CYCLOTOMIC PREPERIODIC POINTS FOR MORPHISMS IN AFFINE SPACES AND PREPERIODIC POINTS WITH BOUNDED HOUSE AND HEIGHT

JORGE MELLO

Abstract. Under special conditions, we prove that the set of preperiodic points for semigroups of self-morphisms of affine spaces falling on cyclotomic closures is not dense, generalising results of Ostafe and Young (2020). We also extend previous results about boundness of house and height on certain preperiodicity sets of higher dimension in semigroup dynamics.

1. Introduction

Let $K$ be a field and $\mathbb{A}^N(K) = K^N$ be the $N$-dimensional affine space over $K$. Let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of distinct morphisms from $\mathbb{A}^N(K)$ to $\mathbb{A}^N(K)$ of degrees $d_i \geq 2$. Defining recursively
\[ \mathcal{F}_1 = \mathcal{F}, \quad \mathcal{F}_k = \bigcup_{i=1}^s F_i(\mathcal{F}_{k-1}) \]
with $F_i(\mathcal{F}_{k-1}) = \{F_i \circ G | G \in \mathcal{F}_{k-1}\}$, $i = 1, \ldots, s$, the set $\bigcup_{k \geq 1} \mathcal{F}_k$ becomes the semigroup generated by $F_1, \ldots, F_s$ under composition.

For $P \in \mathbb{A}^N(\overline{Q})$, one defines $\mathcal{F}_k(P) = \{F(P) | F \in \mathcal{F}_k\}$ for $k \geq 1$ and the so-called forward $\mathcal{F}$-orbit of $P$, namely
\[ O_\mathcal{F}(P) = \bigcup_{k \geq 0} \mathcal{F}_k(P). \]

In [11], when $N = 1$ and the maps are non-special polynomials, Ostafe and Young proved finiteness of the set of algebraic numbers in the cyclotomic closure whose forward $\mathcal{F}$-orbit has a self-intersecting trajectory, namely, such that
\[ F_{i_k} \circ \ldots \circ F_{i_1}(P) \in \mathcal{F}_l(F_{i_k} \circ \ldots \circ F_{i_1}(P)) \text{ for } k \geq 0, l \geq 1, \]
for $i_j \in \{1, \ldots, s\}$. Such finiteness of certain preperiodic points on a cyclotomic closure is a more general version of previous results from Northcott [10], Kawaguchi [7, 8], Dvornicich and Zannier [5], and Chen
and provides another result in arithmetic dynamics that is analogous to results in the classical setting of elliptic curves and their torsion points, for which finiteness over cyclotomic closure was proved by Ribet [12].

In this work, among other results, we seek to extend some results of [11] to higher dimension affine varieties following similar strategies. We prove in Section 3 a result of the following type

Let \( F = \{F_1, \ldots, F_s\} \) be a set of distinct morphisms from \( \mathbb{A}^N(K) \) to \( \mathbb{A}^N(K) \) of degrees \( d_i \geq 2 \), whose components do not have zero common, and that satisfies a certain property (*) defined further below, namely that the degree of iterates composed with rational functions is bounded by the number of non-constant terms. Then the set of cyclotomic preperiodic points

\[
\{ P \in \mathbb{A}^N(K^c) | F_{i_k} \circ \ldots \circ F_{i_1}(P) \in F_l(F_{i_k} \circ \ldots \circ F_{i_1}(P)) \text{ for } k \geq 0, l \geq 1 \}
\]

is not Zariski dense (is contained in the zero set of a finite set of polynomials).

Using some more ideas of [11] and references therein, as well as useful results about the size of elements in orbits discussed in Section 2, we can extend [11, Theorem 1.9] to higher dimension, namely, proving that the coordinates of the elements in the set

\[
\{ P \in \mathbb{A}^N(\overline{K}) | F_n(P) \cap F_m(P) \neq \emptyset \text{ for some } n \neq m \}
\]

have bounded house, and become algebraic integers after multiplied by a certain positive integer depending only on \( F \), provided that the morphisms have the same degree. Such is also done in Section 3. One can also look to such set in the context of more general polarized projective varieties and associated Weil heights. In the case respective to when all the degrees are equal, using properties of canonical heights constructed by Kawaguchi [7, 8], we can show boundness for these heights on such preperiodicity sets, extending works of Call and Silverman [3] and Kawaguchi [7, 8] in Section 4.

Lastly, Section 4 is also place for extending [1, Theorem 1.10]. This result says that if \( f \in K(X) \) has degree at least 2 and \( F(T_1, \ldots, T_k) = \sum_{i=1}^r c_i \prod_{j \in J_i} F_j \) with \( J_1 \cup \ldots \cup J_r = \{1, \ldots, k\} \) a disjoint partition and \( c_1, \ldots, c_k \in K^* \), then the set of \( \alpha \in \overline{K} \) such that there exist positive integers \( n_1 < n_2 < \ldots < n_k \) satisfying \( F(f^{(n_1)}(\alpha), \ldots, f^{(n_k)}(\alpha)) = 0 \) is a set of bounded height, and there are only finitely many possible values of \( n_1, \ldots, n_k \) that satisfy such equation. We extend this result for \( \mathbb{A}^N(\overline{K}) \) as a ring and \( F \) a more general polynomial with split variables. For
NON-DENSITY, BOUNDED HOUSE AND HEIGHT IN PREPERIODICITY 3

this, we use the ideas of [1] with properties from the canonical heights of [7, 8]. We finish Section 4 with simple semigroup extensions of some results of [9] that have some flavour of Section 3, now also related to Lattès maps. For this we use properties of canonical heights and previous results.

2. PRELIMINARIES

2.1. Notation, convention and definitions. Throughout the paper, we use the following notations:

- $\mathbb{U}$ the set of all roots of unity in $\mathbb{C}$.
- $K$ is a number field.
- $M_L$ the finite set of places for the number field $L$.
- $\bar{K}$ is an algebraic closure of $K$.
- $K^c = K(\mathbb{U})$ the cyclotomic closure of $K$.
- $\mathbb{A}^N(L) = L^N$ the $N$-dimensional affine space with coordinates in the field $L$, endowed with the usual sum and product done coordinate by coordinate, multiplicative inversion for elements with non-zero coordinates done by the inversion of each coordinate, and the standard multiplication by scalars from $L$.
- $\mathbb{P}^N(L)$ the $N$-dimensional projective space with coordinates in the field $L$.
- $(a_1, ..., a_N) \mapsto (a_1, ..., a_N, 1)$ a standard embedding from $\mathbb{A}^N(L)$ to $\mathbb{P}^N(L)$.
- $\bar{F} = (f_1, ..., f_N, X_{N+1})$ the standard extension or lift of $F = (f_1, ..., f_N) : \mathbb{A}^N \to \mathbb{A}^N$ to a map from $\mathbb{P}^N$ to $\mathbb{P}^N$, for $\mathbb{P}^N$ with coordinates $X_1, ..., X_{N+1}$.
- $\mathcal{F} = \{F_1, ..., F_s\}$ a set of distinct morphisms from $\mathbb{A}^N(K)$ to $\mathbb{A}^N(K)$ of degrees $d_i \geq 2$, given by $F_i = (f_1, ..., f_N), f_{ij} \in K[X_1, ..., X_N]$ with $\max_j \deg f_{ij} \geq 2, i = 1, ..., s$.
- $\mathcal{F}_1 = \mathcal{F}, \mathcal{F}_k = \bigcup_{i=1}^s F_i(\mathcal{F}_{k-1}), k \geq 2$, and $O_{\mathcal{F}}(P) = \{F(P) | F \in \bigcup_{k \geq 0} \mathcal{F}_k\}$ the orbit of $P \in \mathbb{A}^N(K)$.
- $\mathcal{H}_A$ the set of points in $\mathbb{A}^N(K)$ that have algebraic integer coordinates and with house at most $A$.

**Definition 2.1.** For $P = (x_1, ..., x_N) \in \mathbb{A}^N(\overline{\mathbb{Q}})$ and $\|v\|$ an absolute value on the field of definition $\mathbb{Q}(P)$ of $P$, we define

$$|P|_v := \max\{|x_1|_v, ..., |x_N|_v\}$$

and

$$|\sigma(P)| := \max\{|\sigma(x_1)|, ..., |\sigma(x_N)|\}$$
for each Galois conjugate $\sigma$ acting on the Galois closure of $\mathbb{Q}(P)$ over $\mathbb{Q}$, so that the house of $P$ is defined as

$$[P] = \max_{\sigma} |\sigma(P)|,$$

for $\sigma$ in the same range.

**Definition 2.2.** For $f = \sum I a_I X^I \in \mathbb{Q}[X_1, \ldots, X_N]$, $|.|_v$ an absolute value on the field of definition of $f$, and $\sigma$ Galois conjugate acting on the Galois closure of such field over $\mathbb{Q}$, we define $|f|_v := \max_{I} \{|a_I|_v\}$, $\sigma(f) = \sum I \sigma(a_I) X^I$ and $|\sigma(f)| = \max_{I} \{|\sigma(a_I)|\}$. For $G$ a finite set or uple of polynomials in $\mathbb{Q}[X_1, \ldots, X_N]$, we define $|G|_v := \max_{f \in G} |f|_v$ and $|\sigma(G)| := \max_{f \in G} |\sigma(f)|$.

**Definition 2.3.** We define the set of preperiodic points for $F$ by

$$\Pi(F) := \{ P \in \mathbb{A}^N(\mathbb{Q}) | F_k \circ \ldots \circ F_1 \in F_l(F_k \circ \ldots \circ F_1(P)) \text{ for } k \geq 0, l \geq 1 \}.$$

**Definition 2.4.** The set $\mathcal{F} = \{ F_1, \ldots, F_s \}$ is said to satisfy the property $(\ast)$ if for any $q \in \mathbb{Q}[X, X^{-1}]^N$, and $F = (f_1, \ldots, f_N) \in \mathcal{F}$, we have that

$$\log(\deg F) = O(1)$$

where the implied constant depends only on

$$\max_{i} \#\{ \text{non-constant terms of } f_i \circ q \}.$$

**Remark 2.5.** It is seen in [11] that such property is satisfied in the univariate case for non-special sets of polynomials, making use of [11, Lemma 2.9]. This implies naturally the validity of this property for sets $\mathcal{F}$ of split polynomials of the form $f_1 \times \ldots \times f_N : \mathbb{A}^1 \times \ldots \times \mathbb{A}^1$, with $f_i$ univariate non-special polynomials.

### 2.2. Representation via linear combinations of roots of unity.

**Lemma 2.6.** [5, Theorem L] There exists a number $B$ and a finite set $E \subset K$ with $|E| \leq [K : \mathbb{Q}]$ such that any algebraic integer $\alpha \in K^c$ can be written as $\alpha = \sum_{i=1}^b c_i \xi_i$, where $c_i \in E, \xi_i \in \mathbb{U}$ and $b \leq \#E \cdot \mathcal{R}(\mathbb{R})$, where $\mathcal{R} : \mathbb{R} \to \mathbb{R}$ is any Loxton function.

### 2.3. Size of elements in orbits.

**Lemma 2.7.** Let $F : \mathbb{A}^N \to \mathbb{A}^N$ be a morphism defined over $\overline{\mathbb{Q}}$, so that its extension to $\mathbb{P}^N$ is given by a $(n + 1)$-tuple $\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_{N+1})$ of homogeneous polynomials of degree $d$ having no common zero. Then there are positive constants $C, D$ and a finite set $G_F$ of polynomials defined over $\overline{\mathbb{Q}}$ such that

$$C^{(v)}|G_F|_v^{-1} |P|_v^d \leq |F(P)|_v \leq D^{(v)}|F|_v |P|_v^d.$$
for all $P \in \mathbb{A}^N(\overline{\mathbb{Q}})$, $v \in M_K$ with $K$ a sufficiently large number field in which all the objects above are defined, $\epsilon_v = 0$ if $v$ is finite and $\epsilon_v = 1$ if $v$ is infinite.

Moreover, if the polynomials $\tilde{f}_1, \ldots, \tilde{f}_{N+1}$ have a set $Z$ of common zeros in $\mathbb{P}^N$, and $X \subset \mathbb{A}^N$ is an affine variety over $\overline{\mathbb{Q}}$ whose closure in $\mathbb{P}^N$ does not intersect $Z$ and such that $F(X) \subset X$, then the same kind of result is true for $X$ in place of $\mathbb{A}^N(\overline{\mathbb{Q}})$.

Proof. This fact is shown in the proof of [15, Theorem 5.6], and the last part/paragraph of the statement is shown in the proof of [6, Theorem B.2.5 (b)]. □

Lemma 2.8. Let $F_i : \mathbb{A}^N \to \mathbb{A}^N$, $i = 1, \ldots, s$, be morphisms defined over $K$ of degrees $d_i \geq 2$, and let $P \in \mathbb{A}^N(\overline{\mathbb{Q}})$ such that

$$|P|_v > \max_i \{1, |G_{F_i}|_v\}$$

for some non-archimedean absolute value $|.|_v$ extended to $\overline{\mathbb{Q}}$, where $G_{F_i}$ is given by Lemma 2.7. Then

$$|F_{i_n} \circ \ldots \circ F_{i_1}(P)|_v > |F_{i_{n-1}} \circ \ldots \circ F_{i_1}(P)|_v$$

for all $i_1, \ldots, i_n \in \{1, \ldots, s\}$, $n \geq 1$

Proof. By Lemma 2.7 and the statement hypothesis, we can see that

$$|P|_v < |G_{F_i}|^{-1}_v |P|_v^{d_i} \leq |F_i(P)|_v$$

for each $i$.

Assuming the statement to be true for iterates up to $n - 1$, we have

$$|F_{i_{n-1}} \circ \ldots \circ F_{i_1}(P)|_v > \ldots > |F_{i_1}(P)|_v > |P|_v$$

$$> \max_i \{1, |G_{F_i}|_v\}.$$ 

Applying the argument as above with $F_{i_{n-1}} \circ \ldots \circ F_{i_1}(P)$ instead of $P$ yields the result. □

Lemma 2.9. Under the conditions of Lemma 2.8, let $P \in \mathbb{A}^N(\overline{\mathbb{Q}})$ such that

$$|P|_v > \max_i \{1, C^{-1}_v |G_{F_i}|_v\}$$

for some archimedean absolute value $|.|_v$ extended to $\overline{\mathbb{Q}}$. Then

$$|F_{i_n} \circ \ldots \circ F_{i_1}(P)|_v > |F_{i_{n-1}} \circ \ldots \circ F_{i_1}(P)|_v$$

for all $i_1, \ldots, i_n \in \{1, \ldots, s\}$, $n \geq 1$

Proof. The proof follows the same lines as those of Lemma 2.8. □
Lemma 2.10. Under the conditions of Lemma 2.8, supposing that the polynomials in the sets $G_{F_i}$ are also defined over $K$, let $A \in \mathbb{R}$ be positive and define

$$L = \max \sigma \left\{ \max_i \{1, C^{-1}|\sigma(G_{F_i})|\}, A \right\},$$

where the maximum runs over all the embeddings $\sigma$ of $K$ in $\mathbb{C}$. Let $P \in \mathbb{A}^N(\mathbb{O})$ be such that for some $k \geq 1$, and some $i_1, \ldots, i_k \in \{1, \ldots, s\}$, we have $|F_{i_k} \circ \ldots \circ F_{i_1}(P)| \leq A$. Then $|F_{i_l} \circ \ldots \circ F_{i_1}(P)| \leq L$ for all $l < k$.

Proof. In fact, assume that $|F_{i_l} \circ \ldots \circ F_{i_1}(P)| > L$ for some $l < k$. Then there is a conjugate $\sigma(F_{i_l} \circ \ldots \circ F_{i_1}(P))$ such that $|\sigma(F_{i_l} \circ \ldots \circ F_{i_1}(P))| > L$. Since $\sigma(F_{i_l} \circ \ldots \circ F_{i_1}(P)) = \sigma(F_{i_l} \circ \ldots \circ F_{i_1})(\sigma(P)) = \sigma(F_{i_1}) \circ \ldots \circ \sigma(F_{i_l})(\sigma(P))$. Applying Lemma 2.9 for the morphisms $\sigma(F_{i_k}), \ldots, \sigma(F_{i_{l+1}})$ and the point $\sigma(F_{i_l} \circ \ldots \circ F_{i_1}(P))$, we conclude that $|\sigma(F_{i_l} \circ \ldots \circ F_{i_1}(P))| > A$, which is a contradiction with our assumption. \hfill\Box

Lemma 2.11. Under the conditions of Lemma 2.8, there exists a positive integer $D$, depending only on $F_1, \ldots, F_s$ such that for any $i_1, \ldots, i_n \in \{1, \ldots, s\}$ and $P \in \mathbb{A}^N(K^c)$, if $F_{i_1} \circ \ldots \circ F_{i_n}(P)$ has algebraic integer coordinates, then $DP$ and $DF_{i_r} \circ \ldots \circ F_{i_1}(P), r = 1, \ldots, n - 1$, have algebraic integer coordinates.

Proof. For $P \in \mathbb{A}^N(K^c)$, and $F_{i_n} \circ \ldots \circ F_{i_1}(P)$ with algebraic integer coordinates, we must have

$$|P|, |F_{i_r} \circ \ldots \circ F_{i_1}(P)| \leq \max \left\{1, |G_{F_i}| \right\}$$

for $r = 1, \ldots, n - 1$, for otherwise we would have by Lemma 2.8 that $|F_{i_n} \circ \ldots \circ F_{i_1}(P)| > 1$, but $F_{i_n} \circ \ldots \circ F_{i_1}(P)$ has algebraic integer coordinates. Therefore, choosing a positive integer $D$ such that the products between $D$ and coefficients of the polynomials in $G_{F_i}$ are all algebraic integers for each $i = 1, \ldots, s$, we conclude that $DP$ and $DF_{i_r} \circ \ldots \circ F_{i_1}(P), r = 1, \ldots, n - 1$, have also algebraic integer coordinates. \hfill\Box

3. RESULTS ON POINTS OF BOUNDED HOUSE

Theorem 3.1. Let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of distinct morphisms from $\mathbb{A}^N(K)$ to $\mathbb{A}^N(K)$ of degrees $d_i \geq 2$, whose lift’s components do not have zero em common, and that satisfies property (*) 2.4. Then the set of cyclotomic preperiodic points

$$\Pi(\mathcal{F})(K^c) = \{P \in \mathbb{A}^N(K^c)|F_{i_k} \circ \ldots \circ F_{i_1} \in \mathcal{F}_l(F_{i_k} \circ \ldots \circ F_{i_1}(P)) \text{ for } k \geq 0, l \geq 1\}$$

is not Zariski dense.
Proof. Suppose that $\Pi(\mathcal{F})(K^c)$ is Zariski dense. Then, given any large $M > 0$ integer, we must have infinitely many $P \in \mathbb{A}^N(K^c)$ satisfying $F_i \circ \ldots \circ F_{i_1}(P) = F_{i_{k_1}} \circ \ldots \circ F_{i_1}(P)$ with $n > k$ and $n > M$. Thus, there exist some fixed indices $i_1, \ldots, i_M \in \{1, \ldots, s\}$ such that there are infinitely many $P \in \mathbb{A}^N(K^c)$ with

$$F_{i_k} \circ \ldots \circ F_{i_1}(P) \in \mathcal{F}(F_{i_M} \circ \ldots \circ F_{i_1}(P))$$

for some $k < \ell + M, \ell \geq 1$. Considering such a $P$, we have

$$F_{i_k} \circ \ldots \circ F_{i_1}(P) = F_{i_M+\ell} \circ \ldots \circ F_{i_M+1} \circ F_{i_M} \circ \ldots \circ F_{i_1}(P)$$

with $i_{M+1}, \ldots, i_{M+\ell} \in \{1, \ldots, s\}$.

If $k \leq M$, then composing the above identity with $F_{i_M} \circ \ldots \circ F_{i_{k+1}}$, we obtain that

$$F_{i_M} \circ \ldots \circ F_{i_1}(P) = F_{i_M} \circ \ldots \circ F_{i_{k+1}} \circ F_{i_M+\ell} \circ \ldots \circ F_{i_M+1} \circ F_{i_M} \circ \ldots \circ F_{i_1}(P).$$

Thus $F_{i_M} \circ \ldots \circ F_{i_1}(P) \in \Pi(\mathcal{F})(K^c)$.

Let $L$ be defined as in Lemma 2.10 with $A = 1$. Then we must have $|F_{i_M} \circ \ldots \circ F_{i_1}(P)| \leq L$, for otherwise, as in Lemma 2.10, there would be an embedding $\sigma$ such that $|\sigma(F_{i_M} \circ \ldots \circ F_{i_1}(P))| > L$, and by Lemma 2.9

$$|\sigma(F_{i_M} \circ \ldots \circ F_{i_1}(P))| = |\sigma(F_{i_M} \circ \ldots \circ F_{i_{k+1}} \circ F_{i_M+\ell} \circ \ldots \circ F_{i_M+1} \circ F_{i_M} \circ \ldots \circ F_{i_1}(P))|$$

$$> |\sigma(F_{i_M} \circ \ldots \circ F_{i_1}(P))|,$$

which is a contradiction.

Moreover, similar to Lemma 2.11, for any finite place $v$ of the field of definition of $P$, we have

$$|F_{i_M} \circ \ldots \circ F_{i_1}(P)|_v \leq \max_i \{1, |G_i|_v \},$$

since otherwise $\{ |F_{i_r} \circ \ldots \circ F_{i_1}(P)|_v \}_{r=M}^\infty$ is strictly increasing by Lemma 2.8. Thus, if $D$ is such that the products between $D$ and coefficients of the polynomials in $G_F$ are all algebraic integers for each $i = 1, \ldots, s$, then $DF_{i_M} \circ \ldots \circ F_{i_1}(P)$ has also algebraic integer coordinates.

If $M < k < \ell + M$, then since $F_{i_k} \circ \ldots \circ F_{i_1}(P) \in \Pi(\mathcal{F})(K^c)$, as above we obtain that $|F_{i_k} \circ \ldots \circ F_{i_1}(P)| \leq L$, with $L$ as in Lemma 2.10, and thus $|F_{i_M} \circ \ldots \circ F_{i_1}(P)| \leq L$, for any $M < k$ by Lemma 2.9. Moreover, as above, for any finite place $v$ of a field of definition,

$$|F_{i_k} \circ \ldots \circ F_{i_1}(P)|_v \leq \max_i \{1, |G_i|_v \}.$$
since otherwise Lemma 2.8 would imply

$$|F_{i_k} \circ \ldots \circ F_{i_1}(P)|_v > |F_{i_M} \circ \ldots \circ F_{i_1}(P)|_v > \max_i \{1, |G_i|_v\},$$

a contradiction. Therefore we may obtain a positive integer $D$ such that $DF_{i_M} \circ \ldots \circ F_{i_1}(P)$ has algebraic integer coordinates.

We hence obtained that

(3.1) \# \{P \in \mathbb{A}^N(K) \mid DF_{i_M} \circ \ldots \circ F_{i_1}(P) \in \mathcal{H}_{DL}\} = \infty,

and by Lemmas 2.8 and 2.9, that

\# \{P \in \mathbb{A}^N(K) \mid DF_{r} \circ \ldots \circ F_{i_1}(P) \in \mathcal{H}_{DL}\} = \infty, \quad r = 0, \ldots, M.

Applying Lemma 2.6, there exists a number $B$ and a finite set $E \subset \mathbb{K}$ with $\#E \leq [K : \mathbb{Q}]$ such that for each $P$ in the set above, such $P$ equals to

$$F_{r} \circ \ldots \circ F_{i_1} \left( \sum_{j=1}^B c_{1,j}^{(0)} \zeta_{1,j}^{(0)}, \ldots, \sum_{j=1}^B c_{N,j}^{(0)} \zeta_{N,j}^{(0)} \right) = \left( \sum_{j=1}^B c_{1,j}^{(r)} \zeta_{1,j}^{(r)}, \ldots, \sum_{j=1}^B c_{N,j}^{(r)} \zeta_{N,j}^{(r)} \right)$$

for $r = 1, \ldots, M$, where $c_{i,j}^{(r)} \in E, \zeta_{i,j}^{(r)} \in \mathbb{U}$. We can pick an infinite subset of the infinite set above whose elements have fixed $c_{i,j}^{(r)}$’s, not depending on $P$ from such infinite subset. Thus the variety defined by

$$F_{i_M} \circ \ldots \circ F_{i_1} \left( \sum_{j=1}^B c_{1,j}^{(0)} X_{1,j}^{(0)}, \ldots, \sum_{j=1}^B c_{N,j}^{(0)} X_{N,j}^{(0)} \right),$$

$$= \left( \sum_{j=1}^B c_{1,j}^{(M)} X_{1,j}^{(M)}, \ldots, \sum_{j=1}^B c_{N,j}^{(M)} X_{N,j}^{(M)} \right)$$

inside $\mathbb{Q}_m^{2BN}$ has infinitely many torsion points, and this by the Torsion Points Theorem [2, Theorem 4.2.2] leads to the identity

$$F_{i_M} \circ \ldots \circ F_{i_1} \left( \sum_{j=1}^B c_{1,j}^{(0)} s_{1,j}^{(0)} t^{e_{1,j}}, \ldots, \sum_{j=1}^B c_{N,j}^{(0)} s_{N,j}^{(0)} t^{e_{N,j}} \right),$$

$$= \left( \sum_{j=1}^B c_{1,j}^{(M)} s_{1,j}^{(M)} t^{e_{1,j}}, \ldots, \sum_{j=1}^B c_{N,j}^{(M)} s_{N,j}^{(M)} t^{e_{N,j}} \right),$$

where $c_{i,j}^{(0)}, c_{i,j}^{(M)} \in \mathbb{U}$ for each $i, j$. Denoting the inner $N$-uple in the left-hand side of the expression above by $q(t) \in \mathbb{Q}[t, t^{-1}]^N$, we have by the property (*) as defined in 2.4 that

$$M \ll_B 1.$$
Then $M$ must be bounded, contradicting $M$ being large enough, and then $\Pi(\mathcal{F})(K^c)$ is contained in the zero set of a finite set of polynomials, and thus is not Zariski-dense, deriving a contradiction. \hfill \Box

**Corollary 3.2.** Let $X \subset \mathbb{A}^N$ be an affine variety, and $\mathcal{F} = \{F_1, ..., F_s\}$ be a set of distinct morphisms from $X$ to $X$ of degrees $d_i \geq 2$ whose lift’s components common zero set does not intersect the closure of $X$ in $\mathbb{P}^N$, and that satisfies property (*) 2.4. Then the set of cyclotomic preperiodic points

$$\{ P \in X(K^c) | F_{i_k} \circ ... \circ F_{i_1} \in \mathcal{F}_l(F_{i_k} \circ ... \circ F_{i_1}(P)) \text{ for } k \geq 0, l \geq 1 \}$$

is not Zariski dense.

**Proof.** The proof follows the same procedure of Theorem 3.1’s proof, but making use of the last part of Lemma 2.7 instead. \hfill \Box

**Proposition 3.3.** Let $\mathcal{F} = \{F_1, ..., F_s\}$ be a set of distinct morphisms from $\mathbb{A}^N(K)$ to $\mathbb{A}^N(K)$ of degrees $d_i \geq 2$, whose lift’s components do not have zero em common, and that satisfies property (*) 2.4. Let $A \geq 1$, and suppose that $F_i((K^c)^N) \cap \mathcal{H}_A$ is not Zariski dense for each $i = 1, ..., s$. Then the set

$$\left\{ P \in \mathbb{A}^N(K^c) \mid \bigcup_{k \geq 0} \mathcal{F}_k(P) \cap \mathcal{H}_A \neq \emptyset \right\}$$

is not Zariski dense.

**Proof.** Since $\bigcup_{i \leq s} F_i((K^c)^N) \cap \mathcal{H}_A$ is not Zariski dense by the hypothesis, if $F(P) \in \mathcal{H}_A$ for some $F \in \mathcal{F}_k$, then $P \in F^{-1}(F_i((K^c)^N) \cap \mathcal{H}_A)$, which is not dense in the Zariski topology. Therefore, if the set of the statement is Zariski dense, one may suppose that for any $M$ there are infinitely many $P \in \mathbb{A}^N(K^c)$ such that

$$\mathcal{F}_l(F_{i_m} \circ ... \circ F_{i_1}(P)) \cap \mathcal{H}_A \neq \emptyset$$

for some certain indices $i_1, ..., i_M$ and arbitrarily large $l \geq 1$. By Lemma 2.11, there exists a positive integer $D$ depending only on $F_1, ..., F_s$ such that $DF_{i_m} \circ ... \circ F_{i_1}(P)$ for infinitely many $P \in (K^c)^N$, and as in Lemma 2.10, $DF_{i_m} \circ ... \circ F_{i_1}(P)$ has house at most $DL$ for a certain $L$. Therefore

$$\#\{ P \in \mathbb{A}^N(K^c) | DF_{i_m} \circ ... \circ F_{i_1}(P) \in \mathcal{H}_{DL} \} = \infty$$

and from this point on, we proceed as in 3.1. \hfill \Box

**Definition 3.4.** For $A \geq 1$, and $P \in \mathbb{A}^N(\mathbb{Q})$, we define

$$\mathcal{L}_{n,A}(\mathcal{F}; P) = \left\{ \sum_{k=0}^{n} \sum_{i_1, ..., i_k=1}^{s} \gamma_{i_1, ..., i_k} F_{i_k} \circ ... \circ F_{i_1}(P) \mid \gamma_{i_1, ..., i_k} \in \mathcal{H}_A^{dn} \right\}$$
where \( F_1, \ldots, F_s \) all have degree \( d \) and set

\[
\Sigma_A(F) = \{ P \in \mathbb{A}^N(\mathbb{Q}) | \mathcal{F}_n(P) \cap \mathcal{L}_{n-1,A}(F; P) \neq \emptyset \text{ for some } n \geq 1 \}.
\]

**Theorem 3.5.** Let \( F = \{ F_1, \ldots, F_s \} \) be a set of distinct morphisms from \( \mathbb{A}^N(K) \) to \( \mathbb{A}^N(K) \) of degree \( d \geq 3 \), whose lift’s components do not have zero em common, let \( A \geq 1 \), and suppose that the polynomials in the sets \( G_{F_i} \) are also defined over \( K \). Then \( \Sigma_A(F) \) is a set of bounded house, and there exists a positive integer \( E \), depending only on \( F \) and \( K \), such that \( E \Sigma_A(F) \) is a set of algebraic integers.

**Proof.** We consider \( C \) to be the smallest among the \( C \)’s coming from Lemma 2.7 applied to each \( \sigma(F_i) \), \( i = 1, \ldots, s \), \( \sigma : K \to \mathbb{C} \) embedding, and \( D \) to be the maximum among the \( D \)’s from the same Lemma applied to each \( \sigma(F_i) \) in the same range as well. We also choose \( m \geq 1 \) such that \( C|\sigma(G_{F_i})|^{-1} > 1/m \) for every embedding \( \sigma : K \to \mathbb{C} \) and \( i = 1, \ldots, s \). Making

\[
M := \max_{\sigma : K \to \mathbb{C}} \left\{ 2sm^2A + \max_i \{ D|\sigma(F_i)| \} \right\},
\]

suppose that \( |\sigma(P)| > M \) for certain fixed \( \sigma : K \to \mathbb{C} \) and \( P \in \Sigma_A(F) \).

In this case we have

\[
C|\sigma(G_{F_i})|^{-1}|\sigma(P)|^d \geq |\sigma(P)|^d/m,
\]

implying that

\[
|\sigma(F_i)(P)| \geq C|\sigma(G_{F_i})|^{-1}|\sigma(P)|^d > |\sigma(P)|^d/m > M,
\]

and hence that

\[
|\sigma(F_{i_1} \circ \ldots \circ F_{i_t}(P))| > \frac{|\sigma(P)|^{d_{i_t}}}{m^{1+\ldots+d_{i_t}^{-1}}} > \frac{|\sigma(P)|^{d_{i_t}}}{m^{2d_{i_t}^{-1}}}.\]

On the other hand, \( |\sigma(F_i)(P)| \leq D|\sigma(F_i)||\sigma(P)|^d \), and then

\[
|\sigma(F_{j_k} \circ \ldots \circ F_{j_1}(P))| \leq D|\sigma(F_{j_k})|(D|\sigma(F_{j_{k-1}})|)^{d_{j_{k}}} \ldots (D|\sigma(F_{j_1})|)^{d_{j_{1}}-1} |\sigma(P)|^{d_{j_k} - 1} < |\sigma(P)|^{1+\ldots+d_{j_k}} \leq |\sigma(P)|^{2d_{j_k}}.
\]
We thus obtain
\[
|\sigma(F_{i_n} \circ \ldots \circ F_{i_1}(P))| = \left| \sum_{k=0}^{n-1} \sum_{j_1,\ldots,j_k=1}^s \sigma(\gamma_{j_1,\ldots,j_k}) \sigma(F_{j_k} \circ \ldots \circ F_{j_1}(P)) \right|
\]
\[
\leq A^{d^{n-1}} \sum_{k=0}^{n-1} \sum_{j_1,\ldots,j_k=1}^s |\sigma(\gamma_{j_1,\ldots,j_k})| |\sigma(F_{j_k} \circ \ldots \circ F_{j_1}(P))|
\]
\[
\leq A^{d^{n-1}} \sum_{k=0}^{n-1} \sum_{j_1,\ldots,j_k=1}^s |\sigma(P)|^2d^k
\]
\[
\leq 2A^{d^{n-1}} s^{n-1}|\sigma(P)|^2d^{n-1}.
\]
Noting that, for \(d \geq 3\) and \(|\sigma(P)| > M > 2sm^2A\), one has
\[
\frac{|\sigma(P)|^d}{m^{2d^{n-1}}} = \frac{|\sigma(P)|^{d^{n-2}d^{n-1}}}{m^{2d^{n-1}}} |\sigma(P)|^{2d^{n-1}} \geq \left( \frac{|\sigma(P)|}{m^2} \right)^{d^{n-1}} |\sigma(P)|^{2d^{n-1}}
\]
\[
> (2sA)^{d^{n-1}} |\sigma(P)|^{2d^{n-1}} > 2A^{d^{n-1}} s^{n-1}|\sigma(P)|^{2d^{n-1}},
\]
we have a contradiction, so \([\mathcal{P}]\) is bounded by \(M\), and the first part of the result is proved.

Moreover, suppose that
\[
|P|_v > \max_{i,j=1,\ldots,s} \{|F_i|_v, |G_{F_j}|_v\}
\]
From Lemma 2.7, we similarly have that \(|F_i(P)|_v \geq |G_{F_i}^{-1}|P|_v^d\), and then
\[
|F_{i_n} \circ \ldots \circ F_{i_1}(P)|_v \geq |G_{F_i}|^{-1} |G_{F_i}^{-1} |P|_v^{d-1} |P|_v^d
\]
\[
\geq |G_{F_i}|^{-1} |P|_v \ldots (|G_{F_i}|^{-1} |P|_v)|^d^{n-1} |P|_v^{d^n-d^{n-1}}
\]
\[
> |P|_v^{d^k+1-d^{n-1}}
\]
\[
= (|P|_v^d \ldots |P|_v^d)^{d-1} |P|_v^d
\]
\[
> |F_{j_k}^\circ \ldots \circ F_{j_1}(P)|_v,
\]
for each \(k < n\). On the other hand,
\[
|(F_{i_n} \circ \ldots \circ F_{i_1}(P))|_v = \left| \sum_{k=0}^{n-1} \sum_{j_1,\ldots,j_k=1}^s \gamma_{j_1,\ldots,j_k} F_{j_k} \circ \ldots \circ F_{j_1}(P) \right|
\]
\[
\leq \max_{0 \leq k \leq n-1} \left\{ \max_{1 \leq j_1,\ldots,j_k \leq s} |F_{j_k} \circ \ldots \circ F_{j_1}(P)|_v \right\},
\]
yielding a contradiction, so \(|P_v| \leq \max_{i,j=1,...,s}\{|F_i|_v, |G_j|_v\}\) and it is enough to choose a positive integer \(E\) such that the coefficients of the \(F_i\)'s and of the polynomials from the \(G_j\)'s become algebraic integers when multiplied by \(E\). \(\square\)

4. Canonical heights

Given a projective variety \(X\) over a number field \(K\) and \(L\) a line bundle on \(X\), a height function \(h_{X,L}\) corresponding to \(L\) is fixed. Let \(\mathcal{H}\) be a set of morphisms \(f : X \to X\) over \(K\) such that \(f^* L \cong L \otimes d_f\) for some integer \(d_f \geq 2\). For \(f \in \mathcal{H}\), we set

\[
c(f) := \sup_{x \in X(\bar{K})} \left| \frac{1}{d_f} h_L(f(x)) - h_L(x) \right|
\]

For \(f = (f_i)_{i=1}^{\infty}\) a sequence with \(f_i \in \mathcal{H}\), i.e. \(f \in \prod_{i=1}^{\infty} \mathcal{H}\), we set

\[
c(f) := \sup_{i \geq 1} c(f_i) \in \mathbb{R} \cup \{+\infty\}.
\]

When \(c(f) < +\infty\), the sequence is said to be bounded. The property of being bounded is independent of the choice of height functions corresponding to \(L\).

Let \(\mathcal{B}\) be the set of all bounded sequences in \(\mathcal{H}\), and for \(c > 0\), we define

\[
\mathcal{B}_c := \{f = (f_i)_{i=1}^{\infty} \in \mathcal{B} | c(f) \leq c\}.
\]

It is easy to see that if \(\mathcal{H}\) is a finite set of self-maps on a projective space, then any sequence of maps arising from \(\mathcal{H}\) belongs to \(\mathcal{B}_c\) for some \(c\).

In fact, for \(\mathcal{H} = \{g_1, \ldots, g_k\}\), we set

\[
(4.1) \quad J = \{1, \ldots, k\}, \quad W := \prod_{i=1}^{\infty} J_i \quad \text{and} \quad f_w := (g_{w_i})_{i=1}^{\infty} \quad \text{for} \quad w = (w_i) \in W.
\]

If \(c := \max\{c(g_1), \ldots, c(g_k)\}\), then \(\{f_w | w \in W\} \subset \mathcal{B}_c\).

We also let \(S : \prod_{i=1}^{\infty} \mathcal{H} \to \prod_{i=1}^{\infty} \mathcal{H}\) be the shift map which sends \(f = (f_i)_{i=1}^{\infty}\) to

\[
S(f) = (f_{i+1})_{i=1}^{\infty}.
\]

Then \(S\) maps \(\mathcal{B}\) into \(\mathcal{B}\) and \(\mathcal{B}_c\) into \(\mathcal{B}_c\) for any \(c\).

For \(f = (f_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \mathcal{H}\) and \(x \in X(\bar{K})\), making

\[
f^{(n)} := f_n(f_{n-1}(\ldots(f_1(x)))).
\]

the set

\[
\{x, f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), \ldots\} = \{x, f_1(x), f_2(f_1(x)), f_3(f_2(f_1(x))), \ldots\}
\]
is called the forward orbit of \( x \) under \( f \), denoted by \( O_f(x) \). The point \( x \) is said to be \( f \)-preperiodic if \( O_f(x) \) is finite. If \( f = f_1 = f_2 = \ldots \), then the forward orbit is the forward orbit under \( f \) in the usual sense.

**Lemma 4.1.** [7, Theorem 3.3] Let \( X \) be a projective variety over \( K \), and \( L \) a line bundle on \( X \). Let \( h_L \) be a height function corresponding to \( L \), and \( f = (f_i)_{i=1}^{\infty} \) a bounded sequence over \( K \) such that \( f_i^*L \cong L^\otimes d_i \) for integers \( d_i \geq 2 \).

1. There is a unique way to attach to the sequence \( f = (f_i)_{i=1}^{\infty} \in \mathcal{B} \) a canonical height function

   \[ \hat{h}_f : X(\bar{K}) \to \mathbb{R} \]

   such that

   \[(i) \sup_{x \in X(\bar{K})} |\hat{h}_f(x) - h_L(x)| \leq 2c(f),\]

   \[(ii) \hat{h}_{S(f)} \circ f_1 = d_{f_1} \hat{h}_f. \text{ In particular, } \hat{h}_{S(f)} \circ f_n \circ \ldots \circ f_1 = d_{f_n} \ldots d_{f_1} \hat{h}_{L,f}.\]

2. Assume \( L \) is ample. Then \( \hat{h}_f \) satisfies the following properties:

   \[(i) \hat{h}_f(x) \geq 0 \text{ for all } x \in X(\bar{K}).\]

   \[(iv) \hat{h}_f = 0 \text{ if and only if } x \text{ is } f\text{-preperiodic.}\]

   We call \( \hat{h}_f \) a canonical height function (normalized) for \( f \).

Under similar conditions of the previous lemma, namely, \( X \) is a projective variety over \( K \), \( L \) is a line bundle on \( X \), \( \mathcal{H} = \{g_1, \ldots, g_k\} \), \( g_j^*L \cong L^{\otimes d_j} \), we have

\[ g_1^*L \otimes \ldots \otimes g_k^*L \cong L^{\otimes (d_{g_1} + \ldots + d_{g_k})}. \]

Thus \( (X, g_1, \ldots, g_k) \) becomes a particular case of what we call a dynamical eigensystem for \( L \) of degree \( d_{g_1} + \ldots + d_{g_k} \). For this, Kawaguchi also proved that

**Lemma 4.2.** [8, Theorem 1.2.1] There exists the canonical height function

\[ \hat{h}_{\mathcal{H}} : X(\bar{K}) \to \mathbb{R} \]

for \( (X, g_1, \ldots, g_k, L) \) characterized by the following two properties:

\[(i) \hat{h}_{\mathcal{H}} = h_L + O(1);\]

\[(ii) \sum_{j=1}^{k} \hat{h}_{\mathcal{H}} \circ g_j = (d_{g_1} + \ldots + d_{g_k}) \hat{h}_{\mathcal{H}}.\]

Moreover, if \( L \) is ample, then

\[ \{P \in X(\bar{K})|O_{\mathcal{H}}(P) \text{ is finite} \} = \{P \in X(\bar{K})|\hat{h}_{\mathcal{H}}(P) = 0\}. \]

**Lemma 4.3.** [8, Proposition 4.1] Give \( J = \{1, \ldots, k\} \) the discrete topology (each subset is an open set), and let \( \nu \) be the measure on \( J \) that assigns mass \( \frac{d_{g_j}}{d_{g_1} + \ldots + d_{g_k}} \) to \( j \in J \). Let \( \mu := \prod_{i=1}^{\infty} \nu \) be the product measure on \( W \). Then we have, for \( x \in X(\bar{K}) \), \( \mathcal{H} = \{g_1, \ldots, g_k\} \), that
\[
\hat{h}_H(x) = \int_W \hat{h}_f(x) d\mu(w).
\]

In particular,
\[
|\hat{h}_H(x) - h_L(x)| \leq 2c \quad \text{and} \quad |\hat{h}_H(x) - \hat{h}_f(x)| \leq 4c
\]
for all \(x \in X(\bar{K}), w \in W\), where \(c = \max\{c(g_1), \ldots, c(g_k)\}\).

**Proof.** The only thing different from the original statement of [7] is the last part. The first inequality above is the first inequality after (4.1) in the proof of [7, Proposition 4.1]. The second is derived from the first and from Lemma 4.1. \(\Box\)

**Theorem 4.4.** Let \(X\) be a projective variety over \(K\), and \(L\) a line bundle on \(X\). Let \(h_L\) be a height function corresponding to \(L\), and let \(F = \{F_1, \ldots, F_s\}\) be a set of distinct morphisms from \(X\) to \(X\) over \(K\), such that \(F_i^* L \cong L \otimes d\) an integer \(d \geq 2\). Then the set
\[
\{P \in X(\bar{Q}) | F_n(P) \cap F_m(P) \neq \emptyset \text{ for some } n \neq m\}
\]
is a set of bounded height \(h_L\).

**Proof.** We may suppose that \(\phi \in F_n\) and \(\psi \in F_m\) are iterates of the sequences \(\Phi\) and \(\Psi\) from \(F\) respectively with \(n > m\), satisfying \(\phi(P) = \psi(P)\). Here, we omit the subscript \(L\) for the hecanonical heights given by the previous Lemmas for simplicity. Using Lemma 4.1, we have that
\[
d^m h_{\Phi}(P) = \deg(\phi) \hat{h}_{\Phi}(P) = \hat{h}_{\Psi(\phi, \phi)}(\phi(P)) = h(\phi(P)) + O(1)
\]
is bounded by a constant plus
\[
h(\phi(P)) = h(\psi(P))
\]
\[
\leq \hat{h}_{\Psi(m)}(\psi(P)) + O(1)
\]
\[
= \deg(\psi) \hat{h}_{\Psi}(P) + O(1)
\]
\[
= d^m \hat{h}_{\Psi}(P) + O(1).
\]
implies that
\[
\left[d^n - d^m, \frac{\hat{h}_{\Psi}(P)}{\hat{h}_\Phi(P)}\right] \hat{h}_{\Phi}(P) \leq O(1).
\]
If \(\hat{h}_{\Phi}(P) \geq 12c\), as in Lemma 4.3, then \(\hat{h}_{\Psi}(P) \geq 8c\) by Lemma 4.3 (b), and thus by Lemma 4.1 (1) (i) and Lemma 4.2, \(\frac{\hat{h}_{\Psi}(P)}{\hat{h}_\Phi(P)} \leq \frac{\hat{h}_{\Phi}(P)+4c}{\hat{h}_\Phi(P)} \leq 3/2\), for \(P\) out of a set of bounded Weil height, so that we have that
\[
[d^n - d^m3/2] \hat{h}_\Phi(P) = O(1).
\]
And \(d^n - d^m3/2 \geq d^n - d^{n-1}3/2 \geq d^{n-1}/2 \iff d^n \geq 2d^{n-1}\), in the inequality above, it yields that
\[
d^{n-1} \hat{h}_\Phi(P) = O(1),
\]
and hence, apart from a set of points $P$ of bounded Weil height, we have $\hat{h}_\Phi(P) = O(1)$. This gives the desired boundeness for the Weil height treated, since by Lemma 4.1, $O(1)$ is independent of $\Phi$, depending only on $X, \mathcal{F}$ and $L$. \hfill $\square$

**Remark 4.5.** Apart from a similar set of bounded height with $\hat{h}_F(P) \leq 12c$, as well as discarding preperiodic points with bounded height, 

$$\frac{d^{n-1}}{2} \inf_{\Phi \circ \tau} \hat{h}_\Phi(P) = O(1) \implies n = O(1),$$

so that $n, m$ of the referred set will also be bounded.

**Definition 4.6.** Let $T_i = (T_{i1}, \ldots, T_{iN}), i = 1, \ldots, k$ be $k$ $N$-uples of variables. We define a generalized multilinear polynomial with split variables to be a vector of polynomials

$$F(T_1, \ldots, T_k) = \sum_{i=1}^r c_i \prod_{j \in J_i} T_j \in K[T_1, \ldots, T_k]$$

for some disjoint partition $J_1 \cup \ldots \cup J_r = \{1, \ldots, s\}$ and $c_i \in (K^*)^N, i = 1, \ldots, r$, where the sum and product are the polynomials sum and product coordinate by coordinate.

**Theorem 4.7.** Let $F(T_1, \ldots, T_k) \in K[T_1, \ldots, T_k]$ be a generalized multilinear polynomial with split variables and let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of distinct morphisms from $\mathbb{A}^N(K)$ to $\mathbb{A}^N(K)$ of degrees $d_i > N$, whose components do not have zero in common.

(a) The set of $P \in \mathbb{G}_m^N(K)$ for which there exists a $k$-tuple of distinct non-negative integers $(n_1, \ldots, n_k)$ and a sequence $\Phi$ from $\mathcal{F}$ with $\Phi^{(n_i)}(P) \in \mathbb{G}_m^N(K)$ satisfying

$$F(\Phi^{(n_1)}(P), \ldots, \Phi^{(n_k)}(P)) = 0$$

is a set of bounded height.

(b) If $d_1 = \ldots = d_k = d \geq \frac{3N + 2}{2}$, the set of $P \in \mathbb{G}_m^N(K)$ for which there exists a $k$-tuple of distinct non-negative integers $(n_1, \ldots, n_k)$ and sequences $\Phi_1, \ldots, \Phi_k$ from $\mathcal{F}$ with $\Phi_i^{(n_i)}(P) \in \mathbb{G}_m^N(K)$ satisfying

$$F(\Phi_1^{(n_1)}(P), \ldots, \Phi_k^{(n_k)}(P)) = 0$$

is a set of bounded height.

In particular, if $N = 1$, then the results are true also with $\mathbb{A}^1(K) = \overline{K}$ in place of $\mathbb{G}_m(K) = \overline{K}^*$, when $0$ belongs to the referred orbits.
Proof. By assumption, the polynomial $F$ has the form

$$F(T_1, \ldots, T_k) = \sum_{i=1}^r c_i \prod_{j \in J_i} T_j$$

for some disjoint partition $J_1 \cup \ldots \cup J_r = \{1, \ldots, s\}$ and we can suppose that $n_1 > \ldots > n_k$. Here we use heights and canonical heights with their properties from the Lemmas 4.1 and 4.2 for $L$ the usual hyperplane section, so that $h_L = h$ usual logarithmic height, and we omit $L$ for the associated canonical heights. Hence

$$\Phi(n_1)(P) = \sum_{i=2}^r (-c_i) \prod_{j \in J_i} \Phi(n_j)(P) \prod_{j \in J_i \setminus \{1\}} \Phi(n_j)(P).$$

By some elementary properties of height functions including [6, Proposition 8.7.2], this yields

$$h(\Phi(n_1)(P)) \leq h \left( \sum_{i=2}^r (-c_i) \prod_{j \in J_i} \Phi(n_j)(P) \right) + Nh \left( c_1 \prod_{j \in J_1 \setminus \{1\}} \Phi(n_j)(P) \right)$$

$$\leq \sum_{i=2}^r \left( h(c_i) + \sum_{j \in J_i} \Phi(n_j)(P) \right) + \log(r - 1)$$

$$+ N \left( h(c_1) + \sum_{j \in J_1 \setminus \{1\}} h(\Phi(n_j)(P)) \right)$$

$$\leq N \left( \sum_{j=2}^k h(\Phi(n_j)(P)) + \sum_{i=1}^r h(c_i) \right) + \log(r - 1).$$

Hence,

$$\hat{h}_{S^n \Phi}(\Phi(n_1)(P)) \leq N \left( \sum_{j=2}^k \hat{h}_{S^n \Phi}(\Phi(n_j)(P)) + \sum_{i=1}^r h(c_i) \right) + \log(r - 1) + O(1),$$

and then

$$(\deg \Phi(n_1) \hat{h}_\Phi(P) \leq N \left( \sum_{j=2}^k (\deg \Phi(n_j)) \hat{h}_\Phi(P) + \sum_{i=1}^r h(c_i) \right) + \log(r - 1) + O(1),$$
and

$$\left( \deg \Phi^{(n_1)} - N \sum_{j=2}^{k} (\deg \Phi^{(n_j)}) \right) \hat{h}_\Phi(P)$$

$$\leq N \sum_{i=1}^{r} h(c_i) + \log(r - 1) + O(1).$$

We note that

\[
\deg \Phi^{(n_1)} - N \sum_{j=2}^{k} (\deg \Phi^{(n_j)}) \geq \deg \Phi^{(n_1)} - N \sum_{i=1}^{k-1} (\deg \Phi^{(n_1-i)}) \geq \deg \Phi^{(n_1)} \left[ 1 - N \left( \sum_{i=1}^{k-1} \frac{1}{d_1^{i}} \right) \right] \geq \deg \Phi^{(n_1)} \left[ 1 - N \left( \frac{1 - d_1^{k-1}}{1 - d_1} \right) \right] \geq \frac{(\deg \Phi^{(n_1)})}{d_1^{k-1}} \geq d_1^{n_1-k+1} \geq 1,
\]

since \(d_1 > N\) and thus, \(P\) has bounded height from above, independent of \(\Phi\), but only of \(\mathcal{F}\). Considering \(\inf_P \{ \hat{h}_\Phi(P) \}\), we can see that the \(n_j\)'s can be bounded by a bound depending on \(\Phi\) and similarly as it is said in Remark 4.5, we can see that for \(P\) out of a set of bounded height, \(\inf_P \{ \hat{h}_\Phi(P) \}\) for \(P\) on such range is finite and the bound for the \(n_j\)'s will not depend on \(\Phi\), but only on \(\mathcal{F}\), which proves the first part of the Theorem.

On the other hand, for (b) if for some sequences \(\Phi_1, ..., \Phi_k\) we have

\[
\Phi_1^{(n_1)}(P) = \frac{\sum_{i=2}^{r} (-c_i) \prod_{j \in J_i} \Phi_j^{(n_j)}(P)}{c_1 \prod_{j \in J_1 \setminus \{1\}} \Phi_j^{(n_j)}(P)}
\]
and $n_1 > \ldots > n_k$, then similarly one computes

$$h(\Phi_1^{(n_1)}(P))$$

$$\leq h \left( \sum_{i=2}^{r} (-c_i) \prod_{j \in J_i} \Phi_j^{(n_j)}(P) \right) + Nh \left( c_1 \prod_{j \in J_1 \setminus \{1\}} \Phi_j^{(n_j)}(P) \right)$$

$$\leq \sum_{i=2}^{r} \left( h(c_i) + \sum_{j \in J_i} \Phi_j^{(n_j)}(P) \right) + \log(r - 1)$$

$$+ N \left( h(c_1) + \sum_{j \in J_1 \setminus \{1\}} h(\Phi_j^{(n_j)}(P)) \right)$$

$$\leq N \left( \sum_{j=2}^{k} h(\Phi_j^{(n_j)}(P)) + \sum_{i=1}^{r} h(c_i) \right) + \log(r - 1).$$

Hence,

$$\hat{h}_{S^{n_1} \Phi_1}(\Phi_1^{(n_1)}(P))$$

$$\leq N \left( \sum_{j=2}^{k} \hat{h}_{S^{n_j} \Phi_j}(\Phi_j^{(n_j)}(P)) + \sum_{i=1}^{r} h(c_i) \right) + \log(r - 1) + O(1),$$

and then

$$d_{n_1} \hat{h}_{\Phi_1}(P) \leq N \left( \sum_{j=2}^{k} d_{n_j} \hat{h}_{\Phi_j}(P) + \sum_{i=1}^{r} h(c_i) \right) + \log(r - 1) + O(1).$$

Since $\hat{h}_{\Phi_j}(P) = \frac{\hat{h}_{\Phi_j}(P)}{\hat{h}_{\Phi_1}(P)} \hat{h}_{\Phi_1}(P) \leq (3/2) \hat{h}_{\Phi_1}(P)$ out of a set of bounded height, it follows, on such range, that

$$\left( d_{n_1} - (3N/2) \sum_{j=2}^{k} d_{n_j} \right) \hat{h}_{\Phi_1}(P) \leq N \sum_{i=1}^{r} h(c_i) + \log(r - 1) + O(1).$$
We note that
\[ d^{n_1} - (3N/2) \sum_{j=2}^{k} d^{n_j} \geq \deg \Phi^{(n_1)} - (3N/2) \sum_{i=1}^{k-1} d^{n_{1-i}} \]
\[ \geq d^{n_1} \left[ 1 - (3N/2) \left( \sum_{i=1}^{k-1} \frac{1}{d^i} \right) \right] \]
\[ \geq d^{n_1} \left[ 1 - \frac{3N}{2d} \left( \frac{1}{d^k - 1} \right) \right] \]
\[ \geq \frac{(d^{n_1}}{d^{k-1}} \geq d^{n_{1-k+1}} \geq 1, \]
and thus, \( P \) has bounded height from above, independent of the \( \Phi_j \)'s, but only of \( F \). Considering \( \inf_{F, \Phi} \{ h_{\Phi}(P) \} \), similarly as in Remark 4.5, we can see that for \( P \) out of a set of bounded height, \( \inf_{F, \Phi} \{ h_{\Phi}(P) \} \) for \( P \) is finite and the bound for the \( n_j \)'s will not depend on \( \Phi \), but only on \( F \), which proves the second part of the Theorem.

Finally, if \( N = 1 \) and \( 0 = h(\Phi^n(P)) \) for some sequence \( \Phi \), and \( n \), then one has that
\[ h(P) \leq \hat{h}_{\Phi}(P) + O(1) = (\deg \Phi^n)^{-1} \hat{h}_{\Phi^n}(\Phi^n(P)) + O(1) \]
\[ \leq \hat{h}_{\Phi^n}(0) + O(1) \]
\[ \leq h(0) + O(1) = O(1), \]
concluding the proof. \( \square \)

Let
\[ \mathbb{P}^N(U) := \{(x_0, ..., x_N) \in \mathbb{P}^N(\mathbb{C}) | x_i \in U, i = 0, ..., N \}. \]

**Definition 4.8.** Let \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree at least 2. We say that \( \phi \) is a Lattès map if there are an elliptic curve \( E \) and finite maps \( \epsilon : E \to E \) and \( \pi : E \to \mathbb{P}^1 \) such that \( \pi \circ \epsilon = \phi \circ \pi \).

We shall now derive some special conclusions of the flavour of 3.1 that aim to generalize [9, Theorem’s 27 and 34] and its corollaries to a general semigroup dynamical system of several morphisms.

**Theorem 4.9.** Let \( F = \{F_1, ..., F_s\} \) morphisms over \( \mathbb{C} \) from \( \mathbb{P}^N \) to \( \mathbb{P}^N \) of degrees at least 2 and \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) a Lattès map over \( \mathbb{C} \) of degree at least 2.

(a) If \( \{ P \in \mathbb{P}^N(U) | \mathcal{O}_F(P) \text{ is finite} \} \) is dense and invariant under some map \( F \) of the semigroup of \( F \), then \( F \) has the form
\[ D \circ S \circ \alpha^{\deg F}, \]
where \(D_i\) is a diagonal matrix of finite order, \(S\) is a permutation matrix and \(\alpha^{\deg F}\) is the \(\deg F\)-th power morphism.

In particular, if \(\hat{h}_F = h\), then each \(F_i\) has the form as above.

(b) If \(N = 1\) and \(\{P \in \mathbb{P}^1(\text{Preper}(\phi)) | \mathcal{O}_F(P) \text{ is finite} \}\) is infinite and invariant under some map \(F\) of the semigroup of \(\mathcal{F}\), then \(F\) is also a Lattès map and \(F\) and \(\phi\) are associated with the same elliptic curve.

In particular, if \(\hat{h}_F = \hat{h}_\phi\), then each \(F_i\) is a Lattès map and \(F_i\) and \(\phi\) are associated with the same elliptic curve.

**Proof.** Supposing the hypothesis of (a), we have that

\[
Z = \{P \in \mathbb{P}^N(\mathbb{U}) | \mathcal{O}_F(P) \text{ is finite} \}
\]

is dense, lies inside of \(\mathbb{P}^N(\mathbb{U})\), and \(F(Z) \subset \mathbb{P}^N(\mathbb{U})\). Then [9, Theorem 34] implies the referred form for \(F\). Moreover, if \(\hat{h}_F = h\), then the last part of Lemma 4.2 and the fact that \(h\) is in fact the canonical height associated with any power morphism assures us that

\[
\{P \in \mathbb{P}^N(\overline{\mathbb{Q}})|\mathcal{O}_F(P) \text{ is finite} \} = \mathbb{P}^N(\mathbb{U}),
\]

so that we are under the conditions of the first part of (a) with the invariance condition satisfied for each \(F_i, i = 1, ..., s\), giving the desired result.

Supposing the hypothesis of (b), we have that

\[
Z = \{P \in \mathbb{P}^1(\text{Preper}(\phi)) | \mathcal{O}_F(P) \text{ is finite} \}
\]

is infinite, lies inside of \(\mathbb{P}^1(\mathbb{U})\), and \(F(Z) \subset \mathbb{P}^1(\text{Preper}(\phi))\). Then [9, Theorem 27] implies the referred form for \(F\). Moreover, if \(\hat{h}_F = \hat{h}_\phi\), then the last part of Lemma 4.2 assures us that

\[
\{P \in \mathbb{P}^1(\overline{\mathbb{Q}})|\mathcal{O}_F(P) \text{ is finite} \} = \text{Preper}(\phi),
\]

so that we are under the conditions of the first part of (b) with the invariance condition satisfied for each \(F_i, i = 1, ..., s\), giving the desired result. \(\square\)

**Acknowledgement**

The author is grateful to Alina Ostafe and Igor Shparlinski for helpful discussions. For the research, the author was supported by the Australian Research Council Grant DP180100201.
References

[1] A. Bérczes, A. Ostafe, I. E. Shparlinski, J. Silverman: Multiplicative dependence among iterated values of rational functions modulo finitely generated groups, Internat. Math. Res. Notices, (to appear).

[2] E. Bombieri and W. Gubler: Heights in Diophantine Geometry, Cambridge Univ. Press, Cambridge, 2006.

[3] G. S. Call and J. Silverman: Canonical heights on varieties with morphisms, Compositio Math., 89 (1993), 163-205.

[4] E. Chen: Avoiding algebraic integers of bounded house in orbits of rational functions over cyclotomic closures, Proc. Amer. Math. Soc., 46 (2018) 4189-4198.

[5] R. Dvornicich and U. Zannier: Cyclotomic diophantine problems (Hilbert irreducibility and invariant sets for polynomial maps), Duke Math. J., 139 (2007) 527-554.

[6] M. Hindry, J. Silverman: Diophantine Geometry: An Introduction, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York (2000).

[7] S. Kawaguchi: Canonical heights for random iterations in certain varieties, International Mathematics Research Notices, Volume 2007, 2007, rnm023.

[8] S. Kawaguchi: Canonical heights, invariant currents, and dynamical systems of morphisms associated with line bundles, J. reine angew. Math. 597 (2006), 135-173.

[9] S. Kawaguchi and J. Silverman: Dynamics of projective morphisms having identical canonical heights, Proc. London Math. Soc., 95 (2007), 519-544.

[10] D. G. Northcott: Periodic points on an algebraic variety, Ann. of Math., 51 (1950), 167-177.

[11] A. Ostafe and M. Young: On algebraic integers of bounded house and preperiodicity in polynomial semigroup dynamics, arXiv:1807.11645

[12] K. Ribet: Torsion points of abelian varieties in cyclotomic extensions, Enseign. Math., 27 (1981), 315-319.

[13] J. Silverman: Integer points, Diophantine approximation, and iteration of rational maps, Duke Math. J., 71 (1993), 793-829.

[14] J. Silverman: The arithmetic of dynamical systems, Springer-Verlag, New York, 2007

[15] J. H. Silverman, The arithmetic of elliptic curves, 2nd ed., Springer, Dordrecht, 2009.

[16] J. Silverman: The theory of height functions, Arithmetic geometry, Springer (1986), 151-166.

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

E-mail address: j.mello@unsw.edu.au