Nested Domain Defects

J.R. Morris
Physics Department, Indiana University Northwest,
3400 Broadway, Gary Indiana 46408, USA

An example of a supersymmetric model involving two interacting chiral superfields is presented here which allows for solutions describing string-like “domain ribbon” defects embedded within a domain wall. It is energetically favorable for the fermions within the wall to populate the domain ribbons, and an explicit solution is found for the fermion zero modes. The Fermi gas within ribbons can allow them to stabilize in the form of small loops.

PACS: 11.27.+d, 12.60.Jv, 98.80.Cq

I. INTRODUCTION

Much attention has been given to topological defects, not only because they are interesting nonperturbative solutions in field theories, but also because they may have been physically realized in the early Universe [1,2]. The additional possibility that the early Universe may have existed in a supersymmetric phase provides motivation to investigate possible types of defects that may occur in supersymmetric theories. Here attention is focused on domain defects that can arise from broken discrete symmetries, and in particular, we investigate the case of nested domain defects wherein a string-like, or ribbon-like, defect referred to here as a “domain ribbon” can inhabit the interior of a domain wall. Recently, domain ribbons have been looked at in a nonsupersymmetric theory [3], and in a model [4] that can be seen as the real bosonic sector of a supersymmetric theory. Within this context, the present investigation serves as an extension of these previous studies. Supersymmetric theories with a single chiral superfield can admit domain wall solutions with some interesting properties [5,6], and an inclusion of a second chiral superfield allows nontrivial interactions that can result in a nontrivial internal structure of a domain wall. Furthermore, a supersymmetric theory naturally includes fermions which can interact with the scalar fields in interesting ways.

As an example, we assume a relatively simple superpotential which gives rise to a model which, in the real bosonic sector, admits domain ribbon solutions. The domain ribbons appear because the system stabilizes by forming a real scalar condensate in the wall’s core. However, the condensate formation breaks a discrete $Z_2$ symmetry
so that different condensate domains can form in initially uncorrelated regions of the wall, and these different domains must be separated by a “wall within the wall”, i.e. a domain ribbon. The fermions can respond to the scalar field background by forming zero modes [3], for which analytical solutions are obtained. It becomes energetically favorable for fermions to populate the ribbons, where they are massless. Consequently, in the supersymmetric model, a Fermi gas pressure can exist within the ribbons, modifying the eventual fate of the ribbons. Instead of rapidly fissioning away to nothing, as in a nonsupersymmetric model [3], fermion supported closed ribbons can stabilize in the form of small loops, which may be particle-sized. A domain wall may therefore end up being populated with these small Fermi loops.

A supersymmetric model with two interacting chiral superfields is presented in the next section, and in sec. III the domain wall and domain ribbon solutions are found in the real bosonic sector. The fermion zero modes, which can behave as traveling waves that propagate through the ribbons, are then analyzed in sec. IV. In sec. V we consider an effective one dimensional Fermi gas in a ribbon, and look at the conditions for which a closed ribbon loop can stabilize. We conclude with a brief summary.

II. A MODEL WITH TWO INTERACTING CHIRAL SUPERFIELDS

A. Fields

We consider a supersymmetric theory involving the two chiral superfields \( \Phi \) and \( X \). These chiral multiplets can be displayed as \( \Phi = (\phi, \psi_\phi, F_\phi) \), \( X = (\chi, \psi_\chi, F_\chi) \), where \( \phi \) and \( \chi \) are complex-valued scalar fields, \( \psi_\phi, \psi_\chi \) are Weyl 2-spinors, and \( F_\phi, F_\chi \) are complex-valued scalar auxiliary fields. From a superpotential \( W(\Phi, X) \), a scalar potential \( V(\phi, \chi) \) can be generated describing the interactions between the scalar fields. The Yukawa couplings for the fermions are obtained from the superpotential \( W \). The two chiral superfields \( \Phi \) and \( X \) have superspace representations [8,9]

\[
\Phi(y, \theta) = \phi(y) + \sqrt{2} \theta \psi_\phi + \theta^2 F_\phi(y), \\
X(y, \theta) = \chi(y) + \sqrt{2} \theta \psi_\chi + \theta^2 F_\chi(y),
\]

where \( y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} \). (We use a metric with signature \((+, -, -, -)\). See the Appendix for notation, conventions, and gamma matrices.) The complex scalars \( F_\phi, F_\chi \) are auxiliary fields which will be eliminated. Majorana 4-spinors \( \Psi_{\phi, \chi} \) can be defined in terms of the Weyl 2-spinors:

\[
\Psi_{\phi} = \left( \begin{array}{c} \psi_\phi \\ \psi^\dagger_\phi \end{array} \right), \quad \Psi_{\chi} = \left( \begin{array}{c} \psi_\chi \\ \psi^\dagger_\chi \end{array} \right), \quad \alpha, \dot{\alpha} = 1, 2.
\]
B. Lagrangian

In terms of the chiral superfields, the Lagrangian is

\[ L = (\Phi^* \Phi)|_{\theta^2} + (X^* X)|_{\theta^2} + W|_{\theta^2} + W^*|_{\theta^2}, \]  

(3)

where \( W \) is the superpotential and \( W|_{\theta^2} \) represents the \( \theta^2 \) part of \( W \), etc. In terms of the component fields, \( L \) can be written as

\[ L = L_B^K + L_F^K + L_Y - V, \]

(4)

where

\[ L_B^K = \partial^\mu \phi^* \partial_\mu \phi + \partial^\mu \chi^* \partial_\mu \chi, \]

(5)

\[ L_F^K = \frac{i}{2} \left[ (\partial_\mu \psi_\phi \sigma^\mu \tilde{\psi}_\phi - \psi_\phi \sigma^\mu \partial_\mu \tilde{\psi}_\phi + (\partial_\mu \psi_\chi) \sigma^\mu \tilde{\psi}_\chi - \psi_\chi \sigma^\mu \partial_\mu \tilde{\psi}_\chi \right], \]

(6)

\[ L_Y = -\frac{1}{2} \sum_{i,j} \left[ Y_{ij} \psi_i \psi_j + Y_{ij}^* \tilde{\psi}_i \tilde{\psi}_j \right] = -\frac{1}{2} \left[ Y_{\phi \phi} \psi_\phi \psi_\phi + Y_{\chi \chi} \psi_\chi \psi_\chi + 2Y_{\phi \chi} \psi_\phi \psi_\chi \right] + c.c., \]

(7)

\[ V = |F_\phi|^2 + |F_\chi|^2 = \left| \frac{\partial W}{\partial \phi} \right|^2 + \left| \frac{\partial W}{\partial \chi} \right|^2, \]

(8)

with \( Y_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \), \( F_\phi = -\left( \frac{\partial W}{\partial \phi} \right)^* \), \( F_\chi = -\left( \frac{\partial W}{\partial \chi} \right)^* \).

C. Superpotential and Scalar Potential

Let us consider a superpotential, which written in terms of the scalar fields \( \phi \) and \( \chi \), is given by

\[ W = \lambda (\phi^2 - a^2) \chi + \frac{1}{3} \mu \chi^3. \]

(9)

We then have the Yukawa coupling terms \( Y_{\phi \phi} = 2\lambda \chi \), \( Y_{\chi \chi} = 2\mu \chi \), \( Y_{\phi \chi} = 2\lambda \phi \), and the auxiliary fields are given by \(-F_\phi^* = 2\lambda \phi \chi \), \(-F_\chi^* = \lambda (\phi^2 - a^2) + \mu \chi^2 \). The Yukawa part of the Lagrangian can then be written out as

\[ L_Y = -[\lambda \chi \psi_\phi \psi_\phi + \mu \chi \psi_\chi \psi_\chi + 2\lambda \phi \psi_\phi \psi_\chi] \]

\[ -[\lambda \chi^* \tilde{\psi}_\phi \tilde{\psi}_\phi + \mu \chi^* \tilde{\psi}_\chi \tilde{\psi}_\chi + 2\lambda \phi^* \tilde{\psi}_\phi \tilde{\psi}_\chi]. \]

(10)

The scalar potential is given by

\[ V = 4\lambda^2 |\phi \chi|^2 + |\lambda (\phi^2 - a^2) + \mu \chi^2|^2. \]

(11)

Note that \( V \geq 0 \), and vacuum states for which \( V = 0 \) are supersymmetric vacuum states.
D. Vacuum States

The potential is \( V = F_\phi^* F_\phi + F_\chi^* F_\chi \geq 0 \) and the vacuum states are located by

\[
\frac{\partial V}{\partial \phi^*} = F_\phi^* \frac{\partial F_\phi}{\partial \phi^*} + F_\chi^* \frac{\partial F_\chi}{\partial \phi^*} = 4 \lambda^2 |\chi|^2 \phi + 2 \lambda \phi^* [\lambda (\phi^2 - a^2) + \mu \chi^2] = 0, \\
\frac{\partial V}{\partial \chi^*} = F_\phi^* \frac{\partial F_\phi}{\partial \chi^*} + F_\chi^* \frac{\partial F_\chi}{\partial \chi^*} = 4 \lambda^2 |\phi|^2 \chi + 2 \mu \chi^* [\lambda (\phi^2 - a^2) + \mu \chi^2] = 0. \quad (12)
\]

Supersymmetric vacuum states are solutions of \( V = 0 \), which is equivalent to the conditions \( F_\phi = 0, F_\chi = 0 \). Using \(-F_\phi^* = 2 \lambda \phi \chi \) and \(-F_\chi^* = \lambda (\phi^2 - a^2) + \mu \chi^2 \), we see that there are two possible sets of vacuum states: (1) \( \phi = \pm a, \chi = 0 \), and (2) \( \phi = 0, \chi = \pm \sqrt{\lambda \mu} a \equiv \pm \chi_0 \). These two sets of vacuum states are energetically degenerate and supersymmetric (\( V = 0 \)). We will focus our attention upon the first set of vacuum states where \( \phi = \pm a, \chi = 0 \). A broken \( Z_2 \) symmetry associated with \( \phi \) gives rise to a domain wall, and a discrete \( Z_2 \) symmetry associated with \( \chi \) gets broken in the core of the wall, giving rise to a \( \chi \) condensate and domain ribbons inside the wall.

III. DOMAIN WALL AND DOMAIN RIBBONS

A. Domain Wall

Let us now focus on the real bosonic sector of the model, where \( \psi_\phi = \psi_\chi = 0 \) and \( \text{Im}(\phi) = \text{Im}(\chi) = 0 \), i.e. the scalar fields \( \phi \) and \( \chi \) are real-valued in this sector. Then the field equations for the scalars \( \phi \) and \( \chi \) in the real bosonic sector, obtained, e.g., from \( \Box \phi + \left( \frac{\partial V}{\partial \phi^*} \right) \big|_{\text{Im}(\phi) = \text{Im}(\chi) = 0} = 0 \), etc. are given by

\[
\Box \phi + 4 \lambda^2 \chi^2 \phi + 2 \lambda \phi^* [\lambda (\phi^2 - a^2) + \mu \chi^2] = 0, \quad (13)
\]

\[
\Box \chi + 4 \lambda^2 \phi^2 \chi + 2 \mu \chi^* [\lambda (\phi^2 - a^2) + \mu \chi^2] = 0. \quad (14)
\]

where \( \Box = \partial_0^2 - \nabla^2 \). If we assume that the vacuum states which are realized are given by \( \phi = \pm a, \chi = 0 \), then, when \( \chi \) is set equal to zero, a domain wall solution is admitted for the field \( \phi \), with \( \phi \) interpolating between the asymptotic values \( \phi = \pm a \). The domain wall solution, describing a wall centered on the \( x - y \) plane \( (z = 0) \), is of the form \( \phi(z) = a \tanh \frac{z}{\Delta} \), where \( \Delta \) represents the thickness of the wall. It will often be convenient to approximate the domain wall by a slab of thickness \( \Delta \) inside of which \( \phi = 0 \), with \( \phi = \pm a \) outside.

We now follow the line of reasoning used by Witten [11] to examine the formation of a scalar condensate inside a superconducting cosmic string. We see that an examination of the field \( \chi \) inside the domain wall, where we take \( \phi = 0 \), indicates that, for a certain parameter range, the minimal energy configuration of \( \chi \) is not given by \( \chi = 0 \), but rather by \( \chi = \pm \chi_0 \), where \( \chi_0 = \sqrt{\frac{\Delta}{\mu}} a \). In this case there are two
energetically degenerate ground states given by $\chi = \pm \chi_0$ within the core of the wall. Taking the gradient energy of the field $\chi$ into account, it can be seen that there is a range of parameters for which $\chi = 0$ is, in fact, an unstable solution inside the wall. This follows by considering small fluctuations of $\chi$ about the value $\chi = 0$. Writing $\chi = F(z) \sin(\omega t)$, and applying this to the equation of motion for $\chi$ gives

$$-\partial_z^2 F + (2\mu + 4\lambda^2)[a^2 \tanh^2(\frac{z}{\Delta})]F = (\omega^2 + 2\mu a^2)F \equiv EF, \quad E = (\omega^2 + 2\mu a^2).$$

(15)

Then, for a normalizable bound state for which $E < 2\lambda a^2$, we have $\omega^2 < 0$. We can therefore conclude that there is a parameter range for which the solution $\chi = 0$ is unstable inside the domain wall, and a scalar condensate with $\chi = \pm \chi_0$ tends to form in the core of the wall. It will be assumed that the model parameters do indeed occupy a range for which the condensate formation is energetically favorable.

**B. Domain Ribbons**

When the $\chi$ condensate forms within the domain wall, the field $\chi$ can settle into either a $+\chi_0$ state or a $-\chi_0$ state, since these two states are energetically degenerate. One can expect that domains of these different states form, but the domains will be uncorrelated beyond some coherence length $\xi$; i.e., we expect there to be domains where $\chi = +\chi_0$ and domains where $\chi = -\chi_0$. Two different domains are separated by a region where $\chi = 0$, locating the core of a domain ribbon. The domain ribbon is just a portion of a domain wall within the host domain wall, with the static domain ribbon $(R)$ behaving like $\chi(x)_R \sim \pm \chi_0 \tanh \frac{x}{w_R}$ and the antiribbon ($\bar{R}$) function behaving like $\chi(x)_{\bar{R}} = -\chi_R(x)$, where $w_R$ is the thickness of the ribbon or antiribbon. Domain ribbons form between $\pm \chi_0$ domains and can be in the form of infinite ribbons or in the form of closed ribbon loops. A ribbon loop encloses a $\pm \chi_0$ domain and is surrounded by a $\mp \chi_0$ domain. Self intersecting loops can fission into smaller loops, with $\chi$ particle radiation being emitted from the annihilating ribbon sections. Two different loops can also fuse together to form a larger loop, with the emission of $\chi$ particles. Ribbon loops can also be formed at the intersections of an infinite ribbon and an antiribbon. Oscillating ribbon loops with self intersecting trajectories are expected to decay mainly through $\chi$ particle production, with a negligible fraction of the released energy in the form of gravitational radiation. (Further details can be found in ref. [3])

**C. Fermions**

Let us now look at the response of the fermions to the real scalar field background. Inside the domain wall (but outside of a ribbon or antiribbon), taking $\phi = 0$ and $\chi = +\chi_0$, (10) becomes
\[ L_Y = -\chi_0[\lambda(\psi_\phi\psi_\phi + \bar{\psi}_\phi\bar{\psi}_\phi) + \mu(\psi_\chi\psi_\chi + \bar{\psi}_\chi\bar{\psi}_\chi)]. \]  

(16)

In terms of the Majorana spinors $\Psi_{\phi,\chi}$ this can be written as

\[ L_Y = i\chi_0[\lambda\bar{\Psi}_\phi\Psi_\phi + \mu\bar{\Psi}_\chi\Psi_\chi]. \]  

(17)

The Majorana mass term is of the form $L_{mass} = \frac{1}{2}iM\bar{\Psi}\Psi = -\frac{1}{2}M(\psi\psi + \bar{\psi}\bar{\psi})$ for a Majorana fermion of mass $M$. Therefore, we see that in the domain wall (but outside of a ribbon or antiribbon) the $\Psi_\phi$ fermion mass is $M_\phi = 2\lambda\chi_0$ and the $\Psi_\chi$ fermion mass is $M_\chi = 2\mu\chi_0$.

Also, there is [see (14)] a Dirac fermion in the vacuum state where $\chi = 0$, $\phi = +a$, made from the Weyl spinors $\psi_\phi, \psi_\chi$: $-2\lambda a[\psi_\phi\psi_\chi + \bar{\psi}_\phi\bar{\psi}_\chi] = i2\lambda a\bar{\psi}'\psi'$, where the Dirac spinor $\psi'$ is given in terms of the Weyl two-spinors as $\psi' = \left( \begin{array}{c} \psi_\phi \\ \psi_\chi \end{array} \right)$. (Note that in going from a domain where $\phi = +a$ to one where $\phi = -a$, the spinor mass eigenstates change, i.e. the Weyl 2 spinors undergo a phase rotation $\psi_\phi \rightarrow i\psi_\phi$, etc. and the Majorana 4 spinors undergo a $\gamma_5$ “rotation”, $\Psi_\phi \rightarrow \gamma_5\Psi_\phi$, etc.) The Dirac spinor in the vacuum has a mass $M' = 2\lambda a$. So, we have a Dirac fermion in the vacuum with mass $M' = 2\lambda a$, and Majorana fermions in the domain wall, which have masses $M_\phi = 2\lambda\chi_0$, $M_\chi = 2\mu\chi_0$ outside of a domain ribbon, but become massless inside the core of a domain ribbon.

Now note [see (14) and (17)] that in the core of a domain ribbon, where $\chi \rightarrow 0$, the Majorana fermions become massless: $M_{\phi,\chi} \rightarrow 0$. We can suspect that there are fermion zero modes [7] within the domain ribbons.

The situation and the particle masses can be briefly summarized in the following way. There are three different regions where we can look at field expectation values (DW=domain wall, DR=domain ribbon):

(I) In vacuum: $|\phi| = a$, $\chi = 0$

(II) Inside DW, outside DR: $\phi = 0$, $|\chi| = \chi_0$, $\chi_0 = \sqrt{\frac{\lambda}{\mu}}a$

(III) Inside DR: $\phi = 0$, $\chi = 0$.

The boson particle masses can be examined from the potential $V$ given by (11):

\[ m_\phi^2 = \frac{\partial^2 V}{\partial \phi \partial \phi^*} = (2\lambda)^2 [||\phi||^2 + ||\phi||^2], \]

\[ m_\chi^2 = \frac{\partial^2 V}{\partial \chi \partial \chi^*} = (2\lambda)^2||\phi||^2 + (2\mu)^2||\chi||^2. \]  

(18)

The fermion masses come from $L_Y$, given by (10). For Dirac and Majorana fermions the mass terms are of the form

\[ L_{mass} = -m(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2) = im(\bar{\Psi}\Psi), \quad \Psi = \left( \begin{array}{c} \psi_1 \\ \bar{\psi}_2 \end{array} \right) \quad \text{(Dirac)}, \]

\[ L_{mass} = -\frac{1}{2}m(\alpha\alpha + \bar{\alpha}\bar{\alpha}) = \frac{1}{2}im(\bar{M}M), \quad M = \left( \begin{array}{c} \alpha \\ \bar{\alpha} \end{array} \right) \quad \text{(Majorana)}, \]  

(19)
and these fermion masses have been looked at previously. A summary of particle masses in the various regions is given below:

Region I: \( m_\phi = m_\chi = M' = 2\lambda a \)
Region II: \( m_\phi = M_\phi = 2\lambda \chi_0, \ m_\chi = M_\chi = 2\mu \chi_0 \)
Region III: \( m_\phi = m_\chi = M_\phi = M_\chi = 0 \)

IV. FERMION ZERO MODES INSIDE A DOMAIN RIBBON

A. Reaction of Fermion Fields to Wall and Ribbon Backgrounds

Now let’s consider the effect of the real scalar fields upon the dynamical spinor fields by again looking at the spinors in the background fields described by the domain wall and domain ribbon solutions. For approximation purposes, we neglect field gradients in the wall and ribbons and we take \( \phi = 0 \) inside the domain wall (and inside a ribbon) and \( \chi(x) = \chi_0 \tanh \frac{x}{w} \) for the (static) ribbon. The Majorana fields \( \Psi_{\phi,\chi} \) are given in terms of the Weyl 2-spinors by (2) and the Lagrangian is given by \( L = L^F_K + L^F_K + L_Y - V \). Consider the field equations for the Majorana spinors inside the domain wall, where \( \phi = 0 \). These field equations follow from \( \frac{\partial L}{\partial \bar{\Psi}} = \frac{\partial L^F_K}{\partial \bar{\Psi}} + \frac{\partial L_Y}{\partial \bar{\Psi}} = 0 \), where

\[
L^F_K = \frac{i}{2} \bar{\Psi}_\phi \gamma^\mu \partial_\mu \Psi_\phi + \frac{i}{2} \bar{\Psi}_\chi \gamma^\mu \partial_\mu \Psi_\chi, \tag{20}
\]

\[
L_Y = -\left[\lambda \chi (\psi_\phi \bar{\psi}_\phi + \bar{\psi}_\phi \bar{\psi}_\phi) + \mu \chi (\psi_\chi \bar{\psi}_\chi + \bar{\psi}_\chi \bar{\psi}_\chi)\right] = i \left[\lambda \chi \bar{\Psi}_\phi \Psi_\phi + \mu \chi \bar{\Psi}_\chi \Psi_\chi \right]. \tag{21}
\]

Therefore, the field equations for the Majorana fields, in the background of the domain wall and ribbon fields only [i.e. inside the domain wall where we assume that \( \phi = 0 \) and \( \chi^* = \chi \)] are given by

\[
\gamma^\mu \partial_\mu \Psi_\phi + 2\lambda \chi \Psi_\phi = 0,
\gamma^\mu \partial_\mu \Psi_\chi + 2\mu \chi \Psi_\chi = 0. \tag{22}
\]

[These equations do not include any descriptions of the possible interactions of the fermions with scalar field excitations (where, e.g., we could more generally write \( \phi = \phi_{\text{wall}} + \delta \phi, \ \chi = \chi_{\text{ribbon}} + \delta \chi \).] Again, from the field equations we see that inside the wall, but outside a ribbon (\( \phi = 0, \ \chi = +\chi_0 \)), the Majorana fermion masses are \( M_\phi = 2\lambda \chi_0, \ M_\chi = 2\mu \chi_0 \), and inside a ribbon (\( \phi = 0, \ \chi \to 0 \)) the Majorana fermions become massless, \( M_{\phi,\chi} \to 0 \).
B. Static Zero Modes

To search for static Majorana zero modes inside a ribbon, let’s assume $\Psi_{\phi, \chi} = \Psi_{\phi, \chi}(x)$, and use the fact that $(\gamma^1)^2 = 1$, along with $\chi(x) = \chi_0 \tanh \frac{x}{w_R}$ for the description of a ribbon. Furthermore, since the equations for $\Psi_{\phi}$ and $\Psi_{\chi}$, given by (22) are similar and decoupled, let’s only deal with the equation for $\Psi_{\phi}$ and drop the subscript $\phi$ for now; i.e. $\Psi_{\phi} \rightarrow \Psi$. The equation for $\Psi$ therefore becomes

$$\gamma^1 \partial_x \Psi(x) + 2\lambda \chi(x) \Psi(x) = 0,$$

(23)

where $\gamma^1 = i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$, with $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, and $(\gamma^1)^2 = 1$.

Multiplying (23) by $\gamma^1$ gives

$$\partial_x \Psi = -2\lambda \chi \gamma^1 \Psi.$$

(24)

Let us now write the Majorana 4-spinor $\Psi$ in terms of 2-spinors $\eta$ and $\xi$: $\Psi = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$. We then have $\gamma^1 \Psi = i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} = i \begin{pmatrix} \sigma_1 \xi \\ -\sigma_1 \eta \end{pmatrix}$. Therefore,

$$\partial_x \begin{pmatrix} \eta \\ \xi \end{pmatrix} = -2i \lambda \chi \begin{pmatrix} \sigma_1 \xi \\ -\sigma_1 \eta \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma_1)^2 = 1.$$

(25)

The equations for $\eta$ and $\xi$ can be decoupled by writing

$$\xi = -i \sigma_1 \eta, \quad \eta = i \sigma_1 \xi.$$

(26)

Then, by (23) and (26),

$$\partial_x \eta = -2\lambda \chi \eta, \quad \partial_x \xi = -2\lambda \chi \xi, \quad \Psi = \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ -i \sigma_1 \eta \end{pmatrix}.$$

(27)

A solution is given by

$$\eta = \tau \exp \left[ -2\lambda \int_0^x \chi(x')dx' \right] = \tau \left[ \cosh \frac{x}{w_R} \right]^{-2},$$

(28)

where $\tau$ is an arbitrary constant Weyl 2-spinor and where $w_R = \frac{1}{\lambda \chi_0}$.

The Majorana condition $\Psi_C = -\gamma^2 \Psi^* = \Psi$, (where $\Psi_C$ is the charge conjugate of $\Psi$) i.e.

$$\Psi = \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ i \sigma_2 \eta^* \end{pmatrix},$$

(29)
must also be satisfied. Upon comparing (27) and (29), we have $\sigma_2\eta^* = -\sigma_1\eta$, or $\sigma_1\sigma_2\eta^* = -\eta$, so that with the help of $\sigma_1\sigma_2 = i\sigma_3$, we get $\eta^* = i\sigma_3\eta$. We must therefore require that $\tau^* = i\sigma_3\tau$. We therefore have for our present case the static zero mode solutions

$$
\Psi_\phi = \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad \eta = \tau \exp \left[ -2\lambda \int_0^x \chi(x') dx' \right], \\
\Psi_\chi = \begin{pmatrix} \eta' \\ \xi' \end{pmatrix}, \quad \eta' = \tau' \exp \left[ -2\mu \int_0^x \chi(x') dx' \right],
$$

(30)

where $\xi = -i\sigma_1\eta$, $\xi' = -i\sigma_1\eta'$. These solutions describe static Majorana zero modes localized within the domain ribbon.

C. Traveling Waves

Let us now regard $\Psi$ to be a function of $x$, $y$, and $t$, i.e., $\Psi(x,y,t) = (\eta(x,y,t) - i\sigma_1\eta(x,y,t))$, where $\eta(x,y,t) = \tau(y,t) \left[ \cosh \frac{x}{w} \right]^{-2}$. Then (22) implies that

$$
(\gamma^0\partial_0 + \gamma_2\partial_2) \begin{pmatrix} \tau(y,t) \\ -i\sigma_1\tau(y,t) \end{pmatrix} \left[ \cosh \frac{x}{w} \right]^{-2} = 0,
$$

(31)

which is solved by

$$
(\partial_0 + \sigma_2\partial_2)\tau(y,t) = 0.
$$

(32)

This can be seen by multiplying (31) by $\gamma^1$ and using $\gamma^0\gamma^2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$, so that (31) reduces to the set of equations $(\partial_0 + \sigma_2\partial_2)\tau = 0$, and $(\partial_0 - \sigma_2\partial_2)\sigma_1\tau = 0$, and the second equation is automatically solved when the first equation is solved. Then writing $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ (32) can be written explicitly as

$$
\partial_0\tau_1 - i\partial_2\tau_2 = 0, \quad \partial_0\tau_2 + i\partial_2\tau_1 = 0.
$$

(33)

These can be combined to give

$$
(\partial_0^2 - \partial_2^2)\tau_{1,2} = 0,
$$

(34)

which is solved by $\tau_{1,2}(y,t) = \tau_{1,2}(y \pm t)$. Therefore, $\tau$ can be written as

$$
\tau(y,t) = \begin{pmatrix} \tau_1(y \pm t) \\ \tau_2(y \pm t) \end{pmatrix}.
$$

(35)

so that $\tau$, and hence $\Psi$, can contain a linear combination of “up” and “down” moving waves.
V. FERMI GAS IN DOMAIN RIBBON LOOPS

A. One-Dimensional Fermi Gas

Let’s consider the case where fermions occupy the interior of a domain ribbon, so that in the singular, thin ribbon limit, there is a one-dimensional Fermi gas. (The two Majorana spinors in the domain ribbon can be related to a Dirac spinor.) The number of states of spin 1/2 fermions with momentum between $p_x$ and $p_x + dp_x$ in a length $L$ is

$$\rho(p_x)dp_x = \frac{gL}{2\pi\hbar}dp_x,$$  \hspace{1cm} (36)

where $g = 2$ is the number of spin states for a spin 1/2 fermion. In the ground state there is a momentum spread from $p_x = -p_F$ to $p_x = p_F$, so that the number of fermions in the ground state is

$$N = \int_{-p_F}^{p_F} \rho(p_x)dp_x = \frac{gL}{\pi\hbar}p_F$$  \hspace{1cm} (37)

which implies that

$$p_F = \frac{\pi\hbar N}{gL}.$$  \hspace{1cm} (38)

The total energy of fermions in the ground state is

$$E_F = \int_{-p_F}^{p_F} \rho(p_x)\epsilon(p_x)dp_x,$$  \hspace{1cm} (39)

For the case of massless fermions, $\epsilon = pc = |p_x|c$, and we therefore get

$$E_F = \frac{\pi\hbar cN^2}{2gL} = \frac{\pi\hbar cN^2}{4L},$$  \hspace{1cm} (40)

where we have set $g = 2$ for one species of spin 1/2 fermion.

B. Massless Fermi Gas in a Ribbon Loop

Now consider a ribbon loop of length $L$ to be inhabited by a one-dimensional Fermi gas of massless fermions, with a “ribbon field” mass function $\chi(x)$, which vanishes in the ribbon core at $x = 0$. (We now set $\hbar = c = 1$.) The energy per unit length of ribbon is $\mu_R$ and the energy of the Fermi gas is $E_F$. The total energy for the ribbon loop is therefore

$$E = \mu_R L + E_F = \mu_R L + \frac{\pi N^2}{2gL},$$  \hspace{1cm} (41)
where \( N \) is the total fermion number for the fermions inhabiting the loop. For a fixed value of \( N \), the energy is minimized for

\[
L = \left( \frac{\pi}{2g\mu_R} \right)^{1/2} N
\]  

(42)

which, for a circular ribbon loop with \( L = 2\pi R \), would correspond to a radius of

\[
R = (\pi \mu_R)^{-1/2} \frac{N}{\pi}.
\]

For a loop of length \( L = \sqrt{\pi/\mu_R (N/2)} \), the total energy of the loop is

\[
E = \sqrt{\pi \mu_R N}.
\]  

(43)

However, we can notice that the loop is evidently unstable against flattening, since the configuration energy \( E \) depends upon the loop length \( L \), but is independent of the loop area, which could be decreased while keeping the length \( L \) constant. Therefore the loop may flatten (or have self intersecting trajectories) and subsequently fragment into smaller loops. This process may be continued resulting in the production of many smaller loops, but we expect this process to halt when the thin ribbon approximation breaks down, and the solitonic structure of the ribbon becomes important. (A similar type of reasoning has been used previously by MacPherson and Campbell [11] in the description of the collapse of three dimensional false vacuum bags to form “Fermi balls”.) Let us assume that stable circular loops of radius \( R \) are produced at the end of this fragmentation process. We take the minimal loop radius, where the thin ribbon wall approximation begins to break down, and the solitonic structure of the ribbon becomes important, to be roughly \( R_{\text{min}} \sim \nu w_R \), where \( w_R \) is the ribbon width, or thickness, and a reasonable guess for \( \nu \) may be roughly 1-10.

A static, straight domain ribbon has a profile given by \( \chi_R = \chi_0 \tanh(x/w_R) \), where \( w_R = 1/(\lambda \chi_0) \) is the thickness of the ribbon, and we estimate the energy density of this configuration [3] to be \( T_{\text{eff}} \sim (\partial_x \chi)^2 = \lambda^2 \chi_0^4 \text{sech}^4(x/w_R) \). The energy per unit area of the ribbon is then roughly \( \Sigma \sim (\chi_0^2/w_R)w_R = \lambda \chi_0^3 \). We multiply this by the thickness \( \Delta = 1/(\lambda a) \) of the domain wall to get an estimate of the energy per unit length, \( \mu_R \), of the ribbon:

\[
\mu_R \sim \Sigma \Delta \sim \frac{\chi_0^3}{a} = \left( \frac{\lambda}{\mu} \right)^{3/2} a^2.
\]  

(44)

By (42) \( L \sim N/\sqrt{\mu_R} \) so that upon setting \( L/2\pi \) equal to \( R_{\text{min}} \sim \nu/(\lambda \chi_0) \), we get

\[
N \sim \frac{2\pi \nu}{\lambda} \sqrt{\chi_0 \frac{\chi_0}{a}} = 2\pi \nu \left( \frac{1}{\lambda^3 \mu} \right)^{1/4}
\]  

(45)

as an estimate for the number of fermions that occupy a stabilized ribbon loop.

By (12) the mass of a stabilized loop is roughly \( E \sim N\sqrt{\mu_R} \), which by (14) and (15), gives
\[ E \sim \frac{2\pi\nu}{\lambda a} \chi_0^2 \frac{2\pi\nu}{\mu} a. \]  

(46)

At the GUT scale, the mass of the ribbon loop is roughly \( E \sim (2\pi\nu/\mu)10^{16} \) GeV, while at the electroweak scale, \( E \sim (2\pi\nu/\mu)10^3 \) GeV.

VI. SUMMARY

Topological defects represent interesting nonperturbative field theoretic solutions, but they are also interesting because they may have been physically realized in the early Universe during symmetry breaking phase transitions. This defect production may have taken place when the Universe existed in a supersymmetric phase, and it is therefore of interest to investigate defects within a supersymmetric context. Realistic supersymmetric theories contain interacting chiral superfields, so we have examined an example of a type of supersymmetric model where interactions can yield defects with a nontrivial internal topological structure. This extends some previous work on supersymmetric defects and structured nonsupersymmetric defects. Supersymmetry also couples fermions to the scalar fields, so that there may be fermionic effects introduced, such as the existence of zero modes and degeneracy pressure in defects.

We have considered a model admitting the simplest type of topological defect, a domain wall. The simple superpotential allows interactions between two scalar fields, with the result that a real scalar condensate can form within the wall. There is a distribution of \( \pm \chi_0 \) condensate domains within the wall, and at the interface between two different domains there must exist a topological “domain ribbon”. The ribbon can support fermion zero modes, which have been described analytically. In general, there will be fermionic excitations above the zero mode, describing fermionic particles trapped within the ribbon. These fermions give rise to a Fermi gas pressure, which can help to stabilize ribbon loops so that they do not completely disappear because of fissioning due to self intersecting loop trajectories, but perhaps stabilize in the form of microscopic particle sized loops. However, a complete description of the dynamics of the infinite ribbons and the multiple loops occupying the domain wall may be complicated, with fission and fusion processes taking place due to ribbon and loop interactions. It could be the case that essentially all of the loops in a domain wall eventually annihilate one another away, which could leave a domain wall populated with fermions, depending upon the relative values of the model parameters. At any rate, it can be seen that interactions in supersymmetric field theories of topological defects can give rise to bosonic and fermionic effects that may often be otherwise ignored or overlooked.

Acknowledgement

I wish to thank D. Bazeia for discussions related to this work.
APPENDIX A: CONVENTIONS

Some of the notations and conventions are briefly listed here. A metric $g_{\mu\nu}$ is used with signature $(+, -, -, -)$. Aside from the metric, the notation, conventions, and gamma matrices used conform to those of ref. [8]. The gamma matrices can be written in the form

$$
\gamma^\mu = i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}
$$

(A1)

with

$$
\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}),
$$

(A2)

where $\vec{\sigma}$ represents the Pauli matrices. Then

$$
\gamma^0 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = i \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,
$$

(A3)

and $\gamma_5$ is given by

$$
\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(A4)

The gamma matrices have the properties

$$
\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^\dagger = -\gamma_5, \quad (\gamma_5)^2 = -1.
$$

(A5)

A Majorana 4-spinor $\Psi$ is expressed in terms of the Weyl 2-spinors $\psi$ and $\bar{\psi}$ by

$$
\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix}
$$

and we use the summation conventions for Weyl spinors [with $\bar{\psi}_{\dot{\alpha}} = (\psi^\alpha)^*$]

$$
\xi \psi \equiv \xi^\alpha \psi_{\alpha}, \quad \xi \bar{\psi} \equiv \xi_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}, \quad \alpha = 1, 2, \quad \dot{\alpha} = 1, 2,
$$

(A6)

with $\varepsilon$ metric tensors (for raising and lowering Weyl spinor indices)

$$
(\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot{\alpha}\dot{\beta}}) = i\sigma_2, \quad (\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot{\alpha}\dot{\beta}}) = -i\sigma_2, \quad \varepsilon^{12} = 1 = \varepsilon^{\dot{1}\dot{2}}.
$$

(A7)
See, for example, A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, 1994)

[2] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, 1990)

[3] J.R. Morris, Phys. Rev. D **51**, 697 (1995)

[4] D. Bazeia, R. F. Ribeiro, and M. M. Santos, Phys. Rev. D **54**, 1852 (1996)

[5] M. Cvetic, F. Quevedo, and S-J Rey, Phys. Rev. Lett. **67**, 1836 (1991)

[6] M. Cvetic, S. Griffies, and S-J Rey, Nucl. Phys. B **381**, 301 (1992)

[7] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976)

[8] See, for example, P. Srivastava, *Supersymmetry, Superfields and Supergravity: an Introduction* (Adam Hilger, 1986)

[9] See, for example, J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Second Edition (Princeton University Press, 1992)

[10] E. Witten, Nucl. Phys. B **249**, 557 (1985)

[11] A.L. MacPherson and B.A. Campbell, Phys. Lett. B **347**, 205-210 (1995)