Modular properties of type I locally compact quantum groups

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Abstract

The following paper is devoted to the study of type I locally compact quantum groups. We show how various operators related to the modular theory of the Haar integrals on $G$ and $\hat{G}$ act on the level of direct integrals. Using these results we derive a web of implications between properties such as unimodularity or traciality of the Haar integrals. We also study in detail two examples: discrete quantum group $\hat{SU}_q(2)$ and the quantum $az + b$ group.

1 Introduction

A remarkable feature of the theory of compact quantum groups introduced by Woronowicz ([22, 23]) is the fact that the Haar integral need not be tracial (in such case one says that a compact quantum group $G$ is not of Kac type). Whether $G$ is of Kac type or not, is related to a number of other properties. To name a few, the Haar integral of $G$ is tracial if, and only if its scaling group is trivial and this happens if, and only if the dual discrete quantum group $\hat{G}$ is unimodular (equivalently has tracial integrals). In fact, behind all these objects and properties stands a family $(\rho_\alpha)_{\alpha \in \text{Irr} (G)}$ of positive invertible operators (see [13]) and $G$ is of Kac type if, and only if $\rho_\alpha = 1_{H_\alpha}$ for all $\alpha \in \text{Irr} (G)$.

A theory of locally compact quantum groups was proposed by Kustermans and Vaes ([11]). As general quantum group can be non-unimodular, each quantum group $G$ has two Haar integrals: left $\varphi$ and right $\psi$. It is still possible that these are non-tracial, however now the situation is more complicated and the above simple equivalences from the world of compact quantum groups are no longer valid.

An intermediate step between the theory of compact and general locally compact quantum groups is formed by the so called type I locally compact quantum groups. Roughly speaking, similarly to the classical (i.e. not quantum) setting, these are quantum groups with type I universal group $C^*$-algebras. Their study was initiated in the doctoral dissertation of Desmedt [5] where he has constructed a Plancherel measure on $\text{Irr}(G)$ and described its

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properties. Together with the Plancherel measure come two fields of strictly positive self-adjoint operators, \((D_x)_{x \in \text{Irr}(G)}\) and \((E_x)_{x \in \text{Irr}(G)}\). They can be thought of as replacements of the operators \(\rho_\alpha\) from the compact theory; for example, the Haar integrals of \(\hat{G}\) are tracial if, and only if almost all operators \(D_\pi, E_\pi\) are multiples of the identity. One of the main results of our paper is a theorem which describes a relation between various properties of \(G\) and \(\hat{G}\) (unimodularity, traciality of the Haar integrals, trivial scaling group etc.) and properties of operators \(D_\pi, E_\pi (\pi \in \text{Irr}(G))\) – this is accomplished in Section 6.

In the next section we introduce objects used in the paper and set up the notation. Section 3 is devoted to introducing a notion of matrix coefficients (in type I case) and recalling results of Desmedt ([5]) and Caspers ([3]) which are used later in the paper. In Section 4 we describe the polar decomposition of the map \(T^\prime : \Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*)\) coming from the Tomita-Takesaki theory and as a corollary we get an important relation between unitary operators \(Q_L, Q_R\) from the Desmedt’s theorem. In Section 5 we show how various operators act on the level of direct integrals. We remark that a formula for \(\nabla_{1, \varphi}^\prime\) from Theorem 5.3 was recently used in [8] to deduce that the Toeplitz algebra is not an algebra of continuous functions on a compact quantum group. Finally, in Section 7 we describe two interesting examples of type I locally compact quantum groups: discrete group \(\text{SU}_q(2)\) and the quantum "\(az + b\)" group.

2 Notation

Throughout the paper, \(G\) will be a locally compact quantum group in the sense of Kustermans and Vaes. We refer the reader to papers [11, 20] for an introduction to the subject, here we will recall only necessary facts. Quantum group \(G\) comes together with a number of objects: first of all we have a von Neumann algebra \(L^\infty(G)\), a normal unital \(*\)-homomorphism \(\Delta_G : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)\) called comultiplication and two n.s.f. weights on \(L^\infty(G)\): \(\varphi\) and \(\psi\). They are called respectively the left and the right Haar integral as they satisfy certain invariance conditions. We will write \(\Lambda_\varphi, \Lambda_\psi\) for the GNS maps. The GNS Hilbert spaces \(H_\varphi, H_\psi\) can be identified and will be denoted by \(L^2(G)\). We will write \((\sigma_\varphi^x)_{x \in \mathbb{R}}, \nabla_\varphi, J_\varphi\) for the group of modular automorphisms associated with the weight \(\varphi\), the modular operator and the modular conjugation – an analogous notation will be used for other weights. The predual of \(L^\infty(G)\) will be denoted by \(L^1(G)\). With every locally compact quantum group \(G\) one can associate its dual \(\hat{G}\). The objects associated with \(\hat{G}\) will be decorated with hats. The Hilbert spaces \(L^2(G), L^2(\hat{G})\) can (and will) be identified. We will use a \(C^*\)-algebra \(C_0(G) \subseteq B(L^2(G))\). It is a \(\sigma\)-wot dense subalgebra of \(L^\infty(G)\). An important role in the theory plays the Kac-Takesaki operator \(W \in M(C_0(G) \otimes C_0(\hat{G}))\). It is a unitary operator characterized by the property \(((\omega \otimes \text{id})W^*)\Lambda_\varphi(x) = \Lambda_\varphi((\omega \otimes \text{id})\Delta_G(x)) (\omega \in L^1(G), x \in \mathfrak{M}_\varphi)\). We note that the right leg of \(W\) generates \(C_0(\hat{G})\) – this means that the map \(\lambda : L^1(G) \ni \omega \mapsto (\omega \otimes \text{id})W \in C_0(\hat{G})\) satisfies \(\lambda(L^1(G)) = C_0(\hat{G})\). There is also a unitary \(V \in L^\infty(\hat{G})^* \otimes L^\infty(G)\) related to the

\(^1\)Symbol \(\otimes\) stands for the minimal tensor product of \(C^*\)-algebras or the tensor product of Hilbert spaces.
right Haar integral \( \psi \). With \( \mathbb{G} \) one can associate yet another C*-algebra, \( C^*_0(\mathbb{G}) \) called the universal version of \( C_0(\mathbb{G}) \). It is related to \( C_0(\mathbb{G}) \) via so called reducing morphism \( \Lambda_\mathbb{G}: C^*_0(\mathbb{G}) \to C_0(\mathbb{G}) \). We remark that \( C^*_0(\widehat{\mathbb{G}}) \) plays a role of the full group C*-algebra and its representations are in bijection with unitary representations of \( \mathbb{G} \) (see [10, 16]).

To be more precise, there exists a unitary operator \( \mathcal{W} \in M(C_0(\mathbb{G}) \otimes C^*_0(\widehat{\mathbb{G}})) \) such that every unitary representation of \( \mathbb{G} \) on a Hilbert space \( H_\pi \) is of the form \( U^\pi = (id \otimes \pi)\mathcal{W} \) for a nondegenerate representation \( \pi: C^*_0(\widehat{\mathbb{G}}) \to B(H_\pi) \). This correspondence preserves irreducibility – consequently the spectrum of \( C^*_0(\widehat{\mathbb{G}}) \) will be denoted by \( \text{Irr}(\mathbb{G}) \).

Besides the groups of modular automorphisms \( (\sigma^\varphi_t)_{t \in \mathbb{R}}, (\sigma^\psi_t)_{t \in \mathbb{R}} \) there is also a third group of automorphisms of \( L^\infty(\mathbb{G}) - (\tau_t)_{t \in \mathbb{R}} \), called the scaling group. It is implemented by a strictly positive selfadjoint operator \( P: \tau_t(x) = P^{it}xP^{-it} \) \( x \in L^\infty(\mathbb{G}), t \in \mathbb{R} \). We remark that \( P \) is selffual: we have \( P = P \). The Haar integrals are relatively invariant under the scaling group: we have \( \varphi \circ \tau_t = \nu^{-t}\varphi, \psi \circ \tau_t = \nu^{-t}\psi \) \( t \in \mathbb{R} \) for a number \( \nu > 0 \) called the scaling constant. The scaling constant relates the modular conjugations for \( \varphi \) and \( \psi \): we have \( J_\varphi = \nu^{\frac{1}{2}}J_{\nu^t\varphi} \). An important role in our paper is played by the so called modular element \( \delta \). It is a strictly positive selfadjoint operator affiliated with \( L^\infty(\mathbb{G}) \) which appears as (a part of) the Radon–Nikodym derivative between \( \psi \) and \( \varphi \). There is a plethora of formulas which relates the above objects. Let us end this part of the introduction with a collection of them – we will use it a lot through the paper:

\[
\begin{align*}
J_{\varphi}J_{\tilde{\varphi}} &= \nu^{\frac{1}{2}}J_{\varphi}J_{\tilde{\varphi}}, \\
\nabla^it\psi &= J_{\varphi}\nabla^{-it}\varphi, \\
J_{\varphi}\delta^{it} &= \delta^{it}J_{\varphi}, \\
\nabla^it\psi &= \tilde{\delta}^{-it}P^{-it}, \\
J_{\varphi}\psi &= P^{it}J_{\varphi}, \\
P^{it}\nabla^it\psi &= \nabla^itP^{it}, \\
P^{it}\nabla^it\psi &= \nabla^itP^{it},
\end{align*}
\]

where \( t \) and \( s \) are arbitrary reals numbers. The above properties belong to the standard theory of locally compact quantum groups, their proofs can be found in [10, 11, 20].

Throught the paper, we will extensively use the theory of direct integrals – we refer the reader to [6, 12] for basic notions and properties. Let us mention here only that if \( (H_x)_{x \in X} \) is a measurable field of Hilbert spaces, then \( \int_X^H H_x \, d\mu(x) \) is a Hilbert space which consists of (classes of) measurable vector fields \( \xi = (\xi_x)_{x \in X} \) satisfying \( \int_X \|\xi_x\|^2 \, d\mu(x) < +\infty \). Usually one also writes \( \xi = \int_X^H \xi_x \, d\mu(x) \). For two closed operators \( A \) and \( B \), the symbol \( A \circ B \) will stand for the operator given by \( A \circ B(\xi) = A(B(\xi)) \) on the domain \( \text{Dom}(A \circ B) = \{ \xi \in \text{Dom}(B) | B\xi \in \text{Dom}(A) \} \). Whenever \( A \circ B \) is closable, we will denote its closure by \( AB \).

The complex conjugate of a Hilbert space \( H \) will be denoted by \( \overline{H} \). For an operator \( A \) on \( H \), \( A^T \) will be an operator on \( \overline{H} \) given by \( A^T \xi = A^*\xi (\xi \in H) \). If \( \pi \) is a representation of \( \mathbb{G} \) on \( H_\pi \) then we associate with it a representation \( \pi^c = \iota \circ \pi \circ \overline{\pi} \) on \( \overline{H_\pi} \), where \( \overline{\pi} \) is the unitary antipode on \( C^*_0(\widehat{\mathbb{G}}) \) ([10, 16]). All scalar products are linear on the right.

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2This means that \( (H_x)_{x \in X} \) comes together with a choice of a fundamental sequence \( \{(\xi^n_x)_{x \in X} | n \in \mathbb{N}\} \) and for each \( n \in \mathbb{N} \), the function \( X \ni x \mapsto (\xi^n_x, \xi_x) \in \mathbb{C} \) is measurable. We will neglect mentioning the fundamental sequence and simply say that \( (H_x)_{x \in X} \) is a measurable field of Hilbert spaces.
3 Preliminaries

Let us introduce two notions: we say that $G$ is second countable\(^3\) if $C_0^0(\hat{G})$ separable and type I if $C_0^0(\hat{G})$ is of type I. Our work is based on the work of Desmedt \([5]\) and Caspers and Koelink \([3, 4]\). First of all, we will use the fundamental result of Desmedt which states existence of the Plancherel measure and its properties (see also \([7]\) and discussion therein). We recall only parts that will be used in the paper.

**Theorem 3.1.** Let $G$ be a second countable, type I locally compact quantum group. There exists a standard measure $\mu$ on $\text{Irr}(G)$, a measurable field of Hilbert spaces $(H_\pi)_{\pi \in \text{Irr}(G)}$, measurable field of representations, measurable field of strictly positive self-adjoint operators $(D_\pi)_{\pi \in \text{Irr}(G)}$ and a unitary operator $Q_L: L^2(\hat{G}) \to \int_{\text{Irr}(G)}^\oplus \text{HS}(H_\pi) \, d\mu(\pi)$ such that:

1) For all $\alpha \in L^1(G)$ such that $\lambda(\alpha) \in \mathcal{M}_\phi$ and $\mu$-almost every $\pi \in \text{Irr}(G)$ the operator $(\alpha \otimes \text{id})(U^\pi) \circ D_\pi^{-1}$ is bounded and its closure $(\alpha \otimes \text{id})(U^\pi)D_\pi^{-1}$ is Hilbert-Schmidt.

2) The operator $Q_L$ is the isometric extension of

$$\Lambda_\phi(\lambda(L^1(G)) \cap \mathcal{M}_\phi) \ni \Lambda_\phi(\lambda(\alpha)) \mapsto \int_{\text{Irr}(G)}^\oplus (\alpha \otimes \text{id})(U^\pi)D_\pi^{-1} \, d\mu(\pi) \in \int_{\text{Irr}(G)}^\oplus \text{HS}(H_\pi) \, d\mu(\pi),$$

3) The operator $Q_L$ satisfies the following equations:

$$Q_L(\omega \otimes \text{id})W = \left( \int_{\text{Irr}(G)}^\oplus (\omega \otimes \text{id})U^\pi \otimes 1_{H_\pi} \, d\mu(\pi) \right)Q_L$$

and

$$Q_L(\omega \otimes \text{id})\chi(W) = \left( \int_{\text{Irr}(G)}^\oplus 1_{H_\pi} \otimes \pi^c((\omega \otimes \text{id})W) \, d\mu(\pi) \right)Q_L$$

for every $\omega \in L^1(G)$.

4) Haar integrals on $\hat{G}$ are tracial if and only if almost all $D_\pi$ are multiples of the identity.

5) The operator $Q_L$ transforms $L^\infty(\hat{G}) \cap L^\infty(\hat{G})'$ into diagonalisable operators.

6) We can assume that $(H_\pi)_{\pi \in \text{Irr}(G)}$ is the canonical measurable field of Hilbert spaces.

We have also the right version of the above theorem.

\(^3\)This condition is equivalent to number of other separability assumptions, see \([7, \text{Lemma 14.6}]\). We note that these conditions are satisfied for $G$ if and only they are satisfied for $\hat{G}$.

\(^4\)We use the same symbol $\pi$ for a class of representations and its representative chosen according to a fixed measurable field of representations.

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Theorem 3.2. Let $\mathbb{G}$ be a second countable, type I locally compact quantum group. There exists a standard measure $\mu^R$ on $\text{Irr}(\mathbb{G})$ a measurable field of Hilbert space $(K_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$, measurable field of representations, measurable field of strictly positive self-adjoint operators $(E_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$ and a unitary operator $Q_R : L^2(\mathbb{G}) \to \int_{\text{Irr}(\mathbb{G})} \text{HS}(K_\pi) d\mu^R(\pi)$ such that:

1) For all $\alpha \in L^1(\mathbb{G})$ such that $\lambda(\alpha) \in \mathfrak{N}_\psi$ and $\mu^R$-almost every $\pi \in \text{Irr}(\mathbb{G})$ the operator $(\alpha \otimes \text{id})(U^\pi) \circ E_\pi^{-1}$ is bounded and its closure $(\alpha \otimes \text{id})(U^\pi)E_\pi^{-1}$ is Hilbert-Schmidt.

2) The operator $Q_R$ is the isometric extension of

\[ J_H^\varphi : H_\psi \otimes \overline{H_\psi} \ni \xi \otimes \eta \mapsto \eta \otimes \xi \in H_\psi \otimes \overline{H_\psi} \quad (\pi \in \text{Irr}(\mathbb{G})). \]

3) The operator $Q_R$ satisfies the following equations:

\[ Q_R J_H^\varphi (\omega \otimes \text{id})W = \left( \int_{\text{Irr}(\mathbb{G})} (\omega \otimes \text{id})U^\pi \otimes 1_{H_\psi} d\mu^R(\pi) \right) Q_R J_H^\varphi \]

and

\[ Q_R J_H^\varphi (\omega \otimes \text{id})\chi(V) = \left( \int_{\text{Irr}(\mathbb{G})} 1_{H_\pi} \otimes \pi^c((\omega \otimes \text{id})W) d\mu^R(\pi) \right) Q_R J_H^\varphi \]

for every $\omega \in L^1(\mathbb{G})$.

4) Haar integrals on $\hat{\mathbb{G}}$ are tracial if and only if almost all $E_\pi$ are multiples of the identity.

5) The operator $Q_R$ transforms $L^\infty(\hat{\mathbb{G}}) \cap L^\infty(\hat{\mathbb{G}})'$ into diagonalisable operators.

6) We can choose $\mu^R = \mu$ and $K_\pi = H_\pi$ (and the same field of representations as in Theorem 3.1).

From now on, let $\mathbb{G}$ be a second countable, type I locally compact quantum group and choose all the objects provided by theorems 3.1, 3.2. The last point of the above theorem allows us to assume $\mu^R = \mu$ and $K_\pi = H_\pi$. Let us introduce two strictly positive, selfadjoint operators $D = \int_{\text{Irr}(\mathbb{G})} D_\pi d\mu(\pi)$ and $E = \int_{\text{Irr}(\mathbb{G})} E_\pi d\mu(\pi)$. We will use plenty of times the following easily derived property:

**Proposition 3.3.** Define an antiunitary operator $\Sigma = \int_{\text{Irr}(\mathbb{G})} J_{H_\pi} d\mu(\pi)$, where

\[ J_{H_\pi} : H_\pi \otimes \overline{H_\pi} \ni \xi \otimes \overline{\eta} \mapsto \eta \otimes \overline{\xi} \in H_\pi \otimes \overline{H_\pi} \quad (\pi \in \text{Irr}(\mathbb{G})). \]

We have

\[ \nu^T J_{\hat{\varphi}} = Q_L^* \Sigma Q_L = Q_R^* \Sigma Q_R. \]
Proof. Let \( \hat{\varphi}^u \) be the left Haar weight on the universal \( C^* \)-algebra \( C_0^u(\hat{G}) \). Its GNS construction is \( (L^2(\hat{G}), \Lambda_{\hat{G}}, \Lambda_{\hat{G}} \circ \Lambda_{\hat{G}}) \) (see [10]), hence \( J_{\hat{\varphi}} \) is the modular conjugation for \( \hat{\varphi}^u \). It is transformed to \( \Sigma \) by \( Q_L \) – it is a part of the Desmedt’s result. Similarly, \( Q_{R,0} \) transforms \( J_{\hat{\varphi}} \) to \( \Sigma: J_{\hat{\varphi}} = Q_{R,0} \Sigma Q_{R,0} \). Operator \( Q_R \) is defined as \( Q_R = Q_{R,0} J_{\varphi} J_{\hat{\varphi}} \). Let us now define \( J_{\hat{\varphi}} \) and its commutant. Using the commutation relation \( J_{\hat{\varphi}} J_{\varphi} = \nu^* J_{\varphi} J_{\hat{\varphi}} \) (see equation (2.1)) and formula \( J_{\hat{\varphi}} = \nu^{-\frac{i}{2}} J_{\varphi} \) (the scaling constant of \( \hat{G} \) is \( \nu = \nu^{-1} \)) we arrive at

\[
Q_R \Sigma Q_R = J_{\hat{\varphi}} J_{\varphi} (\nu^{-\frac{i}{2}} J_{\varphi} J_{\hat{\varphi}}) = \nu^{-\frac{i}{2}} \nu^* J_{\varphi} J_{\hat{\varphi}} J_{\varphi} J_{\hat{\varphi}} = J_{\hat{\varphi}}.
\]

Let us note in the next proposition how \( Q_L, Q_R \) transform \( L^\infty(\hat{G}) \) and its comutant.

Proposition 3.4. We have the following equalities of von Neumann algebras:

\[
Q_L L^\infty(\hat{G}) Q_L^* = \int_{\mathcal{L}(G)} B(H_\pi) \otimes \mathbb{1}_{H_\pi} \, d\mu(\pi), \quad Q_L L^\infty(\hat{G})^* Q_L^* = \int_{\mathcal{L}(G)} \mathbb{1}_{H_\pi} \otimes B(H_\pi) \, d\mu(\pi)
\]

\[
Q_R L^\infty(\hat{G}) Q_R^* = \int_{\mathcal{L}(G)} \mathbb{1}_{H_\pi} \otimes B(H_\pi) \, d\mu(\pi), \quad Q_R L^\infty(\hat{G})^* Q_R^* = \int_{\mathcal{L}(G)} B(H_\pi) \otimes \mathbb{1}_{H_\pi} \, d\mu(\pi)
\]

The first part of the above result is a result of Desmedt. The second one can be derived as in the proof of Proposition 3.3 using equation \( Q_R = Q_{R,0} J_{\varphi} J_{\hat{\varphi}} \). Let us now define analogs of the matrix coefficients \( U_{i,j}^\alpha \) used in the theory of compact quantum groups. Elements of this form were already considered in [3].

Definition 3.5. For \( \xi, \eta \in \int_{\mathcal{L}(G)} H_\pi \, d\mu(\pi) \) we define elements of \( L^\infty(\hat{G}) \):

\[
M_{\xi,\eta}^L = \int_{\mathcal{L}(G)} (\text{id} \otimes \omega_{\xi,\eta}) (U^{\pi*}) \, d\mu(\pi), \quad M_{\xi,\eta}^R = \int_{\mathcal{L}(G)} (\text{id} \otimes \omega_{\xi,\eta}) (U^\pi) \, d\mu(\pi).
\]

The above elements will be referred to as left (resp. right) matrix coefficients.

Note that the above (weak) integrals converge in \( \sigma \)-wot and we have \( (M_{\xi,\eta}^L)^* = M_{\eta,\xi}^R \).

Our further reasoning is based on results derived by Caspers and Koelink in [3, 4]. We remark that one needs to be careful when taking equations from these papers as there is a difference in convention: we prefer to use inner products linear on the right and functionals \( \omega_{\xi,\eta} \) defined accordingly. That is why we choose to state explicitly used results with necessary changes, which we do in this section.

First, we can transport a left (resp. right) matrix coefficient via \( Q_L \) (resp. \( Q_R \)). The following is a reformulation of [4] Lemma 3.7, Lemma 3.9.
Lemma 3.6.

1) If $\xi, \eta \in J_{\text{irr}}^\oplus H_\pi \, d\mu(\pi)$, $\xi \in \text{Dom}(D)$ and the vector field $(\eta_\pi \otimes \overline{D_\pi \xi_\pi})_{\pi \in \text{irr}(G)}$ is square integrable, then $M_{\xi,\eta}^L \in \mathfrak{R}_\varphi$ and $Q_L \Lambda_\varphi(M_{\xi,\eta}^L) = J_{\text{irr}}^\oplus \eta_\pi \otimes \overline{D_\pi \xi_\pi} \, d\mu(\pi)$.

2) If $\xi, \eta \in J_{\text{irr}}^\oplus H_\pi \, d\mu(\pi)$, $\xi \in \text{Dom}(E)$ and the vector field $(\eta_\pi \otimes E_\pi \xi_\pi)_{\pi \in \text{irr}(G)}$ is square integrable, then $M_{\xi,\eta}^R \in \mathfrak{R}_\psi$ and $Q_R \Lambda_\psi(M_{\xi,\eta}^R) = J_{\text{irr}}^\oplus \eta_\pi \otimes E_\pi \xi_\pi \, d\mu(\pi)$.

Using the above result and the fact that $Q_L, Q_R$ are unitary, one can easily derive the following density results:

Lemma 3.7.

1) Set $\{\Lambda_\varphi(M_{\xi,\eta}^L)\}$, where $\xi, \eta$ run over vectors in $J_{\text{irr}}^\oplus H_\pi \, d\mu(\pi)$ such that $\xi \in \text{Dom}(D)$ and $(\eta_\pi \otimes \overline{D_\pi \xi_\pi})_{\pi \in \text{irr}(G)}$ is square integrable, is linearly dense in $L^2(G)$.

2) Set $\{\Lambda_\varphi(M_{\xi,\eta}^R)\}$, where $\xi, \eta$ run over vectors in $J_{\text{irr}}^\oplus H_\pi \, d\mu(\pi)$ such that $\xi \in \text{Dom}(E)$ and $(\eta_\pi \otimes \overline{E_\pi \xi_\pi})_{\pi \in \text{irr}(G)}$ is square integrable, is linearly dense in $L^2(G)$.

Consider an antilinear map\footnote{This map appears during a construction of the Radon-Nikodym derivative between $\psi$ and $\varphi$, see [17].}

$$\Lambda_\varphi(\mathfrak{R}_\psi \cap \mathfrak{R}_\varphi^*) \ni \Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*) \in L^2(G)$$

(3.1)

and define $T'$ to be its closure. Let $T' = J'\nabla^{1/2}$ be the polar decomposition of $T'$. It is well known that $J'$ is antiunitary and $\nabla^{1/2}$ is strictly positive and selfadjoint. In the next section we will describe these operators, for now let us recall how they look on the level of direct integrals.

Proposition 3.8. We have $Q_L J' Q_R^* = \Sigma$ and $Q_R \nabla^{1/2} Q_R^* = J_{\text{irr}}^\oplus D_\pi \otimes (E_\pi^{-1})^T \, d\mu(\pi)$.

The above proposition is a combination of [11 Proposition 4.4, Proposition 4.5, Theorem 4.6]. We finish this section with formulas expressing the action of modular automorphism groups on the matrix coefficients.

Proposition 3.9. For each $\xi, \eta \in J_{\text{irr}}^\oplus H_\pi \, d\mu(\pi), t \in \mathbb{R}$ the following holds:

$$\sigma^\psi_t(M_{\xi,\eta}^R) = \nu^{1/2} i t \delta^{it} M_{E^{2it} \xi, D^{2it} \eta}^R, \quad \sigma^\varphi_t(M_{\xi,\eta}^R) = \nu^{1/2} i t M_{E^{2it} \xi, D^{2it} \eta}^R \delta^{it},$$

$$\sigma^\psi_t(M_{\xi,\eta}^L) = \nu^{-1/2} i t M_{D^{2it} \xi, E^{2it} \eta}^L, \quad \sigma^\varphi_t(M_{\xi,\eta}^L) = \nu^{-1/2} i t \delta^{-it} M_{D^{2it} \xi, E^{2it} \eta}^L.$$
4 Relation between $\mathcal{Q}_L$ and $\mathcal{Q}_R$

In this section we will describe the polar decomposition of the closed operator $T': \Lambda_\psi(x) \mapsto \Lambda_\psi(x^*)$ (see equation (3.1)), namely we will derive a equation $T' = (\nu^* J_\varphi)(J_\varphi \nu^* \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi)$. As a corollary we get an important relation between $\mathcal{Q}_L$ and $\mathcal{Q}_R$. Before we do that, let us justify through a formal calculation, why the above formula for $T'$ should hold:

$$T'\Lambda_\psi(x) = \Lambda_\psi(x^*) = J_\varphi \nabla_\varphi^{-\frac{1}{2}} \Lambda_\psi(x) = J_\varphi \nabla_\varphi^{-\frac{1}{2}} J_\varphi (\delta^{-\frac{1}{2}})^* J_\varphi \Lambda_\psi(x) \delta^{\frac{1}{2}}$$

$$= \nabla_\varphi^{-\frac{1}{2}} (\nu^{-\frac{i}{2}} \delta^{-\frac{1}{2}})^* J_\varphi \Lambda_\psi(x) = (\nu^* J_\varphi)(J_\varphi \nu^* \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi) \Lambda_\psi(x).$$

(4.1)

We need to include the factor $\nu^*$ due to the following lemma:

**Lemma 4.1.** For all $s, t \in \mathbb{R}$ operators $\nabla_\varphi \circ \delta^t, \delta^t \circ \nabla_\varphi$ are closed. We have equality $\nu^{ist} \nabla_\varphi \delta^t = \nu^{-ist} \delta^t \nabla_\varphi$ of strictly positive, selfadjoint operators, moreover

$$(\nu^{ist} \nabla_\varphi \delta^t)^{itr} = \nu^{-ist} \delta^t \nabla_\varphi^{itr} = \nu^{ist} \delta^t \nabla_\varphi^{itr} (r \in \mathbb{R}).$$

The above result is a consequence of the commutation relation $\nabla_\varphi^{itr} \delta^t = \nu^{ist} \delta^t \nabla_\varphi^{itr} (s, t \in \mathbb{R})$. Indeed, it follows that operators $\nabla_\varphi, \delta^t$ satisfy the Weyl relation. Then Lemma 4.1 follows from [24] Example 3.1, Theorem 3.1. The next lemma describes the action of the unbounded operator $\delta^t$.

**Lemma 4.2.**

1) Let $t \in \mathbb{R}, x \in \mathfrak{M}_\varphi$ be such that $x \circ \delta^t$ is closable and $x \delta^t \in \mathfrak{M}_\varphi$. Then $J_\varphi \Lambda_\psi(x) \in \text{Dom}(\delta^t)$ and $\nu^* J_\varphi \delta^t J_\varphi \Lambda_\psi(x) = \Lambda_\psi(x \delta^t)$.

2) Let $t \in \mathbb{R}, x \in \mathfrak{M}_\varphi$ be such that $x \circ \delta^t$ is closable and $x \delta^t \in \mathfrak{M}_\varphi$. Then $J_\varphi \Lambda_\psi(x) \in \text{Dom}(\delta^t)$ and $\nu^* J_\varphi \delta^t J_\varphi \Lambda_\psi(x) = \Lambda_\psi(x \delta^t)$.

**Proof.** We prove only the first assertion, the second one can be derived analogously. Take $x \in \mathfrak{M}_\varphi, t \in \mathbb{R}$ which satisfy conditions of the lemma and define

$$x_n = \sqrt{\frac{n}{\pi}} \int_\mathbb{R} e^{-np^2} x \delta^t d\nu \in L^\infty(\mathbb{G}) \quad (n \in \mathbb{N})$$

(the above weak integral converges in $\sigma$-wot). Operator $x_n \circ \delta^t$ is closable and we have

$$x_n \delta^t = \sqrt{\frac{n}{\pi}} \int_\mathbb{R} e^{-np^2} (x \delta^t) \delta^{it} d\nu = \sqrt{\frac{n}{\pi}} \int_\mathbb{R} e^{-n(p-it)^2} x \delta^{it} d\nu. \quad (4.2)$$

Clearly $x_n, x_n \delta^t \in \mathfrak{M}_\varphi$ and due to the Hille’s theorem

$$\Lambda_\psi(x_n) = \sqrt{\frac{n}{\pi}} \int_\mathbb{R} e^{-np^2} \Lambda_\psi(x \delta^t) d\nu = \sqrt{\frac{n}{\pi}} J_\varphi \int_\mathbb{R} e^{-np^2} \nu^{-\frac{i}{2}} \delta^{it} J_\varphi \Lambda_\psi(x) d\nu,$$
similarly thanks to the equation (4.2) we have

\[ \Lambda_{\varphi}(x_n\delta^t) = \sqrt{\frac{n}{\pi}} J_{\varphi} \int_{\mathbb{R}} e^{-np^2} \nu^{-\frac{t}{2}} \delta^{-ip} J_{\varphi} \Lambda_{\varphi}(x) \, dp = \sqrt{\frac{n}{\pi}} J_{\varphi} \int_{\mathbb{R}} e^{-n(p-it)^2} \nu^{-\frac{t}{2}} \delta^{-ip} J_{\varphi} \Lambda_{\varphi}(x) \, dp. \]

Consequently, \( \Lambda_{\varphi}(x_n) \xrightarrow{n \to \infty} \Lambda_{\varphi}(x) \) and \( \Lambda_{\varphi}(x_n\delta^t) \xrightarrow{n \to \infty} \Lambda_{\varphi}(x\delta^t) \). For each \( r \in \mathbb{R} \) we have

\[ \delta^t J_{\varphi} \Lambda_{\varphi}(x_n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-np^2} \nu^{-\frac{t}{2}} \delta^{-ip} J_{\varphi} \Lambda_{\varphi}(x) \, dp = f_n(r), \]

where \( f_n \) is an entire function

\[ f_n: \mathbb{C} \ni z \mapsto \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(p+z)^2} \nu^{-\frac{t}{2}} \delta^{-zp} J_{\varphi} \Lambda_{\varphi}(x) \, dp \in L^2(\mathbb{G}). \]

From the above follows that \( J_{\varphi} \Lambda_{\varphi}(x_n) \in \text{Dom}(\delta^t) \) for all \( z \in \mathbb{C} \) and \( \delta^t J_{\varphi} \Lambda_{\varphi}(x_n) = f_n(-iz) \). Let us show that the sequence \( (\delta^t J_{\varphi} \Lambda_{\varphi}(x_n))_{n \in \mathbb{N}} \) converges to \( \nu^t J_{\varphi} \Lambda_{\varphi}(x\delta^t) \):

\[ \delta^t J_{\varphi} \Lambda_{\varphi}(x_n) = f_n(-it) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(p-it)^2} \nu^{-\frac{t}{2}} \delta^{-ip} J_{\varphi} \Lambda_{\varphi}(x) \, dp = \nu^t J_{\varphi} \Lambda_{\varphi}(x_n\delta^t) \xrightarrow{n \to \infty} \nu^t J_{\varphi} \Lambda_{\varphi}(x\delta^t). \]

Norm closedness of \( \delta^t \) implies \( J_{\varphi} \Lambda_{\varphi}(x) \in \text{Dom}(\delta^t) \) and \( \delta^t J_{\varphi} \Lambda_{\varphi}(x) = \nu^t J_{\varphi} \Lambda_{\varphi}(x\delta^t) \). \( \square \)

In what follows we introduce a space \( D_0 \) of sufficiently nice vectors on which calculation (4.1) is justified and which forms a core for the operators involved. First, define

\[ \delta_{n,z} = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-nt^2} \nu^{zt} \delta^{zt} \, dt \in L^\infty(\mathbb{G}) \quad (n \in \mathbb{N}, z \in \mathbb{C}). \]

Note that for each \( z \in \mathbb{C} \), the sequence \( (\delta_{n,z})_{n \in \mathbb{N}} \) is bounded and converges to \( \mathbb{I} \) in sot. Next, for \( x \in \mathcal{H}_\varphi \cap \mathcal{H}_\varphi^* \cap \mathcal{H}_\psi \cap \mathcal{H}_\psi^*, k \in \mathbb{N}, A = (A_1, A_2) \in \mathbb{C}^2 \) define

\[ x_{k,A} = \frac{k}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-k(t-A_1)^2-k(s-A_2)^2} \sigma_t^x \circ \sigma_s^y(x) \, dt \, ds \in L^\infty(\mathbb{G}). \]

Finally, define a subspace \( D_0 \) via

\[ D_0 = \text{span}\{ \Lambda_\psi(\delta_{n,z} x_{k,A} \delta_{m,w}) \mid x, x^* \in \mathcal{H}_\varphi \cap \mathcal{H}_\psi, n, m, k \in \mathbb{N}, A \in \mathbb{C}^2, z, w \in \mathbb{C} \}. \]

Lemma 4.3.

- The subspace \( D_0 \) is a core for \( \nabla_\varphi^{-\frac{t}{2}} \). Moreover, for \( \xi \in \text{Dom}(\nabla_\varphi^{-\frac{t}{2}}) \) we can find a sequence \( (\xi_p)_{p \in \mathbb{N}} \) in

\[ \{ \Lambda_\psi(x_{k,A} \delta_{m,w}) \mid x, x^* \in \mathcal{H}_\varphi \cap \mathcal{H}_\psi, m, k \in \mathbb{N}, A \in \mathbb{C}^2, w \in \mathbb{C} \} \]

such that \( \xi_p \xrightarrow{p \to \infty} \xi \) and \( \nabla_\varphi^{-\frac{t}{2}} \xi_p \xrightarrow{p \to \infty} \nabla_\varphi^{-\frac{t}{2}} \xi \).
Each element of $\mathcal{D}_0$ can be written as $\Lambda_\psi(x)$ for some $x \in L^\infty(\mathbb{G})$ such that $x, x^* \in \mathcal{M}_\psi \cap \mathcal{M}_\psi \cap \bigcap_{\epsilon \in \mathbb{C}} \text{Dom}(\sigma_\psi^\epsilon)$. Moreover, $\sigma_\psi^\epsilon(x)$ and $\Lambda_\psi(x^*)$ and $\Lambda_\psi(\sigma_\psi^\epsilon(x)) \in \mathcal{D}_0$. Next, $\Lambda_\psi(x) \in \bigcap_{\epsilon \in \mathbb{C}} \text{Dom}(\nabla_\psi^\epsilon)$ and $\nabla_\psi^\epsilon \Lambda_\psi(x) = \Lambda_\psi(\sigma_\psi^\epsilon(x))$.

For all $z, w \in \mathbb{C}, \Lambda_\psi(x) \in \mathcal{D}_0$ the operator $\delta^z \circ x \circ \delta^w$ is closable and after closure belongs to $\mathcal{M}_\psi \cap \mathcal{M}_\psi$.

We have $J_\varphi \mathcal{D}_0 = \mathcal{D}_0$.

A proof of the above lemma requires only standard reasoning, hence will be skipped. In the next two lemmas we prove properties of $\mathcal{D}_0$ which allows us to derive the polar decomposition of $T'$.

**Lemma 4.4.** The subspace $\mathcal{D}_0$ is a core for $\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi$. We have

$$
\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi \Lambda_\psi(x) = \Lambda_\psi(x)
$$

for all $x \in \mathcal{M}_\psi \cap \mathcal{M}_\psi$ such that $x \circ \delta^{-\frac{1}{2}}$ is closable and $x \delta^{-\frac{1}{2}} \in \mathcal{M}_\psi$. Moreover, the operator

$$(J_\varphi \nabla_\psi^\frac{1}{2}) \circ (\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi) = \nu^{\frac{1}{2}} (\nabla_\psi^{-\frac{1}{2}} \circ \delta^{-\frac{1}{2}}) J_\varphi
$$

is closable and $\mathcal{D}_0$ is a core for its closure $\nu^{\frac{1}{2}} \nabla_\psi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi$.

**Proof.** It is clear that $\text{span} \bigcup_{n \in \mathbb{N}} \delta_{n,0} \mathcal{H}^2(\mathbb{G})$ is a core for $\delta^{-\frac{1}{2}}$. Take $\xi = \delta_{n,0} \eta \in \text{Dom}(\delta^{-\frac{1}{2}})$ for some $n \in \mathbb{N}$ and let $(\eta_p)_{p \in \mathbb{N}}$ be a sequence of vectors of the form $\Lambda_\psi(x_k, \lambda_B \delta_{m,w})$ (see the first point of the Lemma 4.3) converging to $\eta$. We have $\delta_{n,0} \eta_p \in \mathcal{D}_0$,

$$
\| \xi - \delta_{n,0} \eta_p \| \leq \| \eta - \eta_p \| \xrightarrow{p \to \infty} 0 \quad \text{and} \quad \| \delta^{-\frac{1}{2}} \xi - \delta^{-\frac{1}{2}} \delta_{n,0} \eta_p \| \leq \| \delta^{-\frac{1}{2}} \delta_{n,0} \| \| \eta - \eta_p \| \xrightarrow{p \to \infty} 0,
$$

which shows that $\mathcal{D}_0$ is a core for $\delta^{-\frac{1}{2}}$. Since $\mathcal{D}_0$ is invariant under $J_\varphi$, it is also a core for $\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi$.

Take $x \in \mathcal{M}_\psi \cap \mathcal{M}_\psi$ such that $x \circ \delta^{-\frac{1}{2}}$ is closable and $x \delta^{-\frac{1}{2}} \in \mathcal{M}_\psi$. Lemma 4.2 gives us $J_\varphi \Lambda_\psi(x) \in \text{Dom}(\delta^{-\frac{1}{2}})$ and $\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi \Lambda_\psi(x) = \Lambda_\psi(x \delta^{-\frac{1}{2}}) = \Lambda_\psi(x)$.

Equality from the claim $(J_\varphi \nabla_\psi^\frac{1}{2}) \circ (\nu^{-\frac{1}{2}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi) = \nu^{\frac{1}{2}} (\nabla_\psi^{-\frac{1}{2}} \circ \delta^{-\frac{1}{2}}) J_\varphi$ is a straightforward consequence of the relation $J_\varphi \nabla_\psi^\frac{1}{2} = \nabla_\psi^{-\frac{1}{2}} J_\varphi$.

To deduce the last assertion let us observe that Lemma 4.4 gives us an equality $\nu^{i/8} \nabla_\psi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} = \nu^{-i/4} \delta^{-\frac{1}{2}} J_\varphi \nabla_\psi^{-\frac{1}{2}}$. It follows that the closure of $\nu^{-i/4} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}}$ is $\nabla_\psi^{-\frac{1}{2}} \delta^{-\frac{1}{2}}$. Take $\xi \in \text{Dom}(\nu^{-i/4} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}})$. For each $n \in \mathbb{N}$ we have $\delta_{n,0} \xi \in \text{Dom}(\nu^{-i/4} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}})$,

$$
\delta_{n,0} \xi \xrightarrow{n \to \infty} \xi \quad (4.3)
$$

and

$$
\nu^{-i/4} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}} (\delta_{n,0} \xi) = \nu^{-i/4} \delta_{n,0} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}} (\xi) = \nu^{-i/4} \delta_{n-1/2} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}} (\xi) \xrightarrow{n \to \infty} \nu^{-i/4} \delta^{-\frac{1}{2}} \circ \nabla_\psi^{-\frac{1}{2}} (\xi). \quad (4.4)
$$
As previously, since $\mathcal{D}_0$ is invariant for $J_\varphi$, it is enough to check that $\mathcal{D}_0$ is a core for $\nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}}$. Take $\xi \in \text{Dom}(\nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}})$. The above reasoning and equations (4.3), (4.4) show that it is enough to take vector of the form $\xi = \delta_{n,0} \eta$ for $\eta \in \text{Dom}(\nu^{-1/4} \delta^{-1/4} \circ \nabla_\varphi^{-1/2})$ and some $n \in \mathbb{N}$. Let $(\eta_p)_{p \in \mathbb{N}}$ be a sequence of vectors of the form $\Lambda_\varphi(x_{k,A,B}\delta_{m,w})$ such that $\eta_p \xrightarrow{p \to \infty} \eta$ and $\nabla_\varphi^{-\frac{1}{2}} \eta_p \xrightarrow{p \to \infty} \nabla_\varphi^{-\frac{1}{2}} \eta$. We have $\delta_{n,0} \eta_p \in \mathcal{D}_0$, $\delta_{n,0} \eta_p \xrightarrow{p \to \infty} \delta_{n,0} \eta = \xi$ and

$$
\left\| \nu^{-1/4} \delta^{-1/4} \circ \nabla_\varphi^{-1/2} (\delta_{n,0} \eta - \delta_{n,0} \eta_p) \right\| = \left\| \delta_{n,-1/2} \delta^{-1/2} \circ \nabla_\varphi^{-1/2} (\eta - \eta_p) \right\|
$$

$$
\leq \left\| \delta_{n,-1/2} \delta^{-1/2} \right\| \left\| \nabla_\varphi^{-1/2} (\eta - \eta_p) \right\| \xrightarrow{p \to \infty} 0.
$$

\[ \]

**Lemma 4.5.** The subspace $\mathcal{D}_0$ is a core for $T'$.

**Proof.** Take $x \in \mathfrak{M}_\varphi \cap \mathfrak{N}_\varphi^*$ and define $x_n$ as $x_n = \frac{1}{4 \pi} \int_\mathbb{R} \int_\mathbb{R} e^{-n(r^2 + s^2)} \sigma_\varphi \circ \sigma_\varphi^*(x_n) \, ds \, dp \quad (n \in \mathbb{N})$. We have $x_n, x_n^* \in \mathfrak{M}_\varphi \cap \mathfrak{N}_\varphi$. Next, define $x_{n,n} = \frac{1}{4 \pi} \int_\mathbb{R} \int_\mathbb{R} e^{-n(r^2 + s^2)} \sigma_\varphi \circ \sigma_\varphi^*(x_{n,n}) \, ds \, dp$. We have $\delta_{n,0} x_{n,n} \delta_{n,0} \in \mathfrak{M}_\varphi \cap \mathfrak{N}_\varphi^*$, $\Lambda_\varphi(\delta_{n,0} x_{n,n} \delta_{n,0}) \in \mathcal{D}_0$, $\Lambda_\varphi(\delta_{n,0} x_{n,n} \delta_{n,0}) \xrightarrow{n \to \infty} \Lambda_\varphi(x)$ and

$$
T' \Lambda_\varphi(\delta_{n,0} x_{n,n} \delta_{n,0}) = \Lambda_\varphi(\delta_{n,0} x_{n,n}^* \delta_{n,0}) \xrightarrow{n \to \infty} \Lambda_\varphi(x^*) = T' \Lambda_\varphi(x).
$$

\[ \]

Now we can derive the main results of this section.

**Proposition 4.6.** We have $(J_\varphi \nabla_\varphi^\pm) \circ (\nu^{-1/4} J_\varphi \delta^{-1/2} J_\varphi) = \nu^{\pm} (\nabla_\varphi^{-1/2} \circ \delta^{-1/2}) J_\varphi$ and after closure

$$
\nu^{\pm} \nabla_\varphi^{-1/2} \delta^{-1/2} J_\varphi = T'.
$$

**Proof.** The first equality was justified in Lemma 4.3. Take $\Lambda_\varphi(x) \in \mathcal{D}_0$. Lemmas 4.3, 4.4 justify the following calculation:

$$(J_\varphi \nabla_\varphi^\pm) \circ (\nu^{-1/4} J_\varphi \delta^{-1/2} J_\varphi) \Lambda_\varphi(x) = J_\varphi \nabla_\varphi^\pm \Lambda_\varphi(x) = \Lambda_\varphi(x^*) = T' \Lambda_\varphi(x).$$

In lemmas 4.3, 4.4 we have shown that $\mathcal{D}_0$ is a core for $T'$ and $\nu^\pm \nabla_\varphi^{-1/2} \delta^{-1/2} J_\varphi$, which shows $T' = \nu^\pm \nabla_\varphi^{-1/2} \delta^{-1/2} J_\varphi$. \[ \]

The above result has a number of interesting corollaries.

**Corollary 4.7.** The polar decomposition of $T'$ is $T' = (\nu^{\pm} J_\varphi)(J_\varphi \nu^{\pm} \nabla_\varphi^{-1/2} \delta^{-1/2} J_\varphi)$. Moreover, we have

$$(J_\varphi \nu^{\pm} \nabla_\varphi^{-1/2} \delta^{-1/2} J_\varphi)^t = \nu^{\pm t} J_\varphi \nabla_\varphi^{t/2} \delta^{t/2} J_\varphi \quad (t \in \mathbb{R}).$$

(4.5)
Proof. The first equality follows directly from Proposition 4.6. Let us justify that it is indeed the polar decomposition. First, it is clear that \( \nu^\frac{i}{2} J_\varphi \) is antiunitary. Next, Lemma 4.1 implies that \( \nu^\frac{i}{2} \nabla_\varphi^{-\frac{1}{2}} J_\varphi \) is selfadjoint and strictly positive. Consequently, the operator \( J_\varphi \nu^\frac{i}{2} \nabla_\varphi^{-\frac{1}{2}} J_\varphi \) has the same properties. Uniqueness of the polar decomposition gives us the first claim. The second formula follows from Lemma 4.1.

\[
(J_\varphi \nu^\frac{i}{2} \nabla_\varphi^{-\frac{1}{2}} J_\varphi)^{it} = f((\nu^\frac{i}{2} \nabla_\varphi^{-\frac{1}{2}} J_\varphi)^{it}) = \nu^{-\frac{i}{2}} J_\varphi, \\
\]

where \( f : a \mapsto J_\varphi a^* J_\varphi \).

Now we combine our polar decomposition of \( T' \) with the result of Caspers (Proposition 3.8 and Proposition 3.3).

Corollary 4.8. We have \( Q_L \nu^\frac{i}{2} J_\varphi Q_R = \Sigma \) and \( Q_R Q_L = Q_L Q_R = \nu^{-\frac{i}{2}} J_\varphi \).

Formula \( Q_R Q_L = \nu^{-\frac{i}{2}} J_\varphi \) is of great importance and will be used numerous times throughout the paper.

5 Operators expressed on the level of direct integrals

In this section we will derive several equations, which express important operators on \( L^2(G) \) via \( Q_L, Q_R \) as direct integrals. The first result of this type comes from the polar decomposition of \( T' \).

Proposition 5.1. For all \( t \in \mathbb{R} \) we have

\[
\nabla_\psi^d \delta^{-it} = J_\varphi \nabla_\psi^d \delta^{-it} J_\varphi = \nu^{-\frac{i}{2} t^2} \mathcal{Q}_L^* \left( \int_{\text{Irr}(G)}^\oplus D^2_{\pi} \otimes (E_{\pi}^{-2it})^T \, d\mu(\pi) \right) \mathcal{Q}_R, \\
J_\varphi \nabla_\psi^d \delta^{-it} J_\varphi = \nabla_\psi^d \delta^{-it} = \nu^\frac{i}{2} \mathcal{Q}_L \left( \int_{\text{Irr}(G)}^\oplus E_{\pi}^{2it} \otimes (D_{\pi}^{-2it})^T \, d\mu(\pi) \right) \mathcal{Q}_R, \\
\nabla_\varphi^{-i t} J_\varphi = \nabla_\varphi^{-i t}, \\
J_\varphi \nabla_\varphi^{-i t} J_\varphi = \nabla_\varphi^{-i t} \mathcal{Q}_R \left( \int_{\text{Irr}(G)}^\oplus D^2_{\pi} \otimes (E_{\pi}^{-2it})^T \, d\mu(\pi) \right) \mathcal{Q}_L.
\]

Proof. First, observe that we have \( \nabla_\psi^d = J_\varphi \nabla_\psi^{-i t} J_\varphi = \delta^{-it} (J_\varphi \delta^{-it} J_\varphi) \nabla_\psi^d \) (see [20] Theorem 5.18 and equation (2.1)). It follows that

\[
\nabla_\psi^d \delta^{-it} = \nu^{-i t^2} \delta^{-it} \nabla_\psi^d = \nu^{-it^2} J_\varphi \delta^{-it} \nabla_\psi^d J_\varphi = J_\varphi \nabla_\varphi^{-i t} J_\varphi,
\]

where
and first equation in each row easily follows. The formula expressing $J_{\varphi} \nabla_{\varphi}^{\mu} \delta^{it} J_{\varphi}$ via direct integral of operators follows from equation \([15] \) combined with Proposition \([3,18] \). The second equation can be found using already derived relation $Q^*_L Q_R = \nu \cdot J_{\varphi} J_{\varphi}$:

$$J_{\varphi} \nabla_{\varphi}^{\mu} \delta^{it} J_{\varphi} = \nu \cdot \nabla_{\varphi}^{\mu} \delta^{it} J_{\varphi} = \nu \cdot \nabla_{\varphi}^{\mu} \delta^{it} J_{\varphi} = \nu \cdot \nabla_{\varphi}^{\mu} \delta^{it} J_{\varphi},$$

which implies $\nabla_{\varphi}^{\mu} \delta^{it} = \nu \cdot \nabla_{\varphi}^{\mu} (\delta^{it} J_{\varphi})$. The last two equations come from applying the operation $J_{\varphi} \cdot J_{\varphi}$ to both sides of already derived formulas. □

Let us now derive an interesting corollary of these results.

**Corollary 5.2.** There exists a unique measurable function $f : \text{Irr}(\hat{G}) \rightarrow \mathbb{R}_{>0}$ such that

$$J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} D^{2it} \otimes 1_{\text{Irr}(\hat{G})} d\mu(\pi) \right) Q_R J_{\varphi} = Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} f(\pi)^i E^{2it} \otimes 1_{\text{Irr}(\hat{G})} d\mu(\pi) \right) Q_R,$$

$$J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} 1_{\text{Irr}(\hat{G})} \otimes (E^{2it})^T d\mu(\pi) \right) Q_R J_{\varphi} = Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} f(\pi)^i 1_{\text{Irr}(\hat{G})} \otimes (E^{2it})^T d\mu(\pi) \right) Q_R,$$

$$J_{\varphi} Q^*_L \left( \int_{\text{Irr}(\hat{G})}^{t} 1_{\text{Irr}(\hat{G})} \otimes (D^{2it})^T d\mu(\pi) \right) Q_L J_{\varphi} = Q^*_L \left( \int_{\text{Irr}(\hat{G})}^{t} f(\pi)^i 1_{\text{Irr}(\hat{G})} \otimes (D^{2it})^T d\mu(\pi) \right) Q_L,$$

$$J_{\varphi} Q^*_L \left( \int_{\text{Irr}(\hat{G})}^{t} E^{2it} \otimes 1_{\text{Irr}(\hat{G})} \right) Q_L J_{\varphi} = Q^*_L \left( \int_{\text{Irr}(\hat{G})}^{t} f(\pi)^i D^{2it} \otimes 1_{\text{Irr}(\hat{G})} \right) Q_L$$

for all $t \in \mathbb{R}$.

We note that the function $f$ might depend on the choice of a measure $\mu$.

**Proof.** Fix $t \in \mathbb{R}$. The first and the third row in Proposition \([5,1] \) implies

$$J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} D^{2it} \otimes (E^{2it})^T d\mu(\pi) \right) Q_R J_{\varphi} = Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} E^{2it} \otimes (D^{2it})^T d\mu(\pi) \right) Q_R.$$

Since $J_{\varphi} L_{\infty}(^{\hat{G}}) J_{\varphi} = L_{\infty}(^{\hat{G}})$, $J_{\varphi} L^{\infty}(^{\hat{G}})$ and the center of $\int_{\text{Irr}(\hat{G})}^{t} 1_{\text{Irr}(\hat{G})} \otimes B(\text{Irr}(\hat{G})) d\mu(\pi)$ is $\int_{\text{Irr}(\hat{G})}^{t} \mathbb{C}1_{\text{Irr}(\hat{G})} d\mu(\pi)$, Proposition \([3,4] \) implies that there exists a measurable function $f_t : \text{Irr}(\hat{G}) \rightarrow \mathbb{T}$ such that

$$J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} D^{2it} \otimes 1_{\text{Irr}(\hat{G})} \right) Q_R J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} E^{2it} \otimes 1_{\text{Irr}(\hat{G})} \right) Q_R$$

$$= J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} 1_{\text{Irr}(\hat{G})} \otimes (E^{2it})^T d\mu(\pi) \right) Q_R J_{\varphi} Q^*_R \left( \int_{\text{Irr}(\hat{G})}^{t} 1_{\text{Irr}(\hat{G})} \otimes (D^{2it})^T d\mu(\pi) \right) Q_R$$

$$= \int_{\text{Irr}(\hat{G})}^{t} f_t(\pi) 1_{\text{Irr}(\hat{G})} \ d\mu(\pi).$$
The above equations imply
\[ J_\varphi Q^*_R \left( \int_{\text{Irr}(G)}^{\oplus} D^{2it}_\pi \otimes 1_{\text{Irr}} \, d\mu(\pi) \right) \ast Q_R J_\varphi = Q^*_R \left( \int_{\text{Irr}(G)}^{\oplus} f_t(\pi) E^{2it}_\pi \otimes 1_{\text{Irr}} \, d\mu(\pi) \right) Q_R \]  
(5.1)
and
\[ J_\varphi Q^*_R \left( \int_{\text{Irr}(G)}^{\oplus} 1_{H_n} \otimes (E^{2it}_\pi)^T \, d\mu(\pi) \right) \ast Q_R J_\varphi = Q^*_R \left( \int_{\text{Irr}(G)}^{\oplus} f_t(\pi) 1_{H_n} \otimes (D^{2it}_\pi)^T \, d\mu(\pi) \right) Q_R. \]  
(5.2)

Equation (5.1) together with relation \( Q^*_L Q_R = \nu^{-\frac{1}{2}} J_{\varphi} J_\varphi \) (Corollary 4.8) gives us
\[ J_\varphi J_\varphi \hat{Q}^*_L \left( \int_{\text{Irr}(G)}^{\oplus} D^{2it}_\pi \otimes 1_{\text{Irr}} \, d\mu(\pi) \right) \ast \hat{Q}_L J_{\varphi} J_{\varphi} = J_\varphi J_{\varphi} \hat{Q}^*_L \left( \int_{\text{Irr}(G)}^{\oplus} f_t(\pi) E^{2it}_\pi \otimes 1_{\text{Irr}} \, d\mu(\pi) \right) \hat{Q}_L J_{\varphi} J_{\varphi}, \]

hence also (thanks to \( Q^*_L J_{\varphi} \hat{Q}^*_L = \Sigma \), see Proposition 3.3)
\[ \hat{Q}^*_L \left( \int_{\text{Irr}(G)}^{\oplus} 1_{H_n} \otimes (D^{2it}_\pi)^T \, d\mu(\pi) \right) \ast \hat{Q}_L = J_\varphi \hat{Q}^*_L \left( \int_{\text{Irr}(G)}^{\oplus} f_t(\pi) 1_{H_n} \otimes (E^{2it}_\pi)^T \, d\mu(\pi) \right) \hat{Q}_L J_{\varphi}. \]

The last equation can be derived from equation (5.2) in a similar manner. Clearly we have \( f_t(\pi) = f(\pi)^t \) for a measurable function \( f : \text{Irr}(G) \to \mathbb{R}_{>0}. \)

In the second part of this section, we will transport operators \( \nabla_{\varphi}^t, \nabla_{\psi}^t, \delta^t (t \in \mathbb{R}) \) to \( \int_{\text{Irr}(G)}^{\oplus} H_S(H_n) \, d\mu(\pi) \). We start with a formula expressing the action of \( (\tau_t)_{t \in \mathbb{R}} \) on matrix coefficients.

**Lemma 5.3.** For \( \xi, \eta \in \int_{\text{Irr}(G)}^{\oplus} H_n \, d\mu(\pi) \) and \( t \in \mathbb{R} \) we have
\[
\tau_t(M^L_{\xi,\eta}) = \nu^{-\frac{1}{2}} \delta^t \left( \int_{\text{Irr}(G)} (\text{id} \otimes \omega_{D^{2it}_\pi(\delta^{-1t}\xi,\varphi_2^{2it}\eta)})(U^{\pi*}) \, d\mu(\pi) \right)
= \nu^{-\frac{1}{2}} \delta^t \left( \int_{\text{Irr}(G)} (\text{id} \otimes \omega_{D^{2it}_\pi(\xi,\varphi_2^{2it}\pi(\delta^{-1t}\eta))})(U^{\pi*}) \, d\mu(\pi) \right),
\]
\[
\tau_t(M^R_{\xi,\eta}) = \nu^{\frac{1}{2}} \delta^t \left( \int_{\text{Irr}(G)} (\text{id} \otimes \omega_{E^{2it}_\pi(\delta^{-1t}\xi,\varphi_2^{2it}\eta)})(U^{\pi}) \, d\mu(\pi) \right)
= \nu^{\frac{1}{2}} \delta^t \left( \int_{\text{Irr}(G)} (\text{id} \otimes \omega_{E^{2it}_\pi(\varphi_2^{2it}\pi(\delta^{-1t}\eta))})(U^{\pi}) \, d\mu(\pi) \right).
\]
Later on in Proposition 5.5, we will get simpler expressions for this action (once we find out what \( \pi(\delta^t_u) \) is).
Proof. The proof is based on several facts from the theory of locally compact quantum groups. First of all, we know that $\delta^it = P^{-it}\nabla_{\psi}^{-it}$ (equation (2.1)). Next, [20 Lemma 5.14] gives us

$$(\sigma_t^\pi \otimes \text{id})W = (1 \otimes P^{-it})W((1 \otimes \nabla_{\psi}^{-it}), (1 \otimes \nabla_{\psi}^{-it})W(1 \otimes P^{-it}),$$

and $(\tau_t \otimes \text{id})W = (\text{id} \otimes \hat{\tau}_t)W$. We note also that $\hat{\delta}^it \in \text{M}(C_0(\hat{G}))$, $\hat{\delta}^{-it} \in \text{M}(C_0^0(\hat{G}))$ and $\Lambda_\sigma(\hat{\delta}^it) = \hat{\delta}^{-it}$ ([23]). Fix $t \in \mathbb{R}$, a representation $\pi \in \text{Irr}(G)$ which factorises through $C_0(\hat{G})$ (i.e. $\pi = \pi' \circ \Lambda_\sigma$ for a representation $\pi' : C_0(\hat{G}) \to B(\mathcal{H})$) and arbitrary vectors $\xi_\pi, \eta_\pi \in \mathcal{H}_\pi$. We have

$$\tau_t((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})) = (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(\tau_t \otimes \text{id})(\text{id} \otimes \pi)(\mathcal{H}^*)$$

$$(\text{id} \otimes \omega_{\xi_\pi, \eta_\pi}) \circ \pi'((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')(\text{id} \otimes \hat{\tau}_t)(W^*)$$

$$= (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((1 \otimes P^{-it})(W^*)(1 \otimes P^it))$$

Now we write the above expression in two different ways: we have

$$\tau_t((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})) = (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((1 \otimes P^{-it}\nabla_{\psi}^{-it})(1 \otimes \nabla_{\psi}^{it})(W^*)(1 \otimes P^it))$$

$$(\text{id} \otimes \omega_{\xi_\pi, \eta_\pi}) \circ \pi'((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')(\text{id} \otimes \hat{\tau}_t)(W^*)$$

$$= (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((1 \otimes \delta^it)(\sigma_t^\pi \otimes \text{id})(W^*)) = (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((1 \otimes \delta^{-it})(\mathcal{H}^*))$$

$$= \sigma_t^\pi((\text{id} \otimes \omega_{\pi(\hat{\delta}^{-it})\eta_\pi})(U^{\pi\pi}))$$

(5.3)

and

$$\tau_t((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})) = (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((1 \otimes P^{-it}\nabla_{\psi}^{-it})(1 \otimes \nabla_{\psi}^{it})(1 \otimes \nabla_{\psi}^{it})(P^it))$$

$$(\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')(\text{id} \otimes \hat{\tau}_t)(W^*)$$

$$= (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')(\text{id} \otimes \hat{\tau}_t)(W^*)$$

$$= (\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi} \circ \pi')(\text{id} \otimes \hat{\tau}_t)(W^*)$$

$$= \sigma_t^\pi((\text{id} \otimes \omega_{\pi(\hat{\delta}^{-it})\eta_\pi})(U^{\pi\pi}))$$

(5.4)

Let now $\xi, \eta$ be vectors in $\int_{\Gamma}(\mathcal{G})\mu(\pi)$. Then fields $(\pi(\hat{\delta}^{-it})\xi, \pi(\hat{\delta}^{-it})\eta) \in \text{Irr}(G)$ are also square integrable. Using equations (5.3), (5.4) and Proposition 3.9, we arrive at

$$\tau_t(M_{\xi, \eta}^L) = \tau_t(\int_{\Gamma}(\mathcal{G})((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})\mu(\pi))$$

$$= \int_{\Gamma}(\mathcal{G})\sigma_t^\pi(((\text{id} \otimes \omega_{\pi(\hat{\delta}^{-it})\xi_\pi, \eta_\pi})(U^{\pi\pi}))\mu(\pi))$$

$$= \int_{\Gamma}(\mathcal{G})\sigma_t^\pi(((\text{id} \otimes \omega_{\pi(\hat{\delta}^{-it})\xi_\pi, \eta_\pi})(U^{\pi\pi}))\mu(\pi))$$

$$= \nu^{-\frac{1}{2}it^2}\delta^{-it}\int_{\Gamma}(\mathcal{G})((\text{id} \otimes \omega_{\pi(\hat{\delta}^{-it})\xi_\pi, \eta_\pi})(U^{\pi\pi})\mu(\pi))$$

and

$$\tau_t(M_{\xi, \eta}^L) = \tau_t(\int_{\Gamma}(\mathcal{G})((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})\mu(\pi))$$

$$= \int_{\Gamma}(\mathcal{G})\sigma_t^\pi(((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi}))\mu(\pi))$$

$$= \int_{\Gamma}(\mathcal{G})\sigma_t^\pi(((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi}))\mu(\pi))$$

$$= \nu^{-\frac{1}{2}it^2}\int_{\Gamma}(\mathcal{G})((\text{id} \otimes \omega_{\xi_\pi, \eta_\pi})(U^{\pi\pi})\mu(\pi))\delta^{-it}.$$
The second pair of equations follow by applying the adjoint.

Now we are ready to obtain the main results of this section. Even though we will prove them together, they are of different nature, hence we prefer to state them separately. First, we have a couple of equations expressing important operators on the level of direct integrals.

**Theorem 5.4.** For every \( t \in \mathbb{R} \) we have

\[
\nabla_{\hat{\psi}}^{-it} = \delta^{-it} P^{-it} = Q_L^*(\int_{\text{Irr}(G)}^\oplus E_\pi^{2it} \otimes (E_\pi^{-2it})^T d\mu(\pi)) Q_L
\]

\[
= Q_R^*(\int_{\text{Irr}(G)}^\oplus D_\pi^{-2it} \otimes (D_\pi^{2it})^T d\mu(\pi)) Q_R;
\]

\[
\nabla_{\hat{\varphi}}^{-it} = J_{\varphi}^{-it} P^{-it} J_{\varphi} = Q_L^*(\int_{\text{Irr}(G)}^\oplus D_\pi^{-2it} \otimes (D_\pi^{2it})^T d\mu(\pi)) Q_L
\]

\[
= Q_R^*(\int_{\text{Irr}(G)}^\oplus E_\pi^{2it} \otimes (E_\pi^{-2it})^T d\mu(\pi)) Q_R.
\]

Next, we show that the modular element for \( \hat{\mathbb{G}} \) can be expressed using operators \((D_\pi)_{\pi \in \text{Irr}(G)}, (E_\pi)_{\pi \in \text{Irr}(G)}\).

**Proposition 5.5.** For all \( t \in \mathbb{R} \) we have

\[
\delta^{-it} = \nu^{-\frac{1}{2}t^2} Q_L^*(\int_{\text{Irr}(G)}^\oplus D_\pi^{2it} E_\pi^{-2it} \otimes 1_{\mathbb{H}_\pi} d\mu(\pi)) Q_L
\]

\[
= \nu^{-\frac{1}{2}t^2} Q_R^*(\int_{\text{Irr}(G)}^\oplus 1_{\mathbb{H}_\pi} \otimes (D_\pi^{-2it} E_\pi^{2it})^T d\mu(\pi)) Q_R.
\]

Moreover, \( \pi(\delta^{-it}) = \nu^{it^2} E_\pi^{-2it} D_\pi^{2it} \) and \( \nu^{it} D_\pi^{2it} E_\pi^{-2it} = E_\pi^{2it} D_\pi^{2it} \) for all \( s,t \in \mathbb{R} \) and almost all \( \pi \in \text{Irr}(G) \). We also get better expressions for the action of \((\tau_t)_{t \in \mathbb{R}}\):

\[
\tau_t(M^L_{\xi,\eta}) = \delta^{-it} M^L_{E^{2it}\xi,E^{2it}\eta} = M^L_{D^{-2it}\xi,D^{-2it}\eta}\delta^{-it},
\]

\[
\tau_t(M^R_{\xi,\eta}) = M^R_{E^{2it}\xi,E^{2it}\eta} \delta^{-it} = \delta^{-it} M^R_{D^{-2it}\xi,D^{-2it}\eta}
\]

for all \( t \in \mathbb{R} \) and \( \xi, \eta \in J_{\text{Irr}(G)}^\oplus \mathbb{H}_\pi d\mu(\pi) \).

**Proof.** Let \( \xi, \eta \) be vector fields satisfying conditions from the first point of Lemma 3.6. Note that vector fields \((D_\pi^{-2it}\xi)_\pi \in \text{Irr}(G), (E_\pi^{-2it}\pi(\hat{\delta}^{-it})\eta)_\pi \in \text{Irr}(G)\) also satisfy conditions of
this lemma. Using the second equation from Lemma 5.3 we get:
\[
\mathcal{Q}_L \mathcal{P}^{it} \Lambda_\varphi(M_{\xi,\eta}^L) = \nu^\frac{i}{2} \mathcal{Q}_L \Lambda_\varphi(\tau_t(M_{\xi,\eta}^L))
\]
\[
= \nu^\frac{i}{2} \mathcal{Q}_L \mathcal{P}^{it} \Lambda_\varphi(\int_{\text{Irr}(G)} (\text{id} \otimes \omega_{\mathcal{D}^{2it},\xi,\mathcal{E}^{2it},(\hat{\delta}^{-it})_{\eta}}(U^{\pi})) \text{d}\mu(\pi) \delta^{it})
\]
\[
= \nu^\frac{i}{2} \mathcal{Q}_L \mathcal{P}^{it} \Lambda_\varphi(\int_{\text{Irr}(G)} (\text{id} \otimes \omega_{\mathcal{D}^{2it},\xi,\mathcal{E}^{2it},(\hat{\delta}^{-it})_{\eta}}(U^{\pi})) \text{d}\mu(\pi))
\]
\[
= \nu^\frac{i}{2} \mathcal{Q}_L \mathcal{P}^{it} \Lambda_\varphi(\int_{\text{Irr}(G)} E^{-2it}_\pi(\hat{\delta}^{-it})_{\eta} \otimes D^{2it}_\pi \text{d}\mu(\pi))
\]
\[
= \nu^\frac{i}{2} \mathcal{Q}_L \mathcal{P}^{it} \Lambda_\varphi(\int_{\text{Irr}(G)} E^{-2it}_\pi(\hat{\delta}^{-it}) \otimes (D^{2it}_\pi)^T \text{d}\mu(\pi)) \mathcal{Q}_L, \tag{5.5}
\]

Since the set of \( \Lambda_\varphi(M_{\xi,\eta}^L) \) with \( \xi, \eta \) as above form a linearly dense set (Lemma 3.7), we get
\[
J_\varphi \delta^{it} J_\varphi \mathcal{P}^{it} = \mathcal{Q}*_{L} \left( \int_{\text{Irr}(G)} (\nu^\frac{i}{2} E^{-2it}_\pi(\hat{\delta}^{-it}) \otimes (D^{2it}_\pi)^T \text{d}\mu(\pi)) \right) \mathcal{Q}_L. \tag{5.5}
\]

Since \( (J_\varphi \delta^{it} J_\varphi \mathcal{P}^{it})_{t \in \mathbb{R}} \) are strongly continuous groups (see equation (2.1)) the same is true for \( (\nu^\frac{i}{2} E^{-2it}_\pi(\hat{\delta}^{-it})_{\eta}) \) (see point 2) of Lemma 8.1. Using relations gathered in equation (2.1) one easily checks that \( J_\varphi \delta^{it} J_\varphi \mathcal{P}^{it} \). Since \( J_\varphi = \mathcal{Q}*_{L} \Sigma \mathcal{Q}_L \), point 3 of Lemma 8.1 implies
\[
\nu^\frac{i}{2} E^{-2it}_\pi(\hat{\delta}^{-it}) = D^{2it}_\pi \Rightarrow \pi(\hat{\delta}^{-it}) = \nu^\frac{i}{2} E^{-2it}_\pi D^{2it}_\pi \quad (\pi \in \text{Irr}(G), t \in \mathbb{R}) \tag{5.6}
\]

Let us choose \( s, t \in \mathbb{R} \) and use the fact that \( (\pi(\hat{\delta})_{\eta})_{t \in \mathbb{R}} \) is a group: we have
\[
\nu^\frac{i}{2} E^{-2it}_\pi(\hat{\delta}^{-it}) = D^{2it}_\pi \Rightarrow \pi(\hat{\delta}^{-it}) = \nu^\frac{i}{2} E^{-2it}_\pi D^{2it}_\pi \quad \text{and formula } \nu^\frac{i}{2} E^{-2it}_\pi D^{2it}_\pi = D^{2it}_\pi E^{-2it}_\pi \text{ easily follows. Equations expressing the action of } \tau_t \text{ on matrix coefficients follows from the equation } \pi(\hat{\delta}^{-it}) = \nu^\frac{i}{2} E^{-2it}_\pi D^{2it}_\pi \text{, commutation relation between } E^{-2it}_\pi \text{ and } D^{2it}_\pi \text{ and Lemma 5.3. Let us now plug in the above results to equation (5.5)}:
\]
\[
J_\varphi \delta^{it} J_\varphi \mathcal{P}^{it} = \nu^\frac{i}{2} \mathcal{Q}*_{L} \left( \int_{\text{Irr}(G)} E^{-2it}_\pi(\hat{\delta}^{-it}) \otimes (D^{2it}_\pi)^T \text{d}\mu(\pi)) \right) \mathcal{Q}_L
\]
\[
= \nu^\frac{i}{2} \mathcal{Q}*_{L} \left( \int_{\text{Irr}(G)} E^{-2it}_\pi \nu^\frac{i}{2} E^{2it}_\pi D^{2it}_\pi \otimes (D^{2it}_\pi)^T \text{d}\mu(\pi)) \right) \mathcal{Q}_L \tag{5.7}
\]
\[
= \mathcal{Q}*_{L} \left( \int_{\text{Irr}(G)} D^{2it}_\pi \otimes (D^{2it}_\pi)^T \text{d}\mu(\pi)) \right) \mathcal{Q}_L,
\]

which is the third equation of Theorem 5.4. If we use formula \( \mathcal{Q}*_{R} \mathcal{Q}_L = \nu^{-\frac{i}{2}} J_\varphi J_\varphi \), we readily get the second equation. Now we can derive the first pair of equations of Proposition 5.5.
Since for all $t \in \mathbb{R}$ we have $\nabla_\psi^t = \delta^{-it} P^{-it}$ and $J_\varphi \delta^it = \delta^it J_\varphi$, it follows that $\delta^it = J_\varphi \delta^it J_\varphi = (J_\varphi P^{-it} \delta^{-it} J_\varphi)(J_\varphi \delta^it \nabla_\psi^{-it} J_\varphi)$, which we can express using equation (5.7) and Proposition 5.1.

$$Q_L\delta^it Q_L = \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} \otimes (D_{\pi}^{-2it})^T \, d\mu(\pi) \right) \nu^{-\frac{1}{2}it} \left( \int_{\text{Irr}(G)}^\oplus E_{\pi}^{-2it} \otimes (D_{\pi}^{2it})^T \, d\mu(\pi) \right)$$

$$= \nu^{-\frac{1}{2}it} \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} E_{\pi}^{-2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right).$$

On the other hand, we also have $\delta^it = (\nabla_\psi^{-it}\delta^it)(\delta^{-it}P^{-it})$, hence

$$Q_R\delta^it Q_R = \nu^{-\frac{1}{2}it} \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{-2it} \otimes (E_{\pi}^{2it})^T \, d\mu(\pi) \right) \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} \otimes (D_{\pi}^{-2it})^T \, d\mu(\pi) \right),$$

which implies the second equation for $\delta^it$ and ends the proof of Proposition 5.5. In order to finish the proof of Theorem 5.4 we have to derive a lemma concerning the function $f$ introduced in Corollary 5.2.

**Lemma 5.6.** For all $t \in \mathbb{R}$ we have

$$J_\varphi Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^it 1_{\text{HS}(H_\pi)} \, d\mu(\pi) \right) Q_L J_\varphi = Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^{-it} 1_{\text{HS}(H_\pi)} \, d\mu(\pi) \right) Q_L,$$

$$J_\varphi Q_R^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^it 1_{\text{HS}(H_\pi)} \, d\mu(\pi) \right) Q_R J_\varphi = Q_R^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^{-it} 1_{\text{HS}(H_\pi)} \, d\mu(\pi) \right) Q_R.$$

**Proof of Lemma 5.6.** Recall that $J_\varphi \delta^it J_\varphi = \delta^it$, hence

$$\nu^{\frac{1}{2}it} J_\varphi Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} E_{\pi}^{-2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi = \nu^{-\frac{1}{2}it} Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} E_{\pi}^{-2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L.$$

Using the above relation and the fourth equation of Corollary 5.2 we get

$$J_\varphi Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi$$

$$= J_\varphi Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} E_{\pi}^{-2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus E_{\pi}^{2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi$$

$$= \nu^{-it} Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus D_{\pi}^{2it} E_{\pi}^{-2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^it D_{\pi}^{2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L$$

$$= Q_L^\ast \left( \int_{\text{Irr}(G)}^\oplus f(\pi)^it E_{\pi}^{2it} \otimes 1_{\mathcal{H}_\pi} \, d\mu(\pi) \right) Q_L,$$
consequently
\[ Q_L^* \left( \int_{\text{Irr}(G)} E^{2it}_\pi \otimes 1_{\text{H}_\pi} \, d\mu(\pi) \right) Q_L = J_\varphi (J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} E^{2it}_\pi \otimes 1_{\text{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi \]
\[ = J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it D^{2it}_\pi \otimes 1_{\text{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi \]
\[ = Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it E^{2it}_\pi \otimes 1_{\text{H}_\pi} \, d\mu(\pi) \right) Q_L J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it 1_{\text{HS}(\text{H}_\pi)} \, d\mu(\pi) \right) Q_L J_\varphi \]
and
\[ J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it 1_{\text{HS}(\text{H}_\pi)} \, d\mu(\pi) \right) Q_L J_\varphi = Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^{-it} 1_{\text{HS}(\text{H}_\pi)} \, d\mu(\pi) \right) Q_L. \]

The second equation can be proved analogously or using equation \( Q_R Q_L = \nu^{-\frac{1}{2}} J_\varphi J_\varphi. \)

Using the above lemma and Corollary 5.2 we can derive the first equation of Theorem 5.4 out of the third one:

\[ \delta^{it} P^{it} = J_\varphi J_\varphi \delta^{it} P^{it} J_\varphi J_\varphi = J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} D^{-2it}_\pi \otimes (D^{2it}_\pi)^T \, d\mu(\pi) \right) Q_L J_\varphi \]
\[ = J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it 1_{\text{HS}(\text{H}_\pi)} \, d\mu(\pi) \right) Q_L J_\varphi Q_L^* \left( \int_{\text{Irr}(G)} (\pi)^{-it} D^{-2it}_\pi \otimes (D^{2it}_\pi)^T \, d\mu(\pi) \right) Q_L J_\varphi \]
\[ = Q_L^* \left( \int_{\text{Irr}(G)} f(\pi)^it 1_{\text{HS}(\text{H}_\pi)} \, d\mu(\pi) \right) Q_L Q_L^* \left( \int_{\text{Irr}(G)} (\pi)^{-it} E^{2it}_\pi \otimes (E^{-2it}_\pi)^T \, d\mu(\pi) \right) Q_L \]
\[ = Q_L^* \left( \int_{\text{Irr}(G)} E^{2it}_\pi \otimes (E^{-2it}_\pi)^T \, d\mu(\pi) \right) Q_L. \]

Now, the last equation of Theorem 5.4 follows as usual from the formula relating \( Q_L \) and \( Q_R \). This concludes the proof of Theorem 5.4 and Proposition 5.5. \( \square \)

The commutation relation \( \nu^{it} D^{2is}_\pi E^{2it}_\pi = E^{2it}_\pi D^{2is}_\pi (t, s \in \mathbb{R}) \) derived in the previous proposition has the following consequence.

**Corollary 5.7.** If \( \nu \neq 1 \) then for almost all \( \pi \in \text{Irr}(G) \), operators \( D_\pi, E_\pi \) have empty point spectrum. In particular, if \( \nu \neq 1 \) then the set of finite dimensional irreducible representations is of measure zero.

## 6 Special cases

In this section we show how the properties of operators \( (E_\pi)_{\pi \in \text{Irr}(G)}, (D_\pi)_{\pi \in \text{Irr}(G)} \) are related to the modular theory of a type I, second countable locally compact quantum group (i.e. properties of the modular element, scaling group, modular automorphism groups etc.).

First, let us mention three lemmas which are probably well known to experts and which hold for a general locally compact group.
Lemma 6.1. The following conditions are equivalent:

1) $P^it \in L^\infty(\mathbb{G})'$ for all $t \in \mathbb{R}$,

2) the scaling group of $\mathbb{G}$ is trivial,

3) $P^it = 1$ for all $t \in \mathbb{R}$.

Proof. Implications 1) $\iff$ 2) $\iff$ 3) follow from the equation $\tau_i(x) = P^it x P^{-it}$ ($x \in L^\infty(\mathbb{G})$).

For all $x \in \mathfrak{N}_\psi$ and $t \in \mathbb{R}$ we have $P^it \Lambda_\psi(x) = \nu_\psi \Lambda_\psi(\tau_i(x))$, hence 2) implies $P^it = \nu_\psi 1$.

Taking the norm of both sides gives us $1 = \nu_\psi^2$ hence $\nu = 1$. \hfill $\Box$

Lemma 6.2.

1) The Haar integrals on $\mathbb{G}$ are tracial if, and only if $P = \hat{\delta} = 1$.

2) $\hat{\mathbb{G}}$ is unimodular if, and only if $\nabla^it = \nabla^{-it} (t \in \mathbb{R})$.

Proof. We will use formulas gathered in equation (2.1). Equality $\nabla^it = \hat{\delta}^{-it} P^{-it} (t \in \mathbb{R})$ shows that $P = \hat{\delta} = 1$ implies $\nabla^it = 1$ and the traciality of $\psi$. Let us prove the converse implication. If $\nabla^it = 1$ then $P^it = \hat{\delta}^{-it} \in L^\infty(\hat{\mathbb{G}})$ for all $t \in \mathbb{R}$. Since $P^it$ commutes with $J_{\hat{\mathbb{G}}}$, we have $P^it = J_{\hat{\mathbb{G}}} P^it J_{\hat{\mathbb{G}}} \in L^\infty(\hat{\mathbb{G}})'$ and by the previous lemma $P^it = 1 = \hat{\delta}^{-it}$. If $\hat{\mathbb{G}}$ is unimodular, then we have $J_{\hat{\mathbb{G}}} \nabla^it J_{\hat{\mathbb{G}}} = \nabla^{-it} = P^{-it}$ for all $t \in \mathbb{R}$. Since $P^{-it}$ commutes with $J_{\hat{\mathbb{G}}}$, it follows that $\nabla^it = \nabla^{-it}$. On the other hand, if $\nabla^it = \nabla^{-it}$ for all $t \in \mathbb{R}$, then

$$\hat{\delta}^{-it} P^{-it} = \nabla^it = \nabla^{-it} = J_{\hat{\mathbb{G}}} \nabla^it J_{\hat{\mathbb{G}}} = J_{\hat{\mathbb{G}}} \hat{\delta}^{-it} P^{-it} J_{\hat{\mathbb{G}}} = J_{\hat{\mathbb{G}}} \hat{\delta}^{-it} J_{\hat{\mathbb{G}}} P^{-it}$$

and we get $\hat{\delta}^{-it} = J_{\hat{\mathbb{G}}} \hat{\delta}^{-it} J_{\hat{\mathbb{G}}}$. This in particular means that $\hat{\delta}^{-it} \in Z(L^\infty(\hat{\mathbb{G}}))$ and [17] Proposition 1.23] implies $\hat{\delta}^{-it} = J_{\hat{\mathbb{G}}} \hat{\delta}^{-it} J_{\hat{\mathbb{G}}}$, unimodularity of $\hat{\mathbb{G}}$ follows. \hfill $\Box$

Although we will not use this result, let us mention here that if $\mathbb{G}$ is unimodular then $\sigma^\hat{\varphi}_{t}(x) = \hat{\tau}_{t}(x) = \hat{\delta}^{-tx} \delta^{tx}$ and $\Delta_{\hat{\mathbb{G}}} (\sigma^\hat{\varphi}_t(x)) = (\sigma^\hat{\varphi}_t \otimes \sigma^\hat{\varphi}_t) \Delta_{\hat{\mathbb{G}}}(x)$ for all $t \in \mathbb{R}, x \in L^\infty(\hat{\mathbb{G}})$. It is a consequence of the formula $P^{-2it} = \delta^{it}(J_{\hat{\mathbb{G}}} \delta^{it} J_{\hat{\mathbb{G}}} \delta^{it}(J_{\hat{\mathbb{G}}} \delta^{it} J_{\hat{\mathbb{G}}})$ and $\Delta_{\hat{\mathbb{G}}}(\delta^{it}) = \delta^{it} \otimes \delta^{it}$ (see [20] Theorem 5.20, Proposition 5.15).

Lemma 6.3. For all $t, s \in \mathbb{R}$, if $\sigma^\varphi_{t} = \sigma^\varphi_{s}$ then $\nabla^it = \nabla^is$. If $(s, t) \neq (0, 0)$ then also $\nu = 1$.

Proof. For all $x \in \mathfrak{N}_\psi$ we have $\nabla^{-is} \nabla^it \Lambda_\psi(x) = \nu^{\frac{s}{2}} \Lambda_\psi(\sigma^\psi_s(\sigma^\varphi_t(x))) = \nu^{\frac{s}{2}} \Lambda_\psi(x)$ (see [20] Remark 5.2 i)), hence $\nabla^{-is} \nabla^it = \nu^{\frac{s}{2}} 1$. Taking the norm of both sides implies $\nu^{\frac{s}{2}} = 1$ and proves the first claim. If $s \neq 0$ then we get $\nu = 1$, if $s = 0$ and $(s, t) \neq (0, 0)$ then $t \neq 0$ and we get $\nabla^it = 1$. Formula $\nabla^it \Lambda_\psi(y) = \nu^{\frac{s}{2}} \Lambda_\psi(\sigma^\varphi_t(y)) = \nu^{\frac{s}{2}} \Lambda_\psi(y) (y \in \mathfrak{N}_\psi)$ implies $\nu = 1$. \hfill $\Box$

The next theorem is the main result of this section. It presents a web of connections between various properties of a type I, second countable locally compact quantum group (and its dual).
Theorem 6.4. Let $G$ be a second countable, type I locally compact quantum group. Consider the following conditions:

1) $D^t_\pi \in C_{1H_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \text{Irr}(G)$,
2) $E^t_\pi \in C_{1H_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \text{Irr}(G)$,
3) the Haar integrals on $\hat{G}$ are tracial (left $\iff$ right $\iff$ both),
4) the Haar integrals on $G$ are tracial (left $\iff$ right $\iff$ both),
5) $\hat{\delta}^t \in Z(L^\infty(\hat{G}))$ for all $t \in \mathbb{R}$,
6) $G$ is unimodular,
7) $E^t_\pi D^{-t}_\pi \in C_{1H_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi$,
8) $E^t_\pi = D^t_\pi$ for all $t \in \mathbb{R}$ and almost all $\pi \in \text{Irr}(G)$,
9) $\hat{\delta}^t \in Z(L^\infty(G))$ for all $t \in \mathbb{R}$,
10) $E^t_\pi D^{-t}_\pi \in C_{1H_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \text{Irr}(G)$,
11) $\sigma_t^\varphi = \sigma_t^\psi$ for all $t \in \mathbb{R}$.

The following implications hold:

1) $\iff$ 2) $\iff$ 3) $\iff$ 4) $\iff$ 5) $\iff$ 6) $\iff$ 7) $\iff$ 8) $\iff$ 9) $\iff$ 10) $\iff$ 11) $\iff$ 12)

Moreover, each of the above conditions implies $\nu = 1$.

Proof. First, let us note that $\varphi$ is tracial if and only $\psi$ is tracial: it is a consequence of the equation $\nabla^t_\psi = J_\varphi \nabla^{-t}_\varphi J_\varphi (t \in \mathbb{R})$. Equivalence 1) $\iff$ 2) $\iff$ 3) is a part of the Desmedt’s theorem, one can also deduce this from formulas for $\nabla_\varphi$, $\nabla_\psi$ – see Theorem 5.4. Equivalence 7) $\iff$ 5) follows from the formula for $\hat{\delta}^t$ in Proposition 5.5 and $Q^L L^\infty(\hat{G}) Q^*_L = \int_{\text{Irr}(G)} \mathbb{B}(H_\pi) \otimes 1_{H_\pi} d\mu(\pi)$ (see Proposition 3.3). Equivalence 8) $\iff$ 9) is a straightforward consequence of Proposition 5.5.
Theorem 5.4 implies \( \delta \) of \( L(E) \) imply \( \delta \). The remaining implications are trivial. Let us now argue why all of the above conditions imply Corollary 6.6.

Proof. The right implication is an easy corollary of Lemma 6.2. Assume that \( G \) and \( \hat{G} \) are unimodular. Equivalences 8) \( \iff \) 9) and 6) \( \iff \) 10) of Theorem 6.4 imply that \( E_\pi = D_\pi \in C_{1,H_\pi} \) for almost all \( \pi \in \text{Irr}(G) \). Then 2) \( \iff \) 3) of the same theorem implies that the Haar integrals on \( \hat{G} \) are tracial. Equalities \( \nabla_\psi^u = \delta^{-iu} P^{-iu}, \ \nabla_\psi^d = \delta^{-iu} P^{-iu} \) \((t \in \mathbb{R})\) end the proof.

Let us now show how certain classes of quantum groups fit into the above diagram. In particular, these examples show that one-sided implications in the above theorem cannot be reversed.

**Proposition 6.5.** Let \( G \) be a type I, second countable locally compact quantum group.

- If \( G \) is classical and non-unimodular, then it satisfies 4) and does not satisfy 6).
- If \( \hat{G} \) is classical and non-unimodular, then \( G \) satisfies 3) and does not satisfy 9).
- If \( G \) is compact and not of Kac type, then it satisfies 6) and does not satisfy 5).
- If \( G \) is discrete and non-unimodular, then it satisfies 9) and does not satisfy 11).

The numbering in the above proposition corresponds to the numbering introduced in Theorem 6.4. Clearly each of the above classes is non-empty: examples are given by the classical \( ax + b \) group, its dual, the \( SU_q(2) \) group and its dual (see Example 7.1). At the end of this section let us derive a corollary of Theorem 6.4.

**Corollary 6.6.** Let \( G \) be a type I, second countable locally compact quantum group. The Haar integrals on \( G \) and \( \hat{G} \) are tracial if, and only if \( G \) and \( \hat{G} \) are unimodular.

Proof. The right implication is an easy corollary of Lemma 6.2. Assume that \( G \) and \( \hat{G} \) are unimodular. Equivalences 8) \( \iff \) 9) and 6) \( \iff \) 10) of Theorem 6.4 imply that \( E_\pi = D_\pi \in C_{1,H_\pi} \) for almost all \( \pi \in \text{Irr}(G) \). Then 2) \( \iff \) 3) of the same theorem implies that the Haar integrals on \( \hat{G} \) are tracial. Equalities \( \nabla_\psi^u = \delta^{-iu} P^{-iu}, \ \nabla_\psi^d = \delta^{-iu} P^{-iu} \) \((t \in \mathbb{R})\) end the proof.
7 Examples

7.1 Group $\widehat{\text{SU}_q(2)}$

Fix a real number $q \in ]-1,1[ \setminus \{0\}$. Let $G = \text{SU}_q(2)$ be the compact quantum group introduced by Woronowicz in [22] and let $\Gamma = \text{SU}_q(2)$. The C*-algebra of continuous functions on the quantum space $\text{SU}_q(2)$, $C(\text{SU}_q(2))$ is the universal unital C*-algebra generated by elements $\alpha, \gamma$ satisfying the following relations:

$$
\begin{align*}
\alpha^* \alpha + \gamma^* \gamma &= 1, \\
\alpha \gamma &= q \gamma \alpha, \\
\alpha \gamma^* &= q^* \gamma^* \alpha,
\end{align*}
\begin{align*}
\alpha^* + q^2 \gamma^* &= 1, \\
\gamma^* &= \gamma^*.
\end{align*}
$$

The Haar integral of $\text{SU}_q(2)$ is faithful on $C(\text{SU}_q(2))$ and we have $C_u(\text{SU}_q(2)) = C(\text{SU}_q(2))$ ($\text{SU}_q(2)$ is coamenable, see [1, Theorem 2.12]). Furthermore, the C*-algebra $C(\text{SU}_q(2))$ is separable and type I (see [22, Theorem A2.3]) hence $\Gamma$ is an interesting example of a second countable, type I discrete quantum group. We will describe the Plancherel measure for this group and show how various operators related to $\Gamma$ act on the level of direct integrals.

Let us start with describing the measurable space $\text{Irr}(\hat{\Gamma})$ (i.e. the spectrum of $C(\text{SU}_q(2))$).

Proposition 7.1. Measurable space $\text{Irr}(\hat{\Gamma})$ can be identified with the disjoint union of two circles $T \sqcup T = \{\psi_1, \rho \mid \rho \in T\} \cup \{\psi_2, \lambda \mid \lambda \in T\}$. Representations $\psi_1, \rho$ are one dimensional and given by

$$
\begin{align*}
\psi_1, \rho(\alpha) &= \rho, \\
\psi_1, \rho(\alpha^*) &= \bar{\rho},
\end{align*}
\begin{align*}
\psi_2, \lambda(\gamma) &= \lambda q^k \phi_k, \\
\psi_2, \lambda(\gamma^*) &= \overline{\lambda q^k} \phi_k, \quad (\rho \in \mathbb{T}, \lambda \in \mathbb{T})
\end{align*}
$$

Representations $\psi_2, \lambda$ act on a separable Hilbert space $H_\lambda = \ell^2(\mathbb{Z}_+)$ with an orthonormal basis $\{\phi_k \mid k \in \mathbb{Z}_+\}$ via

$$
\begin{align*}
\psi_2, \lambda(\alpha)\phi_k &= \sqrt{1 - q^{2k}} \phi_{k-1}, \\
\psi_2, \lambda(\alpha^*)\phi_k &= \sqrt{1 - q^{2(k+1)}} \phi_{k+1}, \\
\psi_2, \lambda(\gamma)\phi_k &= \lambda q^k \phi_k, \\
\psi_2, \lambda(\gamma^*)\phi_k &= \overline{\lambda q^k} \phi_k, \quad (\rho \in \mathbb{T}, \lambda \in \mathbb{T})
\end{align*}
$$

with the convention $\phi_{-n} = 0$ ($n \in \mathbb{N}$).

In the next proposition we calculate the Plancherel measure of $\Gamma$, the unitary operator $Q_L$ and operators $(D_\pi)_{\pi \in \text{Irr}(\Gamma)}$. In what follows, $\varphi, \psi$ are the Haar integrals on $\Gamma$ and $h$ is the Haar integral on $G = \text{SU}_q(2)$.

Proposition 7.2. The Plancherel measure of $\Gamma$ equals 0 on $\{\psi_1, \rho \mid \rho \in \mathbb{T}\}$ and the normalized Lebesgue measure on the second circle $\{\psi_2, \lambda \mid \lambda \in \mathbb{T}\}$. Consequently, we will identify $\text{Irr}(\Gamma)$ with $\mathbb{T}$. Operators $\{D_\lambda \mid \lambda \in \mathbb{T}\}$ are given by

$$
D_\lambda = (1 - q^2)^{-\frac{1}{2}} \text{diag}(1, |q|^{-1}, |q|^{-2}, \ldots) \quad (\lambda \in \mathbb{T})
$$

\[\text{In this section } \Gamma \text{ is the "main" group and } G \text{ is the "dual" one.}\]
with respect to the basis \( \{ \phi_k \mid k \in \mathbb{Z}_+ \} \). Operator \( Q_L \) is given by

\[
Q_L : L^2(\mathbb{G}) \ni \Lambda_h(a) \mapsto \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(a) D_\lambda^{-1} d\mu(\lambda) = \int_{\text{Irr}(\Gamma)}^\oplus \text{HS}(H_\lambda) d\mu(\lambda) \quad (a \in C(SU_q(2))).
\]

**Proof.** Define \( \mu \) to be the normalized Lebesgue measure on the second circle of \( \text{Irr}(\Gamma) = T \sqcup T \) and let \( Q_L \) be the operator given by the above formula. In order to show that these objects are the one given by Desmedt’s theorem, we will use\(^7\) point 7 of \([7\), Theorem 3.3\]. Let us start with showing that \( Q_L \) is well defined and unitary. First, it is clear that for \( a \in C(SU_q(2)) \) the field of operators \( (\psi^{2,\lambda}(a) D_\lambda^{-1})_{\lambda \in T} \) is measurable and square integrable. Consequently, we can introduce a densely defined linear map \( Q_L : \Lambda_h(a) \mapsto \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(a) D_\lambda^{-1} d\mu(\lambda) \). Since \( \| Q_L \Lambda_h(a) \| \leq \| a \| (a \in C(SU_q(2))) \), the linear map \( Q_L \circ \Lambda_h \) is bounded. Let us now show that \( Q_L \) is isometry, i.e. \( \langle Q_L \Lambda_h(a') | Q_L \Lambda_h(a) \rangle = \langle \Lambda_h(a') | \Lambda_h(a) \rangle \) for all \( a, a' \in C(SU_q(2)) \).

\[
\langle Q_L \Lambda_h(a') | Q_L \Lambda_h(a) \rangle = \left\langle \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(a') D_\lambda^{-1} d\mu(\lambda) | \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(a) D_\lambda^{-1} d\mu(\lambda) \right\rangle
\]

\[
= \left\langle \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(1) D_\lambda^{-1} d\mu(\lambda) | \int_{\text{Irr}(\Gamma)}^\oplus \psi^{2,\lambda}(a' a) D_\lambda^{-1} d\mu(\lambda) \right\rangle = \langle Q_L \Lambda_h(1) | Q_L \Lambda_h(a' a) \rangle
\]

and \( \langle \Lambda_h(a') | \Lambda_h(a) \rangle = \langle \Lambda_h(1) | \Lambda_h(a' a) \rangle \), it is enough to consider the case \( a' = 1 \). Next, as maps \( Q_L \circ \Lambda_h, \Lambda_h \) are bounded and linear, it is enough to consider a \( a \) in a basis of \( \text{Pol}(SU_q(2)) \), \( \{ \alpha^{l'} \gamma^n \gamma^m | l, n, m \in \mathbb{Z}_+, l' \in \mathbb{N} \} \) (see \([22\), Theorem 1.2\]).

In order to calculate \( \langle \Lambda_h(1) | \Lambda_h(a) \rangle \) we need to introduce a faithful representation \( \pi_0 : C(SU_q(2)) \to B(\ell^2(\mathbb{Z}_+ \times \mathbb{Z})) \) defined in \([22\). One can express the Haar integral \( h \) as

\[
h(a) = (1 - q^2) \sum_{k=0}^\infty q^{2k} \langle \phi_{k,0} | \pi_0(a) \phi_{k,0} \rangle \quad (a \in C(SU_q(2))),
\]

where \( \{ \phi_{k,p} \mid (k, p) \in \mathbb{Z}_+ \times \mathbb{Z} \} \) is the standard basis of \( \ell^2(\mathbb{Z}_+ \times \mathbb{Z}) \). Now, for \( l, n, m \in \mathbb{Z}_+ \) we have

\[
\langle \Lambda_h(1) | \Lambda_h(\alpha^{l} \gamma^n \gamma^m) \rangle = h(\alpha^{l} \gamma^n \gamma^m) = \delta_{l,0} (1 - q^2) \sum_{k=0}^\infty q^{2k} \delta_{n,m} q^{(n+m)k} = \delta_{l,0} \delta_{n,m} \frac{1 - q^2}{1 - q^2(n+m)}
\]

\( ^7\)This result is formulated only for type I quantum groups with finite dimensional irreducible representations. However, its proof is based on \([7\), Lemma 3.2\] and proof of this lemma works just as well for more general groups with bounded operators \( D_{\pi}^{-1} \), such as second countable, type I discrete quantum groups.
and similarly $\langle \Lambda_h(1) | \Lambda_h(\alpha^4 \gamma^n \gamma^m) \rangle = \delta_{l,0}\delta_{n,m}\frac{1-q^2}{1-q^{2(l+n)}}$. On the other hand

$$\langle \mathcal{Q}_L \Lambda_h(1) | \mathcal{Q}_L \Lambda_h(\alpha^4 \gamma^n \gamma^m) \rangle = \int_{\text{Irr}^2(G)} D^{-1}_\lambda d\mu(\lambda) \int_{\text{Irr}^2(G)} \psi^{2,\lambda}(\alpha^4 \gamma^n \gamma^m) \tilde{D}_\lambda^{-1} d\mu(\lambda)$$

$$= \delta_{l,0}(1-q^2) \sum_{k=0}^{\infty} \langle \phi_k | \alpha_{n-m} q^{(n+m)k} \phi_k \rangle d\mu(\lambda)$$

$$= \delta_{l,0}\delta_{n,m}(1-q^2) \sum_{k=0}^{\infty} q^{(n+m)k} q^{2k} = \delta_{l,0}\delta_{n,m}\frac{1-q^2}{1-q^{2(l+n)}}.$$ 

In an analogous manner we check $\langle \mathcal{Q}_L \Lambda_h(1) | \mathcal{Q}_L \Lambda_h(\alpha^4 \gamma^n \gamma^m) \rangle = \delta_{l,0}\delta_{n,m}\frac{1-q^2}{1-q^{2(l+n)}}$. This shows that $\mathcal{Q}_L$ is isometry and consequently extends to the whole of $L^2(\mathbb{G})$. Let us now argue that $\mathcal{Q}_L$ is surjective. Fix $\lambda \in \mathbb{T}$, $k, l \in \mathbb{Z}_+$. We have $\psi^{2,\lambda}(\gamma^n) \phi_k = q^{2k} \phi_k$, hence $\psi^{2,\lambda}(\chi(q^n \gamma^n) \gamma^n) \phi_k = \delta_{k,l} \phi_k$ (note that operator $\chi(q^n \gamma^n) \gamma^n$ belongs to $C(SU_q(2))$ because $q^\alpha$ is an isolated point in the spectrum of $\gamma^n \gamma^n$). Next, for $n \in \mathbb{Z}_+$ the following holds

$$\psi^{2,\lambda}(\alpha^n \chi(q^n \gamma^n) \gamma^n) \phi_k = \delta_{k,l} \prod_{a=0}^{n-1} (1-q^{2(k-a)\lambda}) \phi_{k-n} = \delta_{k,l} \prod_{a=0}^{n-1} (1-q^{2(k-a)\lambda}) \phi_{l-n}$$

which (together with a similar reasoning for $\alpha^*$) implies that for all $l, n \in \mathbb{Z}_+$ there exists an operator $E_{n,l} \in C(SU_q(2))$ such that $\psi^{2,\lambda}(E_{n,l}) \phi_k = \delta_{l,k} \phi_n$ ($k \in \mathbb{Z}_+, \lambda \in \mathbb{T}$). Next, for $m \in \mathbb{Z}_+$ we have

$$\psi^{2,\lambda}(q^{lm} E_{n,l} \gamma^m) \phi_k = \delta_{l,k} \lambda^m \phi_n, \quad \psi^{2,\lambda}(q^{lm} E_{n,l} \gamma^m) \phi_k = \delta_{l,k} \lambda^{-m} \phi_n \quad (k \in \mathbb{Z}_+, \lambda \in \mathbb{T})$$

and consequently for any polynomial function $P$ in $\lambda, \lambda^k$ and $n, l \in \mathbb{Z}_+$ an operator $\int_{\text{Irr}^2(G)} P(\lambda) \psi^{2,\lambda}(E_{n,l}) \ d\mu(\lambda)$ belongs to the image of $\mathcal{Q}_L$. By density of such polynomials in $L^2(\mathbb{T})$ it follows that for all $f \in L^2(\mathbb{T})$

$$\int_{\text{Irr}^2(G)} f(\lambda) \psi^{2,\lambda}(E_{n,l}) \ d\mu(\lambda) \in \mathcal{Q}_L(L^2(\mathbb{G})). \quad (7.1)$$

We have an isomorphism (given by choice of bases) $\int_{\text{Irr}^2(G)} \text{HS}(H_\lambda) \ d\mu(\lambda) \simeq L^2(\mathbb{T}) \otimes \text{HS}(\ell^2(\mathbb{Z}_+))$, hence it is clear that operators as in $(7.1)$ span a dense subspace in $\int_{\text{Irr}^2(G)} \text{HS}(H_\lambda) \ d\mu(\lambda)$, and consequently $\mathcal{Q}_L$ is unitary. Let us now check the first commutation relation. We have

$$\mathcal{Q}_L \lambda^\gamma(\omega) \mathcal{Q}_L^* = \mathcal{Q}_L \Lambda_h(\alpha^4 \gamma^n \gamma^m) = \int_{\text{Irr}^2(G)} \psi^{2,\lambda}(\lambda^\gamma(\omega) \alpha) \tilde{D}_\lambda^{-1} d\mu(\lambda)$$

$$= \int_{\text{Irr}^2(G)} \psi^{2,\lambda}(\lambda^\gamma(\omega)) \psi^{2,\lambda}(\alpha) \tilde{D}_\lambda^{-1} d\mu(\lambda) = \left( \int_{\text{Irr}^2(G)} \psi^{2,\lambda}(\lambda^\gamma(\omega)) \otimes 1_{\mathbb{H}_\alpha} \ d\mu(\lambda) \right) \mathcal{Q}_L \Lambda_h(a),$$

for all $\omega \in \ell^1(\Gamma)$, $a \in C(SU_q(2))$ where $\lambda^\gamma(\omega) = (\omega \otimes \text{id}) \omega^\gamma$, hence

$$\mathcal{Q}_L \lambda^\gamma(\omega) \mathcal{Q}_L^* = \int_{\text{Irr}^2(G)} \psi^{2,\lambda}(\lambda^\gamma(\omega)) \otimes 1_{\mathbb{H}_\alpha} \ d\mu(\lambda) \quad (\omega \in \ell^1(\Gamma)). \quad (7.2)$$
In order to show the second commutation relation, let us show that $Q_L$ transpots $J_h$ to the direct integral of adjoints. For $a \in \text{Pol}(\text{SU}_q(2))$ we have

$$Q_L J_h A_h(a) = Q_L A_h(\sigma_{h/2}^\lambda(a^*)) = \int_{\text{Irr}(\Gamma)}^{\oplus} \psi^{2\lambda}(\sigma_{-h/2}^\lambda(a^*)) D_\lambda^{-1} d\mu(\lambda).$$

Next, observe that $\psi^{2\lambda}(\sigma_{h}^\lambda(a)) = D_\lambda^{-2it} \psi^{2\lambda}(a) D_\lambda^{2it}$ for all $\lambda \in \mathbb{T}, t \in \mathbb{R}, a \in \text{Pol}(\text{SU}_q(2))$. Indeed, we have $\sigma_{h}^\lambda(\alpha) = |q|^{-2it\alpha}, \sigma_{h}^\lambda(\gamma) = \gamma (t \in \mathbb{R})$ ([3] Example 1.74) and consequently

$$\psi^{2\lambda}(\sigma_{h}^\lambda(\gamma)) = \psi^{2\lambda}(\gamma) = D_\lambda^{-2it} \psi^{2\lambda}(\gamma) D_\lambda^{2it} \quad (t \in \mathbb{R})$$

and similarly for all $k \in \mathbb{Z}_+, t \in \mathbb{R}$

$$D_\lambda^{-2it} \psi^{2\lambda}(\alpha) D_\lambda^{2it} \phi_k = (1 - q^{2k}) \frac{1}{2} |q|^{-2ikt} |q|^{2(k-1)t} \phi_{k-1} = |q|^{-2it} \psi^{2\lambda}(\alpha) \phi_k = \psi^{2\lambda}(\sigma_{h}^\lambda(\alpha)) \phi_k.$$

It follows that for all $a \in \text{Pol}(\text{SU}_q(2))$

$$Q_L J_h A_h(a) = \int_{\text{Irr}(\Gamma)}^{\oplus} D_\lambda^{-1} \psi^{2\lambda}(a^*) D_\lambda^{-1} d\mu(\lambda) = \int_{\text{Irr}(\Gamma)}^{\oplus} (\psi^{2\lambda}(a) D_\lambda^{-1})^* d\mu(\lambda),$$

hence $Q_L J_h Q_L^*$ equals $\Sigma = \int_{\text{Irr}(\Gamma)} J_{H_h} d\mu(\lambda)$. Now we can show the second commutation relation. Formula $\chi(V^\Gamma) = (J_h \otimes J_h)(\chi'(V^\Gamma))^*(J_h \otimes J_h)$ ([20] Proposition 5.9) implies that for all $\omega \in \ell^1(\Gamma)$ we have $(\omega \otimes \text{id}) \chi(V^\Gamma) = J_h((\omega \circ R^\Gamma \otimes \text{id})W^\Gamma)^* J_h$ and consequently

$$Q_L (\omega \otimes \text{id}) \chi(V^\Gamma) Q_L^* = Q_L J_h Q_L^* \left( \int_{\text{Irr}(\Gamma)}^{\oplus} \psi^{2\lambda}(\lambda^\Gamma(\omega \circ R^\Gamma)) \otimes 1_{H_{\lambda}} d\mu(\lambda) \right) = \int_{\text{Irr}(\Gamma)}^{\oplus} (\psi^{2\lambda}(\lambda^\Gamma(\omega)) d\mu(\lambda),$$

which is the second commutation relation. We are left to show

$$Q_L (L^\infty(G) \cap L^\infty(G')) Q_L^* = \text{Diag}(\int_{\text{Irr}(\Gamma)}^{\oplus} \text{HS}(H_{\lambda}) d\mu(\lambda)),$$

let us first argue that

$$Q_L L^\infty(G) Q_L^* = \int_{\text{Irr}(\Gamma)}^{\oplus} B(H_{\lambda}) \otimes 1_{H_{\lambda}} d\mu(\lambda). \quad (7.3)$$

Inclusion $\subseteq$ follows from the commutation relation (7.2). On the other hand, equation (7.2) and reasoning similar to the one showing that $Q_L$ is unitary, implies that for any polynomial $P$ in $\lambda, \overline{\lambda}$ and $n, l \in \mathbb{Z}_+$ we have

$$\int_{\text{Irr}(\Gamma)}^{\oplus} P(\lambda) \psi^{2\lambda}(E_{n,l}) \otimes 1_{H_{\lambda}} d\mu(\lambda) \in Q_L L^\infty(G) Q_L^*.$$
Proposition 7.3. \[\sigma\text{-wot density of polynomials in } L^\infty(\mathbb{T}) \text{ and isomorphism } \int_{\text{Irr}(\mathbb{T})}^\oplus B(\mathbb{H}_\lambda) \otimes 1_{\mathbb{H}_\lambda} d\mu(\lambda) \simeq L^\infty(\mathbb{T}) \otimes B(\ell^2(\mathbb{Z}_+)) \] gives us (7.3). Consequently

\[
\mathcal{Q}_L(L^\infty(\mathbb{G}) \cap L^\infty(\mathbb{G}')) Q_L^* = (\int_{\text{Irr}(\mathbb{T})}^\oplus B(\mathbb{H}_\lambda) \otimes 1_{\mathbb{H}_\lambda} d\mu(\lambda)) \cap (\int_{\text{Irr}(\mathbb{T})}^\oplus 1_{\mathbb{H}_\lambda} \otimes B(\mathbb{H}_\lambda) d\mu(\lambda))
\]

= \text{Diag}(\int_{\text{Irr}(\mathbb{T})}^\oplus \text{HS}(\mathbb{H}_\lambda) d\mu(\lambda)). \]

In the next proposition we find an action of the operator \(P^it\) on the level of direct integrals.

**Proposition 7.3.** For each \(t \in \mathbb{R}\), operator \(Q_L P^it Q_L^*\) acts on \(\int_{\text{Irr}(\mathbb{T})}^\oplus \text{HS}(\mathbb{H}_\lambda) d\mu(\lambda)\) as follows:

\[
Q_L P^it Q_L^*: \int_{\text{Irr}(\mathbb{T})}^\oplus T_\lambda d\mu(\lambda) \mapsto \int_{\text{Irr}(\mathbb{T})}^\oplus T_{\lambda q^{2it}} d\mu(\lambda).
\]

Note that the above result implies that \(Q_L P^it Q_L^*\) is not decomposable.

**Proof.** Let \(\tilde{P}^it\) be the operator in the claim, i.e. \(\tilde{P}^it: \int_{\text{Irr}(\mathbb{T})}^\oplus T_\lambda d\mu(\lambda) \mapsto \int_{\text{Irr}(\mathbb{T})}^\oplus T_{\lambda q^{2it}} d\mu(\lambda)\). Clearly it is well defined and bounded. The scaling group of \(\mathbb{G} = \text{SU}_q(2)\) acts as follows (\([13\text{ Example 1.7.8}]\))

\[
\tau^G_t(\alpha) = \alpha, \quad \tau^G_t(\alpha^*) = \alpha^*, \quad \tau^G_t(\gamma) = |q|^{2it}\gamma, \quad \tau^G_t(\gamma^*) = |q|^{-2it}\gamma^* \quad (t \in \mathbb{R}).
\]

Recall that \(P^it\) satisfies \(P^it \Lambda_h(a) = \Lambda_h(\tau^G_t(a))\) for all \(t \in \mathbb{R}, a \in C(\mathbb{G})\). Fix \(l, k, n, m \in \mathbb{Z}_+, \lambda \in \mathbb{T}\) and corresponding operator \(\alpha^l \gamma^n \gamma^m\) in the basis of \(\text{Pol}(\mathbb{G})\). We have

\[
\psi^{2\lambda}(\alpha^l \gamma^n \gamma^m)\phi_k = \prod_{a=0}^{l-1} (1 - q^{2(k-a)})^{\frac{k}{2^a}} \lambda^{n-m} q^{k(n+m)} \phi_{k-l}
\]

\[
= |q|^{-2it(n-m)} \prod_{a=0}^{l-1} (1 - q^{2(k-a)})^{\frac{k}{2^a}} (\lambda |q|^{2it})^{n-m} q^{k(n+m)} \phi_{k-l}
\]

\[
= |q|^{-2it(n-m)} \psi^{2\lambda q^{2it}}(\alpha^l \gamma^n \gamma^m)\phi_k,
\]

(recall that we use convention \(\phi_{-p} = 0\) for \(p \in \mathbb{N}\)) and consequently

\[
\mathcal{Q}_L P^it \Lambda_h(\alpha^l \gamma^n \gamma^m) = |q|^{2it(n-m)} \mathcal{Q}_L \Lambda_h(\alpha^l \gamma^n \gamma^m)
\]

\[
= \int_{\text{Irr}(\mathbb{T})}^\oplus \psi^{2\lambda q^{2it}}(\alpha^l \gamma^n \gamma^m) D^{-1}_\lambda d\mu(\lambda) = \tilde{P}^it \mathcal{Q}_L \Lambda_h(\alpha^l \gamma^n \gamma^m).
\]

In a similar manner we check \(\mathcal{Q}_L P^it \Lambda_h(\alpha^s \gamma^n \gamma^m) = \tilde{P}^it \mathcal{Q}_L \Lambda_h(\alpha^s \gamma^n \gamma^m)\). The claim follows because \(\Lambda_h(\text{Pol}(\mathbb{G}))\) is dense in \(L^2(\mathbb{G})\). \(\square\)
Remark 7.5. In propositions 7.3, 7.4 we have expressed operators \( P^t \) (\( t \in \mathbb{R} \)) and \( J_\varphi \) on \( \int_{\text{Irr}(\Gamma)} \text{HS}(H_\lambda) \, d\mu(\lambda) \). Theorem 5.4 and Proposition 5.1 allow us to do the same for \( \delta^t, \nabla_\mu^t, \nabla_\psi^t \) (\( t \in \mathbb{R} \)) — operators obtained in this way are not decomposable.
7.2 Quantum group $az + b$

In this section we will describe some aspects of the theory of the quantum $az + b$ group. We begin by introducing a complex number $q$ and an abelian group $\Gamma_q \subseteq \mathbb{C}^\times$. We will consider three cases:

1) $q = e^{\frac{\pi i}{N}}$ for a natural number $N \in 2\mathbb{N} \setminus \{2\}$ and $\Gamma_q = \{q^k r \mid k \in \mathbb{Z}, r \in \mathbb{R}_{>0}\}$,

2) $q$ is a real number in $|0, 1|$ and $\Gamma_q = \{q^{i\theta + k} \mid \theta \in \mathbb{R}, k \in \mathbb{Z}\}$,

3) $q = e^{\frac{1}{r}}$, where $\text{Re}(\rho) < 0$, $\text{Im}(\rho) = \frac{N}{2\pi}$ and $N \in 2\mathbb{Z} \setminus \{0\}$. In this case $\Gamma_q = \{e^{\frac{k+i\mu}{r}} \mid k \in \mathbb{Z}, t \in \mathbb{R}\}$.

It will be more convenient for us to work in the dual picture: let $\hat{G}$ be the quantum $az + b$ group associated with the parameter $q$. We refer the reader to papers [25, 14, 26] for construction of these groups, here we will recall only necessary properties. We treat all three cases simultaneously. The group $\Gamma_q$ has closure given by $\overline{\Gamma_q} = \Gamma_q \cup \{0\}$ and is selfdual. This duality is implemented by a certain bicharacter $\chi': \overline{\Gamma_q} \times \Gamma_q \rightarrow \mathbb{T}$. We choose a Haar measure on $\Gamma_q$ in such a way that the Fourier transform $\mathcal{F}(f)(\gamma) = \int_{\Gamma_q} \chi(\gamma, \gamma')f(\gamma')d\mu(\gamma')$ is a unitary operator on $L^2(\Gamma_q)$. Next, the group $\Gamma_q$ acts on $C_0(\overline{\Gamma_q})$ by translations: $\sigma_\gamma(f)(\gamma') = f(\gamma\gamma')(f \in C_0(\overline{\Gamma_q}), \gamma \in \Gamma_q, \gamma' \in \overline{\Gamma_q})$. Let $C_0(\overline{\Gamma_q}) \rtimes_{\sigma} \Gamma_q \subseteq B(L^2(\Gamma_q))$ be the associated crossed product $C^*$-algebra (note that since $\Gamma_q$ is abelian, the reduced crossed product is universal). It turns out that the $C^*$-algebra $C_0(\widehat{G})$ is isomorphic to the crossed product $C_0(\overline{\Gamma_q}) \rtimes_{\sigma} \Gamma_q$. Furthermore, it is known that $\widehat{G}$ is amenable, indeed, it was pointed in [14, 15]. It follows from an easy observation that the universal property of $C_0(\Gamma_q) \rtimes_{\sigma} \Gamma_q$ together with the trivial representation of $\Gamma_q$ and the character $C_0(\Gamma_q) \ni f \mapsto f(0) \in \mathbb{C}$ give rise to a character of $C_0(\Gamma_q) \rtimes_{\sigma} \Gamma_q \simeq C_0(\widehat{G})$. Then [21, Theorem 3.1] implies that $\widehat{G}$ is amenable.

One easily checks that the quotient space $\overline{\Gamma_q}/\Gamma_q$ consists of two points and is not antidiscrete. Consequently, [21, Proposition 7.30] implies that $G$ is second countable and type I. Using [21, Theorem 8.39] one can describe the spectrum of $C_0(\widehat{G}) \simeq C_0(\Gamma_q) \rtimes_{\sigma} \Gamma_q$: there is a family of one dimensional representations indexed by $\widehat{\Gamma_q}$ and one faithful irreducible infinite dimensional representation given by the inclusion into $B(L^2(\Gamma_q))$. Denote this representation by $\pi$.

**Proposition 7.6.** *The Plancherel measure of $G$ equals the Dirac measure at $\pi$, a representation corresponding to the inclusion $\pi: C_0(\widehat{G}) \xrightarrow{\sim} C_0(\overline{\Gamma_q}) \rtimes_{\sigma} \Gamma_q \hookrightarrow B(L^2(\Gamma_q))$. Consequently we have $Q_L, Q_R: L^2(G) \rightarrow HS(L^2(\Gamma_q))$.***

**Proof.** It is observed in [26] that we have $\hat{\psi} \circ \tau_t^{\widehat{G}} = |q^{-4it}|\hat{\psi}$ for all $t \in \mathbb{R}$, hence the scaling constant of $G$ equals $\nu = \hat{\psi}^{-1} = |q^{-4}|$. In the first and the third case $q$ is not real

\footnote{In fact, $G$ is isomorphic to the quantum group opposite to quantum $az + b$.}
and it follows that \( \nu \) is nontrivial. Corollary 5.7 implies that the set of one dimensional representations is of measure zero, and the claim follows.\footnote{We remark that it was already observed in [19] that in the first case, \( L^\infty(\hat{G}) \) is isomorphic to the algebra of bounded operators on a separable Hilbert space.} Let us now consider the second case, i.e. \( q \in [0,1[. \) It is argued in [19] Section 5, Proposition A.3 that the von Neumann algebra \( L^\infty(\hat{G}) \) is isomorphic to the von Neumann algebra \( M \) associated with a pair \((a,b)\) of admissible normal operators (see [19] Definition 5.1). Moreover, up to an isomorphism \( M \) does not depend on the choice of \((a,b)\), in particular we can take a pair \((a,b)\) introduced in [19] Proposition 5.2. In this case one easily sees that the resulting von Neumann algebra equals the whole \( B(L^2(\mathbb{Z})) \). In particular it is a factor, hence Proposition 5.4 implies that the Plancherel measure of \( \hat{G} \) must be the Dirac measure at \( \pi \).

Now we turn to the problem of identifying operators \( D_\pi, E_\pi \). To simplify the notation, we will call these operators respectively \( D \) and \( E \). Let us start with introducing two normal (unbounded) operators on \( L^2(\Gamma_q) \): \( a \) and \( b \). Operator \( b \) acts by multiplication: \((bf)(\gamma) = \gamma f(\gamma) (f \in \text{Dom}(b), \gamma \in \Gamma_q) \) and has the obvious domain. The second operator \( a \) can be defined as \( a = \mathcal{F}_b \mathcal{F}^* \).

Note that there exists an isomorphism of von Neumann algebras \( \Phi_R \colon L^\infty(\hat{G}) \to B(L^2(\Gamma_q)) \) induced by \( Q_R J_{\varnothing} J_{\varnothing} \), such that \( \Phi_R(x) = \pi(x) \) for \( x \in C_0(\hat{G}) \) (see Theorem 3.2 and Proposition A.4). Under this isomorphism, the right Haar integral \( \hat{\psi} \) is transformed to \( \text{Tr}(E^{-1} \cdot E^{-1}) \) – it follows from the construction of the Plancherel measure in [5]. On the other hand, we have \( \hat{\psi}(x) = \text{Tr}(|b| \pi(x) |b|) \) for all \( x \in C_0(\hat{G})^+ \) (Theorem 3.1). This means that the weights \( \text{Tr}(E^{-1} \cdot E^{-1}), \text{Tr}(|b| \cdot |b|) \) are equal on \( \Phi_R(C_0(\hat{G})) \). Let \( \theta \) be the restriction of these weights to \( \Phi_R(C_0(\hat{G})) \). The modular automorphism group of \( \text{Tr}(E^{-1} \cdot E^{-1}) \) is given by \( \sigma_t^{\text{Tr}E^{-1}}(A) = E^{-2it} AE^{2it} \), similarly \( \sigma_t^{\text{Tr}|b|}(A) = |b|^{2it} A |b|^{-2it} \) \( \text{for } A \in B(L^2(\Gamma_q)), t \in \mathbb{R} \).

Next, the weight \( \theta \) satisfies the KMS condition for both groups \((\sigma_t^{\text{Tr}E^{-1}}|_{\Phi_R(C_0(\hat{G}))}) \) \( \in \mathbb{R} \) and \((\sigma_t^{\text{Tr}|b|}|_{\Phi_R(C_0(\hat{G}))}) \) \( \in \mathbb{R} \) and as this weight is faithful, [9] Corollary 6.35 implies \( E^{-2it} AE^{2it} = |b|^{2it} A |b|^{-2it} \text{ for all } A \in \Phi_R(C_0(\hat{G})), t \in \mathbb{R} \). By the \( \sigma \)-wot density of \( \Phi_R(C_0(\hat{G})) \) in \( B(L^2(\Gamma_q)) \) we get \( E = c |b|^{-1} \) for some \( c > 0 \). Equality \( \text{Tr}(E^{-1} \cdot E^{-1}) = \text{Tr}(|b| \cdot |b|) \) on \( \Phi_R(C_0(\hat{G})) \) forces \( c = 1 \) and consequently \( E = |b|^{-1} \).

The next step is to identify the operator \( D \). Observe that Lemma 5.6 implies \( f(\pi) = 1. \) Recall ([13] Section 6.2), [23] Equation 3.18) that operator \( a \cdot o \cdot b \) is closable and its closure \( a^{-1}b \) is normal. Moreover, we have \( \hat{R}^\pi(\pi^{-1}(b)) = \pi^{-1}(-qa^{-1}b) \). If we combine this information together with Corollary 5.2 and the equality \( E = |b|^{-1} \) we arrive at

\[
\begin{align*}
\mathcal{Q}_L(D^{2it} \otimes 1_{L^2(\Gamma_q)}) \mathcal{Q}_L &= \hat{R}^\pi(\mathcal{Q}_L(E^{2it} \otimes 1_{L^2(\Gamma_q)}) \mathcal{Q}_L) = \hat{R}^\pi(\mathcal{Q}_L(|b|^{-2it} \otimes 1_{L^2(\Gamma_q)}) \mathcal{Q}_L) \\
&= \mathcal{Q}_L(|-qa^{-1}b|^{-2it} \otimes 1_{L^2(\Gamma_q)}) \mathcal{Q}_L - \mathcal{Q}_L(|qa^{-1}b|^{-2it} \otimes 1_{L^2(\Gamma_q)}) \mathcal{Q}_L,
\end{align*}
\]

which implies \( D = |qa^{-1}b|^{-1} \).

**Proposition 7.7.** We have \( D = |qa^{-1}b|^{-1} \text{ and } E = |b|^{-1} \).

\footnote{We remark that it was already observed in [19] that in the first case, \( L^\infty(\hat{G}) \) is isomorphic to the algebra of bounded operators on a separable Hilbert space.}
8 Appendix

Lemma 8.1. Let $H$ be a Hilbert space and $J : H \otimes \overline{H} \to H \otimes \overline{H}$ an antilinear map given by $J : \xi \otimes \overline{\eta} \mapsto \eta \otimes \overline{\xi} \, (\xi, \eta \in H)$.

1) If $x, y \in B(H)$ are unitaries such that $x \otimes (y^*)^T = y \otimes (x^*)^T \in B(H \otimes \overline{H})$, then operators $x^*, y^*$ are selfadjoint.

2) Let $(a_t)_{t \in \mathbb{R}}, (b_t)_{t \in \mathbb{R}}$ be families of unitary operators on $H$. Define $c_t = a_t \otimes b_t^T \,(t \in \mathbb{R})$. If $(b_t)_{t \in \mathbb{R}}$ and $(c_t)_{t \in \mathbb{R}}$ are strongly continuous groups, then $(a_t)_{t \in \mathbb{R}}$ is also a strongly continuous group.

3) Let $(a_t)_{t \in \mathbb{R}}, (b_t)_{t \in \mathbb{R}}$ be strongly continuous groups of unitary operators on $H$. Define $c_t = a_t \otimes b_t^T \,(t \in \mathbb{R})$. If $Jc_t = c_tJ$ for all $t \in \mathbb{R}$ then $a_t = b_{-t} \,(t \in \mathbb{R})$.

Proof. 1) Equality from the assumption gives us $xSy^* = ySx^*$ for all $S \in HS(H)$. We can approximate the unit by Hilbert-Schmidt operators hence $x^*y = y^*x$, i.e. $x^*$ is selfadjoint. Multiplying this equation from the left by $x^*$ and from the right by $x$ gives us $y^*x = x^*y$, i.e. $y^*$ is selfadjoint.

2) For all $t, s \in \mathbb{R}$ we have $a_{t+s} \otimes b_{t+s}^T = c_{t+s} = c_tc_s = a_ta_s \otimes b_t^Tb_s^T$ hence $a_{t+s} = a_ta_s$, i.e. $(a_t)_{t \in \mathbb{R}}$ is a group. Equation $a_t \otimes 1_H = c_t(1_H \otimes b_{t}^T)$ implies that $\mathbb{R} \ni t \mapsto a_t \in B(H)$ is strongly continuous.

3) We have $Jc_tJ = b_{-t} \otimes a_t^T$ for all $t \in \mathbb{R}$. Consequently, for $s, t \in \mathbb{R}, x \in B(H)$ we have

$$c_sJc_tJ(x \otimes 1_H)Jc_{-t}Jc_{-s} = (a_{s}b_{-t} \otimes b_{s}^Tc_{-t})(x \otimes 1_H)(b_{s}a_{-s} \otimes a_{s}^Tb_{s}^T) = a_{s}b_{-t}xb_{t}a_{-s} \otimes 1_H,$$

and on the other hand

$$Jc_sJc_tJ(x \otimes 1_H)Jc_{-t}Jc_{-s} = (b_{-t}a_{s} \otimes a_{s}^Tb_{s}^T)(x \otimes 1_H)(a_{s}b_{t} \otimes b_{t}^Ta_{s}^T) = b_{-t}xa_{-t}b_{-s}b_t \otimes 1_H,$$

hence

$$a_{s}b_{-t}xb_{t}a_{-s} = b_{-t}xa_{-s}b_tb_t \quad \Rightarrow \quad a_{s}b_{s}b_{-t} = b_{-t}xa_{-s}b_{-t}.$$

The above equation holds for all $x \in B(H)$, hence there exists $\lambda_{t, s} \in \mathbb{C}$ such that $a_{s}b_{t}a_{s}b_{-t} = \lambda_{t, s}1_H$ and consequently $a_{s}b_{t} = \lambda_{t, s}a_{s}b_{t}$ for all $t, s \in \mathbb{R}$. Clearly we have $|\lambda_{t, s}| = 1$. Since $a_t = b_t^T = c_t = Jc_t = Jc_tJ = b_{-t} \otimes (a_{-t})^T$, for all $t \in \mathbb{R}$, the first point implies that $a_tb_t, b_ta_t$ are selfadjoint. For $s = -t$ we have $a_tb_t = \lambda_{t, -t}a_{-t}$, and since $a_tb_t, b_ta_t$ are selfadjoint we have $\lambda_{t, -t} \in \mathbb{R} \cap \mathbb{T} = \{-1, 1\}$. As the function $t \mapsto \lambda_{t, -t} \in \mathbb{R}$ is continuous and $\lambda_{0, 0} = 1$, we have $\lambda_{t, -t} = 1$ for all $t \in \mathbb{R}$. Consequently $b_ta_t = a_tb_t = (a_db_t)^* = b_{-t}a_{-t}$ and $b_{2t} = a_{-2t} \quad (t \in \mathbb{R})$.

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