ROTATION SET AND ENTROPY

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Abstract. In 1991 Llibre and MacKay proved that if \( f \) is a 2-torus homeomorphism isotopic to identity and the rotation set of \( f \) has a non empty interior then \( f \) has positive topological entropy. Here, we give a converselike theorem. We show that the interior of the rotation set of a 2-torus \( C^{1+\alpha} \) diffeomorphism isotopic to identity of positive topological entropy is not empty, under the additional hypotheses that \( f \) is topologically transitive and irreducible.

1. Introduction

1.1. History. The theory of dynamical systems began with Henri Poincaré’s approach to studying toral flows. It consists in passing to the first return map on a topological circle. Hence, the initial requirement is replaced by a qualitative study of dynamical properties of a circle map. Let \( f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be a circle homeomorphism and \( \tilde{f}: \mathbb{R} \to \mathbb{R} \) be a lift of \( f \). The Poincaré’s rotation number of \( f \) is defined as

\[
\rho(f) = \lim_{n \to +\infty} \frac{\tilde{f}^n(x) - x}{n} \quad (\text{mod } 1).
\]

It’s easy to see that this limit exists and depends neither on the point \( x \) in \( \mathbb{R} \) nor on the lift \( \tilde{f} \) of \( f \).

From the definition, the formulas \( \rho(f^n) = n \rho(f) \) and \( \rho(h \circ f \circ h^{-1}) = \rho(f) \) hold for any orientation preserving circle homeomorphism \( h \). If \( h \) is orientation reversing then \( \rho(h \circ f \circ h^{-1}) = -\rho(f) \).

The rotation number gives rise to a description of the dynamical behavior of circle homeomorphisms. Poincaré proved that:

**Poincaré’s Theorem.** Let \( f \) be an orientation preserving circle homeomorphism with rotation number \( \rho \). Then

1. the rotation number \( \rho \) is rational if and only if \( f \) has a periodic point;
2. if the rotation number \( \rho \) is irrational, then \( f \) is semi-conjugate to \( R_{\rho} \), the rotation by \( \rho \), that is there exists a continuous degree one monotone circle map \( h \) such that \( h \circ f = R_{\rho} \circ h \).

The most natural generalization of circle homeomorphisms are 2-torus homeomorphisms isotopic to identity.

Let \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \) be the 2-torus and \( \Pi: \mathbb{R}^2 \to \mathbb{T}^2 \) be the natural projection. Let \( f: \mathbb{T}^2 \to \mathbb{T}^2 \) be a continuous map and \( \tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2 \) a lift of \( f \), that is, \( f \circ \Pi = \Pi \circ \tilde{f} \). If \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are two lifts of \( f \), it holds that there exists \( v \in \mathbb{Z}^2 \) such that \( \tilde{f}_1(\vec{x}) = \tilde{f}_2(\vec{x}) + v \) for every
\( \tilde{x} \in \mathbb{R}^2 \), and if \( f \) is isotopic to identity then for every \( v \in \mathbb{Z}^2 \) one has \( \tilde{f}(\tilde{x} + v) = \tilde{f}(\tilde{x}) + v \).

In order to generalize the rotation number, we can consider the sequences of 2-vectors \( \{ \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} \}_{n \in \mathbb{N}} \). But these sequences may not converge and even in the case that the limit exists, it may depend on the point \( \tilde{x} \). To avoid that difficulty, Misiurewicz and Ziemian ([11]) have proposed to define a rotation set as follows.

### 1.2. Definitions of the rotation set and the rotation vectors.

Let \( f \) be a 2-torus homeomorphism isotopic to identity and \( \tilde{f} \) a lift of \( f \) to \( \mathbb{R}^2 \). We call \( \tilde{f} \)-rotation set the subset of \( \mathbb{R}^2 \) defined by

\[
\rho(\tilde{f}) = \bigcap_{i=1}^{\infty} \bigcup_{n \geq i} \left\{ \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n}, \, \tilde{x} \in \mathbb{R}^2 \right\}.
\]

Equivalently, \((a, b) \in \rho(\tilde{f})\) if and only if there exist sequences \((\tilde{x}_i)\) with \( \tilde{x}_i \in \mathbb{R}^2 \) and \( n_i \to \infty \) such that

\[
(a, b) = \lim_{i \to \infty} \frac{\tilde{f}^{n_i}(\tilde{x}_i) - \tilde{x}_i}{n_i}.
\]

Let \( \tilde{x} \) be in \( \mathbb{R}^2 \), the \( \tilde{f} \)-rotation vector of \( \tilde{x} \) is the 2-vector defined by \( \rho(\tilde{f}, \tilde{x}) = \lim_{n \to \infty} \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} \in \mathbb{R}^2 \) if this limit exists.

### 1.3. Some classical properties and results on the rotation set.

Let \( f \) be a 2-torus homeomorphism isotopic to identity and \( \tilde{f} \) be a lift of \( f \) to \( \mathbb{R}^2 \).

- Let \( \tilde{x} \in \mathbb{R}^2 \) such that \( \rho(\tilde{f}, \tilde{x}) \) exists, it holds that \( \rho(\tilde{f}, \tilde{x}) \in \rho(\tilde{f}) \).
- If \( \tilde{f} \) has a fixed point then \((0,0) \in \rho(\tilde{f}) \).
- Misiurewicz and Ziemian (see [11]) have proved that:
  1. \( \rho(\tilde{f}^n) = n \rho(\tilde{f}) \)
  2. \( \rho(\tilde{f} + (p, q)) = \rho(\tilde{f}) + (p, q) \)
  3. the rotation set is a compact convex subset of \( \mathbb{R}^2 \).
- Franks (see [2]) proved that any rational point \( q \) in \( \text{int} \rho(\tilde{f}) \) is the rotation vector of a lift of a \( f \)-periodic point. That is there exists a \( f \)-periodic point \( x \in T^2 \) and a lift \( \tilde{x} \) of \( x \), such that \( \lim_{n \to \infty} \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} = q \).
- The rotation set is not a conjugacy invariant. However, if the conjugating homeomorphism \( h \) is isotopic to identity, then the homeomorphism \( \tilde{f} \) and its conjugate homeomorphism \( \tilde{h} \circ \tilde{f} \circ \tilde{h}^{-1} \) have the same rotation set. Anyway, the property of having a lift with a rotation set of non empty interior does not depend on the choice of the lift and it is a conjugacy invariant.
1.4. **Relationship between the rotation set and the entropy.** An important conjugacy invariant is the **topological entropy**, it can be defined for \( f : X \to X \) as
\[
h_{\text{top}}(f) = \lim_{\epsilon \to 0} h_{\text{top}}(f, \epsilon), \quad \text{where} \quad h_{\text{top}}(f, \epsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log S(f, \epsilon, n)
\]
and \( S(f, \epsilon, n) \) is the cardinality of a minimal \((n, \epsilon)\) spanning set (i.e a set \( E \) such that \( X = \bigcup_{x \in E} B_f(x, \epsilon, n) \)), where \( B_f(x, \epsilon, n) \) are dynamical balls).

A result of Katok (\cite{Katok}) claims that for \( C^{1+\alpha} \) surface diffeomorphisms the topological entropy is upper bounded by the growth rate of periodic points. Therefore, any \( C^{1+\alpha} \) surface diffeomorphism without periodic points has null topological entropy.

In \cite{lilibr00}, Llibre and MacKay proved that any toral homeomorphism, isotopic to the identity and such that the interior of its rotation set is not empty, has positive topological entropy.

1.5. **Remarks, questions and statement.** The converse of this result by Llibre and MacKay does not hold. We will show examples where the rotation set has empty interior but the topological entropy is positive (see Section 5 -examples \cite{Kwapisz95}, \cite{Kwapisz96}, \cite{Kwapisz94} and \cite{Kwapisz95}). So, we are interested in conditions implying that the interior of the rotation set is not empty. In his thesis Kwapisz (\cite{Kwapisz95}) proved that any Pseudo-Anosov homeomorphism relative to a finite set (in the sense of Handel) has rotation set with non empty interior. Our aim is to give dynamical conditions (in addition to positive entropy) to obtain the same conclusion. So, we introduce the following definitions:

**Definitions.**

- A homeomorphism \( f \) on \( M \) is **topologically transitive** if there exists a point \( x_0 \) of \( M \) such that the \( f \)-orbit of \( x_0 \) is dense.
- A homeomorphism \( f \) on \( M \) is **totally transitive** if any iterate of \( f \), \( f^n \), is topologically transitive.

Let \( M \) be a manifold and \( f \) a homeomorphism of \( M \), it is denoted by \((M, f)\).

- A subset \( K \) of \( M \) is **essential** if \( K \) is not contained in a disk and there exists a finite covering \( M_N \) of \( M \) such that \( M_N \setminus P^{-1}(K) \) is not connected, where \( P : M_N \to M \) stands for the natural projection.
- A homeomorphism \((M, f)\) is **irreducible** if there is no compact, \( f \)-invariant, of empty interior set which is essential.

A non null homotopic circle is essential. In section 2, it will be showed that if \( f \) admits a non null homotopic periodic circle then \( f \) is not irreducible.
Our main result is the following:

**Theorem 1.** Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a diffeomorphism isotopic to identity satisfying the following conditions:

1. $f$ is of class $C^{1+\alpha}$;
2. the topological entropy of $f$ is positive;
3. $f$ is topologically transitive;
4. $f$ is irreducible;

then $\text{int}(\rho(\tilde{f})) \neq \emptyset$, where $\tilde{f}$ is a lift of $f$ to $\mathbb{R}^2$.

**Remark 1.1.** The conditions $\mathbf{1}$ and $\mathbf{2}$ will be used for proving that there exists $n$ such that $f^n$ admits a horseshoe. The conditions $\mathbf{3}$ and $\mathbf{4}$ will be used for proving that there exists a suitable finite covering of $(\mathbb{T}^2, f)$ that is totally transitive (it will be proved in lemma $\mathbf{4.1}$). Finally, according to our proof, a torus homeomorphism isotopic to identity, which admits a horseshoe and admits a suitable finite covering that is totally transitive, has a rotation set with non-empty interior.

Any lift to a finite covering of a Pseudo-Anosov map is also Pseudo-Anosov so it is totally transitive and admits horseshoe. As a consequence we get, as it has been already proved by Kwapisz, that $\text{int}(\rho(\tilde{f})) \neq \emptyset$.

In Section 2 we give some properties related with irreducibility and explain how it arises in our context. Roughly speaking, according to a result in $\mathbf{10}$, the existence of non null homotopic $f$-invariant circle implies that the rotation set of $f$ has empty interior. Hence, we have to avoid this case, but not only, as other invariant sets can play a similar role (the pseudo-circles that arise in Anosov-Katok construction (see $\mathbf{4}$), for example). We prove the main result in Section 3 using the total transitivity of $f$ and some finite covering, which is proved in Section 4. In Section 5 we exhibit different examples showing that the hypotheses are necessary.

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### 2. Irreducibility and Invariant Circles

#### 2.1. Properties related with irreducibility

**Proposition 2.1.** Let $f$ be a 2-torus homeomorphism then

1. If $f$ admits a non null homotopic periodic circle $C$ then $f$ is not irreducible
2. If $f$ is an irreducible homeomorphism then any iterate of $f$ is irreducible.
3. If $f$ is an irreducible homeomorphism then any finite covering of $f$ is irreducible.

**Proof.**

1. If $f$ admits a non null homotopic periodic circle $C$ then the orbit of $C$ is a closed invariant subset $O_C$ and it has empty interior. In the double covering of $M$ associated to $C$, the union of the lifts of $C$ and therefore the union of the lifts of $O_C$ is a disconnecting set.
(2) By contradiction, suppose that there is an iterate \( f^n \) of \( f \) that is not irreducible. Then there exist a finite covering \( M_N \) of the torus and a compact set \( K \) of empty interior that is \( f^n \)-invariant, such that \( M_N \setminus P^{-1}(K) = U_1 \cup U_2 \), where \( P : M_N \to \mathbb{T}^2 \) is the natural projection and \( U_1 \) and \( U_2 \) are disjoint non empty open sets.

Let \( K' = \bigcup_{k=0}^{n-1} f^k(K) \). This set is compact, of empty interior and \( f \)-invariant. The set \( P^{-1}(K') \) contains \( P^{-1}(K) \), we write \( P^{-1}(K') = P^{-1}(K) \cup L \), where \( L \) is a compact of empty interior set. Then \( M_N \setminus P^{-1}(K') = M_N \setminus (P^{-1}(K) \cup L) = (U_1 \cup U_2) \setminus L = (U_1 \setminus L) \cup (U_2 \setminus L) \). Since \( L \) is a closed set of empty interior, \( M_N \setminus P^{-1}(K') \) is the disjoint union of non empty open sets. This fact contradicts the irreducibility of \( f \).

(3) Suppose that there exist \( (M_0, f) \) irreducible and \( (M_1, f_1) \) a finite lift of \( (M_0, f) \) that is not irreducible. So there exists a compact set \( K_1 \) that is \( f_1 \)-invariant, of empty interior and essential. Let \( \pi_0 : M_1 \to M_0 \) be the natural projection. We claim that the set \( K_0 = \pi_0(K_1) \) is compact, \( f \)-invariant, of empty interior and essential, proving that \( (M_0, f) \) is not irreducible.

We first prove that the set \( K_0 \) is \( f \)-invariant.

Since \( \pi_0 \circ f_1 = f \circ \pi_0 \), we have \( \pi_0 \circ f_1(K_1) = f \circ \pi_0(K_1) \). Thus \( \pi_0(K_1) \supseteq f(\pi_0(K_1)) \) that is \( K_0 \supseteq f(K_0) \).

The set \( K_0 \) is clearly compact and of empty interior because \( \pi_0 \) is a local homeomorphism.

It remains to prove that it is essential.

Denote by \( M \) the finite covering of \( (M_1, f_1) \) such that \( M \setminus \pi_1^{-1}(K_1) \) is not connected, where \( \pi_1 : M \to M_1 \) is the natural projection. So \( M \setminus \pi_1^{-1}(K_1) \) can be written as the union of two disjoint open sets \( A \) and \( B \). Denote by \( \pi : M \to M_0 \) the natural projection and \( D = \pi^{-1}(K_0) \). We have that \( D = \bigcup_{i=1}^{n} \pi_1^{-1}(\gamma_i(K_1)) \) where \( \{\gamma_i\} \) stands for the finite group consisting in the automorphisms of the covering \( \pi_0 \). It holds that \( D \) is closed and has empty interior.

So \( M \setminus D = (A \cup B) \setminus D \) since \( \pi_1^{-1}(K_1) \subset D \). Consequently, \( M \setminus D = (A \setminus D) \cup (B \setminus D) \) where \( A \setminus D \) and \( B \setminus D \) are non empty open sets. Thus \( M \setminus D \) is not connected, it remains to prove that \( K_0 \) is not contained in a disk. If \( K_0 \) is included in a disk \( D_0 \) then \( K_1 \) is contained in a finite union of disjoint disks, then we can construct a disk \( D_1 \) that contains this union and therefore \( K_1 \subset D_1 \), this contradicts the irreducibility of \( f \).

\[ \square \]

2.2. Invariant circles. Given a homeomorphism \( f \) isotopic to the identity on \( \mathbb{T}^2 \), we are interested in relationships between the existence periodic circles and the interior of the rotation set.

In the case where \( f \) admits a homotopically non-trivial invariant single curve, Llibre-Mac Kay (see [10]) proved that all the rotation vectors of \( f \)-periodic points are collinear, therefore the rotation set has empty interior. In the case where \( f \) admits a homotopically non-trivial periodic single curve, the rotation set has empty interior since \( \rho(f^n) = n \rho(f) \).

On the other hand, is there a relationship between the existence of homotopically trivial \( f \)-invariant single curve and the interior of the rotation set of \( f \)?
Let us show two examples:

- The identity map on $T^2$ fixes every circle and its rotation set is $\{(0,0)\}$.
- In [10], the authors give examples of $T^2$-homeomorphisms having rotation set of non empty interior. Let us consider the particular example $f$ given by one of its lifts $\tilde{f}$ to $\mathbb{R}^2$. Let $\text{frac}(x) = x - \lfloor x \rfloor$ be the fractional part of $x$ (where $\lfloor x \rfloor$ is the floor function of $x$). We define $h, g : \mathbb{R}^2 \to \mathbb{R}^2$ as $h(x, y) = (x, y + \text{frac}(2x))$ if $\text{frac}(x) \in [0, \frac{1}{2})$ and $h(x, y) = (x, y + 2 - \text{frac}(2x))$ if $\text{frac}(x) \in \left[\frac{1}{2}, 1\right)$. Analogously $g(x, y) = (x + \text{frac}(2y), y)$ if $\text{frac}(y) \in [0, \frac{1}{2}]$ and $g(x, y) = (x + 2 - \text{frac}(2y), y)$ if $\text{frac}(y) \in \left[\frac{1}{2}, 1\right)$. Let us define $\tilde{f} = g \circ h$. It holds that its rotation set is $[0, 1]^2$. Actually, the point $(0, 0)$ is fixed by $\tilde{f}$, its rotation vector is $(0,0)$, also $\tilde{f}(\frac{1}{2}, 0) = (\frac{1}{2}, 0) + (0, 1)$, $\tilde{f}(0, \frac{1}{2}) = (0, \frac{1}{2}) + (1, 0)$ and $\tilde{f}(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) + (1, 1)$. Then $\rho(\tilde{f}, (0, 1)) = (0, 1)$, $\rho(\tilde{f}, (0, \frac{1}{2})) = (1, 0)$, and $\rho(\tilde{f}, (\frac{1}{2}, \frac{1}{2})) = (1, 1)$.

We are going to modify this example in order that $f$ have an invariant homotopically trivial curve and that $\tilde{f}$ have still the same rotation set. Let us explain it. The point $(0,0)$ is fixed by $f$, we replace it by a small disk $D$ by blowing up. This construction does not change the rotation set because of the following facts:

- the points in $D$ have the same rotation vector than $(0,0)$ ($D$ is $f$-invariant)
- the three other vertices of the rotation set are unchanged since they are realized by points for which the blow up did not change the orbits.

On the other hand, it holds that $\partial D$ is a homotopically trivial single curve which is invariant by this perturbation of $f$.

Remark 2.1. Since, there is no relationship between the existence of homotopically trivial $f$-invariant single curves and the interior of the rotation set of $\tilde{f}$, we ask for $K$ not to be contained in a disk, in the definition of an essential set.

Remark 2.2. There exist compact sets that are $f$-invariant, with empty interior and essential but that are not circles (they are not even locally connected [4] and they are called pseudo-circles). It is possible to change an invariant non null-homotopic circle by a connected invariant set that disconnects the torus and that is not locally connected. That is the reason why –in the definition of irreducibility– we ask for the non existence of a compact, $f$-invariant, of empty interior set (instead of a circle) which is essential.

Remark 2.3. Other notions related to essential sets for surface homeomorphisms were proposed recently by T. Jäger (called “circloids” in [5]) and by A. Koropecki (called “annular sets” in [9]).

3. Proof of Theorem 1

In this section we prove Theorem 1

Proof. Denote by $T^2_* = \mathbb{R}^2/(2\mathbb{Z})^2$ the 4-1 covering of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Let $F : T^2_* \to T^2$ be a lift of $f$ to $T^2_*$. Let the natural projections be $\Pi : \mathbb{R}^2 \to T^2$, $\Pi_* : \mathbb{R}^2 \to T^2_*$ and $P : T^2_* \to T^2$. Note that $\Pi = P \circ \Pi_*$. Let us endow $T^2$ and $T^2_*$ with their usual flat Riemannian metrics (inherited from the standard Euclidean metric on $\mathbb{R}^2$) and the associated distances.
By hypotheses, \( f \) is a \( C^{1+\alpha} \) diffeomorphism and \( h_{\text{top}}(f) > 0 \) therefore \( F \) is also a \( C^{1+\alpha} \) diffeomorphism and \( h_{\text{top}}(F) > 0 \).

By Katok (see [6]), there exist \( n \in \mathbb{N} \) and a hyperbolic periodic point \( y_0 \) of \( F \) with period \( n \) such that the intersection of the stable and unstable manifolds of \( y_0 \) is transversal. Thus, there exists \( k \in \mathbb{N} \) the minimal positive number such that \( F^{nk}(y_0) = y_0 \) and both eigenvalues of the differential \( DF^{nk}(y_0) \) are positive. In what follows, we denote \( F^{nk} \) by \( f^k \).

Let us denote \( x_0 = P(y_0) \). Since \( P \) is a local diffeomorphism, \( x_0 \) is a hyperbolic fixed point of \( f^{nk} \) and it has the local type of \( y_0 \). Let \( \tilde{x}_0 \in [0,1] \times [0,1] \) be a lift of \( x_0 \) and \( \tilde{f}_* \) be a lift of \( f^{nk} \) to \( \mathbb{R}^2 \) such that \( \tilde{f}_*(\tilde{x}_0) = \tilde{x}_0 \). Note that \( \tilde{f}_* \) is a lift of both \( f^{nk} \) and \( f_* \).

We define \( \tilde{\varepsilon}_1 = \tilde{x}_0, \tilde{\varepsilon}_2 = \tilde{x}_0 + (1,0), \tilde{\varepsilon}_3 = \tilde{x}_0 + (0,1) \) and \( \tilde{\varepsilon}_4 = \tilde{x}_0 + (1,1) \). Since \( \tilde{f}_* \) is isotopic to identity, then \( \tilde{f}_*(\tilde{x}_0 + (a,b)) = \tilde{f}_*(\tilde{x}_0) + (a,b) = \tilde{x}_0 + (a,b) \), for any \( (a,b) \in \mathbb{Z}^2 \). Then every \( \tilde{\varepsilon}_i \) is a fixed point of \( \tilde{f}_* \) and therefore its projection on \( \mathbb{T}_*^2 \) denoted by \( x_i \) is fixed by \( f_* \). Note that there exists \( i \in \{1,2,3,4\} \) such that \( y_0 = x_i \).

**Proposition 1.** There exists \( 0 < \varepsilon < \frac{1}{2} \) such that for \( i \in \{2,3,4\} \) there exist \( n_i \in \mathbb{N} \) and non empty compact sets \( L_i \subset B_{\varepsilon}(x_i) \subset \mathbb{T}_*^2 \) and \( L_1 \subset B_{\varepsilon}(x_1) \) such that \( L_i = f_*^{n_i}(L_1) \) and \( P(L_1) = P(L_i) \).

**Proof of Proposition 1.** By the classical Hartman-Grobman’s theorem, there is an open subset \( U \) of \( \mathbb{T}^2 \) containing \( x_0 \) such that the restriction of \( f^{nk} \) to \( U \) is topologically conjugated to its differential \( Df^{nk}(x_0) \). By conjugating \( f^{nk} \) by a suitable homeomorphism with support in a small compact \( K \supset U \), we may suppose that \( f^{nk} \) is a linear diagonal map in \( U \) with eigenvalues \( 0 < \lambda_1 < 1 < \lambda_2 \).

Fix \( \varepsilon > 0 \) sufficiently small so that the ball \( B_\varepsilon(x_0) \subset U \) and the lifts by \( P \) of it are disjoints \( \varepsilon \)-balls \( B_\varepsilon(x_i) \subset \mathbb{T}_*^2 \), \( i = 1,\ldots,4 \) (this fact is realized by taking \( \varepsilon < \frac{1}{2} \)). Restricted to these balls, \( f_* \) is a linear map.

Let \( x \) be in \( \mathbb{T}_*^2 \), we denote by \( W^s(x) \) [resp. \( W^u(x) \)] the stable [resp. unstable] manifold of \( x \) for \( f_* \). For any \( 0 < \delta \leq \varepsilon \) and \( x \in \mathbb{T}_*^2 \), let’s denote by \( W^s_\delta(x) \) [resp. \( W^u_\delta(x) \)] the connected component of \( W^s(x) \cap B_\delta(x) \) [resp. \( W^u(x) \cap B_\delta(x) \)] containing \( x \).

Since \( W^s(x_1) \) and \( W^u(x_1) \) have a transverse intersection in some point \( p \), there is:

- \( N \in \mathbb{N} \) such that for \( k \geq N \), \( f_*^k(p) \in W^s(x_1) \) (these points converge monotonically to \( x_1 \) when \( k \) goes to \( +\infty \)) and
- \( M \in \mathbb{N} \) such that for \( k \geq M \), \( f_*^{-k}(p) \in W^u(x_1) \) (these points converge monotonically to \( x_1 \) when \( k \) goes to \( -\infty \)).

Consider a small arc \( \Sigma^u \) of \( W^u(x_1) \) containing \( f_*^N(p) \), the segments \( f_*^k(\Sigma^u) \) for \( k \geq N \) become larger and more vertical with \( k \), so there is \( r \geq N \) minimal such that \( d(x_1, f_*^r(p)) \leq \frac{\varepsilon}{2} \) and the arc \( f_*^{r}(\Sigma^u) \) intersects transversally the boundary of \( B_\varepsilon(x_1) \) in two points.

Analogously, consider an arc \( \Sigma^s \) of \( W^s(x_1) \) containing \( f_*^{-M}(p) \), there is \( r' \geq M \) minimal such that \( d(x_1, f_*^{-r'}(p)) \leq \frac{\varepsilon}{2} \) and the arc \( f_*^{-r'}(\Sigma^s) \) (almost horizontal) intersects transversally the boundary of \( B_\varepsilon(x_1) \) in two points.

The arcs \( f_*^r(\Sigma^u) \cap B_\varepsilon(x_1) \) and \( f_*^{-r'}(\Sigma^s) \cap B_\varepsilon(x_1) \) intersect transversally.
Finally, we define a rectangle $R_1$ in $\mathbb{T}_s^2$ whose boundary is the union of arcs $C_{1,s}^j$ for $j = 1,2$ included in $W^s(x_1)$ and arcs $C_{1,u}^j$ for $j = 1,2$ included in $W^u(x_1)$. In fact $x_1$ is a corner of $R_1$ and it is the intersection of the sides $C_{1,s}^1$ and $C_{1,u}^1$ which are included in $W^s_t(x_1)$ and $W^u_s(x_1)$ respectively, the two other sides are $C_{1,s}^2 \subset f^m_s(\Sigma^s) \cap B_s(x_1)$ and $C_{1,u}^2 \subset f^{-m'}_s(\Sigma^u) \cap B_s(x_1)$. By definition, the diameter of $R_1$ is less than $\epsilon$.

Let $R_0 = P(R_1)$ be a rectangle in $\mathbb{T}^2$. For $i = 1 \ldots 4$, denote by $\gamma_i$ the automorphism of the finite covering $P$ such that $\gamma_i(x_1) = x_i$, and let $R_i = \gamma_i(R_1)$ be a rectangle in $\mathbb{T}_s^2$ and for $j = 1,2$ we set $C_{i,u}^j = \gamma_i(C_{1,u}^j)$, $C_{i,s}^j = \gamma_i(C_{1,s}^j)$ (see Figure 1).

Figure 1. Stable and unstable sides of $R_i$.

Fix $i \in \{2,3,4\}$, as $f$ is topologically transitive and irreducible then Corollary 4.1 implies that $f^i_s$ is topologically transitive so there exists $m_i \in \mathbb{N}$ such that $f^{m_i}_s(R_1) \cap \text{int}(R_i) \neq \emptyset$.

It is not possible that $f^{m_i}_s(R_1) \supset R_i$. In fact, projecting via $P$ on $\mathbb{T}^2$, we obtain that $f^{nk m_i}_n(R_0) \supset R_0$. According to the Hartman-Grobman Theorem, the map $f^{nk}$ has a
unique fixed point in $R_0$ which is the hyperbolic saddle point $x_0$. Hence, it is not possible that $f^{nkm}(R_0) \supset R_0$.

Furthermore, it is not possible that $x_i$ belongs to $f^m_i(R_1)$ since $x_i$ is a $f_i$-fixed point and $R_i$ and $R_1$ are disjoint.

As $W^{u}(x_1) \cap W^{u}(x_i) = \emptyset$ for $i \neq 1$ we have that there exists $l_i \geq m_i$ such that $f^{l_i}(C^1_{i,u}) \cap (C^1_{i,s} \cup C^2_{i,s}) \neq \emptyset$ and this intersection is topologically transversal.

There is no loss of generality if we suppose that $f^{l_i}(C^1_{i,u}) \cap C^1_{i,s} \neq \emptyset$, i.e $W^{u}(x_1) \cap W^{s}_{\epsilon/2}(x_i) \neq \emptyset$.

Since $W^{u}(x_1)$ is topologically transversal to $W^{s}_{\epsilon/2}(x_i)$ we can assert that there exists $N_i$ such that at least one connected component of $f^{m_i}(C^1_{1,u}) \cap R_i$ has one end point in $C^1_{i,s}$ and another one in $C^2_{i,s}$ for all $m \geq N_i$.

Let us define $B_i$ as a small subrectangle in $R_i$ whose boundary contains $C^1_{1,u}$ and stable and unstable arcs. We take the stable sides of $B_1$ contained in the stable arcs of the boundary of $R_1$. We choose the other unstable side of $B_1$, $L_u$ close enough to $C^1_{1,u}$ so that a connected component of $f^{m_i}(L_u) \cap R_i$ has one end point in $C^1_{i,s}$ and another one in $C^2_{i,s}$ for all $m \geq N_i$. Hence, one connected component of $f^{m_i}(B_1) \cap R_i$ is a compact set with nonempty intersection with $C^1_{i,s}$ and with $C^2_{i,s}$, for $m \geq N_i$. Let $B_i \subset R_i$ verifying $P(B_i) = P(B_1)$.

There exists $n_i \geq N_i$ such that for all $m \geq n_i$ one connected component denoted by $D_i$ of $f^{m_i}(B_1) \cap B_i$ is a compact set included in $B_i$ with nonempty intersection with $C^1_{i,s}$ and with $C^2_{i,s}$; and with empty intersection with the unstable sides of $B_i$. One can show that the set $f^{m_i}(D_i)$ is connected, compact and contained in $B_1$. It intersects the unstable sides of $B_1$ and it does not intersect the stable sides of $B_1$ (see Fig 2).

It follows that $\bigcap_{j=-N}^{N} f^{njm}(P(D_i))$ has the finite intersection property. Consequently, the compact set (depending on $i$ and $m$) in $\mathbb{T}^2$ defined by:

$$L = \bigcap_{j=-\infty}^{\infty} f^{njm}(P(D_i))$$

is non empty, $f^{nmk}$-invariant and it’s contained in $R_0 = P(R_i) \subset \mathbb{T}^2$.

In what follows, we argue for $m = n_i$. For $j = 1, \ldots, 4$, let $L^i_j = P^{-1}(L) \cap R_j$. The set $\bigcup_{j=1}^{4} L^i_j$ is $f^{n_i}_*$-invariant. Moreover, since $f_*$ is surjective and $L^i_1 \subset f^{n_i}_*(D_i)$, we have that $f^{n_i}_*(L^i_1) = L^i_1$. Therefore, we have proved that there exist an integer $n_i$ and compact sets $L^i_1$ and $L_i := L^i_1$ such that $P(L_i) = P(L^i_1)$ and $f^{n_i}_*(L^i_1) = L_i$. We get the proposition 1.

**Proof of the theorem.** We prove that the proposition 1 implies the theorem.

Since $\bar{x}_0$ is a fixed point of $f_*$ it follows that $\rho(\bar{f}_*, \bar{x}_0) = (0, 0)$. 


By proposition 1 for \( i = 2 \), there exist \( n_2 \) and non empty compact sets \( L_2 \subset R_2 \) and \( L_1 \subset R_1 \) such that \( P(L_2) = P(L_1) \) and \( f^{n_2}(L_1) = L_2 \).

Let us denote \( L_2 \) [resp. \( L_1 \)] a lift of \( L_2 \) [resp. \( L_1 \)] to \( \mathbb{R}^2 \). Then there exist \( k_2 \in (2\mathbb{Z})^2 \) such that \( \tilde{f}^{n_2}(L_1) = L_2 + k_2 \). Since \( P(L_2) = P(L_1) \) and \( L_2 \subset B_{x_2}(\varepsilon) \), we have that necessarily \( L_2 = L_1^2 + (1, 0) \). Therefore

\[
\tilde{f}^{n_2}(L_1^2) = L_1^2 + k_2 + (1, 0).
\]

Hence, \( \tilde{f}^{n_2}(L_1^2 + k_2 + (1, 0)) = \tilde{f}^{n_2}(L_1^2) + (k_2 + (1, 0)) = L_1^2 + 2(k_2 + (1, 0)) \) and for every \( k \in \mathbb{N} \)

\[
\tilde{f}^{kn_2}(L_1^2) = L_1^2 + k(k_2 + (1, 0)).
\]

Let \( \bar{x} \in L_1^2 \). For every \( k \), there exists \( \bar{y}_k \in \mathcal{L}_1^2 \) such that \( \tilde{f}^{kn_2}(\bar{x}) = \bar{y}_k + k(k_2 + (1, 0)) \).

It follows that

\[
\frac{\tilde{f}^{kn_2}(\bar{x}) - \bar{x}}{kn_2} = \frac{\bar{y}_k - \bar{x}}{kn_2} + \frac{k(k_2 + (1, 0))}{kn_2}
\]

Then

\[
\lim_{k \to \infty} \frac{\tilde{f}^{kn_2}(\bar{x}) - \bar{x}}{kn_2} = \frac{k_2 + (1, 0)}{n_2}.
\]
Hence,
\[ \frac{k_2 + (1, 0)}{n_2} \in \rho(\tilde{f}_\ast). \]

Analogously, for \( i = 3 \) there exist integers \( n_3, k_3 \) and a compact set \( L_1^i \) in \( \mathbb{R}^2 \) such that
\[ \tilde{f}_\ast^{n_3}(L_1^i) = L_1^i + k_3 + (0, 1). \]
It comes that
\[ \frac{k_3 + (0, 1)}{n_3} \in \rho(\tilde{f}_\ast). \]
Finally, \( (0, 0) \in \rho(\tilde{f}_\ast) \) and the vectors \( \frac{k_2 + (1, 0)}{n_2} \) and \( \frac{k_3 + (0, 1)}{n_3} \) are linearly independent.
Actually, for \( i = 2, 3 \) let us write \( k_i = (2p_i, 2q_i) \) and compute the determinant:
\[
\det \left( \frac{n_2}{n_2} k_2 + \frac{(1, 0)}{n_2}, \frac{n_3}{n_3} k_3 + \frac{(0, 1)}{n_3} \right) = \begin{vmatrix} 2p_2 + 1 & 2p_3 \\ 2q_2 & 2q_3 + 1 \end{vmatrix} \neq 0,
\]
since it is the difference between an even number and an odd number.
Then, it follows that \( \rho(\tilde{f}_\ast) \) has 3 non colinear points. By convexity (see \[12\]) of \( \rho(\tilde{f}_\ast) \), we have that \( \text{int}(\rho(\tilde{f}_\ast)) \neq \emptyset \), for a lift \( \tilde{f}_\ast \) of \( f^{nk} \) to \( \mathbb{R}^2 \). Thus, this property holds for any lift of \( f^{nk} \) and therefore for any lift of \( f \) to \( \mathbb{R}^2 \). \( \square \)

4. PROOF OF THE TOTAL TRANSITIVITY.

Let \( T^2_h := \mathbb{R}^2/(2\mathbb{Z} \times \mathbb{Z}) \) (resp. \( T^2_v := \mathbb{R}^2/(\mathbb{Z} \times 2\mathbb{Z}) \)) be a 2-1 covering of \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \), and let the natural projection be \( \Pi_h : T^2_h \to T^2 \) (resp. \( \Pi_v : T^2_v \to T^2 \)). Let \( f_h : T^2_h \to T^2_h \) be the lifting of \( f \) to \( T^2_h \) (resp. \( f_v : T^2_v \to T^2_v \) be the lifting of \( f \) to \( T^2_v \)).

**Lemma 4.1.** Let \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) be a torus homeomorphism satisfying that \( f \) is topologically transitive and irreducible then
1. the 2-1 coverings \( f_h \) and \( f_v \) are topologically transitive;
2. \( f \) is totally transitive.

**Corollary 4.1.** Let \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) be a torus homeomorphism topologically transitive and irreducible. Let \( F \) be the lift of \( f \) defined in the proof of the theorem. Then \( F \) is totally transitive.

**Proof of the Corollary.** By definition, \( F = (f_h)_v = (f_v)_h \). According to the previous lemma \( f_h \) is topologically transitive and by the proposition \[2.1\](3) it is irreducible, so we can apply once again this lemma to \( f_h \) and obtain that \((f_h)_v \) is totally transitive. \( \square \)

**Proof of lemma \[4.1\].** *(1)* We will argue by absurd for \( f_h \). We suppose that \( f \) is transitive but not \( f_h \).
Since \( f \) is transitive, there exists \( x_0 \) such that \( O_f(x_0) := \{f^n(x_0) : n \in \mathbb{Z} \} \) is a dense set in \( \mathbb{T}^2 \). Let \( \{x_1, x_2\} \) be the lifts of \( x_0 \) by \( \pi_h^{-1} \). Let \( O(x_i) \) be the \( f_h \)-orbit of \( x_i \). We have that \( O(x_1) \cup O(x_2) = \pi_h^{-1}(O_f(x_0)) \) is a dense set in \( T^2_h \) since \( \pi_h \) is a local homeomorphism. Since \( f_h \) is not transitive, neither \( O(x_1) \) nor \( O(x_2) \) is dense. We claim that \( \text{int}(\overline{O(x_2)}) \cap \overline{O(x_1)} = \emptyset \). Actually if there were a point \( y \) in the intersection, it would exist \( m \in \mathbb{Z} \) such that \( f_h^m(x_1) \) belongs to \( \overline{O(x_2)} \) then \( O(x_1) \subseteq \overline{O(x_2)} \). Thus
\[ \overline{O(x_1)} \subset \overline{O(x_2)} \] so we have \( T^2_h = \overline{O(x_1)} \cup \overline{O(x_2)} = \overline{O(x_2)} \): a contradiction.

Analogously, the symmetric holds and these two equalities imply that \( \partial \overline{O(x_1)} = \partial \overline{O(x_2)} \) and \( T^2_h = \text{int}(\overline{O(x_1)}) \cup \text{int}(\overline{O(x_2)}) \cup \partial \overline{O(x_1)} \), where \( \cup \) denotes disjoint union.

The set \( \partial \overline{O(x_1)} \) is a closed invariant of empty interior subset that disconnects \( T^2_h \).

We are going to prove that it can not be contained in a disk \( D \subset T_h \).

Suppose, by absurd that \( \partial \overline{O(x_1)} \subset D \). First, we prove that \( \text{int} \overline{O(x_1)} \) or \( \text{int} \overline{O(x_2)} \) is included in \( D \). In fact, if both of them intersect the complement \( D^C \) of \( D \), we can take a path in \( D^C \) joining a point of \( \text{int} \overline{O(x_1)} \) and a point of \( \text{int} \overline{O(x_2)} \). By connexity, this path must contain a point of the boundary \( \partial \overline{O(x_1)} \), which contradicts the fact that \( \partial \overline{O(x_1)} \subset D \).

Finally, suppose that \( \text{int} \overline{O(x_1)} \subset D \) then \( \text{int} \overline{O(x_2)} \supset D^C \), but this is not possible since \( \text{int} \overline{O(x_1)} \) and \( \text{int} \overline{O(x_2)} \) are homeomorphic.

We have proved that \( \partial \overline{O(x_1)} \) is a closed invariant of empty interior subset that disconnects \( T^2_h \) and that is not contained in a disk. But this is a contradiction with the fact that \( f \) is irreducible.

(2) We suppose that \( f \) is transitive but there exists a positive integer \( N \) such that \( f^N \) is not.

So, there exists \( x_0 \) such that \( O_f(x_0) \) is a dense set in \( \mathbb{T}^2 \), but \( O_{f^N}(x_0) \) is not. We have
\[
\overline{O_{f^N}(x_0)} \cup \overline{O_{f^N}(f(x_0))} \cup ... \cup \overline{O_{f^N}(f^{N-1}(x_0))} = O_f(x_0) = \mathbb{T}^2.
\]

As in the proof of the first item of this lemma we have that if \( \text{int}(\overline{O_{f^N}(f^i(x_0))}) \cap \overline{O_{f^N}(f^j(x_0))} \neq \emptyset \) then \( \overline{O_{f^N}(f^i(x_0))} \supset \overline{O_{f^N}(f^j(x_0))} \).

Let \( K = \min\{i = 1, ..., N-1 : \text{int}(\overline{O_{f^N}(x_0)}) \cap \overline{O_{f^N}(f^i(x_0))} \neq \emptyset \} \). Then \( \mathbb{T}^2 \) can be decomposed as union of closed sets with disjoint interiors:
\[
\mathbb{T}^2 = \overline{O_{f^N}(x_0)} \cup \overline{O_{f^N}(f(x_0))} \cup ... \cup \overline{O_{f^N}(f^{K-1}(x_0))}.
\]

In the case that \( K \) is an even number then \( \mathbb{T}^2 = A \cup B \), where \( A = \overline{O_{f^N}(x_0)} \cup \overline{O_{f^N}(f(x_0))} \cup ... \cup \overline{O_{f^N}(f^{K}(x_0))} \) and \( B = \overline{O_{f^N}(f^{K+1}(x_0))} \cup ... \cup \overline{O_{f^N}(f^{N-1}(x_0))} \). The sets \( A \) and \( B \) have disjoint interiors, their boundaries coincide and they are \( f^N \)-invariant. Thus \( \mathbb{T}^2 = \text{int}(A) \cup \text{int}(B) \cup \partial B \). According to proposition 2.1 \( f^N \) is irreducible, then \( \partial B \) is contained in a disk. As \( A \) and \( B \) are homeomorphic, an analogous proof to the previous item shows that this can not occur.

In the case that \( K \) is an odd number then \( \mathbb{T}^2 = A \cup B \cup C \), where \( A = \overline{O_{f^N}(x_0)} \cup \overline{O_{f^N}(f(x_0))} \cup ... \cup \overline{O_{f^N}(f^{K+1}(x_0))} \) and \( B = \overline{O_{f^N}(f^{K+2}(x_0))} \cup ... \cup \overline{O_{f^N}(f^{N-2}(x_0))} \) and \( C = \overline{O_{f^N}(f^{N-1}(x_0))} \). The sets \( A \) and \( B \cup C \) have disjoint interiors, their boundaries coincide and they are \( f^N \)-invariant. Thus \( \mathbb{T}^2 = \text{int}(A) \cup \text{int}(B \cup C) \cup \partial A \). According to proposition 2.1 \( f^N \) is irreducible, then \( \partial A \) is contained in a disk \( D \).

By connexity, as in the previous item, we prove that \( A \) or \( B \cup C \) is included in \( D \).

Finally, if \( A \subset D \) then \( B \subset f^{N+1}(D) \) and \( C \subset f^{N-1}(D) \). If \( B \cup C \subset D \) then \( A \subset f^{N+1}(D) \).

Then, in both cases, the torus is the union of at most three disks which is impossible.
So we have proved that $f$ is totally transitive. 

\[\square\]

**Remark 4.1.** The topological transitivity of $f$ is not enough to guarantee that $f_n$ (or $f_v$) is topological transitive. For example, consider a diffeomorphism $f$ of $A = [0, 1] \times S^1$ obtained by the Katok-Anosov process (see [1]) in such a way that:

- $f$ is topologically transitive in $\text{int}(A) = (0, 1) \times S^1$, and
- $f(0, x) = f(1, x)$ for all $x \in S^1$.

We collapse the circles $\{0\} \times S^1$ and $\{1\} \times S^1$ by identifying $(0, x)$ with $(1, x)$ for $x \in S^1$. Then, we have a diffeomorphism of the torus, $\hat{f}$, verifying that $\hat{f}$ is topologically transitive on $T^2$ and $C = \{0\} \times S^1$ is a circle invariant of $\hat{f}$. Let us consider the finite covering $T^2_h$ of the torus $T^2$ and let $\hat{f}_h$ be the lifting of $\hat{f}$ to $T^2_h$. The lifting of $C$ is the union of two circles $C_1$ and $C_2$ that disconnect $T^2_h$, and the $T^2_h \setminus (C_1 \cup C_2)$ is the disjoint union of two cylinders. The orbits of $\hat{f}_h$ are dense in each cylinder but $\hat{f}_h$ is not topologically transitive.

### 5. Examples

In this section, we give examples in order to show that each hypothesis of the theorem is necessary.

1) **Missing hypothesis [2]**

Let $R_{\alpha, \beta}$ be the rotation of vector $(\alpha, \beta)$ with $\alpha$ and $\beta$ irrational, that is, the projection to $T^2$ of the translation of vector $(\alpha, \beta)$ in $R^2$. It is a well known fact that $R_{\alpha, \beta}$ is topologically transitive, it is differentiable and it is irreducible, but its rotation set is $\{(\alpha, \beta)\}$. This example shows that conditions [1] [2] and [4] of the theorem [1] do not ensure that the interior of the rotation set is not empty.

2) **Missing hypothesis [3] and [4]**

Let $f_D: \overline{D^2} \to \overline{D^2}$ be a diffeomorphism such that there exists a horseshoe in the interior of $D^2$ and such that $f_D$ is the identity on $\partial D^2$. It follows that $f_D$ has positive entropy. Let us embed $\overline{D^2}$ in $T^2$ and then extend $f_D$ to $f$ by the identity on $T^2 \setminus \overline{D^2}$. It holds that $f$ has positive entropy and the rotation set has empty interior (because there exist invariant circles homotopically non trivial). This example show that conditions [1] and [2] do not ensure that the conclusion of the theorem is verified.

3) **Missing hypothesis [3]**

We start with an irrational flow $\phi_0$ on $T^2$. By making an appropriate smooth time change vanishing at one point $x_0$ (we replace the vector field $X$ by $g.X$ where $g(x_0) = 0$ and $Dg(x_0) = 0$), we get a new smooth topologically transitive flow $\phi_t^g$ with a fixed point $x_0$. Consider the time one map of this flow, $f$, and replace $x_0$ by a small closed disk $D_0$ by blowing up. The dynamic of the blow up of $f$ on $\partial D_0$ is of the type north-south. We have that $D_0 \setminus \{N, S\}$ is foliated by meridians $\{M_t\}_{t \in [-1, 1]}$ and $\partial D_0 = M_{-1} \cup M_1 \cup \{N, S\}$. Let $\gamma: D_0 \to D_0$ a differentiable map such that $\gamma|M_t = f|M_t$, for $i = -1, 1$, $\gamma(N) = N$, $\gamma(S) = S$, for all $t \in [-1, 1]$ $M_t$ is $\gamma$-invariant and $\gamma|M_t = Id$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$. In the blow up manifold, $T^2$ we define the differentiable map $\Gamma$ as $\Gamma(x) = f(x)$ if $x \in T^2/D_0$ and $\Gamma(x) = \gamma(x)$ if $x \in D_0$. Let $D_1 = \cup_{t \in [-\frac{1}{2}, \frac{1}{2}]} M_t \cup \{N, S\}$, it holds that $\Gamma|D_1 = Id$. 


As in the previous example, we can put a horseshoe in the interior of \( D_1 \). The resulting diffeomorphism satisfies trivially the conditions \( 1, 2 \).

It also verifies the condition \( 3 \). Moreover, an invariant compact set \( K \) of \( \Gamma \) is included either:
- in \( D_0 \), in this case \( K \) is not essential or
- in the complement \( D_0^c \) of \( D_0 \), in this case \( K \) coincides with \( D_0^c \) (since each orbit in \( D_0^c \) is dense in it) hence its interior is not empty.

But it does not satisfy the condition \( 3 \) since it has an invariant disk.

Finally, its rotation set has empty interior. In fact, before the blowing up, the map \( f \) is the time one map of a flow with a fixed point \( x_0 \) so according to Franks and Misiurewicz’s result (see [3]) its rotation set is a line segment containing \((0, 0)\). The blowing up does not change the rotation set because of the following facts:
- the points in \( D_0 \) have the same rotation vector as \( x_0 \) which is \((0, 0)\) (\( D_0 \) is \( \Gamma \)-invariant),
- for the points out of \( D_0 \), the blowing up does not change the orbits so it does not change their rotation vectors.

4) Missing hypothesis \( 1 \)

According to [13] there exists a torus homeomorphism \( f_0 \) isotopic to the identity such that it is minimal and it has positive entropy. Since \( f_0 \) is minimal, all its orbits are dense so it has no periodic points. By [2] we know that if the interior of the rotation set is not empty, then each vector with rational coordinates in the interior of the rotation set is realized as the rotation vector of a periodic point. It follows that \( f_0 \) verifies that \( \text{int}(R(f_0)) = \emptyset \). This example shows that conditions \( 2, 3 \) and \( 4 \) are not enough to guarantee that the interior of the rotation set is not empty.

5) Missing hypothesis \( 4 \)

According to [7] there exists a \( C^\infty \) topologically transitive Bernoulli diffeomorphism \( f_0 : S^2 \to S^2 \) which preserves a smooth positive measure on \( S^2 \). Since \( f_0 \) (or \( f_0^k \)) preserves orientation then it is isotopic to the identity.

As in the construction of [7], there exist \( x_1, x_2 \), two fixed points of \( f_0 \) (or \( f_0^k \)) such that \( Df_0(x_i) = \text{Id}, i = 1, 2 \). We can replace \( x_1 \) and \( x_2 \) by small closed disks \( D_1 \) and \( D_2 \), respectively, by blowing up. The dynamic of the blow up of \( f_0 \) on \( \partial D_1 \) and \( \partial D_2 \) is the identity. By gluing \( \partial D_1 \) and \( \partial D_2 \) we have a smooth map \( f : T^2 \to T^2 \) which is topologically transitive and it has positive entropy but there exists a compact \( f \)-invariant of empty interior set (\( \partial D_1 \)) which is essential. This example fails to be irreducible because of the existence of a non null homotopic invariant circle, then its rotation set has empty interior because of Llibre and MacKay’s result (see [10]).

References

[1] Anosov, D. and Katok, A., New examples in smooth ergodic theory. Ergodic diffeomorphisms., Trans- action of the Moscow Mathematical Society, 1970, vol.23, p.1-35.

[2] Franks, J., Realizing rotation vectors for torus homeomorphisms, Trans. Amer. Math. Soc., 1989, vol. 311, p.107-116.

[3] Franks, J. and Misiurewicz, M., Rotation sets on toral flows, Proc. A.M.S., 1990, vol. 109, p. 243-249.
[4] Herman, M.  *Construction of some curious diffeomorphisms of the Riemann sphere*, J. London Math. Soc., 1986, vol. 134, p. 375-384.

[5] Jäger, T.H., *Linearisation of toral homeomorphisms*, Inventiones Math., 2009, to appear.

[6] Katok, A., *Lyapounov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.É.S., 1980, vol. 51, p. 131-173.

[7] Katok, A., *Bernoulli diffeomorphism on surfaces*, Annals of Math., 1979, vol. 31, p. 529-547.

[8] Kwapisz, J., *Rotation set and entropy*, PhD. Dissertation, State University of New-York at Stony Brook, 1995.

[9] Koropecski, A., *Communication*.

[10] Llibre, J. and MacKay R., *Rotation vectors and entropy for homomorphism of the torus isotopic to the identity*, Erg. Th. & Dyn. Sys., 1991, vol.11, p.115-128.

[11] Misuurewicz, M. and Ziemian, K., *Rotation sets for maps of tori*, J. London Math. Soc., 1989, vol.40, p.490-506.

[12] Misuurewicz, M. and Ziemian, K., *Rotation sets and ergodic measures for torus homeomorphisms*, Fundamenta Mathematic, 1991, vol.137.

[13] Rees, M., *A minimal positive entropy homeomorphism of the 2-torus*, J. London Math. Soc., 1981, vol. 23.

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