Quantum $M^2 \to 2\Lambda/3$ discontinuity for massive gravity with a $\Lambda$ term

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Abstract

In a previous paper we showed that the absence of the van Dam-Veltman-Zakharov discontinuity as $M^2 \to 0$ for massive spin-2 with a $\Lambda$ term is an artifact of the tree approximation, and that the discontinuity reappears at one loop, as a result of going from five degrees of freedom to two. In this paper we show that a similar classical continuity but quantum discontinuity arises in the “partially massless” limit $M^2 \to 2\Lambda/3$, as a result of going from five degrees of freedom to four.

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In a previous paper [1], we showed that the absence [2–4] of the van Dam-Veltman-Zakharov discontinuity [5,6] for massive spin-2 with a Λ term is an artifact of the tree approximation, and that the discontinuity reappears at one loop. This result may be understood as follows. While a generic massive graviton propagates five degrees of freedom, gauge invariance ensures only propagation of the familiar two degrees of freedom of a massless graviton. Although the introduction of Λ̸= 0 allows for a smooth classical $M^2 \to 0$ limit, the mismatch between two and five degrees of freedom cannot be eliminated altogether, and the discontinuity shows up at the quantum level.

Curiously, the presence of a cosmological constant allows for new gauge invariances of massive higher spin theories, yielding a rich structure of "partially massless" theories with reduced degrees of freedom [7]. In particular, for spin-2 a single gauge invariance shows up at the value $M^2 = 2\Lambda/3$, yielding a partially massless theory with four degrees of freedom [8–10]. In this paper we extend the result of [1] to the partially massless theory and show that a discontinuity first arises at the quantum level as $M^2 \to 2\Lambda/3$.

We work in four dimensions with Euclidean signature (+ + ++). As in Refs. [4,1], we take the massive spin-2 theory to be given by linearized gravity with the addition of a Pauli-Fierz mass term. Thus our starting point is the action

$$S[h_{\mu\nu}, T_{\mu\nu}] = S_L[h_{\mu\nu}] + S_M[h_{\mu\nu}] + S_T[h \cdot T],$$

(1)

where $S_L$ is the Einstein-Hilbert action with cosmological constant $S_E = -\frac{1}{16\pi G} \int d^4x \sqrt{\hat{g}}(\hat{R} - 2\Lambda)$, linearized about a background metric $g_{\mu\nu}$ satisfying the Einstein condition, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Taking $\hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}$ where $\kappa^2 = 32\pi G$, this linearized action for $h_{\mu\nu}$ is

$$S_L = \int d^4x \sqrt{g} \left[ \frac{1}{2} \tilde{h}^{\mu\nu} (-g_{\rho\sigma}g_{\nu\sigma} - 2R_{\mu\rho\sigma\nu}) h^{\rho\sigma} - \nabla^\rho \tilde{h}_{\rho\mu} \nabla^\sigma \tilde{h}_{\sigma\mu} \right],$$

(2)

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^{\sigma\sigma}$. All indices are raised and lowered with respect to the metric $g_{\mu\nu}$, and $\nabla_\mu$ is taken to be covariant with $\nabla_\mu g_{\lambda\sigma} = 0$. Furthermore, the source term is given by

$$S_T = \int d^4x \sqrt{\hat{g}} h_{\mu\nu} T^{\mu\nu}. $$

(3)

As in Ref. [1], we apply the simplifying assumption that $T_{\mu\nu}$ is conserved with respect to the background metric, $\nabla_\mu T^{\mu\nu} = 0$.

$S_L$ and $S_T$ together correspond to the linearized massless theory coupled to a conserved source. Each term independently has a gauge symmetry described by a vector $\xi_\mu(x)$:

$$h_{\mu\nu} \to h_{\mu\nu} + 2\nabla_{(\mu} \xi_{\nu)},$$

(4)

corresponding to diffeomorphism invariance of the Einstein theory. Introduction of the Pauli-Fierz spin-2 mass term,

$$S_M = \frac{M^2}{2} \int d^4x \sqrt{g} \left[ h^{\mu\nu} h_{\mu\nu} - (h^\mu_\mu)^2 \right],$$

(5)
breaks the symmetry \((4)\). However, at the critical value \(M^2 = \frac{2\Lambda}{3}\), there remains a residual symmetry

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\nabla(\mu \nabla(\nu) \alpha + \frac{2}{3}\Lambda g_{\mu\nu} \alpha) \tag{6}
\]

parameterized by \(\alpha(x)\). This gauge invariance was first noted in \([8]\), and results in a partially massless de Sitter theory with four degrees of freedom and propagation along the light cone. It also requires that the coupling to matter be via a traceless energy-momentum tensor.

We wish to consider the generating functional

\[
Z[g,T] = \int \mathcal{D}h e^{-\left(S_L[h] + S_M[h] + S_T[h \cdot T]\right)} . \tag{7}
\]

Since the generic theory with mass term has broken gauge invariance and a quadratic action, it may be quantized in a straightforward manner. On the other hand, for the case \(M^2 = 2\Lambda/3\), one would first gauge fix the symmetry \((6)\) before proceeding. However, to make contact with previous results for the pure massless case, we find it useful to reintroduce the gauge symmetry \((4)\) using a St"uckelberg \([11,2]\) formulation. This allows a uniform approach to quantization throughout the \((\Lambda, M^2)\) plane, and provides connection to the operators appearing in Ref. \([12]\) for the massless case, as well as the ones in Ref. \([1]\) for the massive case.

For any value of \(M^2 > 0\), we introduce an auxiliary vector field \(V_\mu\) to restore the gauge symmetry \((4)\). We first multiply \(Z[g,T]\) by an integration \(\int \mathcal{D}V\) over all configurations of this decoupled field, and then perform the shift \(h_{\mu\nu} \rightarrow h_{\mu\nu} - 2M^{-1}\nabla(\mu V(\nu))\). Since \(S_L\) and \(S_T\) are gauge invariant in themselves, the only effect of this shift is to make the replacement

\[
S_M[h_{\mu\nu}] \rightarrow S_M[h_{\mu\nu} - 2M^{-1}\nabla(\mu V(\nu))] \tag{8}
\]
in \((5)\). Thus \(S_M\) becomes a “St"uckelberg mass”, and gauge invariance is restored, yielding the simultaneous shift symmetry

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\nabla(\mu \xi(\nu)) , \quad V_\mu \rightarrow V_\mu + M\xi_\mu . \tag{9}
\]

For generic \(M^2\), this is the only symmetry of theory. However, for \(M^2 = 2\Lambda/3\), the additional symmetry \((6)\) remains even after the St"uckelberg shift. Note that this symmetry is a combination of a Weyl scaling and diffeomorphism [with parameter \(\xi_\mu(x) = \nabla_\mu \alpha(x)\)]. Since the latter has been restored by the addition of \(V_\mu\), we are now able to disentangle the two. The resulting gauge symmetry for the partially massless theory may be written as

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\nabla(\mu \xi(\nu)) + \frac{2}{3}\Lambda g_{\mu\nu} \alpha(x) , \quad V_\mu \rightarrow V_\mu + M[\xi_\mu(x) - \nabla_\mu \alpha(x)] \tag{10}
\]

with parameters \(\xi_\mu(x)\) for diffeomorphisms and \(\alpha(x)\) for Weyl rescalings.

For the partially massless theory, there are five degrees of freedom to gauge fix. As in \([1]\), we make use of diffeomorphisms to identify \(V\) with the longitudinal part of \(\hat{h}\), i.e. \(MV_\mu = \nabla^\rho \hat{h}_{\rho\mu}\). Additionally, the conformal rescaling may be used to make \(h_{\mu\nu}\) traceless.
This choice is made in order to simplify the relevant operators appearing in the action, and is accomplished by adding to the action the gauge-fixing terms

\[ S_{gf} = \int d^4x \sqrt{g} \left( \nabla^\rho \tilde{h}_{\rho\mu} - MV_\mu \right) \left( \nabla^\sigma \tilde{h}_\sigma^\mu - MV^\mu \right) + \frac{2}{3}\Lambda \int d^4x \sqrt{g} h^2. \]  
\[ (11) \]

In conjunction with this gauge fixing, it is necessary to include a Faddeev-Popov determinant connected with the variation of the gauge condition under \((10)\). It is straightforward to show that the appropriate determinant is

\[ \text{Det} \left( \begin{array}{cc} \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - 4\Lambda/3 & 0 \\ 2\nabla_\mu & 8\Lambda/3 \end{array} \right) \]  
\[ (12) \]

corresponding to the set of gauge parameters \((\xi_\mu, \alpha)\). So up to an overall (infinite) constant piece, \(\text{Det}[8\Lambda/3]\), the relevant Faddeev-Popov term is

\[ \text{Det}\left[ \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - 4\Lambda/3 \right] \]  

To highlight the tensor structure of the gauge-fixed action, we decompose the metric fluctuation \(h_{\mu\nu}\) into its traceless and scalar parts: \(\phi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}h^{\sigma\sigma}\), and \(\phi \equiv h^{\sigma\sigma}\). The source may similarly be split into its irreducible components \(j_{\mu\nu}\) and \(j\), so that \(T_{\mu\nu} = j_{\mu\nu} + \frac{1}{4}g_{\mu\nu}j\). The gauge-fixed partially massless action then becomes

\[ \tilde{S} = \int d^4x \sqrt{\tilde{g}} \left[ \frac{1}{2} \phi_{\mu\nu} \left( \Delta \left( 1, 1 \right) - 4\Lambda/3 \right) \phi_{\mu\nu} + V_\mu \left( \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - 4\Lambda/3 \right) V_\mu - (\nabla \cdot V)^2 + \phi_{\mu\nu} j^{\mu\nu} + \frac{1}{4} \phi j \right]. \]  
\[ (13) \]

The second-order spin operators are the scalar Laplacian \(\Delta(0,0) \equiv -\Box\) and the Lichnerowicz operator for symmetric rank-2 tensors \(\Delta(1,1)\phi_{\mu\nu} = -\Box \phi_{\mu\nu} + R_{\mu\nu\tau\rho} \phi^{\tau\rho} + R_{\mu\nu}\phi^{\tau\rho} - 2R_{\mu\nu\tau\rho} \phi^{\tau\rho} \) \[12\].

The St"uckelberg field, \(V_\mu\), in \((13)\) appears as a massive spin-1 field in the Einstein background with an effective mass \(m^2 = -4\Lambda/3\). We now restore vector gauge invariance by repeating the St"uckelberg formalism. Thus we introduce a scalar field \(\chi\) and make the change of variables \(V_\mu \rightarrow V_\mu - M^{-1} \nabla_\mu \chi\). By construction, the resulting action is now invariant under the gauge transformation

\[ V_\mu \rightarrow V_\mu + \nabla_\mu \zeta, \]  
\[ \chi \rightarrow \chi + M \zeta. \]  
\[ (14) \]

One can then choose a gauge-condition to simplify the shifted action. It is useful to associate the longitudinal component of \(V\) with \(\chi\) according to \(M \nabla \cdot V = (-2\Lambda + M^2)\chi\). This is done by adding a gauge-fixing term
\[ S'_{gf} = \int d^4x \sqrt{g} \left( \nabla \cdot V - \frac{-2\Lambda + M^2}{M} \chi \right)^2, \quad (15) \]

along with a corresponding scalar Faddeev-Popov determinant

\[ \text{Det} \left[ \Delta(0, 0) - 2\Lambda + M^2 \right]. \quad (16) \]

The final completely gauged-fixed action for the partially massless graviton now takes the form

\[
\tilde{S} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \phi^{\mu\nu} (\Delta(1, 1) - 2\Lambda + M^2) \phi_{\mu\nu} \\
+ V^\mu (\Delta(\frac{1}{2}, \frac{1}{2}) - 2\Lambda/3) V_\mu \\
- 2\chi (\Delta(0, 0) - 4\Lambda/3) \chi \\
+ \phi_{\mu\nu} j^{\mu\nu} + \frac{1}{4} \phi j \right]. \quad (17)
\]

Along with the addition to the two Faddeev-Popov determinants (12) and (16), this provides a complete description of \( Z \), including couplings to the background metric. This is to be compared with the generic massive case where the corresponding action is given by [1]

\[
\tilde{S} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \phi^{\mu\nu} (\Delta(1, 1) - 4\Lambda/3) \phi_{\mu\nu} \\
- \frac{1}{8} (\frac{-2\Lambda + 3M^2}{-2\Lambda + M^2}) \phi (\Delta(0, 0) - 2\Lambda + M^2) \phi \\
+ V^\mu (\Delta(\frac{1}{2}, \frac{1}{2}) - 2\Lambda + M^2) V_\mu \\
+ (\frac{-2\Lambda + M^2}{M^2}) \chi (\Delta(0, 0) - 2\Lambda + M^2) \chi \\
+ \phi_{\mu\nu} j^{\mu\nu} + \frac{1}{4} \phi j \right]. \quad (18)
\]

Note that in the partially massless case the trace mode \( \phi \) has disappeared except for its coupling to the trace of the energy-momentum tensor. With \( \phi \) now acting as a Lagrange multiplier, this indicates that the theory couples to conformal matter. To compare the massive and partially massless theories at the classical level, therefore, let us assume that the massive theory also couples to matter with \( T^\mu_\mu = 0 \) as well as \( \nabla^\mu T^\mu_\nu = 0 \). Then the tree-level amplitude for the current \( T^\mu_\nu \) can be read from the action (18) directly and is given by

\[ A[T] = \frac{1}{2} \left( \Delta(1, 1) - 2\Lambda + M^2 \right)^{-1} T^\mu_\nu, \]

since there are sources for neither \( V_\mu \) nor \( \chi \). Thus at tree level, there is no discontinuity in taking the \( M^2 \to 2\Lambda/3 \) limit. We note here that there would be sources for the St"uckelberg fields if one were to relax the assumption of a conserved stress tensor or a traceless stress tensor. In this case, one needs only to account for the shifts in \( h_{\mu\nu} \) and \( V_\mu \) to see how \( T^\mu_\nu \) contributes to sources for \( V_\mu \) and \( \chi \).

For the partially massless case, (17), we integrate over all species to find the first quantum correction
\[
Z[g, T] \propto e^{-A[T]} \text{Det} \left[ \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - \frac{4}{3} \Lambda \right] \text{Det} \left[ \Delta(0, 0) - \frac{4}{3} \Lambda \right] \\
\times \text{Det} \left[ \Delta(1, 1) - \frac{4}{3} \Lambda \right]^{-1/2} \text{Det} \left[ \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - \frac{4}{3} \Lambda \right]^{-1/2} \text{Det} \left[ \Delta(0, 0) - \frac{4}{3} \Lambda \right]^{-1/2} 
\]

(19)

where the operator \( \Delta(1, 1) - \frac{4}{3} \Lambda \) arises in the traceless \( \phi^{\mu\nu} \) sector so its determinant refers to traceless modes only. This allows us to compute the one-loop contribution

\[
\Gamma^{(1)}[g] = -\ln Z[g, 0] = -\frac{1}{2} \ln \text{Det} \left[ \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - \frac{4}{3} \Lambda \right] + \frac{1}{2} \ln \text{Det} \left[ \Delta(1, 1) - \frac{4}{3} \Lambda \right] - \frac{1}{2} \ln \text{Det} \left[ \Delta(0, 0) - \frac{4}{3} \Lambda \right] 
\]

(20)

to the effective action for the Einstein background \( g_{\mu\nu} \). This is now to be compared with the one loop contribution in the generic massive case [1].

\[
\Gamma^{(1)}[g] = -\ln Z[g, 0] = -\frac{1}{2} \ln \text{Det} \left[ \Delta \left( \frac{1}{2}, \frac{1}{2} \right) - 2\Lambda + M^2 \right] + \frac{1}{2} \ln \text{Det} \left[ \Delta(1, 1) - 2\Lambda + M^2 \right].
\]

(21)

The difference in these two expressions reflects the fact that 5 degrees of freedom are being propagated around the loop in the massive case and only 4 in the partially massless case. Denoting the dimension of the spin \((A, B)\) representation by \(D(A, B) = (2A+1)(2B+1)\), we count \(D(1, 1) - D(1/2, 1/2) = 5\) for the massive case, while \(D(1, 1) - D(1/2, 1/2) - D(0, 0) = 4\) for the partially massless one.

It remains to check that there is no conspiracy among the eigenvalues of these operators that would make these two expressions coincide. To show this, it suffices to calculate the coefficients in the heat-kernel expansion for the graviton propagator associated with \(S_L + S_M\), and compare it with the massive case given in Ref. [1]. The coefficient functions \(b_k^{(A)}\) in the expansion

\[
\text{Tr} \, e^{-\Delta^{(A)} t} = \sum_{k=0}^{\infty} t^{(k-4)/2} \int d^4x \sqrt{g} \, b_k^{(A)}
\]

(22)

were calculated in Ref. [13] for general “spin operators” \(\Delta^{(A)}(A, B) \equiv \Delta(A, B) - 2\Lambda\) in an Einstein background \(R_{\mu\nu} = \Lambda g_{\mu\nu}\). So, adapted for the generic operators \(\Delta(A, M) \equiv \Delta(A, B) - 2\Lambda + M^2\) appearing in Eq. (21), but still in the same Einstein background \(R_{\mu\nu} = \Lambda g_{\mu\nu}\), the results are:

\[
180(4\pi)^2 b_4^{(A, M)}(1, 1) = 189 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 756 \Lambda^2 + 810 M^4,
180(4\pi)^2 b_4^{(A, M)}(1, \frac{1}{2}, \frac{1}{2}) = -11 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 984 \Lambda^2 - 1200 \Lambda M^2 + 360 M^4,
180(4\pi)^2 b_4^{(A, M)}(0, 0) = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 636 \Lambda^2 - 480 \Lambda M^2 + 90 M^4.
\]

(23)

For the partially massless four degrees of freedom theory, \(M^2 = 2\Lambda/3\), we obtain

\[
180(4\pi)^2 b_4^{(A)}(\text{total}) = 180(4\pi)^2 \left[ b_4^{(A)}(1, 1) - b_4^{(A)}(1, \frac{1}{2}, \frac{1}{2}) - b_4^{(A)}(0, 0) \right] = 199 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 1096 \Lambda^2,
\]

(24)

which differs from the result for the \(M^2 \to 2\Lambda/3\) limit of the massive case,
\[ 180(4\pi)^2 b_4^{(\Lambda,M)} \text{(total)} = 180(4\pi)^2 \left[ b_4^{(\Lambda,M)}(1,1) - b_4^{(\Lambda,M)}(\frac{1}{2},\frac{1}{2}) \right] \]
\[ \rightarrow 200 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 740 \Lambda^2. \quad (25) \]

Even for a de Sitter background with constant curvature

\[ R_{\mu\nu\rho\sigma} = \frac{1}{3} \Lambda (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}), \]
\[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{8}{3} \Lambda^2, \quad (26) \]

there is no cancellation.

Thus we conclude that the absence of a discontinuity between the \( M^2 \rightarrow 2\Lambda/3 \) and \( M^2 = 2\Lambda/3 \) results for massive spin-2 is only a tree-level phenomenon, and that the discontinuity itself persists at one loop. That the full quantum theory is discontinuous is not surprising considering the different degrees of freedom for the two cases. Just as the \( M^2 \rightarrow 0 \) limit is discontinuous at the quantum level as a result of going from five degrees of freedom to two, so the \( M^2 \rightarrow 2\Lambda/3 \) limit is discontinuous as a result of going from five degrees of freedom to four.

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REFERENCES

[1] F. A. Dilkes, M. J. Duff, James T. Liu and H. Sati, Quantum $M \to 0$ discontinuity for massive gravity with a $\Lambda$ term, hep-th/0102093.

[2] A. Higuchi, Forbidden Mass Range For Spin-2 Field Theory In De Sitter Space-Time, Nucl. Phys. B282, 397 (1987).

[3] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, The $m \to 0$ limit for massive graviton in $dS_4$ and $AdS_4$: How to circumvent the van Dam-Veltman-Zakharov discontinuity, Phys. Lett. B 503, 173 (2001) [hep-th/0011138].

[4] M. Porrati, No van Dam-Veltman-Zakharov discontinuity in AdS space, Phys. Lett. B 498, 92 (2001) [hep-th/0011152].

[5] H. van Dam and M. Veltman, Massive and massless Yang-Mills and gravitational fields, Nucl. Phys. B22, 397 (1970).

[6] V.I. Zakharov, JETP Lett. 12, 312 (1970).

[7] S. Deser and A. Waldron, Gauge invariances and phases of massive higher spins in (A)dS, hep-th/0102164.

[8] S. Deser and R. I. Nepomechie, Gauge Invariance Versus Masslessness In De Sitter Space, Annals Phys. 154, 396 (1984).

[9] S. Deser and A. Waldron, Partial Masslessness of Higher Spins in (A)dS, hep-th/0103198.

[10] S. Deser and A. Waldron, Stability of massive cosmological gravitons, hep-th/0103253.

[11] E.C.G. St"uckelberg, Helv. Phys. Acta 30, 209 (1957).

[12] S. M. Christensen and M. J. Duff, Quantizing gravity with a cosmological constant, Nucl. Phys. B170, 480 (1980).