Interior capacities of condensers in locally compact spaces

Natalia Zorii

Abstract. The study is motivated by the known fact that, in the noncompact case, the main minimum-problem of the theory of interior capacities of condensers in a locally compact space is in general unsolvable, and this occurs even under very natural assumptions (e.g., for the Newton, Green, or Riesz kernels in $\mathbb{R}^n$ and closed condensers). Therefore it was particularly interesting to find statements of variational problems dual to the main minimum-problem (and hence providing some new equivalent definitions of the capacity), but always solvable (e.g., even for nonclosed condensers). For all positive definite kernels satisfying B. Fuglede’s condition of consistency between the strong and vague topologies, problems with the desired properties are posed and solved. Their solutions provide a natural generalization of the well-known notion of interior capacitary distributions associated with a set. We give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties. A condenser is treated as a finite collection of arbitrary sets with sing $+1$ or $-1$ prescribed, such that the closures of opposite-signed sets are mutually disjoint.

Mathematics Subject Classification (2000): 31C15.

Key words: Minimal energy problems, interior capacities of a condenser, interior capacitary distributions associated with a condenser, consistent kernels, completeness theorem for signed Radon measures.

1. Introduction

The present work is devoted to further development of the theory of interior capacities of arbitrary (noncompact or even nonclosed) condensers in a locally compact space, started by the author in [Z2, Z3]. For a background of that theory in the compact case, see the study by M. Ohtsuka [O].

The reader is expected to be familiar with the principal notions and results of the theory of measures and integration on a locally compact space; its exposition can be found in [B2, E2] (see also [F1, Z2] for a brief survey).

The theory of interior capacities of condensers provides a natural extension of the well-known theory of interior capacities of sets, developed by H. Cartan [C] and Vallée-Poussin [VP] for classical kernels in $\mathbb{R}^n$ and later on generalized by B. Fuglede [F1] for general kernels in a locally compact space $X$. However, those two theories — for sets and, on the other hand, condensers — are drastically different. To illustrate this, it is enough to note that, in the noncompact case, the main minimum-problem of the theory of interior capacities of condensers...
is in general unsolvable, and this phenomenon occurs even under very natural assumptions (e.g., for the Newton, Green, or Riesz kernels in $\mathbb{R}^n$ and closed condensers); compare with [C, F1]. Necessary and sufficient conditions for the problem to be solvable were given in [Z3, Z5]; see Sec. 4.5 below for a brief survey. Therefore it was particularly interesting to find statements of variational problems dual to the main minimum-problem of the theory of interior capacities of condensers, but in contrast to the last one, always solvable — e.g., even for nonclosed condensers. (When speaking on duality of variational problems, we always mean their extremal values to be equal.)

In all that follows, $X$ denotes a locally compact Hausdorff space, and $\mathcal{M} = \mathcal{M}(X)$ the linear space of all real-valued Radon measures $\nu$ on $X$ equipped with the vague topology, i.e., the topology of pointwise convergence on the class $C_0(X)$ of all real-valued continuous functions on $X$ with compact support. A kernel $\kappa$ on $X$ is meant to be a lower semicontinuous function $\kappa : X \times X \to (-\infty, \infty]$. To avoid some difficulties when defining energies and potentials, we follow [F1] in assuming that $\kappa \geq 0$ unless the space $X$ is compact.

The energy and the potential of a measure $\nu \in \mathcal{M}$ with respect to a kernel $\kappa$ are defined by

$$\kappa(\nu, \nu) := \int \kappa(x, y) \, d(\nu \otimes \nu)(x, y)$$

and

$$\kappa(x, \nu) := \int \kappa(x, y) \, d\nu(y), \quad x \in X,$$

respectively, provided the corresponding integral above is well defined (as a finite number or $\pm \infty$). Let $\mathcal{E}$ denote the set of all $\nu \in \mathcal{M}$ with $-\infty < \kappa(\nu, \nu) < \infty$.

In the present study we shall be concerned with minimal energy problems over certain subclasses of $\mathcal{E}$, properly chosen. For all positive definite kernels satisfying B. Fuglede’s condition of consistency between the strong and vague topologies on $\mathcal{E}$ (see Sec. 2 below), those variational problems are shown to be dual to the main minimum-problem of the theory of interior capacities of condensers (and hence providing some new equivalent definitions of the capacity), but always solvable. See Theorems 2–4 and Corollaries 6, 8.

Their solutions provide a natural generalization of the notion of interior capactitary distributions associated with a set, introduced in [F1]. We shall give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties (see Sec. 6–9).

Condensers and their capacities are treated in a fairly general sense; see Sec. 3 and 4 below for the corresponding definitions.

2. Preliminaries: topologies, consistent and perfect kernels

Recall that a measure $\nu \geq 0$ is said to be concentrated on $E$, where $E$ is a subset
of $X$, if the complement $\overline{E} := X \setminus E$ is locally $\nu$-negligible; or, equivalently, if $E$ is $\nu$-measurable and $\nu = \nu_E$, where $\nu_E$ denotes the trace of $\nu$ upon $E$.

We denote by $\mathcal{M}^+ (E)$ the convex cone of all nonnegative measures concentrated on $E$, and write $\mathcal{E}^+ (E) := \mathcal{M}^+ (E) \cap \mathcal{E}$ To shorten notation, write $\mathcal{E}^+ := \mathcal{E}^+ (X)$. From now on, the kernel under consideration is always assumed to be positive definite, which means that it is symmetric (i.e., $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in X$) and the energy $\kappa(\nu, \nu)$, $\nu \in \mathcal{M}$, is nonnegative whenever defined. Then $\mathcal{E}$ is known to be a pre-Hilbert space with the scalar product

$$\kappa(\nu_1, \nu_2) := \int \kappa(x, y) d(\nu_1 \otimes \nu_2)(x, y)$$

and the seminorm $\|\nu\| := \sqrt{\kappa(\nu, \nu)}$; see [F1]. A (positive definite) kernel is called strictly positive definite if the seminorm $\|\cdot\|$ is a norm.

A measure $\nu \in \mathcal{E}$ is said to be equivalent in $\mathcal{E}$ to a given $\nu_0 \in \mathcal{E}$ if $\|\nu - \nu_0\| = 0$; the equivalence class, consisting of all those $\nu$, will be denoted by $[\nu_0]_{\mathcal{E}}$.

In addition to the strong topology on $\mathcal{E}$, determined by the above seminorm $\|\cdot\|$, it is often useful to consider the weak topology on $\mathcal{E}$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \mu)|$, $\mu \in \mathcal{E}$ (see [F1]). The Cauchy-Schwarz inequality

$$|\kappa(\nu, \mu)| \leq \|\nu\| \|\mu\|, \quad \nu, \mu \in \mathcal{E},$$

implies immediately that the strong topology on $\mathcal{E}$ is finer than the weak one.

In [F1], B. Fuglede introduced the following two properties of consistency between the induced strong, weak, and vague topologies on $\mathcal{E}^+$:

(C) Every strong Cauchy net in $\mathcal{E}^+$ converges strongly to every its vague cluster point;

(CW) Every strongly bounded and vaguely convergent net in $\mathcal{E}^+$ converges weakly to the vague limit;

in [F2], the properties (C) and (CW) were shown to be equivalent.

Definition 1. Following B. Fuglede, we call a kernel $\kappa$ consistent if it satisfies either of the properties (C) and (CW), and perfect if, in addition, it is strictly positive definite.

Remark 1. One has to consider nets or filters in $\mathcal{M}$ instead of sequences, because the vague topology in general does not satisfy the first axiom of countability. We follow Moore’s and Smith’s theory of convergence, based on the concept of nets (see [MS]; cf. also [E2, Ch. 0] and [K, Ch. 2]).

Theorem 1 [F1]. A kernel $\kappa$ is perfect if and only if $\mathcal{E}^+$ is strongly complete and the strong topology on $\mathcal{E}^+$ is finer than the vague one.
Examples. In $\mathbb{R}^n$, $n \geq 3$, the Newton kernel $|x - y|^{2-n}$ is perfect [C]. So is the Riesz kernel $|x - y|^{\alpha-n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ (see [D1, D2]). Furthermore, if $D$ is an open set in $\mathbb{R}^n$, $n \geq 2$, and its generalized Green function $g_D$ exists (see, e.g., [HK, Th. 5.24]), then the Green kernel $g_D$ is perfect as well [E1].

Remark 2. As is seen from Theorem 1, the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over classes of nonnegative measures with finite energy. Indeed, the theory of capacities of sets has been developed in [F1] exactly for those kernels. We shall show below that this concept is still efficient in minimal energy problems over classes of signed measures associated with a condenser. This is guaranteed by the theorem on the strong completeness of proper subspaces of $\mathcal{E}$, to be stated in Sec. 10 below.

3. Condensers. Measures associated with a condenser; their energies and potentials

3.1. Fix natural numbers $m$ and $m + p$, where $p \geq 0$, and write

$$I := \{1, \ldots, m + p\}, \quad I^+ := \{1, \ldots, m\}, \quad I^- := I \setminus I^+.$$

Definition 2. An ordered collection $\mathcal{A} = (A_i)_{i \in I}$ of nonempty sets $A_i \subset X$, $i \in I$, is called an $(m, p)$-condenser in $X$ (or simply a condenser) if

$$\overline{A_i} \cap \overline{A_j} = \emptyset \quad \text{for all} \quad i \in I^+, \quad j \in I^-.$$  \hfill (1)

The sets $A_i$, $i \in I^+$, and $A_j$, $j \in I^-$, are said to be the positive and, respectively, negative plates of an $(m, p)$-condenser $\mathcal{A} = (A_i)_{i \in I}$. Note that any two equal-signed plates of a condenser are allowed to intersect each other.

Let $\mathcal{C}_{m,p} = \mathcal{C}_{m,p}(X)$ denote the collection of all $(m, p)$-condensers in $X$. A condenser $\mathcal{A} \in \mathcal{C}_{m,p}$ is called closed or compact if all the plates $A_i$, $i \in I$, are closed or, respectively, compact. Similarly, we shall call it universally measurable if all the plates are universally measurable — that is, measurable with respect to every nonnegative Radon measure.

Given $\mathcal{A} = (A_i)_{i \in I}$, write $\overline{\mathcal{A}} := (\overline{A_i})_{i \in I}$. Then, due to (1), $\overline{\mathcal{A}}$ is a (closed) $(m, p)$-condenser. In the sequel, also the following notation will be required:

$$A := \bigcup_{i \in I} A_i, \quad A^+ := \bigcup_{i \in I^+} A_i, \quad A^- := \bigcup_{i \in I^-} A_i.$$

3.2. With the preceding notation, write

$$\alpha_i := \begin{cases} +1 & \text{if} \quad i \in I^+, \\ -1 & \text{if} \quad i \in I^- \end{cases}.$$
Given \( A \in C_{m,p} \), let \( \mathcal{M}(A) \) consist of all linear combinations of the form
\[
\mu = \sum_{i \in I} \alpha_i \mu^i, \quad \text{where } \mu^i \in \mathcal{M}^+(A_i) \text{ for all } i \in I.
\]
Any two \( \mu_1 \) and \( \mu_2 \) in \( \mathcal{M}(A) \),
\[
\mu_1 = \sum_{i \in I} \alpha_i \mu^i_1 \quad \text{and} \quad \mu_2 = \sum_{i \in I} \alpha_i \mu^i_2,
\]
are regarded to be identical \((\mu_1 \equiv \mu_2)\) if and only if \( \mu^i_1 = \mu^i_2 \) for all \( i \in I \).
Note that, under the relation of identity in \( \mathcal{M}(A) \) thus defined, the following correspondence is one-to-one:
\[
\mathcal{M}(A) \ni \mu \mapsto (\mu^i)_{i \in I} \in \prod_{i \in I} \mathcal{M}^+(A_i).
\]
We shall call \( \mu \in \mathcal{M}(A) \) a measure associated with a condenser \( A \), and the measure \( \mu^i \), \( i \in I \), its \( i \)-coordinate.
For any \( \mu_1, \mu_2 \in \mathcal{M}(A) \) and \( q_1, q_2 \in \mathbb{R}_+ \), define \( q_1 \mu_1 + q_2 \mu_2 \) to be an element from \( \mathcal{M}(A) \) uniquely determined by the relations
\[
(q_1 \mu_1 + q_2 \mu_2)^i := q_1 \mu^i_1 + q_2 \mu^i_2, \quad i \in I.
\]
Then the set \( \mathcal{M}(A) \) becomes convex.

3.3. Given \( \mu \in \mathcal{M}(A) \), denote by \( R\mu \) the Radon measure uniquely determined by either of the two equivalent relations
\[
R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \quad \text{for all } \varphi \in C_0(X),
\]
\[
(R\mu)^+ = \sum_{i \in I^+} \mu^i, \quad (R\mu)^- = \sum_{i \in I^-} \mu^i. \tag{2}
\]
Of course, the mapping \( R : \mathcal{M}(A) \to \mathcal{M} \) thus defined is in general non-injective, i.e., one can choose \( \mu' \in \mathcal{M}(A) \) so that \( \mu' \not\equiv \mu \), while \( R\mu' = R\mu \). (It would be injective if all \( A_i \), \( i \in I \), were mutually disjoint.) We shall call \( \mu, \mu' \in \mathcal{M}(A) \) equivalent in \( \mathcal{M}(A) \), and write \( \mu \equiv \mu' \), whenever their \( R \)-images coincide.
It follows from (2) that, for given \( \mu, \mu_1 \in \mathcal{M}(A) \) and \( x \in X \),
\[
\kappa(x, R\mu) = \sum_{i \in I} \alpha_i \kappa(x, \mu^i),
\]
\[
\kappa(R\mu, R\mu_1) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j_1),
\]
5
each of the above identities being understood in the sense that any of its sides is well defined whenever so is the other, and then they coincide. We shall call

\[ \kappa(x, \mu) := \kappa(x, R\mu) \]

the value of the potential of \( \mu \) at a point \( x \), and

\[ \kappa(\mu, \mu_1) := \kappa(R\mu, R\mu_1) \]

the mutual energy of \( \mu \) and \( \mu_1 \) — of course, provided the right-hand side of the corresponding relation is well defined. For \( \mu \equiv \mu_1 \) we get the energy \( \kappa(\mu, \mu) \) of \( \mu \).

Since we make no difference between \( \mu \in \mathcal{M}(\mathcal{A}) \) and \( R\mu \) when dealing with their energies or potentials, we shall sometimes call a measure associated with a condenser simply a measure — certainly, if this causes no confusion.

Let \( \mathcal{E}(\mathcal{A}) \) consist of all \( \mu \in \mathcal{M}(\mathcal{A}) \) with finite energy \( \kappa(\mu, \mu) =: \| \mu \|_2^2 \). Then \( \mathcal{E}(\mathcal{A}) \) is convex and can be treated as a semimetric space with the semimetric

\[ \| \mu_1 - \mu_2 \| := \| R\mu_1 - R\mu_2 \|, \quad \mu_1, \mu_2 \in \mathcal{E}(\mathcal{A}); \]

the topology on \( \mathcal{E}(\mathcal{A}) \) defined by means of this semimetric will be called strong.

Two elements of \( \mathcal{E}(\mathcal{A}) \), \( \mu_1 \) and \( \mu_2 \), are called equivalent in \( \mathcal{E}(\mathcal{A}) \) if \( \| \mu_1 - \mu_2 \| = 0 \).

If, in addition, \( \kappa \) is assumed to be strictly positive definite, then the equivalence in \( \mathcal{E}(\mathcal{A}) \) implies that in \( \mathcal{M}(\mathcal{A}) \), namely then \( \mu_1 \equiv \mu_2 \).

3.4. For measures associated with a condenser, it is also reasonable to introduce the following concept of convergence, actually corresponding to the vague convergence by coordinates. Let \( S \) denote a directed set of indices, and let \( \mu_s, s \in S \), and \( \mu_0 \) be given elements of the class \( \mathcal{M}(\overline{\mathcal{A}}) \).

Definition 3. A net \((\mu_s)_{s \in S}\) is said to converge to \( \mu_0 \) \( \mathcal{A}\)-vaguely if

\[ \mu^i_s \to \mu^i_0 \quad \text{vaguely for all} \quad i \in I. \]

Since \( \mathcal{M}(X) \) is a Hausdorff space, an \( \mathcal{A}\)-vague limit in \( \mathcal{M}(\overline{\mathcal{A}}) \) is unique (if exists).

Remark 3. The \( \mathcal{A}\)-vague convergence of \((\mu_s)_{s \in S}\) to \( \mu_0 \) certainly implies the vague convergence of \((R\mu_s)_{s \in S}\) to \( R\mu_0 \). By using the Tietze-Urysohn extension theorem (see, e.g., [E2, Th. 0.2.13]), one can see that, for the converse to be true, it is necessary and sufficient that all \( \overline{A_i}, i \in I, \) be mutually disjoint.

4. Interior capacities of \((m, p)\)-condensers

4.1. Let \( \mathcal{H} \) be a set in the pre-Hilbert space \( \mathcal{E} \) or in the semimetric space \( \mathcal{E}(\mathcal{A}) \), where \( \mathcal{A} \) is a given \((m, p)\)-condenser. In either case, let us introduce the quantity

\[ \| \mathcal{H} \|^2 := \inf_{\nu \in \mathcal{H}} \| \nu \|^2, \]
interpreted as $+\infty$ if $\mathcal{H}$ is empty. If $\|\mathcal{H}\|^2 < \infty$, one can consider the variational problem on the existence of $\lambda = \lambda(\mathcal{H}) \in \mathcal{H}$ with minimal energy
\[
\|\lambda\|^2 = \|\mathcal{H}\|^2;
\]
such a problem will be referred to as the $\mathcal{H}$-problem. The $\mathcal{H}$-problem is said to be solvable if a minimizer $\lambda(\mathcal{H})$ exists.

The following elementary lemma is a slight generalization of [F1, Lemma 4.1.1].

**Lemma 1.** Suppose $\mathcal{H}$ is convex, and $\lambda = \lambda(\mathcal{H})$ exists. Then for any $\nu \in \mathcal{H}$,
\[
\|\nu - \lambda\|^2 \leq \|\nu\|^2 - \|\lambda\|^2.
\]  

**Proof.** Assume $\mathcal{H} \subset \mathcal{E}$. For every $t \in [0, 1]$, the measure $\mu := (1 - t)\lambda + t\nu$ belongs to $\mathcal{H}$, and therefore $\|\mu\|^2 \geq \|\lambda\|^2$. Evaluating $\|\mu\|^2$ and then letting $t$ tend to zero, we get $\kappa(\nu, \lambda) \geq \|\lambda\|^2$, and (4) follows (see [F1]).

Suppose now $\mathcal{H} \subset \mathcal{E}(A)$. Then $R\mathcal{H} := \{R\nu : \nu \in \mathcal{H}\}$ is a convex subset of $\mathcal{E}$, while $R\lambda$ is a minimizer in the $R\mathcal{H}$-problem. What has been shown thus yields
\[
\|R\nu - R\lambda\|^2 \leq \|R\nu\|^2 - \|R\lambda\|^2,
\]
which gives (4) when combined with (3).

We shall be concerned with the $\mathcal{H}$-problem for various specific $\mathcal{H}$ related to the notion of interior capacity of an $(m,p)$-condenser (in particular, of a set); see Sec. 4.2 and Sec. 6 below for their definitions.

4.2. Fix a continuous function $g : X \to (0, \infty)$ and a numerical vector $a = (a_i)_{i \in I}$ with $a_i > 0$, $i \in I$. Given a kernel $\kappa$ and an $(m,p)$-condenser $\mathcal{A}$ in $X$, write
\[
\mathcal{E}(\mathcal{A}, a, g) := \left\{ \mu \in \mathcal{E}(\mathcal{A}) : \int g d\mu^i = a_i \text{ for all } i \in I \right\}.
\]

**Definition 4.** We shall call the value
\[
\text{cap} \mathcal{A} := \text{cap} (\mathcal{A}, a, g) := \frac{1}{\|\mathcal{E}(\mathcal{A}, a, g)\|^2}
\]
the (interior) capacity of an $(m,p)$-condenser $\mathcal{A}$ (with respect to $\kappa$, $a$, and $g$).

Here and in the sequel, we adopt the convention that $1/0 = +\infty$. It follows immediately from the positive definiteness of the kernel that
\[
0 \leq \text{cap} (\mathcal{A}, a, g) \leq \infty.
\]

**Remark 4.** If $I$ is a singleton, any $(m,p)$-condenser consists of just one set, $A_1$, positively signed. If moreover $g = 1$ and $a_1 = 1$, then the notion of interior
capacity of a condenser, defined above, certainly reduces to the notion of interior capacity of a set (see [F1]). We denote it by $C(\cdot)$ as well.

**Remark 5.** In the case of the Newton kernel in $\mathbb{R}^3$, the notion of capacity of a condenser $A$ has an evident electrostatic interpretation. In the framework of the corresponding electrostatics problem, the function $g$ serves as a characteristic of nonhomogeneity of the conductors $A_i$, $i \in I$.

4.3. On $\mathcal{C}_{m,p} = \mathcal{C}_{m,p}(X)$, it is natural to introduce the ordering relation $\prec$ by declaring $A' \prec A$ to mean that $A'_i \subset A_i$ for all $i \in I$. Then $\text{cap}(\cdot, a, g)$ is a nondecreasing function of a condenser, namely

$$\text{cap}(A', a, g) \leq \text{cap}(A, a, g) \quad \text{whenever} \quad A' \prec A.$$  \hspace{1cm} (6)

Given $A \in \mathcal{C}_{m,p}$, denote by $\{K\}_A$ the increasing ordered family of all compact condensers $K = (K_i)_{i \in I} \in \mathcal{C}_{m,p}$ such that $K \prec A$.

**Lemma 2.** If $K$ ranges over $\{K\}_A$, then

$$\text{cap}(A, a, g) = \lim_{K \uparrow A} \text{cap}(K, a, g).$$ \hspace{1cm} (7)

**Proof.** We can certainly assume $\text{cap}(A, a, g)$ to be nonzero, since otherwise the lemma follows at once from (6). Then the set $E(A, a, g)$ must be nonempty; fix $\mu$, one of its elements. For any $K \in \{K\}_A$ and $i \in I$, let $\mu^i_K$ denote the trace of $\mu$ upon $K_i$. Applying Lemma 1.2.2 from [F1], we get

$$\int g \, d\mu^i = \lim_{K \uparrow A} \int g \, d\mu^i_K, \quad i \in I, \quad (8)$$

$$\kappa(\mu^i, \mu^j) = \lim_{K \uparrow A} \kappa(\mu^i_K, \mu^j_K), \quad i, j \in I. \quad (9)$$

Thus for $K \in \{K\}_A$ large enough, $\int g \, d\mu^i_K \neq 0$ for all $i \in I$, and consequently

$$\sum_{i \in I} \frac{\alpha_i a_i}{\int g \, d\mu^i_K} \mu^i_K \in E(K, a, g).$$

Together with (8) and (9), this yields

$$\|\mu\|^2 = \lim_{K \uparrow A} \sum_{i,j \in I} \kappa\left(\int g \, d\mu^i_K, \frac{\alpha_i a_i}{\int g \, d\mu^i_K} \mu^i_K \right) \geq \lim_{K \uparrow A} \|E(K, a, g)\|^2,$$

and hence, in view of the arbitrary choice of $\mu \in E(A, a, g)$,

$$\|E(A, a, g)\|^2 \geq \lim_{K \uparrow A} \|E(K, a, g)\|^2.$$

Since the converse inequality is obvious from (6), the proof is complete.
Let \( \mathcal{E}(A, a, g) \) denote the class of all \( \mu \in \mathcal{E}(A, a, g) \) such that, for every \( i \in I \), the support \( S(\mu^i) \) of \( \mu^i \) is compact and contained in \( A_i \).

**Corollary 1.** The capacity \( \text{cap}(A, a, g) \) remains unchanged if the class \( \mathcal{E}(A, a, g) \) in its definition is replaced by \( \mathcal{E}^0(A, a, g) \). In other words,

\[
\| \mathcal{E}(A, a, g) \|^2 = \| \mathcal{E}^0(A, a, g) \|^2.
\]

**Proof.** We can certainly assume \( \text{cap } A \) to be nonzero, since otherwise the corollary follows immediately from the inclusion \( \mathcal{E}^0(A, a, g) \subset \mathcal{E}(A, a, g) \). Then, by (6) and (7), for every \( \varepsilon > 0 \) there exists a compact condenser \( K \prec A \) such that

\[
\| \mathcal{E}(K, a, g) \|^2 \leq \| \mathcal{E}(A, a, g) \|^2 + \varepsilon.
\]

This leads to the claimed assertion when combined with the relation

\[
\| \mathcal{E}(K, a, g) \|^2 \geq \| \mathcal{E}^0(A, a, g) \|^2 \geq \| \mathcal{E}(A, a, g) \|^2.
\]

**4.4.** Unless explicitly stated otherwise, in all that follows it is assumed that

\[
\text{cap}(A, a, g) > 0. \tag{10}
\]

**Lemma 3.** The assumption \((10)\) is equivalent to the following one:

\[
C(A_i) > 0 \quad \text{for all } i \in I. \tag{11}
\]

**Proof.** Indeed, \( \text{cap}(A, a, g) \) is nonzero if and only if \( \mathcal{E}(A, a, g) \) is nonempty. As \( g \) is positive, for the latter to happen, it is necessary and sufficient that, for every \( i \in I \), there exists a nonzero nonnegative measure of finite energy concentrated on \( A_i \). Since this is equivalent to \((11)\) by \([F1, \text{Lemma 2.3.1}]\), the proof is complete.

In the following assertion, providing necessary and sufficient conditions for \( \text{cap } A \) to be finite, we assume \( g|_A \) to have a strictly positive lower bound (say \( L \)).

**Lemma 4.** For \( \text{cap}(A, a, g) \) to be finite, it is necessary that

\[
C(A_j) < \infty \quad \text{for some } j \in I. \tag{12}
\]

This condition is also sufficient if it is additionally assumed that \( A \) is closed, \( g|_A \) bounded, and \( \kappa \) bounded from above on \( A^+ \times A^- \) and perfect.

**Proof.** Let \( \text{cap } A < \infty \), and assume, on the contrary, that

\[
C(A_i) = \infty \quad \text{for all } i \in I. \tag{13}
\]

Then, for every \( i \), there exist probability measures \( \nu^i_n \in \mathcal{E}^+(A_i) \), \( n \in \mathbb{N} \), of compact support such that

\[
\| \nu^i_n \| \to 0 \quad (n \to \infty).
\]
Since
\[ \mu_n := \sum_{i \in I} \alpha_i a_i \nu_i \in \mathcal{E}(\mathcal{A}, a, g), \quad n \in \mathbb{N}, \]
and
\[ \|\mu_n\| \leq L^{-1} \sum_{i \in I} a_i \|\nu_i\|, \]
we arrive at a contradiction by letting \( n \) tend to \( \infty \).

Assume now all the conditions of the remaining part of the lemma to be satisfied, and let \( (12) \) be true. Then, by \([Z4, \text{Lemma 13}]\), there exists \( \zeta \in \mathcal{E}(\mathcal{A}) \) with the properties that \( \int g d\zeta = a_j \) (hence, \( \zeta \neq 0 \)) and
\[ \|\zeta\|^2 = \|\mathcal{E}(\mathcal{A}, a, g)\|^2. \]

Since \( \kappa \) is strictly positive definite, this yields \( \text{cap } \mathcal{A} < \infty \), as was to be proved.

4.5. Because of \([10]\), we are naturally led to the \( \mathcal{E}(\mathcal{A}, a, g) \)-problem (cf. Sec. 4.1), i.e., the problem on the existence of \( \lambda \in \mathcal{E}(\mathcal{A}, a, g) \) with minimal energy
\[ \|\lambda\|^2 = \|\mathcal{E}(\mathcal{A}, a, g)\|^2; \]
the \( \mathcal{E}(\mathcal{A}, a, g) \)-problem might certainly be regarded as the main minimum-problem of the theory of interior capacities of condensers. The collection (possibly empty) of all minimizing measures \( \lambda \) in this problem will be denoted by \( \mathcal{S}(\mathcal{A}, a, g) \).

If moreover \( \text{cap } (\mathcal{A}, a, g) \) is finite, let us look, as well, at the \( \mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, g) \)-problem. By reasons of homogeneity, both the \( \mathcal{E}(\mathcal{A}, a, g) \)- and the \( \mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, g) \)-problems are simultaneously either solvable or unsolvable, and their extremal values are related to each other by the following law:
\[ \frac{1}{\|\mathcal{E}(\mathcal{A}, a, g)\|^2} = \|\mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, g)\|^2. \quad (14) \]

Assume for a moment that \( \mathcal{A} \) is compact. Since the mapping
\[ \nu \mapsto \int g d\nu, \quad \nu \in \mathcal{M}^+(K), \]
where \( K \subset \mathcal{X} \) is a compact set, is vaguely continuous, \( \mathcal{E}(\mathcal{A}, a, g) \) is compact in the \( \mathcal{A} \)-vague topology. Therefore, if \( \kappa \) is additionally assumed to be continuous on \( A^+ \times A^- \) (which, due to \([11]\), is always the case for either of the classical kernels), then the energy \( \|\mu\|^2 \) is \( \mathcal{A} \)-vaguely lower semicontinuous on \( \mathcal{E}(\mathcal{A}) \), and the solvability of both the problems immediately follows (cf. \([11] \text{ Th. 2.6}]\)).

But if \( \mathcal{A} \) is noncompact, then the class \( \mathcal{E}(\mathcal{A}, a, g) \) is no longer \( \mathcal{A} \)-vaguely compact and the problems become quite nontrivial. Moreover, it has recently been shown
by the author that, in the noncompact case, the problems are in general unsolvable and this phenomenon occurs even under very natural assumptions (e.g., for the Newton, Green, or Riesz kernels in $\mathbb{R}^n$, $n \geq 2$, and closed condensers).

In particular, it was proved in [Z3] that, if $\mathcal{A}$ is closed, $\kappa$ is perfect, and bounded and continuous on $A^+ \times A^-$, and satisfies the generalized maximum principle (see, e.g., [L, Chap. VI]), while $g|_A$ is bounded and has a strictly positive lower bound, then for either of the $\mathcal{E}(\mathcal{A}, a, g)$- and the $\mathcal{E}(\mathcal{A}, a \cap \mathcal{A}, g)$-problems to be solvable for any vector $a$, it is necessary and sufficient that

$$C(A_i) < \infty \quad \text{for all} \quad i \in I.$$ 

If moreover there exists $i_0 \in I$ such that

$$C(A_{i_0}) = \infty,$$

then both the problems are unsolvable for all $a = (a_i)_{i \in I}$ with $a_{i_0}$ large enough.

In [Z5, Th. 1], the last statement was sharpened. It was shown that if, in addition to all the preceding assumptions, for all $i \neq i_0$,

$$C(A_i) < \infty \quad \text{and} \quad A_i \cap A_{i_0} = \emptyset,$$

while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets, then there exists a number $\Lambda_{i_0} \in [0, \infty)$ such that the problems are unsolvable if and only if

$$a_{i_0} > \Lambda_{i_0}.$$ 

**Remark 6.** It was actually shown in [Z5] that

$$\Lambda_{i_0} = \int g \, d\tilde{\lambda}_{i_0},$$

where $\tilde{\lambda}$ is a minimizer (it exists) in the auxiliary $\mathcal{H}$-problem for

$$\mathcal{H} := \left\{ \mu \in \mathcal{E}(\mathcal{A}) : \int g \, d\mu^i = a_i \quad \text{for all} \quad i \neq i_0 \right\}.$$ 

**Remark 7.** The mentioned results were actually obtained in [Z3, Z5] for the energy evaluated in the presence of an external field.

**4.6.** In view of the results reviewed in Sec. 4.5, it was particularly interesting to find statements of variational problems dual to the $\mathcal{E}(\mathcal{A}, a \cap \mathcal{A}, g)$-problem (and hence providing some new equivalent definitions of $\text{cap} \mathcal{A}$), but solvable for any condenser $\mathcal{A}$ (e.g., even nonclosed) and any vector $a$. We have succeeded in this under the following conditions, which will always be tacitly assumed.
From now on, in addition to (10), the following **standing assumptions** are always required: \( \kappa \) is consistent, and either

\[ I^- = \emptyset \quad (\text{i.e., } p = 0), \]

or both the conditions are satisfied

\[ g_{\text{min}} := \inf_{x \in A} g(x) > 0, \quad (15) \]

\[ \sup_{x \in A^+, y \in A^-} \kappa(x, y) < \infty. \quad (16) \]

**Remark 8.** These assumptions on a kernel are not too restrictive. In particular, they all are satisfied by the Newton, Riesz, or Green kernels in \( \mathbb{R}^n \) provided the Euclidean distance between the opposite-signed plates of a condenser is nonzero.

### 5. \( \mathcal{A} \)-vague and strong cluster sets of minimizing nets

To formulate the results obtained, we shall need the following notation.

5.1. Denote by \( \mathbb{M}(\mathcal{A}, a, g) \) the class of all \( (\mu_t)_{t \in T} \subset \mathcal{E}^0(\mathcal{A}, a, g) \) such that

\[ \lim_{t \in T} \|\mu_t\|^2 = \|\mathcal{E}(\mathcal{A}, a, g)\|^2. \quad (17) \]

This class is not empty, which is clear from (10) in view of Corollary 1.

Let \( \mathcal{M}(\mathcal{A}, a, g) \) (respectively, \( \mathcal{M}'(\mathcal{A}, a, g) \)) consist of all limit points of the nets \( (\mu_t)_{t \in T} \in \mathbb{M}(\mathcal{A}, a, g) \) in the \( \mathcal{A} \)-vague topology of the space \( \mathbb{M}(\overline{\mathcal{A}}) \) (respectively, in the strong topology of the semimetric space \( \mathcal{E}(\overline{\mathcal{A}}) \)). Also write

\[ \mathcal{E}(\mathcal{A}, \leq a, g) := \left\{ \mu \in \mathcal{E}(\mathcal{A}) : \int g \, d\mu_i \leq a_i \text{ for all } i \in I \right\}. \]

With the preceding notation and under our standing assumptions (see Sec. 4.6), there holds the following lemma, to be proved in Sec. 11 below.

**Lemma 5.** Given \( (\mu_t)_{t \in T} \in \mathbb{M}(\mathcal{A}, a, g) \), there exist its \( \mathcal{A} \)-vague cluster points; hence, the class \( \mathcal{M}(\mathcal{A}, a, g) \) is nonempty. Moreover,

\[ \mathcal{M}(\mathcal{A}, a, g) \subset \mathcal{M}'(\mathcal{A}, a, g) \cap \mathcal{E}(\overline{\mathcal{A}}, \leq a, g). \quad (18) \]

Furthermore, for every \( \chi \in \mathcal{M}'(\mathcal{A}, a, g) \),

\[ \lim_{t \in T} \|\mu_t - \chi\|^2 = 0, \quad (19) \]

and hence \( \mathcal{M}'(\mathcal{A}, a, g) \) forms an equivalence class in \( \mathcal{E}(\overline{\mathcal{A}}) \).
It follows from (17) – (19) that
\[ \| \zeta \|^2 = \| \mathcal{E}(A, a, g) \|^2 \quad \text{for all} \quad \zeta \in \mathcal{M}(A, a, g). \]

Also observe that, if \( A = \mathcal{K} \) is compact, then moreover \( \mathcal{M}(\mathcal{K}, a, g) \subset \mathcal{E}(\mathcal{K}, a, g) \), which together with the preceding relation proves the following assertion.

**Corollary 2.** If \( A = \mathcal{K} \) is compact, then the \( \mathcal{E}(\mathcal{K}, a, g) \)-problem is solvable. Actually,
\[ S(\mathcal{K}, a, g) = \mathcal{M}(\mathcal{K}, a, g). \] (20)

5.2. When approaching \( A \) by the increasing family \( \{ \mathcal{K} \}_A \) of the compact condensers \( \mathcal{K} \prec A \), we shall always suppose all those \( \mathcal{K} \) to be of capacity nonzero. This involves no loss of generality, which is clear from (10) and Lemma 2.

Then Corollary 2 enables us to introduce the (nonempty) class \( \mathbb{M}_0(A, a, g) \) of all nets \( (\lambda_K)_{K \in \{ \mathcal{K} \}_A} \), where \( \lambda_K \in S(\mathcal{K}, a, g) \) is arbitrarily chosen. Let \( \mathcal{M}_0(A, a, g) \) consist of all \( A \)-vague cluster points of those nets. Since, by Lemma 2,
\[ \mathbb{M}_0(A, a, g) \subset \mathbb{M}(A, a, g), \]
application of Lemma 5 yields the following assertion.

**Corollary 3.** The class \( \mathcal{M}_0(A, a, g) \) is nonempty, and
\[ \mathcal{M}_0(A, a, g) \subset \mathcal{M}(A, a, g) \subset \mathcal{M}'(A, a, g). \]

**Remark 9.** Each of the cluster sets, \( \mathcal{M}_0(A, a, g) \), \( \mathcal{M}(A, a, g) \) and \( \mathcal{M}'(A, a, g) \), plays an important role in our study. However, if \( \kappa \) is additionally assumed to be strictly positive definite (hence, perfect), while \( A_i, i \in I \), are mutually disjoint, then all these classes coincide and consist of just one element.

5.3. Also the following notation will be required. Given \( \chi \in \mathcal{M}'(A, a, g) \), write
\[ \mathcal{M}'_\mathcal{E}(A, a, g) := [R\chi]_{\mathcal{E}}. \]

This equivalence class does not depend on the choice of \( \chi \), which is clear from Lemma 5. Lemma 5 also yields that, for any \( (\mu_t)_{t \in T} \in \mathbb{M}(A, a, g) \) and any \( \nu \in \mathcal{M}'_\mathcal{E}(A, a, g) \), \( R\mu_t \to \nu \) in the strong topology of the pre-Hilbert space \( \mathcal{E} \).

6. Extremal problems dual to the main minimum-problem of the theory of interior capacities of condensers

Throughout Sec. 6, as usual, we are keeping all our standing assumptions, stated in Sec. 4.6.

6.1. A proposition \( R(x) \) involving a variable point \( x \in X \) is said to subsist nearly everywhere (n.e.) in \( E \), where \( E \) is a given subset of \( X \), if the set of all \( x \in E \) for which \( R(x) \) fails to hold is of interior capacity zero. See, e.g., [F1].

13
If $C(E) > 0$ and $f$ is a universally measurable function bounded from below nearly everywhere in $E$, write
\[
\inf_{x \in E} f(x) := \sup \{ q : f(x) \geq q \text{ n.e. in } E \}.
\]
Then
\[
f(x) \geq \inf_{x \in E} f(x) \text{ n.e. in } E,
\]
which is seen from the fact that the interior capacity $C(\cdot)$ is countably subadditive on sets $U_n \cap E$, $n \in \mathbb{N}$, where $U_n$ are universally measurable, whereas $E$ is arbitrary (see Lemma 2.3.5 in [F1] and the remark attached to it).

6.2. Let $\hat{\Gamma} = \hat{\Gamma}(A, a, g)$ denote the class of all Radon measures $\nu \in \mathcal{E}$ such that there exist real numbers $c_i(\nu)$, $i \in I$, satisfying the relations
\[
\alpha_i a_i \kappa(x, \nu) \geq c_i(\nu) g(x) \text{ n.e. in } A_i, \quad i \in I,
\]
\[
\sum_{i \in I} c_i(\nu) \geq 1.
\]
The property of subadditivity of $C(\cdot)$, mentioned above, implies that $\hat{\Gamma}$ is convex. The following assertion, to be proved in Sec. 14 below, holds true.

**Theorem 2.** Under the standing assumptions,
\[
\|\hat{\Gamma}(A, a, g)\|^2 = \text{cap } (A, a, g).
\]
If $\|\hat{\Gamma}(A, a, g)\|^2 < \infty$, we are interested in the $\hat{\Gamma}(A, a, g)$-problem (cf. Sec. 4.1), i.e., the problem on the existence of $\hat{\omega} \in \hat{\Gamma}(A, a, g)$ with minimal energy
\[
\|\hat{\omega}\|^2 = \|\hat{\Gamma}(A, a, g)\|^2;
\]
the collection of all those $\hat{\omega}$ will be denoted by $\hat{\mathcal{G}} = \hat{\mathcal{G}}(A, a, g)$.

A minimizing measure $\hat{\omega}$ can be shown to be unique up to a summand of seminorm zero (and, hence, it is unique whenever the kernel under consideration is strictly positive definite). Actually, the following stronger result holds true.

**Lemma 6.** If $\hat{\omega}$ exists, $\hat{\mathcal{G}}(A, a, g)$ forms an equivalence class in $\mathcal{E}$.

**Proof.** Since $\hat{\Gamma}$ is convex, Lemma 1 yields that $\mathcal{G}$ is contained in an equivalence class in $\mathcal{E}$. To prove that $\mathcal{G}$ actually coincides with that equivalence class, it suffices to show that, if $\nu$ belongs to $\hat{\Gamma}$, then so do all measures equivalent to $\nu$ in $\mathcal{E}$. But this follows at once from the property of subadditivity of $C(\cdot)$, mentioned above, and the fact that the potentials of any two equivalent in $\mathcal{E}$ measures coincide nearly everywhere in $X$ [F1, Lemma 3.2.1].
6.3. Assume for a moment that $\text{cap}\ (A, a, g)$ is finite. When combined with (5) and (14), Theorem 2 shows that the $\hat{\Gamma}(A, a, g)$-problem and, on the other hand, the $E(A, a \text{ cap} A, g)$-problem have the same infimum, equal to the capacity $\text{cap} A$, and so these two variational problems are dual.

But what is surprising is that their infimum, $\text{cap} A$, turns out to be always an actual minimum in the former extremal problem, while this is not the case for the latter one (see Sec. 4.5). In fact, the following statement on the solvability of the $\hat{\Gamma}(A, a, g)$-problem, to be proved in Sec. 14 below, holds true.

**Theorem 3.** Under the standing assumptions, if moreover $\text{cap} A < \infty$, then the class $\hat{G}(A, a, g)$ is nonempty and can be given by the formula

$$\hat{G}(A, a, g) = M'_E(A, a \text{ cap} A, g).$$

(24)

The numbers $c_i(\hat{\omega}), i \in I$, satisfying both (21) and (22) for $\hat{\omega} \in \hat{G}(A, a, g)$, are determined uniquely, do not depend on the choice of $\hat{\omega}$, and can be written in either of the forms

$$c_i(\hat{\omega}) = \alpha_i \text{ cap} A^{-1} \kappa(\zeta^i, \zeta),$$

(25)

$$c_i(\hat{\omega}) = \alpha_i \text{ cap} A^{-1} \lim_{s \in S} \kappa(\mu^i_s, \mu^s_s),$$

(26)

where $\zeta \in M(A, a \text{ cap} A, g)$ and $(\mu^s_s)_{s \in S} \in M(A, a \text{ cap} A, g)$ are arbitrarily given.

The following two assertions, providing additional information about $c_i(\hat{\omega}), i \in I$, can be obtained directly from the preceding theorem.

**Corollary 4.** Given $\hat{\omega} \in \hat{G}(A, a, g)$, it follows that

$$c_i(\hat{\omega}) = \inf_{x \in A_i} \frac{\alpha_i a_i \kappa(x, \hat{\omega})}{g(x)}, \quad i \in I.$$  

(27)

**Corollary 5.** The inequality (22) for $\hat{\omega} \in \hat{G}(A, a, g)$ is actually an equality; i.e.

$$\sum_{i \in I} c_i(\hat{\omega}) = 1.$$  

(28)

**Remark 10.** Assume for a moment that $\text{cap} A = 0$. Then, by Lemma 3, there exists $i \in I$ (say $i = 1$) with $C(A_i) = 0$. Hence, the measure $\nu_0 = 0$ belongs to $\hat{\Gamma}(A, a, g)$ since it satisfies both (21) and (22) with $c_i(\nu_0), i \in I$, where

$$c_1(\nu_0) \geq 1 \quad \text{and} \quad c_i(\nu_0) = 0, \quad i \neq 1.$$  

This implies that the identity (23) actually holds true in the degenerate case $\text{cap} A = 0$ as well, and then $\hat{G}(A, a, g)$ consists of all $\nu \in E$ of seminorm zero. What then, however, fails to hold is the statement on the uniqueness of $c_i(\hat{\omega})$.  

15
6.4. Let \( \hat{\Gamma}_* (A, a, g) \) consist of all \( \nu \in \hat{\Gamma} (A, a, g) \) for which the inequality (22) is actually an equality. By arguments similar to those that have been applied above, one can see that \( \hat{\Gamma}_* (A, a, g) \) is convex, and hence all the solutions to the minimal energy problem over this class form an equivalence class in \( \mathcal{E} \). Combining this with Theorems 2, 3 and Corollary 5 leads to the following assertion.

**Corollary 6.** Under the standing assumptions,

\[
\| \hat{\Gamma}_* (A, a, g) \|^2 = \text{cap} (A, a, g).
\]

If moreover \( \text{cap} A < \infty \), then the \( \hat{\Gamma}_* (A, a, g) \)-problem is solvable and the class \( \hat{\mathcal{G}}_* (A, a, g) \) of all its solutions is given by the formula

\[
\hat{\mathcal{G}}_* (A, a, g) = \mathcal{M}_c (A, a \cap A, g).
\]

**Remark 11.** Theorem 2 and Corollary 6 (cf. also Theorem 4 and Corollary 8 below) provide new equivalent definitions of the capacity \( \text{cap} (A, a, g) \). Note that, in contrast to the initial definition (cf. Sec. 4.2), no restrictions on the supports and total masses of measures from the classes \( \hat{\Gamma} (A, a, g) \) or \( \hat{\Gamma}_* (A, a, g) \) have been imposed; the only restriction involves their potentials. These definitions of the capacity are actually new even in the compact case; compare with [O]. They are not only of obvious academic interest, but seem also to be important for numerical computations.

6.5. Our next purpose is to formulate an \( \mathcal{H} \)-problem such that it is still dual to the \( \mathcal{E} (A, a \cap A, g) \)-problem and solvable, but now with \( \mathcal{H} \) consisting of measures associated with a condenser.

Let \( \Gamma (A, a, g) \) consist of all \( \mu \in \mathcal{E} (\overline{A}) \) for which both the relations (21) and (22) hold (with \( \mu \) in place of \( \nu \)). In other words,

\[
\Gamma (A, a, g) := \{ \mu \in \mathcal{E} (\overline{A}) : \ R\mu \in \hat{\Gamma} (A, a, g) \}.
\]

Observe that the class \( \Gamma (A, a, g) \) is convex and

\[
\| \Gamma (A, a, g) \|^2 \geq \| \hat{\Gamma} (A, a, g) \|^2.
\]

We proceed to show that the inequality (30) is actually an equality, and that the minimal energy problem, if considered over the class \( \Gamma (A, a, g) \), is still solvable.

**Theorem 4.** Under the standing assumptions,

\[
\| \Gamma (A, a, g) \|^2 = \text{cap} (A, a, g).
\]

If moreover \( \text{cap} (A, a, g) < \infty \), then the \( \Gamma (A, a, g) \)-problem is solvable and the class \( \mathcal{G} (A, a, g) \) of all its solutions \( \omega \) is given by the formula

\[
\mathcal{G} (A, a, g) = \mathcal{M}' (A, a \cap A, g).
\]
Proof. We can certainly assume \( \text{cap} \, \mathcal{A} \) to be finite, for if not, (31) is obtained directly from (23) and (30). Then, according to Lemma 5 with \( \text{cap} \, \mathcal{A} \) instead of \( \text{a} \), the class \( \mathcal{M}'(\mathcal{A}, \text{cap} \, \mathcal{A}, g) \) is nonempty; fix \( \chi \), one of its elements. It is clear from its definition and the identity (24) that \( \chi \in \mathcal{E}(\mathcal{A}) \) and \( R\chi \in \hat{\Gamma}(\mathcal{A}, a, g) \). Hence, by (29), \( \chi \in \Gamma(\mathcal{A}, a, g) \), and therefore
\[
\|\hat{\Gamma}(\mathcal{A}, a, g)\|^2 = \|\chi\|^2 \geq \|\Gamma(\mathcal{A}, a, g)\|^2.
\]
In view of (23) and (30), this proves (31) and, as well, the inclusion
\[
\mathcal{M}'(\mathcal{A}, \text{cap} \, \mathcal{A}, g) \subset \mathcal{G}(\mathcal{A}, a, g).
\]
But the right-hand side of this inclusion is an equivalence class in \( \mathcal{E}(\mathcal{A}) \), which is proved by the convexity of \( \Gamma(\mathcal{A}, a, g) \) and Lemma 1 in the same manner as in the proof of Lemma 6. Since, by Lemma 5, also the left-hand side is an equivalence class in \( \mathcal{E}(\mathcal{A}) \), the two sets must actually be equal. The proof is complete.

Corollary 7. If \( \mathcal{A} = \mathcal{K} \) is compact and \( \text{cap} \, (\mathcal{K}, a, g) < \infty \), then any solution to the \( \mathcal{E}(\mathcal{K}, \text{cap} \, \mathcal{K}, g) \)-problem gives, as well, a solution to the \( \Gamma(\mathcal{K}, a, g) \)-problem.

Proof. This follows from (32), when combined with (18) and (20) for \( \text{cap} \, \mathcal{K} \) in place of \( \text{a} \).

Remark 12. Assume \( \text{cap} \, \mathcal{A} < \infty \), and fix \( \omega \in \mathcal{G}(\mathcal{A}, a, g) \) and \( \hat{\omega} \in \hat{\mathcal{G}}(\mathcal{A}, a, g) \). Since, by (24) and (32), \( \kappa(x, \omega) = \kappa(x, \hat{\omega}) \) nearly everywhere in \( X \), the numbers \( c_i(\omega), i \in I \), satisfying (21) and (22) for \( \nu = \omega \), are actually equal to \( c_i(\hat{\omega}) \). This implies that relations (25) – (28) do hold, as well, for \( \omega \) in place of \( \hat{\omega} \).

Remark 13. Observe that, in all the preceding assertions, we have not imposed any restrictions on the topology of \( A_i, i \in I \). So, all the \( \hat{\Gamma}(\mathcal{A}, a, g) \), \( \hat{\Gamma}^*(\mathcal{A}, a, g) \), and \( \Gamma(\mathcal{A}, a, g) \)-problems are solvable even for a nonclosed condenser \( \mathcal{A} \).

Remark 14. If \( I = \{1\} \) and \( g = 1 \), Theorems 2–4 and Corollary 6 can be derived from [F1]. Moreover, then one can choose \( \gamma \in \mathcal{G}(\mathcal{A}, a, g) \) so that
\[
\gamma(X) = a_1 C(A_1),
\]
and exactly this kind of measures was called by B. Fuglede interior capacitary distributions associated with the set \( A_1 \). However, this fact in general can not be extended to the case \( I \neq \{1\} \); that is, in general,
\[
\mathcal{G}(\mathcal{A}, a, g) \cap \mathcal{E}(\mathcal{A}, \text{cap} \, \mathcal{A}, g) = \emptyset,
\]
which can be seen from the unsolvability of the \( \mathcal{E}(\mathcal{A}, \text{cap} \, \mathcal{A}, g) \)-problem.

7. Interior capacitary constants associated with a condenser

7.1. Throughout Sec. 7, it is always required that \( \text{cap} \, (\mathcal{A}, a, g) < \infty \). Due to the uniqueness statement in Theorem 3, the following notion naturally arises.
**Definition 5.** The numbers

\[ C_i := C_i(A, a, g) := c_i(\hat{\omega}), \quad i \in I, \]

satisfying both the relations (21) and (22) for \( \hat{\omega} \in \hat{G}(A, a, g) \), are said to be the (interior) capacitary constants associated with an \((m,p)\)-condenser \( A \).

**Corollary 8.** The interior capacity \( \text{cap} (A, a, g) \) equals the infimum of \( \kappa(\nu, \nu) \), where \( \nu \) ranges over the class of all \( \nu \in \mathcal{E} \) (similarly, \( \nu \in \mathcal{E}(\mathcal{A}) \)) such that

\[ \alpha_i a_i \kappa(x, \nu) \geq C_i(A, a, g) g(x) \quad \text{n. e. in } A_i, \quad i \in I. \]

The infimum is attained at any \( \hat{\omega} \in \hat{G}(A, a, g) \) (respectively, \( \omega \in G(A, a, g) \)), and hence it is an actual minimum.

**Proof.** This follows immediately from Theorems 2–4 and Remark 12.

7.2. Some properties of the interior capacitary constants \( C_i(A, a, g), i \in I \), have already been provided by Theorem 3 and Corollaries 4, 5. Also observe that, if \( I \) is a singleton, then certainly \( C_1(A, a, g) = 1 \) (cf. [11, Th. 4.1]).

**Corollary 9.** \( C_i(\cdot, a, g), i \in I \), are continuous under exhaustion of \( A \) by the increasing family of all compact condensers \( K \prec A \). Namely,

\[ C_i(A, a, g) = \lim_{K \uparrow A} C_i(K, a, g). \]

**Proof.** Under our assumptions, \( 0 < \text{cap} K < \infty \) for every \( K \in \{K\}_A \), and hence there exists \( \lambda_K \in S(K, a \text{cap} K, g) \). Substituting it into (25) yields

\[ C_i(K, a, g) = \alpha_i \text{cap} K^{-1} \kappa(\lambda^i_K, \lambda_K), \quad i \in I. \]  \hfill (33)

On the other hand, by Lemma 2 the net

\[ \text{cap} A \text{cap} K^{-1} \lambda_K, \quad \text{where } K \in \{K\}_A, \]

belongs to the class \( M(A, a \text{cap} A, g) \). Substituting it into (26) and then combining the relation obtained with (33), we get the corollary.

In the following assertion we suppose \( g_{\text{min}} > 0 \). According to our agreement (see Sec. 4.6), this does hold automatically whenever \( I^- \) is nonempty.

**Corollary 10.** Assume \( C(A_j) = \infty \) for some \( j \in I \). Then

\[ C_j(A, a, g) \leq 0. \]  \hfill (34)

Hence, \( C_j(A, a, g) = 0 \) if moreover \( I^- = \emptyset \).
Proof. Assume, on the contrary, that $C_j > 0$. Given $\hat{\omega} \in \hat{G}(A, a, g)$, then
\[ \alpha_j a_j \kappa(x, \hat{\omega}) \geq C_j g_{\min} > 0 \text{ n.e. in } A_j, \]
and therefore, by [F1 Lemma 3.2.2],
\[ C(A_j) \leq a_j^2 \| \hat{\omega} \|^2 C_j^{-2} g_{\min}^{-2} < \infty, \]
which is a contradiction. What is left is to show that $C_j \geq 0$ provided $I^- = \emptyset$. But this is obvious because of (25).

Remark 15. Observe that the necessity part of Lemma 4, which has been proved above with elementary arguments, can also be obtained as a consequence of Corollary 10. Indeed, if (13) were true, then by (34) the sum of $C_i$, where $i$ ranges over $I$, would be not greater than 0, which is impossible.

8. Interior capacitary distributions associated with a condenser

As always, we are keeping all our standing assumptions, stated in Sec. 4.6. Throughout Sec. 8, it is also required that $\text{cap} \mathcal{A} < \infty$.

Our next purpose is to introduce a notion of interior capacitary distributions $\gamma_{\mathcal{A}}$ associated with a condenser $\mathcal{A}$ such that the distributions obtained possess properties similar to those of interior capacitary distributions associated with a set. Fuglede’s theory of interior capacities of sets [F1] serves here as a model case.

8.1. If $\mathcal{A} = \mathcal{K}$ is compact, then, as follows from Theorem 4, Corollary 7 and Remark 12, any minimizer $\lambda_{\mathcal{K}}$ in the $E(\mathcal{K}, a \text{ cap} \mathcal{K}, g)$-problem has the desired properties, and so $\gamma_{\mathcal{K}}$ might be defined as
\[ \gamma_{\mathcal{K}} := \lambda_{\mathcal{K}}, \text{ where } \lambda_{\mathcal{K}} \in S(\mathcal{K}, a \text{ cap} \mathcal{K}, g). \]

However, as is seen from Remark 14, in the noncompact case the desired notion can not be obtained as just a direct generalization of the corresponding one from the theory of interior capacities of sets. Having in mind that, similar to our model case, the required distributions should give a solution to the $\Gamma(\mathcal{A}, a, g)$-problem and be strongly and $\mathcal{A}$-vaguely continuous under exhaustion of $\mathcal{A}$ by compact condensers, we arrive at the following definition.

Definition 6. We shall call $\gamma_{\mathcal{A}} \in E(\overline{\mathcal{A}})$ an (interior) capacitary distribution associated with $\mathcal{A}$ if there exists a subnet $(\mathcal{K}_s)_{s \in S}$ of $(\mathcal{K})_{\mathcal{K} \in \mathcal{K}_{\mathcal{A}}}$ and
\[ \lambda_{\mathcal{K}_s} \in S(\mathcal{K}_s, a \text{ cap} \mathcal{K}_s, g), \quad s \in S, \]
such that $(\lambda_{\mathcal{K}_s})_{s \in S}$ converges to $\gamma_{\mathcal{A}}$ in both the $\mathcal{A}$-vague and the strong topologies. Let $D(\mathcal{A}, a, g)$ denote the collection of all those $\gamma_{\mathcal{A}}$. 
Application of Lemmas 2 and 5 enables us to rewrite the above definition in the following, apparently weaker, form:

\[ D(A, a, g) = M_0(A, a \text{ cap } A, g). \] (35)

**Theorem 5.** \( D(A, a, g) \) is nonempty, contained in an equivalence class in \( E(\overline{A}) \), and compact in the induced \( A \)-vague topology. Furthermore,

\[ D(A, a, g) \subset \mathcal{G}(A, a, g) \cap E(\overline{A}, \leq a \text{ cap } A, g). \] (36)

Given its element \( \gamma := \gamma_A \), then

\[ \|\gamma\|^2 = \text{cap } A, \] (37)

\[ \alpha_i a_i \kappa(x, \gamma) \geq C_i g(x) \quad \text{n. e. in } A_i, \quad i \in I, \] (38)

where \( C_i = C_i(A, a, g), i \in I, \) are the interior capacitary constants. Actually,

\[ C_i = \frac{\alpha_i \kappa(x, \gamma)}{\text{cap } A} = \inf_{x \in A_i} \frac{\alpha_i a_i \kappa(x, \gamma)}{g(x)}, \quad i \in I. \] (39)

If \( I^- \neq \emptyset \), assume moreover that the kernel \( \kappa(x, y) \) is continuous for \( x \neq y \), while \( \kappa(\cdot, y) \to 0 \) (as \( y \to \infty \)) uniformly on compact sets. Then, for every \( i \in I, \)

\[ \alpha_i a_i \kappa(x, \gamma) \leq C_i g(x) \quad \text{for all } x \in S(\gamma_i), \] (40)

and hence

\[ \alpha_i a_i \kappa(x, \gamma) = C_i g(x) \quad \text{n. e. in } A_i \cap S(\gamma_i). \]

Thus, an interior capacitary distribution \( \gamma_A \) is unique if the kernel is additionally assumed to be strictly positive definite and all \( \overline{A}_i, i \in I, \) are mutually disjoint.

**Remark 16.** As is seen from the preceding theorem, the properties of interior capacitary distributions associated with a condenser are quite similar to those of interior capacitary distributions associated with a set (cf. [F1, Th. 4.1]). The only important difference is that the sign \( \leq \) in the inclusion (36) in general can not be omitted — even for a closed, noncompact condenser.

**Remark 17.** Like as in the theory of interior capacities of sets, in general none of the \( i \)-coordinates of \( \gamma_A \) is concentrated on \( A_i \) (unless \( A_i \) is closed). Indeed, let \( X = \mathbb{R}^n, n \geq 3, \kappa(x, y) = |x - y|^{2-n}, g = 1, I^+ = \{1\}, I^- = \{2\}, a_1 = a_2 = 1, \) and let \( A_1 = \{x : |x| < r\} \) and \( A_2 = \{x : |x| > R\} \), where \( 0 < r < R < \infty \). Then it can be shown that

\[ \gamma_A = \gamma_{\overline{A}} = [\theta^+ - \theta^-] \text{ cap } A, \]

where \( \theta^+ \) and \( \theta^- \) are obtained by the uniform distribution of unit mass over the spheres \( S(0, r) \) and \( S(0, R) \), respectively. Hence, \( |\gamma_A|(A) = 0. \)
The purpose of this section is to point out characteristic properties of the interior capacitary distributions and the interior capacitary constants.

**Proposition 1.** Assume \( \mu \in E(A) \) has the properties
\[
\|\mu\|^2 = \text{cap} (A, a, g),
\]
\[
\alpha_i a_i \kappa (x, \mu) \geq \frac{\alpha_i \kappa (\mu, \mu)}{\text{cap} A} g(x) \quad \text{n. e. in } A_i, \quad i \in I.
\]
Then \( \mu \) is equivalent in \( E(A) \) to every \( \gamma_A \in D(A, a, g) \), and for all \( i \in I \),
\[
C_i (A, a, g) = \frac{\alpha_i \kappa (\mu, \mu)}{\text{cap} A} = \inf_{x \in A_i} \frac{\alpha_i a_i \kappa (x, \mu)}{g(x)}.
\]

Actually, there holds the following stronger result, to be proved in Sec. 16 below.

**Proposition 2.** Let \( \nu \in E(A) \) and \( \tau_i \in \mathbb{R}, i \in I \), satisfy the relations
\[
\alpha_i a_i \kappa (x, \nu) \geq \tau_i g(x) \quad \text{n. e. in } A_i, \quad i \in I,
\]
\[
\sum_{i \in I} \tau_i = \frac{\text{cap} A + \|\nu\|^2}{2 \text{cap} A}.
\]
Then \( \nu \) is equivalent in \( E(A) \) to every \( \gamma_A \in D(A, a, g) \), and for all \( i \in I \),
\[
\tau_i = C_i (A, a, g) = \inf_{x \in A_i} \frac{\alpha_i a_i \kappa (x, \nu)}{g(x)}.
\]

Thus, under the conditions of Proposition 1 or 2, if moreover \( \kappa \) is strictly positive definite and all \( A_i, i \in I \), are mutually disjoint, then the measure under consideration is actually the (unique) interior capacitary distribution \( \gamma_A \).

## 9. On continuity of the capacities, capacitary distributions, and capacitary constants

**9.1.** Given \( A_n = (A_i^n)_{i \in I}, n \in \mathbb{N}, \) and \( A \) in \( \mathfrak{c}_{m,p} \), write \( A_n \uparrow A \) if \( A_n \prec A_{n+1} \) for all \( n \) and
\[
A_i = \bigcup_{n \in \mathbb{N}} A_i^n, \quad i \in I.
\]

Following [B1, Chap. 1, §9], we call a locally compact space countable at infinity if it can be written as a countable union of compact sets.

**Theorem 6.** Suppose that either \( g_{\min} > 0 \) or the space \( X \) is countable at infinity. If \( A_n, n \in \mathbb{N}, \) are universally measurable and \( A_n \uparrow A \), then
\[
\text{cap} (A, a, g) = \lim_{n \in \mathbb{N}} \text{cap} (A_n, a, g).
\]
Assume moreover cap \((A, a, g)\) to be finite, and let \(\gamma_n := \gamma_{A_n}, n \in \mathbb{N}\), denote an interior capacitary distribution associated with \(A_n\). If \(\gamma\) is an \(A\)-vague limit point of \((\gamma_n)_{n \in \mathbb{N}}\) (such a \(\gamma\) exists), then \(\gamma\) is actually an interior capacitary distribution associated with the condenser \(A\), and

\[
\lim_{n \to \infty} \| \gamma_n - \gamma \|^2 = 0.
\]

Furthermore,

\[
C_i(A, a, g) = \lim_{n \to \infty} C_i(A_n, a, g), \quad i \in I.
\]

Thus, if \(\kappa\) is additionally assumed to be strictly positive definite (hence, perfect) and all \(A_i, i \in I\), are mutually disjoint, then the (unique) interior capacitary distribution associated with \(A_n\) converges both \(A\)-vaguely and strongly to the (unique) interior capacitary distribution associated with \(A\).

Remark 18. Theorem 6 remains true if \((A_n)_{n \in \mathbb{N}}\) is replaced by the increasing ordered family of all compact condensers \(K\) such that \(K \prec A\). Moreover, then the assumption that either \(g_{\min} > 0\) or \(X\) is countable at infinity can be omitted. Cf., e. g., Lemma 2 and Corollary 9.

Remark 19. If \(I = \{1\}\) and \(g = 1\), Theorem 6 has been proved in [F1, Th. 4.2].

9.2. The remainder of the article is devoted to proving the results formulated in Sec. 5–9 and is organized as follows. Theorems 2, 3, 5, and 6 are proved in Sec. 14, 15, and 17. Their proofs utilize the description of the potentials of measures from the classes \(\mathcal{M}'(A, a, g)\) and \(\mathcal{M}_0(A, a, g)\), to be given in Sec. 12 and 13 by Lemmas 9 and 10. In turn, Lemmas 9 and 10 use the theorem on the strong completeness of proper subspaces of \(\mathcal{E}\), which is a subject of Sec. 10.

10. On the strong completeness

10.1. Keeping all our standing assumptions on \(\kappa, g,\) and \(A\), stated in Sec. 4.6, we consider \(\mathcal{E}(\overline{A}, \leq a, g)\) to be a topological subspace of the semimetric space \(\mathcal{E}(\overline{A})\); the induced topology is likewise called the strong topology.

Theorem 7. Suppose \(A\) is closed. Then the semimetric space \(\mathcal{E}(A, \leq a, g)\) is complete. In more detail, if \((\mu_s)_{s \in S} \subset \mathcal{E}(A, \leq a, g)\) is a strong Cauchy net and \(\mu\) is its \(A\)-vague cluster point (such a \(\mu\) exists), then \(\mu \in \mathcal{E}(A, \leq a, g)\) and

\[
\lim_{s \to S} \| \mu_s - \mu \|^2 = 0.
\]

Assume, in addition, that the kernel is strictly positive definite and all \(A_i, i \in I\), are mutually disjoint. If moreover \((\mu_s)_{s \in S} \subset \mathcal{E}(A, \leq a, g)\) converges strongly to \(\mu_0 \in \mathcal{E}(A)\), then actually \(\mu_0 \in \mathcal{E}(A, \leq a, g)\) and \(\mu_s \to \mu_0 A\)-vaguely.
Remark 20. This theorem is certainly of independent interest since, according to the well-known counterexample by H. Cartan [C], the pre-Hilbert space $E$ is strongly incomplete even for the Newton kernel $|x - y|^{2-n}$ in $\mathbb{R}^n$, $n \geq 3$.

Remark 21. Assume the kernel is strictly positive definite (hence, perfect). If moreover $I^- = \emptyset$, then Theorem 7 remains valid for $E(A)$ in place of $E(A, \leq a, g)$ (cf. Theorem 1). A question still unanswered is whether this is the case if $I^+$ and $I^-$ are both nonempty. We can however show that this is really so for the Riesz kernels $|x - y|^{\alpha-n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ (cf. [Z1, Th. 1]). The proof utilizes Deny’s theorem [D1] stating that, for the Riesz kernels, $E$ can be completed with making use of distributions of finite energy.

10.2. We start by auxiliary assertions to be used in the proof of Theorem 7.

Lemma 7. $E(A, \leq a, g)$ is $A$-vaguely bounded.

Proof. Let $K \subset A_i$, $i \in I$, be compact. Since $g$ is positive and continuous, the inequalities

$$a_i \geq \int g \, d\mu^i \geq \mu^i(K) \min_{x \in K} g(x),$$

where $\mu \in E(A, \leq a, g)$,

yield

$$\sup_{\mu \in E(A, \leq a, g)} \mu^i(K) < \infty,$$

and the lemma follows.

Lemma 8. Suppose $A$ is closed. If a net $(\mu_s)_{s \in S} \subset E(A, \leq a, g)$ is strongly bounded, then its $A$-vague cluster set is nonempty and contained in $E(A, \leq a, g)$.

Proof. We begin by showing that the nets $(\mu^i_s)_{s \in S}$, $i \in I$, are strongly bounded as well, i.e.,

$$\sup_{s \in S} \|\mu^i_s\| < \infty, \quad i \in I. \quad (47)$$

This is obvious when $I^- = \emptyset$ and $\kappa \geq 0$; hence, one can assume that either $I^- \neq \emptyset$, or $I^- = \emptyset$ while $X$ is compact. In any case, both (15) and (16) hold. Since

$$\int g \, d\mu^i_s \leq a_i, \quad i \in I, \quad (48)$$

(15) implies

$$\sup_{s \in S} \mu^i_s(X) \leq a_i \, g_{\min}^{-1} < \infty, \quad i \in I. \quad (49)$$

When combined with (16), this shows that $\kappa(\mu^+_s, \mu^-_s)$ remains bounded from above on $S$, and hence so do $\|\mu^+_s\|^2$ and $\|\mu^-_s\|^2$. Since $\kappa$ is bounded from below on $X \times X$, repeated application of (19) gives (17) as desired.

Moreover, for every $i \in I$, $(\mu^i_s)_{s \in S}$ is vaguely bounded according to the preceding lemma, while $\mathfrak{M}^+(A_i)$ is vaguely closed. Since any vaguely bounded part of $\mathfrak{M}$ is
vaguely relatively compact (see, e.g., \[B2\], Chap. III, § 2, Prop. 9), there exists a vague cluster point of \((\mu_s^i)_{s \in S}\), say \(\mu^i\), and \(\mu^i \in \mathcal{M}^+(A_i)\).

It remains to show that \(\mu^i\) is of finite energy and satisfies (48) with \(\mu^i\) in place of \(\mu_s^i\). To this end, recall that, if \(Y\) is a locally compact Hausdorff space and \(\psi\) is a lower semicontinuous function on \(Y\) such that \(\psi \geq 0\) (unless its support is compact), then the map

\[ \nu \mapsto \int \psi d\nu, \quad \nu \in \mathcal{M}^+(Y), \]

is lower semicontinuous in the induced vague topology (see, e.g., \[F1\]). Applying this to \(Y = A_i \times A_i\), \(\psi = \kappa|_{A_i \times A_i}\) and, subsequently, \(Y = A_i\), \(\psi = g|_{A_i}\), we derive the required properties of \(\mu^i\) from (47) and (48).

10.3. Proof of Theorem 7. Suppose \(A\) is closed, and let \((\mu_s)_{s \in S}\) be a strong Cauchy net in \(\mathcal{E}(A, \leq a, g)\). Since such a net converges strongly to every its strong cluster point, \((\mu_s)_{s \in S}\) can certainly be assumed to be strongly bounded. Then, by Lemma 8, there exists an \(A\)-vague cluster point \(\mu\) of \((\mu_s)_{s \in S}\), and

\[ \mu \in \mathcal{E}(A, \leq a, g). \] (50)

We next proceed to verify (46).

Without loss of generality we can also assume that, for every \(i \in I\),

\[ \mu_s^i \to \mu^i \quad \text{vaguely}. \]

Since, by (47), \((\mu_s^i)_{s \in S}\) is strongly bounded, the property \((CW)\) (see Sec. 2) shows that \(\mu_s^i\) approaches \(\mu^i\) in the weak topology as well, and so

\[ R\mu_s \to R\mu \quad \text{weakly}. \]

This gives

\[ \|\mu_s - \mu\|^2 = \|R\mu_s - R\mu\|^2 = \lim_{l \in S} \kappa(R\mu_s - R\mu, R\mu_s - R\mu_l), \]

and hence, by the Cauchy-Schwarz inequality,

\[ \|\mu_s - \mu\|^2 \leq \|\mu_s - \mu\| \lim_{l \in S} \|\mu_s - \mu_l\|, \]

which proves (46) as required, because \(\|\mu_s - \mu_l\|\) becomes arbitrarily small when \(s, l \in S\) are both large enough.

Suppose now that \(\kappa\) is strictly positive definite, while all \(A_i, i \in I\), are mutually disjoint, and let the net \((\mu_s)_{s \in S}\) converge strongly to some \(\mu_0 \in \mathcal{E}(A)\). Given an vague limit point \(\mu\) of \((\mu_s)_{s \in S}\), then we conclude from (46) that \(\|\mu_0 - \mu\| = 0\), hence \(\mu_0 \equiv \mu\) since \(\kappa\) is strictly positive definite, and finally \(\mu_0 \equiv \mu\) because \(A_i\),
$i \in I$, are mutually disjoint. In view of (50), this means that $\mu_0 \in \mathcal{E}(\mathcal{A}, \le a, g)$, which is a part of the desired conclusion.

Moreover, $\mu_0$ has thus been shown to be identical to any $\mathcal{A}$-vague cluster point of $(\mu_s)_{s \in S}$. Since the vague topology is separated, this implies that $\mu_0$ is actually its $\mathcal{A}$-vague limit (cf. [B1 Chap. I, § 9, n° 1, cor.]), which completes the proof.

11. Proof of Lemma 5

Fix any $(\mu_s)_{s \in S}$ and $(\nu_t)_{t \in T}$ in $M(\mathcal{A}, a, g)$. It follows by standard arguments that

$$\lim_{(s,t) \in S \times T} \|\mu_s - \nu_t\|^2 = 0,$$

where $S \times T$ denotes the directed product of the directed sets $S$ and $T$ (see, e.g., [K Chap. 2, § 3]). Indeed, by the convexity of the class $\mathcal{E}(\mathcal{A}, a, g)$,

$$2 \|\mathcal{E}(\mathcal{A}, a, g)\| \le \|\mu_s + \nu_t\| \le \|\mu_s\| + \|\nu_t\|,$$

and hence, by (17),

$$\lim_{(s,t) \in S \times T} \|\mu_s + \nu_t\|^2 = 4 \|\mathcal{E}(\mathcal{A}, a, g)\|^2.$$

Then the parallelogram identity gives (51) as claimed.

Relation (51) implies that $(\mu_s)_{s \in S}$ is strongly fundamental. Therefore Theorem 7 shows that there exists an $\mathcal{A}$-vague cluster point $\mu_0$ of $(\mu_s)_{s \in S}$, and moreover $\mu_0 \in \mathcal{E}(\mathcal{A}, \le a, g)$ and $\mu_s \rightharpoonup \mu_0$ strongly. This means that $\mathcal{M}(\mathcal{A}, a, g)$ and $\mathcal{M}'(\mathcal{A}, a, g)$ are both nonempty and satisfy the inclusion (18).

What is left is to prove that $\mu_s \rightharpoonup \chi$ strongly, where $\chi \in \mathcal{M}'(\mathcal{A}, a, g)$ is arbitrarily given. But then one can choose a net in $M(\mathcal{A}, a, g)$ converging to $\chi$ strongly, and repeated application of (51) leads immediately to the desired conclusion.

12. Potentials of strong cluster points of minimizing nets

12.1. The aim of this section is to provide a description of the potentials of measures from the class $\mathcal{M}'(\mathcal{A}, a, g)$. As usual, we are keeping all our standing assumptions, stated in Sec. 4.6.

Lemma 9. There exist $\eta_i \in \mathbb{R}$, $i \in I$, such that, for every $\chi \in \mathcal{M}'(\mathcal{A}, a, g)$,

$$\alpha_i a_i \kappa(x, \chi) \geq \alpha_i \eta_i g(x) \text{ n.e. in } A_i, \quad i \in I, \quad \sum_{i \in I} \alpha_i \eta_i = \|\mathcal{E}(\mathcal{A}, a, g)\|^2.$$
These \( \eta_i, i \in I \), are determined uniquely and given by either of the formulas

\[
\eta_i = \kappa(\zeta_i, \zeta),
\]

\[
\eta_i = \lim_{s \in S} \kappa(\mu^i_s, \mu_s),
\]

where \( \zeta \in \mathcal{M}(A, a, g) \) and \( (\mu_s)_{s \in S} \in \mathcal{M}(A, a, g) \) are arbitrarily chosen.

**Proof.** Throughout the proof, we shall assume every net \( (\mu^i_s)_{s \in S} \in \mathcal{M}(A, a, g) \) to be strongly bounded, which certainly involves no loss of generality. Then all the nets \( (\mu^i_s)_{s \in S}, i \in I \), are strongly bounded as well (see the proof of Lemma 8).

Choose \( (\mu^t_t)_{t \in T} \in \mathcal{M}(A, a, g) \) with the property that, for every \( i \in I \), there exists the limit (finite or infinite)

\[
\eta_i := \lim_{t \in T} \kappa(\mu^i_t, \mu_t).
\]

We proceed to show that \( \eta_i, i \in I \), so defined, satisfy both (52) and (53).

Given \( \chi \in \mathcal{M}'(A, a, g) \), suppose, contrary to our claim, that for some \( j \in I \) there exists a set \( E_j \subset A_j \) of interior capacity nonzero such that

\[
\alpha_j a_j \kappa(x, \chi) < \alpha_j \eta_j g(x) \quad \text{for all} \quad x \in E_j.
\]

Then one can choose \( \nu \in \mathcal{E}^+ \) with compact support so that \( S(\nu) \subset E_j \) and

\[
\int g \, d\nu = a_j.
\]

Integrating the inequality in (57) with respect to \( \nu \) gives

\[
\alpha_j \left[ \kappa(\chi, \nu) - \eta_j \right] < 0.
\]

To get a contradiction, for every \( \tau \in (0, 1] \) write

\[
\tilde{\mu}^i_t := \begin{cases} 
\mu^i_t - \tau(\mu^i_t - \nu) & \text{if } i = j, \\
\mu^i_t & \text{otherwise.}
\end{cases}
\]

Clearly,

\[
\tilde{\mu}_t := \sum_{i \in I} \alpha_i \tilde{\mu}^i_t \in \mathcal{E}^0(A, a, g), \quad t \in T,
\]

and consequently

\[
\|\mathcal{E}(A, a, g)\|^2 \leq \|\tilde{\mu}_t\|^2 = \|\mu_t\|^2 - 2\alpha_j \tau \kappa(\mu_t, \mu^j_t - \nu) + \tau^2 \|\mu^j_t - \nu\|^2.
\]

The coefficient of \( \tau^2 \) is bounded from above on \( T \) (say by \( M_0 \)), while by Lemma 5

\[
\lim_{t \in T} \|\mu_t - \chi\|^2 = 0.
\]
From (56) and (59) we therefore obtain

$$0 \leq M_0 \tau^2 + 2\alpha_j \tau \left[ \kappa(\chi, \nu) - \eta_j \right].$$

By letting here \( \tau \) tend to 0, we arrive at a contradiction to (58).

It has thus been proved that \( \eta_i, \; i \in I \), defined by means of (56), satisfy (52).

Note that \( \kappa(\cdot, R\chi) \), being the potential of a measure of finite energy, is finite nearly everywhere in \( X \) (see [F1]), and hence so is \( \kappa(\cdot, \chi) \). Since, by Lemma 3, \( C(A_i) > 0 \) for all \( i \in I \), it follows from (52) that

$$\alpha_i \eta_i < \infty, \; i \in I.$$  

Hence, \( \sum_{i \in I} \alpha_i \eta_i \) is well defined and, by (56),

$$\sum_{i \in I} \alpha_i \eta_i = \lim_{t \in T} \| \mu_t \|^2 = \| \mathcal{E}(A, a, g) \|^2.$$  

This means that \( \eta_i, \; i \in I \), are finite and satisfy also (53) as required.

To prove the statement on uniqueness, consider some other \( \eta'_i, \; i \in I \), satisfying both (52) and (53). Then they are necessarily finite, and for every \( i \),

$$\alpha_i a_i \kappa(x, \chi) \geq \max \{ \alpha_i \eta_i, \alpha_i \eta'_i \} g(x) \; \text{n.e. in} \; A_i,$$  

which follows from the property of subadditivity of \( C(\cdot) \), mentioned in Sec. 6.1.

Since \( \mu_t^i \) is concentrated on \( A_i \) and has finite energy and compact support, application of [F1, Lemma 2.3.1] shows that the inequality in (60) holds \( \mu_t^i \)-almost everywhere in \( X \). Integrating it with respect to \( \mu_t^i \) and then summing up over all \( i \in I \), in view of \( \int g \, d\mu_t^i = a_i \) we have

$$\kappa(\mu_t, \chi) \geq \sum_{i \in I} \max \{ \alpha_i \eta_i, \alpha_i \eta'_i \}, \; t \in T.$$  

Passing here to the limit as \( t \) ranges over \( T \), we get

$$\| \chi \|^2 = \lim_{t \in T} \kappa(\mu_t, \chi) \geq \sum_{i \in I} \max \{ \alpha_i \eta_i, \alpha_i \eta'_i \} \geq \sum_{i \in I} \alpha_i \eta_i = \| \mathcal{E}(A, a, g) \|^2,$$

and hence

$$\max \{ \alpha_i \eta_i, \alpha_i \eta'_i \} = \alpha_i \eta_i, \; i \in I,$$

for the extreme left and right parts of the above chain of inequalities are equal. Applying the same arguments again, but with the roles of \( \eta_i \) and \( \eta'_i \) reversed, we conclude that \( \eta_i = \eta'_i \) for all \( i \in I \), as claimed.

It remains to show that \( \eta_i, \; i \in I \), can be written in the form (54) or (55). To this end, fix \( (\mu_s)_{s \in S} \in \mathcal{M}(A, a, g) \). Then it follows at once from the above reasoning that, for every \( i \in I \), any cluster point of the net \( \kappa(\mu_s^i, \mu_s) \), \( s \in S \), coincides with \( \eta_i \). Hence, there exists \( \lim_{s \in S} \kappa(\mu_s^i, \mu_s) \) and it equals \( \eta_i \).
Passing to a subnet if necessary, by Lemma 8 we can also assume \((\mu_s)_{s \in S}\) to be \(\mathcal{A}\)-vaguely convergent, say to \(\zeta\). The proof will be completed once we prove
\[
\kappa(\zeta^i, \zeta) = \lim_{s \in S} \kappa(\mu^i_s, \mu_s), \quad i \in I.
\] (61)

Since \(\|\mu^i_s\|\) is bounded from above on \(S\) (say by \(M_1\)), while \(\mu^i_s \to \zeta^i\) vaguely, the property (CW) yields that \(\mu^i_s\) approaches \(\zeta^i\) also weakly. Hence, for every \(\varepsilon > 0\),
\[
|\kappa(\zeta^i - \mu^i_s, \zeta)| < \varepsilon
\]
whenever \(s \in S\) is large enough. Furthermore, by the Cauchy-Schwarz inequality,
\[
|\kappa(\mu^i_s, \zeta - \mu_s)| \leq M_1 \|\zeta - \mu_s\|, \quad s \in S.
\]

Since, by Lemma 5, \(\mu_s \to \zeta\) strongly, the last two relations combined give (61).

12.2. In what follows, \(\eta_i := \eta_i(\mathcal{A}, a, g), i \in I\), will always denote the numbers appeared in Lemma 9. They are uniquely determined by relation (52), where \(\chi \in \mathcal{M}'(\mathcal{A}, a, g)\) is arbitrarily chosen, taken together with (53). This statement on uniqueness can actually be strengthened as follows.

**Lemma 9'**. Given \(\chi \in \mathcal{M}'(\mathcal{A}, a, g)\), choose \(\eta'_i, i \in I\), so that
\[
\sum_{i \in I} \alpha_i \eta'_i \geq \|\mathcal{E}(\mathcal{A}, a, g)\|^2.
\]

If there holds (52) for \(\eta'_i\) in place of \(\eta_i\), then \(\eta'_i = \eta_i\) for all \(i \in I\).

**Proof**. This follows in the same manner as the uniqueness statement in Lemma 9.

12.3. The following assertion is specifying Lemma 9 for a compact condenser \(\mathcal{K}\).

**Corollary 11**. Let \(\mathcal{A} = \mathcal{K}\) be compact. Given \(\lambda_\mathcal{K} \in S(\mathcal{K}, a, g)\), then for every \(i\),
\[
\alpha_i a_i \kappa(x, \lambda_\mathcal{K}) \geq \alpha_i \kappa(\lambda^i_\mathcal{K}, \lambda_\mathcal{K}) g(x) \quad n. e. \text{ in } K_i,
\] (62)

and hence
\[
a_i \kappa(x, \lambda_\mathcal{K}) = \kappa(\lambda^i_\mathcal{K}, \lambda_\mathcal{K}) g(x) \quad \lambda^i_\mathcal{K}\text{-almost everywhere.} \quad (63)
\]

**Proof**. In view of (20) and (54), \(\eta_i(\mathcal{K}, a, g), i \in I\), can be written in the form
\[
\eta_i(\mathcal{K}, a, g) = \kappa(\lambda^i_\mathcal{K}, \lambda_\mathcal{K}),
\]
which leads to (62) when substituted into (52). Since \(\lambda^i_\mathcal{K}\) has finite energy and is supported by \(K_i\), the inequality in (62) holds \(\lambda^i_\mathcal{K}\)-almost everywhere in \(X\). Hence, (63) must be true, for if not, we would arrive at a contradiction by integrating the inequality in (62) with respect to \(\lambda^i_\mathcal{K}\).
13. Potentials of $\mathcal{A}$-vague cluster points of minimizing nets

In this section we shall restrict ourselves to measures $\xi$ of the class $\mathcal{M}_0(\mathcal{A}, a, g)$. It is clear from Corollary 3 that their potentials have all the properties described in Lemmas 9 and 9'. Our purpose is to show that, under proper additional restrictions on the kernel, that description can be sharpened as follows.

**Lemma 10.** In the case where $I^- \neq \emptyset$, assume moreover that $\kappa(x, y)$ is continuous for $x \neq y$, while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets. Given $\xi \in \mathcal{M}_0(\mathcal{A}, a, g)$, then for all $i \in I$,

\[
\begin{align*}
\alpha_i a_i \kappa(x, \xi) &\geq \alpha_i \kappa(\xi^i, \xi) g(x) \quad \text{n. e. in } A_i, \quad (64) \\
\alpha_i a_i \kappa(x, \xi) &\leq \alpha_i \kappa(\xi^i, \xi) g(x) \quad \text{for all } x \in S(\xi^i), \quad (65)
\end{align*}
\]

and hence

\[
a_i \kappa(x, \xi) = \kappa(\xi^i, \xi) g(x) \quad \text{n. e. in } A_i \cap S(\xi^i).
\]

**Proof.** Choose $\lambda_K \in S(\mathcal{K}, a, g)$ such that $\xi$ is an $\mathcal{A}$-vague cluster point of the net $(\lambda_K)_{K \in \mathcal{K}}$. Since this net belongs to $\mathcal{M}(\mathcal{A}, a, g)$, from (54) and (55) we get

\[
\eta_i = \kappa(\xi^i, \xi) = \lim_{K \in \mathcal{K}} \kappa(\lambda^i_K, \lambda_K), \quad i \in I.
\]

Substituting this into (52) with $\chi$ in place of $\chi$ gives (64) as required.

We next proceed to prove (65). To this end, fix $i$ (say $i \in I^+$) and $x_0 \in S(\xi^i)$. Without loss of generality it can certainly be assumed that

\[
\lambda_K \to \xi \quad \mathcal{A}\text{-vaguely}, \quad (66)
\]

since otherwise we shall pass to a subnet and change the notation. Then, due to (63) and (66), there exist $x_K \in S(\lambda^i_K)$ with the following properties:

\[
x_K \to x_0 \quad \text{as } K \uparrow \mathcal{A}, \quad (67)
\]

\[
a_i \kappa(x_K, \lambda_K) = \kappa(\lambda^i_K, \lambda_K) g(x_K).
\]

Taking into account that, by [F1, Lemma 2.2.1], the map

\[
(x, \nu) \mapsto \kappa(x, \nu)
\]

is lower semicontinuous on $X \times \mathcal{M}^+$ in the topology of a Cartesian product (where $\mathcal{M}^+$ is equipped with the vague topology), we conclude from what has already been shown that the desired relation (65) will follow once we prove

\[
\kappa(x_0, \xi^i) = \lim_{K \in \mathcal{K}} \kappa(x_K, \lambda^i_K), \quad (68)
\]
where \( j \in I^- \) is arbitrarily chosen.

The case we are thus left with is \( I^- \neq \emptyset \). Then, according to our standing assumptions, \( g_{\text{min}} > 0 \), and therefore there exists \( q \in (0, \infty) \) such that

\[
\lambda^j_k(X) \leq q \quad \text{for all } K \in \{K\}_A.
\]  

(69)

Hence, by (66),

\[
\xi^j(X) \leq q.
\]  

(70)

Fix \( \varepsilon > 0 \). Under the assumptions of the lemma, one can choose a compact neighborhood \( W_{x_0} \) of the point \( x_0 \) and a compact neighborhood \( F \) of the set \( W_{x_0} \) so that

\[
W_{x_0} \cap A_j = \emptyset, \quad F_j := F \cap \overline{A}_j \neq \emptyset,
\]

and

\[
|\kappa(x, y)| < q^{-1}\varepsilon \quad \text{for all } (x, y) \in W_{x_0} \times \overline{F}.
\]  

(71)

In the remainder, \( C_j \) and \( \partial_j \) denote respectively the complement and the boundary of a set relative to \( \overline{A}_j \), where \( \overline{A}_j \) is treated as a topological subspace of \( X \).

Having observed that \( \kappa|_{W_{x_0} \times A_j} \) is continuous, we proceed to construct a function

\[
\varphi \in C_0(W_{x_0} \times \overline{A}_j)
\]

with the following properties:

\[
\varphi|_{W_{x_0} \times F_j} = \kappa|_{W_{x_0} \times F_j},
\]

(72)

\[
|\varphi(x, y)| \leq q^{-1}\varepsilon \quad \text{for all } (x, y) \in W_{x_0} \times C_jF_j.
\]  

(73)

To this end, consider a compact neighborhood \( V_j \) of \( F_j \) in \( \overline{A}_j \), and write

\[
f := \begin{cases} 
\kappa & \text{on } W_{x_0} \times \partial_jF_j, \\
0 & \text{on } W_{x_0} \times \partial_jV_j.
\end{cases}
\]

Note that \( E := (W_{x_0} \times \partial_jF_j) \cup (W_{x_0} \times \partial_jV_j) \) is a compact subset of the Hausdorff and compact, hence normal, space \( W_{x_0} \times V_j \), and \( f \) is continuous on \( E \). By using the Tietze-Urysohn extension theorem (see, e.g., [E2 Th. 0.2.13]), we deduce from (71) that there exists a continuous function \( \hat{f} : W_{x_0} \times V_j \to [-\varepsilon q^{-1}, \varepsilon q^{-1}] \) such that \( \hat{f}|_E = f|_E \). Thus, the function in question can be defined as follows:

\[
\varphi := \begin{cases} 
\kappa & \text{on } W_{x_0} \times F_j, \\
\hat{f} & \text{on } W_{x_0} \times (V_j \setminus F_j), \\
0 & \text{on } W_{x_0} \times C_jV_j.
\end{cases}
\]
Furthermore, since the function $\varphi$ is continuous on $W_{x_0} \times \bar{A_j}$ and has compact support, there exists a compact neighborhood $U_{x_0}$ of $x_0$ such that

$$U_{x_0} \subset W_{x_0}$$

and

$$|\varphi(x, y) - \varphi(x_0, y)| < q^{-1}\varepsilon \quad \text{for all } (x, y) \in U_{x_0} \times \bar{A_j}. \quad (75)$$

Given an arbitrary measure $\nu \in \mathcal{M}^+(\bar{A_j})$ with the property that $\nu(X) \leq q$, we conclude from (71)–(75) that, for all $x \in U_{x_0}$,

$$|\kappa(x, \nu|_{\mathcal{C}F})| \leq \varepsilon, \quad (76)$$

$$\kappa(x, \nu|_{\mathcal{F}}) = \int \varphi(x, y) d(\nu - \nu|_{\mathcal{C}F})(y), \quad (77)$$

$$\left|\int \varphi(x, y) d\nu|_{\mathcal{C}F}(y)\right| \leq \varepsilon, \quad (78)$$

$$\left|\int \left[\varphi(x, y) - \varphi(x_0, y)\right] d\nu(y)\right| \leq \varepsilon. \quad (79)$$

Finally, choose $\mathcal{K}_0 \in \{\mathcal{K}\}_A$ so that, for all $\mathcal{K} \succ \mathcal{K}_0$, there hold $x_\mathcal{K} \in U_{x_0}$ and

$$\left|\int \varphi(x_\mathcal{K}, y) d(\lambda^j_\mathcal{K} - \xi^j)(y)\right| < \varepsilon;$$

such a $\mathcal{K}_0$ exists in view of (66) and (67). Applying now (76)–(79) to each of $\lambda^j_\mathcal{K}$ and $\xi^j$, which is possible due to (69) and (70), for all $\mathcal{K} \succ \mathcal{K}_0$ we therefore get

$$|\kappa(x_\mathcal{K}, \lambda^j_\mathcal{K}) - \kappa(x_0, \xi^j)| \leq |\kappa(x_\mathcal{K}, \lambda^j_\mathcal{K}|_{\mathcal{F}}) - \kappa(x_0, \xi^j|_{\mathcal{F}})| + 2\varepsilon$$

$$\leq \left|\int \varphi(x_\mathcal{K}, y) d\lambda^j_\mathcal{K}(y) - \int \varphi(x_0, y) d\xi^j(y)\right| + 4\varepsilon$$

$$\leq \left|\int \left[\varphi(x_\mathcal{K}, y) - \varphi(x_0, y)\right] d\lambda^j_\mathcal{K}(y)\right| + \left|\int \varphi(x_0, y) d(\lambda^j_\mathcal{K} - \xi^j)(y)\right| + 4\varepsilon$$

$$\leq \varepsilon + \varepsilon + 4\varepsilon = 6\varepsilon,$$

and (68) follows by letting $\varepsilon$ tend to 0. The proof is complete.

14. Proof of Theorems 2 and 3

We begin by showing that

$$\text{cap}(\mathcal{A}, a, g) \leq \|\hat{\Gamma}(\mathcal{A}, a, g)\|^2. \quad (80)$$

To this end, $\|\hat{\Gamma}(\mathcal{A}, a, g)\|^2$ can certainly be assumed to be finite. Then there are $\nu \in \hat{\Gamma}(\mathcal{A}, a, g)$ and $\mu \in \mathcal{E}^0(\mathcal{A}, a, g)$, the existence of $\mu$ being clear from (10)
and Corollary 1. By [F1, Lemma 2.3.1], the inequality in (21) holds \( \mu^i \)-almost everywhere. Integrating it with respect to \( \mu^i \) and then summing up over all \( i \in I \), in view of \( \int g \, d\mu^i = a_i \) we get

\[
\kappa(\nu, \mu) \geq \sum_{i \in I} c_i(\nu),
\]

hence \( \kappa(\nu, \mu) \geq 1 \) by (22), and finally

\[
\|\nu\|^2 \|\mu\|^2 \geq 1
\]

by the Cauchy-Schwarz inequality. The last relation, being valid for arbitrary \( \nu \in \hat{\Gamma}(A, a, g) \) and \( \mu \in \mathcal{E}^0(A, a, g) \), forces (80).

The inequality (80) establishes Theorem 2 in the case where \( \text{cap } A = \infty \).

We are thus left with proving both Theorems 2 and 3 for the case \( \text{cap } A < \infty \). Then the \( \mathcal{E}(A, a, \text{cap } A, g) \)-problem can be considered as well.

Taking (5) and (14) into account, we deduce from Lemmas 5 and 9 with \( a \) replaced by \( \text{cap } A \) that, for every \( \chi \in M'(A, \text{cap } A, g) \),

\[
\|\chi\|^2 = \text{cap } A
\]

and there exist unique \( \tilde{\eta}_i \in \mathbb{R}, i \in I \), such that

\[
\alpha_i a_i \kappa(x, \chi) \geq \tilde{\eta}_i g(x) \quad \text{n.e. in } A_i, \quad i \in I,
\]

\[
\sum_{i \in I} \tilde{\eta}_i = 1.
\]

Actually,

\[
\tilde{\eta}_i = \alpha_i \text{cap } A^{-1} \eta_i(A, \text{cap } A, g), \quad i \in I,
\]

where \( \eta_i(A, \text{cap } A, g), i \in I \), are the numbers uniquely determined in Sec. 12. Using the property of subadditivity of \( C(\cdot) \), mentioned in Sec. 6.1, and the fact that the potentials of equivalent in \( \mathcal{E} \) measures coincide nearly everywhere in \( X \), we conclude from (82) and (83) that

\[
M'_E(A, \text{cap } A, g) \subset \hat{\Gamma}(A, a, g).
\]

Together with (80) and (81), this implies that, for every \( \sigma \in M'_E(A, \text{cap } A, g) \),

\[
\text{cap } A = \|\sigma\|^2 \geq \|\hat{\Gamma}(A, a, g)\|^2 \geq \text{cap } A,
\]

which completes the proof of Theorem 2. The last two relations also yield

\[
M'_E(A, \text{cap } A, g) \subset \hat{G}(A, a, g).
\]

As both the sides of this inclusion are equivalence classes in \( \mathcal{E} \) (see Lemmas 5 and 6), they must actually be equal, and (24) follows.
Applying Lemma 9\' for \( a \cap A \) in place of \( a \), we deduce from (24) that \( c_i(\hat{\omega}) \), \( i \in I \), satisfying (21) and (22) for \( \nu = \hat{\omega} \in \hat{\mathcal{G}}(A, a, g) \), are determined uniquely, do not depend on the choice of \( \hat{\omega} \in I \). Applying Lemma 10 suffices to apply Lemma 10 (with \( a \cap A \) in place of \( a \)), we get (23) and (26). This proves Theorem 3.

15. Proof of Theorem 5

We start by observing that \( D(A, a, g) \) is nonempty, contained in an equivalence class in \( E(\mathcal{A}) \), and satisfies the inclusions

\[
D(A, a, g) \subset M(A, a \cap A, g) \subset M'(A, a \cap A, g) \cap E(\mathcal{A}, \leq a \cap A). \tag{85}
\]

Indeed, this follows from (35), Corollary 3, and Lemma 5, the last two being taken for \( a \cap A \) in place of \( a \). Substituting (32) into (85) gives (36) as required.

Since, by (36), every \( \gamma \in D(A, a, g) \) is a minimizer in the \( \Gamma(A, a, g) \)-problem, the claimed relations (37) and (38) are obtained directly from Theorem 3 and 4 in view of Definition 5. To show that \( C_i(A, a, g), i \in I \), can actually be given by means of (39), one only needs to substitute \( \gamma \) instead of \( \zeta \) into (25) — which is possible due to (35) — and use Corollary 4.

Assume for a moment that, if \( I^- \neq \emptyset \), then \( \kappa(x, y) \) is continuous for \( x \neq y \), while \( \kappa(\cdot, y) \to 0 \) (as \( y \to \infty \)) uniformly on compact sets. In order to establish (40), it suffices to apply Lemma 10 (with \( a \cap A \) in place of \( a \)) to \( \gamma \), which can be done because of (35) and then substitute (39) into the result obtained.

To prove that \( D(A, a, g) \) is \( A \)-vaguely compact, fix \( ( \gamma_s )_{s \in S} \subset D(A, a, g) \). Then the inclusion (36) and Lemma 7 yield that this net is \( A \)-vaguely bounded, and hence \( A \)-vaguely relatively compact. Let \( \gamma_0 \) denote one of its \( A \)-vague cluster points, and let \( ( \gamma_t )_{t \in T} \) be a subnet of \( ( \gamma_s )_{s \in S} \) that converges \( A \)-vaguely to \( \gamma_0 \). In view of (35), the proof will be completed once we show that

\[
\gamma_0 \in M_0(A, a \cap A, g). \tag{86}
\]

By (35), for every \( t \in T \) there exist a subnet \( (K_{s_t})_{s_t \in S_t} \) of the net \( (K)_{K \in \mathcal{K}} \), and

\[
\lambda_{s_t} \in S(K_{s_t}, a \cap A, g), \quad s_t \in S_t,
\]

such that \( \lambda_{s_t} \) approaches \( \gamma_t \) \( A \)-vaguely as \( s_t \) ranges over \( S_t \). Consider the Cartesian product \( \prod \{ S_t : t \in T \} \) — that is, the collection of all functions \( \psi \) on \( T \) with \( \psi(t) \in S_t \), and let \( D \) denote the directed product \( T \times \prod \{ S_t : t \in T \} \) (see, e.g., [K] Chap. 2, §3). Given \( (t, \psi) \in D \), write

\[
K_{(t, \psi)} := K_{\psi(t)} \quad \text{and} \quad \lambda_{(t, \psi)} := \lambda_{\psi(t)}.
\]

Then application of Theorem 4 from [K] Chap. 2] yields that \( (\lambda_{(t, \psi)})_{(t, \psi) \in D} \) converges \( A \)-vaguely to \( \gamma_0 \). Since, as can be seen from the above construction, \( (K_{(t, \psi)})_{(t, \psi) \in D} \) forms a subnet of \( (K)_{K \in \mathcal{K}} \), this proves (86) as required.
16. Proof of Proposition 2

Consider $\nu \in \mathcal{E}(A)$ and $\tau_i \in \mathbb{R}$, $i \in I$, satisfying both the assumptions (41) and (42), and fix arbitrarily $\gamma_A \in \mathcal{D}(A, a, g)$ and $(\mu_i)_{i \in T} \in \mathbb{M}(A, a \cap A, g)$.

Since $\mu_i$ is concentrated on $A_i$ and has finite energy and compact support, the inequality in (41) holds $\mu_i$-almost everywhere. Integrating it with respect to $\mu_i$ and then summing up over all $i \in I$, in view of (37) and (42) we obtain

$$2 \kappa(\mu_i, \nu) \geq \|\gamma_A\|^2 + \|\nu\|^2, \quad t \in T.$$ 

But $(\mu_i)_{i \in T} \in T$ converges to $\gamma_A$ in the strong topology of the semimetric space $\mathcal{E}(A)$, which is clear from (85) and Lemma 5 with $a \cap A$ instead of $a$. Therefore, passing in the preceding relation to the limit as $t$ ranges over $T$, we get

$$\|\nu - \gamma_A\|^2 = 0,$$

which is a part of the conclusion of the proposition. In turn, the preceding relation implies that, actually, the right-hand side in (42) is equal to 1, and that $\nu \in \mathcal{M}'(A, a \cap A, g)$. Since, in view of Theorem 3, the latter means that

$$R\nu \in \hat{\mathcal{G}}(A, a, g),$$

the claimed relation (43) follows.

17. Proof of Theorem 6

To establish (44), fix $\mu \in \mathcal{E}(A, a, g)$. Under the assumptions of the theorem, either $g_{\text{min}} > 0$, and consequently $\mu^i(X) < \infty$ for all $i \in I$, or $X$ is countable at infinity; in any case, every $A_i$, $i \in I$, is contained in a countable union of $\mu^i$-integrable sets. Therefore, by [B2, E2] (cf. the appendix below),

$$\int g \, d\mu^i = \lim_{n \in \mathbb{N}} \int g \, d\mu_{A_n}^i, \quad i \in I,$$

$$\kappa(\mu^i, \mu^j) = \lim_{n \in \mathbb{N}} \kappa(\mu_{A_n}^i, \mu_{A_n}^j), \quad i, j \in I,$$

where $\mu_{A_n}^i$ denotes the trace of $\mu^i$ upon $A_n^i$. Now, applying the same arguments as in the proof of Lemma 2, but with the preceding two relations instead of (8) and (9), we arrive at (44) as required.

By (10) and (44), for every $n \in \mathbb{N}$, cap $(A_n, a, g)$ can certainly be assumed to be nonzero. Suppose moreover that cap $(A, a, g)$ is finite; then, by (5), so is cap $(A_n, a, g)$. Hence, according to Theorem 5, there exists

$$\gamma_n := \gamma_A \in \mathcal{D}(A_n, a, g). \quad (87)$$

Observe that $R\gamma_n$ is a minimizer in the $\hat{\mathcal{G}}(A_n, a, g)$-problem, which is clear from (24), (32), and (36). Since, furthermore,

$$\hat{\mathcal{G}}(A_{n+1}, a, g) \subset \hat{\mathcal{G}}(A_n, a, g),$$
application of Lemma 1 to \( \mathcal{H} = \tilde{\Gamma}(\mathcal{A}_n, a, g) \), \( \nu = R\gamma_{n+1} \), and \( \lambda = R\gamma_n \) gives
\[
\|\gamma_{n+1} - \gamma_n\|^2 \leq \|\gamma_{n+1}\|^2 - \|\gamma_n\|^2.
\]
Also note that \( \|\gamma_n\|^2 \), \( n \in \mathbb{N} \), is a Cauchy sequence in \( \mathbb{R} \), because, by (11), its limit exists and, being equal to \( \text{cap} \mathcal{A} \), is finite. The preceding inequality therefore yields that \( (\gamma_n)_{n \in \mathbb{N}} \) is a strong Cauchy sequence in the semimetric space \( \mathcal{E}(\mathcal{A}) \).

Besides, since \( \text{cap} \mathcal{A}_n \leq \text{cap} \mathcal{A} \), we derive from (36) that
\[
(\gamma_n)_{n \in \mathbb{N}} \subset \mathcal{E}((\mathcal{A}, \leq a \text{ cap} \mathcal{A}, g)).
\]
Hence, by Theorem 7, there exists an \( \mathcal{A} \)-vague cluster point \( \gamma \) of \( (\gamma_n)_{n \in \mathbb{N}} \), and
\[
\lim_{n \in \mathbb{N}} \|\gamma - \gamma_n\|^2 = 0.
\]

Let \( (\gamma_t)_{t \in T} \) denote a subnet of the sequence \( (\gamma_n)_{n \in \mathbb{N}} \) that converges \( \mathcal{A} \)-vaguely and strongly to \( \gamma \). We next proceed to show that
\[
\gamma \in \mathcal{D}(\mathcal{A}, a, g). \tag{88}
\]

For every \( t \in T \), consider the ordered family \( \{\mathcal{K}_t\}_{\mathcal{A}_t} \) of all compact condensers \( \mathcal{K}_t \prec \mathcal{A}_t \). By (37), there exist a subnet \( (\mathcal{K}_t)_{s \in S_t} \) of \( (\mathcal{K}_t)_{t \in \{\mathcal{K}_t\}_{\mathcal{A}_t}} \) and
\[
\lambda_{s_t} \in S(\mathcal{K}_{s_t}, a \text{ cap} \mathcal{K}_{s_t}, g)
\]
such that \( (\lambda_{s_t})_{s_t \in S_t} \) converges both strongly and \( \mathcal{A} \)-vaguely to \( \gamma_t \). Consider the Cartesian product \( \prod \{S_t : t \in T\} \), that is, the collection of all functions \( \psi \) on \( T \) with \( \psi(t) \in S_t \), and let \( D \) denote the directed product \( T \times \prod \{S_t : t \in T\} \). Given \( (t, \psi) \in D \), write
\[
\mathcal{K}_{(t, \psi)} := \mathcal{K}_{\psi(t)} \quad \text{and} \quad \lambda_{(t, \psi)} := \lambda_{\psi(t)}.
\]
Then application of Theorem 4 from [K, Chap. 2] yields that \( (\lambda_{(t, \psi)})_{(t, \psi) \in D} \) converges both strongly and \( \mathcal{A} \)-vaguely to \( \gamma \). Since \( (\mathcal{K}_{(t, \psi)})_{(t, \psi) \in D} \) is easily checked to form a subnet of \( (\mathcal{K})_{\mathcal{K} \in \{\mathcal{K}\}_{\mathcal{A}}} \), this proves (88) as required.

What is finally left is to prove (45). By Corollary 9, for every \( n \in \mathbb{N} \) one can choose a compact condenser \( \mathcal{K}_n^0 \prec \mathcal{A}_n \) so that
\[
|C_i(\mathcal{A}_n, a, g) - C_i(\mathcal{K}_n^0, a, g)| < n^{-1}, \quad i \in I.
\]
This \( \mathcal{K}_n^0 \) can certainly be chosen so large that the sequence obtained, \( (\mathcal{K}_n^0)_{n \in \mathbb{N}} \), forms a subnet of \( (\mathcal{K})_{\mathcal{K} \in \{\mathcal{K}\}_{\mathcal{A}}} \); therefore, repeated application of Corollary 9 yields
\[
\lim_{n \in \mathbb{N}} C_i(\mathcal{K}_n^0, a, g) = C_i(\mathcal{A}, a, g).
\]
This leads to (45) when combined with the preceding relation.
18. Acknowledgments

The author is greatly indebted to Professors W. Hansen, E. Saff, and W. Wendland for several helpful comments concerning this study, and to Professor B. Fuglede for drawing the author’s attention to the articles [F2] and [E1].

19. Appendix

Let $\nu \in \mathcal{M}^+(X)$ be given. As in [E2], Chap. 4, § 4.7, a set $E \subset X$ is called $\nu$-$\sigma$-finite if it can be written as a countable union of $\nu$-integrable sets.

The following assertion, related to the theory of measures and integration, has been used in Sec. 17. Although it is not difficult to deduce it from [B2, E2], we could not find there a proper reference.

**Lemma 11.** Consider a lower semicontinuous function $\psi$ on $X$ such that $\psi \geq 0$ unless the space $X$ is compact, and let $E$ be the union of an increasing sequence of $\nu$-measurable sets $E_n$, $n \in \mathbb{N}$. If moreover $E$ is $\nu$-$\sigma$-finite, then

$$\int \psi \, d\nu_E = \lim_{n \in \mathbb{N}} \int \psi \, d\nu_{E_n}.$$

**Proof.** We can certainly assume $\psi$ to be nonnegative, for if not, we replace $\psi$ by a function $\psi'$ obtained by adding to $\psi$ a suitable constant $c > 0$:

$$\psi'(x) := \psi(x) + c \geq 0,$$

which is always possible since a lower semicontinuous function is bounded from below on a compact space. Then, for every $\nu$-measurable and $\nu$-$\sigma$-finite set $Q$,

$$\int \psi \, d\nu_Q = \int \psi \varphi_Q \, d\nu,$$

where $\varphi_Q(x)$ equals 1 if $x \in Q$, and 0 otherwise. Indeed, this can be concluded from [E2], Chap. 4, § 4.14] (see Propositions 4.14.1 and 4.14.6).

On the other hand, since $\psi \varphi_{E_n}$, $n \in \mathbb{N}$, are nonnegative and form an increasing sequence with the upper envelope $\psi \varphi_E$, [E2], Prop. 4.5.1] gives

$$\int \psi \varphi_E \, d\nu = \lim_{n \in \mathbb{N}} \int \psi \varphi_{E_n} \, d\nu.$$

Applying (89) to both the sides of this equality, we obtain the lemma.
References

[B1] N. Bourbaki, *Topologie générale, Chap. I–II*, Actualités Sci. Ind., 1142, Paris (1951).

[B2] N. Bourbaki, *Intégration, Chap. I–IV*, Actualités Sci. Ind., 1175, Paris (1952).

[C] H. Cartan, *Théorie du potentiel newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. France 73 (1945), 74–106.

[D1] J. Deny, *Les potentiels d’énergie finite*, Acta Math. 82 (1950), 107–183.

[D2] J. Deny, *Sur la définition de l’énergie en théorie du potential*, Ann. Inst. Fourier Grenoble 2 (1950), 83–99.

[E1] R. Edwards, *Cartan’s balayage theory for hyperbolic Riemann surfaces*, Ann. Inst. Fourier 8 (1958), 263–272.

[E2] R. Edwards, *Functional analysis. Theory and applications*, Holt. Rinehart and Winston, New York (1965).

[F1] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. 103 (1960), 139–215.

[F2] B. Fuglede, *Caractérisation des noyaux consistants en théorie du potentiel*, Comptes Rendus 255 (1962), 241–243.

[HK] W. K. Hayman, P. B. Kennedy, *Subharmonic functions*, Academic Press, London (1976).

[K] J. L. Kelley, *General topology*, Princeton, New York (1957).

[L] N. S. Landkof, *Foundations of modern potential theory*, Springer–Verlag, Berlin (1972).

[MS] E. H. Moore, H. L. Smith, *A general theory of limits*, Amer. J. Math. 44 (1922), 102–121.

[O] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 25 (1961), 135–352.

[VP] Ch. de la Valée-Poussin, *Le potentiel logarithmique, balayage et représentation conforme*, Louvain – Paris (1949).

[Z1] N. Zorii, *A noncompact variational problem in the Riesz potential theory. I; II*, Ukrain. Math. Zh. 47 (1995), 1350–1360; 48 (1996), 603–613 (in Russian); English transl. in: Ukrain. Math. J. 47 (1995); 48 (1996).
[Z2] N. Zorii, *Extremal problems in the theory of potentials in locally compact spaces. I; II; III*, Bull. Soc. Sci. Lettr. Łódź 50 Sér. Rech. Déform. 31 (2000), 23–54; 55–80; 81–106.

[Z3] N. Zorii, *On the solvability of the Gauss variational problem*, Comput. Meth. Funct. Theory 2 (2002), 427–448.

[Z4] N. Zorii, *Equilibrium problems for potentials with external fields*, Ukrain. Math. Zh. 55 (2003), 1315–1339 (in Russian); English transl. in: Ukrain. Math. J. 55 (2003).

[Z5] N. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem*, Ukrain. Math. Zh. 57 (2005), 60–83 (in Russian); English transl. in: Ukrain. Math. J. 57 (2005).

Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereshchenkivska Str.
01601, Kyiv-4, Ukraine

E-mail: natalia.zorii@gmail.com