The Cayley isomorphism property for groups of order $8p$

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Abstract

For every prime $p > 3$ we prove that $Q \times \mathbb{Z}_p$ and $\mathbb{Z}_3^2 \times \mathbb{Z}_p$ are DCI-groups. This result completes the description of CI-groups of order $8p$.

1 Introduction

Let $G$ be a finite group and $S$ a subset of $G$. The Cayley graph $Cay(G, S)$ is defined by having the vertex set $G$ and $g$ is adjacent to $h$ if and only if $gh^{-1} \in S$. The set $S$ is called the connection set of the Cayley graph $Cay(G, S)$. A Cayley graph $Cay(G, S)$ is undirected if and only if $S = S^{-1}$, where $S^{-1} = \{ s^{-1} \in G \mid s \in S \}$. Every right multiplication via elements of $G$ is an automorphism of $Cay(G, S)$, so the automorphism group of every Cayley graph on $G$ contains a regular subgroup isomorphic to $G$. Moreover, this property characterises the Cayley graphs of $G$.

It is clear that $Cay(G, S) \cong Cay(G, S^\mu)$ for every $\mu \in Aut(G)$. A Cayley graph $Cay(G, S)$ is said to be a CI-graph if, for each $T \subseteq G$, the Cayley graphs $Cay(G, S)$ and $Cay(G, T)$ are isomorphic if and only if there is an automorphism $\mu$ of $G$ such that $S^\mu = T$. Furthermore, a group $G$ is called a DCI-group if every Cayley graph of $G$ is a CI-graph and it is called a CI-group if every undirected Cayley graph of $G$ is a CI-graph.

It was proved in [5] that $\langle a, z \mid a^p = 1, z^8 = 1, z^{-1}az = a^{-1} \rangle$ is a CI-group, though not a DCI-group. Let $G$ be a DCI-group of order $8p$, where $p$ is odd prime. It can easily be seen that every subgroup of a DCI-group is also a DCI-group. It follows that the Sylow 2-subgroup of $G$ can only be the quaternion group $Q$ of order 8 or $\mathbb{Z}_2^3$.

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If \( p > 8 \) or \( p = 5 \), then by Sylow’s Theorem the Sylow \( p \)-subgroup of \( G \) is a normal subgroup therefore \( G \) is isomorphic to one of the following groups: \( \mathbb{Z}_2^3 \times \mathbb{Z}_p, Q \times \mathbb{Z}_p, \mathbb{Z}_2^3 \times \mathbb{Z}_p \) or \( Q \times \mathbb{Z}_p \). It was proved in \([2]\) in that \( \mathbb{Z}_2^3 \times \mathbb{Z}_p \) is a CI-group with respect to ternary relational structures if \( p \geq 11 \). Moreover, Dobson and Spiga proved in \([3]\) that \( \mathbb{Z}_2^3 \times \mathbb{Z}_p \) is a CI-group with respect to ternary relational structures for all primes \( p \) and it is a CI-group with respect to color ternary relational structures if and only if \( p \neq 3 \) and 7.

Spiga proved in \([6]\) that \( Q \times \mathbb{Z}_3 \) is not a CI-group with respect to colour ternary relational structures and the non-nilpotent group \( Q \times \mathbb{Z}_3 \) is not a CI-group.

If \( p = 7 \), then either the Sylow 7-subgroup is normal, in which case \( G \) is as before, or \( G \) has 8 Sylow 7-subgroups, when \( G \cong \mathbb{Z}_2^3 \times \mathbb{Z}_7 \). The non-nilpotent groups above are not DCI-groups, see \([4]\). We show that the other groups are DCI-groups.

**Theorem 1.** For every prime \( p > 3 \) the groups \( Q \times \mathbb{Z}_p \) and \( \mathbb{Z}_2^3 \times \mathbb{Z}_p \) are DCI-groups.

Our paper is organized as follows. In section \([2]\) we introduce the notation that will be used throughout this paper. In section \([3]\) we collect important ideas that we will use in the proof of Theorem \([1]\). Section \([4]\) contains the proof of Theorem \([1]\) for primes \( p > 8 \) and Section \([5]\) contains the proof of Theorem \([1]\) for \( p = 5 \) and 7.

## 2 Technical details

In this section we introduce some notation. Let \( G \) be a group. We use \( H \leq G \) to denote that \( H \) is a subgroup of \( G \) and by \( N_G(H) \) and \( C_G(H) \) we denote the normalizer and the centralizer of \( H \) in \( G \), respectively.

Let us assume that the group \( H \) acts on the set \( \Omega \) and let \( G \) be an arbitrary group. Then by \( G \wr_H H \) we denote the wreath product of \( G \) and \( H \). Every element \( g \in G \wr_H H \) can be uniquely written as \( hk \), where \( k \in K = \prod_{\omega \in \Omega} G_{\omega} \) and \( h \in H \). The group \( K = \prod_{\omega \in \Omega} G_{\omega} \) is called the base group of \( G \wr_H H \) and the elements of \( K \) can be treated as functions from \( \Omega \) to \( G \). If \( g \in G \wr_H H \) and \( g = hk \) we denote \( k \) by \( (g)_b \). In order to simplify the notation \( \Omega \) will be omitted if it is clear from the definition of \( H \) and we will write \( G \wr_H H \).

The symmetric group on the set \( \Omega \) will be denoted by \( \text{Sym}(\Omega) \). Let \( G \) be a permutation group on the set \( \Omega \). For a \( G \)-invariant partition \( B \) of the set \( \Omega \) we use \( G^B \) to denote the permutation group on \( B \) induced by the action of \( G \) and similarly, for every \( g \in G \) we denote by \( g^B \) the action of \( g \) on the partition \( B \).

For a group \( G \), let \( \hat{G} \) denote the subgroup of the symmetric group \( \text{Sym}(G) \) formed by the elements of \( G \) acting by right multiplication on \( G \). For every Cayley graph \( \Gamma = \text{Cay}(G, S) \) the subgroup \( \hat{G} \) of \( \text{Sym}(G) \) is contained in \( \text{Aut}(\Gamma) \).

**Definition 1.** Let \( G \leq \text{Sym}(\Omega) \) be a permutation group. Let

\[
G^{(2)} = \left\{ \pi \in \text{Sym}(\Omega) \mid \forall a, b \in \Omega \exists g_{a,b} \in G \text{ with } \pi(a) = g_{a,b}(a) \text{ and } \pi(b) = g_{a,b}(b) \right\}.
\]
We say that $G^{(2)}$ is the 2-closure of the permutation group $G$.

**Lemma 1.** Let $\Gamma$ be a graph. If $G \leq \text{Aut}(\Gamma)$, then $G^{(2)} \leq \text{Aut}(\Gamma)$.

### 3 Basic ideas

In this section we collect some results and some important ideas that we will use in the proof of Theorem 1.

We begin with a fundamental lemma that we will use all along this paper.

**Lemma 2** (Babai [1]). $\text{Cay}(G, S)$ is a CI-graph if and only if for every regular subgroup $\hat{\varrho}G$ of $\text{Aut}(\text{Cay}(G, S))$ isomorphic to $G$ there is a $\mu \in \text{Aut}(\text{Cay}(G, S))$ such that $\hat{\varrho}G\mu = \hat{G}$.

We introduce the following definition.

**Definition 2.**

(a) We say that a Cayley graph $\text{Cay}(G, S)$ is a CI$(2)$-graph iff for every regular subgroup $\hat{\varrho}G$ of $\text{Aut}(\text{Cay}(G, S))$ isomorphic to $G$ there is a $\sigma \in \langle \hat{\varrho}G, \hat{G} \rangle^{(2)}$ such that $\hat{\varrho}G\sigma = \hat{G}$.

(b) A group $G$ is called a DCI$(2)$-group if for every $S \subset G$ the Cayley graph $\text{Cay}(G, S)$ is a CI$(2)$-graph.

Let us assume that $A = \text{Aut}(\text{Cay}(G, S)) \leq \text{Sym}(8p)$ contains two copies of regular subgroups, $\hat{Q} \times \hat{Z}_p$ and $\hat{Q} \times \hat{Z}_p$. By Sylow’s theorem we may assume that $\hat{Z}_p$ and $\hat{Z}_p$ are in the same Sylow $p$-subgroup $P$ of $\text{Sym}(8p)$. If $p > 8$, then $P$ is isomorphic to $\mathbb{Z}_p^\ast$. Moreover, $P$ is generated by 8 disjoint $p$-cycles. It follows that both $\hat{Q}$ and $\hat{Q}$ normalize $P$ so we may assume that $\hat{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $N_A(P)$. Let $P_2$ denote a Sylow 2-subgroup of $\text{Sym}(8)$. It is also well known that $P_2$ is isomorphic to the automorphism group of the following graph $\Delta$:

![Graph Delta](image)

Every automorphism of $\Delta$ permutes the leaves of the graph and the permutation of the leaves determines the automorphism, therefore $\text{Aut}(\Delta)$ can naturally be embedded into $\text{Sym}(8)$.

It is easy to see that the same holds if we change $Q \times \mathbb{Z}_p$ to $\mathbb{Z}_2^3 \times \mathbb{Z}_p$.
Lemma 3.  (a) There are exactly two regular subgroups of $P_2$ which are isomorphic to $Q$.

(b) There are exactly two regular subgroups of $P_2$ which are isomorphic to $\mathbb{Z}_2^3$.

Proof.  (a) Let $Q$ be a regular subgroup of $\text{Aut} (\Delta)$ isomorphic to the quaternion group with generators $i$ and $j$. For every $1 \leq m \leq 4$ there is a $q_m \in Q$ such that $q_m(2m - 1) = 2m$. These are automorphisms of $\Delta$ so $q_m(2m) = 2m - 1$ and hence the order of $q_m$ is 2. There is only one involution in $Q$ so $q_m = i^2$ for every $1 \leq m \leq 4$ and this fact determines completely the action of $i^2$ on $\Delta$.

We can assume that $i(1) = 3$. Such an isomorphism of $\Delta$ fixes setwise $\{1, 2, 3, 4\}$ so we have that $i(3) = 2$, $i(2) = 4$ and $i(4) = 1$ since $i$ is of order 4. Using again the fact that $Q$ is regular on $\Delta$ and $i^2(5) = 6$, we get that there are two choices for the action of $i$: $i = (1324)(5768)$ or $i = (1324)(5867)$.

We can also assume that $j(1) = 5$. This implies that $j(5) = j^2(1) = i^2(1) = 2$, and $j(2) = 6$ since $j \in \text{Aut} (\Delta)$ and $j(6) = 1$. The action of $i$ determines the action of $j$ on $\Delta$ since $iji = j$. Applying this to the leaf 3 we get that $j(3) = 8$ if $i = (1324)(5768)$ and $j(3) = 7$ if $i = (1324)(5867)$ so there is no more choice for the action of $j$. Finally, $i$ and $j$ generate $Q$ and this gives the result.

(b) Let us assume that $x \in \mathbb{Z}_2^3$ such that $x(1) = 2$ A fixed point free automorphism of $\Gamma$ of order 2 which maps 1 to 2 will map 3 to 4. There is an $y \in \mathbb{Z}_2^3$ such that $y(1) = 5$. Such an automorphism of $\Gamma$ maps 2 to 6 so we have that $x(5) = 6$ since $x$ and $y$ commute. This determines $x$ completely so we have that $x = (12)(34)(56)(78)$.

We have two possibilities for $y(3)$. If $y(3) = 7$, then $y = (15)(26)(37)(48)$ and if $y(3) = 8$, then $y = (15)(26)(38)(47)$. The third generator of the group $\mathbb{Z}_2^3$ which maps 1 to 3 is determined by $x$ and $y$ since $\mathbb{Z}_2^3$ is abelian.

The previous proof also gives the following.

Lemma 4.  (a) The following two pairs of permutations generate the two regular subgroups of $\text{Aut} (\Delta) \leq \text{Sym}(8)$ isomorphic to $Q$:

\[
i_1 = (1324)(5768), \quad j_1 = (1526)(3748)
\]

and

\[
i_2 = (1324)(5867), \quad j_2 = (1526)(3847)
\]
(b) The elements of these regular subgroups of $\text{Aut}(\Delta)$ are the following:

\[
\begin{array}{ccc}
Q_l : & Q_r : \\
\text{id} & \text{id} \\
(12)(34)(56)(78) & (12)(34)(56)(78) \\
(1324)(5768) & (1324)(5867) \\
(1423)(5867) & (1423)(5768) \\
(1526)(3748) & (1526)(3847) \\
(1625)(3847) & (1625)(3748) \\
(1728)(3546) & (1728)(3645) \\
(1827)(3645) & (1827)(3546)
\end{array}
\]

Using the following identification $Q_l$ and $Q_r$ act on $Q$ by left-multiplication and right-multiplication, respectively:

1 2 3 4 5 6 7 8
1 -1 i -i j -j k -k'

(c) The following permutations generate two regular subgroups of $\text{Aut}(\Delta) \leq \text{Sym}(8)$ isomorphic to $\mathbb{Z}_3^2$.

$A_1$ is generated by:

$x_1 = (12)(34)(56)(78), x_2 = (13)(24)(57)(68), x_3 = (15)(26)(37)(48)$

and $A_2$ is generated by:

$y_1 = (12)(34)(56)(78), y_2 = (13)(24)(58)(67), y_3 = (15)(26)(38)(47)$.

Lemma 5. Let us assume that $G_1 \leq P_2$ is generated by two different regular subgroups $Q_a$ and $Q_b$ of $\text{Aut}(\Delta)$ which are isomorphic to $Q$ and $G_2 \leq P_2$ is generated by two different regular subgroups $A_1$ and $A_2$ of $\text{Aut}(\Delta)$ which are isomorphic to $\mathbb{Z}_3^2$. Then $G_1 = G_2$.

Proof. It is clear that $|P_2| = |\text{Aut}(\Delta)| = 2^7$. One can see using Lemma 4(a) that $G_1$ and $G_2$ are generated by even permutations. Both $G_1$ and $G_2$ induce an action on the set $V = \{A, B, C, D\}$ which is a set of vertices of $\Delta$ and it is easy to verify that every permutation of $V$ induced by $G_1$ and $G_2$ is even. This shows that $G_1$ and $G_2$ are contained in a subgroup of $P_2$ of cardinality $2^5$.

Lemma 4(b) shows that $|Q_a \cap Q_b| = 2$ and one can also check that $|A_1 \cap A_2| = 2$. This gives $|G_1| \geq 2^5$ and $|G_2| \geq 2^5$, finishing the proof of Lemma 5. ■

Proposition 1. (a) The quaternion group $Q$ is a $\text{DCI}^{(2)}$-group.
(b) $\mathbb{Z}_2^3$ is a $\text{DCI}^{(2)}$-group.

Proof. (a) Let $Q_a$ and $Q_b$ be two regular subgroups of $\text{Sym}(8)$ isomorphic to the quaternion group $Q$. By Sylow’s theorem we may assume that $Q_a$ and $Q_b$ lie in the same Sylow 2-subgroup of $H = \langle Q_a, Q_b \rangle$. Since every Sylow
2-subgroup of $H$ is contained in a Sylow 2-subgroup of $\text{Sym}(8)$, we may assume that $Q_a$ and $Q_b$ are subgroups of $\text{Aut}(\Delta)$.

Our aim is to find an element $\pi \in \langle Q_a, Q_b \rangle^{(2)}$ such that $Q_a^\pi = Q_b$ so let us assume that $Q_a \neq Q_b$. Using Lemma 4(a) we may also assume that $Q_a$ and $Q_b$ are generated by the permutations $(1324)(5768)$, $(1526)(3748)$ and $(1324)(5867)$, $(1526)(3847)$, respectively. Lemma 4(b) shows that $H$ contains the following three permutations:

\[
(12)(34) = (1324)(5768)(1324)(5867) \\
(12)(56) = (1526)(3748)(1526)(3847) \\
(12)(78) = (1728)(3546)(1728)(3645).
\]

Now one can easily see that the permutation $(12)$ is in $H^{(2)}$. Finally, it is also easy to check using Lemma 4(b) that $Q_a^{(12)} = Q_b$.

(b) Let $A_1$ and $A_2$ be two regular subgroups of $\text{Sym}(8)$ isomorphic to $\mathbb{Z}_3^2$. Let $H'$ denote the group generated by $A_1$ and $A_2$. Similarly to the previous case we may assume that $A_1$ and $A_2$ are different regular subgroups of $\text{Aut}(\Delta)$. By Lemma 4, $A_1$ and $A_2$ are generated by the permutations $x_1 = (12)(34)(56)(78)$, $x_2 = (13)(24)(57)(68)$, $x_3 = (12)(34)(56)(78)$, $y_1 = (13)(24)(57)(68)$, $y_2 = (13)(24)(58)(67)$, $y_3 = (15)(26)(37)(48)$, respectively.

By Lemma 4 the group $H'$ contains the permutations $(12)(34)$, $(12)(56)$ and $(12)(78)$. Therefore $H'$ contains the permutation $(12)$ which conjugates $A_1$ to $A_2$ since $(12)$ centralizes $x_1$ and we also have $(12)x_2(12) = y_2y_1$ and $(12)x_3(12) = y_1y_3$, finishing the proof of Proposition 1.

\[\square\]

**Definition 3.** Let $\Gamma$ be an arbitrary graph and $A, B \subset V(\Gamma)$ such that $A \cap B = \emptyset$. We write $A \sim B$ if one of the following four possibilities holds:

(a) For every $a \in A$ and $b \in B$ there is an edge from $a$ to $b$ but there is no edge from $b$ to $a$.

(b) For every $a \in A$ and $b \in B$ there is an edge from $b$ to $a$ but there is no edge from $a$ to $b$.

(c) For every $a \in A$ and $b \in B$ the vertices $a$ and $b$ are connected with an undirected edge.

(d) There is no edge between $A$ and $B$.

We also write $A \asymp B$ if none of the previous four possibilities holds.

**Lemma 6.** Let $A, B$ be two disjoint subsets of cardinality $p$ of a graph. We write $A \cup B = \mathbb{Z}_p \cup \mathbb{Z}_p$. Let us assume that $\mathbb{Z}_p$ acts naturally on $A \cup B$ and for a generator $\hat{a}$ of the cyclic group $\mathbb{Z}_p$ the action of $\hat{a}$ is defined by $\hat{a}(a_1, a_2) = (a_1 + b, a_2 + c)$ for some $b, c \in \mathbb{Z}_p$. 

\[6\]
(a) If $b = c$, then the action of $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}_p$ on $A \cup B$ are the same.

(b) If $A \sim B$, then $b = c$.

(c) If $A \sim B$, then every $\pi \in \text{Sym}(A \cup B)$ which fixes $A$ and $B$ setwise is an automorphism of the graph defined on $A \cup B$ if $\pi \mid A \in \text{Aut}(A)$ and $\pi \mid B \in \text{Aut}(B)$.

Proof. These statements are obvious. ■

4 Main result for $p > 8$

In this section we will prove that $Q \times \mathbb{Z}_p$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are DCI-groups if $p > 8$. We will first prove it for $Q \times \mathbb{Z}_p$ and then we will repeat the argument for the case of $\mathbb{Z}_2^3 \times \mathbb{Z}_p$.

Proposition 2. For every prime $p > 8$, the group $Q \times \mathbb{Z}_p$ is a DCI-group.

Our technique is based on Lemma 2 so we have to fix a Cayley graph $\Gamma = \text{Cay}(Q \times \mathbb{Z}_p, S)$. Let $A = \text{Aut}(\Gamma)$ and $\hat{G} = Q \times \hat{\mathbb{Z}}_p$ be a regular subgroup of $A$ isomorphic to $Q \times \hat{\mathbb{Z}}_p$. In order to prove Proposition 2 we have to find an $\alpha \in A$ such that $\hat{G}^\alpha = \hat{G} = Q \times \hat{\mathbb{Z}}_p$ what we will achieve in three steps.

4.1 Step 1

We may assume $\hat{\mathbb{Z}}_p$ and $\mathbb{Z}_p$ lie in the same Sylow $p$-subgroup $P$ of $\text{Sym}(8p)$. Then both $\hat{Q}$ and $\hat{Q}$ are subgroups of $N_{\text{Sym}(8p)}(P) \cap A$ so we may assume that $\hat{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $N_{\text{Sym}(8p)}(P) \cap A$ which is contained in a Sylow 2-subgroup of $A$.

The Sylow $p$-subgroup $P$ gives a partition $B = \{B_1, B_2, \ldots, B_8\}$ of the vertices of $\Gamma$, where $|B_i| = p$ for every $i = 1, \ldots, 8$ and $B$ is $P$-invariant. It is easy to see that $B$ is invariant under the action of $\hat{Q}$ and $\hat{Q}$ and hence $\langle \hat{G}, \hat{G} \rangle \leq \text{Sym}(p) \cdot \text{Sym}(8)$. Moreover, both $\hat{G}$ and $\hat{G}$ are regular so $\hat{Q}$ and $\hat{Q}$ induce regular action on $B$ which we denote by $Q_1$ and $Q_2$, respectively. The assumption that $\hat{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $A$ implies that $Q_1$ and $Q_2$ are in the same Sylow 2-subgroup of $\text{Sym}(8)$.

4.2 Step 2

Let us assume that $Q_1 \neq Q_2$. We intend to find an element $\alpha \in A$ such that $(\hat{Q}^\alpha)^B = Q_2$.

Using Lemma 4(b) we can assume that $\hat{Q}$ is generated by the permutations $i$ and $j$ such that $i$ and $j$ induce the permutations $(B_1B_3B_4)(B_5B_7B_8)$ and $(B_1B_5B_2B_6)(B_3B_7B_4B_8)$, respectively. Similarly, $\hat{Q}$ is generated by $i$ and $j$ with $i^B = (B_1B_3B_2B_4)(B_5B_8B_6B_7)$ and $j^B = (B_1B_3B_2B_6)(B_3B_8B_4B_7)$.
We define a graph $\Gamma_0$ on $\mathcal{B}$ such that $B_i$ is connected to $B_j$ if and only if $B_i \sim B_j$. This is an undirected graph with vertex set $\mathcal{B}$ and both $Q_1$ and $Q_2$ are regular subgroups of $Aut(\Gamma_0)$. It follows that $\Gamma_0$ is a Cayley graph of the quaternion group of order 8.

**Definition 4.** (a) For a pair $(B_i, B_j) \in \mathcal{B}^2$ we write $B_i \equiv B_j$ if either there exists a path $C_1, C_2, \ldots, C_n$ in $\Gamma_0$ such that $C_1 = B_1, C_n = B_2$ or $i = j$. (b) For a pair $(B_i, B_j) \in \mathcal{B}^2$ we write $B_i \not\equiv B_j$ if $B_i \equiv B_j$ does not hold. (c) If both $H$ and $K$ are subsets of the vertices of $\Gamma_0$ such that $H \cap K = \emptyset$ and for every $B_i \in H, B_j \in K$ we have $B_i \not\equiv B_j$, then we write $H \not\equiv K$.

**Observation 1.** (a) The relation $\equiv$ defines an equivalence relation on $\mathcal{B}$. The equivalence classes defined by the relation $\equiv$ will be called equivalence classes. (b) Since $Q_1$ acts transitively on $\mathcal{B}$ we have that the size of the equivalence classes defined by the relation $\equiv$ divides 8.

We can also define a colored graph $\Gamma_1$ on $\mathcal{B}$ by coloring the edges of the complete directed graph on 8 points. $B_i$ is connected to $B_j$ with the same color as $B_i'$ is connected to $B_j'$ in $\Gamma_1$ if and only if there exists a graph isomorphism $\phi$ from $B_i \cup B_j$ to $B_i' \cup B_j'$ such that $\phi(B_i) = B_i'$ and $\phi(B_j) = B_j'$. The graph $\Gamma_1$ is a colored Cayley graph of the quaternion group. Moreover, both $Q_1$ and $Q_2$ act regularly on $\Gamma_1$. Using the fact that $Q$ has property $DCI(2)$ it is clear that there exists an $\alpha' \in \langle Q_1, Q_2 \rangle^{(2)} \leq Aut(\Gamma_1)$ such that $Q_2^{\alpha'} = Q_1$. We would like to lift $\alpha'$ to an automorphism $\alpha$ of $\Gamma$ such that $\alpha_B = \alpha'$.

(a) Let us assume first that $\Gamma_0$ is a connected graph.

**Lemma 7.** (a) $\hat{Q} \times \hat{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$. (b) If $\hat{Q} \times \hat{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$, then for every $\hat{q} \in \hat{Q}$ we have $(\hat{q})_b = id$.

**Proof.** (a) We first prove that $\hat{\mathbb{Z}}_p = \hat{\mathbb{Z}}_p$. Let $x$ and $y$ generate $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$, respectively. We can assume that $x \mid B_1 = y \mid B_1$. Using Lemma 4(b) we get that $x \mid B_1 = y \mid B_1$ if there exists a path in $\Gamma_0$ from $B_1$ to $B_1$. This shows that $x = y$ since $\Gamma_0$ is connected. Moreover, $\hat{Q} \times \hat{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$ since the elements of $\hat{\mathbb{Z}}_p$ and the elements of $\hat{Q}$ commute.

(b) Let $A' = A \cap \hat{\mathbb{Z}}_p \wr Sym(8)$. We have already assumed that $\hat{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $A'$. Let $\hat{q}$ be an arbitrary element of $\hat{Q}$. For every $(a, u) \in Q \times \mathbb{Z}_p$ we have $\hat{q}(a, u) = (b, u+t)$ for some $b \in Q$ and $t \in \mathbb{Z}_p$, where $t$ only depends on $\hat{q}$ and $a$ since $\hat{q} \leq \hat{\mathbb{Z}}_p \wr Sym(8)$. The permutation group $\hat{G}$ is transitive, hence there exist $\hat{q}_1, \hat{q}_2 \in \hat{Q}$ such that $\hat{q}_1(1, u) = (a, u)$ and $\hat{q}_2(b, u + t) = (1, u + t)$. The order of $\hat{q}_2\hat{q}_1$ is a power of 2 since $\hat{q}_2, \hat{q}, \hat{q}_1$ lie in a Sylow 2-subgroup. Therefore $t = 0$ and hence $(\hat{q})_b = id$. 

\[\square\]
Lemma 7 says that if \( \Gamma_0 \) is connected, then \( \langle \hat{Q}, \hat{Q} \rangle \leq \hat{Z}_p \wr Sym(8) \) and \( (q)_b = id \) for every \( q \in \langle \hat{Q}, \hat{Q} \rangle \). Therefore we can define \( \alpha = \alpha' id_B \) to be an element of the wreath product \( \hat{Z}_p \wr Sym(8) \) and clearly \( \alpha' id_B \) is an element of \( A \) with \( \alpha^B = \alpha' \).

(b) Let us assume that \( \Gamma_0 \) is the empty graph.

Then Lemma 4(c) shows that every permutation in \( \langle Q_1, Q_2 \rangle^{(2)} \) lifts to an automorphism of \( \hat{\Gamma} \).

(c) Let us assume that \( \Gamma_0 \) is neither connected nor the empty graph.

Observation 2. If \( Q_1 \neq Q_2 \), then \( \langle \hat{Q}, \hat{Q} \rangle \leq A \) contains \( \beta_1, \beta_2, \beta_3 \) such that

\[
\beta^B_1 = (B_1 B_2)(B_3 B_4), \quad \beta^B_2 = (B_1 B_2)(B_5 B_6), \quad \beta^B_3 = (B_1 B_2)(B_7 B_8).
\]

Proof. By Lemma 4 the elements \( \beta_1, \beta_2, \beta_3 \) can be generated as products of an element of \( \hat{Q} \) and \( \hat{Q} \).

Lemma 8. We claim that \( B_{2k-1} \equiv B_{2k} \) for \( k = 1, 2, 3, 4 \).

Proof. Since \( \Gamma_0 \) is a Cayley graph and \( Q_1 \) is transitive on the pairs of the form \( \langle B_{2k-1}, B_{2k} \rangle \) it is enough to prove that \( B_1 \equiv B_2 \). If \( B_1 \sim B_2 \), then \( B_1 \equiv B_2 \) so we can assume that \( B_1 \sim B_2 \). Since \( \Gamma_0 \) is not the empty graph \( B_1 \) is connected to \( B_l \) for some \( l > 2 \). By Observation 2 there exists \( \beta \in A \) such that \( \beta(B_1) = B_2 \) and \( \beta(B_l) = B_l \). This shows that \( B_2 \sim B_l \) and hence \( B_1 \equiv B_2 \).

Lemma 9. There exists \( \alpha \in A \) such that \( \alpha^B = \alpha' \).

Proof. Let us assume first that \( H_1 = \{B_1, B_2, B_3, B_4\} \). Then we define \( \alpha_1 \) to be equal to \( \beta_2 \) on \( H_1 \) and the identity on \( H_2 \). Using Lemma 4(b) we get that \( \alpha_1 \) is in \( \langle \hat{Q}, \hat{Q} \rangle^{(2)} \).

If \( H_1 = \{B_1, B_2, B_5, B_6\} \) or \( H_1 = \{B_1, B_2, B_7, B_8\} \), then we define \( \alpha_2 \) by \( \alpha_2 \mid H_1 = \beta_1 \) and \( \alpha_2 \mid H_2 = id \). Lemma 4(b) shows again that \( \alpha_2 \in A \).

It is easy to see that \( \alpha^B_1 = \alpha^B_2 = (B_1 B_2) \). Therefore \( A \) contains an element \( \alpha \) such that \( Q_1^{\alpha} = Q_2 \).

We conclude that we can assume that \( Q_1 = Q_2 \).

4.3 Step 3

Let us now assume that \( Q_1 = Q_2 \). We intend to find \( \gamma \in A \) such that \( \hat{Q}^\gamma = \hat{Q} \).

Let \( \hat{x} \) and \( \hat{\hat{x}} \) denote the generators of \( \hat{Z}_p \) and \( \hat{\hat{Z}}_p \), respectively. We may assume that \( \hat{x} \mid B_1 = \hat{x} \mid B_1 \).
Lemma 10. There exists \( \gamma \in A \) such that \( \hat{x}^\gamma = \hat{x} \).

Proof. Let us assume first that \( \Gamma_0 \) is connected. In this case there is only one equivalence class of size 8. It is clear by Lemma 6 (b) that \( \hat{x} = \hat{x} \).

Let us assume that \( \Gamma_0 \) is not connected. In this case there are at least two equivalence classes which we denote by \( \mathcal{C}_1, \ldots, \mathcal{C}_n \). The permutations \( \hat{x} \) and \( \hat{x} \) are elements of the base group of \( \hat{\mathcal{Z}}_p \wr \text{Sym}(8) \) and hence they can be considered as functions on \( B \). By Lemma 10 we may assume that \( \hat{x} \) is constant on every equivalence class of size 8. It is clear by Lemma 6 (b) that \( \hat{x} = \hat{x} \).

For every \( 1 \leq m \leq n \) there exists \( \hat{q}_m \in \hat{Q} \) such that \( \hat{q}_m(\mathcal{C}_1) = \mathcal{C}_m \) and for every \( \hat{q}_m \in \hat{Q} \) there exists \( \hat{q}_m \in \hat{Q} \) such that \( \hat{q}_m^\mathcal{B} = \hat{q}_m^\mathcal{B} \). Let \( \gamma \) be defined as follows:

\[
\begin{align*}
\gamma &\mid \cup \mathcal{C}_1 = id \\
\gamma &\mid \cup \mathcal{C}_m = \hat{q}_m \hat{q}_m^{-1} \text{ for } 2 \leq m \leq n.
\end{align*}
\]

Let \( (b, v) \in \hat{q}_m(B_e) \) with \( B_e \in \mathcal{C}_1 \) and we denote \( \hat{q}_m^{-1}(b, v) \) by \( (a, u) \). Since \( \hat{x} \) is constant on \( \mathcal{C}_m \) we have \( \hat{x}(b, v) = (b, v + c_m s) \) for some \( c_m \) which only depends on \( \mathcal{C}_m \). Thus \( \hat{q}_m(a, u + s) = (b, v + c_m s) \) since \( \hat{x} \) and \( \hat{q}_m \) commute and \( \hat{x} \mid B_e = \hat{x} \mid B_e \). Therefore for every \( w \in \mathbb{Z}_p \) we have

\[
\gamma(b, w) = \hat{q}_m(a, w) = \hat{q}_m(a, u + (w - u)) = (b, v + c_m (w - u))
\]

for every \( (b, w) \in \hat{q}_m(B_e) \). It is easy to verify that \( \gamma^{-1}(b, w) = (b, \frac{w - v + uc_m}{c_m}) \) for every \( w \in \mathbb{Z}_p \) which gives

\[
\gamma^{-1}\hat{x}\gamma(b, w) = \gamma^{-1}\hat{x}(b, wc_m + v - uc_m) = \gamma^{-1}(b, wc_m + v - uc_m + c_m) = (b, w + 1).
\]

It remains to show that \( \gamma \in A \). Let \( y \) and \( z \) be two points of \( Q \times \mathbb{Z}_p \).

If \( y \) and \( z \) are in the same equivalence class \( \mathcal{C}_m \), then either \( \gamma \) is defined on \( y \) and \( z \) by \( \hat{q}_m \hat{q}_m^{-1} \) which is the element of the group \( (\hat{G}, \hat{G}) \leq A \) or \( \gamma(y) = y \) and \( \gamma(z) = z \).

We denote by \( B_y \) and \( B_z \) the elements of \( B \) containing \( y \) and \( z \), respectively. If \( y \) and \( z \) are not in the same equivalence class, then \( B_y \sim B_z \). The definition of \( \gamma \) shows that \( \gamma^B = id \). Using Lemma 6 (c) we get that \( \gamma \mid B_y \cup B_z \) is an automorphism of the induced subgraph of \( \Gamma \) on the set \( B_y \cup B_z \), which proves that \( \gamma \in A \), finishing the proof of Lemma 10.

Using Lemma 10 we may assume that \( \hat{x} = \hat{x} \). Since \( \hat{x} \) and \( \hat{q} \) commute we have \( \hat{Q} \times \mathbb{Z}_p \leq \hat{Z}_p \wr \text{Sym}(8) \). Now we can apply Lemma 6 which gives \( (\hat{q})_b = id \) for every \( \hat{q} \in \hat{Q} \). This proves that \( \hat{Q} = \hat{Q} \) since \( Q_1 = Q_2 \). Therefore \( \hat{G} = \hat{G} \), finishing the proof of Proposition 2.

Our method also gives the analogous result for \( \mathbb{Z}_2^3 \times \mathbb{Z}_p \), what also follows from the theorem of Dobson and Spiga.

Proposition 3. For every prime \( p > 8 \), the group \( \mathbb{Z}_2^3 \times \mathbb{Z}_p \) is a DCI-group.
In order to prove Proposition 3 we will modify the proof of Proposition 2. Let $\Gamma$ be a Cayley graph of $G = \mathbb{Z}_3^2 \times \mathbb{Z}_p$ and let $A = Aut(\Gamma)$. Let $\hat{G} = \mathbb{Z}_3^2 \times \mathbb{Z}_p$ be a regular subgroup of $A$ isomorphic to $\mathbb{Z}_3^2 \times \mathbb{Z}_p$. It is enough to prove that there exists $\alpha \in A$ such that $\hat{G}^\alpha = (\mathbb{Z}_3^2 \times \mathbb{Z}_p)^\alpha = \mathbb{Z}_3^2 \times \mathbb{Z}_p = \hat{G}$.

It is easy to verify that the argument of the first step in subsection 4.1 only uses the fact that $p > 8$. Therefore there exists a $P$-invariant partition $\mathcal{D} = \{D_1, D_2, \ldots, D_8\}$, where $P$ is a Sylow $p$-subgroup of $Sym(8p)$ containing $\hat{Z}_p$ and $\mathbb{Z}_p$. We denote $A_1$ and $A_2$ the regular action on $\mathcal{D}$ induced by $\mathbb{Z}_3^2$ and $\mathbb{Z}_3^2$, respectively.

Let us assume that $A_1 \neq A_2$. We will repeat the argument of Step 2. Similarly to the definition of $\Gamma_1$ one can define a colored graph $\Gamma'_1$ on $\mathcal{D}$. Since $\mathbb{Z}_3^2$ is also a DCI-group there exists $\beta' \in Aut(\Gamma'_1)$ such that $A_2^{\beta'} = A_1$.

One can also define the graph $\Gamma'_0$ using the relation $\equiv$ and similarly to Lemma 7 one can prove that if $\Gamma'_0$ is connected, then there exists $\beta \in A$ such that $\beta^p = \beta'$.

If $\Gamma'_0$ is the empty graph, then every automorphism of $\Gamma'_1$ lifts to an automorphism of $\Gamma$.

Similarly to Observation 2 the automorphism group $A$ contains $\delta_1, \delta_2, \delta_3$ such that

$$\delta_1^p = (D_1D_2)(D_3D_4), \quad \delta_2^p = (D_1D_2)(D_5D_6), \quad \delta_3^p = (D_1D_2)(D_7D_8).$$

since $\langle A_1, A_2 \rangle = \langle Q_1, Q_2 \rangle$ by Lemma 5.

It is straightforward to check that Lemma 8 and Lemma 9 only uses the existence of the involutions $\beta_1, \beta_2, \beta_3$ so the argument can be repeated using $\delta_1, \delta_2$ and $\delta_3$. Therefore we may assume that $A_1 = A_2$.

Finally, the proof of Lemma 10 can also be repeated for $\mathbb{Z}_3^2 \times \mathbb{Z}_p$ which gives that the generators of $\hat{Z}_p$ and $\mathbb{Z}_p$ coincide. Since $A_1 = A_2$ we have $\hat{G} = \hat{G}$, finishing the proof of Proposition 3.

It is straightforward to check that the proof of Proposition 2 and Proposition 3 only uses the fact that $p > 8$ in the first step of the argument. We can formulate this fact in Proposition 4.

**Proposition 4.** Let $\Gamma$ be a Cayley graph of $G = Q \times \mathbb{Z}_p$ or $G = \mathbb{Z}_3^2 \times \mathbb{Z}_p$, where $p$ is an odd prime and let $\hat{G} = Q \times \hat{Z}_p$ or $\hat{G} = \mathbb{Z}_3^2 \times \hat{Z}_p$ be a regular subgroup of $Aut(\Gamma)$ isomorphic to $G$. Let us assume that there exists a $(\hat{G}, \hat{G})$-invariant partition $\mathcal{B} = \{B_1, B_2, \ldots, B_8\}$ of $V(\Gamma)$, where $|B_i| = p$ for every $i \in \{1, \ldots, 8\}$. In addition, we assume that $\hat{Z}_p$ is a subgroup of the base group of $\hat{Z}_p \wr Sym(\mathcal{B})$. Then there is an automorphism $\alpha$ of the graph $\Gamma$ such that $\hat{G}^\alpha = \hat{G}$.

## 5 Main result for $p = 5$ and 7

In this section we will prove that $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_3^2 \times \mathbb{Z}_5$ and $\mathbb{Z}_3^2 \times \mathbb{Z}_7$ are CI-groups.

The whole section is based on the paper [5], so we will only modify the proof of Lemma 5.4 of [5].
Proposition 5. Every Cayley graph of $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-graph.

We denote by $R$ one of the groups $Q$ and $\mathbb{Z}_2^3$. Let $\Gamma$ be a Cayley graph of one of these groups, $A = Aut(\Gamma)$ and $P$ a Sylow $p$-subgroup of $A$ for $p = 5, 7$, respectively. Let us assume that $A$ contains two copies of regular subgroups which we denote by $\hat{G} = \hat{R} \times \hat{\mathbb{Z}}_p$ and $\hat{G} = \hat{R} \times \hat{\mathbb{Z}}_p$. We can assume that $\Gamma$ is neither the empty nor the complete graph and both $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$ are contained in $P$.

It was proved in [5] that the action of $A$ on the points of graph $\Gamma$ cannot be primitive so there is a nontrivial $A$-invariant partition $\mathcal{B} = \{B_0, B_1, \ldots, B_{t-1}\}$ of $V(\Gamma) = G$. The elements of the partition $\mathcal{B}$ have the same cardinality since the action of $A$ is transitive on $\mathcal{B}$ so $|B_i| < p^2$ for every $i = 0, 1, \ldots, t - 1$. The partition $\mathcal{B}$ is $P$-invariant so $P$ acts on $\mathcal{B}$. Since $P$ is a $p$-group, the length of every orbit of $P$ is a power of $p$.

If the length of every orbit of $P$ on $V(\Gamma)$ is $p$, then it is clear from Proposition 4 that $\Gamma$ is a CI-graph. Therefore $P$ has an orbit $\Lambda \subset G$ such that $|\Lambda| = p^2$ since $p^3 > |G|$ and the remaining orbits of $P$ have length $p$ since $2p^2 > 8p$.

Let $\mathcal{C} = \{C_0, C_1, \ldots, C_{s-1}\}$ be an orbit of $P$ on $\mathcal{B}$ such that $\Lambda \subseteq \cup_{i=0}^{s-1} C_i$. We may assume that $B_i = C_i$ for $i = 0, 1, \ldots, s - 1$. It is clear that $s$ is a power of $p$. If $s \geq p^2$, then $|\cup_{i=0}^{s-1} C_i| \geq 2p^2 > 8p$ which is a contradiction. It follows that $1 < s < p^2$ which implies $s = p$.

For every $i < s$ and every $x \in P$ the following equalities hold for some $j < s$

$$(B_i \cap \Lambda)^x = B_i^x \cap \Lambda^x = B_j \cap \Lambda.$$

This implies that

$$|B_0 \cap \Lambda| = |B_i \cap \Lambda|$$

for every $0 \leq i < s$. Therefore

$$p^2 = |\Lambda| = \left|\cup_{i=0}^{p-1} (B_i \cap \Lambda)\right| = s |B_0 \cap \Lambda| = p |B_0 \cap \Lambda|.$$

This gives $|B_0 \cap \Lambda| = p$ so $|B_0| = p$ since $|B_0| t = 8p$ and both $|B_0|$ and $t$ are at least $p$.

If $|B_0| = p$, then $\Lambda$ is the union of $p$ elements of the $A$-invariant partition $\mathcal{B}$ and every orbit $\Lambda'$ of $P$ is an element of the partition $\mathcal{B}$ if $\Lambda' \neq \Lambda$. For every orbit $\Lambda' \neq \Lambda$ of $P$ and for every $y \in \hat{\mathbb{Z}}_p \cup \hat{\mathbb{Z}}_p$ we have $y(\Lambda') = \Lambda'$. By Proposition 4 we may assume that there exists an element $x'$ in $\hat{\mathbb{Z}}_p \cup \hat{\mathbb{Z}}_p$ such that $x'(B_0) \neq B_0$ and clearly $x'(B_7) = B_7$ for every $x' \in \hat{\mathbb{Z}}_p \cup \hat{\mathbb{Z}}_p$. Since both $\hat{G}$ and $\hat{G}$ are regular there exists $a \in C_A(x')$ such that $a(B_0) = B_7$, which contradicts the fact that $\mathcal{B}$ is $A$-invariant and $B_7$ is an orbit of $P$.

Let us assume that $|B_0| = 8$ and let $\hat{x}$ and $\hat{x}$ generate $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$, respectively. Since $\hat{G}$ and $\hat{G}$ are regular we have that neither $\hat{x}^R$ nor $\hat{x}^R$ is the identity, while for every $r \in \hat{R} \cup \hat{R}$ we have $r^R = id$. Since $\hat{x}$ and $\hat{x}$ are in the same Sylow $p$-subgroup of $P$ we may assume that $\hat{x}(B_i) = \hat{x}(B_i) = B_{i+1}$ for $i = 0, 1, \ldots, p - 1$, where the indices are taken modulo $p$. By Proposition 4 we may also assume that $\hat{x} \neq \hat{x}$.
For every $m$ there exists an $l$ such that the action of $\tilde{\tau}^l \tilde{\tau}^{-l}$ is nontrivial on $B_m$ since $\tilde{\tau} \neq \tilde{\tau}$. Therefore $A_{B_m} \upharpoonright B_m$ contains a regular subgroup and a cycle of length $p$ such that $p > \frac{|B_m|}{2}$. A theorem of Jordan says that such a permutation group is 2-transitive and hence the induced subgraph by $B_m$ of $\Gamma$ is the complete or the empty graph for every $m$.

**Lemma 11.** $B_m \sim B_n$ for $0 \leq m \neq n \leq p - 1$.

**Proof.** There exists a unique element $\tilde{g} \in \tilde{\mathbb{Z}}_p \leq P$ such that $\tilde{g}(B_m) = B_n$. We also have a unique element $\tilde{g} \in \tilde{\mathbb{Z}}_p \leq P$ with $\tilde{g}^B = \tilde{g}^B$. Since $\mathbb{Z}_p$ is cyclic and $\tilde{\tau} \neq \tilde{\tau}$ we have $\tilde{g} \neq \tilde{g}$. Moreover, we may also assume that $\tilde{g} \upharpoonright B_m \neq \tilde{g} \upharpoonright B_m$ since $\tilde{g} \neq \tilde{g}$ and the induced subgraphs of $\Gamma$ by $A_{B_m+c} \cup B_{n+c}$ are all isomorphic, where both $m + c$ and $n + c$ are taken modulo $p$.

Clearly, $\tilde{g} = \tilde{g}^{\tilde{g}^{-1}}$ is cycle of length $p$ on $B_n$. The points of $V(\Gamma) \setminus \Lambda$ are contained in $P$-orbits of length $p$ so and $\tilde{g}$ fixes every point of the set $B_m \cup B_n \setminus \Lambda$ since $\tilde{g}^B = id$.

Let $u \in B_m \setminus \Lambda$. It is enough to show that if $u$ is connected to some $v \in B_n$, then $u$ is connected to every point of $B_n$. We will prove that $A$ is transitive on the following pairs: $\{(u, w) \mid w \in B_n\}$.

$A$ is transitive on $\{(u, w) \mid w \in B_n \cap supp(\tilde{g})\} = \{(u, w) \mid w \in B_n \cap \Lambda\}$ since $\tilde{g}$ fixes $u$. Therefore we may assume that $v \in B_n \setminus \Lambda$ and we only have to find an element $u \in A$ such that $a(u) = u$ and $a(v) \in B_n \cap \Lambda$.

The restriction of $\tilde{g}$ to $B_n$ is a cycle of length $p$ which does not commute with $\tilde{r} \upharpoonright B_n$, where $\tilde{r}$ is an involution of $\tilde{R}$. Since $\tilde{r}$ and $\tilde{g}$ commute we have that there is a $u' \in B_n$ such that $\tilde{r}\tilde{g}(u') \neq \tilde{g}\tilde{r}(u')$. Since the action of $\tilde{R}$ is transitive on $B_n$ there exists $\tilde{r} \in \tilde{R}$ such that $\tilde{r}(u) = u'$. Then

\[(\tilde{r}\tilde{r})\tilde{g}(u) = \tilde{r}\tilde{g}\tilde{r}(u) = \tilde{r}\tilde{g}(u') \neq \tilde{g}\tilde{r}(u') = \tilde{r}(\tilde{r})\tilde{g}(u)\]

so there exists $a' \in A$ such that

\[a'\tilde{g}(u) \neq \tilde{g}a'(u).\]  

(1)

Let us assume that $v = \tilde{g}(u)$. Then the inequality $1$ gives $a'(v) \neq \tilde{g}a'(u)$. Since $\tilde{R} \upharpoonright B_m$ is regular on $B_m$ there exists $\tilde{s} \in \tilde{R}$ such that $\tilde{s}(u) = a'(u)$ and since $\tilde{s}$ and $\tilde{g}$ commute we have $\tilde{s}(v) = \tilde{g}\tilde{s}(u) = \tilde{g}a'(u)$. Therefore $\tilde{s}(v) \neq a'(v)$ and hence $\tilde{s}^{-1}a'$ fixes $u$ and $\tilde{s}^{-1}a'(v) \neq v$ so we may assume that $v \neq \tilde{g}(u)$.

If $p = 7$, then $v \in B_n \cap \Lambda$.

If $p = 5$, then there exists $\tilde{t} \in \tilde{R}$ such that $\tilde{t}(u) \in B_m \setminus \Lambda = B_m \setminus supp(\tilde{g})$ while $\tilde{t}(v) \in B_m \cap \Lambda \subset supp(\tilde{g})$ since both $\tilde{R} \upharpoonright B_m$ and $\tilde{R} \upharpoonright B_n$ are regular and $\gcd(8,5) = 1$. The permutations $\tilde{t}^{-1}\tilde{g}\tilde{t}$ fix the point $u$ for every $0 \leq l \leq 4$ and $\tilde{t}^{-1}\tilde{g}\tilde{t}(y) \neq \tilde{t}^{-1}\tilde{g}\tilde{t}(y)$ if $l_1 \neq l_2 \pmod{5}$. At least one of the four elements $\tilde{t}^{-1}\tilde{g}\tilde{t}$, $\tilde{t}^{-1}\tilde{g}^2\tilde{t}$, $\tilde{t}^{-1}\tilde{g}^3\tilde{t}$, $\tilde{t}^{-1}\tilde{g}^4\tilde{t}$ of $A$ fixes $u$ and maps $v$ to an element of $B_n \cap fix(\tilde{g}) = B_n \cap \Lambda$ since $|B_n \setminus supp(\tilde{g})| = 3$, finishing the proof of the fact that $B_m \sim B_n$ for $0 \leq m \neq n \leq 7$.  

\[\bbox{\text{\textbullet\textbullet\textbullet\textbullet\textbullet}}\]
Every permutation of $V(\Gamma)$ which fixes setwise $B_m$ for every $m$ is an automorphism of $\Gamma$ so there is an $a \in A$ such that $\hat{x}^a = \hat{x}$. Applying Proposition 4 we get that there exists $\alpha \in A$ such that $\left(\hat{R} \times \hat{\mathbb{Z}}_p\right)^{\alpha} = \hat{R} \times \hat{\mathbb{Z}}_p$, finishing the proof of Proposition 5.

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