Fundamental building blocks of controlling complex networks: A universal controllability framework

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Abstract

To understand the controllability of complex networks is a forefront problem relevant to different fields of science and engineering. Despite recent advances in network controllability theories, an outstanding issue is to understand the effect of network topology and nodal interactions on the controllability at the most fundamental level. Here we develop a universal framework based on local information only to unearth the most fundamental building blocks that determine the controllability. In particular, we introduce a network dissection process to fully unveil the origin of the role of individual nodes and links in control, giving rise to a criterion for the much needed strong structural controllability. We theoretically uncover various phase-transition phenomena associated with the role of nodes and links and strong structural controllability. Applying our theory to a large number of empirical networks demonstrates that technological networks are more strongly structurally controllable (SSC) than many social and biological networks, and real world networks are generally much more SSC than their random counterparts with intrinsic resilience and adaptability as a result of human design and natural evolution.

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Controlling complex networked systems has become a frontier field of interdisciplinary research. The past few years have witnessed the emergence of two theoretical frameworks: the structural controllability theory (SCT) [1–3] and the exact controllability theory (ECT) [4–6], both aiming to achieve full control by identifying a minimum set of driver nodes to fully control the dynamics of the underlying networked system. Indeed, the issue of controllability has attracted a great deal of attention [6–38].

Despite advances, there are fundamental issues that need to be addressed to achieve full understanding of controlling complex networks. First, at the most basic level, the effects of nodes and links on control and their origins have not been fully understood, which pertain to the problems of nodal and link classifications. To understand the role of links in controllability is of particular interest, in view of the recently developed notion of strongly structurally controllability, i.e., to determine the circumstance under which the network is strongly structurally controllable (SSC) [39–41]. Especially, a network is said to be SSC if all link weights have no effect on the controllability and it is only determined by topology. A SSC network is generally more controllable and robust against the uncertainty and inaccessibility of link weights and external perturbation on links. However, a rigorous criterion has been missing for determining if a network is SSC through controlling a minimum set of driver nodes. Second, in spite of SCT and ECT, we lack a general and efficient controllability framework, because SCT based on Lin’s classic controllability theory [1] is applicable to directed networks only and does not take into account the interaction strengths along the links, while ECT based on the Popov-Belevitch-Hautus rank condition [42] is sensitive to perturbation or computational error (see Supplementary Note 1 for the details of SCT and ECT). In fact, the underlying connection between SCT and ECT has remained unknown. These issues are fundamental to controlling complex networks, but no existing theory can be adopted to address them, calling for a drastically different approach to studying the controllability of complex networks.

Here we develop an efficient controllability framework based on the principle of structural dissection, which is universally applicable to complex networks of arbitrary structures and link weights. The framework goes much beyond SCT and ECT by dissecting a network completely and revealing the role of nodes and links in full control in a comprehensive but straightforward manner. Specifically, the dissection process gives rise to a hierarchy of configurations based solely on the local network structure without requiring any global information. Central to the process is the emergence of a core structure, in which the weights of links play a determining role in controllability, whereas those of the links outside the core have little effect on control. Thus, the hierarchical structure classifies links into two categories in terms of SSC: either essential or spareable. Mathematically, we prove that a network consisting entirely of spareable links must be SSC, a generally sufficient and necessary criterion for SSC. An interesting result is that, for a SSC network, the SCT and ECT give exactly the same results, providing a natural connection between the two well-established controllability frameworks. Application of our dissection framework to real world networks reveals that (i) technological networks are SSC, whereas many social and biological networks are less so, and (ii) real world networks tend to be much more SSC than their randomized counterparts. These results imply that man-made networks intrinsically possess
a higher degree of resilience and adaptability, and natural evolution tends to contribute positively to the emergence of SSC.

Additional results are the following. The hierarchical structure classifies nodes into three categories in an extremely efficient manner: critical, intermittent, and redundant in terms of their contribution to controllability. Various phase transitions pertaining to the significance of links and nodes have been uncovered. For both homogeneous and heterogeneous networks, as a core emerges during the dissection process, a discontinuous phase transition occurs. For random networks, the phase transition point is universal but, for scale-free networks, it depends on the degree distribution. For undirected random networks, associated with nodal classification two phase transitions can arise. A striking phenomenon is that, for different network sizes, the curves characterizing the phase transitions intersect at a single point. Based on the canonical control theory, we obtain a rigorous mathematical proof for the dissection process. The analysis also enables us to make theoretical predictions about the phase transitions associated with full control.

Results

Principle of structural dissection. The structural dissection process (SDP) dissects a network completely to uncover a hierarchical structure that indicates the role of nodes and links in the controllability of the network in an intuitive way. The process consists of two orders (two stages) with respect to nodal and link removal.

For a directed network, the first-order SDP targets leaf nodes and their associated links, as shown schematically in Fig. 1. For an out-leaf node, e.g., the dark red node shown in Fig. 1a with unit out-degree, we remove all the incoming links of the successor (the blue node), and merge the out-leaf node (the dark red node) and its successor (the blue node), and specify the merged node using the out-leaf node’s index. For an in-leaf node with unit in-degree, e.g., the dark red node in Fig. 1b, we remove all the outgoing links of the predecessor (the blue node), merge the in-leaf node (the red node) and its predecessor (the blue one), and use the predecessor’s index to denote the resulting node. This procedure removes a single node and some links (see Supplementary Note 2 and Supplementary Fig. 1 for first-order SDP in matrix representation).

For an undirected network, the first-order SDP entails removing the leaf nodes and their neighbors iteratively from the network. For example, as shown in Fig. 1c, a leaf node (the dark red node) of degree one and its neighbors, together with all their links, are removed. As a result, two nodes and some links are removed from the network.

We repeat the first-order SDP until no leaf-nodes are left. For either a directed or an undirected network, with respect to the end result of the first-order SDP, two situations can arise. First, all remaining nodes are isolated. In this case, the isolated nodes constitute a minimum set of driver nodes and their number is $N_D$. We can prove that the network is SSC in this case (see Supplementary Note 3 for proof of SSC). Second, after the first-order SDP, in addition to the isolated nodes, various cores without leaf-nodes can emerge. As a result, the isolated nodes are the driver nodes, but nodes in the cores may not be. A second-order SDP is then necessary to probe into the cores to identify all the remaining driver nodes.
We can justify that the weights of the removed links by the first-order SDP have no effect on the value of $N_D$. With respect to SSC, it is reasonable to regard the removed links as ”spareable”. Note that a second-order SDP is implemented only when cores are present. Figure 1 schematically illustrates the process, where the aim is to generate leaf nodes in the cores and this can be accomplished by removing some links of the nodes with the smallest degree. As shown in Fig. 1, the link marked as the pink circle is chosen to be removed, which induces a change in the weight of each related link. The weights of the outgoing links of the green node are updated. Specifically, the weights of the green and orange links are changed to $d - a(e/b)$ and $f - c(e/b)$, respectively (see Supplementary Fig. 2 and Supplementary Note 2 for second-order SDP in matrix representation). Analogous to the first-order SDP, the second-order SDP keeps $N_D$ and hence the controllability configuration invariant. After leaf nodes arise, we can apply the first-order SDP continuously to reduce the sizes of the cores until all cores disappear and all remaining nodes are isolated.

The nodal and link removal processes take into account the link weights in the remaining cores, which affect the value of $N_D$. For this reason these links are regarded as ”effective”. If a network contains effective links, it is not SSC.

Any network can be reduced to a set of isolated nodes through repeated and alternative applications of the first- and second-order SDPs. In the end, all the remaining isolated nodes are driver nodes, whose number is nothing but $N_D$. This represents a general and efficient method to identify a set of minimal driver nodes for arbitrary network structure and link weights. Note that both the first- and second-order SDPs require only information about the local network structure and are therefore highly efficient. It is worth emphasizing that a SSC network contains only spareable links, the weights of which have no effect on the value of $N_D$ and the set of driver nodes. Thus, if a network is SSC, the SCT and ECT will lead exactly to the same results of $N_D$ because of the null effect of the link weights, establishing a natural connection between the two existing controllability frameworks for complex networks.

**Controllability, link significance and SSC.** We test SDP on both homogeneous and heterogeneous networks to discern links with respect to SSC and identify driver nodes that determine controllability. We first apply the first-order SDP and denote the fraction of the remaining nodes and links as $n_1$ and $l_1$, respectively. Figure 2a shows, for Erdős-Rényi (ER) random networks, a phase-transition in $n_1$ associated with the emergence of the cores at $\langle k \rangle = e$. For $\langle k \rangle < e$, no core arises after the first-order SDP terminates, indicating that the networks are SSC. A core suddenly emerges at $\langle k \rangle = e$ and grows gradually as $\langle k \rangle$ is increased. The vertical line $\langle k \rangle = e$ thus separates two phases: (a) a phase on the left side of the line, corresponding to SSC and the presence of only spareable links that have no effect on the value of $n_D$, and (b) a phase on the right side of the line with the emergence of cores and effective links, violating the condition of SSC. Figure 2c shows the behavior of $l_1$ for directed ER random networks, where a transition occurs at $\langle k \rangle = 2e$. Figures 2b,d show the results for undirected and directed scale-free (SF) networks, respectively, where cases of different exponents $\gamma$ associated with the power-law degree distribution were considered. The phenomenon of phase transition between SSC and non-SSC persists in all the cases.
but with a difference from the ER networks: the critical transition point now depends on the value of the degree distribution exponent $\gamma$.

To obtain a clearer picture of the SDPs, we illustrate in Fig. 3a the first-order SDP, where we see that a hierarchical structure arises and a core emerges in the central area. During the process, partial driver nodes and all effective links can be distinguished, where the former is the surviving isolated nodes and the latter belong to the core. Repeated and alternative applications of the first- and second-order SDPs for both homogeneous and heterogeneous networks yield all driver nodes for full control. The cores vanish and all the remaining nodes are isolated, whose number is $N_D$, as exemplified in Figs. 2a-d, which agrees exactly with the results from ECT but in a highly efficient way. We were able to obtain theoretical predictions for the phase transitions and the core sizes, which agree well with the numerical results for all cases (see Supplementary Figs. 3 and 4 and Supplementary Note 2).

**Nodal significance.** The significance of the nodes can be measured in terms of their role in controllability. A previous work [15] suggested three types of nodes: critical, redundant and intermittent (the detailed definitions can be found in Supplementary Note 4). A brute-force search requiring global information was introduced for node classification for directed networks. However, the method is not applicable to undirected networks. The emerging hierarchical structure as a result of applying our first-order SDP opens up a new way to classify nodes in networks of arbitrary structure in an extremely efficient manner. Specifically, SDP allows us to establish a directed relation network among nodes based on local information only, where each removed leaf is pointed at by its neighbors’ neighbors, and the category of the nodes is determined by their predecessors in the relation network. The remaining isolated nodes after first-order SDP are intermittent nodes. For the nodes removed during the SDP, their categories can also be determined. In particular, for an uncategorized node, if there is at least one intermittent node in its predecessors in the relation network, it will be classified an intermittent node. We repeat this procedure until no more intermittent nodes are categorized, and then the remaining nodes are redundant.

Figure 3(b) illustrates the relation network built from the outside to insider layers based on the principle illustrated in Fig. 3a. The directions of the relation links are all from the leaves’ neighbors’ neighbors to the removed leaves, and from the insider to the outside layers. Note that the relation network is not unique, but any of its configurations can place each node into a unique category (see Supplementary Note 4 for how to build the relation network and the corresponding mathematical proof). The categories of all nodes are determined by the remaining isolated nodes and nodes in the core through the relation network. The categories of the nodes in the core can be determined by adding a probing node pointing to it (See Supplementary Figs. 5 and 6). As shown in Fig. 3c), if the probing node is added to a redundant node, we will have $N_D \rightarrow N_D + 1$ for the core. But if the probing node is added to an intermittent node, $N_D$ will remain unchanged. In general, the core is much smaller than the whole network, leading to a much higher efficiency than with the brute-force search.

We focus on nodal classification in undirected networks, a feat that has not been achieved prior
to our work. We find some nontrivial phenomena in ER networks, as demonstrated in Figs. 3(d) - (f), where $n_r$, the fraction of redundant nodes, versus the average degree is shown for different values of the network size. We see that $n_r$ increases with the average degree $\langle k \rangle$ when it is small but decreases when $\langle k \rangle$ exceeds a threshold. Two distinct phase transitions can be identified, which occur at $\langle k \rangle = e$ and $\langle k \rangle / \ln N = 1$, respectively. Fig. 3(e) shows the behavior of $n_r$ for $\langle k \rangle = e$, where we see that all curves for different network size $N$ cross each other at the same point - the defining signature of phase transition. If we normalize the average degree $\langle k \rangle$ by $\ln N$, the curves associated with different system sizes cross at $\langle k \rangle / \ln N = 1$, as shown in Fig. 3(f), indicating another phase transition at this critical point. We offer a theoretical explanation for the behavior of $n_r$ versus $\langle k \rangle$, which agree well with the numerical results (see Supplementary Note 4 for the theoretical expression of $n_r$).

**Application of SDP to real networks.** Table 1 and Fig. 4 show the results from applying SDP to a variety of real-world networks, directed or undirected, where $n_2$ and $n'_2$ are the fractions of the remaining nodes after the SDP for identical link weights (equivalent to unweighted networks) and random link weights, respectively (Details about these networks are included in Supplementary Table 1 and Note 5). We see that, for all the networks, the values of $n_2$ and $n_D$ from the ECT [6] are exactly equal to each other. For directed networks with random weights, the values of $n'_2$ are exactly the same as those of $n_D$ from the SCT [3, 6]. The consistency in these results validates our SDP method for characterizing controllability.

In Table 1, $l_1$ is a key index for SSC. In particular, if a network has $l_1 = 0$, then all the links are separable and the network is SSC. In this case, we have $n_1 = n_2$ because of the absence of cores in the real networks and $n_2 = n'_2$ because of the null effect of the link weights. From the values of $l_1$ in Fig. 4(a), we find that almost all technological networks are SSC with their $l_1$ values close to zero. In contrast, most social and biological networks are not SSC because their values of $l_1$ are not close to zero. These results are consistent with the intuition that the man-made networks, due to their design optimization, are generally more SSC than natural and self-organized networks. In addition, as shown in Table 1, non-SSC networks ($l_1 > 0$) with random link weights are always more controllable than unweighted non-SSC networks, i.e., $n'_2 < n_2$. This is because of the elimination of linear correlations by random weights comparing with identical weights.

Performing node classification for the real world networks, we find that the number of critical nodes tends to be much smaller than that of intermittent nodes (see $n_c$, $n_i$ and $n_r$ in Table 1). Except for a few social networks, the fraction of intermittent nodes ($n_i$) is much larger than that of critical nodes ($n_c$), indicating that in general there are many configurations of the driver nodes to realize full control. That is, there is flexibility in controlling real world networks, in spite of the existence of a fraction of inaccessible nodes on which external input signals cannot be imposed. Another result is that, for undirected networks, the quantity $n_D$ is negatively correlated with $n_r$ [the solid symbols in the upper branch in Fig. 4(b)], whereas for directed networks, there exists a bifurcation in the values of $n_D$ from those of $n_r$. There is thus a generic difference between directed and undirected networks. Figure 4(c) shows the relation between the fraction $D_1$ of the remaining
links after the first round of the first-order SDP and \( l_1 \) obtained from all rounds of the process. The results indicate that the quantity \( D_1 \) plays a dominant role in generating SSC as characterized by \( l_1 \). More specifically, when \( D_1 \) is less than a critical value (about 0.4), the values of \( l_1 \) are quite close to zero, indicating that the network is SSC. In contrast, when \( D_1 \) exceeds a critical value, \( l_1 \) increases rapidly and the network is not SSC.

Figure 4(d) shows that the real networks become more SSC when the nodal degrees are fixed but the links are randomized. The reason can be attributed to the existence of the core structures but link randomization among the nodes would destroy the cores so as to enhance the degree of SSC. However, if the nodal degrees are allowed to vary, the result would be drastically different in that the networks can be much more SSC than their completely randomized counterparts, as shown in Fig. 4(e). A plausible reason is that the networks, regardless of whether they are formed through self-organization (e.g., social networks), natural evolution (e.g., biological networks), or intentional design (e.g., technological networks), all have the tendency to be SSC. It is worth noting that, for SSC, degree distribution does not play a significant role [Fig. 4(d)], which is characteristically different from the conventional structural controllability. Instead, \( D_1 \) dominates \( l_1 \) [Fig. 4(c)], which can be explained in terms of the first-order SDP (the results for the randomized networks can be found in Supplementary Fig. 7 and Supplementary Note 5).

**Discussion**

Our structural dissection paradigm provides an effective way to identify the fundamental building blocks in complex networks of arbitrary topology and weights, which determine the network controllability. Comparing with the existing SCT and ECT frameworks, our approach offers a much deeper understanding of control of complex networks. Our study indicates that local structures are sufficient for precisely pinning down the key nodes and links for determining controllability, in contrast to SCT and ECT that require a global search and optimization scheme for nodal classification with respect to controllability. Particularly, based on the local information, structural dissection can be done in an extremely efficient manner, which gives rise to a hierarchical structure that classifies links and nodes into different categories with respect to their role in controllability. In general, two categories of links arise, where the weights of the effective links play a role in controllability but those of the spareable links do not. A network is SSC if it contains only spareable links, a sufficient and necessary criterion for determining SSC efficiently. The criterion naturally establishes an intrinsic connection between SCT and ECT in the sense that, for SSC networks, they give rise to the same controllability result.

Some striking results are obtained when we apply our SSC criterion to real-world networks. In particular, technological networks tend to be more SSC, whereas most social networks and biological networks are less. This indicates that man-made networks, due to the underlying design process to optimize performance, are generally more controllable than networks formed through natural evolution or self-organization. Our structural dissection process can also classify nodes efficiently into three categories in the underlying hierarchical relation structure, regardless of whether the network is directed or undirected. Two phase transitions associated with the re-
dundant nodes in undirected random networks are uncovered, and real directed and undirected networks are drastically different with respect redundant nodes.

There are some outstanding questions that need to be addressed to further advance the field of controlling complex networks. First and foremost, finding a framework to generalize the linear control theory to nonlinear dynamical networks is urgent. For a nonlinear dynamical system, the classical canonical control theory is not applicable. It is thus not clear whether our structural dissection framework can be applied to arbitrary nonlinear networks. Second, in existing works, controllability is referred to as the implementation of control through a minimum set of driver nodes. However, there are other definitions for controllability such as one in terms of the control energy. The trade-off between a smaller number of driver nodes and high energy cost provides a physically more reasonable way to define and realize better control. Third, for very dense networks where zeros no longer dominate the eigenvalue spectrum, knowledge about the eigenvalues is needed to implement the structural dissection process (see Supplementary Note 6 and Supplementary Figs. 8 and 9 for SDP in situations where zero is not the dominant eigenvalue), calling for an improved approach independent of the eigenvalues. In spite of the open issues, our universal controllability framework goes much beyond the existent ones and provides a deeper understanding of the controllability of complex networks at the most fundamental level.

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**Author contribution**

W.-X.W. and Z.S. contribute equally. W.-X.W., Z.S., C.Z., and Y.-C.L. designed research; Z.S. performed computations. Z.S., C.Z., and W.-X.W. did analytic derivations and analyzed data. Z.S., W.-X.W., and Y.-C.L. wrote the paper.

**Additional information**

**Supplementary Information** accompanies this paper as [http://www.nature.com/naturecommunications](http://www.nature.com/naturecommunications)
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FIG. 1: **Illustration of first- and second-order structural dissection process.** First-order SDP for (a) an out-leaf node, (b) an in-leaf node, and (c) a leaf node in an undirected network. (d) Second-order SDP, where the link with weight $e$ is removed, and the weights of the links ($d$ and $f$) are updated. (See Supplementary Figs. 1 and 2 and Supplementary Note 2 for the corresponding adjacency matrix representation and a mathematical proof of the SDP.) Driver nodes and spareable links whose weights have no effect on controllability can be identified. The configuration of the driver nodes is not unique, but the configuration of the spareable links is.
FIG. 2: **Structural dissection of model networks.** The fractions of the remaining nodes after first- and second-order dissection processes, denoted as $n_1$ and $n_2$, respectively, versus the average degree $\langle k \rangle$ for (a) an undirected ER random network and (b) an undirected scale-free network. The fractions of the remaining links ($l_1$) after the first-order dissection process versus the average degree $\langle k \rangle$ are shown for (c) a directed random network, and (d) a directed scale-free network. The unweighted random and scale-free networks of size $N = 15000$ are generated from a static model. The data points are result of averaging over 20 independent realizations. The dashed curves are theoretical predictions from an iterative equation based on the expected degree distribution. The symbols are results from the network subject to structural dissection: open circles and squares indicate the remaining fractions after the first-order and second-order processes, respectively. The solid curves are results obtained from ECT.
FIG. 3: **Structural dissection, hierarchical relation and nodal classification.** (a) First- and second-order structural dissection of an undirected network. A minimum set of driver nodes can be identified. (b) The hierarchical relation network that specifies the node categories, which consists of directed links pointing to the removed leaves from their neighbors’ neighbors. The category of a node is determined by its upstream neighbors in the relation network. A node is intermittent if and only if at least one of its upstream neighbors is an intermittent node. If all the upstream neighbors of a node in the relation network are redundant, the node itself is redundant. (c) Determination of node categories in the core through the addition of an external probing node. After adding a probing node pointing at a node, if the number of the driver nodes in the core is increased by 1, the node pointed at is redundant. Otherwise, if the number of driver nodes does not change, the node is classified as intermittent. Node classification for ER random networks: (d) The fraction $n_r$ of redundant nodes as a function of the average degree $\langle k \rangle$ for undirected ER random networks, (e) the phase transition at $\langle k \rangle = e$ and (f) the phase transition at $\langle k \rangle / \ln N$. At the phase-transition point, the curves of $n_r$ versus $\langle k \rangle$ for different network sizes cross each other at exactly the same point, unequivocal indication of a phase transition. The data points are result of averaging over 500 independent network realizations. Curves are theoretical prediction based on the expected degree distribution in the $N \to \infty$ limit (see Supplementary Note 2 for analytical expression), where the decreasing part is based on the degree distribution for $N = 15000$. 
FIG. 4: Structural dissection analysis of real-world networks. (a) The fractions of remaining links \(l_1\) after the first-order SDP for a number of real-world networks, where SSC can be characterized by the value of \(l_1\). (b) Fraction \(n_r\) of redundant nodes versus \(1 - n_D\). (c) The relation between \(l_1\) and the fraction \(D_1\) of remaining links after the first round of the 1st-order SDP \(D_1\). (d) Fraction of remaining links after 1st-order SDP, \(l^{rand\text{-}degree}_1\), obtained from the degree-preserving randomized network in comparison with the values of \(l_1\). (e) The fraction of remaining links after the first-order SDP, \(l_1^{rand\text{-}ER}\), obtained from fully randomized version in comparison with the values of \(l_1\).
TABLE I: Structural dissection of real world networks. For each network, its type and name, the number of nodes ($N$), and the number of links ($L$) are shown. The quantities $n_1$ and $l_1$ are the fractions of the remaining nodes and edges after first-order dissection, respectively. The quantities $n_2$ and $l_2$ are the fractions of the remaining nodes after second-order dissection for networks with uni-weights and random weights, respectively. The quantities $n_c$, $n_i$ and $n_r$ are the fractions of the critical, intermittent and redundant nodes, respectively. For data sources and references, see Supplementary Table 1 and Supplementary Note 5.

| Name                  | Type | $N$  | $L$    | $n_1$ | $l_1$ | $n_2$ | $l_2$ | $n_c$ | $n_i$ | $n_r$ |
|-----------------------|------|------|--------|-------|-------|-------|-------|-------|-------|-------|
| Electronic circuits   |      |      |        |       |       |       |       |       |       |       |
| S208a                 | D    | 122  | 189    | 0.2377| 0.2377| 0.082 | 0.2541| 0.6639|
| S420a                 | D    | 252  | 399    | 0.2341| 0.2341| 0.0714| 0.2579| 0.6706|
| S838a                 | D    | 512  | 819    | 0.2324| 0.2324| 0.0664| 0.2598| 0.6738|
| Autonomous graphs     |      |      |        |       |       |       |       |       |       |       |
| Oregon1-010526        | U    | 11174| 23409  | 0.7036| 0.0009| 0.7028| 0.7028| 0.7928| 0.2072|
| Oregon2-010526        | U    | 11461| 32730  | 0.6714| 0.0074| 0.6661| 0.6659| 0.7715| 0.2285|
| Internet              |      |      |        |       |       |       |       |       |       |       |
| P2P-3                 | D    | 8717 | 31525  | 0.5845| 0.0273| 0.5778| 0.5778| 0.9235| 0.0675|
| P2P-2                 | D    | 8846 | 31839  | 0.5861| 0.0435| 0.5779| 0.5779| 0.9131| 0.0756|
| Internet1997          | U    | 3015 | 5156   | 0.508 | 0.0829| 0.264 | 0.264 | 0.41   | 0.59   |
| Internet1999          | U    | 5357 | 10328  | 0.6532| 0.0001| 0.6517| 0.6517| 0.7616| 0.2384|
| Internet2001          | U    | 10515| 21455  | 0.6999| 0.0001| 0.6988| 0.6987| 0.7874| 0.2126|
| Polblogs              | D    | 1224 | 19025  | 0.3668| 0.0024| 0.3595| 0.3562| 0.3162| 0.4926|
| Transportation        |      |      |        |       |       |       |       |       |       |       |
| USAtop500             | U    | 500  | 2980   | 0.508 | 0.0829| 0.264 | 0.264 | 0.41   | 0.59   |
| Airline               | U    | 332  | 2126   | 0.5181| 0.2041| 0.1747| 0.1536| 0.3072| 0.6928|
| citation              |      |      |        |       |       |       |       |       |       |       |
| Small-World           | D    | 233  | 994    | 0.6094| 0.004  | 0.6052| 0.6052| 0.8541| 0.1459|
| SciMet                | D    | 2729 | 10413  | 0.4306| 0.0046| 0.4251| 0.4251| 0.6017| 0.2763|
| Kohonen               | D    | 3772 | 12731  | 0.5673| 0.0068| 0.562 | 0.562 | 0.7256| 0.2744|
| language              |      |      |        |       |       |       |       |       |       |       |
| Word-English          | D    | 7381 | 46281  | 0.6345| 0.0003| 0.6739| 0.6734| 0.7223| 0.2345|
| Word-French           | D    | 8325 | 24295  | 0.6739| 0.0003| 0.6739| 0.6734| 0.7223| 0.2345|
| Food Web              |      |      |        |       |       |       |       |       |       |       |
| grassland             | D    | 88   | 137    | 0.5227| 0.0114| 0.7273| 0.7273| 0.0255| 0.9745|
| ythan                 | D    | 135  | 601    | 0.5926| 0.0732| 0.5185| 0.5185| 0.8889| 0.1111|
| silwood               | D    | 154  | 370    | 0.7727| 0.0324| 0.7662| 0.7652| 0.9286| 0.0714|
| stmartin              | D    | 45   | 224    | 0.7111| 0.0179| 0.5333| 0.5333| 0.4444| 0.5556|
| seagrass              | D    | 49   | 226    | 0.6939| 0.4558| 0.3265| 0.2653| 0.4694| 0.5102|
| littlerock            | D    | 183  | 2494   | 0.9781| 0.9403| 0.7541| 0.541 | 0.5774| 0.4372|
| neuronal              |      |      |        |       |       |       |       |       |       |       |
| C.elegans             | D    | 297  | 2345   | 0.8485| 0.7514| 0.165 | 0.165 | 0.9099| 0.0901|
| coauthor              |      |      |        |       |       |       |       |       |       |       |
| CA-ArTh               | U    | 9877 | 25998  | 0.4459| 0.3156| 0.1314| 0.0778| 0.1874| 0.8126|
| CA-GrQc               | U    | 5242 | 14496  | 0.4914| 0.4918| 0.2224| 0.0792| 0.174 | 0.8260|
| coauthorships         | U    | 1461 | 2742   | 0.6578| 0.709 | 0.4613| 0.0253| 0.0684| 0.9316|
| social                |      |      |        |       |       |       |       |       |       |       |
| facebook-combined     | U    | 4039 | 88234  | 0.9445| 0.931 | 0.0255| 0.0191| 0.0248| 0.9752|
| Freeman-1             | D    | 34   | 695    | 1     | 1     | 0.0294| 0.0294| 1      | 0      |
| consulting            | D    | 46   | 879    | 0.9783| 0.9954| 0.0435| 0.0435| 0.9783| 0.0217|
| UCOnline              | D    | 1899 | 20296  | 0.3233| 0.0195| 0.7994| 0.1811| 0.0957| 0.8358|
| prison                | D    | 67   | 182    | 0.5224| 0.4835| 0.1343| 0.1343| 0.1045| 0.0957|
| WikiVote              | D    | 7115 | 103689 | 0.6656| 0.0006| 0.6656| 0.6656| 0.0006| 0.3341|