Approximations of the self-similar solution for blastwave
in a medium with power-law density variation

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Abstract
Approximations of the Sedov self-similar solution for a strong point explosion in a medium with the
power-law density distribution \( \rho^o \propto r^{-m} \) are reviewed and their accuracy are analyzed. Taylor approximation
is extended to cases \( m \neq 0 \). Two approximations of the solution are presented in the Lagrangian
coordinates for spherical, cylindrical and plane geometry. These approximations may be used for the investiga-
tion of the ionization structure of the adiabatic flow, i.e., inside adiabatic supernova remnants.

1 Introduction
Self-similar (Sedov \([17,18]\)) solutions for the strong point explosion in the uniform medium \( \hat{\rho}^o = \text{const} \) or in a
medium with power-law density distribution

\[
\hat{\rho}^o(\hat{r}) = \hat{\rho}^o(0)r^{-m},
\]

where \( \hat{r} \) is the distance from the center of explosion, are widely used for modelling the adiabatic supernova
remnants, solar flares and processes in active galactic nuclei.

Sedov \([17,18]\) has obtained the exact solution solving the system of hydrodynamic differential equation
on the base of dimensional methods. Independently, Taylor \([19]\) has solved the same task in the case of the
uniform medium numerically and in analytical form approximately. The main Taylor’s idea was to approximate
the fluid velocity variation behind the shock front.

Kahn \([11]\) have proposed the approximation of the Sedov solution in the uniform medium. His technic
consists in approximation of mass distribution inside the shocked region. Using Kahn methodology, Cox &
Franco \([4]\) have built the approximation of the exact solution in the power-law medium \([4]\) for \( m < 2 \). With
the same technic, Cox & Anderson \([3]\) have presented the approximation for description of the shocked region
and blastwave motion in uniform medium of finite pressure.

Ostriker & McKee \([16]\) basing on the virial theorem have given a number of approximations for the fluid
characteristic variation as one- or two-power polinoms.

Hnatyk \([9]\) have proposed to approximate firstly the relation between the Eulerian and Lagrangian coor-
dinates of the flow elements.

In present work, Taylor approximation is written for the medium with power-law density variation \([4]\). Using Hnatyk’s approach, we develop also two approximations of the Sedov solution for power-law medium
with \( m \leq 2 \) in Lagrangian coordinates that is useful for investigations of the nonequilibrium ionization processes
in a shocked plasma, e.g., inside the adiabatic supernova remnants. One of the approximations presented here
bases on the approximate hydrodynamic method for description of the nonspherical strong point explosion in
the medium with arbitrary large-scale nonuniformity developed by Hnatyk & Petruk \([10]\). Therefore, it may
also be considered as additional test on this method.

2 Sedov solution and its approximations

2.1 Sedov solution
If strong \((P_s/P_o \rightarrow \infty)\) point \((R_o/R \rightarrow 0)\) explosion with finite energy \(E_o\) becomes in the point with coordinate
\( \hat{r} = 0 \) in time \( t = 0 \), the blastwave creates and propagates with velocity \( D \) in the ambient medium with density
\( \tilde{\rho}(\tilde{r}) \) (\( P_o \) and \( P_0^o \) are pressure of the shocked gas and the gas of the ambient medium at the shock front position, \( R_o \) is the size of the body exploded, \( R \) is the radius of the blastwave). It is also assumed that injected mass is small and no energy lost from the shocked region during the motion.

Such a task is described by a system of hydrodynamic differential equations. Sedov ([18]) gives the analytical self-similar solution for description of the motion of shock front and the distribution of fluid parameters inside the shocked region for a strong point explosion in the uniform ambient medium and in the center of symmetry of a radially stratified medium ([1]).

This solution shows that the strong blastwave in a medium with the power-law density distribution (1) moves with deceleration then \( m < N + 1 \) and accelerates then \( m > N + 1 \). If \( m \geq N + 1 \), both the mass inside any sphere, which contains the center of the symmetry, and kinetic energy equal infinity. We will consider \( m < N + 1 \) cases only.

Radius \( R \) and velocity \( D \) of the strong blastwave in the medium ([1]) with \( m < N + 1 \) are (Sedov [18]):

\[
R = \left( \frac{E_\alpha}{\alpha_A \tilde{\rho}(0)} \right)^{1/(N+3-m)} t^{2/(N+3-m)},
\]

\[
D(R) = \frac{2}{N+3-m+1} \left( \frac{E_\alpha}{\alpha_A \tilde{\rho}(0)} \right)^{1/2} R^{-(N+1-m)/2},
\]

where \( N = 0, 1, 2 \) for plane, cylindrical and spherical wave, respectively, \( \alpha_A \) is a self-similar constant.

Distributions of the fluid characteristics behind the shock front are self-similar, i.e., for any time \( t \) the density \( \tilde{\rho} \), pressure \( \tilde{P} \), fluid velocity \( \tilde{u} \) variations and coordinate \( \tilde{a} \) are

\[
\tilde{\rho}(\tilde{r}, t) = \tilde{\rho}_s(t) \cdot \rho(r),
\]

\[
\tilde{P}(\tilde{r}, t) = \tilde{P}_s(t) \cdot P(r),
\]

\[
\tilde{u}(\tilde{r}, t) = \tilde{u}_s(t) \cdot u(r),
\]

\[
\tilde{a}(\tilde{r}, t) = R(t) \cdot a(r),
\]

where \( r = \tilde{r}/R(t) \), \( \tilde{a} \) is the original position of the fluid mass element and superscript "s" corresponds to values of the parameters at the shock front (Fig. [1]).

Gas occupies whole shocked region (0 \( \leq \tilde{r} \leq R \)) when \( m \leq m_1 \),

\[
m_1 = \frac{1 + 3N + (1 - N)\gamma}{\gamma + 1}.
\]

When \( m \to m_1 \) central pressure \( P(0) \to 0 \). Shock waves in media with steep density gradients (\( m > m_1 \)) develop a cavity around the center of explosion. Such a cavity creates in the uniform medium (\( m = 0 \)) when \( \gamma > \gamma_1 = (1+3N)/(N-1) \). Sedov has also presented a solution for hollow blastwaves. Review of approximations for these cases is given by Ostriker & McKee ([16]). We do not consider \( m > m_1 \) in this paper.

For \( m = m_1 \) (or \( \gamma = \gamma^* \)) in the uniform medium solution has very simple form:

\[
\rho(r) = r^{N-1}, \quad P(r) = r^{N+1},
\]

\[
u(r) = r, \quad a(r) = r^{(\gamma+1)/(\gamma-1)}.
\]

Singularity in the solution also appear with \( m_2 = (N+1)(2 - \gamma) \) and \( m_3 = (2\gamma + N - 1)/\gamma \) then some exponents in the solution equal infinity. Similarity solutions for these cases are deduced by Korobejnikov & Rjazanov ([14]). For \( N = 2 \) and \( \gamma = 5/3 \) \( m_1 = 2 \), \( m_2 = 1 \), \( m_3 = 13/5 \).

Self-similar constant \( \alpha_A = \alpha_A(N, \gamma, m) \) in equations ([3]-[4]) for \( R \) and \( D \) may be found from the energy balance equation with variations of density \( \tilde{\rho} \), pressure \( \tilde{P} \) and mass velocity \( \tilde{u} \) inside the shocked region

\[
\frac{E_\alpha}{\sigma} = \int_0^R \left( \frac{\tilde{\rho}(r, t)\tilde{u}(r, t)}{2} \right)^{2} r^{N} dr + \int_0^R \left( \frac{\tilde{P}(r, t)}{\gamma - 1} \right)^{1/\gamma} r^{N} dr,
\]

where \( \sigma = 4\pi \) for \( N = 2 \), \( \sigma = 2\pi \) for \( N = 1 \) and \( \sigma = 1 \) for \( N = 0 \), generally, \( \sigma = 2\pi N + (N-1)(N-2) \). If we proceed to normalized parameters using ([3]-[4]) and general shock front conditions

\[
\rho_s = \frac{\gamma + 1}{\gamma - 1}\tilde{\rho}_s; \quad \tilde{P}_s = \frac{2}{\gamma + 1}\tilde{\rho}_s^2 D^2; \quad \tilde{u}_s = \frac{2}{\gamma + 1}D
\]
we will obtain that $E_o = \beta_A \cdot MD^2/2$ with
\[
M = \sigma \bar{\rho}^o(0)R^{N+1-m}/(N + 1 - m),
\]
constant shape-factor
\[
\beta_A = \frac{4(N + 1 - m)}{\gamma^2 - 1} \cdot (I_K + I_T)
\]
and constant integrals
\[
I_K = \int_0^1 \rho(r)u(r)^2r^N dr, \quad I_T = \int_0^1 P(r)r^N dr.
\]
Also we will have a self-similar constant
\[
\alpha_A = \frac{2\sigma}{(N + 1 - m)(N + 3 - m)} \cdot \beta_A.
\]
Simple formula gives $\alpha_A(N, \gamma, m_1)$:
\[
\alpha_A = \frac{2\sigma(\gamma + 1)}{(N + 1)(\gamma - 1)((N + 1)\gamma - N + 1)^2}.
\]
The distributions (4)-(7) in the exact solution are parametric functions of an internal parameter. The expressions for the functions are complicated. These factors stimulate developing the approximations of the self-similar solution.

### 2.2 Taylor approximation

Basing on own numerical results, Taylor (19) propose to approximate the velocity variation $u(r)$ behind spherical ($N = 2$) shock front moving into the uniform medium ($m = 0$) as
\[
\frac{\ddot{u}(r, t)}{D} = \frac{r}{\gamma} + \alpha r^n,
\]
where $\alpha$ and $n$ are found to give exact values of $\ddot{u}_s$, $\bar{P}_s$, $\bar{\rho}_s$ and their first derivatives in respect to $r$. Substituting this approximation into the continuity equation and into the equation of state for perfect gas, the approximated distributions of the density and pressure obtain. Taylor do not give the dependence $a(r)$, but it may be taken from the adiabaticity condition $P(a)\rho(a)^{-\gamma} = P(r)\rho(r)^{-\gamma}$ and (70)-(71):
\[
a^{\gamma_m - (N + 1)} = P(r)\rho(r)^{-\gamma},
\]
with approximations for $P(r)$ and $\rho(r)$.

So, Taylor approximation for the variations of density $\rho$, pressure $P$, fluid velocity $u$ and coordinate $a$ are:
\[
\rho(r) = \frac{\rho(r, t)}{\rho_s(t)} = r^{3/(\gamma - 1)} \left(\frac{\gamma + 1}{\gamma} - \frac{\gamma n - 1}{\gamma}\right)^{-p},
\]
\[
P(r) = \frac{P(r, t)}{P_s(t)} = \left(\frac{\gamma + 1}{\gamma} - \frac{\gamma n - 1}{\gamma}\right)^{-q},
\]
\[
u(r) = \frac{u(r, t)}{u_s(t)} = \frac{\gamma + 1}{2} \left(\frac{\gamma + 1}{\gamma} - \frac{\gamma - 1}{\gamma}\right)^{1},
\]
\[
a(r) = \frac{a_o(r)}{R(t)} = r^{\gamma/(\gamma - 1)} \left(\frac{\gamma + 1}{\gamma} - \frac{\gamma n - 1}{\gamma}\right)^{-s},
\]
where $n = (7\gamma - 1)/(\gamma^2 - 1)$, $p = 2(\gamma + 5)/(7 - \gamma)$, $q = (2\gamma^2 + 7\gamma - 3)/(7 - \gamma)$, $s = (\gamma + 1)/(7 - \gamma)$. Self-similar constant $\alpha_A = \alpha_A(2, \gamma, 0)$ goes with Fig. 3 and approximated profiles of $\rho$, $P$ and $u$. Fig. 4 and table 5 demonstrate accuracy of Taylor approximation in comparison with the exact solution.

This approximation is extended to cases $m \neq 0$ in section 3.
Figure 1: a-c. Sedov solution and the accuracy of Taylor and Kahn approximations of the solution in the uniform medium: a exact Sedov solution, b relative differences of Taylor approximation, c relative differences of Kahn approximation. Lines: 1 – $\rho(r)$, 2 – $P(r)$, 3 – $u(r)$, 4 – $a(r)$. $\gamma = 5/3$.

2.3 Kahn approximation

Kahn ([11]) apply his methodology to the strong spherical blastwave ($N = 2$) in uniform medium ($m = 0$) with $\gamma = 5/3$. It is proposed to approximate first the mass distribution

$$
\mu(r) = \frac{M(r,t)}{M_s(t)} = 3 \int_0^r \rho(r)r^2 dr.
$$

(23)

Sedov solution shows that $P_r(r) = 0$ near the centre (subscript "$r"" denotes a partial derivative in respect to $r$). This fact allows to find that $\mu_r/\mu = 15/(2r)$ at $r = 0$. On the base of the equation of motion, $\mu_r/\mu = 12$, $\mu_{rr} = 168$ and $(\mu_r/\mu)_r = 24$ at $r = 1$. Therefore ratio $\mu_r/\mu$ is proposed to be approximated as

$$
\frac{\mu_r}{\mu} = \frac{15}{2r} + \frac{9}{2} r^7.
$$

(24)

This formula satisfies all written boundary conditions at both ends.

The mass distribution finds as integral from (24):

$$
\mu(r) = r^{15/2} \exp \left( \frac{9}{16} (r^8 - 1) \right).
$$

(25)

Density distribution follows from (23) and (25):

$$
\rho(r) = \mu_r / 3r^2.
$$

(26)

Adiabaticity condition gives pressure variation

$$
P(r) = \left( \frac{2}{3} \right)^{5/3} \frac{1}{32} \frac{\mu_r^{5/3}}{\mu r^{10/3}}.
$$

(27)
Velocity deduces from the mass conservation equation

\[ u(r) = \frac{4}{3} r - \frac{4 \mu}{\mu_r}. \]  

(28)

If present location of mass element \(a\) is \(r\), then \(a(r)\) may be found from the condition of mass conservation \(\mu(a) = \mu(r)\) and relation (73) \(\mu(a) = a^3;\)

\[ a(r) = \mu(r)^{1/3}. \]  

(29)

The expressions for Kahn approximation are the same as (30)-(34) with \(m = 0\). The accuracy of this approximation are shown on the Fig. 1.

2.4 Approximation of Cox & Franco

Applying Kahn’s approximation technique, Cox & Franco (3) obtain the approximation of the self-similar solution for an ambient medium with the power-law density distribution (1) with \(m < 2\) for \(\gamma = 5/3\) and \(N = 2\). Approximation of Cox & Franco are:

\[ \rho(r) = \left(\frac{5}{8} + \frac{3}{8} r^{8-4m}\right) \cdot r^{(9-5m)/2} \times \exp\left(\frac{3}{8} \left(\frac{3-m}{2-m}\right) (r^{8-4m} - 1)\right), \]  

(30)

\[ P(r) = \left(\frac{5}{8} + \frac{3}{8} r^{8-4m}\right)^{5/3} \times \exp\left(\frac{3}{8} \left(\frac{3-m}{2-m}\right) (r^{8-4m} - 1)\right), \]  

(31)

\[ u(r) = 4 r \left(1 + \frac{1}{5} \frac{r^{8-4m}}{3 r^{8-4m}}\right), \]  

(32)

\[ a(r) = r^{5/2} \exp\left(\frac{3}{8} \left(\frac{3-m}{2-m}\right) (r^{8-4m} - 1)\right), \]  

(33)

\[ \mu(r) = r^{5(3-m)/2} \exp\left(\frac{3}{8} \left(\frac{3-m}{2-m}\right) (r^{8-4m} - 1)\right). \]  

(34)

Author’s approximation for \(\beta_A\) is

\[ \beta_A = 1.125 \cdot (0.22 + 0.52 \cdot (3 - m)/3). \]  

(35)

The accuracy of Cox & Franco approximation is shown on Fig. 2 and table 3.

2.5 Approximations of Ostriker & McKee

Ostriker & McKee (16) in the frame of the virial theorem approach applied to spherical blastwave (N=2) in the power-law ambient medium (1) and time-dependent energy injection \(E_o(t) \propto t^s\), present a number of approximations for the self-similar solution. We consider further \(s = 0\).

Authors introduce the dimensionless moments of coordinate \(r\) and velocity \(u\):

\[ K_{ij} = l_\mu \int_0^1 r^i u(r)^j \rho(r) r^2 dr, \]  

(36)

where \(l_\mu = (\gamma + 1)(3-m)/(\gamma - 1)\), and consider three types of approximations for \(u(r)\) and \(\rho(r)\): linear velocity approximation (LVA)

\[ u(r) = r, \quad \rho(r) = r^{(6-(\gamma+1)m)/(\gamma-1)}, \]  

(37)

one-power aproximation (OPA)

\[ u(r) = r^{l_u}, \quad \rho(r) = r^{l_\rho}, \]  

(38)

and two-power approximation (TPA)

\[ u(r) = a_u r^{l_{u,1}} + (1-a_u) r^{l_{u,2}}, \]  

(39)
Figure 2: a-c. Accuracy of Cox & Franco approximation of the self-similar solution in the power-law medium \([16]\): a relative differences of the approximation for \(m = -4\), b relative differences for \(m = -2\), c relative differences for \(m = 1\). Lines are the same as on Fig. 1. Fracture in the curves for \(\rho(r)\) and \(a(r)\) is due to very strong dependence of the relevant Sedov distributions on the internal parameter, which changes in these wide intervals of \(r\) on \(10^{-10}\) only.

\[
\rho(r) = a_{\rho} r^{l_{\rho,1}} + (1 - a_{\rho}) r^{l_{\rho,2}},
\]

In such an approach the self-similar constant \(\alpha_A\) as well as exponents \(l_u\) and \(l_{\rho}\) may be expressed in terms of moments \(K_{02}\) and \(K_{11}\). Namely, under self-similarity \(\alpha_A = 2\pi \eta^{2} \beta_A/(3 - m)\), where \(\eta = 2/(5 - m)\) and factor \(\beta_A\) equals

\[
\beta_A = \frac{2}{3} \frac{2K_{02}(3\gamma - 5) + (5 - m)\gamma(1 + K_{11})}{(\gamma^2 - 1)(\gamma + 1)}.
\]

Exponents in OPA are

\[
l_u = \frac{2K_{20} - K_{11}(1 + K_{20})}{(1 - K_{20})K_{11}}, \quad l_{\rho} = \frac{5K_{20} - 3}{1 - K_{20}}.
\]

Derivatives at shock front are used to obtain the moments. So,

\[
K_{ij} = \frac{1}{1 + s_{ij}/l_{ij}},
\]

where \(s_{ij} = i + j\) in LVA and \(s_{ij} = i + j(m_1 - m)/2\) in OPA. Using \([13]\) \(\alpha_A\) may be written in a simple form in LVA:

\[
\alpha_A = \frac{16\pi}{3(5 - m)^2} \frac{11\gamma - 5 - m(\gamma + 1)}{(\gamma^2 - 1)(5\gamma + 1 - m(\gamma + 1))}.
\]

Moments have more complicated form in TPA. In this approach the expression for \(u(r)\) coincides with the approximation \([21]\) of Taylor with \(n = 1 + \gamma(m_1 - m)/(\gamma - 1)\) that equals to Taylor’s \(n\) at \(m = 0\). So, TPA is extension of Taylor approximation of \(u(r)\) to \(m \neq 0\). Contrary to Taylor’s approach to find \(\rho(r)\) and \(P(r)\) from the hydrodynamic equations, Ostriker & McKee find the density variation independently as TPA \([40]\) with

\[
a_{\rho} = \frac{\gamma(m_1 - m)}{10 - \gamma - (\gamma + 2)m},
\]

\[
l_{\rho,1} = \frac{3 - \gamma m}{\gamma - 1}, \quad l_{\rho,2} = \frac{6 + (\gamma + 1)(m_1 - 2m)}{\gamma - 1},
\]

where \(\gamma > 1\). For \(m = 0\) variation \(\rho(r)\) in TPA coincides with the result of Gaffet \([7]\) for case of uniform medium (Ostriker & McKee \([17]\)). Two-power velocity approximation is used to extend Taylor approximation to cases \(m \neq 0\) in section \([3]\).
Figure 3: Relative differences of Ostriker & McKee two-power approximation of the self-similar solution for the uniform medium. $\gamma = 5/3$. Lines are the same as on Fig. 1.

Pressure distribution are also restored independently. It may be found in OPA as a linear pressure approximation (LPA) and for TPA in the frame of pressure-gradient approximation (PGA).

Most of mass is concentrated near the shock front and distribution $u(r)$ is close to a linear function of $r$. Therefore, as noted by Gaffet (7), the right side of Euler equation

$$\frac{\partial \tilde{P}(\tilde{r}, t)}{\partial M(\tilde{r}, t)} = -\frac{1}{4\pi \tilde{r}^2} \frac{d\tilde{u}(\tilde{r}, t)}{dt}$$

is nearly a constant. LPA (Gaffet [7], Ostriker & McKee [16]) use this feature assuming the pressure to be a linear function of the mass fraction $\mu(r)$

$$P(r) = P(0) + \frac{P^*}{l_{\mu}} \mu(r).$$

Logarithmic derivative of pressure at the shock front is

$$P^* = \frac{(d \ln P/d \ln r)_{s}}{(2\gamma^2 + 7\gamma - 3 - \gamma_{m}(\gamma+1))/(\gamma^2-1)).$$

Mass in OPA is

$$\mu(r) = 3 l_{\mu}^{-1} \tilde{u}_{t},$$

$P(0)$ in LPA is (Gaffet [8])

$$P(0) = 1 + \frac{\tilde{u}_{t}}{\omega(3 - m)}$$

where $\tilde{u}_{t} = \tilde{u}_{s} R/D^2 = \omega((4 - 3\omega)(m - 3) + 2(1 - \omega)(4 - 2\omega - m))/2$ (Hnatyk [15]), $\omega = 2/(\gamma + 1)$.

Such an approach (substitution with $\tilde{r}^{-2} \tilde{u}_{t}$ instead of $\tilde{r}^{-2} \tilde{u}_{t}$ in [10]) was also used by Laumbach & Probstein [15] to develop the sector approximation.

In PGA a power-law form for the pressure gradient

$$\frac{dP(r)}{dr} = P^* r^{l_{p,2} - 1}$$

is used to give two-power approximation for the pressure

$$P(r) = P(0) + a_p r^{l_{p,2}}$$

where $a_p = P^*/l_{p,2}$ and

$$P(0) = \frac{(\gamma + 1)^2(m_1 - m)}{3\gamma^2 + 20\gamma + 1 - (\gamma + 1)(3\gamma + 1)m},$$

$$l_{p,2} = \frac{3\gamma^2 + 20\gamma + 1 - (\gamma + 1)(3\gamma + 1)m}{2(\gamma^2 - 1)}.$$ 

Accuracy in determination of $\alpha_A$ and $P(0)$ in approximations of Ostriker & McKee is shown in table 3 and in revealing the flow parameters on Fig. 3.

### 2.6 Cavaliere & Messina approximation of $\alpha_A$

Cavaliere & Messina [3] with a simple technique approximate the equations for the radius and velocity of shock in the power-law medium [3] and $E_o(t) \propto t^s$. For $s = 0$ his approximation gives

$$\beta_A = \frac{4}{\gamma^2 - 1} \left( \frac{\gamma - 1}{\gamma + 1} + \frac{1}{2} \frac{N + 1 - m}{N + 1} \right).$$
Figure 4: Accuracy of Hnatyk approximation of Sedov solution for the uniform medium. Lines: 1 – $\rho(a)$, 2 – $P(a)$, 3 – $u(a)$, 4 – $r(a)$. $\gamma = 5/3$.

2.7 Approximate methods for an explosion in medium with arbitrary large-scale nonuniformity

In this subsection we pointed out a number of approximate methods for description of a point explosion in arbitrary nonuniform medium. These methods may also be applicable for a medium with power-law density variation. Bisnovatyi-Kogan & Silich ([2]) and Hnatyk ([9]) have given the reviews of these methods, their applications and accuracy.

2.7.1 Thin-layer approximation

Thin-layer approximation is firstly introduced by Chernyi ([4]) and used by Kompaneets ([13]) and other authors to find analytical solutions for evolution of the shock front in a number of type of nonuniform media. It is assumed in this approach that all swept-up mass is concentrated in the infinitely thin layer just after shock front and the motion is stimulated with the hot gas inside the shocked region with uniform pressure distribution $P(r) = 0.5$ (excepting $P_s = 1$). Layer of the gas moves with velocity $u_s$. This method was developed to calculate only the shock front dynamics and therefore does not allow to reveal the distribution of the fluid parameters behind the shock front.

Thin-layer approximation gives for spherical blastwave in the uniform medium (Andriankin et al. [1])

$$\alpha_A = \frac{16\pi(3\gamma - 1)}{75(\gamma - 1)(\gamma + 1)^2}.$$  \hspace{1cm} (53)

2.7.2 Sector approximation

In the sector approximation, the characteristics of an one-dimensional flow find as decompositions about the shock front.

Laumbach & Probstein ([15]) have proposed the sector approximation applying it to spherical blastwaves in a plane-stratified exponential medium. Authors use Lagrangian coordinate $a$ and propose to approximate pressure variation in the form equivalent to $P(a) = 1 + P_{\infty}(a - 1)$ (Hnatyk [9]). Density variation is given by the adiabaticity condition and relation $r = r(a)$ by continuity equation. Fluid velocity field is not determined. For shock radius and its velocity Laumbach & Probstein approximation yields in the uniform medium limit

$$\alpha_A = \frac{32\pi(4\gamma^2 - \gamma + 3)}{225(\gamma - 1)(\gamma + 1)^3}.$$  \hspace{1cm} (54)

Gaffet ([7, 8]) uses Lagrangian mass coordinates $\mu$ and finds pressure variation as a linear pressure approximation $P(\mu) = 1 + P_{\infty}(\mu - 1)$. Gaffet ([7, 8]) also propose to improve accuracy of the approximation, taking into account the second order coefficients in the series. Author calculates such coefficients in terms of Lagrangian mass coordinate $\mu$. Hnatyk ([9]), considering different modifications of the sector approximation, presents the coefficients up to the second order in terms of $a$.

2.7.3 Hnatyk approximation

Hnatyk ([9]) introduces also the idea to approximate firstly the relation $\tilde{r} = \tilde{r}(a, t)$ between the Lagrangian $a$ and Eulerian $r$ coordinates of the gas element in each sector of shocked region. Density $\rho$, pressure $P$ and velocity $u$ variation behind the shock front are exactly deduced from this relation. Really, the continuity equation

$$\tilde{\rho}^a(\tilde{a})\tilde{a}^N\tilde{d}\tilde{a} = \tilde{\rho}(\tilde{r})\tilde{r}^N\tilde{d}\tilde{r}$$  \hspace{1cm} (55)
This extended Taylor approximation is compared with the exact solution on Fig. 5.

The derivatives of characteristics do not restore correctly (Fig. 4). This approximation does not take into consideration any situation. This approximation is extended to the central region in subsection 4.5.

In this section Taylor’s technic is applied to the case of a medium with the power-law density distribution \( \rho(a) = \frac{\rho(a, t)}{\rho(R, t)} = \left( \frac{\rho(a)}{\rho(R)} \right)^N \left( \frac{\partial \rho(a, t)}{\partial a} \right)^{-1} \),

the equation of adiabaticity

\[ \tilde{P}(\tilde{a}, t) = K \tilde{\rho}(\tilde{a}, t)^\gamma \]

yields the distribution of pressure

\[ P(a) = \frac{\tilde{P}(\tilde{a}, t)}{P_n(t)} = \left( \frac{\tilde{\rho}(\tilde{a})}{\rho(R)} \right)^{1-\gamma} \left( \frac{D(\tilde{a})}{D(R)} \right)^2 \left( \frac{\tilde{\rho}(\tilde{a}, t)}{\rho(R, t)} \right)^\gamma \]

and relation \( \tilde{r} = \tilde{r}(\tilde{a}, t) \) gives velocity

\[ u(a) = \frac{\tilde{u}(\tilde{a}, t)}{u_n(t)} = \frac{\gamma + 1}{2} \frac{d\tilde{r}(\tilde{a}, t)}{dt} . \]

Author propose to approximate \( r(a) \) as

\[ r(a) = a^\alpha \exp(\beta(a - 1)) \]

with

\[ \alpha = (r_s^a)^2 - r_{aa}^s \quad \text{and} \quad \beta = r_{aa}^s + r_s^a - (r_s^a)^2 . \]

Such an expression ensures the edge condition \( r(0) = 0, r_s = 1 \) and values of the derivatives

\[ r_{aa}^s = 1 - \omega, \]

\[ r_{aa}^s = \omega(1 - \omega)[3B + N(2 - \omega) - m] \]

where \( B = R\ddot{R}/\dot{R}^2, \dot{R} = dR/dt \) is the shock velocity, \( m = -d\ln \rho(R)/d\ln R \), subscript “\( a \)” denotes a partial derivative in respect to \( a \).

This approximation is accurate near the shock front, but around the explosion site (for \( a < 0.1 \) or \( r < 0.4 \)) characteristics do not restore correctly (Fig. 4). This approximation does not take into consideration any derivates of \( r(a) \) near the center and the distributions of \( \rho(a), P(a), u(a) \) do not bind there, causing such a situation. This approximation is extended to the central region in subsection 4.5.

### 3 Extension of Taylor approximation to \( m \neq 0 \)

In this section Taylor’s technic is applied to the case of a medium with the power-law density distribution \( \rho(a) = \frac{\rho(a, t)}{\rho(R, t)} = \left( \frac{\rho(a)}{\rho(R)} \right)^N \left( \frac{\partial \rho(a, t)}{\partial a} \right)^{-1} \), with \( m < m_1 \). Ostriker & McKee ([16]) give the coefficients in the approximation \( \tilde{r} = r(a, t) \) for \( u(r) \) in such a case. Approximated velocity variation is \( [21] \) with \( n = P_s^a - 2 = (7\gamma - 1 - m\gamma(\gamma + 1))/(\gamma^2 - 1) \). Substitution with \( \tilde{r} = r(a, t) \) into the equations of continuity and state gives

\[ \frac{\rho r}{\rho} = \frac{3 + \alpha\gamma(n + 2)r^{n-1} - m\gamma}{(\gamma - 1)r - \alpha\gamma r^n} , \]

\[ \frac{P r}{P} = \frac{\alpha\gamma^2(n + 2)r^{n-1}}{(\gamma - 1)r - \alpha\gamma r^n} . \]

After integration, pressure variation will be expressed with \( [20] \) where

\[ q = \frac{2\gamma^2 + 7\gamma - 3 - m\gamma(\gamma + 1)}{7 - \gamma - m(\gamma + 1)} . \]

Density is

\[ \rho(r) = r^{(3-m\gamma)/(\gamma-1)} \left( \frac{\gamma + 1}{\gamma} - \frac{r^{n-1}}{\gamma} \right)^{-p} , \]

where

\[ p = \frac{2(\gamma + 5 - m(\gamma + 1))}{7 - \gamma - m(\gamma + 1)} . \]

Eq. \( [22] \) gives \( a(r) \) with

\[ s = \frac{\gamma + 1}{7 - \gamma - m(\gamma + 1)} . \]

Exponents \( n = 6 - 5m/2, q = (16 - 5m)/(3(2 - m)), p = (5 - 2m)/(2 - m) \) and \( s = 1/(2 - m) \) for \( \gamma = 5/3 \). This extended Taylor approximation is compared with the exact solution on Fig. 3.
4 Approximations of the Sedov solution in Lagrangian coordinates

In this section we present two analytical approximations of the self-similar solution for a medium with the power-law density distribution expressed in Lagrangian geometric coordinates $a$.

4.1 Flow characteristic distributions

Exact expressions for normalized density $\rho$ and pressure $P$ variations behind the shock front moving into the power-law medium (1) follow from (56) and (58):

$$\rho(a) = \frac{\gamma - 1}{\gamma + 1} \cdot a^{N-m} \cdot (r(a)^N \cdot r_a(a))^{-1}, \quad (70)$$

$$P(a) = \left(\frac{\gamma - 1}{\gamma + 1}\right) \cdot a^{N(\gamma-1)/\gamma} \cdot (r(a)^N \cdot r_a(a))^{-1}. \quad (71)$$

Distribution of the fluid velocity $u(a)$ may be found from (59). Due to $\ddot{r} = R\dot{r} + Rr_a a_t + rD$ (subscript "t" denotes a partial derivative in respect to $t$). We have also that $a_t = -aD/R$ and, in the self-similar case, $r_t = 0$. So,

$$u(a) = \frac{\gamma + 1}{2} \left( r(a) - r_a(a) a \right). \quad (72)$$

The distribution $\mu(a)$ follows from the definition (23) and (55):

$$\mu(a) = a^{(N+1) - m}. \quad (73)$$

4.2 Self-similar constant $\alpha_A$

Self-similar constant $\alpha_A(N, \gamma, m)$ in equations for $R$ and $D$ (2)-(3) obtains from (15) and (13):

$$\alpha_A = \frac{8}{\gamma^2 - 1} \cdot \frac{\sigma}{(3 + N - m)^2}, \quad (I_K + I_T), \quad (74)$$

with

$$I_K = \frac{\gamma^2 - 1}{4} \int_0^1 (r(a) - r_a(a)a)^2 a^{N-m} da, \quad I_T = \left(\frac{\gamma - 1}{\gamma + 1}\right) \int_0^1 \left( r(a)^N r_a(a) \right)^{\gamma} a^{N(\gamma-1)/\gamma} - \frac{1}{\gamma} da. \quad (75)$$
Table 1: \( P(0) \) calculated according to the self-similar solution and \( C \) for a strong point explosion in the power-law medium.

| N | m | \( P(0) \) | \( C_A \) |
|---|---|---|---|
| \( \gamma = 7/5 \) | \( \gamma = 5/3 \) |
| 0 | 0 | 0.3900 | 0.3532 | 1.1429 | 1.1670 |
| 1 | 0 | 0.3729 | 0.3215 | 1.0863 | 1.1112 |
| 2 | 0 | 0.3655 | 0.3062 | 1.0618 | 1.0833 |
| -4 | 0 | 0.4268 | 0.3954 | 1.0233 | 1.0293 |
| -3 | 0 | 0.4193 | 0.3848 | 1.0276 | 1.0350 |
| -2 | 0 | 0.4088 | 0.3696 | 1.0339 | 1.0433 |
| -1 | 0 | 0.3928 | 0.3463 | 1.0438 | 1.0570 |
| 1 | 0 | 0.3087 | 0.2217 | 1.1054 | 1.1556 |
| 2 | 0 | 0.1273 | 0.0000 | 1.3648 | 1.0000 |

### 4.3 Factor \( C \) and exponent \( x \)

In Sedov self-similar solution, if \( r \rightarrow 0 \) then the dependence \( r(a) \) is

\[
 r = C \cdot a^x. \tag{76}
\]

For \( m < m_1 \), if we substitute (76) into (71) we obtain the connection between the factor \( C \) and normalized central pressure \( P(0) \):

\[
 C = \left( \frac{\gamma}{\gamma + 1} \cdot P(0)^{-1/\gamma} \right)^{1/(N+1)}. \tag{77}
\]

We have to put \( x = (\gamma - 1)/\gamma \) during this transformation in order to satisfy condition \( P(0) \neq 0 \). In the case \( m = m_1 \) the exact solution (9) gives \( x \) and \( C = 1 \). General formula for exponent \( x \) is

\[
 x = \left\{ \begin{array}{ll}
 (\gamma - 1)/\gamma & \text{for } m < m_1 \\
 (\gamma - 1)/(\gamma + 1) & \text{for } m = m_1
\end{array} \right. . \tag{78}
\]

Analytical expressions for \( P(0) \) from self-similar solution are presented in Appendix 5. Calculated values of \( P(0) \) and \( C \) for a number of \( N, \gamma \) and \( m \) are shown in table 1.

### 4.4 Derivatives at shock front

Expressions for the derivatives \( r_a^a, r_a^{aa}, r_a^{aaa} \) may be obtained with the technic of Gaffet (4) from the set of hydrodynamic equations for perfect gas and conditions on the shock front (see Hnatyk & Petruk (10) for details). Derivatives \( r_a, r_{aa} \) are given with (62)-(63) and

\[
 r_a^{aaa} = \omega(1 - \omega) \left[ 3(7 - 5\omega)B^2 + 
 + [-5\omega^2 + 4\omega + 8]N + (4\omega - 11)m \right] B +
 + \omega(2\omega^2 - 7\omega + 6)N^2 + (\omega^2 + \omega - 4)Nm -
 - \omega(2 - \omega)N - (\omega - 2)m^2 + (2\omega - 1)m +
 + (2\omega - 1)m' + (6\omega - 4)Q, \tag{79}
\]

where \( Q = R^2 R^{(3)} / \dot{R}^3 \) and \( m' = -dm/d\ln R \).

In the power-law medium (1) \( m' = 0 \). Taking into consideration the equations for the shock radius (2) and shock velocity (3) we may also write

\[
 B = -\frac{N - m + 1}{2}, \quad Q = \frac{(N - m + 1)(N - m + 2)}{2}. \tag{80}
\]

Reduced expressions for the derivatives \( r_a^a, r_a^{aa}, r_a^{aaa} \) are shown in table 2.
Table 2: Derivatives of the relation between Lagrangian and Eulerian coordinates at shock front moving into the power-law medium (1).

| Derivative | \( \gamma \) |
|------------|-------------|
| \( r_s^a = \frac{1}{6} \) | \( \gamma = 7/5 \) |
| \( r_s^{aa} = \frac{5}{2^{3/3}} (-2N + 3m - 9) \) | \( \gamma = 5/3 \) |
| \( r_s^{aaa} = \frac{5}{2^{5/3}} (-2N^2 + 9Nm + 183N - 9m^2 - 270m + 675) \) | |

4.5 Second order approximation

So, to approximate the self-similar solution, we approximate the relation \( r = r(a) \) between Eulerian \( r \) and Lagrangian \( a \) coordinates of flow elements. Following to Hnatyk’s approach (60) and like to relation (33), \( r = r(a) \) may be approximated in the form

\[
r(a) = a^x \exp \left( \alpha(a^\beta - 1) \right)
\]

(81)

with \( x \) given by (78) and

\[
\alpha = \frac{(r_s^a - x)^2}{r_s^{aa} + r_s^a - (r_s^a)^2}, \quad \beta = \frac{r_s^{aa} + r_s^a - (r_s^a)^2}{r_s^a - x},
\]

(82)

or, after substitution with (62)-(63),

\[
\alpha = \frac{2(1 - \omega - x)^2}{\omega(1 - \omega)(N + m - 1 - 2N\omega)},
\]

(83)

\[
\beta = \alpha^{-1}(1 - \omega - x).
\]

Such a second order approximation, besides \( r(0) = 0, r_s = 1, r_s^a, r_s^{aa} \), gives \( (\partial \ln r / \partial \ln a)^0 = x \), and, contrary to Hnatyk approximation, extends description of a flow to the central region.

Variations of \( \rho(a) \), \( P(a) \) and \( u(a) \) follow from (70)-(72). For case \( N = 2 \), \( \gamma = 5/3 \) and \( m < 2 \) these relations give \( \beta = 5(2 - m)/8 \) and

\[
r(a) = a^{2/5} \exp \left( -\frac{6}{25(2 - m)}(a^\beta - 1) \right),
\]

(84)

\[
\rho(a) = \left( \frac{8}{5} - \frac{3}{5}a^\beta \right)^{-1} \cdot a^{(9-5m)/5}
\]

(85)

\[
P(a) = \left( \frac{8}{5} - \frac{3}{5}a^\beta \right)^{-5/3} \exp \left( \frac{6}{5(2 - m)}(a^\beta - 1) \right),
\]

(86)

\[
u(a) = a^{2/5} \left( \frac{4}{5} + \frac{1}{5}a^\beta \right) \exp \left( -\frac{6}{25(2 - m)}(a^\beta - 1) \right).
\]

(87)
Figure 6: a-c. Accuracy of the second order approximation of the self-similar solution in the power-law medium (1) for $\gamma = 5/3$ and $N = 2$: a relative differences for $m = 0$, b relative differences for $m = -2$, c relative differences for $m = 1$. Lines are the same as on Fig. 4.

Approximation (84)-(87) may be considered as an inversion of Cox & Franko approximation (30)-(33). Unfortunately, accuracy of presented formulae is lower (Fig. 6, table 5).

4.6 Third order approximation

In order to improve accuracy, we postulate the approximation $r = r(a)$ to give exact values of two additional derivatives: third order $r_{aaa}$ and $(\partial r/\partial (a^x))^0 = C$. Consideration of $r_{aaa}$ is equivalent to consideration of the second order derivatives $p_{aa}^s$, $P_{aa}^s$, $u_{aa}^s$ in expansion of relevant characteristics into the series near the shock front. This approximation is the same as used in the approximate hydrodynamical method for modelling the asymmetrical strong point explosion in the medium with a large-scale density nonuniformity (Hnatyk & Petruk [10]). Contrary to the method, we take here that both the self-similar constant $\alpha_A$ and factor $C$ are different for different $m$.

Namely, if at time $t$ the shock position is $R(t)$, we approximate a connection $r = r(a)$ as follows

$$r(a) = a^x \cdot (1 + \alpha \cdot \xi + \beta \cdot \xi^2 + \varsigma \cdot \xi^3 + \delta \cdot \xi^4),$$

(88)

where $\xi = 1 - a$. Coefficients $\alpha$, $\beta$, $\varsigma$, $\delta$ and exponent $x$ are choosen from the condition that the partial derivatives $r^s_a$, $r^s_{aa}$, $r^s_{aaa}$ at the shock front ($a = 1$) as well as $(\partial \ln r/\partial \ln a)^0 = x$ and $(\partial r/\partial (a^x))^0 = C$ in the place of explosion ($a = 0$) equal to their exact values:

$$\alpha = -r^s_a + x,$$

$$\beta = \frac{1}{2} \cdot (r^s_{aa} - 2x \cdot r^s_a + x(x + 1)),$$

$$\varsigma = \frac{1}{6} \cdot (-r^s_{aaa} + 3x \cdot r^s_{aa} - 3x(1 + x) \cdot r^s_a + x(x + 1)(x + 2)),$$

$$\delta = C - (1 + \alpha + \beta + \varsigma).$$

(89)

In terms of a relation (88) and its first derivative are

$$r(a) = a^x (B_0 - B_1 a + B_2 a^2 - B_3 a^3 + B_4 a^4),$$

(90)

$$r_a(a) = a^{x-1} (A_0 - A_1 a + A_2 a^2 - A_3 a^3 + A_4 a^4),$$

(91)

with
Table 3: Self-similar constant $\alpha_A(N, \gamma, m)$ calculated with third order approximation of $r(a)$ (88).

| $N$ | $m$ | $\alpha_A$ |
|-----|-----|------------|
|     | 0   | 1.0763 0.6018 |
| 1   | 0   | 0.9841 0.5644 |
| 2   | 0   | 0.8519 0.4944 |
|     | -4  | 0.2295 0.1270 |
|     | -3  | 0.2960 0.1650 |
|     | -2  | 0.3966 0.2232 |
|     | -1  | 0.5598 0.3192 |
| 1   |     | 1.4631 0.8722 |
| 2   |     | 3.3537 1.8235 |

Table 4: Coefficients in approximation (90). $\gamma = 5/3$.

| $N$ | $m$ | $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|-----|-----|-------|-------|-------|-------|-------|
| 0   | 0   | 1.1670 | 0.1333 | -0.1127 | -0.1074 | -0.02833 |
| 1   | 0   | 1.1112 | -0.01510 | -0.24655 | -0.1530 | -0.03276 |
| 2   | 0   | 1.0833 | -0.05189 | -0.2130 | 0.6971 | 0.2213 |
|     | -4  | 1.0293 | 0.2837 | 0.9302 | 0.9766 | 0.3008 |
|     | -3  | 1.0350 | 0.1744 | 0.6152 | 0.6971 | 0.2213 |
|     | -2  | 1.0433 | 0.07170 | 0.3041 | 0.4162 | 0.1406 |
|     | -1  | 1.0570 | -0.01334 | 0.01359 | 0.1452 | 0.06128 |
| 1   |     | 1.1556 | 0.08965 | -0.1753 | -0.1471 | -0.03778 |
| 2   | 1   | 0     | 0     | 0     | 0     |

\[
B_0 = 1 + \alpha + \beta + \varsigma + \delta = C, \quad A_0 = xB_0, \\
B_1 = \alpha + 2\beta + 3\varsigma + 4\delta, \quad A_1 = (1 + x)B_1, \\
B_2 = \beta + 3\varsigma + 6\delta, \quad A_2 = (2 + x)B_2, \\
B_3 = \varsigma + 4\delta, \quad A_3 = (3 + x)B_3, \\
B_4 = \delta, \quad A_4 = (4 + x)B_4.
\]

Distribution of $\rho(a)$, $P(a)$ and $u(a)$ obtain from (71)-(72). Self-similar constant $\alpha_A$ is given with (74). To simplify the procedure, numerical values of $\alpha_A(N, \gamma, m)$ in this approximation are presented in table 3. Table 4 gives ready-calculated values of the coefficients in the approximation (90) for a number of cases.

The accuracy of flow characteristic distributions in this approximation is high for uniform medium (Fig. 7). Approximation coincides with the exact solution (9) for case $m = m_1$. For other $m \neq 0$, differences increase with increasing $|m|$ but maximal errors reveal in the region with low densities (Fig. 8). We compare also numerical values of $\alpha_A$ and $P(0)$ in this approximation with those from exact Sedov solution in table 5. $\alpha_A$ in the approximation is close to the exact values and gives accurate shock radius $R$ and velocity $D$.

5 Conclusions

In this paper, we review approximations of the self-similar solution for a strong point explosion in the power law medium $\rho \propto r^{-m}$ and compare their accuracy with the exact Sedov solution of the problem. Different approaches found on the different basic approximations. Namely, Taylor (19) and Ostriker & McKee (16)
Figure 7: a–c. Accuracy of the third order approximation of the Sedov solution in the uniform medium \((m = 0)\) for \(\gamma = 5/3\): a relative differences of the approximation for \(N = 0\), b relative differences for \(N = 1\), c relative differences for \(N = 2\). Lines are the same as on Fig. 4.

Table 5: Comparision of the self-similar constant \(\alpha_A\) and pressure \(P(0)\) calculated with: S – Sedov \([17]\) solution (Kestenboim et al. \([12]\)); T – Taylor \([19]\) approximation; CF – approximation of Cox & Franco \([6]\); LVA, OPA and TPA of Ostriker & McKee \([16]\); CM – approximation of Cavaliere & Messina \([8]\); TL – thin-layer \([53]\) approximation; LP – approximation \([54]\) of Laumbach & Probstein \([15]\); SOA – second order \([81]\) and TOA – third order \([88]\) approximations. Uniform medium, \(\gamma = 5/3\) and \(N = 2\).

|     | S   | T   | CF  | IVA | OPA/LPA | TPA/PGA | CM  | TL  | LP  | SOA | TOA |
|-----|-----|-----|-----|-----|---------|---------|-----|-----|-----|-----|-----|
| \(\alpha_A\) | 0.4936 | 0.4957 | 0.4930 | 0.5386 | 0.5027 | 0.4957 | 0.5655 | 0.5655 | 0.4398 | 0.4981 | 0.4944 |
| \(P(0)\)    | 0.3062 | 0.2855 | 0.3140 | –   | 0.3333  | 0.3333  | –   | 0.5000 | 0.3333 | 0.2507 | 0.3062 |

approximate firstly the fluid velocity variation behind the shock front. Taylor used approximated \(u(r)\) substituting it into the hydrodynamic equations to obtain full description of the flow. Contrary to this, Ostriker & McKee approximate \(\rho(r)\) and \(P(r)\) independently. Kahn \([11]\) technic, used also by Cox & Franco \([6]\), consists in approximation of the fluid mass variation \(\mu(r)\) and further usage of the system of hydrodynamic equations. Gaffet \([7]\), Laumbach & Probstein \([15]\), Ostriker & McKee \([16]\) base their approaches on the approximation of \(P(\mu)\) or \(P(r)\). Thin layer approximation may also be included into this group. Hnatyik \([9]\) take approximation of the connection between Eulerian and Lagrangian coordinates as basic relation. So, practically all possible approaches are used to have approximation for the self-similar solution.

In this paper we apply Taylor’s methodology to describe a strong point explosion in the power-law medium, extending his approximation written for uniform medium, and write also two approximations expressed in Lagrangian geometric coordinates, approaching \(r(a)\) with different accuracy.

Errors of all approximations are caused only by errors in the basic approximation. When the first approximation has higher accuracy we have more accurate approximation for parameters of the shock and flow.

**Appendix: central pressure \(P(0)\)**

In this appendix, we give exact expression for \(P(0)\) in self-similar solution when \(m \leq m_1\) (Sedov \([18]\)) and when \(m = m_2\) (Korobejnikov & Rjazanov \([14]\)). These relations complete the full set of formulae to build the
Figure 8: a-c. Accuracy of the third order approximation for power-law medium, $\gamma = 5/3$ and $N = 2$: a relative differences for $m = -4$, b relative differences for $m = -2$, c relative differences for $m = 1$. Lines are the same as on Fig. 4.

third order approximation of the Sedov solution for any $\gamma$, $m \leq \min(N + 1, m_1)$ and type of symmetry (plane, cylindrical or spherical blastwave).

$P(0) = 0$ for $m = m_1$.

In the case of $m < m_1$ and $m \neq m_2$

$$P(0) = \left(\frac{1}{2}\right)^{\varepsilon_1} \left(\frac{\gamma + 1}{\gamma}\right)^{\varepsilon_2} \left(\frac{m - m_3}{m - m_1}\right)^{\varepsilon_3},$$  \hspace{1cm} (92)

$$\varepsilon_1 = \frac{2(N + 1)}{N + 3 - m},$$

$$\varepsilon_2 = \frac{2(N + 1)}{N + 3 - m} - \frac{\gamma(N + 1 - m)}{(N + 1)(2 - \gamma) - m},$$

$$\varepsilon_3 = \frac{(N + 1 - m)(N + 3 - m)}{(N + 1)(2 - \gamma) - m} + m - 2,$$

$$\varepsilon_4 = \frac{\gamma + 1}{(N + 1)(\gamma - 1) + 2} - \frac{2}{N + 3 - m} \frac{\gamma - 1}{\gamma(2 - m) + N - 1}.$$  \hspace{1cm} (93)

If $m = m_2$ then

$$P(0) = \left(\frac{1}{2}\right)^{\varepsilon} \left(\frac{\gamma + 1}{\gamma}\right)^{\varepsilon\gamma N/(N + 1)} \exp\left(-\frac{\gamma}{2} \varepsilon\right),$$

$$\varepsilon = \frac{2(N + 1)}{(N + 1)(\gamma - 1) + 2}.$$  \hspace{1cm} (93)

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