Limit Distribution of Two Skellam Distributions, Conditionally on Their Equality

Élie de Panafieu*, François Durand†
Nokia Bell Labs France

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This note provides a proof of the following proposition.

**Proposition 1.** Let $A_n$, $B_n$, $C_n$ and $D_n$ denote independent random Poisson variables of respective parameters $n\tau_A$, $n\tau_B$, $n\tau_C$ and $n\tau_D$. We denote $X_n = (A_n - B_n | A_n - B_n = C_n - D_n)$.

1. If $\tau_A = \tau_B = 0$, or $\tau_B = \tau_C = 0$, or $\tau_C = \tau_D = 0$, or $\tau_D = \tau_A = 0$, then the distribution of $X_n$ is a Dirac measure in 0.

2. Otherwise,

$$E(X_n) = nE + E' + o(1),$$
$$V(X_n) = nV + V' + o(1),$$

where

$$E = \frac{\tau_A \tau_C - \tau_B \tau_D}{\sqrt{(\tau_A + \tau_D)(\tau_B + \tau_C)}},$$
$$E' = -\frac{\tau_A \tau_C - \tau_B \tau_D}{4(\tau_A + \tau_D)(\tau_B + \tau_C)},$$
$$V = \frac{(\tau_A \tau_C + 2\tau_A \tau_B + \tau_B \tau_D)(\tau_A \tau_C + 2\tau_C \tau_D + \tau_B \tau_D)}{2[(\tau_A + \tau_D)(\tau_B + \tau_C)]^2},$$
$$V' = -\frac{(\tau_A \tau_C + \tau_B \tau_D)(\tau_A \tau_B + \tau_C \tau_D) + 4\tau_A \tau_B \tau_C \tau_D}{4[(\tau_A + \tau_D)(\tau_B + \tau_C)]^2}.$$

And the distribution of $(X_n - nE)/\sqrt{nV}$ is asymptotically Gaussian.

Since case 1 is trivial, we only need to prove case 2 and thus assume that $\tau_A + \tau_B > 0$ and $\tau_B + \tau_C > 0$ and $\tau_C + \tau_D > 0$ and $\tau_D + \tau_A > 0$.

Up to exchanging roles between $A_n$ and $B_n$, and between $C_n$ and $D_n$, we also assume that $\tau_A \tau_C \geq \tau_B \tau_D$. Intuitively, it means that in $X_n = (A_n - B_n | A_n - B_n = C_n - D_n)$, the “positive” forces $A_n$ and $C_n$ are stronger than the “negative” ones, $B_n$ and $D_n$. It corresponds to the cases where $E$, the main term in the asymptotic development of the expectation, will be proved nonnegative.

*depanafieuelie@gmail.com  †fradurand@gmail.com
1 Generic Case: All Coefficients Are Positive

For this section, we add the assumption that $\tau_A$, $\tau_B$, $\tau_C$ and $\tau_D$ are all positive. This assumption will be removed in the second section.

Let $P_n(u)$ denote the probability generating function

$$P_n(u) = \sum_{m \in \mathbb{Z}} P(A_n - B_n = m \mid A_n - B_n = C_n - D_n) u^m.$$ 

Introducing the function

$$F_n(u) = \sum_{a,b,c,d \geq 0 \atop a-b=c-d} P(A_n = a)P(B_n = b)P(C_n = c)P(D_n = d) u^{a-b},$$

we obtain

$$P_n(u) = \frac{F_n(u)}{F_n(1)}.$$ 

Expressing the probabilities explicitly, the expression becomes

$$F_n(u) = e^{-n(\tau_A+\tau_B+\tau_C+\tau_D)} \sum_{a,b,c,d \geq 0 \atop a-b=c-d} \frac{(n\tau_A)^a}{a!} \frac{(n\tau_B)^b}{b!} \frac{(n\tau_C)^c}{c!} \frac{(n\tau_D)^d}{d!} u^{a-b}.$$ 

Introducing the variable $g = a - b = c - d$, we obtain

$$F_n(u) = e^{-n(\tau_A+\tau_B+\tau_C+\tau_D)} G_n(u),$$

where

$$G_n(u) = \sum_{b,d \geq 0 \atop g \geq \max(-b,-d)} \frac{(n\tau_A)^{(b+g)}}{(b+g)!} \frac{(n\tau_B)^b}{b!} \frac{(n\tau_C)^c}{c!} \frac{(n\tau_D)^d}{d!} u^g.$$ 

The Stirling approximation is introduced and we define the values $x = \frac{b}{n}$, $y = \frac{d}{n}$, $z = \frac{a}{n}$. Then:

$$G_n(u) = \sum_{b,d \geq 0 \atop g \geq \max(-b,-d)} \psi_n(x,y,z) e^{-n\phi_n(x,y,z)},$$

where

$$\psi_n(x,y,z) = \frac{(n(x+z)+ny)e^{-n(x+z)}}{(nx)!} \frac{(n(y+z))e^{-n(y+z)}}{(ny)!},$$

$$\phi_n(x,y,z) = x \log(x) - x(1 + \log(\tau_B)) + (x + z) \log(x + z) - (x + z)(1 + \log(\tau_A)) + y \log(y) - y(1 + \log(\tau_D)) + (y + z) \log(y + z) - (y + z)(1 + \log(\tau_C)) - z \log(u).$$

The above expression is well defined because we assumed that $\tau_A$, $\tau_B$, $\tau_C$ and $\tau_D$ are all positive.

We will soon see that the main contributions to $G_n(u)$ come from the vicinity of the minimum of $\phi_n$. We will first compute this main contribution, then prove that the rest of the sum is negligible.

The function $\phi_n$ is convex with a unique minimum. Therefore, there is a small enough vicinity $V$ of this minimum such that $\phi_n$ is larger anywhere outside this vicinity than anywhere inside. Without loss of generality, we assume that $V$ does not contain the origin. Thus, uniformly on $V$, we have

$$\psi_n(x,y,z) \sim \frac{\psi(x,y,z)}{(2\pi n)^{3/2}}$$

where

$$\psi(x,y,z) := \frac{1}{\sqrt{x(x+z)y(y+z)}}.$$ 

We apply the following classical lemma (Laplace method) to extract the asymptotics.
Lemma 1 (Laplace Method). Consider a compact set $C$ of $\mathbb{R}^d$ and the series

$$I_n = \sum_{k \in \mathbb{Z}^d \cap \frac{k}{n} \in C} \psi(k/n)e^{-n\phi(k/n)}$$

where $\psi, \phi$ are differentiable functions from $C$ to $\mathbb{R}$. Assume furthermore that $\phi$ has a unique global minimum $\theta^*$ which is located in the interior of $C$, $\phi$ is three-times differentiable in a neighborhood of $\theta^*$, $\text{Hessian}_\phi(\theta^*) > 0$ and $\psi(\theta^*) \neq 0$. Then

$$I_n \sim (2\pi n)^{d/2} \psi(\theta^*) e^{-n\phi(\theta^*)} \sqrt{\text{Hessian}_\phi(\theta^*)}.$$

Proof. There are many variants of this classic result. The one-dimensional case ($d = 1$) is treated by Masoero (2015). There, the approximation of the sum by a Riemann integral is justified. The same transformation applies to the multivariate case. The asymptotics of the triple integral is then obtained by a multivariate Laplace method, see e.g. Pemantle and Wilson (2013).

Denoting $x^*, y^*, z^*$ the minimal point of $\phi_u$, we conclude that the contribution from $V$ to the sum $G_n(u)$ has an asymptotics of the form $e^{-n\phi(x^*, y^*, z^*)}$ multiplied by a polynomial term. Because $\phi_u$ outside of $V$ is larger than $\phi_u(x^*, y^*, z^*) + \epsilon$ for some positive $\epsilon$, we conclude that the contribution of the rest of the sum is exponentially small and therefore negligible in the asymptotics. (Also: outside of $\mathcal{V}$, we use Stirling bounds to bound $\psi_n(x, y, z)$.)

In order to get the asymptotics of $G_n(u)$, what remains to do is to evaluate $\psi(x^*, y^*, z^*)$, $\phi_u(x^*, y^*, z^*)$ and $\text{Hessian}_\phi(x^*, y^*, z^*)$. The minimal point of $\phi_u$ is characterized by the system

$$x^*(x^* + z^*) = \tau_{AB} u,$$
$$y^*(y^* + z^*) = \tau_{CD} u,$$
$$(x^* + z^*)(y^* + z^*) = \tau_{AC} u.$$

The solution is given by:

$$x^* = \frac{\tau_B \sqrt{u} + \frac{\tau_D}{\sqrt{u}}}{\tau_A \sqrt{u} + \frac{\tau_B}{\sqrt{u}}},$$
$$y^* = \frac{\tau_D \sqrt{u} + \frac{\tau_B}{\sqrt{u}}}{\tau_C \sqrt{u} + \frac{\tau_D}{\sqrt{u}}},$$
$$z^* = \frac{\tau_{AC} u - \frac{\tau_D}{\sqrt{u}}}{\left(\tau_A \sqrt{u} + \frac{\tau_B}{\sqrt{u}}\right) \left(\tau_C \sqrt{u} + \frac{\tau_D}{\sqrt{u}}\right)}.$$

We have $x^* > 0$ and $y^* > 0$. Moreover, the assumption $\tau_{AC} \geq \tau_{BD}$ ensures that for $u$ close enough to 1, $z^* > \max(-x^*, -y^*)$. As a consequence, $(x^*, y^*, z^*)$ is on the interior of the integration zone defining $G_n(u)$, which validates the approximation by a Riemann integral mentioned above.

Simple algebra leads to:

$$\psi(x^*, y^*, z^*) = \frac{1}{\sqrt{\tau_{AB} \tau_{CD}}},$$
$$\phi_u(x^*, y^*, z^*) = -2 \left(\frac{\tau_B \sqrt{u} + \frac{\tau_D}{\sqrt{u}}}{\tau_A \sqrt{u} + \frac{\tau_B}{\sqrt{u}}} \right) \left(\tau_C \sqrt{u} + \frac{\tau_D}{\sqrt{u}}\right),$$
$$\text{Hessian}_\phi(x^*, y^*, z^*) = \frac{2}{\tau_{AB} \tau_{CD} \tau_{BD}} \sqrt{\left(\tau_A \sqrt{u} + \frac{\tau_D}{\sqrt{u}}\right) \left(\tau_C \sqrt{u} + \frac{\tau_B}{\sqrt{u}}\right)}.$$
Applying Lemma 1, we then have:

\[ G_n(u) \sim \frac{1}{2\sqrt{\pi n}} \gamma(u)^{-\frac{1}{2}} \exp \left( 2n \sqrt{\gamma(u)} \right), \]

where

\[ \gamma(u) = \left( \frac{\tau_A u + \tau_B}{\sqrt{u}} \right) \left( \frac{\tau_C u + \tau_D}{\sqrt{u}} \right). \]

To obtain the convergence in distribution to a Gaussian law, we will apply the Quasi-powers Theorem, due to Hwang (1998), which proof is also given by Flajolet and Sedgewick (2009) (Lemma IX.1) (we use a slightly weaker version because we are not interested into the speed of convergence).

**Lemma 2** (Quasi-powers). Assume that the Laplace transform \( \mathbb{E}(e^{sX_n}) \) of a sequence of random variables \( X_n \) is analytic in a neighborhood of 0, and has an asymptotics of the form

\[ \mathbb{E}(e^{sX_n}) \sim e^{\beta_n s + g(s)}, \]

with \( \beta_n \to +\infty \) as \( n \to +\infty \), and \( f(s), g(s) \) analytic on a neighborhood of 0. Assume also the condition \( f''(0) \neq 0 \). Under these assumptions, the mean and variance of \( X_n \) satisfy

\[ \mathbb{E}(X_n) = \beta_n f'(0) + o(1), \]
\[ \mathbb{V}(X_n) = \beta_n f''(0) + g''(0) + o(1), \]

and the distribution of \( (X_n - \beta_n f'(0))/\sqrt{\beta_n f''(0)} \) is asymptotically Gaussian.

We apply Lemma 2 to \( X_n = (A_n - B_n | A_n - B_n = C_n - D_n) \). Using the asymptotics of \( G_n \), we have:

\[ \mathbb{E}(e^{sX_n}) = \frac{F_n(e^s)}{F_n(1)} \sim \exp \left[ 2n \left( \sqrt{\gamma(e^s)} - \sqrt{\gamma(1)} \right) - \frac{1}{4} (\log(\gamma(e^s)) - \log(\gamma(1))) \right]. \]

The result of Proposition 1 is then obtained by application of Lemma 2 with

\[ \beta_n = n, \]
\[ f(s) = 2 \left( \sqrt{\gamma(e^s)} - \sqrt{\gamma(1)} \right), \]
\[ g(s) = -\frac{1}{4} (\log(\gamma(e^s)) - \log(\gamma(1))). \]

The assumptions \( \tau_A + \tau_B > 0, \tau_B + \tau_C > 0, \tau_C + \tau_D > 0 \) and \( \tau_D + \tau_A > 0 \) ensure that

\[ f''(0) = \frac{(\tau_A \tau_C + 2\tau_A \tau_B + \tau_B \tau_D)(\tau_A \tau_C + 2\tau_C \tau_D + \tau_B \tau_D)}{2[(\tau_A + \tau_D)(\tau_B + \tau_C)]^{\frac{3}{2}}} \]

is positive.

## 2 Degenerate Case: Some Coefficients Are Zero

We now consider the case where one or several coefficients \( \tau \) vanish. Considering our assumptions \( \tau_A + \tau_B > 0 \) and \( \tau_B + \tau_C > 0 \) and \( \tau_C + \tau_D > 0 \) and \( \tau_D + \tau_A > 0 \) and \( \tau_A \tau_C \geq \tau_B \tau_D \), there are only two cases, up to symmetries:

- \( \tau_B = 0 \) and the other coefficients are positive,
- \( \tau_B = \tau_D = 0 \) and the other coefficients are positive.

In both cases, the proof is based on the same principle as in the first section. The main difference is that the triple sum is replaced by a double sum in the first case, and by a simple sum in the second case.
2.1 \( \tau_B = 0 \) and the other coefficients are positive

The probability generating function becomes

\[
P_n(u) = \sum_{m \in \mathbb{Z}} P(A_n = m \mid A_n = C_n - D_n)u^m.
\]

Introducing the function

\[
F_n(u) = \sum_{a, c, d \geq 0} P(A_n = a)P(C_n = c)P(D_n = d)u^a,
\]

we obtain

\[
P_n(u) = \frac{F_n(u)}{F_n(1)}.
\]

Expressing the probabilities explicitly, the expression becomes

\[
F_n(u) = e^{-n(\tau_A + \tau_C + \tau_D)} \sum_{a, c, d \geq 0} \frac{(n\tau_A)^a (n\tau_C)^c (n\tau_D)^d}{a! c! d!} u^a
\]

and we obtain

\[
F_n(u) = e^{-n(\tau_A + \tau_C + \tau_D)} G_n(u),
\]

where

\[
G_n(u) = \sum_{a, d \geq 0} \frac{(n\tau_A)^a (n\tau_D)^d}{a! (a + d)!} u^a.
\]

The Stirling approximation is introduced and we define the values \( x = \frac{a}{n} \) and \( y = \frac{d}{n} \). Then:

\[
G_n(u) = \sum_{a, d \geq 0} \psi_n(x, y)e^{-n\phi_u(x, y)},
\]

where

\[
\psi_n(x, y) = \frac{(nx)^n e^{-nx} (n(x + y))^n e^{-n(x + y)} (ny)^ne^{-ny}}{(nx)! (n(x + y))! (ny)!},
\]

\[
\phi_u(x, y) = x \log(x) - x(1 + \log(\tau_A)) + (x + y) \log(x + y) - (x + y)(1 + \log(\tau_C)) + y \log(y) - y(1 + \log(\tau_D)) - x \log(u).
\]

The minimum of \( \phi_u \) is obtained for:

\[
x^* = \frac{\tau_A\tau_C u}{\sqrt{\tau_A\tau_C u + \tau_C\tau_D}}, \quad y^* = \frac{\tau_C\tau_D}{\sqrt{\tau_A\tau_C u + \tau_C\tau_D}}.
\]

This leads to:

\[
\psi_n(x^*, y^*) \sim \frac{1}{(2\pi n)^\frac{3}{2}} \frac{(\tau_A\tau_C u + \tau_C\tau_D)^\frac{1}{2}}{\sqrt{\tau_A\tau_C^2\tau_D u} u},
\]

\[
\phi_u(x^*, y^*) = -2\sqrt{\tau_A\tau_C u + \tau_C\tau_D},
\]

\[
\text{Hessian}_{\phi_u}(x^*, y^*) = \frac{2\tau_A\tau_C u + \tau_C\tau_D}{\tau_A\tau_C^2\tau_D u}.
\]
Applying the same reasoning as in the previous section and Lemma 1, we obtain
\[ G_n(u) \sim \frac{1}{2\sqrt{\pi n}} \gamma(u)^{-\frac{1}{4}} \exp \left( 2n \sqrt{\gamma(u)} \right), \]
where \( \gamma(u) \) has the same expression as in Section 1, applied to the particular case \( \tau_B = 0 \). From this point, the end of the proof is the same as in Section 1.

2.2 \( \tau_B = \tau_D = 0 \) and the other coefficients are positive

In this case, we have
\[ F_n(u) = e^{-n(\tau_A + \tau_C)} \sum_{a \geq 0} a \frac{(n \tau_A)^a}{a!} \frac{(n \tau_C)^c}{c!} u^a, \]
which leads to a simple sum (instead of a double or triple sum):
\[ G_n(u) = \sum_{a \geq 0} \frac{a}{a!} \frac{(n \tau_A)^a}{a!} \frac{(n \tau_C)^a}{a!} u^a. \]

As usual, we define the value \( x = \frac{a}{n} \) and obtain
\[ G_n(u) = \sum_{a \geq 0} \psi_n(x)e^{-n\phi_u(x)}, \]
where
\[ \psi_n(x, y) = \left( \frac{(n x)^x e^{-n x}}{(n x)!} \right)^2 \]
\[ \phi_u(x, y) = x(2 \log(x) - 2 - \log(\tau_A \tau_C u)). \]
The minimum of \( \phi_u \) is obtained for \( x^* = \sqrt{\tau_A \tau_C u} \), which leads to:
\[ \psi_n(x^*) \sim \frac{1}{2\sqrt{\pi n}} \frac{1}{\sqrt{\tau_A \tau_C u}}, \]
\[ \phi_u(x^*) = -2\sqrt{\tau_A \tau_C u}, \]
\[ \text{Hessian}_{\phi_u}(x^*) = \frac{2}{\sqrt{\tau_A \tau_C u}}. \]

Applying Lemma 1, we obtain
\[ G_n(u) \sim \frac{1}{2\sqrt{\pi n}} \gamma(u)^{-\frac{1}{4}} \exp \left( 2n \sqrt{\gamma(u)} \right), \]
where \( \gamma(u) \) has the same expression as in Section 1, applied to the particular case \( \tau_B = \tau_D = 0 \). We then conclude like in Section 1.

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