Quantum transport theory with vector interaction

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We derive the relativistic quantum kinetic equation for massless fermions with vector and axial vector interaction using the Wigner function formalism. The vector and axial vector currents are self-consistently treated with corresponding constraint equations. The kinetic equations are derived and the condition for equilibrium is discussed up to the first order of $\hbar$. In addition to the vorticity and shear contributions, the divergence of the local particle number density is also found to contribute to the kinetic equations.

1. INTRODUCTION

Quantum chromodynamics (QCD) predicts the existence of a new state of matter, the quark-gluon plasma (QGP). It is widely accepted that QGP is created in relativistic heavy-ion collisions create [1–3]. In the QGP, the chiral symmetry is restored and light quarks can be considered massless fermions. On the other hand, there are local chiral imbalances due to triangle anomaly. The interaction between the chiral anomaly and the magnetic field or vorticity gives rise to novel transport phenomena such as the chiral magnetic effect (CME) [4] and the chiral vortical effect (CVE) [5]. The vorticity will also result in the polarization of particles which is confirmed by the polarization of the $\Lambda$ hyperion [6–8]. In such a system, the chirality and spin degree of freedom are important. Also, the study of local $\Lambda$ polarization shows that the non-equilibrium effect should be considered [7, 8]. The chiral kinetic theory is among the effective theoretical tools to study such topics. Many results were obtained, such as the anomalous chiral transport equation in heavy-ion collisions [9–11, 13, 39], the quantum kinetic theory for massive fermions under external fields [14–17] and the non-relativistic kinetic of spin-polarized system [18].

Previous works show that vorticity and shear play important roles in the system [19–22]. They can be introduced as $\hbar$ order corrections to the free-streaming part of the kinetic equations. On the other hand, when fermion-fermion interaction is taken into consideration, effective vector and axial vector interaction terms appear in the collision part. It is also necessary to check the self-consistency of the vorticity and shear, as they come from the velocities of the fermion field itself. In the past, many works focused on the Nambu–Jona-Lasinio (NJL) model, such as the spin polarization for massive fermion [23] and the collision term [24]. But to better study the spin-related phenomenon, it might be helpful to directly consider an effective vector and axial vector interaction. Therefore, in this work, we study the relativistic quantum kinetic equation for massless fermions with effective vector axial interaction. This Lagrangian goes back to Fermi’s theory of weak interaction [25]. In QCD, the vector interaction can be used to explain the one-gluon exchange [26] and is used as an effective theory for 4-fermion interaction with axial interaction [27, 28]. Furthermore, the study of vector interaction and axial interaction profit for the derivation of anomalous hydrodynamics [30–33], magnetohydrodynamics [34] and spin hydrodynamics [35–37]. The mean-field approximation is used for simplicity, and the vector and axial vector currents are self-consistently constrained by constraint equations. The semi-classical expansion is taken up to the $\hbar$ order. In the zero-order, we get the on-shell conditions, the kinetic equations, and the equilibrium conditions for left- and right-handed components.

The paper is organized as follows: In Sect.2, we derive the equation of motion for the Wigner function without the collision term. In Sect.3, the $\hbar$ expansion is used to simplify the equations. Finally, we make a brief discussion and conclusion in Sect.4.

2. EQUATION OF MOTION FOR THE WIGNER FUNCTION

We start from a Lagrangian density with effective vector and axial vector interaction terms [25]

$$\mathcal{L} = i\gamma^\mu \bar{\psi} \partial_\mu \psi + G \left[ (\bar{\psi} \gamma^\mu \psi)^2 + (\bar{\psi} \gamma_5 \gamma^\mu \psi)^2 \right].$$

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With the mean-field approximation \((\bar{\psi}\gamma^\mu \psi)^2 = \langle \bar{\psi}\gamma^\mu \psi \rangle \bar{\psi}\gamma^\mu \psi\) and \((\bar{\psi}i\gamma_5 \gamma^\mu \psi)^2 = \langle \bar{\psi}i\gamma_5 \gamma^\mu \psi \rangle \bar{\psi}i\gamma_5 \gamma^\mu \psi\), the Lagrangian density can be reduced as
\[
\mathcal{L} = \bar{\psi} \left( i\gamma^\mu \partial_\mu - \gamma^\mu J_{V_\mu} - \gamma_5 \gamma^\mu J_{A_\mu} \right) \psi,
\]
(2)
where \(J_{V_\mu} = G \langle \bar{\psi}\gamma^\mu \psi \rangle\) and \(J_{A_\mu} = G \langle \bar{\psi}i\gamma_5 \gamma^\mu \psi \rangle\). Here \(J_{V_\mu}\) and \(J_{A_\mu}\) are not to be considered as condensates, but as the vector and axial vector currents of the system that we are interested in. For example, in heavy-ion collisions, the fireball is rapidly expanding, resulting in a radial divergence and current. What is more, the large angular momentum in non-central collisions means there is a modification to the current that has non-zero derivatives. The chiral anomaly, along with possible CME and CVE effects will also give an axial vector current. As the fireball is a thermal system, these expectation values can be treated as ensemble averages. For simplicity, the coupling constant is absorbed into the currents.

From the mean-field effective Lagrangian density, one can obtain the Dirac equation:
\[
\bar{\psi} \left( i\gamma^\mu \partial_\mu - \gamma^\mu J_{V_\mu} - \gamma_5 \gamma^\mu J_{A_\mu} \right) \psi = 0,
\]
(3)
The Wigner function for fermions is
\[
W(x, p) = \int d^4y e^{i p y} \left\langle \psi \left( x + \frac{y}{2} \right) \bar{\psi} \left( x - \frac{y}{2} \right) \right\rangle,
\]
(4)
where we did not consider the effect of gauge fields. Combining the above Dirac equations and the Wigner function, one can get the equation of motion for the Wigner function\[^{[19]}^{[39]}\] \([5\)56\] where
\[
K_\mu = \Pi_\mu + i D_\mu,
\]
\[
K_{5\mu} = \Pi_{5\mu} + i D_{5\mu},
\]
\[
\Pi_\mu = p_\mu + \cos \left( \frac{\hbar}{2} \nabla \right) J_{V_\mu},
\]
\[
D_\mu = \frac{\hbar}{2} \partial_\mu - \sin \left( \frac{\hbar}{2} \nabla \right) J_{V_\mu},
\]
\[
\Pi_{5\mu} = - \cos \left( \frac{\hbar}{2} \nabla \right) J_{A_\mu},
\]
\[
D_{5\mu} = \sin \left( \frac{\hbar}{2} \nabla \right) J_{A_\mu},
\]
\[
\nabla = \partial_x \cdot \partial_p, \quad \text{in which the spatial derivative } \partial_x \text{ only acts on } J_{A_\mu} \text{ or } J_{V_\mu}, \quad \text{but not on the Wigner function } W.
\]
The Wigner function is a \(4 \times 4\) matrix and satisfies the relationship \(\gamma_0 W^+ \gamma_0 = W\). It can be decomposed in terms of 16 independent generators of Clifford algebra\[^{[39]}\]. So the equation of motion can be decomposed to 10 equations\[^{[38]}\]
\[
\Pi_\mu V^\mu + \Pi_{5\mu} A^\mu = 0,
\]
\[
D_\mu A^\mu + D_{5\mu} V^\mu = 0,
\]
\[
\Pi_\mu F + D_\mu S_{\mu\nu} - D_{5\mu} P + \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} \Pi_5^\sigma S^{\nu\rho} = 0,
\]
\[
D_\mu P - \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} \Pi_5^\sigma S^{\nu\rho} - \Pi_{5\mu} F - D_{5\mu} S_{\mu\nu} = 0,
\]
\[
D_\mu V_\nu - D_\nu V_\mu - \varepsilon_{\mu\nu\sigma\rho} \Pi^\sigma A^\rho - \varepsilon_{\mu\nu\sigma\rho} \Pi_5^\sigma V^\rho + D_{5\mu} A_\nu - D_{5\nu} A_\mu = 0,
\]
\[
D_\mu V^\mu + D_{5\mu} A^\mu = 0,
\]
\[
\Pi_\mu A^\mu + \Pi_{5\mu} V^\mu = 0,
\]
\[
D_\mu F - \Pi_\mu S_{\mu\nu} + \Pi_{5\mu} P + \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} D_{5}^\sigma S^{\nu\rho} = 0,
\]
\[
\Pi_\mu P + \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} D^\nu S^{\rho\sigma} + D_{5\mu} F - \Pi_{5\mu} S_{\mu\nu} = 0,
\]
\[
\Pi_\mu V_\nu - \Pi_\nu V_\mu + \varepsilon_{\mu\nu\sigma\rho} D^\sigma A^\rho + \varepsilon_{\mu\nu\sigma\rho} D_5^\sigma V^\rho + \Pi_{5\mu} A_\nu - \Pi_{5\nu} A_\mu = 0.
\]
(6)
Where $V_\mu, A_\mu, F, P, S_{\mu\nu}$ are the vector, axial vector, scalar, pseudoscalar, and tensor components respectively. $\epsilon_{\mu\nu\sigma\rho}$ is the Levi-Civita symbol.

The roles of the currents look similar to gauge fields. But they are different. In some studies\cite{39,40}, the gauge fields are taken as classical and their strength is decided beforehand. In other ones\cite{19}\cite{41}, they are treated as dynamical degrees of freedom, with their own transport equations. In both cases, the gauge fields are not determined by the fermion field. On the contrary, the currents $J_V$ and $J_A$ are by definition part of the fermion field, and they are connected to the vector and axial-vector components.

$$J_V^\mu(x) = G \int d^4p V^\mu(x,p), \quad (7)$$
$$J_A^\mu(x) = G \int d^4p A^\mu(x,p). \quad (8)$$

Similar to the usual constraint equations that determine the value of sigma and pion condensates\cite{23,42,43}, these equations can be used to determine the possible value of particle number density and axial imbalance, which we will discuss later.

It is simpler to consider the chiral components

$$V_{\chi\mu} = V_\mu + \chi A_\mu, \quad (9)$$

where $\chi = \pm 1$ corresponds to right-handed and left-handed components. The equation of motion for $V_{\chi\mu}$ are

$$(\Pi_\mu + \chi S_{\mu\nu}) V_{\chi\mu} = 0,$$
$$(D_\mu + \chi P_{\mu\nu}) V_{\chi\mu} = 0,$$
$$(\Pi_\mu + \chi S_{\mu\nu}) V_{\chi\nu} - (\Pi_\nu + \chi S_{\nu\mu}) V_{\chi\mu} + \chi \epsilon_{\mu\nu\sigma\rho} (D^\sigma + \chi P_\sigma) V_{\chi\rho} = 0. \quad (10)$$

One can see that the chiral components are decoupled from each other as well as the other components.

3. TRANSPORT EQUATION

The equations are still difficult to solve, but the operators and functions can be expanded by $\hbar$ and solved order by order\cite{40}.

3.1. The zeroth order

To the zeroth order, eq.(6) can be written as

$$\left( p_\mu + J_{V\mu}^{(0)} \right) F^{(0)} - \frac{1}{2} \epsilon_{\mu\sigma\rho\nu} J_{A\sigma}^{(0)\nu} S^{(0)}{}_{\sigma\rho} = 0,$$
$$-\frac{1}{2} \epsilon_{\mu\sigma\rho\nu} \left( p^\nu + J_{V\nu}^{(0)} \right) S^{(0)}{}_{\mu\nu} + J_{A\mu}^{(0)} F^{(0)} = 0,$$
$$- \left( p^\nu + J_{V\nu}^{(0)} \right) F^{(0)} - J_{A\nu}^{(0)} P^{(0)} = 0,$$
$$\left( p_\mu + J_{V\mu}^{(0)} \right) P^{(0)} + J_{A\mu}^{(0)} S^{(0)\mu\nu} = 0 \quad (11)$$

Form eq.(11), we can get the on-shell conditions

$$\left( p + J_{V\mu}^{(0)} \right)^2 + J_{A\mu}^{(0)2} F^{(0)} = 0, \quad (12)$$
$$\left( p + J_{V\mu}^{(0)} \right)^2 + J_{A\mu}^{(0)2} P^{(0)} = 0, \quad (13)$$
$$\left( p + J_{V\mu}^{(0)} \right)^2 + J_{A\mu}^{(0)2} S^{(0)\mu\nu} = 0, \quad (14)$$
$$\left( p + J_{V\mu}^{(0)} + \chi J_{A\mu}^{(0)} \right)^2 V_{\chi\nu}^{(0)} = 0. \quad (15)$$

Form eq.(15), we find that even at the 0th order, the on-shell conditions are modified, and they are different for left-hand, right-hand, and other components. This can be viewed as a shift of masses and/or as the modification of
mechanical momenta. The tensor components can be expressed by the scalar and pseudoscalar components

\[ S^{(0)\sigma\rho} = \frac{1}{f_A^{(0)2}} \left( \varepsilon^{\mu\nu\sigma\rho} \left( p_{\mu} + J^{(0)\nu}_{\mu} \right) J_A^{(0)} F^{(0)} - \left( p^\rho + J^{(0)}_{\rho} \right) J_A^{(0)\sigma} F^{(0)} + \left( p^\sigma + J^{(0)}_{\sigma} \right) J_A^{(0)\rho} F^{(0)} \right), \]  

(16)

So among them, the degree of freedom is 2.

We will then focus on the chiral components, the kinetic equation for chiral components are

\[ \bar{p}_{\chi\mu} V_{\chi\mu}^{(0)} = 0, \]
\[ \bar{p}_{\chi\mu} V_{\chi\nu}^{(0)} - \bar{p}_{\chi\nu} V_{\chi\mu}^{(0)} = 0, \]
\[ \nabla_{\chi\mu} V_{\chi\mu}^{(0)} = 0. \]  

(17)

where \( \bar{p}_\chi = p + J_\chi \), \( J_\chi = J_V + \chi J_A \), \( \nabla_{\chi\mu} = \partial_{\beta} J_{\chi\mu}^{(0)} \partial_p^{\beta} \) and the constraint equation for the chiral currents is

\[ J_{\chi\mu} = G \int d^4p V_{\chi\mu}. \]  

(18)

From these equations, the chiral components can be expressed as \( V_{\chi\mu}^{(0)} = \bar{p}_{\chi\mu} f_{\chi}^{(0)} \delta (\bar{p}_\chi^2) \), Putting this into the transport equation we get

\[ \bar{p}_\chi^2 \left( \nabla_{\chi\mu} f_{\chi}^{(0)} \right) \delta (\bar{p}_\chi^2) = 0. \]  

(19)

This is very similar to the free-streaming transport equation. The only difference is that the momentum is shifted. We assume the equilibrium distribution as

\[ f_{\chi\text{eq}}^{(0)} = \frac{1}{e^{\frac{p-n_\chi}{T}+1}}, \]  

(20)

where \( T \) is temperature and \( n_\chi = \frac{f_{\chi}^{(0)}}{\bar{p}_{\chi} f_{\chi}^{(0)}} \) is the fluid velocity, \( n_\chi = \frac{1}{\beta} \sqrt{\frac{f_{\chi}^{(0)} m_{\chi}^{(0)}}{T}} \) is the particle number density. Putting eq. (20) into eq. (19), we can get the equilibrium condition at zeroth order

\[ (p^\mu p^\nu + J_{\chi}^{(0)\mu} J_{\chi}^{(0)\nu}) \left[ \partial_{\mu} \left( \frac{H_{\chi}^{(0)}}{T} \right) + \partial_{\nu} \left( \frac{H_{\chi}^{(0)}}{T} \right) \right] + (2 J_{\chi}^{(0)} - p) \cdot \partial (\frac{H_{\chi}^{(0)}}{T}) = 0. \]  

(21)

We can see that in general, the thermal shear tensor \( \partial_{\mu} \left( \frac{u_{\mu}}{T} \right) + \partial_{\nu} \left( \frac{u_{\nu}}{T} \right) \) is not required to be 0. Only when the particle number density is a constant do we get the usual killing condition.

Finally, we consider the constraint equation. Eq. (17) and eq. (18) give

\[ J_{\chi\mu}^{(0)} = G \int d^4p \bar{p}_{\chi\mu} f_{\chi}^{(0)} \delta (\bar{p}_\chi^2). \]  

(22)

This gives constrain on the current. Especially, if we consider the equilibrium distribution, we have

\[ \frac{n}{T} = \frac{2}{3} \pi^3 \frac{n}{T} GT^2 + \left[ 4 \ln 2 \cdot \pi \frac{n^2}{T^2} GT^2 + 6 \pi (3) GT^2 + 4 \pi GT^2 Li_3 \left( -e^\frac{-\pi}{T} \right) + 4 \pi GT^2 Li_3 \left( -e^{-\frac{T}{\pi}} \right) \right] \text{sgn}(n), \]  

(23)

where \( Li_s( x) \) is the polylogarithm function\footnote{\text{[44]}} and \( \zeta \) is the Riemann zeta function. The chiral index is omitted here because the equation does not depend on chirality, even though the chiral currents can be different. This is because the interaction is symmetric for left- and right-handed components. The quantity \( \frac{n}{T} \) can be viewed as a dimensionless parameter determined by \( GT^2 \). Performing Taylor expansion, we get

\[ \frac{n}{T} \approx GT^2 \left( \frac{2}{3} \pi^3 \frac{n}{T} - \frac{\pi^4}{12} \text{sgn}(n) \right). \]  

(24)

Therefore when \( GT^2 < \frac{3}{2\pi^3} \approx 0.05 \), the constraint equation has only trivial solution \( n = 0 \). When \( GT^2 \geq 0.05 \), there are two additional opposite non-zero solutions whose absolute value is shown in Fig[3][1]

This indicates that without chemical potential, the vector interaction can result in a non-zero net local chiral particle number density at finite temperatures, with equal possibilities of being positive and negative. Therefore in a statistical many-body system with zero chemical potential, these non-zero solutions will serve as local fluctuations, and the expectation value of particle density remains zero.
FIG. 1: The $nT$ as a function of $GT^2$

3.2. The $\hbar$ order

From the $\hbar$ order of the kinetic equation for chiral components, we can get the solution of $V^{(1)}_{\chi\mu}$

$$V^{(1)}_{\chi\mu} = \bar{p}_{\chi\mu} f^{(1)}_{\chi} \delta (\bar{p}_\chi^2) - \frac{\chi}{2 (\bar{p}_\chi \cdot u')} \varepsilon_{\mu\nu\sigma\rho} \bar{p}_\chi^\sigma \left( \nabla_{\chi} f^{(0)} \right) \delta (\bar{p}_\chi^2)$$

$$+ \frac{\chi}{\bar{p}_\chi^2} \varepsilon_{\mu\nu\rho} \bar{p}_\chi^\rho \Omega_{\chi\mu} f^{(0)} \delta (\bar{p}_\chi^2) - \frac{2 \bar{p}_{\chi\mu} \bar{p}_{\chi\nu} J^{(1)\nu}}{\bar{p}_\chi^2} \delta (\bar{p}_\chi^2) + J^{(1)}_{\chi\mu} f^{(0)} \delta (\bar{p}_\chi^2),$$

(25)

where $\nabla_{\chi} = \partial_{\mu} - \nabla J^{(0)}_{\chi\mu}$, $\Omega_{\chi\mu\nu} = \frac{1}{2} \left( \partial_{\mu} J^{(0)}_{\chi\nu} - \partial_{\nu} J^{(0)}_{\chi\mu} \right) = \frac{a_\chi}{2} (\partial_{\mu} u_{\chi\nu} - \partial_{\nu} u_{\chi\mu}) + \frac{1}{2} (u_{\chi\nu} \partial_{\mu} - u_{\chi\mu} \partial_{\nu}) n_\chi$ and $u'$ is an additional time-like vector. One should notice that here the kinetic vorticity instead of the thermal vorticity tensor appears because we have not considered the equilibrium condition. The detailed calculation is in Appendix B.

In the following derivation, we will choose $u'$ to be $J^{(0)}_{\chi\mu}$. According to eq. (25), we can get the transport equation for $f_{\chi} = f^{(0)}_{\chi} + \hbar f^{(1)}_{\chi}$

$$\delta \left( \bar{p}_\chi^2 + \frac{2 \chi \hbar}{\bar{p}_\chi \cdot J^{(0)}_{\chi} \bar{p}_\chi^2 \Omega_{\chi\nu} J^{(0)\nu}} + 2 \hbar \bar{p}_{\chi\alpha} J^{(1)\alpha} \bar{p}_\chi \nabla_{\chi} \right) \left\{ \bar{p}_\chi \cdot \nabla_{\chi} \right\}$$

$$+ \frac{\chi \hbar}{\bar{p}_\chi \cdot J^{(0)}_{\chi}} \left[ \left( \partial_{[\mu} J^{(0)\nu]} \Omega^{\nu]}_{\chi\beta} \right) \bar{p}_{\chi\beta} - \Omega^{\nu]}_{\chi\nu} J^{(0)\beta} \right] \frac{2 \hbar}{\bar{p}_\chi \cdot J^{(0)}_{\chi}} \bar{p}_\chi \nabla_{\chi}$$

$$- \frac{\chi \hbar}{\bar{p}_\chi \cdot J^{(0)}_{\chi}} \Omega_{\chi\nu} \bar{p}_\chi \nabla_{\chi} + \frac{\chi \hbar}{\bar{p}_\chi \cdot J^{(0)}_{\chi}} \bar{p}_\chi \alpha \left( \partial_{\beta} \Omega^{\chi\alpha} \right) \partial_{\beta} \hbar J^{(1)}_{\mu} \Omega^{\mu}_{\chi}$$

$$- \hbar \frac{\bar{p}_\chi \delta \Omega^{\mu\nu}_{\chi\mu} J^{(1)\nu}}{\bar{p}_\chi^2} J^{(1)}_{\chi\mu} \bar{p}_{\chi\nu} \Omega^{\nu}_{\chi\nu} \right\} f_{\chi} = 0.$$  

(26)

Where $\tilde{\Omega}_{\chi\nu} = \frac{1}{2} \tilde{\varepsilon}_{\mu\nu\rho} \Omega^{\rho}_{\chi\mu}$. The derivation of this equation will be shown in Appendix C[10]. From eq. (26), the mass shell is shifted again at the $\hbar$ order by the momentum translation and vorticity.

Then we consider a special condition that is similar to the situation in heavy-ion collisions. In the fireball, chiral currents only come from fluctuations, so we can safely assume $J^{(0)}_{A\mu}$ is zero. In that case, up to the 0th order, $J^{(0)}_{J+} = J^{(0)}_{-} = J$ and $f^{(0)}_{+} = f^{(0)}_{-} = f^{(0)}$, with the same mass shell. For simplicity, we will also assume $J^{(1)}_{\nu}$ to be zero.
So $V_{\chi \mu}$ become

$$V_{\chi \mu} = \tilde{p}_\mu f_\chi \delta \left( \frac{p^2}{\hbar^2} \right) - \frac{\chi \hbar}{2p \cdot u} \varepsilon_{\mu \nu \alpha \sigma} u^\nu p^\alpha \left( \nabla^\sigma f^{(0)} \right) \left( \frac{p^2}{\hbar^2} \right) + \frac{\chi \hbar}{p^2} \varepsilon_{\mu \nu \sigma \rho} p^\nu \Omega^\sigma \Omega^\rho f^{(0)} \delta \left( \frac{p^2}{\hbar^2} \right) \tag{27}$$

$$V_\mu = \frac{1}{2} (V_+ + V_-) = \tilde{p}_\mu f^{(0)} \delta \left( \frac{p^2}{\hbar^2} \right) \tag{28}$$

$$A_\mu = \frac{1}{2} (V_+ - V_-) = \frac{1}{2} \tilde{p}_\mu \left( f^{(1)}_- - f^{(1)}_+ \right) \delta \left( \frac{p^2}{\hbar^2} \right) - \frac{\hbar}{2 p \cdot u} \varepsilon_{\mu \nu \sigma \rho} u^\nu \tilde{p}^\sigma \left( \nabla^\sigma f^{(0)} \right) \left( \frac{p^2}{\hbar^2} \right) + \frac{\hbar}{p^2} \varepsilon_{\mu \nu \sigma \rho} \tilde{p}^\sigma \Omega^\rho f^{(0)} \delta \left( \frac{p^2}{\hbar^2} \right) \tag{29}$$

For all components that become independent of chirality, the chiral indices $\chi$ are dropped. Eq. (29) indicates that spin polarization can be induced by vector and axial vector interaction. In addition to the usual chiral vortical effect, there is a contribution from the divergence of the number density. To show this, we set the vorticity tensor to zero, then

$$A_\mu = \frac{1}{2} \tilde{p}_\mu \left( f^{(1)}_- - f^{(1)}_+ \right) \delta \left( \frac{p^2}{\hbar^2} \right) - \frac{\hbar}{2 p \cdot u} \varepsilon_{\mu \nu \sigma \rho} u^\nu \tilde{p}^\sigma \left( \nabla^\sigma f^{(0)} \right) \left( \frac{p^2}{\hbar^2} \right) + \frac{\hbar}{p^2} \varepsilon_{\mu \nu \sigma \rho} \tilde{p}^\sigma \Omega^\rho f^{(0)} \delta \left( \frac{p^2}{\hbar^2} \right) \tag{30}$$

As mentioned above, in the fireball, the gradient of the number density is in the radial direction, so we will have an axial vector current in the angular direction from the last term. This indicates that there is a local spin distribution, which is qualitatively consistent with experimental findings\cite{7,45,46}. And combining this with eq. 21 one can see that the polarization is related to the shear tensor.

There are also contributions to the transport equations

$$\nabla^\mu V_\mu = \tilde{p}_\mu \nabla^\mu f^{(0)} \delta \left( \frac{p^2}{\hbar^2} \right) = 0 \tag{31}$$

$$\nabla^\mu A_\mu = \frac{1}{2} \tilde{p}_\mu \left( \nabla^\mu f^{(1)}_+ - \nabla^\mu f^{(1)}_- \right) \delta \left( \frac{p^2}{\hbar^2} \right) - \frac{\hbar}{p \cdot J^{(0)}_V} \varepsilon_{\mu \nu \sigma \rho} u^\nu \left( \partial^\rho n \right) J^{(0)}_V \tilde{p}_\sigma \left( \nabla^\sigma f^{(0)} \right) \delta \left( \frac{p^2}{\hbar^2} \right)$$

$$+ \frac{\hbar}{2 p \cdot J^{(0)}_V} \left[ \left( \partial^\mu J^{(0)}_V \right) \varepsilon_{\mu \nu \sigma \rho} \left( \partial^\sigma \partial^\rho \right) - \frac{1}{2} J^{(0)}_V \left( u^\nu \partial^\beta - u^\beta \partial^\nu \right) \right] \varepsilon_{\mu \nu \sigma \rho} J^{(0)}_V \tilde{p}_\sigma \left( \nabla^\sigma f^{(0)} \right) \delta \left( \frac{p^2}{\hbar^2} \right)$$

$$+ \frac{\hbar}{2 p \cdot J^{(0)}_V} \varepsilon_{\mu \nu \sigma \rho} \partial^\beta \left( u^\nu \left( \partial^\rho n \right) \right) J^{(0)}_V \tilde{p}_\sigma \left( \partial^\rho f^{(0)} \right) \delta \left( \frac{p^2}{\hbar^2} \right) = 0 \tag{32}$$

There should also be a constraint equation for the 1st-order currents

$$J^{(1)}_{\chi \mu} = \int d^4 p V^{(1)}_{\chi \mu} \tag{33}$$

But we do not discuss the possible form of the 1st order distribution function $f^{(1)}_\chi$ in this study, so the constraint equation is not solved.

4. SUMMARY

In this work, we derive the relativistic quantum kinetic equation for massless fermions with vector and axial interaction from the Wigner function formalism and carry out the semi-classical expansion up to $\hbar$ order. Our results show that there is a non-trivial mass-shell shift and non-zero particle number density even at the 0th order of $\hbar$. We also show that there can be a non-zero shear tensor and gradient of particle number density in the equilibrium state, and they will contribute to the transport equation. There will be additional contributions to the axial current or the spin polarization. Therefore, the effective vector interaction could be useful in the study of spin polarization in heavy-ion collisions. One can further research the spin polarization which is produced by the vector interaction and axial interaction. For massless Fermions, the direction of spin is not well-defined, but we can find the direction of spin polarization by chirality. Also, as the interaction term caused the mass shell to be shifted, it is possible that now the fermions can be treated as massive, then methods such as the Matrix Valued Spin dependent Distribution can be used\cite{47}.

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Appendix A: The polylogarithm function

In this appendix, we present some formulas used in the calculation of eq. (23). Let us introduce the following integral

\[
\int_0^\infty dp^k \frac{1}{1 + e^{(vp + \mu)/T}} = -\frac{T^{k+1} \Gamma (k + 1)}{e^{k+1}} Li_{k+1} \left( -e^{\pm \mu/T} \right),
\]

where \( \Gamma (n) \) is the gamma function and \( Li_{n+1} (x) \) is the polylogarithm function. It is easy to verify that the RHS of eq. (22) is an off function of \( n \). So the solution of eq. (22) can be written as

\[
\begin{aligned}
\frac{1}{\varepsilon} = & 2\pi G \int_0^\infty dp \frac{p^2}{e^{\frac{p^2}{2\varepsilon}} + 1} + 2\pi G \int_n^\infty dp \frac{p^2}{e^{\frac{p^2}{2\varepsilon}} + 1} - 2\pi G \int_0^\infty dp \frac{p^2}{e^{\frac{p^2}{2\varepsilon}} + 1},
\end{aligned}
\]

where

\[
\begin{aligned}
2\pi G \int_0^n \frac{dp}{e^{\frac{p^2}{2\varepsilon}} + 1} &= \frac{1}{3} \pi^3 n GT^2 + 2ln2 \cdot \pi n^2 GT + 3\pi \zeta (3) GT^3 + 4\pi GT^3 Li_3 \left( -e^{\frac{\mu}{T}} \right), \\
2\pi G \int_n^\infty \frac{dp}{e^{\frac{p^2}{2\varepsilon}} + 1} &= \frac{1}{3} \pi^3 n GT^2 + 2ln2 \cdot \pi n^2 GT + 3\pi \zeta (3) GT^3, \\
2\pi G \int_0^\infty \frac{dp}{e^{\frac{p^2}{2\varepsilon}} + 1} &= -4\pi GT^3 Li_3 \left( -e^{-\frac{\mu}{T}} \right).
\end{aligned}
\]

Appendix B: The solution of \( V_{\chi \mu} \)

In this appendix, we show how to get the solution eq. (25). The transport equations for \( V_{\chi \mu} \) to \( h \) order is

\[
\begin{aligned}
\bar{p}_{\chi \mu} V^{(1)}_{\chi \mu} + J^{(1)}_{\chi \mu} V^{(1)}_{\chi \mu} &= 0, \\
\nabla_{\chi \mu} V^{(1)}_{\chi \nu} - \nabla J^{(1)}_{\chi \mu} V^{(0)}_{\chi \nu} &= 0, \\
\varepsilon_{\mu \nu \rho \sigma} \nabla_{\chi} V^{(0)}_{\chi \rho} &= -2\chi \left( \bar{p}_{\chi \mu} V^{(1)}_{\chi \nu} - \bar{p}_{\chi \nu} V^{(1)}_{\chi \mu} + J^{(1)}_{\chi \mu} V^{(0)}_{\chi \nu} - J^{(1)}_{\chi \nu} V^{(0)}_{\chi \mu} \right).
\end{aligned}
\]

From the last equation of eq. (B1), we can get

\[
\begin{aligned}
\varepsilon_{\mu \nu \rho \sigma} \bar{p}_{\chi} \nabla_{\chi} V^{(0)}_{\chi \rho} &= 4\chi \bar{p}_{\chi \mu} J^{(1)}_{\chi \nu} V^{(0)}_{\chi \nu} + 2\chi \bar{p}_{\chi}^2 V^{(1)}_{\chi \mu} - 2\chi J^{(1)}_{\chi \mu} \bar{p}_{\chi} V^{(0)}_{\chi \nu}, \\
V^{(1)}_{\chi \mu} &= \frac{\chi}{2\bar{p}_{\chi}^2} \varepsilon_{\mu \nu \rho \sigma} \bar{p}_{\chi} \nabla_{\chi} V^{(0)}_{\chi \rho} - \frac{2}{\bar{p}_{\chi}^2} \left( \bar{p}_{\chi} \cdot J^{(1)}_{\chi} \right) V^{(0)}_{\chi \mu} + \frac{J^{(1)}_{\chi \mu} \bar{p}_{\chi} V^{(0)}_{\chi \nu}}{\bar{p}_{\chi}^2}.
\end{aligned}
\]

So the solution of \( V_{\chi \mu} \) to \( h \) order is

\[
\begin{aligned}
V^{(1)}_{\chi \mu} &= \bar{p}_{\chi \mu} f^{(1)}_{\chi} \delta (\bar{p}_{\chi}) + \kappa_{\mu} \delta (\bar{p}_{\chi}) + \frac{\chi}{2\bar{p}_{\chi}^2} \varepsilon_{\mu \nu \rho \sigma} \bar{p}_{\chi} \nabla_{\chi} V^{(0)}_{\chi \rho} - \frac{2}{\bar{p}_{\chi}^2} \left( \bar{p}_{\chi} \cdot J^{(1)}_{\chi} \right) V^{(0)}_{\chi \mu} + \frac{J^{(1)}_{\chi \mu} \bar{p}_{\chi} V^{(0)}_{\chi \nu}}{\bar{p}_{\chi}^2}.
\end{aligned}
\]
Combining the last equation of eq. [B1] and eq. [B4], one can get
\[
\varepsilon_{\mu\nu\rho} \nabla^\sigma \chi V^{(0)\rho} = -2\chi \left[ (\bar{\rho}_{\mu
u} - \bar{\rho}_{\mu\nu}) \delta \left( \bar{p}_X^2 \right) + \frac{\chi}{2\bar{p}_X^2} \left( \bar{\rho}_{\mu
u} \varepsilon_{\nu\mu\alpha\rho} - \bar{\rho}_{\mu\nu} \varepsilon_{\mu\nu\sigma\rho} \right) \bar{p}_X^2 \nabla^\sigma \chi V^{(0)\rho} \right] - 2\chi \left( J^{(1)}_{\chi\nu} V^{(0)}_{\chi\mu} \right) \nabla^\sigma \chi V^{(0)\rho} \nabla^\nu \chi V^{(0)}_{\chi\mu}.
\] (B5)

\[
(\bar{\rho}_{\mu\nu} - \bar{\rho}_{\mu\nu}) \delta \left( \bar{p}_X^2 \right) = \frac{-\chi}{2} \varepsilon_{\mu\nu\rho} \nabla^\sigma \chi V^{(0)\rho} - \frac{\chi}{2\bar{p}_X^2} \left( \bar{\rho}_{\mu\nu} \varepsilon_{\nu\mu\alpha\rho} - \bar{\rho}_{\mu\nu} \varepsilon_{\mu\nu\sigma\rho} \right) \bar{p}_X^2 \nabla^\sigma \chi V^{(0)\rho}.
\] (B6)

The derivatives of $V^{(0)}_{\mu}$ can be expanded
\[
\nabla^\sigma \chi V^{(0)\rho} = F^\sigma_{\mu\nu} f^{(0)}_{\chi} \delta \left( \bar{p}_X^2 \right) + \frac{\chi}{2} \varepsilon_{\mu\nu\rho} \bar{\rho}_X^2 \left( \nabla^\sigma f^{(0)}_{\chi} \right) \delta \left( \bar{p}_X^2 \right).
\] (B7)
\[
\nabla^\sigma \chi V^{(0)\rho} - \nabla^\sigma \chi V^{(0)\sigma} = 2 F^\sigma_{\mu\nu} f^{(0)}_{\chi} \delta \left( \bar{p}_X^2 \right) + \frac{\chi}{2} \varepsilon_{\mu\nu\rho} \bar{\rho}_X^2 \left( \nabla^\sigma f^{(0)}_{\chi} \right) \delta \left( \bar{p}_X^2 \right)
\] (B8)

So
\[
(\bar{\rho}_{\mu\nu} - \bar{\rho}_{\mu\nu}) \delta \left( \bar{p}_X^2 \right) = \frac{-\chi}{2} \varepsilon_{\mu\nu\rho} \bar{\rho}_X^2 \left( \nabla^\sigma f^{(0)}_{\chi} \right) \delta \left( \bar{p}_X^2 \right).
\] (B9)

Then we introduce the auxiliary unit vector $u'$ which is perpendicular to $\kappa_{\mu}$
\[
u^\mu \kappa_{\mu} = 0.
\] (B10)

So we can get the solution of $\kappa_{\mu} \delta \left( \bar{p}_X^2 \right)$
\[
\kappa_{\mu} \delta \left( \bar{p}_X^2 \right) = -\frac{\chi}{2} \varepsilon_{\mu\nu\rho} \bar{\rho}_X^2 \left( \nabla^\sigma f^{(0)}_{\chi} \right) \delta \left( \bar{p}_X^2 \right),
\] (B11)

\[
V^{(1)}_{\chi\mu} = \bar{\rho}_{\mu\nu} f^{(1)}_{\chi} \delta \left( \bar{p}_X^2 \right) - \frac{\chi}{2} \varepsilon_{\mu\nu\rho} \bar{\rho}_X^2 \left( \nabla^\sigma f^{(0)}_{\chi} \right) \delta \left( \bar{p}_X^2 \right)
\] (B12)

\[
\nabla^\sigma \chi V_{\chi\mu} = \left( \partial^\mu - \partial^\beta J^{(1)\mu}_{\chi \beta} \partial^\rho - \partial^\beta J^{(1)\mu}_{\chi \beta} \partial^\rho \right) V_{\chi\mu} = 0.
\] (C1)

When we combine eq. (C1) and eq. [C1], to $h$ order, the equation can be separated into six parts. The first term is
\[
\left( \partial^\mu - \partial^\beta J^{(0)\mu}_{\chi \beta} \partial^\rho \right) \left[ \bar{\rho}_{\mu\nu} f_{\chi} \delta \left( \bar{p}_X^2 \right) \right] = \bar{\rho}_{\mu\nu} \left( \nabla^\mu f_{\chi} \right) \delta \left( \bar{p}_X^2 \right).
\] (C2)
The second term is
\[
\frac{\chi}{2} \varepsilon_{\mu\sigma\rho} \left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ \frac{1}{p_{\chi}^2} \Omega_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right] = \chi \left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ \frac{1}{p_{\chi}^2} \Omega_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right]
\]
\[
= \chi \left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ \frac{1}{p_{\chi}^2} \Omega_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right] + \frac{\chi}{p_{\chi}^2} \left( \partial^{\mu} \Omega_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right)
\]
\[
+ \frac{\chi}{p_{\chi}^2} \Omega_{\chi}^{\rho} \left( \nabla_{\chi}^{\rho} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right).
\]  
(C3)

The third term is
\[
\left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ \frac{\chi}{2} \left( p_{\chi} \cdot u \right)^2 \varepsilon_{\mu\rho\sigma} u_{\rho} u_{\sigma} \left( \nabla_{\chi}^{\mu} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right) \right]
\]
\[
= \frac{\chi}{2} \left( p_{\chi} \cdot u \right)^2 \varepsilon_{\mu\rho\sigma} u_{\rho} u_{\sigma} \left( \nabla_{\chi}^{\mu} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ \frac{\chi}{2} \left( p_{\chi} \cdot u \right)^2 \varepsilon_{\mu\rho\sigma} u_{\rho} \Omega_{\chi}^{\rho} \left( \nabla_{\chi}^{\mu} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ \frac{\chi}{2} \left( p_{\chi} \cdot u \right)^2 \varepsilon_{\mu\rho\sigma} u_{\rho} \Omega_{\chi}^{\rho} \left( \nabla_{\chi}^{\mu} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ \frac{\chi}{2} \left( p_{\chi} \cdot u \right)^2 \varepsilon_{\mu\rho\sigma} u_{\rho} \Omega_{\chi}^{\rho} \left( \nabla_{\chi}^{\mu} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right).
\]  
(C4)

The fourth term is
\[
\left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ \frac{2h}{p_{\chi}^2} \Omega_{\chi}^{\rho} J_{\chi}^{(1)\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right] = \frac{2h}{p_{\chi}^2} \left( \Omega_{\chi}^{\mu\rho} \nabla_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right)
\]
\[
+ \frac{2h}{p_{\chi}^2} \Omega_{\chi}^{\mu\rho} \nabla_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ \frac{2h}{p_{\chi}^2} \Omega_{\chi}^{\mu\rho} \nabla_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right).
\]  
(C5)

The fifth term is
\[
\left( \partial^{\mu} - \partial^{\beta} J_{\chi}^{(0)\mu} \partial_{\beta} \right) \left[ h J_{\chi}^{(1)\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right] = h \left( \partial^{\mu} J_{\chi}^{(1)\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right)
\]
\[
+ h J_{\chi}^{(1)\rho} \left( \nabla_{\chi}^{\rho} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ h J_{\chi}^{(1)\rho} \left( \nabla_{\chi}^{\rho} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right).
\]  
(C6)

The sixth term is
\[
h \partial^{\beta} J_{\chi}^{(1)\mu} \partial_{\beta} \left[ \Omega_{\chi}^{\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right] = h \left( \partial^{\mu} J_{\chi}^{(1)\rho} f^{(0)} \delta \left( \vec{p}_{\chi}^2 \right) \right)
\]
\[
+ h \partial^{\beta} J_{\chi}^{(1)\mu} \left( \nabla_{\chi}^{\beta} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right)
\]
\[
+ 2h \partial^{\beta} J_{\chi}^{(1)\mu} \left( \nabla_{\chi}^{\beta} f^{(0)} \right) \delta \left( \vec{p}_{\chi}^2 \right).
\]  
(C7)

We have the relation
\[
\varepsilon_{\mu\rho\sigma} \left( \nabla_{\chi}^{\mu} \nabla_{\chi}^{\rho} f^{(0)} \right) = \frac{1}{2} \varepsilon_{\mu\rho\sigma} \left[ \nabla_{\chi}^{\mu} \nabla_{\chi}^{\rho} f^{(0)} \right] = -\partial^{\sigma} F_{\chi}^{\rho\sigma} f^{(0)}.
\]
\[
2 \partial^{\rho} \varepsilon_{\mu\rho\sigma} u_{\sigma} \Omega_{\chi}^{\mu\beta} = -2 \left( p_{\chi} \cdot u \right) \tilde{\Omega}_{\chi}^{\mu\alpha} \tilde{p}_{\chi} + 2 \tilde{p}_{\chi}^{2} \Omega_{\chi}^{\alpha\nu} u_{\nu} - 2 \tilde{p}_{\chi} \tilde{\Omega}_{\chi}^{\nu\sigma} u_{\nu} \tilde{p}_{\chi}.
\]  
(C8)
So the transport equation can be written as
\[
\left( \partial^\mu - \partial^\beta J^{(0)\mu}_{\chi} \partial_\beta - \hbar \partial^\beta J^{(1)\mu}_{\chi} \partial_\beta \right) V_{\chi \mu}
\]
\[
= \bar{\rho}_{\chi \mu} (\nabla^\mu f_{\chi}^0) \delta (\vec{p}_{\chi}^2) + \frac{\chi \hbar}{p_{\chi} \cdot u} \bar{\rho}_{\chi \mu} (\nabla^\mu f_{\chi}^0) \delta (\vec{p}_{\chi}^2) \\
+ 2 \hbar \rho_{\chi \mu} J^{(0)\mu}_{\chi} \delta (\vec{p}_{\chi}^2) + \frac{\chi \hbar}{2 (\vec{p}_{\chi} \cdot u)^2} \delta (\vec{p}_{\chi}^2) \\
- \frac{\chi \hbar}{2 p_{\chi}^2} \varepsilon_{\mu \nu \sigma} (\partial^\mu u^\nu) \bar{\rho}_{\chi \mu} (\nabla^\sigma f_{\chi}^0) \delta (\vec{p}_{\chi}^2) + \frac{\chi \hbar}{2 p_{\chi}^2} \varepsilon_{\mu \nu \sigma} \bar{\rho}_{\chi \mu} (\nabla^\sigma f_{\chi}^0) \delta (\vec{p}_{\chi}^2) \\
+ \hbar J^{(1)}_{\chi} \left( \nabla^\mu f_{\chi}^0 \right) \delta (\vec{p}_{\chi}^2) - \hbar \partial^\sigma \bar{\rho}_{\chi \mu} (\nabla^\sigma f_{\chi}^0) \delta (\vec{p}_{\chi}^2) - \hbar \partial^\sigma J^{(1)\mu}_{\chi} \delta (\vec{p}_{\chi}^2) \\
= \left( \frac{\chi \hbar}{p_{\chi} \cdot u} \bar{\rho}_{\chi \mu} (\nabla^\mu f_{\chi}^0) \right) \left\{ \bar{\rho}_{\chi \mu} (\nabla X_{\chi}^0_{\chi}) + \frac{\chi \hbar}{2 (\vec{p}_{\chi} \cdot u)^2} \left[ (\partial^\mu u^\nu) \bar{\rho}_{\chi \mu} - \Omega^\beta_{\mu \chi} \left( \partial^\beta \bar{\rho}_{\chi \mu} \right) \right] \right\} \\
- \frac{\chi \hbar}{2 p_{\chi}^2} \varepsilon_{\mu \nu \sigma} (\partial^\mu u^\nu) \bar{\rho}_{\chi \mu} (\nabla^\sigma f_{\chi}^0) \delta (\vec{p}_{\chi}^2) + \hbar J^{(1)}_{\chi} \left( \nabla^\mu f_{\chi}^0 \right) \delta (\vec{p}_{\chi}^2) - \hbar \partial^\sigma \bar{\rho}_{\chi \mu} (\nabla^\sigma f_{\chi}^0) \delta (\vec{p}_{\chi}^2) \\
- \hbar \partial^\sigma J^{(1)\mu}_{\chi} \delta (\vec{p}_{\chi}^2) f_{\chi} = 0.
\]
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