THE LOW-ENERGY THEOREM OF PION PHOTOPRODUCTION IN SOLITON MODELS OF THE NUCLEON

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ABSTRACT

We derive an analytic expression for the Kroll-Ruderman amplitude up to $O(N_C^{-1})$ for general Skyrme-type models of the nucleon. Due to the degeneracy of intermediate $N$- and $\Delta$-states we find deviations from the standard low-energy theorem for the photoproduction of neutral pions.
1. Introduction.

At low energies the amplitude $F$ for the production of pions, $\pi$, from a photon $a_\mu \sim e_\mu e^{ik \cdot x}$ incident on a nucleon arises from the contributions of S-wave $\pi N$-channels. The charge dependence of the reaction follows from three independent amplitudes $E_0^{(-,0,+)}$:

$$
F = \left\{-i(\pi \times \tau)_3 E_0^{(-)} + \pi \cdot \tau E_0^{(0)} + \pi_3 E_0^{(+)}\right\} i e \cdot \sigma,
$$

where the spin and isospin dependence of the nucleonic degrees of freedom is expressed by Pauli-matrices $\sigma_k$ and $\tau_a$. The amplitude $F$ is defined such that its matrix elements between the initial and final spin-isospin state of the nucleon $|i\rangle$, $|f\rangle$, and the final isospin state $|\alpha\rangle$ of the pion, lead to the differential reaction cross section

$$
d\sigma_{c.m.}/d\Omega_\pi = |q_\pi|/|k_\gamma| \sum_{\text{pol}} |\langle f, \alpha | F | i \rangle|^2.
$$

Current algebra and PCAC fix the first terms of an expansion of the three S-wave amplitudes with respect to the coefficient of the chiral symmetry breaking, i.e. the pion mass squared. This expansion, known as the Kroll-Ruderman theorem, reads

$$
E_0^{(-)} = \frac{|e|}{4\pi} \frac{g_A}{2f_\pi} C\left(\frac{m_\pi}{M}\right) \left[1 + \mathcal{O}\left(\left(\frac{m_\pi}{M}\right)^2\right)\right]
$$

$$
E_0^{(0)} = \frac{|e|}{4\pi} \frac{g_A}{2f_\pi} C\left(\frac{m_\pi}{M}\right) \left[-\frac{1}{2} \frac{m_\pi}{M} + \frac{1}{4} (\mu_p + \mu_n) \left(\frac{m_\pi}{M}\right)^2 + \mathcal{O}\left(\left(\frac{m_\pi}{M}\right)^3\right)\right]
$$

$$
E_0^{(+)} = \frac{|e|}{4\pi} \frac{g_A}{2f_\pi} C\left(\frac{m_\pi}{M}\right) \left[-\frac{1}{2} \frac{m_\pi}{M} + \frac{1}{4} (\mu_p - \mu_n) \left(\frac{m_\pi}{M}\right)^2 + \mathcal{O}\left(\left(\frac{m_\pi}{M}\right)^3\right)\right].
$$

For later convenience we have grouped factors of kinematical origin into

$$
C\left(\frac{m_\pi}{M}\right) = 1 + \frac{1}{2} \frac{m_\pi}{M} + \mathcal{O}\left(\left(\frac{m_\pi}{M}\right)^2\right).
$$

Rightaway, from the first attempt already, it has been clearly visible that the Skyrme model follows the Kroll-Ruderman-theorem closely, at least numerically. Later, it was understood, that the zeroth order term in the pion mass entering the isovectorial $E_0^{(-)}$-amplitude actually follows analytically. Further numerical investigations have confirmed this and shown, that the slope of the $E_0^{(0)}$-amplitude with respect to the pion mass is of the size required, although ref. disagrees on the sign.
In the Skyrme model the isoscalar \( E^{(0)}_{0+} \)-amplitude originates from the Wess-Zumino-anomaly. The third, the isovectorial \( E^{(+)}_{0+} \)-amplitude, finally, remained zero due to the adiabatic approximation to meson-soliton scattering adopted in\[4, 5, 7\]. In an \( N_C \)-counting scheme this amplitude is down by one order relative to the other isovectorial amplitude, because the nucleon mass \( M \) is \( O(N_C) \) and the pion mass \( m_\pi \) is of order one. Thus this amplitude can only arise once rotational effects of the soliton are taken into account in soliton-meson scattering[8].

The purpose of the present work is to derive analytic expressions for all three amplitudes complete up to the order \( O(N_C^{-1}) \) relative to the leading terms in eq.(3). This derivation turns out to be possible for the very general class of chirally symmetric actions atmost quadratic in the time derivatives of the meson fields. The result is different from the conclusions in refs.\[3, 7, 8\] which mutually disagree with each other.

2. Low-energy \( U \)-matrix and current algebra

In Skyrme-type models, the nucleon is based on the field configurations of the hedgehog soliton

\[
U_H(r) = e^{i\tau \cdot \hat{r} \chi(r)}
\]

and it acquires its kinematical and spin degrees of freedom by introduction of collective coordinates. For small velocities such coordinates originate from a Galilean transformation of the center of mass \( X(t) \) and from adiabatically slow rotations \( A(t) \)

\[
U = A U_H(r + X) A^\dagger, \quad A^\dagger \dot{A} = -\frac{i}{2} \tau \cdot \Omega \to 0.
\]

of the soliton.

It was soon noticed[10] that the rotational and translational degrees of freedom in eq.(6) alone violate the commutation relations of current algebra

\[
[Q_a, Q_b] = i\epsilon_{abc} Q_c, \quad [Q_a, Q_5^b] = i\epsilon_{abc} Q_5^c, \quad [Q_5^a, Q_5^b] = i\epsilon_{abc} Q_c
\]

between the vector and axial charges

\[
Q_a = \int V^{0}_a(U) d^3r, \quad Q_5^a = \int A^{0}_a(U) d^3r.
\]

As usual, the charges are defined as integrals over the time-components of vector and axial vector currents

\[
V^{\mu}_a(U) = -i \text{ tr } \frac{\delta \mathcal{L}}{\delta \partial^\mu U} [\tau_a, U], \quad A^{\mu}_a(U) = -i \text{ tr } \frac{\delta \mathcal{L}}{\delta \partial^\mu U} \{\tau_a, U\}.
\]
In seeming contradiction to this, it also became clear that Skyrme-type models do reproduce low-energy theorems, generally based on the current algebra relations, such as the Tomozawa-Weinberg relation for S-wave $\pi N$ scattering\[11, 8\] and the Adler-Weisberger sum rule\[12\], once $O(N_c^{-1})$ rotational effects are properly taken into account. Since the current algebra relations conventionally are the starting point also for our present objective, the low-energy theorem of photoproduction, we will reexamine the commutation relations in eq.(7) taking $O(N_c^{-1})$ effects into consideration.

In order to describe $\pi N$ scattering the ansatz, eq.(6), must be augmented by small amplitude fluctuations around the soliton. In the low-energy region of interest here the configurations with fluctuations may simply be written as

$$U = A e^{i f_\pi \tau \cdot e} U_H (r + X + \frac{3g_A}{2f_\pi M} e) e^{i f_\pi \tau \cdot e} A^\dagger. \tag{10}$$

The collective coordinates $X$, $e$ and the Euler angles contained in $A(t)$ are independent variables. The fluctuation corresponding to the first order term of $U$ with respect to the parameters $e$ represents a linear combination of a chiral rotation of the hedgehog and a translation. In case of chiral symmetry, i.e. when the chiral symmetry breaking mass term

$$L^{(CSB)} = \frac{f_\pi^2 m_\pi^2}{4} \text{tr}(U + U^\dagger - 2) \tag{11}$$

is absent, both modes are zero frequency solutions to the adiabatic equations of motion for small amplitude fluctuations. The special linear combination given here is determined by the fact, that the S-wave scattering solution is orthogonal on the localized, purely translational zero mode\[8\]. The overlap integrals between chiral rotations and translations involve the mass $M = -L[U_H]$ of the soliton and its axial coupling $g_A$,

$$\int A^i_a (A U_H A^\dagger) d^3r = D_{ab}(A) \int A^i_b (U_H) d^3r = -\frac{3}{2} D_{aj}(A) g_A, \tag{12}$$

independent of any specific choice of the total action $L$ or any specific parametrization of the fluctuations. Different parametrizations of the fluctuations lead to different norm-kernels which assure that the overlaps are always given in terms of mass and axial coupling constant.

The $D$-Functions $D_{ab}(A) = \frac{1}{2} \text{tr} \tau_a A \tau_b A^\dagger$ transform from soliton-fixed to physical isospin-axes such that the physical pion fields $\pi$ are related to the soliton-fixed chiral rotation angles $e$ via

$$e_b = \pi_c D_{cb}(A). \tag{13}$$
In the presence of the chiral symmetry breaking the changes of the fluctuations are all $\mathcal{O}(m^2)$ except at low energies. There, the time dependence of the fluctuation shifts to

$$\pi = a_0 e^{-i m \pi t} + a_0^\dagger e^{i m \pi t}$$

(14)

because the chiral symmetry breaking changes the asymptotic dispersion relation of the fluctuations. Thus, even with broken chiral symmetry the matrices in eq.(10) represent the exact low-energy behaviour of the chiral fields up to $\mathcal{O}(m,\pi)$ apart from the modifications to be made concerning the time dependence of the fluctuations.

When one abandons the assumption of adiabaticity of pion-soliton scattering, to lowest order the changes in the fluctuations of order $\mathcal{O}(N_C^{-1})$ are driven by an inhomogeneous term linear in the rotational velocities $\Omega$ and linear in the adiabatic fluctuation$[^8]$. Later, we will transform the photoproduction amplitudes to a form where the inhomogeneity can be inserted directly. Thus, up to order $\mathcal{O}(N_C^{-1})$ and $\mathcal{O}(\frac{\omega}{f})$ the general structure of the chiral fields in eq.(10) together with their time dependence in eq.(14) are sufficient for the low-energy photoproduction amplitude.

As a consequence of the specific linear combination chosen in eq.(10) the translation decouples from the other modes in the collective lagrangian which up to order $\mathcal{O}(N_C^{-1})$ reads

$$L = -M + \frac{1}{2} E \dot{\mathbf{e}}^2 + \frac{1}{2} M \dot{\mathbf{X}}^2 + \frac{1}{2} \Theta \Omega^2 + \left( E - \frac{\Theta}{2 f^2} \right) \Omega \cdot (\mathbf{e} \times \dot{\mathbf{e}}).$$

(15)

The constant $E$ represents the infinite norm of the scattering state and will be of no further significance. The rotation-vibration coupling of order $\mathcal{O}(N_C^{-1})$, last term in eq.(15), involves the moment of inertia $\Theta$ and directly leads to the Tomozawa-Weinberg split of the S11 and S31 scattering lengths$[^{13}]$. The collective lagrangian, eq.(15), fixes the conjugate momenta and angular momenta

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{e}}} = E \dot{\mathbf{e}} + \left( E - \frac{\Theta}{2 f^2} \right) \Omega \times \mathbf{e} =$$

$$\mathbf{R} = \frac{\partial L}{\partial \dot{\Omega}} = \Theta \dot{\Omega} + \left( E - \frac{\Theta}{2 f^2} \right) \mathbf{e} \times \dot{\mathbf{e}} =$$

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{X}}} = MX.$$

(16)

The angular velocities expressed by the conjugate momenta

$$\Omega = \frac{1}{\Theta} (\mathbf{R} - \mathbf{e} \times \mathbf{p})$$

(17)
are of order $O(N_C^{-1})$. The vector and axial vector charges expressed in terms of the collective momenta, eq.(16),

$$Q_a = D_{ab} \left[ \Theta \Omega + \left( E - \frac{\Theta}{2f^2} \right) e \times \dot{e} \right]_b = D_{ab} R_b$$

$$Q^5_a = D_{ab} \left[ -f_\pi E \dot{e} + \frac{3}{2} g_A \dot{X} - f_\pi \left( E - \frac{\Theta}{f^2} \right) \Omega \times e \right]_b$$

$$= D_{ab} \left[ -f_\pi p + \frac{3}{2} g_A \dot{P} + \frac{1}{2f_\pi} (R - e \times p) \times e \right]_b$$

(18)

may now be used to verify the current algebra. Cubic terms in the parameters $e$ have been neglected, since they will not enter the relevant photoproduction amplitudes. With this the low-energy charges in eq.(18) are correct next to leading order in $1/N_C$.

Postulating canonical quantization rules, $[p_a, e_b] = -i \delta_{ab}$, $[P_a, X_b] = -i \delta_{ab}$, $[R_a, R_b] = -i \epsilon_{abc} R_c$, $[R_a, D_{bc}] = -i \epsilon_{ace} D_{be}$, these charges become operators which should be hermitized properly. For these hermitean operators it is then straightforward to verify the commutation relations in eq.(7). Note that the non-adiabatic term in the axial charge, eq.(18), is indispensable, because, if neglected, $[Q^5_a, Q^5_b] = 0$ follows immediately. Thus, the ansatz (10) is in accordance with current algebra if $O(N_C^{-1})$ contributions are taken properly into account. Since the pure translations will not contribute to the photoproduction amplitudes later we omit the collective coordinate $X$ from now on.

3. The photoproduction amplitude up to $O(N_C^0)$.

The photoproduction amplitude $F$ for the creation of one pion after absorption of one photon to lowest order follows from the linear photon vertex

$$L_\gamma = L^V_\gamma + L^S_\gamma = - |e| \int a_\mu \left[ V^\mu_3(U) + \frac{1}{2} B^\mu(U) \right]_{lin} d^3r$$

(19)

where the chiral fields in the vector current $V^\mu_3(U)$ and the winding number current $B^\mu(U)$ must be expanded up to linear order in the fluctuations around the soliton\[4\], i.e. up to linear order in $e$. At pion threshold in the c.m. system kinematical and phase space factors relating the matrixelements of this interaction, eq.(19), to the amplitude $F$, eq.(1), combine to $C \left( \frac{M}{\Lambda} \right)$, eq.(4), if the photon field is normalized to $a = -\frac{1}{4\pi} e^{ik \cdot x}$. So we may directly compare the interaction from eq.(19) with the Kroll-Ruderman theorem omitting the factor $C$ in the latter, eq.(3).

In Coulomb gauge, $a_\mu = (0, -a)$, where the expansion of the low-energy amplitudes in orders of $1/N_C$ is straightforward, we find that the spatial part of the winding
number current already requires one time derivative on the chiral fields and thus will be of $O(m_{\pi})$ or $O(\Omega) = O(N_C^{-1})$. To lowest order we therefore only have contributions from the vector current. For the configurations under consideration, eq.(10), we use an identity for the vector currents of a chirally rotated configuration[5] which is an analogue of the current algebra relations. Taking proper care of the shifted arguments of the hedgehog in eq.(10) the identity reads here

$$V_3^\mu(U) = D_{3a}(A) \left[ V^\mu(U_H(r + \frac{3g_A}{2f_{\pi}M}e)) - \frac{1}{f_{\pi}}e \times A^\mu(U_H(r + \frac{3g_A}{2f_{\pi}M}e)) \right]_a + \{\text{terms with \dot{e}, \dot{A}}\}.$$  

Insertion of this expression into the photocoupling keeping the terms linear in the fluctuations, i.e. linear in $e$, with a photon field normalized to $a = -\frac{1}{4\pi}e e^{ikx}$ immediatly leads to the Kroll-Ruderman amplitude of order $O(m_{\pi}^0)$, eq.(3),

$$L_{V}^{\gamma} |_{N_C^0} = - \frac{|e|}{8\pi f_{\pi}} D_{3a} \int d^3r a_{\mu} \left[ \frac{3g_A}{2f_{\pi}M}e \cdot \nabla V^\mu(U_H) - \frac{1}{f_{\pi}}e \times A^\mu(U_H) \right]_a = - \frac{|e|}{8\pi f_{\pi}} D_{3a}(3g_A e \times e)_a = \frac{|e|}{4\pi} \frac{g_A}{2f_{\pi}} (-i\pi \times \tau)_{3i} e \cdot \sigma.$$  

Up to order $O(m_{\pi}^2)$ the translational part of the fluctuation doesn’t contribute and the integration of the axial current of the hedgehog supplies the factor $g_A$, eq.(12). We have used the substitution $D_{ab} \rightarrow -\frac{1}{3} \tau_a \sigma_b$ for the matrixelements of the $D$-function between nucleon states and replaced the soliton-fixed fluctuation by its laboratory components, eq.(13).

4. The isoscalar amplitude up to $O(N_C^{-1})$.

The adiabatic fluctuations inserted into the photocoupling from the Wess-Zumino term leads to an isoscalar vertex which is of the order $O(N_C^{-1})$ because the winding number current

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\sigma\tau} \text{tr}(U_\nu^\dagger \partial_\sigma U)(U_\tau^\dagger \partial_\tau U)(U_\nu^\dagger \partial_\sigma U).$$  

is down by one order in $N_C$ relative to the isovector current. The chiral rotation in the low-energy fluctuation only contributes to the order $O(m_{\pi}^2)$. The piece stemming from the translation, on the other hand, will be proportional to $g_A$ because of the orthonormalization factors in eq.(10). For winding number $B = 1$ this piece immediately leads to the same expression as in the Kroll-Ruderman theorem for $E_{0_+}^{(0)}$, eq.(3), because the time dependence of a pion in the final state, $\pi = im_{\pi} \pi$, eq.(14), must be
inserted:

\[ L^S_{\gamma} \big|_{N_C^{-1}} = -\left| \frac{e}{2} \right| \int a_j \frac{e^{ijj}}{8\pi^2} \frac{3g_A}{2f_\pi M} \text{tr} (U_H^\dagger \mathbf{\hat{e}} \cdot \nabla U_H U_H^\dagger \partial_i U_H U_H^\dagger \partial_k U_H) d^3r \]

\[ = \left| \frac{e}{8\pi} \frac{3g_A}{2f_\pi M} (\mathbf{e} \cdot \mathbf{e}) B = \left| \frac{e}{4\pi} \frac{g_A}{2f_\pi} (\frac{m_\pi^2}{2M}) \mathbf{\tau} \cdot \mathbf{\pi} \right) i\mathbf{\sigma} \cdot \mathbf{e} . \]  \hspace{1cm} (23)

5. The isovector amplitude in $O(N_C^{-1})$.

Up to this point the low-energy amplitude has been evaluated entirely in the adiabatic approximation to meson-soliton scattering. The soliton model has pieced different factors entering the isoscalar amplitude together to an expression identical to the one obtained by standard methods, but in a completely different way. The addition of non-adiabatic contributions now will necessarily involve the rotational frequencies of the soliton and we anticipate, that the little miracle that has happened in case of the isoscalar amplitudes will continue to happen, i.e. the resulting main corrections to the isovector amplitudes will turn out to be entirely expressible in terms of $|e|, g_A, f_\pi$ and $m_\pi^2$.

To demonstrate this we first transform the expression for the isovector amplitude in eq.(19). The exact equations of motion for the chiral fields are equivalent to the vanishing of the divergence of the vector current

\[ \partial_\mu V^\mu(U) = \partial_0 V_0(U) + \partial_i V_i(U) = 0 . \]  \hspace{1cm} (24)

From this equation we write down the identity

\[ \int a_i V_i(U) d^3r = \int a_j \partial_i (x_j V_i(U)) d^3r + \int a_j x_j \partial_0 V_0(U) d^3r . \]  \hspace{1cm} (25)

Let us first reconsider the case of adiabatic fluctuations. By construction, the vector current linear in the adiabatic fluctuations is divergenceless when all rotational velocities are set to zero. Therefore, the time derivative of the time component of the vector current contains two time derivatives both acting on the adiabatic fluctuation such that at threshold the second term on the right hand side of eq.(25) is already $O(m_\pi^2)$. From the remaining term, upon partial integration, the surface term at infinity just gives the contribution calculated in the third section whereas the rest involves a derivative of the photon field and may be discarded here: the spatial components of the vector current linear in the fluctuations only contain even multipoles in $\hat{r}$ such that the angular integration together with the derivative of the photon field is at least quadratic in the photon momenta and thus $O(m_\pi^2)$. 

If we now consider the non-adiabatic case then the first term on the right hand side of eq.(25) does not add any new contributions: additions to the surface term are zero because changes to the adiabatic chiral rotation due to non-adiabatic terms vanish asymptotically and for the same reasons as just outlined above the term with a derivative on the photon field is at least $\mathcal{O}(m_\pi^2)$. Thus the non-adiabatic contributions up to the order considered here only enter via the time derivative of the time component of the vector current. Again, up to the order considered, one of these time derivatives will furnish a vibrational frequency $\omega = m_\pi$, the other one, necessarily, a rotational frequency $\Omega$. So this term will be $\mathcal{O}(N_C^{-1})$ with adiabatic fluctuations inserted and we may safely drop higher order non-adiabatic corrections to the fluctuations. With other words, we have isolated the inhomogeneous term in the equations of motion which drives the non-adiabatic fluctuations and is proportional to the rotational velocity.

Expanded up to linear order in the adiabatic fluctuations $\mathbf{e}$ there are two contributions to the time derivative of the time component of the vector current, one from the global chiral rotation and one from the orthonormalizing translation. The chiral rotation, using eq.(20), leads to

$$V_3^0(U) = D_{3a}(A) \left[ \tilde{V}_a^0(U_H) - \frac{1}{f_\pi} \left( \mathbf{e} \times \tilde{A}_a^0(U_H) \right)_a \right] - \frac{1}{f_\pi} (\mathbf{e} + \Omega \times \mathbf{e})_c \frac{\partial}{\partial \Omega_a} \tilde{A}_c^0(U_H),$$

(26)

where we are using a somewhat sloppy notation for the body-fixed currents of the rotating hedgehog which are linear in the rotational velocity:

$$D_{ab}(A) \tilde{A}_b^0(U_H) = A^0_a \left( A U_H A^\dagger \right)$$

(27)

$$D_{ab}(A) \tilde{V}_b^0(U_H) = V_0^a \left( A U_H A^\dagger \right).$$

Up to order $\mathcal{O}(\Omega) \cdot \mathcal{O}(m_\pi)$ and linear in the fluctuation we retain the terms

$$\partial_0 V_3^0(U) \big|_{\text{lin}} = -\frac{1}{f_\pi} D_{3a}(A) \left[ \left( \Omega \times \frac{\partial}{\partial \Omega} \right)_a \mathbf{e} \cdot \tilde{A}_a^0(U_H) \right]$$

$$+ \left( \mathbf{e} \times \tilde{A}_a^0(U_H) \right)_a + (\mathbf{e} + \Omega \times \mathbf{e})_c \frac{\partial}{\partial \Omega_a} \tilde{A}_c^0(U_H)$$

(28)

from the time derivative of the time component of the vector current. Only the time component of the axial current of the hedgehog enters into this expression, its general structure being

$$\tilde{A}_a^0(U_H) = -\frac{3}{2} \theta(r) \cot \chi [\Omega \times \hat{r}],$$

(29)
The function $\theta$ is the angular averaged density for the moments of inertia of the soliton. Due to this general structure of the axial current and the relation

$$\dot{e} = -2\Omega \times \dot{e} + \mathcal{O}(\Omega^2) + \mathcal{O}(m_{\pi}^2)$$

which follows from the equations of motion of the non-adiabatic fluctuations or also from eq.(13,14) the time derivative in eq.(28), is exactly zero

$$\partial_0 V_3^0(U) \big|_{\text{lin}} = -\frac{3\theta}{2f_\pi} \cot \chi D_{3a}(A) \left[ \Omega \times (\hat{r} \times \dot{e}) + \dot{e} \times (\Omega \times \hat{r}) \right] - \hat{r} \times (\Omega \times \dot{e}) = 0.$$  

It remains to examine the contributions from the translation which shifts the argument of the hedgehog configuration in eq.(10). Due to this specific structure the linear terms of the isovector current with respect to $e$ are easily calculated:

$$L_\gamma \big|_{N_{C}^{-1}} = -\frac{|e|}{4\pi} \int \frac{r \cdot e V_3^0(U) \big|_{\text{lin},N_{C}^{-1}}}{d^3r}$$

$$= \frac{|e|}{4\pi} \frac{3g_A}{2f_\pi M} D_{3a} \Theta (4\Omega_a \dot{e} \cdot \epsilon - e_a \Omega \cdot \dot{e} - \Omega \cdot \epsilon \dot{e}_a).$$

The occurrence of the moments of inertia of the soliton, $\Theta$, is directly related to the fact that the rotational energy is proportional to the integral of the time component of the vector current. In case of time-dependent isospin rotations of a static soliton this is true for any isospin symmetric action at most quadratic in the time derivatives of the chiral fields.

Insertion of the physical pion-field, eq.(13,14), and elimination of the angular velocities in favor of angular momenta will now complete the derivation. However, at this point we are facing two problems. The first one concerns the relation of the right angular momenta to the angular velocities which up to order $\mathcal{O}(N_{C}^0)$ includes a term bilinear in the fluctuation and its conjugate momentum field. Because of this extra term the action of the angular velocity on the fluctuation $e$ generates cubic terms in the mesonic fields. In principle two of them could be contracted using the completeness relation of the adiabatic fluctuations. From the low-energy fluctuations in eq.(10) the piece given in eq.(17) may be deduced. The unrestricted sum over intermediate scattering states on the other hand leads to a multitude of terms not expressible through $|e|$, $g_A$, $f_\pi$ and $\frac{M_\pi}{M}$ alone. However, all these terms will necessarily be proportional to $D_{3a}(e \times \epsilon)_a$, i.e. they provide $\mathcal{O}(m_{\pi})$ corrections to the $E_{0^+}^{(-)}$-amplitude. Here, we just retain the term sufficient for the correct current
commutators, eq.(17). Upon hermitization its inclusion amounts to the replacement rule

$$\Theta \Omega_a \hat{\epsilon}_b \rightarrow R_a \hat{\epsilon}_b + \frac{i}{2} \epsilon_{abc} \hat{\epsilon}_c$$

(33)

for the angular velocity.

A second problem concerns ordering ambiguities related to the position of the right angular momenta relative to two $D$-functions in the expressions in eq.(32)

$$D_{3a} \Theta \left( 4\Omega_a \hat{\epsilon} \cdot e - \epsilon_a \Omega \cdot \dot{e} - \Omega \cdot \epsilon_a \right) =$$

$$im_{\pi} D_{3a} \left( 4R_a \pi_c D_{cb} \epsilon_b - \epsilon_a R_b \pi_c D_{cb} - \epsilon \cdot R \pi_c D_{ca} + \frac{5}{2} i(\epsilon \times e)_a \right).$$

(34)

Since $R$ is a differential operator with respect to the Euler-angles different orderings are distinguished by terms where $R$ differentiates one of the two $D$-functions. The result of such a differentiation is necessarily of the form $D_{3a} (\epsilon \times e)_a$. Thus, all uncertainties of the calculation presented here reside in the $O(m_{\pi})$ corrections to the $E_{0^+}$-amplitude.

The evaluation of the Euler-angle matrixelements will be given here with respect to the ordering specified on the r.h.s. of eq.(34) which, we think, is actually the correct order: the differentiations apply to the soliton-fixed fluctuation alone, just as in the case of the inhomogeneous term in the equations of motion for the fluctuations. Since the Euler-angle dependence of the laboratory fluctuation $\pi$ must correspond to the final $\pi N$-channel, it is given by $D(\frac{1}{2})$-functions and the $\gamma N$ entrance channel provides another $D(\frac{1}{2})$-function. Therefore we can use the substitution $D_{ab} \rightarrow -\frac{1}{3} \tau_a \sigma_b$ for all the $D$-functions in eq.(34) where no couplings to intermediate states of spins higher than $\frac{1}{2}$ are possible:

$$D_{3a} R_a \pi_c D_{cb} \epsilon_b = L_3 \pi_c D_{cb} \epsilon_b = -\frac{1}{6} [\pi + i\pi \times \tau]_3 \sigma \cdot \epsilon$$

(35)

$$D_{3a} \epsilon_a R_b \pi_c D_{cb} = D_{3a} \epsilon_a L_c \pi_c = -\frac{1}{6} [\pi + i\pi \times \tau]_3 \sigma \cdot \epsilon.$$

Note, that the left operators $L_a = D_{ab} R_b$ correspond to the isospin carried by the Euler angles and the right operators to negative spin. The third matrixelement in eq.(34) allows for intermediate $\frac{3}{2}$-states in baryonic spin and isospin

$$D_{3a} \epsilon \cdot R \pi_c D_{ca} = [D_{3a}, \epsilon \cdot R] \pi_c D_{ca} + \epsilon \cdot R \pi_3 = \left[ -\frac{1}{2} \pi + \frac{i}{3} \pi \times \tau \right]_3 \sigma \cdot \epsilon.$$

(36)

When we sum the four matrixelements together with the appropriate coefficients

$$L_{V}^{\nu} \mid_{N_C^1} = \frac{\mid \epsilon \mid g_A}{4\pi \sqrt{2} f_{\pi}} \left[ 0 \cdot \pi_3 - 0 \cdot i(\pi \times \tau)_3 \right] i(\sigma \cdot \epsilon),$$

(37)
and we find a vanishing $E_{0^+}^{(+)}$-amplitude up to $\mathcal{O}(m_\pi)$ in disagreement with the low-energy theorem, eq.(3). The vanishing correction to the $E_{0^+}^{(-)}$-amplitude, on the other hand, is subject to several uncertainties in the calculation, as has been discussed.

6. Higher order corrections to the low-energy theorem

In the standard formulation of the Kroll-Ruderman theorem, eq.(3), it is possible to derive the $\mathcal{O}\left(\frac{m_\pi}{M}\right)^2$ corrections for the $E_{0^+}^{(0)}$ and $E_{0^+}^{(+)}$ amplitudes. They are given in terms of the anomalous magnetic moments of proton and neutron. Analogous contributions may also be recovered explicitly from Skyrme-type models. There are, however, other contributions to the same order whose form cannot be given analytically and which may not vanish, either.

We start with the isoscalar amplitude $E_{0^+}^{(0)}$. The contributions of $\mathcal{O}\left(\frac{m_\pi}{M}\right)^2$ neglected till now originate from two distinct cases: the time derivative in the spatial components of the winding number current will
(i) act on the rotation matrices $A(t)$ and thus lead to a term of $\mathcal{O}(N_C^{-2})$ with adiabatic fluctuations inserted. Here each factor $m_\pi$ is supplied by the photon momentum.
(ii) act on the non-adiabatic correction to the fluctuation which also produces a term of $\mathcal{O}(N_C^{-2})$.

Case (i) allows an explicit derivation:

$$\delta L_{\gamma}^S = \left| \frac{e}{8\pi} \right| \frac{3g_A}{2f_\pi M} \int \epsilon_j e^{-ik \cdot r} \epsilon \cdot \nabla (\Omega \times r) \cdot B^0(U_H) d^3r$$

$$= \left| \frac{e}{8\pi} \right| \frac{3g_A}{2f_\pi M} \frac{1}{3} \epsilon \cdot (\Omega \times k) (\epsilon \cdot k) \langle r^2 \rangle.$$ 

(38)

The contribution from the pure chiral rotation vanishes locally leaving the piece from the translation, only. The isoscalar mean square radius is related to the isoscalar magnetic moment in hedgehog models

$$\mu^S = \mu_p + \mu_n = \frac{M}{3\Theta} \langle r^2 \rangle,$$

(39)

due to the fact that each factor $m_\pi$ is supplied by the photon momentum.

The translational part of the adiabatic low-energy fluctuation also determines the isovectorial amplitudes of order $\mathcal{O}\left(\frac{m_\pi}{M}\right)^2$. Again, each factor $m_\pi$ is supplied by the
photon momentum. Explicit, straight-forward calculation leads to

$$\delta L^V_\gamma = \left| e \right| \frac{3g_A}{2f_\pi M} \Theta D_{3a}(k \times \epsilon)_a e \cdot k.$$  \hspace{1cm} (41)$$

Now, the anomalous isovectorial magnetic moment appears because hedgehog models always have

$$\mu^V = \mu_p - \mu_n = \frac{2}{3} M \Theta.$$  \hspace{1cm} (42)$$

The evaluation of the Euler-angle matrixelements of this expression involves the steps

$$\pi_i k_b(k \times \epsilon)_a D_{3a} D_{ib} = \frac{k^2}{2 \epsilon_{3ij} \pi_i D_{ja} \epsilon_a} = \frac{k^2}{6} i(\pi \times \tau)_3 i\sigma \cdot \epsilon,$$  \hspace{1cm} (43)$$

and we have used the orthogonality of the photon momentum on its polarization. The resulting correction,

$$\delta L^V_\gamma = \left| e \right| \frac{g_A}{2f_\pi} \left[ \frac{3}{8} \frac{m^2}{M^2} (\mu_p - \mu_n) i(\pi \times \tau)_3 i\sigma \cdot \epsilon \right],$$  \hspace{1cm} (44)$$

resides in the wrong amplitude as compared to the standard expression, eq.(3). However, the origin of the discrepancy is fairly easy to locate: like in other cases[15, 14], the degeneracy of the rotational states in the soliton model up to lowest order in $N_C^{-1}$ which allows nucleons and $\Delta$’s as intermediate states leads to different expressions relative to the case where the $\Delta$’s are excluded entirely from the calculation. We can actually implement the second assumption by excluding intermediate $\frac{3}{2}$-states in baryonic spin and isospin but, unfortunately, this now will lead to ordering ambiguities with respect to the $E^{(+)}_{0^+}$-amplitude. Intermediate $\frac{3}{2}$-states are present in the matrixelement given in eq.(43) and the immediate substitution $D_{ab} \rightarrow -\frac{1}{3} \tau_a \sigma_b$ for all $D$-functions by matrixelements of the $\frac{1}{2}$-representation eliminates contributions of the higher spins. The ordering

$$\pi_i k_b(k \times \epsilon)_a D_{3a} D_{ib} \rightarrow \frac{1}{9} \tau_3 \sigma \cdot (k \times \epsilon) \pi \cdot \tau \sigma \cdot k = \frac{k^2}{9} i(\pi + i\pi \times \tau)_3 i\sigma \cdot \epsilon$$  \hspace{1cm} (45)$$

produces the standard correction to the $E^{(+)}_{0^+}$-amplitude

$$\delta L^V_\gamma = \left| e \right| \frac{g_A}{4\pi} \left[ \frac{1}{4} \frac{m^2}{M^2} (\mu_p - \mu_n)(\pi + i\pi \times \tau)_3 \right] i\sigma \cdot \epsilon,$$  \hspace{1cm} (46)$$

plus higher order corrections to the $E^{(-)}_{0^+}$-amplitude. The latter are not available in the standard form of the theorem, and thus we have shown that we may bring
the Skyrme model expressions into agreement with the standard ones if we eliminate contributions of intermediate $\Delta$-states.

Finishing this section we should emphasize once again, that there are more corrections to the low-energy amplitudes, some of which would only be accessible numerically. The first ones, $O\left(\frac{m_{\pi}}{M}\right)^2$, arise in the isoscalar amplitudes due to the contributions from the case (ii), above. The others, already $O\left(\frac{m_{\pi}}{M}\right)$, originate, as discussed in section 5., from the relation between angular velocities and right angular momenta. We only have kept those terms that guarantee the correct commutation relations between vector and axial charges up to $O\left(N_C^{-1}\right)$. Latter uncertainties all reside in the $E_{0+}^{(-)}$-amplitude, the one which also suffers from ordering ambiguities. Lastly, there are further corrections due to the implicit dependence of the chiral angle on the chiral symmetry breaking. These corrections may all be absorbed into the definition of $g_A$.

7. Discussion and summary

We have derived a low-energy theorem for the photoproduction of pions on nucleons under no other assumption than "baryons are rigidly rotated solitons of a chirally invariant action at most quadratic in the time derivatives with a chiral symmetry breaking of $O\left(m_{\pi}^2\right)$". The unambiguous terms of this theorem in an expansion in $\frac{m_{\pi}}{M}$ are summarized by

\[
E_{0+}^{(-)} = \frac{|e|}{4\pi} \frac{g_A}{2f_{\pi}} C \left(\frac{m_{\pi}}{M}\right) \left[ 1 + O\left(\frac{m_{\pi}}{M}\right) \right]
\]

\[
E_{0+}^{(0)} = \frac{|e|}{4\pi} \frac{g_A}{2f_{\pi}} C \left(\frac{m_{\pi}}{M}\right) \left[ -\frac{1}{2} \frac{m_{\pi}}{M} + \frac{1}{4} (\mu_p + \mu_n) \left(\frac{m_{\pi}}{M}\right)^2 + O\left(\left(\frac{m_{\pi}}{M}\right)^2\right) \right]
\]

\[
E_{0+}^{(+)} = \frac{|e|}{4\pi} \frac{g_A}{2f_{\pi}} C \left(\frac{m_{\pi}}{M}\right) \left[ 0 \cdot \frac{m_{\pi}}{M} + O\left(\left(\frac{m_{\pi}}{M}\right)^3\right) \right],
\]

where the kinematical factors $C$ are given in eq.(4). The leading term in the expansion of the first two amplitudes, $E_{0+}^{(-)}$ and $E_{0+}^{(0)}$, coincides with the standard low-energy theorem\[3, 16\], the third amplitude, $E_{0+}^{(+)}$, does not.

The origin of the discrepancy was suspected to reside in the degeneracy of the rotational states in the soliton model up to lowest order in $N_C^{-1}$ which allows nucleons and $\Delta$’s as intermediate states. Elimination of the contributions of the higher spins and the special ordering

\[
\epsilon \cdot R D_{3a} \pi_c D_{c3} \rightarrow \epsilon \cdot R \frac{1}{3} \tau_3 \pi \cdot \tau
\]

(48)
for the truncated matrixelement in eq.(36) would actually also reproduce the standard prediction for the $E_{0^+}^{(+)}$-amplitude, but we could not find any convincing justification for such a special ordering. Corrections from a non-degenerate $\Delta$ to the photoproduction amplitudes have already been considered a long time ago\cite{17} in the framework of phenomenological lagrangians where the photocoupling of the $\Delta$ was introduced through an effective magnetic dipole operator. The couplings taken from experiment have lead to corrections to the low-energy theorem. However, the connection of this to the way soliton models include higher rotational states remains obscure to us since soliton models only have one local production vertex\cite{4}.

Neither a vanishing nor an infinite nucleon-$\Delta$-split appear to be realistic assumptions thus we do not see any compelling reason of why one version of the low-energy theorem should be more realistic than the other. Indeed, if we confront both with existing data we are not able to find a contradiction to either version: the reaction amplitudes in Walker’s convention\cite{18} for specific charge combinations

\begin{align}
A^{n(\gamma,\pi^-)}_p &= \sqrt{2}(+E_{0^+}^{(0)} - E_{0^+}^{(-)}) \\
A^{p(\gamma,\pi^+)}_n &= \sqrt{2}(-E_{0^+}^{(0)} - E_{0^+}^{(-)}) \\
A^{p(\gamma,\pi^0)}_p &= (E_{0^+}^{(+)} + E_{0^+}^{(0)}) \\
A^{n(\gamma,\pi^0)}_n &= (E_{0^+}^{(+)} - E_{0^+}^{(0)})
\end{align}

(49)

involve $E_{0^+}^{(+)}$ only in case neutral pions are produced. Data for the production of $\pi^0$ on neutrons are not available. A reanalysis of the data for the production of $\pi^0$ on protons vary from\cite{19} $A^{p(\gamma,\pi^0)}_p = -(2.0 \pm .2) \cdot 10^{-3} m_{\pi^+}^{-1}$ to\cite{20} $-(1.5 \pm .3) \cdot 10^{-3} m_{\pi^+}^{-1}$ as may be seen from table 1., where we confront the photoproduction data with low-energy-theorems of different origin.

Here, we concentrate on the production of neutral pions, where the soliton model arrives at conclusions different from more standard approaches:

(i) Up to $\mathcal{O}(\frac{m_\pi}{M})$ the standard version, eq.(3), predicts proton amplitudes which are too large but the next order corrections, $\mathcal{O}((\frac{m_\pi}{M})^2)$, are not small and lead to a number which is only slightly above the Mainz data. The production amplitude on neutrons is predicted to be small.

(ii) The low-energy theorem has also been reconsidered recently in the framework of chiral perturbation theory\cite{16} leading to differences in $\mathcal{O}((\frac{m_\pi}{M})^2)$ with respect to the standard expression. Up to this order, in chiral perturbation theory the threshold amplitude on protons has the wrong sign and only the full one loop amplitude up to all orders in the pion mass leads to an amplitude slightly below the Saclay data.
amplitude for the production on neutrons is larger than the one on protons.

(iii) The soliton model only allowed a unique determination of the amplitudes up to $O(\frac{m_{\pi}}{M})$ giving amplitudes of equal magnitude for the production of $\pi^0$ on protons or neutrons. The amplitude for the production on protons lies between the Mainz and the Saclay data. Given the fact that the uncalculable next order still might lead to substantial changes, no conclusions should be drawn, apart, maybe, from the observation that the big differences between different theorems seem to occur in the unmeasured amplitude for the production of $\pi^0$ on neutrons.

The numbers in table 1. have been determined by using the data everywhere for the masses $M$, $m_{\pi}$, the electromagnetic charge and the $\pi N$-coupling constant $g_{\pi NN}$. Since the Skyrme model also relates the axial charge to the $\pi N$-coupling via the Goldberger-Treiman relation

$$g_{\pi NN} = M \frac{g_A}{f_\pi} + O(m_{\pi}^2),$$

its replacement is correct up to the order indicated. It is amusing to note that the apparent good numerical agreement of Skyrme-type models concerning the photoproduction amplitudes can only be obtained when data are inserted for the corresponding constants. As is well known, no version of the Skyrme model can simultaneously fit $\frac{g_A}{f_\pi}$ and $\frac{m_{\pi}}{M}$ unless the pion mass is roughly doubled from its empirical value.

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Table 1. Kroll-Ruderman amplitudes in units $10^{-3}m_{\pi}^{-1}$ for the cases: (i) the standard low-energy theorem, eq.(3), (ii) chiral perturbation theory [16], CPT, (q: up to quadratic order in $m_{\pi}$, f: full one loop result), (iii) soliton model according to eq.(47). (iv) reanalysed experimental data, M: ref.[19], S: ref.[20].

|                        | standard LET | CPT soliton model | experiment      |
|------------------------|--------------|-------------------|-----------------|
| $A^p(\gamma,\pi^-)_{p}$ | -31.8        | -31.5$^q$         | -31.8           |
|                        |              | -31.1$^f$         |                 |
| $A^p(\gamma,\pi^+)_{n}$ | -27.4        | -26.6$^q$         | -27.4           |
|                        |              | -28.4$^f$         |                 |
| $A^p(\gamma,\pi^0)_{p}$ | -2.5         | 0.9$^q$           | -1.6            |
|                        |              | -1.3$^f$          |                 |
| $A^n(\gamma,\pi^0)_{n}$ | 0.4          | 3.6$^q$           | 1.6             |
|                        |              | 3.6$^f$           |                 |