The tensor reduction and master integrals of the two-loop massless crossed box

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Abstract. We briefly discuss an algorithm for the tensor reduction of the two-loop massless crossed boxes, with light-like external legs, and the computation of the relative master integrals.

INTRODUCTION

The level reached nowadays by the precision measurements in high-energy scattering experiments demands the knowledge of next-to-next-to-leading theoretical amplitudes for \( 2 \rightarrow 2 \) scattering processes.

Very recent results for two-loop scattering amplitudes for massless particle have already appeared in the literature: the maximal-helicity-violating two-loop amplitude for \( gg \rightarrow gg \) [1], \( e^+ e^- \rightarrow \mu^+ \mu^- \) and \( e^+ e^- \rightarrow e^+ e^- \) [2], \( q \bar{q} \rightarrow q \bar{q} \) [3] and \( q \bar{q} \rightarrow q \bar{q} \) [4].

In dealing with these two-loop scattering amplitudes we have to face the problem of the tensor reduction of planar [5] and crossed double boxes [6], plus a plethora of simpler topologies [7, 8], and the computation of the relative master integrals [9, 10].

We denote the generic two-loop tensor crossed (or non-planar) four-point function in \( D \) dimensions of Fig. 1 with seven propagators \( A_i \) raised to arbitrary powers \( \nu \), as

\[
\text{Xbox}^D(\nu_1 \ldots \nu_7; s, t) \left[ 1; k^\mu; l^\nu; k'^\rho l'^\sigma; \ldots \right] = \int \frac{d^D k}{i \pi^{D/2}} \int \frac{d^D l}{i \pi^{D/2}} A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7},
\]

where the propagators are

\[
A_1 = (k + l + p_{34})^2 + i0, \quad A_2 = (k + l + p_{134})^2 + i0,
A_3 = (k + l)^2 + i0, \quad A_4 = l^2 + i0,
A_5 = (l + p_3)^2 + i0, \quad A_6 = k^2 + i0,
A_7 = (k + p_4)^2 + i0.
\]

The external momenta \( p_j \) are in-going and light-like, \( p_j^2 = 0, \ j = 1 \ldots 4 \), so that the only momentum scales are the usual Mandelstam variables \( s = (p_1 + p_2)^2 \) and \( t = (p_2 + p_3)^2 \), together with \( u = -s - t \). For ease of notation, we define \( p_{ij} = p_i + p_j \) and \( p_{ijk} = p_i + p_j + p_k \). In the square brackets we keep trace of the tensor structure that may be present in the numerator.

NOTATION

\[
\begin{array}{c}
\begin{array}{c}
\text{FIGURE 1. The generic two-loop crossed box.}
\end{array}
\end{array}
\]

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From the Schwinger representation of Feynman integrals it easy to see that integrals in $D$ dimensions can be connected with integrals in $D + 2$ dimensions (dimensional-shift) [5, 6].

In this way, tensor integrals can be directly connected to scalar integrals with higher powers of the propagators in $D = 4 - 2\varepsilon$ dimensions.

**THE SCALAR CROSSED-BOX REDUCTION**

The strategy to reduce the generic scalar integral to a linear combination of known ones is based on recurrence identities that relate scalar integrals with different powers of propagators. Some of these identities can be obtained using the integration-by-parts method [12] and exploiting the Lorentz invariance of the Feynman diagram [13].

Following the reduction procedure detailed in Ref. [6], any scalar crossed box with arbitrary powers of the propagators can be written as a linear combination of the following integrals, that, therefore, are called master integrals:

$S_{\text{set}}^D(s) = \begin{array}{cc}
\text{(s)}
\end{array}$

$\text{Tri}^D(s) = \begin{array}{cc}
\text{(s)}
\end{array}$

$\text{Xtri}^D(s) = \begin{array}{cc}
\text{(s)}
\end{array}$

$\text{Bbox}^D_{(s,t)} = \begin{array}{cc}
\text{(s,t)}
\end{array}$

$\text{Dbox}^D_{(s,t)} = \begin{array}{cc}
\text{(s,t)}
\end{array}$

$\text{Xbox}^D_{1(s,t)} = \begin{array}{cc}
\text{(s,t)}
\end{array}$

$\text{Xbox}^D_{2(s,t)} = \begin{array}{cc}
\text{(s,t)}
\end{array}$,

where all the propagators have powers one except for the propagator with the blob, that has power two.

**DIFFERENTIAL EQUATIONS FOR THE TWO MASTER INTEGRALS**

The analytic expansion in $\varepsilon = (4 - D)/2$ for the first master crossed box $\text{Xbox}^D_{1(s,t)}$ was computed in Ref. [10]. We can obtain the analytic form for the second one by writing the derivative of $\text{Xbox}^D_{1(s,t)}$ with respect to one of the two independent physical scales (that we choose to be $t$), as a combination of master integrals, and solving the equation for $\text{Xbox}^D_{2(s,t)}$. Moreover we can verify the correctness of both the expressions of $\text{Xbox}^D_{1(s,t)}$ and $\text{Xbox}^D_{2(s,t)}$, by deriving an analogous differential equation for $\text{Xbox}^D_{2}$, and checking that the obtained differential equation is satisfied.

Starting from the Schwinger representation of the generic crossed box, we can differentiate with respect to $t$, and set the values of $v_j$ to reproduce the two master integrals

$$
\frac{\partial}{\partial t} \text{Xbox}^D_{1(s,t)} = \text{Xbox}^{D+2}(1, 2, 1, 2, 1, 2; s, t) - \text{Xbox}^{D+2}(1, 2, 1, 2, 1, 2; s, t),
$$

$$
\frac{\partial}{\partial t} \text{Xbox}^D_{2(s,t)} = 2 \text{Xbox}^{D+2}(1, 3, 1, 2, 1, 2; s, t) - 2 \text{Xbox}^{D+2}(1, 3, 1, 2, 1, 2; s, t).
$$

Applying the reduction formalism for the scalar integrals and the dimensional-shift, we can rewrite the right-hand sides of the system as a combination of the two master crossed boxes plus other master integrals of simpler topologies, obtained by pinching one or more of the propagators of the crossed box,

$$
\frac{\partial}{\partial t} \text{Xbox}^D_{1(s,t)} = \frac{1}{t-u} \left[ \frac{(D-4)s^2 - 4tu}{2tu} \text{Xbox}^D_{1(s,t)} - \frac{(D-6)s}{2(D-5)} \text{Xbox}^D_{1(s,t)} + \text{pinchings} \right] \tag{1}
$$

$$
\frac{\partial}{\partial t} \text{Xbox}^D_{2(s,t)} = \frac{2}{tu} \left[ \frac{(D-5)s^2}{t} \text{Xbox}^D_{1(s,t)} - \frac{(D-6)(u^2 + t^2)}{tu} \text{Xbox}^D_{2(s,t)} + \text{pinchings} \right] \tag{2}
$$

Inserting the $\varepsilon$ expansion of $\text{Xbox}^D_{1(s,t)}$ computed in Ref. [10] and the $\varepsilon$ expansions of the sub-topologies listed in Refs. [5, 7, 14, 15] into Eq. (1), and solving it with respect to $\text{Xbox}^D_{2(s,t)}$, we obtain, in the physical region $s > 0, t, u < 0$,

$$
\text{Xbox}^D_{2(s,t)} = \Gamma^2(1 + \varepsilon) \left\{ \frac{G_1(t,u)}{s^4} + \frac{G_2(t,u)}{s^2 t^2} + \frac{G_1(u,t)}{s^4 u} + \frac{G_2(u,t)}{s^2 u^2} \right\}
$$

where

$$
G_1(t,u) = s^{-2\varepsilon} \left\{ \frac{6}{\varepsilon^2} + \frac{1}{\varepsilon^2} \left[ 32 - 6T - 6U \right] 
+ \frac{1}{\varepsilon} \left( 1 - 12\pi^2 - 24T + T^2 - 24U + 16TU + U^2 \right) \right\}
$$
\[ -43 - 18T + 13T^2 + \frac{8}{3}T^3 - 18U + 16TU \\
+ 11T^2U + 13U^2 - 20TU^2 + \frac{8}{3}U^3 + \pi^2 \left(17T \\
+ 17U - \frac{112}{3}\right) - 122\zeta(3) + 62TLi_2 \left(-\frac{T}{s}\right) \\
- 62Li_3 \left(-\frac{T}{s}\right) + 62S_{1,2} \left(-\frac{T}{s}\right) \\
+ i\pi \left[\frac{1}{\epsilon} (16 + 6T + 6U) - 34 - 9\pi^2 - 6T - 10T^2 \\
- 6U + 14TU - 10U^2\right]\right\}, \\
G_2(t, u) = s^{-2}\epsilon \left\{\frac{2}{\epsilon} + \frac{1}{\epsilon^3} + \left(-8 + \frac{5}{2}T + \frac{7}{2}U\right) \\
+ \frac{1}{\epsilon^2} \left(-\frac{29}{2} - \frac{5}{12}T^2 + 7T - T^2 + 20U - 4TU \\
- U^2\right) + \frac{1}{\epsilon} \left[-\frac{1}{2} + 17T + 2T^2 - \frac{T^3}{3} + \frac{\pi^2}{6} (14 \\
+ 5T - 29U) + 13U - 28TU - 4U^2 + 3TU^2 \\
- U^3 + \frac{19}{2}\zeta(3) - 2TLi_2 \left(-\frac{T}{s}\right) + 2Li_3 \left(-\frac{T}{s}\right) \\
- 2S_{1,2} \left(-\frac{T}{s}\right) + \frac{37}{2} + \frac{37}{40}\pi^4 + 7T - 5T^2 \\
- \frac{22}{3}T^3 + \frac{2}{3}T^4 + 5U - 20TU + \frac{8}{3}T^3U - 2U^2 \\
+ 24TU^2 - 2T^2U^2 - 8U^3 - \frac{4}{3}TU^3 + \frac{4}{3}U^4 \\
+ \frac{\pi^2}{6} (79 - 22T - 5T^2 - 200U + 76TU + 25U^2) \\
+ (68 - 13T - 33U)\zeta(3) + \left(10\pi^2 - 32T + 17T^2 \\
+ 12TU\right)Li_2 \left(-\frac{T}{s}\right) - 36S_{2,2} \left(-\frac{T}{s}\right) \\
+ (28T - 6U - 32)S_{1,2} \left(-\frac{T}{s}\right) - 26S_{1,3} \left(-\frac{T}{s}\right) \\
+ (32 - 60T - 12U)Li_3 \left(-\frac{T}{s}\right) + 86Li_4 \left(-\frac{T}{s}\right) \\
+ i\pi \left[\frac{2}{\epsilon^3} + \frac{1}{\epsilon^2} (11 - T + U) + \frac{1}{\epsilon} (1 - \frac{31}{6}\pi^2 - 10T \\
- 2T^2 + 4U - 2TU - 2U^2) + 11 + 4T - 2T^2 \\
+ \frac{10}{3}T^3 + \frac{\pi^2}{3} (-65 + 28T - U) + 2U - 8TU \\
- 8U^2 + 2U^3 - 89\zeta(3) + (14T + 18U)Li_2 \left(-\frac{T}{s}\right) \\
- 32Li_3 \left(-\frac{T}{s}\right) + 44S_{1,2} \left(-\frac{T}{s}\right)\right\}\right\}. \\
\]

\[ T = \log(-t/s), \ U = \log(-u/s), \] and where we used Nielsen’s generalized polylogarithms \(S_{n,p}\) defined by

\[ S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1-xt)}{t} \, dt, \quad n, p \geq 1, \quad x \leq 1 \]

Expressions for \(X^{(2)}_0(s, t)\) in the other two kinematic regions, \(t > 0, s, u < 0\) and \(u > 0, s, t < 0\) can be easily obtained through the analytic continuation of the polylogarithms and the logarithms.

It is a strong check of the whole formalism that inserting the analytic expansion of \(X^{(2)}_0(s, t)\) and the derived expression of \(X^{(2)}_0(s, t)\) into Eq. (2), we obtain an equality identically satisfied.

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