PARABOLIC $k$–AMPLE BUNDLES

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Abstract. We construct projectivization of a parabolic vector bundle and a tautological line bundle over it. It is shown that a parabolic vector bundle is ample if and only if the tautological line bundle is ample. This allows us to generalize the notion of a $k$–ample bundle, introduced by Sommese, to the context of parabolic bundles. A parabolic vector bundle $E_*$ is defined to be $k$–ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ is $k$–ample. We establish some properties of parabolic $k$–ample bundles.

1. Introduction

For an algebraic vector bundle $E$ over a complex projective variety, let $\mathbb{P}(E)$ be the projective bundle parametrizing hyperplanes in the fibers of $E$. The tautological quotient line bundle on $\mathbb{P}(V)$ will be denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$. We recall that $E$ is called ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample.

Parabolic vector bundles on curves were introduced by Seshadri in [Se]. In [MY], Maruyama and Yokogawa generalized this notion to higher dimensional varieties. In [Bi2], parabolic analog of ample vector bundles were defined using the following standard criterion for ample vector bundles: A vector bundle $E$ over a complex projective variety $M$ is ample if and only if for any vector bundle $F$ on $M$, there is an integer $n(F)$ such that the vector bundle $S^n(E) \otimes F$ is generated by its global sections for all $n \geq n(F)$. The Le Potier vanishing theorem for ample vector bundles and the Hartshorne’s criterion for ample vector bundles on curves were generalized to parabolic ample bundles (see Theorem 4.4 and Theorem 3.1 of [Bi2]).

Given a parabolic vector bundle $E_*$, here we define $\mathbb{P}(E_*)$, a parabolic analog of the projective bundle. This is done using the notion of ramified principal bundles (see [BBN] and [Bi4] for ramified principal bundles). We also construct a tautological line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ on $\mathbb{P}(E_*)$.

In Proposition 3.3 we show that a parabolic vector bundle $E_*$ is ample if and only if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1) \rightarrow \mathbb{P}(E_*)$ is ample. Proposition 3.3 is deduced using a parabolic analog of the above mentioned criterion for ample vector bundles in terms of vanishing of higher cohomologies (this criterion for parabolic vector bundles is proved in Theorem 2.1).

Sommese introduced the notion of a $k$–ample vector bundle (see Definition 1.3 in page 232 of [So]). We recall that a line bundle $L$ over a complex projective variety $M$ is called

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A vector bundle $E$ is called $k$–ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E)$ is $k$–ample.

The constructions of the projectivization of a parabolic bundle and the tautological line bundle allow us to generalize the notion of $k$–ample bundles to the context of parabolic bundles. More precisely, a parabolic vector bundle $E_*$ is called $k$–ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1) \to \mathbb{P}(E_*)$ is $k$–ample. We prove some properties of parabolic $k$–ample bundles.

2. PARABOLIC AMPLE BUNDLES

Let $X$ be an irreducible smooth projective variety defined over $\mathbb{C}$. Let $D \subset X$ be a simple normal crossing divisor; this means that $D$ is effective and reduced, each irreducible component of $D$ is smooth, and the components intersect transversally. Let

\[ D = \sum_{i=1}^{\ell} D_i \]

be the decomposition of $D$ into irreducible components. The above condition that the irreducible components of $D$ intersect transversally means that if

\[ x \in D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k} \subset D \]

is a point where $k$ distinct components of $D$ meet, and $f_{i_j}, j \in [1,k]$, is the local equation of the divisor $D_{i_j}$ around $x$, then $\{df_{i_j}(x)\}$ is a linearly independent subset of the holomorphic cotangent space $T^*_x X$ of $X$ at $x$. This implies that for any choice of $k$ integers

\[ 1 \leq i_1 < i_2 < \cdots < i_k \leq \ell, \]

each connected component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ is a smooth subvariety of $X$.

Let $E_0$ be an algebraic vector bundle over $X$. For each $i \in [1,\ell]$, let

\[ E_0|_{D_i} = F_1^i \supseteq F_2^i \supseteq F_3^i \supseteq \cdots \supseteq F_{m_i}^i \supseteq F_{m_i+1}^i = 0 \]

be a filtration by subbundles of the restriction of $E_0$ to $D_i$. In other words, each $F_j^i$ is a subbundle of $E_0|_{D_i}$ with $\text{rank}(F_j^i) > \text{rank}(F_{j+1}^i)$ for all $j \in [1,m_i]$.

A quasiparabolic structure on $E_0$ over $D$ is a filtration as above of each $E_0|_{D_i}$ satisfying the following extra condition: Take any $k \in [1,\ell]$ and take integers $\{i_{j,i}\}_{j=1}^{k}$ as in (2.3). If we fix some $F_{n_j}^{i_{j,i}}, n_j \in [1,m_{i_{j,i}}]$, then over each connected component $S$ of $D_{i_{1}} \cap D_{i_{2}} \cap \cdots \cap D_{i_{k}}$, the intersection

\[ \bigcap_{j=1}^{k} F_{n_j}^{i_{j,i}} \subset E_0|_{D_{i_{1}} \cap \cdots \cap D_{i_{k}}} \]

gives a subbundle of the restriction of $E_0$ to $S$. It should be clarified that the rank of this subbundle may depend on the choice of the connected component $S \subset D_{i_{1}} \cap \cdots \cap D_{i_{k}}$. 
For a quasiparabolic structure as above, \textit{parabolic weights} are a collection of rational numbers

\[(2.5) \quad 0 \leq \lambda^i_1 < \lambda^i_2 < \lambda^i_3 < \cdots < \lambda^i_{m_i} < 1,\]

where \(i \in [1, \ell]\). The parabolic weight \(\lambda^i_j\) corresponds to the subbundle \(F^i_j\) in (2.4). A \textit{parabolic structure} on \(E_0\) is a quasiparabolic structure with parabolic weights. A vector bundle over \(X\) equipped with a parabolic structure on it is also called a \textit{parabolic vector bundle}. (See [MY], [Se].)

For notational convenience, a parabolic vector bundle defined as above will be denoted by \(E^*\), while \(E_0\) will be referred to as the underlying vector bundle for \(E^*\). The divisor \(D\) is called the \textit{parabolic divisor} for \(E^*\). We fix \(D\) once and for all, so the parabolic divisor of all parabolic vector bundles on \(X\) considered here will be \(D\).

The definitions of direct sum, tensor product and dual of vector bundles extend naturally to parabolic vector bundles. Similarly, symmetric and exterior powers of parabolic vector bundles are also constructed (see [MY], [Bi2], [Yo]).

We will now recall from [Bi2] the definition of a parabolic ample bundle.

A parabolic vector bundle \(E_*\) is called \textit{ample} if for every vector bundle \(V \to X\), there is an integer \(n(V)\) such that for all \(n \geq n(V)\), the vector bundle \(S^n(E_*)_0 \otimes V\) is generated by it global sections, where \(S^n(E_*)_0\) is the vector bundle underlying the parabolic symmetric product \(S^n(E_*)\). (See Definition 2.3 in page 514 of [Bi2].)

In Definition 2.3 of [Bi2], we take \(V\) to be a coherent sheaf. Since any coherent sheaf is a quotient of a vector bundle, and the quotient of a vector bundle generated by global sections is also generated by global sections, it is enough to check the above criterion for locally free sheaves.

**Theorem 2.1.** A parabolic vector bundle \(E_*\) is ample if and only if for every parabolic vector bundle \(F_*\), there is an integer \(n(F_*)\) such that for all \(n \geq n(F_*)\),

\[H^i(X, (S^n(E_*) \otimes F_*)_0) = 0\]

for all \(i \geq 1\), where \((S^n(E_*) \otimes F_*)_0\) is the vector bundle underlying the parabolic tensor product \(S^n(E_*) \otimes F_*\).

**Proof.** First assume that \(E_*\) is ample. Take any parabolic vector bundle \(F_*\) on \(X\).

In [Bi1], a bijective correspondence between parabolic bundles with parabolic weights of the form \(a/a_0\), where \(a_0\) is a fixed integer, and orbifold vector bundles is established. According to it, there is a finite (ramified) algebraic Galois covering

\[(2.6) \quad \gamma : Y \to X,\]

where \(Y\) is an irreducible smooth projective variety with the following property: there are \(\text{Gal}(\gamma)-\text{linearized} vector bundles \(E'\) and \(F'\) over \(Y\) that correspond to the parabolic vector bundles \(E_*\) and \(F_*\) respectively.

For notational convenience, the Galois group \(\text{Gal}(\gamma)\) will be denoted by \(\Gamma\).
Since $E_*$ is parabolic ample, the vector bundle $E'$ is ample (see Lemma 4.6 in page 522 of [Bi2]). As $E'$ is ample, there is an integer $m_0$ such that for all $n \geq m_0$,

\[(2.7) \quad H^i(Y, S^n(E') \otimes F') = 0\]

for all $i \geq 1$ (see Proposition (3.3) in page 70 of [Ha]). The map $\gamma$ being finite, for any vector bundle $W$ on $Y$, we have a canonical identification $H^i(Y, W) = H^i(X, \gamma_*W)$. Therefore, from (2.7),

\[(2.8) \quad H^i(X, \gamma_*(S^n(E') \otimes F')) = 0\]

for all $i \geq 1$ and all $n \geq m_0$.

The above mentioned correspondence between parabolic vector bundles and $\Gamma$–linearized vector bundles takes the tensor product of any two $\Gamma$–linearized vector bundles to the parabolic tensor product of the corresponding parabolic vector bundles. Therefore, the parabolic vector bundle over $X$ corresponding to the $\Gamma$–linearized vector bundle $S^n(E') \otimes F'$ is the parabolic tensor product $S^n(\gamma_*(S^n(E') \otimes F'))$. Hence

\[(\gamma_*(S^n(E') \otimes F'))^\Gamma = (S^n(E_*) \otimes F_*)_0\]

(see (2.9) in page 310 of [Bi1]), where

\[(2.9) \quad (\gamma_*(S^n(E') \otimes F'))^\Gamma \subset \gamma_*(S^n(E') \otimes F')\]

is the sheaf of $\Gamma$–invariants. We note that the subsheaf in (2.9) is a direct summand (see the proof of Proposition 4.10 in page 525 of [Bi2]). Therefore, from (2.8) it follows that

\[(2.10) \quad H^i(X, (S^n(E_*) \otimes F_*)_0) = 0\]

for all $n \geq m_0$ and all $i \geq 1$.

To prove the converse, assume that given any parabolic vector bundle $F_*$, there is an integer $n(F_*)$ such that for all $n \geq n(F_*)$,

\[(2.11) \quad H^i(X, (S^n(E_*) \otimes F_*)_0) = 0\]

for all $i \geq 1$. We will prove that $E_*$ is parabolic ample.

Take any coherent sheaf $W$ on $X$. We will show that

\[(2.12) \quad H^i(X, S^n(E_*)_0 \otimes W) = 0\]

for all $i \geq 1$ and all $n$ sufficiently large, where $S^n(E_*)_0$ is the vector bundle underlying the parabolic symmetric product $S^n(E_*)$.

The coherent sheaf $W$ is a quotient of some locally free coherent sheaf. Let $V$ be a locally free sheaf on $X$, and let

$\rho : V \rightarrow W$

be a surjective homomorphism. The kernel of $\rho$ will be denoted by $\mathcal{K}$. So we have a short exact sequence

\[(2.13) \quad 0 \rightarrow S^n(E_*)_0 \otimes \mathcal{K} \rightarrow S^n(E_*)_0 \otimes V \xrightarrow{\text{Id} \otimes \rho} (S^n(E_*)_0 \otimes W) \rightarrow 0\]
(since \((S^n(E_*)_0\) is locally free, tensoring with it preserves exactness of a sequence). Equip \(V\) with the trivial parabolic structure (so it has no nonzero parabolic weight). Consequently, \((S^n(E_*) \otimes V)_0 = S^n(E_*)_0 \otimes V\). Invoking the given condition in \((2.11)\), from the long exact sequence of cohomologies associated to the short exact sequence in \((2.13)\) we conclude that given any \(i \geq 1\),
\[
H^i(X, S^n(E_*)_0 \otimes W) = 0
\]
for all \(n\) sufficiently large if \(H^{i+1}(X, S^n(E_*)_0 \otimes \mathcal{K}) = 0\) for all \(m\) sufficiently large. Using this inductively, and noting that all cohomologies of degree larger than \(\dim X\) vanish, we conclude that \((2.12)\) holds.

For any point \(x \in X\), let \(I_x \subset \mathcal{O}_X\) be the ideal sheaf. For any vector bundle \(F\) on \(X\), consider the short exact sequence of coherent sheaves
\[
0 \rightarrow S^n(E_*)_0 \otimes F \otimes I_x \rightarrow S^n(E_*)_0 \otimes F \rightarrow ((S^n(E_*)_0 \otimes F)_x \rightarrow 0.
\]
Using the above observation that \(H^1(X, S^n(E_*)_0 \otimes F \otimes I_x) = 0\) for all \(n\) sufficiently large, from the long exact sequence of cohomologies associated to this short exact sequence we conclude that for all \(n\) sufficiently large, the fiber \(((S^n(E_*)_0 \otimes F)_x\) is generated by the global sections of \(S^n(E_*)_0 \otimes F\). Now using the Noetherian property of \(X\) it follows that for all \(n\) sufficiently large, the vector bundle \(S^n(E_*)_0 \otimes F\) is generated by its global sections (see Proposition 2.1 in page 65 of [Ha]). Therefore, the parabolic vector bundle \(E_*\) is ample. \(\Box\)

### 3. Ramified bundles and ampleness

#### 3.1. Ramified bundles and parabolic bundles.
The complement of \(D\) in \(X\) will be denoted by \(X - D\); we will avoid the more standard notation \(X \setminus D\) in order to avoid any confusion with left quotient which we will frequently need.

Let
\[
\varphi : E_{\text{GL}(r, \mathbb{C})} \longrightarrow X
\]
be a ramified principal \(\text{GL}(r, \mathbb{C})\)–bundle with ramification over \(D\) (see \([BBN], [Bi3], [Bi4]\) for the definition). We briefly recall its defining properties. The total space \(E_{\text{GL}(r, \mathbb{C})}\) is a smooth complex variety equipped with an algebraic right action of \(\text{GL}(r, \mathbb{C})\)

\[
f : E_{\text{GL}(r, \mathbb{C})} \times \text{GL}(r, \mathbb{C}) \longrightarrow E_{\text{GL}(r, \mathbb{C})},
\]

and the following conditions hold:

1. \(\varphi \circ f = \varphi \circ p_1\), where \(p_1\) is the natural projection of \(E_{\text{GL}(r, \mathbb{C})} \times \text{GL}(r, \mathbb{C})\) to \(E_{\text{GL}(r, \mathbb{C})}\),
2. for each point \(x \in X\), the action of \(\text{GL}(r, \mathbb{C})\) on the reduced fiber \(\varphi^{-1}(x)_{\text{red}}\) is transitive,
3. the restriction of \(\varphi\) to \(\varphi^{-1}(X - D)\) makes \(\varphi^{-1}(X - D)\) a principal \(\text{GL}(r, \mathbb{C})\)–bundle over \(X - D\),
(4) for each irreducible component $D_i \subset D$, the reduced inverse image $\varphi^{-1}(D_i)_{\text{red}}$ is a smooth divisor and

$$\hat{D} := \sum_{i=1}^{\ell} \varphi^{-1}(D_i)_{\text{red}}$$

is a normal crossing divisor on $E_{\text{GL}(r,\mathbb{C})}$, and

(5) for any point $x$ of $D$, and any point $z \in \varphi^{-1}(x)$, the isotropy group

\begin{equation}
G_z \subset \text{GL}(r,\mathbb{C}),
\end{equation}

for the action of $\text{GL}(r,\mathbb{C})$ on $E_{\text{GL}(r,\mathbb{C})}$, is a finite group, and if $x$ is a smooth point of $D$, then the natural action of on the quotient line $T_z E_{\text{GL}(r,\mathbb{C})}/T_z \varphi^{-1}(D)_{\text{red}}$ is faithful.

We will note some some properties of the isotropy subgroup $G_z$ in condition (5) of the above list. Let

$$D_{\text{sm}} \subset D$$

be the smooth locus of the divisor. Take any $x \in D_{\text{sm}}$, and take any $z \in \varphi^{-1}(x)$. The group of linear automorphisms of the complex line $T_z E_{\text{GL}(r,\mathbb{C})}/T_z \varphi^{-1}(D)_{\text{red}}$ in condition (5) is $\mathbb{C}^*$, and any finite subgroup of $\mathbb{C}^*$ is a cyclic group. Therefore, we conclude that the isotropy subgroup $G_z$ is a finite cyclic group if $x \in D_{\text{sm}}$ (because $G_z$ acts faithfully on $T_z E_{\text{GL}(r,\mathbb{C})}/T_z \varphi^{-1}(D)_{\text{red}}$). Take any $z' \in E_{\text{GL}(r,\mathbb{C})}$ such that $\varphi(z') = \varphi(z)$. There is an element $g \in \text{GL}(r,\mathbb{C})$ such that $f(z, g) = z'$. Therefore, $G_z$ is isomorphic to $G_{z'}$. As $z$ moves over a fixed connected component of $\varphi^{-1}(D_{\text{sm}})$, the order of the finite cyclic group $G_z$ remains unchanged. Fix an integer $m \geq 2$. Let

$$(S(m)) \subset \text{GL}(r,\mathbb{C})$$

be the subset of all elements of order exactly $m$. This $S(m)$ is a disjoint union of finitely many closed orbits for the the adjoint action of $\text{GL}(r,\mathbb{C})$ on itself. Therefore, as $z$ moves over a fixed connected component of $\varphi^{-1}(D_{\text{sm}})$, the conjugacy class of the subgroup $G_z \subset \text{GL}(r,\mathbb{C})$ remains unchanged.

There is a natural bijective correspondence between the ramified principal $\text{GL}(r,\mathbb{C})$–bundles with ramification over $D$ and the parabolic vector bundles of rank $r$ with $D$ as the parabolic divisor (see \cite{BBN}, \cite{Bi3}). This correspondence is recalled below.

Take the standard $\text{GL}(r,\mathbb{C})$–module $\mathbb{C}^r$; for notational convenience, we will denote this $\text{GL}(r,\mathbb{C})$–module by $V$. Recall that

$$E_{\text{GL}(r,\mathbb{C})}^0 := \varphi^{-1}(X - D) \longrightarrow X - D$$

is a usual principal $\text{GL}(r,\mathbb{C})$–bundle. Let

$$E_V = E_{\text{GL}(r,\mathbb{C})}^0(V) := E_{\text{GL}(r,\mathbb{C})}^0 \times^{\text{GL}(r,\mathbb{C})} V \longrightarrow X - D$$

be the associated vector bundle. This vector bundle has a natural extension to $X$ as a parabolic vector bundle (its construction is similar to the construction of a parabolic vector bundle from an orbifold vector bundle; see \cite{Bi1}). Therefore, starting from a ramified
principal GL($r, \mathbb{C}$)–bundle $\varphi : E_{GL(r, \mathbb{C})} \rightarrow X$ we get a parabolic vector bundles over $X$ with parabolic structure over $D$.

We will give an alternative description of the correspondence.

Let $E_*$ be a parabolic vector bundle of rank $r$ over $X$. Let $E' \rightarrow Y$ be the corresponding $\text{Gal}(\gamma)$–linearized vector bundle, where $\gamma$ is the Galois covering in (2.6). As before, $\text{Gal}(\gamma)$ will be denoted by $\Gamma$. Let

$$E'_{GL(r, \mathbb{C})} \rightarrow Y$$

be the principal GL($r, \mathbb{C}$)–bundle defined by $E'$. So the fiber of $E'_{GL(r, \mathbb{C})}$ over any point $y \in Y$ is the space of all linear isomorphisms from $\mathbb{C}^r$ to the fiber $E'_y$. The $\Gamma$–linearization of $E'$ produces a left action of the Galois group $\Gamma$ on the total space of $E'_{GL(r, \mathbb{C})}$. The actions of $\text{GL}(r, \mathbb{C})$ and $\Gamma$ on $E'_{GL(r, \mathbb{C})}$ commute, so the quotient $\Gamma\backslash E'_{GL(r, \mathbb{C})}$ for the action of $\Gamma$ on $E'_{GL(r, \mathbb{C})}$ has an action of $\text{GL}(r, \mathbb{C})$. The projection of $E'_{GL(r, \mathbb{C})}$ to $Y$ produces a projection of $\Gamma\backslash E'_{GL(r, \mathbb{C})}$ to $\Gamma\backslash Y = X$. This resulting morphism

$$\Gamma\backslash E'_{GL(r, \mathbb{C})} \rightarrow X$$

defines a ramified principal $\text{GL}(r, \mathbb{C})$–bundle.

Conversely, if $F_{GL(r, \mathbb{C})} \rightarrow X$ is a ramified principal $\text{GL}(r, \mathbb{C})$–bundle, then there is a finite (ramified) Galois covering

$$\gamma : Y \rightarrow X$$

such that the normalization $\overline{F_{GL(r, \mathbb{C})} \times_X Y}$ of the fiber product $F_{GL(r, \mathbb{C})} \times_X Y$ is smooth. The projection $\overline{F_{GL(r, \mathbb{C})} \times_X Y} \rightarrow Y$ is a principal GL($r, \mathbb{C}$)–bundle equipped with an action of the Galois group $\Gamma := \text{Gal}(\gamma)$. Let $F_V := \overline{F_{GL(r, \mathbb{C})} \times_X Y} (V)$ be the vector bundle over $Y$ associated to the principal GL($r, \mathbb{C}$)–bundle $F_{GL(r, \mathbb{C})} \times_X Y$ for the standard GL($r, \mathbb{C}$)–module $V := \mathbb{C}^r$. The action of $\Gamma$ on $Y$ induces an action of $\Gamma$ on $\overline{F_{GL(r, \mathbb{C})} \times_X Y}$; this action of $\Gamma$ on $\overline{F_{GL(r, \mathbb{C})} \times_X Y}$ commutes with the action of $\text{GL}(r, \mathbb{C})$ on $\overline{F_{GL(r, \mathbb{C})} \times_X Y}$. Hence the action of $\Gamma$ on $\overline{F_{GL(r, \mathbb{C})} \times_X Y}$ induces an action of $\Gamma$ on the above defined associated bundle $F_V$ making $F_V$ a $\Gamma$–linearized vector bundle. Let $E_*$ be the parabolic vector bundle of rank $r$ over $X$ associated to this $\Gamma$–linearized vector bundle $F_V$.

The above construction of a parabolic vector bundle of rank $r$ from a ramified principal $\text{GL}(r, \mathbb{C})$–bundle is the inverse of the earlier construction of a ramified principal $\text{GL}(r, \mathbb{C})$–bundle from a parabolic vector bundle.

3.2. Projectivization and the tautological line bundle. Let $E_*$ be a parabolic vector bundle over $X$ of rank $r$. Let

$$(3.2) \quad \varphi : E_{GL(r, \mathbb{C})} \rightarrow X$$

be the corresponding ramified principal $\text{GL}(r, \mathbb{C})$–bundle. Let $\mathbb{P}^{r-1}$ be the projective space parametrizing the hyperplanes in $\mathbb{C}^r$. The standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ produces an action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{P}^{r-1}$. Let

$$(3.3) \quad \mathbb{P}(E_*) = E_{GL(r, \mathbb{C})}(\mathbb{P}^{r-1}) := E_{GL(r, \mathbb{C})} \times_{\text{GL}(r, \mathbb{C})} \mathbb{P}^{r-1} \rightarrow X$$
be the associated (ramified) fiber bundle.

**Definition 3.1.** We will call $\mathbb{P}(E_\ast)$ the *projective bundle* associated to the parabolic vector bundle $E_\ast$.

Take a point $x \in D$; it should be clarified that $x$ need not be a smooth point of $D$. Take any $z \in \varphi^{-1}(x)$, where $\varphi$ is the morphism in (3.2). As in (3.1), let $G_z \subset \text{GL}(r, \mathbb{C})$ be the isotropy subgroup for $z$ for the action of $\text{GL}(r, \mathbb{C})$ on $E_{\text{GL}(r, \mathbb{C})}$. We recall that $G_z$ is a finite group. Let $n_x$ be the order of $G_z$ (it is independent of the choice of $z$ because $\text{GL}(r, \mathbb{C})$ acts transitively on the fiber $\varphi^{-1}(x)$). The number of distinct integers $n_x$, $x \in D$, is finite. Let

$$
N(E_\ast) := \text{l.c.m.}\{n_x\}_{x \in D}
$$

be the least common multiple of all these integers $n_x$.

As before, $\mathbb{P}^{r-1}$ is the projective space parametrizing the hyperplanes in $\mathbb{C}^r$. For any point $y \in \mathbb{P}^{r-1}$, let

$$
H_y \subset \text{GL}(r, \mathbb{C})
$$

be the isotropy subgroup for the action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{P}^{r-1}$ constructed using the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. So $H_y$ is a maximal parabolic subgroup of $\text{GL}(r, \mathbb{C})$. Let $\mathcal{O}_{\mathbb{P}^{r-1}}(1) \rightarrow \mathbb{P}^{r-1}$ be the tautological quotient line bundle. The group $H_y$ in (3.5) acts on the fiber of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ over the point $y$.

From the definition of $N(E_\ast)$ in (3.4) it follows for any $z \in \varphi^{-1}(D)$ and any $y \in \mathbb{P}^{r-1}$, the group $G_z \cap H_y \subset \text{GL}(r, \mathbb{C})$ acts trivially on the fiber of the line bundle

$$
\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast)) := \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{\otimes N(E_\ast)}
$$

over the point $y$; the group $G_z$ is defined in (3.1).

Consider the action of $\text{GL}(r, \mathbb{C})$ on the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))$ constructed using the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Let

$$
E_{\text{GL}(r, \mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))) := E_{\text{GL}(r, \mathbb{C})} \times^{\text{GL}(r, \mathbb{C})} \mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast)) \rightarrow X
$$

be the associated fiber bundle. Since the natural projection

$$
\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast)) \rightarrow \mathbb{P}^{r-1}
$$

intertwines the actions of $\text{GL}(r, \mathbb{C})$ on $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))$ and $\mathbb{P}^{r-1}$, it produces a projection between the associated bundles

$$
E_{\text{GL}(r, \mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))) \rightarrow \mathbb{P}(E_\ast),
$$

where $\mathbb{P}(E_\ast)$ is the associated bundle constructed in (3.3).

Using the above observation that $G_z \cap H_y$ acts trivially on the fiber of $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))$ over $y$ it follows immediately that $E_{\text{GL}(r, \mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast)))$ in (3.6) is an algebraic line bundle over the variety $\mathbb{P}(E_\ast)$.

**Definition 3.2.** The line bundle $E_{\text{GL}(r, \mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_\ast))) \rightarrow \mathbb{P}(E_\ast)$ will be called the *tautological line bundle*; this tautological line bundle will be denoted by $\mathcal{O}_{\mathbb{P}(E_\ast)}(1)$. 
For any positive integer \( m \), the line bundle \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1)^{\otimes m} \) (respectively, \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1)^{\otimes m} \ast \)) will be denoted by \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(m) \) (respectively, \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(-m) \)). Also, \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(0) \) will denote the trivial line bundle.

**Proposition 3.3.** A parabolic vector bundle \( E_\ast \) is ample if and only if the tautological line bundle \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1) \rightarrow \mathbb{P}(E_\ast) \) is ample.

*Proof.* First assume that the parabolic vector bundle \( E_\ast \) is ample. Let \( E' \rightarrow Y \) be the corresponding \( \text{Gal}(\gamma) \)-linearized vector bundle over \( Y \), where \( \gamma \) is the covering in \( (2.6) \). The vector bundle \( E' \) is ample because \( E_\ast \) is parabolic ample (Lemma 4.6 in page 522 of \cite{F152}). Hence the tautological quotient line bundle \( \mathcal{O}_{\mathbb{P}(E')}(1) \rightarrow \mathbb{P}(E') \) is ample, where \( \mathbb{P}(E') \) is the space of all hyperplanes in the fibers of \( E' \). Since \( \mathcal{O}_{\mathbb{P}(E')}(1) \) is ample, the line bundle

\[
\mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \rightarrow \mathbb{P}(E')
\]

is ample, where \( N(E_\ast) \) is defined in \( (3.3) \).

The action of \( \text{Gal}(\gamma) \) on \( E' \) produces a left action of \( \text{Gal}(\gamma) \) on \( \mathbb{P}(E') \). Evidently, the variety \( \mathbb{P}(E_\ast) \) in \( (3.3) \) is the quotient

\[
\text{Gal}(\gamma) \backslash \mathbb{P}(E') = \mathbb{P}(E_\ast).
\]

We note that the quotient \( \text{Gal}(\gamma) \backslash \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \) is a line bundle over \( \text{Gal}(\gamma) \backslash \mathbb{P}(E') \) because the isotropy subgroups, for the action of \( \text{Gal}(\gamma) \) on \( \mathbb{P}(E') \), act trivially on the corresponding fibers of \( \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \). We have a natural isomorphism of line bundles

\[
\text{Gal}(\gamma) \backslash \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) = \mathcal{O}_{\mathbb{P}(E_\gamma)}(1).
\]

Since the line bundle \( \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \) is ample, from \( (3.3) \) it follows that the line bundle \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1) \) is ample.

To prove the converse, assume that the tautological line bundle \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1) \) is ample. Take any parabolic vector bundle \( F_\ast \) on \( X \). As in the first part of the proof of Theorem \( 2.1 \) take the covering \( \gamma \) in \( (2.6) \) such that there is a \( \text{Gal}(\gamma) \)-linearized vector bundle \( E' \) (respectively, \( F' \)) over \( Y \) that corresponds to the parabolic vector bundle \( E_\ast \) (respectively, \( F_\ast \)). We will use the criterion in Theorem \( 2.1 \) to show that the parabolic vector bundle \( E_\ast \) is ample.

As the quotient map \( \mathbb{P}(E') \rightarrow \text{Gal}(\gamma) \backslash \mathbb{P}(E') \) is a finite morphism, the pullback of \( \mathcal{O}_{\mathbb{P}(E_\gamma)}(1) \) to \( \mathbb{P}(E') \) is ample. Since this pullback is \( \mathcal{O}_{\mathbb{P}(E')}((N(E_\ast)) \) (see \( (3.3) \)), we conclude that the vector bundle \( E' \) is ample.

Take any \( i \geq 1 \). We have

\[
H^i(Y, S^n(E') \otimes F') = 0
\]

for all \( n \) sufficiently large, because \( E' \) is ample. Since \( \gamma \) is a finite map, this implies that

\[
H^i(X, \gamma_*(S^n(E') \otimes F')) = H^i(Y, S^n(E') \otimes F') = 0
\]

for all \( n \) sufficiently large. Now

\[
H^i(X, (S^n(E_\ast))_0) = 0
\]
for all $n$ sufficiently large because $(S^n(E_*) \otimes F_*)_0$ is a direct summand of $\gamma_*(S^n(E') \otimes F')$ (see (2.10)). Hence $E_*$ is ample by Theorem 2.1.

4. Parabolic $k$–ample bundles

We recall from [So] the definition of a $k$–ample vector bundle. A line bundle $L$ over a complex projective variety $M$ is called $k$–ample if there is a positive integer $t$ such that

- $L^\otimes t$ is base point free, and
- all the fibers of the natural morphism $M \to \mathbb{P}(H^0(M, L^\otimes t))$ are of dimension at most $k$.

A vector bundle $E$ is called $k$–ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E)$ is $k$–ample. (See Definition 1.3 in page 232 of [So].)

**Definition 4.1.** A parabolic vector bundle $E_*$ over $X$ will be called $k$–ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E_*)$ (see Definition 3.2) is $k$–ample.

We will express the cohomology of a $\text{Gal}(\gamma)$–linearized vector bundle in terms of cohomology of some parabolic vector bundle.

Let $E_*$ be a parabolic vector bundle on $X$, and let $E' \to Y$ be the corresponding $\text{Gal}(\gamma)$–linearized vector bundle, where $\gamma : Y \to X$ is a covering as in (2.6). Let $\mathbb{C}[[\Gamma]]$ be the group algebra of the finite group $\Gamma := \text{Gal}(\gamma)$. The diagonal action of $\Gamma$ on $Y \times \mathbb{C}[[\Gamma]]$ constructed using the left action of $\Gamma$ on $\mathbb{C}[[\Gamma]]$ makes the trivial vector bundle

$$\mathcal{V} := Y \times \mathbb{C}[[\Gamma]] \to Y$$

a $\Gamma$–linearized vector bundle. Let

$$\mathcal{V}_* \to X$$

be the parabolic vector bundle $X$ corresponding to this $\Gamma$–linearized vector bundle $\mathcal{V}'$.

**Lemma 4.2.** There is a natural isomorphism

$$\gamma_* E' \sim (E_* \otimes \mathcal{V}_*)_0,$$

where $(E_* \otimes \mathcal{V}_*)_0$ is the vector bundle underlying the parabolic tensor product $E_* \otimes \mathcal{V}_*$.

**Proof.** The group algebra $\mathbb{C}[[\Gamma]]$ is a $\Gamma$–bimodule. For any $\Gamma$–module $M_0$, consider the homomorphism of $\Gamma$–modules

$$\mathbb{C}[[\Gamma]] \otimes_{\mathbb{C}} M_0 \to M_0$$

defined by the action of $\Gamma$ on $M_0$. The composition

$$(\mathbb{C}[[\Gamma]] \otimes_{\mathbb{C}} M_0)^\Gamma \to \mathbb{C}[[\Gamma]] \otimes_{\mathbb{C}} M_0 \to M_0$$

is an isomorphism, where $(\mathbb{C}[[\Gamma]] \otimes_{\mathbb{C}} M_0)^\Gamma \subset \mathbb{C}[[\Gamma]] \otimes_{\mathbb{C}} M_0$ is the space of all $\Gamma$–invariants. Using this, the lemma follows immediately from the definition of $(E_* \otimes \mathcal{V}_*)_0$. \qed
Corollary 4.3. There is a natural isomorphism
\[ H^i(Y, E') \sim H^i(X, (E_\ast \otimes \mathcal{V}_\ast)_0) \]
for every \( i \geq 0 \).

Proof. Since \( \Gamma \) is a finite map, we have
\[ H^i(Y, E') = H^i(X, \gamma^\ast E') . \]
Now Lemma 4.2 completes the proof. \( \square \)

Corollary 4.4. Take any vector bundle \( W \) on \( Y \). Let
\[ \tilde{W} := \bigoplus_{g \in \Gamma} g^\ast W \]
be the direct sum of all translates of \( W \) by the automorphisms of \( Y \) lying in the Galois group. Consider the natural \( \Gamma \)-linearization on \( \tilde{W} \). Let \( F_\ast \) be the parabolic vector bundle on \( X \) corresponding to the \( \Gamma \)-linearized vector bundle \( \tilde{W} \). There is a natural isomorphism
\[ H^i(Y, \tilde{W}) \sim H^i(X, (F_\ast \otimes \mathcal{V}_\ast)_0) \]
for every \( i \geq 0 \). In particular,
\[ H^j(Y, W) = 0 \]
if \( H^j(X, (F_\ast \otimes \mathcal{V}_\ast)_0) = 0 \).

Proof. The first statement in the corollary is a special case of Corollary 4.3: set the parabolic vector bundle \( E_\ast \) in Corollary 4.3 to be \( F_\ast \). The second part of the corollary follows from the first part because \( W \) is a direct summand of \( \tilde{W} \).

The following proposition is the parabolic analog of Proposition 1.7 in page 233 of [So].

Proposition 4.5. Let \( E_\ast \) be parabolic vector bundle on \( X \) such that the line bundle \( \mathcal{O}_{\mathbb{P}(E_\ast)}(1) \) is base point free for some positive integer \( t \). Then \( E_\ast \) is \( k \)-ample if and only if for every parabolic vector bundle \( F_\ast \) on \( X \), there is an integer \( n(F_\ast) \) such that for all \( n \geq n(F_\ast) \),
\[ H^i(X, (S^n(E_\ast) \otimes F_\ast)_0) = 0 \]
for all \( i \geq k+1 \), where \( (S^n(E_\ast) \otimes F_\ast)_0 \) is the vector bundle underlying the parabolic tensor product \( S^n(E_\ast) \otimes F_\ast \).

Proof. First assume that \( E_\ast \) is \( k \)-ample. Let
\[ \psi : \mathbb{P}(E') \longrightarrow \text{Gal}(\gamma) \backslash \mathbb{P}(E') = \mathbb{P}(E_\ast) \]
be the quotient map; see (3.7). As \( E_\ast \) is \( k \)-ample, the line bundle \( \mathcal{O}_{\mathbb{P}(E_\ast)}(1) \) is, by definition, \( k \)-ample. Since \( \psi \) is a finite morphism, the pullback \( \psi^\ast \mathcal{O}_{\mathbb{P}(E_\ast)}(1) \) is also \( k \)-ample. But \( \psi^\ast \mathcal{O}_{\mathbb{P}(E_\ast)}(1) = \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \) (see (3.8)), so \( \mathcal{O}_{\mathbb{P}(E')}(N(E_\ast)) \) is \( k \)-ample. Hence the vector bundle \( E' \) over \( Y \) is \( k \)-ample.
Let \( F \) be a parabolic vector bundle on \( X \), and let \( F' \) be the corresponding \( \Gamma \)-linearized vector bundle on \( Y \). Since \( E' \) is \( k \)-ample, there is an integer \( n(E') \) such that for all \( n \geq n(F') \),

\[
H^i(Y, S^n(E') \otimes F') = 0
\]

for all \( i \geq k + 1 \) (Proposition 1.7 in page 233 of [So]).

We have already seen in the proof of Theorem 2.1 (also in the proof of Proposition 3.3) that \( H^i(X, (S^n(E_\ast) \otimes F_\ast)_0) \) is a subspace of \( H^i(Y, S^n(E') \otimes F') \). Therefore, we conclude that

\[
H^i(Y, S^n(E') \otimes F') = 0
\]

for all \( i \geq k + 1 \) and all \( n \geq n(F') \).

To prove the converse, assume that for every parabolic vector bundle \( F_\ast \), there is an integer \( n(F_\ast) \) such that for all \( n \geq n(F_\ast) \),

\[
H^i(X, (S^n(E_\ast) \otimes F_\ast)_0) = 0
\]

for all \( i \geq k + 1 \).

In view of (3.7) and (3.8), to prove that the parabolic vector bundle \( E_\ast \) is \( k \)-ample, it suffices to show that the corresponding vector bundle \( E' \to Y \) is \( k \)-ample.

Take any vector bundle \( V \to Y \). Set the vector bundle \( W \) in Corollary 4.4 to be \( S^n(E') \otimes V \). From the second part of Corollary 4.4, together with the given condition on \( E_\ast \), we conclude that there is an integer \( n(V) \) such that

\[
H^i(Y, S^n(E') \otimes V) = 0
\]

for all \( i \geq k + 1 \) and \( n \geq n(V) \). Now from Proposition 1.7 in page 233 of [So] it follows that the vector bundle \( E' \) is \( k \)-ample.

We can translate the various vanishing theorems on \( k \)-ample vector bundles to parabolic \( k \)-ample bundles. More precisely, using the correspondence between parabolic vector bundles and \( \Gamma \)-linearized vector bundles, any vanishing result on \( k \)-ample vector bundles will give a corresponding vanishing result on parabolic \( k \)-ample bundles. Here is an example.

If \( V \) is a \( k \)-ample vector bundle of rank \( r \) on a complex smooth projective variety \( M \) of dimension \( d \), then

\[
H^q(M, \Omega^p_M \otimes V) = 0
\]

if \( p + q \geq d + r + k \) (Corollary 5.20 in page 96 of [ShS]). This yields the following:

Let \( E_\ast \) be a parabolic \( k \)-ample vector bundle of rank \( r \) on \( X \); as before, the underlying vector bundle is denoted by \( E_0 \). Assume that \( \lambda_i \) in (2.5) is nonzero for all \( i \in [1, \ell] \). Then

\[
H^q(X, \Omega^p_X(\log D) \otimes E_0) = 0
\]

if \( p + q \geq r + k + \text{dim} X \) (see Theorem 4.4 in page 521 of [Bi2]). If some \( \lambda_i \) are zero, then \( \Omega^p_X(\log D) \otimes E_0 \) should be replaced by the vector bundle \( E_\rho \) in Corollary 4.14 in page 527 of [Bi2].
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