SOME EXPPLICIT ARITHMETIC ON CURVES OF GENUS THREE AND THEIR APPLICATIONS

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Abstract. A Richelot isogeny between Jacobian varieties is an isogeny whose kernel is included in the 2-torsion subgroup of the domain. A Richelot isogeny whose codomain is the product of two or more principally polarized abelian varieties is called a decomposed Richelot isogeny.

In this paper, we develop some explicit arithmetic on curves of genus 3, including algorithms to compute the codomain of a decomposed Richelot isogeny. As solutions to compute the domain of a decomposed Richelot isogeny, explicit formulae of defining equations for Howe curves of genus 3 are also given. Using the formulae, we shall construct an algorithm with complexity $O(p^3)$ (resp. $O(p^4)$) to enumerate all hyperelliptic (resp. non-hyperelliptic) superspecial Howe curves of genus 3.

1. Introduction

Throughout, a curve means a non-singular projective variety of dimension one. The complexity is measured by the number of arithmetic operations in a field. Let $k$ be an algebraically closed field of characteristic $p \neq 2$, and $(A, D)$ a principally polarized abelian variety over $k$. For a curve $C$ of genus $g$ over $k$, its automorphism group is denoted by $\text{Aut}(C)$. We also denote respectively by $J(C)$ and $\Theta_C$ the Jacobian variety of $C$ and the principal polarization on $J(C)$ given by $C$. We say that $J(C)$ is decomposable (resp. completely decomposable) if there exists an isogeny $J(C) \to J(C_1) \times \cdots \times J(C_n)$ (resp. $E_1 \times \cdots \times E_g$) for some curves $C_1, \ldots, C_n$ (resp. $g$ elliptic curves $E_1, \ldots, E_g$), and in this case such an isogeny is called a decomposed (resp. completely decomposed) isogeny. Studying the decomposition of Jacobian varieties is classically important, see e.g., [7], [17], [37], [38]. In particular, a method by Kani-Rosen provided in their celebrated paper [17] enables us to find some curves $C_i$ as above as quotients of $C$ by subgroups of $\text{Aut}(C)$, when $\text{Aut}(C)$ is non-trivial.

This paper focuses on the decomposition of Jacobian varieties by Richelot isogenies. Here, an isogeny $\phi : J(C) \to A$ is called a Richelot isogeny if $\phi^*(\Theta_C) \approx 2D$, where “$\approx$” denotes algebraic equivalence. Richelot isogenies are generalizations of 2-isogenies of elliptic curves, and play important roles to analyze the security of cryptosystems constructed by means of isogeny graphs, see e.g., [4], [6]. For a hyperelliptic curve $C$ of genus 2 (resp. 3), Katsura-Takashima [20] (resp. Katsura [18]) showed that the existence of a decomposed Richelot isogeny from $J(C)$ is equivalent to that of an order-2 (long) automorphism in the reduced automorphism group $\text{Aut}(C)$, where $\text{Aut}(C) := \text{Aut}(C)/\langle \iota_C \rangle$ with the hyperelliptic involution $\iota_C$ of $C$.

They showed also in [21] that any hyperelliptic curve $C$ of given genus with an involution $\sigma \in \text{Aut}(C)$ different from $\iota_C$ (such $\sigma$ is called an extra involution of $C$) has a decomposed Richelot isogeny $J(C) \to J(C/\langle \sigma \rangle) \times J(C/\langle \sigma \circ \iota_C \rangle)$, and that

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any (generalized) Howe curve has a decomposed Richelot isogeny, where a Howe curve is the normalization of a fiber product of hyperelliptic curves.

In this paper, we start with writing down Katsura-Takashima’s construction \cite{21} for the hyperelliptic case as an explicit algorithm, so as to symbolically compute defining equations for the quotient curves $C/\langle \sigma \rangle$ and $C/\langle \sigma \circ C \rangle$ as above. The complexity of the algorithm is also determined. More precisely, we prove the following:

**Theorem 1** (Theorem 3.1.1 below). Let $C$ be a hyperelliptic curve of genus $g \geq 2$ over $k$ with an extra involution $\sigma$. Given $C$, there is an algorithm (explicitly, Algorithm 3) to compute equations defining the quotient curves $C/\langle \sigma \rangle$ and $C/\langle \sigma \circ C \rangle$, which define the codomain of a decomposed Richelot isogeny described above. Moreover, all the computations in the algorithm are done in the quadratic extension of a subfield $K$ of $k$, over which all the Weierstrass points of $C$ are defined. Furthermore, the number of arithmetic operations (in the quadratic extension field) required for executing the algorithm is upper-bounded by a constant.

In the case of genus 3, algorithms for completely decomposed Richelot isogenies are also provided:

**Theorem 2** (Theorems 3.2.1 and 3.3.1 below). Let $C$ be a curve of genus 3 over $k$, say, either of the following:

1. A smooth plane quartic $F(X,Y,Z) = 0$, where $F$ is a ternary quartic form over a subfield $K$ of $k$.

2. A hyperelliptic curve $y^2 = f(x)$ with a square-free octic $f$ over $k$, where the roots of $f$ belong to a subfield $K$ of $k$.

We suppose that $C$ has two order-2 automorphisms $\sigma$ and $\tau$ with $\sigma \neq \tau$ such that $\sigma \tau = \tau \sigma$. We also suppose $k$ is an algebraic closure of a finite field if $C$ is a non-hyperelliptic curve. Given $C$, there exists an algorithm to compute defining equations for $C/\langle \sigma \rangle$, $C/\langle \tau \rangle$, and $C/\langle \sigma \tau \rangle$ as genus-1 curves, which define the codomain of a completely decomposed Richelot isogeny $J(C) \rightarrow J(C/\langle \sigma \rangle) \times J(C/\langle \tau \rangle) \times J(C/\langle \sigma \tau \rangle)$. Moreover, all the computations in the algorithm are done in a finite extension of $K$, where the extension degree does not depend on the characteristic of $k$. Furthermore, the number of arithmetic operations (in the finite extension field) required for executing the algorithm is upper-bounded by a constant.

The inverse problem is also considered: Given $C_1, \ldots, C_n$, compute a defining equation for $C$ such that $J(C)$ is Richelot isogenous to $J(C_1) \times \cdots \times J(C_n)$. In Section 4 we shall give a complete solution for this problem in the case of genus 3. In this case, the codomain of a decomposed Richelot isogeny is either $E \times J(H)$ for a genus-1 curve $E$ and a genus-2 curve $H$ or $E_1 \times E_2 \times E_3$ for genus-1 curves $E_1$, $E_2$, and $E_3$. When $E$ and $H$ share four Weierstrass points or each two elliptic curves $E_1$ and $E_2$ share two Weierstrass points, it is straightforward that we may take $C$ as the normalization of $E \times p^1 H$ or $E_1 \times p^1 E_2$, which is nothing but a Howe curve of genus 3, and the problem has been reduced into finding a defining equation for a Howe curve of genus 3. The former case is solved by a general fact on the fiber product of hyperelliptic curves, see Lemma 4.2.1 below. The following main theorem is our solution for the latter case:

**Main Theorem** (Theorems 4.1.1 and 4.2.1). Let $E_1 : y^2 = x(x-\alpha_2)(x-\alpha_3)$ and $E_2 : y^2 = x(x-\beta_2)(x-\beta_3)$ be elliptic curves, where $\alpha_2, \alpha_3, \beta_2$, and $\beta_3$ are pairwise distinct elements in $k \setminus \{0\}$. An equation defining the normalization $C$ of $E_1 \times p^1 E_2$ is given as $\langle 4.1.1 \rangle$ (resp. $\langle 4.2.1 \rangle$) in the hyperelliptic (resp. non-hyperelliptic) case.

As an important corollary, if we take $\alpha_2 = 1$, $\alpha_3 = \nu$, $\beta_2 = \mu$, and $\beta_3 = \mu \lambda$ with $\nu = \mu^2 \lambda$, any coefficient of $\langle 1.1.2 \rangle$ for superspecial $C$ belongs to $\mathbb{F}_{\sqrt{2}}$, see Corollary 4.2.1 below (cf. a resent result \cite{34} Theorem 1.1) by Ohashi).
Furthermore, computational methods to enumerate superspecial Howe curves of genus 3 will be given in Section 3 and their complexities are also determined:

**Theorem 3.** When \( k = \mathbb{F}_p \) for a prime \( p \), all isomorphism classes of superspecial hyperelliptic (resp. non-hyperelliptic) Howe curves of genus 3 are enumerated in \( O(p^3) \) (resp. \( \tilde{O}(p^4) \)), where Soft-O notation omits logarithmic factors.

In our methods, we apply Main Theorem to computing equations defining superspecial Howe curves, and use some criteria given in Subsection 5.1 to make the isomorphism classification efficient.

We implemented the enumeration methods on Magma [2], and the source code are available at the web page of the second author, see Subsection 5.3 for the url. Computational results obtained by our implementation will be described in Subsection 5.3.

2. Preliminaries

This section is mainly devoted to a review of some basic facts on hyperelliptic curves, generalized Howe curves, and Richelot isogenies between abelian varieties.

2.1. Hyperelliptic curves and their isomorphisms. Let \( k \) be a field of characteristic \( p \) with \( p \neq 2 \). Let \( C \) be a hyperelliptic curve of genus \( g \geq 2 \) over \( k \), and \( \pi_C : C \rightarrow \mathbb{P}^1 \) its structure map of degree 2. Then \( C \) has a unique involution \( \iota_C : C \rightarrow C \) such that \( C/\langle \iota_C \rangle \cong \mathbb{P}^1 \), which is called the hyperelliptic involution of \( C \). Recall that \( \iota_C \) is the unique non-trivial automorphism of \( C \) with \( \pi_C \circ \iota_C = \pi_C \).

We also call an involution of \( C \) different from \( \iota_C \) an extra involution. A Weierstrass point of \( C \) is an invariant point under \( \iota_C \), and there are precisely \( 2g+2 \) Weierstrass points of \( C \). It is well-known that any isomorphism of hyperelliptic curves maps Weierstrass points of the domain to ones of the codomain.

In general, a hyperelliptic curve \( C \) can be represented by gluing the following two curves in \( \mathbb{A}^2 \):

\[
C_1: y^2 = f_1(x), \quad C_2: y^2 = f_2(x)
\]

via the map \( C_1 \setminus \{(0,*)\} \rightarrow C_2 \setminus \{(0,*)\} : (x,y) \mapsto (1/x, y/x^{g+1}) \), where \( f_1 \) and \( f_2 \) are polynomials of degree \( 2g+1 \) or \( 2g+2 \) satisfying \( f_1(1/x)x^{2g+2} = f_2(x) \). Note that the map \( C \rightarrow \mathbb{P}^1 \) given by the projection onto the \( x \)-coordinate is a morphism of degree 2. For simplicity, we usually represent \( C \) by \( y^2 = f_1(x) \) or \( y^2 = f_2(x) \), and in this case the hyperelliptic involution \( \iota_C \) is \((x,y) \mapsto (x,-y)\).

The following important lemma enables us to describe isomorphisms of hyperelliptic curves explicitly:

**Lemma 2.1.1** ([20] §2E1]). Let \( C_1: y^2 = f_1(x) \) and \( C_2: y^2 = f_2(x) \) be hyperelliptic curves of genus \( g \) over \( k \). Any isomorphism \( \sigma : C_1 \rightarrow C_2 \) can be represented by a pair \( (A,\lambda) \in \text{GL}_2(k) \times k^\times \) with \( A = \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22} \) for \( \alpha, \beta, \gamma, \delta \in k \) such that

\[
\sigma(x,y) = \left( \frac{\alpha x + \beta}{\gamma x + \delta}, \frac{\lambda y}{(\gamma x + \delta)^{g+1}} \right),
\]

where \( E_{ij} \) denotes the \( 2 \times 2 \)-matrix with 1 in the \((i,j)\) entry and 0’s elsewhere. The representation is unique up to the equivalence \((A,\lambda) \equiv (\mu A, \mu^{g+1} \lambda) \) for \( \mu \in k^\times \).

When \( k = \overline{F} \) in Lemma 2.1.1 one can always take \( \lambda = 1 \), and \( A \) is unique up to the multiplication by \((g+1)\)-th roots of unity. A generic method deduced from Lemma 2.1.1 to decide whether two hyperelliptic curves \( y^2 = f_1(x) \) and \( y^2 = f_2(x) \) over an algebraically closed field are isomorphic is to compute a Gröbner basis of a (projective zero-dimensional) homogeneous multivariate system derived from coefficients in \( f_1(ax + \beta z, \gamma x + \delta z) = f_2(x,z) \), see e.g., [22]. Since the system has
4 variables $\alpha, \beta, \gamma, \delta$ and since the maximal total degree $\leq 2g + 2$, the complexity is estimated as $O(g^4)$ from [28]. Alternative efficient methods in the case where $p$ does not divide $2g + 2$ are proposed in [30] with complexity $O(g)$, and they were implemented in Magma [2] as IsIsomorphicHyperellipticCurves (cf. [39, §1.1]).

Instead of the above methods, we can use Igusa invariants [16] (and absolute invariants) and Shioda invariants [11] in the genus-2 case and in the genus-3 case respectively. Dihedral invariants are also useful when $\text{Aut}(C) \supset C_2$ (equivalently, $\text{Aut}(C) \supset V_4$), see [11, Section 3] for details. Homogeneous dihedral invariants defined by Lercier–Ritzenthaler–Sijsling [31] classify isomorphisms in the case where $\text{Aut}(C)$ is cyclic of order coprime to the characteristic $p$ of $k$.

2.2. Automorphisms of hyperelliptic curves. For a curve $C$ over a field $k$, we denote by $\text{Aut}_k(C)$ the automorphism group of $C$ over $k$, and if $k = \overline{k}$, we denote it simply by $\text{Aut}(C)$. For a hyperelliptic curve $C : y^2 = f(x)$ over an algebraically closed field, the quotient group $\text{Aut}(C) := \text{Aut}(C)/(\mathfrak{i}_C)$ is called the reduced automorphism group of $C$. As a particular case of Lemma 2.1.1, any $\sigma \in \text{Aut}_k(C)$ is represented as in (2.1.1) for some $(A, \lambda) \in \text{GL}_2(k) \times k^\times$. Throughout the rest of this paper, let $k$ be an algebraically closed field of characteristic $p \neq 2$. As noted in [11, §2] (without proof), any automorphism $\sigma$ of a hyperelliptic curve over $k$ is represented by $\text{diag}(\mu, 1) \in \text{GL}_2(k)$ for a primitive $\ell$-th root of unity, and where $\mu$ is an element satisfying $(\mu')^\ell = 1$ or $-1$. We also have $\mu' = \pm \mu^{g+1}$ if $\deg(f) = 2g + 2$, and $\mu' = \pm \sqrt[2g+1]{\mu^g}$ if $\deg(f) = 2g + 1$. Hence, $\sigma$ is the hyperelliptic involution if and only if $(\mu, \mu') = (1, -1)$ and $\ell = 1$.

Proof. Recall from Lemma 2.1.1 that $\sigma$ is represented by some $(A, \lambda) \in \text{GL}_2(k) \times k^\times$. We may assume $\lambda = 1$ by $(A, \lambda) \equiv (\lambda^{-1}/(g+1) A, 1)$. Since $A' = a_12$ and $\lambda' = \pm a^{g+1}$ for some $a \in k^\times$, there exists a matrix $Q = \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22} \in \text{GL}_2(k)$ with $Q^{-1}AQ = B_0 := b^{-1}\text{diag}(\mu_1, \mu_2)$ in $\text{GL}_2(k)$, where $b \in k$ is an arbitrary element with $b^\ell = a^{-1}$, and where $\mu_1$ and $\mu_2$ are $\ell$th roots of 1. Letting $\rho$ be an isomorphism represented by $(Q, \nu)$ with an arbitrary $\nu \in k^\times$, we have the following commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\sigma} & C \\
\rho \downarrow & & \downarrow \rho^{-1} \\
C' & \xrightarrow{} & C',
\end{array}
\]

where $C'$ is a hyperelliptic curve defined by \( \left( \frac{\nu y}{(x + z)^{g+1}} \right)^2 = f_0 \left( \frac{ax + z}{x + z} \right) \) with an automorphism $\rho^{-1}\sigma\rho$ (of order $\ell$ in the reduced automorphism group $\text{Aut}(C')$) represented by $(B_0, \lambda)$. Putting $B := \text{diag}(\mu, 1)$ with $\mu := \mu_1\mu_2^{-1}$ and $\mu' = (b\mu_2^{-1})^g\lambda$, we furthermore have $(B_0, \lambda) \equiv ((b\mu_2^{-1})B_0, (b\mu_2^{-1})^g\lambda) = (B, \mu')$. Note that $\mu$ is an $\ell$-th root of unity, and that $(\mu')^\ell = ((b\mu_2^{-1})^g\lambda)^\ell = (a^{-1})^g\lambda^\ell = \pm 1$.

We also show the assertion on $\mu'$. For this, let $y^2 = f(x)$ be a hyperelliptic equation of $C'$, and put $F(X, Z) := Z^{2g+2}f(X/Z)$. Since $\rho^{-1}\sigma\rho$ is an automorphism of $C'$, it follows from $F(\mu X, Z) = (\mu')^2F(X, Z)$ that $f(\mu x) = (\mu')^2f(x)$. Comparing
As it is defined in [21], the degree of \(\sigma\) is an involution (i.e., order \(\ell = 2\)) as an element of the (not reduced) automorphism group \(\text{Aut}(C)\). Indeed, we have \(\mu = -1\) and \((\mu')^\ell = 1\) in this case, and it follows from \(\mu^{\deg(f)} = (\mu')^2\) that \(\deg(f)\) must be even.

For any subgroup \(G\) of \(\text{Aut}(C)\), we can consider the quotient curve \(C/G\). Finding an equation for \(C/G\) is not so easy in general, but it is possible under some reasonable assumptions on \(C\) and \(G\) such as in the case of the following lemma:

**Lemma 2.2.3.** Let \(C\) be a hyperelliptic curve \(y^2 = f(x^2)\) of genus \(g\) over \(k\), where \(f(x)\) is a square-free polynomial in \(k[x]\) of degree \(g + 1\) with \(f(0) \neq 0\). Let \(C\) be the hyperelliptic involution \((x, y) \mapsto (x, -y)\), and let \(\sigma_1 : (x, y) \mapsto (-x, y)\) be an involution on \(C\). Then the quotient curves \(C/\langle \sigma_1 \rangle\) and \(C/\langle \sigma_2 \rangle\) with \(\sigma_2 := \sigma_1 \circ \iota_C\) are hyperelliptic curves given by \(y^2 = f(x)\) and \(y^2 = xf(x)\), respectively.

Conversely, let \(C_1\) and \(C_2\) be hyperelliptic curves of genera \(g_1\) and \(g_1 + 1\) (resp. \(g_1\) and \(g_1 + 1\)) over \(k\) respectively defined by \(y^2 = f(x)\) and \(y^2 = xf(x)\), where \(f(x)\) is a square free polynomial of degree \(2g_1 + 1\) (resp. \(2g_1 + 2\)) with \(f(0) \neq 0\). Then the normalization of the fiber product \(C_1 \times_{P^1} C_2\) is isomorphic to the hyperelliptic curve \(C : y^2 = f(x^2)\) of genus \(2g_1\) (resp. \(2g_1 + 1\)).

### 2.3. Generalized Howe curves

In this subsection, we recall the notion of generalized Howe curves. Consider two hyperelliptic curves \(C_1\) and \(C_2\) of genera \(g_1\) and \(g_2\) over an algebraically closed field \(k\) with the following affine models:

\[
\begin{align*}
C_1 : & \quad y_1^2 = f_1(x) = (x-a_1) \cdots (x-a_r)(x-b_1) \cdots (x-b_{g_1+2-r}), \\
C_2 : & \quad y_2^2 = f_2(x) = (x-a_1) \cdots (x-a_r)(x-c_1) \cdots (x-c_{g_2+2-r}),
\end{align*}
\]

where \(a_i, b_i, c_i\) are all distinct elements in \(k\). Assume that \(g_1 \leq g_2\), and let \(\sigma_i : C_i \to P^1\) be the usual double covers. As in [21], assume that there is no isomorphism \(\varphi : C_1 \to C_2\) with \(\tau_2 \circ \varphi = \tau_1\). As it is defined in [21], the normalization \(C\) of the fiber product \(C_1 \times_{P^1} C_2\) is called a generalized Howe curve. Let \(\sigma_i\) be an involution on \(C\) whose quotient map is \(C \to C_i\) for each \(i = 1, 2\), and let \(\sigma_3\) be the quotient curve of \(C\) by the other involution \(\sigma_2 := \sigma_1 \sigma_2\) on \(C\). We can write an equation for \(C_1 \times_{P^1} C_2\) locally as \(y_1^2 - f_1(x) = y_2^2 - f_2(x) = 0\), from which we obtain an equation for \(C_3\) as follows:

\[
C_3 : \quad y_3^2 = f_3(x) = (x-b_1) \cdots (x-b_{g_1+2-r})(x-c_1) \cdots (x-c_{g_2+2-r}),
\]

where we set \(y_3 = \frac{y_1 y_2}{(x-a_1) \cdots (x-a_r)}\) as in [21]. We then have the following commutative diagram:

\[
\begin{array}{ccc}
C_1 & \to & C_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
P^1 & \to & P^1.
\end{array}
\]

We here recall the following proposition on the genus \(g(C)\) of the generalized Howe curve \(C\) and a criterion for \(C\) to be hyperelliptic:

**Proposition 2.3.1 (21 Proposition 1 and Theorem 2).** Let \(C_1\), \(C_2\), and \(C\) be as above. Then we have \(g(C) = 2g_1 + 2g_2 + 1 - r\) and \(r \leq g_1 + g_2 + 1\), and hence \(g(C) \geq g_1 + g_2\). Moreover, if \(g(C) \geq 4\), then \(C\) is hyperelliptic if and only if \(r = g_1 + g_2 + 1\), i.e., \(C_3\) is rational.
As a particular case, the generalized Howe curve $C$ with $g_1 = g_2 = 1$ and $r = 1$ is a non-hyperelliptic curve of genus $g(C) = 4$, and it is called simply a Howe curve, which was originally defined in [12] (see also [20] and [21]). On the other hand, in the case where $g(C) = 3$ with $g_1 = g_2 = 1$ and $r = 2$, all the quotient curves $C_1$, $C_2$, and $C_3$ are of genus one, and this case was studied in [4], [23], [24], and [30]. In particular, Oort [35] used this construction to prove the existence of superspecial curves of genus 3, which we will introduce in Section 5 below. As we will also recall at the beginning of Section 4 Oort also gave a criterion for $C$ of genus 3 to be hyperelliptic (cf. Proposition 2.3.1 for the case of $g \geq 4$). Note that this $C$ of genus 3 is called a Ciani curve if it is non-hyperelliptic, see [5].

As it is described in [21, §4], the hyperelliptic involutions of $C_1$ and $C_2$ lift to order-2 automorphisms on $C_1 \times_{\mathbb{P}^1} C_2$, and thus the automorphism group of $C$ contains a subgroup isomorphic to $V_4 := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By combining Proposition 2.2.1 and Lemma 2.2.3 we here prove the converse for the hyperelliptic case:

**Lemma 2.3.2** (cf. [21, Remark 2]). Let $C$ be a hyperelliptic curve of genus $g$ over $k$. Then $C$ is a generalized Howe curve if and only if $\text{Aut}(C)$ contains a subgroup isomorphic to $V_4$. In this case, $C$ is birational to $C_1 \times_{\mathbb{P}^1} C_2$ for hyperelliptic curves $C_1$ and $C_2$ of genera $g_1$ and $g_2$ as in Lemma 2.2.3 where $g_1 = g_2$ (resp. $g_2 = g_1 + 1$) if $g$ is even (resp. odd).

**Proof.** The “only if”-part was proved in [21, §4], and thus we here prove the “if”-part only. Assume that $\text{Aut}(C)$ contains a subgroup isomorphic to $V_4$. In this case, $C$ has a non-hyperelliptic involution, and thus we may suppose by Proposition 2.2.1 that $C$ is given by $y^2 = f(x^2)$ for some square-free $f \in k[x]$ of degree $g + 1$, with automorphisms $\sigma_1 : (x, y) \mapsto (-x, y)$ and $\sigma_2 : (x, y) \mapsto (-x, -y)$, where $\sigma_2 = \sigma_1 \circ C$. We then obtain two hyperelliptic curves $C_1 := C/\langle \sigma_1 \rangle$ and $C_2 := C/\langle \sigma_2 \rangle$, which are defined respectively by $y_1^2 = f(x)$ and $y_2^2 = x f(x)$ from the first half part of Lemma 2.2.3. By the second half part of Lemma 2.2.3 the fiber product $C_1 \times_{\mathbb{P}^1} C_2$ is birational to $C$.

Note that an argument similar to Lemma 2.3.2 is shown in [37, Theorem 5], where the characteristic of $k$ is assumed to be 0.

### 2.4. Decomposed Richelot isogenies

Let $A$ be an abelian variety and $\hat{A}$ be its dual. For an ample divisor $D$ on $A$, there is an isogeny $\Phi_D : A \to A$ defined by $x \mapsto T_2(L) - L$, where $T_2 : A \to A$ is a translation map $a \mapsto a + x$. If $\Phi_D$ is an isomorphism, we call $D$ a principal polarization on $A$, and a pair $(A, D)$ a principally polarized abelian variety. We define a product of polarized abelian varieties as

$$\prod_{i=1}^n (A_i, D_i) = \left( A_1 \times \cdots \times A_n, \sum_{i=1}^n (A_1 \times \cdots \times A_{i-1} \times D_i \times A_{i+1} \times \cdots \times A_n) \right).$$

Let $C$ be a smooth projective curve over $k$ of genus $g \geq 1$, and let $J(C)$ be a Jacobian variety of $C$. For a divisor $D$ on $C$ of degree 1, there is a natural map $\alpha_C : C \to J(C)$ defined as $P \mapsto (P) - D$. The theta divisor of $J(C)$ is defined as a sum of $g - 1$ copies of $\alpha_C$, say $\Theta_C := \alpha_C(C) + \cdots + \alpha_C(C)$. It is known that $(J(C), \Theta_C)$ is a principally polarized abelian variety. If we have $J(C) \cong E^g$ for some supersingular elliptic curve $E$, we call $C$ a superspecial (s.sp. for short) curve. Recall from [12, Theorem 3.5] that for supersingular elliptic curves $E_1, \ldots, E_{2g}$ with $g \geq 2$, we have $E_1 \times \cdots \times E_{2g} \cong E_{g+1} \times \cdots \times E_{2g}$.

Let $(A, D)$ be a principally polarized abelian variety over an algebraically closed field $k$, and let $C$ be a curve over $k$. If an isogeny $\phi : J(C) \to A$ satisfies $\phi^*(\Theta_C) \approx 2D$, we call $\phi$ a Richelot isogeny, where $\approx$ is an algebraic equivalence. We call $\phi$ a decomposed Richelot isogeny if there exists two principally polarized abelian
varieties \((A_1, D_1)\) and \((A_2, D_2)\) such that
\[
(A, D) \cong (A_1, D_1) \times (A_2, D_2).
\]
If there are elliptic curves \((E_1, O_{E_1}), \ldots, (E_g, O_{E_g})\) satisfying
\[
(A, D) \cong \prod_{i=1}^g (E_i, O_{E_i}),
\]
we call \(\phi\) a completely decomposed Richelot isogeny.

In the following, we recall some results in [18] and [21] on decomposed Richelot isogenies. Katsura-Takashima showed in [21] that any generalized Howe curve always has a decomposed Richelot isogeny:

**Theorem 2.4.1** ([21 Theorem 3]). Let \(C\) be a generalized Howe curve associated with two genus-1 curves \(C_1\) and \(C_2\), and let \(\sigma\) and \(\tau\) be the automorphisms on \(C\) of order 2 obtained by lifting the hyperelliptic involutions \(\iota_{C_1}\) and \(\iota_{C_2}\) respectively, say \(C_1 \cong C/\langle \sigma \rangle\) and \(C_2 \cong C/\langle \tau \rangle\). We also let \(C_3 := C/\langle \sigma \circ \tau \rangle\). Then, there exists a decomposed Richelot isogeny \(\hat{J}(C) \to J(C_1) \times J(C_2) \times J(C_3)\).

Since a hyperelliptic curve with an extra involution is a generalized Howe curve by Lemma 2.3.2, it follows from Theorem 2.4.1 that we have the following corollary:

**Corollary 2.4.2** ([21 Theorem 1]). Let \(C\) be a hyperelliptic curve with an automorphism \(\sigma\) of order 2, which is not the hyperelliptic involution \(\iota_C\). Then, there exists a decomposed Richelot isogeny \(\hat{J}(C) \to J(C/\langle \sigma \rangle) \times J(C/\langle \sigma \circ \iota_C \rangle)\).

**Remark 2.4.3.** It follows from Lemma 2.3.2 that we can take \(C_1 := C/\langle \sigma \rangle\) and \(C_2 := C/\langle \sigma \circ \iota_C \rangle\) in Corollary 2.4.2 to be hyperelliptic curves of genera \(g_1\) and \(g_2\) with \(g_1 = g_2 = g/2\) (resp. \(g_1 = (g - 1)/2\) and \(g_2 = (g + 1)/2\)) if \(g\) is even (resp. odd). Such a decomposition of \(J(C)\) is already given in [21] Theorem 5 by Paulhus based on Kani-Rosen’s decomposition method [17, Theorem B], although she did not prove an isogeny providing the decomposition is a Richelot one.

In the case of genus 3, Katsura [18] proved stronger arguments than Theorem 2.4.1 and Corollary 2.4.2. More precisely, he proved in [18] §4 that the converse of Corollary 2.4.2 also holds for genus-3 hyperelliptic curves:

**Theorem 2.4.4.** Let \(C\) be a hyperelliptic curve of genus 3, and \(\iota_C\) its hyperelliptic involution. Then, \(C\) has an extra involution \(\sigma\) if and only if there exists a decomposed Richelot isogeny \(\hat{J}(C) \to J(C/\langle \sigma \rangle) \times J(C/\langle \sigma \circ \iota_C \rangle)\).

**Proof.** See [21 Theorem 1] and Proposition 2.2.1 \( \square \)

As for Theorem 2.4.1, a (not necessarily hyperelliptic) Howe curve of genus 3 associated with two genus-1 curves has two different automorphisms of order 2. These automorphisms are long automorphisms defined as follows:

**Definition 2.4.5** (Long automorphism, [18 Definition 3.4]). Let \(C\) be a curve of genus \(g\), and \(\sigma\) an order-2 automorphism of \(C\). We denote by \(\sigma^*\) the action on \(H^0(C, \Omega_C^1)\) induced from \(\sigma\), where \(\Omega_C^1\) is the sheaf of differential 1-forms on \(C\). We call \(\sigma\) a long automorphism if the \(g\) eigenvalues of \(\sigma^*\) are 1, \(-1, \ldots, -1\) (the number of \(-1\) is \(g - 1\)).

For an order-2 automorphism of a curve \(C\), it is straightforward that \(\sigma\) is a long automorphism if and only if the dimension of the space \(\Gamma(C/\langle \sigma \rangle, \Omega_C/\langle \sigma \rangle)\) is 1, i.e., the quotient curve \(C/\langle \sigma \rangle\) has genus 1 (this also means that \(C\) is bielliptic). In the case of genus 3, any order-2 automorphism of a non-hyperelliptic curve is a long automorphism (cf. [18 Corollary 5.3]), and moreover we have the following:

**Theorem 2.4.6** ([18 Proofs of Theorem 6.2 and Theorem 6.3]). Let \(C\) be a curve of genus 3 over \(k\). Then the following are equivalent:

1. \(C\) is a Howe curve of genus 3 associated with two genus-1 curves.
2. \(C\) has two long automorphisms \(\sigma\) and \(\tau\) with \(\sigma \neq \tau\) such that \(\sigma \circ \tau = \tau \circ \sigma\).
(3) There is a completely decomposed Richelot isogeny \( J(C) \to E_1 \times E_2 \times E_3 \) for some genus-1 curves \( E_1, E_2, \) and \( E_3. \)

In this case, with \( \sigma \) and \( \tau \) in (2), one can take \( E_1, E_2, \) and \( E_3 \) in (3) as \( E_1 := C/\langle \sigma \rangle, E_2 := C/\langle \tau \rangle, \) and \( E_3 := C/\langle \sigma \circ \tau \rangle \) respectively such that \( C \) is birational to \( E_1 \times_{\mathbb{P}^1} E_2. \)

### 3. COMPUTING DECOMPOSED RICHELOT ISOGENIES

In this section, we present algorithms to explicitly compute the codomain of decomposed Richelot isogenies, based on the theory of \[13\] and \[21\] together with properties of automorphisms described in Subsection \[2,2.4\].

#### 3.1. Case of hyperelliptic curves

In this subsection, we explain a precise algorithm to compute decomposed Richelot isogenies of hyperelliptic curves.

**Theorem 3.1.1.** Let \( C \) be a hyperelliptic curve of genus \( g \) and \( \iota_C \) its hyperelliptic involution. Assume that \( C \) has an extra involution \( \sigma \). For a given \( C \), there is an algorithm (explicitly, Algorithm \[2,\] to compute defining equations for the quotient curves \( C/\langle \sigma \rangle \) and \( C/\langle \sigma \circ \iota_C \rangle \), which define the codomain of a decomposed Richelot isogeny in Theorem \[2,4.3\]. Moreover, all the computations in the algorithm are done in the quadratic extension of a subfield \( K \) of \( k \), over which all the Weierstrass points of \( C \) are defined. Furthermore, the number of arithmetic operations (in the quadratic extension field) required for executing the algorithm is upper-bounded by a constant.

**Proof.** In the general case, it seems hard to compute \( C/\langle \sigma \rangle \) and \( C/\langle \sigma \circ \iota_C \rangle \) for some involution \( \sigma \) from \( C \). However, if \( \sigma \) is the automorphism \( (x, y) \mapsto (-x, y) \), then we can easily compute \( C/\langle \sigma \rangle \) and \( C/\langle \sigma \circ \iota_C \rangle \) from Lemma \[2,2.4\]. Proposition \[2,2.1\] claims that there is an isomorphism \( \rho : C' \to C \) such that \( \rho \circ \iota_C \circ \rho \) is the automorphism \( (x, y) \mapsto (-x, y) \) of \( C' \). By swapping \( \sigma \) and \( \iota_C \) if necessary, it suffices to consider only the case that \( \rho^{-1} \circ \rho \circ \iota_C \) provides the target curves \( C/\langle \sigma \rangle \) and \( C/\langle \sigma \circ \iota_C \rangle \).

Let \( (x_1, 0), \ldots, (x_{2g+2}, 0) \) be the Weierstrass points of \( C \). We have \( (x_1, 0) = \sigma((x_1, 0)) \) and \( (x_2, 0) = \sigma((x_2, 0)) \) for some \( s, t_1, t_2 \in \{1, \ldots, 2g + 2\} \) with \( t_1 \neq 1 \) and \( s \neq t_1, t_2 \neq 1, t_1 s \). From \[3,\] Proposition \( A \), a representation matrix of \( \sigma \) is given by

\[
\begin{bmatrix}
 x_s x_{t_2} - x_1 x_{t_1} & (x_s - x_{t_2} + x_1) x_{t_1} - (x_1 + x_{t_2}) x_s x_{t_2} \\
 (x_s + x_{t_2} - x_1) x_{t_1} - (x_1 + x_{t_2}) x_s x_{t_2}
\end{bmatrix}.
\]

Note that if one of the Weierstrass points of \( C \) is a point at infinity, the above matrix is constructed under a proper projectivization (i.e., use \( (X_s : Z_s) \) with \( X_s/Z_s = x_s \) for representing \( (x_s, 0) \)). Therefore, by diagonalizing the above matrix, we obtain a corresponding matrix of \( \rho \) if \( t_1, t_2, s \) are taken properly. That is, if \( (x_1, 0) = \sigma((x_1, 0)) \) and \( (x_2, 0) = \sigma((x_2, 0)) \), the automorphism \( \rho^{-1} \) can be represented by

\[
\begin{bmatrix}
 x_s x_{t_2} - x_1 x_{t_1} + \lambda & (x_s + x_{t_2}) x_{t_1} - (x_1 + x_{t_2}) x_s x_{t_2} \\
 (x_s x_{t_2} - x_1 x_{t_1}) - (x_1 + x_{t_2}) x_s x_{t_2}
\end{bmatrix},
\]

where

\[
\lambda = \sqrt{(x_s x_{t_2} - x_1 x_{t_1})^2 + ((x_s + x_{t_2}) - (x_1 + x_{t_2}))((x_s + x_{t_2}) x_{t_1} - (x_1 + x_{t_2}) x_s x_{t_2}).
\]

The candidates of \( t_1 \) are in the set \( \{2, \ldots, 2g + 2\} \), and those of \( t_2 \) are in the set \( \{2, \ldots, 2g + 2\} \setminus \{s, t_1\} \). It is enough to take \( s \) as the minimal number other than \( 1 \) and \( t_1 \). Therefore, the number of the candidates of \( \rho \) is at most \( 2(2g + 1)(2g - 1) = 2(4g^2 - 1) \). If \( \rho \) is a correct one, then the image points of Weierstrass points of \( C \) under \( \rho^{-1} \) can be represented by \( (\pm x_1', 0), \ldots, (\pm x_{2g+1}', 0) \) for some \( x_1', \ldots, x_{2g+1}' \in k \).

Conversely, if the image points of Weierstrass points of \( C \) under \( \rho^{-1} \) have forms
Algorithm 1 Compute extra involutions of hyperelliptic curves (Extraliv)

Require: A hyperelliptic curve $C$: $y^2 = (x - x_1) \cdots (x - x_{2g+2})$
Ensure: A set of pairs of a matrix corresponding to an isomorphism $C \to C'$ that diagonalizes an extra involution of $C$ and all Weierstrass points of $C'$

1: $R \leftarrow \emptyset$
2: for all $t_1 \in \{2, \ldots, 2g+2\}$ do
3: \hspace{1em} $s \leftarrow$ the minimum integer in $\{2, \ldots, 2g+2\} \setminus \{t_1\}$
4: \hspace{1em} for all $t_2 \in \{2, \ldots, 2g+2\} \setminus \{t_1, s\}$ do
5: \hspace{2em} $\alpha \leftarrow x_s x_{t_2} - x_{t_1} x_t$
6: \hspace{2em} $\beta \leftarrow (x_s + x_{t_2}) x_{t_1} - (x_t + x_{t_1}) x_s x_{t_2}$
7: \hspace{2em} $\gamma \leftarrow (x_s + x_{t_1}) - (x_t + x_{t_1})$
8: \hspace{2em} $\lambda \leftarrow \sqrt{\alpha^2 + \beta \gamma}$
9: \hspace{2em} $x'_1, \ldots, x'_{2g+2} \leftarrow (\alpha + \lambda)x_1 + \beta, \ldots, (\alpha + \lambda)x_{2g+2} + \beta, (-\alpha + \lambda)x_{2g+2} - \beta$
10: \hspace{1em} if for all $i = 1, \ldots, 2g+2$, there exists $j \neq i$ such that $x'_j = -x'_j$ then
11: \hspace{2em} $R \leftarrow R \cup \left\{ \left( \begin{array}{cc} \alpha + \lambda & \beta \\ -\alpha + \lambda & -\beta \end{array} \right), (x'_1, \ldots, x'_{2g+2}) \right\}$
12: \hspace{1em} end if
13: \hspace{1em} end for
14: \hspace{1em} end for
15: return $R$

of $(\pm x'_1, 0), \ldots, (\pm x'_{g+1}, 0)$, then $C'$ has an automorphism $(x, y) \mapsto (-x, y)$, and $\rho$ is the correct candidate. Therefore, correct candidates of $\rho$ are provided by checking for each candidate whether the images of Weierstrass points under $\rho^{-1}$ are $(\pm x'_1, 0), \ldots, (\pm x'_{g+1}, 0)$ for some $x'_1, \ldots, x'_{g+1} \in k$.

From the above discussions, we can compute an isomorphism $\rho^{-1}$, and this gives an algorithm to compute the codomain of a decomposed Richelot isogeny $\langle \delta \rangle$ from $J(C)$. In Algorithms 1 and 2, we write down a computational method described above in algorithmic format.

We finally determine the complexity of Algorithm 2. Let $K$ be a subfield of $k$ over which all the Weierstrass points of $C$ are defined, and $K'$ its quadratic extension field. As for Algorithm 1 called at the line 1, the number of iterations for each for-loop is bounded by a constant, and clearly all the computations are done in a constant number of arithmetic operations on $K'$. For the same reason, the cardinality of $R$ is also bounded by a constant. Clearly the lines 4–7 require $O(1)$ arithmetic operations on $K'$, and consequently our assertion for the total complexity holds.

3.2. Case of genus-3 hyperelliptic curves having completely decomposed Richelot isogenies. Next, we explain a computational method for completely decomposed Richelot isogenies from Jacobian varieties of hyperelliptic curves of genus 3.

Theorem 3.2.1. Let $C$ be a hyperelliptic curve of genus 3. Assume that $C$ has two long automorphisms $\sigma$ and $\tau$ such that $\sigma \circ \tau = \tau \circ \sigma$. For a given $C'$, there is an algorithm to compute curves constructing the codomain of a completely decomposed Richelot isogeny in Theorem 2.4.6, that is, $\langle \sigma \rangle$, $\langle \tau \rangle$, and $\langle \sigma \circ \tau \rangle$. Moreover, all the computations in the algorithm are done in the quadratic extension of a subfield $K$ of $k$, over which all the Weierstrass points of $C$ are defined. Furthermore, the number of arithmetic operations (in the quadratic extension field) required for executing the algorithm is upper-bounded by a constant.
By using Algorithm 1, we can find □

In this □

Let I

Since

Note that

\( \beta \) and \( e \)igenvalues of the corresponding matrix of \( A \) and thus \( c \)

\( \sigma \) and corresponding \( \tau \)eigenvectors of them. After checking the commutativity of \( \alpha \) and \( \beta \), we can compute \( \sigma \circ \tau \) and \( C/\langle \sigma \circ \tau \rangle \) from diagonalization. Proposition 3.2.2 below provides the eigenvalues of the corresponding matrix of \( \sigma \circ \tau \).

The assertions on the complexity follow immediately from Theorem 3.1.1.

\[ \text{Algorithm 2 Decomposed Richelot isogeny of hyperelliptic genus-} g \text{ curves} \]

\[ \text{Require: A hyperelliptic curve} \, C : y^2 = (x - x_1) \cdots (x - x_{g+2}) \]

\[ \text{Ensure: A set of pairs of a curve} \, C_1 \text{ of genus} \, \lfloor g/2 \rfloor \text{ and a curve} \, C_2 \text{ of genus} \, \lceil g/2 \rceil \]

\[ \text{such that there is a decomposed Richelot isogeny} \, J(C) \rightarrow J(C_1) \times J(C_2) \]

1: \( R \leftarrow \text{ExtraInv}(C) \) (Algorithm 1)
2: \( S \leftarrow \emptyset \)
3: for all \((Q, (x'_1, \ldots, x'_{2g+2})) \in R\) do
4: \( x'_i \leftarrow -x'_{i+(g+1)} \) for \( i = 1, \ldots, g + 1 \) by changing indexes
5: \( f_1(x) \leftarrow (x - x_1^2) \cdots (x - x_{g+1}^2), \quad f_2(x) \leftarrow x \cdot f_1(x) \)
6: \( C_1 : y^2 = f_1(x), \quad C_2 : y^2 = f_2(x) \)
7: \( S \leftarrow S \cup \{(C_1, C_2)\} \)
8: end for
9: return \( S \)

Proof. As shown in the proof of Theorem 3.1.1 it is enough to find a hyperelliptic curve \( C' \) and an isomorphism \( \rho : C' \rightarrow C \) such that \((\rho^{-1} \circ \sigma \circ \rho)(x, y) = (-x, y)\) in order to compute a curve \( C/\langle \sigma \rangle \). Therefore, to compute \( C/\langle \sigma \rangle \) and \( C/\langle \tau \rangle \), we need to find eigenvector matrices of two long automorphisms \( \sigma \) and \( \tau \) satisfying \( \sigma \circ \tau = \tau \circ \sigma \). By using Algorithm 1 we can find \( \sigma \) and \( \tau \) and corresponding eigenvectors of them. After checking the commutativity of \( \sigma \) and \( \tau \), we can compute \( \sigma \circ \tau \) and \( C/\langle \sigma \circ \tau \rangle \) from diagonalization. Proposition 3.2.2 below provides the eigenvalues of the corresponding matrix of \( \sigma \circ \tau \).

For the eigenvalues of \( \sigma \circ \tau \), we have the following proposition.

Proposition 3.2.2. Let \( C \) be a hyperelliptic curve of genus 3. Assume that \( C \) has two long automorphisms \( \sigma \) and \( \tau \). Let \( A \) and \( B \) be matrices in \( \text{GL}_2(k) \) corresponding to \( \sigma \) and \( \tau \) respectively. Denote the eigenvalues of \( A \) and \( B \) by \( \pm \lambda_A \) and \( \pm \lambda_B \) respectively. Then, the eigenvalues of \( AB \) are \( \pm \sqrt{-\lambda_A \lambda_B} \).

Proof. Suppose that \( \sigma \circ \tau = \tau \circ \sigma \). There is a constant value \( c \) such that

\[ AB = cBA. \]

Let \( \alpha \) and \( \beta \) be the eigenvalues of \( AB \). Since \( AB \) is a regular matrix, we have \( \alpha \neq 0 \) and \( \beta \neq 0 \). Since \( AB \) and \( BA \) have same eigenvalues, it holds that

\[ (ca, c\beta) = (\alpha, \beta) \text{ or } (\alpha, c\beta) = (\beta, \alpha), \]

and thus \( c^2 = 1 \). We moreover claim \( c = -1 \). Indeed, if \( c = 1 \) (i.e., \( AB = BA \)), then \( A \) and \( B \) are simultaneously diagonalizable. Thus it follows that \( AB = \pm \lambda_A \lambda_B I_2 \), which contradicts \( \sigma \circ \tau \neq \text{id}_C \). Therefore we have \( c = -1 \) and \( AB = -BA \). Since \( A^2 = \lambda_A^2 I_2 \) and \( B^2 = \lambda_B^2 I_2 \), it holds that

\[ (AB)^2 = ABAB = -A^2 B^2 = -\lambda_A^2 \lambda_B^2 I_2. \]

Therefore, the eigenvalues of \( AB \) are \( \pm \sqrt{-\lambda_A \lambda_B} \).

3.3. Case of smooth plane quartics (non-hyperelliptic curves of genus 3).

In this subsection, we explain a way to compute decomposed Richelot isogenies outgoing from the Jacobian variety of a non-hyperelliptic curve of genus 3. In this subsection, we suppose that \( k \) is an algebraic closure of \( \mathbb{F}_p \) with \( p \geq 5 \). Note that from [18 Corollary 5.2], such isogenies are always completely decomposed.

Let \( C \) be a non-hyperelliptic curve of genus 3 over \( k \), which is isomorphic to a non-singular plane quartic \( F(X, Y, Z) = 0 \), where \( F \) is an irreducible quartic form.
in $X$, $Y$, and $Z$ over $k$. Recall also that any isomorphism between plane non-singular curves over $k$ is represented by an element of $\text{PGL}_3(k)$ uniquely. From Theorem 2.4.6, if there is a completely decomposed Richelot isogeny outgoing from $J(C)$, then the automorphism group of $C$ contains $V_4$ as its subgroup, where $V_4$ is the dihedral group of order 4, i.e., $V_4 = (\mathbb{Z}/2\mathbb{Z})^2$. Namely $C$ has different two long automorphisms $\sigma$ and $\tau$ with $\sigma \circ \tau = \tau \circ \sigma$. Such non-hyperelliptic curves are isomorphic to curves of the following form (cf. [29, Theorem 3.1]):

$$C': a_1x^4 + a_2y^4 + a_3x^2y^2 + a_4x^2 + a_5y^2 + a_6 = 0$$

with $a_i \in k$, which is referred to as standard form (cf. [3], [53]). In this case, we easily find long automorphisms of $C'$, say $\sigma : (x,y) \mapsto (-x, y)$ and $\tau : (x, y) \mapsto (x, -y)$ with $\sigma \circ \tau = \tau \circ \sigma$, and a completely decomposed Richelot isogeny

$$J(C') \to C'/\langle \sigma \rangle \times C'/\langle \tau \rangle \times C'/\langle \sigma \circ \tau \rangle,$$

where $C'/\langle \sigma \rangle$, $C'/\langle \tau \rangle$, and $C'/\langle \sigma \circ \tau \rangle$ are elliptic curves defined as:

$$C'/\langle \sigma \rangle : a_1x^2 + a_2y^2 + a_3xy^2 + a_4x + a_5y + a_6 = 0,$$

$$C'/\langle \tau \rangle : a_1x^4 + a_2y^2 + a_3x^2y^2 + a_4x^2 + a_5y^2 + a_6 = 0,$$

$$C'/\langle \sigma \circ \tau \rangle : a_1x^4 + a_2 + a_3x^2 + a_4x^2y + a_5y + a_6y^2 = 0.$$

Therefore, our problem is reduced into finding a curve $C'$ in standard form (3.3.1) and an isomorphism between $C'$ and $C$ explicitly, in order to compute completely decomposed Richelot isogenies outgoing from $J(C)$. For finding them, it suffices to compute a matrix in $\text{PGL}_3(k)$ representing an isomorphism $\rho : C' \to C$ such that $\rho^{-1} \circ \sigma \circ \rho(X : Y : Z) = (-X : Y : Z)$ and $\rho^{-1} \circ \tau \circ \rho(X : Y : Z) = (X : -Y : Z)$.

**Theorem 3.3.1.** With notation as above, the computation of a matrix in $\text{PGL}_3(k)$ representing $\rho : C' \to C$ is reduced into that of solving a zero-dimensional multivariate system over a finite extension of a subfield $K$ of $k$, where the extension degree does not depend on $p$, and where any coefficient of $F$ belongs to $K$. Furthermore, the number of arithmetic operations (in the finite extension field) required for the computation is upper-bounded by a constant.

**Proof.** In general, any $A$ in $\text{PGL}_3(k)$ of order 2 has eigenvalues 1 and $-1$, so that it can be diagonalized into $\text{diag}(-1, 1, 1)$ by multiplying $A$ by $-1$ if necessary. Let $A_\sigma$ and $A_\tau$ be matrices corresponding to $\sigma$ and $\tau$, respectively, say, $A_\sigma A_\tau = A_\tau A_\sigma$. Since $A_\sigma \neq A_\tau$, there is a matrix $Q$ such that

$$A_\sigma = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q^{-1} \quad \text{and} \quad A_\tau = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q^{-1}.$$

Then, the matrix $Q$ corresponds to the target isomorphism $\rho : C' \to C$, and so it suffices to construct $Q$ for our purpose. Note that if $A_{\sigma\tau}$ is a matrix in $\text{PGL}_3(k)$ corresponding to $\sigma \circ \tau$, then it holds that

$$A_{\sigma\tau} = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Q^{-1}.$$

Therefore, swapping columns of $Q$ is equivalent to swapping three long automorphisms $\sigma$, $\tau$, and $\sigma \circ \tau$. Moreover, we know that multiplying each column of $Q$ by a constant does not affect diagonalizations of $A_\sigma$, $A_\tau$, and $A_{\sigma\tau}$. Hence, it suffices to consider the cases that the matrix $Q$ is defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b & c \\ 1 & e & f \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b & c \\ 1 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & b & c \\ 0 & e & f \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & b & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix},$$
Here, $C'$ is defined by $F((X:Y:Z)\cdot^t Q) = 0$, and it follows from (3.3.1) that the coefficient of any monomial in $F((X,Y,Z)\cdot^t Q)$ other than $X^4$, $Y^4$, $X^2Y^2$, $X^2Z^2$, $Y^2Z^2$, and $Z^4$ should be zero. From this, for each of the above 5 case, we obtain a multivariate system of quartic equations with respect to $a$, $b$, $c$, $d$, $e$, and $f$, and it follows from Lemma 3.3.2 that the number of its roots is bounded by a constant not depending on $p$.

Since each of the systems consists of 9 inhomogeneous polynomials of degree at most 4 in at most 6 variables, it follows from arguments in Remark 3.3.3 below that computing the roots of each system with Gröbner basis algorithms terminates in $\tilde{O}(1)$ arithmetic operations in a finite extension field $K'$ of $K$, where the extension degree does not depend on $p$. It is clear that the other parts of the algorithm requires $O(1)$ arithmetic operations in $K'$.

**Lemma 3.3.2.** The number of roots of each multivariate system constructed in Theorem 3.3.1 is bounded by a constant not depending on $p$.

**Proof.** Since $\rho \circ [(x,y) \mapsto (-x,y)] \circ \rho^{-1}$ and $\rho \circ [(x,y) \mapsto (x,-y)] \circ \rho^{-1}$ are long automorphisms of $C$, each $Q$ determined by a root of the multivariate system simultaneously diagonalizes at least one pair of long automorphisms. Let $\sigma$ and $\tau$ be different long automorphisms of $C$ such that $Q$ simultaneously diagonalizes these long automorphisms. Since the dimension of each common eigenspace of $\sigma$ and $\tau$ is 1, the other matrices determined by roots that simultaneously diagonalize $\sigma$ and $\tau$ are obtained by swapping columns of $Q$. Therefore, the number of candidates of $Q$ that simultaneously diagonalizes $\sigma$ and $\tau$ is at most 6. From [29, Theorem 3.1], the number of long automorphisms of a non-hyperelliptic curve is bounded by a constant not depending on $p$. Hence, our claim on the number of roots holds.

**Remark 3.3.3.** In general, for a zero-dimensional affine system of inhomogeneous polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ with $n \leq m$ whose total degrees are $d_1 \geq \ldots \geq d_m$ respectively, the maximal total degree of elements in the reduced Gröbner basis of $F = \{f_1, \ldots, f_m\}$ with respect to an arbitrary monomial order is upper-bounded by $d_1 \cdots d_n$, see [10, Theorem 5.4]. This implies that the extension degree of an extension field $L$ of $K$ where all the roots in $\overline{K}$ of $F$ are defined is bounded by a function of $d_1, \ldots, d_n$.

A typical way of computing the roots of a zero-dimensional affine system $F$ is as follows: Compute the reduced Gröbner basis $G$ of the homogenization of $F$ with respect to a graded monomial order by e.g., the $F_4$ algorithm [9]. Then, convert $G$ to a Gröbner basis $G'$ with respect to a lexicographic monomial order by the FGLM basis conversion [10]. Substituting 1 for an extra variable for homogenization and reducing it, we obtain the reduced Gröbner basis $G''$ of the input $F$ with respect to an induced lexicographic monomial order. We can compute the roots of $G''$ by repeating the univariate polynomial factorization and evaluations, by extending base fields if necessary. All of these computations are done over $L$ defined as above. It is well-known that the complexity of the $F_4$ algorithm is bounded by a function of the number $n$ of variables, the maximal total degree $d$ of the input $F$, and the number $m$ of the input polynomials. The complexity of the FGLM algorithm is $O(nD^3)$ operations in $K$, where $D$ denotes the number of roots of $F$ in $\overline{K}$ counted with multiplicity. It is also well-known that $D$ is bounded by $d_1 \cdots d_n$.

If $K = \mathbb{F}_p$ and if $n, m, d_1, \ldots, d_n$ are fixed (not depending on the characteristic $p$ of $K$), we can take $L = \mathbb{F}_{q^r}$ for some constant $r$ not depending on $p$. In this case, the complexities of the $F_4$ and FGLM algorithms are both $O(1)$ arithmetic operations in $K$. All the roots in $L^n$ of the input system can be computed by repeating the univariate polynomial factorization and evaluations, which are done in $\tilde{O}(1)$ arithmetic operations in $L$. 

4. Explicit construction of Howe curves of genus 3

Let $k$ be an algebraic closure of $\mathbb{F}_p$ with $p \geq 3$. In this section, we determine defining equations of generalized Howe curves of genus 3 explicitly. Recall that a generalized Howe curve $C$ is a curve birational to the fiber product of two hyperelliptic curves of genera $g_1$ and $g_2$, and it follows from Proposition 2.3.1 that $C$ has genus 3 if and only if $(g_1, g_2) = (1, 1)$ or $(1, 2)$, namely $C$ is birational to either of the following:

- **(Oort type)** $E_1 \times_{\mathbb{F}_p} E_2$ for two genus-1 curves $E_1$ and $E_2$ which share exactly 2 Weierstrass points, or
- **(Howe type)** $E \times_{\mathbb{F}_p} H$ for a genus-1 curve $E$ and a genus-2 curve $H$ which share exactly 4 Weierstrass points.

Note that any $C$ of Howe type is always hyperelliptic by Proposition 2.3.1 and we may assume from Lemma 2.3.2 that $E$, $H$, and $C$ are given as

\[(4.0.1) \quad E: y^2 = f(x), \quad H: y^2 = xf(x), \quad C: y^2 = f(x^2),\]

where $f$ is a quartic polynomial over $k$ with non-zero discriminant.

On the other hand, any $C$ of Oort type can be either hyperelliptic or non-hyperelliptic. Transforming the 2 common Weierstrass points to 0 and $\infty$ by an element of $\text{PGL}_2(k)$, we may assume that $E_1$ and $E_2$ are given as follows:

\[(4.0.2) \quad E_1 : y_1^2 = \phi_1(x) = x(x - \alpha_2)(x - \alpha_3), \quad E_2 : y_2^2 = \phi_2(x) = x(x - \beta_2)(x - \beta_3),\]

where $\alpha_2, \alpha_3, \beta_2$, and $\beta_3$ are pairwise distinct elements in $k \setminus \{0\}$. If necessary, we can replace the equation for $E_1$ by a Legendre form: Considering the transformation $x \mapsto \frac{\alpha_2}{x}$, we can replace $E_1$ and $E_2$ by

\[(4.0.3) \quad E_\nu : y^2 = x(x - 1)(x - \nu) \quad \text{and} \quad E_{\mu, \lambda} : y^2 = x(x - \mu)(x - \mu \lambda)\]

respectively, where $\nu = \frac{\alpha_2}{\alpha_3}$, $\mu = \frac{\alpha_3}{\alpha_2}$, and $\lambda = \frac{\beta_3}{\beta_2}$. In this case, recall from [36, §5.11] that $C$ is hyperelliptic if and only if $\nu = \mu^2 \lambda$, which is equivalent to $\alpha_2 \alpha_3 = \beta_2 \beta_3$ by $\nu = \mu^2 \lambda = \frac{\alpha_2 \alpha_3 \beta_2 \beta_3}{\alpha_2 \alpha_3}$.

In Subsection 4.1 (resp. 4.2) below, we shall find an equation for the normalization $C$ of Oort type in the hyperelliptic (resp. non-hyperelliptic) case. We also study the field of definition, the $\mathbb{F}_p^2$-maximality and the $\mathbb{F}_p^2$-minimality in the case where $C$ is superspecial.

**Remark 4.0.1.** Howe, Leprévost, and Poonen proposed in [13] a formula to compute a curve of genus 3 whose Jacobian variety is Richelot isogenous to a given product of three elliptic curves. Our method is different from their method. In fact, their method is similar to Vélu’s formulas, while our method directly constructs a fiber product of elliptic curves.

**Remark 4.0.2.** As for the case of genus 2, Katsura and Oort [19, Section 2] studied genus-2 (hyperelliptic) curves with automorphism group containing a subgroup isomorphic to $V_4$; such a genus-2 curve is nothing but a genus-2 generalized Howe curve $C$ birational to $E_1 \times_{\mathbb{F}_p} E_2$ with elliptic curves $E_1$ and $E_2$ (cf. Subsection 2.3). In particular, they counted superspecial $C$’s with $\text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$ by investigating the supersingularity of $E_1$ and $E_2$ as quotient curves of $C$.

4.1. Hyperelliptic case. The following theorem determines an equation for a hyperelliptic Howe curve of Oort type:

**Theorem 4.1.1.** Let $\alpha_2, \alpha_3, \beta_2, \beta_3 \in k$ such that these are not 0 and different from each other. Assume that $\alpha_2 \alpha_3 = \beta_2 \beta_3$, and let $E_1$ and $E_2$ be elliptic curves defined...
as in (4.0.1). Then, a hyperelliptic Howe curve $C$ of genus 3 birational to $E_1 \times_{\mathbb{P}^1} E_2$ can be represented by

\[
y^2 = \left( x^2 - \frac{\sqrt{\alpha_2 + \alpha_3 + \sqrt{\beta_2 + \sqrt{\beta_3}}}^2}{(\alpha_2 + \alpha_3 - \beta_2 - \beta_3)} \right) \left( x^2 - \frac{\sqrt{\alpha_2 + \alpha_3 - \sqrt{\beta_2 - \sqrt{\beta_3}}}^2}{(\alpha_2 + \alpha_3 - \beta_2 - \beta_3)} \right).
\]

Proof. Note that $C$ is a hyperelliptic curve from the same discussion in [36, §5.11]. To find an equation defining $C$, it suffices to compute image points of its Weierstrass points under the double cover $C \to \mathbb{P}^1$ of degree 2. Since the Weierstrass points of $C$ are fixed points of the hyperelliptic involution of $C$, constructing the hyperelliptic involution provides the Weierstrass points. From [36, §5.11], there is an order-2 automorphism $s$ of $\mathbb{P}^1$ that corresponds to the hyperelliptic involution of $C$ and satisfies $s(0 : 1) = (1 : 0)$, $s(\alpha_2 : 1) = (\alpha_3 : 1)$, and $s(\beta_2 : 1) = (\beta_1 : 1)$. By a straightforward computation, we have

\[
Q \left( \frac{\alpha_2}{1} \right) = \left( \frac{\alpha_2 \alpha_3}{\alpha_2} \right), \quad Q \left( \frac{\beta_2}{1} \right) = \left( \frac{\alpha_2 \alpha_3}{\beta_2} \right), \quad Q^2 = \alpha_2 \alpha_3 I_2,
\]

where $Q = \left( \begin{array}{cc} 0 & \alpha_2 \alpha_3 \\ 1 & 0 \end{array} \right)$. Therefore, $Q$ corresponds to the automorphism $s$. Next, we find the fixed points of $Q$ in $E_1 \times_{\mathbb{P}^1} E_2$ under the lifted automorphism of $s$. By considering the diagonalization of $Q$, we have

\[
\left( \begin{array}{cc} -\sqrt{\alpha_2 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \\ \sqrt{\alpha_2 \alpha_3} & 1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & \alpha_2 \alpha_3 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} -\sqrt{\alpha_2 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \\ \sqrt{\alpha_2 \alpha_3} & 1 \end{array} \right) = \alpha_2 \alpha_3 \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right).
\]

Hence, there is an automorphism $\rho$ of $\mathbb{P}^1$ such that $(\rho^{-1} \circ s \circ \rho)(x) = -x$. The automorphism $\rho$ leads two isomorphism $\rho_1 : E_1' \to E_1$ and $\rho_2 : E_2' \to E_2$. By a direct computation, it holds that

\[
E_1' : y^2 = (x^2 - 1) \left( x^2 - \left( \frac{\sqrt{\alpha_2 - \sqrt{\alpha_3}}}{\sqrt{\alpha_2 + \sqrt{\alpha_3}}} \right)^2 \right),
\]

\[
E_2' : y^2 = (x^2 - 1) \left( x^2 - \left( \frac{\sqrt{\beta_2 - \sqrt{\beta_3}}}{\sqrt{\beta_2 + \sqrt{\beta_3}}} \right)^2 \right).
\]

More precisely, $E_1'$ and $E_2'$ are curves obtained by gluing two curves as the definition of hyperelliptic curves, respectively. Namely, the curve $E_1'$ is a curve represented by gluing

\[
y^2 = (x^2 - 1) \left( x^2 - \left( \frac{\sqrt{\alpha_2 - \sqrt{\alpha_3}}}{\sqrt{\alpha_2 + \sqrt{\alpha_3}}} \right)^2 \right) \text{ and } y^2 = (1-x^2) \left( 1 - \left( \frac{\sqrt{\alpha_2 - \sqrt{\alpha_3}}}{\sqrt{\alpha_2 + \sqrt{\alpha_3}}} \right)^2 \right)^2
\]

by a map $(x, y) \mapsto (1/x, y/x^2)$. The curve $E_1 \times_{\mathbb{P}^1} E_2$ is isomorphic to

\[
E_1' \times_{\mathbb{P}^1} E_2' : \begin{cases}
y_1^2 = (x^2 - 1) \left( x^2 - \left( \frac{\sqrt{\alpha_2 - \sqrt{\alpha_3}}}{\sqrt{\alpha_2 + \sqrt{\alpha_3}}} \right)^2 \right) \\
y_2^2 = (x^2 - 1) \left( x^2 - \left( \frac{\sqrt{\beta_2 - \sqrt{\beta_3}}}{\sqrt{\beta_2 + \sqrt{\beta_3}}} \right)^2 \right).
\end{cases}
\]

Note that the automorphism $s' : (x, y_1, y_2) \mapsto (-x, y_1, y_2)$ of $E_1' \times_{\mathbb{P}^1} E_2'$ corresponds to the hyperelliptic involution of $C$. The fixed points of $E_1' \times_{\mathbb{P}^1} E_2'$ under $s'$ are points whose $x$-coordinates are zero and points at infinity. These points are image points of Weierstrass points of $C$ under a birational
map between $E'_1 \times \mathbb{P} E'_2$ and $C$. Now, we construct a map $E'_1 \times \mathbb{P} E'_2 \to \mathbb{P}^1$ of degree 2. From the definition of $s'$, we have

$$(E'_1 \times \mathbb{P} E'_2)/(s') \ni \begin{cases} Y'_1 = (X - Z) \left( X - \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}} \right)^2 Z, \\ Y'_2 = (X - Z) \left( X - \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta^2 + \alpha^2}} \right)^2 Z \end{cases}.$$ 

Here, we represent $(E'_1 \times \mathbb{P} E'_2)/(s')$ via projective coordinates $(X : Y_1 : Y_2 : Z)$. Note that the natural surjective map $E'_1 \times \mathbb{P} E'_2 \to (E'_1 \times \mathbb{P} E'_2)/(s')$ is of degree 2. We construct an isomorphism between $(E'_1 \times \mathbb{P} E'_2)/(s')$ and $\mathbb{P}^1$. Let $\alpha = \frac{\sqrt{\beta^2 - \sqrt{\alpha}}}{{\sqrt{\beta^2 + \alpha^2}}}$ and $\beta = \frac{\sqrt{\beta^2 - \sqrt{\alpha}}}{{\sqrt{\beta^2 + \alpha^2}}}$. We have an isomorphism as follows:

$$(E'_1 \times \mathbb{P} E'_2)/(s') \to \mathbb{P}^1 \ \ (S : T : U) \mapsto \left( \sqrt{\alpha^2 - \beta^2} (V^2 + W^2) : \sqrt{\alpha^2 + \beta^2} (V^2 - W^2) : 2VW \right) \longleftrightarrow (V : W).$$

It is well known that a quadratic curve $\frac{\beta^2 - 1}{\beta^2 - \alpha^2} S^2 - \frac{\alpha^2 - 1}{\beta^2 - \alpha^2} T^2 = U^2$ is isomorphic to $\mathbb{P}^1$ by a map

$$\left( S : T : U \right) \mapsto \left( \sqrt{\frac{\alpha^2 - 1}{\beta^2 - \alpha^2}} S + \sqrt{\frac{\beta^2 - 1}{\beta^2 - \alpha^2}} \ V \right).$$

By considering the composition of the above two maps, we obtain a degree-2 map $\hat{x} : E'_1 \times \mathbb{P} E'_2 \to \mathbb{P}^1$. From a direct calculation, the image points of Weierstrass points of $C$ under $C \to E'_1 \times \mathbb{P} E'_2 \to (E'_1 \times \mathbb{P} E'_2)/(s')$ are $(0 : \pm \alpha : \pm \beta : 1)$ and $(1 : \pm 1 : \pm 1 : 0)$. Therefore, the image points of Weierstrass points under $\hat{x}$ are

$$\pm \alpha \sqrt{\frac{\beta^2 - 1}{\beta^2 - \alpha^2}} \pm \sqrt{\frac{\alpha^2 - 1}{\beta^2 - \alpha^2}} = \pm \alpha \sqrt{\alpha^2 - \beta^2 - \beta^2} \pm \sqrt{\beta^2 - \alpha^2},$$

$$\pm \beta \sqrt{\frac{\beta^2 - 1}{\beta^2 - \alpha^2}} \pm \sqrt{\frac{\alpha^2 - 1}{\beta^2 - \alpha^2}} = \pm \beta \sqrt{\alpha^2 - \beta^2 - \beta^2} \pm \sqrt{\beta^2 - \alpha^2}.$$ 

We denote these eight values by $x_1, \ldots, x_8$. Then, the target curve $C$ is defined by $y^2 = (x - x_1) \cdots (x - x_8)$. This completes the proof of Theorem 4.1.1.

As a particular case of Theorem 4.1.1, we obtain the equation (4.1.2) for $H$ as

$$y^2 = \left( x^2 - \frac{(1 + \sqrt{\mu})^2(1 + \sqrt{\mu})^2}{(1 - \mu \lambda)(1 - \mu)} \right) \left( x^2 - \frac{(1 - \sqrt{\mu})^2(1 - \sqrt{\mu})^2}{(1 + \mu \lambda)(1 + \mu)} \right) \left( x^2 - \frac{(1 + \sqrt{\mu})^2(1 - \sqrt{\mu})^2}{(1 - \mu \lambda)(1 - \mu)} \right),$$

when $E_1$ and $E_2$ are given by (4.1.3). With this equation, we obtain a result similar to Ohashi’s one in [34] on the $F_{\nu^2}$-maximality and the $F_{\nu^2}$-minimality of hyperelliptic Howe curves of genus 3:

**Corollary 4.1.2** (cf. [34] Theorem 1.1). Assume that $E_1$ and $E_2$ are given by (4.1.3) with $\nu = \nu^2 \lambda$. Let $C$ be a hyperelliptic Howe curve of genus 3 given in Theorem 4.1.1, put the equation (4.1.2) as $y^2 = \prod_{i=1}^4 (x^2 - A_i)$. If $C$ is superspecial, then each $A_i$ belongs to $F_{\nu^2}$, and moreover it is $F_{\nu^2}$-maximal (resp. $F_{\nu^2}$-minimal) if $p \equiv 3 \pmod 4$ (resp. $p \equiv 1 \pmod 4$).
Lemma 2.3.2 together with Theorem 4.1.1, we can immediately prove the

\[ \mu = \nu^2 \]

is the square of an element in \( \mathbb{F}_p^{\nu, \lambda} \) by \( (4.2.1) \). A straightforward computation shows that \( \mu \) and \( \nu \) are the 4-th powers of elements in \( \mathbb{F}_p^{\nu, \lambda} \). By \( \mu^2 = \nu \), we may assume that \( \mu \) is the square of an element in \( \mathbb{F}_p^{\nu, \lambda} \). Therefore \( \sqrt{\mu} \) and \( \sqrt{\mu \lambda} \) in the equation \( (4.1.1) \) for \( C \) can be taken as elements in \( \mathbb{F}_p^{\nu, \lambda} \), which shows the first argument. Here, \( E_1 \), \( E_2 \), and \( E_3 \) are \( \mathbb{F}_p^{\nu, \lambda} \)-isomorphic to \( y^2 = x(x - 1)(x - \lambda) \), \( y^2 = x(x - 1)(x - \nu) \), \( y^2 = x(x - 1)(x - \lambda') \) with

\[
\lambda' = \frac{(\mu \lambda - 1)(\mu - \nu)}{(\mu \lambda - \nu)(\mu - 1)},
\]

and it follows also from [1] Proposition 2.2] that they are both \( \mathbb{F}_p^{\nu_3} \)-maximal (resp. \( \mathbb{F}_p^{\nu_3} \)-minimal) if \( p \equiv 3 \) (mod 4) (resp. \( p \equiv 1 \) (mod 4)). Since \( C \) is \( \mathbb{F}_p^{\nu_3} \)-maximal (resp. \( \mathbb{F}_p^{\nu_3} \)-minimal) if and only if \( E_1 \), \( E_2 \), and \( E_3 \) are both \( \mathbb{F}_p^{\nu_3} \)-maximal (resp. \( \mathbb{F}_p^{\nu_3} \)-minimal), we have proved the second argument.

From Lemma 2.3.2 together with Theorem 4.1.1 we can immediately prove the following corollary which asserts that any hyperelliptic genus-3 curve of Oort type is of Howe type:

**Corollary 4.1.3.** Let \( C \) be a hyperelliptic Howe curve of genus 3 associated with \( E_1 \times_{P_1} E_2 \). Then \( C \) is birational to \( E \times_{P_1} H \) for some genus-1 and genus-2 curves \( E \) and \( H \). If \( E_1 \) and \( E_2 \) are given as in \( (4.1.3) \), then \( E \) and \( H \) are given respectively by \( y^2 = f(x) \) and \( y^2 = f(x) \), where \( f(x) \) is the unique polynomial such that \( f(x^2) \) is equal to the right hand side of \( (4.1.1) \).

4.2. **Non-hyperelliptic case.** Let \( C \) be a genus-3 Howe curve of Oort type associated with genus-1 curves \( E_1 \) and \( E_2 \) given by \( (4.1.2) \). Recall that any non-hyperelliptic curve of genus 3 is canonically embedded into \( \mathbb{P}^2 \) as a non-singular plane quartic. The aim of this subsection is to find an explicit quartic equation defining a non-hyperelliptic \( C \). We shall deduce such a quartic by computing a result from equations for \( E_1 \) and \( E_2 \). Note that this idea can be applied to higher genus cases, see Remark 4.2.6 below. We set

\[
Q_1 := (x - \alpha_2)(x - \alpha_3) = x^2 - \tau_1 x + \tau_2,
Q_2 := (x - \beta_2)(x - \beta_3) = x^2 - \rho_1 x + \rho_2,
\]

where \( \tau_1 := \alpha_2 + \alpha_3 \), \( \tau_2 := \alpha_2 \alpha_3 \), \( \rho_1 := \beta_2 + \beta_3 \), and \( \rho_2 := \beta_2 \beta_3 \). Putting \( Y := y_1/x \) and \( Z := y_2/x \) and squaring both sides, we obtain \( xY^2 = Q_1 \) and \( xZ^2 = Q_2 \). Put

\[
f_1 := Q_1 - xY^2 = x^2 - (\tau_1 - Y^2)x + \tau_2,
f_2 := Q_2 - xZ^2 = x^2 - (\rho_1 - Z^2)x + \rho_2,
f := \text{Res}_x(f_1, f_2) \in k[Y, Z].
\]

A straightforward computation shows that

\[
f = \rho_2 Y^4 + (\tau_2 - \rho_2)Y^2 Z^2 + \tau_2 Z^4 + (2\tau_1 \rho_2 - \tau_2 \rho_1 - \rho_1 \rho_2)Y^2 + (\tau_1 \tau_2 - \tau_1 \rho_2 + 2\tau_2 \rho_1)Z^2 + \tau_1 \rho_2 - \tau_1 \tau_2 \rho_1 - \tau_1 \rho_1 \rho_2 + \rho_2 \rho_1^2 + 2\tau_2 \rho_2 + \rho_2^2,
\]

which is a quartic both in \( Y \) and \( Z \) by \( \rho_2, \tau_2 \neq 0 \) from \( \alpha_2, \alpha_3, \beta_2, \beta_3 \neq 0 \).
Lemma 4.2.1. With notation as above, we denote by \( b_{ij} \) the coefficient of \( Y^i Z^j \) in \( f \). Then we have the following:

\[
\begin{align*}
\alpha &= \beta_1 - \beta_2, \\
\beta_1 &= \alpha_1 - \beta_2, \\
\beta_2 &= \alpha_2 + \beta_1, \\
\beta_3 &= \alpha_3 + \beta_2.
\end{align*}
\]

This is proved by a straightforward computation.

Proof. Assume for a contradiction that \( f \) is reducible. We denote by \( b_{ij} \) the coefficient of \( Y^i Z^j \) in \( f \). If \( f \) has a linear factor, then it follows from Lemma \( \text{A.0.2} \) that we have the system of equations \( \text{A.0.2} \), which contradicts to the last assertion of Lemma \( \text{A.0.1} \). Therefore, \( f \) is factored into \( f = b_{40}Q_1 Q_2 \) for some monic irreducible quadratic polynomials \( Q_1 \) and \( Q_2 \), whence either of the four cases in Lemma \( \text{A.0.2} \) holds. Among the four cases, the cases (I), (II), and (III) are impossible by Lemma \( \text{A.0.2} \) together with our assumption \( \alpha_2 \alpha_3 \neq \beta_2 \beta_3 \). The only possible case is (IV). In this case, it follows from Lemma \( \text{A.0.2} \) that we can take \( \alpha_2 \alpha_3 = \beta_2 \beta_3 \) and \( (\beta_2 - \beta_3)(\alpha_2 \alpha_3 - \beta_2 \beta_3) \) as square roots of \( b_{22} - 4b_{40}b_{00} \) and \( b_{20} - 4b_{40}b_{00} \) respectively. Here, we have

\[
\begin{align*}
2b_{40}b_{02} - b_{22}b_{20} + \sqrt{b_{22}^2 - 4b_{40}b_{04}} \sqrt{b_{20}^2 - 4b_{40}b_{00}} &= -2\beta_3(\alpha_2 \alpha_3 - \beta_2 \beta_3)^2, \\
2b_{40}b_{02} - b_{22}b_{20} - \sqrt{b_{22}^2 - 4b_{40}b_{04}} \sqrt{b_{20}^2 - 4b_{40}b_{00}} &= -2\beta_2(\alpha_2 \alpha_3 - \beta_2 \beta_3)^2,
\end{align*}
\]

which are both non-zero by \( \alpha_2 \alpha_3 \neq \beta_2 \beta_3 \). This contradicts the equation (IV) in Lemma \( \text{A.0.2} \).

Moreover, we have the following theorem:

Theorem 4.2.3. With notation as above, the projective closure \( \bar{C} \) of \( f = 0 \) in \( \mathbb{P}^2 \) is non-singular, and \( C \) is isomorphic to \( \bar{C} \).

Proof. Denoting by \( b_{ij} \) the \( Y^i Z^j \)-coefficient of \( f \), we have the following:

\[
\frac{\partial f}{\partial Y} = 2Y(2b_{40}Y^2 + b_{22}Z^2 + b_{20}), \quad \frac{\partial f}{\partial Z} = 2Z(2b_{40}Z^2 + b_{22}Y^2 + b_{02})
\]

If \( \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0 \), then we have

\[
\begin{align*}
Y &= 0 \quad \text{or} \quad Y^2 = \frac{-b_{22}Z^2 - b_{20}}{2b_{40}}, \\
Z &= 0 \quad \text{or} \quad Z^2 = \frac{-b_{22}Y^2 - b_{02}}{2b_{40}}.
\end{align*}
\]
whence there are four cases. By a tedious computation for each of the four cases together with Lemma 4.2.1, we obtain

\[ f(Y, Z) = \begin{cases} 
    b_{00} = \prod_{i,j} (\alpha_i - \beta_j) & (Y = 0 \text{ and } Z = 0), \\
    \frac{b_{2m} - 4b_{40}b_{00}}{4b_{40}} & (Y^2 = -\frac{b_{22}Z^2 - b_{20}b_{00}}{-b_{40}}, \text{ and } Z = 0), \\
    \frac{b_3^2 - 4b_{40}b_{00}b_{10} - b_{22}^2 + b_{20}b_{10} + b_{20}b_{40} - 4b_{40}b_{00}b_{10}}{4\beta_2^2} & (Y = 0 \text{ and } Z^2 = -\frac{b_{22}Z^2 - b_{20}b_{00}}{-b_{40}}), \\
    \frac{b_3^2 - 4b_{40}b_{00}b_{10} - b_{22}^2 + b_{20}b_{10} + b_{20}b_{40} - 4b_{40}b_{00}b_{10}}{4\beta_2^2} & (\text{otherwise}), 
\end{cases} \]

where the right hand side in the fourth case is equal to \((\alpha_2\alpha_3 - \beta_2\beta_3)^2\). Therefore, it follows from \(\alpha_2\alpha_3 \neq \beta_2\beta_3\) that \(f(Y, Z) \neq 0\) for each of the four cases, whence \(f(Y, Z) = 0\) is non-singular. We next discuss the singularity of the points at infinity of \(C\). It is easy to see that there is a singular point at infinity if and only if the polynomial \(b_{00}Y^4 + b_{22}Y^2Z^2 + b_{10}Z^4\) has a multiple root. Since \(b_{00}, b_{10} \neq 0\) and \(b_{22}^2 - 4b_{00}b_{10} \neq 0\), it does not have any multiple root, whence \(C\) is non-singular.

Next, we shall prove that \(E_1 \times_{\mathbb{P}^1} E_2\) is birational to \(D : f(Y, Z) = 0\). For any point \((x, y_1, y_2)\) on \(E_1 \times_{\mathbb{P}^1} E_2\), it is straightforward that the corresponding point \((Y, Z)\) lies on \(C\), whence we obtain a rational map:

\[ \Phi : E_1 \times_{\mathbb{P}^1} E_2 \to D ; \ (x, y_1, y_2) \mapsto \left( \frac{y_1}{x}, \frac{y_2}{x} \right), \]

which is well-defined over all the points \((x, y_1, y_2)\) in \(E_1 \times_{\mathbb{P}^1} E_2\) with \(x \neq 0\). We can also construct the inverse rational map as follows: For each point \((Y, Z)\) on \(D\), it follows from \(\text{Res}_x(f_1(x, Y, Z), f_2(x, Y, Z)) = 0\) that there exists a common root of the univariate polynomials \(f_1(x, Y, Z)\) and \(f_2(x, Y, Z)\). Since \(\tau_2 \neq \rho_2\) by our assumption that \(C\) is non-hyperelliptic, we have \(-\tau_1 - Y^2 \neq -\rho_1 - Z^2\), and such the root is uniquely given by

\[ x = \frac{\tau_2 - \rho_2}{(Y^2 + \tau_1) - (Z^2 + \rho_1)} \neq 0. \]

With this root \(x\), we define a rational map

\[ \Psi : D \to E_1 \times_{\mathbb{P}^1} E_2 ; \ (Y, Z) \mapsto (xY, xZ), \]

which is defined over all the points on \(D\). Clearly both \(\Phi \circ \Psi\) and \(\Psi \circ \Phi\) are well-defined, and they are equal to \(\text{id}_D\) and \(\text{id}_{E_1 \times_{\mathbb{P}^1} E_2}\) as rational maps. Since \(C\) and \(\tilde{C}\) are smooth, we have completed the proof.

As a particular case, we can take an elliptic curve in Legendre form as \(E_1\), by setting \((\alpha_2, \alpha_3, \beta_2, \beta_3) = (1, \nu, \mu, \lambda)\) with \(\alpha_2\alpha_3 = \nu \neq \mu^2\lambda = \beta_2\beta_3\). In this case, similarly to the hyperelliptic case, we obtain the following corollary (cf. [33] on the \(\mathbb{F}_{p^2}\)-maximality/minimality of non-hyperelliptic Howe curves of genus 3):

**Corollary 4.2.4 (cf. [33] Theorem 1.1).** Let \(C\) be a non-hyperelliptic Howe curve\( C\) of genus 3 associated with genus-1 curves \(E_1\) and \(E_2\) given by (4.1.2). If \(C\) is superspecial, then it is defined over \(\mathbb{F}_{p^4}\), and moreover it is \(\mathbb{F}_{p^4}\)-minimal.

**Proof.** Let \(E_3\) be a genus-1 curve as defined in Theorem 2.4.6 explicitly given by \(E_3 : y_3^2 = (x - 1)(x - \nu)(x - \mu)(x - \mu\lambda)\) in our case. If \(C\) is superspecial, then \(E_1, E_2,\) and \(E_3\) are supersingular, where the Legendre forms \(E_1, E_2,\) and \(E_3\) are \(E_1 : y_1^2 = x(x - 1)(x - \nu), E_2 : y_2^2 = x(x - 1)(x - \lambda),\) and \(E_3 : y_3^2 = x(x - 1)(x - \lambda')\) with \(\lambda'\) given as in (1.1.3). It follows from [1 Proposition 3.1] that \(\nu, \lambda,\) and \(\lambda'\) are the 4-th powers of elements in \(\mathbb{F}_{p^2}\). We have

\[ (4.2.2) \quad \lambda \cdot \mu^2 + \lambda'(\nu + \lambda) - \frac{1 + \lambda\mu}{1 - \lambda'} = 0. \]

Therefore, it holds that \(\mu \in \mathbb{F}_{p^4}\).
Moreover, it follows also from [1] Proposition 2.2] that $E_p$, $E_{1}$, and $E_{2}$ are both $\mathbb{F}_p^2$-maximal (resp. $\mathbb{F}_p^2$-minimal) if $p \equiv 3 \pmod{4}$ (resp. $p \equiv 1 \pmod{4}$). Therefore, $E_1$, $E_2$, and $E_3$ are $\mathbb{F}_p^2$-minimal. Since $C$ is $\mathbb{F}_p^2$-minimal if and only if $E_1$, $E_2$, and $E_3$ are both $\mathbb{F}_p^2$-minimal, we have proved the second argument.

**Remark 4.2.5.** In the above corollary, the curve $C$ is defined over $\mathbb{F}_p^2$ in general. However, from [33, Theorem 1.1], it can descend to a quartic plane curve defined over $\mathbb{F}_p^2$.

**Remark 4.2.6.** The method presented in this subsection might be extended to the case where a generalized Howe curve has genus $\geq 4$ as follows: Let $C$ be a generalized Howe curve associated with two hyperelliptic curves $C_1$ and $C_2$ of genera $g_1$ and $g_2$ given by (2.3.1), and assume $g_1 = g_2$ for simplicity. We set $\phi := (x - a_1)(x - a_2)\cdots(x - a_r)$, $Q_1 := (x - b_1)(x - b_2\cdots(x - b_{g_2+2-r})$, and $Q_2 := (x - c_1)(x - c_{g_2+2-r})$. Putting $Y := y_1/\phi$ and $Z := y_2/\phi$ and squaring both sides, we obtain $\phi Y^2 = Q_1$ and $\phi Z^2 = Q_2$. Then, we obtain an equation $f(Y, Z) = 0$ by computing $f = \text{Res}_x(f_1, f_2) \in k[Y, Z]$ for $f_1 := Q_1 - \phi Y^2$ and $f_2 := Q_2 - \phi Z^2$.

For the case where $H$ is non-hyperelliptic with $(g_1, g_2, r, g) = (2, 2, 4, 5)$, we prove in [32] that $f$ is absolutely irreducible, and that $C$ is birational to the projective closure of $f = 0$. It is an open problem to prove the irreducibility of $f$ and the birationality for $(g_1, g_2, r, g)$ other than $(1, 1, 2, 3)$ or $(2, 2, 4, 5)$.

5. Enumeration of superspecial generalized Howe curves

This section is devoted to the computational enumeration of superspecial (s.sp. for short) generalized Howe curves. Note that an algorithm in genus 4 is already proposed in [20] (resp. [33]) for the non-hyperelliptic (resp. hyperelliptic) case. We have present computational methods for the case of genus 3, and prove Theorem 3 in Section 4. In our methods, defining equations for produced Howe curves as genus-3 curves are explicitly computed by Theorems 4.1.1 and 4.2.3, and they are used for classifying isomorphism classes of the curves.

5.1. Some lemmas for isomorphism classification. Before constructing computational methods for enumerating s.sp. Howe curves, we start with preparing some lemmas that can make isomorphism classification for Howe curves efficient.

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ and $\mathcal{D} = \{D_1, \ldots, D_n\}$ be sets of curves over $k$. We write $\mathcal{C} \sim \mathcal{D}$ if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $C_i \cong D_{\sigma(i)}$ for each $1 \leq i \leq n$.

The following lemma is a generalization of [33] Lemma 4.1:

**Lemma 5.1.1.** Let $C^{(1)}$ and $C^{(2)}$ be generalized hyperelliptic Howe curves defined as the normalizations of $C_1^{(1)} \times \mathbb{P}^1$, $C_2^{(1)}$ and $C_1^{(2)} \times \mathbb{P}^1$, $C_2^{(2)}$ respectively, where each $C_j^{(i)}$ is a hyperelliptic curve. Assume that $\Aut(C^{(i)}) = V_4$ for each $i$. If $C^{(1)} \cong C^{(2)}$, then we have $\{C_1^{(1)}, C_2^{(1)}\} \sim \{C_1^{(2)}, C_2^{(2)}\}$.

**Proof.** Recall from Subsection 2.3 that $C_1^{(1)}$ and $C_2^{(2)}$ are not isomorphic over $\mathbb{P}^1$. Thus, it follows from $\Aut(C^{(i)}) = V_4$ that $\{C_1^{(1)}, C_2^{(1)}\}$ consists of the quotients of $C^{(i)}$ by non-hyperelliptic involutions, whence the assertion holds by $C^{(1)} \cong C^{(2)}$.

In the case where Howe curves are genus-3 curves of Oort type, similarly to the proof of Lemma 5.1.1, we can prove the following lemma:

**Lemma 5.1.2.** Let $C^{(1)}$ and $C^{(2)}$ be genus-3 Howe curves of Oort type defined as the normalizations of $E_1^{(1)} \times \mathbb{P}^1$, $E_2^{(1)}$ and $E_1^{(2)} \times \mathbb{P}^1$, $E_2^{(2)}$ respectively, where $E_1^{(i)}$ and $E_2^{(i)}$ are genus-1 curves sharing exactly 2 ramified points for each $i = 1, 2$. Note
that $E^{(i)}_j \cong C^{(i)}(\sigma^{(i)}_j)$ for some order-2 distinct elements $\sigma^{(i)}_1, \sigma^{(i)}_2 \in \text{Aut}(C^{(i)})$ with $\sigma^{(i)}_1 \sigma^{(i)}_2 = \sigma^{(i)}_2 \sigma^{(i)}_1$. Let $E^{(i)}$ be the genus-1 curve defined by $C^{(i)}(\sigma^{(i)}_1 \sigma^{(i)}_2)$.

We suppose that $\text{Aut}(C^{(i)}) = V_4$ for each $i$. If $C^{(i)} \cong C^{(2)}$, then one has $\{E^{(1)}_1, E^{(1)}_2, E^{(1)}_3\} \sim \{E^{(2)}_1, E^{(2)}_2, E^{(2)}_3\}.

For $\nu, \mu, \lambda \in \mathbb{F}_p$ with $\nu \notin \{0, 1\}$, $\mu \notin \{0, 1, \nu\}$ and $\lambda \notin \{0, 1, \mu^{-1}, \nu \mu^{-1}\}$, we define three genus-1 curves as $E\nu: y^2 = x(x-1)(x-\nu)$ and $E\mu,\lambda: y^2 = x(x-\mu)(x-\mu\lambda)$ and $E\nu,\mu,\lambda: y^2 = (x-1)(x-\nu)(x-\mu)(x-\mu\lambda)$ as in (4.1.3). In the following lemma, we shall bound (by a constant) the number of triples $(\lambda, \mu, \nu)$ such that the three genus-1 curves have given $j$-invariants:

**Lemma 5.1.3.** Given a set $\{E_1, E_2, E_3\}$ of three genus-1 curves over $\mathbb{F}_p$ with $p \neq 2$, the number of triples $(\lambda, \mu, \nu) \in \mathbb{F}_p^3$ with $\nu \notin \{0, 1\}$, $\mu \notin \{0, 1, \nu\}$, and $\lambda \notin \{0, 1, \mu^{-1}, \nu \mu^{-1}\}$ such that $\{E_1, E_2, E_3\} \sim \{E_{\nu, \mu, \lambda}, E_{\nu, \mu, \lambda}, E_{\nu, \mu, \lambda}\}$ is at most 2592 (not depending on $p$).

**Proof.** By definition of $\sim$, there exists $\sigma \in \mathfrak{S}_3$ such that $E_{\nu} \cong E_{\sigma(1)}$, $E_{\mu, \lambda} \cong E_{\sigma(2)}$, and $E_{\nu, \mu, \lambda} \cong E_{\sigma(3)}$. Here we recall a well-known fact that there are at most 6 values of $\xi$ such that an elliptic curve in Legendre form $y^2 = x(x-1)(x-\xi)$ has a given $j$-invariant. From this, the number of choices for $\nu$ is at most 6. The number of choices for $\lambda$ is also at most 6, since $E_{\nu, \mu, \lambda}$ is isomorphic to $y^2 = x(x-1)(x-\lambda)$. The third genus-1 curve $E_{\nu, \mu, \lambda}$ is isomorphic to $y^2 = x(x-1)(x-\lambda')$ for $\lambda'$ as in (4.1.3), and hence $\mu$ can take $6 \times 2 = 12$ values at most. Since the number of choices for $\sigma$ is $3! = 6$, that for $(\nu, \mu, \lambda)$ is bounded by $6^2 \times 12 \times 6 = 2592$, as desired. □

We also define a genus-2 curve $H_{a,b,c}: y^2 = x(x-1)(x-a)(x-b)(x-c)$ (this equation is called a *Rosenhain form*) for mutually distinct elements $a, b, c \in \mathbb{F}_p$ with $a, b, c \notin \{0, 1\}$. As for the number of isomorphism classes of $H_{a,b,c}$, we have the following lemma:

**Lemma 5.1.4.** Given a genus-2 curve $H$ over $\mathbb{F}_p$ with $p \neq 2$, the number of subsets $\{a, b, c\} \subset \mathbb{F}_p$ of cardinality 3 with $H_{a,b,c} \cong H$ is at most 120 (not depending on $p$!).

**Proof.** The assertion follows from the proof of [35, Lemma 5] and $\#\mathfrak{S}_3 = 6$. □

Note that the bounds given in Lemmas 5.1.3 and 5.1.4 are best possible.

5.2. Enumerating s.s.p. Howe curves of genus three. Recall from the beginning of Section 4 that Howe curves $C$ of genus 3 are either of *Oort type* or *Howe type*, say $E_1 \times \mathbb{P}^1, E_2$ for elliptic curves $E_1$ and $E_2$ as in (4.1.3), or $E \times \mathbb{P}^1, H$ for curves $E$ and $H$ of genera 1 and 2 as in (4.0.1). Note also that any $C$ of Howe type is always hyperelliptic, and recall from Corollary (4.1.3) that any hyperelliptic $C$ of Oort type is of Howe type. In the following, we present methods to enumerate s.s.p. Howe curves $C$ of genus 3, dividing into the hyperelliptic or non-hyperelliptic case:

**Hyperelliptic case:** In this case, recall from Lemma (2.2.2) that a Howe curve $C$ is nothing but a hyperelliptic curve with automorphism group containing $V_4$, and it is always of Howe type, namely $C$ is the normalization of $E \times \mathbb{P}^1, H$ for curves $E$ and $H$ of genera 1 and 2 sharing exactly 4 Weierstrass points. Transforming $3$ among the Weierstrass points of $H$ into $\{0, 1, \infty\}$, we may assume that $E$ and $H$ are given by $E_{a,b,c}: y^2 = (x-1)(x-a)(x-b)(x-c)$ and $H_{a,b,c}: y^2 = x(x-1)(x-a)(x-b)(x-c)$, and by Lemma (2.2.3) that $C$ is given by $C_{a,b,c}: y^2 = (x^2-1)(x^2-a)(x^2-b)(x^2-c)$.

Here is our method to enumerate s.s.p. $C_{a,b,c}$:

**Method 1.** For an input $p \geq 3$, proceed with the following:

0. Set $C \leftarrow \emptyset$ and $\mathcal{I} \leftarrow \emptyset.$
1. Compute the list $S_p$ of all supersingular $j$-invariants in characteristic $p > 0$.
2. Enumerate all s.sp. curves of genus 2 in characteristic $p$, by computing Richelot isogenies (see [35] §5A or [35] §5.1, Step 1 for details).
3. List genus-2 s.sp. curves of the form $H_{a,b,c}$ by the method in [35] §5.1, Step 2). Every when $H_{a,b,c}$ is produced, check the following:
   - The $j$-invariant of an elliptic curve isomorphic to $E_{a,b,c}$ belongs to $S_p$.
   - The sequence $I$ of the Shioda invariants of the genus-3 hyperelliptic curve $C_{a,b,c}$ does not belong to $I$.

   If both the two conditions hold, then set $C ← C \cup \{C_{a,b,c}\}$ and $I ← I \cup \{I\}$.

4. Return $C$.

**Lemma 5.2.1.** Method 1 outputs a list of all isomorphism classes of s.sp. hyperelliptic Howe curves of genus 3, and its complexity is $O(p^3)$.

**Proof.** It suffices to show that each computed $C_{a,b,c}$ is superspecial. Indeed, the $a$-number of each computed $C_{a,b,c}$ associated to $(E_{a,b,c}, H_{a,b,c})$ is three by an isomorphism $J(C_{a,b,c})[p] \cong J(E_{a,b,c})[p] \times J(H_{a,b,c})[p]$ of $p$-kernels.

As for the complexity, the complexity of Step 1 is $O(p)$, see, e.g., [35] §5.1.1, and Step 2 is done in $O(p^3)$, see [35] §5.1, Step 1 for details. In Step 3, we apply the method in [35] §5.1, Step 2, where at most 120 (by Lemma 5.1.3) $H_{a,b,c}$’s are produced in constant time for each isomorphism class of s.sp. genus-2 curves, and the number of produced $H_{a,b,c}$'s is $O(p^3)$, namely $\#C = I = O(p^3)$. For each produced $H_{a,b,c}$, it follows from $\#S_p = O(p)$ that the supersingularity test for $E_{a,b,c}$ is done in $O(log(p))$, and $I$ is computed in constant time with respect to $p$. Since $\#I = O(p^3)$, the cost of testing $I \in I$ is $log(p^3) = O(log(p^3))$. Therefore, the total complexity is $p + p^3 + p^3\log(p^3) = O(p^3)$, as desired. \(\square\)

**Remark 5.2.2.** Note that $\#C$ for an output $C$ of Method 1 is in fact expected to be $O(p^2)$ as follows: Heuristically, we can estimate it as the number of s.sp. $H_{a,b,c}$ generated in Step 3, times the probability that $E_{a,b,c}$ is supersingular, say

$$\approx \frac{p^3}{2880} \times 120 \times \frac{1}{2p} = \frac{p^2}{48} = O(p^2).$$

S.sp. hyperelliptic Howe curves of Oort type can be also enumerated by Method 2 below. Recall that an Oort type-curve $C$ is given as the normalization of $E_1 \times_{p_1} E_2$ for $E_1: y^2 = x(x-1)(x-\nu)$ and $E_2: y^2 = x(x-\mu)(x-\mu\lambda)$ together with $E_3 : y^2 = (x-1)(x-\nu)(x-\mu)(x-\mu\lambda)$, and that an equation defining $C$ is (4.1.2).

**Method 2.** For an input $p \geq 3$, proceed with the following:

0. Set $C ← \emptyset$ and $I ← \emptyset$.

1. Compute the list $S_p$ of all supersingular $j$-invariants in characteristic $p > 0$, and generate the list $T_p$ of $t \in \mathbb{F}_p^\times$ such that $y^2 = x(x-1)(x-t)$ is a supersingular elliptic curve.

2. For each $\nu$ and $\lambda$ in $T_p$, conduct the following:
   2a. Compute the roots $t = \mu^2 - \nu\lambda^{-1}$. (Note that the roots belong to $\mathbb{F}_{p^2}$ as described in the proof of Corollary 4.1.2)
   2b. For each root $t$, check the following:
      - $t \notin \{0, 1, \nu, \lambda^{-1}, \lambda^{-1}\nu\}$
      - The $j$-invariant of an elliptic curve isomorphic to $E_3$ is in $S_p$.
      - The sequence $I$ of the Shioda invariants of the genus-3 hyperelliptic curve $C$ (associated with $(E_1, E_2)$) as in (4.1.2) does not belong to $I$.

   If all the three conditions hold, then set $C ← C \cup \{C\}$ and $I ← I \cup \{I\}$.

3. Return $C$. 

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2.5]. and that the roots \( r \) compute a sextic given in (4.2.1) and let \( \tilde{O} \) given as follows: Step 1: \( \tilde{O}(p^2) \), then the number of \( \mu \)'s for each \( (\nu, \lambda) \) (at most 2), times the probability that \( E_1 \) is supersingular \( \approx 1/(2p) \), and thus \( \approx p/4 \) in total.

The complexity of Method 2 is estimated as \( \tilde{O}(p^2) \), and we expect \#\( C \equiv O(p) \) similarly to Remark 5.2.2.

**Non-hyperelliptic case:** In this case, every non-hyperelliptic Howe curve \( C \) of genus 3 is always of Oort type, namely \( C \) is the normalization of \( E_1 \times_{\mathbb{F}_p} E_2 \) for elliptic curves \( E_1 \) and \( E_2 \) as in (4.1.3) with \( \mu^2 \lambda \neq \nu \). For the enumeration of s.s.p. \( C \)'s, we divide the case into the case of \( \text{Aut}(C) = V_4 \), and the other case. In the other case, the order of \( \text{Aut}(C) \) is either of 8, 16, 24, 48, 96, and 168 (corresponding to \( D_4, G_{16}, S_4, G_{48}, G_{96}, \) and \( G_{168} \) respectively), see e.g., [23 Proposition 2.5]. More precisely, it suffices to proceed with the following Method 3.

**Method 3.** For an input \( p \geq 3 \), proceed with the following:

0. Set \( J \leftarrow \emptyset \) and \( L_r^{(\text{others})} \leftarrow \emptyset \) for each \( r \in \mathcal{R} := \{8, 16, 24, 48, 96, 168\} \).
1. Compute the list \( S_p \) of all supersingular \( j \)-invariants in characteristic \( p > 0 \), and for each \( j \in S_p \), generate the list \( T_{p,j} \) of at most six \( t \in \mathbb{F}_{p^2} \) such that \( y^2 = x(x-1)(x-t) \) is a supersingular elliptic curve with \( j \)-invariant \( j \).
2. For each multi-subset \( J \) of \( S_p \) with cardinality 3, conduct the following:
   2a. Set \( L_J \leftarrow \emptyset \).
   2b. For each permutation \((j_1, j_2, j_3)\) of \( J \), and for each \( \nu \in T_{p,j_1}, \lambda \in T_{p,j_2}, \) and \( \lambda' \in T_{p,j_3}, \) compute \( \mu \in \mathbb{F}_p \) with \( \mu \notin \{0, 1, \nu, \lambda^{-1}, \lambda^{-1} \nu, \pm \sqrt{\lambda^{-1} \nu}\} \) such that the genus-1 curve \( E_3 : y^2 = (x-1)(x-\nu)(x-\lambda)(x-\mu \lambda) \) is isomorphic to \( y^2 = x(x-1)(x-\lambda') \), as the roots of the quadratic equation (4.2.2).
   2c. For each \( \mu \) computed in Step 2b, set \( L_J \leftarrow L_J \cup \{(\nu, \lambda, \mu)\} \).
3. For each multi-subset \( J \) of \( S_p \) with cardinality 3, conduct the following:
   3a. Set \( L_J^{(V_4)} \leftarrow \emptyset \).
   3b. For each \( (\nu, \lambda, \mu) \in L_J \), compute a sextic given in (4.2.1) and let \( C_{\nu,\lambda,\mu} \) be the curve defined by it. Compute \( r := \#\text{Aut}(C_{\nu,\lambda,\mu}) \).
      - If \( r = 4 \), then \( L_J^{(V_4)} \leftarrow L_J^{(V_4)} \cup \{C_{\nu,\lambda,\mu}\} \).
      - Otherwise, \( L_J^{(\text{others})} \leftarrow L_J^{(\text{others})} \cup \{C_{\nu,\lambda,\mu}\} \).
   3c. If \( L_J^{(V_4)} \neq \emptyset \), set \( J \leftarrow J \cup \{J\} \).
   4. Execute the isomorphism classification for each of \( L_J^{(V_4)} \) and \( L^{(\text{others})} \).
   4a. For each \( J \in \mathcal{J} \), initialize \( C_J^{(V_4)} \leftarrow \emptyset \), and conduct the following:
      - For each \( C_{\nu,\lambda,\mu} \in L_J^{(V_4)} \), if it is not isomorphic to any curve in \( C_J^{(V_4)} \), we set \( C_J^{(V_4)} \leftarrow C_J^{(V_4)} \cup \{C_{\nu,\lambda,\mu}\} \).
   4b. Similarly to 4a, classify isomorphisms of curves in each \( L_J^{(\text{others})} \), and let \( C_J^{(\text{others})} \) be the list of obtained isomorphism classes.
   5. Return \( C_J^{(V_4)} \cup C_J^{(\text{others})} \).

Note that the square-root \( \sqrt{\lambda^{-1} \nu} \) in Step 2b can be taken as an element in \( \mathbb{F}_{p^2} \) by [1 Proposition 2.2], and that the roots \( \mu \) for each \( (\nu, \lambda, \lambda') \) belong to \( \mathbb{F}_{p^2} \).

**Lemma 5.2.3.** Method 3 outputs a list of all isomorphism classes of s.s.p. non-hyperelliptic Howe curves of genus 3, and the arithmetic complexity of each step is given as follows: Step 1: \( \tilde{O}(p) \), Step 2: \( O(p^2) \), Step 3: \( \tilde{O}(p^3) \), Step 4a: \( \tilde{O}((\#J)^2) \), and Step 4b: \( \tilde{O}((\#L^{(\text{others})})^2) \), so that the total complexity is upper-bounded by \( \tilde{O}(p^3) + \tilde{O}(\#J) + \tilde{O}((\#L^{(\text{others})})^2) \), where the second term is \( \tilde{O}(p^3) \).
Proof. The correctness immediately follows from Theorem 4.2.3 and Lemma 5.1.2 together with the following: Each produced $C$ associated with $(E_1, E_2, E_3)$ is superspecial since we have an isomorphism $J(C)[p] \cong J(E_1)[p] \times J(E_2)[p] \times J(E_3)[p]$.

Step 1 is done in $O(p)$, and $\#S_p$ follows $O(p)$, which means Step 2 has $O(p^2)$ iterations on $J$. The total number of iterations for Step 2b is upper-bounded by $6 \times \#T_{p,j_1} \times \#T_{p,j_2} \times \#T_{p,j_3} = 6 \times 6^3$, and solving a quadratic equation to obtain $\mu$ is done in $O(1)$. Thus, Step 2 has complexity $O(p^2)$.

In Step 3, the number of iterations is just the total number of $(\nu, \lambda, \mu)$ obtained in Step 2, which is upper-bounded by $(\#S_p)^3 \times 6^3 \times 2 = O(p^3)$, and solving a quadratic equation to obtain $\mu$ is done in $O(1)$. Thus, Step 3 has complexity $O(p^3)$.

As for Step 4, it follows from the construction of Step 2b that the cardinality of each $\mathcal{L}_j(V_i)$ is at most $\#\mathcal{L}_j(V_i) \leq 6 \times 6^3 \times 2 = 2592$, which is nothing but the number of triples $(\nu, \lambda, \mu)$ as in Lemma 5.1.4. Therefore, the complexity of Step 4 is upper-bounded by $\#J \times (\max_{j \in J} \#\mathcal{L}_{V_i}^{(\nu)})^2 \leq (\#S_p)^3 \times 2592^2 = O(p^6)$, whereas that of Step 4b is

$$\sum_{r \in R} (\#\mathcal{L}_j^{(\text{others})})^2 \leq (\#\mathcal{L}^{(\text{others})})^2.$$

Note that each isomorphism test for two plane quartics is done by computing a Gröbner basis in constant time with respect to $p$, say $O(1)$, as in the proof of Theorem 3.3.1. \qed

We remark that the number $\#(\mathcal{L}(V_i) \cup \mathcal{L}^{(\text{others})})$ of $C_{\nu,\lambda,\mu}$’s listed in Steps 2 and 3 is approximately $(p-1)/12 \times 6^3 \times 2 = p^3/4$, and thus $\#(\mathcal{L}(V_i) \cup \mathcal{L}^{(\text{others})}) = O(p^3)$. In practice, we estimate $\#\mathcal{L}^{(\text{others})} = O(p^2)$ as described in Subsection 5.3 below, and hence the total complexity of Method 3 is expected to be $O(p^5)$.

5.3. Implementations and computational results. We implemented the three enumeration methods (Methods 1, 2, and 3) described in Subsection 5.2 on Magma V2.26-10 [2] in a PC with macOS Monterey 12.0.1, at 2.6 GHz CPU 6 Core (Intel Core i7) and 16GB memory. We also implemented decision-versions of the three methods in the same environment; a decision-version terminates once a single s.s.p. curve is found, and outputs “true” together with the s.s.p. curve (otherwise outputs “false”). The source codes are available at the following web page:

http://sites.google.com/view/m-kudo-official-website/english/code/genus3v4

5.3.1. Enumeration results. We executed Methods 1 and 2 (resp. Method 3) for every $3 \leq p \leq 200$ (resp. $11 \leq p \leq 50$) in the hyperelliptic (resp. non-hyperelliptic) case. Our computational results are summarized in the following theorem:

**Theorem 5.3.1.** For every prime $p$ with $3 \leq p \leq 200$ (resp. $11 \leq p \leq 50$), the number of isomorphism classes of s.s.p. hyperelliptic (resp. non-hyperelliptic) Howe curves of genus 3 over $\mathbb{F}_p$ is summarized in Table 1 (resp. Table 2).

Our observation of computational results and timing data. In the hyperelliptic case, we find from Table 1 that s.s.p. Howe curves of type $E \times_{\mathbb{F}_p} H$ exist for every prime $p$ with $p \not\in \{3, 5, 13\}$, and that the number of isomorphism classes of such s.s.p. curves is $O(p^3)$ as estimated in Remark 5.2.2. On the other hand, there are many $p$ for which a s.s.p. Howe curve of Oort type $E_1 \times_{\mathbb{F}_p} E_2$ does not exist, and the number of isomorphism classes is $O(p)$ as we expect at the paragraph just after Method 2. As for timing data, we see that our implementation of Method 1 (resp. Method 2) behaves as $O(p^3)$ (resp. $O(p^2)$) estimated in Subsection 5.2.
Table 1. The number of isomorphism classes of s.sp. hyperelliptic Howe curves of genus 3, and timing data for Methods 1 and 2 for each \(p\) with \(3 \leq p \leq 200\). Note that any hyperelliptic Howe curve of Oort type \(E_1 \times P \circlearrowleft E_2\) is of Howe type \(E \times P \circlearrowleft H\).

| \(p\) | \(p \mod 4\) | Howe type \(E \times P \circlearrowleft H\) | Time (s) for Method 1 | Oort type \(E_1 \times P \circlearrowleft E_2\) | Time (s) for Method 2 |
|------|------------|----------------|------------------|----------------|------------------|
| 3    | 3          | 0              | < 0.1            | 0              | < 0.1            |
| 5    | 1          | 0              | < 0.1            | 0              | < 0.1            |
| 7    | 3          | 1              | 0.2             | 1              | 0.2             |
| 11   | 3          | 1              | < 0.1            | 0              | < 0.1            |
| 13   | 1          | 0              | < 0.1            | 0              | < 0.1            |
| 17   | 1          | 2              | < 0.1            | 1              | < 0.1            |
| 19   | 3          | 0              | 0.1             | 0              | < 0.1            |
| 23   | 3          | 0              | 0.2             | 3              | < 0.1            |
| 29   | 1          | 3              | 0.2             | 0              | < 0.1            |
| 31   | 3          | 14             | 0.4             | 3              | < 0.1            |
| 37   | 1          | 10             | 0.4             | 0              | < 0.1            |
| 41   | 1          | 10             | 0.7             | 1              | < 0.1            |
| 43   | 3          | 22             | 0.7             | 0              | < 0.1            |
| 47   | 3          | 32             | 0.9             | 0              | < 0.1            |
| 53   | 1          | 15             | 0.9             | 0              | < 0.1            |
| 59   | 3          | 45             | 1.1             | 0              | < 0.1            |
| 61   | 1          | 27             | 1.3             | 0              | < 0.1            |
| 67   | 3          | 48             | 1.8             | 0              | < 0.1            |
| 71   | 3          | 87             | 2.4             | 11             | 0.2             |
| 73   | 1          | 55             | 2.4             | 0              | < 0.1            |
| 79   | 3          | 106            | 3.2             | 11             | 0.2             |
| 83   | 3          | 190            | 3.5             | 0              | < 0.1            |
| 89   | 1          | 89             | 3.9             | 0              | < 0.1            |
| 97   | 3          | 55             | 4.8             | 5              | 0.3             |
| 101  | 1          | 112            | 5.6             | 0              | < 0.1            |
| 103  | 3          | 132            | 6.6             | 10             | 0.3             |
| 107  | 3          | 166            | 6.9             | 0              | < 0.1            |
| 109  | 1          | 144            | 7.1             | 0              | < 0.1            |
| 113  | 1          | 194            | 7.8             | 8              | 0.5             |
| 127  | 3          | 182            | 11.2            | 8              | 0.4             |
| 131  | 3          | 253            | 13.3            | 0              | < 0.1            |
| 137  | 1          | 164            | 13.9            | 7              | 0.5             |
| 139  | 3          | 234            | 15.3            | 0              | < 0.1            |
| 149  | 1          | 240            | 18.6            | 0              | < 0.1            |
| 151  | 3          | 341            | 20.3            | 13             | 0.6             |
| 157  | 1          | 194            | 21.6            | 0              | < 0.1            |
| 163  | 3          | 221            | 24.9            | 0              | < 0.1            |
| 167  | 3          | 459            | 32.5            | 29             | 0.9             |
| 173  | 1          | 232            | 29.5            | 0              | 0.7             |
| 179  | 3          | 419            | 32.5            | 0              | 0.7             |
| 181  | 1          | 273            | 33.4            | 0              | < 0.1            |
| 191  | 3          | 629            | 43.0            | 50             | 1.3             |
| 193  | 1          | 360            | 42.1            | 17             | 1.0             |
| 197  | 1          | 492            | 45.3            | 0              | 0.9             |
| 199  | 3          | 547            | 48.1            | 25             | 1.1             |

In the non-hyperelliptic case, Table 2 below implies that the number of isomorphism classes of s.sp. Howe curves of genus 3 enumerated by Method 3 is \(O(p^3)\) as estimated at the paragraph just after the proof of Lemma 5.2.3. We also observe the following from Tables 3–5:

- For each prime \(p\) with \(13 \leq p \leq 47\), the most expensive step is Step 4.
In each case, we measured time for Steps 1 and 2 respectively. Time for Step 1 is within \(O(\frac{1}{p})\) seconds, and it is negligible compared to Step 2. We see from Table 3 that Step 2 behaves as \(O(p^3)\) estimated in Lemma 5.2.3. Table 4 implies that the total number \(#(L'_4(V_4) \cup L'_4(\text{others}))\) of triples \((\nu, \mu, \lambda)\) obtained in Step 2 such that \(C_{\nu,\lambda,\mu}\) is a s.s.p. non-hyperelliptic Howe curve of genus 3 follows \(O(p^3)\), as expected in Subsection 5.2.

We observe from Table 4 that the execution time for Step 3 behaves as \(\tilde{O}(p^3)\). As results of Step 3, the number of \((\nu, \lambda, \mu)\) with \(\text{Aut}(C_{\nu,\lambda,\mu}) \supseteq V_4\) (resp. \(\text{Aut}(C_{\nu,\lambda,\mu}) \supseteq V_4\)) is \(O(p^3)\) (resp. \(O(p^2)\)).

As Lemma 5.1.2 shows, the cardinality \(#(L'_4(V_4))\) is bounded by 2592. As described in Lemma 5.2.3 and its proof, we see from Tables 3 and 5 that Step 4b is costly compared to Step 4a for all \(p\) in our computation. In particular, Table 5 also shows that both \(#(J)\) and \(#(L'_4(\text{others}))\) would follow \(O(p^3)\) and \(O(p^3)\), and that the execution time of Step 4a (resp. 4b) behaves as \(\tilde{O}(\#(J) \times (\max_{J \in J} #(L'_4(V_4))^2) = \tilde{O}(p^3)\) (resp. \(\tilde{O}((#(\text{others}))^2) = \tilde{O}(p^4)\)).

Table 2. The total number of isomorphism classes of s.s.p. non-hyperelliptic Howe curves of genus 3 over \(\mathbb{F}_p\) for small \(p\), and their classification by types of automorphism groups.

| \(p\) | Total | \(V_4\) | \(D_4\) | \(G_{16}\) | \(S_4\) | \(G_{48}\) | \(G_{96}\) | \(G_{168}\) | Total |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 8 | 1 | 1 | 0 | 4 | 1 | 1 | 0 | 7 |
| 13 | 11 | 2 | 4 | 0 | 4 | 0 | 0 | 1 | 9 |
| 17 | 23 | 8 | 8 | 0 | 6 | 0 | 0 | 1 | 15 |
| 19 | 34 | 14 | 11 | 1 | 6 | 0 | 1 | 1 | 20 |
| 23 | 58 | 28 | 17 | 1 | 10 | 1 | 1 | 0 | 30 |
| 29 | 119 | 70 | 35 | 0 | 14 | 0 | 0 | 0 | 49 |
| 31 | 142 | 88 | 38 | 2 | 12 | 0 | 1 | 1 | 54 |
| 37 | 249 | 168 | 63 | 0 | 18 | 0 | 0 | 0 | 81 |
| 41 | 339 | 240 | 80 | 0 | 18 | 0 | 0 | 1 | 99 |
| 43 | 394 | 285 | 85 | 3 | 20 | 0 | 1 | 0 | 109 |
| 47 | 509 | 381 | 103 | 3 | 19 | 1 | 1 | 1 | 128 |

Table 3. Timing data for Method 3. For each \(p\), the total time is shown in bold fonts.

| \(p\) | Total time (s) | Time (s) for each step |
|---|---|---|---|---|---|
| 11 | 59.6 | < 0.1 | 30.3 | 0.7 | 28.6 |
| 13 | 128.8 | 0.1 | 64.0 | 4.2 | 60.5 |
| 17 | 158.6 | 0.1 | 71.8 | 10.4 | 76.2 |
| 19 | 206.1 | 0.2 | 88.1 | 23.6 | 94.2 |
| 23 | 392.7 | 0.3 | 171.1 | 53.5 | 167.8 |
| 29 | 526.8 | 0.6 | 128.3 | 121.8 | 276.1 |
| 31 | 865.5 | 0.6 | 271.1 | 178.7 | 415.0 |
| 37 | 4289.6 | 1.8 | 1158.3 | 1125.9 | 2003.6 |
| 41 | 12090.0 | 2.4 | 864.5 | 1339.3 | 9883.8 |
| 43 | 54463.7 | 3.0 | 667.3 | 11732.3 | 42061.2 |
| 47 | 16166.8 | 4.7 | 1082.7 | 1844.0 | 13235.3 |
Table 4. The total number of \((\nu, \lambda, \mu) \in \mathbb{F}_p\) computed in Step 2 of Method 3 such that \(C_{\nu,\lambda,\mu}\) is a s.s.p. curve over \(\mathbb{F}_p\), and their classification obtained in Step 3 by types of \(\text{Aut}(C_{\nu,\lambda,\mu})\).

| \(p\) | Num. of \((\nu, \lambda, \mu)\)'s for Step 3 | Classification of \((\nu, \lambda, \mu)\)'s by \(\text{Aut}(C_{\nu,\lambda,\mu})\) | \(V_4\) | \(D_4\) | \(G_{16}\) | \(S_4\) | \(G_{48}\) | \(G_{96}\) | \(G_{192}\) |
|-----|------------------|---------------------------------|-----|-----|-----|-----|-----|-----|-----|
| 11  | 780              | \textbf{30.3}                   | 96  | 192 | 0   | 384 | 24  | 84  | 0   |
| 13  | 2592             | \textbf{64.0}                   | 576 | 1152| 0   | 768 | 0   | 0   | 96  |
| 17  | 3744             | \textbf{71.8}                   | 1152| 1728| 0   | 768 | 0   | 0   | 96  |
| 19  | 4860             | \textbf{88.1}                   | 1728| 2112| 72  | 768 | 0   | 84  | 96  |
| 23  | 6688             | \textbf{171.1}                  | 2688| 2688| 72  | 1152| 24  | 84  | 0   |
| 29  | 13632            | \textbf{128.3}                  | 6624| 5472| 0   | 1536| 0   | 0   | 0   |
| 31  | 15492            | \textbf{271.1}                  | 7968| 5664| 144| 1536| 0   | 0   | 0   |
| 37  | 25920            | \textbf{1158.3}                 | 14688| 9312| 0   | 1920| 0   | 0   | 0   |
| 41  | 31968            | \textbf{864.5}                  | 19008| 10944| 0   | 1920| 0   | 0   | 0   |
| 43  | 35964            | \textbf{667.3}                  | 22032| 11712| 0   | 1920| 216| 84  | 0   |
| 47  | 43044            | \textbf{1082.7}                 | 27456| 12960| 72  | 1152| 24  | 84  | 96  |

Table 5. Time comparison between Steps 4a and 4b of Method 3, where \(N_{4a}\) (resp. \(N_{4b}\)) denotes the number of isomorphism tests executed in Step 4a (resp. 4b). The notations \(J(V_4)\), \(L(V_4)\), \(J\) (\(V_4\)), and \(L\) (others) are same as in Method 3.

| \(p\) | \#\(J\) | \#\(L(V_4)\) | \(N_{4a}\) | \(N_{4b}\) | Time (s) | \#\(L\) (others) | \(N_{4b}\) | Time (s) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 11  | 1   | \(\leq 96\) | 95  | 0   | 684 | 1253 | 28.6 |
| 13  | 1   | \(\leq 576\) | 862 | 4.2 | 2016| 4887 | 60.5 |
| 17  | 2   | \(\leq 576\) | 2872| 10.4| 2592| 11601| 76.2 |
| 19  | 3   | \(\leq 864\) | 5746| 23.6| 3132| 16840| 94.2 |
| 23  | 6   | \(\leq 768\) | 8708| 53.5| 4020| 30486| 167.8|
| 29  | 7   | \(\leq 1728\) | 46730| 121.8| 7008| 100367| 276.1 |
| 31  | 9   | \(\leq 1728\) | 5752| 178.1| 7524| 126870| 415.0|
| 37  | 10  | \(\leq 2592\) | 172200| 1844.0| 15588| 726460| 13235.3 |
| 41  | 16  | \(\leq 2592\) | 202800| 1339.3| 12960| 479037| 9883.8 |
| 43  | 19  | \(\leq 2592\) | 241203| 11732.3| 13932| 554135| 42061.2 |
| 47  | 29  | \(\leq 2592\) | 272739| 1844.0| 15588| 726460| 13235.3 |

5.3.2. Existence results and fields of definition. For the hyperelliptic case, we have the following results for every prime \(p\) with \(3 \leq p \leq 200\):

1. We see from Table 1 that there is no s.s.p. hyperelliptic Howe curve of genus 3 if \(p \in \{3, 5, 13\}\), and otherwise such curves do exist.

2. We also examined with Magma that the obtained s.s.p. hyperelliptic curves in Table 1 are all \(\mathbb{F}_p^2\)-maximal (resp. \(\mathbb{F}_p^2\)-minimal) when \(p \equiv 3 \pmod{4}\) (resp. \(p \equiv 1 \pmod{4}\)), as Corollary 4.1.2 implies. In fact, when \(p \equiv 3 \pmod{4}\) with \(p \neq 3\), we succeeded in finding a triple \((a, b, c) \in (\mathbb{F}_p^4)^3\) such that \(C_{a,b,c} : y^2 = (x^2 - 1)(x^2 - a)(x^2 - b)(x^2 - c)\) is a s.s.p. curve of genus 3 defined over the prime field \(\mathbb{F}_p\), namely each \(x^i\)-coefficient in \((x^2 - 1)(x^2 - a)(x^2 - b)(x^2 - c)\) belongs to \(\mathbb{F}_p\).

We also investigate the existence of non-hyperelliptic Howe curves of genus 3 for \(p\) larger than ones in Table 2. For this, we executed the decision-version (described in the first paragraph of Section 5.3) of Method 3 and obtain the following results for every prime \(p\) with \(11 \leq p \leq 20000\):

3. For every \(p\), there exists a s.s.p. non-hyperelliptic Howe curve of genus 3.
(4) When \( p \equiv 3 \pmod{4} \), we succeeded in finding a triple \((\nu, \lambda, \mu) \in (\mathbb{F}_p^*)^3\) such that the corresponding non-hyperelliptic Howe curve of genus 3 is a s.p. curve defined over \( \mathbb{F}_p^* \). However, in the case where \( p \equiv 1 \pmod{4} \), there are many \( p \) such that any non-hyperelliptic s.p. Howe curve of genus 3 constructed as above is not defined over \( \mathbb{F}_p^* \).

Note that the execution time for each \( p \) is at most a few seconds (it takes a few hours in total to terminate for all \( p \)).

The existence result (1) (resp. (3)) examines Oort’s result \[36, \text{Theorem 5.12 (3)}\] in the hyperelliptic case with \( p \equiv 3 \pmod{4} \) (resp. \[36, \text{Theorem 5.12 (1)}\] in the non-hyperelliptic case).

As for the maximality/minimality and the field of definition, Ibukiyama showed in [15] that there exists a s.p. curve \( C \) of genus 3 defined over the prime field \( \mathbb{F}_p^* \) which is \( \mathbb{F}_p^* \)-maximal for each odd prime \( p \), but it is not known whether each \( C \) is hyperelliptic or non-hyperelliptic. As such a curve \( C \), we expect from our computational results (2) and (4) together with Corollary 4.1.2 that we can take it to be either a hyperelliptic curve or a non-hyperelliptic Howe curve when \( p \equiv 3 \pmod{4} \), but for the case \( p \equiv 1 \pmod{4} \), we need to consider a curve that is not realized as a Howe curve.

5.4. Some remarks toward higher genus. Taking \( C_1 \) and \( C_2 \) in Subsection 2.3 to be curves of various genera, construction and enumeration of s.p. Howe curves in higher genus would be also possible, see [12], [26], and [35] for the genus-4 case, and [13] for non-hyperelliptic case of genera 5, 6, and 7. We here focus on the case where the resulting Howe curve is hyperelliptic, and restrict ourselves to the case of genera 4, 5, and 6 for simplicity. Let \( C_1 \) and \( C_2 \) be hyperelliptic curves of genera \( g_1 \) and \( g_2 \) over \( k \), and \( C \) the desingularization of \( C_1 \times \mathbb{P}^1 C_2 \). To construct s.p. hyperelliptic Howe curves in genera 4, 5, and 6, it suffices from Lemma 2.3.2 that we take the following values as \( g_1 \) and \( g_2 \):

- **(I)** \( g_1 = g_2 = 2 \): In this case, \( g = 4 \).
- **(II)** \( g_1 = 2, g_2 = 3 \): In this case, \( g = 5 \).
- **(III)** \( g_1 = g_2 = 3 \): In this case, \( g = 6 \).

For **(I)**, Ohashi-Kudo-Harashita [35] recently proposed an enumeration algorithm with complexity \( O(p^3) \) restricting to the case where \( \text{Aut}(C) = V_4 \), and succeeded in finding s.p. curves in every characteristic \( p \) with \( 19 \leq p \leq 6691 \). The reason why we can efficiently enumerate s.p. Howe curves in this case is that we already have an efficient algorithm [26, §5A], which enumerates all genus-2 s.p. curves in \( O(p^3) \) by producing curves whose Jacobian varieties are Richelot isogenous to those of some given \( \mathbb{F}_p^* \)-maximal curves of genus 2.

Regarding **(II)** and **(III)**, the enumeration is of course possible, but the complexity would becomes exponential due to the following reason: In the case of genus 3, there is no polynomial-time algorithm as in [26, §5A] at this point. A method with exponential complexity for the genus-3 case is the following:

- It requires to search among genus-3 curves of the form
  \[
  D : y^2 = x(x - 1)(x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0)
  \]
  with \( a_i \in \mathbb{F}_p^* \), taking the case \( \text{Aut}(D) \not\supset V_4 \) into account, and it suffices to solve a zero-dimensional multivariate system in \( a_i \) obtained from the condition “the Cartier-Manin matrix of \( D \) is zero” by the Gröbner basis computation as in [22, 24]. However, the complexity of this method would be exponential with respect to \( p \) in the worst case.
If we restrict ourselves to the case of $\text{Aut}(D) \supset V_4$, we can apply the methods in Subsection 5.2 and the total complexity would be bounded by a polynomial function of $p$.

Once s.sp. genus-3 hyperelliptic curves are obtained, we can construct s.sp. curves of genus 5 and 6, as in the method in [35] (we will study details in [23]).

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Appendix A. Some properties of a certain bivariate quartic

Let $f(Y, Z)$ be a bivariate quartic polynomial of the form

$$
f = b_{40}Y^4 + b_{22}Y^2Z^2 + b_{20}Y^2 + b_{04}Z^4 + b_{02}Z^2 + b_{00}
$$

in the variables $Y$ and $Z$ over $k$. Assume $b_{40}, b_{04}, b_{00} \neq 0$.

Lemma A.0.1. With notation as above, assume that $f$ has a linear factor. Then we have the following system of equations:

$$
b_{22}^2 - 4b_{40}b_{04} = b_{20}^2 - 4b_{40}b_{00} = b_{22}b_{20} + 2b_{40}b_{02} = 0.
$$
Proof. Assume that \( f \) has a linear factor \( L_1 \in k[Y, Z] \). From our assumptions \( b_{40}, b_{04}, b_{00} \neq 0 \), we may suppose \( L_1 = Y + a_1 Z + a_2 \), where \( a_1, a_2 \in k \setminus \{0\} \). Then, \( f \) is also divided by

\[
L_2 := -L_1(-Y, Z) = Y - a_1 Z - a_2,
\]
\[
L_3 := L_1(Y, -Z) = Y - a_1 Z + a_2,
\]
\[
L_4 := -L_1(Y, -Z) = Y + a_1 Z - a_2.
\]

If \( L_1, L_2, L_3, \) and \( L_4 \) are all different, then \( f = b_{40} L_1 L_2 L_3 L_4 \), so that

\[
f = b_{40}(Y^2 - (a_1 Z + a_2)^2)(Y^2 - (a_1 Z - a_2)^2)
\]

(A.0.3) \[
b_{40}(Y^4 - 2(a_1^2 Z^2 + a_2^2)Y^2 + (a_1^2 Z^2 - a_2^2)^2)
\]

Comparing the coefficients in \( (A.0.1) \) and \( (A.0.3) \), we obtain the system \((A.0.2)\).

We consider the case where two of \( L_1, L_2, L_3, \) and \( L_4 \) are equal to each other. Without loss of generality, we may suppose that \( L_1 \) is equal to one of the others. In this case, it follows from \( a_1 \neq 0 \) that \( L_1 \neq L_2 \) and \( L_1 \neq L_3 \), whence \( L_1 = L_4 \). Therefore, we have \( a_2 = 0 \), so that \( f \) is divisible by \( Y^2 - a_1^2 Z^2 \). A straightforward computation shows

\[
f = (Y^2 - a_1^2 Z^2)Q + (b_{40} a_1^4 + b_{22} a_1^2 + b_{04})Z^4 + (b_{20} a_1^2 + b_{02})Z^2 + b_{00}
\]

with \( Q = b_{40} Y^2 + (b_{40} a_1^2 + b_{22})Z^2 + b_{20} \), a contradiction to \( b_{00} \neq 0 \). \( \square \)

Lemma A.0.2. With notation as above, if \( f \) is factored into \( f = b_{40} Q_1 Q_2 \) for some monic irreducible quadratic polynomials \( Q_1 \) and \( Q_2 \), then either of the following four cases holds:

\[
(\mathbf{I}) \ b_{22}^2 - 4b_{40}b_{04} = 0.
\]
\[
(\mathbf{II}) \begin{cases} 
 b_{22}^2 - 4b_{40}b_{04} = 0, \\
 b_{20}^2 - 4b_{40}b_{00} = 0.
\end{cases}
\]
\[
(\mathbf{III}) \begin{cases} 
 b_{20}^2 - 4b_{40}b_{00} = 0, \\
 b_{02}^2 - 4b_{40}b_{00} = 0.
\end{cases}
\]
\[
(\mathbf{IV}) \ 2b_{40}b_{02} - b_{22}b_{20} + \sqrt{b_{22}^2 - 4b_{40}b_{04}} \sqrt{b_{20}^2 - 4b_{40}b_{00}} = 0.
\]

Proof. A tedious computation implies that there are four cases on \((Q_1, Q_2)\):

\[
(\mathbf{I}) \begin{cases} 
 Q_1 = Y^2 + a_2 Z^2 + a_3 Y + a_5, \\
 Q_2 = Y^2 - a_2 Z^2 - a_3 Y + a_5,
\end{cases}
\]
\[
(\mathbf{II}) \begin{cases} 
 Q_1 = Y^2 + a_2 Z^2 + a_4 Z + a_5, \\
 Q_2 = Y^2 + a_2 Z^2 - a_4 Z + a_5,
\end{cases}
\]
\[
(\mathbf{III}) \begin{cases} 
 Q_1 = Y^2 + a_1 Y Z + a_2 Z^2 + a_5, \\
 Q_2 = Y^2 - a_1 Y Z + a_2 Z^2 + a_5,
\end{cases}
\]
\[
(\mathbf{IV}) \begin{cases} 
 Q_1 = Y^2 + a_2 Z^2 + a_5, \\
 Q_2 = Y^2 + a_2 Z^2 + a_5,
\end{cases}
\]

In each case, \( f \) is expanded as follows:

\[
(\mathbf{I}) \ f = b_{40}(Y^4 + 2a_2 Y^2 Z^2 + (-a_3^2 + 2a_5)Y^2 + a_5^2 Z^4 + 2a_2 a_5 Z^2 + a_5^2),
\]
\[
(\mathbf{II}) \ f = b_{40}(Y^4 + 2a_2 Y^2 Z^2 + 2a_3 Y^2 + a_5^2 Z^4 + (2a_2 a_5 - a_3^2)Z^2 + a_5^2),
\]
\[
(\mathbf{III}) \ f = b_{40}(Y^4 + (-a_3^2 + 2a_2)Y^2 Z^2 + 2a_3 Y^2 + a_5^2 Z^4 + 2a_3 a_5 Z^2 + a_5^2),
\]
\[
(\mathbf{IV}) \ f = b_{40}(Y^4 + (a_2 + a_5')Y^2 Z^2 + (a_5 + a_5')Y^2 + a_2 a_5' Z^4 + (a_2 a_5' + a_5 a_5')Z^2 + a_5 a_5').
\]

As for the case (IV), we have the following:

- From the coefficients of \( Y^2 Z^2 \) and \( Z^4 \), the elements \( b_{40} a_2 \) and \( b_{40} a_2' \) are the roots of \( X^2 - b_{22} X + b_{40} b_{04} \), whence
  \[
  2b_{40} a_2, 2b_{40} a_2' = b_{22} \pm \sqrt{b_{22}^2 - 4b_{40} b_{04}}.
  \]
- From the coefficients of \( Y^2 \) and 1, the elements \( b_{40} a_5 \) and \( b_{40} a_5' \) are the roots of \( X^2 - b_{20} X + b_{40} b_{00} \), whence
  \[
  2b_{40} a_5, 2b_{40} a_5' = b_{20} \pm \sqrt{b_{20}^2 - 4b_{40} b_{00}}.
  \]
• Once $a_2$, $a_2'$, $a_5$, and $a_5'$ are specified, from the coefficient of $Z^2$, we have $b_{40}(a_2a_5' + a_5a_2') = b_{02}$, whence
\[
4b_{40}b_{02} = 4b_{40}^2(a_2a_5' + a_5a_2') = (2b_{40}a_2)(2b_{40}a_5') + (2b_{40}a_2')(2b_{40}a_5)
\]
\[
= 2 \left( b_{22}b_{20} \pm \sqrt{b_{22}^2 - 4b_{40}b_{04} \sqrt{b_{20}^2 - 4b_{40}b_{00}}} \right),
\]
as desired. \qed