A Newman type bound for $L_p[-1, 1]$-means of the logarithmic derivative of polynomials having all zeros on the unit circle

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Abstract
Let $g_n, n = 1, 2, \ldots$, be the logarithmic derivative of a complex polynomial having all zeros on the unit circle, i.e., a function of the form $g_n(z) = (z-z_1)^{-1} + \cdots + (z-z_n)^{-1}$, $|z_1| = \cdots = |z_n| = 1$. For any $p > 0$, we establish the bound
\[ \int_{-1}^{1} |g_n(x)|^p \, dx > C_p \, n^{p-1}, \]
sharp in the order of the quantity $n$, where $C_p > 0$ is a constant, depending only on $p$. The particular case $p = 1$ of this inequality can be considered as a stronger variant of the well-known estimate $\int_{|z|<1} |g_n(z)| \, dx \, dy > c > 0$ for the area integral of $g_n$, obtained by Newman (Am Math Mon 79(9):1015–1016, 1972). The result also shows that the set $\{g_n\}$ is not dense in the spaces $L_p[-1, 1], p \geq 1$.

Keywords Logarithmic derivative of a polynomial · Polynomials with zeros on a circle · Integral mean on a segment · Chui’s problem

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1 Introduction: Main results
Chui [4] asked whether there existed an absolute constant $c > 0$ such that
\begin{align*}
\iint_{|z|<1} |g_n(z)| \, dx \, dy > c \quad (z = x + iy)
\end{align*}

(1)

for any rational function \( g_n, n = 1, 2, \ldots \), of the form

\begin{align*}
g_n(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}, \quad |z_1| = \cdots = |z_n| = 1
\end{align*}

(2)

\( g_n \) coincides with the logarithmic derivative of a polynomial all of whose zeros lie on the unit circle. The area integral in the left-hand side of (1) shows the average strength in the unit disk of the electrostatic field corresponding to \( n \) given point charges.

Newman [12] has published an elegant solution to Chui’s problem by showing that (1) holds with

\( c := \pi/18. \)

Later on, considering approximation properties of logarithmic derivatives of complex polynomials in the Bers spaces in bounded Jordan domains, Chui [5] has proved, in particular, that for every \( q > 2 \) the set of sums (2) is dense in the space of functions \( f \), analytic in the open unit disk \( D \), endowed with the norm

\[ \| f \|_q := \iint_{|z|<1} |f(z)| (1 - |z|^2)^{q-2} \, dx \, dy. \]

So, Newman’s bound (1) shows that the condition \( q > 2 \) cannot be improved here.

We also recall that it was conjectured [4] that the integral in (1) is minimized by the choice \( z_k = e^{2\pi ki/n} \) as the poles of \( g_n \). This conjecture is still open, however, its weighted \( L_2 \) analogs were recently solved in [1] as well as the problem of denseness of (2) in the corresponding function spaces in the unit disk.

A similar approximation problem for the unit interval was posed in 2014 by S. R. Nasyrov (see [2, Sect. 4]): whether sums (2) are dense in the complex space \( L_2[-1, 1] \). The author [8] has obtained the negative answer to this question by showing the bound

\[ \int_{-1}^{1} |g_n(x)|^2 \, dx > \frac{1}{64} \]

(3)

for any function \( g_n \) of the form (2).

In this paper, using more subtle reasoning, we establish a stronger result, showing, in particular, that \( 1/64 \) in (3) can be replaced by \( n/213 \).

**Theorem 1** For any function \( g_n \) of the form (2) and any \( p > 0 \), we have

\[ \int_{-1}^{1} |g_n(x)|^p \, dx > \int_{-1}^{1} |g_n(x)|^p |x|^p \, dx > C_p \, n^{p-1}, \]

(4)

where \( C_p > 0 \) is a constant, depending only on \( p \). For example, in the cases \( p = 1 \) and \( p = 2 \), the inequality (4) holds with \( C_1 = 1/50 \) and \( C_2 = 1/213 \).
Choosing $p = 1$ in (4), we obtain the Newman type bound for the interval:

$$
\int_{-1}^{1} |g_n(x)| \, dx > \int_{-1}^{1} |xg_n(x)| \, dx > \frac{1}{50}
$$

and conclude that rational functions (2) are not dense even in the space $L_1[-1, 1]$.

Note that the property (5) is stronger than (1) in the sense that Newman’s result (1) (although with a constant $c < \pi / 18$) immediately follows from (5):

$$
\int |z| < 1 |g_n(z)| \, dz = \int_{0}^{2\pi} dt \int_{0}^{1} |g_n(re^{it})| \, rdr = \int_{0}^{\pi} dt \int_{-1}^{1} |rg_n(re^{it})| \, dr > \frac{\pi}{50}.
$$

The proof of Theorem 1 uses the following metric property of functions (2) that is also of independent interest. By $\mu(E)$ we will denote the linear measure of a set $E \subset \mathbb{R}$.

**Theorem 2** For any function $g_n$ of the form (2), the measure $\mu(E)$ of the set

$$
E = E_\delta(g_n) := \{x \in [-1, 1] : |\text{Re}(xg_n(x))| \geq \delta n\}, \quad 0 < \delta < 1/2,
$$

admits the estimate

$$
\mu(E) \geq Kn^{-1}, \quad K = K(\delta) := \frac{8}{9} \frac{1 - 2\delta}{(3 + 4\delta)(1 + 2\delta)} > 0.
$$

Moreover, the set $E$ is concentrated near the endpoints of the segment $[-1, 1]$ in the sense that $\mu(E)$ in (6) can be replaced by $\mu(E \cap \Delta)$, where

$$
\Delta = \Delta_n, \delta := \{x \in [-1, 1] : |x| \geq 1 - \frac{n^{-1}}{1 + 2\delta}\}.
$$

In particular, by the definition of the set $E$, we have

$$
\int_{-1}^{1} |xg_n(x)|^p \, dx > \int_{E} |\text{Re}(xg_n(x))|^p \, dx \geq (\delta n)^p \mu(E) \geq \delta^p K(\delta) n^{p-1} = C_p n^{p-1}
$$

for any fixed $\delta \in (0, 1/2)$, and Theorem 1 follows. Taking, for example, $\delta := 3p/(6p + 11)$, we can write the explicit value of the constant $C_p$ as a function of $p$, $p > 0$:

$$
C_p = \frac{88 \cdot 3^{p-3} p^p}{(6p + 11)p^{-1}(10p + 11)(12p + 11)} \left( C_1 > \frac{1}{50}, \quad C_2 > \frac{1}{213} \right).
$$

Both estimates (4), (6) are sharp in the order of the quantity $n$. Indeed, firstly, for the function $g_n(x) = \tilde{g}_n(x) := nx^{-1}/(x^n + i)$ we have

$$
\int_{-1}^{1} |\tilde{g}_n(x)|^p \, dx = 2n^{p-1} \int_{0}^{1} \frac{(t^{1-\frac{1}{n}})^{p-1} \, dt}{(t^2 + 1)^{p/2}} \leq \tilde{C}_p n^{p-1}, \quad \tilde{C}_p = \int_{0}^{1} \frac{2t^p \, dt}{(t^2 + 1)^{p/2}}
$$
(\(p > 0, \kappa = \min\{p - 1; 0\}\)). Secondly, if \(0 < \delta < 1/2\) then from

\[
\text{Re}(x\tilde{g}_n(x)) = \frac{nx^{2n}}{x^{2n} + 1}, \quad E_{\delta}(\tilde{g}_n) = \{x \in [-1, 1]: |x| \geq (\delta^{-1} - 1)^{-1/(2n)}\}
\]

and \(a^{-t} > 1 - t \ln a\) (\(a > 1\)) we deduce

\[
\mu(E_{\delta}(\tilde{g}_n)) < \tilde{K}n^{-1}, \quad \tilde{K} = \ln(\delta^{-1} - 1) > 0.
\]

**Remark 1** For all \(\delta \geq 1/2\), we have

\[
\mu(E_{\delta}(\tilde{g}_n)) = \mu(\{x \in [-1, 1]: x^{2n} \geq \delta/(1 - \delta)\}) = 0,
\]

therefore the property \(\lim_{\delta \to 1/2} K(\delta) = 0\) (see (6)) is natural. On the other hand, the quantity \(\mu(E_{\delta})\) can be arbitrarily close to 2 for every \(\delta \in (0, 1/2)\). Indeed, if \(\tilde{g}_{2n}(x) := 2nx/(x^2 - 1)\) then

\[
2 - \mu(E_{\delta}(\tilde{g}_{2n})) = \frac{2\sqrt{\delta}}{2n + \delta} \to 0 \text{ as } n \to \infty.
\]

**Remark 2** Estimates of the logarithmic derivatives of polynomials with restricted zeros are well-known. Thus, it was proved in [7] that for polynomials \(p_n\), whose zeros \(z_1, \ldots, z_n\) lie in the half-disk \(U = \{z: |z| \leq 1, \Im z \geq 0\}\), the inequality \(|p_n'(x)/p_n(x)| > \delta n\ (\delta > 0)\) holds for all \(x \in [-1, 1]\) outside an exceptional set \(e_\delta \subset [-1, 1]\) of a measure

\[
\mu(e_\delta) < 70e\delta. \quad (7)
\]

If, moreover, all \(z_1, \ldots, z_n \in [-1, 1]\) then

\[
\mu(e_\delta) \leq (\sqrt{1 + 4\delta^2} - 1)/\delta \quad (8)
\]

(see [7]), and also the following weighted estimate holds (see [9]):

\[
\mu(\{x \in [-1, 1]: (1 - x^2)|p_n'(x)| \leq \delta n|p_n(x)|\}) \leq 2\delta, \quad (9)
\]

whose proof uses Borwein’s result [3].

Estimates (7)–(9) are useful in constructing Turán-type reverse Markov inequalities on a segment (see [6, 9–11]). For example, it was shown by the author [10] that (7) implies

\[
\|p_n'\|_{C[-1, 1]} > A\sqrt{n} \|p_n\|_{C[-1, 1]}, \quad A := 2(3\sqrt{210e})^{-1},
\]

for polynomials \(p_n\) having all \(n\) zeros in the upper unit half-disk \(U\).
2 Lemmas

For any \( z_1, \ldots, z_n \) on the unit circle \(|z| = 1\), by using the identity

\[
 xg_n(x) = \sum_{k=1}^{n} \frac{x}{x - z_k} = -\frac{1}{2} \left( \sum_{k=1}^{n} \frac{z_k + x}{z_k - x} - n \right),
\]

we get

\[
 |\text{Re}(xg_n(x))| = \frac{1}{2} \left| \sum_{k=1}^{n} P(z_k; x) - n \right|, \quad -1 \leq x \leq 1,
\]

where \( P \) is the Poisson kernel

\[
 P(v; x) := \text{Re} \left( \frac{v + x}{v - x} \right) = \frac{1 - x^2}{1 - 2x \text{Re} v + x^2} \geq 0, \quad -1 \leq x \leq 1, \quad |v| = 1.
\]

It can be checked that the set of all points \( z \in \mathbb{C} \), satisfying the inequality \( \text{Re} ((v + z)/(v - z)) \geq h \), coincides with the disk \(|z - vh/(h + 1)| \leq 1/(h + 1)\) for any given \( v \) (\(|v| = 1\)) and \( h > 0 \). We will be interested in the intersection of such disks with the unit interval, therefore, we will investigate the inequalities of the form \( P(v; x) \geq h \).

Fix \( \rho \in (0, 1/3] \) and put

\[
 T(h) = \sqrt{1 + \rho^2 - \frac{2\rho}{h}}, \quad 1 \leq h \leq \frac{2}{3\rho}.
\]

In the interval \( 1 \leq h \leq 2/(3\rho) \), the function \( T(h) \) is well defined \((1 + \rho^2 - 2\rho/h \geq (1 - \rho)^2)\) and strictly increasing. In particular,

\[
 T(h) \leq T(2/(3\rho)) = \sqrt{1 - 2\rho^2} < 1.
\]

**Lemma 1** Let \( |v| = 1, \ \rho \in (0, 1/3], \) and let

\[
 \text{Re} v \geq T(h) \quad \text{for some} \quad h \in [1, 2/(3\rho)].
\]

Then the inequality

\[
 P(v; x) \geq h, \quad x \in S,
\]

holds, where \( S = S(h) \) is the segment

\[
 S(h) = [x_-, x_+], \quad x_{\pm} := \frac{\sqrt{h^2 - 2\rho h + \rho^2 h^2} \pm (1 - \rho h)}{h + 1}.
\]
At the same time, for any \( h \in [1, 2/(3\rho)] \) the inclusion holds:

\[
S(h) \supset S^* := \left[1 - 3\rho/2, 1 - \rho\right], \quad \mu(S^*) = \rho/2.
\]

**Proof** Fix \( h \in [1, 2/(3\rho)] \). The inequality \( P(v; x) \geq h \) is equivalent to

\[
(h + 1)x^2 - 2hx \text{Re} v + h - 1 \leq 0,
\]

or

\[
\left(x - \frac{hd}{h + 1}\right)^2 \leq \frac{h^2(d^2 - 1) + 1}{(h + 1)^2}, \quad d := \text{Re} v. \tag{12}
\]

Here by \( d \geq T = T(h) \) and \( \rho h \leq 2/3 \) we have

\[
h^2(d^2 - 1) + 1 \geq h^2(T^2 - 1) + 1 = (1 - \rho h)^2 \geq 1/9.
\]

Hence, for every individual value of \( d, T \leq d \leq 1 \), the segment (12) is non-degenerate and has the endpoints

\[
A(d) = \frac{hd - \sqrt{h^2(d^2 - 1) + 1}}{h + 1}, \quad B(d) = \frac{hd + \sqrt{h^2(d^2 - 1) + 1}}{h + 1}.
\]

Clear that at points

\[
x \in \bigcap_{T \leq d \leq 1} [A(d), B(d)],
\]

the inequality \( P(v; x) \geq h \) holds for any \( d = \text{Re} v \in [T, 1] \). But

\[
\min\{B(d) : T \leq d \leq 1\} = B(T)
\]

and, because of the equality \( A(d) = (B(d))^{-1}(h - 1)(h + 1)^{-1} \),

\[
\max\{A(d) : T \leq d \leq 1\} = A(T).
\]

Thus,

\[
\bigcap_{T \leq d \leq 1} [A(d), B(d)] = [A(T), B(T)] = [x_-, x_+] = S(h).
\]

Let us prove the second part of Lemma. To do this, we consider \( x_- \) and \( x_+ \), the endpoints of the segment \( S \), as the functions of \( h \). We first have

\[
\frac{d}{dh}(x_-(h)) = \frac{(h - \rho) + \rho h (1 + \rho)}{\sqrt{h^2 - 2\rho h + \rho^2 h^2 (h + 1)^2}} + \frac{1 + \rho}{(h + 1)^2} > 0.
\]
Therefore,

\[ x_-(h) \leq x_-(2/(3\rho)) \quad (1 \leq h \leq 2/(3\rho)), \]

where by \( \sqrt{1-t} < 1-t/2 \) \((t < 1)\) and \( \rho \leq 1/3 \), we have

\[ x_-(2/(3\rho)) = \frac{2\sqrt{1-2\rho^2} - \rho}{2+3\rho} < 1 - 3\rho/2. \]

On the other hand, the following inequality may be checked directly:

\[ x_+(h) \geq 1 - \rho \equiv x_+(1) \quad (1 \leq h \leq 2/(3\rho)). \]

Thus,

\[ \bigcap_{1\leq h \leq 2/(3\rho)} S(h) = [x_-(2/(3\rho)), x_+(1)] \supset [1 - 3\rho/2, 1 - \rho]. \]

Lemma 1 is completely proved.

**Lemma 2** Let \(|v| = 1, \rho \in (0, 1/3] \), and let

\[ |\text{Re } v| < T(h) \quad \text{for some } \quad h \in [1, 2/(3\rho)]. \]

Then for any \( 0 < s < 3h/(2\rho) \) we have

\[ P(v; x) < s, \quad x \in S' := \{x \in [-1, 1] : |x| \geq 1 - 2s/3h \}. \]

**Proof** By the inequality (12), it suffices to show that

\[ \left| x - \frac{s \text{ Re } v}{s + 1} \right| > \frac{\sqrt{1 + s^2(\text{Re}^2v - 1)}}{s + 1} \quad \text{for } \quad x \in S'. \]

But, by \( |\text{Re } v| < T(h), \rho h \leq 2/3 \) and \( \sqrt{1-t} < 1-t/2 \), we have \( \rho^2 \leq 2\rho/(3h) \),

\[ |\text{Re } v| < \sqrt{1 + \rho^2 - \frac{2\rho}{h}} \leq \sqrt{1 - \frac{4\rho}{3h}} < 1 - \frac{2\rho}{3h} \]

and

\[ \sqrt{1 + s^2(\text{Re}^2v - 1)} < \sqrt{1 - \frac{4s^2\rho}{3h}} < 1 - \frac{2s^2\rho}{3h}. \]
Therefore, if \( x \in S' \) then
\[
\left| x - s \Re v \right| \geq |x| - s \left| \Re v \right| \geq \left( 1 - \frac{2s\rho}{3h} \right) - \frac{s}{s+1} \left( 1 - \frac{2\rho}{3h} \right) = \frac{1}{s+1} \left( 1 - \frac{2s^2\rho}{3h} \right),
\]
and Lemma 2 follows.

### 3 Proof of Theorem 2

Given \( n = 2, 3, \ldots \) and \( \delta \in (0, 1/2) \), put
\[
\nu := (1 + 2\delta)n, \quad \rho := \frac{2}{3\nu} = \frac{2}{(3 + 6\delta)n} \quad (\rho < 1/3),
\]
with the corresponding definition of the function \( T = T(h) \) (see (11)).

Let \( m \in \mathbb{N} \), \( \varepsilon_0 := 0 \), and let \( \varepsilon_1, \ldots, \varepsilon_m \) be positive numbers such that
\[
\varepsilon := \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_m < 1.
\]

Then for the quantities
\[
h_j := \nu^{1-\varepsilon_0-\cdots-\varepsilon_j} \quad (j = 0, \ldots, m)
\]
we have
\[
1 < \nu^{1-\varepsilon} = h_m < h_{m-1} < \cdots < h_0 = \nu = 2/(3\rho).
\]

The set of poles \( z_1, \ldots, z_n \) of a given function (2) may be represented in the form
\[
\{z_1, \ldots, z_n\} = I_0 \cup I_1 \cup \cdots \cup I_{m+1},
\]
where the subsets, \( I_j \), are defined as follows:
\[
T(\nu) \leq |\Re z_k| \leq 1 \quad \text{for} \quad z_k \in I_0;
\]
\[
T(h_j) \leq |\Re z_k| < T(h_{j-1}) \quad \text{for} \quad z_k \in I_j \quad (j = 1, \ldots, m);
\]
\[
|\Re z_k| < T(\nu^{1-\varepsilon}) \quad \text{for} \quad z_k \in I_{m+1}
\]
(the fact of monotonicity of the function \( T \) is used here).
We first assume that the set \( I_0 \) contains at least one pole, \( z_1 \) say, and
\[
T(v) \leq \text{Re } z_1 \leq 1.
\]

Then, by using Lemma 1, we get
\[
P(z_1; x) \geq v \quad \text{for } x \in S^* = \left[ 1 - 3\rho/2, 1 - \rho \right],
\]
while \( P(z_2; x) \geq 0, \ldots, P(z_n; x) \geq 0 \). Therefore (see (10))
\[
\left| \text{Re}(xg_n(x)) \right| \geq \frac{1}{2} (P(z_1; x) - n) \geq \frac{1}{2} (v - n) = \delta n, \quad x \in S^*.
\]

Thus, \( S^* \subset E \), and we obtain the desired estimate (6), because
\[
\mu(S^*) = \frac{\rho}{2} = \frac{1}{3(1 + 2\delta)n} > \frac{K(\delta)}{n},
\]
where \( K(\delta) \) is the constant defined in Theorem 2. Besides, the inclusion \( S^* \subset (E \cap \Delta) \) takes a place since \( 1 - 3\rho/2 = 1 - (1 + 2\delta)^{-1}n^{-1} \).

Further let
\[
I_0 = \emptyset.
\]

Define the quantities
\[
\alpha_j := \frac{\#I_j}{v^{\varepsilon_0 + \cdots + \varepsilon_j}} \quad (j = 1, \ldots, m),
\]
where \( \#G \) denotes the number of elements of a subset \( G \subset \{z_1, \ldots, z_n\} \); any two poles \( z_k, z_j \in G, k \neq j \), are considered as different elements, even if they coincide geometrically.

There are two possible cases: either \( \sum \alpha_j \geq 2 \) or \( \sum \alpha_j < 2 \).

Case 1:
\[
\alpha_1 + \cdots + \alpha_m \geq 2. \quad (14)
\]

For every \( j = 1, \ldots, m \), put \( I_j^+ := I_j \cap \{ \text{Re } z > 0 \} \), \( I_j^- := I_j \cap \{ \text{Re } z < 0 \} \),
\[
\alpha_j = \alpha_j^+ + \alpha_j^- , \quad \alpha_j^+ := \frac{\#I_j^+}{v^{\varepsilon_0 + \cdots + \varepsilon_j}} , \quad \alpha_j^- := \frac{\#I_j^-}{v^{\varepsilon_0 + \cdots + \varepsilon_j}}.
\]

If \( \sum \alpha_j^+ < 1 \) and \( \sum \alpha_j^- < 1 \) then \( \sum \alpha_j = \sum \alpha_j^+ + \sum \alpha_j^- < 2 \) contradicts (14). Therefore without loss of generality we will assume that
\[
\alpha_1^+ + \cdots + \alpha_m^+ \geq 1.
\]
By the definition of the sets $I^+_j$, we have

$$\text{Re } z_k \geq T(h_j), \quad z_k \in I^+_j \quad (j = 1, \ldots, m).$$

Hence, by Lemma 1,

$$P(z_k; x) \geq h_j, \quad z_k \in I^+_j \quad (j = 1, \ldots, m), \quad x \in S^*.$$

Let us emphasize that the segment $S^*$ is common for all $z_k \in (I^+_1 \cup \cdots \cup I^+_m)$.

From this and from $P(z_k; x) \geq 0, z_k \notin (I^+_1 \cup \cdots \cup I^+_m)$, we obtain (see (10))

$$|\text{Re}(xgn(x))| \geq \frac{1}{2} \left( \sum_{j=1}^{m} \sum_{z_k \in I^+_j} P(z_k; x) - n \right) \geq \frac{1}{2} \left( \sum_{j=1}^{m} h_j \cdot \# I^+_j - n \right), \quad x \in S^*.$$

But $h_j = v^{1-\varepsilon_0-\cdots-\varepsilon_j}, \# I^+_j = \alpha_j^+ v^\varepsilon_0+\cdots+\varepsilon_j$ and $\sum \alpha_j^+ \geq 1$, therefore

$$|\text{Re}(xgn(x))| \geq \frac{1}{2} \left( \sum_{j=1}^{m} \alpha_j^+ - n \right) \geq \frac{n}{2}((1 + 2\delta) - 1) = \delta n, \quad x \in S^*.$$

Thus, for any $\{\varepsilon_j\}$, in Case 1 we again obtain the estimate (6).

Case 2:

$$\alpha_1 + \cdots + \alpha_m < 2.$$

For every $j = 1, \ldots, m + 1$ we have (see (13))

$$|\text{Re } z_k| < T(h_{j-1}), \quad z_k \in I_j,$$

where $h_{j-1} = v^{1-\varepsilon_0-\cdots-\varepsilon_{j-1}}$. Let us apply Lemma 2 with

$$h = h_{j-1}, \quad s = s_{j-1} := \frac{1 - 2\delta}{3 + 4\delta} v^{\varepsilon_0+\cdots+\varepsilon_{j-1}}.$$

For such a choice of $h$ and $s$, we have for every $j$

$$\frac{2s_{j-1} \rho}{3h_{j-1}} = \frac{4(1 - 2\delta)}{9(3 + 4\delta)v^{2-\varepsilon}} < 1.$$

By Lemma 2,

$$P(z_k; x) < s_{j-1}, \quad z_k \in I_j \quad (j = 1, \ldots, m + 1),$$
for \( x \in S', \) where
\[
S' := \left\{ x \in [-1, 1] : |x| \geq 1 - \frac{4(1 - 2\delta)}{9(3 + 4\delta)\nu^{2-\varepsilon}} \right\}.
\]
From this with \( I_0 = \emptyset, \) \( \#I_j = \alpha_j \nu^{\varepsilon_0 + \cdots + \varepsilon_j} \) \((j = 1, \ldots, m)\) and \( \#I_{m+1} \leq n \) we have
\[
\Sigma(x) := \sum_{k=1}^{n} P(z_k; x) = \sum_{j=1}^{m} \sum_{z_k \in I_j} P(z_k; x) + \sum_{z_k \in I_{m+1}} P(z_k; x) < \sum_{j=1}^{m} s_{j-1} \cdot \#I_j + s_m n = \frac{1 - 2\delta}{3 + 4\delta} \left( \sum_{j=1}^{m} \alpha_j \nu^{\varepsilon_j + \varepsilon_j} + n \right), \quad x \in S'.
\]
Choosing \( \varepsilon_1 > 0, \ldots, \varepsilon_m > 0 \) as the solution of the system \( \varepsilon + \varepsilon_j = 1 \) \((j = 1, \ldots, m)\) or, equivalently,
\[
\left\{ \varepsilon_1 + \cdots + \varepsilon_{j-1} + 2\varepsilon_j + \varepsilon_{j+1} + \cdots + \varepsilon_m = 1 \right\}_{j=1}^{m},
\]
we obtain
\[
\varepsilon_1 = \cdots = \varepsilon_m = \frac{1}{m+1} \left( \varepsilon = 1 - \frac{1}{m+1} \right).
\]
Since yet \( \alpha_1 + \cdots + \alpha_m < 2, \) then for such \( \{\varepsilon_j\} \) we have
\[
\Sigma(x) < \frac{1 - 2\delta}{3 + 4\delta} \left( \sum_{j=1}^{m} \alpha_j \nu^{n} + n \right) < \frac{1 - 2\delta}{3 + 4\delta} (2\nu + n) = n - 2\delta n
\]
and, therefore,
\[
|\text{Re}(xg_n(x))| = (n - \Sigma(x))/2 > \delta n,
\]
\( x \in \tilde{S'} := \left\{ x \in [-1, 1] : |x| \geq 1 - \frac{4(1 - 2\delta)}{9(3 + 4\delta)\nu^{1+1/m+\varepsilon}} \right\}. \]
But \( m \) can be arbitrarily large, hence,
\[
|\text{Re}(xg_n(x))| > \delta n, \quad x \in S^{**} := \left\{ x \in [-1, 1] : |x| > 1 - \frac{1}{2} K(\delta)n^{-1} \right\},
\]
and we again obtain (6), because \( S^{**} \subset E, \mu(S^{**}) = K(\delta)n^{-1}. \) The inclusion \( S^{**} \subset (E \cap \Delta) \) follows from the inequality \( K(\delta)/2 < 1/(1 + 2\delta). \)
Theorem 2 is proved for all functions \( (2) \) with \( n \geq 2 \).

If \( n = 1 \), then the inequality (6) can be checked directly. Without loss of generality, we consider the functions
\[
g_1(x) = \frac{1}{x - x_1 - iy_1}, \quad \text{where} \quad x_1^2 + y_1^2 = 1, \quad x_1 \in [-1, 0].
\]

Then for \( x \in [0, 1] \), we have
\[
|\text{Re}(xg_1(x))| - \delta = \frac{x^2 - x_1x}{x^2 - 2x_1x + 1} - \delta \geq \frac{(1 - \delta)x^2 - \delta}{x^2 - 2x_1x + 1} \quad (\delta \in (0, 1/2)).
\]

Therefore,
\[
\mu(E_\delta(g_1)) \geq \mu \left( \{ x \in [0, 1] : x^2 \geq \frac{\delta}{1 - \delta} \} \right) = 1 - \frac{\sqrt{\delta}}{\sqrt{1 - \delta}} = \frac{1 - 2\delta}{1 - \delta + \sqrt{\delta - \delta^2}} > \frac{1 - 2\delta}{2} > K(\delta).
\]

Theorem 2 is completely proved.

**Remark 3** In particular, it is proved that the set \( E = E_\delta(g_n) \) contains an interval \((\alpha, \beta)\) such that \(|\beta - \alpha| \geq K(\delta)(2n)^{-1}\) and \((\alpha, \beta) \subset \Delta = \Delta_{n, \delta} \).

**References**

1. Abakumov, E., Borichev, A., Fedorovskiy, K.: Chui’s conjecture in Bergman spaces. Math. Ann. 379(3–4), 1507–1532 (2021). https://doi.org/10.1007/s00208-020-02114-1
2. Borodin, P.A.: Approximation by simple partial fractions with constraints on the poles II. Sb. Math. 207(3–4), 331–341 (2016). https://doi.org/10.1070/SM8500
3. Borwein, P.: The size of \( \{ x : r_n^2 / r_n \geq 1 \} \) and lower bounds for \( \| e^{-x} - r_n \| \). J. Approx. Theory 36(1), 73–80 (1982). https://doi.org/10.1016/0021-9045(82)90072-7
4. Chui, C.K.: A lower bound of fields due to unit point masses. Am. Math. Mon. 78(7), 779–780 (1971)
5. Chui, C.K.: On approximation in the Bers spaces. Proc. Am. Math. Soc. 40(2), 438–442 (1973)
6. Erdélyi, T.: Turán-type reverse Markov inequalities for polynomials with restricted zeros. Constr. Approx. 54(1), 35–48 (2021). https://doi.org/10.1007/s00365-020-09509-y
7. Goverov, N.V., Lapenko, Yu.P.: Lower bounds for the modulus of the logarithmic derivative of a polynomial. Math. Notes 23(4), 288–292 (1978). https://doi.org/10.1007/BF01786958
8. Komarov, M.A.: A lower bound for the \( L_2[-1, 1]\)-norm of the logarithmic derivative of polynomials with zeros on the unit circle. Probl. Anal. Issues Anal. 8(2), 67–72 (2019). https://doi.org/10.15393/j3.art.2019.6030
9. Komarov, M.A.: On Borwein’s identity and weighted Turán type inequalities on a closed interval. Trudy Inst. Mat. Mekh. 28(1), 127–138 (2022). https://doi.org/10.21538/0134-4889-2022-28-1-127-138. ([in Russian])
10. Komarov, M.A.: Reverse Markov inequality on the unit interval for polynomials whose zeros lie in the upper unit half-disk. Anal. Math. 45(4), 817–821 (2019). https://doi.org/10.1007/s10476-019-0009-y
11. Komarov, M.A.: The Turán-type inequality in the space \( L_0 \) on the unit interval. Anal. Math. 47(4), 843–852 (2021). https://doi.org/10.1007/s10476-021-00097-3
12. Newman, D.J.: A lower bound for an area integral. Am. Math. Mon. 79(9), 1015–1016 (1972). https://doi.org/10.2307/2318074

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