BALANCED CURVES AND MINIMAL RATIONAL CONNECTEDNESS
ON FANO HYPERSURFACES

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ABSTRACT. On a general hypersurface $X$ of degree $n$ (resp. $< n$) in $\mathbb{P}^n$ and any $e \geq n - 1$ (resp. arbitrarily large $e$), we construct families of rational curves of degree $e$ going through the maximal number of general points. This solves in many cases the problem of rational curve interpolation on $X$.

As is well known, $q$ general points in the plane lie on a rational curve of degree $\left\lceil \frac{q+1}{3} \right\rceil$ and none lower. More generally, given a (polarized, rationally connected) variety $X$, one is interested in the question of rational curve interpolation (aka minimal rational connectivity) on $X$: what is the minimal degree of a rational curve on $X$ through a general collection of $q$ points? At least when $X$ is anticanonically polarized, there is an obvious expected answer: namely, the minimal anticanonical degree of such a curve $C$ is the smallest such that

$$\chi(N_C/X) = C.(-K_X) + \dim(X) - 3 \geq q(\dim(X) - 1),$$

i.e. it equals

$$\min\{C.(-K_X) : C.(-K_X) \geq (q - 1)(\dim(X) - 1) + 2\}.$$

When the expected degree equals the actual degree we say that $X$ is minimally rationally $(q - 1)$-connected. The results of this paper show (in any characteristic) that a general Fano hypersurface $X$ in projective space is (separably) minimally $(q - 1)$-rationally connected for infinitely many values of $q$ and even any $q \geq 1$ if $X$ has index 1; moreover, in these cases the locus of rational curves through $q$ general points is reduced of the expected dimension. We proceed to describe the results precisely.

Here $\mathcal{O}_C(k)$ denotes the line bundle of degree $k$ and $a\mathcal{O}_C(k)$ denotes $\bigoplus^q_1 \mathcal{O}_C(k)$. A rational curve $C \rightarrow X$ is said to be balanced if its normal bundle $N_C$ is a balanced bundle
on $C \simeq \mathbb{P}^1$, which means, setting $n = \dim(X) + 1 = \text{rk}(N_C) + 2$, that

$$N_C \simeq (n - 2 - r^-) \mathcal{O}_C(a^- + 1) \oplus r^- \mathcal{O}_C(a^-) \text{ for some } r^- > 0, a^- \in \mathbb{Z}. $$

In that case

$$a^- = \lfloor \deg(N_C) / (n - 2) \rfloor = \lfloor (C.(-K_X) - 2) / (n - 2) \rfloor. $$

Geometrically, balancedness implies that $C$ is movable to go through the expected- i.e. maximal- number, viz. $(a^- + 1)$, of general points on $X$, hence balancedness is closely related to rational connectedness and its generalizations. A (polarized) variety $X$ is rationally $(q - 1)$-connected, $q \geq 1$, if there is a family of rational curves $C/B \to X$ such that the induced map $C^q/B \to X^q$ is dominant. $X$ is rationally $(q - 1, e)$-connected if the curves can be taken to have polarized degree $e$. The adjective ‘separable’ may be added to these properties if the induced map $C^q/B \to X^q$ is separable (as well as dominant).

This makes most sense if the polarization is $H = (-1/k_X)K_X$ for some index $k_X \in \mathbb{N}$, where $k_X = n + 1 - d$ for a hypersurface $X$ of degree $d \leq n$ in $\mathbb{P}^n$. Thus, the existence of a balanced rational curve of degree $e$ on $X$ is equivalent to separable $(a^-, e)$ - rational connectedness, where $a^- = \lfloor k_Xe/2 \rfloor$. Fixing $n, d$, we say that $e$ is point-minimal when

$$\frac{(n + 1 - d)(e - 1) - 2}{n - 2} < \lfloor (n + 1 - d)e/2 \rfloor. \tag{\star} $$

Whenever $e$ is point-minimal, $(a^-, e)$ rational connectedness implies that $e$ is the minimal, as well as the expected, degree of a rational curve through $a^- + 1$ general points, in which case we say $X$ is minimally rationally $a^-$-connected and that the point-degree $a^- + 1$ is interpolating (for $X$).

Rational connectedness and $q$-connectedness, not necessarily minimal, of all Fano manifolds has been known since the 90s (see Kollár’s book [8]). For general hypersurfaces $X$ of degree $d \leq n$ in $\mathbb{P}^n$, Chen and Zhu [3] and Tian [14] have proven that $X$ is separably rationally connected. Some more precise results on existence of low-degree balanced rational curves (and consequently, minimal rational connectedness in low degrees) for such hypersurfaces are given in [5] and [10] (see also [1], [12]).

In this paper we extend these results to cases of curves of high degree $e$. Our main results are as follows (here ‘general’ refers to an open set, depending on the values of the numerical parameters involved, in the space of hypersurfaces, and having the results valid for all values simultaneously requires ‘very general’):

- For $d = n \geq 4$, $e \geq n - 1$, a general $X$ contains balanced rational curves of degree $e$ (Theorem [20]), hence a very general $X$ is separably minimally rationally $q$-connected for all $q$ (Corollary [21]).
• For each $3 \leq d < n$ and $e, q$ satisfying certain arithmetical conditions, a general $X$ contains balanced rational curves of degree $e$ and is separably minimally rationally $q$-connected (Theorem 31).

For example (cf. Example 32), if $d = n - 1$ then a general $X$ is minimally $((n-1)k + 1)$- (resp. $(n-1)k + 2$)-rationally connected for all $k \geq 1$ if $n$ is even (resp. odd) and $\geq 6$. More generally, a result of M. C. Chang (see the Appendix) shows, for all $2 < d < n - 1$, that the set of $e$ (resp. $q$) satisfying the arithmetical conditions contains about $d(n - d)$ (resp. $(n + 1 - d)/2$) many distinct arithmetic progressions with modulus $d(n - 2)$ (resp. $d(n + 1 - d)$).

To my knowledge these are the first examples (in any characteristic) of high-degree balanced rational curves on Fano hypersurfaces except hypersurfaces of very low degree or dimension $\leq 3$; ditto for the minimal rational connectivity results (separable or not). While our results are essentially optimal for the case $d = n$, they need not be optimal for $d < n$, indeed we see no obstruction to the existence of balanced rational curves of all sufficiently large degrees $e$ on a very general $X$ of degree $d < n$.

The proof for $d = n$, presented in §4, is based on degenerating the hypersurface to a reducible variety $X_1 \cup X_2$ called a fan hypersurface, where $X_1$ is a hypersurface of degree $n$ with a point of multiplicity $n - 1$ blown up at that point, and where $X_2$ is a hypersurface of degree $n - 1$. Using a bundle smoothing result proven in §2 plus in some cases a vanishing theorem of Rathmann [11], we show that a suitable rational curve on $X_1$, glued to a union of some lines on $X_2$, is well behaved and smooths out to a balanced rational curve on $X$.

The proof for $d < n$, presented in §6, is based on a generalization of fan called fang where $X_1$ and $X_2$ are blowups of a degree-$d$ hypersurface containing, respectively, a $(d-1)$-fold $\mathbb{P}^{n-m-1}$ or a $\mathbb{P}^m$, $m \geq 2$.

In §3 we illustrate the fan method by computing the normal bundle of a general rational curve in $\mathbb{P}^n$.

We begin in §2 with a general result on smoothing of bundles on a curve consisting of a ‘body’ together with rational tails. The general thesis is that if the bundle is balanced on each tail and glued to the body in a sufficiently general manner, then a smoothing is no worse, and usually better, than the bundle on the body. Note that a bundle on a rational tree- including the kind envisaged in this result and its applications- need not split as a direct sum of line bundles (see [9], Example 5.6 or Example 5 below). Thus the proof is not just a matter of semi-continuity, but is rather based on bundle modifications on surfaces.

The two ‘preliminaries’ sections §1 and §5 establish some notation, terminology and largely standard general results used in the paper.
An appendix by M.C. Chang analyzes the numerology of curve and hypersurface degrees arising out of the slope-matching condition in Lemma 25.

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1. Preliminaries

We work over an arbitrary algebraically closed field and use Grothendieck’s projective bundle convention. In this section we collect some definitions and facts, mostly well known, for later reference. See §5 for more such.

1.1. Some numerology. It is convenient to set, for fixed $d \leq n$,

\begin{equation}
q_{\text{max}}(e) = \left\lfloor \frac{e(n + 1 - d) - 2}{n - 2} \right\rfloor + 1
\end{equation}

\begin{equation}
e_{\text{min}}(q) = \left\lceil \frac{(q - 1)(n - 2) + 2}{n + 1 - d} \right\rceil.
\end{equation}

Some trivial remarks:

- For $d > 3$ $e$ is point-minimal iff the remainder of $e(n + 1 - d) - 2 \mod n - 2$ is $< n + 1 - d$.
- $q_{\text{max}}(e) - 1$ is the round-down of the slope of the normal bundle of a rational curve $C$ of degree $e$ on a hypersurface $X$ of degree $d$ in $\mathbb{P}^n$, hence $q_{\text{max}}(e)$ equals the expected (i.e. maximum possible) number of general points that may lie on $C$, as well as the actual number of general points on a general $C$ if $C$ is balanced (note that a balanced curve is automatically unobstructed, hence lies on a unique smooth component of the Hilbert scheme of curves in $X$).
- Similarly $e_{\text{min}}(q)$ is the expected (i.e. minimal possible) degree of a rational curve that may go through $q$ general points of $X$ and the expected degree equals the actual degree iff $X$ is minimally $(q - 1)$-connected.
- The two functions above are mutual 'sub' -inverses: If $d > 3$ then $\frac{n + 1 - d}{n - 2} < 1$ so by elementary arithmetic $q_{\text{max}}(e_{\text{min}}(q)) = q$ while $e_{\text{min}}(q_{\text{max}}(e)) \leq e$, with equality iff $q_{\text{max}}(e - 1) < q_{\text{max}}(e)$ iff $e$ is of the form $e_{\text{max}}(q)$ for some $q$ iff $e$ is point-minimal.
- Thus, if $X$ contains a balanced rational curve of degree $e$ which is point-minimal, then $X$ is minimally rationally $(q_{\text{max}}(e) - 1)$-connected (if $e$ is not point-minimal there may exist rational curves of degree $< e$ through $q_{\text{max}}(e)$ general points).
1.2. Fans. [cf. [10], sec. 4] A 2-fan is a variety of the form \( P_1 \cup P_2 \) where \( P_1 \) is a blowup \( B_p \mathbb{P}^n \) with exceptional divisor \( E \simeq \mathbb{P}^{n-1} \) and \( P_2 = \mathbb{P}^n \), so that \( P_1 \cap P_2 \) is embedded as \( E \subset P_1 \) and as a hyperplane in \( P_2 \). For every \( d > e > 0 \) there is a very ample divisor on \( P_1 \cup P_2 \) which is \( dH - eE \) on \( P_1 \) and \( eH \) on \( P_2 \), \( H \) =hyperplane. A divisor of this class is said to be of type \((d,e)\).

A 2-fan is the special fibre \( \pi^{-1}(0) \) in a relative 2-fan \( \pi : \mathcal{P}(2) \to \mathbb{A}^1 \) which is just \( B_{(p,0)} \mathbb{P}^n \times \mathbb{A}^1 \), where \( P_2 \) is the exceptional divisor. The divisor \( p^* (dH) - e P_2 \) induces a divisor of type \((d,e)\) on the special fibre and \( dH \) on other fibres.

We will also use a generalization of 2-fans called fangs, obtained in the above relative setting by blowing up \( \mathbb{P}^r \times 0 \subset \mathbb{P}^n \times \mathbb{A}^1 \) for any \( r \leq n - 2 \) rather than just \((p,0)\). Fangs will be used implicitly in §1 below to give a proof of the balancedness of a general rational curve of degree \( e \geq n \) in \( \mathbb{P}^n \), and again more formally in §5 starting in §5.3.

1.3. Subvarieties of fan(g)s. This actually applies to any ambient variety \( P \) with normal crossing double points and no other singularities. Let

\[
P = P_1 \cup_Q P_2
\]

be a transverse union of two smooth \( n \)-dimensional varieties meeting in a common smooth divisor \( Q \) (e.g. \( P \) could be a fan or fang, see §5.3). Let

\[
X = X_1 \cup_Q X_2 \subset P
\]

be a subvariety consisting of smooth, codimension-\( c \) subvarieties \( X_i \subset P_i \) meeting \( Q \) transversely in a smooth common divisor \( Q_X = X_1 \cap X_2 \). We have a Mayer-Vietoris sequence

\[
0 \to \mathcal{O}_P \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to \mathcal{O}_Q \to 0,
\]

and likewise for \( X \). Locally, \( Q_X \subset Q \) is defined by equations \( u_1, ..., u_c \) on \( Q \) which extend to defining equations \( u_i^j, i = 1, ..., c, j = 1, 2 \) for \( X_j \) on \( P_j, j = 1, 2 \). Here the \( u_i \) are part of a local coordinate system on \( Q \) and \( u_i^j \) their extension to \( P_j \). Then \( u_i^1 \) and \( u_i^2 \) glue together to a function \( v_i \) on \( P \) and \( v_1, ..., v_c \) constitute local defining equations for \( X \subset P \).

In particular, \( X \to P \) is an lc-embedding and

\[
\mathcal{I}_{X/P} \otimes \mathcal{O}_{P_j} = \mathcal{I}_{X_j/P_j}.
\]

Consequently we have for the conormal bundles

\[
\hat{N}_{X/P} \otimes \mathcal{O}_{X_j} = \hat{N}_{X_j/P_j}, j = 1, 2,
\]

and

\[
\hat{N}_{X/P} \otimes \mathcal{O}_{Q_X} = \hat{N}_{Q_X/Q}.
\]
Therefore we have likewise for the normal bundles

\[ N_{X/P} \otimes \mathcal{O}_{X} = N_{X/P} \otimes \mathcal{O}_{j}, \]

and

\[ N_{X/P} \otimes \mathcal{O}_{QX} = N_{QX/Q}. \]

The latter equality may be written suggestively as

\[ N_{X/P} = N_{X_1/P_1} \cup N_{QX/Q} N_{X_2/P_2}. \]

More generally, given coherent sheaves \( E_i \) on \( P_i, i = 1, 2 \), together with an isomorphism

\[ E_1 \otimes \mathcal{O}_Q \rightarrow E_2 \otimes \mathcal{O}_Q \]

there is a uniquely determined coherent sheaf \( E \) on \( P \) restricting to \( E_i \) on each \( P_i, i = 1, 2 \).

1.4. Subvarieties of subvarieties of fan(g)s. Notations as above, given a subvariety \( C = C_1 \cup C_2 \subset X \) where \( C_1 \subset X_1, C_2 \subset X_2 \) are smooth codimension-\( c' \) subvarieties meeting \( Q_X \) transversely in the same subvariety \( Q_C \), then \( C \) is lci in \( X \) with normal bundle

\[ N_{C/X} = N_{C_1/X_1} \cup N_{C_2/X_2} \]

and we have an exact sequence

\[ 0 \rightarrow N_{C/X} \rightarrow N_{C/P} \rightarrow N_{X/P} \otimes \mathcal{O}_C \rightarrow 0. \]

Now suppose we have a smoothing, i.e. a flat family \( \pi : \mathcal{X} \rightarrow T \) with \( \pi^{-1}(0) = X \) and smooth general fibre. Then there is an exact sequence

\[ 0 \rightarrow N_{C/X} \rightarrow N_{C/\mathcal{X}} \rightarrow N_{X/\mathcal{X}}|_C \rightarrow 0 \]

with \( N_{X/\mathcal{X}} \) a trivial bundle. If \( H^1(N_{C/X}) = 0 \) and \( H^1(\mathcal{O}_C) = 0 \) (e.g. \( C \) is a rational tree), then \( H^1(N_{C/\mathcal{X}}) = 0 \) and \( H^0(N_{C/\mathcal{X}}) \rightarrow H^0(N_{X/\mathcal{X}}|_C) \) is surjective. This means that \( C \) is unobstructed as subvariety of either \( X \) or \( \mathcal{X} \) and deforms along with the smoothing \( \mathcal{X} \).

Now as \( C \subset X \) is an inclusion of normal-crossing double point varieties, it induces an isomorphism \( T^1_X \otimes \mathcal{O}_C \simeq T^1_C \). Hence clearly the general fibre of the deformation of \( C \) is smooth because this is true for \( X \). Moreover given smoothings

\[ C/T \subset \mathcal{X}/T \subset \mathcal{P}/T, \]

then, possibly shrinking \( T \), equations for \( X \subset P \) extend to equations for \( \mathcal{X} \subset \mathcal{P} \) and likewise for \( \mathcal{C} \). Thus, \( C \) is lci in \( \mathcal{X} \) and its equations restrict to to equations of \( C \) on \( X \), therefore

\[ N_{C/\mathcal{X}} \otimes \mathcal{O}_C = N_{C/X}. \]
Note that a necessary condition for a smoothing $\mathcal{P}$ to exist is $N_{Q/P_1} \otimes N_{Q/P_2} \sim_{\text{num}} \mathcal{O}_Q$. This is just because

$$N_{Q/P_1} = \mathcal{O}_P(P_2) \otimes \mathcal{O}_Q, \quad N_{Q/P_2} = \mathcal{O}_P(P_1) \otimes \mathcal{O}_Q,$$

and $\mathcal{O}_P(P_1 + P_2) \otimes \mathcal{O}_Q \sim_{\text{num}} \mathcal{O}_Q$. If e.g. $T = \mathbb{A}^1$, numerical equivalence may be replaced by linear equivalence.

1.5. **Reducible normal bundles.** Normal bundles to reducible varieties in a smooth ambient space behave differently to the case of a reducible ambient space:

**Lemma 1.** Let $P$ be a smooth $n$-dimensional variety and $X_1, X_2 \subset P$ smooth codimension-$c$ subvarieties such that $Y = X_1 \cap X_2$ is a smooth divisor in each. Let $X = X_1 \cup X_2$. Then $X$ is lci in $P$ and there are exact sequences

$$0 \to \tilde{N}_{X/P} \otimes \mathcal{O}_{X_1} \to \tilde{N}_{X_1/P} \to \tilde{N}_{Y/X_2} \to 0,$$

$$0 \to N_{X_1/P} \to N_{X/P} \otimes \mathcal{O}_{X_1} \to N_{Y/X_1} \otimes N_{Y/X_2} \to 0.$$

**Proof.** Locally, $X$ has defining equations of the form

$$u_1, \ldots, u_{c-1}, u_c^{(1)}, u_c^{(2)}$$

where $u_1, \ldots, u_{c-1}, u_c^{(1)}$ are defining equations for $X_1$ and $u_c^{(2)}$ is an equation for $Y$ on $X_1$; similarly $u_1, \ldots, u_{c-1}, u_c^{(2)}$ are defining equations for $X_2$ and $u_c^{(1)}$. This shows exactness of the first sequence, and the second is just its ext-dual. □

Thus, unlike in the case $P$ reducible, the conormal $\tilde{N}_{X/P}$ (resp. normal $N_{X/P}$) restricts on $X_i$ to a rank-1 elementary down (resp. up) modification of $\tilde{N}_{X_i/P}$ (resp. $N_{X_i/P}$) (see §1.7).

**Example 2.** Suppose $X_1, X_2$ are smooth curves meeting transversely in a point $Y$. Then we have an exact sequence

$$0 \to N^0_{X/P} \to N_{X/P} \to T^1_X \to 0$$

where $T^1_X$, which is the usual $T^1$ sheaf of $X$, is a 1-dimensional vector space skyscraper at $Y$ and a 1st-order deformation, i.e. an element of $H^0(N_{X/P})$, is smoothing iff is has a nonzero image in $H^0(T^1_X)$. The kernel $N^0_{X/P}$ is the ‘locally trivial’ normal sheaf, which parametrizes local motions of the triple $(X_1, X_2, Y)$ and fits in an exact sequence

$$0 \to N^0_{X/P} \to N_{X_1/P} \oplus N_{X_2/P} \to Z^0 \to 0$$

where $Z^0$ is a skyscraper at $Y$ equal to the Zariski normal space $T_{P,Y}/(T_{X_1,Y} \oplus T_{X_2,Y})$; e.g. if $\text{dim}(P) = 2$ then $Z^0 = 0$. Note $N^0_{X/P}$ is not locally free on $X$. Anyhow the condition
$H^1(N^0_{X/P}) = H^1(N_{X/P}) = 0$ implies that $X$ has unobstructed deformations in $P$ and is smoothable in $P$ (of course the first vanishing implies the second for $X$ a curve).

1.6. Balanced bundles. See [10] for more details. A balanced bundle $E$ or rank $r$ on $\mathbb{P}^1$ has the form

$$E = r^+ \mathcal{O}(a^+) \oplus (r - r^+) \mathcal{O}(a^+ - 1), r^+ > 0,$$

where the uniquely determined subbundle $r^+ \mathcal{O}(a^+)$ is called the upper subbundle and its rank and slope are called the upper rank and degree, respectively. $E$ is said to be perfectly balanced if $r^+ = r$ or equivalently, $E$ is a twist of a trivial bundle. Note that the smaller line bundle degree appearing above (i.e. $a^+ - 1$ if $r^+ < r$, otherwise $a^+$) equals the round-down of the slope, i.e. $\lfloor \deg(E)/r \rfloor$, and that $H^1(E(-t)) = 0$ iff $t \leq \lfloor \deg(E)/r \rfloor + 1$. The fibre of the upper subbundle at a point $p$, which is a subspace of the fibre $E_p = E \otimes k(p)$, is called the upper subspace at $p$.

A lci rational curve $C$ on a smooth variety $X$ is said to be balanced (resp. perfect) if its normal bundle $N_{C/X}$ is balanced (resp. perfectly balanced).

Balancedness of $E$ is equivalent to rigidity, i.e. vanishing of $H^1(\check{E} \otimes E)$, and in particular it is an open property.

For our purposes it is important to consider ‘balanced’ bundles on rational trees. We could define a bundle $E$ on a rational chain $T$ to be balanced if its restriction on any connected subtree is a direct sum of line bundles of total degree $a^+$ or $a^+ - 1$. This condition is satisfied whenever the restriction of $E$ on any component of the chain is balanced and the gluing at the nodes is general ([10], Lemma 2). In particular, if $T$ is a 2-component chain $T_1 \cup_p T_2$ then $E$ is balanced provided $E_{T_1}, E_{T_2}$ are balanced and the upper subspaces if $E_{T_1}$ and $E_{T_2}$ are transverse. If $E$ is balanced on $T$ then in any smoothing of $(T, E)$, the general fibre is balanced (see [10], Lemma 5). A different approach to smoothing is given in §2 below.

1.7. Modifications. Given a vector bundle $E$ on a variety $X$, a Cartier divisor $D$ on $X$, and an exact sequence of locally-free $\mathcal{O}_D$-modules, where $E_D$ denotes $E \otimes \mathcal{O}_D$,

$$0 \to P \to E_D \to Q \to 0,$$

the elementary down modification of $E$ corresponding to $Q$ is an exact sequence

$$0 \to M_Q(E) \to E \to Q \to 0.$$

Then $M_Q(E)$ is a locally free $\mathcal{O}_D$-module containing $E(-D)$ and fits in another exact sequence

$$0 \to E(-D) \to M_Q(E) \to P \to 0.$$

\[^1\text{Notation here differs slightly from the introduction.}\]
which yields

\[ 0 \to E \to M_Q(E)(D) \to P \otimes \mathcal{O}_D(D) \to 0 \]

Set \( M^P(E) := M_Q(E)(D) \), called the elementary up modification of \( E \) corresponding to \( P \).

Locally, if \( t \) is an equation for \( D \), then there is a local basis \( x_1, \ldots, x_r \) of \( E \) such that \( x_1, \ldots, x_s \) (resp. \( x_{s+1}, \ldots, x_r \)) yields a local basis for \( P \) (resp. \( Q \)), then \( tx_{s+1}, \ldots, tx_r, x_1, \ldots, x_s \) is a local basis of \( M_Q(E) \) (resp. \( x_{s+1}, \ldots, x_r, x_1/t, \ldots, x_s/t \) is a local basis for \( M^P(E) \)), adapted in each case to the appropriate exact sequence above. Note in our applications \( Q \) will have constant rank but this is not required.

For restriction on \( D \), we have a exact sequences

\[ 0 \to Q \otimes \mathcal{O}_D(-D) \to M_Q(E) \otimes \mathcal{O}_D \to P \to 0, \]

\[ 0 \to Q \to M^P(E) \otimes \mathcal{O}_D \to P \otimes \mathcal{O}_D(D) \to 0. \]

Thus an elementary down (resp. up) modification turns a sub to a quotient (resp. a quotient to a sub).

A modification of \( E \) is the composition of a sequence of elementary down and elementary up modifications. These constructions apply in particular to the case of a bundle \( E \) on a curve \( C \), in which case a modification may be realized as the composition of a single down and a single up modification (or vice versa). For an elementary modification, corresponding to a smoothly supported reduced divisor \( D = \sum p_\ell \) on \( C \), \( P \) and \( Q \) are just a sub and quotient vector space of \( E_D = \bigoplus E \otimes k(p_\ell) \). The modification is said to be general if \( D \) is reduced and the sub or quotient in question are general. If \( D \) is supported on a unique component \( F \) of \( C \) and \( E \) restricted on \( F \) is balanced, the modification is said to be in general position if the induced map from the upper subbundle (see §1.6)

\[ E^+_F \otimes \mathcal{O}_D \to Q \]

has maximal rank.

**Lemma 3.** Assumptions as above, let \( E \) be a bundle on a curve \( C \) and \( F \) a component of \( C \) such that \( E_F \) is balanced with upper rank \( r^+ \) and upper degree \( a^+ \) and let \( E' = M_Q(E) \subset E \) be an elementary modification of colength \( s \) in general position cosupported on \( F \). Then if \( s < r^+ \), we have

\[ r^+(E') = r^+-s, a^+(E') = a^+(E). \]

Otherwise,

\[ r^+(E') = r + r^+-s, a^+(E') = a^+-1. \]

**Proof.** This follows easily from the fact that the induced map \( E^+_F(p) \to sk(p) \) has maximal rank by generality (here and elsewhere, \( sA \) where \( s \in \mathbb{N} \), denotes \( \bigoplus_1^s A \) wherever this makes sense). \( \square \)
1.8. Blowing up normal bundles. Elementary modifications occur often in the geometry of embedded curves. One example is the following standard result which to save notation we have stated just for a curve \( C \) but with evident modifications is equally valid for any lci subvariety (which will naturally get blown up in the blowup of \( X \)).

**Lemma 4.** Let \( C \) be a lci curve on a smooth variety \( X \) and let \( Y \) be a lci subvariety of codimension \( s \) in \( X \) meeting \( C \) schematically in a unique point \( p \) smooth on \( C \). Let \( \pi : X' \to X \) be the blowup of \( Y \) with exceptional divisor \( E \) and let \( C' \) the birational transform of \( C \) on \( X \).

(i) \( N_{C'/X'} \) is the elementary down modification of colength \( s - 1 \) of \( N_{C/X} \) corresponding to the image of \( T_p Y \) in \( N_{C/X}(p) \).

(ii) Under the natural identification of \( (N_{C'/X'})_p \) with \( T_p E \), the kernel of the map \( (N_{C'/X'})_p \to (N_{C/X})_p \) coincides with the vertical subspace of \( T_p E \), i.e. the tangent space to the fibre of \( E \to Y \).

**Proof.** For convenience we work with conormal bundles denoted \( \tilde{N}_{C/X} \) etc. If \( \pi : X' \to X \) denotes the blowup map, \( \pi^* \tilde{N}_{C/X} \) is clearly a subsheaf of \( \tilde{N}_{C'/X'} \) and coincides with it locally off \( p \), so it suffices to identify the image at \( p \). We can choose local coordinates at \( p \) of the form \( y, x_1, ..., x_{s-1}, x_s, ..., x_n \) so that \( y \) defines \( p \) on \( C \), \( x_1, ..., x_n \) define \( C \) and \( y, x_1, ..., x_{s-1} \) define \( Y \). Then \( x_1, ..., x_n \) yield a local basis for \( \tilde{N}_{C/X} \) while, in a suitable affine open in \( X' \) containing \( C' \) with coordinates \( y, x_1/y, ..., x_{s-1}/y, x_s, ..., x_n \), a basis for \( \tilde{N}_{C'/X'} \) is \( x_1/y, ..., x_{s-1}/y, x_s, ..., x_n \). This proves the dual statement for conormals which is equivalent to assertion (i).

As for (ii), it follows from the above computation or, just as well, from the diagram

\[
\begin{array}{ccc}
T_p E & \simeq & (N_{C'/X'})_p \\
\downarrow & & \downarrow \\
T_p Y & \subseteq & (N_{C/X})_p.
\end{array}
\]

\[\square\]

2. Bundles on caudate curves

The purpose of this section is to prove a general and elementary result about smoothing of vector bundles on curves endowed with multiple tails. This result permits construction of some balanced vector bundles on rational curves and in particular to prove the existence of some balanced rational curves. The result is stated in much greater generality than is needed for the applications to minimal rational curves given in this paper, in the hope that it might enable further such applications. See also [4], [13], [9], [10] for other results on bundles on rational trees.

By definition, a rational tree is a nodal curve that is a tree of smooth rational curves. A broken comb is a connected nodal curve of the form

\[ C = B \cup \bigcup_{i=1}^{10} T_i \]
where $B$, the base (aka the body), is a connected nodal curve and each tooth (aka tail) $T_i$ is a rational tree meeting $B$ in a unique smooth point called its root and meeting no other $T_j$. A broken comb is rational if $B$ is a rational tree. A rational comb is a rational broken comb that is unbroken, i.e. where $B$ and each $T_i$ are $\simeq \mathbb{P}^1$.

Unlike the irreducible case, or for that matter the case of rational chains- see [10]- even nice bundles on rational combs need not split as direct sums of line bundles. The following example is essentially taken from [9].

**Example 5.** Let $C = B \cup \bigcup_{i=1}^{t} T_i$ be a rational comb and let $E$ be a vector bundle on $C$ whose restriction on each $T_i$ is isomorphic to $O \oplus O(-1)$, with $E_{B}$ an arbitrary vector bundle and with general gluing at nodes. Then $h^0(\bar{E} \otimes E) \geq t$ hence, if $t \geq 5$, then $h^0(\bar{E} \otimes E) > 4 = \chi(\bar{E} \otimes E)$, therefore $h^1(\bar{E} \otimes E) > 0$. Consequently, $E$ is not balanced and in fact not a direct sum of line bundles. Nonetheless, as we shall see below in Example 18, Theorem 6 below applies to $E$, showing that a smoothing of $E$ is a deformation of a general down modification of $E_B$ at the nodes, hence is as well-behaved as possible.

If $E_C$ is a bundle on a curve $C$, by a smoothing of $(C, E_C)$ is meant an irreducible surface $S$ endowed with a flat map to a smooth curve $S \to T$ with smooth general fibre and special fibre $C$ (with multiplicity 1), plus a vector bundle $E$ on $S$ that restricts to $E_C$ on $C$. Similarly, for a subset $A \subset C$, a partial smoothing at $A$ assumes only that there is a neighborhood $U$ of $A$ on $S$ such that the intersection of the general fibre with $U$ is smooth.

**Theorem 6.** Let $C = B \cup \bigcup_{i=1}^{t} T_i$ be a broken comb with teeth $T_1, ..., T_k$ and respective roots $p_1, ..., p_k$ and let $E_C$ be a vector bundle on $C$. Assume

(i) on each component of each $T_i$, $E_C$ is balanced;

and for each $T_i$ either

(ii) the gluing at each node $q$ of $C$ on $T_i$ of the restrictions of $E_C$ on the components of $C$ through $q$ is sufficiently general; or

(ii)' the restriction of $E_C$ on $T_i$ is perfectly balanced, i.e. a twist of a trivial bundle, i.e. has the form (vector space) $\otimes$ (line bundle).

Then any partial smoothing of $(C, E_C)$ at $\bigcup T_i$ is the pullback by a birational map of a deformation of a general modification of some twist $E_B \otimes O_B(\sum m_ip_i)$ at $p_1, ..., p_k$.

**Remark 7.** We recall that a general (down) modification of a bundle $E$ at a smooth point $p$ on a curve means the kernel of a map $E \to Q$ to general quotient $Q$ of $E \otimes k(p)$. Ditto for up modification and several points.

**Remark 8.** Here the genus of $C$ is arbitrary but in applications it will be zero. In fact, the only case used here is that of a rational comb, i.e. where $B$ and each $T_i$ are $\mathbb{P}^1$. 

11
Remark 9. Note that the Theorem applies an *arbitrary* 1-parameter partial smoothing rather than just a ’sufficiently general’ one or, for that matter, a multi-parameter smoothing dominating a versal deformation of the curve, where the nodes smooth independently. This feature is crucial for applications to curves on fans because when the curve smooths together with the fan, the nodes lying on the fan’s double locus smooth *simultaneously*, so this smoothing of the curve is never general.

Remark 10. Below we will use the fact that in a partial smoothing, any extremal fibre component contained in the smooth part of the total space must be a (-1) curve. This follows easily from the fact that the fibre is reduced (see below).

Remark 11. Regarding the meaning of ’general gluing’ in terms of the inductive procedure used in the proof below. This proceeds ’from the outside in’ (i.e. toward the ’spine’ B). In practice, general gluing at a node q lying on an ’outer’ component F and an ’inner’ one $F^*$ means that the upper subspace of $E_{F^*}$ at is transverse to the ’distinguished subspace’ of $E_F$ at $p$, the latter being the upper subspace of a modification of the original $E_F$ previously constructed in the course of the proof, which has to do with upper subbundles on components ’further out’ than $F$. It is possible that $E_{F^*}$ is a twist of a trivial bundle, so its upper subspace is all of $E_p$, but after the next modification of $E$ the resulting bundle will generally be nontrivial on $F^*$.

Proof of theorem. Given a partial smoothing $(E,S)$, we first resolve all singularities of the surface $S$ lying on $\bigcup T_i$ to obtain a smoothing with total space that is smooth in a neighborhood of the preimage of $\bigcup T_i$, this at the cost of augmenting the $T_i$ by some further rational trees $K_j$ on which $E$ is trivial. Suitably refreshing notation, we have a fibred surface $\pi : S \to B$ with

$$\pi^{-1}(0) = B \cup \bigcup T_i \cup \bigcup K_j$$

such that $S$ is smooth along $\bigcup T_i \cup \bigcup K_j$, and such that each restriction $E_{T_i}$ balanced and each $E_{K_j}$ is trivial. Moreover, each $K_j$ meets $B \cup \bigcup T_i$ in exactly 2 points $p'_j, p''_j$ at most one of which is on $B$. We may write $E_{K_j} = U \otimes O_{K_j}$ for a vector space $U$. Then letting $Z', Z''$ denote the components other than $K_j$ through $p'_j, p''_j$ respectively, the generality hypothesis (ii) above implies that the identification of the fibres $E_{p'_j}$ and $E_{p''_j}$ with $U$ is general and in particular the upper subspaces of $E_{Z'}$ at $p'_j$ and $E_{Z''}$ at $p''_j$, considered as subspaces of $U$, are mutually in general position.

Now the proof is an inductive procedure on the irreducible components of the ‘multitail’ $\mathcal{T} = \bigcup T_i \cup \bigcup K_j$, proceeding ‘inward’ towards $B$, where each step eliminates by contraction an extremal component. The procedure works separately for each $T_i$ and we will work on a $T_i$ for which (ii) above holds as the case of (ii)' is similar and simpler. We start with the initial step, which is essentially identical to the inductive step. Let $F$ be an
extremal component of $\mathcal{T}$, i.e. $F$ meets the rest of the curve, say $G$, in a single point $p$. Because $G$ is reduced and $F.(F+G) = F.(\text{fibre}) = 0$, we have $F^2 = -F.G = -1$ so $F$ must be a $(-1)$ curve (initially $F$ must be a component of some $T_i$ but this is unimportant). By assumption we can write

$$E_F \simeq r^+ \mathcal{O}_F(d^+) \oplus (r-r^+)\mathcal{O}_F(d^+-1).$$

Replacing $E$ by its twist $E(d^+)$, we may assume $d^+ = 0$. Now if $r^+ = r$, i.e. $E_F$ is a twist of a trivial bundle, we may as well assume $E_F \simeq r\mathcal{O}_F$. If $r^+ < r$, perform an elementary down modification on $E$ corresponding to the quotient

$$E \to (r-r^+)\mathcal{O}_F(-1).$$

This modification yields a subsheaf $E' \subset E$, equal to $E$ off $\pi^{-1}(0)$, with $E'_F \simeq r\mathcal{O}_F$. Moreover if $F^*$ is the unique other component of $\pi^{-1}(0)$ through $p$ then $E'_{F^*}$ is the elementary modification of $E_{F^*}$ at $p$ corresponding to the corresponding pointwise quotient of vector spaces

$$E(p) \to (r-r^+)\mathcal{O}_F(-1)(p).$$

Furthermore, if $G$ is the subcurve of $\pi^{-1}(0)$ complementary to $F \cup F^*$ then $E'_G = E_G$.

Now if the aforementioned $F^*$ is a component of some $T_i$ then by our general gluing hypothesis, $E'_{F^*}$ is balanced and its upper subspace at all nodes on $F^*$, being a sum or intersection of 2 general subsheaves (compare Lemma 3), is general.

In the other case, $F^*$ is a component of some $K_j$ which meets $B \cup \bigcup T_i$ in another point $p' = K_j \cap F^{**}$ where $F^{**}$ is a component of $B \cup \bigcup T_i$ and by general gluing plus the fact that $E_{K_j}$ is a trivial bundle $U \otimes_{\mathbb{C}} \mathcal{O}_{K_j}, U \simeq \mathbb{C}$, ensures as above that $(E_F^*)^+(p)$ and $(E_{F^{**}})^+(p')$, as subspaces of $U$, are general and meet transversely. In particular, $E'_{F^*}$ is balanced.

Next, we blow down $F$. Set $\mathcal{O}_{rf} = \mathcal{O}_S/\mathcal{O}_S(-rF)$ and consider the standard exact sequence

$$0 \to E' \otimes \mathcal{O}_F(-rF) \to E' \otimes \mathcal{O}_{(r+1)f} \to E' \otimes \mathcal{O}_{rf} \to 0.$$ 

Using the fact that $E'_F = r\mathcal{O}_F$ and $\mathcal{O}_F(-F) = \mathcal{O}_F(1)$ it follows easily that if $\hat{F}$ denotes the formal completion of $S$ along $F$, then

$$E' \otimes \mathcal{O}_{\hat{F}} \simeq r\mathcal{O}_F.$$ 

Consequently if we let $f : S \to S_1$ denote the blowing down of the $(−1)$ curve $F$, then by the formal function theorem $f_*(E')$ is locally free near $q = f(F)$ (also $R^1f_*(E') = 0$). Hence if we let

$$E_1 = f_*(E')$$
then identifying the general fibre $Y$ of $S/B$ and $S_1/B$, we have $(E_1)_Y \simeq E'_Y$. Thus we get a modified family with similar properties whose special fibre has a smaller multitail, and we may continue the process until there is no multitail, which is the desired family. 

\[ \square \]

**Remark 12.** Similar results in the case of a rational chain, rather than broken comb, or a comb with few teeth are given in [10], §1.

**Corollary 13.** Let $T$ be a rational tree and let $E_T$ be a vector bundle on $T$ such that for each component $S$ of $T$ either

(i) the restriction $E_S$ is balanced and the gluing at each node on $S$ is general; or

(ii) $E_S$ is a twist of a trivial bundle.

Then any smoothing of $(T, E_T)$ has balanced general fibre.

**Remark 14.** Case (ii) above is ‘almost’ a special case of case (i), except when there is a pair of intersecting components $S_1, S_2$ with $E_{S_1}, E_{S_2}$ trivial; then the gluing of the trivial bundles $E_{S_1}, E_{S_2}$ at $S_1 \cap S_2$ is immaterial.

Note that by Example [18] below, it is possible under the hypotheses of the Corollary to have $h^1(\tilde{E}_T \otimes E_T) > 0$, a condition which for $\mathbb{P}^1$ is equivalent to non-balancedness. The Corollary may be used in lieu of Lemma 2 or Lemma 7 of [10] to show existence of some balanced rational curves of low degree $e$ on general Fano hypersurfaces of degree $d \leq n$, and will be used for a similar purpose in §3 below for the case $d = n$ and $e \geq n - 1$.

**Corollary 15.** Let $f : X \to S$ be a proper flat family of nodal-or-smooth curves with general fibre isomorphic to $\mathbb{P}^1$, over an irreducible variety $S$. Let $\partial S \subset S$ be the locus of singular fibres. Let $E$ be a vector bundle on $X$. Suppose that $T := f^{-1}(s_0)$ together with $E_T$ satisfy the hypotheses of Corollary [13]. Then there is a neighborhood $U$ of $s_0$ in $S$ such that for every $s \in U \cap (S \setminus \partial S)$, $E_{f^{-1}(s)}$ is balanced; equivalently,

$$\text{supp}(R^1 f_*(\tilde{E} \otimes E)) \cap U \subset \partial S \cap U.$$ 

The Corollary is interesting because it applies in situations where standard semi-continuity fails because, with the above notation, one has $H^1(\tilde{E}_T \otimes E_T) \neq 0$- see Example [5] below. Then we conclude that $R^1 f_*(\tilde{E} \otimes E)$ is nontrivial and locally supported on the boundary.

Returning to the general situation of the Theorem, it actually implies more, namely to the effect that, when nontrivial modification get involved, a general smoothing of $(C, E)$ is ‘better behaved’ than $E_B$. To make this precise, it is convenient to use the language of partitions. Suppose $E$ is a vector bundle on $\mathbb{P}^1$ of the form
\[ E \simeq \bigoplus_{i=1}^{s} r_i \mathcal{O}(d_i), \quad d_1 > d_2 > \ldots > d_s. \]

The subbundles
\[ E_j = \sum_{i=1}^{j} r_i \mathcal{O}(d_i) \]
are canonically defined and form the Harder-Narasimhan filtration of \( E \):
\[ E_1 \subset E_2 \subset \ldots \subset E_s = E. \]

We associate to \( E \) the partition \( \Pi(E) \) with blocks of height \( d_i \) and width \( r_i, i = 1, \ldots, s \) and total width \( r \), which may be viewed as usual as a subset of the first quadrant in \( \mathbb{R}^2 \) containing the origin. In addition to being partially ordered by inclusion, these partitions are lexicographically ordered via the degree sequence \( (d_i) \) and if \( E' \) is a general member of a deformation of \( E \) then
\[ \Pi(E') \leq \Pi(E). \]

Given a partition \( \Pi \) of degree \( d \) and width \( r \) and an integer \( k \), the **elementary modification** of type \( k \) of \( \Pi \), denoted \( M_k(\Pi) \), is the lexicographically smallest partition \( \Pi' \) of width \( r \) and degree \( d + k \), such that
\[ \Pi' = \Pi \text{ if } k = 0, \]
\[ \Pi' \supset \Pi \text{ if } k > 0, \]
and
\[ \Pi' \subset \Pi \text{ if } k < 0. \]

One way to define \( M_k(\Pi) \) is inductively as \( M_1(M_{k-1}(\Pi)) \) (\( k > 0 \)) or \( M_{-1}(M_{k+1}(\Pi)) \) (\( k < 0 \)), where \( M_1(\Pi) \) (resp. \( M_{-1}(\Pi) \)) replaces the first (resp. last) column of height \( d_r \) (resp. \( d_1 \)) by a column of height \( d_r + 1 \) (resp. \( d_1 - 1 \)).

A modification corresponding to \( E \to Q = \bigoplus_{\ell=1}^{t} Q_{p_\ell} \) supported on \( D = \sum p_\ell \) is said to be **in general position** if for each \( i \) the induced map
\[ E_i \otimes \mathcal{O}_D \to Q \]
has maximal rank. The following is a simple generalization of Lemma 3.

**Lemma 16.** If \( E' \) is an elementary modification in general position of \( E \) (up or down, at one or more points), and
\[ \deg(E') = \deg(E) + k \]
then
\[ \Pi(E') = M_k(\Pi(E)). \]
Proof. It suffices treat the case of a down modification. Let $j$ be smallest such that $E_j \otimes O_D \rightarrow Q$ is surjective. Then there is an exact sequence

$$0 \rightarrow \bigoplus_{i<j} r_i O(d_i - 1) \oplus r_j' O(d_j - 1) \rightarrow E' \rightarrow (r_j - r_j') O(d_j) \oplus \bigoplus_{i>j} r_i O(d_i) \rightarrow 0$$

with $r_1 + \ldots + r_{j-1} + r_j' = \ell(Q)$, $0 < r_j' \leq r_j$. Such a sequence automatically splits and this suffices to imply that $E'$ has the desired partition. □

Therefore the Theorem implies (compare [10], Lemma 7):

**Corollary 17.** Assumptions as in the Theorem, if $B \simeq P^1$ and $(C', E')$ is a smoothing of $E$ then $\Pi(E') \leq M_k(\Pi(E_B))$, where $k = \sum \deg(E_T)$.

**Example 18.** [Example 5 revisited] Notations as in Example 5. Theorem 6 applies to $E$, showing that a smoothing $E'$ of $E$ is a deformation of a general down modification of $E_B$ at the nodes. Consequently, if $E_B \simeq O(a_1) \oplus O(a_2)$ then $E' \simeq O(b_1) \oplus O(b_2)$ with $|b_1 - b_2| \leq \max(|a_1 - a_2| - t, 1)$. Informally, attaching an $O \oplus O(-1)$ tail works like an elementary down modification.

### 3. Curves in Projective Space

Here as a warmup for fan-like methods we will prove the well-known fact (see [9] for a longer proof):

**Proposition 19.** A general rational curve of degree $e \geq n$ in $P^n$ is balanced.

**Proof.** Case 1 (probably well known): a rational normal curve $C \subset P^n$.

We must show $C_{C/P^n} = (n - 1)O(n + 2)$.

Proof 1 (arbitrary char.): We work inductively, the cases $n = 1, 2$ being easy. Degenerate $C$ to $C_0 = C' \cup_p L$ where $C' \subset P^{n-1}$ is rational normal and $L$ is a transversal line. See the discussion in §1.5. By induction, we have $C_{C'/P^n-1} = (n - 2)O(n + 1)$. The natural exact sequence

$$0 \rightarrow C_{C'/P^{n-1}} \rightarrow C_{C'/P^n} \rightarrow O(1)|_{C'} \rightarrow 0$$

splits because $C'$ is the intersection of a minimal scroll in $P^n$ with $P^{n-1}$, hence

$$C_{C'/P^n} = (n - 2)O(n + 1) \oplus O(n - 1).$$

Then $C_{C_0/P^n}|_{C'}$ is the rank-1 elementary up modification of $C_{C'/P^n}$ at $p$ corresponding to $T_p L$ which is not tangent to $P^{n-1}$, hence

$$C_{C_0/P^n}|_{C'} = (n - 2)O_{C'}(n + 1) \oplus O_{C'}(n),$$

with upper subbundle $C_{C'/P^{n-1}} = (n - 2)O_{C'}(n + 1)$. Similarly,

$$C_{C_0/P^n}|_L = O_L(2) \oplus (n - 2)O_L(1).$$
Now the natural maps
\[ i_1 : N_{C'/\mathbb{P}^n} \rightarrow N_{C_0/\mathbb{P}^n} \otimes \mathcal{O}_{C'}, i_2 : N_L/\mathbb{P}^n \rightarrow N_{C_0/\mathbb{P}^n} \otimes \mathcal{O}_L \]
have the same codimension-1 image at \( p \), namely the hyperplane dual to the element of \( \mathcal{N}_{C_0/\mathbb{P}^n}(p) \) coming from the unique order-2 minimal local generator of \( \mathcal{I}_{C_0} \), i.e. a generator of \( \ker(\mathcal{N}_{C_0/\mathbb{P}^n} \rightarrow \mathcal{N}_{C'/\mathbb{P}^n}) = \ker(\mathcal{N}_{C_0/\mathbb{P}^n} \rightarrow \mathcal{N}_{L/\mathbb{P}^n}) \). Considering \( i_1(p) \), the image in question coincides with the upper subspace of \( \mathcal{N}_{C_0/\mathbb{P}^n}|_{C'} \) at \( p \), coming from the \((n-2)\mathcal{O}(n+1)\) (unique) subbundle. On the other hand, the upper subspace of \( \mathcal{N}_{C_0/\mathbb{P}^n}|_{L} \), coming from the \( \mathcal{O}(2) \), is clearly not in the image of \( i_2(p) \). Since the images of \( i_1(p) \) and \( i_2(p) \) coincide, it follows the respective upper subspaces at \( p = C' \cap L \) are different, i.e. in general position, so \( \mathcal{N}_{C_0/\mathbb{P}^n} \simeq (n-1)\mathcal{O}_{C_0}(n+2) \).

Proof 2 (quicker but maybe valid only in char. 0): The normal bundle \( N = N_{C'/\mathbb{P}^n} \) has degree \((n-1)(n+2)\) and rank \( n-1 \). Textbooks (\cite{7} or \cite{6} p.12) show that there is a unique \( C \) through \( n+3 \) general points. Carefully examining this construction (or else using char. 0) shows that the locus of rational normal curves through \( n+3 \) general points is a reduced singleton. Consequently, this locus has trivial tangent space, i.e. \( H^0(N(-\sum_{i=1}^{n+3} p_i)) = 0 \). Hence \( N \) contains no line bundle of degree \( n+3 \) or more, so \( N \simeq (n-1)\mathcal{O}(n+2) \).

Case 2: \( n < e < 2n \).

Consider the blowup \( \mathcal{X} \) of \( \mathbb{P}_1^n \times \mathbb{A}^1 \) in \( \mathbb{P}_1^e \times \mathbb{P}_2^e - 0 \) (\( \mathbb{P}_b^d \) is a copy of \( \mathbb{P}^d \)), with natural maps \( \pi : \mathcal{X} \rightarrow \mathbb{A}^1, f : \mathcal{X} \rightarrow \mathbb{P}_1^n \). Then
\[ X_0 := \pi^{-1}(0) = X_1 \cup X_2, \]
where
\[ X_1 = B_{\mathbb{P}_1^e \times \mathbb{P}_2^e}, X_2 = B_{\mathbb{P}_2^e}, Z := X_1 \cap X_2 = \mathbb{P}_1^e \times \mathbb{P}_2^{2n-e-1}. \]

(\( \mathbb{P}_1^n, \mathbb{P}_2^n \) are copies of \( \mathbb{P}^n \) and likewise for their subspaces.) \( \mathcal{X} \) is endowed with a relative hyperplane bundle \( H = f^*\mathcal{O}(1)(-X_2) \) which restricts on the general fibre of \( \pi \) to \( \mathcal{O}(1) \), on each \( X_i \) to \( \mathcal{O}(1)(-Z) \) and on \( Z \) to \( \mathcal{O}(1,1) \).

Let \( C_1' \subset X_1, C_2' \subset X_2 \) be respective proper transforms of curves \( C_1 \), a rational normal curve in \( \mathbb{P}_1^n \) and \( C_2 \), a rational normal curve in its span \( S \simeq \mathbb{P}^{e-n+1} \), where \( C_1 \) (resp. \( C_2 \)) meets \( \mathbb{P}_1^e \) (resp. \( \mathbb{P}_2^{2n-e-1} \)), transversely in 1 point, so that \( C_0 = C_1' \cup C_2' \subset X_0 \) is a connected nodal curve. We also assume \( S \) is transverse to the blowup center \( \mathbb{P}_2^{2n-e-1} \). Note that the 'degree', i.e. \( H \)-degree, of \( C_1' \cup C_2' \), is \( e \), e.g. because \( C_1' \cup C_2' \) projects in \( \mathbb{P}_1^n \) to the union of \( C_1 \), of degree \( n \), and the projection \( \tilde{C}_2 \) of \( C_2 \) from \( \mathbb{P}_2^{2n-e-1} \) which meets it in 1 point so that \( \tilde{C}_2 \) has degree \( e-n \). The family of curves \( C \) in \( \mathbb{P}^n \) that we construct
below degenerates to $C_1 \cup C_2 \subset X$, and that degeneration dominates a degeneration of $C$ in $\mathbb{P}^n$ to $C_1 \cup \tilde{C}_2$, so $C$ has degree $e$. Also see Example 24.

We analyze $C_2$ first. Note

$$N_{C_2/S} = (e - n)O(e - n + 3)$$

hence

$$N_{C_2/P_2} \simeq (e - n)O(e - n + 3) \oplus (2n - e - 1)O(e - n + 1).$$

The latter bundle is not balanced, however after the blowup of the transverse $\mathbb{P}^{2n-e-1}$, we get (see Lemma 4)

$$N_{C_2/P_2} \simeq (e - n)O(e - n + 2) \oplus (2n - e - 1)O(e - n + 1),$$

which is balanced and whose upper subspace at $p$ comes from the first summand and, identifying the normal space at $p$ with $T_pZ$, corresponds to the $\mathbb{P}^{e-n}_2$ factor, i.e. kernel of the projection $T_pZ \to T_p\mathbb{P}^{2n-e-1}_2$ (see Lemma 4), which is also the kernel of the differential of the blowdown map $X_2 \to \mathbb{P}^n_2$.

As for $C_1$, we have $N_{C_1/P_1} = (n - 1)O(n + 2)$. Then after blowing up the transverse $\mathbb{P}^{e-n}_1 = \mathbb{P}^{n-(2n-e)}$ to $C_1$ we get

$$N_{C_1/X_1} \simeq (e - n)O(n + 2) \oplus (2n - e - 1)O(n + 1).$$

This bundle is balanced and its upper subspace at $p$ comes from the first summand. I claim that under the identification of the normal space with $T_pZ$, the upper subspace has trivial intersection with $T_p\mathbb{P}^{e-n}_1$. To see this consider the projection $X_1 \to \mathbb{P}^{2n-e-1}$ which identifies $X_1 = \mathbb{P}\mathbb{P}^{2n-e-1}_2$ of $(O(1) \oplus (e - n + 1)O)$ and maps $C_1$ to a general rational curve of degree $n - 1$ in $\mathbb{P}^{2n-e-1}$. Then (see §5.1) the vertical (i.e. part killed by the blowdown map $X_1 \to \mathbb{P}^{2n-e-1}$) subbundle of the normal bundle to $C_1$ has the form $K^*(1)$ where $K$ is the relative tautological subbundle which fits in an exact sequence on $C_1$:

$$0 \to K \to (e - n + 1)O \oplus O(-n + 1) \to O(1) \to 0.$$ 

Because the exceptional divisor $\mathbb{P}((e - n + 1)O)$ meets $C_1$ at $p$ only, it follows that the induced map $(e - n + 1)O \to O(1)$ vanishes at $p$ hence factors through an inclusion $O \to O(1)$. This yields a (locally split) inclusion $(e - n)O \to K$ and it follows easily that $K \simeq (e - n)O \oplus \oplus(-n + 1)$ so

$$K^*(1) \simeq (e - n)O(1) \oplus O(n).$$

Now a nontrivial intersection of the upper subspace above with $T_p\mathbb{P}^{e-n}$ would yield a subbundle of $K^*(1)$ which is a sum of copies of $O(n + 2)$ which is evidently impossible, proving our claim.
Hence, the two upper subspaces of $N_{C_1}/X_1$ and $N_{C_2}/X_2$ at $p$, both of which may be considered as subspaces of $N_{C_0}/X_0$ at $p$, are transverse as such. Therefore $C_0$, which is a locally complete intersection on $X_0$, hence also on $X$, has normal bundle $N_{C_0}/X_0$ that is balanced, positive and has $H^1 = 0$. Therefore as in §1.4 $C_0$ smooths out to a smooth irreducible rational curve of degree $e$ in a nearby fibre $\mathbb{P}^n \times t, t \neq 0$ with balanced normal bundle.

Case 3: $e \geq 2n$.

This case is analogous to Case 2 except that we take $C_2 \subset \mathbb{P}^n_2$ to be a general (hence now nondegenerate) curve of degree $e - n + 1$ with $C_1 \subset \mathbb{P}^n_1$ still a rational normal curve. By induction we may assume $N_{C_2}/\mathbb{P}^n_2$ is balanced so we have

$$N_{C_2}/\mathbb{P}^n_2 \simeq r\mathcal{O}(a^+) \oplus (n - 1 - r)\mathcal{O}(a^+ - 1),$$

$$0 < r \leq n - 1, a^+ = \lceil((e - n + 1)(n + 1) - 2)/(n - 1)\rceil.$$

Then let $X_2$ be the blowup of $\mathbb{P}^n_2$ in a general $\mathbb{P}^{n-r}_2$ meeting $C_2$ transversely in 1 point and $X_1$ be the blowup of $\mathbb{P}^n_1$ in a general $\mathbb{P}^{r-1}_1$ meeting $C_1$ transversely in 1 point, with $C_1', C_2'$ being the birational transforms of $C_1, C_2$ resp. So as above we have

$$N_{C_1}/X_1 = (n - 1 - r)\mathcal{O}(n + 2) \oplus r\mathcal{O}(n + 1)$$

while

$$N_{C_2'}/X_2 = \mathcal{O}(a^+) \oplus (n - 2)\mathcal{O}(a^+ - 1).$$

Note that $a^+ > n + 2$ thanks to $e - n + 1 > n$, so $N_{C_1'}/X_1$ cannot have an $\mathcal{O}(a^+)$ summand. Hence we can argue as above that the upper subspaces of $N_{C_1'}/X_1$ and $N_{C_2'}/X_2$ at $p$ must have trivial intersection.

\[\square\]

The method of proof implicitly uses the notion of fang which will be revisited more explicitly in §5.3.

4. CASE $d = n$

Our result is the following.

**Theorem 20.** Let $X$ be a general hypersurface of degree $n$ in $\mathbb{P}^n$, $n \geq 4$. Then for any $e \geq n - 1$, $X$ contains a nonsingular irreducible balanced rational curve of degree $e$.

**Corollary 21.** Notations as above, $X$ is separably $(\lfloor \frac{e - 2}{n - 2} \rfloor + 1, e)$-rationally connected.
Proof of Corollary. Standard. Let $C/B$ be a component of the universal degree-$e$ rational curve in $X$ containing a good curve as above and $C^q/B$ its $q$-th fibre power, which admits an obvious map

$$f_q : C^q/B \to X^q$$

For $z = (C, p_1, ..., p_q) \in C^q/B$, there is a derivative map

$$df_q : T_z(C^q/B) \to \bigoplus T_{p_i}X$$

taking the vertical part of the tangent space to $\bigoplus T_{p_i}C$, hence inducing $T[C]B \to \bigoplus N_{p_i,C/X}$ which is none other than the evaluation map

$$H^0(N_{C/X}) \to \bigoplus N_{p_i,C/X},$$

with cokernel $H^1(N_{C/X}(-q))$. For $q = \left\lfloor \frac{e-2}{n-2} \right\rfloor + 1 = q_{\text{max}}(e)$, the latter map is surjective by an evident $H^1$ vanishing, hence so is $f_q$ locally. □

Corollary 22. Notations as above, for all $q \geq 2$, the minimal degree of a rational curve in $X$ through $q$ general points is the expected one, viz. $(q-1)(n-2) + 2$, and the locus of such curves is reduced and of the expected dimension.

Proof. By Corollary 21, there exists a rational curve of degree $e = (q-1)(n-2) + 2 = e_{\text{max}}(q)$ through $q$ general points. Because $q_{\text{max}}(e-1) < e$, this $e$ is smallest. □

Proof of Theorem 20. Consider a 2-fan $P = P_1 \cup P_2$ as in §1.2, so $P_1 = B_p\mathbb{P}^n, P_2 = \mathbb{P}^n, E = \mathbb{P}^{n-1}$. On $P$, consider a general hypersurface $X_0$ of type $(n, n-1)$ as in [10], i.e.

$$X_0 = X_1 \cup_F X_2,$$

with $X_2$ is a general hypersurface of degree $n-1$ in $\mathbb{P}^n$ and $X_1 = B_pX_1$ is the blowup at $p$ of a general quasi-cone $\tilde{X}_1$ with quasi-vertex $p$, i.e. a hypersurface in $\mathbb{P}^n$ of degree $n$ and multiplicity $n-1$ at $p$, with exceptional divisor $F$ which is a degree-$n-1$ hypersurface in $E = \mathbb{P}^{n-1}$. Using projection from $p$, which maps $\tilde{X}_1$ birationally onto $\mathbb{P}^{n-1}$, $X_1$ can also be realized as the blowup of $\mathbb{P}^{n-1}$ in a general $(n-1, n)$ complete intersection

$$Y = F_{n-1} \cap F_n, \deg(F_i) = i.$$ 

and in this realization $F$ is the birational transform of $F_{n-1}$ and is isomorphic to the latter, while $Y$ can be identified as the set of lines through $p$ contained in $\tilde{X}_1$. There is a family $\mathcal{X}/\mathbb{A}^1 \subset \mathcal{P}/\mathbb{A}^1$ with general fibre $X \subset \mathbb{P}^n$ a general hypersurface of degree $n$ and special fibre $X_0 \subset P$. Concretely, if $p$ is the point $[1, 0, ..., 0]$ and we write the general degree-$n$ polynomial as $\sum_{i=0}^n f_i x_0^{n-i}$ with $f_i$ of degree $i$ in $x_1, ..., x_n$ then the equation for the
quasi-cone \( X_1 \) is \( f_n + x_0f_{n-1} \), that of \( X_2 \) is \( \sum_{i=0}^{n-1} f_i x_0^{n-i-1} \), that of \( F \) and \( F_{n-1} \) is \( f_{n-1} \), and that of \( F_n \) is \( f_n \).

**Case 1:** \( e \geq (n-1)^2 \).

Write \( e = e_1n - a \) with

\[ e_1 \geq n - 1, a \leq n - 1. \]

To construct a suitable curve in \( X_0 \) we proceed as follows. Let \( C \) be a general rational curve of degree \( e_1 \) in \( \mathbb{P}^{n-1} \). Let \( F_{n-1} \subset \mathbb{P}^{n-1} \) be a general hypersurface meeting \( C \) transversely in \( e_1(n-1) \) points. Note that \( C \cap F_{n-1} \) is in (linearly) general position (i.e. any \( n \) points linearly independent). Choose a subset \( A \) from it with \( |A| = a \) which we may assume consists of coordinate vertices \( p_1, ..., p_a \) (recall that \( a \leq n - 1 \)). We also assume \( x_0, ..., x_{n-1} \) are standard coordinates on \( \mathbb{P}^{n-1} \) (compatible with the \( p_i \)).

Now I claim that that \( \mathcal{I} = \mathcal{I}_{p_1, ..., p_a}/\mathbb{P}^{n-1}(2) \), in fact already \( \mathcal{I}_{p_1, ..., p_a}/\mathbb{P}^{n-1}(2) \), is globally generated. If \( p \) is not in the span \( S = \langle p_1, ..., p_a \rangle \) then a general hyperplane containing \( S \) generates \( \mathcal{I} \) at \( p \). On the other hand if \( p \in \langle p_j : j \in J \rangle \) for some \( J \subset \{1, ..., a\} \) then \( \mathcal{I} \) is globally generated. If \( p \in \langle p_j : j \in J \rangle \) then a general hyperplane containing all \( p_k, k \neq j \) plus another general hyperplane containing only \( p_j \) to generate \( \mathcal{I} \) at \( p \).

Now as \( \mathcal{I} \) is globally generated, we may assume \( F_{n-1} \) will miss any given finite set of points. I claim we can find a hypersurface \( F_n \) through \( A \) and no other points of \( C \cap F_{n-1} \) and with given normal hyperplanes to \( C \) at \( A \), i.e. given image of \( T_pF_n \) in \( N_{C/\mathbb{P}^{n-1}, p} \) for all \( p \in A \). Indeed, a degree-\( n \) form through \( p_i \) has no \( x_i^n \) term and its tangent at \( p_i \) corresponds to a term \( x_i^{n-1}g_i \) with \( g_i \) linear in \( x_j, j \neq i \). The \( g_i \) may be chosen independently of one another in, identifying the normal space to \( C \subset \mathbb{P}^{n-1} \) at \( p_i \) with \( T_{p_i}F_{n-1}, g_i \) specifies a hyperplane in that normal space. Choosing the \( g_i \) generally with this property and setting \( F_n = \sum x_i^{n-1}g_i \) yields the desired \( F_n \); choosing \( F_{n-1} \) general enough with given \( p_1, ..., p_a \) ensures that \( F_{n-1} \cap F_n \cap C = \{ p_1, ..., p_a \} \).

Now blow up

\[ Y = F_{n-1} \cap F_n \subset \mathbb{P}^{n-1} \]

to get \( X_1 \) and let \( C_1 \subset X_1 \) be the birational transform of \( C \). Because \( C_1 \) has balanced normal bundle and \( Y \) has general tangents at \( Y \cap C, C_1 \subset X_1 \) also has balanced normal bundle, and it meets \( F \) transversely in \( e_1(n-1) - a \) points.

Now let \( C_2 \subset X_2 \) be \( e_1(n-1) - a \) general lines so that \( C_2 \cap F = C_1 \cap F \). As \( X_2 \) is a general hypersurface of degree \( n - 1 \) it is easy to check that each of the lines has trivial (i.e. globally free) normal bundle. Now in view of §1.4 Corollary 13 applies and shows that \( C_1 \cup C_2 \) smooths out to a smooth rational curve of degree \( e \) on a general hypersurface \( X \) of degree \( n \) in \( \mathbb{P}^{n} \) with balanced normal bundle.

**Case 2:** \( n - 1 \leq e < (n-1)^2 \).
We consider the same fan hypersurface $X_0 = X_1 \cup X_2$ as in Case 1 and in $X_0$ a curve $C_1 \cup C_2$ constructed as follows. Considering $X_1$ as $\mathbb{P}^{n-1}$ blown up in $Y = F_{n-1} \cap F_n$, we take for $C_1 \subset X_1$ the birational transform of a general rational normal curve $C \subset \mathbb{P}^{n-1}$ meeting $Y$ in $a$ points with $a$ as above. For $C_2$ we take $(n-1)^2 - a$ general lines in $X_2$ corresponding to the points of $C \cap F_{n-1}$ not on $Y$. To conclude as in Case 1 that $C_1 \cup C_2$ will smooth out to a balanced curve on $X_0$, it suffices to show that we can choose $C_1$ to have balanced normal bundle in $X_1$. Let $2C$ be the first order neighborhood of $C$, with ideal sheaf $\mathcal{I}_C^2$, so we have an exact sequence
\[
0 \to \mathcal{N} \to \mathcal{O}_{2C} \to \mathcal{O}_C \to 0
\]
where $\mathcal{N}$ is the dual to
\[
\mathcal{N} = (n-2)\mathcal{O}_C(n+1)
\]
where $\mathcal{O}_C(\ell)$ is the line bundle of degree $\ell$ on $C \simeq \mathbb{P}^1$. Because $\mathcal{O}_C(\ell) = \mathcal{O}_C(n-1)$, this sequence shows that
\[
H^1(\mathcal{O}_{2C}(k\mathcal{H})) = 0, k \geq 2
\]
where $\mathcal{H}$ is a hyperplane. Let $Z = C \cap F_{n-1}, 2Z = 2C \cap F_{n-1}$. Then $\mathcal{I}_{Z/C} = \mathcal{O}_C(-\mathcal{H})$, $\mathcal{I}_{2Z/2C} = \mathcal{O}_{2C}(-\mathcal{H})$, hence we have exact sequences
\[
0 \to \mathcal{I}_{C/\mathbb{P}^{n-1}} \to \mathcal{I}_{Z/\mathbb{P}^{n-1}} \to \mathcal{I}_{Z/C} \to 0
\]
\[
0 \to \mathcal{I}_{C/\mathbb{P}^{n-1}}^2 \to \mathcal{I}_{2Z/\mathbb{P}^{n-1}} \to \mathcal{I}_{2Z/2C} \to 0.
\]
A theorem of Rathmann ca. 1991 (see [11], Prop.4.2 or [13]) shows that
\[
H^1(\mathcal{I}_{C/\mathbb{P}^{n-1}}^2(k\mathcal{H})) = 0, k \geq 3.
\]
Putting (3) and (4) we conclude
\[
H^1(\mathcal{I}_{2Z/\mathbb{P}^{n-1}}^2(k\mathcal{H})) = 0, k \geq n-1
\]
since $n-1 \geq 3$. This shows that the map $\rho : H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(k\mathcal{H})) \to H^0(\mathcal{O}_{2Z}(k\mathcal{H}))$ is surjective. Now the image of $T_pF_n$ in the normal space to $C$ at $p \in C \cap F_n$ is just the hyperplane in the normal space, or equivalently, element in the conormal space, which corresponds to $\rho(F_n)$. By surjectivity of $\rho$, therefore, $F_n$ can be chosen to have general tangent hyperplanes at the points $p \in Y \cap C$ modulo $T_pC$, which makes $N_{C_1/X_1}$ balanced. Therefore again $C_1 \cup C_2$ smooths out to a balanced rational curve on $X$. \[\square\]
5. More preliminaries

In the next section we will construct, for infinitely many degrees $e$, rational curves of degree $e$ with balanced normal bundle on a general hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ with $3 \leq d \leq n - 1$. Here we collect a few more preliminary results and constructions that we will need.

5.1. Curves in projective bundles. Let $G$ be a vector bundle on a variety $B$, with associated projective (quotient) bundle $\mathbb{P}(G) \overset{\pi}{\to} B$. Given a parametrized curve $c_0 : C_0 \to B$, a lifting of $c_0$ to $c : C_0 \to \mathbb{P}(G)$ corresponds to an invertible quotient $G_0 := c^* G \to L_0$. In this case we have

$$c^* \mathcal{O}_{\mathbb{P}(G)}(1) = L_0.$$

There is an exact sequence for the normal bundle

$$0 \to N_{C_0/\mathbb{P}(G)/B} \to N_{C_0/\mathbb{P}(G)} \to c^* N_{C_0/B} \to 0$$

and, setting $K_0 := \ker(G_0 \to L_0)$ (the relative tautological subbundle), the vertical normal bundle, i.e. $N_{C_0/\mathbb{P}(G)/B}$, is given by

$$N_{C_0/\mathbb{P}(G)/B} = K_0^* \otimes L_0.$$

The horizontal normal bundle is by definition $c^* N_{C_0/B}$.

5.2. Some blowups. This construction will be used to construct components of fangs and fang hypersurfaces. Let $b : Z \to \mathbb{P}^n$ denote the blow-up of $\mathbb{P}^n$ in $\mathbb{P}^{n-m-1}$ and $\pi : Z \to \mathbb{P}^m$ the morphism induced by linear projection with center $\mathbb{P}^{n-m-1}$, whose fibres are fibres $\mathbb{P}^{n-m}$. Note that via $\pi$, $Z$ is a projective bundle

$$Z = \mathbb{P}_{\mathbb{P}^m}(\mathcal{O}(1) \oplus (n - m)\mathcal{O}) := \mathbb{P}(1, 0^{n-m})$$

and that the exceptional divisor of $b$ is

$$E = \mathbb{P}_{\mathbb{P}^m}((n - m)\mathcal{O}) = \mathbb{P}^m \times \mathbb{P}^{n-m-1}.$$

A hypersurface $X$ of type $(d, b)$ on $Z$ is an element of the linear system $|b^* \mathcal{O}(d) - bE|$ and $X$ maps to a hypersurface of degree $d$ in $\mathbb{P}^n$ with multiplicity $b$ on $\mathbb{P}^{n-m-1}$ and $X$ meets $E = \mathbb{P}^m \times \mathbb{P}^{m-n-1}$ in a divisor $Y$ of bidegree $(d - b, b)$ on $E$. The fibres of $\pi|_X$ are hypersurfaces of degree $d - b$ in $\mathbb{P}^{n-m-1}$.

Here are the two case of the construction to be used below.
5.2.1. **Special Case 1:** \( b = d - 1 \). This means \( X \) maps to a quasi-cone in \( \mathbb{P}^n \). Then \( X \) is a projective subbundle of \( Z \) of the form \( \mathbb{P}(G) \) where \( G \) fits in an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-(d-1)) \to \mathcal{O}_{\mathbb{P}^n}(1) \oplus (n-m)\mathcal{O}_{\mathbb{P}^n} \to G \to 0.
\]

In terms of coordinates, the image of \( X \) in \( \mathbb{P}^n \), for any \( b \), has an equation of the form

\[
f = \sum_{i=0}^{d-b} \sum_{j} a_{d-i,j}(x)b_{i,j}(y)
\]

where \( x_0, \ldots, x_m, y_1, \ldots, y_{n-m} \) are linear coordinates on \( \mathbb{P}^n \) such that \( x_0, \ldots, x_m \) cut out \( \mathbb{P}^{n-m-1} \), and each \( a_{k,j}, b_{i,j} \) is homogeneous of degree \( k, i \) respectively. The term \( i = d - b \) in the same equation, viewed as bihomogeneous form of bidegree \( (b, d - b) \), also yields the equation of \( Y \) in \( E \). When \( b = d - 1 \), we can write the equation \( f \) in the form

\[
f = a_{d}(x) + \sum_{j=1}^{n-d+1} y_j a_{d-1,j}(x).
\]

The left map in (7) is just given by \((a_d, a_{d-1,1}, \ldots, a_{d-1,n-d+1})\).

5.2.2. **Special Case 2:** \( b = 1, d = n - m \). Then the general fibre of \( \pi_X \) is a hypersurface of degree \( n - m - 1 \) in \( \mathbb{P}^{n-m} \). As is well known, a general such hypersurface is filled up by lines \( L \) with trivial normal bundles (cf. Remark 22 below), and of course \( L \) must meet \( \mathbb{P}^{n-m-1} \). Now the birational transform of \( L \) in \( X \) has normal bundle which is an extension of trivial bundles, hence trivial as well. Consequently:

**Lemma 23.** Notations as above, let \( X \) be a general hypersurface of type \((n-m,1)\) in \( Z \). Then there is a filling family in \( X \) of birational transforms of lines meeting the projection center \( \mathbb{P}^{n-m-1} \), hence contained in fibres of \( \pi \), whose general member has trivial normal bundle in \( X \).

5.3. **Fans generalized.** This is a slight generalization of the fans considered above and in [10], and has already occurred in §3. Let \( \pi_1 : Z_1 = B_{\mathbb{P}^{n-m-1}}\mathbb{P}^n \to \mathbb{P}^m \) be as in §5.2 with blowdown map \( b_1 : Z_1 \to \mathbb{P}^n \) and exceptional divisor \( E_1 \). Let \( \pi_2 : Z_2 = B_{\mathbb{P}^m}\mathbb{P}^n \to \mathbb{P}^{n-m-1} \) be the analogous object, based on blowing up \( \mathbb{P}^m \), with blowdown map \( b_2 \) and exceptional divisor \( E_2 \). Note that both \( Z_1 \) and \( Z_2 \) have exceptional divisor \( E = \mathbb{P}^m \times \mathbb{P}^{n-m-1} \approx E_1 \approx E_2 \). The normal-crossing variety

\[
Z_0 = Z_1 \cup_E Z_2
\]

is called a *generalized fan* or *fang* of type \((n,m)\).

A flat morphism \( Z \to B \) is called a *relative fang* of type \((n,m)\) if each fibre is either \( \mathbb{P}^n \) or a fang of type \((n,m)\). A standard way to construct a relative fang is to blow up
the subvariety \( \mathbb{P}^{n-m-1} \times 0 \) in \( \mathbb{P}^n \times \mathbb{A}^1 \). There \( Z_1 \) and \( Z_2 \) are, respectively, the birational transform of \( \mathbb{P}^n \times 0 \) and the exceptional divisor, and both have projective bundle structure:

\[
Z_1 = \mathbb{P}(G_1), Z_2 = \mathbb{P}(G_2)
\]

where

\[
G_1 = \mathcal{O}_{\mathbb{P}^n}(1) \oplus (n-m)\mathcal{O}_{\mathbb{P}^n}, G_2 = \mathcal{O}(1)_{\mathbb{P}^{n-m-1}} \oplus (m+1)\mathcal{O}_{\mathbb{P}^{n-m-1}}.
\]

We will denote the latter as \( Z_1 = \mathbb{P}_{\mathbb{P}^n}(1,0^{n-m}), Z_2 = \mathbb{P}_{\mathbb{P}^{n-m-1}}(1,0^{m+1}). \) Also each \( Z_i \) is endowed with the \( \mathcal{O}(1) \) induced by from \( G_i \) which is also the pullback of \( \mathcal{O}_{\mathbb{P}^n}(1) \) by the blowdown map \( b_i : Z_i \to \mathbb{P}^n \). We denote the exceptional divisor of \( b_i \) by \( E_i \), and we also have

\[
E_1 = \mathbb{P}((n-m)\mathcal{O}_{\mathbb{P}^n}), E_2 = \mathbb{P}((m+1)\mathcal{O}_{\mathbb{P}^{n-m-1}}),
\]

hence of course

\[
E = E_1 \simeq E_2 \simeq \mathbb{P}^m \times \mathbb{P}^{n-m-1}.
\]

The case \( m = 0 \) or \( m - n - 1 \) reduces to the fan construction used previously. Now assume \( 0 < m < n - 1 \) and \( e < d \). Then the linear system \( |dH - eZ_2| \) on \( Z \), where \( H \) is the pullback of a hyperplane in \( \mathbb{P}^n \), restricts as follows.

- on the general fibre, to \( |dH| \);
- on \( Z_1 \), to \( |db_1^*H_{\mathbb{P}^n} - eE_1| \), i.e. the birational transform on \( Z_1 \) of the system of hypersurfaces of degree \( d \) on \( \mathbb{P}^n \) with multiplicity \( e \) on \( \mathbb{P}^{n-m-1} \);
- on \( Z_2 \) to \( |db_2^*H_{\mathbb{P}^n} - (d-e)E_2| \);
- on \( E = \mathbb{P}^m \times \mathbb{P}^{n-m-1} \) to the linear system of hypersurfaces bidegree \( (e,d-e) \).

**Example 24.** Taking \( d = 1, e = 0 \) shows that a curve of degree \( k \) in \( \mathbb{P}^n \) can specialize to a curve \( C_1 \cup C_2 \subset Z_0 \) where \( C_1 \subset Z_1 \) maps to a curve \( b_1(C_1) \) of degree \( k_1 \) in \( \mathbb{P}^n \) while \( C_2 \) a curve \( b_2(C_2) \subset \mathbb{P}^n \) of degree \( k - k_1 + \ell \) meeting the blown-up \( \mathbb{P}^n \) in \( \ell \) points (hence \( C_2 \) maps to a curve \( \pi_2(C_2) \) of degree \( k - k_1 \) in the base \( \mathbb{P}^{n-m-1} \)). Here \( \ell = C_1.E = C_2.E \) and \( C_1 \cap E = C_2 \cap E \) consists of \( \ell \) points.

Similarly for the dual case \( d = e = 1 \).

The foregoing construction may obviously be extended to the case of more than 2 components but we don’t need this.

### 5.4. Balanced extensions and kernels.

An extension of balanced vector bundles is balanced when their slopes are roughly equal. This is useful for constructing balanced bundles.

**Lemma 25.** Let

\[
0 \to E_1 \to E \to E_2 \to 0
\]
be an exact sequence of vector bundles on \( \mathbb{P}^1 \), of respective slopes \( s_1, s, s_2 \). Assume \( E_1, E_2 \) are balanced and

(8) \[ [s_1] = [s_2]. \]

Then \( E \) is balanced and \( [s] = [s_1] \). Moreover the extension splits.

The proof may be left to the reader.

Balancedness is also inherited by the kernel of a general map to a vector bundle:

**Lemma 26.** Let \( E \) be a balanced bundle on \( \mathbb{P}^1 \) and \( \phi : E \to L \) a sufficiently general surjection to a vector bundle. Then \( \ker(\phi) \) is balanced.

**Proof.** By an obvious induction we may assume \( L \) has rank 1. Because balancedness is open it suffices to prove: given \( E \) balanced of slope \( s \) and an integer \( \ell \geq \lfloor s \rfloor \), there exists a balanced bundle \( K \) with \( c_1(K) = c_1(E) - \ell, \rk(K) = \rk(E) - 1 \) and a locally split injection \( K \to E \). We may assume \( E = r_0 \mathcal{O}(a) \oplus (r_1 - r_0) \mathcal{O}, \) so \( \ell \geq 0 \). Write \( \ell = q(r-1) + p, 0 \leq p < r-1 \).

If \( p \leq r_0 \) we can take

\[ K = ((r_0 - p) \mathcal{O}(1 - q) \oplus (r - r_0 + p) \mathcal{O}(-q). \]

If \( p > r_0 \) we can take

\[ K = (r - p + r_0) \mathcal{O}(-q) \oplus (p - r_0) \mathcal{O}(-q - 1). \]

Clearly, a general map \( K \to E \) is locally split injective. \( \square \)

**Remark 27.** The Lemma yields a quick proof of the fact that a general line \( L \) on a general hypersurface \( X \) has balanced normal bundle: indeed if \( x_2, ..., x_n \) are linear equations for \( L \) and \( \sum_{i=2}^n x_i f_i \) is an equation for \( X \) then \( \mathcal{N}_{L/X} \) is the kernel of the general map \( \mathcal{N}_{L/P^n}^{(x_2, ..., x_n)} \mathcal{O}_{L}(d). \)

In particular, a general line on a general hypersurface of degree \( n - 1 \) in \( \mathbb{P}^n \) has trivial normal bundle.

The following is a close analogue of Lemma 16 showing that a suitable elementary down modification brings a bundle closer to balance.

**Lemma 28.** Let

(9) \[ E = r_0 \mathcal{O}(a) \oplus r_1 \mathcal{O}(a - 1) \oplus ... \oplus r_b \mathcal{O}(a - b), \quad r_0 > 0, r_1, ..., r_b \geq 0 \]

be a bundle on \( \mathbb{P}^1 \) and for \( p \in \mathbb{P}^1 \) denote by \( k_p \) the skyscraper sheaf \( k(p) \) at \( p \). Then there exists a map \( \phi : E \to k_p \) such that \( E_1 = \ker(\phi) \) has the form

\[ E_1 = (r_0 - 1)\mathcal{O}(a) \oplus (r_1 + 1)\mathcal{O}(a - 1) \oplus ... \oplus r_b \mathcal{O}(a - b). \]
The proof is obvious as it suffices to take a map that is nontrivial on one \(O(a)\) summand and zero on all other summands. Then applying the Lemma \(r_0\) times we conclude that there is a map \(E \to \bigoplus_{i=1}^{r_0} \mathbb{K}_{p_i}\) with kernel

\[ E_{r_0} = (r_0 + r_1)O(a - 1) \oplus \ldots \oplus r_bO(a - b). \]

Thus passing from \(E\) to \(E_{r_0}\) decreases by 1 the degree difference between the most positive subbundle and least positive quotient bundle. Continuing in this manner at least \(r_k(E) - r_b\) many times, and using openness of balancedness, we conclude

**Corollary 29.** (i) Notations as above, the kernel of a sufficiently general map \(E \to \bigoplus_{i=1}^{s} \mathbb{K}_{p_i}\) is balanced (resp. perfectly balanced) provided \(s \geq \sum_{i=0}^{b-1} (b - i) r_i\) (resp. \(s = \sum_{i=0}^{b-1} (b - i) r_i\)).

(ii) In particular, if \(E\) is balanced with upper rank \(r^+\) then a sufficiently general map \(E \to \bigoplus_{i=1}^{r^+} \mathbb{K}_{p_i}\) has perfectly balanced kernel.

**Remark 30 (non-essential).** We may define the ‘unbalanced degree’ of \(E\) as above as \(u(E) = \sum r_i(b - i)\) and its ‘lower rank’ \(r^- (E)\) as the rank of its lowest-slope quotient, i.e. \(r_b\). Clearly \(u(E) \geq \text{rk}(E) - r^-(E)\) with equality iff \(E\) is balanced. Then Lemma \([28]\) implies that, unless \(E\) is perfectly balanced, we have \(u(E_1) = u(E) - 1\). Also, \(r^- (E_1) = r^- (E)\) unless \(r^- (E) = \text{rk}(E) - 1\). Since \(u(E) \geq 0\), the modification \(E \mapsto E_1\) yields a perfectly balanced bundle after \(u(E)\) many iterations. This yields another way to deduce Corollary \([29]\).

6. FANGS AND THE CASE \(d < n\)

Here we will prove the results on balanced rational curves on a general hypersurface \(X\) degree \(d < n\). Since it is known by the result of Riedl-Yang that the family of rational curves of degree \(e\) on \(X\) is irreducible at least if \(d < n - 1\), it follows that almost all of these curves have balanced normal bundle. This construction has consequences for minimal rational connectivity.

To state the results we need a definition. Fix \(n, d\) with \(3 \leq d \leq n - 1\). An integer \(e\) is said to be accessible via \(e_0\) if \(e \geq e_0 \geq d - 1\) and

\[ \lfloor \frac{-de_0 + e}{n - d} \rfloor + e = e_0 + \lfloor \frac{2e_0 - 2}{d - 2} \rfloor. \] (10)

\(e\) is accessible if it is accessible via some \(e_0\). Also recall (see \([1,1]\)) that an integer \(e\) is said to be point-minimal if \(q_{\max}(e - 1) < q_{\max}(e)\).
Theorem 31. (i) A general hypersurface $X$ of degree $d < n$ in $\mathbb{P}^n$ contains balanced rational curves of any accessible degree $e$ and is separably rationally $(q_{\max}(e) - 1)$-connected for any accessible $e$.

(ii) For $2 < d < n$, the set of accessible $e$ contains the intersection of $[d - 1, \infty)$ with $a(d, n - d)$ many congruence classes mod $d(n - 2)$ where $a(n, n - d) = (n - d)d - (\text{linear terms})$.

(iii) For $3 < d < n - 1$, the set of $e$ which are both accessible and point-minimal contains the intersection of $[d - 1, \infty)$ with $(n + 1 - d)/2$ many congruence classes mod $d(n - 2)$ and the set of interpolating $q$ contains some ray intersected with $(n + 1 - d)/2$ many congruence classes mod $(n + 1 - d)d$.

Below we prove part (i). Parts (ii), (iii) follow from this by arithmetic: see Example 32 (for $d = n - 1$) or the Appendix by M. C. Chang.

Example 32. Take $d = n - 1, n \geq 4$. Write $e_0 = k(n - 3) + r, k \geq 1, 0 \leq r < n - 3$. Then either

- $n$ even, $n \geq 6$, $0 < r \leq \frac{n - 3}{2}$, $e = \left(\frac{n - 1}{2}\right)k + \frac{nr}{2}$
- $n, r$ both odd, $n \geq 5$, $r \geq \frac{n - 1}{2}$, $e = \left(\frac{n - 1}{2}\right)k + \frac{nr + 1}{2}$
- $n$ odd, $r$ even, $r \leq \frac{n - 3}{2}$, $e = \left(\frac{n - 1}{2}\right)k + \frac{r}{2}$

or else

- $n = 4, r = 0, e = 3k - 1$.

Thus, the accessible degrees cover about $(n - 3)/2$ of the possible congruence classes of $e \mod \left(\frac{n - 1}{2}\right)$.

The point-minimal condition on the accessible $e$ is that the remainder of $2e - 2 \mod n - 2$ should equal 0 or 1. Considering those $e$ that are both accessible and point-minimal yields that $X$ is minimally rationally $(q - 1)$-connected for $q - 1 = (n - 1)k + 1$ if $n$ is even (resp. $q - 1 = (n - 1)k + 2$ if $n$ is odd), for any $k \geq 1$. □

Proof of Theorem 31 (i). The proof is based on a relative fang as in §5.3. Thus, fixing integers $d < n$, let $\mathcal{Z} \to \mathbb{A}^1$ be a relative fang of type $(n, m), m = d - 1$, with general fibre $\mathbb{P}^n$ and special fibre $Z_0 = Z_1 \cup Z_2$. Thus

- $Z_1 = \mathbb{P}^{m}(1, 0^{n-m}), Z_2 = \mathbb{P}_{\mathbb{P}^{n-m-1}}(1, 0^{m+1})$.

Consider a general member of the linear system $|dH - (d - 1)Z_2|$ on $\mathcal{Z}$, whose general fibre over $\mathbb{A}^1$ is a general hypersurface of degree $d$ in $\mathbb{P}^n$, and let

$$X_0 = X_1 \cup X_2$$
be its special fibre over $A^1$. Thus, $X_1 = \mathbb{P}(G) \to \mathbb{P}^m$ as in §5.2.1, while $X_2$ fibres over $\mathbb{P}^{n-m-1}$ with general fibre a general hypersurface of degree $d-1 = (m+1)-1$ in $\mathbb{P}^{m+1}$ §5.2.2 (beware the switch in notation, interchanging $n-m$ and $m+1$). The idea, as in the proof of Theorem 20, is to construct a good curve $C_1 \cup C_2 \subset X_0$ which will smooth out to a balanced curve on $X$. In fact, $C_1 \subset X_1$ will be balanced and $C_2 \subset X_2$ will be perfectly balanced.

We first construct $C_1$. To this end consider a general rational curve $C_0$ of degree $e_0$ in $\mathbb{P}^m$, and let $C_1 \subset X_1$ be a general degree-$e$ lifting of $C_0$, which corresponds to a general surjection

$$\psi : G_{C_0} \to \mathcal{O}(e)$$

Let $K = \ker(\psi)$, i.e. the restriction of the relative tautological subbundle of the projective bundle $\mathbb{P}(G)$. Then $C_1$ meets the exceptional divisor $E = \mathbb{P}^m \times \mathbb{P}^{n-m-1}$, $m = d-1$, in $e-e_0$ points. By Proposition 19 as soon as $e_0 \geq m$, the normal bundle $N_0 = N_{C_0}/\mathbb{P}^m$ is balanced, of slope $s_0 = \frac{(m+1)e_0-2}{m-1}$. By §5.1, the ‘relative’ or vertical normal bundle $N_{C/X_1}/\mathbb{P}^m$ is just $K^*(e)$. Thus, we have an exact sequence

$$0 \to K^*(e) \to N_{C_1/X_1} \to N_0 \to 0.$$

**Lemma 33.** Notations as above, $K$ is balanced.

**Proof.** Assume first that $e_0 \leq n-d+1$. Recall the surjection

$$s : \mathcal{O}(e_0) \oplus (n-m)\mathcal{O} \to G_{C_0}.$$

By generality of $\psi$ we may assume the induced map $\mathcal{O}(e_0) \to \mathcal{O}(e)$, i.e. the restriction of $\psi \circ s$ on the $\mathcal{O}(e_0)$ summand, is injective and general and let $\tau$ be its cokernel, which is a skyscraper of the form $\tau = \bigoplus_{i=1}^{e-e_0} \mathbb{K} p_i$, where $p_1, \ldots, p_{e-e_0} \subset C_0$ are distinct points. Let $G_1$ be the cokernel of the map $\mathcal{O}(e_0) \to G_{C_0}$, so we have an exact sequence

$$0 \to \mathcal{O}(e_0) \to G_{C_0} \to G_1 \to 0.$$

Then from (13) we get another exact sequence on $C_0$:

$$0 \to K \to G_1 \to \tau \to 0,$$

where the map $\psi_1 : G_1 \to \tau$ is general.

Now I claim $G_1$ is balanced. To this end note $G_1$ fits in an exact sequence

$$0 \to \mathcal{O}(-(d-1)e_0) \to (n-d+1)\mathcal{O} \to G_1 \to 0.$$
where the left map is given by restriction of part of the equation in \( \mathbb{P}^n \) of the quasicone \([7]\), image of \( X_1 \), which is general as such. Now by the maximal rank property of general rational curves \([2]\), the restriction maps \( H^0(\mathcal{O}_{\mathbb{P}^m}(\ell)) \to H^0(\mathcal{O}_{C_0}(\ell e_0)) \) are surjective, \( \ell = d, d-1 \), because \( de_0 + 1 \leq d(n - d + 1) + 1 \leq \binom{d+n}{n} \). Therefore the left map in \((15)\) is general, so by Lemma \([26]\) \( G_1 \) is balanced. Finally, since the map \( \psi_1 \) is general, it follows by Corollary \([29]\) that \( K \) is balanced, provided \( e_0 \leq n - d + 1 \).

Now assume \( e_0 > n - d + 1 \) and write \( e_0 = e_1 + (n - d + 1) \) and consider a general connected curve of the form \( C_{11} \cup C_{12} \) where \( C_{11} \) (resp. \( C_{12} \)) is a general rational curve of degree \( e_1 \) (resp. \( n - d + 1 \)). By induction, \( G_{C_{11}} \) is balanced while \( G_{C_{12}} \) is perfectly balanced (twist of a trivial bundle). Then using Theorem \([6]\) case (ii)', it follows that \( C_{11} \cup C_{12} \) smooths to a curve \( C_0 \subset \mathbb{P}^m \) on which \( G \) is balanced. This proves Lemma \([33]\).

Now that we know \( N_0 \) and \( K \) are balanced, we can apply Lemma \([25]\) to the exact sequence \((12)\), to conclude that \( N_{C_1/X_1} \) is balanced provided the matching condition \((10)\) holds, i.e. provided \( e \) is accessible via \( e_0 \).

Note that the curves \( C_1 \) produced above meet \( E \) in \( e - e_0 \) points, say \( p_1, \ldots, p_{e-e_0} \). Let \( L_i \subset X_2 \) be a line in \( X_2 \) through \( p_i \) contained in the fibre through \( p_i \) of \( X_2/\mathbb{P}^{n-m+1} \). Then by Lemma \([23]\) each \( L_i \) has trivial normal bundle in the fibre, hence in \( X_2 \). Now let \( C_2 = \bigcup L_i \). Then \( C_1 \cup C_2 \) is an lci curve in \( X_0 \) with ‘balanced’ normal bundle as in \([34]\), which smooths out to a balanced rational curve of degree \( e \) on a general hypersurface of degree \( d \). This proves Theorem \([31]\) Part (i).

**Remark 34.** The use of the maximal rank property of the curve \( C_0 \) could be avoided at the cost of forcing \( e \) to satify a lower bound. Because by \((15)\), \( G_1 \) has no negative quotient hence its most positive line subbundle is \( \mathcal{O}(a), a \leq e_0(d-1) \) and we can apply Lemma \([28]\) to conclude that \( K \) will be balanced provided \( e \geq de_0 \).

**Remark 35.** Extending the fang method beyond the accessible values of \( e \) would require taking \( m \neq d - 1 \), and hence attaching lines with balanced, but nontrivial normal bundle. One would have to prove a general position property for the upper subspaces of these normal bundles, as in Theorem \([6]\) (i). This seems difficult.
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In this appendix, we prove Theorem A.1 and Theorem A.2 below.

**Theorem A.1.** Let $2 < d < n$ be integers. Then the set of integers $e$ which are accessible, i.e. such that for some $e_0$, $d - 1 \leq e_0 \leq e$, one has

$$\left\lfloor \frac{-de_0 + e}{n-d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 - 2}{d-2} \right\rfloor \quad (A.1)$$

is the intersection of $[d - 1, \infty)$ with a union of at least

$$\min \left\{ (n - d - \frac{1}{2})(d - 3) - \frac{15}{2}, (n - d)(d - 5) - 2, (n - d + 1)(d - 5) \right\}$$

congruence classes mod $d(n - 2)$.

**Remark 1.** For $n, d >> 0$ the density of the accessible $e$ is about $(n - d)/n$.

**Remark 2.** The formula for the lower bound on $N(d, n)$, the number of congruence classes of $e \mod d(n - 2)$ will be obtained by applying Facts 1-3 to count the number of permissible $c \in [0, d - 3]$ in display (A.3) below.

**Proof of Theorem A.1.**

Fix $n, d$ as in Theorem A.1. Let

$$b = n - d + 1. \quad (A.2)$$

Dividing $e_0$ by $d - 2$, then dividing the quotient obtained by $b$, we have

$$e_0 = k(d - 2)b + r(d - 2) + c, \quad (A.3)$$

where

$$0 \leq c \leq d - 3, \text{ and } 0 \leq r \leq b - 1. \quad (A.4)$$

Hence

$$\frac{2e_0 - 2}{d - 2} = 2kb + 2r + \frac{2c - 2}{d - 2}. \quad (A.5)$$

Since we consider the lower bound on the number of e’s, we may assume that $c > 0$.

There are the following two cases of the integral part of $\frac{2e_0}{d-2}$.
Case (a). $0 < c < \frac{d}{2}$.
In this case we have $\frac{2c - 2}{d - 2} < 1$. Therefore, from display (A.5)
\[
\left\lfloor \frac{2e_0 - 2}{d - 2} \right\rfloor = 2kb + 2r.
\] (A.6.a)

Case (b). $c \geq \frac{d}{2}$.
In this case we have $1 \leq \frac{2c - 2}{d - 2} < 2$. Therefore,
\[
\left\lfloor \frac{2e_0 - 2}{d - 2} \right\rfloor = 2kb + 2r + 1,
\] (A.6.b)
and the fractional part of $\frac{2n - 2}{d - 2}$ is
\[
\left\{ \frac{2e_0 - 2}{d - 2} \right\} = \frac{2c - 2 - (d - 2)}{d - 2}.
\] (A.7)

Coming back to display (A.1), we let $\varepsilon$ be the fractional part of $\frac{-de_0 + e}{n - d}$, i.e.,
\[
\frac{-de_0 + e}{n - d} = \left\lfloor \frac{-de_0 + e}{n - d} \right\rfloor + \varepsilon.
\] (A.8)
In particular, $\varepsilon < 1$.

Putting displays (A.1), (A.2), (A.3), (A.6), (A.8) together, we have

Case (a). $e = d(n - 2)k + rd + c + \frac{r(d^2 - 3d) + c(d - 1) + \varepsilon b - 1}{b}$ (A.9)
Case (b). $e = d(n - 2)k + rd + c + 1 + \frac{r(d^2 - 3d) + c(d - 1) - 1 + \varepsilon b - 1}{b}$
for Cases (a) and (b) respectively.

We want to count the values $e$ expressed in display (A.9) with all possible $(c, r) \in [1, d - 3] \times [0, b - 1]$ by counting congruence classes of $e \mod d(n - 2)$.

We will give the argument for Case (a) only, since the argument for Case (b) is identical. Let
\[
E(c, r, \varepsilon) = rd + c + \frac{r(d^2 - 3d) + c(d - 1) + \varepsilon b - 1}{b}.
\]

Claim 1. If $(c, r) \neq (c_1, r_1)$, then $E(c, r, \varepsilon) \neq E(c_1, r_1, \varepsilon_1)$ as real numbers.

Proof of Claim 1.
Assume $E(c, r, \epsilon) = E(c_1, r_1, \epsilon_1)$. Then

$$(r_1 - r)\left(d + \frac{d^2 - 3d}{b}\right) = (c - c_1)\left(1 + \frac{d - 1}{b}\right) + (\epsilon - \epsilon_1)\frac{b - 1}{b}.$$  \hfill (A.10)

We may assume $r_1 - r \geq 1$. Hence the left-hand-side of display (A.10) gives

$$(r_1 - r)\left(d + \frac{d^2 - 3d}{b}\right) \geq 1 \cdot \left(d + \frac{d^2 - 3d}{b}\right),$$

while, by (4) and that $\epsilon, \epsilon_1 \in [0, 1)$, the right-hand-side of display (A.10) gives

$$(c - c_1)\left(1 + \frac{d - 1}{b}\right) + (\epsilon - \epsilon_1)\frac{b - 1}{b} \leq (d - 3) \cdot \frac{b + d - 1}{b} + \frac{b - 1}{b}.$$  

This is a contradiction.

**Claim 2.** $E(c, r, \epsilon) < d(n - 2)$.

This is clear, because again, by displays (A.4) and (A.8)

$$E(c, r, \epsilon) \leq (b - 1)\left(d + \frac{d^2 - 3d}{b}\right) + (d - 3)\left(1 + \frac{d - 1}{b}\right) + \frac{b - 1}{b}$$

$$< d(b + d - 3)$$

$$= d(n - 2).$$

From Claim 1 and Claim 2, we conclude

if $(c, r) \neq (c_1, r_1)$, then $E(c, r, \epsilon) \not\equiv E(c_1, r_1, \epsilon_1) \pmod{d(n - 2)},$

i.e., for $e = e(c, r)$ in display (A.9)

if $(c, r) \neq (c_1, r_1)$, then $e(c, r) \not\equiv e(c_1, r_1) \pmod{d(n - 2)}.$  \hfill (A.11)

Next, we want to count the permissible $(c, r) \in [1, d - 3] \times [0, b - 1]$. As before, we give the argument for Case (a).

Since $e \in \mathbb{Z}$, we may let

$$\frac{r(d^2 - 3d) + c(d - 1)}{b} + \frac{\epsilon b - 1}{34} = m + 1, \text{ for } m \in \mathbb{Z}.$$  \hfill (A.12)
Hence
\[ m < \frac{r(d^2 - 3d) + c(d - 1)}{b} \]
and
\[ \varepsilon = \frac{(m + 1)b - (r(d^2 - 3d) + c(d - 1))}{b - 1}. \]  

By display (A.8), \( \varepsilon < 1 \), which is equivalent to
\[ m < \frac{r(d^2 - 3d) + c(d - 1) - 1}{b}. \]

So we want to rule out those \((c, r)\) such that
\[ \frac{r(d^2 - 3d) + c(d - 1) - 1}{b} = m, \]
i.e., we want to rule out \((c, r) \in [1, d - 3] \times [0, b - 1]\) such that
\[ r(d^2 - 3d) + c(d - 1) \equiv 1 \mod b. \]  

By display (A.2), solving the congruence equation (A.14) is the same as solving
\[ cn \equiv -r(n + 1)(n - 2) + 1 \mod b. \]  

We will use the following facts about the congruence equation
\[ ax \equiv d \mod b \]

**Fact 1.** Equation + is solvable if and only if \( g := \gcd(a, b) \) divides \( a \).

**Fact 2.** Assume \( g|d \), and let \( b' := b/g \). If we consider the solution \( x \) as an integer, then \( x \) is unique in any interval of size \( b' \).

**Fact 3.** For \( C \geq b' \), the number of solutions of + in \([1, C]\) is \( \lfloor C/b' \rfloor \) or \( \lceil C/b' \rceil \) + 1.

Coming back to congruence equation (A.15.a), we let \( g = \gcd(n, b) \). Counting the numbers of \( r \in [0, b - 1] \) such that
\[ g \mid r(n + 1)(n - 2) + 1 \]  
is the same as counting \( r \) satisfying
\[ g|2r + 1 \]
i.e., counting the number of \( r \) such that
\[ 2r \in \{ g - 1, 2g - 1, \ldots, 2b'g - 1 \}, \text{ where } b' = \frac{b}{g}. \]
I.a. Assume $C_a \geq b'$, where $C_a = \left| \left[ 1, \frac{d}{2} \right] \cap \mathbb{Z} \right|$.

Case (I.a.i.) $b$ is odd.
In this case, $g = \gcd(n, b)$ is odd and

$$\{2r : r \in [0, b - 1] \text{ and } r \text{ satisfies } (A.15.a) \} = \{g - 1, 3g - 1, \ldots, (2b' - 1)g - 1\}. \quad (A.19.a)$$

There are at most $(\left\lfloor \frac{C_a}{b'} \right\rfloor + 1) b' \leq C_a + b'$ pairs of $(c, r)$.

Case (I.a.ii.1.) $b$ is even and $g$ is even. (Hence $n$ is even.)
There is no $r$ satisfying display (A.17).

Case (I.a.ii.2.) $b$ is even and $g$ is odd. (Hence $n$ is odd.)

$$\{2r : r \in [0, b - 1] \text{ and } r \text{ satisfies } (A.15.a) \} = \{g - 1, 3g - 1, \ldots, (2b' - 1)g - 1\}. \quad (A.20.a)$$

There are at most $(\left\lfloor \frac{C_a}{b'} \right\rfloor + 1) b' \leq C_a + b'$ pairs of $(c, r)$.

I.b. Assume $C_b \geq b'$, where $C_b = \left| \left[ \frac{d}{2}, d - 3 \right] \cap \mathbb{Z} \right|$. For Case (b), we have

$$cn \equiv -r(n + 1)(n - 2) + 2 \mod b, \quad (A.15.b)$$

and hence the following cases.

Case (I.b.i) $b$ is odd. (Hence $g$ is odd.)

$$\{2r : r \in [0, b - 1] \text{ and } r \text{ satisfies } (A.15.b) \} = \{2g - 2, 4g - 2, \ldots, 2b'g - 2\}. \quad (A.19.b)$$

There are at most $(\left\lfloor \frac{C_b}{b'} \right\rfloor + 1) b' \leq C_b + b'$ pairs of $(c, r)$.

Case (I.b.ii.1) $b$ is even and $g$ is even.

$$\{2r : r \in [0, b - 1] \text{ and } r \text{ satisfies } (A.15.b) \} = \{g - 2, 2g - 2, \ldots, 2b'g - 2\}. \quad (A.21.b)$$

There are at most $(\left\lfloor \frac{C_b}{b'} \right\rfloor + 1) 2b' \leq 2C_b + 2b'$ pairs of $(c, r)$.

Case (I.b.ii.2) $b$ is even and $g$ odd.

$$\{2r : r \in [0, b - 1] \text{ and } r \text{ satisfies } (A.15.b) \} = \{2g - 2, 4g - 2, \ldots, 2b'g - 2\}. \quad (A.20.b)$$

There are at most $(\left\lfloor \frac{C_b}{b'} \right\rfloor + 1) b' \leq C_b + b'$ pairs of $(c, r)$. 

36
II.a. Assume $C_a < b'$.

For each $r$, there is at most one solution $c$ in $[1, C_a]$.

Hence

Case (II.a.i.) $b$ is odd. There are at most $b'$ pairs of $(c, r)$.

Case (II.a.ii.1) $b$ is even and $g$ is even. There is no $r$ satisfying display (A.17).

Case (II.a.ii.2) $b$ is even and $g$ is odd. There are at most $b'$ pairs of $(c, r)$.

II.b. Assume $C_b < b'$.

Case (II.b.i) $b$ is odd. There are at most $b'$ pairs of $(c, r)$.

Case (II.b.ii.1) $b$ is even and $g$ is even. There are at most $2b'$ pairs of $(c, r)$.

Case (II.b.ii.2) $b$ is even and $g$ is odd. There are at most $b'$ pairs of $(c, r)$.

Summing up Cases (a) and (b), and using the facts that

1. $C_a + C_b = d - 3$,

2. $C_a = \frac{d}{2} - 1$, if $d$ is even,
   
   $C_a = \frac{d - 1}{2}$, if $d$ is odd,

3. $C_b = \frac{d}{2} - 2$, if $d$ is even,

4. $C_b = \frac{d + 1}{2} - 3$, if $d$ is odd.

Taking off the bad pair $(c, r)$ from Cases (a) and (b), we have that the number of the permissible $(c, r) \in [1, d - 3] \times [0, b - 1]$ is at least

(Ia & Ib). When $C_a > C_b \geq b'$,

1. $(b - 1)(d - 5) - 2$, if $b$ is odd, or $b$ is even and $g$ is odd,

2. $(b - 1)(d - 5) - 1$, if $b$ is even, $g$ is even, and $d$ is even,

3. $(b - 1)(d - 5)$, if $b$ is even, $g$ is even, and $d$ is odd.

(Ia & IIb). When $C_a \geq b' > C_b$,

1. $(b - \frac{3}{2})(d - 3) - \frac{15}{2}$, if $b$ is odd, or $b$ is even and $g$ is odd,

2. $(b - 1)(d - 3) - 1$, if $b$ is even, $g$ is even, and $d$ is even,

3. $(b - 1)(d - 3) - 2$, if $b$ is even, $g$ is even, and $d$ is odd.
When $b' > C_a > C_b$, the number of the permissible pairs is at least $b(d - 5)$.

Combining the above, we conclude the proof of Theorem A.1. □

**Remark 3.** The estimates can be improved by $b = n - d + 1$, if $C_a$ or $C_b$ is a multiple of $b' = \frac{b}{\gcd(b, n)}$. For example, in $I_a$ & $I_b$ (when $C_a > C_b \geq b'$), the number of permissible pairs $(c, r) \in [1, d - 3] \times [0, b - 1]$ is at least

- $(b - 1)(d - 4) - 1$ for $b$ odd, or $b$ even and $g$ odd,
- $(b - 1)(d - 3) + 1$ for $b$ even, $g$ even, and $d$ even,
- $(b - 1)(d - 3) + 2$ for $b$ even, $g$ even, and $d$ odd.

**Remark 4.** Suppose $b = n + 1 - d \neq 2$. If $c = 1$, then at least half of $r \in [0, b - 1]$ are permissible. This can be seen in equation (A.15.a) and Facts 1 and 3. Fact 1 implies that $n - 2$ and $b$ are relatively prime. If all $r \in [0, b - 1]$ satisfy (A.15.a), then let $r = 0$ and $r = 1$, we have $b = 2$. Together with Fact 3, we see that the number of solutions $r$ is a proper factor of $b$.

**Remark 5.** In $(I_a \& I_b)$ the condition $C_a \geq b' > C_b$ implies that

\[
\frac{d - 3}{2} = b' - \frac{1}{2}, \quad \text{if } d \text{ even,}
\]

\[
\frac{d - 3}{2} = b' - 1, \quad \text{or } b', \quad \text{if } d \text{ odd.}
\]

**Theorem A.2.** (i). For all integers $n, d$, $3 < d < n - 1$, set

\[
q(e) = \left\lfloor \frac{e(n + 1 - d) - 2}{n - 2} \right\rfloor + 1.
\]

There are at least $\frac{n+1-d}{2}$ ray congruence classes (arithmetic progressions) mod $d(n - 2)$ of $e$ which are both accessible and point-minimal, i.e. beside condition (A.1), $e$ also satisfies

\[
q(e - 1) < q(e). \quad \text{(A.22)}
\]

(ii). The set of values $q(e)$ with $e$ accessible and point-minimal contains $(n + 1 - d)/2$ many ray congruence classes mod $(n + 1 - d)d$.

(iii). If integers $n, d$, $3 < d < n - 1$ such that

\[
\gcd((n + 1)(n - 2), n + 1 - d) = 1, \quad \text{(A.23)}
\]
then there are at least $f(n, d)$ ray congruence classes mod $d(n - 2)$ (respectively, mod $d(n + 1 - d)$) of accessible and point-minimal $e$ (resp. of values $q(e)$ over those), where

$$f(n, d) = \min \left\{ \frac{(n - d)(n - d - 1)}{2} - 2, \frac{(d + 1)(d - 6)}{2}, \frac{(n - d + 1)(n - d - 2)}{2} - 3 \right\}. \quad (A.24)$$

**Remark 6.** By definition, the point-minimal condition does not affect the set of $q$-values. Thus,

$$\{q(e) : e \text{ is accessible and point-minimal} \} = \{q(e) : e \text{ is point-minimal} \}.$$

**Remark 7.** For $n, d \in \mathbb{Z}$ the density of the accessible and point-minimal $e$ is about $\frac{n - d}{2n^2}$ in Theorem A.2(i) or $\min \left( \frac{(n-d)^2}{2n^2}, \frac{d}{2n} \right)$ in Theorem A2(iii), and the density of the set of $q$ values is $\frac{1}{2n}$ for (ii) and $\min \left( \frac{(n-d)^2}{2}, \frac{d}{2(n-d)} \right)$ for (iii).

**Proof of Theorem A.2.**

We will use the set up in the proof of Theorem A.1. By the expression of $e$ in display (A.9), inequality (A.22) is equivalent to the following inequality

**Case (a).** $0 < c < \frac{d}{2}$.

$$dbk + rd + \left[ \frac{cn + \varepsilon(b - 1) - 2 - b}{n - 2} \right] < dbk + rd + \left[ \frac{cn + \varepsilon(b - 1) - 2}{n - 2} \right]$$

**Case (b).** $c \geq \frac{d}{2}$.

$$dbk + rd + \left[ \frac{cn + \varepsilon(b - 1) - 3}{n - 2} \right] < dbk + rd + \left[ \frac{cn + b + \varepsilon(b - 1) - 3}{n - 2} \right]$$

Since there is no restriction on $k$, as long as one pair of $(c, r)$ such that inequality holds, there are infinitely many pair. Hence we want to find $(c, r)$ such that

$$0 < c < \frac{d}{2}, \quad \text{and} \quad \left[ \frac{cn + \varepsilon(b - 1) - 2 - b}{n - 2} \right] < \left[ \frac{cn + \varepsilon(b - 1) - 2}{n - 2} \right], \quad (A.25.a)$$

or

$$c \geq \frac{d}{2}, \quad \text{and} \quad \left[ \frac{cn + \varepsilon(b - 1) - 3}{n - 2} \right] < \left[ \frac{cn + b + \varepsilon(b - 1) - 3}{n - 2} \right]. \quad (A.25.b)$$

We will only prove the theorem for Case (a), since Case (b) is similar.
In display (A.25.a), it is straightforward that

\[ L := \frac{cn + \varepsilon(b-1) - 2 - b}{n-2} > c - 1, \quad \text{and} \quad R := \frac{cn + \varepsilon(b-1) - 2}{n-2} > c. \]

The condition \(0 < c < \frac{d}{2}\) implies

\[ R = \frac{cn + \varepsilon(b-1) - 2}{n-2} < c + 1. \]

Hence

\[ \lfloor R \rfloor = \left\lfloor \frac{cn + \varepsilon(b-1) - 2}{n-2} \right\rfloor = c, \quad (A.26.a) \]

and the permissible pairs \((c, r)\) for \(0 < c < \frac{d}{2}\) are exactly those satisfy

\[ L = \frac{cn + \varepsilon(b-1) - 2 - b}{n-2} < c, \]

i.e.

\[ \varepsilon < \frac{b + 2 - 2c}{b - 1}. \quad (A.27.a) \]

It is clear that (A.27.a) holds for \(c = 1\), and for any \(r \in [0, b - 1]\). So does (A.22). On the other hand, by Remark 4, (A.1) holds for at least half of \(r \in [0, b - 1]\). Hence, parts (i) and (ii) are proved, and we may assume \(c \in [2, \frac{d}{2})\) for the proof of part (iii).

By the expression of \(\varepsilon\) in (A.13), display (A.27.a) is equivalent to

\( (m + 1)b - r(d^2 - 3d) - c(d - 1) < b + 2 - 2c, \)  

where \(m\) is as in (A.12) and (A.13). So we want to rule out \((c, r)\) such that

\[ \frac{r(d^2 - 3d) + c(d - 3) + 2}{b} \leq m < \frac{r(d^2 - 3d) + c(d - 1)}{b}, \quad (A.29.a) \]

i.e. we want to rule out \((c, r)\) such that

\[ r(d^2 - 3d) + c(d - 1) \equiv K \mod b, \]  

for \(K = 2, \ldots, 2c - 2,\)

(The case \(K = 1\) was done in Theorem A.1.)

or equivalently

\[ r(n + 1)(n - 2) \equiv K - cn \mod b, \]  

for \(K = 2, \ldots, 2c - 2. \quad (A.30.a) \]
For each $c$, there are $2c - 3$ choices of $K$, and for each $K$, the solution $r$ is unique under assumption (A.23). So the total number of $(c, r) \in [2, C_a] \times [0, b - 1]$ to rule out is

$$1 + 3 + \cdots + (b - 2) + (b - 1)\left(C_a - \frac{b + 1}{2}\right) = (b - 1)C_a - \frac{(b - 1)(b + 3)}{4},$$

if $b$ is odd, and $C_a \geq \frac{b + 1}{2}$,

and

$$1 + 3 + \cdots + (2C_a - 3) = (C_a - 1)^2,$$

if $C_a < \frac{b + 1}{2}$.

For Case (b), $c \geq \frac{d}{2}$, we have the corresponding displays

$$[R_b] = \left\lfloor \frac{cn + b + \varepsilon(b - 1) - 3}{n - 2} \right\rfloor = c,$$ \hspace{1cm} (A.26.b)

$$\varepsilon < \frac{b + d - 2c}{b - 1}.$$ \hspace{1cm} (A.27.b)

$$(m + 1)b - r(d^2 - 3d) - c(d - 1) < b + d - 2c,$$ \hspace{1cm} (A.28.b)

$$\frac{r(d^2 - 3d) + c(d - 3) - 2c + d - 1}{b} \leq m < \frac{r(d^2 - 3d) + c(d - 1) - 1}{b},$$ \hspace{1cm} (A.29.b)

$$r(n + 1)(n - 2) \equiv K - cn \mod b, \text{ for } K = 3, \ldots, 2c - d + 1.$$ \hspace{1cm} (A.30.b)

Summing up Case (a) and Case (b), the number of pairs $(c, r) \in [2, d - 3] \times [0, b - 1]$ to rule out is at most

$(1_a \& 1_b).$When $C_a > C_b \geq \frac{b + 1}{2}$,

(1).$(b - 1)(d - 3) - \frac{(b - 1)(b + 3)}{2}$, for $b$ odd and $d$ even,

(2).$(b - 1)(d - 3) - \frac{(b - 1)(b + 2)}{2}$, for $d$ odd,

(3).$(b - 1)(d - 3) - \frac{b^2 + 2b - 4}{2}$, for $b$ and $d$ even.
When \( C_a \geq \frac{b + 1}{2} > C_b \),
\[
\begin{align*}
(1) & \quad \frac{b^2 - 4b + 5}{2}, \text{ when } b = d - 3, \\
(2) & \quad \frac{b^2 - 5b + 8}{2}, \text{ when } b = d - 2.
\end{align*}
\]

When \( \frac{b}{2} > C_a > C_b \),
\[
\begin{align*}
(1) & \quad \frac{d^2 - 10d + 26}{2}, \text{ if } d \text{ is even,} \\
(2) & \quad \frac{d^2 - 9d + 22}{2}, \text{ if } d \text{ is odd.}
\end{align*}
\]

Note that from displays (A.25.a), (A.25.b), (A.26.a), and (A.26.b), the number of values \( q(e) \mod db \) is the same as the number of permissible pair \((c, r)\).

In each case of \((1_a & 1_b), (1_a & 2_b)\), and \((2_a & 2_b)\) above, we take the maximum and subtract it from the minimum in Theorem A.1. Applying Remark 5, we prove Part (iii). □

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