ORTHOGONALITY PRESERVING TRANSFORMATIONS OF HILBERT GRASSMANNIANS

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Abstract. Let $H$ be a complex Hilbert space and let $G_k(H)$ be the Grassmannian formed by $k$-dimensional subspaces of $H$. Suppose that $\dim H > 2k$ and $f$ is an orthogonality preserving injective transformation of $G_k(H)$, i.e., for any orthogonal $X, Y \in G_k(H)$ the images $f(X), f(Y)$ are orthogonal. If $\dim H = n$ is finite, then $n = mk_i$ for some integers $m \geq 2$ and $i \in \{0, 1, \ldots, k-1\}$ (for $i = 0$ we have $m \geq 3$). We show that $f$ is a bijection induced by a unitary or anti-unitary operator if $i \in \{0, 1, 2\}$ or $m \geq i \geq 3$. As an application, we obtain a statement concerning the case when $H$ is infinite-dimensional.

1. Introduction

By Gleason’s theorem [3], the set of pure states of a quantum mechanical system can be identified with the set of rank-one projections, i.e., the set of rays of a complex Hilbert space. Wigner’s theorem [13] describes symmetries of quantum mechanical systems, it states that every transformation of the set of pure states preserving the transition probability (the trace of the composition of two projections or, equivalently, the angle between two rays) is induced by a linear or conjugate-linear isometry. Various kinds of Wigner-type theorems can be found in [10]. In this note, we consider results concerning orthogonality preserving transformations.

Let $H$ be a complex Hilbert space. For every positive integer $k < \dim H$ we denote by $G_k(H)$ the Grassmannian formed by $k$-dimensional subspaces of $H$. It can be naturally identified with the set of rank-$k$ projections. In the case when $\dim H \geq 2k$, two $k$-dimensional subspaces are orthogonal if and only if the composition of the corresponding projections is zero.

Suppose that $\dim H \geq 3$. Then the bijective version of Wigner’s theorem is a consequence of the following Uhlhorn’s observation [12]: every bijective transformation of $G_1(H)$ preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator. In fact, the latter statement is a reformulation of the Fundamental Theorem of Projective Geometry (see, for example, [10, Proposition 4.8]).

Uhlhorn’s observation was extended on other Grassmannians by Győry [4] and Šemrl [11]: if $\dim H > 2k$, then every bijective transformation of $G_k(H)$ preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator. A simple example shows that the statement fails for $\dim H = 2k$. If $H$ is infinite-dimensional, then the same holds for orthogonality preserving (in both directions) bijective transformations of the Grassmannian formed by subspaces whose dimension and codimension both are infinite [11]. Győry–Šemrl’s theorem is used to

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study transformations preserving the gap metric \([2]\) and commutativity preserving transformations \([7, 8]\).

It was noted in \([11]\) that for the case when \(H\) is infinite-dimensional there are transformations of \(G_k(H)\) which are not induced by linear or conjugate-linear isometries and preserve the orthogonality relation in both directions. If \(H\) is finite-dimensional and \(\dim H > 2k\), then such transformations do not exist, i.e. every transformation of \(G_k(H)\) preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator \([9]\).

In this note, we obtain analogues of the above mentioned results for injective transformations under the assumption that the orthogonality relation is preserved only in one direction.

2. Results

We start from the case when \(H\) is finite-dimensional. Let \(k\) be a positive integer such that \(\dim H > 2k\). Then

\[
\dim H = mk + i
\]

for some integers \(m \geq 2\) and \(i \in \{0, 1, \ldots, k - 1\}\). Note that \(m \geq 3\) if \(i = 0\) (since \(\dim H > 2k\)).

**Theorem 1.** Suppose that one of the following conditions is satisfied:

- \(i \in \{0, 1, 2\}\),
- \(i \geq 3\) and \(m \geq i\).

Then every injective transformation \(f\) of \(G_k(H)\) preserving the orthogonality relation, i.e. for any orthogonal \(X, Y \in G_k(H)\) the images \(f(X), f(Y)\) are orthogonal, is a bijection induced by a unitary or anti-unitary operator on \(H\).

In particular, Theorem \([1]\) shows that every injective transformation of \(G_k(H)\) preserving the orthogonality relation is a bijection induced by a unitary or anti-unitary operator if

\[
k \in \{1, 2, 3\} \text{ or } k \geq 4 \text{ and } \dim H \geq (k - 1)k.
\]

For \(k = 1\) Theorem \([1]\) is a simple consequence of the Fundamental Theorem of Projective Geometry \([10, \text{Remark 49}]\).

Now, we suppose that \(H\) is infinite-dimensional and consider the non-bijective orthogonality preserving (in both directions) transformation of \(G_k(H)\) constructed in \([11]\).

**Example 1.** Let be \(U : H \to H\) be a linear isometry whose image is a proper subspace of \(H\). Then \(U^*\) is surjective and its kernel is the orthogonal complement of the image of \(U\). We fix \(X \in G_k(H)\). Since \(U^*U\) is identity, there is \(X' \in G_k(H)\) which is not contained in the image of \(U\) and such that \(U^*(X') = X\). Denote by \(f\) the transformation of \(G_k(H)\) which sends \(X\) to \(X'\) and every \(Y \in G_k(H) \setminus \{X\}\) to \(U(Y)\). We obtain an injection preserving the orthogonality relation in both directions which is not induced by a linear or conjugate-linear isometry. Let \(H'\) be the smallest closed subspace of \(H\) containing all elements from the image of \(f\). We take any \(Y \in G_k(H) \setminus \{X\}\) non-orthogonal to \(X\) and any family \(\{Z_i\}_{i \in I}\) of mutually orthogonal elements of \(G_k(H)\) whose sum is \(Y^\perp\). Then \(f(Y) = U(Y)\) and \(f(Z_i) = U(Z_i)\) for every \(i \in I\). Therefore, \(U(H)\) is the orthogonal sum of \(f(Y)\) and
all \( f(Z_i) \). Note that \( U(H) \) is a proper subspace of \( H' \), since it does not contain \( f(X) = X' \).

**Corollary 1.** Let \( f \) be an injective transformation of \( \mathcal{G}_k(H) \) and let \( H' \) be the smallest closed subspace of \( H \) containing all elements from the image of \( f \). Suppose that \( f \) satisfies the following condition:

\((^*)\) for any orthogonal \( X, Y \in \mathcal{G}_k(H) \) there is a family \( \{Z_i\}_{i \in I} \) of mutually orthogonal elements of \( \mathcal{G}_k(H) \) such that each \( Z_i \) is orthogonal to \( X + Y \) and \( H' \) is the orthogonal sum of \( f(X), f(Y) \) and all \( f(Z_i) \).

Then \( f \) is induced by a linear or conjugate-linear isometry.

It is clear that \((^*)\) implies that \( f \) is orthogonality preserving and the transformation from Example 1 does not satisfy \((^*)\).

### 3. Proof of Theorem 1 and Corollary 1

#### 3.1. Proof of Theorem 1

First of all, we present some facts which will be exploited to prove Theorem 1. Two \( k \)-dimensional subspaces of \( H \) are called adjacent if their intersection is \((k-1)\)-dimensional. Any two distinct 1-dimensional subspaces of \( H \) are adjacent. Similarly, if \( \dim H = n \) is finite, then any two distinct \((n-1)\)-dimensional subspaces of \( H \) are adjacent. For the remaining cases the adjacency relation is not trivial.

**Theorem 2 (\cite{9}).** Suppose that \( \dim H > 2k > 2 \) and \( f \) is a transformation of \( \mathcal{G}_k(H) \) satisfying the following conditions:

- \( f \) is adjacency preserving, i.e. for any adjacent \( X, Y \in \mathcal{G}_k(H) \) the images \( f(X), f(Y) \) are adjacent;
- \( f \) is orthogonality preserving.

Then \( f \) is induced by a linear or a conjugate-linear isometry.

Also, we will need the following result mentioned in Introduction.

**Theorem 3 (\cite{9}).** If the dimension of \( H \) is finite and greater than \( 2k \), then every transformation of \( \mathcal{G}_k(H) \) preserving the orthogonality relation in both directions is a bijective transformation induced by a unitary or anti-unitary operator on \( H \).

Suppose that \( \dim H = n \) is finite. It was noted above that Theorem 1 holds for \( k = 1 \). We assume that \( k \geq 2 \) and

\[
n = mk + i > 2k,
\]

where \( m \geq 2 \) and \( i \in \{0, 1, \ldots, k-1\} \). Let \( f \) be an injective transformation of \( \mathcal{G}_k(H) \) preserving the orthogonality relation.

**The case** \( i = 0 \). In this case, we have \( m \geq 3 \). Suppose that \( f(X), f(Y) \) are orthogonal for some \( X, Y \in \mathcal{G}_k(H) \) and show that \( X, Y \) are orthogonal. Observe that

\[
\dim(X^\perp \cap Y^\perp) \geq n - 2k = (m-2)k.
\]

In the case when \( m \geq 4 \), there are mutually orthogonal \( k \)-dimensional subspaces

\[
Z_1, \ldots, Z_{m-2} \subset X^\perp \cap Y^\perp;
\]
if \( m = 3 \), then we take any \( k \)-dimensional subspace \( Z_1 \) in \( X^\perp \cap Y^\perp \). Since \( f \) is orthogonality preserving, each \( f(Z_i) \) is orthogonal to \( f(X) + f(Y) \). By our assumption, \( f(X) \) and \( f(Y) \) are orthogonal. Then \( H \) is the orthogonal sum of

\[
f(Z_1), \ldots, f(Z_{m-2}), f(X), f(Y).
\]

The dimension of \( X^\perp \) is equal to \( n - k = (m-1)k \), i.e. \( X^\perp \) contains the unique \( k \)-dimensional subspace \( Z \) orthogonal to all \( Z_i \) and \( H \) is the orthogonal sum of

\[
f(Z_1), \ldots, f(Z_{m-2}), f(X), f(Z).
\]

Therefore, \( f(Y) = f(Z) \). Since \( f \) is injective, we have \( Y = Z \), i.e. \( Y \) is orthogonal to \( X \).

So, \( f \) is orthogonality preserving in both directions and Theorem 3 gives the claim. \( \Box \)

The case \( i \in \{1, 2\} \). It is sufficient to show that \( f \) is adjacency preserving and apply Theorem 2.

Suppose that \( X, Y \in G_k(H) \) are adjacent. Then \( \dim(X + Y) = k + 1. \)

If \( i = 1 \), then

\[
\dim(X + Y)^\perp = n - (k+1) = mk + 1 - (k+1) = (m-1)k
\]

and there are mutually orthogonal \( X_1, \ldots, X_{m-1} \in G_k(H) \) whose sum is orthogonal to \( X + Y \) (in the case when \( m = 2 \), we take the unique \( k \)-dimensional subspace \( X_1 \) orthogonal to \( X + Y \)). Then

\[
f(X_1) + \cdots + f(X_{m-1})
\]

is a subspace of dimension \((m-1)k\) orthogonal to \( f(X) + f(Y) \) and

\[
\dim(f(X) + f(Y)) \leq n - (m-1)k = mk + 1 - (m-1)k = k + 1.
\]

Since \( f \) is injective, \( f(X) \neq f(Y) \) and the latter inequality implies that \( f(X), f(Y) \) are adjacent.

In the case when \( i = 2 \), we have

\[
\dim(X + Y)^\perp = n - (k+1) = mk + 2 - (k+1) = (m-1)k + 1.
\]

Suppose that \( m \geq 3 \) and choose mutually orthogonal \( X_1, \ldots, X_{m-2} \in G_k(H) \) whose sum is orthogonal to \( X + Y \) (if \( m = 3 \), then we take any \( k \)-dimensional subspace \( X_1 \) orthogonal to \( X + Y \)). Then

\[
f(X_1) + \cdots + f(X_{m-2})
\]

is a subspace of dimension \((m-2)k\) orthogonal to \( f(X) + f(Y) \). Consider the subspace

\[
Z = X_1 + \cdots + X_{m-2} + X + Y
\]

whose dimension is \((m-2)k + k + 1 = (m-1)k + 1. \) Since

\[
\dim Z^\perp = n - (m-1)k - 1 = mk + 2 - (m-1)k - 1 = k + 1,
\]

there are distinct \( X', Y' \in G_k(H) \) orthogonal to \( Z \). Then \( f(X') \) and \( f(Y') \) are distinct \( k \)-dimensional subspaces orthogonal to the subspace

\[
Z' = f(X_1) + \cdots + f(X_{m-2}) + f(X) + f(Y),
\]

i.e. the codimension of \( Z' \) is not less than \( k + 1 \). Therefore,

\[
\dim Z' \leq n - (k+1) = (m-1)k + 1 = (m-2)k + k + 1.
\]
The subspace $Z'$ is the orthogonal sum of
\[ f(X) + f(Y) \] and \[ f(X_1) + \cdots + f(X_{m-2}) \]
and the dimension of the second subspace is $(m-2)k$. This implies that
\[ \dim(f(X) + f(Y)) \leq k + 1, \]
i.e. $f(X), f(Y)$ are adjacent (since $f(X) \neq f(Y)$).

For $m = 2$ we have
\[ \dim(X + Y) = n - (k + 1) = 2k + 2 - k - 1 = k + 1 \]
and choose distinct $X', Y' \in \mathcal{G}_k(H)$ orthogonal to $X + Y$. We repeat the latter arguments an establish that $f(X), f(Y)$ are adjacent. \(\qed\)

**The case $i \geq 3$.** We have $m \geq i$ according to the assumption. As above, we show that $f$ is adjacency preserving.

Suppose that $X, Y \in \mathcal{G}_k(H)$ are adjacent. Then $\dim(X + Y) = k + 1$ and
\[ \dim(X + Y) = n - (k + 1) = mk + i - k - 1 = (m - 1)k + i - 1. \]
Without loss of generality, we can assume that $m = i$. In the case when $m - i > 0$, we choose mutually orthogonal $k$-dimensional subspaces
\[ X_1, \ldots, X_{m-i} \subset (X + Y)^{\perp} \]
(if $m - i = 1$, then we take any $k$-dimensional subspace $X_1$ orthogonal to $X + Y$). The subspaces
\[ X_1 + \cdots + X_{m-i} \text{ and } f(X_1) + \cdots + f(X_{m-i}) \]
both are of dimension $(m - i)k$ and the dimension of the orthogonal complements
\[ M = (X_1 + \cdots + X_{m-i})^\perp \text{ and } N = (f(X_1) + \cdots + f(X_{m-i}))^\perp \]
is $n - (m - i)k = mk + i - (m - i)k = ik + i$. The map $f$ sends any $k$-dimensional subspace of $M$ to a $k$-dimensional subspace of $N$, i.e. it induces an orthogonality preserving injection of $\mathcal{G}_k(M)$ to $\mathcal{G}_k(N)$, where $M$ and $N$ are $(ik + i)$-dimensional and $X, Y \subset M$.

From this moment, we assume that $m = i$. Observe that two $k$-dimensional subspaces of $H$ are adjacent if and only if their orthogonal complement are adjacent. In particular, we have
\[ \dim(X^\perp \cap Y^\perp) = n - k - 1 = ik + i - k - 1 = (i - 1)k + i - 1. \]
In the case when $f(X), f(Y)$ are not adjacent, their orthogonal complement also are not adjacent and
\[ \dim(f(X)^\perp \cap f(Y)^\perp) < (i - 1)k + i - 1. \]
We set
\[ M_1 = X^\perp \cap Y^\perp \text{ and } N_1 = f(X)^\perp \cap f(Y)^\perp. \]
Then $f$ sends every $k$-dimensional subspace of $M_1$ to a $k$-dimensional subspace of $N_1$. In other words, $f$ induces an orthogonality preserving injection
\[ f_1 : \mathcal{G}_k(M_1) \to \mathcal{G}_k(N_1), \]
where
\[ \dim N_1 < \dim M_1 = (i - 1)k + i - 1. \]
Now, let us take any adjacent $k$-dimensional subspaces $U, V \subset M_1$ and consider the subspace

$$M_2 = U^\perp \cap V^\perp \cap M_1$$

whose dimension is equal to

$$\dim M_1 - (k + 1) = (i - 2)k + i - 2.$$ 

The map $f$ sends every $k$-dimensional subspace of $M_1$ to a $k$-dimensional subspace contained in

$$N_2 = f(U)^\perp \cap f(V)^\perp \cap N_1.$$ 

Since $\dim N_1 < \dim M_1$ and $f(U), f(V)$ are distinct $k$-dimensional subspaces of $N_1$, we have $\dim N_2 < \dim M_2$.

Recursively, we establish that $f$ induces a sequence of maps

$$f_j : \mathcal{G}_k(M_j) \to \mathcal{G}_k(N_j), \quad j = 1, \ldots, i - 1,$$

where

$$\dim N_j < \dim M_j = (i - j)k + i - j$$

for every $j$. Also, $f_1, \ldots, f_{i-2}$ are orthogonality preserving injections and $f_{i-1}$ is constant. The latter contradicts the injectivity of $f$.

Therefore, $f$ is adjacency preserving. \qed

3.2. Proof of Corollary \[1\]. We will use Faure-Frölicher-Havlicek’s version of the Fundamental Theorem of Projective Geometry \[1\] \[5\] to prove the statement for $k = 1$.

Let $V$ and $V'$ be left vector spaces over division rings $R$ and $R'$, respectively. The dimensions of the vector spaces are assumed to be not less than 3. Denote by $\mathcal{G}_1(V)$ and $\mathcal{G}_1(V')$ the sets of 1-dimensional subspaces of $V$ and $V'$, respectively. A map $L : V \to V'$ is semilinear if

$$L(x + y) = L(x) + L(y) \quad x, y \in V$$

and there is a non-zero homeomorphism $\sigma : R \to R'$ such that

$$L(ax) = \sigma(a)L(x) \quad a \in R, \quad x \in V.$$ 

Every semilinear injection $L : V \to V'$ induces a map between $\mathcal{G}_1(V)$ and $\mathcal{G}_1(V')$ which sends every $P \in \mathcal{G}_1(V)$ to the 1-dimensional subspace containing $L(P)$.

**Theorem 4** (\[1\] \[5\]). Suppose that $f : \mathcal{G}_1(V) \to \mathcal{G}_1(V')$ is an injection satisfying the following conditions:

1. If $P, P', P'' \in \mathcal{G}_1(V)$ and $P \subset P' + P''$, then $f(P) \subset f(P') + f(P'')$;
2. there is no 2-dimensional subspace of $V'$ which contains all elements from the image of $f$.

Then $f$ is induces by a semilinear injection of $V$ to $V'$.

**Lemma 1** (Proposition 4.2 in \[10\]). If an injective semilinear transformation of $H$ sends orthogonal vectors to orthogonal vectors, then it is a scalar multiple of a linear or conjugate-linear isometry.

Suppose that $H$ is infinite-dimensional and $f$ is an injective transformation of $\mathcal{G}_k(H)$ satisfying the condition (*) from Corollary \[1\] i.e. for any orthogonal $X, Y \in \mathcal{G}_k(H)$ there is a family $\{Z_i\}_{i \in I}$ of mutually orthogonal elements of $\mathcal{G}_k(H)$ such that each $Z_i$ is orthogonal to $X + Y$ and $H'$ is the orthogonal sum of $f(X), f(Y)$
and all \( f(Z_i) \), where \( H' \) is the smallest closed subspace containing all elements from the image of \( f \). Then \( f \) is orthogonality preserving.

Suppose that \( k = 1 \) and show that \( f \) satisfies the conditions of Theorem 4. Since \( f \) is orthogonality preserving, \( H' \) is infinite-dimensional and (2) holds. For any distinct \( X, Y \in \mathcal{G}_1(H) \) there is \( Y' \in \mathcal{G}_1(H) \) orthogonal to \( X \) and such that \( X + Y = X + Y' \). By (*) there is a family \( \{Z_i\}_{i \in I} \) of mutually orthogonal elements of \( \mathcal{G}_1(H) \) such that each \( Z_i \) is orthogonal to \( X + Y' \) and \( H' \) is the orthogonal sum of \( f(X), f(Y') \) and all \( f(Z_i) \). Since \( Y \subset X + Y' \) is orthogonal to all \( Z_i \), \( f(Y) \) is orthogonal to every \( f(Z_i) \). This means that \( f(Y) \) is contained in \( f(X) + f(Y') \) (since \( H' \) is the orthogonal sum of \( f(X), f(Y') \) and all \( f(Z_i) \)). By the injectivity of \( f \), we have

\[
f(X) + f(Y) = f(X) + f(Y').
\]

Similarly, for every 1-dimensional subspace \( Z \) contained in \( X + Y = X + Y' \) we establish that \( f(Z) \subset f(X) + f(Y) \), i.e., the condition (1) also is satisfied. So, \( f \) is induced by a semilinear injection and Lemma 3 gives the claim.

In the case when \( k \geq 2 \), it is sufficient to show that \( f \) is adjacency preserving and apply Theorem 2. Suppose that \( X, Y \in \mathcal{G}_k(H) \) are adjacent. There exists \( Y' \in \mathcal{G}_k(H) \) orthogonal to \( X \) and such that

\[ X + Y \subset X + Y'. \]

Let \( \{Z_i\}_{i \in I} \) be a family of mutually orthogonal elements of \( \mathcal{G}_k(H) \) such that each \( Z_i \) is orthogonal to \( X + Y' \) and \( H' \) is the orthogonal sum of \( f(X), f(Y') \) and all \( f(Z_i) \). We fix \( i_0 \in I \) and consider the \( (3k) \)-dimensional subspaces

\[ M = X + Y' + Z_{i_0} \quad \text{and} \quad N = f(X) + f(Y') + f(Z_{i_0}). \]

Every \( k \)-dimensional subspace \( Z \subset M \) is orthogonal to all \( Z_i \) with \( i \neq i_0 \). Then \( f(Z) \) is orthogonal to all \( f(Z_i) \) with \( i \neq i_0 \), i.e., \( f(Z) \) is contained in \( N \) (since \( H' \) is the orthogonal sum of \( N \) and all \( f(Z_i) \) with \( i \neq i_0 \)). Therefore, \( f \) induces an orthogonality preserving injection of \( \mathcal{G}_k(M) \) to \( \mathcal{G}_k(N) \). By Theorem 1 this restriction is induced by a unitary or anti-unitary operator from \( M \) to \( N \), in particular, it is adjacency preserving and \( f(X), f(Y) \) are adjacent.

### 3.3. Final remark.

Recall that the distance between two \( k \)-dimensional subspaces \( X, Y \subset H \) is

\[ d(X, Y) = k - \dim(X \cap Y) = \dim(X + Y) - k; \]

in particular, \( d(X, Y) = 1 \) is equivalent to the fact that \( X, Y \) are adjacent. Suppose that \( n, k, m, i \) are as in Subsection 3.1, but the conditions of Theorem 4 do not hold. Using arguments from the proof of Theorem 4 for the case \( i \in \{1, 2\} \), we can show the following: if \( f \) is an injective transformation of \( \mathcal{G}_k(H) \) preserving the orthogonality relation and \( X, Y \in \mathcal{G}_k(H) \), then

\[ d(X, Y) \leq i \implies d(f(X), f(Y)) \leq i \]

and

\[ d(X, Y) \leq i - 1 \implies d(f(X), f(Y)) \leq i - 1. \]

There is a characterization of the adjacency relation in terms of bounded distance [6], but we cannot use it, since \( f \) needs not to be bounded distance preserving in both directions.
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