Geometrical Nonlinearity of Circular Plates and Membranes

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We apply the well-established theoretical method developed for geometrical nonlinearities of micro/nano-mechanical beams to circular drums. The calculation is developed under the same hypotheses, the extra difficulty being to analytically describe the (coordinate-dependent) additional stress generated in the structure by the motion. We compare our predictions to published experimental results. Generalization of the presented method to Duffing-type mode-coupling should be a straightforward extension of the presented work.

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I. INTRODUCTION

The field of micro- and nano-electro-mechanics (MEMS and NEMS) [1, 3] has been continuously expanding over the last decades. These devices, which transduce motion into electrical signals, have been both developed into sensors (e.g. pressure gauge [4]) and components (e.g. r.f. signal mixer [5]). Beyond the notorious acceleration, mass spectroscopy applications, it even becomes possible today to embed nanomechanical elements into quantum electronic circuits [8, 10].

Within the field, nonlinearities can be both a limitation or a resource. For all systems that build on linear response, nonlinearities of all kinds limit the dynamic range of the device [11]. On the other hand, one can devise efficient schemes that rely on nonlinearities to work: this very rich area includes applications such that e.g. amplification of small signals [12], bit storage [13, 14], and synchronization of oscillators [15] among others.

In both cases, understanding and mastering the sources of nonlinearities is required, in order to tailor them on demand: maximizing, or minimizing them [17–19, 21]. The main feature that impacts the dynamics of MEMS/NEMS is a Duffing-type nonlinear behavior [20]. The basic modeling capturing the physics is a $k x^3$ restoring force inserted in the dynamics equation of the mechanical mode; in practice, other terms may also contribute and be taken into account [22].

Even if the materials are perfectly Hookean, all devices experience nonlinear behavior at large deformations: these arise from purely geometrical considerations. For flexural doubly-clamped beams, it consists of the extra stress stored in the beam under motion because of stretching [20]. This effect has been widely studied experimentally, even beyond the nonlinear features of a single mode: the same effect indeed couples all the flexural modes of the structure [23–25]. The measurements are in very good agreement with the simple theory built on arguments first proposed in Ref. [26]. On the other hand, no analytic description exists to date of geometrical nonlinearities of drums, despite their broad utilization. The point of the present paper is thus to adapt the very same reasoning applied to doubly-clamped beams to the case of circular drum resonators.

Let us start by recalling the basics of the geometrical nonlinear modeling of beams. We first write the Euler-Bernoulli equation that applies to thin-and-long structures [27].

$$E_I \frac{\partial^4 f(z,t)}{\partial z^4} + S_z \frac{\partial^2 f(z,t)}{\partial z^2} = -\rho A_z \frac{\partial^2 f(z,t)}{\partial t^2},$$ (1)

with $E_I$ the Young’s modulus, $I_z$ the second moment of area, $S_z$ the axial force load, $\rho$ the mass density and $A_z$ the section area. The $z$ index refers to the axis pointing along the beam, see Fig. 1. The beam is assumed homogeneous with a constant cross section over its length $L$. The function $f(z,t)$ describes the transverse motion of the structure (in the $x$ direction), with the proper boundary conditions. This equation essentially neglects rotational inertia of beam elementary elements $\delta z$, and all shearing forces.

When dealing with small displacements, Eq. [1] is solved by linear superposition of eigenmodes $f_n(z,t)$:

$$f_n(z,t) = x_n(t) \psi_n(z),$$ (2)

with $\psi_n(z)$ the mode shape of mode $n$ (no units), and corresponding mode resonance frequency $\omega_n$. $x_n(t)$ is the time-dependent motion associated with the mode; by means of a Rotating-Frame Transform, it writes

\[\psi_n(z) = \frac{1}{\sqrt{S_z L}} \int_{S_z} f_n(z,t) \, dz\]
\( a_n(t) \cos(\omega t + \phi) \) with \( a_n(t) \) a slow varying amplitude variable, nonzero only for \( \omega \approx \omega_n \) (resonance condition). Here, \( S_z = S_{z,0} = \sigma_0 A_z \) the initially stored axial load in the structure (from uniaxial stress \( \sigma_0 \)). With this convention, \( S_z \) is negative for a tensile stored stress. Note that the quantitative value of \( x_n(t) \) depends on the normalization choice of \( \psi_n(z) \); in this paper we will always normalize modal functions to the maximum displacement amplitude, such that at this abscissa \( z \) one gets \( \psi_n(z_n) = 1 \).

The stretching of the beam writes \( S_z = S_{z,0} + \Delta S \) with \( |\Delta S| = E_z A_z \Delta L/L \) and \( \Delta L \) the extension [20]:

\[
\Delta L = \frac{1}{2} \int_0^L \left( \frac{\partial f(z,t)}{\partial z} \right)^2 \, dz,
\]

expanded at lowest order in \( f \). Note that from Eq. [2] this expression is quadratic in motion amplitudes \( x_n(t) \), thus a simple Rotating-Wave Approximation leads to an extension \( \Delta L \propto a_n^2/2 \) (the slow variables): the nonlinear stretching is essentially a static effect, which is why there is no time-delay in the relationship between \( \Delta S \) and \( \Delta L \).

The basic nonlinear modeling consists then in re-injecting Eq. [3] into Eq. [1], and neglecting any other alterations due to the large motion amplitude, e.g., higher order terms in the radius of curvature of the distorted shape or the modification of the mode shape \( \psi_n(z) \) itself [20][26]. While the validity of these assumptions is questionable, it has been found experimentally that this modeling describes very well experimental results [24][25].

For a single mode \( f \rightarrow f_n \), the projection of Eq. [1] onto it (i.e., multiplying the equation by \( \psi_n \) and integrating over the beam length) leads to the definition of modal parameters:

\[
m_n = \rho A_z L \int_0^L [\psi_n(z)]^2 \, dz,
\]

\[
k_n = E_z I_z \int_0^L \left[ \frac{\partial^2 \psi_n(z)}{\partial z^2} \right]^2 \, dz - S_{z,0} \int_0^L \left[ \frac{\partial \psi_n(z)}{\partial z} \right]^2 \, dz,
\]

\[
\tilde{k}_n = \frac{E_z A_z}{2L} \left( \int_0^L \left[ \frac{\partial^2 \psi_n(z)}{\partial z^2} \right]^2 \, dz \right),
\]

with \( m_n \) the mode mass, \( k_n \) the mode spring constant and \( \tilde{k}_n \) the Duffing nonlinear parameter. The resonance frequency verifies \( \omega_n = \sqrt{k_n/m_n} \). Including in Eq. [1] a damping and a drive term is straightforward [20]. The obtained equation of motion for \( x_n \) is then the one of a harmonic oscillator plus a purely cubic nonlinear restoring term \( +k_n x_n(t)^3 \). \( \tilde{k}_n \) is always positive, because of stretching (the mode “hardens”); in the steady-state (\( a_n = \) constant), the resonant response measured while sweeping the drive frequency upwards will be pulled up, with the frequency at maximum amplitude \( \omega_n^{max} \) given by \( \omega_n^{res} = \omega_n + \beta_n (a_n^{max})^2 \) with \( \beta_n = \frac{3}{8} \omega_n \tilde{k}_n \) [20][22]. The free-decay solution can also be analytically produced

![FIG. 2: (Color online) Schematic of a drum device, in its fundamental flexure \( (n = 0, m = 0) \) mode. The biaxial force \( 2\pi R_d T_{r,0} \) is here tensile.](image)

\[22\]. Owing to the success of this modeling, we present below its exact analog in two dimensions.

**II. FORMULATION OF THE PROBLEM**

We now develop the same ideas for the case of a 2D circular structure, see Fig. 2. We first remind the reader about the conventional linear theory [3]. The generic formalism applying to thin drums [obtained within the same reasoning as Eq. [1]] is the Kirchhoff-Love equation:

\[
D_r \Delta^2 f(r, \theta, t) + T_{r,0} \Delta f(r, \theta, t) = -\rho h \frac{\partial^2 f(r, \theta, t)}{\partial t^2},
\]

with \( \Delta \cdots = \frac{1}{r \partial r} \left( r \frac{\partial \cdots}{\partial r} \right) + \frac{1}{r^2 \partial \theta^2} \) the Laplacian operator (here in polar coordinates), \( D_r = \frac{1}{12} E_r h^3/(1 - \nu^2) \) the flexural rigidity in the plane of the drum (\( \nu \) being Poisson’s ratio), \( 2\pi R_d T_{r,0} = 2\pi R_d h \sigma_0 \) the tension within the drum, \( h \) its thickness and \( R_d \) its radius. We assume materials properties \( E_r, \nu, \rho, \sigma_0 \) and thickness \( h \) to be homogeneous and isotropic over the device; in Eq. [7], the \( T_{r,0} \) term resulting from the biaxial stress \( \sigma_0 \) is taken negative for tensile load.

In the limit of small displacements, we write:

\[
f_{n,m}(r, \theta, t) = z_{n,m}(t) \psi_{n,m}(r, \theta),
\]

with \( \psi_{n,m}(r, \theta) = \phi_{n,m}(r) \cos(n \theta) \) the mode shapes and \( z_{n,m}(t) \) the motion amplitude; now two indexes are necessary to label all 2D flexural modes of the structure. Two simple limits are considered in this paper: the high-stress case (membranes, \( D_r = 0 \) with \( T_{r,0} < 0 \) here), and the low-stress one (plates, \( T_{r,0} = 0 \)). Using the boundary
FIG. 3: (Color online) Calculated mode shape \( \psi_{n,m}(r, \theta) \) for mode \( \{n = 2, m = 1\} \) (radius \( R_d = 1 \)). Top: high-stress limit. Bottom: low-stress limit. Both are very similar in topography.

Conditions, the solutions write:

\[
\begin{align*}
\phi_{n,m}(r) &= \frac{\text{BesselJ}_n\left(\frac{\lambda_{n,m} r}{R_d}\right)}{\text{BesselJ}_n\left(\frac{\lambda_{n,m} r_{n,m}}{R_d}\right)}, \\
\text{or} \quad \text{BesselI}_n\left(\frac{\lambda_{n,m} r}{R_d}\right) - \frac{\text{BesselI}_n\left(\lambda_{n,m}\right)}{\text{BesselI}_n\left(\lambda_{n,m}\right)} \text{BesselJ}_n\left(\frac{\lambda_{n,m} r}{R_d}\right) - \frac{\lambda_{n,m} r}{R_d} \text{BesselJ}_n\left(\frac{\lambda_{n,m} r_{n,m}}{R_d}\right),
\end{align*}
\]

for high-stress and low-stress respectively. \( \lambda_{n,m} \) is the mode parameter and \( r_{n,m} \) the radial position of the maximum amplitude (occurring for given angles \( \theta \) when \( n \neq 0 \)). We give the first modes \( \lambda_{n,m} \) and \( r_{n,m} \) in Tab. IV (Appendix A); the mode \( \{n = 2, m = 1\} \) is displayed as an example in Fig. 3 for the two limits (top: high-stress, bottom: low-stress).

The stretching in 2D is a change of surface area per unit angle. This writes mathematically:

\[
\frac{\delta S}{\delta \theta} = \frac{1}{2} \int_0^{R_d} \left[ \left( \frac{\partial f(r, \theta, t)}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial f(r, \theta, t)}{\partial \theta} \right)^2 \right] rdr,
\]

at lowest order in \( f \). Geometrically, this quantity is directly linked to the radial strain \( \epsilon = \Delta r/R_d \) experienced by the drum at its edge: \( \delta S = R_d \delta \theta \Delta r \), i.e. \( \delta S(\theta, t) = R_d^2 \epsilon(\theta, t) \) [see Fig. 4]. Injecting the mode shape Eq. (8) into Eq. (10), one obtains:

\[
\epsilon(\theta, t) = \left( \frac{z_{n,m}(t)}{R_d} \right)^2 \times \left[ \frac{C_{n,m}^{(1)} + C_{n,m}^{(2)}}{2} + \frac{C_{n,m}^{(1)} - C_{n,m}^{(2)}}{2} \cos(2n \theta) \right],
\]

where we have defined (constants with no dimensions):

\[
\begin{align*}
C_{n,m}^{(1)} &= \frac{1}{2} \int_0^{R_d} \left( \frac{d\phi_{n,m}(r)}{dr} \right)^2 rdr, \\
C_{n,m}^{(2)} &= \frac{1}{2} \int_0^{R_d} \frac{n^2}{r^2} \phi_{n,m}(r)^2 rdr, \\
C_{0,m}^{(1)} &= C_{0,m}^{(1)}, \\
C_{0,m}^{(2)} &= C_{0,m}^{(2)},
\end{align*}
\]

The \( C_{n,m}^{(1,2)} \) constants of the first modes are given in Tab. IV Appendix C. We omit indexes \( n, m \) in the labeling of \( \epsilon \) for simplicity. The function Eq. (11) is plotted in Fig. 4 for mode \( \{n = 2, m = 1\} \) in the high-stress limit.

For \( n = 0 \), the problem is isotropic and the solution rather straightforward. However for \( n \neq 0 \), the stress...
within the drum has an extra angle-dependent component \( \cos(2n \theta) \). Eq. (7) has thus to be modified to:

\[
D_r \Delta^2 f + \int_{-h/2}^{+h/2} \frac{1}{r} \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \sigma_\theta \frac{\partial f}{\partial \theta} \right) dz
= -\rho \frac{\partial^2 f}{\partial t^2},
\]

with \( \sigma_r(r, \theta, z, t), \sigma_\theta(r, \theta, z, t) \) the superposition of the initial biaxial stress \( \sigma_0 \) plus the elastic response of the drum to the strain \( \epsilon \), Eq. (11). These stress components are defined below. As for beams, we neglect any other nonlinear contribution arising from the large motion amplitude; shear stresses (e.g. \( \sigma_{r,\theta} \) component) are not taken into account in Kirchhoff-Love theory (as in Euler-Bernoulli).

### III. STRESS FIELD

The next step is thus to compute the stress field within the device; this is indeed the extra difficulty that arises in 2D. As for beams, we assume that the stretching is adiabatic, i.e. the stress/strain relation can be treated in a time-independent manner. The total stress field is the sum of a homogeneous contribution, plus the response to the angle-dependent stretching. The former is straightforward (e.g. Appendix [B]):

\[
\sigma_r^{\text{hom.}} = \sigma_0 - E_r \frac{1}{1 - \nu_r} \epsilon_r^{\text{hom}.},
\]

\[
\sigma_\theta^{\text{hom.}} = \sigma_0 - E_r \frac{1}{1 - \nu_r} \epsilon_\theta^{\text{hom}.},
\]

\[
\sigma_z^{\text{hom.}} = 0,
\]

with all shears equal to zero \( \sigma_{r,z} = \sigma_{r,\theta} = \sigma_{\theta,z} = 0 \).

The − sign above comes from our stress convention. In the problem at stake, from Eq. (11) we have \( \epsilon_r^{\text{hom}.} = \left( \frac{z_0}{R_d} \right)^2 \left[ \frac{C_{11,m}^{(1)} + C_{12,m}^{(2)}}{2} \right] \). This stress field component remains biaxial.

To compute the angle-dependent term, we start with an ansatz for the associated displacement field \( \{u_r, u_\theta, u_z\} \):

\[
u_r = R_d f_r(\tilde{r}, \tilde{z}) \epsilon_r^{\text{angl.}}(\theta),
\]

\[
u_\theta = R_d f_\theta(\tilde{r}, \tilde{z}) \frac{\partial \epsilon_r^{\text{angl}.}}{\partial \theta},
\]

\[
u_z = h f_z(\tilde{r}, \tilde{z}) \epsilon_r^{\text{angl.}}(\theta),
\]

with \( \epsilon_r^{\text{angl.}} = \left( \frac{z_0}{R_d} \right)^2 \left[ \frac{C_{11,m}^{(1)} - C_{12,m}^{(2)}}{2} \right] \cos(2n \theta) \). These expressions are then injected in the well-known equilibrium equations of elasticity theory (see e.g. [2]), neglecting inertial terms; these are given for the interested reader in Appendix [B]

Introducing reduced variables \( \tilde{r} = r/R_d \) and \( \tilde{z} = z/h \), one can show that the displacement functions have to be written, at lowest order in \( h/R_d \ll 1 \) (thin structure):

\[
f_r(\tilde{r}, \tilde{z}) = c_r(\tilde{r}) |\tilde{z}| + b_r(\tilde{r}) + a_r(\tilde{r}) (\tilde{z})^2 \frac{h}{R_d} \frac{1}{2},
\]

\[
f_\theta(\tilde{r}, \tilde{z}) = c_\theta(\tilde{r}) |\tilde{z}| + b_\theta(\tilde{r}) + a_\theta(\tilde{r}) (\tilde{z})^2 \frac{h}{R_d} \frac{1}{2},
\]

\[
f_z(\tilde{r}, \tilde{z}) = c_z(\tilde{r}) |\tilde{z}| + b_z(\tilde{r}) + a_z(\tilde{r}) (\tilde{z})^2 \frac{h}{R_d} \frac{1}{2}
\]

For the nine (adimensional) functions \( a_X, b_X, c_X \ (X = r, \theta, z) \) of the \( \tilde{r} \)-variable, we then chose the following ansatz:

\[
b_r(\tilde{r}) = b_{r,0} \tilde{r}^{\alpha},
\]

\[
b_\theta(\tilde{r}) = b_{\theta,0} \tilde{r}^{\alpha},
\]

\[
c_r(\tilde{r}) = c_{r,0} \tilde{r}^{\alpha},
\]

\[
c_\theta(\tilde{r}) = c_{\theta,0} \tilde{r}^{\alpha},
\]

\[
b_z(\tilde{r}) = b_{z,0} \tilde{r}^{\alpha-1},
\]

\[
c_z(\tilde{r}) = c_{z,0} \tilde{r}^{\alpha-1},
\]

\[
a_r(\tilde{r}) = a_{r,0} \tilde{r}^{-2},
\]

\[
a_\theta(\tilde{r}) = a_{\theta,0} \tilde{r}^{-2},
\]

\[
a_z(\tilde{r}) = a_{z,0} \tilde{r}^{-3},
\]

which leads to seven equations linking the above introduced constants. Obviously, \( \alpha \geq 3 \) to guarantee a physical solution.

Three more equations are obtained from the stress boundary conditions on the surface of the drum: \( \sigma_z(r, \theta, z = \pm h/2) = 0, \sigma_r(z, r, \theta, z = \pm h/2) = 0 \) and \( \sigma_{\theta,z}(r, \theta, z = \pm h/2) = 0 \). The last relation is obtained from the stretching on the periphery, equating the radial strain \( \partial u_r / \partial r \) to \( \epsilon_r^{\text{angl.}} \) at \( r = R_d \) (see Fig. 4). Solving the problem under Mathematica [9], we list the constants appearing in Eqs. (25-32) in Tab. 11 Appendix [B] (as a function of \( n \) and \( \nu_r \)). The exponent \( \alpha \) is found to be \( 2n + 1 \), reminding \( n \neq 0 \).

The \( \theta \)-dependent stress field can finally be calculated. The normal components write, in the limit \( h/R_d \approx 0 \):

\[
\sigma_r^{\text{angl.}} = -E_r \eta_r^{(n)}(\nu_r) \left( \frac{r}{R_d} \right)^{\alpha-1} \epsilon_r^{\text{angl.}},
\]

\[
\sigma_\theta^{\text{angl.}} = -E_r \eta_\theta^{(n)}(\nu_r) \left( \frac{r}{R_d} \right)^{\alpha-1} \epsilon_r^{\text{angl.}},
\]

\[
\sigma_z^{\text{angl.}} = 0.
\]
The functions $\eta_r^{(n)}(\nu_r)$ and $\eta_\theta^{(n)}(\nu_r)$ with $n \neq 0$ are defined by:

$$\eta_r^{(n\neq0)}(\nu_r) = \frac{1 + 2n - 2(1 + n)\nu_r}{(1 + 2n)(1 + \nu_r)}, \quad (36)$$

$$\eta_\theta^{(n\neq0)}(\nu_r) = -\frac{3 + 4n}{(1 + 2n)(1 + \nu_r)}. \quad (37)$$

The only nonzero shear stress is $\sigma_{r\theta}$ (see Appendix [B]). It shall be neglected in this modified Kirchhoff-Love theory, as already stated. As an example, the computed (normalized) stress components are displayed in Fig. [5] for mode $\{n = 2, m = 1\}$, in the high-stress limit.

Angle-dependent terms Eqs. [33-35] and homogeneous terms Eqs. [15-17] can be rewritten in a compact form:

$$\sigma_r = \sigma_0 - E_r \left[ \eta_r^{(0)}(\nu_r) \epsilon^\text{hom.} + \eta_r^{(n)}(\nu_r) \left( \frac{r}{R_d} \right)^{2n} \epsilon^\text{angl.} \right], \quad (38)$$

$$\sigma_\theta = \sigma_0 - E_r \left[ \eta_\theta^{(0)}(\nu_r) \epsilon^\text{hom.} + \eta_\theta^{(n)}(\nu_r) \left( \frac{r}{R_d} \right)^{2n} \epsilon^\text{angl.} \right], \quad (39)$$

$$\sigma_z = 0, \quad (40)$$

provided we define $\eta_r^{(0)}(\nu_r) = \eta_\theta^{(0)}(\nu_r) = \eta_r^{(n)}(\nu_r) = 1/(1 - \nu_r)$. The stress is still planar, and independent of $z$, but $\sigma_r \neq \sigma_\theta$ and is neither homogeneous nor isotropic. Injecting these in Eq. [14], we can now solve the problem at hand.

### IV. Mode Parameters

Having found the stress field, we can now project Eq. [14] on a given mode $\{n, m\}$. We thus define modal parameters:

$$\mathcal{M}_{n,m} = \rho h \int_0^{2\pi} \int_0^{R_d} \left[ \psi_{n,m}(r, \theta) \right]^2 r \, dr \, d\theta, \quad (41)$$

$$\mathcal{K}_{n,m} = D_r \int_0^{2\pi} \int_0^{R_d} \left[ \psi_{n,m}(r, \theta) \Delta^2 \psi_{n,m}(r, \theta) \right] r \, dr \, d\theta + T_{r,0} \int_0^{2\pi} \int_0^{R_d} \left[ \psi_{n,m}(r, \theta) \Delta \psi_{n,m}(r, \theta) \right] r \, dr \, d\theta, \quad (42)$$

in a similar fashion to Eqs. [4-5]. The resonance frequencies $\omega_{n,m} = \sqrt{\mathcal{K}_{n,m}/\mathcal{M}_{n,m}}$ reduce to:

$$\omega_{n,m} = \sqrt{\frac{|T_{r,0}|}{\rho h} \left( \frac{\lambda_{n,m}}{R_d} \right)}, \quad (43)$$

or

$$\omega_{n,m} = \sqrt{\frac{D_r}{\rho h} \left( \frac{\lambda_{n,m}}{R_d} \right)^2},$$

in the limit of high-stress and low-stress devices, respectively. We give mass and spring values for the first modes in Tab. [III] Appendix [C].

![Fig. 5: (Color online) Stress components $\sigma_{r\theta}^{\text{angl.}}$ (top) and $\sigma_{\theta\theta}^{\text{angl.}}$ (bottom) computed for mode $\{n = 2, m = 1\}$ in the high-stress limit. The graph has been normalized to $E_r = 1$, $R_d = 1$, and $z_{n,m}/R_d = 1$ (using $\nu_r = +0.3$).](image)

Beyond the usual linear coefficients, the Duffing term analogous to Eq. [6] finally writes:

$$\tilde{K}_{n,m} = -\frac{E_r h}{R_d^2} \left[ \frac{C_{n,m}^{(1)} + C_{n,m}^{(2)}}{2} \eta_r^{(0)}(\nu_r) \int_0^{2\pi} \int_0^{R_d} [\psi_{n,m}(\tilde{r}, \theta) \Delta \psi_{n,m}(\tilde{r}, \theta)] \tilde{r} \, d\tilde{r} \, d\theta \right.$$

$$+ \int_0^{2\pi} \int_0^{R_d} [\psi_{n,m}(r, \theta) \Delta \psi_{n,m}(r, \theta)] r \, dr \, d\theta \\ + \left. \frac{2}{C_{n,m}^{(1)} - C_{n,m}^{(2)}} \right] \left( \frac{\nu_r^{(n)}(\nu_r)}{2} \left[ \int_0^{1} \frac{\phi_{n,m}(\tilde{r})}{\tilde{r}} d\tilde{r} \left( \tilde{r}^{2n+1} \frac{d\phi_{n,m}(\tilde{r})}{d\tilde{r}} \right) \tilde{r} \right] \right) \tilde{r} \, d\tilde{r}$$

$$+ \left. \frac{\nu_r^{(n)}(\nu_r)}{2} \left[ \int_0^{1} \frac{\phi_{n,m}(\tilde{r})}{\tilde{r}} d\tilde{r} \left( \tilde{r}^{2n+1} \frac{d\phi_{n,m}(\tilde{r})}{d\tilde{r}} \right) \tilde{r} \right] \right),$$

with the integrals written in normalized units $\tilde{r} = r/R_d$ (no dimensions).

Numerical values for the integrals defining the coefficients $\tilde{K}_{n,m}$ are listed and discussed for the first modes in Appendix [C] Tabs. [V] and [VI]. For beams, Eq. [6] leads to a scaling of the Duffing parameter $\tilde{k}_n \propto E_z A_z/L^3$. Similarly here, Eq. [44] leads to $\tilde{K}_{n,m} \propto E_r (h 2\pi R_d)/R_d^2$; in both cases, the Duffing effect is a stiffening.
V. CONCLUSION

Following the same methodology as for beams, we present a theory describing the geometrical (stretching) nonlinearity of drum devices. The basic hypotheses are to neglect any other nonlinear features apart from the extra tensile stress, to neglect shearing forces, and to treat the stretching as a static effect. Two limits are considered for numerical values: high-stress (membranes) and low-stress (plates), but the mathematical description is written in a generic fashion. The difficulty lies in the analytical calculation of the stress profile induced in the stretched drum for non-axisymmetric modes; the solution however exists in the limit of a thin structure.

We can compare the theory to experimental data published for modes \( n = 0, m = 0 \). In Ref. [29] the nonlinear behavior of a square silicon nitride drum has been studied. From Fig. 1 (b) of this article, we infer a Duffing parameter normalized to the mode mass of about \( \tilde{K}_{0,0}/M_{0,0} \approx 1.5 \times 10^3 \) m²/s². This fits the data for weak enough excitation; with larger drives, other nonlinear features kick in [29]. Even though the initial stress \( \sigma_0 \) stored in the structure is not very high (110 MPa), the device is well within the membrane limit. From the parameters given in the publication (supplementary material, membrane vice is well within the limit of a thin structure. From parameters quoted in the publication (supplementary material, \( E_r = 240 \) GPa, \( \rho = 3200 \) kg/m³, thickness \( h = 480 \) nm), taking a standard Poisson ratio of \( \nu_r = +0.3 \) and approximating the first mode by a circular shape of radius \( R_d \approx 210 \) μm [consistent with Fig. 2 (a)], we obtain \( \tilde{K}_{0,0}/M_{0,0} \approx 1.2 \times 10^3 \) m²/s². The corresponding mode frequency resonance calculated is 338 kHz, matching also consistently the measured 321 kHz.

In Ref. [28] the nonlinear behavior of a (multilayer) graphene drum has been studied. The reported stored stress is very low (about 5 MPa), and the device is better described in the plate limit. From parameters quoted in the publication (\( E_r \approx 700 \) GPa, \( \rho = 600 \) kg/m³, \( R_d = 2.5 \) μm, \( h = 5 \) nm, neglecting the Poisson ratio) we compute \( \tilde{K}_{0,0}/M_{0,0} \approx 3.3 \times 10^3 \) m²/s² for a resonance frequency of 12.8 MHz. Again, this is in close agreement with measured values of +2.10 m²/s² and 14.5 MHz respectively.

As for beams, the proposed modeling seems to reproduce rather well experimental data. Further comparison should be done with higher modes, especially non-axisymmetric ones (\( n \neq 0 \)). Besides, the presented theory can be in principle extended to mode-coupling [23, 24]: an experimental and theoretical study of this regime would definitely assess the validity of the presented mathematical methods.

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Appendix A: Mode parameters

In the Table below we give the first modes \( \lambda_{n,m} \) and \( r_{n,m} \) parameters for both high-stress (H.S.) and low-stress (L.S.) limits. Inserting these in Eqs. (9) one can easily compute the corresponding mode shapes (see Fig. 3 for an example).

| \( n,m \) | H.S. \( \lambda_{n,m} \) | H.S. \( r_{n,m} \) | L.S. \( \lambda_{n,m} \) | L.S. \( r_{n,m} \) |
|----------|------------------|------------------|------------------|------------------|
| (0, 0)   | 2.4083           | 0                | 3.1962           | 0                |
| (0, 1)   | 5.5200           | 0                | 6.3064           | 0                |
| (1, 0)   | 3.8317           | 0.4805           | 4.6109           | 0.4102           |
| (1, 1)   | 7.0155           | 0.2624           | 7.7992           | 0.2358           |
| (0, 2)   | 8.6537           | 0                | 9.4395           | 0                |
| (2, 0)   | 5.1356           | 0.5947           | 5.9056           | 0.5282           |
| (1, 2)   | 10.1735          | 0.1809           | 10.9581          | 0.1680           |
| (2, 1)   | 8.4172           | 0.3628           | 9.1968           | 0.3319           |

TABLE I: First mode parameters. Left: high-stress (H.S.), right: low-stress (L.S.). For \( n = 0 \) modes, the maximum amplitude is at the center (\( r_{0,m} = 0 \)).

Appendix B: Stress field solution

We remind the reader basics of elasticity theory expressed in cylindrical coordinates. The strain fields can be written in terms of the displacement fields:

\[
\begin{align*}
&\epsilon_r = \frac{\partial u_r}{\partial r}, \\
&\epsilon_\theta = \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \\
&\epsilon_z = \frac{\partial u_z}{\partial z},
\end{align*}
\]

for the normal components, and:

\[
\begin{align*}
2\epsilon_{r,\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta}, \\
2\epsilon_{r,z} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \\
2\epsilon_{\theta,z} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},
\end{align*}
\]
for the shear strains.

For an isotropic homogeneous Hookean material, we have:

\[
\begin{pmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\sigma_{r,\theta} \\
\sigma_{r,z} \\
\sigma_{\theta,z}
\end{pmatrix} = \frac{E}{(1 + \nu_r)(1 - 2\nu_r)} \begin{pmatrix}
\epsilon_r \\
\epsilon_\theta \\
\epsilon_z \\
\epsilon_{r,\theta} \\
\epsilon_{r,z} \\
\epsilon_{\theta,z}
\end{pmatrix}
\]

with \((\mathcal{H}) =

\begin{pmatrix}
1 - \nu_r & \nu_r & 0 & 0 & 0 & 0 \\
0 & 1 - \nu_r & \nu_r & 0 & 0 & 0 \\
\nu_r & 0 & 1 - \nu_r & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - 2\nu_r & 0 \\
0 & 0 & 0 & 0 & 0 & 1 - 2\nu_r
\end{pmatrix}

for the relationship between stresses \((\sigma)\) and strains \((\epsilon)\).

The equilibrium equations then write:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r,\theta}}{\partial \theta} + \frac{1}{r} (\sigma_r - \sigma_\theta) + \frac{\partial \sigma_z}{\partial z} = 0,
\]

\[
\frac{\partial \sigma_{r,\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + 2 \frac{\sigma_r}{r} + \frac{\partial \sigma_{\theta,z}}{\partial z} = 0,
\]

\[
\frac{\partial \sigma_{r,z}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_z}{\partial \theta} + \sigma_{r,z} + \frac{\partial \sigma_z}{\partial z} = 0,
\]

when neglecting the inertial terms.

The solution for the homogeneous stretching component is straightforward. The well-known displacement field simply writes:

\[
f_r(r, z) = r,
\]

\[
f_\theta(r, z) = 0,
\]

\[
f_z(r, z) = \frac{-2\nu_r}{1 - \nu_r} z,
\]

with \(u_r = f_r\epsilon_{\text{hom}}, u_\theta = 0, u_z = f_z\epsilon_{\text{hom}}\) by definition.

Then \(\epsilon_r = \epsilon_\theta = \epsilon_{\text{hom}}\), and \(\epsilon_z = -2\nu_r\epsilon_{\text{hom}}/(1 - \nu_r)\); all other components of the strain field are zero. Clearly, imposing a radial stretching also causes nonzero tangential and vertical strains. The resulting stresses are Eqs. \([15]\).

The case of the angular-dependent component is much more complex. Injecting in the above the ansatz Eqs. \([18,20]\) for the displacement fields, and writing the problem in reduced coordinates, we realize that the solution should be of the type Eqs. \([21,23]\) at lowest order in \(h/R_d\). The symmetry of the drum with respect to \(z \to -z\) has been used. To further reduce the problem, another ansatz is needed for the \(r\)-dependent functions introduced in the writing of the solution: we assume them to be power laws, Eqs. \([25,22]\). Taking into account the boundary conditions (no \(z\)-component stress on the surface of the drum, and fixed radial strain at the periphery), we end up with the constants listed in Tab. \([11]\).

The \(\epsilon_0\) term is simply the prefactor of the angular-dependent strain, \(\epsilon_{\text{angl}} = \epsilon_0 \cos(2n\theta)\). The stresses do depend on \(z^2\). However, in the limit \(h/R_d \to 0\) these terms vanish and the stress components are homogeneous within the thickness of the drum. Also \(\sigma_z = 0\): the stress state is \(\text{planar}\). The two normal components \(\sigma_r^{\text{angl}}, \sigma_\theta^{\text{angl}}\) Eqs. \([33-34]\) are displayed in Fig. \([5]\) for mode \(n = 2, m = 1\) in the high-stress limit.

Furthermore, the only nonzero shear stress component is \(\sigma_{r,\theta}^{\text{angl}}\). It then writes:

\[
\sigma_{r,\theta}^{\text{angl}} = -E_r \frac{n(-3 + \nu_r) + (-2 + \nu_r)(r/R_d)^2}{(1 + 2n)(1 + \nu_r)} \frac{2n}{\epsilon_0} \sin(2n\theta),
\]

with the \(-\) sign matching our stress convention (tensile). It is neglected in the presented modeling.

| Parameter | Expression |
|-----------|-------------|
| \(b_r,0\) | \(\epsilon_0/(1 + 2n)\) |
| \(b_\theta,0\) | \(b_{\theta,0} (1 + n)(2 - \nu_r)/(2n)^2\) |
| \(c_r,0\) | 0 |
| \(c_\theta,0\) | 0 |
| \(b_z,0\) | 0 |
| \(c_z,0\) | \(b_{z,0} 2(1 + n)\nu_r\) |
| \(a_{r,0}\) | \(-b_{r,0} 2(1 + n)(2 - \nu_r) - n^2\) |
| \(a_{\theta,0}\) | \(b_{\theta,0} n(1 - 4(1 + n)(2 - \nu_r)/(2n)^2)\) |
| \(a_{z,0}\) | 0 |
| \(\alpha\) | \(2n + 1\) |
| \(\epsilon_0\) | \(\frac{x_{n,m}}{R_d} \frac{2 c_{n,m}^{\alpha} - c_{n,m}^{\beta}}{2}\) |

TABLE II: Strain coefficients of the angular-dependent contribution, as a function of \(n, \nu_r\). \(\epsilon_0\) is the amplitude of the \(\cos(2n\theta)\) stretching term.

Appendix C: Mass, spring and Duffing parameters

In this Appendix we give numerical estimates for mass, spring constant and nonlinear parameters calculated for the first modes, in the two simple limits of high-stress and low-stress.

For this purpose, we re-write the relevant integrals in an adimensional form such that:

\[
\mathcal{M}_{n,m} = \rho h \pi R_d^2 M_{n,m},
\]

and:

\[
K_{m,n} = \frac{2\pi R_d |T_{r,0}|}{R_d} K_{n,m},
\]

\[
D_r \frac{2\pi R_d}{R_d} K_{n,m},
\]

in the high-stress and low-stress limits, respectively. \(\rho h \pi R_d^2\) is the mass of the drum (in kg), and \(2\pi R_d |T_{r,0}|\)
TABLE III: Mass and spring constant for the first modes (norm. integrals, see text). Left: high-stress (H.S.) and right: low-stress (L.S.).

| mode \{n, m\} | H.S. \(M_{n,m}\) | H.S. \(K_{n,m}\) | L.S. \(M_{n,m}\) | L.S. \(K_{n,m}\) |
|----------------|----------------|----------------|----------------|----------------|
| \{0, 0\}      | 0.269513       | 0.779329      | 0.182834       | 0.540576       |
| \{0, 1\}      | 0.115780       | 1.763983      | 0.101896       | 0.805872       |
| \{1, 0\}      | 0.239561       | 1.758616      | 0.184581       | 41.7156        |
| \{1, 1\}      | 0.133016       | 3.273413      | 0.119933       | 221.883        |
| \{2, 0\}      | 0.073686       | 2.759075      | 0.067543       | 268.132        |
| \{2, 1\}      | 0.243735       | 3.214208      | 0.200046       | 121.669        |
| \{1, 2\}      | 0.092082       | 4.765268      | 0.085466       | 616.168        |
| \{2, 2\}      | 0.155586       | 5.511635      | 0.142446       | 509.546        |

TABLE IV: Nonlinear coefficients \(C^{(1,2)}_{n,m}\) computed for the first modes. Left: high-stress (H.S.) and right: low-stress (L.S.). Note the specificity of \(n = 0\) modes (by definition \(C^{(1)}_{0,m} = C^{(2)}_{0,m}\), no angular dependence of strain/stress).

| mode \{n, m\} | H.S. \(C^{(1)}_{n,m}\) | H.S. \(C^{(2)}_{n,m}\) | L.S. \(C^{(1)}_{n,m}\) | L.S. \(C^{(2)}_{n,m}\) |
|----------------|----------------|----------------|----------------|----------------|
| \{0, 0\}      | 0.389664       | 0.389664      | 0.316669       | 0.316669       |
| \{0, 1\}      | 0.881992       | 0.881992      | 0.851698       | 0.851698       |
| \{1, 0\}      | 1.139994       | 0.618625      | 0.920002       | 0.630536       |
| \{1, 1\}      | 2.60152        | 0.671898      | 2.50136        | 0.682519       |
| \{2, 0\}      | 1.37954        | 1.37954       | 1.34942        | 1.34942        |
| \{2, 1\}      | 1.62609        | 1.58811       | 1.31374        | 1.62645        |
| \{1, 2\}      | 4.07293        | 0.692365      | 3.96830        | 0.696559       |
| \{2, 2\}      | 3.71904        | 1.79259       | 3.55652        | 1.83054        |

From Tabs. IV and VI, the \(C^{(1,2)}_{n,m}\) values of Tab. IV and the expressions of the functions \(\eta_{r,\theta}^{(n)}(\nu_{r})\) [Eqs. 36 and 37], one realizes that the geometrical Duffing nonlinear parameter is dominated by the homogeneous contribution. As a result, \(\hat{K}_{n,m}\) is always positive, as in the beam case. Finally, one can see that the numerical evaluations of \(\hat{K}^{(2,3)}_{n,m}\) are about twice larger in the high-stress limit than in the low-stress case. As such, for identical material parameters \((E, \nu_{r}, \rho)\) except the biaxial stress \(\sigma_{0}\) and identical geometry \((R_{d}, h)\), a membrane Duffing nonlinearity \(\hat{K}_{n,m}\) (H.S.) is approximately twice larger than for a plate (L.S.).
[1] A. N. Cleland and M. L. Roukes, Fabrication of high frequency nanometer scale mechanical resonators from bulk Si crystals, Appl. Phys. Lett. 69, 2653 (1996).
[2] A. N. Cleland, Foundations of Nanomechanics, Springer (2003).
[3] Silvan Schmid, Luis Guillermo Villanueva, Michael Lee Roukes, Fundamentals of Nanomechanical Resonators, Springer (2016).
[4] V. K. Alper, Y.-I. Sohn, H. Atikian, V. Yakhot, M. Loncar, K. L. Ekinci, NanoFluids of Single-Crystal Diamond Nanomechanical Resonators, Nano Letters 15, 12, 8070-8076 (2015).
[5] K. Jensen, J. Weldon, H. Garcia, A. Zettl, Nano Lett. 7, 11, 3508-3511 (2007).
[6] Anesh Koka and Henry A. Sodano, High-sensitivity accelerometer composed of ultra-long vertically aligned barium titanate nanowire arrays, Nature Communications 4, 2682 (2013).
[7] Eric Sage, Marc Sansa, Shawn Frostner, Martial Defoort, Marc Gly, Maksy K. Naik, Robert Morel, Laurent Duraffourg, Michael L. Roukes, Thomas Alava, Guillaume Jourdan, Eric Colinet, Christophe Masselon, Ariel Brenac and Bastien Hentz, Single-particle mass spectroscopy with arrays of frequency-addressed nanomechanical resonators, Nature Communications 9, 3283 (2018).
[8] D. O’Connell, M. Hofheinz, M. Ansmann, Radoslaw Aneesh Koka and Henry A. Sodano, High-sensitivity accelerometer composed of ultra-long vertically aligned barium titanate nanowire arrays, Nature Communications 4, 2682 (2013).
[9] Eric Sage, Marc Sansa, Shawn Frostner, Martial Defoort, Marc Gly, Maksy K. Naik, Robert Morel, Laurent Duraffourg, Michael L. Roukes, Thomas Alava, Guillaume Jourdan, Eric Colinet, Christophe Masselon, Ariel Brenac and Bastien Hentz, Single-particle mass spectrometry with arrays of frequency-addressed nanomechanical resonators, Nature Communications 9, 3283 (2018).
[10] A. N. Cleland, Quantum ground state and single-phonon control of a mechanical resonator, Nature 464, 697-703 (2010).
[11] T. A. Palomaki, J. W. Harlow, J. D. Teufel, R. W. Simmonds, K. W. Lehnert, Coherent state transfer between itinerant microwave fields and a mechanical oscillator, Nature 495, 210 (2013).
[12] J.-M. Pirkkalainen, S. U. Cho, Jian Li, G. S. Paraoanu, P. J. Hakonen and M. A. Sillanpää, Hybrid circuit cavity quantum electrodynamics with a micromechanical resonator, Nature 494, 211 (2013).
[13] H. W. Lulla, R. B. Cousins, A. Venkatesan, M. J. Patton, A. D. Armour, C. J. Mellor and J. R. Owens, Nonlinear modal coupling in a high-stress doubly-clamped nanomechanical resonator, New Journal of Physics 14, 113040 (2012).
[14] M. H. Matheny, L. G. Villanueva, R. B. Karabalin, J. E. Sader, and M. L. Roukes, Nonlinear Mode-Coupling in Nanomechanical Systems, Nano Lett. 13, 1622 (2013).
[15] Olivier Maillet, Xin Zhou, Rasul Gazizulin, Ana Maldonado Cid, Martial Defoort, Olivier Bourgeois, Eddy Collin, Non-linear Frequency Transduction of Nano-mechanical Resonators, Nano Letters 13, 1622 (2013).
[16] Matthew H. Matheny, Matt Grau, Luis G. Villanueva, Rassul B. Karabalin, M. C. Cross, and Michael L. Roukes, Phase Synchronization of Two Anharmonic Nanomechanical Oscillators, Phys. Rev. Lett. 112, 014101 (2014).
[17] I. Kozinsky, H. W. Ch. Postma, I. Bargatin, and M. L. Roukes, Tuning nonlinearity, dynamic range, and frequency of nanomechanical resonators, Appl. Phys. Lett. 88, 253101 (2006).
[18] N. Kacem, J. Arcamone, F. Perez-Murano and S. Hentz, Dynamic range enhancement of nonlinear nanomechanical resonant cantilevers for highly sensitive NEMS gas/mass sensor applications, J. Micromech. Microeng. 20, 045023 (2010).
[19] A. D. O’Connell, M. Hofheinz, M. Ansmann, Radoslaw Aneesh Koka and Henry A. Sodano, High-sensitivity accelerometer composed of ultra-long vertically aligned barium titanate nanowire arrays, Nature Communications 4, 2682 (2013).
[20] A. N. Cleland, Quantum ground state and single-phonon control of a mechanical resonator, Nature 464, 697-703 (2010).
[21] T. A. Palomaki, J. W. Harlow, J. D. Teufel, R. W. Simmonds, K. W. Lehnert, Coherent state transfer between itinerant microwave fields and a mechanical oscillator, Nature 495, 210 (2013).
[22] J.-M. Pirkkalainen, S. U. Cho, Jian Li, G. S. Paraoanu, P. J. Hakonen and M. A. Sillanpää, Hybrid circuit cavity quantum electrodynamics with a micromechanical resonator, Nature 494, 211 (2013).
[23] H. W. Lulla, R. B. Cousins, A. Venkatesan, M. J. Patton, A. D. Armour, C. J. Mellor and J. R. Owens, Nonlinear modal coupling in a high-stress doubly-clamped nanomechanical resonator, New Journal of Physics 14, 113040 (2012).
[24] M. H. Matheny, L. G. Villanueva, R. B. Karabalin, J. E. Sader, and M. L. Roukes, Nonlinear Mode-Coupling in Nanomechanical Systems, Nano Lett. 13, 1622 (2013).
[25] Olivier Maillet, Xin Zhou, Rasul Gazizulin, Ana Maldonado Cid, Martial Defoort, Olivier Bourgeois, Eddy Collin, Non-linear Frequency Transduction of Nano-mechanical Brownian Motion, Phys. Rev. B 96, 165434 (2017).
[26] B. Yurke, D.S. Greywall, A.N. Pargellis, P.A. Bush, Theory of amplifier-noise evasion in an oscillator employing a nonlinear resonator, Phys. Rev. A 51, 4211 (1995).
[27] L.D. Landau and E.M. Lifshitz, Theory of elasticity, Butterworth-Heinemann, Oxford 3rd Ed. (1986).
[28] D. Davidovikj, F. Alijani, S.J. Cartamil-Bueno, H.S.J. van der Zant, M. Amabili P.G. Steeneken, Nonlinear dynamics and Complexity, Ed. by H. G. Schuster, Wiley-VCH (2008).
[29] M. H. Matheny, L. G. Villanueva, R. B. Karabalin, J. E. Sader, and M. L. Roukes, Nonlinear Mode-Coupling in Nanomechanical Systems, Nano Lett. 13, 1622 (2013).
[30] Olivier Maillet, Xin Zhou, Rasul Gazizulin, Ana Maldonado Cid, Martial Defoort, Olivier Bourgeois, Eddy Collin, Non-linear Frequency Transduction of Nano-mechanical Brownian Motion, Phys. Rev. B 96, 165434 (2017).
[31] B. Yurke, D.S. Greywall, A.N. Pargellis, P.A. Bush, Theory of amplifier-noise evasion in an oscillator employing a nonlinear resonator, Phys. Rev. A 51, 4211 (1995).
[32] L.D. Landau and E.M. Lifshitz, Theory of elasticity, Butterworth-Heinemann, Oxford 3rd Ed. (1986).