Super critical problems with concave and convex nonlinearities in $\mathbb{R}^N$

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Abstract

In this paper, by utilizing a newly established variational principle on convex sets, we provide an existence and multiplicity result for a class of semilinear elliptic problems defined on the whole $\mathbb{R}^N$ with nonlinearities involving sublinear and superlinear terms. We shall impose no growth restriction on the nonlinear term and consequently our problem can be super-critical by means of Sobolev spaces.

Keywords: Multiplicity, semilinear elliptic problem, super critical, concave-convex, unbounded domain.

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1. Introduction

In this paper, we aim to prove a multiplicity result for the class of super linear problems of the form,

\[
\begin{aligned}
-\Delta u + V(x)u &= f(u) + \lambda |u|^{q-2}u \quad x \in \mathbb{R}^N, \\
 u &\in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < \infty,
\end{aligned}
\]

where $N \geq 2$, $1 < q < 2$ and $\lambda > 0$ is a real parameter. We study the above problem for the following two cases:

(1) $N \geq 3$ and $f(u) = |u|^{p-2}u$ for $p > 2$.
(2) $N = 2$ with the following two assumptions on the function $f$,

\begin{itemize}
  \item [(f1)] $f : \mathbb{R} \to \mathbb{R}$ is an odd continuous function with $f(t) \geq 0$ for $t \geq 0$ and $f(0) = 0$;
  \item [(f2)] There exists $\nu > 1$ such that $\lim_{t \to 0} \frac{f(t)}{t^\nu} = 0$.
\end{itemize}

We shall also impose the following conditions on the potential $V(x)$,

\begin{itemize}
  \item [(V1)] The function $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $0 < V_0 \leq V(x)$ for all $x \in \mathbb{R}^N$;
  \item [(V2)] The function $1/V \in L^1(\mathbb{R}^N)$.
\end{itemize}

After the eminent work of Ambrosetti-Brezis-Cerami [2], the class of problems under consideration has been studied comprehensively in bounded domains, see [1, 3, 7, 8, 15, 20] and references therein. Using sub and super solution method, authors in [2] have proved the existence of two positive solutions with the nonlinearity $f_\lambda(x, u) = \lambda u^q + u^p$ satisfying $0 < q < 1 < p$. The gravity of results lies in the fact that there
was no control on \( p \) from above. Along with many results, when \( p \leq (N + 2)/(N - 2) \), the existence of infinitely many solution was also established for suitable choice of parameter \( \lambda \). Besides \([2]\), we refer interested readers to see \([4, 10]\) also for concave and convex problems with super critical growth in bounded domains for the existence of infinitely many solutions, where authors have adopted different techniques. In \([4]\) authors have applied a truncation argument while in \([10]\) authors have adopted a new abstract variational principle discussed in \([14]\) (see also \([13]\)).

In case of \( \mathbb{R}^N \), lesser has been explored for the elliptic problem involving concave and convex growth, see \([6, 12, 21]\) with no claim of citing all of them. To begin with, authors in \([6]\) have attempted to give existence results based on the method of successive approximations with no restrictions on the growth of super linear term. With a subcritical control over superlinear term, authors in \([12]\) have proved the existence of infinitely many nodal solutions for Schrödinger equation with concave-convex nonlinearity. A similar class of problem with sign changing weights has been studied in \([21]\) for the existence of multiple positive solutions, using the idea of Nehari manifold.

As far as super critical concave and convex problem on whole \( \mathbb{R}^N \) are concerned, we are only aware of the work in \([6]\) in which the existence of a single solution has been proved. In this paper we shall prove both existence and multiplicity. To be precise, we prove the following results related with the problem \((P_{\lambda})\).

**Theorem 1.1.** Assume that \( 1 < q < 2 < p \) and \( N \geq 3 \). Then there exists \( \Lambda_0 > 0 \) such that for each \( \lambda \in (0, \Lambda_0) \) problem \((P_{\lambda})\) has at least one positive solution with a negative energy.

The next theorem is about the multiplicity result for the super-critical case.

**Theorem 1.2.** Assume that \( 1 < q < 2 < p \) and \( N \geq 3 \). Then there exists \( \Lambda_0 > 0 \) such that for each \( \lambda \in (0, \Lambda_0) \) problem \((P_{\lambda})\) has infinitely many distinct nontrivial solutions with negative energy.

The above results can be considered as an extension of results in \([12]\) in critical and super critical case. Now we state the following result in reference to the two dimensional case.

**Theorem 1.3.** Assume that \( 1 < q < 2 \). Then there exists \( \Lambda_1 > 0 \) such that for each \( \lambda \in (0, \Lambda_1) \) problem \((P_{\lambda})\) has at least one positive solution with a negative energy.

**Remark 1.1.** We remark that the assumptions \((V1)\) and \((V2)\) do not imply that \( V(x) \) is coercive. For example, \( V(x_1, x_2, ..., x_N) = 1 + x_1^2|\sin(2\pi x_1)| + x_2^2 + x_3^2 + ... + x_N^2|^\alpha \) for \( \alpha > N \) satisfies \((V1) - (V2)\) but it is not coercive.

**Remark 1.2.** A typical example satisfying \((f1) - (f2)\) can be

\[
f(t) = t^{2\alpha + 1}\exp(\beta t^2), \quad \alpha \in \mathbb{N} \text{ such that } 2\alpha + 1 > \nu \text{ and } \beta \in \mathbb{R}.
\]

**Remark 1.3.** As one of the applications of the above problem, we remark that the concave-convex problems arise in the study of anisotropic continuous media. We refer readers to see the introduction in \([18]\) for several applications of this kind of problems.

To prove these results we follow an idea based on variational principles on convex sets. One difficulty while dealing with problems in unbounded domains is to choose a suitable convex set which has a required tolerance with the appropriate solution space so that one can apply the abstract result established recently in \([14]\).
The outline of the paper is as follows. In Section 2, we shall recall a new variational principle established in [13, 14] that paves a way to do critical point theory on convex sets and yet to obtain critical points with respect to the whole space. In section 3, we give some preliminary results required for our variational setup. In section 4, we prove the existence result in Theorem 1.1 while Section 5 is devoted to the existence of infinity many solutions and the proof of Theorem 1.2. Finally, we conclude this paper by dealing with the two dimensional case and the proof of Theorem 1.3 in Section 6.

2. A variational principle

Let $V$ be a reflexive Banach space, $V^*$ its topological dual and $K$ be a convex and weakly closed subset of $V$. Assume that $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semi-continuous function which is Gâteaux differentiable on $K$. The Gâteaux derivative of $\Psi$ at each point $u \in K$ will be denoted by $D\Psi(u)$. The restriction of $\Psi$ to $K$ is denoted by $\Psi_K$ and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases}$$ (1)

For a given functional $\Phi \in C^1(V, \mathbb{R})$ denoted by $\Phi'(u) \in V^*$ its derivative and consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u).$$

According to Szulkin [19], we have the following definition for critical points of $I_K$.

**Definition 2.1.** A point $u_0 \in V$ is said to be a critical point of $I_K$ if $I_K(u_0) \in \mathbb{R}$ and if it satisfies the following inequality

$$\langle \Phi'(u_0), u_0 - v \rangle + \Psi_K(v) - \Psi_K(u_0) \geq 0, \quad \forall v \in V,$$

where $\langle ., . \rangle$ is the duality pairing between $V$ and its dual $V^*$.

**Proposition 2.1.** Each local minimum of $I_K$ is necessarily a critical point of $I_K$.

**Proof.** Let $u$ be a local minimum of $I_K$. Using convexity of $\Psi_K$, it follows that for all small $t > 0$,

$$0 \leq I_K((1-t)u + tv) = \Phi((1-t)u + tv) - \Phi(u) + \Psi_K((1-t)u + tv) - \Psi_K(u) \leq \Phi(u + t(v-u)) - \Phi(u) + t(\Psi(v) - \Psi(u)).$$

Dividing by $t$ and letting $t \to 0^+$ we obtain (2). \square

We also recall the notion of point-wise invariance condition from [14].

**Definition 2.2.** We say that the triple $(\Psi, K, \Phi)$ satisfies the point-wise invariance condition at a point $u_0 \in V$ if there exists a convex Gâteaux differentiable function $G : V \to \mathbb{R}$ and a point $v_0 \in K$ such that

$$D\Psi(v_0) + DG(v_0) = \Phi'(u_0) + DG(u_0).$$
We shall now recall the following variational principle established recently in [14] (see also [13]).

**Theorem 2.1.** Let $V$ be a reflexive Banach space and $K$ be a convex and weakly closed subset of $V$. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is Gâteaux differentiable on $K$ and let $\Phi \in C^1(V, \mathbb{R})$. Assume that the following two assertions hold:

(i) The functional $I_K : V \to \mathbb{R} \cup \{+\infty\}$ defined by $I_K(u) := \Psi_K(u) - \Phi(u)$ has a critical point $u_0 \in V$ as in Definition 2.1 and;

(ii) the triple $(\Psi, K, \Phi)$ satisfies the point-wise invariance condition at the point $u_0$.

Then $u_0 \in K$ is a solution of the equation

$$D\Psi(u) = \Phi'(u). \tag{3}$$

We shall now adapt the latter theorem to our case. Consider the Banach space $\mathcal{V} = E \cap L^p(\mathbb{R}^N)$ equipped with the following norm

$$\|u\| := \|u\|_{E_V} + \|u\|_{L^p(\mathbb{R}^N)},$$

where

$$E_V = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < \infty \right\}$$

and

$$\|u\|_{E_V} = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right)^{\frac{1}{2}}.$$

Let $I : \mathcal{V} \to \mathbb{R}$ be the Euler-Lagrange functional related to (P), given as

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q \, dx.$$

Define the function $\Phi : \mathcal{V} \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q \, dx,$$

Note that $\Phi \in C^1(\mathcal{V}; \mathbb{R})$. Define $\Psi : \mathcal{V} \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx.$$

Let $K$ be a convex and weakly closed subset of $\mathcal{V}$. Then the restriction of $\Psi$ over $K$ is denoted by $\Psi_K$ and defined as

$$\Psi_K(u) = \begin{cases} 
\Psi(u), & u \in K, \\
+\infty, & u \not\in K.
\end{cases} \tag{4}$$
Finally, let us introduce the functional $I_K : \mathcal{V} \to (-\infty, +\infty]$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u).$$

(5)

For the convenience of the reader, we shall prove a simplified version of Theorem 2.1 suitable to our problem (P).

**Theorem 2.2.** Let $V = E_V \cap L^p(\mathbb{R}^N)$ as defined before, and let $K$ be a convex and weakly closed subset of $V$. If the following two assertions hold:

(i) The functional $I_K : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ defined in (5) has a critical point $\bar{v} \in \mathcal{V}$ as in Definition 2.1 and;

(ii) there exists $\bar{v} \in K$ such that $-\Delta v + V(x)v = D\Phi(\bar{v}) = \bar{v}\bar{v}^{p-2} + \lambda \bar{v}\bar{v}^{q-2}$ in the weak sense.

Then $\bar{v} \in K$ is a solution of the equation

$$-\Delta u + V(x)u = u|u|^{p-2} + \lambda u|u|^{q-2}.$$  

(6)

**Proof.** Since $\bar{v}$ is a critical point of $I_K(u) = \Psi_K(u) - \Phi(u)$, it follows from Definition 2.1 that

$$\Psi_K(v) - \Psi_K(\bar{v}) \geq \langle D\Phi(\bar{v}), v - \bar{v} \rangle, \quad \forall v \in V.$$  

(7)

where $\langle D\Phi(\bar{v}), v - \bar{v} \rangle = \int_{\mathbb{R}^N} D\Phi(\bar{v})(v - \bar{v}) \, dx$. Which leads to

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 - |\nabla \bar{v}|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)(|v|^2 - |\bar{v}|^2) \, dx \geq \langle D\Phi(\bar{v}), v - \bar{v} \rangle, \quad \forall v \in V.$$  

(8)

It follows from the second assumption in the theorem that there exists $\bar{v} \in K$ such that

$$\int_{\mathbb{R}^N} \nabla \bar{v} \cdot \nabla \eta \, dx + \int_{\mathbb{R}^N} V(x)\bar{v}\eta \, dx = \int_{\mathbb{R}^N} D\Phi(\bar{v})\eta \, dx, \quad \forall \eta \in \mathcal{V}.$$  

(9)

Now putting $\eta = \bar{v} - v$ in (8), we get

$$\int_{\mathbb{R}^N} \nabla \bar{v} \cdot \nabla (\bar{v} - v) \, dx + \int_{\mathbb{R}^N} V(x)(\bar{v} - v) \, dx = \int_{\mathbb{R}^N} D\Phi(\bar{v})(\bar{v} - v) \, dx.$$  

(10)

Now substituting $v = \bar{v}$ in (8) and using (10), we get

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \bar{v}|^2 - |\nabla v|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)(|\bar{v}|^2 - |v|^2) \, dx \geq \int_{\mathbb{R}^N} \nabla \bar{v} \cdot \nabla (\bar{v} - v) \, dx + \int_{\mathbb{R}^N} V(x)(\bar{v} - v) \, dx.$$  

(11)

On the other hand, by the convexity of $\Psi$, we get

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 - |\nabla \bar{v}|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)(|v|^2 - |\bar{v}|^2) \, dx \geq \int_{\mathbb{R}^N} \nabla \bar{v} \cdot \nabla (\bar{v} - v) \, dx + \int_{\mathbb{R}^N} V(x)(\bar{v} - v) \, dx.$$  

(12)

Combining (11) and (12), we get

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \bar{v}|^2 - |\nabla v|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)(|\bar{v}|^2 - |v|^2) \, dx = \int_{\mathbb{R}^N} \nabla \bar{v} \cdot \nabla (\bar{v} - v) \, dx + \int_{\mathbb{R}^N} V(x)(\bar{v} - v) \, dx.$$  

(13)
Indeed the last equation is equivalent to
\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v - \nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|\nabla v|^2 \, dx = 0.
\]
Since \( V(x) > 0 \), we get \( v = u \). Using this observation in (11), we get the required result. This completes the proof. \( \square \)

We remark that the condition \( ii \)) in Theorem 2.2 indeed shows that the triple \( (\Psi, K, \Phi) \) satisfies the point-wise invariance condition at \( u_0 \) given in Definition 2.2. In fact, it corresponds to the case where \( G = 0 \). This is why Theorem 2.2 is a very particular case of the general Theorem 2.1.

3. Preliminary results

In this section we prove some preliminary results required throughout the paper. We have the following result for the compact inclusion of the space \( E_V \) into suitable Lebesgue spaces.

**Lemma 3.1.** Under the assumption \((V1) - (V2)\) and \( N \geq 2 \) the embedding \( E_V \hookrightarrow L^\beta(\mathbb{R}^N) \) is compact for \( \beta \in [1, 2^*) \) where \( 2^* = 2N/(N-2) \) for \( N > 2 \) and \( 2^* = \infty \) for \( N = 2 \).

**Proof.** By \((V1)\) the embedding \( E_V \hookrightarrow H^1(\mathbb{R}^N) \) is continuous. Thus, \( E_V \hookrightarrow L^\beta(\mathbb{R}^N) \) is continuous for \( \beta \in [2, 2^*) \). Moreover, if \( u \in E_V \), we have
\[
\int_{\mathbb{R}^N} |u| \, dx \leq \left( \int_{\mathbb{R}^N} (V(x))^{-1} \, dx \right)^{\frac{1}{2}} \|u\|_{E_V}.
\]
Therefore, by interpolation \( E_V \hookrightarrow L^\beta(\mathbb{R}^N) \) is continuous for \( \beta \in [1, 2^*) \). Now, let \( \{u_n\} \) be a bounded sequence in \( E_V \), i.e. \( \|u_n\|_{E_V} \leq C \) for some \( C > 0 \). Hence, up to a subsequence, \( u_n \rightharpoonup u_0 \) weakly in \( E_V \). Given \( \epsilon > 0 \), we consider \( R > 0 \) such that
\[
\int_{|x|>R} (V(x))^{-1} \, dx \leq \left[ \frac{\epsilon}{2(C + \|u_0\|_{E_V})} \right]^2.
\]
This implies that
\[
\int_{|x|>R} |u_n - u_0| \, dx \leq \left( \int_{|x|>R} (V(x))^{-1} \, dx \right)^{\frac{1}{2}} \|u_n - u_0\|_{E_V}
\]
and since \( E_V \hookrightarrow L^1(B_R) \) is compact, it follows that there exists \( n_0 \) such that for all \( n > n_0 \),
\[
\int_{B_R} |u_n - u_0| \, dx \leq \frac{\epsilon}{2}.
\]
Thus, \( u_n \to u_0 \) in \( L^1(\mathbb{R}^N) \). Next, if \( \beta \in [1, 2^*) \) then choose \( \beta < \beta_0 < 2^* \) and use interpolation inequality, for some \( 0 < \alpha \leq 1 \) to get
\[
\|u_n - u_0\|_{L^\beta(\mathbb{R}^N)} \leq \|u_n - u_0\|_{L^1(\mathbb{R}^N)}^{\beta_0} \|u_n - u_0\|_{L^\beta_0(\mathbb{R}^N)}^{1-\alpha} \to 0
\]
and the proof is complete. \( \square \)
Lemma 3.2. Let \((V1) - (V2)\) be satisfied and let \(g \in L^\infty(\mathbb{R}^N)\) for \(N \geq 2\). If \(u \in E_V\) is a weak solution of the problem,

\[-\Delta u + V(x)u = g(x),\]  \hspace{1cm} (13)

then \(V_0\|u\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}\).

Proof. Take \(h \in C_c^\infty(\mathbb{R}^N)\) and assume that \(v \in E_V\) is a weak solution of \(-\Delta v + V(x)v = h(x)\). We show that \(V_0\|v\|_{L^1(\mathbb{R}^N)} \leq \|h\|_{L^1(\mathbb{R}^N)}\). Let \(\eta \in C^1(\mathbb{R}, \mathbb{R})\) be such that \(\eta(0) = 0, \eta' \geq 0, |\eta| \leq 1\), and \(\eta' \in L^\infty(\mathbb{R})\). It can be easily deduced that \(\eta(v) \in E_V\). Since \(v\) is a weak solution of \(-\Delta v + V(x)v = h(x)\), it follows that

\[
\int_{\mathbb{R}^N} \eta'(v)|\nabla v|^2 \, dx + \int_{\mathbb{R}^N} V(x)v\eta(v) \, dx = \int_{\mathbb{R}^N} h(x)\eta(v) \, dx
\]

and therefore,

\[
V_0 \int_{\mathbb{R}^N} v\eta(v) \, dx \leq \int_{\mathbb{R}^N} h(x)\eta(v) \, dx \leq \|h\|_{L^1(\mathbb{R}^N)}.
\]  \hspace{1cm} (14)

Given \(\epsilon > 0\), let \(\eta(v) = v/\sqrt{\epsilon + v^2}\). It follows from (14) that

\[
V_0 \int_{\mathbb{R}^N} \frac{v^2}{\sqrt{\epsilon + v^2}} \, dx \leq \|h\|_{L^1(\mathbb{R}^N)}\]

Letting \(\epsilon \to 0^+\) and applying Fatou’s Lemma imply that \(V_0\|v\|_{L^1(\mathbb{R}^N)} \leq \|h\|_{L^1(\mathbb{R}^N)}\).

On the other hand \(u \in E_V\) is a weak solution of (13). Thus,

\[
\int_{\mathbb{R}^N} uh \, dx = \int_{\mathbb{R}^N} \nabla v \cdot \nabla u \, dx + V(x)uv \, dx = \int_{\mathbb{R}^N} gv \, dx.
\]

Therefore,

\[
\left| \int_{\mathbb{R}^N} uh \, dx \right| \leq \|v\|_{L^1(\mathbb{R}^N)}\|g\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{V_0}\|h\|_{L^1(\mathbb{R}^N)}\|g\|_{L^\infty(\mathbb{R}^N)}.
\]

Since \(h \in C_c^\infty(\mathbb{R}^N)\) is arbitrary, we obtain that \(V_0\|u\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}\). \(\square\)

Recall that \(\mathcal{V} = E_V \cap L^p(\mathbb{R}^N)\). To prove Theorem 1.1 keeping in mind the continuous inclusion of Lemma 3.1 we have the following construction of the closed set \(K \subset E_V\),

\[
K = K(r) := \{ u \in \mathcal{V} \cap L^\infty(\mathbb{R}^N) : \|u\|_{L^\infty(\mathbb{R}^N)} \leq r \}, \hspace{1cm} (15)
\]

for some \(r > 0\) to be determined later.

In the next Lemma we show that the set \(K\) is weakly closed.

Lemma 3.3. Let \(r > 0\) be fixed then the set \(K(r)\) defined in (15) is weakly closed in \(\mathcal{V}\).

Proof. Take a sequence \(\{u_n\}\) in \(K(r)\) such that \(u_n \rightharpoonup u\) weakly in \(\mathcal{V}\). Since \(\mathcal{V}\) is reflexive, \(u \in \mathcal{V}\). Now it remains to show that \(u \in L^\infty(\mathbb{R}^N)\) and \(\|u\|_{L^\infty(\mathbb{R}^N)} \leq r\). Since \(u_n \rightharpoonup u\) in \(\mathcal{V}\), it converges point wise up to a subsequence, i.e., \(u_n(x) \to u(x)\) almost everywhere in \(\mathbb{R}^N\). This implies that \(|u(x)| = \lim_{n \to \infty} |u_n(x)| \leq r\) for a.e. \(x \in \mathbb{R}^N\). Thus, \(\|u\|_{L^\infty(\mathbb{R}^N)} \leq r\). \(\square\)
4. Proof of Theorem 1.1

We begin with the following elementary result which can be deduced by a straightforward computation.

**Lemma 4.1.** Let $1 < q < 2 < p$ and $V_0 > 0$ is same as defined in (f1). Then there exists $\Lambda_0 > 0$ with the following properties.

1. For each $\lambda \in (0, \Lambda_0)$, there exist positive numbers $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that $r \in [r_1, r_2]$ if and only if $r^{p-1} + \lambda r^{q-1} \leq V_0 r$.
2. For $\lambda = \Lambda_0$, there exists one and only one $r > 0$ such that $r^{p-1} + \lambda r^{q-1} = V_0 r$.
3. For $\lambda > \Lambda_0$, there is no $r > 0$ such that $r^{p-1} + \lambda r^{q-1} = V_0 r$.

The following Lemma is helpful in verifying the condition (ii) in Theorem 2.2.

**Lemma 4.2.** Assume that $1 < q < 2 < p$. Then

$$\|D\Phi(u)\|_{L^\infty(\mathbb{R}^N)} \leq r^{p-1} + \mu r^{q-1}, \quad \forall u \in K(r).$$

**Proof.** By definition of $D\Phi(u)$ we have

$$\|D\Phi(u)\|_{L^\infty(\mathbb{R}^N)} = \|u|u|^{p-2} + \lambda u|u|^{q-2}\|_{L^\infty(\mathbb{R}^N)}$$

$$\leq \|u|u|^{p-2}\|_{L^\infty(\mathbb{R}^N)} + \lambda \|u|u|^{q-2}\|_{L^\infty(\mathbb{R}^N)}.$$

Therefore,

$$\|D\Phi(u)\|_{L^\infty(\mathbb{R}^N)} \leq \|u|^{p-1}\|_{L^\infty(\mathbb{R}^N)} + \lambda \|u|^{q-1}\|_{L^\infty(\mathbb{R}^N)}.$$ 

It follows from $u \in K(r)$ that

$$\|D\Phi(u)\|_{L^\infty(\mathbb{R}^N)} \leq r^{p-1} + \lambda r^{q-1},$$

as desired. \qed

We are now in the position to state the following result addressing condition (ii) in Theorem 2.2.

**Lemma 4.3.** Let $1 < q < 2 < p$. Assume that $\Lambda_0 > 0$ is given in Lemma 4.1 and $\lambda \in (0, \Lambda_0)$. Let $r_1, r_2$ be given in part (1) of Lemma 4.1. Then for each $r \in [r_1, r_2]$ and each $\mathbf{u} \in K(r)$ there exists $v \in K(r)$ such that

$$-\Delta v + V(x)v = \frac{\mu}{|\mathbf{u}|^{p-2}} + \lambda \frac{\mu}{|\mathbf{u}|^{q-2}}.$$ 

**Proof.** Let $g(x) = \frac{\mu}{|\mathbf{u}|^{p-2}} + \lambda \frac{\mu}{|\mathbf{u}|^{q-2}}$. Since $\mathbf{u} \in K(r)$, it follows that $g \in L^\infty(\mathbb{R}^N)$. Since the embedding $E_V \hookrightarrow L^1(\mathbb{R}^N)$ is compact, the functional

$$Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx - \int_{\mathbb{R}^N} g(x)u \, dx,$$
is well-defined on $E_V$ and admits its minimum at some $v \in E_V$ which indeed satisfies
\[
-\Delta v + V(x)v = g(x) = D\Phi(\eta),
\]
(16)
in a weak sense. Since the right hand side is an element in $L^\infty(\mathbb{R}^N)$, it follows from Lemma 3.2 that $V_0\|v\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}$. This together with Lemma 4.2 yield that
\[
V_0\|v\|_{L^\infty(\mathbb{R}^N)} \leq r^{p-1} + \lambda r^{q-1}.
\]
By Lemma 4.1 for each $r \in [r_1, r_2]$ we have that $r^{p-1} + \lambda r^{q-1} \leq V_0 r$. Therefore,
\[
\|v\|_{L^\infty(\mathbb{R}^N)} \leq r,
\]
as desired. \hfill \Box

**Proof of Theorem 1.1.** Let $\Lambda_0$ be as in Lemma 4.3 and $\lambda \in (0, \Lambda_0)$. Also, let $r_1$ and $r_2$ be as in Lemma 4.3 and define
\[
K := \{u \in K(r_2); \ u(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^N\}.
\]
Now we proceed with the proof in the following steps.

**Step 1.** We show that there exists $\eta \in E_V$ such that $\mathcal{I}_K(\eta) = \inf_{u \in E_V} \mathcal{I}_K(u)$. Then by Proposition 2.1 we conclude that $\eta$ is a critical point of $\mathcal{I}_K$.

Set $\eta := \inf_{u \in E_V} \mathcal{I}_K(u)$. So by definition of $\Psi_K$ for every $u \notin K$, we have $\mathcal{I}_K(u) = +\infty$ and therefore $\eta = \inf_{u \in K} \mathcal{I}_K(u)$. It follows that for every $u \in K$
\[
\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q \, dx \\
\leq \frac{r_2^{p-1}}{p} \int_{\mathbb{R}^N} |u| \, dx + \frac{\lambda r_2^{q-1}}{q} \int_{\mathbb{R}^N} |u| \, dx \\
\leq c_1 \|u\|_{E_V},
\]
where we have used the embedding $E_V \hookrightarrow L^1(\mathbb{R}^N)$ due to Lemma 3.1. Thus, for $u \in K$ we have that
\[
\mathcal{I}_K(u) := \Psi_K(u) - \Phi(u) \geq \frac{1}{2} \|u\|_{E_V}^2 - c_1 \|u\|_{E_V},
\]
(17)
from which we obtain that $\eta > -\infty$. Now, suppose that $\{u_n\}$ is a sequence in $K$ such that $\mathcal{I}_K(u_n) \to \eta$. It follows from (17) and the definition of set $K$ that the sequence $\{u_n\}$ is bounded in $E_V \cap L^\infty(\mathbb{R}^N)$. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\bar{u} \in E_V$ such that $u_n \rightharpoonup \bar{u}$ weakly in $E_V$. Moreover $u_n(x) \to \bar{u}(x)$ in $\mathbb{R}^N$ pointwise almost everywhere which implies $\bar{u} \in L^\infty(\mathbb{R}^N)$ with $\|\bar{u}\|_{L^\infty(\mathbb{R}^N)} \leq r_2$. As a consequence $\bar{u} \in K$. We now show that $\Phi(u_n) \to \Phi(\bar{u})$. Indeed, we have that
\[
\frac{1}{p} |u_n|^p + \frac{\lambda}{q} |u_n|^q \leq \frac{r_2^{p-1}}{p} |u_n| + \frac{\lambda r_2^{q-1}}{q} |u_n|.
\]
Therefore, by the strong convergence $u_n \to \bar{u}$ in $L^1(\mathbb{R}^N)$ because of the compact embedding $E_V \hookrightarrow L^1(\mathbb{R}^N)$ as in Lemma 5.1 and the dominated convergence theorem we obtain that $\Phi(u_n) \to \Phi(\bar{u})$. 


Therefore, \( I_K(\varpi) \leq \liminf_{n \to \infty} I_K(u_n) \). So, \( I_K(\varpi) = \eta = \inf_{u \in \mathcal{V}} I_K(u) \), and the proof of Step 1 is complete.

Step 2. In this step we show that there exists \( v \in K \) such that \(-\Delta v + V(x)v = \varpi|\varpi|^{p-2} + \lambda|\varpi|^q - 2\). By Lemma [4.3] together with the fact that \( \varpi \in K(r_2) \) we obtain that \( v \in K(r_2) \). To show that \( v \in K \), we shall need to verify that \( v \) is non-negative almost everywhere. But, this is a simple consequence of the maximum principle and the fact that \(-\Delta v + V(x)v = \varpi|\varpi|^{p-2} + \lambda|\varpi|^q - 2 \geq 0\).

It now follows from Theorem [2.2] together with Step 1 and Step 2 that \( \varpi \) is a solution of \((P_{\lambda})\). To complete the proof we shall show that \( \varpi \) is non-trivial by proving that \( I_K(\varpi) = \inf_{u \in K} I_K(u) < 0 \). Take \( e \in K \). For \( i \in [0,1] \), we have that \( te \in K \) and therefore

\[
I_K(te) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla te|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|te|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |te|^p \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |te|^q \, dx
\]

\[
= t^q \left( \frac{t^2 - q}{2} \int_{\mathbb{R}^N} |\nabla e|^2 \, dx - \frac{t^{2-q}}{2} \int_{\mathbb{R}^N} V(x)|e|^2 \, dx - \frac{p^{1-q}}{p} \int_{\mathbb{R}^N} |e|^p \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |e|^q \, dx \right).
\]

Since \( 1 < q < 2 < p \), \( I_K(te) \) is negative for \( t \) sufficiently small. Thus, we can conclude that \( I_K(\varpi) < 0 \). Thus, \( \varpi \) is a non-trivial and non-negative solution of \((P_{\lambda})\). Finally, it follows from the strong maximum principle that \( \varpi > 0 \).

\[
\square
\]

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. According to [19], say that \( I_K \) satisfies the compactness condition of Palais-Smale type provided,

(PS): If \( \{u_n\} \) is a sequence such that \( I_K(u_n) \to c \in \mathbb{R} \) and

\[
<D\Phi(u_n), u_n - v> + \Psi_K(v) - \Psi_K(u_n) \geq -\epsilon_n \|v - u_n\| \quad \forall v \in \mathcal{V},
\]

where \( \epsilon_n \to 0 \), then \( \{u_n\} \) possesses a convergent subsequence.

We recall an important result about critical points of even functions of the type \( I_K \). We shall begin with some preliminaries. Let \( \Sigma \) be the of all symmetric subsets of \( \mathcal{V} \backslash \{0\} \) which are closed in \( \mathcal{V} \). A nonempty set \( A \subseteq \Sigma \) is said to have genus \( k \) (denoted \( \gamma(A) = k \)) if \( k \) is the smallest integer with the property that there exists an odd continuous mapping \( h : A \to \mathbb{R}^k \backslash \{0\} \). If such an integer does not exist, \( \gamma(A) = \infty \). For the empty set \( \emptyset \) we define \( \gamma(\emptyset) = 0 \).

**Proposition 5.1.** Let \( A \subseteq \Sigma \). If \( A \) is a homeomorphic to \( S^{k-1} \) by an odd homeomorphism, then \( \gamma(A) = k \).

Proof and a more detailed discussion of the notion of genus can be found in [16] and [17]. Let \( \Theta \) be the collection of all nonempty closed and bounded subsets of \( \mathcal{V} \). In \( \Theta \) we introduce the Hausdorff metric distance ([11], §15, VII), given by

\[
\text{dist}(A,B) = \max \left\{ \sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A) \right\}.
\]

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The space \((\Theta, \text{dist})\) is complete \((\Theta, \text{dist})\). Denote by \(\Gamma\) the sub-collection of \(\Theta\) consisting of all nonempty compact symmetric subsets of \(\mathcal{V}\) and let

\[
\Gamma_j = \text{cl}\{A \in \Gamma : 0 \notin A, \gamma(A) \geq j\}
\]

(\(\text{cl}\) is the closure in \(\Gamma\)). It is easy to verify that \(\Gamma\) is closed in \(\Theta\), so \((\Gamma, \text{dist})\) and \((\Gamma_j, \text{dist})\) are complete metric spaces. The following Theorem is proved in \cite{19}.

**Theorem 5.1.** Let \(\Psi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}\) be a convex, lower semi-continuous function, and let \(\Phi \in C^1(\mathcal{V}, \mathbb{R})\) and define \(I = \Psi - \Phi\). Suppose that \(I : \mathcal{V} \to (-\infty, +\infty]\) satisfies \((PS)\), \(I(0) = 0\) and \(\Phi, \Psi\) are even. Define

\[
c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} I(u).
\]

If \(-\infty < c_j < 0\) for \(j = 1, \ldots, k\), then \(I\) has at least \(k\) distinct pairs of nontrivial critical points by means of Definition 2.1.

Now, to prove Theorem 1.2 we have the following construction of the closed set \(K\).

\[
K := \{u \in \mathcal{V} \cap L^\infty(\mathbb{R}^N) : \|u\|_{L^\infty(\mathbb{R}^N)} \leq r_2\},
\]

where \(r_2\) is given in Lemma 4.2. Before proving Theorem 5.1 for our problem, we use the following result.

**Lemma 5.1.** Assume the potential \(V\) satisfies (V1) – (V2). Then the Schrödinger operator, \(-\Delta + V\), is self-adjoint.

**Proof of Theorem 1.2.** Let \(\Lambda_0\) be as in Lemma 4.2 and \(\lambda \in (0, \Lambda_0)\). We first show that the functional \(I_K\) has infinitely many distinct critical points by verifying Theorem 5.1 in our set-up. It is obvious that the function \(\Phi\) is even and continuously differentiable. Also \(\Psi_K\) is a proper, convex and lower semi-continuous even function. We now verify \((PS)\). Let \(\{u_n\}\) be a Palais-Smale sequence for \(I_K\) in \(\mathcal{V}\) such that \(I_K(u_n) \to c\) for some \(c \in \mathbb{R}\) and

\[
\langle D\Phi(u_n), u_n - v \rangle + \Psi_K(v) - \Psi_K(u_n) \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in \mathcal{V},
\]

where \(\epsilon_n \to 0\). Since \(c \in \mathbb{R}\), \(\{u_n\}\) is bounded in \(L^\infty(\mathbb{R}^N)\) \((u_n\) must belong to \(K\) otherwise \(\Psi_K(u_n) = \infty\) which contradicts that \(c \in \mathbb{R}\)). Moreover, it is easy to conclude that \(\{u_n\}\) is bounded in \(E_V\). Now using \(I_K(u_n) \to c\), \(\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq r_2\) and the compact embedding \(E_V \hookrightarrow L^q(\mathbb{R}^N)\), it follows that \(\{u_n\}\) is bounded in \(L^p(\mathbb{R}^N)\) as well. Now since \(\mathcal{V}\) is reflexive, there exists \(\overline{u} \in \mathcal{V}\) such that \(u_n \rightharpoonup \overline{u}\) in \(\mathcal{V}\). Moreover following the previous argument as in Theorem 1.1 \(\overline{u} \in L^\infty(\mathbb{R}^N)\) with \(\|\overline{u}\|_{L^\infty(\mathbb{R}^N)} \leq r_2\). Consequently, it implies that \(\overline{u} \in K\). Now we prove that \(u_n \rightharpoonup \overline{u}\) strongly in \(\mathcal{V}\). Following the similar idea as in proof of Theorem 1.1 combined with Lebesgue dominated convergence theorem and compact embedding of \(E_V \to L^1(\mathbb{R}^N)\), we obtain that

\[
\Phi(u_n) \to \Phi(\overline{u}), \text{ and } \langle D\Phi(u_n), u_n - v \rangle \to 0, \quad \forall v \in \mathcal{V}.
\]

As a result, invoking Brezis-Lieb Lemma, \(u_n \to \overline{u}\) in \(L^p(\mathbb{R}^N)\). Moreover by \(21\)

\[
\Psi(u_n) - \Psi(\overline{u}) - \langle D\Phi(u_n), u_n - u \rangle \leq \epsilon_n \|\overline{u} - u_n\|_V,
\]

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and by the boundedness of \( \{u_n - \pi\} \) in \( \mathcal{V} \) we obtain that

\[
\limsup_{n \to +\infty} (\Psi(u_n) - \Psi(\pi) - \langle D\Phi(u_n), u_n - \pi \rangle) \leq 0. \tag{23}
\]

It now follows from (22) and (23) that

\[
\limsup_{n \to +\infty} \Psi(u_n) \leq \Psi(\pi). \tag{24}
\]

On the other hand by the weak lower semi-continuity of \( \Psi \) we have that

\[
\liminf_{n \to +\infty} \Psi(u_n) \geq \Psi(\pi), \tag{25}
\]

from which together with (24) one has that \( u_n \to \pi \) in \( E_V \). This together with the fact that \( u_n \to \pi \) strongly in \( L^p(\mathbb{R}^N) \) imply that \( u_n \to \pi \) strongly in \( \mathcal{V} \).

For each \( k \in \mathbb{N} \), considering the definition of \( \Gamma_k \) in (19), we define

\[
c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I(u).
\]

We shall now prove that \( -\infty < c_k < 0 \) for all \( k \in \mathbb{N} \). From Lemma 5.1 and the compactness of the embedding, the spectrum of the Schrödinger operator \( -\Delta + V \) on \( L^2(\mathbb{R}^N) \) is discrete and consists of eigenvalues of finite multiplicity, \( 0 < \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \), and \( \mu_k \to \infty \) as \( k \to \infty \). To this, let us denote by \( \mu_j \) the \( j \)-th eigenvalue of \( -\Delta + V(x) \) (counted according to its multiplicity) and by \( e_j \) a corresponding eigenfunction satisfying

\[
\int_{\mathbb{R}^N} \nabla e_i \cdot \nabla e_j \, dx + \int_{\mathbb{R}^N} V(x) e_i e_j \, dx = \delta_{ij}.
\]

As in the proof of Theorem 1.1 we have that \( I_K \) is bounded below. Thus \( c_k > -\infty \) for each \( k \in \mathbb{N} \). Let

\[
A := \left\{ u = \alpha_1 e_1 + \ldots + \alpha_k e_k : \| u \|_{E_V}^2 = \sum_{i=1}^k \alpha_i^2 = \rho^2 \right\},
\]

for small \( \rho > 0 \) to be determined. Then \( A \in \Gamma_k \) because \( \gamma(A) = k \) by Proposition 5.1. Since \( A \) is finite dimensional, all norms are equivalent on \( A \). Thus, for any \( u \in A \), \( \| u \|_{L^\infty(\mathbb{R}^N)} \leq C \| u \|_{E_V} = C \rho \leq r_2 \), for sufficiently small \( \rho > 0 \). Hence \( A \subseteq K \) for suitable choice of \( \rho \). Also there exist positive constants \( c_1, c_2 \) such that \( \| u \|_{L^p(\mathbb{R}^N)} \geq c_1 \| u \|_{E_V} \) and \( \| u \|_{L^q(\mathbb{R}^N)} \geq c_2 \| u \|_{E_V} \) for all \( u \in A \). Therefore,

\[
I_K(u) = \frac{1}{2} \| u \|_{E_V}^2 - \frac{1}{p} \| u \|_{L^p(\mathbb{R}^N)}^p - \frac{\lambda}{q} \| u \|_{L^q(\mathbb{R}^N)}^q \\
\leq \frac{1}{2} \rho^2 - \frac{1}{p} \rho^p - \frac{\lambda}{q} \rho^q \leq \rho^q \left( \frac{1}{2} \rho^2 - \frac{1}{p} \rho^p - \frac{\lambda}{q} \rho^q \right) = \rho^q \left( \frac{1}{2} \rho^2 - \frac{1}{p} \rho^p - \frac{\lambda}{q} \rho^q \right).
\]

Now we can choose \( \rho \) small enough such that \( I_K(u) \leq \rho^q \left( \frac{1}{2} \rho^2 - \frac{1}{p} \rho^p - \frac{\lambda}{q} \rho^q \right) < 0 \) for every \( u \in A \). It then follows that \( c_k < 0 \). Thus, by Theorem 5.1 \( I_K \) has a sequence of distinct critical points \( \{u_k\}_{k \in \mathbb{N}} \) by means of Definition 2.1. Also, by Lemma 2.2 for each critical point \( u_k \) of \( I_K \) there exists \( v_k \in K \) such that

\[
-\Delta v_k + V(x) v_k = D\Phi(u_k). \]

It now follows from Theorem 2.2 that \( \{u_k\} \) is a sequence of distinct solutions of
such that $I_K(u_k) < 0$ for each $k \in \mathbb{N}$. This completes the proof. \hfill \Box

6. Two dimensional case

In order to study $(P_\lambda)$ for $N = 2$, we adopt the truncation idea as follows. Define

$$K(r) = \{u \in E_V \cap L^\infty(\mathbb{R}^2) : \|u\|_{L^\infty(\mathbb{R}^2)} \leq r\}$$

and $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \begin{cases} f(t), & |t| \leq r, \\ \frac{f(t)}{|t|^\nu}, & |t| \geq r. \end{cases}$$

Thus, $g(u) = f(u)$ for $u \in K(r)$.

Therefore, our aim is to find a solution of the following truncated problem

$$\begin{cases} -\Delta u + V(x)u = g(u) + \lambda |u|^{q-2}u & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), & \int_{\mathbb{R}^2} V(x)|u|^2 \, dx < \infty, \end{cases} \quad (T_\lambda)$$

in $K(r)$ for some suitable choice of $r > 0$. We shall apply Theorem 2.1 in the following variational set-up. Let $J : E_V \to \mathbb{R}$ be the Euler-Lagrange functional related to $(T_\lambda)$, given as

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)|u|^2 \, dx - \int_{\mathbb{R}^2} G(u) \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^2} |u|^q \, dx,$$

where $G(t) = \int_0^t g(s) \, ds$ is the primitive of $g(t)$. Now, define the function $\Upsilon : E_V \to \mathbb{R}$ by

$$\Upsilon(u) = \int_{\mathbb{R}^2} G(u) \, dx + \frac{\lambda}{q} \int_{\mathbb{R}^2} |u|^q \, dx.$$

Note that $\Upsilon \in C^1(E_V; \mathbb{R})$. Finally, let us introduce the functional $J_K : E_V \to (-\infty, +\infty]$ defined by

$$J_K(u) := \Psi_K(u) - \Upsilon(u), \quad (26)$$

where $\Psi_K$ is defined as in (4) for

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)|u|^2 \, dx.$$

In the process to verify Theorem 2.1, we need similar Lemmas as in the section 2. Note that the assumption $(f_2)$ implies that for every $\epsilon_0 > 0$, there exists $\delta_0 > 0$ such that $|f(t)| \leq \epsilon_0 |t|^\nu$, whenever $|t| < \delta_0$. In particular, for fixed $\epsilon_0 = 1$ there exists $\delta_1 > 0$ such that

$$|f(t)| \leq |t|^\nu, \text{ whenever } |t| < \delta_1. \quad (27)$$

We will fix this $\delta_1$ to avoid any confusion.
Lemma 6.1. Assume that $1 < q < 2$ and $\delta_1 > 0$ as defined in $[27]$. Then for all $u \in K(r)$ with $0 < r < \delta_1$, we have

$$\|DT(u)\|_{L^\infty(\mathbb{R}^2)} \leq r^\nu + \lambda r^{q-1}. $$

Proof. By definition of $DT(u)$ we have

$$\|DT(u)\|_{L^\infty(\mathbb{R}^2)} = \|g(u) + \lambda u|u|^{q-2}\|_{L^\infty(\mathbb{R}^2)} \leq \|g(u)\|_{L^\infty(\mathbb{R}^2)} + \lambda \|u\|_{L^\infty(\mathbb{R}^2)}^{q-1}. $$

Now, using $[27]$ and choosing $r \leq \delta_1$, we get

$$\|DT(u)\|_{L^\infty(\mathbb{R}^2)} \leq \|u\|_{L^\infty(\mathbb{R}^2)}^{\nu} + \lambda \|u\|_{L^\infty(\mathbb{R}^2)}^{q-1} \leq r^\nu + \lambda r^{q-1}$$

as desired. \qed

We are now in the position to state the following result addressing condition $(ii)$ in Theorem $2.1$ by following the similar idea as in Lemma $6.3$ combined with Lemma $4.2$.

Lemma 6.2. Let $1 < q < 2 < p$. Choose $\Lambda_1 > 0$ in such a way that for each $\lambda \in (0, \Lambda_1)$ there exist positive numbers $r_1, r_2$ with $r_1 < r_2 < \delta_1$ such that $r \in [r_1, r_2]$ if and only if $r^\nu + \lambda r^{q-1} \leq V_0 r$. Then for each $r \in [r_1, r_2]$ and each $\overline{u} \in K(r)$ there exists $v \in K(r)$ such that

$$-\Delta v + V(x)v = g(\overline{u}) + \lambda |\overline{u}|^{q-2}. $$

Proof of Theorem 1.3. Let $\Lambda_1$ be as in Lemma 6.2 and $\lambda \in (0, \Lambda_1)$. Also, let $r_1$ and $r_2$ be as in Lemma 6.2 and define

$$K := \{u \in K(r_2); \ u(x) \geq 0 \ a.e. \ x \in \mathbb{R}^2 \}. $$

Now we continue the proof in the following few steps.

Step 1. We show that there exists $\overline{u} \in K$ such that $\overline{J}_K(\overline{u}) := \inf_{u \in K} J_K(u)$. Then by Proposition 2.1 we conclude that $\overline{u}$ is a critical point of $J_K$.

Set $\sigma := \inf_{u \in E_V} J_K(u)$. So by definition of $\Psi$ for every $u \notin K$, we have $J_K(u) = +\infty$ and therefore $\sigma = \inf_{u \in K} J_K(u)$. For each $u \in K$, it follows from the compact embedding $E_V \hookrightarrow L^1(\mathbb{R}^2)$ as in Lemma 3.1 that

$$\int_{\mathbb{R}^2} G(u) \ dx = \int_{|u| \leq r} F(u) \ dx \leq C \int_{|u| \leq r} |u|^{q+1} \ dx \leq C \|u\|_{L^\infty(\mathbb{R}^2)}^{q+1} \int_{|u| \leq r} |u| \ dx \leq C_1 \|u\|_{E_V}.$$ 

Therefore,

$$\Upsilon(u) \leq C_1 \|u\|_{E_V} + C_2 \|u\|_{E_V}^q.$$ 

Here we have used the embedding $E_V \hookrightarrow L^q(\mathbb{R}^2)$ due to Lemma 3.1. Thus, for $u \in K$ we have that

$$\overline{J}_K(u) := \Psi_K(u) - \Upsilon(u) \geq \frac{1}{2} \|u\|_{E_V}^2 - C_1 \|u\|_{E_V} - C_2 \|u\|_{E_V}^q,$$ 

(28)

from which we obtain that $\sigma > -\infty$. Now, suppose that $\{u_n\}$ is a minimizing sequence in $K$ such that $\overline{J}_K(u_n) \to \sigma$. It follows from (28) and the definition of the set $K$ that the sequence $\{u_n\}$ is bounded in
for the case $N = 2$. Now, using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\underline{\nu} \in E_V$ such that $u_n \rightharpoonup \underline{\nu}$ weakly in $E_V$. Moreover $u_n(x) \to \underline{\nu}(x)$ in $\mathbb{R}^2$ pointwise almost everywhere which implies $\underline{\nu} \in L^\infty(\mathbb{R}^2)$ with $\|\underline{\nu}\|_{L^\infty(\mathbb{R}^2)} \leq r_2$. As a consequence $\underline{\nu} \in K$. We now show that $\Upsilon(u_n) \to \Upsilon(\underline{\nu})$. Indeed, using $u_n \in K$, we have that

$$G(u_n) = \int_0^{u_n} g(t) dt \leq C|u_n|^{\nu+1}.$$ 

Therefore, by the strong convergence $u_n \to \underline{\nu}$ in $L^\beta(\mathbb{R}^2)$ for all $\beta \in [1, \infty)$ as in Lemma B.1 and the dominated convergence theorem we obtain that $\Upsilon(u_n) \to \Upsilon(\underline{\nu})$.

Therefore, $J_K(\underline{\nu}) \leq \liminf_{n \to \infty} J_K(u_n)$. Thus, $J_K(\underline{\nu}) = \sigma = \inf_{u \in E_V} J_K(u)$ and the proof of Step 1 is complete.

**Step 2.** In this step we show that there exists $v \in K$ such that

$$-\Delta v + V(x)v = g(\underline{\nu}) + \lambda \underline{\nu} \underline{\nu}|^{q-2}.$$ 

By Lemma [2.2] together with the fact that $\underline{\nu} \in K(\mathbb{R}^2)$ we obtain that $v \in K(\mathbb{R}^2)$. Also, by the maximum principle and the fact that

$$-\Delta v + V(x)v = g(u) + \lambda \underline{\nu} \underline{\nu}|^{q-2} \geq 0,$$

we obtain that $v \geq 0$. It now follows from Theorem 2.2 together with Step 1 and Step 2 that $\underline{\nu}$ is a solution of the (I$_{\lambda}$). To complete the proof we shall show that $\underline{\nu}$ is non-trivial by proving that $J_K(\underline{\nu}) = \inf_{u \in K} J_K(u) < 0$. Take $0 \leq e \in K$. For $t \in [0,1]$, we have that $te \in K$ and therefore

$$J_K(te) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla te|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \bar{V}(x)|te|^2 dx - \int_{\mathbb{R}^2} G(te) dx - \frac{\lambda}{q} \int_{\mathbb{R}^2} |te|^q dx$$

$$\leq t^q \left( \frac{t^2 - q}{2} \int_{\mathbb{R}^2} |\nabla e|^2 dx + \frac{t^2 - q}{2} \int_{\mathbb{R}^2} \bar{V}(x)|e|^2 dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |e|^q dx \right).$$

Since $1 < q < 2$, $J_K(te)$ is negative for $t$ sufficiently small. Thus, we can conclude that $J_K(\underline{\nu}) < 0$. Thus, $\underline{\nu}$ is a non-trivial and non-negative solution of (I$_{\lambda}$). Moreover, it follows from the strong maximum principle that $\underline{\nu} > 0$. Finally, using the fact that $\underline{\nu} \in K$ implies $\underline{\nu}$ is a positive solution of (I$_{\lambda}$). \qed

We would like to remark that by using the same argument as in the proof of Theorem 1.2, one can also prove multiplicity for the case $N = 2$.

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