GAUGED HYPERINSTANTONS
AND
MONOPOLE EQUATIONS

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Abstract

The monopole equations in the dual abelian theory of the N=2 gauge–theory, recently proposed by Witten as a new tool to study topological invariants, are shown to be the simplest elements in a class of instanton equations that follow from the improved topological twist mechanism introduced by the authors in previous papers. When applied to the N=2 σ–model, this twisting procedure suggested the introduction of the so-called hyperinstantons that are the solutions to an appropriate condition of triholomorphicity imposed on the maps $q : \mathcal{M} \to \mathcal{N}$ from a four–dimensional almost quaternionic world–manifold $\mathcal{M}$ to an almost quaternionic target manifold $\mathcal{N}$. When gauging the σ–model by coupling it to the vector multiplet of a gauge group $G$, one gets instantonic conditions (named by us gauged hyperinstantons) that reduce to the Seiberg–Witten equations for $\mathcal{M} = \mathcal{N} = \mathbb{R}^4$ and $G = U(1)$. The deformation of the self–duality condition on the gauge–field strength due to the monopole–hyperinstanton is very similar to the deformation of the self–duality condition on the Riemann curvature previously observed by the authors when the hyperinstantons are coupled to topological gravity. In this paper the general form of the hyperinstantonic equations coupled to both gravity and gauge multiplets is presented.

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Recently the non-perturbative results obtained by Seiberg and Witten \[1, 2\] on the infrared behavior of N=2 gauge theories for a gauge group $G$ have attracted a lot of interest. The N=2 supersymmetric pure Yang–Mills theory has a moduli–space of vacua, namely it admits flat directions of the scalar potential, and there has been a concentration of efforts on studying the geometry of this space \[1, 2, 3, 4, 5\]. This is done by considering the effective lagrangian which, if N=2 supersymmetry is preserved, must fall into the general form of an N=2 super Yang–Mills lagrangian for the unbroken gauge subgroup $H \subset G$. This is completely encoded in the choice of a flat special Kähler geometry \[6, 7, 8, 9, 3\], namely into a holomorphic section \{\(X_i(z), \frac{\partial F(X)}{\partial X^i}\)\} of a flat $Sp(\dim H, \mathbb{R})$ bundle, determining the kinetic Kähler metric of the vector multiplet scalars $X^i$ via the formula 

\[
g_{ij} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} \left( X^i \partial_i \mathcal{F}(\bar{X}) + X^j \partial_j \mathcal{F}(X) \right).
\]

The non-perturbative determination of the holomorphic section \{\(X_i(z), \frac{\partial F(X)}{\partial X^i}\)\} is performed by relying on duality considerations that connect the infrared and the ultraviolet regimes, by inverting the strength of the gauge coupling constant and exchanging magnetic with electric charges. The implementation of a discrete group of duality transformations leads to a set of Picard–Fuchs equations for \{\(X_i(z), \frac{\partial F(X)}{\partial X^i}\)\}, that are indeed interpreted as periods of suitable holomorphic forms on the moduli space. This is just analogous to what happens in Calabi–Yau compactifications \[11, 12\].

One starts from a microscopic theory that is a pure N=2 gauge–theory for the group $G$ with the choice, for its flat special geometry, of the minimal coupling $F(X) = \sum_{i=1}^{\dim G} (X^i)^2$ and one arrives at an effective dual theory which is also an N=2 gauge theory displaying, however, the following differences.

i) The gauge group is abelian and it is the dual $\tilde{H}$ of the maximal torus $H \subset G$ in the original gauge group.

ii) The self–interaction of the gauge–multiplet is encoded in a non–minimal flat special geometry possessing a discrete group of duality symmetries.

iii) In addition to the gauge–multiplet the theory contains a certain number of N=2 hypermultiplets that represent the monopoles of the original theory.

The last point in this list of properties is the main issue and motivation of the present letter, in conjunction with a recent suggestion by Witten \[13\].

It is well-known that there is a relation between topological Yang-Mills theory \[14, 13\] and the mathematical problem of calculating Donaldson invariants of four–manifolds \[16, 17\]. As a matter of fact, this has longly been believed to be an equivalence relation, in the sense that Donaldson invariants were thought to be the only physical amplitudes of topological Yang-Mills theory. However, it has been recently shown by one of us \[18\] by explicit solving the theory in the case $M = \mathbb{R}^4$, $G = SU(2)$ and unit instanton number, that certain anomalous behaviors are able to enrich the theory with many nonvanishing amplitudes computing link invariants. In this sense, topological Yang-Mills theory can no longer be considered equivalent to Donaldson theory.

Moreover, in \[18\] computations of a third kind of topological invariants of four manifolds were performed. These are some (again anomalous) physical amplitudes of topological gravity, eventually coupled to topological Yang-Mills theory. These invariants
were constructed in \cite{19,20} in a general context. Finally, a fourth type of topological invariants of four manifolds are those related to the topological $\sigma$-model, constructed in \cite{21}.

On the other hand, topological Yang Mills theory, just as topological gravity \cite{19} and the topological sigma model \cite{20,21}, can be obtained by topologically twisting an $N=2$ gauge–theory. Hence, due to the conjectured equivalence \cite{1,2} between the infrared limit of the $N=2$ gauge–theory of $G$ with its dual $N=2$ theory of the abelian group $\tilde{H}$ (coupled to monopoles), the suggestion of \cite{13} is to recast the problem of calculating topological invariants of whatever type into a dual abelian framework. To this purpose, it is worth considering the topological twist of the dual theory. Although Witten did not phrase his argument exactly in these terms, we think that this is more or less the point. Indeed, the whole idea of \cite{13} is that “rather than computing the Donaldson invariants by counting $SU(2)$ instanton solutions, one can obtain the same invariants by counting the solutions of the dual equations, which involve $U(1)$ gauge fields and monopoles”. As a matter of fact, we think that both the original and the dual gauge theory, once topologically twisted, provide a framework for the calculation of topological invariants that are quite worth consideration. Their relation is possibly still a matter of debate, but certainly those associated with the dual theory are of the utmost interest. It is therefore important, in our opinion, to clarify the general meaning of the monopole equations mentioned in the above quotation from Witten’s paper.

Effective lagrangians are not constrained by power counting renormalizability. On the other hand, topological field theories are finite and exactly soluble \cite{22,23,18}. So, they represent a very general and powerful tool for studying topological invariants of four manifolds with the methods of physics \cite{18}.

What are these monopole equations? In \cite{13}, using considerations on spin–bundles and focusing on the dual of the minimal $SU(2)$ theory, namely on an $N=2 U(1)$ gauge–theory coupled to one hypermultiplet, Witten obtained the following equations:

$$F_{\alpha\beta} = \frac{i}{2}(M_{\alpha} \bar{M}_{\beta} + M_{\beta} \bar{M}_{\alpha}), \quad D_{\alpha\beta'} M^{\alpha} = 0. \quad (1)$$

where $M_{\alpha}, \alpha = 1, 2$ are the two complex scalars belonging to the monopole hypermultiplet, while $F_{\alpha\beta}$ is the antisen–dual part of the $U(1)$ gauge field written in a formalism that uses spinor indices.

In the present letter we want to show that:

i) Eq.s (1) are just the instanton equations gauge–fixing the topological symmetry of the topologically twisted dual $N=2$ theory \cite{19,21}. They are produced in an algorithmic fashion by applying the generalization derived in \cite{19,20,21} of the twisting procedure of \cite{14}.

ii) Eq.s (1) are the specialization to a very simple case, namely to the case where both the world–manifold $M$ and the target–manifold $N$ are flat $\mathbb{R}^4$, the gauge group $G$ is $U(1)$ and gravity is external, of a more general set of three equations:

$$R^{-ab} = -\frac{1}{2} f_{ab} q^u q^u,$$
\[
F_{\Lambda}^{ab} = \frac{g}{2} T_{u}^{ab} \mathcal{P}_{\Lambda}^{u},
\]

\[
\mathcal{D}_{\mu} q^{i} - (j_{u})_{\mu}^{\nu} \mathcal{D}_{\nu} q^{j} (J_{u})_{j}^{i} = 0,
\]

the first being the yield of dynamical gravity, the last two being the appropriate generalization of (1). Eq.s (2) are obtained by the twist procedure of [13, 20, 21] and are the gauged version of eq.s (2.1) and (3.12) of [21].

iii) The interpretation of the third of eq.s (2) as a Dirac equation, as it is done in the second of eq.s (1), is a peculiarity of the flat case \( \mathcal{N} = \mathbb{R}^{4} \).

The rest of this letter is devoted to prove the above statements and to explain the symbols appearing in eq.s (2).

As just stated, the puzzling feature of conditions (1) is that scalar fields satisfy a Dirac-type equation. However, as it was shown in [21], the second of eq.s (1) has a different geometrical meaning. In the most general case, the scalars \( M_{a} \) (from now on to be denoted by \( q^{i} \), \( i = 1, \ldots, 4 \) in real notation) describe a sigma model, \( q : \mathcal{M} \to \mathcal{N} \), mapping an almost quaternionic four dimensional manifold \( \mathcal{M}^{4} \) into an almost quaternionic target manifold \( \mathcal{N} \). One can define a condition of triholomorphicity for the map \( q \), that arises naturally from the topological twist of N=2 hypermultiplets [21]. For this reason, the solutions to that condition were named by us hyperinstantons (= instantons of the hypermultiplets). In the simplest case, namely \( \mathcal{M} = \mathcal{N} = \mathbb{R}^{4} \), the instantonic equations reduce to the so called Cauchy-Fueter equations [24], that, after introducing a quaternionic number notation, can be written in a form that resembles the usual holomorphicity condition [21]

\[ \bar{\partial} q = 0. \]  

It was well-known from the literature [24], that these equations can also be written as a Dirac equation on a spinor of definite chirality. However, it was pointed out in [21] that this is more a coincidence and does not correspond to the most significative nature of the equations\(^3\). What turned out to be the most significative interpretation of the equations under consideration is that they are a triholomorphicity condition on the map, as already recalled.

In [21] the most general solution when \( \mathcal{M} = T^{4} \) and \( \mathcal{N} = T^{4}' \) are both four-tori was found and the corresponding topological \( \sigma \)-model was solved, leading to a meaningful partition function, which turns out to be just a \( \theta \)-function. This \( \theta \)-function is characterized by a genus \( g \), being integer valued between 0 and 12. \( g \) measures the degree of commensurability of the two tori. \( g \) is a very nontrivial function on the \( T^{4} \)-moduli space, and it would be very interesting to know it better. In [21] examples with \( g = 0 \), \( g = 12 \) and \( 0 < g < 12 \) where exhibited.

\(^3\)We recall that this is not a restrictive requirement, since any four dimensional Riemannian manifold is almost quaternionic.

\(^4\)Actually, in the flat case there are analogies with other equations of a known type. For example, (3) can be written as the self-duality condition on the field strength of an abelian gauge-field in the Lorentz gauge. Such misleading similarities explain why the generalization of these equations given in [21] was not straightforward.
The reason why a Dirac-like equation makes its appearance was also stressed in [21]: it is the equation for the infinitesimal deformations of the triholomorphic map \(q\) rather than the equation defining the map \(q\) itself. This happens because the deformations of the map are the topological ghosts, that, via topological twist, come from the fermions of the hypermultiplet (the hyperini). Of course, the field equations of such fermions are Dirac-type equations. With a flat target manifold, the triholomorphicity condition is linear in the map \(q\), so that the equations of the deformations of the map have the same form as the equations of the map itself, and that is why they resemble the Dirac equation. Thus, the second of eqs (1) looks like the gauged version of the hyperinstanton equations. Therefore, the solutions to this type of equations will be from now on named gauged hyperinstantons.

It remains to explain the first of eqs (1), that looks more mysterious. Actually, it is much less mysterious if we recall one more result obtained in [21]. There it was shown that when hyperinstantons are coupled to dynamical gravity, they modify the equation \(R^{\mu\nu} = 0\) of gravitational instantons as follows

\[
R^{\mu\nu} = -\frac{1}{2} I^u_{ab} q^* \Omega^u,
\]

which is the generalization of the self-duality condition on the Riemann tensor. Here \(I^u_{ab}, u = 1, 2, 3\) is a triplet of \(4 \times 4\) antiselfdual matrices \((I^u_{ab} = I^u_{ab}^-)\) satisfying the quaternionic algebra (see formula (5)). \(\Omega^u\) will be defined below. If the target manifold is four dimensional, one can also write \(R^{\mu\nu} = \tilde{R}^{\mu\nu}\), where the tilded 2–form is the pull-back of the corresponding target 2–form.

This remark suggests that when hyperinstantons are coupled to Yang-Mills fields, instead of gravity, the Yang-Mills instanton condition \(F^{\mu\nu} = 0\) should be modified in a similar way. Thus, the first of eqs (1) is the gauge-analogue of the modification (1) that is present in the gravitational case. It is clear from the argument developed so far that the most general form of the first of conditions (1) can be found by repeating the twisting exercise of ref. [21] in the gauged case. The rest of this letter is devoted to derive the most general form (2) of the gauged hyperinstanton equations and to show that they reduce to (1) in the simplest case. The interpretation offered here also provides expressions of the topological observables of the theory, since the topological field theory encoded in (1) is nothing else but a particular case of the known topological models (see [21] for more details).

By definition, \(\mathcal{N}\) possesses an almost quaternionic structure, namely three locally

\[5\]In Minkowskian notation, we use \(M^{-ab} = \frac{1}{2} (M^{ab} + \frac{i}{2} \varepsilon^{abcd} M^{cd})\), while in Euclidean notation we use \(M^{-ab} = \frac{1}{2} (M^{ab} - \frac{i}{2} \varepsilon^{abcd} M^{cd})\).

\[6\]We recall that the twisting procedure of ref. [21], which was firstly defined in [20], is a nontrivial generalization of that of [14], since the procedure of [14] could not work on hypermultiplets. In particular, in [21] it was shown that one also has to identify \(SU(2)_L\) with a suitable \(SU(2)\) subgroup of the Lorentz group of the target manifold \(\mathcal{N}\).
defined\(^7\)(1, 1)-tensors \(J^u\), \(u = 1, 2, 3\), satisfying the quaternionic algebra

\[
J^u J^v = -\delta^{uv} + \varepsilon^{uz} J^z.
\]  

Moreover, \(\mathcal{N}\) is endowed with a metric \(h_{ij}\) that is by assumption Hermitean with respect to the almost quaternionic (1, 1)-tensors \(J^u\). One can introduce the generalized Kähler forms

\[
\Omega^u = \lambda h_{ik} (J^u)_j^k dq^i \wedge dq^j. 
\]  

In particular, if the \(J_u\) are globally defined and covariantly constant complex structures then the target manifold \(\mathcal{N}\) is a hyperKähler manifold. In that case, the forms \(\Omega^u\) are closed:

\[
d\Omega^u = 0. 
\]  

On the other hand, if \(\mathcal{N}\) is a quaternionic Kähler manifold, there exist three one-forms \(\omega^u\) that make an \(SU(2)\) connection, with respect to which the forms \(\Omega^u\) are covariantly closed and such that \(\Omega^u\) is the field strength of this connection. To say it in formulæ, we have:

\[
d\Omega^u + \varepsilon^{uz} \omega^u \wedge \Omega^z = 0, \quad d\omega^u + \frac{1}{2} \varepsilon^{uz} \omega^v \wedge \omega^z = \Omega^u. 
\]  

Any quaternionic Kähler manifold is an Einstein manifold. Then, in (3) \(\lambda\) is a real constant that is related to the cosmological constant of \(\mathcal{N}\). When the limit \(\lambda \to 0\) is taken in an appropriate way \cite{21}, then one can go from the quaternionic Kähler to the hyperKähler case.

As far as \(\mathcal{M}\) is concerned, since it is four dimensional, it is sufficient to have a metric \(g_{\mu\nu}\) to endow it with an almost quaternionic structure, as we already recalled. Explicitly, we have

\[
(j_u)_\mu^\nu = (j_u)^{ab} e_{a\mu} e_{b\nu}, 
\]  

e\(_{\mu}\) being the vierbein. The instantons as derived from the topological twist of \cite{20, 21} are given by the following condition on the world metric:

\[
\omega^{-ab} = -\frac{1}{2} j_u^{ab} q^* \omega^u, 
\]  

that is equivalent to (4) in a suitable local Lorentz frame\(^8\), plus the following equations on the map \(q: \mathcal{M} \to \mathcal{N}\),

\[
e^\mu[a e_{i}^{0}b]^{\pm \kappa} \partial_\mu q^i = 0, \quad e^\mu_{a} E_{i}^{ak} \partial_\mu q^i = 0, 
\]

\(^7\)It means that the almost quaternionic (1, 1)-tensors are defined on neighborhoods \(U(\alpha)\) such that on the intersection \(U(\alpha) \cap U(\beta)\) of two neighborhoods the transition functions are \(SO(3)\) matrices \(\Lambda_{uv}^{w}\) \cite{25}.

\(^8\) (10) is the form of (4) as obtained by the twist. It is well-defined only if \(\mathcal{N}\) is quaternionic Kähler (or hyperKähler, in which case the right hand side of (10) is zero). On the other hand, (4) is well-defined in the most general case.
meaning antisymmetrization and self–dualization in the indices $a, b$. The vielbein $E^i_{ab}$ of the target manifold $N$, defined so that $h_{ij} = 2E^i_{ak}E^k_{bj}$, has a Lorentz index that is split into $(a, k)$, $a$ being identified with the Lorentz index of the world manifold and $k$ being an extra index ranging from 1 to $n$, if $\dim N = 4n$. This is the effect of the topological twist of \cite{21}. As a matter of fact, written in this form, (11) are not sufficiently explicit. Introducing the inverse vielbein $E^i_{ak}(E^i_{ak}E^k_{bj} = \delta^i_j, E^i_{ai}E^j_{bl} = \delta^i_j\delta^a_b)$ and the almost quaternionic $(1, 1)$ tensors $(J_u)_{ij} = (I_u)_{kj}E^k_{ai}E^j_{bk}$, (11) becomes (see \cite{21} for the details):

\begin{equation}
\partial_\mu q^i - (J_u)_\mu^\nu \partial_\nu q^i(J_u)_{ji} = 0, \tag{13}
\end{equation}
which appears clearly as a generalization of the Cauchy-Riemann equations\cite{10}. These equations are a condition of triholomorphicity of the maps $\mathcal{M} \to \mathcal{N}$ and that is why we named triholomorphic a map $q$ satisfying eq.s (13).

As a matter of fact, the contraction between the indices $u$ of the almost quaternionic structures on the two manifolds can be performed introducing an arbitrary point–dependent $SO(3)$ matrix $\Lambda^u$, since an almost quaternionic structure is defined up to $SO(3)$ matrices:

\begin{equation}
\partial_\mu q^i - \Lambda^{uw}(j_u)_\mu^\nu \partial_\nu q^i(J_u)_{ji} = 0. \tag{14}
\end{equation}

The solutions can be called $\Lambda$-triholomorphic. Some properties of this ambiguity have been also studied in \cite{21} on explicit examples of isometries for $\mathcal{M} = \mathcal{N} = K3$ (to be precise in the realization of $K3$ as a Fermat surface in $\mathbb{C}P_3$ \cite{4}).

It is clear that when identifying indices of the Lorentz groups of two different manifolds, one has to be careful about covariance. The role of (10) is then to relate the spin connections of the two manifolds consistently: one can no more distinguish an index $u$ for $\mathcal{M}$ and one for $\mathcal{N}$; similarly, the corresponding components of the spin connections of $\mathcal{M}$ and $\mathcal{N}$ are identified, so that it is immaterial which one is used in defining the covariant derivative for $u$-indexed tensors. Stated in a different way, (10) is the condition for making $I^a_{ab}$ covariantly constant:

\begin{equation}
D(I^u)_{ab} = d(I^u)_{ab} - \omega^{-ac}(I^u)^{cb} + \omega^{-bc}(I^u)^{ca} + \varepsilon^{uzq}q^*(\partial_U(I^u))_{ab} = 0. \tag{15}
\end{equation}

The purpose, now, is to gauge the hyperinstanton equations. So, suppose that the target manifold $\mathcal{N}$ admits Killing vectors

\begin{equation}
k_\Lambda(q) = k^i_\Lambda(q)\partial_i, \quad [k_\Lambda, k_\Sigma] = -f^\top_{\Lambda\Sigma}k_\Gamma, \tag{16}
\end{equation}

\footnote{Notice that when in (10) the duality is $-$, then in (11) it is necessarily $+$. This is a consequence of a $U(1)$ symmetry discovered in \cite{20}. It makes the improved topological twist meaningful, since it defines the new ghost number. It is called R-duality and generalizes the R-symmetry to supergravity.}

\footnote{The number of independent conditions contained in (13) is equal to $\dim \mathcal{N}$, as it must be. This follows from a duality condition satisfied identically by the matrix $H^i_\mu = \partial_\mu q^i - (j_u)_\mu^\nu \partial_\nu q^i(J_u)_j^i$, namely $H^i_\mu + \frac{1}{3}(j_u)_\mu^\nu H^j_\nu(J_u)_j^i = 0$ \cite{21}.}
of a certain Lie algebra $\mathcal{G}$ with structure constants $f^I_{\Lambda\Sigma}$. Then, one can introduce the covariant derivatives
\[ \mathcal{D}q^i = dq^i + gA^\Lambda k^i_\Lambda(q) \] (17)
and replace (11) with
\[ e^{\mu[a}E^{b]}_{i}^{j}D_{\mu}q^{i} = 0, \quad e_{\alpha}^{i}E_{i}^{\alpha k}D_{\mu}q^{i} = 0, \] (18)
or, equivalently, (13) with
\[ \mathcal{D}_{\mu}q^{i} - (j_a)_{\mu}^{\nu}D_{\nu}q^{j}(J_a)^{i}_j = 0. \] (19)
(18) is in agreement with the topological twist of [20, 21] when applied to the gauged N=2 supersymmetric $\sigma$-model [8]. The solutions $q$ to (19) can be called gauged triholomorphic maps. The same twisting procedure provides the generalization of the first equation of (1). Here, we shall consider the most general case, in which gravity is dynamical, so that we shall also find the (straightforward) generalization of (4). Following [8], we see that the gauging is achieved with the replacements
\[ \omega^u \rightarrow \hat{\omega}^u = \omega^u + gA^\Lambda P^u_\Lambda, \]
\[ \Omega^u \rightarrow \hat{\Omega}^u = d\hat{\omega}^u + \frac{1}{2} \varepsilon^{uvwz}\hat{\omega}^v\hat{\omega}^w = \Omega^u_{ij}^qD^i_j \wedge D^j_i + gF^\Lambda P^u_\Lambda, \] (20)
where $P^u_\Lambda$ is the momentum map function, while $F^\Lambda$ is the field strength of the gauge-field,
\[ F^\Lambda = dA^\Lambda + \frac{1}{2}gf_{\Sigma\Gamma}^\Lambda A^\Sigma A^\Gamma. \] (21)
Let us then pause for a moment and recall the important notion of momentum map [25].

**Momentum map for hyperKähler manifolds**

Consider a compact Lie group $G$ acting on a hyperKähler manifold $\mathcal{N}$ of real dimension $4n$ by means of Killing vector fields $X$ that are holomorphic with respect to the three complex structures of $\mathcal{N}$; then these vector fields preserve also the Kähler forms:
\[ \mathcal{L}_X g = 0 \leftrightarrow \nabla_{(\mu}X_{\nu)} = 0, \quad \mathcal{L}_X J^u = 0, \quad u = 1, 2, 3 \]
\[ \Rightarrow \quad 0 = \mathcal{L}_X \Omega^u = i_X d\Omega^u + d(i_X \Omega^u) = d(i_X \Omega^u). \] (22)
Here $\mathcal{L}_X$ and $i_X$ denote respectively the Lie derivative along the vector field $X$ and the contraction (of forms) with it. If $\mathcal{N}$ is simply connected, $d(i_X \Omega^u) = 0$ implies the global existence of three functions $P^u_X$ such that
\[ i_X \Omega^u = -dP^u_X. \] (23)
If $\mathcal{N}$ is not simply connected, the functions $P^u_X$ exist only locally. The $P^u_X$ are defined up to a constant, which can be arranged so as to make them equivariant
\[ XP^u_Y = 2 \Omega^u(X, Y) = P^u_{[X, Y]}. \] (24)
The \( \{P^u_X\} \) constitute then a *momentum map*. This can be regarded as a map \( P : \mathcal{N} \rightarrow \mathbb{R}^3 \otimes \mathcal{G}^* \), where \( \mathcal{G}^* \) denotes the dual of the Lie algebra \( \mathcal{G} \) of the group \( G \). Indeed let \( x \in \mathcal{G} \) be the Lie algebra element corresponding to the Killing vector \( X \); then, for a given \( m \in \mathcal{N} \), the functional \( P^u(m) : x \rightarrow P^u_X(m) \in \mathcal{C} \) is a linear functional on \( \mathcal{G} \). In practice, expanding \( X = X^\Lambda k_\Lambda \) on a basis of Killing vectors \( k_\Lambda \) such that (16) holds, we also have \( P^u_X = X^\Lambda P^u_\Lambda, u = 1, 2, 3 \); the \( P^u_\Lambda \) are the components of the momentum map.

**Momentum map for quaternionic Kähler manifolds**

In the case of a quaternionic Kähler manifold where the three 2–forms \( \Omega^u \) are not closed but just covariantly closed with respect to the \( SU(2) \) connection \( \omega \), then the momentum map also exists but equation (23) is replaced by its \( SU(2) \)–covariant analogue:

\[
i_X \Omega^u = -\nabla P^u_X = -(dP^u_X + \varepsilon^{uvz} \omega^v P^z_X).
\]

(25)

Now \( P_X^u \) are fixed uniquely [24]. They satisfy identically the following modified equivariance condition

\[
P^u_X[X, Y] - 2 \Omega^u(X, Y) + \varepsilon^{uvz} P^v_X P^z_X = 0,
\]

(26)

that generalizes (24). This is proven by showing that, calling \( C^u_{\Lambda \Sigma} \) the left hand side of (26), one has \( \nabla C^u_{\Lambda \Sigma} = 0 \) [8]. Then, the result follows from \( 0 = \nabla^2 C^u_{\Lambda \Sigma} = \varepsilon^{uvz} \Omega^v C^z_{\Lambda \Sigma} \).

Equipped with these results (for more details see [23] and [8]) we can now resume our previous discussion. (26) reads

\[
0 = \Omega^u_{ij} k^i_\Sigma k^j_\Gamma + \frac{1}{2} f^\Lambda_{\Sigma \Gamma} P^u_\Lambda - \frac{1}{2} \varepsilon^{uvz} P^v_\Sigma P^z_\Gamma
\]

(27)

and guarantees the consistency of (20) and (23) [8]. The relevant supersymmetry transformation is the one of the right handed gaugino [8]

\[
s\lambda^I_A = \cdots + \frac{1}{2} F^{-ab}_I - \gamma^{ab} \varepsilon_{AB} C^B + ig(\sigma_u)_A^C \varepsilon_{BC} C^B P^Iu.
\]

(28)

In this equation, the scalars of the vector multiplets have been set to zero, since they become ghosts for the ghosts after the twist. Similarly, the graviphoton has to be set to zero [19]. For this reason, the index \( I \) is the same as the index \( \Lambda \). \( C^B \) are the right handed components of the supersymmetry ghosts. Performing the topological twist formulated in [20], one finds the following instantonic condition:

\[
F^{-ab}_\Lambda = -\frac{g}{2} I^a_u P^u_\Lambda,
\]

(29)

which is the desired generalization of the first of (1). The generalization of (4), on the other hand, is obtained by replacing \( \Omega^u \) with \( \hat{\Omega}^u \). Summarizing, the complete set of equations for the gauged hyperinstantons are given by eq.s (2), the first of which can be also expressed, when \( \mathcal{N} \) is quaternionic Kähler, in the form

\[
\omega^{-ab} = -\frac{1}{2} I^a_u q^*_u \hat{\omega}^u.
\]

(30)
The total ghostless twisted lagrangian of the most general N=2 theory\(^{11}\) is:

\[
\mathcal{L} = \varepsilon_{abcd} R_{ab} e^c e^d - \frac{1}{6} \lambda g_{\mu\nu} h_{ij} D_\mu q^i D_\nu q^j \varepsilon_{abcd} e^a e^b e^c e^d - \frac{1}{12} (F^a_\Lambda F^{ab}_\Lambda + 2 g^2 P_u^u \mathcal{P}_u^u) \varepsilon_{abcd} e^a e^b e^c e^d,
\]

(31)

the last term being the scalar potential. A crucial test for conditions (19), (29) and (30) is to show that \(\mathcal{L}\) can be written as the sum of their squares plus a topological term and a total derivative, namely

\[
\mathcal{L} = 4 i \left( \omega^{-ab} + \frac{1}{2} I_u q^* \hat{\omega}^u \right) \wedge \left( \omega_{-ac} + \frac{1}{2} (I_v)_{ac} q^* \hat{\omega}^v \right) e^b e^c
-
\frac{\lambda}{24} g^{\mu\nu} h_{ij} (D_\mu q^i - (j_u)_{\mu} q^i) (D_\nu q^j - (j_v)_\nu q^j) \varepsilon_{cdef} e^d e^e e^f
-
\frac{1}{6} \left( F^a_\Lambda F^{ab}_\Lambda + \frac{g}{2} I_u \mathcal{P}_u^u \right)^2 \varepsilon_{cdef} e^d e^e e^f
-
i F_a F^a - 4 i d \left( \omega^{-ab} + \frac{1}{2} I_u q^* \hat{\omega}^u \right) e^a e^b.
\]

(32)

This proves that the solutions to (2) are solutions to the Einstein-Yang-Mills-matter-coupled field equations. Notice that the last total derivative term of (32) is zero for any hyperinstanton. On the solutions of (2),(30) the action \(S\) is simply

\[
S = -i \int_M \text{tr} [F \wedge F],
\]

(33)
i.e. the Pontrjagin number of the gauge bundle.

The observables encoding the meaningful topological invariants will be not written down explicitly, due to lack of space. They are the observables of the \(\sigma\)-model\([21]\), topological gravity\([19]\) and topological Yang-Mills theory\([14,18]\), coupled together (see\([18]\) for explicit examples of nontrivial couplings).

One can formally turn to the case when supersymmetry is global (\(\mathcal{N}\) hyperKähler) by performing the following replacements\([21]\):

\[
A^\Lambda \rightarrow \lambda^{\frac{1}{2}} A^\Lambda, \quad g \rightarrow \lambda^{-\frac{1}{2}} g, \quad \Omega^u \rightarrow \lambda \Omega^u, \quad \omega^u \rightarrow \mathcal{O}(\lambda), \quad \mathcal{P}_u^u \rightarrow \lambda \mathcal{P}_u^u,
\]

(34)

and simplifying \(\lambda\) wherever possible. At the end one puts \(\lambda = 0\). In this way (8), (23) and (24) become (7), (23) and (24), respectively. The first of (2) becomes \(R^{-ab} = 0\), so that \(\mathcal{M}\) is also hyperKähler. Finally, \(\hat{\Omega}^u\) become closed, \(d\hat{\Omega}^u = 0\). The lagrangian

\[
\mathcal{L} = -\frac{1}{12} (2 g^{\mu\nu} h_{ij} D_\mu q^i D_\nu q^j + F^a_\Lambda F^{ab}_\Lambda + 2 g^2 P_u^u \mathcal{P}_u^u) \varepsilon_{cdef} e^d e^e e^f
\]

(35)

\(^{11}\)This the bosonic lagrangian with the graviphoton and the scalar fields of the vector multiplets equated to zero, since after the twist they become ghost fields.
can then be written as
\[ L = \frac{1}{24} g^\mu \nu h_{ij} \left( D_\mu q^i - (j_u)_{\mu} ^{\rho} D_\rho q^k (J_u)_{k} ^{i} \right) \left( D_\nu q^j - (j_v)_{\nu} ^{\sigma} D_\sigma q^l (J_v)_{l} ^{j} \right) \varepsilon_{cdef} e^c e^d e^e e^f \\
- \frac{1}{6} \left( F_A ^{-ab} + \frac{g}{2} I_u ^{ab} P_A \right) ^2 \varepsilon_{cdef} e^c e^d e^e e^f - i F_A F_A - 2i \Theta^u \hat{\Omega}^u, \]
(36)
where \( \Theta^u = I_u ^{ab} e^a e^b \) are the Kähler forms of \( M \). The last term of (36) is also a topological invariant, like in [21], since both \( \Theta^u \) and \( \hat{\Omega}^u \) are closed. Consequently, (35) is minimized by the second and the third conditions of (2):
\[ F_A ^{-ab} + \frac{g}{2} I_u ^{ab} P_A = 0, \quad D_\mu q^i - (j_u)_{\mu} ^{\nu} D_\nu q^j (J_u)_{j} ^{i} = 0. \]
(37)
From these equations, we can now retrieve eq.s (1) explicitly, choosing \( M = N = \mathbb{R}^4 \) and \( G = U(1) \). Let us consider a Killing vector of the form
\[ k(q) = M_{ij} q^i \frac{\partial}{\partial q^j} = k^i \frac{\partial}{\partial q^i}, \]
(38)
\( M \) being a to-be-determined \( 4 \times 4 \) constant matrix. We have \( \Omega^u = I_u ^{ij} dq^i \wedge dq^j \). In Euclidean notation, we choose the matrices \( I_u ^{ab} \) as follows [21]:
\[ I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \]
(39)
Equation (23) gives
\[ \frac{\partial P^u}{\partial q^i} = -2q^k M^{kj} I_j^u. \]
(40)
The integrability condition, i.e. (22), requires \( MI^u \) to be symmetric \( \forall u \). A good \( M \) is any matrix with the opposite duality of \( I^u \), for example,
\[ \bar{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{I} = \bar{I}^+. \]
(41)
Then one finds
\[ P^u = -q^t \bar{I} I^u q, \quad k = q^t \bar{I} \frac{\partial}{\partial q}, \]
\( t \) meaning transposition. The equivariance condition (24) is trivially satisfied. With the same identification as in [21], namely
\[ \psi = \begin{pmatrix} 0 \\ 0 \\ q^4 - iq^3 \\ q^2 - iq^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ M^1 \\ M^2 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
(43)
the second equation of (37) becomes the Dirac equation
\[ \mathcal{D}_\psi = \gamma^\mu D_\mu \psi = \gamma^\mu (\partial_\mu + igA_\mu) \psi = 0. \] (44)

Instead, the first of (37) becomes
\[ F^-_{12} = \frac{g}{2}([q^4]^2 + (q^3)^2 - (q^2)^2 - (q^1)^2] = \frac{g}{2}(M^1 \bar{M}^1 - M^2 \bar{M}^2), \]
\[ F^-_{13} = g(q^1 q^4 - q^2 q^3) = \frac{g}{2}(M^2 \bar{M}^1 - M^1 \bar{M}^2), \]
\[ F^-_{23} = g(q^1 q^3 + q^2 q^4) = \frac{g}{2}(M^2 \bar{M}^1 + M^1 \bar{M}^2), \] (45)

which is equivalent to the first of (1). Finally, with a triplet \( \bar{I}_u = I_u^+ \) satisfying (3) and \( \mathcal{P}_u = -q^I \bar{I}_u I^u q \), one can consider the case \( \mathcal{M} = \mathcal{N} = \mathbb{R}^4 \), \( G = SU(2) \). Again, (24) is easily checked.

References

[1] E. Witten and N. Seiberg. “Electric–magnetic duality, monopole condensation and confinement in N=2 supersymmetric Yang–Mills theory”. Nucl. Phys., B426:19, (1994).

[2] E. Witten and N. Seiberg. “Monopoles, duality and chiral supersymmetry breaking in N=2 QCD”. hep-th/9408013: (1994).

[3] A. Ceresole, R. D’Auria, and S. Ferrara. “On the geometry of moduli space of vacua in N=2 supersymmetric Yang–Mills theory”. CERN-TH 7384/94 POLFIS-TH, 07/94: (1994).

[4] A. Klemm, W. Lerche, S. Yankielowicz, and S. Theisen. “Simple singularities and N=2 supersymmetric Yang–Mills theory”. hep-th/9411048: (1994).

[5] P. Argyres and A. Faraggi. “The Vacuum structure and spectrum of N=2 supersymmetric SU(N) gauge theory”. hep-th/9411057: (1994).

[6] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit, and L. Girardello. “Vector multiplets coupled to N=2 supergravity: superHiggs effect, flat potentials and geometric structures”. Nucl. Phys., B250:385, (1985).

[7] L. Castellani, R. D’ Auria, and S. Ferrara. “Special Kähler geometry: an intrinsic formulation from N=2 spacetime supersymmetry”. Phys. Lett., 241B:57, (1990).

[8] R. D’Auria, S. Ferrara, and P. Fré. “Special and quaternionic isometries: general couplings in N=2 supergravity and the scalar potential”. Nucl. Phys., B359:705, (1991).
[9] A. Strominger. “Special Geometry”. Comm. Math. Phys., 133:163, (1990).

[10] B. de Wit and A. Van Proeyen. “Special Geometry, cubic polynomials and homogeneous quaternionic spaces”. Comm. Math. Phys., 149:307, (1992).

[11] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes. “A pair of Calabi–Yau manifolds as an exatly soluble superconformal theory”. Nucl. Phys., B359:21, (1991).

[12] A. Ceresole, R. D’ Auria, S. Ferrara, W. Lerche, and J. Louis. “Picard Fuchs equations and special geometry”. Int. Jour. Mod. Phys., 8:79, (1993).

[13] E. Witten. “Monopoles and four manifolds”. IASSNS-HEP-9496 hep-th/9411102: (1994).

[14] E. Witten. “Topological quantum field theories”. Comm. Math. Phys., 117:353, (1988).

[15] L. Baulieu and I.M. Singer. “Topological Yang Mills theory”. Nucl. Phys. B (proc. suppl.), 5B:12, (1988).

[16] S.K. Donaldson. “An application of gauge theories to the topology of four manifolds”. J. Diff. Geom., 18:269, (1983).

[17] M.F. Atiyah and L. Jeffrey. “Topological lagrangians and cohomology”. J. Diff. Geom, 7:119, (1990).

[18] D. Anselmi. “Anomalies in Instanton Calculus”, preprint HUTP-94/A040 and hepth 9411049.

[19] D. Anselmi and P. Fré. “Twisted N=2 supergravity as topological gravity in four dimensions”. Nucl. Phys., B392:401, (1993).

[20] D. Anselmi and P. Fré. “Topological twist in four dimensions, R-duality and hyper-instantons”. Nucl. Phys., B404:288, (1993).

[21] D. Anselmi and P. Fré. “Topological sigma models in four dimensions and triholomorphic maps”. Nucl. Phys., B416:255, (1994).

[22] D. Anselmi. “Removal of divergences with the Batalin-Vilkovisky formalism”. Class. and Quantum Grav. 11:2181, (1994).

[23] D. Anselmi. “More on the subtraction algorithm”. preprint SISSA/ISAS 90/94/EP and hepth 9407023, to appear in Class. and Quantum Grav.

[24] A. Sudbery. “Quaternionic Analysis”. Maths. Proc. Cambridge Philos., 85:199, (1979).

[25] K. Galicki. “A generalization of the momentum mapping construction for quaternionic Kähler manifolds”. Comm. Math. Phys., 108:117, (1987).