KLEIN BOTTLES IN LENS SPACES

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Abstract. Bredon and Wood have given a complete answer to the embeddability question for nonorientable surfaces in lens spaces. They formulate their result in terms of a recursive formula that determines, for a given lens space, the minimal genus of embeddable nonorientable surfaces. Here we give a direct proof that, amongst lens spaces as target manifolds, the Klein bottle embeds into $L(4n, 2n \pm 1)$ only. We describe four explicit realisations of these embeddings.

1. Introduction

A nonorientable closed manifold never embeds as a hypersurface into euclidean space (or a sphere). A strikingly beautiful proof of this fact is due to Samelson [11]. Although it is hardly possible to be more concise than Samelson, here is a précis of his proof.

Suppose $M \subset \mathbb{R}^n$ were a nonorientable hypersurface. Take a simple closed path in $M$ along which the orientation is reversed; then the same must be true for the coorientation. By pushing the path in the normal direction, and then joining initial and end point by a short line segment transverse to $M$, one obtains an embedded loop in $\mathbb{R}^n$ intersecting $M$ in a single point. The embedding of this loop extends to a smooth mapping $D^2 \to \mathbb{R}^n$ of a disc, which can be made (rel boundary) transverse to $M$. But then the preimage of $M$ in $D^2$ is a 1-dimensional closed submanifold with a single boundary point — contradiction. (Notice that $M$ need not be a closed manifold for us to reach this contradiction; it suffices that $M$ be a submanifold without boundary that is a closed subset of $\mathbb{R}^n$.)

Thus, it is a natural question to ask about embeddings of nonorientable surfaces into the ‘most simple’ orientable 3-manifolds beyond the 3-sphere. We take these to be 3-manifolds with a Heegaard splitting of genus 1, that is, lens spaces and $S^1 \times S^2$.

Remark 1.1. By ‘lens spaces’ we shall mean ‘honest’ lens spaces $L(p, q)$ with $p \geq 1$, which are quotients of $S^3$ under a linear $\mathbb{Z}_p$-action. In other words, we do not regard $L(0, 1) = S^1 \times S^2$ as a lens space, but instead discuss it separately.

Explicitly, $L(p, q)$ is defined, as an oriented manifold, as the quotient of $S^3 \subset \mathbb{C}^2$ under the $\mathbb{Z}_p$-action generated by

$$\sigma(z_1, z_2) = \left(e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2\right).$$

Here $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ are assumed to be coprime. Without loss of generality one may require that $1 \leq q \leq p - 1$.

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The following theorem is Corollary 6.4 in [1]. See also [2] for alternative proofs of the results of Bredon and Wood.

**Theorem 1.2.** Amongst lens spaces as target manifolds, the Klein bottle embeds precisely into $L(4n, 2n \pm 1)$, $n \in \mathbb{N}$.

Bredon and Wood derive this statement as a corollary of their much more general results about the minimal genus of a nonorientable surface that can be embedded into a given lens space. (It is clear that one can always increase the genus by 2; simply form the connected sum with a small 2-torus embedded in a ball inside the ambient 3-manifold.) They present a recursive formula that allows one to compute this minimal genus.

The generality of their argument slightly obscures the beautiful geometry behind the embeddings of Klein bottles. Our aim in this note, therefore, is to present a direct and elementary proof of Theorem 1.2. Furthermore, we describe four explicit realisations of these embeddings, some of which do not seem to have appeared in the literature before.

2. **Embeddings of the projective plane**

As a warm-up, we discuss embeddings of the projective plane $\mathbb{R}P^2$ into lens spaces. Obviously, $\mathbb{R}P^2$ embeds into $\mathbb{R}P^3 = L(2, 1)$ by inclusion.

**Proposition 2.1.** The only lens space into which $\mathbb{R}P^2$ embeds is $L(2, 1)$.

**Proof.** Let $\mathbb{R}P^2 \subset L(p, q)$ be an embedded copy of the projective plane. Since $L(p, q)$ is orientable, and $\mathbb{R}P^2$ is not, a (closed) tubular neighbourhood $\nu \mathbb{R}P^2$ is diffeomorphic to the total space of a nontrivial $I$-bundle over $\mathbb{R}P^2$, where $I$ denotes a compact interval.

There is a unique such bundle, which can be seen by regarding $\mathbb{R}P^2$ as being obtained by gluing a 2-disc $D^2$ and a Möbius band along their respective boundary circle. Over the Möbius band, the $I$-bundle must be nontrivial, and since the Möbius band retracts to a circle, this bundle is unique. The boundary circle of the Möbius band is a double cover of its spine, so there the $I$-bundle is trivial, as it is over $D^2$. The gluing of these two bundles is unique, since there are no nontrivial loops in the diffeomorphism group of $I$.

We claim that $\partial (\nu \mathbb{R}P^2)$ is a 2-sphere. To this end, think of $\mathbb{R}P^3$ as a 3-ball $D^3$ with antipodal points on $S^2 = \partial D^3$ identified. Since $S^2$ descends to an embedded $\mathbb{R}P^2$ in the quotient, we see that the total space of the unique nontrivial $I$-bundle over $\mathbb{R}P^2$ is diffeomorphic to $\mathbb{R}P^3$ with a small open 3-ball $B^3$ removed.

Now, every lens space is irreducible [6, Proposition 1.6] — that is, any embedded 2-sphere bounds a 3-ball — since it is covered by the 3-sphere, which is irreducible by Alexander’s theorem [6, Theorem 1.1].

Since $\nu \mathbb{R}P^2 \simeq \mathbb{R}P^2$ is not a ball, we deduce that the ambient space $L(p, q)$ is obtained from $\nu \mathbb{R}P^2 = \mathbb{R}P^3 \setminus B^3$ by attaching a copy of $D^3$. The result of this gluing is unique: up to homeomorphism by the Alexander trick (any homeomorphism of $S^2$ extends to a homeomorphism of $D^3$), up to diffeomorphism by Smale’s theorem [12].

We conclude that $L(p, q) = L(2, 1)$.

**Remark 2.2.** Observe that our argument has shown more than is stated in the proposition. The only information we had to use about the ambient manifold was that any separating 2-sphere bounds a 3-ball, in other words, that the ambient
manifold is prime. Thus, $\mathbb{R}P^3$ is the only orientable prime 3-manifold into which $\mathbb{R}P^2$ can be embedded. In particular, $\mathbb{R}P^2$ does not embed into $S^1 \times S^2$ either.

3. HEegaard splitting of $L(p,q)$

The following result, which is standard textbook material, is crucial for much of our discussion, so we give a brief indication how to prove this statement.

**Proposition 3.1.** Choose $r, s \in \mathbb{Z}$ such that $pr + qs = 1$. Then $L(p,q)$ has a Heegaard splitting of genus 1, where the gluing of the two solid tori is given by

\[
\begin{align*}
\mu_1 &\sim p\lambda_2 - q\mu_2, \\
\lambda_1 &\sim s\lambda_2 + r\mu_2.
\end{align*}
\]

Observe that with respect to the bases $(\mu_i, \lambda_i)$, the gluing map is described by a matrix of determinant $-1$, that is, the gluing map reverses the orientation.

**Proof of Proposition 3.1.** It is convenient to think of $S^3 \subset \mathbb{C}^2$ as the 3-sphere of radius $\sqrt{2}$, that is, $S^3 = \{|z_1|^2 + |z_2|^2 = 2\}$. A genus 1 Heegaard splitting of $S^3 \subset \mathbb{C}^2$ is then given by the solid tori $\tilde{M}_i := \{(z_1, z_2) \in S^3 : z_i \leq 1\}, i = 1, 2$. A diffeomorphism from $S^3 \times D^2$ to $\tilde{M}_2$ is defined by

\[ (e^{i\theta}, re^{iz}) \mapsto (\sqrt{2 - r^2 e^{i\theta}}, re^{iz}). \]

In terms of these coordinates, the generator $\sigma$ of the $\mathbb{Z}_p$-action as in (2) acts by

\[ \sigma|_{\tilde{M}_2} : \theta \mapsto \theta + \frac{2\pi}{p}, \quad \varphi \mapsto \varphi + \frac{2\pi q}{p}. \]

A fundamental domain for this action is $\{0 \leq \theta \leq 2\pi/p\}$, and the quotient $M_2 := \tilde{M}_2/\langle \sigma \rangle$ is again a solid torus.

The meridian $\tilde{\mu}_2$ of $\partial \tilde{M}_2$ is given by a $\varphi$-curve, and this descends to a meridian $\mu_2$ of $\partial M_2$. As longitude $\tilde{\lambda}_2$ on $\partial \tilde{M}_2$ we take a $\theta$-curve, and as longitude $\lambda_2$ on $\partial M_2$ a curve joining $(\theta_0, z_0) := (0,0)$ with $(\theta_1, \varphi_1) := (2\pi/p, 2\pi q/p)$ as shown in Figure 1. Here it is assumed that $\lambda_2$ does indeed turn through an angle $2\pi q/p$ in meridional direction, and we need not require $1 \leq q \leq p - 1$.

The horizontal line segments shown in the figure are two of the $p$ segments that make up $\tilde{\lambda}_2$, after they have all been mapped to the fundamental domain. We then see that $\lambda_2$ descends to $p\lambda_2 - q\mu_2$.

In a similar way one considers $M_1 = \tilde{M}_1/\langle \sigma \rangle$. In the parametrisation of $\tilde{M}_1$ as a solid torus,

\[ (e^{i\theta}, re^{iz}) \mapsto (re^{iz}, \sqrt{2 - r^2 e^{i\theta}}), \]

the roles of $z_1$ and $z_2$ are interchanged, so $\sigma$ acts by

\[ \sigma|_{\tilde{M}_1} : \theta \mapsto \theta + \frac{2\pi q}{p}, \quad \varphi \mapsto \varphi + \frac{2\pi q}{p}. \]

(Beware that, as before, $\varphi$ is the meridional coordinate, and $\theta$ the longitudinal one.) If one replaces the generator $\sigma$ of the $\mathbb{Z}_p$-action by $\sigma^s$, the action is described by

\[ \sigma^s|_{\tilde{M}_1} : \theta \mapsto \theta + \frac{2\pi s}{p}, \quad \varphi \mapsto \varphi + \frac{2\pi s}{p}. \]

Now the discussion continues as before, and we see that there is a choice of longitude $\lambda_1$ on $M_1$ such that $\tilde{\mu}_1$ descends to $\mu_1$, and $\tilde{\lambda}_1$ to $p\lambda_1 - s\mu_1$. 
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Figure 1. A Heegaard (solid) torus of $L(p, q)$.

In the Heegaard splitting of $S^3$, the identification of the two solid tori is determined by

$$\tilde{\mu}_1 \sim \tilde{\lambda}_2 \quad \text{and} \quad \tilde{\lambda}_1 \sim \tilde{\mu}_2.$$ 

This descends to

$$\mu_1 \sim p\lambda_2 - q\mu_2 \quad \text{and} \quad p\lambda_1 - s\mu_1 \sim \mu_2,$$

which is equivalent to the identifications in the proposition. □

4. KLEIN BOTTLES EMBED INTO $L(4n, 2n \pm 1)$ ONLY

Suppose we have an embedded copy $K \subset L(p, q)$ of the Klein bottle. Think of $K$ as being obtained by gluing two Möbius bands along their boundary. As in the preceding section, we see that the normal bundle of $K$ in $L(p, q)$ has to be the unique nontrivial $I$-bundle over either of these Möbius bands, where we take $I$ to be the interval $[-1, 1]$. The boundary (in the fibre direction) of this $I$-bundle is the orientable double cover of the Möbius band, that is, an annulus. It follows that the boundary of a tubular neighbourhood $\nu K$ is a separating 2-torus $T$, the orientable double covering of $K$.

Remark 4.1. We shall see presently that the homomorphism $\pi_1(T) \to \pi_1(\nu K)$ induced by inclusion is given by

$$\langle a, b \mid ab = ba \rangle \quad \mapsto \quad \langle u, v \mid uvu^{-1} = v^{-1} \rangle$$

$$\alpha^n\beta^\ell \quad \mapsto \quad u^{2n}v^\ell.$$

This homomorphism is injective, for otherwise there would be some relation $u^{2n}v^\ell = 1$ in $\pi_1(K)$. However, adding such a relation to the given presentation of $\pi_1(K)$ turns $u$ into a torsion element of order $2n$ for $\ell = 0$, or $v$ into a torsion element of order $2\ell$ for $\ell \neq 0$ (since $uv^\ell u^{-1} = v^{-\ell}$ by the first relation, and $uv^\ell u^{-1} = v^\ell$ by the second). This would contradict the fact that $\pi_1(K)$ is torsion-free, as indeed is the fundamental group of any finite-dimensional CW complex with a contractible universal covering space [5, Proposition 2.45].

4.1. The complement of $\nu K$. We first show that the complement of $\nu K$ in $L(p, q)$ is a solid torus. This observation has been recorded in various places in the literature, e.g. [10, Lemma 2].

Lemma 4.2. The complement $L(p, q) \setminus \text{Int}(\nu K)$ is a solid torus. In other words, $L(p, q)$ is a Dehn filling of $\nu K$.

Proof. The homomorphism $\pi_1(T) \to \pi_1(L(p, q))$ on fundamental groups induced by the inclusion $T \subset L(p, q)$ cannot be injective, so $T$ is compressible [3, Corollary 3.3],
that is, there is an embedded disc \( D \) in \( L(p, q) \) with \( D \cap T = \partial D \) a homotopically nontrivial circle on \( T \). By Remark 4.1, this disc has to lie on the other side of \( T \) than \( \nu K \).

Now thicken \( D \equiv D \times \{0\} \) to a cylinder \( Z := D \times [-\varepsilon, \varepsilon] \) with \( Z \cap T = \partial D \times [-\varepsilon, \varepsilon] \), and replace this part of \( T \) by \( D \times \{ \pm \varepsilon \} \), thus creating a 2-sphere \( S \). This process is called a surgery of \( T \) along \( D \).

In an irreducible 3-manifold (such as a lens space), this 2-sphere \( S \) bounds a ball \( B \). If \( B \cap D = \emptyset \), reversing the surgery creates a solid torus bounded by \( T \) on the other side of \( T \) than \( \nu K \), and we are done.

If \( B \) were on the side of \( S \) containing \( D \), then \( B \) would equal

\[
\nu K \cup_{\partial D \times [-\varepsilon, \varepsilon]} (D \times [-\varepsilon, \varepsilon]).
\]

In particular, this ball would contain a Klein bottle, which is impossible by Samelson’s theorem.

4.2. The fundamental group of the Dehn filling. Next, we wish to describe an explicit model for \( \nu K \) that will allow us to compute the fundamental group of the manifold obtained by any Dehn filling of \( \nu K \).

We think of \( S^1 \) as \( \mathbb{R}/2\pi \mathbb{Z} \) and write the Klein bottle as

\[
K = ([0, 1] \times S^1)/(1, \theta) \sim (0, -\theta).
\]

The tubular neighbourhood \( \nu K \) of \( K \) in any orientable 3-manifold is then given by

\[
\nu K = ([0, 1] \times S^1 \times [-1, 1])/(1, \theta, r) \sim (0, -\theta, -r),
\]

where \( K \subset \nu K \) is defined by \( \{ r = 0 \} \). Indeed, notice that \( K \) decomposes into the two Möbius bands

\[
\left( [0, 1] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \{ 0 \} \right)/\sim
\]

and

\[
\left( [0, 1] \times \left( \frac{3\pi}{2} \right) \times \{ 0 \} \right)/\sim,
\]

and over either Möbius band the \([-1, 1]-bundle \) is nontrivial.

The first homology group of \( \nu K \) is \( H_1(\nu K) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \), where the \( \mathbb{Z} \)-summand may be taken to be generated by \( ([0, 1] \times \{ 0 \} \times \{ 0 \})/\sim \), and the \( \mathbb{Z}_2 \)-summand is generated by \( \left\{ \frac{1}{2} \right\} \times S^1 \times \{ 0 \} \). Notice that the latter circle is isotopic, by sliding it into the \([0, 1]-direction\), to a copy of itself with reversed orientation, so two copies of this circle bound a cylinder.

Now consider the effect of attaching a solid torus \( S^1 \times D^2 \) to \( \nu K \). To compute the first homology of the resulting space, it suffices to consider the attaching of a meridional disc \( D := \left\{ * \right\} \times D^2 \), for the attaching of a solid torus may be thought of as an attaching of a disc, followed by the attaching of a 3-ball.

As remarked before, the boundary of \( \nu K \) is a 2-torus \( T \). In our model, this is

\[
T = \partial(\nu K) = \left( [0, 1] \times S^1 \times \{ \pm 1 \} \right)/\sim.
\]

We take \( H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \) to be generated by \( \left( [0, 1] \times \{ 0 \} \times \{ \pm 1 \} \right)/\sim \) and \( \left( \frac{1}{2} \right) \times S^1 \times \{ 1 \} \). Then the homomorphism on homology induced by the inclusion \( T \rightarrow \nu K \) is described by

\[
H_1(T) \rightarrow H_1(\nu K)
\]

\[
\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2
\]

\[
(n, \ell) \rightarrow (2n, \ell \mod 2).
\]
The attaching circle $\partial D$ of the meridional disc represents an element $(n, \ell) \in H_1(T)$, with $n, \ell$ coprime. Thus, assuming that the corresponding gluing results in a lens space $L = L(p, q)$, the Mayer–Vietoris sequence becomes

$$
\begin{array}{cccc}
H_1(\partial D) & \rightarrow & H_1(\nu K) & \rightarrow \ H_1(L) \rightarrow 0 \\
\mathbb{Z} & \rightarrow & \mathbb{Z} \oplus \mathbb{Z}_2 & \rightarrow \ \mathbb{Z}_p \rightarrow 0 \\
1 & \rightarrow & (2n, \ell \mod 2)
\end{array}
$$

It follows that $\ell$ must be odd, for the quotient $(\mathbb{Z} \oplus \mathbb{Z}_2)/\langle(2n, 0)\rangle$ equals $\mathbb{Z}_{2n} \oplus \mathbb{Z}_2$, and not $\mathbb{Z}_p$, as it should. On the other hand, we have

$$(\mathbb{Z} \oplus \mathbb{Z}_2)/\langle(2n, 1)\rangle \cong \mathbb{Z}_{4n},$$

since $\mathbb{Z} \oplus \mathbb{Z}_2$ is generated by $(2n, 1)$ and $(1, 0)$, and we have $(4n, 0) = 2(2n, 1)$. So with this attaching map the resulting manifold might indeed be a lens space, with $p = 4n$.

To determine the freedom in choosing $\ell$, we compute the fundamental group in a similar fashion. Let $u, v \in \pi_1(\nu K)$ be the elements corresponding to the very circles we chose as generators of $H_1(\nu K)$. These yield the presentation

$$\pi_1(\nu K) = \langle u, v \mid uvu^{-1} = v^{-1} \rangle,$$

and in terms of this presentation, the homomorphism $\pi_1(T) \to \pi_1(\nu K)$ induced by inclusion is as claimed in Remark 4.1. Notice that $u^2v = uv^{-1}u = vu^2$, that is, $u^2$ commutes with $v$. So the map in Remark 4.1 is indeed a group homomorphism.

Thus, $u^{2n}v^\ell$ is the class of the attaching circle $\partial D$ in $\pi_1(\nu K)$. By Seifert–van-Kampen we then have

$$\pi_1(\nu K \cup D) = \langle u, v \mid uvu^{-1} = v^{-1}, u^{2n}v^\ell = 1 \rangle.$$

Since $\ell \neq 0$, this is a presentation of the metacyclic group

$$\mathbb{Z}_{2\ell} \twoheadrightarrow G \twoheadrightarrow \mathbb{Z}_{2n},$$

which is abelian (and then cyclic of order $4n$) if and only if $\ell = \pm 1$, see [3]. The normal subgroup of order $2\ell$ is generated by $v$. Every element of $G$ can be written uniquely in the form $u^iv^k$ with $0 \leq j \leq 2n - 1$ and $0 \leq k \leq 2\ell - 1$, and the projection to $\mathbb{Z}_{2n}$ is given by $u^iv^k \mapsto j$.

To summarise, we have shown that the only Dehn fillings of $\nu K$ that might result in a lens space $L(p, q)$ are those described by an attaching map $(n, \pm 1) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(T)$, and that the resulting lens space would satisfy $p = 4n$.

4.3. Seifert fibrations. We now show that the manifolds obtained as a Dehn filling of $\nu K$ described by an attaching map $(n, \pm 1)$ are Seifert fibred in two different ways. Either fibration will subsequently allow us to show that the Dehn filling is indeed a lens space $L(4n, 2n \pm 1)$.

Observe that the map $(z_1, z_2) \rightarrow (z_1, z_2)$ induces an orientation-reversing diffeomorphism from $L(4n, 2n + 1)$ to $L(4n, 2n - 1)$. Likewise, there is an orientation-reversing diffeomorphism of the Dehn fillings corresponding to $(n, 1)$ and $(n, -1)$, respectively, extending the diffeomorphism $[(t, \theta, r)] \rightarrow [(t, -\theta, r)]$ of $\nu K$. So it suffices to consider the attaching map $(n, 1)$ only, and we may ignore questions of orientation.
4.3.1. A Seifert fibration over \( \mathbb{R}P^2(n) \). The first Seifert fibration will yield a very simple description of an embedded Klein bottle, see Section 5.1.

**Lemma 4.3.** Given an attaching map described by \( (n, 1) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(T) \), the resulting Dehn filling of \( \nu K \) has a Seifert fibration over \( \mathbb{R}P^2 \) with one singular fibre of order \( n \in \mathbb{N} \).

**Proof.** The tubular neighbourhood \( \nu K \) equals the total space of the nontrivial \( S^1 \)-bundle over the M"obius band; the bundle projection is induced by the map

\[
[0, 1] \times S^1 \times [-1, 1] \longrightarrow [0, 1] \times [-1, 1]
\]

\[
(t, \theta, r) \longmapsto (t, r),
\]

which descends to the quotient under the identification \( (1, \theta, r) \sim (0, -\theta, -r) \) and \( (1, r) \sim (0, -r) \), respectively. With our choice of generators for \( H_1(T) \), the class of the \( S^1 \)-fibre \( \{\frac{1}{2}\} \times S^1 \times \{1\} \) in \( \partial(\nu K) = T \) is \( (0, 1) \in \mathbb{Z} \oplus \mathbb{Z} \). Notice that the restriction of the \( S^1 \)-bundle to the boundary of the M"obius band is the trivial bundle \( T \to [0, 1] \times \{ \pm 1 \}/\sim \).

We now look at the gluing map \( \partial(S^1 \times D^2) \to T \). By assumption, the meridian \( \mu := \{\ast\} \times \partial D^2 \subset \partial(S^1 \times D^2) \) is mapped to a curve in \( T \) representing the class \( (n, 1) \in H_1(T) \). The longitude \( \lambda := S^1 \times \{\ast\} \subset \partial(S^1 \times D^2) \) of the solid torus we glue in has to map to a curve that forms a basis of \( H_1(T) \) together with the image of \( \mu \); the most simple choice is \( \lambda \sim (1, 0) \).

With these choices, the fibre class \( (0, 1) \) is identified with \( \mu - n\lambda \). The foliation of \( \partial(S^1 \times D^2) \) by simple closed curves in this class \( \mu - n\lambda \) extends radially in the obvious fashion to a Seifert fibration of \( S^1 \times D^2 \) over \( D^2 \) with one singular fibre (the spine \( S^1 \times \{0\} \)) of order \( n \).

In [4] it was shown that these Seifert fibred manifolds are precisely the lens spaces \( L(4n, 2n \pm 1) \).

4.3.2. A Seifert fibration over \( S^2(2, 2) \). The second Seifert fibration will be shown to translate into a genus 1 Heegaard splitting of the manifold, from which one can read off directly that the Dehn filling is \( L(4n, 2n \pm 1) \).

**Lemma 4.4.** The Dehn filling of \( \nu K \) resulting from an attaching map described by \( (n, 1) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(T) \) has a Seifert fibration over \( S^2 \) with two singular fibres of order 2.

**Proof.** In the model

\[
\nu K = \left( [0, 1] \times S^1 \times [-1, 1] \right) / \sim (0, -\theta, -r),
\]

the pairs of intervals \( [0, 1] \times \{\theta\} \times \{r\} \) and \( [0, 1] \times \{-\theta\} \times \{-r\} \), for \( (\theta, r) \notin \{(0, 0), (\pi, 0)\} \), define circles of length 2. The two exceptional intervals \( [0, 1] \times \{0\} \times \{\pi\} \) and \( [0, 1] \times \{\pi\} \times \{0\} \) define circles of length 1. This foliation by circles defines a Seifert fibration of \( \nu K \) with quotient

\[
S^1 \times [-1, 1] / \sim (-\theta, -r),
\]

which equals \( D^2(2, 2) \), the disc with two orbifold points of order 2, see Figure 2.

On the boundary \( T \) of \( \nu K \), the Seifert fibres lie in the class \( (1, 0) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(T) \). As seen in the proof of Lemma 4.3 this class becomes identified with the longitude \( \lambda \) of the solid torus producing the Dehn filling, so the Seifert fibration extends as the product fibration of \( S^1 \times D^2 \). Therefore, the Dehn filling results in a Seifert fibration over \( S^2(2, 2) \). 

\( \square \)
Remark 4.5. More generally, it is shown in [10] that the irreducible 3-manifolds with finite fundamental group that contain Klein bottles are precisely the Seifert fibrations over $S^2$ with at most three singular fibres of multiplicity $2, 2, p$, respectively, for some $p \in \mathbb{N}$.

The Seifert fibration, restricted to two hemispheres $D^2(2)$ of $S^2$, each containing one of the two orbifold points, defines a Heegaard splitting of genus 1. This shows that the Dehn filling produces a lens space, which we now want to identify.

Proposition 4.6. The Dehn filling of $\nu K$ with the meridian of the solid torus glued to $(n, 1) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(T)$ is the lens space $L(4n, 2n \pm 1)$.

Proof. In $\nu K$ we consider tubular neighbourhoods $V_1, V_2$ of the two singular fibres over the orbifold points $(0, 0)$ and $(\pi, 0)$ in $S^1 \times [-1, 1]/\sim$. In Figure 3 these tubular neighbourhoods are represented as holes in a slice $\{t_0\} \times S^1 \times [-1, 1]$, invariant under the action $(\theta, r) \mapsto (-\theta, -r)$ on $S^1 \times [-1, 1]$. Here the outer boundary of the annulus is given by $\{r = 1\}$, the inner boundary, by $\{r = -1\}$, and the angular coordinate $\theta$ is the usual one in the euclidean plane.
The complement of $V_1 \cup V_2$ in $\nu K$, together with the solid torus $S^1 \times D^2$ producing the Dehn filling, is a thickened 2-torus. In order to identify the lens space resulting from the Heegaard splitting, it suffices to consider a meridian $\mu_1$ on $\partial V_1$ (in Figure 3 on the right), and isotope it in this thickened 2-torus to a curve on $\partial V_2$. Notice that we may think homologically, since two simple closed curves on a surface are isotopic if and only if they are homotopic (Baer’s theorem); and on a 2-torus, with abelian fundamental group, homotopy of curves equals homological equivalence.

In $\nu K \setminus (V_1 \cup V_2)$, the meridian $\mu_1$ is isotopic to the curve $\mu_1'$ as shown. Notice that the two oriented circles in $\mu_1'$ on the boundary $T$ of $\nu K$ correspond to the same class $(0, 1) \in H_1(T)$. On $\partial(S^1 \times D^2)$, each of these curves becomes identified with $\mu - n\lambda$, see the proof of Lemma 4.8. Since we are now allowed to isotope the curves over $S^1 \times D^2$, these are isotopic to two copies of $-n\lambda$. The class $-2n\lambda$, in turn, is identified with $(-2n, 0) \in H_1(T)$.

Next we need to recall that the class $(1, 0) \in H_1(T)$ is represented by a regular Seifert fibre. In the cylinder over the disc with three holes shown in Figure 3 this corresponds to two intervals $[0, 1] \times \{\theta_0\} \times \{r_0\}$ and $[0, 1] \times \{-\theta_0\} \times \{-r_0\}$.

In $\nu K \setminus (V_1 \cup V_2)$, any two such Seifert fibres are isotopic. On $\partial V_i$, this is a curve going twice in longitudinal direction, and we may choose the longitude $\lambda_2$ on $\partial V_2$ such that this becomes $2\lambda_2 + \mu_2$, since the $V_i$ are cylinders over a disc with bottom and top glued by a rotation of the disc through an angle $\pi$.

In conclusion, the two circles $\mu_1' \cap T$ are isotopic in the thickened 2-torus to curves on $\partial V_2$ representing (in total) the class $-2n(2\lambda_2 + \mu_2)$. In addition, as we see from Figure 3 there is a copy of $-\mu_2$ in $\mu_1'$. Thus, the Heegaard splitting of the lens space resulting from the Dehn filling of $\nu K$ has the identification

$$\mu_1 \sim -\mu_2 - 2n(2\lambda_2 + \mu_2) = -4n\lambda_2 - (2n + 1)\mu_2,$$

which is the description of $L(4n, -(2n + 1)) = L(4n, 2n - 1)$. □

5. Explicit embeddings of Klein bottles

Before we turn to lens spaces, for completeness we record a simple construction of an embedding of the Klein bottle in $S^1 \times S^2$. One of the four explicit embeddings in $L(4n, 2n \pm 1)$ will be based on the same idea.

Example 5.1. An embedding of the Möbius band into a solid torus $S^1 \times D^2$ with boundary mapping to $2\lambda \pm \mu$ is given by

$$([(0, 1] \times [-1, 1])/(1, r) \sim (0, -r) \quad \mapsto \quad (t, r) \quad \mapsto \quad (e^{2\pi it}, r e^{\pm \pi it})$$

see Figure 4 which shows the embedding with boundary curve $2\lambda + \mu$. The manifold $S^1 \times S^2$ is obtained from two copies of $S^1 \times S^2$ via the identification $\mu_1 \sim \mu_2$ and $\lambda_1 \sim \lambda_2$. Thus, two Möbius bands embedded as described (with the same choice of sign) will glue together to yield a Klein bottle in $S^1 \times S^2$.

Next we describe four explicit realisations of an embedding $K \subset L(4n, 2n \pm 1)$. We mention in passing that all embeddings of $K$ in a given lens space $L$ are in fact isotopic, as was proved by Rubinstein [9]. A simple homological argument, using the fact that $H_3(L) = 0$, shows that an embedded nonorientable surface of minimal genus must be incompressible, and then [9, Theorem 12] contains the isotopy statement.
5.1. **Embedding into a Seifert fibration.** In Section 4.3.1 we observed that an embedded Klein bottle in a lens space leads to a Seifert fibration over $\mathbb{RP}^2(n)$, the projective plane with one orbifold point of order $n$, for some $n \in \mathbb{N}$. This argument can be reversed.

From [4] we know that the lens spaces $L(4n, 2n \pm 1)$ admit a Seifert fibration over $\mathbb{RP}^2(n)$. Restricted to a simple closed curve in $\mathbb{RP}^2(n)$ (disjoint from the orbifold point) along which the orientation of the projective plane is reversed, this Seifert bundle is the nontrivial $S^1$-bundle over $S^1$, which is a Klein bottle.

5.2. **Embedding two Möbius bands.** According to Proposition 3.1, one can find a Heegaard splitting of $L(4n, 2n \pm 1)$ with $r = n$ and $s = -(2n \mp 1)$ in (2). Then

$$2\lambda_1 + \mu_1 \sim 2(-(2n \mp 1)\lambda_2 + n\mu_2) + (4n\lambda_2 - (2n \pm 1)\mu_2) = \pm(2\lambda_2 - \mu_2).$$

Hence, if we embed a Möbius band in either Heegaard torus as in Example 5.1, but now with the opposite choice of signs, they glue to a Klein bottle in the lens space.

5.3. **Embedding a handle decomposition.** The Klein bottle has a handle decomposition with one 0-handle, two twisted 1-handles, and a single 2-handle. One may now try to place the 0-handle as a meridional disc in one Heegaard torus of $L(4n, 2n \pm 1)$, the 1-handles on the boundary 2-torus, and complete with a meridional disc in the complementary Heegaard torus.

For the lens space $L(4, 1)$, such an embedding has been described in [8, p. 460]. For $L(4n, 2n \pm 1)$ with $n \geq 2$ one needs to modify the above idea. After attaching the first 1-handle, one has to push the handlebody slightly into the Heegaard torus, keeping its boundary fixed on the splitting torus. Only then one can attach the second 1-handle. We shall presently describe this in detail.

A similar description can be found in the unpublished note [7] by M. Iwakura. Beware that on the arXiv this paper is listed under the title ‘Geometrically incompressible non-orientable closed surfaces in lens spaces’, but the file that opens actually carries the title ‘Non-orientable fundamental surfaces in lens spaces’, as listed in our references. There is a published paper carrying that second title, under the joint authorship of Iwakura with C. Hayashi; this is not the paper we are referring to. The paper [14] seems to contain similar ideas, but not the construction we are about to present.

We illustrate the splitting 2-torus in the Heegaard decomposition of $L(4n, 2n \pm 1)$ by a square with opposite sides identified. The horizontal direction corresponds to

\[
\text{Figure 4. A Möbius band in a solid torus with boundary } 2\lambda + \mu.
\]
the meridional direction $\mu_2$; the vertical, to $\lambda_2$. First we attach a twisted 1-handle to a meridional disc in the Heegaard torus $M_2$ as shown in Figure 5. The horizontal line in the centre of the square represents a meridian, i.e. the boundary of a meridional disc. The grey band is the 1-handle attached to this disc. (In spite of the optical illusion, the segments where the 1-handle intersects the top and the bottom of the square really do match.) You may also want to take a peek at Figure 8.

\[ \mu_2 \]
\[ \lambda_2 \]
\[ \mu_2 \]

Figure 5. The first 1-handle on the splitting 2-torus.

The second 1-handle we draw as a single curve; imagine this curve as being thickened into a band. For $L(4,3)$ we attach the second 1-handle as shown in Figure 6. The boundary of the resulting 1-handlebody is a curve that intersects $\mu_2$ in four points, and $\lambda_2$ in three. Taking orientations into account, this shows that the boundary curve lies in the class $4\lambda_2 - 3\mu_2$, which by (2) becomes identified with $\mu_1$. So we can complete the 1-handlebody to a Klein bottle by attaching a meridional disc of $M_1$ as a 2-handle.

For $L(4n, 2n + 1)$ with $n \geq 2$ we need to modify this construction. Again we start with the 1-handlebody obtained by attaching a single twisted 1-handle to a meridional disc in $M_2$. Next we push the interior of this 1-handlebody slightly into $M_2$, keeping the boundary curve on $\partial M_2$ fixed. This will allow us to attach a second 1-handle, lying entirely in $\partial M_2$, as long as we stay away from the boundary curve of the handlebody made up of the meridional disc and the first 1-handle, except at the ends of the second 1-handle, which we attach to the meridional disc. For $L(4n, 2n+1)$ we take the second 1-handle as shown in Figure 7, which illustrates the general principle by the case $n = 5$. Notice that the second 1-handle passes $n - 1$ times over the (original) first 1-handle.

Now the boundary curve of the resulting 1-handlebody intersects $\mu_2$ in $2 + 1 + 2(n - 1) + 1 + 2(n - 1) = 4n$ points, and $\lambda_2$ in $2 + 1 + 2(n - 1) = 2n + 1$ points, so it lies in the class $4n\lambda_2 - (2n + 1)\mu_2$. Again, this is the class of $\mu_1$.

For $L(4n, 2n - 1)$, $n \in \mathbb{N}$, one can draw similar pictures, or one appeals to the orientation-reversing diffeomorphism from $L(4n, 2n + 1)$ to $L(4n, 2n - 1)$. A 3-dimensional visualisation of the 1-handlebody in $L(4,1)$ is shown in Figure 8.
5.4. Embedding into a lens model. Before we specialise to $L(4n, 2n \pm 1)$, we want to describe a fundamental domain for the $\mathbb{Z}_p$-action on the unit sphere $S^3 \subset \mathbb{C}^2$ generated by $\sigma$ as in (1), producing the quotient $L(p, q)$.

In $S^3 \subset \mathbb{C}^2$ we define the 2-disc

$$D := \left\{ (\sqrt{1-r^2}, re^{i\varphi}) : r \in [0, 1], \varphi \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$ 

The images of $D$ under the action of $\mathbb{Z}_p$ are discs that share the boundary with $D$, and which are characterised by $\arg(z_1)$ being an integer multiple of $2\pi/p$. As fundamental domain for the action we may take the ‘lens’

$$B := \{ (z_1, z_2) \in S^3 : \arg(z_1) \in [0, 2\pi/p] \},$$
which is a homeomorphic copy of the 3-ball with boundary sphere $D \cup \partial D \sigma(D)$. The lens space $L(p, q)$ is obtained by identifying points $x \in D$ on the lower hemisphere with $\sigma(x)$ on the upper hemisphere $\sigma(D)$. This boundary identification amounts to a rotation of the lower hemisphere through $2\pi q/p$ followed by vertical projection onto the upper hemisphere.

We now specialise to $p = 4n$, $q = 2n \pm 1$. As a model for the Klein bottle we take

$$K = \left[0, \frac{\pi}{2n}\right] \times [0, \pi] / \sim,$$

where $\sim$ denotes the boundary identification of the rectangle by

$$(\varphi, 0) \sim (\varphi, \pi) \quad \text{and} \quad (0, \theta) \sim \left(\frac{\pi}{2n}, \pi - \theta\right).$$

The embedding of the rectangle into $S^3$,

$$\iota: \left[0, \frac{\pi}{2n}\right] \times [0, \pi] \rightarrow S^3$$

$$(\varphi, \theta) \mapsto \left(\sin \theta e^{i\varphi}, \cos \theta e^{\pm i\varphi}\right),$$

sends the rectangle to the fundamental domain $B$, and its boundary to $\partial B$; see Figure 9 which shows the situation for $L(8, 3)$.
This embedding descends to an embedding of $K$ into $L(4n, 2n \pm 1)$, since
\[
\iota(\varphi, \pi) = (0, -e^{\pm i\varphi}) = \sigma^{2n}(0, e^{\pm i\varphi}) = \sigma^{2n}\iota(\varphi, 0)
\]
and
\[
\iota\left(\frac{\pi}{2n}, \pi - \theta\right) = (\sin \theta e^{\pi i/2n}, -\cos \theta e^{\pm \pi i/2n}) = \sigma(\sin \theta, \cos \theta) = \sigma\iota(0, \theta).
\]

**Remark 5.2.** In [13] this embedding is related explicitly to the embedding into the Seifert fibration and described in terms of the stereographic projection of $S^3$ to $\mathbb{R}^3$.

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**References**

[1] G. E. Bredon and J. W. Wood, Non-orientable surfaces in orientable 3-manifolds, *Invent. Math.* 7 (1969), 83–110.

[2] W. End, Nonorientable surfaces in 3-manifolds, *Arch. Math. (Basel)* 59 (1992), 173–185.

[3] H. Geiges and C. Lange, Seifert fibrations of lens spaces, *Abh. Math. Semin. Univ. Hambg.* 88 (2018), 1–22.

[4] H. Geiges and C. Lange, Correction to: Seifert fibrations of lens spaces, *Abh. Math. Semin. Univ. Hambg.* 91 (2021), 145–150.

[5] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[6] A. Hatcher, *Notes on Basic 3-Manifold Topology* (2007).

[7] M. Iwakura, Non-orientable fundamental surfaces in lens spaces, arXiv:0903.4614v2.

[8] A. S. Levine, D. Ruberman and S. Stolz, Nonorientable surfaces in homology cobordisms, *Geom. Topol.* 19 (2015), 439–494.

[9] J. H. Rubinstein, One-sided Heegaard splittings of 3-manifolds, *Pacific J. Math.* 76 (1978), 185–200.

[10] J. H. Rubinstein, On 3-manifolds that have finite fundamental group and contain Klein bottles, *Trans. Amer. Math. Soc.* 251 (1979), 129–137.

[11] H. Samelson, Orientability of hypersurfaces in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* 22 (1969), 301–302.

[12] S. Smale, Diffeomorphisms of the 2-sphere, *Proc. Amer. Math. Soc.* 10 (1959), 621–626.

[13] N. Thies, Kleinische Flaschen in Linsenräumen, B.Sc. thesis, Universität zu Köln (2022).

[14] C. M. Tsau, A note on incompressible surfaces in solid tori and lens spaces, in: *Knots 90 (Osaka, 1990)*, de Gruyter, Berlin (1992), 213–229.

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