C₀ ESTIMATES FOR HESSION QUOTIENT EQUATIONS
ON HKT MANIFOLDS

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ABSTRACT. We show the C₀ estimate for solutions to Hessian quotient equations on hyperKähler with torsion manifolds without any additional assumption on its hypercomplex structure by a clever use of the cone condition directly.

1. INTRODUCTION

Let (M, I, J, K) be a compact hyperKähler with torsion, later abbreviated as HKT, manifold and g be the hyperhermitian metric of M. We denote by Ω the associated HKT form with respect to g. Fix a q-real smooth closed (2,0)-form Ω₀ on M, we introduce a new q-real (2,0) form for any smooth function u : M → ℝ

$$\Omega_u := \Omega + \partial \partial_J u.$$  

We consider the following Hessian quotient equation on (M, Ω)

$$\Omega_u^k \wedge \Omega^{n-k} = e^F \Omega_u^l \wedge \Omega^{n-l}, \quad 0 \leq l < k \leq n,$$

where F is a real smooth functions on M. For k = n and l = 0, the equation (1.1) is just the quaternionic Monge-Ampère equation associated to the quaternionic Calabi conjecture which was introduced by Alesker-Verbitsky [3] formulated in analogy with the famous complex Calabi conjecture solved by Yau [35]. Later, M. Verbitsky [34] has found a geometric interpretation of this quaternionic Monge-Ampère equation.

Let us now give an overview of the advances towards proving the quaternionic Calabi conjecture. Until now, this conjecture was solved by Alesker in [1] on the HKT manifold with a flat hyperKähler metric, by Gentili-Vezzoni [16] 8-dimensional 2-step nilmanifolds M with an abelian hypercomplex structure and by Dinew-Sroka [12] for compact hyperKähler manifolds. But for compact HKT manifolds, this conjecture is still open.

The nature strategy to face this problem is, of course, to use the continuity method for which a priori estimates are crucial. Here, the C₀ estimate is a very important part in solving this conjecture. Alesker-Verbitsky [3] obtained the C₀ estimate in the case when the canonical bundle is trivial.
by repeating the classical Moser iteration method used by Yau in [35]. This bound was shown to hold in [2] when the hypercomplex structure is locally flat by using the method of BLocki from [8]. However, the $C^0$ estimate can be established in more general settings now. Alesker-Shelukhin [6] obtained the $C^0$ estimate on a compact HKT manifold without any additional assumption on its HKT structure, following the scheme of BLocki [8]. Recently, it is Sroka [24], who provided a simpler proof, by using an analogous approach of Cherrier [11], Tosatti-Weinkove [29], or Zhang [36] on hermitian manifolds.

For $k = n$ and $l = n - 1$, the equation (1.1) is analogous to the $J$-equation on Kähler manifolds proposed by Donaldson [13] in the setting of moment maps. The $J$-equation was solved by Song-Weinkove [26] via $J$-flow after some progresses made in [10, 30, 31, 32].

Recently, Zhang [37] and Gentili [15] investigated a general class of fully non-linear equations on HKT manifolds $(M, I, J, K, g)$

$$G(\Omega_u) = e^F,$$

which includes the Hessian quotient equation (1.1). They solved the equation (1.2) independently on closed HKT manifold with a flat hyperKähler metric under the existence of an admissible $C$-subsolution by adapting the approach of Székelyhidi [23] to the hypercomplex setting. In particular, the $C^0$ estimate for solution of the equation (1.2) was established in [37, 15] by using the Alexandroff-Bakelman-Pucci (ABP) method as in [23] which generalized the scheme of BLocki [8].

Thus, it is a very interesting problem to solve the equation (1.2) on the compact KHT manifold without any additional assumption on its HKT structure. In this paper, we can make some progress on this problem. In details, we can prove the $C^0$ estimate for solutions of the Hessian quotient equation (1.1) with on the compact KHT manifold without any additional assumption on its HKT structure. To statement our main result, we need to introduce the cone condition which is similar to the Kähler setting [26, 14, 18, 27].

**Definition 1.1.** Let $\Omega_0$ be a $q$-real smooth closed $(2,0)$-form on $M$, we say that $\Omega_0$ satisfies the cone condition for the equation (1.1) with respect with $\Omega$ if it satisfies

$$k\Omega_0^{k-1} \wedge \Omega^{n-k} > le^F \Omega_0^{l-1} \wedge \Omega^{n-l}. \quad (1.3)$$

It is easy to see that the cone condition (1.3) is the necessary condition for the solvability of (1.1) in view of (2.11), (2.8) and (2.13). We get the following result.

**Theorem 1.1.** Let $(M, I, J, K, \Omega)$ be a closed HKT manifold of dimension $n$, and $\Omega_0 \in \Gamma_k$ be a smooth $q$-real $(2,0)$-form on $M$ with $\partial \Omega_0 = 0$. Assume $\Omega_0$ satisfies the cone condition (1.3), then for any smooth solution $u$ of the Hessian quotient equation (1.1) with $\sup_M u = 0$, the following bound holds

$$|u|_{C^0(M)} \leq C, \quad \sup_M.$$
where the constant $C$ depending only on the HKT structure and $F$.

The proof we present is strongly motivated by the work of Sun [27] in which he derive $C^0$ estimate for solutions of Hessian quotient equations on Kähler manifolds by a clever use of the cone condition directly. This in turn is based on an inequality obtained originally by Cherrier in [11]. The method emerged in the course of proving the $C^0$ estimate for the complex Monge-Ampère equation and Hessian equations on a compact hermitian manifold [29, 36].

2. Preliminaries

2.1. HyperKähler manifold with torsion. Let us recall the definition of HKT manifolds. A hypercomplex manifold is a smooth manifold $M$ of real dimension $4n$ equipped with a triple of complex structures $(I,J,K)$ satisfying the quaternionic relations

$$I \circ J = -J \circ I = K.$$ 

A hypercomplex manifold $(M,I,J,K,g)$ with a Riemannian metric $g$ is called a hyperhermitian manifold if $g$ is invariant under the three complex structures $(I,J,K)$, i.e.

$$g = g(\cdot, I) = g(\cdot, J) = g(\cdot, K).$$

This action extends uniquely to the right action of the algebra $\mathbb{H}$ of quaternions on $TX$. A hypercomplex manifold admits the whole sphere of complex structures namely

$$S_M = \{ aI + bJ + cK : a,b,c \in \mathbb{R}, \quad a^2 + b^2 + c^2 = 1 \}.$$

and a hyperhermitian metric $g$ is invariant under all of them.

For a given $L \in S_M$ we denote the associated hermitian form by $\omega_L$, i.e.

$$\omega_L = g(\cdot, L),$$

Consider the following differential form

$$\Omega := \omega_J - \sqrt{-1} \omega_K.$$

Definition 2.1. A hyperhermitian manifold $(M,I,J,K,g)$ is called HKT if

$$\partial \Omega = 0,$$

where $\partial$ in the whole paper is the differential operator with respect to $I$. To be more precise,

$$\partial = \frac{1}{2} (d + \sqrt{-1}I \circ d \circ I)$$

and $d$ is the standard exterior differential operator on $M$.

HKT manifolds were introduced in the physical literature by Howe and Papadopoulos [19]. For the mathematical treatment see Grantcharov-Poon [17] and Verbitsky [33]. The original definition of HKT-metrics in [19] was different but equivalent to Definition 2.1 (the latter was given in [33]). The classical hyperKähler metrics (i.e. Riemannian metrics with the holonomy
of the Levi-Civita connection contained in the group $Sp(n)$ form a subclass of HKT-metrics. It is well known that a hyperhermitian metric $g$ is hyperKähler if and only if the form $\Omega$ is closed, or equivalently $\partial \Omega = 0$ and $\overline{\partial} \Omega = 0$.

Let $(M, I, J, K)$ be a hypercomplex manifold. Let us denote by $\Lambda^{p,q}_I(M)$ the vector bundle of differential forms of the type $(p, q)$ on the complex manifold $(M, I)$. By the abuse of notation we will also denote by the same symbol $\Lambda^{p,q}_I(M)$ the space of $C^\infty$-sections of this bundle. The twisted Dolbeault differential operator was introduced in [33] as

$$\overline{\partial}_J := J^{-1} \circ \overline{\partial} \circ J,$$

where $\overline{\partial}_J : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{p,q+1}_I(M)$ is the usual $\overline{\partial}$-differential on differential forms on the complex manifold $(M, I)$.

It is evidently that (see [33])

**Proposition 2.1.**

$$J : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{q,p}_I(M),$$

$$\partial_J : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{p+1,q}_I(M),$$

$$\partial \partial_J = -\partial J \partial.$$

**Definition 2.2.** For each $k = 1, 2, \cdots, n$. A form $\alpha \in \Lambda^{2k,0}_I(M)$ is called $q$-real if

$$J\alpha = \overline{\alpha}$$

under the quaternionic conjugation $\mathbb{H}$ [33].

The space of $q$-real $(2k, 0)$ forms on $(M, I)$ will be denoted by $\Lambda^{2k,0}_{I,\mathbb{R}}(M)$. We have the following lemma [1].

**Lemma 2.1.** Let $(M, I, J, K)$ be a hypercomplex manifold. For $u \in C^2(M, \mathbb{R})$, then $\partial \partial_J u \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$ and we call it the quaternionic Hessian of $u$.

### 2.2. The fundamental symmetric functions

In this subsection, we give some basic properties of elementary symmetric functions, which could be found in [20, 25], and establish some key lemmas.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the $k$-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$  

We also set $\sigma_0 = 1$ and denote by $\sigma_k(\lambda \mid i)$ the $k$-th symmetric function with $\lambda_i = 0$. Recall that the Gårding’s cone is defined as

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}.$$
Proposition 2.2. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( k = 1, \ldots, n \), then
\[
\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall \ 1 \leq i \leq n,
\]
(2.1)
\[
\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda|i) = k \sigma_k(\lambda),
\]
\[
\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n - k) \sigma_k(\lambda).
\]

The generalized Newton-MacLaurin inequality are as follows, which will be used later.

Proposition 2.3. For \( \lambda \in \Gamma_k \) and \( n \geq k > l \geq 0 \), \( r > s \geq 0 \), \( k \geq r \), \( l \geq s \), we have
\[
\left[ \frac{\sigma_k(\lambda)}{C_n^k} \right]^{\frac{1}{k-l}} \leq \left[ \frac{\sigma_r(\lambda)}{C_n^r} \right]^{\frac{1}{r-s}}.
\]
(2.2)

Proposition 2.4. For \( \lambda \in \Gamma_k \) and \( 0 \leq l < k \leq n \), we have
\[
\frac{\partial \left[ \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \right]}{\partial \lambda_i} > 0, \quad \forall \ 1 \leq i \leq n.
\]

Proposition 2.5. For any \( n \geq k > l \geq 0 \),
\[
\left[ \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \right]^{\frac{1}{k-l}}
\]
is a concave function in \( \Gamma_k \).

We recall the Gårding’s inequality (see [9]).

Proposition 2.6. For \( 1 \leq k \leq n \) and \( \lambda, \mu \in \Gamma_k \), then we have
\[
\sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) \geq k [\sigma_k(\mu)] \frac{1}{k} \sigma_k(\lambda)^{1-\frac{1}{k}}.
\]
(2.3)

In particular,
\[
\sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) > 0.
\]

As a corollary of Proposition 2.6, we can obtain by using the equality (2.1).

Proposition 2.7. Assume \( \lambda \in \Gamma_k \) and \( 1 \leq l < k \leq n \), then for any \( i \in \{1, 2, \ldots, n\} \) we have
\[
\frac{\sigma_{k-1}(\lambda|i)}{\sigma_{l-1}(\lambda|i)} > \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}.
\]
(2.4)
2.3. **Hyperhermitian matrices.** In this subsection, we will remind the basic properties of hyperhermitian matrices. First, we shall also use a version of a determinant defined for hyperhermitian quaternionic matrices referring for further details, properties. For a quaternionic \( n \times n \)-matrix \( A \in \text{Mat}_n(\mathbb{H}) \) let us denote by \( A^R \) the realization matrix of \( A \) which is a real \( 4n \times 4n \)-matrix.

Assume \( A = A_0 + iA_1 + jA_2 + kA_3 \), where \( A_0, A_1, A_2, A_3 \) are real \( n \times n \) matrices, then (see Page 10 in [7])

\[
A^R = \begin{pmatrix}
A_0 & -A_1 & -A_2 & -A_3 \\
A_1 & A_0 & -A_3 & A_2 \\
A_2 & A_3 & A_0 & -A_1 \\
A_3 & -A_2 & A_1 & A_0
\end{pmatrix}.
\]

Clearly, if \( A = (a_{ij}) \) is hyperhermitian, i.e. \( \overline{a_{ij}} = a_{ji} \), \( A^R \) is real symmetric.

The following is classical (see [7] for the references).

**Theorem 2.1.** There exists a polynomial \( P \) defined on the space of all hyperhermitian \( n \times n \)-matrices such that for any hyperhermitian \( n \times n \)-matrix \( A \) one has \( \det(A^R) = P^4(A) \) and \( P(\text{Id}) = 1 \). Furthermore \( P \) is homogeneous of degree \( n \).

**Definition 2.3.** For a hyperhermitian matrix \( A \), its Moore determinant is defined as (see (8) in [7])

\[ \det(A) := P(A) \in \mathbb{R}. \]

The explicit formula for Moore determinant was given by Moore [22] (see also Page 14 in [7]).

This notation should not cause any confusion with the usual determinant of real or complex matrices due to part (i) of the next theorem.

**Theorem 2.2.** (i) The Moore determinant of any complex hermitian matrix is equal to the usual determinant.

(ii) For any hyperhermitian matrix \( A \) and any quaternionic matrix \( C \)

\[ \det(C^*AC) = \det(A) \cdot \det(C^*C), \]

where \( C^* = \overline{C}^t \).

(iii) For any hyperhermitian matrix \( A \) can be symplectically diagonalized. That is, we can find \( C \in \text{GL}(n, \mathbb{H}) \) with \( C^*C = \text{Id} \) such that

\[ C^*AC = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\}, \]

where \( \lambda_i \in \mathbb{R} \) for \( 1 \leq i \leq n \).

Let us introduce more notation. Let \( A \) be any hyperhermitian \( (n \times n) \)-matrix. For any non-empty subset \( I \subset \{1, ..., n\} \), the minor \( M_I(A) \) of \( A \) which is obtained by deleting the rows and columns with indexes from the set \( I \), is clearly hyperhermitian. For \( I = \{1, ..., n\} \), let \( \det M_{\{1, ..., n\}} = 1 \). Then, we recall the following important property (see Proposition 1.1.11 in [5]).
Proposition 2.8. For any hyperhermitian \((n \times n)\)-matrix \(A\) and any diagonal real matrix \(T = \text{diag}\{t_1, ..., t_n\}\)

\[
\det(A + T) = \sum_{I \subset \{1, ..., n\}} \left( \prod_{i \in I} t_i \right) \cdot \det M_I(A).
\]

In particular

\[
(2.6) \quad \det(A + t \cdot \text{Id}) = \sum_{I \subset \{1, ..., n\}} t^{|I|} \cdot \det M_I(A).
\]

where \(|I|\) denotes the cardinality of the set \(I\).

Definition 2.4. Let \(A\) be a hyperhermitian \((n \times n)\)-matrix, for \(k = 1, 2, ..., n\), we define

\[
\sigma_k(A) = \sigma_k(\lambda(A)) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},
\]

where \(\lambda(A) = (\lambda_1(A), \lambda_2(A), ..., \lambda_n(A))\) are the real eigenvalues of \(A\) (see (iii) of Theorem 2.2).

Remark 2.1. We can find from Theorem 2.2 and the equality (2.6)

\[
\sigma_k(A) = \frac{1}{(n - k)!} \frac{d^{n-k}}{dt^{n-k}} \det(A + t \cdot \text{Id}).
\]

Thus, \(\sigma_k(A)\) is also the sum of all \((k \times k)\) principle minors of \(A\).

Proposition 2.9. Assume \(\lambda(A) \in \Gamma_k\), then

\[
\begin{bmatrix} \sigma_k(A) \\ \sigma_l(A) \end{bmatrix} \quad \text{is concave.}
\]

Proof. Let \(\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_n(A)\) be the eigenvalues of \(A\). Then, we know from Theorem 2.2 that \(\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_n(A)\) are also the eigenvalues of \(A^R\) with multiplicity four. Applying the same argument for the real symmetric matrix \(A^R\) in Page 277 in [9], we find for \(1 \leq k \leq n\)

\[
\sum_{i=1}^{k} \lambda_i(A)
\]

is concave with respect with \(A\). Then, Proposition 2.5 implies the conclusion (see Page 277 in [9] for details). \(\square\)

Proposition 2.10. Assume \(\lambda(A) \in \Gamma_k\), then \(\lambda(A|i) \in \Gamma_{k-1}\) for any \(1 \leq i \leq n\), where \(A|i\) is the matrix obtained by deleting the \(i\)-row and \(i\)-column of the matrix \(A\).
Proof. Without loss of generality, we only prove $i = n$ and assume $A|n = \text{diag}\{\mu_1, \ldots, \mu_{n-1}\}$. Then, we write

\[(2.7) \quad A = \begin{pmatrix} A|n & \alpha \\ \alpha^t & a_{nn} \end{pmatrix},\]

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})^t$. Then, we get from Remark 2.1

\[\sigma_k(A) = \sigma_k(\mu) + a_{nn}\sigma_{k-1}(\mu) - \sum_{i=1}^{n-1} |\alpha_i|^2 \sigma_{k-2}(\mu|i),\]

where $\mu = (\mu_1, \ldots, \mu_{n-1})$, and the eigenvalues of $A$ satisfy

\[\left(\lambda - a_{nn} - \sum_{i=1}^{n-1} \frac{|\alpha_i|^2}{\lambda - \mu_i}\right) \prod_{i=1}^{n-1} (\lambda - \mu_i) = 0,\]

which implies $\frac{\partial \lambda_l}{\partial a_{nn}} = 0$ for $1 \leq l \leq n - 1$ and $\frac{\partial \lambda_l}{\partial a_{nn}} = \frac{a_{nn}}{1 + \sum_{i=1}^{n-1} \frac{|\alpha_i|^2}{(\lambda - \mu_i)^2}} > 0$.

Thus, we have from Proposition 2.3

\[\sigma_{k-1}(\mu) = \frac{\partial \sigma_k(A)}{\partial a_{nn}} = \frac{\partial \sigma_k(A)}{\partial \lambda_l} \frac{\partial \lambda_l}{\partial a_{nn}} > 0.\]

\[\square\]

**Proposition 2.11.** Assume $\lambda(A) \in \Gamma_k$ and $1 \leq l < k \leq n$, then for any $i \in \{1, 2, \ldots, n\}$ we have

\[(2.8) \quad \frac{\sigma_{k-1}(A|i)}{\sigma_{l-1}(A|i)} > \frac{\sigma_k(A)}{\sigma_l(A)}.\]

Proof. Without loss of generality, we only prove $i = n$ and assume $A|n = \text{diag}\{\mu_1, \ldots, \mu_{n-1}\}$. Then, we write

\[(2.9) \quad A = \begin{pmatrix} A|n & \alpha \\ \alpha^t & a_{nn} \end{pmatrix},\]

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})^t$. Then,

\[\sigma_k(A) = \sigma_k((\mu, a_{nn})) - \sum_{i=1}^{n-1} |\alpha_i|^2 \sigma_{k-2}(\mu|i),\]

where $\mu = (\mu_1, \ldots, \mu_{n-1})$. Applying Proposition 2.10 the inequalities (2.2) and (2.4), we get

\[\frac{\sigma_k(A)}{\sigma_l(A)} \leq \frac{\sigma_k((\mu, a_{nn}))}{\sigma_l((\mu, a_{nn}))}.\]

Thus, the inequality (2.4) follows from (2.1).

\[\square\]

**Proposition 2.12.** For $1 \leq k \leq n$, assume $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$ with $\lambda_1 \leq \ldots \leq \lambda_n$ and $\lambda(B) = (\mu_1, \ldots, \mu_n) \in \Gamma_k$ with $\mu_1 \leq \ldots \leq \mu_n$, then we have

\[(2.10) \quad \sum_{i=1}^{n} B_{ii} \sigma_{k-1}(\lambda|i) \geq \sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) > 0.\]
Proof. For any hyperhermitian matrix $B$, we can find $C \in GL(n, \mathbb{H})$ with $C^*C = Id$ such that

$$C^*BC = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_n\},$$

Then,

$$\text{diag}\{B_{11}, B_{22}, \ldots, B_{nn}\} = T \text{diag}\{\mu_1, \mu_2, \ldots, \mu_n\},$$

where $T = (|c_{ij}|^2)$. Then, the conclusion follows from the same argument Lemma 6.2 in [9]. □

2.4. Hessian quotient operators on HKT manifolds. Let $(M, I, J, K, g)$ be a HKT manifold and $\Omega$ be the associated HKT form of the hyperhermitian metric $g$. For each $z \in M$, there exists a unit orthonormal basis $e_1, e_1^J, \ldots, e_n, e_n^J$ of $T_z^{1,0}M$, i.e.

$$\Omega(e_i, e_j) = 0, \quad \Omega(e_i, e_i^J) = \delta_{ij}.$$ 

Let $\{e^i\}$ be the dual basis of $\{e_j\}$, i.e. $e^i(e_j) = \delta_{ij}$. Then,

$$\Omega = e^1 \wedge J^{-1}(e^1) + \ldots + e^n \wedge J^{-1}(e^n).$$

Given $W \in \Lambda^2_{I, \mathbb{R}}(M)$, we can decompose

$$W = \sum_{i,j=1}^n w_{ij} e^i \wedge J^{-1}(e^j),$$

where $(w_{ij})$ is a hyperhermitian $(n \times n)$-matrix, i.e. $w_{ij} = w_{ji}^\dagger$.

**Definition 2.5.** For $W \in \Lambda^2_{I, \mathbb{R}}(M)$, we define $\sigma_k(W)$ with respect to $\Omega$ as

$$\sigma_k(W) = \sigma_k(\lambda(w_{ij}^\dagger)),$$

where $\lambda(w_{ij}^\dagger)$ are eigenvalues of the hyperhermitian $(n \times n)$-matrix $(w_{ij}^\dagger)$. The definition of $\sigma_k(W)$ is independent of the choice of the unit orthonormal basis $e_1, e_1^J, \ldots, e_n, e_n^J$ of $TM$. In fact, $\sigma_k$ can be defined without the use of the unit orthonormal basis by (see Proposition 4.5 in [9])

$$\sigma_k(W) = C_n^k W^k \wedge \Omega^{n-k} \Omega^n,$$

where $C_n^k = \frac{n!}{(n-k)!k!}$. We also define the Gårding’s cone on $M$ by

$$\Gamma_k(M) = \{W \in \Lambda^2_{I, \mathbb{R}}(M) : \sigma_i(W) > 0, \ \forall \ 1 \leq i \leq k\}.$$

Thus, we can rewrite the equation (1.1) as

$$\sigma_k(\Omega u) = \widetilde{F} \sigma_i(\Omega u),$$

where $\widetilde{F} = \frac{c_n}{C_n^k} e^F$. Under the unit orthonormal basis $e_1, e_1^J, \ldots, e_n, e_n^J$ of $T_z^{1,0}M$, it is easy to see

$$W^i \wedge \Omega^{n-i} \wedge e^l \wedge J^{-1}(e^l)$$

(2.11)

$$= (i-1)! \sigma_{i-1}(W^l) \Omega^n,$$

(2.12)
where $W/l$ is the matrix obtained by deleting the $l$-row and $l$-column of the matrix $(w_{ij})$. Then, we can get a local version of the cone condition (1.3).

**Proposition 2.13.** The cone condition (1.3) is equivalent to

\[ \sigma_{k-1}(\Omega_0|j) - \tilde{F}\sigma_{l-1}(\Omega_0|j) > 0, \quad j = 1, 2, \ldots, n, \]

where $\tilde{F} = \frac{C_k}{C_n} e^F$.

**Proof.** The equality (2.13) follows directly from (2.12) if we observe that coefficient of $(n−1,0)$ form $\prod_{j=1, j \neq s}^n e^j \wedge J^{-1}(e^j)$ in $\Omega_0^{-1} \wedge \Omega^{n-i}$ is

\[ (i-1)!(n-i)!\sigma_{i-1}(\Omega_0|s) = \frac{1}{i} \frac{n!}{C_n^i} \sigma_{i-1}(\Omega_0|s). \]

□

3. C<sup>0</sup> Estimate

In the section, for convenience, we always assume that $(M^n, I, J, K, g)$ is a compact HKT manifold, and $\Omega$ is the associated HKT form of the hyperhermitian metric $g$. Moreover, $\Omega_0$ is a q-real $(2,0)$ form with $\Omega_0 \in \Gamma_k(M)$ and $\partial\partial^* \Omega_0 = 0$, and $u \in C^2(M)$ is a real function with $\Omega_u = \Omega_0 + \partial\partial^* J^u \in \Gamma_k(M)$.

3.1. Some lemmas. Since $\Omega_0 \in \Gamma_k(M)$, we may assume that there is a uniform constant $\tau > 0$ such that

\[ \Omega_0 - \epsilon \Omega \in \Gamma_k(M) \quad \text{and} \quad \Omega - \epsilon \Omega_0 \in \Gamma_k(M). \]

Then, We have the following pointwise inequalities.

**Lemma 3.1.** (1) For $0 \leq t \leq 1$ and $1 \leq i \leq k$, it holds

\[ \Omega_{tu}^{i-1} \wedge \omega^{n-i} \geq (1-t)^{i-1}e^{i-1}\Omega^{n-1}. \]

(2) For $0 < t \leq 1$ and $1 \leq i \leq k$, it holds

\[ \Omega_u^i \wedge \Omega^{n-i} \leq \frac{1}{t} \Omega_{tu}^i \wedge \Omega^{n-i}. \]

Moreover, if $u$ is a solution to the equation (1.1) and $\Omega_0$ satisfies the cone condition (1.3), then there exists a uniform constant $C > 0$ such that for $0 \leq t \leq 1$

\[ k\Omega_{tu}^{k-1} \wedge \Omega^{n-k} - le^F\Omega_{tu}^{l-1} \wedge \Omega^{n-l} > C(1-t)\Omega_{tu}^{l-1} \wedge \Omega^{n-l}, \]

and consequently

\[ k\Omega_{tu}^{k-1} \wedge \Omega^{n-k} - le^F\Omega_{tu}^{l-1} \wedge \Omega^{n-l} > C\epsilon^{l-1}(1-t)\Omega^{n-1}. \]
Proof. For $1 \leq i \leq k$ and $1 \leq j \leq n$, we can find from Propositions 2.9 and 2.10
\begin{equation}
\sigma_i^1(\Omega_tu) \geq (1-t)\sigma_i^1(\Omega_0) + t\sigma_i^1(\Omega_u),
\end{equation}
and
\begin{equation}
\sigma_{i-1}^1(\Omega_tu|j) \geq (1-t)\sigma_{i-1}^1(\Omega_0|j) + t\sigma_{i-1}^1(\Omega_u|j),
\end{equation}
which is just the local form of (3.2). Then, (3.4) follows by substituting (3.9) and (3.11).

Moreover, we have by Proposition 2.11
\begin{equation}
f(\Omega_tu|j) = (1-t)f(\Omega_0|j) + tf(\Omega_u|j).
\end{equation}

In addition, since $\Omega_0$ satisfies the cone condition (1.3), there exists some uniform constant $\delta > 0$ which is independent of $\alpha$ such that
\begin{equation}
f(\Omega_0|j) > \bar{F}(z) + \delta,
\end{equation}
where we write the cone condition in a local version as Proposition 2.13. Substituting (3.10) and (3.11) into (3.9), we get
\begin{equation}
\frac{\sigma_{k-1}(\Omega_tu|j)}{\sigma_{l-1}(\Omega_tu|j)} > (1-t)(\bar{F}(z) + \delta) + t\bar{F}(z),
\end{equation}
this is to say
\begin{equation}
\sigma_{k-1}(\Omega_tu|j) - \bar{F}\sigma_{l-1}(\Omega_tu|j) > \delta(1-t)\sigma_{l-1}(\Omega_tu|j),
\end{equation}
which is just the local form of (3.1). Then, (3.1) follows by substituting (3.12) into it. \hfill \square

**Lemma 3.2.** Let $\Omega_1$ be a strictly positive $(2,0)$ form, i.e. $\Omega_1(X, \overline{X}J) > 0$ for any non zero $(1,0)$ vector $X$, and $\Omega_2$ a $q$-real $(2,0)$ form on $M$. For each $z \in M$ there exists a basis $e_1, \overline{e_1}, ..., e_n, \overline{e_n}$ of $T_z^{1,0} M$ such that for $i \neq j$
\begin{equation}
\Omega_1(e_i, e_j) = \Omega_2(e_i, e_j) = \Omega_1(e_i, \overline{e_j}) = \Omega_2(e_i, \overline{e_j}) = 0.
\end{equation}
Using Lemma 3.2, we can get the following inequality which is a generalization of Lemma 2 in [24].

**Lemma 3.3.** Let $\alpha$ be some $(1,0)$ form and $W$ be a $q$-real $(2,0)$ form with $W \in \Gamma_1(M)$, there exists a positive constant $C$ depending on $\alpha$ such that

$$
\left| \frac{\partial_j u \wedge \alpha \wedge W^{i-1} \wedge \Omega^{n-i}}{\Omega^n} \right| \\
\leq \frac{C}{\delta} \frac{\partial u \wedge \partial_j u \wedge W^{i-1} \wedge \Omega^{n-i}}{\Omega^n} + C\delta \frac{W^{i-1} \wedge \Omega^{n-i+1}}{\Omega^n},
$$

for some $\delta > 0.$

**Proof.** Using Lemma 3.2 for any $z \in M$, we can choose a unit orthonormal basis $e_1, e_2, \ldots, e_n$ of $T_z^{1,0} M$ such that

$$
\Omega = e^1 \land J^{-1}(e^1) + \ldots + e^n \land J^{-1}(e^n)
$$

and

$$
W = w_1 e^1 \land J^{-1}(e^1) + \ldots + w_n e^n \land J^{-1}(e^n).
$$

Let us decompose

$$
\alpha = \sum_{j=1}^n a_{2j-1} e^j + a_{2j} J^{-1}(e^j), \quad \partial u = \sum_{j=1}^n u_{2j-1} e^j + u_{2j} J^{-1}(e^j).
$$

Thus,

$$
\partial_j u = J^{-1}(\partial u) = \sum_{j=1}^n \frac{u_{2j-1}}{\partial_j J^{-1}(e^j)} - \frac{u_{2j}}{\partial_j e^j}.
$$

One can easily check that

$$
W^{i-1} \wedge \Omega^{n-i+1} = \frac{1}{C_{n-1} \sigma_{i-1}(W) \Omega^n}
$$

and

$$
\partial u \wedge \partial_j u \wedge W^{i-1} \wedge \Omega^{n-i} = \frac{(i-1)! (n-i)!}{n!} \sum_{1 \leq j_1 < \ldots < j_i \leq n} \left( \sum_{l \notin \{j_1, \ldots, j_i\}} |u_{2l-1}|^2 + |u_{2l}|^2 \right) w_{j_1} \ldots w_{j_{i-1}} \Omega^n
$$

and

$$
\partial_j u \wedge \alpha \wedge W^{i-1} \wedge \Omega^{n-i} = \frac{(i-1)! (n-i)!}{n!} \sum_{1 \leq j_1 < \ldots < j_i \leq n} \left( \sum_{l \notin \{j_1, \ldots, j_i\}} -\frac{u_{2l-1} a_{2l-1} - u_{2l} a_{2l}}{\partial_j \Omega^n} \right) w_{j_1} \ldots w_{j_{i-1}} \Omega^n.
$$

Moreover,

$$
\partial_j u \wedge \alpha \wedge W^{i-1} \wedge \Omega^{n-i} = \frac{(i-1)! (n-i)!}{n!} \sum_{1 \leq j_1 < \ldots < j_i \leq n} \left( \sum_{l \notin \{j_1, \ldots, j_i\}} -\frac{u_{2l-1} a_{2l-1} - u_{2l} a_{2l}}{\partial_j \Omega^n} \right) \sigma_{i-1}(W|l) \Omega^n.
$$
Noting that \( \sigma_{i-1}(Wl) > 0 \) in view of \( W \in \Gamma_i(M) \), then the conclusion follows directly from Cauchy-Schwartz inequality. □

Next, we derive the following two important inequalities.

**Lemma 3.4.** Suppose that \( \Omega_0 \) satisfies (3.1), then we have the following inequalities for \( 1 \leq i < k \):

\[
(3.14) \quad \epsilon \int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt \leq \frac{k}{i} \int_0^a \Omega_{tu}^{k-1} \wedge \Omega^{n-k+1} dt,
\]

where \( a \) is an arbitrary positive constant and

\[
(3.15) \quad \epsilon \int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt \leq \frac{k}{i} \int_0^a \Omega_{tu}^{k-1} \wedge \Omega^{n-k} \wedge \Omega^n.
\]

**Proof.** Using integration by parts, (3.1) and the inequality (2.10)

\[
\int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt \\
\geq \epsilon \int_0^a \Omega_{tu}^{i-2} \wedge \Omega^{n-i+2} dt + \frac{1}{i - 1} \int_0^a \frac{t}{d} \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt \\
= \epsilon \int_0^a \Omega_{tu}^{i-2} \wedge \Omega^{n-i+2} dt + \frac{a}{i - 1} \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} - \frac{1}{i - 1} \int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt.
\]

Hence,

\[
\frac{i}{i - 1} \int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} dt \geq \epsilon \int_0^a \Omega_{tu}^{i-2} \wedge \Omega^{n-i+2} dt.
\]

Then, the first inequality follows by iteration. The proof of the second inequality is similar to that of the first one. Using integration by parts, (3.1) and the inequality (2.10), it yields

\[
\int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i} \wedge \Omega^n \\
\geq \epsilon \int_0^a \Omega_{tu}^{i-2} \wedge \Omega^{n-i+1} \wedge \Omega^n \\
+ \frac{1}{i - 1} \int_0^a \frac{t}{d} \Omega_{tu}^{i-1} \wedge \Omega^{n-i+1} \wedge \Omega^n \\
\geq \epsilon \int_0^a \Omega_{tu}^{i-2} \wedge \Omega^{n-i+1} \wedge \Omega^n \\
- \frac{1}{i - 1} \int_0^a \Omega_{tu}^{i-1} \wedge \Omega^{n-i} \wedge \Omega^n.
\]

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Thus,
\[
\frac{i}{i-1} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{i-1} \wedge \Omega^{n-i} \wedge \Omega^n
\geq e \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{i-2} \wedge \Omega^{n-i+1} \wedge \Omega^n.
\]
So, we obtain by iteration
\[
\frac{k}{i} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{k-1} \wedge \Omega^{n-k} \wedge \Omega^n
\geq e^{k-i} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{k-1} \wedge \omega^{n-i} \wedge \Omega^n,
\]
which completes the proof.

**Lemma 3.5.** Suppose that \( \Omega_0 \) satisfies (3.1), then we have the following inequality:
\[
\int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{k-1} \wedge \Omega^{n-k+1} \wedge \Omega^n \leq C p \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{k-1} \wedge \Omega^{n-k} \wedge \Omega^n
\]
(3.16) \[+ C \int_M e^{-pu} \Omega^n \wedge \Omega^n.\]

**Proof.** According to the inequality (3.15), we have
\[
\int_0^1 dt \int_M e^{-pu} \Omega_{tu}^{k-1} \wedge \Omega^{n-k+1} \wedge \Omega^n
= p(k-1) \int_0^1 dt \int_0^t ds \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{st}^{k-2} \wedge \Omega^{n-k+1} \wedge \Omega^n
- (k-1) \int_0^1 dt \int_0^t ds \int_M e^{-pu} \partial J u \wedge \Omega_{st}^{k-2} \wedge \Omega^{n-k+1} \wedge \partial \Omega^n
+ \frac{1}{2} \int_M e^{-pu} \Omega_{tu}^{k-1} \wedge \Omega^{n-k+1} \wedge \Omega^n
\leq \frac{p(k-1)}{2} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega_{tu}^{k-2} \wedge \Omega^{n-k+1} \wedge \Omega^n
+ (k-1) \int_0^1 dt \int_M e^{-pu} \partial J u \wedge \Omega_{tu}^{k-2} \wedge \Omega^{n-k+1} \wedge \partial \Omega^n
+ C \int_0^1 dt \int_M e^{-pu} \Omega^n \wedge \Omega^n,
\]
(3.17) \[+ C \int_M e^{-pu} \Omega^n \wedge \Omega^n.\]
Denote by $\partial(\Omega^n) = \beta \wedge \overline{\Omega^n}$ for some $(1,0)$ form $\beta$. Applying the inequality (3.13), we find

$$\left| \frac{\partial J u \wedge \beta \wedge \Omega^{k-2} \wedge \Omega^{n-k+1}}{\Omega^n} \right| \leq \frac{p(k-1)}{2} \frac{\partial u \wedge \partial J u \wedge \Omega^{k-2} \wedge \Omega^{n-k+1}}{\Omega^n} + \frac{C \Omega^{k-2} \wedge \Omega^{n-k+2}}{p \Omega^n}. \tag{3.18}$$

Substituting (3.18) into (3.19) and applying (3.15) give

$$\int_0^{\frac{1}{2}} dt \int_M e^{-p\partial u} \wedge \partial J u \wedge \Omega^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \leq pC \int_0^{\frac{1}{2}} dt \int_M e^{-p\partial u} \wedge \partial J u \wedge \Omega^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \tag{3.19}$$

where we use the inequality (3.14) for $a = \frac{1}{2}$ to get the last inequality. Choosing $p$ sufficiently large, we find

$$\int_0^{\frac{1}{2}} dt \int_M e^{-p\partial u} \wedge \partial J u \wedge \Omega^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \leq C p \int_0^{\frac{1}{2}} dt \int_M e^{-p\partial u} \wedge \partial J u \wedge \Omega^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \tag{\text{□}}$$

3.2. The proof of $C^0$ estimate. According to the argument of Page 13-14 in [24], in order to prove Theorem 1.1, it suffices to show the following Cherrier type inequality.

**Lemma 3.6.** Under the assumptions in Theorem 1.1, we have for $p$ large enough

$$\int_M |\partial e^{-\frac{u}{p}}|^2 (\Omega \wedge \overline{\Omega})^n \leq C p \int_M e^{-pu} (\Omega \wedge \overline{\Omega})^n.$$
Proof. Since

\begin{align*}
& (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \Omega^{n-k}) - e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \\
& = \int_0^1 \partial J_0 u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - le^F \Omega^l_{tu} \wedge \Omega^{n-l}) dt,
\end{align*}

we have

\begin{align*}
(3.20) \quad & \int_M e^{-pu} \left[ (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \omega^{n-k}) \\
& - e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \right] \wedge \Omega^n \\
& = p \int_0^1 dt \int_M e^{-pu} \partial J_0 u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l}) \wedge \Omega^n \\
& + l \int_0^1 dt \int_M e^{-pu} \partial e^F \wedge \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \Omega^n \\
& - \int_0^1 dt \int_M e^{-pu} \partial J_0 u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l}) \wedge \Omega^n.
\end{align*}

Denote by $\partial \Omega^n = \beta \wedge \Omega^n$ for some $(1,0)$ form $\beta$. Applying the inequality \[3.13\], we find

\begin{align*}
(3.21) \quad & \left| \frac{\partial J_0 u \wedge \beta \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l})}{\Omega^n} \right| \\
& \leq \frac{p}{2} \left| \partial u \wedge \partial J_0 u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l}) \right| \\
& \quad + \frac{C \Omega^k_{tu} \wedge \Omega^{n-k+1} + \Omega^l_{tu} \wedge \Omega^{n-l+1}}{\Omega^n}. \\
\end{align*}

and

\begin{align*}
(3.22) \quad & \left| \frac{\partial J_0 u \wedge \partial e^F \wedge \Omega^l_{tu} \wedge \Omega^{n-l+1}}{\Omega^n} \right| \\
& \leq \frac{C}{\delta} \left| \partial u \wedge \partial J_0 u \wedge \Omega^l_{tu} \wedge \Omega^{n-l+1} \right| + C \delta \frac{\Omega^l_{tu} \wedge \Omega^{n-l+2}}{\Omega^n}.
\end{align*}
Then, plugging (3.21) and (3.22) into (3.20), and using (3.14), there is

\[
\int_M e^{-pu} \left[ (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \omega^{n-k}) \\
- e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \right] \wedge \Omega^n
\]

\[
\geq \frac{p}{2} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \left( k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l} \right) \wedge \Omega^n
\]

\[
- \frac{C}{\delta} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \Omega^n
\]

(3.23) \(- (\frac{1}{p} + \delta) C \int_0^1 dt \int_M e^{-pu} \Omega^k_{tu} \wedge \Omega^{n-k+1} \wedge \Omega^n,
\]

Furthermore, applying the inequalities (3.7) and (3.15), we find

\[
\int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \Omega^n
\]

\[
\leq 2^{l-1} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \Omega^n
\]

\[
\leq 2^{l-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \Omega^n
\]

(3.24) \leq C \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^k_{tu} \wedge \Omega^{n-k} \wedge \Omega^n.
\]

and

\[
\int_0^1 dt \int_M e^{-pu} \Omega^k_{tu} \wedge \Omega^{n-k+1} \wedge \Omega^n
\]

\[
\leq 2^{k-1} \int_0^1 dt \int_M e^{-pu} \Omega^k_{tu} \wedge \Omega^{n-k+1} \wedge \Omega^n
\]

\[
\leq 2^{k-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \Omega^k_{tu} \wedge \Omega^{n-k+1} \wedge \Omega^n
\]

\[
\leq C \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^k_{tu} \wedge \Omega^{n-k} \wedge \Omega^n
\]

(3.25) + C \int_M e^{-pu} \Omega^n \wedge \Omega^n.
Taking the inequalities (3.24) and (3.25) into (3.23), we have

\[
\int_M e^{-pu} \left[ (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \omega^{n-k}) - e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \right] \wedge \overline{\Omega^n} \geq p^2 \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l} - e^F (\Omega^l_0 \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \wedge \overline{\Omega^n} \geq (\frac{1}{\delta} + 1 + p\delta)C \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \overline{\Omega^n} \geq \frac{1}{p} \delta C \int_M e^{-pu} \Omega^n \wedge \overline{\Omega^n}.
\]

To cancel the second term on the right side of (3.26), we will use a part of the first term on the right side of (3.26). In details, we can get the following positive term for \(0 \leq t \leq \frac{1}{2}\) from the inequality (3.4)

\[
pe^{-pu} \partial u \wedge \partial J u \wedge \left( k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l} \right) \wedge \overline{\Omega^n} \geq Cpe^{-pu} \partial u \wedge \partial J u \wedge \Omega^k_{tu} \wedge \Omega^{n-k}.
\]

Thus, we first choose \(\delta\) sufficiently small, and then choose \(p\) sufficiently large, the integral of the term (3.27) on \(M\) can kill the second term on the right side of (3.26). Then, (3.26) becomes

\[
\int_M e^{-pu} \left[ (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \omega^{n-k}) - e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \right] \wedge \overline{\Omega^n} \geq \frac{p}{4} \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge (k \Omega^k_{tu} \wedge \Omega^{n-k} - e^F \Omega^l_{tu} \wedge \Omega^{n-l} \wedge \overline{\Omega^n} \geq C \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^{n-1} \wedge \overline{\Omega^n} \geq \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^{n-1} \wedge \overline{\Omega^n} \geq \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^{n-1} \wedge \overline{\Omega^n} \geq C \int_0^1 dt \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^{n-1} \wedge \overline{\Omega^n} \geq C \int_M e^{-pu} \Omega^n \wedge \overline{\Omega^n}.
\]

where we use (3.5) to get the last inequality. Notice that in view of the equation (1.1)

\[
\int_M e^{-pu} \left[ (\Omega^k_u \wedge \Omega^{n-k} - \Omega^k_0 \wedge \omega^{n-k}) - e^F (\Omega^l_u \wedge \Omega^{n-l} - \Omega^l_0 \wedge \Omega^{n-l}) \right] \wedge \overline{\Omega^n} = \int_M e^{-pu} \left( - \Omega^k_0 \wedge \Omega^{n-k} + e^F \Omega^l_0 \wedge \Omega^{n-l} \right) \wedge \overline{\Omega^n} \leq C \int_M e^{-pu} \Omega^n \wedge \overline{\Omega^n}.
\]

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Thus,
\[ C \int_M e^{-pu} \Omega^n \wedge \overline{\Omega}^m \geq p \int_M e^{-pu} \partial u \wedge \partial J u \wedge \Omega^{n-1} \wedge \overline{\Omega}^m. \]
So, our proof is completed. \( \square \)

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