Finite Size Effects for the Ising Model Coupled to 2-D Random Surfaces

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Abstract

Finite size effects for the Ising Model coupled to two dimensional random surfaces are studied by exploiting the exact results from the 2-matrix models. The fixed area partition function is numerically calculated with arbitrary precision by developing an efficient algorithm for recursively solving the quintic equations so encountered. An analytic method for studying finite size effects is developed based on the behaviour of the free energy near its singular points. The generic form of finite size corrections so obtained are seen to be quite different from the phenomenological parameterisations used in the literature. The method of singularities is also applied to study the magnetic susceptibility. A brief discussion is presented on the implications of these results to the problem of a reliable determination of string susceptibility from numerical simulations.

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1 Introduction

Numerical simulations are important for the study of nonperturbative effects which are difficult to handle analytically. In such simulations one necessarily has to work with systems of finite size. The effects of finite size manifest themselves as systematic errors in measurements. These corrections are generically difficult to estimate as they involve dynamical details. A practical approach is thus often resorted to by applying phenomenological parametrisations to finite size effects[1].

In this paper we address the question of finite size corrections by considering the model of Ising spins coupled to two dimensional random surfaces. Since the Ising spin case, with central charge \( c_M = 1/2 \), is exactly solvable by the method of matrix models[2, 3, 4] it provides an important test case for the efficacy of numerical simulations. An understanding of the nature of finite size corrections in this case may be useful in probing the unknown region beyond \( c_M = 1 \).

We approach this issue numerically and analytically by exploiting the parametric solution of the two matrix models[3, 4]. In the numerical approach the inherent quintic equations are solved recursively from which the fixed area partition sums are extracted for various spin couplings. We also extend this approach to the case of nonzero magnetic field and determine the magnetic susceptibility scaling laws. The numerical analysis is augmented by an analytic analysis of the free energy about its singular points. From this we suggest a general ansatz for the form of finite size corrections. We apply our results to different estimates of the string susceptibility and to the minbhu technique[3], demonstrating the difference with the phenomenological estimates previously employed[1]. The method of singularities is also applied to the problem of the finite size behaviour of the magnetic susceptibility. It is found that these agree with the usual finite size scaling laws[2].
2 Numerical Solution of the Two Matrix Model

Matrix models are solved by the method of orthogonal polynomials whereby a parametric solution for the free energy is obtained. The physical couplings are related to the parameter so introduced by what we shall call a “constraint”. From such a solution the relevant critical exponents can be derived. We will restrict our analysis to surfaces topologically equivalent to $S^2$.

For pure gravity (the one matrix model) the constraint equations are quadratic and cubic for the case of quartic and cubic interactions respectively. The constraints are thus solvable by radicals allowing the constraint to be inverted. From this it is possible to derive a closed form for the free energy in the coupling $g$. For a quartic interaction the series solution for the free energy is given by $[2]$:

$$F = -\sum_n \frac{(-12g)^n(2n-1)!}{n!(n+2)!},$$

which can be written in the form $F = \sum_n Z_n g^n$, $Z_n$ being the fixed area partition sum for random surfaces. By exploiting the asymptotic expansion for the gamma function it is a simple matter to explicitly derive the asymptotic form of the fixed area partition sum:

$$Z_n \sim \frac{(-1)^n(48)^n}{\sqrt{4\pi}n^{7/2}(1 - \frac{25}{8n}......)}.$$

The leading order behaviour corresponds to that originally predicted by KPZ [8] and has the generic form $Z_n \sim \exp(\mu n)n^{-b}$, where the string susceptibility is defined as $\gamma = -b+3$, and $\mu$ is the cosmological constant. Similar results can be obtained in the case of cubic interactions as well as for models with tadpole and/or self energy contributions removed [2].

To represent models in which Ising spins have been coupled to the, discretised, random surface a two matrix model is required. The method of solution follows as before. Again the free energy is given as a parametric solution in terms of a parameter $z$. However, we now have the difficulty of a constraint that is a quintic [7], which for quartic interactions is given by (where $c = \exp(-2\beta)$ and $\beta = 1/T$):

$$g(z) \equiv \frac{z}{(1-3z)^2} - c^2z + 3c^2z^3 = g,$$
The general quintic is not solvable by radicals. As such a simple closed form for the free energy cannot be derived, as was possible in the case of pure gravity, and consequently the asymptotic form of the fixed area partition sums cannot be easily extracted.

In this section we will describe our numerical method of solution to this problem where we seek a power series expansion for the free energy in terms of $g$. We will describe in detail the case for quartic interactions. The solution in the quartic case contains all the essential elements, while the cubic case is both algorithmically and algebraically more cumbersome.

To avoid the logarithm in the solution for the free energy we study $d\mathcal{F}/dg$ which is equivalent in this instance to $\partial \mathcal{F}/\partial g$ yielding:

$$\frac{\partial \mathcal{F}}{\partial g} = \left[ -\frac{1}{2g} + \frac{1}{g^2} \int_0^z \frac{dt}{t} g(t) - \frac{1}{g^2} \int_0^z \frac{dt}{t} g(t)^2 \right].$$

We wish to represent this as a simple polynomial in $z$. To remove the inverse powers of $(1 - 3z)$ which arise we employ the quintic constraint (3). $\partial \mathcal{F}/\partial g$ is then a polynomial in $z$ of degree 8 with coefficients depending on the physical parameters $c$ and $g$. This can further be reduced to a polynomial of degree 5 by using the constraint to express $z^8$, $z^7$ and $z^6$ as quintic polynomials in $z$.

To solve the quintic (3) numerically we develop an efficient algorithm to handle the multiple sums inherent in a power series solution of this expression. We require an algorithm which avoids recalculation. We begin by explicitly expressing each power of $z$ as a power series in $g$:

$$z = \sum a_1(n) \, g^n; \quad z^2 = \sum a_2(n) \, g^n; \quad \ldots \quad z^5 = \sum a_5(n) \, g^n .$$

Using (3), and the fact that $a_2(1) = a_3(1) = a_4(1) = a_5(1) = 0$ it is a simple matter to extract $a_1(1) = 1/(1 - c^2)$. Indeed, a recursion relation for $a_1(n)$ for $n \geq 2$ can be derived from (3) which has the general form:

$$a_1(n) = \xi_1 \, a_1(n - 1) + \xi_2 \, a_2(n) + \xi_3 \, a_2(n - 1) + \xi_4 \, a_3(n) + \xi_5 \, a_4(n) + \xi_6 \, a_5(n) .$$

where the $\xi$’s are the coefficients appearing in the quintic. The $a_2(n)$’s, $a_3(n)$’s etc can ultimately be described in terms of the $a_1(n)$’s. For each power of $z$ the coefficients are
dependent on a calculation of those of lower orders such that for example

$$a_2(N) = \sum_{1}^{N-1} a_1(n) \cdot a_1(N-n) .$$  \hspace{1cm} (7)$$

Thus in order to calculate $a_2(N)$ we need to find $a_1(k)$ up to $k = N - 1$. Likewise, $a_3(N) = \sum_{1}^{N-1} a_1(n) \cdot a_2(N-n)$ implies that a calculation of $a_3(N)$ requires $a_2(N-1)$, which requires $a_1(N-2)$ etc.

By this approach the calculation time is significantly reduced (in this case the calculation time goes as $\sim N^2$ as opposed to $\sim N^6$ for a naive approach to multiple sums). Substituting into (4) and integrating with respect to $g$ we can recover the fixed area partition sums up to large orders in a convenient time.

We must, however, be careful to consider the exponential growth in the fixed area partition sums due to the cosmological term. To overcome this problem we estimate the cosmological term by studying the fixed area partition sums up to areas allowed by machine limits. The fixed area sums are then scaled by scaling all the $a(N)$ by this estimate, so that

$$a(N) \rightarrow e^{-\mu N} a(N) ; \quad \xi_1 \rightarrow e^{-\mu} \xi_1 \; , \; \xi_3 \rightarrow e^{-\mu} \xi_3 .$$  \hspace{1cm} (8)$$

We found that in this way sufficiently accurate estimates of the cosmological constant could be obtained for relatively small values of $n (\approx 200)$ which could then be used to extract the scaled fixed area partition sums for even large $n \approx 100000$ easily. This makes it possible to investigate regions corresponding to those typical of numerical simulations via dynamical triangulation, and far beyond.

To investigate the magnetic susceptibility we follow the same general prescription as above. With the introduction of a magnetic field two different coupling constants arise. In a perturbative solution to lowest order in $H$ the constraint (3) becomes

$$g(z, H) = \frac{z}{(1 - 3z)^2} - c^2 z + 3c^2 z^3 + \frac{z^2 H^2}{(1 - 3z)^2(1 + 3z)^2} .$$  \hspace{1cm} (9)$$

Since we require solutions in the limit $H \rightarrow 0$ the power series expansions for $z, z^2, \ldots$ etc in $g$ remain defined as before by (3). The high and low temperature phases corresponding
to the singularities of the free energy with respect to $g$ are defined by the expression $g'(z, H = 0) = 0$. This has five solutions of which only two are physical for $0 < c < 1$:

$$z_0 = -1/3 \quad \text{(Low temp. phase)} \quad \text{and} \quad z_0 = z_0(c) \quad \text{(High temp. phase)}$$ \hspace{1cm} (10)

The critical temperature is at $c = 1/4$.

The magnetic susceptibility is essentially given by the second derivative of $\mathcal{F}$ with respect to $H$ at $H = 0$. In particular we find

$$K(z) \equiv \frac{\partial^2 \mathcal{F}}{\partial H^2}|_{H=0} = \int_0^z \frac{dt}{t} \left[ \frac{g(t, H)}{g^2} - \frac{1}{g} \frac{\partial^2 g}{\partial H^2}|_{H=0} \right].$$ \hspace{1cm} (11)

We can express this as before as a power series in $g$. We thus have an expansion of the form $K(z) = \sum K_n g^n$. The fixed area magnetic susceptibility is then given by

$$\chi_n = \frac{K_n}{nZ_n}.$$ \hspace{1cm} (12)

This is valid at or above the critical temperature. Below the critical temperature we must account for the spontaneous ordering of spins so that the fixed area magnetic susceptibility is given by

$$\chi_n = \frac{K_n}{nZ_n} - n < \sigma >^2,$$ \hspace{1cm} (13)

where $< \sigma >$ is the spontaneous magnetization.

3 Results from the Numerical Analysis

We first exhibit the finite size effects in string susceptibility. For the case at hand $\gamma$ is known exactly: $\gamma = -1/3$ at the critical temperature and $\gamma = -1/2$ off the critical temperature. We estimate the string susceptibility at finite area, $\gamma_{\text{est}}$, by a suitable ratio of fixed area partition sums designed to cancel the cosmological constant:

$$\gamma_{\text{est}} = \left\{ \ln \left( \frac{Z_{n+1}Z_{n-1}}{Z_n^2} \right) / \ln \left( 1 - \frac{1}{n^2} \right) \right\} + 3.$$ \hspace{1cm} (14)

This is a “local” estimate of $\gamma$ in that it involves neighbouring partition sums. Clearly, however, there are many different ways in which to extract such an estimate.
We present the results of $\gamma_{\text{est}}$ for quartic interactions at the critical temperature in Fig(1) and simply note that the cubic case demonstrates the same behaviour. Indeed we observe the following important common features:

(i) For large areas $\gamma_{\text{est}}$ approaches the theoretical expectation both on and off the critical temperature.

(ii) Off the critical temperature $\gamma_{\text{est}}$ approaches $-1/2$ rapidly from above. At the critical temperature $\gamma_{\text{est}}$ drops below $-1/3$ and then slowly converges towards the theoretical value from below.

(iii) For the region at the critical temperature where $\gamma_{\text{est}}$ is less than $-1/3$ the effects of finite size are greatest in the range approximately bounded by $200 < n < 2000$.

That a qualitative difference on and off the critical temperature should appear is consistent with the expectation that finite size effects will be influenced by large Ising spin correlations at the critical point. However, such a difference has not previously been taken into account. We observe that this is, in fact, an important factor which cannot be ignored. Significantly, the effects of such finite size corrections appear most pronounced in regions where previous numerical simulations have concentrated their estimates of $\gamma$, which most typically deal with simulations of the size ranging around $n = 1000 \sim 2000[1]$. We see that finite size effects at the critical temperature do not simply decrease with increasing size but exhibit an important nonlinearity with greatest effect around the simulation sizes chosen. In addition we note that, since a range of values for $\gamma_{\text{est}}$ are recovered, it follows that we can engineer to find many values of $\gamma$. Indeed, at the critical temperature $\gamma_{\text{est}}$ actually passes through the correct value even for small systems. It appears then that extraction of $\gamma$ from small area studies can be misleading.

As we pointed out above, these results are exhibited for a particular choice of form for the estimation of $\gamma$. However, as will be demonstrated below, while some of the features may change, our observations in general remain sound.

The magnetic susceptibility results reproduce those expected from Liouville theory[10] as well as from standard scaling analysis [6]. In particular we find that

$$\chi_n \rightarrow \text{constant} \quad \text{(High temp. phase)}$$
\[ n^{2/3} \quad \text{(on the critical temp.)}, \quad (15) \]

while in the low temperature phase

\[ \chi_n + n < \sigma >^2 \rightarrow n. \quad (16) \]

### 4 Singularity Analysis of Finite Size Effects

We wish now to analytically find the asymptotic form of the fixed area partition sums. This is made possible by the observation that the large \( n \) behaviour of \( Z_n \) is dominated by the singular points of the free energy. As before we will deal explicitly with the case of quartic interactions. We motivate the analysis by applying this approach to the simple case of pure gravity.

The parametric solution for the one matrix model with quartic interactions is given by

\[ F = -\frac{1}{2} \ln z + \frac{1}{24} (z-1)(9-z) \quad \text{with} \quad g(z) \equiv \frac{1-z}{12z^2} = g. \quad (17) \]

The singularity of \( F \) with respect to \( g \) is determined by the condition \( g'(z_0) = 0 \) for which \( z_0 = 2 \), so that \( g_c = -1/48 \). The constraint in (17) can easily be inverted to yield

\[ (z - z_0) = a(g - g_c)^{1/2} + b(g - g_c) + \ldots. \quad (18) \]

where for brevity we have not displayed the values of \( a, b, \ldots \). This in turn can be used to generate an expansion for \( F \) about the critical point \( g_c \). Being careful to retain sufficient terms in the expansion the nonregular terms are found to be

\[ F(g) \sim \frac{12283\sqrt{3}}{5} (g-g_c)^{5/2} + \frac{1769472\sqrt{3}}{7} (g-g_c)^{7/2} + \ldots. \quad (19) \]

The required asymptotic form can be obtained by employing the binomial expansion for \((g-g_c)^\alpha\) and subsequently the asymptotic expansion for the gamma function. Writing \((g-g_c)^\alpha = \sum_n g^n Z_n^\alpha\), one finds

\[ Z_n^\alpha \sim \frac{\sin(\pi\alpha)}{\pi} \Gamma(\alpha + 1) n^{-1+\alpha} \exp(\alpha(1+\alpha)/2n) + \ldots. \quad (20) \]
The term with $\alpha = 5/2$ is seen to reproduce the leading order behaviour found in (2).

We can also find the leading order corrections which involve contributions from both the $(g - g_c)^{5/2}$ and $(g - g_c)^{7/2}$ terms:

$$Z_n \sim Z_n^{\text{leading}} (1 + \frac{35}{8n} - \frac{1769472}{12283} \Gamma(9/2) \Gamma(7/2) \frac{1}{n} + \ldots) \quad (21)$$

which is exactly that expressed in (2). We thus have a method by which finite size corrections may be derived without relying on a closed form for $\mathcal{F}$.

We now apply this approach to the two matrix model. From the parametric representation of the free energy it follows that $\mathcal{F}'(z_0) = d\mathcal{F}/dz|_{z_0} = 0$, where $z_0$ is given in (10).

The behaviour of the second derivatives with respect to $z$ are:

- Off crit. temp. : $\mathcal{F}''(z_0) \neq 0$, $g''(z_0) \neq 0$,
- On crit. temp. : $\mathcal{F}''(z_0) = 0$, $g''(z_0) = 0$, \quad (22)

where the non-vanishing of $\mathcal{F}''(z_0)$ off criticality is crucial to obtaining the correct scaling laws. Hence we differ from [3] in this respect. We must thus consider two regions.

(i) **The case off the critical temperature**, $c \neq c_{\text{crit}}$: As in the case of pure gravity we invert the constraint (3) to give an expansion for $(z - z_0)$ in $(g - g_c)$:

$$z - z_0 = a(g - g_c)^{1/2} + b(g - g_c) + \ldots. \quad (23)$$

Taylor expanding $\mathcal{F}$ around $z_0$ and substituting (23) generates a series of both regular and nonregular terms in $(g - g_c)$. Formally, the lowest exponent of the nonregular terms is $3/2$ but this term vanishes owing to the relation

$$\mathcal{F}''(z_0)ab + \frac{a^3}{6} \mathcal{F}'''(z_0) = 0 \quad (24)$$

explicitly requiring that $\mathcal{F}''(z_0) \neq 0$. The nonregular terms contributing to $\mathcal{F}$ are thus

$$\mathcal{F}(g) \sim \mathcal{A}1(g - g_c)^{5/2} + \mathcal{A}2(g - g_c)^{7/2} + \ldots \quad (25)$$

so that

$$Z_n \sim (-g_c)^{-n} n^{-7/2} \{ 1 + \frac{B_1}{n} + \frac{B_2}{n^2} + \ldots \} \quad (26)$$
again exhibiting the same basic form as that for pure gravity. Some representative values for $B_1$ with quartic interaction are

$$B_1 = -72.69 \quad \text{at} \quad c = 0.20 \quad ; \quad B_1 = -8.76 \quad \text{at} \quad c = 0.36. \quad (27)$$

(ii) At the critical temperature $c = c_{\text{crit}}$: Since now $g''(z_0) = 0$ the expansion of (3) takes the form

$$(g - g_c) = \frac{(z - z_0)^3 g'''(z_0)}{6} + \frac{(z - z_0)^4 g^{IV}(z_0)}{24} + \ldots,$$

which after inversion gives:

$$(z - z_0) = a(g - g_c)^{1/3} + b(g - g_c)^{2/3} + d(g - g_c) + \ldots. \quad (28)$$

Furthermore, since now $F''(z_0) = 0$, the Taylor expansion for $F$ around $z_0$ starts at $(z - z_0)^3$. Substituting (28) into this expansion for $F$ the coefficients of powers of $(g - g_c)$ conspire so that the $(g - g_c)^{4/3}$ and $(g - g_c)^{5/3}$ terms are absent. The leading singular behaviour for $F$ is thus given by $F \sim (g - g_c)^{7/3}$. Consequently, following the same steps as before, the leading behaviour of $Z_n$ will take the form $Z_n \sim (-g_c)^{-n} n^{-10/3}$, from which it follows that the string susceptibility is given by $\gamma = -1/3$. The corrections can be similarly calculated for which we quote the results:

Quartic interaction : $Z_n = Z_n^{\text{leading}} \{1 + \frac{0.4287}{n^{1/3}} - \frac{3.08}{n} - \frac{1.2980}{n^{4/3}} + \ldots\}$

Cubic interaction : $Z_n = Z_n^{\text{leading}} \{1 + \frac{0.286}{n^{1/3}} - \frac{3.05}{n} - \frac{0.936}{n^{4/3}} + \ldots\}. \quad (29)$

The crucial observation here is that the next to leading order correction goes as $1/n^{1/3}$ rather than $1/n$ which is the case with pure gravity. We have verified that these corrections reproduce the observed finite size effect from the numerical analysis (see Fig(2)). We have thus isolated the fundamental difference between these two cases. It is now clear that assuming $1/n$ type corrections both off and on the critical temperature is not justified.

For the magnetic susceptibility the important expression was given in (11) which we can express as the sum of two integrals $K(z) = I_1 + I_2$. We see from (3) that at the critical point $z_0$ the expression $g(z, H)$ is singular. Consequently both $I_1$ and $I_2$ are singular. However, the sum $I_1 + I_2$ is regular at $z_0$.

As with the free energy we can extract the critical behaviour of the magnetic susceptibility by taking account of its behaviour about the singular point. Expanding $K(z)$ about
we find that, in the high temperature phase as well as at the critical temperature,
\[ K(z) \sim \alpha(z-z_0)^2 + \beta(z-z_0)^3 + \ldots, \]
while in the low-temperature phase, \( K(z) \sim (z-z_0). \)

We know from (23) that off the critical temperature \( (z-z_0) \sim a(g-g_c)^{1/2} + b(g-g_c) + \ldots \).

Consequently, the leading nonregular term is \( K(g) \sim (g-g_c)^{3/2} \), where we have verified that this term does not vanish as was the case with the free energy for which we know that \( F \sim (g-g_c)^{5/2} \). Expanding in \( g^n \) it follows that
\[
\chi_n = \frac{K_n}{n\mathcal{Z}_n} \to \text{constant} \ . \tag{30}
\]

Similarly at the critical temperature we have \( (z-z_0) \sim a(g-g_c)^{1/3} + \ldots \) so that \( K(g) \sim (g-g_c)^{2/3} \). We know that \( F \sim (g-g_c)^{7/3} \) so that we find
\[
\chi_n \sim n^{2/3} \ . \tag{31}
\]

In the low temperature phase, it follows that \( K(g) \sim (g-g_c)^{1/3} \). Consequently,
\[
\frac{K_n}{n\mathcal{Z}_n} \sim n \ . \tag{32}
\]

We can thus account for the behaviour expected from standard Fisher-scaling theory as well as from Liouville theory.

5 Alternate Estimates of \( \gamma \)

We now investigate alternate definitions for the estimation of \( \gamma \). From \( \mathcal{Z}_n \) one can introduce an obvious such alternative:
\[
\gamma_{\text{alt. est}} = \gamma_{\text{exact}} + \frac{\ln(1 + \text{finite size corrections})}{\ln(n)} \ , \tag{33}
\]
from which it follows that
\[
\begin{align*}
\gamma_{\text{alt. est}} &= \gamma_{\text{exact}} + \frac{c_1}{\ln(n)n^{1/3}} ; \\
c &= c_{\text{crit}} : \gamma_{\text{alt. est}} \sim \gamma_{\text{exact}} + \frac{c_1}{\ln(n)n^{1/3}} ; \\
c &\neq c_{\text{crit}} : \gamma_{\text{alt. est}} \sim \gamma_{\text{exact}} + \frac{d_1}{\ln(n)n} \ . \tag{34}
\end{align*}
\]
where $c_1 > 0$ and $a_1 < 0$. According to this definition the finite size behaviour predicted is opposite that found from the numerical results for $\gamma_{\text{est}}$. That is, off the critical temperature $\gamma_{\text{alt. est}}$ approaches $-1/2$ from below while on the critical temperature it approaches $-1/3$ from above. There is no real inconsistency here as there are many ways in which to estimate $\gamma$, and the different estimates are only required to coincide asymptotically. For example, Brezin and Hikami [9] use an estimate based on Padé approximation where

$$\gamma_{\text{Pade est}} = 3 - \frac{n(1 + n)(f_n - f_{n-1})}{(1 + n)f_n - nf_{n-1}},$$

and $f_n$ is a Padé approximant to the ratio $Z_n/Z_{n-1}$. We observe that each choice has a different finite size behaviour. We can explicitly demonstrate this where in addition to those results in (34) we have that on the critical temperature, where $Z_n \sim Z_n^{\text{leading}}(1 + c_1/n^{1/3} + c_2/n + ...)$ and $c_1 > 0$,

$$\begin{align*}
\gamma_{\text{est}} &\sim -\frac{1}{3} - \frac{4c_1}{9n^{1/3}} + \frac{5c_1^2}{9n^{2/3}} - \frac{2c_1^3/3 + c_2}{n} + .... \quad \text{and} \\
\gamma_{\text{Pade est}} &\sim -\frac{1}{3} - \frac{4c_1}{9n^{1/3}} + \frac{5c_1^2}{9n^{2/3}} + \frac{-2c_1^3/3 - 2c_2 - 3b + b^2}{n} + ....
\end{align*}$$

(36)

where $b = 3 - \gamma$, while off the critical temperature where $Z_n \sim Z_n^{\text{leading}}(1 + a_1/n + a_2/n^2 + ...)$ and $a_1 < 0$,

$$\begin{align*}
\gamma_{\text{est}} &\sim -\frac{1}{2} - \frac{2a_1}{n} + \frac{3a_1^2 - 6a_2}{n^2} + .... \quad \text{and} \\
\gamma_{\text{Pade est}} &\sim -\frac{1}{2} - \frac{2a_1 + 3b - b^2}{n} + ....
\end{align*}$$

(37)

where we note that $-2a_1 - 3b + b^2$ is positive and that $\gamma_{\text{est}}$ is in fact analytically consistent with the numerical results. The situation is thus similar to the scheme dependence in renormalisation. Any particular prescription for $\gamma$ is suitable but comparison of different definitions is not a meaningful exercise. Thus different models should be compared using the same definition for $\gamma_{\text{est}}$. From these analyses we can summarise the general structure of finite size corrections pertinent to any choice of estimate:

(i) They are not always parameterisable as $1 + \alpha/n + ...$.

(ii) The parameterisation is dependent on the expected value of $\gamma$. 

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(iii) If $\gamma_{\text{expected}} = q/p$ for $q, p \in \mathbb{Z}$ then in general the finite size corrections will be parameterised as $1 + \alpha/n^{1/p} + \ldots$. This is our general ansatz for the form of finite size corrections.

(iv) Some of the coefficients in the expansion may be zero, for instance in the Ising case at the critical temperature we have the expansion $1 + c_1/n^{1/3} + c_2/n + c_3/n^{4/3} + c_4/n^2 + \ldots$

6 Some Comments on Minbu Analysis

A particular approach to extracting the string susceptibility from numerical simulations is given by measuring the distribution of minimum neck baby universes (minbus) on random surfaces of a given fixed total area $A$. An estimation for the average number of minbus of size $B$, $\pi_A(B)$, was given by Jain and Mathur [5] and takes the form

$$\pi_A(B) = \frac{3(B+1)(A-B+1)Z_{B+1}Z_{A-B+1}}{Z_A}. \quad (38)$$

For quartic interactions we should substitute 4 for 3. It follows that (to leading order)

$$\ln(\pi_A(B)) = \text{constant} + (\gamma - 2) \ln(B(1 - B/A)). \quad (39)$$

Thus by measuring the slope of this function we can numerically determine $\gamma[1].$

The form of finite size corrections for pure gravity being $(1 + \alpha/n + \ldots)$ it follows that the leading finite size correction to (39) is

$$\ln(\pi_A(B)) = \text{constant} + (\gamma - 2) \ln(B(1 - B/A)) + (\gamma - 2 + \alpha) \ln(1 + \frac{1}{B(1 - B/A)}) + \ldots, \quad (40)$$

We know from (2) that for pure gravity with quartic interactions the finite size parameter has the value $\alpha = -25/8$. Plotting the minbu distribution (40) and the exact minbu distribution calculated using (1) in Fig(3i) we see that this correction gives excellent agreement with the exact result. Applying this approach to the Ising case off the critical temperature, where the finite size corrections were given in (27) for the high and low temperature phases, we find that these finite size corrections do not accurately mimic the exact minbu distributions, Fig(3ii). Clearly, higher order corrections are more important.
in the Ising case. This behaviour is again evident at the critical temperature, where we must also account for $1/n^{1/3}$ type corrections. Again we find that retaining higher order corrections is necessary in order to obtain a reasonable fit.

Since minbus are measured over a range of volumes finite size effects are unavoidable for small minbus. As these finite size effects are hard to extract for cases where either $\gamma$ is not known or where $\gamma = q/p$ with large integer $p$, it appears that applying the minbus technique to extract meaningful estimates for $\gamma$ from simulations in these interesting cases is fraught with difficulties.

7 Conlusion

A simple parameterisation of finite size effects is a natural first approach to analysing numerical simulations. As we have shown, however, the actual parameterisation is non-trivially dependent on the exact value of the string susceptibility. On this basis we have proposed a general ansatz for the form of finite size corrections. A possible algorithm then is to make a best guess for $\gamma$ and to fit this to the observed data with the finite size corrections correctly included. By performing a $\chi^2$ analysis the best fit for a particular $\gamma$ could be recursively searched.

We have compared different approaches to estimating $\gamma$ and demonstrated that attempts to compare estimates from different definitions can be misleading. These considerations become relevant if we wish to extract reliable numerical estimates beyond the $c_M = 1$ barrier where it is known that large logarithmic corrections also play a role. A clear understanding of the functional form for the finite area estimates are indispensable in these cases.

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Figure 1: $\gamma_{est}$ at the critical temperature for quartic interactions.
Figure 2: Comparison of the analytic and numerical estimates of $\gamma$ at the critical temperature with quartic interactions. The analytic graph includes all the corrections given in (29).
Figure 3: Numerically and analytically derived minbu plots for (i) pure gravity and (ii) the Ising case off the critical temp. \((c = 0.36)\). The calculated values of \(\ln(B(1-B/A))\) in (ii) have been shifted by a constant for clarity. All are with quartic interactions.