DEGENERATIONS OF $\mathbb{C}^n$ AND CALABI-YAU METRICS

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Abstract. We construct infinitely many complete Calabi-Yau metrics on $\mathbb{C}^n$ for $n \geq 3$, with maximal volume growth, and singular tangent cones at infinity. In addition we construct Calabi-Yau metrics in neighborhoods of certain isolated singularities whose tangent cones have singular cross section, generalizing work of Hein-Naber [13].

1. Introduction

Since the seminal work of Yau [34], Calabi-Yau metrics have been studied extensively in Kähler geometry. Beyond the case of compact Kähler manifolds, there have been many constructions of non-compact Calabi-Yau metrics with various behaviors at infinity, by Cheng-Yau [4], Tian-Yau [31, 30] and others. In this paper we are concerned with constructing new non-compact Calabi-Yau manifolds that have Euclidean volume growth at infinity. In this case there exists a tangent cone at infinity $\mathbb{E}$, which is expected to be unique $\mathbb{E}$, and a natural problem is to try constructing complete Calabi-Yau metrics with prescribed tangent cones. There are many such constructions in the literature, with tangent cones that have smooth links $\mathbb{E}$, as well as singular links $\mathbb{E}$.

Our work pushes these methods further, obtaining a large class of new examples on $\mathbb{C}^n$, for $n \geq 3$. In particular these give counterexamples to a conjecture of Tian [29, Remark 5.3], stating that the flat metric is the unique Calabi-Yau metric on $\mathbb{C}^n$ with maximal volume growth. Some of these examples have very recently been independently obtained by Li [22] and Conlon-Rochon [10], using somewhat different techniques. See Section 1.2 for a comparison with our work. In addition we also construct Calabi-Yau metrics in neighborhoods of certain isolated singularities, with singular tangent cones, extending unpublished work of Hein-Naber [18].

Consider the hypersurface $X_1 \subset \mathbb{C}^{n+1}$ given by the equation

$$z + f(x_1, \ldots, x_n) = 0,$$

for a polynomial $f$, so that $X_1$ is biholomorphic to $\mathbb{C}^n$. Suppose in addition that if we let the $x_i$ have weights $w_i > 0$, then $f$ has degree $d > 1$. If we write $F_t(z, x_1, \ldots, x_n) = (tz, t^{w_1}x_1, \ldots, t^{w_n}x_n)$, then $F_t^{-1}X_1$ has the equation

$$t^{1-d}z + f(x_1, \ldots, x_n) = 0,$$

and so if $d > 1$, then $F_t^{-1}X_1 \to X_0$ as $t \to \infty$, where

$$X_0 = \mathbb{C} \times f^{-1}(0).$$

Suppose that $X_0$ admits a (singular) Ricci flat cone metric $\omega_0$ whose homothetic transformations are the maps $F_t$. It is then natural to expect that we can use $\omega_0$ to define an asymptotically Ricci flat metric on $X_1$ whose tangent cone at infinity is $(X_0, \omega_0)$, that we can perturb to a complete Calabi-Yau metric on $X_1$ with the
same tangent cone. This almost fits into the class of problems studied by Conlon-
Hein [8], except for the fact that \( X_0 \) has more than just an isolated singularity at
the origin.

We restrict ourselves to the simplest situation, when \( V_0 = f^{-1}(0) \subset \mathbb{C}^n \) has an
isolated normal singularity at the origin, so that \( X_0 \) is only singular along \( \mathbb{C} \times \{0\} \).
In particular \( n \geq 3 \). If we focus on the slice \( \{1\} \times V_0 \subset X_0 \) near the singular ray,
then the corresponding slice of \( F_t^{-1}X_1 \) is a smoothing

\[
t^{1-d} + f(x_1, \ldots, x_n) = 0
\]

of the cone \( V_0 \). Our strategy is to write down a metric on \( X_1 \) by combining a
perturbation of the singular Calabi-Yau metric on \( X_0 \) with a Calabi-Yau metric
on this smoothing of \( V_0 \) near the singular rays. The Calabi-Yau metric on the
smoothing is obtained using the results of Conlon-Hein [8]. Our main result then
is as follows.

**Theorem 1.** Under the assumption that the hypersurface \( V_0 \) admits a Calabi-Yau
cone metric, there is a complete Calabi-Yau metric on \( \mathbb{C}^n \) whose tangent cone at
infinity is \( \mathbb{C} \times V_0 \).

As an example we can consider the case when \( n = 3 \), and

\[
f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^k
\]

for \( k \geq 2 \). The hypersurfaces \( f^{-1}(0) \) are the \( A_{k-1} \) singularities and they all ad-
mit flat cone metrics, being cyclic quotients of \( \mathbb{C}^2 \). Therefore we obtain infinitely
many complete Calabi-Yau metrics on \( \mathbb{C}^3 \) with tangent cones \( \mathbb{C} \times A_{k-1} \) at infinity.
Simple higher dimensional examples can be obtained by taking products with \( \mathbb{C} \).
The corresponding metric when \( k = 2 \) has recently been constructed by Li [22]
and Conlon-Rochon [10] independently. Another example in higher dim ensions is
obtained with

\[
f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2,
\]

for \( n \geq 3 \), i.e. the \( A_1 \) singularity, which admits the Stenzel cone metric. We
therefore obtain complete Calabi-Yau metrics on \( \mathbb{C}^n \) with tangent cone \( \mathbb{C} \times A_1 \) at
infinity.

Consider now the hypersurface \( X_1 \subset \mathbb{C}^{n+1} \) with an isolated singularity \( 0 \in X_1 \),
given by

\[
z^p + f(x_1, \ldots, x_n) = 0,
\]

where \( p > 1 \), and \( f \) is as above. With \( F_t \) as before, the equation of \( F_t^{-1}X_1 \) is now

\[
t^{p-d}z^p + f(x_1, \ldots, x_n) = 0,
\]

and so if \( p > d \), then we have \( F_t^{-1}X_1 \rightarrow X_0 \) as \( t \rightarrow 0 \), with \( X_0 = \mathbb{C} \times V_0 \). It is
therefore natural to expect that \( X_1 \) admits a metric which is asymptotically Calabi-
Yau as we approach the singular point, and whose tangent cone at \( 0 \) is \( \mathbb{C} \times V_0 \). Using
an argument that is essentially identical to part of the proof of Theorem 1, we prove
that this is the case, and in fact this metric can be perturbed to be Calabi-Yau in
a neighborhood of the singular point. This result generalizes unpublished work of
Hein-Naber [18], which applies to the case when \( f = x_1^2 + \ldots + x_n^2 \), and \( p > 2 \frac{n-1}{n+1} \),
using different techniques.
Theorem 2. If $p > d$, then $X_1$ admits a Calabi-Yau metric on a neighborhood of the singular point $0$, whose tangent cone at $0$ is $\mathbb{C} \times V_0$.

It is likely that such constructions can be applied in a much more general setting, and we focus on these relatively explicit examples for simplicity. In general suppose that $X_0 \subset \mathbb{C}^N$ is a subvariety which admits a possibly singular Calabi-Yau cone metric whose homothetic (Reeb) vector field is $\xi = \sum w_i z_i \partial_{z_i}$ for weights $w_i > 0$. This vector field generates a real one-parameter group of biholomorphisms $F_t$ of $\mathbb{C}^N$. Let $X_1 \subset \mathbb{C}^N$ be another subvariety, and suppose that one of the following two conditions holds:

1. $\lim_{t \to \infty} F_t^{-1} X_1 = X_0$,
2. $\lim_{t \to 0} F_t^{-1} X_1 = X_0$.

Case (1) is the setting of Theorem 1, and here one expects to be able to construct a Calabi-Yau metric on $X_1$ near infinity, with tangent cone $X_0$ at infinity. If $X_1$ is smooth, then one can hope to deform this into a global Calabi-Yau metric on $X_1$ using techniques of Tian-Yau [31]. In case (2), by contrast, we necessarily have $0 \in X_1$, and one expects to be able to construct a Calabi-Yau metric on $X_1$ on a neighborhood of $0$, whose tangent cone at $0$ is $X_0$, as in Theorem 2. It seems likely that our methods can be generalized to prove these expectations whenever the singular behavior of $X_0$ has enough structure, such as the iterated edge spaces of Degeratu-Mazzeo [11].

Note that for a given $X_1$, one typically expects infinitely many possible $X_0$ to fit in case (1) above, as in Theorem 1 but there should be at most one $X_0$ fitting into case (2). This is because the tangent cone of a Calabi-Yau metric on $X_1$ at the singularity is expected to be independent of the metric (see Hein-Sun [20] for a special case of this).

This discussion fits into the framework developed by Donaldson-Sun [13] for analyzing the metric tangent cones of singular Calabi-Yau metrics under certain assumptions. They show that (under additional assumptions) the metric tangent cone $C(Y)$ at $0$ of a Calabi-Yau metric on $X_1$ is an affine algebraic variety which can be obtained from $X_1$ by a “two step” degeneration. In a first step we associate to $X_1$ its weighted tangent cone $W$ at $0$ for a suitable canonical valuation at $0$. This valuation is obtained, roughly speaking, from the rate of growth of germs of holomorphic functions on $X_1$ with respect to the Calabi-Yau metric. The metric tangent cone $C(Y)$ is then obtained by a further degeneration (or test-configuration) of $W$. One can think of these two steps as being analogous to the Harder-Narasimhan and Jordan-Hölder filtrations of an unstable, respectively semistable, vector bundle. Note that in the setting of Theorem 2 both $W$ and $C(Y)$ equal $X_0$ and so it would be interesting to construct examples where $X_1 \neq W \neq C(Y)$.

1.1. Outline. We now give an outline of the proof of Theorem 1. First, in Section 2 we construct a complete Calabi-Yau metric on the smoothing $V_1 = f^{-1}(1)$ of the Calabi-Yau cone $V_0$. This is already contained in work of Conlon-Hein [8], but we give some of the main points since they are a simpler version of what we need later.

In Section 3 we write down a metric $\omega$ on $X_1$, which is approximately Calabi-Yau near infinity, modeled on the product metric on $X_0 = \mathbb{C} \times V_0$. This product is singular along the rays $\mathbb{C} \times \{0\}$, and so near these singular rays we glue in suitable scaled copies of the Calabi-Yau metric on $V_1$ using cutoff functions. The
key technical result is the estimate in Proposition 5 of the Ricci potential of $\omega$, in suitable weighted Hölder spaces.

The next step, in Section 7, is to modify the metric $\omega$ to improve the decay of its Ricci potential. This is analogous to Lemma 2.12 in Conlon-Hein [8] and is based on inverting the Laplacian in suitable weighted spaces. After this we can further deform $\omega$ to a Calabi-Yau metric using a non-compact version of Yau's Theorem developed by Tian-Yau [34]. We use the version of this due to Hein [17].

The technical heart of the paper is in Sections 5 and 6, inverting the Laplacian in suitable weighted spaces. On asymptotically conical manifolds with smooth link at infinity there is a well developed theory for this, going back to Lockhart-McOwen [23], however the tangent cone $X_0$ of $(X_1, \omega)$ has a singular cross section - it is the double suspension of the link of $V_0$, which has a circle of singularities modeled on $V_0$. There have been several works dealing with the Laplacian on similar spaces, most notably the theory of QALE spaces due to Joyce [21], and the more general QAC spaces studied by Degeratu-Mazzeo [11]. On the one hand $(X_1, \omega)$ does not quite fit into the QAC framework, but more crucially, when solving the equation $\Delta u = f$, these works require at least quadratic decay of $f$ (at least away from the singular rays), since they essentially rely on taking a convolution with the Green's function. In our application, however, we also deal with $f$ which have slower decay (this is the case when the degree $d \leq 3$).

In order to solve $\Delta u = f$ for $f$ which do not decay sufficiently fast, instead of analysing the Green's function, we construct an approximate inverse for $\Delta$ on suitable local patches, which we can then glue together using cutoff functions. This is analogous to the method employed in many other geometric gluing problems (see e.g. Donaldson-Kronheimer [14] Chapter 7 or [28]). One of the local models that we have to analyze is the space $X_0 \setminus (C \times \{0\})$ which the part of $X_1$ away from the singular rays of $X_0$ is modeled on. The other model space is $C \times V_1$, on which neighborhoods of those points in $X_1$ are modeled on that are close to the singular rays of $X_0$. The basic strategy for solving $\Delta u = f$ approximately is to first decompose $f$ into pieces that are supported inside the model neighborhoods, then invert the Laplacian on the model spaces, and finally patch the results back together using cutoff functions. We then need to control the errors that are introduced by the cutoff functions on the one hand, and by replacing the metric $\omega$ on the model neighborhoods by the corresponding model metrics.

1.2. Relation to other recent works. As this project was nearing completion, two other works appeared that have significant overlap with our results. One is the paper of Yang Li [22] mentioned above. It deals with the particular case of constructing a metric on $\mathbb{C}^3$ with tangent cone $\mathbb{C} \times A_1$ at infinity. It relies on fairly explicit calculations, exploiting the symmetries of the situation. It is likely that his approach gives better asymptotic understanding of this metric, and this may be important in applications to gluing problems.

The other, even more recent, paper is Conlon-Rochon [10], constructing Calabi-Yau metrics on $\mathbb{C}^n$ with tangent cones $\mathbb{C} \times V_0$ for suitable Calabi-Yau cones $V_0$. It relies on extending the work of Degeratu-Mazzeo [11] to a class of “warped QAC” metrics, and as such it still requires faster than quadratic decay of the Ricci potential of the approximate solution. In terms of our Theorem 1 this is the case when the degree $d > 3$. One important case that this does not cover is when $f(x) = x_1^2 + \ldots + x_n^2$, and $n > 3$, since here $d = 2 \frac{n-2}{n-2} \leq 3$, although in a subsequent
version of [11] the authors have overcome this issue by constructing a metric whose
Ricci potential decays more rapidly. Nevertheless, our approach to inverting the
Laplacian also has the advantage that the same method applies to Theorem 2.

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tions.

2. The smoothing model

Suppose that \( V_0 \subset \mathbb{C}^n \) is a hypersurface given by the equation \( f(x_1, \ldots, x_n) = 0 \),
with an isolated singularity at the origin. Suppose that we have a weight vector
\( \xi = (w_1, \ldots, w_n) \) with positive, possibly non-integral entries, giving rise to the
action

\[
(t \cdot (x_1, \ldots, x_n) = (t^{w_1} x_1, \ldots, t^{w_n} x_n)
\]
on \( \mathbb{C}^n \) for \( t > 0 \). We denote the degree of \( f \) under this action by \( d \), i.e. \( f(t \cdot x) = t^d f(x) \).
This action generates the action of a complex torus \( T^c \) on \( \mathbb{C}^n \), which fixes \( V_0 \).
We write \( z \cdot x \) for \( z \in \mathbb{C}^* \), which means that we choose a branch of \( \log z \) to define the non-integer
powers of \( z \). The choice of branch will not matter.

There is a nowhere vanishing holomorphic \((n-1)\)-form \( \Omega \) on \( V_0 \setminus \{0\} \) given by

\[
\Omega = \frac{dx_2 \wedge dx_3 \wedge \ldots \wedge dx_n}{\partial_{x_1} f}
\]

where \( \partial_{x_1} f \neq 0 \), and by similar expressions where \( \partial_{x_i} f \neq 0 \) for \( i > 1 \). We assume
that \( \Omega \) has degree \( n-1 \) under the action above, which is equivalent to the identity

\[
\sum_{i=1}^{n} w_i = d + n - 1.
\]

Finally we suppose that \( V_0 \) admits a Ricci flat Kähler cone metric \( \omega_{V_0} \), whose
homothetic transformations are given by the action above. Equivalently we have
\( \omega_{V_0}^{-1} = (\sqrt{-1})^{(n-1)/2} \Omega \wedge \overline{\Omega} \). This setup has been studied extensively in the literature,
in particular in relation to Sasakian geometry (see [15]).

Lemma 3. Under these assumptions, unless \( V_0 \cong \mathbb{C}^{n-1} \), we have \( d > 2 \).

Proof. Let us denote by \( w_{\text{min}} \) the smallest weight. The Lichnerowicz obstruction
of Gauntlett-Martelli-Sparks-Yau [15] implies that since \( V_0 \) admits a Calabi-Yau
cone metric, we have \( w_{\text{min}} > 1 \) (if \( w_{\text{min}} = 1 \) then necessarily \( V_0 \cong \mathbb{C}^{n-1} \)). It then
follows that \( d > 2 \), since \( d \geq 2w_{\text{min}} \), unless \( f \) is linear. \( \square \)

Because of this Lemma we can assume throughout that \( d > 2 \). Otherwise \( V_0 \cong \mathbb{C}^{n-1} \) in which case Theorems 3 and 2 are clear. Here we are interested in the
existence of complete Ricci flat metrics on the smoothing \( V_1 \) of \( V_0 \), given by the
equation \( 1 + f(x) = 0 \). This question was addressed by Conlon-Hein [3], and here
we review the main points since our result can be thought of as a generalization of
their work.

The Ricci flat metric on \( V_0 \) is given by \( \omega_{V_0} = \sqrt{-1} \partial \bar{\partial} r^2 \), where \( r \) is the radial
distance from the vertex of the cone. The idea is to use \( r^2 \) to write down a Kähler

potential on $V_1$ near infinity, which is approximately Ricci flat. Since $V_1$ is asymptotic to $V_0$ near infinity, a natural approach, followed by Conlon-Hein, is to simply use an orthogonal projection from $V_1$ to $V_0$ (outside a bounded set) to pull back the potential $r^2$. We follow a slight variant of this method. First let us define the function $R : \mathbb{C}^n \to \mathbb{R}$, by letting $R = 1$ on the Euclidean unit sphere, and extending $R$ to have degree 1 under the action of $F_i$. As shown in He-Sun [19] Lemma 2.2, then the form $\sqrt{-1} \partial \overline{\partial} R^2$ defines a cone metric on $\mathbb{C}^n$. By the homogeneity, its restriction to $V_0$ is uniformly equivalent to $\omega_{V_0}$. In particular the function $R$ restricted to $V_0$ is also uniformly equivalent to the distance function $r$. We now take any smooth extension of $r$ from $V_0$ to $\mathbb{C}^n \setminus \{0\}$, which has degree 1 under the action $\xi$. This can be defined by first extending $r$ on the sphere $R = 1$, and then extending it further by homogeneity. We will continue to write $r$ for this extended function.

We claim the following.

**Proposition 4.** There exists a constant $A > 0$ such that

(a) The form $\omega = \sqrt{-1} \partial \overline{\partial} r^2$ is positive definite on $V_1 \cap \{R > A\}$.

(b) The Ricci potential

$$h = \log \frac{(\sqrt{-1} \partial \overline{\partial} r^2)^{n-1}}{(\sqrt{-1})^{(n-1)^2} \Omega \wedge \Omega}$$

of $(V_1, \omega)$ satisfies

$$\nabla^i h = O(R^{-d-i}),$$

as $R \to \infty$, for all $i \geq 0$, measured with respect to $\omega$.

**Proof.** Given large $K$, let us analyze the region $\{K/2 < R < 2K\}$ by scaling the metric by a factor of $K^{-1}$. At the same time let us introduce rescaled coordinates $\tilde{x} = K^{-1} \cdot x$, and $\tilde{r} = K^{-1} R$, $\tilde{R} = K^{-1} R$. The rescaled metric is given by

$$K^{-2} \omega = \sqrt{-1} \partial \overline{\partial} \tilde{r}^2,$$

and in the rescaled coordinates the equation of $V_1$ is

$$K^{-d} f(\tilde{x}) = 0.$$

Let us write this as $V_{K^{-d}}$. Restricted to the annular region $\{1/2 < \tilde{R} < 2\}$, the submanifolds $V_{K^{-d}}$ converge in $C^\infty$ to $V_0$ uniformly. In addition $\tilde{r}^2$ as a function of $\tilde{x}$ is independent of $K$ by the homogeneity of $r$.

Using the implicit function theorem we can cover $V_{K^{-d}} \cap \{1/2 < \tilde{R} < 2\}$ by a finite number of coordinate balls $B_i$ in each of which $V_{K^{-d}}$ is a hyperplane, and moreover we can find holomorphic functions $G_i : B_i \to \mathbb{C}^n$ such that $G_i(V_{K^{-d}}) \subset V_0$, and $G_i(y) = y + O(K^{-d})$. Since we also control the derivatives of the $G_i$ and $\tilde{r}^2$ is a fixed smooth function, we obtain

$$\tilde{r}^2 - G_i^* \tilde{r}^2 = O(K^{-d})$$

in $C^\infty$, on each of our coordinate balls. We measure derivatives here with respect to a fixed metric $\sqrt{-1} \partial \overline{\partial} \tilde{R}^2$. In particular once $K$ is sufficiently large, $\sqrt{-1} \partial \overline{\partial} \tilde{r}^2$ defines a positive definite form on $V_{K^{-d}}$, and in fact it is uniformly equivalent to $\sqrt{-1} \partial \overline{\partial} \tilde{R}^2$.

To control the Ricci potential, we simply need to compare $\Omega$ with $G_i^* \Omega$, since by assumption we have $\omega_{V_0} = (\sqrt{-1})^{(n-1)^2} \Omega \wedge \Omega$. By the same reasoning we have
\( \Omega - G^* \Omega = O(K^{-d}) \). Since the Ricci potential is invariant under scaling we find that on the annulus \( \{ K/2 < R < 2K \} \) we have

\[ |\nabla^i h|_{K^{-2} \sqrt{-\partial \partial R}^2} = O(K^{-d}), \]

where we indicate that we measure derivatives with respect to \( K^{-2} \sqrt{-\partial \partial R}^2 \). Since \( \sqrt{-\partial \partial R}^2 \) is uniformly equivalent to \( \sqrt{-\partial \partial r}^2 \) (once \( K \) is sufficiently large), this implies the result. \( \square \)

We can now write down a metric \( \omega_{V_1} \) on \( V_1 \), which agrees with \( \sqrt{-\partial \partial r}^2 \) on the set where \( R \) is sufficiently large. One way to do this is to consider the Kähler potential \( C'(1 + |x|^2)^\alpha \) on \( C^n \) for large \( C' \). For sufficiently small \( \alpha > 0 \) this grows slower than \( r^2 \), but if \( C' \) is sufficiently large, then we can ensure that

\[ C'(1 + |x|^2)^\alpha > r^2 \]

on the set where \( R < 3K/2 \), say. We can now define a regularized maximum (see Demailly [12 §5.E])

\[ \Phi = \max \left\{ C'(1 + |x|^2)^\alpha, r^2 \right\}, \]

and let \( \omega_{V_1} = \sqrt{-\partial \partial \Phi}_{V_1} \). This defines a smooth metric on \( V_1 \), which equals \( \sqrt{-\partial \partial r}^2 \) where \( R \) is sufficiently large.

The estimate (2.1) says that the Ricci potential \( h \) of \( \omega_{V_1} \) satisfies \( h \in C_{C^0}^\infty \) in terms of the weighted spaces used by Conlon-Hein [8]. Their Theorem 2.1 then implies that we can perturb \( \omega_{V_1} \) to a Calabi-Yau metric \( \eta_{V_1} = \sqrt{-\partial \partial \Phi} \) on \( V_1 \), where the decay of \( \phi - r^2 \) can be controlled. Since by Lemma [5] we have \( d > 2 \), from [5 Theorem 2.1] it follows that we can ensure \( \phi - r^2 \in C_{C^0}^\infty(V_1) \) for some \( c > 0 \). If \( d > 3 \) then we can even choose \( c > 1 \). We can write this estimate in the following form: there are constants \( C_i \), such that if \( x \) lies in the region where \( 1/2 < R < 2 \), and \( \lambda : x \in V_1 \) for some \( \lambda > 1 \), then

\[ \left| \nabla^i \left[ r^2 - \lambda^{-2} \phi(\lambda \cdot x) \right] \right|_{\sqrt{-\partial \partial r}^2} < C_i \lambda^{-2-c}. \]

Moreover by the uniqueness statement of [8 Theorem 2.1], \( \phi \) is invariant under the action of \( T \).

3. The approximate solutions on \( C^n \)

We now consider \( X_1 \subset C^{n+1} \) given by

\[ z + f(x_1, \ldots, x_n) = 0, \]

where \( f \) is the polynomial from Section 2. Recall that we have a Ricci flat cone metric on \( V_0 = f^{-1}(0) \subset C^n \), whose distance function is \( r \), and we have smoothly extended \( r \) so that it is defined on \( C^n \setminus \{ 0 \} \) and has degree 1 for the action \( \xi \).

The hypersurface \( X_0 = C \times V_0 \subset C^{n+1} \) then has a Ricci flat cone metric \( \sqrt{-\partial \partial |z|^2 + r^2} \). The corresponding homothetic scalings are given by the action with weights \( (1, w_1, \ldots, w_n) \). This metric is uniformly equivalent to \( \sqrt{-\partial \partial \rho^2} \), where

\[ \rho^2 = |z|^2 + R^2, \]

in terms of \( R \) from above.

We would like to define a metric on \( X_1 \) using the potential \( |z|^2 + r^2 \) just as we did for \( V_1 \) above, but now \( X_0 \) and \( |z|^2 + r^2 \) are singular along \( C \times \{ 0 \} \). Our approach is
to use $|z|^2 + r^2$ as the potential away from the singular rays, and a suitable scaling of the potential $\phi$ on $V_1$ near the singular rays.

Let us denote by $\gamma_1(s)$ a cutoff function satisfying

$$\gamma_1(s) = \begin{cases} 1 & \text{if } s > 2 \\ 0 & \text{if } s < 1, \end{cases}$$

and write $\gamma_2 = 1 - \gamma_1$. We then define the approximate solution on $X_1$, at least on the set where $\rho > P$ for sufficiently large $P$, by

$$(3.1) \quad \omega = \sqrt{-1} \partial \overline{\partial} \left( |z|^2 + \gamma_1(R\rho^{-\alpha})^2 + \gamma_2(R\rho^{-\alpha})|z|^{2/d} \phi(z^{-1/d}, x) \right),$$

where $\alpha \in (1/d, 1)$ is to be chosen. As before we must choose a branch of $\log z$ to define $z^{-1/d} \cdot x$, however the value of $\phi(z^{-1/d} \cdot x)$ is independent of this choice since $\phi$ is $T$-invariant. Note moreover that if

$$z + f(x) = 0,$$

then

$$1 + f(z^{-1/d} \cdot x) = 0,$$

and so $z^{-1/d} \cdot x \in V_1$, where $\phi$ is defined. Writing $\phi(x) = r^2 + \phi_{-c}(x)$ we have

$$\omega = \sqrt{-1} \partial \overline{\partial} \left( |z|^2 + r^2 + \gamma_2(R\rho^{-\alpha})|z|^{2/d} \phi_{-c}(z^{-1/d} \cdot x) \right),$$

and the estimate (2.4) implies that the term involving $\phi_{-c}$ is of lower order than $|z|^2 + r^2$. It follows from this that $\omega = \sqrt{-1} \partial \overline{\partial} \Phi$ where the potential $\Phi$ has the same growth rate as $|z|^2 + r^2 = \rho^2$. In particular if $\omega$ is positive definite on the set where $\rho > P$, we can argue as in the construction of $\omega_{V_1}$ above, to construct a metric on $X_1$ that agrees with $\omega$ where $\rho > 2P$.

The holomorphic $n$-form

$$\Omega = \frac{dz \wedge dx_2 \wedge \ldots \wedge dx_n}{\partial x_1 f}$$

restricts to a nowhere vanishing $n$-form on $X_0$ as well as on $X_1$, and as in the previous section we wish to estimate the Ricci potential of $\omega$ with respect to $\Omega$. We have the following in analogy with Proposition 4.

**Proposition 5.** Fix $\alpha \in (1/d, 1)$. The form $\omega$ defines a metric on the subset of $X_1$ where $\rho > P$, for sufficiently large $P$. For suitable constants $\kappa, C_i > 0$ and weight $\delta < 2/d$, the Ricci potential $h$ of $\omega$ satisfies, for large $\rho$,

$$|\nabla^i h|_{\omega} < \begin{cases} C_i \rho^{\delta - 2 - i} & \text{if } R > \kappa \rho \\ C_i \rho^3 R^{-2 - i} & \text{if } R \in (\kappa^{-1} \rho^{1/d}, \kappa \rho) \\ C_i \rho^{\delta - 2 - i/d} & \text{if } R < \kappa^{-1} \rho^{1/d}. \end{cases}$$

If in addition $d > 3$ and $\alpha$ is chosen sufficiently close to 1, then we can even choose $\delta < 0$, i.e. in this case $h$ decays faster than quadratically away from the singular rays in this case.

**Proof.** The proof is similar to that of Proposition 4 but we scale our metric in a different way near the singular rays of $X_0$, and so we have several different regions to study separately.
**Region I**: Consider the region where $R > \kappa \rho$, and $\rho \in (D/2, 2D)$ for some large $D$. Here we are uniformly far away from the singular rays. We study the scaled form $D^{-2}\omega$, in terms of rescaled coordinates

\[
\tilde{z} = D^{-1}z, \\
\tilde{x} = D^{-1} \cdot x,
\]

and we let $\tilde{r} = D^{-1}r$ (recall that here $D^{-1} \cdot x$ is defined using the action with weights $w_1$). On this region, once $D$ is large enough, we have $\gamma_1 = 1, \gamma_2 = 0$, and so

\[
D^{-2}\omega = \sqrt{-1} \partial \bar{\partial} (|\tilde{z}|^2 + \tilde{r}^2).
\]

In addition in terms of these coordinates $X_1$ has equation

\[
D^{1-d} \tilde{z} + f(\tilde{x}) = 0,
\]

using that $f$ has degree $d$, and so $f(D \cdot \tilde{x}) = D^d f(\tilde{x})$. Note that $R \in (\kappa \rho, \rho)$, and $r$ is uniformly equivalent to $R$, and in addition $|\tilde{z}| < 2$. It follows that we are in essentially the same situation as in the proof of Proposition 4. The same arguments show that

\[
|\nabla^i h|_{D^{-2}\omega} \leq C_i D^{1-d},
\]

and so on this region

\[
|\nabla^i h|_{\omega} \leq C_i D^{1-d-i}.
\]

We can therefore choose any $\delta$ such that $\delta - 2 > 1 - d$, i.e. $\delta > 3 - d$. If $d > 3$ we can choose $\delta < 0$, while if $d > 2$, then $3 - d < 2/d$ and so we can choose $\delta < 2/d$. This implies the required estimates on this region.

**Region II**: Suppose now that $R \in (K/2, 2K)$ for some $K < \kappa \rho$, but $K/2 > 2\rho^\alpha$. In addition let $\rho \in (D/2, 2D)$. In this case $\rho$ is comparable to $|z|$, and here we still have $\gamma_1 = 1, \gamma_2 = 0$. We assume that for some fixed $z_0$ we have $|z - z_0| < K$ (so that $z$ will be in a unit ball centered at $z_0$ after scaling). We now scale our form by $K$, and define

\[
(3.2) \quad \tilde{z} = K^{-1}(z - z_0), \quad \tilde{x} = K^{-1} \cdot x, \quad \tilde{r} = K^{-1}r.
\]

In terms of these we have

\[
K^{-2}\omega = \sqrt{-1} \partial \bar{\partial} (|\tilde{z}|^2 + \tilde{r}^2),
\]

and the equation of $X_1$ is

\[
K^{-d}(K \tilde{z} + z_0) + f(\tilde{x}) = 0.
\]

In addition $|\tilde{z}| < 1$. We can still argue essentially like in Proposition 4 and now the errors we obtain are of order $K^{-d}D$ since $|\tilde{z}| < 1$ and $|z_0| \sim D$. The Ricci potential therefore satisfies

\[
|\nabla^i h|_{\omega} \leq C_i DK^{-d-i}.
\]

Since $d > 2$ and $K > 4\rho^\alpha$, we have

\[
DK^{2-d} K^{-2-i} < CD^{1+\alpha(2-d)} K^{-2-i} < C\rho^{1+\alpha(2-d)} R^{-2-i},
\]
We scale in the same way as in Regions II, III, so we make the change of variables
\[ x \mapsto C^2 \rho \] as above. We still want to compare the metric to (3.2) as in Region II. We then have
\[ c > 1, \quad \text{and so} \quad (2 + (3.4) \alpha > 1). \]
If we merely have \( X \) the error introduced in passing from coordinates. The equation of \( K \) from comparing the two equations, as well as a new error \((3.4)\) from the error in the \( \mathcal{K} \)ähler potential. Since now \( \frac{K}{\mathcal{K}} = 0 \), we suppose that \( \rho = \mathcal{K}/2 \mathcal{K} \) but \( \mathcal{K} \) Region IV: We now consider \( R \in (\mathcal{K}/2, 2 \mathcal{K}) \) and \( K \in (\mathcal{K}/2, 2 \mathcal{K}) \) here.
\[
K^{-2} \omega = \sqrt{-1} \partial \bar{\partial} \left( |z|^2 + \gamma_1 r^2 + \gamma_2 K^{-2} |K \bar{z} + z_0|^{2/d} \phi((K \bar{z} + z_0)^{-1/d} K \cdot \bar{x}) \right),
\]
and the cutoff functions \( \gamma_1, \gamma_2 \) have bounded derivatives in terms of these rescaled coordinates. The equation of \( X_1 \) is
\[
K^{-d}(K \bar{z} + z_0) + f(\bar{x}) = 0,
\]
as above. We still want to compare the metric to
\[
\sqrt{-1} \partial \bar{\partial}(|z|^2 + r^2)
\]
on \( X_0 \) which has equation \( f(\bar{x}) = 0 \). We use the estimate (2.2) to get that in these coordinates
\[
(3.3) \quad \nabla^i \left[ K^{-2} |K \bar{z} + z_0|^{2/d} \phi((K \bar{z} + z_0)^{-1/d} K \cdot \bar{x}) - r^2 \right] = O \left( (K^{-1} D^{1/d})^{2+c} \right).
\]
In other words, in the Ricci potential we will have the same error \( K^{-d} D \) as in Region II from comparing the two equations, as well as a new error \((K^{-1} D^{1/d})^{2+c}\)
from the error in the \( \mathcal{K} \)ähler potential. Since now \( K \sim D^\alpha \), this new term can be estimated as follows:
\[
(K^{-1} D^{1/d})^{2+c} < CD^{\frac{2+c}{d} - c \alpha} K^{-2}.
\]
We need \( \delta \) so that \( 2 + c - c \alpha < \delta \). Choosing \( \alpha \) is sufficiently close to 1, we can choose any \( \delta > (2 + c)/d - c \). If in addition \( d > 3 \), then we have seen that we can choose \( c > 1 \), and so \((2 + c)/d - c < 0 \), and \( \delta \) can be chosen negative. In general if we only have \( \alpha > 1/d \), then we still have \((2 + c)/d - \alpha c < 2/d \), and so we can choose \( \delta < 2/d \).

Region IV: We now consider \( R \in (\mathcal{K}/2, 2 \mathcal{K}) \), but \( K \in (\mathcal{K}/2, 2 \mathcal{K}) \). Here \( \gamma_1 = 0, \gamma_2 = 1 \). We suppose that \( \rho \in (D/2, 2 D) \) and so \(|z|\) is comparable to \( \mathcal{D} \). We scale in the same way as in Regions II, III, so we make the change of variables (3.2). We have
\[
K^{-2} \omega = \sqrt{-1} \partial \bar{\partial} \left( |z|^2 + K^{-2} |K \bar{z} + z_0|^{2/d} \phi((K \bar{z} + z_0)^{-1/d} K \cdot \bar{x}) \right),
\]
and the equation of \( X_1 \) is still
\[
K^{-d}(K \bar{z} + z_0) + f(\bar{x}) = 0.
\]
Instead of comparing \( X_1 \) to \( X_0 \), this time we want to compare \( X_1 \) to the product \( \mathcal{C} \times V_1 \), scaled suitably. Note that \( K^{-d}(K \bar{z} + z_0) \) to leading order is \( K^{-d} z_0 \), and so we compare \( X_1 \) to the variety \( \mathcal{C} \times V_{K^{-d} z_0} \) with equation
\[
K^{-d} z_0 + f(\bar{x}) = 0. \tag{3.4}
\]
The error introduced in passing from \( X_1 \) to this variety is of order \( K^{-1-d} \) (since \(|z| < 1 \)).
It remains to study the difference in the Kähler potentials. On the variety with equation (5.4) we have the metric

$$\sqrt{-1}d\bar{\Omega}(|\bar{z}|^2 + K^{-2}|z_0|^{2/d}\phi(K|z_0|^{-1/d} \cdot \bar{x})),$$

and so we need to estimate the difference

$$E = K^{-2}|\bar{z} + z_0|^{2/d}\phi((K\bar{z} + z_0)^{-1/d}K \cdot \bar{x}) - K^{-2}|z_0|^{2/d}\phi(K|z_0|^{-1/d} \cdot \bar{x}).$$

Let us write $\phi = r^2 + \phi_{-c}$. By the homogeneity of $r$, we have

$$E = K^{-2}|\bar{z} + z_0|^{2/d}\phi_{-c}((K\bar{z} + z_0)^{-1/d}K \cdot \bar{x}) - K^{-2}|z_0|^{2/d}\phi_{-c}(K|z_0|^{-1/d} \cdot \bar{x}).$$

In addition since $|\bar{z}| < 1$, $|z_0| \sim D$ and $K \ll D$, we have

$$K(\bar{z} + z_0)^{-1/d} = z_0^{-1/d}K(1 + O(KD^{-1})).$$

Using (2.2) for $\phi_{-c}$, we therefore find that in the rescaled coordinates

$$|\nabla E| < C_1(|z_0|^{-1/d}K)^{-2-c}D^{-1}K = O(K^{-1-c}D^{2+c-1}).$$

Combining the error $K^{1-d}$ from changing the equation of the variety with this error in the Kähler potentials, we find that the Ricci potential is of order $K^{1-d} + K^{-1-c}D^{2+c-1}$. We therefore need to ensure that the choice of $\delta$ satisfies

$$K^{1-d} + K^{-1-c}D^{2+c-1} < CD^\delta K^{-2}.$$ 

Suppose first that $d > 3$, so that also $c > 1$. Then

$$K^{1-d} = K^{3-d}K^{-2} < CD^{3-d-1}K^{-2},$$

and also

$$K^{-1-d}D^{2+c-1} = (KD^{-1/d})^{1-c}D^{\frac{1}{d}-1}K^{-2}.$$ 

Since $3/d - 1 < 0$, we can choose $\delta < 0$ close to zero.

If we only have $d > 2$, so $c > 0$, then

$$K^{-1-c}D^{2+c-1} = (KD^{-1/d})^{1-c}(KD^{-1})D^{2/d}K^{-2}. $$

Since for some $C$ we have $C^{-1}D^{1/d} < K < CD$, it follows that we can choose $\delta < 2/d$.

**Region V:** Finally we suppose that $R < 2\kappa^{-1}\rho^{1/d}$. Again, let $z$ be very close to $z_0$, with $\rho$, and therefore $|z_0|$ comparable to $D$. We now scale by $|z_0|^{1/d}$, and so we change variables to

$$\tilde{z} = z_0^{-1/d}(z - z_0), \quad \tilde{x} = z_0^{-1/d}x, \quad \tilde{r} = |z_0|^{-1/d}r,$$

so that $|\tilde{z}|, \tilde{r} < C$. We have

$$|z_0|^{-2/d}\omega = \sqrt{-1}d\bar{\Omega}(|\tilde{z}|^2 + |z_0|^{-2/d}|z_0|^{1/d}\tilde{z} + z_0|^{2/d}\phi(z_0^{1/d}(z_0^{-1/d}\tilde{z} + z_0)^{-1/d} \cdot \tilde{x}),$$

and the equation of $X_1$ is

$$z_0^{1/d-1}\tilde{z} + 1 + f(\tilde{x}) = 0.$$ 

We want to compare this to $C \times V_1$, with equation

$$1 + f(\tilde{x}) = 0,$$
and metric
\[ \sqrt{-1} \partial \bar{\partial} [ |z|^2 + \phi(\hat{x})] . \]
The difference in the equations results in an error of order \( D^{1/d-1} \). For the Kähler potential note that
\[ z_0^{1/d} (z_0^{1/d} \bar{z} + z_0)^{-1/d} = 1 + O(D^{1/d-1}) , \]
and so
\[ |z_0|^{-2/d} |z_0^{1/d} \bar{z} + z_0|^{2/d} \phi(\frac{z_0^{1/d} \bar{z} + z_0}{z_0^{1/d} \bar{z} + z_0} \cdot \hat{x}) - \phi(\hat{x}) = O(D^{1/d-1}) . \]
In sum the Ricci potential is of order \( D^{1/d-1} \), and we need \( \delta \) that satisfies
\[ D^{1/d-1} < C D^{\delta-2/d} , \]
i.e. \( \delta > 3/d - 1 \). In particular if \( d > 2 \) we can choose \( \delta < 2/d \), while if \( d > 3 \) we can choose \( \delta < 0 \).

4. Weighted spaces on \( X_1 \)

As in the discussion before Proposition 5 we define a metric \( \omega \) on all of \( X_1 \), which agrees with the form defined in (3.1) for sufficiently large \( \rho \). Our eventual goal is to perturb the metric \( \omega \) to a Calabi-Yau metric on the set \( \{ \rho > A \} \) for sufficiently large \( A \), at which point we will be able to apply Hein [17, Proposition 4.1] to construct a global Calabi-Yau metric on \( X_1 \). The main difficulty is to invert the linearized operator, which is the Laplacian on \( (X_1, \omega) \).

The analysis of the Laplacian on asymptotically conical spaces has been studied extensively (see e.g. Lockhart-McOwen [23]), and was used in the work of Conlon-Hein [8] discussed in Section 2 above. The difference is that we now need to invert the Laplacian in a more complicated weighted space that accounts for the singular rays in the tangent cone at infinity. This almost fits into the framework of QAC spaces studied by Degeratu-Mazzeo [11], which in turn generalizes the work of Joyce [21] on QALE manifolds, but unfortunately those results cannot be applied directly in our setting. The issue is that at distance \( D \) along the singular rays our metric \( \omega \) is modeled on a product of \( C \) with a scaling of the metric \( \omega_{V_1} \) by a factor of \( D^{2/d} \). In the QAC or QALE geometries there is no such additional scale factor on the geometry “transverse” to the singular rays. Very recently, Conlon-Rochon [10] have extended the techniques of Degeratu-Mazzeo to such a more general setup, however an additional difficulty in our setting is that we need to invert the Laplacian on functions that decay more slowly than what they consider. Because of this, we follow a different approach, which has the added advantage that our method applies with essentially no changes to constructing certain Calabi-Yau metrics in a neighborhood of an isolated singularity, as we will see in Section 5.

We first define weighted spaces \( C^k_{\delta, \tau} (X_1, \omega) \), and then define \( C^k_{\delta, \tau} (\rho^{-1}[A, \infty), \omega) \) by restricting functions to the set where \( \rho \geq A \). The norm of a function on \( \rho^{-1}[A, \infty) \) is defined to be the infimum of the corresponding norms of its extensions to \( X_1 \).

On the set where \( \rho < P \) for some fixed large \( P \) (we will have \( P < A \)) we use the usual Hölder norms. When \( \rho > P \) then we define the weighted spaces in terms of the functions \( \rho \) (which is essentially the radial distance), and \( R \) (which controls...
the distance from the singular rays). To make the analogy with Degeratu-Mazzeo’s weighted spaces we define a smooth function \( w \) satisfying

\[
\begin{cases}
1 & \text{if } R > 2\kappa, \\
\frac{R}{(\kappa \rho)} & \text{if } R \in (\kappa^{-1} \rho^{1/d}, \kappa \rho), \\
\kappa^{-2} \rho^{1/d-1} & \text{if } R < \frac{1}{2} \kappa^{-1} \rho^{1/d},
\end{cases}
\]

for the same \( \kappa \) as in Proposition 5.

Similarly to Degeratu-Mazzeo, we define the Hölder seminorm

\[
[T]_{0, \gamma} = \sup_{\rho(z) > K} \rho(z)^{\gamma} \sup_{z' \neq z, z' \in B(z, c)} \frac{|T(z) - T(z')|}{d(z, z')^{\gamma}}
\]

for the same \( c > 0 \) is such that \( X_1 \) has bounded geometry on balls of radius \( c > 0 \) and these balls are geodesically convex. \( T \) could be a tensor, in which case we compare \( T(z) \) with \( T(z') \) using parallel transport along a geodesic.

The weighted norm of \( f \) is then defined by

\[
\|f\|_{C^{k, \alpha}} = \|f\|_{C^{k, \alpha}(\rho < 2P)} + \sum_{j=0}^{k} \sup_{\rho(z) > P} \rho^{-\delta + j} w^{-\tau + j} |\nabla^j f| + [\rho^{-\delta + k} w^{-\tau + k} \nabla^k f]_{0, \alpha}
\]

A more concise way to express these weighted norms is in terms of the conformal scaling \( \rho^{-2} w^{-2} \omega \). More precisely for this we should replace \( \rho \) by a smoothing of \( \max\{1, \rho\} \).

The estimate in Proposition 5 can now be stated as saying that the Ricci potential \( h \) is in the weighted space \( C^{k, \alpha}_{0, \gamma} \) with \( \delta \) as in the Proposition. Moreover since \( w \leq 1 \), we also have \( h \in C^{k, \alpha}_{0, \tau} \) for any \( \tau < 0 \).

In the following propositions we compare the geometry of \((X_1, \omega)\) to suitable model spaces in different regions. This will be used to study the Laplacian on \((X_1, \omega)\) in Section 6 below. We will write \( g, g_{X_0} \) for the Riemannian metrics given by \( \omega, \omega_{X_0} \) respectively.

Proposition 6. Given any \( \epsilon > 0 \) we can choose \( \Lambda > \Lambda(\epsilon) \), and \( A > A(\epsilon) \) sufficiently large so that on \( U \) we have

\[
|\nabla^i (G^* g_{X_0} - g)|_g < \epsilon w^{-1} \rho^{-1},
\]
for $i \leq k + 1$. In particular in terms of our weighted spaces we have

$$\|G^i gX_0 - g\|_{c_{\rho,0}} < \epsilon.$$  

**Proof.** The proof is very similar to the analysis of regions I, II, III, IV in the proof of Proposition [5]. Let us first consider region I. Suppose that $\rho \in (D/2, 2D)$, with $D > A$, and $R > \kappa \rho$. Here we have $w \sim 1$, the notation $a \sim b$ meaning that $C^{-1} < a/b < C$ for a constant $C$. The estimate that we need to show is equivalent to

$$|D^{-2}\nabla^i(G^i gX_0 - g)|_{D^{-2}g} < \epsilon.$$  

We can work in the rescaled coordinates $\tilde{z}, \tilde{x}$ as in Proposition [5]. In these coordinates $X_1$ is given by the equation $D^{1-d} \tilde{z} + f(\tilde{x}) = 0$, and $X_0$ by the equation $f(\tilde{x}) = 0$. In Proposition [5] we were only interested in estimating the Ricci potential, and so we compared the two hypersurfaces in local holomorphic charts (see the proof of Proposition [4]). Now, however, we want a more global comparison, using the projection map $G$ which is not holomorphic. This is closer to the approach taken by Conlon-Hein [3]. The end result is the same, since in local charts the difference between $G$ and the identity map is of order $D^{1-d}$. Since $d > 1$ we can ensure that this is less than $\epsilon$ by taking $A$ large.

Regions II, III follow similar calculations to the proof of Proposition [5]. Let us therefore consider region IV, which is a little different, since previously we compared this region to $C \times V_1$, whereas now we are comparing it to $X_0 = C \times V_0$. Suppose that $\rho \in (D/2, 2D)$, and $R \in (K/2, 2K)$ with $\Lambda \rho^{1/d} < K < R^\alpha/2$. In this region $\rho$ is comparable to $|z|$, and we suppose that $z$ is close to some $z_0$, with $|z_0| \in (D/4, 4D)$. Introducing coordinates $\tilde{z}, \tilde{x}$ as before, the equation for $X_1$ is

$$K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0,$$

while the equation for $X_0$ is $f(\tilde{x}) = 0$. As long as $|\tilde{z}| < 1$, the error introduced by orthogonal projection is of order $K^{1-d}$, which can be made arbitrarily small by choosing $D$ large.

The two Kähler potentials that we need to compare are

$$\sqrt{-1} \partial \bar{\partial} \left( |\tilde{z}|^2 + K^{-2} |K\tilde{z} + z_0|^{2/d} \phi((K\tilde{z} + z_0)^{-1/d} K \cdot \tilde{x}) \right),$$

and

$$\sqrt{-1} \partial \bar{\partial} (|\tilde{z}|^2 + \tilde{r}^2).$$

If $|\tilde{z}| < 1$ and $|z_0| \sim D$, by the estimate (2.2) the difference between the Kähler potentials is of order

$$(KD^{-1/d})^{-c-2} < C\Lambda^{-c-2}.$$

By choosing sufficiently large $\Lambda$, we can ensure that this is less than $\epsilon$. □

Next, we focus on the region where $\rho > A$, but $R < \Lambda \rho^{1/d}$. Fix $z_0 \in C$ with $|z_0| > A$, and a large constant $B$. Consider the region $\mathcal{V} \subset X_1$ given by points $(z, x)$ satisfying $|z - z_0| < B|z_0|^{1/d}$ and where $R < \Lambda \rho^{1/d}$. Let us define new coordinates

$$\hat{x} = z_0^{-1/d} \cdot x, \quad \hat{z} = z_0^{-1/d} (z - z_0),$$

where $\hat{z}$ is given by points $(\hat{z}, \hat{x})$ satisfying $|\hat{z}| < 1$ and $|\hat{x}| \sim D$. The equation for $X_1$ in these coordinates is

$$K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0.$$
and let \( \hat{R} = |z_0|^{-1/d}R \). In addition we let \( \hat{\zeta} = \max \{1, \hat{R}\} \). Note that \( (\hat{z}, \hat{x}) \) satisfy the equation

\[
\hat{z}_0^{1/d-1}\hat{z} + 1 + f(\hat{x}) = 0,
\]

and \( |\hat{z}| < B, \ |\hat{R}| < CA \) for some fixed constant \( C \) (since \( \rho \) is comparable to \( |z_0| \)). We define the map

\[
H : V \to C \times V_1
\]

by letting \( H(\hat{z}, \hat{x}) = (\hat{z}, \hat{x}') \), where \( \hat{x}' \) is the nearest point projection from the solutions of (4.2) to solutions of \( 1 + f(\hat{x}) = 0 \).

**Proposition 7.** Given \( \epsilon, \Lambda > 0 \) if \( A > A(\epsilon, \Lambda, B) \), then we have

\[
|\nabla^i (H^* g_{C \times V_1} - |z_0|^{-2/d}g)||_{L^2} < C \hat{\zeta}^{-i},
\]

for \( i \leq k + 1 \). In terms of weighted spaces we then have

\[
\| |z_0|^{2/d} H^* g_{C \times V_1} - g \|_{C^{k,\alpha}} < \epsilon.
\]

**Proof.** This follows the analysis of our metric in regions IV, V in Proposition 3. Let us focus on region IV, which is the more complicated one. As before, we have \( \rho \in (D/2, 2D), R \in (\Lambda/2, 2\Lambda) \), such that \( \kappa^{-1} D^{1/d} < K < \Lambda D^{1/d} \). Here \( |z_0| \) is comparable to \( \rho \), and \( \hat{\zeta} \) is comparable to \( \hat{R} = |z_0|^{-1/d}R \), so \( \hat{\zeta} \sim |z_0|^{-1/d}K \). The estimate we need to show is therefore

\[
|\nabla^i (|z_0|^{2/d} K^{-2} H^* g_{C \times V_1} - K^{-2}g)||_{K^{-2}g} < \epsilon.
\]

We introduce the coordinates \( \tilde{z}, \tilde{x} \) as in Proposition 3 which in terms of \( \hat{x}, \hat{z} \) are

\[
\tilde{x} = K^{-1} \hat{x}_0^{1/d} \cdot \hat{x}, \quad \tilde{z} = K^{-1} \hat{z}_0^{1/d} \hat{z}.
\]

In these coordinates \( X_1 \) is given by

\[
K^{-d}(K \tilde{z} + z_0) + f(\tilde{x}) = 0.
\]

We need to compare the metric \( \omega \) on \( X_1 \) with the product metric \( C \times V_{K^{-d}z_0} \) on the hypersurface with equation

\[
K^{-d}z_0 + f(\tilde{x}) = 0,
\]

under the closest point projection map. By the same calculations as in the analysis of region IV in Proposition 3 we find that the error between the two metrics is of order \( BD^{1/d-1} \), and this can be made arbitrarily small by taking \( D \) large, since \( d > 1 \). \( \square \)

We will need an extension operator \( C_{\delta,\tau}^{0,\alpha}(\rho^{-1}[A, \infty), \omega) \to C_{\delta,\tau}^{0,\alpha}(X_1, \omega) \). More sophisticated methods could be used to deal with the \( C^{k,\alpha} \) spaces for \( k > 0 \) as well (see e.g. Seeley [27]), but for our purposes \( k = 0 \) suffices.

**Proposition 8.** For sufficiently large \( A \), there is a linear extension operator

\[
E : C_{\delta,\tau}^{0,\alpha}(\rho^{-1}[A, \infty), \omega) \to C_{\delta,\tau}^{0,\alpha}(X_1, \omega),
\]

whose norm is bounded independently of the choice of \( A \).
Proof: The basic observation is that a $C^{0,\alpha}$ function $f$ on a half space $\mathbb{R}^n_+ = \{x_n \geq 0\} \subset \mathbb{R}^n$ can be extended to a function on $\mathbb{R}^n$ by reflection, while preserving the $C^{0,\alpha}$-norm. In addition, multiplying by a cutoff function, we can define an extension $E(f)$ supported in the set $\{x_n > -1\}$, such that $\|E(f)\|_{C^{0,\alpha}} \leq C\|f\|_{C^{0,\alpha}(\mathbb{R}^n)}$. The same applies if instead of a half space, $f$ is defined on the set $\{x_n \geq F(x_1, \ldots, x_{n-1})\}$, where $F$ is a $C^{0,\alpha}$ function on $\mathbb{R}^{n-1}$, and the norm of the extension operator will then also depend on the $C^{0,\alpha}$-norm of $F$.

We can globalize this process to extending functions from $\rho^{-1}(A, \infty)$ to $X_1$, using that near each point in $\rho^{-1}(A)$ we control the geometry of $\omega$ at suitable scales. To do this, let $x \in \rho^{-1}(A)$ and consider the ball $B(x, r_x)$, where $r_x$ is defined as follows:

$$r_x = \begin{cases} \kappa A/10, & \text{if } R > \kappa \rho, \\ R/10, & \text{if } \kappa^{-1} \rho^{1/d} < R < \kappa \rho, \\ A^{1/d}, & \text{if } R < \kappa^{-1} \rho^{1/d}. \end{cases}$$

These radii are chosen so that on $B(x, r_x)$ we have $\rho \omega \sim r_x$. Let us analyze these balls at the scale $r_x$.

- If at $x$ we have $R > \kappa \rho$, then from the analysis of Region I in Proposition 5 we know that the scaled down metric $A^{-2} \omega$ on $B(x, r_x)$ converges to the cone metric on $X_0$ as $A \to \infty$ on a ball of radius $A^{-1}r_x = \kappa/10$ in $X_0$. Moreover this ball is centered at a point $\tilde{x}$ satisfying $\rho(\tilde{x}) = 1$, $R(\tilde{x}) > \kappa$. We therefore control the geometry of the boundary $\rho^{-1}(A)$ uniformly, and we can extend functions from $\rho^{-1}((A, \infty)) \cap B(x, r_x)$ to $B(x, r_x)$ as above.

Note in addition that by the definition of the weighted norms we have

$$\|f\|_{C^{0,\alpha}_\delta(B(x, r_x), \omega)} \sim A^{\delta} \|f\|_{C^{0,\alpha}(B(x, r_x), A^{-2} \omega)},$$

and so we can control the weighted Hölder norm of the extension.

- If at $x$ we have $\kappa^{-1} \rho^{1/d} < R < \kappa \rho$, and $R \in (K/2, 2K)$, then from the analysis of Regions II, III, IV in Proposition 3 we know that the scaled down metric $K^{-2} \omega$ approaches the model metric on $C \times V_s$ for some $s$ with $|s| \leq 1$. In each of these spaces we control the geometry of $\rho^{-1}(A)$, and so we can define an extension map. The weighted Hölder norms here are related to the norms with respect to the rescaled metric by

$$\|f\|_{C^{0,\alpha}_\delta(B(x, r_x), \omega)} \sim A^{\delta - \tau} K^\tau \|f\|_{C^{0,\alpha}(B(x, r_x), K^{-2} \omega)},$$

- In a similar way, if $x$ is in the third region $R < \kappa^{-1} \rho^{1/d}$, then the rescaled metric $A^{-2/d} \omega$ approaches the product metric $C \times V_1$, and again we can define an extension map. The relation between the weighted Hölder norm, and the Hölder norm with respect to the rescaled metric is

$$\|f\|_{C^{0,\alpha}_\delta(B(x, r_x), \omega)} \sim A^{\delta - \tau + \tau/d} \|f\|_{C^{0,\alpha}(B(x, r_x), A^{-2/d} \omega)}.$$

Finally, to globalize these local extensions we can use cutoff functions. □

The next result shows that the tangent cone of $(X_1, \omega)$ at infinity is $X_0 = C \times V_0$.

**Proposition 9.** Let $\epsilon > 0$. If $D$ is sufficiently large, then the Gromov-Hausdorff distance between the regions defined by $\rho \in (D/2, 2D)$ in $(X_1, \omega)$, and in $X_0 = C \times V_0$ with the product metric $\omega_0$, is less than $D\epsilon$. 
Proof. It will be useful to introduce the notation
\[ S_\Lambda = \{ x \in X : R < \Lambda \rho^{1/d} \}, \]
which we should think of as a region near the singular rays in \( X_0 \). Let us denote by \( X_1^D, X_0^D \) the two annular regions that we are considering, with their metrics scaled down by a factor of \( D \). Our goal is to define a map \( G : X_1^D \to X_0^D \), such that for sufficiently large \( D \) we have
\[ |d(G(x_1), G(x_2)) - d(x_1, x_2)| < \epsilon, \]
and the image \( G(X_1^D) \) is \( \epsilon \)-dense in \( X_0^D \).

Let \( \Lambda \) be a large constant. On the set \( X_1^D \setminus S_\Lambda \) we define \( G \) by the nearest point projection, as in Proposition \[ \ref{nearest_point} \] while on the set \( X_1^D \cap S_\Lambda \) we define \( G \) by projection onto the \( z \)-axis. From Proposition \[ \ref{rescaled_metrics} \] we know that on \( X_1^D \setminus S_\Lambda \) the rescaled metrics \( D^{-r}\omega \) and \( D^{-r}G^*\omega_0 \) can be made arbitrarily close by choosing \( \Lambda, D \) large. We can therefore assume that for any curve \( \gamma \) in this region
\[ |\text{length}_{D^{-r}\omega}(\gamma) - \text{length}_{D^{-r}G^*\omega_0}(\gamma)| < \epsilon, \]
and in particular if \( x_1, x_2 \) are in this region, then
\[ d_{X_1^D}(x_1, x_2) < d_{X_0^D}(G(x_1), G(x_2)) + \epsilon. \]

Note that the reverse inequality is not yet clear since the shortest curve between \( x_1, x_2 \) in \( X_1^D \) may pass through \( S_\Lambda \).

At the same time, on \( X_1^D \cap S_{2\Lambda} \), Proposition \[ \ref{approximation} \] shows that after suitable scalings we can approximate \( X_1^D \) by the product metric \( C \times V_1 \). In particular, choosing first \( \Lambda \) and then \( D \) sufficiently large, we can assume that the projection \( \pi : X_1^D \cap S_{2\Lambda} \to C \) onto the \( C \) factor satisfies \( |d\pi| < 1 + \epsilon \). It follows from this that for any curve \( \gamma \) in this region
\[ \text{length}_{X_1^D}(\gamma) \geq (1 - \epsilon)\text{length}_C(\pi \circ \gamma). \]

Note that if \( x_1, x_2 \in S_\Lambda \), then the shortest curve between them in \( X_1^D \) will remain in the region \( S_{2\Lambda} \) since on the “annular” region \( S_{2\Lambda} \setminus S_\Lambda \) the metric can be made arbitrarily close to the cone \( X_0^D \) (if \( \Lambda \) and \( D \) are sufficiently large). It follows using \[ \ref{distance_upper_bound} \] that
\[ d_{X_1^D}(x_1, x_2) > d_C(\pi(x_1), \pi(x_2)) - \epsilon. \]

In addition if we fix a reference point \( o \in V_1 \), then for any \( (p, z) \in X_1^D \cap S_\Lambda \) we have
\[ d_{X_1^D}((p, z), (o, z)) < C \Lambda D^{1/d - 1} < \epsilon, \]
for some constant \( C \) (recall that we choose \( D \) after choosing \( \Lambda \)). It then follows that if \( (p_1, z_1), (p_2, z_2) \) are two points in this region, we have
\[ d_{X_1^D}((p_1, z_1), (p_2, z_2)) < |z_1 - z_2| + \epsilon, \]
and similarly
\[ d_{X_0^D}((p_1, z_1), (p_2, z_2)) < |z_1 - z_2| + \epsilon. \]

We have therefore shown that if \( x_1, x_2 \in S_\Lambda \), then
\[ |d_{X_1^D}(x_1, x_2) - d_{X_0^D}(G(x_1), G(x_2))| < \epsilon. \]

It remains to bound the distance from below between points \( x_1, x_2 \in X_1^D \setminus S_\Lambda \). Let \( \gamma \) be the shortest curve from \( x_1 \) to \( x_2 \). We only have to deal with the possibility
that this shortest curve enters the closure of the region $S_\Lambda$. Let $x_1'$ be the first, and $x_2'$ be the last point along the curve in this region. Then by the observations above, the segment of $\gamma$ joining $x_1', x_2'$ must lie in $S_{2\Lambda}$, and therefore we have

$$d_{X_1^D}(x_1', x_2') > d_C(\pi(x_1'), \pi(x_2')) - \epsilon > d_{X_1^D}(G(x_1'), G(x_2')) - 2\epsilon.$$ 

Then by the triangle inequality and the estimate (4.4) we get

$$d_{X_1^D}(x_1, x_2) > d_{X_1^D}(G(x_1), (x_2)) - \epsilon$$

for a larger value of $\epsilon$. This shows (4.3). The fact that $G(X_1^D)$ is $\epsilon$-dense in $X_0^D$ follows from the inequality analogous to (4.6) for $X_0^D$.\)

It follows from this result in particular that if $o \in X_1$ is a fixed basepoint, then the distance function $d(o, \cdot)$ is uniformly equivalent to $\rho$. Later on we will also need that $(X_1, \omega)$ satisfies the following “relatively connected annuli”, or RCA, condition. This is an easy consequence of the fact that the tangent cone at infinity of $(X_1, \omega)$ is a metric cone over a compact connected length space.

**Proposition 10.** For sufficiently large $D$, any two points $x_1, x_2 \in X_1$ with $d(o, x_i) = D$ can be joined by a curve of length at most $CD$, lying in the annulus $B(o, CD) \setminus B(o, C^{-1}D)$, for a uniform constant $C$.

## 5. The Laplacian on the model spaces

Here we consider two model problems, namely the Laplacians on the cone $X_0 = \mathbb{C} \times V_0$, and on the product $\mathbb{C} \times V_1$. These will be the building blocks for inverting the Laplacian on $X_1$ in Section 6. As a preliminary step we consider the Laplacian on the product $\mathbb{C} \times V_0$, but using weighted spaces which involve a weight function only in the $V_0$ factor.

### 5.1. The model space $\mathbb{C} \times V_0$.

Consider the product metric $g = g_\mathbb{C} + g_{V_0}$, and let $r$ be the distance function on $V_0$ as before. We define weighted spaces on $\mathbb{C} \times V_0$ in the usual way, analogously to (4.1) with the weight function given by $r$. A more concise definition is obtained by conformally scaling $g$ to $r^{-2}g$. In terms of this scaled metric our weighted spaces can be written as

$$C^k_{\tau} = r^{-\tau} C^k_{e^{-2\tau}} ,$$

with corresponding norm given by

$$\|f\|_{C^k_{\tau}} = \|r^{-\tau}f\|_{C^k_{e^{-2\tau}}} .$$

Our goal is to show that the kernel of the Laplacian on $(\mathbb{C} \times V_0, g)$ is trivial in the weighted Hölder space $C^k_{\tau}$, for $\tau \in (4 - 2n, 0)$. Note that the (real) dimension of $V_0$ is $m = 2n - 2$, so $\tau \in (2 - m, 0)$ is the usual “good” range of weights for the Laplacian on $V_0$. We use the Fourier transform in the $\mathbb{C}$ direction in a similar way to what was done by Walpuski [33] which in turn is based on Brendle [2] (see also Mazzeo-Pacard [24]), although the details will be slightly different.

For any function $\chi$ on $\mathbb{C} \times V_0$ with sufficient decay in the $\mathbb{C}$ direction we define the Fourier transform

$$\hat{\chi}(\xi, x) = \int_{\mathbb{C}} \chi(z, x) e^{-\sqrt{-1}\xi \cdot z} dz,$$

where we think of $\mathbb{C}$ as $\mathbb{R}^2$. 
Proposition 11. Let \( \chi \) be a smooth function on \( C \times V_0 \), such that \( \hat{\chi} \) has compact support away from \( \{0\} \times V_0 \). In particular \( \chi \) is supported in \( C \times K \) for a compact \( K \subset V_0 \). We can then find \( f \) solving \( \Delta f = \chi \), and moreover

1. \( f \in C^{k,\alpha}_\tau \) for all \( \tau \in (2-m,0) \),
2. In addition \( f \) decays exponentially in the \( C \) direction. More precisely for any \( a > 0 \) there is a constant \( C \) such that \( \| f \|_{C^{k,\alpha}_{\tau}(|z|>A)} < C(1+A)^{-a} \) for any \( A > 0 \) and \( \tau \in (2-m,0) \).

Proof. Taking the Fourier transform, the equation that we need to solve is

\[
\Delta_{V_0} \hat{f}(\xi, x) - |\xi|^2 \hat{f}(\xi, x) = \hat{\chi}(\xi, x).
\]

For each fixed \( \xi \neq 0 \) we can solve this in polar coordinates on \( V_0 \), using the spectral decomposition for the link of \( V_0 \). It reduces to analyzing ODEs of the form

\[
\partial^2_r \hat{f} + \frac{m-1}{r} \partial_r \hat{f} + \frac{1}{r^2} \lambda \hat{f} - |\xi|^2 \hat{f} = \hat{\chi}_\lambda,
\]

where \( \lambda \leq 0 \) are the eigenvalues of the Laplacian on the link of \( V_0 \). Note that the \( \hat{\chi}_\lambda \) have compact support in \( r \). For \( \xi \neq 0 \) the ODE has a fundamental solution which decays exponentially as \( r \to \infty \), and is bounded near \( r = 0 \). Using this we obtain solutions \( \hat{f}(\xi, x) \) of (5.1) decaying exponentially as \( r \to \infty \), and in particular satisfying bounds \( \| \hat{f}(\xi, \cdot) \|_{C_0^0} \leq C \| \hat{\chi}(\xi, \cdot) \|_{C_0^0} \), where \( C \) is uniform as long as \( \xi \) is bounded.

Our results follow by taking the inverse Fourier transform of \( \hat{f} \). Item (1) above follows from the properties of the ODE solutions. Item (2) follows from the fact that if \( \hat{\chi} \) has compact support away from \( \{0\} \times V_0 \), then \( \hat{f} \) is smooth when viewed as a function \( R^2 \to C_0^0(V_0) \). This can be seen by differentiating (5.1) with respect to \( \xi \). We find that any derivative \( \partial^i_\xi \hat{f} \) satisfies an equation of the form

\[
\Delta_{V_0} \partial^i_\xi \hat{f}(\xi, x) - |\xi|^2 \partial^i_\xi \hat{\chi}(\xi, x) + \sum_{|i|<|d|} a_i(\xi) \partial^i_\xi \hat{f}(\xi, x)
\]

in terms of lower order derivatives. Inductively we find that each partial derivative \( \partial^i_\xi \hat{f}(\xi, x) \) is bounded near \( r = 0 \) and decays exponentially as \( r \to \infty \), using that \( \hat{f}(\xi, x) \) is identically zero for \( \xi \) in a neighborhood of 0.

Taking the inverse Fourier transform we obtain exponential decay in \( |z| \) of the \( C_0^0 \) norms of \( f \) in the vertical slices \( \{z\} \times V_0 \). The decay of the derivatives of \( f \) is obtained from this using Schauder estimates since \( \chi \) is compactly supported in the \( V_0 \) direction. \( \square \)

Corollary 12. Suppose that \( \Delta f = 0 \), for some \( f \in C^{k,\alpha}_{\tau}(C \times V_0) \) with \( \tau \in (2-m,0) \). Then \( f = 0 \).

Proof. We will argue by taking the distributional Fourier transform of \( f \), showing that it is supported at the origin. It will be more convenient to use weighted \( L^2 \)-spaces than the Hölder spaces. For this let \( \tau_1, \tau_2 \) be such that \( \tau_1 < \tau < \tau_2 \), and define a weight function \( \sigma \) on \( V_0 \) such that \( \sigma = r^{\tau_2} \) for large \( r \), and \( \sigma = r^{\tau_1} \) for small \( r \). We define the weighted \( L^2_\sigma \) space on \( V_0 \) using the norm

\[
\| F \|_{L^2_\sigma}^2 = \int_{V_0} |F|^2 \sigma^{-2} r^{-m} \, dV.
\]
Our choice of $\tau_1, \tau_2$ ensures that we can view $f$ as a bounded map

$$f : \mathbb{C} \to L^2_{\sigma}(V_0).$$

In addition the dual space under the $L^2$ pairing is $L^2_{\sigma'}$, where $\sigma' = \sigma^{-1}r^{-2m}$. Let $\hat{f}$ be the Fourier transform of $f$ in the $\mathbb{C}$-direction as above, so now $\hat{f}$ is a distribution valued in $L^2_{\sigma}$. If $g$ is a smooth map $\mathbb{C} \to L^2_{\sigma'}$ of compact support, then the pairing $\hat{f}(g)$ is defined as

$$\hat{f}(g) = \int_{\mathbb{C}} \langle f, \hat{g} \rangle \, dz.$$

We first claim that $\hat{f}$ is a distribution of finite order (in fact order at most 4). For this suppose that $g : \mathbb{C} \to L^2_{\sigma'}$ has compact support $K$, and satisfies $|\nabla^i g| \leq A$ for $i \leq 4$, where the derivatives are in the $\mathbb{C}$-direction. Then the usual integration by parts argument shows that $\|\hat{g}(\xi)\| \leq C_{K,A}(1 + |\xi|)^{-4}$, for a constant depending on $K, A$, and so

$$|\hat{f}(g)| \leq \int_{\mathbb{C}} \|f(z, \cdot)\|_{L^2_{\sigma}} \|\hat{g}(z)\|_{L^2_{\sigma'}} \, dz \leq C_{K,A},$$

since $f$ is bounded.

Next we claim that $\hat{f}$ is supported at the origin, i.e. if $g$ is smooth (in the $\mathbb{C}$-directions) and has compact support away from the origin, then $\hat{f}(g) = 0$. We can approximate $g$ in the space $C^4(\mathbb{C}, L^2_{\sigma'})$ with smooth functions that have compact support away from $\{0\} \times V_0$, and so we assume that $g$ is of this type. We can then apply Proposition 11 to the function $\chi = \hat{g}$, and so we find $h$ on $\mathbb{C} \times V_0$ satisfying $\Delta h = \hat{g}$, such that $h \in C^4_{k,\alpha}$, and $h$ decays exponentially in the $\mathbb{C}$-direction. It follows that

$$\hat{f}(g) = \int_{\mathbb{C} \times V_0} f \hat{g} \, dV = \int_{\mathbb{C} \times V_0} f \Delta h \, dV = \int_{\mathbb{C} \times V_0} h \Delta f \, dV = 0,$$

where the integration by parts is justified by the decay properties of $h$.

Since $\hat{f}$ is supported at the origin, it follows that $\hat{f}$ is a linear combination of derivatives of delta functions at the origin, with coefficients in $L^2_{\sigma}$. This follows the usual argument from the scalar case (see Rudin [26, Theorem 6.25]). Therefore we have

$$f(z) = \sum_{i,j=1}^{l} z^i \bar{z}^j f_{ij},$$

for $f_{ij} \in L^2_{\sigma}$. Since $f$ is bounded, we have $f_{ij} = 0$ unless $i = j = 0$, and so $f$ is independent of the $z$-variable. The result now follows since the ODEs (5.2) with $\xi = 0, \chi_{\lambda} = 0$ have no solutions with growth rates in the range $(2 - m, 0)$. □
5.2. The model space $X_0$. We now consider the cone $X_0$, or rather $X_0 \setminus \mathbb{C} \times \{0\}$. In polar coordinates the corresponding metric is

$$g_{X_0} = dr^2 + r^2 h_L,$$

where $h_L$ is the metric on the link $L$. The link $(L, h_L)$ is incomplete (since we remove the singular rays from $X_0$), and its metric completion $\overline{L}$ is the double suspension of the link of the cone $V_0$, since $X_0 = \mathbb{C} \times V_0$. In particular $\overline{L}$ has a circle of singularities modeled on the cone $V_0$. In the language of [11], $\overline{L}$ is a smoothly stratified space of depth 1, and $h_L$ is an iterated edge metric.

We define weighted Hölder norms on $(L, h_L)$ in terms of the weight function $w$ given by smoothing out the distance from the singular stratum. As above, the weighted norms can be expressed concisely in terms of the conformally scaled metric $w^{-2} h_L$:

$$\|f\|_{C^{k,\alpha}_w} = \|w^{-\tau} f\|_{C^{k,\alpha}_{w^{-2} h_L}}.$$

The norm depends on the precise smoothing chosen, but we end up with equivalent norms. The main result about the Laplacian on $L$ that we need is the following.

**Proposition 13.** Let $a \in \mathbb{C}$, and consider the map

$$\Delta_{h_L} + a : C^{k,\alpha}_\tau(L) \to C^{k-2,\alpha}_{\tau-2}(L),$$

for $\tau \in (2 - m, 0)$ (recall $m = 2n - 2$ is the real dimension of $V_0$). If $\text{Im} a \neq 0$, or if $a \in \mathbb{R}$ avoids a discrete set of values, then $\Delta + a$ is invertible.

**Proof.** This follows from the Fredholm theory for edge operators of Mazzeo [24]. It implies that for our range of weights the image of $\Delta + a$ is $L^2$-orthogonal to its kernel, and moreover $\Delta$ has discrete real spectrum. \qed

We now consider the cone $X_0$ over $L$. It is more convenient to conformally scale the cone metric $g_{X_0}$ to a cylinder, using the radial distance function $r$:

$$\tilde{g}_{X_0} = r^{-2} g_{X_0} = dt^2 + h_L,$$

where $t = \ln r$. The equation $\Delta_{g_{X_0}} f = u$ is then equivalent to

$$\Delta_{h_L} f + \partial_t^2 f + (2n - 2) \partial_t f = e^{2t} u.$$

The relevant weighted Hölder norms, compatible with the definition [11] can be formulated in terms of the further conformal scaling $w^{-2} \tilde{g}_{X_0}$, using the weight function $w$ on $L$ above. We then define

$$\|f\|_{C^{k,\alpha}_{\tau}} = \|e^{-\delta t} w^{-\tau} f\|_{C^{k,\alpha}_{w^{-2} \tilde{g}_{X_0}}}.$$

Our goal is to show that the map

$$\mathcal{L} = \Delta_{h_L} + \partial_t^2 + (2n - 2) \partial_t : C^{k,\alpha}_{\delta,\tau} (\mathbb{R} \times L) \to C^{k-2,\alpha}_{\delta-2,\tau-2} (\mathbb{R} \times L)$$

is invertible. It is convenient to conjugate the operator by $e^{\delta t}$, to reduce to the case when $\delta = 0$. Writing $\mathcal{L}_\delta(f) = e^{\delta t} \mathcal{L}(e^{-\delta t} f)$, we have

$$\mathcal{L}_\delta = \Delta_{h_L} + \partial_t^2 + (2n - 2 - 2\delta) \partial_t + (\delta^2 - (2n - 2)\delta),$$

and our main result is the following.
Proposition 14. For $\delta$ avoiding a discrete set of indicial roots, and $\tau \in (2 - m, 0)$, the operator

$$L_\delta : C^{k,\alpha}_\tau(R \times L) \to C^{k-2,\alpha}_{\tau-2}(R \times L)$$

is invertible. Here we write $C^{k,\alpha}_\tau = C^{k,\alpha}_{0,\tau}$ for simplicity.

We first have the following result, analogous to Proposition 11, except we do not need the support to avoid $\{0\} \times L$.

Proposition 15. Suppose that $\delta$ avoids a discrete set of indicial roots. Let $\chi$ be smooth on $R \times L$, such that $\hat{\chi}$ has compact support. We can then find $f$ solving $L_\delta f = \chi$, such that

1. $f \in C^{\tau,\alpha}$ for any $\tau < 0$.
2. For any $a > 0$ there is a constant $C$ such that $\|f(t, x)\|_{C^{\tau,\alpha}(|t| > A)} < C(1 + A)^{-a}$ for any $A > 0$.

The same applies to the operator

$$L_\delta^* = \Delta h_L + \partial_t^2 - (2n - 2 - 2\delta)\partial_t + (\delta^2 - (2n - 2)\delta),$$

which is the adjoint of $L_\delta$.

Proof. The proof is similar to that of Proposition 11. Taking the Fourier transform in the $t$ variable we need to solve the equations

$$\Delta h_L \hat{f} - [\xi^2 \hat{f} - \sqrt{\xi}(2n - 2 - 2\delta) - \delta^2 + (2n - 2)\delta] \hat{f} = \hat{\chi}.$$ 

This equation is of the form $(\Delta h_L + a) \hat{f} = \hat{\chi}$. Suppose that $\delta \neq n - 1$. If $\xi \neq 0$ then we have $\text{Im} a \neq 0$, while if $\xi = 0$, then $a = \delta^2 - (2n - 2)\delta$. Therefore if $\delta$ is generic, we can apply Proposition 13 to find $\hat{f}$ no matter what $\xi$ is. In addition, since $C^{\tau,\alpha}_\tau \subset C^{\tau',\alpha}_{\tau'}$ if $\tau > \tau'$, we have that $f \in C^{\tau}_\tau$, for any $\tau' < 0$. The Schauder estimates then imply that $\hat{f} \in C^{\tau}_\tau$, for any $k > 0$ as well, showing the property (1). To see (2) we claim that $\hat{f}$ is a smooth function of $\xi$. This can be seen by differentiating the equation as we did in Proposition 11. Inductively we find that each derivative $\partial_\xi^k \hat{f}$ satisfies an equation of the form

$$\Delta h_L \partial_\xi^k \hat{f} + a(\xi) \partial_\xi^k \hat{f} = g(\xi),$$

where $g(\xi)$ has compact support in $\xi$, and $g(\xi) \in C^{\tau,\alpha}_\tau$ for all $\tau < 0$. In addition $a(\xi)$ is such that Proposition 13 applies. In this way we can bound the $\xi$-derivatives of $\hat{f}$, and so the inverse Fourier transform $f$ will have the required decay.

This has the following corollary.

Corollary 16. Suppose that $f \in C^{k,\alpha}_\tau$ satisfies $L_\delta f = 0$ for some generic $\delta$, and $\tau \in (2 - m, 0)$. Then $f = 0$.

Proof. Suppose that $f$ is nonzero. We can then find $\chi$ with $\hat{\chi}$ having compact support such that $\int f \chi \neq 0$. We apply Proposition 13 to solve $L_\delta h = \chi$, with $h \in C^{k,\alpha}_{\tau'}$, where $\tau'$ is chosen so that $2 - m - \tau < \tau' < 0$. We then have

$$\int_{R \times L} f \chi dV = \int_{R \times L} f L_\delta^* h dV = \int_{R \times L} (L_\delta f) h dV = 0,$$
where the integration by parts is justified by the decay properties of $h$ and the choice of $\tau'$. This contradicts our choice of $\chi$. \qed

We next use a standard blowup argument to obtain the following.

**Proposition 17.** Let $\tau \in (2 - m, 0)$, and $\delta$ generic. There exists a constant $C$ such that for any $f \in C^{k,\alpha}_\tau(\mathbb{R} \times L)$ we have

$$\|f\|_{C^{k,\alpha}_\tau} \leq C\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}}.$$  

In particular $\mathcal{L}_\delta$ has closed range.

**Proof.** Note first that since the metric $w^{-2}\tilde{g}_{X_0}$ has bounded geometry, we can use the Schauder estimates to obtain

$$\|f\|_{C^{k,\alpha}_\tau} \leq C(\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}} + \|f\|_{C^{0,\alpha}_{\sigma}}).$$

If follows that it is enough to prove that

$$\|f\|_{C^{0,\alpha}_{\sigma}} \leq C\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}}$$

for a uniform constant $C$.

We argue by contradiction. Suppose that we have a sequence of functions $f_i \in C^{k,\alpha}_\tau$ with $\|f_i\|_{C^{0,\alpha}_{\sigma}} = 1$, but $\|\mathcal{L}_\delta f_i\|_{C^{k-2,\alpha}_{\tau-2}} < 1/i$. We can find points $(t_i, x_i) \in \mathbb{R} \times L$ such that

$$|f_i(t_i, x_i)| > \frac{1}{2} w(x_i)^{\tau}.$$  

By translating in the $t$ direction we can assume that $t_i = 0$ for each $i$.

There are two possibilities. If $w(x_i)$ is bounded away from zero, then by choosing a subsequence we can assume that $x_i \to x$ for some $x \in L$. The Schauder estimate \[5.3\] implies that choosing a further subsequence we can assume that the $f_i$ converge locally in $C^{k,\alpha'}$ to a limit $f \in C^{k,\alpha}_\tau$, which then must satisfy $\mathcal{L}_\delta f = 0$. Corollary \[14\] implies that $f = 0$, which contradicts that $|f(x)| \geq \frac{1}{2} w(x)^{\tau}$.

The other possibility is that $w(x_i) \to 0$. Consider the rescaled metrics $w(x_i)^{-2}\tilde{g}_{X_0}$. If we take the pointed limit of $\mathbb{R} \times L$ with these rescaled metrics, based at the points $(0, x_i)$, then (up to choosing a subsequence) we obtain the limit space $\mathbb{R}^2 \times V_0$, with the product metric $\omega_{Euc} + \omega_{V_0}$, with basepoint $(0, x)$ for some $x \in V_0$ at distance 1 from the vertex. This is just the statement that $h_L$ is modeled on $S^1 \times V_0$ near the circle of singularities.

Under taking this pointed limit the rescaled functions $w(x_i)^{-\tau}f_k$ converge locally in $C^{k,\alpha'}$ to a limit $f \in C^{k,\alpha}_\tau(\mathbb{R}^2 \times V_0)$, satisfying $\Delta f = 0$. Corollary \[12\] implies that $f = 0$, contradicting $|f(0, x)| \geq 1/2$. \qed

We can finally prove Proposition \[14\].

**Proof of Proposition \[14\].** It is enough to show that

$$\mathcal{L}_\delta : C^{k,\alpha}_\tau(\mathbb{R} \times L) \to C^{k-2,\alpha}_{\tau-2}(\mathbb{R} \times L)$$

is surjective, since Corollary \[16\] implies that it is injective. While we already know that the image is closed, and moreover Proposition \[14\] provides us with many elements in the image, these functions do not form a dense set in $C^{k-2,\alpha}_{\tau-2}$, so we cannot immediately conclude.
Instead let \( u \in C^{k-2,\alpha}_{\tau^{-2}}(\mathbb{R} \times L) \). We can find a sequence of smooth \( \chi_i \) such that \( \hat{\chi}_i \) has compact support, \( \chi_i \to u \) locally uniformly and moreover
\[
\|\chi_i\|_{C^{k-2,\alpha}_{\tau^{-2}}} < C
\]
for a constant \( C \) depending on \( u \). By Proposition \[15\] we can find \( f_i \in C^{k,\alpha}_{\tau}(\mathbb{R} \times L) \) such that \( L_\delta f_i = \chi_i \), and by Proposition \[17\] we have
\[
\|f_i\|_{C^{k,\alpha}_{\tau}} < C,
\]
for \( C \) independent of \( i \). Up to choosing a subsequence we can take a limit \( f_i \to f \), with the convergence holding locally in \( C^{k,\alpha'} \), and such that \( \|f\|_{C^{k,\alpha}_{\tau}} \leq C \). The limit satisfies \( L_\delta f = u \), and so \( L_\delta \) is surjective. \( \square \)

5.3. The model space \( C \times V_1 \). Let us now move on to the Laplacian on \( C \times V_1 \). Here the relevant weighted spaces are defined in terms of the weight function \( \zeta \) on \( V_1 \), which is a smoothed out version of \( \max\{1, d(\cdot, o)\} \) for a point \( o \in V_1 \). As above, the weighted Hölder spaces can be defined in terms of a conformally scaled version of the product metric \( g = g_{Euc} + g_{V_1} \):
\[
\|f\|_{C^{k,\alpha}_{\tau}} = \|\zeta^{-\tau} f\|_{C^{k,\alpha}_{\zeta^{-\tau}}}.
\]
The weighted norms on \( V_1 \) are defined analogously.

We first have the following result, analogous to Proposition \[13\]

**Proposition 18.** Let \( \tau \in (2 - m, 0) \), \( \lambda \geq 0 \), and let \( u \) be smooth with compact support on \( V_1 \). We can find a smooth function \( f \) on \( V_1 \) such that
\[
\Delta_{V_1} f - \lambda f = u,
\]
and in addition we have an estimate \( |f| \leq C \zeta^\tau \) for a constant that depends on \( \|u\|_{C^{k-2,\alpha}_{\tau^{-2}}} \) and \( \tau \), but not on \( \lambda \). Recall that here \( m = 2n - 2 \) is the dimension of \( V_1 \).

**Proof.** When \( \lambda = 0 \), this follows from the standard theory for the Laplacian acting in weighted spaces on the asymptotically conical manifold \( V_1 \) (see e.g. Lockhart-McOwen \[23\]):
\[
\Delta_{V_1} : C^{k,\alpha}_{\tau}(V_1) \to C^{k-2,\alpha}_{\tau^{-2}}(V_1).
\]
For our choice of weight the Laplacian is self-adjoint, and moreover any decaying element in \( f \in \ker \Delta_{V_1} \) decays at the rate of at least \( d(\cdot, o)^2 \). It follows by integration by parts that \( f \) is constant, since it decays, we have in fact \( f = 0 \). Hence \( \Delta_{V_1} \) is invertible.

When \( \lambda > 0 \), then we can also solve Equation \[57\] using that \( \Delta_{V_1} - \lambda \) is an essentially self-adjoint operator on \( L^2 \), whose kernel is trivial. Moreover using the Schauder estimates it follows that the solution \( f \) decays faster than any inverse power of \( \zeta \). What remains is to obtain a uniform estimate for this decay, independent of \( \lambda \) (in particular as \( \lambda \to 0 \)).

For this, we first let \( b \in C^{k,\alpha}_{\tau} \) be a solution of
\[
\Delta_{V_1} b = -\zeta^{\tau-2}.
\]
By the maximum principle we have that \( b > 0 \). We claim that for sufficiently large \( C \), depending on \( \|u\|_{C^{k-2,\alpha}_{\tau^{-2}}} \), we have an estimate \( |f| < C b \), independent of \( \lambda \). Let us show that \( f < C b \): if this estimate were to fail, then the function \( f - C b \) would
achieve a maximum at some point \(x_{\text{max}}\), using the fast decay of \(f\). In particular 
\(f(x_{\text{max}}) > 0\), and at the same time by the maximum principle
\[
0 \geq \Delta (f - Cb)(x_{\text{max}}) = u(x_{\text{max}}) + \lambda f(x_{\text{max}}) + C\zeta^{\tau-2}(x_{\text{max}})
\]
\(> u(x_{\text{max}}) + C\zeta^{\tau-2}(x_{\text{max}})\).

If \(C\) is chosen large depending on \(\|u\|_{C_0^{\tau-2}}\), this is a contradiction. In a similar way one can prove that \(f > -Cb\) for the same \(C\).

The next result is analogous to Proposition 15.

**Proposition 19.** Suppose that \(\chi\) is a smooth function on \(\mathbf{C} \times V_1\) such that \(\hat{\chi}\) has compact support. We can then solve the equation \(\Delta f = \chi\) such that

1. \(f \in C^{k,\alpha}_\tau\) for any \(\tau > 2 - m\).
2. If \(\hat{\chi}\) is supported away from \(\{(0) \times V_1\), then in addition for any \(a > 0\) there is a constant \(C\) such that \(\|f\|_{C^{k,\alpha}_\tau(|z| > A)} < C(1 + A)^{-a}\) for any \(A > 0\).

**Proof.** The proof is similar to the proofs of Propositions 11, 15. After Fourier transforming, the relevant equations are

\[
(5.5) \Delta V_1 \hat{f} - |\xi|^2 \hat{f} = \hat{\chi}.
\]

Proposition 18 implies that we can solve these equations with uniform estimates on \(\|\hat{f}\|_{C_0^0}\). Taking the inverse Fourier transform we obtain a solution of \(\Delta f = \chi\), with \(f \in C^{k,\alpha}_0\). The Schauder estimates then imply that \(f \in C^{k,\alpha}_\tau\).

In order to get the decay property (2), we can argue as in Propositions 11, 15, by differentiating Equation (5.5). For \(\xi \neq 0\), the solutions of (5.5) decay faster than any inverse power of \(\zeta\) in the \(V_1\)-direction, and so inductively we find that each derivative \(\partial_\xi \hat{f}\) has the same decay. Therefore \(\hat{f}\) is a smooth function \(\mathbf{C} \to C^0(V_1)\) with compact support, and from this we obtain the required decay for \(f\) in the \(C\) direction.

We can next follow the proofs of Corollary 12 and Proposition 17 closely to prove

**Proposition 20.** Let \(\tau \in (2 - m, 0)\). There exists a constant \(C\) such that for any \(f \in C^{k,\alpha}_\tau(\mathbf{C} \times V_1)\) we have

\[
\|f\|_{C^{k,\alpha}_\tau} \leq C\|\Delta f\|_{C^{k-2,\alpha}_{\tau-2}}.
\]

Finally we can prove the following.

**Proposition 21.** The Laplacian

\[
\Delta_{\mathbf{C} \times V_1} : C^{k,\alpha}_\tau(\mathbf{C} \times V_1) \to C^{k-2,\alpha}_{\tau-2}(\mathbf{C} \times V_1)
\]

is invertible for \(\tau \in (2 - m, 0)\).

**Proof.** The proof of this, based on Corollary 12 and Propositions 19 and 20 is essentially identical to the proof of Proposition 14. \(\square\)
In this section we study the mapping properties of the Laplacian in the weighted spaces that we have defined. The main result is the following.

**Proposition 22.** Suppose that we choose \( \tau \in (4 - 2n, 0) \) and \( \delta \) avoids a discrete set of indicial roots. For sufficiently large \( A > 0 \) the Laplacian

\[
\Delta : C^{2,\alpha}_{\delta,\tau}(\rho^{-1}[A, \infty), \omega) \to C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A, \infty), \omega)
\]

is surjective with inverse bounded independently of \( A \).

The proof of this result will take up the remainder of this section. The strategy is to construct an approximate inverse for the Laplacian by localizing the problem on the different regions of \( \rho^{-1}[1, \infty) \) for sufficiently large \( A \), studied in Propositions 6 and 7, and then using the inverses constructed on corresponding model spaces in Section 5.

Suppose that we have a function \( u \in C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A, \infty), \omega) \), with norm \( \|u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} < 1 \). The goal is to construct, once \( A \) is sufficiently large, a function \( f = P u \) on \( X_1 \) with \( \|f\|_{C^{2,\alpha}_{\delta,\tau}(X_1, \omega)} < C \) for a uniform \( C \) such that

\[
\|\Delta f - u\|_{C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A, \infty), \omega)} < \frac{1}{2}.
\]

Then the operator \( P \) is an approximate inverse for \( \Delta \), so \( \Delta P \) is invertible, and as a consequence \( \Delta \) has a bounded right inverse.

Using the extension map defined in Proposition 8, we can assume that \( u \) is actually defined on all of \( X_1 \) and

\[
\|u\|_{C^{0,\alpha}_{\delta-2,\tau-2}(X_1, \omega)} < C,
\]

for a constant \( C \) independent of \( A \). We will decompose \( u \) using cutoff functions into various different pieces. Let us recall the cutoff functions \( \gamma_1, \gamma_2 \) from before, so that \( \gamma_1(s) \) is supported where \( s > 1 \), and \( \gamma_1 + \gamma_2 = 1 \). Let us choose a large number \( \Lambda \), and write \( u = u_1 + u_2 \), where

\[
u_i = \gamma_i(R\Lambda^{-1}\rho^{-1/d})u.
\]

So \( u_1 \) is supported on the set where \( R > \Lambda \rho^{1/d} \). Let us use the notation \( U = \{R > \Lambda \rho^{1/d}\} \cap \{\rho > A\} \) from Proposition 6. We then have a map \( G : U \to X_0 \) such that

\[
\|G^*\omega_{X_0} - \omega\|_{C^{k,\alpha}_{0,0}} < \epsilon.
\]

We can use the map \( G \) to view \( u_1 \) (at least on the set where \( \rho > A \)) as a function on \( X_0 \), supported away from \( C \times 0 \). Proposition 14 implies

**Proposition 23.** As long as \( \tau \in (4 - 2n, 0) \), and \( \delta \) avoids a discrete set of indicial roots, the Laplacian

\[
\Delta_{X_0} : C^{k,\alpha}_{\delta,\tau}(X_0) \to C^{k-2,\alpha}_{\delta-2,\tau-2}(X_0)
\]

is invertible.

We can therefore define \( Pu_1 \) on \( X_0 \) satisfying \( \Delta_{X_0} Pu_1 = u_1 \), and satisfying the estimate

\[
\|Pu_1\|_{C^{2,\alpha}_{\delta,\tau}(X_0)} < C.
\]
for a uniform $C$. In order to transfer this function back to $X_1$, we use another cutoff function

$$\beta_1 = \gamma_1 \left( \frac{\ln(R\Lambda^{-1/2} \rho^{-1/d})}{\ln \Lambda^{1/4}} \right).$$

This has the property that $\beta_1 = 0$ on the region where $R < \Lambda^{3/4} \rho^{1/d}$, while $\beta_1 = 1$ on the support of $\gamma_1(R\Lambda^{-1/2} \rho^{-1/d})$, which is where $u_1$ is supported. Moreover in our weighted spaces we have the estimate

$$\| \nabla \beta_1 \|_{C^{k,\alpha}_{1,1,-1}(\rho^{-1}(1,\infty) \cap X_0)} < \frac{C}{\ln \Lambda}.$$  

We have

$$\Delta_{X_0}(\beta_1 Pu_1) = u_1 + 2\nabla \beta_1 \cdot \nabla (Pu_1) + (\Delta_{X_0} \beta_1) Pu_1,$$

and so using the multiplication properties

$$\| f g \|_{C^{k,\alpha}_{\infty,\infty}^{a,b,c,d}} \leq C \| f \|_{C^{k,\alpha}_{a,\infty}^{b,c}} \| g \|_{C^{k,\alpha}_{b,d}}$$

together with (6.2) we obtain

$$\| \Delta_{X_0}(\beta_1 Pu_1) - u_1 \|_{C^{0,\alpha}_{a-2,r-2}(\rho^{-1}(1,\infty) \cap X_0)} < \frac{C}{\ln \Lambda}.$$  

Using the map $G$ again to view $\beta_1 Pu_1$ as a function on $X_1$, and using (6.1) to compare the Laplacians on $X_0$ and $X_1$, we find that

$$\| \Delta_{\omega}(\beta_1 Pu_1) - u_1 \|_{C^{0,\alpha}_{a-2,r-2}(\rho^{-1}(1,\infty))} < \epsilon,$$

once $\Lambda$ and $A$ are sufficiently large.

We next need to examine the piece $u_2$, which is supported in the region where $R < 2\Lambda \rho^{1/d}$, and we are assuming that in addition $\rho > A$. Note that here we have $\rho \sim |z|$. Geometrically this region can be thought of roughly as a fibration over the set $\{ |z| > A \} \subset \mathbb{C}$, whose fiber over $z$ is the region $\{ r < 2\Lambda \} \subset V_1$, scaled down by a factor of $|z|^{1/d}$. We decompose $u_2$ into pieces whose supports are localized in the $z$-plane. Proposition 21 tells us that suitably scaled, on these regions we can approximate our space with a corresponding region in the product $\mathbb{C} \times V_1$. By Proposition 21 we can invert the Laplacian there.

Let us choose a large $B > 0$. We construct cutoff functions $\chi_i$ on $\mathbb{C}$ such that $\sum \chi_i = 1$ on the set where $|z| > A$ as follows. Consider $(\mathbb{C}, \tilde{g})$, where $\tilde{g} = B^{-2} |z|^{-2/d} g_{Euc}$ is a conformal scaling of the Euclidean metric. Since $d > 1$, on the set where $|z| > A$ for sufficiently large $A$, this metric is close to being Euclidean on larger and larger scales. We can then cover this region with disks of radius 2 (in the metric $\tilde{g}$) centered at points $z_i$, such that the corresponding disks of radius 1 are disjoint, and define the cutoff functions $\chi_i$ supported in the disks of radius 2, and equal to 1 on the disks of radius 1. Scaling back the metric on $\mathbb{C}$ we have that $\chi_i = 1$ on the ball of radius $B|z_i|^{1/d}$ around $z_i$, and $\chi_i = 0$ outside of the ball of radius $2B|z_i|^{1/d}$ around $z_i$. In addition $|\nabla^l \chi_i|_{g_{Euc}} = O(B^{-l}|z_i|^{-l/d})$ for all $l \geq 0$.

We define another set of cutoff functions $\tilde{\chi}_i$ in a similar way, which we will use to transfer our local solutions back to $X_1$ (analogous to $\beta_1$ above). The $\tilde{\chi}_i$ equal 1 on the support of $\chi, i$, and are supported in the balls of radius $3B|z_i|^{1/d}$. Furthermore $|\nabla^l \tilde{\chi}_i| = O(B^{-l}|z_i|^{-l/d})$ as well. An additional important property of these cutoff functions, which can be seen more clearly in terms of the conformally scaled metric
\[ \tilde{g}, \text{ is that any } z \text{ with } |z| > A \text{ is in the support of only a fixed bounded number } N \text{ of the } \tilde{\chi}_i (N \text{ is independent of the choices of large } B, A). \]

We now decompose \( u_2 \) into the pieces \( u_2 = \sum \chi_i u_2 \), at least on the region where \(|z| > A\). By construction the function \( \chi_i u_2 \) is supported on a region where \( z \in B(z_i, 2B |z_i|^{1/d}) \) for a point \( z_i \in C \), and in addition \( R < 2\Lambda |z|^{1/d} \). By Proposition \( 7 \) on this region the scaled metric \( |z|^{-2/d} \omega \) can be approximated by the product metric \( \omega_C \times \omega_{\gamma_i} \) on \( C \times V_1 \), using the map \( H \). Here we have the following result from Section 5.

**Proposition 24.** The Laplacian

\[ \Delta_{C \times V_1} : C^{k, \alpha}(C \times V_1) \rightarrow C^{k-2,\alpha}_{-2}(C \times V_1) \]

is invertible for \( \tau \in (4 - 2n, 0) \).

In order to apply this, we need to view \( \chi_i u_2 \) as a function on \( C \times V_1 \), using the map \( H \), and relate its weighted norm in \( C^{0, \alpha}_{-2} \) on \( C \times V_1 \) to the norm in \( C^{0, \alpha}_{-2, \tau} \) on \( X_1 \). For this, note that on our region we have \( \rho \sim |z_i| \), and so by the definition of the weight function \( \omega \) we have

\[
\begin{align*}
\rho^{-2}w^{-2} & \sim \max\{|z_i|^{1/d}, R\}^{-2} \\
\rho^{\delta-2}w^{\tau-2} & \sim |z_i|^{\delta-\tau} \max\{|z_i|^{1/d}, R\}^{\tau-2}.
\end{align*}
\]

At the same time the weight function \( \zeta \) used to define the weighted spaces on \( C \times V_1 \) (using the notation from Proposition \( 7 \)), is comparable to \( \max\{1, |z_i|^{-1/d} R\} \). It follows that the estimate \( \| u_2 \|_{C^{0, \alpha}_{-2, \tau}} < C \) on \( X_1 \) translates to

\[
(6.5) \quad \| u_2 \|_{C^{0, \alpha}_{-2, \tau}(C \times V_1)} < C|z_i|^{\delta-\tau+\frac{\tau-2}{\alpha}}
\]
on the support of \( \chi_i \). Note also that by construction, on the support of \( u_2 \) we have

\[
(6.6) \quad |\nabla^l \chi_i| < CB^{-l} < CB^{-l} \Lambda |\zeta|^{-l},
\]

thinking of \( \chi_i \) as a function on \( C \times V_1 \) and using the Euclidean metric on \( C \) (since \( \zeta < C \Lambda \)). It follows that once \( B \) is sufficiently large, depending on \( \Lambda \), the estimate \( (6.5) \) implies

\[
\| \chi_i u_2 \|_{C^{0, \alpha}_{-2}(C \times V_1)} < C|z_i|^{\delta-\tau+\frac{\tau-2}{\alpha}}.
\]

We now apply Proposition \( 24 \) but note that \( \Delta_{|z_i|^{-2/d} \omega} = |z_i|^{2/d} \Delta_{\omega} \). We therefore use the Proposition to define \( P(\chi_i u_2) \) by

\[
\Delta_{C \times V_1} P(\chi_i u_2) = |z_i|^{2/d} \chi_i u_2,
\]
satisfying the bound

\[
\| P(\chi_i u_2) \|_{C^{2, \alpha}} < C|z_i|^{\delta-\tau+\frac{\tau}{\alpha}}.
\]

We need to transfer this function back to the manifold \( X_1 \). Note that in terms of the coordinate \( \tilde{z}_i \) from Proposition \( 7 \) on the \( C \) factor and the distance function \( \zeta \) on \( V_1 \), the function \( \chi_i u_2 \) is supported in the region where \( |\tilde{z}_i| < 2B \) and \( \zeta < 4\Lambda \). We use the cutoff functions \( \tilde{\chi}_i \) from above, which equal 1 on the supports of \( \chi_i \), and are supported, in these coordinates, where \( |\tilde{z}_i| < 3B \). In addition we use the cutoff function

\[
\beta_2 = \gamma_2 \left( \frac{\ln(4\Lambda^{-1})}{\ln \Lambda} \right)^\jmath,
\]

where

\[
\gamma_2 = \frac{\ln(4\Lambda^{-1})}{\ln \Lambda}.
\]
which equals 1 where \( \hat{\zeta} < 4\Lambda \), vanishes where \( \hat{\zeta} > 4\Lambda^2 \) and has the property that in our weighted spaces

\[
\|\nabla \beta_2\|_{L^{1,\alpha}(\mathbb{X})} < \frac{C}{\ln \Lambda}.
\]

We now need an estimate analogous to (6.3) for the difference

\[
\Delta_{C \times V_1}(\tilde{\chi}_i \beta_2 P(\chi_i u_2)) - |z_i|^{2/d} \chi_i u_2 = 2\nabla(\beta_2 \tilde{\chi}_i) \cdot \nabla P(\chi_i u_2) + \Delta_{C \times V_1}(\beta_2 \tilde{\chi}_i) P(\chi_i u_2).
\]

Using the estimate (6.7) for \( \beta_2 \) and an estimate analogous to (6.6) for \( \tilde{\chi}_i \) (with \( \Lambda \) replaced by \( \Lambda^2 \) and \( B \) chosen correspondingly larger), we find that

\[
\|\Delta_{C \times V_1}(\tilde{\chi}_i \beta_2 P(\chi_i u_2)) - |z_i|^{2/d} \chi_i u_2\|_{C_{\delta-2,\tau-2}^{\alpha,\alpha}(C \times V_1)} < C\epsilon |z_i|^{-\tau+\frac{\delta}{2}}.
\]

Proposition 7 allows us to estimate the difference between \( |z_0|^{-2/d} \omega \) and the product metric on \( C \times V_1 \) under the identification using the map \( H \), and this leads to

\[
\|\Delta_{|z_i|^{-2/d} \omega}(\tilde{\chi}_i \beta_2 P(\chi_i u_2)) - |z_i|^{2/d} \chi_i u_2\|_{C_{\delta-2,\tau-2}^{\alpha,\alpha}(C \times V_1)} < C\epsilon |z_i|^{-\tau+\frac{\delta}{2}}.
\]

Dividing through by \( |z_i|^{2/d} \), and translating the estimate back to our weighted spaces on \( X_1 \), we get

\[
\|\Delta_{\omega}(\tilde{\chi}_i \beta_2 P(\chi_i u_2)) - \chi_i u_2\|_{C_{\delta-2,\tau-2}^{\alpha,\alpha}} < C\epsilon.
\]

Finally we define

\[
Pu = \beta_1 Pu_1 + \sum_i \beta_2 \tilde{\chi}_i P(\chi_i u_2).
\]

We use the estimates (6.4) and (6.8), together with the observation that any given point is contained in at most a fixed number of our regions, to deduce that

\[
\|\Delta(Pu)-u\|_{C_{\delta-2,\tau-2}^{\alpha,\alpha}} < C\epsilon < 1/2,
\]

for sufficiently small \( \epsilon \), which by the above discussion we can achieve by first choosing \( \Lambda \), then \( B \), and finally \( A \) sufficiently large. In this case \( \Delta P \) is invertible, with \( \|\Delta P\|^{-1} < 2 \). Since by construction the operator \( P \) has bounded norm independent of \( A \), we have constructed a right inverse \( P(\Delta P)^{-1} \) for \( \Delta \) for sufficiently large \( A \), with norm independent of \( A \). This completes the proof of Proposition 22.

7. Calabi-Yau metrics on \( \mathbb{C}^n \)

In Section 3 we wrote down a form \( \omega \) on the hypersurface \( X_1 \), whose Ricci potential decays in a suitable weighted space by Proposition 5. Our goal is to use the linear theory developed in Section 6 to improve the decay of the Ricci potential enough to be able to apply Hein [17, Proposition 4.1] to construct a global Calabi-Yau metric on \( X_1 \). Since we will use a similar method in Section 8 below, we will actually perturb \( \omega \) to a metric \( \tilde{\omega} \) which is Calabi-Yau on the set \( \rho^{-1}[A, \infty) \) for sufficiently large \( A \).

**Proposition 25.** Suppose that \( A \) is sufficiently large, \( \tau < 0 \) is sufficiently close to 0, and \( \delta < 2/d \) is as in Proposition 8. Then there exists a small \( \epsilon \in C_{\delta-\tau}^{\alpha,\alpha} \) such that

\[
(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = (\sqrt{-1})^n \Omega \wedge \bar{\Omega}
\]

on the set \( \rho^{-1}[A, \infty) \).
Proof. Note first that if \( \|u\|_{C^{0,\alpha}_{2,2}} < \epsilon \), then by the definition of the weighted norms we have

\[
|\nabla^2 u|_\omega < C\epsilon \rho^{\delta - 2} w^{\tau - 2}.
\]

Since \( w > C^{-1} \rho^{1/d - 1} \), this implies

\[
|\nabla^2 u|_\omega < C\epsilon \rho^{\delta - 2/d - \tau(1 - 1/d)}.
\]

If \( \tau \) is close to zero, and \( \delta < 2/d \), then for sufficiently small \( \epsilon \), the form \( \omega + \sqrt{-1} \partial \bar{\partial} u \) defines a metric uniformly equivalent to \( \omega \). This is the reason for our requirement that \( \delta < 2/d \) in Proposition 5.

Let us define

\[
\mathcal{B} = \{ u \in C^{\delta,\alpha}_{2,2} : \|u\|_{C^{\delta,\alpha}_{2,2}} \leq \epsilon_0 \},
\]

where \( \epsilon_0 \) is sufficiently small so that \( \omega + \sqrt{-1} \partial \bar{\partial} u \) is uniformly equivalent to \( \omega \). Let us define the operator

\[
F : \mathcal{B} \to C^{\delta,\alpha}_{2,2}(\rho^{-1}[A, \infty))
\]

\[
u \mapsto \log \left( \frac{(\omega + \sqrt{-1} \partial \bar{\partial} u)^n}{(\sqrt{-1})^n \Omega \wedge \Omega} \right) \rho^{-1}[A, \infty).
\]

Our goal is to find \( u \in \mathcal{B} \) such that \( F(u) = 0 \).

Let us write

\[
F(u) = F(0) + \Delta \omega \cdot u + Q(u)
\]

for a suitable nonlinear operator \( Q \). Denoting by \( P \) the right inverse for \( \Delta \) found in Proposition 22, it is enough to solve

\[
F(u) = P(-F(0) - Q(u)),
\]

i.e., we are looking for a fixed point of the map \( N(u) = P(-F(0) - Q(u)) \). Note that we have a uniform bound for \( P \) independent of \( A \) (for sufficiently large \( A \)), since the right inverse for one choice of \( A \) also provides a right inverse for all larger choices of \( A \). In addition either from an explicit formula for \( Q \), or from differentiating Equation (7.1) and estimating the difference \( \Delta \omega + \sqrt{-1} \partial \bar{\partial} u - \Delta \omega + \sqrt{-1} \partial \bar{\partial} u \), we see that as long as \( u, v \in \mathcal{B} \) we have the estimate

\[
\|Q(u) - Q(v)\|_{C^{\delta,\alpha}_{2,2},2} \leq C(\|u\|_{C^{\delta,\alpha}_{2,2}} + \|v\|_{C^{\delta,\alpha}_{2,2}})\|u - v\|_{C^{\delta,\alpha}_{2,2},2}.
\]

For this note that the \( C^{\delta,\alpha}_{2,2} \) norms of \( u \) controls \( \|Q(u)\|_{C^{\delta,\alpha}_{0,0}} \), which in turn is the difference between the metrics \( \omega \) and \( \omega + \sqrt{-1} \partial \bar{\partial} u \).

It follows that there is a constant \( \epsilon_1 \) such that \( N \) is a contraction on \( \mathcal{B} \) as long as

\[
\|u\|_{C^{\delta,\alpha}_{2,2}} < \epsilon_1
\]

for all \( u \in \mathcal{B} \). This holds if \( \epsilon_0 \) in the definition of \( \mathcal{B} \) is sufficiently small, since as above we have

\[
\rho^{\delta} w^\tau \leq C\rho^{\delta - 2 + (\tau - 2)(1/d - 1)} \rho^2 w^2,
\]

which implies

\[
\|u\|_{C^{\delta,\alpha}_{2,2}} \leq C\|u\|_{C^{\delta,\alpha}_{2,2}}.
\]

We can therefore assume that \( \|N(u) - N(v)\| < \frac{1}{2}\|u - v\| \) for \( u, v \in \mathcal{B} \).
Finally we just have to ensure that $N$ maps $B$ to itself. For this first note that by the estimates of Proposition 5 we have $F(0) \in C^{0,\alpha}_{\delta^2/2,\tau/2}$ for some $\delta' < \delta$ that is sufficiently close to $\delta$. It follows that

$$\|F(0)\|_{C^{0,\alpha}_{\delta^2/2,\tau/2}(\rho^{-1}[A,\infty))} < CA^\delta' - \delta,$$

which can be made arbitrarily small by choosing $A$ large. Next, we have that if $u \in B$, then

$$\|N(u)\|_{C^{2,\alpha}_{\delta/2,\tau}} \leq \|N(0)\|_{C^{2,\alpha}_{\delta/2,\tau}} + \|N(u) - N(0)\|_{C^{2,\alpha}_{\delta/2,\tau}}$$

$$\leq C\|F(0)\|_{C^{0,\alpha}_{\delta^2/2,\tau/2}(\rho^{-1}[A,\infty))} + \frac{1}{2}\|u\|_{C^{2,\alpha}_{\delta/2,\tau}}$$

$$\leq CA^\delta' - \delta + \frac{\epsilon_0}{2}.$$

For sufficiently large $A$ we therefore have $N(u) \in B$. Therefore we can find a fixed point of $N$, as required.

We can now complete the proof of Theorem 1 by applying Hein [17, Proposition 4.1] to the metric $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$ constructed in the previous proposition. Note that by elliptic regularity $\tilde{\omega}$ is actually smooth. Since $\tilde{\omega}$ is asymptotically a small perturbation of $\omega$, Proposition 4 shows that the tangent cone at infinity of $(X_1, \tilde{\omega})$ is the cone $X_0$. To apply Hein’s result we need to check that $(X_1, \tilde{\omega})$ satisfies the condition $SOB(2n)$ (see [17, Definition 3.1]), and it has a $C^{3,\alpha}$ quasi-atlas. The condition $SOB(2n)$ includes the connectedness of certain annuli on $X_1$, but for the proof it is actually enough to check the RCA condition used in Degeratu-Mazzeo [11] for instance. Since $\tilde{\omega}$ is uniformly equivalent to $\omega$, this condition holds by Proposition 10. To check that $(X_1, \tilde{\omega})$ satisfies $SOB(2n)$ it is then enough to show that for a constant $C > 0$ and any $s > C$ the volume of the ball $B(x, s)$ satisfies

$$C^{-1}s^{2n} < \text{Vol}(B(x, s)) < Cs^{2n},$$

for any $x \in X_1$. This can be seen for instance by using Colding’s volume convergence theorem [3] under Gromov-Hausdorff limits, and by noting that $\tilde{\omega}$ is Ricci flat outside of a compact set, and has a tangent cone at infinity that is non-collapsed (i.e. has Euclidean volume growth).

The existence of a $C^{3,\alpha}$ quasi-atlas, i.e. charts of a uniform size around each point in which the metric is controlled in $C^{3,\alpha}$, can also be seen using Propositions 6, 7. These results show that for the metric $\omega$ we actually have charts of size $\rho\omega$ around any point, in which the metric is controlled in $C^{9,\alpha}$. Note that since $\omega > \kappa^{-2}\rho^{1/d-1}$ we have $\rho\omega > \kappa^{-2}\rho^{1/d}$, which goes to infinity as $\rho \to \infty$. When we perturb the metric to $\tilde{\omega}$, then by our construction we a priori only control $\tilde{\omega}$ in $C^{2,\alpha}$ in these charts, but elliptic regularity allows us to improve this to $C^{9,\alpha}$.

We can therefore apply Proposition 4.1 from Hein [17] to further perturb $\tilde{\omega}$ to a global Calabi-Yau metric on $X_1$. This perturbation does not change the tangent cone at infinity, and so we obtain the required Calabi-Yau metric on $\mathbb{C}^n \cong X_1$ with tangent cone $X_0 = \mathbb{C} \times V_0$ at infinity.
8. Calabi-Yau metrics in a neighborhood of an isolated singularity

In this section we show that the ideas developed in the previous sections can also be used to construct Calabi-Yau metrics on neighborhoods of certain isolated singularities \((X_1, 0)\), whose tangent cone at the singularity has singular cross section, as described in the introduction. This is a generalization of unpublished work of Hein-Naber \([18]\), who considered the case when \(X_1\) is the hypersurface

\[
(8.1) \quad z^p + x_1^2 + \ldots + x_n^2 = 0,
\]

with \(p > 2 \frac{n-1}{n-2}\).

We consider the situation where \(X_1 \subset \mathbb{C}^{n+1}\) is the hypersurface

\[
z^p + f(x_1, \ldots, x_n) = 0,
\]

where as before, \(V_0 = f^{-1}(0) \subset \mathbb{C}^n\) admits a Calabi-Yau cone metric. As before, we let the degree of \(f\) be \(d\) for the homothetic action with weight \(w = (w_1, \ldots, w_n)\) on \(x\), and this time we require that \(p > d\). For the hypersurface \((8.1)\) the condition \(p > d\) coincides with Hein-Naber’s condition \(p > 2 \frac{n-1}{n-2}\). Our goal is to construct a Calabi-Yau metric in a neighborhood of the singular point \(0 \in X_1\), whose tangent cone at \(0\) is \(\mathbb{C} \times V_0\).

As before, we have the nowhere vanishing holomorphic \(n\)-form

\[
\Omega = \frac{dz \wedge dx_2 \wedge \ldots \wedge dx_n}{\partial_x f}
\]
on \(X_1\), and the first step is to write down a metric \(\omega\) on \(X_1\) whose Ricci potential

\[
h = \log \left( \frac{\omega^n}{(\sqrt{-1})^{n^2} \Omega \wedge \Omega} \right)
\]
decays in a suitable weighted space near the origin. The definition of \(\omega\) is completely analogous to \((3.1)\), given by

\[
\omega = \sqrt{-1} \partial \bar{\partial} \left( |z|^2 + \gamma_1 (R \rho^{-\alpha}) r^2 + \gamma_2 (R \rho^{-\alpha}) |z|^{2p/d} \phi(z^{-p/d}, x) \right),
\]

where this time we choose \(\alpha \in (1, p/d)\), and \(\gamma_1, \rho, R\) are just as before. Note that the potential is asymptotic to \(|z|^2 + r^2\) as \(\rho \to 0\).

In analogy with Proposition 5 we have the following.

**Proposition 26.** The form \(\omega\) defines a metric on \(X_1 \setminus \{0\}\) on the set where \(\rho < P^{-1}\) for sufficiently large \(P\). In addition we can find a weight \(\delta > 2p/d\) for which the Ricci potential \(h\) of \(\omega\) satisfies

\[
|\nabla^i h|_\omega < \begin{cases} 
C_i \rho^{\delta - 2i} & \text{if } R > \kappa \rho, \\
C_i \rho^\delta R^{-2i} & \text{if } R \in (\kappa^{-1} \rho^{p/d}, \kappa \rho), \\
C_i \rho^\delta - 2p/d - ip/d & \text{if } R < \kappa^{-1} \rho^{p/d},
\end{cases}
\]

for suitable \(\kappa, C_i > 0\). Recall that by Lemma 3 we can assume \(d > 2\).

**Proof.** The proof of these estimates is very similar to the proof of Proposition 5 estimating the Ricci potential by comparing \(\omega\) to various model metrics in different regions, by scaling. We will keep the notation the same as in the proof of Proposition 5 to make the similarities apparent. The main difference is that now we are interested in estimating the errors as \(\rho \to 0\) rather than \(\rho \to \infty\).
**Region I:** Consider the region where $R > \kappa \rho$, and $\rho \in (D/2, 2D)$ for small $D$. We scale the metric to $D^{-2} \omega$ and use coordinates
\[ \hat{z} = D^{-1} z, \quad \hat{x} = D^{-1} x, \quad \hat{r} = D^{-1} r. \]
We have
\[ D^{-2} \omega = \sqrt{-1} \partial \bar{\partial} (|\hat{z}|^2 + \hat{r}^2), \]
and the equation of $X_1$ is
\[ D^{p-d} \hat{z}^p + f(\hat{x}) = 0. \]
As before, we obtain
\[ |\nabla^i h|_{D^{-2} \omega} \leq C_1 D^{p-d}. \]
If $p > d > 2$, then $p - d > 2p/d - 2$, and so we can choose $\delta > 2p/d$ satisfying $D^{p-d} < D^{d-2}$.

**Region II:** Here $R \in (K/2, 2K)$ for some $K \in (4\rho^2, \kappa \rho)$, and $\rho \in (D/2, 2D)$. Let $z$ be close to $z_0$ such that $|z_0| \sim D$. In terms of coordinates
\[ \hat{z} = K^{-1}(z - z_0), \quad \hat{x} = K^{-1} x, \quad \hat{r} = K^{-1} r, \]
we have
\[ K^{-2} \omega = \sqrt{-1} \partial \bar{\partial} (|\hat{z}|^2 + \hat{r}^2), \]
and the equation of $X_1$ is
\[ K^{-d}(K \hat{z} + z_0)^p + f(\hat{x}) = 0, \]
where $|\hat{z}| \leq 1$. Arguing as before, when we compare this to the equation $f(\hat{x}) = 0$ we obtain an error of order $K^{-d} D^p$. This implies
\[ |\nabla^i h|_{K^{-2} \omega} \leq C_1 K^{-d} D^p = C_1 K^{2-d} D^p K^{-2}. \]
On this region $K > 4\rho^2$ (and $d > 2$), so we have
\[ K^{2-d} D^p < D^{p+(2-d)\alpha}. \]
Since $\alpha < p/d$ we have $p + (2-d)\alpha > 2p/d$, so we can choose $\delta > 2p/d$.

**Region III:** Here $R \in (K/2, 2K)$ for $K \in (\rho^2, 2\rho^2)$, and $\rho \in (D/2, 2D)$. We consider $z$ close to $z_0$, and since in this region $\rho$ is comparable to $|z|$, we have $|z_0| \sim D$. With the same scaling as in Region II, we have
\[ K^{-2} \omega = \sqrt{-1} \partial \bar{\partial} (|\hat{z}|^2 + \gamma_1 \hat{r}^2 + \gamma_2 K^{-2}(K \hat{z} + z_0)^{2p/d} \phi ((K \hat{z} + z_0)^{-p/d} K \cdot \hat{x})). \]
The equation of $X_1$ is
\[ K^{-d}(K \hat{z} + z_0)^p + f(\hat{x}) = 0. \]
As in Proposition [3] we compare this to the metric
\[ \sqrt{-1} \partial \bar{\partial} (|\hat{z}|^2 + \hat{r}^2) \]
on $X_0$ with equation $f(\hat{x}) = 0$. To compare the potentials, similarly to [3], we have
\[ \nabla^i \left[ K^{-2} |K \hat{z} + z_0|^{2p/d} \phi ((K \hat{z} + z_0)^{-p/d} K \cdot \hat{x}) - \hat{r}^2 \right] = O \left( (K^{-1} D^{p/d})^{2+c} \right). \]
Arguing as before we have an error of order $K^{-d}D^p$ from comparing the two equations, which is bounded in the same way as in Region II. Since in this region $K \sim D^\alpha$, the new error from comparing the Kähler potentials satisfies

$$ (K^{-1}D^{p/d})^{2+c} = K^{-c}D^{\frac{p}{d}(2+c)}K^{-2} = D^{\frac{2p}{d}+c\left(\frac{p}{d} - \alpha\right)}K^{-2}. $$

We need $\delta$ so that this is bounded by $D^\delta K^{-2}$ (as $D \to 0$). Since $\alpha < p/d$, we can choose $\delta > 2p/d$.

**Region IV:** Here $R \in (K/2, 2K)$ with $K \in (\kappa^{-1}\rho^{p/d}, \rho^\alpha/2)$, and $\rho \in (D/2, 2D)$. We choose $z$ close to $z_0$, with $|z_0| \sim D$. We scale the same way as in Regions II, III. As before, we have

$$ K^{-2}\omega = \sqrt{-1}\partial\bar{\partial}\left(|z|^2 + K^{-2}|K\bar{z} + z_0|^{2p/d} \phi(|K\bar{z} + z_0|^{-p/d}K\cdot \bar{x})\right), $$

and the equation of $X_1$ is

$$ K^{-d}(K\bar{z} + z_0)^p + f(\bar{x}) = 0. $$

Similarly to the proof of Proposition we now compare $K^{-2}\omega$ to the product metric on $C \times V_{K-dz_0^p}$ with equation

$$ K^{-d}z_0^p + f(\bar{x}) = 0. $$

The error given by the difference of the equations is of order $K^{1-d}\alpha^{-p-1} = O(K^{1-d}D^{-1})$.

Let us denote by $E$ the difference in Kähler potentials,

$$ E = K^{-2}|K\bar{z} + z_0|^{2p/d}(|K\bar{z} + z_0|^{-p/d}K\cdot \bar{x}) - K^{-2}|z_0|^{2p/d}\phi(Kz_0^{-p/d} \cdot \bar{x}). $$

Since

$$ K(K\bar{z} + z_0)^{-p/d} = z_0^{-p/d}K(1 + O(KD^{-1})), $$

similarly to (3.5), (3.6), (3.7) we get

$$ |\nabla^i E| < C_i(|z_0|^{-p/d}K)^{-2-c}D^{-1}K = O(K^{-1-c}D^{(2+c)p-1}). $$

Combining this with the error of order $K^{1-d}D^{p-1}$, we need $\delta > 2p/d$ such that

$$ K^{1-d}D^{p-1} + K^{-1-c}D^{(2+c)p-1} < CD^\delta K^{-2}. $$

This is equivalent to

$$ K^{3-d}D^{p-1}(1 + (KD^{-p/d})^{d-2-c}) < CD^\delta. $$

We can assume that $c > 0$ is small, so that $d-2-c > 0$ (in fact from the estimate in Proposition we cannot expect that $c > d-2$ is possible). Then since $KD^{-p/d}$ is bounded away from zero, it is enough to choose $\delta$ so that

$$ K^{3-d}D^{p-1}(KD^{-p/d})^{d-2-c} < CD^\delta, $$

i.e.

$$ K^{1-c}D^{\frac{(2+c)p}{d}-1} < CD^\delta. $$

Since $K < D$, in order to be able to choose $\delta > 2p/d$, we need

$$ 1 - c + \frac{(2+c)p}{d} - 1 > \frac{2p}{d}. $$

This is equivalent to $p > d$, which holds in our setting.
**Region V**: Here $R < 2\kappa^{-1}p/d$, $\rho \in (D/2, 2D)$, and $z$ is close to $z_0$ satisfying $|z_0| \sim D$. We rescale the metric by $|z_0|^{p/d}$. We introduce new coordinates

$$
\tilde{z} = z_0^{-p/d}(z - z_0), \quad \tilde{x} = z_0^{-p/d} \cdot x, \quad \tilde{\rho} = |z_0|^{-p/d},
$$
so that $|\tilde{z}| < 1$. We have

$$
|z_0|^{-2p/d} \omega = \sqrt{-1} \bar{\partial} \partial \left[ |\tilde{z}|^2 + |z_0|^{p/d} |\tilde{z}| + |z_0|^{-2p/d} \phi \left( (z_0^{p/d} \tilde{z} + z_0)^{-p/d} \cdot \tilde{\rho} \cdot \tilde{\omega} \right) \right],
$$
and the equation of $X_1$ is

$$
z_0^{-p}(z_0^{p/d} \tilde{z} + z_0)^p + f(\tilde{x}) = 0.
$$

Note that

$$
z_0^{-p}(z_0^{p/d} \tilde{z} + z_0)^p = 1 + O(z_0^{p/d - 1}) = 1 + O(D^{p/d - 1}),
$$
and so comparing to $C \times V_1$ with equation $1 + f(\tilde{x}) = 0$ we introduce an error of order $D^{p/d - 1}$.

The difference in the corresponding Kähler potentials is

$$
E = |z_0|^{p/d} \tilde{z} + z_0 |^{2p/d} |z_0|^{-2p/d} \phi \left( (z_0^{p/d} \tilde{z} + z_0)^{-p/d} z_0^{p/d} \cdot \tilde{\omega} \right) - \phi(\tilde{\omega}).
$$

Since

$$
(z_0^{p/d} \tilde{z} + z_0)^{-p/d} z_0^{p/d} = 1 + O(z_0^{p/d - 1}) = 1 + O(D^{p/d - 1}),
$$
we have $|\nabla^i E|_{\tilde{\omega}/z_0^{-p/d}} = O(D^{p/d - 1})$. Therefore we need to be able to choose $\delta > 2p/d$ for which

$$
D^{p/d - 1} < C D^{\delta - 2p/d}.
$$

This is possible, since we assumed $p/d > 1$.

Abusing notation, let us now denote by $\omega$ a metric on $X_1 \setminus \{0\}$ which agrees with the $\omega$ constructed above on a neighborhood of $0$. The rest of the proof of Theorem 2 is essentially the same as the proof of Proposition 2, in order to find a $u$ such that

$$
(\omega + \sqrt{-1} \bar{\partial} \partial u)^n = (\sqrt{-1})^n \Omega \wedge \overline{\Omega}
$$
on the set where $\rho < A^{-1}$ for sufficiently large $A$. The key ingredient is to find a right inverse for the Laplacian

$$
(8.2) \quad \Delta_\omega : C^{k,\alpha}_{\delta, \tau}(\rho^{-1}(0, A^{-1})] \rightarrow C^{k-2, \alpha}_{\delta-2, \tau-2}(\rho^{-1}(0, A^{-1})],
$$
for sufficiently large $A$, where the weighted spaces are defined analogously to before. Here we use smooth weight functions $\rho, \omega$ satisfying that $\rho^2$ agrees with $|z|^2 + R^2$ near the origin (this is essentially the radial distance from the singular point), and

$$
w = \begin{cases} 
1 & \text{if } R > 2\kappa \rho, \\
R/(\kappa \rho) & \text{if } R \in (\kappa^{-1} \rho^{p/d}, \kappa \rho), \\
\kappa^{-2} p^{p/d - 1} & \text{if } R < \frac{1}{2} \kappa^{-1} \rho^{p/d}.
\end{cases}
$$

The reason for the definition of $w$ is that near the singular rays $C \times \{0\}$ our metric $\omega$ is now modeled on $D^{2p/d} \omega_{V_1}$, transverse to the $C$-factor.
We have results analogous to Propositions 7 and 6. Here we consider two types of regions:

\[ U = \{ \rho < A^{-1}, R > \Lambda \rho^{p/d} \}, \]

for large \( A, \Lambda, \) and

\[ V = \{ |z - z_0| < B |z_0|^{p/d}, \rho < A^{-1}, R < \Lambda \rho^{p/d} \}, \]

where \( z_0, B \) are viewed as fixed. In an identical way to Propositions 7 and 6 we can define maps \( G : U \rightarrow X_0 \) and \( H : V \rightarrow C \times V_1. \) The Riemannian metrics \( g, g_{X_0}, g_{C \times V_1} \) defined by \( \omega, \omega_0 \) and the product metrics on \( C \times V_1 \) then satisfy

\[ \| G^* g_{X_0} - g \|_{C^{k,\alpha}} < \epsilon, \]

if \( \Lambda > \Lambda(\epsilon) \) and \( A > A(\epsilon, \Lambda, B), \)

for any \( z_0, B, \Lambda, \) once \( A > A(\epsilon, \Lambda, B). \)

Just as in Proposition 9, these estimates imply that the tangent cone of \( (X_1, \omega) \) at 0 is given by the cone \( X_0 = C \times V_0. \) Moreover the existence of a right inverse for the Laplacian in (8.2) once \( A \) is sufficiently large follows exactly the argument from Section 6.

Proposition 26 implies that the Ricci potential \( h \) of \( \omega \) satisfies

\[ h \in C^{0,\alpha}_{\delta' - 2, \tau - 2} \]

for some \( \delta' > 2p/d, \) and any \( \tau < 0. \) If follows that for slightly smaller \( \delta > 2p/d \) we have

\[ \| h \|_{C^{0,\alpha}_{\delta - 2, \tau - 2}(\rho^{-1}(0, A^{-1}))} < CA^{\delta - \delta'}, \]

which can be made arbitrarily small by choosing \( A \) large. We can now follow the proof of Proposition 26 to solve Equation 8 with \( u \in B, \)

where

\[ B = \{ u \in C^{2,\alpha}_{\delta, \tau} : \| u \|_{C^{2,\alpha}_{\delta, \tau}} < \epsilon_0 \}, \]

for sufficiently small \( \epsilon_0, \) with \( \tau < 0 \) very close to 0. As in the proof of Proposition 26 the key estimate that we need is that

\[ \| u \|_{C^{2,\alpha}_{\delta, \tau}(\rho^{-1}(0, A^{-1}))} \leq C \| u \|_{C^{2,\alpha}_{\delta, \tau}(\rho^{-1}(0, A^{-1}))}. \]

This follows since the inequality \( w > C^{-1} \rho^{p/d - 1}, \) together with \( \delta > 2p/d \) implies

\[ \rho^{\delta} w^{\tau} < C \rho^2 w^2. \]

In particular if \( u \in B \) with sufficiently small \( \epsilon_0, \) then \( \omega + \sqrt{-1} \partial \bar{\partial} u \) is uniformly equivalent to \( \omega. \) The rest of the argument is identical to the proof of Proposition 26.

Finally, the tangent cone of \( \omega + \sqrt{-1} \partial \bar{\partial} u \) at 0 agrees with the tangent cone of \( \omega, \) since if \( u \in C^{2,\alpha}_{\delta, \tau} \) for our choices of \( \delta, \tau, \) then \( |\sqrt{-1} \partial \bar{\partial} u|_\omega \to 0 \) as \( \rho \to 0. \)

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