In this work we study the symmetry breaking conditions, given by a (anti)de Sitter-valued vector field, of a full (anti)de Sitter-invariant MacDowell-Mansouri inspired action. We show that under these conditions the action breaks down to General Relativity with a cosmological constant, the four dimensional topological invariants, as well as the Holst term. We obtain the equations of motion of this action, and analyze the symmetry breaking conditions.

Keywords  Gauge Theories of Gravity; Torsion.

1 INTRODUCTION

The greatest example of the geometrization of the fundamental interactions is Einstein’s General Relativity (GR). Since the formulation of GR, gravity has gone hand in hand with geometry. Currently the gravitational interaction is identified as a manifestation of a curved space-time. The description of the remaining fundamental interactions is also described by the geometrical theory known as Yang-Mills (YM) theory. Although there are many differences between GR and YM, there have been attempts to unify them in the framework of classical field theory \[1,2\]. Different approaches have been proposed to construct a unified model of the fundamental interactions. For example, one can consider higher dimensional models of gravity, where the metric is a fundamental field from which the particle physics interactions are constructed \[3,4\]. On the other hand, in YM the interactions are described by a connection associated with a symmetry group in a non-dynamical space-time (in contrast to GR). Fortunately, there are several proposals for YM type gauge theories of gravity. These formulations that are known in the literature as pure connection actions for
We will like to finish this section by clarifying some notation. We have labeled $\omega$ where we identify $F$ where use identify $\Lambda$.

Then the Cartan-Killing form can be “cut off” in such a way that we keep the geometry of gravity. This is achieved by considering that the Lie algebra is $so(1, 4) = so(1, 3) \oplus R^{1, 3}$ is a reductive algebra $[13, 14]$. Then the Cartan-Killing form can be “cut off” in such a way that we keep the geometry of gravity. This is achieved by considering that the Lie algebra is $so(1, 4)$ valued fields. In the same spirit, Stelle and West $[15, 16]$ considered an action where a symmetry breaking mechanism is induced by introducing a field $v$. This field represents the coordinates of a point in an internal anti-de Sitter space, where the motions induced by parallel transport takes place. The gravitational vierbein and spin connection are identified from the original $SO(1, 4)$ gauge fields. Consequently, by the symmetry mechanism, Einstein-Cartan theory plus the Euler class are obtained $[11]$. A self-dual spin connection approach for MM has been given by $[17, 18]$, where duality is defined with respect to the corresponding internal symmetry group indices. Also, a formal derivation of the MM formulation from a topological $BF$ theory plus a regular interaction term can be found in the literature (see $[19]$ and references therein).

In this work the dynamical sector leads to Einstein-Cartan gravity with the cosmological term plus the Holst term. The topological sector is given by the Nieh-Yan, Pontryagin, and Euler classes.

The paper is arranged as follows. In section II, we present a brief account of the Stelle-West symmetry breaking approach for the Palatini action with cosmological constant term and Euler class. In section III, we generalize a Stelle-West formulation, which besides the Palatini action has the cosmological constant term. We also get the Holst modification with the Immirzi parameter and three topological terms, Nieh-Yan form, Euler class and $SO(1, 4)$ Pontryagin term. In section IV, without explicitly breaking the symmetry, we give two conditions for obtaining gravity. The first condition yields Einstein’s equations plus the cosmological constant term and torsionless equation. The second condition, allows us to construct a family of topological torsionless field theories. Section V is devoted to discussion and outlook.

We will like to finish this section by clarifying some notation. We have labeled $so(1, 4)$ Lie algebra indices by capital Latin letters $\{A, B, C, \ldots\}$, $so(1, 3)$ Lie algebra indices by the beginning of the Latin alphabet lowercase letters $\{a, b, c, \ldots\}$ and Greek alphabet indices for space-time indices $\{\mu, \nu, \rho, \ldots\}$. Also, we consider the Minkowski metric as $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$, and we establish $\eta_{AB,CD} = \eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}$ and $\eta_{ab,cd} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}$. In addition, we have defined the totally anti-symmetric tensor $\epsilon$ with $\epsilon^{01234} = +1$ and $\epsilon^{01234} = -1$ and, in four dimensions, $\epsilon^{0123} = +1$ and $\epsilon^{0123} = -1$. Finally, we define $G_{(ab)} = G_{ab} + G_{ba}$ and $G_{[ab]} = G_{ab} - G_{ba}$.

2 MacDowell-Mansouri action by Stelle-West Approach

Let us briefly describe the MacDowell-Mansouri (MM) approach to gravity. MM theory is a gauge theory of gravity with the gauge group $G \supset SO(3, 1)$, where $G$ depends on the sign of the cosmological constant $\Lambda$. In this work, we will consider the case when $\Lambda > 0$ and use $SO(1, 4)$ (the case with $\Lambda < 0$ can be calculated straightforward). Moreover, space-time is a 4-dimensional oriented smooth manifold $M = R \times \Sigma$, $\Sigma$ is a compact manifold without boundary and $R$ represents the evolution parameter. Now we choose a principal $SO(1, 4)$-bundle $P$ over $M$. Because MM theory is essentially a pure connection action of gravity $[2, 14]$, the fundamental field for constructing the action is the connection $A$. The connection is an $so(1, 4)$ valued 1-form on $M$. It can be written as $A = A_{\mu}^{\Lambda\beta}t_{AB}dx^\mu$, where $t_{AB}$ are the skew-symmetric generators of the Lie algebra $so(1, 4)$ and satisfy

$$[t_{AB}, t_{CD}] = f_{ABC}{}^D t_{EF} = \frac{1}{4} \eta_{AB,\{C} [E_{\{F]} t_{\}D]} t_{EF}. \quad (1)$$

Because $so(1, 4)$ is a reductive geometry $[14, 13]$, we have $so(1, 4) \cong so(1, 3) \oplus R^{1, 3}$. Then, we can write the gauge field as follows

$$A_{\mu}^{\Lambda\beta} = \begin{pmatrix} A_{\mu}^{ab} & A_{\mu}^{a4} \\ A_{\mu}^{ab} & 0 \end{pmatrix} = \begin{pmatrix} \omega_{\mu}^{ab} & \frac{1}{2} e_{\mu}^a \\ \frac{1}{2} e_{\mu}^a & 0 \end{pmatrix}, \quad (2)$$

where we identify $\omega_{\mu}^{ab}$ with the spin connection and $e_{\mu}^a$ with the tetrad field. Also, $\lambda$ is constant with length dimension introduced for units requirement. The covariant derivative $D$ of the de Sitter group acts over Lie algebra valued fields $\xi = \xi^{\Lambda\beta}t_{AB}$ as follows

$$D\xi = \left[ D\xi^{ab} + \frac{1}{2} \left( e^a \wedge \xi^{b4} - e^b \wedge \xi^{a4} \right) \right] t_{ab} + 2 \left[ D\xi^{a4} - \frac{1}{2} e^b \wedge \xi_{\mu}^a \right] t_{a4}. \quad (3)$$

And we have introduced the Lorentz covariant derivative $D\chi^{ab} = d\chi^{ab} + \omega^{ac} \wedge \chi_{c}^b + \omega^{bc} \wedge \chi_{c}^a$. The field strength is defined as $F = dA + \frac{1}{2} [A, A]$, with

$$F_{AB} = \begin{pmatrix} F_{ab} & F_{a4} \\ F_{4b} & 0 \end{pmatrix} = \begin{pmatrix} R_{ab} - \frac{1}{2} T_{b} \Sigma_{ab} & -\frac{1}{2} T_{a} \\ \frac{1}{2} T_{b} \Sigma_{ab} & 0 \end{pmatrix}, \quad (4)$$
where $T^a = De^a = de^a + \omega^a_e \wedge e^e$ is the torsion and $R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^c_b$ is the curvature. The torsion and the curvature are $so(1,3)$ valued two-forms. We have also defined $\Sigma_{ab} = e^a \wedge e^b$.

Stelle and West considered the action $^{15, 16}$

$$I_{MM} [A,v] = \frac{1}{\beta_1} \int_M \epsilon^{ABCD} F_{AB} \wedge F_{CD} + \rho (v^E v_E - l^{-2}),$$

where $\beta_1 = 4Gl^{-3}$. $v$ is a vector field with dimension of length, $\rho$ is four-form acting as a Lagrange multiplier and $l$ is the same real constant in Eq. (2). The value of $\beta_1$ was chosen in accordance with $^{19}$. To obtain GR we break the $SO(1,4)$ symmetry down to $SO(1,3)$. This is achieved by choosing a preferred direction for the vector field $v^E$ and imposing that in the resulting subgroup $v$ remains fixed. We will refer to these conditions as the symmetry breaking conditions (SBC). Taking $v^a = 0$ in $v^E = (v^a, v^4)$ and imposing the field equation for $\rho$, we get $v^a = 1/l$. Then the action in Eq. (5) takes the form

$$I_{MM} [A] = \frac{1}{\beta_2} \int_M \epsilon^{abcd} F_{ab} \wedge F_{cd} \frac{1}{7}$$

$$= \frac{1}{\beta_2} \int_M \epsilon^{abcd} \left( R_{ab} \wedge R_{cd} - \frac{2}{l^2} R_{ab} \wedge \Sigma_{cd} + \frac{1}{l^2} \Sigma_{ab} \wedge \Sigma_{cd} \right).$$

From this equation, we recognize the Euler class plus the Palatini action with non-vanishing cosmological constant. The cosmological constant is given by $\frac{\beta_1}{\beta_2} = \frac{\Lambda}{2}$.

### 3 A Stelle-West generalization

In this section we propose a generalization to the action in Eq. (5) that not only includes the terms in Eq. (6), but also the Nieh-Yan term and the second Chern class. Furthermore, we want to construct the action in such a way that it can be identified with a pure connection self-dual formulation of gravity. The construction is inspired by the Plebański formulation $^{20}$ and the Capovilla, Dell and Jacobson formulation $^{8}$. We will start by defining a pseudo-projector for $so(1,4)$ as follows

$$\Pi^{ABCD} \xi_{CD} = \frac{1}{2} \left( \alpha \eta^{AB,CD} + \epsilon^{ABCD} v_E \right) \xi_{CD}$$

$$= \frac{1}{2} \left( \alpha \eta^{AB,CD} + \epsilon^{ABCD} \right) \xi_{CD} = \bar{\xi}^{AB},$$

where $\alpha$ is a constant with units of $\text{length}^{-1}$, that will be related to the Immirzi parameter and the cosmological constant. The proposed action is

$$I_{GMM} [A,v] = \frac{1}{\beta_2} \int_M < \bar{F} \wedge \bar{F} > + \rho (v^E v_E - l^{-2}),$$

where $< , >$ is the trace over the Lie algebra generators and $\beta_2 = 4\gamma Gl^{-4}$. We can see that Eq. (8) is a pure connection YM type action and the metric is not explicitly in the action. The action $I_{GMM}$, can be thought as the integral of the Pontryagin density “twisted” by the pseudoprojector $\Pi$. Interestingly, if instead of taking $\bar{F} \bar{F}$ product we consider a $BF$ formulation in Eq. (8), the action gives rise to Einstein conformally flat spaces $^{21}$.

Using Eq. (7), the action can be rewritten as

$$I_{GMM} = \frac{1}{\beta_2} \int_M (\alpha^2 - v^E v_E) F^{AB} \wedge F_{AB} + \alpha \epsilon^{ABCD} F_{AB} \wedge F_{CD}$$

$$- 2 F^{AC} \wedge F^{B} v_{AB} + \rho (v^E v_E - l^{-2}).$$

Using the field equation for $\rho$, together with the SBC and the vector $v$ with vanishing components $v^a$, the action reads

$$I_{GMM} [e, \omega] = \frac{6}{G_A} \int_M \frac{\gamma^2 - 1}{2\gamma} R^{ab} \wedge R_{ab} + \frac{1}{2} \epsilon^{abcd} R_{ab} \wedge R_{cd} + \frac{\Lambda \gamma}{3} (e^a \wedge T_a)$$

\(^{a}\)We work in units where $c, 16\pi \hbar = 1$ and $16\pi G l = l_p$.

\(^{b}\)In this work, the have defined a pseudoprojector $\Pi$ as a projector operator that maps a Lie algebra bi-vector field onto a Lie algebra bi-vector field, and it does not satisfy $\Pi \Pi = \Pi$.

\(^{c}\)The last action is related to the model that was proposed in $^{22}$ where the action is constructed by hand. In that work, the consider the connection one-form $A$ and $v$ as the fundamental fields, combining the two and looking for a consistent action (invariant under the gauge group and polynomial in the basic fields). In our case the action is gauge invariant and is derived from an Yang-Mills type action.
We have separated the action in its topological and dynamical sectors. The first three terms in the action are the second \(\frac{1}{2G} \int_M \left( \epsilon^{abcd} R_{ab} \wedge \Sigma_{cd} \right)\), which led us to understand the role played by \(v\).

To do so, let us first consider Eq. (8) and its equations of motion. From this point forward, we will take \(v\) such that \(v^2 = l^{-2}\) and no longer constant. When considering the action in Eq. (8) with the field equation for \(v\) might have non constant components. Consequently, some terms that are proportional to \(D v\) can contribute and the action can not be written as a total derivative. Since \(v\) is not necessarily a constant vector, we take its covariant derivative and find

\[
\mathcal{D}(v^A v_A) = \mathcal{D}(l^{-2}) \Rightarrow v^A \mathcal{D} v_A = 0 \Rightarrow dv^A = -\frac{1}{v^4} v^a dv_a,
\]

where we have written \(v^A = (v^a, v^4)\) and \(v^4 \neq 0\).

The equations of motion from \(I_{GMM}\), are given by

\[
\epsilon^{ABCDE} F_{AB} \wedge F_{CD} = \frac{4}{\alpha} F^{EA} \wedge F_B^A v_B, \\
\epsilon^{ABCDE} F_{CD} \wedge \mathcal{D} v_E = \frac{2}{\alpha} F^{[A} \wedge \mathcal{D}(v^B) v_C).
\]

The first equation is related to the variation with respect to \(v\), the second to the variation respect to \(A\). The equation of motion for \(A\) implies a dynamical equation for \(v\). Then, our task is to find some solutions for this dynamical behavior which led us to understand the role played by \(v\) in a more fundamental level. In the next subsections we will present two different dynamical solutions for \(v\).

### 4.1 Case \(\mathcal{D} v^E = A_{E4} v_4\)

Let us consider the case \(\mathcal{D} v^E = A_{E4} v_4\), with non-vanishing \(v^4\) for at least one component of \(v\). This gauge implies the following equations

\[
v^A \mathcal{D} v_A = \frac{1}{l} v^a e_a v^4 = 0, \\
D v^a = 0, \\
dv^4 = -\frac{1}{l} v^a e_a.
\]

\(\text{This case was briefly mentioned in [23] but was not explored.}\)
From eq. (13) we identify two cases. If we take \( v^4 = 0 \), from Eq. (17) we get that \( v^a e_a \) vanishes as well. However, since we are considering a non-degenerate tetrad field, \( v^a \) also vanishes. Consequently, we conclude that \( v^4 = 0 \) but this is the trivial case that we are not considering. In the second case we have a non-vanishing \( v^4 \) (at least in some finite regions on the manifold). As in the previous case \( v^a e_a = 0 \), hence \( v^a = 0 \). Since we are considering non-degenerate tetrad fields and considering Eq. (17), we conclude that \( v^4 \) is constant and equal to \( l^{-1} \). Moreover, from the algebraic constraint imposed on \( v \cdot v \) we conclude that this case is equivalent to considering the case with SBC.

Our next step is to consider equations Eq. (13) and Eq. (14), substitute \( v^E = (0, 0, 0, l^{-1}) \). Together with the condition and \( \alpha = 1/(l\gamma) \), it gives the following equations

\[
\begin{align*}
(\gamma\eta^{ab,cd} + e^{abcd})T_e \wedge e_d &= 0, \\
e^{abcd} R_{bc} \wedge e_d - \frac{\Lambda}{3} e^{abcd} e_b \wedge e_c \wedge e_d &= \frac{1}{\gamma} DT^a,
\end{align*}
\]

where the first equation is the well-known zero torsion condition, which allows us to write the spin connection as a function of the tetrad field. The second line describes vacuum Einstein equations. This pair of equations come directly from the condition in Eq. (14).

Also from Eq. (13) and Eq. (14), we obtain

\[
\frac{\Lambda^2}{9} = \frac{e^{abcd} R^{ab} \wedge R^{cd}}{e^{abcd} e^a \wedge e^b \wedge e^c \wedge e^d},
\]

which provides an algebraic equation for the cosmological constant and it can allow us to fix its value [23, 24]. So the gauge given in this subsection for the covariant derivative of \( v \), not only gives us GR with cosmological constant, it fixes the value of the cosmological constant. In some sense, using this gauge is almost equivalent as breaking the symmetry group. It also gives more information by giving an expression for the the cosmological constant. Accordingly, we can argue that \( \Lambda \) is calculated within the theory. Finally, it is possible to calculate the metric by using Eq. (11).

### 4.2 Case \( D v^E = 0 \)

This case is of particular interest since Eq. (13) vanishes identically. Then the metric can not be constructed by considering Eq. (11) as in the previous cases. The gauge condition \( D v^E = 0 \), gives

\[
\begin{align*}
Dv^a &= \frac{1}{l} e^a v^4, \\
dv^4 &= -\frac{1}{l} v^a e_a,
\end{align*}
\]

from Eq. (12) and Eq. (22), we conclude that the tetrad field is completely defined by the vector \( v \) and its derivatives

\[
e^a = \frac{l}{v^4} dv^a.
\]

From the last equation and Eq. (21) we get

\[
\omega^a_a v^b = 0,
\]

then spin connection must be orthogonal to the Minkowski components of \( v \). By applying the \( so(1, 4) \) covariant derivative on the gauge condition \( D v^E = 0 \), we find \( F^{AB} v_B = 0 \). This expression can be decomposed in two expressions

\[
\begin{align*}
F^{ab} v_b &= \left( R^{ab} - \frac{1}{12} \Sigma^{ab} \right) v_b = \frac{1}{l} T^a v_4, \\
T^a v_a &= 0.
\end{align*}
\]

The first equation is an identity (see Appendix B) and gives no further information. In the next subsections we will focus our attention on the second equation. But before we proceed, let us point out that \( v^a \) can not be constant (this can be seen from Eq. (23)).

#### 4.2.1 Zero Torsion condition.

A solution for Eq. (23) is \( T^a = 0 \). Then we can write the spin connection as function of the tetrad field and because the tetrad field depends on \( v \), also \( \omega \) depends on \( v \). \( e \) also depends on \( v \)

\[
\omega = \omega(e) = \omega(v^a, v^b), \quad e = e(v^a, v^b).
\]
We can see from the previous equation that all the fundamental objects are given by the vector field $\nu^A$. Consequently, this vector allows us to construct the frame bundle and (locally) the SO(1,3)-bundle, where the spin connection must be orthogonal to the vector $\nu^a$. Using Eq. (26) and Eq. (24), we get

$$
\chi^a b v_b = 0, \quad \text{where} \quad \chi_{ab} = d\omega_{ab} - \frac{1}{2} \Sigma^{ab}.
$$

Let us note that $\det|\chi| \neq 0$ is not possible, since it will imply that $\nu^a = 0$. We can take $\det|\chi| = 0$ and calculate the null vectors $\nu^a$, but this case goes beyond the scope of this work. The other possibility is to consider $\chi_{ab} = 0$, this implies that $d\omega_{ab} = \Sigma^{ab}$. Thus, by taking $\chi_{ab} = 0$ and the zero torsion condition, we observe that the field equations derived from $\nu$ reduce to

$$
e_{abcd} F_{ab} \wedge F_{cd} = 0 \Rightarrow e_{abcd} \omega^f_a \wedge \omega^g_b \wedge \omega^c_e \wedge \omega^d_f = 0.
$$

This equation vanishes and we don’t have any dynamical equations of motion. Hence, using this gauge in conjunction with $\chi = 0$, gives the conditions to have a topological field theory. Therefore, we conclude that $\nu$ parametrizes a family of topological torsionless field theories.

### 4.2.2 Orthogonal spin connection condition.

We now consider Eq. (26) with $T^a \neq 0$. Then, from Eq. (24) we get

$$
u_e d\nu^e = 0, \tag{30}
$$

but this equation is an identity (see Appendix B). In this case, the equation of motion for $\nu$ can be treated as the zero torsion condition, since it is the only equation that relates the spin connection with the tetrad field. Therefore, we get

$$e_{abcd} F_{ab} \wedge F_{cd} = 0 \Rightarrow e_{abcd} \omega^f_a \wedge \omega^g_b \wedge \omega^c_e \wedge \omega^d_f = 0.
$$

Contracting it with a vector $\nu^d$ and using Eq. (28) we find

$$D \left[ e_a \wedge R_{bc} + e_b \wedge R_{ca} + e_c \wedge R_{ab} - \frac{1}{L^2} e_a \wedge e_b \wedge e_c \right] = 0. \tag{32}
$$

With the help of Eq. (24) it can be rewritten as

$$D \left[ \frac{d\nu_a}{v^4} \wedge R_{bc} + \frac{d\nu_b}{v^4} \wedge R_{ca} + \frac{d\nu_c}{v^4} \wedge R_{ab} - \frac{1}{L^2} \frac{d\nu_a}{v^4} \wedge \frac{d\nu_b}{v^4} \wedge \frac{d\nu_c}{v^4} \right] = 0. \tag{33}
$$

From this equation one obtains $\omega$ as a function of $\nu$ its derivatives $\omega = \omega(\nu, \partial \nu)$. For this particular case, Eq. (33) plays the role of the zero torsion condition, as it relates the tetrad and the connection. Interestingly, the tetrad field and therefore the metric, is not obtained from the Einstein field equations. Instead it is a consequence the imposed gauge, and depends on the functional form of $\nu$.

### 5 Conclusions and Outlook

In this work we have considered a generalized Stelle-West formulation of gravity by means of introducing a pseudoprojector II acting on $so(1,4)$ Lie algebra valued field strength. This projector has the form of a complex (anti)self-dual projector of a Plebanski like formulation. Moreover, as in the original Stelle-West formulation, the vector field $\nu$ hidden into the projector II, plays a central role in breaking the symmetry down to $SO(1,3)$. When considering the SBC, we not only get Einstein-Cartan theory and the Euler class, we have also obtained the Holst modification, the Nieh-Yan and the Pontryagin topological terms. We also considered the action without the SBC. The equations of motion are not trivial, so we introduced constraints on the covariant derivative of $\nu$. In the first case, we imposed a condition that allowed us to recover Einstein’s equations. Furthermore, from the equation for $\nu$ we obtained a condition over the cosmological constant. We find that $\Lambda$ is calculated within the theory, this is consistent with the result presented in [23, 24]. It will be intriguing to compare the relationship between the action in Eq. (8) and the quasitopological principle proposed by Alexander et. al. in [23], for their $\theta$ term and the non-constant $\Lambda$.

For the second case, we were able to construct topological torsionless field theories. In this case, the Minkowski part of the vector $\nu$ is orthogonal to the spin connection and $\chi = 0$. Since the Lie algebra is a reductive algebra and the connection is an $so(1,3)$ valued field, the vector $\nu$ allows us to construct the (local) $SO(1,3)$ bundle and the frame bundle. Furthermore, the tetrad field (as well as the the metric) does not come from the Einstein field equations. Instead, is obtained from the gauge we used and from Eq. (33), we also find that the metric depends on the functional form of $\nu$. Numerical solutions for the field $\nu$ (under the second condition) could be used to reconstruct the metric and characterize the resulting manifolds and their topological structures.
In this paper we have only worked the bosonic case, it will be interesting to consider the the supersymmetric extension for \( \mathcal{N} = 1 \). Also higher dimensional internal symmetry group based on MM approach for pure connection formulations of gravity for real fields, inspired in [3], can be considered in order to obtain a family of torsionless conformally flat Einstein manifolds. These ideas are work in progress and will be reported elsewhere.

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A  GR action from the Stelle-West generalization.

In this appendix we present the derivation of the GR action (including topological terms) from Eq. (8). We start by considering the term \( \langle \tilde{F} \wedge \tilde{F} \rangle \)

\[
\langle \tilde{F} \wedge \tilde{F} \rangle = \frac{1}{4} \left( \alpha \eta^{AB,CD} + \varepsilon^{ABCD}v_E F_{CD} \wedge (\alpha \eta_{AB,FG} + \varepsilon_{ABFGH}v^H) \right) F^{FG}
\]

and use \( \varepsilon^{ABCD} \varepsilon_{ABFGH} = -2 \left[ \delta^{CD}_{FG} \delta^E_H + \delta^{CD}_{GH} \delta^E_F + \delta^{CD}_{HF} \delta^E_G \right] \) to get

\[
\langle \tilde{F} \wedge \tilde{F} \rangle = \alpha^2 v^E F^{AB} \wedge F_{AB} + \alpha \varepsilon^{ABDE} v_E F_{AB} \wedge F_{CD} - 2 F_{AC} \wedge F^{CB} v^A v_B.
\]

We are imposing the conditions \( v^E v_E = l^{-2}, v^E = (0, 0, 0, l^{-1}) \) and \( \alpha = \gamma l^{-1} \). Then for each term in the last equation we have

\[
F^{AB} \wedge F_{AB} = F^{ab} \wedge F_{ab} + F^{a4} \wedge F_{a4} + F^{4a} \wedge F_{4a} = R^{ab} \wedge R_{ab} + 2(\Lambda/3) [d(e_a \wedge T^a)],
\]

\[
\varepsilon^{ABDE} v_E F_{AB} \wedge F_{CD} = l^{-1} \varepsilon^{abcd} [R_{ab} \wedge R_{cd} - 2l^{-2} R_{ab} \wedge \Sigma_{cd} + l^{-4} \Sigma_{ab} \wedge \Sigma_{cd}],
\]

and

\[
2 F_{AC} \wedge F^{CB} v^A v_B = -2l^{-4} T_a \wedge T^a = -2l^{-4} [d(e_a \wedge T^a) + R^{ab} \wedge \Sigma_{ab}].
\]

Finally, by substituting Eq. (35) in Eq. (36) and defining \( l^{-2} = \Lambda / 3 \) we get

\[
\langle \tilde{F} \wedge \tilde{F} \rangle = \frac{2\gamma \Lambda}{3} \left[ \left( \frac{\gamma^2 - 1}{2\gamma} \right) R^{ab} \wedge R_{ab} + \frac{1}{2} \varepsilon^{abcd} R_{ab} \wedge R_{cd} + \frac{\gamma \Lambda}{3} d(e_a \wedge T^a) \right]
\]

\[
- \frac{\Lambda}{3} \left( \varepsilon^{abcd} R_{ab} \wedge \Sigma_{cd} - \frac{1}{\gamma} R^{ab} \wedge \Sigma_{ab} - \frac{\Lambda}{6} \varepsilon^{abcd} \Sigma_{ab} \wedge \Sigma_{cd} \right).
\]

B  Two equations that vanish identically.

In this section, we show the vanishing of two equations derived from \( Dv^E = 0 \).

Let us start with Eq. (25).

\[
\left( R^{ab} - \frac{1}{l^2} \Sigma^{ab} \right) v_b = \left( d\omega^{ab} + \omega^a \wedge e^b - \frac{1}{l^2} e^a \wedge e^b \right) v_b
\]

\[
= d(\omega^{ab} v_b) + \omega^{ab} \wedge dv_b - \frac{1}{l^2} e^a \wedge [-ldv^a],
\]

where we have used \( e^a v_a = -ldv^a \) and \( \omega^{ab} v_b = 0 \).

By considering that \( v^a e^a = dv^a \), we can rewrite the last equation as

\[
\omega_{ab} \wedge \left[ \frac{v^4}{l} \right] e^b + \frac{1}{l} e^a \wedge dv^a = \frac{v^4}{l} \omega^{ab} \wedge e_b + \frac{1}{l} [-d(e^a v^a) + de^a v^a]
\]

\[
= \frac{v^4}{l} T^a - \frac{1}{l} d(v^a) = \frac{v^4}{l} T^a,
\]
So finally we get \( F^{ab}v_b - v^4 / l T^a = 0 \).

We now turn our attention to Eq. (26),

\[
v_a d e^a = v_a \left[ \frac{1}{v^4} \  dv^4 \right] = \frac{v_a}{2(v^4)^2} dv^4 \wedge d(v_a v^a). \tag{40}
\]

Let us now consider the equation \( v^F v_F = v^a v_a + v^4 v_4 = 1 / l^2 \), then the last equation reads

\[
\frac{v_a}{2(v^4)^2} dv^4 \wedge d((v^4)^2) = \frac{1}{v^4} dv^4 \wedge dv^4 = 0. \tag{41}
\]

so finally, \( v_a d e^a = 0 \).

References

[1] Peter Peldan. Gravity coupled to matter without the metric. *Phys. Lett. B*. 248:62–66, 1990.

[2] Kirill Krasnov and Roberto Percacci. Gravity and Unification: A review. *Class. Quant. Grav.*, 35(14):143001, 2018.

[3] D.B. Fairlie. Higgs’ Fields and the Determination of the Weinberg Angle. *Phys. Lett. B*, 82:97–100, 1979.

[4] Yuval Ne’eman, S. Sternberg, and D. Fairlie. Superconnections for electroweak su(2/1) and extensions, and the mass of the Higgs. *Phys. Rept.*, 406:303–377, 2005.

[5] R. Capovilla, T. Jacobson, and J. Dell. A Pure spin connection formulation of gravity. *Class. Quant. Grav.*, 8:59–73, 1991.

[6] Kirill Krasnov. Pure Connection Action Principle for General Relativity. *Phys. Rev. Lett.*, 106:251103, 2011.

[7] J.E. Rosales-Quintero. Anti-self-dual gravity and supergravity from a pure connection formulation. *Int. J. Mod. Phys. A*, 31(12):1650064, 2016.

[8] Ermis Mitsou. Spin connection formulations of real Lorentzian General Relativity. *Class. Quant. Grav.*, 36:045008, 2019.

[9] J. E. Rosales-Quintero. A pure connection formulation with real fields for gravity. *International Journal of Modern Physics D*, 29(16):1–9, 2020.

[10] S. W. MacDowell and F. Mansouri. Unified Geometric Theory of Gravity and Supergravity. *Phys. Rev. Lett.*, 38:739, 1977. [Erratum: Phys.Rev.Lett. 38, 1376 (1977)].

[11] J.C. López-Domínguez, J.E. Rosales-Quintero, and M. Sabido. Comments on MacDowell–Mansouri gravity and torsion. *Int. J. Mod. Phys. D*, 27(03):1850018, 2017.

[12] O. Obregon, M. Ortega-Cruz, and M. Sabido. Immirzi parameter and theta ambiguity in de Sitter MacDowell-Mansouri supergravity. *Phys. Rev. D*, 85:124061, 2012.

[13] Derek K. Wise. Symmetric space Cartan connections and gravity in three and four dimensions. *SIGMA*, 5:080, 2009.

[14] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Class. Quant. Grav.*, 27:155010, 2010.

[15] K. S. Stelle and Peter C. West. DE SITTER GAUGE INVARINANCE AND THE GEOMETRY OF THE EINSTEIN-CARTAN THEORY. *J. Phys. A*, 12:L205–L210, 1979.

[16] K. S. Stelle and Peter C. West. Spontaneously Broken De Sitter Symmetry and the Gravitational Holonomy Group. *Phys. Rev. D*, 21:1466, 1980.

[17] J. A. Nieto, J. Socorro, and O. Obregon. Gauge theory of supergravity based only on a selfdual spin connection. *Phys. Rev. Lett.*, 76:3482–3485, 1996.

[18] H. Garcia-Compean, O. Obregon, and C. Ramirez. Gravitational duality in MacDowell-Mansouri gauge theory. *Phys. Rev. D*, 58:104012, 1998.

[19] Laurent Freidel and Artem Starodubtsev. Quantum gravity in terms of topological observables. *hep-th/0501191*.

[20] Jerzy F. Plebanski. On the separation of Einsteinian substructures. *J. Math. Phys.*, 18:2511–2520, 1977.

[21] Arthur L. Besse. *Einstein Manifolds*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1987.

[22] H. Westman and T. Zlošnik. Exploring Cartan gravity with dynamical symmetry breaking. *Class. Quant. Grav.*, 31:095004–095027, 2014.
[23] Stephon Alexander, Marina Cortês, Andrew R. Liddle, João Magueijo, Robert Sims, and Lee Smolin. Zero-parameter extension of general relativity with a varying cosmological constant. *Phys. Rev. D*, 100(8):083506, 2019.

[24] Stephon Alexander, João Magueijo, and Lee Smolin. The Quantum Cosmological Constant. *Symmetry*, 11(9):1130, 2019.