Exact partition function in $U(2) \times U(2)$ ABJM theory deformed by mass and Fayet-Iliopoulos terms

Jorge G. Russo$^{a,b}$ and Guillermo A. Silva$^{c,d}$

$^a$Institució Catalana de Recerca i Estudis Avançats (ICREA), Pg. Lluis Companys, 23, 08010 Barcelona, Spain
$^b$Department ECM, Institut de Ciències del Cosmos, Universitat de Barcelona, Martí Franquès, 1, 08028 Barcelona, Spain
$^c$Instituto de Física La Plata, CONICET & Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina
$^d$Abdus Salam International Centre for Theoretical Physics, Associate Scheme, Strada Costiera 11, 34151 Trieste, Italy

E-mail: jorge.russo@icrea.cat, silva@fisica.unlp.edu.ar

ABSTRACT: We exactly compute the partition function for $U(2)^k \times U(2)^{-k}$ ABJM theory on $S^3$ deformed by mass $m$ and Fayet-Iliopoulos parameter $\zeta$. For $k = 1, 2$, the partition function has an infinite number of Lee-Yang zeros. For general $k$, in the decompactification limit the theory exhibits a quantum (first-order) phase transition at $m = 2\zeta$.

KEYWORDS: Matrix Models, Supersymmetric gauge theory, AdS-CFT Correspondence, Chern-Simons Theories

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1 Introduction

The dynamics of two coincident M2 branes on the orbifold $\mathbb{R}^8/\mathbb{Z}_k$ is described by ABJM theory, three-dimensional U(2)$_k \times$ U(2)$_{-k}$ supersymmetric Chern-Simons theory with bi-fundamental matter [1]. For this particular gauge group, the ABJM theory has $\mathcal{N} = 8$ superconformal symmetry and is in fact equivalent to Gustavsson-Bagger-Lambert theory [2, 3]. The partition function for the theory on $S^3$ can be computed by supersymmetric localization [4, 5]. This theory can be deformed, preserving $\mathcal{N} = 4$ supersymmetry, by adding mass and Fayet-Iliopoulos (FI) parameters $m, \zeta$, and the localization technique then reduces the full supersymmetric functional integral to the matrix integral [5]

$$Z = \frac{1}{4} \int \frac{d^2 \mu}{(2\pi)^2} \frac{d^2 \nu}{(2\pi)^2} \frac{\sinh^2 \frac{\mu_1 - \mu_2}{2} \sinh^2 \frac{\nu_1 - \nu_2}{2}}{\prod_{i,j} \cosh \left( \frac{\mu_1 - \nu_j + m}{2} \right) \cosh \left( \frac{\mu_i - \nu_2 - m}{2} \right) } \frac{i}{i!} \sum_i (\mu_i^2 - \nu_i^2) - \frac{i}{2} \zeta \sum_i (\mu_i + \nu_i),$$

(1.1)

where $i, j = 1, 2$. The parameter $\zeta$ represents a Fayet-Iliopoulos parameter for the diagonal U(1) subgroup, whereas $m$ corresponds to a mass for the chiral multiplets. The partition function should be understood as a function $Z(2\zeta, m; k)$, but for ease of presentation we will omit its arguments unless needed. For $k = 1$, the theory is mirror dual to $\mathcal{N} = 4$ supersymmetric super Yang-Mills theory with gauge group U(2) coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet [5].

By shifting the integration variables, $x \equiv \mu - \zeta$, $y \equiv \nu + \zeta$, the partition function becomes

$$Z = \frac{1}{4} \int \frac{d^2 x}{(2\pi)^2} \frac{d^2 y}{(2\pi)^2} \frac{\sinh^2 \frac{x_1 - x_2}{2} \sinh^2 \frac{y_1 - y_2}{2}}{\prod_{i,j} \cosh \left( \frac{x_1 - y_j + m_1}{2} \right) \cosh \left( \frac{x_i - y_2 - m_2}{2} \right) } \frac{i}{i!} \sum_i (x_i^2 - y_i^2),$$

(1.2)

where $m_1, m_2$ are

$$m_1 = m + 2\zeta \quad \text{and} \quad m_2 = m - 2\zeta.$$  

(1.3)

Note that $\zeta$ has dimension of mass. We are using units where the radius $R$ of the threesphere is $R = 1$. 

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The purpose of this note is to explicitly carry out the integration in (1.2). In the 
\( m = \zeta = 0 \) case, the integral was computed in [6] (a discussion of the partition function 
in the more general ABJ case can be found in [7]). On the other hand, the \( m, \zeta \)-deformed 
ABJM theory was studied in [8] using the Fermi-gas formulation [9] and at large \( N \) for the \( U(N)_k \times U(N)_{-k} \) gauge group in [10] (with \( \zeta = 0 \)) and in [11] (with general \( m, \zeta \neq 0 \)), 
where phase transitions in the complex parameter space generated by \( m_1, m_2 \) and \( g = 2\pi i/k \) 
were investigated. Our explicit formula will uncover some interesting physical properties 
of the mass-deformed system with gauge group \( U(2)_k \times U(2)_{-k} \).

The partition function (1.2) manifests the \( m_1 \leftrightarrow m_2 \) symmetry or, equivalently, 
\( \zeta \rightarrow -\zeta \). A less obvious symmetry is \( m_2 \rightarrow -m_2 \), or [8, 11]

\[
Z(2\zeta, m; k) = Z(m, 2\zeta; k). \tag{1.4}
\]

For the \( k = 1 \) case, this symmetry already appeared in [5], where it was also explained by 
the fact that the corresponding brane configuration is self-mirror. The symmetry implies, 
in particular, that a FI-deformation \( m = 2\zeta \) on the massless theory is equivalent to a mass-
deformation \( m = 2\zeta \) in the theory with vanishing FI-parameter. The case \( m = 2\zeta \) — 
representing a fixed point of this symmetry — is special, as we shall shortly see. In the 
dual \( \mathcal{N} = 4 \) supersymmetric super Yang-Mills theory, \( m_2 = 0 \) corresponds to coupling the 
theory to a massless adjoint hypermultiplet.

2 Residue integration

The partition function for the \( m, \zeta \)-deformed ABJM theory with \( U(N)_k \times U(N)_{-k} \) gauge 
group can be written in the following form [5, 11]

\[
Z(2\zeta, m; k) = \sum_{\rho} (-1)^\rho \frac{1}{N!} \int d^N \tau \prod_i \cosh(k\pi \tau_i) \cosh(\pi(\tau_i - \tau_{\rho(i)}) - \frac{m_1^2}{2}), \tag{2.1}
\]

where the sum goes over permutations. The derivation uses a trigonometric identity, 
Fourier integrations and only holds for opposite Chern-Simons levels (see section 2 in [11] 
for details). For \( N = 2 \), the formula (2.1) then leads to the following expression

\[
Z = \frac{1}{2}(Z_1 - Z_2), \tag{2.2}
\]

with

\[
Z_1 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1 + \tau_2)}}{\cosh(\pi k \tau_1) \cosh(\pi k \tau_2) \cosh^2(\frac{m_1}{2})}, \tag{2.3}
\]

and

\[
Z_2 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1 + \tau_2)}}{\cosh(\pi k \tau_1) \cosh(\pi k \tau_2) \cosh(\pi(\tau_1 - \tau_2) - \frac{m_1}{2}) \cosh(\pi(\tau_1 - \tau_2) + \frac{m_1}{2})}, \tag{2.4}
\]

Using the identity

\[
\frac{1}{\cosh^2 \frac{m_1}{2}} - \frac{1}{\cosh(\pi \tau - \frac{m_1}{2}) \cosh(\pi \tau + \frac{m_1}{2})} = \frac{\text{sech}^2 \frac{m_1}{2} \sinh^2 \pi \tau}{\cosh(\pi \tau - \frac{m_1}{2}) \cosh(\pi \tau + \frac{m_1}{2})} \tag{2.5}
\]

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and the formula for the Fourier transform \[11\]

\[
\int du \frac{e^{-ikm_2u}}{\cosh \left( \frac{\pi u}{2} (u + v) \right) \cosh \left( \frac{\pi u}{2} (u - v) \right)} = \frac{4 \sin(km_2v)}{k \sinh(\pi ku) \sinh m_2},
\]

we obtain

\[
Z = \frac{1}{k^2 \sinh(m_2) \cosh^2 \frac{m_1}{2}} \int du \frac{\sin(m_2u) \sin^2 \frac{\pi u}{k}}{\sinh(\pi u) \cosh \left( \frac{\pi u}{k} - \frac{m_1}{2} \right) \cosh \left( \frac{\pi u}{k} + \frac{m_1}{2} \right)}.
\]

In the limit \( m_2 \to 0 \), the partition function becomes

\[
Z \big|_{m_2=0} = \frac{1}{k^2 \cosh^2 \frac{m_1}{2}} \int du \frac{u \sin^2 \frac{\pi u}{k}}{\sinh(\pi u) \cosh \left( \frac{\pi u}{k} - \frac{m_1}{2} \right) \cosh \left( \frac{\pi u}{k} + \frac{m_1}{2} \right)}.
\]

In the following, we compute the integrals (2.7), (2.8) by residue integration.

To compute (2.7) we follow the ideas in \[6\], where the partition function was computed in the case \( m = \zeta = 0 \).

Thus we start by writing the integrand as the product of two even functions \( f, g \)

\[
Z = \frac{1}{k^2 \sinh(m_2) \cosh^2 \frac{m_1}{2}} \int du f(u)g(u),
\]

with

\[
f(u) = \frac{\sin m_2u}{\sinh \frac{\pi u}{k}}, \quad g(u) = \frac{\sin^2 \frac{\pi u}{k}}{\cosh \left( \frac{\pi u}{k} - \frac{m_1}{2} \right) \cosh \left( \frac{\pi u}{k} + \frac{m_1}{2} \right)}.
\]

Under the shift \( u \to u + ik \) these functions transform as

\[
f(u) \to (-)^k \cosh(m_2k)f(u) + \text{odd function},
\]

\[
g(u) \to g(u)
\]

These properties imply that the integral in (2.9) along the curve \( u = x + ik \) with \( x \in \mathbb{R} \) will differ from the integration along the real axis by the factor \((-)^k \cosh(m_2k)\). Therefore, the rectangular contour composed by the real axis, two vertical segments and the displaced real axis \( u = x + ik \) becomes appropriate for residue computation in the case \( m_2 \neq 0 \) (see figure 1).\(^1\)

The residues encircled by the contour comprise the ones arising from the poles of \( f(z) \) located at \( z = in \) with \( n = 1, \ldots, k \) and those of \( g(z) \) located at \( z = \pm \frac{m_1}{2\pi} + ik \). The pole located at \( z = ik \) does not contribute due to a double zero in the numerator of \( g(z) \).

Calling \( C \) the closed rectangular contour described above and \( \mathcal{F}(z) = f(z)g(z) \) one finds

\[
\oint_C dz \mathcal{F}(z) = (1 - (-)^k \cosh(m_2k)) \int du \mathcal{F}(u)
\]

\[
= 2\pi i \left[ \sum_{n=1}^{k-1} \text{Res}_{z=im} \mathcal{F}(z) + \text{Res}_{z=\pm} \mathcal{F}(z) \right]
\]

\(^1\)It is easily seen that the vertical contours do not contribute when we push them to infinity.
which gives

\[
\int du \mathcal{F}(u) = \frac{2\pi i}{1 - (-)^k \cosh(m_2 k)} \left[ -i \sum_{n=1}^{k-1} \frac{(-)^n \sin^2 \left( \frac{m_1}{k} \right) \sinh(m_2 n)}{\cosh \left( \frac{m_1}{2} - \frac{m_2 n}{k} \right) \cosh \left( \frac{m_1}{2} + \frac{m_2 n}{k} \right)} + R_k \right]
\]

where

\[
R_k = \begin{cases} 
\left( -k \frac{ik}{\pi} \coth \frac{m_1}{2} \sinh \frac{k m_2}{2} \right) \cos \frac{k m_1 m_2}{2 \pi}, & k \text{ even} \\
\left( -\frac{k+1}{\pi} \frac{ik}{\coth \frac{m_1}{2}} \sin \frac{k m_1 m_2}{2 \pi} \right), & k \text{ odd}
\end{cases}
\]

Case \( m_2 = 0, k \ odd \). It is evident from (2.12) that the \( m_2 \to 0 \) limit of (2.9) is smooth, the result is

\[
Z \big|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} \left[ \sum_{n=1}^{k-1} (-)^n \frac{n \sin^2 \left( \frac{m_1}{k} \right)}{\cosh (m - \frac{mn}{k}) \cosh (m + \frac{mn}{k})} - \left( -\frac{k+1}{\pi} \frac{k^2 m \coth m}{\pi \cosh km} \right) \right], \quad k \ odd
\]

where we have used \( m_1 = 2m \).

Case \( m_2 = 0, k \ even \). The factor multiplying the bracket in (2.12) prevents taking \( m_2 \to 0 \) in the even \( k \) case. To compute the integral in (2.8) we consider

\[
I = \int du \tilde{f}(u) g(u),
\]

with \( g(u) \) as in (2.10) and

\[
\tilde{f}(u) = \frac{i (u - ik/2)^2}{k \sinh \pi u}.
\]
Upon integration, the odd piece in $\tilde{f}$ vanishes against $g(u)$ and therefore the partition function (2.8) can be written as

$$Z|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} I$$

(2.16)

The shift $u \rightarrow u + ik$ in $\tilde{f}(u)$ gives

$$\tilde{f}(u) \rightarrow (-)^{k+1} \tilde{f}(-u).$$

As discussed below (2.11), this property makes the rectangular contour in figure 1 appropriate for computing $I$ by residues.

For the residues analysis we should now consider the pole in $\tilde{f}(z)$ at the origin $z = 0$ but a zero in $g(z)$ eliminates it; along the same lines the residue from $z = ik/2$ is absent since a zero appears for $\tilde{f}$. Calling $\tilde{F}(z) = \tilde{f}(z)g(z)$ one finds

$$\oint_C dz \tilde{F}(z) = 2I,$$

on the other hand

$$\oint_C dz \tilde{F}(z) = 2\pi i \left[ \sum_{n=0}^{k-1} \text{Res}_{z=\pm in} \tilde{F}(z) + \text{Res}_{z=\pm \infty} \tilde{F}(z) \right]$$

$$= 2\pi i \left[ \frac{i}{k\pi} \sum_{n=1}^{k-1} (-)^n \left( \frac{k}{2} - n \right)^2 \frac{\sin^2 \left( \frac{\pi n}{k} \right)}{\cosh \left( m - \frac{\pi n}{k} \right) \cosh \left( m + \frac{\pi n}{k} \right)} + \hat{R}_k \right].$$

(2.17)

where

$$\hat{R}_k = (-)^{\frac{k}{2}} \frac{2i(mk)^2 \coth(m) \sinh mk}{\pi^3 \cosh(2mk) - 1}.$$

The $n = \frac{k}{2}$ term in the sum vanishes as expected. The final result is

$$Z|_{m_2=0} = -\frac{1}{k \cosh^2 m},$$

$$\left[ \sum_{n=1}^{k-1} (-)^n \left( \frac{n}{k} - \frac{1}{2} \right)^2 \frac{\sin^2 \left( \frac{\pi n}{k} \right)}{\cosh \left( m - \frac{\pi n}{k} \right) \cosh \left( m + \frac{\pi n}{k} \right)} + (-)^{\frac{k}{2}} \frac{2m^2k \coth(m) \sinh mk}{\pi^2 \cosh(2mk) - 1} \right]$$

(2.18)

3 Summary of results and limits

Thus we have obtained

$$Z = \frac{2}{k^2 \sinh(m_2)} \frac{1}{1 - (-1)^k \cosh(m_2k)} (J_1 - J_2)$$

(3.1)

where

$$J_1 = \frac{1}{\cosh^2 \left( \frac{m_1}{2} \right)} \sum_{n=1}^{k-1} (-)^n \frac{\sin^2 \left( \frac{\pi n}{k} \right) \sinh(m_2n)}{\cosh \left( \frac{m_1}{2} - \frac{\pi n}{k} \right) \cosh \left( \frac{m_1}{2} + \frac{\pi n}{k} \right)}$$

(3.2)
and

\[
J_2 = \begin{cases} 
(-)^{\frac{k}{2}} \frac{2k \sinh \frac{k m_2}{2}}{\sinh(m_1) \sinh \frac{k m_1}{2}} \cos \frac{k m_1 m_2}{2\pi}, & k \text{ even} \\
(-)^{\frac{k+1}{2}} \frac{2k \cosh \frac{k m_2}{2}}{\sinh(m_1) \cosh \frac{k m_1}{2}} \sin \frac{k m_1 m_2}{2\pi}, & k \text{ odd}
\end{cases}
\] (3.3)

Using

\[
\frac{2}{1 + \cosh \alpha} = \frac{1}{\cosh^2 \left(\frac{\alpha}{2}\right)}, \quad \frac{2}{1 - \cosh \alpha} = -\frac{1}{\sinh^2 \left(\frac{\alpha}{2}\right)}, \quad \alpha = \frac{m_1}{2}
\] (3.4)

we can finally put the partition function in the form

\[
Z|_{k \text{ even}} = -\frac{1}{k^2 \sinh(m_2) \sinh^2 \left(\frac{k m_2}{2}\right)} (J_1 - J_2)
\] (3.5)

\[
Z|_{k \text{ odd}} = \frac{1}{k^2 \sinh(m_2) \cosh^2 \left(\frac{k m_2}{2}\right)} (J_1 - J_2)
\] (3.6)

In the formulas (3.5)–(3.6), the symmetry \(m_1 \leftrightarrow m_2\) — which is manifest in the integral form (1.2) — is hidden. Interestingly, this symmetry is only recovered upon summation over \(n\). On the other hand, the symmetry \(m_2 \rightarrow -m_2\) is manifest.

Note that \(Z\) is real. While this is expected in a unitary theory, it is not generally the case in Chern-Simons theories (for a discussion, see [12]). In the present case, it is related to the fact the theory is a combination of two Chern-Simons theory with opposite levels.\(^2\)

Consider, as particular examples, the important cases \(k = 1, 2\). The partition functions take the form

\[
Z|_{k=1} = \frac{2}{\sinh(m_1) \sinh(m_2) \cosh \left(\frac{m_1}{2}\right) \cosh \left(\frac{m_2}{2}\right)} \sin \left(\frac{m_1 m_2}{2\pi}\right), \quad (3.7)
\]

\[
Z|_{k=2} = \frac{2}{\sin^2(m_1) \sin^2(m_2) \sin^2 \left(\frac{m_1}{2}\right) \sin^2 \left(\frac{m_2}{2}\right)} \sin \left(\frac{m_1 m_2}{2\pi}\right). \quad (3.8)
\]

Now the symmetry \(m_1 \leftrightarrow m_2\) has become manifest.

Note that the partition functions for \(k = 1, 2\) have zeros. Restoring the \(R\) dependence, the zeros are located at

\[
m_1 m_2 R^2 = 2\pi^2 n, \quad n = \pm 1, \pm 2, \ldots \quad (3.9)
\]

They represent Lee-Yang zeros (see, for example, [13]). In the infinite volume, \(R \rightarrow \infty\), the zeros condense in a certain line, and a phase transition should emerge. The fact that the partition function has zeros seems to be related to the fact that the coupling, \(g = 2\pi i/k\), is imaginary for real \(k\). Indeed, from the general expressions (3.2)–(3.3) we see that the arguments of the sine and cosine functions in (3.7), (3.8) contain a factor \(\pi/k\). If the coupling \(g\) is (unphysically) continued to the real line by taking \(k \rightarrow ik\), the partition function zeros would then lie on the imaginary \(g\)-axis, in accordance with the Lee-Yang theorem (see [11] for a related discussion).

For the undeformed ABJM theory, the \(k = 1\) case is of special interest, since it is conjectured to describe the dynamics of two M2 branes in eleven-dimensional Minkowski

\(^2\)We thank Miguel Tierz for comments on this point.
spacetime. An interesting question is what is the origin of these Lee-Yang singularities in the brane realization.

The partition function $Z(2\zeta, m; k)$ does not have any zeros for $k > 2$. For higher values of $k$, the partition function becomes more involved, below we quote explicitly the $k = 3$ and $k = 4$ cases

\begin{align}
Z_{k=3} &= \frac{2}{3} \frac{2 - \sin \left(\frac{3m_1 m_2}{2}\right) \text{csch} \left(\frac{m_1}{2}\right) \text{csch} \left(\frac{m_2}{2}\right)}{(\cosh m_1 + \cosh 2m_1)(\cosh m_2 + \cosh 2m_2)} \\
Z_{k=4} &= \frac{1 - \text{sech}(m_1) - \text{sech}(m_2) + \cos \left(\frac{2m_1 m_2}{\pi}\right) \text{sech}(m_2) \text{sech}(m_1)}{8 \sinh^2 m_1 \sinh^2 m_2} \tag{3.10}
\end{align}

Note that the symmetry under the exchange $m_1 \leftrightarrow m_2$ is manifest.

**Asymptotic formulas.** Let us consider the limit of a large sphere, $mR \gg 1$, at fixed $k$. Assuming $m_1 > 0$, $m_2 > 0$ and restoring the $R$ dependence, we find

\begin{align}
Z \big|_{k=1} &\sim 32 e^{-\frac{3}{2}(m_1 + m_2)R} \sin \left(\frac{m_1 m_2 R^2}{2\pi}\right), \\
Z \big|_{k=2} &\sim 32 e^{-2(m_1 + m_2)R} \sin^2 \left(\frac{m_1 m_2 R^2}{2\pi}\right), \\
Z \big|_{k>2} &\sim \frac{64}{k^2} e^{-2(m_1 + m_2)R} \sin^2 \left(\frac{\pi}{k}\right) \tag{3.12}
\end{align}

The general asymptotic formula with arbitrary sign for $m_2$ and $m_2 \neq 0$, is obtained by replacing $m_2$ by $|m_2|$.

The absolute value implies a discontinuity in the first derivative of $F = -\ln Z$. This indicates a first-order phase transition in the parameter $m_2$ at $m_2 = 0$, i.e., when the two mass scales $m, 2\zeta$ cross. Explicitly, at large $R$, we have

\begin{equation}
F = 2(|m_1| + |m_2|)R + O(1), \quad k > 1. \tag{3.13}
\end{equation}

Hence

\begin{equation}
\frac{d\Delta F}{dm_2} \big|_{m_2=0} = 4R, \quad \Delta F \equiv F_{m_2>0} - F_{m_2<0}. \tag{3.14}
\end{equation}

For $k = 1$ the discontinuity in the first derivative of $\Delta F$ is equal to $3R$, as can be seen from (3.12).

For the general theory with gauge group $\text{U}(N)_k \times \text{U}(N)_{-k}$, large $N$ phase transitions in the complex parameter $Ng = 2\pi i N/k$ were studied in [10, 11]. These phase transitions require taking infinite volume and, at the same time, a strong coupling limit with fixed $kR$ — a limit that already appeared in the context of supersymmetric $\text{U}(N)$ Chern-Simons theory with massive fundamental matter in [14, 15]. It should be noted that such decompactification limit is different from the present (more physical) limit of large $R$ at fixed $k$.

Another interesting aspect of (3.14) is that it is in a form suitable for a weak coupling expansion in powers of $1/k$:

\begin{equation}
Z \big|_{k>2} \sim -\frac{32}{k^2} e^{-2(m_1 + m_2)R} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2\pi}{k}\right)^{2n}. \tag{3.15}
\end{equation}
The perturbative expansion has an infinite radius of convergence. However, the original theory on the three-sphere of finite radius $R$ has an asymptotic perturbative expansion, with $2n!$ asymptotic behavior for the $1/k^{2n}$ term. This can be seen by using the integral form (2.7) and generalizing the study of [16, 17] on the resurgence properties of the perturbation series of ABJM theory. Now, expanding the integrand in (2.7), one finds a series with finite radius of convergence determined by the poles of sech($n\pi u/k \pm m_1/2$) in the complex $u$-plane. The integral over $u$ then adds an extra $(2n)!$, leading to an asymptotic (but Borel summable) perturbation series.

4 The special case $m_2 = 0$

The $m_2 = 0$ case is special and must be considered separately. In particular, it represents the critical point in the phase transitions that arise in the decompactification limit. In section 2 we have obtained the following formulas:

**Odd $k$:**

$$
Z|_{m_2=0} = \frac{1}{k^2 \cosh^2 \frac{m}{k}} \sum_{n=1}^{k-1} (-)^n \frac{n \sin^2 \frac{n\pi}{k}}{\cosh \left( m + \frac{n\pi}{k} \right) \cosh \left( m - \frac{n\pi}{k} \right)} + \frac{(-)^{\frac{k-1}{2}}}{\pi \cosh(km) \sinh(2m)}.
$$

(4.1)

**Even $k$:**

$$
Z|_{m_2=0} = \frac{1}{k \cosh^2 \frac{m}{k}} \sum_{n=1}^{k-1} (-)^{n+1} \frac{\left( \frac{n}{k} - \frac{1}{2} \right)^2 \sin^2 \frac{n\pi}{k}}{\cosh \left( m - \frac{n\pi}{k} \right) \cosh \left( m + \frac{n\pi}{k} \right)} + \frac{(-)^{\frac{k}{2}+1} 4m^2}{\pi^2 \sinh(mk) \left( \cosh(2mk) - 1 \right)}.
$$

(4.2)

In particular,

$Z|_{k=1} = \frac{2m}{\pi \cosh(m) \sinh(2m)}$,

$Z|_{k=2} = \frac{2m^2}{\pi^2 \sinh^2(2m)}$.

(4.3)

Note that the partition function does not have zeros in this case.

**Asymptotic formulas $m_2 = 0$.** Consider again the limit of a large sphere, $mR \gg 1$, at fixed $k$, but now with $m_2 = 0$. We find

$$
Z|_{k=1} \sim \frac{8mR}{\pi} e^{-3mR},
$$

(4.4)

$$
Z|_{k=2} \sim \frac{8}{\pi} m^2 R^2 e^{-4mR},
$$

(4.5)

$$
Z|_{k>2} \sim \frac{4}{k^2} e^{-4mR} \tan^2 \frac{\pi}{k}.
$$

(4.6)
Note that these formulas differ from the asymptotic formulas (3.12)–(3.14) given above for $Z(m_1, m_2)$ at $m_2 = 0$. This is expected, since the latter were obtained by assuming $|m_1 R|, |m_2 R| \to \infty$.

Unlike the $m_2 \neq 0$ case, the perturbation series for this flat-theory limit has now finite radius of convergence $|\pi/k| < \pi/2$, therefore perturbation series is convergent for all $k > 2$, where the formula applies. On the other hand, just like the general $m_2 \neq 0$ case, the theory on a finite-radius $S^3$ has an asymptotic perturbation series with $2n!$ asymptotic behavior.

Finally, it would be interesting to study supersymmetric Wilson loops in the present mass/FI deformed theory, along the lines of [18].

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