An interlacing technique for spectra of random walks and its application to finite percolation clusters

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Abstract: A comparison technique for random walks on finite graphs is introduced, using the well-known interlacing method. It yields improved return probability bounds. A key feature is the incorporation of parts of the spectrum of the transition matrix other than just the principal eigenvalue. As an application, an upper bound of the expected return probability of a random walk with symmetric transition probabilities is found. In this case, the state space is a random partial graph of a regular graph of bounded geometry and transitive automorphism group. The law of the random edge-set is assumed to be stationary with respect to some transitive subgroup of the automorphism group (‘invariant percolation’). Given that this subgroup is unimodular, it is shown that stationarity strengthens the upper bound of the expected return probability, compared with standard bounds derived from the Cheeger inequality.

Keywords: random walks, random walks on random partial graphs, percolation, critical percolation, heat kernel decay, return probability, comparison theorems

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1 Interlacing for Random Walks on Graphs

How does the return probability of a simple random walk (SRW) on a graph change under removal and insertions of edges? The intention of the present paper is to answer this question for another specific random walk on finite graphs. The link to SRW will be provided in the form of inequalities comparing these two Markov processes. Using a circle of results from matrix theory called interlacing, we derive a method for comparing the return probabilities of random walks of the same type, but realized on different graphs. Moreover, we apply these results to invariant percolation with finite clusters. While the first section of this paper is concerned with the presentation of the main result in the context of standard interlacing theory, the second part describes the implications concerning the annealed return probability on finite, percolative subgraphs of transitive, unimodular graphs. The third section is devoted to the proof of the main result and a geometric property of finite trees.

1.1 Introduction

Partial graphs are subgraphs in which only edges are removed, while the set of vertices remains the original one [15]. We will study a certain type of random walk on partial graphs of graphs for which the return probability is known. In the second section, we will consider finite partial graphs of transitive graphs. This will include allowing for connected components consisting of only single vertices. The percolation generating these components will be assumed only invariant. The finite clusters may occur as connected components of either subcritical, supercritical, or critical percolations.

Interlacing refers to a set of techniques concerning the spectrum of matrices under perturbations of known rank. The smaller the rank of the perturbing operator, the more similar the set of eigenvalues of two matrices, one being the perturbed version of the other. The heart of the interlacing methods is the Courant-Fischer variational principle. In the present context, it will be used to show a monotonicity property of the spectrum of a certain regularised form of a given simple random walk, which we will call the regularised random walk (RRW): under removal of edges, the eigenvalues of the transition matrix increase. Furthermore, insertion of edges results in a shift of the $k$-th eigenvalue across less than $k$ intervals of the unperturbed spectrum [29].

A similar property holds for the spectrum of combinatorial Laplacians of graphs [25, 28]. The combinatorial Laplacian of subcritical Bernoulli-bond percolation graphs has been considered in [32] for $\mathbb{Z}^d$, and amenable graphs in [4]. For the adjacency matrix, as well as ‘Laplacian spectra of graphs’, interlacing is successfully applied (e.g. [23, 24]). In [13, 36] interlacing results for normalised Laplacians have been proven. In the present paper, interlacing will be used to find lower bounds for the eigenvalues of the transition kernel of the RRW, which is, up to scaling, equivalent to the Laplacian spectrum of graphs. RRW has been considered (not under this name) in many other contexts. In [12] and [37], RRW is called delayed random walk.
For reversible random walks, it is a fundamental and well-known fact that the spectral properties of the transition probability kernel are directly linked to the quickness of the Markov chain to approach stationarity. In particular, the spectral gap ([41], sec. 2.1.2) as a function of the geometry of the random walk’s state space (given by a finite connected graph, here) plays a characteristic role in the determination of the rate of convergence in terms of the graph’s order (=: size of its vertex set). If \( \lambda \) denotes the spectral gap, then it is well known that the return probability of a reversible random walk \( P_t(o,o) \) of some initial site \( o \) of a connected graph with \( N < \infty \) vertices obeys ([14], chap. 1.5), with \( c \) some constant generally depending on \( N \) and on the degree of \( o \),

\[
|P_t(o,o) - \pi(o)| \leq c e^{-t\lambda},
\]

where \( \pi(\cdot) \) denotes the stationary distribution. A characteristic of the first eigenfunction of a discrete Laplacian with Neumann boundary conditions is \( \pi \equiv 1/N \). Under these circumstances \( \lambda \geq c'/N^2 \), for some \( c' > 0 \), by Cheeger’s inequality (see [16], Corr. 2.1.5).

Our goal in this paper are upper bounds for the annealed return probability of the regularised and simple random walks on a random finite graph. To achieve this, geometric information of the subgraph will be incorporated into spectral estimates. In particular, more information about the spectrum of the transition kernel than just the spectral gap will be used. Instead of comparing each of the eigenvalues with the principal eigenvalue, an estimate is derived by using an interlacing technique. It results in an estimate similar to that in (1), given by Theorem 1.10, however, for the return probability \( \bar{P}_t = (1/N \sum_k P_t(k,k)) \). This is the case of a uniform initial distribution (averaged return probability). For \( t \) from a certain ‘intermediate time-window’, there is an additional polynomial prefactor:

\[
|\bar{P}_t - \pi(o)| \leq a N^{r} t^{s} e^{-t\lambda}.
\]

Here \( a \) is a known constant, and \( r < 2s \). This strict inequality is the reason for an improvement over using (1) to obtain bounds for the annealed return probability. The constants \( a, r, \) and \( s \) depend only on the largest occurring degree, as is \( c \) in (1) for the RRW. The consequences for the annealed return probability of the RRW on finite clusters of invariant percolations with exponential decay of the cluster-size are discussed in section 2. It will be seen that the time-window spans far enough to result in the aforementioned improvement.

As for notation, apart from a finite simple random walk \( X_n, n \in \mathbb{N} \) in discrete time with transition probability matrix \( P \), we will consider the corresponding regularised random walk \( \tilde{X}_n, n \in \mathbb{N} \), with transition probability matrix \( A \). We have chosen this letter to remind the close connection to the adjacency matrix of the regularised graph whose vertex set is the state space. The adjacency matrix of the original graph is denoted by \( A \). By \( D \) we denote the diagonal matrix with entries given by the vertices’ degrees. \( \sigma(A) \) is the set of eigenvalues of \( A \). Since we will need both terms, we distinguish between combinatorial Laplacian \( L = D - A \) of a graph (admittance, [15]), and its normalised
Laplacian \( \mathcal{L} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \) [14]. The Laplacian spectrum of a graph is \( \sigma(L) \) (e.g. [23]). Finally, subtracting an edge from a graph will be written in short form by \( G - e \), which means that the vertex set remains unchanged, while the edge set looses ‘e’. We will reserve the letter \( H \) for finite graphs, while \( G \) will denote infinite, transitive graphs. In particular, \( H \) will be the finite, random connected component containing the root of some \( G \) with bounded geometry of degree \( \delta \).

### 1.2 Interlacing: standard results

In matrix theory, the term *interlacing* refers to the nature of the change of the spectrum of a hermitian matrix under a hermitian perturbation of given rank [29, 34]. In particular, the location of the eigenvalues on the real line of the perturbed matrix may be compared to the eigenvalues of the unperturbed one. These techniques can be extended to the singular values of a general matrix and are used in various contexts.

In graph theory, interlacing has been used to study the spectrum of the adjacency matrix and the Laplacian of a finite graph, e.g. by [25, 28, 20, 10, 23], but also by many others (see [25] for a survey). In [11], interlacing is used in connection with the return time of random walks. In the context of random walks, we like to show that by considering transformations of graphs which connect different components, it is useful to perform comparisons of the return probabilities of reducible random walks.

**Definition 1.1.** Given a finite, simple (single-edged, undirected, loopless; see [45]) graph \( H = \langle V, E \rangle \), let \( \hat{H} \) be its regularisation with loops, where for each loop attached to a vertex, its degree is increased by one. Let \( (\hat{X}_{n}, \mu_{0}) \) be the simple random walk on \( \hat{H} \) with initial distribution \( \mu_{0} \) and call it **regularised random walk** on \( H \) (see Fig. 1).

![Figure 1: a.) Finite connected graph, with highest degree equal to three (\( \delta = 3 \)); b.) regularisation of the graph (RRW is the simple random walk on the regularised graph.)](image)

Many of the properties of the regularised random walk are passed on to the simple random walk, due to the following ‘equivalence’ (see also [12], paragraph 3).

**Definition 1.2.** The return probability of a random walk \( X_{n}, (n \in \mathbb{N}) \) with finite state space and uniform initial distribution into its initial state \( X_{0} \), is called **average return probability**, denoted by \( \mathbb{P}_{\downarrow} \hat{X}_{n} = \hat{X}_{0} \).
The following definitions are standard in the theory of finite random walks [41] and spectral graph theory [14]:

**Definition 1.3.** (Time-continuous random walk) The time-continuous regularised random walk $\tilde{X}_t$ associated with the discrete-time regularised random walk $\tilde{X}_n$ on a graph $H = (V, E)$ with transition matrix $A$ is the random walk performing $n$ discrete independent steps across edges in $E$ within the time interval $[0, t]$ with probability $e^{-t^n}/n!$. Therefore, in the case of $H$ being finite, if $(e, Af) = \sum_{k,l \in V} e(k)A_{kl}f(l)$, the transition probability from $k$ to $l$ in finitely many steps is given by

$$P^k[\tilde{X}_t = l] = \sum_{n=0}^{\infty} e^{-t^{n+1}/n!}(e_k, A^n e_l) = (e_k, e^{-t(1-A)}e_l).$$

The quadratic form $(e_k, e^{-t(1-A)}e_l)$ with $e_k, e_l \in \mathbb{R}^{|V|}$ the unit vectors along the direction corresponding to $k, l \in V$, respectively, is called the heat kernel of $A$. It is the kernel of the time-continuous semigroup $\exp(-t(\mathbb{I} - A))$, $(t \geq 0)$ associated with $A$. The time-continuous simple random walk $X_t$, which is associated with $P$, is defined analogously, with $A$ replaced by $P$.

We denote by $l^2(V, \pi)$ the space $\{ f : V \to \mathbb{C} : (f, f)_\pi < \infty \}$, where $(f, g)_\pi := \sum_{k} f(k)g(k)\pi(k)$, and $\pi \in \mathcal{M}_{+1}(V)$ is some probability distribution on $V$. Naturally, for finite $V$, with $|V| = N \in \mathbb{N}$, this is isomorphic to $\mathbb{C}^N$.

**Definition 1.4.** (Laplacians and heat kernel for the regularised graph) The graph Laplacian of the regularised graph $\tilde{H}$ is denoted by $\tilde{L} = \tilde{D} - \tilde{A}$, where $\tilde{D} = \delta \mathbb{I}$, and the adjacency matrix $\tilde{A}$ accounts for the attached loops: $\tilde{A} = A + \delta \mathbb{I} - D$. Therefore,

$$L = D - A = \tilde{D} - \tilde{A} = \tilde{L},$$

i.e. the graph Laplacian $L$ is invariant under the decoration of $H$ with loops. $D = \text{diag}((\text{deg}(v_1)), \text{deg}(v_2), \cdots, \text{deg}(v_N))$ and $\text{deg}(v_j)$ the degree of vertex $v_j$ of $H$. The normalised Laplacian for the regularised $\tilde{H}$ is $\tilde{L} = \tilde{D}^{-\frac{1}{2}}\tilde{L}\tilde{D}^{-\frac{1}{2}} = (1/\delta) L$. The quadratic forms $(e, \tilde{L}f)_\pi$ and $(\tilde{e}, \tilde{L}\tilde{f})_{\tilde{\pi}}$ with $f, \tilde{f} \in l^2(V, \pi)$ and $\tilde{e}, \tilde{\tilde{f}} \in l^2(V, \tilde{\pi})$ where $\pi$ and $\tilde{\pi}$ are the invariant distributions of $X_t$ and $\tilde{X}_t$ are called Dirichlet-Forms of $L$ and $\tilde{L}$, respectively.

We quote the following comparison theorem for the eigenvalues of the random walks from [41], section 1.2.5 without proof in a simplified form. It is used in the proof of the next lemma.

**Lemma 1.5.** Let $\mathcal{L}$ and $\mathcal{L}'$ be two normalised Laplacians on $l^2(V, \pi)$ and $l^2(V, \pi')$, respectively. Let there be a linear map $l^2(V, \pi) \ni f \mapsto f' \in l^2(V, \pi')$ and $a > 0$ such that for every $f \in l^2(V, \pi)$ it holds that

$$(f, \mathcal{L}f)_\pi = (f', \mathcal{L}'f')_{\pi'} \quad \text{and} \quad a \cdot (f, f)_\pi \leq (f', f')_{\pi'}.$$

Then it holds for all pairs of $j$-th eigenvalues $\lambda_j$ and $\lambda'_j$ of $\mathcal{L}$ and $\mathcal{L}'$, respectively,

$$a \cdot \lambda_j \leq \lambda'_j,$$

if both sets of eigenvalues $(\sigma(\mathcal{L})$ and $\sigma(\mathcal{L}')$ are enumerated in the same way (e.g. $\lambda_j \leq \lambda_{j+1}$ and $\lambda_j' \leq \lambda_{j+1}'$ for all $j \in \{1, \ldots, |V| - 1\}$).
Lemma 1.5, it follows that
\[ \sum \text{degree operators} \] for \( \hat{X}_t \) for the regularised random walk it is given by \( \exp(-t(\mathbb{I} - \hat{A})) \), where \( \hat{A} = \delta^{-1}\hat{A} \), with \( \hat{A} \) the adjacency matrix of the regularised graph \( \hat{H} \). Then, with \(|V| = N:\]
\[
P([\hat{X}_t = \hat{X}_0] = \frac{1}{N} \sum_{k \in V} P^k[\hat{X}_t = k] = \frac{1}{N} \text{Tr} \exp(-t(\mathbb{I} - \hat{A})).
\]

**Remark:** The average return probability of the time-continuous \( X_t \) and \( \hat{X}_t \) relates to the trace of the corresponding heat kernel (see [41, 19]). While \( \exp(-t(\mathbb{I} - P)) \) is the heat kernel of \( X_t \), for the regularised random walk it is given by \( \exp(-t(\mathbb{I} - \hat{A})) \), where \( A = \delta^{-1}\hat{A} \), with \( \hat{A} \) the adjacency matrix of the regularised graph \( \hat{H} \). Then, with \(|V| = N:\]
\[
P([X_t = X_0] \leq P([\hat{X}_t = \hat{X}_0] \leq P([X_{t/\delta} = X_0].
\]

**Proof:** Introduce \( \hat{P} = (1 - 1/\delta)\mathbb{I} + (1/\delta)P \), the transition-matrix of the lazy random walk, which is the simple random walk on the graph \( \hat{H} = (V, \hat{E}, \hat{m}) \), with \( N = |V| \) vertices, and edges \( \hat{E} = E \cup E' \), where \( E' \) are the multiple loops \( \{k\} \in \hat{E} \) with multiplicity
\[
m(\{k\}) = (\delta - 1) \cdot \text{deg}_H(k).
\]
Then the lazy normalized Laplacian is \( \hat{\mathcal{L}} = 1 - \hat{D}^{1/2} \hat{P} \hat{D}^{-1/2} = (1/\delta)\mathcal{L} \), a scaled version of the normalized Laplacian of \( H \), where \( \hat{D} = D + (\delta - 1)D = \delta \hat{D} = \text{diag}(\text{deg}(v_1), \text{deg}(v_2), \cdots, \text{deg}(v_N)) \) is the diagonal degree operator of \( H \) taking account of the loops multiplicity. Note that the combinatorial Laplacian \( L = D - \hat{A} \) (or admittance operator) which doesn’t change under the regularisation (s.a.), also remains invariant under ‘laziness’, i.e.
\[
D - \hat{A} = \delta \mathbb{I} - \hat{A} = \hat{D} - \hat{\mathcal{L}}.
\]
Here, \( \hat{A} \), \( \hat{\mathcal{L}} \), and \( \hat{H} \) are the adjacency matrices of \( H, \hat{H} \) and \( \hat{H} \) taking into account the changed degree due to the loops. \( \delta \mathbb{I} \) and \( \hat{D} = D + (\delta - 1)D = \delta \hat{D} \) are the corresponding ‘degree operators’ for \( \hat{H} \) and \( \hat{H} \).

In this situation, the properties of the normalised Laplacian differ from those of the combinatorial Laplacian. Instead of invariance under ‘loop-decoration’, the spectrum \( \{\lambda_j(\hat{\mathcal{L}})\} \) of the lazy \( \hat{\mathcal{L}} \) is a scaled version of \( \sigma(\mathcal{L}) = \{\lambda_j(\mathcal{L})\} \): for \( j \in \{1, \cdots, N\},
\[
\lambda_j(\hat{\mathcal{L}}) = \frac{1}{\delta} \lambda_j(\mathcal{L}).
\]
On the other hand, the diagonal elements of \( D \) fulfil \( (v, DV) \leq (v, (\delta \mathbb{I})v) \leq (v, (\delta D)v) \). Therefore, while the Dirichlet-forms (for \( L, \hat{L} \), and \( \hat{L} = \delta D - \hat{A} \)) are equal in value, the norms \( \|v\|_\pi \) of \( v \) in \( l^2(V, \pi) \), where \( \pi \) is the invariant distribution of \( P, A \), or \( \hat{P} \), differ:
\[
\|v\|_\pi^2 = (v, v)_\pi = (v, Dv) \leq (v, \hat{D}v) = (v, v)_{\hat{\pi}} \leq (v, \hat{D}v) = (v, v)_{\hat{\pi}} \text{ for } v \in l^2(V, \pi).
\]
By Lemma 1.5, it follows that
\[
\lambda_j(\mathcal{L}) \geq \lambda_j(\hat{\mathcal{L}}) \geq \lambda_j(\hat{\mathcal{L}}).
\]
Therefore, \( \sum_{k \in V} P^k[X_t = k] = \text{Tr}[\exp(-t\mathcal{L})] \leq \text{Tr}[\exp(-t\hat{\mathcal{L}})] \leq \text{Tr}[\exp(-t\hat{\mathcal{L}})] \). □

For completeness, we add the following standard result about the relevant comparison between discrete time random walks:
Lemma 1.7. The average return probabilities for discrete ($\mathbb{P}^d$) and continuous ($\mathbb{P}^c$) time on $H$ of $(X_\cdot, \mu_0)$ and (with $\mu_0$ the uniform distribution on $V$) fulfil

$$\mathbb{P}^d[X_{4n} = X_0] \leq 2 \mathbb{P}^c[X_{2n} = X_0] \leq 2 \mathbb{P}^d[X_n = X_0] + 2e^{-t}.$$  

Proof: This follows from standard methods of comparison between discrete, lazy and continuous-time versions of a given Markov chain (see [46], Lemma 14.2.c [41], and Corollary 1.3.4): Let $\overline{P} = \frac{1}{2}I + \frac{1}{2}P$. With $\text{Tr}[\exp(-t(1-P))] = \text{Tr}[\exp(-t\mathcal{L})]$ it holds due to $(1-x)^n \leq \exp(-nx)$ for $0 \leq x \leq 1$, and $\log(1-y) \geq -2y$ for $0 \leq y \leq \frac{1}{2}$

$$\text{Tr}[P^{4n}] \leq 2 \text{Tr}[\overline{P}^{4n}] \leq 2 \text{Tr}[e^{-4n(1-P)}] = 2 \text{Tr}[e^{-2n\mathcal{L}}] \leq 2(\text{Tr}[P^n] + e^{-t}).$$  

From now on let $\sigma(A) = \{\beta_j\}_{j=1}^N$ be the eigenvalues of $A$ enumerated in a non-increasing way, i.e. $1 = \beta_1 > \beta_2 \geq \beta_3 \geq \beta_4 \ldots \geq \beta_N$. The strict inequality arises from $H$ being connected, with $\beta_1$ being the Perron-Frobenius eigenvalue of $A$. The restriction of the interlacing method to the transition kernel of the continuous time RRW, for which the eigenvalues $\exp(-t(1 - \beta_j))$ are all positive, and therefore easier to handle in comparison techniques, is now feasible. The bounds obtained for this setting can then be transferred to discrete time, and the simple random walk, by the use of the two Lemmata 1.6 and 1.7.

Now, we find bounds for eigenvalues of the RRW on finite graphs with edges removed: Let $A'$ denote the transition probability matrix of the regularised random walk of the graph $H' = H - e$, obtained by removing a single edge and keeping all vertices (see Fig. 2).

![Figure 2: Removing an edge of a (loopless) graph acting as the state space of RRW corresponds to the SRW on a decorated graph with the edge replaced by a pair of loops.](image)

We then have $A' = A + S_e$, with $S_e$ of the following type:

$$S_e = \frac{1}{\delta} \begin{pmatrix}
0 & \ldots & 0 \\
\ldots & +1 & \ldots & -1 & \ldots \\
\ldots & \ldots & \ldots \\
\ldots & -1 & \ldots & +1 & \ldots \\
0 & \ldots & 0
\end{pmatrix},$$

where the non-zero elements appear in the rows and columns indexed by the vertices of the edge $e$ removed. This is $(1/\delta)$ times a projection onto the one-dimensional subspace spanned by $(0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)$, where the non-zero entries occur at the positions indexed by the vertices incident with $e$. It follows that $S_e$ is positive semi-definite, so
\( \beta_j \leq \beta'_j \). Moreover, its rank equals one. Taking away all edges from some subset \( R \subset E \), with \( r = |R| \), leads to

\[
A' = A + \sum_{e \in R} S_e. \tag{7}
\]

This is a perturbation with rank bounded by \( r \). (Note, if \( R \) are the edges removed from the initial graph, the number of linearly independent conditions induced by the boundary conditions may in general be smaller than \( r \).) By the Courant-Fisher variational principle (see [29, 34]), we have, with \( \mathcal{Y} \) a subspace of \( \mathbb{C}^N \) and \( S = \sum_{e \in R} S_e \), if \( j > r \),

\[
\beta'_j = \max_{\mathcal{Y} \subset \mathbb{C}^N \text{ dim(}\mathcal{Y}\text{)=}j} \min_{v \in \mathcal{Y}} \frac{(v, A'v)}{(v, v)} \leq \max_{\mathcal{Y} \subset \mathbb{C}^N \text{ dim(}\mathcal{Y}\text{)=}j} \min_{v \in \mathcal{Y}, v_k = v_l \text{ f.a.}(k,l) \in R} \frac{(v, (A + S)v)}{(v, v)} = \max_{\mathcal{Y} \subset \mathbb{C}^N \text{ dim(}\mathcal{Y}\text{)=}j} \min_{v \in \mathcal{Y}} \frac{(v, Av)}{(v, v)} = \beta_{j-r}. \tag{8}
\]

If \( j \leq r \), since \( A \) and \( A' \) are stochastic matrices, \( \beta'_j \leq \beta_1 = 1 \).

**Theorem 1.8.** (Interlacing for RRW on finite partial graphs) The eigenvalues \( \{\beta'_j\} \) of a regularised random walk on a partial graph \( H' = H - R \) relate to those \( \{\beta_j\} \) of the regularised random walk on the initial graph \( H \), with \( r = |R| \), by

\[
\beta_j \leq \beta'_j \leq \beta_{(j-r)\vee 1}. \tag{9}
\]

**Proof:** Removing the edges in \( R \) is a perturbation of type (7) and \( r \) is an upper bound for its rank. The result follows from the positivity of \( S_e \), (8), and by considering that stochasticity of the matrix is preserved. \( \square \)

The second inequality in (9) gives an upper bound for the change of eigenvalues of the RRW under insertion of edges.

Similar results have been proven for normalised Laplacians of non-regular graphs [13, 36]. In this case the monotonicity is lost due to the loss of semi-definiteness of the corresponding pertubation matrix (instead of \( S_e \)), which, however, is still of rank one.

**Remark:** The condition (constraint) given by \( v_k = v_l \) is the same as the vanishing discrete gradient ‘d’ of \( v \) as a function of the vertices (if some direction of the edge \( e = \{k,l\} \) has been agreed upon):

\[
v_k - v_l = dv(e) = 0.
\]
In this form, the constraint may be considered a Neumann boundary condition for the corresponding normalised Laplacian $L = \mathbb{I} - A$ (comp. with [43]). This is due to the monotonicity of the shifts in the spectrum, under an increase of ‘boundary’, which is negative, similar to Laplacians of functions in continuous function spaces. Note that increasing the boundary is realised in the discrete setting by the removal of edges. However, in the context of induced subgraphs ([14], chapter 8.3), the Neumann eigenvalues of a graph relate to yet another random walk, which shows a different behaviour at ‘corners’ - owing to a special definition of the discrete gradient at ‘irregular’ parts of the boundary. It is easily shown that the Neumann random walk on induced subgraphs also shows the monotonicity of the spectral shift under removal of edges. In the present context, however, the perturbed graphs are partial graphs, i.e. subgraphs of which only edges (not vertices) are removed.

Before stating the main result, we prove a simple well-known interlacing theorem for the Laplacian spectrum of partial graphs [28, 13]:

**Lemma 1.9.** Let $H = (V, E)$ be a finite, possibly unconnected, undirected graph with $N = |V| \text{ vertices. Let } D = \text{diag}(\cdots, \deg(v_j), \cdots) \text{ be the } N \times N \text{ diagonal matrix with the degrees of the vertices as its entries, and } A \text{ the adjacency matrix of } H. \text{ Let } L = D - A \text{ denote the combinatorial Laplacian of } H \text{ and } \sigma(L) = \{ \lambda_j \}_{j=1}^N \text{ its spectrum. Let the eigenvalues be enumerated in an increasing way: } 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$

Then, removing a single edge results in a graph with eigenvalues shifted downwards. In other words, in a transformation $H \rightarrow H' = H - e$, for some edge $e \in E$ and corresponding transformation $L \rightarrow L'$ of the combinatorial Laplacian, the $j$-th eigenvalue in $\sigma(L')$ will obey (with $j \in \{2, \cdots, N\}$):

$$
\lambda_{j-1} \leq \lambda'_j \leq \lambda_j.
$$

**Proof:** We note that decorating a graph with loops leaves the combinatorial Laplacian invariant. Therefore, if $\delta$ is the maximum degree of $H$, regularising $H$ by adding loops to each vertex until every vertex has degree equal to $\delta$, where the degree of a loop counts one, yields the same Laplacian. Call the graph obtained by the loop-decoration $\hat{H}$ and its Laplacian $\hat{L}$. So, $L = \hat{L}$, and, in particular, the $j$-th eigenvalue of $L$ equals the $j$-th eigenvalue of $\hat{L}$.

Removing an edge $e$ of $H$ yields the transformed graph $H' = H - e$. Let the corresponding decorated (regularised) graph be called $\hat{H}'$ and its Laplacian $\hat{L}'$ (again, the hat denotes the decoration with loops): Then the matrix $\hat{L}' - \hat{L}$ is negative semidefinite and of rank one, since it is of the form (6), multiplied by $-\delta$. Since $\hat{L}$ and $\hat{L}'$ are real and symmetric, it follows from $\hat{L} = \delta \mathbb{I} - \delta A$ with $A$ the transition matrix of the RRW that the $j$-th eigenvalues $\hat{\lambda}_j$ and $\hat{\lambda}'_j$ of $\hat{L}'$ fulfil $\hat{\lambda}_j = \delta(1 - \beta_j)$ and $\hat{\lambda}'_j = \delta(1 - \beta'_j)$. Therefore, it follows from Theorem 1.8 for $j \geq 2$,

$$
\lambda_{j-1} \leq \hat{\lambda}'_j \leq \lambda_j.
$$

Since $\hat{L}' = L'$, it follows $\hat{\lambda}'_j = \lambda_j$, the $j$-th eigenvalues of $L'$. 

\end{proof}

9
1.3 Main result

The second inequality in (9) gives upper bounds for eigenvalues of a graph into which edges have been inserted - if $\beta_j'$ is viewed as the ‘unperturbed eigenvalue’. This is the main ingredient of the following theorem. Its proof, given in the last section, uses Theorem 1.11. Before stating the result we define some numerical constants, some of which find repeated use in several of the theorems. Again, given the finite, simple graph $H$ (see Definition 1.1), $N = |V(H)|$ is the size of its vertex set, $\delta$ is the maximum degree. Furthermore, we define the following constants; they will be given again at the places where they are used:

$$\nu := \frac{1}{(\delta - 1) \ln 16}, \quad a = 1 + \frac{\sqrt{\pi \delta}}{21 - \nu}, \quad \hat{t} = 4^{1+\nu} N^{1-\nu}, \quad \hat{
u} = 4^{\frac{(1+\nu)^2}{\nu} N^4}. \quad (11)$$

**Theorem 1.10.** The return probability of the time-continuous RRW $\hat{X}_t$, for $t \in [\hat{t}, \hat{\nu}]$ on a finite, connected, simple graph $H$ of order $N$ and maximum degree $\delta \geq 3$ obeys the bound

$$\mathbb{P}[\hat{X}_t = \hat{X}_0] \leq \frac{1}{N} + a \left( \frac{N^{1-\nu}}{\sqrt{\tilde{t}}} \right)^{\frac{1}{1+\nu}} e^{-4\hat{t}/(\delta N^2)}. \quad (12)$$

**Remark:** For times $N^{2(1-\nu)} < t < N^2$, the exponential factor is still close to one, while the prefactor may already be small. The result is to be compared with the trivial bound

$$\mathbb{P}[\hat{X}_t = \hat{X}_0] = \frac{1}{N} \text{Tr} [-t(1 - A)] \leq \frac{1}{N} + \frac{N-1}{N} e^{-4\hat{t}/(\delta N^2)}, \quad (14)$$

resulting from $\exp(-t(1 - \beta_2)) \leq \exp(-4t/(\delta N^2))$. For times as large as $\hat{t} = O(N^4)$, the improvement of the prefactor is negligible in comparison with the effect of the exponential factor. This justifies picking $t \leq \hat{t}$ from an ‘intermediate time-frame’.

We now derive a theorem on the eigenvalues of a regularised random walk, which is proved by an interlacing technique involving unconnected graphs. It will be used in the proof of Theorem 1.10. It is easy to show that the second largest eigenvalue of the transition kernel of RRW on any finite, connected tree is bounded from above by the second largest eigenvalue of the path ($=: P_N$) with the same number of vertices $N$. With the spectrum of the RRW on $P_N$ given by $\beta_j = 1 - \frac{2}{\delta}(1 - \cos \frac{(j-1)\pi}{N})$, $j \in \{1, ..., N\}$ (see [17], or [41], Ex. 2.1.1), this gives the upper bound for its second largest eigenvalue $\beta^T_2$ (‘T’ for tree):

$$\beta^T_2 \leq \beta_2 \leq 1 - \frac{4}{\delta N^2}. \quad (15)$$

Using interlacing, we now find an upper bound for the eigenvalues of the transition matrix of the RRW on a finite graph. For the proof, a lemma about the geometry of finite trees is used, which is given in the last section.
**Theorem 1.11.** Let \( \sigma(A) = \{ \beta_j \} \) be the spectrum of the transition kernel of the regularised random walk on a finite, connected graph \( H \) with \( N \) vertices, enumerated in a decreasing way. Let \( H \) have largest degree \( \delta \). Then, with \( B \in \{0, \ldots, N-2\} \), and \( \nu \) given in (11)

\[
\beta_{B+2} \leq 1 - \frac{4}{\delta N^2} \left( \frac{B + 1}{2} \right)^{2\nu}. \tag{16}
\]

Before proving this theorem, we make some general observations and give definitions regarding the interlacing inequalities (9) applied to arbitrary spanning trees of \( H \):

Removing an edge shifts the spectrum of the RRW (of the transition probability matrix) **upwards**, by at most one interval between successive eigenvalues. Therefore, if we look at the \( j \)-th eigenvalue of the RRW on \( H (\beta_j) \), it is possible to bound \( \beta_j \) from above by the corresponding eigenvalue \( \beta_j^T \) of RRW on any spanning tree. Likewise, it is possible to bound \( \beta_j^T \) from above by the \( j \)-th eigenvalue of the RRW on any forest resulting from removal of edges: In the case where removal of edges results in several connected components, the multiplicity of the eigenvalue equal to one is equal to the number of connected components.

Comparing the second largest eigenvalue \( \beta_2^T \) with an eigenvalue \( \beta_{2,1}^T \) of a graph resulting from removal of one edge (yielding a disconnected graph with two trees as connected components) only yields \( \beta_2^T \leq 1 \). The reason is that the multiplicity of ‘1’ in the spectrum of the disconnected graph is two and \( \beta_{2,1}^T = 1 \). Define \( \beta_j^{T,k} \) to be the eigenvalue of RRW on the spanning forest resulting from removal of any \( k \) distinct edges from an arbitrary spanning tree of \( H \). Let \( j \mapsto \beta_j^{T,k} \) be enumerated in decreasing order. Then, if the \( j \)-th eigenvalue \( \beta_j^T \) of RRW on the spanning tree of \( H \) for \( j \in \{2, \ldots, k+1\} \) is compared with the \( j \)-th eigenvalue \( \beta_j^{T,k} \), interlacing (9) yields for \( j \in \{2, \ldots, k+1\} \) only \( \beta_j^T \leq 1 = \beta_1^{T,k} = \beta_2^{T,k} = \cdots = \beta_{k+1}^{T,k} \).

**Lemma 1.12.** Removing \( k \) edges from any spanning tree of \( H \), the \( k+2 \)-nd eigenvalue \( \beta_{k+2}^{T,k} \) of the RRW on the resulting spanning forest is known to be strictly smaller than one. In particular, it is the first eigenvalue with this property in this enumeration:

\[
\min \left\{ j \in \{1, \ldots, N\} : \beta_j^{T,k} < 1 \right\} = k + 2. \tag{17}
\]

**Proof:** (Lemma 1.12) The forest resulting from removing \( k \) edges from a spanning tree of \( H \) consists of precisely \( k + 1 \) connected components. Therefore, the eigenvalue with numerical value equal to one has multiplicity \( k + 1 \). \( \Box \)

**Proof:** (Theorem 1.11) Let \( \hat{\beta}_2^o \) be the second largest eigenvalue of RRW on the path \( P_N \), decorated with loops to make it a regular graph of degree \( \hat{\delta} \), where \( \hat{N} \) is the size of the largest tree in the forest resulting from removing any \( k \) edges of any spanning tree of \( H \).

\[
\beta_{k+2} \leq \beta_{k+2}^T \leq \beta_{k+2}^{T,k} \leq \hat{\beta}_2^o \leq 1 - \frac{4}{\delta \hat{N}^2}. \tag{18}
\]
The first and second inequality is the first of (9). By Lemma 1.1, the third and fourth inequality of (18) are (15) with the order \((\hat{N} = N)\) of the tree replaced by \(\hat{N}\).

An optimal bound of \(\hat{N}\) will be made by the help of Lemma 3.1 (compare with [23], where eigenvalues of the adjacency matrix of a graph without loops are estimated in a similar way - also involving interlacing). We take an arbitrary spanning tree of \(H\), and note that we can choose a sequence of edges, which are taken away, such that at each edge, the size reduces by at least a factor of 
\[
q = 1 - \frac{1}{4(\delta - 1)}.
\]

Now, the order \(\hat{N}\) of the largest remaining subtree depends on \(B\), the number of removed edges: If some \(b \in \mathbb{N}\) is such that 
\[
2^b \leq B + 1 < 2^{b+1} \quad \text{(i.e. } b = \lfloor \log_2(B + 1) \rfloor),
\]
then by removing \(B\) edges, the cardinality \(N\) of the vertex set of \(H\) may be reduced by a factor of \(q\) at least \(b\) times. Therefore, since \(q < 1\),
\[
\beta_{B+2} \leq 1 - \frac{4}{\delta N} \left( \frac{B+1}{2} \right)^{2\nu}.
\]

By (18), this implies
\[
\beta_{B+2} \leq 1 - \frac{4}{\delta N} \left( \frac{B+1}{2} \right)^{2\nu}.
\]

The proof yields another form of this theorem, namely, the direct comparison of the spectrum of the RRW on \(H\) by the spectrum of RRW on the path \(P_N\):

**Corollary 1.13.** Under the conditions of the last theorem, let \(\{\beta_j\}_{j=1}^{N}\) be the spectrum (ordered in decreasing order) of the transition kernel RRW on the path \(P_N\), with a number of vertices \(N\). Then, with \(x \mapsto [x] := \max\{m \in \mathbb{N} : m \leq x\}\), the eigenvalues \(\{\beta_j\}\) of the RRW on \(H\) relate to \(\sigma(P_N)\) by

\[
\beta_{B+1} \leq \beta_{\lfloor B/2 \rfloor + 1 \nu + 1}, \quad (B \in \{0, \ldots, N - 2\}).
\]

**Proof:** Follows directly from the first three inequalities of (18), and from

\[
\beta_2^0 = 1 - \frac{2}{\delta} (1 - \cos \pi \frac{1}{\lfloor N(\frac{2}{(B+1)})^\nu \rfloor}) \leq 1 - \frac{2}{\delta} (1 - \cos \pi \frac{2^{-\nu}(B + 1)^\nu}{N}) = \beta_{\lfloor 2^{-\nu}(B+1)^\nu \rfloor + 1},
\]

where \(\beta_2^0\) is the second largest eigenvalue of the path \(P_N\), with \(\hat{N} = [2^\nu N/(B + 1)^\nu]\).

**Remark:** This is a comparison theorem for other elements of the spectrum of \(A\) than \(\beta_2\) and, as such, compares to results concerning 'higher eigenvalues' of the spectrum of the Laplacian \(1 - A\). (See e.g. [41], Theorem 3.3.17, where assumptions about the isoperimetric properties of the graph are made.) The relevant geometric property assumed here is only the uniform bound on the degree of the graph, given by \(\delta\).
2 Invariant Percolation

2.1 The mass transport principle

We apply our results to finite random graphs, by considering the expected return probability (= annealed return probability) of a continuous-time random walk on the finite random partial graphs of a transitive graph of finite degree: namely the finite connected components of an invariant percolation. All parameters and constants of the estimates shall be accessible, i.e. it should be possible to express them as functionals (expected values) of the distribution of the random process generating the subgraphs. Our results can be used in this case, if the random graphs considered here will be restricted to the subgraphs induced by the almost surely finite connected components. Subcritical percolation on the Cayley graphs of finitely generated groups are included as a special case. See [32, 4] for estimates of the integrated density of states (‘cumulative spectral measure’) near the edges of the generator’s spectrum in this case. However, also the critical [6], or supercritical percolation measure, conditioned on the finiteness of the cluster containing the root is a possible setting for the results proven, here. See Theorems 2.4 and 2.5 for this subject.

One of the most important results in percolation theory in the last few years was the fact that critical percolation on non-amenable, unimodular graphs has almost surely finite clusters [6]. The mass transport principle [42] states that for invariant percolation on an infinite, unimodular, transitive graph $G = \langle V, E \rangle$, a function of two vertex-valued arguments, invariant under diagonal action, allows interchange of the arguments under summation of one of them, i.e. for all $f : V \times V \to \mathbb{R}$,

$$
\sum_{v \in V} f(v, w) = \sum_{v \in V} f(w, v).
$$

See [37] for a detailed discussion of the consequences in the theory of invariant percolation.

2.2 Application to the expected return probability

The results concerning the return probability of the RRW were formulated for an initial distribution given by the uniform distribution on the finite cluster. The relation to the return probability to a fixed, ‘deterministic’ starting point in the case of invariant percolation is settled for random partial graphs of unimodular graphs by the following lemma. Consider $H(\omega), \omega \in \Omega$ be a random partial graph of a unimodular transitive graph $G = \langle V, E \rangle$ realised with probability given by an invariant law $\mu$ on the probability space $\Omega = 2^E$, where $F$ is the product $\sigma$-algebra on $\Omega$. Let $\mathcal{C}_o$, depending on $\omega \in \Omega$, be the connected component of $H(\omega)$ containing the root $o \in V$.

**Lemma 2.1.** Let $\mathcal{C}_o$ be $\mu$-a.s. finite. Let $\tilde{X}_t$ be RRW on $H(\omega)$, with initial distribution $\mathbb{P}[X_0 = \cdot]$ given either by the uniform distribution $\mathbb{P}_\perp[X_0 = \cdot]$ on $\mathcal{C}_o$, or by the atom at $o$, denoted by $\mathbb{P}^o[\cdot]$. Then, the annealed return probability is given by

$$
\mathbb{E}_\mu \left[ \mathbb{P}^o[\tilde{X}_n = o] \right] = \mathbb{E}_\mu \left[ \mathbb{P}_\perp[\tilde{X}_n = \tilde{X}_0] \right].
$$
Proof: Let $A_{vw} = P^v[\hat{X}_1 = w]$ be the transition matrix of $\hat{X}$ on $C_o \ni v, w$. For the annealed return probability of the vertex $o$ we have

$$E_\mu^o[\hat{X}_n = o] = E_\mu A^o_{oo} = \sum_{k \in V} E_\mu \left[ A^o_{kk} \frac{\chi(k \in C_o)}{|C_o|} \right].$$

Define

$$f : V \times V \rightarrow \mathbb{R},$$

$$\langle v, w \rangle \mapsto E_\mu \left[ A^v_{ww} \frac{\chi(w \in C_o)}{|C_v|} \right].$$

Applying the mass transport principle, we obtain by interchanging the arguments of $f$,

$$\sum_{k \in V} E_\mu \left[ A^n_{kk} \frac{\chi(o \in C_k)}{|C_k|} \right] = \sum_{k \in V} E_\mu \left[ A^n_{kk} \frac{\chi(k \in C_o)}{|C_o|} \right].$$

But $o \in C_k \Leftrightarrow C_o = C_k \Leftrightarrow C_o \ni k$, so

$$\sum_{k \in V} E_\mu \left[ A^n_{kk} \frac{\chi(o \in C_k)}{|C_k|} \right] = \sum_{k \in V} E_\mu \left[ A^n_{kk} \frac{\chi(k \in C_o)}{|C_o|} \right] = E_\mu \left[ \frac{1}{|C_o|} \sum_{k \in C_o} A^n_{kk} \right].$$

Since $|C_o| < \infty$, the expression on the right is $P_\mu[\hat{X}_n = \hat{X}_0]$. These results equally apply to the continuous time RRW. In this case, the transition kernel $A_{v,w}, (v, w \in C_o)$ is replaced by $(\exp(I - A))_{v,w}$ (see [41], chapter 1.3).

This result gives the opportunity to apply the result of the former section to the annealed return probability of RRW back to the given vertex $o$ in case of specific random partial graphs. Invariant percolation on the Euclidean lattice as well as on homogeneous trees in the subcritical regime shows to have a cluster size $|C_o|$ with exponentially decaying distribution [1] (see the generalisations of this to quasi-transitive graphs in [3]).

In order to study the return probability of RRW in this case, we consider an invariant percolation on a transitive, unimodular graph with distribution of the cluster size on $\{1, 2, 3, \ldots\}$ given by $\mu[|C_o| = m] = C \exp(-m/\hat{N})$, for some $\hat{N} \in \mathbb{N}$. Then $\hat{N} \leq E[|C_o|] \leq \hat{N} + 1$.

Let $a := 1 + \frac{\sqrt{\delta}}{2^{1+r}}$, and $r = \nu/(1 + \nu) = 1/((\delta - 1)\ln 16 + 1)$, using the definitions in (11). Define $c := a \left(1 + e \cdot (1 + (\hat{N} + 1)^{1+2r})\right)$, and $\rho := 4^{-\frac{(1+\nu)^2}{2r}}$.

**Theorem 2.2.** With the definitions given above, for $t \geq \frac{\delta}{4\hat{N}}$, it holds

$$E_\mu^o[\hat{X}_t = o] \leq P_\mu \left[ \frac{1}{|C_o|} \right] + c t^{-\frac{1}{6}(1 + r)} \exp\left\{ -\left( \frac{4t}{\delta \hat{N}^2} \right)^\frac{1}{4} \right\} + \sqrt{\rho} t^{1/2} e^{-\frac{\sqrt{\delta}}{\delta \rho}} + e^{-\frac{r}{16\hat{N}}}.$$
Remark: For large values of the maximal degree $\delta$, the bound retains a prefactor $\sim t^{-1/6}$. For comparison, the trivial bound (14) yields a constant prefactor, along with the same exponential. Moreover, since $\nu \sim 1/\delta$, it improves with decreasing $\delta$. For the constant $c$ it holds: $c \leq 8e^{\sqrt{d}(N + 1)^{1+2r}} < 24\sqrt{d}N$.

Proof: Let $\hat{C} := \sum_{m=1}^{\infty} \exp(-m/N) = (\exp(1/N) - 1)^{-1}$ be the normalising constant in $E[f(|C_o|)] = \hat{C}^{-1} \sum_{m=1}^{\infty} f(m) \exp(-m/N)$. Write $N_o = |C_o|$. As in (11), we let $\hat{t}(N) := 4^{1+\nu} N^{1-\nu}$, and $\hat{t}(N) = 4^{-(1+\nu)/\nu} N^4$. We note that $\hat{t}(N_o) \leq t$ if and only if $N_o \leq \hat{N}_t$, where $\hat{N}_t = 4^{-\frac{1}{1+r}} t^{1/(1-\nu)}$. Likewise, $\hat{t}(N_o) \leq \hat{t}(N)$ if and only if $N_o \leq \hat{N}_t$, with $\hat{N}_t = \sqrt{\rho t}$. Let $b = \frac{1_t}{1+r} = 1/(1 + 2r)$. By Lemma 2.1, the expected return probability is equal to the expected average return probability. By Theorem 1.10, with $I_t(\omega) := \mathbb{P}_e[\hat{X}_t = \hat{X}_0] - 1/C_o(\omega)$, and $i = \hat{t}(N)$, $i = \hat{t}(N)$

$$E[I_t] = E[I_t|t < \hat{t}] E[t < \hat{t}] + E[I_t|t \leq \hat{t}] E[t \leq \hat{t}] + E[I_t|N_o < \hat{N}_t] E[N_o < \hat{N}_t]$$

The last sum results from comparing all eigenvalues of $\exp(-t(I - A))$ with the second largest, as in (14), which is apt for $t > \rho \sim N_o^2$, as given in Theorem 1.10. To summarize, for large $m$ we add all missing terms corresponding to $m \in \mathbb{N}$ and divide it into two new sums, the first with the terms up to $M$, and the second with terms $m = M + 1, M + 2, M + 3, \ldots$. This yields the following upper bound for $E[I_t]$:

$$e^{-\hat{N}_t/N} + a t^{-\frac{1-1}{1+r}} \left( M^b e^{-\frac{4t}{\delta M^2}} + e^{-\frac{Mt}{N}} \hat{C}^{-1} \sum_{m=1}^{\infty} (M + m)^b e^{-m/N} \right) + \hat{N}_t e^{-4t/(\delta \hat{N}_t^2)}$$

$$\leq e^{-\frac{t}{16N}} + a t^{-\frac{1-1}{1+r}} M^b \left( e^{-\frac{4t}{\delta M^2}} + e^{-\frac{Mt}{N}} E\left[ (1 + \frac{|C_o|}{M})^b \right] \right) + \sqrt{\rho} t^{1/4} e^{-\frac{4\sqrt{\rho t}}{3}}.$$

Balancing the two exponential terms inside the parentheses (by setting them equal) would yield $M = (\frac{4t}{\delta \hat{N}})^{1/3}$, but this would work only for some $t$, those for which $M$ is an integer. However, by setting $M_o = (\frac{4t}{\delta \hat{N}})^{1/3}$, we have $M - 1 \leq M_o \leq M$, and therefore $\exp\left(-\frac{4t}{\delta M_o^2}\right) \leq \exp\left(-\frac{4t}{\delta M^2}\right)$, as well as $\exp\left(-\frac{M_o}{N}\right) \leq e^{1/N} \exp\left(-\frac{M}{N}\right) \leq e \exp\left(-\frac{M}{N}\right)$. Since $b < 1$, it follows $(1 + x)^b \leq 1 + bx$ for $x \geq 0$. The statement follows by Jensen’s inequality ($x \mapsto x^b$ is concave) if $t$ obeys $t \geq \frac{\delta}{4N}$, implying $M \geq 1$. \hfill \Box

The corollary to Theorem 2.2 gives the corresponding bound in terms of $\chi_p := E_\mu[|C_o|]$ if the random events of edge removal of the Euclidean lattice of dimension $d$ are independent.

Corollary 2.3. For subcritical Bernoulli bond percolation on the Euclidean lattice in $d$ dimensions, the annealed return probability of the continuous time RRW with $\bar{a} = 4\sqrt{d}$, $\rho = 4^{(1+r)/2}$, and $\bar{c} = 160\sqrt{d} \chi_p^4$ has for $t \geq \max\{d/(4 \chi_p^2), 4^{1+\nu} \chi_p^{2(1-\nu)}\}$ the upper bound

$$E[1/|C_o|] + \bar{c} t^{-\frac{1}{6}(1+r)} e^{-\frac{4\sqrt{\rho}}{3}} + \sqrt{\rho} t^4 e^{-\frac{2\sqrt{\rho}}{3} t} + e^{-\frac{d}{2} \chi_p^2} + a \chi_p^b t^{-\frac{1}{2(1+r)}} e^{-\frac{d}{2} \chi_p^4}.$$
Proof: The only difference if compared with Theorem 2.2 consists of the cluster-size distribution having only an exponentially decaying tail, and isn’t necessarily of geometric (exponential) form in the first terms. We use \( \mathbb{E}[\cdot] := \mathbb{E}_\mu[\cdot] \). The following bound is from [22]. It is explicitly stated, when the specific exponential bound ‘sets in’: if \( m \geq \chi_p^2 =: L \),

\[
P[|C_o| \geq m] \leq 2 \exp(-\frac{m}{2L}).
\]

For \( m \geq L \), this implies \( P[|C_o| = m] \leq 2 \exp(-\frac{m}{2L}) \), and \( \mathbb{E}[f(|C_o|)] = \sum_{m=1}^{\infty} \phi(m)f(m) \), with

\[
\phi(m) \leq C_1 \chi_{<L}(m) + 2 \exp(-\frac{m}{2L}) \chi_{\geq L}(m),
\]

for some non-negative constant \( C_1 \leq 1 \). We mean to use Theorem 1.10 in the calculation of \( \mathbb{E}[\tilde{X}_t = \tilde{X}_0] - \mathbb{E}[1/|C_o|] \leq \mathbb{E}[e^{-\frac{4t}{3\chi_0}}X_{N_0}[S_t,S_t]}(N_o)\] + \( a \sqrt{t} \frac{1}{\sqrt{1+\nu}} \mathbb{E}[N_o e^{-\frac{4t}{3\chi_0}}X_{|S_t,S_t]}(N_o)\), where \( b = \frac{1}{1+\nu} \). First, we restrict \( t \geq t_o \), where \( t_o = t(L) \). Then \( \hat{N}_t \geq L \), which, under \( t < \hat{t}(N_o) \), implies \( |C_o| \geq L \) and \( \mathbb{P}[t \leq \hat{t}(|C_o|)] \leq e^{-\hat{N}_t/\hat{N}} \). Furthermore, we have

\[
\mathbb{E} \left[ |C_o|^b e^{-\frac{4t}{3\chi_o^2}} \right] \leq L^b e^{-\frac{4t}{3\chi_o^2}} + 2 \sum_{m=L+1}^{\infty} m^b \exp(-\frac{m}{2L} - \frac{4t}{\delta m^2}).
\]

From here, the proof follows exactly the lines of the one of the last theorem, except for the normalising constant \( \tilde{C}^{-1} \), which is missing in front of the sum. Using Theorem 2.2, we obtain the upper bound of \( \mathbb{E}[\tilde{X}_t = \tilde{X}_0] - \mathbb{E}[1/|C_o|] \)

\[
e^{-\hat{N}_t/\hat{N}} + \hat{N}_t e^{-\frac{4t}{3\chi_0^2}a} + \frac{a}{\sqrt{\nu} \sqrt{1+\nu}} \left( L^b e^{-\frac{4t}{3\chi_o^2}} + 2K^b e^{-\frac{4t}{3\chi_o^2}} \sum_{m=1}^{K} e^{-\frac{4t}{3\chi_o^2}} + 2 \sum_{m>K} m^b e^{-\frac{4t}{3\chi_o^2}} \right)
\]

\[
\leq e^{-\frac{\hat{N}_t}{\hat{N}}} + \hat{N}_t e^{-\frac{4t}{3\chi_0^2}a} + \frac{a}{\sqrt{\nu} \sqrt{1+\nu}} \left( L^b e^{-\frac{4t}{3\chi_o^2}} + 4LK^b e^{-\frac{4t}{3\chi_o^2}} + 4LK^b(1 + \chi_b^{2}e^{-\frac{4t}{3\chi_o^2}}) \right).
\]

Here, it was used that \( \sum_{K=1}^{\infty} (K+m)^b \exp(-m/(2L)) = \exp(-K/(2L)) K^b \mathbb{E}[(1 + |C_o|/K)^b] / \exp(1/(2L) - 1) \), and \( 1/\exp(1/(2L) - 1) \leq 2L \) together with Jensen’s inequality and the fact that \( x \mapsto (1 + x)^b \) is concave. We determine \( K \) by equating the last two exponentials:

\[
\frac{4t}{\delta K^2} = \frac{K}{2L} \iff K = \left( \frac{8t}{\delta L^2} \right)^{\frac{1}{3}}.
\]

Again, to respect \( K \in \mathbb{N} \), we use \( |K| \), and \( K - 1 \leq |K| \leq K \). We restrict the time parameter by \( t \geq \delta/(8L) \), such that \( K \geq 1 \). With \( \delta = 2d \), this yields the upper bound

\[
e^{-\frac{\hat{N}_t}{\hat{N}}} + m_t e^{-\frac{2d}{\delta m_t}} + a t^{-\frac{1}{1+\nu}} \left( L^b e^{-\frac{2d}{\delta b}} + \frac{4L^{1+b/3}}{(2d)^{b/3}} l_b^{b/3}(1 + \sqrt{\chi_b c/\chi_o^2}) e^{-\frac{2d}{\delta m_t}} \right).
\]

With the definitions of \( \hat{N}_t \), and \( \hat{N}_t \) from the proof of the last theorem, \( a < 4\sqrt{d} =: \bar{a} \), \( \bar{b} = \frac{1}{1+2a} \), and \( a(2\chi_b)^2(1+b/3) \frac{1}{(2d)^{b/3}} < 160\sqrt{d}\chi_o^4 =: \bar{c} \) the result follows. \( \square \)
Now, we turn to critical Bernoulli bond-percolation in two dimensions. Here, it is known that the connected component containing the origin is almost surely finite [26]. Due to the polynomial asymptotic behaviour of the annealed return probability [19, 5], the improvement gained by the interlacing method is more significant than in the sub-critical case. Let \( \Theta > 0 \) be the exponent in a scaling inequality for critical 2d Bernoulli bond percolation, i.e. \( \mathbb{P}[|C_o| \geq m] \leq B \cdot m^{-1/\Theta} \) holds for some \( B > 0 \) ([30]: \( \Theta \) is usually \( \delta \)).

**Theorem 2.4.** For subcritical Bernoulli bond percolation on the Euclidean lattice in \( d = 2 \) dimensions there is some \( \Theta \geq 5 \) and some \( C > 0 \), such that the annealed return probability of the continuous time RRW has an upper bound given by

\[
\mathbb{E}^{\omega}[\hat{X}_t = o] \leq \mathbb{E}\left[\frac{1}{|C_o|}\right] + \frac{C}{tw^{\Theta}}, \quad \text{with} \quad w > \frac{1}{2} \cdot \left(1 + \frac{1}{1 + 9 \log 16}\right).
\]

*Proof:* We have with \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) some given function and with \( I_t(\omega) := \mathbb{P}^{\omega}[\hat{X}_t = o] - 1/|C_o| \) and \( N_o := |C_o| \)

\[
\mathbb{E}[I] = \mathbb{E}[I_t|N_o \leq h(t)]\mathbb{P}[N_o \leq h(t)] + \mathbb{E}[I_t|N_o > h(t)]\mathbb{P}[N_o > h(t)] \\
\leq \mathbb{E}[I_t|N_o \leq h(t)] + \mathbb{P}[N_o > h(t)],
\]

since \( I_t(\omega) \leq 1 \), for all \( \omega \in \Omega = 2^E \). By Theorem 1.10 the first term can be bounded by

\[
a \cdot \mathbb{E}\left[(N^{1-\nu}/\sqrt{t})^H e^{-4t/(\delta N^2)} \right], \quad \text{where} \quad a = 1 + \frac{\sqrt{\pi} \delta}{2t^{1/2}}.
\]

Together with the scaling inequality, \( b = \frac{1}{1 + \nu} \), and the degree \( \delta = 2d = 4 \) this yields \( (\nu = \frac{1}{3 \ln 16} \sim 0.12) \)

\[
\mathbb{E}[I_t] \leq (1 + 2^{\nu} \sqrt{t}) \sqrt{\frac{1}{\nu + 1}} h(t)^b e^{-\frac{t}{h(t)^2}} + \frac{\bar{B}}{h(t)^{1/\Theta}}.
\]

Using \( \exp(-x) \leq 1/x \) for \( x = t/h(t)^2 \), we get by choosing \( h(t) = t^w \) for some \( w > 0 \)

\[
\mathbb{E}[I_t] \leq t^{w(1-\nu)/2 - (1-2w)} + \bar{B}t^{-w/\Theta}.
\]

Setting the exponents equal gives \( w = \frac{1}{2} \cdot \frac{3 + 2\nu}{3 + \nu} > \frac{1}{2}(1 + \frac{\nu}{3 + \nu}) \), and \( C = 1 + 2^{\nu} \sqrt{\pi + \bar{B}}. \) \( \square \)

**Remark:** Using \( I_t(\omega) \leq e^{-t/N_0^\Theta(\omega)} \) in the proof instead of (12) produces only \( w = \frac{1}{2} \).

To display the usefulness of Lemma 2.1, we conclude this section with a lower bound for \( \mathbb{E}[I_t] \) for critical percolation on the binary tree, where also a.s. \( |C_o| < \infty \) (see [7]).

**Theorem 2.5.** The annealed return probability of RRW on critical Bernoulli bond percolation on the binary tree fulfills \( \mathbb{E}[\mathbb{P}^{\omega}[\hat{X}_t = o]] \geq \mathbb{E}[1/|C_o|] + c/t^{3/2} \), for some \( c \geq e^{-4}/15. \)

*Proof:* By the easy part of Cheeger’s inequality (see for example [41], Lemma 3.3.7), there is an upper bound for the spectral gap \( \lambda \) of the continuous time RRW: \( \lambda \leq I := \min_{A \subset C_o, |A| \leq \frac{1}{2}|C_o|} 1/|A| \). By Lemma 3.1 \( , I \leq 4(\delta - 1)/|C_o| = 12/|C_o| \). By the scaling-inequality \( \phi(m) := \mathbb{P}[|C_o| = m] \geq \frac{1}{m} m^{-\frac{3}{4}} \) (see [22], sec.10) with \( I_t(\omega) := \mathbb{P}^{\omega}[\hat{X}_t = o] - 1/|C_o| \)

\[
\mathbb{E}[I_t] = \mathbb{E}\left[\frac{\text{Tr}^{1/2}}{|C_o|} - \frac{1}{|C_o|}\right] \geq \mathbb{E}\left[\frac{e^{-\lambda t}}{|C_o|}\right] = \sum_{m=1}^{\infty} \phi(m) \frac{e^{-t/12m}}{m} \geq \sum_{m\geq t} \frac{0.1}{\sqrt{m^3}} \frac{e^{-t/12m}}{m} \geq e^{-12\sqrt{t}/15}. \quad \square
\]

**Remark:** Compare this with the asymptotic type of the RRW on the homogeneous binary tree: \( \mathbb{P}^{\omega}[\hat{X}_n = o] = \mathbb{P}^{\omega}[X_n = o] \sim \tilde{\rho}^n n^{-3/2} \), where \( \tilde{\rho} = 2\sqrt{2}/3 \) [46]. See also [19], sec.4.
2.3 Discussion

The random field on the edges of the graph is required to be stationary with respect to the action of a transitive, unimodular subgroup of the group of automorphisms. Bernoulli percolation on $\mathbb{Z}^d$ is the simplest example. The asymptotic type (as defined in [46], chap. 14) of symmetric random walks with finite range on groups with polynomial growth of degree $d$ is $n^{-d/2}$. The first who have proven accurate bounds on the return probability of simple random walk on percolation clusters were Mathieu and Remy [38]. These were improved by Barlow [5]: The return probability of the simple random walk on the infinite percolation cluster in $\mathbb{Z}^d$ has the same asymptotic quenched and annealed estimates as the random walk on the original graph (see also Heicklen and Hoffmann[27], in which the upper bound has an extra logarithmic factor). The mixing time of simple random walk on the subgraph of an infinite percolation cluster induced by a large box has been estimated in [38] and [9], the latter of which is valid only for values of $p$ close to one. Most relevant for the present work is the paper by Fontès and Mathieu[19] who consider independent random conductances of networks on $\mathbb{Z}^d$. They determine the asymptotic type of quenched and annealed probabilities of a random walker. Moreover, monotonicity of the annealed return probability under removal of edges is proven. Results of this type are given in [2] for the much more general situation of random subgraphs of unimodular transitive graphs.

The present work focuses on a different subject: random walks on finite clusters of a percolative graph. The convergence rate of random walks on finite sets has been extensively studied. However, the finite percolative clusters don’t exhibit enough symmetry to allow the use of a general, nontrivial estimate for the spectral gap. In general, there are no typical isoperimetric properties, as opposed to the infinite component in the case of Bernoulli percolation on $\mathbb{Z}^d$ [38]. For the random walk restricted to a finite connected component (in the subcritical case of percolation models, or if possible conditioned on finiteness in the supercritical phase), the the quenched estimate is almost surely exponential. This is not the case for the annealed return probability. Unlike the case of the infinite cluster, there are arbitrarily large clusters with bounded edge-connectivity related to a small spectral gap. By the annealing, these large clusters contribute to some degree, and it is natural to ask whether this influences the expected return probability to the extent of a retarding effect. A result by I.Benjamini and O.Schramm used in [27] (Theorem 3.1) on finite subgraphs of percolation clusters implies that under removal of edges the expected return probability is non-decreasing. In particular, it is of interest by how much at most the clusters with a particularly large relaxation time (in terms of the cluster size), such as paths, slow down the walk, i.e. decrease the asymptotic decay of the return probability. This question is answered by the more difficult upper bound (1.8) which results from the second of the two inequalities in (9). Furthermore, it is of interest if the quality of stationarity in an invariant percolation leads to a speed up of the convergence of the simple bounds resulting from assuming for each connected component the geometry with the longest relaxation time. This is answered positively in the case of unimodular graphs, allowing for the application of Lemma 2.1, which states the equivalence of the expected return probability with the expected average return probability. For this quantity Theorem 2.2 testifies an improved
polynomial prefactor $\sim t^{-\frac{1}{6}(1+\nu/(1+\nu))}$ (see (11) for the definition of $\nu$).

Note that the return probability of the simple random walk on (deterministic) Cayley graphs of polycyclic groups with exponential growth (see [46], chapter III., 15), lamplighter groups [40], and Diestel-Leader graphs [8] without drift also show the characteristic $e^{-cn^{1/3}}$ for some $c > 0$. Our estimate concerns the finite percolative subgraphs of all $\delta$-regular graphs which have a *unimodular*, transitive subgroup $\Gamma$ of the automorphism group. Since the lamplighter group is amenable, it is unimodular [44], while among the Diestel-Leader graphs are examples of transitive graphs with a non-unimodular automorphism group (e.g. [8]) and they don’t fall into the present scope.

In [32], Lemmata 2.7, 2.9, it is shown, that the integrated density of states IDS (cumulative spectral function) $E \mapsto N_N(E) = \mathbb{E}(e_o, \mathbb{I}_{[0,\infty]}(E - L)e_o)$ (in the notation of [32]) of combinatorial Laplacians $L$ (the subscript $N$ stands for ‘Neumann Laplacians’; see the remark after Theorem 1.8) of subcritical Bernoulli percolation graphs on a Euclidean lattice $\mathbb{Z}^d$ has ‘Lifshits tails’ of a specific form: there are constants $\alpha^-, \alpha^+$, such that

$$\exp(-\alpha^- E^{-1/2}) \leq N_N(E) - N_N(0) \leq \exp(-\alpha^+ E^{-1/2}),$$

(21)

where $C_o$ is the finite percolation cluster containing the origin of the lattice. No further specification of these constants is given. For the upper bound, we specify a value of $\alpha^+$, for which this estimate is valid.

As was stated in section 1.2, the combinatorial Laplacian $L$ of a graph remains invariant under decoration with loops, so $L = \tilde{L} \equiv \delta \mathbb{I} - \hat{A}$, where the loops are those of the regularisation of definition 1.1. The transition matrix of the RRW on the graph is $A = \delta^{-1}\hat{A}$, and the normalised Laplacian $\hat{L}$, of the regularised graph is given by $\hat{L} = \mathbb{I} - \delta^{-1}(\delta \mathbb{I} - \tilde{L}) = \delta^{-1}L$. Therefore, spectral estimates about the RRW on a graph always also apply to the scaled Laplacian spectrum of the graph, as in Lemma 1.9. In case of Bernoulli percolation on $\mathbb{Z}^d$, the scaling factor is $\delta = 2d$.

The connection with the integrated density of states is given by the following well-known lemma, which states, that the IDS is the expected continuous time return probability of the RRW on $C_o$, the percolation cluster containing the origin.

**Lemma 2.6.** Let $\hat{X}_t$ be RRW on the finite, subcritical percolation cluster $C_o$ of Bernoulli bond percolation on $\mathbb{Z}^d$ containing the origin. Then

$$\int_0^\infty e^{-tE} dN_N(E) = \mathbb{E}_{\mu}^o [\hat{X}_{2dt} = o] - \mathbb{E}_{\mu}^o \left[ \frac{1}{|C_o|} \right].$$

**Proof:** Due to ergodicity ([32], Lemma 1.12), we have weak convergence of

$$N_N^\alpha(E) := \mathbb{E}_{\mu} \left[ \frac{1}{|\Lambda_n|} \text{Tr}_{[0,\infty]}(E - L_n) \right].$$
to \(N_N(E)\) as \(n \to \infty\), where \(L_n\) is the Laplacian belonging to the subgraph of the subcritical percolation graph induced by (restricted to) \(\Lambda_n := \{ -n + 1 \ldots, n \}^d\). Since the function \(x \mapsto \exp(-x)\) is bounded and continuous on \(\mathbb{R}_+\), this implies convergence of \(\int_0^\infty e^{-tE}dN_N^n(E) \) to \(\int_0^\infty e^{-tE}dN_N(E)\) as \(n \to \infty\). Moreover, since there are only finitely many configurations on the edges of the finite graph, the expected value \(\mathbb{E}_\mu\) is just the arithmetic mean over these configurations, so:

\[
\int_0^\infty e^{-tE}dN_N^n(E) = \mathbb{E}_\mu \left[ \int_0^t e^{-Et} \frac{1}{|\Lambda_n|} d\text{Tr}(\mathbb{1}_{[0,\infty]}(E - L_n)) \right].
\] (22)

The integration bound follows from \(\sigma(L^n) \subset [0, 4d]\), see [32], remark 1.10.(ii). Let \(\Pi_{\Lambda_n}\) be the \(l^2(\mathbb{Z}^d)\)-projector onto \(l^2(\Lambda_n)\). Since \(L_n = \Pi_{\Lambda_n}L\Pi_{\Lambda_n}\) is of finite rank, by the spectral theorem for real, symmetric matrices, the right hand side equals \(\mathbb{E}_\mu(1/|\Lambda_n|)\text{Tr}(\exp(-tL_n))\), the average return probability of the RRW on the subgraph of the percolation graph induced by \(\Lambda_n\).

We now use Lemma 1.9. Considering the perturbation \(S_n\) in \(L_n = \Pi_{\Lambda_n}L\Pi_{\Lambda_n} + S_n\), it is clear that \(S_n \geq 0\) and \(\text{rank}(S_n) \leq |\partial \Lambda_n|\) (the edge boundary of \(\Lambda_n\)). Then, by the second inequality of (10),

\[
\mathbb{E}_\mu(e_o, \exp(-tL)e_o) = \mathbb{E}_\mu \frac{1}{|\Lambda_n|} \text{Tr}\Pi_{\Lambda_n} \exp(-tL) \leq \mathbb{E}_\mu \frac{1}{|\Lambda_n|} \text{Tr}\Pi_{\Lambda_n} \exp(-tL_n).
\]

Since \(e^{-x} - e^{-y} \leq y - x\) for \(0 \leq y \leq x\), the right hand side can be further approximated:

\[
\mathbb{E}_\mu \frac{1}{|\Lambda_n|} \text{Tr}\Pi_{\Lambda_n} (\exp(-tL_n) - \exp(-tL)) + \mathbb{E}_\mu \frac{1}{|\Lambda_n|} \text{Tr}\Pi_{\Lambda_n} \exp(-tL) \leq t\frac{|\partial \Lambda_n|}{|\Lambda_n|} + \mathbb{E}_\mu(e_o, \exp(-tL)e_o).
\]

Therefore, due to the amenability of the Euclidean lattice,

\[
\lim_{n \to \infty} \mathbb{E}_\mu \frac{1}{|\Lambda_n|} \text{Tr} \exp(-tL_n) = \mathbb{E}(e_o, \exp(-tL)e_o). \quad (23)
\]

With the arguments given above, this implies \(\int_0^\infty e^{-tE}dN_N(E) = \mathbb{E}_\mu(e_o, e^{-tL}e_o)\).

Now, \(A = \mathbb{1} - \hat{\mathcal{C}} = \delta^{-1}\hat{\Lambda}\) is the transition operator of the RRW on the random connected components. With \(L = \hat{L} = \delta\hat{\mathcal{C}} = \delta(\mathbb{1} - A)\), the right hand side of (23) is \(\mathbb{E}_\mu \mathbb{P}^o[\hat{X}_t = o]\), the expected return probability of the RRW on \(\mathcal{C}_o\), where \(\delta = 2d\). By remark 1.15, (ii), \(N_N(0) = \mathbb{E}[1/|\mathcal{C}_o|]\) (see also [21]).

**Remarks:** Due to results in [35] (see theorem 2), the convergence of \(N_N^p\) to \(N_N(\cdot)\) is even uniform. The application of Lemma 1.9 is similar to the application of Theorem 3.1 to the proof of Theorem 8.1 in [27].

Now, Theorem 2.2 will be applied in such a way as to show: \(\alpha^+ \geq \frac{\chi}{3\sqrt{3}}\chi^{-4}\).
Theorem 2.7. The integrated density of states, $E \mapsto N_N(E)$, of Laplacians belonging to subcritical Bernoulli percolation on the Euclidean lattice in $d$ dimensions fulfills with $0 < E \leq \tilde{E}$, s.t. $\tilde{E} = \min\{ \frac{d^{1/3}}{\lambda^3_{\text{c}}}, \frac{\sqrt{d/\rho}}{2d^{1/3}\lambda^3_{\text{c}}}, \frac{8/3}{2d^{1/3}\lambda^3_{\text{c}}}, \rho \} \}$, $\tilde{c} = 160\sqrt{d}\chi^{4}_{\text{p}}\rho = 4\left(\frac{d^2 + \nu^2}{2\nu}\right)$

$$N_N(E) - N_N(0) \leq AE \exp(-\alpha E^{-1/2}) + BE^{-3/8} \exp(-\beta E^{-3/4}),$$

where $\alpha = 4 \frac{3\sqrt{3}}{\lambda^3_{\text{c}}} \chi^{-4}_{\text{p}}, A = 3\sqrt{3} 2^{-\frac{1}{3}} (1+r) \tilde{c} \left( \frac{d \chi^4_{\text{p}}}{\rho} \eta^{1 \nu} \right), B = 3\sqrt{d} \chi^{-4}_{\text{p}} \chi^{-1}_{\text{p}}, \beta = \frac{2\sqrt{d}}{\left(\sqrt{d/\rho}\right)^{3/2}}.$

Proof: We begin with a lower bound of the Laplace transform of the IDS:

$$\int_{0}^{\infty} e^{-tE} dN_N(E) \geq \int_{0}^{\epsilon} e^{-tE} dN_N(E) \geq e^{-\epsilon t}(N_N(\epsilon) - N_N(0)).$$

Due to Lemma 2.6, we may apply Corollary 2.3 to obtain an upper bound. From the corollary, we see that for all $t \geq t_0 := 2/(d\chi^4_{\text{p}})$ with $r = \nu/(1+\nu)$

$$N_N(\epsilon) - N_N(0) \leq e^{\epsilon t} \left( c \left( \frac{d t}{\rho} \right)^{-\frac{1}{3}} \right) + \sqrt{d} t^{1/4} e^{-\frac{d t}{4\rho}} + e^{-\frac{d t}{4\rho}} \chi^{-2}_{\text{p}} + a\chi^4_{\text{p}} t^{1/3} \chi^{-2}_{\text{p}},$$

for some $c > 1$. Then, $t = \frac{\sqrt{d}}{(c\epsilon)^{3/2}}$. This yields for the exponent

$$t \epsilon - c \epsilon t = -\frac{c - 1}{c^{3/2}} \frac{\sqrt{d}}{\chi^{-2}_{\text{p}}}.$$

We optimise the result by setting $c = 3$. The upper bound of $N_N(\epsilon) - N_N(0)$ follows by inserting the resulting $t$ into the inequality. Restricting $t = t(\epsilon) \geq \max\{ t_0, t_1 \}$ gives the range $[0, \tilde{E}]$ for $\epsilon$, in which the first two terms of the right hand side dominate.

Remark: A different situation prevails in the supercritical case. As shown in [39], the asymptotics of the SRW on the infinite cluster derived in [5] correspond with and characterise a different behaviour of the spectrum of the graph Laplacian in this case. The exponential decay of subcritical Bernoulli percolation clusters on quasi-transitive graphs has been shown in [3].

3 Interlacing as a new technique for comparison theorems

We now introduce a general method to derive bounds for the average return probability of the RRW on a finite graph $H$. It consists of four steps: i.) Interpreting $H$ as a connected component of a larger graph, by adjoining an additional connected component ($\tilde{H}$); ii.) Comparing the graph $H + \tilde{H}$ with a transformation thereof, called $H'$, on which the RRW has known spectrum (here, interlacing is used); iii.) Performing an estimation of the RRW on $H'$; iv.) Optimising the result, e.g. by choosing the optimal size of $\tilde{H}$. 21
3.1 Proof of the main result

We begin by noting that for RRW \( \hat{X} \) on a finite, simple, connected graph \( H \),

\[
P_{\hat{X}}[\hat{X}_t = \hat{X}_0] = \frac{1}{N} \text{Tr}[\exp(-t(1 - A))],
\]

which is the return probability of the RRW on \( H \) in continuous time (Observe: \( A = \delta^{-1} \hat{A} \), where \( \hat{A} \) is the adjacency matrix of the regularized graph). We then have

\[
P_{\hat{X}}[\hat{X}_t = \hat{X}_0] - \frac{1}{N} = \frac{1}{N} \sum_{j=2}^{N} \alpha_j^t,
\]

where \( \alpha_j = \exp(-(1 - \beta_j)) \), with \( \beta_j \) the \( j \)-th element of the spectrum of \( A \), ordered in a non-increasing fashion: \( 1 = \beta_1 > \beta_2 \geq \beta_3 \geq \cdots \geq \beta_N \). Furthermore, \( \alpha_j^t = (\alpha_t)^j \), and \( 1 = \alpha_1 > \alpha_2 \geq \cdots \geq \alpha_N > 0 \).

The idea is to compare RRW on \( H \) with RRW on another graph, a reference graph, about which the spectrum of the transition kernel is known. Interlacing can be usefully employed, whenever the rank of the transformation of \( H \) into the reference graph is small compared to the order of \( H \). We will use the finite path \( P_{N'} \) (for some size \( N' \)) as a reference graph. In general, the transformation of \( H \) into \( P_{N'} \) will entail rearranging a number of edges comparable to \( N \), the order of \( H \). This implies that the rank \( K \) of the transformation of the corresponding transition kernels does not fulfil \( K \ll N \).

To still make interlacing applicable, we remedy this circumstance by adjoining another graph to \( H \) as a different connected component, namely, the finite path \( \tilde{H} = P_{\tilde{N}} \). We then consider the transformation of the reducible graph \( H + P_{\tilde{N}} \) into \( H' := P_{N'} \) as the relevant transformation \( T \) for estimating the eigenvalues \( \beta_j \) by interlacing. The rank of \( T \) divided by the total number of vertices, \( N' = N + \tilde{N} \) will be arbitrarily small if \( \tilde{N} \) is chosen large enough.

In other words, we consider \( H \) as part of a larger graph by adjoining \( P_{\tilde{N}} \), which will not be changed during the transformation (see Fig. 3, a.). The size \( \tilde{N} \) of the adjoined component will then be taken large enough in order to guarantee that the number of edges which have to be rearranged to obtain \( P_{N'} \) is small in comparison to \( N' \). This enables to control the change of the spectrum of the transition kernel of RRW on \( H \) by interlacing, if considered as a perturbation of RRW on \( P_{N'} \). Enlarging \( \tilde{N} \) has the effect that \( N' = N + \tilde{N} \) becomes larger, and the intervals between successive eigenvalues of RRW on \( P_{N'} \) smaller.

In the case \( N = 1 \), there is nothing to prove, so assume \( N \geq 2 \). We proceed in four steps:
i.) To abbreviate notation, let \( \bar{\alpha}_i = \exp(-(1 - \bar{\beta}_i)) \), and \( \bar{\beta}_i \) be the \( i \)th eigenvalue of \( A \oplus \tilde{A} \) with \( \tilde{A} \) the transition kernel of a RRW on the finite path \( P_N \) of length \( \tilde{N} \). Taking some value \( B \in \{0, ..., N - 2\} \), we have

\[
P[\widehat{X}_t = \widehat{X}_0] - \frac{1}{N} \leq \frac{B}{N} \bar{\alpha}_2^t + \frac{1}{N} \sum_{i = \iota(B+2)}^{\iota(N)} \bar{\alpha}_i^t,
\]

where \( \iota : \{1, ..., N\} \to \{1, ..., N'\} \) is the injective map assigning \( \beta_j \) its index in the spectrum of \( A \oplus \tilde{A} \), which is the transition kernel of the reducible RRW on \( H + P_N \). We choose the eigenvalues \( \bar{\beta}_i \) also enumerated in a non-increasing way. This implies the following properties of \( j \mapsto \iota(j) \):

\[
\iota(j) \geq j, \quad \iota(N) \leq N', \quad j \in \{1, ..., N\}.
\]

ii.) We now perform the transformation \( T \) of the state space, the disconnected graph \( H + P_N \), to a connected graph \( G' \) of equal order \( N' = N + \tilde{N} \), namely, the finite path \( P_{N'} \).

![Figure 3: a.) Transformation of a disconnected graph \( H + P_{N'} \) into the path \( P_{N'} \); b.) schematic of the corresponding change of the spectrum: the imbedded \( \beta_j = \bar{\beta}_{\iota(j)} \) shift across less than \( K \) intervals of eigenvalues of RRW on \( P_{N'} \) ((9) in Theorem 1.8).](image)

We may assume \( H \) to be a tree. If \( H \) has cycles, we take out edges, until a spanning tree is achieved. This can only increase \( \alpha_j \). Then, the transformation \( T \) entails pairs of operations, each including a removal and an insertion of an edge. Note, the total number of insertions of edges doesn’t exceed \( N - 1 \). Correspondingly, the rank \( K \) of the negative part of the perturbation \( S \) in \( A' = A \oplus \tilde{A} + S \) is bounded by \( N \) (see Fig. 3 b.).

The following is the crucial step involving interlacing: let \( \alpha'_i = \exp(-(1 - \beta'_i)) \), where \( \beta'_{i+1} = 1 - \frac{2}{\pi}(1 - \cos(\pi i/N')) \) is the 1 + \( i \)th eigenvalue in \( \sigma(A') \). Now, we ask \( N \) (or \( N' \)) to be sufficiently large, such that the interval \( (\beta_{B+2}, 1) \) contains enough elements of the spectrum of \( A' \), such that \( \bar{\alpha}_{\iota(B+2)} \leq \alpha'_{K+X} \), for some integer \( X \geq 1 \), which we specify later. This is always possible, since \( |\sigma(A') \cap [1 - \epsilon, 1]| = O(N') \), for any \( \epsilon > 0 \), as \( N' \to \infty \).

In other words, we set \( N' \) appropriately, such that the condition

\[
\beta_{B+2} < \beta'_{K+X}
\]
holds. Then, also $\alpha_{i(B+2)} < \alpha'_{K+X}$, and $\nu(B+2) \geq K + X + 1$. So, by Theorem 1.8, the second term of the right hand side in (24) can be further bounded from above, and using condition (26), and (25):

$$\frac{1}{N} \sum_{i=B+2}^{\nu(N)} \alpha_i^t \leq \frac{1}{N} \sum_{i=B+2}^{\nu(N)} (\alpha'_{i-K})^t \leq \frac{1}{N} \sum_{i=X+K+1}^{\nu(N)} (\alpha'_{i-K})^t \leq \frac{1}{N} \sum_{i=1}^{\nu(N)} (\alpha_i^t)^t. \quad (27)$$

\[\text{iii.} \] Finally, approximating the cosine by a quadratic polynomial ($\cos(\pi x) \leq 1 - 2x^2$, for $x \in [0, 1]$), and using $\int_{-\infty}^{\infty} \exp(-x^2)dx \leq (\sqrt{\pi}/2) \exp(-\xi^2)$ (see [41], Example 2.1.1), this can be further bounded from above by

$$\frac{1}{N} \sum_{i=X+1}^{\nu(N)} \exp\left(-\frac{4}{\delta} \frac{\xi^2}{N^2}\right) \leq \frac{\nu(N)}{N} \int_{X/N'} \exp\left(-t \frac{4}{\delta} y^2\right) dy \leq \frac{\sqrt{N/N'}}{\sqrt{N}} \exp\left(-\frac{4}{\delta} \frac{X^2}{N^2}\right).$$

Putting it all together yields

$$ \mathbb{P} \{(\hat{X}_t = \hat{X}_0) \} - \frac{1}{N} \leq \frac{B}{N} \exp\left(-\frac{4t}{\delta N^2}\right) + \frac{\sqrt{N/N'}}{4\sqrt{t} \sqrt{N}} \exp\left(-\frac{4X^2}{\delta N^2}\right). \quad (28)$$

\[\text{iv.} \] What remains is to select the parameters $B, N'$, and $X$, such that, on the one hand, condition (26) is fulfilled, and, on the other, the bound becomes optimal.

First, to meet condition (26), we use Theorem 1.11 in the guise of Corollary 1.13. From (20), we see that it is fulfilled, if $\beta'_{[(B+1)/2)^\nu+1} < \beta'_{K+X}$. Equivalently, since $\beta'_{j} = 1 - \frac{2}{\nu}(1 - \cos(\pi j/N))$, and $\beta'_{1} = 1 - \frac{2}{\nu}(1 - \cos(\pi j/N'))$, the claim (26) is true if

$$\frac{[(B+1)/2)^\nu+1]}{N} > \frac{K+X}{N'}. \notag$$

Using $[x] + 1 > x$, and choosing $X = N$, it shows that this condition is met if

$$N' := 2\nu \frac{N(K+N)}{(B+1)^\nu}. \notag$$

Now, we use the observation made above that $K \leq N$, and set $B := [xN]$ with $x \in [0, \frac{1}{2})$ to give an upper bound of the right hand side of (28), using $xN - 1 < B \leq xN$:

$$\mathbb{P} \{(\hat{X}_t = \hat{X}_0) \} - \frac{1}{N} \leq \left( x + \frac{\sqrt{\pi \delta}}{21-\nu \sqrt{t}} \frac{N^{1-\nu}}{x^{\nu}} e^{-\frac{4t}{\pi \delta} (2^{-2(1+\nu)} x^{2\nu} N^{2\nu} - 1)} \right) e^{-\frac{4t}{\pi \delta} x^2}. \notag$$

By setting $x = (N^{1-\nu}/\sqrt{t})^{1/(1+\nu)}$ with $t > \bar{t} = 4^{1+\nu} N^{1-\nu}$ (making $x < \frac{1}{2}$), the exponential inside the parentheses remains less than one for $t \leq \bar{t} = 4^{-(1+\nu)^2/\nu} N^4$, whence the exponent remains negative. Inserting this bound, we obtain the result claimed by Theorem 1.10. \qed
3.2 A lemma on $\delta$-bounded trees

**Lemma 3.1.** Every finite tree $T = (V, E)$ of order $N = |V|$ with at least two vertices and largest degree $\delta \geq 2$ can be divided into two subtrees $T_1$ and $T_2$ by removal of an edge $e \in E$, such that the ratio $m$ of the cardinalities of the vertex sets of $T_1$ and $T_2$ obeys:

$$1 \leq m \leq 4(\delta - 1) - 1.$$

Before giving the proof, we consider some definitions concerning any finite tree $T = (V, E)$:

Let a path in $T$ of length $n \in \mathbb{N}$ be an $n + 1$-tuple of vertices $\bar{v} := (v_0, v_1, ..., v_n) \in V^{n+1}$ with $\{v_k-1, v_k\} \in E$, for $k \in \{1, ..., n\}$. Let $u \leftrightarrow w$ for $u, w \in V$ signify that there is a path $\bar{v}$ in $T$ of length $n$ for some $n \in \mathbb{N}$, such that $v_0 = u$, and $v_n = w$. Say that the path $\bar{v}$ crosses the edge $e \in E$ if $e = \{v_{k-1}, v_k\}$, for some $k \in \{1, ..., n\}$.

$$V_{u, e} := \{ w \in V : u \leftrightarrow w \text{ for some path } \bar{v} \text{ which does not cross } e \}.$$  

In other words, $V_{v, e}$ is the vertex-set of the subtree containing $v$ which results from removing edge $e$. Furthermore, call an edge $e_c = \{v, w\} \in E$ to be a central edge of $T$ if

$$|V_{v, e_c}| = \min_{e \in E} \max_{u \in e} |V_{u, e}|.$$

(29)

The minimum always exists, as long as $V_{v, e_c}$ is non-empty. This is the case for trees of order at least two. $e_c$ is however not unique, as the example of a ball in a regular tree shows. Removing one of the central edges from a tree results in two subtrees, the larger of which is as small as possible. For the order of the subtree $T|_{V_{u, e}}$, write $N_{u, e} := |V_{u, e}|$.

**Proof:** (Lemma 3.1) In the trivial cases $N = 2, 3$, the ratio $m$ is 1, 2, respectively. Let $N \geq 4$. Let $\{u, v\} := e_c$ be a central edge of $T$ with $N_{v, e_c} \geq N/2$. Consider an incident edge $e' = \{v, w\}$ of $e_c$ (where $e' \neq e_c$), such that $N_{w, e'} = \max\{|V_{r, \{v, r\}}| : \{v, r\} \in E\}$. Such an edge always exists, since $T_{v, e_c}$ has more than one vertex. Then, by the Pigeon-hole principle ([45], Lemma 1.3.10), we have

$$N_{w, e'} \geq \frac{N_{v, e_c} - 1}{\delta - 1}. \quad (30)$$

By the definition of the central edge, $N_{u, e_c} \leq N_{v, e_c}$, and $N_{v, e'} > N_{w, e'}$. On the other hand,

$$\frac{N_{v, e'}}{N_{w, e'}} = \frac{N - N_{w, e'}}{N_{w, e'}} = \frac{N}{N_{w, e'}} - 1 \leq (\delta - 1)\frac{N}{N_{v, e_c}} - 1 \leq 4(\delta - 1) - 1.$$

The first inequality is (30). The second inequality follows from the definition (29) of $e_c$, giving $N_{v, e_c} \geq \frac{N}{2}$, and from the assumption $N \geq 4$. $\square$
4 Outlook and Acknowledgement

In a forthcoming paper, the method of obtaining comparison theorems for return probabilities with interlacing is applied to random walks on infinite graphs.

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