Solutions associated with the point symmetries of the hyperbolic
Ernst equation

Sebastian Moeckel

Theoretisch Physikalisches Institut, Friedrich-Schiller-Universität Jena,
Max-Wien-Platz 1, 07743 Jena, Germany

Abstract

The continuous point symmetry algebra of the hyperbolic Ernst equation is presented. In a second step the corresponding group transformations are considered. Accordingly, the solutions of the hyperbolic Ernst equation that are invariant under Lie point symmetries, are constructed from the related invariant surface conditions. Furthermore, all these solutions are revealed to be related to solutions of the Euler-Poisson-Darboux equation by a simple coordinate transformation. The parallels of these results to coordinate transformations, which are important in the context of colliding plane wave space times, are pointed out.

PACS numbers: 02.20.Sv, 02.20.Tw, 02.30.Jr, 04.30.-w

*Electronic address: sebastian.moeckel@uni-jena.de
I. INTRODUCTION

Symmetry based methods can be easily employed for finding explicit solutions of PDEs, especially if no supplementary requirements (e.g. boundary conditions) are posed on the desired solution\(^1\). The Lie symmetries of a given set of PDEs can be used for locally reducing the number of independent coordinates and therefore finding explicit special solutions of the underlying set (an introduction can be found in \([1]\)). This feature appears to be of particular interest for a pair of PDEs depending on only two independent coordinates. Here, a local reduction of independent coordinates reduces the number of independent variables to one - hence one is left with a system of ODEs.

In this article, the Lie point symmetry algebra of the hyperbolic Ernst equation is presented. Furthermore, the solutions related to these Lie generators are derived. Finally, some special remarks on the relation of the so found solutions to the point symmetries of the metric tensor for colliding gravitational plane waves are given.

II. THE HYPERBOLIC ERNST EQUATION AND ITS POINT SYMMETRIES

The hyperbolic Ernst equation can be written in the following form

\[
(Z + \bar{Z}) \left[ 2Z f g + \frac{Z_f + Z_g}{f + g} \right] = 4Z_f Z_g
\]

(1)

where \(Z : \mathbb{R} \times \mathbb{R} \to \mathbb{C}\) is a complex function of the two independent coordinates.

For deriving the Lie point symmetries, it is convenient to separate (1) into a pair of real valued PDEs:

\[
K \left[ 2K f g + \frac{K_f + K_g}{f + g} \right] = 2 (K_f K_g - L_f L_g),
\]

(2)

\[
K \left[ 2L f g + \frac{L_f + L_g}{f + g} \right] = 2 (K_f L_g + K_g L_f).
\]

(3)

where \(Z = K + iL\).

The Lie point symmetry generators of the system (2)-(3) can be determined rather easily by

\(^1\) The method can be also extended to boundary value problems, but here also admitted symmetries of the boundary conditions have to be taken into account.
using an appropriate computer algebra system (the package SYM \[2\] has been used here):

\[
X_1 = \partial f - \partial g, \quad X_2 = f \partial f + g \partial g, \tag{4}
\]

\[
X_3 = \partial L, \quad X_4 = K \partial K + L \partial L, \tag{5}
\]

\[
X_5 = 2KL \partial K + (L^2 - K^2) \partial L, \tag{6}
\]

with the regarding commutator table

| \[\cdot, \cdot\] | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5\) |
|-----------------|------|------|------|------|------|
| \(X_1\) | 0 | \(X_1\) | 0 | 0 | 0 |
| \(X_2\) | \(-X_1\) | 0 | 0 | \(2X_4\) | 0 |
| \(X_3\) | 0 | \(X_3\) | 2 | 0 | 0 |
| \(X_4\) | \(-X_3\) | 0 | \(X_5\) | 0 | 0 |
| \(X_5\) | \(-2X_4\) | \(-X_5\) | 0 | 0 | 0 |

Table I: commutator table of the Lie algebra \((4)-(6)\)

Hence, the Lie algebra decomposes into a two-dimensional non-abelian sub-algebra and another three-dimensional sub-algebra. The Lie group related to the two dimensional sub-algebra can be immediately classified as being isomorphic to \(\text{Aff} (1)\)^2, since there is only one real non-abelian two-dimensional Lie group up to isomorphisms. The group corresponding to the three-dimensional sub-algebra will be classified by considering its group action. The corresponding group actions are obtained as

\[
\begin{pmatrix}
K (f, g) \\
L (f, g)
\end{pmatrix}
\xrightarrow{e^{\delta X_1}}
\begin{pmatrix}
K (f + \beta, g - \beta) \\
L (f + \beta, g - \beta)
\end{pmatrix}, \tag{7}
\]

\[
\begin{pmatrix}
K (f, g) \\
L (f, g)
\end{pmatrix}
\xrightarrow{e^{\alpha X_2}}
\begin{pmatrix}
K (e^{\alpha} f, e^{\alpha} g) \\
L (e^{\alpha} f, e^{\alpha} g)
\end{pmatrix}, \tag{8}
\]

\[
\begin{pmatrix}
K (f, g) \\
L (f, g)
\end{pmatrix}
\xrightarrow{e^{\gamma X_3}}
\begin{pmatrix}
K (f, g) \\
L (f, g) + \gamma
\end{pmatrix}, \tag{9}
\]

\[
\begin{pmatrix}
K (f, g) \\
L (f, g)
\end{pmatrix}
\xrightarrow{e^{\delta X_4}}
\begin{pmatrix}
e^{\delta} K (f, g) \\
e^{\delta} L (f, g)
\end{pmatrix}, \tag{10}
\]

\[2\] The group of invertible affine transformations from \(\mathbb{R}\) to \(\mathbb{R}\).
where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The group action of $X_5$ is a bit more involved

$$K \xrightarrow{e^{x_5}} K\frac{1}{1-2L\epsilon + (K^2 + L^2)\epsilon},$$

$$L \xrightarrow{e^{x_5}} L - (K^2 + L^2)\epsilon\frac{1}{1-2L\epsilon + (K^2 + L^2)\epsilon^2}.$$  \hspace{1cm} (11)\hspace{1cm} (12)

Combining the group actions of $X_1$ and $X_2$ on $Z = K + iL$ gives

$$Z (f, g) \xrightarrow{\alpha X_2} Z (e^\alpha f, e^\alpha g) \xrightarrow{\beta X_1} Z (e^\alpha f + \beta, e^\alpha g - \beta),$$ \hspace{1cm} (13)

where $\alpha, \beta \in \mathbb{R}$.

The action of $X_5$ maps a given solution $Z$ of the Ernst equation to a new solution according to

$$Z \xrightarrow{e^{x_5}} \frac{Z - iZ\bar{Z}\epsilon}{1 + i(Z - Z)\epsilon + ZZ\epsilon^2} = \frac{Z}{1 + i\epsilon Z},$$ \hspace{1cm} (14)

where $\epsilon \in \mathbb{R}$. Finally, combining the actions of $X_3, X_4, \text{and} X_5$ results in

$$Z \xrightarrow{\delta X_4} e^\delta Z \xrightarrow{\gamma X_3} e^\delta Z + i\gamma \xrightarrow{\epsilon X_5} \frac{e^\delta Z + i\gamma}{1 - \epsilon\gamma + i\epsilon e^\delta Z} = \frac{iaZ + ib}{cz + id},$$ \hspace{1cm} (15)

where

$$a = e^{\delta/2}, \ b = \gamma e^{-\delta/2},$$

$$c = -\epsilon e^{\delta/2}, \ d = (1 - \epsilon\gamma) e^{-\delta/2}.$$ \hspace{1cm} (16)\hspace{1cm} (17)

Hence, $ad - bc = 1$ and therefore the Lie group corresponding to $X_3, X_4, \text{and} X_5$ is isomorphic to $SL(2, \mathbb{R})$. As a special remark, the transformation (15) has already been considered by Neugebauer and Kramer in 1969 (\cite{3}). In the context of space-times with two commuting global Killing vector fields they have shown, that the transformation (15) can be derived by a simple $SL(2, \mathbb{R})$ rotation of the two Killing vector fields.

Hence, the Ernst equation (2)-(3) is invariant under a Lie group, whose algebra is isomorphic to $aff(1) \oplus sl(2, \mathbb{R})$. 

4
III. SPECIAL SOLUTIONS ASSOCIATED TO THE POINT SYMMETRIES

A. Solutions related to $X_1 = \partial_g - \partial_f$

The invariant surface conditions for the solutions $K$ and $L$ of the system of PDEs (2)-(3) with respect to the infinitesimal generator $X_1 = \partial_g - \partial_f$ read

$$\frac{\partial K}{\partial g} = \frac{\partial K}{\partial f}, \quad (18)$$

$$\frac{\partial L}{\partial g} = \frac{\partial L}{\partial f}. \quad (19)$$

Plugging the constraints (18)-(19) into the system of PDEs (2)-(3) yields the following system of (parametric) ordinary differential equations in $K$ and $L$ with the independent coordinate $f$ and the parameter $g$

$$K \left[ K_{ff} + \frac{K_f}{f + g} \right] = K_f^2 - L_f^2, \quad (20)$$

$$K \left[ L_{ff} + \frac{L_f}{f + g} \right] = 2K_f L_f. \quad (21)$$

The general solution of (20)-(21) can be found as follows.
First, it is necessary to write (20) in solved form

$$K_{ff} = \frac{K_f^2 - L_f^2}{K} - \frac{K_f}{f + g}. \quad (22)$$

Furthermore, solving (20) for $L_f^2$ yields

$$L_f^2 = K_f^2 - K \left( K_{ff} + \frac{K_f}{f + g} \right), \quad (23)$$

and solving (21) for $L_{ff}$ gives

$$L_{ff} = L_f \left( 2 \frac{K_f}{K} - \frac{1}{f + g} \right). \quad (24)$$

Differentiating (22) with respect to $f$ leads to

$$K_{fff} = -K_f \left( \frac{K_f^2 - L_f^2}{K^2} \right) + 2 \left( \frac{K_f K_{ff} - L_f L_{ff}}{K} \right) - \frac{K_{ff}}{f + g} + \frac{K_f}{(f + g)^2}. \quad (25)$$

Now replacing $L_{ff}$ and $L_f^2$ according to (24) and (23) yields the following third order nonlinear ordinary differential equation in $K$

$$K_{fff} = F(f, K, K_f, K_{ff}) = -4 \frac{K_f^3}{K^2} - 3 \frac{K_{ff}}{f + g} - \frac{K_f}{(f + g)^2} + 5 \frac{K_f K_{ff}}{K} + 5 \frac{K_f^2}{K(f + g)}. \quad (26)$$
Note that $g$ appears only as a parameter in (26). The method of determining integrating factors for generating corresponding first integrals (see the appendix and [1] for details) is one possible way for obtaining the general solution of (26). For this purpose, the surface generated by equation (26) in the 2+3 dimensional jet-space \( \mathbb{J} \) with coordinates \((f, K, K_1, K_2, K_3)\) is considered

\[
K_3 = F(f, K, K_1, K_2) = -4 \frac{K_1^3}{K^2} - 3 \frac{K_2}{f + g} - \frac{K_1}{(f + g)^2} + 5 \frac{K_1 K_2}{K} + 5 \frac{K_2}{K (f + g)}. \tag{27}
\]

Since (27) does not involve higher order terms of \(K_2\), the following ansatz for the integrating factor seems to be promising

\[
\Lambda(f, K, K_1) = \alpha(f, K) + K_1 \beta(f, K). \tag{28}
\]

Plugging (28) into the integrating-factor determining equation (71) immediately yields \(\beta(f, K) \equiv 0\). Plugging the remaining expression for \(\Lambda\) into the second determining equation (72) yields an overdetermined system of six linear PDEs for \(\alpha\)

\[
\begin{align*}
9 \frac{\alpha}{K^2} + 15 \frac{\alpha}{K} + 3 \frac{\alpha}{K K} &= 0, \tag{29}
-6 \frac{\alpha}{K^3} - 2 \frac{\alpha}{K^2} + 5 \frac{\alpha}{K K} + \alpha_{K K K} &= 0, \tag{30}
- \frac{10}{(f + g) K} \alpha - \frac{6}{f + g} \alpha_K + 5 \frac{\alpha}{K f} + 3 \alpha_{f K} &= 0, \tag{31}
- \frac{8}{(f + g)^3} \alpha + \frac{7}{(f + g)^2} \alpha_f - \frac{3}{f + g} \alpha_{f f} + \alpha_{f f f} &= 0, \tag{32}
\frac{5}{(f + g) K^2} \alpha - \frac{5}{(f + g) K} \alpha_K - \frac{3}{f + g} \alpha_{K K} + \frac{2}{K^2} \alpha_f + 10 \frac{\alpha_{f K}}{K} + 3 \alpha_{f f K} &= 0, \tag{33}
\frac{10}{(f + g) K^2} \alpha + \frac{6}{(f + g)} \alpha_K - \frac{10}{(f + g) K} \alpha_f - \frac{6}{(f + g)} \alpha_{f K} + \frac{5}{K} \alpha_{f f} + 3 \alpha_{f f K} &= 0. \tag{34}
\end{align*}
\]

Solving (29) yields

\[
\alpha(f, K) = \frac{C_1(f)}{K} + \frac{C_2(f)}{K^3}, \tag{35}
\]

where \(C_1(f)\) and \(C_2(f)\) are arbitrary functions of \(f\). Furthermore, the expression (35) solves (30) identically. Plugging (35) into (31) yields the following determining equations for \(C_1(f)\)

\(\text{The jet-space is constructed by interpreting the independent variable } f \text{ and the dependent variable } K \text{ together with all derivatives appearing in the ODE under consideration (i.e. } K_f, K_{ff}, \text{ and } K_{fff} \text{) as independent coordinates of a 2+3 dimensional space. See [1] for details.} \)
and \( C_2 (f) \)

\[
2C_1 (f) - (f + g) C_1' (f) = 0, \tag{36}
\]
\[
2C_2 (f) - (f + g) C_2' (f) = 0, \tag{37}
\]

which yield the solutions

\[
C_1 (f) = c_1 (f + g)^2, \tag{38}
\]
\[
C_2 (f) = c_2 (f + g)^2, \tag{39}
\]

where \( c_1, c_2 \in \mathbb{C} \) for fixed \( g \). Hence \( \alpha \) reads

\[
\alpha (f, K) = c_1 \frac{(f + g)^2}{K} + c_2 \frac{(f + g)^2}{K^3}. \tag{40}
\]

Expression (40) also solves (32)-(34) identically. Hence, two functionally independent integrating factors for equation (26) are obtained with

\[
\Lambda_1 (f, K) = \frac{(f + g)^2}{K}, \quad \Lambda_2 (f, K) = \frac{(f + g)^2}{K^3}. \tag{41}
\]

Now plugging \( \Lambda_1 \) and \( \Lambda_2 \) into the line integral formula (74) for determining the corresponding first integrals give

\[
\psi_1 (f, K, K_1, K_2) = \frac{(f + g)^2}{K^2} \left\{ K_1 [K - 2 (f + g) K_1] + (f + g) K K_2 \right\}, \tag{42}
\]
\[
\psi_2 (f, K, K_1, K_2) = \frac{(f + g)^2}{K^4} \left\{ K_1 [K - (f + g) K_1] + (f + g) K K_2 \right\}, \tag{43}
\]

where the path of integration has been chosen parallel to the axis of \( \mathbb{R}^4 \) from \((0, K, 0, 0)\) to \((f, K, K_1, K_2)\), such that the singularity at \( K = 0 \) is avoided. Since (42) and (43) are first integrals of (26), every solution of (26) satisfies \( \psi_1 (f, K, K_f, K_{ff}) = c_1 = \text{const.}, \) together with \( \psi_2 (f, K, K_f, K_{ff}) = c_2 = \text{const.} \) for arbitrary constants \( c_1, c_2 \in \mathbb{C} \) (for fixed \( g \)). Therefore, setting \( \psi_1 = c_1 \) and \( \psi_2 = c_2 \) results in an effective reduction of the third order equation (26) to the first order equation

\[
\frac{c_1 K^2 + (f + g)^2}{K^4} K_{f} = c_2. \tag{44}
\]

Because (44) is separable, a general solution can be obtained by quadrature, leading to the general solutions

\[
K_I (f, g) = \pm \sqrt{\frac{c_1 (g)}{c_2 (g)} \csc \left[ \sqrt{c_1 (g)} (c_3 (g) - \ln (f + g)) \right]}, \tag{45}
\]
\[
K_{II} (f, g) = \pm \sqrt{\frac{c_1 (g)}{c_2 (g)} \csc \left[ \sqrt{c_1 (g)} (c_3 (g) + \ln (f + g)) \right]}, \tag{46}
\]
where \( c_1, c_2 \) and \( c_3 \) are still arbitrary functions of \( g \) (recall that ODE (26) is parametric in \( g \)). Inserting (45) and (46) into (23) yields the following expressions for \( L(f, g) \)

\[
L_I(f, g) = \pm \sqrt{-\frac{c_1(g)}{c_2(g)}} \cot \left[ \sqrt{c_1(g)} (c_3(g) - \ln(f + g)) \right] + c_4(g), \tag{47}
\]

\[
L_{II}(f, g) = \pm \sqrt{-\frac{c_1(g)}{c_2(g)}} \cot \left[ \sqrt{c_1(g)} (c_3(g) + \ln(f + g)) \right] + c_4(g). \tag{48}
\]

Plugging the expressions for \( L \) and \( K \) back into the invariant surface conditions (18)-(19) allows for the determination of the remaining functions \( c_1(g), c_2(g), c_3(g) \) and \( c_4(g) \). As a result, all functions \( c_1(g), c_2(g), c_3(g) \) and \( c_4(g) \) need to be simple constants, thus generating the following solutions

\[
K_I(f, g) = \pm \sqrt[4]{\frac{c_1}{c_2}} \csc \left[ \sqrt{c_1} (c_3 - \ln(f + g)) \right], \tag{49}
\]

\[
K_{II}(f, g) = \pm \sqrt[4]{\frac{c_1}{c_2}} \csc \left[ \sqrt{c_1} (c_3 + \ln(f + g)) \right], \tag{50}
\]

\[
L_I(f, g) = \pm \sqrt{-\frac{c_1}{c_2}} \cot \left[ \sqrt{c_1} (c_3 - \ln(f + g)) \right] + c_4, \tag{51}
\]

\[
L_{II}(f, g) = \pm \sqrt{-\frac{c_1}{c_2}} \cot \left[ \sqrt{c_1} (c_3 + \ln(f + g)) \right] + c_4, \tag{52}
\]

where \( c_1, c_2, c_3, c_4 \in \mathbb{C} \).

For interpreting \( K \) and \( L \) as the real- and imaginary parts of the Ernst potential \( Z = K + iL \), it is necessary to transform the expressions (49)-(52) to real valued functions. One possibility for achieving this goal is to choose \( c_3 = \frac{\pi}{\sqrt{c_1} 2}, \sqrt{c_1} = iA, \sqrt{c_2} = iB, c_4 = C \) with \( A, B, C \in \mathbb{R} \), resulting in

\[
K(f, g) = \frac{2A}{B} \frac{(f + g)^A}{1 + (f + g)^{2A}}; \tag{53}
\]

\[
L(f, g) = \frac{A 1 - (f + g)^{2A}}{B 1 + (f + g)^{2A}} + C, \tag{54}
\]

where a possible factor of \(-1\) has been absorbed in the constants.
B. Solutions related to $X_2 = f\partial_f + g\partial_g$

The invariant surface conditions for the solutions $K$ and $L$ of the system of PDEs (2)-(3) with respect to the infinitesimal generator $X_2 = f\partial_f + g\partial_g$ read

\[
\frac{g}{\partial g} \frac{\partial K}{\partial g} = -f \frac{\partial K}{\partial f}, \quad (55) \\
\frac{g}{\partial g} \frac{\partial L}{\partial g} = -f \frac{\partial L}{\partial f}. \quad (56)
\]

Now, following exactly the same steps as for the solutions corresponding to $X_1$, leads to the final solution

\[
K(f, g) = \frac{A}{B} \text{sech} \left[ 2A \text{arctan} \left( \sqrt{\frac{f}{g}} \right) \right], \quad (57)
\]

\[
L(f, g) = \frac{A}{B} \text{tanh} \left[ 2A \text{arctan} \left( \sqrt{\frac{f}{g}} \right) \right] + C, \quad (58)
\]

where $A, B, C \in \mathbb{R}$.

IV. RELATION TO GRAVITATIONAL PLANE WAVE COLLISIONS

The two group invariant solutions obtained in the preceding section appear to be surprisingly similar, since they are both of the form

\[
K(f, g) = \text{sech} \left[ F(f, g) \right], \quad (59)
\]

\[
L(f, g) = \text{tanh} \left[ F(f, g) \right], \quad (60)
\]

where $F(f, g)$ is an arbitrary real function of $f$ and $g$ and all pre-factors have been omitted, which can be re-introduced later on by using the apparent scaling symmetry of the Ernst equation.

Plugging the ansatz (59)-(60) into the separated form of the Ernst equation (2) and (3) leads to the following result

\[
\frac{\text{sech} \left[ F \right] \text{tanh} \left[ F \right]}{f + g} \left\{ 2 (f + g) F_{fg} + F_f + F_g \right\} = 0, \quad (61)
\]

\[
\frac{\text{sech} \left[ F \right]}{f + g} \left\{ 2 (f + g) F_{fg} + F_f + F_g \right\} = 0, \quad (62)
\]

Hence, if $F(f, g)$ satisfies the EPD equation

\[
2 (f + g) F_{fg} + F_f + F_g = 0, \quad (63)
\]
\(K(f,g)\) and \(L(f,g)\) solve the Ernst equations \((2)-(3)\).
Thus, every real function \(F(f,g)\) satisfying the EPD equation \((63)\) generates a complex valued Ernst potential by means of

\[
Z(f,g) = \text{sech} \left[ F(f,g) \right] + i \tanh \left[ F(f,g) \right],
\]

which is a solution of the Ernst equation.

\(F\) can be interpreted as the scalar potential of a real valued Ernst potential \(Z_o = \exp (2F)\).
Therefore, \(Z_o\) is possibly the potential of a collinear colliding gravitational plane wave solution, if certain conditions are satisfied (see \([4]\) for a detailed discussion of colliding gravitational plane wave space-times).

Writing \((64)\) in terms of \(2F = \ln Z_o\) leads to the following form of the transformation

\[
Z = \frac{1 + iZ_o}{i + Z_o},
\]

revealing \((64)\) to be just a special case of the transformation induced by the Lie sub-algebra generated by \(X_3, X_4\) and \(X_5\) (c.f. \((15)\)).

As an additional feature, transformation \((15)\) leaves the plane-wave conditions invariant (cf. \([4]\)). Therefore colliding wave solutions are mapped to colliding wave solutions by transformation \((65)\).

Now, recalling that \((15)\) can be interpreted as a simple rotation of the two globally commuting Killing vector fields\(^4\), this leads to the conclusion that the two group-invariant solutions \((53)-(54)\) and \((57)-(58)\), as well as all solutions generated by using \((15)\), cannot be considered as inherently non-collinear\(^5\), since they can be reduced to collinear solutions by a simple coordinate transformation. It follows that the solutions of the Ernst equation, which are invariant under continuous point transformations can all be reduced to collinear solutions.

This relation has already been known for the point-symmetries generated by \(X_3, X_4\) and \(X_5\) (cf. \([3]\)), but has not been obvious for the solutions which have been generated from the invariant surface condition with respect to \(X_1\) and \(X_2\).

\(^4\) Colliding gravitational plane wave space-times are equipped with a pair of space-like globally commuting Killing vector fields.

\(^5\) Solutions that lead to a reduction of the Ernst equation to the linear Euler-Poisson-Darboux equation are classified as collinear solutions (see \([4]\) for more details).
V. CONCLUSIONS AND OUTLOOK

As a major result, it has been shown that Lie’s method for determining symmetry invariant solutions, is well suited for generating explicit solutions of the hyperbolic Ernst equation. Furthermore, the important relation between the class of solutions generated from the closed sub-algebra \( \langle X_1, X_2 \rangle \) and solutions that are obtained by only using transformations of the closed sub-algebra \( \langle X_3, X_4, X_5 \rangle \) has been revealed. Moreover, the relation to simple coordinate transformations in the context of colliding plane wave space-times has been established.

A future perspective for generating new group-invariant solutions is to consider contact- and higher order symmetries of the Ernst equation. The resulting solutions can be obtained by invoking a similar procedure as has been carried out for the point symmetries (cf. [1] and [2]). In addition, the group-invariant solutions of the coupled pair of hyperbolic Ernst equations, appearing in the context of coupled electromagnetic and gravitational plane wave collisions, can be considered.

Appendix

Taking the definition from [1], a first integral of an \( n \)th-order ODE

\[
y^{(n)} = F \left( x, y, y', \ldots, y^{(n-1)} \right)
\]

is a function \( \psi \left( x, y, y', \ldots, y^{(n-1)} \right) \) with an essential dependence on \( y^{(n-1)} \) satisfying

\[
\frac{d\psi}{dx} = 0, \quad \text{when} \quad y^{(n)} = F,
\]

i.e. \( \psi \left( x, y, y', \ldots, y^{(n-1)} \right) \) is constant for every solution \( y = \Theta (x) \) of the ODE (66).

Hence, a first integral is a conserved quantity for each solution and therefore the knowledge of \( r \) functionally independent first integrals leads to an effective reduction of the order of the given ODE to \( n - r \).

Furthermore, an integrating factor of ODE (66) is defined as a function \( \Lambda \left( x, y, y', \ldots, y^{(l)} \right) \neq 0 \) \( (0 \leq l \leq n-1, l \) is the order of the integrating factor), such that

\[
\Lambda \left( x, y, y', \ldots, y^{(l)} \right) \left( y^{(n)} - F \left( x, y, y', \ldots, y^{(n-1)} \right) \right) = \frac{d\psi}{dx} \left( x, y, y', \ldots, y^{(n-1)} \right),
\]

(68)
for a function $\psi (x, y, y', \ldots, y^{(n-1)})$ with an essential dependence on $y^{(n-1)}$.

Considering a general third order ODE

$$y''' = F(x, y, y', y'') \tag{69}$$

represented by the surface

$$y_3 = F(x, y, y_1, y_2). \tag{70}$$

provides the explicit determining system for integrating factors:

$$2\Lambda_{y_1} + \Lambda_{y_2x} + y_1\Lambda_{y_2y} + y_2\Lambda_{y_2y_1} + (F\Lambda)_{y_2y_2} = 0, \tag{71}$$

$$\begin{align*}
3y_2\Lambda_{xy} + 3y_1y_2\Lambda_{yy} + 3y_2^2\Lambda_{y_1y_1} + \Lambda_{xxx} + y_1^3\Lambda_{yyy} + y_2^2\Lambda_{y_1y_1y_1} \\
+ 3y_1\Lambda_{xxy} + 3y_2\Lambda_{xyy} + 3y_2^2\Lambda_{xyy_1} + 3y_1^2y_2\Lambda_{y_2y_1} \\
+ 3y_1y_2\Lambda_{y_2y_1} + 6y_1y_2\Lambda_{y_2y_2} + (F\Lambda)_y - (F\Lambda)_{xy_1} - y_1 (F\Lambda)_{y_1} \\
- y_2 (F\Lambda)_{y_1y_1} + y_2 (F\Lambda)_{yy_2} + (F\Lambda)_{xxy_2} + y_1^2 (F\Lambda)_{y_2y_2} \\
+ y_2^2 (F\Lambda)_{y_1y_1y_2} + 2y_1 (F\Lambda)_{xyy_2} + 2y_2 (F\Lambda)_{xyy_2} + 2y_1y_2 (F\Lambda)_{y_2y_1y_2}
\end{align*} \tag{72}$$

For a general integrating factor $\Lambda (x, y, y_1, y_2)$, the system (71) and (72) admits an infinite number of solutions, but restricting $\Lambda$ to special ansatzes yields an overdetermined system of linear PDEs that has at most a finite number of independent solutions. For instance, if $F(x, y, y_1, y_2)$ is linear in $y_2$, the following ansatz leads to some remarkable simplifications

$$\Lambda (x, y, y_1) = \alpha (x, y) + y_1\beta (x, y). \tag{73}$$

Plugging (73) into (71) immediately yields $\beta \equiv 0$, and (72) decomposes into several linear PDEs for determining $\alpha$.

When an integrating factor of (69) has been obtained, a corresponding first integral can be calculated by means of the following line integral formula (11)

$$\psi = \int_C \left\{ - (F\Lambda)_{y_1} + (F\Lambda)_{xy_2} + y_1 (F\Lambda)_{yy_2} + y_2 (F\Lambda)_{y_1y_2} + y_2\Lambda_y \\
+ \Lambda_{xx} + y_1^2\Lambda_{yy} + y_2^2\Lambda_{y_1y_1} + 2y_1\Lambda_{xy} + 2y_2\Lambda_{xy_1} + 2y_1y_2\Lambda_{y_2y_1} \right\} (dy - y_1dx)$$

$$- \left[ (F\Lambda)_{y_2} + \Lambda_x + y_1\Lambda_y + y_2\Lambda_{y_1} \right] (dy_1 - y_2dx) + \Lambda (dy_2 - f\,dx) \right\}, \tag{74}$$
where the path $C$ can be chosen arbitrarily from any point $(\tilde{x}, \tilde{y}, \tilde{y}_1, \tilde{y}_2)$ to $(x, y, y_1, y_2)$, such that all singularities are avoided.

[1] G. Bluman and S. Anco, *Symmetry and integration methods for differential equations* (Springer, 2002).

[2] S. Dimas and D. Tsoubelis, Proceedings of the 10th International Conference in Modern Group Analysis, 64 (2005).

[3] G. Neugebauer and D. Kramer, Annalen der Physik 479, 62 (1969).

[4] J. Griffiths, *Colliding plane waves in general relativity* (Clarendon Press, 1991).

[5] G. Bluman, A. Cheviakov, and S. Anco, *Applications of symmetry methods to partial differential equations* (Springer, 2010).