Filter functions in quantum phase-space representations

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Representation of quantum states in terms of phase-space quasiprobability distributions provides practical tools for identifying features such as the nonclassicality of quantum states. In this setting, filter functions are commonly used to regularize or smooth the phase-space quasiprobability distributions, in particular, the Glauber-Sudarshan P-function. We show that the quantum map associated with a filter function is completely positive and trace-preserving if and only if the Fourier transform of the filter function is a probability density distribution. In this case, filtering the quasiprobability distributions of a quantum state can be viewed as applying a random displacement operation on the quantum state according to the Fourier transform of the filter function. We derive a lower bound on the fidelity between the input and output states of a quantum filtering map. We illustrate several examples of filter functions corresponding to physical and nonphysical maps, in particular, a class of positive but not completely positive maps. We also discuss interesting applications of our results in estimating the output state of unknown quantum channels and estimating the outcome probabilities of quantum measurements.

I. INTRODUCTION

The well-developed theory of phase-space quasiprobability distributions (PQDs) plays an indispensable role in representing the quantum states of optical systems with infinite-dimensional Hilbert spaces. Well-known examples of PQDs are the Wigner function [1], the Husimi Q-function [2], and the Glauber-Sudarshan P-function (GSP) [3, 4], which are, in fact, the special cases of the s-ordered phase-space quasiprobability distributions ([s]-PQD) [5, 6]. A useful application of the PQDs is to identify nonclassical quantum states [7] whose GSP in terms of nonclassical states onto a regular function with some boundedness theorem. This theorem implies that the filtering procedure transforms the (s)-PQD of another quantum state, i.e., completely positive trace-preserving (CPTP) linear map, resulting in a physical Wigner function.

Despite the wide applications of filter functions in quantum optics, a general formalism describing how filters preserve the physicality of a quantum state is lacking. In particular, bounding the distance between a quantum state and its filtered version is of great interest for various applications in quantum information.

In this paper, we close this gap by providing the necessary and sufficient condition for a filter function to preserve the physicality of the PQDs. Specifically, we show that the filtering procedure transforms the (s)-PQD of a quantum state to the (s)-PQD of another quantum state, if and only if the filter function is the Fourier transform of a probability density distribution. Then the filtering procedure corresponds to a quantum CPTP map, which is a random application of the displacement operator on the initial state according to the Fourier transform of the
filter function. Using this condition, we check the physicality of several examples of filtering maps. We show that the filtering map associated with the Klauder’s filter function is not CPTP and, in general, does not preserve the physicality of quantum states. Our condition also enables us to identify a class of positive but not completely positive maps. Moreover, we derive a lower bound on the fidelity between the input and output states of the filtering map. This bound, in particular, is useful in estimating how much error is introduced by a filtering process, and later we discuss its application in estimating the output of a quantum channel, and also the outcome probabilities of a general measurement given the heterodyne record.

The outline of this paper is as follows. We start with reviewing the PQDs and the filtering procedure in Section II. In Section III we introduce the necessary and sufficient condition for a filtering map - induced by the filter function - to be a physical quantum process. Examples of various filter functions are discussed in Section IV. Later we derive a lower bound on the fidelity between the filtered and unfiltered states in Section IV and discuss its applications in Section VII. The paper is concluded in Section VIII.

II. PHASE-SPACE QUASIPROBABILITY DISTRIBUTIONS AND FILTER FUNCTIONS

A density operator describing the physical state of a single-mode bosonic system can be expressed as

$$\rho = \frac{1}{\pi} \int d^2 \xi \Phi(\xi) D(-\xi)$$

where the integral is over the entire complex plane, \(D(\xi) = \exp(\xi a^\dagger - \xi^* a)\) is the displacement operator with \(a^\dagger\) and \(a\) being the creation and annihilation operators, respectively, and \(\Phi(\xi) = \text{Tr}[\rho D(\xi)]\) is the characteristic function. The physicality conditions of density operators (\(\rho \geq 0\) and \(\text{Tr}[\rho] = 1\)) translates into the following conditions for physical characteristic functions: \(\Phi(\xi)\) is continuous, \(\Phi(0) = 1\), and for every finite set of points \(\{\xi_i\}\) the matrix \(M_{jk} = \Phi(\xi_j - \xi_k)\) is nonnegative.

Eq. (2) can also be formulated as

$$\rho = \int d^2 \alpha W^{(s)}(\alpha) T^{(-s)}(\alpha),$$

where the operators \(T^{(-s)}(\alpha)\) are given by

$$T^{(-s)}(\alpha) = \frac{1}{\pi} \int d^2 \xi e^{-s|\xi|^2/2} D(-\xi) e^{s\xi^* - \xi \alpha^*},$$

and

$$W^{(s)}(\alpha) = \frac{1}{\pi^2} \int d^2 \xi \Phi(\xi) e^{s|\xi|^2/2} e^{s\xi^* - \xi \alpha^*}$$

is the \((s)\)-PQD of density operator \(\rho\). Here, \(s\) is the order parameter, which is \(-1\) for the Husimi \(Q\) function, \(0\) for the Wigner function, and \(1\) for GSP with \(T^{(-1)}(\alpha) = |\alpha\rangle\langle \alpha|\). Notice that, by using Eq. (5) and the physicality conditions for \(\Phi(\xi)\), one can derive the necessary and sufficient conditions for \(W^{(s)}(\alpha)\) to be physical.

The filtering procedure is defined by multiplying the characteristic function with a filter function \(\Omega(\xi)\)

$$\Phi_{\Omega}(\xi) = \Phi(\xi) \Omega(\xi).$$

In this case, \(\Phi_{\Omega}(\xi)\) can be thought of as the characteristic function of another operator \(\rho_{\Omega}\) that is formally given by a linear map \(E_{\Omega}\)

$$\rho_{\Omega} = E_{\Omega}(\rho) = \int d^2 \alpha \tilde{\Omega}(\alpha) D(\alpha) \rho D^\dagger(\alpha),$$

where \(\tilde{\Omega}(\alpha)\) is the Fourier transform of the filter function. One can verify this equation by using Eq. (2) and \(D(\alpha) D(-\xi) D^\dagger(\alpha) = \exp(\xi \alpha^* - \alpha \xi^*) D(-\xi)\). Notice that \(\rho_{\Omega}\) may not be necessarily a density operator, as we discuss in the next section. By taking the Fourier transformation of Eq. (6) the \((s)\)-PQD of operator \(\rho_{\Omega}\) reads

$$W_{\Omega}^{(s)}(\alpha) = W^{(s)} * \tilde{\Omega}(\alpha)$$

$$= \int d^2 \beta W^{(s)}(\alpha - \beta) \tilde{\Omega}(\beta),$$

which is the convolution of the \((s)\)-PQD of density operator \(\rho\) and Fourier transform of the filter function.

III. NECESSARY AND SUFFICIENT CONDITION FOR PHYSICALITY OF A FILTERING MAP

A quantum process, described by a CPTP map, transforms density operators to density operators and, in principle, can be realized in the laboratory. However, in general, the filtering map generated by the filter function \(\Omega(\xi)\) may not be a physical quantum process. Here, we introduce the necessary and sufficient condition for a filter function to generate a CPTP map, preserving the physicality of quantum states.

Theorem. The necessary and sufficient condition for the filtering map \(E_{\Omega}\) to be a CPTP map is that the filter function \(\Omega(\xi)\) must be the Fourier transform of a probability density function.

Proof. If the Fourier transform of the filter function, \(\tilde{\Omega}(\alpha)\), is a probability density, then according to Eq. (7) \(E_{\Omega}(\rho)\) is a statistical mixture of displaced density operators \(D(\alpha) \rho D^\dagger(\alpha)\), which is a valid density operator.
Therefore, $\tilde{\Omega}(\alpha)$ being a probability density function is sufficient for $\mathcal{E}_\Omega$ to be a CPTP map. To prove that this condition is also necessary, we show in the following that if $\tilde{\Omega}(\alpha)$ takes on negative values, then $\mathcal{E}_\Omega$ will not be a completely-positive map.

Suppose our system is a subsystem $S$ of a bipartite system in a two-mode squeezed vacuum state,

$$|\psi_{SE}\rangle = \sqrt{1 - \chi^2} \sum_{n=0}^{\infty} \chi^n |n\rangle_S \otimes |n\rangle_E,$$

where $|n\rangle$ are the number states and $0 \leq \chi < 1$ is the squeezing parameter. We apply $\mathcal{E}_\Omega$ on subsystem $S$ and the identity $I$ on subsystem $E$, resulting the operator $\sigma_{SE} = (\mathcal{E}_\Omega \otimes I)|\psi_{SE}\rangle\langle \psi_{SE}|$. Then, by using Eq. (7) we calculate the fidelity between the output operator $\sigma_{SE}$ and the state $|\phi_{SE}\rangle = D(\beta) \otimes I|\psi_{SE}\rangle$

$$\langle \phi_{SE}| \sigma_{SE} | \phi_{SE}\rangle = \int d^2 \alpha \tilde{\Omega}(\alpha) \times |\langle \psi_{SE}| (D(\alpha - \beta) \otimes I)|\psi_{SE}\rangle|^2.$$

By using $\langle n|D(\gamma)|n\rangle = \exp(-|\gamma|^2/2)L_n(|\gamma|^2)$ with $L_n(x)$ being the Laguerre polynomial of the order $n$ and its generating function $\sum_{n=0}^{\infty} t^n L_n(x) = 1/(1 - t \exp(-tx/(1 - t))$, we have

$$\langle \psi_{SE}|(D(\gamma) \otimes I)|\psi_{SE}\rangle = (1 - \chi^2) \sum_{n=0}^{\infty} \chi^{2n} \langle n|D(\gamma)|n\rangle$$

$$= \exp\left(-\frac{|\gamma|^2}{2} - \frac{\chi^2|\gamma|^2}{1 - \chi^2}\right).$$

Therefore, the fidelity becomes

$$\langle \phi_{SE}| \sigma_{SE} | \phi_{SE}\rangle = \int d^2 \alpha \tilde{\Omega}(\alpha) e^{-(1+2\tilde{n})|\alpha - \beta|^2}$$

where $\tilde{n} = \chi^2/(1 - \chi^2)$ is the mean photon number of the reduced density operator of $|\psi_{SE}\rangle$ that is a thermal state. By noting that the Dirac-delta function can be defined as the limit of a Gaussian function, $\delta^2(\gamma) = \delta(\text{Re}(\gamma))\delta(\text{Im}(\gamma)) = \lim_{n \to \infty} (2n/\pi) \exp(-2\tilde{n}|\gamma|^2)$ we can see that if $\tilde{\Omega}(\alpha_0) < 0$ for some $\alpha_0$, by choosing $\beta = \alpha_0$ in Eq. (12) we get

$$\lim_{n \to \infty} \frac{2\tilde{n}}{\pi} \langle \phi_{SE}| \sigma_{SE} | \phi_{SE}\rangle = \tilde{\Omega}(\alpha_0) < 0.$$  

This implies that by choosing a sufficiently large value of $\tilde{n}$ the fidelity $\langle \phi_{SE}| \sigma_{SE} | \phi_{SE}\rangle$ becomes negative, indicating that $\sigma_{SE}$ is not a positive operator. Therefore, if the Fourier transform of the filter function takes on negative values the filtering map does not preserve the physicality of the entangled state $|\psi_{SE}\rangle$ and is not completely positive. Also, for the filtering map $\mathcal{E}_\Omega$ to be trace preserving $\Omega(\alpha)$ must also be normalized, as $\text{Tr}[\mathcal{E}_\Omega(\rho)] = \int d^2 \alpha \tilde{\Omega}(\alpha) = \Omega(0) = 1$. Therefore, the filtering map is CPTP if and only if the function $\Omega(\alpha)$ is a probability density distribution.

If this condition is satisfied, as it is evident from Eq. (7), the filtering process can be realized by applying the displacement operator chosen randomly according to the probability density distribution $\tilde{\Omega}(\alpha)$. Therefore, in this sense, the filtered state $\rho_S$ is a noisy version of the original state $\rho$. In practice, to implement a displacement operation on a quantum state, one can overlap the state on a highly transmissive beamsplitter with a coherent state.

### IV. EXAMPLES OF FILTER FUNCTIONS

In this section, we consider several examples of filter functions that have been previously used in the literature. We show that the associated filtering map for some of them is not a CPTP map.

#### A. Gaussian filters

One simple example of filter functions corresponding to CPTP maps is the Gaussian function $\Omega_r(\xi) = \exp(-r|\xi|^2/2)$ where $r$ is a positive number. The Fourier transform is a Gaussian probability density $2 \exp(-2|\xi|^2/r)/(\pi r)$. By using this filter function and Eqs (15) and (18), we have

$$W_\Omega^{(s)}(\alpha) = W^{(s-r)}(\alpha) = W^{(s)} \ast \frac{2}{\pi r} \exp\left(-\frac{-2|\alpha|^2}{r}\right).$$

This equation implies that the $(s-r)$-PQD of density operator $\rho$ can be thought of as the $(s)$-PQD of the filtered state $\rho_S$ that is obtained by applying random displacement operator on $\rho$ according to the Gaussian probability density. For example, for $r = 1$, the Wigner function of a density operator can be thought of as the GSP of a noisy version of the same density operator. Also, note that if the $(s)$-PQD of $\rho$ is equal to the $(s-r)$-PQD of another state, the state $\rho$ is mixed.

#### B. Nonclassicality filters

As another class of filter functions that can be described by CPTP maps, we can consider the nonclassicality filter functions that can regularize the singular GSPs of all nonclassical states. A nonclassicality filter is designed as the auto-correlation of an infinitely differentiable function $\omega_L(\xi)$.

$$\Omega_L(\xi) = \int d^2 \zeta \omega_L(\zeta) \omega_L(\xi - \zeta),$$

where $0 \leq L \leq \infty$ is the width parameter such that $\lim_{L \to \infty} \Omega_L(\xi) = \Omega_L(0) = 1$. Therefore, the Fourier transform of the filter function $\tilde{\Omega}_L(\xi)$ is always guaranteed to be a probability density, and the negativity in the
regularized GSP, \( P_{\Omega}(\alpha) \), which is the GSP of \( \rho_{\Omega} \), indicates the nonclassicality of the state \( \rho \).

A class of nonclassicality filters is defined using

\[
\omega_L(\xi) = \frac{1}{L^{21/4}} \sqrt{\frac{q}{2\pi \Gamma(2/q)}} \exp\left(-\frac{|\xi|^q}{L^q}\right),
\]

where \( 2 < q < \infty \) is a parameter characterizing the analytic form of \( \omega_L(\xi) \), and \( \Gamma \) is the gamma function. Notice that the case \( q = 2 \) corresponds to the Gaussian filters, in which case the filter function does not decay fast enough to regularize the GSP for all quantum states.

### C. Klauder’s filter

An example of a filter function whose filtering map is not CPTP, is the filter function introduced by Klauder to approximate a density operator with a bounded operator with infinite differentiable GSP. We show that the associated filtering map is not physical and therefore the bounded operators obtained by this particular filtering map may not be a valid density operator. Klauder’s filter is defined as

\[
\Omega_{\omega}^K(u, v) = e^{-[f(u-L)+f(-u-L)+f(v-L)+f(-v-L)]}
\]

where \( (u, v) = \sqrt{2}(\text{Re}(\xi), \text{Im}(\xi)) \) and \( f(x) \) is defined as

\[
f(x) = \begin{cases} \frac{x^q e^{-1/x^2}}{q} & x > 0 \\ 0 & x \leq 0. \end{cases}
\]

This filter function decays rapidly such that it regularizes the GSP of all quantum states.

According to Bochner’s theorem, the Fourier transform of the filter function, \( \Omega(\alpha) \), is probability density if and only if for any finite set of points \( \{\xi_i\} \) the matrix \( F_{jk} = \Omega(\xi_j - \xi_k) \) is positive-semidefinite. Using a set of points \( \{u_1 = -L, u_2 = 0, u_3 = L\} \) on the real axis, we can see that \( \Omega_{\omega}^K(u_j - u_k, 0) \) is not positive-semidefinite because its determinant is negative,

\[
\det \begin{pmatrix} 1 & 1 & \Omega_{\omega}^K(-2L, 0) \\ 1 & 1 & \Omega_{\omega}^K(2L, 0) \\ \Omega_{\omega}^K(-2L, 0) & \Omega_{\omega}^K(2L, 0) & 1 \end{pmatrix} = -(\Omega_{\omega}^K(2L, 0) - 1)^2,
\]

where we used \( \Omega_{\omega}^K(-2L, 0) = \Omega_{\omega}^K(2L, 0) \). Therefore, the Fourier transform of Klauder’s filter, \( \Omega_{\omega}^K(\xi) \), is not a probability density and the associated filtering map is not CPTP.

This filter function was used for quantum process tomography using coherent states. In this method the process tensor in the Fock basis, \( \langle k | \mathcal{E}(|n\rangle |m\rangle) |k\rangle \) is reconstructed through finding the action of the process on coherent state, using a regularized GSP in Eq. (11) for operators \(|n\rangle |m\rangle \). However, as we just observed, the filtering map does not necessarily preserve the physicality of quantum states and this may lead to a nonphysical process tensor. The need for using a filter function was eliminated in an improved version of coherent state quantum process tomography.

### D. Wigner smoothing filters

In the Wigner function smoothing, one convolves the Wigner function of a quantum state with another Wigner function, known as smoothing kernel

\[
W_{\text{sm}}(\alpha) = W \ast W_{\text{ker}}(\alpha).
\]

By comparing with Eq. (8), we can see that the characteristic function of the smoothed Wigner function is given by \( \Phi_{\text{sm}}(\xi) = \Phi(\xi)\Phi_{\text{ker}}(\xi) \) and the smoothing procedure is, in fact, a filtering map associated with the filter function \( \Omega(\xi) = \Phi_{\text{ker}}(\xi) \). It can be shown that \( W_{\text{sm}}(\alpha) \geq 0 \) over all the phase space. However, for an arbitrary choice of the smoothing kernel \( W_{\text{ker}}(\alpha) \), it is not guaranteed that \( \Phi_{\text{sm}}(\xi) \) satisfies the physicality conditions, which are discussed in Section (II).

To be more specific, it is necessary to introduce the concept of Narcowich-Wigner spectrum. Consider a function \( f(\xi) \), the set of all real parameters \( \eta \) for which the matrix

\[
F_{jk} = f(\xi_j - \xi_k) \exp \left( \frac{q}{4} (\xi_j^* - \xi_k^*) (\xi_j - \xi_k) \right) \geq 0,
\]

is positive semi-definite for any finite set of complex numbers \( \{\xi_i\} \), is called the “Narcowich-Wigner spectrum” of \( f(\xi) \) and is represented by \( \mathcal{W}\{f(\xi)\} \). Based on this definition, one can verify the following properties: if \( \eta \in \mathcal{W}\{f(\xi)\} \) then \( \eta \in \mathcal{W}\{f(\xi)\} \); if \( \eta_1 \in \mathcal{W}\{f(\xi)\} \) and \( \eta_2 \in \mathcal{W}\{g(\xi)\} \), then \( \eta_1 + \eta_2 \in \mathcal{W}\{f(\xi)g(\xi)\} \).

As discussed in Section (II), the characteristic function of a density operator must be continuous, \( \Phi(0) = 1 \), and \( 2 \in \mathcal{W}\{\Phi(\xi)\} \). In addition, based on the Bochner’s theorem, our necessary and sufficient condition for a continuous filter function \( \Omega(\xi) \) with \( \Omega(0) = 1 \) to generate a physical filtering map implies that \( 0 \in \mathcal{W}\{\Omega(\xi)\} \). By using the above properties, one can show that if either \( \{0, 2\} \in \mathcal{W}\{\Phi_{\text{ker}}(\xi)\} \) or \( \{2, 4\} \in \mathcal{W}\{\Phi_{\text{ker}}(\xi)\} \) is satisfied, then we always have \( \{0, 2\} \in \mathcal{W}\{\Phi_{\text{sm}}(\xi)\} \) and therefore \( W_{\text{sm}}(\alpha) \) is a physical and point-wise nonnegative Wigner function.

However, as we have shown, if \( 0 \notin \mathcal{W}\{\Phi_{\text{ker}}(\xi)\} \) then the filtering map is not completely positive. Thus, we can identify a class of filter functions whose filtering maps are positive but not completely positive. Any filter function that has 4 but not 0 in its Narcowich-Wigner spectrum belongs to this class. An example of such a filter function is

\[
\Phi_{\text{ker}}(\xi) = \left( 1 - \frac{3}{2} |\xi|^2 \right) e^{-|\xi|^2},
\]

whose Fourier transform takes on negative values.

### V. Bound on the distance between original and filtered states

Of great practical interest is the distance between the density operator obtained by a physical filtering procedure, \( \rho_{\Omega} \), and the original state \( \rho \). In this section we derive a lower bound on the fidelity between the two states,
which are in fact the output and input states of the filtering map $\mathcal{E}_\Omega$.

The fidelity between two states is given by $F(\rho_1, \rho_2) = \max(|\mu_1|, |\mu_2|)$ where the maximum is taken over all possible purifications $|\mu_1\rangle$ and $|\mu_2\rangle$ of $\rho_1$ and $\rho_2$, respectively \[32\]. Suppose $|\mu_{SE}\rangle = \sum_j \sqrt{\lambda_j} |s_j\rangle_S \otimes |e_j\rangle_E$ is a purification of $\rho$ where $\lambda_j$ are its eigenvalues and $|s_j\rangle_S$ and $|e_j\rangle_E$ are orthogonal bases for subsystems $S$ and $E$, respectively. Using this state and the filtering map $\mathcal{E}_\Omega$ given in Eq. \[7\] we can calculate the entanglement fidelity \[33\].

$$F_c(\rho, \mathcal{E}) = \langle \mu_{SE} | (\mathcal{E}_\Omega \otimes \mathcal{I}) (|\mu_{SE}\rangle \langle \mu_{SE}|) | \mu_{SE}\rangle$$

$$= \int d^2 \alpha \, \tilde{\Omega}_L(\alpha) \left| \langle \mu_{SE} | (\mathcal{D}(\alpha \otimes \mathcal{I}) | \mu_{SE}\rangle \right|^2$$

$$= \int d^2 \alpha \, \tilde{\Omega}_L(\alpha) |\Phi(\alpha)|^2,$$

where in the last line we have used the characteristic function of $\rho$, $|\mu_{SE}| (\mathcal{D}(\alpha \otimes \mathcal{I}) | \mu_{SE}\rangle = \mathcal{I}[\rho \mathcal{D}(\alpha)] = |\Phi(\alpha)|$.

Here, the width parameter $L$ of the filter function can be defined such that $\lim_{L \to \infty} \tilde{\Omega}_L(\alpha) = \delta^2(\alpha)$, as for example in the nonclassicality filters given by Eqs. \[15\] and \[16\]. In this case, by choosing sufficiently large values of $L$ the entanglement fidelity \[22\] can be made arbitrary close to $|\Phi(0)|^2 = 1$.

The entanglement fidelity is always a lower bound on the fidelity between the states before and after a quantum process \[35\]. Therefore, Eq. \[22\] provides a lower bound on the fidelity between the filtered and unfiltered states, $F_c(\rho, \mathcal{E}) \leq F(\rho, \rho_1)$, which can be adjusted by using the width parameter $L$. For any $\epsilon > 0$ one can choose $L$ such that

$$1 - \epsilon \leq \int d^2 \alpha \, \tilde{\Omega}_L(\alpha) |\Phi(\alpha)|^2 \leq F(\rho, \rho_1). \tag{23}$$

This relation can, in particular, be useful in estimating the error associated with the regularization of the GSP of a quantum state, as we discuss in the following section. Notice that if the unfiltered state is pure, $\rho = |\psi\rangle \langle \psi|$, we have an exact expression for the fidelity

$$F(\rho, \rho_1) = |\langle \psi | \mathcal{E}(|\psi\rangle \langle \psi|) | \psi\rangle| = \int d^2 \alpha \, \tilde{\Omega}_L(\alpha) |\Phi(\alpha)|^2, \tag{24}$$

which can be simply verified using Eq. \[4\].

We also know that the fidelity between two states $\rho_1$ and $\rho_2$ provides an upper bound on the trace distance, $D(\rho_1, \rho_2) \leq \sqrt{1 - F(\rho_1, \rho_2)^2}$, where $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} \left( |\rho_1 - \rho_2| \right)$ with $|A| \equiv \sqrt{A^\dagger A}$ \[32\]. Hence, our lower bound on the fidelity Eq. \[22\] can also be used to obtain an upper bound on the trace distance between filtered and unfiltered states,

$$D(\rho, \rho_1) \leq \sqrt{1 - F_c(\rho, \mathcal{E}_\Omega)^2}, \tag{25}$$

where, as discussed, the bound can be made arbitrarily small by choosing a sufficiently large value of the width parameter $L$.

VI. APPLICATIONS

Our physicality condition for filtering maps has interesting applications in estimating the output of a quantum process based on using coherent probe states, and also estimating the outcome probability distribution of a general quantum process using the nonclassicality filters given by Eqs. \[15\] and \[16\]. The great advantage here is that coherent states, as the probe states, are readily available from a laser source. However, for many quantum states, such as squeezed states or cat states, the GSP exists as a highly-singular distribution and hence using this expression for the output state is not useful, in general. To avoid this problem, it was shown that by having $\mathcal{E}(\langle \alpha | \langle \alpha |)$ one can construct the elements of the process tensor in the Fock basis $\mathcal{E}_{jk}^{mn} = \langle j | \mathcal{E}(|m\rangle \langle n|) | k\rangle$ which can be used to obtain the matrix elements of the output states over a finite-dimensional subspace of the Hilbert space for any given input state \[20\]. In this method, the truncation of the Hilbert space may introduce an error in the estimation of the output state.

Our formalism provides an alternative way for estimating the output state within an arbitrary accuracy by approximating the input state $\rho$ with a filtered state $\rho_\Omega$ with a regular GSP. For this purpose, one can use a filter function, such as the nonclassicality filter, whose Fourier transform is a probability density and its width can be adjusted by the parameter $L$. The corresponding output state is then given by

$$\mathcal{E}(\rho_\Omega) = \int d^2 \alpha \, P_\Omega(\alpha) \mathcal{E}(|\alpha\rangle \langle \alpha|). \tag{26}$$

In general, for any CPTP map the trace distance between the two output states is upper bounded by the trace distance between the corresponding input states $D(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \leq D(\rho_1, \rho_2)$ \[32\]. Therefore, by using Eqs. \[25\] and \[22\] we can see that for any $\delta > 0$ there exists an $L$ such that

$$D(\mathcal{E}(\rho), \mathcal{E}(\rho_1)) \leq \sqrt{1 - F_c(\rho, \mathcal{E}_\Omega)^2} \leq \delta. \tag{27}$$

As a consequence, the error associated with this approximation, i.e., using the filtered state $\rho_1$ instead of the actual input state $\rho$ can be made arbitrarily small by choosing an appropriate filter function.

This formalism, in particular provides a simple way for estimating the output states when the action of
the quantum process on all coherent states can be simply described. For example, considering a loss channel with transmissivity $\eta$, we have $\mathcal{E}_{\text{Loss}}(|\alpha\rangle\langle\alpha|) = |\eta\alpha\rangle\langle\eta\alpha|$. Hence any quantum state under a loss channel can be approximated by

$$\mathcal{E}_{\text{Loss}}(\rho) = \frac{1}{\eta^2} \int d^2\alpha P_{\Omega}(\alpha/\eta) |\alpha\rangle\langle\alpha|,$$  \hspace{1cm} (28)

where $P_{\Omega}(\alpha)$ is the regularized GSP of the input state $\rho$. One can similarly consider other examples such as squeezing and amplification channels whose action on the coherent states are known.

B. Estimating the outcome probabilities of measurements using heterodyne measurement

In general, a quantum measurement is described by a positive-operator valued measure (POVM) $\{\Pi_n\}$, where measurement operators satisfy $\Pi_n \geq 0$ and $\sum_n \Pi_n = \mathbb{I}$. Given a quantum system in the state $\rho$, the probabilities of measurement outcomes are given by the Born rule, $p(n|\rho) = \text{Tr}[\rho \Pi_n]$. If the state is unknown, however, one has to perform a set of informationally-complete measurements on an ensemble of identically prepared systems in order to estimate $\rho$, through a procedure known as quantum state tomography. In general, quantum state tomography requires many measurement settings that, in particular, for systems with infinite-dimensional Hilbert spaces is very challenging. A readily available, informationally-complete measurement is heterodyne whose POVM elements are proportional to coherent states $|\alpha\rangle\langle\alpha|/\pi$ and by which one directly samples from the Husimi $Q$-function of the state $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$. In principle, by using the Husimi $Q$-function, one can reconstruct the density matrix in the Fock basis by evaluating derivatives of $Q(\alpha)$ at the origin [36]. However, due to the finite sampling in the experiment, full characterization of the density matrix is not possible, and an approximate estimation of the actual state would only be available.

Another interesting application of our formalism is to estimate probabilities of outcomes of a quantum measurement, such as photon counting, for an unknown quantum state using heterodyne measurement. For a given measurement with a POVM $\{\Pi_n\}$, the outcome probabilities $p(n|\rho)$ can be approximated by

$$p(n|\rho) = \text{Tr}[\mathcal{E}_\Omega(\rho) \Pi_n] = \text{Tr}[\rho \mathcal{E}_\Omega^*(\Pi_n)]$$

$$= \pi \int d^2\alpha P_{\Omega}(n|\alpha) Q(\alpha),$$  \hspace{1cm} (29)

where the dual of the filtering map, $\mathcal{E}_{\Omega}^*$, is equal to $\mathcal{E}_{\Omega}$, assuming that $\Omega(-\alpha) = \Omega(\alpha)$. In the equation above, we have used Eq. (7) for the CPTP filtering map and $P_{\Omega}(n|\alpha)$ is the regularized GSP of the POVM element $\Pi_n$. For a given measurement, the trace distance between quantum states Eq. (25) is an upper bound on the trace distance between the corresponding probabilities [32]

$$\frac{1}{\pi} \sum |p(n|\rho) - p(n|\rho_\Omega)| \leq \sqrt{1 - F_\varepsilon(\rho, \mathcal{E}_\Omega)^2} \leq \delta. \hspace{1cm} (30)$$

Therefore, by adjusting the width parameter of the filter function, one can achieve a desired level of accuracy $\delta$ in the probability estimation.

VII. CONCLUSION

We have studied phase-space filter functions in the context of quantum maps on the space of operators in an infinite-dimensional Hilbert space associated with a bosonic system. We have shown that the necessary and sufficient condition for such a map to be a quantum process, meaning completely positive and trace-preserving, is positive semi-definiteness of the filter function. This condition guarantees that the output of the filtering map is always a physical density operator, and more importantly it can be implemented by displacements drawn randomly from a probability distribution that is the Fourier transform of the filter function. An example of such filter function is the nonclassicality filter, whose Fourier transform is always nonnegative by definition.

As an example, we studied Klauder’s filter function and showed that the associated filtering map does not meet the requirement derived in this work. Therefore, in practical applications, such as quantum process tomography, using this filter alone can lead to nonphysical states, requiring further steps to impose the physicality [19]. In addition, by applying our criterion to the Wigner function smoothing procedure, we identified a class of filter functions which are positive but not completely positive. This class of filtering maps can be useful in witnessing entanglement of quantum states [37]. We leave this as a subject for future research.

We have derived a lower bound on the fidelity between the states before and after filtering. As we have shown, this bound enables us to use physical filtering maps to approximate an operator with another operator with a regular Glauber-Sudarshan $P$-function, to an arbitrary precision tuned by the filter function characteristics. This, in turn, leads to interesting applications of the filtering maps in estimating the output state of an unknown quantum process by knowing its action on coherent states, or in estimating output probabilities of any measurement by measuring the given state in the coherent state basis.

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