1}{2} a^3 \left( \int (\partial_t \nabla \phi) \cdot \nabla u \mathbf{e} \mathbf{e} \right) + \frac{1}{2} a^3 \left( \int \nabla \phi \cdot (\nabla \partial_t \mathbf{e} \mathbf{e} + \nabla \mathbf{e} \partial_t \mathbf{e}) \right).
\] (5.15)
Using the bounds (5.5), we estimate
\[ |\partial_t \nabla \tilde{\phi}| \leq |a_t| + a \frac{\lambda_t}{\lambda} \leq 1, \quad |\partial_t \Delta \tilde{\phi}| \leq \lambda^{-1} \] (5.16)

Hence, by [14, Lemma F.1], we deduce that
\[ \Im \left( (\partial_t \nabla \tilde{\phi}) \cdot \nabla \tilde{\epsilon} \right) \leq \| \tilde{\epsilon} \|^2_{H^{1/2}} + \lambda^{-1} \| \tilde{\epsilon} \|^2. \] (5.17)

Using (5.3), a calculation yields that
\[ \frac{1}{2} \Im \left( \int \nabla \tilde{\phi} \cdot (\nabla \partial_t \tilde{\epsilon} + \nabla \tilde{\epsilon} \partial_t \tilde{\epsilon}) \right) \]
\[ = -\frac{1}{4} \Re \left( \int \bar{\epsilon} \left[ -iD, \nabla \tilde{\phi} \cdot (-i\nabla) + (-i\nabla) \cdot \nabla \tilde{\phi} \right] \right) \]
\[ - a \Re \left( \int (|u|u - |Q|Q) \nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \tilde{\epsilon} \right) \]
\[ - \frac{1}{2} \Re \left( \int (|u|u - |Q|Q) \Delta \phi \left( \frac{x}{A\lambda} \right) \cdot \tilde{\epsilon} \right) \]
\[ - a \Re \left( \int \psi \nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \tilde{\epsilon} \right) - \frac{1}{2} \Re \left( \int \psi \Delta \phi \left( \frac{x}{A\lambda} \right) \tilde{\epsilon} \right). \] (5.18)

Next, we rewrite the commutator by using some identities from functional calculus. Here, we recall the known formula
\[ x^\beta = \frac{\sin (\pi \beta)}{\pi} \int_0^\infty s^{\beta-1} \frac{x}{x+s} ds \]
for \( x > 0 \) and \( 0 < \beta < 1 \). From spectral calculus applied to the self-adjoint operator \(-\Delta\), we have the Balakrishnan’s formula
\[ (-\Delta)^\beta = \frac{\sin (\pi \beta)}{\pi} \int_0^\infty s^{\beta-1} \frac{-\Delta}{-\Delta+s} ds. \]

Next, we note the formal identity
\[ \left[ \frac{A}{A+s}, B \right] = \left[ 1 - \frac{s}{A+s}, B \right] = - \left[ \frac{s}{A+s}, B \right] = s \frac{1}{A+s} [A, B] \frac{1}{A+s} \]
for operators \( A \geq 0 \), and \( B \), where \( s > 0 \) is the positive constant. We obtain the formal commutator identity
\[ [(-\Delta)^m, B] = \frac{\sin (\pi m)}{\pi} \int_0^s s^{m-1} \frac{1}{-\Delta+s} [-\Delta, B] \frac{1}{-\Delta+s} ds. \]

In particular, we deduce that
\[ [D, \nabla \tilde{\phi} \cdot (-i\nabla) + (-i\nabla) \cdot \nabla \tilde{\phi}] \]
\[ = \frac{1}{\pi} \int \sqrt{s} \frac{1}{-\Delta+s} [-\Delta, \nabla \tilde{\phi} \cdot (-i\nabla) + (-i\nabla) \cdot \nabla \tilde{\phi}] \frac{1}{-\Delta+s} ds. \]
Next, we recall the known formula
\[
\left[ -\Delta, \nabla \tilde{\phi} \cdot (-i \nabla) + (-i \nabla) \cdot \nabla \tilde{\phi} \right] = -4 \nabla \cdot (\Delta \tilde{\phi} \cdot (-i \nabla)) + i \Delta^2 \tilde{\phi},
\]
for any smooth function \( \phi \).

We now define the auxiliary function
\[
\tilde{e}_s(t, x) := \sqrt{\frac{2}{\pi}} \frac{1}{-\Delta + s} \tilde{e}(t, x) \quad \text{for } s > 0.
\]

Hence, by construction, we have that \( \tilde{e}_s \) solves the elliptic equation
\[
-\Delta \tilde{e}_s + s \tilde{e}_s = \sqrt{\frac{2}{\pi}} \tilde{e}.
\]

Note that the integral kernel for the resolvent \((-\Delta + s)^{-1}\) is explicitly given by
\[
G^s(x) = \int_0^\infty (4\pi t)^{-1} \exp \left\{ \frac{-|x|^2}{4t} - st \right\} dt.
\]

Hence, we remark that we have the convolution formula
\[
\tilde{e}_s = \sqrt{\frac{2}{\pi}} G^s(x) * \tilde{e}(t, x).
\]

Recalling that \( \nabla \tilde{\phi}(t, x) = aA \nabla \phi \left( \frac{x}{A t} \right) \) and using that \((-\Delta + s)^{-1}\) is self-adjoint and the definition of \( \tilde{e}_s \), as well as Fubini’s theorem, we conclude that
\[
\mathcal{R} \left( \int \tilde{e} \left[ -\Delta, \nabla \tilde{\phi} \cdot (-i \nabla) + (-i \nabla) \cdot \nabla \tilde{\phi} \right] \tilde{e} \right)
= \mathcal{R} \left( \int \int \frac{1}{\pi} \sqrt{s} \frac{1}{-\Delta + s} \left[ -\Delta, \nabla \tilde{\phi} \cdot (-i \nabla) + (-i \nabla) \cdot \nabla \tilde{\phi} \right] \frac{1}{-\Delta + s} ds \tilde{e} dx \right)
= \frac{1}{\pi} \mathcal{R} \left( \int \int \sqrt{s} \tilde{e} \frac{1}{-\Delta + s} \left( 4 \nabla \cdot (\Delta \tilde{\phi} \cdot \nabla) + \Delta^2 \tilde{\phi} \right) \frac{1}{-\Delta + s} \tilde{e} dx ds \right)
= \frac{1}{2} \mathcal{R} \left( \int \int \sqrt{s} \tilde{e}_s \left( 4 \nabla \cdot (\Delta \tilde{\phi} \cdot \nabla) + \Delta^2 \tilde{\phi} \right) \tilde{e}_s dx ds \right)
= -\frac{2a}{\lambda} \int_0^{+\infty} \sqrt{s} \int_{\mathbb{R}^2} \Delta \phi \left( \frac{x}{A \lambda} \right) |\nabla \tilde{e}_s|^2 dx ds + \frac{a}{2A^2 \lambda^3} \int_0^{+\infty} \sqrt{s} \int_{\mathbb{R}^2} \Delta^2 \phi \left( \frac{x}{A \lambda} \right) |\tilde{e}_s|^2 dx ds.
\]

(5.19)

Next, we estimate the other term in (5.18). Using the bound (5.2), (5.4) and (5.5), we find that
This completes the proof of lemma.
6. Backwards propagation of smallness

We now apply the energy estimate of the previous section in order to establish a bootstrap argument that will be needed in the construction of ground state mass blowup solution.

Let \( u = u(t,x) \) be a solution to (1.1) defined in \([t_0, 0)\). Assume that \( t_0 < t_1 < 0 \) and suppose that \( u \) admits on \([t_0, t_1]\) a geometrical decomposition of the form

\[
u(t, x) = \frac{1}{\lambda(t)} \left[ Q_{1,2} + \epsilon \left( s, \frac{x}{\lambda(t)} \right) e^{i\eta(t)} \right], \quad (6.1)
\]

where \( \epsilon = \epsilon_1 + i \epsilon_2 \) satisfies the orthogonality condition (4.4) and \( a^2 + \| \epsilon \|_{H^{1/2}}^2 \ll 1 \) holds. We set

\[
\tilde{\epsilon}(t, x) = \frac{1}{\lambda(t)} \epsilon \left( s, \frac{x}{\lambda(t)} \right) e^{i\eta(t)}.
\]

Suppose that the energy satisfies \( E_0 = E(u_0) > 0 \) and define the constant

\[
A_0 = \sqrt{\frac{\epsilon_1}{E_0}},
\]

with the constant \( \epsilon_1 = \frac{1}{2} (L_1 - S_1, S_1) > 0 \).

Now we claim that the following backwards propagation estimate holds.

**Lemma 6.1.** (Backwards propagation of smallness) Assume that, for some \( t_1 < 0 \) sufficiently close to 0, we have the bounds

\[
\left| ||u||^2 - ||Q||^2 \right| \leq \lambda^2(t_1),
\]

\[
\left| ||D^k\tilde{\epsilon}(t_1)||^2_{L^2} + \frac{\|\tilde{\epsilon}(t_1)\|^2_{L^2}}{\lambda(t_1)} \right| \leq \lambda(t_1),
\]

\[
\left| \lambda(t_1) - \frac{t_1^2}{4A_0^2} \right| \leq \lambda^2(t_1), \quad \left| \frac{a(t_1)}{\lambda^2(t_1)} \right| \leq \lambda(t_1),
\]

where \( A_0 \) is defined in (6.3). Then there exists a time \( t_0 < t_1 \) depending on \( A_0 \) such that \( \forall t \in [t_0, t_1], \) it holds that

\[
\left| ||D^k\tilde{\epsilon}(t)||^2_{L^2} + \frac{\|\tilde{\epsilon}(t)\|^2_{L^2}}{\lambda(t)} \right| \leq \left| ||D^k\tilde{\epsilon}(t_1)||^2_{L^2} + \frac{\|\tilde{\epsilon}(t_1)\|^2_{L^2}}{\lambda(t_1)} \right| \leq \lambda(t),
\]

\[
\left| \lambda(t) - \frac{t^2}{4A_0^2} \right| \leq \lambda^2(t), \quad \left| a(t) \lambda^2(t) - \frac{1}{A_0} \right| \leq \lambda(t).
\]

**Proof.** By assumption, we have \( u \in C^0([t_0, t_1]; H^{1/2+\delta}_\text{rad} (\mathbb{R}^2)) \). Hence, by this continuity and the continuity of the functions \( \{\lambda(t), a(t)\} \), there exists a time \( t_0 < t_1 \) such that for any \( t \in [t_0, t_1] \) we have the bounds

\[
\|\tilde{\epsilon}(t)\|^2_{L^2} \leq K\lambda^2(t), \quad \|\tilde{\epsilon}(t)\|^2_{H^{1/2}} \leq K\lambda(t), \quad (6.4)
\]
\[
\left| \lambda(t) - \frac{t^2}{4A_0^2} \right| \leq K\lambda^2(t), \quad \left| \frac{a(t)}{\lambda^2(t)} - \frac{1}{A_0} \right| \leq K\lambda(t),
\]
(6.5)

with some constant \(K > 0\). We now claim that the bounds stated in this lemma hold on \([t_0, t_1]\), hence improving (6.4) and (6.5) on \([t_0, t_1]\) for \(t_0 = t_0(A_0) < t_1\) small enough but independent of \(t_1\). We divide the proof into the following steps.

**Step 1 Bounds on energy and \(L^2\)– norm.** We set
\[
\tilde{Q}(t, x) = \frac{1}{\lambda(t)} Q_{P(t)} \left( \frac{x}{\lambda(t)} \right) e^{\gamma(t)}.
\]
(6.6)

Let \(J_A\) be given by above section. Applying Lemma 5.1, we claim that we obtain the following coercivity estimate:
\[
\frac{dJ_A}{dt} \geq \frac{a}{\lambda^2} \int |\epsilon|^2 + \mathcal{O} \left( ||\tilde{\epsilon}||_{H^{1/2}}^2 + K^4\lambda^2 \right),
\]
(6.7)

Assume (6.7) holds. By the Sobolev embedding and small of \(\epsilon\), we deduce the upper bound
\[
|J_A| \leq ||D^2\tilde{\epsilon}||_2^2 + \frac{1}{\lambda} ||\tilde{\epsilon}||_2^2
\]
(6.8)

Here we use the following inequality
\[
\left| \mathfrak{N} \left( \int A\nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \tilde{\epsilon} \tilde{\epsilon} \right) \right| \leq ||D^2\tilde{\epsilon}||_2^2 + \frac{1}{\lambda} ||\tilde{\epsilon}||_2^2,
\]

where we can see [14, Lemma F.1]. Furthermore, due to the proximity of \(Q_P\) to \(Q\), we derive the lower bound
\[
J_A = \frac{1}{2} \int |D^2\tilde{\epsilon}|^2 + \frac{1}{2} \int \frac{|\tilde{\epsilon}|^2}{\lambda} - \int \left[ F(u) - F(w) - F'(w) \cdot \tilde{\epsilon} \right]
\]
\[
+ \frac{a}{2} \mathfrak{N} \left( \int A\nabla \phi \left( \frac{x}{A\lambda} \right) \cdot \nabla \tilde{\epsilon} \tilde{\epsilon} \right)
\]
\[
= \frac{1}{2\lambda} \left[ (L_+\epsilon_1, \epsilon_1) + (L_-\epsilon_2, \epsilon_2) + o(||\epsilon||_{H^{1/2}}^2) \right]
\]
\[
\geq \frac{C_0}{\lambda} \left[ ||\epsilon||_{H^{1/2}}^2 - (\epsilon_1, Q)^2 \right],
\]
(6.9)

using the orthogonality conditions (4.4) satisfied by \(\epsilon\) and the coercivity estimate for the linearized operator \(L = (L_+, L_-)\). On the other hand, using the conservation of the \(L^2\)– mass and applying Lemma 4.1, we combine the assumed bounds to conclude that
\[
|\Re(\epsilon, Q_P)| \leq ||\epsilon||_2^2 + \lambda^2(t) + \int \left| u \right|^2 - \int |Q|^2 \leq ||\epsilon||_2^2 + K^2\lambda^2(t).
\]

This implies
\[
(\epsilon_1, Q)^2 \leq o(||\epsilon||_{H^{1/2}}^2) + K^4\lambda^4(t).
\]
(6.10)
Next, we define

\[ H(t) := \|D^2 \tilde{e}(t)\|_2^2 + \frac{1}{\lambda(t)} \|\tilde{e}(t)\|_2^2. \]

By integrating (6.7) in time and using (6.8), (6.9) and (6.10), we find that

\[
H(t) \leq H(t_1) + K^4 \lambda^3(t) + \int_{t}^{t_1} \left( \|\tilde{e}\|_{H^{1/2}}^2 + K^4 \lambda^3(t) \right) d\tau
\]

\[
\leq H(t_1) + K^4 \lambda^3(t) + \int_{t}^{t_1} H(\tau) d\tau
\]

for \( t \in [t_0, t_1] \) with some \( t_0 = t_0(C_0) < t_1 \) close enough to \( t_1 < 0 \). By Gronwall’s inequality, we deduce the desired bound for \( H(t) \). In particular, we obtain

\[
H(t) := \|D^2 \tilde{e}(t)\|_2^2 + \frac{1}{\lambda(t)} \|\tilde{e}(t)\|_2^2 \leq \lambda(t), \quad \text{for} \quad t \in [t_0, t_1],
\]

(6.11)

and closes the bootstrap for (6.4).

**Step 2 Controlling the law for the parameters.** From Lemma 4.2 and using (6.5), we deduce

\[
\left| a_s + \frac{1}{2} a^2 \right| + \left| \frac{\lambda_s}{\lambda} + a \right| \leq \lambda^2.
\]

(6.12)

As a direct consequence of this bound, we obtain that

\[
\left( \frac{a}{\lambda^2} \right)_s = \frac{a_s + \frac{1}{2} a^2}{\lambda^2} + \frac{a}{2\lambda^2} \left( \frac{\lambda_s}{\lambda} + a \right) \leq \lambda^2.
\]

Hence, for any \( s < s_1 \), we have

\[
\frac{1}{A_0} - \frac{a}{\lambda^2}(s) \leq \frac{1}{A_0} - \frac{a}{\lambda^2}(s_1) + \int_s^{s_1} \lambda^3(\tau) d\tau \leq \lambda(s).
\]

(6.13)

Here we used \( \lambda(t) \sim t^2 \) and the relation \( dt = \lambda^{-1} ds \), as well as the assumed initial bound for \( \left| \frac{a(t)}{\lambda^2(t)} - \frac{1}{A_0} \right| \) at time \( t = t_1 \). Next, by following the calculations in the proof of Lemma 4.1 and recalling that \( a^2 \sim \lambda \) thanks to (6.5) and \( \|\tilde{e}\|_{H^{1/2}}^2 \leq \lambda \), we deduce

\[
a^2 e_1 = \lambda E_0 + \left( \int |u|^2 - \int |Q|^2 \right) + O(\lambda^2),
\]

where \( e_1 = \frac{1}{2} (L_1 S_1, S_1) > 0 \) is a constant. Since \( \int |u|^2 - \int |Q|^2 = O(\lambda^2) \) and recalling the definition of (6.3), we deduce that

\[
\frac{a^2}{\lambda} - \frac{1}{A_0} = \left( \frac{a}{\lambda^2} - \frac{1}{A_0} \right) \left( \frac{a}{\lambda^2} + \frac{1}{A_0} \right) = O(\lambda).
\]
Furthermore, from (6.13) we see that $\frac{a}{\lambda^2} \geq 1$. Hence, we obtain the desired bound
\[ \left| \frac{a}{\lambda^2} - \frac{1}{A_0} \right| \leq \lambda. \]

From (6.4) and (6.12), we conclude that
\[-\lambda_t = a + O(\lambda^2) = \frac{\lambda_x}{A_0} + O\left(\lambda^2 + t^4\right) = \frac{\lambda_x}{A_0} + O(t^4).\]

Dividing the above equality by $\lambda_x$, and integrating in $[t, t_1]$ and using the boundary value at $t_1$, we have
\[ \left| \lambda_x(t) - \frac{t}{2A_0} \right| \leq \left| \lambda_x(t_1) - \frac{t_1}{2A_0} \right| + O(t^3) \leq t^2, \]
and the bound for $\lambda$ is obtained. This completes the proof of Step 2.

**Step 3 Proof of the coercivity estimate.** Let $K_A(\tilde{e})$ denote the terms in $\tilde{e}$ on the right-hand side in Lemma 5.1, that is, we have
\[ K_A(\tilde{e}) = -\frac{1}{\lambda} (\tilde{e}, f'(\tilde{Q})\tilde{e}) + \frac{a}{2\lambda} \int_{\mathbb{R}^2} \frac{|\tilde{e}|^2}{\lambda} \, ds \]
\[ + \frac{2a}{\lambda} \int_0^{+\infty} \sqrt{s} \left[ \Delta \phi \left( \frac{x}{A\lambda} \right) |\nabla \tilde{e}| \, ds \right] \, dx \, ds \]
\[ + \frac{a}{2^2 \lambda} \int_0^{+\infty} \sqrt{s} \left[ \Delta \phi \left( \frac{x}{A\lambda} \right) |\tilde{e}| \, ds \right] \, dx \, ds \]
\[ + a \Re \left( \int_{\mathbb{R}^2} A \nabla \phi \left( \frac{x}{A\lambda} \right) \left( \frac{\lambda}{4} |\tilde{Q}|^{-1} |\tilde{e}|^2 \tilde{Q} + \frac{1}{4} |\tilde{Q}|^{-1} \tilde{Q} \cdot \nabla \tilde{Q} \right) \right). \]

Recalling that the function $\tilde{e}_s = \tilde{e}_s(t, x)$ with the parameter $s > 0$ was defined in Lemma 5.1 to be $\tilde{e}_s = \sqrt{\frac{2}{\pi}} - \frac{1}{f(x) \tilde{Q}} \tilde{e}$ and $\tilde{e} = \frac{1}{\lambda} \tilde{e}(t, \tilde{x})$, we now claim that the following estimate holds:
\[ K_A(\tilde{e}) \geq \frac{C}{\lambda^2} \int |\tilde{e}|^2 + O(K^4 \lambda^2), \] (6.15)
where $C > 0$ is some positive constant.

Indeed, from the Lemma 4.2 and the estimate (6.4) we obtain that
\[ |\text{Mod}(t)| \leq K^2 \lambda^4(t). \] (6.16)

We find that $\tilde{Q}$ satisfies
\[ \tilde{e}_t \tilde{Q} = e^{i(t)} \int \frac{1}{\lambda} \left[ -\frac{\lambda_t}{\lambda} Q + \frac{\lambda_t}{\lambda} \text{Q} + a_i \frac{\partial Q_p}{\partial a} \right] \left( \frac{x}{\lambda} \right) \]
\[ = \left( \frac{i}{\lambda} + \frac{a}{\lambda} \right) \tilde{Q} + a \left( \frac{x}{\lambda} \right) \cdot \nabla \tilde{Q} + O(K \lambda^{-1}), \]
where we use the uniform bounds $||\tilde{Q}_p|| \leq 1$, and the fact that $|a_t| \leq K$, which can be seen from (6.16) and (6.5). Hence
\[-\Re \int \partial_t \bar{Q}|\bar{\epsilon}|^2 = -\frac{1}{\lambda} \Re \int \bar{Q}|\bar{\epsilon}|^2 - \frac{a}{\lambda} \Re \int \bar{Q}|\bar{\epsilon}|^2 \]
\[-a \Re \left( \left( \frac{x}{\lambda} \right)|\bar{\epsilon}|^2 \cdot \nabla \bar{Q} + O(K\lambda^{-1}||\bar{\epsilon}||_2^2) \right).\]

By the definition of $K_A(\bar{\epsilon})$ and expressing everything in terms of $\epsilon(t,x) = \lambda \bar{\epsilon}(t,\lambda x)$, we conclude that

\[K_A(\bar{\epsilon}) \geq \frac{a}{2\lambda^2} \left\{ \int_0^\infty \sqrt{s} \int \Delta \left( \frac{x}{A} \right)|\nabla \epsilon|^2 \, dx \, ds + \int |\epsilon|^2 - 2 \int Q_1P(\epsilon_1^2 + \epsilon_2^2) - \frac{1}{2A^2} \int_0^\infty \sqrt{s} \int \Delta^2 \phi \left( \frac{x}{A} \right)|\epsilon|^2 \, dx \, ds \right. \]
\[+ 2\Re \left( \int A \nabla \phi \left( \frac{x}{A} \right) \left( \frac{3}{4} |Q_p|^{-1} |\epsilon|^2 Q_p + \frac{1}{4} |Q_p|^{-1} |\epsilon^2 Q_p| \cdot \nabla \bar{Q}_p \right) \right) - 2 \int x \cdot (|\epsilon|^2 \nabla \bar{Q}_p) \right\} + O(K\lambda^{-1}||\bar{\epsilon}||_2^2).\] (6.17)

Next, we note that the definition of $\phi$ and we estimate

\[\left| \Re \int A \nabla \phi \left( \frac{x}{A} \right) \left( \frac{3}{4} |Q_p|^{-1} |\epsilon|^2 Q_p + \frac{1}{4} |Q_p|^{-1} |\epsilon^2 Q_p| \cdot \nabla \bar{Q}_p - x \cdot |\epsilon|^2 \nabla \bar{Q}_p \right) \right| \]
\[\leq ||(A + |\epsilon|) \nabla \bar{Q}_p||_{L^\infty(|x| \geq A)}||\epsilon||_2^2 \approx \frac{1}{A} ||\epsilon||_2^2,\]
where we use the uniform decay estimate of $\nabla \bar{Q}_p$. Furthermore, thanks to Lemma C.3, we have

\[\left| \int_{s=0}^{+\infty} \sqrt{s} \int \Delta^2 \phi_A |\epsilon|^2 \, dx \, ds \right| \approx \frac{1}{A} ||\epsilon||_2^2.\] (6.18)

Recalling the definitions of $L_{+,A}$ and $L_{-,A}$ in (C.1) and (C.2), we deduce that

\[K_A(\bar{\epsilon}) = \frac{a}{2\lambda^2} \left\{ (L_{+,A}\epsilon_1, \epsilon_2) + (L_{-,A}\epsilon_2, \epsilon_2) + O \left( \frac{1}{A} ||\epsilon||_2^2 \right) \right\} + \frac{1}{\lambda^2} O \left( K\lambda^{-\frac{3}{2}} ||\bar{\epsilon}||_2^3 \right).\] (6.19)

Next, we recall that $a \sim \lambda^\frac{3}{2}$ due to the above. Hence, by Lemma C.2 and choosing the $A > 0$ sufficiently large, we deduce from previous estimates that

\[K_A(\bar{\epsilon}) \geq \frac{1}{\lambda^2} \left\{ \int |\epsilon|^2 - (\epsilon_1, Q_2)^2 \right\} \approx \frac{1}{\lambda^2} \int |\epsilon|^2 + O(K^4\lambda^{\frac{3}{2}}).\] (6.20)

**Step 4 Controlling the remainder terms in $\frac{d}{dt}J_A$.** We now control the terms that appear in Lemma 5.1 and contain $\psi$. Here we recall that the definition of $\bar{Q}$ and (5.3), which yields

\[\psi = \frac{1}{\lambda^2} \left[ i \left( a \lambda + \frac{1}{\lambda^2} \right) \partial_x Q_P - i \left( \frac{\lambda}{\lambda} + a \right) \lambda Q_P + \hat{\gamma}_s Q_P + \Phi_P \right] \left( \frac{x}{\lambda} \right)^{\bar{\epsilon}/2}.\]

Here $\Phi_P$ is the error term given in Lemma 3.1. In fact, by the estimate for $Q_P$ and $\Phi_P$ from Lemma 3.1 and recalling (6.16), we deduce the rough pointwise bounds:
\[ |\nabla^k \psi(x)| \leq \frac{1}{\lambda^{2+k}} \left( \frac{x}{\lambda} \right)^{-3} K^2 \lambda^2, \quad \text{for } k = 0, 1. \quad (6.21) \]

Hence
\[ ||\nabla^k \psi||_2 \leq K^2 \lambda^{1-k}, \quad \text{for } k = 0, 1. \quad (6.22) \]

In particular, we obtain the following bounds
\[ ||\psi||_2^2 \leq K^4 \lambda^2, \quad (6.23) \]
\[ \left| \Im \left( \int \left[ iA \nabla \psi \left( \frac{x}{\lambda} \right) \cdot \nabla \psi + i \frac{a}{2\lambda} \Delta \phi \left( \frac{x}{\lambda} \right) \psi \right] \right) \right| \]
\[ \leq \lambda^{\frac{3}{2}} ||\nabla \psi||_2 ||\bar{e}||_2 + \lambda^{-\frac{1}{2}} ||\psi||_2 ||\bar{e}||_2 \quad (6.24) \]
\[ \leq K^2 \lambda^{\frac{1}{2}} ||\bar{e}||_2 \leq o \left( \frac{||\bar{e}||_2^2}{\lambda^2} \right) + K^4 \lambda^\frac{1}{2}. \]

Write \( \psi = \psi_1 + \psi_2 \) with \( \psi_2 = O(\mathcal{P}|\text{Mod}|^{1} + a^5) = O(\lambda^{\frac{1}{2}}) \), that is, we denote
\[ \psi_1 = \frac{1}{\lambda^2} \left[ -\left( a_s + \frac{1}{2} a^2 \right) S_1 - i \left( \frac{\lambda_2}{\lambda} + a \right) \Lambda Q + \tilde{\gamma} Q \right] \left( \frac{x}{\lambda} \right) e^{iy}. \]

Let us first deal with estimating the contributions coming from \( \psi_2 \). Indeed, since \( a^2 \sim \lambda \) we note that \( \psi_2 = O(\lambda^{\frac{1}{2}}) \) satisfies the pointwise bound
\[ |\nabla^k \psi_2(x)| \leq \frac{1}{\lambda^{2+k}} \left( \frac{x}{\lambda} \right)^{-3} K^2 \lambda^2, \quad \text{for } k = 0, 1. \quad (6.25) \]

Hence
\[ ||\nabla^k \psi_2||_2 \leq K^2 \lambda^{1-k}, \quad \text{for } k = 0, 1. \quad (6.26) \]

Therefore, we obtain that
\[ \left| \Re \left( \int \left[ -D \psi_2 - \frac{\psi_2}{\lambda} + \frac{3}{2} |\bar{Q}||\psi_2 + \frac{1}{2} |\bar{Q}|^{-1} \bar{Q}^2 \psi_2 \right] \bar{e} \right) \right| \]
\[ \leq \left( ||\nabla \psi_2||_2 + \lambda^{-1} ||\psi_2||_2 + ||\bar{Q}||^{1/2} ||\nabla \bar{Q}||^{1/2} ||\psi_2||_2^{1/2} ||\nabla \psi_2||_2^{1/2} \right) ||\bar{e}||_2 \quad (6.27) \]
\[ \leq K^2 \lambda^{\frac{1}{2}} ||\bar{e}||_2 \leq o \left( \frac{||\bar{e}||_2^2}{\lambda^2} \right) + K^4 \lambda^\frac{1}{2}, \]

which is acceptable. Here we used the Hölder inequality and Sobolev inequality. We finally use the fact that \( \psi_1 \) belongs to the generalized null space of \( L = (L_+, L_-) \) and hence an extra factor of \( O(\mathcal{P}) \) is gained using the orthogonality conditions obeyed by \( e = e_1 + e_2 \). Indeed, we find the following bound
\begin{align*}
\Re \left( \int \left[ -D\psi_1 - \frac{\psi_1}{\lambda} + \frac{3}{2} |\tilde{Q}|\psi_1 + \frac{1}{2} |\tilde{Q}|^{-1} Q^2 \tilde{\psi}_1 \bar{e} \right] \right) \\
\leq \frac{\|\text{Mod}(t)\|}{\lambda^2} \left[ \| (\varepsilon_2, L_S) \| + \| (\varepsilon_2, L_0) \| + O(\|\varepsilon\|_2) \right] \\
+ \frac{1}{\lambda^2} \left[ \frac{\lambda_2}{\lambda} + a \| (\varepsilon_1, LQ) \| \right] \\
\leq K^2 \lambda^\frac{5}{2} \|\varepsilon\|_2 + \frac{K^2 \lambda}{\lambda^2} \left( \lambda^\frac{2}{3} \|\varepsilon\|_2 + K^2 \lambda^2 \right) \\
\leq o \left( \frac{\|\varepsilon\|_2^2}{\lambda^2} \right) + K^4 \lambda^\frac{3}{2},
\end{align*}

Here we used (6.16) once again and \( |\mathcal{P}| \leq \lambda^\frac{1}{2} \), as well as \( (\varepsilon_2, L_S) = (\varepsilon_2, \Lambda Q) = O(\|\varepsilon\|_2) \) thanks to the orthogonality conditions for \( \varepsilon \). Moreover, we used that \( L_0 \Lambda Q = -Q \) together with the improved bound in Lemma 4.2, combined with the fact that \( |(\varepsilon_1, Q)| \leq \lambda^\frac{1}{2} \|\varepsilon\|_2 + K^2 \lambda^2 \), which follows from \( \|\varepsilon\|_2 \leq \lambda \) and the conservation of \( L^2 \)-norm. And the proof of this lemma is complete. \( \square \)

7. Existence of ground state mass blowup solutions

In this section, we prove the following result.

Theorem 7.1. Let \( \gamma_0 \in \mathbb{R} \) and \( E_0 > 0 \) be given. Then there exist a time \( t_0 < 0 \) and a solution \( u \in C([-t_0, 0); H^{1/2+\delta}(\mathbb{R}^2)) \), \( \delta \in (0, \frac{1}{2}) \), of (1.1) such that \( u \) blowup at time \( T = 0 \) with \( E(u) = E_0 \) and \( \|u\|_2^2 = \|Q\|_2^2 \).

Furthermore, we have \( \|D^2 u\|_2 \sim t^{-1} \) as \( t \to 0^- \), and \( u \) is of the form

\[
\begin{align*}
\begin{array}{c}
\left. u(t, x) = \frac{1}{\lambda(t)} \left[ Q_{\mathcal{P}(t)} + \varepsilon \right] \right| e^{i\mathcal{P}(t)} = \tilde{Q} + \bar{e},
\end{array}
\end{align*}
\]

where \( \mathcal{P}(t) = a(t) \), and \( \varepsilon \) satisfies the orthogonality condition (4.4). Finally, the following estimate hold:

\[
\begin{align*}
\|\varepsilon\|_2 \leq \lambda, \|\varepsilon\|_{H^{1/2}} \leq \lambda^\frac{1}{2},
\lambda(t) - \frac{t^2}{2A_0^2} = O(\lambda^2), \quad \frac{a}{\lambda^3}(t) - \frac{1}{A_0} = O(\lambda), \quad \gamma(t) = -\frac{4A_0^2}{t} + \gamma_0 + O(\lambda^\frac{3}{2}),
\end{align*}
\]

for \( t \in [t_0, 0) \) and \( t \) sufficiently close to 0. Here \( A_0 > 0 \) is the constant defined in (6.3).

Proof. Step 1. Backwards uniform bounds.

Let \( t_n \to 0^- \) be a sequence of negative times and let \( u_n \) be the radial solution to (1.1) with initial data at \( t = t_n \) given by

\[
\begin{align*}
u_n(t_n, x) = \frac{1}{\lambda_n(t_n)} Q_{\mathcal{P}_n(t_n)} \left( \frac{x}{\lambda_n(t_n)} \right) e^{i\mathcal{P}_n(t_n)},
\end{align*}
\]

where the sequence \( \mathcal{P}_n(t_n) = a_n(t_n) \) and \( \{\lambda_n(t_n)\} \) are given by
\[ a_n(t_n) = -\frac{t_n}{2A_0}, \lambda_n(t_n) = \frac{t_n^2}{4A_0^2}, \gamma_n(t_n) = \gamma_0 - \frac{4A_0^2}{t_n}. \] (7.2)

By Lemma 3.2, we have
\[ \int |u_n(t_n)|^2 \leq \int |Q|^2 + O(t_n^4), \] (7.3)
and \( \bar{\epsilon}(t_n) = 0 \) by construction. Thus \( u_n \) satisfies the assumptions of Lemma 6.1. Hence we can find a backwards time \( t_0 \) independent of \( n \) such that for all \( t \in [t_0, t_n) \) we have the geometric decomposition
\[ u_n(t, x) = \frac{1}{\lambda_n(t)} Q_{P_n(t)} \left( \frac{x}{\lambda_n(t)} \right) + \bar{\epsilon}_n(t, x), \] (7.4)
with the uniform bounds given by
\[ ||D^2 \bar{\epsilon}_n(t)||_2^2 + ||\bar{\epsilon}_n(t)||_2^2 \leq \lambda_n(t), \] (7.5)
\[ \lambda_n(t) - \frac{t^2}{4A_0^2} \leq K\lambda_n^2(t), \quad \frac{\lambda_n(t)}{\lambda_n^2(t)} - \frac{1}{A_0} \leq K\lambda_n(t). \] (7.6)
and
\[ ||\bar{\epsilon}_n(t)||_{L^{1+\theta}} \leq \lambda_n^{\frac{1}{2} - \theta}, \quad \theta \in \left( 0, \frac{1}{2} \right). \] (7.7)
which we prove in step 2.

Next, we conclude that \( \{u_n(t_0)\}_{n=1}^{\infty} \) converges strongly in \( H^{s}_{rad}(\mathbb{R}^2) \) (after passing to a subsequence if necessary), where \( s \in [0, \frac{1}{2} + \theta) \). Indeed, from the uniform bound \( ||\bar{\epsilon}_n(t_0)||_{L^{1+\theta}} \leq 1 \) we can assume (after passing to a subsequence if necessary) that \( u_n(t_0) \rightarrow u_0 \) weakly in \( H^{s}_{rad}(\mathbb{R}^2) \) for any \( s \in [0, \frac{1}{2} + \theta] \). Moreover, we note the uniform bound
\[ \left| \frac{d}{dt} \int \chi_R |u_n|^2 \right| = \left| \int \chi_R ( (u_n)_t \bar{u}_n + u_n (\bar{u}_n)_t ) \right| \]
\[ = \left| \int u_n [\chi_R, iD] \bar{u}_n \right| \]
\[ \leq ||\nabla \chi_R||_\infty ||u_n||_2 \leq \frac{1}{R}, \] (7.8)
with a smooth cutoff function \( \chi_R(x) = \chi(\frac{x}{R}) \) where \( \chi(x) \equiv 0 \) for \( |x| \leq 1 \) and \( \chi(x) \equiv 1 \) for \( |x| \geq 2 \). Note that we used the commutator estimate (which we can see [24])
\[ ||[\chi_R, D]||_{L^2 \rightarrow L^2} \leq ||\nabla \chi_R||_\infty. \]

By integrating the previous bound from \( t_1 \) to \( t_0 \) and using the previous estimate (7.1) and (7.2), we derive that for every \( \tau > 0 \) there is a radius \( R > 0 \) such that
\[
\int_{|x| \geq R} |u_n((t_0))|^2 \leq \tau \text{ for all } n \geq 1.
\]

Combining this fact with the weak convergence of \(\{u_n(t_0)\}_{n=1}^{\infty}\) in \(H^s_{\text{rad}}(\mathbb{R}^2)\), we deduce that

\[ u_n(t_0) \rightarrow u_0(t_0) \text{ strongly in } H^s_{\text{rad}}(\mathbb{R}^2) \text{ for every } s \in \left[0, \frac{1}{2} + 0\right). \tag{7.9} \]

Thus, by local well-posedness in \(H^\frac{1+\delta}{2}_{\text{rad}}(\mathbb{R}^2)\) (see [13]), we can solve the Cauchy problem (1.1) and find

\[ u \in C([t_0, T); H^\frac{1+\delta}{2}_{\text{rad}}(\mathbb{R}^2)), \quad \text{where } 0 < \delta < \theta, \]

and obtain

\[ u_n(t) \rightarrow u(t) \text{ strongly in } H^s_{\text{rad}}(\mathbb{R}^2) \text{ for } t \in [t_0, T), \tag{7.10} \]

where \(0 > T > t_0\) is the lifetime of \(u\) on the right. Moreover, \(u\) for \(t < \min\{T, 0\}\) a geometrical decomposition of the form state in above with

\[ a_n(t) \rightarrow a(t), \lambda_n(t) \rightarrow \lambda(t), \gamma_n(t) \rightarrow \gamma(t). \tag{7.11} \]

Furthermore, we deduce that

\[ ||\bar{e}(t)||_2 \leq \lambda \text{ and } ||\bar{e}(t)||_{H^{1/2}} \leq \lambda^{\frac{3}{2}}. \]

for \(t \in [t_0, T)\). In particular, this implies that \(u(t)\) blows up at time \(T=0\) such that

\[ ||D^1 u||_2^2 \sim \lambda^{-1}(t) \sim |t|^{-2} \text{ as } t \to 0^-. \]

In addition, we deduce from \(L^2\)-mass conversation and the strong convergence that

\[ ||u||_2 = \lim_{n \to \infty} ||u_n(t_n)||_2 = ||Q||_2. \]

As for the energy, we note that

\[ E(u(t)) = \frac{a^2}{\lambda} \epsilon_1 + o(1) \rightarrow E_0 \text{ as } t \to 0^-, \]

by the choice of \(A_0, a_n(t_n)\) and \(\lambda_n(t_n)\). By energy conversation, this implies that

\[ E(u) = E_0. \]

Next, we recall that rough bound

\[ |\tilde{\gamma}_s| \leq \lambda_n. \]

Therefore, using that \(\frac{d\tilde{\gamma}}{dt} = \lambda^{-1}\) and the estimates for \(\lambda_n\)

\[ \left| \frac{d}{dt} \left( \gamma_n + \frac{4A_0^2}{t} \right) \right| = \frac{1}{\lambda_n} \left| (\gamma_n)_s - \frac{4A_0^2\lambda_n}{t^2} \right| = \frac{1}{\lambda_n} \left| (\tilde{\gamma}_n)_s - \frac{4A_0^2\lambda_n}{t^2} + 1 \right| \leq 1. \]
Integrating this bound and using (7.2) and \( \lambda \sim t^2 \), we find
\[
\gamma_n(t) + \frac{4A_3^2}{t} = \gamma_0 + O(\lambda^3),
\]
whence the claim for \( \gamma \) follows, since we have \( \lambda \sim t^2 \).

**Step 2.** \( H^{1/2+0} \) bound.

It remains to prove the \( H^{1/2+0} \) bound (7.7). Our point of departure is again the identity
\[
i\partial_t \epsilon_n = D\tilde{\epsilon}_n - \psi_n - F_n,
\]
where
\[
F_n = |\tilde{Q}_n + \tilde{\epsilon}_n|(\tilde{Q}_n + \tilde{\epsilon}_n) - |\tilde{Q}_n|\tilde{Q}_n.
\]
We plan to obtain a \( H^{1/2+0} \)-bound on \( \tilde{\epsilon}_n \), taking advantage of the a priori bounds at time \( t_n \sim \lambda_n^{1/2} \) and those assumed for \( t \in [t_0, t_n] \). We make partition of the interval \([t_0, t_n]\) into
\[
t_0 = s_0 < s_1 < \cdots < s_N = t_n, s_j - s_{j-1} = h, j = 1, \ldots, N,
\]
where
\[
h \sim \lambda_n^3, N \sim (t_n - t_0)/h \sim (\lambda_n)^{-3}.
\]
We can obtain estimates in each interval \( \Delta_j = [s_{j-1}, s_j] \). For the purpose we follow [12] and for any time interval \( I \) we use the space \( X_\delta \) with norm
\[
||u||_{X_\delta} = ||u(t, x)||_{L^\infty_t H^1(R^2)} + ||[x]^{-\delta/2} \nabla_x u(t, x)||_{L^2_t L^2(R^2)} + ||[x]^{-1/2} u(t, x)||_{L^2_t L^2(R^2)},
\]
where \( q, \bar{q} > 2 \) and \( \delta \in (0, 1) \) satisfies
\[
1 - \delta q + \frac{1 - \delta}{\bar{q}} = \frac{1}{2} \quad \text{and} \quad [x]_\delta = |x|^{1+\delta} + |x|^{1-\delta}.
\]

To more concrete, we take \( q = \bar{q} \in (2, 4), \delta > 0 \) small so that \( q = 4(1 - \delta) \). Using the same argument as the one in [12, section 2.3], in interval \( \Delta_N = [s_{N-1}, s_N] \), we can obtain
\[
||\tilde{\epsilon}_n||_{X_\delta N} \leq h^{-1/2} (||\tilde{Q}_n||_{L^\infty_t H^1(R^2)} + ||\tilde{\epsilon}_n||_{L^\infty_t H^1(R^2)}) ||\tilde{\epsilon}_n||_{X_\delta N} + h||\psi_n||_{L^\infty_t H^1(R^2)}
\]
\[
\leq h^{-1/2} (||\tilde{\epsilon}_n||_{X_\delta N} (\lambda_n^{-1} + ||\tilde{\epsilon}_n||_{X_\delta N}) + h||\psi_n||_{X_\delta N})
\]
\[
\leq h^{1/2-\delta}\lambda_n^{-\delta} ||\tilde{\epsilon}_n||_{X_\delta N} + h||\psi_n||_{X_\delta N}.
\]

Here we used \( \tilde{\epsilon}(t_n) = 0, ||\tilde{\epsilon}_n||_{H^1} \) is small, \( h \sim \lambda_n^3 \). From above we have
\[
||\tilde{\epsilon}_n||_{X_\delta N} \leq \lambda_n^3.
\]

The term \( ||\psi_n||_{X_\delta N} \) can be estimated by the aid of interpolation between (6.21) and (6.22) so that we have
\[
||\psi_n||_{L^m} \leq \lambda_n^{-1+2/m}, \ m \in [2, \infty]
\]
so that
In fact, taking \( m = 4 \) we can write
\[
\left\| \left[ x_0 \right]^{-1/4} \nabla \psi_n \right\|_{L^2(|x| \leq 1)} \leq h^{1/4} \left\| \left[ x_0 \right]^{-1/4} \nabla \psi_n \right\|_{L^1(|x| \leq 1)} \lesssim \lambda_n^{3/4} - 1/4 \ll 1.
\]

In a similar way we estimate
\[
\left\| \left[ x_0 \right]^{-1/4} \nabla \psi_n \right\|_{L^2(|x| \leq 1)} \lesssim \left\| \nabla \psi_n \right\|_{L^q} h^{1/4} \lesssim 1, \quad k = 0, 1.
\]

Hence we arrive at (7.12).

For the other intervals \( \Delta_j, j = 1, \ldots, N - 1 \), we have
\[
\left\| \epsilon_n \right\|_{X_{\Delta_j}} \lesssim \left\| \epsilon_n(s_j) \right\|_{H^1} + h^{1/4} \frac{1}{4} \left( \left\| \tilde{Q}_n \right\|_{L^\infty H^1(\mathbb{R}^2)} + \left\| \tilde{E}_n \right\|_{L^\infty H^1(\mathbb{R}^2)} \right) \left\| \epsilon_n \right\|_{X_{\Delta_j}} + \left. h \right\| \psi_n \|_{L^\infty H^1(\mathbb{R}^2)}.
\]

By the similar argument as before, we deduce
\[
\left\| \epsilon_n \right\|_{X_{\Delta_j}} \lesssim \left\| \epsilon_n(s_j) \right\|_{H^1} + \lambda_n^3 \lesssim \left\| \epsilon_n \right\|_{X_{\Delta_{j+1}}} + \lambda_n^3.
\]

and inductively we find
\[
\left\| \epsilon_n \right\|_{X_{\Delta_j}} \lesssim (N - j) \left\| \epsilon_n \right\|_{H^1} + \lambda_n^3 \lesssim (N - j + 1) \lambda_n^3.
\]

Therefore, we have
\[
\left\| \epsilon_n \right\|_{X_{[\epsilon_1, \epsilon_2]}} \lesssim \sup_{1 \leq j \leq N} \left\| \epsilon_n \right\|_{X_{\Delta_j}} \lesssim N \lambda_n^3 \lesssim 1.
\]

This means
\[
\left\| \nabla \epsilon_n \right\|_{L^2} \lesssim 1.
\]

By using the interpolation inequality, we can obtain
\[
\left\| D_{\lambda_n} \psi_n \right\|_{L^2} \lesssim \left\| \nabla \psi_n \right\|_{L^2} \frac{1}{\lambda_n} \lesssim \lambda_n^{-1/2}.
\]

Hence, we can obtain (7.7).

Now the proof of this Theorem is complete. \( \square \)

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A. Appendix A

In this section, we collect some regularity and decay estimates concerning the linearized operators \( L_+ \) and \( L_- \).

**Lemma A.1.** Let \( f, g \in H^k(\mathbb{R}^2) \) for some \( k \geq 0 \) and suppose \( f \perp Q \) and \( g \perp \partial_x Q \), where \( j = 1, 2 \). Then we have the regularity bounds

\[
||L_+^{-1}g||_{H^{k+1}} \leq ||g||_{H^k}, \quad ||L_-^{-1}f||_{H^{k+1}} \leq ||f||_{H^k},
\]

and the decay estimates

\[
||\langle x \rangle^3 L_-^{-1}g||_{\infty} \leq ||\langle x \rangle^3 g||_{\infty}, \quad ||\langle x \rangle^3 L_+^{-1}f||_{\infty} \leq ||\langle x \rangle^3 f||_{\infty}.
\]

**Proof.** It suffices to prove the lemma for \( L_-^{-1}g \), since the estimate for \( L_+^{-1}f \) follow in the same fashion.

To show the regularity bound, we can assume that \( k \in \mathbb{N} \) is an integer. Let \( L_-^{-1}g = h \), and thus

\[ Dh + h = Qh + g. \]

Note that \( Q \in H^2(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2) \) for any \( k \in \mathbb{N} \) by Sobolev embeddings and the fact that \( Q \in H^{1/2}(\mathbb{R}^2) \). Applying \( \nabla^k + 1 \) to the equation above and using the Leibniz rule and Hölder, we find that

\[
||h||_{H^{k+1}} \sim ||(\nabla^k + 1)(Dh + h)||_2 \leq ||Q||_{W^{k,\infty}}||h||_{H^k} + ||g||_{H^k}.
\]

(A.1)

Note, in particular, that \( ||h||_2 = ||L_-^{-1}g||_2 \leq ||g||_2 \) holds, since \( L_- \) has a bound inverse on \( Q^\perp \). Hence (A.1) shows that the desired regularity estimates are true for \( k=0 \). By induction, we obtain the desired estimate \( ||L_-^{-1}g||_{H^{k+1}} \leq ||g||_{H^k} \) for any integer \( k \in \mathbb{N} \).

To show the decay estimate, we argue as follows. Assume that \( ||\langle x \rangle^{-3}g||_\infty < +\infty \), because otherwise there is nothing to prove. As above, let \( L_-^{-1}g = h \) and rewrite the equation satisfied by \( h \) as resolvent form:

\[ h = \frac{1}{D} Qh + \frac{1}{D} g. \]

Let \( R(x-y) = F^{-1}\left(\frac{1}{|D+1|}\right)(x-y) \) denote the associated kernel of the resolvent \( (D+1)^{-1} \). From [15] we recall the standard fact that \( R \in L^p(\mathbb{R}^2) \) for any \( p \in [1, \infty] \) with \( 1 - \frac{1}{p} < \frac{1}{2} \). Since \( h \in L^2(\mathbb{R}^2) \), this implies that \( (R \ast h)(x) \) is continuous and vanishes as \( |x| \to \infty \). Moreover we have the pointwise bound
First, we show that the parameters \( \{ \lambda, \gamma \} \) are uniquely determined if \( \epsilon = \epsilon_1 + i \epsilon_2 \in H^{1/2}(\mathbb{R}^2) \) is sufficiently small and satisfies the orthogonality conditions (4.4). Indeed, this follows from an implicit function argument, which we detail here.

For \( \delta > 0 \), let \( W_\delta = \{ w \in H^{1/2}(\mathbb{R}^2) : ||w - Q||_{H^{1/2}} < \delta \} \). Consider approximate blowup profiles \( Q_p \) with \( |p| = |a| < \eta \), where \( \eta > 0 \) is a small constant. For \( w \in W_\delta, \lambda_1 > 0, \gamma \in \mathbb{R} \) and \( |p| < \eta \), we define

\[
\epsilon_{\lambda_1, \gamma_1, a}(y) = e^{t \lambda_1} w(\lambda_1 y) - Q_p.
\]

Consider the map \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) define by

\[
\begin{align*}
\sigma^1 &= ((\epsilon_{\lambda_1, \gamma_1, a})_2, \Lambda Q_1) - ((\epsilon_{\lambda_1, \gamma_1, a})_1, \Lambda Q_2) , \\
\sigma^2 &= ((\epsilon_{\lambda_1, \gamma_1, a})_2, \partial_a Q_1) - ((\epsilon_{\lambda_1, \gamma_1, a})_1, \partial_a Q_2) , \\
\sigma^3 &= ((\epsilon_{\lambda_1, \gamma_1, a})_1, \rho_2) - ((\epsilon_{\lambda_1, \gamma_1, a})_1, \rho_1) .
\end{align*}
\]

Recall that \( p = p_1 + ip_2 \) was defined in (4.5). Taking the partial derivatives at \( (\lambda_1, \gamma_1, a, w) = (1, 0, 0) \) yields that

\[
\begin{align*}
\frac{\partial \epsilon_{\lambda_1, \gamma_1, a}}{\partial \lambda_1} &= \Lambda w, \\
\frac{\partial \epsilon_{\lambda_1, \gamma_1, a}}{\partial \gamma_1} &= iw, \\
\frac{\partial \epsilon_{\lambda_1, \gamma_1, a}}{\partial a} &= -\partial_a Q_p |_{p=0} = -i S_1,
\end{align*}
\]

where we recall that \( L S_1 = \Lambda Q \). Note that \( S_1 \) is an radial function. At \( (\lambda_1, \gamma_1, a, w) = (1, 0, 0, Q) \), the Jacobian of the map \( \sigma \) is hence given by

\[
\begin{align*}
\frac{\partial \sigma^1}{\partial \lambda_1} &= 0, & \frac{\partial \sigma^1}{\partial \gamma_1} &= 0, & \frac{\partial \sigma^1}{\partial a} &= -(S_1, L S_1) , \\
\frac{\partial \sigma^2}{\partial \lambda_1} &= -(S_1, L S_1) , & \frac{\partial \sigma^2}{\partial \gamma_1} &= 0, & \frac{\partial \sigma^2}{\partial a} &= 0 , \\
\frac{\partial \sigma^3}{\partial \lambda_1} &= 0, & \frac{\partial \sigma^3}{\partial \gamma_1} &= -(Q, \rho_1) , & \frac{\partial \sigma^3}{\partial a} &= 0.
\end{align*}
\]

Note that we used here that \( Q \) and \( S_1 \) are the radial symmetry functions. Moreover, we note

\[
-(Q, \rho_1) = (L_+, \Lambda Q, \rho_1) = -(\Lambda Q, L_+ \rho_1) = -(\Lambda Q, S_1) = -(L_+ S_1, S_1) .
\]

Since \( (L_+ S_1, S_1) > 0 \), hence the determinant of the functional matrix is nonzero. By the implicit function theorem, we obtain existence and uniqueness for \( (\lambda_1, \gamma_1, a, w) \) in some neighborhood around \( (1, 0, 0, Q) \).
B.2. Estimates for the modulation equations

To conclude this section, we collect some estimates needed in the discussion of the modulation equations in section 4.

Lemma B.1. The following estimates hold

\[
\begin{align*}
(M_+ \psi) - a\Lambda_1, \Lambda Q_{1^p}) + (M_+(\psi) + a\Lambda_2, \Lambda Q_{1^p}) &= -\Re(\psi, Q_p) + O(P^2||\psi||_2), \\
(M_- \psi) - a\Lambda_1, \partial_\psi Q_{2^p}) + (M_-(\psi) + a\Lambda_2, \partial_\psi Q_{1^p}) &= O(P^2||\psi||_2), \\
(M_- \psi) - a\Lambda_1, \psi_2) + (M_+(\psi) + a\Lambda_2, \psi_1) &= O(P^2||\psi||_2).
\end{align*}
\] (B.1) (B.2) (B.3)

Proof. First, we recall that

\[
M_+(\psi) = L_+ \psi_1 - |Q_p|^{-1}Q_{1^p}Q_{2^p}\psi_2 + O(P\psi),
M_-(\psi) = L_- \psi_2 - |Q_p|^{-1}Q_{1^p}Q_{2^p}\psi_1 + O(P\psi).
\]

We have notice the identity

\[L_- \Lambda S_1 = -S_1 + 2(\Lambda Q)S_1 + \Lambda Q + \Lambda^2 Q. \] (B.4)

To see this relation, we recall that \[L_- S_1 = \Lambda Q \] and hence

\[
\begin{align*}
L_- \Lambda S_1 &= [L_-, \Lambda]S_1 + \Lambda L_- S_1 = DS_1 + (x \cdot \nabla Q)S_1 + \Lambda^2 Q \\
&= -S_1 + |Q|S_1 + \Lambda Q + (x \cdot \nabla Q)S_1 + \Lambda^2 Q \\
&= -S_1 + \Lambda Q + (\Lambda Q)S_1 + \Lambda^2 Q.
\end{align*}
\]

Hence, the identity (B.4) is hold.

Next, we recall that

\[\Lambda Q_{1^p} = \Lambda Q + O(P^2), \quad \Lambda Q_{2^p} = a\Lambda S_1 + O(P^2).\]

Using (B.4) and \[L_+ \Lambda Q = -Q, \] we find that

left-hand side of (B.1)

\[
\begin{align*}
&= a(\psi_2, L_- \Lambda S_1) - |Q_p|^{-1}(Q_{1^p}Q_{2^p}\psi_1, a\Lambda S_1) \\
&\quad + (a\Lambda_1, a\Lambda S_1 + (\psi_1, L_+ \Lambda Q) + (a\Lambda_2, \Lambda Q) \\
&\quad - |Q_p|^{-1}(Q_{1^p}Q_{2^p}\psi_2, \Lambda Q) + O(P^2||\psi||_2) \\
&\quad = -(\psi_1, Q) - a(\psi_2, S_1) + a(\psi_2, \Lambda Q) + O(P^2||\psi||_2) \\
&\quad = -\Re(\psi, Q_p) + O(P^2||\psi||_2).
\end{align*}
\]

Here we used that \[a(\psi_2, \Lambda Q) = O(P^2||\psi||_2), \] which follows from the orthogonality condition (4.4).

Estimate (B.2). From Lemma 3.2 we recall that

\[\partial_\psi Q_{1^p} = 2aT_2 + O(P^2), \partial_\psi Q_{2^p} = S_1 + O(P^2),\]

where

\[L_+ T_2 = \frac{1}{2}S_1 - \Lambda S_1 + \frac{1}{2}|S_1|^2.\]
Using this fact, we have

left-hand side of (B.2)

\[ (e_2, L_- S_1) - |Q_p|^{-1}(Q_1pQ_2p \varepsilon_1, S_1) + a(e_1, \Lambda S_1) \]

\[ + 2a(e_1, L_+ T_2) + O(P^2||\varepsilon||_2) \]

\[ = (e_2, \Lambda Q) - |Q_p|^{-1}(aQS_1 \varepsilon_1, S_1) + a(e_1, \Lambda S_1) \]

\[ + 2a(e_1, \frac{1}{2} S_1 - \Lambda S_1 + \frac{1}{2} |S_1|^2) + O(P^2||\varepsilon||_2) \]

\[ = (e_2, \Lambda Q) - a(e_1, \Lambda S_1) + O(P^2||\varepsilon||_2) \]

\[ = (e_2, \Lambda Q_1p) - (e_1, \Lambda Q_2p) + O(P^2||\varepsilon||_2). \]

Estimate (B.3). Indeed, by the definition of \( \rho = \rho_1 + i\rho_2 \), we have

left-hand side of (B.3)

\[ = (e_2, L_- \rho_2) + (e_1, L_+ \rho_1) - |Q_2|^{-1}(aQS_1 \varepsilon_2, \rho_1) \]

\[ - a(e_2, \Lambda \rho_1) + O(P^2||\varepsilon||_2) \]

\[ = a(e_2, S_1 \rho_1) + a(e_2, \Lambda \rho_1) - 2a(e_2, T_2) \]

\[ + (e_1, S_1) - |Q_2|^{-1}((aQS_1, \rho_1) - a(e_2, \Lambda \rho_1)) + O(P^2||\varepsilon||_2) \]

\[ = -2a(e_2, T_2) + (e_1, S_1) + O(P^2||\varepsilon||_2) \]

\[ = -(e_2, \partial_a Q_1p) + (e_1, \partial_a Q_2p) + O(P^2||\varepsilon||_2) \]

\[ = O(P^2||\varepsilon||_2), \]

where we use the orthogonality condition (4.4). Hence we proven this lemma.

\[ \square \]

C. Coercivity estimate for the localized energy

In the following, we assume that \( A > 0 \) is a sufficiently large constant. Let \( \phi : \mathbb{R}^2 \to \mathbb{R} \) be the smooth cutoff function introduced in (5.6), Section 5. For \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^{1/2}(\mathbb{R}^2) \), we consider the quadratic forms

\[ L_{+, A}(\varepsilon_1) := \int_{\mathbb{R}^2} \sqrt{s} \int \Delta \phi_A |\nabla (\varepsilon_1)|^2 dxds + \int |\varepsilon_1|^2 - 2 \int Q|\varepsilon_1|^2 \]  

\[ (C.1) \]

\[ L_{-, A}(\varepsilon_2) := \int_{\mathbb{R}^2} \sqrt{s} \int \Delta \phi_A |\nabla (\varepsilon_2)|^2 dxds + \int |\varepsilon_2|^2 - \int Q|\varepsilon_2|^2, \]  

\[ (C.2) \]

where \( \Delta \phi_A = \Delta(\phi(\frac{1}{\sqrt{s}})) \). As in Lemma 5.1, we denote

\[ u_s = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\pi - \Delta + s}} u, \text{ for } s > 0. \]

\[ (C.3) \]

We start with the following simple identity.

For \( u \in H^{1/2}(\mathbb{R}^2) \), we have

\[ \int_{\mathbb{R}^2} \sqrt{s} |\nabla u|^2 dxds = ||D^{1/2}u||_2^2. \]  

\[ (C.4) \]

Indeed, by applying Fubini’s theorem and using Fourier transform, we find that

\[ \int_{\mathbb{R}^2} \sqrt{s} |\nabla u|^2 dxds = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\sqrt{sd}}{(\xi^2 + s)^2} |\xi|^2 |\hat{u}(\xi)|^2 d\xi = ||D^{1/2}u||_2^2. \]
In general, we have
\[
\frac{2}{\pi} \int_0^\infty \sqrt{s} \int_{\mathbb{R}^2} |(-\Delta)^{3/2} u_s|^2 \, dx \, ds = ||D^{3/2} u||_2^2. \quad (C.5)
\]

Next, we establish a technical result, which show that, when taking the limit \(A \to +\infty\), the quadratic form \(\int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla u_s|^2 \, dx \, ds + ||u||_2^2\) defines a weak topology that serves as a useful substitute for weak convergence in \(H^{1/2}(\mathbb{R}^2)\). The precise statement reads as follows.

**Lemma C.1.** Let \(A_n \to \infty\) and suppose that \(\{u_n\}_{n=1}^\infty\) is a sequence in \(H^{1/2}(\mathbb{R}^2)\) such that
\[
\int_0^\infty \sqrt{s} \int \Delta \phi_{A_n} |\nabla (u_n)|^2 \, dx \, ds + ||u_n||_2^2 \leq C,
\]
for some constant \(C > 0\) independent of \(n\). Then, after possibly passing to a subsequence of \(\{u_n\}_{n=1}^\infty\), we have that \(u_n \to u\) weakly in \(L^2(\mathbb{R}^2)\) and \(u_n \to u\) strongly in \(L^2_{\text{loc}}(\mathbb{R}^2)\), and \(u \in H^{1/2}(\mathbb{R}^2)\). Moreover, we have the bound
\[
||D^{1/2} u||_2^2 \leq \liminf_{n \to \infty} \int_0^\infty \sqrt{s} \int \Delta \phi_{A_n} |\nabla (u_n)|^2 \, dx \, ds.
\]

**Proof.** Let \(\eta \in C^\infty(\mathbb{R}^2)\) be a smooth cutoff function in Fourier space that satisfies
\[
\hat{\eta}(\xi) = \begin{cases} 
1 & \text{for } |\xi| \leq 1, \\
0 & \text{for } |\xi| \geq 2.
\end{cases}
\]

For any \(u \in H^{1/2}(\mathbb{R}^2)\), we write \(u = u^l + u^h\) with
\[
u^l = \hat{\eta} \hat{u}, \quad \hat{u}^h = (1 - \hat{\eta}) \hat{u}.
\]

Recall the definition of \(u_s\), we readily notice that the relations
\[
(u^l)_s = (u_s)^l, \quad (u^h)_s = (u_s)^h.
\]

**Step 1 control of \(u^h\).** Let \(\chi \in C^\infty(\mathbb{R}^2)\) be a smooth cutoff function such that
\[
\chi(x) = \begin{cases} 
1 & \text{for } |x| \leq 1, \\
0 & \text{for } |x| \geq 2.
\end{cases}
\]

For any \(R > 0\) given, we set
\[
\chi_R(x) = \chi \left( \frac{x}{R} \right).
\]

We now claim the following control: For any \(R > 0\), there exist constants \(C_R > 0\) and \(A_0 = A_0(R) > 0\) such that for any \(A \geq A_0\) and \(u \in H^{1/2}(\mathbb{R}^2)\), we have
\[
\int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla u_s^h|^2 \, dx \, ds + ||u||_2^2 \leq C_R \left[ \int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla u_s^h|^2 \, dx \, ds + ||u||_2^2 \right]. \quad (C.6)
\]

Indeed, from definition (C.3) we see that
\[
-\Delta (\chi_R u^h)_s + s (\chi_R u^h)_s = \sqrt{\frac{2}{R}} \chi_R u^h. \quad (C.7)
\]
On the other hand, an elementary calculation shows that
\[-\Delta (\chi_R(u_s)^h) + s \chi_R(u_s)^h = \chi_R(-\Delta (u_s)^h + s (u_s)^h) - 2 \nabla \chi_R \cdot \nabla (u_s)^h - (u_s)^h \Delta \chi_R\]
\[= \sqrt{\frac{2}{\pi}} \chi_R(u_s)^h - 2 \nabla \chi_R \cdot \nabla (u_s)^h - (u_s)^h \Delta \chi_R.\]

Therefore, the function
\[w := \sqrt{\frac{\pi}{2}} \left\{ (\chi_R u_s)^h - \chi_R(u_s)^h \right\}\]
satisfies the equation
\[-\Delta w + sw = \sqrt{\frac{\pi}{2}} \left\{ 2 \nabla \chi_R \cdot \nabla (u_s)^h + (u_s)^h \Delta \chi_R \right\}.

Hence, we deduce the bound
\[\int_{\mathbb{R}^2} |\nabla w|^2 + s \int_{\mathbb{R}^2} |w|^2 \leq \int_{\mathbb{R}^2} \left\{ |\nabla \chi_R||\nabla (u_s)^h| + |(u_s)^h||\Delta \chi_R| \right\} |w|.\]

By using the Cauchy-Schwarz inequality, we conclude that
\[\int_{\mathbb{R}^2} |\nabla w|^2 + s \int_{\mathbb{R}^2} |w|^2 \leq C_R \left\{ \int_{|x| \leq 2R} |\nabla (u_s)^h|^2 + \int |(u_s)^h|^2 \right\} \text{ for } s \geq 1,
\[\int_{\mathbb{R}^2} |\nabla w|^2 + s \int_{\mathbb{R}^2} |w|^2 \leq C_R \frac{s}{s} \left\{ \int |\nabla (u_s)^h|^2 + \int |(u_s)^h|^2 \right\} \text{ for } 0 < s < 1.

Next, we apply identity (C.5) and note that \(\hat{u}_h(\xi) = 0\) for \(|\xi| \leq 1\). For some sufficiently large \(A > A_0(R)\), we obtain
\[\int_1^\infty \sqrt{s} \int_{\mathbb{R}^2} |\nabla w|^2 dxds \leq C_R \int_0^\infty \sqrt{s} \left\{ \int_{|x| \leq 2R} |\nabla (u_s)^h|^2 + \int |(u_s)^h|^2 \right\} ds
\[\leq C_R \int_0^\infty \sqrt{s} \int \Delta \phi_{\lambda} |\nabla (u_s)^h|^2 dxds + ||D^{-1}u_h||_2^2
\[\leq C_R \int_0^\infty \sqrt{s} \int \Delta \phi_{\lambda} |\nabla (u_s)^h|^2 dxds + ||u||_2^2\]

and
\[\int_0^1 \sqrt{s} \int_{\mathbb{R}^2} |\nabla w|^2 dxds \leq C_R \int_0^1 \sqrt{s} \left\{ \frac{(1 + |\xi|^2)}{s + |\xi|^2} |\hat{u}_h|^2 \right\} d\xi ds\]
\[\leq C_R \int_0^1 \frac{1}{\sqrt{s}} \left\{ \frac{1 + |\xi|^2}{|\xi|^4} |\hat{u}_h|^2 \right\} d\xi ds\]
\[= C_R \int_0^1 \frac{1}{\sqrt{s}} \int_{|\xi| \geq 1} \frac{1 + |\xi|^2}{|\xi|^4} |\hat{u}_h|^2 d\xi ds\]
\[\leq C_R ||u||_2^2.\]
Using (C.4) and the previous bounds, we find that
\[
\|D^2(\chi_R u^h)\|_2^2 = \int_0^\infty \sqrt{s} \int |\nabla (\chi_R u^h)_s|^2 dxds \\
\leq \int_0^\infty \sqrt{s} \int |\nabla w|^2 dxds + \int_0^\infty \sqrt{s} \int |\nabla (\chi_R (u^h))_s|^2 dxds \\
\leq C_R \left( \int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla (u^h)_s|^2 dxds + \|u^h\|_2^2 \right) + \int_0^\infty \sqrt{s} \int |u^h|^2 dxds \\
\leq C_R \left( \int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla (u^h)_s|^2 dxds + \|u^h\|_2^2 \right),
\]
which shows the claim (C.6).

**Step 2 Conclusion** Let \( \{u_n\}_{n=1}^\infty \) satisfy the assumption in this lemma. By (C.4), we have for all \( A > 0 \) that
\[
\int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla (u^h_n)_s|^2 dxds \leq \liminf_{n \to +\infty} \int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla (u^h_n)_s|^2 dxds.
\]
Here we used the definition of \( u^h \). Thus the assumed bound in this lemma ensures that
\[
\int_0^\infty \sqrt{s} \int \Delta \phi_A |\nabla (u^h_n)_s|^2 dxds \leq C.
\]

Therefore we conclude from (C.6) that, for all \( R > 0 \), the \( \{u_n\}_{n=0}^\infty \) is a bounded sequence in \( H^{1/2}(B_R) \) and \( L^2(\mathbb{R}^2) \). Hence, by passing to a subsequence if necessary, we can find that
\( u_n \rightharpoonup u \) in \( L^2(\mathbb{R}^2) \) and \( u_n \to u \) in \( H^{1/2}(\mathbb{R}^2) \) for all \( R > 0 \).

By the compactness of the Sobolev embedding \( H^{1/2}(\mathbb{R}^2), \rightarrow L^2_{loc}(\mathbb{R}^2) \), we also have
\( u_n \to u \) in \( L^2_{loc}(\mathbb{R}^2) \).

It remains to show the “weak lower semicontinuity property” given by
\[
\|D^{1/2}u\|_2^2 = \int_0^\infty \sqrt{s} \int |\nabla u_s|^2 dxds \leq \liminf_{n \to +\infty} \int_0^\infty \sqrt{s} \int \Delta \phi_{A_n} |\nabla (u^h_n)_s|^2 dxds.
\]
Indeed, we first note that
\[
\nabla (u^h_n)_s(x) = \sqrt{\frac{2}{\pi}} \int \nabla (G^h(x - y))u_n(y)dy.
\]
Since \( u_n \rightharpoonup u \) weakly in \( L^2(\mathbb{R}^2) \) and \( \nabla G^h(x - y) \in L^2_{loc}(\mathbb{R}^2) \) for any \( x \in \mathbb{R}^2 \), we thus obtain
\( \nabla (u^h_n)_s(x) \rightharpoonup \nabla u_s(x) \) pointwise on \( \mathbb{R}^2 \) for any \( s > 0 \).

Next, by the Cauchy-Schwarz inequality, we derive the uniform pointwise bound
\[
|\nabla (u^h_n)_s| \leq \|\nabla G^h\|_2 \|(u^h_n)_s\|_2 \leq C.
\]
Let \( 0 < \epsilon < 1 \) and \( B > 0 \) be given. By the dominated convergence theorem, we deduce that
\[
\int_{s=\epsilon}^{1/\epsilon} \sqrt{s} \int_{|x| \leq B} |\nabla u_s| dxds = \lim_{n \to +\infty} \int_{s=\epsilon}^{1/\epsilon} \sqrt{s} \int_{|x| \leq B} |\nabla (u^h_n)_s| dxds \\
\leq \liminf_{n \to +\infty} \int_0^\infty \sqrt{s} \int \Delta \phi_{A_n} |\nabla (u^h_n)_s| dxds,
\]
where in the last step we used Fatou’s lemma and the fact that \( \Delta \phi_{A_n} \geq 0 \) satisfies
\lim_{n \to +\infty} \phi_{A_n} = 1 \quad \text{for all } x \in \mathbb{R}^2. \] Since the previous bound holds for arbitrary \( 0 < \epsilon < 1 \) and \( B > 0 \), we conclude that

\[ \|D^{1/2}u\|_2^2 = \int_0^\infty \sqrt{s} \int |\nabla u_n|^2 \, dx \, ds \leq \liminf_{n \to +\infty} \int_0^\infty \sqrt{s} \int \Delta \phi_{A_n} \nabla (u_n) \, dx \, ds. \]

The proof of Lemma C.1 is complete.

**Lemma C.2.** Let \( L_{+,A}(\epsilon_1) \) and \( L_{-,A}(\epsilon_2) \) be the quadratic forms defined (C.1) and (C.2), respectively. Then there exist constants \( C_0 > 0 \) and \( A_0 > 0 \) such that, for all \( A \geq A_0 \) and all \( \epsilon = \epsilon_1 + i \epsilon_2 \in H^{1/2}_rad(\mathbb{R}^2) \), we have the coercivity estimate

\[ (L_{+,A}(\epsilon_1, \epsilon_1) + (L_{-,A}(\epsilon_2, \epsilon_2) \geq C_0 \left| \epsilon \right|^2 - \frac{1}{C_0} \left\{ (\epsilon_1, Q)^2 + (\epsilon_1, S_1)^2 + (\epsilon_2, \rho_1)^2 \right\}. \]

Here \( S_1 \) is the unique functions such that \( L_{-,} = AQ \) with \( (S_1, Q) = 0 \), and the function \( \rho_1 \) is defined in (4.5).

**Proof.** It suffices to prove the coercivity bound

\[ (L_{-,A}(\epsilon_2, \epsilon_2)) \geq C_0 \int |\epsilon|^2 - \frac{1}{C_0} |(\epsilon_2, \rho_1)|^2, \tag{C.8} \]

since the corresponding estimate for \( L_{+,A} \) follows by the same argument.

To prove (C.8), we argue by contradiction as follows. Suppose that there exists a sequence of functions \( \{u_n\}_{n=1}^\infty \subset H^{1/2}(\mathbb{R}^2) \) with

\[ \|u_n\|_2^2 = 1, (u_n, \rho_1) = 0, \]

as well as a sequence \( A_n \to +\infty \) such that

\[ \int_0^{+\infty} \sqrt{s} \int \Delta \phi_{A_n} |u_n|^2 \, dx \, ds + \int |u_n|^2 - \int |Q||u_n|^2 \leq o(1) \int |u_n|^2, \tag{C.9} \]

where \( o(1) \to 0 \) as \( n \to +\infty \). By applying Lemma C.1, we find that (after passing to a subsequence if necessary)

\[ u_n \to u \quad \text{weakly in } L^2(\mathbb{R}^2) \quad \text{and} \quad u_n \to u \quad \text{strongly in } L^2_{loc}(\mathbb{R}^2). \]

But since \( Q(x) \to 0 \) as \( |x| \to +\infty \), we easily check that \( \int |Q||u_n|^2 \to \int |Q||u|^2 \). Moreover, from (C.9) and \( \|u_n\|_2^2 = 1 \), we deduce that \( \int |Q||u|^2 \geq 1 \) must hold. In particular, the weak limit \( u \neq 0 \) is nontrivial. However, by the weak lower semicontinuity inequality in Lemma C.1 and the fact that \( \liminf_{n \to +\infty} \int |u_n|^2 \geq \int |u|^2 \), we deduce that

\[ (L_{-,} u, u) = \int D^2u + \int |u|^2 - \int Q|u|^2 \leq 0, (u, \rho_1) = 0. \]

Since \( u \neq 0 \), this bound contradicts the coercivity estimate for \( L_- \) stated in below.

**Lemma C.3.** For any \( u \in L^2(\mathbb{R}^2) \), we have the bound

\[ \left| \int_0^{+\infty} \sqrt{s} \int \Delta^2 \phi_A |u_n|^2 \, dx \, ds \right| \leq \frac{1}{A} \|u\|_2^2. \]

**Proof.** First, we recall that \( \Delta \phi_A(x) = \Delta \phi(\frac{x}{A}) \) and hence \( \Delta^2 \phi(x) = \frac{1}{A^2} \Delta^2 \phi(\frac{x}{A}) \). Now we consider the following integral

\[ \frac{1}{A^2} \int_0^\infty \int \sqrt{s} \Delta^2 \phi \left( \frac{x}{A} \right) |u_n|^2 \, dx \, ds =: I + II, \]
where
\[ I = \frac{1}{A^2} \int_0^\tau \sqrt{s} \Delta^2 \phi \left( \frac{x}{A} \right) |u_s|^2 ds, \quad II = \frac{1}{A^2} \int_{\tau}^{\infty} \sqrt{s} \Delta^2 \phi \left( \frac{x}{A} \right) |u_s|^2 ds. \]

Here \( \tau > 0 \) is some given number. We can integrate by parts twice and use the Hölder inequality to deduce that
\[
|I| \leq ||\Delta \phi||_\infty \int_0^\tau \sqrt{s}(|\Delta u_s|_2) |u_s|_2 + ||\nabla u_s|_{\infty}^2 ds \\
\leq \int_0^\tau \sqrt{s} \left( \left| \frac{-\Delta}{-\Delta + s} u_s \right|_2 \left| \frac{1}{-\Delta + s} u_s \right|_2 + \left| \nabla \frac{1}{-\Delta + s} u_s \right|_2 \right) ds \\
\leq \int_0^{\tau} \frac{1}{s^{1/2}} ds ||u||_2^2 \leq \sqrt{\tau} ||u||_2^2.
\]
where we use the bound \( ||u_s||_2 \leq s^{-1/2} ||u||_2 \). To estimate \( II \),
\[
|II| \leq \frac{1}{A^2} ||\Delta \phi||_\infty \int_{\tau}^{\infty} \frac{1}{s^{1/2}} ds ||u||_2^2 \leq \frac{1}{A^2 \sqrt{\tau}} ||u||_2^2.
\]
Thus, we have shown that
\[
\frac{1}{A^2} \int_0^{\infty} \sqrt{s} \Delta^2 \phi \left( \frac{x}{A} \right) |u_s|^2 ds \leq \left( \sqrt{\tau} + \frac{1}{A^2 \sqrt{\tau}} \right) ||u||_2^2 \leq \frac{1}{A} ||u||_2^2.
\]
In the last step, we minimizing this bound with respect to \( \tau \).

**Lemma C.4.** There exist a constant \( C_1 > 0 \) such that for all \( \mathbf{e} = \mathbf{e}_1 + i \mathbf{e}_2 \in H^{1/2}(\mathbb{R}^2) \), we have the coercivity estimate
\[
(L+ \mathbf{e}_1, \mathbf{e}_1) + (L- \mathbf{e}_2, \mathbf{e}_2) \geq C_1 \int |\mathbf{e}|^2 - \frac{1}{C_1} \{ (\mathbf{e}_1, Q)^2 + (\mathbf{e}_1, S_1)^2 + |(\mathbf{e}_2, \rho_1)|^2 \}.
\]
Here \( S_1 \) is the unique functions such that \( L_- S_1 = \Lambda Q \) with \( (S_1, Q) = 0 \) and the function \( \rho_1 \) is defined in (4.5).

**Proof.** From [15] we recall that the key fact that the null space of \( L_+ \) and \( L_- \) are given by
\[
\ker L_+ = \text{span}\{\nabla Q\} = \text{span}\{\partial_1 Q, \partial_2 Q\}, \quad \ker L_- = \text{span}\{Q\}.
\]
Moreover, \( L_+ \) has a unique negative eigenvalue, while \( L_- \geq 0 \).
If we consider the minimization problem
\[
\inf_{g \in H^{1/2}, \ g \perp Q} (L_- g, g)_{L^2},
\]
then we have two possibilities: or there exists \( g \neq 0, g \in H^{1/2}(\mathbb{R}^2), g \perp Q \) such that
\[
L_- g = aQ
\]
or
\[
(L_- g, g) \geq C ||g||_{H^{1/2}}^2, \forall g \perp Q. \tag{C.11}
\]
The first possibility leads easy to a contradiction. Indeed, multiplying by \( Q \) we find \( a = 0 \) and then we arrive at a contradiction with (C.10). Therefore, remains (C.11) and this estimate implies
\[
(L_- \mathbf{e}_2, \mathbf{e}_2) \geq C ||\mathbf{e}_2||_{H^{1/2}}^2 - \frac{1}{C} |(\mathbf{e}_2, Q)|^2. \tag{C.12}
\]
In a similar way, we consider the minimization problem
\[
\inf_{g \in H^{1/2}_{\text{rad}}, g \perp \varphi, g \perp \nabla Q} (L_+ g, g)_{L^2},
\]
and using the argument of sections 7.1 and 7.2 in deduce
\[
(L_+ g, g) \geq C\|g\|_{H^{1/2}}^2, \forall g \perp \varphi, g \perp \nabla Q.
\] (C.13)
As before this estimate implies
\[
(L_+ \epsilon_1, \epsilon_1) \geq C\|\epsilon_1\|_{H^{1/2}}^2 - \frac{1}{C_1} \{(\epsilon_1, \varphi)^2 + (\epsilon_1, \nabla \varphi)^2 + (\epsilon_2, Q)^2\}
\] (C.14)
From (C.12) and (C.14) we find
\[
(L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) \geq C_1\|\epsilon_1\|_{H^{1/2}}^2 - \frac{1}{C_1} \{(\epsilon_1, \varphi)^2 + (\epsilon_1, \nabla \varphi)^2 + (\epsilon_2, Q)^2\}
\] (C.15)
for all \(\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}_{\text{rad}}(\mathbb{R}^2)\), where \(C_1 > 0\) is some constant. Here \(\varphi = \varphi(x) > 0\) with \(\|\varphi\|_2^2 = 1\) denotes the unique ground state eigenfunction of \(L_+\), and we have \(L_+ \varphi = e\varphi\) with some \(e < 0\).
To derive the coercivity estimate in this lemma from an estimate of the form (C.15), we can use some arguments that can be found, for example, in [26]. For the reader’s convenience, we provide the details of the adaptation to our case. To prove the desired coercivity estimate, we can assume \(\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}(\mathbb{R}^2)\) satisfies
\[
(\epsilon_1, S_1) = (\epsilon_2, \rho_1) = 0.
\]
and let the auxiliary function \(\tilde{\epsilon} = \epsilon_1 + i\epsilon_2\) satisfy
\[
\tilde{\epsilon} = \epsilon - a\Lambda Q - ibQ,
\]
where \(a, b, c\) are chosen such that
\[
(\epsilon_1, \varphi) = (\epsilon_2, Q) = 0.
\]
That is
\[
a = \frac{(\epsilon_1, \varphi)}{(\Lambda Q, \varphi)}, \quad b = \frac{(\epsilon_2, Q)}{(Q, Q)},
\]
where we also used that \((\Lambda Q, \nabla Q) = 0\) holds and \((\varphi, \nabla Q) = 0\) since \(\nabla Q \in \ker L_+\) and \(\varphi \in \text{ran} L_+\). Next, recall that \(L_+ \varphi = e\varphi\) with \(e < 0\) and \(L_+ \Lambda Q = -Q\). Hence \((\Lambda Q, \varphi) = -(\varphi, Q) > 0\), by the strict positivity of \(Q > 0\) and \(\varphi > 0\). On the other hand, the orthogonality conditions satisfied by \(\epsilon = \epsilon_1 + i\epsilon_2\) imply that
\[
a = \frac{(\tilde{\epsilon}_1, S_1)}{(\Lambda Q, S_1)}, \quad b = \frac{-\tilde{\epsilon}_2, \rho_1}{(Q, \rho_1)},
\]
where we use that \((\nabla Q, S_1) = 0\), since \(Q\) and \(S_1\) are radial symmetry. Note that \(L_- S_1 = \Lambda Q\) and hence \((\Lambda Q, S_1) = (L_- S_1, S_1) \neq 0\). Furthermore, recall that \(L_+ \rho_1 = S_1\). Thus \((Q, \rho_1) = -(\Lambda Q, S_1) = (L_- S_1, S_1) > 0\) again. In summary, we find that
\[
\frac{1}{K} \|\epsilon\|_{H^{1/2}} \leq \|\tilde{\epsilon}\|_{H^{1/2}} \leq K \|\epsilon\|_{H^{1/2}}
\]
with some constant \(K > 0\). Now, since \((\Lambda Q, Q) = 0\) and \(L_+ \Lambda Q = -Q\) as well as \(L_- Q = 0\), we obtain
\[
(\tilde{\epsilon}_1, Q) = (\epsilon_1, Q), \quad (L_- \tilde{\epsilon}_2, \epsilon_2) = (L_- \epsilon_2, \epsilon_2)
\]
\[
(L_+ \epsilon_1, \epsilon_1) = (L_+ \epsilon_1, \epsilon_1) + a(\epsilon_1, Q).\]
By the previous relations and estimate (C.15), we conclude
\[
(L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) = (L_+ \tilde{\epsilon}_1, \tilde{\epsilon}_1 + (L_- \tilde{\epsilon}_2, \tilde{\epsilon}_2) - a(\epsilon_1, Q)
\geq C_1 \| \tilde{\epsilon} \|_{H^{1/2}}^2 - a(\epsilon_1, Q) \geq C_0 \| \epsilon \|_{H^{1/2}}^2 - \frac{1}{C_0} (\epsilon_1, Q)^2,
\]
with some sufficiently small constant $C_0 > 0$. \qed