Deep-inelastic structure functions: Reconstruction from Mellin moments

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We summarize the method of calculating Mellin moments of deep-inelastic structure functions in perturbative QCD. We briefly discuss all steps to a complete analytical reconstruction of the perturbative corrections in x-space.

1. INTRODUCTION

The precision measurement of structure functions and their scale evolution in deep inelastic scattering is of basic importance for testing perturbative QCD. By now, experimental data allows for an accurate determination of the parton distribution functions, which serve as input for numerous other hard scattering processes. Therefore, in particular with respect to the upcoming LHC experiments, it is highly desirable to further reduce the theoretical uncertainties on these very fundamental quantities. This task requires the calculation of higher order perturbative QCD corrections. In order to perform a consistent next-to-next-to-leading order analysis, which is expected to significantly reduce the theoretical uncertainties, one has to calculate the still unknown anomalous dimensions of deep-inelastic structure functions at three loops.

The method of calculating Mellin moments of deep-inelastic structure functions is directly connected with the early days of QCD. Since then, it has been applied several times to determine higher order perturbative corrections, some of them related to sum rules. The method rests on the operator product expansion (OPE), which allows the calculation of either all even or all odd integer moments of the anomalous dimensions and coefficient functions. This information by itself then uniquely determines the quantity under consideration for general N in Mellin space or, equivalently its corresponding expression in x-space.

In the following, we will summarize the calculation of the anomalous dimensions and coefficient functions in terms of harmonic sums and the subsequent analytical reconstruction based on an inverse Mellin transformation to harmonic polylogarithms in x-space. This procedure as detailed in has recently been used to check the original calculation of the two-loop coefficient functions, which was performed with conventional techniques. Currently, work is in progress to obtain the anomalous dimensions and coefficient functions at three loops by this method.

The present approach of analytical reconstruction nicely complements the use of Bernstein polynomials for a direct fit of Mellin moments to experimental data. The latter method is particularly useful if only numerical results for a finite number of fixed moments are available, as it is presently the case.

2. METHOD

Deep-inelastic structure functions are defined by the commutator of two local electroweak currents \( j(y) \) and \( j(z) \), sandwiched between hadronic states and Fourier transformed into momentum space. In the Bjorken limit, for \( Q^2 \to \infty \) and \( x \) fixed, the OPE allows to express this current product in an asymptotic expansion around the lightcone \( (y-z)^2 \sim 0 \) into a series of local composite flavour non-singlet quark \( O^q \), singlet quark \( O^q \) and gluon operators \( O^g \) of leading twist and spin \( N \). The same OPE also holds for the forward Compton amplitude \( T \) of boson-hadron scattering. Standard perturbation theory applies to \( T \), which can be written as a series in \( 1/x^N \) in the Euclidean region. Then, by means of the optical theorem one obtains a direct relation between the parameters of the OPE and the Mellin moments.
of the structure functions. For $F_2$ we can write

$$F_2^N(Q^2) = \frac{1}{x} \int_0^x x^{N-2} F_2(x, Q^2)$$  \hspace{1cm} \text{(1)}$$

and similar relations define $F_3^N$ and $F_L^N$. Here $C_{2,j}^N$ denote the coefficient functions and $A_{p,N}^{\mu}$ the spin averaged hadronic matrix elements of the operators $O^p$, $O^q$ and $O^g$.

In the dispersive approach the derivation of eq.(1) uses symmetry properties of the Compton amplitude $T$ under exchange $x \rightarrow -x$. As a consequence, eq.(1) is restricted to only either the even or the odd Mellin moments of $F_2$. By analytic continuation on the other hand, all moments in the complex $N$-plane are fixed if the infinite set of either even or odd moments is known. To be precise, for unpolarized electron-proton scattering eq.(1) determines the even moments of $F_2^p$, while for neutrino-proton scattering the odd moments of $F_2^p$-$\overline{p}$ and $F_3^p$+$\overline{p}$ and the even moments of $F_2^p$+$\overline{p}$ and $F_3^p$-$\overline{p}$ are fixed.

The coefficient functions and the renormalized operator matrix elements in eq.(1) both satisfy renormalization group equations. Due to current conservation they are governed by the same anomalous dimensions,

$$\sum_{k=\alpha,q,g} \left[ \left\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s(\mu^2)) \frac{\partial}{\partial \alpha_s(\mu^2)} \right\} \delta_{jk} + \gamma_{jk}(\alpha_s(\mu^2)) \right] A_{p,N}^{\mu}(\mu^2) = 0,$$

$$\sum_{k=\alpha,q,g} \left[ \left\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s(\mu^2)) \frac{\partial}{\partial \alpha_s(\mu^2)} \right\} \delta_{jk} - \gamma_{jk}(\alpha_s(\mu^2)) \right] C_{2,k}^N \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = 0,$$

where $j = \alpha, q, g$, while $\beta$ and $\gamma_{jk}$ represent the QCD $\beta$-function and the anomalous dimensions. Both are calculable order by order in $\alpha_s$ in perturbative QCD as well as the coefficient functions $C_{2,j}^N$.

The anomalous dimensions $\gamma_{jk}$ in eqs.(2), (3) determine the scale evolution of deep-inelastic structure functions and, dependent on the particular scattering process, one considers operator matrix elements corresponding to the quark flavour non-singlet and singlet distributions $q_{ns}^\pm$, $q_{ns}^V$ and $q_s$. Their scale evolution is governed by linear combinations,

$$q_{ns}^\pm \rightarrow \gamma_{qq}^V q_{qq}^V + \gamma_{qq}^S q_{qq}^S$$  \hspace{1cm} \text{(4)}$$

$$q_{ns}^V \rightarrow \gamma_{qq}^V - \gamma_{qq}^S + n_f \left( \gamma_{qq}^S - \gamma_{qq}^S \right)$$  \hspace{1cm} \text{(5)}$$

$$q_s \rightarrow \gamma_{qq}^V + \gamma_{qq}^S + n_f \left( \gamma_{qq}^S + \gamma_{qq}^S \right)$$  \hspace{1cm} \text{(6)}$$

and the flavour singlet quark distribution $q_s$ mixes with the gluon distribution.

The actual calculation of the anomalous dimensions $\gamma_{jk}$ and the coefficient functions $C_{2,j}^N$ is performed in dimensionally regularized perturbation theory with the external hadron states replaced by quark and gluon states of momentum $p$. By means of the OPE, the forward Compton amplitude for the scattering off a parton can be expressed in terms of the coefficient functions of eq.(1) and renormalized partonic operator matrix elements $A_{p,N}^k$. To be specific, we write for the flavour singlet case,

$$T_{2,p}(x, Q^2, \alpha_s, \epsilon) = \sum_{N, j,k=\alpha,q,g} \left( \frac{1}{2x} \right)^N C_{2,j}^N \left( \frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right)$$

$$\times Z_{jk} \left( \alpha_s, \frac{1}{\epsilon} \right) A_{p,N}^k \left( \alpha_s, \mu^2, \epsilon \right) + O(p^2),$$

where $p = q, g$ and $O(p^2)$ denotes higher twist contributions. The $Z_{jk}$ represent the matrix of renormalization factors which contain all poles in $1/\epsilon$. They relate bare to renormalized operators according to $O^{\text{bare}} = Z O^{\text{ren}}$ and provide the anomalous dimensions via,

$$\gamma = \left( \frac{d}{d \ln \mu^2} Z \right) Z^{-1}.$$

The gauge invariant operators $O^q$ and $O^g$ in eq.(6) mix under renormalization with unphysical operators \cite{26}, \cite{27}, which are BRST variations of some operators or else disappear by use of the equations of motion. However, if one calculates quantities related to physical $S$-matrix elements with physical polarization and on-shell momenta.
such as in eq.\((\ref{eq:1})\), these unphysical operators vanish and have therefore been omitted in eq.\((\ref{eq:1})\).

The coefficient function $C_{2,j}^N$ and the renormalization factors $Z^{jk}$ are calculated from eq.\((\ref{eq:1})\) using the method of projection \cite{28,29}. This method relies on dimensional regularization and amounts to the application of the projection operator,

$$
P_N \equiv \left[ \frac{q^{(\mu_1 \cdots \mu_N)}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right]_{p=0}, \quad (9)
$$

where $q^{(\mu_1 \cdots \mu_N)}$ is symmetrical and traceless part of the tensor $q^{\mu_1 \cdots q^{\mu_N}}$.

It is obvious, that on the right hand side of eq.\((\ref{eq:1})\) the $N$-th order differentiation in $P_N$ projects on the precisely the $N$-th moment, which is the coefficient of $1/(2\pi)^N$ being $(q^2/Q^2)^N$. All higher powers of $p \cdot q/Q^2$ vanish after nullifying the momentum $p$. Furthermore, $P_N$ does not act on the renormalization constants $Z^{jk}$ and the coefficient functions being only functions of $N$, $\alpha_s$ and $\epsilon$. However, the nullification of the momentum $p$ in $P_N$ effects the partonic matrix elements $A_{\mu \nu N}^k$. There, it eliminates all diagrams containing loops, as they become massless tadpole diagrams, which are put to zero in dimensional regularization. Only the tree level operator matrix elements $A_{\mu \nu N}^{\text{tree}}$ survive.

Additionally in the flavour singlet case this decouples the operator mixing. Thus, eq.\((\ref{eq:1})\) provides two independent identities to separately determine $Z^{qg}$, $Z^{gq}$ and $Z^{gg}$, $Z^{qs}$. We should be aware however, that an $l$-loop calculation eq.\((\ref{eq:1})\) alone does not suffice to give the full information about the renormalization constants $Z^{qg}$ and $Z^{qs}$. They are determined only up to $(l-1)$-loops. This limitation is due to the gluonic coefficient function $C_{2,g}^N$ being zero at tree level, as photons couple directly to quarks only. To resolve this situation and to extract the anomalous dimension $\gamma_{qg}$ and $\gamma_{gs}$ to $l$-loops we also calculate Green’s functions in which the photon is replaced by an external scalar particle $\phi$ that couples directly only to gluons \cite{15}. These Green’s functions are obtained from a gauge invariant interaction term $\phi F_{\mu \nu} F^{\mu \nu}$ in the QCD Lagrangian, $F_{\mu \nu}$ being the QCD field strength. They can be expressed in partonic invariants $T_{\phi,p}$ and satisfy an OPE similar to eqs.\((\ref{eq:1}), \ref{eq:1}\) with the same singlet operators $O_q^i$ and $O_g^i$ but with different coefficient functions $C_{\phi,p}^N$, where the gluonic coefficient function $C_{\phi,g}^N$ starts now already tree level. Thus the invariants $T_{\phi,p}$ provide us with the necessary renormalization constants $Z^{qg}$ and $Z^{qs}$ of the singlet operators to $l$-loops and complete the determination of all anomalous dimensions in the flavour singlet case to the desired order.

Use of the method of projection to calculate all Feynman diagrams contributing to the forward partonic Compton amplitude, that is to say the application of the projection operator $P_N$ to the integrands on the left hand side of eq.\((\ref{eq:1})\) proceeds as follows. All ultraviolet divergences as well as those of infrared origin as the momentum $p \to 0$ are dimensionally regularized. At the same time, the nullification of $p$ drastically simplifies the topological structure of the corresponding graphs, which is reduced to the type of self-energy diagrams. Although the latter appear with symbolic powers of scalar products in the numerator and denominator, they are by far easier to calculate.

In practice \cite{19,20} this is performed by means of integration-by-parts identities and scaling identities for the $N$-th Mellin moment of the diagram to be calculated. Combined these identities give rise to difference equations, which relate the $N$-th Mellin moment of the diagram under consideration to lower Mellin moments and diagrams of a simpler topology. Thereby they introduce a natural hierarchy in the set of diagrams. All difference equations can be summed recursively and the answer is expressible in terms of a basis of harmonic sums \cite{14,17} of a given weight. The whole approach exploits powerful summation algorithms \cite{16} for a large class of single parameter sums which are reducible to the basis of harmonic sums.

As a demonstration of the method discussed thus far and as a check on \cite{21,22}, we have performed a recalculation of the two-loop coefficient functions. The contributing Feynman diagrams have been generated with QGRAF \cite{30} and all recursion relations for the evaluation of the individual topologies have been programmed.
in FORM \cite{31}. The nested sums which one encounters in this way are solved with the SUMMER algorithm \cite{13} in terms of the basis of harmonic sums. In addition, there is the possibility to perform checks at all stages of the calculation by means of the standard MINCER routine \cite{32}. This last feature is very important from a practical point of view as the debugging is greatly simplified.

To choose a specific example, we consider the case of initial state quarks in $D = 4 - 2\epsilon$ dimensions and calculate with the help of eqs.\((4)\)–\((6)\) the parts, allowing for electroweak interactions, one can by means of the standard MINCER routine \cite{32}.

To perform checks at all stages of the calculation, the quantities of interest are defined for either even or odd integer values in $x$-space, which then serves as an analytical continuation to all non-negative integer values in $N$.

In particular, we are then ready to determine individually all anomalous dimensions which control the scale evolution of the structure functions in the flavour non-singlet case as entering in eqs.\((4)\)–\((6)\). This is easily done now, as we obtain from the even moments of $F_{2}^{P}$ the sums $\gamma_{qq}^{V}$ and $\gamma_{qq}^{+S}$. The odd moments of $F_{2}^{P}$ on the other hand determine the differences $\gamma_{qq}^{-V}$ and $\gamma_{qq}^{-S}$, the latter being zero up to two loops. As emphasized above, the analytical continuation provides us with expressions valid for all non-negative integer values of $N$, such that at this point we can simply take the sums and the differences of $\gamma_{qq}^{+V}$ and $\gamma_{qq}^{+S}$.
and $\gamma_{qq}^{V}$, $\gamma_{q\bar{q}}^{V}$, $\gamma_{qq}^{S}$ and $\gamma_{q\bar{q}}^{S}$ of eqs. (3), (4).

Similar reconstructions in Mellin space and $x$-space may be carried out for the two loop coefficient functions in eqs. (13), (14). To summarize, we find complete agreement with the published results for the anomalous dimensions [5–10,33,34] and for the coefficient functions [21–24].

3. CONCLUSION

The calculation of higher order perturbative corrections to deep-inelastic structure functions is vitally important to improve quantitative predictions for many hard scattering processes in QCD.

In the past, this has been done either in $x$-space or in Mellin space. Using the method of projection, which enables us to obtain anomalous dimensions and coefficient functions at the same time, we perform the calculation in Mellin space. Due to new insight into the mathematical properties of harmonic sums and their interplay with harmonic polylogarithms, we are first of all able to solve all nested sums in Mellin space. Secondly we can reconstruct the complete analytical expressions of the results in $x$-space by means of an inverse Mellin transformation. Thereby we exploit all advantages of working in Mellin space, such as the use of difference equations in the Mellin moment $N$ to solve the resulting loop integrals and the possibility for independent checking with the MINCER routine.

Finally, we would like to remark, that the method presented allows for a direct application to the calculation of the three loop anomalous dimensions, as well as a generalization to polarized deep inelastic scattering [35].

REFERENCES

1. D. J. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973).
2. D. J. Gross and F. Wilczek, Phys. Rev. D9, 980 (1974).
3. H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
4. W. L. van Neerven and A. Vogt, Nucl. Phys. B568, 263 (2000), hep-ph/9907472.
5. E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. B129, 66 (1977).
6. E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. B139, 545 (1978).
7. E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. B152, 493 (1979).
8. A. Gonzalez-Arroyo, C. Lopez, and F. J. Yndurain, Nucl. Phys. B153, 161 (1979).
9. A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B166, 429 (1980).
10. C. Lopez and F. J. Yndurain, Nucl. Phys. B183, 157 (1981).
11. D. I. Kazakov and A. V. Kotikov, Nucl. Phys. B307, 721 (1988).
12. D. I. Kazakov and A. V. Kotikov, Nucl. Phys. B345, 299 (1990), Erratum.
13. S. A. Larin, F. V. Tkachev, and J. A. M. Vermaseren, Phys. Rev. Lett. 66, 862 (1991).
14. S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B259, 345 (1991).
15. S. A. Larin, P. Nogueira, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. B492, 338 (1997), hep-ph/9605317.
16. J. A. M. Vermaseren, Int. J. Mod. Phys. A14, 2037 (1999), hep-ph/9806280.
17. J. Blumlein and S. Kurth, Phys. Rev. D60, 014018 (1999), hep-ph/9810214.
18. E. Remiddi and J. A. M. Vermaseren, (1999), hep-ph/9905237.
19. S. Moch and J. A. M. Vermaseren, Nucl. Phys. B573, 853 (2000), hep-ph/9912353.
20. J. A. M. Vermaseren and S. Moch, (2000), hep-ph/0004233, these proceedings.
21. W. L. van Neerven and E. B. Zijlstra, Phys. Lett. B272, 127 (1991).
22. E. B. Zijlstra and W. L. van Neerven, Phys. Lett. B273, 476 (1991).
23. E. B. Zijlstra and W. L. van Neerven, Nucl. Phys. B383, 525 (1992).
24. E. B. Zijlstra and W. L. van Neerven, Phys. Lett. B297, 377 (1992).
25. J. Santiago and F. J. Yndurain, Nucl. Phys. B563, 45 (1999), hep-ph/9904344.
26. R. Hamberg and W. L. van Neerven, Nucl. Phys. B379, 143 (1992).
27. Y. Matiounine, J. Smith, and W. L. van Neerven, Phys. Rev. D57, 6701 (1998), hep-ph/9801224.
28. S. G. Gorishnii, S. A. Larin, and F. V. Tkachev, Phys. Lett. 124B, 217 (1983).
29. S. G. Gorishnii and S. A. Larin, Nucl. Phys. B283, 452 (1987).
30. P. Nogueira, J. Comput. Phys. 105, 279 (1993).
31. J. A. M. Vermaseren, *Symbolic Manipulation with FORM* (Computer Algebra Nederland, Kruislaan 413, 1098 SJ Amsterdam, 1991), ISBN 90-74116-01-9.
32. S. A. Larin, F. V. Tkachev, and J. A. M. Vermaseren, NIKHEF-H-91-18.
33. G. Curci, W. Furmanski, and R. Petronzio, Nucl. Phys. B175, 27 (1980).
34. W. Furmanski and R. Petronzio, Phys. Lett. 97B, 437 (1980).
35. S. Moch and J. A. M. Vermaseren, in preparation.