Maximal Spread of Coherent Distributions: a Geometric and Combinatorial Perspective

Stanisław Cichomski

Faculty of Mathematics, Informatics and Mechanics,
University of Warsaw

December 8, 2020

Abstract

We discuss some open problems concerning the maximal spread of coherent distributions. We prove a sharp bound on $\mathbb{E}|X - Y|^\alpha$ for $(X, Y)$ coherent and $\alpha \leq 2$, and establish a novel connection between coherent distributions and such combinatorial objects as bipartite graphs, conjugate partitions and Ferrer diagrams. Our results may turn out to be helpful not only for probabilists, but also for graph theorists, especially for those interested in mathematical chemistry and the study of topological indices.

1 Introduction

1.1 Background and contributions

How radically different and contradictory can opinions, stated by two experts or specialists be, while based on distinct sources of information? This question, as in [4], can be formalised using the notion of conditional probability. Firstly, both experts must agree upon a basic model of reality, which can be understood as accepting common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Inconsistent sources of information shall then be identified with different sub $\sigma$-fields $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$. Consequently, opinions involved with judging odds of an event $A \in \mathcal{F}$, will be expressed as random variables $X, Y$, defined by

$$X = \mathbb{P}(A|\mathcal{G}),$$

$$Y = \mathbb{P}(A|\mathcal{H}).$$

Based on [1], we shall refer to such $(X, Y)$ random vectors as coherent, or alternatively, we might occasionally say that their joint distribution on $[0, 1]^2$ is coherent. However, this ambiguity will not lead to any
misunderstanding. Note that the characterisation of coherence presented, admits a straightforward extension
to vectors of length \( n > 2 \). For notational convenience, hereinafter, we write
\[
(X, Y) \in C,
\]
or
\[
(X_1, X_2, \cdots, X_n) \in C,
\]
whenever we want to indicate, that the vector \((X, Y)\) or \((X_1, X_2, \cdots, X_n)\) is coherent. Thus, answering
initial question, concerning the maximal spread of coherent opinions, is equivalent to finding (or at least
bounding) such quantities, as
\[
\sup_{(X,Y) \in C} E|X - Y|^r,
\]
for \( r \in \mathbb{R}_+ \), or the strongly related quantity
\[
\sup_{(X,Y) \in C} P(|X - Y| > \delta),
\]
for \( \delta \in [0, 1] \), with further variants assuming independence, fixing \( P(A) \), e.t.c.. Let us highlight, that this
formalism should be regarded as taking supremum over all probability spaces \((\Omega, \mathcal{F}, P)\), all events \( A \in \mathcal{F} \)
and all sub \( \sigma \)-fields \( \mathcal{G}, \mathcal{H} \subseteq \mathcal{F} \). It may seem surprising, but despite the fundamental nature of these problems,
they have not been studied in depth, at least until lately. In [5], expressed differently, it was proved that

**Theorem 1.1.** For all \( n \in \mathbb{Z}_+ \), and any \((X_1, X_2, \cdots, X_n) \in C\) with \( EX_1 = p \), we have
\[
E \max_{1 \leq i \leq n} X_i \leq \frac{p(n - p)}{1 + p(n - 2)},
\]
but only recently, in [1], the following result was established.

**Theorem 1.2.**
\[
\sup_{(X,Y) \in C} E|X - Y| = \frac{1}{2}.
\]

**Proof:** Fix any \((X,Y) \in C\) and let \( p = E\, X \). We use the identity
\[
|X - Y| = 2 \cdot \max(X,Y) - X - Y.
\]
Thus, by Theorem 1.1 with \( n = 2 \), we have
\[
E|X - Y| \leq 2p(2 - p) - 2p = 2p(1 - p) \leq \frac{1}{2}.
\]
To attain the equality, consider \( X' = 1_A \) and \( Y' = E\, 1_A \) for arbitrary \( A \in \mathcal{F} \), with \( P(A) = \frac{1}{2} \). Then
\[
E|X' - Y'| = E\left|1_A - \frac{1}{2}\right| = \frac{1}{2}. \quad \square
\]
It is however doubtful, whether this line of reasoning could be pushed further in order to find

$$\sup_{(X,Y) \in C} \mathbb{E}|X - Y|^r,$$

for $r > 1$. In fact, one of the main contributions of this thesis is establishing that

$$\sup_{(X,Y) \in C} \mathbb{E}|X - Y|^\alpha = 2^{-\alpha},$$

for all $\alpha \in [0, 2]$, which is achieved using only the $L^2$-norm and an elementary geometric framework. Based on this premise, one might suspect that $2^{-r}$ must turn out to be a true bound in a general setting. Unfortunately, it is quite easy to construct counterexamples to this hypothesis for $r > 3$. It seems clear, that progression on this problem for higher exponents, will be associated with establishing some new perspective on theory of coherent opinions. In fact, there are known alternative characterisations of coherent distributions, some of which we shall recall in the next section. Having said that, let us quote [11] on:

For reasons we do not understand well, these general characterisations seem to be of little help in establishing the evaluations of $\epsilon(\delta)$ [i.e. $\mathbb{P}(|X - Y| > 1 - \delta)$] discussed above, or in settling a number of related problems about coherent distributions [...].

It is our belief, that this is indeed so, because of underlying combinatorial nature of those problems. Let us define

$$C_I = \{(X, Y) : X, Y \in C, \ X \perp Y\},$$

as a family of those coherent distributions, which are additionally independent. Our second important result shows, that for all $k \in \mathbb{Z}_+, \ k \geq 2$, we have

$$\sup_{(X,Y) \in C_I} \mathbb{E}|X - Y|^k = \sup_{n \in \mathbb{Z}_+} \sup_{B(n,n)} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^k,$$

where $B(n,n)$ stands for the set of all bipartite graphs with two $n$ element groups of vertices, i.e.

$$V = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\},$$

and $\deg(v)$ is a degree of vertex $v$. This connection may turn out to be helpful not only for probabilists, but also for graph theorists, especially for those interested in mathematical chemistry and the study of topological indices. For example, based on [10], for a simple graph $G$ (i.e. without loops or multiple edges), one defines its total irregularity measure by

$$\text{irr}_{k,t}(G) = \frac{1}{2} \sum_{u,v \in V(G)} |\deg(u) - \deg(v)|^k.$$

Thus, finding supremum of $\text{irr}_{k,t}$ over all graphs with prescribed number of vertices, seems to be a highly related problem. To the best of our knowledge, it had not yet arisen broad interest, with exception of
$k = 1, 2$. Note that mentioned graph functionals depend on the choice of particular graph only through the degree sequence. In this context, we turn our attention to the study of coherent distributions related to conjugate partitions of integers and Ferrer diagrams. This should not come as a surprise, since the relation between degree sequences and conjugate partitions is well understood; for a comprehensive overview of the topic, see [9]. Prior to giving formal definitions, let us only mention, that Ferrer diagrams, nowadays attracting growing attention, provide a useful, graphical representations of conjugate partitions. As a slight generalisation of Ferrer diagrams, we define

$$ F_f = \{(u, v) \in [0, 1]^2 : v < f(u)\}, $$

where

$$ f : [0, 1] \to [0, 1], $$

is any weakly decreasing step function that takes finitely many different values. We also denote the set of such step functions as \(\text{STEP}\). Subsequently, for any diagram \(F_f\), we define a corresponding distribution \((X_f, Y_f) \in C_I\), while ensuring that

$$ \sup_{f \in \text{STEP}} \mathbb{E}|X_f - Y_f|^k = \sup_{(X,Y) \in C_I} \mathbb{E}|X - Y|^k, $$

for all \(k \in \mathbb{Z}_+\), \(k \geq 3\). Our third and last contribution, is a novel application of those combinatorial ideas to demonstrate that

$$ \sup_{f \in \text{STEP}} \mathbb{P}(|X_f - Y_f| > \delta) = 2\delta(1 - \delta), $$

for \(\delta \in (\frac{1}{2}, 1]\). This, at least partially, answers the question raised by Burdzy and Pitman in [1], were they have formulated the following conjecture.

**Conjecture 1.1.** For \(\delta \in (\frac{1}{2}, 1]\), we have

$$ \sup_{(X,Y) \in C_I} \mathbb{P}(|X - Y| \geq \delta) = 2\delta(1 - \delta). $$

As a direct consequence, we also obtain a new upper bound, namely

$$ \sup_{(X,Y) \in C_I} \mathbb{E}|X - Y|^k \leq 2 \cdot \frac{k}{(k+1)(k+2)} + 2^{-k} - 2^{-k-1} \cdot \frac{k(k+3)}{(k+1)(k+2)}. $$

### 1.2 Alternative characterisations

In this section, we provide a short collection of alternative characterisations of coherent distributions, although these are not referred to elsewhere in the paper. All of them can be found in [1], and we refer the interested reader to this excellent resource.

**Proposition 1.1.** Let \(X, Y\) be random variables defined on a probability space \((\Omega, \mathcal{F}, \mathcal{P})\), on which one can also define

$$ U \sim \mathcal{U}[0, 1], \quad U \perp (X, Y). $$

Then the following conditions are all equivalent:

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1. \((X, Y) \in \mathcal{C}\)

2. \(0 \leq X, Y \leq 1\), and for some \(A \in \mathcal{F}\) we have

\[
X = \mathbb{P}(A|X), \\
Y = \mathbb{P}(A|Y).
\]

3. one can define random variable \(Z\), with \(0 \leq Z \leq 1\), such that

\[
\mathbb{E}[Zg(X)] = \mathbb{E}[Xg(X)], \\
\mathbb{E}[Zg(Y)] = \mathbb{E}[Yg(Y)],
\]

for all bounded, measurable functions \(g\) with domain \([0, 1]\).

4. there exists a measurable function \(\phi: [0, 1]^2 \to [0, 1]\) such that

\[
\mathbb{E}[\phi(X,Y)g(X)] = \mathbb{E}[Xg(X)], \\
\mathbb{E}[\phi(X,Y)g(Y)] = \mathbb{E}[Yg(Y)],
\]

for all bounded, measurable functions \(g\) with domain \([0, 1]\).

2 Reduction to bipartite graphs

In this chapter we will restate the problem of finding

\[
\sup_{(X,Y) \in \mathcal{C}} \mathbb{E}|X - Y|^2
\]

in the language of bipartite graphs. Making use of a graph-theoretic topological index - namely the first Zagreb index \(M_1(G)\) - we will establish that the solution to the reformulated problem is \(\frac{1}{4}\).

2.1 Reformulation of the problem

We start with the definition of independent \(\sigma\)-fields.

**Definition 2.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \(\mathcal{F}_1\) and \(\mathcal{F}_2\) be two sub \(\sigma\)-fields of \(\mathcal{F}\). Then \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are said to be independent if for any events \(F_1 \in \mathcal{F}_1\) and \(F_2 \in \mathcal{F}_2\):

\[
\mathbb{P}(F_1 \cap F_2) = \mathbb{P}(F_1) \cdot \mathbb{P}(F_2).
\]

The following proposition allows us to perform the discretization.
Proposition 2.1. For all \( n \in \mathbb{Z}_+ \) and any \((X, Y) \in C\), there exists \((X_n, Y_n) \in C\), such that \( X_n \) and \( Y_n \) both take at most \( n \) different values, and
\[
|X - X_n| \leq \frac{1}{n},
\]
\[
|Y - Y_n| \leq \frac{1}{n}.
\]
Moreover, if \( X \perp Y \), then we may always choose \( X_n \perp Y_n \).

Proof: Fix any \((X, Y) \in C\) and assume that it is is defined on the probability space \((\Omega, \mathcal{H}, P)\). Let \( F \) and \( G \) be two sub-\( \sigma \)-fields of \( \mathcal{H} \), such that
\[
X = P(A|F) = \mathbb{E}(1_A|F),
\]
\[
Y = P(A|G) = \mathbb{E}(1_A|G),
\]
for some measurable set \( A \in \mathcal{H} \). Then if \( \sigma_X \) is a \( \sigma \)-field generated by \( X \) and \( \sigma_Y \) is a \( \sigma \)-field generated by \( Y \), we have
\[
\sigma_X \subset F, \quad \sigma_Y \subset G.
\]
By the tower property we have
\[
X = \mathbb{E}(X|\sigma_X) = \mathbb{E}\left(\mathbb{E}(1_A|F)|\sigma_X\right) = \mathbb{E}(1_A|\sigma_X),
\]
and similarly
\[
Y = \mathbb{E}(1_A|\sigma_Y).
\]
Let \( \sigma_X^n \) be the \( \sigma \)-field generated by
\[
P_X^n := \left\{ \left\{ X \in \left[0, \frac{1}{n}\right] \right\}, \left\{ X \in \left(\frac{1}{n}, \frac{2}{n}\right] \right\}, ..., \left\{ X \in \left(\frac{n-1}{n}, 1\right] \right\} \right\}.
\]
Set
\[
X_n = \mathbb{E}(X|\sigma_X^n) = \mathbb{E}\left(\mathbb{E}(1_A|\sigma_X)|\sigma_X^n\right) = \mathbb{E}(1_A|\sigma_X^n),
\]
and similarly
\[
Y_n = \mathbb{E}(1_A|\sigma_Y^n),
\]
where the last equality in the first line follows from the tower property and the fact that \( \sigma_X^n \subset \sigma_X \). Firstly note that, from the above we have \((X_n, Y_n) \in C\). Secondly, since \( P_X^n \) is \( n \)-element disjoint partition of \( \Omega \), \( X_n \) can take at most \( n \) different values. Thirdly, by elementary considerations we get \( |X - X_n| \leq \frac{1}{n} \). Finally, independence of \( X_n \) and \( Y_n \) corresponds to independence of \( \sigma_X^n \) and \( \sigma_Y^n \). But \( \sigma_X^n \subset \sigma_X, \sigma_Y^n \subset \sigma_Y \) imply \( \sigma_X^n \perp \sigma_Y^n \) whenever \( \sigma_X \perp \sigma_Y \). \( \square \)

Definition 2.2. Let \( C(n) \) be the set of \((X, Y) \in C\), such that \( X \) takes at most \( n \) different values, and \( Y \) takes at most \( n \) different values.
**Definition 2.3.** Let $C_{I}(n)$ be the set of $(X, Y) \in C_{I}$, such that $X$ takes at most $n$ different values, and $Y$ takes at most $n$ different values.

**Proposition 2.2.** We have

$$\sup_{(X,Y)\in C} \mathbb{E}|X-Y|^2 = \sup_{n\in \mathbb{Z}_+} \sup_{(X,Y)\in C(n)} \mathbb{E}|X-Y|^2.$$

**Proof:** For given $(X, Y)$ and $n \in \mathbb{Z}_+$ choose $(X_n, Y_n)$ as in previous proposition. Note that

$$\mathbb{E}|X-Y|^2 = \mathbb{E}|X-X_n + X_n - Y_n + Y_n - Y|^2 \leq \mathbb{E}\left( |X-X_n| + |X_n - Y_n| + |Y_n - Y| \right)^2$$

$$= \mathbb{E}\left( |X-X_n| + |Y-Y_n| \right)^2 + 2|X_n - Y_n|\left( |X-X_n| + |Y-Y_n| \right) + |X_n - Y_n|^2$$

$$\leq \frac{4}{n^2} + \frac{4}{n}\mathbb{E}|X_n - Y_n| + \mathbb{E}|X_n - Y_n|^2 \leq \frac{4}{n^2} + \frac{4}{n} + \mathbb{E}|X_n - Y_n|^2.$$  

We can now write

$$\mathbb{E}|X-Y|^2 \leq \limsup_{n\to\infty} \left( \frac{4}{n^2} + \frac{4}{n} + \mathbb{E}|X_n - Y_n|^2 \right)$$

$$\leq \limsup_{n\to\infty} \left( \frac{4}{n^2} + \frac{4}{n} \right) + \limsup_{n\to\infty} \mathbb{E}|X_n - Y_n|^2,$$

so $\mathbb{E}|X-Y|^2 \leq \limsup_{n\to\infty} \mathbb{E}|X_n - Y_n|^2$, and as a result

$$\sup_{(X,Y)\in C} \mathbb{E}|X-Y|^2 \leq \sup_{n\in \mathbb{Z}_+} \sup_{(X,Y)\in C(n)} \mathbb{E}|X-Y|^2.$$

The inequality in the other direction is clear. □

Repeating the same reasoning with the restriction of independence, gives the following result.

**Corollary 2.1.** We have

$$\sup_{(X,Y)\in C_{I}} \mathbb{E}|X-Y|^2 = \sup_{n\in \mathbb{Z}_+} \sup_{(X,Y)\in C_{I}(n)} \mathbb{E}|X-Y|^2.$$

**Proposition 2.3.** For every $n \in \mathbb{Z}_+$ we have

$$\sup_{(X,Y)\in C(n)} \mathbb{E}|X-Y|^2 = \sup_{A,B} \sum_{b_{ij} \neq 0} b_{ij} \left| \sum_{j} a_{ij} - \sum_{j} b_{ij} \right|^2,$$

where the supremum is taken over all $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$, such that

$$\forall ij \ 0 \leq a_{ij} \leq b_{ij}, \quad \sum_{ij} b_{ij} = 1. \quad (1)$$
Proof: Fix $(X, Y) \in \mathcal{C}(n)$, and for the time being assume that $X$ and $Y$ both take exactly $n$ different values, namely

$$X(\Omega) = \{x_1, x_2, \ldots, x_n\}, \quad Y(\Omega) = \{y_1, y_2, \ldots, y_n\}.$$  

Now, we can write

$$\sigma_X = \sigma\left(\{X^{-1}(x_1), X^{-1}(x_2), \ldots, X^{-1}(x_n)\}\right),$$

$$\sigma_Y = \sigma\left(\{Y^{-1}(y_1), Y^{-1}(y_2), \ldots, Y^{-1}(y_n)\}\right),$$

which means that $\sigma$-fields generated by $X$ and $Y$ are also generated by given two, disjoint partitions. Hence, for all $1 \leq i, j \leq n$ and $\omega \in X^{-1}(x_i) \cap Y^{-1}(y_j)$, we have

$$X(\omega) = E\left(\mathbb{1}_A | \sigma_X\right)(\omega) = \frac{\mathbb{P}(A \cap \{X = x_i\})}{\mathbb{P}(X = x_i)},$$

$$Y(\omega) = E\left(\mathbb{1}_A | \sigma_Y\right)(\omega) = \frac{\mathbb{P}(A \cap \{Y = y_j\})}{\mathbb{P}(Y = y_j)}.$$  

Thus, setting

$$\forall_{ij} \quad a_{ij} = \mathbb{P}\left(A \cap \{X = x_i\} \cap \{Y = y_j\}\right),$$

$$\forall_{ij} \quad b_{ij} = \mathbb{P}\left(\{X = x_i\} \cap \{Y = y_j\}\right),$$

gives

$$E|X - Y|^2 = \sum_{b_{ij} \neq 0} b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2.$$  

Furthermore, for any $(X, Y) \in \mathcal{C}(n)$ with $X$ or $Y$ taking less than $n$ different values, we can always begin by setting redundant rows or columns of $A$, $B$ to zero, and assigning the others as described above. In that way, we have just shown that

$$\sup_{(X, Y) \in \mathcal{C}(n)} E|X - Y|^2 \leq \sup_{A, B} \sum_{b_{ij} \neq 0} b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2.$$  

To prove the opposite inequality, we start by fixing $A, B$ such that (1) holds. We will give an explicit construction of $(X', Y') \in \mathcal{C}(n)$, such that

$$E|X' - Y'|^2 = \sum_{b_{ij} \neq 0} b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2,$$

defined on the probability space $([0, 1], \mathcal{L}, \lambda)$, where $\lambda$ is the lebesgue measure on $[0, 1]$ and $\mathcal{L}$ is the $\sigma$-field of $\lambda$-mesurable subsets of $[0, 1]$. Start by dividing $[0, 1]$ into a family of disjoint intervals $\{I_{ij}\}_{1 \leq i, j \leq n}$, such that

$$\forall_{1 \leq i, j \leq n} \quad \lambda(I_{ij}) = b_{ij}.$$  

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For every $1 \leq i, j \leq n$, chose a subinterval $A_{ij} \subset I_{ij}$, such that 
\[
\forall 1 \leq i,j \leq n \quad \lambda(A_{ij}) = a_{ij}.
\]
Construction of $\{I_{ij}\}_{1 \leq i,j \leq n}$ and $\{A_{ij}\}_{1 \leq i,j \leq n}$ is clearly possible by the (1) condition. Set
\[
A = \bigcup_{1 \leq i,j \leq n} A_{ij},
\]
\[
\forall 1 \leq i \leq n \quad G_i = \bigcup_{1 \leq j \leq n} I_{ij},
\]
\[
\forall 1 \leq j \leq n \quad H_j = \bigcup_{1 \leq i \leq n} I_{ij}.
\]
Thus $(G_i)_{1 \leq i \leq n}$ and $(H_j)_{1 \leq j \leq n}$ are disjoint partitions of $[0, 1]$, satisfying
\[
\forall ij \quad a_{ij} = P(A \cap G_i \cap H_j),
\]
\[
\forall ij \quad b_{ij} = P(G_i \cap H_j).
\]
In this setup, for
\[
X' = E[1_A | \sigma((G_i)_i)],
\]
\[
Y' = E[1_A | \sigma((H_j)_j)],
\]
we get
\[
E|X' - Y'|^2 = \sum_{b_{ij} \neq 0} b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2,
\]
which completes the proof. □

Repeating similar reasoning with the modification of independence, leads to

**Corollary 2.2.** We have
\[
\sup_{(X,Y)\in C_2(n)} E|X - Y|^2 = \sup_{A,B} \sum_{b_{ij} \neq 0} b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2,
\]
where supremum is taken over all $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$, such that
\[
\forall ij \quad 0 \leq a_{ij} \leq b_{ij},
\]
for which there exists $R = (r_i), C = (c_j) \in \mathbb{R}^n$, satisfying
\[
\forall i \quad 0 \leq r_i, \quad \sum_i r_i = 1, \tag{2}
\]
\[
\forall j \quad 0 \leq c_j, \quad \sum_j c_j = 1,
\]
\[
B = RC^T.
\]
Definition 2.4. Let us define $\Phi_n : [0, 1]^{n \times n} \times [0, 1]^{n \times n} \rightarrow \mathbb{R}$ as

$$\Phi_n(A, B) = \sum_{ij} \mathbb{1}(b_{ij} \neq 0) \cdot b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2,$$

and let $\mathcal{S}_n \subset [0, 1]^{n \times n} \times [0, 1]^{n \times n}$ denote the set of pairs $(A, B)$ described in Corollary 2.2, i.e. satisfying conditions (1) and (2).

Proposition 2.4. The set $\mathcal{S}_n$ is compact. The function $\Phi_n$ is continuous on $\mathcal{S}_n$.

Proof: It is straightforward to see that $\mathcal{S}_n$ is closed and bounded. To check continuity of $\Phi_n$ it will be enough to verify that

$$\forall_{ij} \quad \phi^{ij}_n(A, B) := \mathbb{1}(b_{ij} \neq 0) \cdot b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2,$$

is continuous on $\mathcal{S}_n$. It is clear, that $\phi^{ij}_n$ is continuous at any $(A, B) = (a_{kl}, b_{kl})_{kl}$ with $b_{ij} \neq 0$. Therefore, let us consider

$$(A^{(m)}, B^{(m)}) = (a^{(m)}_{kl}, b^{(m)}_{kl})_{kl} \xrightarrow{m \rightarrow \infty} (a_{kl}, b_{kl})_{kl},$$

with $b_{ij} = 0$. For $m$ satisfying $b_{ij}^{(m)} = 0$, we have $\phi^{ij}_n(A^{(m)}, B^{(m)}) = \phi^{ij}_n(A, B) = 0$. On the other hand, if $b_{ij}^{(m)} \neq 0$, then the given fractions are well defined, and

$$b_{ij}^{(m)} \cdot \left| \frac{\sum_j a_{ij}^{(m)}}{\sum_j b_{ij}^{(m)}} - \frac{\sum_i a_{ij}^{(m)}}{\sum_i b_{ij}^{(m)}} \right|^2 \leq b_{ij}^{(m)} \cdot 1 \xrightarrow{m \rightarrow \infty} b_{ij} = 0,$$

which completes the proof. □

Corollary 2.3. We have

$$\sup_{(X,Y)\in C_I(n)} \mathbb{E}|X - Y|^2 = \sup_{(A,B)\in \mathcal{S}_n} \Phi_n(A, B).$$

Proposition 2.5. Without loss of generality, we have

$$\sup_{\mathcal{S}_n} \Phi_n = \sup_{\mathcal{S}_Q n} \Phi_n,$$

where $\mathcal{S}_Q n \subset \mathcal{S}_n$ is the set of those $(A, B) \in \mathcal{S}_n$, that satisfy

$$B = RC^T,$$

for some $R = (r_i), C = (c_j) \in \mathbb{Q}_n$, with

$$\forall_i \quad 0 \leq r_i, \quad \sum_i r_i = 1,$$

$$\forall_j \quad 0 \leq c_j, \quad \sum_j c_j = 1.$$
Proof: From Proposition 2.4 we see, that \( \Phi_n \) is uniformly continuous on \( S_n \). Therefore

\[
\forall k \in \mathbb{Z}_+ \ \exists \delta_k > 0 \ : \ \forall x,y \in S_n \ |x - y| < \delta_k \implies |\Phi_n(x) - \Phi_n(y)| < \frac{1}{k}.
\]

For given \((A, B) \in S_n\) and \(k \in \mathbb{Z}_+\) choose \((A_k, B_k) \in S Q_n\) satisfying

\[
||(A, B) - (A_k, B_k)|| < \delta_k.
\]

The set \( S Q_n \) is clearly dense in \( S_n \) and thus, such \((A_k, B_k)\) can be found. Hence, we have

\[
\Phi_n(A, B) < \Phi_n(A_k, B_k) + \frac{1}{k} \implies \Phi_n(A, B) \leq \limsup_k \Phi_n(A_k, B_k),
\]

and therefore

\[
\sup_{S_n} \Phi_n \leq \sup_{S Q_n} \Phi_n.
\]

The inequality in the other direction is clear. \( \square \)

We will sometimes omit the subscript and write \( \Phi(x) \) for \( \Phi_n(x) \). By convention, we will also write \( \Phi(A, B) \) for \( A, B \in \mathbb{R}^{m \times n} \) with \( m \neq n \): we just start by making \((A, B)\) square matrices first, adding by default zero rows or columns, as needed.

**Definition 2.5.** For pairs \( A, B \in \mathbb{R}^{m \times n} \) we define the operation \( \Delta_r \) of row slicing, as follows:

\[
(A, B) = \begin{pmatrix}
    a_1 \\
    \vdots \\
    a_{i-1} \\
    a_i \\
    a_{i+1} \\
    \vdots \\
    a_m
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_{i-1} \\
    b_i \\
    b_{i+1} \\
    \vdots \\
    b_m
\end{pmatrix}
\implies \begin{pmatrix}
    a_1 \\
    \vdots \\
    a_{i-1} \\
    a_i \\
    a_{i+1} \\
    \vdots \\
    a_m
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_{i-1} \\
    b_i \\
    b_{i+1} \\
    \vdots \\
    b_m
\end{pmatrix} = \Delta_r^{i,j}(A, B),
\]

where \( a_i = \cdots = a_{i_j} = \frac{a_i}{T} \) and \( b_i = \cdots = b_{i_j} = \frac{b_i}{T} \). We also define the operation \( \Delta_c \) of column slicing similarly.

**Lemma 2.1.** Fix \( A, B \in \mathbb{R}^{m \times n} \). We have

\[
\forall 1 \leq l \leq m \ \forall i \in \mathbb{Z}_+ \quad \Phi(A, B) = \Phi(\Delta_r^{i,j}(A, B)),
\]

\[
\forall 1 \leq j \leq n \ \forall i \in \mathbb{Z}_+ \quad \Phi(A, B) = \Phi(\Delta_c^{i,j}(A, B)).
\]
Proof: We will only prove the first part. Just as before, let us write \( \Phi(A, B) = \sum_{ij} \phi_{ij}(A, B) \), where

\[
\phi_{ij}(A, B) = 1(b_{ij} \neq 0) \cdot b_{ij} \left| \frac{\sum_j a_{ij}}{\sum_j b_{ij}} - \frac{\sum_i a_{ij}}{\sum_i b_{ij}} \right|^2.
\]

Start by noting, that row slicing preserves sums of all columns of \( A \) and \( B \). On the other hand, row slicing can change the sum of a row, only if this particular row was sliced. In this second case, both the corresponding rows of \( A \) and \( B \) have been reduced by the same factor, leaving their proportion unchanged. Therefore we have

\[
\forall j, i \neq i_0 \quad \phi_{ij}(A, B) = \phi_{ij}(\Delta_{i_0}^l(A, B)),
\]

\[
\forall j, i = i_0 \quad \forall 1 \leq t \leq l \quad \phi_{ij}(A, B) \cdot \frac{1}{l} = \phi_{ij}(\Delta_{i_0}^l(A, B)),
\]

and hence, for \( i = i_0 \)

\[
\forall j \quad \phi_{ij}(A, B) = \sum_{t=1}^l \phi_{ij}(\Delta_{i_0}^l(A, B)).
\]

Therefore, summation over the full ranges concludes the proof. \( \square \)

Let us use \( \mathbf{1}_n \) as notation for \( n \)-dimensional vector of ones. The following proposition allows us to eliminate the \( b_{ij} \) coefficients.

**Proposition 2.6.** We have

\[
\sup_{n \in \mathbb{Z}^+} \sup \left\{ \Phi_n(A, B) : (A, B) \in \mathcal{S} \mathcal{Q}_n \right\}
\]

\[
= \sup_{n \in \mathbb{Z}^+} \sup \left\{ \Phi_n(A, B) : (A, B) \in \mathcal{S} \mathcal{Q}_n, \ B = \frac{1}{n^2} \mathbf{1}_n \mathbf{1}_n^T \right\}.
\]

**Proof:** Fix \((A, B) \in \mathcal{S} \mathcal{Q}_n\) and \( R = (r_i), C = (c_j) \in \mathbb{Q}^n\), such that

\[
\forall i \quad 0 \leq r_i, \quad \sum_i r_i = 1,
\]

\[
\forall j \quad 0 \leq c_j, \quad \sum_j c_j = 1,
\]

\[
B = RC^T.
\]

Since \( \{r_1, \ldots, r_n\} \cup \{c_1, \ldots, c_n\} \) is a set of rational numbers, there is a common denominator \( D \) and natural numbers \( N_{r,1}, \ldots, N_{r,n}, N_{c,1}, \ldots, N_{c,n} \), such that

\[
(r_1, \ldots, r_n) = \left( \frac{N_{r,1}}{D}, \ldots, \frac{N_{r,n}}{D} \right),
\]

\[
(c_1, \ldots, c_n) = \left( \frac{N_{c,1}}{D}, \ldots, \frac{N_{c,n}}{D} \right).
\]
Let us now

- slice every \( i \)-th row of initial \((A, B)\) matrices exactly \(N_{r,i}\) times,
- slice every \( j \)-th column of initial \((A, B)\) matrices exactly \(N_{c,j}\) times,

where slicing row or column 0 times is to be understood as removing it.

Execution of those operations, leaves us with \((\tilde{A}, \tilde{B})\), such that

\[
(\tilde{A}, \tilde{B}) \in \mathcal{S}Q_n, \\
\tilde{B} = \frac{1}{\tilde{n}} \frac{T^{T}}{\tilde{n}}, \\
\tilde{n} = \sum_{i=1}^{n} N_{r,i} = \sum_{j=1}^{n} N_{c,j}.
\]

From Lemma 2.1, it is apparent that \(\Phi(A, B) = \Phi(\tilde{A}, \tilde{B})\). This proves the inequality in one direction. The other direction is clear. \(\square\)

With the analysis so far, we have successfully removed coefficients \(B = (b_{ij})\) from our optimisation problem. Collecting all the pieces together, gives us

**Corollary 2.4.** We have

\[
\sup_{(X, Y) \in \mathcal{L}} \mathbb{E}[X - Y]^2 = \sup_{n \in \mathbb{Z}_+} \sup_{A \in [0,1]^{n \times n}} \frac{1}{n^4} \cdot \sum_{i,j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} - \sum_{j=1}^{n} a_{ij} \right|^2.
\]

**Definition 2.6.** Let us define \(\Xi_n : [0, 1]^{n \times n} \rightarrow \mathbb{R}\) as

\[
\Xi_n(A) = \sum_{i,j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} - \sum_{j=1}^{n} a_{ij} \right|^2.
\]

**Proposition 2.7.** For all \(n \in \mathbb{Z}_+\), we have

\[
\sup_{A \in [0,1]^{n \times n}} \Xi_n(A) = \sup_{A \in \{0,1\}^{n \times n}} \Xi_n(A).
\]

**Proof:** Function \(\Xi_n\) is continuous on the compact set \([0, 1]^{n \times n}\) and hence it attains its maximum. Let us choose

\[
\bar{A} = (\bar{a}_{ij}) \in \arg \max_{[0,1]^{n \times n}} \Xi_n.
\]

For any fixed pair \((i, j)\) let us set \(\xi_{ij} : [0, 1] \rightarrow \mathbb{R},\)

\[
\xi_{ij}(a_{ij}) = \Xi_n(\bar{A} \setminus \bar{a}_{ij}, a_{ij}).
\]
The notation means that we use all but one variables of the $\bar{A}$; we replace $\bar{a}_{ij}$ with $a_{ij}$. Of course we have 
\[ \bar{a}_{ij} \in \arg \max_{[0,1]} \xi_{ij}. \]

If $\bar{a}_{ij} \notin \{0,1\}$, then $\xi'_{ij}(\bar{a}_{ij}) = 0$ and $\xi''_{ij}(\bar{a}_{ij}) \leq 0$. After some basic calculations, with slight abuse of notation, we get
\[ \xi'_{ij}(a_{ij}) = 2 \cdot \left[ n \sum_{j=1}^{n} a_{ij} + n \sum_{i=1}^{n} a_{ij} - 2 \sum_{i,j=1}^{n} a_{ij} \right], \]
and
\[ \xi''_{ij}(a_{ij}) = 2 \cdot [n + n - 2] \geq 0. \]
This proves that, apart from the trivial case $n = 1$, we cannot have $\bar{a}_{ij} \in (0,1)$. □

Now, after Proposition 4.2 we can finally explain the connection of our initial problem with bipartite graphs.

**Definition 2.7.** An undirected graph $G$ is defined as a pair
\[ G = (V,E), \]
where $V$ is a finite set of vertices and $E$ is a set of edges, i.e. unordered pairs of elements of $V$.

**Definition 2.8.** A simple graph is any undirected graph $G$, without loops or multiple edges.

**Definition 2.9.** A bipartite graph is any simple graph $G = (V,E)$, for which $V$ can be split into two disjoint sets $V_1$ and $V_2$, such that each edge $e \in E$ joins a vertex in $V_1$ to a vertex in $V_2$.

For every $n \in \mathbb{Z}_+$ and $A \in \{0,1\}^{n \times n}$ consider the graph $G = (V,E)$ such that
\[ V = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}, \]
\[ \{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \emptyset, \]
\[ (x_i,y_j) \in E \iff a_{ij} = 1. \]
This leaves us with:

**Corollary 2.5.**
\[ \sup_{(X,Y) \in \mathcal{C}_I} \mathbb{E}|X - Y|^2 = \sup_{n \in \mathbb{Z}_+} \sup_{B(n,n)} \frac{1}{n^4} \cdot \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^2, \]
where $B(n,n)$ stands for the set of all bipartite graphs with two $n$ element groups of vertices, and $\deg(v)$ is degree of vertex $v$. 

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2.2 Solution of the graph problem

We now show the following:

**Theorem 2.1.** For all \( n \in \mathbb{Z}_+ \), we have

\[
\sup_{B(n,n)} \sum_{i,j} |\deg(x_i) - \deg(y_j)|^2 \leq \frac{n^4}{4}.
\]

Let us start by little simplification

\[
\sum_{i,j} |\deg(x_i) - \deg(y_j)|^2 \leq \frac{n^4}{4} \iff n \cdot \left( \sum_{i=1}^{n} \deg^2(x_i) + \sum_{j=1}^{n} \deg^2(y_j) \right)
\]

\[
\leq \frac{n^4}{4} + 2 \cdot \sum_{i,j=1}^{n} \deg(x_i) \deg(y_j) = \frac{n^4}{4} + 2 \cdot |E|^2,
\]

where the last equality follows from \( G = (V, E) \) being bipartite.

**Definition 2.10.** For any graph \( G = (V, E) \), we define first Zagreb index \( M_1(G) \), as

\[
M_1(G) = \sum_{v \in V} \deg^2(v).
\]

A comprehensive overview, of the state of the art of knowledge of \( M_1(G) \), can be found in [7]. In particular we can find there the following

**Theorem 2.2.** Fix \( n, e, q \in \mathbb{Z}_+ \), \( e \leq n^2 \) and let \( e = q \cdot n + r \), where \( 0 \leq r < n \). Let \( B^1(n,n,e) \) be such a bipartite graph \( G = (V, E) \), that \( V = X \cup Y, |X| = |Y| = n, |E| = e, \) and \( q \) vertices from \( Y \) are adjacent to all the vertices in \( X \) and one more vertex from \( Y \) is adjacent to \( r \) vertices in \( X \). \( B^1(n,n,e) \) has its maximum \( M_1 \) among all \( B(n,n) \) with \( e \) edges.

**Proof of Theorem 2.2** Take any \( G = (V, E) \in B(n,n) \) with \( |E| = e = q \cdot n + r \) as above. We want to prove that

\[
n \cdot M_1(G) \leq \frac{n^4}{4} + 2(qn + r)^2.
\]

From Theorem 2.2 we can see, that

\[
M_1(G) \leq (n - r)q^2 + r(q + 1)^2 + qn^2 + r^2,
\]

and we simply need to check if

\[
n \cdot \left[ (n - r)q^2 + r(q + 1)^2 + qn^2 + r^2 \right] \leq \frac{n^4}{4} + 2(qn + r)^2
\]

\[
\iff 0 \leq \frac{n^4}{4} - qn^3 + q^2n^2 + nr(2q - 1 - r) + 2r^2
\]
\[ \iff 0 \leq \left( \frac{n}{2} - q \right)^2 + \left[ r \frac{2q - 1 - r}{n} \right] + 2 \left( \frac{r}{n} \right)^2 \iff 0 \leq \left( \frac{n}{2} - q \right)^2 + \left[ r \frac{2q - n + n - 1 - r}{n} \right] + 2 \left( \frac{r}{n} \right)^2 \iff 0 \leq \left( q - \frac{n}{2} \right)^2 + r \frac{n}{n} (n - 1 - r) + \left( \frac{r}{n} \right)^2. \]

The last expression is nonnegative, because \( r + 1 \leq n \) from assumption. □

### 3 Solutions for \( \mathbb{E}XY \) and \( \mathbb{E}|X - Y|^2 \)

In this chapter we obtain tight bounds on

\[
\sup_{X,Y \in C(A)} \mathbb{E}XY \text{ and } \sup_{X,Y \in C(A)} \mathbb{E}|X - Y|^2,
\]

where \( C(A) \) is defined for all \( A \in \mathcal{F} \), by

\[
C(A) = \{ \mathbb{E}(\mathbb{1}_A|\mathcal{G}) : \mathcal{G} \subset \mathcal{F} \}.
\]

Note that, if \( X, Y \in C(A) \), then \( (X, Y) \) is clearly coherent. We shall also use

\[
C_I(A) = \{ (X, Y) : X, Y \in C(A), X \perp Y \}.
\]

#### 3.1 Two simple bounds on \( \mathbb{E}XY \)

To get a better understanding of the definitions, let us start by two exercise-level problems.

**Proposition 3.1.** We have

\[
\sup_{(X,Y) \in C_I(A)} \mathbb{E}XY = \mathbb{P}(A)^2.
\]

**Proof:** From independence and the tower property of conditional expectation, we get

\[
\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y = \mathbb{P}(A) \cdot \mathbb{P}(A). \quad \square
\]

**Proposition 3.2.** We have

\[
\sup_{(X,Y) \in C(A)} \mathbb{E}XY = \mathbb{P}(A).
\]

**Proof:** Clearly, for all \( (X, Y) \in C(A) \) we have \( \mathbb{E}XY \leq \mathbb{E}X = \mathbb{P}(A) \). Now, note that

\[
\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A|\mathcal{F}) \in C(A),
\]

and hence, putting \( X = Y = \mathbb{1}_A \), we get

\[
\mathbb{E}XY = \mathbb{E}X^2 = \mathbb{E}\mathbb{1}_A^2 = \mathbb{E}\mathbb{1}_A = \mathbb{P}(A). \quad \square
\]
3.2 General bound on $E|X - Y|^2$

We will start by crystallising the basic geometric intuition in the setting of abstract Hilbert spaces.

![Right triangle inscribed in a circle](image)

**Figure 1:** right triangle inscribed in a circle

**Lemma 3.1.** Let $A, B, C \in L^2(\Omega)$, $x = A - C$, $y = B - C$. If $\langle x, y \rangle = 0$, then for $M = B + \frac{x - y}{2}$ we have $||B - M|| = ||C - M|| = ||A - M||$.

**Proof:** From definition, we have $A - B = (A - C) - (B - C) = x - y$ and therefore

$$A - M = (A - B) - (M - B) = (x - y) - \frac{x - y}{2} = \frac{x - y}{2}.$$  

This proves that $||B - M|| = ||A - M||$. Now, the condition $\langle x, y \rangle = 0$ yields

$$||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2 = ||x||^2 - 2\langle x, y \rangle + ||y||^2 = ||x - y||^2. \quad (3)$$

Hence, we get

$$||M - C|| = \left\| (M - B) + (B - C) \right\| = \left\| \frac{x - y}{2} + y \right\| = \left\| \frac{x + y}{2} \right\| = ||B - M||. \quad \Box$$

**Proposition 3.3.** For any $\alpha \geq 0$, we have

$$\sup_{(X,Y) \in \mathcal{C}} E|X - Y|^\alpha \geq \sup_{(X,Y) \in \mathcal{C}_x} E|X - Y|^\alpha \geq 2^{-\alpha}.$$  

**Proof:** It is enough to set $X = 1_A$ and $Y = E1_A$ for arbitrary $A \in \mathcal{F}$, with $P(A) = \frac{1}{2}$. In such a situation, $Y$ is a constant and therefore $X \perp Y$. Then

$$E|X - Y|^\alpha = E\left| 1_A - \frac{1}{2} \right|^\alpha = \frac{1}{2^\alpha}. \quad \Box$$

**Theorem 3.1.** We have

$$\sup_{(X,Y) \in \mathcal{C}} E|X - Y|^2 = \frac{1}{4}.$$  

Proof: Fix the probability space \((\Omega, \mathcal{F}, P)\) and \(A \in \mathcal{F}\). We show that
\[
\sup_{X,Y \in \mathcal{C}(A)} \mathbb{E}|X - Y|^2 \leq P(A)(1 - P(A)) \leq \left(\frac{P(A) + (1 - P(A))}{2}\right)^2 = \frac{1}{4}.
\]
Start by choosing any two \(\sigma\)-fields \(G, H \subset \mathcal{F}\), and consider
\[
X = \mathbb{E}(\mathbb{1}_A|G), \quad Y = \mathbb{E}(\mathbb{1}_A|H),
\]
\[
\mathbb{E}\mathbb{1}_A = \mathbb{E}(X|\{\emptyset, \Omega\}) = \mathbb{E}(Y|\{\emptyset, \Omega\}).
\]

Figure 2: \(\mathcal{C}(A)\) and containing it sphere

We now check that
\[
\left\langle (\mathbb{1}_A - X), (X - \mathbb{E}\mathbb{1}_A) \right\rangle = 0,
\]
namely
\[
\left\langle (\mathbb{1}_A - X), (X - \mathbb{E}\mathbb{1}_A) \right\rangle = \mathbb{E}\left[(\mathbb{1}_A - X)(X - \mathbb{E}\mathbb{1}_A)\right]
\]
\[
= \mathbb{E}\left[\mathbb{1}_AX\right] - \mathbb{E}\left[\mathbb{1}_A\mathbb{E}(\mathbb{1}_A)\right] - \mathbb{E}\left[X^2\right] + \mathbb{E}\left[X\mathbb{E}(\mathbb{1}_A)\right]
\]
\[
= \mathbb{E}\left[\mathbb{1}_A\mathbb{E}(\mathbb{1}_A|G)\right] - \mathbb{E}\left[\mathbb{1}_A\mathbb{E}(\mathbb{1}_A)\right] - \mathbb{E}\left[\mathbb{E}(\mathbb{1}_A|G)^2\right] + \mathbb{E}\left[\mathbb{E}(\mathbb{1}_A|G)\mathbb{E}(\mathbb{1}_A)\right]
\]
\[
= \mathbb{E}\left[\mathbb{E}(\mathbb{1}_A|G)^2\right] - P(A)^2 - \mathbb{E}\left[\mathbb{E}(\mathbb{1}_A|G)^2\right] + P(A)^2 = 0.
\]
Similarly
\[
\left\langle (\mathbb{1}_A - Y), (Y - \mathbb{E}\mathbb{1}_A) \right\rangle = 0.
\]
We have
\[
\left\| X - \frac{\mathbb{1}_A + \mathbb{E}\mathbb{1}_A}{2} \right\| = \left\| \frac{1}{2}(X - \mathbb{1}_A) + \frac{1}{2}(X - \mathbb{E}\mathbb{1}_A) \right\|
\]
\[
\left\| \frac{1}{2}(X - \mathbb{1}_A) - \frac{1}{2}(X - \mathbb{E}\mathbb{1}_A) \right\| = \frac{1}{2} \cdot \left\| \mathbb{1}_A - \mathbb{E}\mathbb{1}_A \right\|,
\]
were we have flipped the sign by observation (3). Similarly
\[
\left\| Y - \frac{\mathbb{1}_A + \mathbb{E}\mathbb{1}_A}{2} \right\| = \frac{1}{2} \cdot \left\| \mathbb{1}_A - \mathbb{E}\mathbb{1}_A \right\|.
\]
Applying the triangle inequality, we get
\[
\left\| X - Y \right\| \leq \left\| X - \frac{\mathbb{1}_A + \mathbb{E}\mathbb{1}_A}{2} \right\| + \left\| Y - \frac{\mathbb{1}_A + \mathbb{E}\mathbb{1}_A}{2} \right\| = \frac{1}{2} \cdot \left\| \mathbb{1}_A - \mathbb{E}\mathbb{1}_A \right\| + \frac{1}{2} \cdot \left\| \mathbb{1}_A - \mathbb{E}\mathbb{1}_A \right\| = \left\| \mathbb{1}_A - \mathbb{E}\mathbb{1}_A \right\|,
\]
resulting in
\[
\mathbb{E}|X - Y|^2 \leq \mathbb{E}|\mathbb{1}_A - \mathbb{E}\mathbb{1}_A|^2 = (1 - \mathbb{P}(A))^2 \cdot \mathbb{P}(A) + \mathbb{P}(A)^2 \cdot (1 - \mathbb{P}(A))
\]
\[
= \mathbb{P}(A)(1 - \mathbb{P}(A)),
\]
which completes the proof. □

The following corollary is immediate from the analysis above.

**Corollary 3.1.** For fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and an event \(A \in \mathcal{F}\), we have
\[
\mathcal{C}(A) \subset \left\{ X \in L_2(\Omega) : \left\| X - \frac{\mathbb{1}_A + \mathbb{E}\mathbb{1}_A}{2} \right\| = \frac{\mathbb{P}(A)(1 - \mathbb{P}(A))}{2} \right\}.
\]

**Corollary 3.2.** For all \(\alpha \in [0, 2]\), we have
\[
\sup_{(X,Y)\in \mathcal{C}} \mathbb{E}|X - Y|^\alpha = \sup_{(X,Y)\in \mathcal{C}_X} \mathbb{E}|X - Y|^\alpha = 2^{-\alpha}.
\]

**Proof:** Thanks to Proposition 3.3, we only need to verify that
\[
\sup_{(X,Y)\in \mathcal{C}} \mathbb{E}|X - Y|^\alpha \leq 2^{-\alpha}.
\]
Clearly \(\alpha \in [0, 1]\), and thus \(f(x) = x^{\frac{\alpha}{2}}\) is concave on \(\mathbb{R}_+\). By Jensen inequality, we get
\[
\mathbb{E}|X - Y|^\alpha \leq \left( \mathbb{E}|X - Y|^2 \right)^{\frac{\alpha}{2}} \leq \left( \frac{1}{4} \right)^{\frac{\alpha}{2}} = 2^{-\alpha},
\]
for all \((X,Y) \in \mathcal{C}\). □
3.3 Geometry of the multivariate case

In this section we will obtain an upper bound on

$$\sup_{(X_1, \ldots, X_n) \in C(A)} \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \mathbb{E}|X_i - X_j|^2.$$ 

By Theorem 3.1 we could simply write

$$\sup_{(X_1, \ldots, X_n) \in C(A)} \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \mathbb{E}|X_i - X_j|^2 \leq \frac{n(n-1)}{2} \cdot \mathbb{P}(A)(1 - \mathbb{P}(A)),$$

but it turns out that by using geometric tools, we can improve it by a factor of roughly 2:

$$\sup_{(X_1, \ldots, X_n) \in C(A)} \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \mathbb{E}|X_i - X_j|^2 \leq \frac{n^2}{4} \cdot \mathbb{P}(A)(1 - \mathbb{P}(A)).$$

We start with the observation, that enables us to work in the much more intuitive space $\mathbb{R}^n$, rather than the abstract Hilbert space $L^2(\Omega)$.

**Proposition 3.4.** For any $(X_1, X_2, \ldots, X_n) \in C(A)$, there are $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$, such that

a) for all $1 \leq i, j \leq n$ we have $\mathbb{E}|X_i - X_j|^2 = ||x_i - x_j||^2$,

b) the set $\{x_1, x_2, \ldots, x_n\}$ lies on a sphere with radius $\sqrt{\frac{\mathbb{P}(A)(1 - \mathbb{P}(A))}{2}}$.

**Proof:** The random variables $\{X_1, X_2, \ldots, X_n, \frac{1 + \mathbb{E}(1 \cdot A)}{2}\}$ are at most $n+1$ different points in the Hilbert space $L_2(\Omega)$. Therefore they must lie on an $n$-dimensional affine subspace $H$. Since it is finite dimensional, $H$ is isometric to the euclidean space $\mathbb{R}^n$. Let $x_1, x_2, \ldots, x_n$ be the respective images of $X_1, X_2, \ldots, X_n$ under this isometry. Point a) then follows automatically and point b) is a direct consequence of Corollary 3.1. □

By Proposition 3.4 we get the following geometric restriction

**Corollary 3.3.** We have

$$\sup_{(X_1, \ldots, X_n) \in C(A)} \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \mathbb{E}|X_i - X_j|^2 \leq \sup_{x_1, \ldots, x_n \in S} \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n ||x_i - x_j||^2,$$

where $S \subset \mathbb{R}^n$ is a sphere with a radius $\sqrt{\frac{\mathbb{P}(A)(1 - \mathbb{P}(A))}{2}}$. 

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Definition 3.1. Let \( \mathcal{M} = \{x_1, \ldots, x_m\} \) be a finite multiset of points in \( \mathbb{R}^n \). We will say, that \( \bar{x} \in \mathbb{R}^n \) is a mass centre of \( \mathcal{M} \), if
\[
\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i.
\]

We now recall Definition 3.2 and Theorem 3.2, which can be found in [6].

Definition 3.2. Let \( \mathcal{M} \) be a multiset of points on an \((n-1)\)-sphere in \( \mathbb{R}^n \). We define a chord of \( \mathcal{M} \) to be a line segment whose endpoints belong to \( \mathcal{M} \).

Theorem 3.2. Let \( \mathcal{M} \) be a multiset of \( m \) points on a unit \((n-1)\)-sphere, and let \( \mathcal{C} \) be the multiset of the lengths of all the chords between them. Then
\[
\sum_{c \in \mathcal{C}} c^2 = m^2(1 - d^2),
\]
where \( d \) is the distance between the mass centre of \( \mathcal{M} \) and the centre of the unit \((n-1)\)-sphere.

Let us emphasise an important feature of Theorem 3.2 with the following remark.

Remark 3.1. The sum \( \sum_{c \in \mathcal{C}} c^2 \) depends on the configuration of the \( \{x_1, \ldots, x_m\} = \mathcal{M} \) only through the number of points \( m \) and mass centre \( \bar{x} \). It does not depend on the affine dimension of \( \mathcal{M} \).

Theorem 3.3. We have
\[
\sup_{(X_1, \ldots, X_n) \in \mathcal{C}(A)} \frac{1}{2} \sum_{i,j=1}^{n} \mathbb{E} |X_i - X_j|^2 \leq \frac{n^2}{4} \cdot \mathbb{P}(A)(1 - \mathbb{P}(A)).
\]

Proof: By Corollary 3.3, it suffices to show that
\[
\frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{n} ||x_i - x_j||^2 \leq \frac{n^2}{4} \cdot \mathbb{P}(A)(1 - \mathbb{P}(A)),
\]
for all \( x_1, \ldots, x_n \in \mathbb{R}^n \) lying on a sphere with a radius \( r = \frac{\sqrt{\mathbb{P}(A)(1 - \mathbb{P}(A))}}{2} \), which is an immediate consequence of Theorem 3.2 and scaling by a factor \( r^2 \). \( \square \)

It is not clear whether the inequality in Theorem 3.3 can be replaced by an equality sign for all \( n \), it is however straightforward to attain equality for even \( n = 2k \); for \( i \in \{1, 2, \ldots, 2k\} \), set
\[
X_i = \begin{cases} 
\mathbb{I}_A & \text{for } 2 \mid i \\
\mathbb{P}(A) & \text{for } 2 \nmid i,
\end{cases}
\]
which can be thought of as placing an equal number of points on each side of the diameter.
4 Relationship with Ferrer diagrams

In this chapter, exploiting the graphical representation introduced earlier, we will establish a connection between

$$\sup_{(X,Y) \in C} |X - Y|^k,$$

for \(k \in \mathbb{Z}_+, k \geq 3\), and well studied conjugate partitions of integers.

4.1 Reduction to bipartite graphs: \(k \geq 3\)

Theorem 4.1. For all \(k \in \mathbb{Z}_+, k \geq 3\), we have

$$\sup_{(X,Y) \in C} |X - Y|^k = \sup_{n \in \mathbb{Z}_+} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^k.$$

To prove this result it is enough to reconsider argumentation presented in Chapter 1. Only Proposition 2.2 and Proposition 2.7 used the assumption \(k = 2\) explicitly. In the rest of this section we will show that analogous statements hold for any \(k\).

Proposition 4.1. We have

$$\sup_{(X,Y) \in C} |X - Y|^k = \sup_{n \in \mathbb{Z}_+} \sup_{(X,Y) \in C(n)} |X - Y|^k.$$

Proof: For given \((X,Y)\) and \(n \in \mathbb{Z}_+\) choose \((X_n, Y_n)\) as in Proposition 2.1. Note that

$$E|X - Y|^k = E|X - X_n + X_n - Y_n + Y_n - Y|^k \leq E\left(|X - X_n| + |X_n - Y_n| + |Y_n - Y|\right)^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} E\left[\left(|X - X_n| + |Y - Y_n|\right)^{k-j} |X_n - Y_n|^j\right]$$

$$\leq \sum_{j=0}^{k-1} \binom{k}{j} E\left[\left(|X - X_n| + |Y - Y_n|\right)^{k-j} |X_n - Y_n|^j\right] + E|X_n - Y_n|^k.$$

We can now write

$$E|X - Y|^k \leq \limsup_{n \to \infty} \left(\sum_{j=0}^{k-1} \binom{k}{j} \cdot \left(\frac{2}{n}\right)^{k-j} E|X_n - Y_n|^k\right)$$

$$\leq \limsup_{n \to \infty} \left[\sum_{j=0}^{k-1} \binom{k}{j} \cdot \left(\frac{2}{n}\right)^{k-j}\right] + \limsup_{n \to \infty} E|X_n - Y_n|^k,$$
so $\mathbb{E}|X - Y|^k \leq \limsup_{n \to \infty} \mathbb{E}|X_n - Y_n|^k$ and as a result

$$\sup_{(X,Y) \in \mathcal{C}} \mathbb{E}|X - Y|^k \leq \sup_{n \in \mathbb{Z}_+} \mathbb{E}|X - Y|^k.$$ 

Inequality in the other direction is clear. □

**Definition 4.1.** Let us define $\Xi^k_n : [0, 1]^{n \times n} \to \mathbb{R}$ as

$$\Xi^n_k(A) = \sum_{i,j=1}^n \left| \sum_{i=1}^n a_{ij} - \sum_{j=1}^n a_{ij} \right|^k.$$ 

For $i, j \in \{1, 2, \cdots, n\}$ we introduce the abbreviation

$$A_{i\bullet} = \sum_{j=1}^n a_{ij},$$

$$A_{\bullet j} = \sum_{i=1}^n a_{ij}.$$ 

We can now write

$$\Xi^n_k(A) = \sum_{i,j=1}^n |A_{i\bullet} - A_{\bullet j}|^k.$$ 

**Lemma 4.1.** For any $x \in \mathbb{R}$ and $k \geq 3$ we have

$$\frac{\partial}{\partial x} |x|^k = k|x|^{k-2} \cdot x,$$

$$\frac{\partial}{\partial x} (|x|^{k-2} \cdot x) = (k-1)|x|^{k-2}.$$ 

**Proof:**

$$\frac{\partial}{\partial x} |x|^k = \begin{cases} 
\frac{\partial}{\partial x} x^k : x \geq 0 & = \begin{cases} 
kx^{k-1} : x \geq 0 
(-1)^{k-1}x^{k-1} : x \leq 0
\end{cases} = k|x|^{k-2} \cdot x,
\end{cases}$$

$$\frac{\partial}{\partial x} (|x|^{k-2} \cdot x) = \begin{cases} 
\frac{\partial}{\partial x} x^{k-1} : x \geq 0 & = \begin{cases} 
(k-1)x^{k-2} : x \geq 0 
(-1)^{k-2}(k-1)x^{k-2} : x \leq 0
\end{cases} = (k-1)|x|^{k-2},
\end{cases}$$

which ends the proof. □

**Proposition 4.2.** For all $n \in \mathbb{Z}_+$, we have

$$\sup_{[0,1]^{n \times n}} \Xi^n_k = \sup_{\{0,1\}^{n \times n}} \Xi^n_k.$$
Proof: The function $\Xi^k_n$ is continuous on the compact set $[0, 1]^{n \times n}$, and hence it attains its maximum. Let us choose 
$$\bar{A} = (\bar{a}_{ij}) \in \arg \max_{[0,1]^{n \times n}} \Xi^k_n.$$
For any fixed pair $(i, j)$, let us put $\xi_{ij} : [0, 1] \to \mathbb{R}$, 
$$\xi_{ij}(a_{ij}) = \Xi^k_n(\bar{A} \setminus \bar{a}_{ij}, a_{ij}),$$
meaning that we have all but one of the variables as in $\bar{A}$, we replace $\bar{a}_{ij}$ with $a_{ij}$. With a slight abuse of notation, we have 
$$\frac{\partial}{\partial a_{ij}} \xi_{ij}(a_{ij}) = \frac{\partial}{\partial a_{ij}} \left( \sum_{p \neq j} |A_{i \bullet} - A_{p \bullet}|^k + \sum_{p \neq i} |A_{j \bullet} - A_{p \bullet}|^k + |A_{i \bullet} - A_{j \bullet}|^k \right).$$
Note that $a_{ij}$ cancels out in $|A_{i \bullet} - A_{j \bullet}|$. By Lemma 4.1, we now have 
$$\frac{\partial}{\partial a_{ij}} \xi_{ij}(a_{ij}) = k \left[ \sum_{p \neq j} |A_{i \bullet} - A_{p \bullet}|^{k-2}(A_{i \bullet} - A_{p \bullet}) - \sum_{p \neq i} |A_{j \bullet} - A_{p \bullet}|^{k-2}(A_{p \bullet} - A_{j \bullet}) \right],$$
and hence $\xi_{ij}$ is a convex function. Since maximum of a convex function on compact, convex set is attained on the boundary, we can without loss of generality assume, that $\bar{a}_{ij} \in \{0, 1\}$ for all $i, j$. □

4.2 Degree sequences and majorization

For $n \in \mathbb{Z}_+$ and given two integer sequences $a = (a_i)_{i=1}^n$, $b = (b_j)_{j=1}^n$, with 
$$n \geq a_1, a_2, \ldots, a_n \geq 0,$$
$$n \geq b_1, b_2, \ldots, b_n \geq 0,$$
we might wonder if there exists a bipartite graph $G \in B(n, n)$ with degree sequences $a$ and $b$ in each part, respectively. We shall call such $(a, b)$ pairs bigraphic. This question can be answered by famous Theorem 4.2 (Gale-Ryser), see for example [8] or [9].

Definition 4.2. For $n \in \mathbb{Z}_+$ and any integer sequence $b = (b_i)_{i=1}^n$, with 
$$n \geq b_1, b_2, \ldots, b_n \geq 0,$$
we define its conjugate partition $b^* = (b^*_i)_{i=1}^n$, by 
$$b^*_k = |\{i : b_i \geq k\}|,$$
for all $k \in \{1, 2, \ldots, n\}$.  

Definition 4.3. For real sequences \( x = (x_i)_{i=1}^n, y = (y_j)_{j=1}^n \), we say that \( x \) majorizes \( y \), and write \( x \succ y \), if

\[
x_{\pi(1)} \geq y_{\sigma(1)},
\]

\[
x_{\pi(1)} + x_{\pi(2)} \geq y_{\sigma(1)} + y_{\sigma(2)},
\]

\[
\cdots
\]

\[
x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(n)} \geq y_{\sigma(1)} + y_{\sigma(2)} + \cdots + y_{\sigma(n-1)},
\]

\[
x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(n)} = y_{\sigma(1)} + y_{\sigma(2)} + \cdots + y_{\sigma(n)},
\]

where \( \pi \) and \( \sigma \) are such permutations of \( \{1, 2, \cdots, n\} \), that

\[
x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)},
\]

\[
y_{\sigma(1)} \geq y_{\sigma(2)} \geq \cdots \geq y_{\sigma(n)}.
\]

Theorem 4.2. A pair \((a, b)\), where \( n \in \mathbb{Z}_+ \), \( a = (a_i)_{i=1}^n \), \( b = (b_j)_{j=1}^n \), and

\[
n \geq a_1, a_2, \cdots, a_n \geq 0,
\]

\[
n \geq b_1, b_2, \cdots, b_n \geq 0,
\]

is bigraphic, if and only if \( b^* \succ a \).

The next lemma is well known as Karamata’s or majorization inequality, see for instance [11].

Lemma 4.2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a convex function and assume that two real sequences \( x = (x_i)_{i=1}^n \), \( y = (y_j)_{j=1}^n \) satisfy \( x \succ y \). Then we have

\[
\sum_{i=1}^n f(x_i) \geq \sum_{j=1}^n f(y_j).
\]
Theorem 4.3. For all $k \in \mathbb{Z}_+$, $k \geq 3$, we have

$$\sup_{n \in \mathbb{Z}_+} \sup_{B(n,n)} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^k = \sup_{n \in \mathbb{Z}_+} \sup_{(b_j)_{j=1}^{n} \in \{0,1,\ldots,n\}^n} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |b_i^* - b_j|^k.$$ 

Proof: From Theorem 4.2 we know that

$$\sup_{n \in \mathbb{Z}_+} \sup_{B(n,n)} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^k = \sup_{n \in \mathbb{Z}_+} \sup_{(a_i)_{i=1}^{n} \in \{0,1,\ldots,n\}^n} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |a_i - b_j|^k.$$ 

Let us fix $n, a, b$ for the time being. After rearrangement, we get

$$\sum_{i,j=1}^{n} |a_i - b_j|^k = \sum_{j=1}^{n} \sum_{i=1}^{n} |\{p : b_p = j\}| \cdot |a_i - j|^k = \sum_{j=1}^{n} (b_j - b_{j+1}) \sum_{i=1}^{n} |a_i - j|^k,$$

where we put $b_{n+1}^* = 0$ for convenience. Note that $f_j(x) = |x - j|^k$ is a convex function in $x$ for all $k \geq 3$ and $b^* \succ a$ from the assumption. By Lemma 4.2 for all $j \in \{1, 2, \ldots, n\}$, we have

$$\sum_{i=1}^{n} |a_i - j|^k \leq \sum_{i=1}^{n} |b_i^* - j|^k.$$

Summation over $j$ yields

$$\sum_{i,j=1}^{n} |a_i - b_j|^k \leq \sum_{i,j=1}^{n} |b_i^* - b_j|^k,$$

and hence we obtain

$$\sup_{n \in \mathbb{Z}_+} \sup_{B(n,n)} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |\deg(x_i) - \deg(y_j)|^k \leq \sup_{n \in \mathbb{Z}_+} \sup_{(b_j)_{j=1}^{n} \in \{0,1,\ldots,n\}^n} \frac{1}{n^{2+k}} \sum_{i,j=1}^{n} |b_i^* - b_j|^k.$$ 

To prove the opposite inequality, it is enough to verify that for every $n \in \mathbb{Z}_+$ and every

$$(b_j)_{j=1}^{n} \in \{0,1,\ldots,n\}^n,$$

the pair $(b^*, b)$ is bigraphic. From Theorem 4.2 this is equivalent to

$$b^* \succ b^*,$$

which is clearly true. □

4.3 Ferrer diagrams and an upper bound

Let us start by introducing the following notation: for $1 \geq \delta \geq 0,$

$$\epsilon(\delta) = \sup_{(X,Y) \in C} \mathbb{P}(|X - Y| \geq \delta).$$
In [2], Theorem 18.1, it was proved that for all $\delta \in [0, 1]$

$$
\epsilon(\delta) \leq [2(1 - \delta)] \land 1. \quad (4)
$$

By the Fubini’s theorem we have the so-called ”layer-cake” representation

$$
\mathbb{E}|X - Y|^k = \int_0^1 ku^{k-1} \cdot P(|X - Y| \geq u) \, du. \quad (5)
$$

Using (4) and (5), we see that for all $(X, Y) \in C$ and $k > 0$

$$
\mathbb{E}|X - Y|^k \leq \int_0^1 ku^{k-1} \cdot \epsilon(u) \, du = \frac{2 - 2^{-k}}{1 + k}. \quad (6)
$$

The upper bound (6) has been considered already by Burdzy and Pitman in [1]. In this section we will reproove this result with additional assumption of independence. Hence, our result is weaker, but the approach we take is different. The reader should treat this section as a soft introduction to the combinatorial ideas that will be studied further in the next chapter.

Definition 4.4. For $n \in \mathbb{Z}_+$ and any integer sequence $b = (b_i)_{i=1}^n$, with

$$
n \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0,
$$

we define the corresponding Ferrer diagram as the $n \times n$ binary matrix, such that

- its column sums, starting from the left, are $b_1, \ldots, b_n$, respectively,
- for every fixed column, all ones are below all zeros.

Rather then thinking in terms of $0-1$ matrices, we will visualise Ferrer diagrams as square grids with empty or filled cells. For example, the Ferrer diagram of the sequence $b = (5, 4, 3, 3, 2)$ is illustrated in Figure 3.

Note, that the conjugate sequence $b^*$ can now be easily interpreted as the row sums of the Ferrer diagram of $b$. In the example given above, we have $b^* = (5, 5, 4, 2, 1)$.

Theorem 4.4. For all $k \in \mathbb{Z}_+, k \geq 2$ we have

$$
\sup_{(X,Y)\in \mathcal{C}_Z} \mathbb{E}|X - Y|^k \leq \frac{2 - 2^{-k}}{1 + k}.
$$

Proof: Fix any $n \in \mathbb{Z}_+$ and a decreasing, integer sequence $b = (b_i)_{i=1}^n$, just as in Definition 4.4. By Theorems 4.3 and 4.1 it is enough to check that

$$
\frac{1}{n^{2+k}} \sum_{i,j=1}^n |b_i^* - b_j|^k = \frac{1}{n^2} \sum_{i,j=1}^n \left| \frac{b_i^* - b_j}{n} \right|^k \leq \frac{2 - 2^{-k}}{1 + k}.
$$

We will start by constructing $(X, Y) \in \mathcal{C}_Z$, such that

$$
\mathbb{E}|X - Y|^k = \frac{1}{n^2} \sum_{i,j=1}^n \left| \frac{b_i^* - b_j}{n} \right|^k. \quad (7)
$$
For this purpose, take the Ferrer diagram of \( b \), and rescale it so that it is contained in the unit square

\[ I = \left\{ (u, v) : u, v \in [0, 1] \right\}. \]

We shall, from now on, work with the probability space

\[ \left( I, \mathcal{L}_{\lambda \otimes \lambda}(I), \lambda \otimes \lambda \right), \]

where \( \lambda \) is simply one-dimensional lebesgue measure on \([0, 1]\), \( \lambda \otimes \lambda \) stands for the product measure on \( I \) and \( \mathcal{L}_{\lambda \otimes \lambda}(I) \) is an appropriate product \( \sigma \)-field. Set

\[ U(u, v) = u, \]
\[ V(u, v) = v. \]

Therefore \( U, V \sim U[0, 1] \), and \( U \perp V \). Moreover, by \( A \) let us denote the staircase-shaped region obtained by uniting all of the filled cells in the rescaled diagram. We can now define

\[ X(u, v) \equiv x(U(u, v)) := \lambda \left( A \setminus \text{bd}(A) \right) \cap \left( u \times [0, 1] \right), \]
\[ Y(u, v) \equiv y(V(u, v)) := \lambda \left( A \setminus \text{bd}(A) \right) \cap \left( [0, 1] \times v \right), \]

where \( \text{bd}(A) \) stands for boundary of \( A \). This notation indicates that the random variables \( X \) and \( Y \) can be also treated as a deterministic (borel) functions

\[ x, y : [0, 1] \rightarrow [0, 1], \]

of random \( U \) and \( V \). Clearly \( X \perp Y \) and the condition (7) holds. Finally, one can also check that

\[ X(u, v) = \mathbb{E}(\mathbb{1}_{A \setminus \text{bd}(A)} | U = u), \]
\[ Y(u, v) = \mathbb{E}(\mathbb{1}_{A \setminus \text{bd}(A)} | V = v), \]

but we postpone the formal verification till the next section.
In this setting we will show the (4) inequality, namely
\[ P(|X - Y| > \delta) \leq 2(1 - \delta), \]
for all \( \delta \in [0, 1] \). This will also establish (6) and conclude the proof. We start by writing
\[ P(|X - Y| > \delta) = P[X > (Y + \delta)] + P[Y > (X + \delta)]. \]

By the symmetry of the problem, it is sufficient to demonstrate that
\[ P[X > (Y + \delta)] \leq 1 - \delta. \]

For any \( \tau \in [0, 1] \), considering the intersection with \( \{ Y < \tau \} \), gives
\[ P[X > (Y + \delta)] = P[X > (Y + \delta), Y < \tau] + P[X > (Y + \delta), Y \geq \tau] \leq P[Y < \tau] + P[X > (\tau + \delta)]. \]

Let us now consider the following linear function of \( v \)
\[ f(v) = v - \delta \]
and let \( \tau \) be such, that \( \tau + \delta \) is an argument at which the graph of \( f \) and the "staircase" part of \( \text{bd}(A) \) intersect - see Figure 5.
Clearly, we have two possible scenarios. Firstly, graph of $f$ may intersect vertical part of the boundary. In this case, let us note that

$$y(\tau + \delta) = f(\tau + \delta) = (\tau + \delta) - \delta = \tau.$$  

On the other hand, assume that $f$ intersects horizontal part of the "staircase". Having this in mind, recall that we have omitted $bd(A)$ in the construction of $(X,Y)$. This leaves us with

$$y(\tau + \delta) = \lambda\left(\left(A \setminus bd(A)\right) \cap \left([0,1] \times (\tau + \delta)\right)\right) \leq f(\tau + \delta) = \tau.$$  

Thus, either way, we get $y(\tau + \delta) \leq \tau$. Again, from construction of $(X,Y)$ and omission of boundary, we have

$$\mathbb{P}[X > (\tau + \delta)] = y(\tau + \delta) \leq \tau.$$  

Hence, it remains to check, that

$$\mathbb{P}[Y < \tau] \leq 1 - \tau - \delta,$$

or equivalently, that

$$\mathbb{P}[Y \geq \tau] \geq \tau + \delta.$$  

Luckily, observe that

$$\mathbb{P}[Y \geq \tau] = \mathbb{P}[y(V) \geq \tau],$$  

and $y(v) \geq \tau$ for all $v < \tau + \delta$. \qed

5 Upper bounds and a novel approach

We will continue exploiting combinatorial nature of Ferrer diagrams. We shall start by introducing more flexible definitions.
5.1 Generalization of Ferrer diagrams

Definition 5.1. From now on, by (generalised) Ferrer diagram, we shall mean a set

\[ F_f = \{(u, v) \in [0, 1]^2 : v < f(u)\}, \]

where

\[ f : [0, 1] \to [0, 1], \]

is any weakly decreasing step function that takes finitely many different values; let us denote the set of such step functions as \( \text{STEP} \).

In the next definition, we formalise an idea used in the proof of Theorem 4.4 - compare Figure 4.

Definition 5.2. For any Ferrer diagram \( F_f \) we define a pair of associated random variables \( (X_f, Y_f) \) on a probability space \( (I, \mathcal{L}, \lambda \otimes \lambda) \), where

\[ I = \{(u, v) : u, v \in [0, 1]\}, \]

by

\[ X_f(u, v) \equiv x_f(U(u, v)) := \lambda\left( (F_f \setminus \text{bd}(F_f)) \cap (u \times [0, 1]) \right), \]  

\[ Y_f(u, v) \equiv y_f(V(u, v)) := \lambda\left( (F_f \setminus \text{bd}(F_f)) \cap ([0, 1] \times v) \right), \]

where

\[ U(u, v) = u, \]

\[ V(u, v) = v. \]

We shall prove that \( (X_f, Y_f) \) defined by (8) and (9) does satisfy \( (X_f, Y_f) \in \mathcal{C}_I \).

Proof: Clearly \( U, V \sim U[0, 1] \) and \( U \perp V \). This gives \( X \perp Y \). It is therefore enough to check that

\[ X_f(u, v) \equiv x_f(U(u, v)) = \mathbb{E}(1_{F_f \setminus \text{bd}(F_f)}|U = u), \]

\[ Y_f(u, v) \equiv y_f(V(u, v)) = \mathbb{E}(1_{F_f \setminus \text{bd}(F_f)}|V = v). \]

This gives \( (X_f, Y_f) \in \mathcal{C} \). We limit ourselves to showing (10). It is straightforward to check that \( x_f \) is a borel function. Thus by (8) we have

\[ X_f = x_f(U), \]

and \( X_f \) is \( \sigma(U) \) measurable. It remains to verify that for every \( A \in \sigma(U) \), we have

\[ \int_A X_f \, d\mathbb{P} = \int_A 1_{F_f \setminus \text{bd}(F_f)} \, d\mathbb{P}. \]
The condition $A \in \sigma(U)$ is equivalent to $A = \tilde{A} \times [0, 1]$, for some $\tilde{A} \in \mathcal{L}([0, 1])$. We can write

$$x_f(u) = \lambda \left( \left( F_f \setminus \text{bd}(F_f) \right) \cap (u \times [0, 1]) \right) = \int_0^1 1_{F_f \setminus \text{bd}(F_f)}(u, v) \, d\lambda(v),$$

and hence, by Fubini’s theorem, we have

$$\int_A 1_{F_f \setminus \text{bd}(F_f)} \, d\mathbb{P} = \int_A \left[ \int_{[0,1]} 1_{F_f \setminus \text{bd}(F_f)}(u, v) \, d\lambda(v) \right] \, d\lambda(u) = \int_A x_f(u) \, d\lambda(u) = \int_A X_f(u, v) \, d\lambda_2(u, v) = \int_A X_f \, d\mathbb{P},$$

as required. □

Although it is rather obvious, let us state the following

**Proposition 5.1.** For all $k \in \mathbb{Z}_+$, $k \geq 3$, we simply have

$$\sup_{f \in \text{STEP}} \mathbb{E}|X_f - Y_f|^k = \sup_{(X,Y) \in \mathcal{C}} \mathbb{E}|X - Y|^k.$$

*Proof:* For all $f \in \text{STEP}$, by definition

$$(X_f, Y_f) \in \mathcal{C},$$

and thus, the inequality

$$\sup_{f \in \text{STEP}} \mathbb{E}|X_f - Y_f|^k \leq \sup_{(X,Y) \in \mathcal{C}} \mathbb{E}|X - Y|^k,$$

is clear. The opposite inequality follows from the same argument as the proof of Theorem 4.4. □

### 5.2 Sharpening a layer-cake upper bound

In this section we will continue our analysis of upper bounds generated by the layer-cake representation. Let us start with a brief overview of the relevant results. As already mentioned, in [2] it was proved that for $\delta \in \left( \frac{1}{2}, 1 \right]$, we have

$$\sup_{(X,Y) \in \mathcal{C}} \mathbb{P}(|X - Y| \geq \delta) \leq 2(1 - \delta).$$

This result has been lately greatly improved by Burdzy and Pal. In [3] they proved that for $\delta \in \left( \frac{1}{2}, 1 \right]$: 

$$\sup_{(X,Y) \in \mathcal{C}} \mathbb{P}(|X - Y| \geq \delta) = \frac{2(1 - \delta)}{2 - \delta}.$$ 

Moreover, for all $\delta$ in this range, one can find such pairs $(X_\delta, Y_\delta) \in \mathcal{C}$ for which the equality is attained. It is however important to note that for $\delta < 1$, those variables turn out to be dependent. It is relatively easy to check that for $\delta \in \left( \frac{1}{2}, 1 \right]$: 

$$\sup_{(X,Y) \in \mathcal{C}_2(2)} \mathbb{P}(|X - Y| \geq \delta) = 2\delta(1 - \delta).$$

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Based on this premise, Burdzy and Pitman conjectured in [1], that for $\delta \in (\frac{1}{2}, 1]$, we have

$$\sup_{(X,Y) \in C_I} \mathbb{P}(|X - Y| \geq \delta) = 2\delta(1 - \delta). \quad (12)$$

In this chapter we will prove a result, that is strikingly similar. Namely, for $\delta \in (\frac{1}{2}, 1]$, we have

$$\sup_{f \in \text{STEP}} \mathbb{P}(|X_f - Y_f| > \delta) = 2\delta(1 - \delta). \quad (13)$$

In this place let us highlight that, thanks to Proposition 5.1, both (12) and (13) generate exactly the same layer-cake bound on

$$\sup_{(X,Y) \in C_I} E|X - Y|^k,$$

for all $k \in \mathbb{Z}_+, k \geq 3$. We begin with the simple, but useful observation.

**Lemma 5.1.** Fix $f \in \text{STEP}$ and consider the associated $(X_f, Y_f) \in C_I$. For any $s, t \in [0, 1]$, there are only two possible scenarios:

- either $(x_f(s) \leq t) \land (y_f(t) \leq s)$ or $(x_f(s) \geq t) \land (y_f(t) \geq s)$.

**Proof:** This is a direct consequence of the definitions - see Figure 6.

**Theorem 5.1.** For all $\delta \in (\frac{1}{2}, 1]$, we have

$$\sup_{f \in \text{STEP}} \mathbb{P}(|X_f - Y_f| > \delta) = 2\delta(1 - \delta).$$

Proof of this theorem will be based on two lemmas, which follow below.
Lemma 5.2. Fix $\delta \in (\frac{1}{2}, 1]$ and consider any $f \in \text{STEP}$, for which one of conditions:

\[
\begin{align*}
\{ & x_f(\delta) \geq \delta, \\
& y_f(\delta) \geq \delta, \\
\} \quad (14)
\end{align*}
\]

or

\[
\begin{align*}
\{ & x_f(1 - \delta) \leq 1 - \delta, \\
& y_f(1 - \delta) \leq 1 - \delta, \\
\} \quad (15)
\end{align*}
\]

is satisfied. Then we have

\[\mathbb{P}(\left|X_f - Y_f\right| > \delta) \leq 2\delta(1 - \delta).\]

Proof: Let us first consider point (14). We start by writing

\[\mathbb{P}(\left|X_f - Y_f\right| > \delta) = \mathbb{P}[X_f > (Y_f + \delta)] + \mathbb{P}[Y_f > (X_f + \delta)].\]

By independence, we can evaluate

\[
\begin{align*}
\mathbb{P}[X_f > (Y_f + \delta)] & \leq \mathbb{P}[X_f > \delta] \cdot \mathbb{P}[Y_f \leq \delta] = y_f(\delta)(1 - x_f(\delta)), \\
\mathbb{P}[Y_f > (X_f + \delta)] & \leq \mathbb{P}[Y_f > \delta] \cdot \mathbb{P}[X_f \leq \delta] = x_f(\delta)(1 - y_f(\delta)).
\end{align*}
\]

Summing up, we get

\[
\mathbb{P}(\left|X_f - Y_f\right| > \delta) \leq y_f(\delta)(1 - x_f(\delta)) + x_f(\delta)(1 - y_f(\delta)) \leq \sup_{x,y \in [\delta, 1]} g(x, y),
\]

where $g(x, y) = y(1 - x) + x(1 - y)$. Since $\delta > \frac{1}{2}$, we get

\[
\frac{\partial g}{\partial x}(x, y) = 1 - 2y < 0,
\]

\[
\frac{\partial g}{\partial y}(x, y) = 1 - 2x < 0,
\]

and hence

\[\mathbb{P}(\left|X_f - Y_f\right| > \delta) \leq g(\delta, \delta) = 2\delta(1 - \delta),\]

which was to be proved.

The proof of point (15) is similar and we will only sketch it. One can evaluate

\[
\begin{align*}
\mathbb{P}(\left|X_f - Y_f\right| > \delta) & \leq \mathbb{P}[X_f > 1 - \delta] \cdot \mathbb{P}[Y_f \leq 1 - \delta] + \mathbb{P}[Y_f > 1 - \delta] \cdot \mathbb{P}[X_f \leq 1 - \delta] \\
& = y_f(1 - \delta)(1 - x_f(1 - \delta)) + x_f(1 - \delta)(1 - y_f(1 - \delta)) \\
& \leq \sup_{x,y \in [0,1-\delta]} g(x, y) = 2\delta(1 - \delta),
\end{align*}
\]

where the last line is a consequence of $1 - \delta < \frac{1}{2}$. □
Lemma 5.3. For all $\delta \in (\frac{1}{2}, 1]$ and any $f \in \text{STEP}$, if

$$1 - \delta \leq y_f(\delta) \leq \delta,$$

then

$$\Pr(X_f > Y_f + \delta) \leq \delta(1 - \delta).$$

The same holds true for $X_f$ and $Y_f$ with switched roles.

Proof: Just as previously, due to independence, we can write

$$\Pr(X_f > Y_f + \delta) \leq \Pr[X_f > \delta] \cdot \Pr[Y_f < 1 - \delta].$$

Firstly, note that

$$\Pr[X_f > \delta] = y_f(\delta) \leq \delta.$$

Secondly, thanks to monotonicity of $Y_f$, we have

$$y_f(\omega) \geq y_f(\delta) \geq 1 - \delta \quad \text{for all} \quad \omega \leq \delta,$$

so $\{y_f < 1 - \delta\} \subset (\delta, 1]$, and hence

$$\Pr[Y_f < 1 - \delta] \leq 1 - \delta,$$

which completes the proof.  \qed

Proof of Theorem 5.1: Fix $\delta \in (\frac{1}{2}, 1]$ and any $f \in \text{STEP}$. It is enough to check, that

$$\Pr(|X_f - Y_f| > \delta) \leq 2\delta(1 - \delta).$$

By Lemmas 5.1 and 5.2 (14), we can assume, that

$$\begin{cases}
  x_f(\delta) \leq \delta, \\
  y_f(\delta) \leq \delta.
\end{cases} \quad (16)$$

By the Lemma 5.1 again, there are only 4 possible scenarios:

$$\begin{cases}
  x_f(1 - \delta) \geq \delta, \\
  y_f(\delta) \geq 1 - \delta,
\end{cases} \quad \text{and} \quad \begin{cases}
  y_f(1 - \delta) \geq \delta, \\
  x_f(\delta) \geq 1 - \delta.
\end{cases} \quad (17)$$

$$\begin{cases}
  x_f(1 - \delta) \geq \delta, \\
  y_f(\delta) \geq 1 - \delta,
\end{cases} \quad \text{and} \quad \begin{cases}
  y_f(1 - \delta) \leq \delta, \\
  x_f(\delta) \leq 1 - \delta.
\end{cases} \quad (18)$$

$$\begin{cases}
  x_f(1 - \delta) \leq \delta, \\
  y_f(\delta) \leq 1 - \delta,
\end{cases} \quad \text{and} \quad \begin{cases}
  y_f(1 - \delta) \geq \delta, \\
  x_f(\delta) \geq 1 - \delta.
\end{cases} \quad (19)$$
\[
\begin{cases}
x_f(1 - \delta) \leq \delta, \\
y_f(\delta) \leq 1 - \delta,
\end{cases}
\quad \text{and} \quad
\begin{cases}
y_f(1 - \delta) \leq \delta, \\
x_f(\delta) \leq 1 - \delta.
\end{cases}
\tag{20}
\]

We continue by inspection, one by one.

\[\text{(17). In view of (16) we have} \]
\[
1 - \delta \leq y_f(\delta) \leq \delta,
\]
\[
1 - \delta \leq x_f(\delta) \leq \delta.
\]

By a double use of Lemma 5.3, we get
\[
\mathbb{P}(|X_f - Y_f| > \delta) = \mathbb{P}[X_f > (Y_f + \delta)] + \mathbb{P}[Y_f > (X_f + \delta)]
\leq \delta(1 - \delta) + \delta(1 - \delta) = 2\delta(1 - \delta). \quad \triangle
\]

\[\text{(18). In view of (16) we have} \]
\[
1 - \delta \leq y_f(\delta) \leq \delta.
\]

From Lemma 5.3, we have
\[
\mathbb{P}[X_f > Y_f + \delta] \leq \delta(1 - \delta).
\]

Furthermore, since to \( \delta \in \left(\frac{1}{2}, 1\right] \), we can evaluate
\[
\mathbb{P}[Y_f > X_f + \delta] \leq \mathbb{P}(Y_f > \delta) \cdot \mathbb{P}(X_f < \delta).
\]

By monotonicity, we have
\[
y_f(v) \leq y_f(1 - \delta) \leq \delta \quad \text{for all} \quad v \geq 1 - \delta,
\]
\[
x_f(u) \geq x_f(1 - \delta) \geq \delta \quad \text{for all} \quad u \leq 1 - \delta,
\]
so
\[
\{y_f > \delta\} \subset [0, 1 - \delta),
\]
\[
\{x_f < \delta\} \subset (1 - \delta, 1],
\]
and hence
\[
\mathbb{P}[Y_f > X_f + \delta] \leq (1 - \delta)\delta. \quad \triangle
\]

\[\text{(19). This scenario is analogous to (18). It is sufficient to change roles of } X_f \text{ and } Y_f. \quad \triangle
\]
Let us start by repeating again the bounds for this scenario. We have

\[
\begin{align*}
\{ x_f(1 - \delta) & \leq \delta, & y_f(1 - \delta) & \leq \delta, \\
x_f(\delta) & \leq 1 - \delta, & y_f(\delta) & \leq 1 - \delta.
\end{align*}
\] (20)

At this point, it is beneficial to graph the constraints given by (20).

![Figure 7: example diagram $F_f$ meeting constraints given by (20).](image)

As in Figure 7, every diagram $F_f$ meeting constraints discussed in scenario (20), must be a subset of hatched region. For any such diagram $F_f$, let us now define

\[ F'_f = F_f \setminus [1 - \delta, \delta]^2. \]

Put differently, we are removing the (possibly empty) intersection $F_f \cap [1 - \delta, \delta]^2$ - see Figure 8.

![Figure 8: diagram $F'_f$ is obtained from $F_f$ by removing $[1 - \delta, \delta]^2$.](image)
Note that, by construction, the transformation $F_f \rightarrow F_f'$ fulfils both
\[ x_f' \leq x_f, \]
\[ y_f' \leq y_f, \]
and
\[ x_f'(u) = x_f(u) \quad \text{for all} \quad u \in \{ u : x_f(u) > \delta \}, \]
\[ y_f'(v) = y_f(v) \quad \text{for all} \quad v \in \{ v : y_f(v) > \delta \}. \]
Thus, it is straightforward to see, that
\[ \mathbb{P}(X_f > Y_f + \delta) \leq \mathbb{P}(X_f' > Y_f' + \delta), \]
\[ \mathbb{P}(Y_f > X_f + \delta) \leq \mathbb{P}(Y_f' > X_f' + \delta), \]
and
\[ \mathbb{P}(|X_f - Y_f| > \delta) \leq \mathbb{P}(|X_f' - Y_f'| > \delta), \]
as a result. To complete the proof, it is enough to show, that
\[ \mathbb{P}(|X_f' - Y_f'| > \delta) \leq 2\delta(1 - \delta), \]
but this is a direct consequence of Lemma 5.2 (15).

By Proposition 5.1 and Theorem 5.1, we get the following corollary directly.

**Corollary 5.1.** For all $k \in \mathbb{Z}_+, k \geq 3$, we have
\[ \sup_{(X,Y) \in \mathcal{C}_Z} \mathbb{E}|X - Y|^k \leq \int_0^{\frac{1}{2}} kt^{k-1}dt + \int_{\frac{1}{2}}^1 kt^{k-1} \cdot 2t(1-t)dt, \]
that is
\[ \sup_{(X,Y) \in \mathcal{C}_Z} \mathbb{E}|X - Y|^k \leq 2 \cdot \frac{k}{(k+1)(k+2)} + 2^{-k} - 2^{-k-1} \cdot \frac{k(k+3)}{(k+1)(k+2)}.\]

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