A correlation game for unsupervised learning yields computational interpretations of Hebbian excitation, anti-Hebbian inhibition, and synapse elimination

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Abstract

Much has been learned about plasticity of biological synapses from empirical studies. Hebbian plasticity is driven by correlated activity of presynaptic and postsynaptic neurons. Synapses that converge onto the same neuron often behave as if they compete for a fixed resource; some survive the competition while others are eliminated. To provide computational interpretations of these aspects of synaptic plasticity, we formulate unsupervised learning as a zero-sum game between Hebbian excitation and anti-Hebbian inhibition in a neural network model. The game formalizes the intuition that Hebbian excitation tries to maximize correlations of neurons with their inputs, while anti-Hebbian inhibition tries to decorrelate neurons from each other. We further include a model of synaptic competition, which enables a neuron to eliminate all connections except those from its most strongly correlated inputs. Through empirical studies, we show that this facilitates the learning of sensory features that resemble parts of objects.

Much has been learned about plasticity of biological synapses from empirical studies. One line of research has explored Hebbian plasticity, which is triggered by correlated presynaptic and postsynaptic activity. Many kinds of excitatory synapses are strengthened by correlated activity, and are said to be Hebbian [Bi and Poo 2001]. Strengthening of inhibitory synapses by correlated activity has also been observed [Gaiarsa et al., 2002]. The phenomenon is sometimes said to be anti-Hebbian, because strengthening an inhibitory synapse is like making a negative number more negative. Another line of research has shown that synapses converging onto the same neuron often behave as if they compete for shares of a fixed “pie”; some survive the competition while others are eliminated [Lichtman and Colman 2000]. According to this viewpoint, Hebbian plasticity does not increase or decrease the pie of synaptic resources; it only allocates resources across convergent synapses [Miller 1996].

Theoretical neuroscientists have proposed a number of computational functions for Hebbian excitation, anti-Hebbian inhibition, and synaptic competition/elimination. Hebbian excitation has long been invoked as a mechanism for learning features from sensory input [Von der Malsburg 1973]. Lateral inhibition has been used to sparsen neural activity, thereby facilitating Hebbian feature learning [Von der Malsburg 1973, Fukushima 1980, Rumelhart and Zipser 1985, Kohonen 1990]. Endowing the lateral inhibition with anti-Hebbian plasticity can give more robust control over sparseness of activity. Földiák [1990] demonstrated this with numerical experiments, but did not provide an interpretation in terms of an optimization principle. Less relevant here are anti-Hebbian models without reference to sparse activity in linear [Foldiak 1989, Rubner and Tavan 1989, Rubner and Schulten 1990] and nonlinear [Carlson 1990, Girolami and Fyfe 1997] networks. Also less relevant is the application of anti-Hebbian plasticity to feedforward rather than lateral connections [Hyvärinen and Oja 1998].
Leen [1991] performed a stability analysis for a linear network with Hebbian feedforward connections and anti-Hebbian lateral connections. Plumbley [1993] derived a linear network with anti-Hebbian lateral inhibition (but no plasticity of feedforward connections) from the principle of information maximization with a power constraint. Pehlevan et al. [2015] showed that a linear network with Hebbian feedforward connections and anti-Hebbian lateral inhibition can be interpreted as online gradient optimization of a “similarity matching” cost function. Pehlevan and Chklovskii [2014] and Hu et al. [2014] went on to extend the similarity matching principle to derive nonlinear neural networks for unsupervised learning. Synaptic competition and elimination have been studied in models of cortical development, and have been shown to play an important role in the emergence of feature selectivity [Miller, 1996].

The subject of the present work is a mathematical formalism that provides computational interpretations of Hebbian excitation, anti-Hebbian inhibition, and synaptic competition/elimination in nonlinear neural networks. We start by formulating unsupervised learning as the maximization of output-input correlations subject to upper bound constraints on output-output correlations. We motivate our formulation by describing its relation to previous theoretical frameworks for unsupervised learning, such as maximization [Linsker, 1988, Atick and Redlich, 1990, Plumbley, 1993, Bell and Sejnowski, 1995] or minimization of mutual information [Hyvärinen and Oja, 2000], (2) projection onto a subspace that maximizes a moment-based statistic such as variance [Oja, 1982, Linsker, 1988] or kurtosis [Huber, 1985], and the (3) similarity matching principle [Pehlevan et al., 2015].

To solve our constrained maximization problem, we introduce Lagrange multipliers. This Lagrangian dual formulation of unsupervised learning can in turn be solved by a nonlinear neural network with Hebbian excitation and anti-Hebbian inhibition. The network is very similar to the original model of Foldiák [1990], differing mainly by its use of rectification rather than sigmoidal nonlinearity. (The latter can also be handled by our formalism, as shown in Appendix B.) Lagrange multipliers were also used to study anti-Hebbian plasticity by Plumbley [1993], but only for linear networks.

Effectively, excitation and inhibition behave like players in a game, and the inhibitory connections can be interpreted as Lagrange multipliers. The game is zero-sum, in that excitation tries to maximize a payoff function and inhibition tries to minimize exactly the same payoff function. Roughly speaking, however, one could say that excitation aims to maximize the correlation of each output neuron with its inputs, while inhibition aims to decorrelate the output neurons from each other. Our term “correlation game” is derived from this intuitive picture.

Within our mathematical formalism, we also consider a dynamics of synaptic competition and elimination that is drawn from models of cortical development [Miller and MacKay, 1994]. Competition between the excitatory synapses convergent on a single neuron is capable of driving the strengths of some synapses to zero. In numerical experiments with the MNIST dataset, we show that synapse elimination has the computational function of facilitating the learning of features that resemble “parts” of objects. Theoretical analysis shows that the surviving synapses converging onto an output neuron come from its most strongly correlated inputs; synapses from weakly correlated inputs are eliminated.

Our correlation game is closely related to the similarity matching principle of Pehlevan and Chklovskii [2014] and Pehlevan et al. [2015]. The major novelty is the introduction of decorrelation as a constraint for the optimization. Paralleling our work, Pehlevan et al. [2017] have shown that the similarity matching principle leads to a game theoretic formulation through Hubbard-Stratonovich duality. Again our novelty is the use of decorrelation as a constraint, which leads to our correlation game through Lagrangian duality.

Our model of synaptic competition and elimination was borrowed with slight modification from the literature on modeling neural development [Miller and MacKay, 1994]. It can be viewed as a more biologically plausible alternative to previous unsupervised learning algorithms that sparsen features. For example, Hoyer [2004] is similar to ours because it can be interpreted as independently sparsening each set of convergent synapses, rather than applying a global L1 regularizer to all synapses.
1 Primal formulation

Our neural networks will learn to transform a sequence of input vectors $\mathbf{u}(1), \ldots, \mathbf{u}(T)$ into a sequence of output vectors $\mathbf{x}(1), \ldots, \mathbf{x}(T)$. Both input and output will be assumed nonnegative.

Define the input matrix $U = [\mathbf{u}(1), \ldots, \mathbf{u}(T)]$ as the matrix containing input vectors $\mathbf{u}(t)$ as its columns. The element $U_{at}$ is the $a$th component of $\mathbf{u}(t)$. Similarly, define the output matrix $X = [\mathbf{x}(1), \ldots, \mathbf{x}(T)]$ as containing output vectors $\mathbf{x}(t)$ as its columns. Both input and output will be assumed nonnegative. We define the output-input correlation matrix as

$$
\frac{XU^T}{T} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t) \mathbf{u}(t)^T
$$

Its $ia$ element is the time average of $x_i u_a$, or $\langle x_i u_a \rangle$. Similarly, we define the output-output correlation matrix

$$
\frac{XX^T}{T} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t) \mathbf{x}(t)^T
$$

Its $ij$ element is the time average of $x_i x_j$, or $\langle x_i x_j \rangle$. Note that we use “correlation matrix” to mean second moment matrix rather than covariance matrix. In other words, our correlation matrix does not involve subtraction of mean values. We believe that this is natural for sparse nonnegative variables, but covariance matrices may be substituted in other settings (see Appendix B).

Problem 1 (Primal formulation). We define the goal of unsupervised learning as the constrained optimization

$$\max_{X \geq 0} \Phi^* \left( \frac{XU^T}{T} \right) \text{ such that } \frac{XX^T}{T} \leq D \quad (1)$$

where $D$ is a fixed matrix and $\Phi^*$ is a scalar-valued function of a matrix argument. We will assume that $\Phi^*(C)$ is monotone nondecreasing as a function of every element of $C$, which allows us to interpret the objective of Eq. (1) as maximization of correlations between inputs and outputs. Later on, it will also be convenient to assume that $\Phi^*$ is convex.

The inequality constraint in Eq. (1) applies to all elements of the matrices $XX^T/T$ and $D$. Its role is to prevent two kinds of degenerate optima. The diagonal elements of the constraint, $\langle x_i^2 \rangle \leq D_{ii}$, limit the power in the outputs, as in Plumbley [1993]. This prevents the trivial solution of maximizing output-input correlations by scaling up $x_i$. The off-diagonal elements, $\langle x_i x_j \rangle \leq D_{ij}$, insure that the outputs remain distinct from each other. Suppose for example that all diagonal elements of $D$ are $q^2$ and all off-diagonal elements of $D$ are $p^2$,

$$
D_{ij} = \begin{cases} 
p^2, & i \neq j, 
q^2, & i = j
\end{cases} \quad (2)
$$

If a pair of outputs saturates the inequality constraints, then their cosine similarity is

$$
\frac{\langle x_i x_j \rangle}{\sqrt{\langle x_i^2 \rangle} \sqrt{\langle x_j^2 \rangle}} = \frac{p^2}{q^2} \quad (3)
$$

If the outputs are nonnegative, the cosine similarity lies between 0 and 1. A cosine similarity of $p/q = 1$ means that every output is exactly the same. A cosine similarity of $p/q = 0$ means a “winner-take-all” representation in which only one output is active for any given stimulus. We will generally work in the intermediate regime where $p^2/q^2$ is positive but much less than one. In this regime, we can say that the constraint in Eq. (1) requires that the outputs be “decorrelated” or “desimilarized.” The outputs end up sparse but distributed, as will be seen empirically later on. To summarize the above, we can say that Eq. (1) maximizes output-input correlations while decorrelating the outputs.
Our Problem 1 may look unfamiliar, but in fact it is similar to other formulations of unsupervised learning that are already well-accepted: (1) maximization [Linsker, 1988, Atick and Redlich, 1990, Bell and Sejnowski, 1995] or minimization [Hyvärinen and Oja, 2000] of mutual information, (2) projection onto a subspace that maximizes a moment-based statistic such as variance or kurtosis [Huber, 1985], and (3) the similarity matching principle [Pehlevan et al., 2015]. These connections are discussed in Appendix A. One might ask whether a wider class of objective functions admits a similar analysis to the one given in this paper. An example of the flexibility of our approach is given in Appendix B.

2 Lagrangian dual and the correlation game

While Problem 1 has connections with traditional unsupervised learning objectives, a dual formulation makes its connections to neural networks manifest. We make use of two duality transforms to introduce auxiliary variables \( W_{ia} \) and \( L_{ij} \), which will respectively turn out to be feedforward and lateral synaptic weight matrices of a neural network.

First, we rewrite \( \Phi^* \) in terms of its Legendre-Fenchel transform \( \Phi \),

\[
\Phi^*(C) = \max_W \left\{ \sum_{ia} W_{ia} C_{ia} - \Phi(W) \right\} \tag{4}
\]

\[
= \max_{W \geq 0} \left\{ \sum_{ia} W_{ia} C_{ia} - \Phi(W) \right\} \tag{5}
\]

The first equality holds since \( \Phi^* \) is convex, and the second equality holds since \( \Phi^* \) is monotonic as a function of each element of \( C \). We will often take \( \Phi(W) = \infty \) outside a convex set \( B \). This effectively restricts \( W \) and lets us replace \( \max_{W \geq 0} \) with \( \max_{W \in B} \).

Second, we introduce Lagrange multipliers \( L_{ij} \) to solve the constrained optimization of Eq. (1). The optimization is equivalent to

\[
\max_{X \geq 0} \min_{L \geq 0} \left\{ \Phi^* \left( \frac{XU^T}{T} \right) - \frac{1}{2} \sum_{ij} L_{ij} \left( \frac{1}{T} \sum_t X_{it} X_{jt} - D_{ij} \right) \right\} \tag{6}
\]

(The outer maximum must choose \( X \) such that \( XX^T / T \leq D \) because otherwise the minimum with respect to \( L \) is \(-\infty\).) Switching the order of maximum and minimum yields the following unconstrained optimization.

**Problem 2** (Dual formulation). An alternative to Problem 1 is to define unsupervised learning as

\[
\min_{L \geq 0} \max_{X \geq 0} \left\{ \Phi^* \left( \frac{XU^T}{T} \right) - \frac{1}{2} \sum_{ij} L_{ij} \left( \frac{1}{T} \sum_t X_{it} X_{jt} - D_{ij} \right) \right\} \tag{7}
\]

By the minimax inequality, this is an upper bound for Eq. (6). A maximization with respect to \( W \) is implicit in the definition (4) of the convex conjugate \( \Phi^* \). We can switch the order of \( W \) and \( X \) maximizations to obtain the following equivalent problem.

**Problem 3** (Correlation game). Define the payoff function

\[
R(W, L) = \max_{X \geq 0} \left\{ \frac{1}{T} \sum_t \left[ \sum_{ia} W_{ia} X_{it} U_{it} - \Phi(W) - \frac{1}{2} \sum_{ij} L_{ij} (X_{it} X_{jt} - D_{ij}) \right] \right\} \tag{8}
\]

Then Eq. (7) is equivalent to

\[
\min_{L \geq 0} \max_{W \in B} R(W, L) \tag{9}
\]
which can be interpreted as a zero-sum game played between excitation and inhibition. The goal of excitation is to maximize the payoff function \( (8) \), and the goal of inhibition is to minimize the same payoff function. Roughly speaking, however, one could say that excitation aims to maximize output-input correlations while inhibition aims to decorrelate the outputs. This is our rationale for referring to \( (9) \) as the “correlation game.”

3 Neural network algorithm

The optimizations in Eq. \( 9 \) could be performed by many methods. Here we consider an iterative online method, which is based on one input vector at each time step. First we describe the optimization with respect to \( X \) in Eq. \( (8) \). The objective function is nondecreasing under coordinate ascent with nonnegativity constraints,

\[
X_{it} := \frac{1}{L_{ii}} \left[ \sum_a W_{ia} U_{at} - \sum_{j,j\neq i} L_{ij} X_{jt} \right]^+ \tag{10}
\]

This can be interpreted as the dynamics of a neural network (Fig. 1). The conjugate variables \( W_{ia} \) in the Legendre-Fenchel transform of Eq. \( 4 \) are now feedforward connections from the input to the output. The Lagrange multipliers \( L_{ij} \) are now lateral connections between the outputs. The lateral connections are assumed to be symmetric \( (L_{ij} = L_{ji}) \), which guarantees that the dynamics will converge to a local maximum of the objective function if there is no runaway instability [Hahnloser et al., 2003]. If the diagonal elements of \( L \) have a positive lower bound and the off-diagonal elements are nonnegative, then \( L \) is copositive definite, \( x^T L x \geq 0 \) for \( x \geq 0 \) with equality only for \( x = 0 \). It follows that the dynamics of Eq. \( (10) \) exhibits no runaway instability [Hahnloser et al., 2003].

The coordinate ascent dynamics of Eq. \( (10) \) was previously considered by Pehlevan and Chklovskii [2014], who added nonnegativity constraints to the similarity matching principle. Coordinate ascent is a particularly simple way of optimizing Eq. \( (8) \) with respect to \( X \); many other “neural” algorithms could be used. Appendix B extends our formalism to neural networks with sigmoidal rather than rectification nonlinearity.

For the other optimizations in Eq. \( (9) \), we perform gradient ascent with respect to \( W \) and gradient descent with respect to \( L \). Stochastic projected gradient ascent on \( W \) is

\[
\Delta W_{ia} \propto X_{it} U_{at} - \frac{\partial \Phi}{\partial W_{ia}} \tag{11}
\]

followed by projection of \( W \) onto the convex domain \( B \). Stochastic projected gradient descent on \( L \) is

\[
\Delta L_{ij} \propto X_{it} X_{jt} - D_{ij} \tag{12}
\]

followed by rectification of \( L \). This update preserves symmetry of \( L \), provided that \( D \) is symmetric. Gradient ascent-descent is not generally guaranteed to converge to a steady state, but may display other dynamical behaviors such as limit cycles. We have found in practice that convergence to a steady state is not difficult to obtain, as will be shown by empirical results later on.


4 Biological interpretation

We now elaborate on the biological interpretation of the above learning algorithm, using specific choices for \( \Phi(W) \), convex domain \( B \), and the desired correlation matrix \( D \). We define

\[
\Phi(W) = \frac{\kappa}{2} \sum_i \left( \sum_a W_{ia} - \rho \right)^2 \tag{13}
\]

which is minimized when the row sums of \( W \) are equal to \( \rho \). We define \( B \) by the bound constraints, \( 0 \leq W_{ia} \leq \omega \). Since it is online, the algorithm will be rewritten below omitting time \( t \).

Given a stimulus vector \( u \), update the activities \( x_i \) of the output neurons using

\[
x_i := \frac{1}{L_{ii}} \left[ \sum_a W_{ia} u_a - \sum_{j,j \neq i} L_{ij} x_j \right]^+ \tag{14}
\]

The dynamics may cycle through the neurons in a fixed or random order. The input neurons \( u_a \) try to activate the output neurons \( x_i \) through the excitatory connections \( W_{ia} \). The output neurons try to turn each other off through the lateral inhibitory connections \( L_{ij} \). Due to the (half-wave) rectification of Eq. (14), denoted by \([z]^+ = \max\{z,0\}\), neural activities are nonnegative. This is consistent with the interpretation of \( x_i \) as neural activity defined by rate of action potential firing (a rate is nonnegative by definition).

After convergence of \( x \), the connection strengths are updated. The excitatory connections change as

\[
\Delta W_{ia} = \eta W \left[ x_i u_a - \kappa \left( \sum_b W_{ib} - \rho \right) \right] \tag{15}
\]

The first term is Hebbian, as it causes strengthening of \( W_{ia} \) when \( x_i \) and \( u_a \) are coactive. The second term weakens \( W_{ia} \) when other synapses converging onto the same neuron are strengthened. It has the effect of creating competition between convergent synapses. For large \( \kappa \), the plasticity rule drives the connections to satisfy \( \sum_b W_{ib} \approx \rho \). In words, the connections converging onto a neuron behave as if they are competing for a fixed resource \( \rho \). Hebbian plasticity determines how strength is allocated between the connections, but does not change the overall sum of strengths [Miller, 1996].

After the update (15), the connections are thresholded to the range \([0, \omega]\) via

\[
W_{ia} := \max\{0, W_{ia}\} \quad W_{ia} := \min\{\omega, W_{ia}\} \tag{16}
\]

The first thresholding prevents negative connections, preserving the interpretation of \( W \) as excitatory connectivity. The second thresholding ensures that connections do not exceed the upper bound \( \omega \). As will be shown later, the outcome of competition is simple when \( \kappa \) is large and the ratio \( \rho / \omega \) is a positive integer. Then the number of surviving synapses converging on each neuron is \( \rho / \omega \), and all survivors have maximal strength \( \omega \). The remaining synapses have strength zero, i.e., they have been eliminated. Similar models of synapse elimination have been used previously in theoretical studies of neural development [Miller and MacKay, 1994]. This model of synaptic competition is presented because it is especially simple, but other models produce similar results. For example, Appendix D presents an alternative model in which the surviving synapses are not fixed in number, and have graded, analog strengths. Note that there are many other models of synaptic competition that do not eliminate synapses. Included in this category are models that hold fixed the Euclidean norm of convergent synapses but lack nonnegativity constraints [Oja, 1982].

The off-diagonal \((i \neq j)\) elements of \( L \) are lateral connections, mediating competitive interactions between the output neurons. They are updated via

\[
\Delta L_{ij} = \eta L \left( x_i x_j - \rho^2 \right) \tag{17}
\]
In the first term, coincident activation of presynaptic and postsynaptic neurons causes strengthening of $L$. This sometimes called “anti-Hebbian” plasticity since the $L$ connections are inhibitory [Foldiak, 1989, Rubner and Tavan, 1989, Foldiak, 1990, Rubner and Schulten, 1990]. The plasticity rule (17) is symmetric with respect to interchange of $i$ and $j$. Therefore, $L$ remains a symmetric matrix for all time, assuming that its initial condition is symmetric. In the absence of activity, the second term of Eq. (17) causes weakening of lateral connections. The anti-Hebbian update (17) is followed by rectification,

$$L_{ij} := \max\{L_{ij}, 0\}$$

(18)

This constrains the matrix $L$ to be nonnegative, so that the lateral connections remain inhibitory (note the minus sign preceding $L$ in the dynamics of Eq. 14).

The diagonal elements of $L$ are updated via

$$\Delta L_{ii} = \eta_L (x_i^2 - q^2)$$

(19)

$$L_{ii} := \max\{L_{ii}, 0\}$$

(20)

The $L_{ii}$ update seeks to maintain the activity of neuron $i$ at a level set by $q$. Homeostatic regulation of neural activity has been observed experimentally, and is mediated by more than one mechanism. The diagonal element $L_{ii}$ is not a synaptic connection of neuron $i$ to itself (though such connections, known as autapses, are known to exist). Rather, it has two biological interpretations. First, it could be regarded as a uniform scaling factor applied to all synapses, whether feedforward or lateral, that converge on neuron $i$. Such homeostatic synaptic scaling has been extensively studied for biological synapses [Turrigiano, 2012]. Second, $L_{ii}$ could be regarded as dividing the activation function in (14), acting like a “gain” parameter for neuron $i$. Homeostatic regulation of intrinsic excitability has also been studied for biological synapses [Zhang and Linden, 2003].

After the connection strengths are updated, the dynamics (14) is run for the next stimulus vector, and so on. The above neural network learning algorithm is very similar to ones proposed by Foldiak [1990] and Pehlevan and Chklovskii [2014]. The activity dynamics (14) uses rectification nonlinearity, as in Pehlevan and Chklovskii [2014]. Appendix B describes how our formalism can be extended to the sigmoidal nonlinearity of the Foldiak [1990] model. The plasticity rule (17) for the lateral inhibitory connections is exactly the one in Foldiak [1990]. The plasticity rule (15) for the feedforward excitatory connections includes synaptic competition with elimination. This is novel relative to Foldiak [1990] and Pehlevan and Chklovskii [2014] and is drawn from previous models of neural development [Miller and MacKay, 1994, Miller, 1996].

In Eq. (14), the effective synaptic strengths are given by $W_{ia}/L_{ii}$ and $L_{ij}/L_{ii}$. In both ratios, the numerator is Hebbian or anti-Hebbian and the denominator is homeostatic. A similar decomposition of Hebbian and homeostatic plasticity into multiplicative factors was recently proposed in a model of neural development [Toyoizumi et al., 2014].

5 Empirical results with MNIST

To illustrate the properties of the algorithm, we ran it on the MNIST dataset. Figure 2 shows that sparse activity patterns emerge from learning. We define density as the fraction of neurons with nonzero activity. The lower the density, the more sparse the representation is. The density starts out at 1, and decreases as learning proceeds. As expected, the final density is smaller when $p/q$ is smaller.

The learned connectivity is shown in Figure 3. Most connection strengths have been driven to zero; these synapses have effectively been eliminated. Each vector of convergent connections is displayed as an image. The features can be interpreted as “parts” of handwritten digits. Increasing $\rho/\omega$ increases the density of the connectivity. Of course, many other unsupervised learning algorithms have been used to learn features that resemble parts of objects [Lee and Seung, 1997, 1999, Hoyer, 2004]. The main distinction of the present algorithm is its biological plausibility. It should be noted
Figure 2: Output neuron activity is sparse and decorrelated. (a) Fraction of active neurons (activity density) versus time as learning progresses for 60,000 time steps. The density starts at one, decreases rapidly at first, and then more slowly. The final value of the density is lower for smaller $p/q$. Each point on the curve represents density averaged over 100 time steps. (b) Cosine similarity of activities as defined on the left side of Eq. (3). The vertical coordinate is the number of neuron pairs with that cosine similarity. Cosine similarity was computed over the last 10,000 time steps of the learning. In each case, the cosine similarity is clustered around the value $p^2/q^2$, as predicted by Eq. (3). Simulations used $\kappa = 1$, $\rho = 1$, $\rho/\omega = 10$, $q = 0.09$, $\eta_L = 0.1$, $\eta_W = 0.001$. In these experiments and all others, the elements of the $W$ matrix were chosen randomly from a uniform distribution on $[0,1]$ and then each row was normalized so that it summed to $\rho$. The matrix $L$ was initialized to the identity matrix.

Figure 3: Learned connectivity is sparse, as competition between synapses leads to elimination. The rows of $W$ contain 64 features learned from MNIST and are displayed as images. Experiments for (a) $\rho/\omega = 10$ and (b) $\rho/\omega = 20$ demonstrate that this parameter controls the sparsity of $W$ after learning. Other parameter settings were $p = 0.03$, $q = 0.09$, $\kappa = 1$, $\rho = 1$, $\eta_L = 0.01$, $\eta_W = 0.001$. Many features consist of spatially contiguous pixels; this is an outcome of learning as the algorithm has no prior knowledge of which pixels are near each other.
that synapse elimination by competition is essential for sparse connectivity in the present model. Without synaptic competition, the Hebbian update of Eq. (15) would cause each weight vector to be a nonnegative superposition of input vectors. The learned features would resemble whole objects rather than object parts.

How are the lateral connections related to feedforward excitation? If two neurons receive similar feedforward weights, their activities will be highly correlated in the absence of lateral inhibition. To weaken the correlation, the learning algorithm is expected to strengthen the inhibitory connection between the two neurons. Such a trend is indeed seen empirically in Figure 4. At the same time, there are many neurons with completely different weight vectors, but are still connected by strong lateral inhibition. In the absence of lateral inhibition, the activities of these neurons would presumably be highly correlated due to the correlations present in the input. Hence the learning algorithm also strengthens inhibition to reduce the correlations.

6 Theoretical analysis of synapse elimination

Figure 3 shows that $\rho/\omega$ controls the number of surviving synapses. For theoretical insight into this phenomenon, suppose that $\Phi(W) = \sum_i \phi(w_i)$ where $w_i$ is the $i$th column of $W$. Then $\Phi^*(C) = \sum_i \phi^*(c_i)$ where $c_i$ is the $i$th column of $C$. For the penalty function of Eq. (13), we have

$$
\phi(w) = \frac{\kappa}{2} \left( \sum_a w_a - \rho \right)^2
$$

$$
\phi^*(c) = \max_{w \in B} \left\{ \sum_a w_a c_a - \frac{\kappa}{2} \left( \sum_a w_a - \rho \right)^2 \right\}
$$

For simplicity, we consider the limit of infinite $\kappa$, when $w$ lies on the simplex $\mathcal{S}^\rho$ satisfying $\sum w_a = \rho$, so that

$$
\phi^*(c) = \max_{w \in B \cap \mathcal{S}^\rho} \left\{ \sum_a w_a c_a \right\}
$$

Suppose also that $\rho$ is an integral multiple of $W_{\text{max}}$, i.e., $\rho = k\omega$. Assume without loss of generality that $c_1 \geq c_2 \geq \ldots \geq c_N$. Then the maximum is achieved by setting $w_1 = \cdots = w_k = \omega$ and $w_{k+1} = \cdots = w_N = 0$, and

$$
\phi^*(c) = \omega \sum_{a=1}^k c_a
$$

In other words, the objective function $\Phi^*$ in Eqs. (6) and (7) is proportional to the sum of the top $k$ elements of each row of the output-input correlation matrix.
Recall that for a single neuron, the correlation game reduces in Eq. \( \text{23} \) to finding a projection with maximal second moment, subject to the penalty function \( \phi \). For the specific case of the penalty function above, this is a version of sparse PCA \([\text{Moghaddam et al., 2006, Zass and Shashua, 2007}]\) in which the projection vector \( \mathbf{w} \) is required to have \( k \) elements equal to one and all other elements equal to zero. Effectively, this selects the set of \( k \) inputs that have maximal correlations with each other.

7 Discussion

Our correlation game is a new formulation of unsupervised learning that is convenient for understanding biologically plausible synaptic plasticity rules in neural networks. We have considered several intuitions about the computational functions of synaptic plasticity:

1. Hebbian feedforward excitation enables neurons to learn features from sensory inputs.
2. Anti-Hebbian inhibition serves to sparsen activity, facilitating the learning of features by Hebbian excitation.
3. By sparsening connectivity, synaptic competition and elimination facilitate the learning of features that resemble parts of objects.

These long-standing intuitions have been made mathematically precise using the correlation game, and are illustrated by the numerical experiments of Figs. 2 and 3. The neural network learns by fostering competition (1) between neurons mediated by lateral inhibition, (2) between synapses converging onto the same neuron, and (3) between excitation and inhibition in the correlation game.

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A Relations with other unsupervised learning principles

Unsupervised learning has been formulated as maximization of the mutual information between output and input \[ \text{[Linsker, 1988]} \text{[Atick and Redlich, 1990]} \text{[Bell and Sejnowski, 1995]} \], or minimization of the mutual information between outputs \[ \text{[Hyvärinen and Oja, 2000]} \]. Our output-input correlations can be viewed as a proxy for mutual information between output and input. Our constraint keeps the output-output correlations small, which is similar to limiting the redundancy of the outputs, or the mutual information between the outputs. We prefer correlations rather than entropies in our unsupervised learning principle, because we would like to derive Hebbian plasticity rules. There is no explicit model of the relationship between input and output in Eq. (1); the relationship will emerge from solving the optimization problem. In contrast, most information theoretic formulations explicitly assume a linear relationship between output and input.
Unsupervised learning has been formulated as projection onto a subspace that maximizes a moment-based statistic like variance [Oja, 1982; Linsker, 1988] or kurtosis [Huber, 1985]. Our Problem 1 is equivalent for the case of a single output, $X_t = x_t$. To demonstrate this, combine Eqs. (1) and (4) to yield
\[
\max_{w \in B} \max_{X \succeq 0} \left\{ \frac{1}{T} \sum_t x_t \sum_a w_a U_{at} - \phi(w) \right\} \text{ such that } \frac{1}{T} \sum_t x_t^2 \leq q^2
\]  
(The notation has switched to $\Phi$ from $\Phi$ to indicate a function of a vector $w$ rather than a matrix $W$, and the scalar $q^2$ denotes the single element of the $1 \times 1$ matrix $D$.) The maximum with respect to $X$ is $x_t \propto \sum_a w_a U_{at}$, i.e., the output is proportional to a projection of the input along the direction defined by $w$. Then the remaining optimization depends on the input through the second moment $(\langle w \cdot u \rangle^2)$.
\[
\max_{w \in B} \left\{ q \sqrt{\frac{1}{T} \sum_t \left( \sum_a w_a U_{at} \right)^2} - \phi(w) \right\} = \max_{w \in B} \left\{ q \sqrt{\langle w \cdot u \rangle^2} - \phi(w) \right\}
\]  
If we choose $\phi(w) = 0$ and $B$ the unit ball, this amounts to finding a one-dimensional projection with maximal second moment. For zero-centered inputs, this is equivalent to maximizing variance, which yields the principal component.

It is well-known that a single linear neuron with Hebbian synapses learns the principal component of the input distribution [Oja, 1982]. A problem with generalizing this idea to more than one principal component is that linear neurons have the tendency to all learn the same principal component rather than the top $k$ distinct principal components [Sanget, 1989]. Preventing such similarity could be regarded as the motivation for Eq. (3) with $p/q < 1$, and was the original intuition behind the model of Földiák [1990]. In the general case of multiple outputs, the relationship between $X$ and $U$ is nonlinear, and will be derived below.

Finally, our Problem 1 can be viewed as a generalization of the similarity matching principle of Pehlevan et al. [2015]. In classical multidimensional scaling (CMDS), the objective is to minimize the sum of squared differences between output and input similarities at all pairs of times,
\[
\frac{1}{2} \sum_{t \neq t'} \| x(t) \cdot x(t') - u(t) \cdot u(t') \|^2 = \frac{1}{2} \| XX^T - U^T U \|^2
\]
When $x$ has lower dimensionality than $u$, the CMDS cost function is minimized by $X$ that is a projection of $U$ onto the subspace spanned by the principal components of the input [Williams, 2002]. Pehlevan et al. [2015] used this idea to derive a neural network learning algorithm for implementing PCA. In the special case of $\Phi(W) = \sum_a W_{at}^2 / 2$ and $B$ the nonnegative orthant, the maximization of Eq. (4) yields $\Phi^*(C) = \sum_a C_{at}^2 / 2$, since $C$ is assumed nonnegative. Given the constraint $XX^T/T \leq D$, the CMDS cost function can be upper bounded by
\[
\frac{1}{2} \| XX^T - U^T U \|^2 = \frac{1}{2} \| XX^T \|^2 - \| XX^T \|^2 + \frac{1}{2} \| UU^T \|^2 
\leq \frac{T^2}{2} \| D \|^2 - \| XX^T \|^2 + \frac{1}{2} \| UU^T \|^2
\]  
Therefore Eq. (1) can be viewed as the minimization of an upper bound for the CMDS cost function (24) given the constraint $XX^T/T \leq D$.

### B Extension to sigmoidal nonlinearity

Földiák [1990] used sigmoidal rather than rectification nonlinearity in his neural network model. Our formalism can be extended to sigmoidal nonlinearity as follows. Problem 1 is replaced by
\[
\max_X \Phi^* \left( \frac{1}{T} X U^T \right) - \frac{1}{T} \sum_t F(X_t)
\]
such that
\[ \frac{1}{T} \sum_{t} X_{it} = p \text{ for all } i \]
\[ \frac{1}{T} \sum_{t} X_{it} X_{jt} \leq p^2 \text{ for all } i \neq j \]

Then Lagrangian duality yields the analog of Problem 2,
\[
\max_{X} \min_{L \geq 0} \theta \left\{ \Phi' \left( \frac{1}{T} X U^T \right) - \frac{1}{T} \sum_{it} F(X_{it}) - \theta \left( \frac{1}{T} \sum_{t} X_{it} - p \right) - \sum_{i<j} L_{ij} \left( \frac{1}{T} \sum_{t} X_{it} X_{jt} - p^2 \right) \right\}
\]
or equivalently
\[
\frac{1}{T} \max_{X} \min_{L \geq 0} \theta \left( \sum_{t} \left[ \sum_{ia} W_{ia} X_{it} U_{at} - \Phi(W) - \sum_{i} F(X_{it}) - \theta \left( X_{it} - p \right) - \sum_{i<j} L_{ij} \left( X_{it} X_{jt} - p^2 \right) \right] \right)
\]
The coordinate ascent dynamics (10) is replaced by
\[
X_{it} := f \left( \sum_{ia} W_{ia} U_{at} - \sum_{j, j \neq i} L_{ij} X_{jt} - \theta \right)
\] (25)

where \( F' = f^{-1} \). The updates (11) and (12) for \( W \) and the off-diagonal elements of \( L \) remain the same. However, the update rule for \( L_{ii} \) is replaced by an update for the threshold \( \theta_i \) in Eq. (25),
\[
\Delta \theta_i \propto X_{it} - p
\] (26)

which can be viewed as an alternate model for homeostatic regulation of activity.

In the Földiák [1990] model, when combined with homeostatic plasticity as in Eq. (26), the anti-Hebbian update of Eq. (17) acts to drive the covariances of all pairs of neural activities to zero. In other words, the neural activities are approximately statistically independent. Consistent with this idea, Falconbridge et al. [2006] argued that the Földiák [1990] algorithm implements a kind of independent component analysis. Our algorithm makes the cosine similarities small when \( p/q \) is small. Therefore one might call it “desimilarizing” rather than decorrelating. Another difference is that we impose nonnegativity constraints on both feedforward and lateral connections, while Földiák [1990] only imposed constraints on the latter.

### C Relation to contrastive Hebbian learning

Of the existing neural net learning algorithms, the one that hews most closely to Hebbian principles is the Boltzmann machine or contrastive Hebbian learning. The inhibitory plasticity of the present model is actually a form of contrastive Hebbian learning. This can be seen by defining

\[
E(X) = \frac{1}{T} \sum_{t} \left( \frac{1}{2} \sum_{ij} L_{ij} X_{it} X_{jt} - \sum_{ia} W_{ia} X_{it} U_{at} \right)
\]

and rewriting Eq. (8) as

\[
R(W, L) = \left[ E(X^0) - \min_{X \geq 0} E(X) \right] + \frac{1}{T} \sum_{ia} \left[ W_{ia} X_{it}^0 U_{at} - \Phi(W) \right]
\] (27)

where the “ghost activity” matrix \( X^0 \) satisfies \( X^0 X^0^T / T = D \). The gradient update for \( L \) involves the first term only, since the second term does not depend on \( L \). The first term of Eq. (27) is the cost function for contrastive Hebbian learning. The columns of the ghost activity matrix represent the “clamped” or desired patterns for the learning, while the vectors \( \hat{X} \) represent the “free” patterns.
Figure 5: Synaptic competition with elimination still occurs after replacing the upper bound on synaptic strength (Eq. 15) with a simple weight decay (Eq. 28). The rows of $W$ contain 64 features learned from MNIST and are displayed as images. Experiments for (a) $\gamma = 0.1$ and (b) $\gamma = 0.5$ demonstrate that this parameter controls the sparsity of $W$ after learning. The results are qualitatively similar to those of Fig. 3, except that synaptic weights are graded rather than saturated at upper and lower bounds. Other parameter settings were $p = 0.03$, $q = 0.09$, $\kappa = 1$, $\rho = 1$, $\eta_L = 0.1$, $\eta_W = 0.001$.

The preceding makes clear that the plasticity rule for the inhibitory connections is the same as in contrastive Hebbian learning [Movellan, 1991, Baldi and Pineda, 1991], except here the nonlinearity is rectification rather than sigmoidal. Plasticity of inhibition is trying to make the second order statistics of the neural activities match those of the ghost activities.

### D Synapse elimination without upper bounds

Eq. (16) contains an upper bound $\omega$ for all excitatory synaptic strengths. The number of surviving synapses is set by $\rho/\omega$. If we replace Eq. (15) by a more complex model of synaptic competition

$$
\Delta W_{ia} \propto x_i u_{ia} - \gamma W_{ia} - \kappa \left( \sum_\beta W_{i\beta} - \rho \right)
$$

then synapse elimination is possible without the upper bound $\omega$ (though there is still an upper bound on summed synaptic strength due to $\rho$). As can be seen from Fig. 5, the results of learning are qualitatively similar. The parameter $\gamma$ controls the sparsity of $W$, with smaller values producing “parts” and larger values producing “wholes.” The synaptic weights are graded, rather than saturated at upper and lower bounds as in Fig. 3.

A theoretical analysis of this analog model of synaptic competition is more complex, but the end result is qualitatively similar to the previous bound-constrained model. As we will see below, the competition eliminates connections from weakly correlated inputs, focusing the objective function on strongly correlated inputs. The $W$ update of Eq. (28) can be derived from the penalty function

$$
\Phi (W) = \frac{\gamma}{2} \sum_{ia} W_{ia}^2 + \frac{\kappa}{2} \sum_i \left( \sum_a W_{ia} - \rho \right)^2
$$

The convex conjugate can be written as

$$
\tilde{\phi} (c) = \max_{w \geq 0} \left\{ \sum_a w_a c_a - \frac{\gamma}{2} \sum_a w_a^2 - \frac{\kappa}{2} \left( \sum_a w_a - \rho \right)^2 \right\}
$$
summed over the rows of the matrix $C$. According to the KKT conditions for the optimum, either

$$w_a \geq 0 \text{ and } c_a - \gamma w_a - \kappa \left( \sum_b w_b - \rho \right) = 0 \quad (31)$$

or

$$w_a = 0 \text{ and } c_a - \gamma w_a - \kappa \left( \sum_b w_b - \rho \right) \leq 0 \quad (32)$$

must be satisfied for all $a$. The conditions can be combined into an expression of the form $\gamma w_a = \left[ c_a - \theta \right]^+$ where the threshold is given by $\theta = \kappa \left( \sum_b w_b - \rho \right)$. Assume without loss of generality that $c_1 \geq c_2 \geq \ldots \geq c_N$. Then the solution of the KKT conditions is

$$\gamma w_a = \left[ c_a - \theta_k \right]^+$$

where

$$\theta_k = \kappa \left( \sum_{a=1}^k w_a - \rho \right) = \frac{1}{k + \gamma / \kappa} \left( \sum_{a=1}^k c_a - \rho \gamma \right)$$

satisfies the inequality

$$c_k > \theta_k \geq c_{k+1} \quad (33)$$

At the solution, $w_1, \ldots, w_k$ are positive while $w_{k+1}, \ldots, w_N$ vanish. In the context of the learning algorithm, this means that $k$ synapses survive while the rest are eliminated. The surviving synapses come from the inputs that are most correlated with the output.

In the limit of infinite $\kappa$, Eq. (33) yields some simple conditions for synapse elimination. At least one synapse is eliminated if and only if

$$\rho \gamma < \sum_{b=1}^N c_b - Nc_N = N(\bar{c} - c_N)$$

In other words, synapse elimination is triggered if and only if the quotient $\rho \gamma / N$ is smaller than the difference between the mean $\bar{c}$ and the minimum $c_N$ of the elements of $c$. It can also be shown that the solution of the inequality (33) is $k = 1$ for sufficiently small $\rho \gamma$. This is winner-take-all competition: all synapses are eliminated except for a single winner.

What is the value of the objective function $\phi$ at the optimum? If $\kappa = 0$, the objective function reduces to

$$2\gamma \phi (c) = ||c||^2 = \sum_a c_a^2 \quad (34)$$

In this case, learning is trying to maximize the sum of the squares of the output-input correlations. This is the objective function used by the similarity matching principle of Pehlevan et al. [2015].

For $\kappa > 0$, the result of the optimization is

$$2\gamma \phi (c) = \sum_{a=1}^k \left( c_a^2 - \theta_k^2 \right) - \frac{\gamma}{\kappa} \theta_k^2$$

where $k$ is the number of surviving synapses. Now the sum only includes the top correlations, those corresponding to surviving synapses. This can also be written as

$$2\gamma \phi (c) = \sum_a \left[ c_a^2 - \theta_k^2 \right]^+ - \gamma \theta_k^2,$$

so effectively the correlations are being thresholded.