Point Process-based estimation of $k^{th}$-order moment

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Abstract This paper addresses the issue of estimating the $k^{th}$-order moment of a real-valued random variable that may have heavy tailed distribution. While Monte-Carlo estimator is widely used for this purpose its main disadvantage is its relatively high — and possibly infinite — variance. We introduce here a new framework to address this issue. Using recent results on Point Processes related to real-valued random variables, we derive a new unbiased estimator for $k^{th}$-order moment that has finite variance and asymptotic normal distribution as soon as there exists a moment of order $k'$ for some $k' > k$. It remains versatile, requiring only the continuity of the cumulative distribution function of the random variable, and intrinsically parallel.

Keywords Heavy tails · moment estimation · trimmed mean · tail index estimation

1 Introduction

Monte-Carlo method (Metropolis and Ulam 1949) is based on the rather intuitive idea that repeating a random experience several times would lead to some statistical knowledge about it (Eckhardt 1987). It finds its mathematical support combining the Strong Law of Large Numbers (SLLN) and the Central Limit Theorem (CLT); given $n$ independent and identically distributed (iid) copies of a real-valued random variable $X$, the SLLN says:

$$\hat{X}_n \overset{\text{def}}{=} \frac{X_1 + X_2 + \cdots + X_n}{n} \overset{a.s.}{\rightarrow} \mathbb{E}[X]$$

while the CLT gives a convergence in law:

$$\frac{\hat{X}_n - \mu}{\sigma/\sqrt{n}} \overset{\text{d}}{\rightarrow} \mathcal{N}(0, 1)$$

where $\mu = \mathbb{E}[X]$, $\sigma^2 = \text{var}[X] \in (0, \infty)$. One of the main drawbacks of this method is that it requires $\sigma < \infty$ to get a finite variance for the estimator $\hat{X}_n$. This limitation may be critical when dealing with heavy-tailed distributions, like Pareto or Pareto type distributions.

In this scope, variance-reduction techniques such as Importance Sampling (IS) can circumvent this limitation. Indeed, let $f_X$ be the probability density function (pdf) of $X$ and $q$ be another pdf such that $q(x) = 0 \Rightarrow f_X(x) = 0$, IS makes use of $n$ (iid) copies $(Y_i)_i$ with pdf $q$ and computes the following estimator:

$$m_{IS} \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} Y_i \frac{f_X(Y_i)}{q(Y_i)}$$

This estimator is unbiased and has a finite variance as soon as $\mathbb{E}_q \left[ X^2 (f_X/q)(X)^2 \right] = \mathbb{E}_X \left[ X^2 (f_X/q)(X) \right] < \infty$. It is then possible to choose $q$ such that $\text{var}[m_{IS}] < \infty$. However it requires the knowledge of the likelihood ratio $f_X/q$ and the capacity to generate samples according to $q$ (see Robert and Casella 2004) or (Glynn and Iglehart 1989) for further details on Importance Sampling.

Other work has been devoted to order statistics, tail index estimation and trimmed sums (Bickel et al 1965; Stigler 1973; Griffin 1988; Berkes and Horváth 2012). While producing finite-variance estimators that are unbiased or with a controlled bias, they always assume parametric forms for the distribution functions and give only limited results; see (Beirlant et al 2012) for a comprehensive overview of tail index estimation, and (Peng 2001; Johansson 2003; Necir et al 2010; Hill 2013) for references on mean estimation for heavy-tailed random variables.
Using recent results on Point Processes related to a real random variable with continuous cdf (Walter 2014) we introduce a new general estimator for $k^{th}$-order moment. Our estimator is unbiased with finite variance as soon as a moment of order $k'$ exists for some $k' > k$, and supports also a CLT. Furthermore, it only requires to be able to generate samples according to $\mu^X$ (for instance the output of a complex numerical code with random inputs).

The outline of this paper is as follows: Section 2 presents the new ideal (not practically implementable) estimator and compare it with the usual Monte Carlo one’s; Section 3 develops two possible estimator based on the ideal one and then Section 4 deals with the technical issues of implementing the estimator and provides examples of its efficiency; then an Appendix gathers all the proofs.

2 Ideal estimator

First of all, let us recall that for a real-valued random variable $X$, one can write $X = X_+ - X_-$ with $X_+$ and $X_-$ non-negative random variables. Then, $E[X] = E[X_+] - E[X_-]$. Thus in the sequel and without loss of generality we assume that $X$ is a non-negative random variable with law $\mu^X$. We also assume that $X$ has a continuous cdf $F$ and we write $p_x$ instead of $P[X > x]$, for any $x \in \mathbb{R}^+$. In this section, we further assume that it is possible to generate random variables with distribution $\mathcal{F}(X | X > x)$ for any $x \in \mathbb{R}_+$. (we explain how to build a simulator for the conditional distributions in Section 4.1).

2.1 Extreme event simulation

In this section we recall the results of (Walter 2014).

Definition 1 (Point Process associated with $X$) Let $X_0 = 0$ and define recursively the Markov sequence $(X_n)_n$ such that:

$$\forall n \in \mathbb{N} : P[X_{n+1} \in A | X_0, \ldots, X_n] = \frac{\mu^X(A \cap (X_n, +\infty))}{\mu^X((X_n, +\infty))}$$

In other words $(X_n)_n$ is a strictly increasing sequence where each element is generated conditionally greater than the previous one. Considering the sequences $(X_n)_{n \geq 1}$ and $(T_n)_{n \geq 1}$, $T_n = - \log(1 - F(X_n))$, it can be shown that $(T_n)_{n \geq 1}$ is distributed as the arrival times of a Poisson Process with parameter 1. Thus, the counting random variable of the number of events before $x$: $M_x = \text{card}\{n \geq 1 | X_n \leq x\}$ follows a Poisson law with parameter $- \log p_x$.

These results lead to the construction of a new estimator for a probability of exceeding a threshold. Indeed, simulating $N \geq 2$ independent point processes $(X_n^{(i)})_{n \geq 1, i = 1..N}$, one can consider the marked Poisson Process whose counting variables $M_t = \text{card}\{ (i,n) | i = 1..N, n \geq 1, X_n^{(i)} < x\}$ follow the Poisson law with parameter $-N \log p_x$ and so introduce the following estimator:

$$\hat{p}_x = \left( 1 - \frac{1}{N} \right)^{M_t} \quad (1)$$

Proposition 1 (Statistical properties of $\hat{p}_x$)

$$E[\hat{p}_x] = p_x \quad \text{var} [\hat{p}_x] = p_x^2 \left( p_x^{-1/N} - 1 \right)$$

This estimator has been shown to have a smaller variance than a Monte-Carlo one, exhibits a logarithmic efficiency and almost achieves the Cramer-Rao bound $-p_x^2 \log p_x / N$ (Walter 2014).

2.2 Definition of the moment estimator

While the purpose of (Walter 2014) was only the estimation of extreme probability and quantile, we introduce here a new $k^{th}$-order moment estimator based on (1). Let $k \in (0, \infty)$ be such that $m_k \overset{df}{=} E[X^k] < \infty$. One has:

$$m_k = \int_0^\infty k x^{k-1} p_x dx \quad (2)$$

This latter formulation brings the idea of using (1) to build an estimator for $m_k$.

From now on we will consider that $N \geq 2$ point processes have been simulated and denote by $(M^k_t)_t$ the random counting variables associated with the marked Poisson Process with parameter $N$: $\forall x > 0, M^k_t \sim \mathcal{P}(-N \log p_x)$. The sequence $(X_n)_{n \geq 1}$ is the cumulated one, i.e the combination of the $N$ Markov Chains sorted in increasing order; then the associated $(T_n)_{n \geq 1}$ are the times of the marked Poisson Process with parameter $N$. We set $X_0 = 0$ and then consider the following estimator:

$$\hat{m}_k = \int_0^\infty k x^{k-1} \left( 1 - \frac{1}{N} \right)^{M_t} dx = \sum_{i=0}^{\infty} \left( X_{i+1}^k - X_i^k \right) \left( 1 - \frac{1}{N} \right)^i \quad (3)$$

While the first form is easier to analyse as the law of $(M^k_t)_t$ is well determined, the second one paves the way for the practical implementation (see Section 3 and 4).

Proposition 2 (Statistical properties of $\hat{m}_k$)

$$E[\hat{m}_k] = m_k \quad \text{var} [\hat{m}_k] = 2 \int_0^\infty \int_0^x k^2 x^{k-1} \left( x^{(k-1)} p_x p_x^{-1/N} dx \right) dx - m_k^2$$

We thus have defined an unbiased estimator for $m_k$. We now focus on its variance as this is the critical point as mentioned in the Introduction.
Proposition 3 (Finiteness of $\text{var} [\hat{m}_k]$) The variance of $\hat{m}_k$ can be bounded from above by $C_{k,N} \text{E}[X^{2k}]^{2k/k'}$ for any $k' > \max((kN - 1)/(N - 1), k + 1/N)$ and $C_{k,N}$ a constant depending only on $k$ and $N$.

Corollary 1 (Value of $N$) Let $k' > k$, if $\text{E} \left[ X^{k'} \right] < \infty$ then for any $N > \max((k' - 1)/(k' - k), 1/(k' - k))$, $\hat{m}_k$ has a finite variance.

While usual Monte-Carlo estimator requires the finiteness of $\text{E} [X^{2k}]$ to have a finite variance, our estimator only requires the finiteness of a moment of order $k' > k$.

2.3 Comparison with Monte-Carlo methods

We now intend to compare our estimator with the classical Monte-Carlo one. As the finiteness condition of the variance is much weaker for our estimator, one can expect a globally lower variance. This result is shown in Proposition 4. We first recall the crude Monte-Carlo estimator of the $k^{th}$-order moment:

$$\hat{m}_{MC} \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} X_i^k$$

with $(X_i), N$ (iid) random variables with law $\mu^X$ and the Importance Sampling estimator:

$$\hat{m}_{IS} \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \frac{X_i^k f_X(X_i)}{q(X_i)}$$

with $(X_i), N$ (iid) random variables with importance density $q$.

Proposition 4 For any $N \geq 2$, $\text{var} [\hat{m}_k] \leq \text{var} [\hat{m}_{MC}]$.

Thus this estimator is always better than Monte-Carlo in terms of variance. As for this class of problems Pareto-type distributions are often considered, we then present an explicit result for a distribution $P[X > x] = 1 - x^{-a}, a > 1$. In the case of an IS estimator, the pdf $q$ is chosen to be a Pareto distribution with parameter $b > 0$.

Proposition 5 (Results for a Pareto distribution) For a Pareto distribution, one has $m_k = a/(a - k)$ and the variances write:

$$\forall k < \frac{a}{2}, \text{var} [\hat{m}_{MC}] = \frac{m_k (m_k - 1)^2}{N}$$

$$\forall k < a \left( 1 - \frac{1}{2N} \right), \text{var} [\hat{m}_k] = \frac{m_k (m_k - 1)^2}{2N - m_k}$$

$$\forall k < a - \frac{b}{2}, \text{var} [\hat{m}_k] = \frac{m_k^2 (B - 1)^2}{N(2B - 1)}$$

with $B = (a - k)/b \in (1/2, \infty)$.

On this latter case it is clearly visible that the Monte-Carlo estimator needs a $2k^{th}$-order moment while $\hat{m}_k$ only requires $a - k > a/(2N)$ and $\hat{m}_{IS}$ requires $a - k > b/2$. The optimal value $b = a - k$ (ie $B = 1$) cancels out $\text{var} [\hat{m}_{IS}]$. It is well known that there is an optimal density $q$ for IS that cancels out the variance of the IS estimator. It turns out that the optimal density is a Pareto density when the original distribution is Pareto. However the IS strategy is intrusive as it requires to simulate the random variable with a biased distribution and it also requires the knowledge of the likelihood ratio.

Remark 1 (Limit distribution of Monte-Carlo estimator) In the case of Pareto distribution, for $2k < a$ the Central Limit Theorem gives the limit law of the estimator while for $1 < a/k < 2$ the Generalised Central Limit Theorem (see for example (embrechts et al 1997)) states that $\sum_i X_i^k$ is in the domain of attraction of a stable law with parameter $\alpha = a/k \in (1, 2)$:

$$N^{1-\alpha} \left( \frac{1}{N} \sum_{i=1}^{N} X_i^k - m_k \right) \frac{1}{C_{\alpha}} \xrightarrow{\mathcal{L}} X_{\alpha}$$

with the characteristic function of $X_{\alpha}, \phi_{X_{\alpha}}$, writing $\phi_{X_{\alpha}}(t) = \exp \left[ -|t|^\alpha \left( 1 - i \left( \tan(\pi\alpha/2) \right) \text{sign}(t) \right) \right]$ and $C_{\alpha}$ a normalising constant $C_{\alpha} = \pi^{1/\alpha} (2^{1-\alpha} \Gamma(\alpha) \sin \pi\alpha/2)^{-1/\alpha}$.

To conclude this section, one can stress out the fact that our estimator is always better than the usual Monte-Carlo estimator and especially does not require the finiteness of the $2k^{th}$-order moment to have a finite variance. This is especially useful when working with heavy-tailed random variables. Unlike Importance Sampling techniques it does not require the choice of a biased distribution and the knowledge of the likelihood ratio.

3 Randomised unbiased estimator

The estimator defined Section 2 is not directly usable as it requires to simulate the infinite sum (3). Using recent results on paths simulation, we propose a randomised unbiased estimator.

3.1 Definition

We are facing the issue of estimating $\text{E}[\hat{m}_k]$ while it is not possible to generate such a $\hat{m}_k$ in a finite computer time. This problem is well identified in the field of Stochastic Differential Equations (SDE) where one often intends to compute the expectation of a path functional while only discrete-time approximations are available. Recently there have been two major breakthroughs that address this issue: first the Multilevel Monte-Carlo (MLMC) method (giles 2008) has
introduced the idea of combining intelligently different biased estimators (levels of approximations) to speed up the convergence and reduce the bias; then (McLeish 2011) and (Rhee and Glynn 2013) have introduced a general approach to constructing unbiased estimator based on a family of biased ones. Basically in our context it randomises the number of simulated steps of the Markov chain, and combines the different approximations to remove the bias of the final estimator.

More precisely let us consider the randomly truncated estimators \( \hat{m}_k^n \) \( n \geq 1 \):

\[
\hat{m}_k^n = \int_0^{X_k} k x^{k-1} \left( 1 - \frac{1}{N} \right)^{M_k} dx \\
= \sum_{i=0}^{n-1} \left( X_{k+1}^i - X_k^i \right) \left( 1 - \frac{1}{N} \right)^i
\]

and \( T \) a non-negative integer-valued random variable independent of \( (X_k)_{n \geq 1} \) such that \( \forall i \in \mathbb{N}, P[T \geq i] \equiv \beta_i > 0 \); one builds the following estimator (with \( \hat{m}_k^n = 0 \)):

\[
\hat{Z}_k = \sum_{n=0}^{\infty} \hat{m}_k^{n+1} - \hat{m}_k^n, \quad T \geq n = \sum_{n=0}^{T} \hat{m}_k^{n+1} - \hat{m}_k^n \\
\hat{Z}_k = \sum_{n=0}^{\infty} \left( X_{k+1}^i - X_k^i \right) \left( 1 - \frac{1}{N} \right)^n \frac{\mathbb{I}_{T \geq n}}{P[T \geq n]} \tag{5}
\]

**Proposition 6 (Statistical properties of \( \hat{Z}_k \))**

\[
\begin{align*}
E[\hat{Z}_k] &= m_k \\
\text{var}[\hat{Z}_k] &= \sum_{i=0}^{\infty} q_i \beta_i^{-1} - m_k^2 \\
\end{align*}
\]

with:

\[
q_i = c_{k,N,t} \int_0^\infty \int_0^{\infty} x^{k-1} \lambda(x) \parallel f(x) \parallel \left[ -N \log p(x) \right]^{i} \frac{dx}{x^i} \tag{6}
\]

and \( c_{k,N,t} = k^2 (1 - 1/N)^2 \).

One can rewrite the \( (q_i) \); assuming \( X \) has a density \( f_X \). Indeed in this context \( X_k \) has a density \( f_k \) such that:

\[
\forall k \geq 1, \quad f_k(x) = N f_k^{N-1}(x) \frac{\left( -N \log p(x) \right)^{k-1}}{(k-1)!} f_X(x)
\]

which brings for the \( (q_i) \) (with \( k = 1 \)):

\[
\forall i \in \mathbb{N}, \quad q_i = \frac{2}{N} \left( 1 - \frac{1}{N} \right)^{2i} E[S(X_{i+1})]
\]

with \( S(x) = E[X \mid X > x] / f_X(x) \). The function \( S \), depending only on the distribution of \( X \), can be seen as a measure of interest of moving forward from a given position \( x \): if \( x \) is highly not likely and high values of \( X \) are expected above \( x \), then \( S(x) \) will be large, meaning the process should go forward; on the opposite if \( x \) is relatively highly probable and nothing large is expected above, then there is no need to go further and one should rather stop the process to avoid useless simulations.

**Lemma 1** The sequence \( (q_i) \) decreases at exponential rate. Furthermore, if \( X \) has density \( f_X \) such that \( \| f_X \|_\infty < \infty \), it is also bounded from below by an exponentially decreasing sequence.

As we will see below, this rate of convergence allows for light tail randomising variables, which means that the number of simulated samples for each \( \hat{Z}_k \) can remain low. This is especially interesting as throughout the paper we consider that the computational cost for generating a realisation of \( \hat{Z}_k \) is the number of simulated samples. In this section, it is the number of calls to a simulator of a conditional law.

**Proposition 7** Let \( \tau \) be the random variable of the number of samples required to generate \( \hat{Z}_k \). One has \( \tau = N + T \).

**Corollary 2 (Convergence rate of \( \hat{Z}_k \))** For any randomising variable \( T \), one has:

\[
E[\tau] \cdot \text{var}[\hat{Z}_k] > 2q_1(N = 2) + O \left( \frac{1}{N} \right), \quad N \to \infty
\]

Thus, depending on the order of magnitude of \( E[\tau] \cdot \text{var}[\hat{Z}_k] \) as \( N \to \infty \), \( \hat{Z}_k \) has a canonical square-root convergence rate (as a function of the computational cost) or not. To circumvent this limitation, the idea behind randomised estimators is to average several replicas of \( \hat{Z}_k \) because it will average the quantities \( \mathbb{I}_{T \geq n} / P[T \geq n] \) in (5). More precisely, let \( \Gamma(c) \) be the random variable of the number of simulations of \( \hat{Z}_k \) one can afford with a computational budget \( c \):

\[
\Gamma(c) = \max \{ n \geq 0 \mid \sum_{i=1}^{n} \tau_i \leq c \}
\]

where \( \tau_i \) is the computational effort required to generate the \( i^{th} \)-sample \( \hat{Z}_k^i \). One considers the following estimator:

\[
\hat{\alpha}(c) = \frac{1}{\Gamma(c)} \sum_{i=1}^{\Gamma(c)} \hat{Z}_k^i
\]

In this setting (Glynn and Whitt 1992) showed a CLT-like result:

\[
\sigma^{1/2} \left( \hat{\alpha}(c) - E[\hat{Z}_k] \right) \xrightarrow{c \to \infty} N \left( 0, \frac{1}{N} \right)
\]

Hence in our context one has to tune \( (\beta_i) \) and \( N \) to minimise the product \( E[\tau] \cdot \text{var}[\hat{Z}_k] \).
3.2 Optimal randomisation

Since $T$ is a non-negative random variable, one has $\beta_0 = 1$. Let $\mathcal{F} = \{(\beta_i) \in (0,1]^n | \beta_0 = 1$ and $\forall i \in \mathbb{N}, \beta_{i+1} \leq \beta_i\}$; we intend to solve the optimisation problem:

$$\arg\min_{(\beta_i) \in \mathcal{F}} \left( N - 1 + \sum_{i=0}^{\infty} \beta_i \right) \left( \sum_{i=0}^{\infty} q_i \beta_i^{-1} - m_i^2 \right) \quad \text{(7)}$$

where the $(q_i)$ depend on $N$ and are given by (6). We further assume that $(q_i)$ is decreasing, which will be the case for a Pareto random variable and at least for any distribution for which $\mathcal{F}$ is non-increasing like exponential and uniform distributions. In this context Proposition 8 gives the optimal distribution for $T$ for a given $N$.

**Proposition 8 (Optimal distribution for $T$)** If $(q_i)_{i \geq 1}$ is decreasing then the optimal distribution $(\beta_i^*)$, for $T$ is given by:

$$\forall i \in [0, i_0], \beta_i^* = 1$$

$$\forall i > i_0, \beta_i^* = \sqrt{\frac{N + i_0}{S_0}} \sqrt{\frac{q_i}{\beta_i}}$$

with $i_0 = \min\{i \in \mathbb{N} | \sum_{j=0}^{i} q_j - m_i^2 > (N+i)q_{i+1}\}$ and $S_0 = \sum_{j=0}^{i_0} q_j - m_i^2$.

It is part of the proof in the appendix that $i_0$ is well defined and so it appears that the optimal distribution enforces the estimator to go at least until the $i_0$th event. Recalling $(X_n)_n$ is the cumulated Markov Chain (associated with the marked Point Process with parameter $N$), this can be understood in the sense that one requires at least $N$ events to use at least one time each process; and this can be even more if $X$ is heavy-tailed as the first event of a given Point Process can then arrive very late. Even if the link between $i_0$ and $N$ is not that straightforward, one can then conjecture that $\lim_{N \to \infty} i_0 = \infty$.

**Corollary 3 (Bounds on $\beta_i^*$)** For all $i > i_0$, one has:

$$\sqrt{\frac{q_i}{q_{i+1}}} > \beta_i^* \geq \sqrt{\frac{q_i}{q_{i_0}}}$$

Thus the tail of the optimal distribution $(\beta_i^*)$ is exponentially decreasing by Lemma 1. From these bounds on the $(\beta_i)$, one can also derive bounds on the variance:

$$q_{i_0+1} E[\tau^2] < E[\tau] \cdot \var\left[\hat{Z}_k\right] \leq q_{i_0} E[\tau]^2$$

Assuming $\lim_{N \to \infty} i_0 = \infty$ and using the lower bound on $q_i$ from Lemma 1, one could show that $\lim_{N \to \infty} E[\tau] \cdot \var\left[\hat{Z}_k\right] = \infty$, which implies the existence of an optimal $N$. We now present an exact resolution of this optimisation problem for a Pareto random variable.

**Proposition 9 (Result for a Pareto distribution)** If $X$ is a Pareto random variable with parameter $a > k$, then:

$$\forall i \in \mathbb{N}, q_i = 1 + \frac{a + k}{a - k} + \frac{c_k N}{(a - k)(aN - 2k)} \left[ \frac{aN}{aN - 2k} \right]^i$$

Hence for a Pareto distribution $(q_i)$ is decreasing. One can then look for $i_0$, the solution of the problem $i_0 = \min\{i \in \mathbb{N} | \sum_{j=0}^{i} q_j - m_i^2 > (N+i)q_{i+1}\}$. Whilst an exact solution can be expressed using the lower branch of the Lambert W function (see for example (Corless et al. 1996)), the following proposition gives an asymptotic approximation when $N \to \infty$ to precise the growth rate of $i_0$.

**Proposition 10** If $X$ is a Pareto random variable, then:

$$i_0 = \frac{Nm_k}{2} \left( \log N + \log \log N - \log(m_k/2) + o(1) \right)$$

**Corollary 4 (Order of magnitude of $E[\tau] \cdot \var\left[\hat{Z}_k\right]$)**

$$E[\tau] \cdot \var\left[\hat{Z}_k\right] \sim N \to \infty \left( \frac{m_k(m_k-1)}{2} \right)^2 \log N$$

$E[\tau] \cdot \var\left[\hat{Z}_k\right] \to \infty$ when $N \to \infty$ so there is an optimal value for $N$; a numerical resolution for several values of $a/k$ from 1 to 3 and for $N \in [2, 1000]$ was performed and the result is displayed in Figure 1a. We also present in Figure 1b a comparison between this optimal variance (with the optimal distribution $(\beta_i^*)$, and optimal $N$) and a Monte-Carlo one. There we can see that for $a/k \leq 2.5$ this new estimator performs better in terms of variance; especially for $a/k < 2$ it remains finite while $\var\left[\hat{Z}_{\text{MC}}\right] = \infty$.

Finally, we have presented in this section the framework for an optimal resolution of Problem (7) and proven existence of a solution under reasonable assumptions $(q_i)$ is decreasing and $\lim\inf i_0 = \infty$. Furthermore the comprehensive resolution of this problem in the case of a Pareto distribution legitimises them. Generally speaking, if $(q_i)_{i \geq 1}$ is not decreasing the optimisation has to be performed over all the decreasing sub-sequences of $(q_i)$, which turns it into a combinatorial problem, see (Rhee and Glynn 2013) Theorem 3 for more details.

3.3 Geometric randomisation

On the one hand the computation of the optimal distribution for $T$ can be quite demanding in computer time; and on the other hand the geometric law plays a key role as for any distribution $(q_i)_i$ decreases at exponential rate and the optimal randomising distribution (when $(q_i)_i$ is decreasing) is somehow a shifted geometric law. Therefore we study the parametric case where $P[T \geq n] = e^{-\beta n}, \beta > 0$ and tune $\beta$ and $N$ to minimise $E[\tau] \cdot \var\left[\hat{Z}_k\right]$. 


Proposition 11 (Variance of $\hat{Z}_K$) If $P[T \geq n] = e^{-\beta n}$, $\beta > 0$, then:
\[ \text{var} \left[ \hat{Z}_K \right] = 2 \int_0^\infty \int_0^\infty k^2 e^{-k(x+y)} p_\tau e^{-\frac{1}{\gamma(N)}} \text{d}x \text{d}y = \frac{1}{\gamma(N)^2} \] (8)

with $\gamma(\beta, N)$ as defined in Proposition 10 (3).

This expression is indeed the same as the one of Proposition 2 with the function $\gamma(\beta, N)$ instead of $N$; choosing $\beta = 0$, $i.e.$ a non-truncated sum, gives $\text{var} \left[ \hat{Z}_K \right] = \text{var} [\tilde{m}_K]$. Hence one has directly all the results from Section 2.2, especially the finiteness conditions for the variance given in Proposition 3 and Corollary 1, replacing $N$ by $\gamma(\beta, N)$. Furthermore, using the exponential power series in $\text{var} \left[ \hat{Z}_K \right]$, the optimisation problem (7) becomes:
\[ \min_{\beta > 0} \left( N + \frac{1}{e^\beta - 1} \right) \left( \sum_{i=0}^\infty q_i(N = 2) \left( \frac{2}{\gamma(\beta, N)} \right)^i - m_2^2 \right) \] (9)

Proposition 12 There exists a global minimiser ($\beta_{\text{opt}}, N_{\text{opt}}$) to Problem (9).

However there are no closed-form expressions for $\beta_{\text{opt}}$ and $N_{\text{opt}}$. In this context the following results give an idea of the link between $\beta$, $N$ and the total variance.

Corollary 5 ($\beta_{\text{opt}}, N_{\text{opt}}$) satisfies the relationship:
\[ \beta_{\text{opt}} = \log \left( 1 + \frac{2}{(N_{\text{opt}}^2 - 1) + \sqrt{(N_{\text{opt}}^2 - 1) + 4N_{\text{opt}}}} \right) \] (10)

Corollary 6 (Order of magnitude of $E[\tau] \cdot \text{var} \left[ \hat{Z}_K \right]$) If $\beta = \Theta(1/N^{1+\varepsilon})$, $\varepsilon > 0$ then:
\[ \begin{align*}
E[\tau] \cdot \text{var} \left[ \hat{Z}_K \right] = \Theta(N) & \quad \varepsilon \in [0, 1] \\
E[\tau] \cdot \text{var} \left[ \hat{Z}_K \right] = \Theta(N^{\varepsilon}) & \quad \varepsilon > 1
\end{align*} \]

Since the finiteness conditions on $\text{var} \left[ \hat{Z}_K \right]$ are expressed as lower bounds on $\gamma$ and $\gamma$ decreases with $\beta$, choosing $\beta = \Theta(N^{-2})$ seems to be a conservative choice to insure the finiteness of the variance without increasing it drastically. Moreover, one also has $\beta_{\text{opt}} = \Theta(N_{\text{opt}}^{-2})$.

Remark 2 While there is no value of $\beta$ cancelling $\text{var} \left[ \hat{Z}_K \right]$ at a given $N$ (the smaller $\beta$ the smaller the variance of the randomised estimator $\hat{Z}_K$), there is an optimal value for $N$ at a given $\beta$, $i.e.$ for a given finite computational budget: $N = \sqrt{1 + E[T]}$. One can invert this relation, which gives:
\[ \beta_{\text{opt}} = \log \left( 1 + 1/(N^2 - 1) \right) \] (11)

This relation is almost the same as the one of Corollary 5 and that is why we call it $\beta_{\text{opt}}$. Furthermore, for all $N \geq 2$, one has: $\gamma(\beta_{\text{opt}}, N) = (N + 1)/2$, which makes the optimisation problem (9) simpler to handle as we shall see further in the Pareto case.

Proposition 13 (Result for a Pareto distribution) If one has $P[X > x] = 1/(1 + x^{-a})$, $a > k$, then:
\[ \text{var} \left[ \hat{Z}_K \right] = \frac{m_k(m_k - 1)^2}{2\gamma(\beta, N) - m_k} \]
and
\[ \beta_{\text{opt}} = \log \left( \frac{1}{B_+} + 1 \right) \] (12)

where $B_+$ is the positive root of the quadratic polynomial $P(B)$:
\[ P(B) = \frac{2N_{\text{opt}} - m_k}{(N_{\text{opt}}^2 - 1)^2} B^2 - 2m_kB - (m_k(N_{\text{opt}} - 1)^2 + 2N_{\text{opt}}^2) \]

With this relation and the one of Corollary 5 one can derive the optimal parameters ($\beta_{\text{opt}}, N_{\text{opt}}$). Figure 1a shows a numerical resolution of this problem for several values of $a/k \in (1, 3]$.

Furthermore, if one considers the approximation of the optimisation problem with relation (11) instead of (10), one has to minimise $N \mapsto (N^2 + N - 1)m_k(m_k - 1)^2/(N + 1 - m_k)$. Denoting $N_{\text{app}}$ this minimiser, one has:
\[ N_{\text{app}} = \max \left( m_k - 1 + \sqrt{m_k^2 - m - 1, 2} \right) \] (13)

This approximation is the red dotted-dashed line of Figure 1a. As we can see, it is in good agreement with the optimal resolution, both for the parameter $N$ and for the global variance (see further Section 3.4 and Figure 1b).

Hence there is always an optimal solution to Problem (9), meaning this parametrisation is meaningful. The question remains as to whether this implementation is close to the optimal one in terms of variance or not.

3.4 Comparison of the estimators

We have seen in Section 3.2 and 3.3 two ways of implementing the ideal estimator $\tilde{m}_K$ defined in Section 2.2. These two ways involve a truncation of the infinite sum (3) by an integer-valued random variable $T$. In the first implementation the distribution of $T$ and the number $N$ of point processes are optimised in order to minimise the estimator variance. In the second implementation, the distribution of $T$ is enforced to be geometric and its parameter as well as $N$ are optimised.

While the first implementation is optimal in terms of variance, it requires to solve a combinatorial problem, which
can turn it into a poorer algorithm in terms of computational time. In this scope, the parametric algorithm constraining the randomising variable $T$ to be geometric with parameter $\beta$ is much simpler to implement. The aim of this section is to benchmark these two implementations and to challenge the optimal parameters against the fixed ones we will suggest.

More precisely, while both optimisations ended up with optimal parameters depending on the distribution of $X$, we also consider the parametric algorithm with parameter $\beta_{\text{app}}$ given by (11) and $N = N_{\text{opt}} = 2$, 5 or 10. We focus on the Pareto case because we are especially interested in heavy-tailed random variables and full resolution as well as exact sampling is possible in this case.

Figure 1b shows the relative increase of the standard deviations due to the suboptimal implementations for a given computational budget, ie for a given number of generated samples. It also shows the standard deviation ratios between the optimal implementation, the Monte-Carlo estimator (4) and $\hat{m}_2$ given by (3). For this latter, it is assumed that its computation cost is $N$, ie that it costs 1 to simulate a Markov Chain while it requires an infinite number of simulated samples. This calls for certain comments:

– the parametric implementation with optimised parameters $(\beta_{\text{opt}}, N_{\text{opt}})$ remains competitive against the optimal implementation (solid black line going from $\approx 1.3$ to $\approx 1.1$);

– the parametric implementation with parameters $\beta_{\text{app}}$ and $N_{\text{app}}$ is almost not distinguishable from the parametric implementation with optimal parameters $\beta_{\text{opt}}$ and $N_{\text{opt}}$. This means that it is not necessary to strive to estimate the parameters $(\beta_{\text{opt}}, N_{\text{opt}})$;

– the Monte-Carlo estimator is better than the optimal implementation as soon as $a/k \gtrsim 2.5$ and better than the parametric implementation as soon as $a/k \gtrsim 2.3$; this confirms that this new estimator is especially convenient for heavy-tailed random variables;

– the standard deviation of $\hat{m}_2$ illustrates the efficiency of the ideal estimator compared to the Monte-Carlo one (cf. Proposition 5), with a standard deviation at least twice as small;

– generally speaking and without any knowledge on the distribution of $X$, $N$ should not be set too small as the variance increases much faster when it is smaller than the optimal value; especially with $\beta = \beta_{\text{app}}$ finiteness condition of the variance writes $a/k > 1 + 1/N$.

Given these results we can consider that the parametric implementation is a good trade-off between minimal variance estimation and complexity, especially when no information on the distribution of $X$ is provided. In the next section we develop the technical issues related to practical implementation and present a real test-case on which the parametric estimator is challenged against the Monte-Carlo one.

4 Practical implementation

4.1 Simulating conditional distributions

This is the main requirement of our estimator. To do so we propose to use Markov Chain simulation, which only requires to be able to generate samples according to $\mu^X$.

Generally speaking, let us assume that the random variable of interest $X$ writes $X = g(U)$ with $g$ a deterministic (computationally expensive) function and $U$ a random vector with known distribution. The estimator requires to generate the output $X$ according to the conditional law $\mu^X(\cdot \mid X > x)$, ie to generate the input $U$ according to $\mu^U(\cdot \mid g(U) > x)$ for a given $x$. A general idea is to use convergence properties of an ergodic Markov Chain to its unique invariant probability to sample from a given distribution. Assuming $\mu^U$ has a pdf $f_U$, it means we intend to generate a Markov Chain with stationary pdf $\frac{1}{g(U)} f_U(u) / \mathbb{P}[g(U) > x]$.

This implementation is rather simple when a reversible transition kernel $K$ is available. We present two ways of getting it: the Metropolis-Hastings method (Hastings 1970) and a direct construction of the reversible kernel for standard Gaussian input space (Cérou et al 2012; Guyader et al 2011). Because the goal is to reach the stationary state of the Markov Chain, several transitions have to be done to ensure independence between the starting point and the final sample and adequacy with the targeted distribution. This number of transitions is referred to as a burn-in parameter $b$. Eventually the last generated sample is kept. In theory, one can start from any point provided the burn-in is large.

### Algorithm 1 Metropolis-Hastings method

**Require:** initial state $u$

**repeat**

Pick $W$ from a standard multivariate Gaussian distribution or a symmetric Uniform distribution over a compact set of $\mathbb{R}^d$

$U^* \leftarrow u + \sigma W$

$\rho \leftarrow \min(1, f_U(U^*) / f_U(u) \cdot \mathbb{P}[g(U^*) > x])$

With probability $\rho$, do $u \leftarrow U^*$

until $b \geq 1$ states are generated

**return** $u$

### Algorithm 2 Direct construction for $U \sim \mathcal{N}(0, I)$

**Require:** initial state $u$

**repeat**

Pick $W$ from a standard multivariate Gaussian distribution

$U^* \leftarrow \frac{u + \sigma W}{\sqrt{1 + \sigma^2}}$

if $g(U^*) > x$ then

$u \leftarrow U^*$

end if

until $b \geq 1$ states are generated

**return** $u$
Furthermore, we assume the optimal resolution is referred to (Rhee and Glynn 2013). Binomial problem in a general case and so we present here as explained previously, we do not intend to solve the combinatorial problem in a general case and so we present here.

**Remark 3** The burn-in parameter increases the cost of an estimator because it needs several simulations for only one sample. In this context, the computational cost defined in Proposition 7 becomes $\tau = N + bT$ and is the number of calls to $\mu^U$.

**Remark 4** If one considers that $X$ is an exact Pareto random variable with parameter $a$, then $P[X > x | X > x_0] = (x_0/x)^a$, which gives: $\forall n \geq 1$, $X_{n+1} \sim X_n \cdot Z_n$ where $X_n$ a random variable distributed according to $\mu^X$ and independent of $(X_i)_{i=1,..n}$. Therefore the Markov chain can be generated by multiplying (iid) samples.

**4.2 Pseudo-code**

As explained previously, we do not intend to solve the combinatorial problem in a general case and so we present here a pseudo-code for the parametric case. Reader interested in the optimal resolution is referred to (Rhee and Glynn 2013). Furthermore, we assume $N$ and $\beta$ are given, being optimised (with previous knowledge or simulations) or not. We then present in Algorithm 3 how to compute $\hat{Z}_k$ and in Algorithm 4 how to compute $\hat{\alpha}(c)$. Basically, Algorithm 4 is just a wrap-up of Algorithm 3 with an update of the remaining computational budget. If one intends to use Markov Chain simulation as presented in Section 4.1 then one has to take

**Algorithm 3 Pseudo-code for $\hat{Z}_k$**

Require: $N, \beta$

Generate $T$ according to $P[T \geq n] = e^{-\beta n}$

Generate $N$ random variables $(X_i)_{i=1..n}$ according to $\mu^X$

$\text{times}[0] \leftarrow 0$; $\text{delta}[0] \leftarrow 0$

for $i$ in $1:T$

$\text{ind} \leftarrow \text{argmin}_j X_j$

$\text{times}[i] \leftarrow X_{\text{ind}}$

$\text{delta}[i] \leftarrow (\text{times}[i] - \text{times}[i - 1]) \cdot \frac{1 - 1/N}{e^{-\beta i}}$

Generate $X' \sim \mu^X(\cdot | X > X_{\text{ind}})$

$X_{\text{ind}} \leftarrow X'$

end for

$\text{ind} \leftarrow \text{argmin}_j X_j$

$\text{times}[T + 1] \leftarrow X_{\text{ind}}$

$\text{delta}[T] \leftarrow (\text{times}[T + 1] - \text{times}[T]) \cdot \frac{1 - 1/N}{e^{-\beta T}}$

$\hat{Z}_k = \sum_0^T \text{delta}[i]$
into account the burn-in and update \( c \) in Algorithm 4 as follows: \( c = c - (N + T) \). Note also that it is not necessary to consider only the minimum of the \( N \) samples in Algorithm 3; however in the context of Markov Chain drawing it is better to select the starting point in a relatively big population following already the targeted distribution.

### 4.3 Test cases

We first apply our new algorithm to a Pareto random variable and benchmark the different implementations against the Monte-Carlo estimator. Then we study the capacity of our new estimator on a test case from Lamboni et al (2013) and Iooss and Lemaître (2015), namely the model of a dam under flood risk.

**Test with an exact Pareto random variable**

In this section we consider \( X \) is an exact Pareto random variable with parameter \( a \) and we try to estimate a moment of order \( k \) such that \( a/k = 1.3 \). According to Remark 4, we know exactly how to sample from the conditional distribution at each step. Thus the goal is (1) to check the loss in term of variance between the optimal implementation and the parametric case with \( \beta_{opt} \) and \( N = 10 \); (2) to evaluate the efficiency of the Markov chain sampling as described in Section 4.1 with \( b = 20 \) and \( \sigma = 0.45 \) is efficient: the boxplot is almost centred and the increase of the variance corresponds to the expected increase due to the budget loss in the burn-in step.

**Flood uncertainty quantification**

**Risk modelling** Figure 3 and Table 1 present the different parameters of the problem, with \( \mathcal{W} \) standing for a Gumbel distribution, \( \mathcal{F} \) for a triangular distribution and \( \mathcal{U} \) for the uniform distribution. Moreover the Gumbel and Gaussian random variables are truncated at 0 for physical reasons.

| Variable | Physical meaning | PDF          |
|---------|------------------|--------------|
| \( Q \) | River flow rate  | \( \mathcal{W}(1013,558) \) |
| \( K_s \) | Stricker coefficient (friction) | \( \mathcal{N}(30,20^2) \) |
| \( Z_c \) | Riverbed elevation at bottom | \( \mathcal{F}(49,50,51) \) |
| \( Z_m \) | Riverbed elevation at top | \( \mathcal{F}(54,55,56) \) |
| \( H_d \) | Dam Height | \( \mathcal{U}(7,9) \) |
| \( C_b \) | Elevation of the bank | \( \mathcal{F}(55,55,5,56) \) |
| \( L \) | Length of the studied part | \( \mathcal{F}(4990,5000,5010) \) |
| \( B \) | River width | \( \mathcal{F}(295,300,305) \) |

---

### Algorithm 4 Pseudo-code for \( \hat{\alpha}(c) \)

**Require**: \( c, N, \beta \)

\[
\Gamma \leftarrow 0; \hat{\alpha} \leftarrow 0
\]

while \( c > 0 \) do

Generate \( T \) according to \( P[T \geq n] = e^{-\beta n} \)

\( c = c - (N + T) \)

Start Algorithm 3 from step 2.

\( \Gamma = \Gamma + 1 \); \( \hat{\alpha} = \hat{\alpha} + Z_k \)

end while

\( \hat{\alpha} = \hat{\alpha}/\Gamma \)

---

We are here interested in quantifying a risk of flood, ie estimating the mean value of \( S \) the difference between dam height and water height:

\[
S = H_d + C_b - Z_c - H \quad \text{with} \quad H = \left( \frac{Q}{BK_s \sqrt{Z_m Z_c}} \right)^{0.75}
\]

We are in the general setting of Section 4.1 as the real random variable \( S \) can be written \( S = g(U) \) with \( U \) a random
Moments estimation The criticality of this problem comes from the Stricker coefficient $K^{-0.75}$, meaning the left-hand tail is heavy. There is no analytical solution to the problem but the random variables are all independent and so we make use of quadrature approximations to estimate each moment: this brings $m_{tot} \approx 8.345$. We then run 1000 simulations of crude Monte-Carlo estimator and ours with a budget of 400000 computational units. Since we want to focus on the left-hand tail, we are going to split the total budget $c$ into a right-hand tail budget $c_f$ and a left-hand tail budget $c_l$. Because the right-hand tail is not expected to be heavy, we will use a crude Monte-Carlo estimator on this part. Then the procedure is described in Algorithm 5. Results are displayed Figure 4 as boxplots, whiskers extending to the extreme values. As one could expect from the analytical expression of $S$, the Monte-Carlo estimator is unreliable and the same behaviours as in the previous example are visible. Hence these examples illustrate the efficiency of our new algorithm for parallel non-parametric estimation of moments of heavy-tailed random variables.

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5 Conclusion

To estimate the $k^{th}$-order moment of a heavy-tailed random variable when the $2k^{th}$-order moment does not exist, usual strategies work with an (iid) sampling and make use of some parametric assumptions on the distribution function (Pareto-type especially). We have proposed here a new approach to

![Figure 2: Boxplots of 1000 estimation of $E[X^k]$ with computational budget $c = 400000$ when $P[X > x] = 1 \wedge x^{-a}$ and $a/k = 1.3$. (MC): Monte Carlo; $(\beta, N)$: parametric implementation with $\beta$ and $N$; (MH): with Markov Chain sampling as described in Section 4.1; $(\beta_{opt}, N_{opt})$ as in Proposition 12 and $\beta_{app}$ as in Equation (11). For (MC) and (MH), the cost is the number of calls to a simulator of $\mu^U$ while for the two other it is the number of calls to a simulator of conditional law (cf Proposition 7 and Remark 3)](image)

**Algorithm 5** Pseudo-code for the flood test-case

```
N ← 10; β ← β_{simu}(N = 10); b ← 20; c_0 ← 20; γ ← 1
Generate $T[\Gamma]$ according to $P[T \geq n] = e^{-\beta_n}$
bud = bud − $\Gamma b$
γ = γ + 1
while bud > 0 do
    Generate $T[\Gamma]$ according to $P[T \geq n] = e^{-\beta_n}$
bud = bud − $(N + b T[\Gamma])$
    γ = γ + 1
end while
Generate $N_{tot} = NT + c_f$, random variables according to $\mu^U$ and sort them: $(U_i)_{i=1,N_{tot}}$, $g(U_i) \leq g(U_j) \leq \cdots \leq g(U_{N_{tot}})$
$\hat{m}_{MC}(c_i) ← \sum_{n=0}^{N_{tot}} g(U_i) / c_i$
\hat{α} ← 0
for i in 1 : $\Gamma$
do
    Get current stopping time $T[i] ← T[i]$
    Get starting points $(U^{(n)})_{i=1,N} = (U_{i+1}, \cdots, U_{N})$
    Start Algorithm 3 from step 3 with $T[i]$ and $(U^{(n)})_{i=1,N}$
    $\hat{α} = \hat{α} + Z_k$
eンド
end for
\bar{α} = \hat{α} / \Gamma
\bar{m} = \bar{α} NT / N_{tot} + \hat{m}_{MC}(c_i) / N_{tot}
```

Fig. 2: Boxplots of 1000 estimation of $E[X^k]$ with computational budget $c = 400000$ when $P[X > x] = 1 \wedge x^{-a}$ and $a/k = 1.3$. (MC): Monte Carlo; $(\beta, N)$: parametric implementation with $\beta$ and $N$; (MH): with Markov Chain sampling as described in Section 4.1; $(\beta_{opt}, N_{opt})$ as in Proposition 12 and $\beta_{app}$ as in Equation (11). For (MC) and (MH), the cost is the number of calls to a simulator of $\mu^U$ while for the two other it is the number of calls to a simulator of conditional law (cf Proposition 7 and Remark 3)
Proof of Proposition 2

one has:

\[ \text{Appendix} \]

vices and suggestions.

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Remark 4). Section 2.1 and, for example, Pareto-type distributions (cf

to explicit the link between the Markov chain presented in

only (iid) samples are available, further work has to be done

MetropolisHastings algorithm can overcome this issue. If

put of a complex computer code, Markov Chain drawing like

tional laws. When the random variable of interest is the out-

assumption on the distribution.

framework lets us define a new unbiased estimator for the

order moment exists, for any \( k’ > k \), and does not make any

Practically speaking, it is totally parallel and only re-

output of a complex computer code, Markov Chain drawing like

MetropolisHastings algorithm can overcome this issue. If

only (iid) samples are available, further work has to be done
to explicit the link between the Markov chain presented in
Section 2.1 and, for example, Pareto-type distributions (cf
Remark 4).

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ices and suggestions.

Appendix

Proof of Proposition 2

one has:

\[
\begin{align*}
E[\hat{m}_k] &= \int_0^\infty k x^{k-1} E \left[ \left( 1 - \frac{1}{N} \right)^{M_1} \right] \, dx = \int_0^\infty k x^{k-1} p_x \, dx \\
\text{var}[\hat{m}_k] &= 2 \int_0^\infty \int_0^\infty x^{k-1} x'^{(k-1)} \left( 1 - \frac{1}{N} \right)^{M_1} \left( 1 - \frac{1}{N} \right)^{2M_{\epsilon}} \, dx \, dx' 
\end{align*}
\]

For the variance, one uses the fact that \( M_1 \) and \( M_\epsilon \) are independent to expand \( E[\hat{m}_k^2] \):

\[
E[\hat{m}_k^2] = 2 \int_0^\infty \int_0^\infty k x^{k-1} x'^{(k-1)} \left( 1 - \frac{1}{N} \right)^{M_1 + M_\epsilon} \, dx \, dx' \\
= \int_0^\infty \int_0^\infty k x^{k-1} x'^{(k-1)} \left( 1 - \frac{1}{N} \right)^{M_1 - M_\epsilon} \left( 1 - \frac{1}{N} \right)^{2M_{\epsilon}} \, dx \, dx' 
\]

Proof of Proposition 3

Starting from the expression of the variance found in Proposition 2:

\[
\begin{align*}
\text{var}[\hat{m}_k] &= 2 \int_0^\infty k x^{k-1} p_x \int_0^\infty x'^{(k-1)} p_{\epsilon'} dx' dx - E \left[ X^k \right]^2 \\
\text{we make use of Hölder’s inequality:} \\
\int_0^\infty x^{(k-1)N/(N-1)} p_{\epsilon'} \, dx' \\
&\leq \left( \int_0^\infty x^N \, dx' \right)^{1/N} \left( \int_0^\infty x'^{(k-1)N/(N-1)} p_{\epsilon'} \, dx' \right)^{1-1/N} \\
&\leq x^{1/N} \left( \int_0^\infty x'^{(k-1)N/(N-1)+1} p_{\epsilon'} \, dx' \right)^{1-1/N} \\
&\leq x^{1/N} \left( \frac{E \left[ X^{(k-1)N/(N-1)+1} \right]}{(k-1)N/(N-1)+1} \right)^{1-1/N} \\
\end{align*}
\]

And therefore:

\[
\begin{align*}
\text{var}[\hat{m}_k] &\leq K E \left[ X^{(kN-1)/(N-1)} \right]^{1-1/N} E \left[ X^{k+1/N} \right]
\end{align*}
\]

Fig. 4: Boxplots of 1000 estimations of \( E[S] \) with computation budget \( c = 400000 \). Monte Carlo estimator (4); \( \hat{\alpha}(\beta_{\text{app}}, N = 10) \): parametric implementation with \( \beta_{\text{app}} \) as in Equation (11) and \( N = 10 \)
with \( K \) a constant depending only on \( k \) and \( N \). Now consider \( k' > (kN - 1)/(N - 1) \vee (k + 1/N) \); using Hölder's inequality again, one gets:

\[
\text{var} [\hat{m}_k] \leq KE \left[ X'^2 \right] \frac{1}{(k + \frac{1}{k})} \]

\[
\text{var} [\hat{m}_k] \leq KE \left[ X'^2 \right] \frac{2}{k^2}
\]

\( \square \)

**Proof of Proposition 4** On the one hand one has:

\[
N \text{var} [\hat{m}_{MC}] + m_k^2 = 2 \int_0^\infty kx^{k-1} p_k dx
\]

and on the other hand one can write:

\[
N \text{var} [\hat{m}_k] + m_k^2 = 2 \int_0^\infty kx^{k-1} p_k \left[ N(p_x^{-1/N} - 1) + 1 \right] dx' dx
\]

Thus we have:

\[
p_x \left[ N(p_x^{-1/N} - 1) + 1 \right] \leq 1 \Rightarrow \text{var} [\hat{m}_k] \leq \text{var} [\hat{m}_{MC}]
\]

Considering \( f : p \mapsto p \left[ N(p_x^{-1/N} - 1) + 1 \right] \), we have \( f(1) = 1 \) and

\[
f'(p) = (N - 1)(p_x^{-1/N} - 1) \geq 0, \forall p \in [0, 1].
\]

Thus \( \forall p \in [0, 1], f(p) \leq 1. \) \( \square \)

**Proof of Proposition 5** For the first equality:

\[
E \left[ X^k \right] = k \int_0^\infty x^{k-1} p_k dx = \frac{a}{a - k}
\]

\[
\text{var} [\hat{m}_{MC}] = \frac{1}{N} \left( E \left[ X^{2k} \right] - E \left[ X^k \right]^2 \right) = \frac{ak^2}{N(a - 2k)(a - k)^2}
\]

\[
= \frac{m_{2k}(m_k - 1)^2}{N}
\]

For the second one:

\[
E \left[ \hat{m}_k^2 \right] = 2 \int_0^\infty \int_0^\infty x^{k-1}p_{\hat{m}_k} \frac{1}{k^N} dx' dx
\]

\[
= 2 \left[ \int_0^1 \int_0^{x-1} \int_1^{x-1} \int \frac{2k}{a - k} \right] + \frac{2k^2}{(a - k)(2(a - k) - a/N)}
\]

\[
\text{var} [\hat{m}_k] = \frac{N(a - k)^2(2(a - k) - a/N)}{ak^2}
\]

For the third one:

\[
\text{var} [\hat{m}_k] = \frac{1}{N} \left[ \int_1^\infty x^{2k} \frac{a^2}{B} x^{-a+b} dx - \frac{a^2}{(a - k)^2} \right]
\]

\[
\text{var} [\hat{m}_k] = \frac{a^2}{(a - k)^2} \left( \frac{1}{B(2 - B)} - 1 \right)
\]

with \( B = b/(a - k). \) \( \square \)

**Proof of Proposition 6** \( T \) and \( (X_i)_i \) are independent. Then one has:

\[
E \left[ \hat{Z}_k \right] = \int_0^\infty kx^{k-1} E \left[ \frac{1 - 1}{N} \right] dx = m_k
\]

For the 2nd-order moment the same reasoning as for \( E \left[ \hat{m}_k^2 \right] \) apply: given \( x > x' \), the random variables \( M_x, M_{x'}, M_x \) and \( T \) are independent, which brings:

\[
E \left[ \left( 1 - 1/N \right)^{M_x + M_{x'}} \right] \frac{1}{T \geq M_x} \frac{1}{T \geq M_{x'}}
\]

\[
= E \left[ \left( 1 - 1/N \right)^{M_x - M_{x'}} \left( 1 - 1/N \right)^{2M_x} \right] \frac{1}{T \geq M_x}
\]

\[
= E \left[ \left( 1 - 1/N \right)^{M_x - M_{x'}} \left( 1 - 1/N \right)^{2M_x} \right] \frac{1}{T \geq M_{x'}}
\]

\[
= E \left[ \left( 1 - 1/N \right)^{M_x - M_{x'}} \right] \frac{1}{P \geq M_x}
\]

Then using this equality in \( E \left[ \hat{Z}_k^2 \right] \) gives the solution. \( \square \)

**Proof of Lemma 1** Let \( N \geq 2 \) and \( i \geq 0 \). According to Corollary 1: \( \exists N' \in \mathbb{R} \) such that \( N' - N \) and \( \text{var} [\hat{m}_k] (N') \ll \infty. \) Furthermore, given \( x \) and \( x' \) one can write:

\[
p_x p_{x'}^{N' - 1}(-\log p_{x'}) \leq p_x p_{x'}^{1 - N'/N} \frac{1}{\sqrt{2\pi i}} \]

Moreover the function \( p : (0, 1) \mapsto p^{N'/N - 2}(-\log p) \) is bounded above by \( e^{-i}(N + 1/N' - 2)^{-1} \). Using the Stirling formula \( i! \sim e^{-i}/\sqrt{2\pi i} \) lets write:

\[
p_x p_{x'}^{N' - 1}(-\log p_{x'}) \leq p_x p_{x'}^{1 - N'/N} \frac{1}{\sqrt{2\pi (N + 1/N' - 2)}}
\]

Finally, this inequality brings:

\[
i \leq \text{var} [\hat{m}_k] (N') \left( N + 1/N' - 2 \right)
\]

and \( (N + 1/N - 2)/(N + 1/N' - 2) < 1 \), which concludes the first part of the proof.

Let us now assume that \( X \) has a density \( f_x \). For \( k = 1 \), one has:

\[
i \leq 2 \int_0^x \int_0^x p_x p_{x'} \left[ -N \log p_{x'} (1 - N'/N)^2 \right] dx' dx
\]

Denote \( x_L \) the left end point of \( X \). Then:

\[
i \leq \sum_{i=0} \left( x_L^2 + 2x_L (m - x_L) \right)
\]

\[
+ 2 \int_{x_L}^x \int_{x_L}^x p_x p_{x'} \left[ -N \log p_{x'} (1 - N'/N)^2 \right] dx' dx
\]

\( \square \)
We then consider the change of measure \( u = - \log p_x \) and \( u' = - \log p_{x'} \). Denote \( Q \) the generalised inverse function of \( p_x = P[X > x] = \tilde{F}(x) \); for all \( i \geq 1 \) one has:

\[
q_i \geq \frac{2}{\|f_x\|_2^2} \left( 1 - \frac{1}{N} \right)^{2i} \int_0^\infty e^{-2u} \int_0^u e^{-N^u} (Nu)^i \frac{dudv}{i!} \\
\geq \frac{2}{\|f_x\|_2^2} \left( 1 - \frac{1}{N} \right)^{2i} \int_0^\infty e^{-2u} \int_0^\infty e^{-N^u} (Nu)^k \frac{dudv}{k!} \\
\geq \frac{2}{\|f_x\|_2^2} \frac{1}{N} \left( 1 - \frac{1}{N} \right)^{2i} \sum_{k=i+1}^\infty \left( \frac{N}{N+2} \right)^k \\
q_i \geq \frac{1}{(N+2) \|f_x\|_2^2} \left[ \frac{N}{N+2} \left( 1 - \frac{1}{N} \right)^2 \right]^i 
\]

Proof of Proposition 7 If \( T = 0 \) then no other simulation is done than the first element of each Markov chain, i.e. \( N \) simulations are done. Then each step requires the simulation of the next stopping time, i.e. one simulation. Finally, this brings \( \tau = N + T \)

Proof of Corollary 2 First one has \( \mathbb{E} \left[ \hat{Z}_k \right] > \text{var} [\hat{m}_k] \) because \( \mathbb{E} \left[ \hat{Z}_k \right] = \text{var} [\hat{m}_k] \Leftrightarrow \forall i \in \mathbb{N}, P[T \geq i] = 1 \) and \( \mathbb{E} [\tau] > N \) because \( \mathbb{E} [\tau] = N \Leftrightarrow \mathbb{E} [T] = 0 \) while \( \forall i \in \mathbb{N}, P[T \geq i] > 0 \). Furthermore, the power series of the exponential and the dominated convergence theorem let us rewrite \( \text{var} [\hat{m}_k] \):

\[
\text{var} [\hat{m}_k] = \sum_{i=1}^\infty 2k^2 \int_0^\infty \int_0^\infty x^{k-1} x^{(k-1)} p_x p_{x'} \left( - \log p_{x'} \right)^i N! dxdx' \\
\text{var} [\hat{m}_k] = \sum_{i=1}^\infty q_i (N = 2) \left( \frac{2}{N} \right)^i 
\]

which brings: \( \text{var} [\hat{m}_k] = q_i (N = 2) \cdot 2 / N + O \left( 1 / N^2 \right) \). All together, these inequalities complete the proof.

Proof of Proposition 8 First one shows that \( i_0 \) is well determined. The sequence \((\Delta_i)\) defined by:

\[
\forall i \in \mathbb{N}, \Delta_i = \sum_{j=0}^i q_j - m_k^2 - (N+i)q_{i+1} 
\]

is increasing:

\[
\Delta_{i+1} - \Delta_i = q_{i+1} - (N+i+1)q_{i+2} + (N+i)q_{i+1} = (N+i+1)(q_{i+1} - q_{i+2}) > 0 
\]

Furthermore \( \Delta_0 = q_0 - m_k^2 \leq 0 < Nq_i \) and \( \Delta_i \rightarrow \text{var}[\hat{m}_k] \) when \( i \rightarrow \infty \) because \( (q_i) \) decreases at exponential rate. So there exists \( i_0 \in \mathbb{N} \mid \Delta_{i_0-1} \leq 0 \) and \( \Delta_{i_0} > 0 \).

Let us now consider the auxiliary problem:

\[
\arg\min_{\substack{\beta \geq 0 \\ \beta_i > 0}} \left( \beta + \sum_{i=1}^\infty \beta_i \right) \left( q + \sum_{i=1}^\infty q_i \beta_i^{-1} \right) 
\]

with \( \beta > 0 \) and \( q \in \mathbb{R} \). We show that it has a solution if and only if \( q > 0 \). Let \( i \geq 1 \), cancelling the partial derivatives brings:

\[
\forall i \geq 1, 0 = \left( q + \sum_{j=0}^i q_j \beta_j^{-1} \right) + \left( \beta + \sum_{j=0}^\infty \beta_j \right)^{-q_i \beta_i^{-1}} 
\]

Then the solution should be of the form: \( \forall i \in [1, \infty), \beta_i = c_0 \sqrt{q_i} \) for some \( c_0 > 0 \). Solving now the problem with \( c_0 \), the derivative writes \( q - \beta / c_0^2 \). If \( q \leq 0 \) then it is strictly decreasing and there is no global minimiser. On the contrary, \( q > 0 \) brings \( c_0 = \sqrt{q} / q \) and \( \forall i \geq 1, \beta_i = c_0 \sqrt{q} \).

Thus, in our context with the constraint \( \forall i \in \mathbb{N}, \beta_i \leq 1 \), this means that solving the optimisation problem will set globally \( \beta_i = 1 \) until the minimiser is feasible, i.e. until \( i_0 = \min \{ i \in \mathbb{N} \mid \sum_{j=0}^i q_j - m_k^2 > (N+i)q_i \} \). Then the solution will be given by:

\[
\forall i \in [1, i_0], \beta_i = 1 \\
\forall i > i_0, \beta_i = \sqrt{q_i} \sqrt{\frac{\sum_{j=0}^{i_0} q_j \beta_j^{-1}}{\sum_{j=0}^\infty \beta_j}} 
\]

Proof of Corollary 3 By definition of \( i_0 \), one has \( \sum_{j=0}^{i_0} q_j - m_k^2 \leq (N+i_0-1)q_{i_0} + \sum_{j=0}^{i_0} q_j - m_k^2 \leq (N+i_0-1)q_{i_0} + q_{i_0} \)

which concludes the proof.

Proof of Proposition 9 Let \( i \geq 0 \), one has:

\[
\int_{x_0}^{x_0} \int_{x_0}^{x_0} x^{k-1} x^{(k-1)} p_x p_{x'} \left( - \log p_{x'} \right)^i N! dxdx' = \\
\frac{aN(1 - 1/N)^{2i}}{i!} \int_{x_0}^{x_0} \int_{x_0}^{x_0} x^{k-1} x^{(k-1)} (log x')^{i} dxdx' \\
= \frac{aN(1 - 1/N)^{2i}}{i!} \int_{x_0}^{x_0} \int_{x_0}^{x_0} x^{k-a-1} (log x')^{i} dxdx' \\
= \frac{aN(1 - 1/N)^{2i}}{i!} \frac{1}{(a-k)!} \int_{x_0}^{x_0} x^{2k-a-1} (log x')^{i} dxdx' \\
= \frac{1}{(a-k)(aN-2k)} aN(1 - 1/N)^{2i} 
\]

with \( \Gamma \) standing here for the Gamma function. One then verifies that \( (q_i) \), is decreasing:

\[
\frac{aN}{aN-2k} \left( 1 - \frac{1}{N} \right)^2 < 1 \Leftrightarrow k < a \left( 1 - \frac{1}{2N} \right) 
\]

which is indeed the condition for the finiteness of \( \text{var}[\hat{m}_k] \) already stated in Proposition 5.
Proof of Proposition 10 The problem can be rewritten:
\[
\min \left\{ i \geq 1 \mid \frac{1}{1 - \beta} + \frac{aN - 2k}{2(a - k)} > \beta^{i+1} \left( N + i + \frac{1}{1 - \beta} \right) \right\}
\]

Furthermore one has:
\[
\frac{1}{1 - \beta} = \frac{Nm_k}{2} + \frac{(a - 2k)^2}{4(a - k)^2} + o(1)
\]

which brings the left hand term equal to \((m_k/2)^2 + o(1)\). Writing \(i = N(k_0 + k_1 \log N + k_2 \log \log N)\) brings:
\[
\beta^{i+1} = e^{-\frac{2a}{m_k} N - \frac{2a}{m_k} (\log N) - \frac{2a}{m_k} (1 + o(1))}
\]

Hence one has to choose \(k_0, k_1\) and \(k_2\) such that the right hand term also equals \((m_k/2)^2 + o(1)\), which gives the solution.
\[
\square
\]

Proof of Corollary 4 Using the asymptotic expansion of \(i_0\) one finds \(q_{i_0} \sim (N^2 \log N)^{-1} (m_k - 1)^2\). Furthermore, one has \(E[\tau] \sim i_0\). Finally, the use of \(E[\tau] \cdot \var [\hat{Z}_k] \sim q_{i_0} E[\tau]^2\) gives the result.
\[
\square
\]

Proof of Proposition 11 Let \(\alpha > 0\) be such that \((1 - 1/N) = e^{-\alpha}\). The argument is the same one as in Proposition 6. One has:
\[
E[\hat{Z}_k^2] = 2 \int_0^\infty \int_0^x k^2 x^{k-1} x'(k-1) E \left[ e^{-\alpha(M_e - M') e^{(\beta - 2\alpha)M'_e}} \right] dx' dx
= 2 \int_0^\infty \int_0^x k^2 x^{k-1} x'(k-1) p_e p_e^{-1} \frac{1}{\var \gamma} dx' dx
\]

with:
\[
\frac{N}{\gamma(\beta, N)} = 2N - N^2 + e^\beta (N - 1)^2
= 1 + (N - 1)^2 (e^\beta - 1)
\]

\[
\square
\]

Proof of Proposition 12 Denote:
\[
Q_N(\beta) = \left( N + \frac{1}{e^\beta - 1} \right) \left( \sum_{i=0}^\infty q_i(N = 2)/(2/\gamma)^i - m_k^2 \right)
\]

Then \(Q_N\) is continuous on \((0, \beta_\infty)\) with infinite limits on 0 and \(\beta_\infty\), so it reaches its minimum on \((0, \beta_\infty)\). Or \(\exists \beta_\infty \in (0, \infty)\) such that:
\[
\left\{ \begin{array}{ll}
Q_N(\beta) < \infty & \forall \beta \in (0, \beta_\infty) \\
Q_N(\beta) = \infty & \forall \beta > \beta_\infty
\end{array} \right.
\]

\[
\square
\]

Proof of Corollary 5 Denote \(B = E[T] = 1/(e^\beta - 1)\); therefore \(E[\tau] = N + B\). We write the partial derivatives of \(E[\tau] \cdot \var [\hat{Z}_k]\) against \(B\) and \(N\):
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial B} \left( E[\tau] \cdot \var [\hat{Z}_k] \right) = \var [\hat{Z}_k] + E[\tau] \frac{\partial \var [\hat{Z}_k]}{\partial \gamma} \frac{\partial \gamma}{\partial B} \\
\frac{\partial}{\partial N} \left( E[\tau] \cdot \var [\hat{Z}_k] \right) = \var [\hat{Z}_k] + E[\tau] \frac{\partial \var [\hat{Z}_k]}{\partial \gamma} \frac{\partial \gamma}{\partial N}
\end{array} \right.
\]

At point \((\beta_{opt}, N_{opt})\), both equations are cancelled, which gives:
\[
\frac{\partial \gamma}{\partial N}(B_{opt}, N_{opt}) = \frac{\partial \gamma}{\partial B}(B_{opt}, N_{opt})
\]

Recalling \(\gamma(B, N) = NB/(B + (N - 1)^2)\), this gives the equation:
\[
B_{opt} - (N_{opt}^2 - 1)B_{opt} - N_{opt} = 0.
\]

One can solve it in \(B_{opt}\) and keep the positive root, which gives the solution.
\[
\square
\]

Proof of Proposition 13 One gets the expression of the variance directly from Section 2.2 with \(\gamma(N, B)\) instead of \(N\). Then, denoting \(B = 1/(e^\beta - 1)\), one solves the problem:
\[
\frac{\partial}{\partial B} \left( (N + B) \left( \frac{a/k}{2(a/k - 1)\gamma - a/k} \right) \right) = 0
\]

\[
\square
\]
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