Can one determine cosmological parameters from multi-plane strong lens systems?

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ABSTRACT

Strong gravitational lensing of sources with different redshifts has been used to determine cosmological distance ratios, which in turn depend on the expansion history. Hence, such systems are viewed as potential tools for constraining cosmological parameters. Here we show that in lens systems with two distinct source redshifts, of which the nearest one contributes to the light deflection towards the more distant one, there exists an invariance transformation which leaves all strong lensing observables unchanged (except the product of time delay and Hubble constant), generalizing the well-known mass-sheet transformation in single plane lens systems. The transformation preserves the relative distribution of mass and light, so that a ‘mass-follows-light’ assumption does not fix the MST. All time delays (from sources on both planes) scale with the same factor – time-delay ratios are therefore invariant under the MST. Changing cosmological parameters, and thus distance ratios, is essentially equivalent to such a mass-sheet transformation. As an example, we discuss the double source plane system SDSSJ0946+1006, which has been recently studied by Collett and Auger, and show that variations of cosmological parameters within reasonable ranges lead to only a small mass-sheet transformation in both lens planes. Hence, the ability to extract cosmological information from such systems depends heavily on the ability to break the mass-sheet degeneracy.

Key words. cosmological parameters – gravitational lensing: strong

1. Introduction

Strong gravitational lensing by galaxies is a powerful tool for cosmological studies, in particular regarding precise mass estimates of the inner region of galaxies, the angular structure of the mass distribution (e.g., ellipticity and orientation), and mass substructure (see, e.g., Kochanek 2006; Treu 2010; Bartelmann 2010, and references therein). In his pioneering paper,Refsdal (1964) pointed out the possibility to determine cosmological parameters from lensing, specifically the Hubble constant from measurements of time delays in lens systems. Whereas time delays have been determined in some 20 lens systems by now, the accuracy of the corresponding values of \( H_0 \) is difficult to judge, due to the difficulty to reliably constrain the mass distribution of the lens: On the one hand, the number of observational constraints can be insufficient to constrain the mass distribution to sufficient accuracy. On the other hand, Falco et al. (1985) have found that there exists a transformation of the mass distribution of the lens which leaves all observables invariant, except the product of time delay and \( H_0 \). Thus, although very detailed studies of some lens systems have been conducted (see, e.g., Suyu et al. 2013), this mass-sheet transformation (MST) poses a fundamental limitation on the accuracy of the derived value for \( H_0 \) from gravitational lensing alone. Though the degeneracy caused by the MST can be broken with additional observations, such as stellar dynamics in the lens or some additional information about the properties of the source (such as its luminosity), the current accuracy on these quantities leads to uncertainties of \( H_0 \) larger than those from other methods, and the method may be biased. In practice, the degeneracy due to the MST is broken by assumptions about the mass distribution, e.g., that it follows a power law. A recent discussion of the impact of the MST on \( H_0 \) determination can be found in Schneider & Sluse (2013).

A different approach to cosmology by strong lens systems involves the relative lensing strength for two sources at two different redshifts. It is known that the ’classical’ MST is an exact transformation of the lens mass distribution only for sources at a single distance, and can be broken in principle when sources at several redshifts are employed (Bradac et al. 2004). However, it must be stressed that this degeneracy breaking assumes that there is no second deflector along the line-of-sight to the sources.

The recent discovery of a strong galaxy-scale lens with two extended multiple image arc structures from two sources at vastly different redshifts (SDSSJ0946+1006; Gavazzi et al. 2008) opened up the possibility to study a multi-plane lens system in great detail. Collett & Auger (2014; hereafter CA14) performed a detailed analysis of this system, constructing a model which involves both the main lens in the foreground of both sources, and a smaller-mass deflector associated with the lower-redshift source. The relative lens strength \( \beta \) of the main lens on the two sources depends on distance ratios, which in turn depend on the redshifts of lenses and sources involved as well as on the distance-redshift relation in the universe. Since the latter is sensitive to the density parameters and the equation of state parameter \( w \) of dark energy, CA14 were able to obtain constraints on \( w \) from this lens system.

In this letter, we investigate whether this method can yield robust results. Specifically, we will show in Sect. 2 that an analog of the MST also exists for the case of two lens and two source planes, as is the case for SDSSJ0946+1006. We then demonstrate in Sect. 3 that a change of cosmological parameters, leading to a change of the expected value of \( \beta \), is equivalent to such a
generalized MST. We then study the amplitude of this equivalent MST for a range of values of \( w \), concluding that the transformation amplitude across a plausible range of \( w \) is indeed very small. We thus conclude that this method heavily relies on assumptions regarding the shape of the mass profiles of the lenses.

## 2. The two lens plane mass-sheet transformation

In this section we first summarize the lensing equations for the multi-plane case (Sect. 2.1) and specify our notation, then recall the MST in the single plane case (Sect. 2.2), before deriving the new MST in the case of two lens/source planes.

### 2.1. The multi-plane lens equations

We consider lenses and source distributed along nearly the same line-of-sight at \( N \) different distances from us, characterized by their redshifts \( z_i \), or angular diameter distances \( D_i \) from us, \( 1 \leq i \leq N \) (we largely follow the notation of Schneider et al. 1992, where more details of the derivation are given). Perpendicular to the line-of-sight, we consider ‘planes’ in which the sources at these distances are located, and onto which the mass distribution of the deflecting masses are projected (lens/source planes). We denote by \( D_i \) the angular diameter distance of the \( j \)-th plane as seen from the \( i \)-th plane, with \( 1 \leq i < j \leq N \). The projected mass distribution in the \( i \)-th plane is characterized by its surface mass density \( \Sigma_i(\xi) \) and gives rise to a deflection angle \( \hat{\alpha}_i(\xi) \), where

\[
\hat{\alpha}_i(\xi) = \frac{4G}{c^2} \int d^2\xi' \Sigma_i(\xi') \frac{\xi - \xi'}{(|\xi - \xi'|)^2},
\]

and \( \xi = D_i \theta_i \) is a transverse separation vector in the \( i \)-th plane, with the corresponding unlensed angular position \( \alpha_i \). The resulting propagation equations of a light ray follows solely from geometry and the definition of angular diameter distances,

\[
\theta_j = \theta - \sum_{i=1}^{j-1} \frac{D_j}{D_i} \hat{\alpha}_i(D_i \theta_i) = \theta - \sum_{i=1}^{j-1} \beta_i \alpha_i(\theta_i),
\]

where we set \( \theta \equiv \theta_1 \), scaled the deflection angles \( \hat{\alpha}_i \) to the final source plane at \( i = N \), i.e.,

\[
\alpha_i(\theta_i) = \frac{D_N}{D_i} \hat{\alpha}_i(D_i \theta_i), \quad \text{and defined} \quad \beta_i = \frac{D_N}{D_i} \frac{D_j}{D_N},
\]

as coefficients of relative distance ratios. Accordingly, we define the dimensionless surface mass densities \( \kappa_i(\theta_i) = 4\pi G D_i D_N \Sigma_i(D_i \theta_i)/(c^2 D_N) \) which satisfy \( \nabla \cdot \alpha_i = 2\kappa_i \).

### 2.2. Summary of the single plane MST

We briefly recall the MST in the single lens-plane case, for which \( N = 2 \) (see Falco et al. 1985; Schneider & Sluse 2013, and references therein). Hence, we have a single lens plane at redshift \( z_1 \) with deflection \( \alpha(\theta) \equiv \alpha_1(\theta_1) \) and dimensionless surface mass density \( \kappa(\theta) \equiv \kappa_1(\theta_1) \), and a single source plane at redshift \( z_2 \).

If a mass distribution \( \kappa(\theta) \) can explain all the observed lensing features, such as image positions, flux ratios, relative image shapes, and time-delay ratios, of a source with unlensed brightness profile \( I'(\theta_2) \), then the whole family of mass distributions and corresponding scaled deflection angles

\[
\kappa_N(\theta) = \lambda \kappa(\theta) + (1 - \lambda); \quad \alpha_N(\theta) = \lambda \alpha(\theta) + (1 - \lambda) \theta
\]

explains the lensed features equally well, for a source of brightness profile \( I'_N(\theta_2) = \lambda I'(\theta_2)/(\lambda) \). Hence, the transformed brightness distribution in the source plane is rescaled by a factor \( \lambda \). This rescaling of the source plane implies that the magnification of images is changed, \( \mu_1 = \mu/\lambda^2 \), but that could only be observed if the luminosity or physical size of the source was known (standard candle or standard rod). In most cases, this information is unavailable, so that the magnification cannot be obtained from observations, whereas magnification (and thus flux) ratios are invariant under the MST. The time delay between pairs of images is changed under the MST, so that a time delay measurement can be employed to break the degeneracy caused by the MST, provided the Hubble constant (and the density parameters of our Universe) are assumed to be known.

### 2.3. The MST for two source/lens planes

We will now study whether a similar invariance transformation exists in the case that one has sources at two different redshifts \( z_1 \) and \( z_2 \), with lenses at redshift \( z_1 \) and \( z_2 \). In particular, we consider that the closer source at \( z_2 \) is associated with mass which deflects light rays from the more distant source. Specializing the equations of Sect. 2.1 to the case \( N = 3 \), we get the pair of relations

\[
\theta_j = \theta - \beta \alpha_j(\theta) \quad \text{;} \quad \theta_j = \theta - \alpha_j(\theta) - \alpha_2(\theta_2),
\]

where we defined \( \beta \equiv \beta_{12} \) (see Eq. 3). A mass-sheet transformation is defined as a change of the deflection angles (and correspondingly the lensing mass distributions) such that the lens equations stay invariant, with only a uniform isotropic scaling in the source planes. Considering the first of (5), a MST for the first source plane is obtained by applying the single lens plane MST to the ‘effective’ deflection \( \beta \alpha_1 \), i.e., by setting

\[
\alpha'_1(\theta) = \lambda \alpha_1(\theta) + \frac{1}{\beta} \theta,
\]

after which the transformed position at \( z_2 \) becomes

\[
\theta'_2 = \lambda \theta - \lambda \beta \alpha_1(\theta) = \lambda \theta_2,
\]

i.e., the required uniform isotropic scaling. The question now is whether we can find a transformation of the deflection angle \( \alpha_2 \) such that the second of (5) also remains invariant up to a scaling of \( \theta_2 \). If \( \alpha'_2(\theta_2) \) denotes the transformed deflection angle, then the transformed lens equation reads

\[
\theta'_1 = \theta - \lambda \alpha'_1(\theta) - \frac{1}{\lambda} \theta - \alpha'_2(\lambda \theta_2).
\]

Requiring that \( \theta'_1 = \nu_1 \theta_1 \), corresponding to uniform scaling with the factor \( \nu_1 \), we obtain

\[
\beta + \frac{1}{\beta} \theta - \lambda \alpha_1(\theta) - \alpha'_2(\lambda \theta_2) = \nu_1 [\theta - \alpha_1(\theta) - \alpha_2(\theta_2)].
\]

Without assuming any functional form for the deflection angles, the term \( \alpha'_2(\theta_2) \) on the l.h.s. of (9) must contain the term \( \nu_2 \alpha_2(\theta_2) \). Therefore, we set

\[
\alpha'_2(\theta_2) = \nu_2 \alpha_2(\theta_2/\lambda) + K_2 \theta_2 = \nu_2 \alpha_2(\theta_2'/\lambda) + K_2 \lambda \theta - \beta \alpha_1(\theta_2' \lambda)
\]

where in the second step we used \( \theta'_2 = \lambda \theta_2 \) and the first of (5). Here, \( K_2 \) is a constant, to be constrained later. Inserting (10) into (9), we see that the terms involving \( \alpha_2 \) vanish. Comparing the
remaining terms proportional to $\alpha_1$ and $\theta$, we find the pair of constraints $\lambda - K_2\beta = \nu_3$ and $\beta + \lambda - 1 - K_2\beta = \beta\nu_3$, which have the unique solution
\[ \nu_3 = 1; \quad K_2 = \frac{\lambda - 1}{\alpha \beta}. \]

Hence we obtain a solution of (9) and accordingly a transformation of the mass distributions in both lens planes which lead to at most a uniform isotropic scaling of the source planes.

The transformation of the first lens plane is a ‘normal’ MST, in that the deflection angle (and thus the surface mass density) is scaled by an overall factor $\lambda$, and a uniform density $K_2(1 - \lambda)/\beta$ is added, leading to a scaling of the first source plane by a factor $\lambda$. The same scaling must also be applied to the mass distribution in that same plane (i.e., the second lens plane) as seen by (10). Thus, after the MST, both the light and the mass in the plane $i = 2$ are transformed in exactly the same way. In particular this means that if the original model has a mass component centered on a light component, the same remains true after the MST, but both are located at a position that differs by a factor of $\lambda$.

The condition (10) furthermore implies that the transformed deflection in the second lens plane is a scaled version of the original deflection, plus a contribution from a uniform mass sheet $K_2$. The implication of the scaling of the first term in (10) on the corresponding mass distribution can be seen as follows: If
\[ \alpha(\theta|\kappa(\theta)) = \frac{1}{\pi} \int d^3\theta' \kappa(\theta') \frac{\theta - \theta'}{\theta' - \theta} \theta', \]

is the deflection caused by the mass distribution $\kappa$, then
\[ \alpha(\theta|\lambda \kappa(\theta)) = \alpha(\theta|\lambda^{-1} \kappa(\lambda \theta)). \]

Hence, the transformed mass distribution $\kappa'_2$ is a scaled version of the original one, multiplied by a factor $\lambda^{-1}$, plus a uniform mass sheet. It must be stated here that the phrase ‘adding a uniform mass sheet’ corresponds to a global interpretation of the MST; however, only a relatively small inner region of the lens is probed by strong lensing, and hence the MST needs to apply only locally. Its main effect is the change of the local slope of the mass profile near the Einstein radius of a lens, making it flatter (steeper) for $\lambda < 1$ ($\lambda > 1$). The global interpretation of the mass sheet as an ‘external convergence’, relating it to the large-scale environment of the lens which can be probed by the observed density field of galaxies (e.g., Wong et al. 2011; Collett et al. 2013; Greene et al. 2013), constitutes an extrapolation over a vast range of scales.

Surprisingly, the second source plane remains unscaled under this MST; due to $\nu_3 = 1$. Hence, the MST implies no change in the mapping of the second source plane, including no change in the magnification matrix. Whereas simple algebra has straightforwardly led to this result, the geometrical reason for this appears unclear.

If the MST is such that the scaling in the first lens plane corresponds to an additional focusing, which means $\lambda < 1$, so that the uniform mass sheet has positive convergence, then the mass sheet in the second lens plane has negative convergence, since the sign of $K_2$ is opposite to that of $\lambda - 1$. In particular, $\lambda = 1$ implies $K_2 = 0$, so there is no MST which leaves the first lens plane invariant and only affects the second one. It must be stressed that the MST leaves the global lens mapping invariant, and thus applies to sources of (in principle) arbitrary extent. This is quite different from other modifications of the lens mass distribution (e.g., Coe et al. 2008; Liesenborgs et al. 2008; Liesenborgs & De Rijcke 2012, and references therein) which apply to a discrete set of isolated ‘small’ images of sources.

2.4. The time delay

We now consider the impact of the MST on the time delay. For sources at $z_2$, the MST is a ‘normal’ single plane MST, and the time delays are changed by a factor $\lambda$. For sources at $z_3$, we use the expression for the light travel time from $\theta_2$ via $\theta_2$ and $\theta_3$ to the observer, as given in Schneider et al. (1992),
\[ T(\theta, \theta_2, \theta_3) = \sum_{i=1}^{n} \frac{1 + z_i}{c} \left( \frac{D_i D_{i+1}}{D_{i+1}} \right) T_{\nu_{i+1}, i+1}(\theta_i, \theta_{i+1}), \]

where $T_{\nu_{i+1}, i+1} = (\theta_i - \theta_{i+1})^2/2 - \beta_{i+1} |\psi_i(\theta_i)|$ is the Fermat potential corresponding to neighboring planes, with $\beta_{i+1} = \beta$ and $\beta_{23} = 1$, and $\psi_i(\theta_i)$ is the deflection potential with $\nabla \psi_i = \alpha \theta_i$. We now consider how $T$ behaves under a MST. The scalings (6) and (10) of $\alpha_i$ imply that
\[ \psi'_i(\theta) = \lambda \psi_i(\theta) + \frac{1 - \lambda}{2\beta} \theta_i^2; \quad \psi'_2(\theta_2) = \lambda \psi_2(\theta_2) + K_2 \theta_2^2. \]

Together with $\theta'_2 = \lambda \theta_2$ and $\theta'_3 = \theta_3$, we get
\[ \begin{align*}
\tau'_{12} &= \frac{(\theta - \lambda \theta_2)^2}{2 \beta} - \beta \left( \lambda \psi_1(\theta) + \frac{1 - \lambda}{2\beta} \theta_i^2 \right) \\
&= \lambda \left( \frac{(\theta - \lambda \theta_2)^2}{2} - \beta \psi_1(\theta) \right) + \frac{\lambda(1 - \lambda)}{2} \theta_i^2; \\
\tau'_{23} &= \frac{(\theta_2 - \lambda \theta_3)^2}{2} - \lambda \psi_2(\theta_2) - K_2 \lambda \theta_2^2 \\
&= \lambda \left( \frac{(\theta_2 - \lambda \theta_3)^2}{2} - \psi_2(\theta_2) \right) + \frac{\lambda^2 - \lambda}{2} \theta_2^2 - \frac{1 - \lambda}{2} \theta_3^2.
\end{align*} \]

Hence, the light travel time function transforms as
\[ T'(\theta, \theta'_2, \theta'_3) = \lambda T(\theta, \theta_2, \theta_3) + \frac{1 + z_3}{c} \left( \frac{D_2 D_3}{D_2 D_3} \right) \frac{(1 - \lambda)}{2} \theta_3^2 + \frac{\lambda(1 - \lambda)}{2} \theta_2^2 \left( 1 + z_3 \right) D_2 D_3 \left( 1 - \frac{1}{\beta} \right). \]

The second term in (17) depends only on the source position $\theta_3$; therefore, this term does not contribute to the time delay, which is obtained as the difference of $T$ between images. The third term of (17) vanishes, since the expression in the bracket is zero due to relations between the D’s – see Schneider et al. (1992). We thus find that all time delays in a double source plane lens scale with $\lambda$ under a MST, for sources on both source planes. Hence, time delay ratios do not break the degeneracy of the MST; on the other hand, any time delay from a source in either source plane breaks it, provided $H_0$ is assumed to be known.

3. Cosmology from double source plane lensing?

3.1. Impact of the MST

In the light of the new MST, we now consider the possibility to constrain cosmological parameters from double source plane lenses. In the approach of CA14, the sensitivity to cosmology is due to the distance ratio parameter $\beta$, which depends on the density parameters and the dark energy e.o.s. parameter $w$. Different cosmological models yield different distance-redshift relations, and thus different $\beta$. Thus let $\beta$ correspond to a fiducial cosmological model, and $\beta'$ to a model with different parameters. We will next see whether we can find deflection angles $\alpha'_i(\theta'_i)$ such that the lens mappings are unchanged between the fiducial and...
modified models, up to a uniform scaling (by a factor $v_1$) in the lens/source planes. Thus we require for the first lens equation

$$\theta_2' = \theta - \beta \alpha_1'(\theta) = v_2 \theta_2 = v_3 \theta_3 = v_3 \{\theta - \alpha_1(\theta) - \alpha_2(\theta_2)\}. \quad (18)$$

With the ansatz $\alpha_i'(\theta) = \lambda \alpha_i(\theta) + K_i \theta$, we obtain by comparing terms in (18) proportional to $\alpha_1$ and $\theta$ the two relations $\beta \lambda = v_3 \beta$ and $1 - \beta \alpha_1' = v_3 \lambda$, with solutions $v_3 = \beta \lambda / \beta \alpha_1' \lambda$ and $K_1 = 1/\beta \alpha_1 - \lambda / \beta$. The requirement for the second lens equation reads

$$\theta_1' = \theta - \beta \alpha_1'(\theta) = v_1 \theta_1 = v_3 \{\theta - \alpha_1(\theta) - \alpha_2(\theta_2)\}. \quad (19)$$

Inserting the ansatz $\alpha_i'(\theta) = v_1 \alpha_2(\theta_2 / v_3) + K_2 \theta_2 = v_3 \alpha_2(\theta_2 / v_3) + K_2 v_3 \theta - \beta \alpha_1(\theta)\}$, into (19), the comparison of terms proportional to $\alpha_1$ and $\theta$ yields the equations $\lambda = K_2 v_3 \beta = v_3$ and $1 - K_1 - K_2 v_3 = v_3$, which have the solutions

$$v_3 = \frac{\beta}{\beta \lambda} \frac{1 - \beta}{1 - \beta}; \quad K_2 = \frac{1}{\beta} \left(1 - \frac{1 - \beta}{\lambda \beta} - 1\right). \quad (20)$$

Hence we see that a change of cosmology changes the equations in a similar way as a MST. Even with a change in $\beta$, we retain the freedom of a one-parameter family of mass models which leave the lens mapping invariant, up to a uniform scaling. In contrast to the MST discussed in Sect. 2.3, here a non-trivial distortion in the second lens plane is implied, where $v_3$ solely depends on change of $\beta$.

Three special choices of $\lambda$ are worth to discuss separately: (1) No mass sheet in first lens plane (NMS1): For $\lambda = 1/\beta \alpha_1'$, $K_1 = 0$ and $v_2 = 1$, so that there is no mass sheet in the first lens and no scaling in the source/lens plane is implied. The mass sheet in the second lens plane then becomes $K_2 = (\beta - \beta \lambda / \beta \alpha_1') (1 - \beta \lambda)$). For four different variations around a fiducial cosmological model, all with zero spatial curvature, we give the values of $\beta \lambda$, $v_3$ and $K_2$ in Table 1, using the redshifts for the system SDSSJ0946+1006 (setting the uncertain redshift of the second source to be $z_3 = 2.4$). (2) No mass sheet in second lens plane (NMS2): Setting $\lambda = v_1$, we get $K_2 = 0$, $v_2 = (1 - \beta)/(1 - \beta \lambda)$, and $K_1 = (\beta - \beta \lambda)/[\beta \alpha_1'](1 - \beta \lambda)$. Hence, in this case, there is no mass sheet in the second lens plane, and the one in the first lens plane is the same as $K_2$ was in the case NMS1. (3) Equal mass scaling (EMS): Choosing equal mass scaling in both lens planes, we get $\lambda = (\beta \lambda / \beta \alpha_1') (1 - \beta \lambda)/[\beta \alpha_1'](1 - \beta \lambda)$, $K_1 = 2 - (1 - \sqrt{1 - (1 - \beta)/(1 - \beta \lambda)})$, $K_2 = (1 - \sqrt{1 - (1 - \beta)/(1 - \beta \lambda)}) / \beta \alpha_1'$, and $v_3 = (1 - \sqrt{1 - (1 - \beta)/(1 - \beta \lambda)}) / \beta \alpha_1'$. Also for these choices of $\lambda$, we list some quantities for different cosmological parameters in Table 1.

From the table, we see that variations of the cosmological parameters within generous plausible ranges yield only small deviations of $\beta \lambda$ from $\beta$. Correspondingly, the required MSTs are small, for the three special choices discussed above they imply mass sheets with $|K_i| < 0.06$. In particular, in the case EMS, the mass sheets are less than 0.03. Such small values of $K_i$ are perfectly acceptable, even if fairly accurate measurements of the velocity dispersion of the lenses were available (see Schneider & Sluse 2013, for more discussion on this issue). In particular, if the original mass distribution was a power law, the transformed ones $\kappa_i'$ would deviate only very little from a power law over the range were multiple images are formed.

4. Discussion

We have shown that a double lens/double source plane lens admits a mass-sheet transformation which leaves the lens mapping invariant, up to a uniform scaling in the source and lens plane(s).

Furthermore, we demonstrated in the previous section that a change of cosmological parameters is essentially equivalent to a MST. For the particular case of SDSSJ0946+1006 (Gavazzi et al. 2008), we have shown that the changes of the mass distribution in the second lens plane are very small when changing the cosmological parameters within currently acceptable ranges.

As explicitly stated in CA14, the cosmological constraints they obtain are based on the assumption of a power-law mass distribution in either lens planes. Indeed, the case NMS1 discussed above implies no added mass sheet in the first lens plane, thus preserving the power-law property, whereas the case EMS corresponds to only small $K_i$, yielding only marginal modifications of the power-law mass profile. In any case, the existence of a MST also for multiple source/lens planes provides a degeneracy in the determination of lens mass distributions and cosmological distance ratios in lens systems, and needs to be accounted for in future studies.

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Table 1. For the redshifts of the double lens plane system SDSSJ0946+1006, i.e., $z_1 = 0.222$, $z_2 = 0.609$, $z_3 = 2.4$, we consider four different variations of the fiducial flat cosmological model (first block), for which $Q_{0m} = 0.3$, $w = -1$, and thus $\beta = 0.708$. The second block lists $\beta$ and $v_1$ for these cosmologies. Then three special choices of $\lambda$ are considered, as described in the text, and listed in block 3 (NMS1), 4 (NMS2) and 5 (EMS)

| $\Omega_m$ | $w$ | $\beta$ | $v_3$ | $K_2$ | $\lambda$ | $K_1$ | $v_2$ |
|----------|-----|--------|--------|------|--------|------|------|
| 0.3      | -0.5 | 0.697  | 1.058  | 0.983 | 1.058  | -0.058 | 1.040 |
| 0.3      | -1.5 | 0.720  | 0.944  | 1.017 | 1.002  | 0.056  | 0.960 |
| 0.2      | 0.710 | 0.993 | 0.993  | 0.007 | 0.997  | 0.007  | 0.995 |
| 0.3      | -1    | 0.706  | 1.010  | -0.010 | 1.010  | -0.005 | 1.007 |
| 0.4      | -1    | 1.006  | 0.996  | 0.006 | 0.996  | 0.003  | 1.003 |