Quantum revivals in free field CFT

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Abstract

The recent work by Cardy (arXiv:1603.08267) on quantum revivals and higher dimensional CFT is revisited and enlarged upon for free fields. The expressions for the free energy used here are those derived some time ago. The calculation is extended to spin–half fields for which the power spectrum involves the odd divisor function. An explanation of the rational revivals for odd spheres is given in terms of wrongly quantised fields and modular transformations. Comments are made on the equivalence of operator counting and eigenvalue methods, which is quickly verified.

Keywords: quantum revivals, CFT, modular transformations

(Some figures may appear in colour only in the online journal)

1. Introduction and summary

In a discussion of the topic of quantum revivals, on quenching, in the context of conformal field theory, Cardy [1] extends his previous 2d analysis [2] to higher dimensions. In doing so he encounters, for free fields, what is essentially finite temperature theory on a generalised Einstein universe—i.e. a ‘generalised torus’; more particularly a generalised cylinder $I \times \mathcal{M}$. $I$ is an interval and $\mathcal{M}$ here is, specifically, a $d$–sphere $^1$.

The precise revival object sought is the return amplitude $A(t, \epsilon) = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle$ for some quenching initial state, $| \psi_0 \rangle$. $A$ is a function of the evolution time, $t$, and whatever parameters $| \psi_0 \rangle$ depends on. $H$ is the (free) field Hamiltonian.

For the particular, conformal choice of $| \psi_0 \rangle = e^{-iH/4} | D \rangle$, $| D \rangle$ being a Dirichlet boundary state, $A(t, \beta)$ is determined by the partition function, $Z$, on a generalised cylinder, with Dirichlet conditions $I(\beta/2) = [0, \beta/2]$ on the interval. The specific expression is

$$
\log A(t, \beta) = \frac{1}{2} \Re \left( \log Z(\mathcal{M} \times I(\beta/2 + i \epsilon)) - \log Z(\mathcal{M} \times I(\beta/2)) \right),
$$

(1)

$^1$ My $d$ differs from that in [1] by 1.
For fixed $\beta$, the second term can be omitted for graphical purposes. The parameter $\beta$ is ultimately interpreted as an inverse temperature and I will formally treat it as such. The approach to thermality of $|\psi_0\rangle$ is analysed in [1].

To further the evaluation of $A$, I wish to draw attention to earlier calculations [3–5] of $\log Z$ as the log determinant of the relevant propagating operator, $\mathcal{O}$, on the cylinder:

$$\frac{1}{2} \log \det \mathcal{O} = -\frac{1}{2} \zeta'(0).$$

(2)

I have used the $\zeta$-function definition of a functional determinant.

The free energy has also been derived in [6].

In this paper, I present some technical remarks, in the free field setting—taking this earlier work into account—and make some connections which might be interesting. Extensions to the analysis are also undertaken, as indicated in the following paragraph.

The fundamental equations are recalled in section 2, closely following [1], and the return amplitude evaluated using earlier found expressions for the partition function. The results, of course, are the same as in [1]. Section 3 discusses the consequences of modular invariance, introduced a little differently to [1] with the same conclusion but for all dimensions. In section 4, these calculations are repeated for the spin-half, fermion field—a case not considered in [1]. The power spectrum is given in section 5, and found to involve the odd divisor function. Section 6 treats the situation in which the spatial $d$-sphere is quotiented by a cyclic group. An explanation of the sign reversed revivals using ‘wrongly quantised’ fields is made in section 7, using modular behaviour. In an appendix, the equivalence of the operator counting and eigenvalue methods is verified.

2. The calculations

The first step in the earlier evaluation of the determinant was to rearrange the interval modes. In [3–5] this was neatly expressed in $\zeta$-function terms as the relation

$$\zeta(I \times \mathcal{M}) = \frac{1}{2}(\zeta(S^1 \times \mathcal{M}) \mp \zeta(\mathcal{M})),
$$

between, on the left, the $\zeta$-function on the generalised cylinder, and, on the right those on the generalised torus, $\zeta(S^1 \times \mathcal{M})$, and cross section manifold, $\zeta(\mathcal{M})$. The sign $\mp$ gives Neumann and Dirichlet conditions on the interval.

Inserted into (1), the second term cancels, being $\beta$-independent. This conclusion is also reached by Cardy a little differently.

Equation (1) can therefore be replaced by

$$\log A(t, \beta) = \frac{1}{4} \text{Re} (\log Z(\mathcal{M} \times S^1(\beta/2 + it)) - \log Z(\mathcal{M} \times S^1(\beta/2))).$$

(3)

As noted and used in [3–5], the main problem then reduces to a thermal one on the Einstein universe, which is a topic with a history, (e.g. [7–10]). $S^1$ can be referred to as the thermal circle. Consult also [11].

In general, in $\zeta$-function regularisation on any manifold, the effective action, essentially $\log Z$, consists of a divergence, with an associated logarithm, plus a finite part which has the form (2).

The divergence and logarithmic parts are controlled by the conformal anomaly on $\mathcal{M} \times I$ and a simple argument shows that for conformal coupling (for odd dimensional spheres) this anomaly is zero. It is automatically zero for even spheres.
This being so, on the basis of $\zeta$-function regularisation one can set, in (3), for bosons,

$$\log Z = \frac{1}{2} \zeta'_0(0),$$

where $O$ is the thermal propagation operator.

Notationally, from now on, in order to avoid confusion, I set $\Xi = \log Z$. This includes the zero temperature part, $\Xi_0$, and totally $\Xi = \Xi_0 + \Xi'$, $\Xi'$ being the finite temperature correction. The reason for this definition is that in [1] and elsewhere, $Z$ refers to the usual partition function, the sum over Fock space states—i.e. $\log Z$ is just $\Xi'$. Actually it does not matter whether $\Xi$ or $\Xi'$ is used in (1) as $\Xi_0$ cancels on taking the difference and real part$^2$.

The general statistical sum equation (31) or, equivalently (55), in [12] can be written to give

$$\beta F = \beta E_0 - \sum_{m=1}^{\infty} \frac{1}{m} K^{1/2}(m\beta).$$  

(4)

$F$ is the conventional free energy, $\beta F = -\Xi$ and $K^{1/2}$ is the degeneracy generating function or, equivalently, the ‘cylinder kernel’ for the pseudo operator (Hamiltonian) $\sqrt{D}$, $D$ being the propagating operator on $\mathcal{M}$. In the present instance this is related to the conformally invariant (Yamabe–Penrose) Laplacian, $\mathcal{Y}$, by $D = \mathcal{Y} + 1/4$. $K^{1/2}$ is a single particle sum-over-states partition function. $E_0$ is the zero temperature, Casimir energy.

For the present spherical situation, the expression is given explicitly in [6], equation (78), for $\mathcal{M}$ an orbifold quotient of the sphere, $S^d$, in particular for a hemisphere, and thence, by addition, for a full sphere. The expressions for this latter case can also, conveniently, just be read off from [3–5].

Since, notationally, I am generally adhering to [1], I give the relation with the parameters used in [3–5] (shown first),

$$L = \frac{\beta}{2}, \quad a = L/\pi.$$  

$L$ is the sphere radius; In order to simplify the exposition, I set now $a = 1$ i.e. $L = 2\pi$.

The expressions given in [3–5] and [6] imply that$^3$

$$\Xi'_d(\beta) = \frac{1}{2d} \sum_{m=1}^{\infty} \frac{1}{m} \cosh(m\beta/2) \coth(d(m\beta/2)),$$  

(5)

which holds for odd and even sphere dimensions.

As remarked, it is sufficient to use (5) to display the return amplitude, (1). Figures 1 and 2 plot $\Xi'_d(\beta + 2\pi)$ for $d = 2$ and $d = 3$. $s$ equals $2\pi a$.

These are also both plotted in [1] but I have extended the horizontal range a little in order to make the periodicity clearer.

3. Modular invariance

Cardy relates the partial revivals at rational values of $s$, evidenced by the maxima in the curves, to the modular properties of the free energy. Actually, it is the internal energy (including the Casimir term) that has the simpler behaviour (for odd $d$). In the Einstein universe, this was early recognised [7, 13], and also was discussed by Cardy [14] in a conformal field theory context—specific higher dimensions being considered. A few comments occur in [15], and a

$^2$This being the case, one does not really need the full apparatus of $\zeta$-function regularisation.

$^3$To save space, I do not give any derivation of these expressions but an outline of one can be found in the appendix.
more extensive analysis was given in [16] for odd dimensional spheres. Further calculations are provided in [17] for different field contents and an extension to $\text{AdS}_d$.

I recapitulate a few details of [16]. The mode structure (eigenlevels and their degeneracies) on spheres is an ancient topic and needs no explanation. The upshot is that, in the usual description, the conformal eigenlevels are squares of integers, say $n^2$, with degeneracies that are polynomials in $n^2$.

It is best, for present purposes, to treat the terms in this polynomial individually and then, if required, reconstitute the full expression. Selecting the $(2l - 2)$ power, the standard statistical formula gives for the corresponding ‘partial’ energy

$$
\epsilon_l(\xi) = \epsilon_l(0) + \sum_{n=1}^{\infty} \frac{n^{2l-2}q^{2n}}{1 - q^{2n}}
\equiv \epsilon_l(0) + \epsilon'(\xi). 
$$

Figure 1. Log return amplitude, $d = 2$.

Figure 2. Log return $d = 3$.

(6)

Here $\epsilon'$ is the energy finite temperature correction.

I have rescaled the energy by the sphere radius $a (=1$ here), introduced the dimensionless parameter $\xi = 2\pi a l / \beta$ and defined $q = \exp(-\pi l / \xi)$. The quantity $\epsilon_l(0)$ is the (scaled) partial Casimir energy. Explicitly,

$$
\epsilon_l(0) = -\frac{B_{2l}}{4l},
$$

(7)
in terms of Bernoulli numbers, $B$.

The easiest way of showing the inversion symmetry,
\[
\frac{1}{\xi^l} \epsilon_l(\xi) = (-1)^l \xi^l \epsilon_l(1/\xi),
\]

is to relate \(\epsilon\) to a (holomorphic) Eisenstein series,

\[
G_l(\omega_1, \omega_2) \equiv \sum_{m_1, m_2 = -\infty}^{\infty} \frac{1}{(m_1 \omega_1 + m_2 \omega_2)^{2l}},
\]

by

\[
\epsilon_l(\xi) = (-1)^l C(l) G_l(1, il\xi)
\]

\((C(l)\) is an inessential constant). This connection is a basic result in analytic number theory.
The inversion symmetry, (8), follows immediately. More generally, the expression is invariant under the modular group action on the periods \(\omega_1, \omega_2\). The translational generator is \(b \rightarrow b - i\); for convenience and to agree with a previous notation \([18]\), I have put \(b \equiv \xi^{-1}\).

Note that the Casimir energy appears naturally. If it is extracted according to (6), equation (8) reads

\[
\frac{1}{\xi^l}(\epsilon_l(0) + \epsilon_l(\xi)) = (-1)^l \xi^l (\epsilon_l(0) + \epsilon_l(1/\xi)).
\]

In this form, the identity can be traced back at least to Ramanujan, see \([16]\).

To relate the high and low temperature regimes, let \(\xi\) become large in (10). From its form, \(\epsilon'(1/\xi)\) tends to zero exponentially fast\(^4\)—so, up to the terms

\[
\epsilon_l(\xi) + \epsilon_l(0) \approx (-1)^l \epsilon_l(0) \xi^{2l} \equiv \sigma_l \xi^{2l} \sim \sigma_l T^{2l},
\]

connecting high temperature on the left, to low temperature (the \(\epsilon_l(0)\)) on the right. The right-hand side is the (partial) Planck term\(^5\) and \(\sigma_l\) is a (partial) Stefan–Boltzmann constant, which is positive, using (7).

For the \(d\)-dimensional sphere, the actual energy, \(aE\), is a sum of \(\epsilon_l\) where \(l\) runs from 2 to \((d + 1)/2\) and the high temperature behaviour is a sum of terms like (11). That there are only a finite number agrees with the general expression for the high temperature limit in terms of the heat-kernel coefficients \([12, 19]\). This is because the conformal heat-kernel expansion terminates on odd spheres (up to exponential corrections). The inversion properties of \(E\) are therefore not straightforwardly expressed, apart from the three-sphere when \(l = 2\). However, for many purposes, the dominant term is given by \(l = (d + 1)/2\) and suffices.

More logically, the expansion for the free energy \((\beta F = -\Xi)\) would be derived, as in \([12]\), from first principles—the energy following by differentiation \((E = \partial (\beta F) / \partial \beta)\). To utilise the inversion behaviour, (8), this procedure is reversed (see \([1]\)).

According to the expression (1) for the return amplitude, one needs to make the replacement \(\xi^{-1} \rightarrow \xi^{-1} + i\) in the thermodynamic quantities. The analysis is eased by taking \(\xi\) large. Then the revivals at rational \(s\) are more pronounced because the initial value (the Planck term) is large when \(\xi \rightarrow \infty\) and \(s \rightarrow 0\). In this case it is more convenient to use \(b\). The standard Planck contribution to the free energy then says that the initial (partial) log return amplitude, (1), is

\[
\log A_l(s) \approx C_l \text{Re} \frac{1}{(b + is)^{2l-1}}, \quad s \text{ small},
\]

where the constant \(C_l = 2\pi \sigma_l/(2l - 1)\).

\(^4\)This is generally true, for finite systems \([12]\).

\(^5\)This corresponds to the Weyl universal term in the asymptotic distribution of eigenvalues.
The total amplitude is a linear combination of the $\log A_i$. For small enough $b$ and $s$ this is dominated by the first term, i.e. the one with the largest $l_i = (d+1)/2^s$. It can be checked numerically that this gives a good approximation to the complete quantity obtained from (5).

Because of the translational invariance under $b \rightarrow b \pm i$, there will be identical copies of the initial behaviour, (12), around integral $s$. Multiple translations, combined with inversion replicates this behaviour, with reduced amplitude, at rational $s$. The argument is as follows [1].

The modular relation, (10), is written in terms of the partial finite temperature correction part of $\Xi$, $\Xi'$, which is the quantity plotted. Then,

$$\frac{\partial}{\partial \xi} \Xi'(\xi) = (1)^{n+1} \zeta 2n-2 \frac{\partial}{\partial \xi} \Xi'(1/\xi) + (1)^{n+1} 2\pi \epsilon(0) ((1)^{n+1} - \xi^{-1})$$

(13)

The simplest case is one inversion combined with a translation of $\xi^{-1}$ by $im$, with $m$ integral. This converts the region around $s = 1/m$ to that around $s = 0$. The first region is accessed by setting $\xi^{-1} \approx \frac{\beta}{2\pi} + im + i\epsilon$ with $\epsilon$ (and $\beta$) small. One requires the left-hand side in this region. This is provided by the right-hand side. Then one has $\xi \approx -im + \frac{\beta}{2\pi} m^2 + i\epsilon m^2$. Translating away the $im$ gives $\xi \rightarrow m^2(1/2\pi + i\epsilon)$. Since this is small, the right-hand side is well approximated by the high temperature form (11), on ignoring the last term. Hence

$$\Xi'_{\xi=1/m+\epsilon} \sim (1)^{n+1} (1/m)^{2n-2} \frac{1}{(b + i\epsilon)^{2n-1}m^{d-2}}$$

This shows that the profile around $s = 1/m$ is the same as that around $s = 0$, except for a reduction in amplitude by a factor of $1/m^{d+1}$. For the dominant term this equals $1/m^{d+1}$ which can be checked numerically from the complete expression, (5).

Repeating this procedure [2] allows the revivals at the rational points $n/m$ to be obtained. The result is the same, with the amplitude reduction still $1/m^{d+1}$.

This analysis is just for odd spheres, but the complete expressions, (5), hold for all $d$ and it is these that have been plotted in the previous figures. It is noticed empirically that there are also revivals for even $d$, but the structure is not straightforwardly analysed.

4. Spin-half

The analysis can be repeated for spin-$1/2$ fields. Mixed boundary conditions are conformally invariant. There are two such boundary conditions, which yield identical spectral results and are effectively fermionic (i.e. antisymmetric on the thermal circle). The relevant formulae on the generalised cylinder and torus are given in [3, 4].

The fermion effective action leads to,

$$\Xi'_d = \beta E'_d + \Xi'_d(\beta),$$

(14)

where $E'_d$ is the classic Dirac Casimir energy on the $d$-sphere and the correction $\Xi'_d$ has been found to be,

$$\Xi'_d(\beta) = \frac{2}{2(d-1/2)} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \csc \frac{m}{2}}{m}$$

$$= \frac{2}{2(d-1/2)} \sum_{m=1}^{\infty} \frac{1}{m} \left[ \csc \frac{m}{2} - \csc \frac{m}{2} \right]$$

(15)

It is conventional, in a general manifold, to refer to just this term as the Planck term, although it is not strictly thermal. Cardy refers to it as the Casimir term.
As explained before, just the second term on the right-hand side of (14) is plotted. As examples, revivals, partial and complete, can be seen for the $d = 1$, $d = 2$ and $d = 3$ cases shown in figures 3–5.

As for the scalar field, these can be traced to the modular invariance of the spinor partial total internal energy, $\eta(\xi)$, which is related, for odd $d$, to a doubly sign-modulated Eisenstein series,

$$H_d(\omega_1, \omega_2) \equiv \sum_{m_1, m_2 = -\infty}^{\infty} \frac{(-1)^{m_1 + m_2}}{(m_1 \omega_1 + m_2 \omega_2)^{\beta}},$$

through [16].
\[ \eta_l(\xi) = (-1)^{l+1} C_l(l) H_l(1, il\xi). \]

\(C_l\) is an unrequired constant.

\(\eta_l(\xi)\) therefore enjoys the same inversion properties, for odd spheres, as the scalar quantity, \(\epsilon_l(\xi)\). The translation behaviour is altered because of the twisting. For odd spheres the periodicity is now 2. This also follows from the total expression (15). In figure 5, the complete period is obtained by reflecting in the line \(s = 1\).

Because of this periodicity, the attenuations to \(s = 1/m\) from \(s = 0\), for \(m\) even, and from \(s = 1\), for \(m\) odd, both equal \(1/m^{l+1}\). A more extended analysis is presented in section 7.

The maximum at \(s = 1\) can be investigated from the forms (15) which show, in addition, that the periodicity is 1 for even spheres.

The underlying mechanism giving rise to revivals at the rationals for even spheres has yet to be elucidated.

5. The fermion power spectrum

Since Cardy provides a treatment of the scalar case, I need present only the fermion analysis.

Working in terms of the partial quantities (therefore only odd spheres are covered) the Fourier series form of the \(q\)-series for \(\eta_l\) is standard elliptic fare. Glaisher [21, 20] conveniently has the requisite lists. The fermion series is (see [16] equation (20))

\[ \sum_{n=0}^{\infty} (2n + 1)^{2l-1} \frac{q^{2n+1}}{1 + q^{2n+1}} = \sum_{n=1}^{\infty} (-1)^{n-1} \Delta_{2l-1}(n) q^n, \]  \hspace{1cm} (17)

where \(\Delta_k(n)\) is the odd divisor function related to the usual one, \(\sigma_k\), by

\[ \Delta_k(n) = \sigma_k(n) - 2^k \sigma_k(n/2), \]

with \(\sigma_k\) at a half-integer defined zero.

In the case under consideration, \(q\) takes the form \(q = e^{-\beta/2 + i\pi x}\).

Equation (17) refers to the energy. To find \(\Xi'\) an integration with respect to \(\beta\) yields the factor \(2/n\). Then, taking the real part gives

\[ \text{Re} \Xi' \approx 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \Delta_{2l-1}(n) e^{-\beta n^2/2} \cos(\pi ns), \]

implying the power spectrum amplitude

\[ \frac{2}{n} \Delta_{2l-1}(n) e^{-\beta n^2/2}. \]

Figure 6 plots this out for \(l = 2\), corresponding to the three-sphere. The apparently different curves are due to the behaviour of the odd divisor function.

\(n\) measures the frequency in units of 1. Even frequencies are suppressed, some severely. The curves for the higher spheres are similar in shape but more extreme.

6. Spherical factors

Taking quotients of the sphere does not destroy any conformal properties and one can pursue the same path to find the return amplitude. However, the exact inversion behaviour is lost [22].

The free energy, and hence \(\Xi\), were given in [6] for the quotients by a regular solid symmetry group, \(\Gamma\). In particular for the cyclic case (\(\Gamma = Z_B, B \in Z\)) the formula easily gives (see also [22])
\[ \Xi_{d}(\beta, B) = -\beta E_{d,0}(B) + \Xi'_{d}(\beta, B), \]

where,

\[ \Xi'_{d}(\beta, B) = \frac{1}{2d} \sum_{m=1}^{\infty} \frac{1}{m} \coth(mb/2) \cosech^{d-1}(m/2). \]

Again, the Casimir term, \( E_{0} \), can be ignored for plotting and, as an example, the log return for an odd dimensional \( B = 3 \) periodic lune is shown in figure 7. The period is 1 and there are returns at the rationals, just as for the full sphere, \( B = 1 \). Only the vertical scale changes as \( B \) varies, despite the lack of inversion symmetry for \( B \neq 1 \). In fact the attenuation factor is independent of \( B \).

Taking the \( B \to \infty \) limit one finds

\[ \Xi'_{d}(\beta, \infty) = \frac{1}{2d} \sum_{m=1}^{\infty} \frac{1}{m} \cosech^{d-1}(m/2), \]

which is, to a factor, the \( \Xi' \) for a bosonic spinor in one fewer dimension. (See next section). At the moment, I have no explanation for this fact.

7. Spin-zero fermions and spin-half bosons

Cardy [1] explained the integer revivals at \( s = 1, 3, \ldots \) for even spheres, figure 1, in terms of a fermionic scalar \( \log Z \). A more extensive treatment is possible for odd spheres, as I now show.
First I present some facts which result from an inspection of Glaisher’s tables in [21]. Referring to that on p 64, I compare the q-series second from top with the one second from bottom, viz.,

\[ X'(q) = \sum_{n=1}^{\infty} \frac{(2n+1)2^{l-1}q^{2n+1}}{1 - q^{2n+1}} \]

\[ Y'(q) = \sum_{n=0}^{\infty} \frac{(2n+1)2^{l-1}q^{2n+1}}{1 - q^{2n+1}}. \]

\[ X' \] corresponds to the (partial) internal energy correction of a kinematic spin-zero field thermalised as a fermion, and \[ Y' \] to that for a kinematic spin-half field thermalised as a boson.

For convenience, I give the corresponding standard boson and fermion series discussed above (as in (6) and (17)):

\[ B'(q) = \sum_{n=1}^{\infty} \frac{(2n-1)2^{l-1}q^{2n}}{1 - q^{2n}} \]

\[ F'(q) = \sum_{n=0}^{\infty} \frac{(2n+1)2^{l-1}q^{2n+1}}{1 + q^{2n+1}}. \]

Under translation, \( q \to -q \), it is easily seen that \( F'(-q) = -Y'(q) \). This explains the inverted revivals depicted in figure 5, as I now show in detail.

The modular behaviour under inversion \( q \to \tilde{q} = e^{-\mu \pi / \xi} \log_l 2 \) can be determined from the double sum representations given in the table. These are singly twisted Eisenstein series for \( X' \) and \( Y' \).

Glaisher’s manipulations allow one to write (with \( \mu = \pi / \xi \))

\[ \frac{(2l - 1)!}{4} \sum_{n=-\infty}^{\infty} \frac{(1)^{n}q^{n+1}}{(s + r + \xi n)^{2l}} = X'(q) + 2^{l-1}B_{2l}/4 = \frac{e^{(f)(\xi)} + e^{(f)(0)}}{2^{l-1}} \]

\[ \frac{(2l - 1)!}{4} \sum_{n=-\infty}^{\infty} \frac{(1)^{n}q^{n+1}}{(s + r + \xi n)^{2l}} = Y'(q) + (2^{l-1} - 1)B_{2l}/4l = \frac{e^{(b)(\xi)} + e^{(b)(0)}}{2^{l-1}} \]

\[ \frac{(2l - 1)!}{4} \sum_{n=-\infty}^{\infty} \frac{(1)^{n}q^{n+1}}{(s + r + \xi n)^{2l}} = F'(q) - (2^{l-1} - 1)B_{2l}/4l = \frac{e^{(f)(\xi)} + e^{(f)(0)}}{2^{l-1}} \]

\[ \frac{(2l - 1)!}{4} \sum_{n=-\infty}^{\infty} \frac{(1)^{n}q^{n+1}}{(s + r + \xi n)^{2l}} = B'(q) - 2^{l-1}B_{2l}/4l = \frac{e^{(b)(\xi)} + e^{(b)(0)}}{2^{l-1}}. \]

To accord with my previous notation I have \( \mu = \pi / \xi \), and on the right have dropped the \( l \) label to avoid overload. Also \( f \) means fermionic, \( b \) bosonic, \( s \) scalar and \( d \) Dirac spinor.

As the notation indicates, \( 2^{l-1}B' \) is the finite temperature correction part, \( e' \), of the (partial) internal energy for the conventional scalar, see (6). \( F' \) is the corresponding quantity, \( \eta' \), for the conventional spinor. Similarly, \( 2^{l-1}X' \) and \( Y' \) are the finite temperature corrections for an anti-commuting scalar and commuting spinor respectively. The quantities on the right-hand side of (18) are, to a factor, the total internal energies—and it is these that satisfy the simplest modular behaviour, as I show below. I denote them by \( X, Y, F \) and \( B \). The vacuum energies of normal and odd fields with the same kinematics are equal and opposite, as expected.
\[
X'_q(q) + 2^{2l-1} \frac{B_{2l}}{4l} = -\left( 2l - 1 \right) \frac{1}{4} \sum_{-\infty}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^r}{(s\mu + r\pi)^{2l}}
\]
\[
= -\left( 2l - 1 \right) \frac{1}{4} \mu \sum_{-\infty}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^r}{(s\pi + r\mu)^{2l}}
\]
\[
= -\left( 2l - 1 \right) \frac{1}{4} \left( \mu \pi \right) \sum_{-\infty}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^r}{(s\pi + r\mu)^{2l}}
\]
\[
= -\left( 2l - 1 \right) \frac{1}{4} \left( \mu \pi \right) \sum_{-\infty}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^r}{(r\pi + s\mu)^{2l}}
\]
\[
= -\left( -1 \right) \left( \mu \pi \right) \left[ \frac{1}{2} \right] \left( \frac{B_{2l}}{4l} \right). \quad (19)
\]

Therefore, the finite temperature corrections are related by\(^7\).

\[
\left( \frac{\pi}{\mu} \right) X'_q(q) = -\left( \frac{\mu}{\pi} \right) Y'_q(q) - \left[ 2^{2l-1} \left( \frac{\pi}{\mu} \right)^l + \left( \frac{\mu}{\pi} \right)^l \left( 2^{2l-1} - 1 \right) \right] \frac{B_{2l}}{4l} \quad (20)
\]

or

\[
\xi^l X'\xi^{-1} = -\left( -1 \right) \left( \xi^{-1} \right) \frac{1}{\xi^l} Y'\xi^{-1} - \left[ 2^{2l-1} \xi^l + \left( -1 \right) \frac{1}{\xi^l} \left( 2^{2l-1} - 1 \right) \right] \frac{B_{2l}}{4l}, \quad (21)
\]

where I have reverted to \( \xi = \pi/\mu \) and, for ease, use \( X(\xi) \equiv X(q) \) etc.

The effect of translations (\( q \to -q \) or \( 1/\xi \to 1/\xi + i \) or \( \mu \to \mu + \pi i \) or \( s \to s + 1 \)) follows easily from the \( q \)-series

\[
B_l(\xi) = B_l(q), \quad X_l(-q) = X(q) \quad \text{and} \quad F_l(-q) = -Y(q). \quad (22)
\]

The most interesting is the final transformation, which interchanges commuting and anti-commuting spinors with a sign change.

As before, the inversion relation (21) leads to the high temperature behaviour of \( X \) and \( Y \). Letting \( \xi \to \infty \) and \( \xi \to 0 \) in turn gives

\[
Y'\xi^{-1} \to -\left( -1 \right) \frac{1}{\xi^l} Y(\xi) - \left[ 2^{2l-1} \xi^l + \left( -1 \right) \frac{1}{\xi^l} \left( 2^{2l-1} - 1 \right) \right] \frac{B_{2l}}{4l},
\]

\[
X'\xi^{-1} \to -\left( -1 \right) \left( 2^{2l-1} - 1 \right) \frac{B_{2l}}{4l} \xi^l. \quad (23)
\]

These results explain the revivals at the rationals for the fermion plots. The mechanism runs exactly parallel to that given by Cardy [1], which was outlined above, and the details need not be repeated. Taking the revivals at \( s = 1/m \), as before, the explanation involves an inversion, followed by a translation through \( m \). It is easiest to treat \( m \) even and \( m \) odd separately for then the relevant translation in (22) reads,

\[
F_l(-1)^{2s} q = F_l(q), \quad F_l((-1)^{2s+1} q) = -Y_l(q). \quad (23)
\]

\(^7\)This is one of class of identities referred to by Berndt [23], as ‘hybrid’. See his theorem 5.6.
The first equality means that the previous scalar analysis holds unchanged for $F$ and the revival attenuation is again $1/m^{d+1}$, for $m$ even. The same argument applies to $Y(q)$. Shifting the origin to $s = 1$, $Y(q)$ has revivals at $1/m'$ where $m'$ is even. $m$ and $m'$ are related by $m' = m - 1$, because of the shift. Hence from (23), $F$ will exhibit sign reversed revivals of $Y$ at $s = 1/m$ for $m$ odd. The attenuation for these two series of revivals is $1/m^{d+1}$ from $F(s = 0)$ and $F(s = 1)$.

The relevant ratio in sizes between these two sets of revivals is given by the ratio of the corresponding Stefan–Boltzmann constants at the two starting points, $s = 0,1$, i.e.

$$(1 - 2^{-2l+1}).$$

For $l = 2$, which is a good approximation for the three-sphere, this ratio equals $7/8$ which can be verified numerically from the complete expression for $\Xi' \sim \log Z$, also plotted in figure 3.

These results are for the partial energies, but, for any $d$, the complete expressions are available for the partition function, $\Xi'$, (5) and (15), for normal bosons and fermions. Then, for example, the normal spinor can be translated from $s = 0$ to $s = 1$ to give

$$-\sum_{m=1}^{\infty} (-1)^m \frac{1}{m} \cosh^{d}(m\beta + 2\pi i/2)$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{1}{m} (-1)^m \cosh^{d}(m\beta/2)$$

$$= -\sum_{m=1}^{\infty} \frac{1}{m} \cosh^{d}(m\beta/2), \quad d \text{ odd}, \quad (24)$$

and this is minus the $\Xi'$ for a spinor boson, in agreement with (22), for the energy.

If $d$ is even, the utility of partial quantities disappears. Only the complete expressions are available, and then just for translations. For scalars, one has,

$$\Xi_d(\beta + 2\pi i) = \frac{1}{2d} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cosh^{d+1}(m\beta/2) \cosh^{d}(m\beta/2). \quad (25)$$

Hence, if $d$ is even,

$$\Xi_d(\beta + 2\pi i) = \frac{1}{2d} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cosh^{d+1}(m\beta/2) \cosh^{d}(m\beta/2), \quad (26)$$

which is the negative of a fermionic scalar.

In this way, the integer sign reversed revivals in, say, figure 1 can be understood, as noted in [1] (obtained slightly differently).

Glaisher also gives the power spectra of these two ‘systems’. That for the bosonic spinor is the same as that for the normal spinor, figure 6. The fermionic scalar spectrum for $l = 2$ is given in figure 8.

I recast some of the above more abstractly in terms of the $S$ (inversion) and $T$ (translation) generators of the modular group. For this purpose, it is algebraically convenient to absorb the factors of $\xi^2$ in (8) by defining

$$\tau_1(\xi) = \frac{1}{\xi^2} \sigma(\xi)$$

so that $\tau_1(1/\xi) = (-1)^l \tau_1(\xi)$, and likewise for $X_l, Y_l, F_l$ and $B_l$.

The actions of the modular group can be summarised as
so that the matrix representations are
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

I separate \(\overline{B}\) as it does not mix.

It is seen that, in addition to the usual relations, \((\text{viz. } S^2 = 1, (ST)^3 = 1)\), one has \(T^2 = 1\).

I note that \(X = ST F\).

The behaviour around a rational \(s = p_1/p_2\) can be obtained from the modular transformation connecting the origin to the point \(s = p_1/p_2\). This is achieved, as said in [2], using the transformation \(T^n S T^n S \ldots\) where the \(a_i\) are the integers in the continued fraction form of \(p_1/p_2\). (The order of the operators is such as to take \(s = p_1/p_2\) back to \(s = 0\).) It is clear from the above actions that one gets either \(F\), \(-Y\) or \(X\), depending on the particular rational. The behaviour around any rational, if it can be discerned, could therefore be considered as an attenuated revival of one of these three functions, evaluated at its origin. Visible examples in figure 5 are at \(s = 1/2\) (giving \(F\)), at \(s = 1/3\) (giving \(-Y\)) and at \(s = 2/3\) (giving \(X\)).

I finally note that \(B,F,X\) and \(Y\) correspond to the set of elliptic functions, \(zs,ds,cs\) and \(ns\), which are grouped together by Glaisher [21].

8. Discussion and conclusion

This paper enlarges upon, with additions, a recent work of Cardy [1], concerning quantum revivals, after quenching, in higher dimensional, spherical cylinders, \(I \times S^d\). Explicit expressions emerge, for free fields, on choice of a particular, conformal quenching initial state. The revivals in the return amplitude, as time goes on, at the rationals can be explained, for odd

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8I am taking the same connection between the energy and partition function that allows (13) to be derived.
dimensional spheres, by modular behaviour and have here been related to ‘wrongly quantised’
fields, for want of a better term. Together with the normal fields, these can be associated with
the elliptic functions zs, ds, cs and ns.

The calculations have been extended to the spin-half field for which the periodicity is
altered because of the fermionic signs.

A fact that needs properly elucidating is that, even though there is no simple, exact modular
property for even spheres, rational revivals can still be detected. The same remark holds also
for the case when the sphere, \( S^d \), is replaced by an orbifold quotient.

The restriction to free fields is a severe one, but, in higher dimensions, it is difficult to find
systems with the requisite conformal behaviour. Free fields do, however, lead to explicit for-
malae and to quantitative information that is probably generic.

Appendix. The partition function, an observation

It is well known that there are two equivalent ways of evaluating the partition function. One
relies on the CFT operator counting method (in flat space) and the other on solving the energy
eigenvalue problem on the spatial section of the conformally related, curved manifold. Here, I
would like to make a few technical remarks on this equivalence in the present set up, restrict-
ing myself to the scalar, boson field.

The most appropriate form of the operator counting method for me is that outlined in [14]
and [1]. I have to repeat some of this known material in order to make my point. See also
Kutasov and Larsen [24].

Because of the conformal relation, the time translation generator on the (Euclidean)
Einstein universe, \( S^d \times S^d \), is proportional to the scale generator on the flat \( \mathbb{R}^{d+1} \). This implies
that the energy \( E \) of a state equals the scaling dimension of the corresponding field, which
means that the partition function (actually just the sum over states part) is the generating func-
tion for the complete set of modular weights, \( \Delta \), i.e.

\[
\Xi(\beta) = \sum_{\Delta} e^{-\beta\Delta}.
\]

The total set of independent operators in \( \mathbb{R}^{d+1} \) is

\[
\prod_{j} \prod_{i=1}^{d+1} \partial_{\theta_j}^{(i)} \phi,
\]

modulo the equation of motion, \( \partial_{i} \partial^{i} \phi = 0 \), a requirement that can be implemented by restrict-
ing \( n_{d+1}^{(i)} \) to 0 and 1, as explained in [1, 14]. This leaves \( n_1 \ldots n_d \) as unrestricted, non-negative
integers.

The scaling dimension of the operator, (A.1), then splits into two,

\[
\Delta = \sum_{j} ((d - 1)/2 + n_1^{(j)} + \ldots + n_d^{(j)}) + \sum_{j} ((d + 1)/2 + n_1^{(j)} + \ldots + n_d^{(j)}),
\]

and the partition function factorises, implying

\[
\Xi_d(\beta) = \sum_{\mathbf{n}=0}^{\infty} \frac{1}{1 - \exp - \beta(a_\mathbf{n} + \mathbf{n}.1)} + \sum_{\mathbf{n}=0}^{\infty} \frac{1}{1 - \exp - \beta(a_D + \mathbf{n}.1)},
\]

with \( a_\mathbf{n} = (d - 1)/2, \quad a_D = a_\mathbf{n} + 1 \).
The significance of this is that the quantities $a_{n,D} + nI$ are recognised as the conformal single particle energies (the eigenvalues of $\sqrt{D}$) on a $d$–sphere with Neumann and Dirichlet conditions on the rim$^9$. Uniting these gives the full sphere result—and so the equivalence of the two approaches has been verified with no work. There is no need to perform the combinatorics in, say, (A.2) in order to obtain the degeneracies of the eigenlevels, nor any group theory likewise (in this simple case).

Equation (4) provides an alternative to (A.3) and, after very slight algebra, reads

$$\Xi'_{\beta}(\beta) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \cosh(m\beta/2) \operatorname{cosech}(m\beta/2),$$

(A.4)
as used above, (5). Expression (A.4) is derived in [6]$^{10}$ and also in [3–5].

This particular equivalence of approaches has been discussed recently in some detail by Beccaria et al [25], who employ degeneracies and apply the harmonic condition, $\partial\partial^i\phi = 0$, differently, so that there is no split, (A.3). They retrieve (A.4).

References

[1] Cardy J 2016 Quantum revivals in conformal field theories in higher dimensions (arXiv:1603.08267)
[2] Cardy J 2014 Thermalization and revivals after a quantum quench in conformal field theory Phys. Rev. Lett. 112 220401
[3] Apps J S 1996 The effective action on a curved space and its conformal properties PhD Thesis University of Manchester
[4] Dowker J S and Apps J S 1995 Further functional determinants Class. Quantum Grav. 12 1363
[5] Dowker J S and Apps J S 1996 Functional determinants on certain domains Int. J. Mod. Phys. S 799
[6] Chang P and Dowker J S 1993 Vacuum energy on orbifold factors of spheres Nucl. Phys. B 395 407
[7] Dowker J S and Critchley R 1977 Vacuum stress tensor in an Einstein universe: finite temperature effects Phys. Rev. D 15 1484
[8] Kennedy G 1981 Topological symmetry restoration Phys. Rev. D 23 2884
[9] Unwin S D 1980 Selected quantum field theory effects in multiply connected space times PhD Thesis University of Manchester
[10] Altaie M B and Dowker J S 1978 Spinor fields in an Einstein universe: finite temperature effects Phys. Rev. D 18 3557
[11] Cappelli A and Costa A 1989 On the stress tensor of conformal field theories in higher dimensions Nucl. Phys. B 314 707
[12] Dowker J S and Kennedy G 1978 Finite temperature and boundary effects in static space-times J. Phys. A: Math. Gen. 11 895
[13] Candelas P and Dowker J S 1979 Field theories on conformally related space-times: some global considerations Phys. Rev. D 19 2902
[14] Cardy J 1991 Operator content and modular properties of higher dimensional conformal field theories Nucl. Phys. B 366 403
[15] Dowker J S 2003 Zero modes, entropy bounds and partition functions Class. Quantum Grav. 20 L105
[16] Dowker J S and Kirsten K 2002 Elliptic functions and temperature inversion on spheres Nucl. Phys. B 638 405
[17] Gibbons G W, Perry M J and Pope C N 2006 Partition functions, the Bekenstein bound, temperature inversion in Anti-de Sitter space, its conformal boundary Phys. Rev. D 74 084009
[18] Dowker J S 2008 Modular properties of Eisenstein series and statistical physics (arXiv:0810.0537)

$^9$It is interesting to remark that the pseudo-operator, $H_d = \sqrt{D}$, on the $d$-sphere can be defined recursively by $H_d = L \oplus H_{d-1}$ with $L$ realised as the operator $-i\partial^i\partial^j$ acting on $Z$ twisted fields on the circle. $L$ has eigenvalues $n + 1/2$ and iterating from $H_0 = \mp i\partial$ gives the eigenvalues on the $N,D$ hemisphere mentioned above. $H_0^2 = 1/4$ means that the Yamabe–Penrose operator vanishes in dimension 0, consonant with its form.

$^{10}$The spatial geometry was, more generally, an orbifolded sphere.
[19] Dowker J S 1984 Finite temperature and vacuum effects in higher dimensions *Class. Quantum Grav.* 1 359

[20] Glaisher J W L 1886 On certain sums of products of quantities depending on the divisors of a number *Messenger Math.* 15 1

[21] Glaisher J W L 1889 On the series which represent the twelve elliptic and four zeta functions *Messenger Math.* 18 1

[22] Dowker J S and Kirsten K 2008 Elliptic aspects of statistical mechanics on spheres *J. Math. Phys.* 49 113513

[23] Berndt B C 1978 Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan *J. Angew. Math.* 303–4 332

[24] Kutasov D and Larsen F 2001 Partition sums, entropy bounds in weakly coupled CFT *J. High Energy Phys.* JHEP01(2001)001

[25] Beccaria M, Bekaert X and Tseytlin A A 2014 Partition function of free conformal higher spin theory *J. High Energy Phys.* JHEP08(2014)113