Characterization theorems for $Q$-independent random variables with values in a locally compact Abelian group

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Abstract. Let $X$ be a locally compact Abelian group, $Y$ be its character group. Following A. Kagan and G. Székely we introduce a notion of $Q$-independence for random variables with values in $X$. We prove group analogues of the Cramér, Kac–Bernstein, Skitovich–Darmois and Heyde theorems for $Q$-independent random variables with values in $X$. The proofs of these theorems are reduced to solving some functional equations on the group $Y$.

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1. Introduction

It is well known that if a random variable $\xi$ has a Gaussian distribution and $\xi$ is a sum of two independent random variables $\xi = \xi_1 + \xi_2$, then $\xi_j$ are also Gaussian (Cramér’s theorem). Let $\xi_1, \ldots, \xi_n$ be independent random variables. Consider the linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$, where the coefficients $\alpha_j, \beta_j$ are nonzero real numbers. Then the independence of the linear forms $L_1$ and $L_2$ implies that the random variables $\xi_j$ are Gaussian (Skitovich–Darmois’s theorem). A theorem similar to the Skitovich–Darmois theorem was proved by Heyde. In this theorem a Gaussian distribution is characterized by the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$. On the one hand, in the article [12] A. Kagan and G. Székely introduced a notion of $Q$-independence and proved, in particular, that the classical Cramér and Skitovich–Darmois theorems hold true if instead of independence $Q$-independence is considered. On the other hand, in the articles [2], [3], [6], [8], see also [7], the locally compact Abelian groups $X$ for which group analogues of the Cramér, Skitovich–Darmois and Heyde theorems for independent random variables with values in $X$ where described.

In this article we introduce the notion of $Q$-independence for random variables taking values in a locally compact Abelian group. We prove that if we consider $Q$-independence instead of independence, then group analogues of theorems by Cramér, Skitovich–Darmois and Heyde hold true for the same classes of groups. We also consider a group analogue of the Kac–Bernstein theorem for $Q$-independent random variables. The proofs of the corresponding theorems are reduced to solving some functional equations on the character group of the initial group in the class of continuous positive definite functions. Standard results on abstract harmonic analysis (see e.g. [9]) will be used.

In this paper we suppose $X$ is a second countable locally compact Abelian group. Denote by $Y$ the character group of $X$, and by $(x, y)$ the value of a character $y \in Y$ at $x \in X$. If $\xi$ is a random variable with values in the group $X$, then denote by $\mu_{\xi}$ its distribution and by

$$\hat{\mu}_{\xi}(y) = \mathbf{E}[(\xi, y)] = \int_X (x, y) d\mu_{\xi}(x), \quad y \in Y,$$

the characteristic function of the distribution $\mu_{\xi}$. We will also call $\hat{\mu}_{\xi}(y)$ the characteristic function of the random variable $\xi$. Let $f(y)$ be a function on the group $Y$, and let $h \in Y$. Denote by $\Delta_h$ the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y).$$
A function $f(y)$ on $Y$ is called a polynomial if
\[ \Delta_h^{n+1}f(y) = 0 \]
for some $n$ and for all $y, h \in Y$. The minimal $n$ for which this equality holds is called the degree of the polynomial $f(y)$.

Let $\xi_1, \ldots, \xi_n$ be random variables with values in the group $X$. Following A. Kagan and G. Székely ([12]) we say that the random variables $\xi_1, \ldots, \xi_n$ are $Q$-independent if their joint characteristic function is represented in the form
\[
\hat{\mu}_{(\xi_1, \ldots, \xi_n)}(y_1, \ldots, y_n) = \mathbb{E}[(\xi_1, y_1) \cdots (\xi_n, y_n)] = \\
\left( \prod_{j=1}^{n} \hat{\mu}_{\xi_j}(y_j) \right) \exp\{q(y_1, \ldots, y_n)\}, \quad y_j \in Y,
\]
where $q(y_1, \ldots, y_n)$ is a continuous polynomial on the group $Y^n$. We will also assume that $q(0, \ldots, 0) = 0$.

Denote by $M^1(X)$ the convolution semigroup of probability distributions on the group $X$. We remind that a distribution $\gamma \in M^1(X)$ is called Gaussian (see [14, Chapter IV]), if its characteristic function is represented in the form
\[
\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad y \in Y,
\]
where $x \in X$, and $\varphi(y)$ is a continuous non-negative function on the group $Y$ satisfying the equation
\[
\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.
\]
Denote by $\Gamma(X)$ the set of Gaussian distributions on $X$. We note that according this definition the generated distributions are Gaussian.

Denote by $\text{Aut}(X)$ the group of topological automorphisms of the group $X$, and by $I$ the identity automorphism of a group. If $\alpha \in \text{Aut}(X)$, then the adjoint automorphism $\tilde{\alpha} \in \text{Aut}(Y)$ is defined as follows $(x, \tilde{\alpha}y) = (\alpha x, y)$ for all $x \in X, y \in Y$. Note that $\alpha \in \text{Aut}(X)$ if and only if $\tilde{\alpha} \in \text{Aut}(Y)$. Denote by $\mathbb{R}$ the group of real numbers, by $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ the circle group (the one dimensional torus), and by $\mathbb{Z}$ the group of integers. Let $n$ be an integer. Denote by $f_n$ the mapping of $X$ into $X$ defined by the formula $f_nx = nx$. Put $X^{(n)} = \text{Ker}f_n$ and $X^{(n)} = f_n(X)$. A group $X$ is called a Corwin group if $X^{(2)} = X$.

2. Cramér’s theorem for $Q$-independent random variables

In the article [2], see also [7, Theorem 4.6], the following analogue of the classical Cramér theorem for the random variables with values in a locally compact Abelian group was proved.

**Theorem A.** Let $X$ be a locally compact Abelian group. Assume that $X$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $Y$ be the character group of $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$. If a random variable $\xi = \xi_1 + \xi_2$ has a Gaussian distribution, then $\xi_j, j = 1, 2$, are also Gaussian. To put it in another way, if $\mu_{\xi} \in \Gamma(X)$ and
\[
\hat{\mu}_{\xi}(y) = \hat{\mu}_{\xi_1}(y)\hat{\mu}_{\xi_2}(y),
\]
then $\hat{\mu}_{\xi_j}(y), j = 1, 2$, are the characteristic functions of Gaussian distributions.
We prove that Theorem A remains true if we change the condition of independence for \( Q \)-independence. The following statement is valid.

**Proposition 1.** Let \( X \) be a locally compact Abelian group. Assume that \( X \) contains no subgroup topologically isomorphic to the circle group \( T \). Let \( \xi_1 \) and \( \xi_2 \) be \( Q \)-independent random variables with values in the group \( X \). If a random variable \( \xi = \xi_1 + \xi_2 \) has a Gaussian distribution, then \( \xi_j, j = 1, 2, \) are also Gaussian.

To prove Proposition 1 we need the following lemma which is a group analogue of the classical Marcinkiewicz theorem.

**Lemma 1** ([5], see also [7, Theorem 5.11]). Let \( X \) be a locally compact Abelian group. Assume that \( X \) contains no subgroup topologically isomorphic to the circle group \( T \). Let \( Y \) be the character group of \( X \) and \( f(y) \) be a characteristic function on the group \( Y \). If \( f(y) \) is of the form

\[
f(y) = \exp\{P(y)\}, \quad y \in Y,
\]

where \( P(y) \) is a continuous polynomial, then \( P(y) \) is a polynomial of degree \( \leq 2 \), and \( f(y) \) is the characteristic function of a Gaussian distribution.

**Proof of Proposition 1.** Since the random variables \( \xi_1 \) and \( \xi_2 \) are \( Q \)-independent, we have

\[
\hat{\mu}_\xi(y) = E[(\xi, y)] = E[(\xi_1 + \xi_2, y)] = E[(\xi_1, y)(\xi_2, y)] = \frac{\alpha_1}{\xi_1}(y)\frac{\alpha_2}{\xi_2}(y)\exp\{q(y, y)\}, \quad y \in Y,
\]

where \( q(y_1, y_2) \) is a continuous polynomial on the group \( Y^2 \). It follows from the definition of a polynomial on a group that \( q(y, y) \) is a continuous polynomial on the group \( Y \). By the condition the characteristic function \( \hat{\mu}_\xi(y) \) is represented in the form \( \hat{\mu}_\xi(y) = \exp\{P(y)\} \). Then it follows from \( \hat{\mu}_\xi(y) = \exp\{P(y)\} \) that

\[
(-x, y)\frac{\alpha_1}{\xi_1}(y)\frac{\alpha_2}{\xi_2}(y) = \exp\{P(y)\}, \quad y \in Y,
\]

where \( P(y) = -\varphi(y) - q(y, y) \). Since \( \varphi(y) \) is a continuous polynomial on the group \( Y \), \( P(y) \) is also a continuous polynomial on the group \( Y \). The left-hand side of \( \hat{\mu}_\xi(y) = \exp\{P(y)\} \) is a characteristic function. Since the group \( X \) contains no subgroup topologically isomorphic to the circle group \( T \), by Lemma 1 the left-hand side of \( \hat{\mu}_\xi(y) = \exp\{P(y)\} \) is the characteristic function of a Gaussian distribution. This implies by Theorem A that the random variables \( \xi_j, j = 1, 2, \) are also Gaussian. \( \square \)

We note that since independent random variables are \( Q \)-independent, and Theorem A fails if a locally compact Abelian group \( X \) contains a subgroup topologically isomorphic to the circle group \( T \) (see e.g. [7, §4]), the condition on the group \( X \) in Proposition 1 can not be weaken.

3. Skitovich–Darmois’s theorem for \( Q \)-independent random variables

In the article [6], see also [7, §10], the analogue of the Skitovich–Darmois theorem for random variables with values in a locally compact Abelian group was proved.

**Theorem B.** Let \( X \) be a locally compact Abelian group. Assume that \( X \) contains no subgroup topologically isomorphic to the circle group \( T \). Let \( \xi_1, \ldots, \xi_n \) be independent random variables with values in the group \( X \) such that their characteristic functions do not vanish. Consider the linear forms \( L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n \) and \( L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n \), where coefficients \( \alpha_j, \beta_j \in \text{Aut}(X) \). If the linear forms \( L_1 \) and \( L_2 \) are independent, then the random variables \( \xi_j \) are Gaussian.

We prove that Theorem B remains true if we change the condition of independence of \( \xi_1, \ldots, \xi_n \) and \( L_1, L_2 \) for \( Q \)-independence. The following statement holds true.
Theorem 1. Let $X$ be a locally compact Abelian group. Assume that $X$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $\xi_1, \ldots, \xi_n$ be $Q$-independent random variables with values in the group $X$ such that their characteristic functions do not vanish. Consider the linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$, where coefficients $\alpha_j, \beta_j \in \text{Aut}(X)$. If the linear forms $L_1$ and $L_2$ are $Q$-independent, then the random variables $\xi_j$ are Gaussian.

To prove Theorem 1 we need the following lemmas.

Lemma 2. Let $X$ be a locally compact Abelian group and $Y$ be its character group. Let $\xi_1, \ldots, \xi_n$ be $Q$-independent random variables with values in the group $X$, and let $\alpha_j, \beta_j \in \text{Aut}(X)$. The linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ are $Q$-independent if and only if the characteristic functions $\hat{\mu}_{\xi_j}(y)$ satisfy the equation

$$
\prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \left( \prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\alpha}_j u) \right) \left( \prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\beta}_j v) \right) \exp\{q(u, v)\}, \quad u, v \in Y,
$$

(6)

where $q(u, v)$ is a continuous polynomial on the group $Y^2$, $q(0, 0) = 0$.

Proof. On the one hand, since the random variables $\xi_1, \ldots, \xi_n$ are $Q$-independent, the join characteristic function of $L_1$ and $L_2$ is of the form

$$
\hat{\mu}_{(L_1, L_2)}(u, v) = E[(L_1, u)(L_2, v)] = E[(\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n, u)(\beta_1 \xi_1 + \cdots + \beta_n \xi_n, v)] = \hat{\mu}_{\xi_1}(\tilde{\alpha}_1 u + \tilde{\beta}_1 v) \cdots (\tilde{\alpha}_n u + \tilde{\beta}_n v)] = \left( \prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\alpha}_j u + \tilde{\beta}_j v) \right) \exp\{q_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v, \ldots, \tilde{\alpha}_n u + \tilde{\beta}_n v)\}, \quad u, v \in Y,
$$

(7)

where $q_1(y_1, \ldots, y_n)$ is a continuous polynomial on the group $Y^n$. On the other hand, since the random variables $\xi_1, \ldots, \xi_n$ are $Q$-independent, we have

$$
\hat{\mu}_{L_1}(y) = E[(L_1, y)] = E[(\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n, y)] = \left( \prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\alpha}_j y) \right) \exp\{q_1(\tilde{\alpha}_1 y, \ldots, \tilde{\alpha}_n y)\}, \quad y \in Y,
$$

(8)

$$
\hat{\mu}_{L_2}(y) = E[(L_2, y)] = E[(\beta_1 \xi_1 + \cdots + \beta_n \xi_n, y)] = \left( \prod_{j=1}^n \hat{\mu}_{\xi_j}(\tilde{\beta}_j y) \right) \exp\{q_1(\tilde{\beta}_1 y, \ldots, \tilde{\beta}_n y)\}, \quad y \in Y.
$$

(9)

Assume that the linear forms $L_1$ and $L_2$ are $Q$-independent. Then the join characteristic function of $L_1$ and $L_2$ can be written in the form

$$
\hat{\mu}_{(L_1, L_2)}(u, v) = \hat{\mu}_{L_1}(u)\hat{\mu}_{L_2}(v) \exp\{q_2(u, v)\}, \quad u, v \in Y,
$$

(10)

where $q_2(u, v)$ is a continuous polynomial on the group $Y^2$. Put

$$
q(u, v) = -q_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v, \ldots, \tilde{\alpha}_n u + \tilde{\beta}_n v) + q_1(\tilde{\alpha}_1 u, \ldots, \tilde{\alpha}_n u) + q_1(\tilde{\beta}_1 v, \ldots, \tilde{\beta}_n v) + q_2(u, v).
$$

(11)

It follows from the definition of a polynomial on a group that $q(u, v)$ is a continuous polynomial on the group $Y^2$. Obviously, (6) follows from (11). If (6) holds, then (10) follows from (1)–(9), where $q_2(u, v)$ is defined by formula (11) and, obviously, $q_2(u, v)$ is a continuous polynomial on the group $Y^2$. 


where $k$

Lemma 3. Let $Y$ be an Abelian group and $b_j$ be automorphisms of the group $Y$ such that $b_i \neq b_j$ for $i \neq j$. Consider on the group $Y$ the equation

$$
\sum_{j=1}^{n} \psi_j(u + b_jv) = P(u) + Q(v) + R(u,v), \quad u, v \in Y, \tag{12}
$$

where $\psi_j(y)$, $P(y)$, $Q(y)$ are functions on $Y$, and $R(u,v)$ is a polynomial on $Y^2$. Then $P(y)$ is a polynomial on $Y$.

Proof. In proving lemma we use the finite-difference method. Let $h_1$ be an arbitrary element of the group $Y$. Put $k_1 = -b_n^{-1} h_1$. Then $h_1 + b_n k_1 = 0$. Substitute $u + h_1$ for $u$ and $v + k_1$ for $v$ in equation (12). Subtracting equation (12) from the resulting equation we obtain

$$
\sum_{j=1}^{n-1} \Delta_{l_j} \psi_j(u + b_jv) = \Delta_{h_1} P(u) + \Delta_{k_1} Q(v) + \Delta_{(h_1,k_1)} R(u,v), \quad u, v \in Y, \tag{13}
$$

where $l_j = h_1 + b_j k_1 = (b_j - b_n) k_1$, $j = 1, 2, \ldots, n - 1$. Let $h_2$ be an arbitrary element of the group $Y$. Put $k_2 = -b_{n-1}^{-1} h_2$. Then $h_2 + b_{n-1} k_2 = 0$. Substitute $u + h_2$ for $u$ and $v + k_2$ for $v$ in equation (13). Subtracting equation (13) from the resulting equation we find

$$
\sum_{j=1}^{n-2} \Delta_{l_j} \Delta_{h_1} \psi_j(u + b_jv) = \Delta_{h_2} \Delta_{h_1} P(u) + \Delta_{k_2} \Delta_{k_1} Q(v) + \Delta_{(h_2,k_2)} \Delta_{(h_1,k_1)} R(u,v), \quad u, v \in Y, \tag{14}
$$

where $l_j = h_2 + b_j k_2 = (b_j - b_{n-1}) k_2$, $j = 1, 2, \ldots, n - 2$. Arguing similarly as above we get the equation

$$
\Delta_{l_{n-1}} \Delta_{l_{n-2}} \cdots \Delta_{l_1} \psi_1(u + b_1v) = \Delta_{h_{n-1}} \Delta_{h_{n-2}} \cdots \Delta_{h_1} P(u) +
\Delta_{k_{n-1}} \Delta_{k_{n-2}} \cdots \Delta_{k_1} Q(v) + \Delta_{(h_{n-1},k_{n-1})} \Delta_{(h_{n-2},k_{n-2})} \cdots \Delta_{(h_1,k_1)} R(u,v), \quad u, v \in Y, \tag{15}
$$

where $h_m$ are arbitrary elements of the group $Y$, $k_m = -b_{n-m+1}^{-1} h_m$, $m = 1, 2, \ldots, n - 1$, $l_{mj} = h_m + b_j k_m = (b_j - b_{n-m+1}) k_m$, $j = 1, 2, \ldots, n - m$. Let $h_n$ be an arbitrary element of the group $Y$. Put $k_n = -b_1^{-1} h_n$. Then $h_n + b_n k_n = 0$. Substitute $u + h_n$ for $u$ and $v + k_n$ for $v$ in equation (15). Subtracting equation (15) from the resulting equation we obtain

$$
\Delta_{h_n} \Delta_{h_{n-1}} \cdots \Delta_{h_1} P(u) + \Delta_{k_n} \Delta_{k_{n-1}} \cdots \Delta_{k_1} Q(v) +
\Delta_{(h_n,k_n)} \Delta_{(h_{n-1},k_{n-1})} \cdots \Delta_{(h_1,k_1)} R(u,v) = 0, \quad u, v \in Y. \tag{16}
$$

Let $h_{n+1}$ be an arbitrary element of the group $Y$. Substitute $h_{n+1}$ for $u$ in equation (16). Subtracting equation (16) from the resulting equation we find

$$
\Delta_{h_{n+1}} \Delta_{h_n} \Delta_{h_{n-1}} \cdots \Delta_{h_1} P(u) + \Delta_{(h_{n+1},0)} \Delta_{(h_n,k_n)} \Delta_{(h_{n-1},k_{n-1})} \cdots \Delta_{(h_1,k_1)} R(u,v) = 0, \quad u, v \in Y. \tag{17}
$$

We note that if $h$ and $k$ are arbitrary elements of the group $Y$, by the condition

$$
\Delta_{h,k} R(u,v) = 0, \quad u, v \in Y, \tag{18}
$$
for some \( l \). Since \( h_m, m = 1, 2, \ldots, n + 1 \) are arbitrary elements of the group \( Y \), we can put in (17) \( h_1 = \cdots = h_{n+1} = h \), and apply to the both sides of the resulting equation the operator \( \Delta^{l+1}_{(h,k)} \). Taking into account (18), we get

\[
\Delta^{l+n+2}_h P(u) = 0, \quad u, h \in Y.
\]

So, the lemma is proved. \( \square \)

Remark 1. Let \( l \) be the degree of the polynomial \( R(u, v) \). Using some properties of polynomials on groups (see e.g. [7, §5]), it is not difficult to obtain from (17) that the degree of the polynomial \( P(u) \) does not exceed \( \max \{n, l\} \).

Proof of Theorem 1. By Lemma 2 the characteristic functions \( \hat{\mu}_{\xi_j}(y) \) satisfy equation (6). Put \( g_j(y) = |\hat{\mu}_j(\tilde{\alpha}_j y)|^2 \), \( b_j = \tilde{\alpha}_j^{-1} \tilde{\beta}_j \), \( R(u, v) = 2\text{Re} \ q(u, v) \). It follows from (6) that

\[
\prod_{j=1}^n g_j(u + b_j v) = \left( \prod_{j=1}^n g_j(u) \right) \prod_{j=1}^n g_j(b_j v) \exp \{R(u, v)\}, \quad u, v \in Y.
\]

(19)

Put \( \psi_j(y) = \log g_j(y) \). Assume first that \( b_i \neq b_j \) for \( i \neq j \). It follows from (19) that

\[
\sum_{j=1}^n \psi_j(u + b_j v) = P(u) + Q(v) + R(u, v), \quad u, v \in Y,
\]

(20)

where

\[
P(y) = \sum_{j=1}^n \psi_j(y), \quad Q(y) = \sum_{j=1}^n \psi_j(b_j y).
\]

By Lemma 3 \( P(y) \) is a polynomial on \( Y \). Obviously, the polynomial \( P(y) \) is continuous. We have

\[
\prod_{j=1}^n g_j(y) = \exp \{P(y)\}, \quad y \in Y.
\]

(21)

Since the group \( X \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \) and the left-hand side in (21) is a characteristic function, by Lemma 1 the left-hand side in (21) is the characteristic function of a Gaussian distribution. This implies by Theorem A that all \( g_j(y) \) are characteristic functions of Gaussian distributions. Applying Theorem A again we get that \( \hat{\mu}_{\xi_j}(\tilde{\alpha}_j y) \) are characteristic functions of Gaussian distributions, and hence, \( \hat{\mu}_{\xi_j}(y) \) are characteristic functions of Gaussian distributions, i.e. the random variables \( \xi_j \) are Gaussian.

Assume now that not all \( b_j \) are different automorphisms. Changing if necessary functions numbering \( g_j(y) \), we can assume that \( b_1 = \cdots = b_l_1, \ldots, b_{h_{k-1}+1} = \cdots = b_{h_k} = b_n \). Put \( h_1(y) = g_1(y) \cdots g_i(y), \ldots, h_k(y) = g_{h_{k-1}+1}(y) \cdots g_i(y) \). Reasoning as above we get that all \( h_l(y) \) are characteristic function of Gaussian distributions. By Theorem A this implies that all \( g_j(y) \), and hence all \( \hat{\mu}_{\xi_j}(y) \) are characteristic functions of Gaussian distributions, i.e. the random variables \( \xi_j \) are Gaussian. \( \square \)

The following statement results from the proof of Theorem 1.

Corollary 1. Let \( X \) be a locally compact Abelian group. Assume that \( X \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Let \( Y \) be the character group of \( X \). Let \( \xi_1, \ldots, \xi_n \) be independent random variables with values in the group \( X \) such that their characteristic functions do not vanish and satisfy equation (6), where \( \tilde{\alpha}_j, \tilde{\beta}_j \in \text{Aut}(Y) \). Then \( \hat{\mu}_{\xi_j}(y) \) are characteristic functions of Gaussian distributions.
Remark 2. It should be noted that since independent random variables are $Q$-independent and in the proof of Theorem 1 we do not use Theorem B, Theorem B follows from Theorem 1.

Remark 3. It is well known, see e.g. [7, Lemma 7.8], if a locally compact Abelian group $X$ contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$, then there exist independent random variables $\xi_1$ and $\xi_2$ with values in the group $X$ such that their characteristic functions do not vanish, $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, and $\mu_{\xi_1}, \mu_{\xi_2} \notin \Gamma(X)$. Since independent random variables are $Q$-independent, the condition on the group $X$ in Theorem 1 cannot be weakened.

4. Heyde’s theorem for $Q$-independent random variables

The following group analogue of the well-known Heyde theorem on characterization of a Gaussian distribution on the real line (see [10], see also [11, §13.4]), was proved in [8].

**Theorem C.** Let $X$ be a locally compact Abelian group. Assume that $X$ contains no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ with non-vanishing characteristic functions. The symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\xi_1$ and $\xi_2$ are Gaussian if and only if $\alpha$ satisfies the condition

$$\text{Ker}(I + \alpha) = \{0\}. \quad (22)$$

We prove that Theorem C remains true if we change the condition of independence of random variables $\xi_1$ and $\xi_2$ for $Q$-independence. The following statement is valid.

**Theorem 2.** Let $X$ be a locally compact Abelian group. Assume that $X$ contains no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be $Q$-independent random variables with values in $X$ with non-vanishing characteristic functions. The symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\xi_1$ and $\xi_2$ are Gaussian if and only if $\alpha$ satisfies condition (22).

To prove Theorem 2 we need the following lemma.

**Lemma 4.** Let $X$ be a locally compact Abelian group, $Y$ be its character group, and $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be $Q$-independent random variables with values in $X$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the characteristic functions $\hat{\mu}_{\xi_j}(y)$ satisfy the equation

$$\hat{\mu}_{\xi_1}(u + v)\hat{\mu}_{\xi_2}(u + \tilde{\alpha} v) = \hat{\mu}_{\xi_1}(u - v)\hat{\mu}_{\xi_2}(u - \tilde{\alpha} v) \exp\{q(u, v)\}, \quad u, v \in Y, \quad (23)$$

where $q(u, v)$ is a continuous polynomial on the group $Y^2$, $q(0, 0) = 0$.

**Proof.** Since the random variables $\xi_1$ and $\xi_2$ are $Q$-independent, the joint characteristic function of $L_1$ and $L_2$ is of the form

$$\hat{\mu}_{(L_1, L_2)}(u, v) = \mathbb{E}[(L_1, u)(L_2, v)] = \mathbb{E}[(\xi_1 + \xi_2, u)(\xi_1 + \alpha \xi_2, v)] =$$

$$= \mathbb{E}[(\xi_1 + \xi_2, u + \tilde{\alpha} v)] = \hat{\mu}_{\xi_1}(u + v)\hat{\mu}_{\xi_2}(u + \tilde{\alpha} v) \exp\{q_1(u + v, u + \tilde{\alpha} v)\}, \quad u, v \in Y, \quad (24)$$

where $q_1(u, v)$ is a continuous polynomial on the group $Y^2$, $q_1(0, 0) = 0$. Similarly, the joint characteristic function of $L_1$ and $-L_2$ is of the form

$$\hat{\mu}_{(L_1, -L_2)}(u, v) = \hat{\mu}_{\xi_1}(u - v)\hat{\mu}_{\xi_2}(u - \tilde{\alpha} v) \exp\{q_1(u - v, u - \tilde{\alpha} v)\}, \quad u, v \in Y. \quad (25)$$
The symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ means that the random vectors $(L_1, L_2)$ and $(L_1, -L_2)$ are identically distributed, i.e. the joint characteristic functions () and (25) are the same. Put \( q(u, v) = -q_1(u + v, u + \alpha v) + q_1(u - v, u - \alpha v) \). It is obvious that \( q(u, v) \) is a continuous polynomial on the group $Y^2$, and (23) follows from () and (25). □

**Proof of Theorem 2.** Since independent random variables are $Q$-independent, the necessity follows from Theorem C.

Sufficiency. We will use some ideas from the articles [8] and [13]. By Lemma 4 the characteristic functions \( \hat{\mu}_y(y) \) satisfy equation (23). Put $b = \alpha$. Let $w, z \in Y$. Substituting in (23) $u = bw, v = -w$ and $u = z, v = -z$, we get

\[
\hat{\mu}_1((b-I)w) = \hat{\mu}_1((I-b)w)\hat{\mu}_2(2bw)\exp\{q(bw, -w)\}, \quad w \in Y. \tag{26}
\]

Substituting in (23) $u = bw + z, v = w + z$, we obtain

\[
\hat{\mu}_1((I-b)w)\hat{\mu}_2(-(b-I)z) = \hat{\mu}_1((I+b)z)\exp\{q(w+bz, w+z)\}, \quad w, z \in Y. \tag{27}
\]

It follows from (26)–() that

\[
\hat{\mu}_1((I+b)w + 2z)\hat{\mu}_2(2bw + (I+b)z) = \hat{\mu}_1((I+b)w)\hat{\mu}_2(2bw)\exp\{q(bw, -w)\} \times
\]

\[
\hat{\mu}_1(2z)\hat{\mu}_2((I+b)z)\exp\{q(z, -z)\} \exp\{q(bw + z, w + z)\}, \quad w, z \in Y. \tag{29}
\]

Put \( p(u, v) = q(bu, -u) + q(v, -v) + q(bu + v, u + v) \). Obviously, \( p(u, v) \) is a continuous polynomial on the group $Y^2$, \( p(0, 0) = 0 \).

Thus, the characteristic functions \( \hat{\mu}_y(y) \) satisfy the equation

\[
\hat{\mu}_1((I+b)u + 2v)\hat{\mu}_2(2bu + (I+b)v) =
\]

\[
\hat{\mu}_1((I+b)u)\hat{\mu}_2(2bu)\hat{\mu}_2(2v)\hat{\mu}_1((I+b)v)\exp\{p(u, v)\}, \quad u, v \in Y. \tag{30}
\]

Since the group $X$ contains no elements of order 2, $X$ contains no subgroup topologically isomorphic to the circle group $T$. In the case when \( I + \alpha \) and \( f_2 \) are topological automorphisms of the group $X$, the statement of the theorem follows from () and Corollary 1. In general, \( I + \alpha \) and \( f_2 \) are only continuous monomorphisms, and we reason as follows.

Put \( g_j(y) = |\hat{\mu}_j(y)|^2 \), $j = 1, 2$, $R(u, v) = 2\Re p(u, v)$. It follows from () that the functions $g_j(y)$ satisfy the equation

\[
g_1((I + b)u + 2v)g_2(2bu + (I + b)v) =
\]

\[
g_1((I + b)u)g_2(2bu)g_2(2v)g_1((I + b)v)\exp\{R(u, v)\}, \quad u, v \in Y. \tag{31}
\]

Put \( \psi_j(y) = \log g_j(y) \), $j = 1, 2$. Then () implies that the functions \( \psi_j(y) \) satisfy the equation

\[
\psi_1((I + b)u + 2v) + \psi_2(2bu + (I + b)v) = P(u) + Q(v) + R(u, v), \quad u, v \in Y, \tag{32}
\]

where

\[
P(y) = \psi_1((I + b)y) + \psi_2(2by), \quad Q(y) = \psi_1(2y) + \psi_2((I + b)y). \tag{33}
\]

In solving the equation (32) we use the finite-difference method. Let $h_1$ be an arbitrary element of the group $Y$. Substitute in (32) $u + (I + b)h_1$ for $u$ and $v - 2bh_1$ for $v$. Substituting equation (32) from the resulting equation we obtain

\[
\Delta_{(I-b)h_1}\psi_1((I + b)u + 2v) = \Delta_{(I+b)h_1}P(u) + \Delta_{-2bh_1}Q(v) + \Delta_{((I+b)h_1,-2bh_1)}R(u, v), \quad u, v \in Y. \tag{34}
\]
Let \( h_2 \) be an arbitrary element of the group \( Y \). Substitute in (34) \( u + 2h_2 \) for \( u \) and \( v - (I + b)h_2 \) for \( v \). Subtracting equation (34) from the resulting equation we get

\[
\Delta_{2h_2} \Delta_{(I+b)h_1} P(u) + \Delta_{-(I+b)h_2} \Delta_{-2bh_1} Q(v) + \\
+ \Delta_{(2h_2, -(I+b)h_2)} \Delta_{((I+b)h_1, -2bh_1)} R(u, v) = 0, \quad u, v \in Y.
\]

Let \( h_3 \) be an arbitrary element of the group \( Y \). Substitute in (34) \( u + h_3 \) for \( u \). Subtracting equation (34) from the resulting equation we find

\[
\Delta_{h_3} \Delta_{2h_2} \Delta_{(I+b)h_1} P(u) + \Delta_{(h_3, 0)} \Delta_{(2h_2, -(I+b)h_2)} \Delta_{((I+b)h_1, -2bh_1)} R(u, v) = 0, \quad u, v \in Y.
\]

We note that if \( h \) and \( k \) are arbitrary elements of the group \( Y \), then (18) holds. Applying to the both sides of equation (36) the operator \( \Delta_{(h,k)}^{l+1} \) we obtain

\[
\Delta_{(h,k)}^{l+1} \Delta_{h_3} \Delta_{2h_2} \Delta_{(I+b)h_1} P(u) = 0, \quad u, v \in Y.
\]

Since the group \( X \) contains no elements of order 2, the subgroup \( Y^{(2)} \) is dense in \( Y \) (\( [4] \) (24.22)). Since \( \alpha \) satisfies condition (22), the subgroup \( (I + b)(Y) \) is also dense in \( Y \) (\( [9] \) (24.41)). Taking into account that \( h_1, h_2, h_3 \) are arbitrary elements of the group \( Y \), (18) and (37) imply that

\[
\Delta_{h}^{l+4} P(u) = 0, \quad u, h \in Y,
\]

i.e. \( P(y) \) is a continuous polynomial. It follows from (33) that \( \exp\{P(y)\} \) is a characteristic function.

Since the group \( X \) contains no elements of order 2, \( X \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Then, by Lemma 1 \( P(y) \) is a continuous polynomial of degree \( \leq 2 \), and \( \exp\{P(y)\} = g_1((I + b)y)g_2(2by) \) is the characteristic function of a Gaussian distribution. By Theorem A, this implies that \( g_1((I + b)y) \) and \( g_2(2by) \) are the characteristic functions of Gaussian distributions. Since the subgroups \( (I + b)(Y) \) and \( Y^{(2)} \) are dense in \( Y \), \( g_1(y) \) and \( g_2(y) \) are also the characteristic functions of Gaussian distributions. Applying Theorem A again we get that \( \xi_1 \) and \( \xi_2 \) are Gaussian. □

**Remark 4.** Let \( X \) be a locally compact Abelian group, \( Y \) be its character group, and \( \alpha \) be a topological automorphism of the group \( X \). Let \( \xi_1 \) and \( \xi_2 \) be random variables with values in \( X \). Consider the linear forms \( L_2 = \xi_1 + \alpha \xi_2 \) and \( L_1 = \xi_1 + \xi_2 \). Note that the symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) means that the random vectors \((L_1, L_2)\) and \((L_1, -L_2)\) are identically distributed. Let \( \eta_1 \) and \( \eta_2 \) be random variables with values in the group \( X \). Following A. Kagan and G. Székely (\( [12] \)) we say that \( \eta_1 \) and \( \eta_2 \) are \( Q \)-identically distributed if

\[
\mu_{\eta_1}(y) = \mu_{\eta_2}(y) \exp\{q(y)\}, \quad y \in Y,
\]

where \( q(y) \) is a continuous polynomial on the group \( Y \). We will also assume that \( q(0) = 0 \). It is easy to see that Lemma 4 remains true if we change in the lemma the condition of symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) for the condition that the random vectors \((L_1, L_2)\) and \((L_1, -L_2)\), i.e. the random variables with values in the group \( X^2 \), are \( Q \)-identically distributed. Since in the proof of Theorem 2 we only use equation (23), Theorem 2 remains true if we change in Theorem 2 the condition of symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) for weaker condition: the random vectors \((L_1, L_2)\) and \((L_1, -L_2)\) are \( Q \)-identically distributed.

5. Kac-Bernstein’s theorem for \( Q \)-independent random variables
To prove the main theorem of this section we need some facts on structure of locally compact Abelian groups and duality theory (see [9] Chapter 6). Any locally compact Abelian group $X$ is topologically isomorphic to a group of the form $\mathbb{R}^m \times G$, where $m \geq 0$, and the group $G$ contains a compact open subgroup. Let $Y$ be a character group of the group $X$. We will also denote the character group of the group $X$ by $X^\ast$. If $K$ is a closed subgroup of $X$, denote by $A(Y,K) = \{ y \in Y : (x,y) = 1 \text{ for all } x \in K \}$ its annihilator. Note that $A(X,A(Y,K)) = K$, and the following topological automorphisms $K^\ast \cong Y/A(Y,K)$ and $(X/K)^\ast \cong A(Y,K)$ hold. Moreover, if $B$ is a closed subgroup of $Y$, then any character of the group $B$ is of the form $y \mapsto (x,y)$ for some $x \in X$. We also note that $A \left( Y, X_{(n)} \right) = \overline{Y(n)}$ for any natural $n$.

Let $\mu \in M^1(X)$. Denote by $\sigma(\mu)$ the support of $\mu$. Denote by $E_x$ the degenerate distribution concentrated at an element $x \in X$. If $K$ is a compact subgroup of the group $X$, then denote by $m_K$ the Haar distribution of the group $K$. We note that the characteristic function of a distribution $m_K$ is of the form

$$
\hat{m}_K(y) = \begin{cases} 
1, & \text{if } y \in A(Y,K), \\
0, & \text{if } y \notin A(Y,K).
\end{cases}
$$

(39)

Denote by $I_B(X)$ the set of Haar distributions $m_K$ of compact Corwin subgroups $K$ of the group $X$.

In the article [4], see also [7] Theorem 7.10, the following analogue of the Kac-Bernstein theorem for random variables with values in a locally compact Abelian group was proved.

**Theorem D.** Let $X$ be a locally compact Abelian group. Assume that the connected component of zero of the group $X$ contains no elements of order 2. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Then $\mu_{\xi_1}, \mu_{\xi_2} \in \Gamma(X) \ast I_B(X)$. Moreover,

$$
\mu_{\xi_1} = \mu_{\xi_2} \ast E_x,
$$

(40)

where $x \in X$.

We note that in contrast to Theorems B and C, in Theorem D we do not assume that the characteristic functions of considering random variables do not vanish. We will prove that Theorem D, except, generally speaking, statement [4],[7], remains true if we substitute the condition of independence for $Q$-independence. The following statement holds true.

**Theorem 3.** Let $X$ be a locally compact Abelian group. Assume that the connected component of zero of the group $X$ contains no elements of order 2. Let $\xi_1$ and $\xi_2$ be $Q$-independent random variables with values in $X$ such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are $Q$-independent. Then $\mu_{\xi_j} = \gamma_j \ast m_W$, where $\gamma \in \Gamma(X)$, $j = 1, 2$, and $W$ is a compact Corwin subgroup.

To prove Theorem 3 we need the following lemmas.

**Lemma 5** (see e.g. [7] Proposition 5.7). Let $Y$ be a compact Abelian group and $f(y)$ be a continuous polynomial on $Y$. Then $f(y) \equiv \text{const}$.

We formulate as a lemma the following well-known statement.

**Lemma 6.** Let $X$ be a locally compact Abelian group and $Y$ be its character group. Let $\mu \in M^1(X)$. Put $E = \{ y \in Y : \hat{\mu}(y) = 1 \}$. Then $\sigma(\mu) \subset A(X,E)$.

**Lemma 7** ([7] §7). Let $X$ be a locally compact Abelian group of the form $X = \mathbb{R}^m \times K$, where $m \geq 0$, and $K$ is a compact Corwin group. Assume that the connected component of zero of the group $X$ contains no elements of order 2. Let $Y$ be the character group of $X$. Then $X$ and $Y$ are groups with unique division by 2.
Proof of Theorem 3. We will use the scheme of the proof of Theorem 7.10 in [7]. Taking into account the structure theorem for locally compact Abelian groups, we can assume without loss of generality, that \( X = \mathbb{R}^m \times G, \ Y = \mathbb{R}^m \times H, \) where \( m \geq 0, \ H \cong G^*, \) and each of the groups \( G \) and \( H \) contains a compact open subgroup. Denote by \( L \) a compact open subgroup in \( H. \) Put

\[
N_1 = \{ y \in Y : \hat{\mu}_{\xi_1}(y) \neq 0 \}, \quad N_2 = \{ y \in Y : \hat{\mu}_{\xi_2}(y) \neq 0 \}, \quad N = N_1 \cap N_2.
\]

By Lemma 2, the characteristic functions \( \hat{\mu}_{\xi_j}(y) \) satisfy equation (11) which takes the form

\[
\hat{\mu}_{\xi_1}(u + v)\hat{\mu}_{\xi_2}(u - v) = \hat{\mu}_{\xi_1}(u)\hat{\mu}_{\xi_2}(u)\hat{\mu}_{\xi_1}(v)\hat{\mu}_{\xi_2}(v) \exp\{ q(u, v) \}, \quad u, v \in Y,
\]

where \( q(u, v) \) is a continuous polynomial on the group \( Y^2, \ q(0, 0) = 0. \) It follows from (11) that \( N \) is a subgroup of \( Y. \) Obviously, \( N \) is an open subgroup. Consider the intersection \( B = N \cap L. \) Since any open subgroup is closed, \( B \) is a compact open subgroup of \( H. \) By Lemma 5 \( q(u, v) = 0, \ u, v \in B. \) This implies that the restriction of equation (11) to the subgroup \( B \) is of the form

\[
\hat{\mu}_{\xi_1}(u + v)\hat{\mu}_{\xi_2}(u - v) = \hat{\mu}_{\xi_1}(u)\hat{\mu}_{\xi_2}(u)\hat{\mu}_{\xi_1}(v)\hat{\mu}_{\xi_2}(v), \quad u, v \in B.
\]

Put \( A = B^*. \) Since \( B \) is a compact group, \( A \) is a discrete group, and hence, \( A \) is totally disconnected. In particular, the connected component of zero of the group \( A \) contains no elements of order 2. Taking into account that Gaussian distributions on an arbitrary locally compact totally disconnected Abelian group are degenerated (see [14, Chapter IV]), and the characteristic functions \( \hat{\mu}_{\xi_j}(y) \) do not vanish on \( B, \) by Theorem D applying the group \( A \) we obtain

\[
\hat{\mu}_{\xi_1}(y) = (x_1, y), \quad \hat{\mu}_{\xi_2}(y) = (x_2, y), \quad y \in B,
\]

where \( x_j \in X, \ j = 1, 2. \) Consider the new random variables \( \xi_j' = \xi_j - x_j. \) It is obvious that the random variables \( \xi_j' \) are \( Q \)-independent, and \( \xi_1' + \xi_2' \) and \( \xi_1' - \xi_2' \) are also \( Q \)-independent. Therefore, passing from the random variables \( \xi_j \) to the random variables \( \xi_j', \) we can prove the theorem assuming from the beginning that

\[
\hat{\mu}_{\xi_1}(y) = \hat{\mu}_{\xi_2}(y) = 1, \quad y \in B,
\]

is fulfilled. Then, by Lemma 6 \( \sigma(\hat{\mu}_{\xi_j}) \subset A(X, B), \ j = 1, 2. \) We have \( A(X, B) \cong (Y/B)^* = ((\mathbb{R}^m \times H)/B)^* \cong \mathbb{R}^m \times (H/B)^*. \) Put \( F = (H/B)^*. \) Since \( B \) is an open subgroup of \( H, \) the factor-group \( H/B \) is discrete, and hence, \( F \) is a compact group.

Thus, we reduced the proof of the theorem to the case, when \( X = \mathbb{R}^m \times F, \ Y = \mathbb{R}^m \times D, \) where \( F \) is a compact group, and \( D \cong F^* \) is a discrete group. Let \( D_2 \) be the subgroup of \( D \) consisting of all elements \( y \in D \) such that the order of \( y \) is a power of 2. It follows from Lemma 5 that \( q(u, v) = 0, \ u, v \in D_2. \) Hence, the restriction of equation (11) to the subgroup \( D_2 \) is of the form (12). Substituting in equation (12) \( u = v = y \) and \( u = -v = y, \) we obtain

\[
\hat{\mu}_{\xi_1}(2y) = (\hat{\mu}_{\xi_1}(y))^2|\hat{\mu}_{\xi_2}(y)|^2, \quad \hat{\mu}_{\xi_2}(2y) = |\hat{\mu}_{\xi_1}(y)|^2(\hat{\mu}_{\xi_2}(y))^2, \quad y \in D_2.
\]

It follows from (13) that for any natural \( n \) the following identities

\[
|\hat{\mu}_{\xi_j}(2^ny)| = |\hat{\mu}_{\xi_1}(y)|\hat{\mu}_{\xi_2}(y)|^{2^{n-1}}, \quad y \in D_2, \quad j = 1, 2,
\]

hold true. Let \( y \in D_2. \) Then \( 2^ny = 0 \) for some natural \( n, \) and (11) implies that \( |\hat{\mu}_{\xi_1}(y)| = |\hat{\mu}_{\xi_2}(y)| = 1 \) for \( y \in D_2. \) It follows from this that there exist elements \( x_j' \in X, \ j = 1, 2, \) such that

\[
\hat{\mu}_{\xi_1}(y) = (x_1', y), \quad \hat{\mu}_{\xi_2}(y) = (x_2', y), \quad y \in D_2.
\]
Reasoning as above we can prove the theorem assuming that the identities
\[ \hat{\mu}_\xi(y) = \mu_\xi(y) = 1, \quad y \in D_2, \]
are fulfilled from the beginning. Then, by Lemma 6 \( \sigma(\mu_\xi) \subset A(X, D_2) \), \( j = 1, 2 \). Put \( M = A(X, D_2) \).
It is obvious that \( M = \mathbb{R}^m \times K \), where \( K \) is a compact group. We have \( M^* \cong Y/D_2 \). It is clear that the factor-group \( Y/D_2 \) contains no elements of order 2. This implies that \( M^{(2)} = M \). Since \( M^{(2)} = M^{(2)} \), it means that \( M \) is a Corwin group, and hence, \( K \) is also a Corwin group.

Thus, we reduced the proof of the theorem to the case, when \( X = \mathbb{R}^m \times K \), where \( K \) is a compact Corwin group. Since the connected component of zero of the group \( X \) contains no elements of order 2, by Lemma 7 and \( X \) are groups with unique division by 2. We will check that this implies the equality \( N_1 = N_2 \).

Substituting in equation (41) \( u = v = y \) and \( u = -v = y \), we get
\[ \hat{\mu}_\xi(2y) = (\hat{\mu}_\xi(y))^2|\hat{\mu}_\xi(y)|^2 \exp\{q(y, y)\}, \quad \hat{\mu}_\xi(2y) = |\hat{\mu}_\xi(y)|^2(\hat{\mu}_\xi(y))^2 \exp\{q(y, -y)\}, \quad y \in Y. \] (45)
Assume for definiteness that there exists an element \( y_0 \in Y \) such that \( \hat{\mu}_\xi(y_0) \neq 0, \hat{\mu}_\xi(y_0) = 0 \). Since \( Y \) is a group with unique division by 2, we have \( y_0 = 2y' \). Substituting \( y = y' \) in (45), we get from the one hand, \( \hat{\mu}_\xi(y') \hat{\mu}_\xi(y') \neq 0 \), and from the other hand, either \( \hat{\mu}_\xi(y') = 0 \) or \( \hat{\mu}_\xi(y') = 0 \). This contradiction implies that \( N_1 = N_2 = N \). Since \( Y \) is a group with unique division by 2, \( (45) \) implies that \( N \) is also a group with unique division by 2. Put \( W = A(X, N) \). It is not difficult to check that \( W \) is a compact Corwin group. Since \( (X/W)^* \cong N \), this implies that \( X/W \) is a group with unique division by 2. Obviously, the group \( X/W \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Consider the restriction of equation (41) to the group \( N \) and apply Corollary 1 to the group \( X/W \). We obtain the representation
\[ \hat{\mu}_\xi(y) = (x_1, y) \exp\{-\varphi_1(y)\}, \quad \hat{\mu}_\xi(y) = (x_2, y) \exp\{-\varphi_2(y)\}, \quad y \in N, \] (46)
where \( x_j \in X \), and \( \varphi_j(y) \) are continuous non-negative functions on \( N \) satisfying equation (3). Extend the functions \( \varphi_j(y) \) from the subgroup \( N \) to the group \( Y \) in such a way that the extended functions are continuous non-negative and also satisfy equation (3) (see e.g. [7, Lemma 3.18]). We retain the notation \( \varphi_j(y) \) for the extended functions. Let \( \gamma_j \) be the Gaussian distributions on the group \( X \) with the characteristic functions
\[ \hat{\gamma}_j(y) = (x_j, y) \exp\{-\varphi_j(y)\}, \quad y \in Y. \] (47)
Since \( N = A(Y, W) \), it follows from (39) that the characteristic function of the Haar distribution \( m_W \) is of the form
\[ \hat{m}_W(y) = \begin{cases} 1, & \text{if} \quad y \in N, \\ 0, & \text{if} \quad y \not\in N. \end{cases} \] (48)
It follows from (46)–(48) that \( \hat{\mu}_\xi(y) = \hat{\gamma}_j(y) \hat{m}_W(y) \). Hence, \( \mu_\xi = \gamma_j * m_W, \ j = 1, 2. \)

**Remark 5.** Let \( X \) be a locally compact Abelian group and \( Y \) be its character group. Assume that the connected component of zero of \( X \) is non-zero and contains no elements of order 2. Then, in contrast to Theorem D, we can not state in Theorem 3 that \( \mu_\xi = \mu_{\xi_2} * E_x \) for some \( x \in X \). Indeed, since the connected component of zero of \( X \) is non-zero, there exists a non-degenerate Gaussian distribution on \( X \) (see [13, Chapter IV]). Let \( \gamma_j \) be a non-degenerate Gaussian distributions on the group \( X \) with the characteristic functions
\[ \hat{\gamma}_j(y) = \exp\{-\varphi_j(y)\}, \quad y \in Y, \quad j = 1, 2, \] (49)
where \( \varphi_j(y) \) are continuous non-negative functions on \( Y \), satisfying equation (3). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) such that \( \mu_{\xi_j} = \gamma_j \). Put \( \eta_1 = \xi_1 + \xi_2, \)
\( \eta_2 = \xi_1 - \xi_2 \). On the one hand, taking into account that \( \varphi_j(-y) = \varphi_j(y) \) and independence of the random variables \( \xi_j \), the characteristic functions of the random variables \( \eta_j \) are of the form
\[
\hat{\mu}_{\eta_j}(y) = E[(\eta_1, y)] = \gamma_1(y) \gamma_2(y), \quad y \in Y, \quad j = 1, 2.
\]
(50)

On the other hand, taking into account independence of the random variables \( \xi_j \) and \( \eta_j \), the join characteristic function of the random variables \( \eta_1 \) and \( \eta_2 \) can be written in the form
\[
\hat{\mu}_{(\eta_1, \eta_2)}(u, v) = E[(\eta_1, u)(\eta_2, v)] = E[(\xi_1 + \xi_2, u)(\xi_1 - \xi_2, v)] = E[(\xi_1, u + v)(\xi_2, u - v)] = \\
= \gamma_1(u + v) \gamma_2(u - v) = \hat{\mu}_{\eta_1}(u) \hat{\mu}_{\eta_2}(v) \exp\{-\varphi_1(u + v) - \varphi_2(u - v) + \varphi_1(u) + \varphi_2(u) + \varphi_1(v) + \varphi_2(v)\}.
\]

Put
\[
q(u, v) = -\varphi_1(u + v) - \varphi_2(u - v) + \varphi_1(u) + \varphi_2(u) + \varphi_1(v) + \varphi_2(v).
\]
Then \( q(u, v) \) is a continuous polynomial on \( Y^2 \). It is easy to see that \( q(u, v) \equiv 0 \) if and only if \( \varphi_1(y) \equiv \varphi_2(y) \). Thus, if \( \varphi_1(y) \not\equiv \varphi_2(y) \), then the random variables \( \eta_1 \) and \( \eta_2 \) are not independent, but \( Q \)-independent.

From what has been said it follows that if \( \varphi_1(y) \not\equiv \varphi_2(y) \), then the random variables \( \xi_1 \) and \( \xi_2 \) are \( Q \)-independent, they even are independent, the random variables \( \xi_1 + \xi_2 \) and \( \xi_1 - \xi_2 \) are \( Q \)-independent, i.e. the conditions of Theorem 3 are fulfilled, whereas the distributions \( \mu_{\xi_1} = \gamma_1 \) and \( \mu_{\xi_2} = \gamma_2 \) are not shifts one another.

Remark 6. It is interesting to observe that there exist \( Q \)-independent random variables with values in a locally compact Abelian group \( X \) such that they are not independent if and only if the connected component of zero of the group \( X \) is non-zero. Indeed, let \( X \) be a totally disconnected group and \( Y \) be its character group. Then all elements of the group \( Y \) are compact. Assume that \( \xi_1, \xi_2, \ldots, \xi_n \) are \( Q \)-independent random variables with values in \( X \). By Lemma 5 any continuous polynomial on \( Y \) is a constant on the subgroup of all compact elements of the group \( Y \). Hence, in \((q(y_1, \ldots, y_n) \equiv 0 \). It means that the random variables \( \xi_1, \xi_2, \ldots, \xi_n \) are independent.

Assume that the connected component of zero of a group \( X \) is non-zero. Let \( \xi_1 \) and \( \xi_2 \) be independent non-degenerated Gaussian random variables with values in the group \( X \) such that \( \mu_{\xi_1} = \gamma_1 \) and \( \mu_{\xi_2} = \gamma_2 \), where the characteristic functions of the distributions \( \gamma_j \) are of the form \( \langle \langle 49 \rangle \rangle \). If \( \varphi_1(y) \not\equiv \varphi_2(y) \), then as noted in Remark 5, the random variables \( \eta_1 = \xi_1 + \xi_2 \) and \( \eta_2 = \xi_1 - \xi_2 \) are \( Q \)-independent, but not independent.

Remark 7. Compare Theorems 1, 2 and 3 with Theorems B, C and D. We see that Theorems B, C and D remain true for the corresponding groups if we change the condition of independence for \( Q \)-independence. It turns out that either a characterization theorem remains true or not after such change depends on the group. We give an example of a characterization theorem on the circle group \( \mathbb{T} \) which fails if we change the condition of independence for \( Q \)-independence. Moreover, actually a weak analogue of this theorem fails too.

The following theorem results from the article [1], see also [7, Theorem 9.9]. It characterizes a Gaussian distribution on the circle group \( \mathbb{T} \) (compare with Theorem B).

**Theorem E.** Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in the circle group \( \mathbb{T} \) such that their characteristic functions do not vanish. Assume that \( \xi_1 + \xi_2 \) and \( \xi_1 - \xi_2 \) are independent. Then \( \mu_{\xi_j} \in \Gamma(\mathbb{T}), \quad j = 1, 2. \)

The character group of the circle group \( \mathbb{T} \) is topologically isomorphic to \( \mathbb{Z} \). We will assume that \( \mathbb{T}^* = \mathbb{Z} \). We construct independent identically distributed random variables \( \xi_1 \) and \( \xi_2 \) with values in
the circle group $\mathbb{T}$ and with distribution $\mu_{\xi_j} = \gamma$, $j = 1, 2$, such that the random variables $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are $Q$-independent, and the characteristic function $\hat{\gamma}(n)$ is represented in the form

$$\hat{\gamma}(n) = \exp\{-\varphi(n)\}, \quad n \in \mathbb{Z},$$

where $\varphi(n)$ is a polynomial on the group of integers $\mathbb{Z}$ of sufficiently arbitrary form.

Let $\varphi(n)$ be a polynomial on the group of integers $\mathbb{Z}$ such that $\varphi(0) = 0$, $\varphi(-n) = \varphi(n)$, $n \in \mathbb{Z}$, and

$$\sum_{n \in \mathbb{Z}} \exp\{-\varphi(n)\} < 2.$$

This implies that

$$\rho(t) = \sum_{n \in \mathbb{Z}} \exp\{-(\varphi(n) - \text{int})\} > 0, \quad t \in \mathbb{R}.$$ 

It is also obvious that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t) dt = 1.$$ 

Let $\gamma$ be a distribution on the circle group $\mathbb{T}$ with the density $r(e^{it}) = \rho(t)$ with respect to $m_{\mathbb{T}}$. Then $\hat{\gamma}(n) = \exp\{-\varphi(n)\}$, $n \in \mathbb{Z}$.

Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in the circle group $\mathbb{T}$ and with distribution $\mu_{\xi_j} = \gamma$. Reasoning as in Remark 5, it is easy to make sure that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are $Q$-independent.
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