Abstract

A hypersurface is said to be quasihomogeneous if in suitable coordinates with assigned weights, its equation becomes weighted homogeneous in its variables. For an irreducible quasihomogeneous plane curve, the equation necessarily becomes a two term equation of the form $aY^n + bX^m$ where $n, m$ are necessarily coprime. Zariski, in a short paper, established a criterion for an algebroid curve to be quasihomogeneous \cite{Z1} and a celebrated theorem of Lin and Zaidenberg gives a global criterion for quasihomogeneity \cite{LZ}. The Lin-Zaidenberg theorem does not have a simple proof, despite having three different proofs using function theory, topology and algebraic surface theory respectively. We give here a global version of the Zariski result. As a consequence we give a proof of a slightly weaker version of the Lin-Zaidenberg Theorem, namely that a rational curve with one place at infinity is unibranch and locally quasihomogeneous if and only if it is globally quasihomogeneous, provided the ground field is algebraically closed of characteristic zero. Our method of proof leads to some interesting questions about the change in the module of differentials when we go to the integral closure.
On Quasihomogeneous Curves

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1 Introduction and notation

Let \( k \) be an algebraically closed field of characteristic 0 and let \( A = k^{[2]} = k[X,Y] \) be a polynomial ring in two variables over \( k \). An irreducible polynomial \( f = f(X,Y) \in A \) defines an irreducible plane curve. We shall denote its coordinate ring by \( R = A/(f) \) and the canonical homomorphism \( A \rightarrow R \) shall be denoted by \( \phi_f \). In general we write \( \phi_f(X) = x, \phi_f(Y) = y \).

For any \( h = h(X,Y) \in A \) we shall denote by \( J_f(h) = J_f(h(X,Y)) \) its jacobian with \( f \), namely \( h_X f_Y - h_Y f_X \). The ideal in \( A \) generated by all such jacobians is denoted by \( J_f = (f_X, f_Y) \).

We also need the extended ideal \( Jac_f = (f, J_f) \). Note that for any \( u = g/h \in qt(A) \) the quotient field of \( A \), we can easily extend the definition to calculate \( J_f(u) \), by the usual rules of derivatives. We shall denote by \( J_f \) the ideal \( \phi_f(J_f) \subset R \).

We may often drop the reference to \( f \) if it is clear during a discussion and we may drop explicit reference to \( \phi_f \) as follows. We may write \( J_f(h(x,y)) \) or simply \( J(h(x,y)) \) to denote the image \( \phi_f(h_X(X,Y) f_Y - h_Y(X,Y) f_X) \). If we consider \( u(X,Y) = g(X,Y)/h(X,Y) \) where \( \phi_f(h(X,Y)) = h(x,y) \neq 0 \), we have a well defined element of \( k(f) \): \( \phi_f(J_f(u)) = \phi_f(J_f(g))/\phi_f(h) - \phi_f(gJ_f(h))/\phi_f(h)^2 \). We have thus defined \( J_f(u(x,y)) \) for any \( u(x,y) \in k(f) \).

We introduce the following additional notations for future use. Define \( \chi(f;Q) \) to be the number of branches of \( f \) at \( Q \), and let \( \overline{\chi}(f;Q) = \chi(f;Q) - 1 \). Let \( \overline{R} \) denote the integral closure of \( R \) in its quotient field, i.e. the function field \( k(f) \) of the curve \( f \). We define the conductor ideal \( \mathfrak{C}(\overline{R}, R) = \{ u \in R \mid u\overline{R} \subset R \} \) and may shorten the notation to \( \mathfrak{C}_f \) or simply \( \mathfrak{C} \) for convenience.

Let also \( \mathfrak{C}(f;Q) \) be the conductor ideal of \( R_{Q'} \) in its integral closure (denoted as \( \overline{R}_{Q'} \)), where \( Q' = \phi_f(Q) \). Let \( C(f;Q) \) be the length \( l(\overline{R}_{Q'}/\mathfrak{C}(f;Q)) \) as \( R \)-modules.

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We define $C(f)$ to be the sum $\sum Q C(f; Q)$ over all points $Q$ such that $f$ passes thru $Q$.

As is well known, $\mathfrak{c}_f$ is an ideal in $R$ as well as $\overline{R}$ and has the property $l(\overline{R}/\mathfrak{c}) = 2l(R/\mathfrak{c})$. Further this is clearly equal to $C(f)$.

We have the usual module $\Omega(R, k) = \Omega(R)$ of 1-differentials generated by $dx, dy$ and subject to the relation $f_x dx + f_y dy = 0$. If $h(x, y) \in k(f)$, such that $dh \neq 0$, then the differential $dh(x, y)/J_f(h(x, y))$ is easily seen to be independent of the choice of $h$ when viewed as a member of $\Omega(k(f), k)$. We will denote it by $\omega$ and call it the fundamental differential of $f$. In our zero characteristic case, the condition $dh \neq 0$ can be replaced by “$h$ is a non constant”.

Given non constant $u \in \overline{R}$, we have $du \neq 0$. For any valuation $w$ of the function field $k(f)$ over $k$, recall that the value $w(du)$ of the differential $du$ is defined as follows. Choose a parameter $t$ for the valuation $w$ which is a separating transcendence (separability being automatic in our characteristic zero). We have $du/J_f(u) = dt/J_f(t)$ and we define $w(du) = w(J_f(u)/J_f(t))$. It is then known that this is independent of the choice of the parameter $t$. In short, this can be also described as $w(du) = w(h)$ where $h$ is chosen such that $du = hdt$ for some (separating) uniformizing parameter $t$ for $w$.

## 2 Generalized Berger formula

**Lemma 2.1** Let $f \in A$ be a non constant irreducible polynomial and using the above notation, let

$$J^* = (\{J_f(u) \mid u \in \overline{R}\})\overline{R}$$

Then we have:

$$J^* = \mathfrak{c} = \mathfrak{c}_f.$$

**Proof.** Let $w$ be any valuation at finite distance (i.e. the valuation ring of $w$ contains $R$). By the Dedekind formula for the conductor and differential [AS](28.15.1) we know that $w(J_f(u)) = w(\mathfrak{c}) + w(du)$.

Since $u \in \overline{R}$ we get $w(du) \geq 0$ and hence $w(J_f(u)) \geq w(\mathfrak{c})$. Thus $w(J^*) \geq w(\mathfrak{c})$. Further, if we choose $u$ to be itself a uniformizing parameter for $w$ in $\overline{R}$, then $w(J_f(u)) = w(\mathfrak{c})$ and thus the minimum of values of elements of $J^*$ at $w$ is actually equal to $w(\mathfrak{c})$. Thus $w(J^*) = w(\mathfrak{c})$ for all valuations dominating the normal domain $\overline{R}$ and hence $J^* = \mathfrak{c}$.

**Corollary 2.2** With the same setup as in Lemma 2.1, we have: $\mathfrak{c}_\omega = \Omega(\overline{R}, k)$, where we recall that $\omega$ is the fundamental differential of $f$ as described above.
Proof. Note that $\Omega(R, k)$ consists of sums of elements $adu$ with $a, u \in \overline{R}$. We have $adu = aJ_f(u)\omega \in J^*\omega$. Thus $\Omega(R, k) \subset J^*\omega$. Similarly $J^*\omega$ consists of sums of elements $aJ_f(u)\omega$ for $a, u \in \overline{R}$. But $aJ_f(u)\omega = aJ_f(u)(du/J_f(u)) = adu \in \Omega(R, k)$. Thus $J^*\omega \subset \Omega(R, k)$.

By a similar argument we get the following:

**Lemma 2.3** With the same setup as in Lemma 2.1, we have: $\overline{J_f}\omega = \Omega(R, k)$.

**Proof.** Just imitate the argument of the above corollary.

We can now state the following:

**Proposition 2.4** We the notations as explained above, we have:

$$l(A/Jac_f) = l(R/\overline{J_f}) = l(R/\mathcal{C}) + l(\mathcal{C}/\overline{J_f}) = l(\overline{R}/R) + l(\Omega(R, k)/\Omega(R, k))$$

**Proof.** The first equality is obvious and the second is clear since $\overline{J_f}$ is contained in $\mathcal{C}$.

The multiplication by the fundamental differential $\omega$ clearly gives an injective homomorphism of $\mathcal{C}$ into $\Omega(R, k)$ and we get:

$$l(\mathcal{C}/\overline{J_f}) = l(\mathcal{C}\omega/\overline{J_f}\omega)$$

We use Corollary 2.2. and Lemma 2.3 to get our quantity equal to $l(R/\mathcal{C}) + l(\Omega(R, k)/\Omega(R, k))$.

Finally, we use the well known fact that $l(R/\mathcal{C}) = l(\overline{R}/R)$ to finish the proof.

**Remark 2.5** We now connect the above calculations with Zariski’s concept of the torsion module of differentials. Let $T(R)$ be the torsion module of (the $R$-module) $\Omega(R, k)$, i.e., the submodule consisting of the elements of $\Omega(R, k)$ which have a non vanishing annihilator. Explicitly, we see that:

$$T(R) = \{adx + bdy \mid a, b \in R, uadx + ubdy = 0 \text{ for some nonzero } u \in R\}.$$

Zariski showed that in case $R$ is the local ring of an analytically irreducible plane curve, the length of the torsion module satisfies:

$$l(T(R)) = l(R/\overline{J_f}).$$

where $f$ is the local equation. Zariski’s arguments remain valid for our global case as well and we get that this result continues to hold.

Zariski further used the Berger formula giving a local result similar to our global statement of Proposition 2.4.
Finally, Zariski noticed that the term \( l(\Omega(R, k)/\Omega(R, k)) \) depends on the structure of the equation and is at most equal to the adjacent term (the length of the integral closure over the coordinate ring or equivalently the length of the conductor ideal in the coordinate ring). He showed that it reaches this maximum value if and only if under a suitable local analytic change of coordinates the curve becomes \( y^m - x^n \) for coprime natural numbers \( m, n \).

The result depends on the analytic irreducibility, i.e. the property of having only one place at the point. The main theme of our paper is to globalize this result by showing that it continues to hold under the assumption that the global curve has only one place at infinity.

Saito has generalized the Zariski result in a different direction. His result states that a local hypersurface with an isolated singularity is (locally) quasihomogeneous if and only if its equation belongs to the jacobian ideal. \( [SA] \) Unfortunately, his function theory proof has not yet been transformed to an algebraic form.

### 3 Algebraic invariants associated with the curve \( f \)

Let the notations be as in Section 2. In the following, we shall introduce invariants associated with the curve \( f \). Let \( Q \in j\text{-}Spec(A) \) (i.e. \( Q \) is a maximal ideal in \( A \). We also describe it as a point in the plane). We recall the intersection multiplicity of \( f \) and \( g \) at \( Q \), denoted by \( \text{int}(f, g; Q) \), to be the length of the \( A_Q \)-module \( A_Q/(f,g)A_Q \). Note that this length is finite exactly when \( f, g \) don’t have a common factor passing thru \( Q \), or equivalently, \( (f,g)A_Q \) is primary for \( QA_Q \). Also, it is positive, if and only if both the curves pass thru the point. We recall the total intersection multiplicity of \( f \) with \( g \) in \( A \), denoted by \( \text{int}(f, g; A) = \sum \text{int}(f, g; Q) \) taken over all the points in the plane.

**Milnor numbers**: Let \( f \in A \) be non constant. We define

\[
\mu(f; Q) = l(A_Q/J_fA_Q) = \text{int}(fX, fY; Q).
\]

and call it the (local) Milnor number of \( f \) at \( Q \). Note that this is finite if and only if \( f \) is reduced, i.e. does not have a multiple factor thru \( Q \). We also define \( \mu(f) \) the affine Milnor number of \( f \) as the sum of the local Milnor numbers over all points \( Q \) such that \( f \) passes thru \( Q \), i.e. \( f \in Q \). Note that this is finite if and only if \( f \) has no multiple factors. We define the Milnor number of the pencil \( \{f-a\}_{a \in k} \) by \( \mu(f; A) = \sum_{a \in k} \mu(f-a) \). Clearly, \( \mu(f; A) = l(A/J_f) \) and \( \mu(f; A) = \mu(f-a; A) \) for all \( a \in k \). Note that this is finite if and only if all members of the pencil \( \{f-a\} \) are reduced.

**Tjurina numbers**: Let \( f \in A \) be irreducible (non constant). We define

\[
\nu(f; Q) = l(A_Q/Jac_fA_Q) \simeq l(R_{\phi_f(Q)}/(J_f)R_{\phi_f(Q)})
\]

and call it the local Tjurina number. We also define \( \nu(f) \) the affine Tjurina number of \( f \) as the sum over all points \( Q \) such that \( f \) passes thru \( Q \). Clearly, \( \nu(f) = l(A/Jac_fA) = l(R/J_fR) \).
The Tjurina numbers can be also defined for reduced $f$ and will still be finite.\footnote{We could also define a Tjurina number for a pencil, but so far no useful consequences of such a definition are known, hence we refrain from making such a definition. For the same reason, defect numbers below are also not defined for pencils.}

In view of Zariski’s work and our observations above, the Tjurina numbers could also be defined as lengths of the corresponding torsion modules of the module of differentials (locally as well as globally).

**Defect numbers:** Let $f \in A$ be irreducible (non constant). Since $J_f \subset Jac_f$, we define

$$\delta(f; Q) = \mu(f; Q) - \nu(f; Q) = l(J_f A_Q / Jac_f A_Q)$$

and call it the local defect number of $f$. We define $\delta(f)$ the affine defect of $f$ as the sum of local defects $\delta(f; Q)$ taken over all the points $Q$ such that $f$ passes thru $Q$.

Thus: $\mu(f) = \delta(f) + \nu(f)$, i.e. Milnor=Tjurina+defect.

**Zariski numbers:** Let $f \in A$ be non constant. We define $Z(f; Q) = l(C(f; Q) / J_f R_Q')$ where $Q' = \phi_f(Q)$. We call it the local Zariski number of $f$ at $Q$. We also define the affine Zariski number $Z(f)$ of $f$, as the sum of local Zariski numbers $Z(f; Q)$ taken over all the points $Q$ such that $f$ passes thru $Q$. Note that then $Z(f) = l(C(f) / J_f)$.

We now observe some useful relations between these numbers.

**Lemma 3.1** Suppose $f \in A$ is irreducible (non constant) and passes thru a point $Q$. Then

$$\mu(f; Q) + \chi(f; Q) = C(f; Q).$$

**Proof.** The result is well known. We indicate an outline.

First choose suitable local coordinates so that $Q$ is the maximal ideal generated by $X, Y$ and that $X$ is not tangential to $f$ at $Q$, i.e. at every valuation of $f$ centered at $Q$, the image $x = \phi_f(X)$ has minimal value in the ideal $(x, y)$. From the Dedekind formula we deduce that

$$E_1. \quad C(f; Q) = \text{int}(f, f_Y; Q) - \text{int}(f, X; Q) + \chi(f; Q)$$

Next, by considering the coordinate ring of $f_Y$ and noticing that modulo any irreducible factor of $f_Y$ we have $f_x = \frac{df}{dx}$, we deduce that

$$E_2. \quad \text{int}(f, f_Y; Q) = \text{int}(f_X, f_Y; Q) + \text{int}(X, f_Y; Q)$$

By nontangentiality of $X$, we further get that

$$E_3. \quad \text{int}(X, f_Y; Q) = \text{int}(X, f; Q) - 1$$

The result follows by combining these equations.

**Lemma 3.2** If $f \in A$ is irreducible (non constant) and passes thru a point $Q$ then we have

$$\mu(f; Q) = \nu(f; Q) + \delta(f; Q) = C(f; Q)/2 + Z(f; Q) + \delta(f; Q).$$
Proof. The first equality is by definition, while the second follows the obvious fact that
\[ \nu(f; Q) = l(R_{\phi f}(Q)/\mathfrak{c}(f; Q)) + Z(f; Q). \]
and the fact that the first term is half the length \( C(f; Q) \) of the conductor ideal in the integral closure.

Lemma 3.3 If \( f \in A \) is unibranch and passes thru \( Q \) then we have \( \mu(f; Q) = C(f; Q) \) and further
\[ Z(f; Q) + \delta(f; Q) = C(f; Q)/2. \]
In particular, the maximum possible value for the local Zariski number is \( C(f; Q)/2 \) and it is attained exactly when \( \delta(f; Q) = 0 \).

Proof. By Lemma 3.1, we get that \( \mu(f; Q) = C(f; Q) \) and now Lemma 3.2 gives the result.

Lemma 3.4 If \( f \in A \) is an irreducible (non constant) curve, then we have the following:
\[ C(f) = \mu(f) + \chi(f) = C(f)/2 + Z(f) + \delta(f) + \chi(f). \]
In particular,
\[ C(f)/2 - Z(f) = \delta(f) + \chi(f). \]

Proof. This is simply obtained by adding up the results from Lemmas 3.1 and 3.2 over all points of the curve \( f \).

Remark 3.5 Zariski’s result can now be described as a necessary and sufficient condition for maximality of \( Z(f; Q) \) or vanishing of the defect. To describe this, let us change coordinates so that \( Q = (X, Y) \) and consider \( f \in A \subset k[[X, Y]] \). We say that \( f \) is formally quasihomogeneous at \( Q \) if there is a change of variables \( k[[X, Y]] = k[[X', Y']] \) such that after giving some positive weights to \( X', Y' \) a local equation becomes a sum of equal weight monomials in \( X', Y' \). It is easy to see that by irreducibility of \( f \) and a further change of coordinates if necessary, we may assume that \( f = u((X')^m - (Y')^n) \) where \( u \) is a unit in the power series ring and \( m, n \) are (necessarily comprime) positive integers. We can now restate

Zariski Theorem \( \delta(f; Q) = \chi(f; Q) = 0 \) if and only if \( f \) locally irreducible and formally quasihomogeneous at \( Q \).

We shall be imitating Zariski’s original proof in a global setting.
4 One place curves

Let the notations be as in Section 1. The curve $f$ is said to have one place at infinity if there is only one valuation of its function field $k(f)$ over $k$ which does not contain the coordinate ring $R$. Several important properties of such curves are well known from the Abhyankar-Moh theory. We summarize them briefly.

4.1 Setup

Suppose that $f(X, Y) \not\in k[X]$ has one place at infinity (or equivalently $f(X, Y) \in k((X))[Y]$ is irreducible). Then it is essentially monic in all coordinate systems and can be written as

$$f(X, Y) = 0 - Y^n + a_1(X)Y^{n-1} + \cdots + a_n(X) \in k[X, Y]$$

where $0 -$ is the well known "Abhyankar non zero", a symbol denoting some non zero constant in $k$. Be aware that this symbol can denote different quantities even in the same expression. The highest $X$-degree term of $f$ can then be described as $0 - X^m$. We can and usually do assume that $f$ is actually monic in $Y$.

Let $v$ denote the unique valuation at infinity and denote by $\Gamma(f)$ the semigroup consisting of all $v(h)$ as $h$ varies over non zero elements of $R$. Note that $\Gamma(f)$ is a subset of $-N$ the set of negatives of natural numbers. The structure and generation of this semigroup is the center piece of the Abhyankar-Moh theory.

4.2 Characteristic quantities

There is a well defined sequence of numbers $r_i \in \Gamma(f)$ for $0 \leq i \leq h$ where

$$r_0 = v(x) = -n, r_1 = v(y) = -\deg_X(f(X, Y)).$$

Denoting the GCD($r_0, \cdots, r_{i-1}$) by $d_i$ we get that $d_1 = n \geq d_2 > \cdots d_h > d_{h+1} = 1$. Further these $d_i$ successively divide the earlier ones and we define $n_i = d_i/d_{i+1}$ for $i = 1, \cdots h$. We additionally make two special definitions $d_0 = m$ and $n_0 = d_0/d_2$.

There is a natural sequence of (essentially ) monic polynomials

$$g_0 = X, g_1 = Y, g_2(X, Y), \cdots, g_h(X, Y)$$

such that the $Y$-degrees of $g_i$ are $d_1/d_i = n/d_i$ for $i = 1, \cdots h$. Besides this degree condition, we also have that $v(g_i(x, y)) = r_i$ for $i = 0, \cdots h$.

Having made $f$ monic in $Y$, the $g_i$ may be chosen to be the approximate $n/d_i$-th roots of $f$, which means, the unique polynomial of $Y$-degree $n/d_i$ such that $\deg_Y(f - g_i^{d_i}) < n - n/d_i$. In general, the chosen polynomials are called pseudo-approximate roots and are said to form an associated $g$-sequence.
4.3 Admissible expansions

The polynomials give a standard basis for $\mathbb{R}$ over $k$ in the following sense.

A sequence $a = (a_0, \cdots, a_h)$ of integers is said to be admissible if it satisfies the following two conditions of admissibility:

**Higher level condition:** $n_i > a_i \geq 0$ for $i = 1, \cdots, h$ and

**Level zero condition:** $a_0 \geq 0$

Then by a standard $g$-monomial we mean an expression $g^a = g_0^{a_0} \cdots g_h^{a_h}$, where $a = (a_0, \cdots, a_h)$ is admissible.

The main theorem of the Abhyankar-Moh theory is that the set of standard monomials form a $k$-basis for $\mathbb{R}$ and moreover, the values $v(g^a) = a_0 r_0 + \cdots + a_h r_h$ are distinct for distinct admissible $a$.

In particular

$$\Gamma(f) = \{a_0 r_0 + \cdots + a_h r_h \mid (a_0, a_1, \cdots, a_h) \text{ is admissible} \}.$$  

It can be shown by simple properties of integers that every integer $w$ has a unique expression $w = \sum_0^h (b_i r_i)$, where the sequence $b = (b_0, \cdots, b_h)$ satisfies higher level admissibility and $w \in \Gamma(f)$ if and only if $b_0 \geq 0$ i.e. the zero level admissibility holds as well.

4.4 Properties of the Semigroup of values

Define an integer:

$$\Theta = \Theta(f) = -r_0 + \sum_1^h (n_i - 1) r_i.$$  

As a consequence of the above description, easy calculations show that given integers $p, q$ with $p + q = \Theta$ we have that exactly one of $p, q \in \Gamma(f)$.

In particular, since $\Gamma(f)$ has no positive integers, all integers less than $\Theta$ are in $\Gamma(f)$ while $\Theta \not\in \Gamma(f)$. The integer $-1 + \Theta$ is called the conductor of the semigroup $\Gamma(f)$ and is necessarily an even number, since exactly half the integers between 0 and it are not in the semigroup. We shall denote it by $C(\Gamma(f))$ and we have the formula:

$$C(\Gamma(f)) = -1 - r_0 + \sum_1^h (n_i - 1) r_i.$$  

4.5 A conductor formula

Let us observe that this conductor of the semigroup is connected with the length of the conductor ideal of the coordinate ring of the curve by the formula:

$$C(\Gamma(f)) = -C(f) - 2P_g(f)$$
where $P_g(f)$ denotes the geometric genus of the function field $k(f)$. Among its various equivalent definitions, let us choose the one which says that it is the integer which equals 1 plus half the degree of the divisor of any nonzero differential in the function field. Explicitly, given any nonzero differential $adb$ we know that $2P_g(f) - 2 = \sum w(adb)$ where the sum is taken over all valuations of the function field $k(f)$ over $k$ and the values $w(adb)$ are as described in section 1.

For our affine curve, we can conveniently choose the fundamental differential $\omega = dx/f_y$ as described in section 2. The sum of values for valuations at finite distance is easily seen to be $-C(f)$ by the Dedekind formula and hence the order of $\omega$ at the valuation $v$ at infinity is seen to be $C(f) + 2P_g(f) - 2$.

On the other hand, we can compute it explicitly thus: Since $v(x) = r_0$ we have $v(dx) = r_0 - 1$ and $v(f_y)$ is well known by the Abhyankar-Moh theory to be equal to $\sum (n_i - 1) r_i$. Thus we get
\[ v(<\omega>) = -1 + r_0 - \sum_{i=1}^{h} (n_i - 1) r_i = C(f) + 2P_g(f) - 2. \]

Thus we have as claimed:
\[ -C(f) - 2P_g(f) = -1 - r_0 + \sum_{i=1}^{h} (n_i - 1) r_i = C(\Gamma(f)). \]

4.6 A formula for the Milnor number of the pencil.

We recall that the Milnor number for the pencil $(f-a)_{a \in k}$ is defined as $\mu(f; A) = l(A/J_f)$. By an argument similar to our proof in Lemma 3.1 we can argue that
\[ int(f_X, f_Y; A) = int(f, f_Y; A) - int(X, f_Y; A) = - \sum_{i=1}^{h} (n_i - 1) r_i + 1 + r_0. \]

For proof, we note that the left hand side can be interpreted as the negative of the sum of values at valuations at infinity for $f_Y$, with suitable adjustment made for reducible $f_Y$. The fact that $int(f-a, f_Y; A)$ is independent of $a$, being purely in terms of the characteristic terms common to all translates $f-a$ and the fact that $f$ is necessarily essentially monic in $Y$ guarantee that $f, x$ have negative orders at each valuation at infinity of $f_Y$ and hence our reductions are simply valid. Thus we get the global Milnor number
\[ \mu(f; A) = \sum_{a \in k} \mu(f-a) = 1 + r_0 - \sum_{i=1}^{h} (n_i - 1) r_i = -C(\Gamma(f)). \]

**Proposition 4.1** If $f \in A$ is a curve with one place at infinity, then we have
\[
\mu(f; A) = C(f) + 2P_g(f) = \mu(f) + \chi(f) + 2P_g(f) = \nu(f) + \delta(f) + \chi(f) + 2P_g(f)
\]

\[ \text{This is worked out by starting with an NP expansion as described later in this section and explicitly determining the t-order of the resultant Resultant}(f, f_Y, Y) \text{ interpreted as a product } \prod (\eta(t) - \eta(\zeta t)), \text{ where } \zeta \text{ varies over non identity } n\text{-th roots of 1.} \]
Proof. This follows by using already obtained results from Lemmas in Section 3 as well as conclusions of Sections 4.5 and 4.6.

Corollary 4.2 If \( f \) is a curve with one place at infinity such that \( \mu(f; A) = \nu(f) \) then we have that \( f \) is a rational curve which is unibranch and locally quasihomogeneous at all its points. Further all translates of \( f \) are nonsingular. Moreover, \( Z(f) = l(\Omega(R, k)/\Omega(R, k)) = C(f)/2 \).

Proof. It follows from the Proposition 4.1 that \( \delta(f) + \chi(f) + 2P_2(f) = 0 \) and since all the quantities are nonnegative, each is zero. This proves the first assertion. It follows that \( \mu(f; A) = C(f) = \mu(f) \). Thus \( \mu(f - a) = 0 \) for all \( a \neq 0 \) giving that all translates of \( f \) are nonsingular. The last result follows from \( C(f) = \nu(f) \) using Proposition 2.4.

4.7 Connection with the NP (Newton-Puiseux) expansion.

All the above quantities can either inductively be described and calculated from the known valuation, or they can be deduced from the NP expansion which defines the valuation at infinity.

The NP expansion is a power series parametrization:

\[
x = \tau^{-n}, \quad y = \eta(\tau) = 0 - \tau^{-m} + \cdots \in k((\tau)).
\]

The valuation \( v \) can then be described as \( v(p(x, y)) = \text{ord}_\tau(p(\tau^{-n}, \eta(\tau))) \).

We refer the reader to look up any of the following sources: [A1], [SS], [S], [AM1], [AM2], for the definition of the associated characteristic sequences and calculations.

For future use, we prove:

Lemma 4.3 Suppose that \( f \in A \) has one place at infinity and an NP expansion

\[
x = \tau^{-n}, \quad y = \eta(\tau) = 0 - \tau^{-m} + \cdots \in k((\tau)).
\]

Then there are no terms of the form \( 0 - \tau^{ns} \) in the expansion of \( \eta(\tau) \) for \( s > 0 \).

Proof. Write \( \eta(\tau) = \sum_{-m}^{\infty} a_i \tau^i \). We know that

\[
f = \prod_{\zeta^n = 1} (Y - \eta(\zeta \tau)).
\]

and the coefficient of \( Y^{n-1} \) is then easily seen to be the trace \( -\sum_{\zeta^n = 1} \eta(\zeta \tau) \) where we substitute \( X \) for \( \tau^{-n} \). Suppose if possible, \( s > 0 \) and \( a_{ns} \neq 0 \). Then we get that the term \( Y^{n-1}X^{-s} \) has the nonzero coefficient \( -a_{ns} \sum_{\zeta^n = 1} (\zeta^{ns}) = -a_{ns}(n) \). Since the coefficient of \( Y^{n-1} \) must be a polynomial in \( X \), we get a contradiction, thereby proving the result.
5 Rational one place curves

Let the notations be as in the Section above. Assume that $f$ defines a rational curve. It follows by a modification of the Lüroth theorem that it is a “polynomial curve” which means that there are two polynomials $X(T), Y(T)$ in a new indeterminate $T$, such that $f(X(T), Y(T)) = 0$ and at least one of $X(T), Y(T)$ is a non constant.

As before, let us take the image $\phi_f(T) = t$ and write $x(t) = \phi_f(X(T)), y(t) = \phi_f(Y(T))$, so that $f(x(t), y(t)) = 0$. By choosing minimal $t$-degrees for $x, y$, we may assume $k(f) = k(t)$ and hence $R = k[t]$. It follows that $P_g(f) = 0$ and hence $C(\Gamma(f)) = -C(f)$. Also we have

$$l(\overline{R}/R) = l(R/\mathfrak{m}) = C(f)/2 = -C(\Gamma(f))/2.$$ 

The (unique) valuation $v$ at infinity of $f$ is now seen to be the $1/t$-adic valuation which can be simply described as $v(a(t)/b(t)) = \deg_t(b(t)) - \deg_t(a(t))$. The order $v(a(t)db(t))$ can now be seen as $v(a(t)) + v(b(t)) - 1$ as long as $a(t)d(b(t)) \neq 0$, i.e. $0 \neq a(t) \in k(t)$ and $b(t) \in k(t) \setminus k$.

We easily see that the set

$$\{v(h) \mid 0 \neq h \in \overline{R} = k[t]\} = \{-N\}$$

the set of all negative natural numbers. The semigroup of the curve is a subset of this with exactly $(-1/2)C(\Gamma(f)) = (1/2)C(f)$ missing values.

We now prove:

**Lemma 5.1** With the notation as above, let $\Gamma^*(f) = \{v(p) \mid p \in \Omega(R, k)\}$ the set of values of elements in $\Omega(R, k)$ and $\Gamma'(f) = \{v(u) - 1 \mid u \in R \setminus k\}$, the set of values of exact differentials $du \in \Omega(R, k)$. Further, set: $M = -N \setminus \Gamma^*(f)$. We have the following:

1. $\Gamma'(f) \subseteq \Gamma^*(f)$.

2. $Z(f) = the number of elements of M$.

3. The cardinality of $-N \setminus \Gamma'(f)$ is clearly $-C(\Gamma(f))/2$ and hence $Z(f) \leq -C(\Gamma(f))/2$ and the inequality is strict if and only if $\Gamma^*(f)$ is strictly bigger than $\Gamma'(f)$ or equivalently there is some element $p \in \Omega(R, k)$ which is not exact.

**Proof.** The first statement is obvious.

Given any element $a \in \Omega(\overline{R}, k) = k[t]dt$ we claim that

- $a \in \Omega(R, k)$ or,
- there is an element $b \in \Omega(R, k)$ such that $v(a - b) \in M$. 

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If \( a \not\in \Omega(R, k) \), then we can choose \( b \in \Omega(R, k) \) which maximizes the value of \( v(a - b) \), since all values in question are bounded above by 0. If \( v(a - b) \in M \) then we are done. Otherwise, there exists \( b^* \in \Omega(R, k) \) such that \( v(a - b) = v(b^*) \). Then \( v(a - b - \lambda b^*) > v(a - b) \) for some \( \lambda \in k \), contradicting the choice of \( b \).

Hence \( v(a - b) \in M \) as claimed.

This shows that we can make a basis of \( \Omega(\overline{R}, k)/\Omega(R, k) \) consisting of elements with distinct values in \( M \), proving our first second assertion.

The last assertion easily follow.

We now summarize the results.

**Proposition 5.2** Let \( f \in A \) be a curve with one place at infinity. Then \( \mu(f; A) = \nu(f) \) if and only if \( f \) is a rational curve which is locally unibranch at all its points and all differentials in \( \Omega(R, k) \) are exact.

**Proof.** We have already proved the only if part in view of Corollary 4.2 except for the claim of exactness of differentials. This claim follows from Lemma 5.1, since under our hypothesis we know that \( -C(\Gamma(f)) = C(f) \).

The if part easily follows from the Proposition 4.1, since our hypothesis implies that the quantities \( P_g(f), \overline{\chi}(f), \delta(f) \) are all zero.

We now define the following terms. Let \( z_1, z_2 \) be non constant elements of the function field \( k(f) \) of an irreducible curve and let \( w \) be any valuation of \( k(f) \) over \( k \). The differential \( \beta(z_1, z_2; w) = w(z_1)z_1dz_2 - w(z_2)z_2dz_1 \) is defined to be the basic differential associated with the pair \((z_1, z_2)\) corresponding to the valuation \( w \).

We also define the gap of the basic differential \( \beta(z_1, z_2; w) \) to be

\[
w(\beta(z_1, z_2; w)) - w(z_1) - w(z_2) + 1.
\]

In case the basic differential is zero the gap is defined to be \( \infty \).

If we take \( x, y \) to be the generators of the coordinate ring, then we will show in the next Lemma 5.3 that the gap of \( \beta(x, y; v) \) is completely determined by the NP expansion associated with \( x, y \). We shall say that the pair \((x, y)\) has a maximal gap if for all automorphisms \( \sigma \) of \( A \) such that \( v(\sigma(x)) = v(x), v(\sigma(y)) = v(y) \), the gap of the basic differential \( \beta(\sigma(x), \sigma(y); v) \) is at most equal to the gap of \( \beta(x, y; v) \). Note that an infinite gap is clearly maximal.

We now prove an auxiliary Lemma.

**Lemma 5.3** Suppose that \( f \in A \) has one place at infinity and an NP expansion

\[
x = \tau^{-n}, \; y = \eta(\tau) = \tau^{-m} + \theta \tau^{-m+q} + \text{higher terms}.
\]

Let \( v \) denote the associated valuation at infinity.
Assume that \((n, m)\) is non principal, i.e. neither divides the other.

Then the gap of \(\beta(x, y; v)\) is exactly \(q\).

Suppose that the pair \((x, y)\) has a maximal gap as defined above.

Then we have the following:

1. \(-m + q\) is not a multiple of \(n\).
2. \(-n + q\) is not a multiple of \(m\).
3. An equation \(-m - n + q = -an - bm\) does not hold for non-negative integers \(a, b\).

Proof. The calculation of the gap of the basic differential is straightforward substitution. It follows that the value of \(q\) does not change by interchanging the role of \(x, y\).

The proofs are by contradiction. We show that when the desired conditions on \(q\) fail, we can make an automorphism \(\sigma\) on \(x, y\) so that the gap of the basic differential \(\beta(\sigma(x), \sigma(y); v)\) is bigger than \(q\) but \(v(\sigma(x)) = v(x)\) and \(v(\sigma(y)) = v(y)\). This would give a contradiction.

Consider the first case and suppose is possible \(-m + q = -ln\). First, we claim that \(l \geq 0, \) for otherwise, we have a contradiction by Lemma 4.3. Now consider new variables \(X' = X, Y' = Y + uX^l\) where \(u \in k\). Let \(x', y'\) be their respective images modulo \(f\) and note that:

\[
v(x) = v(x'), \ v(y) = v(y'), \beta(x', y'; v) = \beta(x, y, v) + (v(x)l - v(y))ux^l dx.
\]

It is easy to verify that the leading term of \(\beta(x', y'; v)\) is \((0 - 0 - u)\tau^{-ln-n-1}d\tau\) and so by choosing \(u \in k\) to kill this term we can make its order bigger than \(-ln-n-1 = -m-n+q-1\). It follows that the gap gets bigger than \(q\). This is a contradiction to the assumed maximality of \(q\).

The second case has a similar argument.

Now in the third case, we already know that if \(-m - n + q = -an - bm\) then both \(a, b\) are non zero. it follows that the right hand side is less than or equal to \(-n - m\) giving \(q \leq 0\). Since \(q > 0\) we have a contradiction!

**Corollary 5.4** Suppose that \(f \in A\) has one place at infinity and an NP expansion

\[x = \tau^{-n}, y = \eta(\tau) = \tau^{-m} + 0 - \tau^{-m+q} + \text{higher terms}.
\]

Further assume that \((n, m)\) is non principal.

Let \(v\) denote the associated valuation at infinity. Then there is an automorphism \(\sigma\) of \(A\) such that \(v(\sigma(x)) = v(x), v(\sigma(y)) = v(y)\) and the pair \((\sigma(x), \sigma(y))\) has a maximal gap as defined above.

Further, the automorphism \(\sigma\) is “very elementary” which means that it is either of the form \(\sigma(X) = 0 - X, \sigma(Y) = 0 - Y + l(X)\) or of the form \(\sigma(X) = 0 - X + l(Y), \sigma(Y) = 0 - Y\) with \(l(X) \in k[X]\).
Proof. From known structure of automorphisms of curves with one place at infinity as in [AM3] and [ASI], we see that automorphisms which preserve that $v$-values of $x, y$ are necessarily very elementary.

Considering how the equation $f(X, Y)$ is related to the NP expansion, we see that either $q = \infty$ and we have the desired maximal gap or $q$ is bounded above by $mn - 1$.

The proof of Lemma 5.3 shows that the gap of the basic differential $\beta(x, y; v)$ is precisely $q$ and since this is bounded above there is a clear maximum value for this $q$ under all possible very elementary automorphisms.

We simply choose an automorphism which gives such a maximal value.

We now prove:

**Theorem 5.5 Generalized Theorem of Zariski.** Let $f \in A$ be a curve with one place at infinity. Then $\mu(f; A) = \nu(f)$ if and only if there is an automorphism $\sigma : A \rightarrow A$ such that $f = \sigma(Y)^b - \sigma(X)^a$ for non negative integers $a, b$. Moreover, in view of irreducibility of $f$, we must have $\text{GCD}(a, b) = 1$.

Proof. It is easy to check that when $f$ has the indicated form $\sigma(Y)^b - \sigma(X)^a$, we have

$$\mu(f; A) = l(A/\langle \sigma(X)^{a-1}, \sigma(Y)^{b-1} \rangle)A = (a-1)(b-1) = \nu(f).$$

All the remaining properties of $f$ are easy to verify.

Now for the only if part, we note that Proposition 5.2 already gives that $f$ is a rational unibranch curve with $\delta(f) = 0$ and that all differentials in $\Omega(R, k)$ are exact. We use this to prove the existence of the automorphism.

We can write after an adjustment of coordinates:

$$f(X, Y) = Y^n - X^m + \sum f_{ij}X^iY^j$$

where $n > m$ and the terms of the summation are naturally restricted by the condition that $f$ has one place at infinity.

If $m = 0$, then we have $y \in k$ and after a translation in $Y$ we get $f = Y$. This is the case when $a = 1, b = 0$.

Now assume that $n > m > 0$ such that $(n, m)$ is non principal, i.e. neither divides the other. This assumption is standard and ensured by performing elementary automorphisms on $X, Y$ if one of $n, m$ divides the other.

We shall now assume the notations and details about the Abhyankar-Moh theory in Section 4 without further explanation.

In particular, $r_0 = -n, r_1 = -m, d_1 = n, d_2 = \text{GCD}(m, n) > 1$ and by our arrangement $n > m > d_2$.
Let an NP expansion be written as:

\[ x = \tau^{-n}, \quad y = \eta(\tau) = \tau^{-m} + \text{higher terms} \]

and let \( v \) denote the corresponding valuation at infinity.

If there are no higher terms, then we have \( y^n = x^m \) and clearly \( f = Y^n - X^m \). By irreducibility of \( f \) we get \( \text{GCD}(m, n) = 1 \) finishing the proof.

So, now assume that:

\[ x = \tau^{-n}, \quad y = \eta(\tau) = \tau^{-m} + 0 - \tau^{-m+q} + \text{higher terms} \]

As in Corollary 5.4, we assume that the gap \( q \) of the basic differential \( \beta(x, y; \nu) \) can be chosen to be maximal so that we can assume that the conditions established in Lemma 5.3 hold.

By exactness of all differentials in \( \Omega(R, k) \), we deduce that \( -m - n + q \in \Gamma(f) \).

We have two possible cases and we show that both fail.

1. Case 1: \(-m + q\) is the second characteristic exponent, i.e. \( q = q_2 \). Since \( r_2 = n_1 r_1 + q_2 \), we have

\[ -m - n + q = r_0 + r_1 + r_2 - n_1 r_1 = r_0 + r_1 + r_2 - n_0 r_0 = (1 - n_0) r_0 + r_1 + r_2. \]

This is an admissible expression and hence is in \( \Gamma(f) \) if and only if \( 1 - n_0 \geq 0 \). Further, this is possible only if \( n_0 = 1 \), i.e. \( r_0 = -n \) divides \( r_1 = -m \). This is a contradiction.

2. Case 2: \(-m + q\) is not a characteristic term, so \(-m + q\) and hence \(-m - n + q\) is divisible by \( d_2 \) and we have \(-m - n + q = a_0 r_0 + a_1 r_1\) where \( 0 \leq a_1 < n_1 \) and \( 0 \leq a_0 \).

Thus \(-m - n + q = -a_0 n - a_1 m\) in contradiction with condition 3 of Lemma 5.3.

### 6 Link with Lin-Zaidenberg Theorem

Let \( f \in A \) be an irreducible polynomial. In [LZ], Lin and Zaidenberg proved the following:

**Theorem 6.1 Lin-Zaidenberg** If \( f \) is a rational curve with one place at infinity which is locally unibranch at all its points then there is an automorphism \( \sigma : A \to A \) such that \( f = \sigma(Y)^b - \sigma(X)^a \) for non negative integers \( a, b \). Moreover, in view of irreducibility of \( f \), we must have \( \text{GCD}(a, b) = 1 \).

Moreover, the hypothesis of the theorem can be alternatively stated as: \( f \in A \) is a curve with one place at infinity such that \( \mu(f; A) = \mu(f) \).

Their results generalized the celebrated Abhyankar-Moh Epimorphism Theorem which can be stated as follows:
Theorem 6.2 Abhyankar-Moh

If $f$ is a rational nonsingular curve with one place at infinity then there is an automorphism $\sigma : A \rightarrow A$ such that $f = \sigma(Y)$.

Remark 6.3

Note that the hypothesis of the Lin-Zaidenberg Theorem implies in our notation: $P_y(f) = 0 = \chi(f)$. Then, in view of sections 4.5, 4.6 we have

$$\mu(f; A) = C(f) = \mu(f) = \nu(f) + \delta(f).$$

In turn, the alternate hypothesis $\mu(f; A) = \mu(f)$ is easily seen to imply all their conditions as in our proof of Corollary 4.2.

The Epimorphism Theorem is a special case in view of the facts that $\mu(f) = 0$ by the non singularity of the curve while the formula in 4.6 gives: $\mu(f; A) = -C(\Gamma(f)) = 0$. Thus it is now a special case of the Lin-Zaidenberg Theorem. We note that, since the proofs of the Lin-Zaidenberg theorem do use the Epimorphism Theorem, they do not provide an alternative proof for it.

It follows from the conclusion of the Lin-Zaidenberg theorem that further, $\delta(f) = 0$. All known proofs of the Lin-Zaidenberg theorem are rather involved, using respectively complex analysis (Lin and Zaidenberg [LZ]), topology (Neumann and Rudolph [NR]), and finally algebraic geometry using surface theory (Gurjar and Miyanishi [GM]).

Our original aim was to provide a much simpler proof of this important theorem using the techniques of the Abhyankar-Moh theory.

Our theorem is a weaker version of the Lin-Zaidenberg theorem since we additionally need to assume that $\delta(f) = 0$ and this is clearly the condition that $J_f = \text{Jac}_f$ or $f \in J_f$. The Lin-Zaidenberg hypothesis easily implies that $f \in \sqrt{J_f}$ and the Lin-Zaidenberg theorem implies that $f \in J_f$. We have not yet succeeded in establishing this independently using our simpler techniques. This condition can be also described as the condition of local quasihomogeneity explained earlier in Remark 3.5.

7 Estimating independent non exact differentials

7.1 Preamble

Let $f \in A$ be a rational curve with one place at infinity. We know from Proposition 4.1 and Lemma 3.4 that for such a curve $C(f) = -C(\Gamma(f))$ and $C(f)/2 - Z(f) = \delta(f) + \chi(f)$ measures how far the curve fails to be quasihomogeneous. Also the quantity $C(f)/2 - Z(f)$ is seen to be the number of independent non exact differentials by calculations of Lemma 5.1. \(^3\)

In this section we establish some estimates of this quantity.

\(^3\)In the notation of the Lemma 5.1 this is the cardinality of the difference $\Gamma^*(f) \setminus \Gamma'(f)$. 

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In the above notation we assume: $f \in A$ is a rational curve with one place at infinity with NP expansion
$$X = \tau^{-n}, \quad Y = \tau^{-m} + 0 - \tau^{-m+q} + \cdots$$
and let $v$ as usual denote the resulting valuation. Note that we are implicitly assuming that $q \neq \infty$, in other words, our curve is not globally quasihomogeneous.

We further assume that $n > m > \text{GCD}(n, m)$ and the gap $q$ is chosen to be as large as possible for the given coprime degrees.

We already know that the basic differential $\Delta = \beta(x, y; v)$ associated with $x, y$ is not exact in view of our proof of Theorem 5.5.

We investigate the set of non exact differentials which are multiples of this $\Delta$.

By the two cases analyzed in the proof of our Theorem 5.5, we see that we have:

1. Case 1: $-m + q = -m + q_0$ the second characteristic number and $-m - n + q = (1 - n_0) r_0 + r_1 + r_2 \notin \Gamma(f)$ since $n_0 > 1$.

2. Case 2: $-m + q$ is not a characteristic term and hence $-m - n + q = a_0 r_0 + a_1 r_1$ where $0 \leq a_1 < n_1$ and $a_0 < 0$.

### 7.2 Number of distinct values of inexact differentials

We now prove

**Lemma 7.1** Assume the setup of the preamble above and assume that we have case 1. Let $w_b$ be any monomial $\prod g_i^{b_i}$ where $b = (b_0, \cdots, b_h)$ is an admissible sequence satisfying
$$0 \leq b_0 < n_0 - 1, 0 \leq b_1 < n_1 - 1, 0 \leq b_2 < n_2 - 1, 0 \leq b_i < n_i \text{ for } 3 \leq i \leq h.$$ Then the differentials $\{w_b \Delta\}$ are inexact members of $\Omega(R, k)$ with distinct values. In particular, the number of these is at least $(n_0 - 1)(n_1 - 1)(n_2 - 1) \prod_{3}^{h} n_i \geq 2^{h-1}$.

**Proof.** The differentials are inexact because their values augmented by 1 are not in $\Gamma(f)$ as evident from their standard representation and it also gives the distinctness of the values. The estimate on the count is simply by counting the possible values of $b$’s. For the last estimate note that each $n_i$ is at least 2. Moreover the first two $n_0, n_1$ are coprime to each other and hence at least one of them is 3 or bigger. This gives that at least $h - 1$ of the $h + 1$ factors are at least 2 and finishes the proof.

**Lemma 7.2** Assume the setup of the preamble above and assume that we have case 2. Let $w_b$ be any monomial $\prod g_i^{b_i}$ where $b = (b_0, \cdots, b_h)$ is an admissible sequence satisfying
$$0 \leq b_0 < -a_0, 0 \leq b_1 < n_1 - a_1, 0 \leq b_i < n_i \text{ for } 2 \leq i \leq h.$$
Then the differentials \( \{w_b \Delta\} \) are inexact members of \( \Omega(R, k) \) with distinct values. In particular, the number of these is at least \((-a_0)(n_1 - a_1) \prod_{i=2}^{h} n_i \geq 2^{h-1}\).

**Proof.** Everything except the last statement follows exactly as in Lemma 7.1. The last estimate follows from the fact that each \( n_i \) is at least 2 and we have \( h - 1 \) terms \( n_2, \ldots, n_h \) as factors of our estimate.

We now investigate the case when the number of values of inexact differentials is exactly 1. We prove:

**Proposition 7.3** Suppose that \( f \in A \) is a curve with one place at infinity such that \( \mu(f; A) = \nu(f) + 1 \). Then we have the following:

1. \( f \) is a rational curve with \( \delta(f) = 0 \) and \( \chi(f) = 1 \).
2. There is exactly one value of an inexact differential in \( \Omega(R, k) \).
3. All such curves have \( h = 1 \) or equivalently \( n, m \) are coprime. Moreover, we have exactly one of the following three situations:

   - **Situation 1:** \( m = 2, n = 2p + 1 \) for some \( p = 1, 2, \ldots \).
   - **Situation 2:** \( m = 3, n = 4 \).
   - **Situation 3:** \( m = 3, n = 5 \).

   Each of these situations will be explicitly described below.

**Proof.** From the given condition we deduce from Proposition 4.1 that

\[
\delta(f) + \chi(f) + 2P_g(f) = 1.
\]

It follows that \( P_g = 0 \), i.e. the curve is rational. If \( \chi(f) = 0 \), then we could apply the Lin-Zaidenberg Theorem 6.1 and deduce that \( \delta(f) = 0 \) giving a contradiction. This gives the first part.

The second part follows from Lemma 3.4.

From the lower bound \( 2^{h-1} \) in Lemma 7.1, 7.2 we deduce that \( h = 1 \). Further, this means that we can apply the setup of Lemma 7.2 and in the notation of the preamble we get that

\[
\text{GCD}(m, n) = 1, a_0 = -1, a_1 = n_1 - 1 = n - 1 \text{ and } -m - n + q = (-1)(-n) + (n - 1)(-m).
\]

By adding 4 to both sides and rearranging, the last equation reduces to

\[
nm - 2n - 2m + 4 = (n - 2)(m - 2) = 4 - q
\]
Since $n > m$ by the preamble, we see that $m = 2, 3$, since otherwise $4 < (n-2)(m-2) = 4-q < 4$ is a contradiction.

If $m = 2$ then we get $q = 4$ and $n = 2p + 1$ for some $p = 1, 2, \ldots$ since it is coprime with $m$.

If $m = 3$, then $n = 4, q = 2$ and $n = 5, q = 1$ give the only possibilities, since $n > 5$ leads to $4 \leq (n-2)(m-2) = 4-q < 4$ a contradiction again!

We shall now give the explicit description of each of these cases below in separate Lemmas.

**Lemma 7.4** Assume that we have Situation 1 as described in Proposition 7.3. Then by an automorphism we can arrange that $f = g(Y) - X^2$ where $g(Y) \in k[Y]$ is monic of degree $2p+1$ and has exactly two distinct linear factors. In turn, every such curve satisfies the hypothesis of Proposition 7.3.

**Proof.** Since $m = 2$ we know that $f$ is of degree 2 in $X$ and has a non zero constant coefficient for $X^2$. Thus the form of $f$ can easily be arranged by translating $X$ by a polynomial in $Y$ to kill the $X$-term. Now let $s$ be the number of distinct linear factors of $g(Y)$. We see:

$$Jac_f = (f, f_Y, f_X)A = (f, g'(Y), -2X)A = (X, g(Y), g'(Y))A.$$  

It is not hard to see that the length of the last ideal is $\nu(f) = 2p + 1 - s$ and since this is required to be $\mu(f; A) - 1 = 2p - 1$ we get that $s = 2$. In turn, this also shows that all such curves satisfy the hypothesis of Proposition 7.3.

**Lemma 7.5** Assume that we have Situation 2 as described in Proposition 7.3. Then we can arrange the parametric form of the curve as

- either $x = t^4 + at^2, y = t^3 - 3t^2, a = 0, -8, -9$ in case there is at least one singular (cuspidal) branch at finite distance
- or $x = (t^2 - 1)(t^2 + a), y = (t^2 - 1)(t + b)$, where $a = -1$ and $b \neq \pm 1$ in case there is no singular branch.

**Proof.** The proof is done by using Maple and we will only outline the strategy.  

First, we assume that the singular branch is at $t = 0$ and then we can easily assume that either $y = t^3$ or $y = t^3 + 0 - t$.

In the first case, we have $x = t^4 + 0 - t^2$ and a detailed calculation leads to impossible subcases.

In the second case, by a suitable scaling we may arrange $y = t^3 - 3t^2$.

By singularity of the branch, $x$ has order at least 2 and we can remove its cubic term by adding a multiple of $y$. 

---

4A Maple file with actual calculations will be provided if needed.
We next calculate the Taylor Resultant (TRES) introduced by Abhyankar and described thus: Given a parametric curve \( x = u(t), y = v(t) \) the branches at the singular points are determined by calculating the

\[
TRES(u(t), v(t)) = \text{Resultant}\left(\frac{u(t) - u(s)}{t - s}, \frac{v(t) - v(s)}{t - s}, s\right).
\]

This resultant has precisely the degree equal to \( C(f) \) and the multiplicities of its factors give the exact order of the conductor ideal at various branches i.e. valuations. Indeed, for a rational one place curve, it gives the exact generator of the conductor ideal \( C_f \) in the ring \( k[t] \).

Under our hypothesis of \( \chi(f) = 1 \), we see that this TRES has degree 6 and has at most two simple roots and thus a maximum of 4 distinct roots. Systematic use of the discriminant lets us determine the conditions on the coefficients \((a, b)\) leading to the announced values.

For the case when we don’t have a singular (cuspidal) branch, we assume that two branches (which can be arranged without loss of generality to be \( t = \pm 1 \) are centered at a singular point arranged to be the origin. We then get the announced parametrization after killing out the cubic term by adding a multiple of \( y \) to \( x \). We now note that there cannot be another singular point, for it necessarily will be unibranch and will give a singular branch reducing to the first case. Also, at the origin, we must have just the two branches and three nodes in successive neighborhoods. Calculations with usual quadratic transformations give the announced result.

**Lemma 7.6** Assume that we have Situation 3 as described in Proposition 7.3. Then we can arrange the parametric form of the curve as

- either there is at least one singular (cuspidal) branch at finite distance and
  \[
  x = t^5 + at^4 + bt^2, \quad y = (t^3 - 3t^2)
  \]
  where the values of \((a, b)\) come from the finite set:
  \[
  \{(-15, 0), (-3, 0)(-5/2, 0), (-3/2, 0), (-3, 4), (-6, 27), (-7, 36), (-15/2, 40)\};
  \]
- or there is at least one singular (cuspidal) branch at finite distance and
  \[
  x = t^5 + as^4 + bs^2, q = s^3 \text{ with } b = 0, a \neq 0;
  \]
- or
  \[
  x = (t^2 - 1)(t^3 \pm \sqrt{2}t^2 + 1), y = (t^2 - 1)(t + 1 \pm \sqrt{2})
  \]
  in case there is no singular branch.
Proof. The arguments here are similar to Lemma 7.5, except we get an extra variable in each case.

In case of at least one singular branch, we argue as before and both the subcases now lead to indicated solutions.

In case there is no singular branch, we again make a similar argument and end up with only a finite set of solutions as indicated. Note that in this case, we need four successive nodes at the origin (instead of the three nodes in Lemma 7.5) and hence the loss of a free parameter is expected, so we end up with only a finite set of solutions.

The reduction in the first case is identical and the argument in the second case that the origin must be the only singular point still holds, except we need to find four nodes in successive neighborhoods.

Remark 7.7 It seems difficult to generalize the characterization above to one place rational curves with small $\mu(f; A) - \nu(f)$, for example, if $\mu(f; A) - \nu(f) = 2$, then Lemma 7.1. and Lemma 7.2. show that $h = 1$ or 2. Every case contains many subcases. This makes the (already technical) work hard to realize.
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