Let $x_1, \ldots, x_n$ be a fixed sequence of real numbers. At each stage, pick two indices $I$ and $J$ uniformly at random and replace $x_I, x_J$ by $(x_I + x_J)/2, (x_I + x_J)/2$. Clearly all the coordinates converge to $(x_1 + \cdots + x_n)/n$. We address the rate of convergence in various norms. This answers a question of Jean Bourgain.

1. Introduction

About 1980, Jean Bourgain asked one of us (personal communication to P. D.) the question in the abstract. It recently resurfaced via a question in quantum computing (thanks to Ramis Movassagh). We record some convergence theorems.

Fix $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in \mathbb{R}^n$. Define a Markov chain as follows: Given $x_k$, pick two distinct coordinates $I$ and $J$ uniformly at random, and replace both $x_{k,I}$ and $x_{k,J}$ by $(x_{k,I} + x_{k,J})/2$, keeping all other coordinates the same, to obtain $x_{k+1}$.

Let $x_0 = \frac{1}{n} \sum_{i=1}^{n} x_i$. In Section 4 we show:

- $\mathbb{E}(\sum_{i=1}^{n} (x_{k,i} - x_0)^2) = (1 - \frac{1}{n-1})^k \sum_{i=1}^{n} (x_{0,i} - x_0)^2$, giving convergence in $L^2$ for $k \gg n$.
- $x_k = (x_0, x_0, \ldots, x_0)$ for all large $k$ (for any choice of $x_0$) if and only if $n$ is a power of 2.
- Let $T_k = \sum_{i=1}^{n} \left| x_{k,i} - x_0 \right|$. Then,
  - for $x_0 = (1, 0, 0, \ldots, 0)$ and $k = \frac{1}{2} n \log n - cn$ for some $c > 0$, $\mathbb{E}(|T_k| - 2) \leq 2e^{-2c}$; and
  - for any $x_0$ and $k = n \log n + cn$ for some $c > 0$, $\mathbb{E}(T_k) \leq e^{-c/2} (\sum_{i=1}^{n} (x_{0,i} - x_0)^2)^{1/2}$, giving convergence in $L^1$ for any $x_0$ when $k$ is slightly bigger than $n \log n$, and non-convergence in $L^1$ for at least one value of $x_0$ when $k$ is slightly less than $\frac{1}{2} n \log n$.

It is natural to ask if there is a ‘cut off’ associated with the behavior of $T_k$, especially since the underlying random transpositions chain has a cutoff at $\frac{1}{2} n \log n$. We simulated this for large $n$ and the data does not seem to
Figure 1. Graph of $T_k$ against $k/n$, with $n = 10^7$ and $x_0 = (1, 0, 0, \ldots, 0)$. Note that $\frac{1}{2} \log n \approx 8.06$ and $\log n \approx 16.12$ in this example, and this is almost the same as the interval where $T_k$ decreases from 2 to 0.

support a cutoff. Figure 1 shows results for $n = 10^7$. The decay clearly starts at $\frac{1}{2} n \log n$ but finishes only at $n \log n$. The size of the cutoff window is about $8n$. Since $8$ is about $\frac{1}{2} \log n$, a cutoff seems unlikely. Similar results were found for smaller values of $n$.

When $x_0$ lies in the positive orthant, the behavior of $T_k$ has an amusing interpretation in terms of a toy model of reduction in wealth inequality in a socialist regime. Suppose that there are $n$ individuals (or entities) in the population, and the coordinates of $x_k$ denote the wealths of these $n$ individuals at time $k$. The socialist regime tries to redistribute wealth by picking two individuals uniformly at random at each time point, and making them equally distribute their wealths among themselves. This is our repeated averages process. It is not hard to show that at time $k$, $\frac{1}{2} T_k$ is the amount of wealth that remains to be redistributed to attain perfect equality. This is actually a well-known measure of wealth inequality in the economics literature, known as the Hoover index [15] or Schutz index [21].

What our results show is that if we start from the initial configuration where one individual has all the wealth, then for a long time this index of inequality does not decrease to any appreciable degree, and then starts decreasing gradually to zero. On the other hand, if we start from a wealth distribution where the wealth of the wealthiest individual is comparable to the average wealth, then the Hoover index decreases much faster, as shown
by the following calculation. Suppose that the total wealth is 1 (so that the average wealth is $1/n$), and the maximum wealth is $C/n$ for some $C \geq 1$. Then by the results stated before,

$$E(T_k) \leq E \left[ \left( \sum_{i=1}^{n} (x_{k,i} - \bar{x}_0)^2 \right)^{1/2} \right]$$

$$\leq \left[ n E \left( \sum_{i=1}^{n} (x_{k,i} - \bar{x}_0)^2 \right) \right]^{1/2}$$

$$\leq \sqrt{n} \left( 1 - \frac{1}{n-1} \right)^{k/2} \left( \sum_{i=1}^{n} (x_{0,i} - \bar{x}_0)^2 \right)^{1/2}$$

$$\leq \sqrt{n} e^{-k/2n} \left( \frac{C}{n} \sum_{i=1}^{n} |x_{0,i} - \bar{x}_0| \right)^{1/2} \leq \sqrt{Ce^{-k/2n}}.$$

A numerical example of this second scenario is shown in Figure 2.

Section 2 gives background and a literature review and acknowledgments are in Section 3. All results and proofs are in Section 4.

2. Background

Bourgain asked this question because of our previous work on the random transpositions Markov chain. This evolves on the symmetric group $S_n$ by repeatedly picking $I, J$ uniformly at random and transposing these two
labels in the current permutation. In joint work with Shahshahani [10], we showed \( \frac{1}{2}n \log n \) steps are necessary and sufficient for convergence to the uniform distribution in both \( L^1 \) and \( L^2 \). Map the symmetric group into the set of \( n \times n \) doubly stochastic matrices by sending \((i,j)\) to the matrix 

\[
m_{ab} = \begin{cases} 
\frac{1}{2} & \text{if } (a,b) = (i,i), (j,j), (i,j), (j,i), \\
1 & \text{if } a = b \neq i \text{ or } j, \\
0 & \text{otherwise}.
\end{cases}
\]

The successive images of the random transformations are exactly our averaging operators.

It seemed difficult to transform the results for the random transpositions walk into useful results for random averages. It is worth noting that very sharp refinements have recently been proved for transpositions; sharp numerical bounds like

\[
\|Q^k - U\|_{TV} \leq 2e^{-c} \quad \text{for } k = \frac{1}{2}n(log n + c)
\]

are in [20] (where \( Q^k \) is the law of walk after \( k \) steps, \( U \) is the uniform distribution on \( S_n \), and TV is the total variation norm) and even the limiting shape of the error is now understood [25]. Good results for random \( k \)-cycles [5] suggest results for averaging over larger random sets which should be accessible with present techniques. Finally, the ‘shape’ of the non-randomness for random transpositions if shuffling only \( O(n) \) steps is of current interest because of its connection to spatial random permutations and the ‘exchange process’ of mathematical physics (see [4, 22]).

The \( L^2 \) convergence of a more general version of our repeated averaging process was studied by Aldous and Lanoue [2] a few years ago. Aldous and Lanoue worked on an edge weighted graph with numbers at the vertices. At each step, an edge is picked with probability proportional to its weight and the two numbers on the vertices of the edge are replaced by their average. The ‘random transpositions’ version of this process was treated in [9].

Related processes, under the name of gossip algorithms, have been studied by Shah [23]. Such processes are also known as distributed consensus algorithms [18]. The Deffuant model from the sociology literature is a closely related model where averaging takes place only if the two values differ by less than a specified threshold [3, 12, 16]. Acemoğlu et al. [1] analyze a model where some agents have fixed opinions and other agents update according to an averaging process.

Iterated local averages have a long tradition in the actuarial literature going back to Charles Peirce and de Forest. See [8] for a survey. Replacing ‘averages of 3’ by ‘median of 3’ gives the ‘3RSSH smoother’. William Feller [11, p. 333, p. 425] studies repeated averages for examples of the renewal theorem and Markov chains. An interesting literature on getting experts to reach consensus is surveyed and developed by Chatterjee and Seneta [6]. Finally, our work can be set in the space of random walk on the
The Kac walk has a semigroup of doubly stochastic matrices \( [14] \). This subject does not seem to focus on the rates of convergence. The present note suggests there is much to do.

One mathematical use of iterated averages appears in summability theory \([13]\). Let \( x_1, x_2, \ldots \) be a real sequence. Let \( c^1_n(x) = \frac{1}{n}(x_1 + \cdots + x_n) \) — the first Cesaro average — and let \( c^{k+1}_n(x) = \frac{1}{n}(c^1_n(x) + \cdots + c^n_k(x)) \). Often \( x_n \) is 1 or 0 as \( n \) ∈ \( A \) or not, where \( A \subseteq \{1, 2, \ldots\} \). If \( \lim c^n_k(x) \) exists this assigns a density to \( A \). For \( k \geq 1 \), it can be shown that if \( \{c^{k+1}_n(x)\}_{n=1}^{\infty} \) has a limit then \( \{c^n_k(x)\}_{n=1}^{\infty} \) has a limit. However, \( \liminf c^n_k(x) \) is increasing in \( k \) and \( \limsup c^n_k(x) \) is decreasing in \( k \). If these meet as \( k \to \infty \), \( \{x_n\}_{n=1}^{\infty} \) is called \( H_\infty \) summable. The sequence

\[
x_n = \begin{cases} 
1 & \text{if lead digit of } n \text{ is } 1, \\
0 & \text{otherwise}
\end{cases}
\]

is not \( c^k \) summable for any \( k \) but has \( H_\infty \) density \( \log_{10}(2) \approx 0.301 \). For proofs and references see \([7]\).

The repeated averages process has an interesting connection with Schur convexity. The majorization partial order on \( \mathbb{R}^n \) is defined as follows. Call \( x = (x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) = y \) if \( \sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i) \) for all \( 1 \leq k \leq n \), where \( x(1) \geq x(2) \geq \cdots \geq x(n) \) is a decreasing rearrangement of \( x_1, \ldots, x_n \) and \( y(1) \geq y(2) \geq \cdots \geq y(n) \) is a decreasing rearrangement of \( y_1, \ldots, y_n \). If all the entries are nonnegative and sum to \( s \), the largest vector is \( (s, 0, \ldots, 0) \) and the smallest vector is \( (s/n, s/n, \ldots, s/n) \). An encyclopedic treatise on majorization is in \([17]\). A function \( f: \mathbb{R}^n \to \mathbb{R} \) is called Schur convex if \( x \preceq y \) implies \( f(x) \leq f(y) \). It is easy to see that a symmetric convex function is Schur convex and that one moves down in the order by replacing \( x_i, x_j \) by \( (x_i + x_j)/2, (x_i + x_j)/2 \). Therefore the relevance for the present paper is clear: each step of the Markov chain moves down in the order. Moreover, for any symmetric convex function \( f \), \( f(x_{k+1}) \leq f(x_k) \) for all \( k \). Thus, for instance, \( \sum_{i=1}^{n} \left|x_{k,i}\right|^p \) is monotone decreasing in \( k \) for any \( p \geq 1 \).

The repeated averages process has similarities with two familiar Markov chains. The first is the Kac walk — a toy model for the Boltzmann equation that is widely studied in the physics and probability literatures. The walk proceeds on the unit sphere \( \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\} \). From \( x \in \mathbb{S}^{n-1} \), choose distinct \( I \) and \( J \) uniformly at random and replace \( x_I \) and \( x_J \) by \( x_I \cos \theta + x_J \sin \theta \) and \( -x_I \sin \theta + x_J \cos \theta \), with \( \theta \) chosen uniformly from \([0, 2\pi]\). This is a surrogate for “random particles collide and exchange energy at random”. This Markov chain has a uniform stationary distribution on \( \mathbb{S}^{n-1} \). Following a long series of improvements, the current best results on the rate of convergence of this walk are due to Pillai and Smith \([19]\). They show that order \( n \log n \) steps are necessary and sufficient for mixing in total variation distance (indeed, \( \frac{1}{2} n \log n \) is not enough and \( 200 n \log n \) is enough). Aside from differences in state space and dynamics, the Kac walk has a
uniform stationary distribution while repeated averaging is absorbing at a single point.

The second Markov chain that is similar to repeated averaging is the Gibbs sampler for the uniform distribution on the simplex $\Delta_{n-1} = \{ x \in \mathbb{R}^n : x_i \geq 0 \ \forall i, \ x_1 + \cdots + x_n = 1 \}$. The explicit description of this chain is as follows. From $x \in \Delta_{n-1}$, choose $I$ and $J$ uniformly at random and replace $x_I, x_J$ by $x'_I, x'_J$, with $x'_I$ chosen uniformly from $[0, x_I + x_J]$ and $x'_J = x_I + x_J - x'_I$. Settling a conjecture of Aldous, Aaron Smith [24] showed that order $n \log n$ steps are necessary and sufficient for convergence in total variation. Cutoff remains an open problem.

3. Acknowledgments

We thank David Aldous, Laurent Miclo, Evita Nestoridi and Perla Sousi for comments. Our revival of this project stems from a quantum computing question of Ramis Movassagh — roughly, to develop a convergence result with $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ replaced by the parallel quantum spin operator — we hope to develop this in joint work with him. Finally, we acknowledge the great, late Jean Bourgain. He sent us a typed draft of a preliminary manuscript. Alas, 30 years later, this proved difficult to find. We will be grateful if a copy can be located.

4. Results and proofs

Define a Markov chain $\{x_k\}_{k=0}^\infty$ as in Section 1. It is not difficult to see that without loss of generality, we can assume $x_0 = 0$. We will work under this assumption throughout this section. For each $k$, let

$$S_k := n \sum_{i=1}^n x_{k,i}^2.$$  

Let $\mathcal{F}_k$ be the $\sigma$-algebra generated by the history up to time $k$.

**Proposition 4.1.** For any $k \geq 0$,

$$\mathbb{E}(S_{k+1}|\mathcal{F}_k) = \left(1 - \frac{1}{n-1}\right) S_k.$$  

**Proof.** For any $i$, the probability that it is one of the chosen coordinates is $2/n$. If it is chosen, then the other coordinate is uniformly chosen among the remaining coordinates. Therefore

$$\mathbb{E}(x_{k+1,i}^2|\mathcal{F}_k) = \left(1 - \frac{2}{n}\right) x_{k,i}^2 + \frac{2}{n(n-1)} \sum_{1 \leq j \leq n, j \neq i} \left(\frac{x_{k,i} + x_{k,j}}{2}\right)^2$$

$$= \left(1 - \frac{2}{n}\right) x_{k,i}^2 + \frac{1}{2n} x_{k,i}^2 + \frac{1}{2n(n-1)} \sum_{1 \leq j \leq n, j \neq i} \left(x_{k,j}^2 + 2x_{k,i}x_{k,j}\right).$$
Since \( \sum_{i=1}^{n} x_{k,i} = 0 \), we have
\[
\sum_{1 \leq j \leq n, j \neq i} x_{k,j} = -x_{k,i}.
\]

Thus, we get the further simplification
\[
\mathbb{E}(x_{k+1,i}^2 | F_k) = \left( 1 - \frac{2}{n} + \frac{1}{2n(n-1)} \right) x_{k,i}^2 + \frac{1}{2n(n-1)} \sum_{1 \leq j \leq n, j \neq i} x_{k,j}^2
\]
\[
= \left( 1 - \frac{2}{n} + \frac{1}{2n(n-1)} - \frac{1}{2n(n-1)} \right) x_{k,i}^2 + \frac{1}{2n(n-1)} S_k
\]
\[
= \left( 1 - \frac{3}{2(n-1)} \right) x_{k,i}^2 + \frac{1}{2n(n-1)} S_k.
\]

Summing over \( i \), we get the required identity. \( \square \)

**Corollary 4.2.** Let \( \tau := 1 - \frac{1}{n-1} \). Then \( \mathbb{E}(S_k) = \tau^k S_0 \). Moreover, \( \lim S_k/\tau^k \) exists and is finite almost surely.

**Proof.** The claim \( \mathbb{E}(S_k) = \tau^k S_0 \) is immediate by Proposition 4.1 and induction. Proposition 4.1 also shows that \( M_k := S_k/\tau^k \) is a nonnegative martingale, and so its limit exists and is finite almost surely. \( \square \)

Incidentally, the argument works for more general methods of averaging. For example, if \( x_I \) and \( x_J \) are replaced by \( \theta x_I + \theta x_J \) and \( \theta x_I + \theta x_J \), where \( 0 < \theta < 1 \) and \( \bar{\theta} = 1 - \theta \), Proposition 4.1 becomes
\[
\mathbb{E}(S_{k+1}|F_k) = \left( 1 - \frac{4\theta\bar{\theta}}{n-1} \right) S_k.
\]

Corollary 4.2 shows that \( S_k \) is small when \( k/n \gg 1 \). One can also ask whether \( S_k \) eventually attains the value 0. Of course, this is trivially true if the initial vector is zero. But in general this is impossible unless \( n \) is a power of 2.

**Proposition 4.3.** Suppose that \( n \) is not a power of 2. Then there is a vector \( x_0 \) such that if \( x_k \) is defined as above, then \( x_k \neq 0 \) for all \( k \).

**Proof.** Let \( x_0 = (1 - \frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}) \). We claim that for any \( k \) and any \( i \), \( x_{k,i} \) equals \( m/2^l - 1/n \) for some nonnegative integers \( m \) and \( l \) where \( m \) is odd. This is true for \( k = 0 \) by definition. Suppose that this holds for some \( k \). To produce \( x_{k+1} \), suppose that we choose two coordinates \( i \) and \( j \). Suppose that \( x_{k,i} = m/2^l - 1/n \) and \( x_{k,j} = m'/2^{l'} - 1/n \). Without loss of generality, suppose that \( l \geq l' \). Then
\[
x_{k+1,i} = x_{k+1,j} = \frac{1}{2} \left( \frac{m}{2^l} + \frac{m'}{2^{l'}} \right) - \frac{1}{n}
\]
\[
= m + \frac{m(2^l-2^{l'})}{2^{l+1}} - \frac{1}{n}.
\]
If \( l > l' \), then \( m + 2^{l-l'}m' \) is odd and our claim is proved. If \( l = l' \), then the above expression reduces to \( m''/2^l - 1/n \), where \( m'' = (m + m')/2 \). Since \( m'' = 2^jr \) for some \( j \) and some odd \( r \), this expression becomes \( r/2^{l-j} - 1/n \). Note that \( l-j \geq 0 \), because otherwise \( x_{k+1,i} \) would be greater than 1, which is impossible because we are always averaging quantities that are in \([-1, 1]\). This completes the induction step. This also completes the proof of the lemma, because a quantity like \( m/2^l - 1/n \), where \( m \) is odd, cannot be zero unless \( n \) is a power of 2. □

On the other hand, if \( n \) is a power of 2, then \( x_k \) eventually becomes zero for any starting state.

**Proposition 4.4.** If \( n \) is a power of 2, then with probability 1, \( x_k = 0 \) for all large enough \( k \).

**Proof.** If \( n \) is a power of 2, then it is easy to see that there is a particular sequence of steps that produces the vector of all 0’s starting from any \( x_0 \). For example, consider the case \( n = 4 \). We can first average coordinates 1 and 2, and then 3 and 4, and then 1 and 3 and then 1 and 4, which will render all coordinates equal and hence zero. The scheme for \( n \) equal to a general power of 2 is similar: We do averages so that coordinates in successive blocks of size \( 2^l \) become equal, for \( l = 1, 2, \ldots \), until all coordinates become equal. Note that this scheme has nothing to do with the initial state. Since the above scheme has a fixed number of steps, it occurs sooner or later as we go along. Thus, sooner or later, \( x_k = 0 \). □

Let us now investigate the evolution of the \( L^1 \) norm of \( x_k \). For each \( k \), let

\[
T_k := \sum_{i=1}^{n} |x_{k,i}|.
\]

The following theorem shows that if \( k \) is less than \( \frac{1}{2}n \log n \) then \( T_k \) need not be small, whereas if \( k \) is bigger than \( n \log n \) then \( T_k \approx 0 \).

**Theorem 4.5.** If \( x_0 = (1 - \frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}) \) and \( k = \frac{1}{2}n \log n - cn \) for some \( c > 0 \), then

\[
\mathbb{E}|T_k - 2| \leq 2e^{-2c}.
\]

On the other hand, if \( k = n \log n + cn \) for some \( c > 0 \), then for any \( x_0 \),

\[
\mathbb{E}(T_k) \leq e^{-c/2} \left( \sum_{i=1}^{n} x_{0,i}^2 \right)^{1/2}.
\]

**Proof.** Define a sequence of subsets of \( \{1, \ldots, n\} \) as follows. Let \( A_0 := \{1\} \). Take any \( k \geq 0 \). Let \( i \) and \( j \) be the indices chosen to produce \( x_{k+1} \) from \( x_k \). If \( i \) and \( j \) are both in \( A_k \), or both outside \( A_k \), let \( A_{k+1} := A_k \). Otherwise, when one of \( i \) and \( j \) is in \( A_k \) and the other is outside, include the one outside to produce \( A_{k+1} \).

Our first claim is that for any \( k \), if \( i \notin A_k \), then \( x_{k,i} = -1/n \). This is true for \( k = 0 \). Suppose that this holds for some \( k \). Take any index \( l \notin A_{k+1} \).
Let $i$ and $j$ be the indices chosen for producing $x_{k+1}$. If $i$ and $j$ are both in $A_k$, then $A_{k+1} = A_k$, and $x_{k+1,l} = x_{k,l} = -1/n$. If $i$ and $j$ are both outside $A_k$, then again $A_{k+1} = A_k$, and $x_{k+1,l} = (x_{k,i} + x_{k,j})/2 = -1/n$. Lastly if $i \in A_k$ and $j \notin A_k$, then $A_{k+1} = A_k \cup \{j\}$, and hence $l$ must have been untouched by the process of updating from $x_k$ to $x_{k+1}$. Therefore in this case too, $x_{k+1,l} = x_{k,l} = -1/n$. This completes the induction.

Now define

$$Q_k := \sum_{i \in A_k} x_{k,i}.$$  

Our second claim is that for any $k$,

$$Q_k = 1 - \frac{|A_k|}{n}.$$  

This is true for $k = 0$. Suppose that this holds for some $k$. Let $i$ and $j$ be indices chosen for updating $x_k$ to $x_{k+1}$. If $i, j \in A_k$ then clearly $A_{k+1} = A_k$ and $Q_{k+1} = Q_k$. Same holds if $i, j \notin A_k$. Now suppose that $i \in A_k$ and $j \notin A_k$. Then $x_{k,j} = -1/n$, and hence

$$Q_{k+1} = x_{k,i} + x_{k,j} + \sum_{l \in A_k \setminus \{i\}} x_{k,l}$$

$$= -\frac{1}{n} + \sum_{l \in A_k} x_{k,l} = -\frac{1}{n} + Q_k = 1 - \frac{|A_{k+1}|}{n}.$$  

Due to the above two claims, we get the lower bound

$$T_k = \sum_{i \in A_k} |x_{k,i}| + \sum_{i \notin A_k} |x_{k,i}|$$

$$\geq Q_k + n - \frac{|A_k|}{n} = 2 - \frac{2|A_k|}{n}.$$  

But by the triangle inequality, it is clear that $T_{k+1} \leq T_k$ for all $k$, and hence $T_k \leq T_0 = 2 - 2/n$. Thus, for all $k$,

$$|T_k - 2| \leq \frac{2|A_k|}{n}. \tag{4.1}$$  

Now suppose that $|A_k| = s$ for some $k$ and $s$. Then $|A_{k+1}| = s + 1$ with probability

$$f(s) := \frac{2s(n-s)}{n(n-1)},$$

independent of whatever happened in the past, and $|A_{k+1}| = s$ with probability $1 - f(s)$. Thus, if $\mathcal{F}_k$ is the $\sigma$-algebra generated by the history up to time $k$, then

$$\mathbb{E}(|A_{k+1}| | \mathcal{F}_k) = f(|A_k|)(|A_k| + 1) + (1 - f(|A_k|))|A_k|$$

$$= |A_k| + f(|A_k|)$$

$$\leq |A_k| + \frac{2|A_k|}{n}.$$
This gives
\[ E|A_{k+1}| \leq \left(1 + \frac{2}{n}\right) E|A_k|. \]
Using this and induction, we get
\[ E|A_k| \leq \left(1 + \frac{2}{n}\right)^k \leq e^{2k/n}. \]
Combining with (4.1), we get
\[ E|T_k - 2| \leq \frac{2e^{2k/n}}{n}. \tag{4.2} \]
On the other hand, by the Cauchy–Schwarz inequality and Corollary 4.2,
\[ E(T_k) \leq \sqrt{n E(S_k)} \leq \sqrt{n \left(1 - \frac{1}{n-1}\right)^{k/2}} \sqrt{S_0} \]
\[ = \sqrt{n^{-1} \left(1 - \frac{1}{n-1}\right)^{k/2}} \]
\[ \leq \sqrt{n^{-1} e^{-k/2(n-1)}} \leq \sqrt{n} e^{-k/2n}. \tag{4.3} \]
Taking \( k = \frac{1}{2} n \log n - cn \) in (4.2) and \( k = n \log n + cn \) in (4.3), we get the required bounds.

\[ \square \]

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