Classical Hamiltonian Reduction

On $D(2|1; \alpha)$ Chern-Simons Gauge Theory and Large $N = 4$ Superconformal Symmetry

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Abstract

3d Chern-Simons gauge theory has a strong connection with 2d CFT and link invariants in knot theory. We impose some constraints on the $D(2|1; \alpha)$ CS theory in the similar context of the hamiltonian reduction of 2d superconformal algebras. There Hilbert states in $D(2|1; \alpha)$ CS theory are partly identified with characters of the large $N = 4$ SCFT by their transformation properties.
1 Introduction

A decade before, a great deal of insights on 2d-3d field theoretic correspondence was exposed by Witten\cite{1}. In his celebrated paper, it is indicated that there is a certain relation between 2-dimensional rational conformal field theory (2d RCFT) and 3-dimensional Chern-Simons gauge theory (3d CSGT). This correspondence, called Chern-Simons-Witten theory (CSW), makes it possible to create not only topological invariants as a set of Jones polynomials, also new link invariants of knot theory\cite{2}. This means effectiveness of field theoretic methods on knot theory and has promoted a large amount of studies on this subject\cite{3, 4, 5, 6, 7}.

Most of these works have not dealt with 2d superconformal field theories (SCFT’s) and applications on them, which recently developed by Ennes et al.\cite{6}. Here we formulate the $D(2|1;\alpha)$ CSGT and present a way to its connection with 2d large $N = 4$ SCFT, adopting the idea of hamiltonian reduction technique (HR method)\cite{8, 9}, which is well-known in 2d CFT.

On the other hand, 1+1-dimensional $N = 4$ SCFT’s were shown to be good tools for describing a low energy structure of IIB string theory compactified on $K3$ to six spacetime dimensions. It has also been conjectured by Witten et al. and confirmed in the system of $D1 + D5$ brane configurations\cite{10}. In addition, among recent studies of $AdS_{d+1}/CFT_d$ correspondence on string theory, there is such a 2d-3d correspondence on $AdS_3$ string theory background as CSW theory reveals\cite{11}. It seems to follow that superalgebra-valued CSGT’s are associated with operators of the boundary SCFT through the hamiltonian reduction method. Though $AdS_{d+1}/CFT_d$ correspondence is not directly connected to our case, these should nevertheless be a part of our motivation.

In this paper we perform an anti-holomorphic quantization procedure on CSGT. It has been applied to the $osp(1|2)$ case where it leads to the $N = 1$ SCFT\cite{8}, but not to the cases of $N > 1$ SCFT yet. In what follows, we are going to extend this formalism to the platform with the basic Lie superalgebras $D(2|1;\alpha)$, which contains eight fermionic generators, and then apply HR method to it. Finally we explicitly show the partly correspondence between 2d large $N = 4$ SCFT and 3d Lie superalgebra-valued Chern-Simons gauge theory (CSGT) in the context of HR method. It should be the basic background for further investigations on them, and there should be possibilities for future applications which is shortly discussed in the end of this paper. This formulation can also be applied
to the other Lie superalgebras[12].

2 Anti-holomorphic quantization of CSGT

With an invariant bilinear form, ( , )defined in sect.3, we can write the action of $D(2|1;\alpha)$ CSGT as

$$S_{CS,k} = \frac{k}{4\pi} \int_{\mathcal{M}} \left( A, dA + \frac{1}{3} [A, A] \right),$$

where $A$ is a $D(2|1;\alpha)$-valued one form over an arbitrary three-dimensional $\mathcal{M}$. $[,]$ is a $\mathbb{Z}_2$-graded commutator defined on the superalgebra $D(2|1;\alpha)$. The variation of this action leads to the Gauss law constraint $F \approx 0$ ($F = dA + A^2$). If $\mathcal{M}$ has a boundary, some 2-dimensional CFT is realized on it, but this is not our case.

There are two ways of formulations through quantization procedures on 2d Riemann surface $\Sigma$, which is made in our case by cutting $\mathcal{M}$ into two pieces. One way is that: first the constraints due to the gauge invariance are imposed, then quantization of the reduced phase space produces a projectively flat vector bundle over the moduli space of complex structures of $\Sigma$. The other is called anti-holomorphic quantization procedure. One constructs quantum mechanical wave functionals, then impose gauge constraints on the unconstrained wave functionals[4, 5]. As we know, CSW theory arises in the identification of conformal blocks of 2d RCFT defined on $\Sigma$, with Hilbert states of 3d CSGT on $\mathcal{M}$. It is better to take the latter quantization scheme for our purpose, since it enables us concretely to suppose that CSGT Hilbert states should be SCFT characters, generating functionals for 2d current correlator blocks.

![Fig.1: $\mathcal{M}$ is divided into two pieces which share a common 2-dimensional Riemann surface $\Sigma.$](image)

Let $\mathcal{M}$ to be a manifold without boundary and cut it by 2-dimensional Riemann surface $\Sigma$ of genus $g$, then we get two 3-dimensional manifolds, $\mathcal{M}_1, \mathcal{M}_2$ (Fig.1). Each

\[\text{The supertrace operation, seen in the basic definitions of superalgebras amounts to zero-killing form in this case}[13].\]

\[\text{Labastida et al. construct operator formalism in the context of the latter one}[5].\]
three-manifold has a boundary corresponding to the cut and each one can be identified with each other via homeomorphisms. Taking $\mathcal{M}$ locally to be $\mathbb{R} \times \Sigma$, we set $\mathbb{R}$ to be a time-axis and impose the time-axial gauge $A_t = 0$. In this set-up, we can canonically quantize the CSGT, introducing an appropriate complex structure on $\Sigma$. It complexifies the gauge group and sets $[A^a(z, \bar{z}), A^b(w, \bar{w})] = \pi/k \delta^2(z - w) g^{ab}/2$, where $[ , ]$ is also $\mathbb{Z}_2$-graded one, $g^{ab}$ is a metric on group manifold constructed from the bilinear form. This canonical commutation relation induces a condition $A_z = A^\dagger_{\bar{z}}$. We follow the anti-holomorphic quantization procedure developed by Labastida\cite{5}. Let us assume that the states of Hilbert space $\Phi(A_{\bar{z}}), \Phi(A_z)$ on $\Sigma$ are spanned by holomorphic and anti-holomorphic functions in terms of $A_{\bar{z}}, A_z (\equiv \overline{A_z})$. Then anti-holomorphic quantization is accomplished by introducing an inner product on the functional space,

$$\langle \Phi_2(A_z) | \Phi_1(A_{\bar{z}}) \rangle = \int D A_z D A_{\bar{z}} e^{\frac{2k}{\pi} \int_{\Sigma} (A_z, A_{\bar{z}}) } \overline{\Phi_2(A_{\bar{z}})} \Phi_1(A_z).$$ (2)

Wilson line operators provide natural framework of this topological quantum field theory, as gauge invariant observables. These operators are gathered into the usual Feynman path integral expression and have vacuum expectation value. In the anti-holomorphic quantization we represent these vev’s as inner products on the unconstrained Hilbert space, and impose the Gauss constraint on the states and gauge invariance on the inner product as physical conditions. One may realize that the corresponding conformal blocks of the SCFT satisfy these conditions.

3 Conventions of the Basic Lie Superalgebra $D(2|1; \alpha)$

We are now in a position to fix the conventions of gauge field $A$ as an element of the superalgebra(SA) $D(2|1; \alpha)$. Elements parameterizing the gauge field are defined by the use of root and weight systems, and Cartan matrix of the algebra. The basic Lie superalgebra $\mathcal{G} \equiv D(2|1; \alpha)$ is rank $(\mathcal{G}) = 3$ and dual Coxeter number $h^\vee = 0$. By definition, $\mathcal{G}$ is decomposed into two parts up to $\mathbb{Z}_2$-grading, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$. The even subalgebra $\mathcal{G}_0$ is a Lie algebra $\mathcal{G}_0 = \mathcal{A}_1 \oplus \mathcal{A}_1 \oplus \mathcal{A}_1$, while the odd subalgebra $\mathcal{G}_1$ is set to be a fundamental representation of $\mathcal{G}_0$, in other words, $\mathcal{G}_0$ on $\mathcal{G}_1$ is $sl_2 \otimes sl_2 \otimes sl_2$.

In general, the basic Lie SA’s are equipped with an invariant bilinear form which induces an inner product on the root space. Even roots have positive or negative length squared, while odd ones have zero length squared. It should be noted that, in contrast
to the usual case of Lie algebras, Cartan matrix of Lie SA is not uniquely determined. There are several possible inequivalent choices of simple roots. [ For more information on Lie superalgebras see e.g. [13] ]

In our case, simple roots are chosen to be $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}$, and their inner products are given by

$$
\alpha_{(1)}^2 = 0, \quad \alpha_{(2)}^2 = -2\gamma, \quad \alpha_{(3)}^2 = -2(1 - \gamma),
$$

$$
\alpha_{(1)} \cdot \alpha_{(2)} = \gamma, \quad \alpha_{(1)} \cdot \alpha_{(3)} = 1 - \gamma, \quad \alpha_{(2)} \cdot \alpha_{(3)} = 0,
$$

(3)

where $\gamma \equiv \frac{1}{1+\alpha}$ ($\gamma \neq 0, \pm \infty$). The positive even roots, conventionally denoted by $\Delta_0^+$, are mutually orthogonal to each other, $\alpha_+ \equiv \alpha_{(2)}, \alpha_- \equiv \alpha_{(3)}, \alpha_{\theta} \equiv 2\alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)}$, and the positive odd roots $\Delta_1^+$ are spanned by four elements, $\beta_- \equiv \alpha_{(1)}, \beta_{-K} \equiv \alpha_{(1)} + \alpha_{(3)}, \beta_+ \equiv \alpha_\theta - \beta_-, \beta_{+K} \equiv \alpha_\theta - \beta_{-K}$. With these roots, we can set the even part of this algebra to be a real form $su(2)_+ \oplus su(2)_- \oplus su(2)_\theta$. One of these three $su(2)$, explicitly denoted by $su(2)_\theta$, will be constrained later in the context of hamiltonian reduction.

A set of canonical basis is given by $\{E_\alpha, e_\beta, h^i\}$ where $\mathcal{G}_0$ and $\mathcal{G}_1$ are generated by $\{E_\alpha, h^i\}$ and $\{e_\beta\}$, respectively. Their commutation relations are given by

$$
\begin{align*}
\left[ E_{\alpha_i}, E_{\alpha_j} \right] &= \frac{2\alpha_i \cdot h}{\alpha_i^2} \delta_{\alpha_i + \alpha_j, 0}, \\
\left\{ e_{\beta_\mu}, e_{\beta_\nu} \right\} &= N_{\beta_\mu, \beta_\nu} E_{\beta_\mu + \beta_\nu} + \beta_\mu \cdot h \delta_{\beta_\mu + \beta_\nu, 0} \quad \text{for} \quad \beta_\mu, \beta_\nu \in \Delta_0^+, \\
\left[ e_{\beta_\mu}, E_{\alpha_i} \right] &= N_{\beta_\mu, \alpha} e_{\beta_\mu + \alpha_i} \quad \text{for} \quad \beta_\mu + \alpha_i \in \Delta_1, \\
\left[ h^i, E_{\alpha_j} \right] &= \alpha_j^i E_{\alpha_j}, \quad \left[ h^i, e_{\beta_\nu} \right] = \beta_\mu^i e_{\beta_\mu},
\end{align*}
$$

(4)

where $\alpha_i, \alpha_j \in \Delta_0$, and $\beta_\mu, \beta_\nu \in \Delta_1$. The suffix $i$ of $\{h^i\}$ takes a value of $\{+, -, \theta\}$, $\alpha_j^i$ and $\beta_\mu^i$ are components in the $h^i$-direction.

On $D(2|1; \alpha)$, Killing form, naturally defined by supertrace, is a cumbersome degenerate, zero-Killing form. Instead of the above definition one may replace it to a non-degenerate Killing form with a non-degenerate invariant bilinear form on $D(2|1; \alpha)$, letting the algebra contragredient. Then the Killing form is defined by

$$
\begin{align*}
\left( E_{\alpha_i}, E_{\alpha_j} \right) &= \frac{2}{\alpha^2} \delta_{\alpha_i, -\alpha_j}, \quad \left( h^i, h^j \right) = \delta^{ij}, \\
\left( e_{\beta_\mu}, e_{-\beta_\nu} \right) &= -\left( e_{-\beta_\mu}, e_{\beta_\nu} \right) = \delta_{\beta_\mu, \beta_\nu} \quad \text{for} \quad \beta_\mu, \beta_\nu \in \Delta_0^+.
\end{align*}
$$

(5)

This Killing form $\left( \right)$ on $\mathcal{G}$ enables us to identify the dual space $\mathcal{G}^\vee$ of $\mathcal{G}$ with $\mathcal{G}$ itself.
Now we can express an element of the algebra in terms of the canonical basis,

$$A(z, \bar{z}) = \sum_{\alpha \in \Delta_0} A^\alpha(z, \bar{z}) E_\alpha + \sum_{\gamma \in \Delta_1} A^\gamma(z, \bar{z}) e_\gamma + \sum_{i=+, -, \theta} A^i(z, \bar{z}) \tilde{h}_i,$$  \hspace{1cm} (6)

where the Cartan subalgebra of $G$ is spanned by $\{\tilde{h}_i\}$ ($\tilde{h}_i \equiv \frac{\alpha_i \cdot h_i}{(\alpha_i)^2}$). Infinitesimal gauge transformation of the $D(2|1; \alpha)$ current $J$ is written down as

$$\delta_{\text{gauge}} A(z, \bar{z}) = [\Lambda(z, \bar{z}), A(z, \bar{z})] + \partial \Lambda(z, \bar{z}),$$

$$\Lambda(z, \bar{z}) = \sum_{\alpha \in \Delta_0} \epsilon^\alpha(z, \bar{z}) E_\alpha + \sum_{\gamma \in \Delta_1} \epsilon^\gamma(z, \bar{z}) e_\gamma + \sum_{i=+, -, \theta} \epsilon^i(z, \bar{z}) \tilde{h}_i.$$ \hspace{1cm} (7)

If we impose holomorphy on this current, central extension of $G$ is realized so that coadjoint action on its dual space provide the above gauge transformation, setting its level $k = 1$ \cite{9}.

4 Determination of the Hilbert States in $D(2|1; \alpha)$ CSGT

In eq.\((2)\) we define the inner product on Hilbert space of $D(2|1; \alpha)$ CSGT. The transition from this unconstrained Hilbert space to the physical one is obtained by imposing the spatial gauge invariance on $\Sigma$. This physical space satisfies the Gauss law constraint automatically. This is done in two steps. First a subspace of wave functionals is selected by the requirement that inner product be totally gauge invariant, which can generate some constraints on the Hilbert states. Then this vector space is endowed with the inner product by restricting the $A$-integration to a subspace intersecting every gauge orbit once.

When $M_1(M_2)$ is a solid ball (i.e. $\Sigma = S^2$), there is no new constraint on the Hilbert space except for Gauss law constraint. Thus, this case cannot contain so much information on the Hilbert space that establishes connections with SCFT\cite{5}. In the following, we consider a genus-1 handlebody case as in \cite{3}, namely, $M_1(M_2)$ is a solid torus and $\Sigma = T^2$.

Flat connection admits the following parameterizations of the gauge fields on $\Sigma = T^2$,

$$A_\bar{z} = (u_a u)^{-1} \partial_\bar{z} (u_a u), \quad A_z = (u_a \bar{u})^{-1} \partial_z (u_a \bar{u}),$$

where $u$ is a single-valued map: $\Sigma \rightarrow G^C$, and $u_a$ contains non-trivial global information associated with the fundamental group on $\Sigma$\cite{14},

$$u_a(z, \bar{z}) = \exp \left[ \frac{i \pi}{\text{Im } \tau} \left\{ \int_{z}^{z'} \omega(z') \cdot \tilde{h} - \int_{\bar{z}}^{\bar{z}'} \omega(z') \cdot \bar{\tilde{h}} \right\} \right].$$ \hspace{1cm} (8)
The space of the one-forms over $\mathcal{M}$ allows the Hodge parameterization where the (anti-)holomorphic one-form $\omega(z)(\bar{\omega}(z))$ is taken to be $\int_\alpha \omega = 1$, $\int_\beta \omega = \tau$, and $\int \omega \wedge \bar{\omega} = \text{Im} \tau$. $\alpha$ is a contractable homology cycle in the solid torus and $\beta$ is a non-contractable one. “$a$” is a 3-dimensional vector and $\bar{h}$ is in the Cartan subalgebra. Contraction of them is introduced in such a way that $a \cdot \bar{h} = \sum_{i=\theta,+,-} a^i \bar{h}_i$. The condition $A^\dagger_z = A_z$ causes relations on parameters, $\bar{u}^{-1} = u^\dagger$ and $u_a^{-1} = u_a^\dagger$.

This parameterization induces a change of the variables in Haar measure from $A_z, A^\dagger_z$ into $u_a, u^\dagger_a, u$ and $\bar{u}$, denoted by $F$. It generates the Jacobian $j(a, \bar{a}, u)$ as below.

$$F^*(\text{Haar measure}) = j(a, \bar{a}, u) du_a du^\dagger_a du d\bar{u},$$

$$j(a, \bar{a}, u) = C (\text{Im} \tau)^3 \frac{\text{Det}'(\bar{D}^\dagger_{A_z} D_{A_z})}{\det (\epsilon_i, \epsilon_j) \det (a_i, a_j)},$$

where $C$ arises from the normalization up to group manifold and $\bar{D}_{A_z}$ is the map from $\mathcal{G} \subset \mathbb{C}$-valued function to $\mathcal{G} \subset \mathbb{C}$-valued one-forms,$$
\bar{D}^\dagger_{A_z} : \mathcal{E} \longrightarrow \mathcal{A}, \quad \bar{D}_{A_z} \epsilon = \partial \epsilon + [A_z, \epsilon].$$

$\mathcal{D}^\dagger_{A_z}$ is its adjoint map. In order to explain the rest of the unknown variables, some conventions should be introduced, $\epsilon \in \mathcal{E}$ ($\mathcal{E}$ is a space of $\mathcal{G} \subset \mathbb{C}$-valued functions on $\Sigma$), $a \in \mathcal{A}$ ($\mathcal{A}$ is a space of $\mathcal{G} \subset \mathbb{C}$-valued one-forms on $\Sigma$) and kernels of the above two maps are denoted by $\ker \bar{D}_{A_z} = \{\epsilon_i\}$, $\ker \bar{D}^\dagger_{A_z} = \{a_i\}$ ($i = +, -, \theta$). Matrix $(\epsilon_i, \epsilon_j)$ is defined by a projection on the space $\ker \bar{D}_{A_z} = \{\epsilon_i\}$. Det’ is $\zeta$-function regularized, $\text{Det}' \mathcal{O} = \lim_{\epsilon \to 0} \exp[\int_\epsilon^\infty \frac{dt}{t} \text{Tr}' \exp\{-t \mathcal{O}\}]$, where $\text{Tr}'$ is a trace over $\mathcal{E}$ excluding the kernels. The above definition gives rise to the famous chiral anomaly\cite{14, 15}.

$$\frac{\text{Det}'(\bar{D}^\dagger_{A_z} D_{A_z})}{\det (\epsilon_i, \epsilon_j) \det (a_i, a_j)} = \frac{\text{Det}'(\bar{D}_{A_z} D_{A_z})}{\det (\epsilon_i, \epsilon_j) \det (a_i, a_j)} \bigg|_{u=\text{const.}} S_{G,2h^\vee}(u\bar{u}^{-1}, A_z)_{u=\text{const.}}(9)$$

In general, chiral anomalies are expressed by gauged WZW(Wess-Zumino-Witten) action $S_{G,k}(g, A)$ with some gauge group $G$, its element $g$, a gauge field $A$, and its Kac-Moody level $k$. Note that in the second line of eq.(9), the gauged WZW action vanishes due to the dual Coxeter number of $\mathcal{G}$ ($h^\vee = 0$). It means the absence of the well-known quantum shift $k \rightarrow k + h^\vee$ in this theory. It persists everywhere in what follows.

At this stage, we refer to the hamiltonian reduction and apply to our case. The hamiltonian reduction on basic classical Lie SA’s is realized as a constraint on lowering (or
raising) current of $A_n$ among (semi-)simple components of the even subalgebras. As the constraint is retained along the whole gauge orbit, gauge transformations lead to a Poisson bracket structure and operator product expansion (OPE) relations of superconformal algebra (SCA) or $W_n$ algebra.

Along this strategy, we introduce the analogous constraint for our case, which is imposed on one of the $su(2)_\theta$ current in eq.(6), so that

$$J_{-\theta} = 1.$$ 

This constraint on the gauge symmetry turns out to be translated into constraints on the gauge parameters, $\Lambda$, by extracting the $J_{-\theta}$ part of the gauge transformation.

$$\epsilon^{-\theta} = \epsilon^i \cdot \alpha_\theta = 0, \epsilon^{-\gamma} = 0 \ (\gamma \in \Delta^+_1).$$

(10)

However, with these constraints, there is still left the following gauge group $\mathcal{N}$ (equal to $su(2)_+ \oplus su(2)_- \oplus n_1$). $n_1$ is generated by

$$\Lambda_{n_1} = \epsilon_\theta E_\theta + \sum_{\gamma \in \Delta_1^+} \epsilon_\gamma e_\gamma.$$ 

Leaving the $su(2)_+ \oplus su(2)_-$ gauge symmetry, this residual gauge symmetry must be subtracted from the functional integral eq.(2). We will discuss this point later. With an appropriate gauge fixing procedure for this residual one, for example, Drinfeld-Sokolov gauge[16], one can obtain the complete OPE relations isomorphic to the non-linear large $N = 4$ SCA in two dimensions[17].

Now, it is clear how to impose HR constraints on our $D(2|1;\alpha)$ CSGT, on the inner product of its Hilbert states. Let us realize eq.(14) as delta functions of HR constraints and append them into the integrand of eq.(4). Its restriction of the gauge transformations determines properties of the Hilbert states. Finite gauge transformation of the gauge field $A$ is represented by

$$gA = g^{-1}Ag + g^{-1}\partial g.$$ 

(11)

The HR method restricts the above $g$ to be an exponential map with constrained parameters,

$$g = \exp[\Lambda], \ \Lambda = \epsilon_\theta E_\theta + \sum_{\alpha \in \Delta^-_\alpha} \epsilon_\alpha E_\alpha + \sum_{\gamma \in \Delta_1^+} \epsilon_\gamma e_\gamma + \sum_{i=+,-} \epsilon^i h^i.$$ 

(12)
where $\widetilde{G}$ denotes $su(2)_+ \oplus su(2)_-$. Note that $\Lambda$ includes $\Lambda_{n_1}$ corresponding to $n_1$. We apply this expression to all the gauge transformations.

The gauge transformation, eq. (11), can be simply rewritten in terms of the $u$-parameter transformation, which we denote type (i) transformation as in $[5]$, $u \to u g$. By this type (i) gauge transformation, the integrand apart from a bilinear term of Hilbert states, is transformed and generates additional factors so that

$$e^{\frac{2k}{\pi} \int_{\Sigma}(A_z, A_{\bar{z}})} \to e^{\frac{2k}{\pi} \int_{\Sigma}(A_z, A_{\bar{z}}) \epsilon^k (2\Gamma(g) + \langle u_a u g \rangle + \langle u a u g \rangle)} ,$$

where $\Gamma(g)$ denotes WZW-action and $\langle u, g \rangle$ is given by

$$\langle u, g \rangle = 2\pi \int_{\Sigma}(u^{-1} \partial_{\bar{z}} u, \partial_{z} g g^{-1}).$$

Type (i) gauge invariance of the integrand leads to the next transformation properties of the Hilbert states,

$$\Phi(A_{\bar{z}}) \to e^{-k(\Gamma(g) + \langle u_a u g \rangle)} \Phi(A_{\bar{z}}) ,$$

Thus, wave functional $\Phi(A_{\bar{z}})$ is expected to be decomposed into the following three parts

$$\Phi(A_{\bar{z}}) = \kappa \Psi_k(u a u) \Xi(u_a)$$

$$= \kappa \exp[-k(\Gamma(u) + \langle u_a u \rangle)] \Psi_k(u_a) \Xi(u_a) ,$$

(13)

where constant $\kappa$ and $\Xi(u_a)$ should be determined up to the orthonormality of Hilbert states.

Substituting eq. (9), eq. (13) into eq. (2), HR constrained inner product is thus obtained,

$$\langle \Phi_1 | \Phi_2 \rangle = \bar{\kappa}_1 \kappa_2 \int du_a d\bar{u}_a [K(\tau, a)] \exp[-k \langle u_a, u_a^{-1} \rangle] \times \overline{\Psi_{1,k}(u_a)} \Psi_{2,k}(u_a) ,$$

$$[K(\tau, a)] \equiv (\text{Im} \, \tau)^{-3} \text{Det}'(\overline{D}_{A_z} D_{A_{\bar{z}}})|_{u=\text{const.}} \overline{\Xi_1(u_a)} \Xi_2(u_a)$$

$$\times \int \frac{dud\bar{u}}{[\text{gauge volume of } n_1]} \exp[-k \Gamma(u \bar{u}^{-1}, B)] \delta(\text{HR-constraints}) ,$$

(14)

In eq. (13), Hilbert states are decomposed into three parts, $\Psi_k(u_a)$, $\Xi(u_a)$, and an exponential factor. This factor with its anti-holomorphic partner and a $A_{\bar{z}} - A_z$ potential term is rewritten in the form of a gauged WZW action and a new potential term in terms of $u_a$, $u_a^{-1}$. While the gauged WZW action is alone to be integrated over $u$-variable, it has to be treated carefully. The determinant at a point of a constant $u$ can be thought of a kind of gauge fixing. Trivial $u = 1$ gauge leaves only Cartan generators in the form of $A_{\bar{z}}, A_z$, and easily enables us to construct an effective quantum mechanics, integrating out the
u-parameter dependence. In this thought, u-parameter space is also regarded as a gauge orbit going through the point of the constant u and its integration must be restricted along the HR context of 2d SCFT by an appropriate delta function. Assembling the results and discussions, \([K(\tau, a)]\) could be the form in eq.(14) and provide a part of large \(N = 4\) character. Moreover, it should be noted that the gauge volume of \(n_1\) must be subtracted from u-integration for getting a proper inner product, since integration over \(n_1\) with delta function of HR simply gives rise to the gauge volume for \(n_1\)-symmetry.

Next, in order to construct an explicit form of \(\Psi_k(u_a)\), we consider another gauge transformation (ii)

\[
u_a \longrightarrow u_a\hat{g}, \; u \longrightarrow \hat{g}^{-1}u,
\]

where \(\hat{g}\) is a map \(\hat{g} : \Sigma \rightarrow [\text{Maximal torus of the group}], \) parameterized by

\[
\hat{g}_{n,m} = \exp \left[ \frac{i\pi}{1 \text{Im } \tau} \left\{ (n + m\tau) \cdot \bar{h} \int \omega(z') - (n + m\tau) \cdot \bar{h} \int \omega(z') \right\} \right] \quad (n, m \in \tilde{\Lambda}_R).
\]

“\(\tilde{\Lambda}_R\)” represents three dimensional half-integer lattice which can be interpreted as a normalized root lattice spanned by the normalized even roots \(\{\tilde{\alpha}(i)\}\). According to this interpretation, a set of points on the lattice is thought to be \(\{\sum_{\tilde{\alpha} \in \tilde{\Delta}_+} n_\tilde{\alpha} | n_\alpha \in \mathbb{Z} \text{ for } \tilde{\alpha} \in \tilde{\Delta}_+; n_\alpha = \{-1/2, 0, 1/2\} \text{ for } \tilde{\alpha} \in \tilde{\Delta}_0^+\}\), while \(\tilde{\Delta}_+\) is a set of normalized positive roots. \(\tilde{\Delta}_0^+\) and \(\tilde{\Delta}_1^+\) are, respectively, the normalized even roots and the normalized odd roots which is normalized after projected onto the root space of the even subalgebra. As is seen in eq.(12), since coefficient of the \(h_\theta\) has to vanish in the power of exponential, we may put \(n_{az}\) and \(\theta\)-components of odd roots be zero. With the expression \(n + m\tau = \sum_{i=+,-,0}(n^i + m^i\tau)\tilde{\alpha}(i)\), it turns into \(n + m\tau = \sum_{i=+,-}\{(n^i - \frac{1}{2}n_1^i) + \tau(m^i - \frac{1}{2}m_1^i)\}\tilde{\alpha}(i)\), where \((n_1^+, n_1^-), (m_1^+, m_1^-) = \{(-2,0), (0,-2), (\pm1, \pm1), (0,0), (\pm1, \mp1), (2,0), (0,2)\}\). It gives the next variation for the field \(\Psi_k(u_a)\), taking eq.(13) into account.

\[
\Psi_k(u_a\hat{g}_{n,0}) = \Psi_k(u_{a+n}) = \exp \left[ -\frac{\pi}{1 \text{Im } \tau} \sum_{i=+,-} \left\{ k^i \left( (n^i - \frac{1}{2}n_1^i)^2 + 2\alpha^i(n^i - \frac{1}{2}n_1^i) \right) \right\} \right] \Psi_k(u_a),
\]

\[
\Psi_k(u_a\hat{g}_{0,m}) = \Psi_k(u_{a+m\tau}) = \exp \left[ -\frac{\pi}{1 \text{Im } \tau} \sum_{i=+,-} \left\{ k^i \left( \tau\bar{\alpha}(m^i - \frac{1}{2}m_1^i)^2 + 2\bar{\alpha}^i(m^i - \frac{1}{2}m_1^i) \right) \right\} \right] \Psi_k(u_a),
\]

where \(k^i = \frac{-2h}{(\alpha(i))}\), that is, \((k^+, k^-) = (k\gamma^{-1}, k(1 - \gamma)^{-1})\). Note that \(k = k^+k^-/(k^+ + k^-)\).
For simplicity we restrict ourselves to special cases \( n_1, m_1 = 0, \pm 2 \), where we obtain one of the solutions of above two equations

\[
\Psi_{k,p}(\tau, a) = \exp \left\{ -\sum_{i=0,\pm} \frac{n_i^2}{2\text{Im} \tau} (a_i^2)^2 \right\} \Theta_{k^+,p^+}(\tau, a^+) \Theta_{k^-,p^-}(\tau, a^-), \tag{15}
\]

where \( \Theta_{k,p}(\tau, a) \) are the \( su(2) \)-theta functions of level \( k \) and \( p \in \Lambda_W/k\Lambda_R \)

\[
\Theta_{k,p}(\tau, a) \equiv \sum_{\mu \in \Lambda_R} \exp \left[ i\pi \tau k \left( \mu + \frac{p}{k} \right)^2 + 2\pi i k \left( \mu + \frac{p}{k} \right) a \right],
\]

\( \Lambda_W \) and \( \Lambda_R \) being the weight and root lattice of \( su(2) \), respectively. We can show these solutions satisfy the Gauss law constraint. Noting that the most general solution is any linear combination of the functions in eq.(15), we can take a Weyl anti-symmetrized combination of the theta functions. Together with an appropriate \( \Xi(u_a) \), it will provide two orthogonal \( su(2) \) parts of large \( N = 4 \) characters revealed by Petersen et al. [18].

5 Conclusions and discussions

In this paper, we formulate the anti-holomorphic quantization of \( D(2|1; \alpha) \) CSGT, giving an inner product of the Hilbert states and the Jacobian coming from a parameterization of the gauge field. After that, we extend the correspondence between CSGT Hilbert states and SCFT characters to the \( n_1, m_1 = 0, \pm 2 \) cases of large \( N = 4 \) SCFT, using the HR constraints in eq.(10). If we clarify a full contribution of the odd parts, \( n_1^i, m_1^i \) in \( \bar{\Lambda}_R \), a requirement of the modular invariance on the inner product will establish an entire identification of the characters and the Hilbert states. It will lead to form a modular invariant combination of large \( N = 4 \) characters at the same time. With those of large \( N = 4 \) characters, Wilson line operators lying on \( \Sigma \) construct braiding and fusion matrices on them and provide link invariants on \( \mathcal{M} \). Finally we remark that this procedure will also be applicable to the case of small \( N = 4 \) SCFT. These will be discussed elsewhere [12].

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