Lie Algebroid Yang Mills with Matter Fields

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Abstract

Lie algebroid Yang-Mills theories are a generalization of Yang-Mills gauge theories, replacing the structural Lie algebra by a Lie algebroid $E$. In this note we relax the conditions on the fiber metric of $E$ for gauge invariance of the action functional. Coupling to scalar fields requires possibly nonlinear representations of Lie algebroids. In all cases, gauge invariance is seen to lead to a condition of covariant constancy on the respective fiber metric in question with respect to an appropriate Lie algebroid connection.

The presentation is kept in part explicit so as to be accessible also to a less mathematically oriented audience.

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1 Introduction

In Ref. [1] pure Yang Mills (YM) gauge theories have been generalized to a setting where the structural Lie algebra is replaced by a Lie algebroid. This is a vector bundle $E \rightarrow M$ with, among others, a Lie algebra structure on its sections, thus reducing to a Lie algebra for $M$ being a point, in which case also the Lie Algebroid Yang Mills (LAYM) gauge theory reproduces just an ordinary YM theory in $d$ spacetime dimensions. Simultaneously, it constitutes a nonlinear type of gauge theory, which in contrast to topological prototypes like the Poisson Sigma Model [13, 14] has propagating degrees of freedom. At least on the classical level, moreover, these propagating degrees seem to be those of ordinary YM theories, albeit of potentially different type and with potentially different structure groups, glued together over some finite dimensional moduli space [1].

In this paper we first reconsider these LAYM theories, using a second type of gauge symmetries (one that is induced by an auxiliary connection chosen on $E$). For actions of type $F^2_{(1)} + F^2_{(2)}$, where $F_{(1)}$ and $F_{(2)}$ denote the 1-form and 2-form field strengths of the gauge field, respectively, and the square is understood as denoting an appropriate norm square, we find that gauge invariance restricts $E$ to be an action Lie Algebroid, which from the physical point of view corresponds to ordinary YM theory coupled to “Higgs fields” possibly taking values in some curved target manifold. We then show that the action [1] of the form $B F^2_{(1)} + F^2_{(2)}$, $B$ denoting Lagrange multiplier fields, is gauge invariant if the respective fiber metric on $E$ is covariantly constant w.r.t. a Lie algebroid or $E$-connection (induced by the auxiliary ordinary one on $E$) that (at least when flat) can be thought of generalizing the adjoint representation of a Lie algebra.

Then we turn to the main subject of the paper, the coupling of scalar matter fields to the LAYM theory. We first assume that these scalar fields take values in some vector bundle $V \rightarrow M$. Starting with some elementary ansatz for gauge transformations of the scalar fields, we are lead rather directly to flat $E$-connections on $V$. This is the mathematical generalization of a Lie algebra representation on a vector space, to which it reduces for $M$ being a point. Gauge invariance of the kinetic term requires that the fiber metric on $V$, needed for its construction, is covariantly constant w.r.t. this $E$-connection.

This is then, in a second step, generalized to scalar fields with target space an arbitrary bundle $p: \tilde{M} \rightarrow M$ over the base of the Lie algebroid. In this
context it is helpful to observe that all the needed data can be reassambled into a Lie algebroid structure on $\tilde{E} = p^*E$. It is the generalization of an action Lie algebroid to the context of Lie algebroids. Gauge invariance of the more general kinetic term in this sigma model context now requires a fiber metric on $V\tilde{M}$ (the bundle of vertical vectors on $\tilde{M}$) that is covariantly constant w.r.t. a canonically induced flat $\tilde{E}$-connection on $V\tilde{M}$.

This perspective suggests a reinterpretation of the scalar fields. Namely, one may have started right away by considering the Lie algebroid $\tilde{E} \to \tilde{M}$. The previously constructed coupled LAYM-matter Lagrangian is then seen as a functional of a (pure, if one likes) LAYM gauge theory for $\tilde{E}$ of the form that only part of the 1-form field strengths enter the functional with Lagrange multipliers while the “remaining” ones\footnote{This is formulated more intrinsically in the last section of the present article.} are squared by means of an appropriate symmetric covariant two-tensor on $\tilde{M}$ (a partially degenerate “metric” tensor on $\tilde{M}$).

Finally, we exploit generalized Bianchi identities to further relax the condition on the fiber metric $Eg$ on $E$ in a pure LAYM-theory of the type considered in \cite{1}. In fact, for gauge invariance it turns out to be sufficient that the restriction of $Eg$ to the kernel of the anchor map of $E$ is invariant w.r.t. a canonical Bott-type $E$-connection.

## 2 Lie Algebroid Yang Mills revisited

Here we start recalling the basic elements of a Lie algebroid Yang-Mills (YM) theory in a rather explicit, elementary fashion. The $d$-dimensional spacetime manifold we denote as $(\Sigma, h)$, where $h$ is a fixed (possibly pseudo-) Riemannian metric. The structural Lie algebra entering the construction of an ordinary YM algebra is generalized to a Lie algebroid $(E \to M, [\cdot, \cdot], \rho)$, the basic definition of which (together with other background material) can be found in Appendix A. Using some local coordinates $x^i$ on $M$ and a local frame $e_a$ of $E$, all the structural quantities of $E$ can be described by functions $\rho^i_a(x)$ and $C^c_{ab}(x)$, satisfying the differential equations

$$\rho^i_a \rho^j_b, - \rho^j_b \rho^i_a = C^c_{ab} \rho^i_c, \quad C^e_{ad} C^d_{bc} + \rho^i_a C^e_{bc,i} + \text{cycl}(abc) = 0.$$  

Clearly, if $M$ is a point, thus $C^c_{ac}$ not depending on $x^i$, and $\rho^i_a \equiv 0$, one reobtains the structure constants of a Lie algebra $\mathfrak{g}$.
For the fields we take 1-form fields $A^a = A^a_\mu(u^\mu) du^\mu$, where $u^\mu$ are coordinates on $\Sigma$, together with 0-form fields $X^i(u^\mu)$. The latter ones describe a map $X^i$ from $\Sigma$ to $M$, $X$ and $A$ together a vector bundle map $a : T\Sigma \to E$ (cf. [2] or [1] for further details). Associated to these “gauge fields” are the “field strengths”

$$F^i = dX^i - \rho^i_a A^a,$$

$$F^a = dA^a + \frac{1}{2} \epsilon^a_{bc} A^b \wedge A^c + \Gamma^a_i F^i \wedge A^b,$$

where $\Gamma^a_i$ are the coefficients of a fixed background connection $\nabla_\mu$ in $E$; they are necessary if one wants to define the 2-form field strengths $F^a$ covariantly with respect to $E$-frame changes. This becomes most transparent when rewriting the second equation according to

$$F^a = (D_\nabla A)^a - \frac{1}{2} T^a_{bc} A^b \wedge A^c,$$

where

$$(D_\nabla A)^a \equiv dA^a + \Gamma^a_i \rho^i A^i \wedge A^b,$$

$$T^c_{ab} \equiv -C^c_{ab} + \rho^i_a \rho^i_b - \rho^i_b \Gamma^c_i.$$

Here $D_\nabla A$ is the exterior covariant derivative on $A \in \Omega(\Sigma, X^* E)$ and $T$ is the $E$-torsion of the $E$-connection $\nabla_{\rho^i}$, both being induced by the chosen connection $\nabla$ on $E$ (cf. Appendix A for further details); the 2-form field strength is then an element in $\Omega^2(\Sigma, X^* E)$. In the specific case described at the end of the previous paragraph, $E = \mathfrak{g}$, one is back to the usual YM setting (with a Lie algebra valued 2-form curvature and no 1-form field strength).

This also applies to the gauge transformations, which we will now address.

Infinitesimally the gauge transformations are taken to be of the form

$$\delta_\epsilon X^i = \rho^i_a \epsilon^a,$$

$$\delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c + \Gamma^a_i \epsilon^b F^i,$$

Both field strengths together can be given a meaning also without introducing a connection (cf., e.g. [15]): it is only the separation of the 2-form part which requires the connection.—The fixed connection on $E$ is not to be confused with the gauge fields, which, in the case of an ordinary YM theory are connections in a principal bundle; the former ones correspond to structures needed to be fixed for defining a functional, while the latter ones are dynamical, i.e. they are the argument of that functional.

We discuss trivial bundles over $\Sigma$ here only, cf. [15] for how to generalize to nontrivial ones.
where the same connection coefficients were used that entered already the definition of $F^a$. There is also an alternative, geometrically motivated, off-shell closed version of gauge symmetries, not using an auxiliary connection and also generalizing the usual YM ones (cf. [2, 1]): as mentioned already in the Introduction, in this note we want instead to focus on this connection-induced type of gauge symmetries. In any case, in the variation of $A^a$ a term proportional to $F^i$ is needed for $E$-covariance again. Note, however, that the terms in (7) do not combine completely into covariant objects following the pattern of (3):

$$\delta_c A^a = D_c \epsilon^a - T_{ba}^a A^b \epsilon^c - \rho_i^c \epsilon^a T_{ib}^a A^b.$$  

The reason is that infinitesimal gauge transformations are a derivative-type object and the extra term is needed for compatibility with (6), cf. [2] as well as the likewise discussion following Eq. (21) below.

On the $A$-fields the variations (7) close only modulo a term proportional to $F^i$:

$$\left( [\delta_{\epsilon_1}, \delta_{\epsilon_2}] - \delta_{\epsilon_3} \right) A^a = \epsilon_1^b \epsilon_2^c F^i S_{ibc}^a,$$

$$\epsilon_3^a \equiv C_{bc}^a \epsilon_1^b \epsilon_2^c$$

$$S_{ibc}^a \equiv \nabla_i T_{bc}^a + \rho_i^b R_{ij}^a - \rho_i^b R_{ijc}^a$$

where $\nabla_i$ denotes the covariant derivative with respect to the fixed background connection on $E$ and $R_{ij}^b$ its curvature. As a consequence, the gauge symmetries provide a representation of the Lie algebroid on the fields $X^i$, $A^a$ only if either $F^i = 0$ or $S_{ibc}^a = 0$. For later use we provide the gauge variation of the field strengths:

$$\delta_i F^i = \epsilon^a \left( \nabla_j \rho_i^a \right) F^j,$$

$$\delta_i F^a = -\epsilon^c \left( \epsilon_{bc}^a + \Gamma_i^c \rho_i^b \right) F^b,$$

$$+ \frac{1}{2} \epsilon^b R_{ijb}^a F^i \wedge F^j + \epsilon^c S_{ibc}^a F^i \wedge A^b.$$  

where $\nabla_j \rho_i^a \equiv \rho_{a,j}^i - \Gamma_{ja}^b \rho_b^i$ denotes the covariant derivative w.r.t. the index $a$ only.

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4These equations hold in a frame where the parameters $\epsilon^a$ depend on coordinates of $\Sigma$ only, but not also on the fields $X$ or even $X$ and $A$. We intend to provide a more coordinate independent interpretation elsewhere.
With these ingredients it is now easy to provide a generalization of BF-theories to the setting of Lie algebroids (cf. [3, 1, 4]):

\[ S_{LABF} = \int_{\Sigma} B_i \wedge F^i + B_a \wedge F^a, \]  

(14)

where \( B_i \) and \( B_a \) are \((d - 1)\) and \((d - 2)\)-form fields, respectively, the transformations of which can be adjusted to render the action invariant under the above gauge transformations. The field equations \( F^i = 0 \) and \( F^a = 0 \) require \((X, A)\) to correspond to a Lie algebroid morphism from \( T\Sigma \) to \( E \), while the above gauge transformations reduce to Lie algebroid homotopies in that case (cf. [2] for further details).

So as to construct a gauge invariant Lie Algebroid YM action, one would naturally be lead to square both field strengths,

\[ \int_{\Sigma} -\frac{1}{2} F^a \wedge \ast F^b g_{ab} - \frac{1}{2} F^i \wedge \ast F^j g_{ij}, \]  

(15)

using a metric \( g \sim g_{ij} \) on \( M \), a fibre metric \( g^\Sigma \sim g_{ab} \) on \( E \), and the metric \( h \) on \( \Sigma \) for the Hodge dual of differential forms. The condition of gauge invariance of the action should then imply some meaningful conditions on the additional structures \( g_{ij}, g_{ab} \) (generalizing ad-invariance of the metric on the Lie algebra in the ordinary YM case) and their existence then possibly a restriction on the possible Lie algebroids \( E \) (quadratic Lie algebras in the YM situation).

In the context of the above functional, however, the restrictions turn out to be enormous, bringing one back implicitly to the realm of ordinary YM gauge theories: The variation of the field strength \( F^a \) in the first term produces terms proportional to \( F^i \wedge F^j \wedge \ast F^a \) and \( F^i \wedge A^a \wedge \ast F^b \), both of which cannot be compensated for by variations of other parts of the actions and thus have to vanish individually. The vanishing of the first term implies that \( \nabla \) is a flat connection on \( E \), \( R_{ij}^a = 0 \), the second constraint, \( S_i^a = 0 \) (cf. Eqs. (11) and (5)). This in turn implies that we can identify \( E \) with \( M \times g \), \( g \) being the Lie algebra with the respective structure constants \( C^a_{bc} \) and \( \rho: E \to TM \) can be identified with a representation of it on \( M \) (cf. Eq. (1)). From a physical perspective, then, the theory reduces to standard YM theory (first term in (15)) with structural Lie algebra \( g \), coupled

\[ ^5 \text{Such an } E \text{ is called an action Lie algebroid.} \]
to a Higgs-type sigma model with the Higgs fields taking values in $M$ (the second term in (15) reduces to the usual kinetic term of such a theory).

In fact, part of these conditions, namely $S_{abc} = 0$, can already be deduced from (9), taking into account that obviously $F^i = 0$ are not field equations for the action functional (15). This consideration, however, provides also a hint for a way to avoid the above no-go-type result: One may want to ensure that $F^i = 0$ are part of the field equations of the strived for generalization of the YM-action (note that for an ordinary YM theory, $M$ is a point and $F^i$ vanishes identically). In this way one is lead to

$$S_{LAYM} = \int_{\Sigma} B_i \wedge F^i - \frac{1}{2} F^a \wedge \star F^b g_{ab},$$  

(16)

$B_i$ being $(d-1)$ forms on $\Sigma$ like in (14) above.

Now again we ask for the conditions on the structural ingredients, i.e. $E$, $\nabla_i$, and $g_{ab}$, such that the above functional is gauge invariant w.r.t. the symmetries generated by Eqs. (6, 7) (for some transformation induced on the $B_i$-fields). The action functional (16) is gauge invariant w.r.t. those gauge transformations, if the fiber metric $E g \sim g_{ab}$ is covariantly constant w.r.t. a certain Lie algebroid ("$E$-"") connection $^{E}E\nabla$

$$^{E}E\nabla E g = 0.$$  

(17)

This $E$-connection is one induced by the ordinary connection $\nabla$ on $E$ and defined via

$$^{E}E\nabla_{\psi} \bar{\psi} = \nabla_{\rho(\bar{\psi})} \psi + [\psi, \bar{\psi}].$$  

(18)

In local components the coefficients of this $E$-connection read as $\Gamma^a_{bc} = \rho^a_{\epsilon} \Gamma^\epsilon_{ib} + e^a_{\epsilon} = \rho^a_{\epsilon} \Gamma^\epsilon_{ib} - T^a_{ib}$. From the first equality one obtains (17) at once, observing that the first line of (13) contains precisely $\Gamma^a_{bc}$ (while the two terms in the second line, which resulted in the unwanted severe restriction on $E$ in the case of (15), now can be absorbed by the variation of $B_i$ since they are both proportional to $F^i$); the second equality shows that $^{E}E\nabla$ differs from the more obvious $E$-connection $\nabla_{\rho(\cdot)}$ by subtraction of its own $E$-torsion.

Note that for $M$ being a point, the first term in (18) is absent since $\rho$ vanishes and one reobtains the usual condition of an ad-invariant metric

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6 This geometric interpretation was observed already shortly after completion of [1] and reported e.g. in [2].

7 The concept of a Lie algebroid connection and corresponding generalizations of curvature and torsion is recalled in Appendix A.
on the Lie algebra. The *existence* of an ordinary connection $\nabla$ and a fiber metric $E g$ such that (17) is fulfilled, poses a restriction on $E$. In the case of integrable Lie algebroids and for $E g$ having definite signature, this restriction is conjectured by Fernandes to precisely give Lie algebroids $E$ coming from *proper* Lie groupoids (a notion coinciding with compactness in the Lie group case) [6]. It is amusing that this particular $E$-connection pops out naturally from invariance of the functional (16) and the simple ansatz (6,7) for the gauge symmetries.

In fact, it turns out that a condition like (17) (or the likewise one found in [1]) is sufficient but not also necessary for gauge invariance of the action functional $S_{LAYM}$. We will discuss this issue in detail in section 5 below.

Before closing this section, we make a remark on some geometric interpretation of the tensor (11); in fact it is related to the $E$-curvature of $E \tilde{\nabla}_a$ by contraction with the anchor map $\rho$ (cf. Appendix A):

$$E \tilde{R}_{abc}^d = \rho_c^i S_{iab}^d.$$

Hence, if the gauge transformations close off-shell, i.e. if $S_{iab}^d = 0$, then $E \tilde{\nabla}_a$ is flat. The converse statement is not true. We will encounter flat $E$-connections in the subsequent section when considering the issue of coupling matter fields to the above action functional $S_{LAYM}$. Flat $E$-connections on vector bundles over $M$ are the natural generalization of a (linear) representation of a Lie algebra to the context of Lie algebroids (cf. Appendix A) [9]. A flat $E$-connection $E \tilde{\nabla}$ on $E$ can then be considered as a possible generalization of the adjoint representation of a Lie algebra.

### 3 Matter Fields with values in vector bundles

In this section we address the issue of coupling scalar fields to the YM-type theory of the previous section. Since we address trivial bundles over $\Sigma$ only within this note, in the ordinary YM situation this would correspond to some functions on $\Sigma$ taking values in a vector space which carries a representation of the structural Lie algebra. Representations of Lie algebroids are known in the mathematical literature as flat $E$-connections. Here we will, however,

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8We will in the following section, however, *not* assume familiarity with such a mathematical concept. Instead, we will start in a pedestrian style for the construction of a coupling to matter fields and be lead automatically to the mathematical concepts by means of gauge invariance.
adopt a more pedestrian, physics oriented route which will lead us there by itself. In fact, following the same route we will be lead to a more general setting, permitting also non-linear representations.

For this purpose we start with a set \( \phi^\sigma \) of functions on \( \Sigma \). We expect/want formulas to be covariant w.r.t.

\[
\phi^\sigma \rightarrow \phi^\sigma \equiv M^\sigma_\tau \phi^\tau
\]

(20)

for arbitrary matrices \( M^\sigma_\tau \). In the usual YM setting this corresponds to a change of basis in the representation space of the Lie algebra. In the present more general setting the gauge fields contain not only 1-forms \( A \) on \( \Sigma \), but also 0-forms \( X^i \) and it is thus natural to permit \( M^\sigma_\tau \) to depend on \( x \). More abstractly, this implies that the Higgs-type scalar fields \( \phi^\sigma \) correspond to sections of \( \mathcal{X}^*V \), where \( V \) is a vector bundle over \( M \), the same base as the Lie algebroid \( E \) (and \( \mathcal{X} \) the previous map from \( \Sigma \) to \( M \)).

Now we make the following ansatz for infinitesimal gauge transformations:

\[
\delta_\epsilon \phi^\sigma = -\epsilon^a \Gamma_a^\sigma_\tau \phi^\tau,
\]

(21)

where \( \Gamma_a^\sigma_\tau \) are some at this point not further specified fixed parameters depending on \( X \). Covariance restricts them further, however: We want that for \( \tilde{\phi}^\sigma \) we have a likewise formula. On the other hand, using Eq. (6) and the fact that \( \delta_\epsilon (M^\sigma_\tau \phi^\tau) = \delta_\epsilon (M^\sigma_\tau) \phi^\tau + M^\sigma_\tau \delta_\epsilon \phi^\tau \), we can determine the transformation property of the above coefficients,

\[
\tilde{\Gamma}_a^\sigma_\tau = M^\sigma_\sigma' \Gamma_a^\sigma_\tau' M^{-1}\tau' - \rho^a_\sigma M_{\sigma',i}^\sigma M^{-1}\tau' .
\]

(22)

This implies that these coefficients have the geometrical interpretation of an \( E \)-connection \( E\nabla \) on the vector bundle \( V \).

Finally we demand that the gauge transformations close on the newly introduced fields,

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \phi^\sigma = \delta_{\epsilon_3} \phi^\sigma ,
\]

(23)

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9We usually drop the extra upper \( E \) in the \( E \)-connection coefficients, since their indices already make clear of what nature they are. Solely with the respective \( E \)-curvatures we keep it for clarity also in the components. Note that in the present section ordinary connections as well as \( E \)-connections always refer to the vector bundle \( V \rightarrow M \), in contrast to the previous section where they both referred to \( E \rightarrow M \) itself—for notational simplicity we use the same symbols. The representation space \( V \) can be chosen as \( E \) itself, certainly; the notations are chosen such that they coincide in that particular case.
where $\epsilon_3$ is given by formula (10). Note that in this case it is not natural to permit a contribution proportional to $F^i$ as in (9), although using the metric $h$ on $\Sigma$ one could produce also 0-form contributions from $F^i$. The condition (23) is equivalent to $E\nabla$ having vanishing $E$-curvature. Thus with these physical considerations we indeed find that to couple matter fields to a Lie algebroid Yang-Mills theory $S_{LAYM}$ for some given structural Lie algebroid $E \to M$ we need a vector bundle $V \to M$ carrying a flat $E$-connection $E\nabla$, a “Lie algebroid representation” on $V$ in the mathematical sense. Note that vanishing $E$-curvature by definition means $[E\nabla_\psi, E\nabla_\phi] = E\nabla_{[\psi,\phi]}$, with the Lie algebroid bracket on the r.h.s.; thus this indeed implies that the differential operators $E\nabla_\psi$ are a representation of the Lie algebra defined by the Lie algebroid bracket.

The generalization of a covariant derivative on Higgs fields in ordinary YM-theory takes the form

$$D\phi^\sigma = d\phi^\sigma + \Gamma^\sigma_a \phi^a + \Gamma^\sigma_i \phi^i F^i,$$  \hspace{1cm} (24)

where $\Gamma^\sigma_a \phi^a$ and $\Gamma^\sigma_i \phi^i$ are coefficients of an ordinary connection $\nabla$ and the above $E$-connection $E\nabla$, respectively, both defined on $V$ (it is certainly their pullback by $X$ that enters in such an expression, $\phi$ being a section in $X^*V$—following physics conventions such identifications are understood). The first two terms are familiar ones if, following Eq. (21), one identifies $\Gamma^\sigma_a \phi^a$ with the coefficients of a representation; the contribution proportional to $F^i$ is again needed for covariance (under changes of $E$- and $V$-frames). Indeed, the terms in (24) may be recombined into

$$(D\phi)^\sigma = (D\Gamma\phi)^\sigma - A^a T^\sigma_{a\tau} \phi^\tau$$  \hspace{1cm} (25)

where $(D\Gamma\phi)^\sigma = d\phi^\sigma + dX^i \Gamma^\sigma_i \phi^i$ is the canonical exterior covariant derivative in $X^*V$ induced by the connection $\nabla$ on $V$. $T$, on the other hand, is (the pullback by $X$ of) a section in $E^* \otimes \text{End}(V)$, defined, for any $\psi \in \Gamma(E)$, by means of the difference of two $E$-connections (on $V$), namely

$$T_\psi = \nabla_{\rho(\psi)} - E\nabla_\psi.$$  \hspace{1cm} (26)

In the particular case of $V = E$ and $E\nabla = E\tilde{\nabla}$ it coincides with the $E$-torsion tensor of $\nabla_{\rho(\cdot)}$, cf. the text following Eq. (18). Thus $D\phi$ is indeed a section of $T^*\Sigma \otimes X^*V$, as it should be $^{10}$

$^{10}$For a more general ansatz of a covariant derivative in the Lie algebroid setting cf. [7], with however the same result.
Now we can compute the gauge transformation of (25). In this context we will adopt a slightly vague, but, for practical purposes, still quite useful point of view: We find that the covariant derivative $D$ “commutes” with gauge transformations, but only modulo a term proportional to the field strength $F^i$,

$$\left(\delta_i\phi\right)^\sigma - (D\delta_i\phi)^\sigma = \epsilon^a F^i \tilde{\Sigma}_{ai\tau} \phi^{\tau} + \epsilon^a A^b R_{ba\tau} \phi^{\tau}. \quad (27)$$

Indeed, the second term vanishes identically since $E\nabla$ is a flat $E$-connection. Here $S_{ai\tau} \equiv (\nabla_i T_a - \rho^j_a R_{ij})_{a\tau}$. We remark in parenthesis that for the adjoint $E$-connection $\tilde{E}\nabla$ on $E$ the tensor $S$ parametrizing the non-closure of gauge transformations on $A^I$, cf. Eq. (9), and $\tilde{S}$ do, for $R_{ij\,c} a \neq 0$, not coincide, $S_{ai\,b} = \tilde{S}_{ai\,b} + \rho^j_a R_{ij\,c}$.

The action of LAYM theory coupled to matter fields $\phi^\sigma$ is then the sum of the LAYM action and a kinetic term for the matter fields,

$$S_{\text{LAYM + matter}} = S_{\text{LAYM}} - \int_{\Sigma} \frac{1}{2} (D\phi)^\sigma \wedge \ast (D\phi)^\tau \ g_{\sigma\tau}(X), \quad (28)$$

where $g_{\sigma\tau} \sim Vg$ is (the pullback by $X$ of) a non-degenerate metric on $V$. The action is invariant under the gauge symmetries, if $g_{\sigma\tau}$ is compatible with the $E$-connection $E\nabla$ on $V$, i.e. if

$$E\nabla(Vg) = 0. \quad (29)$$

The terms proportional to $\tilde{S}$, coming from the variation of the kinetic term by use of eq. (27), are proportional to $F^i$ and thus can be absorbed by redefining $\delta B_i$ correspondingly.

It is easy to add e.g. a mass term for $\phi$ to this, using $Vg$: $\int_{\Sigma} \phi^\sigma \phi^\tau g_{\sigma\tau} \mathrm{vol}_{\Sigma}$ is already by itself invariant under gauge transformations (here $\mathrm{vol}_{\Sigma}$ denotes the volume form on $\Sigma$ induced by $h$). This can be generalized in a straightforward manner to higher powers in $\phi$, including thus self-interactions of the scalar fields, by means of completely symmetric tensors $I_{\sigma_1...\sigma_n} \sim I \in \Gamma(S^n V)$ which are $E$-covariantly constant, $E\nabla(I) = 0$:

$$\sum_n \int_{\Sigma} \phi^{\sigma_1} \ldots \phi^{\sigma_n} I_{\sigma_1...\sigma_n}(X) \mathrm{vol}_{\Sigma}. \quad (30)$$

Another way of obtaining a coupling of a LAYM theory to scalar fields is to perform a Kaluza-Klein dimensional reduction from $\Sigma_d$ to $\Sigma_{d-1}$.
along a circle $S^1$, which we may take to be along the direction $\mu = 0$. Then
the vector field $A^a$ on $\Sigma_d$ decomposes into a vector field $\hat{A}^a$ on $\Sigma_{d-1}$ and into a
scalar field, $\phi^a$ coming from the 0-component of $A^a$. As shown in Appendix B,
the dimensional reduction of both, the gauge symmetries and the action,
shows that $\phi^a$ transforms according to $E\nabla_a$, Eq. (18); moreover the $(0, m)$
component of $F^a$ coincides with the covariant derivative $D$ for the particular
$E$-covariant derivative $E\nabla_I = E\nabla_I$. One might ask how the condition of
a flat $E$-connection found in this Section is compatible with dimensional
reduction where $E\nabla$ is arbitrary. However, the dimensional reduction of the
zero-component of $F$ restricts $\phi^a$ to $\ker \rho$ on-shell—as a relict from the $B_iF^i$
term in $S_{LAYM}$—where the curvature of $E\nabla$ vanishes, cf. (19). Dimensional
reduction therefore leads to a rather restricted setting. The $E$-connection is
permitted to be nonflat, but at the price of restricting the scalar fields to
taking values in $\ker \rho$ only.

4 Matter fields of sigma model type

One of the possible perspectives on a LAYM theory is that it generalizes
ordinary YM gauge theories to the realm of sigma models, cf., e.g., [10].
In the usual YM setting, scalar fields, like the Higgs field, take values in
vector bundles associated to the principal bundle in which the gauge fields
are connections. From the present perspective, such a restriction to linearity,
as present in the formulas (20), (21), (24) for example, seems unnecessary and
non-natural. In the ordinary Lie algebra situation, $E = \mathfrak{g}$, this corresponded
to linear representations of the Lie algebra on a vector space $V$ (used in the
construction of the associated bundle). However, we may be interested also in
nonlinear, sigma-model like couplings of the scalar fields to the LAYM-part.

Towards this goal it is useful to note that the data used in the previous
section, a Lie algebroid $E \to M$ together with a flat $E$-connection $E\nabla$ on
$p: V \to M$ can be put together into a bigger Lie algebroid $\tilde{E}$: As a vector
bundle this Lie algebroid is just $\tilde{E} \equiv p^*E \to V$, i.e. $E$ considered as
living over the bundle $V$ as base manifold. The Lie bracket between sections
coming from sections of $E$ is the old one, $[p^*\psi_1, p^*\psi_2] := p^*[\psi_1, \psi_2]$. It
remains to define what happens when $p^*\psi_2$ is multiplied by a function
over $V$ that is fiber-linear (the rest follows by the Leibniz rule), i.e. by sections
$\alpha \in \Gamma(V^*)$. It is here where the $E$-connection enters: $[p^*\psi_1, \alpha p^*\psi_2] :=
\alpha p^*[\psi_1, \psi_2] + (E\nabla_{\psi_1}\alpha) p^*\psi_2$. The flatness condition of $E\nabla$ comes in when
checking the Jacobi condition of that bracket.

Now it is straightforward to generalize to the nonlinear setting. Let us just replace the vector bundle $p: V \to M$ by a general fiber bundle $p: \tilde{M} \to M$. Again we can consider the vector bundle $\tilde{E} := p^* E \to \tilde{M}$. Instead of a representation on $V$ we want to consider the structure of a Lie algebroid defined on $\tilde{E}$, satisfying an appropriate compatibility condition with $E$:

There is always a natural projection $\pi: p^* E \to E$ induced by $p: \tilde{M} \to M$.

\[
\begin{array}{ccc}
\tilde{E} & \equiv & p^* E \\
\downarrow \pi & & \downarrow p \\
E & \to & M
\end{array}
\]  

(31)

One can check that in the linear situation above, $\pi$ is a Lie algebroid morphism (cf. e.g. [2] for a convenient way of checking this). This is what we now want to require also in the present more general situation: by definition, an $E$-action on $\tilde{M}$ is a Lie algebra structure on $p^* E$ such that the projection $\pi$ is a morphism of Lie algebroids.

It is important in this context that $\tilde{E}$ really is the bundle $p^* E$ and not just isomorphic to it and that $\pi$ is the corresponding canonical projection. One can check, furthermore, that for $M$ being a point the Lie algebroid $\tilde{E} \to \tilde{M}$ reduces to the action Lie algebroid $\tilde{E} = \mathfrak{g} \times \tilde{M}$ of a Lie algebra action $\mathfrak{g}$ on a manifold $\tilde{M}$. So, $\tilde{E} = E \times_M \tilde{M}$ is the “action Lie algebroid” of a Lie algebroid $(E \to M)$-action on $\tilde{M} \to M$.

Part of the Lie algebroid morphism property of $\pi: \tilde{E} \to E$ is the commutativity of the following diagram

\[
\begin{array}{ccc}
p^* E & \xrightarrow{\tilde{\rho}} & T\tilde{M} \\
\downarrow \pi & & \downarrow p^* \\
E & \xrightarrow{\rho} & TM
\end{array}
\]  

(32)

This permits us to identify the anchor map $\tilde{\rho}: \tilde{E} \to T\tilde{M}$ with an $E$-connection on the fiber bundle $p: \tilde{M} \to M$, which, by definition as given in [11], is precisely a map $\tilde{\rho}$ such that the above diagram is commutative. A map $\tilde{\rho}$ permits to lift a vector $\psi_x \in \mathfrak{e}_x$ at the point $x \in M$ to the corresponding vector in $T_u\tilde{M}$ at the point $u \in \tilde{M}$ with $p(u) = x$. Commutativity of the diagram means that this “horizontal lift” should be such that the projection down to $M$ by

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\( p_\ast \) of the lifted vector agrees with the vector \( \rho(\psi_x) \). For \( E = TM, \rho = \text{id} \), the standard Lie algebroid, this reproduces the standard condition of an ordinary connection in \( p: \tilde{M} \to M \) that the composition of the projection with the lift is the identity on \( T_x M \).

Since \( \tilde{E} \) is a Lie algebroid, its anchor is a morphism of Lie brackets,

\[
\tilde{\rho}([\tilde{\psi}_1, \tilde{\psi}_2]) - [\tilde{\rho}(\tilde{\psi}_1), \tilde{\rho}(\tilde{\psi}_2)] = 0 \tag{33}
\]

which, for \( \tilde{\rho} \) being viewed as an \( E \)-connection on \( \tilde{M} \), is tantamount to its flatness. In fact, a flat \( E \)-connection \( \tilde{\rho}: p^\ast E \to T\tilde{M} \) on \( \tilde{M} \) can be seen to be equivalent to our definition of an \( E \)-action on \( \tilde{M} \). In this formulation we easily reproduce the results of the previous section, where the connection was further restricted to respect the linear structure on the bundle \( \tilde{M} = V \).

We now put this into explicit formulæ, generalizing the respective ones of the previous section. In bundle coordinates \((X^i, \phi^\sigma)\) on \( \tilde{M} \), the anchor of \( \tilde{E} \) applied to the \((\tilde{M}\)-fiberwise constant) basis \( \tilde{e}_a := p^\ast e_a \) induced by a local basis of sections on \( E \), takes the form

\[
\tilde{\rho}(\tilde{e}_a) = \rho^i_a(x) \frac{\partial}{\partial x^i} + \tilde{\rho}_a^\sigma(x, \phi) \frac{\partial}{\partial \phi^\sigma}, \tag{34}
\]

where instead of \( \tilde{\rho}_a^\sigma \) we could have written also \( \Gamma_a^\sigma \), stressing the interpretation of these components as an \( E \)-connection on \( \tilde{M} \). Equation (21) for the gauge transformations now turns into

\[
\delta \phi^\sigma = -e^a \tilde{\rho}_a^\sigma(X, \phi), \tag{35}
\]

while for the exterior covariant derivative of \( \phi \in C^\infty(\Sigma, A^\ast \tilde{M}) \) we get

\[
D\phi^\sigma = d\phi^\sigma + \tilde{\rho}_a^\sigma(X, \phi) A^a + \Gamma_i^\sigma(X, \phi) F^i. \tag{36}
\]

Here \( \Gamma_i^\sigma \) denote the components of an ordinary connection on \( p: \tilde{M} \to M \). Requiring linearity in \( \phi \), we recover the context of the previous section in all these formulæ.

Now we are in the position of considering the coupling of a kinetic sigma model term to the pure gauge part of the action. This gives

\[
S_{LAYM+\text{matter}} = S_{LAYM} - \int_\Sigma \frac{1}{2} (D\phi)^\sigma \wedge * (D\phi)^\tau g_{\sigma\tau}(X, \phi). \tag{37}
\]
The allegedly small change of permitting \( g_{\sigma\tau} \) to depend also on \( \phi \) in comparison to (28) implies some conceptual complications: Before, \( g_{\sigma\tau} \) corresponded to a fiber metric on \( V \), which we could also view as a quadratic function on \( V = \tilde{M} \). Now, \( Vg \) is a fiber metric on \( V\tilde{M} \subset T\tilde{M} \), the subbundle over \( \tilde{M} \) consisting of vertical tangent vectors. A condition of the type (29) does not yet make any sense thus, we first need a Lie algebroid-connection on \( V\tilde{M} \), which can be viewed also as the foliation Lie algebroid \( T\mathcal{F} \) of the foliation/fibration of \( \tilde{M} \) by its fibers.

However, in fact there is a canonical lift of the flat \( E \)-connection \( \tilde{\rho}: \tilde{E} \rightarrow T\tilde{M} \) to a flat \( \tilde{E} \)-connection \( \tilde{\rho}: \tilde{E} \rightarrow T(V\tilde{M}) \) with \( \tilde{E} = \tilde{p}^*\tilde{E} \) and \( \tilde{\rho}: V\tilde{M} \rightarrow \tilde{M} \):

\[
\begin{align*}
\tilde{E} & \equiv \tilde{p}^*\tilde{E} \xrightarrow{\tilde{\rho}} T(V\tilde{M}) \\
\tilde{\pi} \downarrow & \quad \downarrow \tilde{p} \\
\tilde{E} & \equiv p^*E \xrightarrow{\tilde{\rho}} T\tilde{M} \\
\pi \downarrow & \quad \downarrow p \\
E & \xrightarrow{\rho} TM
\end{align*}
\]  

In other words, there exists a Lie algebroid structure on \( \tilde{E} = \tilde{p}^*\tilde{E} \) such that \( \tilde{\pi}: \tilde{E} \rightarrow \tilde{E} \) is a Lie algebroid morphism:

\[
\begin{align*}
\tilde{p}^*\tilde{E} & \xrightarrow{\tilde{\rho}} V\tilde{M} \\
\tilde{\pi} \downarrow & \quad \downarrow \tilde{\rho} \\
\tilde{E} & \xrightarrow{\rho} \tilde{M}
\end{align*}
\]  

In order to show that the Lie algebroid structure on \( \tilde{E} \) induces a Lie algebroid structure on \( \tilde{E} \), it suffices to specify the anchor \( \tilde{\rho} \) of the latter, since the bracket on \( \tilde{E} \) is fixed already uniquely by means of the bracket on \( \tilde{E} \) or \( E \) when applied to sections coming from \( \tilde{E} \) and \( E \), respectively. Denoting by \( \varphi^\sigma = d\phi^\sigma \) fiber linear coordinates on \( V\tilde{M} \), the anchor map \( \tilde{\rho} \) of \( \tilde{E} \) applied to \( \tilde{e}_a := \tilde{p}^*\tilde{e}_a \equiv \tilde{p}^*p^*e_a \) reads as

\[
\tilde{\rho}_a = \rho^i_a(x) \frac{\partial}{\partial x^i} + \tilde{p}_a^\sigma(x, \phi) \frac{\partial}{\partial \phi^\sigma} + \frac{\partial}{\partial \phi^\sigma} \varphi^\tau \frac{\partial}{\partial \varphi^\tau}.
\]
This also corresponds to a flat \( \tilde{E} \)-connection on \( VM \) with components \( \tilde{\Gamma}_{\alpha \tau}^\sigma(x, \phi) = \frac{\partial}{\partial \phi^\sigma} \tilde{\rho}_a(x, \phi) \).

Let us now provide a coordinate independent construction of this canonical lift, which also shows that its definition is independent of the chosen basis in \( E \), and that the construction depends crucially on restriction to vertical vector fields on \( \tilde{M} \) (equipped itself with a flat \( \tilde{E} \)-connection). We want to define a bundle map \( \tilde{\rho}: \tilde{E} \rightarrow T(VM) \). Extend a point \( \tilde{\psi}_0 \in \tilde{E} \) to some fiber-wisely constant section \( \tilde{\psi} \in \Gamma(\tilde{E}) \) coming from a section \( \psi \in \Gamma(E) \); so, in the previously introduced local basis of sections in \( \tilde{E} \), \( \tilde{\psi} = \psi_a \tilde{e}_a \) with \( \psi_a \) depending on coordinates \( x^i \) of \( M \) only and with \( \tilde{\psi} \) evaluated at the projection of \( \tilde{\psi}_0 \) to \( M \) agreeing with \( \tilde{\psi}_0 \). This induces also a section \( \psi = \psi_a \tilde{e}_a \) in \( E \), whose image with respect to \( \tilde{\rho} \) gives a vector field on \( \tilde{M} \). Consider the (local) flow \( \Phi^t_\psi \) of this vector field and lift it to \( T\tilde{M} \) by means of the pushforward map \( (\Phi^t_\psi)_*: T\tilde{M} \rightarrow T\tilde{M} \), a vector bundle morphism covering the flow \( \Phi^t_\psi \) on \( \tilde{M} \). This lift is thus generated by a vector field on \( T\tilde{M} \) covering the vector field \( \tilde{\rho}(\tilde{\psi}) \). We can restrict the vector field viewed as a section in \( T(T\tilde{M}) \) to the submanifold \( V\tilde{M} \) of \( T\tilde{M} \). Two things happen in this context: Firstly, while the vector field on \( T\tilde{M} \) is not \( C^\infty(M) \) linear in \( \psi \in \Gamma(E) \) in general, the restriction has this property (which is essential for having the result being independent on the extension of \( \tilde{\psi}_0 \) to an at least locally defined section \( \tilde{\psi} \) or \( \psi \)). Secondly, the restriction is tangent to \( V\tilde{M} \subset T\tilde{M} \) (here the fact that \( \tilde{\rho}(\tilde{\psi}) \) is projectable to \( M \), covering \( \rho(\psi) \), cf. diagram (32), enters crucially) and can thus be viewed as a vector field on \( V\tilde{M} \). Evaluate this vector field at the point in \( V\tilde{M} \) living under \( \tilde{\psi}_0 \in \tilde{E} \) and call this \( \tilde{\rho}(\tilde{\psi}_0) \). By a straightforward calculation one may check that this geometric construction indeed yields (40).

With these ingredients at hand, we are now in the position to formulate a condition on the fiber metric \( Vg \) on \( V\tilde{M} \) as entering the functional (37). The functional becomes invariant w.r.t. gauge transformations if the following condition on \( g \) is satisfied (in addition to the conditions to be placed on \( E_g \)):

\[
\tilde{E} \nabla(g) = 0, \tag{41}
\]

where \( \tilde{E} \nabla \) is the flat \( \tilde{E} \)-connection corresponding to (40) and described in the sentence after that formula.
In the present more general framework than in the previous section, formulating the conditions on some selfinteraction for the scalar fields, cf. Eq. (30) and the corresponding discussion, becomes simpler: We can add to (37) any term of the form
\[ \int_{\Sigma} W(X, \phi) \text{vol}_\Sigma, \] (42)
provided only that \( W \) is a function on \( \widetilde{M} \) invariant along the orbits generated by \( \widetilde{\rho} \), i.e. if \((\rho^\alpha_i \partial_i + \rho^\alpha_\sigma \partial_\sigma)W = 0\).

If the Lie algebroid \( E \) permits an integration to an source-simply connected Lie groupoid \( G \rightrightarrows M \) (cf. [12, 9] for the necessary and sufficient conditions), the above considerations have the following global reinterpretation: First, given \( G \) we can consider its action on \( p: \widetilde{M} \rightarrow M \), where \( p \) is usually called the moment map in this context. An action is then given by a map \( \varphi: G \times_M \widetilde{M} \rightarrow \widetilde{M} \) which is compatible with the structural maps on \( G \). In particular this means that any \( g \in G \) with source \( x \) and target \( y \) is lifted to an isomorphism of fibers, \( \varphi_g: p^{-1}(x) \rightarrow p^{-1}(y) \). This can again be made into a new groupoid \( \widetilde{G} \rightrightarrows \widetilde{M} \) whose elements \( \widetilde{g} \) consist of the maps \( \varphi_g \) mapping one point in \( \widetilde{M} \) (the source of \( \widetilde{g} \)) to another one (the target of \( \widetilde{g} \)). Finally, the diffeomorphisms of \( \widetilde{M} \)-fibers \( \varphi_g \) can be lifted to isomorphisms of their tangent bundles. This induces canonically a Lie groupoid \( \widetilde{G} \rightrightarrows T\mathcal{F} \). As already anticipated by the notations, these two groupoids are the integrations of \( \widetilde{E} \) and \( \widetilde{E} \), respectively, as we recommend the reader to check as an exercise. The condition (41) now just states that the maps \( (\varphi_g)_*: T(p^{-1}(x)) \rightarrow T(p^{-1}(y)) \), corresponding to a collection of elements in \( \widetilde{G} \), are also isomorphisms (isometries) of \( T\mathcal{F} \) equipped with the fiber metric \( V_g \).

5 Continuative Discussion

In this concluding section we want to discuss two more aspects of the topics presented in this article up to here. First of all this concerns the pure gauge field system (16), relaxing the conditions on the tensor \( E_g \) needed for squaring the 2-form field strength. Afterwards we come back to the coupled matter gauge field system, discussing it from a slightly more unified perspective. We now turn to the first issue.
The field strengths $F^i$, $F^a$ satisfy some generalized version of Bianchi identities \[1\]. In what follows in particular the first one of those will play an important role, for which reason we display it explicitly:

\[
dF^i - \rho^i_{a;j} A^a \wedge F^j + \rho^i_a F^a = 0.
\] (43)

Primarily, this leads to a second independent gauge symmetry \[1\]. Suppose that we transform $B_i$ according to

\[
d\lambda_i + \rho^i_{a;i} A^a \wedge \lambda_j.
\] (44)

Then it is easy to see that $S_{LAYM}$ is invariant w.r.t. such transformations up to boundary contributions (resulting from a partial integration), if $\lambda_i \rho^i_a = 0$—implying, more geometrically, that $\lambda$, instead of taking values arbitrarily in $T^* M$, is restricted to the conormal bundle of the tangent distribution to the orbits generated by $\rho$. In fact, here, and also in what is to follow, we will consider only regions of $M$ where the rank of $\rho$ is constant. Further investigations of what happens more precisely at regions where the rank of $\rho$ jumps would be interesting though.

One may employ Eq. (43) in another direction also, however: The contraction of $\rho$ with the 2-form field strength can be expressed in terms proportional to the 1-form field strengths (and its derivative). Since, on the other hand, any term proportional to $F^i$ in (16) can be dropped by an appropriate redefinition of the field $B_i$, one finds that there is an equivalence relation between fiber metrics on $E$ yielding physically equivalent gauge theories—in fact, the tensors $E^a g$ can even become partially degenerate by such redefinitions. Let $e^a$ denote a local frame in $E^*$, then $E g = g_{ab} e^a e^b$. Consider replacing $g_{ab}$ by $\bar{g}_{ab} = g_{ab} + \rho^i_a \beta_{ib} + \rho^i_b \beta_{ia}$ for some collection $\beta_a$ of 1-forms on $M$. Since in the action functional $E g$ is contracted with $F^a$s, the terms proportional to $\beta_{ai}$ can be absorbed completely: we replace $\rho(F(2))$ by the corresponding two terms according to (43), perform a partial integration in the term with $dF^i$, and then absorb all prefactors of the newly introduced terms proportional to $F^i$ by redefining $B_i$ appropriately. This means that a redefinition $g_{ab} \mapsto \bar{g}_{ab}$ can be compensated by a local diffeomorphism on the field space of the functional (16). In other words, on the physical level, there is an equivalence
between two functionals (43) induced by an equivalence relation between its $E$-2-tensors $Eg$

$$g_{ab} \sim g_{ab} + \rho^i_a \beta_{ib} + \rho^i_b \beta_{ia}$$

(45) for arbitrary choices of $\beta \in \Omega^1(M, E^*)$. The quotient of $\Gamma(M, S^2 E^*) \ni E g$ by these orbits is in one-to-one correspondence to fiber metrics on the subbundle $\ker \rho \subset E$\footnote{This is true, when restricting to orbits that have at least one non-degenerate representative $Eg$. Recall also that $E$ was assumed to be regular for the moment so that under this assumption its kernel really defines a subbundle of $E$.} Denote the restriction of $E g$ to $\ker \rho$ by $\rho g$; it is one-one to some equivalence class $[E g]$ of a fiber metric on $E$.

The bundle $\ker \rho \to M$ carries a canonical $E$-connection. Let $\psi \in \Gamma(E)$ and $\tilde{\psi} \in \Gamma(\ker \rho)$ and define

$$\rho \nabla \psi \tilde{\psi} := [\psi, \tilde{\psi}]$$

(46) Since $\tilde{\psi}$ is in the kernel of $\rho$, this is indeed $C^\infty(M)$-linear in $\psi$. This connection is sometimes called the $E$-Bott connection. Comparing with equations (17) and (18), it is now obvious that

$$\rho \nabla \rho g = 0$$

(47) is sufficient for gauge invariance of (16). In contrast to (17), this condition is not only independent of any auxiliary connection $\nabla$ on $E$, it is certainly also a weaker condition on $E g$, needing $E g$ only to be in some orbit characterized by its restriction $\rho g$ to $\ker \rho$ such that (17) holds true.

We now turn to the second issue, the coupled matter gauge field system. The emphasis on a new Lie algebroid $\tilde{E} \to \tilde{M}$ governing linear or nonlinear actions of Lie algebroids $E \to M$ on bundles $\tilde{M} \to M$ corresponding to matter field target spaces also resides in a possible reinterpretation of the gauge invariant coupled matter-LAYM functional (37). Who forbids one to consider all the coordinates on $\tilde{M}$ on the same footing to start with. We had the kind of no-go theorem around (15), where a squaring of all 1-form and 2-form field strengths was taken. Eq. (37) from this perspective shows that squaring some of the 1-form field strengths, keeping the others included via Lagrange multipliers, does not necessarily lead to likewise strong restrictions on admissible Lie algebroids. At the same time, some of the coordinates of the target Lie algebroid are promoted into propagating degrees of freedom typical for scalar fields from the physical point of view.
We make this point more explicit by rewriting (37) in this spirit. First, we denote by \( x^I = (x^i, \phi^\sigma) \) collectively all coordinates on \( \tilde{M} \) and, correspondingly, by \( \tilde{X} \) the map from \( \Sigma \) to all of \( \tilde{M} \). Then, the corresponding 1-form field strengths \( F^{(1)}_I \) split into \( F^i_1 \), agreeing with the respective previous expression (2) except that, for clarity, we added an index in brackets to emphasize to form character of the field strength, and, by the same formula, one now has \( F^{(1)}_I = d\phi^\sigma - \tilde{\rho}_\sigma^a A^a \). Note that geometrically \( F^i_1 \) corresponds to elements tangent to \( M \) and \( F^\sigma_1 \) tangent to fibers of \( \tilde{M} \to M \). \( F^{(1)} \) should be an element of \( \Omega^1(\Sigma, \tilde{X}^*T\tilde{M}) \) on the other hand, i.e. a vector on \( \tilde{M} \). The two components cannot be combined intrinsically or coordinate independently into a meaningful vector on \( \tilde{M} \) without a connection on that bundle. Let \( (d\phi^\sigma + \Gamma^\sigma_i dx^i)^{\frac{\partial}{\partial \phi^\sigma}} \in \Omega^1(\tilde{M}, V\tilde{M}) \) be such a connection 1-form on \( \tilde{M} \), its kernel determining what is horizontal in \( T\tilde{M} \).

\[
V\tilde{M} \oplus H\tilde{M} = T\tilde{M} \quad \exists \quad v = v^\text{ver} + v^\text{hor}.
\]

We now see that the vertical part of \( F^{(1)} \), \( F^\text{ver}^{(1)} = F^\sigma_1 + \Gamma^\sigma_i F^i_1 \), reproduces precisely eq. (36). On the other hand, the horizontal part is always proportional to \( F^i_1 \) (for any choice of \( \Gamma_i^\sigma \)), thus the first term in (16) constrains \( F^\text{hor}^{(1)} \) to vanish. Likewisely, we could map \( F^{(1)} \in \Omega^1(\Sigma, \tilde{X}^*T\tilde{M}) \) by \( p_* \circ \tilde{X} \) to a tangent component on the base \( M \) of \( \tilde{M} \) and interpret the first LAYM-term in this way within the present setting, the Lagrange multiplier living in \( T^*M \) then as before. Preferring the first option, a coordinate independent, geometrical form of the total action, using \( \tilde{E} \to \tilde{M} \) as starting Lie algebroid and a split of \( T\tilde{M} \) into the two subbundles as above in (48), one finds for the combined matter-gauge field action (37) the following form:

\[
\int_{\Sigma} \langle B \wedge F^{\text{hor}}_1 \rangle - \frac{1}{2} \left( V\tilde{M} g \circ \tilde{X} \right) \left( F^{\text{ver}}_1 \wedge \star F^{\text{ver}}_1 \right) - \frac{1}{2} \left( \tilde{E} g \circ \tilde{X} \right) \left( F_2 \wedge \star F_2 \right),
\]

where \( B \) is a \((d-1)\)-form taking now values in (the pullback by \( \tilde{X} \) of) \( H^*\tilde{M} \).

This shows that partially squaring some of the 1-form field strengths is compatible with Lie algebroids different from mere Lie algebras. Gauge invariance of such a functional will certainly also heavily restrain the starting Lie algebroid \( \tilde{E} \to \tilde{M} \). What we showed constructively is that such a functional is compatible with a Lie algebroid structure on \( \tilde{E} \) coming from a Lie algebroid \((E \to M)\)-action on \( p: \tilde{M} \to M \) for some \( E \) and \( M \) such that one
has the diagram (31) with $\pi$ being a Lie algebroid morphism. In the language of [8] this corresponds to a Q-bundle $\pi: \tilde{E}[1] \to E[1]$ (which is locally trivial only in the sense of graded but not in the category of Q-manifolds), where in any local chart on the total space there exists a canonical isomorphism of its degree one variables with the degree one variables on the base.

In the extreme context of squaring all 1-form field strengths we showed that one is necessarily lead to the Lie algebroid of a Lie algebra action on its base. This corresponds to the situation of $M$ being a point in the discussion above. It may be interesting to see if in a generalization of this observation a functional of the form (49) with Lie algebroid $\tilde{E}$ always leads to the scenario as in (31) above.

### A Some formulas on Lie algebroids

A Lie Algebroid consists of a vector bundle $E \to M$ over a manifold $M$, a Lie algebra bracket, $[\cdot, \cdot]$, between sections $\psi$ of $E$, and of a bundle map $\rho: E \to TM$, called the anchor map. The bracket satisfies a Leibnitz rule,

$$[\psi_1, f \psi_2] = f[\psi_1, \psi_2] + \rho_{\psi_1}(f)\psi_2, \quad f \in C^\infty, \quad \psi_1, \psi_2 \in \Gamma(E). \quad (50)$$

In local coordinates $X^i$ on $M$ and a local basis $e_a$ of $\Gamma(E)$, this data is encoded in structural functions $C^{c}_{ab}, \rho^i_a \in C^\infty(M)$, such that the bracket and the anchor map take the form $[e_a, e_b] = C^{c}_{ab} e_c$, and $\rho(e_a) = \rho^i_a \partial_i$. As a consequence of the definitions above, the anchor map is a morphism wrt. the bracket, i.e.

$$[\rho(e_a), \rho(e_b)] = \rho([e_a, e_b]). \quad (51)$$

Examples of Lie Algebroids include a bundle Lie Algebras ($\rho = 0$), $TM$ ($\rho = id$), and Poisson manifolds.

In order to talk about $E$-connections $E\nabla$, we need to specify the Leibnitz rule:

$$E\nabla_{\psi_1}(f \psi_2) = f \ E\nabla_{\psi_1} \psi_2 + \rho_{\psi_1}(f)\psi_2, \quad (52)$$

Any connection $\nabla$ on the vector bundle $E$ can be lifted to an $E$-connection using the anchor map: $\nabla_{\rho(\cdot)}$.

Now that we have the concept of an $E$-connection on a Lie Algebroid, we can translate concepts involving connections on vector bundles to the realm of Lie Algebroids. For a connection $\nabla$ on a vector bundle, the curvature is
defined as
\[ R(\partial_i, \partial_j) = \nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[\partial_i, \partial_j]} . \] (53)

Analogously, we define the corresponding \( E \)-curvature as
\[ E R(\psi_1, \psi_2) = E \nabla_{\psi_1} E \nabla_{\psi_2} - E \nabla_{\psi_2} E \nabla_{\psi_1} - E \nabla_{[\psi_1, \psi_2]} . \] (54)

By the morphism property of the anchor map the \( E \)-curvature of an induced \( E \)-connection satisfies
\[ E R(\psi_1, \psi_2) = R(\rho(\psi_1), \rho(\psi_2)) . \] (55)

Given any \( E \)-connection \( E \nabla \), we can form a tensor involving the structure functions \( C_{ab}^c \). This tensor \( T \in \Omega^1(E) \otimes \Gamma(End(E)) \) is called the the \( E \)-torsion tensor corresponding to the \( E \)-connection \( E \nabla \) and is defined as
\[ T(\psi_1)\psi_2 = [\psi_2, \psi_1] + E \nabla_{\psi_1} \psi_2 - E \nabla_{\psi_2} \psi_1 , \] (56)
which in components takes the form
\[ T(e_a)e^c = T_a^c e^b , \quad T_a^c = -C_{ab}^c + \Gamma_{ab}^c - \Gamma_b^c \] (57)

Finally, we derive an identity which involving the \( E \)-torsion of an induced \( E \)-connection. In components, the induced connection is given by \( \Gamma_{ab}^c = \rho_b^i \Gamma_{ib}^c \). As a consequence of the Jacobi identity, the \( E \)-torsion corresponding to this induced connection satisfies the identity:
\[ T_a^d T_c^e + \text{cycl}(abc) = \rho_c^i \nabla_i T_a^e + \rho_b^j \rho_c^j R_{ij} R_{ab}^e + \text{cycl}(abc) , \] (58)
which can be used to show that \( E \)-curvature of the “adjoint connection” \( E \nabla_a \) (cf. Eq. (13)) reduces to
\[ E \tilde{R}_{abc}^d = \rho_b^i S_{iab}^d , \quad S_{iab}^d = \nabla_i T_{ab}^d + \rho_b^j R_{ij} - \rho_a^j R_{ijb}^d . \] (59)

**B Dimensional Reduction of LAYM**

In ordinary YM theory scalar fields in the adjoint representation can be obtained my performing a Kaluza-Klein dimensional reduction along a circle \( S^1 \). Here, we perform this dimensional reduction for the LAYM theory (6), (7), (16). Starting with a \( d \)-dimensional world sheet \( \Sigma_d \) we perform a dimensional reduction to a \( (d-1) \)-dimensional world sheet \( \Sigma_{d-1} \) by splitting
\[ \Sigma_d = \Sigma_{d-1} \times S^1 \] and shrinking the radius of the circle \( S^1 \) to zero. Then the components of \( A \) along the \( S^1 \)-direction become scalar fields in the lower-dimensional theory.

On \( \Sigma_d \) we decompose the world-sheet indices \( \mu = 0 \ldots (d-1) \) into \( (\mu) = (0, m) \) where \( \mu = 0 \) denotes the direction along the \( S^1 \) and \( m = 1 \ldots (d-1) \) the directions perpendicular to that. The 1-form fields \( A^a \) split into \( (A^a_\mu) = (A^a_0, A^a_m) \). After the dimensional reduction, the zero components of \( A^a_0 \) become scalar fields \( \phi^a \) on \( \Sigma_{d-1} \). The zero-component of the gauge transformation of \( A^I \) reduces to the gauge variations of \( \phi^a \):

\[
\delta A^a_0 \rightarrow \delta \phi^a = \epsilon^a_{bc} \phi^b \epsilon^c - \Gamma^a_{ib} \epsilon^b \rho^i \phi^c = -\epsilon^b \tilde{\Gamma}^a_{bc} \phi^c ,
\]

the reduction of the zero component of the \( F^i = 0 \) field equations, \( F^i_0 \rightarrow -\rho^i_0 \phi^a \), constrains \( \phi^a \) to be in \( \ker \rho \), and the reduction of the \((0, m)\) component of the field strength \( F^a \) becomes the covariant derivative for \( \phi^a \):

\[
F^a_{0, m} du^m \rightarrow \frac{1}{2} (D\phi)^a .
\]

Hence, the gauge transformations and the covariant derivative of \( \phi^a \) obtained by dimensional reduction coincide with the gauge transformations and covariant derivative of a scalar field which takes values in \( E \) and transforms according to the adjoint connection \( \tilde{\nabla} \). The difference between the two constructions is that the fields generated by dimensional reduction are always in the adjoint representation, and that they are constrained to taking values in \( \ker \rho \).

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