Fourier series expansion type of spectral collocation method for vibration analysis of cylindrical shells

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Abstract: An analysis method using a spectral collocation method for the vibration of cylindrical shells is proposed. Conventional spectral collocation methods have difficulty applying boundary conditions to fourth-order differential equations such as vibration equations of cylindrical shells. In this paper, an Hermite differentiation matrix is developed such that the proposed spectral collocation method can treat flexibly various boundary conditions. Since the vibration displacement of a cylindrical shell is periodic in the circumferential direction, it is solved semi-analytically using the Fourier series expansion. It is shown that the proposed method can offer more accurate solutions at a smaller number of unknowns, in less computation time and required memory than a finite element method.

Keywords: Spectral collocation method, Hermite interpolation, Fourier series expansion, Cylindrical shell, Vibration analysis

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1. INTRODUCTION

Several vibration fields cannot be solved analytically, even if they have simple geometries, because of their complexities of boundary conditions. For instance, although the vibration of cylindrical shells with both ends simply supported can be solved analytically, one with an end clamped cannot be solved analytically. In such cases, numerical analysis methods such as finite difference methods or finite element methods (FEMs) have to be applied.

Spectral methods are known as high-accuracy numerical analysis methods for smooth problems in simple geometries [1]. In spectral methods, a solution of a problem under consideration is approximated by a linear combination of global interpolating polynomials. These polynomials enable spectral methods to offer greater accuracy than FEMs.

Conventional spectral collocation methods, however, have a problem when solving vibration equations of shells. They include a fourth-order differentiation, thus two boundary conditions have to be imposed at boundary nodes. However, since only one equation per one node can basically be introduced in the conventional spectral collocation method that uses differentiation matrices based on Lagrange interpolating polynomials, two boundary conditions cannot be imposed at one node. Therefore, several approaches to consider more than two boundary conditions at each node have been proposed.

Huang has proposed a form of an interpolating polynomial whose values and derivatives are zeros at boundary nodes so that clamped boundary conditions can be imposed [2]. It also has been introduced in Trefethen’s literature [1]. This approach, however, cannot be applied to other boundary conditions such as simply supported conditions. One of approaches used broadly is to introduce equations expressing boundary conditions at boundary nodes instead of original equations at nodes next to the boundary nodes [3,4]; the original equations at the nodes next to the boundary nodes are completely ignored. It is simple and easy to implement but not exact. The approach introducing fictitious nodes in or out of a domain and using the nodal values as additional unknowns has been introduced in Forberg’s literatures; the effectiveness has been shown in them [5,6]. However, it is not a simple way because of addition of nodes that are different from regular nodes. Mai-
Duy has formulated Chebyshev integration matrices [7,8]. In this strategy, integration constants that arise in the process of constructing the integration matrices can be used as additional unknowns, thus the same number of boundary conditions as the integration constants can be imposed. Although this approach is exact, Chebyshev coefficients are used as unknowns; they have to be transformed into physical values after the unknowns are obtained. The purpose of this study is to formulate a spectral collocation method for vibration analysis of cylindrical shells and to verify its accuracy. In order to overcome the difficulty described above of imposing various boundary conditions, a differentiation matrix of an Hermite interpolating polynomial formulated by Chirikov [9] is adopted. This approach can more easily and simply deal with two boundary conditions at one boundary node than other approaches referred for above. The idea is close to Hermitian elements used in FEMs. The Hermite interpolation is expressed using not only nodal values but also derivatives. Differentiation and discretization of the interpolating polynomial yield the differentiation matrix. Derivatives at boundary nodes can be considered as additional unknowns, thus two boundary conditions can naturally be imposed at each boundary node. Although the entries of the matrices were derived by Chirikov, any applications of them have not been published yet. We apply the differentiation matrix of an Hermite interpolating polynomial to vibration analysis of cylindrical shells.

First, vibration displacements and external force are expanded in the Fourier series since they are periodic in the circumferential direction. Using the orthogonality of trigonometric functions, vibration equations reduced to a sequence of one-dimensional problems in the axial direction are obtained. These equations are solved using the spectral collocation method including an Hermite differentiation matrix. Note that application of the Hermite differentiation matrix requires special attention to eliminate spurious eigenvalues, which is described in the formulation of the proposed method. Next, the accuracy, computation time and required memory of the proposed method are examined through comparison with those of the FEM with axisymmetric shell elements. Finally, it is shown that the proposed method can effectively solve a cylindrical shell problem with boundary conditions that cannot be solved analytically.

2. PROPOSED METHOD

2.1. Vibration Equations and Fourier Series Expansion

Let us consider a cylindrical shell like Fig. 1. Many theories and equations of cylindrical shells have been proposed. In this paper, we adopt the Donnell-Mushtari equations [10]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{1}{2a^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{2a} \frac{\partial v}{\partial \theta} + \frac{v}{a} \frac{\partial w}{a} - \frac{1}{c_L^2} \frac{\partial^2 u}{\partial t^2} = 0, \tag{1a}
\]

\[
\frac{1}{2a} \frac{\partial^2 u}{\partial \theta} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{a} \frac{\partial w}{a} - \frac{1}{c_L^2} \frac{\partial^2 v}{\partial t^2} = 0, \tag{1b}
\]

\[
\frac{v}{a} \frac{\partial u}{\partial x} + \frac{1}{a^2} \frac{\partial v}{\partial \theta} + \frac{1}{a} \frac{\partial w}{a} + \beta^2 \nabla^4 w + \frac{1}{c_L^2} \frac{\partial^2 w}{\partial t^2} = \frac{p(1 - \nu^2)}{Eh}, \tag{1c}
\]

with

\[
\nabla^4 \equiv a^2 \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{1}{a^2} \frac{\partial^4}{\partial \theta^4}, \tag{2}
\]

where, \( u, v \) and \( w \) are the vibration displacements in the \( x-, \theta- \) and \( r- \) direction, respectively; \( p \) is the external force with the units of pressure in the \( r- \) direction. Parameters \( a, \nu, c_L \) and \( \beta \) are the radius, Poisson’s ratio, propagation velocity of the longitudinal wave and stiffness factor, respectively. The propagation velocity \( c_L \) and stiffness factor \( \beta \) are given by \( c_L = \sqrt{E/\rho(1-\nu^2)} \) and \( \beta^2 = \rho^2/(12a^2) \), where \( E, \rho \) and \( h \) are the Young’s modulus, density and thickness.

Since all the vibration displacements are periodic in the \( \theta- \) direction, those and external force can be expanded in the Fourier series

\[
u(x, \theta, t) = \sum_{n=0}^{\infty} v_n(x) \sin n\theta e^{j\omega t}, \tag{3b}
\]

\[
w(x, \theta, t) = \sum_{n=0}^{\infty} w_n(x) \cos n\theta e^{j\omega t}, \tag{3c}
\]

\[
p(x, \theta, t) = \sum_{n=0}^{\infty} p_n(x) \cos n\theta e^{j\omega t}, \tag{3d}
\]

where \( n \) is the expansion order and \( u_n, v_n, w_n \) and \( p_n \) are the \( n- \) th expansion coefficients. Substituting the vibration displacements and external force given above into Eqs. (1a), (1b) and (1c), and using the orthogonality of trigonometric functions, we get the following equations

\[\text{Fig. 1 The analysis model of a cylindrical shell.}\]
The other choice is to use Legendre-Gauss-Lobatto nodes, which are the zeros of the derivative of a Legendre polynomial \( L_{\alpha}(x) \). Since the zeros cannot be expressed explicitly, an iteration method such as Newton’s method must be used to obtain their values. Neither of two types of nodes is always better than the other and their accuracies depend on problems [11]. Then \( u_n(x) \) is approximated by the following \( M \)-th order Hermite interpolating polynomial with values \( w_{ni} \) at all nodes, derivatives \( w'_{n0} \) and \( w'_{nM} \) at boundary nodes,

\[
w_n(x) \approx w'_{n0}H_0(x) + \sum_{i=0}^{M} w_{ni}H_i(x) + w'_{nM}H_M(x),
\]

where \( H_0(x) \), \( H_i(x) \) and \( H_M(x) \) are Hermite basis functions [9]. Differentiating Eq. (8) with respect to \( x \) and substituting \( x = x_0, x_1, \ldots, x_M \) into that, an Hermite differentiation matrix \( D_H \) is obtained. The entries of \( D_H \) are written in the appendix. Using \( D_H \), derivatives of the vibration displacements at nodes are expressed as follows:

\[
\{w'_n\} = [D_H]\{w_n\},
\]

where

\[
\{w_n\} = \{w'_{n0} \ w_{n0} \ w_{n1} \ \cdots \ w_{nM} \ w'_{nM}\}^T, \quad (10)
\]

\[
\{w'_n\} = \{w'_{n0} \ w_{n0} \ w'_{n1} \ \cdots \ w'_{nM} \ w_{nM}\}^T. \quad (11)
\]

The number of rows/columns of the Hermite differentiation matrix is larger than the number of nodes by two. Therefore two boundary conditions can naturally be imposed at each boundary node.

### 2.3. Matrix Equation and Boundary Condition

In this section, a matrix equation governing the vibration of a cylindrical shell is constructed with differentiation matrices of spectral collocation methods. Equations (4a) and (4b) are second-order differential equations with respect to \( u_n \) and \( v_n \), thus a conventional Lagrange differentiation matrix, which we describe as \( D_L \), is applied to the terms of \( u_n \) and \( v_n \). The entries of the Lagrange differentiation matrix are given by [1]

\[
(D_L)_{ij} = \frac{a_i}{a_i(x_i - x_j)}, \quad i \neq j,
\]

\[
(D_L)_{ij} = \sum_{k=0}^{M} (x_j - x_k)^{-1},
\]

where

\[
a_j = \sum_{k=0 \ k \neq j}^{M} (x_j - x_k).
\]

Applying the differentiation matrices \( D_H \) and \( D_L \) to Eqs. (4a), (4b) and (4c) yields their discretized version, as a matrix equation expressed as follows:

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
w_n \\
v_n \\
w_{nK}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\{p_n\}
\end{bmatrix},
\]

where

\[
L_{11} = -D^2_L + \frac{(1 - v)\eta^2}{2a^2} I,
\]

\[
L_{ij} = -D_{ij}^2 + \frac{(1 - v)\eta^2}{2a^2} I, \quad i \neq j.
\]
It should be noted that an entry $L_{33}$ includes different types of differentiation matrices. Actually, that is a necessary operation to eliminate spurious eigenvalues. It is known that physically spurious eigenvalues occur when solving fourth-order differential equations with spectral methods. A strategy to avoid the phenomenon is to use polynomials of different degrees [2,12]. For this reason, a displacement strategy to avoid the phenomenon is to use polynomials of fourth-order differential equations with spectral methods. A boundary nodes in the equations above can be calculated using the first or last row of the Lagrange differentiation matrix. For instance, a derivative $(\partial u_n/\partial x)_{x=x_0}$ at node $x_0$ are expressed as
\[ \left( \frac{\partial u_n}{\partial x} \right)_{x=x_0} = [(D_l)_{0a}](u_n) \]
where $(D_l)_{0a}$ is the first row of the Lagrange differentiation matrix. The first and $(M+1)$-th equations of Eq. (15) should be replaced by Eq. (17). Similarly, the $(M+2)$-th and $(2M+2)$-th equations should be replaced by Eq. (18). In regard to $w$, the $(2M+3)$-th, $(2M+4)$-th and last two equations of Eq. (15) should be replaced by Eqs. (19a) and (19b).

3. NUMERICAL RESULTS

3.1. Verification of Accuracy in Eigenvalue Problem

In this section, the eigenvalue problem of a cylindrical shell is solved by the proposed spectral collocation method and the accuracy is compared with that of the FEM with axisymmetric shell elements. We calculated the eigen frequencies of a cylindrical shell with both ends simply supported, whose length $l = 2.0 \text{ m}$, radius $a = 1 \text{ m}$, thickness $h = 0.001 \text{ m}$, Young’s modulus $E = 205 \text{ GPa}$, density $\rho = 7,900 \text{ kg/m}^3$ and Poisson’s ratio $\nu = 0.3$. The simply supported boundary conditions are mathematically expressed as
\[ N_{xx,n} = v_n = w_n = M_{xx,n} = 0 \text{ at } x = 0, l. \]

The number of considered eigen modes was 20 in each of the axial and circumferential directions, i.e., 400 eigen frequencies in total were calculated. Then the averaged relative error of them $\varepsilon$ was calculated by
\[ \varepsilon = \frac{1}{400} \sum_{m=0}^{19} \sum_{n=1}^{20} \frac{|f_{mn} - \tilde{f}_{mn}|}{|f_{nm}|}, \]
where $f_{mn}$ and $\tilde{f}_{mn}$ are exact eigen frequencies and ones calculated by the proposed spectral collocation method or the FEM, respectively.

Figure 2 shows the averaged relative errors of eigen frequencies calculated by three methods, the spectral collocation method with Chebyshev nodes, the spectral collocation method with Legendre-Gauss-Lobatto nodes, and the FEM with axisymmetric shell elements. The errors of the spectral collocation methods rapidly decrease with the number of unknowns increased, while that of the FEM slowly decrease. The error of Chebyshev nodes is less than that of Legendre-Gauss-Lobatto nodes but the difference is small. In practice, Chebyshev nodes are
easier to use because they are explicitly calculated with Eq. (7).

Figure 2 indicates that, when the number of unknowns is less than 70, the errors of the spectral collocation methods are larger than that of the FEM. This feature is attributed to higher eigen frequencies, as is described below. Figures 3(a) and 3(b) display the convergence of the relative errors of (0, 1) and (0, 20) eigen frequencies, respectively. One can see that, for the spectral collocation methods, the number of unknowns for (0, 20) eigen frequency at which the errors start converging is larger than that for (0, 1) eigen frequency; the errors of (0, 20) eigen frequency are larger than that of the FEM at a small number of unknowns.

The relative errors of each mode number (each eigen frequency) are shown in Figs. 4(a) and 4(b) at the numbers of unknowns of 48 and 192, respectively. The horizontal axes of the figures display axial direction mode number. Figure 4(a) demonstrates that the errors of the spectral collocation methods are much less than those of the FEM at lower mode numbers, i.e., lower eigen frequencies. However, those of the spectral collocation methods are larger than the FEM at higher eigen frequencies. Figure 4(b) indicates that higher eigen frequencies of the spectral collocation methods as well as lower eigen frequencies converge. On the other hand, the trend of the error of the FEM is almost unchanged for 48 and 192 unknowns. Based on this result, it can be said that the spectral collocation method, interestingly, puts out the errors to higher frequencies for reducing the errors of lower frequencies. The similar property of spectral collocation methods is reported in [5,13]. Such interesting property may result from Chebyshev nodes and Legendre-Gauss-Lobatto nodes, which are unevenly spaced nodes generally used in spectral collocation methods. It is also described in the literatures [5,13] that intermediate nodes between those unevenly spaced nodes and equispaced nodes generally used in FEMs and a mapping technique can distribute the errors evenly from low order modes to high order modes. Using that technique, the accuracy of high order modes improves. However, the accuracy of low order modes deteriorates. We recommend adopting that technique depending on a situation; whether one needs very high accuracy at only a few low order modes, or needs acceptable accuracy from low to high order, many modes.

3.2. Verification of Accuracy in Forced Vibration Problem

Next, the forced vibration problem of a cylindrical shell is solved. The cylindrical shell that has the same parameters as the previous section is driven by an external force whose spatial distribution is given by

$$p(x, \theta) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-x_s)^2}{2\sigma^2}\right) \delta(\theta - \theta_s).$$  

The standard variation $\sigma = 0.1$ and excitation point $(x_s, \theta_s) = (1.9, 0)$ were provided, then the distribution of
vibration displacements using 199 receiving points on the line \( \theta = 0 \) was calculated. The maximum order of the Fourier series of the proposed spectral collocation method, FEM and analytical solution was set to 20.

Here, if averaged relative errors of the distribution are calculated with expression like Eq. (26), they may not be evaluated correctly in the case that an exact solution is nearly equal to zero. Therefore we adopted the discrete relative \( L_2 \) error defined as

\[
\varepsilon = \sqrt{\frac{\sum_{j=0}^{J} [w(x_j) - \tilde{w}(x_j)]^2}{\sum_{j=0}^{J} w^2(x_j)}},
\]

(28)

where \( w(x_j) \), \( \tilde{w}(x_j) \) and \( J (= 199) \) are exact solutions at the \( j \)-th receiving point, numerical solutions at the \( j \)-th receiving point and the total number of receiving points, respectively.

Figures 5(a), 5(b) and 5(c) show the discrete relative \( L_2 \) errors of the spectral collocation method and the FEM at frequencies of 125 Hz, 250 Hz and 500 Hz, respectively. Chebyshev nodes were adopted as nodes of the spectral collocation method. As in the eigenvalue problem, the proposed spectral collocation method can offer more accurate solutions than the FEM for the forced vibration problem. While the FEM needs more than 10,000 unknowns to obtain a relative \( L_2 \) error of \( 10^{-3} \) at these frequencies, the proposed method can offer equivalent accuracies with only about 100 unknowns.

Although the proposed spectral collocation method can obtain highly-accurate solutions at a small number of

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**Fig. 4** Relative errors of each mode number.

**Fig. 5** Relative errors of displacement distribution.
unknowns, comparison of only the number of unknowns is not sufficient. Since the coefficient matrix of the proposed method is dense and not symmetry, sparse solvers, which have been developed to efficiently solve sparse matrices appearing in FEMs, cannot be applied. Thus, it is reasonable that the computation time and required memory of the proposed method not using any sparse solvers are compared with those of the FEM using a sparse solver. Sparse solvers have their variations of direct method and iterative method. It has been mentioned in a previous paper that, for problems of shells, iterative methods fail to converge or are too slow to obtain competitive solutions with direct methods because coefficient matrices of shell elements are very ill-conditioned [14]. Some optimization of a preconditioning technique and parameters are necessary for applying iterative methods to matrices of shell elements. We consider that such an issue is beyond the scope of this study. In this paper, only a sparse solver of direct type is used as the solver for the FEM.

The matrix equations were solved with the commercial numerical library NAG. The matrix equation of the proposed spectral collocation method was solved with the LU factorization. For the matrix equation of the FEM, the approximate minimum degree column ordering algorithm COLAMD [15], which is an ordering technique considering sparsity, was executed before the LU factorization. All the calculations were executed on a workstation with Intel Xeon 2.00 GHz CPU.

Figures 6(a), 6(b) and 6(c) show relations between computation time and relative $L_2$ errors at frequencies of 125 Hz, 250 Hz and 500 Hz, respectively. The numbers of unknowns correspond to Figs. 5(a), 5(b) and 5(c). The number of data displayed in Figs. 6(a), 6(b) and 6(c) of the proposed spectral collocation method is fewer than that displayed in Figs. 5(a), 5(b) and 5(c). It is because the computation time at a small number of unknowns is less than the time resolution of measurement thus such computation times are omitted. For instance, if one needs to suppress a relative error to about $10^{-3}$ at a frequency of 125 Hz, the calculation of the proposed method finishes instantly while the FEM requires approximately 100 s to obtain the same degree of accuracy. It is observed that the proposed spectral collocation method can obtain high accuracy in much less time than the FEM also at frequencies of 250 Hz and 500 Hz.

Additionally, required memories for each method are shown in Figs. 7(a), 7(b) and 7(c). As in the case with the computation time, the numbers of unknowns in Figs. 7(a), 7(b) and 7(c) correspond to Figs. 5(a), 5(b) and 5(c), respectively. Required memory means the size of memory to store the entries of coefficient matrix. For the spectral collocation method, all entries of the coefficient matrix are stored as a usual two-dimensional array. Therefore the size of required memory for the spectral collocation method increases almost at a rate of square of the number of unknowns. On the other hand, for the FEM, only nonzero entries and the indices are stored. In the figures, required memories after factorization of the matrix are also shown because it changes the structure of the matrix and causes fill-in.
Note that, both the proposed spectral collocation method and FEM have already reduced the dimension of the problem due to the semi-analytical technique using the Fourier series expansion in the $C_1$-direction. Thus, the size of required memory is basically small. Each result in Figs. 7(a), 7(b) and 7(c) shows that the proposed spectral collocation method is efficient compared to the FEM, at frequencies of 125 Hz, 250 Hz and 500 Hz. However, the differences between the proposed spectral collocation method and FEM become smaller as the frequency is raised. As for memory requirement, the proposed spectral collocation method is probably inferior to the FEM in the higher frequency range.

3.3. Forced Vibration Problem of a Cylindrical Shell with Both Ends Clamped

Finally, a numerical example of a cylindrical shell with both ends clamped, which cannot be solved analytically, is shown. The clamped boundary conditions are mathematically expressed as follows:

$$u_n = v_n = w_n = \frac{\partial w_n}{\partial x} = 0 \text{ at } x = 0, l.$$  \hspace{1cm} (29)

The parameters of a cylindrical shell and external force are the same as the previous section. Since the exact solution in this case cannot be obtained, a relative error cannot be estimated. Thus the numerical solution calculated by the FEM with very large degrees of freedom is considered a reference solution. Suppose we want to suppress the relative error to $10^{-2}$ at a frequency of 250 Hz. From Fig. 5(b) one can estimate that the FEM needs approximately 1,600 unknowns. On the other hand, the proposed spectral collocation method needs only about 60 unknowns.

Figure 8 displays frequency response functions at a receiving point $(x, \theta) = (1.9, \pi)$ calculated by the spectral collocation method and the FEM. The numbers in brackets are the numbers of unknowns used in the calculations. The frequency response function calculated by the proposed spectral collocation method coincides with the reference solution calculated by the FEM with 1,600 unknowns. The error of the frequency response function calculated by the FEM using the same number of unknowns as the proposed method, on the other hand, increases with frequency. It is shown that the proposed method can effectively analyze...
The entries are

\[
\begin{bmatrix}
H_{00}^0(x_0) & H_{00}^1(x_0) & H_{00}^2(x_0) & \cdots & H_{00}^{M-1}(x_0) & H_{00}^M(x_0) \\
1 & 0 & 0 & \cdots & 0 & 0 \\
H_{01}^0(x_1) & H_{01}^1(x_1) & H_{01}^2(x_1) & \cdots & H_{01}^{M-1}(x_1) & H_{01}^M(x_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_{01}^{M-1}(x_{M-1}) & H_{01}^M(x_{M-1}) & H_{01}^{M+1}(x_{M-1}) & \cdots & H_{01}^{2M-1}(x_{M-1}) & H_{01}^{2M}(x_{M-1}) \\
0 & 0 & 0 & \cdots & 0 & 1 \\
H_{00}^0(x_M) & H_{00}^1(x_M) & H_{00}^2(x_M) & \cdots & H_{00}^{M-1}(x_M) & H_{00}^M(x_M)
\end{bmatrix}
\]

Appendix: Entries of Hermite Differentiation Matrix

The entries of an Hermite differentiation matrix can be obtained by differentiating Eq. (8) and substituting \( x = x_0, x_1, \ldots, x_M \) into it. As a result, the Hermite differentiation matrix is written as

\[
H_{00}^0(x_0) = 2(b_{00} - 1/L),
\]

\[
H_{01}^0(x_1) = 2b_{00} + b_{01}/L - 2/L^2,
\]

\[
H_{00}^0(x_M) = 2b_{00}/L - 1/M.
\]
\[ H'_{M1}(x_k) = b_{KM}c_{c_{KM}}/L, \quad (A-8) \]
\[ H'_{00}(x_k) = a_{00}b_{LM}p_0(x_k), \quad (A-9) \]
\[ H'_{M0}(x_k) = a_{KM}b_{LM}p_M(x_k), \quad (A-10) \]
\[ H'_{i0}(x_k) = [(c_{i0} + c_{iM})\delta_{ik} + b_{ki}c_{c_{kiM}}]/c_{i0}c_{iM}, \quad i, k = 1, 2, \ldots, M - 1, \quad (A-11) \]
\[ H'_{01}(x_M) = -2b_{M0}, \quad (A-12) \]
\[ H'_{M1}(x_M) = 2(b_{MM} + 1/L), \quad (A-13) \]
\[ H'_{00}(x_M) = -2b_{M0}p_0(x_M)/L, \quad (A-14) \]
\[ H'_{i0}(x_M) = b'_{MM} - 2b^2_{MM} - 2b_{MM}/L - 2/L^2, \quad (A-15) \]
\[ H'_{00}(x_M) = 2b_{MM}/c_{i0}c_{iM}, \quad (A-16) \]

with
\[ i = 1, 2, \ldots, M - 1, \quad (A-16) \]
\[ L = x_M - x_0, \quad (A-17) \]
\[ a_{i0} = -(x_i - x_M)/L, \quad (A-18) \]
\[ a_{iM} = (x_i - x_M)/L, \quad (A-19) \]
\[ c_{ij} = x_i - x_j, \quad (A-20) \]
\[ p_0(x) = 1 - (b_{00} - 1/L)(x - x_0), \quad (A-21) \]
\[ p_M(x) = 1 - (b_{MM} + 1/L)(x - x_M), \quad (A-22) \]

where \( b_{ij} \) and \( b'_{ij} \) are the entries of the first order Lagrange differentiation matrix \((D_{L})_{ij}\) (Eq. (12) and Eq. (13)) and the second order Lagrange differentiation matrix.

where \( b_{ij} \) and \( b'_{ij} \) are the entries of the first order Lagrange differentiation matrix \((D_{L})_{ij}\) (Eq. (12) and Eq. (13)) and the second order Lagrange differentiation matrix.