Research on the Weighted Regression Method of Artillery Trajectory Fitting Computation

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Abstract—In order to improve the precision of the heteroscedastic artillery firing table, a fitting computation method for artillery firing table compilation without the assumption of homologous variance is proposed in the paper. The principle of the method is to elevate the regression effect of heteroscedasticity model with using appropriate weights. First, a reasonable method to estimate the variance of disturbance term is explored. On this basis, the variance function of disturbance term is fitted by polynomial basis and B-spline basis respectively. Finally, weighted B-spline regression was performed on the heteroscedasticity model with the variance function as the weight. This method does not need to ignore the existence of heteroscedasticity, nor is it limited by the general assumption in the past. It has positive significance to reflect the operational performance of the new weapon system objectively which is still in the stage of scientific research and design finalization. In addition, the method can be applied in both homoscedastic and heteroscedastic populations.

1. Introduction

As the basis of the design of the fire control system of conventional weapons, the precision of the artillery firing table is very important. In the compilation of artillery firing table, the data processing method is an important factor directly affecting the precision of the artillery firing table. In the previous fitting computation, it was always assumed that the firing data samples were normally distributed with a constant variance. Besides, the samples that did not belong to the normal population were eliminated by using the extreme deviation criterion, Smirnov criterion or Dixon criterion [1]. Although, it was emphasized that the elimination must be prudent, it has been inevitable to ignore the statistical significance that the excluded firing test data should have revealed.

There could be a better idea on the statistical analysis. For those outliers that are found to be caused by definite intrinsic factors, correction can be made by eliminating intrinsic factors. For those data that cannot be corrected by analysis of environment and ATS, it is better to keep the data and treat the sample points more fairly to reveal the statistic quality of the data. This objective revelation lays more reliable theoretical foundation for the follow-up research and performance verification of the new weapon system.

Although the conclusion that the variance of firing test data is heteroscedastic has been proved quite well in many literatures[2], in the traditional artillery firing table technique , it is more likely to maintain
the normality assumption with constant variance by eliminating outliers. A new method to solve the heteroscedasticity problem in designing artillery firing table has been explored in the paper.

2. THE FITTING COMPUTATION AND THE HETEROSCEDASTICITY PHENOMENON

The fitting computation is the core work in data processing after firing tests. According to the fitting computation, the theoretical ballistic trajectory should be consistent with the real ballistic trajectory. For example, in order to conform to the coordinates of drop point, the resistance coincidence before the zero resistance parameter and the lift coefficient before the lift parameter are led in. For the purpose of fitting the ballistic elements at other points on the trajectory, more different coefficients should be led in [1]. For a particular powder charge type, the fitting coefficient is a function of the firing angle. For the fixed firing angle $\theta$, the fitting coefficient is a function of the powder charge type. As the fitting coefficient includes many factors that are not explicitly considered by the model, it is treated as the random variable.

According to the traditional fitting computation, it is assumed that the fitting coefficient and the firing angle satisfy the unary polynomial regression model. The model is as follows.

$$K = \sum_{j=0}^{m} a_j \theta^j + r$$  \hspace{1cm} (1)

Where $\theta$ is the firing angle, $K$ is the fitting coefficient, $a_0,\ldots,a_m$ are the parameters to be estimated, $r$ is the random disturbance term. $r$ is assumed to be independent identically distributed, where $r \sim N(0, \sigma^2)$.

The sum of squares of deviation is as follows.

$$Q(a_0,\ldots,a_m) = \sum_{i=1}^{n} (\sum_{j=0}^{m} a_j \theta^j_i - K_i)^2$$  \hspace{1cm} (2)

Where $\hat{a}_0,\ldots,\hat{a}_m$ is the estimate of $a_0,\ldots,a_m$, the sum of the squares of the deviations is minimized.

$$\frac{\partial Q(a_0,\ldots,a_m)}{\partial a_k} = 2 \sum_{i=1}^{n} (\sum_{j=0}^{m} a_j \theta^j_i - K_i) \theta^k_i$$  \hspace{1cm} (3)

By solving the system of equations, there are

$$\sum_{i=1}^{n} (\sum_{j=0}^{m} a_j \theta^j_i - K_i) = 0$$

$$\sum_{i=1}^{n} (\sum_{j=0}^{m} a_j \theta^j_{i+1} - K_i \theta_i^1) = 0$$

$$\vdots$$

$$\sum_{i=1}^{n} (\sum_{j=0}^{m} a_j \theta^j_i - K_i \theta^n_i) = 0$$  \hspace{1cm} (4)

The estimates $\hat{a}_0,\ldots,\hat{a}_m$ are got. Besides, the regression equation is obtained.

$$K = \sum_{j=0}^{m} \hat{a}_j \theta^j$$  \hspace{1cm} (5)

In the polynomial regression model, it is worth noting that the disturbance term is assumed to be homoscedastic. In fact, $\sigma^2$ may not be a constant. For heteroscedasticity phenomenon, a definite treatment is to apply the weighted least square method with the inverse of $\sigma^2$ as the weight. By such a model transformation, a heteroscedasticity model is transformed into a homoscedasticity model. In fact, limited prior information is known about the variance at the compilation stage of the firing table design. How to make use of less information to find a better way for fitting computation of various artillery systems is the problem this paper aims to solve.
3. Weighted regression method

3.1. B-spline regression

As the primary task, the outliers are discriminated according to the current statistical analysis of firing table data. The outliers which are not considered to be part of the homoscedastic normal population are excluded. In view of the common criterions, extreme deviation criterion which is of the highest priority requires knowing the variance [1]. For many types of ammunition, the variance is unknown, furthermore the variance which was already recorded was usually derived from the homoscedasticity assumption. These two strong constraint conditions are exactly what this paper aims to break through. The Smirnov criterion or Dixon criterion based on variance are obviously less reliable than the extreme deviation method [1]. In fact, whether the observation values that deviate significantly from other values are caused by the accidental destruction of standard test conditions and serious errors in the observation and calculation, or by the characteristics of the tested weapon itself is hard to distinguish.

Since polynomial regression model is applied in the fitting computation, when the outliers are not removed, there will be badly Runge phenomenon, therefore the regression curve cannot be used as the base for further ballistic trajectory calculation or subsequent theoretical analysis of the artillery system.

To resolve this contradiction, B-spline regression will be applied in this paper. The most important reason is that B-spline function is compact supported.

B-spline basis is defined as follows.

\[ B_{i,j}(t) = \begin{cases} 1, & t_j \leq x \leq t_{j+1} \\ 0, & \text{otherwise} \end{cases} \]

For \( m \leq 4 \), where \( i = 1, \ldots, k + 8 - m \), there are

\[ B_{i,m} = \frac{x - t_i}{t_{i+m-1} - t_i} B_{i,m-1} + \frac{t_{i+m} - x}{t_{i+m} - t_{i+1}} B_{i+1,m-1} \]

When the denominator is zero, the function is considered to be zero.

Thus it can be seen that an \( M \) order B-spline curve is only determined by \( M + 1 \) points. One point change merely affects the adjacent \( M + 1 \) segments of the curve, not the entire B-spline curve. With the ability of compact support, B-spline function allows adopting the elimination strategy with extremely harsh conditions in this paper. First, outliers are no longer specially considered by reason of large deviations in value. Secondly, it is important to pay attention to the samples obtained after changes in test conditions, even if they are not significantly large or small in value. As far as possible, the observed values are corrected by excluding the inherent factors identified and existing calculation errors. In the end, only explicit error data that cannot be corrected due to serious damage to test conditions or failed to be correctly collected by observation equipment are eliminated. Such approach retains the integrity of the original data to the maximum extent.

Moreover, an \( M \) order B-spline function is \( M - 1 \) order continuous. The smoothness of function benefits for studying the monotonicity, concavity and other properties of fitting coefficient regression curve.

3.2. Regression rule

3.2.1. Least squares rule

The least squares rule is used to determine the coefficients of model (1). Minimize the following formula.
\[ Q_0 = \sum_{i=1}^{n} [K_i - \sum_{j=1}^{k+4} \mu_j B_j(\theta_i)]^2 \]  \hspace{1cm} (8)

The matrix form is as follows.
\[ Q_1 = (K - CB)^T (K - CB) \]  \hspace{1cm} (9)

Where \( K, C, B \) is the matrix form of \( \{K_i\}, \{\mu_j\}, \{B_j(\theta_i)\} \).

Differentiate \( C \), then,
\[ 2BB^T C - 2BK = 0 \]  \hspace{1cm} (10)

Thus, the least squares estimate of \( C \) is
\[ \hat{C} = (B^T B)^{-1} B^T K \]  \hspace{1cm} (11)

The estimate of \( K \) is
\[ K = B\hat{C} = B(B^T B)^{-1} B^T K \]  \hspace{1cm} (12)

The estimates of \( \mu_j \) are \( \hat{\mu}_j \), where \( j = 1, \cdots, k + 4 \). Thus, B-spline estimate of regression function \( g(\theta) \) is obtained.
\[ \hat{g}(\theta) = \sum_{j=1}^{k+4} \hat{\mu}_j B_j(\theta) \]  \hspace{1cm} (13)

However, the above results are got under the assumption that the disturbance term is independently and identically distributed. When the disturbance term is heteroscedastic, the weighted least squares rule is applied to find the minimum value of the following formula.
\[ Q_2 = (K - CB)^T W(K - CB) \]  \hspace{1cm} (14)

Where \( W \) is the weight matrix, furthermore, it equals to the inverse of covariance matrix of the disturbance term \( \sigma \).

The estimates of the coefficients are as follows.
\[ \hat{C} = (B^T WB)^{-1} B^T WK \]  \hspace{1cm} (15)

3.2.2 Rough punishment criterion

Estimating merely by the least squares rule will lead to over-fitting. For this reason, introduce the rough punishment term, which acts as a balance between fitting and smoothing.

The absolute value of the second derivation of a function (i.e. curvature) can only measure how much the curve bends at the point. If the second derivative of function \( g(\theta) \) is differentiable, and the second derivative is squared integrated over the domain, then the integral of squared curvature can be used to measure the overall smoothness of the function \( g(\theta) \).

\[ P = \int [\gamma''(\theta)]^2 ds \]  \hspace{1cm} (16)

The bigger \( P \) is, the rougher \( g(\theta) \) is. When \( P = 0 \), the graph of \( g(\theta) \) is a line. More generally, to wonder whether \( g(\theta) \) makes \( L \theta = 0 \), \( L \) can be defined as the generalized smoothness.

\[ P^* = \int [\gamma''(\theta)]^2 ds \]  \hspace{1cm} (17)

To minimize
\[ Q_3 = (K - CB)^T W(K - CB) \]  \hspace{1cm} (18)

It is to maximize the degree of fitting.

The minimization of the formula (17) is the maximization of smoothness. By combining the two, over-fitting can be prevented.

The new loss function is
\[ Q_3 = \sum_{i=1}^{n} \frac{1}{\sigma_i^2(\theta)} [K_i - g(\theta)]^2 + \lambda \int [\gamma''(\theta)]^2 d\theta \]  \hspace{1cm} (19)

The matrix form is as follows.
\[ Q_4 = (K - CB)^T W (K - CB) + \lambda \int [LC^T B(\theta)]^2 d\theta \]  

(20)

Where \( \lambda \int [LC^2 B(\theta)]^2 d\theta \) is the smooth term, \( \lambda \) is the smoothing coefficient. The bigger \( \lambda \) is, the smoother the fitting function tends to be. The smaller \( \lambda \) is, the more likely to get a better fitting. In this paper, the generalized cross validation method is used to select smooth parameter \( \lambda \) [4].

Now calculate the estimates when \( \lambda \) is already determined.

\[
\int [Lg(\theta)]^2 d\theta = \int [LC^T B(\theta)]^2 d\theta
= C^T [\int [LB(\theta)]^2 d\theta] C = C^T RC
\]

(21)

Where

\[ R = \int [LB(\theta)]^2 d\theta \]

(22)

It is the smooth matrix. Then

\[ Q_i = (K - CB)^T W (K - CB) + \lambda C^T RC \]

(23)

Take the derivative of \( C \), it follows that

\[ \hat{C} = (B^T WB + \lambda R)^{-1} B^T WK \]

(24)

### 3.2.3. Heteroscedasticity treatment principle

In order to deal with heteroscedasticity, the least square method should be improved.

The heteroscedasticity model is

\[ K = g(\theta) + r \]

(25)

Where \( r \sim N(0, \sigma_i^2(\theta)) \), by using the ordinary least square method, to estimate \( g(\theta) \) that minimizes \( \sum_i [K_i - g(\theta_i)]^2 \).

This shows that the importance of the terms in the loss function should not be the same. The sum of the squares corresponding to disturbance term with big variance could be super large, for the same reason, the loss function is affected too much by this term. In order to balance the influence of each item in loss function, the key is to add proper weights. The principle is to make the weighted loss function reflect the influence of each item more objectively. Specifically, control the item of large disturbance with small weight to reduce its importance in loss function and use large weight to the disturbance item of small variance to increase its influence.

The greater the disturbance term variance is, the less reliable the data is. In other words, the penalty on them is greater and the regression line is allowed to be a little further away from the sample points here. It happens to dispel the concern that the outliers will seriously change the regression curve.

Thus, the loss function of the improved ordinary least squares is

\[ \sum_i W_i [K_i - g(\theta_i)]^2 \]

(26)

The purpose of balancing can be achieved by selecting appropriate weights, which are theoretically equal to the inverse of the disturbance term variance [5].

\[ W_i = \frac{1}{\sigma_i^2} \]

(27)

### 3.2.4. A piecewise strategy to estimate variance

By adding the right weights, heteroscedasticity can be solved perfectly. However, there is no way to apply weighted least square method without knowing \( \sigma_i^2 \). Actually, at the stage of firing table design, it’s unlikely to know \( \sigma_i^2 \) in advance. In this paper, a piecewise estimate strategy is explored to find \( \sigma_i^2 \), then naturally, \( \frac{1}{\sigma_i^2} \) can be used as the weight in weighted regression for heteroscedastic fitting computation.
Assuming that the disturbance term variance in fitting computation model is heteroscedastic, it can be considered that the variance changes little within the same segment, and can be approximately treated as homoscedasticity.

In order to get the disturbance term, the ordinary B-spline regression based on rough punishment criterion was first performed to get the regression function $g(\theta)$. The residual $e_i = K_i - g(\theta)$ is used as the disturbance term to piecewise estimate. Specifically, $n$ residuals were equally divided into $m$ segments, where $m$ was taken as the number of angles in firing table shooting test. In segment $p$, $(p = 1, 2, \cdots, m)$

$$Var(e_i) = \sigma_p^2$$  \hspace{1cm} (28)

The focus issue is how to estimate $\sigma_p^2$ properly. First, all residuals are calculated by regression model and firing data. Secondly, the residuals are grouped and the variances are estimated by group. Finally, fit the variance function.

Where $e_i$ are the residuals corresponding to the sample points in piecewise $p$, $(i = (p-1)m + 1, \cdots, pm)$ and $e_i \sim N(0, \sigma_p^2)$, the residual variance in piecewise is as follows.

$$\sigma_p^2 = \frac{1}{m-1} \sum_{i=(p-1)m+1}^{pm} (e_i - \bar{e})^2$$  \hspace{1cm} (29)

Since $e_i \sim N(0, \sigma_p^2)$, then $\bar{e}_p = E(e_p) = 0$, and that gives the residual sample variance.

$$\sigma_p^2 = \frac{1}{m-1} \sum_{i=(p-1)m+1}^{pm} e_i^2$$  \hspace{1cm} (30)

There is the estimate of $\sigma_p^2$.

$$\hat{\sigma}_p^2 = \frac{1}{m-1} \sum_{i=(p-1)m+1}^{pm} e_i^2$$  \hspace{1cm} (31)

3.2.5. The fitting of the variance function of disturbance term

3.2.5.1. Polynomial fitting
Assume $(\theta_p, \sigma_p^2)$ is $M_p$, $p = 1, 2, \cdots, m$, taking $M_1, \cdots M_m$ as the sample points for fitting the variance function of the disturbance term, do $l$ order polynomial fit.

Consider a unary polynomial regression model.

$$\hat{\sigma}^2(\theta) = \sum_{j=0}^{l} b_j \theta^j + \varepsilon$$  \hspace{1cm} (32)

Where $b_0, \cdots b_l$ are the coefficients to be estimated, $\varepsilon$ is the random disturbance term, and $\varepsilon \sim N(0, \sigma^2(\theta))$.

There is the sum of the squares of the deviations.

$$Q_n = \sum_{p=1}^{m} \left[ \sum_{j=0}^{l} b_j \theta_p^j - \sigma_p^2 \right]^2$$  \hspace{1cm} (33)

Solve the normal equation set.
Minimize the sum of squares of deviations, estimates \( \hat{b}_0, \cdots, \hat{b}_l \) are got.

There is the fitting equation of variance function.

\[
\sigma^2(\theta) = \sum_{j=0}^{l} \hat{b}_j \theta^j
\]  

(35)

### 3.2.5.2. B-spline fitting

The variance function of the disturbance term can also be fitted by B-spline basis. Assuming that the variance function is smooth and choose 3-order B-spline basis, the estimated variance function will be a two-order continuous differentiable smooth function. Where \( M_1, \cdots, M_m \) are taken as the nodes to fit \( m-4 \) sections of the cubic B-spline curve, the B-spline regression function \( \sigma^2(x) \) can be got according to the way given in B. 1).

The inverse of the variance function is used in model (25) as the weight. Hence, the problem is transformed into minimizing the following function.

\[
\sum_{i=1}^{n} \frac{1}{\sigma^2(\theta)} \left[ (K_i - \sum_{j=1}^{k+4} \phi_j B_j(\theta))^2 + \lambda \int \left( \sum_{j=1}^{k+4} \phi_j B_j(\theta) \right)^2 d\theta \right] (36)
\]

According to equation (24), the estimates of \( \phi_j \) are \( \hat{\phi}_j, j = 1, \cdots, k+4 \). Therefore, the B-spline estimate of regression function \( g(\theta) \) is

\[
\hat{g}(\theta) = \sum_{j=1}^{k+4} \hat{\phi}_j B_j(\theta)
\]  

(37)

### 4. The numerical example analysis

In order to verify the validity of the proposed method, the firing table test data of a howitzer is taken as an example for statistical analysis. First, the shooting data were imported, and the sample points were assigned to two sets alternately. The first set would be used as the training set for regression. The second set would be used as the test set for testing and verifying the regression effect.

First, the training set is utilized to do B-spline regression. Regression function \( g(\theta) \) was got by running the program. Then get the residuals by \( e_i = g(\theta_i) - K_i \). Furthermore, the variance is estimated by using formula (30).

Spearman’s rank correlation coefficient test was used to verify heteroscedasticity, and the result of the program was heteroscedasticity.

For the sake of contrast, the variance function of the disturbance term was fitted by the third order polynomial basis and cubic B-spline basis respectively.
Then, do the weighted B-spline regression. The comparison of the final regression curves is shown in figure 3.

In the process of numerical experiments, with different choice of order, the polynomial fitting variance was negative sometimes, which led to the error of program. On the contrary, B-spline function is non-negative, so that kind of error never happens. Polynomial fitting brings about terrible fluctuation of the regression curve, which surely holds back the following study. For the same reason, to predict the trend of fitting coefficient regression curve seems not possible as well.

For the sake of contrast, the weighted regression curve of disturbance term variance function fitted by cubic B-spline basis function was compared with the B-spline regression curve without weight.
Figure 4. Weighted B-spline regression and unweighted B-spline regression diagram

As can be seen from figure 4, due to heteroscedasticity, the unweighted regression curve runs from extremely low to extremely high at the extreme points or vice versa. This incapacitates the unweighted regression curve to reflect the rule of fitting coefficient changing with the shooting angle. The weighted B-spline regression curve is much smoother, so it can be used for subsequent calculations and trend prediction. Obviously, the weighted regression is better.

For the sake of contrast, the weighted B-spline regression curve with the disturbance term variance function fitted by cubic B-spline basis as the weight was compared with the polynomial regression curve.

Figure 5. Weighted B-spline regression and polynomial regression diagram

For heteroscedasticity firing table test data, the polynomial regression over-fitted seriously, which often happened before. From figure 5, the weighted B-spline regression method solves heteroscedasticity in firing table design pretty well.

In order to know whether the regression results reflect the statistical rule between the coincidence coefficient and the firing table, three indicators are led in.

The root-mean-square error represents the mean deviation between the sample points and the regression, which is expressed as follows.

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (K_i - \hat{K}_i)^2}$$  \hspace{1cm} (38)

The mean relative error represents the precision of the regression curve.

$$\text{MRE} = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{K_i - \hat{K}_i}{K_i} \right|$$  \hspace{1cm} (39)

Another measure of regression effect is FVU. The expression is as follows.
\[
FVU = \frac{\sum_{j=1}^{n}[\hat{g}(\theta_j') - K_j']^2}{\sum_{j=1}^{n}(\bar{K} - K_j')^2}
\]

(40)

Where \(\bar{K} = \frac{1}{h} \sum_{j=1}^{n} K_j'\), \(\hat{g}(\theta_j')\) is the regression value of \(\theta_j'\).

It is perfect fitted at all the points when \(FVU=0\). Besides, the fit is worse than just averaging the points when \(FVU>1\)\(^6\).

Calculate three regression indicators respectively with the data in the test set.

|                  | Weighted B-spline regression | Unweighted B-spline regression | Polynomial regression |
|------------------|-----------------------------|-------------------------------|-----------------------|
| RMSE             | 0.1960                      | 0.2717                        | 0.300                 |
| MRE              | 0.4379                      | 0.5745                        | 1.1358                |
| FVU              | 0.4201                      | 0.8946                        | 0.6782                |

The comparison shows that the weighted B-spline regression is closer to the test set data.

Taking both regression images and regression indicators into account, weighted B-spline regression is satisfactory. In addition, when different heteroscedasticity tests are used for the same data, it is worth noting that the result of the Spearman’s rank correlation coefficient test is heteroscedasticity, while the result of White test is homoscedasticity. In fact, in order to avoid the misjudgment of heteroscedasticity test, it is necessary to comprehensively consider the model and the test in choosing the test method. However, it is difficult to make a wise choice without enough prior information. From another point of view, consider to deal with homoscedastic shooting data with the algorithm in this paper. It is nothing but to add the same weights to each term of the sum of squares of deviation, which is the same to no weight. Therefore, the Weighted B-spline regression method can be used for fitting computation of firing table compilation without discrimination, which means heteroscedasticity test can be omitted.

5. Conclusion

In this paper, the heteroscedasticity phenomenon in the fitting computation of firing table compilation is considered. An improved weighted B-spline regression method based on the combination of least squares criterion and rough punishment criterion is proposed. The calculation method of weight is emphatically analyzed, which uses piecewise strategy to estimate residual variance, fits variance function with B-spline basis, and takes the inverse of variance function as the weight of weighted regression. Combined with theoretical analysis and numerical experimental results, the work is summarized as follows.

- An improved weighted regression method is proposed to solve heteroscedasticity in fitting computation for firing table compilation. Compared with the traditional polynomial regression, it needs fewer assumptions and can be wide applied. It will benefit in future tasks of various new weapon systems.
- In the traditional method, outliers are eliminated with subjective factors involved. The method in this paper is no longer based on the degree of numerical deviation as the basis for elimination. In this case, the original shooting data can be retained the maximum extent to describe the statistical characteristics of the operational performance of the tested artillery system.
- The results of numerical experiments and regression analysis show that the weighted B-spline regression method can significantly improve the data processing precision, thus improve the precision of the firing table. The weighted regression curve is smooth and the trend is clear, which will make significant contribution to the follow-up research and prediction.
• Since the weighted regression method applied to homoscedastic data is the same to no weight, heteroscedasticity test is not indispensable in the subsequent firing table compilation work, which saves the effort of discriminating.

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