ON THE SHAPE FIELDS FINITENESS PRINCIPLE

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ABSTRACT. In this paper, we improve the finiteness constant for the finiteness principles for $C^m(\mathbb{R}^n, \mathbb{R}^d)$ and $C^{m-1}\{\mathbb{R}^n, \mathbb{R}^D\}$ selection proven in [19] and extend the more general shape fields finiteness principle to the vector-valued case.

1. INTRODUCTION

Suppose we are given integers $m \geq 0, n \geq 1, D \geq 1$. We write $C^m(\mathbb{R}^n, \mathbb{R}^D)$ to denote the space of all functions $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^D$ whose derivatives $\partial^\alpha \tilde{F}$ (for all $|\alpha| \leq m$) are continuous and bounded on $\mathbb{R}^n$, equipped with the norm

$$\|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} := \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \|\partial^\alpha \tilde{F}(x)\|_{\infty}.$$

Here and below, we view $\partial^\alpha \tilde{F}(x) = (\partial^\alpha F_1(x), \ldots, \partial^\alpha F_D(x))$ as a vector in $\mathbb{R}^D$.

We write $\tilde{C}^m(\mathbb{R}^n, \mathbb{R}^D)$ to denote the vector space of $m$-times continuously differentiable $\mathbb{R}^D$-valued functions whose $m$-th order derivatives are bounded, equipped with the seminorm

$$\|\tilde{F}\|_{\tilde{C}^m(\mathbb{R}^n, \mathbb{R}^D)} := \max_{|\alpha|=m} \sup_{x \in \mathbb{R}^n} \|\partial^\alpha \tilde{F}(x)\|_{\infty}.$$

We write $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$ to denote the space of all $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^D$ whose derivatives $\partial^\alpha \tilde{F}$ (for all $|\alpha| \leq m-1$) are bounded and Lipschitz on $\mathbb{R}^n$. When $D = 1$, we write $C^m(\mathbb{R}^n)$ and $C^{m-1,1}(\mathbb{R}^n)$ in place of $C^m(\mathbb{R}^n, \mathbb{R}^1)$ and $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^1)$.

We write $\tilde{P}$ to denote the vector space $\bigoplus_{j=1}^D P_j$, where $P_j$ is the space of polynomials on $\mathbb{R}^n$ with degree no greater than $m-1$. Note that $\dim \tilde{P} = D \cdot \binom{n+m-1}{m-1}$.

Quantities $c(m, n)$, $C(m, n)$, $k(m, n)$, etc., denote constants depending only on $m, n$; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by $c(m, n, D)$, $k(m, n, D)$, etc.

If $S$ is any finite set, then $|S|$ denotes the number of elements in $S$.

In [19], the authors proved the following

**Theorem 1.1** (Finiteness Principle for Smooth Selection). For large enough $k^\sharp = k(m, n, D)$ and $C^\# = C(m, n, D)$, the following hold.

(A) $C^m$ FLAVOR: Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, let $K(x) \subset \mathbb{R}^D$ be convex. Suppose that for each $S \subset E$ with $|S| \leq k^\sharp$, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ with norm $\|\tilde{F}^S\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq 1$, such that $\tilde{F}^S(x) \in K(x)$ for all $x \in S$.

Then there exists $\tilde{F} \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ with norm $\|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq C^\#$, such that $\tilde{F}(x) \in K(x)$ for all $x \in E$. 

The conjectured by Y. Brudnyi and P. Shvartsman in [6]. An approach is inspired by [1]. The corresponding to the finiteness principle proven by C. Fefferman in [12]. Our present work builds on the results of Bierstone and P. Milman [1] in the case where each selection in [1] with (K(x))_{x \in E} is a map \( \tilde{F} : \mathbb{R}^n \to \mathbb{R}^D \) such that \( \tilde{F}(x) \in K(x) \) for all \( x \in E \). Therefore, Theorem 1.3 tells us when there exists a \( C^{m-1,1} \) selection \( \tilde{F} \) of \((K(x))_{x \in E}\) for the case of infinite \( E \) and provides estimates for the \( C^m \)-norm of a selection for finite \( E \).

The number \( k^2 \) in Theorem 1.1 is called the finiteness number. The \( k^2 \) obtained in [19] is \( k^2 = 100 + |D + 2| + 100 \), where \( l_n = \binom{m+n}{n} \).

Here, we give a sharper bound on \( k^2 \). Our first result is the following.

**Theorem 1.2.** The \( k^2 \) found in Theorem 1.1 may be taken to be \( k^2 = 2^{\dim \overline{P}} \), where \( \dim \overline{P} = D \cdot \left( \frac{n+m-1}{m-1} \right) \).

A few remarks on Theorem 1.2 are in order. For \( D = 1 \), our result coincides with the one proven by P. Shvartsman [27]. A similar result was obtained by E. Bierstone and P. Milman [1] in the case where each \( K(x) \) consists of a single point, corresponding to the finiteness principle proven by C. Fefferman in [12]. Our present approach is inspired by [1].

In the case \( D = 1 \) and \( m = 2 \), Theorem 1.2 gives \( k^2 = 4 \cdot 2^{n-1} \). This is comparable to the finiteness constant \( 3 \cdot 2^{n-1} \) given by Shvartsman [27], which he shows to be optimal. See also [9].

To prove Theorem 1.2, we will need to extend the finiteness principle proven in [19] to the vector valued case.

**Theorem 1.3.** The following holds for \( \mathcal{X} = C^m(\mathbb{R}^n, \mathbb{R}^D) \) and \( \mathcal{X} = \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D) \).

Let \( S \subset \mathbb{R}^n \) be a finite set of diameter at most 1. For each \( x \in S \), let \( \tilde{G}(x) \subset \overline{P} \) be convex. Suppose that for every subset \( S' \subset S \) with \( |S'| \leq 2^\dim \overline{P} \), there exists \( F^{S'} \in \mathcal{X} \) such that \( \|F^{S'}\|_\infty \leq 1 \) and \( \partial_x F^{S'} \in \tilde{G}(x) \) for all \( x \in S' \).

Then, there exists \( F \in \mathcal{X} \) such that \( \|F\|_\infty \leq \gamma \) and \( \partial_x F \in \tilde{G}(x) \) for all \( x \in S \).

Here, \( \gamma \) depends only on \( m, n, D, \) and \( |S| \).

Because the constant \( \gamma \) depends on the number of points in \( S \), following [27], we will refer to Theorem 1.3 as a “weak finiteness principle”.

To conclude the introduction, we give an overview of how we prove Theorems 1.2 and 1.3. The proof of Theorem 1.1 given in [19] is via a more general finiteness principle for shape fields, see Theorem 2.4 below. Using Theorem 1.3, we will show an improved bound for \( k^2 \) in the finiteness principle for shape fields (i.e., Theorem 2.4); we can then deduce the bound for \( k^2 \) in Theorem 1.1 obtaining the bound asserted in Theorem 1.2. The heart of the matter therefore lies in Theorem 1.3. To put things in perspective, we would like to point out that one can’t directly apply the techniques from [1] because of the nonlinear structure in the selection.
problem and that the result in [27] is for scalar-valued functions. To prove our main theorem (Theorem 1.3), we will adapt the strategy from [1] with some new ingredients: Instead of linear structure, we will handle general convex structure using the duality theorem of linear programming to describe the relevant convex sets.

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney’s seminal work [29]–[31], and including fundamental contributions by G. Glaeser [22], Y. Brudnyi and P. Shvartsman [4]–[9], [25], [26], [28], and E. Bierstone, P. Milman, and W. Pawlucki [11–3], and C. Fefferman [10, 18, 21].

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2. Background and main results

2.1. Polynomial and Whitney fields. We write $\mathcal{P}$ to denote the vector space of polynomials on $\mathbb{R}^n$ with degree no greater than $m-1$.

For $x \in \mathbb{R}^n$, let $F$ be the $(m-1)$-times differentiable at $x$. We identify the $(m-1)$-jet of $F$ at $x$ with the $(m-1)$st-degree Taylor polynomial of $F$ at $x$:

$$\mathcal{J}_x F(y) := \sum_{|\alpha| \leq m-1} \frac{\partial^\alpha F(x)}{\alpha!} (y-x)^\alpha.$$ 

For $P, Q \in \mathcal{P}$ and $x \in \mathbb{R}^n$, we define

$$P \odot_x Q := \mathcal{J}_x (PQ).$$

The operation $\odot_x$ turns $\mathcal{P}$ into a ring, which we denote by $\mathcal{R}_x$.

We define

$$\vec{\mathcal{P}} := \mathcal{P} \oplus \cdots \oplus \mathcal{P}_D.$$ 

Thus, every $\vec{P} \in \vec{\mathcal{P}}$ has the form $\vec{P} = (P_1, \cdots, P_D)$, with $P_j \in \mathcal{P}$ for $j = 1, \cdots, D$.

Let $\vec{F} = (F_1, \cdots, F_D)$ be a $\mathbb{R}^D$-valued function $(m-1)$-times differentiable at $x \in \mathbb{R}^n$. We define

$$\mathcal{J}_x \vec{F} := (\mathcal{J}_x F_1, \cdots, \mathcal{J}_x F_D) \in \vec{\mathcal{P}}.$$ 

We will also use the $\mathcal{R}_x$-module structure on $\vec{\mathcal{P}}$, whose multiplication is given by

$$R \odot_x \vec{P} := (R \odot_x P_1, \cdots, R \odot_x P_D) \in \vec{\mathcal{P}},$$

for $x \in \mathbb{R}^n$, $\vec{P} = (P_1, \cdots, P_D) \in \vec{\mathcal{P}}$, and $R \in \mathcal{R}_x$.

Let $S \subseteq \mathbb{R}^n$ be a finite set. A Whitney field is an array $(\vec{P}_x)_{x \in S}$ parameterized by points in $S$, where $\vec{P}_x \in \vec{\mathcal{P}}$ for $x \in S$. We write $\mathcal{W}^m(S)$ to denote the space of Whitney fields on $S$.

Given $(\vec{P}_x)_{x \in S} \in \mathcal{W}^m(S)$, we define

$$(2.1) \quad \|((\vec{P}_x)_{x \in S})\|_{\mathcal{W}^m(S)} := \max_{x \in S} \|\mathcal{J}_x \vec{P}_x(x)\|_\infty + \max_{x, y \in S, x \neq y} \|\mathcal{J}_x (\vec{P}_x - \vec{P}_y)(x)\|_\infty.$$
Note that $\|\cdot\|_{W^m(S)}$ is a norm on $W^m(S)$.

We will also be using the seminorm

$$\|(\tilde{P}^x)_{x \in S}\|_{W^m(S)} := \max_{x, y \in S, x \neq y |\alpha| \leq m-1} \frac{\|\partial^\alpha (\tilde{P}^x - \tilde{P}^y)\|_{\infty}}{|x - y|^{m-|\alpha|}}. \quad (2.2)$$

We use $\tilde{P}^*$ to denote the dual of $\tilde{P}$. We use $W^m(S)^*$ to denote the dual of $W^m(S)$. An element $\xi \in W^m(S)^*$ has the form $\xi = (\xi_x)_{x \in S}$. We use $\xi[(\tilde{P}^x)_{x \in S}]$ to denote the action of $\xi \in W^m(S)^*$ on $(\tilde{P}^x)_{x \in S} \in W^m(S)$.

**Lemma 2.1.** Fix $m, n, D \in \mathbb{N}$. There exists $C = C(m, n) < \infty$ such that the following hold.

1. Let $S \subset \mathbb{R}^n$ be a finite set.
   (a) For all $\tilde{F} \in C^m(\mathbb{R}^n, \mathbb{R}^D)$, $\|(\partial_x \tilde{F})_{x \in S}\|_{W^m(S)} \leq C\|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)}$.
   (b) Let $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ denotes the space of $\mathbb{R}^D$-valued functions on $\mathbb{R}^n$ with continuous derivatives up to order $m$. For all $\tilde{F} \in C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$, $\|(\partial_x \tilde{F})_{x \in S}\|_{W^m(S)} \leq C\|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)}$.

2. Let $S \subset \mathbb{R}^n$ be a finite set. There exists a linear map $T_w^x : W^m(S) \rightarrow C^m(\mathbb{R}^n, \mathbb{R}^D)$ such that
   (a) $\|T_w^x[(\tilde{P}^x)_{x \in S}]\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq \|(\tilde{P}^x)_{x \in S}\|_{W^m(S)}$,
   (b) $\|T_w^x[(\tilde{P}^x)_{x \in S}]\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq \|(\tilde{P}^x)_{x \in S}\|_{W^m(S)}$,
   (c) $\partial^\alpha x T_w^x[(\tilde{P}^x)_{x \in S}] = \tilde{P}^x$ for each $x \in S$.

Lemma 2.1(1) is simply Taylor’s theorem. Lemma 2.1(2) is an $\mathbb{R}^D$-valued version of the Whitney Extension Theorem (for finite sets), the proof of which follows the $\mathbb{R}$-valued case by component-wise treatment. The proof for the classical Whitney Extension Theorem can be found in e.g. [23].

### 2.2 Shape fields

In this section, we generalize a key object introduced in [19].

**Definition 2.2.** Let $S \subset \mathbb{R}^n$ be finite. For each $x \in S$, $0 < M < \infty$, let $\bar{\Gamma}(x, M) \subset \tilde{P}$ be a (possibly empty) convex set. We say that $(\bar{\Gamma}(x, M))_{x \in S, M > 0}$ is a vector-valued shape field if for all $x \in S$ and $0 < M' \leq M < \infty$, we have $\bar{\Gamma}(x, M') \subset \bar{\Gamma}(x, M)$.

When $D = 1$, we write $\Gamma(x, M)$ instead of $\bar{\Gamma}(x, M)$, and we omit the adjective “vector-valued”.

**Definition 2.3.** Let $C_{\Gamma, M, \delta_{\max}}$ be positive real numbers. We say that a vector-valued shape field $(\bar{\Gamma}(x, M))_{x \in S, M > 0}$ is $(C_{\Gamma, M, \delta_{\max}})$-convex if the following condition holds:

Let $0 < \delta \leq \delta_{\max}$, $x \in S$, $0 < M < \infty$, $\tilde{P}_1, \tilde{P}_2 \in \tilde{P}$, $Q_1, Q_2 \in \mathcal{P}$. Assume that

1. $\bar{\Gamma}(x, M) \subset \bar{\Gamma}(x, M')$;
2. $\|\partial^\alpha (\tilde{P}_1 - \tilde{P}_2)(x)\|_{\infty} \leq M\delta^{m-|\alpha|}$ for $|\alpha| \leq m - 1$;
3. $\|\partial^\alpha Q_i(x)\|_{\infty} \leq \delta^{-|\alpha|}$ for $|\alpha| \leq m - 1$ and $i = 1, 2$;
4. $Q_1 \circ x Q_2 = Q_1 + Q_2$.

Then

5. $\tilde{P} := \sum_{i = 1}^2 (Q_i \circ x Q_i) \circ x P_i \in \bar{\Gamma}(x, C_{\Gamma, M})$. Here, in each summand, the first multiplication is the ring multiplication in $\mathcal{R}_x$, and the second is the action of $\mathcal{R}_x$ on the $\mathcal{R}_x$-module.
2.3. Main technical results. The main technical results are the following two theorems. The first is the Finiteness Principle for vector-valued shape fields, and second improves the finiteness constant.

**Theorem 2.4.** There exists \( k^2 = k^2(m,n,D) \) such that the following holds.

Let \( E \subset \mathbb{R}^n \) be a finite set and \( (\tilde{\bar{f}}(x, M))_{x \in E, M > 0} \) be a \((C_w, \delta_{\text{max}})\)-convex vector-valued shape field. Let \( Q_0 \subset \mathbb{R}^n \) be a cube of side length \( \delta_{Q_0} \leq \delta_{\text{max}} \) and \( x_0 \in E \cap 5Q_0 \) and \( M_0 > 0 \) be given.

Suppose that for each \( S \subset E \) with \( |S| \leq k^2 \), there exists a Whitney field \((\bar{p}^z)_{z \in S}\) such that

\[
\| (\bar{p}^z)_{z \in S} \|_{\tilde{W}^m(S)} \leq M_0
\]

and

\[
\bar{p}^z \in \tilde{\Gamma}(z, M_0) \text{ for all } z \in S.
\]

Then, there exist \( \bar{p}^0 \in \tilde{\Gamma}(x_0, M_0) \) and \( \bar{f} \in C^m(Q_0, \mathbb{R}^D) \) such that

- \( J_z \bar{f} \in \tilde{\Gamma}(z, CM) \) for all \( z \in E \cap 5Q_0 \).
- \( \| \bar{f} \|_{\tilde{W}^m(S)} \leq CM_0 \delta_{Q_0}^{m-1} \) for all \( x \in Q_0, |\alpha| \leq m \).
- In particular, \( \| \bar{f} \|_{\tilde{W}^m(S)} \leq CM_0 \) for all \( x \in Q_0, |\alpha| = m \).

The case of scalar-valued shape fields \((D = 1)\) was proven in [19]. In this paper, we will use the \( D = 1 \) case to prove the more general Theorem 2.4 stated above using a gradient trick, inspired by [19,20].

**Theorem 2.5.** One may take \( k^2 = 2^{\text{dim} \cdot \bar{p}} \) in Theorem 2.4.

*Proof of Theorem 2.5 via Theorem 2.3.* Take as given the hypotheses for Theorem 2.4 but with \( k^2 = 2^{\text{dim} \cdot \bar{p}} \). This means that for each \( S' \subset E \) with \( |S'| = 2^{\text{dim} \cdot \bar{p}} \), there exists \( (\bar{p}^z)_{z \in S'} \) such that

\[
\| (\bar{p}^z)_{z \in S'} \|_{\tilde{W}^m(S')} \leq M_0
\]

and

\[
\bar{p}^z \in \tilde{\Gamma}(z, M_0) \text{ for all } z \in S.
\]

Recall that in the definition of shape field, we require \( \Gamma(x, M) \) be convex for all \( x \in S \) and \( M > 0 \).

Let \( S \subset E \) with \( |S| \leq k^2 \), where \( k^2 \) is as initially stated in Theorem 2.4 (and coming from [19] and our gradient trick for \( D \geq 2 \)). Then, the above holds for all \( S' \subset S \) with \( |S'| = 2^{\text{dim} \cdot \bar{p}} \), so by the homogeneous version of Theorem 1.3 there exists \( \bar{f} \in C^m(\mathbb{R}^n, \mathbb{R}^D) \) such that

\[
\| \bar{f} \|_{\tilde{C}^m(\mathbb{R}^n, \mathbb{R}^D)} \leq \gamma M_0
\]

and

\[
J_x \bar{f} \in \tilde{\Gamma}(x, M_0) \text{ for all } x \in S.
\]

By (2.7), we have

\[
\| (J_x \bar{f}^x)_{x \in S} \|_{\tilde{W}^m(S)} \leq C \gamma M_0.
\]

Thus, the hypotheses for Theorem 2.4 with the \( k^2 \) from the initial statement are satisfied. \( \square \)
At this point, we have shown that the shape fields finiteness principle holds with an improved value of $k^2$ (Theorem 2.5); the next step is to show that the selection problem of Theorems 1.2 and 1.3 may be described through shape fields.

**Proof of Theorem 1.2 via Theorem 2.5.** Let

$$
\bar{f}(x, M) := \left\{ \bar{P} \in \bar{P} : \|\phi \bar{P}(x)\|_{\infty} \leq M, \bar{P}(x) \in K(x) \right\}.
$$

It suffices to observe that $(\bar{f}(x, M))_{x \in E, M > 0}$ is a $(C, 1)$-convex shape field when $K(x)$ is convex for each $x \in E$.

Let $\delta \in (0, 1), x \in E, M \in (0, \infty), \bar{P}_1, \bar{P}_2 \in \bar{P}$, and $Q_1, Q_2 \in \mathcal{P}$ be given, such that

1. $\bar{P}_1, \bar{P}_2 \in \bar{f}(x, M)$ as in (2.10);
2. $||\alpha(\bar{P}_1 - \bar{P}_2)(x)||_{\infty} \leq M\delta^{m-\alpha}$ for $m \leq m - 1$;
3. $|\alpha| \leq m - 1, Q_1 \circ \alpha Q_1 + Q_2 \circ \alpha Q_2 = 1$.

We set

$$
\bar{P} := \sum_{i=1,2} Q_1 \circ \alpha Q_1 \circ x \bar{P}_i.
$$

We want to show that $\bar{P} \in \bar{f}(x, CM)$ for some $C = C(m, n, D)$.

It is clear from (C1) and (C4) that $\bar{P}(x) = K(x)$. It remains to show that $||\alpha \bar{P}(x)||_{\infty} \leq CM$.

Using the product rule, we have, for $|\alpha| \leq m - 1$,

$$
\alpha \bar{P}(x) = \sum_{i=1,2} \sum_{\beta \leq \alpha, \gamma \leq \beta} C_{\alpha, \beta, \gamma} \cdot \alpha Q_1 (x) \cdot \alpha Q_2 (x) \cdot \alpha \bar{P}_1 (x).
$$

By (C4), we have $\delta \alpha(Q_2 \circ \alpha Q_1) = -\delta \alpha(Q_1 \circ \alpha Q_1)$ for $|\alpha| > 0$. It follows from (C2) and (C3) that $||\alpha \bar{P}(x)||_{\infty} \leq CM$.

Thus, it remains to establish Theorem 1.3. This will be done in Section 4.

3. Whitney norm and dual norm on clusters

In this section, we review the data structure in [1], and prove a series of results that allows us to reduce the size of supports for linear functionals on $W^m(S)^*$.

We write $|S|$ to denote the cardinality of a finite set $S \subset \mathbb{R}^n$.

If $X, Y \subset \mathbb{R}^n$, we define

$$
diam(X) := \max_{x, x' \in X} |x - x'| \quad \text{and} \quad \text{dist}(X, T) := \min_{x \in X, y \in Y} |x - y|.
$$

A rooted tree ("tree" for short) is an undirected graph with a distinct node (i.e., the root) in which any two nodes are connected by exactly one path. A leaf of a tree is any non-root node of degree one.

Let $S \subset \mathbb{R}^n$ be a finite set. We consider trees $T$, each node of which corresponds to a subset of $S$, that satisfy the following properties.

1. The root of $T$, $R(T) = S$.
2. If $V$ is a node, then $V$ corresponds to a subset of $S$. The children of any node $V$ form a partition of $V$ (unless $V$ is a leaf).
(T3) The nodes of any given level correspond to a partition of $S$. The leaves of $T$ are the singletons $\{x\}$, with $x \in S$.

(T4) The number of nodes of level $\ell = 0, 1, \ldots$ is a strictly increasing function of $\ell$.

A collection of points $\mathbf{x} = \{x_V \in S : V \in T \setminus \text{leaves}(T)\}$ is called a set of reference points for $T$ if, for each $V$, $x_V \in V$ and $x_V = x_W$ for some child $W$ of $V$. We adopt the convention $x_{\{\}} := x$ in the last level.

Let $\mathbf{x}$ be a set of reference points of $T$. For each $V \in T \setminus \text{leaves}(T)$, define $V(\mathbf{x}) := \{x_W : W \text{ is a child of } V\}$.

Suppose $x \in S \setminus \{x_S\}$. Then there is a unique node $V$ of highest level such that $x \in V \setminus x_V$. We set $\text{ref}(x) := x_V$.

We also set $U(x) :=$ the oldest ancestor of $U$ such that $x = x_U$.

A trunk $T$ of $T$ denotes a directed path from the root $S$ to level $\text{height}(T) - 1$. Let $T$ be a trunk of $T$. We define the set of branch nodes $B(T)$ as the set of nodes of $T$ which are adjacent to $T$.

Next, we define the notion of “clustering”.

**Definition 3.1.** Let $S \subset \mathbb{R}^n$ be finite. Let $T$ be a tree of subsets of $S$ that satisfies (T1) to (T4). We say that $T$ is a clustering of $S$ if $T$ has a set of reference points $\mathbf{x} = \{x_V\}$ such that for each $\ell = 0, 1, \ldots, \text{height}(T-1)$, the set

$$\Pi := \{V(\mathbf{x}) : \text{level}(V) = \ell\}$$

forms a partition of

$$\{a_W : \text{level}(W) = \ell + 1\}$$

and satisfies

$$|x - y| \geq c_{\mathbf{x}} \cdot \text{diam}(S) \text{ for each } S \in \Pi \text{ and } x \neq y \text{ in } S$$

(3.1)

$$\text{dist}(S, S') \geq c_{\mathbf{x}} \cdot \text{diam}(S) \text{ for all } S, S' \in \Pi, S \neq S'.$$

Here, $0 < c_{\mathbf{x}}(m, n, |S|) \leq 1$ is called the clustering constant.

We write $C = C(T, \mathbf{x})$ to denote a clustering $T$ of $S$ together with a set of reference points $\mathbf{x}$.

**Lemma 3.2** (Lemma 2.4 of [1]). Given a finite set $S \subset \mathbb{R}^n$, we can always find a clustering $T$ of $S$ such that for any set of reference points $\mathbf{x}$ for $T$, the condition of Definition 3.1 is satisfied with some $0 < c_{\mathbf{x}} \leq 1$, where $c_{\mathbf{x}}$ depends only on $m, n$, and $\#(S)$.

**Definition 3.3.** Let $C = C(T, \mathbf{x})$ be a clustering of $S$ with a set of reference points $\mathbf{x}$. We define the $C^m$-clustering norm $\|\cdot\|_C$ on $W^m(S)$ to be

$$\|(\tilde{p}^x)_{x \in S}\|_C := \max \left\{ \|\tilde{p}^x\|_C , \|\tilde{p}^{x_S}\|_{x_S} \right\},$$
where
\[ \|\bar{P}\|_c := \max_{x \in S \setminus \{x\}} \frac{\|\alpha(\bar{p}^x - \bar{p}^y)(x)\|_{\infty}}{|x - y|^{|m - |\alpha||}} \quad \text{and} \quad \|\bar{p}^x_s\|_{S} := \max_{|\alpha| \leq m} \|\alpha\bar{p}^{sx}(x_s)\|_{\infty} \]

**Lemma 3.4** (Proposition 3.2 of [1]). Let \( S \subseteq \mathbb{R}^n \) be a finite set, and let \( C = C(T, x) \) be a clustering of \( S \) with a set of reference points \( x \) and clustering constant \( c_x \). Then
\[ (3.2) \quad \|\bar{p}^s\|_{W^m(S)} \leq C \|\bar{p}^s\|_{C}. \]
Here, \( C(c_x, m, n, |S|, B) \), where \( B \) is an upper bound on \( \text{diam}(S) \).

Next, we characterize linear functionals on clusters.

Let \( S \subseteq \mathbb{R}^n \) be finite, and let \( \xi = (\xi_x)_{x \in S} \in W^m(S)^* \).

Let \( C(T, x) \) be a clustering of \( S \). For each node \( V \in T \), we define \( \bar{\xi}_V \) by the formula
\[ (3.3) \quad \bar{\xi}_V := \sum_{x \in V} \xi_x. \]
It is shown in Lemma 5.1 of [1] that the action of \( \bar{\xi} \in W^m(S)^* \) has the form:
\[ (3.4) \quad \xi_x[(\bar{p}^x)_{x \in S}] = \sum_{x \in S \setminus \{x\}} \xi_{U\{x\}}(\bar{p}^x - \bar{p}^{\text{ref}(x)}) + \xi_S(\bar{p}^x_s). \]
As a consequence, we can compute the cluster dual norm using the formula:
\[ (3.5) \quad \|\bar{\xi}\|_{C^*} = \sum_{|\alpha| \leq m} \left| \xi_S \left( 0, \ldots, 0, \frac{(-x_S)^{\alpha}}{\alpha!}, 0, \ldots, 0 \right) \right| \]
\[ + \sum_{x \in S \setminus \{x\}} \left| x - \text{ref}(x) \right|^{m - |\alpha|} \left| \xi_{U\{x\}} \left( 0, \ldots, 0, \frac{(-x)^{\alpha}}{\alpha!}, 0, \ldots, 0 \right) \right|. \]

In the above, the nontrivial expression in the arguments of \( \xi_S \) and \( \xi_{U\{x\}} \) are in the \( j \)-th coordinates.

**Lemma 3.5.** Let \( S \subseteq \mathbb{R}^n \) be a finite set, and let \( \Phi : S \times \bar{P}^n \rightarrow \mathbb{R} \) be a function that is positively homogeneous with degree one on the fibers and vanishes along the zero section. Let \( T \) be a clustering of \( S \). Let \( \bar{\xi}, \in W^m(S)^* \). For each \( V \in T \), define \( \bar{\xi}_V \) as in (3.3). Define
\[ \Phi(\bar{\xi}_V) := \sum_{x \in V} \Phi_x(\bar{\xi}_x), \]
and set \( \bar{\xi}_V := (\xi_V, \Phi(\bar{\xi}_V)) \in \bar{P}^n \oplus \mathbb{R} \). Let \( T \) be a trunk of \( T \), and let \( \Xi(T) \) denote the linear span of \( \{\bar{\xi}_V : V \in B(T)\} \) in \( \bar{P}^n \oplus \mathbb{R} \). Assume
\[ \dim \Xi(T) < \#(B(T)). \]
Then there exists \( \eta \in W^m(S)^* \) such that the following hold.
(1) For all \( V \in T \setminus T \), \( \eta_V = \theta_V \xi_V \) for some \( 0 \leq \theta_V \leq 2 \).
(2) For some \( V \in B(T) \), \( \eta_v = 0 \) for all \( x \in V \).
(3) \( \sum_{x \in S} \bar{\xi}_x = \sum_{x \in S} \eta_x \) as elements of \( \bar{P}^n \).
(4) \( \sum_{x \in S} \Phi_x(\xi_x) = \sum_{x \in S} \Phi_x(\eta_x). \)
Moreover, for such $\eta$, we have
\begin{equation}
\|\eta\|_{C^*} \leq 2\|\eta\|_{C^*}.
\end{equation}

**Proof.** We modify the proof of Lemma 6.1 of \cite{1}.

Since $\dim \Xi(T) \leq \#(B(T))$, $\{\xi_V, W \in B(T)\}$ is not linearly independent, so we may find $V \in B(T)$ such that
\[
\tilde{\xi}_V = \sum_{W \in B(T) \setminus V} \lambda_{VW} \cdot \xi_W \text{ where all } |\lambda_{VW}| \leq 1.
\]

For each $x \in S$, we set $\eta_x := \theta_x \cdot \xi_x$, where
\[
\theta_x := \begin{cases} 
0 & \text{if } x \in V, \\
1 + \lambda_{VW} & \text{if } x \in W \text{ and } W \in B(T) \setminus V, \\
1 & \text{otherwise}.
\end{cases}
\]

Conditions (1) and (2) then follow by construction.

Now we prove (3) and (4). First we make the following crucial observation. Thanks to our assumption on $\Delta$ and the conditions on the $\lambda_{VW}$’s, we see that
\begin{equation}
\Phi_x ((1 + \lambda_{VW})\xi_x) = \Phi_x(\xi_x) + \lambda_{VW}\Phi_x(\xi_x).
\end{equation}

Therefore,
\[
\sum_{x \in S} \eta_x = \sum_{x \in S} \tilde{\xi}_x - \sum_{x \in V} \tilde{\xi}_x + \sum_{W \in B(T) \setminus V} \lambda_{VW} \sum_{x \in W} \tilde{\xi}_x
\]
\[
= \sum_{x \in S} \tilde{\xi}_x - \left( \sum_{W \in B(T) \setminus V} \lambda_{VW} \tilde{\xi}_W \right)
\]
\[
= \sum_{x \in S} \tilde{\xi}_x.
\]

We see that (3) and (4) follow.

Lastly, \[(3.6)\] follows from \[(3.5)\] and conditions (1) and (3).

Let $S \subset \mathbb{R}^n$ and let $\mathcal{T}$ be a clustering of $S$. For any subset $S' \subset S$, $\mathcal{T}$ determines a clustering $\mathcal{T}'$ of $S'$ by restriction.

The main result of the section is the following.

**Lemma 3.6.** Let $k \geq 2$. Under the hypotheses of Lemma \[\[\ref{lemma3.5}\]\] if $|S| \leq k$, then there exists $S' \subset S$ satisfying the following.

1. Let $\mathcal{T}'$ be the clustering of $S'$ determined by $\mathcal{T}$. For every trunk $T'$ of $\mathcal{T}'$, let $\Xi(T')$ denote the linear span of $\{\xi_V : V \in B(T')\}$ in $P^* \oplus \mathbb{R}$. Then we have
   \[
   \#(B(T')) \leq \dim \Xi(T').
   \]

2. There exists $\eta \in W^m(S)^*$ such that the following hold.
   a. $\eta_x$ is a multiple of $\xi_x$ for each $x \in S$, and $\eta_x = 0$ for $x \in S \setminus S'$.
   b. $\|\eta\|_{W^m(S)^*} \leq C\|\xi\|_{W^m(S)^*}$, where $C = C(m,n,k,B)$ with $B$ being an upper bound for $\text{diam}(S)$.
   c. $\sum_{x \in S} \xi_x = \sum_{x \in S} \eta_x$.
   d. $\sum_{x \in S} \Phi_x(\xi_x) = \sum_{x \in S} \Phi_x(\eta_x)$, with $\Phi_x$ as in Lemma \[\[\ref{lemma3.5}\]\].
convex sets $\Gamma$ constants arising in our proof will be independent of this number. require an arbitrarily increasing number of linear constraints to describe, the con-

Lemma 4.1. is nonnegative. Then

$\| \cdot \|_V$ and let $K \subset V$ be convex. Given $\delta > 0$, there exists a convex polytope $K_\delta$ such that $K \subset K_\delta \subset B_\delta(K)$, where $B_\delta(K)$ is the $\delta$-neighborhood of $K$ under the metric determined by $\| \cdot \|_V$.

Proof. We first address the case where $V = \mathbb{R}^d$, where the norm is the $\ell^\infty$ norm given by $\| (x_1, \ldots, x_d) \| = \max_{1 \leq j \leq d} |x_j|$.

Let $Q$ be the set of cubes of the form

$Q = [k_1\delta, (k_1 + 1)\delta] \times \cdots \times [k_d\delta, (k_d + 1)\delta],$

where $k_1, \ldots, k_d \in \mathbb{Z}$. Define

$K' = \bigcup_{Q \in Q, Q \cap K' \neq \emptyset} Q,$

and let $K'' = \text{Conv}(K')$, where $\text{Conv}(\cdot)$ is used to denote the convex hull of a set. Thus, $K''$ is a convex polytope. By definition, $K \subset K'$.

Let $x \in K''$. Then, there exist $y', z' \in K'$ such that $x$ is on the line segment from $y'$ to $z'$. Since $y', z' \in K'$, there exist $y, z \in K$ such that $\| y - y'\|, \| z - z'\| < \delta$.

Consider the function $f(t) = \| t(y' - y) + (1-t)(z' - z)\|$. Then $f(0), f(1) < \delta$ and $f$ is a convex, nonnegative function, so $f(t) < \delta$ for all $t \in [0, 1]$. Pick $t_0 \in [0, 1]$ such that $x = t_0y' + (1-t_0)z'$. Then, $f(t_0) < \delta$ means that $\| x - [ty' + (1-t)y']\| < \delta$. Since $K$ is convex, $ty' + (1-t)y' \in K$, so $x$ is within distance $\delta$ of $K$. Thus, $K' \subset B_\delta(K)$, completing the proof in the case $V = \mathbb{R}^d$.

Now suppose that $V$ is an arbitrary $d$-dimensional, normed space. Since any two norms on a finite-dimensional space are equivalent, there exists $M < \infty$ such that $M^{-1}\|v\|_V \leq \|T^{-1}(v)\|_{\ell^\infty(\mathbb{R}^d)} \leq M\|v\|_V$.

Let $T: \mathbb{R}^d \to V$ be a linear isomorphism and let $K \subset \mathbb{R}^n$ be a polytope satisfying $T^{-1}(K) \subset \hat{K} \subset B_\epsilon(T^{-1}(K))$, where $\epsilon > 0$ is to be determined. It follows that
\( K \subset T(\tilde{K}) \subset B_{M\varepsilon}(K) \) and that \( T(\tilde{K}) \) is a polytope in \( V \). (To see the latter, observe that for linear functionals \( \xi \) on \( \mathbb{R}^d \), \( \xi(v) \leq c \) if and only if \( \xi \circ T^{-1}(T(v)) \leq c \) and \( \xi \circ T^{-1} \in V^* \).) Thus, choosing \( \varepsilon = \delta / M \), we see that \( T(\tilde{K}) \) is the desired polytope. \( \Box \)

4.1. **Theorem 1.3** with \( X = C^m(\mathbb{R}^n, \mathbb{R}^D) \).

**Proof of Theorem 1.3** with \( X = C^m(\mathbb{R}^n, \mathbb{R}^D) \). Given \( E \subset \mathbb{R}^n \) and \( K(x) \subset \tilde{P} \) for each \( x \in E \), we define

\[
\| (K(x))_{x \in E} \|_{W^m(E)} := \inf \{ \| (\tilde{P}^x)_{x \in E} \|_{W^m(E)} : \tilde{P}^x \in K(x) \text{ for all } x \in E \}.
\]

While not strictly a norm, the above notation allows for a concise description of a quantity which is the main focus of the proof.

Our goal is to show there exists \( C = C(m,n,D,B) \) such that for any finite \( S \subset \mathbb{R}^n \) satisfying \( |S| \leq B \), there exists \( S' \subset S \) with \( |S'| \leq 2 \dim P \) such that

\[
C^{-1} \| (G(x))_{x \in S'} \|_{W^m(S')} \leq \| (G(x))_{x \in S} \|_{W^m(S)} \leq C \| (G(x))_{x \in S'} \|_{W^m(S')}.
\]

If so, it follows that

\[
\| (G(x))_{x \in S'} \|_{W^m(S')} \leq 1 \text{ for all } S' \subset S \text{ satisfying } |S'| = 2 \dim P
\]

implies

\[
\| (G(x))_{x \in S} \|_{W^m(S)} \leq C
\]

for all \( S \subset E \) satisfying \( |S| = k^2 \).

We now make the following reduction: it suffices to prove Theorem 1.3 in the case that each \( G(x) \) is a polytope.

If not, replace each \( G(x) \) with \( G(x)_\delta \) for sufficiently small \( \delta > 0 \), where \( G(x)_\delta \) is the polytope guaranteed by Lemma 4.1. By taking \( \delta > 0 \) small enough, one may approximate both \( \| (G(x))_{x \in S} \|_{W^m(S)} \) and \( \| (G(x))_{x \in S'} \|_{W^m(S')} \) within a factor of 2, as these norms are continuous with respect to the relevant metrics.

To this end, we replace each \( G(x) \) with \( G(x)_\delta \), which will now be denoted \( K_x \), as \( \delta \) is fixed. For each \( x \), we write \( K_x = \{ \tilde{P} : \Omega_x \tilde{P} \leq \tilde{c}_x \} \) for some linear map \( \Omega_x : \tilde{P} \to \mathbb{R}^{m_x} \), where \( m_x \in \mathbb{N} \). We will occasionally write \( \Omega : W^m(S) \to \prod_x \mathbb{R}^{m_x} \) to denote the mapping which sends \( (\tilde{P}^x)_{x \in S} \) to \( (\Omega_x \tilde{P}^x)_{x \in S} \).

We begin by writing \( \| (K_x)_{x \in S} \|_{W^m(S)} \) as the solution to a linear programming problem:

\[
\| (K_x)_{x \in S} \|_{W^m(S)} = \inf \left\{ \| (\tilde{P}^x)_{x \in S} \|_{W^m(S)} : (\Omega_x \tilde{P}^x)_{x \in S} \leq (\tilde{c}_x)_{x \in S} \right\}
\]

\[
\| (K_x)_{x \in S} \|_{W^m(S)} = \inf \left\{ \| (\tilde{P}^x)_{x \in S} \|_{W^m(S)} : (\Omega_x \tilde{P}^x)_{x \in S} \leq (\tilde{c}_x)_{x \in S} \right\}
\]

By Lemma 4.1, the unit ball in \( W^m(S)^* \) may be approximated within a factor of 2 by a polytope, written as \( \{ (\xi_x)_{x \in S} : L(\xi_x)_{x \in S} \leq 1_k \} \) for some \( k \in \mathbb{N} \) and linear map \( L : W^m(S)^* \to \mathbb{R}^k \). Thus, we may rewrite (4.2) as

\[
\| (K_x)_{x \in S} \|_{W^m(S)} = \inf \left\{ \| (\tilde{P}^x)_{x \in S} \|_{W^m(S)} : (\Omega_x \tilde{P}^x)_{x \in S} \leq (\tilde{c}_x)_{x \in S} \right\}
\]

for some linear map \( L : W^m(S)^* \to \mathbb{R}^k \) and some \( k \in \mathbb{N} \).

The advantage of this formulation is that it becomes possible to apply the LP Duality Theorem (Lemma A.2 in Appendix) to the supremum above, giving us
\[(K_x)_{x \in S} \|_{W^m(S)} \approx \inf_{(\Omega, P) \leq \varepsilon_x} \inf_{y \geq 0} \sup_{L^T y = \langle P \rangle} 1_k \cdot y\]
\[\sim \inf_{(\Omega, P) \leq \varepsilon_x} \sup_{y \geq 0} \inf_{L^T y = \langle P \rangle} 1_k \cdot y \sim \inf_{y \geq 0} -\Omega^T y - c\]

Note the referenced linear programming problem is feasible, as its solution corresponds to finding the smallest norm of a vector in a closed set.

Applying the Duality Theorem again, one obtains

\[(4.5) \quad \| (K_x)_{x \in S} \|_{W^m(S)} \approx \sup_{\sum x_{x \geq 0} z_x} \sum x_{x \in S} -\varepsilon_x \cdot z_x \]

\[(4.6) \quad = \sup_{L(\xi_{x \in S}) \leq 1} \sup_{\xi_{x \in S}} \sum x_{x \in S} -\varepsilon_x \cdot z_x \]

\[(4.7) \quad \approx \sup_{\sum x_{x \in S} f_x(\xi_x)} \frac{\sum x_{x \in S} f_x(\xi_x)}{\| (K_x)_{x \in S} \|_{W^m(S)}} , \]

where

\[f_x(\xi_x) = \sup_{x \geq 0} \sum x_{x \in S} -\varepsilon_x \cdot z_x \]

Fix \((\xi_x)_{x \in S}\) such that \(\xi_x \in (-\Omega^T_x)_{R^{m+}}\) for all \(x \in S\). We see that \(f\) satisfies the hypotheses of Lemmas 3.5 and 3.6. By Lemma 3.7, we may apply Lemma 3.6 repeatedly until the conclusion satisfies \(|S'| \leq 2^{\dim P}\).

Let \((\eta_x)_{x \in S}\) be as guaranteed in the conclusion of the lemma and recall \(S' = \{x \in S : \eta_x \neq 0\}\). Thus,

\[\| (\eta_x)_{x \in S} \|_{W^m(S')} = \| (\eta_x)_{x \in S} \|_{W^m(S)} \leq \| (\xi_x)_{x \in S} \|_{W^m(S')} \]

and \(|S'| \leq 2^{\dim P}\). Note that each \(\eta_x\) is obtained by multiplying some \(\xi_x\) by a nonnegative scalar; thus, \(\eta_x \in (-\Omega^T_x)_{R^{m+}}\) for all \(x \in S\).

By this reasoning and (4.5) applied both as written above and with \(S'\) in place of \(S\),

\[\| (K_x)_{x \in S} \|_{W^m(S)} \approx \sup_{\xi_{x \in S \in \Omega_{x \in S} \in R^{m+}}} \frac{\sum x_{x \in S} f_x(\xi_x)}{\| (\xi_x)_{x \in S} \|_{W^m(S)}} \approx \| (K_x)_{x \in S} \|_{W^m(S')}, \]

□
4.2. **Theorem 1.3** with $\mathcal{X} = \hat{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^D)$. In this section, we point out the modifications needed in order to prove Theorem 1.3 for the case $\mathcal{X} = \hat{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^D)$.

Let $S \subset \mathbb{R}^n$ be a finite set. Recall the definition of $\|\|_W^m(S)$ in (2.2). Recall Section 3. We write $\dot{\mathcal{L}}(S) \subset W^m(S)^*$ to denote the vector subspace spanned by

$$\{\xi_{\alpha, y, z} : y \in S \setminus \{x_S\}, z = \text{ref}(y), |\alpha| \leq m - 1\} \subset W^m(S)^*,$$

where

$$\xi_{\alpha, y, z}(\tilde{P}^x)_{x \in S} = \frac{\partial^\alpha(\tilde{P}^y - \tilde{P}^z)(y)}{|y - z|^m - |\alpha|}.$$

We are now ready to state the necessary modifications.

- **Cluster seminorm.** Mirroring Definition 3.3, we define

$$\|\|_{\dot{\mathcal{L}}}^m(S) := \|\|_{\tilde{\mathcal{L}}}^m(S),$$

with $\|\|_{\tilde{\mathcal{L}}}^m(S)$ as in Definition 3.3. Mirroring Lemma 3.3, we have

$$\|\|_{\dot{\mathcal{L}}}^m(S) \leq C \|\|_{\tilde{\mathcal{L}}}^m(S),$$

with $C$ depending only on $m, n, D, |S|$, the clustering constant for $C$, and an upper bound on $\text{diam}(S)$. It follows from Remark 3.3 of [1] that

$$\|\|_{\dot{\mathcal{L}}}^m(S) \approx \sup_{\xi \in \dot{\mathcal{L}}^m(S)} \xi(\tilde{P}^x)_{x \in S}.\quad (4.8)$$

The constant of equivalence depends on $m, n, D, |S|$, and an upper bound on $\text{diam}(S)$.

- **Linear functionals.** Mirroring (3.4), the action of an element $\xi \in \dot{\mathcal{L}}(S)$ on $W^m(S)$ has the form

$$\xi(\tilde{P}^x)_{x \in S} = \sum_{x \in S \setminus \{x_S\}} \xi(U(x))(\tilde{P}^x - \tilde{P}^\text{ref}(x)).\quad (4.9)$$

It follows that the dual cluster norm of an element $\xi \in \dot{\mathcal{L}}(S)$ is given by

$$\|\xi\|_{\dot{\mathcal{L}}^m(S)\ast} = \sum_{x \neq \text{ref}(x)} |x - \text{ref}(x)|^m - |\alpha| \left| \sum_{|\alpha| \leq m} \sum_{1 \leq j \leq D} \xi(U(x)) \left( 0, \ldots, 0, \frac{(x - j)^\alpha}{\alpha!}, 0, \ldots, 0 \right) \right|. \quad (4.10)$$

In (4.10), the nontrivial expression in the arguments of $\xi(U(x))$ are in the $j$-th coordinates.

- **The key lemmas.** We use $\dot{\mathcal{L}}(S)$ in place of $W^m(S)^*$ in Lemma 3.3. Correspondingly, in Lemma 3.6(2), the condition $\eta \in W^m(S)^*$ is replaced by $\eta \in \dot{\mathcal{L}}(S)$.

- **Proof of the main theorem.** Mirroring (1.1), we define the selection “seminorm” to be

$$\|\|_{K^m(S)} := \inf_{\tilde{P}^x \in K(x)} \|\|_{W^m(S)}(\tilde{P}^x),$$

We repeat proof of Theorem 1.3 with $\mathcal{X} = C^m(\mathbb{R}^n, \mathbb{R}^D)$ in the previous section, but with the following tweaks.

- We use $\|\|_{W^m(S)}$ in place of $\|\|_{W^m(S)}$ (both the Whitney seminorm and the selection “seminorm”).
Theorem 2.4 using the $R_1$ by definition, and for 
Proof of Theorem 2.4.

Let $X = \hat{C}^m(R^n, R^D)$.
Thus, we conclude the proof of Theorem 1.3

5. Vector-Valued Shape Fields Finiteness Principle

In this section we use what is colloquially known as the “gradient trick” to prove Theorem 2.4 using the $D = 1$ case proven in [19]. (See [19, 20].)

The following proof will require working in both $R^n$ and $R^{n+D}$, so we provide a brief introduction to some of the notation.

The variable for $R^n$ will be $x$, while $R^{n+D}$ will be viewed as $z = (x, \xi)$ with $x \in R^n, \xi \in R^D$. The appropriate level of regularity for $R^{n+D}$ will be $C^{m+1}$, so let $P^+$ denote the vector space of $R$-valued, $m$-degree polynomials over $R^{n+D}$. (Recall that $P$ is the vector space of $R^P$-valued, $(m-1)$-degree polynomials over $R^n$.)

\textbf{Proof of Theorem 2.4.}

Let $E, Q_0 \subset R^n, (\bar{f}(x, M))_{x \in E, M > 0}, C_w, 0 < \delta Q_0 \leq \delta_{\max}, x_0 \in E \cap 5Q_0$ as in the hypotheses of Theorem 2.4 be given.

Let $E^+ = \{x, 0\} : x \in E \} \subset R^{n+D}$. For $(x_0, 0) \in E^+$, define

\begin{equation}
\Gamma((x_0, 0), M) = \{P \in P^+ : P(x, 0) = 0, \nabla_{\xi} P(x, 0) \in \bar{f}(x_0, M)\}.
\end{equation}

We now show that $(\Gamma(z, M))_{x \in E^+}$ satisfies the hypotheses of the $D = 1$ case of Theorem 2.4.

Let $S^+ \subset E^+$ with $|S^+| \leq k^2$. By definition, $S^+$ is of the form $\{(x, 0) : x \in S\}$ for some $S \subset E$ with $|S| \leq k^2$.

By hypothesis of Theorem 2.4, there exist $(\bar{P}^x)_{x \in S}$ such that

\begin{equation}
\|P^x\|_{W_m(S)} \leq M_0.
\end{equation}

and

\begin{equation}
\bar{P}^x \in \bar{f}(x, M_0) \text{ for all } x \in S.
\end{equation}

For $z = (x_0, 0) \in E^+$, define

\begin{equation}P^z(x, \xi) = P^{(x_0, 0)(x, \xi)} := \sum_{j=1}^{D} \xi_j P_j(x).
\end{equation}

Clearly, $P^{(x_0, 0)}(x_0, 0) = 0$ and $\nabla_{\xi} P^{(x_0, 0)} = \bar{P}^x$, so $P^z \in \Gamma(z, M_0)$ for all $z \in E^+$.

Let $(x_0, 0), (y_0, 0) \in E^+$. Then,

\begin{equation}
\partial^\alpha_x P^{(x_0, 0)}(x, 0) = 0 \text{ and } \partial^\alpha_x \partial^\beta_x P^{(x_0, 0)}(x, 0) = 0 \text{ for } |\beta| \geq 2
\end{equation}

by definition, and for $1 \leq j \leq D$,

\begin{equation}\left|\partial^\alpha_x \partial^\beta_x \left(P^{(x_0, 0)} - P^{(y_0, 0)}\right)(x_0, 0)\right| \leq C|x_0 - y_0|^{m-|\alpha|}\end{equation}

\begin{equation}\leq C|x_0 - y_0|^{(m+1)-(|\alpha|+1)}.
\end{equation}

Thus, $(P^z)_{z \in S^+}$ satisfy (2.3).
To demonstrate \((C_w, \delta_{\text{max}})\)-convexity, let \(0 < \delta \leq \delta_{\text{max}}, x \in S^+, M < \infty, P_1, P_2, Q_1, Q_2 \in \mathcal{P}^+\) be as in Definition 2.2. If \(P := Q_1 \circ (x_0, 0) Q_1 \circ (x_0, 0) P_1 + Q_2 \circ (x_0, 0) Q_2 \circ (x_0, 0) P_2\), then \(P(x_0, 0) = 0\) and
\[
\nabla Q(x, 0) = [Q_1 \circ (x_0, 0) Q_1 \circ (x_0, 0) \nabla Q_1(x, 0)] + [Q_2 \circ (x_0, 0) Q_2 \circ (x_0, 0) \nabla Q_2(x, 0)],
\]
which lies in \(\Gamma(x, C_w M)\) by the \((C_w, \delta_{\text{max}})\)-convexity of the \(\tilde{f}(x, M)\).

Let \(Q'\) be the unit cube in \(\mathbb{R}^D\). By the \(D = 1\) case of Theorem 2.4 applied to \(E \subset \mathbb{R}^{n+D}, (\Gamma(z, M))_{z \in \mathbb{R}^{n+D}, M > 0, (x_0, 0), Q_0 \times Q', \text{we have the following. There exist } F \in C^{m+1}(\mathbb{R}^{n+D}, \mathbb{R}) \text{ and } P^0 \in \Gamma((x_0, 0), C M_0) \text{ such that}
\[
J_{(x_0, 0)} F \in \Gamma((x_0, 0), C M)
\]
for all \((x, 0) \in E^+;\)
\[
|\partial^n_x \partial^0_y (F - P^0)(x, \xi)| \leq C M_0 \text{ for all } (x, \xi) \in Q_0 \times Q', |\alpha| + |\beta| \leq m + 1;
\]
and
\[
|\partial^n_x \partial^0_y (F - P^0)(x, \xi)| \leq C M_0 \text{ for } |\alpha| + |\beta| = m + 1.
\]

Define \(\bar{G}(x) := \nabla Q(x, 0)\) and \(\bar{Q}^0(x) = \nabla Q^0(x, 0)\). We claim \(\bar{G} \in C^m(\mathbb{R}^n, \mathbb{R}^D)\) and \(\bar{Q}^0 \in \Gamma(x, C M)\) are the desired function and jet, respectively, found in the conclusion of Theorem 2.4.

First, by (5.10),
\[
J_x G(y) = \nabla Q_{(y, 0)} F(x, 0) \in \tilde{f}(x, C M)
\]
because \(J_{(y, 0)} F(x, 0) \in \tilde{f}((x, 0), C M)\).

Next, for any \(|\alpha| \leq m\) and \(1 \leq j \leq D,\)
\[
|\partial^n_x (G_j - Q^0_j)(x)| = |\partial^n_x (\partial^n_j F - \partial^n_j P^0)(x, 0)|
\]
\[
\leq C M_0 \delta_0^{m+1-|\alpha|} = C M_0 \delta_0^{m-|\alpha|}
\]
by (5.11).

Lastly, for \(|\alpha| = m,\)
\[
|\partial^n_x G_j (x)| = |\partial^n_x \partial^0_y F| \leq C M_0
\]
via (5.12). \(\square\)

**Appendix A. Linear Programming and Duality**

**Lemma A.1 (LP Duality Theorem).** Let \(p, q \) be positive integers. Let \(c \in \mathbb{R}^p\) and \(b \in \mathbb{R}^q\). Let \(A : \mathbb{R}^p \to \mathbb{R}^q\) be a linear map. Consider the following two optimization problems.

(A.1) \[
\text{Maximize } c^T \cdot x \text{ subject to } Ax \leq b.
\]

(A.2) \[
\text{Minimize } b^T \cdot y \text{ subject to } A^T y = c \text{ and } y \geq 0.
\]

Suppose one of (A.1) or (A.2) has a feasible solution, then both have feasible and optimal solutions. Moreover, if \(x_0\) optimizes (A.1) and \(y_0\) optimizes (A.2), then \(c^T \cdot x_0 = b^T \cdot y_0\), i.e. the maximum of (A.1) equals the minimum of (A.2).

The same conclusion holds if we replace “\(Ax \leq b\)” by “\(Ax \leq b\) and \(x \geq 0\)” in (A.1) and “\(A^T y = c\)” by “\(A^T y \geq c\)” in (A.2).

See [23] for a proof.

We generalize the theorem above to finite dimensional normed spaces.
Lemma A.2. Let $V$ be a finite-dimensional normed vector space with norm $\|\cdot\|_V$ and dual $V^*$. Let $L : V^* \to \mathbb{R}^d$ be a linear map and let $L^* : \mathbb{R}^d \to V$ be the dual operator of $L$ defined by
\[
x^T \cdot L(\phi) = \langle \phi, L^* x \rangle \quad \text{for all } x \in \mathbb{R}^d \text{ and } \phi \in V^*.
\]
Let $b \in \mathbb{R}^d$. Suppose there exists $\phi_0 \in V^*$ such that $L(\phi_0) \leq b$. Then
\[
(A.3) \quad \sup_{L(\phi) \leq b} \langle \phi, v \rangle = \inf_{y \geq 0} b^T \cdot y.
\]

Proof. Let $p = \dim V < \infty$. There exists a linear isomorphism $J : V \to \mathbb{R}^p$. Let $J^* : \mathbb{R}^p \to V^*$ denote its dual. Note that $J^*$ is also a linear isomorphism. We have the following diagram.

\[
\begin{array}{ccc}
\mathbb{R}^p & \overset{J}{\longrightarrow} & V \\
\downarrow & & \downarrow \overset{L^*}{\longrightarrow} \\
\mathbb{R}^p & \overset{J^*}{\longrightarrow} & V^*
\end{array}
\]

For each $v \in V$ and $\phi \in V^*$, there exist unique $c, x \in \mathbb{R}^p$ such that $J^{-1}(p) = v$ and $J^*(x) = \phi$. Thus, thanks to LP Duality Theorem (see Appendix), we have
\[
\begin{align*}
\sup_{L(\phi) \leq b} \langle \phi, v \rangle &= \sup_{L \circ J^*(x) \leq b} \langle J^*(x), J^{-1}(c) \rangle \\
&= \sup_{L \circ J^*(x) \leq b} c^T \cdot x \\
&= \inf_{L \circ J^* \leq b} \langle c, y = b \rangle. 
\end{align*}
\]

Notice that $(L \circ J^*)^T = J \circ L^*$. Moreover, since $J$ is an isomorphism, the equality $J \circ L^*y = c$ is equivalent to $L^*y = J^{-1}(c) = v$. \(\Box\)

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\footnote{1Here we identify the dual of any Euclidean space with itself via the dot product.}
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