Abstract. This paper contains the first two parts (I-II) of a three-part series concerning the scalar wave equation $\Box g \psi = 0$ on a fixed Kerr background $(M, g_0, M)$. We here restrict to two cases: (II$_1$) $|a| \ll M$, general $\psi$ or (II$_2$) $|a| < M$, $\psi$ axisymmetric. In either case, we prove a version of ‘integrated local energy decay’, specifically, that the 4-integral of an energy-type density (degenerating in a neighborhood of the Schwarzschild photon sphere and at infinity), integrated over the domain of dependence of a spacelike hypersurface $\Sigma$ connecting the future event horizon with spacelike infinity or a sphere on null infinity, is bounded by a natural (non-degenerate) energy flux of $\psi$ through $\Sigma$. (The case (II$_1$) has in fact been treated previously in our Clay Lecture notes: Lectures on black holes and linear waves, arXiv:0811.0354.) In our forthcoming Part III, the restriction to axisymmetry for the general $|a| < M$ case is removed. The complete proof is surveyed in our companion paper The black hole stability problem for linear scalar perturbations, which includes the essential details of our forthcoming Part III. Together with previous work (see our: A new physical-space approach to decay for the wave equation with applications to black hole spacetimes, in XVIth International Congress on Mathematical Physics, Pavel Exner ed., Prague 2009 pp. 421–433, 2009, http://arxiv.org/abs/0910.4957), this result leads, under suitable assumptions on initial data of $\psi$, to polynomial decay bounds for the energy flux of $\psi$ through the foliation of the black hole exterior defined by the time translates of a spacelike hypersurface $\Sigma$ terminating on null infinity, as well as to pointwise decay estimates, of a definitive form useful for nonlinear applications.

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1. Introduction

The Kerr metrics $g_{M,a}$ constitute a remarkable two-parameter family of explicit solutions to the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

and describe spacetimes corresponding to rotating stationary black holes with mass $M$ and angular momentum $aM$, as measured at infinity.

The family was discovered in local coordinates in 1963 by Kerr [53], and within the subsequent decade, its salient local and global geometric features were definitively understood, especially through work of Carter [16]. It is widely expected that the exterior region of the Kerr family is dynamically stable in the context of the Cauchy problem (see [21]) for (1), in fact, that for a wide range of initial data for (1), not necessarily close to data arising from $g_{M,a}$, the solution metric $g$ in an appropriate region asymptotically settles down to a member of the family. See the discussion in [56]. Indeed, it is this expectation which lies at the basis for the centrality of the Kerr metric in our current astrophysical world-view. At present, however, a mathematical resolution of even the stability question (i.e. the dynamics for spacetimes $g$ initially very near $g_{M,a}$) remains a great open problem in classical general relativity. Many of the important difficulties of this problem are already manifest at the linear level, for the conjectured stability mechanism would rest fundamentally on the dispersive properties of waves (an essentially linear phenomenon) in the black hole exterior region. One must certainly address, thus, the following central mathematical question:

**Do waves indeed disperse on Kerr backgrounds, and how does one properly quantify this?**

1.1. Overview. Motivated by the above, we consider the scalar homogeneous wave equation

$$\Box_g \psi = 0$$

on exactly Kerr spacetimes $(M, g_{M,a})$. The study of (2) can be thought of as a poor-man’s linear theory associated to (1), neglecting, in particular, its tensorial structure. For results pertinent to the latter, see Holzegel [51]. In the present paper, we shall restrict to either of the following cases:

- The parameters of the Kerr metric satisfy $|a| \ll M$.
- The parameters of the Kerr metric are allowed to lie in the entire subextremal range $|a| < M$, but $\psi$ is assumed axisymmetric.

The significance of the restriction $|a| \ll M$ is that the effect of superradiance (to be discussed below) is weak, and this weakness can be exploited as a small parameter. In the axisymmetric case, superradiance is in fact entirely absent. A forthcoming paper [39] will consider non-axisymmetric solutions for the general subextremal case $|a| < M$, thus completing the study of the problem at hand. We provide in our companion paper [38] the essential details of the proof in [39].

(Note: Part I of the present paper (consisting of Sections 1–7) already provides certain preliminary results relevant to the entire series. With the completion of Part III, Part I will be further extended so as to serve as an introduction to the entire series and will be broken off from part...
II (Sections 8–10), which will retain the proofs in the cases (II$_1$), (II$_2$) above.)

1.1.1. **Boundedness.** The most primitive global question concerning solutions to (2) which one must address is that of boundedness. The essential difficulty of this problem lies in the phenomenon of ‘superradiance’. Briefly put, superradiance means that, since the stationary Killing field $T$ becomes spacelike at some points of the exterior region for all Kerr spacetimes with $|a| \neq 0$, the energy radiated to infinity can be greater than the initial energy on a Cauchy hypersurface. In fact, it is not a priori obvious that the energy radiated to infinity (and hence the solution pointwise, etc.) can be controlled at all.

This long-standing open problem of energy and pointwise boundedness has been resolved previously in our [35] for solutions to (2) on a general class of axisymmetric, stationary black hole exterior spacetimes sufficiently close to the Schwarzschild metric, a class which includes as a special case the Kerr family $g_{M,a}$ for $|a| \neq 0$, as well as the Kerr-Newman family $g_{M,a,Q}$ for $|a| \ll M, |Q| \ll M$.

Recall that the Schwarzschild family, discovered already in 1916, is precisely the subfamily of Kerr corresponding to $a = 0$. For the Schwarzschild case, superradiance is absent as the stationary Killing field $T$ is in fact non-spacelike everywhere in the exterior. Thus, the boundedness statement is much easier, though still non-trivial if one desires uniform control up to the horizon where $T$ becomes null; for this, results go back to Wald [85] and Kay–Wald [52]. See the discussion in [36] for many further references.

The fact that under the assumptions of [34], superradiance is weak and can be treated as a small parameter is fundamental for the boundedness proof. It is in fact more natural to discuss briefly the strategy of this proof in the following section, after discussing some of the ideas involved in proving decay, in particular because the main theorem of the present paper will retrieve the boundedness statement in the special case of the Kerr family with $|a| \ll M$.

We note finally that boundedness for axisymmetric solutions of (2) on Kerr exteriors in the entire subextremal range $|a| < M$ also follows from [35]. This latter case shares the feature with Schwarzschild that superradiance is absent, as the energy defined by $T$ is nonnegative definite in the exterior when restricted to axisymmetric $\psi$.

1.1.2. **Integrated local energy decay.** The present paper thus concerns the problem of decay for $\psi$ precisely in the cases for which boundedness was previously known. Our main theorems (Theorem 1.1 and 1.2 below) state that, for the two cases outlined above:

The spacetime integral of an energy-type density integrated over the domain of development of an appropriate surface $\Sigma$ is bounded by an initial energy-type flux through $\Sigma$ (see (4)).

The above spacetime energy density is non-degenerate at the horizon, but degenerates at infinity and in a region to which “trapped null geodesics” (see Section 3.3) approach. Both these degenerations are necessary. The boundedness of a spacetime integral degenerating only at infinity can then be obtained (see (7)) in terms of a higher order initial energy type quantity. Estimates (4) and (7) should be thought of as statements of dispersion which often go by the name “integrated local energy decay”.
Let us briefly put these results in perspective. In Minkowski space, the analogue of estimate (4), with degeneration only at infinity, was first shown in seminal work of Morawetz [69], via use of an identity associated to a virial-type current. In that case, the analogous estimate applied to $\Sigma = \{ t = 0 \}$ would take the form

$$\int_{\mathbb{R}^4} (1+r)^{-1-\delta}|\partial_t \psi|^2 + (1+r)^{-1-\delta}|\partial_r \psi|^2 + r^{-1}|\nabla \psi|^2 \leq C_\delta \int_{t=0} |\partial_t \psi|^2 + |\partial_r \psi|^2 + |
abla \psi|^2.$$  

for any $\delta > 0$, where $r$ is a standard spherical coordinate and $\nabla$ denotes the induced gradient on the constant $(r, t)$ spheres. Such estimates were extended to solutions of the wave equation outside of non-trapping obstacles in [70]. On the other hand, it was shown by Ralston [76] that in the presence of trapping, the integrand of the left-hand side of such an estimate must degenerate near trapped rays. Analogues of (4) in the Schwarzschild case $a = 0$, exhibiting for the first time the expected degeneration associated with the photon sphere of trapped null geodesics (see Section 3.3), were pioneered by [60, 11], with unnecessary additional degeneration on the horizon, however. This degeneration was related to the fact that naively, the null generators of the horizon may seem to be subject to the trapping obstruction of Ralston, if one does not remark that their tangent vectors shrink exponentially with respect to the stationary Killing field. This is essentially the redshift effect. The degeneration at the horizon was then overcome in [32] by using a new vector field multiplier current which quantifies this celebrated effect. As we shall see, this is essential for the stability of the construction. Analogues of (4), (7) for $a = 0$ have by now been obtained, extended and refined by many authors [32, 12, 34, 2, 13, 64]. See [36] for a detailed review.

In passing from Schwarzschild to Kerr, two major difficulties appear immediately. One is the superradiance effect mentioned earlier, the other is the more complicated structure of trapped null geodesics. In view of the previous work described just above in Section 1.1.1, the former difficulty has been completely resolved in the cases under consideration. The latter difficulty was beautifully expounded upon by Alinhac [2], who explicitly showed that currents of the form of [32] can never yield nonnegative definite spacetime estimates on Kerr spacetimes for $|a| \neq 0$, even in the high frequency approximation; this is essentially related to the fact that there are trapped null geodesics for an open range of $r$, whereas the construction of the currents of [32] implicitly rely on the fact that in Schwarzschild, all such geodesics approach the codimension-1 hypersurface $r = 3M$.

It turns out that the “exceptionalism” of Schwarzschild from this point of view is in some sense merely an accident of the projection of geodesic flow to the spacetime manifold. On the tangent bundle, the dynamics of geodesic flow near trapped null geodesics is stable upon passing from $a = 0$ to $|a| \ll M$. This stability is in turn related to the separability and thus complete integrability of geodesic flow for Kerr metrics, first discovered by Carter [17].

The geometric origin of the separability of geodesic flow is subtle. Besides Killing fields $T$ and $\Phi$, the Kerr geometry admits a nontrivial independent Killing tensor [86]. It turns out that using the Ricci flatness, this implies both separability of geodesic flow and separability of the wave equation [18]. The latter provides a convenient way to frequency-localise the virial currents $J^{X,w}$ of [32] and subsequent papers, in a way intimately tied to both the local and global geometry of the solution. This localisation allows us to capture the obstruction to integrated decay provided by trapped null geodesics, overcoming the difficulty described in [2].
The properties of geodesic flow are only suggestive of the high frequency regime of solutions $\psi$ of \((2)\). To retrieve the original spacetime integral estimates proven in the Schwarzschild case with the Morawetz-type current, one must understand the behaviour of such currents on all frequencies, including low frequencies outside of the “trapped” regime. For the case $|a| \ll M$, one can essentially infer positivity by stability considerations from the previous constructions on Schwarzschild. In the general case $|a| < M$, however, the existence of suitable currents providing positivity would depend on global geometric features of the Kerr metric. The geometric frequency-localisation we have employed is particularly suitable for exploiting this. This is all the more important to make contact with our forthcoming part III \([39]\), where low frequencies in the non-trapped but superradiant regime introduce a new important difficulty. See the discussion in our companion paper \([38]\).

It is perhaps useful at this point to recall how boundedness was proven in \([35]\) for the wave equation on suitable axisymmetric, stationary perturbations of Schwarzschild. After applying appropriate cutoffs and taking Fourier expansions associated to the Killing directions $T$ and $\Phi$, one can partition general solutions into their superradiant and non-superradiant part, where the latter part is distinguished by the positivity of its energy flux through the horizon. The crucial observation is that for sufficiently small perturbations of Schwarzschild space, the superradiant part is non-trapped, and a Morawetz-type current could be constructed without degeneration, essentially from the original Schwarzschild construction and stability considerations. Adding, a small part of the ‘red-shift vector field’ component of the current as first introduced in \([32]\), and in addition, the stationary Killing vector field $T$, the boundary terms of this current could also be shown to be positive, without destroying the positivity of the spacetime integral. This observation (together with a new robust argument to show boundedness in the case where the energy flux is positive through the horizon) was sufficient to prove boundedness for the sum of the superradiant and non-superradiant part.

Turning back to the problem at hand, in view of the fact that boundedness has already been proven, we need not worry here about the sign of boundary terms in our construction. Of course, the observation due to \([32]\) that the boundary terms can be made positive by adding a small amount of the red-shift current can be applied directly here. Thus, the proof of Theorem 1.1 in fact reproves the boundedness statement, when specialised to the case of Kerr spacetimes with $|a| \ll M$, and we have indeed set things up so that the boundedness statement (see \((11)\)) is retrieved rather than used. This emphasizes a philosophical point: when superradiance is small, one need not distinguish the superradiant part from the nonsuperradiant part, provided one is indeed proving dispersion for the total solution. This distinction, however, is an important aspect of the general non-axisymmetric case $|a| < M$. In particular, as in the general boundedness theorem \([35]\), it is essential to construct multipliers separately for the superradiant and non-superradiant regimes. See the discussion in Section 5.3. These constructions are already given in detail in Section 11 of our companion paper \([38]\).

1.1.3. Red-shift commutation and higher derivatives. Non-linear applications make it imperative to understand boundedness and decay properties for higher order quantities. In the usual application of the vector field method, these are typically proven via commutation with Killing fields, or, on dynamical geometries, vector fields which become Killing asymptotically in time. It is here essential that the
set of commutators span a timelike direction, so that all derivatives can then be recovered by elliptic estimates.

Already in the Schwarzschild case, one is faced with the difficulty that the span of the Killing fields does not include a timelike direction at points on the event horizon. This difficulty was circumvented by Kay and Wald in their celebrated [52] for the purpose of proving pointwise estimates for $\psi$ itself on the horizon. Unfortunately, the method of [52] was very fragile. In addition, the method was not able, for instance, to uniformly bound transversal (to the horizon) derivatives of $\psi$.

It was first shown in the context of our general boundedness theorem of [35], referred to above, that commutation by a suitable transverse vector field to the horizon, though not Killing, has the property that the most dangerous error terms which arise have a favourable sign. This is yet another manifestation of the redshift, and, as shown in Section 7 of [36], is in fact true for all Killing horizons with positive surface gravity. Using this commutation, one can obtain higher order decay results which do not degenerate at the horizon. We have included these statements in our results. See Theorem 1.3.

1.1.4. Further decay. On the surface, the local integrated decay statement of Theorems 1.1 and 1.2 appears to be a very weak statement of dispersion, seemingly far from the type of statements necessary to obtain non-linear stability results as in [25]. It turns out, however, that it is the essential ingredient in obtaining further decay.

This fact was already partially apparent from work in the Schwarzschild case. In addition to proving a version of Theorem 1.1 for $a = 0$ incorporating for the first time a vector field estimate capturing the red-shift effect, our [32] gave a method for then inferring quantitative decay of energy through a suitable foliation, through use of a suitably constructed conformal Morawetz multiplier. Independently, a related argument was developed in [14]. The construction of the conformal Morawetz multiplier appeared however to still be sensitive to the global geometry all the way up to the event horizon. In [37] and our companion [40], however, it is shown that for a wide class of metrics, (i) a local integrated decay statement as in Theorem 1.1, (ii) a uniform boundedness statement as in [35], and (iii) an additional estimate whose validity depends only on the asymptotic behaviour of the metric at null infinity are together in fact sufficient to prove quantitative decay of energy (through a suitable foliation) and pointwise decay of the form useful for non-linear stability problems. We note that this argument has other potential applications unrelated to the black hole case. In particular, one need not assume the metric to be stationary (or even asymptotically stationary as $t \to \infty$; see the recent nonlinear application of [88]).

It is fundamental, however, that the local energy decay statement be nondegenerate at horizons, if the latter are present. For completeness, we will include in this paper a version of the resulting decay estimates (Theorem 1.4) for the Kerr case. In this context, we note also an independent Fourier based method of obtaining refined pointwise decay estimates [81] from Theorem 1.1 relying heavily however on exact stationarity.

1.1.5. Final remarks. A version of Theorem 1.1 and its proof first appeared as Proposition 5.3.1 of our Clay lecture notes [36], where, in addition, an earlier method for turning this into energy and pointwise decay (Theorem 5.2), closely following [52], was also presented. The original method of proof of Theorem 5.2
has been extended to yield additional decay on Kerr by Luk [63] via commutation
by the scaling vector field, following a method he introduced in the Schwarzschild
case [62]. Furthermore, this additional decay is then used in [63] to obtain a stabil-
ity result for the wave-map equation on Kerr for $|a| \ll M$. Note that this additional
decay is also retrieved in the statement of Theorem 1.4 and its corollaries.

We give here a new proof of Theorem 1.1 in order (i) to give a self-contained
presentation in paper form not depending on the previous delicate constructions
on Schwarzschild (see [32] [12] [34] and subsequent papers), (ii) to unify it with the
proof of Theorem 1.2 which appears here for the first time, and in addition, (iii) to
introduce constructions and notations which will be used in the forthcoming Part
III [39] concerning the case of non-axisymmetric solutions for the general parameter
range $|a| < M$. See also our companion [38] which gives a complete survey, including
the essential details of the constructions of [39].

We mention finally that in parallel with the results of [36] concerning decay, there
are two independent other approaches to the problem for $|a| \ll M$ which have since
appeared, due to Tataru–Tohaneanu [82], and, more recently, Andersson–Blue [3].
The former replaces frequency localisation via Carter’s separation with a technique
based on the standard pseudo-differential calculus, whereas the latter introduces
an innovative alternative technique for frequency localisation via commutation by
higher order operators related to the Killing tensor, but requires a high degree of
regularity and fast decay of initial data near spatial infinity. Another important
related development is due to Tohaneanu [83], who, starting from integrated decay
estimates in [82], shows Strichartz estimates with nice applications to semi-linear
problems.

1.2. Statement of the main theorems.

1.2.1. Notations. The statement of the main theorem will depend on a number of
definitions and notations, which we briefly summarise here. These notions will be
developed formally further in the paper.

The reader familiar with Carter–Penrose diagrammatic representations may wish
to refer to the figure below:

Let $\mathcal{R}$ denote the underlying manifold of Section 2.1 and, for parameters $|a| < M$,
let $g_{M,a}$ denote the Kerr metric as defined in Section 2.2 (Note that $\mathcal{R}$ corresponds
to an exterior region including as its boundary the future event horizon $\mathcal{H}^+$ but not
the past event horizon.) Integrals on general spacelike or timelike submanifolds of
$\mathcal{R}$ will always be taken with respect to the induced volume form from a $g_{M,a}$ with
parameters to be specified at each instance.

Let $T$ and $\Phi$ denote the vector fields of Sections 2.1 2.2 and let $\varphi_\tau$ denote the
1-parameter family of diffeomorphisms generated by $T$. Note that for all $g_{M,a}$, $T$
The definition of admissibility depends on the parameters $M > a_0$, and applies for all metrics $g_{M,a}$ with $|a| \leq a_0$. What is depicted in the diagramme is an admissible hypersurface “of the second kind”.

The following notions will depend on the metric $g_{a,M}$, i.e. on both the parameters $|a| < M$ chosen: Let $r$ denote the coordinate of Section 2.2. Let $Z^*$ denote the unique smooth extension to $\mathcal{R}$ of the coordinate vector field $\partial_r$ defined with respect to Kerr-star coordinates, whereas let $Z$ denote the unique smooth extension to $\text{int}(\mathcal{R})$ of $\partial_r$ defined with respect to Boyer-Lindquist coordinates (see Section 2.4). Note that $Z$ is transverse to $\mathcal{H}^+$, whereas $Z$ would not extend smoothly to $\mathcal{H}^+$. Note also that for $|a| \ll M$, we have $Z^* = Z$ in $r \geq 5M/2$, by our definition of Kerr star coordinates.

1.2.2. The statement for $|a| \ll M$.

**Theorem 1.1.** Fix $M > 0$. There exists a constant $\epsilon = \epsilon(M) > 0$ such that for all $0 \leq a_0 \leq \epsilon$, the following statement holds:

There exist smoothly depending positive values $s^4(a_0,M)$ with $s^4(0,M) = 0$ such that the following holds. Let $\Sigma$ be an arbitrary admissible hypersurface in $\mathcal{R}$ (of the first or second kind) for parameter $M$. Then, for all $\delta > 0$, there exists a constant $B = B(\Sigma,\delta) = B(\varphi_r(\Sigma),\delta)$ such that for all $|a| \leq a_0$, considering the metric $g_{M,a}$ on $\mathcal{R}$, $r_* < 5M/2 < 3M - S^-$, and such that for all sufficiently regular solutions $\psi$ of (2) with $g = g_{M,a}$ on $D^+(\Sigma)$, the following inequalities hold (all metric dependent quantities referring to $g = g_{M,a}$, as described above) for $\psi$

$$
\int_{D^+(\Sigma)} \left( r^{-1}(1 - \eta_{(3M-r^-3M+s^+)}(1 - 3M/r^2)(|\nabla\psi|^2 + r^{-\delta}((\square\psi)^2)) \\
+ r^{-1-\delta}(Z^*\psi)^2 + r^{-3-\delta}(\psi - \psi_\infty)^2 \right) \\
\leq B \int_{\Sigma} \mathbf{J}_\mu^N[\psi]n^\mu_{\Sigma},
$$

(4)

$$
\int_{\nu_r(\Sigma)} \mathbf{J}_\mu^N[\psi]n^\mu_{\nu_r} \leq B \int_{\Sigma} \mathbf{J}_\mu^N[\psi]n^\mu_{\Sigma}, \quad \forall \tau \geq 0,
$$

(5)

$$
\int_{\hat{\mathcal{H}} \cap D^+(\Sigma)} \mathbf{J}_\mu^N[\psi]n^\mu_{\hat{\mathcal{H}}} + (\psi - \psi_\infty)^2 \leq B \int_{\Sigma} \mathbf{J}_\mu^N[\psi]n^\mu_{\Sigma},
$$

(6)

where $4\pi\psi_\infty^2 = \lim_{r \to \infty} \int_{\Sigma \cap \{r=r^\ast\}} r^{-2}|\psi|^2$, and $\eta$ denotes the indicator function.

The term 'sufficiently regular' above just means solutions of $\Box_g\psi$ on $D^+(\Sigma)$ such that on any compact spacelike hypersurface-with-boundary $\hat{\Sigma} \subset D^+(\Sigma)$, then $\psi$ and $n_{\Sigma}\psi$ have well-defined traces on $\hat{\Sigma}$ which are in $H^1_{\text{loc}}(\Sigma), L^2_{\text{loc}}(\Sigma)$ respectively. There exists a unique such solution for appropriately defined initial data along $\Sigma$. 

will correspond to the stationary Killing field, and $\Phi$ will correspond to the Killing field generator of axisymmetry.
See Section [1.5]. Note that under these regularity assumptions, $\psi_\infty$ is well defined and is finite if the right hand side of (4) is finite.

Statement (4) is the analogue for Kerr of Morawetz’s estimate (3) for Minkowski space and the integrated decay estimate of [32] for Schwarzschild. The inclusion of the $(1 - 3M/r)^2$ factor is so as to obtain uniform constants as $a_0 \to 0$, so as to retrieve in particular the Schwarzschild result, for which Theorem 1.1 provides an independent proof.

Let us note that (5) followed from our previous boundedness theorem of [35], when the statement of the latter is restricted to the Kerr family. As discussed above, our theorem gives an independent proof of this statement in the Kerr case.

The theorem of [35] also gave an inequality similar to but weaker than (6), with $J^N[\psi]n^\mu_{\hat{H}^+}$ replaced by $|J^N[\psi]|n^\mu_{\hat{H}^+}$ and without the 0'th order term.

We note immediately:

**Corollary 1.1.** Under the assumptions of the above theorem,

$$\int_{D^+(\Sigma)} r^{-1-\delta} J^N[\psi] n^\mu + r^{-3-\delta}(\psi - \psi_\infty)^2 \leq B \int_\Sigma (J^N[\psi] + J^N[T\psi]) n^\mu_{\Sigma}.$$  

Let us note finally that the proof in fact yields an estimate for solutions to the inhomogeneous equation

$$\Box g \psi = F,$$

which we here omit for reasons of brevity.

1.2.3. The statement for axisymmetric solutions. Our second main theorem of the present paper concerns the entire subextremal range $|a| < M$ but is restricted to axisymmetric solutions.

**Theorem 1.2.** Fix $M > 0$. Then for each $0 \leq a_0 < M$, $\delta > 0$, and each $\Sigma$ admissible for $a_0$, $M$, then for all $|a| \leq a_0$, there exist values $s^\pm(a)$, with $r_+(a, M) < 3M - s^-$, a constant $B = B(a_0, \delta, \Sigma)$, and a cutoff function $\chi_a(r)$, with $\chi_a = 1$ for $r \geq 3M - s^-$ and $\chi_a = 0$ for $r \leq (r_+ + 3M - s^-)/2$, such that defining $\tilde{Z}^* = \chi_a Z + (1 - \chi_a) Z^*$, then the following statement holds.

For all $|a| \leq a_0$, and all sufficiently regular solutions $\psi$ of (2) with $g = g_{M,a}$ on $D^+(\Sigma)$, which moreover satisfy

$$\Phi(\psi) = 0,$$

the inequalities (4)–(7) hold, with $\tilde{Z}^*$ in place of $Z^*$.

The restriction (9) is removed in our companion paper [39].

A posteriori, we note that one could have defined Kerr star coordinates so as for the distinction between $\tilde{Z}^*$ and $Z^*$ to be unnecessary, but this would require making the definition depend on certain constants that only appear in the proof of our theorem.

1.2.4. The higher order statement.

**Theorem 1.3.** Let $M$, $a_0$, $a$, $g_{M,a}$, $s^\pm$, $\psi$ be as in Theorem 1.1 or 1.2.
Then, for all $\delta > 0$ and all integers $j \geq 1$, there exists a constant $B = B(\Sigma, \delta, j) = B(\phi_r(\Sigma), \delta, j)$ such that the following inequalities hold (all metric dependent quantities referring to $g = g_{M,a}$, as described above) for $\psi$

$$\int_{D^+\Sigma} r^{-1-\delta} (1-\eta(3M-s, 3M+s+1)) (1-3M/r)^2 \sum_{i_1+i_2+i_3 \leq j} |\nabla^i T^{i_2}(Z^* i_3) \psi|^2 \leq B \sum_{0 \leq i \leq j-1} \int_{\Sigma} \mathcal{J}_\mu^\Sigma [N^i \psi] n_\Sigma^\mu,$$

(10)

$$\int_{\varphi_r(\Sigma)} \sum_{1 \leq i \leq j} |\nabla^i T^{i_2}(Z^* i_3) \psi|^2 \leq B \sum_{0 \leq i \leq j-1} \int_{\varphi_r(\Sigma)} \mathcal{J}_\mu^\Sigma [N^i \psi] n_\Sigma^\mu,$$

(11)

$$\leq B \sum_{0 \leq i \leq j-1} \int_{\Sigma} \mathcal{J}_\mu^\Sigma [N^i \psi] n_\Sigma^\mu, \quad \forall \tau \geq 0$$

(12)

where again, $Z^*$ should be replaced by $\tilde{Z}^*$ in the case of Theorem 1.2.

For given fixed $j \geq 1$, the regularity assumption on $\psi$ implicit in the above statement is that on any compact spacelike hypersurface-with-boundary $\tilde{\Sigma} \subset D^+\Sigma$, then $\psi$ and $n_\Sigma \psi$ have well-defined traces on $\tilde{\Sigma}$ which are in $H^j_{loc}(\tilde{\Sigma}), H^j_{loc}(\tilde{\Sigma})$ respectively.

Note that the first line of (11) is simply an elliptic estimate whereas the second asserts uniform boundedness of non-degenerate higher order energies. Again, (11) in the case of the assumptions of Theorem 1.1 followed from our previous boundedness theorem of [25], when the statement of the latter is restricted to the Kerr family.

1.2.5. The statement of further decay. In view of Theorem 1.1 and the results of [37, 40], we may obtain further decay estimates, precisely of the form in principle compatible with non-linear applications.

We give here only an example of the type of statement that can be shown. The formulation of the result will require expanding the set of multiplier and commutator vector fields: Let $\Omega_i, i = 1, 2, 3$, denote a set of angular momentum operators for the ambient Schwarzschild metric $g_M$ (see Section 2.8), let $L$ denote the outgoing null vector

$$L = \frac{\rho^2}{\sqrt{\Delta (\rho^2 - 2Mr)}} T + Z$$

and let $\tilde{L} = \chi L$ where $\chi = 0$ for $r \leq 3M$ say, and $\chi = 1$ for $r \geq 5M$. We have

**Theorem 1.4.** Under the assumptions of the above theorems and $\Sigma$ an admissible hypersurface of the second kind, then there exists a constant $B = B(\Sigma, \delta, j) = B(\phi_r(\Sigma), \delta, j)$, and, for each $\delta > 0$ a constant $B_\delta = B(\Sigma, \delta) = B(\phi_r(\Sigma), \delta)$, such that for all sufficiently regular solutions $\psi$ of (2) on $D^+\Sigma$,

$$\int_{\varphi_r(\Sigma)} \mathcal{J}_\mu^\Sigma [\psi] n_\Sigma^\mu \leq B \tau^{-2} \sum_{0 \leq k \leq 2} \int_{\Sigma} \mathcal{J}_\mu^{N + r^2 \tilde{L}} [T^k \psi],$$

$$\int_{\varphi_r(\Sigma)} \mathcal{J}_\mu^\Sigma [N^\mu \psi] n_\Sigma^\mu \leq B \tau^{-4+\delta} \sum_{i_1+i_2 \leq 1} \sum_{0 \leq k \leq 2} \int_{\Sigma} \mathcal{J}_\mu^{N + r^2 \tilde{L}} [N^{i_1 \Omega_{i_2}^2 T^k \psi} + \mathcal{J}_\mu^{N + r^4 \tilde{L}} [T^k \psi]] n_\Sigma^\mu$$
Note that the above statement, together with a re-application of Theorem 1.1—applied now with \( \Sigma \) replaced by \( \phi_\tau(\Sigma) \)—yields decay results (in \( \tau \)) for spacetime integral on the left hand side of (4). This type of statement is in fact used in the course of the proof of Theorem 1.4.

From the above results, applied to \( \psi, T\psi, \) etc., and standard application of the vector field method, elliptic estimates, Sobolev inequalities, together with the additional red-shift commutation of Section 5.4, one can obtain pointwise decay results of the form:

**Corollary 1.2.** Under the assumptions of the above, we have

\[
\sup_{\varphi_\tau(\Sigma)} |\psi - \psi_\infty| \leq B \sqrt{E} \tau^{-1/2},
\]

\[
\sup_{\varphi_\tau(\Sigma) \cap \{r \leq R\}} |\psi - \psi_\infty| \leq B_{R,\delta} \sqrt{E} \tau^{-3/2+\delta},
\]

\[
\sup_{\varphi_\tau(\Sigma) \cap \{r \leq R\}} [J^N_{\mu}[\psi]] N_{\mu} \leq B_{R,\delta} E \tau^{-4+\delta},
\]

where in each inequality, \( E \) denotes an appropriate quadratic integral quantity defined on \( \Sigma \).

All these estimates have extensions to arbitrary derivatives of the solution (of arbitrary order), including derivatives transverse to the horizon.

From the point of view of applications to quasilinear problems, past experience would suggest that the above type of results should be viewed as the definitive statements of decay (cf. the role of decay results in the stability of Minkowski space [25]). Less decay may not be enough to absorb error terms in stability proofs, whereas more refined decay statements (e.g. the familiar Price law tails from the physics literature, see [29]), requiring of course more restrictive assumptions on data, are potentially unstable in the context of the dynamical geometries of interest, and in any case, do not yield any added benefit for non-linear stability proofs.

In fact, it is immediate from the identity of Proposition 5.1.1 extended (simply by continuity!) to a region \( r \geq r_+ - \epsilon(a, M) \), that any boundedness and suitable decay result holding uniformly up to the horizon is propagated for instance to \( r_+ - \epsilon \).

More interestingly, this observation (first due to [28]) is stable in the consequence of dynamical spacetimes where the event horizon and apparent horizon do not in general coincide. The interesting question for black hole interiors is not, however, one of stability, but instability as one approaches the Cauchy horizon. This problem has a long tradition of study: see [74] and [27, 28] in the context of the Einstein-Maxwell-scalar field system under spherical symmetry. (In the Kerr solution, the Cauchy horizon corresponds to a hypersurface \( r = r_- \) in maximally extended Kerr.) Further discussion of the behaviour \( \psi \) in the black hole interior is thus best left to such a context.

### 1.3. Other decay-type statements.

There are many other important statements (mode stability, some partial nonquantitative results on azimuthal modes) which have appeared in the literature, at various levels of rigour. As a representative sample, we mention [57, 75, 48, 43, 44]. See our [36] for a discussion of this literature.
1.4. Related spacetimes and equations. We note finally that there has also been interesting work on the Dirac equation \[42, 47\] on Kerr backgrounds, for which superradiance does not occur, and the Klein-Gordon equation in \[8, 46\], in the case of non-superradiant modes. In the Schwarzschild case, there are a number of papers on related equations. We note especially the work of Blue \[10\] on the Maxwell equations.

All questions considered have analogues in higher dimensions. For the wave equation on higher dimensional Schwarzschild, Schlue \[79\] has recently obtained both integrated energy decay and decay bounds for the energy flux and the solution pointwise. See also \[61\].

When a cosmological constant $\Lambda$ is added to the Einstein equations to give

$$R_{\mu\nu} = \Lambda g_{\mu\nu},$$

there corresponds a related family of solution spacetimes, which in the case $\Lambda > 0$ is known as Kerr-de Sitter, whereas in the case $\Lambda < 0$ is known as Kerr-anti de Sitter. For the $\Lambda > 0$ case, results have been obtained for the Schwarzschild-de Sitter subcase by \[33, 15\] and recently extended in \[65, 66\]. Putting together the methods described here with \[33\] should yield a generalisation to the Kerr-de Sitter case for small $a$, but it would be nice to work this out explicitly. In the case of Kerr-anti de Sitter, we note the recent work of Holzegel \[50\].

2. The Kerr spacetime

2.1. The fixed manifold-with-boundary $\mathcal{R}$. Let $\mathcal{R}$ denote the manifold with boundary

(13) $$\mathcal{R} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^2.$$ 

We define standard coordinates $y^*$ for $\mathbb{R}^+$, $t^*$ for $\mathbb{R}$, and standard spherical coordinates $(\theta^*, \phi^*)$. Strictly speaking, the latter is only a coordinate system on the subset of $\mathbb{S}^2$ corresponding to $(0, \pi) \times (0, 2\pi)$, but we shall extend these functions and the associated coordinate vector fields to all of $\mathbb{S}^2$, in the usual fashion.

The collection $(y^*, t^*, \theta^*, \phi^*)$ define a coordinate system on $\mathcal{R}$, global modulo the degeneration of the spherical coordinates remarked upon above. We will refer to these as fixed coordinates.

Let us introduce the notation $\mathcal{H}^+ = \partial \mathcal{R} = \{y^* = 0\}$. We will refer to $\mathcal{H}^+$ as the event horizon.

Let us denote $T = \partial_{t^*}$, $\Phi = \partial_{\phi^*}$, where, as remarked above, the latter is to be understood as the extension of the coordinate vector field to all of $\mathbb{S}^2$ in the usual way.

Let $\varphi_T$ denote the one-parameter family of diffeomorphisms generated by $T$.

We have defined fixed coordinates precisely so that the differential structure of the ambient spacetimes $\mathcal{R}$, the vector fields which are to be Killing, and the location of the horizon, are all independent of the Kerr parameters to be introduced in what follows.

2.2. Kerr-star coordinates. Let $\mathcal{P} \subset \mathbb{R}^2$ denote the subset $\{(x_1, x_2) : 0 \leq x_1 < x_2\}$. Define a smooth map

$$r : \mathcal{P} \times (0, \infty) \to (x_2 + \sqrt{x_2^2 - x_1^2}, \infty)$$

where $x_2$ is the first coordinate on $\mathcal{P}$.
such that $r|_{(x_1,x_2) \times (0,\infty)}$ is a diffeomorphism $(0,\infty) \to (x_2 + \sqrt{x_2^2 - x_1^2}, \infty)$ which moreover restricts to the identity map restricted to $\{(x_1,x_2)\} \times (3x_2,\infty)$.

Now, for each fixed parameters $0 \leq a < M$, the collection $(r(a,M,y^*), t^*, \theta^*, \phi^*)$ determines a coordinate system on $\mathcal{R}$, global modulo the well-known degeneration of the spherical coordinates on $\mathbb{S}^2$. These will be known as *Kerr-star coordinates*. In what follows, we shall denote $r(a,M,y^*)$ simply as $r$. As opposed to our fixed coordinates on $\mathcal{R}$, this latter coordinate depends on the parameters, although in the region $y^* \geq 3M$, then $r = y^*$ and is thus independent of parameter $a$. The coordinate vector fields $\partial_{t^*}$ and $\partial_{\phi^*}$ of Kerr-star coordinates always correspond to the coordinate vector fields of the original fixed coordinate system, i.e. for all $a,M$ we have $\partial_{t^*} = T$, $\partial_{\phi^*} = \Phi$.

Let us define $r_\pm = M \pm \sqrt{M^2 - a^2}$;

after suitably choosing a value for $r^* = 0$. For definiteness, let us say $r^*(3M) = 0$. Note that $\Delta$ vanishes to first order on $\mathcal{H}^+$. The coordinate range $r > r_+$ corresponds to the range $r^* > -\infty$.

2.3. The coordinate $r^*$. Given parameters $|a| < M$, we define a rescaled version of the $r$ coordinate on $r > r_+$ by

$$
\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta},
$$

where

$$
\Delta = r^2 - 2Mr + a^2,
$$

and $\Delta$ is chosen so as to satisfy everywhere

$$
\frac{d(r^*-\bar{t})}{dr} > 0, \quad \frac{d(r^*-\bar{t})}{dr} > 0.
$$

2.4. Boyer-Lindquist coordinates. Let $|a| < M$ be fixed parameters and $(t^*,r,\theta^*,\phi^*)$ be an associated system of Kerr star coordinates on $\mathcal{R}$. In $\text{int}(\mathcal{R})$, i.e. for $r > r_+$, we define

$$
t(t^*,r) = t^* - \bar{t}(r)
$$

$$
\phi(\phi^*,r) = \phi^* - \bar{\phi}(r) \mod 2\pi
$$

$$
\theta = \theta^*
$$

where

$$
\bar{t}(r) = r^*(r) - r - r^*(9M/4) + 9M/4, \quad \text{for} \quad r_+ \leq r \leq 15M/8,
$$

and $\bar{t}$ is chosen so as to satisfy everywhere

$$
\frac{d(r^*-\bar{t})}{dr} > 0, \quad \frac{d(r^*-\bar{t})}{dr} > 0.
$$

One easily constructs such a $\bar{t}$ by smoothing the function defined by (16) for $r \leq 9M/4$ and (17) for $r \geq 9M/4$.

The coordinates $(t,r,\theta,\phi)$ map $\text{int}(\mathcal{R})$ to $(-\infty,\infty) \times (r_+,\infty) \times [0,\pi] \times [0,2\pi)$ and are global modulo the degeneration of the spherical coordinates; we shall call these...
Boyer-Lindquist local coordinates. Note that all coordinates except $\theta$ depend in fact on the parameters $a, M$.

As before, the Killing fields $T$ and $\Phi$ correspond to the coordinate vector fields $\partial_t$ and $\partial_\phi$. We apply here the standard abuse of notation in considering these coordinates at $\theta = 0, \pi$, and at $\phi = 0$, where they are not regular.

Let us define $Z$ to be (the extension to $\text{int}(\mathcal{R})$ of) the Boyer-Lindquist coordinate vector field $\partial_r$. This vector field is significant as it will define the directional derivative that does not degenerate in the integrated decay estimate due to trapping.

2.5. The Kerr metric. Given parameters $|a| < M$, with the help of the associated (parameter-dependent) Boyer–Lindquist coordinates, we may now define the Kerr metric $g_{M,a}$. Let us first define the function

$$\rho^2 = r^2 + a^2 \cos^2 \theta.$$ 

This notation, along with the notation $\Delta$ from (15), are traditional. The Kerr metric $g_{M,a}$ is then defined as the unique smooth extension to $\mathbb{R}$ of the tensor given on the Boyer–Lindquist chart by

$$g_{M,a} = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (a dt - (r^2 + a^2) d\phi)^2.$$ 

In Boyer-Lindquist coordinates, one immediately recognizes the standard Schwarzschild form

$$-(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

of the metric when $a = 0$. The significance of Boyer-Lindquist coordinates is that it is with respect to these that the covariant wave operator $\Box_g$ separates.

That the expression (19) indeed extends to a smooth metric on $\mathcal{R}$ is clear from examining its form in Kerr-star coordinates, in the region $r \leq 15M/8$ where $\tilde{t}$ is given by exactly (16). There we compute

$$g_{M,a} = -\left(1 - \frac{2Mr}{\rho^2}\right)(dt^*)^2 + \frac{4Mr}{\rho^2} dt^* dr + \left(1 + \frac{2Mr}{\rho^2}\right) dr^2$$

$$- \frac{4aMr \sin^2 \theta}{\rho^2} dt^* d\phi^* + \rho^2 (d\theta^*)^2$$

$$+ \frac{[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta]}{\rho^2} \sin^2 \theta (d\phi^*)^2 - \frac{2a(2Mr + \rho^2 \sin^2 \theta)}{\rho^2} dr d\phi^*.$$ 

We note from the explicit form of the metric $g_{M,a}$ that the vector fields $T$ and $\Phi$ are manifestly Killing. Moreover, we easily see from the form of the metric that when $a \neq 0$, $T$ is spacelike on the horizon $\mathcal{H}^*$, except where $\theta = 0, \pi$, i.e. on the so-called axis of symmetry. We will have more to say about this in Section 3.2.

Let us note that we have arranged the definition of Kerr-star coordinates so that the hypersurfaces $t^* = c$ are spacelike (see the conditions (18)). In the region $r \leq 15M/8$, we have in fact

$$g(\nabla t^*, \nabla t^*) = -1 - \frac{2Mr}{\rho^2}.$$
A final remark: It would be possible to define the Kerr family on the fixed $\mathcal{R}$ so that one has smooth dependence up to and including the extremal case $a = M$, but one would have to set things up slightly differently.

2.6. The Carter-Penrose diagramme. For the reader familiar with Carter-Penrose representations, then the region $\mathcal{R}$ corresponds to

![Carter-Penrose diagram](image)

Given appropriate definitions of asymptotic structure, then

$$\mathcal{R} = \text{clos}(J^- (\mathcal{I}^+)) \cap J^+ (\mathcal{I}^-),$$

interpreted in the topology of maximally extended Kerr (see [49]), where $\mathcal{I}^+, \mathcal{I}^-$ are connected components of future and past null infinity corresponding to the same end. It is in this sense that $\mathcal{R}$ corresponds to a domain of outer communications (including future event horizon) of a spacetime containing both a black hole and a white hole region. We shall not attempt here a formal development of these notions.

Alternatively, as is common in the formulation of black hole uniqueness theory (see [1]), one can characterize $\mathcal{R}$ as follows: Let $\mathcal{U}$ denote a connected component of the subset of points $x$ such that $T(x)$ is timelike, future pointing. Then

$$\mathcal{R} = \text{clos}(J^- (\mathcal{U})) \cap J^+ (\mathcal{U}).$$

See

![Alternative characterization](image)

Here $\mathcal{U}$ is the lightest shaded region and $\mathcal{R}$ is the union of the two lighter shaded regions.

2.7. The event horizon $\mathcal{H}^+$ as a Killing horizon. For each $g_{M,a}$ with $|a| < M$, the Killing vector field

$$K = T + \frac{a}{r^2 + a^2} \Phi$$

is null and normal to $\mathcal{H}^+$. Thus $\mathcal{H}^+$ is a Killing horizon. The vector field $K$ is sometimes known as the Hawking vector field. We note the identity

$$\nabla_K K = \kappa K,$$

where $\kappa$ is...
where
\begin{equation}
\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} > 0.
\end{equation}

The quantity \( \kappa \) is known as the surface gravity. We note that \( \kappa \) vanishes in the extremal case \( |a| = M \).

We remark finally that the vector \( K \) restricted to the horizon coincides with the smooth extension of the coordinate vector field \( \partial_{r^*} \) of the \((r^*, t, \theta, \phi)\) coordinate system.

2.8. **Asymptotics and angular momentum operators.** Besides the smooth dependence of \( g_{M,a} \) on \( a \), it will be useful to remark that for fixed \( a \) and large \( r \) values, \( g_{M,a} \) is very close to \( g_M \).

We may define a standard basis \( \Omega_1, \Omega_2, \Omega_3 \) of angular momentum operators corresponding to \( g_M \), such that moreover, \( \Omega_1 = \Phi \) say. These are just a standard set of generators for the Lie algebra \( \text{so}(3) \) corresponding to the spherical symmetry of \( g_M \).

We note that \( \Omega_i \) span the tangent space to the \( S^2 \) factors of the differential-topological product \( (13) \).

For future reference, let us introduce here the notation \( \mathcal{g}, \nabla \) to denote the induced metric and covariant derivative from \( g_{a,M} \) on the \( S^2 \) factors of \( \mathcal{R} \) in the differential-topological product \( (13) \).

2.9. **The volume form.** We shall often go back and forth between divergence identities which arise geometrically and those which are computed explicitly in various coordinates. For this the following remarks may be useful.

We note that the volume form in Boyer-Lindquist coordinates satisfies
\begin{equation}
DV = v(r, \theta) \, dt \, dr \, dV_{\mathcal{g}} \quad \text{with} \quad v \sim 1
\end{equation}
using the alternative \( r^* \) coordinate,
\begin{equation}
DV = v(r^*, \theta) \, dt^* \, dr^* \, dV_{\mathcal{g}} \quad \text{with} \quad v \sim \Delta/r^2
\end{equation}
and in Kerr-star coordinates
\begin{equation}
DV = v(r, \theta^*) \, dt^* \, dr \, dV_{\mathcal{g}} \quad \text{with} \quad v \sim 1.
\end{equation}

Let \( \gamma \) denote the standard unit metric on the sphere in \((\theta, \phi)\) coordinates. We have that \( \mathcal{g} \sim r^2 \gamma \), and thus we may replace \( dV_{\mathcal{g}} \) in the above using
\begin{equation}
DV_{\mathcal{g}} = v(r, \theta) \, r^2 \sin \theta \, d\theta \, d\phi \quad \text{with} \quad v \sim 1.
\end{equation}

3. **The geometry of Kerr**

The geometry of the Kerr solution has been treated at length in the literature, see for instance [72]. We will discuss here only those features that will be particularly relevant to the considerations of this paper.
3.1. **Surface gravity and the redshift.** An important stabilising mechanism for the behaviour of waves near black hole event horizons is what we shall here call the “horizon-localised” red-shift effect. Recall that this is the red-shift relating two observers \( A \) and \( B \), where \( B = \varphi_\tau(A) \) for \( \tau > 0 \), both crossing the event horizon:

![Diagram of black hole event horizon and observers A and B](image)

At horizon crossing time, the frequency of waves received by \( B \) (measured with respect to proper time) from \( A \) are damped (in comparison to the frequency measured by \( A \)) by a factor exponentially decaying in \( \tau \).

The above effect depends in fact only on the positivity of the **surface gravity**. Recall that we have already defined this notion in the case of a Kerr geometry in Section 2.7. More generally, if \( H^+ \) is a Killing horizon and \( K \) is the distinguished Killing null generator of such a horizon, then the surface gravity is defined to be the function \( \kappa \) such that

\[
\nabla_K K = \kappa K.
\]

(24)

Under various conditions on the ambient spacetime, the function \( \kappa \) can be shown to be constant.

We leave it to the reader to directly relate the positivity of \( \kappa \) (see (24)) with frequency measurements of observers \( A \) and \( B \). More relevant for the present considerations, as shown in [36], the condition \( \kappa > 0 \) translates into a positivity property near \( H^+ \) in the energy identity associated to a suitably constructed vector field multiplier \( N \). In view of (23), this construction is thus applicable in the Kerr case. See Section 5. It is in this manifestation that the red-shift effect plays a role.

We have already remarked that the surface gravity \( \kappa \) vanishes precisely in the extremal case \( |a| = M \). It should be clear from this discussion that an additional difficulty that would arise in the extremal case \( |a| = M \) is the failure of the analogous identity for \( N \). Similar phenomena occur in the extremal case of the spherically symmetric Reissner-Nordström family [49], i.e. the case of parameters \( Q = M \). In very recent work, Aretakis [4] has constructed an analogue of the vector field \( N \) for the extremal Reissner-Nordström case, which yields a spacetime integral estimate in which transversal null derivatives of \( \psi \) degenerate at the horizon. From this, the sharp boundedness and decay results for waves on such a background are obtained. In contrast to the non-extremal case, it is shown moreover in [4] that decay for translation-invariant transversal derivatives to the horizon cannot hold for general initial data, and in fact, in general, higher transversal derivatives blow up as advanced time progresses along the horizon.

3.2. **The ergoregion and superradiance.** In general, the subset \( E \subset \mathcal{R} \) where \( T \) is spacelike is known as the **ergoregion**. The boundary \( \partial E \) of \( E \) (in the topology of \( \mathcal{R} \)) is called the **ergosphere**.

---

1We take typically a generator of the form \( T + c\Phi \), where \( T \) is the stationary Killing field.
Under these conventions, we see easily that in the Schwarzschild case \( a = 0 \), \( \mathcal{E} = \emptyset \). If \( a \neq 0 \), then \( \mathcal{E} \neq \emptyset \), and \( \partial\mathcal{E}\cap\mathcal{H}^+ = \{\theta^* = 0, \pi\} \cap \mathcal{H}^+ \) under our standard abuse of notation.

For all \( 0 < |a| < M \), we have

\[
\sup_{\mathcal{E}} r = 2M,
\]

and thus

\[
\lim_{a \to 0} \sup_{\mathcal{E}} r - r_+ = \lim_{a \to 0} (2M - r_+) = 0.
\]

It follows that, for small \( |a| \ll M \), the ergoregion lies very close to the horizon, in particular, it is contained in the region where the red-shift mechanism operates, in the sense of Section 5. More on this below.

The ergosphere allows for a particle “process”, originally discovered by Penrose [73], for extracting energy out of a black hole. This came to be known as the Penrose process. In his thesis, Christodoulou [22] discovered the existence of a quantity—the so-called irreducible mass of the black hole—which he showed to be always nondecreasing in a Penrose process. The analogy between this quantity and entropy developed into a collection of ideas known as “black hole thermodynamics” [9]. This remains the subject of intense investigation from the point of view of high energy physics.

In the context of solutions \( \psi \) of the wave equation, if \( p \in \mathcal{E} \), then it is no longer the case that the energy density \( J^T_{\mu}[\psi]\xi^{\mu}(p) \) (see the notation of Section 4.2) is necessarily non-negative for \( \xi^{\mu} \) a future-directed timelike vector at \( p \). Thus, the conservation law for \( J^T_{\mu}[\psi] \) no longer yields a priori control of a non-negative definite quantity in the derivatives of \( \psi \). The flux of energy to null infinity can thus be larger than the initial energy. This is the phenomenon of superradiance, first discussed by Zeldovich [89]. Moreover, a priori, this flux can be infinite. (Bounds for the strength of superradiance were first derived heuristically by Starobinsky [80] in various asymptotic regimes.) As discussed in the introduction, it is for this reason that the problem of obtaining any sort of boundedness property, even away from the horizon, is so difficult in Kerr. This was the difficulty which was first overcome in [35].

As in [35], the present paper exploits the fact that the “strength” of superradiance can be treated as a small parameter. Essentially this means that given the vector field \( N \) referred to in the section above, and given an arbitrary \( \epsilon > 0 \), then for sufficiently small \( a_0 > 0 \), the vector field \( T + \epsilon N \) is timelike with respect to all \( g_{M,a} \) with \( |a| \leq a_0 \). This is of course related to (25). See Section 5.3

Let us note that in the case of arbitrary \( a_0 < M, |a| \leq a_0 \), superradiance is absent if \( \psi \) is assumed axisymmetric, i.e. \( \Phi(\psi) = 0 \). This is because, although \( T \) fails to be everywhere causal, the flux \( J^T_{\mu}[\psi]\xi^{\mu}_{H^+} \) through the horizon is nonnegative. This in turn is related to the fact that the null generator \( n_{H^+} \) of the horizon is of the form \( T + c(a, M)\Phi \).

In the general case of arbitrary \( a_0 < M, |a| \leq a_0 \), without axisymmetry, the effect of superradiance cannot be treated as a small parameter and is now a priori strongly coupled to the other difficulties. In particular, there are trapped null geodesics (see Section 3.3) that remain in the ergoregion for all positive affine time. Nonetheless, it turns out that the situation is much better than it initially appears and the difficulty of superradiance can again be resolved, combining the ideas of [22] with those of the present paper. See further comments in Section 5.3 and our companion
paper \[38\], where the necessary constructions of our forthcoming Part III \[39\] are given in detail.

3.3. **Separability of geodesic flow and trapped null geodesics.** The high frequency behaviour of solutions to wave equations is intimately related to the properties of geodesic flow.

In the Schwarzschild case $a = 0$, geodesic flow is easily understood because the Hamilton-Jacobi equations separate. This in turn follows immediately from the dimensionality of the span of the Killing fields $T, \Omega_1, \Omega_2, \Omega_3$. Geodesic flow on the Schwarzschild metric is described in detail in many textbooks. One sees from the resulting equations that there are null geodesics which for all affine time remain on the hypersurface $r = 3M$, the so-called *photon sphere*. With reference to suitable asymptotic notions, we can make the following more general statement: If $\gamma(s)$ is a future-directed inextendible null geodesic in Schwarzschild $(\mathcal{R}, g_{M,0})$ with $\gamma(s) \in \mathcal{R} \setminus \mathcal{H}^+$ for all $s > s_0$, such that moreover for all $p \in J^+(\mathcal{I}^+_\mathcal{H})$, $\exists$ $s$ such that $\gamma(s) \notin J^-(p)$, then $\lim_{s \to \infty} r(\gamma(s)) = 3M$.

As already mentioned in the introduction, in view of general results due to Ralston \[76\], the above property restricts the type of decay statement that one can hope to prove. The constructions of the energy currents of \[14, 32, 34, 12\] and subsequent papers, used to prove integrated decay, are thus intimately related to trapped null geodesics, in particular, the vector field multiplier $X$ related to the currents $J_{X,w}$ applied there, vanishes precisely at $r = 3M$.

Turning now to the general Kerr case, remarkably, as discovered by Carter \[17\], geodesic flow admits, besides the conserved quantities associated to the Killing fields $T$ and $\Phi$, a third non-trivial conserved quantity (the ‘Carter constant’). Thus, geodesic flow remains separable and can be completely understood. The dynamics is described in some detail in \[19\].

In contrast to the Schwarzschild case, there are now null geodesics with constant $r$ for an open range of Boyer-Lindquist-$r$ values. However, restricting to geodesics sharing a fixed triple of the nontrivial conserved quantities, there is at most one $r$-value, depending on the triple, to which all null geodesics neither crossing $\mathcal{H}^+$, nor approaching $J^+(\mathcal{I}^+)$, must necessarily asymptote to towards the future.

The above suggests that the underlying dynamics of geodesic flow is similar (for the entire range $|a| < M$) to the Schwarzschild case, but to capture this in the high frequency regime, the proper generalisation of the currents $J_X$ must be frequency-localised so as to vanish at an $r$-value depending on quantised versions of the conserved quantities corresponding to trapped null geodesics.

There is a convenient way of doing phase space analysis in Kerr spacetimes which is strongly linked to their geometry and the dynamics of geodesic flow. Namely, as discovered again by Carter \[18\], the wave equation itself can be formally separated, and the separation introduces frequencies $\omega, m, \ell$ which can be indeed thought of as quantised versions of the conserved quantities associated to geodesic flow.

None of these separability properties is in fact accidental! Walker and Penrose \[86\] showed shortly thereafter that both the complete integrability of geodesic flow and the formal separability of the wave equation have their fundamental origin in the presence of a *Killing tensor*. It turns out that, in view of Ricci flatness,
all three properties, i.e. separability of wave equation, separability of Hamilton-Jacobi, and existence of a Killing tensor, are in fact equivalent. See \cite{20, 59} for recent higher-dimensional generalisations of this statement.

Carrying out the separation involves various technical issues as one must take the Fourier transform in time, something which a priori is not possible. On the other hand, it allows for a clean way to deal not only with frequencies in the trapped regime (see Section 9.5), but with low frequencies (see Section 9.3). The flexibility provided by this geometric frequency localisation is particularly important in our forthcoming Part III \cite{39} where stability arguments from delicate Schwarzschild constructions cannot be carried over. (The reader can already refer to the discussion of Section 7 of our companion paper \cite{38} and the detailed constructions of Section 11 of \cite{38} for the general $|a| < M$ case.) We have thus taken this opportunity to formulate our constructions here so as to yield an independent proof of the Schwarzschild case, which, though having the disadvantage of being dependent on frequency localisation, has the advantage that it does not require “fine-tuning” parameters in the sense of all previous constructions \cite{32, 34, 12, 64}.

4. Preliminaries

4.1. Generic constants and fixed parameters. We shall use $B$ to denote potentially large positive constants and $b$ potentially small positive constants, depending only on $M$, and, in the case of Theorem 1.2, also on $a_0$, potentially blowing up in the extremal limit $a_0 \to M$. Note the algebra of constants: $B + B = B$, $BB = B$, $Bb = B$, $B^{-1} = b$, etc.

Constants which additionally depend on other objects will be expressed for example as follows: $B(\Sigma)$.

Our constructions will depend on various parameters\footnote{For the convenience of the reader, all constants and parameters are referred to in the index, with reference to the page on which they are originally defined.} whose choice is often deferred to late in the paper, for instance the frequency parameters $\omega_1$, $\omega_3$, etc., introduced in Section 9.2, or the parameter $q$ of Section 9.3.1. Until a parameter is selected, e.g. the parameter $\omega_1$, we shall use the notation $B(\omega_1)$, etc., to denote constants depending on $\omega_1$ in addition to $M$ and $a_0$. The final choices of parameters can be made to depend only on $M$ in the case of Theorem 1.1 and, on $M$, $a_0$ in the case of Theorem 1.2. Once such choices are made, $B(\omega_1)$ is replaced by $B$, following our conventions.

We note finally that in the case of Theorem 1.1, the parameter $a_0$ will be constrained to be small at various points in the paper, and the final smallness assumption can be taken to the minimum of these constraints. All these constraints depend finally only on $M$, in accordance with the statement of the Theorem.

4.2. Vector field multipliers and commutators. We recall the notation of \cite{35}. Given a metric $g$, let $\Psi$ be sufficiently regular and satisfy

$$\Box_g \Psi = F.$$ 

We define

$$T_{\mu\nu}[\Psi] = \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi.$$ 

Given a vector field $V_\mu$ and a function $w$ on $\mathcal{R}$, we will define the currents

$$J^V_\mu[\Psi] = T_{\mu\nu}[\Psi] V^\nu.$$ 

\footnote{For the convenience of the reader, all constants and parameters are referred to in the index, with reference to the page on which they are originally defined.}
\[
J^V_w[\Psi] = J^V_\mu[\Psi] + \frac{1}{8} w \partial_\mu(\Psi^2) - \frac{1}{8}(\partial_\mu w)\Psi^2
\]
\[
K^V_\mu[\Psi] = T_{\mu\nu}[\Psi]\nabla^\mu V^\nu
\]
\[
K^V_\mu[\Psi] = K^V[\Psi] - \frac{1}{8} \Box_g w(\Psi^2) + \frac{1}{4} w \nabla^\alpha \Psi \nabla_\alpha \Psi
\]
\[
\mathcal{E}^V[\Psi] = FV^\nu \Psi^\nu
\]
\[
\mathcal{E}^V_\mu[\Psi] = \mathcal{E}^V(\Psi) - \frac{1}{4} w \Psi F
\]

Applying the divergence identity between two homologous spacelike hypersurfaces \(\Sigma_1,\Sigma_2\), bounding a region \(B\) with \(\Sigma_2\) in the future of \(\Sigma_1\), we obtain
\[
\int_{\Sigma_2} J^V_\mu[\Psi] n^\mu_{\Sigma_2} + \int_B K^V[\Psi] + \mathcal{E}^V[\Psi] = \int_{\Sigma_1} J^V_\mu[\Psi] n^\mu_{\Sigma_1},
\]
where \(n_{\Sigma_2}\) denotes the future directed timelike unit normal.

Let us note finally that for a fixed smooth function \(\Psi : U \to \mathbb{R}, U \subset \mathcal{R}\), then \(T_{\mu\nu}[\Psi]\) for \(g_{M,a}\) will depend smoothly on the parameters \(a,M\), i.e. it is smooth as a function \(\mathcal{P} \times \mathcal{R} \to \mathbb{R}\). Similarly, if \(w\) is a fixed function and \(V\) is a fixed smooth vector field (or more generally, smooth maps \(\mathcal{R} \to \mathbb{R}, \mathcal{P} \times \mathcal{R} \to T\mathcal{R}\)), then \(J^V_\mu[\Psi], K^V_\mu[\Psi],\mathcal{E}^V_\mu[\Psi]\), etc., are smooth in the above sense. This remark is useful, for instance, in carrying over positive definitivity properties from the Schwarzschild case, and more generally, in applying certain continuity arguments in our forthcoming paper.

4.3. Hardy inequalities. At various points we shall refer to Hardy inequalities (see e.g. before Propositions 5.2.1 and 6.1) to estimate a weighted \(L^2\) norm (spacetime or spacelike) of \(\psi\) from energy quantities. In view of our comments concerning the volume form (see Section 2.9), the reader can easily derive these from the one-dimensional inequalities
\[
\int_0^2 x^{-1} |\log x|^{-2} f^2(x) \leq C \int_0^2 \left( \frac{df}{dx} \right)^2 (x) \, dx + C \int_1^2 f^2(x) \, dx,
\]
\[
\int_1^\infty f^2(x) \leq C \int_1^\infty x^2 \left( \frac{df}{dx} \right)^2 (x) \, dx,
\]
where the latter holds for functions \(f\) of compact support.

4.4. Admissible hypersurfaces. We shall here define admissible hypersurfaces of the first and second kind. The first case will represent spacelike hypersurfaces connecting the future event horizon to spacelike infinity, and the latter case, to future null infinity.

The prototype for admissible hypersurfaces of the first kind is the hypersurface
\[
\Sigma_1 \doteq \{ t^* = 0 \}.
\]

The prototype for admissible hypersurfaces of the second kind is defined as follows. Let \(\chi(z)\) be a nonnegative cutoff function such that \(\chi = 0\) for \(z \leq 0\) and \(\chi(z) = 1\) for \(z \geq 1\). For fixed \(M\), we choose an \(R > 0\) sufficiently large and an \(\alpha > 0\) sufficiently small, depending on \(M\), and consider the hypersurface
\[
\Sigma_2 \doteq \{ t^* = \chi(\alpha \cdot (y^* - R)) \cdot (r + M \log r) \}.\]
Note that $\Sigma_1$ and (for suitable choice\footnote{Let these choices be made once and for all.} of $R$, $\alpha$) $\Sigma_2$ are spacelike for all $g_{M,a}$ where $0 \leq a_0 < M$, $|a| \leq a_0$. Notice also that for all choices of $a$, the hypersurface $\Sigma_2$ asymptotes to null infinity with respect to usual constructions of asymptotic structure. For reasons concerning dependence of constants on $a_0$ in the statements of our theorems, it is convenient to require these properties for all hypersurfaces to be considered. This will thus be encorporated into the definition below:

**Definition 4.1.** Let $0 \leq a_0 < M$. We will call a hypersurface-with-boundary $\Sigma$ admissible of the first kind with respect to $a_0$, $M$, if the following hold:

1. $\Sigma$ is spacelike with respect to $g_{M,a}$ for all $|a| \leq a_0$.
2. $\Sigma \subset R$ is closed as a subset, and $\partial \Sigma \subset H^+$ is compact and thus a topological sphere.
3. There exists a $\tau \in (-\infty, \infty)$ such that for all $\epsilon > 0$, there exists an $y^*_\epsilon > 0$ such that
   
   $$ \Sigma \cap \{ y^* \geq y^*_\epsilon \} \subset \bigcup_{|\tau| \leq \epsilon} \varphi_{\tau+\tau'}(\Sigma_1) $$

We call $\Sigma$ admissible of the second kind with respect to $a_0$, $M$, if the above hold where $\Sigma_1$ is replaced by $\Sigma_2$.

We will say that $\Sigma$ is admissible . . . with respect to $M$ if there exists an $a_0 > 0$ sufficiently small such that the above is true.

Note that for an admissible $\Sigma$, its future domain of dependence with respect to all $g_{M,a}$ is given by

$$D^+(\Sigma) = \bigcup_{\tau \in [0, \infty)} \varphi_{\tau}(\Sigma)$$

The main theorem of the present paper do not in fact distinguish between the type of admissible hypersurfaces. These definitions are useful more to keep track of the dependence of constants. In the theorem of \cite{10}, stated here as Theorem 1.4, it is essential to consider admissible hypersurfaces of the second kind.

4.5. Well-posedness. First let us quote a general well-posedness statement.

**Proposition 4.5.1.** Let $s \geq 1$, $M > a_0 \geq 0$ be fixed, and let $\Sigma$ be an arbitrary admissible hypersurface (of either kind) in $R$. Let $\psi \in H^s_{\text{loc}}(\Sigma)$, $\psi' \in H^{s-1}_{\text{loc}}(\Sigma)$. Then for all $|a| \leq a_0$, there exists a unique solution $\psi$ of $\Box_{\psi} = 0$ with $g = g_{M,a}$ in $D^+(\Sigma)$ such that

$$ \psi \in C^0_{\tau\in[0,\infty)}(H^s_{\text{loc}}(\varphi_{\tau}(\Sigma))) \cap C^1_{\tau\in[0,\infty)}(H^{s-1}_{\text{loc}}(\varphi_{\tau}(\Sigma))) $$

and $\psi|_{\Sigma} = \psi$, $(n_{\Sigma}\psi)|_{\Sigma} = \psi'$. Moreover, if $X \subset \Sigma$ is open and $\tilde{\psi} \in H^s_{\text{loc}}(\Sigma)$, $\tilde{\psi}' \in H^{s-1}_{\text{loc}}(\Sigma)$ such that $\psi = \tilde{\psi}$ and $\psi' = \tilde{\psi}'$ in $X$, then the corresponding unique solutions $\tilde{\psi}$, $\tilde{\psi}'$ satisfy $\tilde{\psi} = \tilde{\psi}'$ in $D^+(X)$.

Finally the map $(\psi, \psi') \times a \mapsto \psi_a$, where $\psi_a$ denotes $\psi$ above with $g = g_{M,a}$ is $C^0 \times C^\infty$.

The above theorem implies that solutions of the wave equation arising from appropriate initial data exist globally, and their regularity in $L^2$-based Sobolev spaces is preserved on any Cauchy surface, and moreover, the solutions depend continuously on initial data and smoothly on the parameters.
4.6. A reduction. First let us note that by extending initial data, an easy domain of dependence argument, and the fact that \( T \) is timelike for all Kerr metrics near infinity, one can show the following:

**Proposition 4.6.1.** Fix \( M > |a_0| \geq 0 \). Let \( \Sigma \) be admissible with respect to \( M, a_0 \), of the first or second kind. Then there exists a constant \( B = B(\Sigma, M) = B(\phi_r(\Sigma), M) \) such that the following holds:

Let \( \psi \) be a solution of the wave equation \((2)\) for \( g = g_{M,a} \) on \( D^+(\Sigma) \), with \( |a| \leq a_0 \), such that \( \psi, n_\Sigma \psi \) are smooth and compactly supported on \( \Sigma \). Then there exists a \( \tau' \) and a smooth solution \( \tilde{\psi} \) of \((2)\) on \( \mathcal{R} \) such that \( D^+(\Sigma) \subset \{ t^* \geq \tau' \} \),

\[ \tilde{\psi} = \psi \]

in \( D^+(\Sigma) \), \( \tilde{\psi}, \partial^{\tau} \tilde{\psi} \) are of compact support on \( \{ t^* = \tau' \} \) and

\[ B^{-1} \int_\Sigma J^N_\mu [\psi] n^\mu_\Sigma \leq \int_{t^* = \tau'} J^N_\mu [\tilde{\psi}] n^\mu_\Sigma, \]

where \( N \) is the vector field of Section 5.

Taking the difference \( \psi - \psi_\infty \), it follows easily from Proposition 4.6.1 a Hardy inequality and a density argument that without loss of generality, we may assume in Theorems 1.1, 1.2 that \( \Sigma = \{ t^* = 0 \} \) and \( \psi \) is smooth and of compact support. We shall assume this reduction starting in Section 8.

4.7. Uniform boundedness. Let us note that the results of [35] and the above proposition yield in particular

**Theorem 4.1.** Fix \( M > 0 \). Let \( \Sigma \) be admissible with respect to \( M \). Then there exists a constant \( B = B(\Sigma, M) = B(\phi_r(\Sigma), M) \) and a positive constant \( \epsilon = \epsilon(M) \), such that for all \( |a| \leq \epsilon \), if \( \psi \) be a sufficiently regular solution of \((2)\) on \( D^+(\Sigma) \) with \( g = g_{M,a} \), then

\[ \int_{\phi_r(\Sigma)} J^N_\mu [\psi] n^\mu_{\phi_r(\Sigma)} \leq B \int_\Sigma J^N_\mu [\psi] n^\mu_\Sigma, \]

\[ \int_{\mathcal{H}^* \cap D^+(\Sigma)} J^{N,\mathcal{H}}_\mu [\psi] n^\mu_{\mathcal{H}^*} \leq B \int_\Sigma J^{N}_\mu [\psi] n^\mu_\Sigma, \]

where \( N \) is the vector field of Section 5. The same statement holds with arbitrary \( |a| \leq a_0 < M \) if \( \Sigma \) is assumed admissible with respect to \( M, a_0 \), for \( \psi \) with \( \Phi \psi = 0 \).

We referred to \( N \) of Section 5 simply to fix the vector field once and for all. If one does not wish to refer to that particular vector field, one can alternatively take an arbitrary \( \phi_r \)-invariant timelike vector field \( \tilde{N} \) such that \( \tilde{N} = T \) for sufficiently large \( r \). The constant \( B \) will then depend also on the choice of that \( \tilde{N} \).

We shall avoid appealing to Theorem 4.1 in the case of the proof of Theorem 1.1, in order to have a self-contained proof. See the discussion of Section 5.3.

In the case of Theorem 1.2, the proof of Theorem 4.1 should be considered as more elementary than the considerations of the present paper in view precisely of the absense of superradiance, so, in particular, we shall freely use it.

This distinction between the proofs of Theorem 1.1 and 1.2 foreshadows the principle (essential in our forthcoming Part III [39]) that the superradiant and nonsuperradiant modes must be treated separately in the general \( |a| < M \) case. See the discussion in Section 7.3 of our companion [38].
5. A $J^N$ Multiplier Current and the Red-shift

In this section, we define a timelike vector field $N$ whose multiplier current $J^N$ captures the red-shift effect. The existence of this current is in fact a general property of stationary black hole spacetimes with Killing horizons of positive surface gravity. We shall discuss in Section 5.3 how the properties of this current can be used to overcome the difficulty of superradiance in the small $a_0$ case. We shall then recall in Section 5.4 the good commutation properties of the wave operator $\Box_g$ with a related vector field $Y$ (used in the construction of $N$) on $\mathcal{H}^+$, properties which again arise from the positivity of the surface gravity. Finally, in Section 5.5, we shall apply commutations with $T$ and $Y$ to infer Theorem 1.3 from Theorems 1.1 and 1.2.

5.1. The construction of $N$. First, the completely general statement specialised to a given Kerr metric: Theorem 7.1 of [36] yields

**Proposition 5.1.1.** Let $|a| < M$, $g_{M,a}$ be the Kerr metric and $\mathcal{R}$, etc., be as before. There exist positive constants $b = b(a,M)$ and $B = B(a,M)$, parameters $r_1(a,M) > r_0(a,M) > r_+$, and a $\varphi$-invariant timelike vector field $N = N(a,M)$ on $\mathcal{R}$ such that

1. $K^N [\Psi] \geq b J^N [\Psi] N^\mu$ for $r \leq r_0$
2. $-K^N [\Psi] \leq B J^N [\Psi] N^\mu$, for $r \geq r_0$
3. $T = N$ for $r \geq r_1$,

where the currents are defined with respect to $g_{M,a}$.

**Proof.** We recall the proof briefly: Given a Kerr metric $g = g_{M,a}$ and a constant $\sigma > 0$, we first note that we may define a vector field $Y$ such that on $\mathcal{H}^+$ we have:

(a) $Y$ is future directed null with $g(Y,K) = -2$,
(b) $\nabla Y Y = -\sigma (Y + K)$ and
(c) $\mathcal{L}_Y Y = \mathcal{L}_K Y = 0$,

where $K$ is the Hawking vector field of Section 2.7. If we choose $\sigma$ sufficiently large and $r_0(a,M)$ sufficiently close to $r_+$, then, defining

$$N = K + Y, \quad \text{in} \quad r \leq r_0(a,M),$$

then, using (23), one sees easily that requirement 1 will hold, and moreover $N$ is timelike in this region. The choice of $Y$, $r_0(a,M)$ can clearly be made to smoothly depend on $a$. If $r_1(a,M) > r_0(a,M)$ is such that $T$ is timelike for $r \geq r_1(a,M)$ it suffices to smoothly extend $N$ as a timelike vector field invariant to the Lie flow of $\Phi$ and $T$ such that $N = T$ in $r \geq r_1(a,M)$, satisfying requirement 3. Requirement 2 then follows by compactness.

Moreover, in view of (25) we clearly have the following

**Proposition 5.1.2.** Fix $M > 0$. Then a $\varphi$-invariant timelike vector field $N$ can be defined on the manifold $\mathcal{R}$ such that for $a_0$ sufficiently small, and $|a| \leq a_0 < M$, then 1, 2 and 3 of Proposition 5.1.1 hold where then $b$ and $B$, $r_1$, $r_0$ can be chosen to depend only on $M$ with $r_1 \leq 5M/2$, and 3 can be strengthened to the statement

$$-K^N [\psi] \leq B J^T [\Psi] T^\nu, \quad T \text{ is timelike for } r \geq r_0.$$

---

4In Theorem 7.1 of [36] we in fact constructed $Y$ to be only invariant under the flow defined by $K$, but in the presence of our two Killing fields $T$ and $\Phi$, one can clearly arrange also for this.
Let us note that in the case of $|a| \ll M$, both the above propositions follow directly from the existence of a vector field $N$ on Schwarzschild satisfying the above statements and smooth dependence of the Kerr family on $a$ (see Section 12), as first observed and exploited in [32]. See also [33–35]. In this case then, we need thus not appeal to the more general construction of [36].

In what follows, in the setting of Theorem 1.1 we shall always assume $a_0$ sufficiently small so that Proposition 5.1.2 applies. Thus, in the setting of Theorem 1.1 we have in particular that $T$ is timelike for $r \geq r_0$. In the setting of Theorem 1.2 all we know is that $T$ is timelike for $r \geq r_1$.

### 5.2. The red-shift estimate.

Given arbitrary $\tilde{r}$ satisfying $r_+ < \tilde{r} \leq r_0$ and $\tilde{\delta} > 0$, then applying the energy identity of $\mathbf{J}^N$ in $D^+(\Sigma) \cap J^-(\varphi_{\tilde{r}}(\Sigma))$, where $\chi(r)$ is a cutoff function with $\chi = 1$ for $r \leq \tilde{r}$ and $\chi = 0$ for $r \geq \tilde{r} + \tilde{\delta}$, and a Hardy inequality we obtain the following

**Proposition 5.2.1.** Let $g = g_{M,a}$ for $|a| \leq a_0 < M$, and let $r_0$ be as in the above Propositions. Then the following is true. Let $\Sigma$ be an admissible hypersurface for $M, a_0$. For all $r_+ \leq \tilde{r} \leq r_0$ and $\tilde{\delta} > 0$, there exists a positive constant $B = B(\Sigma, \tilde{r}, \tilde{\delta}) = B(\varphi_{\tilde{r}}(\Sigma), \tilde{r}, \tilde{\delta})$, such that for all solutions $\psi$ of (2) on $D^+(\Sigma)$ with $g = g_{M,a}$, then

$$
\int_{D^+(\Sigma) \cap J^-(\varphi_{\tilde{r}}(\Sigma)) \cap \{r \leq \tilde{r}\}} (\mathbf{J}_\mu^N[\psi]N^\mu + |\log(|r - r_+|)|r - r_+|^{-1}\psi^2) \\
+ \int_{H^+ \cap D^+(\Sigma) \cap J^-(\varphi_{\tilde{r}}(\Sigma))} \mathbf{J}_\mu^N[\psi]n^\mu + \int_{\varphi_{\tilde{r}}(\Sigma) \cap \{r \leq \tilde{r}\}} \mathbf{J}_\mu^N[\psi]n^\mu \\
\leq B \int_{\Sigma} \mathbf{J}_\mu^N[\psi]n^\mu + B \int_{D^+(\Sigma) \cap J^-(\varphi_{\tilde{r}}(\Sigma)) \cap \{r \leq \tilde{r} + \tilde{\delta}\}} (\mathbf{J}_\mu^N[\psi]N^\mu + \psi^2)
$$

Recall that the additional dependence of $B$ on $M$ and $a_0$ is now implicit according to our conventions.

### 5.3. The red-shift vs. superradiance.

Superradiance concerns the fact that $T$ is not everywhere timelike. The smallness of superradiance with respect to the redshift in the setting of Theorem 1.1 can be quantified by the following

**Proposition 5.3.1.** There exists an $a_0 > 0$ and a differentiable nonnegative function $e_0(a)$ defined for $|a| \leq a_0$, such that $e_0(0) = 0$, $a \cdot (e_0)'(a) \geq 0$, and such that $T + \tilde{e}N$ is timelike for $\tilde{e} > e_0$ in $r > r_+$, and such that for all $\tau' \geq \tau$, the following inequalities hold:

$$
\int_{\Sigma_{\tau'}} \mathbf{J}_\mu^{T + \tilde{e}N}[\psi]N^\mu \leq \int_{\Sigma_{\tau}} \mathbf{J}_\mu^{T + \tilde{e}N}[\psi]N^\mu + B\tilde{e} \int_{D^+(\Sigma_{\tau}) \cap J^-(\varphi_{\tilde{r}}(\Sigma)) \cap \{r \leq \tilde{r} + \tilde{\delta}\}} \mathbf{J}_\mu^N[\psi]N^\mu \\
\int_{H^+ \cap \{r \leq \tau' \leq \tau\}} \mathbf{J}_\mu^{T + \tilde{e}N}[\psi]n^\mu \leq \int_{\Sigma_{\tau'}} \mathbf{J}_\mu^{T + \tilde{e}N}[\psi]N^\mu \\
+ B\tilde{e} \int_{D^+(\Sigma_{\tau'}) \cap J^-(\varphi_{\tilde{r}}(\Sigma)) \cap \{r \leq \tilde{r} + \tilde{\delta}\}} \mathbf{J}_\mu^N[\psi]N^\mu,
$$

with $B = B(\Sigma, \tilde{r}, \tilde{\delta})$, $r_+ < \tilde{r} \leq r_0$, $\tilde{\delta} > 0$, as in the previous proposition.

**Proof.** The existence of an $e_0(a)$ as described so that $T + \tilde{e}N$ is timelike is clear. Now just add $\tilde{e}$ times the previous proposition to the energy identity of $\mathbf{J}^T$, for an $\tilde{e}$ such that $T + \tilde{e}N$ is timelike, and use that $T$ is Killing, i.e. $\mathbf{K}^T = 0$, and that $T$ is timelike for $r \geq r_0$, and thus $\mathbf{J}_\mu^{T + \tilde{e}N}[\psi]N^\mu \sim \mathbf{J}_\mu^N[\psi]N^\mu$ on $r \geq r_0$. □
The above proposition is of course implied by Theorem 4.1 of [35] but we include it as an independent statement so as to circumvent appeal to this theorem in the case of the proof of Theorem 1.1. For this, we shall also need the following

**Proposition 5.3.2.** Given arbitrary \( \varepsilon > 0 \), there exists an \( a_0 > 0 \) depending on \( \varepsilon \), and \( 1 \geq \hat{e} \geq e_0(a) \),

\[
\int_{\mathcal{D}^{+}(\Sigma) \cap J^{-}(\varphi_{\varepsilon^{-1}}(\Sigma)) \cap (r_0 \leq r \leq r_1)} J_{\mu}^N[\psi] N^\mu \leq B \int_{\Sigma} J_{\mu}^T[\psi] N^\mu,
\]

\( \forall 0 \leq \tau \leq \varepsilon^{-1}. \)

**Proof.** The first inequality follows immediately from the second. The second inequality follows from the fact that (27) holds for the Schwarzschild metric \( g_M \) for all \( \varepsilon \) (note the restriction on \( r_1 \) in Proposition 5.1.2 which guarantees that the domain of integration on the left hand side is separated from the Schwarzschild photon sphere) together with the stability considerations of Section 4.2.

We note that the necessary Schwarzschild result was proven in [32]. The current paper can be read independently of [32], as follows: While the present proposition is appealed to in the proof of Theorem 1.1 if the latter is specialised to the case \( a_0 = 0 \), then, just as in the \( m = 0 \) case, the use of the present proposition is in fact not necessary, as appeal has only been made to circumvent use of Theorem 4.1 in the case \( a \neq 0 \). Thus, the proof of Theorem 1.1 specialised to \( a_0 = 0 \) yields the present proposition, which can then be used in the proof of Theorem 1.1 for the general \( \mathcal{D} \) case.

The significance of the above two propositions is the following: As in the proof of Theorem 4.1 of [35], we shall be able to absorb the boundary terms of our virial identities for sufficiently small \( a \) using solely Propositions 5.3.1 and 5.3.2 exploiting the fact that one can take \( \hat{e} \) very close to 0. It is the use of this argument that mathematically represents one of the insights first used in [35], namely:

*If superradiance is sufficiently weak, it can be overcome provided one can construct a virial current with positive definite divergence, whose boundary terms can be controlled with the positive definite current formed by adding a small amount of the red-shift current \( J^N \) to the conserved energy current \( J^T \).*

What we are essentially able to avoid using here is the second main insight of [35], summarised as follows:

*To obtain just the boundedness statement, it suffices to construct a virial current as above for the projection of the solution to its superradiant frequencies. Such a projection can be defined for arbitrary axisymmetric stationary black hole spacetimes whose Killing fields span the null generator of the horizon. Moreover, for suitably small such perturbations of Schwarzschild, this projection is not “trapped”, thus the existence of the virial current follows from stability considerations from Schwarzschild!*
the special case of the Kerr solution, we are indeed proving decay, that it is to say, we are indeed producing such a virial current for the total solution.

It is to highlight this distinction that we have been careful to avoid use of our general boundedness theorem [35] and its superradiant projection. The reader hoping that one can understand the Kerr family without understanding superradiance will be severely disappointed however! In particular, the superradiant projection of [35] is used heavily in our forthcoming [39] which considers the general case $|a| < M$, precisely because superradiance can no longer be treated simply as a small parameter and the above insight is not sufficient. See the discussion in Section 7.3 of our companion paper [38], and Section 11 of [38] for the details of the necessary constructions in all frequency ranges for the general case.

5.4. Red-shift commutation. We specialise Theorem 7.2 of [36] to the Kerr case.

**Proposition 5.4.1.** Let $g = g_{M,a}$ and $Y$ be as in Proposition 5.1.1. Then, on $\mathcal{H}^+$, extending $K$ to a translation invariant standard null frame $E_1, E_2, K, Y$, then for all $k \geq 0$ and multi-indices $m = (m_1, m_2, m_3, m_4)$,

$$\Box_g [Y^k \Psi] = \kappa_k Y^{k+1} \Psi + \sum_{|m| \leq k+1, m_4 \leq k} c_m E_1^{m_1} E_2^{m_2} L^{m_3} Y^{m_4} \Psi$$

where $\kappa_k > 0$.

The above proposition, which is another manifestation of the red-shift effect, effectively allows us not only to apply a transversal vectorfield to the horizon as a multiplier, but also as a commutator. This is fundamental for retrieving higher order statements as in Theorem 1.3.

5.5. The higher order statement. Commuting the wave equation with $T$ and $Y$, in the latter case applying Proposition 5.4.1, we show the statement of Theorem 1.3 given Theorems 1.1 and 1.2.

For $j = 1$: We commute the equation first with $T$. It follows that the estimates of Theorem 1.1 or 1.2 hold with $T \psi$ in place of $\psi$. Now we commute also with $Y$ and follow [35, 36]. The quantity $Y \psi$ satisfies a wave equation with an inhomogeneous term, but by Proposition 5.4.1 the most dangerous term arising in $\mathcal{E}^N[Y \psi]$ (recall the notation of Section 4.2) has a good sign sufficiently near the horizon. In the case of the assumptions of Theorem 1.1, we may assume that $T$ is strictly timelike outside this region. Elliptic estimates then suffice to bound all remaining terms in $\mathcal{E}^N[Y \psi]$ from the good terms in $\mathcal{K}^N[Y \psi]$ and those already bounded by the commutation by $T$. In the case of $m = 0$, $T$ is effectively timelike everywhere except the horizon from the perspective of axisymmetric solutions, in the sense that for every $x \in \mathcal{R} \cap \mathcal{H}^+$ there is a $T + c(x) \Phi$ which is timelike in a neighborhood of $x$. Thus, again, all terms not manifestly controlled can be easily obtained by elliptic estimates in view of the fact that $NN \psi$ is controlled locally in $L^2$.

The case of higher $j$ is similar and the required positivity of the most dangerous additional terms appearing in $\mathcal{E}^N[Y^j \psi]$ is precisely that provided by Proposition 5.4.1 above for $j = k \geq 2$.

Recall that in our forthcoming Part III [39], it will be shown that the statement of Theorem 1.2 is true without the restriction (9). The above arguments can be extended to show that, in general, given parameters $M > a$ for which the statement of Theorem 1.2 holds without the restriction (9), one can infer the validity of Theorem 1.3 for those parameters $M, a$. Let us first note that Proposition 5.4.1...
still applies, yielding positivity for the most dangerous terms in $\mathcal{C}[Y\psi]$ in a small neighborhood of the horizon. The only additional difficulty is that one can no longer appeal to smallness of $a_0$ (as in the case of Theorem 1.1) to argue that $T$ is timelike outside this region and thus, outside this region, all second derivatives can be controlled by the integrated decay bound applied to $T\psi$ (a statement which, as we saw above, can similarly be inferred in the case of Theorem 1.2 using the assumption of axisymmetry). In the more general setting, in addition to commutations with $T$ and $Y$ as above, we must simply also commute with $\chi\Phi$ (where $\chi = 1$ for $r \leq R$, and $\nabla\chi$ is thus supported away from trapping), and exploit the fact that the span of $T$ and $\Phi$ is timelike away from the horizon. The error terms resulting in the commutation with $\chi\Phi$ are easily absorbed with the integrated decay statement. Thus, commutations with $Y$, $T$ and $\chi\Phi$ ensure that one always has a timelike direction in the span of commutators up to and including the horizon. Application of elliptic estimates allows one as before to absorb all error terms and obtain all second derivatives. One then proceeds by induction on $j$. We shall insert the details of this more general argument with the completion of Part III [39].

6. A CURRENT FOR LARGE $r$

We construct here a current $J^{X,w}$ as follows. Recall (see e.g. [35]) that in the Schwarzschild geometry $g_M$, defining

$$X = f(r^*) \partial_r$$

$$w = 2f'(r^*) + 4 \frac{1 - 2M/r}{r} f - 2\delta \frac{1 - 2M/r}{r^{1+\delta}} f,$$

we have

$$K^{X,w}_{g_M}[\Psi] = \left( \frac{f'}{1 - 2M/r} - \frac{f\delta}{2r^{1+\delta}} \right)(\partial_r \Psi)^2 + \frac{f\delta}{2r^{1+\delta}} (\partial_r \Psi)^2$$

$$+ f \left( \frac{r - 3M}{r^2} - \frac{\delta (r - 2M)}{2r^{2+\delta}} \right) |\Psi|^2 - \frac{1}{2} \square w |\Psi|^2$$

where $'$ denotes differentiation with respect to $r^*$. Let $\chi$ be a cutoff function such that $\chi = 1$ for $r \geq R$, $\chi = 0$ for $r \leq R - 1$ and let $\delta > 0$. Choose now

$$f = \chi(1 - r^{-\delta}) \partial_r.$$

We note that for $R$ sufficiently large, we have

$$K^{X,w}_{g_M}(\Psi) \geq b(\delta) \left( r^{-1-\delta} (\partial_r \Psi)^2 + r^{-1-\delta} (\partial_t \Psi)^2 + r^{-1} |\Psi|_{g_M}^2 + r^{-3-\delta} \Psi^2 \right)$$

and that this inequality is preserved when $X$, $w$, are defined for $g_{M,a}$ again as above, where $|\Psi|_{g_M}^2$ is now replaced by $|\Psi|_{g_{M,a}}^2$ defined in the sense of Section 2.9.

From the energy identity of $J^{X,w}$ and a Hardy inequality to control 0’th order terms on the boundary, we have

**Proposition 6.1.** Fix $M > 0$, let $\Sigma$ be an admissible hypersurface for $M$, and let $\epsilon_0$ be as above. For each $\delta > 0$, there exist positive values $R_0 < R$, and positive constants $B = B(\delta, \Sigma) = B(\delta, \varphi_T(\Sigma))$ such that for all Kerr metrics $g_{M,a}$ with $|a| \leq a_0 < M$, if
\( \psi \) denotes a solution of \( \Delta \) with \( g = g_M \) and \( \psi_\infty = 0 \), then for all \( \tau \geq 0 \)
\[
\int_{D^*(\Sigma) \cap J^-(\varphi^{-1}(\Sigma)) \cap \{r \geq R\}} r^{-1}(r^{-\delta}|\partial_r \psi|^2 + r^{-\delta}|\partial_{\varphi} \psi|^2 + |\nabla \psi|_{g}^2 + r^{-2-\delta}|\psi|^2)
\leq B \int_{\Sigma} J_{\mu}^{\tau+\epsilon_0}[\psi] n_{\Sigma_0}^{\epsilon} + B \int_{\varphi^{-1}(\Sigma)} J_{\mu}^{\tau+\epsilon_0}[\psi] n_{\Sigma_0}^{\epsilon}
+ B \int_{D^*(\Sigma) \cap J^-(\varphi^{-1}(\Sigma)) \cap \{R_0 \leq r \leq R\}} (|\partial_r \psi|^2 + |\partial_{\varphi} \psi|^2 + |\nabla \psi|_{g}^2 + |\psi|^2).
\]

7. Separation

As described in Section 3.3, we will exploit Carter’s separation of the wave equation \( [18] \) to frequency-localise our virial type estimates in a manner particularly suited to the local and global geometry of Kerr. The separation of \( \square_g \psi = 0 \) requires taking the Fourier transform in \( t \), and then expanding into what are known as oblate spheroidal harmonics. Since, a priori, we do not know that solutions \( \psi(t, \cdot) \) of our problem are \( L^2(dt) \), we must in fact apply this separation to solutions of the inhomogeneous equation
\[
(28) \quad \square_g \Psi = F,
\]
where \( \Psi \) is related to \( \psi \) by the application of a suitable cutoff. We will defer the discussion of such cutoffs to Section 8 and will in the meantime in Section 7.2 below give a general discussion of solutions of \( (28) \) in \( r > r_+ \) for which one can indeed take the Fourier transform (see Section 7.2) in \( t \). First, however, we quickly review the classical oblate spheroidal harmonics.

7.1. Oblate spheroidal harmonics. Let \( L^2(\sin \theta d\theta d\phi) \) denote the space of complex-valued \( L^2 \) functions on the sphere with standard spherical coordinates \( \theta, \phi \). Let \( L^2_m \)
denote the eigenspace of \( \partial_\phi \) with eigenvalue \( m \). Recall that
\[
L^2 = \bigoplus_{m \in \mathbb{Z}} L^2_m
\]
orthogonally, and that
\[
(29) \quad L^2_m = L^2(\sin \theta d\theta d\phi) \otimes \mathbb{C} e^{im\phi}.
\]
For \( \xi \in \mathbb{R} \), define the operator \( P(\xi) \) on a suitable dense subset of \( L^2(\sin \theta d\theta d\phi) \) by
\[
P(\xi) f = -\frac{1}{\sin \theta \frac{\partial}{\partial \theta}} \left( \sin \theta \frac{\partial}{\partial \theta} f \right) - \frac{\partial^2 f}{\partial \phi^2 \sin^2 \theta} - \xi^2 \cos^2 \theta f.
\]
Note that \( P(0) \) is the standard spherical Laplacian, whereas \( P(\xi) \) in fact corresponds to the Laplacian on an oblate spheroid. We have certainly that \([P(\xi), \partial_\phi] = 0 \) and thus \( P(\xi) \) preserves \( L^2_m \), and under the identification \( (29) \), \( P(\xi) \) induces an operator \( P_m(\xi) \) defined on a dense subset of \( L^2(\sin \theta d\theta) \) by
\[
P_m(\xi) f = -\frac{1}{\sin \theta \frac{d}{d\theta}} \left( \sin \theta \frac{d}{d\theta} f \right) + \frac{m^2}{\sin^2 \theta} f - \xi^2 \cos^2 \theta f.
\]

The following proposition is classical and can be distilled from standard elliptic theory and the form of \( P_m \):

**Proposition 7.1.1.** Let \( \xi \in \mathbb{R} \). There exists a complete basis of \( L^2(\sin \theta d\theta d\phi) \) of eigenfunctions of \( P(\xi) \) of the form \( S_m(\xi, \cos \theta)e^{im\phi} \), with real eigenvalues \( \lambda_m(\xi) \):
\[
P(\xi) S_m(\xi, \cos \theta)e^{im\phi} = \lambda_m(\xi) S_m(\xi, \cos \theta)e^{im\phi}
\]
where for each \( m \in \mathbb{Z} \), the collection \( S_m(\xi, \cos \theta) e^{im \phi} \) are a basis in \( L^2_m \) of real eigenfunctions of \( P_m(\xi) \):

\[
P_m(\xi) S_m(\xi, \cos \theta) = \lambda_m(\xi) S_m(\xi, \cos \theta)
\]

indexed by the set \( \ell \geq |m| \). The functions \( S_m(\xi, \cos \theta) \) are smooth in \( \xi \) and \( \theta \), and, \( \lambda_m(\xi) \) is smooth in \( \xi \). Finally, for \( \xi = 0 \), these reduce to spherical harmonics, i.e. \( S_m(0, \cos \theta) e^{im \phi} = Y_m, \) where \( Y_m \) denote the standard spherical harmonics, and \( \lambda_m(0) = \ell(\ell + 1) \).

The functions \( S_m \) are known classically as oblate spheroidal harmonics.

Let us introduce the notation

\[
\alpha_{m\ell}(\xi) = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos \theta) \alpha(\theta, \phi) e^{-im \phi} S_m(\xi, \cos \theta).
\]

The completeness and orthogonality conditions of Proposition 7.1.1 are given explicitly below:

\[
\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos \theta) e^{im \phi} S_m(\omega, \cos \theta) e^{-im' \phi} S_{m'}(\xi, \cos \theta) = \delta_{mm'} \delta_{\ell \ell'},
\]

(31)

\[
\alpha(\theta, \phi) = \sum_{m} \alpha_{m\ell}(\xi) S_{m}(\xi, \cos \theta) e^{im \phi}.
\]

The statement (31) is to be understood in \( L^2(\sin \theta d\theta d\phi) \), but, if \( \alpha(\theta, \phi) \) is in fact, say, a smooth function on the standard sphere, then (31) holds pointwise in the topology of the sphere. (We shall not require such pointwise statements, however.)

In the application to the Kerr geometry, \( \xi = a \omega \) for \( \omega \in \mathbb{R} \), where \( \omega \) will be a frequency associated to the Fourier transform with respect to \( t \) of a cutoff version of the solution \( \psi \). We note that the case of complex \( \xi \) is also of interest in the formal mode analysis associated to this problem (see for instance [87]). In view of our setting, however, we shall only need to consider real \( \xi \) here.

Let us note that from the smooth dependence statement of Proposition 7.1.1 (applied with the application \( \xi = a \omega \) in mind) one immediately obtains

**Proposition 7.1.2.** Given any \( \omega_1 > 0, \lambda_1 > 0, \) and \( \epsilon > 0, \) then there exists an \( a_1 > 0 \) such that for \( |a| < a_1, |\omega| \leq \omega_1, \) and \( \lambda_m \leq \lambda_1 \),

\[
|\lambda_m(a \omega) - \ell(\ell + 1)| \leq \epsilon.
\]

Rewriting the equation for the oblate spheroidal function

\[
- \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} S_m \right) + \frac{m^2}{\sin^2 \theta} S_m = \lambda_m S_m + a^2 \omega^2 \cos^2 \theta S_m,
\]

the smallest eigenvalue of the operator on the left hand side of the above equation is \( |m|(|m| + 1) \). This implies that

**Proposition 7.1.3.**

\[
\lambda_m(a \omega) \geq |m|(|m| + 1) - a^2 \omega^2.
\]

The above propositions will be all that we require about \( \lambda_m \). For a more detailed analysis of \( \lambda_m, \) see [45].

Let us finally note that by integration by parts and the above orthogonality we obtain
Proposition 7.1.4.

\[ \int |\gamma \nabla \alpha|^2 \sin \theta \, d\theta \, d\phi - \int \xi^2 \cos^2 \theta |\alpha|^2 \sin \theta \, d\theta \, d\phi = \int (P(\xi)\alpha) \sin \theta \, d\theta \, d\phi = \sum_{m\ell} \lambda_{m\ell}(\xi)|\alpha(\xi)|^2. \]

Here, \( \gamma \) denotes the standard unit metric defined by standard spherical coordinates \((\theta, \phi)\) on the fixed \((r, t)\) spheres, and \( \gamma \nabla \) denotes the covariant derivative with respect to this metric.

7.2. Carter’s separation. Let \( \Psi(t, r, \theta, \phi) \), \( \Upsilon(t, r, \theta, \phi) \), be functions with the property that \( \Psi, \Upsilon \) are smooth and of Schwartz class in \( t \), smooth in \( r > r_+ \), and smooth on the \((\theta, \phi)\) spheres.

Let \( \hat{\Psi}(\omega, r, \theta, \phi) \) denote the Fourier transform with respect to \( t \), i.e.

\[ \hat{\Psi}(\omega, r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(t, r, \theta, \phi) e^{i\omega t} \, dt. \]

Note that under our assumptions, \( \hat{\Psi} \) is smooth and of Schwartz class in \( \omega \in \mathbb{R} \), smooth in \( r > r_+ \), and smooth on the spheres.

Given \( a \in \mathbb{R} \), for fixed \( \omega, r \), we may define the coefficients \( \hat{\Psi}^{(a\omega)}(\omega, r) \) by (30), choosing the value \( \xi = a\omega \). Let us agree to drop the \( \hat{\sim} \) from the notation, and replace the argument pair \((\omega, r)\) by the singlet \( r \), since the \( \omega \)-dependence is also implicit in the superscript, and denote these coefficients

\[ \Psi^{(a\omega)}_{m\ell}(r). \]

We may now state a suitable version of Carter’s celebrated separation of the wave operator on Kerr.

**Theorem 7.1** (Carter [18]). Let \( g_{a,M} \) be a Kerr metric for \(|a| < M\), let \( t, r, \theta, \phi \) be the Boyer-Lindquist coordinates defined in Section 2.4, and let \( \Psi(t, r, \theta, \phi) \) be Schwartz class in \( t \), smooth in \( r > r_+ \), and smooth on the \((\theta, \phi)\) spheres, with \( \Box_g \Psi = F \).

Then defining the coefficients \( \Psi^{(a\omega)}_{m\ell}(r) \), \( F^{(a\omega)}_{m\ell}(r) \) as above, the following holds:

\[ \Delta \frac{d}{dr} \left( \Delta \frac{d\Psi^{(a\omega)}_{m\ell}}{dr} \right) + \left( a^2 m^2 + (r^2 + a^2)\omega^2 - \Delta(\lambda_{m\ell} + a^2 \omega^2) \right) \Psi^{(a\omega)}_{m\ell} = (r^2 + a^2) \Delta F^{(a\omega)}_{m\ell}. \]

(33)

7.3. Basic properties of the decomposition. The coefficients can be related back to \( \Psi \) by

\[ \hat{\Psi}(\omega, r, \theta, \phi) = \sum_{m\ell} \Psi^{(a\omega)}_{m\ell}(r) S_{m\ell}(a\omega, \cos \theta) e^{i m \phi}. \]

The above equality is to be interpreted in \( L^2(\sin \theta d\theta d\phi) \). (It in fact also holds pointwise under the smoothness assumptions given here, but we shall not need to
make use of this.) From Plancherel’s formula, and Proposition \[7.1.1\], we have
\[
\int_0^{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} |\Psi|^2(t, r, \theta, \varphi) \sin \theta \, d\varphi \, d\theta \, dt = \int_{-\infty}^{\infty} \sum_{m, \ell} |\Psi^{(\omega)}_{m, \ell}(r)|^2 \, d\omega,
\]
\[
\int_0^{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \Psi \cdot \Upsilon \sin \theta \, d\varphi \, d\theta \, dt = \int_{-\infty}^{\infty} \sum_{m, \ell} \Psi^{(\omega)}_{m, \ell} \cdot \Upsilon^{(\omega)}_{m, \ell} \, d\omega.
\]

With respect to Boyer-Lindquist coordinates, we clearly have for \(r > r^*\),
\[
\frac{d}{dr} \Psi^{(\omega)}_{m, \ell} = (\partial_r \Psi^{(\omega)}_{m, \ell}).
\]

On the other hand, from well-known properties of the Fourier transform,
\[
(\partial_t \Psi^{(\omega)}_{m, \ell}) = -i \omega \Psi^{(\omega)}_{m, \ell}.
\]

In particular, we have
\[
\int_0^{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} |\partial_r \Psi|^2(t, r, \theta, \varphi) \sin \theta \, d\varphi \, d\theta \, dt = \int_{-\infty}^{\infty} \sum_{m, \ell} \left| \frac{d}{dr} \Psi^{(\omega)}_{m, \ell}(r) \right|^2 \, d\omega,
\]
\[
\int_0^{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} |\partial_t \Psi|^2(t, r, \theta, \varphi) \sin \theta \, d\varphi \, d\theta \, dt = \int_{-\infty}^{\infty} \sum_{m, \ell} \omega^2 |\Psi^{(\omega)}_{m, \ell}(r)|^2 \, d\omega.
\]

Finally, let us note that from Proposition \[7.1.4\] and the above, we obtain
\[
\int_{-\infty}^{\infty} \sum_{m, \ell} \lambda_{m, \ell}(\omega) \left| \Psi^{(\omega)}_{m, \ell}(r) \right|^2 \, d\omega = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\pi} \left| \gamma \nabla \Psi \right|^2 \sin \theta \, d\varphi \, d\theta \, dt
\]
\[
- a^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\pi} |\partial_t \Psi|^2 \cos^2 \theta \sin \theta \, d\varphi \, d\theta \, dt,
\]
where \(\gamma, \gamma \nabla\) are defined previously.

8. Cutoffs

From now onwards, we shall assume the reduction of Section \[4.6\].

We are considering always solutions \(\psi\) of \[3\] for \(g = g_{M,a}\) for parameters to be constrained at each instance. To apply the results of Section \[7\] to \(\psi\), we must first cut off in time.

Let \(\xi(z)\) be a smooth cutoff function such that \(\xi = 0\) for \(z \leq 0\), \(\xi = 1\) for \(z \geq 1\). Let \(\epsilon > 0\). Let \(\tau' \geq 2\epsilon^{-1}\) be fixed, we define \(\xi_{\tau', \epsilon}(t^*) = \xi(\epsilon t^*)\xi(\epsilon(\tau^* - t^*))\) and
\[
\psi_\infty(t^*, r, \theta, \phi) = \xi_{\tau', \epsilon}(t^*) \psi(t^*, r, \theta, \phi).
\]

with respect to coordinates \((t^*, r, \theta, \phi)\). We note that \(\psi_\infty : \mathcal{R} \to \mathbb{R}\) is smooth and supported in \(0 \leq t^* \leq \tau'\). The function \(\psi_\infty\) is a solution of the inhomogeneous equation
\[
\Box_g \psi_\infty = F, \quad F = 2\nabla^\alpha \xi_{\tau', \epsilon} \nabla_\alpha \psi + (\Box_g \xi_{\tau', \epsilon}) \psi.
\]
Note that \(\nabla \xi_{\tau', \epsilon}\) and \(F\) are supported in
\[
\{0 \leq t^* \leq \epsilon^{-1}\} \cup \{\tau' - \epsilon^{-1} \leq t^* \leq \tau'\}
\]

\(^5\)We introduce the parameter \(\epsilon\) solely for the purpose of avoiding reference to Theorem \[4.1\] in proving Theorem \[4.4\]. If we were to allow appeal to Theorem \[4.1\] one could simply take \(\epsilon = 1\). This remark should be considered in the context of the \(m = 0\) or \(a = 0\) case, where Theorem \[4.1\] is a much more elementary statement.
and that, with respect to Kerr-star coordinate derivatives:

\[ |\Box g \xi_{r,\varepsilon}| \leq \varepsilon^2 B \]
\[ |\nabla^\alpha \xi_{r,\varepsilon} \nabla_a \psi| \leq B \varepsilon (|\partial_r \psi|^2 + |\partial_\psi |^2 + |\nabla \psi|^2). \]

Finally, we note that restricted to \( r > r_+ \), the function \( \psi_\infty \) is smooth in Boyer-Lindquist coordinates and for each fixed \( r > r_+ \) is compactly supported in \( t \). In what follows, we may thus apply Theorem 7.1 to \( \Psi = \psi_\infty \).

9. The frequency localised multiplier estimates

We shall construct in this section analogues of the \( J^{X,w} \) currents used in \([32, 34]\), localised however to each \( \Psi_m^{(m\ell)} \).

9.1. The separated current templates. To describe the analogue of multipliers of the form \( J^{X,w} \) localised to frequency triplet \((\omega, m, \ell)\), it will be convenient to define the following current templates.

First, we may recast the ode \((33)\) (applied to \( \Psi = \psi_\infty \)) in a more compact form as follows: Recall the definition \((14)\) of \( r^* \) and set

\[ u_m^{(\omega)}(r) = (r^2 + a^2)^{1/2} \Psi_m^{(\omega)}(r), \quad H_m^{(\omega)}(r) = \frac{\Delta F_m^{(\omega)}(r)}{(r^2 + a^2)^{1/2}}. \]

Then \( u \) satisfies

\[ \frac{d^2}{(dr^*)^2} u_m^{(\omega)} + (\omega^2 - V_m^{(\omega)}(r)) u_m^{(\omega)} = H_m^{(\omega)} \]

where

\[ V_m^{(\omega)}(r) = \frac{4Mrama - a^2 m^2 + \Delta(\lambda_m + \omega^2 a^2)}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4}. \]

Observe the Schwarzschild case:

\[ V_m^{(\omega)}(r) = (r - 2M) \left( \frac{\lambda_m}{r^3} + \frac{2M}{r^4} \right). \]

In what follows, let us suppress the dependence of \( u, H \) and \( V \) on \( \omega, m, \ell \) in our notation\(^6\).

Now, with the notation \( \dot{\gamma} = \frac{d}{dr^*} \), for arbitrary functions \( h(r^*) \), \( y(r^*) \), let us define the currents

\[ i_y^{\dot{\gamma}}[u] = y \left( |u|^2 + (\omega^2 - V)|u|^2 \right), \]
\[ Q^{\dot{\gamma}}[u] = h \text{Re}(u^* \dot{u}) - \frac{1}{2} h'|u|^2. \]

We compute:

\[ (i_y^{\dot{\gamma}}[u])' = y' \left( |u|^2 + (\omega^2 - V)|u|^2 \right) - yV' |u|^2 + 2y \text{Re}(u^* \dot{H}). \]

\(^6\)Before suppressing the dependence, it might be useful to remark that in fact, the functions \( u, H, V \), etc., depend on both \( \omega m \) and \( a \), whereas the parameters \( \lambda_m \) depend only on \( \omega \). For the former 3, will view reference to the \( a \)-dependence as implicit in the reference to \( a \) in \( \omega \). Hence, our writing \( 0\omega \) instead of \( 0 \) in \([35]\).
The currents we shall employ will be various combinations of \( \mathcal{Q}, \mathcal{Y} \) with suitably selected functions \( h, y \). Note that the choice of these functions may depend on \( a,\omega,m,\ell \), but, again, we temporarily suppress this from the notation.

A particular combination of \( \mathcal{Q} \) and \( \mathcal{Y} \) shall appear so often that we shall give it a new name; consideration of this current is motivated by removing the \( \omega^2 \) term from the right hand side of (37), (38). Let us define

\[
Q^f[u] = \mathcal{Q}^f[u] + \mathcal{Y}^f[u] = f(|u'|^2 + (\omega^2 - V)|u|^2) + f'\Re(u'u) - \frac{1}{2} f''|u|^2.
\]

We see immediately

\[
(Q^f[u])' = 2f'|u'|^2 - fV'|u|^2 + \Re(2f'Hu' + f'Hu) - \frac{1}{2} f''|u|^2.
\]

A word of warning: For fixed \( g_{\text{M,a}} \), we shall often refer to \( r^* \)-ranges by their corresponding \( r \) ranges, and functions appearing in most estimates will be written in terms of \( r \). Moreover, given an \( r \)-parameter such as \( R \), then \( R^* \) will denote \( r^*(R) \). It is important to remember at all times that \( ' \) always means \( \frac{d}{dr} \).

9.2. The frequency ranges. Let \( \omega_1, \lambda_1 \) be (potentially large) parameters to be determined, and \( \lambda_2 \) be a (potentially small) parameter to be determined. We define the frequency ranges \( \mathcal{F}_b, \mathcal{F}_y, \mathcal{F}, \mathcal{F}_\#: \)

\[
\cdot \mathcal{F}_b = \{ (\omega, m, \ell) : |\omega| \leq \omega_1, \lambda_{mt}(\omega a) \leq \lambda_1 \}
\]

\[
\cdot \mathcal{F}_y = \{ (\omega, m, \ell) : |\omega| \leq \omega_1, \lambda_{mt}(\omega a) > \lambda_1 \}
\]

\[
\cdot \mathcal{F} = \{ (\omega, m, \ell) : |\omega| \leq \omega_1, \lambda_{mt}(\omega a) \geq \lambda_2 \omega^2 \}
\]

\[
\cdot \mathcal{F}_\# = \{ (\omega, m, \ell) : |\omega| \geq \omega_1, \lambda_{mt}(\omega a) < \lambda_2 \omega^2 \}
\]

The nature of our constructions will be of quite different philosophy for each of the above ranges.

9.3. The \( \mathcal{F}_b \) range (bounded frequencies). This is a compact frequency range and, in view of the fact that we have already constructed multipliers in the Schwarzschild case (see [32, 33]), by stability considerations, their positivity properties carry over for \( |a| \ll M \). See our proof in [30].

To give a completely self-contained presentation in this paper which allows one to treat Theorem [1.1] and Theorem [1.2] together, we will present here a different construction with a greater range of validity. These constructions will also be useful for our companion paper [39] where one must further decompose into superradiant and nonsuperradiant frequencies.

First some preliminaries: Let us define

\[
V_{\text{new}} = \frac{4Mr a \omega - a^2 m^2}{(r^2 + a^2)^2},
\]

\[
V_+ = V - V_{\text{new}} = \frac{\Delta}{(r^2 + a^2)^2} \left( (\lambda_{mt} + \omega^2 a^2)(r^2 + a^2)^2 + (2Mr^3 + a^2 r^2 + a^4 - 4Mar^2) \right).
\]
Remark that by Proposition 7.1.3 the first term of (40) is nonnegative, whereas the second term is easily seen to be strictly positive for \( r > r_+ \) and all \( |a| < M \). Thus, we have in particular
\[
V_+ > 0
\]
for \( r > r_+ \). We have, moreover, according to our conventions, for all \( |a| \leq a_0 \),
\[
B(\Delta/r^2)(\lambda_{mt} + a^2\omega^2)r^{-2} + Br^{-3} \geq V_r \geq (\Delta/r^2)b(\lambda_{mt} + a^2\omega^2)r^{-2} + b(\Delta/r^2)r^{-3}.
\]
In the case of Theorem 1.1 we may now fix an arbitrary \( r_0 > r_c > 2M \), where \( r_0 \) is as in Proposition 5.1.2 and it follows that for sufficiently small \( a_0 \) depending in particular on this choice, we have for \( |a| \leq a_0 \) and \( (\omega, m, \ell) \in \mathcal{F}_b \) the following inequality in the region \( r \geq r_c \):
\[
(41) \quad B(\Delta/r^2)(\lambda_{mt} + a^2\omega^2)r^{-2} + Br^{-3} \geq V \geq (\Delta/r^2)b(\lambda_{mt} + a^2\omega^2)r^{-2} + b(\Delta/r^2)r^{-3},
\]
whereas, for all \( r > r_+ \), we have
\[
(42) \quad |V'(r^\ast)| \leq B(\Delta/r^2)(\lambda_{mt} + a^2\omega^2)r^{-3} + B(\Delta/r^2)r^{-4}.
\]
We are using here Proposition 7.1.3. Finally, again choosing \( a_0 \) sufficiently small, it follows that for all \( r \leq r_c \)
\[
(43) \quad V'(r^\ast) \geq b(\Delta/r^2)(\lambda_{mt} + a^2\omega^2),
\]
whereas there exists a constant \( R_6 \) such that for \( r \geq R_6 \),
\[
(44) \quad -V'(r^\ast) \geq (\Delta/r^2)b(\lambda_{mt} + a^2\omega^2)r^{-3} + b(\Delta/r^2)r^{-4}
\]
in this frequency range.

In the general case \( a_0 < M, |a| \leq a_0 \), but \( m = 0 \) (Theorem 1.2), the above four inequalities similarly hold for \( (\omega, m, \ell) \in \mathcal{F}_b \), with (41) holding in fact for all \( r > r_+ \), and with (43) holding in a region \( r \leq r_c \), where \( r_c = r_+ + s_c \) for some \( s_c \) depending only on \( a_0 \), with \( r_c \to 0 \) as \( a_0 \to M \). Let the choice of \( s_c \) and thus \( r_c(a) \) be now fixed.

We will now split the frequency range \( \mathcal{F}_b \) into two subcases, considering each separately.

9.3.1. The subrange \( |\omega| \leq \omega_3 \) (the near-stationary subcase). The motivation for the current to be constructed here is that in the Schwarzschild or \( m = 0 \) case, applying \( \mathcal{Q}^h \) with \( h = 1 \) immediately excludes nontrivial stationary solutions \( \omega = 0 \).

We will fix an \( \omega_3 > 0 \) which will be constrained in this subsection to be small. Because \( \omega_3 \) is not exactly 0 the naive current \( \mathcal{Q}^h \) with \( h = 1 \) must be modified.

We begin with a \( \mathcal{Y} \) current which will be defined with non-constant seed function \( h = h(r^\ast) \).

In the case of Theorem 1.1, the function \( h \) will be independent of the parameters \( \omega, m, \ell \), but will depend on \( a_0 \). In the case of Theorem 1.2 the choice of \( \omega_3 \) will potentially depend on \( a_0 \). Always remember that in this latter case, constants \( b, B \) will in general depend also on \( a_0 \), following our conventions.

Note first that given arbitrary \( a_0 < M \) and \( q > 0 \), \( p > 0 \), \( R_3 > 0 \) such that \( e^{-p-1}R_3 \) is sufficiently large and \( p \) sufficiently small, for each \( |a| \leq a_0 \), we can define a function \( h(r^\ast) \), such that the following hold: For \( r \leq r_c \),
\[
0 \leq h \leq R_3^{-2}, \quad |h''(r^\ast)| \leq q/|r^\ast|^2,
\]
\footnote{This latter dependence could easily be removed by altering the construction slightly. It arises for instance because the values of \( r_c^\ast \) depends on \( a \), even though \( r_c \) does not.}
with \( h(r^*) \) moreover of compact support when restricted to \(-\infty < r^* \leq r_c^* \), whereas for \( r_c \leq r \leq e^{-p^{-1}} R_3 \),

\[
h \geq R_3^{-2} \Delta/r^2, \quad h''(r^*) \leq 0,
\]

whereas for \( e^{-p^{-1}} R_3 \leq r \leq R_3 \),

\[
|h'(r^*)| \leq 4R_3^{-2}p/r, \quad |h''(r^*)| \leq 4R_3^{-2}p/r^2,
\]

whereas, finally, for \( r \geq R_3 \),

\[
h = 0.
\]

This \( h \) will be useful in view of the positivity of \( V_+ \). We note finally that \( h \) can be chosen so that \( h \) restricted to \( r \geq r_c \) is independent of the parameter \( q \).

Let us also consider a current \( \psi \) defined with \( y = y(r^*) \). Like \( h \), the function \( y \) will be independent of \( \omega, m, \) and \( f \) in the allowed range. Given the parameters \( R_3, p \), with \( e^{-p^{-1}} R_3 \) sufficiently large and \( p \) sufficiently small, the function will satisfy the following properties: We set \( y(r_c^*) = 0 \), and for \( r \leq r_c \),

\[
y'(r^*) = h,
\]

noting that \( y \) is bounded below in \( r \leq r_c \) by a potentially large negative constant depending on \( q \), in view of the fact that \( h \) is identically 0 for sufficiently low \( r^* \). For \( r_c \leq r \leq e^{-p^{-1}} R_3 \), we require

\[
y'(r^*) \geq 0, \quad (yV)'(r^*) \leq \frac{1}{2} R_3^{-2} \Delta/r^2,
\]

whereas for \( e^{-p^{-1}} R_3 \leq r \leq R_3 \)

\[
y'(r^*) \geq bR_3^{-2}, \quad -(yV)'(r^*) \geq bR_3^{-2}/r^2,
\]

whereas, finally, for \( r \geq R_3 \),

\[
y = 1.
\]

In general, for sufficiently large \( e^{-p^{-1}} R_3 \) and small \( p \), we can indeed construct such a \( y \) in view essentially of (41), (42) and (44). Let us add that \( y \) restricted to \( r \geq r_c \) can be chosen independently of the choice of \( h \).

Consider now the current \( Q^h + i \psi \). In \( r \leq r_c \), recall the one-sided bound

\[
yV' \geq 0
\]

which follows from (43). It follows that in \( r \leq r_c \),

\[
Q'(r^*) + i'(r^*) \geq -q|r^*|^{-2}|u|^2 + h\text{Re}(u\bar{H}) + 2y\text{Re}(u'\bar{H}).
\]

In the case of Theorem 1.1 for \( r_c \leq r \leq e^{-p^{-1}} R_3 \), choosing \( p \) appropriately, choosing \( \omega_3, a_0 \) sufficiently small

\[
Q' + i' \geq R_3^{-2}(\Delta/r^2)|u|^2 + bV_+|u|^2 + h\text{Re}(u\bar{H}) + 2y\text{Re}(u'\bar{H}).
\]

In the case of Theorem 1.2 given arbitrary \( a_0 < M \), one can again choose \( \omega_3 \) so that the above holds for all \( |a| \leq a_0 \).

Now in the case of both theorems, for \( e^{-p^{-1}} R_3 \leq r \leq R_3 \), we have for \( p, \omega_3 \) suitably small

\[
Q' + i' \geq bR_3^{-2}(\Delta/r^2)|u|^2 + h\text{Re}(u\bar{H}) + 2y\text{Re}(u'\bar{H}).
\]

Finally, for \( r \geq R_3 \) we have

\[
Q' + i' = i' \geq 2y\text{Re}(u'\bar{H}).
\]
We will choose in particular $R_3 \geq R$. We obtain finally for $r_\infty \geq R_3$, $r_{\infty} \leq r_c$,

\[
b(\lambda_1) \int_{r_c}^{R^*} \frac{1}{2} (\Delta/r^2)|u'|^2 + (\Delta/r^2) r^{-3} (1 + (\lambda_m + a^2 \omega_3^2)) |u|^2 \, dr^*
\]

\[
\leq \int_{r_{\infty}}^{r_c} q|r^*|^{-2}|u|^2 \, dr^*
\]

\[
+ \int_{r_{\infty}}^{r_c^*} \left(2y \text{Re}(u' \hat{H}) + h \text{Re}(\hat{H} u)\right) \, dr^*
\]

\[
+ \nabla^*(r_{\infty}^*) - (q + \nabla)(r_{\infty}^*).
\]

The above will be the prototype of the type of inequality we shall derive for all frequencies. We note that the bad dependence of the constant $b(\lambda_1)$ as $\lambda_1 \to \infty$ arises only from the bound $(\lambda_m + a^2 \omega_3^2) \leq \lambda_1 + a^2 \omega^2$ which we have used to introduce the term $(\lambda_m + a^2 \omega^2)$. We choose this particular combination because it is manifestly nonnegative (see Proposition 7.13) and will in particular bound the angular derivatives upon summation.

The constants $\omega_3$, $R_3$ and $p$ are now chosen, but not yet $q$. In accordance with our conventions, the constant $b$ in the inequality above is in particular independent of $q$. It is worth warning, however, that $y$ restricted to $r \leq r_c$ still depends on the choice of $q$, and that the boundary term $\lim_{r_{\infty} \to -\infty} \nabla^*(r_{\infty}^*) < \infty$, but diverges as $q \to 0$. We must thus be careful to absorb this boundary term properly.

9.3.2. The subrange $|\omega| \geq \omega_3$ (the non-stationary subcase). The construction of this section will yield a positive current for $\omega_3$ arbitrarily small and $\omega_1$, $\lambda_1$ arbitrarily large, if, in the case of Theorem 7.1, we are prepared to restrict to $|a| \leq a_0$, with $a_0$ depending on these choices. The choice of $\omega_3$ has in fact already been determined but $\omega_1$, $\lambda_1$ are determined later. The relevant constants arising grow as $\omega_3 \to 0$, $\omega_1 \to \infty$, $\lambda_1 \to \infty$ and we shall for now continue to track these dependences.

Let us note that in fact, the construction of this section is quite general and depends only on the asymptotic properties of $V$ (modulo ‘small’ terms), and not, in particular, on the sign of $V'$, which will be crucial in Section 9.5. We note that similar constructions have a long tradition in spectral theory and are typically used to prove continuity of the spectrum away from $\omega = 0$.

First, the idea: We search for a $\nabla^*$-current with seed function $y$. Recall that, dropping the terms arising from the cutoff, we have

\[
\nabla^*(r^*) = y'|u'|^2 + \omega^2 y'|u|^2 - (y V)'|u|^2.
\]

In both the Schwarzschild case and the $m = 0$ case we have that $V > 0$. If we restrict to $y$ with $y' > 0$ such that $y$ is moreover bounded at the ends, we have in these cases

\[
\int_{-\infty}^{\infty} (y V)'|u|^2 = -\int_{-\infty}^{\infty} y V (u\bar{u} + u' \bar{u}) \leq \frac{1}{2} \int_{-\infty}^{\infty} y'|u|^2 + 2 \int_{-\infty}^{\infty} (y^2 V^2 / y')|u|^2.
\]

Thus, to control the $(y V)'|u|^2$ term from the $y'|u|^2$ term, it suffices if

\[
4y^2 V^2 / y' \leq \omega^2 y'.
\]

If we assume in addition $y > 0$, we may rewrite this condition as

\[
y'/y \geq 2\omega^2 V
\]

which is ensured if we define, say, $y = e^{2\omega^2 V r^*}$, $\frac{1}{4} \int_{-\infty}^{r_*} V dr^*$. Note that in these two special cases, $y$ is well-defined and bounded in view of the asymptotics of $V$ as $r^* \to \pm \infty$.\]
We adapt the above heuristic to our situation. We shall argue somewhat differently, however. First of all, we do not want to choose \( y \) to depend on \( \omega \), and secondly, for technical reasons related to how we sum the terms arising from the inhomogeneity \( F \) (see Section 10.3), the above choice of \( y \) would not be sufficiently flat at infinity.

Given arbitrary \( a_0 < M \) in the case \( m = 0 \) (Theorem 1.2) or sufficiently small \( a_0 \) in the case of Theorem 1.1 and given in addition an arbitrary sufficiently small parameter \( \epsilon_2 > 0 \) and an arbitrary \( R_2 \geq R \), using the properties (41), (42) and (43), it follows that for all \( |a| \leq a_0 \), we may decompose \( V \) yet again as

\[
V = V_{r,\text{flat}} + V_{\text{junk}}
\]

where

\[
V_{r,\text{flat}} \geq 0
\]

for all \( r \geq r_+ \), and specifically,

\[
V_{r,\text{flat}} = 0, \quad V_{\text{junk}}' < 0
\]

for \( r \geq \epsilon_2^{-1} R_2 \), whereas

\[
V_{\text{junk}} = 0
\]

for \( R \leq r \leq R_2 \), whereas

\[
(46) \quad b(\Delta/r^2)r^{-3} \leq V_{r,\text{flat}} \leq B(\lambda_1 + a_0^2 \omega_1^2)(\Delta/r^2)r^{-2},
\]

\[
|V_{\text{junk}}| \leq B a_0(\lambda_1 + a_0^2 \omega_1^2)r^{-3}, \quad |V_{\text{junk}}'| \leq B a_0(\lambda_1 + a_0^2 \omega_1^2)(\Delta/r^2)r^{-4}
\]

for \( r \leq R_2 \), and whereas finally,

\[
|V_{\text{junk}}| \leq B(\lambda_1 + a_0^2 \omega_1^2)r^{-2}, \quad |V_{\text{junk}}'| \leq B \epsilon_2(\lambda_1 + a_0^2 \omega_1^2)r^{-2}
\]

for \( r \geq R_2 \). Note that the decomposition (45) in the region \( r \leq R \) can be chosen independently of the parameters \( \epsilon_2, R_2 \).

In the case of Theorem 1.2, we may further impose that in fact

\[
V_{\text{junk}} = 0
\]

for \( r \leq R_2 \).

Finally we may define \( V_{\text{ind}}(r^*) \geq 0 \) to be independent of \( \omega, m, \ell \) in this range such that

\[
B(\lambda_1 + a_0^2 \omega_1^2)(\Delta/r^2)r^{-2} \geq V_{\text{ind}} \geq V_{r,\text{flat}} \geq 0
\]

in \( r \leq \epsilon_2^{-1} R_2 \), whereas

\[
V_{\text{ind}} = 0
\]

for \( r \geq \epsilon_2^{-1} R_2 \). Let us also note that \( V_{\text{ind}} \) can be chosen independent of the parameters \( \epsilon_2, R_2 \) in the region \( r \leq R \) and that

\[
\int_{r^*}^{\infty} V_{\text{ind}}dr^* \geq b(\lambda_1 + a_0^2 \omega_1^2)r^{-1}
\]

for \( r^* \leq R^* \), where we are using (46) and the properties of the decompositions. In particular, there is no dependence on \( \epsilon_2, R_2, \omega_3 \) in the above inequality. We have on the other hand for all \( r^* > -\infty \),

\[
\int_{r^*}^{\infty} V_{\text{ind}}dr^* \leq B(\lambda_1 + a_0^2 \omega_1^2)
\]

and \( V_{\text{ind}} \) can be chosen such that there is no dependence of \( B \) on \( \epsilon_2, R_2 \).
Using (45) we may now write
\[ \begin{align*}
\gamma' &= y' |u'|^2 + \omega^2 y' |u|^2 - (yV)' |u|^2 + 2y \text{Re}(u' \bar{H}) \\
(48) &= y' |u'|^2 + \omega^2 y' |u|^2 - (yV_{+, \text{flat}})' |u|^2 - y' V_{\text{junk}} - yV_{\text{junk}}' + 2y \text{Re}(u' \bar{H}).
\end{align*} \]

Note that for general \( y \geq 0, \ y' > 0, \) we have
\[ \begin{align*}
\int_{r^* - \infty}^{r^*} (yV_{+, \text{flat}})' |u|^2 &= -\int_{r^* - \infty}^{r^*} yV_{+, \text{flat}}(u \bar{u}' + u' \bar{u}) + yV_{+, \text{flat}} |u|^2(r^* - \infty) - yV_{+, \text{flat}} |u|^2(r^* - \infty) \\
&\leq \frac{1}{2} \int_{r^* - \infty}^{r^*} y'|u'|^2 + 2(0 - \infty) (yV_{+, \text{flat}}/y)|u|^2 \\
&\leq \frac{1}{2} \int_{r^* - \infty}^{r^*} y'|u'|^2 + 2(0 - \infty) (yV_{\text{ind}}/y)|u|^2 \\
&\leq yV_{\text{ind}} |u|^2(r^* - \infty).
\end{align*} \]

We now define
\[ y = e^{-\omega_3^{-1} \int_{r^* - \infty}^{r^*} V_{\text{ind}} dr}. \]

Note that in this case, we have that
\[ y' = 2\omega_3^{-1} V_{\text{ind}} e^{-\omega_3^{-1} \int_{r^* - \infty}^{r^*} V_{\text{ind}} dr}. \]

We thus have for this choice of \( y \) that
\[ \begin{align*}
\int_{r^* - \infty}^{r^*} (yV_{+, \text{flat}})' |u|^2 &\leq \frac{1}{2} \int_{r^* - \infty}^{r^*} y'|u'|^2 + \frac{1}{2} \int_{r^* - \infty}^{r^*} \omega_3^2 y'|u|^2 \\
&\leq yV_{\text{ind}} |u|^2(r^* - \infty).
\end{align*} \]

We certainly have
\[ 0 \leq y \leq 1, \quad y' \geq 0 \]
for all \( r > r^* \). In \( r_c \leq r \leq R \), we have
\[ y'(r^*) \geq b(\Delta/r^2)r^{-3} e^{-2\omega_3^{-1} b r^{-1}(\lambda_1 + a_0 \omega_1^2)}, \]
while for general \( r \leq R \) we have
\[ y'(r^*) \leq B\omega_3^{-1}(\lambda_1 + a_0 \omega_1^2)(\Delta/r^2)r^{-2}. \]

For \( r \geq \epsilon_2^{-1} R_2 \), we have
\[ y = 1, \quad y' = 0. \]

Finally, in \( R_2 \leq r \leq \epsilon_2^{-1} R_2 \), we have
\[ 0 \leq y'(r^*) \leq B\omega_3^{-1}(\lambda_1 + a_0 \omega_1^2)r^{-2}. \]

Putting everything together, integrating (48), and in the case of Theorem 1.1 restricting to sufficiently small \( a_0 \) (depending on the final choices of \( \omega_1, \lambda_1 \) and \( \omega_3 \)),
we obtain finally for \( r_\infty \geq r_+^* \frac{1}{c_2} R_2, \ r_\infty \leq r_c \), the inequality:

\[
b(\omega_3, \omega_1, \lambda_1) \int_{r_c}^{r_\infty} (\Delta/r^2)|u'|^2 + (\Delta/r^2)(1 + (\lambda_1 + a_2^2 \omega_2) + \omega_2^2)|u|^2 \leq \int_{r_\infty}^{r_c} B_0(\Delta/r^2)|u|^2 + \int_{r_c}^{r_\infty} B(\epsilon_2 r^{-2} + r^{-3})|u|^2 + \int_{r_\infty}^{r_c} 2y \text{Re}(u' \bar{H}) + \gamma(r_\infty) + y V_{\text{ind}}|u|^2(r_\infty) - \gamma(r_\infty).
\]

As before, the integrals are with respect to \( dr^* \). Note that the first term on the right hand side above is absent in the case of Theorem 1.2, in view of (47). Note also that the final appeal to small integrals is with respect to \( dr^* \) and such that for all \( f \) satisfying

\[
f'(r^*) \geq 0
\]

for \( r > r_+ \),

\[
f'(r^*) \geq c_1
\]

for \( r_+ \leq r \leq R_6 \)

\[
f = 1
\]

for \( r \geq R_6 + 1 \), and near the horizon

\[
f \sim -1 + (r - r_+),
\]

and such that for all \( |a| \leq a_0 \) and all \( (\omega, m, \ell) \in \mathcal{F}_B \), we have

\[
f V' \leq c_2 V
\]

for \( r_{3M-} \leq r \leq r_{3M+} \), whereas

\[
f V' < 0
\]

for \( r \leq r_{3M-}, r \geq r_{3M+} \). Moreover, the constants \( r_{3M-} < 3M < r_{3M+} \) can be chosen arbitrarily close to \( 3M \) as \( a_0 \to 0 \).

In the case of Theorem 1.2 recalling that \( V > 0 \) and the asymptotics of \( V' \), there similarly exist such \( f, c_1, c_2, r_{3M-}, r_{3M+} \).

Given \( a_0 < M \), we choose constants

\[-\infty < r_{mp1}^* < r_{mp2}^* < r_{3M-}^* < r_{3M+}^* < r_{at1}^* < r_{at3}^* < \infty,
\]

such that \( f \leq -\frac{1}{2} \) for \( r \leq r_{mp2}, f \geq \frac{1}{2} \) for \( r \geq r_{at1} \), and we fix a function \( h \) such that

\[
h = 0 \text{ in } r \leq r_{mp1}, \ h = c_2 \text{ in } r_{mp2} < r_{at1}, \ h = 0 \text{ in } r \geq r_{at3}.
\]

Now we may choose \( \lambda_1 \) depending on \( \omega_1 \) sufficiently large so that in \( r \leq r_{mp2}, \ r \geq r_{at1} \)

\[
-\frac{1}{2} f V' \geq -\frac{1}{2} f''' - \frac{1}{2} h''
\]
in the $\mathcal{F}_A$ range. Note that this is possible in view of the asymptotics of $f$ and $V$ and the choice of large $\lambda_1$, no matter what the details of the choices of $f$, $h$.

It now follows by construction that the current $Q + \mathcal{Q}$ satisfies

$$Q' + \mathcal{Q}' \geq 2f'[u] r^2 + b(\Delta/r^2) r^{-3} (\lambda_m \ell + a^2 \omega^2) |u|^2 + \Re(2f'H u' + f'H u) + h \Re(uH).$$

We obtain thus in particular (for $r_\infty^* < r_{m_1}$, $r_\infty^* \geq R^*$)

$$\int_{r_\infty^*}^{R^*} b|u|^2 + b(\Delta/r^2) r^{-3} (1 + (\lambda_m \ell + a^2 \omega^2 + \omega^2)|u|^2$$

$$\leq \int_{r_\infty^*}^{r_\infty^*} 2f \Re(u' \bar{H}) + (f' + h) \Re(\bar{H} u)$$

$$+ (Q + \mathcal{Q})(r_\infty^*) - Q(r_\infty^*).$$

(We note finally that an alternative approach to the construction of a current for this range would be considerations similar to Section 9.3.1.)

9.5. The $\mathcal{F}_h$ range (trapped frequencies). This is the frequency range of trapping. Here the current seed function will in general depend on $\omega$, $m$, $\ell$, and $a$. The dependence is smooth in $\omega$ and $a$.

Given an arbitrary choice of $\lambda_2$, this section will require $\omega_1$ to be sufficiently large (specifically, $1 \ll \lambda_2 \omega_1^2$). Note that this constraint on the largeness becomes tighter in the limit $\lambda_2 \to 0$. The frequency parameter $\lambda_2$ will be chosen in Section 9.6.

For $(\omega, m, \ell) \in \mathcal{F}_h$, we have

$$(\lambda_m \ell + \omega^2 a^2) \geq (\lambda_2 + a^2) \omega^2 \geq (\lambda_2 + a^2) \omega_1^2.$$

We set

$$V_0 = (\lambda_m \ell + \omega^2 a^2) \frac{r^2 - 2Mr + a^2}{(r^2 + a^2)^2}$$

so that

$$V_1 = V - V_0 = \frac{4Mr \omega - 2a^2 m^2}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4}.$$  

We easily see that

$$r^3 |V'| + \left(\frac{(a^2 + \omega^2)}{\Delta r^2} - V_0 '\right) r^3 \leq B \Delta r^{-2} (a^2 m^2 + |am| + 1).$$

On the other hand

$$V_0'(r_*) = 2 \frac{\Delta}{(r^2 + a^2)^4} (\lambda_m \ell + \omega^2 a^2) \left((r - M)(r^2 + a^2) - 2(r^2 - 2Mr + a^2)\right)$$

$$= -2 \frac{\Delta^2}{(r^2 + a^2)^4} \left(3r^2 + a^2 - 3M + M \frac{r^2}{r^2}\right).$$

We notice that for all $|a| < M$, the function $s(r) = r^3 - 3Mr^2 + 2a^2 r + a^2 M$, and thus $V_0'$, has a unique simple zero on $(r_+, \infty)$, which we denote $z_{a,M}$. To see this, in view of the fact that $s'(r)$ has at most two zeros, it suffices to remark that $\lim_{r \to \infty} s(r) = \infty$ and,

$$s(r_+) = -M(r_+^2 - a^2) < 0, \quad \frac{d}{dr}s(r_+) = 3r_+^2 - 6Mr_+^2 + a^2 = -2a^2 < 0.$$

We easily see moreover that $|z_{a,M} - 3M| \leq Ba^2$. 
Given any \( r \)-neighborhood of \( z_{a,M} \), it follows from the inequality (51) (applied with the first term on the left hand side in mind) and the identity (52), taking into account also the asymptotics of \( V'_0 \) as \( r \to r_+ \), \( r \to \infty \), and using Proposition 7.1.3 and the smallness of \( a_0 \) in the case of Theorem 1.1 and the vanishing of \( m \) in the case of Theorem 1.2 that, given arbitrary \( \lambda_2 \), then for sufficiently large \( \omega_1 \) (depending on \( \lambda_2 \)), so that the right hand side of (50) is sufficiently large, for frequencies in \( \mathcal{F}_h \), \( V' \) has at least one zero in this neighborhood, and no zeros outside this neighborhood.

To see now that \( V' \) has in fact a unique simple zero in this neighborhood, we note first that there exists a positive \( c = c(a, M) \) such that

\[
\left( \frac{(r^2 + a^2)^4}{\Delta r^2} V'_0 \right)' (z_{a,M}) \leq -c \Delta r^{-2}(\lambda_{m\ell} + \omega^2 a^2).
\]

To see this, we compute

\[
\left( \frac{(r^2 + a^2)^4}{\Delta r^2} V'_0 \right)' (z_{a,M}) = -2 \Delta z_{a,M}^{-2} (\lambda_{m\ell} + a^2 \omega^2) \left( 1 - \frac{a^2}{z_{a,M}^2} - \frac{2Ma^2}{z_{a,M}^3} \right).
\]

Since \( s(z_{a,M}) = 0 \), we have

\[
1 - \frac{a^2}{z_{a,M}^2} - \frac{2Ma^2}{z_{a,M}^3} = 2 + a^2 \frac{2}{z_{a,M}^2} - 3Mz_{a,M} = z_{a,M}^{-2} (2z_{a,M}^2 - 3Ma^2) \geq 1 - \frac{M}{z_{a,M}} > 0,
\]

which yields (53).

From (53) and (51) (now with the second term on the left hand side in mind), it follows that given \( \lambda_2 \) arbitrary, then for \( \omega_1 \) sufficiently large (depending on \( \lambda_2 \)), and, in the case of Theorem 1.1 for \( |a| \leq a_0 \) with \( a_0 \) sufficiently small, we have for frequencies in \( \mathcal{F}_h \):

\[
\left( \frac{(r^2 + a^2)^4}{\Delta r^2} V' \right)' \leq -(c/2) \Delta r^{-2}(\lambda_{m\ell} + \omega^2 a^2)
\]

in a neighborhood of \( z_{a,M} \) as described previously containing any zero of \( V' \). (We have used here Proposition 7.1.3 to estimate \( a^2m^2 \) from \( a^2(\lambda_{m\ell} + a^2 \omega^2) \), and it is here that the smallness of \( a_0 \) is relevant. In the case of Theorem 1.2 \( m = 0 \), and we need not restrict to small \( a_0 \).)

The relation (54) in the neighborhood of possible zeros shows now that \( V' \) has exactly one simple zero, in fact:

**Proposition 9.5.1.** Under the assumptions of Theorem 1.1 or 1.2, then for \( \omega_1 \) sufficiently large (depending on \( \lambda_2 \)), \( V' \) has a unique zero \( r_{m\ell}^{(0)} \) for \( (\omega, m, \ell) \in \mathcal{F}_h \) depending smoothly on the parameters. For fixed \( m, \omega \),

\[
\lim_{\ell \to \infty} r_{m\ell}^{(0)} = z_{a,M}.
\]

There exist \( r_{m\ell}^- (a_0, M, \omega_1, \lambda_2) < r_{m\ell}^+(a_0, M, \omega_1, \lambda_2) \) with \( r_{m\ell}^+ \to 3M \) as \( a_0 \to 0 \) such that

\[
-V'(r^*) \geq b \chi_3 (r^* - ((r_{m\ell}^-)^*) + (r_{m\ell}^+)^*)/2 \Delta r^2 (\lambda_{m\ell} \omega_1^2 + a\omega_1^2)
\]

for \( r \leq r_{m\ell}^- \), and

\[
V'(r^*) \geq b \chi_3 (r^* - ((r_{m\ell}^-)^*) + (r_{m\ell}^+)^*)/2 r^{-3} (\lambda_{m\ell} \omega_1^2 + a\omega_1^2)
\]

for \( r \geq r_{m\ell}^+ \), where \( \chi_3 \) is a fixed function such that \( \chi_3(x) = |x| \) for \( |x| \leq .5 \), \( \chi_3(x) \geq .5 \) for \( |x| \geq .5 \) and \( \chi_3 = 1 \) for \( |x| = 1 \).
In general, limit points of the collection \( r_m^{(a\omega)} \) correspond to \( r \)-values which admit trapped null geodesics. Let us note moreover that as \( a_0 \to M \), we have \( r^{-}_M \to M = r_{+}(M, M) \).

Note that we have included the \( \chi_3(r^{*} - ((r^{-}_M)^* + (r^{+}_M)^*)/2) \) merely so that our estimates do not degenerate in the limit \( a_0 \to 0 \), in accordance with our convention that \( b, B \) can be independent of \( a \) for Theorem 1.1

One can now clearly construct functions \( f \) whose properties are summarised in the proposition below:

**Proposition 9.5.2.** Let \( \lambda_2 \) be given, \( \omega \) be sufficiently large (depending on \( \lambda_2 \)), and, in the case of Theorem 1.1, let \( |a| \leq a_0 \) for sufficiently small \( a_0 \), whereas, in the case of Theorem 1.2, let \( |a| \leq a_0 \) for arbitrary \( a_0 < M \). Then, for each \( (m, \ell, \omega) \in \mathcal{F}_h \), there exists a function \( f = f_m^{(a\omega)}(r^*, a) \) depending smoothly on \( a\omega \) and \( a \) such that

1. \( f(r^*) \geq 0 \) for all \( r^* \), and \( f' \geq b(\Delta/r^2)r^{-2} > 0 \) for \( r_\ell \leq r \leq R \)
2. \( f < 0 \) for \( r < r_m^{(a\omega)} \) and \( f > 0 \) for \( r > r_m^{(a\omega)} \),
3. \( -f'' - \frac{1}{2}f''' \geq b(\Delta/r^2)r^{-3} > 0 \)
4. \( fV'' \geq b\chi_3^{(2)}(r^{*} - ((r^{-}_M)^* + (r^{+}_M)^*)/2)(\Delta/r^2)(\lambda_2\omega_1^2 + a^2\omega_2^2) \) in \( r \leq r^{-}_M \), \( fV'' \geq b\chi_3^{(2)}(r^{*} - ((r^{-}_M)^* + (r^{+}_M)^*)/2)r^{-3}(\lambda_2\omega_1^2 + a^2\omega_2^2) \) in \( r \geq r^{+}_M \).
5. \( \lim_{r^* \to -\infty} f(r^*) = -1 \),
6. \( f(r^*) \geq R_4^*, f = 1, \) for some \( R_4 \).

As always, the convention is that \( b \) depends only on \( M \), and, in the case of Theorem 1.2 also on \( a_0 \).

We apply now the energy identity corresponding to the current \( Q^f \). We obtain, for \( r^{*}_\infty < r^{*}_c, r^{*}_\infty > R^{*} \),

\[
b \int_{r^{*}_\infty}^{R^{*}} (\Delta/r^2)r^{-2}|u'|^2 + (\Delta/r^2)r^{-2}|u|^2 + \frac{\Delta}{r^2}r^{-3}\chi_3^{(2)}(r^{*} - ((r^{-}_M)^* + (r^{+}_M)^*)/2)(1 - \chi_{[r^{-}_M, r^{+}_M]}(\lambda_m^{\ell} + a^2\omega^2 + \omega^2)|u|^2 \\
\leq \int_{r^{*}_c}^{r^{*}_\infty} 2f\text{Re}(u'\bar{H}) + f'\text{Re}(\bar{H}u) + Q(r^*) - Q(r^{*}_\infty),
\]

where \( \chi_{[r^{-}_M, r^{+}_M]} \) denotes the indicator function of \( [r^{-}_M, r^{+}_M] \).

In essentially replacing \( f \) by \( (1 - \chi_{[r^{-}_M, r^{+}_M]})\chi_3(r^{*} - ((r^{-}_M)^* + (r^{+}_M)^*)/2) \), we have thrown away some information. This is because this extra positivity cannot be characterized by a differential operator after summation, whereas for convenience, we have stated our main theorem as a classical integrated energy estimate. For a more refined “pseudodifferential” statement, one merely should retain the frequency dependent \( f \) multiplying the term \( (\lambda_m^{\ell} + a^2\omega^2) + \omega^2 \). It is this sum which is proven to be bounded.

It is worth adding here that in the case \( m = 0 \), or alternatively, in the Schwarzschild case \( a = 0 \), by a slight variant of the above construction, one in fact could have chosen \( f \) independently of \( \omega, m, \ell \), centred always at \( z_{a,M} \). Cf. the construction of Section 4.1.1 of [36] for Schwarzschild.
9.6. The $\mathcal{F}_0$ range (time-dominated frequencies). First some general facts: Note that for all $|a| < M$, we have

\begin{equation}
\frac{\Delta a^2}{(r^2 + a^2)^2} < c < 1
\end{equation}

for all $r > r_*$. For small enough $\lambda_2$ and large enough $\omega$, we have, using also (55) and Proposition 7.1.3 say

$$\omega^2 - V \geq \frac{1 - c}{2} \omega^2,$$

for frequencies in $\mathcal{F}_0$. Finally, there exists an $R_5$ such that, for all $|a| < M$ and all frequencies (not just in $\mathcal{F}_0$), we have $V'(r) < 0$ for $r \geq R_5$. We may take $R_5 \geq R$.

We shall define here a $\gamma'$-current.

In the case of Theorem 1.2, we note that under the decomposition $V = V_0 + V_1$ of the previous section, since $m = 0$, then $V_1$ is independent of $\omega$, and say

$$|V_1'| \leq C\Delta/r^2 r^{-3}$$

in $\mathcal{F}_0$, for $\omega_1$ sufficiently large. $V = V_*$. Recalling $z_{a,M}$ from the previous section, we may now choose, for $\omega_1$ sufficiently large, a $y = y(r^*)$ such that $y' \geq 0$ everywhere, $y(z_{a,M}) = 0$, $y' \geq 2C \omega_1^{-1} \Delta/r^2 r^{-2}$ in $(r_*, R_5]$ and $y = 1$ for $y \geq R_5 + 1$ say, without loss of generality we have selected $R_5$ such that also $V_1' \leq 0$ for $r \geq R_5$.

In the case of Theorem 1.1 one does not in general have a unique vanishing point for $V_1'$ in this frequency range. Let us note, however, that for $(\omega, m, \ell) \in \mathcal{F}_0$, we have

$$|V'| \leq B(\Delta/r^2) r^{-3}((\lambda_{m\ell} + a^2 \omega^2) + 1).$$

We may now define a function $y$ such that say $y \geq 0$, $y' \geq B\Delta/r^2 r^{-2}$ in $(r_*, R_5]$ and $y = 1$ for $y \geq R_5 + 1$.

We apply now (37) with $\gamma'$. For $r \leq r_c$, we have that

$$\gamma' \geq 2y\text{Re}(u' \bar{H})$$

where, in the case of Theorem 1.1, we must possibly further restrict $\omega_1$ to be large.

On the other hand, for $r_c \leq r \leq R_5$, we have, choosing $\omega_1$, sufficiently large, $\lambda_2$ sufficiently small, and in the case of Theorem 1.1 choosing $a_0$ sufficiently small,

$$\gamma' \geq b(\Delta/r^2) r^{-2}((\omega^2 + (\lambda_{m\ell} + a^2 \omega^2) + 1)|u|^2 + 2y\text{Re}(u' \bar{H})].$$

Finally, for $r \geq R_5$, we have

$$\gamma' \geq 2y\text{Re}(u' \bar{H}).$$

In view of the inequality $R_5 \geq R$, we obtain thus for $r_*^+ > R^*$, $r_-^+ < r_*$

$$\int_{r_*^+}^{R^*} b\Delta/r^2 r^{-2} \left(|u|^2 + (\omega^2 + (\lambda_{m\ell} + a^2 \omega^2) + 1)|u|^2\right)$$

\begin{align*}
\leq & \int_{r_-^+}^{r_*^+} 2y\text{Re}(u' \bar{H}) \\
+ & \gamma'_0(r_*^+ - \gamma'_0(r_-^+)).
\end{align*}
10. Summing

We now wish to reinstate the dropped indices \( m, \ell, a, \omega \). For all \( r_{\pm}^* \), we have obtained an identity which we may write as

\[
\int_{r_c^*}^{R^*} M_{m\ell}^{(a\omega)} \, dr^* \leq \left( \int_{r_{-\infty}^*}^{r_c^*} \right) \left( \int_{r_{-\infty}^*}^{R^*} \right) E_{m\ell}^{(a\omega)} \, dr^*
+ \int_{r_{-\infty}^*}^{R^*} C_{m\ell}^{(a\omega)} \, dr^*
+ B_{m\ell}^{(a\omega)}(r_{-\infty}^*) - B_{m\ell}^{(a\omega)}(r_{-\infty}^*).
\]

Here \( C \) represents the terms containing the inhomogeneous term \( F \) arising from the cutoff. Summing over \( m, \ell \) and integrating over \( \omega \), we obtain

\[
\int_{-\infty}^{\infty} \int_{r_c^*}^{R^*} \sum_{m\ell} M_{m\ell}^{(a\omega)} \, d\omega \, dr^* \leq \int_{-\infty}^{\infty} \left( \int_{r_{-\infty}^*}^{r_c^*} \right) \left( \int_{r_{-\infty}^*}^{R^*} \right) E_{m\ell}^{(a\omega)} \, d\omega \, dr^*
+ \int_{-\infty}^{\infty} \int_{r_{-\infty}^*}^{R^*} \sum_{m\ell} C_{m\ell}^{(a\omega)} \, d\omega \, dr^*
+ \int_{-\infty}^{\infty} \sum_{m\ell} \left( B_{m\ell}^{(a\omega)}(r_{-\infty}^*) - B_{m\ell}^{(a\omega)}(r_{-\infty}^*) \right) \, d\omega,
\]

where we have used the regularity properties of the relevant functions to interchange \( \int_{-\infty}^{r_c^*} \) with \( \sum_{m\ell} \).

We thus have that the liminf of the left hand side as \( r_{-\infty}^* \to \pm \infty \) is less than equal to the limsup of the right hand side. This will be the main inequality.

Let us note finally that we may now consider the entirety of our frequency parameters \( (\omega_1, \lambda_1, \lambda_2, \omega_3) \) to have been chosen, with the choices depending only on \( M \), and, in the case of Theorem 1.2, possibly \( a_0 \). Thus, constants which depend only on these may in what follows be denoted simply by \( B, b \).

10.1. The main term. For all \( r_{-\infty}^* \) sufficiently negative and \( r_{+\infty}^* \) sufficiently positive we have in view of the properties of Section 7.3 that

\[
b \int_{\{r, t \leq R\}} r^{-3} \left( (\partial_r \psi \overline{\psi})^2 + \overline{\psi}^2 \right)
+ r^{-3} (1 - \chi_{[r, r_{+\infty}^*)}) \lambda^2 \left( (r_{-\infty}^*)^2 + (r_{+\infty}^*)^2 \right) \left( (\partial_t \psi \overline{\psi})^2 + |\overline{\psi}|^2 \right)
\leq \int_{r_{-\infty}^*}^{r_{+\infty}^*} \sum_{m\ell} M_{m\ell}^{(a\omega)} \, d\omega \, dr^*.
\]

In accordance with our conventions, the integral on the left is now with respect to the volume form. In particular, the inequality holds for the liminf of the right
hand side. We obtain immediately
\[ b \int_{[0 \lesssim t \lesssim r']} \int_{(r \leq r')} r^{-3} ((\partial_t \psi)^2 + \psi^2) \]
\[ + r^{-3} (1 - \chi_{[r_0^{-}, r_0^{+}]}(r^*)^2 (r^* - (r^0)^{+}) (r^0)^{+} / 2) ((\partial_t \psi)^2 + |\nabla \psi|_{\gamma})^2 \]
\[ \leq \liminf_{r_s \to \infty} \int_{r_s}^\infty \int_{r_s}^\infty \sum_{m \leq t} M_{m \ell}^{(a \omega)} d\omega dr \]
\[ + B(r_c, R) \varepsilon^{-1} \int_{\{t = r - \varepsilon, r < -1\}} J_{\mu}^T [\psi] \eta_{t = 0} \]
\[ + B(r_c, R) \varepsilon^{-1} \int_{\{t = 0\}} J_{\mu}^T [\psi] \eta_{t = 0} \tag{57} \]
where the last term arises because the domain of integration on the left hand side
includes \([0, \varepsilon^{-1}]\) and \([r' - \varepsilon^{-1}, r']\), and (in the case of Theorem 1.1) we have appealed
to (the first inequality of Proposition 5.3.2) (In the case of Theorem 1.2 we may
define \(c_0\) and \(\varepsilon\) to be 1 and appeal to Theorem 4.1)
\[ \text{10.2. \textbf{Error terms near the horizon and infinity}. We have} \]
\[ \int_{-\infty}^\infty \int_{r_s}^\infty \sum_{m \leq t} E_{m \ell}^{(a \omega)} d\omega dr \leq B \cdot (a_0 + q) \int_{(r \leq r') \cap (0 \lesssim t \lesssim r')} J_{\mu}^N [\psi \chi] \eta_{t = 0} \]
\[ + \|\log (r - r_0^+)\|^{-2} (r - r_0^+)^{-1} \psi_{\infty}^2 \]
\[ \leq B \cdot (a_0 + q) \int_{(r \leq r') \cap (0 \lesssim t \lesssim r')} J_{\mu}^N [\psi] \eta_{t = 0} \]
\[ + \|\log (r - r_0^+)\|^{-2} (r - r_0^+)^{-1} \psi_{\infty}^2 \tag{58} \]
Here \(q\) is the parameter of Section 9.3.1 which remains to be chosen. In the case
of Theorem 1.2 this estimate holds without the \(a_0\) term.

On the other hand, we have
\[ \int_{-\infty}^\infty \int_{R^*}^\infty \sum_{m \leq t} E_{m \ell}^{(a \omega)} d\omega dr \leq B \int_{(r_2 \leq r \leq r_2 + R_2) \cap (0 \lesssim t \lesssim r')} (\varepsilon_2 r^{-2} + r^{-3}) J_{\mu}^T [\psi \chi] T^\mu \]
\[ \leq B \delta \varepsilon^{-1} (\varepsilon_2 + R_2^{-1}) \int_{(t = 0)} J_{\mu}^T [\psi] \eta_{t = 0} + \int_{(t = r - \varepsilon, r')} J_{\mu}^T [\psi \chi] \eta_{t = 0} \]
\[ + B \delta (\varepsilon_2 R_2^{-1} + R_2^{-1}) \int_{(r_2 \leq r) \cap (0 \lesssim t \lesssim r')} T_{\mu} \eta_{t = 0} \tag{59} \]
where the spatial integrals arise in estimating the 0' th order term (via Hardy)
which arises from the band where \(\psi \) and \(\psi_{\infty}\) do not coincide. We have used the
first inequality of Proposition 5.3.2 in the case of Theorem 1.1 while we have simply
appealed to Theorem 4.1 in the case of Theorem 1.2 taking in the latter case \(c_0 = 1\).

One should perhaps note in advance that these terms will be absorbed in Section
10.5 by the main term of Section 10.1 after appealing to Propositions 5.2.1, 5.3.1
and 6.1 and choosing the parameters accordingly.

\[ \text{10.3. \textbf{Error terms from the cutoff}. These are the terms containing} \ H. \]

Let \(R_7 \geq R\) be such that for \(r \geq R_7\), the current seed functions satisfy \(y = f = 1\)
for all frequencies. We may take \(R_7\) to be specifically:
\[ R_7 = \max \{R_3, R_2, R_4, R_5 + 1, R_5 + 2, R_6 + 1, R_5 + 1\}. \]
Note that all these parameters have already been selected with the exception of $\epsilon_2$, $R_2$.

We now split the error into two parts:

$$
\int_{-\infty}^{\infty} \int_{r_0}^{\infty} \sum_{m\ell} C_{m\ell}^{(aw)} \, d\omega \, dr^* = \int_{-\infty}^{\infty} \int_{R_7}^{\infty} \sum_{m\ell} C_{m\ell}^{(aw)} \, d\omega \, dr^* + \int_{-\infty}^{\infty} \int_{r_0}^{R_7} \sum_{m\ell} C_{m\ell}^{(aw)} \, d\omega \, dr^*.
$$

(61)

The integrand in $r^*$ of the second term of (61) can be written (where we have used the properties of Section 7.3)

$$
\int_{-\infty}^{\infty} \sum_{m,\ell} e_{m\ell}^{(aw)} (r) \Re(F_{m\ell}^{(aw)} (r) \Psi_{m\ell}^{(aw)} (r)) + d_{m\ell}^{(aw)} (r) \Re(F_{m\ell}^{(aw)} (r) (\partial_r \Psi_{m\ell}^{(aw)} (r)))\, d\omega
$$

$$
\leq \int_{-\infty}^{\infty} \sum_{m,\ell} \epsilon_3^{-1} (\Delta/r^2) r^2 v(r) |F_{m\ell}^{(aw)}|^2 (r) + \epsilon_3 (\Delta/r^2) r^2 (r^{-3} |\Psi_{m\ell}^{(aw)}|^2 + r^{-3} |(\partial_r \Psi_{m\ell}^{(aw)})|^2) \, d\omega
$$

$$
= \epsilon_3^{-1} (\Delta/r^2) r^2 v(r) \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F^2 \sin\theta \, d\phi \, d\theta \, dt + \epsilon_3 (\Delta/r^2) r^2 \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} \psi^2 + (\partial_r \psi)^2 \sin\theta \, d\phi \, d\theta \, dt
$$

$$
\leq \epsilon_3^{-1} B(\Delta/r^2) v(r) r^2 \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F^2 \sin\theta \, d\phi \, d\theta \, dt + B\epsilon_3 (\Delta/r^2) r^2 \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} \psi^2 + (\partial_r \psi)^2 \sin\theta \, d\phi \, d\theta \, dt,
$$

where the $\partial_r$ derivative is still in ($t, r, \theta, \phi$) coordinates, where $\epsilon_3 > 0$ can be chosen arbitrarily, and where $v(r)$ is a nonnegative function depending on the choice of $R_7$ and $q$, with

$$
\sup_{r \geq r_0} v(r) \leq B(R_7, q).
$$

The reader should note that unfavourable powers of $r$ have all been incorporated in the definition of $v$ in view of the fact that in the domain of integration $r \leq R_7$.

Integrating thus with respect to $r^*$, and recalling our comments concerning the volume form in Section 2.9, and using the estimates for $F$ in Section 8 we obtain

$$
\int_{-\infty}^{\infty} \int_{r_0}^{R_7} \sum_{m\ell} C_{m\ell}^{(aw)} \, d\omega \, dr^* \leq B(R_7, q) \epsilon_3^{-1} \epsilon \left( \int_{\{0 \leq t^* \leq t^*-\epsilon\} \cap \{r \leq R_7\}} + \int_{\{t^* \geq t^*-\epsilon\} \cap \{r \leq R_7\}} \psi^2 + J_\mu^{N} [\psi]^N \right)
$$

$$
+ B\epsilon_3 \int_{\{0 \leq t^* \leq t^*\} \cap \{r \leq R_7\}} r^{-3} (\psi^2 + (\partial_r \psi)^2)
$$

$$
\leq B(R_7, q) \epsilon_3^{-1} \epsilon \left( \int_{\{t^* = 0\} \cap \{r \leq R_7\}} + \int_{\{t^* \geq t^*-\epsilon\} \cap \{r \leq R_7\}} \right) J_\mu^{N} [\psi]^N
$$

$$
+ B\epsilon_3 \int_{\{0 \leq t^* \leq t^*\} \cap \{r \leq R_7\}} r^{-3} (\psi^2 + (\partial_r \psi)^2)
$$

(62)

We have appealed in the above to the first inequality of Proposition 5.3.2. Note that it is $J^N$ (equivalently $J^{T+N}$) and not $J^{T+\epsilon_0 N}$ which appeared above, and this is why it is essential that we have the extra smallness parameter $\epsilon$, arising from the estimate of $F$. 

We turn to the first term of (64). In view of the fact that \( f(r) = 1 \) (or \( y(r) = 1 \)) is independent of \( \omega, m, \ell \), we have in fact that this term equals precisely

\[
\int_{R_7^*}^{r_1^*} \int_{-\infty}^{\infty} \sum_{m, \ell} \Re((r^2 + a^2)^{1/2}\Psi_{m, \ell}^{(\omega)})(\Delta(r^2 + a^2)^{-1/2}F_{m, \ell}^{(\omega)})d\omega dr^*
\]

\[
= \int_0^1 \int_{R_7^*}^{r_1^*} \int_{-\infty}^{\infty} \partial_{\nu^*}(\omega,r^2 + a^2)^{1/2}\psi_{\nu^*}) (r^2 + a^2)^{-1/2}F \sin \theta d\phi d\theta dt^* dr^*,
\]

(63)

where we have here used the properties of Section 7.3. This integral is supported only in \( \{0 \leq t^* \leq \epsilon^{-1}\} \cup \{\tau' - \epsilon^{-1} \leq t^* \leq \tau'\} \cap \{r \geq R_7\} \). Recalling that

\[
F = 2\nabla^\alpha \xi_{\tau', \epsilon} \nabla_\alpha \psi + (\Box_g \xi_{\tau', \epsilon}) \psi,
\]

and noting that \( \partial_{\nu^*}\psi_{\nu^*} = \xi_{\tau', \epsilon} \partial_{\nu^*}\psi \) for sufficiently large \( r \), we see immediately that the contribution to (63) of the first term of (64) can be controlled (using the first inequality of Proposition 5.3.2 and a Hardy inequality) by

\[
B \left( \int_{\{t^* = 0\}} + \int_{\{t^* = \tau' - \epsilon^{-1}\}} \right) \mathbf{J}^T_{\mu} \mathbf{e}_N[\psi] n^\mu.
\]

(65)

One should note in the estimate above that the \( \epsilon^{-1} \) factor coming from the integral in time cancels the \( \epsilon \) factor appearing in the estimate for \( F \).

We also note that \( \Delta(r^2 + a^2)^{-1/2} \) differs from \( (r^2 + a^2)^{1/2} \) by terms which are lower order in \( r \) and can as above be bounded by (65). We are thus left to bound

\[
\int_0^1 \int_0^1 \int_{-\infty}^{\infty} (\partial_{\nu^*}(r^2 + a^2)^{1/2}\xi_{\tau^*, \epsilon}) (r^2 + a^2)^{1/2}G_{\partial\xi_{\tau^*, \epsilon}} \psi \sin \theta d\phi d\theta dt^* dr^*,
\]

which we may rewrite as

\[
\int_0^1 \int_0^1 \int_{-\infty}^{\infty} (\partial_{\nu^*}(r^2 + a^2)^{1/2}\xi_{\tau^*, \epsilon}) (r^2 + a^2)^{1/2}\xi_{\tau^*, \epsilon} G_{\partial\xi_{\tau^*, \epsilon}} \psi \sin \theta d\phi d\theta dt^* dr^*.
\]

It follows that we may integrate the above by parts with respect to \( r^* \) say, to yield a boundary term supported only in \( \{0 \leq t^* \leq \epsilon^{-1}\} \cup \{\tau' - \epsilon^{-1} \leq t^* \leq \tau'\} \cap \{r \geq R_7\} \), easily bound by (65), a vanishing boundary term at infinity for sufficiently large \( r^* \) in view of the compactness of the support, and finally a term

\[
\frac{1}{2} \int_0^1 \int_0^1 \int_{-\infty}^{\infty} (\partial_{\nu^*}(r^2 + a^2)^{1/2}\xi_{\tau^*, \epsilon})^2 (\xi_{\tau^*, \epsilon} G_{\partial\xi_{\tau^*, \epsilon}} \psi) \sin \theta d\phi d\theta dt^* dr^*,
\]

which, in view of the bound in \( r \geq R_7 \):

\[
|\partial_{\nu^*} \xi G_{\partial\xi_{\tau^*, \epsilon}}| + |\xi_{\tau^*, \epsilon} G_{\partial\xi_{\tau^*, \epsilon}}| \leq B \epsilon r^{-2}
\]

is also easily bound by (65). (In fact the above bound holds with \( \epsilon^2 \) but this is not here necessary.)

Thus we have obtained that

\[
\lim_{r^*_1, \epsilon \to +\infty} \lim_{r^*_1, \epsilon \to +\infty} \int_0^1 \int_{-\infty}^{\infty} \sum_{m, \ell} \mathbf{C}^{(\omega)}_{m, \ell} d\omega dr^* \leq B \left( \int_{\{t^* = 0\}} + \int_{\{t^* = \tau' - \epsilon^{-1}\}} \right) \mathbf{J}^T_{\mu} \mathbf{e}_N[\psi] n^\mu.
\]

(66)
10.4. **Boundary terms.** By the properties of Section 7.3 we obtain the estimate
\[
\int_{-\infty}^{\infty} \sum_{m,l} B^{(\alpha\omega)}_{m,l}(r_{+\infty}) d\omega \\
\leq B(q) \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} \left( (\partial_\tau \psi_r)^2 + (\partial_r \psi_r)^2 + |a(\partial_\phi \psi_r)|^2 \right) (r_{+\infty}) \sin \theta \, dt \, d\phi \, d\theta \\
+ B(q)(|r_{+\infty}| + 1)^{-1} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} (\psi_r^2 + |\psi_\phi|^2)(r_{+\infty}) \sin \theta \, dt \, d\phi \, d\theta.
\]
Note that we have absorbed factors of \(r\) in \(B\). The factor \((|r_{+\infty}| + 1)^{-1}\) could in fact be replaced by an exponential.

Taking the limit supremum as \(r_{+\infty} \to -\infty\), we have that the second term on the right side tends to 0, while the integral in the first term has a well-defined limit which can be expressed as an integral on the event horizon. We have thus
\[
\limsup_{r_{+\infty} \to -\infty} \int_{-\infty}^{\infty} \sum_{m,l} B^{(\alpha\omega)}_{m,l}(r_{+\infty}) d\omega \\
\leq B(q) \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} \left( (\partial_\tau \psi_r)^2 + (\partial_r \psi_r)^2 + |a(\partial_\phi \psi_r)|^2 \right) (r_{+\infty}) \sin \theta \, dt \, d\phi \, d\theta \\
+ B(q)(|r_{+\infty}| + 1)^{-1} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} (\psi_r^2 + |\psi_\phi|^2)(r_{+\infty}) \sin \theta \, dt \, d\phi \, d\theta.
\]

Here we have used the fact (see Section 2.7) that on \(\mathcal{H}^+, \partial_\tau, -\partial_\phi = C(M, a)\partial_\phi\), with \(|C(M, a)| \leq B_0 a_0\), Proposition 5.3.2 and a Hardy inequality to deal with the 0'th order horizon term.

Note that in the case of Theorem 1.2 the above inequality holds formally setting \(e_0 = 0\) to 0.

The other boundary term vanishes for sufficiently large \(r_{+\infty}\) in view of the compactness of the support as assumed in the reduction of Section 4.6.

10.5. **Finishing the proof.** We now add \(\epsilon\) times the estimate of Propositions 5.2.1 and 5.6.1 (applied to the original \(\psi\) with \(\Sigma = \{t^* = 0\}\) and \(\tau = \tau'\)) to the limit of (56), choosing \(\epsilon\) sufficiently small, \(\tilde{r}\) sufficiently close to \(r_+\), and \(\tilde{\delta}\) sufficiently small so that \(\tilde{r} + 2\tilde{\delta} \leq \tilde{r}_{\tilde{h}}\), so that the bulk terms on the right hand side of the former two propositions are absorbed in the lower bound for the main term of (56), as derived in Section 10.1. The constant \(\epsilon\) is now fixed, and we arrive at an inequality where in particular
\[
b \int_{\{0 \leq t^* \leq \tau\}} e^{-1-\delta} J^N[\psi] + (\epsilon |\log (r - r_+)\|^{-2}(r - r_+)^{-1} + r^{-3-\delta}) \psi^2
\]
appears on the left hand side, together with the nonnegative boundary integrals on \(\mathcal{H}^+\) and \(\{t^* = \tau\}\). The constant \(\epsilon\) being chosen, we may in what follows encode...
it into constants $b, B$, in accordance with our usual conventions. On the right hand side are error terms which must be controlled, as well as a term of the form
\[(69)\]
\[B(q, R_\tau, \epsilon_3) \int_{(\tau^* = 0)} J^N[\psi^N] N^\mu,\]
our data term. We wish to absorb the right hand side error terms into (68), producing only more terms (69) on the right.

Choosing $q > 0$ sufficiently small (and, in addition, $a_0 > 0$ sufficiently small in the Case of Theorem 1.2) it follows that one can absorb the error terms arising from the right hand side of (58) in Section 10.2 with (68). We choose $\epsilon_3$ so that the second term on the right hand side of (62) is absorbed also by (68). We may choose $\epsilon_2, R_2$ so that the second term of (59) is absorbed by (68). Note that the choice of $R_2, \epsilon_2$ determines $R_7$ in view of the comments after (60).

In the case of Theorem 1.2 since we can safely appeal to Theorem 4.1, choosing $\varepsilon = 1$, all remaining terms on the right hand side are controlled by (69) where we now drop the dependence of $B$ on the already-chosen constants, and the proof of (4) is complete.

In the case of Theorem 1.1: Restricting to sufficiently small $a_0$, we may choose $\epsilon_0 < \tilde{\epsilon} = e' < \epsilon$ such that the spacetime term on the right hand side of the first inequality of Proposition 5.3.1 (applied for $\tilde{\tau}$ in place of $\tau'$, and $0$ in place of $\tau$, with $0 \leq \tilde{\tau} \leq \tau'$ arbitrary, and with $\tilde{\tau}, \tilde{\delta}$ chosen as above) can also be absorbed. (The other term is of the form (69).) We choose $\epsilon$ sufficiently small so that the first term of (62) and the third term of (67) are absorbed by the positive boundary terms just generated. Given now arbitrary $B_0 > 0$, by restricting to smaller $a_0$, there exists an $\epsilon_0 < \tilde{\epsilon} = e'' < \epsilon$ such that, adding $B_0$ times the inequalities of Proposition 5.3.1 the spacetime term on the right hand side can still be absorbed by (68). It suffices to choose $B_0$ large enough so that the left hand side absorbs the remaining future boundary terms (the term of (57), the term of (59), the term of (66), and the first two terms of (67)). We have obtained (4) for Theorem 1.1 for $j = 0$.

We retrieve the boundedness statement (5) as well as the statement (6) by recalling that we still have the boundary terms on the left hand side of the inequality arising from our application above of Proposition 5.2.1 (alternatively, by revisiting Proposition 5.3.1).

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