NUMERICAL AND ANALYTICAL SOLUTIONS OF NEUTRINO-DOMINATED ACCRETION FLOWS WITH A NON-ZERO TORQUE BOUNDARY CONDITION AND ITS APPLICATIONS IN GAMMA-RAY BURSTS

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ABSTRACT
A stellar-mass black hole (BH) surrounded by a neutrino-dominated accretion flow (NDAF) has been discussed in a number of works as the central engine of gamma-ray bursts (GRBs). It is widely believed that NDAF cannot liberate enough energy for bright GRBs. However, these works have been based on the assumption of a “no torque” boundary condition, which is invalid when the disk is magnetized. In this paper, we present both numerical and analytical solutions for NDAFs with non-zero boundary stresses and reexamine their properties. We find that an NDAF with such a boundary torque can be powerful enough to account for those bright short GRBs, energetic long GRBs, and ultra-long GRBs. The disk becomes viscously unstable, which makes it possible to interpret the variability of GRB prompt emission and the steep decay phase in the early X-ray afterglow. Finally, we study the gravitational waves radiated from a processing BH-NDAF. We find that the effects of the boundary torque on the strength of the gravitational waves can be ignored.

Key words: accretion, accretion disks – gamma-ray burst: general – magnetic fields – neutrinos

1. INTRODUCTION
The leading model of gamma-ray burst (GRB) central engines is a hyper-accreting stellar-mass black hole (BH). The typical accretion rate is extremely high (e.g., 0.01–1M⊙ s⁻¹), leading to a very dense and hot flow. Under such conditions, photons become trapped and are inefficient in cooling the disk. The gravitational energy in the accretion flow is mainly carried by neutrinos and antineutrinos, which annihilate and power GRB jets. These disks are therefore named “neutrino-cooling-dominated accretion flows,” or neutrino-dominated accretion flows (NDAFs; e.g., Popham et al. 1999, hereafter PWF99; Kohri & Mineshige 2002).

The NDAF has been extensively investigated and is usually compared with the magnetic mechanism (e.g., Narayan et al. 2001, hereafter NPK01; Di Matteo et al. 2002, hereafter DPN02; Kohri & Mineshige 2002; Janiuk et al. 2004, 2007; Chen & Beloborodov 2007; Gu et al. 2006; Liu et al. 2007; Lei et al. 2008, 2009, 2013b; Janiuk & Yuan 2010). It was long considered as an inefficient model for GRBs. This conclusion was first made by Popham et al. (1999), and enhanced by Di Matteo et al. (2002). Fan et al. (2005) found that the NDAF model was not favorable to explain the X-ray flares of GRB afterglows. Recently, detailed studies by Liu et al. (2015c) show that some bright short GRBs (SGRBs) are difficult to explain with NDAF. More recently, Song et al. (2016) argued that NDAF may not be the central engine for some extremely high energy long GRBs (LGRBs).

It is worth noting that these works are based on the assumption of zero torque at the inner edge of the accretion disk. This condition has been argued based on the fact that a small amount of mass in the plunging region could hardly be expected to exert a force on the far heavier disk proper, or rapidly become causally disconnected from the disk (Novikov & Thorne 1973, hereafter NT73). However, as recognized by Page & Thorne (1974, hereafter PT74), neither of these arguments applies to magnetic stress. This issue has become increasingly important with the realization that angular momentum transport in disks is entirely due to turbulence generated via the magnetorotational instability (Balbus & Hawley 1991). Krolik (1999) and Gammie (1999) argued that the dominant role of this magnetic stress in angular momentum transport in the disk body should actually lead to stresses near the marginally stable orbit. Based on these considerations, Agol & Krolik (2000) studied a relativistic thin disk with non-zero torque at its inner edge. As a consequence, the additional magnetic stresses have strong effects that change the fundamental properties of the accretion flow. A more complete magnetohydrodynamical (MHD) model of a magnetized thin disk has been developed by Gammie (1999), a numerical MHD simulation has been carried out by Reynolds & Armitage (2001) using the ZEUS code (Stone & Norman 1992a, 1992b), and they verified the existence of torque at a marginally stable radius due to the coupling of the plunging region to the disk through magnetic fields. The accretion of a magnetized torus in Kerr metric has been studied by Gammie et al. (2003) and De Villiers et al. (2003) by using general relativistic magnetohydrodynamical (GRMHD) codes. It is found that the disk would be significantly altered by the additional stress, with wide-ranging observational consequences (Agol & Krolik 2000; Zimmerman et al. 2005). Some authors suggested that the episodic jets in GRBs could be reproduced by magnetized NDAF (e.g., Yuan & Zhang 2012; Cao et al. 2014). The magnetic energy within an episodic jet is possibly dissipated via internal-collision-induced magnetic reconnection and turbulence (Zhang & Yan 2011). These works motivate us to investigate the NDAF with boundary stresses. We refer to this model as non-zero torque NDAF (nztNDAF), and refer to the previous NDAF model with zero boundary torque as NDAF.

This paper is organized as follows. In Section 2, we describe the NDAF model with boundary stress and general relativistic corrections. A free parameter η is introduced to account for the magnitude of the unknown stress at the inner edge of nztNDAF. In Section 3, we study the properties of the disk by solving the set of equations. Based on the solutions, we investigate the stability and total neutrino annihilation luminosity of nztNDAF. In Section 4, we apply the nztNDAF model to GRBs. In Section 5, we find that the variability of the
GRB prompt emissions and the steep decay phase in the early X-ray afterglow can be well explained by the viscous instability in nztNDAF. In Section 6, we investigate the effect of inner boundary torque on the gravitational waves radiated from a processing disk. We summarize and discuss the results of this work in Section 7.

2. NDAF WITH BOUNDARY STRESS

As argued by Krolik (1999), a sizable torque would be exerted on the disk’s inner edge if matter inside the marginally stable orbit \( r_{ms} \) remains magnetically connected to the disk. Hereafter, the subscript “ms” indicates the quantity at the marginally stable orbit. There is no characteristic or “nature” magnitude that one can select for the torque. In principal, additional local dissipation must accompany the additional boundary torque. In the context of a standard thin disk (SSD), such extra dissipation leads to an increment in disk radiation. For this reason, in Agol & Krolik (2000), the significance of the boundary torque is described with an additional radiative efficiency \( \Delta \epsilon \). In the case of NDAF, the gases are cooled via neutrino losses and advection, \( \Delta \epsilon \) is no longer a proper parameter matching the extra stresses. Hence, we introduce a factor \( \eta \) to quantify the non-zero torque at \( r_{ms} \):

\[
g_{ms} = \eta M L_{ms},
\]

where \( L_{ms} = 2GM(3\lambda_{ms} - 2a_s)/\sqrt{3} c\lambda_{ms} \), is the specific angular momentum of a particle in the disk, in which \( \lambda_{ms} = \sqrt{c^2 r_{ms}/GM} \) (NT73).

In the fluid frame for a time-steady, geometrically thin, relativistic accretion disk, \( \eta \) is related to \( \Delta \epsilon \) by

\[
\eta = \Delta \epsilon \frac{c^2}{\Omega_{ms} L_{ms}}.
\]

For a Newtonian disk, this relation is reduced to \( \eta = \Delta \epsilon r_{ms}/r_g \), where \( r_g = GM/c^2 \) denotes the gravitational radius. The angular velocity of the disk at \( r_{ms} \) is \( \Omega_{ms} = ((r_{ms}/GM)^{1/2} + a_s GM/c^2)^{-1} \).

By using the numerical simulation with Pseudo-Newtonian potential, Hawley & Krolik (2002) show \( \eta \sim 0.05-0.1 \). However, as argued in Krolik (1999) and Gammie (1999), in a Kerr metric, the efficiency \( \Delta \epsilon \) would normally become greater than unity because the accumulated spin energy of the BH is being tapped. This suggests a link between \( \Delta \epsilon \) (as well as \( \eta \)) and the BH spin. As we known, spin energy and angular momentum can be transferred from the BH to the disk via a large-scale closed magnetic field (Blandford 1999; van Putten 1999; Li 2000, 2002; Li & Paczynski 2000; Wang et al. 2002, 2003; Gan et al. 2007; Lei et al. 2007, 2009). This mechanism is called the magnetic coupling process (MC). We can thus put a constraint on \( \eta \) by equating the boundary torque \( g_{ms} \) to the total MC torque \( T_{MC} \). As shown in Appendix A, \( \eta_{max} \) can reach \( \sim 10 \) for a rapidly spinning BH.

Our nztNDAF model is based on the context given by DPN02, and the general relativistic corrections are adopted from Riffert & Herold (1995; hereafter RH95). The equation for angular momentum for nztNDAF is written as (details for the derivation are displayed in Appendix B)

\[
Mr^2 \frac{\sqrt{GM} D}{r^3} A + g_{ms} \frac{A_{ms}}{A} = -4\pi r^2 r_g h,
\]

where \( A_{ms} \) is the value of factor \( A \) at \( r_{ms} \). The second term on the left side of Equation (4) vanishes for NDAF with the assumption of a “zero torque” boundary condition. \( A, B, C, D, \) and \( E \) are the relativistic correction factors for a thin accretion disk around a Kerr BH given by RH95 as

\[
A = 1 - 2\frac{GM}{c^2 r} + \left( \frac{GM a_s}{c^2 r} \right)^2,
\]

\[
B = 1 - 3\frac{GM}{c^2 r} + 2a_s \left( \frac{GM}{c^2 r} \right)^{3/2},
\]

\[
C = 1 - 4a_s \left( \frac{GM}{c^2 r} \right)^{3/2} + 3 \left( \frac{GM a_s}{c^2 r} \right)^2,
\]

\[
D = \int_{r_{ms}} r_{ms} \frac{c^4}{\sigma^2} \left( \frac{\lambda_{ms}}{\sigma} - 2 a s^2 \frac{1}{\rho} \right)^2 \frac{dx}{\rho^2},
\]

\[
E = 1 - 6\frac{GM}{c^2 r} + 8a_s \left( \frac{GM}{c^2 r} \right)^{3/2} - 3a_s^2 \left( \frac{GM}{c^2 r} \right)^2.
\]

The \( \alpha \)-prescription for the viscous shear \( \tau_{\alpha,\eta} \), as well as the expression for the disk half-thickness \( h \), are corrected as

\[
\tau_{\alpha,\eta} = -\alpha P \frac{A}{\sqrt{BC}},
\]

\[
h = \frac{Pr^3}{\sqrt{\rho GM}} \frac{B}{C},
\]

where \( P \) is the total pressure, including the gas pressure \( P_\text{gas} \), radiation pressure \( P_\text{rad} \), degeneracy pressure \( P_\text{deg} \), neutrino pressure \( P_\nu \), and the magnetic pressure \( P_\text{B} \):

\[
P = P_\text{gas} + P_\text{rad} + P_\text{deg} + P_\nu + P_\text{B}.
\]

here, we assume that the magnetic pressure accounts for a fraction of the total pressure as \( P_\text{B} = \beta P \). Other terms are expressed in Appendix C.

According to Riffert & Herold (1995; see their Equation (19)), the viscous heating rate is

\[
Q_{\text{vis}}^+ = \frac{3}{4} \sqrt{\frac{GM}{A}} \frac{B}{r} \int_{r_{ms}}^h \tau_{\alpha,\eta} dz;
\]

substituting Equation (45) into Equation (12), we have

\[
Q_{\text{vis}}^+ = \frac{3GMM D}{8\pi r^3} B + \frac{3g_{ms}}{8\pi r^2} \sqrt{\frac{GM}{A}} A_{ms} B,
\]

where the factor \( D/B \) is equal to zero at \( r_{ms} \) and approaches unity at large radii. As stated in Janiuk & Yuan (2010), this asymptotic behavior of \( D/B \) is the same as for the boundary condition derived in NT73 and Chen & Beloborodov (2007), who used a more complex formalism in the Kerr metric. The second term on the right side of the equation is the contribution of the non-zero torque at \( r_{ms} \). This term is non-zero at the inner edge, which will increase the disk luminosity.

The equation for the energy balance is

\[
Q_{\text{vis}}^+ = Q_\nu^- + Q_\text{photo} + Q_\text{adv},
\]

where \( Q_\nu^- \) is the total cooling rate due to neutrino losses, \( Q_\text{photo} \) is the photodisintegration, and \( Q_\text{adv} \) is the advective cooling rate. Detailed expressions for \( Q_\text{photo} \) and \( Q_\text{adv} \), and the bridging formula for \( Q_\nu^- \) are given in DPN02 (see also Appendix C).
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3. THE PROPERTIES OF NON-ZERO TORQUE NDAF MODEL

We are interested primarily in the properties of the inner accretion flow, where the neutrino process is important. As argued in PWF99, NPK01, and DPN02, the flows are fully advection-dominated for \( r > 100r_g \), where neutrino cooling is not important and photons are completely trapped. Therefore, we focused on the region from \( r_{\text{ms}} \) to \( r_{\text{max}} = 100r_g \). In the calculation, we do not include the cooling term arising from the photodisintegration \( Q_{\text{photo}} \) because it is much less than the neutrino cooling rate in the inner region (Janiuk et al. 2004). The strength of the non-zero torque is described by the parameter \( \eta \) referred to in Equation (1). Throughout the paper, we take \( \alpha = 0.1 \) as a typical value, for the detailed effects of \( \alpha \), refer to previous studies (e.g., Chen & Beloborodov 2007; Liu et al. 2010; Lin et al. 2016).

3.1. The Structure of nzTNDAF

We solve numerically Equations (3)–(16) to find the disk temperature \( T \) and density \( \rho \) versus the disk radius with a typical model parameters \( \alpha = 0.1, M = 7M_\odot, a_* = 0.9, \) and \( \dot{m} = 1.0 \) (\( \dot{m} \equiv \dot{M}/M_\odot \) s\(^{-1} \)). The solutions are shown in Figure 1. In order to study the effects of the boundary torque, we calculate the solutions for \( \eta = 0 \) (black solid lines), 0.1 (red dashed lines), 0.5 (green dotted lines), 1.0 (blue dash-dotted lines), and 3.0 (cyan dash-dotted lines). Curves with solid lines (\( \eta = 0 \)) in Figure 1 exhibit the solutions for NDAF without a boundary torque. In Appendix C, we have made an effort to obtain the analytic solutions of nzTNDAF to better understand the main results the numerical calculation exhibited here.

From Figure 1, we find that the boundary torque has strong effects on the properties of inner disk. For example, as shown in Figures 1(a)–(e), the temperature \( T \), density \( \rho \), pressure \( P \), height \( h \), and neutrino optical depth \( \tau_\nu \) become non-zero at \( r_{\text{ms}} \) due to the existence of such a boundary torque. According to Equation (15), a disk with a greater boundary torque will produce more heat in the inner region, leading to a higher temperature, as shown in Figure 1(a). As a result, the disk pressure \( P \), height \( h \), and neutrino optical depth \( \tau_\nu \) increase with the increasing \( \eta \). In Figure 1(f), the drop of advection parameter \( f = Q_{\text{adv}}/Q_{\text{in}} \) in inner region reflects that this additional heating indeed ignites efficient neutrino cooling. However, as discussed in DPN02, the cooling rate due to neutrino emission will be suppressed if \( \tau_\nu \) is too large. This is also illustrated in Figure 1(f). The advection becomes important (\( f > 0.5 \)) in the inner region if \( \eta \) becomes significantly larger than 1. From Figure 1, we also find that the boundary torque weakly affects the outer disk. This is because the additional heating term (the second term in the right side of Equation (15)) scales as \( r^{-7/2} \) at large \( r \) rather than \( r^{-3} \) as in the standard viscous heating term (the first term).

It is shown in Figure 1(f) that advection (denoted by \( f \)) dominates at large radii for both NDAF and nzTNDAF. An equivalent statement is that the cooling timescale is much longer than the accretion timescales, so the energy is advected inward before it can be radiated away. As shown in Figure 1(f), the advection parameter \( f \) slowly decreases as the gas continues to fall inward, since the increasing temperature and density produce a rapid increase in the neutrino cooling rate. For NDAF, the densities and temperatures near the inner edge are too small to ignite significant neutrino cooling, and then \( f \) goes to unity again. As discussed in Appendix C, NDAF generally
consists of four regions, as shown in Figure 2 (see also Figure 10 in Chen & Beloborodov (2007):

(Region I) At large radii, densities and temperatures are too small for neutrino cooling to be significant, and the disk is simply an advection-dominated flow (ADAF).
(Region II) At this region, neutrino emission switches on. The neutrino opacity is not important, so this region is referred as transparent ADAF.
(Region III) Disk becomes opaque for neutrinos, but neutrino cooling is still dominated. We call this region opaque ADAF.
(Region VI) Near \( r_{\text{ms}} \), the flow returns to ADAF due to the low temperature and density.

The inner structure for nztdNTA is quite different from NDAF. There will be two regions inside region III:

(Region IV) In this region, the temperature is very high due to the additional heating driven by boundary torque. The disk is thus dominated by radiation pressure, but is still an opaque NDAF. In Section 3.2, we show that this region is viscously unstable, and for this reason, it is named the unstable NDAF.

(Region V) Since huge amounts of heat are produced near \( r_{\text{ms}} \), the neutrino optical depth is so high that even neutrinos cannot escape, resulting in an advection-cooling flow (corresponding to \( r < 4r_g \) for \( \eta = 3.0 \) in Figure 1(f)).

The analytical solutions for each region are given in detail in Appendix C. To show the strength of these analytical solutions, we compare them with the numerical ones. In Figures 11 and 12 of Appendix C, it is clearly shown that they are consistent with our numerical solutions. These studies suggest that the analytical solutions can capture the main feature of the disk.

3.2. Stability Analysis: Viscous Instability

NDAF are said to be stable under most cases (NPK01, DPN02). As shown in the Section 3.1, by introducing the boundary torque, the nztdNTA behaves quite differently with NDAF. The inner disk will become viscously unstable if a strong magnetic stress is applied on its edge (see to Appendix C for a better understanding). For a disk with high accretion and large \( \eta \), the temperature is significantly increased due to the additional heating driven by a boundary torque. As a result, the flow becomes radiation-pressure and neutrino-pressure dominated, which is unstable according to the viscous instability criterion \( d\dot{m}/d\Sigma < 0 \) (where \( \Sigma \) is the surface density). As an example, for \( \dot{m} = 1.0 \) and \( \beta = 0.0 \), the instability will occur when \( \eta \gtrsim 0.45 \) (refer to Figure 4). The corresponding magnetic field near \( r_{\text{ms}} \) will be \( \gtrsim 4.6 \times 10^{15} \) Gauss, which is estimated by equating the magnetic torque \( \eta ML_{\text{ms}} \) to \( 2\pi r_{\text{ms}}^2 \cdot 2h_{\text{ms}} \cdot (B)^2 \), where \( (B) \) denotes the magnitude of the magnetic field.

Figure 3 shows the \( \dot{m} - \Sigma \) profile for different radius \( r \) and different \( \eta \). The cyan line is a critical radius \( r_{\text{cr}} \) beyond which the solution will be stable for all accretion rates \( \dot{m} \). For \( \dot{m} = 1.0 \) and \( \beta = 0.0 \), this cyan line locates at \( r = 3.43r_g \), but it may vary with \( \eta \) and \( \beta \).

Figure 4 shows the unstable region that depends on \( \dot{m}, \eta, \) and \( \beta \), and the unstable zones are shown as the shaded regions. For fixed \( \eta \) and \( \dot{m} \), the disk might only be unstable in a radius range. With increasing \( \eta \) and \( \dot{m} \), the unstable region moves further out in the flow. Take the scenario \( \dot{m} = 1.0 \) and \( \beta = 0.0 \) as an example, the whole disk is viscously stable if \( \eta \) is not too large (not greater than 0.45), however, the disk will become viscously unstable in the inner region \( r < 2.5r_g \) when \( \eta \sim 0.5 \). In addition, the unstable region expands outwards to \( \sim 3r_g \) if \( \eta \sim 1 \). Note that there is a critical radius \( r_{\text{cr}} \) for the unstable region for each \( \eta \), for example, the cyan line in Figure 3. At

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**Figure 2.** Schematic picture of NDAF (left) and nztdNTA (right). Detailed explanations and analytical solutions for these regions are given in Appendix C.

**Figure 3.** \( \dot{m} - \Sigma \) profiles at different disk radius \( r \). The thick cyan curve denotes the last viscously stable radius \( r_{\text{ms}} \) for any \( \dot{m} \), which is located at 3.43\( r_g \).
Figure 4. (a). Viscous unstable regions are indicated by the shaded regions for different inner edge torque $\eta = 1.0$, 2.0, and 5.0, with $M = 7 M_\odot$, $a_\star = 0.9$, and $m = 1$. Note that for each fixed disk radius in the unstable region, there are two critical values of accretion rate $m_{\text{cr},1}$ and $m_{\text{cr},2}$, the flow at $r$ will be unstable when $m_{\text{cr},1} < m < m_{\text{cr},2}$. The upper and lower border lines of each of the shaded regions separately denote the critical accretion rate $m_{\text{cr},1}$ and $m_{\text{cr},2}$ for different disk radii. (b). The right panel plots the unstable region vs. $\eta$ for different magnetic pressure components $\beta = 0.0, 0.6, 0.7, 0.75$, and $0.77$, with fixed $m = 1.0$. Note that the unstable region is only significantly affected by $\beta$ when it is greater than $\beta > 0.6$.

$\nu < r_{\text{fr}}$, the viscous instability takes place only in a certain accretion rate range, like $0.32 < \dot{m} < 1.94$ if $\eta = 1.0$. Therefore, one conclusion is that under a larger inner edge torque, the disk can be viscously stable only if the accretion rate is relatively low or extremely high, while the disk with a moderate accretion rate may suffer instability. This statement can be understood as follows. If the accretion rate is relatively low, the temperature cannot be high enough and consequently the radiation pressure cannot take the dominant role; on the other hand, if the accretion is extremely high, then the neutrinos will be trapped due to the extremely high neutrino optical depth and the flow will become advection-cooling dominated, both of those scenarios are stable according to the instability criterion.

3.3. Neutrino Annihilation Luminosity

Inspecting Equation (15), the non-zero torque applied on the inner edge results in huge energy dissipation, which could lead to a more powerful neutrino radiation, as well as a greater neutrino annihilation luminosity. The total neutrino luminosity from the accretion flow is expressed as

$$L_\nu = 4\pi \int_{r_{\text{in}}}^{r_{\text{max}}} Q_\nu \, r \, dr,$$

where we adopt $r_{\text{max}} = 100 r_g$ as discussed above. Our method for calculating neutrino annihilation is similar to PWF99 and Rosswog et al. (2003). The disk is modeled as a grid of cells in the equatorial plane. A cell $k$ has its neutrino mean energy $\bar{\varepsilon}_\nu$ and luminosity $\dot{L}_\nu^k$, and the height above (below) the disk is $d_k$. The angle at which neutrinos from cell $k$ encounter antineutrinos from another cell $k'$ is denoted as $\theta_{kk'}$. Then the neutrino annihilation luminosity is given by the summation over all pairs of cells,

$$l_{\nu\bar{\nu}} = A_1 \sum_k \frac{\dot{L}_\nu^k}{d_k^2} \sum_{k'} \frac{\dot{L}_{\bar{\nu}}^{k'} + \dot{L}_{\nu}^{k'}}{d_{k'}^2} (1 - \cos \theta_{kk'})^2$$

$$+ A_2 \sum_k \frac{\dot{L}_\nu^k}{d_k^2} \sum_{k'} \frac{\dot{L}_{\bar{\nu}}^{k'} + \dot{L}_{\nu}^{k'}}{d_{k'}^2} (1 - \cos \theta_{kk'})^2$$

Figure 5. $L_\nu$ vs. $\eta$ with different $m$. Other parameters are $M = 7 M_\odot$, $a_\star = 0.9$, $\alpha = 0.1$, $\beta = 0.0$.

where $A_1 \approx 1.7 \times 10^{-44}$ cm erg$^{-2}$ s$^{-1}$ and $A_2 \approx 1.6 \times 10^{-56}$ cm erg$^{-2}$ s$^{-1}$.

The total neutrino annihilation luminosity is obtained by integrating over the whole space outside the BH and the disk,

$$L_{\nu\bar{\nu}} = 4\pi \int l_{\nu\bar{\nu}} \, r \, dr \, dz.$$
10^{54} \text{ erg s}^{-1} \text{ for the accretion rate } \dot{m} \approx 5.0. \text{ This implies that the effect of neutrino optical depth becomes important.} 

We have fixed the magnetic pressure parameter \( \beta \) to zero so far, and this parameter could be important and deserves discussion. One consideration is that the accretion flow might be magnetized, and the magnetic pressure can account for certain parts of the total pressure. Figure 6 shows the structure of the disk with different magnetization \( \beta \). We find that the disk will become thicker when a significant magnetic pressure \( (\beta \gtrsim 0.5) \) is involved. Consequently, the viscous instability and neutrino luminosity are expected to be suppressed by the strong magnetic pressure, as illustrated by Figures 4(b) and 7. However, if \( \beta \) is not too large \( (\beta \lesssim 0.3) \), the structure and the neutrino annihilation luminosity of the disk are weakly affected. For \( \eta < 10 \), our estimated magnetic parameter \( \beta \) is generally less than 0.3 (see also the Figure 6 in Cao et al. 2014). Therefore, we just take \( \beta = 0 \) as a good approximation.

4. INTERPRETING THE LUMINOSITIES OF BRIGHT SGRBS, LGRBS, AND ULGRBS

The prevailing opinion about the progenitors of SGRBs and LGRBs is that they are separately the result of compact binary mergers and the collapse of massive stars. For accreting BH central engines, the limited total material mass that can be supplied during two such types of events sets a concrete constraint to the accretion model. The related work from this perspective has been carried out by previous authors (Liu et al. 2015c; Song et al. 2016). Because of the relatively low output power and corresponding unreasonably high requirements of the total amount of accreting mass, the NDAF model has difficulty interpreting certain bright SGRBs and powerful LGRBs. In this section, we show that once the inner edge torque is considered, the NDAF model still works well for those “problem samples” identified by previous authors. Meanwhile, using the same method as Liu et al. (2015c), we investigate our nztNDAF model for ultralong GRBs (ULGRBs) in the frame of blue supergiant (BSG) progenitors.
**SGRBs:** Based on the limitation for the accretion disk mass after the compact binary coalescence given by numerical simulations (Kluźniak & Lee 1998; Ruffert & Janka 1998, 2001; Lee & Kluźniak 1999; Popham et al. 1999; Liu et al. 2012), Liu et al. (2015c) argued that some SGRBs could not be explained by the common NDAF model, since the mass of the remnant disk required by the model significantly exceeds the reasonable range given by previous numerical simulations. Specifically, for merging neutron star binaries (NS+NS), the reasonable mass of the remaining disk is likely in the range of 0.1–0.2\(M_\odot\) (Ruffert & Janka 1998, 2001), while for the coalescence of a neutron star and black hole (NS+BH), the survived disk mass is not likely larger than 0.5\(M_\odot\) (Kluźniak & Lee 1998; Lee & Kluźniak 1999; Popham et al. 1999; Liu et al. 2012). However, as shown by Liu et al. (2015c), for several SGRBs (such as GRB 050724, 051221A, 090426, and 120804A), if a common NDAF disk is taken as the model of central engines, then the disk mass constrained by combining the observational data with the common NDAF model will easily exceed the value limiting above for a large range of parameters, such as \(m\) and \(a_k\), except for some extreme values of those parameters, i.e., an extremely low BH mass \(m\) of order 10\(^{-5}\)\(M_\odot\) or an extremely high spin \(a_k\).

As discussed in Section 3.2, the neutrino annihilation luminosity of nztNDAF is much larger than NDAF without an inner edge torque, and the energy problem addressed above may be solved by the nztNDAF model. The same observed luminosity will require a smaller mass accretion rate and consequently a smaller disk mass under a fixed time duration in nztNDAF. Here, we will estimate the value of \(\eta\) required to fit those “problem SGRBs.”

Considering the conversion efficiency, the output power from the NDAF central engine is calculated as follows:

\[
\dot{E} = \eta_{ed} \dot{L}_{\nu,\nu},
\]

in which \(\eta_{ed}\) is the conversion factor (Aloy et al. 2005; Fan & Wei 2011; Liu et al. 2012, 2015c). Meanwhile, from the observation’s point view, the output power can be evaluated as follows:

\[
\dot{E} \approx \frac{(1+z)(E_{\gamma,\text{iso}} + E_{k,\text{iso}})\theta_j^2}{2T_{90}},
\]

where \(E_{\gamma,\text{iso}}\) is the isotropic energy of prompt emission, \(E_{k,\text{iso}}\) is the isotropic kinetic energy of the fireball constrained by fitting the afterglow emission, \(z\) is the redshift, \(T_{90}\) is the time duration of the prompt emission, and \(\theta_j\) is the jet angle.

For a typical set of parameters \((m, a_k, \eta, \eta_{ed})\), we can estimate the mass accretion rate and then the disk mass \(m_{\text{disk}} = \dot{m}T_{90}/(1+z)\) by adopting Equation (18) to a GRB \((E_{\gamma,\text{iso}}, E_{k,\text{iso}}, \theta_j, T_{90}, z)\). For comparison, the required disk mass based on the nztNDAF model (\(\eta = 0\)) and NDAF (\(\eta = 0\)) are listed in Table 1 for the four “problem SGRBs” referred above. We study two possible scenarios, i.e., NS+NS merging and NS+BH merging, and adopt different BH mass and maximum disk masses. For NS+NS merging, \(m_{\text{disk}} < 0.2M_\odot\) and \(m = 3\). For NS+BH merging, we take larger values, i.e., \(m_{\text{disk}} < 0.5M_\odot\) and \(m = 7\). Meanwhile, we take a moderate BH spin \(a_k = 0.5\). The results are summarized in the top panel of Table 1.

From Table 1, we conclude that the inner boundary torque can exactly reduce the requirement of the disk mass so as to bring it back to the reasonable range and solve the problem addressed in Liu et al. (2015c; which uses a traditional NDAF model). Taking GRB 050724 as an example, for the NS+BH case with the NDAF model, the required disk mass should be 1.5\(M_\odot\), which is much larger than the upper limit 0.5\(M_\odot\) (see Liu et al. (2015c) for the same conclusion). However, if we take the nztNDAF model with \(\eta = 0.23\), \(m_{\text{disk}}\) can be reduced to the reasonable value 0.5\(M_\odot\). The disk mass can be even smaller if we increase \(\eta\). The analyses to other samples are similar, and nztNDAF works well for those SGRBs.

**LGRBs:** Using a similar method to Liu et al. (2015c) for SGRBs, Song et al. (2016) investigated the mass distribution of the NDAF disk for 48 LGRBs; \(5M_\odot\) is thought to be a fiducial amount of the remnant material of massive collapsers. Their work showed that NDAF may not be suitable for some extremely high energy LGRBs because they require an unusually large disk mass \(>5M_\odot\). Here, we fit these “problem LGRBs” with nztNDAF, the results are listed in the middle panel of Table 1. BH mass \(m = 3\) and spin \(a_k = 0.9\) are adopted in the fits.

As shown in Table 1, for most LGRBs, the values of \(\eta\) are smaller than unity, except for GRB 050820A. However, \(\eta = 1.14\) is only slightly larger than 1. We thus conclude that all of these energetic LGRBs can be well fitted by nztNDAF.

**ULGRBs:** The BSGs are considered as possible progenitors of ULGRBs, which possess a time duration of about 10\(^5\) s, or even longer (Nakauchi et al. 2013). The masses of the BSGs are about several tens to hundreds of solar mass. Here, we investigate the possibility of powering ULGRBs with NDAF. Due to low metallicities, the progenitor envelopes are considered to form BHs without significant mass loss (Heger et al. 2003), thereby we take 50\(M_\odot\) as the reference value of the accreted material. The core may be rapidly rotating at collapse, we thus set the BH spin as \(a_k = 0.9\). For simplicity, we take BH mass 3\(M_\odot\) without considering it gradually increasing.

The fitting results for 3 ULGRBs are listed in Table 1. We find that NDAF is not suitable for ULGRBs because the needed disk mass seriously exceed 50\(M_\odot\). One thus needs our nztNDAF model to interpret ULGRBs. Note that the fit for GRB 111209A requires a rather large boundary torque \(\eta \approx 3\). As shown in Section 2 and Appendix A, the maximum value of boundary torque parameter \(\eta_{\text{max}}\) can be greater than 10. So, this large \(\eta\) is still acceptable.

5. INTERPRETING THE VARIABILITY OF PROMPT EMISSION FOLLOWED A SLOPE DECAY PHASE IN X-RAY AFTERGLOW

GRBs present remarkable time variability in their prompt emission and a steep decay phase followed by a shallow decay phase (or plateau) in their X-ray afterglow light curves. The steep decay and the plateau phases are often thought to be the result of shutting-off and reactivation of the central engine, respectively. In this section, we try to interpret the variability timescale with the viscous instability timescale, which is induced by the inner boundary torque. Meanwhile, we proposed an alternative scenario for the steep decay based on the instability analysis discussed above. It is worth noting that
the similar connection between the time variability and the disk instability has even been proposed in previous works for some X-ray binaries and active galactic nuclei (AGNs; e.g., Matsumoto et al. 1989; Honma et al. 1992; Ohsuga 2006, 2007; Oda et al. 2009).

We focus here on LGRBs, which are believed to be the results of the collapse of a massive star. An accretion disk will form after the collapse. As argued by Kumar et al. (2008), the mass feeding rate at the outer edge of the disk $m_{\text{acc}}$ decreases with time. For a fixed disk radius in the unstable region of nztNDAF, there are two critical accretion rates, i.e., a low one $m_{\text{cr},l}$ (e.g., the lower turning-point in the S-curve of Figure 8) and a high one $m_{\text{cr},h}$ (e.g., the top turning-point in the S-curve of Figure 8). The flow is unstable when $m_{\text{cr},l} < m_{\text{acc}} < m_{\text{cr},h}$. Once $m_{\text{acc}}$ reduces to the value below the critical rate $m_{\text{cr},l}$, the flow quickly switches to a low accretion rate state ($m < m_{\text{cr},l}$). If the mass feeding rate $m_{\text{acc}}$ at outer edge is greater than $m_{\text{cr},l}$, the disk accretion rate will increase gradually until reaching $m_{\text{cr},l}$. The disk will then jump to the high accretion rate state with $m > m_{\text{cr},h}$. If $m_{\text{cr},l} < m_{\text{acc}} < m_{\text{cr},h}$, the flow behaves in a limit cycle pattern. A sequence of such cycles makes up a series of individual pulses observed in the prompt emission. On the other hand, once the mass feeding rate $m_{\text{acc}}$ significantly decreases to $m_{\text{acc}} < m_{\text{cr},l}$, the disk drops to the low accretion rate state, but cannot jump back again. These features can explain the variability in prompt emission and the followed steep decay.

As shown in Figure 8, we take GRB 080607 (the redshift $z \approx 3.04$) as an example, in which we use an nztNDAF model with $\eta = 3$. The viscous instability is triggered once the accretion rate decreases to $1.22 M_\odot \text{s}^{-1}$, and then it oscillates between $3.7 M_\odot \text{s}^{-1}$ and $0.085 M_\odot \text{s}^{-1}$, resulting in the change of $L_{\text{vis}}$ between $2.05 \times 10^{54} \text{erg s}^{-1}$ and $1.33 \times 10^{52} \text{erg s}^{-1}$. The oscillating timescale is around 100 ms (strictly speaking, the instability timescale is evaluated as viscous timescale $t_{\text{vis}}$ that is between 6 ms and 33 ms, corresponding to the unstable domain of the mass accretion rate $1.22 \sim 0.35 M_\odot \text{s}^{-1}$, this numerical value of the viscous timescale is consistent with that derived from an analytical solution (see Appendix C for more details). The observed variation timescale can be estimated as $(1+z)t_{\text{vis}} \sim 24\sim 133$ ms. A series of limit cycles produces a series of individual pulses, which can explain the variability in prompt emission. Finally, as the feeding rate evolves to $m_{\text{acc}} < m_{\text{cr},l}$, there will be a sharp drop in luminosity, which can be considered as an natural interpretation for the steep decay phase.

It is worth noting that the value of $\eta = 3$ seems rather large compared to most of the $\eta$-values in Table 1. According to Appendix A, this value is still acceptable. We use such a large $\eta$ to reproduce a significant variation in light curve. However, the emission originates from jet instead of disk. The amplitude of the variability may be modulated by various mechanisms, such as magnetic dissipation in the jet (e.g., Deng et al. 2015, 2016), precession and relativistic boosting effect (e.g., Portegies Zwart et al. 1999; Lei et al. 2007, 2015), and so on. If we relax the constraint from the oscillation amplitude and only keep the requirement of timescale, then a much smaller $\eta$
still works. For example, if we take \( \eta = 1 \), an accretion rate of \( \dot{m} = 0.5 \) can also trigger the instability, in such cases, the neutrino annihilation luminosity is about \( 10^{53} \text{ erg s}^{-1} \), and the instability timescale is about \( (1 + z) \times 200 \text{ ms} = 800 \text{ ms} \), which is still consistent with the observations. To trigger an unstable NDAF, \( \eta \) cannot be too small, otherwise it may require an extremely large \( \dot{m} \) and therefore an unreasonable large disk mass (Liu et al. 2015c). So, we do not expect this model can explain all the GRBs.

6. THE GRAVITATIONAL WAVE RADIATION FROM A PRECESSING NDAF WITH BOUNDARY TORQUE

As catastrophic explosion events, GRBs are believed to be promising sources of gravitational waves that might be detected by current and future detectors such as LIGO, VIRGO, DECIGO, LISA, etc. The accretion disks in GRBs may precess with time (Blackman et al. 1996; Portegies Zwart et al. 1999; Reynoso et al. 2006; Lei et al. 2007; Liu et al. 2010; Romero et al. 2010). For a BH–NS binary system, if the spin axis of the BH is misaligned with the angular momentum of the binary system, the accretion disk formed after the merge would precess. For collapsar model, a inclined disk is also possible after an asymmetrical collapse. In both cases, the BH can force the misaligned disk around it to precess via a Bardeen–Petterson effect (Bardeen & Petterson 1975). Precession also exist in tidal disruption events (e.g., Stone & Loeb 2012; Lei et al. 2013a) and AGNs (e.g., Wu et al. 2013). Sun et al. (2012) suggested that the gravitational waves from a precessing NDAF disk might be detected by DECIGO and BBO, which supplies a new probability to carry out multi-mesagger detection for GRBs.

The + (plus) and × (cross) polarization of the gravitational wave are given as (Zimmermann & Szedenis 1979; Maggiore 2008)

\[
h_+ (t) = h_0 \sin 2\theta \cos (\Omega t) \sin \iota \cos \iota + 2h_0 \sin^2 \theta \cos (2\Omega t)(1 + \cos^2 \iota),
\]

\[
h_\times (t) = h_0 \sin 2\theta \sin (\Omega t) \sin \iota + 4h_0 \sin^2 \theta \sin (2\Omega t) \cos \iota,
\]

where

\[
h_0 = \frac{G}{c^3} \left( J_3 - I_3 \right) \Omega^2 d,
\]

in which, \( d \) is the distance of the GRB, \( \theta \) is the angle between the jet and the spin axis of the BH, \( \iota \) denotes the LOS (line of sight) and BH spin axis. \( I_1, I_2, \) and \( I_3 \) are the eigenvalues of the rotary inertia tensor of the inner processing part of the disk; they separatively denote the inertia along the principal axis \( X, Y, Z \) and can be expressed as

\[
I_1 = I_2 = \int_{r<p} \rho (x^2 + z^2) \, dx \, dy \, dz = \pi \int_{r<p} \Sigma r^2 (r^2 + 2H^2) \, dr,
\]

\[
I_3 = \int_{r<p} \rho (x^2 + y^2) \, dx \, dy \, dz = 2\pi \int_{r<p} \Sigma r^3 \, dr.
\]

The precession angular velocity of the accretion disk \( \Omega \) is expressed as (Sarazin et al. 1980; Lu 1990)

\[
\Omega = \frac{2G J_b}{c^2 \rho_c},
\]

here, \( J_b = GM^2 a_0 c \) denotes the BH angular moment. \( \rho_c \) is a critical radius, which is determined by equating the angular moment of this inner part with that of the BH, i.e., \( J_1/p = J_b \). A typical angular momentum of the disk is \( J = 2\pi r^3 \Sigma v_0, \) where \( v_0 = r\Omega_d \) is the angular velocity of disk (Sarazin et al. 1980; Sun et al. 2012).

During GRBs, the accretion disk can only exist for several to tens of seconds, suggesting that the gravitational radiation should appear as a gravitational wave burst. The gravitational waveform should be expressed as (Maggiore 2008)

\[
h(t) = [h_+ (t) + h_\times (t)] \exp \left( -\frac{t^2}{2\delta^2} \right),
\]

where \( \delta \) is the duration of the GRBs. We take \( \delta \approx 20\tau \) as a typical value in the calculations. The root-sum-square (rss) amplitude of the gravitational wave is used to estimate the detectability (Acernese et al. 2008; Maggiore 2008),

\[
h_{rss} = \sqrt{\int_{-\infty}^{\infty} \left( h_+^2 (t) + h_\times^2 (t) \right) \, dt}.
\]
The gravitational radiating power of the precession disk is expressed as

\[ P_{GW} = \frac{2G}{5c^3} (l_i - l_0)^2 \Omega^6 \sin^2 \theta (1 + 15 \sin^2 \theta). \]

We recalculate the GW strength and frequency from the NDAF according to the same procedure of Sun et al. (2012). In our calculations, we adopt a one-zone approximation for the vertical structure of the disk, which is embodied in the deducing process of Equation (22). In this section, we aim to investigate the significance of the effects of the inner edge torque on gravitational wave radiation.

Figure 9 shows the gravitational wave from a precessing NDAF disk with a different inner edge torque. Although the inner edge torque significantly changes the disk properties, it has little effect on the gravitational wave radiation from the precessing disk. This result is reasonable, since the boundary torque only weakly change the mass distribution of the outer disk. With the increase of \( g_{\text{inst}} \), the gravitational wave’s frequency, amplitude, and radiation power slightly represents the increment.

7. CONCLUSIONS AND DISCUSSIONS

We revise the NDAF model by including a boundary stress. Based on numerical and analytical solutions, we study the properties of nztNDAF. The disk becomes much hotter and denser due to the non-zero boundary torque. The properties in the inner region are significantly different from those of NDAF. As a result, we find that the disk becomes unstable if \( \dot{m} \) and \( \eta \) are great enough (e.g., when the parameter \( \eta > 0.45 \) for \( \dot{m} = 1.0 \)). The neutrino annihilation luminosity is greatly enhanced by the boundary stress. The luminosity of nztNDAF varies from \( 7.8 \times 10^{46} \text{erg s}^{-1} \) to \( 2.4 \times 10^{48} \text{erg s}^{-1} \) for \( 0.01 < \dot{m} < 10 \) with \( \eta = 1 \), which is much greater than NDAF (its range is from \( 2.3 \times 10^{45} \text{erg s}^{-1} \) to \( 4.6 \times 10^{43} \text{erg s}^{-1} \)).

We then apply the nztNDAF model to GRBs. For some bright SGRBs and powerful LGRBs, the NDAF model is challenged when interpreting the limited mass of the accretion disk after the compact binaries coalescence or massive collapsar (Liu et al. 2015c; Song et al. 2016). For the same reason, NDAF is not expected to explain ULGRBs. However, in this paper we argued that the NDAF model could still be a feasible model for those issued GRBs, as long as the inner boundary torque is considered. In addition, we extend the method of Liu et al. (2015c) to ultra-long GRBs and find that the nztNDAF model is also suitable under the frame of BSG-progenitor.

Viscous instability may occur nztNDAF in the inner region. When it happens, the disk will transit between two stable bruch with different accretion rates, leading to a variable jet luminosity. The timescale for the instability is about 10 ms (estimated by the viscous timescale at the inner disk). These results can explain the variability in GRB light curves. The steep decay following the prompt emission occurs when the mass feeding rate at the outer edge of the disk reduces to a value lower than a critical accretion rate. Finally, we find that the effects of a boundary torque on gravitational wave can be ignored.

In this work, we describe the boundary torque with a parameter \( \eta \). The properties of nztNDAF strongly depend on the value of \( \eta \). However, there is no characteristic or “natural” magnitude that one can select for the torque (Zimmerman et al. 2005). Numerical simulation performed by Penna et al. (2012) indicated the stress at the inner edge to be directly proportional to the disk thickness. They then argued that a zero-stress boundary condition is valid for thin disks in the limit \( h \to 0 \). However, the GRMHD simulations by Noble et al. (2010) with a different thickness found a large stress at the inner edge, even in the limit of a vanishing disk opening angle \( h \to 0 \). The GRMHD simulations by Krolik et al. (2005) and Beckwith et al. (2008) also found that the torque can reach a very high value in the plunging region. Due to these uncertainties, we take \( \eta \) as a free parameter in the calculations. In these works, the magnetic stress in the plunging region is likely the origin of the non-zero torque at the inner disk edge. For simplicity, we roughly take the magnetic coupling torque exacted by the BH as the upper limit of such boundary torques (Lei et al. 2009). Nonetheless, there are two differences at least between the nztNDAF model in this work and the MCNDAF model in Lei et al. (2009). First, the MCNDAF model essentially adopted the zero-stress assumption at the inner edge of the disk, and the MC torque is a resultant effect of the magnetic stresses differentially distributed in a limited disk region, which is coupled with the BH by ordered large-scale magnetic field lines (see the integrated angular momentum Equation (18) in Lei et al. 2009). While in the nztNDAF model, the non-zero stress is just exerted on the inner edge rather than any other location, this inner edge stress originates from the angular momentum transport between the plunging region and the disk through magnetized turbulence. Second, for the same magnitude of the

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**Figure 9.** Effects of the inner edge torque on the gravitational wave radiated from a precessing disk. The upper-left panel shows one of the polarization modes of the gravitational wave from a precessing disk with different inner edge parameter \( \eta \), the upper-right panel shows the power of gravitational radiation vs. accretion rate with different inner edge parameter \( \eta \), the lower panel shows the root-sum-square amplitude vs. the frequency of the gravitational wave with different inner edge parameter \( \eta \). Other parameters are \( m = 7 \), \( \alpha = 0.1 \); \( \beta = 0.0 \), \( \dot{m} = 1.0 \), \( \theta = 20^\circ \), \( \varphi = 20^\circ \), and \( d = 1 \) Mpc.
two different types of extra torque, the extra viscous heating rate in the inner region caused by the inner edge torque in nztNDAF is much higher than that of the MCNDAF model; consequently, the structure near the inner edge of the nztNDAF model will be changed with a more significant extent than MCNDAF. Those arguments deserve further studying using a GRMHD simulation.

For simplicity, we just consider radial direction in this work, adopting the one-zone approximation in the vertical direction. Our current results show that the inner side of the disk will expand a lot due to the extra heating by the non-zero torque (Figure 1(d)). Especially when $\eta \gtrsim 1$, the ratio of height to radius $h/r$ can approach unity. The larger disk height will aggravate the extent to which the neutrinos are trapping in the disk, this is one of the reasons why the innermost disk becomes advection-cooling dominated. In addition, the increment of the disk height due to the inner edge torque indicates the necessity of further consideration of the vertical structure. According to Gu & Lu (2007), when $h/r \gtrsim 0.2$, the Hōshi form of the gravitational potential (Hōshi 1977), used for deriving the vertical static equilibrium, cannot be satisfactory any more. There are a number of works on the vertical structure of NDAF (e.g., Sawyer 2003; Liu et al. 2010, 2012, 2013, 2014, 2015a; Pan & Yuan 2012). We may further explore the effects of the inner edge torque by considering the vertical structure in the future.

Another widely discussed GRB central engine model is Blandford–Znajek mechanism (Blandford & Znajek 1977; Lee et al. 2000; Lei et al. 2013b; Liu et al. 2015b). Lei et al. (2013b) studied the baryon loading in NDAF and Blandford–Znajek jets. It is found that the Blandford–Znajek mechanism can produce “clean” jets. The NDAF-driven “fireball” is typically too “dirty” to account for GRBs. In our nztNDAF model, however, the existence of the magnetic field may help to suppress baryons from disk, and thus lead to a clean central engine.

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APPENDIX A

CONSTRAINS ON THE INNER BOUNDARY TORQUE

In nztNDAF, the magnetic stress in the plunging region is likely the origin of the non-zero torque at the inner disk edge. As we know, the BH can exert a torque on the disk through the magnetic coupling mechanism (MC) and transfer energy and angular momentum to the disk (Blandford 1999; van Putten 1999; Li & Paczynski 2000; Li 2002; Wang et al. 2002; Lei et al. 2009). Based on this scenario, we can put an upper limit for $\eta$.

The magnetic field at the BH horizon could be derived by using the equipartition relation as follows (McKinney 2005):

$$\frac{B^2}{8\pi} = \rho_{0,\text{disk}} c^2,$$

(27)

where $\rho_{0,\text{disk}} \equiv \dot{M}t_e/r_g^3$, $t_e = GM/c^3$, and $r_g = GM/c^2$.  

The MC torque is expressed as (Wang et al. 2002)

$$T_{MC}/T_0 = f(a_\ast; n)$$

$$= 4a_\ast (1 + q) \int_0^{\pi/2} \left(1 - \beta \sin^2 \theta \right) d\theta$$

(28)

in which $q \equiv \sqrt{1 - a^2}$, $T_0 = B_0^2 (GM/c^3)^3$, $\beta$ denotes the ration of the angular velocity of the disk to that of the BH horizon, which is defined by

$$\beta \equiv \Omega_D/\Omega_H,$$

(29)

where $\Omega_D = c^3/GM (\chi^3 + a_\ast)$, $\Omega_H = a_\ast c^3/2GM (1 + q)$, and $\chi \equiv \sqrt{r/r_g}$. Combining Equations (27) and (28), one gets the MC torque

$$T_{MC} = \frac{8\pi GM M}{c} f(a_\ast; n).$$

(30)

Based on the conservation of magnetic flux, Wang et al. (2003) proposed the mapping relation between the angular coordinate $\theta$ on the horizon and the radial coordinate $\xi$ (defined as $\xi \equiv r/r_{ms}$):

$$\cos \theta = \int_1^{\xi} \sqrt{1 + a^2 \chi^{4} \xi^{-2} + 2a_\ast \chi^{2} \xi^{-3}} d\xi,$$

(31)

$$\Theta(a_\ast; \xi, n) = \frac{4^{\frac{1}{4}} n^{-\frac{1}{4}}}{2(1 + q)} \sqrt{1 + a_\ast^{2} \chi^{4} \xi^{-2} + 2a_\ast \chi^{2} \xi^{-3}}.$$  

(32)

On the other hand, we have introduced a parameter $\eta$ to reflect the strength of the inner boundary torque of the disk, which we rewrite as follows:

$$T_{MC} = \eta \dot{M} L_{\text{ms}}.$$  

(33)

Combining Equations (30) and (33), we have

$$\eta = \frac{8\pi f(a_\ast; n)}{\Theta(a_\ast; n)}.$$  

(34)
where $\zeta(a_*) = L_{ms}c/GM$.

Figure 10 shows the variation of $\eta$ versus the BH spin $a_*$ for different magnetic field configuration parameter $n$. We find that $\eta$ rapidly increases with $a_*$, but is not very sensitive to $n$.

APPENDIX B

THE ANGULAR MOMENTUM CONSERVATION EQUATION

The differential equation of the conservation of angular momentum is given by Riffert & Herold (1995) as

$$\frac{\partial \tau_{\phi\theta}}{\partial r} = -\frac{2}{r A} \left(1 - \frac{M}{r}\right) \tau_{\phi\theta} - \frac{\sqrt{M}}{2r^{3/2}} \frac{E}{A} r \mu r'$$, \hspace{1cm} (35)

here we use natural units $G = c = 1$, $u'$ is the radial component of the four velocity. Integrating over the disk height, Equation (35) becomes

$$\frac{d}{dr} \left( 2\pi \int_{-\infty}^{\infty} \tau_{\phi\theta} dz \right) = -\frac{2}{r A} \left(1 - \frac{M}{r}\right) 2\pi \int_{-\infty}^{\infty} \tau_{\phi\theta} dz$$

$$- \frac{\sqrt{M}}{2r^{3/2}} \frac{E}{A} 2\pi \mu r' \int_{-\infty}^{\infty} \rho dz.$$ \hspace{1cm} (36)

The continuity equation is

$$\dot{M} = -2\pi \mu r' \int_{-\infty}^{\infty} \rho dz = -2\pi (2h) \mu r'$$,

Inserting Equation (37) into Equation (36), we have

$$\frac{d\Lambda}{dr} + P(r)\Lambda = Q(r),$$ \hspace{1cm} (38)

where

$$\Lambda = 2\pi \int_{-\infty}^{\infty} \tau_{\phi\theta} dz,$$

$$P(r) = \frac{2}{r A} \left(1 - \frac{M}{r}\right),$$

$$Q(r) = \frac{\sqrt{M}}{2r^{3/2}} \frac{E}{A} \dot{M}.$$ \hspace{1cm} (39)

Note that $\Lambda$ is related to the torque $g$ by $g = r^2 \Lambda$.

To solve Equation (38), we introduce a function

$$\xi(r) = e^{\int P(r) dr}.$$ \hspace{1cm} (40)

Substituting the expression of $P(r)$ in Equation (39), Equation (40) can be rewritten as

$$\xi(r) = r^2 \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right) = r^2 A.$$ \hspace{1cm} (41)

Multiplying both sides of Equation (38) by the factor $\xi(r)$, we have

$$\frac{d\xi \Lambda}{dr} = \xi(r) Q(r).$$ \hspace{1cm} (42)

Integrating Equation (42) from $r_{ms}$ to $r$, one gets

$$\xi(r) \Lambda(r) - \xi(r_{ms}) \Lambda(r_{ms}) = \int_{r_{ms}}^{r} \xi(r) Q(r) dr,$$

which can be reduced to

$$\Lambda(r) = M \sqrt{\frac{M}{r^3} A + \frac{r_{ms}^2 A_{ms}}{r^2 A}} \Lambda_{ms},$$ \hspace{1cm} (44)

where $D = \frac{1}{2} \int_{r_{ms}}^{r} \frac{E}{r B} dr$.

Dividing both sides of Equation (44) by $4\pi h$, we have

$$\tau_{\phi\theta}(r) = \frac{M}{4\pi h} \left[ \sqrt{\frac{M}{r^3} A + \frac{r_{ms}^2 A_{ms}}{r^2 A}} \right] \tau_{\phi\theta}(r_{ms}).$$ \hspace{1cm} (45)

Multiplying Equation (45) by $4\pi r^2 h$ and using $g = 4\pi r^2 h \tau_{\phi\theta}$, we have

$$g(r) = M r^2 \left[ \sqrt{\frac{M}{r^3} A + \frac{A_{ms}}{A}} \right] g_{ms}.$$ \hspace{1cm} (46)

APPENDIX C

ANALYTICAL SOLUTIONS OF NDAF WITH A NON-ZERO BOUNDARY TORQUE

We dedicate this section to obtain analytical solutions of the nztNDAF model under the indication of previous numerical results. These analytical solutions are helpful to understand the characteristics of nztNDAF.

For convenience of illustration, here we rewrite some dynamical equations and other details. Combining Equations (3), (9), and (10), one gets an expression for the total pressure as

$$P = \left[ \frac{GM \rho^{1/2} C \varphi}{4\pi \alpha r^3} \right]^{2/3},$$ \hspace{1cm} (47)

where $\varphi \equiv D \left(1 + \frac{\Gamma_{ms}}{r_{ms} A_{ms}} \right)$. Note that the dynamical impact of the inner edge (i.e., $\eta$) is embedded in symbol $\varphi$.

As stated before, the total pressure $P$ is composed of several components such as the gas pressure, radiation pressure, degeneracy pressure, neutrino pressure, and magnetic pressure; this is expressed as

$$P = P_{gas} + P_{rad} + P_{deg} + P_{\nu} + P_{B};$$ \hspace{1cm} (48)

each term on the right side of the above expression is separatively given below:

$$P_{rad} = \frac{11}{12} \frac{\rho kT}{m_p},$$ \hspace{1cm} (49a)

$$P_{gas} = \frac{\rho k T}{m_p} \left(1 + \frac{3e_{max}}{4}\right),$$ \hspace{1cm} (49b)

$$P_{deg} = \frac{2\pi \hbar c}{3} \left(\frac{3}{8\pi m_p}\right)^{4/3} \left(\frac{\rho}{\mu_e}\right)^{4/3},$$ \hspace{1cm} (49c)

$$P_{\nu} = \frac{1}{3} \mu_{\nu},$$ \hspace{1cm} (49d)

$$P_{B} = \beta P.$$ \hspace{1cm} (49e)

Equation (48) attached with Equation (49) is known as the equation of state of the accretion flow.

The energy equation is expressed as

$$Q_{vis} = Q_{\gamma} + Q_{\gamma \gamma} + Q_{\gamma \gamma \gamma},$$ \hspace{1cm} (50)

in which the viscous heating rate is

$$Q_{vis} = \frac{3GM M \varphi}{8\pi r^3 B}.$$ \hspace{1cm} (51)
The cooling rate due to neutrino losses $Q_\nu$, photodisintegration $Q_{\text{photo}}$, and advection $Q_{\text{adv}}$ are expressed as

$$Q_\nu = \sum_i \frac{(7/8)\sigma T^4}{(3/4)(\tau_{\nu i} + 1/\sqrt{3} + 1/3\tau_{\bar{\nu} i})},$$

$$Q_{\text{photo}} \approx 10^{29} \rho_{10}^3 v_r dX_{\text{nuc}} / dr,$$

$$Q_{\text{adv}} = \sum \xi_i T \frac{d\xi_i}{dr} \approx \xi_i \frac{H}{r} \left( \frac{11}{3} a T^3 + \frac{3}{2} \frac{\rho k}{m_p} \frac{1 + X_{\text{nuc}}}{4} + \frac{4}{3} u_r \right),$$

where $\tau_{\nu i} = \tau_{\nu i} + \tau_{\bar{\nu} i}$ is the sum of the absorption and scattering optical depth for each neutrino flavor ($\nu_e$, $\nu_x$, $\bar{\nu}_e$). The absorption optical depth includes the contributions from the interaction of neutrinos with one another $\tau_{\nu_i \bar{\nu}_j}$ and the absorption onto protons or neutrons $\tau_{\nu_i \nu}$.

The expressions for these optical depth are

$$\tau_{\nu_i \nu} \approx 2.5 \times 10^{-7} T_1^2 h,$$

$$\tau_{\nu_i \nu} \approx 4.5 \times 10^{-7} T_1^2 X_{\text{nuc}} \rho_{10} h,$$

$$\tau_{\nu_i} \approx 2.7 \times 10^{-7} T_1^2 \rho_{10} h.$$  

Inspecting our numerical solutions, we found that the whole disk can be divided into several different characteristic regions, as shown in Figure 2 and Table 2. We will obtain the analytical solutions for each region in the following subsections.

C.1. Region I—Radiation-pressure-dominated ADAF

At large radii, the disk could be dominated by advection cooling, since the radiation cooling timescale is much longer than the accretion timescales. The mass density in this region is relatively small and the temperature is still so high that the radiation pressure dominates the accretion equation. The equation of state is $P \approx P_\text{rad} + P_\text{b} = (1 - \beta^{-1}) P_\text{rad}$ and the energy conservation equation is $Q_{\text{vis}} \approx Q_{\text{adv}}$. Therefore, we have

$$\left( \frac{G M M \rho^{1/2} C \varpi}{4 \pi \alpha r^3 A^2} \right)^{2/3} = (1 - \beta)^{-1/3} \frac{11}{12} a T^4;$$

$$\left( \frac{3 G M M \varpi}{8 \pi r^3 B} \right)^{2/3} = \frac{M}{4 \pi \alpha r^3} \frac{11}{3} a T^4.$$  

Note that the magnetic pressure is always kept in the deducing process for the convenience to analyze the effect of parameter $\beta$. However, we take $\beta = 0$ in the calculation later on due to its limited influence on the disk structure. From the above equations we can obtain an analytical solution of $\rho$ and $T$. Furthermore, substituting the expressions of $\rho$ and $T$ into Equations (10) and (37), one gets the solution of disk scalar height $h$ and advection velocity $u_r$. The expressions for these parameters are

$$\rho = 1.05 \times 10^{12} (1 - \beta)^{3/2} A^{-2} B^{3/2} \times C \varpi^{-1/2} \alpha^{-1/2} m^{-2} R^{-3/2} \text{ g cm}^{-3},$$

$$T = 4.76 \times 10^{11} (1 - \beta)^{3/2} A^{-1/2} B^{1/3} C^{1/3} \times \varpi^{-1/2} \alpha^{-1/2} \text{ cm}^{-3/2} \text{ K},$$

$$H = 0.61 \varpi^{-1/2} \alpha^{-1/2} (1 - \beta)^{-1/2} \varpi^{-1/2} R,$$

$$u_r = 1.12 \times 10^{10} (1 - \beta)^{-1} A^{3/2} R^{-3/2} C^{-1/2} \alpha^{-1/2} \text{ cm s}^{-1},$$

where $H \equiv h/r_\text{s}$ is the dimensionless disk height. The radial profile for these parameters are shown in Figures 11 and 12 (see Region I).

C.2. Region II—Transparent NDAF

Just inside region I (ADAF), the disk is an NDAF region since the temperature and density are high enough there to ignite neutrino cooling. The neutrino opacity is not important in this region, so it is a $\nu$-transparent NDAF. For such thin disks, cooling by pair capture on nucleons $Q_\text{nuc}$ should dominate over by electron–positron pair annihilation; the neutrino cooling rate in Equation (52) can be reduced to

$$Q_{\nu} \approx Q_{\text{nuc}} \approx \frac{(7/8)\sigma T^4}{(3/4)(1/3\tau_{\nu i})} = C_1 \rho T^4 X_{\text{nuc}} h \text{ erg cm}^{-3} \text{ s}^{-1},$$

where $C_1 = 9.0 \times 10^{-43}$, and we approximately take $X_{\text{nuc}} \approx 1$ hereafter. The pressure is dominated by gas pressure, i.e., $P \approx (1 - \beta)^{-1} P_\text{gas}$, and we have

$$\left( \frac{G M M \rho^{1/2} C \varpi}{4 \pi \alpha r^3 A^2} \right)^{2/3} = (1 - \beta)^{-1} \frac{\rho k T}{m_p}.$$
Figure 11. Radial distribution of pressure, neutrino optical depth, and the cooling rate for different mass accretion rates (\(\dot{m} = 0.1, 1\)) and inner edge torques (\(h = 0, 3\)), the rest parameters are \(m = 7, a_0 = 0.9, \alpha = 0.1\), and \(\beta = 0\). The numerical and analytical results are separately denoted by gray lines and black lines. The vertical dashed lines denote the transition between two neighboring regions. The inner radiation-pressure-dominated ADAF (region V) can appear only if the unstable NDAF region exists in the disk due to a high inner edge torque, which can be certified from the bottom-right graphic above, in which the region V starts from \(r = r_{0.2}\), it is too close to \(r_{	ext{ms}}\) to be exhibited in the figure. Similarly, the width of region VI existing in the first and third row is nearly zero so that our analytical solution can hardly capture it. For a much clearer recognition of region V and VI, one can refer to Figures 13 and 14, corresponding to much more extreme parameters.
The neutrino cooling term dominates now, therefore $Q_{\nu e} = Q_{\nu e}^o$, or,

$$\frac{3GMM}{8\pi r^3} \cdot \frac{\mathcal{D}}{B} = C_1 \rho T^6 \frac{(1 - \beta)^{-1}P_{\text{gas}}r^3}{\rho GM} \sqrt{B/C} = C_1 \rho T^6 \frac{(1 - \beta)^{-1}kT^3}{GMm_p} \sqrt{B/C}. \quad (58)$$

The solutions can be worked out in the same way as for region I. The expressions are collected as follows:

$$\rho = 2.02 \times 10^{14}(1 - \beta)^{9/5}A^{-13/5}B^{9/20}C^{23/20} \times \mathcal{D}^{-1/10}m^{-1/10}R^{-51/20} \text{ g cm}^{-3}, \quad (59a)$$

$$T = 1.23 \times 10^1 \frac{(1 - \beta)^{-1/5}A^{2/5}B^{-3/10}}{C^{-1/10}m^{1/5}R^{-3/10}} \text{ K}. \quad (59b)$$
In our analytical calculations, the transition between region II and region I is roughly determined by $P_{\text{rad}}/P = 0.2$; this value is chosen so as to minimize the gap from region I to

$$H = 0.11 (1 - \beta)^{-3/5} A^{1/5} B^{7/20} C^{-11/20} \alpha^{1/10} m^{-1/10} R^{27/20},$$

$$P_{\text{rad}}/P = 7.70 \times 10^{-4} (1 - \beta)^{-7/5} A^{19/5} B^{-27/20}$$

$$\times C^{-29/20} \alpha^{-1/10} m^{11/10} R^{33/20},$$

$$\tau_{\text{vis}} = 3.44 \times 10^{2} (1 - \beta)^{3/5} A^{-8/5} B^{4/5}$$

$$\times C^{2/5} \alpha^{-4/5} m^{-6/5} R^{-9/5},$$

$$Q_{\text{vis}}^- = Q_{\text{vis}}^+ = 9.79 \times 10^{42} B^{-1} \alpha^{-4/5} m^{-2} R^{-3} \text{ erg cm}^{-2} \text{ s}^{-1},$$

$$\dot{m} = \dot{m}_{\text{vis}} = 3.8 \times 10^{-3} B^{-1} \alpha^{-4/5} m^{-1} R^{-1} \text{ erg cm}^{-2} \text{ s}^{-1}.$$
region II. For the same reason, we take the position where \( \tau_\nu = 2.5 \) as the transition between region II (transparent NDAF) and region III (opaque NDAF).

C.3. Region III—Opaque NDAF

Going further inward, the mass density and temperature gradually increase until the neutrino opacity becomes important. The disk enters region III, i.e., a \( \nu \)-opaque NDAF. The gas pressure still dominates the flow in this region. As the neutrino optical depth is so high, Equation (52) can be reduced to

\[
Q_\nu^- = \frac{7}{3} \sum_i \sigma T_i^4 \sigma T_i^4 \left( \frac{1}{\tau_{a, \nu}} + \frac{1}{\tau_{s, \nu}} + \frac{1}{\tau_{\nu}} \right)
\]

Note that we ignore \( \tau_{a, \nu} \) here. Substituting Equations 53(b) and (c) into Equation (60), we get

\[
Q_\nu^- = 1.16 \times 10^{35} \rho^{-1} T^2 h^{-1} \text{ erg s}^{-1}.
\]

From the above discussions, we have the equations \( P \approx (1 - \beta)^{-1} P_{\text{gas}} \) and \( Q_{\text{vis}} = Q_\nu^- \) in this region. They can be rewritten as

\[
\left( \frac{G M M \rho^{5/2} C D}{4 \pi \alpha r^3} \right)^{2/3} \approx (1 - \beta)^{-1} \frac{b k T}{m_p},
\]

\[
\left( \frac{3 G M M D}{8 \pi r^3} \right) \approx 1.16 \times 10^{35} \rho^{-1} T^2 h^{-1}
\]

\[
\approx 1.16 \times 10^{35} \rho^{-1} T^2 \left( \frac{G M M p_C}{r^3 (1 - \beta)^{-1} k T B} \right)^{1/2}.
\]

The solutions in this region are collected as follows:

\[
\rho = 1.52 \times 10^2 (1 - \beta) A^{-1} B^{1/4}
\]

\[
\times C^{3/4} \alpha^{-1/2} m^{-1/2} R^{-3/4} \text{ g cm}^{-3},
\]

\[
T = 3.20 \times 10^2 (1 - \beta)^{1/3} A^{-2/3} B^{-1/6}
\]

\[
\times C^{1/6} \alpha^{-3/2} \alpha^{-1/3} m^{-1/3} R^{-3/4} \text{ K},
\]

\[
H = 0.54 (1 - \beta)^{-1/3} A^{-1/3} B^{5/12}
\]

\[
\times C^{-5/12} \alpha^{1/3} m^{-1/3} R^{1/4},
\]

\[
u' = 8.82 \times 10^9 (1 - \beta)^{-1/2} A^{-1/3} B^{-2/3}
\]

\[
\times C^{-1/3} D^{1/3} m^{1/3} R^{-1} \text{ cm s}^{-1},
\]

\[
Q_\nu^- = Q_{\text{vis}}^+ = 9.79 \times 10^{32} B^{-1} m^{-2} R^{-3} \text{ erg cm}^{-2} \text{ s}^{-1},
\]

\[
P_{\text{rad}} / P = 6.61 \times 10^2 (1 - \beta) A^{-1} B^{-3/4}
\]

\[
\times C^{-1/4} D^{1/2} \alpha^{-1/2} m^{-5/4} m^{-2} R^{-15/4},
\]

\[
P_{\text{des}} / P_{\text{gas}} = 2.12 \times 10^{-2} A^{1/4}
\]

\[
\times C^{1/12} D^{1/2} \alpha^{-1/8} m^{1/6} m^{-2} R^{5/4}.
\]

If there is a significant non-zero torque at the inner edge, a new solution (region IV) exists inside region III. In this region IV (see the next subsection), the disk is too hot to be dominated by radiation pressure, and thus becomes unstable. The dividing line between region IV and region III is the position satisfying \( P_{\text{rad}} / P = 0.2 \). But, when the boundary torque vanishes or is too small, there will be another solution (region VI) just inside region III, where the degeneracy pressure becomes comparable with the gas pressure. The dividing line between region VI and region III is given by \( P_{\text{des}} / P_{\text{gas}} = 1 \).

C.4. Region IV—Unstable NDAF

From the results of numerical solutions, we found that a viscously unstable region will appear in the inner vicinity of the gray NDAF if the inner edge torque is large enough (e.g., see \( m = 1, \eta = 3 \) compared with \( m = 1, \eta = 0 \) in Figures 11 and 12), since the temperature in this region is so efficiently increased due to the additional viscous heating due to the boundary torque that the radiation pressure dominates the neutrino cooling flow. The neutrino energy density, i.e., \( u_\nu = (7/8) a T^4 \sum (\tau_\nu/2 + 1/\sqrt{3})/(\tau_\nu/2 + 1/\sqrt{3} + 1/3 \tau_{s, \nu}) \) is reduced to 21aT^4/8. Thus, the neutrino pressure in this region can be simply written as \( P_\nu = u_\nu / 3 \approx 7a T^4 / 8 \), which is comparable to the radiation pressure. In this case, the equation of state \( P \approx (1 - \beta)^{-1} (P_{\text{gas}} + P_{\nu}) \) and energy conservation \( Q_{\text{vis}} \approx Q_\nu^- \) are expressed as

\[
\left( \frac{G M M \rho^{1/2} C D}{4 \pi \alpha r^3} \right)^{2/3} \approx (1 - \beta)^{-1} \frac{43}{24} a T^4,
\]

\[
\frac{3 G M M D}{8 \pi r^3} \approx 1.16 \times 10^{35} \rho^{-1} T^2 h^{-1}
\]

\[
\approx 1.16 \times 10^{35} \rho^{-1} T^2 \left( \frac{24 G M M p_C}{43 a T^4 r^3} \right)^{1/2}.
\]

The solutions for this unstable NDAF are

\[
\rho = 4.27 \times 10^5 (1 - \beta) B C D^{2} m^{-1} m^{-2} R^{3} \text{ g cm}^{-3},
\]

\[
T = 2.10 \times 10^{11} (1 - \beta)^{1/3} A^{-1/3} B^{1/12}
\]

\[
\times C^{1/4} \alpha^{-1/6} m^{-1/8} R^{-1/4} \text{ K},
\]

\[
H = 8.28 (1 - \beta)^{-1/3} A^{-2/3} B^{-1/6} C^{-1/2} D^{1/3} \alpha^{-1/3} m^{-1} R^{-1/2},
\]

\[
u' = 2.05 \times 10^{12} (1 - \beta)^{-2/3} A^{2} B^{-7/6}
\]

\[
\times C^{-1/2} D^{1/3} m^{2} R^{-7/2} \text{ cm s}^{-1},
\]

\[
Q_\nu^- = Q_{\text{vis}}^+ = 9.79 \times 10^{32} (1 - \beta)^{-2/3} B^{-1}
\]

\[
\times D^{1/3} m^{2} R^{-3} \text{ erg cm}^{-2} \text{ s}^{-1},
\]

\[
\Sigma = 1.04 \times 10^{15} (1 - \beta)^{-2/3} A^{2} B^{7/6} C^{1/2}
\]

\[
\times D^{-1/2} m^{1/3} R^{-1/2} \text{ g cm}^{-2},
\]

\[
\partial M / \partial \Sigma = -1.91 \times 10^{18} (1 - \beta)^{-2/3} A^{2} B^{-7/6}
\]

\[
\times C^{-1/2} D^{1/3} m^{2} R^{-5/2} \text{ cm s}^{-2},
\]

\[
t_{\text{vis}} = 7.19 \times 10^8 (1 - \beta)^{-2/3} A^{2} B^{-1/3}
\]

\[
\times C^{-2} D^{-1/2} m^{2} R^{-1/2} \text{ s},
\]

\[
P_{\text{gas}} / P = 2.81 \times 10^{-4} (1 - \beta) A B^{3} C^{1/4}
\]

\[
\times D^{-2} m^{2} R^{5/4}.
\]

\[
f \equiv Q_{\text{adv}} / Q_{\text{vis}}^+ = 1.23 \times 10^{2} (1 - \beta)^{-2/3} A^{-4/3}
\]

\[
\times B^{1} D^{1/2} m^{2} R^{-3}.
\]

We find that \( \Sigma \approx m^{-1} \) and \( \partial M / \partial \Sigma < 0 \). Therefore, this region is viscously unstable. The timescale of the instability \( t_{\text{vis}} \) in Equation 65(h) has been evaluated as the viscous timescale.
i.e., $t_{\text{vis}} \sim r^2/\nu = (\alpha \Omega_0)^{-1}(h/r)^{-2}$. For $\eta = 3$ and $r = 3r_g$, $t_{\text{vis}} \approx 6.2$ ms and is 24.7 ms for $\dot{m} = 0.5$, which are consistent with the numerical results in Section 5.

Inside this unstable region, there is a new solution (region V) where the advection cooling dominates over radiation cooling. The solution transits from IV to V at the location satisfying $Q_{\text{adv}}/Q_{\text{vis}} = 0.9$.

C.5. Region V—Radiation and Neutrino Pressures Dominated ADAF

Inside region IV, there is a region where the neutrino optical depth is so high that the neutrinos are trapped, resulting in an advection-cooling-dominated flow. The radiation and neutrino pressures are still dominated due to the high temperature and large neutrino opacity. The analytical solutions are nearly same as those in region I, excepting the discrepancy in the coefficients due to the consideration of neutrino pressure.

\[
\rho = 5.84 \times 10^{11}(1 - \beta)^{3/2}A^{-2/3}B^{3/2} \\
\times C^{-1/2}\alpha^{-1}m^{-2}mR^{-3/2} \text{ g cm}^{-3},
\]

\[
T = 3.83 \times 10^{11}(1 - \beta)^{-2/3}B^{1/3}C^{1/4} \\
\times \sqrt{3}/\alpha^{-1/2}m^{-1/2}mR^{1/4}R^{-5/8} \text{ K}.
\]

\[
H = 0.75(1 - \beta)^{-1/2}C^{-1/2}g^{1/2}R,
\]

\[
\dot{u} = 1.67 \times 10^{10}(1 - \beta)^{-1}A^{1/2}B^{-1/2}C^{-1/2}\alpha^{-1}R^{-1/2} \text{ cm s}^{-1},
\]

\[
f_{\nu} = \frac{Q_{\nu}}{Q_{\text{vis}}} = 2.70 \times 10^{-2}(1 - \beta)^{-1/4} \\
\times AB^{-1/4}g^{1/4}\alpha^{1/2}m^{-3/2}R^{9/4}.
\]

This region is usually too narrow to be picked out (e.g., see the panels of Figures 11 and 12 in which $\dot{m} = 1, \eta = 3$, as an example, the region V starts from $r = 26r_g$). To show this region V, we solve a disk with extremely large $\dot{m}$ and $\eta$, as illustrated in Figure 13.

C.6. Region VI—Gas and Degeneracy Pressure-dominated ADAF

Note that regions IV and V usually exist when the inner edge torque is great enough. For NDAF without a boundary torque, these two regions will be replaced with another ADAF region dominated by degeneracy pressure as well as gas pressure. In order to grasp the key property of such an innermost region analytically, we just keep the degeneracy pressure in the equation of state, i.e., $P \approx (1 - \beta)^{-1}P_{\text{deg}}$. While as for the energy equation $Q_{\nu}^+ \approx Q_{\text{adv}}$, we note that the degeneracy pressure has no contribution to the advection term. So one has

\[
\left(\frac{GMM}{A^2} \rho^{1/2} C \sqrt{A^2}{\frac{4\pi G}{3}} \right)^{2/3} \approx 4.89 \times 10^{14}(1 - \beta)^{-1/4},
\]

\[
\frac{3GMM}{B} \approx \frac{M}{8\pi^2} \frac{3}{2} \rho K T \frac{3}{4} \frac{m}{m_p}.
\]

The solutions of this region are

\[
\rho = 7.15 \times 10^{13}(1 - \beta)A^{4/3}C^{2/3} \\
\times g^{2/3}\alpha^{-2/3}m^{-4/3}m^{2/3}R^{-2} \text{ g cm}^{-3},
\]
From Figure 14, we find that the width of region VI is close to zero, even at extremely large accretion rates, so this region is hard to exhibit in most cases.

C.7. Effects of The Inner Edge Torque

First, the existence of the inner edge torque can reduce the ignition accretion rate, as shown in the first column of Figure 12. At relative lower accretion rate $\dot{m} = 0.01$, the entire disk is ADAF when there is no inner edge torque. However, once the inner edge torque increases to $\eta = 0.5$, an NDAF region emerges in the inner disk. Furthermore, the NDAF region expands outward, accompanying the increase of $\eta$, since the increase of $\eta$ leads to a larger mass density (see Equation 65(a)) or a higher temperature (see Equation 63(b)).

Second, the inner edge torque can enhance the neutrino luminosity effectively. This conclusion can be found by examining Equations 59(e), 63(e), and 65(e).

Third, the inner edge torque can cause the NDAF to be viscously unstable. As shown in the fourth column of Figure 12, for $\dot{m} = 1$, when the inner edge torque $\eta$ increases to $\eta = 0.5$, an unstable region starts to emerge in the inner disk if $\eta > 0.5$ and expands with the increase of $\eta$.

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