Onsager Theory of Turbulence, the Josephson-Anderson Relation, and the D’Alembert Paradox

Hao Quan¹ and Gregory L. Eyink¹,²

¹ Department of Applied Mathematics & Statistics
The Johns Hopkins University, Baltimore, MD 21218, USA
E-mail: haoquan@jhu.edu
² Department of Physics and Astronomy
The Johns Hopkins University, Baltimore, MD 21218, USA
E-mail: eyink@jhu.edu

July 5, 2022

Abstract The Josephson-Anderson relation, valid for the incompressible Navier-Stokes solutions which describe flow around a solid body, instantaneously equates the power dissipated by drag to the flux of vorticity across the flow lines of the potential Euler solution considered by d’Alembert. Its derivation involves a decomposition of the velocity field into this background potential-flow field and a solenoidal field corresponding to the rotational wake behind the body, with the flux term describing transfer from the interaction energy between the two fields and into kinetic energy of the rotational flow. We establish the validity of the Josephson-Anderson relation for the weak solutions of the Euler equations obtained in the zero-viscosity limit, with one transfer term due to inviscid vorticity flux and the other due to a viscous skin-friction anomaly. Furthermore, we establish weak forms of the local balance equations for both interaction and rotational energies. We define nonlinear spatial fluxes of these energies and show that the asymptotic flux of interaction energy to the wall equals the anomalous skin-friction term in the Josephson-Anderson relation. However, when the Euler solution satisfies suitably the no-flow-through condition at the wall, then the anomalous term vanishes. In this case, we can show also that the asymptotic flux of rotational energy to the wall must vanish and we obtain in the rotational wake the Onsager-Duchon-Robert relation between viscous dissipation anomaly and inertial dissipation due to scale-cascade. In this way we establish a precise connection between the Josephson-Anderson relation and the Onsager theory of turbulence, and we provide a novel resolution of the d’Alembert paradox.

1. Introduction

1.1. Josephson-Anderson Relation for Flow Around a Solid Body. The Josephson-Anderson relation was first derived for voltage-drops in superconductors [15] and...
chemical potential differences in superfluids [1], relating these to transverse motion of quantized vortices. A “detailed relation” of Huggins [13] further connected superfluid dissipation to flux of vortices across streamlines of the background potential flow and was applied by Huggins also to classical turbulent channel flow described by the incompressible Navier-Stokes equation [14].

Recently, the detailed Josephson-Anderson relation was extended to incompressible Navier-Stokes solutions describing flow around a smooth, solid body [12]. This result made a direct connection with the classical paradox raised by work of d’Alembert [4,5], who showed that the potential Euler solution \( \mathbf{u}_\phi = \nabla \phi \) for flow around the body \( B \) predicts no drag. See Fig. 1 for the context. Following the earlier work on superfluids, [12] introduced a “rotational velocity” \( \mathbf{u}_\omega := \mathbf{u}^\nu - \mathbf{u}_\phi \) which accounts for all flow vorticity \( \omega^\nu = \nabla \times \mathbf{u}^\nu \) and which satisfies the equation

\[
\partial_t \mathbf{u}^\nu = \mathbf{u}^\nu \times \omega^\nu - \nu \nabla \times \omega^\nu - \nabla \left( p^\omega + \frac{1}{2} |\mathbf{u}^\omega|^2 + \mathbf{u}^\omega \cdot \mathbf{u}_\phi \right),
\]

expressing local conservation of the “vortex momentum” \( P^\omega = \rho \int_\Omega \mathbf{u}^\omega \cdot dV \). Here the pressure \( p^\omega \) is to be determined by the divergence-free constraint \( \nabla \cdot \mathbf{u}^\omega = 0 \) and \( \rho \) is mass density of the fluid. The formulation [1] of the incompressible Navier-Stokes equation gives a precise mathematical description of the rotational wake that develops behind the body. In this setting, the Josephson-Anderson relation follows by considering balance equations for the interaction energy between potential and rotational flow,

\[
E^\nu_{\text{int}} = \rho \int_\Omega \left[ \left( p^\nu + \frac{1}{2} |\mathbf{u}^\nu|^2 + \mathbf{u}^\nu \cdot \mathbf{u}_\phi \right) \mathbf{u}^\nu \right] dV,
\]

that is:

\[
\partial_t \left( \mathbf{u}_\phi \cdot \mathbf{u}^\nu \right) + \nabla \cdot \left[ \left( p^\nu + \frac{1}{2} |\mathbf{u}^\nu|^2 + \mathbf{u}^\nu \cdot \mathbf{u}_\phi \right) \mathbf{u}^\nu \right] = + \mathbf{u}_\phi \cdot \left( \mathbf{u}^\nu \times \omega^\nu - \nu \nabla \times \omega^\nu \right)
\]

(2)

and

\[
\partial_t \left( \frac{1}{2} |\mathbf{u}^\nu|^2 \right) + \nabla \cdot \left[ \left( p^\nu + \frac{1}{2} |\mathbf{u}^\nu|^2 + \mathbf{u}^\nu \cdot \mathbf{u}_\phi \right) \mathbf{u}^\nu - \nu \mathbf{u}^\nu \times \omega^\nu \right] = - \mathbf{u}_\phi \cdot \left( \mathbf{u}^\nu \times \omega^\nu - \nu \nabla \times \omega^\nu \right) - \nu |\omega^\nu|^2.
\]

(3)

Clearly, the space-integrated expression

\[
\mathcal{T}^\nu = - \rho \int_{\partial B} \mathbf{u}_\phi \cdot \left( \mathbf{u}^\nu \times \omega^\nu - \nu \nabla \times \omega^\nu \right) dA
\]

(4)

represents a transfer from the interaction energy to the energy of rotational flow, which is thereafter disposed by viscosity, and it can be interpreted physically in terms of the flux of vorticity across the streamlines of the potential flow [12,13]. This quantity has furthermore a remarkable instantaneous relation to the drag force of the fluid acting on the body:

\[
\mathbf{F}^\nu = \rho \int_{\partial B} (-p^\omega \mathbf{n} + \mathcal{T}^\omega) dA
\]

(5)
where viscous skin friction or wall stress is given by \( \tau^\nu = 2\nu S^\nu \mathbf{n} = \nu \omega^\nu \times \mathbf{n}, \mathbf{n} \) is the unit normal on \( \partial B \) pointing into \( \Omega \), and \( S^\nu_{ij} = (1/2)(\partial_i u^\nu_j + \partial_j u^\nu_i) \) is the strain-rate tensor. The Josephson-Anderson relation derived in [12] states that the transfer \( \mathcal{D}^\nu := \mathbf{F}^\nu \cdot \mathbf{V} \), the power consumption from drag, instantaneously in time:

\[
\mathcal{D}^\nu = -\rho \int_{\Omega} \mathbf{u}_\phi \cdot (\mathbf{u}^\nu \times \omega^\nu - \nu \nabla \times \omega^\nu) \, dV
\]

\[
= -\rho \int_{\Omega} \nabla \mathbf{u}_\phi : \mathbf{u}^\nu \omega^\nu \mathbf{\otimes} \mathbf{u}^\nu \omega^\nu \, dV + \rho \int_{\partial \Omega} \mathbf{u}_\phi \cdot \tau^\nu_w \, dA \quad (6)
\]

The expression for \( \mathcal{T}^\nu \) in the second line is obtained from the first by simple vector calculus identities and spatial integration by parts.

As already remarked in [12], the latter form of the Josephson-Anderson relation should be valid even for weak Euler solutions \( \mathbf{u} \) obtained as a zero-viscosity limit of the Navier-Stokes solution \( \mathbf{u}^\nu \). It is the purpose of the present paper to derive relation (6) rigorously in the limit \( \nu \to 0 \), under reasonable hypotheses which are sufficient for a weak Euler solution to exist [8]. This approach is obviously connected with the “ideal turbulence” theory of Onsager [20], who posited that infinite Reynolds-number turbulent flows will be described by weak Euler solutions dissipating kinetic energy. For reviews, see [6,11]. In particular, we shall follow the approach of Duchon & Robert [9] to show that the local energy balance equations (2), (3) remain valid in the distributional sense for the inviscid limit. The sink term \( Q^\nu(\mathbf{u}^\nu) = \nu |\omega|^2 \) which appears in Eq. (5) and which describes the viscous dissipation in the rotational wake behind the body will be shown to converge to a non-negative distribution \( Q(\mathbf{u}) \) which corresponds to “inertial energy dissipation” or nonlinear energy cascade in the Euler solution. We thus derive a precise connection in the inviscid limit between the Josephson-Anderson relation and Onsager’s dissipative anomaly.
1.2. Prior Work. Our analysis here will depend essentially upon the approach and results in our preceding paper [21], which studied the conditions necessary for a momentum anomaly to exist in wall-bounded turbulence, a possibility conjectured already by Taylor [24]. As remarked at the end of [21], all of the main results of that work carry over to the local balance of vortex momentum given by Eq. (1). As there, we assume that those equations admit strong solutions for arbitrarily large Reynolds numbers. Then, Theorem 1 of [21] implies in this context that distributional limits of the wall shear stress $\tau_w^\nu$ and the rotational pressure stress $p_{\omega,w}^\nu$ at the wall exist for $\nu \to 0$:

$$\tau_w^\nu \xrightarrow{\nu \to 0} \tau_w$$  \hspace{1cm} (7)  

$$p_{\omega,w}^\nu \xrightarrow{\nu \to 0} p_{\omega,w}$$  \hspace{1cm} (8)

More precisely, these limits exist as distributional sections of the tangent and normal bundles, respectively, of the space-time manifold $((\partial B)_T, T(\partial B)_T)$, where we assume that $B \subset \mathbb{R}^3$ is closed, bounded, and connected, the complement $\Omega = \mathbb{R}^3 \setminus B$ is also connected, and the common boundary $\partial \Omega = \partial B$ is a $C^\infty$ manifold embedded in $\mathbb{R}^3$. See [21], section 2 for our notations and conventions regarding distribution theory on manifolds. The limit results (7), (8) require hypotheses regarding the strong convergence $(u_w^\nu, \rho_w^\nu) \to (u_w, \rho_w)$ as $\nu \to 0$, given precisely in the Theorem below, and the limits are then weak solutions of the inviscid version of Eq. (1):

$$\partial_t u_w = -\nabla \cdot (u_w \otimes u_w + u_w \otimes u_\phi + u_\phi \otimes u_w) - \nabla p_w, \quad \nabla \cdot u_w = 0.$$  \hspace{1cm} (9)

The second main result of [21] had to do with the spatial cascade of momentum and its matching to stress at the wall. In the present context, the spatial momentum-flux or stress consists of the advective contribution:

$$T_w : = u_w \otimes u_w + u_w \otimes u_\phi + u_\phi \otimes u_w$$  \hspace{1cm} (10)

and the isotropic pressure contribution $p_w I$. A turbulent cascade of rotational momentum is defined for the infinite-$Re$ Euler solution by introducing fields $T_{\omega,\ell} = G_\ell \ast T_w$, $p_{\omega,\ell} = G_\ell \ast p_w$, spatially coarse-grained at length-scale $\ell$ with a standard mollifier $G$ and a smooth window function $\eta_{h,\ell}(x) = \theta_{h,\ell}(d(x))$ which =0 when distance to the boundary $d(x) < h$ for $h > \ell$. The turbulent flux of momentum toward the wall is thus defined by the quantity

$$- (\nabla \eta_{h,\ell} \cdot T_{\omega,\ell} + p_{\omega,\ell} \nabla \eta_{h,\ell}).$$  \hspace{1cm} (11)

To justify this physical interpretation it is important to note that for sufficiently small $\epsilon < \eta(\Omega)$, for $\Omega_\epsilon := \{x \in \Omega : d(x) < \epsilon\}$, and for any $x \in \Omega_\epsilon$, there exists a unique point $\pi(x) \in \partial \Omega$ such that

$$d(x) = |x - \pi(x)|, \quad \nabla d(x) = n(\pi(x)).$$  \hspace{1cm} (12)

Thus, $\nabla \eta_{h,\ell}(x) = \theta_{h,\ell}'(d(x))n(\pi(x))$ is the product of an approximate delta function $\theta_{h,\ell}'(d(x)) \sim \delta(d(x) - h)$ and the wall normal vector $n(x) := n(\pi(x))$ extended into the interior, so that (11) indeed measures momentum transfer toward the wall at distance $h$. See section 2 in [21] for a detailed discussion of
these results. Note also for later use that the unit normal vector \( \mathbf{n} \), projection map \( \pi \) and distance function \( d \) all belong to \( C^\infty(\Omega_\epsilon) \) for each \( \epsilon < \eta(\Omega) \).

With this background, Theorem 2 of [21] states that the wall-parallel component of the flux \( \tau \) converges in the limit \( \ell < h \to 0 \) to the wall tangential stress \( \tau_w \) and that the wall-normal component converges to the wall pressure stress \( -p_{\omega,w} \mathbf{n} \). For a precise statement, one must be able to interpret \( \tau \) as a sectional distribution of the tangent bundle and also of the normal bundle. For this purpose, [21] introduced a non-standard space of test functions

\[
\mathcal{D}(\Omega \times (0,T)) := \left\{ \varphi = \phi|_{\Omega \times (0,T)} : \phi \in C^\infty_c(\mathbb{R}^3 \times (0,T)), \quad \text{supp}(\phi) \cap (\Omega \times (0,T)) \neq \emptyset \right\}
\]  

(13)

which are non-vanishing on \( \partial \Omega \), and also sets \( \mathcal{E}_T \) and \( \mathcal{E}_N \) of extension operators where \( \text{Ext} \in \mathcal{E}_T \) is a continuous, linear map \( \text{Ext} : \psi \in \mathcal{D}((\partial B)_T, T^*(\partial B)_T) \mapsto \phi \in \mathcal{D}(\Omega \times (0,T), \mathbb{R}^3) \) such that the restriction \( \phi|_{\partial B}_T \) agrees with \( \psi \), and likewise \( \text{Ext} \in \mathcal{E}_N \) is a continuous, linear map \( \text{Ext} : \psi \in \mathcal{D}((\partial B)_T, N^*(\partial B)_T) \mapsto \phi \in \mathcal{D}(\Omega \times (0,T), \mathbb{R}^3) \) such that \( \phi|_{\partial B}_N = \psi \). The precise statement of Theorem 2 in [21] applied to vortex momentum is that for all \( \text{Ext} \in \mathcal{E}_T \)

\[
- \lim_{h, \ell \to 0} \text{Ext}^*(\nabla \eta_{h,\ell} \cdot \mathbf{T}_{\omega,\ell} + \bar{p}_{\omega,\ell} \nabla \eta_{h,\ell}) = \tau_w \quad \text{in} \quad D'(\partial B)_T, T(\partial B)_T)
\]  

(14)

and for all \( \text{Ext} \in \mathcal{E}_N \)

\[
- \lim_{h, \ell \to 0} \text{Ext}^*(\nabla \eta_{h,\ell} \cdot \mathbf{T}_{\omega,\ell} + \bar{p}_{\omega,\ell} \nabla \eta_{h,\ell}) = -p_{\omega,w} \mathbf{n} \quad \text{in} \quad D'(\partial B)_T, N(\partial B)_T)
\]  

(15)

The final main result of [21] was that, under some additional tenable hypotheses, the advective flux of vortex momentum to the wall must vanish. The added assumptions were, for some \( \epsilon > 0 \), the near-wall boundedness property

\[
\mathbf{u}_w \in L^2((0,\epsilon), L^\infty(\Omega_\epsilon))
\]  

(16)

with \( \Omega_\epsilon := \{ x \in \Omega : \text{dist}(x, \partial B) < \epsilon \} \) and the no-flow-through condition

\[
\lim_{\delta \to 0} \| \mathbf{n} \cdot \mathbf{u}_w \|_{L^2((0,\epsilon), L^\infty(\Omega_\epsilon))} = 0.
\]  

(17)

Then Theorem 3 of [21] implies that, for all \( \text{Ext} \in \mathcal{E}_T \),

\[
- \lim_{h, \ell \to 0} \text{Ext}^*(\nabla \eta_{h,\ell} \cdot \mathbf{T}_{\omega,\ell}) = \tau_w = 0 \quad \text{in} \quad D'(\partial B)_T, T(\partial B)_T)
\]  

(18)

and for all \( \text{Ext} \in \mathcal{E}_N \),

\[
- \lim_{h, \ell \to 0} \text{Ext}^*(\bar{p}_{\omega,\ell} \nabla \eta_{h,\ell}) = -p_{\omega,w} \mathbf{n} \quad \text{in} \quad D'(\partial B)_T, N(\partial B)_T).
\]  

(19)

Here \( \mathcal{E}_T \) is a class of natural extensions which consists of those \( \text{Ext} \in \mathcal{E}_T \) such that \( \forall \psi \in \mathcal{D}((\partial B)_T, T^*(\partial B)_T), \phi = \text{Ext}(\psi) \) satisfies

\[
\| \phi \cdot \mathbf{n} \|_{L^\infty((\Omega_{\epsilon<h^\ell} \times \Omega_h) \times (0,T))} = O(\ell), \quad h, \ell \to 0,
\]  

(20)

which allows to infer that \( \bar{p}_{\omega,\ell} \nabla \eta_{h,\ell} \) gives vanishing contribution to the tangential wall stress in \( [14] \), as would be naturally expected. It follows then from \( [18] \) that there is no viscous momentum anomaly, \( \tau_w = 0 \), so that all drag originates
asymptotically from pressure forces and, furthermore, from \( \epsilon > 0 \) the pressure of the Euler solution away from the wall matches onto the pressure at the wall.

The present work will build on these these results for vortex momentum and extend them to rotational kinetic energy and to interaction energy of potential and rotational flow, obtaining the infinite Reynolds-number limit of the Josephson-Anderson relation in the process. Because kinetic energies are scalar quantities, we can simplify somewhat the approach of \([21]\). Following \([25]\), we extend the set of continuous, linear maps denoted by \( D \) to \( \mathbb{R} \) and by \( D'((\partial B)_T) \) the sectional distributions of that bundle. We refer to these simply as spaces of scalar test functions and of scalar distributions on \((\partial B)_T\), respectively. We then define the class of scalar extensions \( \mathcal{E} \) as the set of continuous, linear maps \( \text{Ext} : \psi \in D((\partial B)_T) \mapsto \phi \in D(\Omega \times (0, T)) \) such that the restriction \( \phi|_{(\partial B)_T} = \psi \). This set \( \mathcal{E} \) is defined more precisely in the following section \([2]\) where it is shown also that \( \mathcal{E} \neq \emptyset \) by construction of a concrete example. We shall need further in our proof the fact that the stationary Euler solution describing potential flow around the body \( B \) at infinity satisfies \( u_\phi \in C^\infty(\Omega) \), which follows from the known smoothness of the solution of the Laplace equation with zero Neumann conditions on a domain \( \Omega \) with smooth boundary \( \partial \Omega \). We shall review these results also in section \([2]\).

Since \( u_\phi \cdot n = 0 \) on \( \partial B \) and since \( \partial B \) is compact, it follows that we may interpret \( u_\phi|_{\partial B} \in D((\partial B)_T, T^* (\partial B)_T) \). Thus, the dot product with the distribution \( \tau_w \in D'((\partial B)_T, T(\partial B)_T) \) obtained by Theorem 1 of \([21]\) can be defined with \( u_\phi \cdot \tau_w \in D'((\partial B)_T) \) by setting

\[
(u_\phi \cdot \tau_w, \psi) := \langle \tau_w, \psi u_\phi|_{\partial B} \rangle, \quad \forall \psi \in D((\partial B)_T). \tag{21}
\]

1.3. Results on Interaction Energy and Josephson-Anderson Relation. We can now state the first main result of the present work, which is analogous to Theorem 1 of \([21]\) for vortex momentum, but here on the infinite Reynolds-number limits of the local balance of the interaction energy \([2]\) and of the Josephson-Anderson relation \([0]\):

**Theorem 1.** We make the following assumptions (which are same as those in Theorem 1 of \([21]\)): Let \((u^\nu_\omega, p^\nu_\omega)\) be strong solutions of Eq. (11) on \( \Omega \times (0, T) \) for \( \nu > 0 \). Assume that \((u^\nu_\omega)_{\nu > 0}\) strongly converges to \( u_\omega \) in \( L^2((0, T), L^2_{\text{loc}}(\Omega)) \):

\[
u \to 0 \quad u^\nu_\omega \xrightarrow{\nu \to 0} u_\omega. \tag{22}
\]

and that \((p^\nu_\omega)_{\nu > 0}\) strongly converges to \( p_\omega \) in \( L^1((0, T), L^1_{\text{loc}}(\Omega)) \):

\[
u \to 0 \quad p^\nu_\omega \xrightarrow{\nu \to 0} p_\omega. \tag{23}
\]

Assume furthermore for some \( \epsilon > 0 \) arbitrarily small

\[
u^\epsilon \text{ uniformly bounded in } L^2((0, T), L^2(\Omega_\epsilon)) \tag{24}\]

\[
u^\epsilon \text{ uniformly bounded in } L^1((0, T), L^1(\Omega_\epsilon)). \tag{25}\]
Then, the limit \((u_\phi, p_\phi)\) is a weak solution of \((9)\) and for this same sequence and for all \(\text{Ext} \in \mathcal{E}\),

$$
\lim_{\nu \to 0} \text{Ext}^*[u_\phi \cdot \nu \nabla \times \omega^\nu] = u_\phi \cdot \tau_w \quad \text{in } D'(\partial B_T)
$$

(26)

Furthermore, the local balance Eq.\((2)\) for the interaction energy holds distributionally in the sense that for all \(\varphi \in D(\Omega \times (0,T))\), with \(\psi = \varphi|_{\partial B}\),

$$
- \int_0^T \int_{\Omega} \partial_t \varphi (u_\omega \cdot u_\phi) \, dV \, dt \\
- \int_0^T \int_{\Omega} \nabla \varphi \cdot \left[ (u_\omega \cdot u_\phi) u + \frac{1}{2} |u_\phi|^2 u_\omega + p_\omega u_\phi + p_\phi u_\omega \right] \, dV \, dt \\
= - (u_\phi \cdot \tau_w, \psi) + \int_0^T \int_{\Omega} \varphi (\nabla u_\phi : u_\omega \otimes u_\omega) \, dV \, dt
$$

(27)

Finally, if \((22)\) is strengthened to global \(L^2\) convergence

$$
u \to 0 \quad \lim_{\nu \to 0} L^2((0,T), L^2(\Omega)) 
= u_\omega,
$$

(28)

then the Josephson-Anderson relation \((9)\) holds also in the inviscid limit, in the sense that \(D = \lim_{\nu \to 0} F^\nu \cdot \nu \) exists in \(D'((0,T))\) with

$$
(D, \chi) = - \rho \int_{\Omega} \chi \nabla u_\phi : u_\omega \otimes u_\omega \, dV \, dt + \rho (u_\phi \cdot \tau_w, \chi \otimes 1)
$$

(29)

for all \(\chi \in D((0,T))\).

**Remark 1.** Although we have so far presented our results by assuming a fluid velocity at infinity \(V\) which is time-independent, the proof presented below should allow any function \(V(t)\) which is \(C^\infty\) in time, corresponding to smooth translational motion of the body \(B\). In that case the potential Euler solution \(u_\phi\) is of course also time-dependent. See footnote [91] in [12] and section 2.2 below for more discussion. It would be interesting to generalize these results further to allow for solid-body dynamics of \(B\) including possible rotation, as in [23].

**Remark 2.** Just as in [21], Theorem 1, condition \((23)\) on convergence of pressure can be replaced with any assumption guaranteeing that along a suitable subsequence of \(\nu, p_\omega^\nu \to p_\omega \in L^1((0,T), L^1_{loc}(\Omega))\) in the sense of distributions. E.g.

$$
p_\omega^\nu \text{ uniformly bounded in } L^q((0,T), L^q_{loc}(\Omega)) \text{ for some } q > 1
$$

(30)

would suffice. See [21], Remark 2.

Our next result is analogous to Theorem 2 in [21], which involved spatial momentum cascade, but considering now the spatial flux of interaction energy. Similar to the definition \((1)\) of momentum flux, we apply spatial coarse-graining and windowing to Eq.\((1)\) for \(u_\omega^\nu\) and to the Euler equation for \(u_\phi\), yielding the balance for interaction energy in length scales \(> \ell\) and wall distances \(> h\) :

$$
\partial_t (\eta_{h,\ell} u_\omega^\nu \cdot u_\phi) + \nabla \cdot (\eta_{h,\ell} J_\omega^\nu) = \nabla \cdot q_{h,\ell} + \eta_{h,\ell} \nabla \cdot (u_\omega^\nu \cdot u_\phi^\nu) + \nabla \cdot \eta_{h,\ell} u_\phi^\nu \\
+ \eta_{h,\ell} \nabla u_\phi^\nu \cdot (\tau_\ell (u_\omega, u_\phi^\nu) + \tau_\ell (u_\phi, u_\omega^\nu) + \tau_\ell (u_\phi^\nu, u_\phi^\nu)) - \eta_{h,\ell} u_\phi^\nu \cdot (\nabla \cdot (u_\phi, u_\phi))
$$

(31)
Here \( \tau_\ell(f,g) := (\overline{fg})_\ell - f_\ell g_\ell \) for any functions \( f, g \) on \( \Omega^\ell := \{ x \in \Omega : d(x) > \ell \} \) and spatial flux of interaction energy is given by

\[
J^\nu_{\phi,\ell} = (\overline{u^\nu_{\phi,\ell} \cdot \overline{u_{\phi,\ell}}} + p^\nu_{\phi,\ell} u^\nu_{\phi,\ell} + \frac{1}{2} |u^\nu_{\phi,\ell}|^2 + \overline{p_{\phi,\ell} u^\nu_{\phi,\ell}} - \nu \overline{u_{\phi,\ell} \times \omega^\nu_{\ell}} + (\tau_\ell(u^\nu_{\phi,\ell}, u^\nu_{\phi,\ell}) + \tau_\ell(u^\nu_{\phi,\ell}, u^\nu_{\phi,\ell}) + \tau_\ell(u^\nu_{\phi,\ell}, u^\nu_{\phi,\ell})) \cdot \overline{u_{\phi,\ell}} \tag{32}
\]

The balance equation \((31)\) holds also for the limiting fields \((u_\omega, p_\omega)\) of Theorem 1 as stated by the following Proposition, whose easy proof is left to the reader:

**Proposition 1.** Under the assumptions \((22)-(23)\) of Theorem 1, the following equation holds pointwise for \( x \in \Omega \) and distributionally for \( t \in (0, T) \):

\[
\partial_t (\eta_{h,\ell} u_\omega \cdot u_{\phi,\ell}) + \nabla \cdot (\eta_{h,\ell} J_{\phi,\ell}) = \nabla \eta_{h,\ell} \cdot J_{\phi,\ell} + \eta_{h,\ell} \nabla u_{\phi,\ell} \cdot \nabla u_{\phi,\ell} : + \eta_{h,\ell} \cdot \tau_\ell(\nabla u_\omega, u_\omega) + \tau_\ell(\nabla u_\omega, u_\omega) - \eta_{h,\ell} u_{\phi,\ell} : \left( \nabla \cdot \tau_\ell(u_\omega, u_\omega) \right).
\tag{33}
\]

where \( J_{\phi,\ell} \) is given by the formula \((32)\) with \( \nu \mapsto 0\), \((u^\nu_{\phi,\ell}, p^\nu_{\ell}) \mapsto (u_\omega, p_\omega)\).

Our second main result is then:

**Theorem 2.** Assume conditions \((23)-(25)\) in Theorem 1. Then

\[
- \lim_{h,\ell \to 0} \text{Ext}^\dagger [\nabla \eta_{h,\ell} \cdot J_{\phi,\ell}] = u_\phi \cdot \tau_w \quad \text{in } D'(\partial B_T)
\tag{34}
\]

for all \( \text{Ext} \in \mathcal{E} \).

This theorem states that the spatial cascade of interaction energy to the wall equals the skin friction term in the inviscid Josephson-Anderson relation \((33)\).

Finally, Theorem 3 of \((21)\) showed that momentum cascade to the wall must vanish when the limiting Euler solution satisfies the no-flow-through condition at the wall in a suitable sense. An exact analogue of this result for interaction energy is given by the following:

**Theorem 3.** Let \((u_\omega, p_\omega)\) be the weak solution of Eq.\((9)\) from Theorem 1, with \( u_\omega \in L^2(0,T; L^2(\Omega)) \cap L^2(0,T; L^2(\Omega)) \), \( p_\omega \in L^2((0,T);L^1(\Omega)) \) and for some \( \epsilon > 0 \). Assume further the stronger near-wall boundedness property

\[
u_{\ell} \quad \text{in } D'(\partial B_T)
\tag{37}
\]

Then, for all \( \text{Ext} \in \mathcal{E} \)

\[
- \lim_{h,\ell \to 0} \text{Ext}^\dagger [\nabla \eta_{h,\ell} \cdot J_{\phi,\ell}] = 0 \quad \text{in } D'(\partial B_T)
\tag{36}
\]

and the no-flow-through condition at the boundary in the sense

\[
\lim_{\ell \to 0} |n \cdot u_\omega|_{L^2(0,T; L^\infty(\Omega_\ell))} = 0.
\tag{35}
\]
so that $u_\phi \cdot w = 0$. Thus, an inviscid form of the balance Eq. (2) for the interaction energy holds distributionally, in the sense that for all $\varphi \in \mathcal{D}(\bar{\Omega} \times (0,T))$

$$
\begin{align*}
-\int_0^T \int_{\Omega} \partial_t \varphi (u_\omega \cdot u_\omega) dV dt \\
- \int_0^T \int_{\Omega} \nabla \varphi \cdot \left( [u_\omega \cdot u_\phi] u + \frac{1}{2} |u_\phi|^2 u_\omega + p_\omega u_\phi + p_\phi u_\omega \right) dV dt \\
= \int_0^T \int_{\Omega} \varphi (\nabla u_\phi : u_\omega \otimes u_\omega) dV dt.
\end{align*}
$$

(38)

If convergence to the inviscid limit holds in the global $L^2$-sense (28), then the Josephson-Anderson relation (6) is valid in the inviscid form

$$
D = -\rho \int_{\Omega} \nabla u_\phi : u_\omega \otimes u_\omega dV.
$$

(39)

The implication is that cascade of interaction energy to the wall must vanish if the strong near-wall boundedness and no-flow-through conditions are satisfied.

1.4. Results on Energy Balance of the Rotational Flow. We next wish to show that results hold analogous to those of Duchon & Robert on turbulent kinetic energy balance [9], but now for the energy in the rotational wake. We have:

**Theorem 4.** Let $(u_\nu^\omega, p_\nu^\omega)$ be strong solutions of equations (1) on $\bar{\Omega} \times (0,T)$. Assume that $(u_\nu^\omega)_{\nu > 0}$ strongly converges to $u_\omega$ in $L^3((0,T),L^3_{loc}(\Omega))$,

$$
\lim_{\nu \to 0} u_\nu^\omega \rightharpoonup u_\omega \quad \text{in} \quad L^3((0,T),L^3_{loc}(\Omega)),
$$

(40)

and

$$
p_\nu^\omega \text{ uniformly bounded in } L^{\frac{3}{2}}((0,T),L^2_{loc}(\Omega)).
$$

(41)

Assume further for some $\epsilon > 0$

$$
u_\omega \text{ uniformly bounded in } L^3((0,T),L^3(\Omega_\epsilon))
$$

(42)

and

$$
p_\omega \text{ uniformly bounded in } L^{\frac{3}{2}}((0,T),L^{\frac{3}{2}}(\Omega_\epsilon)).
$$

(43)

Then, along a suitable subsequence $p_\nu^\omega \to p_\omega \in L^{\frac{3}{2}}((0,T),L^{\frac{3}{2}}_{loc}(\Omega))$ distributionally, so that $(u_\omega, p_\omega)$ is a weak solution of the inviscid equation (9), satisfying the results (26), (27) of Theorem 1. Also, $Q^\nu = \nu |\omega^\nu|^2$ converges for this subsequence to a positive linear functional $Q$ on $\mathcal{D}(\bar{\Omega} \times (0,T))$, in the sense that $\forall \varphi \in \mathcal{D}(\bar{\Omega} \times (0,T))$,

$$
\lim_{\nu \to 0} \nu_0 \int_0^T \int_{\Omega} \varphi Q^\nu dV dt = \langle Q, \varphi \rangle
$$

(44)
with $\langle Q, \varphi \rangle \geq 0$ for $\varphi \geq 0$. Finally, an inviscid version of the balance equation for rotational energy holds in the sense that for all $\varphi \in \mathcal{D}(\Omega \times (0, T))$, $\psi = \varphi|_{\partial B}$,

$$\begin{align*}
- \int_0^T \int_\Omega \frac{1}{2} \partial_t \varphi |u_\omega|^2 \, dV \, dt - \int_0^T \int_\Omega \nabla \varphi \cdot \left[ \frac{1}{2} |u_\omega|^2 u + p_\omega u_\omega \right] \, dV \, dt \\
= \langle u_\psi, \tau_w, \psi \rangle - \langle Q, \varphi \rangle - \int_0^T \int_\Omega \varphi \nabla u_\psi : u_\omega \otimes u_\omega \, dV \, dt
\end{align*}
$$

(45)

Remark 3. The result (44) is analogous to Proposition 4 of Duchon-Robert [9], which showed that the inviscid limit of viscous dissipation exists as a non-negative distribution. In fact, the linear functional $Q$ in our theorem restricted to the subspace $D(\Omega \times (0, T))$ is a non-negative distribution. Thus, by standard representation theory for distributions ([2], Example 12.5; [22], Exercise 6.4), there is a non-negative Radon measure on $\Omega \times (0, T)$, which we shall also write as $Q$ by an abuse of notation, so that

$$\langle Q, \varphi \rangle = \int_{\Omega \times (0, T)} \varphi(x, t) Q(dx \, dt), \quad \forall \varphi \in D(\Omega \times (0, T)).
$$

(46)

We have not shown that a Radon measure $Q$ exists on $\Omega \times (0, T)$ such that the representation (46) holds for all $\varphi \in \mathcal{D}(\Omega \times (0, T))$, but it is plausible to conjecture that this is true. If so, another natural question is whether $Q((\partial B)_T) > 0$ or = 0. The latter question is related to the famous theorem of Kato [16, 23], who showed that, if a smooth Euler solution $u$ exists over the time interval $(0, T)$, then $u^\nu \to u$ in $L^2((0, T), L^2(\Omega))$ when $u_0^\nu \to u_0$ in $L^2(\Omega)$ and $Q^\nu(\Omega_{\omega_0} \times (0, T)) \to 0$ for some constant $c > 0$. Under these specific assumptions, a dissipative Euler solution obtained in the inviscid limit plausibly must have $Q((\partial B)_T) > 0$.

Remark 4. Under the stronger assumptions of Theorem 1, then also the conclusions of Theorem 2 hold, so that one can combine Eq. (27) of that theorem with the present result Eq. (45) to obtain the combined balance:

$$\begin{align*}
- \int_0^T \int_\Omega \partial_t \varphi \left[ (u_\omega \cdot u_\phi) + \frac{1}{2} |u_\omega|^2 \right] \, dV \, dt \\
- \int_0^T \int_\Omega \nabla \varphi \cdot \left[ \left( (u_\omega \cdot u_\phi) + \frac{1}{2} |u_\omega|^2 + p_\omega \right) u + \left( \frac{1}{2} |u_\phi|^2 + p_\phi \right) u_\omega \right] \, dV \, dt \\
= - \langle Q, \varphi \rangle
\end{align*}
$$

(47)

This combined quantity was termed the “relative kinetic energy” in [12], because it measures the total energy of the fluid relative to the energy of the potential Euler solution and it is thus locally conserved when $Q = 0$.

Our next result is a close analogue of Theorem 2 but considering now the spatial flux of rotational flow energy. By applying spatial coarse-graining and windowing to Eq. (1) for $u_\nu^\nu$, we obtain a balance equation for rotational energy in length scales $> \ell$ and wall distances $> h$:

$$\begin{align*}
\partial_t \left( \frac{1}{2} \eta_{h, \ell} |u_\nu^\nu|_\ell^2 \right) + \nabla \cdot (\eta_{h, \ell} \mathbf{j}_\nu^\nu) = \nabla \eta_{h, \ell} \cdot \mathbf{j}_\nu^\nu - \eta_{h, \ell} \nabla u_\phi, \ell : u_\nu^\nu, \ell \\
+ \eta_{h, \ell} \nabla u_\nu^\nu, \ell : (\tau_\ell(u_\nu^\nu, u_\nu^\nu) + \tau_\ell(u_\phi, u_\nu^\nu) + \tau_\ell(u_\nu^\nu, u_\phi)) - \nu \eta_{h, \ell} |\omega_\nu^\nu|_\ell^2
\end{align*}
$$

(48)
with spatial flux of rotational energy given by

\[ J_{\omega,\ell}^\nu = \frac{1}{2} |\bar{u}_{\omega,\ell}^\nu|^2 \bar{u}_{\omega,\ell}^\nu + \bar{p}_{\omega,\ell} u_{\omega,\ell}^\nu - \nu \bar{u}_{\omega,\ell}^\nu \times \bar{\omega}_{\ell}^\nu \]

The balance equation (48) holds also for the limiting fields \((u_\omega, p_\omega)\) of Theorem 4, as stated by the following easy Proposition, with proof left to the reader:

**Proposition 2.** Under the assumptions (40)-(44) of Theorem 4, the following equation holds pointwise for \(x \in \Omega\) and distributionally for \(t \in (0, T)\):

\[
\partial_t \left( \frac{1}{2} \eta_{h,\ell} |u_{\omega,\ell}^\nu|^2 \right) + \nabla \cdot (\eta_{h,\ell} J_{\omega,\ell}^\nu) = \nabla \eta_{h,\ell} \cdot J_{\omega,\ell}^\nu - \eta_{h,\ell} \nabla u_{\phi,\ell} : \bar{u}_{\omega,\ell}^\nu u_{\omega,\ell}^\nu + \eta_{h,\ell} \nabla u_{\omega,\ell} : (\tau_\ell(u_{\omega,\ell}), u_{\omega}) + \tau_\ell(u_{\phi,\ell}, u_{\omega})
\]

where \(J_{\omega,\ell}^\nu\) is given by the formula (49) with \(\nu \mapsto 0, (u_{\omega,\ell}^\nu, p_{\omega,\ell}^\nu) \mapsto (u_\omega, p_\omega)\).

Our next main result then relates the spatial cascade of rotational energy toward the wall and the skin friction term in the inviscid Josephson-Anderson relation (29). However, the statement is more complex than our previous Theorem 2 for interaction energy, because it involves also the quantity \(\eta_{h,\ell} \nabla u_{\omega,\ell} : \tau_\ell(u_{\omega}, u_\omega)\) which represents scale-cascade of rotational energy and as well the anomalous energy dissipation \(Q\) in the rotational wake. Thus, we find:

**Theorem 5.** Let \((u_\omega, p_\omega)\) be the limiting weak solutions of the inviscid equation (9) obtained in Theorem 4. For \(0 < \ell < h\), and \(\forall \varphi \in \mathcal{D}(\Omega \times (0, T))\) with \(\psi = \varphi|_{\partial B}\),

\[
\begin{align*}
- \lim_{h,\ell \to 0} \int_0^T \int_\Omega & \left( \varphi \nabla \eta_{h,\ell} \cdot J_{\omega,\ell} + \eta_{h,\ell} \nabla u_{\omega,\ell} : \tau_\ell(u_{\omega}, u_\omega) \right) \, dV \, dt \\
& = -\langle u_\phi \cdot \tau_\omega, \psi \rangle + \langle Q, \varphi \rangle.
\end{align*}
\]

The result (51) is not entirely satisfactory, because one would expect that the first and second terms on the lefthand side equal respectively the first and second terms on the righthand side, and not only when added together. We can obtain this physically more natural statement, but only by assuming conditions under which the first terms of each side of (51) vanish:

**Theorem 6.** Let \((u_\omega, p_\omega)\) be the limiting weak solutions of the inviscid equation (9) obtained in Theorem 4 with \(u_\omega \in L^3((0, T), L^3_\text{loc}(\Omega)) \cap L^3((0, T), L^3(\Omega))\) and \(p_\omega \in L^{\frac{3}{2}}((0, T), L^{\frac{3}{2}}_\text{loc}(\Omega)) \cap L^{\frac{3}{2}}(0, T; L^{\frac{3}{2}}(\Omega))\) for some \(\epsilon > 0\). Assume further the stronger near-wall boundedness property

\[
u \omega_0 \in L^3((0, T), L^\infty(\Omega_\epsilon)), \quad p_\omega \in L^{\frac{3}{2}}((0, T), L^\infty(\Omega_\epsilon))
\]

and the no-flow-through condition at the boundary in the sense

\[
\lim_{\delta \to 0} ||n \cdot u_\omega||_{L^3((0, T), L^\infty(\Omega_\delta))} = 0.
\]

Then \(u_\phi \cdot \tau_\omega = 0\) in \(D'(\partial B)^*\) and for all \(\text{Ext} \in \mathcal{E}\),

\[
\lim_{h,\ell \to 0} \text{Ext}^* \left[ \nabla \eta_{h,\ell} \cdot J_{\omega,\ell} \right] = 0.
\]
In consequence, an inviscid form of the balance equation \(2\) for rotational energy holds distributionally, in the sense that for all \(\varphi \in D(\Omega \times (0, T))\),

\[
- \int_0^T \int_\Omega \frac{1}{2} \partial_t \varphi |u_\omega|^2 \, dV \, dt - \int_0^T \int_\Omega \nabla \varphi \cdot \left[ \frac{1}{2} |u_\omega|^2 u + p_{\omega} u_\omega \right] \, dV \, dt = -(Q, \varphi) - \int_0^T \int_\Omega \varphi \nabla u_\phi : u_\omega u_\omega \, dV \, dt
\]

and furthermore

\[
- \lim_{h, \ell \to 0} \int_0^T \int_\Omega \varphi \eta_h, \ell \nabla u_{\omega, \ell} : \tau_\ell(u_{\omega, \ell}, u_\omega) \, dV \, dt = (Q, \varphi)
\]

Remark 5. The results \((55)\) and \((56)\) of Theorem \([6]\) correspond exactly to those in Proposition 2 of Duchon & Robert \([9]\), who considered \(L^3\) weak solutions of incompressible Euler equation on a torus and derived a local energy balance containing a term due to “inertial energy dissipation,” analogous to the left-hand side of our Eq.\((56)\). Furthermore, the equality in Eq.\((56)\) corresponds to the result in Proposition 4 of \([9]\), stating that the “inertial dissipation” coincides with the zero-viscosity limit of the viscous energy dissipation, assuming convergence to the weak Euler solution in strong \(L^3\) sense as the viscosity tends to zero.

Remark 6. Theorem \([6]\) implies a strong-weak uniqueness result for any viscosity weak solution \(u_\omega \in L^\infty((0, T), L^2(\Omega))\). In fact, the quantity

\[
E_\omega(t) := \frac{1}{2} \|u_\omega(t)\|_{L^2(\Omega)}^2 = \frac{1}{2} \|u(t) - u_\phi(t)\|_{L^2(\Omega)}^2
\]

coincides with the “relative energy” used by Wiedemann \([26]\) to prove strong-weak uniqueness. We do not give full details here, but just sketch main ideas. Under the assumptions of Theorem \([6]\) only we obtain that for a.e. \(t \in (0, T)\)

\[
E_\omega(t) - E_\omega(0) \leq \langle u_\phi \cdot \tau_w, 1 \rangle - \int_0^t \int_\Omega \nabla u_\phi : u_\omega \otimes u_\omega \, dV \, ds
\]

This inequality follows from the result \((15)\) of Theorem \([4]\) by substituting \(\varphi = \chi_{[0, t]}^\epsilon \chi_R^\eta\) where \(\chi_{[0, t]}^\epsilon\) is an \(\epsilon\)-smoothed characteristic function of \([0, T]\) and \(\chi_R^\eta\) is an \(\eta\)-smoothed characteristic function of \(U_\eta = \Omega \cap B(0, R)\) and by then taking the limits \(\epsilon \to 0\) and \(R \to \infty\). Therefore, if we use the consequence \(u_\phi \cdot \tau_w = 0\) of Theorem \([6]\) then it easily follows from \((58)\) that

\[
E_\omega(t) \leq E_\omega(0) + C \rho \int_0^t \|\nabla u_\phi(s)\|_{L^\infty(\Omega)} E_\omega(s) \, ds
\]

and thus by Gronwall inequality

\[
E_\omega(t) \leq E_\omega(0) \exp \left( C \rho \int_0^t \|\nabla u_\phi(s)\|_{L^\infty(\Omega)} \, ds \right).
\]

In particular, we see that if \(u_0 = u_\phi\), then \(u(t) = u_\phi\) for a.e. \(t \in [0, T]\). Note that \((58)\) can be derived also when initial data \(u_\phi^0 \to u_\phi\) strongly in \(L^2(\Omega)\), by taking \(\varphi = \chi_{[0, t]}^\epsilon \chi_R^\eta\) in \((11)\) below for \(\nu > 0\), and then taking the successive limits \(\epsilon \to 0\), \(\nu \to 0\), and \(R \to \infty\). Thus, \(u_\phi \cdot \tau_w = 0\) implies strong-weak uniqueness even if the initial data satisfy \(u_\phi^0 = 0\) at \(\partial \Omega\) because of a thin boundary layer.
2. Preliminaries

We present briefly here some background material needed for our proofs.

2.1. Extension operators. Extension operators of the type discussed in the introduction are defined more precisely by

\[ \text{Ext} : \psi \in D((\partial B)_T) \mapsto \varphi \in \bar{D}(\Omega \times (0, T)) \]  

as linear and continuous operators, satisfying pointwise equality \( \phi|_{(\partial B)_T} = \psi \). Here \( \text{Ext} \) is continuous in the sense that for a multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) with \( |\alpha| \leq N \), \( \forall (x, t) \in \Omega \times (0, T) \)

\[ |D^\alpha \varphi(x, t)| = |D^\alpha \text{Ext}(\psi)(x, t)| \lesssim \sup_{i \in I} p_{N, m, i}(\psi) \]  

where \( \lesssim \) denotes inequality with constant prefactor depending on the domain \( (\partial B)_T \) and the extension operator \( \text{Ext} \). Here, \( (p_{N, m, i}) \) are the seminorms associated with the Fréchet space \( D((\partial B)_T) \), where \( i \in I \) index a system of smooth charts \( \bigcup_{i \in I} (V_i, \phi_i) \), \( \phi_i : V_i \subset (\partial B)_T \to \mathbb{R}^3 \), and \( m \) is the index of a fundamental increasing sequence \( (K^{(i)}_m) \) of compact subsets of \( \phi_i(V_i) \). See section 2 in [21].

We denote by \( \mathcal{E} \) the set of all such linear and continuous operators from \( D((\partial B)_T) \) to \( \bar{D}(\Omega \times (0, T)) \). We show that \( \mathcal{E} \) is non-empty by constructing an extension operator explicitly. For \( 0 < \epsilon < \eta(\Omega) \), we define

\[ \text{Ext}^0 : \psi \in D((\partial B)_T) \mapsto \varphi \in \bar{D}(\Omega \times (0, T)) \]  

by means of the explicit expression

\[ \varphi(x, t) = \begin{cases} \exp \left( -\frac{d(x)}{\epsilon - d(x)} \right) \psi(\pi(x), t), & d(x) < \epsilon \\ 0, & d(x) \geq \epsilon \end{cases} \]  

for any \( 0 < \epsilon < \eta(\Omega) \). \( \text{Ext}^0 \) is clearly linear in \( \psi \) by definition. \( \varphi \) is smooth by the smoothness of distance function \( d \) and projection \( \pi \). One can easily obtain the bound (62) for \( \text{Ext}^0 \) by product rule and chain rule in calculus.

We use extension operators to identify \( F \in D'(([0, T), C^\infty(\Omega))] \) with distributions on the space-time boundary \( (\partial B)_T \). For each \( \text{Ext} \in \mathcal{E} \), we define

\[ \text{Ext}^* : D'([0, T), C^\infty(\Omega))] \to D'((\partial B)_T) \]  

\[ F \mapsto \text{Ext}^*(F) \]  

as follows:

\[ \langle \text{Ext}^*(F), \psi \rangle := \langle F, \text{Ext}(\psi) \rangle \]  

for all \( \psi \in D((\partial B)_T) \). The linearity and continuity properties of \( \text{Ext}^*(F) \) follow from those of \( \text{Ext} \in \mathcal{E} \), so that \( \text{Ext}^*(F) \in D'((\partial B)_T) \). This identification of functions on the whole domain with distributions in \( D'((\partial B)_T) \) depends of course on the choice of the extension operator \( \text{Ext} \).
2.2. Potential flow solution. The potential-flow solution \( u_\phi \) of the incompressible Euler equation is given by \( u_\phi = \nabla \phi \) for the velocity potential \( \phi \) which solves the following Neumann problem for the Laplace equation:

\[
\Delta \phi = 0 \\
\left. \frac{\partial \phi}{\partial n} \right|_{\partial B} = 0, \quad \phi \sim |x| \to \infty \quad V(t)x
\]

(67)

allowing for the moment a time-dependent external velocity field \( V(t) = V(t)x \).

The Euler pressure \( p_\phi \) is then obtained from the Bernoulli equation

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + p_\phi = c(t)
\]

(68)

and is unique up to the arbitrary space-independent constant \( c(t) \). For our proofs below it is essential that \( \phi \in C^\infty(\bar{\Omega} \times (0, T)) \) and thus possesses smoothness up to the boundary \( \partial \Omega \).

In the general case of smoothly time-dependent velocity \( V(t) \), this property will depend upon differentiability of the solution of the Neumann problem (67) in the boundary data. Here, for simplicity, we shall consider only time-independent flow at infinity, so that the required smoothness follows from the following result:

Proposition 3. The Neumann problem (67) of the Laplace equation for the case \( V(t) \equiv V \) has a unique time-independent solution \( \phi \in C^\infty(\bar{\Omega}) \).

Proof. This result is essentially classical but we provide a proof here for completeness. By the mapping \( \tilde{\phi} := \phi - Vx \), any solution \( \phi \) of (67) is transformed into a solution \( \tilde{\phi} \) of the following inhomogeneous Neumann problem:

\[
\Delta \tilde{\phi} = 0 \\
\left. \frac{\partial \tilde{\phi}}{\partial n} \right|_{\partial B} = -V\hat{x} \cdot \hat{n}, \quad \tilde{\phi} \sim |x| \to \infty 0
\]

(69)

From classical potential theory, a unique solution \( \tilde{\phi} \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) of (69) exists. See e.g. [17, Theorem 6.10.6; [19, Theorem 2.1. Since \( \tilde{\phi} \) is harmonic in the open set \( \Omega \), necessarily \( \tilde{\phi} \in C^\infty(\Omega) \). See e.g. [10]. Thus, \( \phi = \tilde{\phi} + Vx \) is also a smooth function in \( C^\infty(\Omega) \) and in \( C^3(\bar{\Omega}) \).

It then suffices to show \( \phi \) is smooth in a neighborhood of the boundary \( \partial \Omega \). Since \( B \) is compact in \( \mathbb{R}^3 \), \( B \subseteq B(0, R) \) for some large \( R > 0 \). Define \( U := B(0, R) \cap \Omega \), which has \( C^\infty \) smooth boundary \( \partial U = \partial \Omega \cup \partial B(0, R) \). For some sufficiently small \( r \), we have \( \Omega_2 \subset U \). To further localize near the boundary, we consider a smooth non-increasing function \( \tilde{\theta} : \mathbb{R} \to [0, 1] \) such that \( \tilde{\theta} = 1 \) in \((-\infty, r]\) and 0 in \([2r, +\infty)\). We denote \( \xi(x) := \tilde{\theta}(d(x)) \). As we reviewed in the introduction, the distance function \( d \) is \( C^\infty \) because of \( C^\infty \) smoothness of \( \partial \Omega \).

Thus, \( \xi \in C^\infty(\bar{\Omega}) \), is supported in \( \Omega_2 \) and \( \xi \equiv 1 \) in \( \Omega \). Since \( \phi|_U \in C^1(U) \), then \( \xi \phi \in H^1(U) \) is a weak solution of the following homogeneous Neumann problem for the Poisson equation:

\[
\Delta(\xi \phi) = \phi \Delta \xi + 2 \nabla \phi \cdot \nabla \xi := f \quad \text{in } U
\]

(70)

\[
\left. \frac{\partial (\xi \phi)}{\partial n} \right|_{\partial U} = 0 \quad \text{on } \partial U.
\]

(71)
By definition, \( f \in C^\infty(\Omega) \), since \( \phi \) is \( C^\infty \) in \( \Omega \) and \( \Delta \xi = \nabla \xi = 0 \) in \( \Omega_r \). Furthermore \( \int_\Omega f \, dV = 0 \). Since \( \partial U \) is \( C^\infty \), we can deduce from boundary elliptic regularity theory, such as Theorem 4 in section 4.2 of \cite{18}, that

\[
\|\phi\|_{H^{k+2}(\Omega_r)} \leq \|\xi \phi\|_{H^{k+2}(U)} \lesssim \|f\|_{H^k(U)} < \infty \tag{72}
\]

for all integers \( k \geq 1 \). Here, the inequalities \( \lesssim \) above with constant prefactors depend only on \( \Omega, R, r \). The Sobolev embedding theorem implies \( \phi \in C^k(\Omega_r) \) for any integer \( k \geq 0 \), and thus \( \phi \in C^\infty(\Omega_r) \). Together with the interior smoothness of \( \phi \), we conclude that \( \phi \in C^\infty(\bar{\Omega}) \).

\[\square\]

3. Proof of Theorems 1-3

3.1. Proof of Theorem 1

We take an arbitrary \( \varphi \in \mathcal{D}(\bar{\Omega} \times (0,T)) \) and let \( \psi = \varphi|_{\partial B} \). Testing the interaction energy equation (2) against \( \varphi \) yields

\[
\begin{align*}
- \int_0^T \int_\Omega \partial_t \varphi (u_\varphi \cdot u_\varphi) \, dV \, dt \\
- \int_0^T \int_\Omega \nabla \varphi \cdot \left[ (u_\varphi \cdot u_\varphi) u_\varphi + \frac{1}{2} |u_\varphi|^2 u_\varphi + p_\varphi u_\varphi + p_\varphi u_\varphi \right] \, dV \, dt \\
= \int_0^T \int_\Omega \varphi (u_\varphi \cdot \nu \Delta u_\varphi) \, dV \, dt + \int_0^T \int_\Omega \varphi (\nabla u_\varphi : u_\varphi \otimes u_\varphi) \, dV \, dt
\end{align*}
\]  \tag{73}

As \( \varphi \in \mathcal{D}(\bar{\Omega} \times (0,T)) \), there exists a compact subset \( K_\varphi \subset \mathbb{R}^3 \) such that

\[
\text{supp}(\varphi) \subset K_\varphi \times (0,T) \subset \bar{\Omega} \times (0,T) \tag{74}
\]

It follows by Green’s identity and stick boundary conditions for the velocity \( u_\varphi \) that

\[
- \int_0^T \int_\Omega \varphi u_\varphi \cdot \nu \Delta u_\varphi \, dV \, dt \\
= \int_0^T \int_{\partial B \times \mathbb{R}^3} \psi u_\varphi \cdot \tau_\varphi \, dS \, dt - \int_0^T \int_\Omega \nu \Delta (\varphi u_\varphi) \cdot u_\varphi \, dV \, dt \tag{75}
\]

with \( \tau_\varphi = \nu \partial u_\varphi / \partial n \). We obtain an upper bound by Cauchy-Schwarz

\[
\left| \int_0^T \int_\Omega \nu \Delta (\varphi u_\varphi) \cdot u_\varphi \, dV \, dt \right| \leq \nu \left( \int_0^T \int_{K_\varphi} |u_\varphi|^2 \, dV \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |\Delta (\varphi u_\varphi)|^2 \, dV \, dt \right)^{\frac{1}{2}} \tag{76}
\]

which vanishes as \( \nu \to 0 \). By Proposition 3, \( u_\varphi \) is smooth everywhere in \( \bar{\Omega} \times (0,T) \) and tangential at the boundary \( \partial B \), so that \( \varphi u_\varphi \in \mathcal{D}(\bar{\Omega} \times (0,T), \mathbb{R}^3) \), with \( \langle \varphi u_\varphi \rangle|_{\partial B} = \psi u_\varphi|_{\partial B} \) and \( u_\varphi|_{\partial B} \cdot n \equiv 0 \). We can thus identify \( \psi u_\varphi|_{\partial B} \) as a smooth section of the cotangent bundle \( D((\partial B)_T, T^* (\partial B)_T) \), via its graph as the natural embedding

\[
\psi u_\varphi|_{\partial B}(x, t) \mapsto (x, t, \psi u_\varphi|_{\partial B}(x, t), 0). \tag{77}
\]
Thus by the convergence of skin friction \( \int \), we have
\[
\int_0^T \int_{\partial B} \psi \mathbf{u}_\phi \cdot \tau_\nu^w \, dS \, dt = \langle \tau_\nu^w, \psi \mathbf{u}_\phi |_{\partial B} \rangle \xrightarrow{\nu \to 0} \langle \tau_\nu^w, \psi \mathbf{u}_\phi |_{\partial B} \rangle = \langle \mathbf{u}_\phi \cdot \tau_\nu^w, \psi \rangle. \tag{78}
\]
Combining (75)-(78) gives
\[
- \int_0^T \int_{\Omega} \phi \mathbf{u}_\phi \cdot \nu \triangle \mathbf{u}^w \, dV \, dt \xrightarrow{\nu \to 0} \langle \mathbf{u}_\phi \cdot \tau_\nu^w, \psi \rangle. \tag{79}
\]
Note that for every extension operator \( \text{Ext} \in \mathcal{E} \), we can define \( \mathbf{u}_\phi \cdot (\nu \nabla \times \omega^w) = -\mathbf{u}_\phi \cdot (\nu \triangle \mathbf{u}^w) \) as a distribution in \( D'((\partial B)_T) \) via
\[
\langle \text{Ext}^*(\mathbf{u}_\phi \cdot (\nu \nabla \times \omega^w)), \psi \rangle = -\int_0^T \int_{\Omega} \phi (\mathbf{u}_\phi \cdot (\nu \triangle \mathbf{u}^w)) \, dV \, dt \tag{80}
\]
for all \( \psi \in D((\partial B)_T) \) with \( \varphi = \text{Ext}(\psi) \in \tilde{D}(\tilde{\Omega} \times (0, T)) \). Thus, as \( \nu \to 0 \)
\[
\text{Ext}^*(\mathbf{u}_\phi \cdot (\nu \nabla \times \omega^w)) \rightarrow \mathbf{u}_\phi \cdot \tau_\nu^w \text{ in } D'((\partial B)_T) \tag{81}
\]
Just as in Theorem 1 and Lemma 1 of [21], we can deduce from the assumptions [22, 25] that
\[
\mathbf{u}_\omega \in L^2((0, T), L^2(\Omega_\epsilon)), \quad p_\omega \in L^1((0, T), L^1(\Omega_\epsilon)) \tag{82}
\]
and as \( \nu \to 0 \)
\[
\| \mathbf{u}_\omega - \mathbf{u}_\nu \|_{L^1((0, T) \times K_\omega)} \leq \| \mathbf{u}_\omega - \mathbf{u}_\nu \|_{L^2((0, T), L^2(K_\omega))} \rightarrow 0 \tag{83}
\]
This implies that as \( \nu \to 0 \)
\[
- \int_0^T \int_{\Omega} \partial_t \phi (\mathbf{u}_\nu \cdot \mathbf{u}_\phi) \, dV \, dt \rightarrow - \int_0^T \int_{\Omega} \partial_t \phi (\mathbf{u}_\omega \cdot \mathbf{u}_\phi) \, dV \, dt \tag{84}
\]
All of the terms in (73) converge along the same sequence as \( \nu \to 0 \) to their inviscid limit, by very similar arguments as above. Thus, we obtain the local balance equation [21] for interaction energy.

Finally, we can write the Josephson-Anderson relation [3] in the form
\[
\int_0^T dt \chi(t) \mathbf{F}^\nu(t) \cdot \mathbf{V} \, dt = -\rho \int_0^T \int_{\Omega} \chi \nabla \mathbf{u}_\phi : \mathbf{u}_\omega^w \otimes \mathbf{u}_\nu^w \, dV \, dt + \rho (\mathbf{u}_\phi \cdot \tau_\nu^w, \chi \otimes 1) \tag{85}
\]
after smearing with \( \chi \in D((0, T)) \). Because the velocity potential \( \phi \) in Proposition 3 is harmonic in \( \mathbb{R}^3 \setminus B \), it follows that \( |\nabla \mathbf{u}_\phi| = O(|x|^{-3}) \) as \( |x| \to \infty \); e.g. see [17], Proposition 2.17.3. Since also \( \phi \in C^\infty(\tilde{\Omega}) \), it holds immediately that \( \nabla \mathbf{u}_\phi \in L^\infty((0, T) \times \Omega) \). Together with assumption [28], we get that as \( \nu \to 0 \)
\[
\int_0^T \int_{\Omega} \chi \nabla \mathbf{u}_\phi : \mathbf{u}_\omega^w \otimes \mathbf{u}_\nu^w \, dV \, dt \rightarrow \int_0^T \int_{\Omega} \chi \nabla \mathbf{u}_\phi : \mathbf{u}_\omega \otimes \mathbf{u}_\omega \, dV \, dt
\]
and the righthand side is well-defined because \( u_\omega \in L^2((0,T), L^2(\Omega)) \). Using also the convergence (7) of the skin friction \( \tau_{w_\omega} \), we conclude that

\[
\lim_{\nu \to 0} \int_0^T dt \chi(t) F^*(t) \cdot V dt = -\rho \int_0^T \chi \nabla u_\phi : u_\omega \otimes u_\omega \ dV dt + \rho (u_\phi \cdot \tau_w, \chi \otimes 1).
\]

3.2. Proof of Theorem 2

Fix an arbitrary \( \psi \in D((\partial B)\tau) \) and let \( \phi = \text{Ext}(\psi) \) for some arbitrary \( \text{Ext} \in \mathcal{E} \). Testing the inviscid balance equation (33) of coarse-grained interaction energy against \( \phi \) and rearranging yields

\[
-(\text{Ext}^* \nabla \eta_{h,\ell} \cdot J_{\phi,\ell}, \psi) = \int_0^T \int_\Omega \varphi (\nabla \eta_{h,\ell} \cdot J_{\phi,\ell}) dV dt \tag{86}
\]

\[
\tag{86}
= \int_0^T \int_\Omega [\partial_t \varphi] \eta_{h,\ell} (u_{\omega,\ell} \cdot u_{\phi,\ell}) + \nabla \varphi \cdot (\eta_{h,\ell} J_{\phi,\ell})] dV dt \\
+ \int_0^T \int_\Omega \eta_{h,\ell} \nabla u_{\omega,\ell} : (\tau_{\ell}(u_{\omega,\ell}, u_{\omega}) + \tau_{\ell}(u_{\phi,\ell}, u_{\omega}) + \tau_{\ell}(u_{\omega,\ell}, u_{\phi})) dV dt \\
- \int_0^T \int_\Omega \varphi \eta_{h,\ell} u_{\omega,\ell} \cdot (\nabla \cdot \tau_{\ell}(u_{\phi,\ell})) dV dt + \int_0^T \int_\Omega \varphi \eta_{h,\ell} u_{\omega,\ell} \cdot (\nabla u_{\omega,\ell}) dV dt
\]

Our strategy is to prove that all the terms on the right hand side of (86) converge as \( h, \ell \to 0 \) and then to compare with the balance (27) obtained in Theorem 1.

The proof of the convergence relies on the following lemma:

**Lemma 1.** If \( f_i \in L^{p_i}((0,T), L^{p_i}_{loc}(\Omega)) \cap L^{p_i}((0,T), L^{p_i}(\Omega)) \) for \( i = 1, \ldots, k \), with some integer \( k \), and \( p_i \in [1, \infty] \) and \( \sum_{i=1}^k \frac{1}{p_i} = 1 \), for some arbitrarily small \( \epsilon > 0 \), then for \( 0 < \ell < h \), and some compact set \( K \subset \Omega \)

\[
\eta_{h,\ell} \prod_{i=1}^k \tilde{f}_i \xrightarrow{\ell \to 0} L^1((0,T), L^1(K)) \to \prod_{i=1}^k f_i \tag{87}
\]

This is a simple extension of Lemma 2 in [21], easily proved by induction on \( k \).

Note that \( u_{\phi,\ell} \) is smooth everywhere in \( (\partial B)T \) and (22, 25) imply that

\[
u_{\omega} \in L^2((0,T), L^2_{loc}(\Omega)) \cap L^2((0,T), L^2(\Omega)) \tag{88}
\]

\[
u_{\phi,\ell} \in L^1((0,T), L^1_{loc}(\Omega)) \cap L^1((0,T), L^2(\Omega)) \tag{89}
\]

Thus, we obtain by Lemma 1 that

\[
\eta_{h,\ell} u_{\omega,\ell} \cdot u_{\phi,\ell} \xrightarrow{\ell \to 0} L^1((0,T), L^1(K_{\phi})) \to u_\omega \cdot u_\phi \tag{90}
\]

\[
\eta_{h,\ell} \nabla u_{\omega,\ell} \cdot u_{\phi,\ell} \xrightarrow{\ell \to 0} L^1((0,T), L^1(K_{\phi})) \to u_\omega \cdot u_\phi \tag{91}
\]

and likewise

\[
\int_0^T \int_{K_{\phi}} |\eta_{h,\ell} \tau_{\ell}(u_{\omega,\ell})| dV dt \leq \int_0^T \int_{K_{\phi}} |\eta_{h,\ell} (u_\omega \otimes u_\omega)_{\ell} - u_\omega \otimes u_\omega| dV dt
\]
To obtain a bound on the term containing \( \nabla \cdot \) the equation (86) containing \( \nabla \cdot u \) of equation (86), we conclude that \( \delta f \) where

\begin{equation}
\eta_{h,\ell} \tau_{h,\ell} \mathbf{u}_\omega \to 0, \quad \eta_{h,\ell} \tau_{h,\ell} \mathbf{u}_\phi \to 0 \quad \text{as } h, \ell \to 0 \tag{93}
\end{equation}

Thus, all the terms in (86) that contains these cumulants \( \tau_{h,\ell}(\mathbf{u}_\omega, \mathbf{u}_\omega) \), \( \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\phi) \), \( \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\omega) \), vanish in the limit as \( h, \ell \to 0 \). We deduce from these vanishing cumulants and Lemma 4 that

\begin{equation}
\eta_{h,\ell} \mathbf{J}_{\phi,\ell} \mathbf{u}_\omega \to 0 \tag{94}
\end{equation}

To obtain a bound on the term containing \( \nabla \cdot \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\phi) \), we use a general commutator estimate on gradients of cumulants from [7], Proposition 4:

\begin{equation}
\| \nabla \cdot \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\phi) \|_{L^2(0,T) \times (K_\phi \setminus K_h)} \lesssim \frac{1}{\ell} \| \delta \mathbf{u}_\phi (\mathbf{r}) \|_{L^2(0,T) \times (K_\phi \setminus K_h)}^2 = O(\ell) \tag{95}
\end{equation}

where \( \delta f(\mathbf{r}; \mathbf{x}) := f(\mathbf{x} + \mathbf{r}) - f(\mathbf{x}) \) and the last bound follows from smoothness of \( \mathbf{u}_\phi \). Using this estimate and the \( L^2 \)-bound (82) on \( \mathbf{u}_\omega \), we see that the term in the equation (86) containing \( \nabla \cdot \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\phi) \) also vanishes in the limit as \( h, \ell \to 0 \).

From the \( L^2 \) convergence of all the integrands on the right hand side of the equation (86), we conclude that

\begin{equation}
- \langle \text{Ext}^+ (\nabla \eta_{h,\ell} \mathbf{J}_{\phi,\ell}), \psi \rangle \to 0 \tag{96}
\end{equation}

\begin{align*}
&= \int_0^T \int_{\Omega} \nabla \varphi \cdot \left[ (\mathbf{u}_\omega \cdot \mathbf{u}_\phi) \mathbf{u} + p_\omega \mathbf{u}_\phi + \left( \frac{1}{2} |\mathbf{u}_\phi|^2 + p_\phi \right) \mathbf{u}_\omega \right] dV dt \\
&= \langle \mathbf{u}_\phi \cdot \tau_\omega, \psi \rangle
\end{align*}

where the final equality follows by comparison with (27). This convergence for all \( \psi \in D'((\partial B)\Omega) \) yields the stated result.

### 3.3. Proof of Theorem 3

The proof is similar to that for Theorem 3 in [21], by bounding directly the momentum flux:

\begin{equation}
\nabla \eta_{h,\ell} \mathbf{J}_{\phi,\ell} = \delta_{h,\ell}^\prime \n \cdot \left[ (\mathbf{u}_\omega \cdot \mathbf{u}_\phi) \mathbf{u} + p_\omega \mathbf{u}_\phi + \left( \frac{1}{2} |\mathbf{u}_\phi|^2 + p_\phi \right) \mathbf{u}_\omega \right] \tag{97}
\end{equation}

\begin{align*}
+ (\tau_{h,\ell}(\mathbf{u}_\omega, \mathbf{u}_\omega) + \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\omega) + \tau_{h,\ell}(\mathbf{u}_\phi, \mathbf{u}_\phi)) \cdot \mathbf{u}_\phi \end{align*}
which is supported in \( \Omega_{h+\ell} \setminus \Omega_h \subset \Omega_{3h} \subset \Omega \). For all \( x \in \Omega_{h+\ell} \setminus \Omega_h \), a.e. \( t \in (0,T) \), we write

\[
\mathbf{n}(\pi(x)) \cdot \mathbf{u}_t(x,t) = \int_{\mathbb{R}^3} G_\ell(r)[\mathbf{n}(\pi(x)) - \mathbf{n}(\pi(x+r))] \cdot \mathbf{u}(x+r,t) \, V(dr) \tag{98}
\]

\[+ \int_{\mathbb{R}^3} G_\ell(r) \mathbf{n}(\pi(x+r)) \cdot \mathbf{u}(x+r,t) \, V(dr) \]

Since \( \mathbf{n} \circ \pi \) is smooth in \( \dot{\Omega}_c \), \( \forall \delta > 0 \), \( \exists \rho(\delta) > 0 \) s.t.

\[
|\mathbf{n}(\pi(x)) - \mathbf{n}(\pi(x+r))| \leq \delta \tag{99}
\]

for all \( x \in \Omega_{h+\ell} \setminus \Omega_h \) and \( |r| < \ell < \rho \). Then it follows that

\[
|\mathbf{n}(\pi(x)) \cdot \mathbf{u}_t(x,t)| \leq \delta \|\mathbf{u}(t)\|_{L^\infty(\Omega_c)} + \|\mathbf{n} \cdot \mathbf{u}(t)\|_{L^\infty(\Omega_{3h})} \tag{100}
\]

Obviously

\[
\|\mathbf{u}(t)\|_{L^\infty(\Omega_c)} \leq \|\mathbf{u_\omega}(t)\|_{L^\infty(\Omega_c)} + \|\mathbf{u_\phi}(t)\|_{L^\infty(\Omega_c)} \tag{101}
\]

and thus \( \mathbf{u} \in L^2((0,T),L^\infty(\Omega_c)) \). Bounds analogous to (100) hold for \( \mathbf{u_\omega}, \mathbf{u_\phi} \).

Recalling the definition of \( \tau_\ell(\mathbf{u_\omega}, \mathbf{u_\omega}) \) we obtain in similar fashion that

\[
\tau_\ell(\mathbf{u_\omega}, \mathbf{u_\omega}) \cdot \mathbf{n}(\pi(x)) \leq \left( \delta \|\mathbf{u_\omega}(t)\|_{L^\infty(\Omega_c)} + \|\mathbf{n} \cdot \mathbf{u_\omega}(t)\|_{L^\infty(\Omega_{3h})} \right) \|\mathbf{u_\omega}(t)\|_{L^\infty(\Omega_c)} \tag{102}
\]

and likewise that

\[
\tau_\ell(\mathbf{u_\phi}, \mathbf{u_\omega}) \cdot \mathbf{n}(\pi(x)) \leq \left( \delta \|\mathbf{u_\omega}(t)\|_{L^\infty(\Omega_c)} + \|\mathbf{n} \cdot \mathbf{u_\omega}(t)\|_{L^\infty(\Omega_{3h})} \right) \|\mathbf{u_\phi}(t)\|_{L^\infty(\Omega_c)} \tag{103}
\]

Consider then some \( \psi \in D((\partial B)_T) \) and \( \mathbf{Ext} \in \mathcal{E}_c \), and let \( \varphi = \mathbf{Ext}(\psi) \in D(\Omega \times (0,T)) \). Using the bounds above, together with \( \|\theta_{h,\ell}'(d(x))\|_{L^\infty} \leq \frac{C}{T} \) and \( \|\mathbf{u}\|_{L^\infty(\Omega_{h+\ell} \setminus \Omega_h)} \leq C' \), we obtain that

\[
\int_0^T \int_B \varphi \theta_{h,\ell}'(\mathbf{u_\omega}, \mathbf{u_\phi}) \partial_t (\mathbf{u_\omega} \cdot \mathbf{n}) \, dV \, dt \leq \|\psi\|_{L^\infty((\partial B)_T)} \int_0^T \int_{\Omega_{h+\ell} \setminus \Omega_h} \|\theta_{h,\ell}'(\mathbf{u_\omega}, \mathbf{u_\phi}) \partial_t (\mathbf{u_\omega} \cdot \mathbf{n})\| \, dV \, dt
\]

\[
\leq \|\psi\|_{L^\infty((\partial B)_T)} \times \|\mathbf{u_\omega}\|_{L^2((0,T),L^\infty(\Omega_c))} \times \|\mathbf{u_\phi}\|_{L^\infty(\Omega_c \times (0,T))}
\]

\[
\times \left[ \delta \|\mathbf{u}\|_{L^2((0,T),L^\infty(\Omega_c))} + \|\mathbf{n} \cdot \mathbf{u}\|_{L^2((0,T),L^\infty(\Omega_{3h}))} \right] \tag{104}
\]
Similarly,
\[
\left| \int_0^T \int_\Omega \varphi_{h,\ell} (\frac{1}{2} |\hat{u}_{\phi,\ell} |^2 + \overline{p}_\phi ) (\hat{u}_{\nu,\ell} \cdot n) \, dV \, dt \right| \\
\lesssim \| \psi \|_{L^\infty((\partial B)_T)} \times \left[ \frac{1}{2} \| u_{\phi} \|_{L^\infty(\Omega, \times (0,T))}^2 + \| p_{\phi} \|_{L^\infty(\Omega, \times (0,T))} \right] \\
\times \left[ \delta \| u_{\omega} \|_{L^2(\Omega, \times (0,T)), L^\infty(\Omega, \times (0,T))} + \| n \cdot u_{\omega} \|_{L^2(\Omega, \times (0,T)), L^\infty(\Omega, \times (0,T))} \right]
\]
(105)
\[
\left| \int_0^T \int_\Omega \varphi_{h,\ell} \cdot n \cdot (\tau_{\ell}(u_{\omega}, u_{\omega}) + \tau_{\ell}(u_{\phi}, u_{\omega}) + \tau_{\ell}(u_{\omega}, u_{\phi})) \cdot \hat{u}_{\phi,\ell} \, dV \, dt \right| \\
\lesssim \| \psi \|_{L^\infty((\partial B)_T)} \times \left[ \delta \| u_{\phi} \|_{L^2((0,T), L^\infty(\Omega,))} + \| n \cdot u_{\phi} \|_{L^2((0,T), L^\infty(\Omega,))} \right] \\
\times \| u_{\phi} \|_{L^2((0,T), L^\infty(\Omega,))} \times \left[ \delta \| u_{\omega} \|_{L^2((0,T), L^\infty(\Omega,))} + \| n \cdot u_{\omega} \|_{L^2((0,T), L^\infty(\Omega,))} \right] \\
\times \| u_{\omega} \|_{L^2((0,T), L^\infty(\Omega,))} \| u_{\phi} \|_{L^\infty(\Omega, \times (0,T))} 
\]
(106)
Thus, by the assumptions on the near-wall boundedness of $u_{\omega}, p_{\omega}$ (35), the no-flow-through condition (108) for $u_{\omega}$, and corresponding properties of $u_{\phi}$,
\[
\lim_{h,\ell \to 0} \int_0^T \int_\Omega \varphi_{h,\ell} \cdot J_{\phi,\ell} \, dV \, dt = 0.
\]
(108)
from which we conclude that
\[
\lim_{h,\ell \to 0} \text{Ext}^* (\nabla \eta_{h,\ell} \cdot J_{\phi,\ell} ) = 0 \quad \text{in } D'((\partial B)_T)
\]
(109)
and thus $u_{\phi} \cdot \tau_w = 0$ by Theorem 2. More generally, all of the above bounds are valid for any $\varphi \in D(\Omega \times (0,T))$ with $\psi := \varphi_{\ell,(\partial B)_T}$ and the local balance equation (38) then follows directly from (27) in Theorem 1. Finally, with the further assumption (28) of global $L^2$-convergence, then (29) of Theorem 1 holds and reduces to (30) since $u_{\phi} \cdot \tau_w = 0$.

4. Proof of Theorems 4-6

4.1. Proof of Theorem 4. From the uniform bound (11), we can use the same type of diagonal argument as in (21). Remark 2, to extract a subsequence $\nu_j \to 0$ so that $p_{\omega}^j \to p_{\omega} \in L^\infty((0,T), L^\infty(\Omega))$ distributionally, with $(u_{\omega}, p_{\omega})$ a weak
solution of \( \Phi \). We then take an arbitrary \( \phi \in \bar{D}(\Omega \times (0,T)) \) with \( \psi = \phi|_{\partial B} \), and test the energy equation (3) of the rotational flow to obtain:

\[
-\int_0^T \int_{\Omega} \frac{1}{2} \partial_t \phi |u_\phi| ^2 \, dV \, dt - \int_0^T \int_{\Omega} \nabla \phi \cdot \left[ \frac{1}{2} |u_\omega| ^2 u_\omega + p_\omega u_\omega \right] \, dV \, dt
\]

\[
= \int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\omega \times \omega) \, dV \, dt - \int_0^T \int_{\Omega} \nu \phi |\omega|^2 \, dV \, dt
\]

\[
- \int_0^T \int_{\Omega} \phi \nabla u_\phi \cdot u_\omega \, dV \, dt
\]

(110)

Note that

\[
\int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\omega \times \omega) \, dV \, dt
\]

\[
= - \int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\phi \times \omega) \, dV \, dt + \int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\phi' \times \omega) \, dV \, dt
\]

(111)

Because of the identity \( \nabla \cdot (u_\nu \times \omega_\nu) = \nabla \otimes \nabla : (u_\nu \otimes u_\nu) - \Delta \left( \frac{1}{2} |u_\nu|^2 \right) \), we have by integration by parts and the no-slip condition on \( u_\nu \) that

\[
\int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\nu' \times \omega_\nu) \, dV \, dt
\]

\[
= -\nu \int_0^T \int_{\Omega} \left[ (\nabla \otimes \nabla) \phi : u_\nu' \otimes u_\nu' - (\Delta \phi) \frac{1}{2} |u_\nu'|^2 \right] \, dV \, dt
\]

(112)

It follows that

\[
\left| \int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\nu' \times \omega_\nu) \, dV \, dt \right| \leq \nu \| \nabla \otimes \nabla \phi \|_{L^\infty((0,T) \times \Omega)} \| u_\nu' \|_{L^2((0,T),L^2(K_\phi))} \to 0
\]

(113)

Next, from Theorem 1, we have

\[
\int_0^T \int_{\Omega} \phi \nabla \cdot (\nu u_\phi' \times \omega_\nu) \, dV \, dt \xrightarrow{\nu \to 0} -(u_\phi \cdot \tau_w, \psi)
\]

(114)

The convergence of all of the rest of the terms in the equation (110) can be easily deduced by Lemma 1 in [21] in conjunction with the assumptions (40-43), except for the term involving \( Q_\nu \). We thus deduce from (110) that the limit (44) of \( \langle Q_\nu, \phi \rangle \) must exist as well with

\[
\langle Q_\nu, \phi \rangle := \int_0^T \int_{\Omega} \partial_t \phi \cdot \frac{1}{2} |\omega_\nu| ^2 \, dV \, dt + \int_0^T \int_{\Omega} \nabla \phi \cdot \left[ \frac{1}{2} |\omega_\nu| ^2 \omega + p_\omega \omega \right] \, dV \, dt
\]

\[
+ \langle u_\phi \cdot \tau_w, \psi \rangle - \int_0^T \int_{\Omega} \phi \nabla u_\phi \cdot u_\omega \otimes u_\omega \, dV \, dt.
\]

(115)

From this definition, the balance equation (45) trivially follows. Furthermore, \( Q \) defined by (115) is clearly a linear functional on \( \bar{D}(\Omega \times (0,T)) \). Finally, since \( \langle Q_\nu, \phi \rangle \geq 0 \) for all \( \nu > 0 \) when \( \phi \geq 0 \), then the limit functional also satisfies the corresponding inequality \( \langle Q, \phi \rangle \geq 0 \). Thus, \( Q \) is non-negative.
4.2. Proof of Theorem. We take an arbitrary \( \varphi \in D(\Omega \times (0, T)) \) with \( \psi = \varphi|_{\partial \Omega} \). Testing the inviscid coarse-grained balance equation (50) for energy in rotational \( \langle \cdot \rangle \) by a standard density argument, based on the Constantin-E-Titi identities [3]: here a few details, discussing only the term

\[
\ell = - \int_0^T \int_\Omega \varphi \nabla \eta_{h, \ell} \cdot \mathbf{J}_{\omega, \ell} : \mathbf{\tau}_\ell(\mathbf{u}_\omega, \mathbf{u}_\omega) \, dV \, dt
\]

(116)

Let \( K_\omega \subset \mathbb{R}^3 \) be a compact set such that \( \text{supp}(\varphi) \subset K_\omega \times (0, T) \subset \Omega \times (0, T) \). Similar to the proof of Theorem 2, one can use Lemma 1 to obtain

\[
\frac{1}{2} \eta_{h, \ell} |\mathbf{u}_{\omega, \ell}|^2 \mathbb{L}^2((0, T), L^2(K_\omega)) \rightarrow \frac{1}{2} |\mathbf{u}_\omega|^2, \quad \eta_{h, \ell} \overline{\rho}_{\omega, \ell} \mathbb{L}^1((0, T), L^1(K_\omega)) \rightarrow p_\omega \mathbf{u}_\omega
\]

(117)

and

\[
\frac{1}{2} \eta_{h, \ell} |\mathbf{u}_{\omega, \ell}|^2 \mathbb{L}^1((0, T), L^1(K_\omega)) \rightarrow \frac{1}{2} |\mathbf{u}_\omega|^2 \mathbf{u}
\]

(118)

Next we note that

\[
\lim_{h, \ell \to 0} \int_0^T \int_\Omega \varphi \eta_{h, \ell} \nabla \mathbf{u}_{\omega, \ell} : \mathbf{\tau}_\ell(\mathbf{u}_\omega, \mathbf{u}_\omega) \, dV \, dt = 0
\]

(121)

\[
\lim_{h, \ell \to 0} \int_\Omega \varphi \eta_{h, \ell} \nabla \mathbf{u}_{\omega, \ell} : \mathbf{\tau}_\ell(\mathbf{u}_\omega, \mathbf{u}_\omega) \, dV \, dt = 0
\]

(122)

by a standard density argument, based on the Constantin-E-Titi identities [3]:

\[
\nabla \mathbf{u}_{\omega, \ell}(x) = - \frac{1}{\ell} (\nabla G)_\ell \delta \mathbf{u}_\omega(; x)
\]

(123)

\[
\tau_\ell(\mathbf{u}_\omega, \mathbf{u}_\varphi)(x) = (G_\ell \delta \mathbf{u}_\omega(; x) \otimes \delta \mathbf{u}_\varphi(; x)) - (G_\ell \delta \mathbf{u}_\omega(; x)) \otimes (G_\ell \delta \mathbf{u}_\varphi(; x))
\]

(124)

where we use \( \langle \cdot \rangle \) as an abbreviated notation for spatial integral over \( \mathbb{R}^3 \). We give here a few details, discussing only the term

\[
\frac{1}{\ell} \int_0^T \int_\Omega \eta_{h, \ell} \varphi (\nabla G)_\ell \delta \mathbf{u}_\omega(; x) : (G_\ell \delta \mathbf{u}_\omega(; x) \otimes \delta \mathbf{u}_\varphi(; x)) \, dV \, dt,
\]

(125)
as all others can be treated in the same way. Since \( \mathbf{u}_\omega \in L^3((0, T), L^3(K_\varphi)) \) from (10) and (12), then for all \( \epsilon > 0 \), there exists \( \mathbf{v}^\epsilon \in C^\infty((0, T) \times \text{Int}(K_\varphi)) \) such that \( \| \mathbf{u}_\omega - \mathbf{v}^\epsilon \|_{L^3((0, T), L^3(K_\varphi)))} < \epsilon \). Defining \( \Delta^\epsilon := \mathbf{u}_\omega - \mathbf{v}^\epsilon \), we can then substitute \( \mathbf{u}_\omega = \mathbf{v}^\epsilon + \Delta^\epsilon \) into (125) to obtain two integrals. The first satisfies the bound

\[
\left| \frac{1}{T} \int_0^T \int_{\Omega} \eta_{h, \ell} \varphi ((\nabla G) \delta \mathbf{u}_\omega (:; \mathbf{x})) : (G \delta \mathbf{v}^\epsilon (:; \mathbf{x}) \otimes \delta \mathbf{u}_\phi (:; \mathbf{x})) dV dt \right| \\
\lesssim \ell \| \varphi \|_{L^\infty(\mathbb{R} \times (0, T))} \| \mathbf{u}_\omega \|_{L^3((0, T), L^3(K_\varphi)))} \| \nabla \mathbf{v}^\epsilon \|_{L^\infty(K_\varphi \times (0, T))} \| \nabla \mathbf{u}_\phi \|_{L^\infty(K_\varphi \times (0, T))}
\]

(126)

where the factor \( \ell \) is obtained from the smoothness of \( \mathbf{v}^\epsilon \) and \( \mathbf{u}_\omega \). Here, \( \lesssim \) is inequality with prefactor depending on \( K_\varphi \times (0, T) \) and the mollifier \( G \). Thus, the first integral vanishes as \( h, \ell \to 0 \), with \( \epsilon \) fixed. The second satisfies

\[
\left| \frac{1}{T} \int_0^T \int_{\Omega} \eta_{h, \ell} \varphi ((\nabla G) \delta \mathbf{u}_\omega (:; \mathbf{x})) : (G \delta \Delta^\epsilon (:; \mathbf{x}) \otimes \delta \mathbf{u}_\phi (:; \mathbf{x})) dV dt \right| \\
\lesssim \epsilon \| \varphi \|_{L^\infty(\mathbb{R} \times (0, T))} \| \mathbf{u}_\omega \|_{L^3((0, T), L^3(K_\varphi)))} \| \nabla \mathbf{u}_\phi \|_{L^\infty(K_\varphi \times (0, T))}
\]

for any \( \epsilon > 0 \). Thus, we obtain that (125) vanishes in the limit of \( h, \ell \to 0 \).

Since the terms considered in (117)-(120) all converge in \( L^1((0, T), L^1(K_\varphi)) \), we conclude from (116) that

\[
\lim_{h, \ell \to 0} \int_0^T \int_{\Omega} \varphi \left( \nabla \eta_{h, \ell} \cdot \mathbf{J}_{\omega, \ell} + \eta_{h, \ell} \nabla \mathbf{u}_\omega, \ell : \tau_\ell (\mathbf{u}_\omega, \mathbf{u}_\omega) \right) dV dt = - \int_0^T \int_{\Omega} \partial_\tau \varphi \cdot \frac{1}{2} |\mathbf{u}_\omega|^2 dV dt + \int_0^T \int_{\Omega} \varphi \nabla \mathbf{u}_\phi : \mathbf{u}_\omega \otimes \mathbf{u}_\omega dV dt \\
- \int_0^T \int_{\Omega} \nabla \varphi \cdot \left[ \frac{1}{2} |\mathbf{u}_\omega|^2 \mathbf{u} + p_\omega \mathbf{u}_\omega \right] dV dt
\]

(128)

for all \( \varphi \in \dot{D}(\Omega \times (0, T)) \). By comparison with the inviscid energy balance equation (45) for rotational flow in Theorem 4, we obtain (51).

4.3. Proof of Theorem 5. The near-wall boundedness property (52) and no-flow-through condition (53) imply those in Theorem 3 which immediately gives

\[
\mathbf{u}_\phi \cdot \tau_w = 0 \quad \text{in} \ D'((\partial B)_T).
\]

(129)

Thus, the local balance equation for rotational energy (45) reduces to

\[
- \int_0^T \int_{\Omega} \frac{1}{2} \partial_\tau \varphi |\mathbf{u}_\omega|^2 dV dt - \int_0^T \int_{\Omega} \nabla \varphi \cdot \left[ \frac{1}{2} |\mathbf{u}_\omega|^2 \mathbf{u} + p_\omega \mathbf{u}_\omega \right] dV dt = - \int_0^T \int_{\Omega} \varphi Q dV dt - \int_0^T \int_{\Omega} \varphi \nabla \mathbf{u}_\phi : \mathbf{u}_\omega \otimes \mathbf{u}_\omega dV dt
\]

(130)
Using the same arguments as in the proof of Theorem 3, one can obtain upper bounds on the spatial flux of kinetic energy of the rotational flow, defined as
\[
\nabla \eta_{h,\ell}(x) \cdot \mathbf{J}_{\omega,\ell}(x) = \theta_h\theta_{\ell}(d(x)) \left( \frac{1}{2} |\mathbf{u}_{\omega,\ell}|^2 \mathbf{n}(\pi(x)) + \bar{p}_{\omega,\ell} \mathbf{u}_{\omega,\ell} \cdot \mathbf{n}(\pi(x)) \right) + \mathbf{n}(\pi(x)) \cdot (\tau_{\ell}(\mathbf{u}_{\omega,\ell}) + \tau_{\ell}(\mathbf{u}_{\omega,\ell})) \cdot \mathbf{u}_{\omega,\ell},
\]
which are analogous to the bounds (104)-(107) on the spatial flux of the interaction energy. We need estimates \(L^3\) in time here since (131) is cubic in \(\mathbf{u}_{\omega}\).

It then follows from these upper bounds and the assumptions (52), (53) that
\[
\lim_{h,\ell \to 0} \int_0^T \int_\Omega \mathbf{J}_{\omega,\ell} \cdot \nabla \eta_{h,\ell} \, dV \, dt = 0 \quad (132)
\]
for all \(\varphi \in \mathcal{D}(\Omega \times (0, T))\) and this implies that for all \(\text{Ext} \in \mathcal{E}\),
\[
\lim_{h,\ell \to 0} \text{Ext}^* \left[ \nabla \eta_{h,\ell} \cdot \mathbf{J}_{\omega,\ell} \right] = 0. \quad (133)
\]
By comparison with (51), we obtain the equality (56) between inertial energy dissipation and zero-viscosity limit of viscous energy dissipation.

Acknowledgements. We are grateful to Samvit Kumar, Charles Meneveau, and Tamer Taki for discussions of this problem. This work was funded by the Simons Foundation, via Targeted Grant in MPS-663054.

References
1. Philip W Anderson. Considerations on the flow of superfluid helium. Reviews of Modern Physics, 38(2):298, 1966.
2. Gustave Choquet. Lectures on analysis. Benjamin, New York, 1969.
3. Peter Constantin, Edriss S Titi, and Weinan E. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. Communications in Mathematical Physics, 165(1):207, 1994.
4. Jean le Rond d’Alembert. Théorie résistente que patitur corpus in fluido motum, ex principiis omnino novis et simplicissimis deducta, habita ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis compressionis partium fluidi. manuscript at Berlin-Brandenburgische Akademie der Wissenschaften, Akademie-Archiv call number: I-M478, 1749.
5. Jean le Rond d’Alembert. Paradoxe proposé aux géomètres sur la résistance des fluides. in: Opuscules mathématiques, vol. 5 (Paris), Memoir XXXIV, Section I, 132–138, 1768.
6. Camillo De Lellis and László Székelyhidi. On turbulence and geometry: from Nash to Onsager. Notices of the American Mathematical Society, 66(5):677—685, 2019.
7. Theodore D Drivas and Gregory L Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. Communications in Mathematical Physics, 359(2):733—763, 2018.
8. Theodore D Drivas and Huy Q Nguyen. Remarks on the emergence of weak Euler solutions in the vanishing viscosity limit. Journal of Nonlinear Science, 29(2):709—721, 2019.
9. Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. Nonlinearity, 13(1):249, 2000.
10. Lawrence C. Evans. Partial Differential Equations. Graduate studies in mathematics. American Mathematical Society, 2010.
11. Gregory L Eyink. Review of the Onsager ‘ideal turbulence’ theory. arXiv preprint arXiv:1808.02223, 2018.
12. Gregory L. Eyink. Josephson-Anderson relation and the classical D’Alembert paradox. Phys. Rev. X, 11:031054, Sep 2021.
13. Elisha R Huggins. Energy-dissipation theorem and detailed Josephson equation for ideal incompressible fluids. Physical Review A, 1(2):332, 1970.
14. Elisha R Huggins. Vortex currents in turbulent superfluid and classical fluid channel flow, the Magnus effect, and Goldstone boson fields. Journal of low temperature physics, 96(5):317–346, 1994.
15. Brian D. Josephson. Potential differences in the mixed state of type II superconductors. Phys. Lett., 16:242–243, 1965.
16. Tosio Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In S. S. Chern, editor, Seminar on nonlinear partial differential equations, volume 2 of Mathematical Sciences Research Institute Publications, pages 85–98, New York, 1984. Springer.
17. Dagmar Medková. The Laplace Equation: Boundary Value Problems on Bounded and Unbounded Lipschitz Domains. Springer International Publishing, 2018.
18. Valentin P. Mikhailov. Partial differential equations. Mir Publishers, 1978.
19. Michael Neudert and Wolf von Wahl. Asymptotic behaviour of the div-curl problem in exterior domains. Advances in Differential Equations, 6(11):1347–1376, 2001.
20. Lars Onsager. Statistical hydrodynamics. Il Nuovo Cimento (1943-1954), 6(2):279–287, 1949.
21. Hao Quan and Gregory L. Eyink. Inertial momentum dissipation for viscosity solutions of Euler equations. I. Flow around a smooth body. in preparation, 2022.
22. Walter Rudin. Functional Analysis. McGraw-Hill, 1991.
23. Franck Sueur. A Kato type theorem for the inviscid limit of the Navier-Stokes equations with a moving rigid body. Communications in Mathematical Physics, 316(3):783–808, 2012.
24. Geoffrey I. Taylor. Eddy motion in the atmosphere. Philos. Trans. R. Soc. London, Ser. A, 215(1):1–26, 1915.
25. Peter Wagner. Distributions supported by hypersurfaces. Applicable Analysis, 89(8):1183–1199, 2010.
26. E. Wiedemann. Weak-strong uniqueness in fluid dynamics. In C.L. Fefferman, J.C. Robinson, and J.L. Rodrigo, editors, Partial differential equations in fluid mechanics, volume 452 of London Math. Soc. Lecture Note Ser., page 289–326, 2018.

Communicated by ???