Octonion X-product Orbits

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Abstract

The octonionic X-product gives to the octonions a flexibility not found in the other real division algebras (reals, complexes, quaternions). The pattern of this flexibility is investigated here.

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1. Introduction.

The inspiration for this article arose from three sources [1] [2] [3]. In [1] the octonionic X-product was introduced, and it was pointed out that although the 7-sphere (S\(^7\)) is the unique parallelizable manifold not also a group manifold, with the aid of the X-product S\(^7\) gains an almost group structure.

In [2] the X-product was applied to the 480 renumberings of the octonionic basis, and it was pointed out that this set of renumberings actually splits into two sets of 240 renumberings via an X-product equivalence. These two sets were dubbed opposites. In both [1] and [2] the ultimate goal was an application of the octonions to string theory, in which context the octonions play a natural role.

In [3] I presented my own view of division algebra theory and how it connects to physics. The presentation of octonion theory in that monograph is pragmatic, a kind of get-down-and-get-dirty mathematics that I find comprehensible and useful. This article is a result of the application of my methods to the octonionic X-product.

2. Four Octonionic Basis Numberings.

In all that follows the symbols \(e_a, \ a = 1, ..., 7\), will represent an orthogonal basis for the 7-dimensional imaginary (pure) octonions, and \(e_0 = 1\) will be the identity. There are 7! permutations of the indices of the pure octonions, and each gives rise to a modified copy of \(O\) (the real octonion division algebra) with an altered, but still octonionic, multiplication table. As it turns out, however, these index rearrangements are not all unique. In the end we will find that there are only 480 distinct multiplication tables for which

\[ e_a e_b = \pm e_c \]

for all \(a, b \in \{0, ..., 7\}\), and some \(a, b\)-dependent \(c\).

Of all these 480 distinct copies of \(O\), there are 4 that are singled out for their elegance and symmetry. These four arise from the following \(8 \times 8\) array of binary numbers (see [3]):

\[
O = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}
\] (1)
Let \( OR_a \) be the \( a^{th} \) row of \( O \), and \( OC_a \) the \( a^{th} \) column. The set of rows and the set of columns are individually closed under binary vector addition (denoted \( \oplus_2 \)). For example,
\[
OR_1 \oplus_2 OR_2 = OR_6; \\
OC_1 \oplus_2 OC_2 = OC_4.
\]
Taking advantage of this closure, I now let the set of rows, or the set of columns, be bases for 8-dimensional real algebras, and define the following 4 products therefrom \((a \in \{0, 1, \ldots, 7\})\):
\[
\begin{align*}
OR_a \circ OR_b &= (-1)^{O_{ab}}(OR_a \oplus_2 OR_b); \\
OR_a \ast OR_b &= (-1)^{O_{ba}}(OR_a \oplus_2 OR_b); \\
OC_a \circ OC_b &= (-1)^{O_{ab}}(OC_a \oplus_2 OC_b); \\
OC_a \ast OC_b &= (-1)^{O_{ba}}(OC_a \oplus_2 OC_b).
\end{align*}
\]
(2)

The power of \((-1)\) out front determines the sign of the result.

Each of the four products in (2) defines an 8-dimensional real algebra from the array \( O \), and each is isomorphic to the octonion algebra, \( O \). Let \( e_a \), be a basis for \( O \), \( a \in \{0, 1, \ldots, 7\} \). Given any of the products in (2), the \( e_a \) automatically satisfy the following useful properties for distinct indices \( a, b, c \in \{1, \ldots, 7\}\):
\[
\begin{align*}
\text{if } e_a e_b &= \pm e_c, \text{ then } e_{a+1} e_{b+1} &= \pm e_{c+1}; \\
\text{if } e_a e_b &= \pm e_c, \text{ then } e_{2a} e_{2b} &= \pm e_{2c};
\end{align*}
\]
(3)
(4)
where the indices in (3) and (4) are understood to cycle from 1 to 7 modulo 7. The multiplication laws (3) (index cycling) and (4) (index doubling) will only both be valid for octonion multiplication rules derived from (2).

Some more general laws, valid for all the 480 renumberings of the \( e_a \) we will consider here, are
\[
\begin{align*}
e_a e_b &= \pm e_c \implies e_c e_a = \pm e_b
\end{align*}
\]
(5)
(that is, \( \{e_a, e_b, e_c\} \) are a quaternionic triple in this case), and
\[
\begin{align*}
e_a^2 &= -1,
\end{align*}
\]
(6)
where again we are assuming \( a, b, c \in \{1, \ldots, 7\} \) in (5) and (6).

The multiplication laws (3 - 6) are enough to completely determine the octonion multiplication tables resulting from the following four product rules
(which arise from the four respective products rules defined in (2)):

\[
\begin{align*}
\mathbf{O}^{+5} &: \ e_a e_{a+1} = e_{a+5}; \\
\mathbf{O}^{-5} &: \ e_a e_{a+1} = -e_{a+5}; \\
\mathbf{O}^{+3} &: \ e_a e_{a+1} = e_{a+3}; \\
\mathbf{O}^{-3} &: \ e_a e_{a+1} = -e_{a+3}.
\end{align*}
\]

The four copies of \(\mathbf{O}\) that result from the rules (7) and the laws (3 - 6) are the cornerstones upon which I shall built the tower of 480 renumberings, and the cement holding it all together will be the \(X\)-product \([1]\).

3. The 480 Renumberings.

Given any of the 480 distinct copies of \(\mathbf{O}\), a complete multiplication table is determined once one has listed 7 quaternionic index triples. For example, for \(\mathbf{O}^{+5}\), \(e_a e_b = e_c\) if \((abc)\) is one of the following triples, or any cyclic permutation thereof:

\[\mathbf{O}^{+5} \text{ triples } : \{(126), (237), (341), (452), (563), (674), (715)\}.
\]

There are simpler, more schematic ways of indicating the same information. A common method uses septagons, but I find the idea of making such a figure with Latex daunting, so I will use the following more concise diagrams:

\[
\begin{align*}
\mathbf{O}^{+5} &: \ 1 2 3 4 5 6 7; \\
\mathbf{O}^{-5} &: \ 7 6 5 4 3 2 1; \\
\mathbf{O}^{+3} &: \ 1 2 3 4 5 6 7; \\
\mathbf{O}^{-3} &: \ 7 6 5 4 3 2 1.
\end{align*}
\]

In each case, the quaternionic index triples of the four respective octonionic algebras are obtained via a cyclic shifting of the pattern of boxes given for the algebra. So for \(\mathbf{O}^{+5}\), (134) is followed by (245), (356), (467), (571), (612), (723), and back to (134).

So, we have 7 index triples, and each triple has 3 cyclic permutations, so there are \(3 \times 7 = 21\) pairs of distinct indices \(a, b \in \{1, \ldots, 7\}\) such that

\[e_a e_b = +e_c\]
for some $c \in \{1, \ldots, 7\}$. For each such pair $a$ and $b$ there is a boxed sequence beginning with that pair from which the algebra multiplication table is derivable. For example, in $O^{+5}$, we have $e_5e_7 = e_1$, and the sequence

$$O^{+5} : \begin{array}{cccccc}
5 & 7 & 2 & 1 & 6 & 1 & 3
\end{array};$$

results in the same set of index triples for $O^{+5}$ as does that in (8).

Therefore, via this type of rearrangement it is always possible to begin the boxed sequences of any of the 480 rearrangements of the indices of $O$ with either the pair $(1,2,\ldots)$ or $(2,1,\ldots)$. There are

$$\frac{7!}{21} = 240 = 2(5!)$$

such reorderings. And there are two inequivalent ways of boxing the triples (modulo cyclic shifts), those shown in (8). So there are

$$2(240) = 480$$
different multiplication tables resulting from rearrangements of the indices of the $e_a$. As was pointed out in [2], these fall into two groups of 240 (different groups than those above related by box pattern) related by the octonionic X-product [1].

I will consider two other boxed sequences before finishing this section. In [2] the following septuplet of index triples is introduced:

$$O^{[2]} : \{(123), (145), (167), (264), (257), (347), (356)\}. \quad (9)$$

Once you get the hang of it, determining the boxed sequence from this is easy:

$$O^{[2]} : \begin{array}{cccccc}
1 & 2 & 6 & 3 & 4 & 5 & 7
\end{array}. \quad (10)$$

In [4] (which was my introduction to the octonions), a frequently encountered octonion multiplication is used with the following septuplet of triples:

$$O^{[4]} : \{(123), (174), (275), (376), (165), (246), (354)\}. \quad (11)$$

Its boxed sequence is

$$O^{[4]} : \begin{array}{cccccc}
1 & 2 & 7 & 4 & 6 & 3 & 5
\end{array}. \quad (12)$$
4. The X-Product.

In what follows I shall use the $O^{+5}$ product as a starting point for all calculations and X-product variations unless explicitly stated otherwise.

Let $A, B, X \in O$, with $X$ a unit octonion ($XX^\dagger = 1$) \[^1\] \[^2\]. Define

$$A \circ_X B = (AX)(X^\dagger B) = (A(BX))X^\dagger = X((X^\dagger A)B),$$

(13)

the X-product of $A$ and $B$. Because of the nonassociativity of $O$, $A \circ_X B \neq AB$ in general. But remarkably, for fixed $X$, the algebra $O_X$ (O endowed with the X-product) is isomorphic to $O$ itself. Modulo sign change each $X$ gives rise to a distinct copy of $O$, so the orbit of copies of $O$ arising from any given starting copy is

$$S^7/\mathbb{Z}_2 = \mathbb{R}P^7,$$

(14)

the manifold obtained from the 7-sphere by identifying opposite points. Moreover, composition of X-products is yet another X-product. That is, if $X, Y \in S^7 \subset O$,

then

$$(A \circ_X Y) \circ_X (Y^\dagger \circ_X B) = (AX)(X^\dagger B) \stackrel{Y}{\rightarrow} [((AX)(X^\dagger Y))X][X^\dagger((Y^\dagger X)(X^\dagger B))]$$

$$= [(A(YX))X^\dagger Y][X^\dagger((X^\dagger Y)B)]$$

$$= A \circ_Y (Y^\dagger X) B,$$

(15)

using the fact that for $X \in S^7$,

$$(UX^\dagger)X = U = X^\dagger(XU)$$

for all $U \in O$. (This would seem to endow $\mathbb{R}P^7$ with a Lie group structure, but in composing with yet a third element of $S^7$ one runs into the nonassociativity of $O$, which spoils the game.)

Clearly in general the result of the X-product $e_a \circ_X e_b$, $0 \neq a \neq b \neq 0$, will be a linear combination of $e_c$, $c \in \{1, \ldots, 7\}$ (it is not difficult to prove that such a product can not have any terms linear in the identity). There are some $X$, however, such that for all $a, b \in \{1, \ldots, 7\}$, there will be a particular $c \in \{1, \ldots, 7\}$ satisfying

$$e_a \circ_X e_b = e_c.$$
Starting from $O^{+5}$, any such $X$ resides in one of the following sets:

\[
\begin{align*}
\Xi^{+5}_0 &= \{ \pm e_a \}, \\
\Xi^{+5}_1 &= \{(\pm e_a \pm e_b)/\sqrt{2} : a, b \text{ distinct}\}, \\
\Xi^{+5}_2 &= \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct, } e_a(e_b e_c e_d) = \pm 1\}, \\
\Xi^{+5}_3 &= \{(\sum_{a=0}^7 \pm e_a)/\sqrt{8} : \text{odd number of } +\text{'s}\},
\end{align*}
\]

\begin{itemize}
\item $a, b, c, d \in \{0, \ldots, 7\}, e_a, e_b, e_c, e_d \in O^{+5}$.
\end{itemize}

\[(16)\]

- NOTE: These sets not general. They will work for $O^{\pm 5}$, and certain of their X-product variants. See section 5. (Quite frankly, I don’t know how many such sets there are for the total of 480 $O$’s, but not 480.)

In [3] I explicitly calculated the effect of the X-product (13). For example, for a general

\[X = X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7 \in S^7,\]

\[
e_1 \circ_X e_2 =
\begin{align*}
&((X^0)^2 + (X^1)^2 + (X^2)^2 + (X^6)^2 - (X^3)^2 - (X^4)^2 - (X^5)^2 - (X^7)^2)e_6 \\
&+2(X^0 X^5 + X^1 X^7 - X^2 X^4 + X^3 X^6)e_3 \\
&+2(-X^0 X^7 + X^1 X^5 + X^2 X^3 + X^4 X^6)e_4 \\
&+2(-X^0 X^3 - X^1 X^4 - X^2 X^7 + X^5 X^6)e_5 \\
&+2(X^0 X^4 - X^1 X^3 + X^2 X^5 + X^7 X^6)e_7.
\end{align*}
\]

\[(17)\]

Because (17) arises from $O^{+5}$, all the other possible $e_a \circ_X e_b$ are derivable from (17) via index cycling and doubling (in both cases, the index 0 is left out, and only the indices $a = 1, \ldots, 7$ are subject to the cycling and doubling operations). Some other products that will prove useful later are:
### Table of X-Products for \( O^5 \)

| X-Product | Expression                                                                 |
|-----------|-----------------------------------------------------------------------------|
| \( e_3 \circ X e_4 \) | \(((X^0)^2 + (X^3)^2 + (X^4)^2 + (X^1)^2 - (X^2)^2 - (X^5)^2 - (X^6)^2 - (X^7)^2)e_1 \)  
  + 2(X^0X^7 + X^3X^2 - X^4X^6 + X^5X^1)e_5  
  + 2(-X^0X^2 + X^3X^7 + X^4X^5 + X^6X^1)e_6  
  + 2(-X^0X^5 - X^3X^6 - X^4X^2 + X^7X^1)e_7  
  + 2(X^0X^6 - X^3X^5 + X^4X^7 + X^2X^1)e_2. |
$O^{+5}$ X-Products continued

| $e_1 \circ_X e_3$ | $((X^0)^2 + (X^1)^2 + (X^3)^2 + (X^4)^2 - (X^2)^2 - (X^5)^2 - (X^6)^2 - (X^7)^2)e_4$ |
|-------------------|-----------------------------------------------------------------|
|                   | $+ 2(X^0 X^2 + X^1 X^6 - X^3 X^7 + X^5 X^4)e_5$                  |
|                   | $+ 2(-X^0 X^6 + X^1 X^2 + X^3 X^5 + X^7 X^4)e_7$                 |
|                   | $+ 2(-X^0 X^5 - X^1 X^7 - X^3 X^6 + X^2 X^4)e_2$                 |
|                   | $+ 2(X^0 X^7 - X^1 X^5 + X^3 X^2 + X^6 X^4)e_6$.                |

| $e_2 \circ_X e_4$ | $((X^0)^2 + (X^2)^2 + (X^4)^2 + (X^5)^2 - (X^1)^2 - (X^3)^2 - (X^6)^2 - (X^7)^2)e_5$ |
|-------------------|-----------------------------------------------------------------|
|                   | $+ 2(X^0 X^3 + X^2 X^7 - X^4 X^1 + X^6 X^5)e_6$                  |
|                   | $+ 2(-X^0 X^7 + X^2 X^3 + X^4 X^6 + X^1 X^5)e_1$                 |
|                   | $+ 2(-X^0 X^6 - X^2 X^1 - X^4 X^7 + X^3 X^5)e_3$                 |
|                   | $+ 2(X^0 X^1 - X^2 X^6 + X^4 X^3 + X^7 X^5)e_7$.                |

| $e_3 \circ_X e_5$ | $((X^0)^2 + (X^3)^2 + (X^5)^2 + (X^6)^2 - (X^1)^2 - (X^2)^2 - (X^4)^2 - (X^7)^2)e_6$ |
|-------------------|-----------------------------------------------------------------|
|                   | $+ 2(X^0 X^4 + X^3 X^1 - X^5 X^2 + X^7 X^6)e_7$                  |
|                   | $+ 2(-X^0 X^1 + X^3 X^4 + X^5 X^7 + X^2 X^6)e_2$                 |
|                   | $+ 2(-X^0 X^7 - X^3 X^2 - X^5 X^1 + X^4 X^6)e_4$                 |
|                   | $+ 2(X^0 X^2 - X^3 X^7 + X^5 X^4 + X^1 X^6)e_1$.                |

| $e_4 \circ_X e_6$ | $((X^0)^2 + (X^4)^2 + (X^6)^2 + (X^7)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^5)^2)e_7$ |
|-------------------|-----------------------------------------------------------------|
|                   | $+ 2(X^0 X^5 + X^4 X^2 - X^6 X^3 + X^1 X^7)e_1$                  |
|                   | $+ 2(-X^0 X^2 + X^4 X^5 + X^6 X^1 + X^3 X^7)e_3$                 |
|                   | $+ 2(-X^0 X^1 - X^4 X^3 - X^6 X^2 + X^5 X^7)e_5$                 |
|                   | $+ 2(X^0 X^3 - X^4 X^1 + X^6 X^5 + X^2 X^7)e_2$.                |
Clearly this is not a complete set, but it will suffice to construct a few examples.

• EXAMPLE 1: \( X = \left( e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 \right) / \sqrt{8} \).

Note that because this \( X \) is invariant under both index cycling and index doubling, the product \( e_a \circ_X e_b \) will have the index cycling and doubling properties shared by the products (7). Using the tables given above we can see that the index triples associated with this modification of the \( O X^+5 \) product are those of \( O X^+3 \). That is,

\[
\text{if } X = \left( e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 \right) / \sqrt{8}, \text{ then } O X^+_5 = O X^+_3. \tag{18}
\]

Since \( XX^\dagger = 1 \),

\[
\text{if } X = \left( e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 \right) / \sqrt{8}, \text{ then } O X^+_3 = O X^+_5. \tag{19}
\]

Therefore \( O X^+_5 \) and \( O X^+_3 \) are part of the same orbit of octonion \( X \)-product variants.

• EXAMPLE 2: \( X = (e_1 - e_2 - e_4 - e_7)/2 \).

In this case,

\[ X^0 = X^3 = X^5 = X^6 = 0; \quad X^1 = -X^2 = -X^4 = -X^7 = \frac{1}{2}. \]

Plug these values into the general form for \( e_1 \circ_X e_2 \) in (17) and get

\[ e_1 \circ_X e_2 = -e_3. \]

Therefore (132) is a quaternionic index triple for this \( O_X \). Likewise, from the general \( e_4 \circ_X e_5 \) table,

\[ e_4 \circ_X e_5 = -e_1, \]

so (154) is another such triple. Carrying on in this way leads to a complete set, with corresponding boxed sequence:

\[ (321), (541), (761), (462), (752), (743), (653). \tag{20} \]

Compare this set of triples to the set (9) for \( O^{[2]} \). Each of the triples in (20) is reversed with respect to a corresponding triple in (9). Following [2] I shall call the copy of \( O \) generated from (20) the opposite of that generated from (9) (\( O^{[2]} \)). Let’s denote it \( O^{[2]} \), the underline signifying opposite.
So

\[ O^{[2]} = O_{X}^{+ 5}. \]

That is, they are a part of the same orbit. What about \( O^{[2]} \) itself? It turns out that \( O^{+5} \) and \( O^{[2]} \) are not in the same orbit. Consider \( O^{+5} \) and \( O^{-5} \) as an example. The quaternionic triples of \( O^{-5} \) are the reverse of those for \( O^{+5} \). (So \( O^{+5} = O^{-5} \).)

Therefore,

\[ e_1 e_2 = e_6 \text{ in } O^{+5} \text{ implies } e_1 e_2 = -e_6 \text{ in } O^{-5}. \]

For this to happen via an X-product, we can see from (17) that

\[
\begin{align*}
(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^6)^2 - (X^3)^2 - (X^4)^2 - (X^5)^2 - (X^7)^2 &= -1, \\
X^0 X^5 + X^1 X^7 - X^2 X^4 + X^3 X^6 &= 0, \\
-X^0 X^7 + X^1 X^5 + X^2 X^3 + X^4 X^6 &= 0, \\
-X^0 X^3 - X^1 X^4 - X^2 X^7 + X^5 X^6 &= 0, \\
X^0 X^4 - X^1 X^3 + X^2 X^5 + X^7 X^6 &= 0.
\end{align*}
\]

These equations can be satisfied in several ways. For example, set \( X^7 = 1 \), and \( X^a = 0, \ a = 0, ..., 6 \). But whatever value of \( X \) we choose must reverse not only this product, but every other \( O^{+5} \) product. In particular, this implies that each of the following eight equations must be satisfied:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
(X^0)^2 \\
(X^1)^2 \\
(X^2)^2 \\
(X^6)^2 \\
(X^3)^2 \\
(X^4)^2 \\
(X^5)^2 \\
(X^7)^2
\end{bmatrix}
= \begin{bmatrix}
1 \\
-1 \\
-1 \\
-1 \\
-1 \\
-1 \\
-1 \\
-1
\end{bmatrix}.
\]

The inverse of that square matrix is \( \frac{1}{8} \) times its transpose. Therefore a solution is easily obtained, and in particular it implies that

\[ (X_0)^2 = -\frac{3}{4}, \]

which of course is not possible for the real algebra \( O \). (As it turns out, even the complexification of \( O \) wouldn’t help in the end.)
• EXAMPLE 3: \( X = (e_0 - e_1 - e_2 - e_3 - e_4 - e_5 + e_6 + e_7)/\sqrt{8} \).
In this case, using (17) and the other \( \mathbf{O}^{+5} \) X-product tables, we find:
\[
e_1 \circ_X e_2 = -e_3, \quad e_3 \circ_X e_4 = e_5, \quad e_5 \circ_X e_6 = e_1, \quad e_6 \circ_X e_7 = e_3, \quad e_2 \circ_X e_4 = -e_6.
\]
By cyclically shifting the \( e_4 \circ_X e_6 \) table up by 1, and the \( e_1 \circ_X e_2 \) table down by 1, we get
\[
e_5 \circ_X e_7 = e_2, \quad e_7 \circ_X e_1 = e_4.
\]
Therefore, a complete set of quaternionic triples for this case, with corresponding boxed sequence, is:
\[
(321), (471), (572), (673), (561), (642), (453),
\]
\[
\boxed{2 \ 1 \ 5 \ 3 \ 6 \ 4 \ 7}.
\]
These are all opposite those for \( \mathbf{O}^{[4]} \) listed in (11). Therefore, in this case,
\[
\mathbf{O}^{+5}_X = \mathbf{O}^{[4]}.
\]

• NOTE: In general, if \( \mathbf{O} \) and \( \mathbf{O}' \) are in the same X-product orbit, then \( \mathbf{O} \) and \( \mathbf{O}' \) are not. There are two orbits all together, one arising from \( \mathbf{O}^{+5} \) and containing \( \mathbf{O}^{+3} \) (\( \text{Orbit}^+ \)), and one arising from \( \mathbf{O}^{-5} \) and containing \( \mathbf{O}^{-3} \) (\( \text{Orbit}^- \)).

5. Two X-Product Orbits.

Look again at the sets \( \Xi_a^{\pm 5}, \ a = 0, 1, 2, 3, \) elements of which cause index rearrangements of \( \mathbf{O}^{\pm 5} \) via X-product variation. Modulo sign change,
\[
\begin{align*}
\text{order } \Xi_0^{\pm 5} &= 8, \\
\text{order } \Xi_1^{\pm 5} &= 56, \\
\text{order } \Xi_2^{\pm 5} &= 112, \\
\text{order } \Xi_3^{\pm 5} &= 64.
\end{align*}
\]
So there are 240 rearrangements arising from these elements all together. Hence each of the two X-product orbits, \( \text{Orbit}^{\pm} \), contains 240 of the 480 octonion index
rearrangements. In each orbit there are 120 rearrangements using the $O^{+5}$ (and $O^{-3}$) pattern of boxes (denote these $\text{Orbit}_0^{\pm}$, a discrete subset of $\text{Orbit}^{\pm}$),

$$
\begin{array}{c}
\begin{array}{c}
\text{Orbit}_0^{+} \\
\uparrow \\
\text{Orbit}_0^{-}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_1^{+5} \cup \Xi_3^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
O^{+5} \\
O^{-5}
\end{array}
\end{array}
\begin{array}{c}
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\begin{array}{c}
\text{Orbit}_1^{+} \\
\text{Orbit}_1^{-}
\end{array}
$$

(23)

and 120 using the $O^{-5}$ (and $O^{+3}$) pattern of boxes (denote these $\text{Orbit}_1^{\pm}$),

$$
\begin{array}{c}
\begin{array}{c}
\text{Orbit}_1^{+} \\
\uparrow \\
\text{Orbit}_1^{-}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_1^{+5} \cup \Xi_3^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
O^{-5} \\
O^{+5}
\end{array}
\end{array}
\begin{array}{c}
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\begin{array}{c}
\text{Orbit}_0^{+} \\
\text{Orbit}_0^{-}
\end{array}
$$

(24)

Starting from $O^{+5}$, the 120 distinct X-products arising from elements of $\Xi_0^{+5} \cup \Xi_2^{+5}$ will result in variants $O^{+5}_X$ sharing $O^{+5}$'s box pattern (23). For example, the boxed sequence of $O^{+5}_X$, $X = (e_1 - e_2 - e_4 - e_7)/2 \in \Xi_2^{+5}$, given in (2), has the same box pattern as $O^{+5}$ itself. (Remember that for any given reordering of the indices only one of the box patterns is possible.)

Starting from $O^{+5}$, the 120 distinct X-products arising from elements of $\Xi_1^{+5} \cup \Xi_3^{+5}$ will result in variants $O^{+5}_X$ having the box pattern (24). For example, the boxed sequence of $O^{+5}_X = O^{+3}, X = (1 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7)/\sqrt{8} \in \Xi_3^{+5}$, has pattern (24).

Schematically,

$$
\begin{array}{c}
\begin{array}{c}
\text{Orbit}_1^{+} \\
\uparrow \\
\text{Orbit}_1^{-}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Xi_1^{+5} \cup \Xi_3^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
O^{+5} \\
O^{-5}
\end{array}
\end{array}
\begin{array}{c}
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5} \\
\Xi_0^{+5} \cup \Xi_2^{+5}
\end{array}
\begin{array}{c}
\text{Orbit}_0^{+} \\
\text{Orbit}_0^{-}
\end{array}
$$

(25)

• NOTE: If $X \notin \Xi_0^{+5} \cup \Xi_1^{+5} \cup \Xi_2^{+5} \cup \Xi_3^{+5}$, then $O^{+5}_X$ is not any of the simple index rearrangements of $O^{+5}$, which are after all just discrete points in the full $RP^7$ orbit of $O^{+5}$.

The set $\Xi_0^{+5} \cup \Xi_1^{+5} \cup \Xi_2^{+5} \cup \Xi_3^{+5}$ is $O^{+5}$ specific. Starting from a general $O^{+5}_X$, for example, we know that

$$(O^{+5}_X)^{X^\dagger} = O^{+5}_{(X^\dagger X)} = O^{+5},$$

and

$$((O^{+5}_X)^Y)^{Y^\dagger} = O^{+5}_Y = (O^{+5}_X)^{(Y^\dagger X)}$$

(see (15)). Therefore, for $(O^{+5}_X)^Z$ to be an index rearrangement of $O^{+5}_X$, $Z$ must satisfy

$$Z \in (\Xi_0^{+5} \cup \Xi_1^{+5} \cup \Xi_2^{+5} \cup \Xi_3^{+5})X^\dagger.$$

(26)
Therefore, for example, had we started with $O_{X}^{+5} = O_{X}^{+5}$, where $X = (1 - e_1 - ... - e_7)/\sqrt{8}$, so that

$$X^\dagger = (1 + e_1 + ... + e_7)/\sqrt{8},$$

then the set $(\Xi_{0}^{+5} \cup \Xi_{1}^{+5} \cup \Xi_{2}^{+5} \cup \Xi_{3}^{+5})X^\dagger$ can be broken up into subsets

| $\Xi_{0}^{+3}$ | $\{\pm e_a\}$ |
| $\Xi_{1}^{+3}$ | $\{\pm e_a \pm e_b)/\sqrt{2} : a, b$ distinct\} |
| $\Xi_{2}^{+3}$ | $\{\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d$ distinct, $e_a(e_b(e_c(e_d))) = \pm 1\}$ |
| $\Xi_{3}^{+3}$ | $\{\sum_{a=0}^{7} \pm e_a)/\sqrt{8} : \text{even number of } +$'s\} |

$a, b, c, d \in \{0, ..., 7\}, e_a, e_b, e_c, e_d \in O^{+3}$.

(27)

Everything that was said for $O^{+5}$ and the $\Xi_{m}^{+5}$ holds true in an analogous way for $O^{+3}$ and the $\Xi_{m}^{+3}$.

6. Conclusion.

In the case of the quaternions $Q$ there are also "opposites", but because $Q$ is associative there are no X-product variations. The quaternion basis with multiplication table determined by

$$q_1 q_2 = q_3$$

has an opposite representation with a multiplication table determined by

$$q_2 q_1 = -q_1 q_2 = q_3.$$

So we drop from 480 variations down to 2.

There is obviously no way to vary the complex numbers $C$, the smallest of the three hypercomplex real division algebras.

Finally, in [1][2][3] interest in the division algebras arose from their evidently intimate connection to our physical reality. In a future article I will investigate the potential and consequences of gauging the X-product.
References

[1] M. Cederwall, C.R. Preitschopf, $S^7$ and $\tilde{S}^7$, hep-th-9309030.

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[3] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).

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