ON THE FAMILY OF NON-TOPOLOGICAL SOLUTIONS FOR THE ELLIPTIC SYSTEM ARISING FROM A PRODUCT ABELIAN GAUGE FIELD THEORY

JANN-LONG CHERN* AND SZE-GUANG YANG

Department of Mathematics, National Central University
Chung-Li 32001, Taiwan

ZHI-YOU CHEN

Department of Mathematics, National Changhua University of Education
Changhua 500, Taiwan

CHIH-HER CHEN

Department of Mathematics, National Central University
Chung-Li 32001, Taiwan

Dedicated to Professor Wei-Ming Ni in honor of his 70th birthday

Abstract. In this paper, by constructing a family of approximation solutions and applying a specific version of the Implicit Function Theorem (please see, e.g., [18]), we prove the existence of non-topological solutions for the elliptic system arising from a product Abelian gauge field theory.

1. Introduction. In this paper, we consider the following system of equations:

\[
\begin{align*}
\Delta u &= e^\eta (4\zeta e^u - 2(\zeta + \tilde\zeta)e^v - 2(\zeta - \tilde\zeta)) + 4\pi \sum_{s=1}^n \delta_{z_{q,s}}, \\
\Delta v &= e^\eta (-2\zeta e^u + 2(\zeta + \tilde\zeta)e^v - 2\tilde\zeta) + 4\pi \sum_{s=1}^{\tilde n} \delta_{z_{p,s}},
\end{align*}
\]

in which \(\zeta, \tilde\zeta > 0\) are parameters with \(\zeta > \tilde\zeta\), \(\delta_p\) denotes the Dirac distribution concentrated at \(p \in \mathbb{R}^2\); \(n, \tilde n\) are positive integers, \(z_{q,s}, z_{p,s} \in \mathbb{R}^2\) and \(e^\eta\) is given by

\[
e^\eta = \lambda \left( e^{\zeta(u-e^u)+(\zeta+\tilde\zeta)(v-e^v)} \prod_{s=1}^n |x - z_{q,s}|^2 \right)^{-\zeta} \left( \prod_{s=1}^{\tilde n} |x - z_{p,s}|^2 \right)^{-\tilde\zeta},
\]

where \(\lambda > 0\) is an arbitrary constant and \(G\) is the universal gravitational constant.

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1.1. **Background.** In order to derive Equation (1), we sketch the theory as follows. Here we use the Einstein summation convention, the Greek letters, $\mu$ and $\nu$ run from 0 to 3 while the roman letters $j$ and $k$ are from 1 to 3. Following [19], we consider two Abelian gauge fields, $A_\mu$ and $\tilde{A}_\mu$, generated from the product gauge group $U(1) \times U(1)$, and use $q$ and $p$ to denote two charged scalars carrying charge (+1, 1) and (0, +1), respectively, so that the gauge-covariant derivatives are given by

$$D_\mu q = \partial_\mu q - i\tilde{A}_\mu q, \quad D_\mu p = \partial_\mu p - i\tilde{A}_\mu p.$$ 

Using the spacetime metric tensor $g_{\mu\nu}$ of signature $(+---)$ to raise and lower indices, we rewrite the Lagrangian density of the Tong-Wong model [19] as

$$\mathcal{L} = -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} + g^{\mu\nu} D_\mu q D_\nu \bar{q} + g^{\mu\nu} D_\mu p D_\nu \bar{p}$$

$$- \frac{1}{2} (|q|^2 - \zeta)^2 - \frac{1}{2} (|p|^2 - \tilde{\zeta})^2,$n

where $\zeta, \tilde{\zeta} > 0$ are parameters, $\tilde{F}_{\mu\nu}, \tilde{F}_{\mu\nu}$ are electromagnetic fields induced from gauge potentials $\tilde{A}_\mu, \tilde{A}_\mu$. Taking the variation of the Einstein-Hilbert action

$$S = \int \left\{ \frac{1}{16\pi G} (R_g - 2\Lambda) + \mathcal{L} \right\} \sqrt{-g} \, dx,$n

where $R_g$ is the Ricci scalar curvature of the metric $g_{\mu\nu}$, $G > 0$ the universal gravitational constant, and $\Lambda$ the cosmological constant, the equations of the motion of the action (4) are

$$G_{\mu\nu} + Ag_{\mu\nu} = -8\pi GT_{\mu\nu}, \quad (5)$$

$$\frac{1}{\sqrt{-g}} D_\mu (g^{\mu\nu} \sqrt{-g} D_\nu q) = (|q|^2 - \zeta) q, \quad (6)$$

$$\frac{1}{\sqrt{-g}} D_\mu (g^{\mu\nu} \sqrt{-g} D_\nu p) = (-|q|^2 + |p|^2 - \tilde{\zeta}) p, \quad (7)$$

$$\frac{1}{\sqrt{-g}} \partial_\nu (g^{\mu\nu} g^{\rho\sigma} \sqrt{-g} \tilde{F}_{\mu\nu}) = i g^{\mu\nu} (q D_\nu q - \bar{q} D_\nu q), \quad (8)$$

$$\frac{1}{\sqrt{-g}} \partial_\nu (g^{\mu\nu} g^{\rho\sigma} \sqrt{-g} \tilde{F}_{\mu\nu}) = i g^{\mu\nu} (p D_\nu p - \bar{p} D_\nu p) - i g^{\mu\nu} (q D_\nu q - \bar{q} D_\nu q), \quad (9)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy–momentum tensor of the matter and gauge field sector given by

$$T_{\mu\nu} = -g^{\nu\rho} \tilde{F}_{\mu\rho} \tilde{F}^{\rho\nu} - g^{\nu\rho} \tilde{F}_{\mu\rho} \tilde{F}^{\rho\nu} + D_\mu q D_\nu \bar{q} + D_\mu p D_\nu \bar{p} - g_{\mu\nu} \mathcal{L}. \quad (10)$$

As in [13, 17] we look for straight time independent cosmic string solutions so that the spacetime is uniform along the time axis $x^0 = t$ and the $x^3$-direction and the line element takes the form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - (dx^3)^2 - g_{jk} dx^j dx^k$ ($j, k = 1, 2$) where now $g_{jk}$ is the Riemannian metric tensor of an orientable 2-surface $\mathcal{M}$ with local coordinates $x^1, x^2$. Within such a metric the only nontrivial components of the Einstein tensor are $G_{03} = -G_{33} = -K_g$, where $K_g$ is the Gauss curvature of the surface $(\mathcal{M}, \{g_{jk}\})$, which imposes constraints to the form of the energy–momentum tensor via (5). On the other hand, it is natural and compatible to assume that the gauge and scalar fields depend on the coordinates on $\mathcal{M}$ only and that the 0- and 3-components of the gauge fields are zero. Thus, we have $T_{03} = T_{0j} = T_{3j} = 0$.
and \( T_{00} = -T_{33} = \mathcal{H} \) immediately, where \( \mathcal{H} \) is the Hamiltonian of the Tong–Wong model (3) given by

\[
\mathcal{H} = -\mathcal{L} = \frac{1}{4} g^{jj'} g^{kk'} \mathcal{F}_{jk} \mathcal{F}_{jk'} + \frac{1}{4} g^{jj'} g^{kk'} \mathcal{F}_{jk} \mathcal{F}_{jk'} - g^{jk} D_j q D_k q - g^{jk} D_j p D_k p + \frac{1}{2} (|q|^2 - \zeta)^2 + \frac{1}{2} (|q|^2 + |p|^2 - \bar{\zeta})^2. \tag{11}
\]

To facilitate our computation, recall that the associated current densities

\[
J_k^{(q)} = \frac{1}{2} (q D_k q - q D_k q), \quad J_k^{(p)} = \frac{1}{2} (p D_k p - p D_k p), \tag{12}
\]

satisfy the identities

\[
\partial_j J_k^{(q)} - \partial_k J_j^{(q)} = i (D_j q D_k q - D_j q D_k q) - \hat{F}_{jk} |q|^2 + \hat{F}_{jk} |q|^2, \tag{13}
\]

\[
\partial_j J_k^{(p)} - \partial_k J_j^{(p)} = i (D_j p D_k p - D_j p D_k p) - \hat{F}_{jk} |p|^2. \tag{14}
\]

Applying (13) and (14), we may rewrite (11) as

\[
\mathcal{H} = \frac{1}{4} g^{jj'} g^{kk'} (\hat{F}_{jk} \pm \epsilon_{jk}(|q|^2 - \zeta)) (\hat{F}_{jk'} \pm \epsilon_{jk'}(|q|^2 - \zeta)) + \frac{1}{4} g^{jj'} g^{kk'} (\hat{F}_{jk} \pm \epsilon_{jk}(-|q|^2 + |p|^2 - \bar{\zeta})) (\hat{F}_{jk'} \pm \epsilon_{jk'}(-|q|^2 + |p|^2 - \bar{\zeta})) + g^{jk} (D_j q \pm i \epsilon_j^k D_j q) (D_k q \pm i \epsilon^k_j D_k q) + g^{jk} (D_j p \pm i \epsilon_j^k D_j p) (D_k p \pm i \epsilon^k_j D_k p) + \frac{1}{2} \epsilon^{jk} \hat{F}_{jk} \pm \frac{1}{2} \hat{\zeta} \epsilon^{jk} \hat{F}_{jk} \pm \nabla_j (\epsilon^{jk} J_k^{(q)}) \pm \nabla_j (\epsilon^{jk} J_k^{(p)}), \tag{15}
\]

where \( \nabla_j \) is the covariant derivative with respect to the metric \( g_{jk} \) over \( \mathcal{M} \) and \( \epsilon_{jk} \) is the Kronecker skew-symmetric tensor with \( \epsilon_{12} = \sqrt{|g|} \) in which \( |g| = \det(g_{jk}) \).

In view of [19], we designate \( \hat{U}(1) \) and \( \bar{U}(1) \) to be the magnetic fluxes given by

\[
\frac{1}{4\pi} \int_{\mathcal{M}} \epsilon^{jk} \hat{F}_{jk} \sqrt{|g|} \ dx = \hat{k}, \quad \frac{1}{4\pi} \int_{\mathcal{M}} \epsilon^{jk} \hat{F}_{jk} \sqrt{|g|} \ dx = \bar{k},
\]

where \( \hat{k}, \bar{k} \) are integers. we obtain from integrating (15) the energy lower bound

\[
E = \int_{\mathcal{M}} \mathcal{H} \sqrt{|g|} \ dx \geq 2\pi (|\hat{k}| + |\bar{k}|). \tag{16}
\]

The lower bound in (16) is attained when the equations

\[
\begin{align*}
\hat{F}_{jk} \pm \epsilon_{jk}(|q|^2 - \zeta) &= 0, \\
\hat{F}_{jk} \pm \epsilon_{jk}(-|q|^2 + |p|^2 - \bar{\zeta}) &= 0, \\
D_j q \pm i \epsilon_j^k D_j q &= 0, \\
D_j p \pm i \epsilon_j^k D_j p &= 0,
\end{align*} \tag{17}
\]

are satisfied for \( j, k = 1, 2 \) with \( \hat{k} = \pm|\hat{k}|, \bar{k} = \pm|\bar{k}| \). As a consequence of (17), it is direct to check that \([15, 19, 20]\) \( T_{jk} = 0 \) \( (j, k = 1, 2) \). Inserting this result into (5) we arrive at the vanishing cosmological constant condition \( \Lambda = 0 \). Therefore, Equation (5) becomes

\[
K_g = 8\pi G \mathcal{H}. \tag{18}
\]
In view of (11) and (17), we have
\[ \mathcal{H} = \frac{1}{2} \zeta e^{ik} \tilde{F}_{jk} + \frac{1}{2} \zeta e^{jk} \tilde{F}_{jk} \pm \nabla_j (e^{jk} j_k^{(q)}) \pm \nabla_j (e^{jk} j_k^{(p)}) \]
\[ = -\zeta (|q|^2 - \zeta) - \tilde{\zeta} (-|p|^2 + |p|^2) \pm \frac{1}{2} \Delta_g |q|^2 \pm \frac{1}{2} \Delta_g |p|^2, \tag{19} \]
where \( \Delta_g \) is the Laplace-Beltrami operator induced from covariant derivative \( \nabla_j \) such that \( \Delta_g f = \nabla^j \nabla_j f = (1/\sqrt{|g|}) \partial_j (g^{jk} \sqrt{|g|} \partial_k f) \). Let \( n, \tilde{n} \geq 0 \) denote the winding numbers of \( q \) and \( p \), respectively. Then \( n, \tilde{n} \) are the algebraic numbers of zeros of \( q, p \), respectively, which are related to the magnetic flux numbers \( k, \tilde{k} \) by \( n = k - \tilde{k}, \tilde{n} = \tilde{k} \) as indicated in [19]. The sets of zeros of \( q, p \) are denoted as
\[ Z(q) = \{ z_{q,1}, \ldots, z_{q,n} \} \quad Z(p) = \{ z_{p,1}, \ldots, z_{p,\tilde{n}} \}. \]
Set \( u = \log |q|^2 \) and \( v = \log |p|^2 \). It is standard that (17) may be recast into
\[ \begin{cases} 
\Delta_g u = 4e^u - 2e^v - 2(\zeta - \tilde{\zeta}) + 4\pi \sum_{s=1}^{n} \delta_{z_{q,s}}, \\
\Delta_g v = -2e^u + 2e^v - 2\zeta + 4\pi \sum_{s=1}^{\tilde{n}} \delta_{z_{p,s}},
\end{cases} \tag{20} \]
where \( \delta_z \) denotes the Dirac function defined over the 2-surface \( \mathcal{M}, \{ g_{jk} \} \) and concentrated at the point \( z \in \mathcal{M} \).

Let \( g_{jk} \) be a unknown metric which is conformal to a known one, \( g_{0,jk} \), so that \( g_{jk} = e^\eta g_{0,jk} \) \((j, k = 1, 2)\). Then we have the relations
\[ -\Delta_g \eta + 2K_{g_0} = 2K_g e^\eta, \quad \Delta_g = e^{-\eta} \Delta_{g_0}. \tag{21} \]
In view of the second relation in (21), we see that the system (20) becomes
\[ \begin{cases} 
\Delta_{g_0} u = e^\eta (4e^u - 2e^v - 2(\zeta - \tilde{\zeta})) + 4\pi \sum_{s=1}^{n} \delta_{z_{q,s}}, \\
\Delta_{g_0} v = e^\eta (-2e^u + 2e^v - 2\zeta) + 4\pi \sum_{s=1}^{\tilde{n}} \delta_{z_{p,s}},
\end{cases} \tag{22} \]
where the Dirac functions are defined over the 2-surface \( \mathcal{M}, \{ g_{0,jk} \} \) instead. From (18), (19) and (21), we have
\[ \frac{1}{2 \pi^G} K_g = e^{-\eta} \frac{1}{8 \pi G} \left( -\frac{1}{2} \Delta_{g_0} \eta + K_{g_0} \right) = \mathcal{H} \]
\[ = -\zeta (e^u - \zeta) - \tilde{\zeta} (-e^u + e^v - \tilde{\zeta}) + \frac{1}{2} e^{-\eta} \Delta_{g_0} e^u + \frac{1}{2} e^{-\eta} \Delta_{g_0} e^v. \tag{23} \]
We consider the situation when \( \mathcal{M} \) is non-compact. If \( \mathcal{M} \) is compact, we refer the readers to the results of [8] and the references therein. For simplicity, we assume that \( \mathcal{M} = \mathbb{R}^2 \) and \( g_{jk} = e^\eta \delta_{jk} \). Now \( g_{0,jk} = \delta_{jk} \) and \( \Delta_{g_0} = \Delta \) is the usual Laplace operator on \( \mathbb{R}^2 \). Equation (22) becomes
\[ \begin{cases} 
\Delta u = e^\eta (4e^u - 2e^v - 2(\zeta - \tilde{\zeta})) + 4\pi \sum_{s=1}^{n} \delta_{z_{q,s}}, \\
\Delta v = e^\eta (-2e^u + 2e^v - 2\zeta) + 4\pi \sum_{s=1}^{\tilde{n}} \delta_{z_{p,s}},
\end{cases} \tag{24} \]
where the Dirac \( \delta \)-functions are defined over \( \mathbb{R}^2 \). Note that, in (23), we have \( K_{g_0} = 0 \). Hence we obtain
\[ \frac{1}{2} e^{-\eta} \Delta \left( \frac{1}{8 \pi G} \eta + e^u + e^v \right) = \zeta (e^u - \zeta) + \tilde{\zeta} (-e^u + e^v - \tilde{\zeta}). \tag{25} \]
In order to facilitate the description of the asymptotic behavior of the fields at infinity, one may take the substitution
\[ u \mapsto u + \log \zeta, \quad v \mapsto v + \log(\zeta + \tilde{\zeta}). \tag{26} \]
So (24) is updated into
\begin{align}
\begin{cases}
\Delta u = e^n(4\zeta e^u - 2(\zeta + \tilde{\zeta})e^v - 2(\zeta - \tilde{\zeta})) + 4\pi \sum_{s=1}^n \delta_{z_q,s}, \\
\Delta v = e^n(-2\zeta e^u + 2(\zeta + \tilde{\zeta})e^v - 2\tilde{\zeta}) + 4\pi \sum_{s=1}^{\tilde{n}} \delta_{z_{p',s}},
\end{cases}
\end{align}
(27)

Correspondingly, (25) is modified into
\begin{align}
\frac{1}{2} e^{-n} \Delta \left( \frac{1}{8\pi G} \eta + \zeta e^u + (\zeta + \tilde{\zeta})e^v \right) = \zeta^2 (e^u - 1) + \tilde{\zeta} (-\zeta e^u + (\zeta + \tilde{\zeta})e^v - \tilde{\zeta}).
\end{align}
(28)

From (27) and (28), it is not hard to see that the gravitational conformal factor \( e^v \) is exactly determined by the expression (2).

In (1), there are three types of natural boundary conditions for solutions \((u, v)\) at infinity, namely,
\begin{align}
\lim_{|x| \to \infty} u(x) &= 0, \quad \lim_{|x| \to \infty} v(x) = 0, \\
\lim_{|x| \to \infty} u(x) &= -\infty, \quad \lim_{|x| \to \infty} v(x) = -\infty, \\
\lim_{|x| \to \infty} u(x) &= \infty, \quad \lim_{|x| \to \infty} v(x) = \infty.
\end{align}
(29)\text{--}(31)

We note that a solution \((u, v)\) satisfying boundary condition (29) is called a topological solution; \((u, v)\) is called a non-topological solution if it satisfies other two boundary conditions.

To consider the radially symmetric case, we let \( z_{q,s} = z_{p',s'} = 0 \) (Origin) for \( s = 1, \ldots, n \) and \( s' = 1, \ldots, \tilde{n} \), and reduce Equation (1) to
\begin{align}
\begin{cases}
\Delta u = e^n(4\zeta e^u - 2(\zeta + \tilde{\zeta})e^v - 2(\zeta - \tilde{\zeta})) + 4\pi n\delta_0, \\
\Delta v = e^n(-2\zeta e^u + 2(\zeta + \tilde{\zeta})e^v - 2\tilde{\zeta}) + 4\pi \tilde{n}\delta_0,
\end{cases}
\end{align}
(32)

where \( e^n \) is given by
\begin{align}
e^n = \lambda \left( e^{\zeta(u-e^u)+(\zeta+\tilde{\zeta})(v-e^v)} |x|^{-2\zeta n-2(\zeta+\tilde{\zeta})} \right)^{8\pi G}.
\end{align}

1.2. Main results. Let \( A_1 = 4\pi G(n+1) \) and \( A_2 = 4\pi G(\tilde{n}+1) \). Define
\begin{align}
D_1 &= \left\{ (\zeta, \tilde{\zeta}) \in \mathbb{R}^2: \left( \zeta - \frac{1}{2A_1} \right)^2 + \left( \tilde{\zeta} - \frac{1}{2A_1} \right)^2 < \frac{1}{2A_1^2} \right\}, \\
D_2 &= \left\{ (\zeta, \tilde{\zeta}) \in \mathbb{R}^2: \zeta^2 + \left( \tilde{\zeta} - \frac{1}{2A_2} \right)^2 < \frac{1}{4A_2^2} \right\}.
\end{align}
(33)\text{--}(34)

Theorem 1.1. Let \( n, \tilde{n} \geq 1 \) be given and satisfy
\begin{align}
(n+1)(n-2\tilde{n}-1) < (2\sqrt{2} + 1)(\tilde{n}+1)^2.
\end{align}
(35)

Assume \((\zeta, \tilde{\zeta}) \in D_1 \cap D_2 \). Then:
\begin{enumerate}
\item Equation (32) has a family of (radially symmetric) solutions \( u_{\varepsilon}, v_{\varepsilon}, \varepsilon > 0 \), with
\begin{align}
u_{\varepsilon}(x) &\to -\infty \quad \text{and} \quad v_{\varepsilon}(x) \to -\infty \quad \text{as} \quad |x| \to \infty.
\end{align}
(36)
\end{enumerate}
(ii) The solution \((u_\varepsilon, v_\varepsilon)\) can be chosen, depending on \(\varepsilon\), such that the fluxes

\[
\beta_1 = \int_0^\infty e^\eta \left[ 4\xi e^{\varepsilon v} - 2(\zeta + \tilde{\zeta})e^{\varepsilon v} - 2(\zeta - \tilde{\zeta}) \right] r \, dr
\]

\[
= \frac{\zeta - \tilde{\zeta}}{\zeta^2 + \tilde{\zeta}^2} \frac{1}{2\pi G} + o(1),
\]

\[
\beta_2 = \int_0^\infty e^\eta \left[ -2\xi e^{\varepsilon u} + 2(\zeta + \tilde{\zeta})e^{\varepsilon v} - 2\tilde{\zeta} \right] r \, dr
\]

\[
= \frac{2}{\zeta^2 + \tilde{\zeta}^2} \frac{1}{2\pi G} + o(1),
\]  

as \(\varepsilon \to 0\).

After Theorem 1.1, there is an intersecting problem about finding the sharp range of flux for all solutions of (1) satisfy one of (29), (30) and (31). Recently, there are many well-known results about this problem for another single and coupled equations in [1, 9, 10, 11, 12, 16].

The present paper is organized as follows. In the following section we construct a family of the non-topological solutions of (32) and carry out the main theorem from an approximation viewpoint.

2. Reduction to a single equation. Consider the equation

\[
\Delta u = -r^{-2bN} h(e^u) + 4\pi N_0 \delta \quad \text{in } \mathbb{R}^2,
\]

where \(b\) is a positive constant, \(N \geq 0\) and \(h\) is a function given by \(h(t) = \lambda t e^{-bt} (1-t)\), \(t > 0\). We take account of the radially symmetric solutions of (39) with \(u = u(r)\), \(r = |x|\); in this case, (39) is written as the following:

\[
\left\{
\begin{array}{ll}
ru'(r) & = -r^{-2bN+1} h(e^{u(r)}), \\
u(r) & = 2N \log r + s + o(1),
\end{array}
\right.
\]

where \(s \in \mathbb{R}\). The function \(h: \mathbb{R}^+ \to \mathbb{R}\) is bounded, so \(u\) must be an entire solution (which does not blow up at a finite \(r\)). Let \(\Omega\) denote the solution set of (40). We divide \(\Omega\) into three classes that

\[
\begin{align*}
\mathcal{G}^+ & = \{ u \in \Omega : \lim_{r \to \infty} u(r) = \infty \}; \\
\mathcal{G}^* & = \{ u \in \Omega : \lim_{r \to \infty} u(r) = 0 \}; \\
\mathcal{G}^- & = \{ u \in \Omega : \lim_{r \to \infty} u(r) = -\infty \}.
\end{align*}
\]

Then the set of real numbers is split up into the components:

\[
\begin{align*}
J^+ & = \{ s \in \mathbb{R} : u(r; s) \text{ is of the class } \mathcal{G}^+ \}; \\
J^* & = \{ s \in \mathbb{R} : u(r; s) \text{ is of the class } \mathcal{G}^* \}; \\
J^- & = \{ s \in \mathbb{R} : u(r; s) \text{ is of the class } \mathcal{G}^- \}.
\end{align*}
\]

Remark 2.1. By the maximum principle, we see that \(u(r)\) does not exist a non-negative(resp., non-positive) local maximum(resp., minimum) in \((0, \infty)\). Hence, if there exists a constant \(\hat{r} \in (0, \infty)\) such that \(u(\hat{r}) > 0\) (resp., \(< 0\)) and \(u'(\hat{r}) > 0\) (resp., \(< 0\)), then by (40), \(ru'(r) \geq \hat{r}u'(\hat{r})\) (resp., \(ru'(r) \leq \hat{r}u'(\hat{r})\)). Thus, \(\mathbb{R} = J^+ \cup J^* \cup J^-\).
Now we consider the functions $G(\zeta)$, $p_c(t)$, $q(r)$ and $f(t)$ given by

\[
\begin{align*}
G(\zeta) &= H(e^\zeta) - (1/b)h(e^\zeta), \\
p_c(t) &= bct^2 + [hm - c(2b + 1)]t - b(m - c), \\
q(r) &= \lambda r^{-2bN}e^{b(u-e^\cdot\eta)}(r), \\
f(t) &= -bt^2 + (2b + 1)t - b,
\end{align*}
\]

where $m = 2(1 - bN)/b > 0$ and $w_c(r) = ru; (r) + c$ satisfying

\[
w_{c}''(r) + \frac{1}{r}w_{c}'(r) = f(e^{u(r)})q(r)w_c(r) + q(r)p_c(e^{u(r)}), \quad r > 0.
\]

Then by applying the similarly argument of proofs as in [6, 7], the structure of above functions have similarly properties, and hence we obtain the following result.

**Theorem 2.1.** Let $u = u(r; s)$ be the solution of (40) with $s \in \mathbb{R}$ and

\[
\beta(s) = \beta(u(\cdot, s)) = \int_{0}^{\infty} r^{-2bN+1}h(e^{u(r;s)}) \, dr.
\]

Then the following statements are valid.

(i) If $bN = 1$, then $\beta(s) \in \{0, 2N, 4N\}$.

(ii) If $bN \geq 2$, then $u(r; s) \to \infty$ as $r \to \infty$ for all $s \in \mathbb{R}$. Moreover,

\[0 < \beta(s) < \frac{4}{b}.
\]

(iii) If $bN < 1$, then there is a point $s^* \in \mathbb{R}$ such that $J^- = (-\infty, s^*)$, $J^+ = \{s^*\}$ and $J^+ = (s^*, \infty)$. Moreover, $\beta'(s)$ exists, $\beta'(s) > 0$ for $s \in \mathbb{R} \setminus \{s^*\}$ and

\[
\lim_{s \to s^-} \beta(s) = \infty, \quad \lim_{s \to s^+} \beta(s) = -\infty.
\]

3. **Approximation of non-topological solutions.** Consider the situation that $z_{n,s} = z_{n,s'} = 0$ for $s = 1, \ldots, n$ and $s' = 1, \ldots, n$. We rewrite (27) and (28) as the following equations

\[
\begin{align*}
\Delta u &= e^{\eta}[4\zeta(e^u - 1) - 2(\zeta + \tilde{\zeta})(e^u - 1)] + 4\pi n\delta_{0}, \\
\Delta v &= e^{\eta}[-2(e^u - 1) + 2(\zeta + \tilde{\zeta})(e^u - 1)] + 4\pi n\delta_{0}, \\
\Delta \{\frac{\eta}{a} + \zeta e^{u} + (\zeta + \tilde{\zeta})e^v\} &= 2e^{\eta}\tilde{\zeta}(\zeta - \tilde{\zeta})(e^u - 1) + \tilde{\zeta}(\zeta + \tilde{\zeta})(e^u - 1),
\end{align*}
\]

where $a = 8\pi G$. Let $u_0, v_0$ and $\zeta_0$ be functions which satisfy the equations

\[
\begin{align*}
\Delta u_0 &= -2(\zeta - \tilde{\zeta})e^{\eta} + 4\pi n\delta_{0}, \\
\Delta v_0 &= -2\tilde{\zeta}e^{\eta} + 4\pi n\delta_{0}, \\
\Delta \eta_0 &= -2a(\zeta^2 + \tilde{\zeta}^2)e^{\eta}.
\end{align*}
\]

Assume $u(|x|), v(|x|), \eta(|x|)$ and $u_0(|x|), v_0(|x|), \eta_0(|x|)$ are radially symmetric solutions of (43) and (44) respectively. We set

\[
\begin{align*}
\chi_1(r) &= \frac{1}{\varepsilon^2}u(\frac{r}{\varepsilon}) - \frac{1}{\varepsilon^2}u_0(r) - w_1(r) - \frac{2}{\varepsilon^2} \log \varepsilon, \\
\chi_2(r) &= \frac{1}{\varepsilon^2}v(\frac{r}{\varepsilon}) - \frac{1}{\varepsilon^2}v_0(r) - w_2(r) - \frac{2}{\varepsilon^2} \log \varepsilon, \\
\chi_3(r) &= \frac{1}{\varepsilon^2}\eta(\frac{r}{\varepsilon}) - \frac{1}{\varepsilon^2}\eta_0(r) - w_3(r) - \frac{2}{\varepsilon^2} \log \varepsilon, \quad r = |x|, \quad \varepsilon > 0,
\end{align*}
\]
where $w_1, w_2, w_3$ are functions yet to be determined. By a direct computation,

$$
\Delta \chi_1 = \frac{1}{\varepsilon^4} (\Delta u) \left( \frac{r}{\varepsilon} \right) - \frac{1}{\varepsilon^2} \Delta u_0 (r) - \Delta w_1 (r)
$$

$$
= \frac{1}{\varepsilon^4} \left\{ e^\eta [4\xi (e^\nu - 1) - 2(\xi + \eta)(e^\nu - 1)] \left( \frac{r}{\varepsilon} \right) + 2(\xi - \eta) \frac{1}{\varepsilon^2} e^{\eta_0} (r) - \Delta w_1 (r) \right\}
$$

$$
= 4\xi \frac{1}{\varepsilon^4} e^{\eta(r/\varepsilon)} (e^{u(r/\varepsilon)} - 1) - 2(\xi + \eta) \frac{1}{\varepsilon^2} e^{\eta(r/\varepsilon)} (e^{u(r/\varepsilon)} - 1)
$$

$$
+ 2(\xi - \eta) \frac{1}{\varepsilon^2} e^{\eta_0} (r) - \Delta w_1 (r)
$$

$$
= 4\xi \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(\chi_3 + w_3)} \left( e^2 e^{u_0 + \varepsilon^2(\chi_1 + w_1)} - 1 \right) (r)
$$

$$
- 2(\xi + \eta) \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(\chi_3 + w_3)} \left( e^2 e^{u_0 + \varepsilon^2(\chi_2 + w_2)} - 1 \right) (r)
$$

$$
+ 2(\xi - \eta) \frac{1}{\varepsilon^2} e^{\eta_0} (r) - \Delta w_1 (r),
$$

which implies that

$$
\Delta \chi_1 = 4\xi e^{u_0 + \eta_0 + \varepsilon^2(\chi_1 + \chi_3 + w_1 + w_3)} - 2(\xi + \eta) e^{u_0 + \eta_0 + \varepsilon^2(\chi_2 + \chi_3 + w_2 + w_3)}
$$

$$
- 2(\xi - \eta) \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(\chi_3 + w_3)} + 2(\xi - \eta) \frac{1}{\varepsilon^2} e^{\eta_0} - \Delta w_1. \quad (48)
$$

Similarly, we have

$$
\Delta \chi_2 = -2\xi e^{u_0 + \eta_0 + \varepsilon^2(\chi_1 + \chi_3 + w_1 + w_3)} + 2(\xi + \eta) e^{u_0 + \eta_0 + \varepsilon^2(\chi_2 + \chi_3 + w_2 + w_3)}
$$

$$
- 2\xi \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(\chi_3 + w_3)} + 2\xi \frac{1}{\varepsilon^2} e^{\eta_0} - \Delta w_2. \quad (49)
$$

Furthermore, from the third equation of (43) and with the substitution of $\chi_1, \chi_2, \chi_3$,

$$
\Delta \left\{ \left[ \frac{\eta}{a} + \xi e^{u} + (\xi + \eta) e^{u} \right] \left( \frac{r}{\varepsilon} \right) \right\}
$$

$$
= 2e^{-2} \left\{ e^\eta [\xi \left( \xi - \eta \right)(e^\nu - 1) + \eta \left( \xi + \eta \right)(e^\nu - 1)] \right\} \left( \frac{r}{\varepsilon} \right)
$$

$$
= 2\xi (\xi - \eta) \left\{ e^{\eta_0 + \varepsilon^2(\chi_1 + w_1)} \left[ e^2 e^{u_0 + \varepsilon^2(\chi_1 + w_1)} - 1 \right] \right\} (r)
$$

$$
+ 2\xi (\xi - \eta) \left\{ e^{\eta_0 + \varepsilon^2(\chi_1 + w_1)} \left[ e^2 e^{u_0 + \varepsilon^2(\chi_2 + w_2)} - 1 \right] \right\} (r)
$$

$$
= 2\xi \varepsilon^2 (\xi - \eta) e^{\eta_0 + \eta_0 + \varepsilon^2(\chi_1 + \chi_3 + w_1 + w_3)}
$$

$$
+ 2\xi \varepsilon^2 (\xi + \eta) e^{\eta_0 + \eta_0 + \varepsilon^2(\chi_2 + \chi_3 + w_2 + w_3)}
$$

$$
- 2(\xi^2 + \xi^2) e^{\eta_0 + \varepsilon^2(\chi_3 + w_3)}.
$$

On the other hand,

$$
\Delta \left\{ \left[ \frac{\eta}{a} + \xi e^{u} + (\xi + \eta) e^{u} \right] \left( \frac{r}{\varepsilon} \right) \right\}
$$

$$
= \frac{1}{a} \Delta \left\{ \eta_0 + \varepsilon^2(\chi_3 + w_3) \right\} (r)
$$

$$
+ \varepsilon^2 \Delta \left\{ \xi e^{u_0 + \varepsilon^2(\chi_1 + w_1)} + (\xi + \eta) e^{u_0 + \varepsilon^2(\chi_2 + w_2)} \right\}.
$$
The last two formulas together with (44) give

\[
\Delta \left\{ \frac{x_3}{a} + \zeta e^{u_0 + \varepsilon^2(x_1 + u_1)} + (\zeta + \tilde{\zeta}) e^{v_0 + \varepsilon^2(x_2 + u_2)} \right\} = \nonumber
\]

\[
= 2\zeta(\zeta - \tilde{\zeta}) e^{v_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + 2(\zeta + \tilde{\zeta}) e^{v_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + \nonumber
\]

\[
+ 2\zeta(\zeta + \tilde{\zeta}) e^{v_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} - 2(\zeta^2 + \tilde{\zeta}^2) \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(x_3 + u_3)} + \nonumber
\]

\[
- 2(\zeta^2 + \tilde{\zeta}^2) \frac{1}{\varepsilon^2} e^{\eta_0} - \frac{1}{a} \Delta w_3. \quad (50)
\]

In view of (48), (49) and (50), we introduce the following mappings:

\[
\mathcal{P}_1(x_1, x_2, x_3, \varepsilon) = \Delta x_1 - 4\zeta e^{u_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + 2(\zeta + \tilde{\zeta}) e^{v_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + \nonumber
\]

\[
+ 2(\zeta - \tilde{\zeta}) \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(x_3 + u_3)} - 2(\zeta - \tilde{\zeta}) \frac{1}{\varepsilon^2} e^{\eta_0} + \Delta w_1, \quad (51)
\]

\[
\mathcal{P}_2(x_1, x_2, x_3, \varepsilon) = \Delta x_2 + 2\zeta e^{u_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} - 2(\zeta + \tilde{\zeta}) e^{v_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + \nonumber
\]

\[
+ 2\zeta \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(x_3 + u_3)} - 2\zeta \frac{1}{\varepsilon^2} e^{\eta_0} + \Delta w_2, \quad (52)
\]

\[
\mathcal{P}_3(x_1, x_2, x_3, \varepsilon) = \nonumber
\]

\[
\Delta \left\{ x_3 + a\zeta e^{u_0 + \varepsilon^2(x_1 + u_1)} + a(\zeta + \tilde{\zeta}) e^{v_0 + \varepsilon^2(x_2 + u_2)} \right\} - 2a\zeta(\zeta - \tilde{\zeta}) e^{u_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + \nonumber
\]

\[
- 2a\zeta(\zeta + \tilde{\zeta}) e^{u_0 + v_0 + \varepsilon^2(x_1 + x_3 + u_1 + u_3)} + \nonumber
\]

\[
+ 2a(\zeta^2 + \tilde{\zeta}^2) \frac{1}{\varepsilon^2} e^{\eta_0 + \varepsilon^2(x_3 + u_3)} - 2a(\zeta^2 + \tilde{\zeta}^2) \frac{1}{\varepsilon^2} e^{\eta_0} + \Delta w_3. \quad (53)
\]

Now we let the functions \(w_1, w_2, w_3\) be given as follows:

\[
\begin{align*}
\Delta w_1 &= -2(\zeta - \tilde{\zeta}) e^{v_0} w_3 + 4\zeta e^{u_0 + v_0} - 2(\zeta + \tilde{\zeta}) e^{v_0 + v_0}, \\
\Delta w_2 &= -2\zeta e^{v_0} w_3 - 2e^{u_0 + v_0} + 2(\zeta + \tilde{\zeta}) e^{v_0 + v_0}, \\
\Delta w_3 &= -2a(\zeta^2 + \tilde{\zeta}^2) e^{v_0} w_3 + 2a(\zeta - \tilde{\zeta}) e^{v_0 + v_0} + 2a(\zeta + \tilde{\zeta}) e^{v_0 + v_0} + \nonumber
\end{align*}
\]

\[
- 2a(\zeta^2 + \tilde{\zeta}^2) \frac{1}{\varepsilon^2} e^{\eta_0} \Delta w_3. \quad (54)
\]

**Function Spaces.** Define the scalar products \(\langle \cdot, \cdot \rangle_{X_\alpha}\) and \(\langle \cdot, \cdot \rangle_{Y_\alpha}\), \(0 < \alpha < 1/2\), for functions in the spaces \(L_{\text{loc}}^2(\mathbb{R}^2)\) and \(W_{\text{loc}}^{2,2}(\mathbb{R}^2)\):

\[
\langle u, v \rangle_{X_\alpha} = \int_{\mathbb{R}^2} (1 + |x|^{2+2\alpha}) uv \, dx, \quad \text{if} \; u, v \in L_{\text{loc}}^2(\mathbb{R}^2),
\]

\[
\langle u, v \rangle_{Y_\alpha} = \langle \Delta u, \Delta v \rangle_{X_\alpha} + \int_{\mathbb{R}^2} \frac{uv}{1 + |x|^{2+2\alpha}} \, dx, \quad \text{if} \; u, v \in W_{\text{loc}}^{2,2}(\mathbb{R}^2).
\]

Let \(X_\alpha, Y_\alpha\) be the completion of \(L_{\text{loc}}^2(\mathbb{R}^2)\) and \(W_{\text{loc}}^{2,2}(\mathbb{R}^2)\) by the norm \(\| \cdot \|_{X_\alpha}\) and \(\| \cdot \|_{Y_\alpha}\), respectively, where

\[
\| u \|_{X_\alpha} = \sqrt{\langle u, u \rangle_{X_\alpha}}, \quad \| u \|_{Y_\alpha} = \sqrt{\langle u, u \rangle_{Y_\alpha}}.
\]
Let $X_3^\alpha = X_\alpha \times X_\alpha \times X_\alpha$ and $Y_3^\alpha = Y_\alpha \times Y_\alpha \times Y_\alpha$ be equipped with inner products $\langle u, v \rangle_{X_3^\alpha}$ and $\langle u, v \rangle_{Y_3^\alpha}$ given by

$$\langle f, g \rangle_{X_3^\alpha} = \sum_{i=1}^3 \langle f_i, g_i \rangle_{X_\alpha}, \quad \langle u, v \rangle_{Y_3^\alpha} = \sum_{i=1}^3 \langle u_i, v_i \rangle_{Y_\alpha},$$

where $f = (f_1, f_2, f_3), g = (g_1, g_2, g_3) \in X_3^\alpha$ and $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in Y_3^\alpha$. Note that $X_\alpha \hookrightarrow L^1(\mathbb{R}^2)$ and $Y_\alpha \subset C^0_{\text{loc}}(\mathbb{R}^2)$ from Hölder's inequality as well as the local regularity of the Laplace operator. Moreover, there exists $C > 0$ such that

$$|u(x)| \leq C \|u\|_{Y_\alpha}(\log^+ |x| + 1), \quad x \in \mathbb{R}^2, \quad (59)$$

for all $u \in Y_\alpha$, where $\log^+ |x| = \max\{0, \log |x|\}$ (please see [2]). We denote the subspaces consisting of radially symmetric functions by $X_\alpha^r$ and $Y_\alpha^r$. 

**Lemma 3.1.** Let $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ where $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are given in (53)-(57) for $\chi_1, \chi_2, \chi_3 \in Y_\alpha^\varepsilon$ and $\varepsilon > 0$. Let

$$\Omega_\delta = \{ (\chi_1, \chi_2, \chi_3) \in (Y_\alpha^\varepsilon)^3 : \|\chi_1\|_{Y_\alpha}^2 + \|\chi_2\|_{Y_\alpha}^2 + \|\chi_3\|_{Y_\alpha}^2 < \delta \}.$$ 

Then

(i) $\mathcal{P} : \Omega_\delta \times (0, 1) \to (X_\alpha^\varepsilon)^3$, provided that $\delta > 0$ is chosen sufficiently small.

(ii) $\mathcal{P}$ can be extended continuously to $\Omega_\delta \times [0, 1)$; in particular, $\mathcal{P}(0, 0, 0, 0) = 0$.

**Proof.** From (44), we have the estimate ([4]) that

$$\eta_0(r) = -4 \log r + O(1), \quad r \to \infty, \quad (60)$$

and thus,

$$\int_0^\infty r e^{\eta_0(r)} dr = \frac{2}{a(\zeta^2 + \tilde{\zeta}^2)},$$

which implies that

$$u_0(r) = \left(2n - \frac{\zeta - \tilde{\zeta}}{\zeta^2 + \tilde{\zeta}^2} \frac{4}{a}\right) \log r + O(1), \quad (61)$$

$$v_0(r) = \left(2\hat{n} - \frac{\tilde{\zeta}}{\zeta^2 + \tilde{\zeta}^2} \frac{4}{a}\right) \log r + O(1). \quad (62)$$

Therefore, $e^{\eta_0} \approx r^{-4}$ near infinity. Since $(\zeta_1, \tilde{\zeta}) \in \mathcal{D}_1 \cap \mathcal{D}_2$ with the assumptions (33) and (34), it follows that

$$2n - \frac{\zeta - \tilde{\zeta}}{\zeta^2 + \tilde{\zeta}^2} \frac{4}{a} < -2, \quad 2\hat{n} - \frac{\tilde{\zeta}}{\zeta^2 + \tilde{\zeta}^2} \frac{4}{a} < -2.$$ 

So $e^{\eta_0} \approx r^{-2-\sigma_1}$ and $e^{\eta_0} \approx r^{-2-\sigma_2}$ (near infinity). By (59) we may pick a small $\delta > 0$ such that $e^{\chi_i} \approx r^{\sigma}$ with $\sigma > 0$ sufficiently small. We estimate $w_1, w_2, w_3$. In
view of (58),

\[
|w_3(r)| \leq |w_3(0)| + \int_0^r \! s \log \left( \frac{r}{s} \right) \left\{ 2a(\zeta^2 + \tilde{\zeta}^2)e^{\eta_0}|w_3| \right\} (s) \, ds \\
+ \int_0^r \! s \log \left( \frac{r}{s} \right) \left\{ 2a(\zeta - \tilde{\zeta})e^{u_0+\eta_0} + 2a(\zeta + \tilde{\zeta})e^{v_0+\eta_0} \right\} (s) \, ds \\
\leq C_1 + C_2 \log r + \int_0^r \! s \log \left( \frac{r}{s} \right) \left\{ 2a(\zeta^2 + \tilde{\zeta}^2)e^{\eta_0}|w_3| \right\} (s) \, ds \\
\leq C'_1 + C_2 \log r + \exp \left( \int_0^r \! s \log \left( \frac{r}{s} \right) e^{\eta_0(s)} \right) \, ds \\
\leq C_2 \log r + C(r_0) \exp \left( \int_{r_0}^r \! s \log \left( \frac{r}{s} \right) s^{-4} \right) \, ds \\
\leq C \log r,
\]

in which \( r_0 > 0 \) is chosen sufficiently large and \( r > r_0 \). Similarly, \( |w_1(r)|, |w_2(r)| \leq C \log r \). From the above estimates, it is not hard to see that \( \mathcal{P} \) maps from \( \Omega_\delta \times (0, 1) \) to \( (X^r_\alpha)^3 \). Define

\[
\begin{align*}
\mathcal{P}_1(\chi_1, \chi_2, \chi_3, 0) &= \Delta \chi_1 + 2(\zeta - \tilde{\zeta})e^{\eta_0}\chi_3, \\
\mathcal{P}_2(\chi_1, \chi_2, \chi_3, 0) &= \Delta \chi_2 + 2\tilde{\zeta}e^{\eta_0}\chi_3, \\
\mathcal{P}_3(\chi_1, \chi_2, \chi_3, 0) &= \Delta \chi_3 + 2a(\zeta^2 + \tilde{\zeta}^2)e^{\eta_0}\chi_3.
\end{align*}
\]

Hence

\[
\begin{align*}
\mathcal{P}_1(\chi_1, \chi_2, \chi_3, \varepsilon) - \mathcal{P}_1(\chi_1, \chi_2, \chi_3, 0) &= -4\zeta e^{u_0+\eta_0} \left[ e^{\varepsilon^2(\chi_1 + \chi_3 + w_1 + w_3)} - 1 \right] \\
&\quad + 2(\zeta + \tilde{\zeta})e^{\eta_0} \left[ e^{\varepsilon^2(\chi_2 + \chi_3 + w_2 + w_3)} - 1 \right] \\
&\quad + 2(\zeta - \tilde{\zeta}) \frac{1}{\varepsilon^2} e^{\eta_0} \left[ e^{\varepsilon^2(\chi_3 + w_3)} - 1 - \varepsilon^2(\chi_3 + w_3) \right] \\
&= -4\zeta e^{u_0+\eta_0} e^2(\chi_1 + \chi_3 + w_1 + w_3) e^{\vartheta_1} \\
&\quad + 2(\zeta + \tilde{\zeta})e^{\eta_0} e^2(\chi_2 + \chi_3 + w_2 + w_3) e^{\vartheta_2} \\
&\quad + 2(\zeta - \tilde{\zeta}) e^{\eta_0} e^2(\chi_3 + w_3)^2 e^{\vartheta_3},
\end{align*}
\]

in which \( \vartheta_1, \vartheta_2, \vartheta_3 \) lie between 0 and \( \varepsilon^2(\chi_1 + \chi_3 + w_1 + w_3) \), \( \varepsilon^2(\chi_2 + \chi_3 + w_2 + w_3) \) and \( \varepsilon^2(\chi_3 + w_3) \) respectively. It follows that

\[
\lim_{\varepsilon \to 0} \| \mathcal{P}_1(\chi_1, \chi_2, \chi_3, \varepsilon) - \mathcal{P}_1(\chi_1, \chi_2, \chi_3, 0) \|_{X_\alpha} = 0.
\]

Similarly, \( \mathcal{P}_2(\chi_1, \chi_2, \chi_3, \varepsilon) \to \mathcal{P}_2(\chi_1, \chi_2, \chi_3, 0) \) and \( \mathcal{P}_3(\chi_1, \chi_2, \chi_3, \varepsilon) \to \mathcal{P}_3(\chi_1, \chi_2, \chi_3, 0) \) as \( \varepsilon \to 0 \). Especially, \( \mathcal{P}(0, 0, 0, 0) = 0. \)

We compute the partial Fréchet derivatives of \( \mathcal{P} \). Let \( L = \partial \mathcal{P}(\chi, \varepsilon)/\partial \chi \) at \( (\chi, \varepsilon) = (0, 0) \) where \( \chi \in (Y^r_\alpha)^3 \). Note that \( L \) is a linear mapping, from \((Y^r_\alpha)^3\) into \((X^r_\alpha)^3\), given by

\[
L: \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \mapsto \begin{pmatrix} \Delta \theta_1 + 2(\zeta - \tilde{\zeta})e^{\eta_0}\theta_3 \\ \Delta \theta_2 + 2\tilde{\zeta}e^{\eta_0}\theta_3 \\ \Delta \theta_3 + 2a(\zeta^2 + \tilde{\zeta}^2)e^{\eta_0}\theta_3 \end{pmatrix}, \quad \theta_1, \theta_2, \theta_3 \in Y^r_\alpha.
\]

\textbf{Lemma 3.2.} \( L \) is a surjective mapping.
Proof. Denote \( \text{Im} L = \{ g \in (X^r_\alpha)^3 : \exists f \in (Y^r_\alpha)^3, Lf = g \} \). We show that \( \text{Im} L = (X^r_\alpha)^3 \). Note that the Laplace operator \( \Delta : (\chi_1, \chi_2, \chi_3) \mapsto (\Delta \chi_1, \Delta \chi_2, \chi_3) \) is surjective (for a proof, please see [2, Proposition 2.1]). Moreover, it is easy to check \( K = L - \Delta \) is a compact operator from \( (Y^r_\alpha)^3 \) into \( (X^r_\alpha)^3 \) by decomposing \( R^3 \) into a bounded and an exterior domains, and using the Rellich-Kondrachev compactness lemma for the bounded domain. From [14, Lemma 5.1] it follows that \( \text{Im} L \) is closed. So we may decompose \( (X^r_\alpha)^3 = \text{Im} L \oplus (\text{Im} L)^\perp \). Suppose \( \text{Im} L \neq (X^r_\alpha)^3 \). There exists a non-zero element \( \phi = (\phi_1, \phi_2, \phi_3) \) of \( (X^r_\alpha)^3 \) such that \( \phi \in (\text{Im} L)^\perp \). Consequently,

\[
0 = \langle L(\chi_1, \chi_2, \chi_3), \phi \rangle = \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) \left\{ \left[ \Delta \chi_1 + 2(\zeta - \tilde{\zeta})e^{\eta_0} \chi_3 \right] \phi_1 + \left[ \Delta \chi_2 + 2\tilde{\zeta} e^{\eta_0} \chi_3 \right] \phi_2 \\
+ \left[ \Delta \chi_3 + 2a(\zeta^2 + \tilde{\zeta}^2) e^{\eta_0} \chi_3 \right] \phi_3 \right\} dx
\]

for all \( \chi_1, \chi_2, \chi_3 \in Y^r_\alpha \) in which \( \psi_i = (1 + |x|^{2+\alpha}) \phi_i, \ i = 1, 2, 3 \). Since \( C^\infty_0(\mathbb{R}^2) \subset Y_\alpha \), it follows from the elliptic regularity that \( \psi_1, \psi_2, \psi_3 \) are \( C^2 \)-functions. Using integration by parts, we have

\[
0 = \int_{\mathbb{R}^2} \chi_1 \Delta \psi_1 + \chi_2 \Delta \psi_2 + \chi_3 \left[ \Delta \psi_3 + 2(\zeta - \tilde{\zeta})e^{\eta_0} \psi_1 + 2\tilde{\zeta} e^{\eta_0} \psi_2 + 2a(\zeta^2 + \tilde{\zeta}^2) e^{\eta_0} \psi_3 \right] dx,
\]

whenever \( \chi_1, \chi_2, \chi_3 \in C^\infty_0(\mathbb{R}^2) \). Then

\[
\begin{align*}
\Delta \psi_1 &= 0, \\
\Delta \psi_2 &= 0, \\
\Delta \psi_3 + 2(\zeta - \tilde{\zeta})e^{\eta_0} \psi_1 + 2\tilde{\zeta} e^{\eta_0} \psi_2 + 2a(\zeta^2 + \tilde{\zeta}^2) e^{\eta_0} \psi_3 &= 0.
\end{align*}
\]

In particular, \( \psi_1, \psi_2, \psi_3 \in Y^r_\alpha \). We select the functions \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) which solve the following equations

\[
\begin{align*}
\Delta \tilde{x}_1 + 2(\zeta - \tilde{\zeta})e^{\eta_0} \tilde{x}_3 &= (1 + r)^{-4} \psi_1, \\
\Delta \tilde{x}_2 + 2\tilde{\zeta} e^{\eta_0} \tilde{x}_3 &= (1 + r)^{-4} \psi_2, \\
\Delta \tilde{x}_3 + 2a(\zeta^2 + \tilde{\zeta}^2) e^{\eta_0} \tilde{x}_3 &= (1 + r)^{-4} \psi_3.
\end{align*}
\]

Since \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) are at most of logarithmic growth near infinity, we have \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in Y^r_\alpha \). Putting \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) into (64), we arrive at the inequality

\[
0 = \int_{\mathbb{R}^2} \left[ \Delta \tilde{x}_1 + 2(\zeta - \tilde{\zeta})e^{\eta_0} \tilde{x}_3 \right] \psi_1 + \left[ \Delta \tilde{x}_2 + 2\tilde{\zeta} e^{\eta_0} \tilde{x}_3 \right] \psi_2 \\
+ \left[ \Delta \tilde{x}_3 + 2a(\zeta^2 + \tilde{\zeta}^2) e^{\eta_0} \tilde{x}_3 \right] \psi_3 dx
= \int_{\mathbb{R}^2} (1 + r)^{-4}(\psi_1^2 + \psi_2^2 + \psi_3^2) dx > 0,
\]

which is a contradiction. Therefore, \( \text{Im} L = (X^r_\alpha)^3 \). \( \square \)

Proof of Theorem 1.1. Let \( n, \tilde{n} \geq 1 \) be given and satisfy (35). Then, by \( A_1 = 4\pi G(n + 1), A_2 = 4\pi G(\tilde{n} + 1) \) and (33)-(34), it is easy to see that \( D_1 \cap D_2 \) is not
empty. Assume \((\zeta, \tilde{\zeta}) \in D_1 \cap D_2\). Apply a specific version of the Implicit Function Theorem (please see, e.g. [18]) to \(P\), it follows from Lemmas 3.1 and 3.2 that there exists a continuous mapping \(h : \varepsilon \mapsto (\chi_{1,\varepsilon}, \chi_{2,\varepsilon}, \chi_{3,\varepsilon})\) from a neighborhood \((0, \varepsilon_0)\) of the origin, with \(0 < \varepsilon_0 < \varepsilon\), to \((Y_0^\varepsilon)^3\) such that \(h(0) = \mathbf{0}\) and \(P(\varepsilon, \chi_{1,\varepsilon}, \chi_{2,\varepsilon}, \chi_{3,\varepsilon}) = 0\) for all \(\varepsilon \in (0, \varepsilon_0)\). We recover (45)-(47) and obtain a family of non-topological solutions solving (43) as follows:

\[
\begin{align*}
 u(r) &= u_{0,\varepsilon}(r) + \varepsilon^2 w_1(\varepsilon r) + \varepsilon^2 \chi_{1,\varepsilon}(\varepsilon r), \\
 v(r) &= v_{0,\varepsilon}(r) + \varepsilon^2 w_2(\varepsilon r) + \varepsilon^2 \chi_{2,\varepsilon}(\varepsilon r), \\
 \eta(r) &= \eta_{0,\varepsilon}(r) + \varepsilon^2 w_3(\varepsilon r) + \varepsilon^2 \chi_{3,\varepsilon}(\varepsilon r),
\end{align*}
\]

where \(u_{0,\varepsilon}(r) = u_0(\varepsilon r) + 2\log \varepsilon\), \(v_{0,\varepsilon}(r) = v_0(\varepsilon r) + 2\log \varepsilon\) and \(\eta_{0,\varepsilon}(r) = \eta_0(\varepsilon r) + 2\log \varepsilon\). Accordingly, from (43), (61) and (67) we have

\[
\sum_{s=1}^2 \beta_s = \int_0^\infty r e^\eta \left[ 4\zeta (e^u - 1) - 2(\zeta + \tilde{\zeta})(e^v - 1) \right] dr
\]

in which the limit

\[
\lim_{s \to +\infty} sw'_1(s) = \int_0^{\infty} r \left[ 2(\zeta - \tilde{\zeta}) e^{\eta_0} w_3 + 4\zeta e^{u_0 + \eta_0} - 2(\zeta + \tilde{\zeta}) e^{v_0 + \eta_0} \right] dr
\]

diverges; On the other hand, we use the property \(X_\alpha \hookrightarrow L^1(\mathbb{R}^2)\) to yield

\[
\lim_{s \to +\infty} s \chi'_{1,\varepsilon}(s) = \int_0^\infty r \Delta \chi_{1,\varepsilon} dr
\]

\[
(1/2\pi) \parallel \Delta \chi_{1,\varepsilon} \parallel_{L^1(\mathbb{R}^2)} \leq C \parallel \Delta \chi_{1,\varepsilon} \parallel \parallel X_\alpha \parallel \leq C \parallel \chi_{1,\varepsilon} \parallel_{Y_0}.
\]

Similarly,

\[
\beta_2 = \int_0^\infty r e^\eta \left[ -2\zeta (e^u - 1) + 2(\zeta + \tilde{\zeta})(e^v - 1) \right] dr
\]

\[
= 2\bar{u} - \lim_{r \to +\infty} rv'(r)
\]

\[
= \frac{4}{\zeta^2 + \tilde{\zeta}^2 a} - \lim_{r \to +\infty} \left[ \varepsilon^3 rw'_2(\varepsilon r) + \varepsilon^3 r \chi'_{2,\varepsilon}(\varepsilon r) \right]
\]

\[
= \frac{4}{\zeta^2 + \tilde{\zeta}^2 a} - \varepsilon^2 \lim_{s \to +\infty} sw'_2(s) - \varepsilon^2 \lim_{s \to +\infty} s \chi'_{2,\varepsilon}(s)
\]

\[
= \frac{4}{\zeta^2 + \tilde{\zeta}^2 a} + o(\varepsilon),
\]

as \(\varepsilon \to 0\). We conclude the theorem. \(\Box\)
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E-mail address: chern@math.nctu.edu.tw
E-mail address: sgyang@math.nctu.edu.tw
E-mail address: zhiyou@cc.ncue.edu.tw
E-mail address: chenzh@math.nctu.edu.tw