ENDOTRIVIAL MODULES FOR THE GENERAL LINEAR LIE SUPeralgebra

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Abstract. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra over an algebraically closed field $k$ of characteristic 0, the notion of an endotrivial module has recently been extended to $\mathfrak{g}$-modules by defining $M$ to be endotrivial if $\text{Hom}_k(M, M) \cong k_{ev} \oplus P$ as $\mathfrak{g}$-supermodules. Here, $k_{ev}$ denotes the trivial module concentrated in degree 0 and $P$ is a $(U(\mathfrak{g}), U(\mathfrak{g}_0))$-projective supermodule. In the stable module category, these modules form a group under the tensor product. If $T(\mathfrak{g})$ denotes the group of endotrivial $\mathfrak{g}$-modules, it is interesting and useful to identify this group for a given Lie superalgebra $\mathfrak{g}$. In this paper, a classification is given in the case where $\mathfrak{g} = \mathfrak{gl}(m|n)$ and it is shown that $T(\mathfrak{gl}(m|n)) \cong k \times \mathbb{Z} \times \mathbb{Z}_2$ and is generated by the one parameter family of one dimensional modules $k_{\lambda}$ where $\lambda \in k$, $\Omega^1(k_{ev})$, which denotes the first syzygy of $k_{ev}$, and the parity change functor.

1. Introduction

Endotrivial modules were first defined by Dade in 1978 for $kG$-modules where $G$ is a finite group and $k$ is a field of characteristic $p$ where $p$ divides the order of $G$. A module $M$ is called endotrivial if there is a $kG$-module isomorphism $\text{Hom}_k(M, M) \cong k \oplus P$ where $k$ is the trivial module and $P$ is a projective module. Dade’s study of this class of module arose through study of endopermutation modules in [14], and in [15] Dade showed that, in the case when $G$ is an abelian $p$-group, any endotrivial module is of the form $\Omega^n(k) \oplus P$ where $\Omega^n(k)$ is the $n$th syzygy of the trivial module $k$ and $P$ is a projective module. Syzygies, which are sometimes called Heller shifts or operators, are discussed in Definition 2.2.

An interesting aspect of the set of endotrivial modules is that they form a group in the stable module category where the group operation is the tensor product, $[M] + [N] = [M \otimes N]$. Puig showed in [20] that the group of endotrivial $kG$-modules, denoted $T(G)$, is finitely generated for any finite group $G$. Carlson and Thévenaz gave a complete classification of $T(G)$ when $G$ is an arbitrary $p$-group in [12] and [13]. Carlson, Mazza, and Nakano have continued the study of $T(G)$ by giving classifications when $G$ is the symmetric or alternating group for certain cases in [8] and Carlson, Hemmer, and Mazza furthered those results in [6].

The definition of an endotrivial module has been extended beyond $kG$-modules and has been successfully implemented and studied in a number of other areas of representation theory. Carlson, Mazza, and Nakano have studied endotrivial modules over finite groups of Lie type in the defining characteristic in [7] and non-defining characteristic in [9]. Carlson and Nakano also introduced this definition in the study of modules for finite group schemes in [10] where they prove that the endotrivial modules for a unipotent abelian group scheme are of the form $\Omega^n(k) \oplus P$. Although it is not known whether the group of endotrivial modules over a finite group scheme is finitely generated, Carlson and Nakano proved in a subsequent

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paper [11] that for an arbitrary finite group scheme, the number of isomorphism classes of endotrivial modules of a fixed dimension is finite.

The author began the study of endotrivial modules of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over an algebraically closed field $k$ of characteristic 0 in [23] working in the category $\mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)$. When $\mathfrak{g}$ is classical $\mathcal{F} = \mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)$ denotes the category of finite dimensional $\mathfrak{g}$-modules which are completely reducible over $\mathfrak{g}_0$. This is an important category which has been of significant interest recently and has been studied in [3], [5], [16], [21], and [22], among others. The category $\mathcal{F}$ has enough projectives ([3]), is self-injective ([4]), meaning that a module is projective if and only if it is injective, and for Type I classical Lie superalgebras, e.g. $\mathfrak{gl}(m|n)$, $\mathcal{F}$ is a highest weight category ([2]).

In this context, a $\mathfrak{g}$-supermodule $M \in \mathcal{F}$ is called endotrivial if there is a supermodule isomorphism $\text{Hom}_k(M, M) \cong k_{ev} \oplus P$ where $k_{ev}$ denotes the trivial supermodule concentrated in degree 0 and $P$ is a projective module in $\mathcal{F}$ (discussed in Section 2.3). As has been noted, such modules are an interesting and natural object to study since they form a group denoted as $T(\mathfrak{g})$ and tensoring with such modules gives a self equivalence of the stable module category. Thus, identifying $T(\mathfrak{g})$ may lead to a better understanding of $\mathcal{F}$ or the Picard group of the stable module category via techniques such as those in [1].

The author showed in [23] that, for a detecting subalgebra of type $\mathfrak{e}$ or $\mathfrak{f}$ (introduced in [3]) whose rank is greater than 1, denoted generically as $\mathfrak{a}$, there is an isomorphism $T(\mathfrak{a}) \cong \mathbb{Z} \times \mathbb{Z}_2$. By definition, $\mathfrak{a}$ is isomorphic to a direct sum of copies of either $\mathfrak{q}(1)$ or $\mathfrak{sl}(1|1)$. The detecting subalgebras are analogous to elementary abelian subgroups in modular representation theory in the sense that they detect cohomology and are a natural starting point for the study of endotrivial modules in $\mathcal{F}$. In the same paper, it is also shown that under certain restrictions, the number of endotrivial modules of a fixed dimension $n$ is finite, giving a result similar to the one mentioned in [11]. However, this statement cannot hold in general by observing that, even for small cases like $\mathfrak{gl}(1|1)$, there are infinitely many one dimensional modules which are necessarily endotrivial, forming a subgroup isomorphic to the field $k$.

This paper seeks to build on these results by giving a classification of the group of endotrivial for the general linear Lie superalgebra, $\mathfrak{gl}(m|n)$. The main result, stated in Theorem 5.3, is that

$$T(\mathfrak{gl}(m|n)) \cong k \times \mathbb{Z} \times \mathbb{Z}_2.$$  

This is achieved by defining an extension of $\mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)$, for non-classical Lie superalgebras $\mathfrak{g}$ and then working through an intermediate parabolic subalgebra, denoted as $\mathfrak{p}$. Endotrivial $\mathcal{F}(\mathfrak{p}, \mathfrak{p}_0)$ modules are in a sense easier to understand and in Theorem 5.1 it is shown that there is an isomorphism

$$T(\mathfrak{p}) \cong k^{r+s} \times \mathbb{Z} \times \mathbb{Z}_2$$

where $r = \min(m, n)$ and $s = |m - n|$ when $\mathfrak{p} \subseteq \mathfrak{gl}(m|n)$. Furthermore, by using a geometric induction functor defined in [17], Corollary 4.3 shows there is an injection

$$T(\mathfrak{gl}(m|n)) \hookrightarrow T(\mathfrak{p})$$

and the image is computed directly, yielding the main theorem. The classification of $T(\mathfrak{p})$ results from studying the restriction map $T(\mathfrak{p}) \to T(\mathfrak{f})$ given by $M \mapsto M|_{\mathfrak{f}}$ and identifying the kernel. This is more approachable than restriction from $\mathfrak{gl}(m|n)$ to $\mathfrak{f}$ because $\mathfrak{p}$ has a smaller and more easily handled set of weights.
2. Notation and Preliminaries

2.1. The Distinguished Parabolic. In [3], the category \( \mathcal{F}_{(g, t)} \) is defined for a classical Lie superalgebra \( g \) where \( t \subseteq g \) is a subalgebra of \( g \). In this paper, we wish to consider a compatible notion beyond the classical case (Section 2.3). To motivate this, consider the following.

The classical Lie superalgebra \( \mathfrak{gl}(m|n) \) can be defined as \( (m+n) \times (m+n) \) matrices with standard basis vectors \( e_{i,j} \) where \( 1 \leq i, j \leq m+n \). The usual grading is that the even part is defined to be matrices where the only nonzero entries are in the \( m \times m \) and \( n \times n \) block diagonal and the odd part is defined to be matrices where the only nonzero entries are in the off block diagonal, i.e.

\[
(g\mathfrak{l}(m|n))_{\Sigma} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad (g\mathfrak{l}(m|n))_{\Sigma} = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}
\]

where \( A \in M_{m,m}(k) \), \( D \in M_{n,n}(k) \), \( B \in M_{m,n}(k) \), and \( C \in M_{n,m}(k) \). It can be verified directly that \( g\mathfrak{l}(m|n) \cong (g\mathfrak{l}(m|n))_{\Sigma} \oplus (g\mathfrak{l}(m|n))_{\Sigma} \) as a \( \mathbb{Z}_2 \) graded algebra via matrix multiplication. If \( Z \in (g\mathfrak{l}(m|n))_i \) is homogeneous then define \( |Z| = i \), and define a bilinear multiplication in \( g\mathfrak{l}(m|n) \) by the supercommutator bracket \( [X, Y] := XY - (-1)^{|X||Y|}YX \) for homogeneous elements \( X, Y \in g\mathfrak{l}(m|n) \). The definition is extended to all elements by linearity and this standard construction gives the matrices the structure of a Lie superalgebra under the bracket operation.

Let \( p \) denote the distinguished parabolic subalgebra of \( g\mathfrak{l}(m|n) \) defined as follows. Let \( p_{\Sigma} \subseteq (g\mathfrak{l}(m|n))_{\Sigma} \cong g\mathfrak{l}(m) \oplus g\mathfrak{l}(n) \) be generated by the upper triangular matrices of \( g\mathfrak{l}(m) \) and \( g\mathfrak{l}(n) \). Define \( p_{\Sigma} \subseteq (g\mathfrak{l}(m|n))_{\Sigma} \) as the \( m \times n \) and \( n \times m \) matrices whose entries are all on or above the odd diagonal. That is, if \( B \) and \( C \) are as above, in the standard basis vectors \( p_{\Sigma} \) is generated by \( B' \in M_{m,n}(k) \) where the only nonzero entries are \( e_{i,j} \) where \( j \geq i + m \) and \( C' \in M_{n,m}(k) \) where \( j + m \geq i \). Then \( g\mathfrak{l}(m|n) \) and \( p \) share a maximal torus \( t_{\Sigma} \) of the (even) diagonal matrices.

Note that \( p \) is not classical however, and in fact \( p_{\Sigma} \) is a solvable Lie algebra. This requires an extension of the definition of \( \mathcal{F}_{(g, t)} \) and while the following is written in a general context, \( p \) is the primary example to keep in mind.

2.2. Relative Projectivity. Before defining the category \( \mathcal{F}_{(g, \mathfrak{h})} \), the notion of relatively projective modules is considered, as detailed in [18, Appendix D]. If \( G \) is a superalgebra and \( H \subseteq G \) a subsuperalgebra, a sequence of \( G \)-supermodules

\[
\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \rightarrow \cdots
\]

where each \( f_i \) is even, i.e. preserves the grading of the modules, is called \( (G, H) \)-exact if it is exact as a sequence as \( G \)-supermodules and when the sequence is considered as \( H \)-supermodules, \( \operatorname{ker} f_i \) is a direct summand of \( M_i|_H \) for all \( i \). A \( G \)-supermodule is called \( (G, H) \)-projective if for any \( (G, H) \)-exact sequence

\[
0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{g} M_3 \rightarrow 0
\]

and \( G \)-supermodule map \( h : P \rightarrow M_3 \) there is a \( G \)-supermodule map \( \tilde{h} : P \rightarrow M_2 \) such that \( g \circ \tilde{h} = h \). Note that any projective \( G \)-module is necessarily \( (G, H) \)-projective. Relatively injective modules are defined in a dual way.
The particular case of interest will be \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F})))\)-projective modules. By \cite[Lemma D.2]{18}, any \(U(\mathfrak{g})\)-supermodule \(M\) has a \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F})))\)-projective module which surjects onto \(M\) given by \(U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mathcal{F}) M\). Dually, any such \(M\) also has an injective module into which \(M\) injects given by \(\text{Hom}_{U(\mathfrak{g}_\mathcal{F})}(U(\mathfrak{g}), M)\).

### 2.3. The Relative Category

When \(\mathfrak{g}\) is not classical, we define \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\) to be finite dimensional \(U(\mathfrak{g})\)-modules which are completely reducible over a fixed maximal semisimple torus \(t_\mathcal{F} \subseteq \mathfrak{g}_\mathcal{F}\), and the morphisms are all even \(U(\mathfrak{g})\)-module homomorphisms. Note, as in \cite{3}, \cite{4}, and \cite{23} the projective (respectively, injective) objects in this category will be \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-projective (respectively, \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-injective) modules. Furthermore, we can also define \(\text{Ext}^1_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}(M, N)\) and \(H^1(\mathfrak{g}, \mathfrak{g}_\mathcal{F}; M)\) whose constructions are given in \cite{3}.

Since the category \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\) is used extensively in \cite{23}, for the sake of compatibility, we make the following assumptions on \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\) when \(\mathfrak{g}\) is a stable Lie superalgebra (see \cite{3} for the definition). For stable \(\mathfrak{g}\), there exists a detecting subalgebra \(\mathfrak{f} \subseteq \mathfrak{g}\) with maximal torus \(t_\mathfrak{f} \subseteq \mathfrak{f}\). Let \(t_\mathcal{F} \subseteq \mathfrak{g}\) be a torus for the Lie algebra \(\mathfrak{g}_\mathcal{F}\) such that \(t_\mathfrak{f} \subseteq t_\mathcal{F}\). Then \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\) modules are assumed to be completely reducible over the torus \(t_\mathfrak{f}\) such that \(t_\mathfrak{f} \subseteq t_\mathcal{F}\).

A few preliminary results are given to establish the theory of endotrivial modules in \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\), which is by convention denoted simply as \(\mathcal{F}\) when there is no ambiguity. The following proposition gives a concrete description of the projective and injective modules in the category, as well as some important properties of \(\mathcal{F}\).

**Proposition 2.1.** Let \(M, P,\) and \(I\) be modules in \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\).

(a) A module \(P\) is \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-projective if and only if it is a direct summand of \(U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mathcal{F}) N\) for some \(U(\mathfrak{g}_\mathcal{F})\)-module \(N\).

(b) For the module \(M\), there exists a projective module \(P\) and an injective module \(I\) such that there are homomorphisms of \(\mathcal{F}\) modules \(\pi : P \rightarrow M\) and \(\iota : M \rightarrow I\).

(c) A module \(P\) is projective in \(\mathcal{F}\) if and only if it is an injective module in \(\mathcal{F}\).

**Proof.** For (a), first assume that \(P\) is projective in \(\mathcal{F}\). The following sequence is, by construction, \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-exact

\[
0 \longrightarrow \ker \mu \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mathcal{F}) \longrightarrow P \longrightarrow 0
\]

and is split by using the \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-projectivity of \(P\) to extend the identity map on \(P\) in the standard way.

Now, let \(P\) be a direct summand of \(U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mathcal{F}) N\) for some \(U(\mathfrak{g}_\mathcal{F})\)-module \(N\). Then

\[
\text{Ext}^1_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}(P, R) \hookrightarrow \text{Ext}^1_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}(U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mathcal{F}) N, R) = \text{Ext}^1_{(\mathfrak{g}_\mathcal{F}, \mathfrak{g}_\mathcal{F})}(N, R) = 0
\]

for any module \(R\) in \(\mathcal{F}\). Thus, \(P\) is \((U(\mathfrak{g}), U(\mathfrak{g}_\mathcal{F}))\)-projective and so it is projective in \(\mathcal{F}\).

Part (b) follows from \cite[Lemma D.2]{18} by noting that the extra condition of complete reducibility holds and the proof given in \cite[Propositions 2.2.2]{4} holds for \(\mathcal{F}\) which proves (c). \(\Box\)

**Definition 2.2.** Let \(\mathfrak{g}\) be a Lie superalgebra and let \(M\) be a module in \(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_\mathcal{F})}\). Let \(P\) be a minimal projective module in \(\mathcal{F}\) which surjects on to \(M\) (called the projective cover), with the map \(\psi : P \rightarrow M\). The first syzygy of \(M\) is defined to be \(\ker \psi\) and is denoted by \(\Omega^1(\mathfrak{g})\). This is also referred to as a Heller shift (or Heller operator) in some literature. Inductively, define \(\Omega^{n+1}(\mathfrak{g}) := \Omega^1(\Omega^n(\mathfrak{g}))\).
Similarly, given $M$, let $I$ be the injective hull of $M$ with the inclusion $ι : M \hookrightarrow I$, and define $Ω^{-1}_g(M) := \text{coker} ι$. This is extended by defining $Ω^{-n}_g(M) := Ω^{-1}_g(Ω^{-n-1}_g(M))$.

Finally, define $Ω^0_g(M)$ to be the compliment of the largest $(U(\mathfrak{g}), U(\mathfrak{g}))$-projective direct summand of $M$. In other words, we can write $M = Ω^0_g(M) \oplus Q$ where $Q$ is projective in $\mathcal{F}$ and maximal with respect to this property. Thus, the $n$th syzygy of $M$ is defined for any integer $n$.

**Convention.** When there is no ambiguity, $Ω^n_g(M)$ may be denoted as $Ω^n(M)$.

2.4. **Endotrivial Modules.** With a better understanding of projective modules in $\mathcal{F}$, we now define the object of interest in this paper.

**Definition 2.3.** A module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g})}$ is called endotrivial if $\text{End}_k(M) \cong k_{ev} \oplus P$ as $U(\mathfrak{g})$-modules for some projective module $P$ in $\mathcal{F}$.

This definition is equivalent to defining $M$ to be endotrivial if $M \otimes M^* \cong k_{ev} \oplus P_M$ by using the isomorphism $\text{Hom}(V, W) \cong W \otimes V^*$. One of the interests in studying endotrivial modules is that they form a group in the stable module category.

**Definition 2.4.** Given a category of modules, $\mathcal{F}$, consider the category with the same objects as the original category and an equivalence relation on the morphisms given by $f \sim g$ if $f - g$ factors through a projective module in $\mathcal{F}$. This is called the **stable module category** of $\mathcal{F}$ and is denoted by $\text{Stmod}(\mathcal{F})$.

**Definition 2.5.** Let $\mathfrak{g}$ be a Lie superalgebra. The set of endotrivial modules in $\text{Stmod}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g})})$

$$T(\mathfrak{g}) := \{[M] \in \text{Stmod}(\mathcal{F}) \mid M \otimes M^* \cong k_{ev} \oplus P_M \text{ for some } P_M \text{ which is projective in } \mathcal{F} \}$$

forms a group in the stable module category of $\mathcal{F}$ under the operation $[M] + [N] := [M \otimes N]$. This group is called the **group of endotrivial $\mathfrak{g}$-supermodules**.

More details on syzygies and this group are given in [23]. One such observation is that if $M$ is any endotrivial module, then $Ω^n(M)$ is endotrivial as well for any $n \in \mathbb{N}$. An additional result is stated here relating syzygies relative to different Lie superalgebras and will be useful throughout this work.

**Lemma 2.6.** Let $\mathfrak{g}$ be a Lie superalgebra with torus $\mathfrak{t}_g$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra with torus $\mathfrak{t}_h$ such that $\mathfrak{t}_h \subseteq \mathfrak{t}_g$ and that for each projective module $Q$ in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g})}$, $Q|_h$ is projective in $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h})}$. Let $M$ be a module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g})}$, then $Ω^n_\mathfrak{g}(M)|_h \cong Ω^n_\mathfrak{h}(M|_h) \oplus P$ for all $n \in \mathbb{Z}$ where $P$ is a projective module in $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h})}$.

**Proof.** Let $M$ be as above and let

$$0 \longrightarrow Ω^1_\mathfrak{g}(M) \longrightarrow Q \longrightarrow M \longrightarrow 0$$

be the short exact sequence of modules in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g})}$ defining $Ω^1_\mathfrak{g}(M)$. Then

$$0 \longrightarrow Ω^1_\mathfrak{g}(M)|_h \longrightarrow Q|_h \longrightarrow M|_h \longrightarrow 0$$

is an exact sequence and the module $Q|_h$ is projective in $\mathcal{F}_{(\mathfrak{h}, \mathfrak{h})}$ (although perhaps not minimal). Then by definition, $Ω^1_\mathfrak{h}(M)|_h \cong Ω^1_\mathfrak{h}(M|_h) \oplus P$. This argument applies to $Ω^{-1}_\mathfrak{g}(M)$ as well and so by induction, $Ω^n_\mathfrak{g}(M)|_h \cong Ω^n_\mathfrak{h}(M|_h) \oplus P$ for all $n \in \mathbb{Z}$.
3. The Distinguished Parabolic

As noted, the primary purpose of extending the definition of $F_{(0,m)}$ beyond classical Lie superalgebras, is to consider the case for $p$, the distinguished parabolic subalgebra of $gl(m|n)$ defined in Section 2.1. There is a relationship between the groups $T(gl(m|n))$ and $T(p)$ and understanding $T(p)$ will eventually lead to a classification of $T(gl(m|n))$, the main goal of this paper.

Note that if $f$ is defined to be the subalgebra of $p$ generated by elements which are strictly on the odd diagonal (plus a torus of dimension $|m-n|$ which is described in the proof of Theorem 5.1), then $f \subseteq p \subseteq gl(m|n)$ and $t_f \subseteq t_p = t_T$. Given this set up, we will relate $T(f)$, $T(p)$, and $T(gl(m|n))$.

The main reasons to study $p$ are that the set of weights relative to the torus for $f$ are well behaved and that it has a $Z$ grading which is consistent with the $Z_2$ grading. This $Z$ grading allows for results analogous to those in [4] and [19] to be extended to $p$.

In the following two sections, we exploit this $Z$ grading to derive results about projectivity when restricting from $gl(m|n)$ to $p$ to $f$. First, we establish that restriction to each of these subalgebras takes projectives to projectives in order to have well defined maps between the groups which are defined in the stable module category. Second, it is shown that if a module in $F_{(p,p\mathfrak{g})}$ is projective when restricted to $F_{(f,f\mathfrak{g})}$, then it is projective in $F_{(p,p\mathfrak{g})}$ as well.

3.1. Restriction and Projectivity. Because $g = gl(m|n)$ is a Type I Lie superalgebra, $g$ has a $Z$ grading of the form $g = g_{-1} \oplus g_0 \oplus g_1$ which is consistent with the standard $Z_2$ grading. This gives a consistent $Z$ grading on $p \subseteq g$ by defining $p_i = p \cap g_i$ for $i \in Z$, and so $p = p_{-1} \oplus p_0 \oplus p_1$. Given this grading, define $p^+ := p_0 \oplus p_1$ and $p^- := p_{-1} \oplus p_0$. Similarly, we may decompose $f = f_{-1} \oplus f_0 \oplus f_1$ and define $f^+ := f_0 \oplus f_1$ and $f^- := f_{-1} \oplus f_0$.

Following the work in [4], define $F(p_{\pm1})$ to be the category of finite dimensional $p_{\pm1}$-modules. For the objects in $F(p_{\pm1})$, define the support variety $V_{p_{\pm1}}(M)$ as in [4] and the rank variety

$$V_{p_{\pm1}}^{rank}(M) = \{ x \in p_{\pm1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module} \} \cup \{0\}.$$  

Since $p_1$ and $p_{-1}$ are both abelian Lie superalgebras, both are well defined and identified by the canonical isomorphism detailed in [3].

Consider $X(t_T) \subseteq t_T^*$, the set of weights relative to a fixed maximal torus $t_T \subseteq p_T$. It will be very useful to have a partial ordering on these weights. Let $d = \dim t_T$. If we fix the dual basis of $t_T$ to be the basis for $X(t_T)$, the weights can be parameterized by the set $k^d$ so any $\lambda \in X(t_T)$ can be thought of as an ordered $d$-tuple $(\lambda_1, \ldots, \lambda_d)$. For two weights $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$, we say that $\lambda \geq \mu$ if and only if for each $k = 1, \ldots, d$,

$$(3.1) \quad \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$$

and equality holds if and only if $\lambda = \mu$. This ordering is referred to as the dominance ordering and will allow the use of highest weight theory.

A module $M \in F_{(p,p\mathfrak{g})}$ is called a highest weight module if in the weight decomposition $M \cong \bigoplus_{\lambda \in X(t_T)} M_\lambda$, there exists a weight $\lambda_0$ such that $\lambda_0 \geq \mu$ for each nonzero weight space $M_\mu$ of $M$.

Proposition 3.2. If $S \in F_{(p,p\mathfrak{g})}$ is simple, then $S$ is a highest weight module in $F_{(p,p\mathfrak{g})}$. 

Proof. Because $S$ is finite dimensional, there exists a weight $\lambda_0 \in X(t_{\widetilde{\mathfrak{g}}})$ such that $\mu \not\succ \lambda_0$ for all nonzero weight spaces $S_{\mu}$ of $S$. Note that this means all weights are either less than or equal to or not comparable to $\lambda_0$.

For any element $p$ of $p_1$, $p.S_\lambda \subseteq S_\mu$ implies that $\mu \succ \lambda$ in $X(t_{\widetilde{\mathfrak{g}}})$. This yields that $p.S_{\lambda_0} = 0$ for any $p \in p_1$. Since $S$ is simple, for $v \in S_{\lambda_0}$, $v$ generates $S$ and $p.v = 0$.

Thus, $S = U(p_{-1})U(p_+).v$ but since any element of $p^+$ either stabilizes or kills $v$, it follows that $S = U(p_{-1}).v$. It is now clear, because $v \in S_{\lambda_0}$ that any element of $S_{\mu} \neq 0$ is equal to $cy.v$ for some $y \in U(p_{-1})$ and $c \in k$, where $\lambda_0 \geq \mu$. Thus $S$ is a highest weight module. □

Since $p_{\mathfrak{g}}$ is a solvable Lie algebra, the only simple modules are one dimensional modules $k_\lambda$ where the torus acts by weight $\lambda \in X(t_{\mathfrak{g}})$. Because $p_1 \subseteq p^+$ and $p_{-1} \subseteq p^-$ are ideals, the module $k_\lambda$ can be considered as a simple $p^\pm$-module by inflation via the canonical quotient map $p^\pm \rightarrow p_{\mathfrak{g}}$. By construction, $p_1$ and $p_{-1}$ act by 0 on $k_\lambda$. Define

\[ K(\lambda) = U(p) \otimes_{U(p^-)} k_\lambda \quad \text{and} \quad K^-(\lambda) = \text{Hom}_{U(p^-)}(U(p), k_\lambda) \]

to be the Kac module and the dual Kac module, respectively.

The Kac module $K(\lambda)$ has several useful properties. First, by construction it is a highest weight module in $F_{(p,p\mathfrak{g})}$. Since $K(\lambda)$ is generated by a highest weight vector, it has a simple head. Also, if $S$ is any simple module in $F_{(p,p\mathfrak{g})}$ where $S$ has highest weight $\lambda$ for some weight $\lambda$ of $S$, and $v \in S_\lambda$, there is a surjective homomorphism $K(\lambda) \twoheadrightarrow S$ given by $u \otimes 1 \mapsto u.v$.

Furthermore, $K(\lambda)/\text{Rad}(K(\lambda)) \cong S$ and is denoted $L(\lambda)$. Note that this surjective homomorphism is in fact valid for any highest weight module and in this sense, the Kac module is universal.

Dually, simple modules in $F_{(p,p\mathfrak{g})}$ are lowest weight modules and if $L(\lambda)$ has lowest weight $\mu$, then $K^-(\mu)$ has a simple socle which is isomorphic to $L(\lambda)$ as well and $\mu$ is the lowest weight of $K^-(\mu)$.

Now we define two useful filtrations of a module $M$ in $F_{(p,p\mathfrak{g})}$. $M$ is said to admit a Kac filtration if there is a filtration

\[ \{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M \]

of the module $M$ such that for $i = 1, \ldots, t$, $M_i/M_{i-1} \cong K(\lambda_i)$ for some $\lambda_i \in X(t_{\mathfrak{g}})$. Similarly, if $M$ has a filtration as above such that for $i = 1, \ldots, t$, $M_i/M_{i-1} \cong K^-(\lambda_i)$, then $M$ is said to admit a dual Kac filtration.

By the same reasoning in [4], modules in $F_{(p,p\mathfrak{g})}$ satisfy the following.

Theorem 3.3. Let $M$ be a module in $F_{(p,p\mathfrak{g})}$. Then the following are equivalent.

1. $M$ has a Kac filtration;
2. $\text{Ext}^1_{F_{(p,p\mathfrak{g})}}(M, K^-(\mu)) = 0$ for all $\mu \in X(t_{\mathfrak{g}})$;
3. $\text{Ext}^1_{F_{(p,-1)}}(M, k) = 0$;
4. $\mathcal{V}_{p_{-1}}(M) = 0$.

Theorem 3.4. Let $M$ be a module in $F_{(p,p\mathfrak{g})}$. Then the following are equivalent.

1. $M$ has a dual Kac filtration;
2. $\text{Ext}^1_{F_{(p,p\mathfrak{g})}}(K(\mu), M) = 0$ for all $\mu \in X(t_{\mathfrak{g}})$;
3. $\text{Ext}^1_{F_{(p,1)}}(k, M) = 0$;
4. $\mathcal{V}_{p_1}(M) = 0$. 
These two theorems can be used to show the following powerful condition relating projectivity in $\mathcal{F}(p,p)$ and the support varieties of $p_{\pm 1}$.

**Theorem 3.5.** Let $M$ be in $\mathcal{F}(p,p)$. Then $M$ is projective in $\mathcal{F}(p,p)$ if and only if $V_{p_1}(M) = V_{p_{-1}}(M) = \{0\}$.

**Corollary 3.6.** A projective module in $\mathcal{F}(g,0)$ is also projective in $\mathcal{F}(p,p)$ and thus, there is a well defined map

$$\text{res}_{T(p)}: T(g) \rightarrow T(p)$$

given by $M \mapsto M|_p$. Moreover, this map is a homomorphism of groups.

**Proof.** Let $P$ be a projective module in $\mathcal{F}(g,0)$. Then by [1] Theorem 3.5.1, $V_{g_1}(M) = V_{g_{-1}}(M) = \{0\}$. Using the rank variety description, we see that $V_{p_1}(M) \subseteq V_{g_1}(M) = \{0\}$ and $V_{p_{-1}}(M) \subseteq V_{g_{-1}}(M) = \{0\}$, and so by Theorem 3.5, $M|_p$ is projective in $\mathcal{F}(p,p)$.

With this conclusion, the restriction map now descends to a well defined map on each of the respective stable module categories, and in particular, if $M \in \mathcal{F}(g,0)$, such that $M \otimes M^* \cong k e_v \oplus P$, then $(M \otimes M^*)|_p \cong k e_v \oplus P|_p$.

Furthermore, since restriction commutes with the tensor product, this is also a group homomorphism.

The following maps, which will be useful in Section 5, now follow easily.

**Proposition 3.7.** Let $M$ be a projective module in $\mathcal{F}(p,p)$. Then $M|_{p^+}$ and $M|_{p^-}$ and $M|_f$ are all projective in their respective categories.

**Proof.** Let $M$ be as above. Then, by Proposition 2.1 [2], $M$ is a summand of $U(p) \otimes_{U(p)} N$ for some $U(p)$-module $N$. Because

$$U(p) \otimes_{U(p)} N \cong U(p^+)U(p^-) \otimes_{U(p)} N \cong U(p^+) \otimes_{U(p)} [U(p^-) \otimes N]$$

where the second isomorphism is given on basis elements by

$$u^+ u^- \otimes_{U(p)} n \mapsto u^+ \otimes_{U(p)} [u^- \otimes n],$$

any summand of $U(p) \otimes_{U(p)} N$ is also a summand of $U(p^+) \otimes_{U(p)} N'$ for some $p^+$-module $N'$. Thus, if $M$ is projective in $\mathcal{F}(p,p)$, $M$ is also $(U(p^+),U(p^-))$-projective as well, or projective in $\mathcal{F}(p^+,p^-)$. By a similar argument, $M$ is also projective in $\mathcal{F}(f,f)$.

Since the rank varieties $V_{p_1}(M)$ and $V_{p_{-1}}(M)$ measure projectivity, both varieties are $\{0\}$. Additionally, $V_{f_{\pm 1}}(M) \subseteq V_{p_{\pm 1}}(M) = \{0\}$, and so by [1] Theorem 3.5.1, $M|_f$ is projective in $\mathcal{F}(f,f)$.

**Corollary 3.8.** A projective module in $\mathcal{F}(p,p)$ is also projective in $\mathcal{F}(f,f)$ and thus, there is a well defined map

$$\text{res}_{T(f)}: T(p) \rightarrow T(f)$$

given by $M \mapsto M|_f$. Moreover, this map is a homomorphism of groups.

3.2. Detecting Projectivity. The second primary result about the relationship between $\mathcal{F}(p,p)$ and $\mathcal{F}(f,f)$ is that if a module in $\mathcal{F}(p,p)$ is projective when restricted to $\mathcal{F}(f,f)$, then it is also projective in $\mathcal{F}(p,p)$. This result is derived from [19] by observing that the triangular decomposition of $p$ will afford the same results as those given for $g$ a classical Type I Lie superalgebras.
One of the steps in the original proof relies on an invariant theory result about the action of the reductive group \(G_0\) where \(\text{Lie}(G_0) = g_0\) on the subalgebra \(f_{\pm 1}\). Namely that if \(\phi \in \text{Hom}_{G_0}(g_{\pm 1}, k)\) such that \(\phi|_{f_{\pm 1}} \equiv 0\) then \(\phi \equiv 0\). This result is shown for \(p\) directly in the following lemma.

**Lemma 3.9.** Let \(p\) be the distinguished parabolic subalgebra with triangular decomposition \(p_{-1} \oplus p_0 \oplus p_1\) and \(P_0\) be the algebraic group such that \(\text{Lie}(P_0) = p_0\). Let \(f_1 = f \cap p_i\) and \(\psi \in \text{Hom}_{P_0}(p_{\pm 1}, k)\) such that \(\psi|_{f_{\pm 1}} \equiv 0\). Then \(\psi \equiv 0\).

**Proof.** Without loss of generality, the proof is given for \(f_1\) where \(p \subseteq gl(m|n)\) with \(n > m\) as all other cases follow in a similar way.

In this instance, consider the given bases for these subalgebras. As defined in 2.1, \(f_1 \subseteq p_1\) can be thought of as elements of \(M_{m,n}(k)\) embedded in the block upper triangular corner of \((m + n) \times (m + n)\) matrices. So if \(0_{i,j}\) is an \(i \times j\) matrix of zeros, and

\[
M = \begin{pmatrix} 0_{m,m} & N \\ 0_{n,m} & 0_{n,n} \end{pmatrix}
\]

where \(N\) is of the form

\[
\begin{pmatrix}
  x_1 & a_2 & a_3 & \ldots & a_n \\
  0 & x_2 & b_3 & \ldots & b_n \\
  0 & 0 & x_3 & \ldots & c_n \\
  \vdots & \vdots & \ddots & \ldots & \vdots \\
  0 & 0 & \ldots & 0 & x_m & \ldots & d_n \\
\end{pmatrix}
\]

then \(M \in f_1\) if the only (possibly) nonzero entries are the \(x_i\)'s, i.e. on the odd diagonal, and \(M\) is in \(p_1\) if arbitrary variables shown above are allowed to be nonzero.

Now let \(\psi\) be as above and for a fixed element \(P \in p_1\), let \(X\) be the matrix whose \(x_i\) entries are those in \(P\) and all others zero, \(A\) be the matrix whose \(a_i\) entries are those in \(P\) (all others zero), etc. and so we can decompose \(P\) by writing \(P = X + A + B + \cdots + D\). By definition, \(\psi(X) = 0\).

We proceed by contradiction and an iterated argument using the number of rows \(m\). So without loss of generality, assume that \(\psi(P) \neq 0\) and that the matrix \(A \neq 0\) (otherwise proceed to the next iteration).

If \(T_i\) denotes (not strictly) upper triangular \(i \times i\) matrices, then \(P_0 \cong T_m \oplus T_n\) and the action on \(p_1\) is given by \((G, H) \cdot M = GMH^{-1}\) for \((G, H) \in P_0\) and \(M \in p_1\). By assumption, \(\psi\) is a \(P_0\) invariant function so the action of \(P_0\) on \(p_1\) does not change the value of the function. Let \(I_{i,j}(c)\) denote the \(i \times j\) identity matrix where the \(j\)th diagonal entry is replaced by the constant \(c \in k\) and \(\psi\) is invariant. Then \(I_{1,c} \cdot f_{\ell} \cdot P = X + cA + B + \cdots + D\) and by iterating the action \(\ell\) times, \((I_{1,c})^\ell \cdot P = X + c^\ell A + B + \cdots + D\). Since \(\psi\) is \(P_0\) invariant,

\[
\psi((I_{1,c})^\ell \cdot P) = \psi(P) = c^\ell \neq 0
\]

and so \((I_{1,c})^\ell \cdot P \in \psi^{-1}(c^\ell)\) for all \(\ell > 0\). Furthermore, \(c^\ell \in k\) is a closed set and \(\psi\) is continuous so \(\psi^{-1}(c^\ell)\) is closed in the Zariski topology of \(p_1\) and contains its limit points under the action of \(P_0\). We conclude that

\[
X + B + \cdots + D = \lim_{\ell \to \infty} (I_{1,c})^\ell \cdot P \in \psi^{-1}(c^\ell)
\]

and so \(\psi(X + B + \cdots + D) = c^\ell\).
Now this argument may be repeated by considering the action of $I_{2,c}$ on $X + B + \cdots + D$, and so on until the action of $I_{m,c}$ on $X + D$ yields that $X = \lim_{\ell \to \infty} (I_{n,c})^\ell \cdot (X + D) \in \psi^{-1}(c')$, and thus $\psi(X) = c' \neq 0$. This is a contradiction and we conclude that the assumption was false. So $\psi(P) = 0$ for any $P \in p_1$ and $\psi \equiv 0$.

**Theorem 3.10.** For all $M \in \mathcal{F}_{(p,p_\Sigma)}$ and $n \neq 0$ the restriction map

$$H^n(p, p_\Sigma, M) \to H^n(f, f_\Sigma, M)$$

is injective.

The proof is the same as in [19] since $p_{\pm 1}$ is an ideal of $p$ and by use of Proposition 3.7 and Lemma 3.9. This powerful result will be used in the form of the following corollary.

**Corollary 3.11.** Let $M \in \mathcal{F}_{(p,p_\Sigma)}$ such that $M|_f$ is projective in $\mathcal{F}_{(i,i_\Pi)}$. Then $M$ is projective in $\mathcal{F}_{(p,p_\Sigma)}$.

**Proof.** Let $S$ be a simple module in $\mathcal{F}_{(p,p_\Sigma)}$. Then

$$H^1(p, p_\Sigma, M \otimes S^*) \hookrightarrow H^1(f, f_\Sigma, M \otimes S^*) \cong \text{Ext}^1_{\mathcal{F}_{(i,i_\Pi)}}(S, M) = 0$$

since $M|_f$ is projective in $\mathcal{F}_{(i,i_\Pi)}$. Thus $\text{Ext}^1_{\mathcal{F}_{(p,p_\Sigma)}}(S, M) = 0$ as well and $M$ is projective in $\mathcal{F}_{(p,p_\Sigma)}$. \qed

4. Restriction from $T(g)$ to $T(p)$

Let $g = \mathfrak{gl}(m|n)$ and $p \subseteq g$ be the distinguished parabolic. Since restriction from $T(g)$ to $T(p)$ is well defined (Corollary 3.6), properties of this map can be exploited to relate a classification of one to the other. An important step in understanding the relationship between these two groups is an induction functor from $p$ to $g$.

In [17, Section 3], the geometric induction functor $\Gamma_0$ is defined. The functor $\Gamma_0$ is from $p$-modules to $g$-modules and will be denoted $\text{Ind}^{\Gamma_0}_p$ since the geometric structure will not be emphasized in this paper. This functor is of particular interest because it will allow us to show that restriction map

$$\text{res}^{T(g)}_{T(p)} : T(g) \to T(p)$$

given by $M \mapsto M|_p$ is injective by checking that $\ker \left( \text{res}^{T(g)}_{T(p)} \right) = \{k_{ev}\}$ since $k_{ev}$ is the identity in $T(g)$.

The first step in the proof is to show that $\text{Ind}^{\Gamma_0}_p k_{ev} = k_{ev}$. This is done by considering [17, Lemma 3] and its proof. In particular, the authors observe that if $L_\mu$ (respectively $L_\mu(a)$) is the simple $g$-module (respectively $a$-module) with highest weight $\mu$, then if $L_\mu$ occurs in $\text{Ind}^{\Gamma_0}_p k_\lambda$, then $L_\mu((g_\Sigma)^*)$ occurs in $H^0(G_0/P_0, L_\lambda^*(p) \otimes S^*(g/(g_\Sigma \oplus p_\Sigma))^*)$.

The case when $k_\lambda$ is the trivial module $k_{ev}$ is of particular interest as noted above. Thus we consider $H^0(G_0/P_0, S^*(g/(g_\Sigma \oplus p_\Sigma))^*)$, and more specifically, the dominant weights in $S^*(g/(g_\Sigma \oplus p_\Sigma))^*$. In order for such a weight to be dominant, it must have positive inner product with $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m$ and $\delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-1} - \delta_n$. The weights of $S^*(g/(g_\Sigma \oplus p_\Sigma))^*$ are positive linear combinations of the weights of the form $\varepsilon_i - \delta_j$ and $\delta_i - \varepsilon_j$ where $i > j$.

**Proposition 4.1.** Let $p \subseteq g = \mathfrak{gl}(n|n)$. No weight of $S^*(g/(g_\Sigma \oplus p_\Sigma))^*$ is dominant.
Proof. This will be proven by induction on \( n \). The first case is trivial since when \( n = 1 \), \( \mathfrak{p}_\mathfrak{r} = \mathfrak{g}_\mathfrak{r} \) and \( \mathfrak{g} = \mathfrak{g}_\mathfrak{r} \oplus \mathfrak{g}_\mathfrak{r} \).

The first nontrivial base case is when \( n = 2 \). If the weights \( \varepsilon_2 - \delta_1 \) and \( \delta_2 - \varepsilon_1 \) are represented as \((0,1|1,0)\) and \((-1,0|0,1)\) respectively, then a positive linear combination of such weights \( r(\varepsilon_2 - \delta_1) + s(\delta_2 - \varepsilon_1) \) is represented as \((-s,r| -r,s)\). We compute

\[
\begin{align*}
\langle(1,-1|0,0),(-s,r|-r,s)\rangle &= -s-r \\
\langle(0,0|1,-1),(-s,r|-r,s)\rangle &= -s-r
\end{align*}
\]

and so any nonzero weight has negative inner product and thus, is not dominant.

Now let \( n > 2 \). In order for a positive linear combination of weights to be dominant, there is a set of conditions which must be satisfied. Let \( \lambda \) be an arbitrary weight and let \( a_{i,j} \) be the coefficient for the weight \( \varepsilon_i - \delta_j \) and \( b_{k,l} \) be the coefficient of the weight \( \delta_k - \varepsilon_l \), where \( i > j \) and \( k > l \). Then

\[
\lambda = \left( \sum_{i > j} a_{i,j}(\varepsilon_i - \delta_j) \right) + \left( \sum_{k > l} b_{k,l}(\delta_k - \varepsilon_l) \right)
\]

or if we denote \( \alpha_{i,j} = a_{i,j}(\varepsilon_i - \delta_j) \) and \( \beta_{k,l} = b_{k,l}(\delta_k - \varepsilon_l) \), then \( \lambda = \sum_{i > j}(\alpha_{i,j} + \beta_{i,j}) \).

Note that

\[
\begin{align*}
\langle \varepsilon_s - \varepsilon_{s+1}, \alpha_{i,j} \rangle &= \delta_{s,i}a_{i,j} - \delta_{s+1,i}a_{i,j} \\
\langle \varepsilon_s - \varepsilon_{s+1}, \beta_{i,j} \rangle &= -\delta_{s,j}b_{i,j} + \delta_{s+1,j}b_{i,j} \\
\langle \delta_s - \delta_{s+1}, \alpha_{i,j} \rangle &= -\delta_{s,j}a_{i,j} + \delta_{s+1,j}a_{i,j} \\
\langle \delta_s - \delta_{s+1}, \beta_{i,j} \rangle &= \delta_{s,j}b_{i,j} - \delta_{s+1,j}b_{i,j}
\end{align*}
\]

where \( \delta_{s,t} \) is the Kronecker delta. We note that the conditions \( \langle \varepsilon_s - \varepsilon_{s+1}, \lambda \rangle \geq 0 \) and \( \langle \delta_s - \delta_{s+1}, \lambda \rangle \geq 0 \) for each \( s = 1, \ldots, n - 1 \) give \( 2(n - 1) \) inequalities which the coefficients \( a_{i,j} \) and \( b_{i,j} \) must satisfy.

The important step in this proof is to add all the given inequalities together to produce one inequality,

\[
\sum_{s=1}^{n-1} \left( \langle \varepsilon_s - \varepsilon_{s+1}, \lambda \rangle + \langle \delta_s - \delta_{s+1}, \lambda \rangle \right) = \sum_{s=1}^{n-1} \sum_{i > j} \left( \langle \varepsilon_s - \varepsilon_{s+1}, \alpha_{i,j} + \beta_{i,j} \rangle + \langle \delta_s - \delta_{s+1}, \alpha_{i,j} + \beta_{i,j} \rangle \right) \geq 0.
\]

Next, observe that each \( a_{i,j} \) and \( b_{i,j} \) appears exactly twice as a negative term in the inequality. Furthermore, each term \( a_{k,l} \) and \( b_{k,l} \) with \( 1 < k, l < n \) appears twice as a positive term and \( a_{i,1}, a_{n,j}, b_{i,1}, \) and \( b_{n,j} \) appear at most once as a positive term (with \( a_{1,n} \) and \( b_{1,n} \) being the terms which do not appear at all). Rearranging the inequality then yields

\[
0 \geq \sum_{s=1}^{n} (a_{s,1} + a_{n,s} + b_{s,1} + b_{n,s})
\]

and so each coefficient of this form is forced to be zero in order for a weight to be dominant. However, by induction, we have now reduced to a weight whose nonzero coefficients come from a lower diagonal \( (n-2) \times (n-2) \) matrix which has no dominant weights by the inductive hypothesis. Thus, the claim is proven. \( \square \)
In order to handle the general case of classifying $T(\mathfrak{gl}(m|n))$ when $m \neq n$, a slight modification to the previous argument must be made. Although the following proof suffices in general, the previous special case is included as it is helpful in clarifying this argument.

Define $r = \min(m, n)$ and $s = |m - n|$. When $m \neq n$, the parabolic subalgebra $\mathfrak{p}$ is now a bit different. As detailed in Section 2.1 The even component still consists of upper triangular matrices (now of different sizes) but the odd component is structurally different. Since $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are no longer square (they are $m \times n$ and $n \times m$), the entries above the odd diagonal are no longer symmetric. In particular, $\dim(\mathfrak{p}_1) \neq \dim(\mathfrak{p}_{-1})$. There is a subalgebra of $\mathfrak{p}$ isomorphic to the distinguished parabolic of $\mathfrak{gl}(r|r)$ and the previous argument can be applied to this subalgebra with only a slight modification.

**Proposition 4.2.** Let $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{gl}(m|n)$. No weight of $S^*(\mathfrak{g}/(\mathfrak{g}_1 \oplus \mathfrak{p}_1)^*)$ is dominant.

As before, let $\lambda$ be an arbitrary weight and let $a_{i,j}$ be the coefficient for the weight $\varepsilon_i - \delta_j$ and $b_{k,l}$ be the coefficient of the weight $\delta_k - \varepsilon_l$, where $i > j$ and $k > l$ and so

$$\lambda = \left( \sum_{i>j} a_{i,j} (\varepsilon_i - \delta_j) \right) + \left( \sum_{k>l} b_{k,l} (\delta_k - \varepsilon_l) \right).$$

Before, we considered all the conditions of dominance all at once and this was sufficient for the previous case. Now, the conditions will be considered in a particular order to achieve the result.

As noted, there is a canonical subalgebra $\mathfrak{gl}(r|r) \subseteq \mathfrak{gl}(m|n)$ which we will denote as $\mathfrak{g}_r$. Furthermore, $\mathfrak{g}_r$ contains a distinguished parabolic subgroup as well which will be denoted $\mathfrak{p}_r$. These subalgebras will provide a useful reduction in this proof.

There are two main steps in the proof. The first is to use an induction argument to eliminate the possibility of dominant roots in the portion of $\mathfrak{g}/(\mathfrak{g}_1 \oplus \mathfrak{p}_1)$ isomorphic to $\mathfrak{g}_r/(\mathfrak{g}_r)_{1} \oplus (\mathfrak{p}_r)_{-1}$ and so a variation on the previous argument used here. We proceed by induction on $r$.

If $r = 1$, then this first reduction step is trivial since $(\mathfrak{p}_r)_{-1} = (\mathfrak{g}_r)_{-1}$ and so $\mathfrak{g}/(\mathfrak{g}_1 \oplus \mathfrak{p}_1)$ is isomorphic to $\mathfrak{g}_1/(\mathfrak{g}_1)_{1}$, if $m > n$ and $\mathfrak{g}_{-1}/(\mathfrak{g}_{-1})_{-1}$ if $m < n$ which is the desired reduction.

If $r > 1$, then consider the conditions imposed by $\langle \varepsilon_t - \varepsilon_{t+1}, \lambda \rangle \geq 0$ if $m > n$, and consider $\langle \delta_t - \delta_{t+1}, \lambda \rangle \geq 0$ if $m < n$, where $1 \leq t \leq r - 1$ in both cases. For brevity, only the case where $m > n$ will be discussed, as the proof for $m < n$ is very similar.

If $\lambda$ is an arbitrary weight as above, the inner products $\langle \varepsilon_t - \varepsilon_{t+1}, \lambda \rangle$ for $1 \leq t \leq r - 1$ have nontrivial interaction only with the part of the weights which lies in the subalgebra $\mathfrak{g}_r/(\mathfrak{g}_r)_{1} \oplus (\mathfrak{p}_r)_{-1}$. Now, nearly the same technique as in Proposition 4.1 can be applied.

As before,

$$\langle \varepsilon_t - \varepsilon_{t+1}, a_{i,j}(\varepsilon_i - \delta_j) \rangle = \delta_{i,j} a_{i,j} - \delta_{i+1,j} a_{i,j}$$

and the inequalities given by $\langle \varepsilon_t - \varepsilon_{t+1}, \lambda \rangle \geq 0$ are again summed to yield

$$\sum_{t=1}^{r-1} \langle \varepsilon_t - \varepsilon_{t+1}, \lambda \rangle = \sum_{t=1}^{r-1} \sum_{i>j} \langle \varepsilon_t - \varepsilon_{t+1}, a_{i,j} + b_{i,j} \rangle \geq 0.$$
$b_{k,l}$ with $i < r$ and $l > 1$ appear once as positive terms. Rearranging the inequality gives

$$0 \geq \sum_{t=1}^{n} (a_{r,t} + b_{1,t})$$

which reduces to a case isomorphic to showing that there are no dominant weights in $\mathfrak{g}_{r-1}/((\mathfrak{g}_{r-1})_0 \oplus (\mathfrak{p}_{r-1})_T)$ which contains no dominant weights by the inductive hypothesis which completes the first step.

Now, if a weight $\lambda$ is dominant, it must be a weight for $\mathfrak{g}_1/(\mathfrak{g}_r)_1$ since all the other coefficients of $\lambda$ have been show to be 0. This step is significantly easier since now

$$\lambda = \sum_{r < i \leq m} a_{i,j} (\varepsilon_i - \delta_j)$$

and the condition $\langle \varepsilon_r - \varepsilon_{r+1}, \lambda \rangle \geq 0$ implies that

$$0 \geq \sum_{m+1 < j \leq m+n} a_{r+1,j}$$

and so each of these coefficients is 0. This process is repeated stepwise for the conditions $\langle \varepsilon_t - \varepsilon_{t+1}, \lambda \rangle \geq 0$ for $r < t < m-1$ which shows that $a_{t,j} = 0$ for all $m+1 < j \leq m+n$, and finally, that $\lambda = 0$, which is not a dominant weight. Thus, no weight of $S^*(\mathfrak{g}/(\mathfrak{g}_0 \oplus \mathfrak{p}_T))^*$ is dominant and the proof is complete.

**Corollary 4.3.** Let $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{gl}(m|n)$, then $\text{Ind}_{\mathfrak{p}}^\mathfrak{g} k \cong k$.

**Proof.** Since $S^*(\mathfrak{g}/(\mathfrak{g}_0 \oplus \mathfrak{p}_T))^*$ has no dominant weights by the previous lemma,

$$H^0(G_0/P_0, S^*(\mathfrak{g}/(\mathfrak{g}_0 \oplus \mathfrak{p}_T))^*) \cong k.$$ 

Furthermore, note that the induction functor does not change the parity of the module, so the degree (either even or odd) is fixed and the result is proven. \qed

**Corollary 4.4.** The restriction map

$$\text{res}^{T(\mathfrak{g})}_{T(\mathfrak{p})} : T(\mathfrak{g}) \to T(\mathfrak{p})$$

given by $M \mapsto M|_\mathfrak{p}$ is injective.

**Proof.** By Corollary 3.6, it is sufficient to check that $\text{ker} \left( \text{res}^{T(\mathfrak{g})}_{T(\mathfrak{p})} \right) = \{k_{ev}\}$.

Let $M \in T(\mathfrak{g})$ be an indecomposable endotrivial module in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ such that $M|_\mathfrak{p} \cong k_{ev} \oplus P$. Then

$$\text{Ind}_{\mathfrak{p}}^\mathfrak{g} M|_\mathfrak{p} \cong \text{Ind}_{\mathfrak{p}}^\mathfrak{g} (k_{ev} \oplus P) \cong \text{Ind}_{\mathfrak{p}}^\mathfrak{g} k_{ev} \oplus \text{Ind}_{\mathfrak{p}}^\mathfrak{g} P \cong k_{ev} \oplus \text{Ind}_{\mathfrak{p}}^\mathfrak{g} P.$$ 

However, since $M$ is already a $\mathfrak{g}$-module, by the tensor identity given in [17, Lemma 1],

$$\text{Ind}_{\mathfrak{p}}^\mathfrak{g} M|_\mathfrak{p} \cong \text{Ind}_{\mathfrak{p}}^\mathfrak{g} (M|_\mathfrak{p} \otimes k_{ev}) \cong M \otimes \text{Ind}_{\mathfrak{p}}^\mathfrak{g} k_{ev} \cong M \otimes k_{ev} \cong M$$

and so we have that, as $\mathfrak{g}$-modules, $M \cong k_{ev} \oplus \text{Ind}_{\mathfrak{p}}^\mathfrak{g} P$. Since $M$ is indecomposable, $M \cong k_{ev}$ and thus the map $\text{res}^{T(\mathfrak{g})}_{T(\mathfrak{p})}$ is injective. \qed
5. Classification of $T(\mathfrak{gl}(m|n))$

5.1. Classification of $T(\mathfrak{p})$. Using Corollary 4.4 we will use the classification of $T(\mathfrak{p})$ to yield a classification of $T(\mathfrak{g})$. The computation of $T(\mathfrak{p})$ is derived by considering the kernel of the restriction map given in Corollary 3.8.

For the remainder of the section let $\mathfrak{g} = \mathfrak{gl}(m|n)$ with maximal torus $\mathfrak{t}_\mathfrak{g}$ and let $\mathfrak{p}$ be the distinguished parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{f} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ and $\mathfrak{f}_\mathfrak{g} = \mathfrak{t}_\mathfrak{g} \subseteq \mathfrak{t}_\mathfrak{p} = \mathfrak{t}_\mathfrak{p}$. Let $r = \min(m, n)$ and $s = |m - n|$. 

**Theorem 5.1.** There are isomorphisms of groups

1. $T(\mathfrak{f}) \cong k^s \times \mathbb{Z} \times \mathbb{Z}_2$;
2. $T(\mathfrak{p}) \cong k^r+s \times \mathbb{Z} \times \mathbb{Z}_2$.

**Proof.** First, we recall some details about $\mathfrak{f}$. It is important to note that in this more general setting, the detecting subalgebra $\mathfrak{f}$ may not be isomorphic to $\mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1)$ as is assumed in [23] because there are more entries on the even diagonal. If $m \neq n$, then define $\mathfrak{f} \cong \mathfrak{f}_r \oplus \mathfrak{t}_s$ where $\mathfrak{f}_r$ is a direct sum of $r$ copies of $\mathfrak{sl}(1|1)$, the detecting subalgebra of $\mathfrak{gl}(r|r) \subseteq \mathfrak{gl}(m|n)$, and $\mathfrak{t}_s$ is an $s$ dimensional torus generated by the remaining diagonal entries whose span does not intersect the torus of $\mathfrak{f}_r$. Thus, the classification of $\mathfrak{f}$-endotrivial modules in [23] must be modified slightly.

First let $L$ be an indecomposable endotrivial module in $\mathcal{F}(\mathfrak{f}_r|\mathfrak{p})$. The classification given in [23, Theorem 6.2] indicates that $L|_{\mathfrak{f}_r} \cong \Omega_{\mathfrak{f}_r}^i(k|_{\mathfrak{f}_r}) \oplus P'$ where $P'$ is some projective $\mathfrak{f}_r$-module, $k|_{\mathfrak{f}_r}$ is either $k_{\text{ev}}$ or $k_{\text{od}}$, and $n \in \mathbb{Z}$. Since $\mathfrak{f}$ is just $\mathfrak{f}_r$ with an enlarged torus, the structure of $\mathfrak{f}$-modules is not fundamentally different, there are just more weights in $\mathfrak{f}$. So an endotrivial $\mathfrak{f}$-module is an endotrivial $\mathfrak{f}_r$-module by restriction, which has an action of an $s$ dimensional torus that may act by any weight because $[\mathfrak{f}_r, \mathfrak{f}_r] \cap \mathfrak{t}_s = 0$ (see [23, Proposition 7.4] for more on this topic). We may then conclude that $L \cong \Omega_{\mathfrak{f}}^i(k_{\mathfrak{f}}|_{\mathfrak{f}}) \oplus P'$ where $P'$ is a projective $\mathfrak{f}$-module, and $\lambda \in X((\mathfrak{f}_r|\mathfrak{p}) \oplus \mathfrak{t}_s)$ of the form $(0, \ldots, 0, \lambda_{2r+1}, \ldots, \lambda_{n+m})$, i.e., $k_{\mathfrak{f}}|_{\mathfrak{f}} \cong k$ the trivial module concentrated in even or odd degree. Thus, $T(\mathfrak{f}) \cong k^s \times \mathbb{Z} \times \mathbb{Z}_2$ in this setting.

Given this observation, consider the map $\text{res}^{T(\mathfrak{p})}_{T(\mathfrak{f})} : T(\mathfrak{p}) \to T(\mathfrak{f})$ and in particular, its kernel. Let $M$ be an indecomposable endotrivial module in $\mathcal{F}(\mathfrak{p}, \mathfrak{p})$ such that $M|_{\mathfrak{f}} \cong k_{\text{ev}} \oplus P$. Now, consider the weight space decomposition of $M|_{\mathfrak{f}}$ relative to the weights of $\mathfrak{t}_\mathfrak{f}$. Then

\begin{equation}
M \cong \bigoplus_{\lambda \in X(\mathfrak{t}_\mathfrak{f})} M_{\lambda}
\end{equation}

is not only a direct sum over $\mathfrak{t}_\mathfrak{f}$, but as a module in $\mathcal{F}(\mathfrak{f}_r, \mathfrak{p})$, since $[\mathfrak{f}_r, \mathfrak{f}_r] = 0$. Thus, by comparing the direct summands of $M|_{\mathfrak{f}} \cong k_{\text{ev}} \oplus P$ with those in Equation 5.2, we note that the only non-projective summand $k_{\text{ev}}$ occurs in the block $M_0$ and thus $M_\lambda$ is projective in $\mathcal{F}(\mathfrak{f}_r, \mathfrak{p})$ when $\lambda \neq 0$.

Define $u = \mathfrak{p}/\mathfrak{t}_\mathfrak{p}$ and consider the action of $u$ on $M$ relative to the direct sum decomposition of $M$ over $\mathfrak{f}$. For $u \in u$, $u.M_\lambda \subseteq M_\mu$ implies that $\mu > \lambda$ in the dominance ordering of $X(\mathfrak{t}_\mathfrak{f})$ defined in Equation 3.11. Define

\[ \hat{M} = \bigoplus_{\lambda \in X(\mathfrak{t}_\mathfrak{f}), \lambda \neq 0} M_{\lambda} \]

then $\hat{M}$ is a $\mathfrak{p}$-submodule of $M$ by construction. Observe that if $\hat{M} \neq 0$, then $\hat{M}$ is a module in $\mathcal{F}(\mathfrak{p}, \mathfrak{p})$ such that $\hat{M}|_{\mathfrak{f}}$ is projective in $\mathcal{F}(\mathfrak{f}_r, \mathfrak{t}_\mathfrak{g})$ and so by Corollary 3.11, $\hat{M}$ is projective in
\( \mathcal{F}(p,p) \). Since \( \mathcal{F}(p,p) \) is self injective, projective modules are also injective and so this gives a splitting \( M \cong \tilde{M} \oplus \tilde{M}^c \) which is a contradiction since \( M \) was assumed to be indecomposable and so \( \tilde{M} = 0 \).

Then \( M \) decomposes as

\[
M \cong \bigoplus_{\lambda \in X(t_f)} M_{\lambda}
\]

in \( \mathcal{F}(f,f_g) \). Note that \( M_0 \) is a \( p \)-submodule of \( M \) and define \( \tilde{M} = M/M_0 \). Again, \( \tilde{M} \) is a \( p \)-module which is projective when restricted to \( f \). Thus \( M \cong \tilde{M} \oplus \tilde{M}^c \) and we conclude \( \tilde{M} = 0 \) as well.

So it must be that \( M = M_0 \) and so \( u.M = 0 \). Since \( M \) is in \( \mathcal{F}(p,p) \), which by definition has a weight space decomposition relative to \( t_p \), and since \( p \cong t_p \oplus u \), the decomposition \( M|_f \cong k_{ev} \oplus P \) is also a decomposition over \( p \). Thus, \( M|_f \cong k_{ev} \) and so \( M \) is a one dimensional module whose weights over \( t_p \) collapse to the trivial weight when restricted to \( t_f \).

So \( M \cong k_{\lambda} \) where \( \lambda = (\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{2r}, 0, \ldots, 0) \) and \( \lambda_i = -\lambda_{r+i} \) for \( i = 1, \ldots, r \)

and we see that \( \ker(\text{res}_{T_{(f)}(p)}^T(M)) \cong k^r \).

Recalling the notation \( f \cong f_0 \oplus t_s \), we now have shown that \( T(p) \cong k^{r+s} \times \mathbb{Z} \times \mathbb{Z}_2 \), generated by \( \Omega_p^j(k_{ev}) \), \( k_{\lambda} \) such that \( k_{\lambda}|_{f_0} \cong k_{ev} \) and the parity change funtor.

5.2. Classification of \( T(\mathfrak{gl}(m|n)) \). The final step in the classification results from making a few observations about the results of Corollary 4.4 and Theorem 5.1.

**Theorem 5.3.** There is an isomorphism of groups

\[
T(\mathfrak{g}) \cong k \times \mathbb{Z} \times \mathbb{Z}_2.
\]

**Proof.** It was shown that the map \( \text{res}_{T(p)}^T(M) : T(\mathfrak{g}) \to T(p) \) is injective, but now that \( T(p) \) has been classified, the image of the injection can be computed directly.

By Lemma 2.6

\[
\Omega^i_p(M)|_p \cong \Omega^i_p(M|_p) \oplus P
\]

where \( P \) is a projective module in \( \mathcal{F}(p,p) \) and so in the respective stable module categories (where \( T(\mathfrak{g}) \) and \( T(p) \) are defined), the syzygy operation commutes with restriction. Additionally, the parity change functor commutes as well and so it is clear that \( \Omega^i_p(k_{ev}) \) and parity change functor generate a subgroup of \( T(\mathfrak{g}) \) isomorphic to \( \mathbb{Z} \times \mathbb{Z}_2 \).

The last factor of \( T(p) \) is \( k^{r+s} \) which arises from the one dimensional modules in \( \mathcal{F}(p,p) \).

There are fewer one dimensional modules in \( \mathfrak{g} \) (except when \( m = n = 1 \) in which case \( \mathfrak{g} = p \)) because \([p,p] \cap t_{\mathfrak{g}} \leq [\mathfrak{g},\mathfrak{g}] \cap t_{\mathfrak{g}} \). In \( \mathfrak{g} \) there is only one parameter of one dimensional modules whose weights relative to \( t_{\mathfrak{g}} \) are \( (\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}) \) where \( \lambda_i = -\lambda_j \) for all \( 1 \leq i \leq m < j \leq m + n \).

Noting that \( p \) and \( \mathfrak{g} \) share the same torus, \( k_{\lambda} \in \mathcal{F}(\mathfrak{g},\mathfrak{g}) \) restricts to \( k_{\lambda} \in \mathcal{F}(p,p) \), and we now have a complete description of the image of the restriction map. An arbitrary indecomposable endotrivial module in \( \mathcal{F}(\mathfrak{g},\mathfrak{g}) \) is of the form \( \Omega^i_p(k_{\lambda}) \) for \( \lambda \in X(t_{\mathfrak{g}}) \) such that \( \lambda = (\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}) \) where \( \lambda_i = -\lambda_j \) for all \( 1 \leq i \leq m < j \leq m + n \) and

\[
T(\mathfrak{g}) \cong k \times \mathbb{Z} \times \mathbb{Z}_2 \hookrightarrow T(p) \cong k^{r+s} \times \mathbb{Z} \times \mathbb{Z}_2
\]

where the injection is given by restriction. Thus, \( T(\mathfrak{g}) \) is generated by \( \Omega^i_p(k_{ev}) \), \( k_{\lambda} \) concentrated in even degree where \( \lambda \) is as above, and the parity change funtor. \( \square \)
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