INFINITE-DIMENSIONAL SUPERMANIFOLDS VIA MULTILINEAR BUNDLES

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Abstract. In this paper, we provide an accessible introduction to the theory of locally convex supermanifolds in the categorical approach. In this setting, a supermanifold is a functor $\mathcal{M} : \text{Gr} \to \text{Man}$ from the category of Grassmann algebras to the category of locally convex manifolds that has certain local models, forming something akin to an atlas. We give a mostly self-contained, concrete definition of supermanifolds along these lines, closing several gaps in the literature on the way.

If $\Lambda_n \in \text{Gr}$ is the Grassmann algebra with $n$ generators, we show that $\mathcal{M}_{\Lambda_n}$ has the structure of a so called multilinear bundle over the base manifold $\mathcal{M}_{\mathbb{R}}$. We use this fact to show that the projective limit $\varprojlim \mathcal{M}_{\Lambda_n}$ exists in the category of manifolds. In fact, this gives us a faithful functor $\varprojlim : \text{SMan} \to \text{Man}$ from the category of supermanifolds to the category of manifolds. This functor respects products, commutes with the respective tangent functor and retains the respective Hausdorff property. In this way, supermanifolds can be seen as a particular kind of infinite-dimensional fiber bundles.

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1. Introduction

The first rigorous definition of infinite-dimensional supermanifolds, and also the one we will use in this work, is the categorical approach suggested by Molotkov in [17]. In this approach supermanifolds are defined to be functors from the category of finitely generated Grassmann algebras $\text{Gr}$ to the category of manifolds $\text{Man}$ with additional local information contained in an ‘atlas’ consisting of certain natural transformations. Let us briefly relate this to the usual sheaf theoretic approach by Berezin and Le\v{c}it\v{s} [4] in the case of finite-dimensional supermanifolds. In the latter, the functor of points (i.e., the Yoneda embedding) has long been known to be a useful tool (see for example [15]). Moreover, to fully understand the functor of points, it suffices to consider supermanifolds whose base manifold is a single point, the so called superpoints. The superpoints are parametrized by the Grassmann algebras and for every superpoint $\mathcal{P}$ the set of morphisms $\text{Hom}_{\text{Man}}(\mathcal{P}, \mathcal{M})$ to a given supermanifold $\mathcal{M}$ can be turned into a smooth manifold. In this way one obtains a functor $\text{Gr} \to \text{Man}$. Shvarts [24] and Voronov [25] had the idea to use such functors to define finite-dimensional supermanifolds and Molotkov extended this definition to infinite-dimensional supermanifolds.\footnote{Throughout this work, we will cite the more readily available and slightly updated article [18].}

We call this the categorical approach.

Because of its close relation to the functor of points, some of the intuition from the finite-dimensional situation carries over to the infinite-dimensional setting. For example, the definition of an internal Hom and the related superdiffeomorphisms are obtained quite easily in this way (see [18, 8.2, p.415 and 8.4, p.417]). Using this, Hanisch [12] was able to endow the inner Hom object of two finite-dimensional supermanifolds with a supermanifold structure in Molotkov’s framework. Another nice feature of the categorical approach is that the definition of finite-dimensional and infinite-dimensional supermanifolds, along with their morphisms and their tangent bundles, is exactly the same. No special topological considerations are necessary. Similarly, as has been shown in [1], it lends itself to easy generalization beyond the real or complex case. What is more, many constructions and calculations can essentially be done pointwise, i.e., for every Grassmann algebra. This means that for finite-dimensional supermanifolds one often only has to deal with finite-dimensional ordinary manifolds.

Despite these advantages, the categorical approach has rarely been used and even where it appears, it is usually only applied half-heartedly. For

\footnote{Most statements in [18] are made for Banach supermanifolds but many can be easily transferred to Fréchet or locally convex supermanifolds (compare [18, 8.5, p.418]).}
instance, when superspaces of morphisms between supermanifolds are considered, the morphisms are usually expressed in the sheaf theoretic language (see for example [22], [12] and [7]). The reason for this lack of interest appears to be twofold. For one, the categorical language of natural transformations, Grothendieck topologies, sheaves in categories and so on is rather abstract and not part of the usual toolbox employed in the field of analysis. This is then exacerbated by the fact that Molotkov’s foundational article [17] (resp. [18]) contains almost no proofs. While some proofs for Molotkov’s statements were subsequently offered by Sachse in [20] and [21], he often falls back to the sheaf theoretic approach so that the statements are not shown in their original generality and one obtains little intuition for the categorical approach.

We attempt to remedy both of these problems in this work. On the one hand, we give a complete definition of infinite-dimensional supermanifolds and their morphisms, proving all statements that we use (with the rare exception where the proof in the literature can directly be applied to our situation and is relatively straightforward). On the other hand, we simplify the categorical language as much as possible. As it turns out, one can develop the categorical approach in fairly concrete terms closely resembling the definition of ordinary manifolds. In this way, we completely avoid dealing with more involved questions like representability.

Remarkably, this concrete point of view leads to a canonical faithful functor from the category of supermanifolds to the category of manifolds. This functor has good properties such as respecting products (i.e., mapping Lie supergroups to Lie groups), commuting with the respective tangent functor and retaining the respective Hausdorff property. It can be turned into an equivalence of categories if one considers a specific type of fiber bundles on the right-hand side. In other words, we may consider supermanifolds as ordinary manifolds with a particular kind of atlas in a canonical, well-behaved way. All non-trivial supermanifolds are at best mapped to Fréchet manifolds and one wonders whether techniques of infinite-dimensional analysis could prove useful in finite-dimensional superanalysis.

To streamline our work, we only consider supermanifolds over the base field \( \mathbb{R} \). However, many of our constructions derive from [1] and [5], where much more general fields and even rings are considered. We have consciously formulated our proofs in such a way that they can easily be generalized where possible. The only noteworthy obstacles to such generalizations are Batchelor’s Theorem (which necessitates a partition of unity) and combinatorial formulas which do not allow for base rings with positive characteristic. For the latter, we indicate ways around the problem.

Many standard constructions are beyond the scope of this paper, but we hope to have provided the reader with the tools to rectify this with relative ease. While equivalences between certain categories of supermanifolds in the sheaf theoretic, the concrete and the categorical approach have been discussed in some detail in [11], it is not immediately obvious how objects like vector fields can be translated between the different point of views. More work to this effect will be critical to enable one to pick and choose effectively which approach is most suitable for the problem at hand. One
1.1. Overview and Main Results. A Grassmann algebra is a free associative \( \mathbb{R} \)-algebra \( \Lambda_n := \mathbb{R}[\lambda_1, \ldots, \lambda_n] \), where the generators satisfy the relation \( \lambda_i \lambda_j = -\lambda_j \lambda_i \). There exists a natural grading \( \Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1 \) and the set of objects \( \{ \mathbb{R}, \Lambda_1, \Lambda_2, \ldots \} \) together with the graded morphisms form the category \( \text{Gr} \) of Grassmann algebras. Generators of Grassmann algebras behave infinitesimally in the sense that \( \lambda_i^2 = 0 \) and we will see that for this reason (together with functoriality) the structure of supermanifolds has many similarities to the structure of higher tangent bundles. This enables us to make heavy use of the techniques developed by Bertam in [5] for dealing with higher tangent bundles, higher tangent groups and higher order diffeomorphism groups.

As mentioned, we want to define supermanifolds as functors from the category of Grassmann algebras to the category of manifolds with certain local models. In analogy to ordinary manifolds, we begin by describing the differential calculus on the model space:

1. Instead of a vector space, the model space of a supermanifold is a functor of the form

\[
\mathcal{E}: \text{Gr} \to \text{Top}, \quad \Lambda \mapsto \mathcal{E}_\Lambda := (E_0 \otimes \Lambda_0^\oplus) \oplus (E_1 \otimes \Lambda_0^-),
\]

where \( E = E_0 \oplus E_1 \) is a \( \mathbb{Z}_2 \)-graded Hausdorff locally convex vector space and \( \mathcal{E}_\Lambda \) is given the obvious product topology. Then \( \mathcal{E}_\Lambda \) is a \( \Lambda_0^- \)-module and the functor \( \mathcal{E} \) has the structure of a so called \( \mathbb{R} \)-module in the category \( \text{Top}^{\text{Gr}} \).

2. Open subsets of the model space correspond to open subfunctors, i.e. functors

\[
\mathcal{U}: \text{Gr} \to \text{Top}
\]

such that \( \mathcal{U}_\Lambda \subseteq \mathcal{E}_\Lambda \) is open for all \( \Lambda \in \text{Gr} \) and the inclusion is a natural transformation. We call such functors \textit{super domains}. One can show that superdomains have the form

\[
\mathcal{U}_\Lambda = \mathcal{U}_\mathbb{R} \times (E_0 \otimes \Lambda_0^+) \times (E_1 \otimes \Lambda_0^-),
\]

where \( \Lambda_0^+ \) is the nilpotent part of \( \Lambda_0^- \).

3. Smooth functions correspond to supersmooth morphisms, i.e. natural transformations

\[
f: \mathcal{U} \to \mathcal{F}
\]

such that \( f_\Lambda \) is smooth for all \( \Lambda \in \text{Gr} \) and

\[
df_\Lambda: \mathcal{U}_\Lambda \times \mathcal{E}_\Lambda \to \mathcal{F}_\Lambda
\]

is \( \Lambda_0^- \)-linear in the second component.

Using the infinitesimal behavior of the generators, one obtains an “exact Taylor expansion” for supersmooth morphisms. This can then be used to identify a supersmooth morphism \( f: \mathcal{U} \to \mathcal{F} \) with its \textit{skeleton}, i.e., a family \( (f_k)_{k \in \mathbb{N}_0} \) of maps \( f_k: \mathcal{U}_k \to \text{Alt}^k(E_1, F_k \mod 2) \) that are smooth in an appropriate sense. Skeletons are of utmost importance for many proofs and the description of spaces of supersmooth morphisms.
A supermanifold is defined to be a functor \( \mathcal{M} : \text{Gr} \to \text{Man} \) such that there exists an atlas of natural transformations \( \varphi^\alpha : \mathcal{U}^\alpha \to \mathcal{M} \) from superdomains \( \mathcal{U}^\alpha \) to \( \mathcal{M} \) for which any change of charts is supersmooth. If \( \varepsilon_{\Lambda_n} : \Lambda \to \mathbb{R} \) denotes the natural projection, we show that \( \mathcal{M}_{\varepsilon_{\Lambda_n}} : \mathcal{M}_{\Lambda_n} \to \mathcal{M}_{\mathbb{R}} \) gives \( \mathcal{M}_{\Lambda_n} \) the structure of a so-called multilinear bundle of degree \( n \) over the base manifold \( \mathcal{M}_{\mathbb{R}} \) (compare [5]). What is more, we show in Theorem 3.37 that the family \( (\mathcal{M}_\Lambda)_{\Lambda \in \text{Gr}} \) gives one an inverse system of such bundles and that the limit \( \varprojlim_n \mathcal{M}_{\Lambda_n} \) exists in the category of manifolds. This provides us with the functor

\[ \varprojlim_n : \text{SMan} \to \text{Man} \]

from the category of supermanifolds to the category of manifolds mentioned above. Multilinear bundles and their limits are discussed in Appendix A.

In the sheaf theoretic approach every manifold together with its sheaf of functions is clearly a supermanifold. For us the situation is a bit more complicated since a manifold is not a functor \( \text{Gr} \to \text{Man} \). However, there exists a natural embedding

\[ \iota : \text{Man} \to \text{SMan} \]

introduced by Molotkov in [16]. In Proposition 3.42, we give a description of \( \iota(M) \) via higher tangent bundles of the manifold \( M \), which is particularly useful for understanding Lie supergroups. Similarly, Molotkov constructed a faithful functor

\[ \iota^1_\infty : \text{VBun} \to \text{SMan} \]

from the category of vector bundles to the category of supermanifolds. He showed in [17] that any supermanifold whose base manifold allows a partition of unity is (non-canonically) isomorphic to a supermanifold that comes from a vector bundle. Since this result, generally known as Batchelor’s Theorem, is important for us and [17] is rather difficult to find, we briefly summarize its proof.

2. Preliminaries and Notation

We set \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), respectively. Let \( k \in \mathbb{N}_0 \). Throughout this work, we will write \( \overline{k} := k \mod 2 \in \{0, 1\} \).

A locally convex super vector space \( E \) is a locally convex vector space \( E \) together with a fixed decomposition \( E = E_0 \oplus E_1 \), where \( E_0 \) and \( E_1 \) are locally convex vector spaces and the direct sum is a topological. A continuous linear map \( f : E \to F \) between locally convex super vector spaces is a morphism of locally convex super vector spaces if \( f(E_i) \subseteq F_i \) holds for \( i \in \{0, 1\} \). We denote by \( \text{SVec}_{\text{lc}} \) the category of Hausdorff locally convex super vector spaces and their morphisms.

We denote by \( \mathfrak{S}_k \) the symmetrical group of order \( k \) and let \( \text{sgn}(\sigma) \in \{1, -1\} \) be the sign of a permutation \( \sigma \in \mathfrak{S}_k \). If \( E_1, \ldots, E_k, E \) and \( F \) are locally convex spaces, we let \( \mathcal{L}^k(E_1, \ldots, E_k; F) \) be the \( \mathbb{R} \)-vector space of continuous \( k \)-multilinear maps

\[ f : E_1 \times \cdots \times E_k \to F. \]
On $\mathcal{L}^k(E; F) := \mathcal{L}^k(E, \ldots, E; F)$, $\mathcal{S}_k$ acts from the left via

$$f \circ \sigma(v) := f(v^\sigma) := f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for $f \in \mathcal{L}^k(E; F)$, $\sigma \in \mathcal{S}_k$ and $v = (v_1, \ldots, v_k) \in E^k$. We denote by $\mathcal{A}lt^k(E; F) \subseteq \mathcal{L}^k(E; F)$ the space of continuous alternating $k$-multilinear maps.

We let $\mathcal{L}^0_k(E; F) = \mathcal{A}lt^0(E; F) := F$ and define the projection

$$\mathfrak{q}^k : \mathcal{L}^k(E; F) \to \mathcal{A}lt^k(E; F), \quad f \mapsto \sum_{\sigma \in \mathcal{S}_k} \frac{\text{sgn}(\sigma)}{k!} f \circ \sigma.$$

2.1. Partitions. We largely use the notation of [5] for partitions. Let $A$ be a finite set. A partition of $A$ is a subset $\nu = \{\nu_1, \ldots, \nu_\ell\}$ of the power set $\mathcal{P}(A)$ of $A$ such that the sets $\nu_i$, $1 \leq i \leq \ell$, are non-empty, pairwise disjoint and their union is $A$. In this situation, we call $A$ the total set of $\nu$ and let $\nu := A$. We define the length of the partition $\nu$ as $\ell(\nu) := |\nu|$. Furthermore, we denote by $\mathcal{P}(A)$ the set of all partitions of $A$ and by $\mathcal{P}_l(A)$ the set of all partitions of $A$ of length $\ell$. If $|A|$ is even, then we define $\mathfrak{p}(A)_\mathcal{P}$ as those partitions which only contain sets of even cardinality and $\mathfrak{p}(A)_\mathcal{P}$ as the partitions from $\mathfrak{p}(A)_\mathcal{P}$ of length $\ell$.

For $k \in \mathbb{N}$, we define $\mathfrak{p}^k := \mathcal{P}(\{1, \ldots, k\})$ and $\mathfrak{p}^k_\mathcal{P} := \mathfrak{p}^k \setminus \{\emptyset\}$. Occasionally, it will be convenient to consider only subsets of even, resp. odd, cardinality and we define $\mathfrak{p}^0_\mathcal{P} := \{A \in \mathfrak{p}^k : |A| \text{ even}\}$, $\mathfrak{p}^1_\mathcal{P} := \{A \in \mathfrak{p}^k : |A| \text{ odd}\}$ as well as $\mathfrak{p}^k_\mathcal{P} := \mathfrak{p}^k \setminus \{\emptyset\}$. As a convention, $\{i_1, \ldots, i_\nu\} \subseteq \{1, \ldots, k\}$ is understood to imply $i_1 < \ldots < i_\nu$. With this, the lexicographic order induces a total order on the power set of $\mathfrak{p}^k$ and every partition $\nu$ can be viewed as an ordered $\ell(\nu)$-tuple, which we will do in the sequel (compare [5 MA.4, p.170]). There is another total order on $\mathfrak{p}^k$ that will be useful for us: On $\mathfrak{p}^0_\mathcal{P}$ and $\mathfrak{p}^1_\mathcal{P}$, we use the order induced by $\mathfrak{p}^k$ but for all $B \in \mathfrak{p}^0_\mathcal{P}$ and all $C \in \mathfrak{p}^1_\mathcal{P}$, we let $B < C$. We will specify whenever we want to use this order which we will call the graded lexicographic order. For a partition $\nu = (\nu_1, \ldots, \nu_\ell)$, we define $e(\nu)$, resp. $o(\nu)$, as the number of sets in $\nu$ with even, resp. odd, cardinality. In other words, in the graded lexicographic order, we have

\[
\begin{align*}
\nu_1 < \ldots < \nu_{e(\nu)} < \nu_{e(\nu)+1} < \ldots < \nu_{e(\nu)+o(\nu)}.
\end{align*}
\]

Let $A$ be a finite set and $\nu, \omega \in \mathfrak{p}(A)$. We call $\nu$ a refinement of $\omega$, or $\omega$ coarser than $\nu$, and write $\omega \preceq \nu$ if for every set $L \in \nu$ there exists a set $O \in \omega$ such that $L \subseteq O$. For $\omega \preceq \nu$ and $O \in \omega$, we define the $\nu$-induced partition of $O$ by

$$O|\nu := \{L \in \nu | L \subseteq O\} \in \mathfrak{p}(O).$$

In this situation, $\{\omega_1|\nu, \ldots, \omega_{\ell(\omega)}|\nu\}$ is a partition of the finite set $\nu$. One easily checks that this defines a one-to-one correspondence between partitions that are coarser than $\nu$ and $\mathfrak{p}(\nu)$. 
2.2. The Category of Grassmann Algebras. For any \( k \in \mathbb{N}_0 \), we let \( \Lambda_k := \mathbb{R}[\lambda_1, \ldots, \lambda_k] \) be the associative algebra freely generated by the generators \( \lambda_i \) with the relation \( \lambda_i \lambda_j = -\lambda_j \lambda_i \) for all \( i, j \in \mathbb{N} \). Note that this implies \( \lambda_i \lambda_i = 0 \). For \( I = \{i_1, \ldots, i_r\} \subseteq \mathbb{N} \) with \( 1 \leq i_1 < \ldots < i_r \leq k \), we set \( \lambda_I := \lambda_{i_1} \cdots \lambda_{i_r} \). These so-called Grassmann algebras have a natural \( \mathbb{Z}_2 \)-grading given by \( \Lambda_{k;\mathbb{Z}} := \prod_{I \in \mathbb{P}^k} \lambda_I \mathbb{R} \) and \( \Lambda_{k;\mathbb{R}} := \prod_{I \in \mathbb{P}^k} \lambda_I \mathbb{R} \) which, with the product topology, turns them into topological \( \mathbb{R} \)-algebras. A morphism \( \varphi: \Lambda \to \Lambda' \) between two Grassmann algebras is a morphism of unital \( \mathbb{R} \)-algebras that is even in the sense that

\[
\varphi(\Lambda_I) \subseteq \Lambda'_I \quad \text{for} \quad i \in \{0, 1\}.
\]

We denote by \( \text{Gr} \) the category of Grassmann algebras, and for every \( n \in \mathbb{N}_0 \), we let \( \text{Gr}^{(n)} \) be the full subcategory containing only the objects \( \Lambda_0, \ldots, \Lambda_n \). For the sake of convenience, we let \( \text{Gr}^{(\infty)} := \text{Gr} \).

We denote the subalgebra of nilpotent elements of \( \Lambda \) by \( \Lambda^+ \) and set \( \Lambda^+_{\mathbb{R}} := \Lambda^+ \cap \Lambda_{\mathbb{R}} \). For every \( m \geq n \geq 0 \), we fix morphisms \( \varepsilon_{m,n}: \Lambda_m \to \Lambda_n \) and \( \eta_{m,n}: \Lambda_m \to \Lambda_n \) by setting

\[
\varepsilon_{m,n}(\lambda_k) := \begin{cases} 
\lambda_k & \text{if } k \leq n \\
0 & \text{otherwise}
\end{cases}
\]

and \( \eta_{m,n}(\lambda_k) = \lambda_k \) for \( 1 \leq k \leq n \). In the special case \( n = 0 \), we let \( \varepsilon_{\Lambda_m} := \varepsilon_{m,0}: \Lambda_m \to \mathbb{R} \) and \( \eta_{\Lambda_n} := \eta_{0,m}: \mathbb{R} \to \Lambda_m \).

2.3. Locally Convex Manifolds. All locally convex vector spaces in this thesis are meant to be Hausdorff locally convex \( \mathbb{R} \)-vector spaces.

2.3.1. Differential calculus in locally convex spaces. A very general differential calculus for topological modules was developed in \cite{1}. We follow this approach but restrict ourselves to the case of Hausdorff locally convex \( \mathbb{R} \)-vector spaces. In this situation, the \( C^k \)-maps coincide with the classical \( C^k \)-maps in the sense of Michal-Bastiani \cite{2} (also known as Keller’s \( C^k \)-maps, see \cite{13}). However, it is useful to keep the more general setting in mind since large parts of this work can be easily generalized without substantial changes. See also [\cite{5}, Chapter I, p.14ff.] for a concise overview.

**Definition 2.1.** Let \( E, F \) be locally convex spaces, \( U \subseteq E \) be open and \( f: U \to F \) continuous. We define the open set \( U^{[1]} := \{(x, v, t): x \in U, x + tv \in U\} \subseteq U \times E \times \mathbb{R} \) and say that \( f \) is \( C^1 \) if there exists a continuous map

\[
f^{[1]}: U^{[1]} \to F
\]

such that

\[
f(x + tv) - f(x) = t \cdot f^{[1]}(x, v, t)
\]

for \( (x, v, t) \in U^{[1]} \). The differential of \( f \) at \( x \in U \) is then defined as

\[
df(x): E \to F, \quad v \mapsto df(x)(v) := f^{[1]}(x, v, 0).
\]

We also use the notation \( df(x, v) := df(x)(v) \). Inductively, we say \( f \) is \( C^{k+1} \) if \( f^{[1]} \) is \( C^k \) for \( k \in \mathbb{N} \). If \( f \) is \( C^k \) for every \( k \in \mathbb{N} \), we call \( f \) smooth or \( C^\infty \).

\footnote{To be precise: Hausdorff topological modules over Hausdorff topological rings whose unit group is dense.}
The usual rules for differentials apply and we sum them up and fix our notation in the following remark.

**Remark 2.2.** In the situation of the definition, the map \( f^{[1]} \) is unique and \( df(x)(v) \) is linear in \( v \). If \( f \) is \( C^2 \), then for every \( v \in E \) the partial map \( \partial_v f := df(\cdot, v) \) is \( C^1 \) and we define \( d^k f(x)(v_1, \ldots, v_k) := \partial_{v_1} \cdots \partial_{v_k} f(x) \) if \( f \) is \( C^k \). The map \( d^k f(x): E^k \to F \) is continuous, \( \mathbb{R}^k \)-multilinear and symmetric. In particular Schwarz’s theorem holds in this setting. If \( U \subseteq F \) is open and \( f \) is \( C^k \), then the restriction \( f|_U \) is so.\(^4\) If \( g \) and \( f \) are \( C^k \) and composable, then \( g \circ f \) is \( C^k \) and we have the the chain rule

\[ d(g \circ f)(x, v) = dg(f(x), df(x, v)). \]

If \( h: U_1 \times U_2 \to F \) is \( C^1 \) we define \( d_1 h(x_1, x_2)(v_1) := dh(x_1, x_2)(v_1, 0) \) and \( d_2 h(x_1, x_2)(v_2) := dh(x_1, x_2)(0, v_2) \) and we have the rule of partial differentials

\[ dh(x_1, x_2)(v_1, v_2) = d_1 h(x_1, x_2)(v_1) + d_2 h(x_1, x_2)(v_2). \]

If \( f \) is of the form \( (f_1, f_2) \) or \( f_1 \times f_2 \), then \( f \) is \( C^k \) if and only if \( f_1 \) and \( f_2 \) are \( C^k \) and it holds that \( df = (df_1, df_2) \) or \( df = df_1 \times df_2 \), respectively.

The following lemma is well-known. As it is instrumental for the rest of the work, we give a quick proof nevertheless. Clearly, the proof works in the most general setting as well.

**Lemma 2.3.** Let \( n \in \mathbb{N} \) and \( E_1, \ldots, E_n \) and \( F \) be locally convex spaces. Each continuous \( \mathbb{R}^n \)-multilinear map \( f: E_1 \times \cdots \times E_n \to F \) is automatically \( C^1 \) and thus smooth by induction. In this case, we have

\[ df(x)(v) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, v_i, x_{i+1}, \ldots, x_n) \]

for \( x = (x_1, \ldots, x_n), v = (v_1, \ldots, v_n) \in E_1 \times \cdots \times E_n \).

**Proof.** Let \( 0y_i := x_i \) and \( 1y_i = v_i \) for \( 1 \leq i \leq n \). With this we calculate

\[ f(x + tv) - f(x) = t \cdot \sum_{j \in \{0,1\}^n, \ell_j \geq 1} t^{\ell_j - 1} f_{j_1 y_1, \ldots, j_n y_n}(x, v; t), \]

where \( \ell_j := j_1 + \cdots + j_n \). With \( t = 0 \), the statement follows. \( \square \)

2.3.2. **Products and Inverse Limits.**

**Lemma 2.4 (\cite{[9]} Lemma 1.3, p.24).** Let \( E, F \) be locally convex spaces, \( U \subseteq E \) open and \( f: U \to F \). If \( f(U) \subseteq F' \) for a closed vector subspace \( F' \subseteq F \), then \( f \) is smooth if and only if its co-restriction \( f|_{F'}: U \to F' \) is smooth.

Let \( J \) be a set and \( (F_j)_{j \in J} \) be a family of locally convex spaces. Then the product \( \prod_{j \in J} F_j \) equipped with the product topology is a Hausdorff locally convex space.

\(^4\)This does not hold in general over rings (compare \cite{[5]} 2.4, p.21).
Lemma 2.5 (10). Let $E$ be a locally convex space, $U \subseteq E$ open and $(F_j)_{j \in J}$ be a family of locally convex spaces. Let $F := \prod_{j \in J} F_j$ and let $\text{pr}_j : F \to F_j$ be the projection onto the $j$-th component. A map $f : U \to F$ is smooth if and only if $f_j := \text{pr}_j \circ f : U \to F_j$ is smooth for every $j \in J$. In this case, we have
\[
df(x,y) = (df_j(x,y))_{j \in J} \quad \text{for all } x \in U \text{ and } y \in E.
\]

Let $J$ be a totally ordered index set. The inverse limit
\[
\varprojlim_{j \in J} F_j := \left\{ (x_j)_{j \in J} \in \prod_{j \in J} F_j : \text{pr}_i(x_j) = x_i \text{ for all } i \leq j \right\}
\]
of an inverse system $((F_j)_{j \in J}, (\varphi_j^i)_{i \leq j})$ of locally convex spaces, with continuous maps $\varphi_j^i : F_j \to F_i$ for all $i \leq j$, is a closed subset of $\prod_{j \in J} F_j$ and thus a Hausdorff locally convex space. A direct consequence of this is the following lemma.

Lemma 2.6 (10). Let $E, F$ be locally convex spaces, $U \subseteq E$ open and $f : U \to F$ be a map. Assume that $F = \varprojlim_{i \in J} F_i$ for an inverse system $((F_i)_{i \in J}, (\varphi_i^j)_{j \leq i})$ of locally convex spaces and continuous linear maps $\varphi_i^j : F_j \to F_i$, with limit maps $\varphi_i : F \to F_i$. Then $f$ is smooth if and only if $\varphi_i \circ f : U \to F_i$ is smooth for each $i \in J$. In this case, we have
\[
df(x,y) = (df_i((x,y)))_{i \in J} \quad \text{for all } x \in U \text{ and } y \in E.
\]

2.3.3. Manifolds. With the above, the definition of manifolds over locally convex spaces is analogous to the finite-dimensional case (see also [8, Section 8, p.253] or [5, Section 2, p.20]). We fix a locally convex space $E$ and let $M$ be a topological space. A set $A := \{ \varphi_\alpha : U_\alpha \to V_\alpha : \alpha \in A \}$ such that $U_\alpha \subseteq M$ and $V_\alpha \subseteq E$ are open, $\varphi_\alpha$ is a homeomorphism, $\bigcup_{\alpha \in A} U_\alpha = M$ and $\varphi_{\alpha \beta} := \varphi_\alpha \circ \varphi^{-1}_\beta \mid_{\varphi_\beta(U_\alpha \cap U_\beta)} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ and its inverse $\varphi_{\beta \alpha}$ are smooth is called a (smooth) atlas of $M$ and the elements of $A$ are called charts of $M$. Two atlases of $M$ are equivalent if and only if their union is again an atlas. Together with an equivalence class of atlases, $M$ is called a (smooth) manifold modelled on $E$ and $E$ is the model space of $M$. We usually only mention a representative atlas of the equivalence class. Moreover, we will generally assume manifolds to be Hausdorff.$^5$ A manifold is called paracompact, resp. $\sigma$-compact, if it is so as a topological space and finite-dimensional if its model space is finite-dimensional. If $M$ and $N$ are manifolds with the atlases $\{ \varphi_\alpha : \alpha \in A \}$ and $\{ \psi_\beta : \beta \in B \}$, then $\{ \varphi_\alpha \times \psi_\beta : \alpha \in A, \beta \in B \}$ is an atlas of $M \times N$.

A map $f : M \to N$ between two manifolds is a morphism of (smooth) manifolds if for any charts $\varphi : U \to \varphi(U)$ of $M$ and $\psi : W \to \psi(W)$ of $N$ the map
\[
\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(W)) \to \psi(W)
\]

---

$^5$The index set $A$ is just to simplify our notation, it does not belong to the data of the atlas $A$.

$^6$This assumption is necessary to guarantee the existence of smooth partitions of unity for finite-dimensional $\sigma$-compact manifolds.
is smooth. This property is independent of the choice of atlases. If $M$ and $N$ are manifolds, we denote by $\mathcal{C}^\infty(M,N)$ the set of all smooth maps $M \to N$ and we denote by $\mathbf{Man}$ the category of Hausdorff manifolds and their morphisms.

The definition of vector bundles or more general fiber bundles, their morphisms and their products is similar to the finite-dimensional case and for it, we refer to [5, Section 3, p.22]. The particular charts of a bundle are called bundle charts and they are elements of a bundle atlas. We write $\mathbf{VBun}$ for the category of vector bundles.

The definition of the tangent bundle $\pi_M: TM \to M$ of a manifold $M$ via equivalence classes of smooth curves works as in the finite-dimensional case. For locally convex $\mathbb{R}$-vector spaces, this is equivalent to the more general definition in [6, p.254] and [5, Section 3, p.22] (see [10]). For the elements of the tangent bundle, we occasionally write $[t \mapsto v_t] \in T_{\alpha_0}M$, where $t \mapsto v_t$ denotes some curve in $M$. In this notation one has

$$Tf[t \mapsto v_t] := [t \mapsto f(v_t)] \in T_{f(v_t)}N,$$

if $f: M \to N$ is a smooth map between manifolds. If $F$ is a locally convex space, one has a natural isomorphism $TF \cong F \times F$ and if $g: M \to F$ is smooth, we also write $dg: TM \to F$ for $pr_2 \circ Tg$ with the projection $pr_2: F \times F \to F$ onto the second component. Like in the finite-dimensional case, the above defines a functor $T: \mathbf{Man} \to \mathbf{VBun}$ and considering $\mathbf{VBun}$ as a subcategory of $\mathbf{Man}$, we define $T^0 = \text{id}_{\mathbf{Man}}$ and $T^n := T \circ T^{n-1}: \mathbf{Man} \to \mathbf{Man}$ for $n \in \mathbb{N}$. Finally, if $\{\varphi_\alpha: \alpha \in A\}$ is an atlas of $M$, then $\{T\varphi_\alpha: \alpha \in A\}$ is a bundle atlas of $TM$ and there is a natural isomorphism of vector bundles $T(M \times N) \cong TM \times TN$.

Smooth partitions of unity. A smooth partition of unity of a manifold $M$ is an open covering $(U_i)_{i \in I}$ of $M$ together with smooth maps $h_i: M \to \mathbb{R}$, such that

(a) For all $x \in M$, we have $h_i(x) \geq 0$.
(b) The support of $h_i$ is contained in $U_i$ for all $i \in I$.
(c) The covering is locally finite.
(d) For each $x \in M$, we have $\sum_{i \in I} h_i(x) = 1$.

In this situation, we say that $h_i$ is a partition of unity that is subordinate to $(U_i)_{i \in I}$. We say that a manifold $M$ admits partitions of unity if it is paracompact and for every locally finite cover $(U_i)$ of $M$, we find smooth maps $h_i: M \to \mathbb{R}$ that constitute a partition of unity subordinate to $(U_i)$ (see [13, p.34]). Paracompact (and in particular $\sigma$-compact) finite-dimensional manifolds always admit partitions of unity (compare [14, Corollary 3.8, p.38]).

2.4. Categories. We follow [23] in the standard definitions. Let us give a brief overview to fix our notations. Throughout, we fix a universe $\mathcal{U}$ (see [23, 3.2.1, p.17]) that contains the natural numbers $\mathbb{N}$ as an element. Sets are then elements of $\mathcal{U}$ and classes are subsets of $\mathcal{U}$. A category $\mathcal{C}$ consists of a class of objects $|\mathcal{C}|$ and a set of morphisms $\text{Hom}_\mathcal{C}(A,B)$ for any objects $A,B$ such that we have a composition map

$$\text{Hom}_\mathcal{C}(B,C) \times \text{Hom}_\mathcal{C}(A,B) \to \text{Hom}_\mathcal{C}(A,C), \quad (f,g) \mapsto f \circ g$$
(where $C \in C$) that satisfies the usual conditions. In particular, we have a unique identity morphism $\text{id}_A \in \text{Hom}_C(A, A)$. For $f \in \text{Hom}_C(A, B)$ we also write $f: A \to B$ and we call $f$ an isomorphism if there exists $f^{-1} \in \text{Hom}_C(B, A)$ such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$. As a shorthand, we write $A \in C$ instead of $A \in |C|$. A small category is a category whose objects form a set.

We denote by $\text{Set}$ the category whose objects are sets and whose morphisms are maps between sets. The category $\text{Top}$ has topological spaces as objects and continuous maps between them as morphisms.  

2.4.1. Functors and Functor Categories. Let $C$ and $D$ be categories. A functor $T: C \to D$ assigns to each $A \in C$ an object $T(A) \in D$ and to each morphism $f \in \text{Hom}_C(A, B)$ a morphism $T(f) \in \text{Hom}_D(T(A), T(B))$ such that $T(\text{id}_A) = \text{id}_{T(A)}$ and $T(f \circ g) = T(f) \circ T(g)$ hold for all $A, B, C \in C$ and all $f \in \text{Hom}_C(B, C)$, $g \in \text{Hom}_C(A, B)$. Let $S: C \to D$ be another functor. A natural transformation $\alpha: S \to T$ consists of morphisms $\alpha_A: S(A) \to T(A)$ for every $A \in C$ such that for every $f \in \text{Hom}_C(A, B)$, we have $T(f) \circ \alpha_A = \alpha_B \circ S(f)$. We always have the natural transformation $\text{id}_T: T \to T$ defined by $(\text{id}_T)_A = \text{id}_{T(A)}$ and if $U: C \to D$ is another functor and $\beta: T \to U$ is a natural transformation, then the object-wise composition $\beta_A \circ \alpha_A$ defines a natural transformation $\beta \circ \alpha: S \to U$.

If $C$ is a small category, then the functors $C \to D$ are the objects and the natural transformations are the morphisms of a category which we denote by $D^C$ (see [23, Proposition 3.4.3, p.19]).

3. Supermanifolds

3.1. Open Subfunctors. Open subfunctors of functors from $\text{Top}^{\text{Gr}}$ will play the same role as open subsets of topological spaces in ordinary differential geometry. The following definitions of intersections, restrictions, open covers and so on are intuitive and even provide one with a Grothendieck topology on $\text{Top}^{\text{Gr}}$ (see [1, Definition 3.17, p.591f.]).

Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F} \in \text{Top}^{\text{Gr}(k)}$. For $\Lambda, \Lambda' \in \text{Gr}^{(k)}$ and $\varrho \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda')$, we set $\mathcal{F}_\Lambda := \mathcal{F}(\Lambda)$ and $\mathcal{F}_\varrho := \mathcal{F}(\varrho)$.

**Definition 3.1.** Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F}, \mathcal{F}' \in \text{Top}^{\text{Gr}(k)}$. We call $\mathcal{F}'$ a subfunctor of $\mathcal{F}$ if for every $\Lambda \in \text{Gr}^{(k)}$, we have $\mathcal{F}'_\Lambda \subseteq \mathcal{F}_\Lambda$ and these inclusions define a natural transformation $\mathcal{F}' \to \mathcal{F}$. In this situation, we write $\mathcal{F}' \subseteq \mathcal{F}$. A subfunctor $\mathcal{F}'$ of $\mathcal{F}$ is called open if every $\mathcal{F}'_\Lambda$ is open in $\mathcal{F}_\Lambda$.

**Lemma/Definition 3.2.** Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{F} \in \text{Top}^{\text{Gr}(k)}$. For an open subset $U \subseteq \mathcal{F}_\emptyset$, we define the restriction $\mathcal{F}|_U$ by setting

$$\mathcal{F}|_U(\Lambda) := \mathcal{F}_\Lambda^{-1}(U) \quad \text{for} \quad \Lambda \in \text{Gr}^{(k)}$$

and $\mathcal{F}|_U(\varrho) := \mathcal{F}_\varrho|_{\mathcal{F}|_U(\Lambda)}$ for morphisms $\varrho: \Lambda \to \Lambda'$. Then $\mathcal{F}|_U$ is an open subfunctor of $\mathcal{F}$.

Schubert, [23], uses the notations $[A, B]_{c}$ for $\text{Hom}_C(A, B)$, $1_A$ for $\text{id}_A$ and $\text{Ens}$ for $\text{Set}$. 

Proof. Let \( x \in F|_U(\Lambda) \) and \( \Lambda \in \text{Gr}^{(k)} \). Then \( F_{\epsilon_{\Lambda'}} \circ F_\varrho(x) = F_{\epsilon_{\varrho \circ \Lambda}}(x) = F_{\epsilon_{\Lambda}}(x) \in U \) holds for all morphisms \( \varrho: \Lambda \to \Lambda' \) since \( \epsilon_{\Lambda'} \circ \varrho = \epsilon_{\Lambda} \). Because \( F_{\epsilon_{\Lambda}} \) is continuous, \( F_{\epsilon_{\Lambda}}^{-1}(U) \) is open. \( \square \)

Lemma/Definition 3.3. Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), \( F \in \text{Top}^{\text{Gr}^{(k)}} \) and \( F', F'' \) be open subfunctors of \( F \). Then \( (F' \cap F'')_{\Lambda} := F'_\Lambda \cap F''_\Lambda \) and \( (F' \cap F'')_\varrho := F_{\varrho}(F' \cap F'')_\Lambda \) for \( \Lambda, \Lambda' \in \text{Gr}^{(k)} \) and \( \varrho \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda') \) defines an open subfunctor \( F' \cap F'' \subseteq F \).

Proof. By definition \( F'_\Lambda \cap F''_\Lambda \) is open in \( F_\Lambda \). If \( x \in F'_\Lambda \cap F''_\Lambda \), then by functoriality \( F_\varrho(x) \in F'_{\varrho}(\Lambda) \cap F''_{\varrho}(\Lambda) \), which shows that \( F' \cap F'' \) is a functor and that the inclusion is a natural transformation. \( \square \)

Definition 3.4. Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( F, F' \in \text{Top}^{\text{Gr}^{(k)}} \). A natural transformation \( f: F' \to F \) is called an open embedding if \( f_{\Lambda}: F'_{\Lambda} \to F_{\Lambda} \) is an open embedding for every \( \Lambda \in \text{Gr}^{(k)} \).

Lemma/Definition 3.5. Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), \( F, F' \in \text{Top}^{\text{Gr}^{(k)}} \) and \( f: F' \to F \) be a natural transformation. Let \( \Lambda, \Lambda' \in \text{Gr}^{(k)} \) and \( \varrho \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda') \) be arbitrary:

(a) Setting \( f^{-1}F_{\Lambda} := f_{\Lambda}^{-1}(F_{\Lambda}) \) and \( f^{-1}F_\varrho := F_{\varrho}(f^{-1}F_{\Lambda}) \) defines an open subfunctor \( f^{-1}F \subseteq F' \).

(b) If \( f \) is an open embedding, then \( f(F'_{\Lambda}) := f_{\Lambda}(F'_{\Lambda}) \) and \( f(F')_\varrho := F_{\varrho}(f(F')_{\Lambda}) \) define an open subfunctor \( f(F') \subseteq F \).

(c) Let \( U \subseteq F' \) be an open subfunctor. Then \( f|_U(\Lambda) := f_{\Lambda}|_U \) defines a natural transformation \( f|_U: U \to F \).

Proof. (a) Because \( f_{\Lambda} \) is continuous, \( f^{-1}F_{\Lambda} \) is open. For \( x \in f^{-1}F_{\Lambda} \), naturality of \( f \) implies \( f_{\Lambda}(F_\varrho(x)) = F_{\varrho}(f_{\Lambda}(x)) \) and therefore \( f^{-1}F_\varrho(x) \in f^{-1}F_{\Lambda'} \).

(b) Because \( f \) is an open embedding, \( f(F'_{\Lambda}) \) is open. For \( x \in f(F'_{\Lambda}) \), naturality of \( f \) implies \( f(F')_\varrho(x) \in f(F')_{\Lambda'} \).

(c) This is obvious. \( \square \)

Definition 3.6. We call a set \( \{ f_\alpha: F^\alpha \to F: \alpha \in A \}^8 \) of open embeddings a covering if \( \bigcup_{\alpha \in A} f_\alpha(F^\alpha) = F \) holds for all \( \Lambda \in \text{Gr}^{(k)} \). In this situation, we define for all pairs \( \alpha, \beta \in A \) an open subfunctor \( F^{\alpha \beta} \subseteq F^\alpha \) by \( F^{\alpha \beta}_\Lambda := (f_\alpha)^{-1}(f_\alpha(F^\alpha_\Lambda)) \) and \( F^{\alpha \beta}_\varrho := F_{\varrho}(F^{\alpha \beta}_\Lambda) \) as well as natural transformations \( f^{\alpha \beta}: F^{\alpha \beta} \to F^{\beta \alpha} \) by \( f^{\alpha \beta}_\Lambda := (f^{\beta \alpha}_\Lambda)^{-1}(f^{\beta \alpha}_\Lambda(F^{\beta \alpha}_\Lambda)) \) and \( f^{\alpha \beta}_\varrho := f^{\beta \alpha}_\varrho \) for all \( \Lambda, \Lambda' \in \text{Gr}^{(k)} \) and all morphisms \( \varrho: \Lambda \to \Lambda' \).

Definition 3.7. For \( k \in \mathbb{N}_0 \cup \{ \infty \} \) a functor \( F \in \text{Top}^{\text{Gr}^{(k)}} \) is called Hausdorff if \( F_{\Lambda} \) is Hausdorff for every \( \Lambda \in \text{Gr}^{(k)} \).

3.2. Superdomains. Superdomains take the role of open subsets of vector spaces in ordinary analysis. Together with appropriately defined smooth morphisms between them, they enable us to define supermanifolds from local data much in the same way as for manifolds. The main result in

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8As with atlases before, the index set \( A \) is just used for the sake of an easier notation and not part of the data of a covering.
this section is the description of supersmooth morphisms through so called skeletons in Proposition 3.20. Since skeletons will be our main tool for concrete calculations, other important results are a formula for the composition (see Proposition 3.23) and a formula for the inversion (see Lemma 3.25) in terms of skeletons. We follow [11] in this section, with only small additions to accommodate \( k \)-superdomains (i.e., certain functors \( \text{Gr}^{(k)} \to \text{Top} \)). With the exception of a concrete inversion formula, these results have already been stated in [16].

**Lemma/Definition 3.8.** For every \( E \in \text{SVec}_{lc} \) and \( k \in \mathbb{N}_0 \cup \{ \infty \} \), we get a functor \( \overline{E}^{(k)} : \text{Gr}^{(k)} \to \text{Top} \) by setting
\[
\overline{E}^{(k)}(\Lambda) := (E_0 \otimes \Lambda_\mathfrak{g}) \oplus (E_1 \otimes \Lambda_\mathfrak{r}),
\]
equipped with the natural locally convex topology induced by \( E_0 \otimes \Lambda_\mathfrak{g} = \prod_{I \in p^0} \lambda_I E_0 \), respectively \( E_1 \otimes \Lambda_\mathfrak{r} = \prod_{I \in p^1} \lambda_I E_1 \). For a morphism \( \varphi : \Lambda \to \Lambda' \) of Grassmann algebras, we let \( \overline{E}^{(k)}_\varphi := \overline{E}^{(k)}(\varphi) := (\text{id}_E \otimes \varphi) |_{\overline{E}^{(k)}} \). Then \( \overline{E}^{(k)}_\Lambda \) is a topological \( \Lambda_\mathfrak{g} \)-module and \( \overline{E}^{(k)}_\varphi \) is a morphism of modules (with a change of rings). In other words, \( \overline{E}^{(k)}_\Lambda \) is an \( \mathbb{R}^{(k)} \)-module in the category \( \text{Top}^{\text{Gr}^{(k)}} \) (see [18] for details). We abbreviate \( \overline{E} := \overline{E}^{(\infty)} \), \( \overline{E}_0^{(k)} := E_0 \oplus \{ 0 \}^{(k)} \) and \( \overline{E}_1^{(k)} := \{ 0 \} \oplus \overline{E}_1^{(k)} \) and let \( \mathbb{R}^{(k)} := \mathbb{R} \oplus \{ 0 \}^{(k)} \), i.e., \( \mathbb{R}^{(k)}_\Lambda = \Lambda_\mathfrak{g} \).

If \( \Delta \subseteq \Lambda \) is an \( \mathbb{R} \)-vector subspace, we set \( \overline{E}^{(k)}_\Delta := (E_0 \otimes \Delta_\mathfrak{g}) \oplus (E_1 \otimes \Delta_\mathfrak{r}) \), where \( \Delta_\mathfrak{g} := \Delta \cap \Lambda_\mathfrak{g} \) and \( \Delta_\mathfrak{r} := \Delta \cap \Lambda_\mathfrak{r} \). For \( n \leq m \leq k \), we will always consider the natural embedding \( \overline{E}^{(k)}_\Delta \subseteq \overline{E}^{(k)}_\Lambda \) via \( \overline{E}^{(k)}_\Delta \).

**Proof.** It is easy to see that \( \overline{E}^{(k)}_\Lambda \) is a functor. Since every \( \Lambda \in \text{Gr} \) is a \( \Lambda_\mathfrak{g} \)-algebra, \( \overline{E}^{(k)}_\Lambda \) is a \( \Lambda_\mathfrak{r} \)-module with the obvious multiplication. This multiplication is smooth in the locally convex case because its components are simply finite linear combinations. For the same reason, \( \overline{E}^{(k)}_\varphi \) is continuous and linear. That we have \( \overline{E}^{(k)}_\varphi (x) \cdot \overline{E}^{(k)}_\varphi (v) = \overline{E}^{(k)}_\varphi (x \cdot v) \) for all \( x \in \mathbb{R}^{(k)} \) and \( v \in \mathbb{R}^{(k)}_\Lambda \), follows directly from the definition of the multiplication. \( \square \)

In the definition of super manifolds the functors \( \overline{E} \) will play the same role as vector spaces do for regular manifolds. Accordingly, we need a notion of open subfunctors and appropriate “smooth morphisms” between open subfunctors. All open subfunctors of \( \mathcal{U} \subseteq \overline{E} \) for \( E \in \text{SVec}_{lc} \) are uniquely determined by \( \mathcal{U}_\mathcal{G} \).

**Lemma 3.9.** Let \( k \in \mathbb{N} \cup \{ \infty \} \) and \( E \in \text{SVec}_{lc} \). Recall the restriction from Lemma/Definition 3.2. Every open subfunctor \( \mathcal{U} \subseteq \overline{E}^{(k)} \) arises as such a restriction, i.e., we have \( \overline{E}^{(k)}|_{\mathcal{U}_\mathcal{G}} = \mathcal{U} \).

**Proof.** For \( k = \infty \) this is just [20, Proposition 3.5.8, p. 61]. The same proof holds for \( k \in \mathbb{N}_0 \) if one only considers \( \Lambda \in \text{Gr}^{(k)} \) (see also [18, Section 3.1, p.388 f.]). \( \square \)

**Definition 3.10.** Let \( E, F \in \text{SVec}_{lc} \) and \( k \in \mathbb{N}_0 \cup \{ \infty \} \). We call an open subfunctor \( \mathcal{U} \subseteq \overline{E}^{(k)} \) a \( k \)-superdomain. In the case of \( k = \infty \) we simply
call it a superdomain. A natural transformation \( f : \mathcal{U} \to \mathcal{V} \) of superdomains \( \mathcal{U} \subseteq \mathcal{E}(k) \) and \( \mathcal{V} \subseteq \mathcal{F}(k) \) is called supersmooth if for all \( \Lambda \in \Gr(k) \) the map \( f_\Lambda : \mathcal{U}_\Lambda \to \mathcal{V}_\Lambda \) is smooth and the derivative

\[
df_\Lambda : \mathcal{U}_\Lambda \times \mathcal{E}(k) \to \mathcal{F}(k)
\]

is \( \Lambda \mathfrak{f} \)-linear in the second component, i.e., for any \( x \in \mathcal{U}_\Lambda \), the map

\[
df_\Lambda(x, \bullet) : \mathcal{E}(k) \to \mathcal{F}(k) \quad v \mapsto df_\Lambda(x)(v)
\]

is \( \Lambda \mathfrak{f} \)-linear. We denote by \( SC^\infty(\mathcal{U}, \mathcal{V}) \) the set of all supersmooth morphisms \( f : \mathcal{U} \to \mathcal{V} \).

It is obvious from the usual chain rule that the \( k \)-superdomains together with the supersmooth natural transformations form a category, which we denote by \( S\text{Dom}^{(k)} \). In the case of \( k = \infty \), we also use the notation \( S\text{Dom} \).

Note that for \( \mathbb{R} \)-linear maps, it suffices to check \( \Lambda \mathfrak{f} \)-linearity on the generators: For \( E, F \in S\text{Vec}_{lc} \) and an \( \mathbb{R} \)-linear map \( L : \mathcal{E}_\Lambda \to \mathcal{F}_\Lambda \) with \( L(\lambda t x) = \lambda t L(x) \) for all \( x \in \mathcal{E}_\Lambda \) and \( \lambda t \in \Lambda_{0, \mathfrak{f}} \), we have

\[
L(t \cdot x) = L\left( \sum_{I \in \mathcal{P}_0} \lambda_I t_I \cdot x \right) = \sum_{I \in \mathcal{P}_0} \lambda_I t_I \cdot L(x) = t \cdot L(x),
\]

where \( t = \sum_{I \in \mathcal{P}_0} \lambda_I t_I \in \Lambda_{0, \mathfrak{f}}, t_I \in \mathbb{R} \). As it turns out, even natural transformations that are merely “smooth” already have very convenient properties.

**Lemma 3.11.** Let \( E, F \in S\text{Vec}_{lc}, \ k \in \mathbb{N}_0 \cup \{ \infty \}, \mathcal{U} \subseteq \mathcal{E}(k) \) be an open subfunctor and \( f : \mathcal{U} \to \mathcal{F}(k) \) be a natural transformation such that \( f_\Lambda \) is smooth for all \( \Lambda \in \Gr(k) \). Then for all \( n \in \mathbb{N}_0 \), the maps \( d^n f_\Lambda \) define a natural transformation

\[
d^n f : \mathcal{U} \times \mathcal{E}(k) \times \cdots \times \mathcal{E}(k) \to \mathcal{F}(k).
\]

**Proof.** Let \( \Lambda, \Lambda' \in \Gr(k) \) and \( \varrho : \Lambda \to \Lambda \) a morphism. For all \( x \in \mathcal{U}_\Lambda \), \( v \in \mathcal{E}_\Lambda(k) \) and \( t \in \mathbb{R} \), we have

\[
f^{[1]}_{\Lambda'}(\varrho(x), \mathcal{E}(k)(y), t) \cdot t = f_{\Lambda'}(\mathcal{E}_\varrho(k)(x + tv)) - f_{\Lambda'}(\mathcal{E}_\varrho(k)(x))
= \mathcal{E}_\varrho(k)(f_\Lambda(x + tv) - f_\Lambda(x))
= \mathcal{E}_\varrho(k)(f^{[1]}_\Lambda(x, v, t)) \cdot t.
\]

In particular, it follows that \( df_{\Lambda'}(\varrho(x))(\mathcal{E}_\varrho(k)(y)) = \mathcal{E}_\varrho(k)(df_\Lambda(x)(v)) \). Thus \( g_\Lambda(x, v) := df_\Lambda(x)(v) \) defines a natural transformation. For any \( u \in \mathcal{E}_\Lambda(k) \), we have \( d^2 f_\Lambda(x)(u, v) = dg_\Lambda(x, v)(u, 0) \). Clearly, this restriction of \( dg_\Lambda \) defines a natural transformation and thus the lemma follows by induction. (compare [19] Lemma 3.6.5, p.812f.] and [1] Lemma 2.15, p.577).

In the situation of the lemma, we write \( df \) for the natural transformation defined by \( df_\Lambda \).

**Lemma 3.12.** Let \( E, F \in S\text{Vec}_{lc}, \ k \in \mathbb{N}_0 \cup \{ \infty \}, \mathcal{U} \subseteq \mathcal{E}(k) \) be an open subfunctor and \( f : \mathcal{U} \to \mathcal{F}(k) \) be a natural transformation such that \( f_\Lambda \) is
smooth for all \( \Lambda \in \text{Gr}^{(k)} \). For \( n, m \in \mathbb{N} \), \( \Lambda_m \in \text{Gr}^{(k)} \) let \( x \in \mathcal{U}_\mathbb{R} \subseteq \mathcal{U}_{\Lambda_m} \) and \( y_i \in \lambda_i E_{\mathcal{U}_\mathbb{R}} \subseteq E_{\Lambda}^{(k)} \), where \( I_i \in \mathbb{P}^{m_i} \) and \( 1 \leq i \leq n \). Then, we have
\[
d^n f_\Lambda(x)(y_1, \ldots, y_n) \in \lambda_1 \cdots \lambda_{m_n} F_{\mathcal{U}_\mathbb{R}} \subseteq F_{\Lambda}^{(k)},
\]
for \( \ell := \sum_{i=1}^n I_i \). If the sets \( I_i \) are not pairwise disjoint then we have \( d^n f_\Lambda(x)(y_1, \ldots, y_n) = 0 \).

**Proof.** Consider \( d^n f_\Lambda \) as a map into \( \prod_{p \in \mathcal{P}_m} \lambda_p F_{\mathcal{U}_\mathbb{R}} \). Let \( I := \bigcup_{i=1}^n I_i, p \in I \) and define \( \varrho: \Lambda \to \Lambda \) by \( \varrho(\lambda_p) = 0 \) and \( \varrho(\lambda_j) = \lambda_j \) for \( j \neq p \). By Lemma 3.11 we have
\[
0 = d^n f_\Lambda(\mathcal{U}_{\varrho}(x))(\mathcal{E}_\varrho^{(k)}(y_1), \ldots, \mathcal{E}_\varrho^{(k)}(y_n)) = F_{\varrho}^{(k)}(d^n f_\Lambda(x)(y_1, \ldots, y_n)).
\]
In other words, all components that do not contain \( \lambda_p \) are zero. Conversely, let \( p' \notin I \) and let \( \varrho': \Lambda \to \Lambda \) be a morphism given by \( \varrho'(\lambda_p') = 0 \) and \( \varrho'(\lambda_j) = \lambda_j \) for \( j \neq p' \). Then, we have
\[
d^n f_\Lambda(\mathcal{U}_{\varrho'}(x))(\mathcal{E}_{\varrho'}^{(k)}(y_1), \ldots, \mathcal{E}_{\varrho'}^{(k)}(y_n)) = d^n f_\Lambda(x)(y_1, \ldots, y_n),
\]
but all components of \( F_{\varrho'}^{(k)}(d^n f_\Lambda(x)(y_1, \ldots, y_n)) \) that contain \( \lambda_{p'} \) vanish. It follows that \( d^n f_\Lambda(x)(y_1, \ldots, y_n) \in \lambda_1 \cdots \lambda_{m_n} F_{\mathcal{U}_\mathbb{R}} \). Finally, assume that the sets \( I_i \) are not pairwise disjoint, for instance let \( p'' \) occur in \( r > 1 \) sets. For \( c \in \mathbb{R} \), we define a morphism \( \varrho'': \Lambda \to \Lambda \) by \( \varrho''(\lambda_{p''}) := c \lambda_{p''} \) and \( \varrho''(\lambda_j) := \lambda_j \) for \( j \neq p'' \). We have
\[
d^n f_\Lambda(\mathcal{U}_{\varrho''}(x))(\mathcal{E}_{\varrho''}^{(k)}(y_1), \ldots, \mathcal{E}_{\varrho''}^{(k)}(y_n)) = c d^n f_\Lambda(x)(y_1, \ldots, y_n).
\]
But we also have \( \mathcal{F}_{\varrho''}^{(k)}(d^n f_\Lambda(x)(y_1, \ldots, y_n)) I = c (d^n f_\Lambda(x)(y_1, \ldots, y_n)) I \), which implies \( d^n f_\Lambda(x)(y_1, \ldots, y_n) = 0 \).

The next lemma, a variation of [11, Proposition 2.16, p.578], is one of the rare cases where the proof for superdomains does not automatically translate to \( k \)-superdomains. In a sense, it shows the infinitesimal character of the generators \( \lambda_i \).

**Lemma 3.13.** Let \( E, F \in \text{SVec}_{\mathcal{C}}, k \in \mathbb{N}_0 \cup \{ \infty \}, \mathcal{U} \subseteq F_{\mathcal{C}}^{(k)} \) be an open subfunctor and \( f: \mathcal{U} \to F_{\mathcal{C}}^{(k)} \) be a natural transformation such that \( f_\Lambda \) is smooth for all \( \Lambda \in \text{Gr}^{(k)} \). Let \( 1 \leq p \leq k, x \in \mathcal{U}_\mathbb{R} \setminus F_{\mathcal{C}}^{(k)}_{\lambda_p} \) and \( y \in F_{\mathcal{C}}^{(k)}_{\lambda_p} \). Then, we have
\[
f_\Lambda(x + y) = f_\Lambda(x) + df_\Lambda(x)(y).
\]

**Proof.** Let \( c \in \mathbb{R} \). We define a morphism \( \varrho_c: \Lambda \to \Lambda \) by \( \varrho_c(\lambda_p) := c \lambda_p \) and \( \varrho_c(\lambda_i) := \lambda_i \) for \( i \neq p \). Then \( \mathcal{U}_{\varrho_c}(x) = x \) and \( \mathcal{F}_{\varrho_c}^{(k)}(y) = cy \). Therefore, we have
\[
f_\Lambda(\mathcal{F}_{\varrho_c}^{(k)}(x + y)) - f_\Lambda(\mathcal{F}_{\varrho_c}^{(k)}(x)) = 0 = \mathcal{F}_{\varrho_c}^{(k)}(f_\Lambda(x + y) - f_\Lambda(x))
\]
and thus \( f_\Lambda(x + y) - f_\Lambda(x) \in F_{\mathcal{C}}^{(k)}_{\lambda_p} \). It follows that
\[
c \cdot f_\Lambda^{(1)}(x, y, c) = f_\Lambda(\mathcal{F}_{\varrho_c}^{(k)}(x + y)) - f_\Lambda(\mathcal{F}_{\varrho_c}^{(k)}(x)) = \mathcal{F}_{\varrho_c}^{(k)}(f_\Lambda(x + y) - f_\Lambda(x)) = c \cdot f_\Lambda^{(1)}(x, y, 1).
\]
Taking the limit \( c \to 0 \), we see that \( f^{[1]}_\Lambda(x, y, 0) = f^{[1]}_\Lambda(x, y, 1) \) or in other words \( f_\Lambda(x + y) - f_\Lambda(x) = df_\Lambda(x, y) \) (compare \[1\] Proposition 2.16, p.578).

Accordingly, we get the following variation of \[1\] Corollary 2.17, p.579).

**Proposition 3.14.** Let \( E, F \in \mathbf{SVec}_{lc} \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( \mathcal{U} \subseteq E^{(k)} \) be an open subfunctor and \( f: \mathcal{U} \to F^{(k)} \) be a natural transformation such that \( f_\Lambda \) is smooth for all \( \Lambda \in \mathbf{Gr}^{(k)} \). For \( x := x_0 + \sum_{i \in \mathbb{P}_+} x_i \in \mathcal{U}_n \), where \( n \leq k \), \( x_0 \in \mathcal{U}_R \) and \( x_1 \in \lambda_1 E^{(k)}_\mathcal{U} \), we have

\[
f_\Lambda(x) = f_\Lambda(x_0) + \sum_{I \in \mathbb{P}^+} \sum_{\omega \in \mathcal{P}(I)} d^{(\omega)} f_\Lambda(x_0)(x_{\omega_1}, \ldots, x_{\omega_\ell(\omega)}).
\]

**Proof.** We first define a suitable partition of \( \mathbb{P}^+ \). Let \( \mathcal{I}_1 := \{\{1\}\} \) and \( \mathcal{I}_j := \mathbb{P}^+ \setminus \mathbb{P}^+_j \) for \( 1 < j \leq n \), i.e., \( \mathcal{I}_j \) contains all subsets that contain \( j \) but no larger index. Set \( x_{\mathcal{I}_j} := \sum_{I \in \mathcal{I}_j} x_i \) then we can write \( x = x_0 + \sum_{j=1}^n x_{\mathcal{I}_j} \). We prove the proposition by induction on the largest index of an odd generator appearing in \( x \). Lemma 3.13 gives us the induction basis. Assume that the formula holds for \( 1 \leq m < n \), i.e., assume that

\[
f_\Lambda(x_0 + \sum_{j=1}^m x_{\mathcal{I}_j}) = f_\Lambda(x_0) + \sum_{I \in \mathbb{P}^+} \sum_{\omega \in \mathcal{P}(I)} d^{(\omega)} f_\Lambda(x_0)(x_{\omega_1}, \ldots, x_{\omega_\ell(\omega)}).
\]

With this, differentiating in the direction of \( x_{\mathcal{I}_{m+1}} \) gives us

\[
df_\Lambda(x_0 + \sum_{j=1}^m x_{\mathcal{I}_j})(x_{\mathcal{I}_{m+1}}) = df_\Lambda(x_0)(x_{\mathcal{I}_{m+1}}) + \sum_{I \in \mathbb{P}^+} \sum_{\omega \in \mathcal{P}(I)} d^{(\omega)+1} f_\Lambda(x_0)(x_{\omega_1}, \ldots, x_{\omega_\ell(\omega)}, x_{\mathcal{I}_{m+1}})
\]

\[
= \sum_{I \in \mathbb{P}^+} \sum_{\omega \in \mathcal{P}(I)} d^{(\omega)} f_\Lambda(x_0)(x_{\omega_1}, \ldots, x_{\omega_\ell(\omega)}).
\]

It follows from Lemma 3.13 that the addition of both equations results in the desired formula for \( f_\Lambda(x_0 + \sum_{j=1}^m x_{\mathcal{I}_j}) \) (compare \[18\] Section 10.2, p.421).

The proposition can be rewritten in the following way.

**Lemma 3.15.** Let \( E, F \in \mathbf{SVec}_{lc} \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( \mathcal{U} \subseteq E^{(k)} \) be an open subfunctor and \( f: \mathcal{U} \to F^{(k)} \) be a natural transformation such that \( f_\Lambda \) is smooth for all \( \Lambda \in \mathbf{Gr}^{(k)} \). For \( \Lambda \in \mathbf{Gr}^{(k)} \) fix \( x \in \mathcal{U}_R \), \( n_0 \in \mathbb{E}^{(k)}_{\Lambda_0} \) and \( n_1 \in \mathbb{E}^{(k)}_{\Lambda_1} \). Then

\[
f_\Lambda(x + n_0 + n_1) = \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \cdot d^{m+l} f_\Lambda(x)(n_0, \ldots, n_0, n_1, \ldots, n_1)_{m \text{ times}}_{l \text{ times}}
\]

\[
= \sum_{i=0}^{\infty} \frac{1}{i!} \cdot d^i f_\Lambda(x)(n_0 + n_1, \ldots, n_0 + n_1).
\]
Proof. Let \( \Lambda = \Lambda_n \). By Lemma 3.12 the sums are finite and after multilinear expansion we only need to consider the summands that consist of partitions. For any partition \( I \in \mathcal{P}(n) \), \( I \in \mathcal{P}^+ \) in graded lexicographic order containing \( m \) even and \( l \) odd sets, there appear exactly \( m!l! \) copies of the term
\[
d^{m+l} f_\Lambda(x)(n_0, \ldots, n_l, 0, \ldots, 0)
\]
in the first sum because we must consider all permutations of \( n_0, \ldots, n_l, 0, \ldots, 0 \), resp. of \( n_l, \ldots, n_l, 0, \ldots, 0 \). The first equality follows then from Proposition 3.14 (see also [1, Proposition 2.21, p.582]). The second equality holds because multilinear expansion of \( d^{m+l} f_\Lambda(x)(n_0 + n_1, \ldots, n_0 + n_1) \) leads to \( \left( \begin{array}{c} m+l \\ l \end{array} \right) \) copies of \( d^{m+l} f_\Lambda(x)(n_0, \ldots, n_0, n_1, \ldots, n_1) \) \((m \text{ times } n_0 \text{ and } l \text{ times } n_1)\)
\[
\frac{1}{(m+l)!} = \frac{1}{m!l!}.
\]
\[\square\]

**Corollary 3.16.** Let \( E, F \in \mathbf{SVec}_{lc} \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( \mathcal{U} \subseteq \mathcal{E}(k) \) be an open subfunctor and \( f: \mathcal{U} \rightarrow \mathcal{F}(k) \) be a natural transformation such that \( f_\Lambda \) is smooth for all \( \Lambda \in \mathbf{Gr}(k) \). If additionally \( df_\Lambda(x_0): \mathcal{E}(k) \rightarrow \mathcal{F}(k) \) is \( \Lambda_\mathcal{F} \)-linear for all \( x_0 \in \mathcal{U}_\mathcal{F} \), then \( f \) is supersmooth.

Proof. Let \( \Lambda = \Lambda_n \). Because \( df_\Lambda(x_0) \) is \( \Lambda_\mathcal{F} \)-linear it follows by symmetry of the higher derivatives that \( d^m f_\Lambda(x_0): \mathcal{E}(k)^m \rightarrow \mathcal{F}(k)^m \) is \( \Lambda_\mathcal{F} \)-linear for all \( m \in \mathbb{N} \). Let \( x = x_0 + \sum_{I \in \mathcal{P}_n} x_I \) and \( y = y_0 + \sum_{I \in \mathcal{P}_n} y_I \) where \( x_I, y_I \in \lambda_I \mathcal{E}^{(m)} \) and \( t \in \Lambda_\mathcal{F} \). With Proposition 3.14 we calculate
\[
df_\Lambda(x)(ty) = d\left( f_\Lambda(x) + \sum_{I \in \mathcal{P}_n} \sum_{\omega(I)} d^{l(\omega)} f_\Lambda(x)(x_\omega + \ldots, x_\omega(\omega)) \right)(ty)
\]
\[
= df_\Lambda(x)(ty) + \sum_{I \in \mathcal{P}_n} \sum_{\omega(I)} d^{l(\omega) + 1} f_\Lambda(x)(x_\omega + \ldots, x_\omega(\omega), ty)
\]
\[
= t \left( df_\Lambda(x)(y) + \sum_{I \in \mathcal{P}_n} \sum_{\omega(I)} d^{l(\omega) + 1} f_\Lambda(x)(x_\omega + \ldots, x_\omega(\omega), y) \right).
\]
\[\square\]

This was already stated in [13, Theorem 3.3.2, p.391] without proof. The corollary simplifies some calculations considerably. A small example is the next lemma.

**Lemma 3.17.** Let \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( E, F \in \mathbf{SVec}_{lc} \) and \( \mathcal{U} \subseteq \mathcal{E}(k) \) be an open subfunctor. If \( f: \mathcal{U} \rightarrow \mathcal{F}(k) \) is supersmooth, then \( df: \mathcal{U} \times \mathcal{E}(k) \rightarrow \mathcal{F}(k) \) is supersmooth as well.

Proof. By Corollary 3.16 it suffices to calculate
\[
d \left( df_\Lambda(x_0, y_0) \right)(t \cdot u, t \cdot v) = d^2 f_\Lambda(x_0)(y_0, t \cdot u) + df_\Lambda(x_0)(t \cdot v)
\]
\[
= t \cdot d^2 f_\Lambda(x_0)(y_0, u) + t \cdot df_\Lambda(x_0)(v) = t \cdot \left( d \left( df_\Lambda(x_0, y_0) \right)(v, u) \right),
\]
for \( (x_0, y_0) \in \mathcal{U}_\mathcal{F} \times \mathcal{E}^{(m)} \), \( (v, u) \in \mathcal{E}^{(m)} \times \mathcal{E}^{(m)} \) and \( t \in \Lambda_\mathcal{F} \).
\[\square\]

By induction, it follows that all higher derivatives of supersmooth maps are supersmooth again. A more general but also more involved version was proved in [1, Proposition 2.18, p.580].
We will now give an explicit description of supersmooth morphisms as so called skeletons, which is essential for almost all applications. It was already stated in [18, Proposition 3.3.3, p.391] and proofs can be found in [21, Theorem 4.11, p.20] or in higher generality in [1, Proposition 3.4, p.584].

**Definition 3.18.** Let \( n \in \mathbb{N} \), let \( E_0, \ldots, E_n \) and \( F \) be locally convex spaces and \( U \subseteq E_0 \) open. Denote by \( C^\infty(U, \mathcal{L}^n(E_1, \ldots, E_n; F)) \) the set of maps \( f: U \to \mathcal{L}^n(E_1, \ldots, E_n; F) \) such that

\[
 f^\wedge: U \times (E_1 \times \cdots \times E_n) \to F, \quad f^\wedge(x, v) := f_n(x)(v)
\]

is smooth. In this situation, we define

\[
 d^m f(x)(w, v) := \partial_{(w_0, \ldots, 0)} \cdots \partial_{(w_1, 0)} f^\wedge(x, v),
\]

for \( m \in \mathbb{N} \), \( x \in U \) and \( v, w = (w_1, \ldots, w_m) \in E_1 \times \cdots \times E_n \). Analogously, we define \( C^\infty(U, \text{Alt}^n(E_1; F)) \) as the set of maps \( f: U \to \text{Alt}^n(E_1; F) \) that are smooth in the above sense.

**Definition 3.19.** Let \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( E, F \in \text{SVec}_k \) and \( U \subseteq E_0 \) open. A \((k-)\text{skeleton}\) is a family of maps \((f_n)_{0 \leq n \leq k+1}\) such that \( f_n \in C^\infty(U, \text{Alt}^n(E_1; F_1)) \).

It will be convenient to set \( d^k f_n := f_n \) and let \( d^0 f_n(x)(w_1, \ldots, w_m, v) := d^0 f_n(x)(v) \) as well as \( d^m f_0(x)(w, v) := d^m f_0(x)(w) \) for \( x \in U \), \( v \in E_1^n \) and \( w = (w_1, \ldots, w_m) \in E_1^m \).

**Proposition 3.20** ([1, Proposition 3.4, p.584]). Let \( E, F \in \text{SVec}_k \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), \( U \subseteq \overline{E}^{(k)} \), \( V \subseteq \overline{F}^{(k)} \) be open subfunctors and \( f \in \mathcal{SC}^\infty(U, V) \). Then the equation

\[
 f_{\Lambda_k}(x + \sum_{l=1}^k \lambda_l y_l) = f_0(x) + \sum_{l=0}^k \sum_{\{i_1, \ldots, i_l\} \in \mathcal{P}_k} \lambda_i f_l(x)(y_{i_1}, \ldots, y_{i_l}),
\]

where \( x \in U_\mathbb{R} \) and \( y_l \in E_1 \), defines a \( k \)-skeleton \((f_n)_n \). For this skeleton, we have

\[
 (1) \quad f_{\Lambda_N}(x + n_0 + n_1) = \sum_{m, l=0}^\infty \frac{1}{m! l!} \cdot d^m f_l(x)(n_0, \ldots, n_0, n_1, \ldots, n_1),
\]

where \( x \in U_\mathbb{R} \), \( n_0 \in \overline{E}_{\Lambda_N}^{(k)} \) and \( n_1 \in \overline{E}_{\Lambda_N}^{(k)} \). Here it is understood that

\[
 d^m f_l(x)(\lambda_{i_1} v_{i_1}, \ldots, \lambda_{i_{m+l}} v_{i_{m+l}}) = \lambda_{i_1} \cdots \lambda_{i_{m+l}} d^m f_l(x)(v_1, \ldots, v_m)
\]

for \( v_1, \ldots, v_m \in E_0 \), \( v_{m+1}, \ldots, v_{m+l} \in E_1 \) and \( \{i_j\} \) even if \( 1 \leq j \leq m \) and odd if \( m+1 \leq j \leq m+l \). Conversely, every \( k \)-skeleton defines a supersmooth map via formula (1) and the skeleton of this map is the original one.

**Proof.** Using Lemma 3.15 instead of [1, Proposition 2.21, p.582], the proof follows in the same way as [1, Proposition 3.4, p.584]. For the reader’s convenience, we will sketch the steps using our notation. By Proposition 3.14 we have

\[
 f_{\Lambda_k}(x + \sum_{l=1}^k \lambda_l y_l) = \sum_{l=0}^k \sum_{\{i_1, \ldots, i_l\} \in \mathcal{P}_k} d^l f_{\Lambda_k}(x)(\lambda_{i_1} y_{i_1}, \ldots, \lambda_{i_l} y_{i_l}).
\]
The maps on the right-hand side are symmetric in $\lambda_i, y_j$ but swapping two odd generators leads to a sign change by the natural transformation property. With Lemma 3.12 one sees that this determines alternating maps in $y_{ij}$, where it is understood that the odd generators can be pulled out in order of their appearance. Now, one applies Proposition 3.14 to derive formula (1). Note that by supersmoothness, the alternating maps defined above determine $d^{m+l} f_{\Lambda_k}$ completely.

To see that the right-hand side of formula (1) defines a natural transformation for a given skeleton is straightforward and supersmoothness then follows directly or with Corollary 3.16. This supersmooth map has the original skeleton by a combinatorial argument similar to the one used for Lemma 3.15.

Remark 3.21. In the situation of the proposition above, we can use Proposition 3.13 instead of Lemma 3.15 to get

$$f_{\Lambda_k}(x + n_0 + n_1) = \sum_{\ell \in \Pi} \sum_{\omega \in \mathcal{P}(I)} \lambda_{\omega} d^{(\mathcal{E}(\omega))} f_{\mathcal{E}(\omega)}(x)(n_\omega),$$

where the partitions $\omega$ are in graded lexicographic order, $\lambda_{\omega} = \lambda_{\omega_1} \cdots \lambda_{\omega_{\ell(\omega)}}$ and $n_\omega := (n_0, \omega_1, \ldots, n_0, \omega_{\ell(\omega)}, n_1, \omega_{\ell(\omega)+1}, \ldots, n_1, \ell(\omega))$.

Remark 3.22. Let $f : \mathcal{U} \to \mathcal{V}$ be as in Proposition 3.20. We have already seen that $df : \mathcal{U} \times \mathcal{E}^{(k)} \to \mathcal{F}^{(k)}$ is supersmooth. For $\Lambda \in \text{Gr}^{(k)}$, $x \in \mathcal{U}_R$, $y \in E_0$ and $x_i, y_i \in E_i \otimes \Lambda_i^+$ set $u := x + x_0 + x_1$ and $v := y + y_0 + y_1$. Then use the proposition to calculate

$$df_{\Lambda}(u)(v) = \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \left( d^{m+1} f_1(x)(y, x_0, \ldots, x_0, x_1, \ldots, x_1) ight)$$

$$+ m \cdot d^m f_1(x)(y_0, x_0, \ldots, x_0, x_1, \ldots, x_1)$$

$$+ l \cdot d^m f_1(x)(x_0, \ldots, x_0, y_1, x_1, \ldots, x_1)$$

$$= \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \left( d^{m+1} f_1(x)(y + y_0, x_0, \ldots, x_0, x_1, \ldots, x_1) \right)$$

$$+ \sum_{m,l=0}^{\infty} \frac{1}{m!l!} \left( d^m f_{l+1}(x)(x_0, \ldots, x_0, y_1, x_1, \ldots, x_1) \right).$$

We see that the skeleton of $df$ is given by

$$(df)_n = df_n(\text{pr}_{\mathcal{U}_R}, \text{pr}_{E_0})(\text{pr}_1, \ldots, \text{pr}_1) + n \cdot \mathfrak{X}^n f_n(\text{pr}_{\mathcal{U}_R})(\text{pr}_2, \text{pr}_1, \ldots, \text{pr}_1),$$

with the projections $\text{pr}_{\mathcal{U}_R} : \mathcal{U}_R \times E_0 \to \mathcal{U}_R$, $\text{pr}_{E_0} : \mathcal{U}_R \times E_0 \to E_0$, the projection to the first component $\text{pr}_1 : E_1 \times E_1 \to E_1$ and the projection to the second argument $\text{pr}_2 : E_1 \times E_1 \to E_1$.

In the sequel, we will not differentiate between supersmooth morphisms and their skeletons. In other words, if $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are $k$-superdomains and $f \in \mathcal{S}^{\mathcal{C}^\infty}(\mathcal{U}, \mathcal{V})$ has the skeleton $(f_n)_n$, we will write $(f_n)_n : \mathcal{U} \to \mathcal{V}$. If additionally $g \in \mathcal{S}^{\mathcal{C}^\infty}(\mathcal{V}, \mathcal{W})$ has the skeleton $(g_n)_n$ we let $(f_n)_n \circ (g_n)_n$ be the skeleton of $g \circ f$. For this composition the concrete formula is given as follows.
Proposition 3.23 (compare [1] Proposition 3.7, p.586). Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $E \in \text{SVec}_c$, $\mathcal{U} \subseteq E^{(k)}$ be an open subfunctor and $\mathcal{V}, \mathcal{W} \in \text{SDom}^{(k)}$. For two supersmooth morphisms $(f_r)_r : \mathcal{U} \to \mathcal{V}$, $(g_r)_r : \mathcal{V} \to \mathcal{W}$ the skeleton $(h_n)_n := (g_r)_r \circ (f_r)_r$ is given by $h_0 := g_0 \circ f_0$ for $n = 0$ and otherwise by

$$h_n(x)(v) = \sum_{m,l,\sigma \in \mathcal{G}_n, (\alpha, \beta) \in I_{m,l}^n} \frac{\text{sgn}(\sigma)}{m!l!\alpha!\beta!} d^n_{\alpha \beta}(f_0(x))((f_{\alpha} \times f_{\beta})(x)(v^\sigma))$$

for $x \in \mathcal{U}_k$ and $v = (v_1, \ldots, v_n) \in E_1^n$, where $v^\sigma := (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$,

$$I_{m,l}^n := \left\{ (\alpha, \beta) \in (2\mathbb{N})^m \times (2\mathbb{N} + 1)^l \mid \forall j : \alpha_j > 0, |\alpha| + |\beta| = n \right\}.$$

$f_{\alpha} := f_{\alpha_1} \times \cdots \times f_{\alpha_m}$, $f_{\beta} := f_{\beta_1} \times \cdots \times f_{\beta_l}$ and

$$\alpha! = \alpha_1! \cdots \alpha_m!, \quad \beta! = \beta_1! \cdots \beta_l!.$$

Proof. By Proposition 3.20 $(h_n)_n$ is defined by

$$g_{\Lambda}(f_{\Lambda}(x + y)) = \sum_{i=0}^{\infty} \frac{1}{i!} h_i(x)(y, \ldots, y)$$

for all $\Lambda \in \mathcal{G}_k$. For $i \in \{0, 1\}$, we let

$$n_i := \sum_{i \in 2\mathbb{N} \setminus \{i\}} \frac{1}{i!} f_i(x)(y, \ldots, y).$$

Together with Proposition 3.20 this implies

$$g_{\Lambda}(f_{\Lambda}(x + y)) = \sum_{m,l=0}^{\infty} \frac{1}{m!l!} d^n_{\alpha \beta}(f_0(x))(n_0, \ldots, n_0, n_1, \ldots, n_1).$$

Since in formula (2), $h_n$ only depends on $(f_r)_{r \leq n}$ and $(g_r)_{r \leq n}$, it suffices to compare the component containing all odd generators of $\Lambda = \Lambda_n$, i.e., the component $I := \{1, \ldots, n\}$. The formula follows then by trivial induction. Multilinear expansion of the $n_i$ in formula (3) shows that exactly those summands contribute, where the indices of all occurring $f_i$ add up to $n$. In other words exactly those containing $f_\alpha \times f_\beta$ with $(\alpha, \beta) \in I_{m,l}^n$. Applying multilinear expansion to $y$, we see that for every $(\alpha, \beta) \in I_{m,l}^n$ exactly all permutations $\lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} \frac{1}{m!l!} (f_\alpha \times f_\beta)(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ for $\sigma \in \mathcal{G}_n$ appear in formula (3) since equal indices cancel each other out. The sign in the formula is explained by $\lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} = \text{sgn}(\sigma) \lambda_I$. \qed

Remark 3.24. Formula (2) was already stated in [13] Proposition 3.3.3, p.91 f.] but the first proof in the literature was [1] Proposition 3.7, p.586. Unfortunately, the proof is incomplete and there is a small mistake in the formula (the original one in [13] is correct), which is why we decided to give the proof in its entirety. To see that our formula differs from the one proposed in [1], consider that in the situation of Proposition 3.23 the latter leads to

$$\sum_{\sigma \in \mathcal{G}_2} \frac{1}{2} dg_1(f_0(x))(f_2(x)(v^\sigma), f_1(x)(v^\sigma)) (v_1, v_2, v_3) = 0.$$
while in general
\[
\sum_{\sigma \in \mathcal{G}_n} \frac{\text{sgn}(\sigma)}{2} dg_1(f_0(x))(f_2(x)(\bullet), f_1(x)(\bullet))(v_1, v_2, v_3)^\sigma \neq 0.
\]

**Lemma 3.25.** Let \( k \in \mathbb{N} \cup \{\infty\} \), \( E, F \in \text{SVec}_{lc} \) and \( \mathcal{U} \subseteq \mathcal{F}^{(k)}_c, \mathcal{V} \subseteq \mathcal{F}^{(k)}_c \) be open subfunctors. A supersmooth morphism \( f: \mathcal{U} \to \mathcal{V} \) is an isomorphism in \( \text{SDom}^{(k)} \) if and only if \( f_{\Lambda_1}: \mathcal{U}_{\Lambda_1} \to \mathcal{V}_{\Lambda_1} \) is a diffeomorphism. In this case, using the same notation as in formula (2), the inverse \( g \) has the skeleton
\[
g_0: \mathcal{V}_R \to \mathcal{U}_R, \quad g_0(x') := f_0^{-1}(x'),
\]
\[
g_1: \mathcal{V}_R \to \text{Alt}^1(F_1; E_1), \quad g_1(x') := f_1(g_0(x'))^{-1} \quad \text{and}
\]
\[
g_n: \mathcal{V}_R \to \text{Alt}^n(F_1; E_1),
\]
\[
g_n(x')(v') := \sum_{m, l < n, (\alpha, \beta) \in I_{m, l, t}} \frac{\text{sgn}(\sigma)}{m! l! \alpha! \beta!} d^m g_l(f_1(x))(f_\alpha \times f_\beta)(g_0(x'))(v^\sigma),
\]
where \( n > 1 \), \( v' = (v_1', \ldots, v_n') \in F_1^n \) and \( v := (g_1(x')(v_1'), \ldots, g_1(x')(v_n')) \in E_1^n \).

**Proof.** If a supersmooth morphism \( f: \mathcal{U} \to \mathcal{V} \) is invertible, then clearly \( f_{\Lambda} \) is a diffeomorphism for every \( \Lambda \in \text{Gr}^{(k)} \). Conversely, let \( f_{\Lambda_1} \) be a diffeomorphism. Then \( f_{\Lambda_1}(x + \lambda_1 v) = f_0(x) + \lambda_1 f_1(x)(v) \) for all \( x \in \mathcal{U}_R \) and \( v \in E_1 \). A direct calculation shows that \( g_{\Lambda_1}(x' + \lambda_1 v') := g_0(x') + \lambda_1 g_1(x')(v') \) is the inverse of \( f_{\Lambda_1} \). With the supersmooth morphism \( (g_n)_n: \mathcal{V} \to \mathcal{U} \), we calculate
\[
((g_r)_r \circ (f_r)_r)_n(x)(v) = \sum_{m, l < n, (\alpha, \beta) \in I_{m, l, t}} \frac{\text{sgn}(\sigma)}{m! l! \alpha! \beta!} d^m g_l(f_0(x))(f_\alpha \times f_\beta)(x)(v^\sigma)
\]
\[
+ \sum_{\sigma \in \mathcal{G}_n} \frac{1}{n!} g_n(f_0(x))(f_1 \times \cdots \times f_1)(x)(v^\sigma),
\]
for \( n > 1 \), \( x \in \mathcal{U}_R \), \( v \in E_1^n \). Note that in the second summand the sum over \( \mathcal{G}_n \) together with the factor \( \frac{1}{n!} \) can be omitted because the expression is already alternating. With \( (f_1(x)(v_1), \ldots, f_1(x)(v)) := v' \) and \( f_0(x) := x' \) it follows from the definition of \( g_n \) that \( ((g_r)_r \circ (f_r)_r)_n = 0 \). This implies \( (g_r)_r \circ (f_r)_r = (\text{id}_{E_1}, 0, 0, \ldots) \), which is the skeleton of the identity \( \text{id}_U: \mathcal{U} \to \mathcal{U} \). Thus, \( (f_n)_n \) has a left inverse. Since the same construction also works for \( (g_n)_n \), the left inverse of \( (f_n)_n \) also has a left inverse. Therefore, \( (g_n)_n \) is the inverse of \( (f_n)_n \) and \( f \) is invertible in \( \text{SDom}^{(k)} \). \( \square \)

In general, it is quite difficult to check that smooth bijective maps between locally convex spaces are diffeomorphisms. However, if the map has the form of \( f_{\Lambda_1} \) in the above lemma, a result of Hamilton can be directly generalized to the locally convex case. We do not need this result in the sequel but since it might be of interest for inverting supersmooth maps, we state it nevertheless.
Lemma 3.26 (compare [11] Theorem 5.3.1, p.102). Let \( E_0, E_1 \) and \( F_1 \) be locally convex spaces, \( U \subseteq E_0 \) open and \( f: U \times E_1 \rightarrow F_1 \) be smooth such that \( f_x := f(x, \cdot): E_1 \rightarrow F_1 \) is linear for all \( x \in U \). If \( f_x \) is invertible for all \( x \in U \) and \( g: U \times F_1 \rightarrow E_1, (x, v) \mapsto f_x^{-1}(v) \) is continuous, then \( g \) is smooth. Moreover, we have
\[
d_1g(x, v)(u) = -g(x, d_1f(x, g(x, v))(u))
\]
for \( x \in U, v \in F_1 \) and \( u \in E_0 \).

Proof. For \( x \in U, v \in F_1, u \in E_0 \) and \( t \in ]0, \infty[ \) such that \( x + tu \in U \) for all \( 0 < s \leq t \), we calculate
\[
g(x + tu, v) - g(x, v) = f_x^{-1}(v) - f_x^{-1}(v)
\]
\[
f_x^{-1}(f_x(f_x^{-1}(x))) - f_x^{-1}(f_x(f_x^{-1}(x)))
\]
\[
f_x^{-1}(f_x(f_x^{-1}(v)) - f_x^{-1}(f_x(f_x^{-1}(v)))
\]
\[
g(x + tu, f_x(f_x^{-1}(v)) - f(x + tu, g(x, v)))
\]
\[
= -g(x + tu, f(x + tu, g(x, v)) - f(x, g(x, v)))
\]
\[
\xrightarrow{t \rightarrow 0} -g(x, d_1f(x, g(x, v))(u)) = d_1g(x, v)(u).
\]

Because \( g \) is continuous, so is \( d_2g(x, v)(w) = g(x, w) \) and thus \( dg \) is continuous. From this, it follows inductively that \( g \) is smooth. \( \square \)

It is easy to generalize this to the situation where additionally a diffeomorphism \( f_0: U \rightarrow V \) between open sets of locally convex spaces is involved.

3.2.1. Generalizations. One obvious generalization is to consider a differential calculus for other base fields (or even rings) than \( \mathbb{R} \). A robust framework for this is provided by [6] and then further developed for the super case in [11]. In the most general case, one has a unital commutative Hausdorff topological ring \( R \) such that the group of units \( R^\times \) is dense, i.e., integers need not necessarily be invertible. For simplicity’s sake, we formulated our results over \( \mathbb{R} \) but we made a conscious effort to make them easily adaptable to more general situations.

In this way Lemma 3.11 through Proposition 3.14 can easily be shown to hold in the most general case. While Corollary 3.16 and Lemma 3.17 also translate, our definition of supersmoothness just means \( C^1_{MS} \) (together with smoothness over \( \mathbb{R} \)) in the terminology of [11]. Note however that \( C^1_{MS} \) is equivalent to \( C^1_{MS} \) if \( R \) is an \( \mathbb{Q} \)-algebra and one has smoothness over \( R \) (see [11] Proposition 2.18, p.580]). In this case Lemma 3.15 Proposition 3.20, Proposition 3.23 and Lemma 3.23 carry over as well.

It should be noted that Remark 3.21 enables us to show an analog to Proposition 3.20 if not all integers are invertible in \( R \), i.e., supersmooth maps are given by something like skeletons even in the most general case. The resulting analog to the composition formula from Proposition 3.23 can
be obtained with general results about multilinear bundles (compare Remark A.3) and a similar induction as in Lemma 3.25 leads to an inversion formula (compare [5, Theorem MA.6(2), p.172]).

The second apparent generalization is to define morphisms of finite differentiability order \( n \in \mathbb{N}_0 \). Given only \( k \)-superdomains with \( k \leq n \), one can simply define \( k \)-skeletons where the differentiability class of the components is appropriately chosen. For a more detailed discussion see [18, 10.1, p.420f.].

3.3. Supermanifolds. The construction of supermanifolds from superdomains is conceptually very close to the respective construction of manifolds. In the categorical approach proposed by Molotkov in [18], one defines a Grothendieck topology on \( \text{Top}^{\text{Gr}} \) that takes the same role as the usual topology in the manifold case. As model space one uses functors of the form \( E \) for \( E \in \text{SVec}_{lc} \) with open subfunctors \( U \) as the open subsets (respectively functors isomorphic to such functors). A supermanifold is then a functor \( \mathcal{M} \in \text{Man}^{\text{Gr}} \) together with an atlas consisting of natural transformations \( \varphi : U \rightarrow \mathcal{M} \), such that the change of charts is supersmooth. Here a technical problem arises. In this approach, the intersection of two chart domains in \( \mathcal{M} \) is defined as a fiber product in the category \( \text{Man}^{\text{Gr}} \), which is not guaranteed to be a superdomain. This has to be demanded in the definition. We avoid this and other technicalities by using concrete definitions of the model spaces. For a concise version of the categorical approach see [1, p.591 ff.].

We introduce \( k \)-supermanifolds in the same way by considering functors \( \text{Man}^{\text{Gr}}(k) \) and obtain respective categories \( \text{SMan}^{(k)} \) for \( k \in \mathbb{N}_0 \cup \{ \infty \} \). One has the restriction functors \( \pi_{mn} : \text{SMan}^{(m)} \rightarrow \text{SMan}^{(n)} \) for \( n \leq m \) and the inclusion functors \( \iota_k : \text{SMan}^{(0)} \rightarrow \text{SMan}^{(k)} \) and \( \iota_k : \text{SMan}^{(1)} \rightarrow \text{SMan}^{(k)} \), which play an important part in understanding the structure of supermanifolds. Note in particular that \( \text{SMan}^{(0)} \cong \text{Man} \) and \( \text{SMan}^{(1)} \cong \text{VBun} \).

These statements are not particularly difficult to prove and were already stated in [16]. Noteworthy new results include the following. For any supermanifold \( \mathcal{M} \), we show that \( \mathcal{M}^{\Lambda_n} \) has the natural structure of a so called multilinear bundle of degree \( n \) over \( \mathcal{M}_R \). What is more, \( \mathcal{M}^{\Lambda_n} \) forms an inverse system of multilinear bundles which enables us to obtain a functor 

\[
\text{SMan} \rightarrow \text{Man}, \quad \mathcal{M} \mapsto \lim_{\leftarrow n} \mathcal{M}_{\Lambda_n}
\]

in Theorem 3.37. As already mentioned, this functor has good properties such as respecting products. Another important result is the characterization of purely even supermanifolds in terms of higher tangent bundles of the base manifold in Proposition 3.42.

**Definition 3.27.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), \( E \in \text{SVec}_{lc} \) and \( \mathcal{M} \in \text{Top}^{\text{Gr}(k)} \). Recall Definition 3.6. We call a covering \( A := \{ \varphi^\alpha : \mathcal{U}^\alpha \rightarrow \mathcal{M} : \alpha \in A \} \) of \( \mathcal{M} \) such that all \( U^\alpha \) are open subfunctors of \( E^{(k)} \) an atlas of \( \mathcal{M} \) if the natural transformations

\[
\varphi_{\alpha\beta} := (\varphi^\beta)^{-1} \circ \varphi^\alpha|_{U^\alpha \beta} : U^\alpha \beta \rightarrow U^\beta \alpha
\]
are supersmooth for all $\alpha, \beta \in A$. Two atlases $\mathcal{A}$ and $\mathcal{B}$ are called 
\emph{equivalent} if their union $\mathcal{A} \cup \mathcal{B}$ is again an atlas. As with ordinary manifolds, 
this clearly defines an equivalence relation and we call the pair $(\mathcal{M}, [\mathcal{A}])$ a
\emph{k-supermanifold modelled on $E$}. If $k = \infty$ we also simply call $\mathcal{M}$ a \emph{supermanifold}.
We will usually omit $[\mathcal{A}]$ from our notation and if we talk about 
an atlas of a supermanifold, it is meant to belong to this equivalence class.

An element of any of the equivalent atlases will be called a \emph{chart}.

An atlas of a supermanifold, it is meant to belong to this equivalence class.

This clearly defines an equivalence relation and we call the pair $(\mathcal{M}, [\mathcal{A}])$ an atlas.

A \emph{morphism} $f : \mathcal{M} \rightarrow \mathcal{N}$ of $k$-supermanifolds $\mathcal{M}$ and $\mathcal{N}$ is a natural
transformation $f : \mathcal{M} \rightarrow \mathcal{N}$ such that for any chart $\varphi : \mathcal{U} \rightarrow \mathcal{M}$
and any chart $\psi : \mathcal{V} \rightarrow \mathcal{N}$

$$
\psi^{-1} \circ f \circ \varphi^{-1}(\psi(V)) : (f \circ \varphi)^{-1}(\psi(V)) \rightarrow \mathcal{V}
$$
is supersmooth.

Note that the definition of morphisms between $k$-supermanifolds is independent
of the atlases, because change of charts satisfies the cocycle condition.

Thus, we get for every $k \in \mathbb{N}_0 \cup \{\infty\}$ the category $\text{SMan}^{(k)}$ of
$k$-supermanifolds. As always, we set $\text{SMan} := \text{SMan}^{(\infty)}$. For two $k$-
supermanifolds $\mathcal{M}, \mathcal{N}$, we denote by $\text{SC}^{\infty}(\mathcal{M}, \mathcal{N})$ the set of supersmooth
morphisms $f : \mathcal{M} \rightarrow \mathcal{N}$.

**Definition 3.28.** A $k$-supermanifold $\mathcal{M}$ modelled on $E \in \text{SVec}_{lc}$ is a \emph{finite-
dimensional}, Banach or Fréchet $k$-supermanifold if $E$ is so. If $E_1 = \{0\}$,
then $\mathcal{M}$ is purely even and if $E_0 = \{0\}$, then $\mathcal{M}$ is purely odd. We call $\mathcal{M}_{\mathbb{R}}$
the \emph{base manifold} of $\mathcal{M}$ and say that $\mathcal{M}$ is $\sigma$-\emph{compact} if $\mathcal{M}_{\mathbb{R}}$ is $\sigma$-
compact.

**Remark 3.29.** Since the objects of $\text{Man}$ are defined to be Hausdorff, every
supermanifold is Hausdorff when considered as a functor to $\text{Top}$. If one
allows non-Hausdorff manifolds in the definition, it is easily seen that a
supermanifold $\mathcal{M}$ is Hausdorff if and only if its base manifold is Hausdorff.
In fact, this follows because $\mathcal{M}_\Lambda$ is a fiber bundle over $\mathcal{M}_{\mathbb{R}}$ whose typical
fiber is Hausdorff by Theorem 3.37 below.

To get some intuition for supermanifolds, we start with several simple
observations.

**Lemma 3.30.** Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \text{SMan}^{(k)}$ with atlas
$\{\varphi^\alpha : \mathcal{U}^\alpha \rightarrow \mathcal{M} : \alpha \in A\}$.

(a) For every $\Lambda \in \text{Gr}^{(k)}$, \{$(\varphi^{\alpha}_{\Lambda, \mathcal{U}^\alpha_{\Lambda}})^{-1} : \varphi^\alpha_{\Lambda}(\mathcal{U}^\alpha_{\Lambda}) \rightarrow \mathcal{U}^\alpha_{\Lambda} : \alpha \in A\}$ is
an atlas of $\mathcal{M}_\Lambda$.

(b) For $n \leq m < k + 1$, the inclusions $\mathcal{M}_{\jmath_{m,m}} : \mathcal{M}_{\jmath_{n,m}} \rightarrow \mathcal{M}_{\jmath_{m,m}}$
are \emph{topological embeddings} and $\mathcal{M}_{\jmath_{m,m}}$ is a \emph{closed submanifold}
of $\mathcal{M}_{\jmath_{m,m}}$.

(c) For $n \leq m < k + 1$, the projections $\mathcal{M}_{\varepsilon_{m,n}} : \mathcal{M}_{\jmath_{m,m}} \rightarrow \mathcal{M}_{\jmath_{m,m}}$
are \emph{surjective}.

**Proof.** (a) This is obvious from the definition of a supermanifold, since the
sets $\varphi^\alpha_{\Lambda}(\mathcal{U}^\alpha_{\Lambda})$ form an open cover of $\mathcal{M}_\Lambda$; $\varphi^\alpha_{\Lambda}(\mathcal{U}^\alpha_{\Lambda})$
is a homeomorphism and the change of charts is smooth.
(b) Let \( \mathcal{M} \) be modelled on \( E \in \text{SVec}_{c} \). In the charts defined by \( \varphi_{\Lambda_{n}}^{a} \) and \( \varphi_{\Lambda_{m}}^{a} \) as in (a), the map \( \mathcal{M}_{\eta_{m}} \) has the form \( U_{\eta_{m}} \), and we have \( U_{\eta_{n}}^{a} \cong U_{\eta_{m}}^{a}(U_{\Lambda_{n}}^{a}) = U_{\Lambda_{m}}^{a} \cap \mathcal{E}_{\Lambda_{n}}^{a} \). By naturality, we have \( \varphi_{\Lambda_{n}}^{a}(U_{\eta_{n}}^{a}(U_{\Lambda_{n}}^{a})) = \mathcal{M}_{\Lambda_{m}} \cap \mathcal{M}_{\eta_{m}}(\varphi_{\Lambda_{n}}^{a}(U_{\Lambda_{n}}^{a})) \).

(c) In the charts defined by \( \varphi_{\Lambda_{n}}^{a} \) and \( \varphi_{\Lambda_{m}}^{a} \) as in (a), the map \( \mathcal{M}_{\epsilon_{m}} \) has the form \( U_{\epsilon_{m}} \), which clearly defines a surjective map. \( \Box \)

Part (c) of this lemma already suggests that \( \mathcal{M}_{\Lambda_{m}} \) is some kind of fiber bundle over \( \mathcal{M}_{\Lambda_{n}} \). As we discuss below, this fiber bundle structure can be accurately described via multilinear bundles. Like ordinary manifolds, supermanifolds and morphisms thereof arise from local data.

Proposition 3.31 (see [1] Proposition 3.23, p.593). Let \( k \in \mathbb{N}_{0} \cup \{0\} \) and \( E \in \text{SVec}_{c} \). Let further \( (U^{a})_{\alpha \in A} \) be a family of open subfunctors of \( E^{(k)} \) and \( U^{a} \subseteq U^{a} \) be open subfunctors for \( \alpha, \alpha' \in A \) such that \( U^{aa} = U^{a} \). Finally, let \( \varphi_{aa}^{a}, \varphi_{aa'}^{a} \rightarrow U^{a} \) be isomorphisms in \( \text{SDom}^{(k)} \) such that we have \( \varphi_{aa}^{a} = \text{id}_{U^{a}} \) and \( \varphi_{aa''}^{a} = \varphi_{aa'}^{a} \circ \varphi_{aa''}^{a} \) on \( U^{a} \cap U^{a} \), for all \( \alpha, \alpha', \alpha'' \in A \).

Then there exists a unique superisomorphism, unique \( k \)-supermanifold \( \mathcal{M} \) with an atlas \( \{ \varphi^{a} : U^{a} \rightarrow \mathcal{M} : \alpha \in A \} \) such that the change of charts coincides with the \( \varphi_{aa}^{a} \) defined above.

Moreover, let \( \mathcal{N} \in \text{SMAN}^{(k)} \) have the atlas \( \{ \psi^{b} : V^{b} \rightarrow \mathcal{N} : \beta \in B \} \) and let \( \hat{U}^{ab} \subseteq U^{a} \) for \( \alpha \in A \) and \( \beta \in B \) such that \( \bigcup_{\beta \in B} \hat{U}^{ab} = U^{a} \). If \( f^{ab} : \hat{U}^{ab} \rightarrow V^{b} \) is a family of supersmooth maps such that \( \psi^{b} = f^{ca} \circ f^{ab} \circ \varphi_{aa}^{a} \), then there exists a unique supersmooth morphism \( f : \mathcal{M} \rightarrow \mathcal{N} \) with \( f^{ab} = (\psi^{b})^{-1} \circ f \circ \varphi_{aa}^{a} \).

Proof. This follows exactly as in [1] Proposition 3.23, p.593. Essentially, we use the well-known equivalent statement for ordinary manifolds for every \( \Lambda \in \text{Gr}^{(k)} \) to construct \( \mathcal{M} \), resp. \( f_{\Lambda} \), and the rest follows from naturality. Note that \( \bigcup_{\beta \in B} \hat{U}^{ab} = U^{a} \) implies \( \bigcup_{\beta \in B} \hat{U}_{\Lambda}^{ab} = U_{\Lambda}^{a} \) for all \( \Lambda \in \text{Gr}^{(k)} \). \( \Box \)

Lemma 3.32 ([8] Corollary 6.2.2, p.409). Let \( k \in \mathbb{N} \cup \{\infty\} \) and \( \mathcal{M}, \mathcal{N} \in \text{SMAN}^{(k)} \). A supersmooth morphism \( f : \mathcal{M} \rightarrow \mathcal{N} \) is an isomorphism in \( \text{SMAN}^{(k)} \) if and only if \( f_{\Lambda} : \mathcal{M}_{\Lambda} \rightarrow \mathcal{N}_{\Lambda} \) is a diffeomorphism.

Proof. Clearly, \( f : \mathcal{M} \rightarrow \mathcal{N} \) is an isomorphism if and only if \( f_{\Lambda} : \mathcal{M}_{\Lambda} \rightarrow \mathcal{N}_{\Lambda} \) is bijective and the maps \( f_{\Lambda}^{-1} \) define a supersmooth natural transformation for every \( \Lambda \in \text{Gr}^{(k)} \). In particular, \( f_{\Lambda} \) is a diffeomorphism in this situation.

Let \( \{ \varphi^{a} : U^{a} \rightarrow \mathcal{M} : \alpha \in A \} \) be an atlas of \( \mathcal{M} \) and \( \{ \psi^{b} : V^{b} \rightarrow \mathcal{N} : \beta \in B \} \) be an atlas of \( \mathcal{N} \). Let \( f : \mathcal{M} \rightarrow \mathcal{N} \) be supersmooth such that \( f_{\Lambda} \) is a diffeomorphism.

For all \( \alpha \in A \) and \( \beta \in B \) we define \( \hat{U}^{ab} := (f \circ \varphi^{a})^{-1}(\psi^{b}((\psi^{b})) \subseteq U^{a} \) and \( \hat{V}^{ab} := f^{ab}(f \circ \varphi^{a})^{-1}(\psi^{b}((\psi^{b})) \subseteq V^{b} \) and let

\[
\hat{f}^{ab} := (\psi^{b})^{-1} \circ f \circ \varphi^{a} |_{\hat{U}^{ab}} : \hat{U}^{ab} \rightarrow \hat{V}^{ab}.
\]

Since \( f_{\Lambda} \) is also a diffeomorphism, the sets \( \hat{V}^{ab} \) cover \( \mathcal{N}_{\Lambda} \) and because every \( f_{\Lambda}^{-1} \) is a diffeomorphism, there exist unique supersmooth inverse morphisms \( (f_{\Lambda})^{-1} : \hat{V}^{ab} \rightarrow \hat{U}^{ab} \) by Lemma 3.25. For every \( \alpha, \alpha' \in A \) and \( \beta, \beta' \in B \), we have \( (\psi^{b})^{-1} \circ f \circ \varphi^{a} = f^{ab} \) on \( \hat{U}^{ab} \cap (\varphi^{a})^{-1}(\hat{U}^{ab}) \). Therefore, \( (f_{\Lambda})^{-1} = (\varphi^{a})^{-1} \circ f \circ \varphi^{a} \) on \( \hat{V}^{ab} \cap (\psi^{b})^{-1}(\hat{V}^{ab}) \) and...
the morphisms lead to a unique supersmooth morphism \( f^{-1}: \mathcal{N} \to \mathcal{M} \) by Proposition 3.31. That it is inverse to \( f \) follows from the local description of \( f^{-1} \circ f \) and \( f \circ f^{-1} \).

**Definition 3.33.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( \mathcal{M} \in \text{SMan}^{(k)} \) be modelled on \( E \in \text{SVec}_{lc} \). A subfunctor \( \mathcal{N} \) of \( \mathcal{M} \) is called a sub-supermanifold of \( \mathcal{M} \) if for every \( x \in \mathbb{N}_\mathbb{R} \) there exists a chart \( \varphi^x : U^x \to \mathcal{M} \) with the atlas \( \mathcal{U} = \{ U^x : \alpha \in A \} \) such that \( \varphi^x(U^x \cap \mathcal{T}^{(k)}) = \varphi^x(U^x) \cap \mathcal{N} \), where \( F := F_0 \oplus F_1 \in \text{SVec}_{lc} \).

We call \( \varphi^a(U^x \cap \mathcal{T}^{(k)}) \) a sub-supermanifold chart of \( \mathcal{N} \). Taking all sub-supermanifold charts of \( \mathcal{N} \) as the atlas turns \( \mathcal{N} \) into a supermanifold and we always give \( \mathcal{N} \) this structure.

**Lemma 3.34.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), \( \mathcal{M} \in \text{SMan}^{(k)} \) and \( \mathcal{N} \) be a sub-supermanifold of \( \mathcal{M} \). Then the inclusion \( i: \mathcal{N} \to \mathcal{M} \) is supersmooth.

**Proof.** By definition of a subfunctor, the inclusion is a natural transformation. Let \( \mathcal{M} \) be modelled on \( E \in \text{SVec}_{lc} \), \( \mathcal{N} \) be modelled on \( F \subseteq E \) and \( \{ \varphi^x : U^x \to \mathcal{M} : \alpha \in A \} \) be a collection of charts such that \( \{ \varphi^x|_{U^x \cap \mathcal{T}^{(k)}} : \alpha \in A \} \) is an atlas of \( \mathcal{N} \). In these charts the inclusion is just the inclusion \( U^x \cap \mathcal{T}^{(k)} \to U^x \), which is obviously supersmooth.

**Lemma/Definition 3.35.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( \mathcal{M} \in \text{SMan}^{(k)} \). For every open subfunctor of \( U \subseteq \mathcal{M} \), we have \( \mathcal{U} = \mathcal{M}|_U \). In this case \( \mathcal{U} \) is a sub-supermanifold of \( \mathcal{M} \) and if \( f: \mathcal{M} \to \mathcal{N} \) is a supersmooth morphism to \( \mathcal{N} \in \text{SMan}^{(k)} \), then so is \( f|_U : \mathcal{U} \to \mathcal{N} \). We call such sub-supermanifolds open sub-supermanifolds.

**Proof.** That \( \mathcal{U} = \mathcal{M}|_U \) holds for \( k = \infty \) follows directly from [20] Corollary 3.5.9, p. 62 and the same proof works for \( k \) if only considers \( \Lambda \in \text{Gr}^{(k)} \). Let \( \{ \varphi^x: U^x \to \mathcal{M} : \alpha \in A \} \) be an atlas of \( \mathcal{M} \). Then \( \{ \varphi^x|_{\mathcal{U}^x \cap \mathcal{T}^{(k)}} : \alpha \in A \} \) is an atlas of \( \mathcal{U} \). With these charts, the supersmoothness of \( f|_U \) is obvious.

**Definition 3.36.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( \mathcal{M}, \mathcal{N} \in \text{SMan}^{(k)} \) be modelled on \( E, F \in \text{SVec}_{lc} \) with atlases \( \{ \varphi^x: U^x \to \mathcal{M} : \alpha \in A \} \) and \( \{ \psi^x: V^x \to \mathcal{N} : \beta \in B \} \). We define the product \( \mathcal{M} \times \mathcal{N} \) of \( \mathcal{M} \) and \( \mathcal{N} \) as the functor \( \Lambda \to \mathcal{M}_\Lambda \times \mathcal{N}_\Lambda \), resp. \( \varrho \to \mathcal{M}_\varrho \times \mathcal{N}_\varrho \) for \( \Lambda, \Lambda' \in \text{Gr}^{(k)} \) and \( \varrho \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda') \). We will always give \( \mathcal{M} \times \mathcal{N} \) the structure of a \( k \)-supermanifold modelled on \( E \times F \) defined by the atlas \( \{ \varphi^a\times \psi^b: U^a \times V^b \to \mathcal{M} \times \mathcal{N} : (\alpha, \beta) \in A \times B \} \).

Clearly, the projections \( \pi_M : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \), \( \pi_N : \mathcal{M} \times \mathcal{N} \to \mathcal{N} \) and the inclusions \( \mathcal{M} \to \mathcal{M} \times \mathcal{N} \), \( \mathcal{N} \to \mathcal{M} \times \mathcal{N} \) are supersmooth morphisms.

Recall the definition of multilinear bundles and inverse systems of multilinear bundles from Appendix A. The following theorem shows that for a supermanifold \( \mathcal{M} \), the manifolds \( \mathcal{M}_\Lambda \) are multilinear bundles of degree \( n \) over \( \mathcal{M}_\mathbb{R} \) and that \( (\mathcal{M}_m, \mathcal{M}_{m,n}) \) is an inverse system of multilinear bundles. This lets us consider supermanifolds as ordinary manifolds.

**Theorem 3.37.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), \( \mathcal{M}, \mathcal{N} \in \text{SMan}^{(k)} \) and \( f: \mathcal{M} \to \mathcal{N} \) be supersmooth. If \( \mathcal{M} \) is modelled on \( E \in \text{SVec}_{lc} \) with the atlas \( \{ \varphi^x : \alpha \in A \} \), then \( \mathcal{M}_\Lambda \) is a multilinear bundle of degree \( n \) over \( \mathcal{M}_{\mathbb{R}} \).
\( E_{\Lambda} \) and the bundle atlas \( \{ \varphi_{\alpha}^\Lambda : \alpha \in A \} \) for every \( \Lambda \in \text{Gr}^{(k)} \). Moreover, \( f_{\Lambda} : M_{\Lambda} \to \mathcal{N}_{\Lambda} \) is a morphism of multilinear bundles of degree \( n \). With this, we obtain a faithful functor
\[
\text{SMan}^{(k)} \to \text{MBun}^{(k)},
\]
defined by \( \mathcal{M} \mapsto \mathcal{M}_{\Lambda_k} \) and \( f \mapsto f_{\Lambda_k} \) for \( k \in \mathbb{N}_0 \). Furthermore, if \( k = \infty \), then \( (\mathcal{M}_{\Lambda_m}, \mathcal{M}_{\varepsilon_{m,n}}) \) is an inverse system of multilinear bundles with the adapted atlas \( \{(\varphi_{\alpha}^\Lambda)^{-1} : n \in \mathbb{N}_0, \alpha \in A \} \)
\[
\lim_{\longleftarrow} \text{SMan} \to \text{MBun}^{(\infty)},
\]
defined by \( \mathcal{M} \mapsto \lim_{\longleftarrow} \mathcal{M}_{\Lambda_n} \) and \( f \mapsto \lim_{\longleftarrow} f_{\Lambda_n} \), is a faithful functor. Along the forgetful functor, we have thus constructed faithful functors
\[
\text{SMan}^{(k)} \to \text{Man}
\]
for \( k \in \mathbb{N}_0 \cup \{ \infty \} \). All these functors respect products.

**Proof.** Let \( \mathcal{M} \) be modelled on \( E \in \text{SVec}_{\mathbb{R}_c} \). We start by showing that \( \{ \varphi_{\alpha}^\Lambda : \mathcal{U}_{\alpha}^\Lambda \to \mathcal{M}_{\alpha} : \alpha \in A \} \) is indeed a bundle atlas of a multilinear bundles of degree \( n \). Let the change of charts \( \varphi^\alpha_{\beta} \) be defined by the skeleton \( (\varphi^\alpha_{\beta}) \).

We consider \( \mathcal{U}_{\alpha}^\Lambda = \mathcal{U}_{\mathbb{R}}^\alpha \times \prod_{l \in \mathcal{P}^n_+} \lambda_l \mathcal{E}_{\mathbb{R}^l} \) as a trivial multilinear bundle over the \( n \)-multilinear space \( (E_l) \) with \( E_l := \lambda_l \mathcal{E}_{\mathbb{R}^l} \). By naturality, we have
\[
(\varphi_{\alpha}^\Lambda)^{-1}(\mathcal{U}_{\alpha}^\Lambda|_{\mathcal{N}_{\alpha\beta}}(\{x\})) = (\mathcal{U}_{\alpha}^\Lambda|_{\mathcal{N}_{\alpha\beta}}(\{x\})) \quad \text{for all } x \in \varphi_{\alpha}(\mathcal{U}_{\alpha}^\Lambda).
\]

In other words, the projection \( \mathcal{M}_{\varepsilon_{\alpha}} : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta} \) turns \( \mathcal{M}_{\alpha} \) into a fiber bundle with typical fiber \( E_{\Lambda^+} \). Recall the sign of a partition defined in Remark A.6. Then \( \lambda_{\omega_1} \cdots \lambda_{\omega_{\ell}(\omega)} \) is indeed an adapted atlas.

Let the chart representation of \( \varphi_{\alpha}^\Lambda \) be defined by the skeleton \( (\varphi_{\alpha}^\Lambda) \).

In the notation of multilinear bundles, the change of chart is thus given by the sum of maps of the form \( (\varphi_{\alpha}^\Lambda)^{\omega}(x) := sgn(\omega) d(\varepsilon(\omega)) \varphi_{\alpha}^\Lambda(x)(x\omega) \)
\[
x_{\omega} := (\lambda_{\omega_1}x_{\omega_1}, \ldots, \lambda_{\omega_{\ell}(\omega)}x_{\omega_{\ell}(\omega)}).
\]

In the inverse system of multilinear bundles, the change of chart is thus given by the sum of maps of the form \( (\varphi_{\alpha}^\Lambda)^{\omega}(x) := sgn(\omega) d(\varepsilon(\omega)) \varphi_{\alpha}^\Lambda(x)(x\omega) \)
\[
x_{\omega} := (\lambda_{\omega_1}x_{\omega_1}, \ldots, \lambda_{\omega_{\ell}(\omega)}x_{\omega_{\ell}(\omega)}).
\]

In bundle charts, we can make the exact same argument as above to see that \( f_{\Lambda_n} \) is a morphism of multilinear bundles.

It follows that we have a functor \( \text{SMan}^{(k)} \to \text{MBun}^{(k)} \) as described in the theorem for \( k \in \mathbb{N}_0 \). Next, we show that \( (\mathcal{M}_{\varepsilon_{\alpha}}^\alpha, \mathcal{M}_{\varepsilon_{m,n}}) \) defines an inverse system of multilinear bundles if \( \mathcal{M} \in \text{SMan} \). We have \( \mathcal{M}_{\varepsilon_{m,n}} \circ \varphi_{\alpha}^\Lambda = \varphi_{\alpha}^\Lambda \circ \mathcal{U}_{\varepsilon_{m,n}} \) for all \( n \leq m \) and therefore \( \mathcal{U}_{\varepsilon_{m,n}} \) is the chart representation of \( \mathcal{M}_{\varepsilon_{m,n}} \).

Hence, in terms of multilinear bundles, \( \mathcal{M}_{\varepsilon_{m,n}} \) is exactly the projection defined in Lemma A.21. It follows that \( \mathcal{M}_{\varepsilon_{\alpha}}^{\varepsilon_{m,n}} = \mathcal{M}_{\varepsilon_{m,n}}(\mathcal{M}_{\varepsilon_{\alpha}}) = \mathcal{M}_{\varepsilon_{m,n}} \circ \varphi_{\alpha}^\Lambda \) shows that \( \{(\varphi_{\varepsilon_{\alpha}}^\Lambda)^{-1} : m \in \mathbb{N}_0, \alpha \in A \} \) is indeed an adapted atlas.
On morphisms \( f : M \to N, \ N \in \text{SMan} \), we have likewise \( f_{\Lambda_m} \circ M_{\varepsilon_{m,n}} = N_{\varepsilon_{m,n}} \circ f_{\Lambda_m} \) which shows that \( (f_{\Lambda_m})_{m \in \mathbb{N}_0} \) is a morphism of inverse systems of multilinear bundles.

It is clear from the definitions that products of supermanifolds correspond to products of inverse systems. □

Remark 3.38. In [18, Remark 3.3.1, p.392] Molotkov constructs a functor \( \text{Man}^{\text{Gr}} \to \text{Man} \) by taking the disjoint union of the \( M_{\Lambda} \) for \( M \in \text{Man}^{\text{Gr}} \) and \( \Lambda \in \text{Gr} \). He also considers this as a functor \( \text{SMan} \to \text{Man} \) along the forgetful functor. For one, this functor relies on a more general definition of manifolds where the model spaces of different connected components may be non-isomorphic. More critically, this functor does not respect products, leading Molotkov to state that "Lie supergroups (groups of the category \( \text{SMan} \)) are not groups at all (considered in \( \text{Set} \) [18, Ibid.]). We hope to have convinced the reader with the above theorem that Lie supergroups can be seen not only as groups but even as Lie groups in a natural way.

Remark 3.39. Consider the following type of fiber bundles. For \( E \in \text{SVec}_{lc} \) let the base manifold \( M \) be modelled on \( E_0 \), let the typical fiber be \( \lim_{\leftarrow n} E_{\Lambda_n}^+ \) and let the transition functions come from the limit of skeletons as in Theorem 3.37. The morphisms of such bundles shall locally also come from limits of skeletons. Obviously, these bundles are elements of \( \text{MBun}^{(\infty)} \) and restricting to this subcategory turns the functor \( \lim_{\leftarrow} \) into an equivalence of categories. If \( E \) is finite-dimensional, we have that \( \lim_{\leftarrow n} E_{\Lambda_n}^+ \) is a Fréchet space. Consequently, non-trivial finite-dimensional supermanifolds are mapped to Fréchet manifolds under \( \lim_{\leftarrow} \).

One can reconstruct the original supermanifold \( M \) from \( \lim_{\leftarrow} M \) if one keeps track of any atlas of \( \lim_{\leftarrow} M \) coming from the limit of an atlas of \( M \). An interesting problem is whether one can at least recover the isomorphism class of a supermanifold without a specific atlas.

**Problem.** Is the functor \( \lim_{\leftarrow} : \text{SMan} \to \text{MBun}^{(\infty)} \) injective on isomorphism classes, i.e., do we have \( \lim_{\leftarrow} M \cong \lim_{\leftarrow} N \) in \( \text{MBun}^{(\infty)} \) if and only if we have \( M \cong N \) in \( \text{SMan} \)?

If \( M_\mathbb{R} \) admits a smooth partition of unity, then it follows from Batchelor’s Theorem 3.47 below that this is the case because \( \lim_{\leftarrow} M \cong \lim_{\leftarrow} N \) in \( \text{MBun}^{(\infty)} \) implies \( M_{\Lambda_1} \cong N_{\Lambda_1} \) in \( \text{VBun} \). The functor \( \lim_{\leftarrow} : \text{SMan} \to \text{Man} \) is not injective on isomorphism classes. For example

\[
\lim_{\leftarrow} \mathbb{R}^{[0]} \cong \prod_{\substack{I \subseteq \mathbb{N}, |I| < \infty, \ |I| \text{ even}}} \mathbb{R} \cong \prod_{n \in \mathbb{N}} \mathbb{R} \cong \mathbb{R} \times \prod_{\substack{I \subseteq \mathbb{N}, |I| < \infty, \ |I| \text{ odd}}} \mathbb{R} \cong \lim_{\leftarrow} \mathbb{R}^{[0]} \]

in the category \( \text{Man} \).

We have seen in Theorem 3.37 how to embed the category of supermanifolds into the category of manifolds. Conversely, one can also embed the category \( \text{Man} \) into the category \( \text{SMan} \). For this, let \( \text{Dom} \) denote the category consisting of pairs \( (U, E_0) \) where \( E_0 \) is a Hausdorff locally convex space and \( U \subseteq E_0 \) is open and where the morphisms are smooth maps between the open subsets.
Proposition 3.40 ([18 cf. Proposition 4.2.1, p.396]). Let \( k \in \mathbb{N}_0 \cup \{\infty\} \).

We define a functor

\[
i_k^0: \text{Dom} \to \text{SDom}^{(k)}
\]

by setting \( \iota_k^0(U) := E^{(k)}|_U \) and \( \iota_k^0(f_0) := (f_0, 0, 0, \ldots) \) for \( (U, E_0) \in \text{Dom} \) and \( E := E_0 \oplus \{0\} \in \text{SVec}_0 \). This functor extends to a fully faithful functor

\[
i_k^0: \text{Man} \to \text{SMan}^{(k)}.
\]

In case of \( k = \infty \) we also write \( \iota: \text{Man} \to \text{SMan} \). The functor \( \iota_k^0: \text{Man} \to \text{SMan}^{(0)} \) is an equivalence of categories. All of these functors respect products.

Proof. It follows from the composition formula in Proposition 3.23 that \( \iota_k^0: \text{Dom} \to \text{SDom}^{(k)} \) is a functor. Let \( M \) be a manifold modelled on \( E_0 \) with atlas \( \{\varphi_\alpha: V_\alpha \to U_\alpha: \alpha \in A\} \). Applying this functor to the change of charts \( \varphi_{\alpha \beta}: U_{\alpha \beta} \to U_{\beta \alpha} \), defines an (up to unique isomorphism) unique supermanifold \( M \) modelled on \( E_0 \oplus \{0\} \) with the atlas \( \{\iota_k^0((\varphi_\alpha)^{-1}): \iota_k^0(U_\alpha) \to M: \alpha \in A\} \) by Proposition 3.31. If \( N \in \text{Man} \) has the atlas \( \{\psi_\beta: V_\beta \to U_\beta: \beta \in B\} \) and \( f: M \to N \) is a smooth map then the same proposition applied to \( f_\alpha \beta = \psi_\beta \circ f \circ (\varphi_\alpha)^{-1} \) leads to a unique morphism \( \iota_k^0(f): \iota_k^0(M) \to \iota_k^0(N) \) such that \( (\iota_k^0(f))^{\alpha \beta} = \iota_k^0(f_{\alpha \beta}) \). Functoriality follows again by the local definition of the composition of supersmooth morphisms.

The uniqueness of this construction shows that \( \iota_k^0 \) is faithful. On the other hand, every supersmooth map \( g: \iota_k^0(M) \to \iota_k^0(N) \) is determined by its local chart descriptions \( g^{\alpha \beta} \), whose skeletons have the form \( (g_0^{\alpha \beta}, 0, 0, \ldots) \) since \( \iota_k^0(M) \) is purely even. Clearly, the maps \( g_0^{\alpha \beta} \) define a unique smooth map \( M \to N \) whose image under \( \iota_k^0 \) is \( g \). We already know from Theorem 3.37 that \( M \mapsto \text{M}_\mathbb{R} \) and \( f \mapsto f_\mathbb{R} \) defines a functor \( \pi_0^0: \text{SMan}^{(0)} \to \text{Man} \) and the above shows that \( \pi_0^0 \circ \iota_k^0 \cong \text{id}_{\text{Man}} \) and that \( \iota_k^0 \circ \pi_0^0 \cong \text{id}_{\text{SMan}} \).

It is obvious that the functor \( \iota_k^0: \text{Dom} \to \text{SDom}^{(k)} \) preserves products and from this it follows immediately that \( \iota_k^0: \text{Man} \to \text{SMan}^{(k)} \) also preserves products.

\[\square\]

Lemma 3.41. Let \( k \in \mathbb{N}_0 \cup \{\infty\} \). For every supermanifold \( M \in \text{SMan}^{(k)} \), we have that \( \iota_k^0(\text{M}_\mathbb{R}) \) is a sub-supermanifold of \( M \). If \( M \) is purely even, we have \( \iota_k^0(\text{M}_\mathbb{R}) \cong M \).

Proof. Let \( M \) be modelled on \( E \in \text{SVec}_0 \) and let \( \{\varphi_\alpha: U_\alpha \to M: \alpha \in A\} \) be an atlas of \( M \). If the changes of charts \( \varphi_{\alpha \beta} \) have the skeletons \( (\varphi_{\alpha \beta}) \), then the skeletons \( (\varphi_0^{\alpha \beta}, 0, 0, \ldots) \) define \( \iota_k^0(\text{M}_\mathbb{R}) \) by Proposition 3.40. Because \( \varphi_0^{\alpha \beta} \) has the skeleton \( (\varphi_0^{\alpha \beta}, 0, 0, \ldots) \), it follows that \( \iota_k^0(\text{M}_\mathbb{R}) \) is a sub-supermanifold of \( M \). If \( M \) is purely even, then changes of charts have the form \( (\varphi_0^{\alpha \beta}, 0, 0, \ldots) \) to begin with and it follows \( \iota_k^0(\text{M}_\mathbb{R}) \cong M \) by Proposition 3.31.

\[\square\]

Purely even supermanifolds \( M \) can be described in terms of higher tangent bundles of \( M_\mathbb{R} \). This will be particularly important for the theory of Lie supergroups.
Proposition 3.42. Let $M$ be a manifold. Recall Example A.24. Using Lemma A.17 and Theorem 3.37, there are isomorphisms

$$\Gamma^k_n : \iota^k_0(M)_{\Lambda_n} \to T^k M|_{p_{0,+}}^{-}$$

of multilinear bundles of degree $n$ for every $n \leq k < \infty$. These isomorphisms are natural in $k$ and $n$ in the sense that

$$(\iota^k_0(M)\Lambda_k, \iota^0(M)_{\varepsilon_k,n}) \cong (T^k M|_{p_{0,+}}^{-}, \pi_n^k|_{p_{0,+}}^{-})$$

holds as inverse systems of multilinear bundles. It follows that $\Lambda_k \mapsto T^k M|_{p_{0,+}}^{-}$ can be made into a supermanifold isomorphic to $\iota(M)$.

Proof. To show that $\iota^k_0(M)_{\Lambda_n} \cong T^n M|_{p_{0,+}}^{-}$ holds, we simply compare the change of charts. Let $M$ be modelled on $E_0$ and $\{\varphi_\alpha : V_\alpha \to U_\alpha : \alpha \in A\}$ be an atlas of $M$. For a change of charts $\varphi^{\alpha \beta} : U_\alpha \to U_\beta$, we have $\iota^k_0(\varphi^{\alpha \beta}) = (\varphi^{\alpha \beta}, 0, 0, \ldots)$ and thus

$$\iota^k_0(\varphi^{\alpha \beta})_{\Lambda_n}(x + \sum_{I \in P_{0,+}^n} \lambda_I x_I) =$$

$$\varphi^{\alpha \beta}(x) + \sum_{I \in P_{0,+}^n} \sum_{\omega \in \mathcal{P}(I)} \lambda_I \text{sgn}(\omega) d(e(\omega)) \varphi^{\alpha \beta}_{\omega I}(x)(x_\omega),$$

where $x \in U_{\alpha \beta}$ and $x_\omega = (x_{\omega_1}, \ldots, x_{\omega_{\ell(\omega)}}) \in E_0^{e(\omega)}$. On the other hand, we know from Example A.15(b) and the Lemma A.17 that the change of charts for $T^k M|_{p_{0,+}}^{-}$ is given by

$$T^k \varphi^{\alpha \beta}|_{p_{0,+}}^{-} (x + \sum_{I \in P_{0,+}^n} \varepsilon_I x_I) =$$

$$\varphi^{\alpha \beta}(x) + \sum_{I \in P_{0,+}^n} \sum_{\omega \in \mathcal{P}(I)} \varepsilon_I \text{sgn}(\omega) d(e(\omega)) \varphi^{\alpha \beta}_{\omega I}(x)(x_\omega),$$

for the same $x \in U_{\alpha \beta}$ and $x_\omega \in E_0^{e(\omega)}$. Therefore, there exists an isomorphism $\Gamma^k_n : \iota^k_0(M)_{\Lambda_n} \to T^k M|_{p_{0,+}}^{-}$ such that $(\Gamma^k_n)^{\alpha} := T^k \varphi^{\alpha \beta}|_{p_{0,+}}^{-} \circ \Gamma^k_n \circ \iota^k_0(\varphi^{\alpha})_{\Lambda_n}$ is given by the obvious isomorphisms of trivial $k$-multilinear bundles

$$(\Gamma^k_n)^{\alpha} : \iota^k_0(U_\alpha)_{\Lambda_n} = U_\alpha \times \prod_{I \in P_{0,+}} \lambda_I E_0 \to U_\alpha \times \prod_{I \in P_{0,+}} \varepsilon_I E_0 = T^k|_{p_{0,+}^n}(U_\alpha)^{-}.$$

Note that for any $k$-multilinear bundle $F$, we have $(F|_{p_{0,+}^n})|_{p_n} = F|_{p_{0,+}^n}$ for $n \leq k$ and it follows from the local description in Lemma A.24 that $(\Phi^k_n)^{-} : F|_{p_{0,+}^n}^{-} \to F|_{p_{0,+}^n}^{-}$ is just the respective projection of $F|_{p_{0,+}^n}^{-}$. This shows that $(T^k M|_{p_{0,+}^n}^{-}, \pi_n^k|_{p_{0,+}^n})$ is indeed an inverse system of multilinear bundles. It is clear from the local description that $\Gamma^k_n \circ \iota(M)_{\varepsilon_k,n} = \pi_n^k|_{p_{0,+}^n} \circ \Gamma^k_n$, which shows $(\iota(M)_{\Lambda_k}, \iota(M)_{\varepsilon_k,n}) \cong (T^k M|_{p_{0,+}^n}^{-}, \pi_n^k|_{p_{0,+}^n}^{-})$. 

To turn $\Lambda_k \mapsto T^k M^{\|}_{\partial_{b,+}}$ into a supermanifold, one simply defines

$$(T^k M^{\|}_{\partial_{b,+}})_\varphi : T^k M^{\|}_{\partial_{b,+}} \to T^n M^{\|}_{\partial_{b,+}}$$

for every morphism $\varphi : \Lambda_k \to \Lambda_n$ via $\iota(M)_{\Lambda_k} \to \iota(M)_{\Lambda_n}$ and the above isomorphisms. The charts are then given by $(T^k \varphi \circ \iota(\varphi^{-1})_{\Lambda_k})_{\Lambda_k}$. □

**Proposition 3.43** ([15] cf. Proposition 4.2.1, p.396). Let $k \in \mathbb{N} \cup \{\infty\}$. There is a faithful functor

$$i^1_k : \text{VBun} \to \text{SMan}^{(1)}.$$

The functor $i^1_k : \text{VBun} \to \text{SMan}^{(1)}$ is an equivalence of categories. All these functors respect products.

**Proof.** The proof is very similar to the proof of Proposition 3.40. Let $\pi : F \to M$ be a vector bundle with typical fiber $E_1$ and bundle atlas $\{\varphi_{\alpha} : V_{\alpha} \to U^\alpha \times E_1 : \alpha \in A\}$. The change of bundle charts $\varphi_{\alpha\beta} : U_{\alpha\beta} \times E_1 \to U_{\beta\alpha} \times E_1$ has the form $(\varphi_{0\alpha\beta}^{\gamma}, \varphi^{\beta\gamma}_{1})$, where $\varphi_{0\beta}^{\alpha} : U_{\alpha\beta} \to U_{\beta\alpha}$ is a smooth and $\varphi_{1\alpha\beta}^{\gamma} : U_{\alpha\beta} \times E_1 \to E_1$ is smooth and linear in the second component. Note that there exists an atlas of $M$ such that the change of charts is given by $\varphi_{0\alpha\beta}^{\gamma}$. Let $M$ be modelled on $E_0$. We define the super vector space $E := E_0 \oplus E_1$ and let $i^1_k(U_{\alpha\beta}) := U^\alpha := E^{(k)}|_{U_{\alpha\beta}}$, as well as $U^{\alpha\beta} := E^{(k)}|_{U_{\alpha\beta}}$, for all $\alpha, \beta \in A$. Then $i^1_k(\varphi_{0\alpha\beta}^{\gamma} := (\varphi_{0\alpha\beta}^{\gamma}, \varphi_{1\alpha\beta}^{\gamma}, 0, 0, \ldots) : U^{\alpha\beta} \to U^{\beta\alpha}$ defines isomorphisms that satisfy the conditions of Proposition 3.31 because by the composition formula from Proposition 3.23 we have

$$(\varphi_0^{\beta\gamma}, \varphi_1^{\beta\gamma}, 0, 0, \ldots) \circ (\varphi_{0\alpha\beta}^{\gamma}, \varphi_{1\alpha\beta}^{\gamma}, 0, 0, \ldots) = (\varphi_0^{\beta\gamma} \circ \varphi_{0\alpha\beta}^{\gamma}, \varphi_1^{\beta\gamma} \circ \varphi_{1\alpha\beta}^{\gamma}, 0, 0, \ldots) = (\varphi_{0\alpha\beta}^{\gamma}, \varphi_{1\alpha\beta}^{\gamma}, 0, 0, \ldots) = i^1_k(\varphi^{\beta\gamma}).$$

where defined. We let $i^1_k(F)$ be the supermanifold $M$ defined in Proposition 3.31 by the given change of charts.

Morphisms $f : F \to F'$ of vector bundles have the local form $(f_0^{\alpha\beta}, f_1^{\alpha\beta})$, where $f_0^{\alpha\beta}$ is linear in the second component and define skeletons of the form $(f_0^{\alpha\beta}, f_1^{\alpha\beta}, 0, 0, \ldots)$ that satisfy Proposition 3.31. In this way, we obtain a unique supersmooth morphism $i^1_k(f) : i^1_k(F) \to i^1_k(F')$. By the same argument as above, this construction is functorial and, by uniqueness, the resulting functor is faithful. □

**Lemma 3.44** ([15] cf. Proposition 4.2.1, p.396). Let $k, n \in \mathbb{N}_0 \cup \{\infty\}$ and $n \leq k$. The restriction of functors $\text{Man}^{Gr^{(k)}}$ to functors $\text{Man}^{Gr^{(n)}}$ leads to functors

$$\pi^n_k : \text{SMan}^{(k)} \to \text{SMan}^{(n)}, \quad \pi^n_k(M) := M^{(n)}.$$

On morphisms, we write $\pi^n_k(f) := f^{(n)}$. These functors respect products and $\pi^n_m = \pi^n_k \circ \pi^n_m$ holds for all $m \leq n$. Identifying $\text{SMan}^{(0)}$ with $\text{Man}$ and $\text{SMan}^{(1)}$ with $\text{VBun}$ via Proposition 3.40 and Proposition 3.43 we have $\pi^n_k \circ i^1_k \cong \text{id}_\text{Man}$ and $\pi^n_k \circ i^1_k \cong \text{id}_\text{VBun}$ if $k > 0$. 


Proof. Let $\mathcal{M}, \mathcal{N} \in \text{SM}^{(k)}$. It follows directly from the definition that $\Lambda \to \mathcal{M}_\Lambda$ for $\Lambda \in \text{Gr}^{(n)}$ defines an $n$-supermanifold $\mathcal{M}^{(n)}$ with the obvious restricted atlas. Likewise, for morphisms $f: \mathcal{M} \to \mathcal{N}$, one defines $f^{(n)}$ by $f^{(n)}_\Lambda := f_\Lambda$ for $\Lambda \in \text{Gr}^{(n)}$. This construction is clearly functorial, respects products and satisfies $\pi^k_m = \pi^k_m \circ \pi^m_n$ for all $m \leq n$. To see $\pi^0_m \circ \iota^0_k \cong \text{id}_{\text{Man}}$ and $\pi^1_1 \circ \iota^1_k \cong \text{id}_{\text{VBun}}$, one simply checks that on the level of skeletons this composition does not change anything. □

One can understand the projections $\pi^k_m: \text{SM}^{(k)} \to \text{SM}^{(m)}$ and the embeddings $\iota^0_k: \text{Man} \to \text{SM}^{(k)}$ and $\iota^1_k: \text{VBun} \to \text{SM}^{(k)}$ completely in terms of skeletons. The former simply cuts skeletons $(f_0, \ldots)$ down to $(f_0, \ldots, f_n)$. The latter two extend skeletons $(f_0)$, resp. $(f_0, f_1)$, to $(f_0, 0, \ldots)$, resp. $(f_0, f_1, 0, \ldots)$. Proposition 3.23 ensures that the composition of two such skeletons is again of this form, which is why these embeddings are well-defined.

A natural question is now whether two $k$-supermanifolds $\mathcal{M}^{(k)}$ and $\mathcal{N}^{(k)}$ such that $\mathcal{M}^{(n)} \cong \mathcal{N}^{(n)}$ holds for $1 < n < k$ are automatically isomorphic as well. In other words, whether a supersmooth isomorphism $f^{(n)}: \mathcal{M}^{(n)} \to \mathcal{N}^{(n)}$ can be lifted to an isomorphism $f^{(k)}: \mathcal{M}^{(k)} \to \mathcal{N}^{(k)}$. This will be discussed in the following section on Batchelor’s Theorem.

Definition 3.45. We denote by $p$ the supermanifold modelled on $\{0\} \oplus \{0\}$ that consists for every $\Lambda \in \text{Gr}$ of a single point. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A point of a $k$-supermanifold $\mathcal{M}$ is a morphism $x: p^{(k)} \to \mathcal{M}$. We also write $x_\Lambda := x_\Lambda(p^{(k)}_\Lambda)$.

Lemma 3.46. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{M} \in \text{SM}^{(k)}$. For every point $x: p^{(k)} \to \mathcal{M}$ and every $\Lambda \in \text{Gr}^{(k)}$, we have $x_\Lambda = \mathcal{M}_{\eta\Lambda}(x_\mathbb{R})$. Conversely, for every $x_\mathbb{R} \in \mathcal{M}_\mathbb{R}$ the assignment $x_\Lambda := \mathcal{M}_{\eta\Lambda}(x_\mathbb{R})$ defines a point.

Proof. For every $\Lambda \in \text{Gr}^{(k)}$, we have

$$\mathcal{M}_{\eta\Lambda}(x_\mathbb{R}) = x_\Lambda \circ p^{(k)}_{\eta\Lambda} \circ (p^{(k)}_\mathbb{R}) = x_\Lambda.$$ 

Conversely, let $x_\mathbb{R} \in \mathcal{M}_\mathbb{R}$ be given and $x_\Lambda := \mathcal{M}_{\eta\Lambda}(x_\mathbb{R})$. Then $\rho \circ \eta\Lambda = \eta\Lambda$ and therefore $\mathcal{M}_{\rho}(x_\Lambda) = \mathcal{M}_{\eta\Lambda}(x_\mathbb{R}) = x_\Lambda$ holds for $\rho \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda')$. □

Hence, the points of a supermanifold can be identified with the usual points of the base manifold.

3.3.1. Connection to the Sheaf Theoretic Approach. The full subcategory of finite-dimensional supermanifolds in the categorical approach is equivalent to the category of supermanifolds in the sheaf theoretic approach. This was already discussed in [25] and [16] but a more thorough and general proof can be found in [11]. Let us briefly sketch the idea behind the equivalence.

Let $p, q \in \mathbb{N}_0$ and $\mathcal{U} \subseteq \mathbb{R}^{p,q}$ be an open subfunctor. In terms of skeletons, we have

$$\mathcal{S}C^\infty(\mathcal{U}, \mathbb{R}^{n,1}) = C^\infty\left(\mathcal{U}_R, \bigoplus_{i=0}^q \text{Alt}^i(\mathbb{R}^q; \mathbb{R}) \right) \cong C^\infty(\mathcal{U}_R, \mathbb{R}) \otimes \Lambda_q.$$
Therefore, for any supermanifold $M$ modelled on $\mathbb{R}^{p|q}$, the sheaf

$$U \mapsto \mathcal{SC}^\infty(M|_U, \mathbb{R}^{1|1}), \ U \subseteq M \text{ open},$$

is locally isomorphic to the sheaf $\mathcal{C}^\infty_{R^{p}} \otimes \Lambda_q$ as needed. One then checks that morphisms of supermanifolds lead to appropriate morphisms of these sheaves along the pullback.

### 3.4. Generalizations.

Many of the generalizations for $k$-superdomains mentioned in 3.2.1 can be applied to supermanifolds without much difficulty such that the results in this section carry over. Additional care is necessary only if the base ring is not a field because restrictions of smooth maps to open subsets need not be smooth in that case (see [5, 2.4, p.21]). As already mentioned, one can consider non-Hausdorff supermanifolds by simply extending the category $\text{Man}$ to non-Hausdorff manifolds. Analytic supermanifolds can be defined by demanding that the skeletons are analytic in an appropriate sense.

One should also note that many structural results do not rely on super-smoothness. In view of Proposition 3.14 and Lemma 3.12 one can define a subcategory of $\text{Man}^{Gr(k)}$ of functors locally isomorphic to some $\mathcal{E}^{(k)}$, $E \in \text{SVec}_{k}$ where the changes of charts are simply natural transformations such that every component is smooth. Then an analog to Theorem 3.37 still holds and one obtains a geometry combining commuting and anticommuting coordinates with less stringent symmetry conditions than for supermanifolds.

### 3.5. Batchelor’s Theorem.

The classical version of Batchelor’s Theorem (see [3]) states that any supermanifold, defined as a sheaf $(M, \mathcal{O}_M)$, is isomorphic to the supermanifold $(M, \Gamma(\wedge^* F))$, where $\wedge^* F$ is the exterior bundle of a vector bundle $F$ that is determined by $\mathcal{O}_M$. The isomorphism is not canonical because its construction involves a partition of unity.

Molotkov transfers this result to supermanifolds in our sense and generalizes it to infinite-dimensional supermanifolds $\mathcal{M}$ in [17]. In his version, the vector bundle $\mathcal{M}_A$, takes the role of $F$ in the classical version. Molotkov only considers Banach supermanifolds, but as we will see, his methods generalize to locally convex supermanifolds. It appears that [17] is not well-known and since it is not readily available, we describe his arguments in detail below. A closer look is also worthwhile because the techniques employed are close to the ones used in [3] and might make it easier to translate between the sheaf theoretic and the categorical approach.

**Theorem 3.47** ([17, Corollary 4, p.279]). Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{M} \in \text{SMan}^{(k)}$ be such that $\mathcal{M}_\mathbb{R}$ admits a smooth partition of unity. If $\mathcal{M}' \in \text{SMan}^{(k)}$ is a $k$-supermanifold such that $\mathcal{M}^{(1)}$ and $\mathcal{M}'^{(1)}$ are isomorphic, then $\mathcal{M}$ is isomorphic to $\mathcal{M}'$. In particular $\mathcal{M} \cong \iota^1_k(\mathcal{M}^{(1)})$.

In other words, if one restricts the categories $\text{VBun}$ and $\text{SMan}^{(k)}$ to the respective subcategories over finite-dimensional paracompact bases, the restricted functor $\iota^1_k$ from Proposition 3.43 becomes essentially surjective.
Definition 3.48. Let $k \in \mathbb{N}$. We call $k$-supermanifolds of the form $\iota^1_k(M^{(1)})$, where $M^{(1)}$ is a vector bundle, supermanifolds of Batchelor type. An isomorphism $f: \mathcal{N} \to \iota^1_k(M^{(1)})$ is called a Batchelor model of $\mathcal{N}$. We say an atlas $\mathcal{A} := \{\varphi^\alpha: \alpha \in A\}$ of a supermanifold is of Batchelor type if all changes of charts have the form $\varphi^{\alpha\beta} = (\varphi^0_0, \varphi^1_0, \varphi^0_1, 0, \ldots)$.

Remark 3.49. It follows from Proposition 3.31 that a supermanifold is of Batchelor type if and only if it has an atlas of Batchelor type. For a supermanifold of Batchelor type, the union of two atlases of Batchelor type is again of Batchelor type because the atlases define the same vector bundle. This does not need to be the case for arbitrary supermanifolds, which implies that there is no canonical choice of a Batchelor model in general.

One can reformulate this result as follows. In the situation of the theorem, any isomorphism $f^{(n)}: M^{(n)} \to M^{(n)}$ can be lifted to an isomorphism $f^{(n+1)}: M^{(n+1)} \to M^{(n+1)}$ for $1 \leq n < k$ (see [17, Theorem 1(a), p.273]). It is not difficult to see that one may assume $M' = M$, which we will do in the sequel to simplify our explanations (compare [17, Proposition 2, p.277]).

Let us introduce some notation for this section. For $M \in \text{SM}^{(k)}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ consider the group $\text{Aut}_{id_k}(M)$ of automorphisms $f: M \to M$ such that $f_\mathbb{R} = id_{M_k}$. We denote by $\mathcal{G}(M)$ the sheaf of groups over $M_\mathbb{R}$ defined by $U \mapsto \text{Aut}_{id_k}(M|_U)$ for every open $U \subseteq M_\mathbb{R}$. The restriction morphisms are given in the obvious way by Lemma/Definition 3.5(c). The restrictions are morphisms of groups because we only consider automorphisms over the identity on the base manifold. The functor $\pi^n_m: \text{SM}^{(n)} \to \text{SM}^{(m)}$ from Lemma 3.44 leads to morphisms $\varphi^n_m: \mathcal{G}(M^{(n)}) \to \mathcal{G}(M^{(m)})$ of sheaves of groups for $m \leq n < k+1$. Locally, $(\varphi^n_m)_U: \text{Aut}_{id_k}(M^{(n)}|_U) \to \text{Aut}_{id_k}(M^{(m)}|_U)$ just maps skeletons $(id, f_1, \ldots, f_n)$ to $(id, f_1, \ldots, f_m)$. We define

$$\Theta^n_m(M) := \ker \varphi^n_m.$$ 

The elements of $\Theta^n_m(M)$ are exactly those which locally have the form $(id, c_{id}, 0, \ldots, 0, f_{m+1}, \ldots, f_n)$. In particular, we get a short exact sequence of sheaves of groups

$$1 \to \Theta^{n+1}_n(M) \to \Theta^n_0(M) \to \Theta^n_0(M) \to 1$$

(see [17, Theorem 1, p.273f.]). Note that $\Theta_0^{n+1}(M) = \mathcal{G}(M^{(n+1)})$. We sum up the most relevant results from [17] about the structure of $\Theta^{n+1}_n(M)$ in the next lemma and give a sketch of the proof.

Lemma 3.50 (compare [17, Theorem 1(d), p.274]). Let $k \in \mathbb{N} \cup \{\infty\}$, $n < k$ and $M \in \text{SM}^{(k)}$. Then $\Theta^{n+1}_n(M)$ is a sheaf of abelian groups and a $C^\infty_{M_\mathbb{R}}$-module. If $M$ is a Banach supermanifold, then there exist canonical isomorphisms of $C^\infty_{M_\mathbb{R}}$-modules

$$\Theta^{n+1}_n(M) \cong \Gamma(\text{Alt}^{n+1}(M_{A_1}; T_{M_\mathbb{R}})) \quad \text{if } n+1 \text{ is even and}$$

$$\Theta^{n+1}_n(M) \cong \Gamma(\text{Alt}^{n+1}(M_{A_1}; A_{M_\mathbb{R}})) \quad \text{if } n+1 \text{ is odd.}$$

Proof. Let $\{\varphi^\alpha: U^\alpha \to M^{(n+1)}: \alpha \in A\}$ be an atlas of $M^{(n+1)}$ and $U \subseteq M_\mathbb{R}$ be open. For $f \in \Theta^{n+1}_n(M)|_U$, we set $f^n := (\varphi^n)^{-1} \circ f \circ \varphi^n$, where we may assume after restriction that $\varphi^\alpha$ is a chart of $M^{(n+1)}|_U$. Locally, $f$ has the
form \( f^\alpha = (\text{id}_{\mathcal{U}}^*, c_{id}, 0, \ldots, 0, f^\alpha_{n+1}) \). For \( g \in \mathcal{O}^{n+1}_n(M)_U \), we use Proposition 3.23 to calculate

\[
(f^\alpha \circ g^\alpha) = (f \circ g)^\alpha = (\text{id}_{\mathcal{U}}^*, c_{id}, 0, \ldots, 0, f^\alpha_{n+1} + g^\alpha_{n+1}).
\]

Thus, \( \mathcal{O}^{n+1}_n(M) \) is a sheaf of abelian groups. Let \( \varphi^{\beta\alpha} \) be a change of charts of \( M^{(n+1)}|_U \). We again use Proposition 3.23 to get

\[
(\varphi^{\beta\alpha})^{-1} \circ f^\alpha \circ \varphi^{\beta\alpha} = (\text{id}_{\mathcal{U}}^*, c_{id}, 0, \ldots, 0, \tilde{f}^\beta_{n+1}),
\]

where

\[
f^\beta_{n+1}(\varphi^\alpha(x))(\bullet) = d\varphi^\alpha(x)(f^\alpha_{k+1}(x)(\varphi^{\beta\alpha}(x)(\bullet), \ldots, \varphi^{\beta\alpha}(x)(\bullet)))
\]

for \( n + 1 \) even, \( x \in \mathcal{U}^{\beta\alpha}_R \) and

\[
f^\beta_{n+1}(\varphi^\alpha(x))(\bullet) = \varphi^{\alpha\beta}(x)(f^\alpha_{k+1}(x)(\varphi^{\beta\alpha}(x)(\bullet), \ldots, \varphi^{\beta\alpha}(x)(\bullet)))
\]

for \( n + 1 \) odd, \( x \in \mathcal{U}^{\beta\alpha}_R \). If \( \mathcal{M}_A \) is a Banach vector bundle, this describes the change of charts for a section \( f_{n+1}: \mathcal{M}_R|_U \rightarrow \mathcal{M}_{\mathcal{A}}^{n+1}(\mathcal{M}_A|_U; \mathcal{M}_{\mathcal{R}}|_U) \), resp. \( f_{n+1}: \mathcal{M}_R|_U \rightarrow \mathcal{M}_{\mathcal{A}}^{n+1}(\mathcal{M}_A|_U; \mathcal{M}_A|_U) \). It is easy to see from these formulas that for a smooth map \( h: \mathcal{M}_R|_U \rightarrow \mathbb{R} \) with the local description \( h^\alpha := h \circ \varphi^\alpha_0 \), the multiplication \( h \cdot f \), defined by

\[
(h \cdot f)^\alpha = (\text{id}_{\mathcal{U}}^*, c_{id}, 0, \ldots, 0, h^\alpha \cdot f^\alpha_{n+1})
\]

leads to a \( C^\infty_{\mathcal{M}_R} \)-module structure on \( \mathcal{O}^{n+1}_n(M) \). This structure corresponds to the \( C^\infty_{\mathcal{M}_R} \)-module structure of the sheaves of sections defined above in the Banach case.

We will now return to finding a lift for an isomorphism \( f^{(n)}: \mathcal{M}^{(n)} \rightarrow \mathcal{M}^{(n)} \) to an isomorphism \( f^{(n+1)}: \mathcal{M}^{(n+1)} \rightarrow \mathcal{M}^{(n+1)} \). Let \( \mathcal{M} \in \text{SMan}^{(k)} \) be modelled on \( E \in \text{SVect}_k \) with atlas \( \{ \varphi^\alpha: U^\alpha \rightarrow \mathcal{M}: \alpha \in A \} \) and \( n < k \). Locally, we have isomorphisms \( f^{(n),\alpha}: U^{\alpha}(n-1) \rightarrow U^{\alpha}(n) \) of the form \( (\text{id}_{\mathcal{U}}^*, f^{(n),\alpha}, \ldots, f^{(n),\alpha}) \). By Lemma 3.23, these can be lifted to isomorphisms \( \tilde{f}^{(n),\alpha}: U^{\alpha}(n-1) \rightarrow U^{\alpha}(n) \) in \( \text{SDom}^{(n+1)} \) of the form

\[
(\text{id}_{\mathcal{U}}^*, f^{(n),\alpha}_1, \ldots, f^{(n),\alpha}_n, \tilde{f}^{(n+1),\alpha}),
\]

with \( \tilde{f}^{(n+1),\alpha}: U^\alpha_{\mathcal{R}} \rightarrow \mathcal{A}^{n+1}((E_1; E_{n+1})) \) an arbitrary map that is smooth in the sense of skeletons.

The morphisms \( \tilde{f}^{(n+1),\alpha} \) and \( \varphi^{\alpha\beta}(\cdot) \circ \tilde{f}^{(n+1),\alpha} \circ \varphi^{\beta\alpha}(\cdot) \) differ on \( U^{\beta\alpha}(n+1) \) only in the \( (n+1) \)-th components of their skeletons because higher components do not affect the composition of any lower components. The difference is given by a unique element \( h^{\beta\alpha} \in \mathcal{O}^{n+1}_n(U^{\beta\alpha}) \) such that

\[
\tilde{f}^{(n+1),\alpha} = \varphi^{\alpha\beta}(\cdot) \circ \tilde{f}^{(n+1),\alpha} \circ \varphi^{\beta\alpha}(\cdot) \circ h^{\beta\alpha}
\]

and one checks that these \( h^{\beta\alpha} \) define a cocycle in \( \mathcal{O}^{n+1}_n(M) \) via \( h^{\beta\alpha} := \mathcal{O}^{n+1}_n(M) \circ h^{\beta\alpha} \circ (\varphi^{\beta\alpha}(\cdot))^{-1} \) on \( \mathcal{O}^{n+1}_n(U^{\beta\alpha}(n+1)) \). This cocycle describes the obstacle to lifting \( f^{(n)} \) to \( f^{(n+1)} \) (see [17, Theorem 3, p.277]). If \( \mathcal{M}_R \) admits a smooth partition of unity, then \( \mathcal{O}^{n+1}_n(M) \) is a fine sheaf by Lemma 3.50 and thus acyclic. Therefore, the cocycle constructed above vanishes and a lift exists.
Remark 3.51. Mirroring the proof of the fact that for fine sheaves the higher Čech cohomologies vanish, one can directly construct the lift via a partition of unity. In the situation above, we assume that \((\varphi^\alpha(U^R_\alpha))_{\alpha \in A}\) is a locally finite cover of \(M_\mathbb{R}\) and that \((\rho^\alpha)_{\alpha \in A}\) is a partition of unity subordinate to this cover. With the module structure from Lemma 3.50 we define

\[ f^{(n+1),\alpha}_{n+1} := \left( (\varphi^\alpha)^{-1} \circ \left( \sum_{\beta \in A} \rho^\beta \cdot \tilde{h}^\beta_{\alpha} \right) \circ \varphi^\alpha \right)_{n+1}, \]

where \(\rho^\beta \cdot \tilde{h}^\beta_{\alpha}\) is continued to \(\varphi^\alpha, (U^{n+1})(U^{n+1})\) by zero. It is elementary to check that the change of charts is well-defined for the resulting local descriptions of \(f^{(n+1)}\).

3.6. Super Vector Bundles. In analogy to our definition of supermanifolds, we give a definition of super vector bundles as supermanifolds with a particular kind of bundle atlas. In this, we follow [18, Definition 5.1, p.29]. See also [22]. While a bit cumbersome, it will be useful to describe the change of bundle charts and the local form of bundle morphisms in terms of skeletons.

Definition 3.52 (compare [18] Subsection 1.3, p.5). Let \(E, F, H \in \text{SVec}_I, \ k \in \mathbb{N}_0 \cup \{\infty\}\) and \(U \subseteq \overline{H}^{(k)}\) be an open subfunctor. A supersmooth morphism \(f : U \times \overline{E}^{(k)} \to \overline{F}^{(k)}\) is called an \(U\)-family of \(\overline{R}\)-linear morphisms if for every \(\Lambda \in \text{Gr}^{(k)}\) and every \(u \in U_\Lambda\), the map

\[ f_\Lambda(u, \cdot) : \overline{E}^{(k)}_\Lambda \to \overline{F}^{(k)}_\Lambda \]

is \(\Lambda_{\overline{R}}\)-linear.

Lemma 3.53. In the situation of Definition 3.52 let \(f : U \times \overline{E}^{(k)} \to \overline{F}^{(k)}\) be a supersmooth morphism. Then \(f\) is an \(U\)-family of \(\overline{R}\)-linear morphisms if and only if for all \(\Lambda \in \text{Gr}^{(k)}\), we have

\[ df_\Lambda((u,0))(0,v) = f_\Lambda(u,v) \text{ for } u \in U_\Lambda, \ v \in \overline{E}^{(k)}_\Lambda. \]

Proof. If the equality holds, then \(f\) is an \(U\)-family of \(\overline{R}\)-linear morphisms because the derivative \(df_\Lambda\) is \(\Lambda_{\overline{R}}\)-linear at every \(u \in U_\Lambda\). The converse is true because any \(\Lambda_{\overline{R}}\)-linear map is in particular \(\overline{R}\)-linear and thus the derivative of such a map is the map itself.

Lemma 3.54. In the situation of Definition 3.52 let \(f : U \times \overline{E}^{(k)} \to \overline{F}^{(k)}\) be a supersmooth morphism with the skeleton \((f_n)_n\). We set \(U := U_\mathbb{R}\). Then \(f\) is an \(U\)-family of \(\overline{R}\)-linear morphisms if and only if every \(f_n\) has the form

\[ f_n = f_n(p_{U, \overline{E}_0}((pr_1,0), \ldots, (pr_1,0)) + n \cdot \mathbb{A}^n f_n(pr_1,0)((0, pr_2), (pr_1,0), \ldots, (pr_1,0)), \]

with \(f_n(p_{U, \overline{E}_0}((pr_1,0), \ldots, (pr_1,0))\) linear in the second component and where \(p_{U, \overline{E}_0} : U \times \overline{E}_0 \to U, \ pr_{\overline{E}_0} : U \times \overline{E}_0 \to \overline{E}_0, \ pr_1 : H_1 \times E_1 \to H_1\) and \(pr_2 : H_1 \times E_1 \to E_1\) are the respective projections.

Proof. Let \(f : U \times \overline{E}^{(k)} \to \overline{F}^{(k)}\) be an \(U\)-family of \(\overline{R}\)-linear morphisms. Choosing \(u := x + \sum_{i=1}^k \lambda_i y_i, \ x \in U\) and \(y_i \in H_1\), Proposition 3.20 implies that
we have

\( f_n(x, v)((y_1, 0), \ldots, (y_n, 0)) \) must be linear in \( E_0 \oplus E_1 \oplus \cdots \oplus E_1 \). We use the multilinearity of \( f_n(x, v)(\bullet) \) to calculate

\[
\begin{align*}
& f_n(x, v)((y_1, w_1), \ldots, (y_n, w_n)) = \\
& f_n(x, v)((y_1, 0), \ldots, (y_n, 0)) + \sum_{i=1}^{n} f_n(x, v)((y_1, 0), \ldots, (0, w_i), \ldots, (y_n, 0)) \\
& = (f_n(x, v)((pr_1, 0), \ldots, (pr_1, 0)) + \\
& n \cdot \mathcal{A}^n f_n(x, 0)((0, pr_2), (pr_1, 0), \ldots, (pr_1, 0))((y_1, w_1), \ldots, (y_n, w_n))
\end{align*}
\]

for \( v \in E_0 \) and \( w_i \in E_1 \). The second equality follows because for \( v' \in E_0 \), we have

\[
\begin{align*}
& f_n(x, v + v')((y_1, 0), \ldots, (0, w_i), \ldots, (y_n, 0)) = \\
& f_n(x, v)((y_1, 0), \ldots, (0, w_i), \ldots, (y_n, 0)) \\
& + f_n(x, v')((y_1, 0), \ldots, (0, 0), \ldots, (y_n, 0))
\end{align*}
\]

Conversely, let \( (f_n)_n \) have the aforementioned form. Let \( \Lambda \in \text{Gr}^{(k)} \), \( (x, y) \in U \times E_0 \) and \( (x_i, y_i) \in (H_i \oplus E_i) \oplus \Lambda^\perp_i, i \in \{0, 1\} \). To simplify our notation, we consider \( \overline{H}_\Lambda(k) \subseteq \overline{H} \oplus \overline{E}_\Lambda(k) \) and \( \overline{E}_\Lambda(k) \subseteq \overline{H} \oplus \overline{E}_\Lambda(k) \) in the obvious way. One sees

\[
\begin{align*}
d^m f_i(x, y)((x_0, y_0), \ldots, (x_0, y_0), (x_1, y_1), \ldots, (x_1, y_1)) =
& d^m f_i(x, y)(x_0, \ldots, x_0, x_1, \ldots, x_1) \\
& + m \cdot d^{m-1} f_i(x, y_0)(x_0, \ldots, x_0, x_1, \ldots, x_1) \\
& + l \cdot d^m f_i(x)(x_0, \ldots, x_0, y_1, x_1, \ldots, x_1),
\end{align*}
\]

where the last two summands are understood to be zero for \( m = 0 \) and \( l = 0 \), respectively. For \( u = x + x_0 + x_1 \) and \( v = y + y_0 + y_1 \), we use Remark \textbf{3.22} to get

\[
df_\Lambda(u)(v) = \sum_{m, l=0}^{\infty} \frac{1}{m!} \bigg( d^m f_i(x, y)(x_0, \ldots, x_0, x_1, \ldots, x_1) \\
& + m \cdot d^{m-1} f_i(x, y_0)(x_0, \ldots, x_0, x_1, \ldots, x_1) \\
& + l \cdot d^m f_i(x)(x_0, \ldots, x_0, y_1, x_1, \ldots, x_1) \bigg)
\]

Comparing the terms, Proposition \textbf{3.20} implies that \( df_\Lambda(u)(v) = f_\Lambda(u, v) \) and the result follows from Lemma \textbf{3.33}. \hfill \Box

\textbf{Definition 3.55.} Let \( E, F \in \text{SVec}_{k, c} \), \( k \in \mathbb{N}_0 \cup \{\infty\} \) and let \( E, M \in \text{SMAn}^{(k)} \) be such that \( E \) is modelled on \( E \oplus F \) and \( M \) is modelled on \( E \) together with a supersmooth morphism \( \pi : E \to M \) such that \( \pi_\Lambda : E_\Lambda \to M_\Lambda \) is a vector bundle with fiber \( F^\perp_\Lambda \) for all \( \Lambda \in \text{Gr}^{(k)} \). A bundle atlas of \( E \) is an atlas \( A := \{ \Psi^\alpha : U^\alpha \times F^{(k)} \to E : U^\alpha \subseteq \overline{E}^{(k)}, \alpha \in A \} \) such that \( \{ \Psi^\alpha_\Lambda : \alpha \in A \} \) is a bundle atlas of \( E_\Lambda \) and the change of two charts \( \Psi^\alpha \) and \( \Psi^\beta \) has the form \( \Psi^\alpha_\Lambda : U^\alpha_\Lambda \times \overline{F}^{(k)} \to \overline{U}^{\beta_\Lambda} \times \overline{F}^{(k)} \) with \( \Psi^\alpha_\beta = (\phi^\alpha_\beta, \psi^\alpha_\beta) \), where

\begin{enumerate}
\item \( \phi^\alpha_\beta : U^\alpha_\beta \to U^{\beta_\Lambda} \) and
\item \( \psi^\alpha_\beta : U^\beta_\Lambda \times \overline{F}^{(k)} \to \overline{F}^{(k)} \) is an \( U^{\beta_\Lambda} \)-family of \( \mathbb{R} \)-linear maps.
\end{enumerate}
The elements of $\mathcal{A}$ are called bundle charts. Two bundle atlases are equivalent if their union is again a bundle atlas. We call $\pi: E \to M$ together with an equivalence class of bundle atlases a $k$-super vector bundle over the base $M$ with typical fiber $F$. The morphism $\pi$ is called the projection to the base.

Let $E'$ be another $k$-super vector bundle with typical fiber $F' \in SV\text{Vec}_\mathbb{C}$ and base $N$. A smooth morphism $f: E \to E'$ is a morphism of super vector bundles, if in bundle charts $\Psi^\alpha$ of $E$ and $\Psi'^\alpha$ of $E'$, it has the form $(h^{\alpha\alpha'}, g^{\alpha\alpha'})$, where

1. $h^{\alpha\alpha'}: U^{\alpha\alpha'} \to U^{\alpha'}$, $U^{\alpha\alpha'} \subseteq U^\alpha$ and
2. $g^{\alpha\alpha'}: U^{\alpha\alpha'} \times F^{(k)} \to F'^{(k)}$ is an $U^{\alpha\alpha'}$-family of $\mathbb{R}$-linear maps.

Clearly, the $h^{\alpha\alpha'}$ define a supersmooth morphism $h: M \to N$ such that $\pi_N \circ f = h \circ \pi_M$. We say that $f$ is a morphism over $h$. The $k$-super vector bundles and their morphisms form a category, which we denote by $SV\text{Bun}^{(k)}$, resp. $SV\text{Bun}$ if $k = \infty$.

By this definition, a $k$-super vector bundle can be seen as a functor $Gr^{(k)} \to VBun$. This point of view is taken by Molotkov in [18].

**Remark 3.56.** It follows from Proposition 3.31 that, if one has a collection of change of charts that satisfy the conditions of Definition 3.55, then one gets a (up to unique isomorphism) unique super vector bundle. In the notation of the definition, the $\phi^{\alpha\beta}$ then define the base supermanifold $M$ and the bundle projection is locally given by

$$(\phi^\alpha)^{-1} \circ \pi \circ \Psi^\alpha := pr_{U^\alpha} : U^\alpha \times F^{(k)} \to U^\alpha.$$

In the same way, morphisms of super vector bundles are determined by their local description.

**Lemma/Definition 3.57.** Let $k \in \mathbb{N}_0 \cup \{\infty\}$, let $E$ be a $k$-super vector bundle with typical fiber $F \in SV\text{Vec}_\mathbb{C}$ over $M$ modelled on $E \in SV\text{Vec}_\mathbb{C}$ and let $x: p^{(k)} \to M$ be a point of $M$. We define $E_x$, the fiber of $E$ at $x$, by setting $(E_x)_\Lambda := ((\pi_M)^{-1})(\{x_\Lambda\})$ for every $\Lambda \in Gr^{(k)}$. Then $E_x$ is a subsupermanifold of $E$ and $E_x$ is, in a canonical way, an $\mathbb{R}$-module such that $E_x \cong F^{(k)}$.

**Proof.** Let $\{\Psi^\alpha : U^\alpha \times F^{(k)} \to E: \alpha \in A\}$ be a bundle atlas of $E$ with corresponding atlas $\{\phi^\alpha : U^\alpha \to M: \alpha \in A\}$ of $M$. Let $\phi^\alpha : U^\alpha \to M$ be such that $x_\mathbb{R} \in \phi^\alpha_\mathbb{R}(U^\alpha_\mathbb{R})$. We may assume that $0 \in U^\alpha_\mathbb{R}$ and that $\phi^\alpha_\mathbb{R}(0) = x_\mathbb{R}$, because the translation defined by $E^{(k)}_{\Lambda} \to E^{(k)}_{\Lambda}$, $v \mapsto v - (\phi^\alpha_\mathbb{R})^{-1}(x_\mathbb{R})$ is clearly an isomorphism in $SV\text{Man}^{(k)}$. Let $\Lambda \in Gr^{(k)}$. We have $y_\Lambda \in (E_x)_\Lambda$ if and only if $(\pi_M)_\Lambda(y_\Lambda) = x_\Lambda$ holds and thus if and only if $y_\Lambda \in \Psi^\alpha_\Lambda((0)_\Lambda^{(k)} \times F^{(k)}_{\Lambda})$ holds. Therefore, $E_x$ is a subsupermanifold of $E$. We define an $\mathbb{R}$-module structure on $E_x$ via the isomorphism

$$\Psi^\alpha \circ (0, id_{F^{(k)}}) : F^{(k)} \to E_x.$$ 

The $\mathbb{R}$-module structure on $E_x$ does not depend on $\Psi^\alpha$ because a change of bundle charts leads to an isomorphism of $\mathbb{R}$-modules in the second component. □
**Lemma 3.58.** Let \( k \in \mathbb{N}_0 \cup \{\infty\} \) and let \( \mathcal{E} \) and \( \mathcal{E}' \) be \( k \)-super vector bundles over \( \mathcal{M} \) and \( \mathcal{N} \). If \( f: \mathcal{E} \to \mathcal{E}' \) is a morphism of \( k \)-super vector bundles over \( g: \mathcal{M} \to \mathcal{N} \), then for every point \( x \) of \( \mathcal{M} \) the morphism

\[
(f|_x): \mathcal{E}_x \to \mathcal{E}'_{g(x)}
\]

is a well-defined morphism of \( \mathbb{R} \)-modules (and in particular supersmooth).

**Proof.** Let \( \Lambda \in \text{Gr}^{(k)} \) and \( y_\Lambda \in (\mathcal{E}_x)_\Lambda \). We have

\[
y_\Lambda \xrightarrow{f_\Lambda} (\mathcal{E}')_\Lambda \\
 \downarrow (\pi_\Lambda)_\Lambda \\
 x_\Lambda \xrightarrow{g_\Lambda} y_\Lambda (x_\Lambda)
\]

and \( g_\Lambda (x_\Lambda) = (g \circ x)_\Lambda \) implies that the morphism is well-defined. In charts the second component of \( f \) is \( \mathbb{R} \)-linear. Thus, \( f|_x \) is a morphism of \( \mathbb{R} \)-modules. \( \square \)

**Lemma 3.59.** The functors \( \iota_0^k \) and \( \iota_1^k \) of Proposition 3.40, Proposition 3.43 and Lemma 3.44 extend to functors

\[
iota_0^k: \text{SVBun}^{(0)} \to \text{SVBun}^{(k)} \text{ for } k \in \mathbb{N}_0 \cup \{\infty\},
\]

\[
iota_1^k: \text{SVBun}^{(1)} \to \text{SVBun}^{(k)} \text{ for } k \in \mathbb{N} \cup \{\infty\} \text{ and }
\]

\[
\pi_n^k: \text{SVBun}^{(k)} \to \text{SVBun}^{(n)} \text{ for } k \in \mathbb{N}_0 \cup \{\infty\}, 0 \leq n \leq k.
\]

With these functors, we have \( \pi_n^k \circ \iota_0^k \cong \text{id}_{\text{SVBun}^{(0)}} \) and \( \pi_1^k \circ \iota_1^k \cong \text{id}_{\text{SVBun}^{(1)}} \).

**Proof.** Let us consider \( \iota_0^k \) and \( \iota_1^k \) as functors \( \text{SMan}^{(0)} \to \text{SMan}^{(k)} \) and \( \text{SMan}^{(1)} \to \text{SMan}^{(k)} \). Let \( k \in \mathbb{N}_0 \cup \{\infty\}, E \in \text{SVec}_{lc}, \mathcal{U} \subseteq \mathcal{E}^{(k)} \). We see from the concrete description in Lemma 3.54 that every \( \mathcal{U} \)-family of \( \mathbb{R} \)-linear morphisms is mapped to an \( \mathcal{U} \)-family of \( \mathbb{R} \)-linear morphisms under the original functors. From this, the result follows. \( \square \)

Note also that for any super vector bundle \( \mathcal{E} \) with base \( \mathcal{M} \) and typical fiber \( F \), the inverse limit \( \lim_{\leftarrow} \mathcal{E} \) is in a natural way a vector bundle over \( \lim_{\leftarrow} \mathcal{M} \) with typical fiber \( \lim_{\leftarrow} F \).

### 3.6.1. The Change of Parity Functor

The space of sections of a super vector bundle can be turned into a vector space, as we will see below. However, in a sense this describes only the even sections. To incorporate odd sections and obtain a super vector space of sections, we need the so-called change of parity functor. On super vector spaces this functor simply swaps the even and odd parts. Doing this fiberwise, one gets the change of parity functor for super vector bundles. As before, it will be useful to express this functor in terms of skeletons.

**Definition 3.60.** Let \( E,F \in \text{SVec}_{lc} \) and \( f: E \to F \) be a morphism. We define a functor \( \Pi: \text{SVec}_{lc} \to \text{SVec}_{lc} \) by setting \( \Pi(E)_i := E_{i+1}^{\mathbb{R}^{\neq}} \) and \( \Pi(f)_i := f_{i+1}^{\mathbb{R}^{\neq}} \) for \( i \in \{0,1\} \). Now, let \( \Lambda = \Lambda_n \in \text{Gr}, n \in \mathbb{N} \) and \( g: \overline{E}_\Lambda \to \overline{F}_\Lambda \) be \( \mathbb{R} \)-linear such that there exist linear maps

\[
g_{(0)}: E_0 \to \overline{F}_\Lambda \quad \text{and} \quad g_{(1)}: E_1 \to \overline{\Pi(F)}_\Lambda
\]
with \( g(\lambda_I v_I) = \lambda_I g(v_I) \) for \( I \in \mathcal{P}_n \), \( v_I \in E_i \), \( i \in \{0,1\} \). We call such a map \( g \) \textit{parity changeable}. We define a parity changeable map
\[
\Pi_A(g) : \Pi(E)_A \to \Pi(F)_A
\]
by setting \( \Pi_A(g)(i) := g_{(i\mapsto i)} \) for \( i \in \{0,1\} \).

Note that in the situation \( g \) is automatically \( A \)-linear and we have \( \Pi_A(\Pi_A(g)) = g \). What is more, with \( f_0 := f_0 \) and \( f_1 := f_1 \), we see that \( f_A \) is parity changeable and it follows \( \Pi_A(f_A) = \Pi(f_A) \).

**Lemma 3.61.** Let \( E, F, H \in \textbf{SVec}_{ic} \), \( A = A_n \in \textbf{Gr} \) with \( n \in \mathbb{N} \) and let \( f : E \to F \), \( g : F \to H \) be parity changeable. Then \( g \circ f \) is also parity changeable and we have
\[
\Pi_A(g \circ f) = \Pi_A(g) \circ \Pi_A(f).
\]

**Proof.** Let \( f_0 \), \( f_1 \) and \( g_0 \), \( g_1 \) be as in Definition 3.60. For \( I \in \mathcal{P}_n \), \( v \in E_i \), \( i \in \{0,1\} \) let
\[
f(\lambda_I v) = \lambda_I f(i)(v) = \lambda_I \sum_{J \in \mathcal{P}_n} \lambda_J w_J,
\]
where \( w_J \in F_{|J|+|I|} \). It follows that
\[
(g \circ f)(\lambda_I v) = \sum_{J \in \mathcal{P}_n} \lambda_I \lambda_J g_{(J|I+|I|)}(w_J).
\]
This implies that \( g \circ f \) is parity changeable with \( (g \circ f)_0 = g \circ f_0 \) and \( (g \circ f)_1 = \Pi_A(g) \circ f_1 \). Thus, \( \Pi_A(g \circ f)_0 = \Pi_A(g) \circ f_1 \) and \( \Pi_A(g \circ f)_1 = g \circ f_0 \). Applying this to \( \Pi_A(g) \circ \Pi_A(f) \), we get
\[
(\Pi_A(g) \circ \Pi_A(f))_0 = \Pi_A(g) \circ \Pi_A(f)_0 = \Pi_A(g \circ f)_0
\]
and
\[
(\Pi_A(g) \circ \Pi_A(f))_1 = \Pi_A(\Pi_A(g)) \circ \Pi_A(f)_1 = g \circ f_0
\]
and therefore
\[
\Pi_A(g \circ f) = \Pi_A(g) \circ \Pi_A(f).
\]

**Lemma 3.62.** Let \( E, F, H \in \textbf{SVec}_{ic} \), \( k \in \mathbb{N} \cup \{\infty\} \), \( U \subseteq \mathcal{H}^{(k)} \) be an open subfunctor and let \( f : U \times E^{(k)} \to F^{(k)} \) be an \( U \)-family of \( F \)-linear morphisms. For \( n \in \mathbb{N} \), \( A = A_n \in \textbf{Gr}^{(k)} \) and \( u \in U \), the map \( f_A(u, \bullet) : E^{(k)}_A \to F^{(k)}_A \) is parity changeable. Defining \( (\Pi(f))(u, \bullet) := \Pi_A(f_A(u, \bullet)) \) leads to an \( U \)-family of \( F \)-linear morphisms
\[
\Pi(f) : U \times \Pi(E)^{(k)}_0 \to \Pi(F)^{(k)}.
\]
The skeleton of \( \Pi(f) \) has the components
\[
\hat{f}_0 = f_1(pr_U)(pr_{\Pi(E)_0}) \quad \text{and}
\]
\[
\hat{f}_I = f_{I+1}(pr_U)(pr_{\Pi(E)_0}, pr_1, \ldots, pr_1) + l \cdot \mathfrak{A}^l f_{I-1}(pr_U, pr_2)(pr_1, \ldots, pr_1)
\]
for \( l > 0 \), where we consider
\[
\mathfrak{A}^l f_{I-1}(pr_U, pr_2)(pr_1, \ldots, pr_1) : U \times \Pi(E)_0 \to \text{Alt}^l(H_1 \oplus \Pi(E)_1 ; \Pi(F)_F)
\]
and

\[ f_{l+1}(\text{pr}_U)(\text{pr}_{\Pi(E)_0}, \text{pr}_1, \ldots, \text{pr}_1): U \times \Pi(E)_0 \to \text{Alt}^l(H_1 \oplus \Pi(E)_1; \Pi(F)_\mathbb{R}) \]

with the projections \( \text{pr}_U: U \times \Pi(E)_0 \to U, \text{pr}_{\Pi(E)_0}: U \times \Pi(E)_0 \to \Pi(E)_0, \)
\( \text{pr}_1: H_1 \times \Pi(E)_1 \to H_1 \) and \( \text{pr}_2: H_1 \times \Pi(E)_1 \to \Pi(E)_1. \)

**Proof.** Let \( U : = \mathcal{U}_\mathbb{R}. \) We set \( \overline{\Pi(f)}_\mathbb{R} := \overline{\Pi(f)}_{\Lambda_1} \mid_{\mathcal{U}_\mathbb{R} \times \overline{\Pi(E)}_\mathbb{R}^{(k)} \to \overline{\Pi(F)}_\mathbb{R}^{(k)}} \) so that \( \overline{\Pi(f)}_\Lambda \) is defined for all \( \Lambda \in \text{Gr}^{(k)}. \) To simplify our notation, we consider \( \overline{\Pi(f)}_\Lambda \subseteq H \oplus E_\Lambda \) and \( \overline{\Pi(E)}_\Lambda \subseteq H \oplus E_\Lambda \) in the obvious way. Let \( x \in U, x_0 \in \overline{\Pi_0}_\Lambda, x_1 \in \overline{\Pi_1}_\Lambda \) and \( y_0 \in \overline{E}_0_\Lambda, y_1 \in \overline{E}_1_\Lambda. \) For \( u = x + x_0 + x_1 \) and \( v = y_0 + y_1, \) we use formula (4) to get

\[
\begin{align*}
&f_\Lambda(u, v) = \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(y_0, x_0, \ldots, x_0, x_1, \ldots, x_1) + \\
&\quad \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \ldots, x_0, y_1, x_1, \ldots, x_1).
\end{align*}
\]

Therefore, \( f_\Lambda(u, \cdot) \) is parity changeable with

\[
\begin{align*}
(f_\Lambda(u, \cdot))(0) &= \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(\cdot, x_0, \ldots, x_0, x_1, \ldots, x_1) \quad \text{and} \\
(f_\Lambda(u, \cdot))(1) &= \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \ldots, x_0, \cdot, x_1, \ldots, x_1).
\end{align*}
\]

In the next step, we show that \( (\tilde{f}_n)_n \) is the skeleton of \( \overline{\Pi(f)}. \) Let \( \tilde{y} \in \overline{\Pi(E)_0}, \)
\( \tilde{y}_0 \in \overline{\Pi(E)_0}_\Lambda \) and \( \tilde{y}_1 \in \overline{\Pi(E)_1}_\Lambda. \) We calculate

\[
\begin{align*}
d^m f_l(x, \tilde{y})(x_0, \tilde{y}_0, \ldots, (x_0, \tilde{y}_0), (x_1, \tilde{y}_1), \ldots, (x_1, \tilde{y}_1)) &= \\
d^m f_{l+1}(x)(x_0, \ldots, x_0, \tilde{y}, x_1, \ldots, x_1) + \\
&\quad m \cdot d^{m-1} f_{l+1}(x)(x_0, \ldots, x_0, \tilde{y}_0, x_1, \ldots, x_1) + \\
&\quad l \cdot d^m f_{l-1}(x, \tilde{y}_1)(x_0, \ldots, x_0, x_1, \ldots, x_1),
\end{align*}
\]

where the last two summands are zero for \( m = 0 \) and \( l = 0, \) respectively. Note that

\[
\begin{align*}
l \cdot d^m f_{l-1}(x, \tilde{y}_1)(x_0, \ldots, x_0, x_1, \ldots, x_1) \\
&\quad = l \cdot d^{m+1} f_{l-1}(x)(\tilde{y}_1, x_0, \ldots, x_0, x_1, \ldots, x_1)
\end{align*}
\]

holds because of Lemma 4.3.1. If \( \tilde{f}: \mathcal{U} \times \overline{\Pi(E)}_\Lambda^{(k)} \to \overline{\Pi(F)}_\Lambda^{(k)} \) is the morphism defined by \( (\tilde{f}_n)_n, \) then it follows

\[
\begin{align*}
\tilde{f}_\Lambda(u, \tilde{v}) &= \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^m f_{l+1}(x)(x_0, \ldots, x_0, \tilde{y} + \tilde{y}_0, x_1, \ldots, x_1) + \\
&\quad \sum_{m, l=0}^{\infty} \frac{1}{m!l!} d^{m+1} f_l(x)(\tilde{y}_1, x_0, \ldots, x_0, x_1, \ldots, x_1)
\end{align*}
\]
for \( \tilde{v} = \tilde{y} + \tilde{y}_0 + \tilde{y}_1 \). This is exactly \( (\Pi(f))_\Lambda(u, \tilde{v}) \).

**Corollary 3.63.** Let \( E, E', F, F', H, H' \in \text{SVect}_{lc} \), \( k \in \mathbb{N} \cup \{ \infty \} \) and \( \mathcal{U} \subseteq H \), \( \mathcal{V} \subseteq H'^{(k)} \) be open subfunctors. Moreover, let \( f : E \to F \) be an \( \mathcal{U} \)-family of \( \mathbb{R} \)-linear morphisms, \( g : \mathcal{V} \to \mathcal{V}' \) be an \( \mathcal{V} \)-family of \( \mathbb{R} \)-linear morphisms and \( h : \mathcal{U} \to \mathcal{V} \) be supersmooth. Then \( g \circ (h : f) \in \mathcal{U} \times \mathcal{V} \to \mathcal{V}' \) is an \( \mathcal{U} \)-family of \( \mathbb{R} \)-linear morphisms and we have

\[
\Pi(g \circ (h : f)) = \Pi(g) \circ (h, \Pi(f)).
\]

In addition, we have \( \Pi(\Pi(f)) = f \).

**Proof.** This follows from the pointwise definition of \( \Pi \) in Lemma 3.62 and Lemma 3.61. \( \square \)

**Proposition 3.64.** For \( k \in \mathbb{N} \cup \{ \infty \} \) let \( \pi : \mathcal{E} \to \mathcal{M} \) be a k-super vector bundle with typical fiber \( F \in \text{SVect}_{lc} \), bundle atlas \( \{ \Psi^\alpha : \mathcal{U} \times F^{(k)} \to \mathcal{E} : \alpha \in \alpha \} \) and the respective change of charts \( \Psi^{\alpha \beta} = (\phi^{\alpha \beta}, \psi^{\alpha \beta}) \). Then the morphisms \( (\phi^{\alpha \beta}, \Pi(\psi^{\alpha \beta})) \) define a k-super vector bundle \( \Pi(\mathcal{E}) \) over \( \mathcal{M} \) with typical fiber \( \Pi(F) \).

Let \( \pi' : \mathcal{E}' \to \mathcal{M}' \) be another k-vector bundle and \( f : \mathcal{E} \to \mathcal{E}' \) be a morphism of k-super vector bundles over \( h : \mathcal{M} \to \mathcal{M}' \). If \( f \) has the local form \( (g^{\alpha', \alpha}, \varphi^{\alpha'}) \), then \( (g^{\alpha', \alpha}, \Pi(\varphi^{\alpha})) \) defines a morphism \( \Pi(f) : \Pi(\mathcal{E}) \to \Pi(\mathcal{E}') \) over \( h \). This construction is functorial and defines an equivalence of categories

\[
\Pi : \text{SVBun}^{(k)} \to \text{SVBun}^{(k)}.
\]

**Proof.** In light of Remark 3.56 it follows from Corollary 3.63 that the morphisms \( (\phi^{\alpha \beta}, \Pi(\psi^{\alpha \beta})) \) define a super vector bundle. That \( \Pi \) is well-defined on morphisms and functorial follows by the same argument. The corollary also implies that \( \Pi(\Pi(\mathcal{E})) \cong \mathcal{E} \) and \( \Pi(\Pi(f)) \cong f \) hold under this identification, which shows that \( \Pi \) is an equivalence of categories. \( \square \)

### 3.7. The Tangent Bundle of a Supermanifold

In this section, we expand on the definition of the tangent functor \( \mathcal{T} \) for supermanifolds given by Molotkov (see [13], Section 5.3, p. 404f.).

Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( \mathcal{M} \in \text{Man}^{\text{Gr}}^{(k)} \). We define a functor \( \mathcal{T}_{\mathcal{M}} \in \text{Man}^{\text{Gr}}^{(k)} \) by setting \( \mathcal{T}_{\mathcal{M}} \lambda := \mathcal{T}_{\mathcal{M}} \lambda \) for all \( \lambda \in \text{Gr}^{(k)} \) and \( \mathcal{T}_{\mathcal{M}} := \mathcal{T}_{\mathcal{M} \phi} : \mathcal{T}_{\mathcal{M} \lambda} \to \mathcal{T}_{\mathcal{M} \lambda} \) for all \( g \in \text{Hom}_{\text{Gr}^{(k)}}(\Lambda, \Lambda') \). It follows from the functoriality of \( \mathcal{T} : \text{Man} \to \text{Man} \) that \( \mathcal{T}_{\mathcal{M}} \) is indeed a functor. By the same argument, the bundle projections \( \pi^{\mathcal{T}_{\mathcal{M}}_{\Lambda}} : \mathcal{T}_{\mathcal{M} \lambda} \to \mathcal{M}_{\lambda} \) define a natural transformation \( \pi^{\mathcal{T}_{\mathcal{M}}_{\Lambda}} : \mathcal{T}_{\mathcal{M}} \to \mathcal{M} \).

If \( \mathcal{N} \in \text{Man}^{\text{Gr}}^{(k)} \) and \( f : \mathcal{M} \to \mathcal{N} \) is a natural transformation, then it is easy to see that setting \( \mathcal{T}_{\mathcal{M}} := \mathcal{T}_{\mathcal{M} \lambda} : \mathcal{T}_{\mathcal{M} \lambda} \to \mathcal{T}_{\mathcal{N} \lambda} \) for all \( \lambda \in \text{Gr}^{(k)} \) defines a natural transformation \( \mathcal{T}_{\mathcal{M}} : \mathcal{T}_{\mathcal{M}} \to \mathcal{T}_{\mathcal{N}} \) and that this gives us a functor \( \mathcal{T} : \text{Man}^{\text{Gr}}^{(k)} \to \text{Man}^{\text{Gr}}^{(k)} \). We obviously have \( \pi^{\mathcal{T}_{\mathcal{N}}} \circ \mathcal{T}_{\mathcal{M}} = f \circ \pi^{\mathcal{T}_{\mathcal{M}}} \).

**Lemma 3.65.** Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \) and \( \mathcal{M} \in \text{SM} \) be modelled on \( E \in \text{SVect}_{lc} \) with the atlas \( \{ \varphi^\alpha : U^\alpha \to \mathcal{M} : \alpha \in \alpha \} \). Then \( \mathcal{T}_{\mathcal{M}} \) is a k-super vector bundle over \( \mathcal{M} \) with typical fiber \( E \), the bundle atlas \( \{ \varphi^\alpha : U^\alpha \to \mathcal{T}_{\mathcal{M}} : \alpha \in \alpha \} \) and the projection \( \pi^{\mathcal{T}_{\mathcal{M}}} \). If \( f : \mathcal{M} \to \mathcal{N} \) is a morphism of
$k$-supermanifolds, then $Tf: TM \rightarrow TN$ is a morphism of $k$-super vector bundles and the above defines a functor

$$T: S\text{Man}^{(k)} \rightarrow SV\text{Bun}^{(k)}.$$\)

Proof. That $\{ T\varphi^\alpha: TU^\alpha \rightarrow TM: \alpha \in A \}$ is a covering is obvious. By functoriality, we have

$$T(\varphi^\beta)^{-1} \circ T\varphi^\alpha = T\varphi^\alpha \beta$$
onumber

on $TU^\alpha\beta = U^{\alpha\beta} \times \bar{E}^{(k)}$ for all $\alpha, \beta \in A$ and by definition, we have

$$T\varphi^\alpha\beta = (\varphi^\alpha\beta, d\varphi^\alpha\beta): U^{\alpha\beta} \times \bar{E}^{(k)} \rightarrow U^{\beta\alpha} \times \bar{E}^{(k)},$$

which is a supersmooth morphism because of Lemma 3.17. Clearly, each $\pi^{TM}_{k}: TM \rightarrow \Lambda$ is a vector bundle and we have that

$$(\varphi^\alpha)^{-1} \circ \pi^{TM} \circ T\varphi^\alpha: U^\alpha \times \bar{E}^{(k)} \rightarrow U^\alpha$$

is simply the projection and thus supersmooth. Since, by definition, $d\varphi^\alpha\beta$ is an $U^{\alpha\beta}$-family of $\mathbb{R}$-linear morphisms, the above atlas is indeed a bundle atlas for $TM$. In such charts, $Tf$ has locally the form $(f^{ab}, df^{ab})$ and therefore is a morphism of $k$-super vector bundles for the same reason. Functoriality follows from the functoriality of $T$ as a functor $\text{Man}^{Gr^{(k)}} \rightarrow \text{Man}^{Gr^{(k)}}$. □

In the situation of the lemma, we call $TM$ the tangent bundle of $M$. We will write $\pi^{TM}_{x}: TM \rightarrow M$ for the bundle projection and $T_{x}M$ instead of $(TM)_{x}$ for the fiber of $TM$ at a point $x$ of $M$.

Lemma 3.66. For every $M \in S\text{Man}$, we have

$$\lim \rightarrow TM \cong \lim \rightarrow M$$

in $MB\text{un}^{(\infty)}$ with the functor $\lim \rightarrow$ from Theorem 3.37. Moreover, $\lim \pi^{TM} = \pi \lim \rightarrow M$ holds for the bundle projections $\pi^{TM}: TM \rightarrow M$ and $\pi^{T\rightarrow M}: TM \rightarrow M$. For morphisms $f: M \rightarrow N$ of supermanifolds, we have

$$\lim \rightarrow Tf = T \lim \rightarrow f$$

under the above identification.

Proof. This follows from the definition of $TM$, Lemma A.25 and the definition of $\lim \rightarrow$ in Theorem 3.37. □

Remark 3.67. In view of Lemma 3.66, it seems likely that one can describe higher tangent bundles, higher jet bundles and higher tangent Lie supergroups analogously to [5].

Lemma 3.68. Let $k \in \mathbb{N}_{0} \cup \{\infty\}$ and $M \in S\text{Man}^{(k)}$. With the functors from Lemma 3.59, we have $T\imath^{n}_{k}(M) \cong \imath^{n}_{k}(TM)$ for $n \in \{0, 1\}, n \leq k$ in $SV\text{Bun}^{(k)}$ and $T\pi^{n}_{k}(M) \cong \pi^{n}_{k}(TM)$ for $0 \leq n \leq k$ in $SV\text{Bun}^{(k)}$.

Proof. With any atlas $A := \{ \varphi^\alpha: \alpha \in A \}$ of $M$ it is obvious that applying $T \circ \imath^{n}_{k}$ and $\imath^{n}_{k} \circ T$ to a change of charts leads to the same morphism. The same is true for $T \circ \pi^{k}_{n}$ and $\pi^{k}_{n} \circ T$. □
Appendix A. Multilinear Bundles

Multilinear bundles were introduced in [5] to describe higher order tangent bundles. As it turns out, the structure of supermanifolds is closely related to the structure of multilinear bundles. One important addition introduced below, is the inverse limit of multilinear bundles.

For this section, we fix the infinitesimal generators \( \varepsilon_k, k \in \mathbb{N} \), with the relations \( \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \) and \( \varepsilon_i \varepsilon_i = 0 \). As usual, we set \( \varepsilon_I := \varepsilon_{i_1} \cdots \varepsilon_{i_r} \) for \( I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\} \).

A.1. Multilinear Spaces.

Definition A.1 ([5, MA.2, p.169]). Let \( k \in \mathbb{N} \). A (locally convex) \( k \)-dimensional cube is a family \((E_I)_{I \in \mathcal{P}_k} \) of (locally convex) \( \mathbb{R} \)-vector spaces with the total space

\[
E := \bigoplus_{I \in \mathcal{P}_k} E_I.
\]

We denote the elements of \( E \) by \( v = \sum_{I \in \mathcal{P}_k} v_I \) or by \( v = (v_I)_{I \in \mathcal{P}_k} \) with \( v_I \in E_I \). By abuse of notation, we will call \( E \) a \( k \)-dimensional cube as well.

For convenience, we let a 0-dimensional cube be defined by the total space \( \{0\} \). The spaces \( E_I \) are called the axes of \( E \).

Let \((E_I) \) and \((E'_I) \) be \( k \)-dimensional cubes. For each partition \( \nu \in \mathcal{P}(I) \), \( I \in \mathcal{P}_k \), let \( f^\nu \) be an \( \mathbb{R} \)-multilinear map

\[
f^\nu: E_\nu := E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \to E'_I,
\]

\[
v_\nu := (v_{\nu_1}, \ldots, v_{\nu_{\ell(\nu)}}) \mapsto f^\nu(v_\nu).
\]

A morphism of (locally convex) \( k \)-dimensional cubes \( E \) and \( E' \) is a (continuous) map of the form

\[
f: E \to E', \quad \sum_{I \in \mathcal{P}_k} v_I \mapsto \sum_{I \in \mathcal{P}_k} \sum_{\nu \in \mathcal{P}(I)} f^\nu(v_\nu).
\]

The composition of two morphisms is simply the composition of maps. We define the product \( E \times E' \) by \((E \times E')_I := E_I \times E'_I \).

Clearly, a morphism \( f \) of \( k \)-multilinear bundles (that are also locally convex \( k \)-multilinear bundles) is continuous if and only if all \( f^\nu \) are continuous.

Theorem A.2.

(a) The (locally convex) \( k \)-dimensional cubes and their (continuous) morphisms form a category, which we will call the category of (locally convex) \( k \)-multilinear spaces.

(b) A morphism \( f: E \to E' \) of \( k \)-dimensional cubes is invertible if and only if \( f^\nu \) is a bijection for all partitions of the form \( \nu = \{I\} \), \( I \in \mathcal{P}_k \), i.e., for all partitions of length one. In this case \( f^{-1} \) is again a morphism of \( k \)-dimensional cubes.

(c) If \( f: E \to E' \) is a morphism of locally convex \( k \)-dimensional cubes such that \( f^{\{I\}} \) is bijective with continuous inverse for all \( I \in \mathcal{P}_k \), then \( f \) is invertible.

Proof. Points (a) and (b) are just [5, MA.6, p.172] and (c) follows from the inductive construction in that proof. \( \square \)
We denote the category of \( k \)-multilinear spaces by \( \text{MSpace}^{(k)} \).

**Remark A.3.** It is calculated in the proof of [5 Theorem MA.6, p.172] that the composition of morphisms \( f : E \to E' \), \( g : E' \to E'' \) of \( k \)-dimensional cubes \( E, E' \) and \( E'' \) is given by

\[
(g \circ f)^\nu(v_\nu) = \sum_{\omega \leq \nu} g^\omega \left( f^{\omega_1|\nu}(v_{\omega_1|\nu}), \ldots, f^{\omega_{\ell(\omega)|\nu}(v_{\omega_{\ell(\omega)}|\nu})} \right).
\]

Of course the sets \( \omega_1|\nu, \ldots, \omega_{\ell(\omega)}|\nu \) need not be in (graded) lexicographic order but by abuse of notation, we also write \( g^\omega \) for the map that arises from permuting the factors.

**Definition A.4.** Let \((E_I)\) be a \( k \)-dimensional cube. For \( P \subseteq \mathcal{P}_+^k \), we define the restriction \((E|_P)_I\) of \((E_I)\) by

\[
(E|_P)_I := \begin{cases} E_I & \text{if } I \in P \\ \{0\} & \text{if } I \in \mathcal{P}_+^k \setminus P. \end{cases}
\]

It has the total space

\[
E|_P = \bigoplus_{J \in P} E_J \oplus \bigoplus_{I \in \mathcal{P}_+^k \setminus P} \{0\}.
\]

When convenient, we identify the restriction \( E|_P \) with the respective \( n \)-dimensional cube in the obvious way if \( P = \mathcal{P}^n_+ \subseteq \mathcal{P}_+^k \) for \( n \leq k \). If \( E|_{\mathcal{P}_0^k} = E \) holds, i.e., if \( E_I = \{0\} \) for \( I \in \mathcal{P}_+^k \), we call \( E \) purely even.

**Lemma A.5.** Let \( P \subseteq \mathcal{P}_+^k \) be a subset such that \( \sum_{I \in I} a_I : I \in P, a_I \in \mathbb{R} \) is a subalgebra of \( \mathbb{R}[\varepsilon_1, \ldots, \varepsilon_k] \). Then every morphism of \( k \)-multilinear spaces \( f : E \to E' \) can be restricted in a natural way to a morphism \( f|_P : E|_P \to E'|_P. \) This restriction defines a functor

\[
\text{MSpace}^{(k)} \to \text{MSpace}^{(k)}
\]

that respects products.

**Proof.** Let \( E, E' \) be \( k \)-dimensional cubes and \( f : E \to E' \) be a morphism given by

\[
f^\nu : E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \to E'_I,
\]

for \( \nu \in \mathcal{P}(I) \) and \( I \in \mathcal{P}_+^k \). If \( I \notin P \), then there exists \( 1 \leq i \leq \ell(\nu) \) such that \( \nu_i \notin P \), which implies \( f(E|_P) \subseteq E'|_P \). Therefore, we can define \( f|_P \) by setting \( (f|_P)^\nu := f^\nu \) if \( \nu_1, \ldots, \nu_{\ell(\nu)} \in P \) and \( (f|_P)^\nu := 0 \) else.

Let \( E'' \) be another \( k \)-dimensional cube and \( g : E' \to E'' \) a morphism. Since \( g(E'|_P) = g|_P(E'|_P) \) holds, functoriality follows. That the restriction respects products is obvious. \( \square \)

The purely even \( k \)-dimensional cubes clearly form a full subcategory of \( \text{MSpace}^{(k)} \) which we denote by \( \text{MSpace}^{(k)}_0 \). It follows from Lemma A.5 that we have an essentially surjective restriction functor

\[
\text{MSpace}^{(k)} \to \text{MSpace}^{(k)}_0, \quad E \mapsto E|_{\mathcal{P}_0^k} \quad \text{and} \quad f \mapsto f|_{\mathcal{P}_0^k}
\]

for \( E, E' \in \text{MSpace}^{(k)} \) and \( f : E \to E' \) a morphism.
Definition A.6. Let $I \in \mathcal{P}_k^+ \setminus \emptyset$ and \(\nu = \{\nu_1, \ldots, \nu_\ell\} \in \mathcal{P}(I)\). We define a tuple \((\nu_1 | \cdots | \nu_\ell)\) by concatenating the elements of \(\nu_1, \ldots, \nu_\ell\) in ascending order and define \(\text{sgn}(\nu)\), the sign of \(\nu\), as the sign of the permutation needed to bring this tuple into strictly ascending order.

This definition depends on the order one chooses on the partitions. However, the sign of a partition taken with regard to the lexicographic order is the same as when one takes it with regard to the graded lexicographic order, because changing the position of sets with even cardinality does not change the sign. We will only use these two orders in the following.

Example A.7. Let \(\nu = \{\{2\}, \{1, 3\}\}\). Then we get the tuple \((\nu_1 | \nu_2) = (2, 1, 3)\) and to permute this tuple to \((1, 2, 3)\), the permutation \(\sigma = (1, 2)\) is needed. Thus \(\text{sgn}(\nu) = -1\).

Lemma A.8. Let \(E, E' \in \mathbf{MSpace}_\mathcal{P}^{(k)}\) and \(f : E \to E'\) be a morphism defined by the family \((f^\nu)_{\nu \in \mathcal{P}(\{1, \ldots, k\})}\). Setting \(E^- := E\), we let the morphism \(f^- : E^- \to E'^-\) be given by \((\text{sgn}(\nu)f^{\nu})_{\nu \in \mathcal{P}(\{1, \ldots, k\})}\). This defines a functor

\[
- : \mathbf{MSpace}_\mathcal{P}^{(k)} \to \mathbf{MSpace}_\mathcal{P}^{(k)}.
\]

The functor is inverse to itself and respects products.

Proof. Let \(E, E', E'' \in \mathbf{MSpace}_\mathcal{P}^{(k)}\) and let \(f : E \to E'\), \(g : E' \to E''\) be morphisms. To check functoriality, it suffices to assume \(I = \{i_1, \ldots, i_s\} \in \mathcal{P}_0^+ \setminus \emptyset\) and \(\nu \in \mathcal{P}(I)\) because if any \(|\nu_j|\) is odd, then \(f^\nu = 0\) holds. Recall formula (5) from Remark A.3. On the one hand, we have

\[
((g \circ f)^-)^{\nu}(v_\nu) = \sum_{\omega \in \mathcal{P}(I), \omega \subseteq \nu} \text{sgn}(\nu) g^\omega \left( f^{\omega_1|\nu}(v_{\omega_1|\nu}), \ldots, f^{\omega_\ell(\omega)|\nu}(v_{\omega_\ell(\omega)|\nu}) \right).
\]

On the other hand, we calculate

\[
(g^- \circ f^-)^{\nu}(v_\nu) = \sum_{\omega \in \mathcal{P}(I), \omega \subseteq \nu} \text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_\ell(\omega)|\nu) g^\omega \left( f^{\omega_1|\nu}(v_{\omega_1|\nu}), \ldots, f^{\omega_\ell(\omega)|\nu}(v_{\omega_\ell(\omega)|\nu}) \right).
\]

The sign \(\text{sgn}(\nu)\) describes the reordering of the tuple \((\nu_1| \cdots | \nu_\ell(\nu))\) to \((i_1, \ldots, i_s)\).

For \(\omega \subseteq \nu\) let \(\omega_j|\nu = \{\nu_1, \ldots, \nu_\ell_j\} \in \mathcal{P}(\omega_j)\). Then

\[
\text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_\ell(\omega)|\nu)
\]

gives the sign of the reordering of the tuple \((\nu_1| \cdots | \nu_\ell_j| \cdots | \nu_1| \cdots | \nu_\ell(\nu))\) to \((i_1, \ldots, i_s)\). Since we only need to consider \(\nu_j\) with even cardinality, reordering \((\nu_1| \cdots | \nu_\ell_j| \cdots | \nu_1| \cdots | \nu_\ell(\nu))\) to \((\nu_1| \cdots | \nu_\ell(\nu))\) does not change the sign and it follows \(\text{sgn}(\nu) = \text{sgn}(\omega) \cdot \text{sgn}(\omega_1|\nu) \cdots \text{sgn}(\omega_\ell(\omega)|\nu)\). This implies \(g^- \circ f^- = (g \circ f)^-\). That the functor respects products is obvious. \(\square\)

The motivation for the lemma is essentially to substitute the infinitesimal generators \(\varepsilon_i\) with \(\lambda_i\). For more details see Remark A.18 below.
A.2. Multilinear Bundles.

**Definition A.9** (compare [15.4, p.81]).

(a) Let $E$ be a locally convex $k$-dimensional cube. A multilinear bundle (with base $M$, of degree $k$) is a smooth fiber bundle $F$ over a manifold $M$ with typical fiber $E$ together with an equivalence class of bundle atlases such that the change of charts leads to an isomorphism of locally convex $k$-dimensional cubes on the fibers.

(b) Let $F$ and $F'$ be multilinear bundles of degree $k$ with base $M$, resp. $M'$. A morphism of multilinear bundles is a smooth fiber bundle morphism $f : F \to F'$ that locally (i.e., in bundle charts) leads to a morphism of the respective $k$-dimensional cubes in each fiber.

We identify multilinear bundles of degree zero with their base manifold in the obvious way.

It follows from Theorem A.2 that the multilinear bundles form a category which we denote by $\text{MBun}$. Multilinear bundles of degree $k$ form a full subcategory denoted by $\text{MBun}(k)$.

**Remark A.10.** The above definition means that in bundle charts a morphism $f : F \to F'$ of multilinear bundles of degree $k$ with fiber $E$, resp. $E'$, has the form

$$U \times E \to U' \times E', \quad (x, (v_I)) \mapsto \left(\varphi(x), \sum_{I \in \mathcal{P}^k} \sum_{\nu \in \mathcal{P}(I)} f^I_{\nu}(v_{\nu})\right),$$

where $\varphi : U \to U'$ is the local representation of the morphism induced by $f$ on the base manifolds and

$$f^I_{\nu} : E_{\nu_1} \times \ldots \times E_{\nu_{\ell(I)}} \to E'_{I},$$

is a multilinear map for each $x \in U$, $I \in \mathcal{P}^k$ and $\nu \in \mathcal{P}(I)$. A function of this form is smooth if and only if $\varphi : U \to U'$ is smooth and the maps $(x, v_{\nu}) \mapsto f^I_{\nu}(v_{\nu})$ are all smooth. This can be easily checked by restricting to the closed subspaces of $E$ defined by a given partition.

**Definition A.11.** Let $F$ and $F'$ be multilinear bundles of degree $k$ over $M$ and $M'$, with typical fiber $E$ and $E'$. Further, let $\{\varphi_\alpha : V_\alpha \to U_\alpha \times E : \alpha \in A\}$ and $\{\psi_\beta : V'_\beta \to U'_\beta \times E' : \beta \in B\}$ be bundle atlases of $F$ and $F'$. We let the product bundle $F \times F'$ be the multilinear bundle of degree $k$ over $M \times M'$ with typical fiber $E \times E'$ given by the bundle atlas

$$\{\varphi_\alpha \times \psi_\beta : V_\alpha \times V'_\beta \to (U_\alpha \times U'_\beta) \times (E \times E') : \alpha \in A, \beta \in B\}.$$
the obvious way (compare to Definition A.4). Any morphism \( f : F \to F' \) of multilinear bundles of degree \( k \) restricts to a morphism \( f|_P : F|_P \to F'|_P \).

This restriction defines a functor

\[
\text{MBun}^{(k)} \to \text{MBun}^{(k)}
\]

that respects products.

**Proof.** Applying Lemma A.5 pointwise to transition maps and morphisms of multilinear bundles in their chart representation shows that \( F|_P \) and \( f|_P \) are well-defined and that the restriction is functorial. That the restriction respects products is obvious. \( \square \)

Note that in the situation of the definition, the identification of \( F|_P \) with a bundle of degree \( n \) is not a morphism of multilinear bundles but only a diffeomorphism of manifolds. There are cases where a subbundle is a multilinear bundle of lesser degree in a natural way that are not contained in the above definition. One important example is the following.

**Lemma/Definition A.13.** Let \( \pi : F \to M \) be a multilinear bundle of degree \( k \) with typical fiber \( E \). For each \( I \in P^k_+ \), we have a subbundle \( F|_{\{I\}} \) which has the structure of a vector bundle with fiber \( E_I \) in a natural way. The \( 2^k - 1 \) vector bundles obtained in this way are called the axes of \( F \).

**Proof.** For any change of bundle charts \( \varphi_{\alpha \beta} : U_{\alpha \beta} \times E \to U_{\beta \alpha} \times E \) of \( F \), we have that the corresponding change of bundle charts

\[
\varphi_{\alpha \beta}|_{\{I\}} : U_{\alpha \beta} \times E_I \to U_{\beta \alpha} \times E_I
\]

of \( F|_{\{I\}} \) is linear in the second component. Thus the restricted charts define a vector bundle with typical fiber \( E_I \). \( \square \)

Bertram uses the above fact in [5, 15.4, p.81] to define multilinear bundles by letting these axes take an analogous role to the axes in cubes. It is easy to see that both definitions are equivalent but our definition via bundle charts makes the relation of multilinear bundles to supermanifolds more direct.

**Definition A.14.** A purely even multilinear bundle is a multilinear bundle \( F \) of degree \( k \) such that \( F|_{\mathbb{P}^k_{0,+}} = F \). The purely even multilinear bundles form a full subcategory of \( \text{MBun} \) (resp. \( \text{MBun}_7 \)) and we denote by \( \text{MBun}^{(k)}_0 \) (resp. \( \text{MBun}_7^{(k)} \)) and we have the essentially surjective restriction functor

\[
\text{MBun} \to \text{MBun}^{(k)}_0, \quad F \mapsto F|_{\mathbb{P}^k_{0,+}} \quad \text{and} \quad f \mapsto f|_{\mathbb{P}^k_{0,+}}
\]

for \( F, F' \in \text{MBun}^{(k)} \) and \( f \in \text{Hom}_{\text{MBun}^{(k)}}(F, F') \) (resp. \( \text{MBun}^{(k)} \to \text{MBun}^{(k)}_0 \)).

**Example A.15.** Let \( k \in \mathbb{N} \).

(a) Let \( U \subseteq E \) be an open subset of a locally convex vector space \( E \). Define inductively \( TU := U \times \varepsilon_1 E, T^2U = T(U \times \varepsilon_1 E) = U \times \varepsilon_1 E \times \varepsilon_2 E \times \varepsilon_1 \varepsilon_2 E \) and so on. Then \( T^k U = U \times \bigoplus_{I \in P^k_+} \varepsilon_I E \) is a trivial multilinear bundle over \( U \) of degree \( k \). The axes are the trivial vector bundles \( U \times \varepsilon_I E \to U \).
(b) Let $M$ be a manifold with the atlas $\{ \varphi_\alpha : V_\alpha \to U_\alpha : \alpha \in A \}$. Then $T^kM$ is a multilinear bundle over $M$ of degree $k$ with the bundle atlas $\{ T^k\varphi_\alpha : T^kV_\alpha \to T^kU_\alpha : \alpha \in A \}$. Let $\varphi_{\alpha\beta}$ be a change of charts. Using (a), the corresponding change of bundle charts is given by

$$T^k\varphi_{\alpha\beta}(x, \sum_{I \in \mathcal{P}_k} \varepsilon_I v_I) = \left( \varphi_{\alpha\beta}(x), \sum_{m=1}^k \sum_{|I|=m} \varepsilon_I \sum_{\nu \in \mathcal{P}(I)} d^m \varphi_{\alpha\beta}(x)(v_\nu) \right)$$

(see [5, Theorem 7.5, p.47]). The axes of $T^kM$ are thus all isomorphic to $TM$ and we write $\varepsilon_I TM$ to differentiate between them. It also follows from [5, Theorem 7.5, p.47] that for each smooth map $f : M \to N$ between manifolds, $T^k f$ is a morphism of multilinear bundles and we get a functor $T^k : \text{Man} \to \text{MBun}^{(k)}$ in this way.

**Lemma A.16.** Let $k \in \mathbb{N}$ and let $f : M \to N$ be a smooth map between manifolds. For each $I \in \mathcal{P}_k$, we have

$$T^k f(\varepsilon_I TM) \subseteq \varepsilon_I TN$$

with the notation of Example A.15(b).

**Proof.** This is obvious because $T^k f$ is a morphism of multilinear bundles. □

**Lemma A.17.** Applying the functor from Lemma A.8 pointwise to transition maps and local chart representations of morphisms, we get a functor

$$- : \text{MBun}^{(k)}_\sigma \to \text{MBun}^{(k)}_\sigma, \quad F \mapsto F^- \quad \text{and} \quad h \mapsto h^-,$$

where $F, F' \in \text{MBun}^{(k)}_\sigma$ and $h \in \text{Hom}_{\text{MBun}^{(k)}_\sigma}(F, F')$. This functor is an equivalence of categories and so are the restrictions $\text{MBun}^{(k)}_\sigma \to \text{MBun}^{(k)}_\sigma$. All these functors respect products.

**Proof.** Locally this is obvious in view of Remark A.10 and Lemma A.8. By functoriality, applying this to all the change of charts of $F$ leads to new cocycles that define a bundle $F^-$. Likewise, applying it pointwise to the chart representation of a morphism $h : F \to F'$ of purely even multilinear bundles leads in a functorial way to a morphism $h^- : F^- \to F'^-$. Obviously $(F^-)^- \cong F$ and $(h^-)^- = h$ under this identification, which shows that the functor is an equivalence of categories. That these functors respect products also follows because it is true locally. □

**Remark A.18.** The intuition behind the above equivalence of categories is as follows. One can take the case of higher tangent bundles as exemplary and define $k$-dimensional cubes as families ($\varepsilon_I E_I$) of vector spaces. A morphism $f : E \to E'$ of $k$-multilinear spaces consists then as before of maps

$$f^\nu : E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \to E'_{I'}$$

for $I \in \mathcal{P}^k_+$, $\nu \in \mathcal{P}(I)$, where it is understood that

$$f^\nu(\varepsilon_{\nu_1} v_{\nu_1}, \ldots, \varepsilon_{\nu_{\ell(\nu)}} v_{\nu_{\ell(\nu)}}) = \varepsilon_{\nu_1} \cdots \varepsilon_{\nu_{\ell(\nu)}} f^\nu(v_{\nu_1}, \ldots, v_{\nu_{\ell(\nu)}}),$$

with the notations of Example A.15(b).
Because of the relations of the infinitesimal generators, this point of view also explains why one only considers partitions for the morphisms and why the order of the partitions can usually be disregarded.

We would like to substitute the generators \( \varepsilon_i \) with the odd generators \( \lambda_i \). One immediately sees that the order of the partition now plays a role, as a change of signs might occur. However, as we have shown in Lemma A.8 in the case of purely even multilinear bundles a consistent choice can be made such that this substitution leads to well-defined bundles and morphisms. In general this is not the case. With the notation of Lemma A.8 one could define a new composition law

\[
(g^{-1} \circ f^{-1})^\nu := \sum_{\omega \leq \nu} \text{sgn}(\sigma_{\omega|\nu}) \text{sgn}(\omega | \nu) \prod_{i=1}^{\nu} \text{sgn}(\omega_{\ell(\nu)} | \nu) g^\omega \left( f^{\omega_1 | \nu}, \ldots, f^{\omega_{\ell(\nu)} | \nu} \right),
\]

where \( \sigma_{\omega|\nu} \in S_I \) is the permutation that reorders the tuple \((\nu_1 | \nu)\) to \((\nu_1^* | \nu)^{\ell(\nu)}\). If all \( \nu_i \) have even cardinality then \( \text{sgn}(\sigma_{\omega|\nu}) = 1 \) and we get the same definition as above. In general the formula does not appear to lead to natural manifold structures though there is one interesting case where it does: If only those \( \nu^\nu \), where \( \nu \) contains at most one set of odd cardinality, are not zero, the same argument as before applies. This means that for a supermanifold \( M \) of Batchelor type, at least \( \mathcal{M}_\mathbb{A} \) would be well-defined. However, morphisms remain problematic.

A.2.1. The tangent bundle of a multilinear bundle. Let \( F \) be a multilinear bundle of degree \( k \) over \( M \) with typical fiber \( E \). Assume that \( M \) is modelled on \( E_0 \) and let \( \varphi : U \times E \to V \times E \) be a change of bundle charts. Then by definition

\[
\varphi(x,(v_I)_{I \in \mathcal{P}_+^k}) = \varphi_0(x) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} b^\nu(x,v_\nu),
\]

where \( \varphi_0 : U \to V \) is a diffeomorphism and \( b^\nu(x,\bullet) : E_{\nu_1} \times \cdots \times E_{\nu_{\ell(\nu)}} \to E_I \) are multilinear maps for \( x \in U, \nu \in \mathcal{P}(I) \). For \( y \in E_0 \) and \((w_I)_{I \in \mathcal{P}_+^k} \in E^k \), we calculate

\[
d\varphi(x,(v_I)_{I \in \mathcal{P}_+^k}),(y,(w_I)_{I \in \mathcal{P}_+^k}) =
\]

\[
d\varphi_0(x,y) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} d_1 b^\nu(x,y,v_\nu) + \sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} \sum_{i=1}^{\ell(\nu)} b^\nu(x,\widehat{v}^i_\nu),
\]

where \( \widehat{v}^i_\nu := (v_{\nu_1}, \ldots, v_{\nu_{i-1}}, w_{\nu_i}, v_{\nu_{i+1}}, \ldots, v_{\nu_{\ell(\nu)}}) \in E_{\nu_i} \). The corresponding change of charts for the tangent bundle \( TF \) is given by

\[
(\varphi,d\varphi) : (U \times E_0) \times E^2 \to (V \times E_0) \times E^2.
\]

For \( I \in \mathcal{P}_+^k \) let \( \text{pr}_I^1 : E_I \times E_I \to E_I \) be the projection to the first and \( \text{pr}_I^2 : E_I \times E_I \to E_I \) be the projection to the second component. Then

\[
(\varphi,d\varphi)((x,(v_I)_{I \in \mathcal{P}_+^k}),(y,(w_I)_{I \in \mathcal{P}_+^k})) = (\varphi_0(x),d\varphi_0(x,y)) +
\]

\[
\sum_{I \in \mathcal{P}_+^k} \sum_{\nu \in \mathcal{P}(I)} \left( \text{pr}_1^I(b^\nu(x,v_\nu)) + \text{pr}_2^I(d_1 b^\nu(x,y,v_\nu) + \sum_{i=1}^{\ell(\nu)} b^\nu(x,\widehat{v}^i_\nu)) \right)
\]
holds. Thus, $TF$ can be seen as a multilinear bundle of degree $k$ over $TM$ with typical fiber $E \times E$. The exact same calculation shows that for a morphism of multilinear bundles $f: F \to F'$, the tangent map $Tf: TF \to T F'$ is also a morphism of multilinear bundles. We have thus shown:

**Lemma A.19.** For each $k \in \mathbb{N}_0$, the tangent functor $T: \text{Man} \to \text{Man}$ restricts to a functor

$$T: \text{MBun}^{(k)} \to \text{MBun}^{(k)}.$$ 

The functor $T: \text{MBun}^{(k)} \to \text{MBun}^{(k)}$ commutes with restrictions of bundles:

**Lemma A.20.** Let $k \in \mathbb{N}_0$, $F \in \text{MBun}^{(k)}$ and $P \subseteq \mathbb{P}^k$ as in Lemma/Definition A.14. Then $(TF)|_P \cong T(F|_P)$ holds as multilinear bundles. If $f: F \to F'$ is a morphism of multilinear bundles, then

$$(T f)|_P = T(f|_P): T(F|_P) \to T(F|_P)$$

holds under the above identification.

**Proof.** Let $F$ have typical fiber $E$ and let the base $M$ of $F$ be modelled on $E_0$. Since each change of charts $\varphi_{\alpha\beta}: U_{\alpha\beta} \times E \to U_{\beta\alpha} \times E$ of $F$ restricts to a map

$$\varphi_{\alpha\beta}|_P: U_{\alpha\beta} \times E|_P \to U_{\beta\alpha} \times E|_P,$$

we have that $d\varphi_{\alpha\beta}$ restricts to

$$d\varphi_{\alpha\beta}|_P = d(\varphi_{\alpha\beta}|_P): (U_{\alpha\beta} \times E_0) \times (E|_P \times E|_P) \to U_{\beta\alpha} \times E|_P.$$

It follows that $(\varphi_{\alpha\beta}, d\varphi_{\alpha\beta})|_P = (\varphi_{\alpha\beta}|_P, d\varphi_{\alpha\beta}|_P)$ holds. We can repeat the same argument for morphisms. \hfill $\square$

By using this lemma, we shall simply write $TF|_P$, resp. $T f|_P$, for the respective restrictions in the sequel.

### A.3. Inverse Limits of Multilinear Bundles.

**Lemma A.21.** Let $k \in \mathbb{N}_0$ and $F$ be a multilinear bundle of degree $k$ with typical fiber $E$ and the bundle atlas $\{\varphi_\alpha: V_\alpha \to U_\alpha \times E: \alpha \in A\}$. For $n \leq k$, the projections

$$(q^k_n)_\alpha: U_\alpha \times E \to U_\alpha \times E|_{y^n_\alpha}, \quad (x, (v_I)_{I \in \mathbb{P}^k_+}) \mapsto (x, (v_I)_{I \in \mathbb{P}^k_+})$$

define a smooth surjective morphism $q^k_n: F \to F|_{y^n_+}$ with $\varphi_\alpha|_{y^n_+} \circ q^k_n \circ \varphi_\alpha^{-1} = (q^k_n)_\alpha$.

**Proof.** We only need to show that $q^k_n$ is well-defined, then smoothness and surjectivity follow immediately. Let $\alpha, \beta \in A$, $x \in U_{\alpha\beta}$ and $(v_I)_{I \in \mathbb{P}^k_+} \in E|_{y^n_+}$. Then $\varphi_{\alpha\beta}(x, (v_I)_I) = \varphi_{\alpha\beta}|_{y^n_+}(x, (v_I)_I)$ holds for the change of bundle charts $\varphi_{\alpha\beta}$. In particular, we have $\varphi_{\beta\alpha}|_{y^n_+} \circ \varphi_{\alpha\beta}(x, (v_I)_I) = (x, (v_I)_I)$. It follows

$$\varphi_{\beta\alpha}|_{y^n_+} \circ (q^k_n)_\beta \circ \varphi_{\alpha\beta} = (q^k_n)_\alpha$$

on $U_{\alpha\beta} \times E$. With this, the lemma follows from the local description of smooth maps between manifolds. \hfill $\square$
Definition A.22. Let $(F_k)_{k \in \mathbb{N}_0}$ be a family of multilinear bundles $F_k$ of degree $k$ with typical fiber $E^{(k)}$ and respective bundle atlas $\{\varphi^{(k)}_\alpha : V^{(k)}_\alpha \to U_\alpha \times E^{(k)} : \alpha \in A\}$ such that for all $n \leq k$, we have $E^{(k)}|_{\mathbb{P}^{n}_+} = E^{(n)}$ and $\varphi^{(k)}_\alpha|_{\mathbb{P}^{n}_+} = \varphi^{(n)}_\alpha$ with the identifications from Definition A.14 and Lemma/Definition A.12. In particular $F_k|_{\mathbb{P}^{n}_+} = F_n$ and all $F_k$ are bundles over $F_0$. Then the family

$$((F_k)_{k \in \mathbb{N}_0}, (q^n_k)_{n \leq k}),$$

where $q^n_k$ is defined as in Lemma A.21, is called an inverse system of multilinear bundles. We shall simply write $(F_k, q^n_k)$ in this situation. We call $\{\varphi^{(k)}_\alpha : k \in \mathbb{N}_0, \alpha \in A\}$ an adapted atlas of $(F_k, q^n_k)$. Two adapted atlases of $(F_k, q^n_k)$ are equivalent if they lead to equivalent atlases for each $F_k$.

Let $(F_k, q^n_k)$ and $(F'_k, q'^{n}_k)$ be inverse systems of multilinear bundles. A morphism of inverse systems of multilinear bundles is a family $(f_k)_{k \in \mathbb{N}_0}$ of morphisms $f_k : F_k \to F'_k$ of multilinear bundles such that $q'^n_k \circ f_k = f_n \circ q^n_k$ for all $n \leq k$. We write $(f_k)_{k \in \mathbb{N}_0} : (F_k, q^n_k) \to (F'_k, q'^{n}_k)$.

Proposition A.23. The inverse system of multilinear bundles with their morphisms is a subcategory of the category of inverse systems of topological spaces and their morphisms. Let $(F_k, q^n_k)$ be an inverse system of multilinear bundles and $\{\varphi^{(k)}_\alpha : k \in \mathbb{N}_0, \alpha \in A\}$ be an adapted atlas of $(F_k, q^n_k)$. Then $\{\lim_k \varphi^{(k)}_\alpha : \alpha \in A\}$ is an atlas of $\lim_k F_k$. Equivalent adapted atlases of $(F_k, q^n_k)$ lead to equivalent atlases of $\lim_k F_k$. With this manifold structure, $\lim_k f_k : \lim_k F_k \to \lim_k F'_k$ is smooth for morphisms $(f_k)_{k \in \mathbb{N}_0} : (F_k, q^n_k) \to (F'_k, q'^{n}_k)$ of inverse systems of multilinear bundles.

Proof. Let $F_k$ have the typical fiber $E^{(k)}$ and let $F_0$ be modelled on $E_0$. It is clear from the local definition in Lemma A.21 that $q^m_n \circ q^n_k = q^m_k$ for all $m \leq n \leq k$. It then follows from the definition that inverse systems of multilinear bundles, resp. morphisms thereof, are inverse systems, resp. morphisms thereof, in the usual sense. Clearly, the composition of two morphisms of inverse systems of multilinear bundles is again a morphism of this type. Let $\{\varphi^{(k)}_\alpha : V^{(k)}_\alpha \to U_\alpha \times E^{(k)} : k \in \mathbb{N}_0, \alpha \in A\}$ be an adapted atlas of $(F_k, q^n_k)$. By definition $E^{(k)}_I = E^{(n)}_I$ holds for all $n \leq k$ and $I \in \mathbb{P}^{n}_+$. Thus, for each $\alpha \in A$, the local projection

$$(q^n_k)_\alpha : U_\alpha \times \prod_{I \in \mathbb{P}^{n}_+} E^{(k)}_I \to U_\alpha \times \prod_{I \in \mathbb{P}^{n}_+} E^{(n)}_I$$

is just the usual projection and

$$\lim_k (U_\alpha \times E^{(k)}) = U_\alpha \times \prod_{I \in \mathbb{N}, 0 < |I| < \infty} E^{(\max(I))}_I,$$

which is an open subset of the locally convex space

$$E_0 \times \prod_{I \in \mathbb{N}, 0 < |I| < \infty} E^{(\max(I))}_I.$$

Also by definition, $q^n_k \circ \varphi^{(k)}_\alpha = \varphi^{(n)}_\alpha \circ (q^n_k)_\alpha$ holds for all $\alpha \in A, n \leq k$ and therefore $\lim_k \varphi^{(k)}_\alpha : \lim_k (U_\alpha \times E^{(k)}) \to \lim_k F_k$ is well-defined and a
homeomorphism because each \( \varphi^{(k)}_\alpha \) is so. We have already seen in Lemma \ref{lem:chart_changes} that the changes of charts \( \varphi^{(k)}_{\alpha\beta} : U_{\alpha\beta} \times E^{(k)} \to U_{\beta\alpha} \times E^{(k)} \) define a morphism of inverse systems of multilinear bundles and that we have

\[
\lim_{\leftarrow k} \varphi^{(k)}_\beta \circ \lim_{\leftarrow k} (\varphi^{(k)}_\alpha)^{-1}|_{U_{\alpha\beta} \times E^{(k)}} = \lim_{\leftarrow k} \varphi^{(k)}_{\alpha\beta}.
\]

Clearly, \( \lim_{\leftarrow k} (U_{\alpha\beta} \times E^{(k)}) \) is an open subset of \( \lim_{\leftarrow k} (U_\alpha \times E^{(k)}) \) and \( \lim_{\leftarrow k} \varphi^{(k)}_{\alpha\beta} \) because \( \varphi^{(k)}_{\alpha\beta} \) is smooth for each \( k \in \mathbb{N}_0 \), so is \( \lim_{\leftarrow k} \varphi^{(k)}_{\alpha\beta} \) by Lemma \ref{lem:smooth_limits}.

It remains to be seen that \( \lim_{\leftarrow k} F_k \) is covered by \( \{ \lim_{\leftarrow k} \varphi^{(k)}_\alpha : \alpha \in A \} \). Because the index set \( \mathbb{N}_0 \) is countable and the maps \( q^k_\alpha \) are all surjective, the projections \( q_\alpha : \lim_{\leftarrow k} F_k \to F_n \) are also surjective (see for example Exercise 7.6.10, p. 269)). For each \( n \in \mathbb{N}_0 \) and \( \alpha \in A \), we have \( (q^0_\alpha)^{-1}((\varphi_\alpha^{(0)})^{-1}(U_\alpha)) = (\varphi_\alpha^{(n)})^{-1}(U_\alpha \times E^{(n)}) \) which implies

\[
(q^0_\alpha)^{-1}((\varphi_\alpha^{(0)})^{-1}(U_\alpha)) = \lim_{\leftarrow n} (\varphi_\alpha^{(n)})^{-1}(\lim_{\leftarrow n} (U_\alpha \times E^{(n)})).
\]

Since the sets \( (\varphi_\alpha^{(0)})^{-1}(U_\alpha) \) cover \( F_0 \), the result follows. The change of charts with an adapted atlas leads to smooth maps in the same way. Because \( q_\alpha : \lim_{\leftarrow k} F_k \to F_0 \) is surjective and for each \( x \in F_0 \), we have that \( q^0_\alpha^{-1}\{x\} \) is homeomorphic to the Hausdorff space \( \lim_{\leftarrow k} E^{(k)} \), it follows that \( \lim_{\leftarrow k} F_k \) is Hausdorff.

Now, let \( (f_k)_{k \in \mathbb{N}_0} : (F_k, q^k_\alpha) \to (F'_k, q^{(k)}_\alpha) \) be a morphism of inverse systems of multilinear bundles and \( \{ \psi^{(k)}_\beta : V^{(k)}_\beta \to U^{(k)}_\beta \times E^{(k)} : k \in \mathbb{N}_0, \beta \in B \} \) be an adapted atlas of \( (F'_k, q^{(k)}_\alpha) \). We define

\[
f^{\alpha\beta}_k := \psi^{(k)}_\beta \circ (f_k \circ (\varphi^{(k)}_\alpha)^{-1}|_{U^{(k)}_\beta})^{-1}|_{V^{(k)}_\beta}
\]

for \( \beta \in B \) and \( \alpha \in A \). Because \( f_k \) is a morphism of multilinear bundles, we have \( \varphi^{(k)}_\alpha \circ f_k^{-1}(V^{(k)}_\beta) = (\varphi^{(0)}_\alpha \circ f_0^{-1}(V^{(0)}_\beta)) \times E^{(k)} \) for all \( k \in \mathbb{N}_0 \). By definition,

\[
(q^{(k)}_\alpha \beta \circ f_k )^{-1} = f^{\alpha\beta}_k \circ q^{(k)}_\alpha
\]

holds and thus

\[
\lim_{\leftarrow k} \psi^{(k)}_\beta \circ \lim_{\leftarrow k} f_k \circ (\lim_{\leftarrow k} \varphi^{(k)}_\alpha)^{-1} = \lim_{\leftarrow k} f^{\alpha\beta}_k
\]

holds on \( \lim_{\leftarrow k} ((\varphi^{(0)}_\alpha \circ f_0^{-1}(V^{(0)}_\beta)) \times E^{(k)}) \) for all \( \alpha \in A, \beta \in B \). These maps are smooth by the same argument as above.

We denote by \( \text{MBun}^{(\infty)} \) the category of all manifolds arising as such a limit (together with an equivalence class of atlases that come from limits of equivalent adapted atlases) and morphisms that come from a respective limit of morphisms. Taking the inverse limit gives us a functor from the category of inverse systems of topological spaces to the category of topological spaces that respects products. By the above, if we restrict this functor to the subcategory of inverse systems of multilinear bundles (and the respective morphisms), we get a functor into the category \( \text{MBun}^{(\infty)} \). We also get a functor to \( \text{Man} \) along the forgetful functor.
Example A.24. Let $\mathcal{M}$ be a manifold modelled on the locally convex space $E$ with the atlas $\{\varphi_\alpha : V_\alpha \to U_\alpha : \alpha \in A\}$. For $n \in \mathbb{N}_0$ we set $\pi_n^\alpha := \text{id}_{T^n M}$ and we have the natural projection $\pi_n^{n+1} : T^{n+1} M \to T^n M$. For $n < k$, we define inductively $\pi_n^k := \pi_{k-1} \circ \cdots \circ \pi_n^{n+1} : T^k M \to T^n M$. Continuing from Example A.15(b), one easily sees that $(T^k M, \pi_n^k)$ is an inductive system of multilinear bundles and $\pi \in \text{mor}(\text{ind sys of multilinear bundles})$ with the atlas $\mathcal{F}$ a morphism of inductive systems of multilinear bundles and $\pi$ is a manifold with the atlas $\{\lim_k T^k \varphi_\alpha : \alpha \in A, k \in \mathbb{N}_0\}$. It follows from Proposition A.23 that $T^\infty M := \lim_k T^k M$ is a manifold with the atlas $\{\lim_k \varphi_\alpha : \alpha \in A\}$. For any smooth map $f : M \to N$ between manifolds, one obviously has $\pi_k^g \circ T^k f = T^n f \circ \pi_n^g$ if $\pi_k^g : T^k N \to T^n N$ denotes the projection. Thus $(T^k f)_k \in \mathbb{N}_0$ is a morphism of inductive systems of multilinear bundles and $T^\infty f := \lim_k T^k f : T^\infty M \to T^\infty N$ is smooth. Moreover, for any Lie group $(G, \mu, i, e)$, we get a Lie group $(T^\infty G, T^\infty \mu, T^\infty i, e)$ because the inverse limit preserves products.

Lemma A.25. If $(F_k, q_n^k)$ is an inverse system of multilinear bundles, then so is $(T F_k, T q_n^k)$ and

$$T \lim_k ((F_k, q_n^k)) \cong \lim_k (T F_k, T q_n^k)$$

holds as manifolds. If $(f_k)_k \in \mathbb{N}_0 : (F_k, q_n^k) \to (F'_k, q'_n^k)$ is a morphism of inverse systems of multilinear bundles, then so is $(T f_k)_k$ and we have

$$\lim_k T f_k = T \lim_k f_k : T \lim_k ((F_k, q_n^k)) \to T \lim_k ((F'_k, q'_n^k))$$

under the above identification. Thus, we may consider $T \lim_k ((F_k, q_n^k))$ as an object in $\text{MBun}^{(\infty)}$ in a natural way.

Proof. One easily sees from the local description of $q_n^k$ in Lemma A.21 that $T q_n^k$ is the projection $T F \to T|_{\mathcal{F}^g}$. If $\{\varphi_{\alpha}^{(k)} : k \in \mathbb{N}_0, \alpha \in A\}$ is an adapted atlas of $F$, it follows by functoriality of the tangent functor that $\{T \varphi_{\alpha}^{(k)} : k \in \mathbb{N}_0, \alpha \in A\}$ is an adapted atlas of $(T F_k, T q_n^k)$. For the same reason $(T f_k)_k \in \mathbb{N}_0 : (T F_k, T q_n^k) \to (T F'_k, T q'_n^k)$ is again a morphism. By Lemma 2.7 $d \lim_k \varphi_{\alpha}^{(k)} = \lim_k d \varphi_{\alpha}^{(k)}$ holds for any change of charts $\varphi_{\alpha}^{(k)}$. Thus, the change of charts of $T \lim_k ((F_k, q_n^k))$ and $\lim_k (T F_k, T q_n^k)$ is the same. The same argument works for morphisms.

In other words, taking the inverse limit commutes with the tangent functor.

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