The shock convergence problem in Euler and Lagrangian coordinates

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Abstract. The analytical solution of the problem of converging shock compression of a cold gas sphere was represented in this paper in the following statement. At the initial time a cold gas velocity is zero, and at the external border of the gas sphere the negative velocity is set. In other words, velocity discontinuity is set. After breakdown of a discontinuity the shock will spread from this point into the symmetry center. The boundary moves under the particular law which conforms to the movement of the shock. It moves in Euler coordinates, but the boundary trajectory is a vertical line in Lagrangian coordinates. Generally speaking, all the trajectories of the particles in Lagrangian coordinates are vertical lines. The value of entropy which appeared on the shock retains along each of this lines. Equations that determine the structure of the gas flow between the shock front and the boundary as a function of time and the Euler or Lagrangian coordinate are obtained, as well as the dependence of the entropy on the shock velocity. The problem was solved using original method which dimmers from the generally accepted ones for constructing self-similar solution.

1. Introduction

In different years, the self-similar solutions of the shock-focusing problem in an ideal gas were received based on dimension and similarity theory, which actively developed since the beginning of the last century. The first results were obtained by Guderley. In his work \cite{1} it was assumed that the amplitude of the shock as it approached the center of symmetry unlimited increases. This problem was also solved independently by Sedov \cite{2} and Stanyukovich \cite{3} in 1945. The review of the work on the focusing shocks and cavities in an ideal gas are described in the work of Brushilinskii and Kazhdan \cite{4}. In subsequent years, Lazarus \cite{5}, Hafner \cite{6} calculated self-similarity coefficients for arbitrary adiabatic indexes. Self-similar solutions, as well as self-similarity coefficients for various regimes of unstressed compression of an ideal gas and the collapse of a spherical cavity with the formation of a shock are obtained by Kraiko \cite{7}. The review and analysis of papers on the Guderley problem, both in the classical formulation and for various special cases, were carried out in Ramsey’s paper \cite{8}. In the Guderley problem, the shock wave moves to the center from $r = \infty$, which makes it difficult to use this solution to verify the numerical methods. In work \cite{8}, several methods are proposed for specifying the boundary condition, which fixes the “outer” boundary of the sphere. In our papers \cite{9,10}, a fixed volume of gas is taken, the motion of the boundary of which is subject to a certain law.
self-similar solution is constructed in accordance with the technique described in Kuropatenko’s monograph [11], a similar mechanism was proposed in work [8]. In this paper, comparing of analytical solutions of the problem of converging shock compression of a gas sphere in Euler and Lagrangian coordinates is represented [9, 10].

2. Formulation of the problem
At \( t = t_0 \) initial parameters for the gas \( \rho_0 = \text{const}, \ U_0 = 0, \ P_0 = 0, \ E_0 = 0 \), where \( \rho \) is density, \( U \) is velocity, \( P \) is pressure, \( E \) is specific internal energy. The index "0" hereinafter denotes parameters at the initial point of time \( t_0 \). Velocity on the boundary is \( U_\text{g0} < 0 \). In other words, velocity jumps on the boundary, producing a shock wave which moves into the sphere. The first independent variable is the time \( t \). The second variable in Lagrangian coordinates is a spherical mass \( M \) or radius \( r \) in Euler coordinates. Parameters of the gas between the shock wave and the boundary are determined by the system of Euler–Helmholtz conservation laws. Take the equation of state for ideal gas in two forms

\[
P = (\gamma - 1) \rho E, \quad P = F(s) \rho^\gamma,
\]

where \( F(s) \) is entropy function.

Conservation laws on the shock are of the form [11]

\[
\rho_w (D - U_w) - \rho_0 D = 0, \quad (2)
\]
\[
\rho_0 D U_w - P_w = 0, \quad (3)
\]
\[
\rho_0 D (E_w + U_w^2/2) - P_w U_w = 0. \quad (4)
\]

The index “\( w \)” indicates values on the shock wave, \( D \) is the shock velocity.

3. Shock in a gas sphere in Euler coordinates
For ideal gas (1), equations (2)–(4) on the shock are simplified

\[
\rho_w = \frac{\gamma + 1}{\gamma - 1} \rho_0, \quad U_w = \frac{2}{\gamma + 1} D, \quad P_w = \rho_0 D U_w. \quad (5)
\]

Substituting \( P_w \) and \( \rho_w \) from (5) into (1), we obtain the entropy dependence on the shock velocity

\[
F = \frac{2}{\gamma + 1} \rho_0^{1-\gamma} D^2 \left( \frac{\gamma - 1}{\gamma + 1} \right)^\gamma. \quad (6)
\]

At the time when the shock converges, \( t_f \), its coordinate \( r_w \) is zero. The equation of motion which satisfies this condition is

\[
r_w = r_0 \left( \frac{t_f - t}{t_f - t_0} \right)^n \quad (7)
\]

for \( n > 0 \). Where \( r_0 \) is initial radius of the sphere. Its differentiation gives shock velocity

\[
D = D_0 \left( \frac{t_f - t}{t_f - t_0} \right)^{n-1}, \quad (8)
\]

where

\[
D_0 = -r_0 n / (t_f - t_0). \quad (9)
\]

Flow parameters are defined by

\[
\frac{\partial p}{\partial t} + U \frac{\partial p}{\partial r} + \rho \frac{\partial U}{\partial r} + \frac{2 \rho U}{r} = 0, \quad \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} = 0. \quad (10)
\]
\[
\frac{\partial E}{\partial t} + U \frac{\partial E}{\partial r} - \frac{P}{\rho^2} \left( \frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial r} \right) = 0. \tag{11}
\]

For ideal gas (1), we transform equation (11) to the form

\[
\frac{\partial P}{\partial t} + U \frac{\partial P}{\partial r} + \gamma P \left( \frac{\partial U}{\partial r} + \frac{2U}{r} \right) = 0. \tag{12}
\]

For solving the problem we change from the variables \( t \) and \( r \) to variables \( t \) and \( \xi(t, r) \). The function \( \xi(t, r) \) is taken such as to remain constant on the shock. Its simplest form reads as

\[
\xi = \frac{r}{r_0} \left( \frac{t_f - t_0}{t_f - t} \right)^n. \tag{13}
\]

Derivatives

\[
\frac{\partial \xi}{\partial t} = \frac{r n}{r_0 (t_f - t)} \left( \frac{t_f - t_0}{t_f - t} \right)^n, \quad \frac{\partial \xi}{\partial r} = \frac{t}{r_0} \left( \frac{t_f - t_0}{t_f - t} \right)^n
\]

together with the relation \( r(t, \xi) \) following from (13)

\[
r = r_0 \xi \left( \frac{t_f - t}{t_f - t_0} \right)^n
\]

substituting in (10)–(11)

\[
\frac{\partial \rho}{\partial t} + \frac{n \xi}{t_f - t} \frac{\partial \rho}{\partial \xi} + \left( U \frac{\partial U}{\partial \xi} + \rho \frac{\partial U}{\partial \xi} \right) \frac{1}{r_0} \left( \frac{t_f - t_0}{t_f - t} \right)^n + \frac{2 \rho U}{r_0 \xi} \left( \frac{t_f - t_0}{t_f - t} \right)^n = 0, \tag{15}
\]

\[
\frac{\partial U}{\partial t} + \frac{n \xi}{t_f - t} \frac{\partial U}{\partial \xi} + \left( U \frac{\partial U}{\partial \xi} + \rho \frac{\partial U}{\partial \xi} \right) \frac{1}{r_0} \left( \frac{t_f - t_0}{t_f - t} \right)^n = 0, \tag{16}
\]

\[
\frac{\partial P}{\partial t} + \frac{n \xi}{t_f - t} \frac{\partial P}{\partial \xi} + \left( U \frac{\partial U}{\partial \xi} + \gamma P \frac{\partial U}{\partial \xi} \right) \frac{1}{r_0} \left( \frac{t_f - t_0}{t_f - t} \right)^n + \frac{2 \gamma PU}{r_0 \xi} \left( \frac{t_f - t_0}{t_f - t} \right)^n = 0. \tag{17}
\]

Now express \( P, \rho \) and \( U \) as functions of time multiplied by functions of \( \xi \):

\[
P = \alpha_p(t) \Pi(\xi), \quad \rho = \alpha_\rho(t) \delta(\xi), \quad U = \alpha_u(t) M(\xi). \tag{18}
\]

Choose \( \Pi(\xi), \delta(\xi), M(\xi) \) such that to allow them at \( \xi = 1 \) take the values

\[
\delta_w = \frac{\gamma + 1}{\gamma - 1}, \quad M_w = \frac{2}{\gamma + 1}, \quad \Pi_w = \frac{2}{\gamma + 1}. \tag{19}
\]

With these \( \delta_w, w, M_w \) the functions \( \alpha_\rho, \alpha_u \) and \( \alpha_p \) take the forms

\[
\alpha_\rho = \rho_0, \quad \alpha_u = D_0 \left( \frac{t_f - t_0}{t_f - t} \right)^{1-n}, \quad \alpha_p = \rho_0 D_0^2 \left( \frac{t_f - t_0}{t_f - t} \right)^{2(1-n)}. \tag{20}
\]

After appropriate manipulation for conversion to the functions \( \Pi, \delta, M \) and variables \( t, \xi \), we obtain equations for functions which only depend on \( \xi \):

\[
(M - \xi) \delta' + \delta M' + \frac{2 \delta M}{\xi} = 0, \quad \frac{n - 1}{n} \delta M + M' \delta (M - \xi) + \Pi' = 0, \tag{21}
\]

\[
\frac{2}{n} (n - 1) \Pi + \Pi' (M - \xi) + \gamma \Pi M' + \frac{2 \gamma \Pi M}{\xi} = 0. \tag{22}
\]

For \( M', \delta', \Pi' \), these equations give a system of linear homogeneous equations. If its determinant

\[
Z = (M - \xi) \left( \gamma \Pi - \delta (M - \xi)^2 \right) \tag{23}
\]
Table 1. The corresponding values of $\gamma$ and $n$ in Euler coordinates.

| $\gamma$ | 1.1 | 1.2 | 4/3 | 1.4 | 5/3 |
|----------|-----|-----|-----|-----|-----|
| $n$      | 0.795973 | 0.757142 | 0.729259 | 0.717175 | 0.688377 |

is nonzero, the system has a unique solution:

$$M' = \frac{(R_1 - 2\gamma n \Pi)}{R_2},$$

$$\delta' = \frac{\delta (2M \delta n (M - \xi)^2 - R_1)}{R_2 (M - \xi)},$$

$$\Pi' = \frac{\delta \Pi (2 (n \gamma M + \xi (n - 1)) (M - \xi) - (n - 1) \gamma \xi M)}{R_2},$$

where

$$R_1 = (n - 1) \xi ((M - \xi) \delta M - 2 \Pi),$$

$$R_2 = n \xi \left(\gamma \Pi - (M - \xi)^2 \delta \right).$$

At a point $\xi_*$ with $Z = 0$, the matrix of coefficients and the augmented matrix of coefficients should be considered. At this point their ranks are identical and equal to 2, and all third-order minors are zero, hence the system of equations (21) and (22) has a unique solution.

From equations (24) we find the appropriate values of $n$ for each $\gamma$. Values $n$ and $\gamma$ at which the sum of the numerators from (24) and denominator $R_2$ vanishes simultaneously are given in table 1.

4. Shock in a gas sphere in Lagrangian coordinates

The Lagrangian coordinate $M_w$ of the shock associated by the equation with its coordinate Euler $r_w$

$$M_w = \frac{4}{3} \pi \rho_0 r_w^3. \quad (25)$$

The shock velocity $W$ in Lagrangian coordinates and velocity $D$ in Euler coordinates associated by the equation

$$W = (3M_w)^{2/3} (4\pi \rho_0)^{1/3} D. \quad (26)$$

Expressing in (26) $D$ through $W$ and $M_w$ and substituting in (2)–(4), we obtain the conditions on the shock wave, containing $W$ and $M_w$

$$\left(\frac{1}{\rho_w} - \frac{1}{\rho_0}\right) W + (4\pi)^{1/3} \left(\frac{3M_w}{\rho_0}\right)^{2/3} U_w = 0, \quad (27)$$

$$U_w W - (4\pi)^{1/3} \left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w = 0, \quad (28)$$

$$(E_w + 0.5 U_w^2) W - (4\pi)^{1/3} \left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w U_w = 0. \quad (29)$$
From (1), (27)–(29) ensue expression for $\rho_w$, $U_w$ and $P_w$

$$\rho_w = \frac{\gamma + 1}{\gamma - 1} \rho_0,$$

$$U_w = \frac{2W}{(\gamma + 1) (4\pi \rho_0)^{1/3} (3M_w)^{2/3}},$$

$$P_w = \frac{2\rho_0^{1/3} W^2}{(\gamma + 1) (4\pi)^{2/3} (3M_w)^{4/3}}. \quad (30)$$

Substituting $\rho_w$ and $P_w$ into (1), we obtain

$$F_w = \frac{2}{\gamma + 1} \left(\frac{\gamma - 1}{\gamma + 1}\right)^\gamma \frac{W^2 \rho_0^{(1-3\gamma)/3}}{(4\pi)^{2/3} (3M_w)^{4/3}}. \quad (31)$$

We define the shock trajectory in the form

$$M_w = M_0 \left(\frac{t_f - t}{t_f - t_0}\right)^n. \quad (32)$$

The shock velocity in Lagrangian coordinates depends on the time

$$W = W_1 \left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}, \quad (33)$$

where $W_0 = -M_0 n/(t_f - t_0)$.

Adiabatic flow parameters between the shock and the gas boundary are defined by

$$\left(\frac{\partial v}{\partial t}\right)_M - U = 0, \quad \left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi \rho^2 \frac{\partial (\rho^2 U)}{\partial M} = 0, \quad (34)$$

$$\left(\frac{\partial U}{\partial t}\right)_M + 4\pi r^2 \frac{\partial (F \rho^\gamma)}{\partial M} = 0. \quad (35)$$

These equations contain three desired functions $r$, $\rho$ and $U$. Let us pass in (34), (35) to new desired functions

$$R = r^3, \quad C = r^2 U. \quad (36)$$

After the pass to the functions $R$ and $C$, equations (34)–(35) take the form

$$\left(\frac{\partial R}{\partial t}\right)_M - 3C = 0, \quad \left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi \rho^2 \frac{\partial C}{\partial M} = 0, \quad (37)$$

$$\left(\frac{\partial C}{\partial t}\right)_M + 4\pi R^2 \frac{\partial (F \rho^\gamma)}{\partial M} - 2C^2 R^{-1} = 0. \quad (38)$$

From (25), (30) and (32)–(33) follow the dependence of $R_w$ and $C_w$ on $M_w$

$$R_w = R_0 \frac{M_w}{M_0}, \quad C_w = C_0 \left(\frac{M_w}{M_0}\right)^{(n-1)/n}. \quad (39)$$

The values $R_0$, $C_0$ are functions of (36) at the initial time. Equations (37) and (38) are essential for finding $R$, $C$ and $\rho$ in the area of the integration $M_w \leq M \leq M_0$, $t_0 \leq t \leq t_f$.

Let us proceed from the variables $t$, $M$ to variables $t$, $\xi(t, M)$. With the help of the equations for the derivatives

$$\left(\frac{\partial}{\partial t}\right)_M = \left(\frac{\partial}{\partial t}\right)_\xi + \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M, \quad \left(\frac{\partial}{\partial M}\right)_t = \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t$$
we transform equations (37)–(38)

\[
\frac{\partial R}{\partial t} + (\frac{\partial R}{\partial \xi})_t (\frac{\partial \xi}{\partial t})_M - 3C = 0,
\]

\[
(\frac{\partial \rho}{\partial t})_\xi + (\frac{\partial \rho}{\partial \xi})_t (\frac{\partial \xi}{\partial t})_M + 4\pi \rho^2 (\frac{\partial C}{\partial \xi})_t (\frac{\partial \xi}{\partial M})_t = 0,
\]

\[
(\frac{\partial C}{\partial t})_\xi + (\frac{\partial C}{\partial \xi})_t (\frac{\partial \xi}{\partial t})_M - \frac{2C^2}{R} \\
+ 4\pi R^\delta \left[ \rho^\gamma \left( \frac{\partial F}{\partial \xi} \right)_t (\frac{\partial \xi}{\partial M})_t + \gamma F \rho^\gamma \left( \frac{\partial \rho}{\partial \xi} \right)_t (\frac{\partial \xi}{\partial M})_t \right] = 0.
\]

We define the dependence of \( T(\xi, t) \) in the form

\[
\xi = \frac{M}{M_0} \left( \frac{t_f - t}{t_f - t_0} \right)^\eta.
\]

It follows from (32) and (44) that on the shock wave at \( M = M_w \) will be \( \xi = 1 \).

Now express \( R, \rho, \) and \( C \) as functions of time multiplied by functions of \( \xi \):

\[
R = \alpha_R (t) T(\xi), \quad \rho = \alpha_\rho (t) \delta(\xi), \quad C = \alpha_C (t) Z(\xi).
\]

Since \( \xi = 1 \) on the shock, the values \( T_w(1), \delta_w(1), Z_w(1) \) must be constant. The dependence \( R_w(\xi, t), C_w(\xi, t) \) is obtained at \( \xi = 1 \) from (39) and (44)

\[
R_w = R_0 \left( \frac{t_f - t}{t_f - t_0} \right)^n, \quad C_w = C_0 \left( \frac{t_f - t}{t_f - t_0} \right)^{n-1}.
\]

To eliminate the arbitrary choice in the separation of functions \( R, \rho, C \) in (45) we require that \( T(\xi), \delta(\xi), Z(\xi) \) would be dimensionless. Then from (45) and (46) we go to

\[
T_w = T_1, \quad \alpha_R (t) = R_0 \left( \frac{t_f - t}{t_f - t_0} \right)^n T_1^{-1}.
\]

Similarly for \( \rho \) and \( C \) we obtain the relations

\[
\delta_w = \delta_1, \quad \alpha_\rho = \rho_0 \left( \frac{\gamma + 1}{\gamma - 1} \right) \delta_1^{-1},
\]

\[
Z_w = Z_1, \quad \alpha_C (t) = C_0 \left( \frac{t_f - t}{t_f - t_0} \right)^{n-1} Z_1^{-1}.
\]

By substituting (45)–(49) into (40)–(43), we obtain three equations for \( T, \delta \) and \( Z \):

\[
\xi T' = A_1, \quad \delta_1 B_1 Z' - \xi Z_1 \delta'_1 = 0,
\]

\[
-\xi Z_1^{-1} Z' + C_1 \gamma \xi \delta_1^{-1} \delta' = C_2,
\]

where the prime indicates differentiation of \( \xi \),

\[
A_1 = T - \frac{2ZT_1}{(\gamma + 1) Z_1}, \quad B_1 = \frac{2\delta^2}{(\gamma - 1) \delta_1^2}, \quad C_1 = \frac{T_1^{4/3} \delta_1^{-1} \xi^{-(n+6)/3n}}{T_1^{4/3} \delta_1^{-1} \xi^{-(n+6)/3n}},
\]

\[
C_2 = \frac{4Z^2 T_1}{3(\gamma + 1) Z_1^2 T - \frac{(n-1)Z}{nZ_1} - C_1 \frac{2(n-3)\delta}{3n\delta_1}}.
\]

Equations (50)–(52) are the system of linear inhomogeneous equations regarding to the \( T', \delta', Z' \). The determinant of the system is the following

\[
\Delta = B_1 C_1 \gamma \xi - \xi^2.
\]
Table 2. The corresponding values of $\gamma$ and $n$ in Lagrangian coordinates.

| $\gamma$ | 1.1 | 1.2 | 4/3 | 1.4 | 5/3 |
|----------|-----|-----|-----|-----|-----|
| $n$      | 2.387916 | 2.271434 | 2.183068 | 2.151532 | 2.065135 |

If $\Delta \neq 0$, the solution of the system (50)–(52) has the form

$$T' = \frac{A_1}{\xi}, \quad \delta' = \frac{B_1 C_2 \delta_1}{\Delta}, \quad Z' = \frac{\xi C_2 Z_1}{\Delta}.$$  

Calculations show that there exists some value of $n$ such that the determinant vanishes. In this case, there is a solution, if $C_2$ is also vanishes. The values $n$ corresponding to the values $\gamma$ are given in the table 2.

5. Solution

The functions in systems of ordinary differential equations (24) or (51) are calculated at certain value $n$ by integration in the interval $1 \leq \xi \leq \infty$. But to verify calculation methods we have to set limited area. In Euler coordinates the boundary coordinate of gas sphere $r_g$ at time $t$ is found from the law of conservation of mass. The total gas sphere mass at moment $t_0$ is $M_0 = \frac{4}{3}\pi \rho_0 r_0^3$.

At $t \geq t_0$ the sphere mass is conserved. It equals to the sum of the mass of the undisturbed gas between the symmetry center and shock and the gas mass between the shock and sphere boundary $r_g$

$$M_0 = \frac{4}{3}\pi r_g^3 \rho_0 + \int_{r_w}^{r_g} 4\pi r^2 \rho(r, t) dr.$$  

We pass from integration over $r$ to $\xi$. As a result, we obtained equation to determine self-similar boundary coordinate $\xi_g$ at moment $t$

$$3 \int_1^{\xi_g} \delta \xi^2 d\xi = \left( \frac{t_f - t_0}{t_f - t} \right)^{3n} - 1.$$

Then, coordinate $r_g(t)$ is found from the value of $\xi_g$ for fixed $t$ using equation (13).

In Lagrangian coordinates this calculation is not required, since the sphere mass is conserved, and the self-similar boundary coordinate $\xi_g$ is determined from equation (44) at $M = M_0$.

Further, the required functions are found from (18) or (45) in the region $r_w \leq r \leq r_g$ occupied by the gas at time $t$.

For example, we set the following initial conditions: a cold gas sphere of radius $r_{g0} = 1$ with parameters $P_0 = 0$, $\rho_0 = 1$, $U_0 = 0$, $U_{g0} = -1$, $\gamma = 1.4$, $t_0 = 0$. The transition to dimensionless variables was conducted by the following formulas:

$$P = \frac{P_d}{\rho_0 c_0^2}, \quad \rho = \frac{\rho_d}{\rho_0}, \quad U = \frac{U_d}{c_0}, \quad r = \frac{r_d}{l_0}, \quad t = \frac{t_0 c_0}{l_0},$$

where the subscript $d$ denotes a dimensional quantity. The unit size of a region $l_0 = 1$ m, the speed of sound $c_0 = 330$ m/s and the density $\rho_0 = 1.29$ kg/m$^3$ of air under normal conditions were used for dimensionless measurements. Generally speaking, substitution to the dimensionless variables can be carried out in any convenient way, and dimensional quantities can be directly used in the formulation of the problem. Pressure, density and velocity profiles at $t = 0.35, 0.45, 0.55$ are shown in figure 1. The solution in Euler and Lagrangian coordinates are identical and it is impossible to separate them in these graphs.
Figure 1. The pressure (a), the density (b) and the velocity (c) as functions of the radius at three moments of time.

6. Conclusion
The analytical solution of the converging shock problem for a gas sphere with the impermeable wall was constructed in Euler and Lagrangian coordinates. It is shown by a physically founded method that there is no movement of matter across the boundary. In addition, corresponding self-similar coefficients for a wide range of adiabatic indexes of an ideal gas were found. This solution may be used for verification of shock calculation methods.

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