Singularity formation to the Cauchy problem of the two-dimensional non-baratropic magnetohydrodynamic equations without heat conductivity *

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Abstract

We study the breakdown of strong solutions to the two-dimensional (2D) Cauchy problem of the non-baratropic compressible magnetohydrodynamic equations without heat conductivity. It is proved that the strong solution exists globally if the gradient of the velocity and the pressure satisfy
\[ \|\nabla u\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} < \infty. \]
In particular, the criterion is independent of the magnetic field and the vacuum in the solutions is allowed.

Keywords: non-baratropic compressible magnetohydrodynamic equations; 2D Cauchy problem; blow-up criterion.
Math Subject Classification: 76W05; 35B65

1 Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a domain, the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in \( \Omega \) can be described by the non-baratropic compressible MHD equations
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P &= b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \\
c_v[(\rho \theta)_t + \text{div}(\rho u \theta)] + P \text{div} u - \kappa \Delta \theta &= 2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2 + \nu |\nabla \times b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div} u &= \nu \Delta b, \\
\text{div} b &= 0.
\end{align*}
\]
(1.1)

Here, \( t \geq 0 \) is the time, \( x \in \Omega \) is the spatial coordinate, and \( \rho, u, P = R \rho \theta \) (\( R > 0 \)), \( \theta, b \) are the fluid density, velocity, pressure, absolute temperature, and the magnetic field respectively; \( \mathcal{D}(u) \) denotes the deformation tensor given by
\[
\mathcal{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{tr}).
\]
The constant viscosity coefficients \( \mu \) and \( \lambda \) satisfy the physical restrictions
\[
\mu > 0, \quad \mu + \lambda \geq 0.
\]
(1.2)

Positive constants \( c_v, \kappa \), and \( \nu \) are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient.

There is huge literature on the studies about the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for the compressible MHD system due to the physical importance, complexity, rich phenomena and mathematical challenges, refer

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to [3, 7, 9, 17, 24, 25, 38] and references therein. However, many physical important and mathematical fundamental problems are still open due to the lack of smoothing mechanism and the strong non-linearit.

Kawashima [16] first obtained the global existence and uniqueness of classical solutions to the multi-dimensional compressible MHD equations when the initial data are close to a non-vacuum equilibrium in $H^3$-norm. When the initial density allows vacuum, the local well-posedness of strong solutions to the initial boundary value problem of 3D nonisentropic MHD equations has been obtained by Fan-Yu [3]. For general large initial data, Hu-Wang [8, 9] proved the global existence of weak solutions with finite energy in Lions’ framework for compressible Navier-Stokes equations [4, 20] provided the adiabatic exponent is suitably large. Recently, Li-Xu-Zhang [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible MHD system in 3D with smooth initial data which are of small energy but possibly large oscillations and vacuum, which generalized the result for compressible Navier-Stokes equations obtained by Huang-Li-Xin [14]. Very recently, Hong-Hou-Peng-Zhu [7] improved the result in [17] to allow the initial energy large as long as the adiabatic exponent is close to 1 and $\nu$ is suitably large. Furthermore, Lü-Shi-Xu [25] established the global existence and uniqueness of strong solutions to the 2D MHD equations provided that the smooth initial data are of small total energy. Nevertheless, it is an outstanding challenging open problem to investigate the global well-posedness for general large strong solutions with vacuum.

Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong (or classical) solutions to the compressible MHD equations. The pioneering work can be traced to [5], where He and Xin proved Serrin’s criterion for strong solutions to the incompressible MHD system, that is,

$$\lim_{T \to T^*} \|u\|_{L^s(0,T;L^r)} = \infty, \text{ for } \frac{2}{s} + \frac{3}{r} = 1, \ 3 < r \leq \infty,$$  

(1.3)

here $T^*$ is the finite blow up time. For the Cauchy problem of 2D compressible isentropic MHD system, Wang [30] obtained the following criterion.

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty.$$   

(1.4)

This criterion asserts that the concentration of density must be responsible for the loss of regularity in finite time. For the IBVP of 2D full compressible MHD system, Fan-Li-Nakamura [2] proved that

$$\lim_{T \to T^*} \left( \|\text{div} \ u\|_{L^1(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^\infty)} \right) = \infty.$$   

(1.5)

Later on, Lu-Chen-Huang [21] extended (1.5) with a refiner form

$$\lim_{T \to T^*} \|\text{div} \ u\|_{L^1(0,T;L^\infty)} = \infty.$$   

(1.6)

The criterion (1.6) is the same as [31] for 2D compressible full Navier-Stokes equations, which shows that the mechanism of blow-up is independent of the magnetic field. Recently, for the Cauchy problem and the IBVP of 3D full compressible MHD system, Huang-Li [10] established the following Serrin type criterion

$$\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^{r'})} \right) = \infty, \text{ for } \frac{2}{s} + \frac{3}{r} \leq 1, \ 3 < r \leq \infty.$$   

(1.7)

There are also some interesting blow-up criteria for the compressible MHD system, see [23, 35]. For more information on the blow-up criteria of compressible Navier-Stokes flows, we refer to [1, 11, 13, 28, 32] and the references therein.

It should be noted that all the results mentioned above on the blow-up of strong (or classical) solutions of viscous, compressible, and heat conducting MHD flows are for $\kappa > 0$. Recently, for the 3D non-isentropic compressible Navier-Stokes equations with $\kappa = 0$, Huang-Xin [15] showed that

$$\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty.$$   

(1.8)
under the assumption
\[ \mu > 4\lambda. \]  
(1.9)

Later on, for the MHD flows, the author \[37\] obtained
\[ \lim_{T \to T^*} \left( \|\mathfrak{D}(u)\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty \]
(1.10)
provided that
\[ 3\mu > \lambda. \]  
(1.11)

Very recently, for the 2D Navier-Stokes flows, Zhong \[36\] established the following criterion
\[ \lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty. \]
(1.12)

It is worth mentioning that in a well-known paper \[33\], Xin considered non-isentropic compressible Navier-Stokes equations with \( \kappa = 0 \) in multidimensional space, starting with a compactly supported initial density. He first proved that if the support of the density grows sublinearly in time and if the entropy is bounded from below then the solution cannot exist for all time. One key ingredient in the proof is a differential inequality on some integral functional (see \[33\] Proposition 2.1 for details). As an application, any smooth solution to the full compressible Navier-Stokes equations for polytropic fluids in the absence of heat conduction will blow up in finite time if the initial density is compactly supported. Recently, based on the key observation that if initially a positive mass is surrounded by a bounded vacuum region, then the time evolution remains uniformly bounded for all time, Xin-Yan \[34\] improved the blow-up results in \[33\] by removing the assumptions that the initial density has compact support and the smooth solution has finite energy, but the initial data only has an isolated mass group. Thus it seems very difficult to study globally smooth solutions of full compressible Navier-Stokes equations without heat conductivity in multi-dimension, the same difficulty also arises in multi-dimensional MHD equations. These motivate us to study a blow-up criterion for the system (1.1) with zero heat conduction. In fact, this is the main aim of this paper.

When \( \kappa = 0 \) and without loss of generality, take \( c_v = R = 1 \), the system (1.1) can be written as

\[
\begin{align*}
\rho_t + \mathrm{div}(\rho u) &= 0, \\
(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P &= b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \\
P_t + \text{div}(P u) + P \text{div} u &= 2\mu |\mathfrak{D}(u)|^2 + \lambda (\text{div} u)^2 + \nu |\nabla \times b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div} u &= \nu \Delta b, \\
\text{div} b &= 0.
\end{align*}
\]

(1.13)

The present paper is aimed at giving a blow-up criterion of strong solutions to the Cauchy problem of the system (1.13) with the initial condition
\[
(\rho, \rho u, P, b)(x, 0) = (\rho_0, \rho_0 u_0, P_0, b_0)(x), \quad x \in \mathbb{R}^2,
\]
(1.14)

and the far field behavior
\[
(\rho, u, P, b)(x, t) \to (0, 0, 0, 0), \quad \text{as } |x| \to +\infty, \quad t > 0.
\]
(1.15)

Before stating our main result, we first explain the notations and conventions used throughout this paper. For \( r > 0 \), set
\[
B_r \triangleq \{ x \in \mathbb{R}^2 | |x| < r \}, \quad \int \cdot \, dx \triangleq \int_{\mathbb{R}^2} \cdot \, dx.
\]

For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), the standard Sobolev spaces are denoted by:
\[
L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = H^{k,2}(\mathbb{R}^2), \quad D^{k,p} = \{ u \in L^1_{\text{loc}} | \nabla^k u \in L^p \}.
\]

Now we define precisely what we mean by strong solutions to the problem (1.13)–(1.15).
Definition 1.1 (Strong solutions) \((\rho, u, P, b)\) is called a strong solution to (1.13)–(1.15) in \(\mathbb{R}^2 \times (0,T)\), if for some \(q_0 > 2\) and \(a > 1\),

\[
\begin{cases}
\rho \geq 0, \quad \rho \bar{x}^a \in C([0,T]; L^1 \cap H^1 \cap W^{1,q_0}), \quad \rho_t \in C([0,T]; L^{q_0}), \\
(u, b) \in C([0,T]; D^{1,2} \cap D^{2,2}) \cap L^2(0,T; D^{2,q_0}), \quad b \in C([0,T]; H^2), \\
(\mathbf{u}_t, \mathbf{b}_t) \in L^2(0,T; D^{1,2}), \quad (\sqrt{\rho_0} \mathbf{u}_t, \mathbf{b}_t) \in L^\infty(0,T; L^2), \\
P \geq 0, \quad P \in C([0,T]; L^1 \cap H^1 \cap W^{1,q_0}), \quad P_t \in C([0,T]; L^{q_0}),
\end{cases}
\]

and \((\rho, u, P, b)\) satisfies both (1.13) almost everywhere in \(\mathbb{R}^2 \times (0,T)\) and (1.14) almost everywhere in \(\mathbb{R}^2\). Here

\[
\bar{x} \triangleq (e + |x|^2) \frac{1}{2} \log 1 + q_0 (e + |x|^2)
\]

and \(q_0\) is a positive number.

Without loss of generality, we assume that the initial density \(\rho_0\) satisfies

\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]

which implies that there exists a positive constant \(N_0\) such that

\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int \rho_0 dx = \frac{1}{2}.
\]

Our main result reads as follows:

**Theorem 1.1** In addition to (1.17) and (1.18), assume that the initial data \((\rho_0 \geq 0, u_0, P_0 \geq 0, b_0)\) satisfies for any given numbers \(a > 1\) and \(q > 2\),

\[
\begin{cases}
\rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \quad \sqrt{\rho_0} u_0 \in L^2, \quad \nabla u_0 \in H^1, \\
P_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad b_0 \bar{x}^\frac{a}{2} \in H^1, \quad \nabla b_0 \in L^2, \quad \text{div} \ b_0 = 0,
\end{cases}
\]

and the compatibility conditions

\[
-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div} u_0 + \nabla P_0 - b_0 \cdot \nabla b_0 + \frac{1}{2} \nabla |b_0|^2 = \sqrt{\rho_0} g
\]

for some \(g \in L^2(\Omega)\). Let \((\rho, u, P, b)\) be a strong solution to the problem (1.13)–(1.15). If \(T^* < \infty\) is the maximal time of existence for that solution, then we have

\[
\lim_{T \to T^*} \left( \|\nabla u\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty.
\]

**Remark 1.1** The local existence of a strong solution with initial data as in Theorem 1.1 was established in [22]. Hence, the maximal time \(T^*\) is well-defined.

**Remark 1.2** It is worth noting that the blow-up criteria (1.21) is independent of the magnetic field. Moreover, according to (1.21), the upper bound of the temperature \(\theta\) is not the key point to make sure that the solution \((\rho, u, P, b)\) is a global one, and it may go to infinity in the vacuum region within the life span of our strong solution.

**Remark 1.3** Compared with [37], where the author investigated a blow-up criterion for the 3D Cauchy problem of non-isentropic magnetohydrodynamic equations with zero heat conduction, there is no need to impose additional restrictions on the viscosity coefficients \(\mu\) and \(\lambda\) except the physical restrictions (1.22).
We now make some comments on the analysis of this paper. We mainly make use of continuation argument to prove Theorem 1.1. That is, suppose that (1.21) were false, i.e.,

$$\lim_{T \to T_*} \left( \|\nabla u\|_{L^1(0, T; L^\infty)} + \|P\|_{L^\infty(0, T; L^{\infty})} \right) \leq M_0 < \infty.$$ 

We want to show that

$$\sup_{0 \leq t \leq T_*} \left( \|\rho, P\|_{H^1 \cap W^{1,q}} + \|\rho \bar{x}^a\|_{L^1 \cap W^{1,q}} + \|\nabla u\|_{H^1} + \|b\|_{H^2} + \|b \bar{x}^2\|_{H^1} \right) \leq C < +\infty.$$ 

It should be pointed out that the crucial techniques of proofs in [2,21] cannot be adapted directly to the situation treated here, since their arguments depend crucially on the boundedness of the $L^\infty$-norm of $u$ just in terms of $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ for any $p \geq 1$. 

To overcome these difficulties mentioned above, some new ideas are needed. Inspired by [18,36], we first observe that if the initial density decays not too slow at infinity, i.e., $\rho_0 \bar{x}^a \in L^1(\mathbb{R}^2)$ for some positive constant $a > 1$ (see (1.19)), then for any $\eta \in (0, 1)$, we can show that (see (3.30))

$$u \bar{x}^{-\eta} \in L^{p_0}(\mathbb{R}^2), \text{ for some } p_0 > 1.$$ 

Then, motivated by the technique of Hoff [6], in order to get the $L^\infty L^2_x$-norm of $\sqrt{\rho}u$, we first show the desired a priori estimates of the $L^\infty L^2_x$-norm of $\nabla u$ and $\nabla b$, which is the second key observation in this paper (see Lemma 3.10). Next, to finish the higher order estimates, our new observation is to obtain the $L^\infty L^2_x$-norm of $\bar{x}^2 b$ and $\bar{x}^2 \nabla b$ (see Lemma 3.7). The a priori estimates on the $L^\infty L^2_x$-norm of $(\nabla \rho, \nabla P)$ can be obtained (see Lemma 3.9) by solving a logarithm Gronwall inequality based on a logarithm estimate for the Lamé system. Finally, with the help of (1.22), we can get the spatial weighted estimate of the density (see Lemma 3.10).

The rest of this paper is organized as follows. In Section 2 we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gronwall’s inequality, which plays a central role in proving a priori estimates on strong solutions $(\rho, u, P, b)$.

**Lemma 2.1** Suppose that $h$ and $r$ are integrable on $(a, b)$ and nonnegative a.e. in $(a, b)$. Further assume that $y \in C[a, b], y' \in L^1(a, b), and$

$$y'(t) \leq h(t) + r(t)y(t) \text{ for a.e } t \in (a, b).$$

Then

$$y(t) \leq \left[ y(a) + \int_a^t h(s) \exp \left(- \int_a^s r(\tau)d\tau \right) ds \right] \exp \left( \int_a^t r(s)ds \right), \text{ for } t \in [a, b].$$

**Proof.** See [29] pp. 12–13. □

Next, the following Gagliardo-Nirenberg inequality (see [26]) will be used later.

**Lemma 2.2** (Gagliardo-Nirenberg) For $p \in [2, \infty), r \in (2, \infty),$ and $s \in (1, \infty), there exists some generic constant $C > 0$ which may depend on $p, r,$ and $s$ such that for $f \in H^1(\mathbb{R}^2)$ and $g \in L^r(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2),$ we have

$$\|f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)} ,$$

$$\|g\|_{C(\mathbb{R}^2)} \leq C\|g\|_{L^r(\mathbb{R}^2)}^{s(r-2)/(2r+s(r-2))} \|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+s(r-2))}.$$
The following weighted \( L^m \) bounds for elements of the Hilbert space \( \tilde{D}^{1,2}(\mathbb{R}^2) \triangleq \{ v \in H^1_{\text{loc}}(\mathbb{R}^2)|\nabla v \in L^2(\mathbb{R}^2) \} \) can be found in [19, Theorem B.1].

**Lemma 2.3** For \( m \in [2, \infty) \) and \( \theta \in (1 + m/2, \infty) \), there exists a positive constant \( C \) such that for all \( v \in \tilde{D}^{1,2}(\mathbb{R}^2) \),

\[
\left( \int_{\mathbb{R}^2} \frac{|v|^m}{e + |x|^2} \left( \log \left( e + |x|^2 \right) \right)^{-\theta} \, dx \right)^{1/m} \leq C\|v\|_{L^2(B_1)} + C\|
abla v\|_{L^2(\mathbb{R}^2)}. \tag{2.1}\]

The combination of Lemma 2.3 and the Poincaré inequality yields the following useful results on weighted bounds, whose proof can be found in [18, Lemma 2.4].

**Lemma 2.4** Let \( \bar{x} \) be as in (1.15). Assume that \( \rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) is a non-negative function such that

\[
\|\rho\|_{L^1(B_{N_1})} \geq M_1, \quad \|\rho\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} \leq M_2,
\]

for positive constants \( M_1, M_2, \) and \( N_1 \geq 1 \). Then for \( \varepsilon > 0 \) and \( \eta > 0 \), there is a positive constant \( C \) depending only on \( \varepsilon, \eta, M_1, M_2, \) and \( N_1 \), such that every \( v \in \tilde{D}^{1,2}(\mathbb{R}^2) \) satisfies

\[
\|v \bar{x}^{-\eta}\|_{L^{2+\varepsilon/\eta}(\mathbb{R}^2)} \leq C\|\sqrt{\rho}v\|_{L^2(\mathbb{R}^2)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.2}\]

with \( \bar{\eta} = \min\{1, \eta\} \).

Next, for \( \nabla \perp \triangleq (\partial_2, \partial_1) \), denoting the material derivative of \( f \) by \( \dot{f} \triangleq f_t + u \cdot \nabla f \), then we have the following \( L^p \)-estimate (see [25, Lemma 2.5]) for the elliptic system derived from the momentum equations (1.13):

\[
\Delta F = \text{div}(\rho \dot{u} - \text{div}(b \otimes b)), \quad \mu \Delta \omega = \nabla^\perp \cdot (\rho \dot{u} - \text{div}(b \otimes b)), \tag{2.3}\]

where \( F \) is the effective viscous flux, \( \omega \) is vorticity given by

\[
F = (\lambda + 2\mu) \text{div} u - P - \frac{1}{2}\|b\|^2, \quad \omega = \partial_1 u_2 - \partial_2 u_1. \tag{2.4}\]

**Lemma 2.5** Let \( (\rho, u, P, b) \) be a smooth solution of (1.13). Then for \( p \geq 2 \) there exists a positive constant \( C \) depending only on \( p, \mu \) and \( \lambda \) such that

\[
\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|b\|\nabla b\|_{L^p}), \tag{2.5}\]

\[
\|F\|_{L^p} + \|\omega\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|b\|\nabla b\|_{L^2})^{1-\frac{2}{p}} (\|\nabla u\|_{L^2} + \|P\|_{L^2} + \|b\|_{L^2})^{\frac{2}{p}}, \tag{2.6}\]

\[
\|\nabla u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|b\|\nabla b\|_{L^2})^{1-\frac{2}{p}} (\|\nabla u\|_{L^2} + \|P\|_{L^2} + \|b\|_{L^2})^{\frac{2}{p}} + C\|P\|_{L^p} + \|b\|_{L^{2p}}. \tag{2.7}\]

Finally, the following Beale-Kato-Majda type inequality (see [13, Lemma 2.3]) will be used to estimate \( \|\nabla \rho, \nabla P\|_{L^q} \) \( (q > 2) \).

**Lemma 2.6** For \( q \in (2, \infty) \), there is a constant \( C(q) > 0 \) such that for all \( \nabla v \in L^2 \cap D^{1,q} \), it holds that

\[
\|\nabla v\|_{L^q} \leq C(\|\text{div} v\|_{L^q} + \|\text{curl} v\|_{L^q}) \log(e + \|\nabla^2 v\|_{L^q}) + C\|\nabla v\|_{L^2} + C. \tag{2.8}\]

### 3 Proof of Theorem 1.1

Let \( (\rho, u, P, b) \) be a strong solution described in Theorem 1.1. Suppose that (1.21) were false, that is, there exists a constant \( M_0 > 0 \) such that

\[
\lim_{T \to T^*} (\|\nabla u\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)}) \leq M_0 < \infty. \tag{3.1}\]

First, the estimate on the \( L^\infty(0,T;L^p) \)-norm of the density could be deduced directly from (1.13) and (3.1) (see [13, Lemma 3.4]).
Lemma 3.1 Under the condition (3.1), it holds that for any $T \in [0, T^*)$,

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^1 \cap L^\infty} \leq C,$$

where and in what follows, $C, C_1, C_2$ stand for generic positive constants depending only on $M_0, \lambda, \mu, \nu, T^*$, and the initial data.

Next, we have the following standard estimate.

Lemma 3.2 Under the condition (3.1), it holds that for any $T \in [0, T^*)$,

$$\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|P\|_{L^1 \cap L^\infty} \right) + \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) dt \leq C. \quad (3.3)$$

Proof. It follows from (1.13) that

$$P_t + u \cdot \nabla P + 2P \text{ div } u = F = 2\mu|\Sigma(u)|^2 + \lambda|\text{div } u|^2 + \nu|\nabla \times b|^2 \geq 0. \quad (3.4)$$

Define particle path before blowup time

$$\begin{cases}
\frac{d}{dt} X(x, t) = u(X(x, t), t), \\
X(x, 0) = x.
\end{cases}$$

Thus, along particle path, we obtain from (3.4) that

$$\frac{d}{dt} P(X(x, t), t) = -2P \text{ div } u + F,$$

which implies

$$P(X(x, t), t) = \exp \left( -2 \int_0^t \text{div } u ds \right) \left[ P_0 + \int_0^t \exp \left( 2 \int_0^s \text{div } u d\tau \right) F ds \right] \geq 0. \quad (3.5)$$

Next, multiplying (1.13) by $u$ and $b$ respectively, then adding the two resulting equations together, and integrating over $\mathbb{R}^2$, we obtain after integrating by parts that

$$\frac{1}{2} \frac{d}{dt} \int \left( \rho|u|^2 + |b|^2 \right) dx + \int \left[ \mu|\nabla u|^2 + (\lambda + \mu)|\text{div } u|^2 + \nu|\nabla b|^2 \right] dx = \int P \text{ div } u dx. \quad (3.6)$$

Integrating (1.13) with respect to $x$ and then adding the resulting equality to (3.6) give rise to

$$\frac{d}{dt} \int \left( \frac{1}{2} \rho|u|^2 + \frac{1}{2} |b|^2 + P \right) dx = 0, \quad (3.7)$$

which combined with (3.5), (1.19), and (3.1) leads to

$$\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|P\|_{L^1 \cap L^\infty} \right) \leq C. \quad (3.8)$$

This together with (3.6) and Cauchy-Schwarz inequality yields

$$\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \mu \|\nabla u\|_{L^2}^2 + \nu \|\nabla b\|_{L^2}^2 \leq C. \quad (3.9)$$

So the desired (3.3) follows from (3.8) and (3.9) integrated with respect to $t$. This completes the proof of Lemma 3.2.

Inspired by [5], we have the following higher integrability of the magnetic field $b$. 

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Lemma 3.3  Under the condition \((3.1)\), it holds that for any \(p \in [2, \infty)\) and \(T \in [0, T^*)\),

\[
\sup_{0 \leq t \leq T} \|b\|_{L^p} \leq C. \tag{3.10}
\]

Proof. Multiplying \((1.13)\) by \(q |b|^{q-2}b\) \((q \geq 2)\) and integrating the resulting equation over \(\mathbb{R}^2\), we derive

\[
\frac{d}{dt} \int |b|^q dx + \nu q(q - 1) \int |b|^{q-2} |\nabla b|^2 dx = q \int (b \cdot \nabla u - u \cdot \nabla b - \text{div} u) \cdot |b|^{q-2} b dx. \tag{3.11}
\]

By the divergence theorem and \((1.13)\), we get

\[-q \int (u \cdot \nabla) b \cdot |b|^{q-2} b dx = \int \text{div} |b|^q dx,
\]

which together with \((3.11)\) yields

\[
\frac{d}{dt} \int |b|^q dx + \nu q(q - 1) \int |b|^{q-2} |\nabla b|^2 dx \leq (2q + 1) \|\nabla u\|_{L^\infty} \int |b|^q dx. \tag{3.12}
\]

Consequently, from \(q \geq 2\) and \((3.12)\), we immediately have

\[
\frac{d}{dt} \|b\|_{L^q} \leq \frac{2q + 1}{q} \|\nabla u\|_{L^\infty} \|b\|_{L^q} \leq 3 \|\nabla u\|_{L^\infty} \|b\|_{L^q}.
\]

Then Gronwall’s inequality and \((3.1)\) imply that for any \(q \geq 2\),

\[
\sup_{0 \leq t \leq T} \|b\|_{L^q} \leq C, \tag{3.13}
\]

where \(C\) is independent of \(q\). Thus, letting \(q \rightarrow \infty\) in \((3.13)\) leads to the desired \((3.10)\) and finishes the proof of Lemma 3.3.

The following lemma gives the estimates on the spatial gradients of both the velocity and the magnetic field, which are crucial for deriving the higher order estimates of the solution.

Lemma 3.4  Under the condition \((3.1)\), it holds that for any \(T \in [0, T^*)\),

\[
\sup_{0 \leq t \leq T} \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) + \int_0^T \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \right) dt \leq C. \tag{3.14}
\]

Proof. Multiplying \((1.13)\) by \(\dot{u}\) and integrating the resulting equation over \(\mathbb{R}^2\) give rise to

\[
\int \rho |\dot{u}|^2 dx = - \int \dot{u} \cdot \nabla P dx + \mu \int \dot{u} \cdot \Delta u dx + (\lambda + \mu) \int \dot{u} \cdot \nabla \text{div} u dx
\]

\[+ \int \dot{u} \cdot b \cdot \nabla b dx - \frac{1}{2} \int \dot{u} \cdot \nabla |b|^2 dx \triangleq \sum_{i=1}^{5} I_i. \tag{3.15}
\]

By \((1.13)\) and integrating by parts, we derive from \((3.1)\) and Garliardo-Nirenberg inequality that

\[
I_1 = \int [(\text{div} u) P - (u \cdot \nabla u) \cdot \nabla P] dx
\]

\[= \frac{d}{dt} \int P \text{div} u dx + \int \left[ P(\text{div} u)^2 - 2\mu \text{div} u \cdot \nabla^2 \text{div} u - \lambda(\text{div} u)^3 - \nu \text{div} u \cdot \nabla |b|^2 \right] dx
\]

\[+ \int P \partial_j u_i \partial_i u_j dx
\]

\[\leq \frac{d}{dt} \int P \text{div} u dx + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla b\|_{L^3}^3.
\]

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\[
\int L \leq \frac{d}{dt} \int P \text{div} u dx + C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^3}^3 + C \| \nabla b \|_{L^2}^4 + \frac{\nu}{4} \| \nabla^2 b \|_{L^2}^2.
\] (3.16)

It follows from integration by parts that

\[
I_2 = \mu \int (u_t + u \cdot \nabla u) \cdot \Delta u dx
- \frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 - \mu \int \partial_i u_j \partial_i (u_k \partial_k u_j) dx
\leq - \frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^3}^3.
\] (3.17)

Similarly to \( I_2 \), one gets

\[
I_3 = - \frac{\lambda + \mu}{2} \frac{d}{dt} \| \text{div} u \|_{L^2}^2 - (\lambda + \mu) \int \text{div} u \text{div} (u \cdot \nabla u) dx
\leq - \frac{\lambda + \mu}{2} \frac{d}{dt} \| \text{div} u \|_{L^2}^2 + C \| \nabla u \|_{L^3}^3.
\] (3.18)

By virtue of (1.13)\ref{2} and (1.13)\ref{3}, one deduces from integration by parts and Gagliardo-Nirenberg inequality that

\[
I_4 = \int b \cdot \nabla b \cdot u_t dx + \int b \cdot \nabla b \cdot (u \cdot \nabla u) dx
- \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (b_t \cdot \nabla u \cdot b + b \cdot \nabla u \cdot b_t) dx - \int b \cdot \nabla (u \cdot \nabla u) \cdot b dx
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (b_t \cdot \nabla u \cdot b + b \cdot \nabla u \cdot b_t) dx
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (b \cdot \nabla u \cdot b + b \cdot \nabla u \cdot b_t) dx
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (\nabla b \cdot u_k + \nabla b_k \cdot u \cdot \nabla b_k \cdot \partial_i u \cdot b dx)
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (\nabla b \cdot u_k + \nabla b_k \cdot u \cdot \nabla b_k \cdot \partial_i u \cdot b dx)
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (\nabla b \cdot u_k + \nabla b_k \cdot u \cdot \nabla b_k \cdot \partial_i u \cdot b dx)
\leq - \frac{d}{dt} \int b \cdot \nabla u \cdot b dx + C \int \| \nabla u \|_{L^3}^3 + C \| \nabla b \|_{L^2}^3 + \frac{\nu}{8} \| \Delta b \|_{L^2}^2.
\] (3.19)

Applying (1.13)\ref{2}, (1.13)\ref{3}, and Gagliardo-Nirenberg inequality, we have

\[
I_5 = \frac{1}{2} \int |b|^2 \text{div} u dx + \frac{1}{2} \int |b|^2 \text{div} (u \cdot \nabla u) dx
= \frac{1}{2} \frac{d}{dt} \int |b|^2 \text{div} u dx - \frac{1}{2} \int |b|^2 (\text{div} u)^2 dx + \frac{1}{2} \int |b|^2 \partial_i u_j \partial_j u_i dx
\leq \frac{1}{2} \frac{d}{dt} \int |b|^2 \text{div} u dx + C \int |b|^2 |\nabla u|^2 dx + \frac{\nu}{8} \| \Delta b \|_{L^2}^2.
\]
This completes the proof of Lemma 3.4. due to (3.22). Thus the desired (3.14) follows from (3.24), (3.25), (3.3), and Gronwall’s inequality. Putting (3.16)–(3.20) into (3.15), we obtain from (2.7) and (3.1) that

\[
\Psi'(t) + \|\sqrt{\rho}u\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^1}^3 + C\|\nabla b\|_{L^2}^2 + \frac{3\nu}{4}\|\Delta b\|_{L^2}^2,
\]

where

\[
\Psi(t) \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda + \mu}{2}\|\nabla u\|_{L^2}^2 - \int P \, \text{div} \, u \, dx + \int b \cdot \nabla u \cdot b \, dx - \frac{1}{2} \int |b|^2 \, \text{div} \, u \, dx
\]

due to (3.3) and (3.10).

Next, multiplying (1.13) by $\Delta b$ and integrating by parts, one deduces that

\[
\frac{d}{dt} \int |\nabla b|^2 \, dx + 2\nu \int |\Delta b|^2 \, dx \leq C \int |\nabla u| |\nabla b|^2 \, dx + C \int |\nabla u| |\Delta b| \, dx
\]
\[
\leq C\|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^2}^2 + C\|\nabla u\|_{L^1}^3 \|\nabla b\|_{L^2}^2 + C\|\nabla u\|_{L^2} \|\Delta b\|_{L^2}
\]
\[
\leq C\|\nabla u\|_{L^2}^3 + C\|\nabla b\|_{L^2}^4 + \frac{\nu}{4}\|\Delta b\|_{L^2}^2.
\]

Adding (3.23) to (3.24), we then derive from (2.7) and (3.3) that

\[
\frac{d}{dt} \int |\nabla u|^2 \, dx + 2\nu \int |\Delta u|^2 \, dx \leq C\|\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2
\]
\[
\leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C\|\nabla b\|_{L^2}^2
\]
\[
\leq C\|\nabla u\|_{L^2}^2 + C(\|\rho\nabla u\|_{L^2} + \|b\| \|\nabla b\|_{L^2}) (\|\nabla u\|_{L^2} + 1)^2 + C + C\|\nabla b\|_{L^2}^2
\]
\[
\leq \frac{1}{2}\|\nabla u\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2}) + C,
\]

where

\[
\frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 - C \leq B(t) \triangleq \Psi(t) + \|\nabla b\|_{L^2}^2 \leq \mu \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 - C
\]
due to (3.22). Thus the desired (3.14) follows from (3.24), (3.25), (3.3), and Gronwall’s inequality. This completes the proof of Lemma 3.4.

The following spatial weighted estimate on the density showed in [36, Lemma 3.5] plays a crucial role in deriving the higher order derivatives of the solutions ($\rho, u, P, b$), we sketch it here for completeness.

**Lemma 3.5** Under the condition (3.1), it holds that for any $T \in [0, T^*)$,

\[
\sup_{0 \leq t \leq T} \|\rho^a\|_{L^1} \leq C(T).
\]

**Proof.** First, for $N > 1$, let $\varphi_N \in C_0^\infty(\mathbb{R}^2)$ satisfy

\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = \begin{cases} 1, & |x| \leq N/2, \\ 0, & |x| \geq N, \end{cases} \quad |\nabla \varphi_N| \leq CN^{-1}.
\]

It follows from (1.13) that

\[
\frac{d}{dt} \int \rho \varphi_N \, dx = \int \rho b \cdot \nabla \varphi_N \, dx
\]
\[ \geq -CN^{-1} \left( \int \rho dx \right)^{1/2} \left( \int \rho |u|^2 dx \right)^{1/2} \geq -\tilde{C}N^{-1}, \]  

(3.28)

where in the last inequality one has used \( (\ref{eq:2.2}) \) and \( (\ref{eq:2.3}) \). Integrating \( (\ref{eq:3.28}) \) and choosing \( N = N_1 \triangleq 2N_0 + 4\tilde{C}T \), we obtain after using \( (\ref{eq:1.10}) \) that

\[ \inf \limits_{0 \leq t \leq T} \int_{B_{N_1}} \rho dx \geq \inf \limits_{0 \leq t \leq T} \int \rho \varphi_{N_1} dx \]
\[ \geq \int \rho_0 \varphi_{N_1} dx - \tilde{C}N_1^{-1}T \]
\[ \geq \int_{B_{N_0}} \rho_0 dx - \frac{\tilde{C}T}{2N_0 + 4\tilde{C}T} \]
\[ \geq 1/4. \]  

(3.29)

Hence, it follows from \( (\ref{eq:3.29}) \), \( (\ref{eq:3.2}) \), \( (\ref{eq:2.2}) \), \( (\ref{eq:3.3}) \), and \( (\ref{eq:3.14}) \) that for any \( \eta \in (0,1] \) and any \( s > 2 \),

\[ \| u \bar{x}^{-\eta} \|_{L^s/\eta} \leq C \left( \| \sqrt{\rho} u \|_{L^2} + \| \nabla u \|_{L^2} \right) \leq C. \]  

(3.30)

Multiplying \( (\ref{eq:1.13}) \) by \( \bar{x}^a \) and integrating the resulting equality by parts over \( \mathbb{R}^2 \) yield that

\[ \frac{d}{dt} \int \rho x^a dx \leq C \int \rho |u|^a dx \leq C \int \rho |u|^a - 1 \log^2 (e + |x|^2) dx \]
\[ \leq C \int \rho |u|^a - \frac{1}{s+a} \| u \bar{x}^{-\frac{1}{s+a}} \|_{L^{s+a}} \]
\[ \leq C \int d x + C, \]

which along with Gronwall’s inequality gives \( (\ref{eq:3.26}) \) and finishes the proof of Lemma 3.5. \( \square \)

**Remark 3.1** Similarly to \( (\ref{eq:3.30}) \), one infers from \( (\ref{eq:2.2}) \), \( (\ref{eq:3.29}) \), and \( (\ref{eq:3.2}) \) that for any \( \eta \in (0,1] \) and any \( s > 2 \),

\[ \| u \bar{x}^{-\eta} \|_{L^s/\eta} \leq C \left( \| \sqrt{\rho} u \|_{L^2} + \| \nabla u \|_{L^2} \right). \]  

(3.31)

**Lemma 3.6** Under the condition \( (\ref{eq:3.1}) \), it holds that for any \( T \in [0,T^*] \),

\[ \sup \limits_{0 \leq t \leq T} \| \sqrt{\rho} u \|_{L^2}^2 + \int_0^T \| \nabla u \|_{L^2}^2 dt \leq C. \]  

(3.32)

**Proof.** Operating \( \partial_t + \text{div}(u \cdot) \) to \( j \)-th component of \( (\ref{eq:1.13})_2 \) and multiplying the resulting equation by \( \dot{u}_j \), one gets by some calculations that

\[ \frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx = \mu \int \dot{u}_j (\partial_t \Delta u_j + \text{div}(u \Delta u_j)) dx + (\lambda + \mu) \int \dot{u}_j (\partial_t \partial_j (\text{div} u) + \text{div} (u \partial_j (\text{div} u))) dx \]
\[ - \int \dot{u}_j (\partial_t P + \text{div} (u \partial_j P)) dx - \frac{1}{2} \int \dot{u}_j (\partial_t \partial_j |b|^2 + \text{div} (u \partial_j |b|^2)) dx \]
\[ + \int \dot{u}_j (\partial_t (b \cdot \nabla b_j) + \text{div} (u (b \nabla b_j))) dx \triangleq \sum \limits_{i=1}^{5} J_i. \]  

(3.33)

Integration by parts leads to

\[ J_1 = -\mu \int (\partial_i \dot{u}_j \partial_i \partial_j u_j + \Delta u_j u \cdot \nabla \dot{u}_j) dx \]
\[ = -\mu \int (|\nabla \dot{u}|^2 - \partial_i \dot{u}_j u_k \partial_k \partial_i u_j - \partial_i \dot{u}_j u_k \partial_{ik} \partial_k u_j + \Delta u_j u \cdot \nabla \dot{u}_j) dx \]
\[-\mu \int (|\nabla \hat{u}|^2 + \partial_t \hat{u}_j \partial_k u_k \partial_t u_j - \partial_t \hat{u}_j \partial_t u_k \partial_k u_j - \partial_t u_j \partial_t u_k \partial_k \hat{u}_j)dx \leq \frac{3\mu}{4} \|\nabla \hat{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4. \quad (3.34)\]

Similarly, one has
\[J_2 \leq -\frac{\lambda + \mu}{2} \|\div \hat{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4. \quad (3.35)\]

It follows from integration by parts, (1.13), (3.1), and (3.14) that
\[J_3 = \int (\partial_j \hat{u}_j P_l + \partial_j P \div \hat{u}_j)dx \]
\[= \int \partial_j \hat{u}_j (2\mu |\mathcal{D}(u)|^2 + \lambda (\div u)^2 + \nu |\nabla \times b|^2 - \div (P \div u) - P \div u) \ dx \]
\[= \int \partial_j \hat{u}_j (2\mu |\mathcal{D}(u)|^2 + \lambda (\div u)^2 + \nu |\nabla \times b|^2 - \div (P \div u) - P \div u) \ dx \]
\[\leq C \int |\nabla \hat{u}| (|\nabla u|^2 + |\nabla b|^2 + 1)dx \]
\[\leq \frac{\mu}{4} \|\nabla \hat{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla b\|_{L^4}^4 + C. \quad (3.36)\]

From (1.13)\_4, (1.13)\_5, and (3.10), we arrive at
\[J_4 = \int \partial_j \hat{u}_j |\mathcal{B}|^2dx + \frac{1}{2} \int u \cdot \nabla \hat{u}_j \partial_j |\mathcal{B}|^2dx \]
\[= \int \partial_j \hat{u}_j |\mathcal{B}|^2dx + \frac{1}{2} \int \partial_j \hat{u}_j \div u |\mathcal{B}|^2dx + \frac{1}{2} \int \partial_j u \cdot \nabla \hat{u}_j |\mathcal{B}|^2dx \]
\[\leq C \int |\nabla \hat{u}| |\nabla u| |\mathcal{B}|^2dx + C \int |\nabla \hat{u}| |\Delta \mathcal{B}| |\mathcal{B}|dx \]
\[\leq \frac{\mu}{8} \|\nabla \hat{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\Delta \mathcal{B}\|_{L^2}^2 + C. \quad (3.37)\]

Similarly to \(J_4\), we infer that
\[J_5 \leq \frac{\mu}{8} \|\nabla \hat{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\Delta \mathcal{B}\|_{L^2}^2 + C. \quad (3.38)\]

Inserting (3.31) into (3.38) yields
\[\frac{d}{dt} \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \mu \|\nabla \hat{u}\|_{L^2}^2 \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla b\|_{L^4}^4 + C \|\Delta \mathcal{B}\|_{L^2}^2 + C. \quad (3.39)\]

Now, we shall estimate the first and second terms on the right-hand side of (3.39). On the one hand, it follows from (2.7), (3.1), (3.3), (3.10), and (3.14) that
\[\|\nabla u\|_{L^4}^4 \leq C (\|\nabla \hat{u}\|_{L^2}^2 + \|b\| \|\nabla b\|_{L^2}^2)^2 + C \leq C (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|b\|_{L^\infty} \|\nabla b\|_{L^2}^2) + C \leq C \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C. \quad (3.40)\]
On the other hand, it holds from Garliardo-Nirenberg inequality and (3.14) that
\[ \|\nabla b\|^4_{L^4} \leq C \|\nabla b\|^2_{L^2} \|\nabla^2 b\|^2_{L^2} \leq C \|\nabla^2 b\|^2_{L^2}. \tag{3.41} \]
Thus, putting (3.40) and (3.41) into (3.39), one has
\[ \frac{d}{dt}\|\sqrt{\rho}u\|^2_{L^2} + \mu \|\nabla \sqrt{\rho}u\|^2_{L^2} \leq C \|\sqrt{\rho}u\|^2_{L^2} + C \|\nabla^2 b\|^2_{L^2} + C. \]

This together with Gronwall’s inequality and (3.14) implies the desired (3.32). The proof of Lemma 3.6 is completed. \(\Box\)

**Remark 3.2** From (2.7), (3.32), (3.2), (3.3), (3.10), and (3.14), we have for any \(p \geq 2\),
\[ \sup_{0 \leq t \leq T} \|\nabla u\|_{L^p} \leq C. \tag{3.42} \]

**Lemma 3.7** Under the condition (3.1), and let \(a > 1\) be as in Theorem 1.1, then it holds that for any \(T \in [0, T^*)\),
\[ \sup_{0 \leq t \leq T} \|\nabla b x^a\|^2_{L^2} + \int_0^T \|\nabla b x^a\|^2_{L^2} dt \leq C, \tag{3.43} \]
\[ \sup_{0 \leq t \leq T} \|\nabla^2 b x^a\|^2_{L^2} + \int_0^T \|\nabla^2 b x^a\|^2_{L^2} dt \leq C. \tag{3.44} \]

**Proof.** Multiplying (1.13) by \(bx^a\) and integrating by parts give rise to
\[ \frac{1}{2} \frac{d}{dt} \int |b|^2 \dot{x}^a dx + \nu \int |\nabla b|^2 \dot{x}^a dx = \nu \int |b|^2 \Delta \dot{x}^a dx + \int (b \cdot \nabla) u \cdot b \dot{x}^a dx \]
\[ - \frac{1}{2} \int \text{div} u |b|^2 \dot{x}^a dx + \frac{\nu}{2} \int |b|^2 u \cdot \nabla \dot{x}^a dx \]
\[ \triangleq \sum_{i=1}^{4} K_i. \tag{3.45} \]

Direct calculations lead to
\[ |K_1| \leq C \int |b|^2 \dot{x}^a x^{-2} \log^{2(1-\mu)} (e + |x|^2) dx \leq C \int |b|^2 \dot{x}^a dx, \tag{3.46} \]
and
\[ |K_2| + |K_3| \leq C \int |\nabla u| |b|^2 \dot{x}^a dx \]
\[ \leq C \|\nabla u\|_{L^2} \|b \dot{x}^a\|^2_{L^4} \]
\[ \leq C \|\nabla u\|_{L^2} \|b \dot{x}^a\|^2_{L^2} \left( \|\nabla b \ddot{x}^a\|^2_{L^2} + \|b \nabla \dot{x}^a\|^2_{L^2} \right) \]
\[ \leq C \|b \dot{x}^a\|^2_{L^2} + \frac{\nu}{4} \|\nabla b \ddot{x}^a\|^2_{L^2}. \tag{3.47} \]

It follows from Hölder’s inequality, Gagliardo-Nirenberg inequality, and (3.30) that
\[ |K_4| \leq C \int |b|^2 \dot{x}^a x^{-\frac{3}{2}} |u| x^{-\frac{1}{2}} \log^{(1-\mu)} (e + |x|^2) dx \]
\[ \leq C \|b \dot{x}^a\|^2_{L^4} \|b \ddot{x}^a\|^2_{L^2} \|u\|_{L^4} \]
\[ \leq C \|b \dot{x}^a\|^2_{L^2} + \frac{\nu}{4} \|\nabla b \ddot{x}^a\|^2_{L^2}. \tag{3.48} \]

Putting (3.46)–(3.48) into (3.45) and using Gronwall’s inequality, we obtain the desired (3.43).
Now we show \((3.44)\). To this end, multiplying \((1.13)\) by \(\Delta b \bar{x}^a\) and integrating the resulting equations yield that

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^2 b|^2 \bar{x}^a dx + \nu \int |\Delta b|^2 \bar{x}^a dx
\]

\[
\leq C \int |\nabla b||b||\nabla u||\nabla \bar{x}^a| dx + C \int |\nabla b|^2 |u||\nabla \bar{x}^a| dx + C \int |\nabla b||\Delta b||\nabla \bar{x}^a| dx
\]

\[
+ C \int |b||\nabla u||\Delta b|\bar{x}^a dx + C \int |\nabla u| |\nabla b|^2 \bar{x}^a dx \equiv \sum_{i=1}^5 J_i. \quad (3.49)
\]

Applying Gagliardo-Nirenberg inequality, \((3.32)\), and \((3.33)\), we have

\[
|J_1| \leq C \int |\nabla b||b||\nabla u|\bar{x}^a(|\bar{x}^{-1}|\nabla \bar{x}|) dx
\]

\[
\leq C ||b\bar{x}^\frac{1}{2}\|_{L_4}^4 + C ||\nabla b\bar{x}^\frac{1}{2}\|_{L_2}^2 + C ||\nabla u||_{L_4}^4
\]

\[
\leq C ||b\bar{x}^\frac{1}{2}\|_{L_4}^4 + C \left( ||\nabla u||_{L_4}^4 + ||b\bar{x}^\frac{1}{2}\|_{L_2}^2 \right) + C ||\nabla b\bar{x}^\frac{1}{2}\|_{L_2}^2 + C; \quad (3.50)
\]

\[
|J_2| \leq C \int |\nabla b|^{\frac{4n-1}{2}} \bar{x}^{\frac{4n-1}{2}} |\nabla \bar{x}| \frac{1}{\bar{x}} |u| \bar{x}^{-\frac{1}{2}} |\nabla \bar{x}| dx
\]

\[
\leq C ||\nabla b||_{L_n} \bar{x}^{\frac{4n-1}{2}} ||\nabla \bar{x}| ||\nabla b||_{L_4} ||u| \bar{x}^{-\frac{1}{2}} |\nabla \bar{x}| dx
\]

\[
\leq C ||\nabla b||_{L_4}^2 + C ||\nabla b||_{L_2}^2
\]

\[
\leq C ||\nabla b||_{L_4}^2 + C ||\nabla b||_{L_2}^2 + C; \quad (3.51)
\]

\[
|J_3| + |J_4| \leq C ||\Delta b\bar{x}^\frac{1}{2}||_{L_2} ||\nabla \bar{x}^\frac{1}{2}||_{L_2} + C ||\Delta b\bar{x}^\frac{1}{2}||_{L_2} ||\nabla \bar{x}^\frac{1}{2}||_{L_2} + ||\nabla b||_{L_4}^2 + C; \quad (3.52)
\]

\[
|J_5| \leq C ||\nabla u||_{L_\infty} ||\nabla \bar{x}^\frac{1}{2}||_{L_2}^2. \quad (3.53)
\]

Inserting \((3.50)\), \((3.53)\) into \((3.49)\), and noting the following fact

\[
\int |\nabla^2 b|^2 \bar{x}^a dx = \int ||\Delta b||^2 \bar{x}^a dx - \int \nabla \nabla \nabla b \cdot \nabla u \bar{x}^a + \int \nabla b \nabla b \cdot \nabla \nabla \bar{x}^a dx + \int \nabla b \nabla b \cdot \nabla \bar{x}^a dx
\]

\[
\leq \int ||\Delta b||^2 \bar{x}^a dx + \frac{1}{2} \int |\nabla^2 b|^2 \bar{x}^a dx + C \int |\nabla b|^2 \bar{x}^a dx,
\]

we derive that

\[
\frac{d}{dt} \int |\nabla^2 b|^2 \bar{x}^a dx + \int |\nabla^2 b|^2 \bar{x}^a dx \leq C(1 + ||\nabla u||_{L_\infty}) ||\nabla \bar{x}^\frac{1}{2}||_{L_2}^2 + C(1 + ||\nabla^2 b||_{L_2}^2), \quad (3.54)
\]

which along with Gronwall’s inequality, \((3.1)\), and \((3.1)\) leads to the desired \((3.44)\). Thus the proof of Lemma 3.7 is completed. \(\square\)

**Lemma 3.8** Under the condition \((3.1)\), it holds that for any \(T \in [0, T^*]\),

\[
\sup_{0 \leq t \leq T} (||b_t||_{L_2}^2 + ||\nabla b_t||_{L_2}^2) + \int_0^T ||\nabla b_t||_{L_2}^2 dt \leq C. \quad (3.55)
\]

**Proof.** Differentiating \((1.13)\) with respect to \(t\), we have

\[
b_t - \nu \Delta b_t = b_t \cdot \nabla u - u \cdot \nabla b_t - b_t \text{ div } u + b \cdot \nabla u - u_t \cdot \nabla b - b \text{ div } u. \quad (3.56)
\]

Multiplying \((3.50)\) by \(b_t\) and integrating by parts lead to

\[
\frac{1}{2} \frac{d}{dt} \int |b_t|^2 dx + \nu \int |\nabla b_t|^2 dx = \int (b_t \cdot \nabla u - u \cdot \nabla b_t - b_t \text{ div } u) \cdot b_t dx
\]
Combining (3.60) and (3.61), one gets (3.55). Hence we complete the proof of Lemma 3.8.

Inserting (3.58) and (3.59) into (3.57), we have

\begin{align*}
\frac{d}{dt}||b_t||^2_{L^2} + \nu||\nabla b_t||^2_{L^2} &\leq C \left(||\nabla u||_{L^\infty} + 1\right) ||b_t||^2_{L^2} + C(||\nabla u||^2_{L^2} + 1),
\end{align*}

which combined with Gronwall's inequality, (3.1), and (3.32) that

\begin{align*}
L_2 = \int (b \cdot \nabla \dot{u} - \dot{u} \cdot \nabla b - b \cdot \nabla \dot{u}) \cdot b_t dx
\end{align*}

It follows from integration by parts that

\begin{align*}
L_1 = \int (b_t \cdot \nabla u \cdot b_t - \frac{1}{2} |b_t|^2 \nabla u) dx \leq C ||\nabla u||_{L^\infty} ||b_t||^2_{L^2}.
\end{align*}

Moreover, one gets from Hölder's inequality, (3.10), (3.11), (3.32), (3.33), (3.34), and (3.42) that

\begin{align*}
\frac{d}{dt}||b_t||^2_{L^2} + \nu||\nabla b_t||^2_{L^2} &\leq C \left(||\nabla u||_{L^\infty} + 1\right) ||b_t||^2_{L^2} + C(||\nabla u||^2_{L^2} + 1),
\end{align*}

which combined with Gronwall's inequality, (3.1), and (3.32) yields

\begin{align*}
\sup_{0 \leq t \leq T} ||b_t||^2_{L^2} + \int_0^T ||\nabla b_t||^2_{L^2} dt \leq C.
\end{align*}

Applying the standard \(L^2\)-estimate to (1.13)4 yields

\begin{align*}
||\nabla^2 b||_{L^2} &\leq C \left(||b_t||_{L^2} + ||u|| ||\nabla b||_{L^2} + ||b|| ||\nabla u||_{L^2}\right)
\end{align*}

\begin{align*}
&\leq C \left(||b_t||_{L^2} + ||u|| ||\nabla b||^2_{L^2} ||\nabla b||^2_{L^4} + ||\nabla u||^2_{L^2} + ||\nabla u||_{L^\infty} ||\nabla u||_{L^2}\right)
\end{align*}

\begin{align*}
&\leq C ||\nabla b||^2_{L^2} ||\nabla b||_{L^4} + C
\end{align*}

\begin{align*}
&\leq \frac{1}{2} ||\nabla^2 b||_{L^2} + C.
\end{align*}

Thus

\begin{align*}
\sup_{0 \leq t \leq T} ||\nabla^2 b||^2_{L^2} \leq C.
\end{align*}

Combining (3.60) and (3.61), one gets (3.55). Hence we complete the proof of Lemma 3.8.

The following lemma will treat the higher order derivatives of the solutions which are needed to guarantee the extension of local strong solution to be a global one.

**Lemma 3.9** Under the condition (3.1), and let \(q > 2\) be as in Theorem 1.1, then it holds that for any \(T \in [0, T^*)\),

\begin{align*}
\sup_{0 \leq t \leq T} \left(||(\rho, P)||_{H^1 \cap W^{1,q}} + ||\nabla u||_{H^1}\right) + \int_0^T ||\nabla^2 u||^2_{L^q} dt \leq C.
\end{align*}
Proof. It follows from the mass equation (1.13) that $\nabla \rho$ satisfies for any $r \in [2, q]$,

$$
\frac{d}{dt} \|\nabla \rho\|_{L^r} \leq C(r)(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^r} + C(r)\|\nabla^2 u\|_{L^r}
$$

$$
\leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^r} + C(\|\rho \dot{u}\|_{L^r} + \|\nabla P\|_{L^r} + \|\nabla u\|_{L^r} + \|\nabla u\|_{L^r} + \|\rho \|_{L^r})
$$

(3.63)

due to

$$
\|\nabla^2 u\|_{L^r} \leq C(\|\rho \dot{u}\|_{L^r} + \|\nabla P\|_{L^r} + \|\nabla b\|_{L^r}),
$$

(3.64)

which follows from the standard $L^r$-estimate for the following elliptic system

$$
\begin{aligned}
\mu \Delta u + (\lambda + \mu) \nabla \mathrm{div} u &= \rho \dot{u} + \nabla P - b \cdot \nabla b + \frac{1}{2} \nabla |b|^2, \quad x \in \mathbb{R}^2, \\
\mu u &\to 0, \quad \text{as } |x| \to \infty.
\end{aligned}
$$

Similarly, one deduces from (1.13) that $\nabla P$ satisfies for any $r \in [2, q]$,

$$
\frac{d}{dt} \|\nabla P\|_{L^r} \leq C(r)(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^r} + \|\nabla^2 u\|_{L^r} + C(\|\rho \dot{u}\|_{L^r} + \|\nabla b\|_{L^r} + \|\nabla b\|_{L^r})
$$

$$
\leq C(1 + \|\nabla u\|_{L^\infty})(\|\rho \dot{u}\|_{L^r} + \|\nabla P\|_{L^r} + \|\nabla b\|_{L^r}) + C(\|\rho \dot{u}\|_{L^r} + \|\nabla b\|_{L^r}).
$$

(3.65)

Taking $r = 2$ in (3.63) and (3.65), one gets from (3.2), (3.10), (3.14), (3.32), (3.55), and Gagliardo-Nirenberg inequality that

$$
\frac{d}{dt} (\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2}) \leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2} + 1) + C\|\nabla b\|_{L^\infty}
$$

$$
\leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2} + 1) + C\|\nabla b\|_{L^2}^q \|\nabla^2 b\|_{L^q}^q
$$

$$
\leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2} + 1) + C\|\nabla b\|_{L^2}^q \|\nabla^2 b\|_{L^q}^q.
$$

(3.66)

Applying the standard $L^q$-estimate to (1.14),

$$
\|\nabla^2 b\|_{L^q} \leq C(\|b\|_{L^q} + \|u\|_{L^q} \|\nabla b\|_{L^q} + \|b\|_{L^q} \|\nabla u\|_{L^q})
$$

$$
\leq C \left( \|b\|_{L^1} + \|u\|_{L^2}^q \|\nabla^2 b\|_{L^2}^q \|\nabla^2 u\|_{L^2}^q + \|b\|_{L^\infty} \|\nabla u\|_{L^q} \right)
$$

$$
\leq C \|\nabla b\|_{L^2} + C,
$$

(3.67)

which combined with (3.70), Gronwall’s inequality, (3.65), and the fact $\frac{q}{2q - 2} \in (0, 1)$ leads to

$$
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2}) \leq C.
$$

(3.68)

Then one derives from (3.64), (3.2), (3.32), (3.68), (3.10), and (3.14) that

$$
\sup_{0 \leq t \leq T} \|\nabla^2 u\|_{L^2} \leq C.
$$

(3.69)

Next, one gets from (2.4), Gagliardo-Nirenberg inequality, (3.1), (3.10), (2.5), (3.32), (3.14), and (3.55) that

$$
\|\nabla u\|_{L^\infty} + \|\omega\|_{L^\infty} \leq C\|P\|_{L^\infty} + C\|F\|_{L^\infty} + C\|b\|^2_{L^\infty} + \|\omega\|_{L^\infty}
$$

$$
\leq C + C(q)\frac{q}{2q - 2} \|\nabla F\|^q_{L^q} \|\nabla F\|_{L^q}^{2q-2} + C(q)\|\omega\|_{L^q} \|\omega\|_{L^q}^{q-2} \|\nabla \omega\|_{L^q}^{2q-2} \|\nabla \omega\|_{L^q}^{q-2}
$$

$$
\leq C + C\|\rho \dot{u}\|_{L^q}^{2q-1} + C\|b\|_{L^2} \|\nabla b\|_{L^q}^{q-1}
$$

$$
\leq C + C\|\rho \dot{u}\|_{L^q}^{2q-1} + C\|b\|_{L^2} \|\nabla b\|_{L^q}^{q-1}
$$

$$
\leq C + C\|\rho \dot{u}\|_{L^q}^{2q-1},
$$

(3.70)
which together with Lemma 2.4, (3.64), and (3.14) yields that
\[ \|\nabla u\|_{L^\infty} \leq C (\|\text{div } u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + C \|\nabla u\|_{L^q} + C \]
\[ \leq C \left( 1 + \|\rho \hat{u}\|_{L^q}^{\frac{4}{q-2}} \right) \log(e + \|\rho \hat{u}\|_{L^q} + \|\nabla P\|_{L^q}) + C \]
(3.71)
owing to
\[ \sup_{0 \leq t \leq T} \|b\|_{L^q} \leq C \left( \sup_{0 \leq t \leq T} \|b\|_{L^\infty} \right) \left( \sup_{0 \leq t \leq T} \|\nabla b\|_{H^1} \right) \leq C. \]
(3.72)

It follows from (3.29), (3.1), (2.2), and (3.26) that for any \( q \in (0,1] \) and any \( s > 2, \)
\[ \|\rho^q v\|_{L^q} \leq C \|\rho^q x\|_{L^\infty}^{\frac{2q-4}{q-2}} \|v\|_{L^q}^{\frac{2q-4}{q-2}} \]
\[ \leq C \|\rho\|_{L^\infty}^{\frac{2(q-1)}{q-2}} \|\rho^q x\|_{L^\infty}^{\frac{2(q-1)}{q-2}} (\|\nabla v\|_{L^q} + \|v\|_{L^q}) \]
\[ \leq C \left( \|\nabla v\|_{L^q} + \|v\|_{L^q} \right), \]
(3.73)
which along with Hölder’s inequality, (3.32), and (3.2) shows that
\[ \|\rho \hat{u}\|_{L^q} \leq C \|\rho \hat{u}\|_{L^q}^{\frac{2(q-2)}{q-2}} \|\rho \hat{u}\|_{L^q}^{\frac{2(q-2)}{q-2}} \]
\[ \leq C \left( 1 + \|\nabla \hat{u}\|_{L^q}^{\frac{2(q-2)}{q-2}} \right), \]
(3.74)

Then we derive from (3.71) and (3.74) that
\[ \|\nabla u\|_{L^\infty} \leq C (1 + \|\nabla \hat{u}\|_{L^q}) \log(e + \|\nabla \hat{u}\|_{L^q} + \|\nabla P\|_{L^q}) + C \]
(3.75)
due to \( \frac{q(q^2-2q)}{(2q-2)(q^2-2)} \in (0,1) \). Consequently, substituting (3.74) and (3.75) into (3.63) and (4.63), we get after choosing \( r = q \) that
\[ f'(t) \leq C g(t) f(t) \log f(t) + C g(t) f(t) + C g(t), \]
(3.76)
where
\[ f(t) \triangleq e + \|\nabla \rho\|_{L^q} + \|\nabla P\|_{L^q}, \]
\[ g(t) \triangleq (1 + \|\nabla \hat{u}\|_{L^q}) \log(e + \|\nabla \hat{u}\|_{L^q}) + \|\nabla b\|_{L^q}^2. \]

This yields
\[ (\log f(t))' \leq C g(t) + C g(t) \log f(t) \]
(3.77)
due to \( f(t) > 1 \). Thus it follows from (3.77), (3.32), (3.55) and Gronwall’s inequality that
\[ \sup_{0 \leq t \leq T} \|\nabla \rho, \nabla P\|_{L^q} \leq C. \]
(3.78)

Choosing \( r = q \) in (3.64), we obtain from (3.74), (3.75), and (3.72) that
\[ \|\nabla^2 u\|_{L^q} \leq C (\|\rho \hat{u}\|_{L^q} + \|\nabla P\|_{L^q} + \|b\|_{L^q} + \|\nabla b\|_{L^q}) \]
\[ \leq C \|\nabla \hat{u}\|_{L^q}^{\frac{2(q-2)}{q-2}} + C. \]
This along with (3.32) gives
\[ \int_0^T \|\nabla^2 u\|_{L^q}^2 \, dt \leq C \]
(3.79)
due to \( \frac{q(q-2)}{q^2-2} \in (0,1) \). Consequently, the desired (3.64) follows from (3.63), (3.78), (3.69), (4.14), and (3.79). The proof of Lemma 3.9 is finished.

The following higher order spatial weighted estimate on the density can be proved similarly as in [30] Lemma 3.7, and we omit the details.
Lemma 3.10 Under the condition \( (3.1) \), it holds that for any \( T \in [0,T^*) \),
\[
\sup_{0 \leq t \leq T} \| \rho \bar{x}^a \|_{L^1 \cap H^1 \cap W^{1,q}} \leq C. \tag{3.80}
\]

With Lemmas 3.1–3.10 at hand, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1** We argue by contradiction. Suppose that \( (1.21) \) were false, that is, \( (3.1) \) holds. Note that the general constant \( C \) in Lemmas 3.1–3.10 is independent of \( t < T^* \), that is, all the a priori estimates obtained in Lemmas 3.1–3.10 are uniformly bounded for any \( t < T^* \).

Hence, the function \((\rho, u, P, b)(x, T^*) \triangleq \lim_{t \to T^*} (\rho, u, P, b)(x, t)\) satisfy the initial condition \( (1.19) \) at \( t = T^* \).

Furthermore, standard arguments yield that \( \rho \dot{u} \in C([0, T]; L^2) \), which implies \( \rho \dot{u}(x, T^*) = \lim_{t \to T^*} \rho \dot{u} \in L^2 \).

Hence,
\[
-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P - b \cdot \nabla b + \frac{1}{2} \nabla |b|^2 \big|_{t = T^*} = \sqrt{\rho}(x, T^*) g(x)
\]

with
\[
g(x) \triangleq \begin{cases} 
\rho^{-1/2}(x, T^*)(\rho \dot{u})(x, T^*), & \text{for } x \in \{ x | \rho(x, T^*) > 0 \}, \\
0, & \text{for } x \in \{ x | \rho(x, T^*) = 0 \}, 
\end{cases}
\]
satisfying \( g \in L^2 \) due to \( (3.62) \). Therefore, one can take \((\rho, u, P, b)(x, T^*)\) as the initial data and extend the local strong solution beyond \( T^* \). This contradicts the assumption on \( T^* \).

Thus we finish the proof of Theorem 1.1. \( \square \)

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**References**

[1] J. Fan, S. Jiang and Y. Ou, A blow-up criterion for compressible viscous heat-conductive flows, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 337–350.

[2] J. Fan, F. Li, and G. Nakamura, A blow-up criterion to the 2D full compressible magnetohydrodynamic equations, Math. Methods Appl. Sci., 38 (2015), 2073–2080.

[3] J. Fan and W. Yu, Strong solution to the compressible magnetohydrodynamic equations with vacuum, Nonlinear Anal. Real World Appl., 10 (2009), 392–409.

[4] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2004.

[5] C. He and Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations, 213 (2005), 235–254.

[6] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Differential Equations, 120 (1995), 215–254.

[7] G. Hong, X. Hou, H. Peng and C. Zhu, Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum, SIAM J. Math. Anal., 49 (2017), 2409–2441.
[8] X. Hu and D. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, Comm. Math. Phys., 283 (2008), 255–284.

[9] X. Hu and D. Wang, Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows, Arch. Ration. Mech. Anal., 197 (2010), 203–238.

[10] X. D. Huang and J. Li, Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and magnetohydrodynamic flows, Comm. Math. Phys., 324 (2013), 147–171.

[11] X. D. Huang, J. Li and Y. Wang, Serrin-type blowup criterion for full compressible Navier-Stokes system, Arch. Ration. Mech. Anal., 207 (2013), 303–316.

[12] X. D. Huang, J. Li and Z. Xin, Blowup criterion for viscous barotropic flows with vacuum states, Comm. Math. Phys., 301 (2011), 23–35.

[13] X. D. Huang, J. Li and Z. Xin, Serrin-type criterion for the three-dimensional viscous compressible flows, SIAM J. Math. Anal., 43 (2011), 1872–1886.

[14] X. D. Huang, J. Li and Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, Comm. Pure Appl. Math., 65 (2012), 549–585.

[15] X. D. Huang and Z. Xin, On formation of singularity for non-isentropic Navier-Stokes equations without heat-conductivity, Discrete Contin. Dyn. Syst., 36 (2016), 4477–4493.

[16] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamic, PhD thesis, Kyoto University, 1983.

[17] H. Li, X. Xu and J. Zhang, Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum, SIAM J. Math. Anal., 45 (2013), 1356–1387.

[18] J. Li and Z. Xin, Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum, [http://arxiv.org/abs/1310.1673](http://arxiv.org/abs/1310.1673)

[19] P. L. Lions, *Mathematical topics in fluid mechanics, vol. I: incompressible models*, Oxford University Press, Oxford, 1996.

[20] P. L. Lions, *Mathematical Topics in Fluid Mechanics, vol. II: Compressible Models*, Oxford University Press, Oxford, 1998.

[21] L. Lu, Y. Chen and B. Huang, Blow-up criterion for two-dimensional viscous, compressible, and heat conducting magnetohydrodynamic flows, Nonlinear Anal., 139 (2016), 55–74.

[22] L. Lu and B. Huang, On local strong solutions to the Cauchy problem of the two-dimensional full compressible magnetohydrodynamic equations with vacuum and zero heat conduction, Nonlinear Anal. Real World Appl., 31 (2016), 409–430.

[23] M. Lu, Y. Du and Z. A. Yao, Blow-up criterion for compressible MHD equations, J. Math. Anal. Appl., 379 (2011), 425–438.

[24] B. Lü and B. Huang, On strong solutions to the Cauchy problem of the two-dimensional compressible magnetohydrodynamic equations with vacuum, Nonlinearity, 28 (2015), 509–530.

[25] B. Lü, X. Shi and X. Xu, Global well-posedness and large time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum, Indiana Univ. Math. J., 65 (2016), 925–975.
[26] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, 13 (1959), 115–162.

[27] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier-Stokes equations, J. Math. Pures Appl., 95 (2011), 36–47.

[28] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows, Arch. Ration. Mech. Anal., 201 (2011), 727–742.

[29] T. Tao, Nonlinear dispersive equations. Local and global analysis, American Mathematical Society, Providence, R.I., 2006.

[30] T. Wang, A regularity criterion of strong solutions to 2D compressible magnetohydrodynamic equations, Nonlinear Anal. Real World Appl., 31 (2016), 100–118.

[31] Y. Wang, One new blowup criterion for the 2D full compressible Navier-Stokes system, Nonlinear Anal. Real World Appl., 31 (2016), 100–118.

[32] H. Wen and C. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum, Adv. Math., 248 (2013), 534–572.

[33] Z. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, Comm. Pure Appl. Math., 51 (1998), 229–240.

[34] Z. Xin and W. Yan, On blowup of classical solutions to the compressible Navier-Stokes equations, Comm. Math. Phys., 321 (2013), 529–541.

[35] X. Xu and J. Zhang, A blow-up criterion for 3D compressible magnetohydrodynamic equations with vacuum, Math. Models Methods Appl. Sci., 22 (2012), 1150010.

[36] X. Zhong, Singularity formation to the 2D Cauchy problem of the full compressible Navier-Stokes equations with zero heat conduction, https://arxiv.org/abs/1705.05161

[37] X. Zhong, On formation of singularity of the full compressible magnetohydrodynamic equations with zero heat conduction, https://arxiv.org/abs/1705.06606, to appear in Indiana Univ. Math. J., 2018.

[38] X. Zhong, Strong solutions to the Cauchy problem of the two-dimensional non-baratropic non-resistive magnetohydrodynamic equations with zero heat conduction, https://arxiv.org/abs/1801.07589