Sparse Multipartite Graphs as Partition Universals for Graphs of Bounded-Degrees *

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Abstract

For graphs $G$ and $H$, let $G \to (H, H)$ signify that any red/blue edge coloring of $G$ contains a monochromatic $H$ as a subgraph, and $H(\Delta, n) = \{H : |V(H)| = n, \Delta(H) \leq \Delta\}$. For fixed $\Delta$ and $n$, we say that $G$ is a partition universal graph for $H(\Delta, n)$ if $G \to (H, H)$ for every $H \in H(\Delta, n)$.

In 1983, Chvátal, Rödl, Szemerédi and Trotter proved that for any $\Delta \geq 2$ there exists a constant $B$ such that, for any $n$, if $N \geq Bn$ then $K_N$ is partition universal for $H(\Delta, n)$. Recently, Kohayakawa, Rödl, Schacht and Szemerédi proved that the complete graph $K_N$ in above result can be replaced by sparse graphs. They obtained that for fixed $\Delta \geq 2$, there exist constants $B$ and $C$ such that if $N \geq Bn$ and $p = C(\log N/N)^{1/\Delta}$, then a.a.s. $G(N, p)$ is partition universal graph for $H(\Delta, n)$, where $G(N, p)$ is the standard random graph on $N$ vertices with $\mathbb{P}(e) = p$ for each edge $e$. From some results of Bollobás and Luczak, we know that a.a.s. $\chi(G(N, p)) = \Theta((N/\log N)^{1-1/\Delta})$.

In this paper, we shall show that the $G(N, p)$ in above result can be replaced by random multipartite graph. Let $K_r(N)$ be the complete $r$-partite graph with $N$ vertices in each part, and $G_r(N, p)$ the random spanning subgraph of $K_r(N)$, in which each edge appears with probability $p$. It is shown that for fixed $\Delta \geq 2$ there exist constants $r, B$ and $C$ depending only on $\Delta$ such that if $N \geq Bn$ and $p = C(\log N/N)^{1/\Delta}$, then a.a.s. $G_r(N, p)$ is partition universal graph for $H(\Delta, n)$. The proof mainly uses the sparse multipartite regularity lemma.

Keywords: Partition universal; Sparse multipartite regularity lemma; Ramsey number

1 Introduction

For a family $\mathcal{H}$ of graphs, a graph $G$ is said to be an $\mathcal{H}$-universal if it contains $H$ as a subgraph for each $H \in \mathcal{H}$. The construction of sparse universal graphs for various families arises in the study of VLSI circuit design, and has attracted much of attention. It is pointed out in [22] that the problem of designing an efficient single circuit, specialized for a variety of other circuits, can be viewed as constructing a small universal graph. For the references of universal graphs, one can see, e.g., [11, 2, 3, 15, 7, 11, 12, 14, 15, 17, 13, 19, 20, 21, 25, 29, 32, 37].

Let $G(n, p)$ be the standard random graph on $n$ vertices with $\mathbb{P}(e) = p$ for each edge $e$. Let $K_r(n)$ be the complete $r$-partite graph with $n$ vertices in each part, and $G_r(n, p)$ the random spanning subgraph

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of $K_r(n)$ with $\mathbb{P}(e) = p$ for each edge $e$ of $K_r(n)$. Let us set

$$H(\Delta, n) = \{H \subseteq K_n : \Delta(H) \leq \Delta\}, \quad \text{and} \quad H(\Delta, n, n) = \{H \subseteq K_{n,n} : \Delta(H) \leq \Delta\},$$

where $H \subseteq K_n$ and $H \subseteq K_{n,n}$ mean that $H$ is a spanning subgraph of $K_n$ and $K_{n,n}$, respectively. By counting all unlabeled $\Delta$-regular graphs on $n$ vertices, it is shown [1] that each $H(\Delta, n)$-universal graph has at least $\Theta(n^{2-2/\Delta})$ edges, where and henceforth $\Theta(f(n))$ to signify a function that differs from $f(n)$ up to a multiplicative positive constant. This lower bound was almost matched that from [2, 3], which constructed $H(\Delta, n)$-universal graphs of order $n$ with at most $\Theta(n^{2-2/\Delta}(\log n)^{4/\Delta})$ edges. It is shown that a.a.s. $G(n, p)$ is $H(\Delta, n)$-universal with $p = c((\log n)/n)^{1/\Delta}$ in [27], and a.a.s. $G_2(n, p)$ is $H(\Delta, n)$-universal with $p = c((\log n)/n)^{1/\Delta}$ in [24, 26].

For graphs $G$ and $H$, let $G \to (H, H)$ signify that any red/blue edge coloring of $G$ contains a monochromatic $H$ as a subgraph. Thus the Ramsey number $R(H) = \min\{N : K_N \to (H, H)\}$. Furthermore, we say that a graph $G$ is partition universal for $\mathcal{H}(\Delta, n)$ if $G \to (H, H)$ for each $H \in \mathcal{H}(\Delta, n)$.

A well-known result of Chvátal, Rödl, Szemerédi and Trotter [23] is as follows, which implies that for any fixed integer $\Delta \geq 2$, there exists a constant $B = B(\Delta)$ such that if $N \geq Bn$ then $K_N$ is partition universal for $\mathcal{H}(\Delta, n)$.

Theorem 1 (23) For any fixed integer $\Delta \geq 2$, there exists a constant $B = B(\Delta)$ such that if $N \geq Bn$ then $K_N$ is partition universal for $\mathcal{H}(\Delta, n)$.

The size Ramsey number $\bar{r}(H)$ is defined to be $\min\{|e(G) : G \rightarrow (H, H)\}$ in [28], where $e(G)$ is the size of the edge set of $G$. Recently, by using the sparse regularity lemma (see [33, 34]), an elegant result of Kohayakawa, Rödl, Schacht and Szemerédi [35] strengthens Theorem 1 by replacing $K_N$ in Theorem 1 with sparse sparse graphs, implying $\bar{r}(H) \leq \Theta(n^{2-1/\Delta}(\log n)^{1/\Delta})$ for $H \in \mathcal{H}(\Delta, n)$, and confirming the upper bound in a conjecture of Rödl and Szemerédi [39].

Theorem 2 (35) For fixed $\Delta \geq 2$, there exist constants $B = B(\Delta)$ and $C = C(\Delta)$ such that if $N \geq Bn$ and $p = C(\log N/N)^{1/\Delta}$, then

$$\lim_{n \to \infty} \mathbb{P}(G(N, p) \text{ is partition universal for } \mathcal{H}(\Delta, n)) = 1.$$
\(K_r(N)\). Suppose \(0 < p, \eta \leq 1\). For subsets \(X \subseteq V^{(i)}\) and \(Y \subseteq V^{(j)}, i \neq j\), let \(e_{G_r}(X, Y)\) be the number of edges of \(G_r\) between \(X\) and \(Y\), and let
\[
d_{G_r,p}(X, Y) = \frac{e_{G_r}(X, Y)}{p|X||Y|},
\]
which is referred to as the \(p\)-density of the pair \((X, Y)\). We say that \(G_r\) is an \((\eta, \lambda)\)-bounded graph with respect to \(p\)-density if any pair of sets \(X \subseteq V^{(i)}\) and \(Y \subseteq V^{(j)}\) \((i \neq j)\) with \(|X| \geq \eta|V^{(i)}|, |Y| \geq \eta|V^{(j)}|\) satisfy
\[
d_{G_r,p}(X, Y) < \lambda.
\]
For fixed \(\epsilon > 0\), we say such a pair \((X, Y)\) is \((\epsilon, p)\)-regular if for all \(X' \subseteq X\) and \(Y' \subseteq Y\) with
\[
|X'| \geq \epsilon |X|\text{ and }|Y'| \geq \epsilon |Y|,
\]
we have
\[
|d_{G_r,p}(X, Y) - d_{G_r,p}(X', Y')| \leq \epsilon.
\]
Note that for \(p = 1\) we get the well-known definition of \(\epsilon\)-regularity. The following is a variant of the Szemerédi’s regularity lemma \([40]\) for sparse multipartite graphs, developed independently by Kohayakawa and Rödl, see \([33, 34]\). We restated it as follows.

**Theorem 4 (Sparse multipartite regularity lemma)** For any fixed \(\epsilon > 0, 0 < p < 1\), \(t_0 \geq 1\), and \(r \geq 2\), there exist \(T_0, \eta\) and \(N_0\), such that each \(r\)-partite graph \(G_r(N) = G_r(V^{(1)}, V^{(2)}, \ldots, V^{(r)})\) with \(N \geq N_0\) that is \((\eta, \lambda)\)-bounded with respect to density \(p\) with \(0 < p \leq 1\), has a partition \(\{V_0^{(i)}, V_1^{(i)}, \ldots, V_t^{(i)}\}\) for each \(V^{(i)}\), where \(t\) is same for each part \(V^{(i)}\) and \(t \leq T_0\), such that
\[
(i) \quad |V_0^{(i)}| \leq \epsilon N\text{ and }|V_1^{(i)}| = |V_2^{(i)}| = \cdots = |V_t^{(i)}|\text{ for }1 \leq i \leq r;
\]
\[
(ii) \quad \text{All but at most } \epsilon^2 t^2\text{ pairs } (V_a^{(i)}, V_b^{(j)})\text{ are } (\epsilon, p)\text{-regular for }1 \leq i \neq j \leq r\text{ and }1 \leq a, b \leq t.
\]
In the following, let us introduce the hereditary nature of sparse regularity which specially holds for \(r\)-partite graphs, see \([50]\).

**Definition 1** Let \(\alpha, \epsilon > 0, 0 < p \leq 1\) be given and let \(G_r(N)\) be defined above. For sets \(X \subseteq V^{(i)}\) and \(Y \subseteq V^{(j)}\) \((i \neq j)\), we say that the pair \((X, Y)\) is \((\epsilon, \alpha, p)\)-dense if for all \(X' \subseteq X\) and \(Y' \subseteq Y\) with
\[
|X'| \geq \epsilon |X|\text{ and }|Y'| \geq \epsilon |Y|,
\]
we have
\[
d_{G_r,p}(X', Y') \geq \alpha - \epsilon.
\]
It follows immediately from the definition that \((\epsilon, \alpha, p)\)-denseness is inherited on large sets, i.e., that for an \((\epsilon, \alpha, p)\)-dense pair \((X, Y)\) with \(|X'| \geq \mu|X|\text{ and }|Y'| \geq \mu|Y|\) the pair \((X', Y')\) is \((\epsilon/\mu, \alpha, p)\)-dense. The following result from \([39]\) states that this “denseness-property” is even inherited on randomly chosen subsets of much smaller size with overwhelming probability.

**Theorem 5 (\([30]\), Theorem 3.6)** For every \(\alpha, \beta > 0\) and \(\epsilon' > 0\), there exist \(\epsilon_0 = \epsilon_0(\alpha, \beta, \epsilon') > 0\) and \(L = L(\alpha, \epsilon')\) such that, for any \(0 < \epsilon \leq \epsilon_0\) and \(0 < p < 1\), if \((X, Y)\) is an \((\epsilon, \alpha, p)\)-dense pair in a graph \(G\), then the number of sets \(X' \subseteq X\) with \(|X'| = w \geq L/p\) such that \((X', Y)\) is an \((\epsilon', \alpha, p)\)-dense pair is at least \((1 - \beta^w)\left(\frac{1}{w}\right)^{(X)}\).

The following corollary is a direct consequence of Theorem 5 obtained by applying the theorem first to \(X\) and then to \(Y\).

**Corollary 1 (\([30]\), Corollary 3.8)** For every \(\alpha, \beta > 0\) and \(\epsilon' > 0\), there exist \(\epsilon_0 = \epsilon_0(\alpha, \beta, \epsilon') > 0\) and \(L = L(\alpha, \epsilon')\) such that, for any \(0 < \epsilon \leq \epsilon_0\) and \(0 < p < 1\), every \((\epsilon, \alpha, p)\)-dense pair \((X, Y)\) in a graph \(G\) has the following property: the number of pairs \((X', Y')\) with \(X' \subseteq X\) and \(Y' \subseteq Y\) with \(|X'| = w_1 \geq L/p\) and \(|Y'| = w_2 \geq L/p\) such that \((X', Y')\) is an \((\epsilon', \alpha, p)\)-dense pair is at least \((1 - \beta^\min\{w_1, w_2\})\left(\frac{1}{w_1}\right)^{(X')}\left(\frac{1}{w_2}\right)^{(Y')}\).
3 Properties of the random multipartite graph

In this section, we shall dedicate in establishing the properties for \( G_r(N, p) \). Recall that \( K_r(N) \) has \( r \) parts \( V^{(1)}, V^{(2)}, \ldots, V^{(r)} \) with \( |V^{(i)}| = N \) for \( 1 \leq i \leq r \), and \( G_r(N, p) \) is the random \( r \)-partite spanning subgraph of \( K_r(N) \) with probability \( p \) for edge appearance.

**Definition 2** For an integer \( N \) and \( 0 < p \leq 1 \), we say that a graph \( G_r(N) \) has the property \( \mathcal{U}_{N, p} \) if for \( 1 \leq i \neq j \leq r \), all \( U^{(i)} \subseteq V^{(i)}, U^{(j)} \subseteq V^{(j)} \) with \( |U^{(i)}|, |U^{(j)}| \geq N/\log N \) satisfy

\[
|d_{G_r(N, p)}(U^{(i)}, U^{(j)})| \leq \frac{1}{\log N}.
\]

The Chernoff’s inequality (see e.g., [16] [31]) will be useful for the proof of the following fact.

**Lemma 1** Let \( X \) be a binomial random variable. If \( 0 < \delta \leq 1 \), then

\[
\mathbb{P}(\{|X - \mathbb{E}(X)| \geq \delta \mathbb{E}(X)\}) \leq 2 \exp \left( -\frac{\delta^2 \mathbb{E}(X)}{3} \right).
\]

The first fact we will obtain as follows implies that a.a.s. the edge number for all sufficiently large pair defined as above are concentrated on the expectation for suitable probability \( p \).

**Fact 1** If \( p \geq 12(\log N)^4/N \), then \( \mathbb{P}(G_r(N, p) \not\in \mathcal{U}_{N, p}) \rightarrow 1 \) as \( N \rightarrow \infty \).

**Proof.** The graph \( G_r(N, p) \not\in \mathcal{U}_{N, p} \) means that there exists a pair \( (U^{(i)}, U^{(j)}) \) (for some \( 1 \leq i \neq j \leq r \)) such that \( |d_{G_r(N, p)}(U^{(i)}, U^{(j)})| > \frac{1}{\log N} \), i.e., \( |e_{G_r(N, p)}(U^{(i)}, U^{(j)})| - p|U^{(i)}||U^{(j)}| > p|U^{(i)}||U^{(j)}|/\log N \).

Hence, Lemma 1 implies the probability that the graph \( G_r(N, p) \not\in \mathcal{U}_{N, p} \) is at most

\[
\binom{r^2}{2} \left( \frac{N}{|U^{(i)}|} \right) \left( \frac{N}{|U^{(j)}|} \right) \cdot 2 \exp \left( -\frac{p|U^{(i)}||U^{(j)}|}{3\log^2 N} \right)
\]

\[
\leq r^2 \cdot \exp \left( \frac{|U^{(i)}| \log N + |U^{(j)}| \log N - p|U^{(i)}||U^{(j)}|}{3\log^2 N} \right),
\]

which will tends to zero as \( N \rightarrow \infty \) since \( p \geq 12(\log N)^4/N \).

For a \( r \)-partite graph \( G_r(N) \) and integers \( 1 \leq k \leq \ell < r \), let \( K \) be the family consists of all \( k \)-subset \( K \) of \( \bigcup_{i=1}^{\ell} V^{(i)} \) such that \( |K \cap V^{(i)}| \leq 1 \) for \( 1 \leq i \leq \ell \). Define the auxiliary bipartite graph \( \Gamma = \Gamma(k, G_r(N)) \) with color classes \( K \) and \( \bigcup_{i=1}^{\ell} V^{(i)} \), where \( (K, v) \in E(\Gamma) \) if and only if \( \{w, v\} \in E(G_r(N)) \) for all \( w \in K \). Here \( E(\Gamma) \) is the edge set of \( \Gamma \).

**Definition 3** Let integers \( N \) and \( k \geq 1 \) and reals \( \xi > 0 \) and \( 0 < p \leq 1 \) be given. We say that a graph \( G_r(N) \) has the property \( \mathcal{C}^k_N(\xi) \) if for every \( U \subseteq V^{(\ell+1)} \) and every family \( \mathcal{F}_k \subseteq K \) of pairwise disjoint \( k \)-sets with

(i) \( |\mathcal{F}_k| \leq \xi N \) and
(ii) \( |U| \leq |\mathcal{F}_k| \),
we have \( e_{\Gamma}(\mathcal{F}_k, U) < 6p|U||\mathcal{F}_k| \).

The following fact tells that for \( G_r(N, p) \), a.a.s. the corresponding graph \( \Gamma(k, G_r(N, p)) \) has no dense subgraph.

**Fact 2** For every integer \( k \geq 1 \) and real \( \xi > 0 \), there exists \( C > 1 \) such that if \( p > C \left( \frac{\log N}{N} \right)^{1/k} \), then \( \mathbb{P}(G_r(N, p) \not\in \mathcal{C}^k_N(\xi)) = 1 - o(1) \).

The following corollary follows immediately from Fact 2 since \( \left( \frac{\log N}{N} \right)^{1/\Delta} \geq \left( \frac{\log N}{N} \right)^{1/k} \) for \( 1 \leq k \leq \Delta \).
Corollary 2 For every integer $\Delta \geq 1$ and real $\xi > 0$, there exists $C > 1$ such that if $p > C\left(\frac{\log N}{\Delta}\right)^{1/\Delta}$, then $\mathbb{P}(G_r(N, p) \notin \mathcal{C}_{N,p}^\Delta(\xi)) = 1 - o(1)$.

Proof of Fact 2 Let $\mathcal{F}_k$ and $U$ be defined as in Definition 3. Note that if $X$ is a binomial random variable $\text{Bi}(M, q)$ then $\mathbb{P}(X \geq t) \leq q^t \binom{M}{t} \leq (eqM/1)^t$. So for fixed pair $(\mathcal{F}_k, U)$, we have

$$\mathbb{P}(c_t(\mathcal{F}_k, U) \geq 66\xi Np^k|\mathcal{F}_k)| \leq \left(\frac{e}{6}\right)^{66\xi Np^k|\mathcal{F}_k|}.$$ 

Moreover, the number of choices for the pair $(\mathcal{F}_k, U)$ is at most

$$\sum_{f=1}^{\frac{\xi N}{f}} \sum_{u=1}^{f} \binom{\frac{\xi N}{f}}{f} \binom{N}{u}.$$ 

Therefore, the probability $G_r(N, p) \notin \mathcal{C}_{N,p}^\Delta(\xi)$ can be bounded from above by

$$\sum_{f=1}^{\frac{\xi N}{f}} \sum_{u=1}^{f} \exp \left( kf \log(rN) + u \log N - 6\xi Np^k f \log(2\xi N/u) \right) \leq \sum_{f=1}^{\frac{\xi N}{f}} \sum_{u=1}^{f} \exp \left( (2k + 1 - 6\xi C^k) f \log N \right),$$ 

which will tends to 0 as $N \to \infty$ if we choose $C$ such that $C^k > \frac{4k+2}{\xi^k}$. \hfill \Box

Now, let us tend to discuss the last property we need for the random $r$-partite graph $G_r(N, p)$. Let us first give the definitions for classes $B_p^I$ and $B_p^{II}$ of “bad” tripartite graphs.

Definition 4 Let integers $m_1, m_2$ and $m_3$ and reals $\alpha, \epsilon, \epsilon, \mu > 0$ and $0 < p \leq 1$ be given.

- Let $B_p^I(m_1, m_2, m_3, \alpha, \epsilon, \epsilon, \mu)$ be the family of tripartite graphs with three color classes $X, Y$ and $Z$, where $|X| = m_1, |Y| = m_2$ and $|Z| = m_3$, satisfying
  
  (a) $(X, Y)$ and $(Y, Z)$ are $(\epsilon, \alpha, p)$-dense pairs and
  
  (b) there exists $X' \subseteq X$ with $|X'| \geq \mu |X|$ such that for every $x \in X'$ the pair $(N(x) \cap Y, Z)$ is not $(\epsilon, \alpha, p)$-dense.

- Let $B_p^{II}(m_1, m_2, m_3, \alpha, \epsilon, \epsilon, \mu)$ be the family of tripartite graphs with three color classes $X, Y$ and $Z$, where $|X| = m_1, |Y| = m_2$ and $|Z| = m_3$, satisfying
  
  (a) $(X, Y), (X, Z)$ and $(Y, Z)$ are $(\epsilon, \alpha, p)$-dense pairs and
  
  (b) there exists $X' \subseteq X$ with $|X'| \geq \mu |X|$ such that for every $x \in X'$ the pair $(N(x) \cap Y, N(x) \cap Z)$ is not $(\epsilon, \alpha, p)$-dense.

In the following, we define the family of graph $\mathcal{D}_{N,p}^\Delta$ that contains no element of $B_p^I \cup B_p^{II}$.

Definition 5 For integers $N$ and $\Delta \geq 2$ and reals $\alpha, \gamma, \epsilon, \epsilon, \mu > 0$ and $0 < p \leq 1$ we say that a graph $G_r(N)$ has the property $\mathcal{D}_{N,p}^\Delta(\gamma, \alpha, \epsilon, \epsilon, \mu)$ if $G_r(N)$ contains no member from

$$B_p^I(m_1^I, m_2^I, m_3^I, \alpha, \epsilon, \epsilon, \mu) \cup B_p^{II}(m_1^{II}, m_2^{II}, m_3^{II}, \alpha, \epsilon, \epsilon, \mu)$$

with $(m_1^I, m_2^I, m_3^I) \geq \gamma p^{\Delta-1} N$ and $(m_1^{II}, m_2^{II}, m_3^{II}) \geq \gamma p^{\Delta-2} N$ as a subgraph.

The following fact tells us that the random $r$-partite graph $G_r(N, p)$ has the property $\mathcal{D}_{N,p}^\Delta$ with high probability for suitable $p$. 

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**Fact 3** For every integer $\Delta \geq 2$ and positive reals $\alpha, \epsilon'$ and $\mu$ there exists $\epsilon = \epsilon(\Delta, \alpha, \epsilon', \mu) > 0$ such that for every $\gamma > 0$ there exists $C = C(\Delta, \alpha, \epsilon', \mu, \gamma) > 1$ such that if $p > C(N/\log N)^{1/\Delta}$, then $\mathbb{P}(G_r(N,p) \in \mathcal{D}_{N,p}^{\Delta}(\gamma, \alpha, \epsilon', \epsilon, \mu)) = 1 - o(1)$.

Before giving a proof for Fact 3 we have the following corollary.

**Corollary 3** For all integers $\Delta, \Delta \geq 2$ and all reals $\alpha, \gamma$ and $\epsilon^* > 0$, there exists $C \geq 1$ such that for every $\gamma > 0$ there exists $\epsilon = \epsilon(\Delta, \alpha, \epsilon^*, \mu, \gamma) > 0$ such that if $p > C(N/\log N)^{1/\Delta}$, then $\mathbb{P}(G_r(N,p) \in \mathcal{D}_{N,p}^{\Delta}(\gamma, \alpha, \epsilon^*, \epsilon, \mu)) = 1 - o(1)$.

**Proof.** Let $\Delta, \Delta \geq 2$ and all reals $\alpha, \gamma$ and $\epsilon^* > 0$ be given. First, set $\epsilon_{\Delta} = \epsilon^*$. For $\Delta > k \geq 1$, set recursively that $\epsilon_{k-1} = \min\{\epsilon_k, \epsilon(\Delta, \alpha, \epsilon_k, \mu)\}$, where $\epsilon(\Delta, \alpha, \epsilon_k, \mu)$ is given by Fact 3. Then set $\Delta$ to be the maximum of all $\Delta(\Delta, \alpha, \epsilon_k, \gamma)$ for $k = 1, \ldots, \Delta$. Thus, Fact 3 guarantees that a.a.s. $G_r(N,p) \in \mathcal{D}_{N,p}^{\Delta}(\gamma, \alpha, \epsilon_k, \mu, \gamma)$ for $1 \leq k \leq \Delta$.  

We first consider Fact 3 for the special case. Define

$$B_p^I(\alpha, \alpha, \epsilon', \epsilon, \mu) = B_p^I(pm, pm, \alpha, \alpha, \epsilon', \epsilon, \mu)$$

and

$$B_p^{II}(\alpha, \alpha, \epsilon', \epsilon, \mu) = B_p^{II}(m, m, \alpha, \alpha, \epsilon', \epsilon, \mu)$$

Similarly, for integers $N$ and $\Delta \geq 2$ and reals $\alpha, \gamma, \epsilon', \epsilon, \mu > 0$ and $0 < p \leq 1$, we say that a graph $G_r(N)$ has the property $\mathcal{D}_{N,p}^{\Delta}(\gamma, \alpha, \epsilon', \epsilon, \mu)$ if $G_r(N)$ contains no member from $B_p^I(\alpha, \alpha, \epsilon', \epsilon, \mu) \cup B_p^{II}(\alpha, \alpha, \epsilon', \epsilon, \mu)$ as a subgraph.

**Lemma 2** For $\Delta \geq 2$ and $\alpha, \epsilon', \mu \in (0, 1]$ there exists $\epsilon > 0$ such that for every $\gamma \in (0, 1]$ there exists $C \geq 1$ such that if $p > C(N/\log N)^{1/\Delta}$, then $\mathbb{P}(G_r(N,p) \in \mathcal{D}_{N,p}^{\Delta}(\gamma, \alpha, \epsilon', \epsilon, \mu)) = 1 - o(1)$.

**Proof.** Let $\Delta \geq 2$ and $\alpha, \epsilon', \mu \in (0, 1]$ be given, and let

$$\beta = \frac{\alpha^2}{4\epsilon^2} \left( \frac{1}{\epsilon} \right)^{8/(\alpha \mu)} .$$

For $\alpha, \beta$ and $\epsilon'$, there exist $\epsilon_1$ and $L_1$, and $\epsilon_2$ and $L_2$ according to Theorem 5 and Corollary 1 respectively. Fix $\epsilon = \min\{\epsilon_2/4, \epsilon_1, \epsilon_2\}$, and for every $\gamma > 0$ we set $C = (4/\gamma)^{1/\Delta}$, and let $N$ be sufficiently large.

First we shall show that a.a.s. $G_r(N,p)$ contains no element from

$$B_p^I(\alpha, \alpha, \epsilon', \epsilon, \mu).$$

Let $T$ be the tripartite graph with color classes $X$, $Y$, and $Z$ from $B_p^I(\alpha, \alpha, \epsilon', \epsilon, \mu)$. We will show that such a graph $T$ is unlikely to appear in $G_r(N,p)$. From (a) of (1), the bipartite subgraphs $T[X, Y]$ and $T[Y, Z]$ of $T$ contain at least $(\alpha - \epsilon)p^2m^2$ edges each. Furthermore, there is a set $X' \subseteq X$ with $|X'| \geq \mu/m$ such that for every $x \in X$ the pair $(N_T(x) \cap Y, Z)$ is not $(\epsilon', \alpha, p)$-dense. Set

$$X'' = \{x \in X' : |N_T(x) \cap Y| \geq \alpha m/2\} .$$

(2)

Since the pair $(X, Y)$ is $(\epsilon, \alpha, p)$-dense, we have $(X', Y)$ is $(\epsilon/\mu, \alpha, p)$-dense and, hence,

$$|X''| \geq (1 - \epsilon/\mu)|X'| \geq |X'|/2 \geq \mu m/2 .$$

Choose a subset $X''' \subseteq X''$ with

$$|X'''| = \mu m/2 .$$

(3)
Fix $x \in X''$. There exists a set $Y'_x \subseteq N_T(x) \cap Y$ with $|Y'_x| \geq \epsilon' \alpha m/2$ such that $d_{T,p}(Y'_x, Z) < \alpha - \epsilon'$ as $(N_T(x) \cap Y, Z)$ is not $(\epsilon', \alpha, p)$-dense. On average, we can suppose $|Y'_x| = \epsilon' \alpha m/2$. Now let $Y_x$ be such that $Y'_x \subseteq Y_x \subseteq N_T(x) \cap Y$ with $|Y_x| = \alpha m/2$. Then, clearly, $(Y_x, Z)$ is not $(\epsilon', \alpha, p)$-dense. Thus, we have a family $\{(Y_x, Z) : x \in X''\}$ such that each pair in which is not $(\epsilon', \alpha, p)$-dense. However, it is unlikely to occur in $G_r(N, p)$.

Firstly, we can fix the sets $X''$, $Y, Z$ and the edges of the bipartite graph $T[Y, Z]$ in at most

$$\sum_{t \geq (\alpha - \epsilon)p^2m^2} r \left( \frac{N}{m} \right) \left( \frac{N}{pm} \right)^2 \left( \frac{pm^2}{t} \right)$$

ways. Moreover, note that $m = \gamma p^{\Delta - 2} N$, so we can choose large $N$ such that $|Y_x| = \alpha m/2 > L_1/p$ for fixed $\Delta, \alpha, \gamma$ and $L_1$. Applying Theorem 4 to $T[Y, Z]$, we have that there are at most

$$\left( \frac{\epsilon' \alpha m}{2p^2m^2/4} \right)^{\mu p^2m^2/4}$$

ways for choosing the sets $Y_x$ for $x \in X''$. Note also that there are at least $\mu \alpha p^2m^2/4$ edges between $X''$ and $Y$ from 2 and 3. Therefore, the probability that $T[X'', Y, Z]$ appears in $G_r(N, p)$ can be bounded from above by

$$\sum_{t \geq (\alpha - \epsilon)p^2m^2} r \left( \frac{N}{m} \right) \left( \frac{N}{pm} \right)^2 \left( \frac{pm^2}{t} \right) p^{\frac{3}{t}N^{3m}} \left( \frac{p^2m^2}{t} \right)^{\frac{2\epsilon' \alpha m}{t}} \left( \frac{\mu \alpha p^2m^2/4}{t} \right)^{\frac{2\epsilon' \alpha m}{t}}$$

$$< \sum_{t \geq (\alpha - \epsilon)p^2m^2} r^3 N^{3m} \left( \frac{p^2m^2}{t} \right)^{\frac{2\epsilon' \alpha m}{t}} \left( \frac{\mu \alpha p^2m^2/4}{t} \right)^{\frac{2\epsilon' \alpha m}{t}}$$

$$< m^2 r^3 N^{3m} \left( \frac{2\epsilon' \alpha m}{t} \right)^{\frac{2\epsilon' \alpha m}{t}}$$

where the last inequality holds since the function $f(t) = (p^2m^2/t)^t$ is maximized for $t = p^2m^2$. Notice from 11 that $e(2\epsilon' \alpha m)^{\alpha / t} = \frac{1}{2} \frac{2\epsilon' \alpha m}{t} \leq 1/\epsilon$, and $p^2m^2 \geq 4m \log N$ since $p > C(N/\log N)^{\frac{1}{\Delta}}$, $m = \gamma p^{\Delta - 2} N$ and $C = (4/\gamma)^{1/\Delta}$. Hence, the right-hand side of the last inequality tends to 0 as $N \to \infty$, i.e., a.a.s. $G_r(N, p)$ contains no graph from $B_f^q(m, \alpha, \epsilon', \epsilon, \mu)$ as a subgraph.

It remains to show that a.a.s. $G_r(N, p)$ also contains no element from $B_f^q(m, \alpha, \epsilon', \epsilon, \mu)$. The proof of this case is similar as above.

Let $T$ be the bipartite graph with color classes $X, Y$ and $Z$ from $B_f^q(m, \alpha, \epsilon', \epsilon, \mu)$. In graph $T$, the bipartite subgraphs $T[X, Y], T[X, Z]$ and $T[Y, Z]$ of $T$ contain at least $(\alpha - \epsilon)p^2m^2$ edges each. Also, there is a set $X' \subseteq X$ with $|X'| \geq \mu |X|$ such that for every $x \in X'$ the pair $(N_T(x) \cap Y, N_T(x) \cap Z)$ is not $(\epsilon', \alpha, p)$-dense. Set

$$X'' = \{ x \in X' : |N_T(x) \cap Y| \geq \alpha m/2 \text{ and } |N_T(x) \cap Z| \geq \alpha m/2 \}.$$ 

Since $(X, Y)$ and $(X, Z)$ are $(\epsilon, \alpha, p)$-dense, we have $|X''| \geq (1 - 2\epsilon/\mu) |X'| \geq |X'|/2 \geq \mu \alpha m/2$. Choose a subset $X''' \subseteq X''$ with $|X'''| = \mu \alpha m/2$.

Fix $x \in X'''$. Similarly, there are sets $Y'_x \subseteq N_T(x) \cap Y$ and $Z'_x \subseteq N_T(x) \cap Z$ of size $\epsilon' \alpha m/2$ such that $d_{T,p}(Y'_x, Z'_x) < \alpha - \epsilon'$. Now let $Y_x$ and $Z_x$ be such that $Y'_x \subseteq Y_x \subseteq N_T(x) \cap Y$ and $Z'_x \subseteq Z_x \subseteq N_T(x) \cap Z$ with $|Y_x| = |Z_x| = \alpha m/2$. Then, $(Y_x, Z_x)$ is not $(\epsilon', \alpha, p)$-dense. Thus, we have a family $\{(Y_x, Z_x) : x \in X''\}$ such that each pair in which is not $(\epsilon', \alpha, p)$-dense.
Note that $\alpha pm/2 \geq L_2/p$ if $N$ is large. Hence, similarly, apply Corollary 1 to $T[Y, Z]$, the probability that $T[X^{\alpha}, Y, Z]$ appears in $G_r(N, p)$, can be bounded from above by

$$\sum_{t \geq (\alpha - \epsilon)pm^2} \binom{r}{3} \binom{n}{m} \left(\frac{pm^2}{t}\right)^p \left(\frac{m}{\alpha pm/2}\right)^{\mu pm^2/2} \leq m^2 \cdot N^{3n} \left(e^{2p/\alpha}\right)^{\mu pm^2}$$

as the function $f(t) = (pm^2e/t)^t$ is maximized for $t = pm^2$. Finally, since $e(\frac{2p}{\alpha})^{\mu pm^2} = 1/e$ and $pm^2$ is much larger than $m \log N$, we have the right-hand side of the last inequality tends to 0 as $N \to \infty$. $\square$

Now, Fact 3 follows immediately from Lemma 2 and the following result obtained in (35, Claim 18 and the remark afterwards).

**Lemma 3** For an integer $\Delta \geq 2$ and positive reals $\alpha, \epsilon', \mu$ and $\gamma$ there exists $\epsilon > 0$ such that for every $\gamma > 0$ there exists $C > 1$ and $N_0$ such that if $N \geq N_0$ and $p > C(\log N/N)^{1/\Delta}$, then every tripartite graph $T \in \mathcal{B}_p(m_1, m_2, m_3, \alpha, \epsilon', \mu) \cup \mathcal{B}_p^{\prime \prime \prime}(m_1, m_2, m_3, \alpha, \epsilon', \mu)$ with $m_1, m_2, m_3 \geq \gamma p^{\Delta - 1}N$ and $m_1^I, m_2^I, m_3^I \geq \gamma p^{\Delta - 2}N$ contains a subgraph $\tilde{T} \in \mathcal{B}_p(m, \alpha, \epsilon', \mu) \cup \mathcal{B}_p^{\prime \prime \prime}(m, \alpha, \epsilon', \mu)$.

### 4 Sparse partition universal multipartite Graphs for $\mathcal{H}(\Delta, n)$

In this section, we will show Theorem 3 i.e., for fixed $\Delta \geq 2$, we can choose suitable constants $r = r(\Delta)$, $B = B(\Delta)$ and $C = C(\Delta)$ such that for any $H \in \mathcal{H}(\Delta, n)$ if $N \geq Bn$ and $p = C(\log N/N)^{1/\Delta}$, then a.a.s. the random graph $G_r(N, p) \to (H, H)$.

**Lemma 4** For every $\Delta \geq 2$ there exist $\Delta \geq 2$ and positive constants $\mu, \epsilon, \xi$ and $\gamma > 0$ and $B > 1$ and $n_0$ such that for every $\epsilon_0, \ldots, \epsilon_\Delta$ satisfying $0 < \epsilon_0 \leq \cdots \leq \epsilon_\Delta \leq \epsilon^*$ and for every $n \geq n_0$ the following holds. If $G_r(N)$ is a $r$-partite graph with $r$-color classes $V^{(1)}, V^{(2)}, \ldots, V^{(r)}$ with $|V^{(i)}| = N \geq Bn$ $(1 \leq i \leq r)$ such that for some $0 < p \leq 1$ we have

(i) $G_r(N) \in \mathcal{H}_{\Delta, p}$,

(ii) $G_r(N) \in \mathcal{C}^\Delta_{\Delta, p}(\xi)$ for every $k = 1, \ldots, \Delta$, and

(iii) $G_r(N) \in \mathcal{D}^\Delta_{\Delta, p}(\gamma, \alpha, \epsilon_k, \epsilon_{k-1}, \mu)$ for every $k = 1, \ldots, \Delta$,

then $G_r(N) \to (H, H)$ for $H \in \mathcal{H}(\Delta, n)$.

Before we prove Lemma 4, we deduce Theorem 3 from it.

**Proof of Theorem 3** Let $\Delta \geq 2$ be given, Lemma 4 yields constants $\Delta \geq 2$ and $\mu, \alpha, \epsilon, \xi$ and $\gamma > 0$ and $B > 1$ and $n_0$. Moreover, from Fact 1, Corollary 2 and Corollary 3 there exists a constant $C$ such that for $p > C(\log N/N)^{1/\Delta}$ the random $r$-partite graph $G_r(N, p)$ a.a.s. satisfies the assumptions (i), (ii) and (iii). Therefore, Lemma 4 asserts that a.a.s. $G_r(N, p) \to (H, H)$ for every $H \in \mathcal{H}(\Delta, n)$ whenever $N \geq Bn$, which completes the proof of Theorem 3. $\square$

In order to prove Lemma 4 we also need the following result which relates to Turán number for $K_r$ in complete $r$-partite graph $K_r(k)$.

**Lemma 5** For integers $k \geq 1$ and $r \geq 2$, let $t_r(k)$ be the maximum number of edges in a subgraph of $K_r(k)$ that contains no $K_r$. We have

$$t_r(k) = \left[\frac{r}{2}\right] k^2.$$
Proof. The lower bound for $t_r(k)$ follows by deleting all edges between a pair of color classes of $K_r(k)$. On the other hand, we shall prove by induction of $k$ that if a subgraph $G = G(V^{(1)}, \ldots , V^{(r)})$ of $K_r(k)$ contains no $K_r$, then $e(G) \leq \left( \frac{k}{2} \right) - 1 \right) k^2$. Suppose $k \geq 2$ and $r \geq 3$ as it is trivial for $k = 1$ or $r = 2$.

Now, suppose that $G$ has the maximum possible number of edges subject to this condition. Then $G$ must contain $K_r - e$ as a subgraph, otherwise we could add an edge and the resulting graph would still not contain $K_r$. Denote the vertex set of this $K_r - e$ by $X$, we have $|X \cap V^{(i)}| = 1$ for $i = 1, 2, \ldots , r$. Without loss of generality, suppose $e = \{v_1, v_2\}$, where $v_1 \in V^{(1)}$ and $v_2 \in V^{(2)}$. Note that $G$ contains no $K_r$, we have for $i = 1, 2$ there is no vertex in $V^{(i)} \setminus \{v_i\}$ is adjacent to all the vertices of $X \setminus \{v_i\}$. Thus, together with the induction hypothesis, we can deduce that there are at least

$$(k - 1)^2 + 2(k - 1) + 1 = k^2$$

edges should be deleted from $K_r(k)$, which completes the induction step hence the proof.

Now, we are ready to give a proof for Lemma [3]

Proof of Lemma [3]. The proof consists of four parts. Firstly, we fix all constants needed in the proof. Secondly, we consider the given $r$-partite graph $G_r(N)$ along with a fixed 2-coloring of its edges. In order to embed every graph $H \in \mathcal{H}(\Delta, n)$ into one of the two monochromatic subgraphs of $G_r(N)$, we first prepare the graph $G_r(N)$ and here the sparse multipartite regularity lemma will be the key tool. Thirdly, we shall prepare a given graph $H \in \mathcal{H}(\Delta, n)$ for the embedding. Finally, we will embed $H$ into a monochromatic subgraph of $G_r(N)$.

Constants. Let $\Delta \geq 2$ be fixed, and let $\Delta = \Delta^4 + \Delta$ and $r = R(K_\Delta)$. The constants $\mu, \alpha, \epsilon^*, \xi, \gamma, B$ and $n_0$ involved in the proof of Lemma [3] are defined as follows. Set

$$\mu = \frac{1}{4\Delta^2}, \quad \alpha = \frac{1}{3}, \quad \text{and} \quad \epsilon^* = \frac{1}{6\Delta}. \tag{4}$$

Let

$$\epsilon = \frac{\epsilon_0}{2}, \quad \lambda = 2, \quad \text{and} \quad t_0 = 1 \tag{5}$$

be given, there exist $T_0, \eta$ and $N_0$ that are guaranteed by the sparse multipartite regularity lemma, Theorem [4]. Finally, we set

$$\xi = \frac{1}{6 \cdot 4\Delta + 1 \cdot T_0}, \quad \gamma = \frac{1 - \epsilon}{4\Delta - 1 T_0}, \quad B = \frac{1}{\xi}, \tag{6}$$

and

$$n_0 = \max \left\{ \frac{N_0}{B}, e^{\gamma \eta}, e^{\frac{T_0}{\epsilon \eta} (1 - \epsilon)} \right\}. \tag{7}$$

Preparing $G_r(N)$. Now, let $\epsilon_0, \ldots , \epsilon_\Delta$ satisfy

$$0 < \epsilon_0 \leq \cdots \leq \epsilon_\Delta \leq \epsilon^* = \frac{1}{6\Delta} \tag{8}$$

and let $n \geq n_0$ be given. Let $G_r(N)$ be the $r$-partite graph which has $r$-color classes $V^{(1)}, \ldots , V^{(r)}$ with $|V^{(i)}| = N$ (1 $\leq i \leq r$), where $N \geq Bn \geq N_0$, satisfies assumptions (i)-(iii) of Lemma [3]. Denote $V = \bigcup_{i=1}^{r} V^{(i)}$. Consider an arbitrary red/blue edge coloring of $G_r(N)$, and let $G_R = (V, E_R)$ and $G_B = (V, E_B)$ be the corresponding monochromatic subgraphs.

In the following, we apply the sparse multipartite regularity lemma with $\epsilon = \epsilon_0/2$, $\lambda = 2$, $t_0 = 1$, and some $p$ to $G_R$. From property (i) of Lemma [3] the graph $G_r(N)$ is $(1/\log N, 1 + \log N)$-bounded (see Definition [2]). Since $G_R \subseteq G_r(N)$ and $1/\log N \leq \eta \leq 1$ from (7), we have $G_R$ is $(\eta, \lambda)$-bounded.
Consequently, from Theorem 4, we have an partition \( \{V_0^{(i)}, V_1^{(i)}, \ldots, V_t^{(i)}\} \) for each \( V^{(i)} \), where \( t \) is same for each \( V^{(i)} \) and \( t_0 \leq t \leq T_0 \), such that

\[
\text{(i)} \ |V_0^{(i)}| \leq \epsilon N \text{ and } |V_1^{(i)}| = \cdots = |V_t^{(i)}| \text{ for } 1 \leq i \leq r;
\]

\[
\text{(ii)} \ \text{All but at most } ct^2 \text{ pairs } (V_a^{(i)}, V_b^{(j)}) \text{ are } (\epsilon, p) - \text{regular for } 1 \leq i, j \leq r \text{ and } 1 \leq a, b \leq t.
\]

Let \( F \) be the subgraph of \( r \)-partite complete graph \( K_r(t) \), whose vertices are \( \{V_a^{(i)} | 1 \leq a \leq t, 1 \leq i \leq r\} \) in which a pair \( (V_a^{(i)}, V_b^{(j)}) \) for \( i \neq j \) is adjacent if and only if the pair is \((\epsilon, p)\)-regular in \( G_R \). Then the number of edges of \( F \) is at least

\[
t^2 \left( \frac{r^2}{2} \right) - ct^2 > \left[ \left( \frac{r}{2} \right)^2 - 1 \right] t^2 = t_r(t).
\]

Hence, Lemma 5 implies that \( F \) contains a complete graph \( K_r \) with \( r \) vertices. Without loss of generality, assume that \( V_1^{(i)}, \ldots, V_t^{(i)} \) are pairwise \((\epsilon, p)\)-regular for \( G_R \). For convenience, let us denote \( V_1^{(i)}, \ldots, V_t^{(i)} \) by \( A_1, \ldots, A_r \). Moreover, since \( G_r(N) \in \mathcal{H}_N \), we have \( |d_{G_r,p}(A_i, A_j) + d_{G_r,p}(A_i, A_j) - 1| \leq 1/ \log N \).

Note from (7) that \( N/ \log N \leq (1 - \epsilon) N/T_0 \leq \epsilon A_r \), we have \( (A_i, A_i) \) is \((\epsilon/2, \log N/p)\)-regular for the graph \( G_R \). From (5) and (7), we have \( \epsilon + 2/ \log N \leq \epsilon_0 \) and, hence, \( (A_i, A_j) \) is \((\epsilon_0, p)\)-regular both for \( G_R \) and \( G_B \) for all \( i, j \in \{1, \ldots, r\} \). Therefore,

\[
\max \{d_{G_R,p}(A_i, A_j), d_{G_R,p}(A_i, A_j)\} \geq \frac{1}{2} - \frac{1}{2 \log N} \geq \frac{1}{3}
\]

for every \( \{i, j\} \in \binom{[r]}{2} \).

Color an edge \( \{i, j\} \in \binom{[r]}{2} \) red if \( d_{G_B,p}(A_i, A_j) \geq d_{G_B,p}(A_i, A_j) \) and blue otherwise. Since \( r \geq R(K_\Delta) \), we can suppose without loss of generality that there exists a monochromatic red clique \( K_\Delta \) with vertex set \( [\Delta] \subseteq [r] \) satisfying

\[
(A_i, A_j) \ \text{is } (\epsilon_0, p)\text{-regular for } G_{R} \ \text{and } d_{G_B,p}(A_i, A_j) \geq 1/3 \ \text{for all } i, j \in \binom{[\Delta]}{2}, \tag{9}
\]

and we shall show that \( G_R \) induced on \( \bigcup_{i \in [\Delta]} A_i \) contain any \( H \in \mathcal{H}(\Delta, n) \).

**Prepping H.** Fix \( H = (W, E(H)) \in \mathcal{H}(\Delta, n) \). Define the third power \( H^3 = (W, E(H^3)) \) of \( H \) such that \( \{w, w'\} \in E(H^3) \) if and only if \( w \neq w' \) and there exists a \( w-w' \)-path with at most three edges in \( H \). Since \( \Delta(H) \leq \Delta \), we have \( \Delta(H^3) \leq \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 = \Delta^3 - \Delta^2 + \Delta \). Hence, \( H^3 \) can not be a complete graph as \( |W| = n \) is large. Consequently, \( \chi(H^3) \leq \Delta(H^3) \leq \Delta^3 - \Delta^2 + \Delta \). Let “\( f \)” be a \((\Delta^3 - \Delta^2 + \Delta)\)-vertex coloring of \( H^3 \). We say two vertices \( w \) and \( w' \) are equivalent according to “\( f \)” if \( f(w) = f(w') \), and

\[
|x \in N_H(w) : f(x) < f(w)| = |x \in N_H(w) : f(x) < f(w')|,
\]

i.e., with same “left-degrees” — the number of neighbors with colors of smaller number. This equivalence relation partitions \( W \) into at most \((\Delta^3 - \Delta^2 + \Delta)(\Delta + 1) = \Delta \) classes as each vertex has “left-degrees” at most \( \Delta \). Denote the partition classes by \( W_1, \ldots, W_\Delta \) (may have empty classes) and let \( g : W \to [\Delta] \) be the corresponding partition function, i.e., \( g(w) = j \) if and only if \( w \in W_j \). Thus, if \( w, w' \in W_j \), then \( |x \in N_H(w) : g(x) < g(w)| = |x \in N_H(w) : g(x) < g(w')| \). For \( \ell \leq g(w) \), denote the “left-degree” of \( w \) with respect to \( g \) and \( \ell \) by

\[
\ldeg^g_\ell(w) = |x \in N_H(w) : g(x) \leq \ell|.
\]

Note that if \( w, w' \in W_j \), then their distance in \( H \) is at least four, which implies

\[
|N_H(w) \cap N_H(w')| = 0 \text{ and } e_H(N_H(w), N_H(w')) = 0. \tag{10}
\]
Embedding of $H$ to $G_R$. Now, we will embed the vertex class $W_\ell$ into $A_\ell$ one at a time, for $\ell = 1, \ldots, \Delta$.
Indeed, we shall inductively verify the following statement ($S_\ell$) for $\ell = 0, 1, \ldots, \Delta$.

($S_\ell$) There is an embedding $\varphi_\ell$ of $H[^4_{j=1} W_j]$ into $G_R[^4_{j=1} A_j]$ such that for every $z \in \bigcup_{j=\ell+1}^\Delta A_j$, there exists a candidate set $C_\ell(z) \subseteq A_g(z)$ given by

$$(a) \ C_\ell(z) = \left( \bigcap_{x \in N_H(z), \ g(x) \leq \ell} N_{G_R}(\varphi_\ell(x)) \right) \cap A_g(z),$$

satisfying

(b) $|C_\ell(z)| \geq (p/4)^{\deg(z)} m$, where $m = |A_g(z)| \geq (1 - \epsilon)N/t$, and

c) $(C_\ell(z), C_\ell(z'))$ is $(\epsilon, 1/3, p)$-dense in $G_R$ for any edge $\{z, z'\} \in E(H)$ with $g(z), g(z') > \ell$.

Remark. The definition of $C_\ell(z)$ implies that if we embed $z$ into $C_\ell(z)$, then its image will be adjacent to all vertices $\varphi_\ell(x)$ with $x \in N_H(z) \cap (W_1 \cup \cdots \cup W_\ell)$.

Note that ($S_\Delta$) gives an embedding $\varphi_\Delta$ of $H[^4_{j=1} W_j]$ into $G_R[^4_{j=1} A_j]$, so $G_R$ contains $H$ as a subgraph. Thus, verifying ($S_\ell$) inductively for $\ell = 0, 1, \ldots, \Delta$ completes the proof of Lemma 3. In the following, we shall use the letter “$x$” for vertices that have been embedded, “$y$” for that will be embedded in the current step, and “$z$” for that shall we embed at a later step.

Let us verify ($S_0$) at first. In this case, $\varphi_0$ is the empty mapping and for every $z \in W$, we have $C_0(z) = A_g(z)$ according to (a) as there is no vertex $x \in N_H(z)$ with $g(x) \leq 0$. So property (b) follows directly; and property (c) follows from 4.

For the inductive step, suppose ($S_\ell$) holds for $\ell < \Delta$, and we shall verify ($S_{\ell+1}$) by embedding $W_{\ell+1}$ into $A_{\ell+1}$ with the required properties. Note from 10 that $|N_H(z) \cap W_{\ell+1}| \leq 1$ for every $z \in \bigcup_{j=\ell+2}^\Delta W_j$.

Hence, for every “right-neighbor” $z \in \bigcup_{j=\ell+2}^\Delta W_j$ of $y \in W_{\ell+1}$, the new candidate set will be $C_{\ell+1}(z) = C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$. For each $y \in W_{\ell+1}$, we should find a suitable subset $C(y) \subseteq C_\ell(y)$ such that if $\varphi_{\ell+1}(y)$ is chosen from $C(y)$, then the new candidate set $C_{\ell+1}(z)$ will satisfy properties (b) and (c) of ($S_{\ell+1}$). However, since in general $|C(y)| \leq |C_\ell(y)| = O(|A_g(y)|)$ if $\deg(z) \geq 1$, we should select $\varphi_{\ell+1}(y)$ from $C(y)$ carefully such that if $y \neq y'$ then $\varphi_{\ell+1}(y) \neq \varphi_{\ell+1}(y')$. Here we shall apply Hall’s condition, a similar idea was used in 5, 35, 35. The details are contained in the following two claims.

Fix $y \in W_{\ell+1}$ and denote the “right neighbors” of $y$ by

$N_{\ell+1}^H(y) = \{ z \in N_H(y) : g(z) > \ell + 1 \}$.

For a vertex $v \in C_\ell(y)$, let $\tilde{C}_\ell(z) = N_{G_R}(v) \cap C_\ell(z)$ if $z \in N_{\ell+1}^H(y)$ and $\tilde{C}_\ell(z) = C_\ell(z)$ if $z \notin N_{\ell+1}^H(y)$.

Claim 1. For every $y \in W_{\ell+1}$, there exists a subset $C(y) \subseteq C_\ell(y)$ with $|C(y)| \geq (1 - \Delta_{\ell} - \Delta_{\ell}^2)p|C_\ell(y)|$ such that for every $v \in C_\ell(y)$, the following (b') and (c') hold.

(b') $|N_{G_R}(v) \cap C_\ell(z)| \geq (p/4)^{\deg(z)}|A_g(z)|$ for every $z \in N_{\ell+1}^H(y)$.

c') $(\tilde{C}_\ell(z), \tilde{C}_\ell(z'))$ is $(\epsilon, 1/3, p)$-dense for all edges $\{z, z'\}$ of $H$ with $g(z), g(z') > \ell + 1$ and $\{z, z'\} \cap N_{\ell+1}^H(y) \neq \emptyset$.

Proof. Fix $z \in N_{\ell+1}^H(y)$. Since $(C_\ell(y), C_g(y))$ is an $(\epsilon, 1/3, p)$-dense pair from (c) of ($S_\ell$), we have there exist at most $\epsilon|C_\ell(y)|$ vertices $v \in C_\ell(y)$ such that

$|N_{G_R}(v) \cap C_\ell(z)| < \left( \frac{1}{3} - \epsilon_{\ell} \right)p|C_\ell(y)|$.

Note that $|N_{\ell+1}^H(y)| \leq \Delta$, we have from (b) and (c) of ($S_\ell$) that all but at most $\Delta_{\ell}p|C_\ell(y)|$ vertices $v \in C_\ell(y)$ satisfy that for every $z \in N_{\ell+1}^H(y)$,

$|N_{G_R}(v) \cap C_\ell(z)| \geq \left( \frac{1}{3} - \epsilon_{\ell} \right)p\left( \frac{p}{4} \right)^{\deg(z)}|A_g(z)| \geq \left( \frac{p}{4} \right)^{\deg(z)}|A_g(z)|$.  

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Now, fix an edge $e = \{z, z'\}$ with $g(z), g(z') > \ell + 1$ and with at least one end vertex in $N_{H}^{\ell+1}(y)$. Clearly, there are at most $\Delta(\Delta - 1) < \Delta^2$ such edges.

If both vertices $z$ and $z'$ are neighbors of $y$, i.e., $z, z' \in N_{H}^{\ell+1}(y)$, then
\[
\max\{\deg_{g}^{f}(y), \deg_{g}^{f}(z), \deg_{g}^{f}(z')\} \leq \Delta - 2,
\]
since $y, z,$ and $z'$ have at least two neighbors in $W_{\ell+1} \cup \cdots \cup W_{\Delta}$. Hence, property (b) of (S1) implies
\[
\min\{|C_{\ell}(y)|, |C_{\ell}(z)|, |C_{\ell}(z')|\} \geq \left(\frac{p}{4}\right)^{\Delta-2} (1 - \epsilon) \frac{N}{T_{0}} \geq \gamma p^{\Delta-2} N.
\]
Recall that $\alpha = 1/3$, see [1]. Hence $G_{R} \subseteq G_{c}(N)$ and $G_{c}(N) \in \mathcal{P}_{N}^{\Delta}(\gamma, \alpha, \epsilon_{\ell+1}, \epsilon_{\ell}, \mu)$ imply that there are at most $\mu|C_{\ell}(y)|$ vertices $v \in C_{\ell}(y)$ such that the pair $(N_{G_{R}}(v) \cap C_{\ell}(z), N_{G_{R}}(v) \cap C_{\ell}(z'))$ is not $(\epsilon_{\ell+1}, 1/3, \mu)$-dense.

If, on the other hand, say, only $z \in N_{H}^{\ell+1}(y)$ and $z' \notin N_{H}^{\ell+1}(y)$, then
\[
\max\{\deg_{g}^{f}(y), \deg_{g}^{f}(z'), \deg_{g}^{f}(z')\} \leq \Delta - 1 \text{ and } \deg_{g}^{f}(z) \leq \Delta - 2.
\]
Hence, similarly, we have
\[
\min\{|C_{\ell}(y)|, |C_{\ell}(z')|\} \geq \gamma p^{\Delta-1} N \text{ and } |C_{\ell}(z)| \geq \gamma p^{\Delta-2} N.
\]
Thus, the fact $G_{c}(N) \in \mathcal{P}_{N}^{\Delta}(\gamma, \alpha, \epsilon_{\ell+1}, \epsilon_{\ell}, \mu)$ implies that there are at most $\mu|C_{\ell}(y)|$ vertices $v \in C_{\ell}(y)$ such that the pair $(N_{G_{R}}(v) \cap C_{\ell}(z), C_{\ell}(z'))$ is not $(\epsilon_{\ell+1}, 1/3, \mu)$-dense.

Therefore, for either case, deleting all “bad” vertices from $C_{\ell}(y)$, we find a subset $C(y) \subseteq C_{\ell}(y)$ with size $|C(y)| \geq (1 - \Delta\epsilon_{\ell} - 2\mu)|C_{\ell}(y)|$ such that for every $v \in C(y)$, the following (b') and (c') hold.

Now, let us turn to the second part of the inductive step. We shall choose $\varphi_{\ell+1}(y) \in C(y)$ such that $\varphi_{\ell+1}(y) \neq \varphi_{\ell+1}(y')$ for two vertices $y, y' \in W_{\ell+1}$. This can be achieved from the following claim.

Claim 2. There is an injective mapping $\psi : W_{\ell+1} \to \bigcup_{y \in W_{\ell+1}} C(y)$ such that $\psi(y) \in C(y)$ for every $y \in W_{\ell+1}$.

Proof. It suffices to verify Hall’s condition that for every $Y \subseteq W_{\ell+1}$,
\[
|Y| \leq \left| \bigcup_{y \in Y} C(y) \right|.
\]
Recall that $\deg_{g}^{f}(y) = \deg_{g}^{f}(y')$ for any two distinct vertices $y, y' \in W_{\ell+1}$ and so we can set
\[
k = \deg_{g}^{f}(y) \text{ for all } y \in W_{\ell+1}.
\]
From Claim 1, property (b) of (S2), and [3] and [4], we have that
\[
|C(y)| \geq (1 - \Delta\epsilon_{\ell} - \Delta^2\mu)|C_{\ell}(y)| \geq (1 - \Delta\epsilon_{\ell} - \Delta^2\mu)\left(\frac{p}{4}\right)^{k} (1 - \epsilon) \frac{N}{T_{0}} \geq \frac{1}{4k+1} p^{k} \frac{N}{T_{0}}.
\]
Hence, the assertion [11] holds if $|Y| \leq 4^{-k-1} p^{k} N / T_{0}$. Thus, we suppose that $|Y| > 4^{-k-1} p^{k} N / T_{0}$.
Denote the $k$-tuple $K(y) = \{u_{1}(y), \ldots, u_{k}(y)\}$ by the neighbors of $y$ that have been embedded already.
Clearly, $K(y) = N_{H}(y) \setminus N_{H}^{\ell+1}(y)$. From [10], we have that
\[
|K(y) \cap W_{i}| \leq 1, \text{ for } 1 \leq i \leq \ell;
\]
\[
K(y) \cap K(y') = \emptyset, \text{ for distinct vertices } y, y' \in W_{\ell+1}.
\]
Thus, the sets of already embedded vertices $\varphi_{\ell}(K(y))$ and $\varphi_{\ell}(K(y'))$ are disjoint as well and, therefore, $F_{k} = \{\varphi_{k}(K(y)) : y \in Y\} \subseteq \mathcal{V}_{T_{0}/k}$ is a family of pairwise disjoint $k$-sets with $|\varphi_{k}(K(y)) \cap A_{i}| \leq 1$ for $1 \leq i \leq \ell$. Note that
\[
C(y) \subseteq C_{\ell}(y) = \left( \bigcap_{x \in K(y)} N_{G_{R}}(\varphi_{\ell}(x)) \right) \cap A_{\ell+1}.
\]
Let $U = \bigcup_{y \in Y} C(y) \subseteq A_{\ell+1}$. Suppose to the contrary that
\[ |U| < |Y| = |\mathcal{F}_k|. \] (14)

Note that $|Y| \leq \ell + 1$ will contradict (10).

Then property (ii) of Lemma 4 follows directly from property (i) of (S\ell) and $\epsilon_\ell \geq \epsilon$. If $N_H(z) \cap W_{\ell+1} = \emptyset$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then property (c) of (S\ell) follows directly from property (c) of (S\ell) and $\epsilon_{\ell+1} \geq \epsilon$. If $N_H(z) \cap W_{\ell+1} = \emptyset$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then property (c) of (S\ell) follows from (c') of Claim 1. Consequently, property (c) of (S\ell) follows from (c') of Claim 1.

In conclusion, this completes the induction step and hence the proof of Lemma 4.

\[ \square \]

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