Algebraic sets defined by the commutator matrix

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Abstract

In this paper we study algebraic sets of pairs of matrices defined by the vanishing of either the diagonal of their commutator matrix or its anti-diagonal. We find a system of parameters for the coordinate rings of these two sets and their intersection and show that they are complete intersections. Moreover, we prove that these algebraic sets are $F$-pure over a field of positive prime characteristic and the algebraic set of pairs of matrices with the zero diagonal commutator is $F$-regular.

Keywords: commutator matrix, system of parameters, complete intersection, $F$-pure, $F$-regular

1 Introduction

Let $X = (x_{ij})_{1 \leq i, j \leq n}$ and $Y = (y_{ij})_{1 \leq i, j \leq n}$ be square matrices of size $n \geq 3$ with indeterminate entries over a field $K$ and $R = K[X,Y]$ be the polynomial ring over $K$ in $\{x_{ij}, y_{ij} \mid 1 \leq i, j \leq n\}$. Let $m$ be the homogeneous maximal ideal of $R$. Let $C = XY - YX = (c_{ij})$ be the commutator matrix and $I$ be the ideal of $R$ generated by the diagonal and anti-diagonal entries of $C$. Let
Let $I$ be the ideal of $R$ generated by the diagonal entries of $C$ and $J$ be the ideal of $R$ generated by the anti-diagonal entries of $C$, that is, $\mathcal{I} = I + J$. Let $X_0$ and $Y_0$ be the square matrices of size $(n-2)$ obtained by removing the first and last rows and columns of $X$ and $Y$, respectively. We denote by $\mathcal{I}_0$ the ideal of $R_0 = K[X_0,Y_0]$ generated by the diagonal and anti-diagonal entries of $C_0 = X_0Y_0 - Y_0X_0 = (c_{ij}^0)$.

**Remark 1.** Observe that if $n = 2$, then $\mathcal{I}$ is the ideal generated by all the entries of the commutator matrix $C$. Therefore, the vanishing set defines the variety of commuting matrices for $n = 2$ and also a determinantal ideal, [5], [1], [6]. Since the properties of rings that we study in this paper have been extensively studied in this case, we focus on matrices of size $n \geq 3$.

**Remark 2.** Since the trace of the commutator matrix $C$ is equal to 0, from the main diagonal of $C$ it is sufficient to use any $n - 1$ entries. Therefore, the ideal $\mathcal{I}$ can be generated by either $2n - 2$, if $n$ is odd, and $2n - 1$, if $n$ is even, elements. In fact, as the results of the paper show, this defines the minimal number of generators for the ideal.

Below is a summary of the results that we prove in the paper.

**Theorem 1.1.** 1. The rings $R/\mathcal{I}$, $R/I$ and $R/J$ are complete intersections, that is, the ideals $\mathcal{I}$, $I$ and $J$ are generated by a regular sequence.

2. $R/\mathcal{I}$ and $R/J$ are F-pure rings when the characteristic of the field $K$ is positive prime. In particular, in this case $\mathcal{I}$ and $J$ are radical ideals.

3. $R/I$ is F-regular when the characteristic of the field $K$ is positive prime. In particular, in this case $R/I$ is an integral domain.

**Remark 3.** Notice that the fact $R/I$ is a complete intersection is first proved by H.W. Young in [7], here we obtain this fact as a corollary of Theorems 2.1, 2.2 and 2.3. Moreover, Young proved that $R/I$ is reduced when $K$ is a field of characteristic 0.

## 2 A system of parameters

In this section we find a homogeneous system of parameters for $R/\mathcal{I}$. We do the proof by induction on $n$ and with the induction step equal to 2. Therefore, we prove the results separately for odd and even values of $n$. Moreover, when $n$ is even we consider two cases: when the characteristic of $K$ is not equal to
2 and when the characteristic of $K$ is 2. There is an appendix at the end of the paper which the reader may find helpful in understanding the pattern of the system of parameters as $n$ varies.

**Theorem 2.1.** Let $n = 2k + 1$ be an odd integer with $k \geq 1$. Let

$$
\Lambda = \{(i, j) | 1 \leq i \leq k, 1 \leq j \leq 2k + 1\} \cup \{(k + 1, k + 1)\} \cup \{(i, j) | k + 2 \leq i \leq 2k + 1, k + 2 \leq j \leq 2k + 1\} \cup \{(i, j) | k + 3 \leq i \leq 2k + 1, 1 \leq j \leq k\}
$$

Then

$$
x_{ij}, y_{ji} \text{ with } (i, j) \in \Lambda,
$$

$$
x_{k+1,s} - y_{2k+2-s,k+1} \text{ with } 1 \leq s \leq 2k + 1
$$

(notice $s = k + 1$ is included above in $\Lambda$)

$$
x_{s,k+1} - y_{k+1,s} \text{ with } k + 2 \leq s \leq 2k + 1
$$

$$
x_{k+2,s} - y_{s,k+2} \text{ with } 1 \leq s \leq k
$$

is a homogeneous system of parameters for $R/I$.

**Proof.** We prove the theorem by showing that the generators of $I$, $n - 1$ entries of the main diagonal and $n - 1$ entries of the anti-diagonal of $C$, together with the ring elements from the statement of the theorem form a system of parameters for $R$. We achieve this by showing that the ideal they generate is $m$-primary. It is not difficult to compute that their number is exactly equal to $2n^2$, the dimension of $R$. Moreover, we use induction on $k$.

Let $k = 1$. We prove the base of the induction by showing that the radical of the ideal generated by $I$ and the above sequence has Krull dimension 18, the dimension of $R$ in this case. Observe, that $I$ is generated by 4 elements and there are 14 elements in the sequence. This will show that the generators of $I$ and the sequence elements form a system of parameters for $R$.

Factor out $R$ by the ideal generated by the sequence elements. Then we have the following matrices

$$
\overline{X} = \begin{bmatrix}
0 & 0 & 0 \\
x_{21} & 0 & x_{23} \\
x_{31} & x_{32} & 0
\end{bmatrix}
$$

and

$$
\overline{Y} = \begin{bmatrix}
0 & x_{23} & x_{31} \\
0 & 0 & x_{32} \\
0 & x_{21} & 0
\end{bmatrix}
$$
and the commutator matrix becomes

\[
XY - YX = \begin{bmatrix}
-x_{21}x_{23} - x_{31}^2 & * & -x_{23}^2 \\
* & 2x_{21}x_{23} - x_{32}^2 & * \\
-x_{21}^2 & * & x_{31}^2 + x_{32}^2 - x_{21}x_{23}
\end{bmatrix}
\]

Therefore,

\[
\text{Rad } \mathcal{I} = \text{Rad}(-x_{23}^2, -x_{21}^2, -x_{21}x_{23} - x_{31}^2, 2x_{21}x_{23} - x_{32}^2, x_{31}^2 + x_{32}^2 - x_{21}x_{23}) = (x_{21}, x_{23}, x_{31}, x_{32})
\]

is the homogeneous maximal ideal in \( K[x_{21}, x_{23}, x_{31}, x_{32}] \). Thus we have the result for \( n = 3 \).

Now we prove the induction step. We claim that a similar pattern persists for all positive integer values of \( k \). We apply the induction hypothesis in the ring \( R_0 = K[X_0, Y_0] \). Factor out \( R \) by the ring elements in the set defined in the statement of the theorem. Then we have the following matrices

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{k+1,1} & x_{k+2,1} & \ldots & x_{k+1,2k+1} & 0 \\
x_{k+1,1} & x_{k+2,1} & \ldots & x_{k+1,2k+1} & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]
and

\[
(k + 1)\text{-column}
\]

\[
\mathbf{Y} = \begin{bmatrix}
0 & 0 & \ldots & 0 & x_{k+1,2k+1} & x_{k+1,2k+1} & 0 & \ldots & 0 & 0 \\
0 & \vdots & \ddots & \vdots & 0 & 0 & \vdots & \ddots & 0 & 0 \\
0 & 0 & \vdots & \vdots & 0 & 0 & \vdots & \ddots & 0 & 0 \\
0 & 0 & \vdots & \vdots & 0 & 0 & \vdots & \ddots & 0 & 0 \\
0 & 0 & \ldots & 0 & x_{k+1,1} & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{Y}_0
\]

\[
\leftarrow (k + 1)\text{-row}
\]

We start off by writing out the generators of \( \mathcal{I} \) in the factor ring.

\[
\overline{c}_{11} = -x_{k+1,1}x_{k+1,2k+1} - x_{k+2,1}^2
\]

\[
\overline{c}_{1,2k+1} = -x_{k+1,2k+1}^2
\]

\[
\overline{c}_{2k+1,1} = -x_{k+1,1}^2
\]

\[
\overline{c}_{2k+1,2k+1} = x_{2k+1,k+1}^2 - x_{k+1,1}x_{k+1,2k+1}
\]

For \( 2 \leq i \leq k \),

\[
\overline{c}_{ii} = \sum_{j=1, j \neq i}^{2k+1} (x_{ij}y_{ji} - x_{ji}y_{ij}) = -x_{k+1,i}y_{i,k+1} - x_{k+2,i}y_{i,k+2} =
\]

\[
-x_{k+1,i}x_{k+1,2k+2-i} - x_{k+2,i}^2 = \overline{c}_{i-1,i-1}^0
\]

\[
\overline{c}_{k+1,k+1} = \sum_{j=2, j \neq k+1}^{2k} (x_{j+1,j}y_{j,k+1} - x_{j,k+1}y_{j+1,k}) +
\]

\[
x_{k+1,1}y_{1,k+1} - x_{k+1,2k+1}y_{2k+1,k+1} - x_{2k+1,1}y_{k+1,2k+1} =
\]

\[
= \overline{c}_{k,k}^0 + 2x_{k+1,1}x_{k+1,2k+1} - x_{2k+1,k+1}^2
\]

\[
\overline{c}_{k+2,k+2} = \sum_{j=2}^{2k} (x_{ij}y_{ji} - x_{ji}y_{ij}) + x_{k+2,1}^2 = \overline{c}_{k+1,k+1}^0 + x_{k+2,1}^2
\]
For \( k + 3 \leq i \leq 2k \)
\[
\overline{c}_{ii} = \sum_{j=2, j \neq i}^{2k} (x_{ij}y_{ji} - x_{ji}y_{ij}) = \overline{c}_{i-1,i-1}
\]

For \( 2 \leq i \leq 2k \) and \( i \neq k + 1 \)
\[
\overline{c}_{i,2k+2-i} = \overline{c}_{i-1,2k+1-i}
\]

Now we have that
\[
\overline{I} = (\overline{c}_{ii}, \overline{c}_{i,2k+2-i})_{1 \leq i \leq 2k+1} =
\]
\[
(-x_{k+1,1}x_{k+1,2k+1}, -x_{k+1,2k+1}, -x_{k+1,1},
\overline{c}_{k,k}^0 + 2x_{k+1,1}x_{k+1,2k+1} - x_{2k+1,k+1}^2,
\overline{c}_{k+1,k+1}^0 + x_{k+2,1}^2, x_{2k+1,k+1}^2 - x_{k+1,1}x_{k+1,2k+1} + \sum_{1 \leq i \leq 2k-1, i \neq k,k+1} \overline{c}_{i,i}^0)\bigcup \sum_{2 \leq i \leq 2k, i \neq k+1} \overline{c}_{i,i}^0
\]

Then
\[
\text{Rad}(\overline{I}) = (x_{k+1,2k+1}, x_{k+2,1}, x_{k+1,1}, x_{2k+1,k+1}) + \text{Rad}(\overline{c}_{ii}, \overline{c}_{i,2k-i})_{1 \leq i \leq 2k-1}.
\]

By induction hypothesis, \( \text{Rad}(\overline{c}_{ii}, \overline{c}_{i,2k-i})_{1 \leq i \leq 2k-1} \) is the homogeneous maximal ideal in \( \overline{R}_0 = K[\overline{X}, \overline{Y}] \). Therefore, we have that \( \text{Rad}(\overline{I}) \) is the homogeneous maximal ideal in \( \overline{R} = K[\overline{X}, \overline{Y}] \).

**Theorem 2.2.** Let \( n = 2k \) be an even integer with \( k \geq 2 \). Let \( K \) be a field of characteristic not equal to 2. Let
\[
\Omega = \{(i,j) | 1 \leq i \leq k - 1, 1 \leq j \leq 2k\} \cup
\{(k,k)\} \cup \{(k,k+1)\} \cup \{(k+1,k+1)\} \cup
\{(k+1,j) | k + 3 \leq j \leq 2k\} \cup
\{(i,k+1) | k + 3 \leq i \leq 2k\} \cup
\{(i,j) | k + 2 \leq i \leq 2k, 1 \leq j \leq k - 1\} \cup
\{(i,j) | k + 2 \leq i \leq 2k, k + 2 \leq j \leq 2k\}
\]
Then

\[ x_{ij}, \ y_{ji} \ \text{with} \ (i,j) \in \Omega, \]

\[ x_{k,s} - y_{2k+1-s,k} \]

with \(1 \leq s \leq 2k\) (notice the cases \(s = k, k + 1\) are included above in \(\Omega\)),

\[ x_{k+1,s} - y_{s,k+1} \]

with \(1 \leq s \leq k + 2\) (notice the case \(s = k + 1\) is included above in \(\Omega\)),

\[ x_{k+2,k} - y_{k+1,k+2}, \quad x_{k+2,k+1} - y_{k,k+2}, \]

\[ x_{s,k} - y_{k,s} \ \text{with} \ k + 3 \leq s \leq 2k \]

is a homogeneous system of parameters for \(R/I\).

Proof. As in the previous theorem for matrices of odd size, we prove the theorem by showing that the generators of \(I\), \(n - 1\) entries of the main diagonal and \(n\) entries of the anti-diagonal of \(C\), together with the ring elements from the statement of the theorem form a system of parameters for \(R\). We achieve this by showing that the ideal they generate is \(m\)-primary. It is not difficult to compute that their number is exactly equal to \(2n^2\), the dimension of \(R\). Moreover, we use induction on \(k\).

Let \(k = 2\). The idea of the proof is similar to that of the proof of the case \(n = 3\). Factor out by the ideal generated by the elements of the sequence. Then we have the following matrices

\[
\bar{X} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
x_{21} & 0 & 0 & x_{24} \\
x_{31} & x_{32} & 0 & x_{34} \\
0 & x_{42} & x_{43} & 0
\end{bmatrix}
\]

and

\[
\bar{Y} = \begin{bmatrix}
0 & x_{24} & x_{31} & 0 \\
0 & 0 & x_{32} & x_{43} \\
0 & 0 & 0 & x_{42} \\
0 & x_{21} & x_{34} & 0
\end{bmatrix}
\]

and the commutator matrix becomes

\[
\bar{X} \bar{Y} - \bar{Y} \bar{X} =
\]

\[
\begin{bmatrix}
-x_{21}x_{24} - x_{31}^2 & * & * & -x_{31}x_{34} - x_{24}^2 \\
2x_{21}x_{24} - x_{32}^2 - x_{42}x_{43} & x_{21}x_{31} - x_{24}x_{34} - x_{31}^2 & x_{24}x_{34} - x_{42}x_{43} & * \\
* & x_{21}x_{31} - x_{21}x_{34} - x_{42}^2 & x_{31}^2 + x_{32}^2 + x_{34}^2 - x_{42}x_{43} & * \\
-x_{31}x_{34} - x_{21}^2 & * & 2x_{42}x_{43} - x_{21}x_{24} - x_{34}^2 & *
\end{bmatrix}
\]
We show that the ideal generated by the diagonal and anti-diagonal entries of the matrix above is primary to the maximal ideal in the corresponding polynomial ring $K[x_{21}, x_{24}, x_{31}, x_{32}, x_{34}, x_{42}, x_{43}]$. We achieve this by using homogeneous Nullstellensatz theorem. That is, we show that the zero set of the ideal is trivial.

First, observe that $x_{21}^2 = x_{24}^2$ and $x_{31}^4 = x_{21}^2 x_{24}^2$. Hence, $x_{31}^4 = x_{31}^2 x_{34}^2$ and either $x_{31} = 0$ or $x_{31}^2 = x_{34}^2$. In the first case, it is not hard to see that this implies that we get the trivial solution. In the second case we have that $0 = 2x_{32}^2 + x_{31}^2 + x_{34}^2 - 2x_{21} x_{24} = 2x_{32}^2 - 4x_{31}^2$. Since the characteristic of the field is not 2, we obtain that $x_{32} = 2x_{31}$. Then $x_{42} x_{43} = 4x_{31}^2$ and we get that $0 = 2x_{42} x_{43} - x_{21} x_{24} - x_{34}^2 = 8x_{31}^2$, which implies that $x_{31} = 0$ and again we have the trivial solution only.

Now we are ready to prove the general statement of the theorem. From now assume that $k \geq 3$ and for all pairs of matrices of size smaller than $2k$, the statement of the theorem holds. Factor $R$ by the ideal generated by the ring elements from the statement. Then we have the following matrices

$$X = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{k,1} & 0 & \ldots & 0 & x_{k,2k} \\
x_{k+1,1} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}$$
and

\[
Y = \begin{pmatrix}
0 & 0 & \ldots & 0 & x_{k,2k} & x_{k+1,1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \vdots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & x_{2k,k} & 0 & \vdots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 0 & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 0 & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & x_{k,1} & 0 & \ldots & 0 & \ldots & 0 \\
\end{pmatrix}
\]

For \(2 \leq i \leq k - 1\)

\[
\overline{c}_{ii} = \overline{c}_{0,i-1,i-1}
\]

\[
\overline{c}_{kk} = 2x_{k,1}x_{k,2k} - x_{2k,k}^2 + \overline{c}_{0,k-1,k-1}
\]

\[
\overline{c}_{k+1,k+1} = x_{k+1,k+1}^2 + \overline{c}_{0,k,k}
\]

For \(k + 2 \leq i \leq 2k - 1\),

\[
\overline{c}_{ii} = \overline{c}_{0,i-1,i-1}.
\]

For \(2 \leq i \leq k - 1\) and \(k + 2 \leq i \leq 2k - 1\),

\[
\overline{c}_{i,2k+1-i} = \overline{c}_{0,i-1,2k-i}.
\]

\[
\overline{c}_{k,k+1} = x_{k,1}x_{k+1,1} + \overline{c}_{0,k-1,k},
\]

\[
\overline{c}_{k+1,k} = x_{k+1,1}x_{k,2k} + \overline{c}_{0,k,k-1}.
\]
Now we have that
\[
\mathcal{I} = ( -x_{k,1}x_{k,2k} - x_{k+1,1}^2, -x_{k,2k}^2, -x_{k,1}^2, x_{2k,k}^2 - x_{k,1}x_{k,2k}, x_{k,1}x_{k,1+1} + c^{0}_{k-1,k},
\]
x_{k+1,1}x_{k,2k} + c^{0}_{k-1,k-1}) + (c^{0}_{ii})_{1 \leq i \leq k-2, k+1 \leq i \leq 2k-2} + (c^{0}_{i,2k+1-i})_{1 \leq i \leq k-2, k+1 \leq i \leq 2k-2}.
\]

Then \( \text{Rad}(\mathcal{I}) = (x_{k,2k}, x_{k,1}, x_{2k,k}, x_{k+1,1}) + \text{Rad}(c^{0}_{ii}, c^{0}_{i,2k+1-i})_{1 \leq i \leq 2k-2} \). By induction hypothesis, \( \text{Rad}(c^{0}_{ii}, c^{0}_{i,2k+1-i})_{1 \leq i \leq 2k-2} \) is the homogeneous maximal ideal in \( R_0 = K[X_0, Y_0] \). Therefore, we have that \( \text{Rad}(\mathcal{I}) \) is the homogeneous maximal ideal in \( R = K[X, Y] \).

\[ \square \]

**Theorem 2.3.** Let \( n = 2k \) be an even integer with \( k \geq 2 \). Let \( K \) be a field of characteristic 2. Let
\[
\Omega_2 = \{(i, j) \mid 1 \leq i \leq k - 1, 1 \leq j \leq 2k\} \cup
\{(k, j) \mid k - 1 \leq j \leq k + 1\} \cup
\{(k + 1, k + 1), (k + 2, k + 2)\} \cup
\{(k + 1, j) \mid k + 3 \leq j \leq 2k\} \cup
\{(k + 2, j) \mid 1 \leq j \leq k - 2\} \cup
\{(k + 2, j) \mid k + 2 \leq j \leq 2k\} \cup
\{(i, j) \mid k + 3 \leq i \leq 2k, 1 \leq j \leq k - 1\} \cup
\{(i, j) \mid k + 3 \leq i \leq 2k, k + 1 \leq j \leq 2k\}
\]

Then
\[
x_{ij}, y_{ji} \text{ with } (i, j) \in \Omega_2,
x_{k,s} - y_{2k+1-s,k} \text{ with } 1 \leq s \leq k - 2 \text{ and } k + 3 \leq s \leq 2k,
x_{k+1,s} - y_{s,k+1} \text{ with } 1 \leq s \leq k - 2,
x_{s,k} - y_{k,s} \text{ with } k + 3 \leq s \leq 2k,
x_{k,k+2} - y_{k+2,k}, x_{k+1,k-1} - y_{k+2,k+1}, x_{k+1,k} - y_{k,k+1}
\]
x_{k+1,k+2} - y_{k-1,k+1}, x_{k+2,k-1} - y_{k-1,k+2}, x_{k+2,k} - y_{k+1,k+2}, x_{k+2,k+1} - y_{k,k+2}
is a homogeneous system of parameters for \( R/I \).
Proof. We use the same approach as in the previous two theorems.
First, we show that the theorem is true for \( k = 2 \).
After factoring out \( R \) by the elements of the sequence we obtain the following matrices

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{24} \\
x_{31} & x_{32} & 0 & x_{34} \\
x_{41} & x_{42} & x_{43} & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & x_{34} & x_{41} \\
0 & 0 & x_{32} & x_{43} \\
0 & 0 & 0 & x_{42} \\
0 & x_{24} & x_{31} & 0
\end{bmatrix}
\]

and the commutator matrix becomes

\[
\begin{bmatrix}
x_{31}x_{34} + x_{41}^2 \\
x^2_{24} + x^2_{32} + x_{42}x_{43} \\
x_{24}x_{34} + x_{42}^2 + x_{43} \\
x_{31}^2 \\
x^2_{32} + x_{42}x_{43} \\
x_{41}^2 + x^2_{24} + x_{31}x_{34}
\end{bmatrix}
\]

Similarly to the above proofs, it is not difficult to show that \( \text{Rad}(I) \) is equal to the homogeneous maximal ideal in the quotient ring of \( R \) modulo the ideal generated by the sequence elements.

The induction step follows exactly the same lines as the induction step in Theorem 2.2.

Theorem 2.4. The ideal \( I \) is generated by a regular sequence, that is, \( R/I \) is a complete intersection of dimension \( 2n^2 - 2n + 2 \), when \( n \) is odd, and \( 2n^2 - 2n + 1 \), when \( n \) is even.

Proof. The ideal \( I \) is generated by \( 2n - 2 \) and \( 2n - 1 \) elements when \( n \) is odd and even, respectively. In the above three theorems we found a system of parameters on \( R/I \) of length \( 2n^2 - 2n + 2 \) and \( 2n^2 - 2n + 1 \), respectively. Therefore, the height of \( I \) is equal to the number of its generators. Thus the result.

Assumption. For the rest of the paper, when we say ”part of a system of parameters for \( R/I \)” we mean part of a system of parameters from one of the Theorems 2.1, 2.2, 2.3 such that those ring elements from the system which identify/associate \( x \)'s and \( y \)'s are omitted.
3 F-purity

In this section we prove that the algebraic set of pairs of matrices defined by the vanishing of the main diagonal and the main anti-diagonal of their commutator matrix is $F$-pure for matrices of all sizes and in all positive prime characteristics. We do so by using the fact that the algebraic set is a complete intersection and that $F$-purity deforms for Gorenstein rings, [2], Theorem 3.4. That is, we prove the result by showing that the ring $R/I$ is $F$-pure once we factor it by a regular sequence.

Similarly to the above section, we split our proof into two cases based on the parity of $n$ and proceed by induction on the size of the matrices. Therefore, we first state several results for small values of $n$.

**Proposition 3.1.** Let $n = 3$ and $K$ be a field of positive prime characteristic $p$. Then $R/I$ is an $F$-pure ring.

**Proof.** By factoring out part of the system of parameters from Theorem 2.1 we obtain the following matrices.

$$X = \begin{bmatrix} 0 & 0 & 0 \\ x_{21} & 0 & x_{23} \\ x_{31} & x_{32} & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & y_{12} & y_{13} \\ 0 & 0 & y_{23} \\ 0 & y_{32} & 0 \end{bmatrix}$$

and the commutator matrix becomes

$$XY - YX = \begin{bmatrix} -x_{21}y_{12} - x_{31}y_{13} & * & -x_{23}y_{12} \\ * & x_{21}y_{12} + x_{23}y_{32} - x_{32}y_{23} & * \\ -x_{21}y_{32} & * & x_{31}y_{13} + x_{32}y_{23} - x_{23}y_{32} \end{bmatrix}.$$ 

Therefore, the image of the ideal $\mathcal{I} = (x_{21}y_{32}, x_{23}y_{12}, x_{21}y_{12} + x_{31}y_{13}, x_{31}y_{13} + x_{32}y_{23} - x_{23}y_{32})$. Then since $\mathcal{I}$ is a complete intersection ideal, we have that

$$\mathcal{I}[p] : \mathcal{I} = (x_{21}y_{32}, x_{23}y_{12}, x_{21}y_{12} + x_{31}y_{13}, x_{31}y_{13} + x_{32}y_{23} - x_{23}y_{32}) ^{p-1} + \mathcal{I}[p] =$$

$$\begin{bmatrix} x_{21}^{p-1}y_{12}^{p-1}y_{32}^{p-1}(x_{21}y_{12} + x_{31}y_{13})^{p-1}(x_{31}y_{13} + x_{32}y_{23} - x_{23}y_{32})^{p-1} + \mathcal{I}[p] \end{bmatrix}$$

since

$$x_{21}^{p-1}y_{12}^{p-1}y_{32}^{p-1}(x_{21}y_{12} + x_{31}y_{13})^{p-1}(x_{31}y_{13} + x_{32}y_{23} - x_{23}y_{32})^{p-1}$$
The monomial term
\[(x_{21}x_{23}y_{12}y_{32}x_{31}y_{13}x_{32}y_{23})^{p-1}\]
has a monomial term
\[(x_{21}x_{23}y_{12}y_{32}x_{31}y_{13}x_{32}y_{23})^{p-1}\]
with coefficient 1. This is true since there is a unique way to obtain this term. Thus by Fedder’s criterion [2] (Theorem 1.12), \(R/\mathcal{I}\) is \(F\)-pure and so is \(R/\mathcal{I}\).

\[\square\]

**Theorem 3.2.** Let \(n = 2k + 1\) be an odd integer with \(k \geq 1\) and let \(K\) be a field of positive prime characteristic \(p\). Then \(R/\mathcal{I}\) is an \(F\)-pure ring.

**Proof.** We prove by induction on \(k\). Base of the induction is proved in the previous proposition.

Now let us assume that for all odd integers \(n \leq 2k - 1\), after we factor out by part of a system of parameters from Theorem 2.1 we have that \(R/\mathcal{I}\) is \(F\)-pure, that is, a deformation of \(R/\mathcal{I}\) is \(F\)-pure.

Factor out \(R\) by part of a system of parameters from Theorem 2.1. Then we obtain the following matrices

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & \vdots & & \vdots & \vdots \\
x_{k+1,1} & 0 & \ldots & x_{k+1,2k+1} & 0 \\
x_{k+2,1} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & x_{2k+1,k+1} & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & 0 & \ldots & y_{1,k+1} & y_{1,k+2} & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \vdots & \vdots \\
0 & \vdots & \ddots & 0 & 0 & \vdots & \vdots \\
0 & 0 & \ldots & y_{k+1,2k+1} & 0 & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \vdots & \vdots \\
\end{bmatrix}
\]
Therefore, the commutator matrix is
\[ XY - YX = \begin{vmatrix}
-x_{k+1,1}y_{1,k+1} - x_{k+2,1}y_{1,k+2} & \cdots & -x_{k+1,2k+1}y_{1,k+1} \\
\vdots & \overline{C}_0 + Z & \vdots \\
-x_{k+1,1}y_{2k+1,k+1} & \cdots & x_{2k+1,k+1}y_{k+1,2k+1} - x_{k+1,2k+1}y_{2k+1,k+1}
\end{vmatrix} \]
where the entries of the matrix \( Z \) are in the ideal
\[ L = (x_{k+1,2k+1}, y_{1,k+1}y_{2k+1,k+1}, x_{k+1,1}, y_{2k+1,k+1}, y_{1,k+2}, y_{1,1}, y_{1,2}, y_{1,2k+1}). \]

Now the generators of \( \overline{I} \) are the entries of the main diagonal and anti-diagonal of \( \overline{C}_0 + Z \) together with the entries at the corners of the above commutator matrix. Denote by \( \Pi \) the product of the \((2k - 2)\) entries of the main diagonal and \( 2k - 2 \) entries of the anti-diagonal of \( \overline{C}_0 + Z \). Then
\[ \overline{I}^{[p]} : \overline{I} = (x_{k+1,2k+1}y_{1,k+1}x_{k+1,1}y_{2k+1,k+1} - x_{k+1,2k+1}y_{2k+1,k+1})^{p-1} \cdot \\
((x_{2k+1,k+1}y_{k+1,2k+1} - x_{k+1,2k+1}y_{2k+1,k+1})\Pi)^{p-1} + \overline{I}^{[p]} \cdot \]

By induction hypothesis, if we factor out by the ideal \( \mathcal{L} \), \((\Pi)^{p-1}\) has a monomial term in the entries of \( \overline{X}_0 \), \( \overline{Y}_0 \) with non-zero coefficient (equal to 1) modulo \( p \) so that every indeterminate that appears in it has degree \( p - 1 \). Therefore, \((\Pi)^{p-1}\) also has such a term in the entries of \( \overline{X}_0 \) and \( \overline{Y}_0 \). Denote it by \( \mu \).

We claim that
\[ \mu(x_{k+1,1}x_{k+1,2k+1}x_{k+2,1}x_{2k+1,k+1}y_{1,k+1}y_{1,k+2}y_{k+1,2k+1}y_{2k+1,k+1})^{p-1} \]
is a nonzero monomial term of a generator \( \overline{I}^{[p]} : \overline{I} \). Clearly, such a term exists. It has a non-zero coefficient since it can be obtained in a unique way from
\[ (x_{k+1,2k+1}y_{1,k+1}x_{k+1,1}y_{2k+1,k+1}(x_{k+1,1}y_{1,k+1} - x_{k+2,1}y_{1,k+2})(x_{2k+1,k+1}y_{k+1,2k+1} - x_{k+1,2k+1}y_{2k+1,k+1})\Pi)^{p-1}. \]
Therefore, \( R/\overline{I} \) is \( F \)-pure by Fedder’s criterion and hence so is \( R/\overline{I} \).

Next we focus on the case of matrices of even size.
Lemma 3.3. Let $p > 2$ be any prime number. Then

$$\sum_{b=0}^{\frac{p-1}{2}} (-1)^b \sum_{a=b}^{p-1-b} \binom{a+b}{a} \binom{a}{b} \equiv_{mod \ p} (-1)^{\frac{p+1}{2}}.$$ 

Proof. Let $A_b = \sum_{a=b}^{p-1-b} \binom{a+b}{a} \binom{a}{b}$. We prove the lemma by showing that $A_b \equiv_{mod \ p} 0$ for all $0 \leq b \leq (p-3)/2$ and $A_{(p-1)/2} \equiv_{mod \ p} (-1)^{(p-1)/2}$.

$$A_b = \sum_{a=b}^{p-1-b} \frac{(a+b)!}{b!(a-b)!} = \binom{2b}{b} \left( \sum_{a=b+1}^{p-1-b} \frac{(2b+1)(2b+2)\ldots(a+b)}{(a-b)!} + 1 \right) = \binom{2b}{b} \sum_{c=0}^{p-1-2b} \binom{2b+c}{2b} = \binom{2b}{b} \sum_{d=0}^{p-1-2b} \binom{p-1-d}{2b}.$$ 

Claim 3.1. For any prime number $p$ we have that

$$\binom{p-i}{k} \equiv_{mod \ p} (-1)^{i-1} \binom{p-1-k}{i-1} \binom{p-1}{k},$$

for all $1 \leq i \leq p-1$ and all $0 \leq k \leq p-i$.

We prove the claim by induction on $i$. If $i = 1$, the claim is clear. We have that

$$\binom{p-i-1}{k} = \binom{p-i}{k} \frac{p-i-k}{p-i}$$

by induction hypothesis modulo $p$ this equals to

$$(-1)^{i-1} \binom{p-1-k}{i-1} \binom{p-1}{k-1} \binom{p-1-k}{p-1} \frac{p-i-k}{p-i} = (-1)^i \binom{p-1-k}{i} \binom{p-1}{k}$$

Now we are ready to finish the proof of the lemma. Let $0 \leq b < (p-3)/2$

$$A_b = \binom{2b}{b} \sum_{d=0}^{p-1-2b} \binom{p-1-d}{2b} \equiv_{mod \ p}$$
Using Newton’s binomial theorem we obtain

\[-\binom{2b}{b} \binom{p-1}{2b} \sum_{d=0}^{p-1-2b} (-1)^{d+1} \binom{p-1-2b}{d} = 0.\]

And

\[A_{(p-1)/2} = \binom{p-1}{(p-1)/2} \equiv_{\text{mod } p} (-1)^{(p-1)/2}.\]

**Proposition 3.4.** Let \( n = 4 \) and let \( K \) be a field of positive prime characteristic \( p \neq 2 \). Then \( R/I \) is \( F \)-pure.

**Proof.** After factoring \( R \) by part of a system of parameters from Theorem 2.2 we have the following matrices

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
x_{21} & 0 & 0 & x_{24} \\
x_{31} & x_{32} & 0 & x_{34} \\
0 & x_{42} & x_{43} & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & y_{12} & y_{13} & 0 \\
0 & 0 & y_{23} & y_{24} \\
0 & 0 & 0 & y_{34} \\
0 & y_{42} & y_{43} & 0 \\
\end{bmatrix}
\]

Therefore, by computing the commutator matrix, we have that the product of the generators of \( I \) is

\[\omega = (x_{21}y_{12} + x_{24}y_{42} - x_{32}y_{42} - x_{42}y_{24}) \]
\[(x_{21}y_{13} + x_{24}y_{43} - x_{34}y_{24})(x_{31}y_{12} + x_{34}y_{42} - x_{42}y_{34}) \]
\[(x_{31}y_{13} + x_{32}y_{23} + x_{34}y_{43} - x_{43}y_{34})(x_{21}y_{12} + x_{31}y_{13}) \]
\[(x_{24}y_{12} + x_{34}y_{13})(x_{21}y_{42} + x_{31}y_{43}).\]

and since \( I \) is a complete intersection,

\[\mathcal{I}^{[p]} : \mathcal{I} = (\omega)^{p-1} + \mathcal{I}^{[p]}.\]

To prove \( F \)-purity of \( R/I \) is sufficient to show that \( \omega^{p-1} \) has a non-zero monomial term which does not lie in \( \mathfrak{m}^{[p]} \), that is, its indeterminates have degree at most \( p - 1 \).
We claim that \( \omega^{p-1} \) has a monomial term which is the product of all the indeterminates in \( X \) and \( Y \) raised to the power \( p-1 \) with non-zero coefficient modulo \( p \). Using Newton’s multinomial theorem, we obtain that each term of \( \omega^{p-1} \) has the form

\[
\binom{p-1}{\alpha_1 \beta_1 \gamma_1 \delta_1} (x_{21}y_{12})^{\alpha_1} (x_{24}y_{42})^{\beta_1} (-x_{32}y_{42})^{\gamma_1} (-x_{42}y_{24})^{\delta_1}
\]

\[
\binom{p-1}{\alpha_2 \beta_2 \gamma_2} (x_{21}y_{13})^{\alpha_2} (x_{24}y_{43})^{\beta_2} (-x_{34}y_{24})^{\gamma_2}
\]

\[
\binom{p-1}{\alpha_3 \beta_3 \gamma_3} (x_{31}y_{12})^{\alpha_3} (x_{34}y_{42})^{\beta_3} (-x_{42}y_{34})^{\gamma_3}
\]

\[
\binom{p-1}{\alpha_4 \beta_4 \gamma_4 \delta_4} (x_{31}y_{13})^{\alpha_4} (x_{32}y_{23})^{\beta_4} (x_{34}y_{43})^{\gamma_4} (-x_{43}y_{34})^{\delta_4}
\]

\[
\binom{p-1}{\alpha_5 \beta_5} (x_{21}y_{12})^{\alpha_5} (x_{31}y_{13})^{\beta_5}
\]

\[
\binom{p-1}{\alpha_6 \beta_6} (x_{24}y_{12})^{\alpha_6} (x_{34}y_{13})^{\beta_6}
\]

\[
\binom{p-1}{\alpha_7 \beta_7} (x_{21}y_{42})^{\alpha_7} (x_{31}y_{43})^{\beta_7}
\]

Denote by \( A_{ij} \) and \( B_{ij} \) the degrees of \( x_{ij} \) and \( y_{ij} \), respectively. Then

\[
A_{21} = \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7 \quad B_{12} = \alpha_1 + \alpha_3 + \alpha_5 + \alpha_6
\]

\[
A_{24} = \alpha_6 + \beta_1 + \beta_6 \quad B_{13} = \alpha_2 + \alpha_4 + \beta_5 + \beta_6
\]

\[
A_{31} = \alpha_3 + \alpha_4 + \beta_5 + \beta_7 \quad B_{23} = \beta_4 + \gamma_1
\]

\[
A_{32} = \beta_4 + \gamma_1 \quad B_{24} = \gamma_2 + \delta_1
\]

\[
A_{34} = \beta_3 + \beta_5 + \beta_7 \quad B_{43} = \gamma_4
\]

\[
A_{42} = \beta_6 + \gamma_4 
\]

\[
A_{43} = \gamma_2 + \delta_4 
\]

In addition, let

\[
C_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 
\]

\[
C_2 = \alpha_2 + \beta_2 + \gamma_2 
\]

\[
C_3 = \alpha_3 + \beta_3 + \gamma_3 
\]
\[ C_4 = \alpha_4 + \beta_4 + \gamma_4 + \delta_4 \]
\[ C_5 = \alpha_5 + \beta_5 \]
\[ C_6 = \alpha_6 + \beta_6 \]
\[ C_7 = \alpha_7 + \beta_7 \]

Therefore, we consider a system of linear equations \( A_{ij} = p - 1 \), \( B_{ij} = p - 1 \), \( C_k = p - 1 \) for all the values of \( i, j, k \). We look for all solutions in the set of non-negative integers. First, observe that the system cannot have a unique solution as the sum of the equations defined by \( A_{ij} \) is equal to the sum of the equations defined by \( B_{ij} \). Next,

\[
\begin{align*}
A_{21} + A_{31} &= \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7 + \alpha_3 + \alpha_4 + \beta_5 + \beta_7 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + C_5 + C_7 \\
\end{align*}
\]

Therefore, \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \). Since \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are non-negative integers, we must have that each term is 0. That is, \( \alpha_i = 0 \) for all \( 1 \leq i \leq 4 \). Then using standard techniques from linear algebra, we obtain the following solution to our system of linear equations.

\[
\begin{align*}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &= (0, \beta_1, p - 1 - (\beta_1 + \beta_2), \beta_2) \\
(\alpha_2, \beta_2, \gamma_2) &= (0, \beta_2, p - 1 - \beta_2) \\
(\alpha_3, \beta_3, \gamma_3) &= (0, \beta_2, p - 1 - \beta_2) \\
(\alpha_4, \beta_4, \gamma_4, \delta_4) &= (0, \beta_1, p - 1 - (\beta_1 + 2\beta_2), \beta_2) \\
(\alpha_5, \beta_5) &= (\beta_1 + \beta_2, p - 1 - (\beta_1 + \beta_2)) \\
(\alpha_6, \beta_6) &= (p - 1 - (\beta_1 + \beta_2), \beta_1 + \beta_2) \\
(\alpha_7, \beta_7) &= (p - 1 - (\beta_1 + \beta_2), \beta_1 + \beta_2) \\
\end{align*}
\]

Therefore, the coefficient of our monomial term is

\[
\sum_{\beta_2=0}^{p-1-2\beta_2} \sum_{\beta_1=0}^{p-1-\beta_2} (-1)^{\beta_1+\beta_2} \left( \begin{array}{cc}
\beta_1 + \beta_2 & p - 1 \\
\beta_1 & p - 1 - (\beta_1 + \beta_2)
\end{array} \right) \left( \begin{array}{c}
p - 1 \\
\beta_2
\end{array} \right)^2 \left( \begin{array}{c}
p - 1 \\
\beta_1 + \beta_2
\end{array} \right)^3 \\
\left( \begin{array}{cc}
\beta_1 + \beta_2 & p - 1 - \beta_1 - 2\beta_2 \\
\beta_1 + \beta_2 & p - 1 - \beta_1 - 2\beta_2
\end{array} \right) \equiv_{\text{mod } p}
\]
\[
\sum_{\beta_2=0}^{p-1} \sum_{\beta_1=0}^{p-1-2\beta_2} (-1)^{\beta_1+\beta_2} \binom{p-1}{\beta_1+\beta_2} \binom{p-1}{\beta_1 \beta_2} \binom{p-1}{p-1-\beta_1-2\beta_2} \equiv_{\text{mod } p} 0
\]

\[
\sum_{\beta_2=0}^{p-1} \sum_{\beta_1=0}^{p-1-2\beta_2} (-1)^{\beta_1+\beta_2} \binom{p-1}{\beta_1+\beta_2}^2 \binom{p-1}{\beta_1 \beta_2} \binom{p-1}{p-1-\beta_1-2\beta_2} \equiv_{\text{mod } p} 0
\]

\[
\sum_{\beta_2=0}^{p-1} \sum_{\beta_1=0}^{p-1-2\beta_2} (-1)^{\beta_1+\beta_2} \binom{p-1}{\beta_1+\beta_2} \binom{p-1}{\beta_1+2\beta_2} \binom{\beta_1 \beta_2}{\beta_1 \beta_2} \equiv_{\text{mod } p} 0
\]

Make a substitution \( \beta_1 + \beta_2 = a, \beta_2 = b \), then we have

\[
\sum_{b=0}^{p-1} (-1)^b \sum_{a=b}^{p-1-b} \binom{a+b}{a} \binom{a}{b}
\]

which is equal to \((-1)^\frac{p-1}{2}\) modulo \( p \) due to Lemma 3.3.

Thus, we have that the coefficient of our monomial term is equal to \( \pm 1 \) modulo \( p \) and \( R/I \) is \( F \)-pure by Fedder’s criterion.

\[\square\]

**Proposition 3.5.** Let \( n = 4 \) and let \( K \) be a field of characteristic 2. Then \( R/I \) is \( F \)-pure.

**Proof.** First, we factor \( R \) by part of a system of parameters from Theorem 2.3. Then

\[
I = (x_{31}y_{13} + x_{41}y_{14}, x_{32}y_{23} + x_{42}y_{24} + x_{24}y_{42}, x_{31}y_{13} + x_{32}y_{23} + x_{43}y_{34} + x_{34}y_{43}, x_{34}y_{13} + x_{43}y_{24} + x_{42}y_{34} + x_{34}y_{42}, x_{31}y_{13}).
\]
Since $\mathcal{I}$ is a complete intersection, we have $\mathcal{I}^{[2]} : \mathcal{I} = (\omega) + \mathcal{I}^{[2]}$, where $\omega$ is the product of the generators of $\mathcal{I}$. Using Macaulay2 \cite{3}, we compute this product and find that there is exactly one monomial term such that every indeterminates in its support has degree 1:

$$x_{24}x_{31}x_{32}x_{41}x_{42}x_{43}y_{13}y_{14}y_{23}y_{24}y_{34}y_{42}y_{43}.$$ 

Thus, by Fedder’s criterion, $R/\mathcal{I}$ is $F$-pure.

\[ \square \]

**Theorem 3.6.** Let $n = 2k$ be an even integer with $k \geq 2$ and let $K$ be a field of positive prime characteristic $p$. Then $R/\mathcal{I}$ is $F$-pure.

**Proof.** We prove by induction on $k$. Base of the induction was proved in the previous two propositions. We assume that for all $n \leq 2k - 2$ after we factor by part of a system of parameters from Theorem 2.2 and Theorem 2.3 we have that $R/\mathcal{I}$ is $F$-pure and if $\omega$ is the product of the generators of $\mathcal{I}$, then $\omega^{p-1}$ has a monomial term $\mu \notin m^{[p]}$. Then we have that

$$X = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ x_{k1} & x_{k1}^0 & \ldots & x_{k1}^{2k} & 0 \\ x_{k+1,1} & x_{k+1,1}^0 & \ldots & x_{k+1,1}^{2k} & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & 0 & \ldots & 0 & y_{1k} & y_{1,k+1} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & y_{1k}^0 & y_{1,k+1}^0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & y_{k,2k} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & y_{k,2k} & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & y_{k,2k} & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & y_{k,2k} & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{k,2k} & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{k,2k} & \ldots & 0 \end{bmatrix}.$$
Therefore, the commutator matrix is

\[
\begin{pmatrix}
-x_{k1}y_{1k} - x_{k+1,1}y_{1,k+1} & \cdots & -x_{k,2k}y_{1k} \\
\vdots & \ddots & \vdots \\
-x_{k1}y_{2k,k} & \cdots & x_{2k,k}y_{k,2k} - x_{k,2k}y_{2k,k}
\end{pmatrix}
\]

\[\mathcal{C}_0 + Z\]

where the entries of the matrix \(Z\) are in the ideal

\[\mathcal{L} = (x_{k1}, x_{k+1,1}, x_{k,2k}, x_{2k,k}, y_{1k}, y_{1,k+1}, y_{k,2k}, y_{2k,k}).\]

By induction hypothesis, the \((p-1)\)st power of the product of any \(2k-3\) entries on the main diagonal and all the entries of the main antidiagonal of \(\mathcal{C}_0 + Z\) has a non-zero monomial term \(\mu\) in the entries of \(X_0, Y_0\) which is not in \(m^p\), (it exists after we factor out by the ideal \(\mathcal{L}\)). Therefore, the \((p-1)\)st power of the generators of \(I\) has a non-zero monomial term

\[\mu(x_{k1}x_{k+1,1}x_{k,2k}x_{2k,k}y_{1k}y_{1,k+1}y_{k,2k}y_{2k,k})^{p-1} \notin m^p.\]

\[\square\]

### 4 Zero diagonal commutator

In this section we study the algebraic set defined by the vanishing of the diagonal of the commutator matrix \(C\). Recall that \(I\) is the ideal of \(R\) generated by the entries of the main diagonal of \(C\). Since the trace of \(C\) is 0, we can choose the generators of \(I\) to be the first \(n-1\) entries of the main diagonal counting from the upper left corner.

In his thesis \([7]\), H-W. Young has proved that \(I\) is a complete intersection. We recover his result from our proof that \(I\) is generated by a regular sequence.

**Theorem 4.1.** \([7]\) (Theorem 5.3.1) \(R/I\) is a complete intersection ring of dimension \(2n^2 - n + 1\).

Moreover, Young showed the following properties of \(I\).
Theorem 4.2. 1. [7] (Theorem 5.3.2) $R/I$ is reduced when the characteristic of the field $K$ is 0.

2. [7] (Theorem 5.3.3) $R/I$ is irreducible when $n = 2$ and $n = 3$ in all characteristics. Therefore, $R/I$ is a domain in characteristic 0 when $n = 2$ and $n = 3$.

Next we find a homogeneous system of parameters for $R/I$ and show that $R/I$ is $F$-regular when $K$ is a field of positive prime characteristic $p$.

4.1 A system of parameters

We already have one system of parameters on $R/I$ which is obtained from the one we got for $R/I$. However, the one we offer here is simpler and will allow us to prove $F$-regularity of $R/I$.

Theorem 4.3. A set of elements

$$x_{11},$$

$$x_{ij}, \text{ where } 2 \leq i \leq n, 1 \leq j \leq n,$$

$$y_{ij}, \text{ where } 1 \leq i \leq n, 2 \leq j \leq n,$$

$$x_{1j} - y_{j1}, \text{ where } 1 \leq j \leq n,$$

is a homogeneous system of parameters and a regular sequence on $R/I$.

Proof. The dimension of $R/I$ is $2n^2 - n + 1$ and the number of elements in the above set is also $2n^2 - n + 1$. It is necessary and sufficient to prove that if we factor out by the ideal they generate together with $I$, the resulting ring has Krull dimension 0. We obtain that this factor ring is isomorphic to the following ring

$$K[x_{12}, x_{13}, \ldots, x_{1n}] / (x_{12}^2, x_{13}^2, \ldots, x_{1n}^2),$$

which is clearly 0-dimensional. \qed
4.2 F-regularity

Theorem 4.4. Let $K$ be a field of positive prime characteristic $p$. Then $R/I$ is $F$-regular.

Proof. In this proof we use the fact that $F$-regularity deforms for Gorenstein rings \cite{4} (Corollary 4.7). We factor $R/I$ by part of a system of parameters from above: we annihilate $x_{11}$, $y_{11}$, the entries of $X$ below the first row and the entries of $Y$ to the right of the first column. Denote the factor ring $\overline{R}$.

Let

$$Z_i = \begin{bmatrix} x_{1,i} & x_{1,i} \\ y_{1,i} & y_{1,i} \end{bmatrix}$$

where $2 \leq i \leq n$.

Then we have that

$$\overline{R} \simeq \frac{K[\{Z_i^i\}_{i=2}^n]}{(\det Z_i)_{i=2}^n} \simeq \bigotimes_{i=2}^n \frac{K[Z_i]}{(\det Z_i)}.$$

where $K[Z_i]/(\det Z_i)$ is a determinantal ring, known to be $F$-regular, \cite{5}, Theorem 7.14. Using Theorem 7.45 in \cite{4}, we obtain that $\overline{R}$ is $F$-regular, and hence so is $R/I$. \hfill \qed

Corollary 4.5. $R/I$ is a domain and hence $I$ is a prime ideal when $K$ is a field of positive prime characteristic.

Next we observe the following result.

Lemma 4.6. The ring $R/I$ deforms to a Stanley-Reisner ring.

Proof. Factor $R/I$ by part of a system of parameters from Theorem 4.3. Then we have that the image of $I$ is generated by the square-free monomials $\{x_{1j}y_{j1} | 2 \leq j \leq n\}$. \hfill \qed

5 Zero anti-diagonal commutator

In this section we study the algebraic set defined by the vanishing of the anti-diagonal entries of the commutator matrix $C$. Recall that the ideal generated by these entries is denoted by $J$.

Theorem 5.1. $R/J$ is a complete intersection ring of dimension $2n^2 - n$. 23
Proof. Since \( I \) is generated by a regular sequence, then so is \( J \).

Next we find a system of parameters for \( R/J \). We already have one obtained from a system of parameters for \( R/I \). However, the one we give here is simpler.

**Theorem 5.2.** A set of elements

\[
\begin{align*}
x_{ij}, & \text{ where } 2 \leq i \leq n, 1 \leq j \leq n, \\
y_{ij}, & \text{ where } 1 \leq i \leq n, 2 \leq j \leq n, \\
x_{1j} - y_{n-j+1,j}, & \text{ where } 1 \leq j \leq n
\end{align*}
\]

is a homogeneous system of parameters and a regular sequence on \( R/J \).

**Proof.** If we factor out by the ideal generated by the set elements, we obtain the following matrices

\[
\begin{align*}
\mathbf{X} &= \begin{bmatrix}
x_{11} & x_{12} & \ldots & x_{1n} \\
0 & 0 & \ldots & 0 \\
\vdots & & & \\
0 & 0 & \ldots & 0
\end{bmatrix} \quad \text{and} \quad \mathbf{Y} &= \begin{bmatrix}
x_{1n} & 0 & \ldots & 0 \\
x_{1,n-1} & 0 & \ldots & 0 \\
\vdots & & & \\
x_{11} & 0 & \ldots & 0
\end{bmatrix}
\end{align*}
\]

Then the generators for the image of \( J \) are \( x_{11}^2, x_{12}^2, \ldots, x_{1n}^2 \). Thus, the factor ring of \( R/J \) by the ideal generated by the set elements has Krull dimension 0. Hence the result.

**Theorem 5.3.** Let \( K \) be a field of positive prime characteristic \( p \). Then \( R/J \) is \( F \)-pure.

**Proof.** We have that \( I = I + J \) is generated by a regular sequence, therefore, \( R/I \) is a deformation of \( R/J \). Since \( R/I \) is \( F \)-pure, then so is \( R/J \).

**Corollary 5.4.** \( R/J \) is a reduced ring, that is, \( J \) is a radical ideal, when \( K \) is a field of positive prime characteristic.

Here we also observe that the ring \( R/J \) has a deformation to a Staynley-Reisner ring.

**Lemma 5.5.** The ring \( R/J \) deforms to a Staynley-Reisner ring.

**Proof.** Factor \( R/J \) by part of a system of parameters from Theorem 5.2. Then we have that the image of \( J \) is generated by the square-free monomials \( \{x_{1j}y_{j1} | 1 \leq j \leq n \} \).
6 Conjectures

In this section we state conjectures that naturally arise in the context of the algebraic sets defined by the ideals $I$, $I$ and $J$.

**Conjecture 6.1.** Let $K$ be a field of positive prime characteristic. Then $R/I$ and $R/J$ are $F$-regular rings. In particular, they are integral domains.

**Conjecture 6.2.** The ideals $I$, $I$ and $J$ are prime ideals in all characteristics.

7 Appendix

The aim of this section is to display how matrices $X$ and $Y$ look like after the ring $R$ is factored out by a system of parameters from Theorems 2.1, 2.2, 2.3. The base cases $n = 3$ and $n = 4$ are respectively highlighted in the center of each of the matrices.

When $n = 7$, we have

$$
X = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{41} & x_{42} & x_{43} & 0 & x_{45} & x_{46} & x_{47} \\
x_{51} & x_{52} & x_{53} & x_{54} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{64} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{74} & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
Y = \begin{bmatrix}
0 & 0 & 0 & x_{47} & x_{51} & 0 & 0 \\
0 & 0 & 0 & x_{46} & x_{52} & 0 & 0 \\
0 & 0 & 0 & x_{45} & x_{53} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{54} & x_{64} & x_{74} \\
0 & 0 & 0 & x_{43} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{42} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{41} & 0 & 0 & 0
\end{bmatrix}
$$

When $n = 8$ and the characteristic of $K$ is not 2, we have
When $n = 8$ and the characteristic of $K$ is 2, we have

\[ X = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{41} & x_{42} & x_{43} & 0 & 0 & x_{46} & x_{47} & x_{48} \\
x_{51} & x_{52} & x_{53} & x_{54} & 0 & x_{56} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{64} & x_{65} & 0 & 0 \\
0 & 0 & 0 & x_{74} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{84} & 0 & 0 & 0 & 0
\end{bmatrix} \]

and

\[ Y = \begin{bmatrix}
0 & 0 & 0 & 0 & x_{48} & x_{51} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{47} & x_{52} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{46} & x_{53} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{54} & x_{65} & x_{74} & x_{84} \\
0 & 0 & 0 & 0 & 0 & x_{64} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{43} & x_{56} & 0 & 0 \\
0 & 0 & 0 & x_{42} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{41} & 0 & 0 & 0 & 0
\end{bmatrix} \]
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