Persistent currents in a Bose-Einstein condensate in the presence of disorder

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We examine bosonic atoms that are confined in a toroidal, quasi-one-dimensional trap, subjected to a random potential. The resulting inhomogeneous atomic density is smoothed for sufficiently strong, repulsive interatomic interactions. Statistical analysis of our simulations show that the gas supports persistent currents, which become more fragile due to the disorder.

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I. INTRODUCTION

Superfluidity is one of the most fascinating problems in physics. Superfluidity includes a whole class of phenomena that appear in various quantum systems [1], such as liquid He, superconductors, and nuclei. More recently, a lot of effort has been put on the study of superfluid properties of dilute vapors of trapped atoms, including, for example, vortex states [2, 3, 4, 5], transport properties [6], etc. Although these gases are very dilute, they are superfluid because of their extremely low temperature. Their diluteness makes cold gases of atoms an ideal system for testing superfluid properties. For example, their theoretical description is much easier as compared to other superfluids, like e.g., liquid helium. Furthermore, they can be manipulated in numerous ways, including the form of the trapping potential, and their coupling constant.

In the present study we examine the stability of persistent currents in trapped superfluid atoms that are confined in toroidal, quasi-one-dimensional traps. While conservation laws imply trivially the conservation of momentum/angular momentum even in a moving/rotating classical system, the crucial question is the robustness of the current-carrying state(s) against impurities or anisotropies; it is exactly this feature that distinguishes a quantum from a classical system.

Many theoretical studies have examined this problem, see, e.g., Refs. [7, 8, 9, 10, 11, 12, 13, 14]. Experimentally, numerous recent studies have considered this question in connection with trapped atomic gases [15, 16, 17, 18]. In another class of experiments – which are closely connected with the present problem – the propagation of Bose-Einstein condensed atoms in magnetic waveguides has also been examined thoroughly [19].

Interestingly, the problem that we study here is not only a toy model that provides answers on some non-trivial questions, but also it is directly applicable to experiments that have created Bose-Einstein condensates of atoms in toroidal traps [20]. Furthermore, the stability of supercurrents in these mesoscopic systems may have revolutionary technological applications [21]; similar arguments may also be applicable to other (nanoscale/mesoscopic) systems, such as quantum rings and quantum dots [22].

In what follows we first describe our model in Sec. II. In Sec. III we consider the effect of a random potential on the atomic density of a static cloud. In Sec. IV we discuss how one may understand the existence of persistent currents as a result of the metastability of the current-carrying state. Section V presents the results for the effect of disorder on the stability of superflow within the mean-field approximation. In this section, we consider many random potentials and analyze statistically our results. In Sec. VI we solve the same problem within the Bogoliubov approximation. Finally, in Sec. VII we discuss our results and give some general conclusions.

II. MODEL

In our model we consider a tight toroidal trap, which freezes the transverse degrees of freedom of the motion of the atoms. Further, we assume for the atom-atom collisions the usual contact potential,

\[ V_{\text{int}}(r-r') = U_0 \delta(r-r') \]

with \( U_0 = 4\pi \hbar^2 a_{\text{sc}} / M \). Here, \( a_{\text{sc}} \) is the scattering length for elastic atom-atom collisions and \( M \) is the atomic mass. Since the transverse degrees of freedom are frozen, the three-dimensional field operator may be written as

\[ \sqrt{N/(RS)} \hat{\Psi}(\theta), \]

where \( \hat{\Psi}(\theta) \) is the field operator associated with the motion of the atoms along the torus. Here \( \theta \) is the azimuthal angle, \( N \gg 1 \) is the atom number, \( R \) is the radius of the torus, and \( S \) is the cross section of the torus (with \( R \gg \sqrt{S} \)). Therefore, the Hamiltonian of our quasi-one-dimensional system is

\[ \hat{H}/N = \int \hat{\Psi}^\dagger(\theta) \left[ -\frac{\hbar^2}{2MR^2} \partial^2 + V(\theta) \right] \hat{\Psi}(\theta) d\theta + \frac{1}{2} 2\pi n_0 U_0 \int \hat{\Psi}^\dagger(\theta) \hat{\Psi}(\theta) \hat{\Psi}(\theta) \hat{\Psi}(\theta) d\theta, \]

(1)

where \( V(\theta) \) is the external potential, which is taken to be piecewise constant and random (see bottom graph in Fig.1). Finally, \( n_0 \) is the average atom density, \( n_0 = N/(2\pi RS) \).

Within the mean-field approximation the condensate order parameter \( \Psi \) satisfies the nonlinear equation

\[ -\partial^2\Psi / \partial \theta^2 + V(\theta)\Psi(\theta) + 2\pi \gamma |\Psi(\theta)|^2 \Psi = \mu \Psi, \]

(2)

where we have set \( \hbar = 2M = R = 1 \); \( V(\theta) \) and the chemical potential \( \mu \) are measured in units of the kinetic energy.
\[ T = \hbar^2/(2MR^2) \]. The ratio between the interaction energy and the kinetic energy is equal to \( \gamma = n_0 U_0 / T \). This parameter is also equal to \( \gamma = 4N a_{sc} R / S \).

The mean-field approximation is valid when the dimensionless quantity \( \gamma \) is \( \ll N^2 \) \[25\]. Further, the limit of weak interactions is achieved when \( \gamma \ll 1 \), while the Thomas-Fermi limit is achieved when \( 1 \ll \gamma \ll N^2 \). For the typical values of \( \gamma \) that we consider in the present study, the mean-field approximation is applicable, and the healing length \( \xi \) is on the order of the radius of the torus, since \( \xi / R = \gamma^{-1/2} \). The opposite Tonks-Girardeau limit of impenetrable bosons is achieved when \( \gamma \) becomes of order \( N^2 \) \[25\], where \( \xi / R \sim 1/N \).

### III. DENSITY PROFILE IN THE PRESENCE OF A RANDOM POTENTIAL

As a first step, we consider the static profile of the gas in the presence of the random potential \( V(\theta) \). We choose the length scale \( d \) of variation of \( V(\theta) \) to be \( R / 10 \), as in the experiment of Ref. \[15\]. In this experiment, the axial size of the condensate was \( \approx 110 \) \( \mu \)m, the radial size was \( \approx 11 \) \( \mu \)m, the smallest length scale of each “speckle” was \( \approx 10 \) \( \mu \)m, and the average distance between speckles was \( \approx 20 \) \( \mu \)m.

There are four energy scales in our problem, namely the kinetic energy associated with the torus \( \hbar^2/(2MR^2) \), the characteristic depth of each separate potential \( V_0 \), the zero-point kinetic energy \( \hbar^2/(2Md^2) \), and the typical interaction energy \( n_0 U_0 \). Since we choose \( d = R / 10 \), therefore \( \hbar^2/(2Md^2) \) is two orders of magnitude larger than \( \hbar^2/(2MR^2) \). We also choose \( V_0 \) to be of the same order as \( \hbar^2/(2MR^2) \), which implies that the lowest state, as well as the low-lying excited (eigen)states of the non-interacting problem are localized around the minima of the random potential, but there is also a significant overlap between the maxima in the density, as shown in Fig. 1 for \( \gamma = 0 \). If one chooses the zero of the energy to be at the maximum of the random potential, quantum mechanics implies that in the effectively one-dimensional problem that we consider here there is always at least one bound state, and thus at least one exponentially localized/delocalized state \[26, 27\].

Under these conditions, depending on \( \gamma \), there are three different regimes. For zero/weak interactions, \( \gamma \ll 1 \), the order parameter is equal/close to the lowest eigenstate of the external potential. As \( \gamma \) increases, the density variations get suppressed. When \( \gamma \gg 1 \), the density becomes homogeneous, as shown in Fig. 1.

### IV. STABILITY OF PERSISTENT CURRENTS

We turn now to the stability of persistent currents. Before we consider any disorder, it is instructive to see how one understands the existence of persistence currents for a constant potential \( V(\theta) \).

The basic idea is that for sufficiently strong interactions, the dispersion relation \( \mathcal{E}(l) \), i.e., the energy per atom as function of the angular momentum per atom, develops a barrier between the states with \( l = 1 \) and \( l = 0 \) \[28, 29, 30\]. Recalling that when \( l = 1 \), there is a vortex state that is located at the center of the torus, while for \( l = 0 \) the vortex is at an infinite distance away, in order for the vortex to exit the torus, there has to be
that has an expectation value of the angular momentum per atom equal to $l$. Here $\Phi_m = e^{im\theta}/\sqrt{2\pi}$ and $\lambda$ is a phase that is determined below. This order parameter gives an energy per atom that satisfies the equation

$$\frac{\mathcal{E}}{N} - \frac{\gamma}{2} - \kappa V_0 = (1 + \gamma)l - \gamma l^2 - 2\kappa |V_c| \sqrt{l(l-1)},$$

where $V_c = \int_{-\pi}^{\pi} V(\theta) \cos \theta \, d\theta/(2\pi)$, and $V_0 = \int_{-\pi}^{\pi} V(\theta) \sin \theta/(2\pi)$ is the zero component of the Fourier transform of $V(\theta)$. If $V_c$ is negative/positive, the phase $\lambda$ is chosen to be $\lambda = 0/\pi$, in order for the last term in Eq. (4) to be always negative. Since, according to Eq. (4), when $\kappa = 0$, $\mathcal{E}(l)$ develops a local minimum at $l = 1$ for a value of $\gamma = 1$, the critical value of the coupling $\gamma_c$ that gives metastability of the current is $\gamma_c = 1$ for the specific $\Psi_{tr}$. According to Eq. (4), for $\kappa \neq 0$, the last term may destabilize the current, forcing $\gamma_c$ to increase with $\kappa$, as shown in the dashed curve of Fig. 2. In other words, the coupling has to be sufficiently strong in order for the state with nonzero circulation to be stable. Clearly this curve is qualitatively, but not quantitatively correct.

Turning to the numerical results, we attack this problem via the method of imaginary time propagation [33]. We choose different realizations of random potentials of the form shown in Fig. 1, multiplied by $\kappa$. For each specific potential, the values of $V(\theta)$ at each subinterval are chosen independently from a uniform random distribution. For each specific $V(\theta)$, we identify the critical value of the coupling $\gamma_c$ that is necessary to obtain stability for a state with unit circulation for a specific $\kappa$, starting with $\Phi_1$ as an initial condition. The solid curve in Fig. 2 shows $\gamma_c(\kappa)$ for the specific random potential shown in Fig. 1. For $\kappa \to 0^+$, our results imply that $\gamma_c = 3/2$; this result is analyzed further below. We also perform a statistical analysis of the values of $\gamma_c$ that we get using this method, i.e., we compute the average value $\langle \gamma_c \rangle$ and the standard deviation $\sigma(\gamma_c)$, for 1000 different random potentials for each value of $\kappa$. Figure 2 shows the result of these calculations. The values of $\gamma_c$ that we get from the random potentials that we consider are approximately Gaussian distributed.

The states that come out of our calculation are different than $\Phi_1$ or $\Phi_0$, as they are not homogeneous, because of the (random) potential. Figure 3 shows the calculated density of the gas, for a circulation equal to zero (solid curve) and unity (dashed curve), for the specific random potential shown in Fig. 1 (with $\kappa = 1$). To get these densities, we use the initial condition $\Phi_0 = 1/\sqrt{2\pi}$ and $\Phi_1 = e^{i\theta}/\sqrt{2\pi}$, respectively.

V. PERSISTENT CURRENTS WITHIN THE MEAN-FIELD APPROXIMATION

We start our analysis of the stability of persistent currents within the mean-field approximation, for relatively strong disorder, where $V_0 \sim \hbar^2/(2MR^2) \sim n_0 U_0$. In the presence of any spatially-dependent potential $V(\theta)$, the angular momentum is not a good quantum number, and even more so in this case, where the disorder is not treated perturbatively. Still, one may examine the energetic stability of a state with one unit of circulation [31]; energetic stability also guarantees dynamic stability [32]. Clearly such a state can only be metastable and not the lowest-energy state of the system, since the non-rotating state has a lower energy.

Before we describe our numerical results, it is instructive to present a toy model that describes the problem qualitatively. We thus consider the random potential shown in the lowest graph of Fig. 1, multiplied by some dimensionless constant $\kappa$, i.e., $\kappa V(\theta)$; essentially $\kappa$ is the “strength” of the disorder. Although this argument can be generalized, for the sake of simplicity we consider a (very limited) truncated order parameter

$$\Psi_{tr}(\theta) = \sqrt{1-l} \Phi_0 + e^{i\lambda} \sqrt{l} \Phi_1,$$

FIG. 2: The dots show the average value $\langle \gamma_c \rangle$ and the bars show the standard deviation $\sigma(\gamma_c)$, of the critical coupling $\gamma_c$, versus the “strength” of the random potential $\kappa$. These values are calculated from the examination of 1000 different random potentials. For each specific potential, the values of $V(\theta)$ at each subinterval are chosen independently from a uniform random distribution. The solid curve shows $\gamma_c$ for the specific potential shown in Fig. 1. The dashed curve shows $\gamma_c$ that results from Eq. (4), i.e., from the toy model described in the text, for the same potential of Fig. 1.

a node in the density of the atoms. However, this node costs interaction energy, if the coupling is strong enough (since for repulsive interactions the interaction energy is minimized for a homogeneous density). The energy barrier that separates the rotating state with $l = 1$ from the lowest-energy state with $l = 0$ thus allows for the existence of persistent currents, making the timescale for the decay of the current exponentially long.

VI. BOGOLIUBOV APPROACH FOR WEAK DISORDER

Another way to attack this problem, incorporating the interactions, as well as the disorder, is via the Bogoliubov transformation [24]. This method gives the whole
excitation spectrum and also the depletion of the condensate, which is assumed to reside at the single-particle state \( \Phi \equiv e^{i\theta}/\sqrt{2\pi} \). This assumption, as well as the assumption for the existence of a Bose-Einstein condensate requires that the disorder is not too strong.

Following the usual tricks, we replace the creation and annihilation operators of atoms with angular momentum \( m = 1, c_1^\dagger \) and \( c_1 \), with \( \sqrt{N_1} \), where \( N_1 \) is the number of atoms occupying the \( m = 1 \) state. Then, since \( N = N_1 + \sum_{m\neq 1} c_m^\dagger c_m \), one can write for the Hamiltonian

\[
\hat{H} - N - \gamma N/2 = \sum_{m>0} (m^2 - 2m + \gamma) c_{-m}^\dagger c_{-m} + (m^2 + 2m + \gamma) c_m^\dagger c_m + \epsilon_m (c_{-m} + c_{m}^\dagger) + \epsilon_m^* (c_{m} + c_{-m}^\dagger),
\]

where \( \epsilon_m = \sqrt{N}v_m \). The last four terms describe the processes where atoms scatter between the condensate and the states \( \Phi_{1\neq m} \) because of the (random) potential, whose Fourier transform is denoted as \( V_m \). We define the new operators

\[
\alpha_m = u_m c_{-m} + v_m c_{m}^\dagger + \epsilon_m (m + 2) u_m + (m - 2) v_m \quad \mu_n \quad \beta_m = u_m c_{m} + v_m c_{-m}^\dagger + \epsilon_m^* (m - 2) u_m + (m + 2) v_m,
\]

and set \( u_m = \cosh \theta_m \) and \( v_m = \sinh \theta_m \), with \( \tanh(2\theta_m) = \gamma/(m^2 + \gamma) \), in order to eliminate the off-diagonal terms. Then, the Hamiltonian is written in the diagonal form

\[
\hat{H} - N - \gamma N/2 = \sum_{m>0} (-2m + m \sqrt{m^2 + 2\gamma}) \alpha_m^\dagger \alpha_m + (2m + m \sqrt{m^2 + 2\gamma}) \beta_m^\dagger \beta_m + m \sqrt{m^2 + 2\gamma} - (m^2 + \gamma) - 2N|V_m|^2/m^2 + 2\gamma - 4.
\]

In order for the excitation spectrum to be positive, the coefficient of the operator \( \alpha_m^\dagger \alpha_m \) implies that \( \gamma > 2 - m^2/2 \). The most unstable mode is the one with \( m = 1 \), and therefore the critical coupling constant for the existence of persistent currents is \( \gamma_c = 3/2 \). This result was first derived in Refs. [23, 24], within the truncated basis set of the states with \( m = 0, 1, \) and \( 2 \). According to the argument given above, this is the exact value of \( \gamma_c \) for the stability of persistent currents, as we have also found numerically within the mean-field approximation.

The eigenstates \( |n_{\alpha,m}, n_{\beta,m} \rangle \) of the diagonalized Hamiltonian of Eq. (7) (i.e., of the number operators \( \alpha_m^\dagger \alpha_m \) and \( \beta_m^\dagger \beta_m \)) carry an angular momentum \( L = N - \sum_{m>0} m(n_{\alpha,m} - n_{\beta,m}) \), i.e., \( L \sim N \pm O(1) \). From Eq. (7) we also see that the disorder potential enters the Hamiltonian only via the last term, which can also be derived perturbatively. As a result, one cannot see the dependence of \( \gamma_c \) on \( \kappa \) within this method. In order to see this dependence, one would have to consider a condensate of non-uniform density, or equivalently a condensate with at least two single-particle states macroscopically-occupied, e.g., of the form \( |0^{\nu_0}, 1^{\nu_1} \rangle \), as in Eq. (3).

Equation (7) can also give the depletion of the condensate \( \sum_{m>0} |n_{\alpha,m} = 0, n_{\beta,m} = 0| c_{-m}^\dagger c_{m}^\dagger c_{m} + c_{m}^\dagger c_{-m} |n_{\alpha,m} = 0, n_{\beta,m} = 0 \rangle \), which consists of two parts, the one resulting from the interactions and the other resulting from the disorder,

\[
\Delta N_{\text{int}} = \sum_{m>0} 2\nu_m^2 = \sum_{m>0} \frac{\gamma + m^2}{m \sqrt{m^2 + 2\gamma}} - 1,
\]

\[
\Delta N_{\text{disorder}} = 2N \sum_{m>0} \frac{|V_m|^2 (m + 2)^2}{m^2 (m^2 + 2\gamma - 4)^2}.
\]

VII. CONCLUDING REMARKS

According to the results of the present study, one may conclude very generally that disorder destabilizes states with nonzero circulation. Physically, any kind of disorder results in an inhomogeneous density distribution of the atoms. As compared to the density \( n_0 \) of the atoms in the absence of any disorder (which is homogeneous for repulsive interactions), the minimum and maximum density, \( n_{\text{min}} \) and \( n_{\text{max}} \), in the presence of disorder is thus both higher, as well as lower than \( n_0 \), \( n_{\text{min}} < n_0 < n_{\text{max}} \), as the conservation of the number of atoms implies.

In other words, in the presence of disorder, there exist points along the torus where the local density \( n(\theta) \) is lower than \( n_0 \). These points (where the density is lower than \( n_0 \)) give the cloud the chance to get rid of its circulation at a lower energy expense as compared to the homogeneous case. Higher values of \( \kappa \) enhance the inhomogeneities in the density, and thus make it easier for the vortex to slip out of the torus. Therefore, the higher the strength of the disorder, the higher the interaction strength that is necessary to make the compressibility high enough, in order for the current to become stable.
The above arguments are rather general and apply to any external potential.

The situation we consider here may be realized experimentally in two different ways: (i) the gas is prepared in a rotating state at a temperature above the condensation temperature, and then is cooled down to zero temperature [5], or (ii) starting from a gas at zero temperature, a phase is imprinted [34]. Finally, one could get evidence for the presence of current/circulation in such systems by, for example, interference experiments [35].

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[1] A. J. Leggett, Rev. Mod. Phys. 71, S318 (1999).
[2] K. W. Madison et al., Phys. Rev. Lett. 84, 806 (2000).
[3] K. W. Madison et al., Phys. Rev. Lett. 86, 4443 (2001).
[4] J. R. Abo-Shaeer et al., Science 292, 476 (2001).
[5] P. C. Haljan et al., Phys. Rev. Lett. 87, 210403 (2001).
[6] R. Onofrio et al., Phys. Rev. Lett. 85, 2228 (2000).
[7] M. Ma et al., Phys. Rev. B 34, 3136 (1986).
[8] M. P. A. Fisher et al., Phys. Rev. B 49, 12938 (1994).
[9] K. Huang and H.-F. Meng, Phys. Rev. Lett. 69, 644 (1992).
[10] S. Giorgini et al., Phys. Rev. B 49, 12938 (1994).
[11] G. E. Astrakharchik et al., Phys. Rev. A 66, 023603 (2002).
[12] M. Kobayashi and M. Tsubota, Phys. Rev. B 66, 174516 (2002).
[13] P. Navez et al., Appl. Phys. B 86, 395 (2007).
[14] L. Sanchez-Palencia, Phys. Rev. A 74, 053625 (2006).
[15] J. E. Lye et al., Phys. Rev. Lett. 95, 070401 (2005).
[16] C. Fort et al., Phys. Rev. Lett. 95, 170410 (2005).
[17] D. Clément et al., Phys. Rev. Lett. 95, 170409 (2005).
[18] D. Clément et al., New Journal of Physics 8, 165 (2006); P. Lugan et al., Phys. Rev. Lett. 98, 170403 (2007).
[19] See, e.g., J. H. Thywissen et al., Eur. Phys. J. D 7, 361 (1999), and references therein.
[20] S. Gupta et al., Phys. Rev. Lett. 95, 143201 (2005).
[21] See, e.g., T. Mukai, C. Hufnagel, A. Kasper, T. Meno, A. Tsukada, K. Semba, and F. Shimizu, e-print cond-mat/0702142
[22] S. M. Reimann and M. Manninen, Rev. Mod. Phys. 74, 1283 (2002).
[23] R. Kanamoto et al., Phys. Rev. A 68, 043619 (2003).
[24] G. M. Kavoulakis, Phys. Rev. A 69, 023613 (2004).
[25] G. M. Kavoulakis et al., Europhys. Lett. 76, 215 (2006).
[26] J. Goldstone and R. L. Jaffe, Phys. Rev. B 45, 14100 (1992).
[27] A. D. Jackson and G. M. Kavoulakis, Phys. Rev. A 74, 065601 (2006).
[28] S. J. Putterman et al., Phys. Rev. Lett. 9, 546 (1972).
[29] D. Rokhsar, e-print cond-mat/9709212.
[30] A. J. Leggett, Rev. Mod. Phys. 73, 307 (2001).
[31] Although the gas may support states with higher (integer) values of the circulation, we have examined the state with one unit of circulation just as an example. In addition, states of higher circulation would require a higher coupling constant to become stable.
[32] A. D. Jackson et al., Phys. Rev. A 72, 053617 (2005).
[33] See, e.g., S. A. Chin and E. Krotscheck, Phys. Rev. E 72 036705 (2005).
[34] A. E. Leanhardt et al., Phys. Rev. Lett. 89, 190403 (2002).
[35] S. Inouye, S. Gupta, T. Rosenband, A. P. Chikkatur, A. Gorlitz, T. L. Gustavson, A. E. Leanhardt, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 87, 080402 (2001).