Chiral two-loop pion-pion scattering parameters from crossing-symmetric constraints

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Constraints on the parameters in the one- and two-loop pion-pion scattering amplitudes of standard chiral perturbation theory are obtained from explicitly crossing-symmetric sum rules. These constraints are based on a matching of the chiral amplitudes and the physical amplitudes at the symmetry point of the Mandelstam plane. The integrals over absorptive parts appearing in the sum rules are decomposed into crossing-symmetric low- and high-energy components and the chiral parameters are finally related to high-energy absorptive parts. A first application uses a simple model of these absorptive parts. The sensitivity of the results to the choice of the energy separating high and low energies is examined with care. Weak dependence on this energy is obtained as long as it stays below ∼ 560 MeV. Reliable predictions are obtained for three two-loop parameters.
I. INTRODUCTION

A method for the determination of the parameters appearing in the pion-pion scattering amplitudes produced by chiral perturbation theory has been proposed in. We demonstrate its practicability in the present work. The method is based on sum rules derived from exact analyticity properties combined with crossing symmetry. A similar approach has been developed and applied in. Two features distinguish our method:

(1) Our constraints on the chiral parameters are obtained by matching the true amplitudes and their chiral approximations at the symmetry point \( s = t = u = 4M_\pi^2/3 \) of the \((s, t, u)-space\) \((s, t, u)\) are the standard Mandelstam variables). More precisely we equate coefficients in the Taylor expansions of both amplitudes at the symmetry point.

(2) The Taylor coefficients are expressed by means of crossing-symmetric sum rules as integrals over absorptive parts. These integrals are decomposed into exactly crossing-symmetric low- and high-energy components. If \( \Lambda \) is the energy separating low and high energies, we assume that \( \Lambda \) can be chosen in such a way that the absorptive parts can be approximated by chiral absorptive parts below \( \Lambda \) and that they are obtained from reliable experimental data above \( \Lambda \).

The last assumption does not fix \( \Lambda \) precisely, whereas the form of the conditions constraining the chiral parameters depends explicitly on \( \Lambda \). Our method is consistent only if these conditions lead to values of the parameters depending weakly on \( \Lambda \). This stability of the parameters with respect to a variation of \( \Lambda \) is one of our main concerns. To settle this point we work with a simple model of the true absorptive parts defined for all energies and at least qualitatively consistent with the data. The separating energy \( \Lambda \) can be pushed down to the elastic threshold \( 2M_\pi \). We have constructed the constraints for the parameters of the one- and two-loop pion-pion amplitudes of standard chiral perturbation theory for values of \( \Lambda^2 \) ranging from \( 4M_\pi^2 \) to \( 40M_\pi^2 \). Our assumptions turn out to be valid as long as \( 4M_\pi^2 < \Lambda^2 \lesssim 16M_\pi^2 \ (= (560 \text{ MeV})^2) \).

Before we continue we have to define our notation and the chiral parameters we are
using. The s-channel isospin \( I \) amplitude \( T^I(s, t, u) \) has the partial wave expansion

\[
T^I(s, t, u) = \sum_l (2l + 1) t^I_l(s) P_l \left( 1 + \frac{2t}{s - 4} \right), \quad I = 0, 1, 2. \quad (1.1)
\]

The pion mass has been set equal to 1, consequently \( s + t + u = 4 \). Below the inelastic threshold

\[
t^I_l(s) = \sqrt{\frac{s}{s - 4}} e^{i\delta^I_l(s)} \sin \delta^I_l(s), \quad (1.2)
\]

where the phase shift \( \delta^I_l(s) \) is real.

The three amplitudes \( T^I \) are obtained from a single function \( A(s; t, u) \):

\[
T^0(s, t, u) = 3A(s; t, u) + A(t; u, s) + A(u; s, t),
\]

\[
T^1(s, t, u) = A(t; u, s) - A(u; s, t),
\]

\[
T^2(s, t, u) = A(t; u, s) + A(u; s, t). \quad (1.3)
\]

In chiral perturbation theory \( A \) is the sum of a polynomial \( A^{\text{pol}} \) and an analytic function \( A^{\text{cut}} \). At 6th order in the momenta and the quark masses, each component of \( A \) can be written as

\[
A^{(6)} = \lambda^2 A_2 + \lambda^4 A_4 + \lambda^6 A_6, \quad (1.4)
\]

where \( \lambda = M_\pi / F_\pi \). In standard chiral perturbation theory we have

\[
A^{\text{pol}}_2(s) = \frac{1}{32\pi} (s - 1),
\]

\[
A^{\text{pol}}_4(s; t, u) = \frac{1}{32\pi} \left[ b_1 + b_2 s + b_3 s^2 + b_4 (t - u)^2 \right],
\]

\[
A^{\text{pol}}_6(s; t, u) = \frac{1}{32\pi} \left[ b_5 s^3 + b_6 (t - u)^3 \right]. \quad (1.5)
\]

The coefficients \( b_i \) are the parameters we want to determine. Their relationship to the coupling constants in the 4th- and 6th-order effective chiral Lagrangian is given in. We do not use these relations here, remarking only that \( b_1, b_2, b_3 \) and \( b_4 \) are sums of 4th- and 6th-order terms whereas \( b_5 \) and \( b_6 \) are 6th-order parameters.

The analytic part of \( A^{(6)} \) has the form
\[ A_{\text{cut}}^{(6)}(s; t, u) = \frac{1}{32\pi} \sum_{\alpha=1}^{5} \{ R_\alpha(s)f_\alpha(s) + S_\alpha(t)f_\alpha(t) + S_\alpha(u)f_\alpha(u) \] 

\[ + s [T_\alpha(t)f_\alpha(t) + T_\alpha(u)f_\alpha(u)] \} \]  

(1.6)

where \( R_\alpha(s), S_\alpha(s) \) and \( T_\alpha(s) \) are polynomials containing 4th- and 6th-order terms and \( f_\alpha(s) \) are analytic functions with a right-hand cut \([4, \infty)\). The polynomials are obtained from formulæ (2) and (3) in 2, the functions used there have been relabelled \( f_1(s) = \bar{J}(s) \), \( f_\alpha(s) = K_{\alpha-1}(s), \alpha = 2, \ldots, 5 \). The coefficients of the polynomials in (1.6) depend linearly on \( b_i \).

Our basic sum rules are obtained from subtracted dispersion relations and contain unknown subtraction constants. For this reason the number of useful constraints is smaller than the number of parameters, two constraints at 4th order and four constraints at 6th order. They are used for the determination of \( b_3^{(4)}, b_4^{(4)}, b_3^{(6)}, b_4^{(6)}, b_5^{(6)} \) and \( b_6^{(6)} \) at fixed values of \( b_1^{(4)} \) and \( b_2^{(4)} \). Four parameters fulfil the criterion of near \( \Lambda \)-independence: \( b_4^{(4)}, b_3^{(6)}, b_4^{(6)} \) and \( b_6^{(6)} \). In spite of the fact that our input modelling the experimental data is rather crude, the values we get for the stable parameters are entirely compatible with the results derived in 4.

The construction of our constraints is outlined in the next two Sections. The inputs are defined in Section IV and the chiral parameters are evaluated in Section V. The results are discussed in Section VI.

II. SUM RULES FOR TAYLOR COEFFICIENTS AT THE SYMMETRY POINT

The full amplitudes \( T^I(s, t, u) \) as well as the chiral amplitudes \( T^I_\chi(s, t, u) \) are real and regular in the Mandelstam triangle \( 0 < s < 4, 0 < t < 4, 0 < u < 4 \). This implies that their Taylor expansions at the symmetry point \( s = t = u = 4/3 \) converge in a neighbourhood of that point. We assume that the \( 2n \)-th order chiral amplitudes approximate the full amplitudes in that neighbourhood up to \( O \left( p^{2n+1} \right) \) corrections. If this assumption is correct
we can fix the parameters appearing in the chiral amplitudes by requiring that $T^I$ and $T^I_\chi$ have identical conveniently truncated Taylor expansions at the symmetry point. We adopt this strategy and explore its implications.

To write down Taylor expansions for the $T^I$, $s$, $t$ and $u$ have to be replaced by two independent variables. Generally the series one obtains are constrained by crossing symmetry. It is advisable to resort to a procedure which ensures that these constraints are automatically fulfilled. To this end the three amplitudes $T^I$ are replaced by three totally symmetric amplitudes $G_i$ ($i = 0, 1, 2$) expressed in terms of two new variables $x$ and $y$ which are totally symmetric and homogeneous in $s$, $t$ and $u$:

$$x = -\frac{1}{16}(st + tu + us), \quad y = \frac{1}{64}stu.$$  \hspace{1cm} (2.1)

The $G_i$ have been introduced by Roskies and are defined as follows:

$$G_0(x, y) = \frac{1}{3} \left[ T^0(s, t, u) + T^0(u, t, s) + T^0(u, s, t) \right]$$

$$G_1(x, y) = \frac{T^1(s, t, u)}{t - u} + \frac{T^1(t, u, s)}{u - s} + \frac{T^1(u, s, t)}{s - t}$$  \hspace{1cm} (2.2)

$$G_2(x, y) = \frac{1}{s - t} \left[ \frac{T^1(s, t, u)}{t - u} - \frac{T^1(t, s, u)}{s - u} \right] + \frac{1}{t - u} \left[ \frac{T^1(t, u, s)}{u - s} - \frac{T^1(u, t, s)}{t - s} \right]$$

$$+ \frac{1}{u - s} \left[ \frac{T^1(u, s, t)}{s - t} - \frac{T^1(s, u, t)}{u - t} \right].$$

The $T^I$ can be obtained from the $G_i$ (see eq. (2) in [1]); their symmetry implies crossing symmetry of the $T^I$.

The $G_i$ are regular around the image $(x_s, y_s)$ of the symmetry point in the $(x, y)$-space ($x_s = -1/3$, $y_s = 1/27$) and have convergent Taylor expansions in $x$ and $y$ at that point. They are not constrained by crossing symmetry. We equate their first coefficients with the corresponding Taylor coefficients of the symmetric chiral amplitudes $G^\chi_i$, the latter being given by (2.2) with $T^I$ replaced by $T^I_\chi$.

The Taylor expansions of the amplitudes $G_i^{(2n)}$ obtained from the $O(p^{2n})$ amplitudes $T^I_{(2n)}$ are truncated at an order obtained by power counting from the definitions (2.2). When identifying truncated Taylor series at a given order one has to remember that a chiral
Taylor coefficient gets contributions from all higher orders of the chiral expansion. Therefore the conditions one obtains for the $2n$-th order amplitudes hold only up to higher order corrections.

For the fourth- and sixth-order amplitudes we get the following constraints:

\begin{align}
    G_i^{(4)}(x_s, y_s) + O(p^6) &= G_i(x_s, y_s), \quad i = 0, 1, 2, \quad (2.3a) \\
    \frac{\partial G_i^{(4)}}{\partial x}(x_s, y_s) + O(p^6) &= \frac{\partial G_0}{\partial x}(x_s, y_s), \quad (2.3b) \\
    G_i^{(6)}(x_s, y_s) + O(p^8) &= G_i(x_s, y_s), \quad i = 0, 1, 2, \quad (2.4a) \\
    \frac{\partial G_i^{(6)}}{\partial x}(x_s, y_s) + O(p^8) &= \frac{\partial G_i}{\partial x}(x_s, y_s), \quad i = 0, 1, \quad (2.4b) \\
    \frac{\partial G_0^{(6)}}{\partial y}(x_s, y_s) + O(p^8) &= \frac{\partial G_0}{\partial y}(x_s, y_s). \quad (2.4c)
\end{align}

The number of constraints obtained in this way at $2n$-th order is equal to the number of free parameters appearing in the $2n$-th order chiral amplitudes (4 parameters and 4 equations (2.3) at 4th order, 6 parameters and 6 equations (2.4) at 6th order). An obvious drawback of conditions (2.3) and (2.4) is that, the symmetry point being unphysical, the right-hand sides are not directly measurable quantities. Fortunately, there are dispersion relations to get rid of this difficulty. These relations are consequences of the exact analyticity properties of the $G_i$ as functions of $x$ and $y$. They can be written in the following way:

\begin{equation}
    G_i(x, y) = (1 - \delta_{i2})G_i(x_0, y_0) + \frac{1}{\pi} \int_{4}^{\infty} d\sigma \left[ \frac{1}{K(\sigma, x, y)} - \frac{1 - \delta_{i2}}{K(\sigma, x_0, y_0)} \right] B_i(\sigma, \tau). \quad (2.5)
\end{equation}

There are once-subtracted dispersion relations for $G_0$ and $G_1$, the subtraction being performed at $(x_0, y_0)$, whereas $G_2$ obeys an unsubtracted dispersion relation. The integration variable $\sigma$ is an energy squared. The denominator function $K(\sigma, x, y)$ has a simple expression in terms of the Mandelstam variables:

\begin{equation}
    K(\sigma, x, y) = (\sigma - s)(\sigma - t)(\sigma - u) = \sigma^2(\sigma - 4) - 16\sigma x - 64y, \quad (2.6)
\end{equation}

if $(s, t, u)$ is a pre-image of $(x, y)$. The second expression results from the first one and the definitions (2.1). The $B_i$ are linear combinations of $s$-channel absorptive parts $A_i$: 6
\[ B_0(s,t) = \frac{1}{3} (s-t)(2s-4+t) \left[ A^0(s,t) + 2A^2(s,t) \right], \]
\[ B_1(s,t) = \frac{1}{6} (3s-4) \left[ 2A^0(s,t) - 5A^2(s,t) \right] + \left[ \frac{(s-t)(2s-4+t)}{2t-4+s} - \frac{1}{2}(2t-4+s) \right] A^1(s,t) \]
\[ B_2(s,t) = -\frac{1}{2} \left[ 2A^0(s,t) - 5A^2(s,t) \right] + \frac{3}{2} \frac{3s-4}{2t-4+s} A^1(s,t). \] (2.7)

They have to be evaluated in (2.5) at a \( \sigma \)-dependent squared momentum transfer \( \tau \):
\[ \tau(\sigma) = -\frac{1}{2} \left\{ (\sigma-4) - \left[ (\sigma-4)^2 - \frac{16}{\sigma+4a}(a\sigma-4) - 16(ax_0-y_0) \right] \right\}^{\frac{1}{2}} \] (2.8)
where \( a \) is the slope of the straight line connecting the point \((x,y)\) to the subtraction point \((x_0,y_0)\):
\[ a = \frac{y-y_0}{x-x_0}. \] (2.9)

Details on the derivation of (2.5) are given in\[.\] At fixed \((x_0,y_0)\), (2.5) holds true if the slope \(a\) belongs to a complex neighbourhood of the origin. For our purpose \((x_0,y_0)\) has to be chosen in such a way that (2.5) provides a representation of \(G_i(x,y)\) in a suitable neighbourhood of the symmetry point \((x_s, y_s)\). According to\[.\] this is the case if \(y_0 = y_s\) and \(-72 < x_0 < 3x_s/2\).

If \(\sigma\) is large enough, \(\tau(\sigma)\) is real and the point \((s=\sigma, t=\tau(\sigma))\) belongs to the physical \(s\)-channel. If \(\sigma\) is close to 4, \(\tau(\sigma)\) can be complex but the point \((s=\sigma, t=\tau(\sigma))\) is inside the large Lehmann ellipse. This implies that the absorptive part \(A^I(\sigma, \tau)\) is either a physical quantity or is obtained from physical partial wave absorptive parts through a convergent partial waves expansion. Consequently the dispersion integral in (2.5) is itself a physical quantity and this relation produces a representation of \(G_2(x,y)\) in terms of measurable absorptive parts. This does not hold for \(G_0(x,y)\) and \(G_1(x,y)\) because their values at the subtraction point which enter into (2.5) are unphysical with our choice of \((x_0,y_0)\). However, by computing derivatives at fixed \((x_0,y_0)\), (2.5) gives expressions for \(\partial G_i/\partial x\) and \(\partial G_i/\partial y\) at \((x_s, y_s)\) which do not involve \(G_i(x_0, y_0)\). Therefore the dispersion relations provide expressions in terms of observable absorptive parts for the right-hand sides of eqs (2.3a, i=2), (2.3b), (2.4a, i=2), (2.4b) and (2.4c).
If one wants to keep eqs (2.3a) and (2.3b) for \(i = 0, 1\), one can use a dispersion relation connecting \(G_i(x_0, y_0)\) to the value of \(G_i\) at the elastic threshold \(x = y = 0\) (image of \(s = 4, t = u = 0\)). This value is determined by the S-wave scattering lengths, and one obtains expressions for \(G_i(x_s, y_s)\), \(i = 0, 1\), in terms of physical quantities. However, as the experimental scattering lengths are poorly known at present, this would not lead to stringent constraints on the chiral parameters. Therefore we restrict our discussion to the constraints involving only dispersion integrals over absorptive parts.

We end this Section with the expressions of the derivatives \(\partial G_i/\partial x\), \(i = 0, 1\), and \(\partial G_0/\partial y\) resulting from (2.5):

\[
\begin{align*}
\frac{\partial G_i}{\partial x}(x_s, y_s) &= \frac{32}{\pi} \int_4^\infty d\sigma \frac{\sigma^2(\sigma - 2) + 32y_s}{K(\sigma, x_s, y_s)^2} \frac{1}{(\sigma - \tau)(2\sigma - 4 + \tau)} B_i(\sigma, \tau), \quad i = 0, 1, \\
\frac{\partial G_0}{\partial y}(x_s, y_s) &= \frac{64}{\pi} \int_4^\infty d\sigma \frac{1}{K(\sigma, x_s, y_s)} \left\{ \frac{\sigma(3\sigma - 8) - 16x_s}{K(\sigma, x_s, y_s)} \frac{1}{3}(A^0 + 2A^2)(\sigma, \tau) \right. \\
&\quad - \left. \frac{2(\sigma^2(\sigma - 2) + 64y_s)}{\sigma^{3/2}[\sigma(4 - \sigma)^2 - 256y_s]^{1/2}} \frac{1}{3} \left( \frac{\partial}{\partial t}(A^0 + 2A^2) \right)(\sigma, \tau) \right\}.
\end{align*}
\]

In these formulae, \(\tau\) is given by (2.8) at \(a = 0\). As a consequence, the integrands do not depend on \(x_0\), the position of the subtraction point on the line \(y = y_s\). This is an advantage of our choice \(y_0 = y_s\).

**III. LOW- AND HIGH-ENERGY COMPONENTS**

Our constraints on the chiral pion-pion parameters involve integrals over absorptive parts. We have good experimental information in the range \(600\ \text{MeV} < \sqrt{s} < 2\ \text{GeV}\) but large uncertainties below this interval. As we cannot rely on experimental data in this low energy region we need some theoretical input. One way is to exploit the fact that the chiral amplitudes are meant to describe low energy scattering correctly. Therefore the low energy chiral absorptive parts can be used as approximations of the full absorptive parts. This idea was proposed in (3) and we apply it here. A more refined procedure is applied in (4) where the absorptive parts are computed from unitarized chiral S- and P-wave amplitudes.
In a first step the dispersion integral in (2.5) is split into a low energy integral $L_i$, from $\sigma = 4$ to $\sigma = \Lambda^2$ and a high energy integral, $\sigma > \Lambda^2$, $H_i$. The representation (2.5) becomes

$$G_i(x, y) = (1 - \delta_{i2})G_i(x_0, y_0) + L_i(x, y) + H_i(x, y). \tag{3.1}$$

This decomposition induces a crossing-symmetric decomposition of the amplitudes $T^I$ into low- and high-energy components.

We assume now that $\Lambda^2$ can be chosen in such a way that the $2n$-th order chiral absorptive parts $A_i^I(\sigma, \tau)$ approximate the full absorptive parts for $4 \leq \sigma \leq \Lambda^2$ up to $O(p^{2(n+1)})$ corrections. Furthermore we have already assumed in Section II that $G_i$ is approximated by $G_i^\chi$ in the neighbourhood of the symmetry point. Thus, if $(x, y)$ is close to $(x_s, y_s)$, our assumptions allow us to rewrite eq. (3.1) as follows:

$$G_i^\chi(x, y) = (1 - \delta_{i2})G_i(x_0, y_0) + L_i^\chi(x, y) + H_i(x, y) + O(p^{2(n+1)}). \tag{3.2}$$

$H_i$ is given by the integral in (2.5) extended over $\sigma > \Lambda^2$; $L_i^\chi$ is given by the same integral restricted to $4 \leq \sigma \leq \Lambda^2$ and with $B_i$ replaced by $B_i^\chi$.

A key observation is that $G_i^\chi$ and $L_i^\chi$ have the same discontinuities across their cuts for $4 < \sigma < \Lambda^2$. We show in the Appendix that, up to 6th order, this implies that the difference $(G_i^\chi - L_i^\chi)$ has the following form:

$$G_i^\chi(x, y) - L_i^\chi(x, y) = P_i(x, y) + H_i^\chi(x, y). \tag{3.3}$$

$P_i(x, y)$ is a polynomial of first degree and $H_i^\chi$ is a high-energy component given by a dispersion integral starting at $\sigma = \Lambda^2$. Equation (3.2) becomes

$$P_i(x, y) + H_i^\chi(x, y) = (1 - \delta_{i2})G_i(x_0, y_0) + H_i(x, y) + O(p^{2(n+1)}). \tag{3.4}$$

The left-hand side is entirely determined by the chiral amplitudes whereas the right-hand side is determined, apart from $G_i(x_0, y_0)$, by absorptive parts above $\Lambda^2$.

Those equations (2.3) and (2.4) which do not involve $G_i(x_0, y_0)$ are now replaced by
\[(P_2 + H_2^X)^{(4)}(x_s, y_s) = H_2(x_s, y_s) + O(p^6), \quad (3.5)\]
\[\frac{\partial}{\partial x}(P_0 + H_0^X)^{(4)}(x_s, y_s) = \frac{\partial H_0}{\partial x}(x_s, y_s) + O(p^6).\]
\[(P_2 + H_2^X)^{(6)}(x_s, y_s) = H_2(x_s, y_s) + O(p^8), \quad (3.6)\]
\[\frac{\partial}{\partial x}(P_i + H_i^X)^{(6)}(x_s, y_s) = \frac{\partial H_i}{\partial x}(x_s, y_s) + O(p^8), \quad i = 0, 1, \]
\[\frac{\partial}{\partial y}(P_0 + H_0^Y)^{(6)}(x_s, y_s) = \frac{\partial H_0}{\partial y}(x_s, y_s) + O(p^8).\]

The two equations (3.5) constrain the four 4th-order parameters and the four equations (3.6) constrain the six 6th-order parameters. These equations clearly reduce to eqs (2.3) and (2.4) if \(\Lambda^2 = 4\). Both sides of the equations depend on \(\Lambda^2\). However, the restrictions on the chiral parameters they imply should not depend on \(\Lambda^2\), up to higher-order corrections, as long as \(\Lambda^2\) is in an interval where our assumption on the absorptive parts is valid. Thus we see that eqs (3.5) and (3.6) provide at the same time a tool for the determination of the chiral parameters and a verification of the validity of the chiral expansion. We elaborate on these two aspects in the following Sections.

**IV. INPUTS FOR THE PARAMETER CONSTRAINTS**

After these lengthy preliminaries we are ready for a feasibility test of our programme. To this end we need an Ansatz for the pion-pion absorptive parts and explicit forms for the chiral pion-pion amplitudes. We start with a definition of an Ansatz modelling the absorptive parts down to the elastic threshold.

For \(4 < s < 51\) (280 MeV < \(\sqrt{s}\) < 1 GeV) we retain only S- and P-wave contributions. A parametrisation proposed by Schenk\(^7\) is used for the corresponding phase shifts:

\[\tan \delta^I_l(s) = \left(\frac{1}{4}(s - 4)\right)^l \left(\frac{s - 4}{s}\right)^{\frac{1}{2}} \left[\frac{4 - s_I}{s - s_I}\right] \left[a^I_l - \frac{1}{4}(s - 4)\tilde{b}^I_l\right], \quad (4.1)\]

where \(l = 0\) for \(I = 0\), 2 and \(l = 1\) for \(I = 1\), \(a^I_l\) is a scattering length and

\[\tilde{b}^I_l = b^I_l - \frac{4a^I_l}{s_I - 4} + (a^I_l)^3, \quad (4.2)\]
where \( b_I \) is an effective range. In (4.1), \( s_1 = M^2_\rho = 30.4 \) and we adopt one of Schenk’s choices for the remaining poles: \( s_0 = 38 = (865 \text{ MeV})^2, \ s_2 = -43 = -(920 \text{ MeV})^2 \). Scattering lengths and effective ranges are chosen in accordance with the results of 4th-order standard chiral perturbation theory:

\[
a_0^0 = 0.20, \quad a_0^2 = -0.042, \quad a_1^1 = 0.037, \\
b_0^0 = 0.24, \quad b_0^2 = -0.075, \quad b_1^1 = 0.005. \tag{4.3}
\]

For \( s > 51 \) we adopt an Ansatz which has been used recently in. In the interval \( 51 < s < 110 \) (1 GeV < \( \sqrt{s} < 1.5 \text{ GeV} \)) we take only the \( f_0 \) contribution

\[
A_{f_0}^0(s,t) = 5\pi \Gamma_{f_0} M_{f_0} \sqrt{M_{f_0}^2 - 4} P_2 \left( 1 + \frac{2t}{M_{f_0}^2 - 4} \right) \delta(s - M_{f_0}^2), \tag{4.4}
\]

with \( \Gamma_{f_0} = 0.896 \) (= 125 MeV), \( M_{f_0} = 9.092 \) (= 1269 MeV).

The high-energy region \( s > 110 \) is described in terms of Pomeron exchange and a degenerate \((\rho + f_0)\) Regge trajectory. If \( A^{(I)}(s,t) \) designates the isospin \( I \) \( t \)-channel absorptive part, the Pomeron exchange is specified by

\[
A_{\text{Pom}}^{(0)}(s,t) = \frac{3x_0}{2\pi M_{\pi}^2} \frac{1}{e^b t} \left( \frac{s}{x_0} \right)^{1+\alpha'_{\rho,t}}, \tag{4.5}
\]

with \( x_0 = 10 \text{ GeV}^2, \ b = 10 \text{ GeV}^{-2}, \ \alpha'_{\rho} = 0.4 \text{ GeV}^{-2}, \ A_{\text{Pom}}^{(1)} = A_{\text{Pom}}^{(2)} = 0 \). The \((\rho + f_0)\) exchange is represented by the following absorptive parts:

\[
A_{\rho,f_0}^{(0)}(s,t) = A_{\rho,f_0}^{(1)}(s,t) = 0.6 \sin(\pi \alpha(t)) \Gamma(1 - \alpha(t)) \left( \frac{s}{2M_{\rho}^2} \right)^{\alpha(t)}, \tag{4.6}
\]

where \( \alpha(t) = 1/2 + t/(2M_{\rho}^2) \) and \( A_{\rho,f_0}^{(2)} = 0 \).

The question we are now asking is whether chiral perturbation theory can produce amplitudes which are consistent with the absorptive parts defined in (4.1), (4.4), (4.5) and (4.6) in the sense that they satisfy the conditions (3.5) and (3.6) in a suitable range of values of \( \Lambda^2 \).

According to eqs (1.4-1.6) and (2.2) the totally symmetric one- and two-loop chiral amplitudes entering into our constraints are of the form
\[ G_0^\chi(s, t, u) = Q_0(s, t, u) + \sum_{\alpha=1}^{5} \left[ U_\alpha(s)f_\alpha(s) + U_\alpha(t)f_\alpha(t) + U_\alpha(u)f_\alpha(u) \right], \]
\[ G_1^\chi(s, t, u) = Q_1(s, t, u) + \sum_{\alpha=1}^{5} \left\{ 3 \left[ T_\alpha(s)f_\alpha(s) + T_\alpha(t)f_\alpha(t) + T_\alpha(u)f_\alpha(u) \right] \right. \]
\[ \left. + \left[ \frac{1}{s - t} (W_\alpha(s)f_\alpha(s) - W_\alpha(t)f_\alpha(t)) + \text{permutations} \right] \right\}, \quad (4.7) \]
\[ G_2^\chi(s, t, u) = Q_2 + \sum_{\alpha=1}^{5} \left\{ \frac{1}{s - t} \left[ \frac{1}{t - u} (W_\alpha(t)f_\alpha(t) - W_\alpha(u)f_\alpha(u)) \right. \right. \]
\[ \left. \left. - \frac{1}{u - s} (W_\alpha(u)f_\alpha(u) - W_\alpha(s)f_\alpha(s)) \right] + \text{permutations} \right\}. \]

In these expressions \( Q_0 \) and \( Q_1 \) are polynomials, \( Q_2 \) is a constant, \( U_\alpha, T_\alpha \) and \( W_\alpha \) are polynomials obtained from those in (1.6) whereas the \( f_\alpha \) are the analytic functions appearing there. In standard chiral perturbation theory the \( Q_i, U_\alpha, T_\alpha \) and \( W_\alpha \) depend linearly on the parameters \( b_1, \ldots, b_6 \).

V. EVALUATING 4TH- AND 6TH-ORDER PARAMETERS

The ingredients collected in the last Section allow us to compute the left- and right-hand sides of eqs (3.5) and (3.6). The 4th-order equations (3.5) involve only two parameters, \( b_3 \) and \( b_4 \):

\[ \frac{3}{32 \pi^4} \lambda^4 \left[ -b_3 + 3b_4 + C_1(\Lambda^2) \right]^{(4)} = H_2(\Lambda^2) + O(p^6), \]
\[ \frac{1}{\pi} \lambda^4 \left[ b_3 + 3b_4 + C_2(\Lambda^2) \right]^{(4)} = \frac{\partial H_0}{\partial x}(\Lambda^2) + O(p^6). \]  

The \( \Lambda^2 \) dependencies are indicated explicitly and every reference to the symmetry point is dropped. For instance \( H_2(\Lambda^2) \) is the \( \Lambda^2 \)-dependent value of \( H_2(x_s, y_s) \) and \( C_1(\Lambda^2) \) is the term of \((P_2 + H_2^\chi)^{(4)}(x_s, y_s)\) which is does not depend on the \( b_i \).

For the time being it is convenient to work with the combinations \((-b_3+3b_4)\) and \((b_3+3b_4)\) rather than \( b_3 \) and \( b_4 \). Figure 1 displays the values obtained form eqs (5.1) ignoring the \( O(p^6) \) corrections. The variations of \( H_2 \) and \( \partial_x H_0 \) as functions of \( \Lambda^2 \) are compensated to a large extent by the \( \Lambda^2 \)-dependence of \( C_1 \) and \( C_2 \), but a sizeable \( \Lambda^2 \)-dependence remains. One may
ask if it could be compensated by $\Lambda^2$-dependent $O(p^6)$ corrections. It is delicate to guess the order of magnitude of such corrections but we do not have to speculate at this point: as we shall see, the $O(p^6)$ corrections we are looking for are obtained from the 6th-order equations (3.6).

Two of the equations (3.6) are corrected versions of the 4th-order eqs (5.1). They have the form:

$$\lambda^4 \left(-b_3 + 3b_4 + C_1(\Lambda^2)\right)^{(6)} + \lambda^6 \left[-4(b_5 - b_6) + g_1(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)}\right] = \frac{32\pi}{3} H_2(\Lambda^2) + O(p^8),$$

(5.2)

$$\lambda^4 \left(b_3 + 3b_4 + C_2(\Lambda^2)\right)^{(6)} + \lambda^6 \left[6b_5 - 3b_6 + g_2(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)}\right] = \pi \frac{\partial H_0}{\partial x}(\Lambda^2) + O(p^8).$$

In these equations

$$b_i^{(6)} = b_i^{(4)} + \lambda^2 \delta b_i, \quad i = 3, 4, \quad C_k^{(6)} = C_k^{(4)} + \lambda^2 \delta C_k, \quad k = 1, 2,$$

(5.3)

where $\delta b_i$ and $\delta C_k$ are 6th-order contributions and $\delta C_k$ is known.

In addition to (5.3) we have two true 6th-order equations:

$$\lambda^4 C_3^{(6)}(\Lambda^2) + \lambda^6 \left[b_5 - 3b_6 + g_3(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)}\right] = \pi \frac{\partial H_0}{\partial y}(\Lambda^2) + O(p^8),$$

(5.4)

$$\lambda^4 C_4^{(6)}(\Lambda^2) + \lambda^6 \left[3(b_5 + b_6) + g_4(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)}\right] = 2\pi \frac{\partial H_1}{\partial x}(\Lambda^2) + O(p^8).$$

The $g_k$ in (5.2) and (5.4) are known linear and homogeneous combinations of $b_1, b_2, b_3$ and $b_4$ with $\Lambda^2$-dependent coefficients. Fourth-order values have to be inserted.

The constants $C_3$ and $C_4$ are of the form indicated in (5.3): they are sums of 4th-order terms and 6th-order corrections. We meet here a peculiarity of our programme. Although equations (5.4) do not appear at 4th-order level because they come from $O(p^6)$ terms in the expansions of the $T_I^f \chi$ at the symmetry point, genuine 4th-order terms enter into these equations. This happens because each coefficient in the Taylor series of $H_I^\chi$ has an expansion starting with a 4th-order term, in contrast with $\partial P_0/\partial y$ and $\partial P_1/\partial x$ which contain no such terms. In practice, the presence of 4th-order contributions in eqs (5.4) is harmless because

13
these contributions are small, especially for large enough \( \Lambda^2 \), so that they have effectively the same order of magnitude as the true 6th-order terms.

Eliminating the two 6th-order parameters \( b_5 \) and \( b_6 \) from eqs (5.2) and (5.4) gives two equations relating \( b_1, b_2, b_3 \) and \( b_4 \):

\[
\lambda^4 \left( -b_3 + 3b_4 + C_5(\Lambda^2) \right)^{(6)} + \lambda^6 g_5(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)} = \frac{\pi}{3} \left[ 32H_2(\Lambda^2) + \frac{\partial H_0}{\partial y}(\Lambda^2) + 12 \frac{\partial H_1}{\partial x}(\Lambda^2) \right] + O(p^8),
\]

\[
\lambda^4 \left( b_3 + 3b_4 + C_6(\Lambda^2) \right)^{(6)} + \lambda^6 g_6(b_1, b_2, b_3, b_4, \Lambda^2)^{(4)} = \frac{\pi}{3} \left[ 3 \frac{\partial H_0}{\partial x}(\Lambda^2) - \frac{\partial H_0}{\partial y}(\Lambda^2) - 12 \frac{\partial H_1}{\partial x}(\Lambda^2) \right] + O(p^8).
\]

As the 4th-order constraints (5.1) do not provide well defined values for \( b_3^{(4)} \) and \( b_4^{(4)} \) we insert \( b_3^{(6)} \) and \( b_4^{(6)} \) into \( g_5 \) and \( g_6 \). This amounts to a redefinition of the \( O(p^8) \) corrections. We retain \( b_1^{(4)} \) and \( b_2^{(4)} \) and drop the indices labelling orders in the chiral expansion from now on. To illustrate the quantitative content of eqs (5.5), we write them explicitly for \( \Lambda^2 = 10 \), without mention of the \( O(p^8) \) corrections:

\[
- (1 - [0.14])b_3 + (3 + [0.24])b_4 + 4.17 \cdot 10^{-4} - [6.12 \cdot 10^{-3}]b_1 - [4.26 \cdot 10^{-3}]b_2 + [4.78 \cdot 10^{-4}] = 1.81 \cdot 10^{-2}
\]

\[
(1 - [0.12])b_3 + (3 + [0.10])b_4 - 3.18 \cdot 10^{-3} + [1.39 \cdot 10^{-3}]b_1 - [1.29 \cdot 10^{-2}]b_2 - [4.56 \cdot 10^{-4}] = 7.28 \cdot 10^{-3}.
\]

Both sides of eqs (5.5) have been divided by \( \lambda^4 \). The square brackets indicate two-loop terms: numbers without square brackets are 4th-order terms.

For \( \Lambda^2 = 10 \), the 4th-order version of eqs (5.6), i.e. eqs (5.1), is

\[
- b_3 + 3b_4 - 1.52 \cdot 10^{-3} = 1.54 \cdot 10^{-2}
\]

\[
b_3 + 3b_4 - 6.50 \cdot 10^{-4} = 1.15 \cdot 10^{-2}.
\]
These equations are not identical to eqs (5.6) with the square brackets dropped. This is due to the 4th-order contributions of $\partial H_0^x/\partial y$ and $\partial H_1^x/\partial x$, mentioned previously, in eqs (5.6).

Equations (5.5) are treated as determining $b_3$ and $b_4$ in terms of $b_1$ and $b_2$, using the values of these parameters obtained from 1-loop standard perturbation theory:

$$b_1 = \frac{1}{96\pi^2} \left[ 8\bar{l}_1 - 3\bar{l}_3 - 12\bar{l}_4 + \frac{13}{3} \right]$$  \hspace{1cm} (5.8)

$$b_2 = \frac{1}{24\pi^2} \left[ -2\bar{l}_1 + 3\bar{l}_4 - \frac{1}{3} \right].$$

With the central values $\bar{l}_1 = -1.7, \bar{l}_3 = 2.9, \bar{l}_4 = 4.6$ we get

$$b_1 = -7.7 \cdot 10^{-2}, \quad b_2 = 7.1 \cdot 10^{-2}. \hspace{1cm} (5.9)$$

Inserting these values into the $\Lambda^2 = 10$ equations (5.6) gives

$$(-b_3 + 3b_4)^{(6)} = 1.63 \cdot 10^{-2}, \quad (b_3 + 3b_4)^{(6)} = 1.12 \cdot 10^{-2}. \hspace{1cm} (5.10)$$

The corresponding 4th-order values are $1.70 \cdot 10^{-2}$ and $1.21 \cdot 10^{-2}$, quite close to the 6th-order values (5.10). The variation of $(\mp b_3 + 3b_4)$ at 4th and 6th order as functions of $\Lambda^2$ obtained from eqs (5.1) and (5.5) is shown in Fig. 1. The striking feature is that $(\mp b_3 + 3b_4)^{(6)}$ is independent of $\Lambda^2$ within 4% for $4 < \Lambda^2 < 16$. On the other hand, the difference between the 6th- and 4th-order values can be substantial, especially for $(b_3 + 3b_4)$ and $\Lambda^2$ between 4 and 10.

The values of $b_3$ and $b_4$ as obtained from $(-b_3 + 3b_4)$ and $(b_3 + 3b_4)$ are displayed in Fig. 2: $b_4^{(4)}$ and $b_4^{(6)}$ are practically constant, their difference due to a 6th-order correction is of the order of 4%. The parameter $b_3$ is not as robust as $b_4$. The way it is obtained amplifies the uncertainties. At 4th order, they must have the same magnitude as the computed values. The situation improves at 6th order and $b_3$ becomes independent of $\Lambda^2$ between 4 and 16 within 6%.

Ignoring all uncertainties in the input quantities $H_i$ and taking the variations with $\Lambda^2$ for $4 < \Lambda^2 < 16$ into account leads to the following 6th-order values of $b_3$ and $b_4$:
\[
b_3 = (-2.55 \pm 0.20) \cdot 10^{-3}, \quad b_4 = (4.55 \pm 0.15) \cdot 10^{-3}.
\] (5.11)

The only clear 4th-order result is \(b_4^{(4)} = (4.85 \pm 0.15) \cdot 10^{-3}\).

Once \(b_3\) and \(b_4\) are known, eqs (5.4) determine \(b_5\) and \(b_6\). With (5.9) and the central values in (5.11), this leads to \(b_6\) depending weakly on \(\Lambda^2\), as shown in Fig. 3. In contrast, \(b_5\) exhibits a strong \(\Lambda^2\)-dependence: it decreases from \(1.4 \cdot 10^{-4}\) to \(0.66 \cdot 10^{-4}\) as \(\Lambda^2\) varies between 4 and 16. Comparing Fig. 2 and Fig. 3, we observe that \(b_4^{(4)}\) and \(b_6^{(6)}\) behave similarly, and so do \(b_3^{(4)}\) and \(b_5^{(6)}\). This is possibly due to the similar roles which the pairs \((b_3, b_4)\) and \((b_5, b_6)\) play in \(A_{\text{pol}}(s; t, u)\) (eq. (1.5)). Furthermore, the 4th-order absorptive parts do not depend on \((b_3, b_4)\) and the 6th-order absorptive parts do not depend on \((b_5, b_6)\). One may expect that \(b_5\) would become stable at 8th order in the same way as \(b_3\) is stabilized at 6th order.

Ignoring again the uncertainties in the \(H_i\), Fig. 3 gives

\[
b_6 = (9.2 \pm 0.3) \cdot 10^{-5}.
\] (5.12)

The values of \(b_3\), \(b_4\) and \(b_6\) indicated in (5.11) and (5.12) are compatible with the results obtained in \(4\). Concerning \(b_5\), no large instability shows up in \(4\). This is probably related to the use of unitarized absorptive parts. In any case the order of magnitude we obtain is the same as in \(4\).

VI. DISCUSSION

The last Section provides evidence that our programme is working. It produces three 6th-order parameters which are stable in the sense that they depend weakly on \(\Lambda\), the energy separating low and high energies as long as \(2 < \Lambda \lesssim 4\). The following comments summarise the salient features of our findings.

1. The fact that \(b_3\), \(b_4\) and \(b_6\) are nearly independent of \(\Lambda^2\) at 6th order is not the result of a weak \(\Lambda^2\)-dependence of the individual terms in eqs. (5.2) and (5.4). In fact they vary strongly with \(\Lambda^2\): for instance \(H_2(\Lambda^2)\) in eq. (5.2) increases from \(1.01 \cdot 10^{-2}\) to \(1.69 \cdot 10^{-2}\)
when $\Lambda^2$ goes from 4 to 16. Very delicate compensations are at work and are responsible for practically constant 6th-order parameters.

2. The choice of the values (5.8) for $b_1$ and $b_2$ plays a role in the constancy of $b_3^{(6)}$, $b_4^{(6)}$ and $b_6^{(6)}$. Substantially different values would destroy the compensations mentioned under 1. In fact, the constancy of $b_3^{(6)}$ and $b_4^{(6)}$ at the level displayed in Fig. 2 is certainly lost if $(b_1, b_2)$ is outside the square $(-2 \cdot 10^{-1}, 0) \times (0, 2 \cdot 10^{-1})$.

3. One of our basic assumptions is that low-energy absorptive parts can be approximated by chiral absorptive parts. To check this we have computed the 4th- and 6th-order S- and P-wave absorptive parts. The 4th-order absorptive parts do not depend on the $b_i$: the central values (5.9) and (5.11) have been used in the 6th-order calculations. Figure 4 shows that in the case of $\text{Im} f_0^0(s)$ there is good agreement between the 6th-order chiral absorptive part and our Ansatz (4.1) if $4 < s < 18$. There is also a spectacular improvement with respect to the 4th-order result. The agreement is not as good for $\text{Im} f_1^1(s)$ and $\text{Im} f_0^2(s)$. This does not affect our results because these absorptive parts are small at low energies and the contributions of $\text{Im} f_0^0(s)$ dominate.

4. According to eqs (5.5), $b_3$ and $b_4$ depend additively on $C_1$, $C_2$, $b_2$, $H_2$, $\partial H_0/\partial x$, $\partial H_1/\partial x$ and $\partial H_0/\partial y$. This means that their values are dictated not only by the inputs $H_2$, $\partial H_0/\partial x$, ... but also by the structure of the chiral amplitudes. The inputs produce contributions given in Table 1 for four values of $\Lambda^2$. This table also displays the relative sizes of the contributions coming from the three energy intervals ($\Lambda$, $1 \text{ GeV}$), ($1 \text{ GeV}$, $1.5 \text{ GeV}$) and ($1.5 \text{ GeV}$, $\infty$) appearing in the input defined in Section IV. In this table, $b_3[s_1, s_2]$ and $b_4[s_1, s_2]$ are the contributions of the $H_i$ to $b_3^{(6)}$ and $b_4^{(6)}$ coming from integrals restricted to $s_1 < \sigma < s_2$.

We are working here with a crude Ansatz for the absorptive parts and we are primarily interested in estimates of the $b_i$ rather than precise determinations including error evaluations. Table 1 indicates how our input should be improved to favour a precise determination of $b_3$ and $b_4$. Although the major parts of $b_3[\Lambda^2, \infty]$ and $b_4[\Lambda^2, \infty]$ come from energies below 1 GeV, the energies above 1 GeV contribute to $b_4[\Lambda^2, \infty]$ at the 10% level and there
is a strong cancellation at work for $b_3[\Lambda^2, \infty]$. We use a Regge Ansatz on $(1.5 \text{ GeV}, \infty)$ and it would be difficult to estimate the errors associated with this. The cut at $s = 110 (= (1.5 \text{ GeV})^2)$ should be replaced by a cut at $s = 200 (= (2 \text{ GeV})^2)$, experimental data used on $(51, 200)$, the Regge Ansatz being restricted to $(200, \infty)$. This would approximately halve the Regge contribution.

5. The implications of our findings for the 6th-order coupling constants are beyond the scope of the present work. We observe only that the 4th-order relation

$$
\bar{l}_2 = 96 \cdot \pi^2 \cdot b_4^{(4)} + \frac{5}{6} \tag{6.1}
$$

allow a determination of $\bar{l}_2$ (notice that $\bar{l}_2$ does not enter into our evaluation of $b_1$ and $b_2$, eqs (5.8)). This gives $\bar{l}_2 = 5.4$ which is below the central value $\bar{l}_2 = 6.1$ obtained from the analysis of $K_{14}$ decays\textsuperscript{[14]}. A low value such as this has also been extracted from pion-pion scattering in\textsuperscript{[13]} and\textsuperscript{[14]}. Still at 4th order we have

$$
b_3^{(4)} = \frac{1}{96\pi^2} \left[ 2\bar{l}_1 + \bar{l}_2 - \frac{7}{3} \right]. \tag{6.2}
$$

With the value $\bar{l}_1 = -1.7$ used in eqs (5.8), and $\bar{l}_2 = 5.4$, this gives $b_3^{(4)} = -1.6 \cdot 10^{-3}$. This is compatible with the range of variation of $b_3^{(4)}$ displayed in Fig. 2 and confirms the credibility of our approach.

ACKNOWLEDGMENTS

Discussions with P. Büttiker, J. Gasser, H. Leutwyler, J. Stern and D. Toublan are gratefully acknowledged. D. Toublan was helpful in the preparation of the figures.

DETERMINATION OF LOW-ENERGY POLYNOMIALS

We have to show that the terms $P_i$ in eq. (3.3) are polynomials. We do this in detail for $P_0$. 

18
The functions $f_\alpha$ vanish at the origin and can be obtained from a once-subtracted dis-

persion relation:

$$ f_\alpha(s) = \frac{s}{\pi} \int_4^\infty \frac{d\sigma}{\sigma} \frac{1}{\sigma - s} \text{Im} \, f_\alpha(\sigma). \quad (1) $$

By writing the right-hand side as the sum of a low-energy and a high-energy integral, $f_\alpha$ is
decomposed into a low- and a high-energy component. Introducing this decomposition into
the expression (4.7) of $G_0$ gives

$$ G_0^\chi(s, t, u) = Q_0(s, t, u) + \frac{1}{\pi} \sum_\alpha \int_4^{\Lambda^2} \frac{d\sigma}{\sigma} \left[ \frac{sU_\alpha(s)}{\sigma - s} + \frac{tU_\alpha(t)}{\sigma - t} + \frac{uU_\alpha(u)}{\sigma - u} \right] \text{Im} \, f_\alpha(\sigma) + \sum_\alpha \int_4^{\Lambda^2} \frac{d\sigma}{\sigma} \left[ \frac{1}{\sigma - s} + \frac{1}{\sigma - t} + \frac{1}{\sigma - u} \right] U_\alpha(\sigma) \text{Im} \, f_\alpha(\sigma), \quad (2) $$

where

$$ U_\alpha(s, \sigma) = \frac{sU_\alpha(s) - \sigma U_\alpha(\sigma)}{s - \sigma} \quad (4) $$
is a polynomial. On the other hand, (4.7) and (2.7) tell us that $B_0^\chi = \sum_\alpha B_{0,\alpha}^\chi$ with

$$ B_{0,\alpha}(s, t) = (s - t)(2s - 4 + t)U_\alpha(s) \text{Im} \, f_\alpha(s). \quad (5) $$

According to (2.5) and (2.6), we have $L_0^\chi = \sum_\alpha L_{0,\alpha}^\chi$, where

$$ L_{0,\alpha}(s, t, u) = \frac{1}{\pi} \int_4^{\Lambda^2} d\sigma \left[ \frac{1}{(\sigma - s)(\sigma - t)(\sigma - u)} - \frac{1}{(\sigma - s_0)(\sigma - t_0)(\sigma - u_0)} \right] B_{0,\alpha}(\sigma, \tau). \quad (6) $$

It turns out that the difference between the second integral in (3) and $L_{0,\alpha}^\chi$ is equal to
this integral with $(s, t, u)$ replaced by $(s_0, t_0, u_0)$. Consequently (3.3) is true for $i = 0$ with

$$ P_0(s, t, u) = Q_0(s, t, u) - \frac{1}{\pi} \sum_\alpha \int_4^{\Lambda^2} \frac{d\sigma}{\sigma} \left[ U_\alpha(s, \sigma) + U_\alpha(t, \sigma) + U_\alpha(u, \sigma) \right] \text{Im} \, f_\alpha(\sigma) $$

$$ + \frac{1}{\pi} \int_4^{\Lambda^2} d\sigma \left[ \frac{1}{\sigma - s} + \frac{1}{\sigma - t} + \frac{1}{\sigma - u} \right] U_\alpha(\sigma) \text{Im} \, f_\alpha(\sigma). \quad (7) $$
In terms of $x$ and $y$, and at fixed $(x_0, y_0)$, $P_0(x, y)$ is a polynomial of first degree.

Similarly,

$$P_1(s, t, u) = Q_1(s, t, u) - \frac{1}{\pi} \sum_{\alpha} \int_{A} d\sigma \frac{\lambda^2}{\sigma} \left\{ 3 \left[ T_{\alpha}(s, \sigma) + T_{\alpha}(t, \sigma) + T_{\alpha}(u, \sigma) \right] + \text{permutations} \right\} + \text{const.}$$  \hspace{1cm} (8)

$$P_2 = Q_2 - \frac{1}{\pi} \sum_{\alpha} \int_{A} d\sigma \frac{\lambda^2}{\sigma} \left\{ \frac{1}{s-t} \left[ \frac{1}{t-u} (W_{\alpha}(t, \sigma) - W_{\alpha}(u, \sigma)) ight] - \frac{1}{u-s} (W_{\alpha}(u, \sigma) - W_{\alpha}(s, \sigma)) \right\} + \text{permutations}$$  \hspace{1cm} (9)

The polynomials $W_{\alpha}(s, \sigma)$ and $T_{\alpha}(s, \sigma)$ are obtained in the same way as $U_{\alpha}(s, \sigma)$ in (7).

It turns out that $P_1(x, y)$ is linear in $x$, independent of $y$ and $P_2$ is a constant.
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FIGURES

FIG. 1. The variations of the combinations \((\pm b_3 + 3b_4)\) as functions of \(\Lambda^2\). The 4th-order values (dashed lines) are obtained from eqs (5.1) and the 6th-order values (full lines) come from eqs (5.5) with \(b_1 = -7.7 \cdot 10^{-2}\), \(b_2 = 7.1 \cdot 10^{-2}\).

FIG. 2. The values of \(b_3\) and \(b_4\) at 4th order (dashed lines) and 6th order (full lines) resulting from Fig. 1.

FIG. 3. The \(\Lambda^2\)-dependent values of \(b_5\) and \(b_6\) obtained from eqs (5.4) with \(b_1 = -7.7 \cdot 10^{-2}\), \(b_2 = 7.1 \cdot 10^{-2}\), \(b_3 = -2.55 \cdot 10^{-3}\) and \(b_4 = 4.55 \cdot 10^{-3}\).

FIG. 4. Comparison of the S- and P-wave absorptive parts defined by the Ansatz (4.1) (full lines) with the chiral 4th-order (dashed lines) and 6th-order (dashed-dotted lines) absorptive parts. The latter are obtained from eq. (1.6) with the same values of \(b_1, b_2, b_3\) and \(b_4\) as in Fig. 3.
TABLE I. The $\Lambda^2$-dependent contributions to $b_3$ and $b_4$ at 6th order coming from the input quantities $H_2$, $\partial H_0/\partial x$, $\partial H_1/\partial x$ and $\partial H_0/\partial y$ according to eqs (5.5). The full contributions are given in the second column. The relative sizes of the contributions of the three energy intervals $\Lambda^2 < s < 51$, $51 < s < 110$ and $s > 110$ appear in columns 3-5.

| $\Lambda^2$ | $b_3 [\Lambda^2, \infty]$ | $b_3 [\Lambda^2, 51]$ | $b_3 [51, 110]$ | $b_3 [110, \infty]$ |
|-------------|-----------------------------|------------------------|-----------------|---------------------|
| 4           | $-5.94 \cdot 10^{-3}$       | 1.011                  | $-0.060$        | 0.049               |
| 8           | $-5.78 \cdot 10^{-3}$       | 1.004                  | $-0.060$        | 0.055               |
| 12          | $-7.60 \cdot 10^{-3}$       | 1.004                  | $-0.049$        | 0.045               |
| 16          | $-8.74 \cdot 10^{-3}$       | 1.007                  | $-0.046$        | 0.038               |

| $\Lambda^2$ | $b_4 [\Lambda^2, \infty]$ | $b_4 [\Lambda^2, 51]$ | $b_4 [51, 110]$ | $b_4 [110, \infty]$ |
|-------------|-----------------------------|------------------------|-----------------|---------------------|
| 4           | $4.94 \cdot 10^{-3}$        | 0.905                  | 0.017           | 0.078               |
| 8           | $4.49 \cdot 10^{-3}$        | 0.900                  | 0.018           | 0.082               |
| 12          | $4.02 \cdot 10^{-3}$        | 0.894                  | 0.019           | 0.088               |
| 16          | $3.37 \cdot 10^{-3}$        | 0.888                  | 0.019           | 0.093               |
\[ (b_3 + 3b_4)^{(6)} \]

\[ (-b_3 + 3b_4)^{(4)} \]
Fig. 2G. Wanders Phys. Rev. D

\begin{align*}
\bar{b}_3^{(6)} & \approx 10^{-3} \\
\bar{b}_3^{(4)} & \approx 10^{-3} \\
\bar{b}_4^{(6)} & \approx 10^{-3} \\
\bar{b}_4^{(4)} & \approx 10^{-3}
\end{align*}
Fig. 4G. Wanders Phys. Rev. D