Intersections of subcomplexes in non-positively curved 2-dimensional complexes

Feng Ji and Shengkui Ye

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Abstract

Let $X$ be a contractible 2-complex which is a union of two contractible subcomplexes $Y$ and $Z$. Is the intersection $Y \cap Z$ contractible as well? In this note, we prove that the inclusion-induced map $\pi_1(Y \cap Z) \to \pi_1(Z)$ is injective if $Y$ is $\pi_1$-injective subcomplex in a locally $\text{CAT}(0)$ 2-complex $X$. In particular, each component in the intersection of two contractible subcomplexes in a $\text{CAT}(0)$ 2-complex is contractible.

1 Introduction

As a motivation, we consider the following problem.

Problem 1.1 Let $X$ be a contractible 2-complex which is a union of two contractible subcomplexes $Y$ and $Z$. Is the intersection $Y \cap Z$ contractible as well?

A higher-dimensional version of this problem is already studied by Begle [2], which is related to the work of Aronszajn and Borsuk [1]. Begle [2] constructs a 3-dimensional contractible simplicial complex $X = Y \cup Z$ whose subcomplexes $Y, Z$ are both contractible but the intersection $Y \cap Z$ is not simply connected. He left open the question as to whether or not there are similar counter-examples in dimension two. We study a more general problem as the following.

Problem 1.2 Let $X$ be a 2-dimensional aspherical simplicial complex (i.e. the universal cover $\tilde{X}$ is contractible) and $Y$ be any $\pi_1$-injective subcomplex. For any subcomplex $Z \subset X$, is the map $\pi_1(Z \cap Y) \to \pi_1(Z)$ induced by the inclusion injective?

Note that any subcomplex $Y$ of a contractible 2-complex $X$ has vanishing second homology group $H_2(Y; \mathbb{Z}) = 0$, by considering the long exact sequence of homology groups for the pair $(X, Y)$. When $X$ is contractible and $Z \subset X$ is also contractible, the triviality of $\pi_1(Z \cap Y)$ would imply that each connected component in $Z \cap Y$ is contractible by the Whitehead theorem. This shows that a positive answer to Problem 1.2 gives a positive answer to Problem 1.1. We define a subcomplex $Y$ of a 2-dimensional complex $X$ to be strongly $\pi_1$-injective if for any subcomplex $Z$ of $X$, the inclusion-induced map $\pi_1(Y \cap Z) \to \pi_1(Z)$ is injective (cf. Definition 2.1). The 2-complex $X$ is said to have strong $\pi_1$-injectivity if any $\pi_1$-injective subcomplex $Y$ is strongly $\pi_1$-injective.

We will give a positive answer to Problem 1.2 for locally $\text{CAT}(0)$ 2-complexes by showing that locally $\text{CAT}(0)$ 2-complexes have strong $\pi_1$-injectivity, as the following.
Theorem 1.3 Let $X$ be a proper nonpositively curved 2-complex and $Y$ a $\pi_1$-injective subcomplex. For any subcomplex $Z$, the inclusion induces an injection $\pi_1(Z \cap Y) \to \pi_1(Z)$. In other words, $X$ has strong $\pi_1$-injectivity.

Corollary 1.4 Let $X$ be a CAT(0) 2-complex. For any two contractible subcomplexes $Y,Z$, each component in the intersection $Y \cap Z$ is contractible.

Theorem 1.3 leads to the following observation: when $X$ is a finite collapsible 2-complex and both $Y$ and $Z$ are contractible, each component in the intersection $Z \cap Y$ is contractible (cf. Corollary 2.5). This is already known by Segev [5] using a different approach.

Notation: All complexes are assumed to be connected simplicial complexes, unless otherwise stated. We use $\pi_1(X)$ to denote the fundamental group of $X$ with a based point in a connected component.

2 Strong $\pi_1$-injectivity

We first give the following definition.

Definition 2.1 A subcomplex $Y$ of a 2-dimensional complex $X$ is strongly $\pi_1$-injective if for any subcomplex $Z$ of $X$, the inclusion-induced map $\pi_1(Y \cap Z) \to \pi_1(Z)$ is injective. The 2-complex $X$ has strong $\pi_1$-injectivity if any $\pi_1$-injective subcomplex $Y$ is strongly $\pi_1$-injective.

Not every 2-complex has strong $\pi_1$-injectivity. For a simple counter-example, let $X$ be the sphere $S^2$. Since $S^2$ is a union of two disks with the circle $S^1$ as the intersection, the upper disk in the sphere $S^2$ is not strongly $\pi_1$-injective.

Lemma 2.2 Any $\pi_1$-injective graph (i.e. 1-simplicial subcomplex) is strongly $\pi_1$-injective in any 2-complex.

Proof. Let $X$ be a 2-complex with a $\pi_1$-injective 1-dimensional subcomplex $Y$. Shrinking a contractible tree in $Y$, we see that the fundamental group of $Y$ is free. For a subcomplex $K \subset Y$, the fundamental group of $K$ is still free. If there is a non-nullhomotopic closed loop in $K$, the loop represents a nontrivial element in $Y$. This implies that the composite $\pi_1(K) \to \pi_1(Y) \to \pi_1(X)$ is injective and thus $Y$ is strongly $\pi_1$-injective.

Next, we study the relation between strong $\pi_1$-injectivity and taking covering space.

Lemma 2.3 Let $X$ be a 2-complex with a $\pi_1$-injective subcomplex $K$. Suppose that $p : \tilde{X} \to X$ is the universal cover. If $p^{-1}(K)$ is strongly $\pi_1$-injective in $\tilde{X}$, then the complex $K$ is strongly $\pi_1$-injective in $X$.

Proof. Suppose that there is a subcomplex $Z$ in $X$ such that $\pi_1(Z \cap K) \to \pi_1(Z)$ is not injective. Let $f : S^1 \to Z \cap K$ be a map whose homotopic class in $\pi_1(Z \cap K)$ is nontrivial, but trivial in $\pi_1(Z)$. The map $f$ has a lifting $f' : S^1 \to \tilde{X}$, since $[f] = 1 \in \pi_1(X)$. Moreover, $[f'] \neq 1 \in \pi_1(p^{-1}(K \cap Z), \ast)$ for the base point in any connected component of $p^{-1}(K \cap Z)$. Since $\pi_1(K) \to \pi_1(X)$ is injective, the complex (each connected component) $p^{-1}(K)$ is simply connected. By assumption, the induced map $\pi_1(p^{-1}(K) \cap p^{-1}(Z)) \to \pi_1(p^{-1}(Z))$ is injective. Therefore, the homotopy class $[f'] \neq 1 \in \pi_1(p^{-1}(Z))$. This is a contradiction, since $\pi_1(p^{-1}(Z)) \to \pi_1(Z)$ is injective.
Lemma 2.3 implies that a 2-complex $X$ has strong $\pi_1$-injectivity if its universal cover $\tilde{X}$ does.

Let $X$ be a 2-complex and $K$ be a closed triangle (2-simplex). The 2-complex $X \cup K$ obtained by identifying two edges of $K$ with those of $X$ is called an elementary extension of $X$, while $X$ is called an elementary collapse of $X \cup K$ (cf. [3]). Denote by $e$ the third edge of $K$, which is not in $X$. A 2-complex $X$ is called collapsible if $X$ could be deformed to be a point by finite steps of elementary extensions, collapses and contracting or adding free edges.

**Theorem 2.4** The 2-complex $X$ has strong $\pi_1$-injectivity if and only if so does an elementary extension $X \cup K$.

**Proof.** Suppose that the elementary extension $X \cup K$ has strong $\pi_1$-injectivity. For any subcomplex $Y$ with injective fundamental group, we see that

$$\pi_1(Y) \to \pi_1(X) \xrightarrow{\cong} \pi_1(X \cup K)$$

is injective as well. Therefore, for any subcomplex $Z \subset X$, the map $\pi_1(Z \cap Y) \to \pi_1(Z)$ is injective.

Conversely, suppose that $X$ has strong $\pi_1$-injectivity. Let $Y \subset X \cup K$ be any $\pi_1$-injective subcomplex and $Z \subset X \cup K$ any subcomplex. We divide the proof into several cases.

**Case 1** $Y \supset K$.

1.1 $Z \supset K$. For convenience, let $Y \setminus K$ denote the subcomplex of $Y$ obtained by deleting the interior of $K$ and the open edge $e$. We see that

$$Y \cap Z = (Y \setminus K \cap Z \setminus K) \cup K.$$

Note that $Y$ is an elementary extension of $Y \setminus K$ and $Y \cap Z$ is also an elementary extension of $Y \setminus K \cap Z \setminus K$. Therefore, we get by the hypothesis on $X$ that

$$\pi_1(Y \cap Z) = \pi_1(Y \setminus K \cap Z \setminus K) \hookrightarrow \pi_1(Z \setminus K) = \pi_1(Z).$$
1.2 $Z \nsubseteq K$ but $Z \supset e$. Let $Z\setminus e$ denote the complex obtained by removing the interior of $e$ from $Z$. We have that
\[ Y \cap Z = (Y \setminus K \cap Z\setminus e) \cup e. \]

If two ends of $e$ are both in the same component of $Y \setminus K \cap Z\setminus e$, let $P$ be a path in $Y \setminus K \cap Z\setminus e$ connecting the two ends. Choose a base point $x_0 \in P$. Contracting the path $P$, we have that (note the injectivity of the first free factor follows from the hypothesis on $X$)
\[ \pi_1(Y \cap Z, x_0) = \pi_1(Y \setminus K \cap Z\setminus e, x_0) \ast Z \hookrightarrow \pi_1(Z\setminus e, x_0) \ast Z = \pi_1(Z, x_0). \]

If the two ends of $e$ lie in two different components $Y_1, Y_2$ of $Y \setminus K \cap Z\setminus e$, choose the path $F$ consisting of the two attaching edges in $K$. Note that $F$ is not in $Z$. Since $X$ has strong $\pi_1$-injectivity, there is an injection
\[ \pi_1(Y \setminus K \cap (Z\setminus e \cup F), x_0) \hookrightarrow \pi_1(Z\setminus e \cup F, x_0) \]
where the base point $x_0$ is one end of $e$. Therefore, we have that
\[ \pi_1(Y \cap Z, x_0) = \pi_1(Y_1) \ast \pi_1(Y_2) = \pi_1((Y \setminus K \cap Z\setminus e) \cup F, x_0) \hookrightarrow \pi_1(Z\setminus e \cup F, x_0) = \pi_1(Z, x_0). \]

1.3 $Z \subset X$. We have that $Y \cap Z = Y \setminus K \cap Z$ and thus
\[ \pi_1(Y \cap Z) = \pi_1(Y \setminus K \cap Z) \hookrightarrow \pi_1(Z). \]

Case 2 $Y \nsubseteq K$ but $Y \supset e$, where $e$ is the closed edge of $K$ not in $X$.

2.1 $Z \supset K$. We have that $Y \cap Z = (Y\setminus e \cap Z\setminus K) \cup e$. Since $\pi_1(Y) \rightarrow \pi_1(X \cup K)$ is injective, the path $F$ consisting of the two attaching edges of $K$ does not lie in $Y$. If the two ends of $e$ lie in the same component of $Y\setminus e \cap Z\setminus K$, the edge $e$ is a part of a loop in $Y \cap Z$. Then the path $F$ is part of a loop in $Y\setminus e \cap Z\setminus K$ by replacing $e$ with $F$. If the two ends of $e$ lie in different components, then the edge $e$ will not contribute to the fundamental group. In any case, we have an injection
\[ \pi_1(Y \cap Z) \hookrightarrow \pi_1((Y\setminus e \cup F) \cap Z\setminus K). \]
Since $\pi_1(Y) = \pi_1(Y\setminus e \cup F) \hookrightarrow \pi_1(X \cup K) = \pi_1(X)$, the subcomplex $Y\setminus e \cup F$ is also $\pi_1$-injective. Considering that $X$ has strong $\pi_1$-injectivity, there is an injection
\[ \pi_1((Y\setminus e \cup F) \cap Z\setminus K) \hookrightarrow \pi_1(Z\setminus K) = \pi_1(Z). \]
This proves that the inclusion induces an injection $\pi_1(Y \cap Z) \hookrightarrow \pi_1(Z)$.

2.2 $Z \nsubseteq K$ but $Z \supset e$. Since $\pi_1(Y) \rightarrow \pi_1(X \cup K)$ is injective, the path $F$ consisting of the two attaching edges of $K$ does not lie in $Y$. For the same reason as that of the case 2.1, we have an injection
\[ \pi_1(Y \cap Z) = \pi_1((Y\setminus e \cap Z\setminus e) \cup e) \hookrightarrow \pi_1((Y\setminus e \cap Z\setminus e) \cup e) \]
\[ = \pi_1((Y\setminus e \cup F) \cap (Z\setminus e \cup F)). \]
Note that $\pi_1(Y\setminus e \cup F) = \pi_1(Y) \hookrightarrow \pi_1(X \cup K) = \pi_1(X)$. Since $X$ has strong $\pi_1$-injectivity, the inclusion induces an injection $\pi_1((Y\setminus e \cup F) \cap (Z\setminus e \cup F)) \hookrightarrow \pi_1(Z\setminus e \cup F)$. If $F \nsubseteq Z$, we have that $\pi_1(Z\setminus e \cup F) = \pi_1(Z)$. If $F \subset Z$, we have that $\pi_1(Z\setminus e \cup F) \ast Z = \pi_1(Z)$. In both cases, there is an injection $\pi_1(Z\setminus e \cup F) \hookrightarrow \pi_1(Z)$. Therefore, the map $\pi_1(Y \cap Z) \rightarrow \pi_1(Z)$ is injective.
2.3 $Z \subset X$. We have that $Y \cap Z = Y \setminus e \cap Z$. For the same reason as that of the case 2.1, the path $F$ is not in $Y$ and there is an injection $\pi_1(Y \setminus e \cup F) \hookrightarrow \pi_1(X)$. We have that
\[
\pi_1(Y \cap Z) = \pi_1(Y \setminus e \cap Z) \hookrightarrow \pi_1((Y \setminus e \cup F) \cap Z) \hookrightarrow \pi_1(Z).
\]

Case 3 $Y \subset X$.

3.1 $Z \supset K$. We have that $Y \cap Z = Y \cap (Z \setminus K)$. Since $Z \setminus K$ is a collapse of $Z$, we get from the hypothesis on $X$ that
\[
\pi_1(Y \cap Z) = \pi_1(Y \cap (Z \setminus K)) \hookrightarrow \pi_1(Z \setminus K) = \pi_1(Z).
\]

3.2 $Z \nsubseteq K$ but $Z \supset e$. In this case, $\pi_1(Z) = \pi_1(Z \setminus e) \ast Z$. The hypothesis on $X$ implies that $\pi_1(Y \cap Z) = \pi_1(Y \cap Z \setminus e)$ injects into $\pi_1(Z \setminus e)$. Therefore, we have an injection
\[
\pi_1(Y \cap Z) \hookrightarrow \pi_1(Z \setminus e) \hookrightarrow \pi(Z \setminus e) \ast Z = \pi_1(Z).
\]

3.3 $Z \subset X$. This subcase follows directly from the hypothesis of $X$.

All the cases are included and the proof is complete.

It is already known by Segev [5] (4.3) that when $X$ is a finite collapsible 2-complex and both $Y$ and $Z$ are contractible, each connected component in the intersection $Z \cap Y$ is contractible. This is a special case of the following.

**Corollary 2.5** A collapsible 2-complex has strong $\pi_1$-injectivity. In particular, each connected component in the intersection $Y \cap Z$ of two contractible subcomplexes $Y, Z$ in a collapsible 2-complex $X$ is contractible.

**Proof.** A collapsible 2-complex is deformed to a point by a finitely many elementary collapse or extensions. The first part is thus implied by Theorem 2.4. When $Y$ and $Z$ are contractible, the intersection $Y \cap Z$ is $\pi_1$-injective in $Z$ and thus simply connected. A simply connected subcomplex of a contractible 2-complex is acyclic by the relative homology exact sequence. Therefore, each connected component in the intersection $Y \cap Z$ is contractible by the Whitehead theorem.

### 3 Non-positively curved complexes

Recall the notion of non-positively curved complexes from Bridson and Haefliger [4] (Chapter II. 1.2). Let $(X,d)$ be a geodesic metric space. A geodesic triangle $\Delta(x,y,z)$ consists of three vertices $x,y,z \in X$ and three geodesics $[x,y], [y,z], [x,z]$ connecting these vertices. A comparison triangle $\Delta(\bar{x},\bar{y},\bar{z})$ (or denoted by $\Delta(x,y,z)$) is an Euclidean triangle in the plane $\mathbb{R}^2$ with three vertices $\bar{x}, \bar{y}, \bar{z}$ and edges of lengths $d(x,y), d(y,z), d(x,z)$ respectively.

**Definition 3.1** A geodesic metric space $X$ is CAT(0) if for any geodesic triangle $\Delta(x,y,z)$ and any two points $p,q \in \Delta(x,y,z)$, we have
\[
d(p,q) \leq d_{\mathbb{R}^2}(\bar{p},\bar{q}),
\]
where $\bar{p}, \bar{q}$ are the corresponding points of $p,q$ in the comparison triangle $\Delta(\bar{x},\bar{y},\bar{z})$. 
A Euclidean cell is the convex hull of a finite number of points in $\mathbb{R}^n$, equipped with the standard Euclidean metric. A Euclidean cell complex $X$ is a space formed by gluing together Euclidean cell-complexes via isometries of their faces. It has the piecewise Euclidean path metric. Precisely, for any $x, y \in X$, let $x = x_0, x_1, \ldots, x_n = y$ be a path such that each successive $x_i, x_{i+1}$ is contained in a Euclidean simplex $S_i$. Define the distance (called path metric) $d_X(x, y) = \inf \sum_{i=0}^{n-1} d_{S_i}(x_i, x_{i+1})$, where the infimum is taken over all such paths. Note that a metric space $X$ is proper if any closed ball $B(x, r) \subset X$ is compact.

Definition 3.2 A Euclidean cell complex $X$ is non-positively curved if it is locally CAT(0), i.e. for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$ with the induced metric is a CAT(0).

Let $X$ be a Euclidean cell complex and $v \in X$. The (geometric) link $Lk(x, X)$ is the set of unit tangent vectors ("directions") at $x$ in $X$. Precisely, let $S$ be the set of all geodesics $[x, y]$ with $y$ in a simplex containing $x$. Two geodesics are called equivalent if one is contained in the other. The link $Lk(x, X)$ is the set of equivalence classes of geodesics in $S$. If $X$ is one $n$-dimensional Euclidean cell, the link $Lk(x, X)$ is part of $S^{n-1}$ and thus the topology on $Lk(x, X)$ is defined as the "angle" topology. In general, the topology on $Lk(x, X)$ is defined as the path metric coming from each cell.

We will need the following facts about 2-complexes from [4].

Lemma 3.3 (1) A finite CAT(0) 2-complex is collapsible.

(2) (Link condition) A 2-dimensional Euclidean cell complex $X$ is non-positively curved if and only if each link $Lk(x, X)$ contains no injective loops of length less than $2\pi$.

(3) A simply connected non-positively curved complex is CAT(0).

Proof. The first claim (1) is [4], 5.34(2), while the second claim (2) is [4], 5.5 and 5.6 in Chapter II.5. The last claim follows the Cartan-Hadamard theorem (see [4], II. 4.1).

Proof of Theorem 1.3. Since the universal cover of a non-positively curved complex is CAT(0), it suffices to prove that a CAT(0) 2-complex $X$ has strong $\pi_1$-injectivity by Lemma 2.3. By Corollary 2.5 any collapsible 2-complex has strong $\pi_1$-injectivity. As a finite CAT(0) 2-complex is collapsible (see Lemma 3.3 (1)), it has strong $\pi_1$-injectivity. Suppose that a simply connected subcomplex $Y \subset X$ is not strongly $\pi_1$-injective. Let $Z \subset X$ be a subcomplex such that $\pi_1(Y \cap Z) \to \pi_1(Z)$ is not injective. Choose $f : S^1 \to Y \cap Z$ such that the homotopy class $[f] \in \ker \pi_1$ is not trivial. Since $\text{Im} f$ is compact and any homotopy $h$ between $f$ and a constant map has compact image in $Z$, we could choose finite subcomplexes $Y' \subset Y$ containing $\text{Im} f$ and $Z' \subset Z$ containing $\text{Im} h$. The link condition implies that the subcomplex $Y$ is non-positively curved (cf. Lemma 3.3 (2)). Since $Y$ is simply connected, the subcomplex $Y$ is CAT(0) by Lemma 3.3 (3). The finite subcomplex $Y'$ is contained in a ball $B_Y(x, r) \subset Y$ of sufficiently large radius, for some point $x \in Y$ and sufficient large $r$. When $X$ is proper, the closed ball $B_Y(x, r)$ is compact. Since the ball $B_Y(x, r)$ is contractible (cf. [4], II.1.4), we may choose a finite contractible subcomplex $Y'' \subset Y$ containing $Y'$ (for example, take $Y'' = B_Y(x, r)$). By the construction, the inclusion-induced map $\pi_1(Y'' \cap Z') \to \pi_1(Z')$ is not injective. Since both $Y''$ and $Z'$ are finite, we may choose a ball $B_X(x, r')$ of sufficiently large radius containing $Y''$ and $Z'$. Therefore, the subcomplexes $Y''$ and $Z'$ is contained in a finite CAT(0) 2-complex $X'$. The strong $\pi_1$-injectivity of $X'$ implies that the map $\pi_1(Y'' \cap Z') \to \pi_1(Z')$ is injective, which gives a contradiction. This finishes the proof.
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Infinitus, Nanyang Technological University, 50 Nanyang Ave, S2B4b05, Singapore 639798. E-mail: jifeng@ntu.edu.sg

Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu, China 215123. E-mail: Shengkui.Ye@xjtlu.edu.cn