THE MANAKOV SYSTEM AS TWO MOVING
INTERACTING CURVES

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The two time-dependent Schrödinger equations in a potential \( V(s, u) \), \( u \) denoting time, can be interpreted geometrically as a moving interacting curves whose Fermi-Walker phase density is given by \( -\frac{\partial V}{\partial s} \). The Manakov model appears as two moving interacting curves using extended da Rios system and two Hasimoto transformations.

Keywords: Soliton curves and surfaces, Manakov model, Periodic solitons

1. Introduction

In recent years, there has been a large interest in the applications of the Frenet-Serret equations¹,² for a space curve in various contexts, and interesting connections between geometry and integrable nonlinear evolution equations have been revealed. The subject of how space curves evolve in time is of great interest and has been investigated by many authors. Hasimoto³ showed that the evolution of a thin vortex filament regarded as a moving space curve can be mapped to the nonlinear Schrödinger equation (NLSE):

\[
i\Psi_u + \Psi_{ss} + \frac{1}{2} |\Psi|^2 \Psi = 0,
\]

Here, \( u \) and \( s \) are time and space variables, respectively, subscripts denote partial derivatives. Lamb⁴ used Hasimoto transformation to connect other motions of curves to the mKdV and sine-Gordon equations. Langer and Perline⁵ showed that the dynamics of non-stretching vortex filament in \( \mathbb{R}^3 \) leads to the NLS hierarchy. Motions of curves on \( S^2 \) and \( S^3 \) were considered.
by Doliwa and Santini.\(^6\) Lakshmanan\(^7\) interpreted the dynamics of a nonlinear string of fixed length in \(\mathbb{R}^3\) through the consideration of the motion of an arbitrary rigid body along it, deriving the AKNS spectral problem without spectral parameter. Recently, Nakayama\(^8\) showed that the defocusing nonlinear Schrödinger equation, the Regge-Lund equation, a coupled system of KdV equations and their hyperbolic type arise from motions of curves on hyperbola in the Minkowski space. Recently the connection between motion of space or plane curves and integrable equations has drawn wide interest and many results have been obtained.\(^9,10,13–17\)

2. Preliminaries

2.1. The Manakov model

Time-dependent Schrödinger equation in potential \(V(s, u)\)

\[
i\Psi_u + \Psi_{ss} - V(s, u)\Psi = 0, \tag{2}\]

goes into the NLS eq. (1) if the potential \(V(s, u) = -1/2|\psi(s, u)|^2\). Similarly, a set of two time-dependent Schrödinger equations:

\[
i\Psi_{1,u} + \Psi_{1,ss} - V(s, u)\Psi_1 = 0, \quad i\Psi_{2,u} + \Psi_{2,ss} - V(s, u)\Psi_2 = 0, \tag{3}\]

where \(V(s, u) = -|\Psi_1|^2 - |\Psi_2|^2\), can be viewed as the Manakov system:

\[
i\Psi_{1,u} + \Psi_{1,ss} + (|\Psi_1|^2 + |\Psi_2|^2)\Psi_1 = 0, \quad \tag{4}\]
\[
i\Psi_{2,u} + \Psi_{2,ss} + (|\Psi_1|^2 + |\Psi_2|^2)\Psi_2 = 0. \quad \tag{5}\]

It is convenient to use two Hasimoto transformations\(^3\)

\[
\Psi_i = \kappa_i(s, u) \exp \left[ i \int s \tau_i(s', u) ds' \right], \quad i = 1, 2, \tag{6}\]

in Eqs. (4), (5). Equating imaginary and real parts, this leads to the coupled partial differential equations (extended daRios system\(^18\))

\[
\kappa_{i,u} = -(\kappa_i \tau_i)_s - \kappa_{i,s} \tau_i, \quad i = 1, 2, \tag{7}\]
\[
\tau_{i,u} = \left[ \frac{\kappa_{i,ss}}{\kappa_i} - \tau_i^2 \right]_s - V(s, u)_s, \tag{8}\]

where

\[
V(s, u) = -|\Psi_1|^2 - |\Psi_2|^2 = -\kappa_1^2 - \kappa_2^2. \quad \tag{9}\]
2.2. Soliton curves

A three dimensional space curve is described in parametric form by a position vectors \( \mathbf{r}_i(s) \), where \( s \) is the arc length. Let \( \mathbf{t}_i = (\partial \mathbf{r}_i / \partial s) \), \( i = 1, 2 \) be the two unit tangent vectors along the two curves. At a given instant of time the triads of unit orthonormal vectors \( \{ \mathbf{t}_i, \mathbf{n}_i, \mathbf{b}_i \} \), where \( \mathbf{n}_i \) and \( \mathbf{b}_i \) denote the normals and binormals, respectively, satisfy the Frenet-Serret equations for two curves

\[
\mathbf{t}_i, s = \kappa_i \mathbf{n}_i, \quad \mathbf{n}_i, s = -\kappa_i \mathbf{t}_i + \tau_i \mathbf{b}_i, \quad \mathbf{b}_i, s = -\tau_i \mathbf{n}_i, \quad i = 1, 2,
\]

where \( \kappa_i \) and \( \tau_i \) denote, respectively the two curvatures and torsions of the curves.

A moving curves are described by \( \mathbf{r}_i(s,u) \), where \( u \) denote time. The temporal evolution of two triads corresponding to a given value \( s \) can be written in the general form as

\[
\mathbf{t}_{i,u} = g_i \mathbf{n}_i + h_i \mathbf{b}_i, \quad \mathbf{n}_{i,u} = -g_i \mathbf{t}_i + \tau_i^0 \mathbf{b}_i, \quad \mathbf{b}_{i,u} = -h_i \mathbf{t}_i - \tau_i^0 \mathbf{n}_i, \quad i = 1, 2,
\]

where the coefficients \( g_i, h_i \) and \( \tau_i^0 \), as well as \( \kappa_i \) and \( \tau_i \), are functions of \( s \) and \( u \).

Introduce

\[
L_i = \begin{pmatrix}
0 & \kappa_i & 0 \\
-\kappa_i & 0 & \tau_i \\
0 & -\tau_i & 0
\end{pmatrix}, \quad M_i = \begin{pmatrix}
0 & g_i & h_i \\
-g_i & 0 & \tau_i^0 \\
0 & -h_i - \tau_i^0 & 0
\end{pmatrix}
\]

and \( \Delta \mathbf{t}_i \equiv (\mathbf{t}_{i,su} - \mathbf{t}_{i,us}), \Delta \mathbf{n}_i \equiv (\mathbf{n}_{i,su} - \mathbf{n}_{i,us}), \) and \( \Delta \mathbf{b}_i \equiv (\mathbf{b}_{i,su} - \mathbf{b}_{i,us}). \) Then

\[
\begin{pmatrix}
\Delta \mathbf{t}_i \\
\Delta \mathbf{n}_i \\
\Delta \mathbf{b}_i
\end{pmatrix} = (\partial_s M_i - \partial_u L_i + [L_i, M_i])
\begin{pmatrix}
\mathbf{t}_i \\
\mathbf{n}_i \\
\mathbf{b}_i
\end{pmatrix} = \begin{pmatrix}
0 & \alpha_i^1 & \alpha_i^2 \\
-\alpha_i^1 & 0 & \alpha_i^3 \\
-\alpha_i^2 & -\alpha_i^3 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{t}_i \\
\mathbf{n}_i \\
\mathbf{b}_i
\end{pmatrix},
\]

where

\[
\alpha_i^1 = \kappa_{i,u} g_i + h_i \tau_i, \quad \alpha_i^2 = -h_i s + \kappa_i \tau_i^0 - g_i \tau_i, \quad \alpha_i^3 = \tau_{i,u} - \tau_{i,s} - \kappa_i h_i,
\]

\[
\kappa_{i,u} = g_i - h_i \tau_i, \quad \tau_i^0 = (h_i s + g_i \tau_i) / \kappa_i.
\]
A generic curve evolution must satisfy the geometric constraints

$$\Delta t_i \cdot (\Delta n_i \times \Delta b_i) = 0,$$

(16)
i.e., $\Delta t_i$, $\Delta n_i$ and $\Delta b_i$ must remain coplanar vectors under time involution. Further, since Eq. (16) is automatically satisfied for $\Delta t_i = 0$, we see that $\Delta n_i$ and $\Delta b_i$ need not necessarily vanish. In addition, we see from (16) that $\Delta t_i = 0$ implies $\alpha_1^i = \alpha_2^i = 0$, so that

$$\Delta n_i = \alpha_3^i \Delta b_i, \quad \Delta b_i = \alpha_3^i \Delta n_i$$

(17)

Substituting these in the second equation in (15) gives

$$\tau_i^0 = \left[ \frac{\kappa_i \alpha_{1,s} - \tau^2_i}{\kappa_i} \right],$$

(18)
hence Eq. (7) yields $(\tau_i^0 - \tau^0_{i,s}) = -V(s,u)_s = (\kappa_1^2 + \kappa_2^2)_s$. Next there is an underlying angle anholonomy$^{19,20}$ or 'Fermi-Walker phase' $\delta \Phi_{FW} = (\tau_{i,u} - \tau_{i,s}^0) ds du$ with respect to its original orientation, when $s$ and $u$ change along an infinitesimal closed path of area $ds du$.

3. Integration of the extended da Rios system

The coupled nonlinear equations (7),(8) constitute the extended Da Rios system as derived in$^{18}$. The solutions of (7),(8) with $\kappa(\xi)$ and $\tau = \tau(\xi)$, where $\xi = s - cu$ are simple. On substitution, we obtain

$$c \kappa_i, \xi = 2 \kappa_i, \xi \tau_i + \kappa_i \tau_i, \xi, \quad \xi = s - cu, i = 1, 2,$$

(19)

$$-c \tau_i, \xi = \left[ -\tau^2_i + \frac{\kappa_i \xi}{\kappa_i} + \kappa_1^2 + \kappa_2^2 \right] \xi, \quad \tau_i = \frac{c}{2}$$

(20)

where we use the boundary condition $\kappa_i \to 0, i = 1, 2$ as $s \to \infty$. Hence $\kappa_i$ obey the nonlinear oscillator equations

$$\kappa_i, \xi + \left( \sum_{j=1}^{2} \kappa_j^2 \right) \kappa_i = a_i \kappa_i, \quad i = 1, 2.$$  

(21)

where $a_i, i = 1, 2$ are arbitrary constants.

Example 1 One soliton solutions of the Manakov system are given by

$$\Psi_1 = \sqrt{2a} \epsilon_1 e^{i \left( \frac{c}{2} (s - s_0) + (a - \frac{1}{4} c^2 u) \right) \text{sech}(\sqrt{a} (s - s_0 - ct))}$$

(22)

$$\Psi_2 = \sqrt{2a} \epsilon_2 e^{i \left( \frac{c}{2} (s - s_0) + (a - \frac{1}{4} c^2 u) \right) \text{sech}(\sqrt{a} (s - s_0 - ct))},$$

(23)

and $|\epsilon_1|^2 + |\epsilon_2|^2 = 1$
We first note that for Manakov system, the expressions for the curvatures $\kappa_i, i = 1, 2$ and the torsions $\tau_i, i = 1, 2$ for the moving curves corresponding to a one soliton solutions of the Manakov system are given by

$$\kappa^2 = \kappa_1^2 + \kappa_2^2 = \sqrt{2a} \sech(\sqrt{a}(s - s_0 - ct)), \quad \tau_1 = \tau_2 = \frac{1}{2}c.$$  \hspace{1cm} (24)

and

$$\kappa_1 = \sqrt{2a} \epsilon_1 \sech(\sqrt{a}(s - s_0 - ct)), \quad \kappa_2 = \sqrt{2a} \epsilon_2 \sech(\sqrt{a}(s - s_0 - ct)).$$

Fig. 1. Two curves (28) of one soliton solution of Manakov system, $\epsilon_1 = \sqrt{2}/\sqrt{3}, \quad \epsilon_2 = 1/\sqrt{3}$

**Example 2** One special solution of Manakov system is written by

$$\kappa_1 = C_1 \cn(\alpha \xi, k), \quad \kappa_2 = C_2 \cn(\alpha \xi, k), \quad (25)$$

where

$$\alpha^2 = \frac{a_1}{2k^2 - 1}, \quad C_1^2 + C_2^2 = 2\alpha^2 k^2, \quad a_1 = a_2 = a.$$  \hspace{1cm} (26)
In the limit $k \to 1$ we obtain the well known Manakov soliton solution

$$
\Psi_1 = \frac{\sqrt{2a} \epsilon_1 \exp \{i \left( 2 \epsilon_1 (s - s_0) + (a - \frac{1}{4}c^2)u \right) \}}{\text{ch} \left( \sqrt{a}(s - s_0 - ct) \right)},
$$

$$
\Psi_2 = \frac{\sqrt{2a} \epsilon_2 \exp \{i \left( 2 \epsilon_2 (s - s_0) + (a - \frac{1}{4}c^2)u \right) \}}{\text{ch} \left( \sqrt{a}(s - s_0 - ct) \right)}.
$$

Here we introduce the following notations

$$
|\epsilon_1|^2 + |\epsilon_2|^2 = 1, \quad \zeta_1 = \frac{1}{2}c + i\sqrt{a} = \xi_1 + i\eta_1, \quad (27)
$$

where $s_0$ is the position of soliton, $(\epsilon_1, \epsilon_2)$ are the components of polarization vector. The real part of $\zeta_1$ i.e. $c/2$ gives us the soliton velocity while the imaginary part of $\zeta_1$, i.e. $\sqrt{2a}$, gives the soliton amplitude and width.

**Example 3** Integrating (10) for two unit tangent vectors along the curves $t_i = (\partial r_i / \partial s), i = 1, 2$ for position vectors $r_i(s), i = 1, 2$ we obtain

$$
r_j = \begin{pmatrix}
\frac{s_j}{2} - \frac{\epsilon_j}{\epsilon_j^2 + \epsilon_k^2} \text{tanh} (\epsilon_j (s - cu)) \\
- \frac{\epsilon_k}{\epsilon_j^2 + \epsilon_k^2} \text{sech}(\epsilon_j (x - cu)) \cos \left( \frac{1}{2}cs + (\epsilon_j^2 - \frac{1}{4}c^2)u \right) \\
- \frac{\epsilon_j}{\epsilon_j^2 + \epsilon_k^2} \text{sech}(\epsilon_j (x - cu)) \sin \left( \frac{1}{2}cs + (\epsilon_j^2 - \frac{1}{4}c^2)u \right)
\end{pmatrix}, \quad j = 1, 2 \quad (28)
$$

and $\epsilon_1 = \cos \alpha, \epsilon_2 = \sin \alpha$, where $\alpha$ is arbitrary positive number.

**Example 4** Let $u(x) = 6\psi(\xi + \omega')$ be the two-gap Lamé potential with simple periodic spectrum (see for example (21))

$$
\lambda_0 = -\sqrt{3g_2}, \quad \lambda_1 = -3e_0, \quad \lambda_2 = -3e_1, \quad \lambda_3 = -3e_2, \quad \lambda_4 = \sqrt{3g_2}. \quad (29)
$$

and the corresponding Hermite polynomial have the form

$$
F(\psi(\xi + \omega'), \lambda) = \lambda^2 - 3\psi(\xi + \omega')\lambda + 9\psi^2(\xi + \omega') - \frac{9}{4}g_2. \quad (30)
$$

Consider the genus 2 nonlinear anisotropic oscillator (21) with Hamiltonian

$$
H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{4} (\kappa_1^2 + \kappa_2^2)^2 - \frac{1}{2} (a_1 \kappa_1^2 + a_2 \kappa_2^2), \quad (31)
$$

where $(\kappa_i, p_i), i = 1, 2$ are canonical variables with $p_i = \kappa_{i,x}$ and $a_1, a_2$ are arbitrary constants. The simple solutions of these system are given in terms of Hermite polynomial

$$
\kappa_1^2 = 2 \frac{F(x, \lambda_1)}{\lambda_2 - \lambda_1}, \quad \kappa_2^2 = 2 \frac{F(x, \lambda_2)}{\lambda_1 - \lambda_2}, \quad (32)
$$

Let us list the corresponding solutions

(A) Periodic solutions in terms of single Jacobian elliptic function
The nonlinear anisotropic oscillator admits the following solutions:

\[ \kappa_1 = C_1 sn(\alpha \xi, k), \quad \kappa_2 = C_2 cn(\alpha \xi, k). \]  

(33)

Here the amplitudes \( C_1, C_2 \) and the temporal pulse-width \( 1/\alpha \) are defined by the parameters \( a_1 \) and \( a_2 \) as follows:

\[ \alpha^2 k^2 = a_2 - a_1, \quad C_1^2 = a_2 + \alpha^2 - 2\alpha^2 k^2, \quad C_2^2 = a_1 + \alpha^2 + \alpha^2 k^2, \]  

(34)

where \( 0 < k < 1 \).

Following our spectral method it is clear, that the solutions (33) are associated with eigenvalues \( \lambda_2 = -e_2 \) and \( \lambda_3 = -e_3 \) of one-gap Lamé potential.

(B) Periodic solutions in terms of products of Jacobian elliptic functions

Another solution is defined by

\[ \kappa_1 = C dn(\alpha \xi, k) sn(\alpha \xi, k), \quad \kappa_2 = C dn(\alpha \xi, k) cn(\alpha \xi, k), \]  

(35)

where \( sn, cn, dn \) are the standard Jacobian elliptic functions, \( k \) is the modulus of the elliptic functions \( 0 < k < 1 \). The wave characteristic parameters: amplitude \( C \), temporal pulse-width \( 1/\alpha \) and \( k \) are related to the physical parameters and, \( k \) through the following dispersion relations

\[ C^2 = \frac{2}{5} (4a_2 - a_1), \quad \alpha^2 = \frac{1}{15} (4a_2 - a_1), \quad k^2 = \frac{5(a_2 - a_1)}{4a_2 - a_1}. \]  

(36)

We have found the following solutions of the nonlinear oscillator

\[ \kappa_1 = C a^2 k^2 cn(\alpha \xi, k) sn(\alpha \xi, k), \quad \kappa_2 = C a^2 dn^2(\alpha \xi, k) + C_1 \]  

(37)

where \( C, C_1, \alpha \) and \( k \) are expressed through parameters \( a_1 \) and \( a_2 \) by the following relations

\[ C^2 = \frac{18}{a_2 - a_1}, \quad \alpha^2 = \frac{1}{10} \left( 2a_2 - 3a_1 + \sqrt{\frac{5}{3}(a_2^2 - a_1^2)} \right), \]

\[ k^2 = \frac{2\sqrt{\frac{5}{3}(a_2^2 - a_1^2)}}{\sqrt{\frac{5}{3}(a_2^2 - a_1^2) + 2a_2 - 3a_1}}, \quad C_1 = \frac{C}{30} (4a_1 - a_2). \]  

(38)

(C) Periodic solutions associated with the two-gap Treibich-Verdier potentials. Below we construct the two periodic solutions associated with the Treibich-Verdier potential. Let us consider the potential

\[ u(x) = 6\varphi(\xi + \omega') + \frac{2(e_1 - e_2)(e_1 - e_3)}{\varphi(\xi + \omega') - e_1}, \]  

(39)
and construct the solution in terms of Lamé polynomials associated with the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_1 > \tilde{\lambda}_2$

$$
\begin{align*}
\tilde{\lambda}_1 &= e_2 + 2e_1 + 2\sqrt{(e_1 - e_2)(7e_1 + 2e_2)}, \\
\tilde{\lambda}_2 &= e_3 + 2e_1 + 2\sqrt{(e_1 - e_3)(7e_1 + 2e_3)}.
\end{align*}
$$

(40)

The finite and real solutions $q_1, q_2$ have the form

$$
\begin{align*}
\kappa_1 &= \tilde{C}_1 \text{sn}(\xi, k) \text{dn}(\xi, k) + C_2 \text{dn}(\xi, k), \\
\kappa_2 &= C_3 \text{cn}(\xi, k) \text{dn}(\xi, k) + C_4 \text{cd}(\xi, k),
\end{align*}
$$

(42)

where $\tilde{C}_1, C_1, C_2$ are given in.24,25

In a similar way we can find the elliptic solution associated with the eigenvalues

$$
\begin{align*}
\tilde{\lambda}_1 &= e_2 + 2e_1 + 2\sqrt{(e_1 - e_2)(7e_1 + 2e_2)}, \\
\tilde{\lambda}_2 &= -6e_1,
\end{align*}
$$

(41)

We have

$$
\begin{align*}
\kappa_1 &= \tilde{C}_1 \text{dn}^2(\xi, k), \\
\kappa_2 &= C_1 \text{sn}(\xi, k) \text{dn}(\xi, k) + C_2 \text{sd}(\xi, k),
\end{align*}
$$

(42)

where $\tilde{C}_1, C_1, C_2$ are given in.24,25

The general formula for elliptic solutions of genus 2 nonlinear anisotropic oscillator is given by24

$$
\begin{align*}
\kappa_1^2 &= \frac{1}{\lambda_2 - \lambda_1} \left( 2\tilde{\lambda}_1^2 + 2\lambda_1 \sum_{i=1}^{N} \varphi(\xi - x_i) \\
&+ 6 \sum_{1 \leq i < j \leq N} \varphi(\xi - x_i) \varphi(\xi - x_j) - \frac{Ng_2}{4} + \sum_{1 \leq i < j \leq 5} \lambda_i \lambda_j \right), \\
\kappa_2^2 &= \frac{1}{\lambda_1 - \lambda_2} \left( 2\tilde{\lambda}_2^2 + 2\lambda_2 \sum_{i=1}^{N} \varphi(\xi - x_i) \\
&+ 6 \sum_{1 \leq i < j \leq N} \varphi(\xi - x_i) \varphi(\xi - x_j) - \frac{Ng_2}{4} + \sum_{1 \leq i < j \leq 5} \lambda_i \lambda_j \right),
\end{align*}
$$

where $x_i$ are solutions of equations $\sum_{i \neq j} \varphi'(x_i - x_j) = 0, j = 1, \ldots, N.$

4. Extended da Rios-Betchov system

Following Betchov we can derive the system of equations, which may be reduced to those for a two fictitious gases with negative pressures accompanied with two complicated nonlinear dispersive stresses. Introducing four
new variables $\rho_1 = \kappa_1^2$, $\rho_1 = \kappa_2^2$, $u_1 = 2\tau_1$, $u_2 = 2\tau_2$ using extended Da Rios system (7), (8) we obtain

\[
\frac{\partial \rho_1}{\partial u} + \frac{\partial (\rho_1 u_1)}{\partial s} = 0, \quad \frac{\partial \rho_2}{\partial u} + \frac{\partial (\rho_2 u_2)}{\partial s} = 0,
\]

\[
\frac{\partial (\rho_1 u_1)}{\partial u} + \frac{\partial}{\partial s} \left[ \rho_1 u_1^2 - (\rho_1^2 + \rho_2^2) - \rho_1 \frac{\partial^2 \log \rho_1}{\partial s^2} \right] = 0,
\]

\[
\frac{\partial (\rho_2 u_2)}{\partial u} + \frac{\partial}{\partial s} \left[ \rho_2 u_2^2 - (\rho_1^2 + \rho_2^2) - \rho_2 \frac{\partial^2 \log \rho_2}{\partial s^2} \right] = 0.
\]

5. HF system is gauge equivalent to Manakov system

The vector nonlinear Schrödinger equation is associated with type $\text{A.III}$ symmetric space $\text{SU}(n+1)/\text{S(U(1)⊗U(n))}$. The special case $n = 2$ of such symmetric space is associated with the famous Manakov system.\[26\]

Let us first fix the notations and the normalizations of the basis of $g$. By $\Delta_+$ ($\Delta_-$) we shall denote the set of positive (negative) roots of the algebra with respect to some ordering in the root space. By $\{E_\alpha, H_i\}$, $\alpha \in \Delta, i = 1 \ldots r$ we denote the Cartan–Weyl basis of $g$ with the standard commutation relations.\[27\] Here $H_i$ are Cartan generators dual to the basis vectors $e_i$ in the root space. The root system is invariant under the action of the Weyl group $W(g)$ of the simple Lie algebra $g$.\[27\]

Let us now consider the gauge equivalent systems. The notion of gauge equivalence allows us to associate with the vector nonlinear Schrödinger equation an equivalent equation solvable by the ISM for the gauge equivalent linear problem:\[28\]

\[
\tilde{L}\tilde{\psi}(x,t,\lambda) = \left( i \frac{d}{dx} - \lambda S(x,t) \right) \tilde{\psi}(x,t,\lambda) = 0,
\]

\[
\tilde{M}\tilde{\psi}(x,t,\lambda) = \left( i \frac{d}{dt} - \lambda^2 S - \lambda S_x S(x,t) \right) \tilde{\psi}(x,t,\lambda) = 0,
\]

(43)

where

\[
\tilde{\psi}(x,t,\lambda) = \psi_0^{-1}\psi(x,t,\lambda), \quad S(x,t) = \sum_{\alpha=1}^{r} (S_\alpha E_\alpha + S^\alpha_\alpha E_{-\alpha}) + \sum_{j=1}^{r} S_j H_j,
\]

(44)

\[
S(x,t) = \text{Ad}_{\tilde{\psi}_0} J \equiv \psi_0^{-1} J \psi_0(x,t), \quad J = \sum_{s=1}^{n} H_s,
\]

and $\psi_0 = \psi(x,t,0)$ is the Jost solution at $\lambda = 0$. The zero-curvature condition $[\tilde{L}, \tilde{M}] = 0$ is equivalent to $iS_t - [S, S_{xx}] = 0$, with $S^2 = I_n$.\[28\]
6. Conclusions

In this paper the Manakov model is interpreted as two moving interacting curves. We derive new extended Da Rios system and obtain the soliton, one-, and two-phase periodic solution of two thin vortex filaments in an incompressible inviscid fluid. The solution was explicitly given in terms of Weierstrass and Jacobian elliptic functions.

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