Mechanism design and kinematics simulation of 3-DOF parallel mechanism with low coupling degree

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Abstract. According to the topological design theory of parallel mechanism (PM) based on the position and orientation characteristic (POC), a new PM with two translational degree of freedom (DOF) and one rotational DOF (2T1R) was proposed. The topological characteristics of the PM was analyzed, including POC set, DOF and coupling degree. Based on analytic method, the solving modeling was established, and the forward and inverse solutions were obtained. Then the correctness of the position solution was verified by Matlab. Velocity and acceleration formulas of PM were calculated by Jacobian matrix, and simulation curves of velocity and acceleration was put forward. Kinematic analysis shows that the 2T1R PM has both translational and rotational characteristics, moreover, the moving platform has stable output and good dynamic characteristics.

1. Introduction
The 2T1R PM with has the characteristics of both movement and rotation. Due to large stiffness, small inertia and high precision, it has attracted the attention and research of many scholars. Mao¹ proposed the RRC+RRRR+SSR type of 2T1R parallel mechanism; Wang² gave the positive and negative solution equations and mechanism velocity formulas of 2-CU+1PU parallel mechanism; Xie³ studied the application of configuration synthesis of 2T1R parallel mechanism in manufacturing; Ni⁴ proposed a new 2T1R parallel mechanism and obtained the error Jacobian matrix. Mazare⁵ calculated the global dexterity index of the 3-[P2(US)] PM. Qu⁶ and Liu⁷ carried out configuration synthesis of 2T1R mechanism. Deng⁸ studied (RPa ǁ 3R)-R+RSS type parasitic motion parallel mechanism.

Most of the existing 2T1R parallel mechanisms have high coupling degree, which brings difficulties to the mechanism design and kinematic analysis. According to the problem mentioned above, a new type of 2T1R parallel mechanism was proposed, which can realize the translation along the x and y axes and rotation about y axes in space. The PM is designed with weak coupling, and the forward and inverse solutions can be obtained by one-dimensional search. In addition, the kinematics characteristics of the PM are analyzed respectively.

2. Topology analysis
According to parallel mechanism topology design theory, and based on position and orientation characteristic⁹, the 2T1R PM including the moving platform I, static platform 0 and single open chain I, II, III was put forward.
As shown in figure 1, the topology of PM is PRR+2-PRRU. And the single open chain(SOC) I(P₁∥R₁∥R₂-U₁) structure is equivalent to (P₁∥R₁∥R₂∥R₁₁⊥R₁₂). Similarly, SOC II(P₂∥R₃∥R₄-U₂) structure is equivalent to (P₂∥R₃∥R₄∥R₂₁⊥R₂₂). And SOC III structure is (P₃∥R₅∥R₆). The relationship of axis layout in each prismatic pair on the static platform 0 is as follows: the prismatic pairs P₁, P₂ and P₃ are arranged along the x, z and y axis respectively. In addition, the axes of the prismatic pairs P₁, P₂ and P₃ are mutual perpendicular.

The POC equation of parallel mechanism are as follows:

\[ M_{ps} = \bigcup_{j=1}^{k} M_{sj} \]  

(1)

\[ M_{ps} = \bigcap_{i=1}^{n} M_{bi} \]  

(2)

where:

- \( M_{ps} \) —— POC set of mechanism moving platform
- \( M_{bi} \) —— The POC set at the end of the \( i \) th chain
- \( M_{sj} \) —— The POC set of the \( j \)th SOC in the branched chain

(1) Topology of the mechanism
As shown in figure 1, the topology structure of SOC\( j \) (\( j=1,2,3 \)) composed of the PM is:
- SOC₁ (−P₁∥R₁∥R₂−U₁−)
- SOC₂ (−P₂∥R₃∥R₄−U₂−)
- SOC₃ (−P₃∥R₅∥R₆−)

On the static platform 0, kinematic pairs P₁, P₂ and P₃ are perpendicular each other; and choose the end point O’ of SOC III as the basis for analysis.

(2) Determine each chain POC set
Equation (1) shows that:

\[ M_{bi} = \begin{bmatrix} t^1(∥P_i) \\ r^0 \end{bmatrix} \bigcup \begin{bmatrix} t^1(⊥R_i) \bigcup t^1(⊥R_2) \\ r^i(∥R_1) \end{bmatrix} \bigcup \begin{bmatrix} t^2(⊥P_i) \\ r^2(∥P_3) \end{bmatrix} \]

\[ = \begin{bmatrix} t^1 \\ r^2(∥P_3) \end{bmatrix} \]
\[
M_{b3} = \begin{bmatrix}
 t^1 (P_1) \\
 r^0 \\
 t^2 (\perp P_1) \\
 r^1 (\perp R_1) \\
 r^2 (P_1) \\
 r^1 (P_1) \\
 r^2 (P_1) \\
 r^1 (P_1) \\
 r^2 (P_1) \\
 r^1 (P_1) \\
 r^2 (P_1) \\
 r^1 (P_1) \\
 r^2 (P_1) \\
 r^1 (P_1) \\
 r^2 (P_1) 
\end{bmatrix}
\]

Freedom degree formula\(^{[9]}\) of PM is:

\[
F = \sum_{i=1}^{m} f_i - \sum_{j=1}^{n} \xi_{ij} \tag{3}
\]

Among them:

\[
\sum_{j=1}^{n} \xi_{ij} = \dim \left( \bigcup_{j=1}^{n} M_{b_j} \right) \bigcup M_{b_{(j+1)}} \] \tag{4}

where:

- \(F\) — freedom in mechanism
- \(f_i\) — the \(i\) th degree of kinematic pair
- \(m\) — number of kinematic pair
- \(n\) — number of component
- \(\nu\) — independent loop number, where \(\nu = m - n + 1\)
- \(\xi_{ij}\) — the number of independent equations of the \(j\) th loop

The subordinative parallel mechanism POC set composed of the \(j\) th chain

\[\bigcap_{i=1}^{n} M_{b_i}\]

The POC set of the terminal component of \(j+1\) chain

\[M_{b_{(j+1)}}\]

The mechanism can be decomposed into two independent loops:

SOC\(_I\)\(-P_1 \parallel R_1 \parallel R_2 = U_1 - P_3 \parallel R_3 \parallel R_6 -\)

SOC\(_II\)

subordinative PM consisted by SOC I and SOC III is the first independent loop, It can be obtained from equation(4):

\[
\xi_{i,1} = \dim \{(M_{b_i} \cup M_{b_3})
\]

\[
= \dim \left[ \begin{bmatrix} t^1 \\ r^2 (\perp P_1) \end{bmatrix} \bigcup \begin{bmatrix} t^2 \\ r^1 (P_1) \end{bmatrix} \right] = 5
\]

The POC set of the subordinative mechanism is:

\[
M_{\text{sub(1-3)}} = M_{b_1} \bigcap M_{b_3}
\]

\[
= \begin{bmatrix} t^1 \\ r^2 (\perp P_1) \end{bmatrix} \bigcap \begin{bmatrix} t^2 \\ r^1 (P_1) \end{bmatrix}
\]

The subordinative PM and the SOC II constitute the second independent loop, It can be obtained from equation(4):
\[ \xi_{l_2} = \dim \{ M_{p_0l} \cup M_{l_2} \} \]
\[ = \dim \left[ \left\{ \begin{array}{c} r^1 \\ r^1(\| P_1 \|) \end{array} \right\} \cup \left\{ \begin{array}{c} r^2 \\ r^2(\| P_2 \| P_3) \end{array} \right\} \right] = 5 \]

(2) Determine the DOF of the PM
The DOF of the parallel mechanism can be obtained from equation (4):
\[ F = \sum f_i - \sum \xi_{l_j} = 13 - 10 = 3 \]

Therefore, the DOF of the PM is 3, and kinematic pairs \( P_1 \), \( P_2 \) and \( P_3 \) on the static platform 0 are selected as the driving pair according to the criteria for driving pair subexistence\(^9\).

3. Coupling degree \( \kappa \)
According to the mechanism composition principle based on SOC, arbitrary components can be decomposed into several basic kinematics chains (BKC). The BKC of \( v \) independent loops can be further decomposed into SOC(\( \Delta_j \)) \((j=1,2,\ldots,v)\), and the constraint degree of the \( j \) th SOC\( i \) is defined as:
\[ \Delta_j = \sum f_i - I_j - \xi_{l_j} \]  

Among them:
\[ \begin{align*} 
\Delta_j &= 1,2,\ldots \\
\Delta_j &= 0 \\
\Delta_j &= -1,-2,\ldots 
\end{align*} \]
where: \( m_j \) —— the prismatic pairs of the \( j \) th SOC\( j \)
\( I_j \) —— the number of driving pairs of the \( j \) th SOC\( j \)
\( \Delta_j \) has positive, zero and negative forms, and must meet \( \sum \Delta_j = 0 \). Therefore, the coupling degree of BKC is defined as:
\[ \kappa = \frac{1}{2} \min \sum |\Delta_j| \]  

Where, BKC can be decomposed into several allocation schemes, \( \min \sum |\Delta_j| \) should be taken as the smallest.
Coupling degree\(^9\) \( \kappa \) reflects the correlation and dependence degree between variables in BKC. The higher value of \( \kappa \) is, the more complex kinematics and dynamics problem is. when \( \kappa = 0 \), kinematics and dynamics analysis of the BKC can be solved separately. when \( \kappa > 0 \), the kinematics and dynamics analysis of the BKC requires simultaneous solution of multiple loops.

According to section 1, the number of independent equations of the two loops is \( \xi_{l_1} = \xi_{l_2} = 5 \), and the number of driving pairs of each loop is \( I_1 = 2 \), \( I_2 = 1 \), so the constraint degree \( \Delta_1, \Delta_2 \) are:
\[ \Delta_1 = \sum f_i - I_1 - \xi_{l_1} = 8 - 2 - 5 = 1 \]
\[ \Delta_2 = \sum f_i - I_2 - \xi_{l_2} = 5 - 1 - 5 = -1 \]
\[ \Delta_1 + \Delta_2 = 0 \rightarrow BKC = 1 \]
Therefore, the institution only contains one BKC. According to equation (6), the coupling degree \( \kappa \) is 1; So the parallel mechanism can obtain all real solutions of the position problem by one-dimensional search method.
4. Kinematic analysis
As shown in figure 2, both static platform 0 and moving platform 1 are rectangular. They are arranged in parallel. The moving platform 1 is located directly above the static platform 0, and the distance between them is \( L_0 \); Moving coordinate system \( P-\{uvw\} \) is established with center \( P \) of moving platform 1 as the origin. The \( u \) axis is parallel to the length direction of moving platform and points to \( A_2 \). The \( w \) axis is vertically upward and perpendicular to the moving platform. And the \( v \) axis is determined by the right hand rule. Similarly, the fixed coordinate system \( O-\{xyz\} \) is established with center \( O \) of the static platform 0 as the origin.

The size parameters of the parallel mechanism are as follows: the static platform 0 is \( 2l_1 \) in length and \( 2l_2 \) in width; the moving platform 1 is \( 2l_3 \) in length and \( 2l_4 \) in width. Other dimension parameters are: \( B_3C_3=l_5 \), \( A_3B_3=l_6 \), \( A_2B_2=l_7 \), \( B_2C_2=l_8 \), \( C_2D_2=l_9 \).

\[ \text{Figure 2. kinematics model of 2T1R parallel mechanism} \]

4.1 Forward kinematics
The forward kinematics of PM means that get the input parameters \((x, y, z_i)\) to solve the position and pose parameters \((x, y, \beta)\).

(1) Solution of \( x \) and \( y \)
As shown in figure 1, during the movement of PM, the displacement of the prismatic pair \( P_1 \) and \( P_3 \) is the displacement of moving platform along the \( u \) and \( v \) axis, then:

\[
\begin{align*}
    x &= x_i \\
    y &= y_i
\end{align*}
\]

(7)

(2) Solution of \( \beta \)
As shown in figure 3, According to the closed loop\(^{[10]}\) consisted by SOC II and SOC III, the following equations can be established:

\[
\begin{align*}
    l_k \cos \theta + l_1 \cos \beta + l_5 + l_y &= 2l_1 \\
    l_k \sin \theta + l_1 \sin \beta + l_5 &= L_0 + z_i
\end{align*}
\]

(8)

Let \( a_i = 2l_1 - l_7 - l_8 - l_y \), add the sum of squares of equation (8):

\[
\begin{align*}
    l_k^2 + a_i^2 + (l_0 + z_i - l_8)^2 - l_7^2 - 2a_i l_1 \cos \beta - 2l_1 (l_0 + z_i - l_8) \sin \beta &= 0
\end{align*}
\]

(9)

Further, equation (9) can be obtained:

\[
A \cos \beta + B \sin \beta + C = 0
\]

(10)

Where:
\[
\begin{align*}
A &= -2l_1a_1 \\
B &= -2l_1(l_0 + z_1 - l_3) \\
C &= l_5^2 + a_1^2 + (l_0 + z_1 - l_4)^2 - l_6^2
\end{align*}
\]

Arranged equation (10):

\[
B^2 \left(1 - \cos^2 \beta \right) = A^2 \cos^2 \beta + C^2 + 2AC \cos \beta 
\]

Further, equation (11) can be obtained:

\[
a \cos^2 \beta + b \cos \beta + c = 0
\]

Where:

\[
\begin{align*}
a &= A^2 + B^2 \\
b &= 2AC \\
c &= C^2 - B^2
\end{align*}
\]

Solution of equation (12) can be obtained as follows:

\[
u = \cos \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]

That is:

\[
\beta = \arccos u
\]

\[\text{Figure 3. kinematics model of closed loop}\]

**4.2 Inverse kinematics**

The inverse kinematics of PM means that get the position and pose parameters \((x, y, \beta)\) to solve the input parameters \((x_1, y_1, z_1)\).

(1) solution of \(x_1\) and \(y_1\)

\[
\begin{align*}
x_1 &= x \\
y_1 &= y
\end{align*}
\]

(2) solution of \(z_1\)

According to equation (9), it can be known that:

\[
(l_0 + z_1 - l_4)^2 + d(l_0 + z_1 - l_4) + e = 0
\]

Where:
\[
\begin{align*}
    d &= -2l_3 \sin \beta \\
    e &= l_3^2 + a_1^2 - l_2^2 - 2a_1l_1 \cos \beta
\end{align*}
\]

Solution of equation (16) can be obtained as follows:

\[
(L_0 + z_i - l_1) = \frac{-d + \sqrt{d^2 - 4e}}{2} = u
\]

That is:

\[
z_i = u + l_1 - L_0
\]

**4.3 Numerical example**

The structure parameters of PM are set as follows:  
\[l_1=400\text{mm}, \ l_2=370\text{mm}, \ l_3=150\text{mm}, \ l_4=75\text{mm}, \ l_5=185\text{mm}, \ l_6=350\text{mm}, \ l_7=165\text{mm}, \ l_8=60\text{mm}, \ l_9=210\text{mm}, \ L_0=415\text{mm}.\]

Matlab software was used to verify the correctness of the forward and inverse kinematics equations. And the simulation steps were as follows:

(1) set input parameters \((x, y, z_i)\)

(2) substitute the input parameters \((x, y, z_i)\) into the forward solution model and calculate the forward solution parameters

(3) substitute the forward solution parameters into the inverse solution model and calculate the inverse solution parameters \((x, y, z_i)\)

(4) calculate the difference between the inverse solution parameters \((x, y, z_i)\) and the input parameters \((x, y, z_i)\)

(5) according to the difference judge the correctness of kinematic analysis

After above simulation steps, the numerical example of kinematic analysis can be obtained as shown in table 1.

| serial number | input parameters \((x, y, z_i)\) | inverse solution parameters \((x, y, z_i)\) | Difference |
|---------------|---------------------------------|---------------------------------|------------|
| 1             | (165,90,55)                     | (165,90,49,9999)               | (0,0,0.0001)|
| 2             | (170,95,55)                     | (170,95,55)                    | 0          |
| 3             | (175,100,60)                    | (175,100,60,0001)             | (0,0,0.0001)|
| 4             | (180,105,65)                    | (180,105,64,9999)             | (0,0,0.0001)|
| 5             | (185,110,70)                    | (185,110,70,0001)             | (0,0,0.0001)|

As shown in table 1, It can be found that the inverse solution parameters \((x, y, z_i)\) are approximately equal to the input parameters \((x, y, z_i)\), which verifies the correctness of the position analysis.

**5. Velocity and acceleration analysis**

**5.1 Solution of velocity formula**

By taking the first derivative of equation (7) and equation (9) by time, the velocity formulas of the three driving sliders can be obtained:

\[
\begin{align*}
    \dot{x} &= \dot{x}_1 \\
    \dot{y} &= \dot{y}_1 \\
    D\dot{\beta} &= E\dot{z}_1
\end{align*}
\]

Where:

\[
\begin{align*}
    D &= a_1l_1 \sin \beta - l_1 \cos \beta (L_0 + z_i - l_1) \\
    E &= l_1 \sin \beta - (L_0 + z_i - l_1)
\end{align*}
\]

According to equation (19), the formula based on Jacobian matrix can be obtained:
\[ J_p \dot{p} = J_q \dot{q} \]  
(20)

Where \( \dot{p} = (\dot{x}, \dot{y}, \dot{\beta})^T \) is the output velocity vector, \( \dot{q} = (\dot{x}_i, \dot{y}_i, \dot{z}_i)^T \) is the input velocity vector, \( J_p \) and \( J_q \) are the forward and inverse Jacobian matrix of PM respectively.

Among them:

\[
J_p = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & D
\end{bmatrix}
\]

\[
J_q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & E
\end{bmatrix}
\]

When there is no singular position of the mechanism, namely \( J_p \) reversible, the velocity solution of PM is:

\[ \ddot{p} = J_p^{-1} J_q \ddot{q} \]  
(21)

5.2 Solution of acceleration formula

The first derivative of each variable in equation (20) by time, the acceleration formulas can be obtained as follows:

\[ \dot{J}_p \dot{p} + J_p \ddot{p} = \dot{J}_q \dot{q} + J_q \ddot{q} \]  
(22)

Where:

\[
\dot{J}_p = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \dot{D}
\end{bmatrix}
\]

\[
\dot{J}_q = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \dot{E}
\end{bmatrix}
\]

\[
\dot{D} = a_l \beta \cos \beta + l_i (L_o + z_i - l_j) \dot{\beta} \sin \beta - l_i z_i \cos \beta
\]

\[
\dot{E} = l_i \beta \cos \beta - z_i
\]

When there is no singular position of the mechanism, namely \( J_p \) reversible, the acceleration solution of PM can be obtained:

\[ \ddot{p} = J_p^{-1} \left( \dot{J}_q \dot{q} + J_q \ddot{q} - \dot{J}_p \ddot{p} \right) \]  
(23)

5.3 Simulated analysis

In order to further understand the output characteristic of moving platform, the 3D model of PM built by Ug software imported into Adams software.

According to the simulation, the translational velocity \( \dot{x}, \dot{y} \) and angular velocity \( \dot{\beta} \) were obtained as shown in figure 4. As shown in figure 5, translational acceleration \( \ddot{x}, \ddot{y} \) and angular acceleration \( \ddot{\beta} \) were also obtained.
It can be found that the velocity and acceleration curves of the PM change steadily without sharp points and have good dynamic performance.

6. Conclusion
To simplify kinematics analysis of PM, a new 2T1R weakly coupled PM is designed, which can realize the translation along the x and y axes and rotation along the y axes. The weakly coupled design of the PM can reduce the number of simultaneous kinematic branches required for the analysis of the forward and inverse kinematics solutions, and it simplify the complexity of kinematics analysis.

Moreover, kinematics modeling of the PM was carried out to obtain the forward and inverse solutions. The correctness of kinematic analysis of the mechanism was verified by Matlab software. The simulation analysis of velocity and acceleration shows that the PM has good dynamic performance.

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