Quantum speed limit for thermal states

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What is quantum speed limit (QSL)?

closed quantum system (in general, in a mixed state):

\[ i\partial_t \rho_t = [H, \rho_t] \]

trace distance: \[ D_{tr}(\rho_1, \rho_2) \equiv (1/2) \text{tr} |\rho_2 - \rho_1| \]

Mandelstam-Tamm QSL (1945):

\[ D_{tr}(\rho_0, \rho_t) \leq \Delta E t \quad \text{out-of-equilibrium evolution} \]

\[ \Delta E \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2}, \quad \langle A \rangle \equiv \text{tr} \rho_0 A \quad \text{observable in the initial (often equilibrium) state} \]
A remark on good and bad distances

In the many-body setting not all distances are equally meaningful!

Good distances (faithfully measure the distinguishability of states):
- trace distance
- Bures distance
- Hellinger distance

Bad distance:
- Hilbert-Schmidt distance – can nearly vanish for orthogonal states

see more in Markham *et al*, Phys. Rev. A 77, 042111 (2008)
Zoo of quantum speed limits

Mandelstam, Tamm (MT, 1945): \[ D_{tr}(\rho_0, \rho_t) \leq \Delta E t \quad \Delta E \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2} \]

Margolus, Levitin (ML, 1998): \[ D_{tr}(\rho_0, \rho_t) \leq \sqrt{2 \bar{E}} t \quad \bar{E} \equiv \langle H \rangle - E_{gs} \]

Mondal, Datta, Sazim (MDS, 2016): \[ D_{tr}(\rho_0, \rho_t) \leq \delta E t \]
\[ \delta E \equiv \sqrt{-\text{tr} \left[ \sqrt{\rho_0} H \right]^2} \]
Thermal initial state

\[ \rho_0 = e^{-\beta H_0} / Z_0, \quad Z_0 = \text{tr} e^{-\beta H_0} \]

Initial Hamiltonian

\[ H = H_0 + V \]

Perturbation

\[ i\partial_t \rho_t = [H_0 + V, \rho_t] \]

V is not assumed to be small
MT and ML QSLs for many-body thermal states

Mandelstam-Tamm: \[ D_{tr}(\rho_0, \rho_t) \leq \Delta E \, t \quad \Delta E \equiv \sqrt{\langle (H_0 + V)^2 \rangle - \langle H_0 + V \rangle^2} \]

\[ \Delta E \sim \sqrt{N} \quad \text{in the thermodynamic limit (where N is the system size)} \]

Margolus-Levitin: \[ D_{tr}(\rho_0, \rho_t) \leq \sqrt{2 \, \overline{E}} \, t \quad \overline{E} \equiv \langle H_0 + V \rangle - E_{gs} \]

\[ \sqrt{\overline{E}} \sim \sqrt{N} \quad \text{in the thermodynamic limit} \]
Quantum speed limit for a thermal initial state

\[ \rho_0 = e^{-\beta H_0} / Z_0, \quad Z_0 = \text{tr} e^{-\beta H_0} \]

\[ i \partial_t \rho_t = [H_0 + V, \rho_t] \]

Thermal QSL:

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\beta t} \sqrt{4 - 2 \langle [H_0, V]^2 \rangle_\beta} \]

\[ \langle A \rangle_\beta \equiv \text{tr} \rho_0 A \]

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T-QSL vs generic QSLs: infinite temperature

Finite Hilbert space with dimension $d$, $\beta = 0$

\[
\begin{align*}
    i\partial_t \rho_t &= [H_0 + V, \rho_t] \\
    \rho_0 &= d^{-1} \mathbb{1} \\
\end{align*}
\]

\[
\begin{align*}
    \rho_t &= \rho_0 = d^{-1} \mathbb{1} \\
    D_{\text{tr}}(\rho_0, \rho_t) &= 0
\end{align*}
\]
T-QSL vs generic QSLs: infinite temperature

\[ \beta = 0 \quad \rho_t = \rho_0 = d^{-1} \mathbf{1} \quad D_{tr}(\rho_0, \rho_t) = 0 \]

\begin{align*}
\text{MT} & \quad D_{tr}(\rho_0, \rho_t) \leq \Delta E t \\
\Delta E & \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2} \sim \sqrt{N} \quad \times
\end{align*}

\begin{align*}
\text{MDS} & \quad D_{tr}(\rho_0, \rho_t) \leq \delta E t \\
\delta E & \equiv \sqrt{-\text{tr} [\sqrt{\rho_0} V]^2} = 0 \quad \checkmark
\end{align*}

\begin{align*}
\text{ML} & \quad D_{tr}(\rho_0, \rho_t) \leq \sqrt{2 \overline{E}} t \\
\sqrt{E} & \equiv \sqrt{\langle H \rangle - E_{gs}} \sim \sqrt{N} \quad \times
\end{align*}

\begin{align*}
\text{thermal} & \quad D_{tr}(\rho_0, \rho_t) \leq \sqrt{\beta t} \sqrt[4]{-2 \langle [H_0, V]^2 \rangle_{\beta}} \quad \checkmark
\end{align*}
T-QSL vs generic QSLs: trivial perturbation

trivial perturbation: \[ [V, H_0] = 0 \]

\[
i \partial_t \rho_t = \left[ H_0 + V, \rho_t \right]
\]

\[
\rho_0 = e^{-\beta H_0 / Z_0}
\]

\[
\rho_t = \rho_0
\]

\[
D_{\text{tr}}(\rho_0, \rho_t) = 0
\]
T-QSL vs generic QSLs: trivial perturbation

\[ [V, H_0] = 0 \quad \rho_t = \rho_0 \quad D_{\text{tr}}(\rho_0, \rho_t) = 0 \]

\text{MT} \quad D_{\text{tr}}(\rho_0, \rho_t) \leq \Delta E t

\Delta E \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2} \sim \sqrt{N} \quad \times

\text{MDS} \quad D_{\text{tr}}(\rho_0, \rho_t) \leq \delta E t

\delta E \equiv \sqrt{-\text{tr} [\sqrt{\rho_0}, V]^2} = 0 \quad \checkmark

\text{ML} \quad D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{2 E} t

\sqrt{E} \equiv \sqrt{\langle H \rangle - E_{\text{gs}}} \sim \sqrt{N} \quad \times

\text{thermal} \quad D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\beta t} \frac{4}{4} - 2 \langle [H_0, V]^2 \rangle_{\beta} \quad \checkmark
MDS QSL: computability issue

\[
\text{MDS: } D_{\text{tr}}(\rho_0, \rho_t) \leq \delta E t
\]

\[
\delta E^2 = -\text{tr} [\sqrt{\rho_0}, V]^2 = 2 \text{tr} \rho_0 V^2 - 2 \text{tr} \sqrt{\rho_0} V \sqrt{\rho_0} V \leq 2 \text{tr} \rho_0 V^2
\]

hard to compute in nontrivial cases

Modified MDS QSL: \[
D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{2 \langle V^2 \rangle_\beta} t
\]
Local perturbation

Local perturbation: $\langle V^2 \rangle_\beta$ is finite in the thermodynamic limit.

For finite-range interactions, if $\langle V^2 \rangle_\beta$ is finite, $\langle [H_0, V]^2 \rangle_\beta$ is also finite.
T-QSL vs generic QSLs: local perturbation

\[ \langle V^2 \rangle_\beta = O(1)_{N \to \infty} \]

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \Delta E t \]

\[ \Delta E \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2} \sim \sqrt{N} \]

\[ \times \]

MT

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{2 \langle V^2 \rangle_\beta} t = O(1)_{N \to \infty} \]

\[ \checkmark \]

MDS

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\langle H^2 \rangle - E_{gs}} \sim \sqrt{N} \]

\[ \times \]

ML

\[ \sqrt{E} \equiv \sqrt{\langle H \rangle - E_{gs}} \sim \sqrt{N} \]

\[ \times \]

thermal

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\beta t} \sqrt[4]{-2 \langle [H_0, V]^2 \rangle_\beta} \]

\[ = O(1)_{N \to \infty} \]

\[ \checkmark \]
Finitely disturbing perturbation

Finitely disturbing perturbation: $\langle [H_0, V]^2 \rangle_\beta$ is finite in the thermodynamic limit

not every finitely disturbing perturbation is local
Example 1: spin-boson model

\[ H_0 = \Omega \sigma^z + \frac{1}{\sqrt{N}} \sigma^x \sum_k g_k (a_k^\dagger + a_k) + \sum_k \omega_k a_k^\dagger a_k \]

\[ V = \sum_k \delta \omega \ a_k^\dagger a_k \]

**non-local, but finitely disturbing**

**MDS QSL:** \[ D_{tr}(\rho_0, \rho_t) \leq \sqrt{2} \delta \omega t \bar{n}_\beta N \]

**T-QSL:** \[ D_{tr}(\rho_0, \rho_t) \leq \sqrt{\delta \omega \tilde{g} \beta t} \sqrt{2(1 + 2 \bar{n}_\beta)} \]

\[ \bar{n}_\beta \equiv \sum_k \langle a_k^\dagger a_k \rangle_\beta / N \]

\[ \tilde{g}^2 \equiv \sum_k g_k^2 / N \]

\[ \bar{n}_\beta \equiv \sum_k g_k^2 \langle a_k^\dagger a_k \rangle_\beta / \sum_k g_k^2 \]

finite in the thermodynamic limit
Example 2: mobile impurity model

\[ H_0 = H_f + \frac{P^2}{2m} + H_{imp-f} \]

Hamiltonian of a fluid, mobile impurity particle with mass \( m \), momentum \( P \) of the impurity, linear potential felt by the impurity.

\[ H_f \] describes a fluid in a 1D box of length \( L \), with particle number \( N \) and particle density \( n=n/L \).

\[ V = FX \]

Linear potential felt by the impurity, coordinate \( X \) of the impurity.
Example 2: mobile impurity model

\[ H_0 = H_f + P^2/(2m) + H_{\text{imp-f}} \]

\[ V = FX \quad \text{non-local, but finitely disturbing} \]

MDS QSL: \[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{2/3} NFt/n \]

T-QSL: \[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\beta t} \sqrt{(F/m) \sqrt{2\langle P^2 \rangle_{\beta}}} \]

finite in the thermodynamic limit
Performance of T-QSL vs general QSLs in the many-body setting: summary

|                          | Mandelstam-Tamm | Margolus-Levitin | Mondal-Datta-Sazim | thermal |
|--------------------------|-----------------|------------------|--------------------|---------|
| zero temperature         | loose           | loose            | exact              | exact   |
| trivial perturbation     | loose           | loose            | exact              | exact   |
| local perturbation       | loose           | loose            | tight              | tight   |
| finitely disturbing      | loose           | loose            | loose              | tight   |
| nonlocal perturbation    |                 |                  |                    |         |
Generalization: time-dependent perturbation

\[ i\partial_t \rho_t = [H_0 + V_t, \rho_t] \]

\[ \rho_0 = e^{-\beta H_0} / Z_0, \quad Z_0 = \text{tr} e^{-\beta H_0} \]

\[ D_{\text{tr}}(\rho_0, \rho_t) \leq \sqrt{\beta} \int_0^t dt' \sqrt{-2 \langle [H_0, V_{t'}]^2 \rangle_{\beta}} \]
Summary

• a new quantum speed limit for initially thermal states derived

• it explicitly exploits structure of the thermal state and depends on temperature

• can be dramatically tighter than generic QSLs in the many-body setting

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Thank you for your attention!