Minimum-cost matching in a regular bipartite graph with random costs

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Abstract
Let $G$ be an arbitrary $d$-regular bipartite graph on $2N$ vertices. Suppose that each edge $e$ is given an independent uniform exponential rate one cost. Let $C(G)$ denote the expected length of the minimum cost perfect matching. We show that if $d = d(N) \to \infty$ as $N \to \infty$, then $EC(G) = (1 + o(1)) \frac{N}{d} \frac{e^2}{6}$. This generalises the well-known result for the case $G = K_{N,N}$.

1 Introduction

There are many results concerning the optimal value of combinatorial optimization problems with random costs. Sometimes the costs are associated with $n$ points generated uniformly at random in the unit square $[0,1]^2$. In which case the most celebrated result is due to Beardwood, Halton and Hattersley [3] who showed that the minimum length of a tour through the points a.s. grew as $\beta n^{1/2}$ for some still unknown $\beta$. For more on this and related topics see Steele [15].

The optimisation problem in [3] is defined by the distances between the points. So, it is defined by a random matrix where the entries are highly correlated. There have been many examples considered where the matrix of costs contains independent entries. Aside from the Travelling Salesperson Problem, the most studied problems in Combinatorial Optimization are perhaps, the shortest path problem; the minimum spanning tree problem and the matching problem. As a first example, consider the shortest path problem in the complete graph $K_n$ where the edge lengths are independent exponential random variables with rate 1. We denote the exponential random variable with rate $\lambda$ by $E(\lambda)$. Thus $\Pr(E(\lambda) \geq x) = e^{-\lambda x}$ for $x \in R$. Janson [9] proved (among other things) that if $X_{i,j}$ denotes the shortest distance between vertices $i, j$ in this model then $\mathbb{E}[X_{1,2}] = \frac{H_n}{n}$ where $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

As far as the spanning tree problem is concerned, the first relevant result is due to Frieze [6]. He showed that if the edges of the complete graph are given independent uniform $[0,1]$ edge weights,

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then the (random) minimum length of a spanning tree $L_n$ satisfies $E[L_n] \rightarrow \zeta(3) = \sum_{i=1}^{\infty} \frac{1}{i^3}$ as $n \rightarrow \infty$. Further results on this question can be found in Janson [8], Beveridge, Frieze and McDiarmid [4], Frieze, Ruszinko and Thoma [7] and Cooper, Frieze, Ince, Janson and Spencer [5].

In the case of matchings, the nicest results concern the minimum cost of a perfect matching when edges are given independent exponential $E(1)$ random variables then the story begins with Walkup [16] who proved that $E[C_n] \leq 3$. Later Karp [10] proved that $E[C_n] \leq 2$. Aldous [1], [2] proved that $\lim_{n \rightarrow \infty} E[C_n] = \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Parisi [13] conjectured that in fact $E[C_n] = \sum_{k=1}^{\infty} \frac{1}{k^2}$. This was proved independently by Linusson and Wästlund [11] and by Nair, Prabhakar and Sharma [12]. A short elegant proof was given by Wästlund [14].

In the paper [4] on the minimum spanning tree problem, the complete graph was replaced by a $d$-regular graph $G$. Under some mild expansion assumptions, it was shown that if $d \rightarrow \infty$ then $\zeta(3)$ can be replaced asymptotically by $\frac{1}{d^2} \zeta(3)$. It is the aim of this paper to prove a similar result in the context of bipartite matching.

Therefore, consider a $d$-regular bipartite graph $G$ on $2N$ vertices. Here $d = d(N) \rightarrow \infty$ as $N \rightarrow \infty$. Each edge $e$ is assigned a cost $X_e$, each independently chosen according to the exponential distribution $E(1)$. Denote the total cost of the minimum-cost perfect matching by $C(G)$.

**Theorem 1.** Suppose $d = d(N) \rightarrow \infty$ as $N \rightarrow \infty$. For any $d$-regular bipartite $G$,

$$E[C(G)] = (1 + o(1)) \frac{N \pi^2}{d^2}.$$  

Here the $o(1)$ term goes to zero as $N \rightarrow \infty$.

As an example, consider the $N$-cube $Q_N$ with vertex set $\{0,1\}^N$ and where there is an edge $\{x, y\}$ if the Hamming distance between $x$ and $y$ is precisely one.

**Corollary 1.**

$$E[C(Q_N))] = (1 + o(1)) \frac{2^N \pi^2}{N}.$$  

### 2 Proof of Theorem 1

We first prove a structural lemma. After this we will adapt Wästlund’s proof [14] to arbitrary $d$-regular graphs.

**Lemma 1.** Let $\varepsilon > 0$, and let $G = (V, E)$ be a $d$-regular bipartite graph on $|V| = 2N$ vertices. Then for $d$ sufficiently large, there exists a sequence $V_N \subseteq V_{N+1} \subseteq \cdots \subseteq V_{2N} = V$ such that $|V_t| = t$ and for $G_t = G[V_t]$ we have

$$|\deg_{G_t}(v) - \frac{dt}{2N}| \leq \varepsilon d, \quad \forall v \in V_t$$  

for $t = N, N + 1, \ldots, 2N$.

**Proof.** Assign a value $0 \leq p_v \leq 1$ to each $v \in V$, chosen independently and uniformly at random from $[0, 1]$. With probability one, $p_v \neq p_w$ for all pairs of vertices $v \neq w$. Define $V_p = \{v \in V : p \geq$
1 − \(p_v\) and \(G_p = G[V_p]\). For \(t = N, N + 1, \ldots, 2N\) define \(V_t = V_{p(t)}\) where \(p(t) = \min\{p : |V_p| = t\}\. Then \(V_N \subseteq \cdots \subseteq V_{2N}\) and \(|V_t| = t\) for all \(t\).

Fix \(0 < \delta < 1/10\) for the remainder of the proof. The following fact will be used repeatedly. Let \(X_1, \ldots X_n\) be drawn from \([0,1]\) uniformly at random. Relabel the variables so that \(X_1 \leq \cdots \leq X_n\). Then as \(n \to \infty\), for any \(\delta > 0\)

\[
\Pr\left\{ \exists i \in \left\{ \frac{n}{4}, \ldots, n \right\} : \left| X_i - \frac{i}{n} \right| > \frac{\delta i}{n} \right\} \leq e^{-\Omega(n)}
\]  

(3)

Indeed, for \(i \geq n/4\), the Chernoff bounds imply that

\[
\Pr \left\{ X_i < (1 - \delta) \frac{i}{n} \right\} = \Pr \left\{ \text{Bin} \left( n, \left(1 - \frac{\delta}{n} \right) \frac{i}{n} \right) \geq i \right\} \leq \exp \left\{ - \frac{\delta^2 n}{3} \right\}
\]

(4)

and

\[
\Pr \left\{ X_i > (1 + \delta) \frac{i}{n} \right\} = \Pr \left\{ \text{Bin} \left( n, \left(1 + \frac{\delta}{n} \right) \frac{i}{n} \right) \leq i \right\} \leq \exp \left\{ - \frac{\delta^2 n}{8} \right\}
\]

(5)

Summing over \(i = n/4, \ldots, n\) gives

\[
\Pr \left\{ \exists i : \left| X_i - \frac{i}{n} \right| > \frac{\delta i}{n} \right\} \leq ne^{-\delta^2 n/3} + ne^{-\delta^2 n/8} = e^{-\Omega(n)}
\]

(6)

Fix a vertex \(v \in V\). Its degree in \(G_p\) is given by \(|N_G(v) \cap V_p|\), and can be written as \(X = \sum_{w \in N_G(v)} X_w\), where \(X_w\) is the indicator variable for \(\{w \in V_p\} = \{w : p \geq 1 - p_w\}\). Let \(B_v\) be the event that for some \(p \in [1/3, 1]\), \(|\deg_{V_p}(v) - dp| > 2\delta d\). Order \(\{p_w : w \in N_G(v)\}\) as \(p_1 \leq \cdots \leq p_d\), write \(p_0 = 0\), and condition on the event \(C_v = \{|p_i - i/d| \leq \delta i/d, \; i = d/4, \ldots, d\}\). The degree of \(v\) in \(G_p\) is given by

\[
\deg_{G_p}(v) = d - \max\{i : p < 1 - p_i\}
\]

(7)

Let \(i \geq d/4\). Since \(p_i \geq (1 - \delta)i/d\, if \, 1 - p_i > p\) then

\[
p < 1 - (1 - \delta)\frac{i}{d} \Rightarrow i < d(1-p)(1+2\delta)
\]

(8)

and similarly \(i \geq d(1-p)(1-\delta)\, if \, 1 - p_i \leq p\). So if \(p < 1 - p_i\) while \(p \geq 1 - p_{i+1}\) for some \(i \geq d/4\), then

\[
d(1-p)(1-\delta) - 1 \leq i < d(1-p)(1+2\delta)
\]

(9)

and

\[
|\deg_{G_p}(v) - dp| = |d - i - dp| \leq 2\delta d.
\]

(10)

It remains to note that if \(p \geq 1/3\), then \(p_{d/4} \leq (1+\delta)/4 < 1/3\). So \(i \geq d/4\) and (10) holds for all \(p \geq 1/3\), conditioning on \(C_v\). In other words \(C_v\) implies \(B_v\) and \(\Pr \{B_v\} \leq \Pr \{C_v\} = e^{-\Omega(d)}\), by (3).

The event \(B_v\) is generated by the \(d\) variables \(\{X_w : w \in N_G(v)\}\). Define a dependency graph \(D\) on the events \(B_v, v \in V\) by letting \(B_u, B_v\) be adjacent if and only if \(N_G(u) \cap N_G(v) \neq \emptyset\). The maximum degree of \(D\) is \(d^2\) since \(G\) is \(d\)-regular. Let \(x_v = 1/d^2\) for each \(v \in V\). Then, for large \(d\),

\[
\Pr \{B_v\} = e^{-\Omega(d)} \leq \frac{1}{d^2} \left( 1 - \frac{1}{d^2} \right)^{d^2} = x_v \prod_{B_u \sim B_v} (1 - x_u)
\]

(11)
and so by the Lovasz Local Lemma,
\[
\Pr\left\{ \bigcap_{v \in V} \mathcal{B}_v \right\} \geq \left(1 - \frac{1}{d^2}\right)^{2N} \geq \exp\left\{-\frac{N}{d^2}\right\}
\] (12)

Now let \( \mathcal{A} \) be the event that for some \( t \geq N/2 \), \(|p(t) - t/2N| > \delta t/2N \). By (3) we have \( \Pr\{\mathcal{A}\} = e^{-\Omega(N)} \). Since \( e^{-\Omega(N)} = o(e^{-N/d^2}) \), for large \( N \)
\[
\Pr\left\{ \mathcal{A} \cap \bigcap_{v \in V} \mathcal{B}_v \right\} > 0
\] (13)

This means that there exists a choice of \( \{p_v : v \in V\} \) such that \(|p(t) - t/2N| \leq \delta t/2N \) for \( t = N, \ldots, 2N \) and \(|\deg_{G_{p}}(v) - dp| \leq 2\delta d \) for all \( v \) and all \( p \in [1/3, 1] \). Recall that \( G_t = G_{p(t)} \) by definition, and note that \( p(N) \geq (1 - \delta)/2 > 1/3 \). So for all \( t \geq N \) and all \( v \),
\[
\left| \deg_{G_t}(v) - \frac{dt}{2N} \right| \leq \left| \deg_{G_{p(t)}}(v) - dp(t) \right| + \left| dp(t) - \frac{dt}{2N} \right|
\]
\[
\leq 2\delta d + \frac{\epsilon t}{2N}
\]
\[
\leq 3\delta d
\]

Using Lemma 1, we find that the proof in [14] can be adapted to our current situation.

Fix a small \( \epsilon > 0 \) and a sequence \( G_N, G_{N+1}, \ldots, G_{2N} = G \) as in Lemma 1. Fix \( t \in \{N + 1, N + 2, \ldots, 2N\} \). Say \( G_t \) has bipartition \( A_t \cup B_t \) with \( A_t = \{a_1, \ldots, a_m\} \), \( B_t = \{b_1, \ldots, b_n\} \), \( m + n = t \).
We have \( \sum_{v \in A_t} \deg(v) = \sum_{w \in B_t} \deg(v) \) in \( G_t \), so by Lemma 1 we can assume that
\[
m(1 - \epsilon)\frac{dt}{2N} \leq \sum_{v \in A_t} \deg(v) = \sum_{w \in B_t} \deg(w) \leq n(1 + \epsilon)\frac{dt}{2N}
\] (14)

which implies \( m/n \leq (1 + \epsilon)/(1 - \epsilon) \leq 1 + 3\epsilon \) for small \( \epsilon \). By symmetry, \( n/m \leq 1 + 3\epsilon \), and so \( m + n = t \) implies
\[
|m - t/2| \leq 2\epsilon t \quad \text{and} \quad |n - t/2| \leq 2\epsilon t.
\] (15)

Each edge of \( G \) is assigned an independent \( E(1) \) cost. Edges in \( G_t \) are assigned the same cost as in \( G \). With probability one, no two distinct sets of edges are assigned the same total cost. In \( G_t \), let \( \sigma_t \) denote the minimal \( r \)-assignment, \( r \leq \min\{m, n\} \). In this context, an \( r \)-assignment is a matching of size \( r \). Define \( C_{r,t} \) to be the total cost of the minimal \( r \)-assignment in \( G_t \). We say that a vertex \( v \) participates in \( \sigma_r \) if it is adjacent to an edge of \( \sigma_r \).

**Lemma 2.** Let \( r < \min\{m, n\} \). Every vertex that participates in \( \sigma_r \) also participates in \( \sigma_{r+1} \).

**Proof.** The proof is copied verbatim from [14]. Let \( H \) be the symmetric difference \( \sigma_r \triangle \sigma_{r+1} \) of \( \sigma_r \) and \( \sigma_{r+1} \), in other words the set of edges that belong to one of them but not to the other. Since no vertex has degree more than 2, \( H \) consists of paths and cycles. We claim that \( H \) consists of a single path. If this would not be the case, then it would be possible to find a subset \( H_1 \subseteq H \) consisting of one or two components of \( H \) (a cycle or two paths) such that \( H_1 \) contains equally many edges.
from \( \sigma_r \) and \( \sigma_{r+1} \). As observed, with probability one, the edge sets \( H_1 \cap \sigma_r \) and \( H_1 \cap \sigma_{r+1} \) cannot have equal total cost. Therefore either \( H_1 \triangle \sigma_r \) has smaller cost than \( \sigma_r \), or \( H_1 \triangle \sigma_{r+1} \) has smaller cost than \( \sigma_{r+1} \), a contradiction. The fact that \( H \) is a path implies the statement of the lemma. \( \square \)

Define \( T \subseteq \{N + 1, \ldots, 2N\} \) to be the set of \( t \) such that \( |A_t| = |A_{t-1}| \) and \( |B_t| = |B_{t-1}| + 1 \). For \( t \in T \), consider a modified version of \( G_t \), obtained by adding a special vertex \( a_{m+1} \) to \( A_t \), with edges to all \( n \) vertices of \( B_t \). Each edge adjacent to \( a_{m+1} \) is assigned an \( E(\lambda) \) cost independently, \( \lambda > 0 \).

**Lemma 3.** Suppose \( r < \min\{m, n\} \). Condition on the event that \( a_{m+1} \) does not participate in \( \sigma_r \). Then the probability that it participates in \( \sigma_{r+1} \) is

\[
\frac{\lambda}{dN^{-1}(m-r)(1+e_r) + \lambda}
\]

where \( |e_r| \leq 5\varepsilon \).

**Proof.** The proof again follows [14] closely. Suppose w.l.o.g. that \( a_1, \ldots, a_r \) participate in \( \sigma_r \). Form a contraction \( G'_t \) of \( G_t \) by identifying the vertices \( a_{r+1}, \ldots, a_{m+1} \) to a vertex \( a'_{r+1} \). By Lemma 2, \( a_1, \ldots, a_r \) all participate in \( \sigma_{r+1} \), so there is a unique edge from \( B_t \) to \( a'_{r+1} \) in \( \sigma_{r+1} \). In \( G_t \), this edge must correspond to a unique edge from \( \{a_{r+1}, \ldots, a_{m+1}\} \) to \( B_t \).

By Lemma 1, there are \( (m-r)dt(1+e_1)/2N \) edges from \( \{a_{r+1}, \ldots, a_m\} \) to \( B_t \), \( |e_1| \leq \varepsilon \). Each edge has cost \( E(1) \), and the \( n \) edges from \( a_{m+1} \) to \( B_t \) have cost \( E(\lambda) \). It follows that the probability that the unique edge from \( B_t \) to \( \{a_{r+1}, \ldots, a_{m+1}\} \) is adjacent to \( a_{m+1} \) is given by

\[
\frac{\lambda}{dN^{-1}(m-r)(1+e_1) + \lambda} = \frac{\lambda}{dN^{-1}(m-r)(1+e_1)(1+e_2) + \lambda}
\]

where we write \( t/2n = 1 + e_2 \). Now (15) implies \( |e_2| = |t/2n - 1| \leq 3\varepsilon \), so \( (1+e_1)(1+e_2) = 1+e_r \), \( |e_r| \leq 5\varepsilon \) for small \( \varepsilon \). \( \square \)

**Corollary 2.** The probability that \( a_{m+1} \) participates in \( \sigma_k \) is given by

\[
\frac{N}{d} \left( \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{m-k+1} \right) (1+e_k)\lambda + O(\lambda^2)
\]

as \( \lambda \to 0 \), where \( |e_k| \leq 5\varepsilon \).

**Proof.** Let \( \nu(j) = dN^{-1}(m-j)(1+e_j) \), \( |e_j| \leq 5\varepsilon \). Then the probability is given by

\[
1 - \frac{\nu(0)}{\nu(0) + \lambda} \cdot \frac{\nu(1)}{\nu(1) + \lambda} \cdots \frac{\nu(k-1)}{\nu(k-1) + \lambda}
= 1 - \left( 1 + \frac{\lambda}{\nu(0)} \right)^{-1} \cdots \left( 1 + \frac{\lambda}{\nu(k-1)} \right)^{-1}
= \left( \frac{1}{\nu(0)} + \frac{1}{\nu(1)} + \cdots + \frac{1}{\nu(k-1)} \right) \lambda + O(\lambda^2)
= \frac{N}{d} \left( \frac{1}{m(1+e_0)} + \frac{1}{(m-1)(1+e_1)} + \cdots + \frac{1}{(m-k+1)(1+e_{k-1})} \right) \lambda + O(\lambda^2)
\]

and each error factor satisfies \( |1 - 1/(1+e_j)| \leq 5\varepsilon \). \( \square \)
Lemma 4. If \( t \leq \min\{m, n\} \) and \( t \in T \),

\[
E[C_{k,t} - C_{k-1,t-1}] = \frac{N}{dn} \sum_{j=0}^{k-1} \frac{1}{m-j} (1 + e_k)
\]

where \(|e_k| \leq 5\varepsilon\).

Proof. Let \( X \) be the cost of the minimum \( k \)-assignment in \( G_t \), and let \( Y \) be the cost of the minimum \((k-1)\)-assignment in \( G_{t-1} \). The variables \( X \) and \( Y \) are essentially the same as \( C_{k,t} \) and \( C_{k-1,t-1} \), but also coupled in a natural way, since the cost of any \( e \in E(G_t) \cap E(G_{t-1}) \) is the same in both \( G_t \) and \( G_{t-1} \).

Assume w.l.o.g. that \( B_{t-1} = \{b_1, \ldots, b_{n-1}\} \) and \( B_t = \{b_1, \ldots, b_n\} \). Let \( w \) denote the cost of the edge \( \{a_{m+1}, b_n\} \), and let \( I \) denote the indicator variable for the event that the cost of the cheapest \( k \)-assignment that contains this edge is smaller than the cost of the cheapest \( k \)-assignment that does not use \( a_{m+1} \). In other words, \( I \) is the indicator variable for the event \( \{Y + w < X\} \).

It follows from Corollary 2 that the probability that \( \{a_{m+1}, b_n\} \) participates in the minimum \( k \)-assignment \( \sigma_k \) of \( G_t \) is given by

\[
\frac{N}{dn} \left( \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{m-k+1} \right) (1 + e_k) \lambda + O(\lambda^2)
\]

as \( \lambda \to 0 \), since each edge adjacent to \( a_{m+1} \) is equally likely to participate in \( \sigma_k \). If \( \{a_{m+1}, b_n\} \) participates in \( \sigma_k \), then \( w < X - Y \). Conversely, if \( w < X - Y \) and no other edge from \( a_{m+1} \) has cost smaller than \( X - Y \), then \( \{a_{m+1}, b_n\} \) participates in \( \sigma_k \), and when \( \lambda \to 0 \), the probability that there are two distinct edges from \( a_{m+1} \) of cost smaller than \( X - Y \) is of order \( O(\lambda^2) \).

On the other hand, \( w \) is \( E(\lambda) \) distributed, so

\[
E[I] = \Pr \{w < X - Y\} = E \left[ 1 - e^{-\lambda(X-Y)} \right] = 1 - E \left[ e^{-\lambda(X-Y)} \right].
\]

Hence \( E[I] \), regarded as a function of \( \lambda \), is essentially the Laplace transform of \( X - Y \). In particular \( E[X - Y] \) is the derivative of \( E[I] \) evaluated at \( \lambda = 0 \), so

\[
E[X - Y] = \frac{d}{d\lambda} E[I] \bigg|_{\lambda=0} = \frac{N}{dn} \left( \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{m-k+1} \right) (1 + e_k)
\]

where \(|e_k| \leq 5\varepsilon\). Since \( X, Y \) are distributed as \( C_{k,t}, C_{k-1,t-1} \) respectively, the lemma follows.

It follows by symmetry that Lemma 3, Corollary 2 and Lemma 4 hold for \( t \notin T \) with \( m \) and \( n \) interchanged, so

\[
E[C_{r,k} - C_{r-1,k-1}] = \frac{N}{dm} \sum_{j=0}^{k-1} \frac{1}{n-j} (1 + e_k), \quad t \notin T
\]

where \(|e_k| \leq 5\varepsilon\).
We use a small trick to finish the proof. Consider $H = K_{N,N}$ and consider the graph sequence $H_t = H[V_t], t = N, N + 1, \ldots, 2N$. Then, according to [14],

$$E[C_{k,t}(H) - C_{k-1,t-1}(H)] = \frac{1}{n} \sum_{j=0}^{k-1} \frac{1}{m-j}, \quad t \in T,$$

$$E[C_{k,t}(H) - C_{k-1,t-1}(H)] = \frac{1}{m} \sum_{j=0}^{k-1} \frac{1}{n-j}, \quad t \notin T.$$

So

$$\sum_{j=1}^{N} \frac{1}{j^2} = E[C_{N,2N}(H)]$$

$$= E[C_{N,2N}(H) - C_{0,N}(H)]$$

$$= \sum_{t=N+1}^{2N} E[C_{t-N,N}(H) - C_{t-N-1,t-1}(H)]$$

$$= \sum_{t \in T} \frac{1}{|A_t|} \sum_{j=0}^{t-N-1} \frac{1}{|B_t| - j} + \sum_{t \notin T} \frac{1}{|B_t|} \sum_{j=0}^{t-N-1} \frac{1}{|A_t| - j}.$$

Here we use the fact that $\min\{|A_t|, |B_t|\} \geq t - N$ for $t = N + 1, \ldots, 2N$. Applying the same calculation on $G$, Lemma 4 and (23) imply

$$\frac{d}{N} E[C_{N,2N}(G)] = \sum_{t \in T} \frac{1 + e_t}{|A_t|} \sum_{j=0}^{t-N-1} \frac{1}{|B_t| - j} + \sum_{t \notin T} \frac{1 + e_t}{|B_t|} \sum_{j=0}^{t-N-1} \frac{1}{|A_t| - j}.$$  \hfill (24)

so since $|e_t| \leq 5\varepsilon$ for all $t$,

$$\left| \frac{d}{N} E[C_{N,2N}(G)] - \sum_{j=1}^{N} \frac{1}{j^2} \right| \leq 5\varepsilon \zeta(2)$$ \hfill (25)

for $N$ large enough.

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