ON THE SECTION CONJECTURE AND BRAUER-SEVERI VARIETIES

GIULIO BRESCIANI

ABSTRACT. J. Stix proved that a curve of positive genus over \( \mathbb{Q} \) which maps to a non-trivial Brauer-Severi variety satisfies the section conjecture. We prove that, if \( X \) is a curve of positive genus over a number field \( k \) and the Weil restriction \( R_k/\mathbb{Q}X \) admits a rational map to a non-trivial Brauer-Severi variety, then \( X \) satisfies the section conjecture. As a consequence, if \( X \) maps to a Brauer-Severi variety \( P \) such that the corestriction cor\(_{k/\mathbb{Q}}([P]) \in Br(\mathbb{Q}) \) is non-trivial, then \( X \) satisfies the section conjecture.

Let \( X \) be a geometrically connected variety over a field \( k \) with separable closure \( \bar{k} \), there is a short exact sequence of étale fundamental groups

\[
0 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \to \text{Gal}(\bar{k}/k) \to 0.
\]

Grothendieck’s section conjecture predicts that, if \( X \) is a smooth, projective curve of genus at least 2 and \( k \) is a number field, the set of rational points \( X(\mathbb{Q}) \) is in natural bijection with sections of the sequence above modulo the action of \( \pi_1(X_{\bar{k}}) \) by conjugation.

Thanks to an idea of Tamagawa [Tam97] [Sti13, Corollary 102] it is sufficient to prove the conjecture for curves with no rational points, and some results have been proved about such curves. The section conjecture holds for \( X \) if

- the number field \( k \) has a real place \( k \hookrightarrow \mathbb{R} \) such that \( X_{\mathbb{R}}(\mathbb{R}) = \emptyset \), see [Moc03, Corollary 3.13], or
- the class of \( \text{Pic}^0_X \in H^1(k, \text{Pic}^0_X) \) is not divisible, see [HS09, Theorem 1.2], or
- \( k = \mathbb{Q} \) and \( X \) maps to a non-trivial Brauer-Severi variety, see [Sti10a, Corollary 18].

The last condition, which is due to Stix, holds over any number field \( k \), but with an additional hypothesis: for every prime number \( p \), it is required that \( X \) has bad reduction at most at one place \( p \) of \( k \) over \( p \), see [Sti10a, Theorem 17]. We provide a different generalization based on Weil’s restriction of scalars.

Recall that, given a finite separable extension \( k/h \) of fields and a quasi-projective variety \( X \) over \( k \), the Weil restriction \( R_{k/h}X \) is a quasi-projective variety over \( h \) characterized by a functorial bijection \( \text{Hom}(S, R_{k/h}X) \simeq \text{Hom}(S_k, X) \) for schemes \( S \) over \( h \). In particular, \( k \)-rational points of \( X \) are in natural bijection with \( h \)-rational points of \( R_{k/h}X \).

If \( X \) is a curve of genus \( g \) over a number field \( k \), then \( (R_{k/\mathbb{Q}}X)_\mathbb{Q} \) is a product of \([k : \mathbb{Q}]\) curves of genus \( g \), so passing to the Weil restriction is basically a trade-off between the complexity of the base field and the complexity of the variety. We prove the following.

**Theorem 1.** Let \( X \) be a smooth, projective, geometrically connected curve of positive genus over a number field \( k \). Assume that \( R_{k/\mathbb{Q}}X \) admits a rational map to a non-trivial Brauer-Severi variety. Then the

---

The author is supported by the DFG Priority Program "Homotopy Theory and Algebraic Geometry" SPP 1786.
section conjecture holds for $X$. Equivalently, if $\pi_1(X) \to \text{Gal}(\bar{k}/k)$ admits a section, then the map $\text{Br}(\mathbb{Q}) \to \text{Br}(R_{k/\mathbb{Q}}X)$ is injective.

As a consequence, we get the following corollaries.

**Corollary 2.** Let $X$ be a smooth projective curve of positive genus over a number field $k$ and $P$ a Brauer-Severi variety such that the corestriction $\text{cor}_{k/\mathbb{Q}}[P] \in \text{Br}(\mathbb{Q})$ is non-trivial. If there exists a morphism $X \to P$, then the section conjecture holds for $X$. Equivalently, if $\pi_1(X) \to \text{Gal}(\bar{k}/k)$ admits a section, the kernel of $\text{Br}(k) \to \text{Br}(X)$ is contained in the kernel of $\text{cor}_{k/\mathbb{Q}} : \text{Br}(k) \to \text{Br}(\mathbb{Q})$.

**Corollary 3.** Let $k$ be a number field and $P$ a Brauer-Severi variety over $\mathbb{Q}$ with $[k : \mathbb{Q}][P] \neq 0 \in \text{Br}(\mathbb{Q})$. If $X$ is a smooth projective curve over $\mathbb{Q}$ of positive genus with a morphism $X \to P$, then the section conjecture holds for the base change $X_k$.

Our argument is analogous to Stix’s one and we rely heavily on his results. Our contribution consists essentially of two things: we realized that such a generalization was possible and we overcame the lack, for higher dimensional varieties, of a sufficiently strong analogue of Lichtenbaum’s theorem about the period and index of a curve over a $p$-adic field, which is an essential ingredient of Stix’s proof.

We mention that it is possible to prove Corollary 2 (and thus Corollary 3) analogously to Stix’ theorem over $\mathbb{Q}$ [Sti10a, Corollary 18] without using Weil’s restriction of scalars. The proof is basically the same plus the observation that, if $\alpha \in \text{Br}(k)$ is a Brauer class over the number field $k$, the Hasse invariant of $\text{cor}_{k/\mathbb{Q}}(\alpha)$ at $p$ is the sum of the Hasse invariants of $\alpha$ at places over $p$.

1. **Weil restriction and the section conjecture**

The behaviour of the étale fundamental group and the section conjecture with respect to the Weil restriction of scalars has been studied by J. Stix in [Sti10b]. Let $k/h$ be a finite, separable extension of fields and $X$ is a geometrically connected variety over $k$. Assume either that $X$ is proper or that $k$ has characteristic 0. Stix describes explicitly the étale fundamental group of $R_{k/h}X$ in terms of the one of $X$, and uses this description to show that the section conjecture holds for $X$ if and only if it holds for $R_{k/h}X$. We give here an alternative treatment based on étale fundamental gerbes.

Recall that A. Vistoli and N. Borne have introduced the étale fundamental gerbe $X \to \Pi_{X/k}$ of a geometrically connected scheme, see [BV15, Section 9] and [Bre21, Appendix]. The set of Galois sections of the étale fundamental group is in natural bijection with the isomorphism classes of $\Pi_{X/k}(k)$. We show here that the étale fundamental gerbe and the Weil restriction commute.

**Proposition 4.** Let $k/h$ be a finite separable extension of fields, and $X$ a geometrically connected quasi-projective variety over $k$. Assume either that $X$ is proper or that char $k = 0$.

Then $R_{k/h}X$ is geometrically connected and the natural morphism $R_{k/h}X \to R_{k/h}\Pi_{X/k}$ induces a natural isomorphism

$$\Pi_{R_{k/h}X/h} \simto R_{k/h}\Pi_{X/k}.$$  

**Proof.** Let $\bar{h}/h$ be a separable closure. We have natural isomorphisms $(R_{k/h}X)_{\bar{h}} = \prod \sigma^*X$ and $(R_{k/h}\Pi_{X/k})_{\bar{h}} \simeq \prod \sigma^*\Pi_{X/k} = \prod \Pi_{\sigma^*X/k}$ where the product runs over the $h$-linear embeddings $\sigma : k \subset \bar{h}$, see [Wei82, Theorem 1.3.2] (Weil’s original work deals only with varieties, but his proof easily generalizes to any fibered category). In particular $R_{k/h}X$ is geometrically connected and
Let $X$ be a smooth, projective variety over a field $k$ of characteristic $p$.

Let $\pi: \overline{k} \rightarrow k$ be the étale homotopy type of $X$ and $\pi = \pi_X / k$.

If $s \in \Pi_{X / k}(k)$ is a Galois section, denote by $s^Q = \Pi_{R_{k/q}X/Q}(Q) = R_{k/q} \Pi_{X/Q}(Q)$ the induced section. Recall that an étale neighbourhood of $s$ is a finite étale cover $Y \rightarrow X$ such that $s$ lifts to $\Pi_{Y / k}(k)$.

**Corollary 7.** Let $X$ be a smooth, projective curve of positive genus over a number field $k$ and $s \in \Pi_{X / k}(k)$ a Galois section. If $Y \rightarrow X$ is an étale neighbourhood of $s$, then $R_{k/q}Y \rightarrow R_{k/q} X$ is an étale neighbourhood of $s^Q$. The étale neighbourhoods of this form are cofinal in the system of all étale neighbourhoods of $s^Q$. 

## 2. Morphisms to Brauer-Severi Varieties

If $X$ is a scheme over $k$, denote by $Br(X / k)$ the kernel of $Br(k) \rightarrow Br(X)$. If $X$ is a regular variety, the restriction map $Br(X) \rightarrow Br(k(X))$ is injective [Mil80, Corollary IV.2.6] and thus $Br(X / k) = Br(k(X) / k)$. In particular, a Brauer class $[P] \in Br(k)$ of a Brauer-Severi variety $P$ is in $Br(X / k)$ if and only if there exists a rational map $X \dashrightarrow P$.

If $X$ is a smooth, projective variety, the Leray spectral sequence in étale cohomology for the map $X \rightarrow \text{Spec} \, k$ gives a short exact sequence

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Pic}_X(k) \rightarrow Br(X / k) \rightarrow 0$$

where $\text{Pic}_X$ is the Picard scheme of $X$ and $\text{Pic}(X) = H^1(X, G_m)$ is the Picard group. Let us call $\beta$ the homomorphism $\text{Pic}_X(k) \rightarrow Br(X / k)$.

**Lemma 8.** Let $X$ be a smooth, projective variety over a field $k$ of characteristic 0, $s \in \Pi_{X / k}(k)$ a section, $b \in Br(X / k)$ a Brauer class split by $X$. Assume that the second étale homotopy group of $X_k$ is trivial. For every positive integer $n$, there exists an étale neighbourhood $f: Y \rightarrow X$ of $s$ such that $f^* b \in n \text{Br}(Y/k)$.

**Proof.** Let $L \in \text{Pic}_X(k)$ be such that $\beta(L) = b \in Br(X / k)$, and $L_K \in \text{Pic}(X_K)$ the associated line bundle over $X_K$.

We have an exact sequence

$$H^1(X_K, \mu_n) \rightarrow \text{Pic}(X_K) \xrightarrow{\mu_n} \text{Pic}(X_K) \xrightarrow{\beta} H^2(X_K, \mu_n).$$

Let $\text{Et}(X_K)$ be the étale homotopy type of $X_K$ and $\text{cosk}(\text{Et}(X_K))$ its third coskeleton, since $\pi_2^{et}(X_K)$ is trivial we have $\text{cosk}(\text{Et}(X_K)) = K(\pi_1^{et}(X_K), 1)$. Therefore, we have

$$H^2(X_K, \mu_n) = H^2(\text{Et}(X_K), \mu_n) = H^2(\text{cosk}(\text{Et}(X_K)), \mu_n) = H^2(\pi_1^{et}(X_K), \mu_n),$$
see [AM69, Corollary 9.3] for the first equality. Since the base change to $\bar{k}$ of the étale neighbourhoods of $s$ are cofinal in all finite étale covers of $X_k$, there exists an étale neighbourhood $g : X' \to X$ of $s$ such that $g_k^*\delta(L_k) = 0$ and thus $g_k^*L_k \in \text{Pic}(X'_k)$ is divisible by $n$.

Choose $M \in \text{Pic}(X'_k)$ such that $M^n = g_k^*L_k$. Since the Picard scheme of $X'$ is locally of finite type, the residue field of $\text{Spec } k \xrightarrow{M} \text{Pic}_{X'}$ is finite over $k$ and thus the Galois orbit of $M$ is finite. If $\sigma \in \text{Gal}(\bar{k}/k)$ is an element, since $g_k^*L_k$ is Galois invariant then $M \otimes \sigma M^{-1}$ is $n$-torsion, and thus it comes from $H^1(X'_k, \mu_n) = \text{Hom}(\pi_1^{\text{et}}(X'_k), \mu_n)$. It follows that there exists an étale neighbourhood $h : Y \to X'$ of $s$ such that $h_k^*M \in \text{Pic}(Y_k)$ is Galois-invariant and thus descends to an element $N \in \text{Pic}_Y(k)$.

Let $f : Y \to X$ be the composition, we have $N^n = f^*L$ and hence $n\beta(N) = \beta(f^*L) = f^*b$. □

Recall that a group $G$ is good in the sense of Serre if $H^i(\hat{G}, M) \to H^i(G, M)$ is an isomorphism for every finite $G$-module $M$, see [Ser94, p. I.2.6]. Fundamental groups of complex curves are good [GJZ08, Proposition 3.6].

**Lemma 9.** Let $X$ be a variety over $\mathbb{C}$. Assume that $\pi_2^{\text{top}}(X^{an})$ is trivial and that $\pi_1^{\text{top}}(X^{an})$ is good in the sense of Serre. Then $\pi_2^{\text{et}}(X)$ is trivial.

**Proof.** Write $\pi_i = \pi_i^{\text{et}}(X)$. The hypothesis implies that the natural homomorphism $H^2(\pi_1, M) \to H^2(X, M)$ is bijective for every finite $\pi_1$-module $M$.

Assume by contradiction that $\pi_2$ is not trivial, then there exists a finite homotopy type $F$ with $\pi_2(F) = 0$ for $n \geq 3$ and a map $\text{Et}(X) \to F$ such that $\pi_2 \to \pi_2(F)$ is non-trivial. Up to passing to finite étale coverings of $X$ and $F$, we may assume that $\pi_2(F)$ is trivial and hence $F = K(M, 2)$ for some finite abelian group $M$ (the fundamental group of the covering of $X$ is still good thanks to [GJZ08, Lemma 3.2]).

We thus have a map $X \to K(M, 2)$ inducing a non-trivial homomorphism $\pi_2 \to M$. This defines a cohomology class $\alpha \in H^2(X, M)$ not in the image of $H^2(\pi_1, M) \to H^2(X, M)$, and this is absurd. □

**Lemma 10.** Let $k/h$ be a finite separable extension and $P/k$ a Brauer-Severi variety. There exists a Brauer-Severi variety $Q/h$ with $[Q] = \text{cor}_{k/h}([P]) \in \text{Br}(h)$ and a closed embedding $R_{k/h}P \to Q$.

**Proof.** Let $\bar{h}/h$ be a separable closure, then

$$R_{k/h}P = \left( \prod_\sigma \sigma^*P \right) / \text{Gal}(\bar{h}/h)$$

where the product runs over $h$-linear embeddings $\sigma : k \to \bar{h}$. The Galois group $\text{Gal}(\bar{h}/h)$ permutes the factors and the stabilizer $\text{Gal}(\bar{h}/\sigma k)$ of $\sigma^*P$ acts on it. Note that, even though $\sigma^*P$ is a projective space over $\bar{h}$, the action of $\text{Gal}(\bar{h}/\sigma k)$ is non-standard.

The external tensor product $\boxtimes_\sigma O(1) \in \text{Pic}(\prod_\sigma \sigma^*P)$ is naturally $\text{Gal}(\bar{h}/h)$-equivariant and thus the Segre embedding

$$S : \prod_\sigma \sigma^*P \to \text{P}(\text{H}^0(\boxtimes_\sigma O(1)))$$
is naturally $\text{Gal}(\bar{h}/h)$-equivariant. The quotient $Q = \mathbb{P}(H^0(\mathbb{X}_rO(1))) / \text{Gal}(\bar{h}/h)$ is a Brauer-Severi variety over $h$.

Using the fact that summation in the Brauer group can be computed using the Segre embedding of Brauer-Severi varieties [Art82, §4] and the fact that the corestriction homomorphism is the derived augmentation homomorphism, it is easy to show that that $[Q] = \text{cor}_{k/h}([P])$. Moreover, the Segre embedding $S$ descends to a closed embedding $R_{k/h}P \hookrightarrow Q$ since it is $\text{Gal}(\bar{h}/h)$-equivariant. \hfill \square

3. Proof of the main theorem

Let us now prove Theorem 1. Let $X$ be a smooth projective curve over a number field $k$ such that $R_{k/Q}X$ admits a rational map to a non-trivial Brauer-Severi variety, we want to show that $\Pi_{X/k}(k)$ is empty. Assume by contradiction that there exists a section $s \in \Pi_{X/k}(k)$ and let $b \in \text{Br}(R_{k/Q}X/Q)$ be a non-trivial Brauer class. Let $s^Q \in \Pi_{R_{k/Q}X/Q}(Q)$ be the associated section.

Since $(R_{k/Q}X)_k$ is a product of curves and the fundamental group of a curve over $\bar{k}$ is good in the sense of Serre [GJZ08, Proposition 3.6], then Lemma 9 implies that $\pi^\text{et}_2((R_{k/Q}X)_k)$ is trivial and we may thus apply Lemma 8 to $R_{k/Q}X$ and $s^Q$. If we apply Lemma 8 together with Corollary 7, for every $N > 0$ we may find an étale neighbourhood $X_N \rightarrow X$ of $s$ and a Brauer class $b_N \in \text{Br}(R_{k/Q}X_N/Q)$ such that $N\text{Br}_N = b \in \text{Br}(Q)$ is non-trivial.

Let $l/k/Q$ be a Galois closure. Up to replacing $X$ with $X_{2[l:Q]}$ and $b$ with $b_{2[l:Q]}$, we may assume that $2[l : Q]b \in \text{Br}(Q)$ is non-trivial.

Fix a prime number, let us show that the order of the Brauer class $2[l : Q]b_{Q_p}$ is a power of $p$. Let $L$ be the completion of $l$ at some place over $p$, we have that $L/Q_p$ is a Galois extension such that $[L : Q_p]$ divides $[l : Q]$, it is enough to show that the order of $[L : Q_p]b_{Q_p}$ is a power of $p$. Let $\Sigma$ be the set of embeddings $k \rightarrow L$, we have

$$ (R_{k/Q}X)_L = \prod_{\sigma \in \Sigma} \sigma^* X. $$

The section $s \in \Pi_{X/k}(k)$ induces a section $\sigma^* s \in \Pi_{\sigma^* X/L}(L)$ for every embedding $\sigma : k \rightarrow L$. By [Sti13, Theorem 15], this implies that the index of $\sigma^* X$ is a power of $p$ for every $\sigma$. Let $D_\sigma \in \text{Z}_0(X_\sigma)$ be a 0-cycle whose degree is a power of $p$, then $\otimes_\sigma D_\sigma$ is a 0-cycle on $\prod_\sigma \sigma^* X$ whose degree is a power of $p$. It follows that the index of $\prod_\sigma \sigma^* X$ is a power of $p$, too.

Since $Q(R_{k/Q}X)$ splits $b$, there exists a Brauer-Ševeri variety $P$ with $[P] = b$ and a smooth projective variety $Y/Q$ birational to $R_{k/Q}X$ with a morphism $Y \rightarrow P$. Since the index is a birational invariant, the index of $Y_L$ is a power of $p$, it follows that the index of $b_L = [P_L]$ is a power of $p$. This implies that the order (i.e. the period) of $b_L \in \text{Br}(L)$ is a power of $p$, and finally the same holds for $\text{cor}_{L/Q_p} b_L = [L : Q_p]b_{Q_p}$.

We thus have that the order of $2[l : Q]b_{Q_p}$ is $p$-primary for every $p$, and clearly $2[l : Q]b_{R} = 0 \in \text{Br}(R) = \mathbb{Z}/2\mathbb{Z}$.

The rest of the argument is analogous to Stix’s one. Let $\alpha_p \in Q/Z$ be the Hasse invariant of $2[l : Q]b_{Q_p}$, by the Brauer-Hasse-Noether theorem we have $\sum_p \alpha_p = 0 \in Q/Z$. Since $\alpha_p$ is $p$-primary for every $p$, it follows that $\alpha_p = 0$ for every $p$ and thus $2[l : Q]b \in \text{Br}(Q)$ is trivial, which is absurd. This concludes the proof of Theorem 1.
Corollaries. Theorem 1 implies Corollary 2 using Lemma 10. Since the composition \( \text{cor}_{k/Q} \circ \text{res}_{k/Q} : \text{Br}(Q) \rightarrow \text{Br}(Q) \) is multiplication by \([k : Q]\), Corollary 2 implies Corollary 3.

Acknowledgements. Lemma 8 was found during joint work with A. Vistoli. I would like to thank an anonymous referee for many useful remarks.

REFERENCES

[AM69] M. Artin and B. Mazur. *Etale Homotopy*. Lecture Notes in Mathematics 100. Springer-Verlag, 1969.

[Art82] M. Artin. “Brauer-Severi varieties”. In: *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*. Vol. 917. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 194–210.

[Bre21] G. Bresciani. “Some implications between Grothendieck’s anabelian conjectures”. In: *Algebr. Geom.* 8.2 (2021), pp. 231–267. ISSN: 2313-1691. DOI: 10.14231/ag-2021-005.

[BV15] N. Borne and A. Vistoli. “The Nori fundamental gerbe of a fibered category”. In: *Journal of Algebraic Geometry* 24 (2 Apr. 2015), pp. 311–353.

[GJZ08] F. Grunewald, A. Jaikin-Zapirain, and P. A. Zalesskii. “Cohomological goodness and the profinite completion of Bianchi groups”. In: *Duke Math. J.* 144.1 (2008), pp. 53–72.

[HS09] D. Harari and T. Szamuely. “Galois sections for abelianized fundamental groups”. In: *Math. Ann.* 344.4 (2009). With an appendix by E. V. Flynn, pp. 779–800. ISSN: 0025-5831. DOI: 10.1007/s00208-008-0327-z.

[Mil80] J. S. Milne. *Étale cohomology*. Vol. 33. Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1980, pp. xiii+323. ISBN: 0-691-08238-3.

[Moc03] S. Mochizuki. “Topics surrounding the anabelian geometry of hyperbolic curves”. In: *Galois groups and fundamental groups*. Vol. 41. Math. Sci. Res. Inst. Publ. Cambridge Univ. Press, Cambridge, 2003, pp. 119–165.

[Ser94] J.-P. Serre. *Cohomologie galoisienne*. Fifth. Vol. 5. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1994, pp. x+181. ISBN: 3-540-58002-6. DOI: 10.1007/BFb0108758.

[Sti10a] J. Stix. “On the period-index problem in light of the section conjecture”. In: *American Journal of Mathematics* 132.1 (2010), pp. 157–180.

[Sti10b] J. Stix. “Trading degree for dimension in the section conjecture: The non-abelian Shapiro lemma”. In: *Mathematical Journal of Okayama University* 52 (2010), pp. 29–43.

[Sti13] J. Stix. *Rational points and arithmetic of fundamental groups*. Lecture Notes in Mathematics 2054. Springer, 2013.

[Tam97] A. Tamagawa. “The Grothendieck conjecture for affine curves”. In: *Compositio Mathematica* 109.2 (1997), pp. 135–194.

[Wei82] A. Weil. *Adeles and algebraic groups*. Vol. 23. Progress in Mathematics. With appendices by M. Demazure and Takashi Ono. Birkhäuser, Boston, Mass., 1982, pp. iii+126. ISBN: 3-7643-3092-9.