BOUNDARY ENTROPY OF A HYPERBOLIC GROUP

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Abstract. We show that the entropy of a hyperbolic group acting on its ideal boundary is closely related to the exponential rate of its growth.

0. Introduction. Hyperbolic groups in the Gromov’s sense [Gro] play an important role in geometric group theory (see [GrH] and the references there). In particular, any non-elementary hyperbolic group has exponential growth and the compact boundary of positive finite Hausdorff dimension ([GhH], pp. 126 and 157). Also, the group itself acts on its boundary via Lipschitz quasi-conformal maps (ibidem, p. 127). The dynamics of this action is of great interest. For instance, it has been shown [CP] that this action is finitely presented, i.e. it is semiconjugate to a subshift of finite type in such a way that the fibres of the conjugating map are finite of bounded length and the equivalence relation determined by this map (two points are related whenever their images are equal) is another subshift of finite type. Also, one can consider the topological entropy of this action in the sense of [GLW]. In this article we prove the following.

Theorem. The topological entropy (with respect to a finite symmetric generating set) of a hyperbolic group $G$ acting on its ideal boundary lies between the exponential rate of growth of $G$ relative to suitable bounds depending on the geometry of the group and the exponential rate of growth of $G$ (with respect to the same generating set).

The precise description of the bounds mentioned in the Theorem can be found in Section 4 which contains also the proof of the Theorem and some final remarks. In the first section, we recall the notion of the topological entropy of a group action. In Section 2, we define the exponential rate of growth relative to given constants. In Section 3, we provide a short review on hyperbolic groups and spaces.

1. Entropy. Let $G$ be a finitely generated group of homeomorphisms of a compact metric space $(X,d)$ and $S$ be a finite symmetric ($e \in S, S^{-1} = S$) set generating $G$. Equip $G$ with the word metric $d_S$ induced by $S$ and let $B(n), n \in \mathbb{N}$, denote the ball in $G$ of radius $n$ and centre $e$. Two points $x$ and $y$ of $X$ are said to be $(n, \epsilon)$-separated ($\epsilon > 0, n \in \mathbb{N}$) whenever

$$d(gx, gy) \geq \epsilon$$

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for some \( g \in B(n) \). Since \( X \) is compact, the maximal number \( N(n, \epsilon) \) of pairwise \((n, \epsilon)\)-separated points of \( X \) is finite. Also, there exist finite \((n, \epsilon)\)-spanning subsets of \( X \): A subset \( A \) of \( X \) is \((n, \epsilon)\)-spanning whenever for any \( y \in X \) there exists \( x \in A \) such that

\[
d(gx, gy) < \epsilon
\]

for all \( g \in B(n) \). Let \( N'(n, \epsilon) \) denote the minimal cardinality of an \((n, \epsilon)\)-spanning subset of \( X \).

Similarly to the case of classical dynamical systems ([Wa], p. 169), the families \( N(n, \epsilon) \) and \( N'(n, \epsilon) \) of functions have the same type of growth [Eg], more precisely, they have the same rate of exponential growth and the topological entropy \( h(G, S) \) of \( G \) (w.r.t. \( S \)) can be defined by the formula

\[
h(G, S) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N'(n, \epsilon).
\]

If all the maps of \( G \) are Lipschitz and \( X \) has finite Hausdorff dimension, then \( h(G, S) \) is finite for any \( S \) (compare [GLW], Prop.2.7). Also, if \( h(G, S) = 0 \) for some \( S \), then \( h(G, S') = 0 \) for any other generating set \( S' \). Therefore, one can distinguish between groups of positive and vanishing entropy without referring to generating sets.

**2. Growth.** Let us keep the notation of the previous section and recall that the exponential *rate of growth* of \( G \) (with respect to \( S \)) is defined as

\[
gr(G, S) = \lim_{n \to \infty} \frac{1}{n} \log N(n) = \lim_{n \to \infty} \frac{1}{n} N_0(n),
\]

where \( N(n) = \#B(n) \) and \( N_0(n) = \#S(n) \) is the cardinality of the sphere \( S(n) \) of radius \( n \) and centre \( e \). Also, if \( \epsilon > 0 \) and \( N_0(n; \epsilon) \) is the maximal cardinality of an \( \epsilon \)-separated subset \( A \) of \( S(n) \), then

\[
gr(G, S) = \lim_{n \to \infty} \frac{1}{n} \log N_0(n; \epsilon).
\]

In fact, \( N_0(n; \epsilon) \leq N_0(n) \) and, if \( A \) is such a subset of \( S(n) \), then \( \cup_{x \in A} B(x, \epsilon) \supset S(n) \) and, therefore, \( N_0(n; \epsilon) N(\epsilon) \geq N_0(n) \) for any \( n \in \mathbb{N} \).

Moreover, if \( m \in \mathbb{N} \) and \( A_n \) is a maximal \( \epsilon \)-separated subset of \( S(mn) \), \( n = 1, 2, \ldots \), then for any \( x \in A_n \) we can find a sequence \( \{x_0, x_1, \ldots, x_n\} \) of elements of the group \( G \) for which \( x_k \in A_k \), \( x_n = x \), \( x_0 = e \) and \( d(x_k, x_{k+1}) \leq m + \epsilon \) for all \( k \). To construct such a sequence one can begin with \( x_n = x \), join \( x \) to \( e \) by a geodesic segment \( \gamma_x \), find the point \( x'_{n-1} \) of intersection of \( \gamma_x \) with the sphere \( S((n-1)m) \) and a point \( x_{n-1} \in A_{n-1} \) such that \( d(x_{n-1}, e) \) and continue by the induction. If \( y \) is another point of \( A_n \), \( \{y_0, y_1, \ldots, y_n\} \) is a corresponding sequence and \( k \) is the maximal natural number such that \( x_k = y_k \), then \( d(x_{k+1}, y_{k+1}) \geq \epsilon \). This motivates the following definition.

Let us fix \( m \in \mathbb{N} \), \( \epsilon > 0 \) and \( \lambda \in (0, 1) \), and denote by \( N_0(n; m, \epsilon, \lambda) \) the maximal cardinality of a subset \( A \) of \( S(mn) \) satisfying the following condition:

\[
(*) \text{ If } x \text{ and } y \text{ lie in } A, \text{ then there exist sequences } \{x_0, x_1, \ldots, x_n\} \text{ and } \{y_0, y_1, \ldots, y_n\} \text{ of elements of } G \text{ such that } x_k, y_k \in S(km), x_0 = y_0 = e, x_n = x, y_n = y,
\]
d(x_j, x_{j+1}) \leq T = m + \lambda \epsilon \text{ for all } j \text{ and } d(x_{k+1}, y_{k+1}) \geq \epsilon \text{ when } k \text{ is the maximal index for which } x_k = y_k \text{ (Figure 1).}

The number

\[ \text{gr}^{\text{rel}}(G, S; m, \epsilon, \lambda) = \limsup_{n \to \infty} \frac{1}{mn} \log N_0(n; m, \epsilon, \lambda) \]

will be called the exponential rate of growth of \( G \) relative to \( m, \epsilon \) and \( \lambda \). Finally, if \( \mu : (0, 1) \times \mathbb{R}_+ \to \mathbb{N} \) and \( \tau : (0, 1) \to \mathbb{R}_+ \) are arbitrary functions, then we define the rate of growth of \( G \) relative to \( \mu \) and \( \tau \) by

\[ \text{gr}^{\text{rel}}(G, S; \mu, \tau) = \sup \{ \text{gr}(G, S; m, \epsilon, \lambda); m > \mu(\lambda, \epsilon), \epsilon > \tau(\lambda), \lambda \in (0, 1) \} \]

Since \( N_0(n; m, \epsilon, \lambda) \leq N_0(mn, \epsilon) \) for all \( m, n, \epsilon \) and \( \lambda \), we have

\[ \text{gr}^{\text{rel}}(G, S; \mu, \tau) \leq \text{gr}(G, S) \]

for all \( \mu \) and \( \tau \) as above. For the free group \( F_k \) generated by the set \( S_k \) of \( k \) free generators we have always

\[ \text{gr}^{\text{rel}}(F_k, S_k; \mu, \tau) = \text{gr}(F_k, S_k) \]

This is because \( F_k \) has no ”dead ends” (see [GrH] for the definition and some information about some related problems) and in this case one can arrange \( \epsilon \)-separated subsets \( A_n \) of the spheres \( S(mn) \) in such a way that \( \text{dist}(x, A_n) = m \) for any \( x \in A_{n+1} \). In general, one can expect that a relative rate of growth is strictly less than the ”true” rate of growth.

3. **Hyperbolic spaces and groups.** Let \((X, d)\) be a metric space. A curve \( \gamma : [a, b] \to X \) is a geodesic segment when

\[ d(\gamma(t), \gamma(s)) = |t - s| \]

for all \( t, s \in [a, b] \). The space \( X \) is geodesic when any two points of \( X \) can be joined by a geodesic segment. For any finitely generated group \( G \) and any finite symmetric set \( S \) generating \( G \), the Cayley graph \( C(G, S) \) is geodesic.

Given three points \( x_0, y \) and \( z \) of a metric space \( X \), the (based at \( x_0 \)) Gromov product of \( y \) and \( z \) is given by

\[ (y|z)_{x_0} = \frac{1}{2} (d(x_0, y) + d(x_0, z) - d(y, z)). \]
The space $X$ is said to be hyperbolic (more precisely, $\delta$-hyperbolic with $\delta \geq 0$) whenever the inequality
\[
(x|z)_{x_0} \geq \min\{(x|y)_{x_0}, (y|z)_{x_0}\} - \delta
\]
holds for arbitrary points $x_0, x, y$ and $z$ of $X$. Clearly, the Cayley graph of any free group $F_k$ ($k = 1, 2, \ldots$) generated by the set $S_k$ of $k$ free generators is a tree, so becomes 0-hyperbolic. A finitely generated group is said to be hyperbolic whenever its Cayley graph with respect to some (equiv., any) generating set is hyperbolic. Free groups and fundamental groups of compact Riemannian manifolds of negative sectional curvature are hyperbolic.

Assume that $X$ is geodesic, take three points $x_1, x_2$ and $x_3$ of $X$ and connecting them geodesic segments $\gamma_1, \gamma_2$ and $\gamma_3$. The union
\[
\Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3
\]
is a geodesic triangle with vertices $x_i$. The triangle $\Delta$ is $\eta$-thin ($\eta \geq 0$) when the canonical isometry $f_\Delta$ mapping $\Delta$ onto a tripod (i.e. the union of three segments with common origin) $T_\Delta$ satisfies the condition
\[
d(x, y) \leq d(f_\Delta(x), f_\Delta(y)) + \eta
\]
for all $x$ and $y$ of $\Delta$.

In the proof of the Theorem we shall use the following.

**Lemma 1.** ([GhH], p. 41) Let $X$ be a geodesic metric space. If $X$ is $\delta$-hyperbolic, then all the geodesic triangles of $X$ are $4\delta$-thin. Conversely, if all the geodesic triangles of $X$ are $\eta$-thin, then $X$ is $2\eta$-hyperbolic. □

To construct the boundary $\partial X$ of a hyperbolic space $X$ let us fix a base point $x_0$ and say that a sequence $(x_n)$ diverges to infinity whenever
\[
\lim_{m,n \to \infty} (x_m|x_n) = \infty,
\]
where $(\cdot|\cdot)$ denotes the Gromov product based at $x_0$. Two such sequences $(x_n)$ and $(y_m)$ are equivalent whenever
\[
\lim_{m,n \to \infty} (x_m|y_n) = \infty.
\]

The boundary $\partial X$ of $X$ consists of all the equivalence classes of sequences diverging to infinity. Note that $\partial X$ can be described also in terms of equivalence classes of geodesic rays (i.e., maps $\gamma : [0, \infty) \to X$ such that $\gamma|[0, b]$ is a geodesic segment for any $b > 0$) or in terms of equivalence classes of quasirays (i.e. quasi-isometric maps of $[0, \infty)$ into $X$): Two such rays (or, quasirays) $\gamma$ and $\sigma$ are equivalent whenever their Hausdorff distance $d_H(\gamma, \sigma)$ is finite. The boundary point corresponding to the equivalence class of such $\gamma$ is that determined by the sequence $x_n = \gamma(n), n \in \mathbb{N}$. The equivalence of these constructions follows from the following fact which will be used later.
Lemma 2. ([GhH], p. 87) Let $X$ be a $\delta$-hyperbolic geodesic metric space. For any $c \geq 1$ there exists $D \geq 0$ such that any $c$-quasigeodesic segment $\gamma$ (i.e. any map $\gamma : [a, b] \to X$ (resp., $\gamma : [a, b] \cap \mathbb{Z} \to X$) such that the inequality $-c - c|s - t| \leq d(\gamma(s), \gamma(t)) \leq c|s - t| + c$ holds for all $s$ and $t$ of $[a, b]$ (resp., of $[a, b] \cap \mathbb{Z}$)) and any geodesic segment $\sigma$ joining $\gamma(a)$ to $\gamma(b)$ satisfy the inequality

$$d_H(\gamma, \sigma) \leq D. \quad \Box$$

The set $\partial X$ can be equipped with the metric structure as follows. First, for any $\xi$ and $\zeta$ of $\partial X$ put

$$(\xi \| \zeta) = \sup \liminf_{m, n \to \infty} (x_m \| y_n),$$

where $(x_m)$ and $(y_n)$ run over the set of all diverging to infinity sequences representing, respectively, $\xi$ and $\zeta$. Note that if $X$ is $\delta$-hyperbolic, then

$$(\xi \| \zeta) - 2\delta \leq \liminf_{m, n \to \infty} (x_m \| y_n) \leq (\xi \| \zeta)$$

for all sequences $(x_m)$ and $(y_m)$ representing $\xi$ and $\zeta$. Next, choose $\eta > 0$ and put

$$\rho_{\eta}(\xi, \zeta) = \exp(-\eta \cdot (\xi \| \zeta)).$$

Finally, let

$$d_{\eta}(\xi, \zeta) = \inf \{ \sum_{i=0}^{k} \rho_{\eta}(\xi_i, \xi_{i+1}); \xi_i \in \partial X, \xi_0 = \xi \text{ and } \xi_{k+1} = \zeta, k \in \mathbb{N} \}.$$

If $\eta > 0$ is small enough, then $d_{\eta}$ is a distance function on $\partial X$ and $(\partial X, d_{\eta})$ becomes a compact metric space of finite Hausdorff dimension ([GhH], pp. 122 - 126). Moreover, the inequalities

$$(1 - 2\eta')\rho_{\eta}(\xi, \zeta) \leq d_{\eta}(\xi, \zeta) \leq \rho_{\eta}(\xi, \zeta), \quad \xi, \zeta \in \partial X,$$

hold with

$$\eta' = \exp(\eta \delta) - 1.$$

4. Proof of the Theorem. Let again $G$ be a finitely generated group, $S$ - a finite symmetric set generating $G$ and consider $C(G, S)$, the Cayley graph of $G$ equipped with the distance function $d$ satisfying

$$d(g_1, g_2) = |g_1^{-1}g_2|, \quad g_1, g_2 \in G,$$

where $| \cdot |$ is the length function on $G$ determined by $S$, and making the edges of $C(G, S)$ isometric to Euclidean segments of length 1. Assume that $C(G, S)$ is $\delta$-hyperbolic and let $\partial G = \partial C(G, S)$ be its boundary equipped, as in Section 3, with the distance function $d_{\eta}$, $\eta > 0$ being small enough.

The group $G$ acts on $C(G, S)$ via isometries which, when restricted to $G \subset C(G, S)$, reduce to left translations $L_g$, $G \ni h \mapsto gh$. Therefore, each $L_g$ extends to a Lipschitz homeomorphism, denoted by $L_g$ again, of the boundary $\partial G$. Given
\( \theta > 0 \) denote by \( N(n, \theta; \partial G) \) the maximal number of points of \( \partial G \) pairwise \((n, \theta)\)-separated by this action of \( G \). Also, let \( N'(n, \theta; \partial G) \) be the minimal cardinality of an \((n, \theta)\)-spanning subset of \( \partial G \) and \( h(G, S; \partial G) \) - the corresponding entropy.

Since \( G \) is hyperbolic, there exists a constant \( c_0 > 1 \) such that for any \( g \in G \) there exists \( g' \in G \) such that \( d(g, g') \leq c_0 \) and \( |g'| = |g| + 1 \) (see [GrH], p. 60). Let \( D \) be a corresponding constant such that any \( c_0 \)-quasigeodesic segment lies at most \( D \)-apart (in the Hausdorff distance) from a true geodesic (compare Lemma 2). Let

\[
\tau(\lambda) = \frac{4(D + \delta)}{1 - \lambda} \quad \text{and} \quad \mu(\lambda, \epsilon) = \frac{\lambda \epsilon}{c_0 - 1}.
\]

We shall show that

\[
(\ast\ast) \quad \text{gr}_{\text{rel}}(G, S; \mu, \tau) \leq h(G, S; \partial G) \leq \text{gr}(G, S).
\]

We begin by the proof of the second inequality in \((\ast\ast)\).

Choose \( \theta > 0 \) and a natural number \( k \) for which the inequality

\[
\exp(-\eta \cdot k) < \theta
\]

holds. For any \( n \in \mathbb{N} \) and any \( g \in S(n+k) \) choose, if only possible, a point \( \xi_g \subset \partial G \) which can be connected to \( e \) by a geodesic ray, say \( \gamma_g \), which passes through \( g \). We claim that the set

\[
A_n = \{ \xi_g ; g \in S(n+k) \}
\]

is \((n, \epsilon)\)-spanning in \( \partial G \). Indeed, if \( \zeta \in \partial G, \sigma : [0, \infty) \rightarrow X \) is a geodesic ray connecting \( e \) to \( \zeta \), \( h \in S(n) \) and \( g = \sigma(n+k) \), then \( g \in S(n+k) \), the corresponding ray \( \gamma_g \) exists and satisfies the conditions

\[
d(x_j, y_i) \leq i + j - 2(n+k)
\]

and

\[
2(h^{-1}y_i|h^{-1}x_j) \geq (i - n) + (j - n) - (i + j) + 2(n+k) = 2k,
\]

where \( x_j = \sigma(j) \) and \( y_i = \gamma(i) \) for all \( i \) and \( j \) sufficiently large (Figure 2).
Therefore,
\[ d_\eta(L_{h^{-1}}\zeta, L_{h^{-1}}\xi_g) \leq \rho_\eta(L_{h^{-1}}\zeta, L_{h^{-1}}\xi_g) < e^{-kn} < \theta. \]
This shows the inequalities
\[ N'(n, \theta; \partial G) \leq \#A_n \leq \#S(n + k) \leq N(n + k) \]
which imply immediately the required inequality in (**).

The proof of the first inequality in (**) is a bit more complicated.

Fix \( \lambda \in (0, 1) \), \( \epsilon > \tau(\lambda) \) and \( m > \mu(\lambda, \epsilon) \). For any \( n \in \mathbb{N} \) choose a maximal subset \( A_n \) of \( S(mn) \) satisfying condition (*) of Section 2. For any \( x \in A_n \) set \( x_n = x \), choose a point \( x_{n-1} \in A_{n-1} \) such that \( d(x, x_{n-1}) \leq m + \lambda \epsilon \), then a point \( x_{n-2} \in A_{n-2} \) for which \( d(x_{n-1}, x_{n-2}) \leq m + \lambda \epsilon \) and so on. Finally, put \( x_0 = e \).

The map
\[ \{0, m, \ldots, mn\} \ni j \mapsto x_j/m \]
is \( c \)-quasi-isometric with \( c = (m + \lambda \epsilon)/m < c_0 \).

Each map considered above can be extended to a \( c_0 \)-quasi-isometric map
\[ \gamma_x : \mathbb{N} \ni j \mapsto x'_j \in G \]
such that \( x'_i = x_i \) for \( i = 0, 1, \ldots, n \) and the sequence \( (x_j) \) converges to a point \( \xi_x \) of \( \partial G \). We are going to show that the set
\[ \{\xi_x; x \in A_n\} \]
is \((n, \theta)\)-separated under the action of \( G \) for some \( \theta \) independent of \( n \).

To this end, let us take arbitrary points \( x \) and \( y \) of \( A_n \), \( x \neq y \), choose sequences \((x_0, x_1, \ldots, x_n)\) and \((y_0, y_1, \ldots, y_n)\) as above, and let \( k \) be the maximal element of \( \{0, 1, \ldots, n\} \) for which \( x_k = y_k \). Denote this common value of \( x_k \) and \( y_k \) by \( g \) and consider the quasi-geodesic rays \( \tilde{\gamma}_x \) and \( \tilde{\gamma}_y \) obtained from \( \gamma_x \) and \( \gamma_y \) by restricting their domains to \( \{mk, mk + 1, \ldots\} \). Then, \( \tilde{\gamma}_x \) and \( \tilde{\gamma}_y \) originate at \( g \) and converge to \( \xi_x \) and \( \xi_y \), respectively. By Lemma 2, there exist geodesic rays \( \check{\gamma}_x \) and \( \check{\gamma}_y \) originated at \( e \) and within the Hausdorff distance \( D \) from \( L_{g^{-1}} \circ \tilde{\gamma}_x \) and \( L_{g^{-1}} \circ \tilde{\gamma}_y \), respectively. Then, for any \( j \in \mathbb{N} \) there exist positive real numbers \( s_j \) and \( t_j \) such that
\[ d(\check{\gamma}_x(s_j), g^{-1}\tilde{\gamma}_x(j)) \leq D \quad \text{and} \quad d(\check{\gamma}_y(t_j), g^{-1}\tilde{\gamma}_y(j)) \leq D. \]
In particular,
\[ d(\check{\gamma}_x(s_m), g^{-1}x_{k+1}) \leq D \quad \text{and} \quad d(\check{\gamma}_y(t_m), g^{-1}y_{k+1}) \leq D. \]
Clearly,
\[ m - D \leq s_m, t_m \leq m + D + \lambda \epsilon, \ |s_m - t_m| \leq 2D + \lambda \epsilon \]
and
\[ d(\check{\gamma}_x(s_m), \check{\gamma}_y(t_m)) \geq \epsilon - 2D. \]
Assume that \( (\check{\gamma}_x(s_j)\check{\gamma}_y(t_j)) > m + D + \lambda \epsilon \) for some \( j \in \mathbb{N} \). The isometry \( f_\Delta \) corresponding to the geodesic triangle \( \Delta \) with vertices \( e, \check{\gamma}_x(s_j), \check{\gamma}_y(t_j) \) (compare
Lemma 1) maps the points $\tilde{\gamma}_x(s_m)$ and $\tilde{\gamma}_y(t_m)$ onto some points of the originated at $e$ edge of the tripod $f_\Delta(\Delta)$ and, therefore, satisfies the condition
\[
d(\tilde{\gamma}_x(s_m), \tilde{\gamma}_y(t_m)) \leq d(f_\Delta(\tilde{\gamma}_x(s_m)), f_\Delta(\tilde{\gamma}_y(t_m))) + 4\delta
= |s_m - t_m| + 4\delta \leq 2D + \lambda\epsilon + 4\delta < \epsilon - 2D.
\]
Comparing the inequalities above we obtain a contradiction which shows that
\[
(\tilde{\gamma}_x(s_j)|\tilde{\gamma}_y(t_j)) \leq m + D + \epsilon\lambda
\]
for all $j \in \mathbb{N}$. This inequality proves that
\[
d_\eta(g^{-1}\xi_x, g^{-1}\xi_y) \geq (1 - 2\eta')\rho_\eta(g^{-1}\xi_x, g^{-1}\xi_y)
\geq (1 - 2\eta')\exp(-\eta(m + D + \lambda\epsilon + 2\delta)),
\]
i.e. that the points $\xi_x$ and $\xi_y$ are $(nm, \theta)$-separated with
\[
\theta = (1 - 2\eta')\exp(-\eta(m + D + \lambda\epsilon + 2\delta)).
\]

The above argument implies the inequality
\[
N(nm, \theta; \partial G) \geq \#A_n \geq N_0(G, S; m, \epsilon, \lambda).
\]
which holds for all $n$. Passing to suitable limits when $n \to \infty$ yields the required inequality in (**) □

In [Fr], Friedland defined the minimal entropy $h_{\min}(G)$ of a finitely generated group $G$ of homeomorphisms of a compact metric space $X$:
\[
h_{\min}(G) = \inf_S h(G, S),
\]
where $S$ ranges over all finite symmetric sets generating $G$. Similarly, the minimal rate of growth $gr_{\min}(G)$ of any finitely generated group $G$ can be defined as follows (compare [GrH]):
\[
gr_{\min}(G) = \inf_S gr(G, S).
\]
The reader can define the minimal relative rate of growth $gr_{\min}^{rel}(G)$ appropriately.

If $G$ is hyperbolic, then $id_G$ induces a Hölder homeomorphism of boundaries of $G$ obtained from different generating sets ([GhH], page 128). Therefore, the boundary entropy of such $G$ (w.r.t. a given finite symmetric generating set $S$) does not depend on the choice of a generating set used in the construction of $\partial G$ and our Theorem implies immediately the following.

**Corollary.** For any hyperbolic group $G$ the equalities
\[
gr_{\min}^{rel}(G) \leq h_{\min}(G, \partial G) \leq gr_{\min}(G)
\]
hold. □

This answers partially the following question asked by Friedland in [Fr]: Find a geometric interpretation of the minimal entropy of a Kleinian group acting on the ideal boundary of a hyperbolic space $H^n$. 
For the free group $F_k$ with $k$ generators the above discussion and an argument of [GLP] (p. 70) imply the equality

$$h_{\text{min}}(F_k, \partial F_k) = \text{gr}_{\text{min}}(F_k) = \text{gr}_{\text{min}}^{\text{rel}}(F_k) = \log(2k - 1).$$

In fact, if $S$ is any finite symmetric set generating $F_k$, then the elements of $S$ represent members of a set $S'$ generating $\mathbb{Z}^k$, the abelianization of $F_k$. $S'$ contains a symmetric set $R'$ such that $\#R' = 2k$ and the subgroup of $\mathbb{Z}^k$ generated by $R'$ has finite index. The corresponding subset $R$ of $S$ consists also of $2k$ elements and generates the free group isomorphic to $F_k$. Therefore,

$$h(F_k, S; \partial F_k) \geq h(F_k, R; \partial F_k) = h(F_k, S_k; \partial F_k) \geq \text{gr}^{\text{rel}}(F_k, S_k) = \text{gr}(F_k, S_k) = \log(2k - 1).$$

The opposite inequality is obvious.

The other natural case, that of the fundamental group $\Gamma_g$ of a closed oriented surface of genus $g > 1$ is more complicated: The minimal rate of growth of $\Gamma_g$ is still unknown even if some estimates exist: $\text{gr}_{\text{min}}(\Gamma_g) \geq 4g - 3$, $\text{gr}_{\text{min}}(\Gamma_g) \leq \text{gr}(\Gamma_g, S_g) \approx 4g - 1 - \epsilon_g$, where $S_g$ is the canonical set of generators of $\Gamma_g$ and $\epsilon_g$ is a small constant found numerically. In particular, $5 \leq \text{gr}_{\text{min}}(\Gamma_2) \leq 6.9798$. The calculation or estimation of the value of the minimal relative rate of growth of $\Gamma_g$ is yet more difficult.

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