On quantization of affine Jacobi varieties of spectral curves.

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Abstract. A quantum integrable model related to $U_q(\widehat{sl}(N))$ is considered. A reduced model is introduced which allows interpretation in terms of quantized affine Jacobi variety. Closed commutation relations for observables of reduced model are found.
1 Classical case.

Consider classical integrable model with the $l$-operator which is an $N \times N$ matrix depending on the spectral parameter $z$:

$$l(z) = l^+(z) + l^0(z) + zl^-(z), \quad (1)$$

$l^\pm(z)$ are polynomials of degree $n - 1$, $l^0(z)$ is polynomial of degree $n$, $l^+(z)$ ($l^-(z)$) is upper (lower)-triangular, $l^0(z)$ is diagonal. The classical algebra of observables $\mathcal{A}$ is generated by the coefficients of polynomials giving the matrix elements of $l(z)$. The algebra $\mathcal{A}$ is Poisson algebra, Poisson structure being given by the $r$-matrix relations:

$$\{l(z) \otimes l(z')\} = [r(z, z'), l(z) \otimes l(z')]$$

where the classical $r$-matrix is

$$r(z, z') = \frac{(z + z')}{2(z - z')} \sum_i E^{ii} \otimes E^{ii} +$$

$$+ \frac{z}{z - z'} \sum_{j > i} E^{ji} \otimes E^{ij} + \frac{z'}{z - z'} \sum_{j < i} E^{ij} \otimes E^{ji} -$$

with

$$E^{ij} = \begin{pmatrix}
    0 & \cdots & 0 \\
    \cdots & 1 & \cdots \\
    0 & \cdots & 0
\end{pmatrix}_{i, j}$$

Let us introduce the polynomials $t_k(z)$:

$$\det(wI + l(z)) = \sum_{k=0}^{N} w^{n-k} t_k(z)$$

and there coefficients $t^{(j)}_k$ defined by

$$t_k(z) = \sum_{j=0}^{kn} t^{(j)}_k z^{nk-j}$$

The form of the Poisson brackets implies that all coefficients of characteristic polynomial are in involution:

$$\{\det(wI + l(z)), \det(w'I + l(z'))\} = 0$$

Moreover, part of them belongs to the center of the Poisson brackets: these are

$$t^{(j)}_N \quad \forall j$$

because of

$$\{\det(l(z)) \otimes l(z')\} = 0$$

and

$$t^{(kn)}_k = t_k(0) \quad \forall k$$

because $r(z, z')$ becomes a constant matrix for $z = 0$. 
Let us fix the center. Then the remaining phase space $\mathcal{M}$ is of dimension:

$$\dim(\mathcal{M}) = nN(N - 1)$$

The number of remaining integrals of motion (of coefficients $t_k^{(j)}$ except the central ones) is

$$\frac{1}{2} nN(N - 1) = \frac{1}{2} \dim(\mathcal{M})$$

So, the number of integrals of motion in involution is exactly one half of the dimension of the phase space and, hence, we are dealing with an integrable model.

However, there is one subtlety here. We consider some specific real form of the model (without going into the details at this point). For the real form in question the level of integrals looks as follows:

Figure 1.

The integrals are divided into “compact” and “non-compact” ones. The typical integral curve of Hamiltonian vector-field corresponding to “compact” integral is the spiral on Figure 1 while the one corresponding to “non-compact” integral is strictly vertical. We would like to reduce the model in order that the “non-compact” integrals disappear and the integral curves of “compact” ones close. Such reduction is important for several reasons:

1. To consider average and adiabatic approximation we need compact levels of integrals.
2. In the quantum case non-compact directions must correspond to continuous spectrum which we would like to eliminate.
3. The integrable model in question can be viewed as lattice approximation of CFT. Non-compact directions correspond to zero-modes. We would like to separate them from the rest of degrees of freedom in order to have clear identification of primary fields.
4. There is an algebra-geometric reason to eliminate the non-compact directions on which we would like to make some detailed comments.

It is well known that the integrable model in question is closely related to the algebraic geometry through the study of the spectral curve:

$$X : r(w, z) \equiv \det(wI + l(z)) = 0$$
of genus: \( g = \frac{1}{2}(N - 1)(Nn - 2) \). The relation is as follows. Consider the Jacobi variety of the curve \( X \) i.e. the complex torus:

\[
J = \frac{\mathbb{C}^g}{\mathbb{Z}^g \times B\mathbb{Z}^g}
\]

where \( B \) is the period matrix. Introduce corresponding Riemann Theta-function which satisfies:

\[
\theta(\zeta + m + Bn) = \exp 2\pi i \left( -\frac{1}{2} t^n Bn - t^n \zeta \right) \theta(\zeta), \quad \forall m, n \in \mathbb{Z}^g
\]

The relation between integrable models and algebraic geometry is normally based on the following “identity”:

\[
\text{Liouville torus} = \text{Real part of Jacobi variety}.
\]

However, this relation only holds in the usual cases because the levels of the “non-compact” integrals are usually fixed from the very beginning in the particular integrable model under consideration, so these integrals rarely, if ever, appear explicitly in the discussion (the problem of having “extra” degrees of freedom resulting from imposing some additional implicit constraints is well-known though).

But in our case this “identity” cannot be correct as can be easily seen from the comparison of dimensions:

\[
\dim(\mathcal{M}) = 2g + 2(N - 1)
\]

In fact, the real part of Jacobi variety describe only the compact part of the level of integrals and, clearly, to use the usual algebra-geometric formulation we need to eliminate \( N - 1 \) non compact integrals. Let us discuss this point in some more details.

The integrable model under consideration allows a complexification. Complexification of the compact part of the level of integrals should give the Jacobi variety. More precisely, the complexification gives the Jacobi variety from which the following divisor is cut off:

\[
D = \{ \zeta \in J | \theta(\zeta + \rho_1) \cdots \theta(\zeta + \rho_N) = 0 \}
\]

where \( \rho_1, \cdots, \rho_N \) are images under Abel map of \( N \) points of \( X \) which project onto the point \( \infty \) on \( z \)-plane. In other words the observables considered as functions on the Jacobi variety possess singularities (only) on \( D \). Generally we have:

\[
\text{Functions on the Level of Integrals of Motion} = (\text{Functions on } J_{\text{aff}}) \times (\text{Sections})
\]

where \( J_{\text{aff}} \equiv J - D \) stands for affine Jacobi variety. \textit{Sections} correspond exactly to non-compact directions of the level of integrals. In algebra-geometric language they are given by expressions of the form

\[
\frac{\theta(\zeta + \rho_i)}{\theta(\zeta + \rho_j)}
\]

Our goal is to reduce the model on the sub-manifold of the phase space which does not contain the non-compact directions. There is a general Dirac procedure to do that: we have to fix \( N - 1 \) first kind constraints (the integrals of motion \( t^n_0 \)), the non-compact coordinates being the “auxiliary” relations. However, in the case under consideration there is a very direct way of describing the reduced model which allows quantum analogue. To be precise, there is an \( N \times N \) matrix \( s \) whose matrix elements are dynamical variables such that the similarity transformation of \( l(z) \):

\[
m(z) = s l(z) s^{-1}
\]

is of the form:

\[
m(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},
\]
where $d(z)$ is $(N - 1) \times (N - 1)$ matrix, $c(z)$ is $(N - 1)$-vector, $b(z)$ is $(N - 1)$-covector and $a(z)$ is a scalar. The similarity transformation (2) is such that $a(z)$ is a polynomial of degree $n - 2$, $c(z)$ is a polynomial of degree $n - 1$,

$$b(z) = z^{n-1}e_{N-1} + \tilde{b}(z), \quad d(z) = z^n d_0 + \tilde{d}(z)$$

where $\tilde{b}(z)$ is a polynomial of degree $n - 2$, $\tilde{d}(z)$ is a polynomial of degree $n - 1$, the matrix elements of $d_0$ are in the center. Here and later we use the notation:

$$e_j = (0 \cdots 1 \cdots 0)$$

Moreover, the coefficients of matrix elements of $m(z)$ have closed Poisson brackets. Thus the matrix $m(z)$ defines the reduced phase space $\mathcal{M}_r$ and the reduced algebra of observables $\mathcal{A}_r$ as $l(z)$ defines $\mathcal{M}$ and $\mathcal{A}$. We do not write down corresponding Poisson brackets because they can be obtained from quantum commutation relations described later. In the classical case a similar construction for linear Poisson brackets is given in [1].

It is easy to calculate that

$$\dim(\mathcal{M}_r) = 2g$$

That is why the algebra-geometric parametrization is well defined for the reduced model. Namely, consider the symmetric power of the spectral curve:

$$X(g) = X^g/S_g$$

where $S_g$ is the symmetric group. The points on $X(g)$ are divisors: $P = \{p_1, \cdots, p_g\}$ where $p_j = (z_j, w_j)$ are points on the spectral curve. Consider the Abel transformation:

$$X(g) \to J$$

This transformation is one-to-one on non compact varieties:

$$X(g) - \tilde{D} \simeq J_{aff}$$

With every $1 \leq i \leq g$ we associate two numbers $k, l$ defined as follows. $k$ is such that

$$\frac{1}{2}(k - 1)(kn - 2) < i \leq \frac{1}{2}k((k + 1)n - 2)$$

and $l$ is defined by

$$l = i - \frac{1}{2}(k - 1)(kn - 2)$$

The polynomial $f_i(z, w)$ is defined by

$$f_i(z, w) = w^{k-1}z^{l-1}$$

(3)

The meaning of $f_i(z, w)$ is clear: the holomorphic differentials on the spectral curve are given by

$$\omega_i = \frac{f_i(z, w)dz}{\partial w r(w, z)} \quad \forall i = 1, \ldots g$$

The divisor $\tilde{D}$ is defined in terms of polynomials $f_i(z, w)$. A point on $X(g)$ for which $p_i \neq p_j \forall i \neq j$ belongs to $\tilde{D}$ if

$$\det(f_i(z_j, w_j)) = 0$$
When there are coinciding points the definition needs some changes, but we shall not go into details here.

There is an explicit construction:

$$m(z) \rightarrow X(g)$$

such that

$$\{z_i, w_j\} = \delta_{i,j} z_i w_i$$

Thus the variables $z_j, w_j$ describe separated variables for the reduced integrable model. It is important that the inverse map (algebraic) exists with singularities on $\tilde{D}$ only. So, the level of integrals of the complexified reduced model give exactly the affine Jacobian.

### 2 Quantum reduction.

Consider a quantum integrable model described by the $l$-operator $L(z)$ satisfying standard commutation relations

$$R(z, z')(L(z) \otimes I)(I \otimes L(z')) = (I \otimes L(z'))(L(z) \otimes I)R(z, z') \quad (4)$$

The quantum $R$-matrix is given by:

$$R(z, z') = zR_{12}(q) - z'R_{21}(q)^{-1}$$

where

$$q = e^{i\gamma},$$

$\gamma$ plays the role of Plank constant. The constant $R$-matrix is given by

$$R_{12}(q) = \sum_{j=1}^{N} q^{E^{jj}} \otimes q^{E^{jj}} + (q - q^{-1}) \sum_{j>i} E^{ji} \otimes E^{ij}$$

Define the quantum determinant of $L(z)$:

$$q\text{-det}(L(z)) = \sum_{\pi} (-q)^{l(\pi)} L_{1\pi(1)}(zq^{N-1}) L_{2\pi(2)}(zq^{N-2}) \cdots L_{N\pi(N)}(zq^{-N-1})$$

where $l(\pi)$ is the minimal number of transpositions of nearest neighbors in $\pi$. Integrals of motion and elements of center appear in

$$q\text{-det}(wI + L(z))$$

Now we have to be a little bit more specific. Like in classics (1) we shall assume that the leading coefficient of $L(z)$ is lower-triangular:

$$L(z) = z^n \mu + O(z^{n-1})$$

where

$$\mu = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\mu_{21} & \mu_{22} & 0 & \cdots & 0 \\
\mu_{31} & \mu_{32} & \mu_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{N1} & \mu_{N2} & \mu_{N3} & \cdots & \mu_{NN}
\end{pmatrix}$$

such form is consistent with the commutation relations (4). Setting $\mu_{11} = 0$ is very convenient for us, it corresponds to some special choice of integrable model.
Take the first row of $L(z)$:
\[ e_1 L(z) = z^{n-1} \nu + O(z^{n-2}), \]
and consider the $N \times N$ matrix whose matrix elements are operators:
\[
S = \begin{pmatrix}
e_1 \\
v \mu^N \\
\vdots \\
v \mu \\
v
\end{pmatrix}
\]
Let us show that
\[ S \mu = U S \]
where
\[
U = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & -t_1 & \cdots & -t_{N-2} & -t_{N-1} \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]
and $t_j$ are defined by
\[
\prod_{i=2}^{N} (x - q^{-1} \mu_{jj}) = \sum_{j=0}^{N-1} x^{N-1-j} t_j
\]
Obviously, the only thing we have to prove is
\[ \sum_{k=0}^{N-1} t_k \nu \mu^{N-1-k} = 0 \quad (5) \]
where $t_0 = 1$. Notice two circumstances. First, the relations (6) imply that
\[
\mu_{jj} q^{E_{jj}} \mu = \mu q^{E_{jj}} \mu_{jj} \\
\nu \mu_{jj} = \mu_{jj} \nu
\]
Second, the expression $\mu^k$ contains
\[
\mu_{j_1 j_2} \mu_{j_2 j_3} \mu_{j_3 j_4} \cdots
\]
with $j_1 \geq j_2 \geq j_3 \geq j_4 \geq \cdots$. From (6) one finds:
\[
\nu \mu^l \mu_{jj} E^{jj} = \nu \mu^l \mu_{jj} q^{E_{jj}-1} E^{jj} = q^{-1} \mu_{jj} \nu \mu^l E^{jj}
\]
So, in the expression $\mu^k$ one can move to the left all the $\mu_{jj}$ replacing them by $q^{-1} \mu_{jj}$, the rest consists of expressions of the form (6) with strictly ordered indices. Hence the equation (6) follows from the corresponding classical equation: a matrix with commuting entries satisfies its characteristic equation.

Introduce the matrix $M(z)$:
\[ M(z) = S L(z) S^{-1} \]
We have shown that
\[ M(z) = z^n U + O(z^{n-1}) \]
Moreover, since \( \nu \) is the last row of \( S \) we have
\[
e_1 M(z) = z^{n-1}e_N + O(z^{n-2})
\]

It is important that \( t_j \) commute with elements of \( M(z) \).

Similarly to the classical case introduce the notations
\[
M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},
\]
where \( D(z) \) is \( (N-1) \times (N-1) \) matrix etc. We have the same behavior in \( z \) as in classics: \( A(z) \) is a polynomial of degree \( n-2 \), \( C(z) \) is a polynomial of degree \( n-1 \),
\[
B(z) = z^{n-1}e_{N-1} + \tilde{B}(z), \quad D(z) = \tilde{U}z^n + \tilde{D}(z)
\]
where \( \tilde{B}(z), \tilde{D}(z) \) are respectively polynomials of degrees \( n-2 \) and \( n-1 \), \( \tilde{U} \) is \( (N-1) \times (N-1) \) matrix obtained from \( U \) by omitting first row and first column.

The algebra \( \mathcal{A}_q \) is generated by coefficients of matrix elements of \( M(z) \), \( t_j \) can be considered as \( \mathbb{C} \)-number parameters of \( \mathcal{A}_q \).

We want to find closed commutation relations for \( M(z) \). Introduce the following object:
\[
\hat{S}_{12} = qE^{11} \otimes q^{E^{11}} + \sum_{i=2}^{N}(\nu \otimes I)(R_{12}(q)(\mu \otimes I))^{N-i}(E^{ii} \otimes I)
\]
which is a matrix in the tensor product \( \mathbb{C}^N \otimes \mathbb{C}^N \) with non-commuting entries. It is possible to prove the auxiliary commutation relations:
\[
Y_{12}(S \otimes I)\hat{S}_{21} = (I \otimes S)\hat{S}_{12}R_{12}(q), \\
(S \otimes I)\hat{S}_{21}(T(z) \otimes I) = Z_{12}(z)(M(z) \otimes I)(S \otimes I)\hat{S}_{21}R_{21}(q)
\]
where we use the usual notation \( X_{21} = PX_{12}P \) with \( P \) being the permutation matrix acting in \( \mathbb{C}^N \otimes \mathbb{C}^N \). The matrices \( Y_{12} \) and \( Z_{12}(z) \) have \( \mathbb{C} \)-number matrix elements. They are defined as follows. Consider the matrix
\[
C_{12} = (I - E^{11}) \otimes I + \sum_{j=1}^{\infty} V^j \otimes U^j
\]
where
\[
V = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
The matrices \( Y_{12} \) and \( Z_{12}(z) \) are defined using \( C_{12} \):
\[
Y_{12} = qI - (q - q^{-1})C_{12}(I - P), \\
Z_{12}(z) = I - (q - q^{-1})z (I \otimes U)C_{12}(E^{N,1} \otimes I)
\]
where \( P \) is permutation.

It is easy to see that the operator \( C_{12}(I - P) \) is a projector:
\[
(C_{12}(I - P))^2 = C_{12}(I - P)
\]
which explains the following simple formula for the inverse of $Y_{12}$:

$$Y_{12}^{-1} = q^{-1} I + (q - q^{-1}) C_{12}(I - P)$$

Now we are ready to find the closed commutation relations in question. Take the relation (4) and multiply it by $(I \otimes S) \tilde{S}_{12}$ from the left and by $((S \otimes I) \tilde{S}_{21})^{-1}$ from the right. After some calculations using (8) one finds

$$\tilde{R}(z_1, z_2) K_{12}(z_1)(M(z_1) \otimes I) K_{21}(z_2)(I \otimes M(z_2)) = K_{21}(z_2)(I \otimes M(z_2)) K_{12}(z_1)(M(z_1) \otimes I) \tilde{R}(z_1, z_2)$$

where

$$\tilde{R}(z_1, z_2) = z_1 Y_{12} - z_2 Y_{21}^{-1},$$

$$K_{12}(z) = Y_{12}^{-1} Z_{12}(z)$$

3 Discussion.

Similarly to [2, 3, 4] it is possible to construct quantum separated variables:

$$M(z) \rightarrow (z_1, w_1), \cdots, (z_g, w_g)$$

such that

$$w_i z_i = q^2 z_i w_i$$

It should be possible to show that every element of $A_q$ can be expressed in terms of these separated variables. This is a complicated statement which we did not prove yet, but exactly this property of $A_q$ is the reason for introducing reduced model. An analogue of singularities on $\tilde{D}$ that we had in classical has to manifest itself after quantization.

The original Hilbert space where the elements of $A_q$ act should be unitary equivalent to the Hilbert space: functions of

$$\zeta_j = \frac{1}{2} \log(z_j)$$

The operator $w_j$ is shift of $\zeta_j$ by $i\gamma$. To finish the definition of the Hilbert space we need to define the scalar product. The duality of quantum integrable model is to be emphasized. There is dual integrable model which is similar to the original one but for which the Plank constant changes

$$\gamma \rightarrow \frac{\pi^2}{\gamma}$$

Consider the dual operators

$$Z_j = \exp\left(\frac{2\pi}{\gamma} \zeta_j\right), \quad W_j = \exp\left(\pi i \frac{\partial}{\partial \zeta_j}\right)$$

It is easy to see that formally $Z_j, W_j$ commute with $z_j, w_j$. But, again, to make real sense of the commutativity we have to define the Hilbert space where the operators act. We would conjecture that the scalar product in the space of functions of $\zeta_j$ is given by

$$\langle F | G \rangle = \int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_g \frac{F(\zeta_1, \cdots, \zeta_g)}{\text{det}(f_i(\zeta_j, w_j)) \text{det}(f_i(Z_j, W_j)) G(\zeta_1, \cdots, \zeta_g)}$$

where $f_i$ are the polynomials defining the holomorphic differentials (3). Notice that $w_j, W_j$ act on $G(\zeta_1, \cdots, \zeta_g)$. There are two reasons why we believe this formula to be true. First of them is quasiclassics: the determinant $\text{det}(f_i(\zeta_j, w_j))$ enters Liouville measure rewritten in the separated variables. The second is known case $N = 2$ [3]. We shall comment more on this point in later publications.
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