String Webs and Curves of Marginal Stability in Five-Dimensional $E_N$ theories on $S^1$

Yukiko Ohtake

Department of Physics, University of Tokyo
Tokyo 113-0033, Japan

Abstract

We study curves of marginal stability (CMS) in five-dimensional $\mathcal{N} = 1$ $E_N$ theories compactified on a circle using the D3-brane probe realization. In this realization, BPS states correspond to string webs in the affine $E_N$ 7-brane background and junction positions of the webs determine CMS. We find that there exist string webs involving infinitely many junctions. Consequently the $E_N$ theories have infinitely many CMS. We also find that there exists a transition from open strings to string webs involving loops. The transition describes a new phenomenon occurring on CMS.
1 Introduction

It is well known that BPS spectrum of four-dimensional $\mathcal{N}=2$ gauge theories jumps non-perturbatively as the moduli of the theory are varied. This phenomenon was first studied in $SU(2)$ Yang–Mills theory and the study has been extended to $SU(2)$ QCD\cite{1, 2}. In addition the phenomenon has been investigated using stringy realizations of the gauge theories\cite{3}. Especially the D3-brane probe realization of $SU(2)$ Yang–Mills theory gives us a transparent understanding of the phenomenon\cite{4, 5, 6}.

In the D3-brane probe realization, $SU(2)$ Yang–Mills theory arises as the world-volume theory on a D3-brane in a background of an O7-plane\cite{7, 8}. The position of the D3-brane on $\mathbf{P}^1$ which is the space transverse to the O7-plane corresponds to the moduli parameter. When the D3-brane is located at the O7-plane, gauge symmetry is enhanced to $SU(2)$. Thus the position of the O7-plane corresponds to the singularity in the classical moduli space. In accordance with the singularity splitting in field theory, the O7-plane splits into two mutually non-local 7-branes. Hence there appear the 3-string junctions connecting these 7-branes and the D3-brane. These string webs correspond to dyons and W-bosons in field theory. The position of the junction should be located on a circle passing through the 7-branes to balance the string tensions\cite{7, 8}. When the D3-brane is located on the circle, 3-string junctions become marginally stable. When the D3-brane is located inside the circle, 3-string junctions are no longer stable and they decay into open strings emanating from each 7-brane. These behaviors of string webs reproduce the jump of BPS spectrum.

When we add $N_f \leq 4$ D7-branes to the O7-plane background, $SU(2)$ gauge theory with $N_f$ fundamental matters arises on the D3-brane. The positions of D7-branes correspond to masses of the matter fields. If the masses are turned on, D7-branes separate from other 7-branes. Thus string webs connecting three or more 7-branes and the D3-brane generally involve multiple 3-string junctions. Consequently infinitely many curves of marginal stability (CMS) appear in the theories\cite{7}.

In this paper we will extend the study to five-dimensional $SU(2)$ gauge theory with $5 \leq N_f \leq 7$ fundamental massless matters compactified on $S^1$. Especially we will consider the strong coupling limit of the theory whose global symmetry $SO(2N_f) \times U(1)$ is enhanced to $E_{N_f+1}$\cite{10}. Classically the five-dimensional theory is realized on a D4-brane in the
background of an O8-plane and \( N_f \) D8-branes. Compactifying the system on \( S^1 \) along the D4-brane and taking T-dual, we see that the theory on \( S^1 \) appears on a D3-brane in the background of two O7-planes and \( N_f \) D7-branes. As each O7-plane splits into two 7-branes, the background includes \( (N_f + 4) \) 7-branes. \( (N_f + 3) \) of them collapse to realize the \( E_{N_f+1} \) symmetry. It is known that the 7-brane background realizes the affine \( E_{N_f+1} \) algebra\[11, 12\].

We will derive the CMS of string webs in the 7-brane background. Since the string webs are dual to D-branes in IIA theory compactified on a Calabi–Yau 3-fold with a shrinking del Pezzo 4-cycle\[13, 14\], the CMS would add our understanding of D-brane stability\[15\]. In addition we will find that some string webs draw a loop around the 7-branes and intersect themselves. The vertex of strings is located on a curve of marginal stability similar to the junction position of a 3-string junction. When the D3-brane is located outside the curve, the string web has a modulus which corresponds to the size of the loop. The loop disappears from the string web when the D3-brane is located inside the curve. Thus the moduli space of a BPS state jumps as the moduli of the theory are varied.

The organization of this paper is as follows. In section 2 we review the D3-brane probe realization of \( E_N \) theories and the duality map between IIA D-branes and string webs. In section 3 we construct 3-string junctions and determine their CMS. In addition we observe that some string webs intersect themselves. In section 4 we study the behavior of the self-intersecting string webs. Consequently we find that the moduli space of a string web changes as the position of the D3-brane is varied. In section 5 we deform 3-string junctions using the Hanany–Witten effect. As a result we find string webs involving multiple junctions. Their CMS are determined similar to those of 3-string junctions.

## 2 D3-brane realizations of \( E_N \) theories

The affine \( E_N \) 7-brane backgrounds are described by elliptic curves,

\[
\begin{align*}
E_8 & : y^2 = x^3 + R^2 u^2 x^2 - 2u^5, \\
E_7 & : y^2 = x^3 + R^2 u^2 x^2 + 2u^3 x, \\
E_6 & : y^2 = x^3 + R^2 u^2 x^2 - 2Riu^3 x - u^4,
\end{align*}
\]

(1)
where $R$ is a parameter with the mass dimension $-1$ and $u$ is a complex coordinate of $\mathbb{P}^1$ which is the space transverse to 7-branes. The positions of the 7-branes on $\mathbb{P}^1$ are represented by zeros of the discriminant

$$
E_8 : \Delta(u) = -4u^{10}(2R^6u - 27),
$$

$$
E_7 : \Delta(u) = -4u^9(R^4u - 8),
$$

$$
E_6 : \Delta(u) = -iu^8(4R^3u + 27i).
$$

The positions are $z = 0$ and $z = 1$ in the variable

$$
z = \frac{2}{27}R^6u, \quad \frac{1}{8}R^4u, \quad \frac{4i}{27}R^3u,
$$

for $E_8$, $E_7$, $E_6$, respectively. Following the convention in [14], we identify $(N + 2)$ 7-branes located at $z = 0$ with $[1, 0]^N [3, -1][3, -2]$ and a 7-brane at $z = 1$ with $[0, 1]$. The 7-branes at $z = 0$ realize the $E_N$ algebra and the 7-brane at $z = 1$ is responsible for extending $E_N$ to the affine $E_N$ [16, 11, 12].

At each point on the $z$-plane the curve (1) describes a torus. The torus periods are given by

$$
\tau(z) = ie^{-\phi(z)} + \chi(z) = \frac{\varpi_D(z)}{\varpi(z)},
$$

$$
ds^2 = \text{Im} \tau(z) |\varpi(z)|^2 dz^2.
$$

The explicit forms of $\varpi_D$ and $\varpi$ are calculated in [14]. For the later use we give here the results for $|z| < 1$;

$$
\left( \begin{array}{c}
\varpi_D(z) \\
\varpi(z)
\end{array} \right) = \frac{\pi R^{1-\alpha}}{2(2\alpha - 1)} \left( \begin{array}{cc}
\frac{2\xi_2}{\sin \pi \alpha} & -\frac{2\xi_1}{\sin \pi \alpha} \\
\frac{\omega}{\sin \pi \alpha} & \frac{\bar{\omega}}{\sin \pi \alpha}
\end{array} \right) \left( \begin{array}{c}
z^{-(1-\alpha)}F(\alpha, \alpha; 2\alpha; z) \\
z^{-\alpha}F(1 - \alpha, 1 - \alpha; 2(1 - \alpha); z)
\end{array} \right),
$$

where $\alpha = 1/6, 1/4, 1/3$ for $E_8$, $E_7$, $E_6$, and

$$
\omega = e^{i\pi(\frac{1}{2} - \alpha)}, \quad \xi_1 = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}, \quad \xi_2 = \frac{\Gamma(2 - 2\alpha)}{\Gamma^2(1 - \alpha)}.
$$
Figure 1: A string web in the affine $E_N$ 7-brane background

$F(\alpha, \beta; \gamma; z)$ is the hypergeometric function defined by

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)z^n}{\Gamma(\gamma+n)\ n!}.$$ (9)

The branch cut of the function is taken to be along the real axis of the $z$-plane to $\infty$.

Now we introduce a D3-brane probe parallel to the 7-branes. The world volume theory on the D3-brane is the five-dimensional $E_N$ theory compactified on $S^1$ [19]. The radius of $S^1$ is identified with $R$ in (1). The moduli parameter corresponds to the position of the D3-brane on the $z$-plane. $U(1)$ gauge symmetry is apparently realized on the D3-brane and $E_N$ global symmetry comes from the 7-branes at $z = 0$. BPS states of the $E_N$ theory correspond to string webs connecting the 7-branes and the D3-brane.

BPS states of $E_N$ theory are characterized by $(N+3)$ integers $\{P, Q, n, \lambda_i\}$ where $P$ and $Q$ are the electric and the magnetic charges, $n$ represents the Kaluza–Klein modes of the $S^1$ compactification, and $\{\lambda_i\} (i = 1, 2, \ldots, N)$ is the $E_N$ Dynkin label. These integers are identified with the charges of string webs as depicted in Fig.1 where $(p, q)$ denotes an open string which can end on $[p, q]$ 7-branes and $(mp, mq)$ denotes $m(p, q)$-strings. A charge vector $(P, Q)$ represents the type of a string ending on the D3-brane and $n$ is the number of strings emanating from the 7-brane at $z = 1$. The Dynkin label $\{\lambda_i\}$ is carried by strings ending on the $E_N$ 7-branes. The string web becomes marginally stable when the D3-brane is located at the junction. Thus the possible positions of the junction determine the marginal stability curve.

On the other hand the D3-brane–7-brane system is dual to IIA theory compactified on a Calabi–Yau 3-fold with a shrinking del Pezzo four-cycle $B_N$ [13, 14]. The position of
the D3-brane corresponds to the Kähler modulus of $B_N$. The large radius limit of $B_N$ is taken by letting the D3-brane to $z = \infty$ and $B_N$ collapses when the D3-brane is located at $z = 0$. String webs correspond to D4, D2, D0-branes wrapped on $B_N$. The RR charges $Q_4, Q_2, Q_0$ are related to $P, Q, n$ as follows;

$$Q_4 = P, \quad Q_2 = Q, \quad Q_0 = n - P - \frac{1}{2}Q.$$  

The homology lattice of $B_N$ contains the root lattice of the $E_N$ algebra, and $\{\lambda_i\}$ is associated with D2-branes wrapped on the 2-cycles constructing the $E_N$ root lattice. These duality map enables us to determine D-brane stability using the D3-brane probe realization.

3 String webs in affine $E_N$ 7-brane backgrounds

In this section we construct 3-string junctions in the affine $E_N$ backgrounds. The junction positions are determined by the BPS condition of string webs. It is known that the BPS condition includes at least two necessity conditions; the self-intersection number condition and the geodesic condition [5, 6, 20]. We start with reviewing these conditions.

3.1 BPS conditions

The self-intersection number of a string web with the charges $\{P, Q, n, \lambda_i\}$ is given by

$$\langle J, J \rangle_{(P,Q)_n} = \frac{1}{9 - N} \left( P^2 + Q^2 + PQ - 2nQ - \sum_{i,j=1}^{N} \lambda_i C^{ij}\lambda_j \right),$$  

where $C^{ij}$ is the inverse of the $E_N$ Cartan matrix [20]. There is an another representation,

$$\langle J, J \rangle_{(P,Q)_n} = GCD(P, Q) - 2 + 2g,$$  

where $GCD$ denotes the greatest common divisor and $g$ is the number of deformations which preserve the mass of the string web. The deformation parameters have been identified with the sizes of string loops in the case of the flat background [21, 22, 8]. In case

---

1 This self-intersection number is valid when the string web ends on the D3-brane located in the region $|z| \gg 1$. If it is not the case, we deform the string web by moving the D3-brane to $|z| \gg 1$ along a contour which does not cross the string web.
of the curved backgrounds, however, it is difficult to find the deformations of string webs. Thus in this paper we assume that the existence of $g$ deformations and we construct a representative of string webs. Since $g \geq 0$, the charges of a BPS string web satisfy the condition

$$\frac{1}{9 - N} P^2 + Q^2 + PQ - 2nQ - \sum_{i,j=1}^{N} \lambda_i C^{ij} \lambda_j - \text{GCD}(P, Q) \geq -2.$$  

(13)

In the following we concentrate on string webs satisfying (13).

The additional condition comes from the fact that the mass of a BPS string web is minimized\cite{5,6}. For example, let us consider a $(p,q)$-string stretched along a curve $C$ connecting $z = z_0$ and $z = z_1$. The mass is given by

$$M_{p,q} = \int_C ds T_{p,q},$$

$$\geq \left| \int_{z_0}^{z_1} (p \varpi(z) - q \varpi_D(z)) dz \right|,$n

$$= \left| pa(z_1) - qa_D(z_1) - pa(z_0) + qa_D(z_0) \right|,$$  

(14)

where $ds$ is (1), $T_{p,q} = |p - q \tau(z)|/\sqrt{\text{Im} \tau(z)}$ is the tension of a $(p,q)$-string, and $a_D$ and $a$ are the Seiberg–Witten periods

$$a_D(z) = \int_0^z dz' \varpi_D(z'), \quad a(z) = \int_0^z dz' \varpi(z').$$  

(15)

To minimize the mass, all points on $C$ must satisfy

$$\text{Arg} \left[ pa(z) - qa_D(z) - s_1 \right] = \phi,$$  

(16)

where $s_1 = pa(z_0) - qa_D(z_0)$, and $\phi$ is a constant between 0 and $2\pi$. The argument $\phi$ determines the direction of the $(p,q)$-string at $z = z_0$.

Since the mass of a string web is given by the sum of the masses of all element strings, strings constructing a BPS web must satisfy (16). In addition, all the strings obey (16) with the same $\phi$. To see this we consider a 3-string junction constructed from a $(p,q)$-string stretching between $z = 0$ and $z = z_J$, $n(0,1)$-strings between $z = 1$ and $z = z_J$ and a $(P,Q)$-string between $z = z_J$ and $z = u$ where the D3-brane is located at. From (16) the trajectories of these strings are given by

$$\text{Arg} \left[ pa(z) - qa_D(z) \right] = \phi_1,$$  

6
\[
\arg[-a_D(z) - s] = \phi_2, 
\]
\[
\arg[Pa(z) - Qa_D(z) - Pa(z_J) + Qa_D(z_J)] = \phi_3, 
\]

where we have used \( s \equiv -a_D(1) \) and \( a(0) = a_D(0) = 0 \). Then the mass of the string web is given by

\[
M_{(P,Q)_n} = M_{(p,q)} + |n| M_{(0,1)} + M_{(P,Q)}, 
\]
\[
= |pa(z_J) - qa_D(z_J)| + |na_D(z_J) - ns| 
\]
\[
+ |Pa(u) - Qa_D(u) - Pa(z_J) + Qa_D(z_J)|, 
\]
\[
\geq |pa(z_J) - qa_D(z_J) - na_D(z_J) - ns 
\]
\[
+ Pa(u) - Qa_D(u) - Pa(z_J) + Qa_D(z_J)|, 
\]
\[
= |Pa(u) - Qa_D(u) - ns|. 
\]

where we have used the charge conservation condition at \( z = z_J \), \((P, Q) = (p, q) + n(0, 1)\).

To minimize the mass \( \phi_i \) (i = 1, 2, 3) must satisfy \( \phi_1 = \phi_2 = \phi_3 \). The result is generalized to more complicated string webs.

The condition \( \phi_1 = \phi_2 = \phi_3 \) simplifies the geodesic condition (17). From the first and the second equations of (17) we find that the junction position \( z = z_J \) satisfies

\[
\text{Im} \frac{pa(z_J) - qa_D(z_J)}{-a_D(z_J) - s} = 0. 
\]

This gives the marginal stability curve on which the state \((P, Q)_n\) decays into \((p, q)_0\) and \(n(0, 1)\). In addition the last equation of (17) becomes

\[
\arg[Pa(z) - Qa_D(z) - ns] = \phi. 
\]

Similarly one can show that the element strings of a BPS string web satisfy (20). We call a string satisfying (20) a \((P, Q)_n\)-string throughout this paper. In this notation \((p, q)_0\) represents a \((p, q)\)-string from \( z = 0 \) and \( (0, 1) \) represents a \((0, 1)\)-string from \( z = 1 \).

Recall here that \( a \) and \( a_D \) have a branch cut on the positive real axis of the \( z \)-plane. In [14] it has been shown that \( \Pi(z) = (a_D, a, s)^t \) satisfies \( \Pi(z_0 + i\epsilon) = M \Pi(z_0 - i\epsilon) \) where \( 0 < \epsilon \ll 1 \) and

\[
M = \begin{pmatrix}
1 & 9 - N & 0 \\
-1 & N - 8 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 9 - N & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} 
\]

(21)
for $0 < z_0 < 1$ and $z_0 > 1$, respectively. To preserve the form (20) the charge vector $(P, Q, n)^t$ also jumps on the branch cut to $M'(P, Q, n)^t$ where

\[
M' = \begin{pmatrix}
1 & 9 - N & 0 \\
-1 & N - 8 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 9 - N & 0 \\
0 & 1 & 0 \\
1 & 9 - N & 1
\end{pmatrix}
\] (22)

for $0 < z_0 < 1$ and $z_0 > 1$.

### 3.2 BPS string webs and CMS

We can determine trajectories of strings (19) and marginal stability curves (20) numerically using the explicit forms of $a$ and $a_D$ given in [14]. The results are understood qualitatively as follows.

First we consider the vicinity of $z = 0$. From (7) we find

\[
(\frac{\varpi_D(z)}{\varpi(z)}) \sim \frac{\pi R^{\frac{1}{\alpha}}}{2(2\alpha - 1)} z^{-(1-\alpha)} \left( \frac{2}{\sin \frac{\pi \alpha}{2}} \right). \tag{23}
\]

Thus $\tau = \varpi_D/\varpi$ becomes a constant $\tau_0 = e^{2\pi i/3}, \sqrt{2} e^{3\pi i/4}, \sqrt{3} e^{5\pi i/6}$ for $E_8, E_7, E_6$, and the curve (20) becomes

\[
\text{Arg} [(P - Q\tau_0)w - ns] = \phi, \tag{24}
\]

where $w = c_0 z^\alpha$ with $c_0 = \frac{-\omega}{\sin \frac{\pi \alpha}{2}} \frac{R^{\frac{1}{\alpha}}}{2(2\alpha - 1)} \xi_2$. Thus a $(P, Q)$-string stretches along a straight line on the $w$-plane with the angle

\[
\text{Arg}[w(p,q)] = \phi - \text{Arg}[P - Q\tau_0]. \tag{25}
\]

The relative angles of various $(p, q)$-strings obeying (24) with the same $\phi$ are shown in Fig.2.

If we set $c_0 = 1$, the branch cut on the $z$-plane is mapped to the lines $\text{Arg}[w^\frac{1}{\alpha}] = 0$ as shown in Fig.1. Note that an angle between a $(P, Q)$-string and the branch cut changes from $\text{Arg}[w_{(P,Q)}]$ to $\text{Arg}[w_{(P,Q)}] - 2\alpha \pi$ when the string crosses the branch cut (See Fig.3). From (23) this means that the $(P, Q)$-string changes to a $(P - (9 - N)Q, -P + (N - 8)Q)$-string. This reproduces the effect of $M'$ for $0 < z_0 < 1$ given in (22).

Next we introduce a $[0, 1]$ 7-brane at $z = 1$. BPS strings are not along straight lines on the $w$-plane, however, the order of $(P, Q)_{0}$-strings shown in Fig.2 is unchanged unless the
Figure 2: Directions of \((P, Q)\)-strings with the same \(\phi\)

Figure 3: BPS strings on the \(w\)-plane
strings hit or encircle the \([0, 1]\) 7-brane. If the order of \((p_1, q_1)_0\) and \((p_2, q_2)_0\) is changed, two strings cross at a point \(z = z_0 \neq 0\). In addition if the string does not encircle the \([0, 1]\) 7-brane, we can move the branch cut not to cross the strings. Thus \((p_i, q_i)\) \((i = 1, 2)\) are unchanged. Note that a \((p_1, q_1)\)-string passing through \(z = z_1\) is given by

\[
\text{Arg}\left[p_1a(z) - q_1a_D(z) - p_1a(z_1) + q_1a_D(z_1)\right] = \phi.
\]

(26)

This is parallel to \((p_1, q_1)_0\) with the same \(\phi\) and it sweeps the \(z\)-plane as we varies \(z = z_1 \in i\mathbb{R}\). Thus there exists a \((p_1, q_1)\)-string which touches \((p_2, q_2)_0\) at a point \(z = z_J\) between \(z = 0\) and \(z = z_0\). Since the direction of a \((p, q)\)-string is locally given by

\[
\text{Arg}[p - q\tau(z)] = \phi,
\]

(27)

\(z_J\) satisfies

\[
\text{Arg}[p_1 - q_1\tau(z_J)] = \text{Arg}[p_2 - q_2\tau(z_J)].
\]

(28)

Therefore \(\tau(z_J)\) must be a rational number when \((p_1, q_1)\) are not parallel to \((p_2, q_2)\). \(\tau(z_J) \in \mathbb{Q}\) is mapped to \(i\infty\) by the \(SL(2, \mathbb{Z})\) transformation of the torus thus there exists a 7-brane at \(z = z_J\). Hence the order of the strings is preserved unless the strings hit the 7-brane.

On the basis of the observations we will construct 3-string junctions. 3-string webs are constructed from \((p, q)_0\) and \(n(0, 1)_1\) and also \((p, q)_0\) and \(n(N - 9, 1)_1\) which is the \(n\) strings emanating from the \([0, 1]\) 7-brane at \(z = 1\) to the lower half \(z\)-plane. From [13] we see that the trajectory of \((P, Q)_n\) with \(\phi\) is the same with that of \((-P, -Q)_{-n}\) with \(\phi + \pi\). Thus we only consider \((P, Q)_n\)-string webs with \(n \geq 0\). In what follows we will only show the analysis of the \(E_8\) theory since the analysis is identical with that of the \(E_7\) and \(E_6\) theories.

First we consider a \((P, Q)_n\)-string web constructed from \((p, q)_0\) and \(n(0, 1)_1\). Note that the strings \((P, Q)_0\), \((p, q)_0\) and \((0, 1)_0\) are approximately represented by straight lines from \(w = 0\) with the direction given by [23] as shown in Fig.4a and b. Since \((0, 1)_1\) is parallel to \((0, 1)_0\), the trajectory is qualitatively determined as depicted in Fig.4c. We see that \((0, 1)_1\) can cross \((p, q)_0 \in A \cup B \cup C\) in the Fig.4a and can construct a junction. Adding a line parallel to \((P, Q)_0\) to the junction we have a \((P, Q)_n\)-string web as shown in Fig.4d. Note that \((P, Q)_0\) runs between \((0, 1)_0\) and \((p, q)_0\) thus \((P, Q) \in A \cup B \cup C\).
Varying the direction of \((p, q)_0\) we obtain the orbit of the junction position \([19]\) as depicted in Fig.4. As a result we see that the marginal stability curves are classified by the charges \((p, q)\). For a web with \((p, q) \in A\) in Fig.2, the curve is an arc from \(z = 1\) to \(z = 0\) which does not cross the branch cut as depicted in Fig.4a. The curve for \((p, q) \in B\) is an arc from \(z = 1\) to \(z = 0\) which crosses the branch cut between \(z = 0\) and \(z = 1\) as shown in Fig.4b. The boundary of the regions \(A\) and \(B\) is \((p, q) = (1, -1)\). This is because \((1, -1)_0\) stretches along \(\text{Arg}[w] = 0\) when \((0, 1)_1\) stretches along \(\text{Arg}[w] = \frac{5}{3}\pi\). This might be changed by the effect of the \([0, 1]\) 7-brane, however, it is not the case. Using the exact form of the periods \([7]\) we see that \((0, 1)_1\) goes along \(z = x + i\epsilon\) where \(0 < x < 1\) and \(0 < \epsilon \ll 1\) when \(\phi = 0\). This means that \((1, -1)\)-strings with \(\phi = 0\) stretches along \(z = x - i\epsilon\). These strings are in agreement with the two lines in the \(w\)-plane.

In addition the curve for \((p, q) \in C\) starts from \(z = 1\), encircles \(z = 0\) and crosses the branch cut in \(z \geq 1\) as depicted in Fig.4c. The boundary of the region \(B\) and \(C\) is \((p, q) = (1, 0)\). This is because \((1, 0)_0\) stretches along \(\text{Arg}[w] = 0\) when \((0, 1)_1\) stretches along \(\text{Arg}[w - e^{-\pi i}] = \frac{\pi}{3}\), thus \((1, 0)_0\) meets \((0, 1)_1\) at \(z = 1 - i\epsilon\). Actually we can show
that the value of $\phi$ of $(1, 0)_{0}$ passing through $z = 1 - i\epsilon$ is the same with that of $(0, 1)_{1}$ passing through $z = 1 - i\epsilon$;

$$\text{Arg}[a(1 - i\epsilon)] = \text{Arg}[-a_{D}(1 - i\epsilon) - s].$$

(29)

Here we have used (22) and $a_{D}(1 + i\epsilon) = -s$.

The analysis is generalized to the string webs constructed from $(p, q)_{0}$ and $n(-1, 1)_{1}$. In this case $(p, q)_{0} \in A, B', C'$ can cross $(-1, 1)_{n}$ and construct 3-string junctions. We can see that the trajectory of a $(P, Q)_{n}$-string web is the mirror image of that of a $(-P - Q, Q)_{n}$-string web constructed from $(-p - q, q)_{0}$ and $n(0, 1)_{1}$ with respect to the real axis in the $z$-plane.

Now we determine transitions of string webs. First let us consider a $(P, Q)_{n}$-string web constructed from $n(0, 1)_{1}$ and $(p, q)_{0}$ with $(p, q) \in A \cup B$ and $(P, Q) = (p, q + n) \notin A$. For a certain value of $\phi$ the $(P, Q)_{n}$-string of the web crosses the branch cut in $|z| > 1$ and changes to $(P', Q')_{n'} = (P + Q, Q)_{P+Q+n}$. As we increase $\phi$, the string hits the $[0, 1]$ 7-brane at $z = 1$ and the Hanany–Witten effect creates $(P + Q) (0, 1)_{1}$-strings as depicted in Fig.6a[23]. Thus the marginal stability curve of $(P', Q')_{n'}$ is connected to that of $(P, Q)_{n}$. 

Figure 6: Marginal stability curves of 3-string webs
Figure 7: Transitions of string webs

The curve of \((P', Q')_{n'}\) is determined from the junction of \((0, 1)_1\) and \((P', Q')_{n'}\);

\[
\text{Im} \left[ \frac{P'a(z_J) - Q'a_D(z_J) - ns}{-a_D(z_J) - s} \right] = 0.
\] (30)

This is equal to (19) with \((p, q) = (P', Q') - n'(0, 1)\), that is, the marginal stability curve of a 3-string junction constructed from \(n'(0, 1)_1\) and \((P', Q' - n')_0\).

Second we consider a \((P, Q)_{n'}\)-string web constructed from \(n(0, 1)_1\) and \((p, q) \in C\). At a certain value of \(\phi\) \((P, Q)_n\), \((p, q)_0\) and \((0, 1)_1\) cross the branch cut in \(|z| > 1\) and change to \((P', Q')_{n'} = (P + Q, Q)_{P + Q + n}, (p', q')_l = (p + q, q)_{P + q}\) and \((1, 1)_2\). As we increases \(\phi\) \((p, q)_0\) hits the 7-brane at \(z = 1\) and the Hanany–Witten effect creates \((0, 1)_1\)-strings as depicted in Fig.7b. Thus the marginal stability curve of \((P', Q')_n\) is given by

\[
\text{Im} \left[ \frac{P'a(z_J) - Q'a_D(z_J) - n's}{P'a(z_J) - q'a_D(z_J) - ls} \right] = 0.
\] (31)
This curve is connected to the marginal stability curve for \((P, Q)_n\) depicted in Fig. 6c.

Finally we consider a \((P, Q)_n\)-string web with \((P, Q) \in A\). When the string web is constructed from \(n(0, 1)_1\) and \((p, q)_0\), the \((P, Q)_n\)-string of the web does not cross the branch cut as shown in Fig. 6c. On the other hand the strings hit the 7-branes at \(z = 0\) at a certain value of \(\phi\). As we increase \(\phi\) the web changes to the \((P, Q)_n\)-string web constructed from \(n(-1, 1)_1\) and \((P + n, Q - n)_0 \in A\). The marginal stability curve is the mirror image of Fig. 6a. Thus the marginal stability curve of \((P, Q)_n\) becomes a circle passing through \(z = 0\) and \(z = 1\).

These transitions give us a set of marginal stability curves which are connected as was shown in four-dimensional \(SU(2)\) theories. In this case, however, there appear two unusual behaviors. First, if \((P, Q) \in B \cup C\), the \((P, Q)_n\)-string of a web constructed from \(n(0, 1)_1\) and \((p, q)_0 \in A \cup B\) passes through the branch cut on \(0 < z < 1\) and intersects \((0, 1)_1\) of the web. Are such unusual string webs actually BPS? We will approach the question in the next section. Second, two or more marginal stability curves would appear for a state. For example, a \((3, 2)_4\)-string web constructed from \(4(0, 1)_1\) and \((3, -2)_0\) has a marginal stability curve depicted in Fig. 6b. On the other hand a marginal stability curve of \((1, 2)_1\)-string web constructed from \((0, 1)_1\) and \((1, 1)_0 \in C\) is connected to that of \((3, 2)_4\). The curve starts from \(z > 1\) thus it is different from the curve depicted in Fig. 6b. We will resolve the multiplicity in section 5 by generalizing the self-intersection number condition.

4 Open string – string loop transition

In the previous section we have seen that some string webs intersect themselves. The simplest example is a \((0, 1)_1\) depicted in Fig. 8 for the \(E_8\) case. In this section we will show that the string web is BPS.

For this purpose recall that the affine \(E_8\) 7-brane system is T-dual to an O8-plane and seven D8-branes compactified on \(S^1\). In the 8-brane background there exists a string winding on \(S^1\). The string is dual to a \((1, 0)\)-string drawing a loop around the 7-branes. Thus the loop string is BPS at least in the semi-classical region \(|z| \gg 1\). The trajectory
Figure 8: An open string intersecting itself

Figure 9: Relation between the loop string and the (1, 0)\textsubscript{1}-string web

of the loop string crossing the positive real axis of the z-plane at \( z = z_0 > 1 \) is given by

\[
\text{Arg}[-a(z) + a(z_0 + i\epsilon)] = \text{Arg}[-a(z_0 - i\epsilon) + a(z_0 + i\epsilon)] = \text{Arg}[s],
\]

where \( 0 < \epsilon \ll 1 \). In the second equality we have used (21). The explicit form of the curve is determined by numerical calculations. The result is a loop as shown in Fig.9a. As we decrease the value of \( z_0 \), the size of the loop string becomes smaller. From (14) and (21) we obtain the mass of the loop string,

\[
M_{\text{loop}} = |a(z_0 + i\epsilon) - a(z_0 - i\epsilon)| = |s|.
\]

The mass is independent of \( z_0 \). This reflects the fact that the mass of the string winding on \( S^3 \) is independent of the distance from 8-branes.

As we decreases \( z_0 \) further, the loop string hits the 7-brane at \( z = 1 \). Then Hanany–Witten effect creates a (0, 1)-string ending on \( z = 1 \), and the loop string becomes the
string web as depicted in Fig.9c. When the loop around \( z = 0 \) crosses the branch cut \( z = z_0 < 1 \), the mass is given by

\[
M_{\text{loop}} = |a(z_0 + i\epsilon) - a(z_0 - i\epsilon)| + |a_D(z_0 + i\epsilon) + s|,
\]

\[
\leq |a(z_0 + i\epsilon) - a(z_0 - i\epsilon) + a_D(z_0 + i\epsilon) + s|,
\]

\[
= |s|.
\]

Thus the mass is preserved under the creation of the \((0, 1)_1\)-string and independent of the loop size.

Now we introduce a D3-brane probe on the \((0, 1)\)-string. Then we find that a \((0, 1)_0\)-string depicted in Fig.9d exists. The string web involves a loop and the mass of the web is independent of the loop size. Next we move the D3-brane anti-clockwisely and make it cross the branch cut in \( |z| < 1 \). The \((0, 1)_0\) undergoes the effect of \( M' \) given in (22) and becomes \((1, 0)_0\) involving a loop as depicted in Fig.9e. Finally we move the D3-brane further and make it cross the branch cut in \( |z| > 1 \). The \((1, 0)_0\)-string undergoes the effect of \( M' \) and becomes a \((1, 0)_1\)-string. When the \((1, 0)_1\)-string hits the 7-brane at \( z = 1 \), a \((0, 1)_1\)-string is created by the Hanany–Witten effect as depicted in Fig.9f. We find that the string depicted in Fig.8 appears when the loop size is maximal. Thus the string web shown in Fig.8 is BPS.

Note that the string web has a deformation parameter corresponding to the size of the loop. Such parameters have been related to the zero-modes of a BPS state in the field theory\[25\]. As we move the D3-brane along the \((1, 0)_1\)-string web in Fig.8 and locate the D3-brane at the vertex of two string, the web becomes marginally stable. The marginal stability curve is given by (19) with \((p, q) = (1, -1)\). If the size of the loop is not maximal, the string web disappears from the BPS spectrum inside the curve. On the other hand the string web at the maximal loop size can change to an open string emanating from \( z = 1 \) inside the curve\[24\]. In this transition the string web loses the deformation parameter thus the transition would be detected as the annihilation of zero-modes of the BPS state.

The number of deformation parameters \( g \) is determined by (11) and (12). The results reproduce the analysis above. In addition \( g \) of an arbitrary self-intersecting string web

\[\text{Inside the marginal stability curve the masses of states } (1, 0)_1, (1, -1)_0 \text{ and } (0, 1)_1 \text{ satisfy the inequality } M_{(1,0)} > M_{(1, -1)} + M_{(0, 1)} \text{. The } (1, 0)_1 \text{ state, however, still has a BPS representative of string webs.}\]
decreases when the string web loses the self-intersection. For example $g$ of a $(P,Q)_n$ self-intersecting string web involving $(0,1)_1$-strings is given by

$$2g = (J,J)_{(P,Q)n} - \text{GCD}(P,Q) + 2. \quad (34)$$

When the D3-brane moves across the branch cut in $|z| < 1$ from the upper-half $z$-plane, the self-intersection of the web disappears. Simultaneously the number of the deformation parameter $g'$ becomes

$$2g' = (J,J)_{(-Q,P+Q)n} - \text{GCD}(-Q,P+Q) + 2 = 2g - 2nP. \quad (35)$$

Since $2nP < 0$, we see that the self-intersection number decreases. Thus the interpretation of the transition would be understood as that of the $(1,0)_1$-string web.

5 Local self-intersection number condition and relevant CMS

We have noted in section 3 that the existence of all the 3-string junctions causes multiplicity of marginal stability curves. The multiplicity is resolved if we apply the self-intersection number condition on all parts of a string web. The reason why the additional condition is needed can be seen by introducing additional D3-brane probes. When the D3-branes are located on a string web, the string web is divided into many parts. Each of them can freely moves along the world-volume of the D3-branes. Thus any parts of the string web must be BPS and satisfy the self-intersection number condition.

A 3-string junction constructed from $n(0,1)_1$ and $(p,q)_0$ is divided into two parts by introducing an extra D3-brane on $(p,q)_0$. The self-intersection number condition for $(p,q)_0$ is satisfied if the web satisfies (13). The self-intersection number of the other part is given in [26] as $(J,J) = n(P - n)$, and the self-intersection number condition becomes

$$n(P - n) \geq -2 + \text{GCD}(P,Q) + \text{GCD}(p,q), \quad (36)$$

where $(P,Q)_n = (p,q)_0 + n(0,1)_1$. One can prove that the condition is equivalent to

$$n(P - n) \geq 0, \quad (37)$$
for the string webs with \( \text{GCD}(P,Q,n) = 1 \). For the webs with \( n > 0 \) \( (37) \) is rewritten as \( 0 < n \leq P \). Other string webs with the charges \( P > n \) are constructed by deforming the 3-string webs with \( 0 < n \leq P \) as we have done in the last part of section 3. One can show that a \((P,Q)_n\)-string web is uniquely determined in this way if \((P_l,Q_l,n_l) = M' - l(P,Q,n)\) with \(M'\) in \( (22) \) for \(|z_0| > 1\).

For example we explicitly construct a \((1,0)_3\)-string web in the \(E_8\) theory. When the web is in the fundamental representation of \(E_8\), the web corresponds to a Kaluza-Klein mode of a quark in the five-dimensional theory. We see that \((P_l,Q_l,n_l) = M' - l(P,Q,n)\) satisfies the condition \((37)\) when \( l = 2 \) and \((P,Q,n) = (1,0,1)\). Thus the \((1,0)_3\) is constructed by deforming \((1,0)_1\). The \((1,0)_1\)-string web is a 3-string junction constructed from \((0,1)_1\) and \((1,-1)_0\) as depicted in Fig.10a. The marginal stability curve is as shown in Fig.10a. As we increases \(\phi\) the \((1,0)_1\)-string crosses the branch cut in \(|z| > 1\) and the \((1,0)_1\)-string becomes a \((1,0)_2\)-string. When the string hits the 7-brane at \(z = 1\) the Hanany–Witten effect creates a \((0,1)_1\)-string as depicted in Fig.10b. The \((1,0)_2\)-string web decays into \((0,1)_1\) and \((1,-1)_1\) when the D3-brane is located at the junction. The

![Figure 10: \((1,0)_n\)-string webs](image-url)
marginal stability curve is given by (19) with \((p, q) = (1, -2)\) and determined as shown in Fig.6a. As \(\phi\) increases further, we can construct the \((1,0)_3\)-string web as shown in Fig.10c. The marginal stability curve is given by (19) with \((p, q) = (1, -3)\). Similarly we can construct a \((1,0)_n\)-string web involving \((n+1)\) junctions. The length of the \((0,1)_1\)-string created in the \(n\)-th Hanany–Witten effect is shorter than that of \((n-1)\). Thus the marginal stability curve of the state \((1,0)_{n+1}\) is nearer to the real axis of the \(z\)-plane than that of the state \((1,0)_{n}\).

Mapped the string webs to the \(z\)-plane we find that string webs involve inner-surfaces which look like loops (See Fig10). The sizes of the inner surfaces, however, are fixed. To enable to change the size of inner surfaces, all strings must be merged at junctions, but this breaks the self-intersection number condition. The same phenomenon has been already known in the D3-brane probe realization of four-dimensional \(SU(2)\) QCD[9].

Similarly we can find the condition \(0 < n \leq -(P + Q)\) of a \((P,Q)_n\) 3-string web constructed from \(n(-1,1)_1\) and \((p,q)_0\). The explicit forms of string webs generated from the 3-string webs are determined by the mirror image of that of a \((-P - Q,Q)_n\)-string web involving \((0,1)_1\)-strings with respect to the real axis in the \(z\)-plane.

**Acknowledgements**

I would like to thank S. K. Yang for helpful discussions. This work was supported by JSPS Research Fellowships for Young Scientists.

**References**

[1] N. Seiberg and E. Witten, Monopole Condensation, and Confinement in \(\mathcal{N}=2\) Supersymmetric Yang–Mills Theory, Nucl. Phys. B426 (1994) 19, [hep-th/9407087](http://arxiv.org/abs/hep-th/9407087).

Monopoles, Duality and Chiral Symmetry Breaking in \(\mathcal{N}=2\) Supersymmetric QCD, Nucl. Phys. B431 (1994) 484, [hep-th/9408099](http://arxiv.org/abs/hep-th/9408099).

[2] A. Bilal and F. Ferrari, The Strong-Coupling Spectrum of the Seiberg-Witten Theory, Nucl. Phys. B469 (1996) 387, [hep-th/9602082](http://arxiv.org/abs/hep-th/9602082).

Curves of Marginal Stability and Weak and Strong-Coupling BPS Spectra in \(\mathcal{N}=2\) Supersymmetric QCD, Nucl. Phys. B480
(1996) 589, hep-th/9605101; The BPS Spectra and Superconformal Points in Massive $\mathcal{N}=2$ Supersymmetric QCD, Nucl. Phys. B516 (1998) 175, hep-th/9706145.

[3] J. Schulze and N. P. Warner, BPS Geodesics in $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory, Nucl. Phys. B498 (1997) 101, hep-th/9702012; A. D. Shapere and C. Vafa, BPS Structure of Argyres-Douglas Superconformal Theories, hep-th/9910182; W. Lerche, On a Boundary CFT Description of Nonperturbative $\mathcal{N}=2$ Yang-Mills Theory, hep-th/0006100; B. Fiol, The BPS Spectrum of $\mathcal{N}=2$ $SU(N)$ SYM and Parton Branes, Phys. Lett. B481 (2000) 365, hep-th/001207.

[4] A. Fayyazuddin, Results in Susy Field Theory from 3-brane Probe in F-theory, Nucl. Phys. B497 (1997) 101, hep-th/9701185.

[5] O. Bergman and A. Fayyazuddin, String Junctions and BPS States in Seiberg-Witten Theory, Nucl. Phys.B531 (1998) 108, hep-th/9802033.

[6] A. Mikhailov, N. Nekrasov and S. Sethi, Geometric Realizations of BPS States in $\mathcal{N}=2$ Theories, Nucl. Phys. B531 (1998) 345, hep-th/9803142.

[7] A. Sen, F-theory and Orientifolds, Nucl. Phys. B475 (1996) 562, hep-th/9605150.

[8] T. Banks, M. Douglas and N. Seiberg, Probing F-theory with Branes, Phys. Lett. B387 (1996) 278, hep-th/9605199.

[9] Y. Ohtake, String Junctions and the BPS Spectrum of $\mathcal{N}=2$ $SU(2)$ Theory with Massive Matters, Prog. Theor. Phys. 102 (1999) 671, hep-th/9812227.

[10] N. Seiberg, Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics, Phys. Lett. B388 (1996) 753, hep-th/9608111.

[11] O. DeWolfe, Affine Lie Algebras, String Junctions and 7-Branes, Nucl. Phys. B550 (1999) 622, hep-th/9809026.

[12] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, Uncovering Infinite Symmetries on $[p,q]$ 7-branes: Kac–Moody Algebras and Beyond, Adv. Theor. Math. Phys. 3 (1999) hep-th/9812203.
[13] T. Hauer and A. Iqbal, Del Pezzo Surfaces and Affine 7-Brane Backgrounds, JHEP 0001 (2000) 043, hep-th/9910054.

[14] K. Mohri, Y. Ohtake and S.-K. Yang, Duality Between String Junctions and D-Branes on Del Pezzo Surfaces, Nucl. Phys. B595 (2001) 138, hep-th/0007243.

[15] M.R. Douglas, D-Branes and $\mathcal{N}=1$ Supersymmetry, hep-th/0105014 and references therein.

[16] A. Johansen, A Comment on BPS States in F-theory in 8 Dimensions, Phys. Lett. B395 (1997) 36, hep-th/9608180.

[17] B.R. Greene, A. Shapere, C. Vafa and S.T. Yau, Stringy Cosmic Strings and Non-compact Calabi–Yau Manifolds, Nucl. Phys. B337 (1990) 1.

[18] A. Sen, BPS States on a Three Brane Probe, Phys. Rev. D55 (1997) 2501, hep-th/9608003.

[19] Y. Yamada and S.-K. Yang, Affine 7-brane Backgrounds and Five-Dimensional $E_N$ Theories on $S^1$, Nucl. Phys. B566 (2000) 642, hep-th/9907134.

[20] O. DeWolfe and B. Zwiebach, String Junctions for Arbitrary Lie Algebra Representations, Nucl. Phys. B541 (1999) 509, hep-th/9804210.

[21] O. Bergman, Three-Pronged Strings and 1/4 BPS States in $\mathcal{N}=4$ Super-Yang-Mills Theory, Nucl. Phys. B525 (1998) 104, hep-th/9712211.

[22] K. Hashimoto, H. Hata and N. Sasakura, 3-String Junction and BPS Saturated Solutions in $SU(3)$ Supersymmetric Yang-Mills, Phys. Lett. B431 (1998) 303, hep-th/9803127.

[23] M.R. Gaberdiel and B. Zwiebach, Exceptional Groups from Open Strings, Nucl. Phys. B518 (1998) 151, hep-th/9709013.

[24] O. Bergman and A. Fayyazuddin, String Junction Transitions in the Moduli Space of $\mathcal{N}=2$ SYM, Nucl. Phys. B535 (1998) 139, hep-th/9806011.
[25] D. Bak, K. Hashimoto, B.-H. Lee, H. Min and N. Sasakura, Moduli Space Dimensions of Multi-Pronged Strings, Phys. Rev. D60 (1999) 046005, hep-th/9901107.

[26] A. Iqbal, Self-Intersection Number of BPS Junctions in Backgrounds of Three and Seven-Branes, JHEP 9910 (1999) 032, hep-th/9807117.