Lens Rigidity in Scattering by Unions of Strictly Convex Bodies in $\mathbb{R}^2$

Lyle Noakes\textsuperscript{1} and Luchezar Stoyanov\textsuperscript{2}

Abstract. It was proved in [NS1] that obstacles $K$ in $\mathbb{R}^d$ that are finite disjoint unions of strictly convex domains with $C^3$ boundaries are uniquely determined by the travelling times of billiard trajectories in their exteriors and also by their so called scattering length spectra. However the case $d = 2$ is not properly covered in [NS1]. In the present paper we give a separate different proof of the same result in the case $d = 2$.

Keywords Scattering by obstacles - Billiard flow - Scattering ray - Travelling times spectrum - Trapped trajectory

Mathematics Subject Classification 37D20 - 37D40 - 53D25 - 58J50

1 Introduction

In scattering by an obstacle in $\mathbb{R}^d$ ($d \geq 2$) the obstacle $K$ is a compact subset of $\mathbb{R}^d$ with a $C^3$ boundary $\partial K$ such that $\Omega_K = \mathbb{R}^d \setminus K$ is connected. A scattering ray in $\Omega_K$ is an unbounded in both directions generalized geodesic (in the sense of Melrose and Sjöstrand [MS1], [MS2]). Most of these scattering rays are billiard trajectories with finitely many reflection points at $\partial K$. In this paper we consider the case when $K$ has the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_k,$$

where $K_i$ are strictly convex disjoint domains in $\mathbb{R}^d$ with $C^3$ smooth boundaries $\partial K_i$. Then all scattering rays in $\Omega_K$ are billiard trajectories, and the so called generalized Hamiltonian (or bicharacteristic) flow $F_t^{(K)} : S^*(\Omega_K) \to S^*(\Omega_K)$ coincides with the billiard flow (see [CFS]).

Given an obstacle $K$ in $\mathbb{R}^d$, consider a large ball $M$ containing $K$ in its interior, and let $S_0 = \partial M$ be its boundary sphere. For any $q \in \partial K$ let $\nu_K(q)$ the outward unit normal to $\partial K$. For $q \in S_0$ we will denote simply by $\nu(q)$ the inward unit normal to $S_0$ at $q$. Set

$$S^*_+(S_0) = \{ x = (q,v) : q \in S_0 , \, v \in S^{d-1}, \, \langle v, \nu(q) \rangle \geq 0 \}.$$  

Given $x \in S^*_+(S_0)$, define the travelling time $t_K(x) \geq 0$ as the maximal number (or $\infty$) such that $\text{pr}_1(F_t^{(K)}(x))$ is in the interior of $\Omega_K \cap M$ for all $0 < t < t_K(x)$, where $\text{pr}_1(p,v) = p$ (see Figure 1). For $x = (q,v) \in S^*_+(S_0)$ with $\langle \nu(q), v \rangle = 0$ set $t(x) = 0$. The set $\{ t_K(x) : x \in S^*_+(S_0) \}$ will be called the travelling times spectrum of $K$.

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It is natural to ask what information about the obstacle $K$ can be derived from its travelling times spectrum. For example: what is the relationship between two obstacles $K$ and $L$ in $\mathbb{R}^d$ if they have (almost) the same travelling times spectra? We say that $K$ and $L$ have almost the same travelling times if there exists a subset $R$ of full Lebesgue measure in $S_+^*(S_0)$ such that $t_K(x) = t_L(x)$ for all $x \in R$. Similar questions can be asked about the so called scattering length spectrum of an obstacle. We refer the reader to [St2] for the definition of the latter and to [PS] (see e.g. Ch.11 there) for its relationship with the scattering kernel related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial \Omega_K$.

Figure 1

Similar inverse problems concerning metric rigidity have been studied for a very long time in Riemannian geometry – see [SU], [SUV] and the references there for more information. It appears that some of the methods used in this area, e.g. those in [Gu], [DGu], could be applied to obstacle scattering as well.

More recently various results have been established concerning inverse scattering by obstacles – see [St1], [St2], [NS1] - [NS3], [St3]. It turns out that some kind of obstacles are uniquely recoverable from their travelling times spectra. For example, if was shown in [NS1] that if $K$ and $L$ are finite disjoint unions of strictly convex bodies in $\mathbb{R}^d$ with $C^3$ boundaries and $K$ and $L$ have almost the same travelling times spectra, the $K = L$. However the argument in [NS1] does not work in the case $d = 2$. We are grateful to Antoine Gansemer who pointed this to us. As he showed in [Gan], when $d = 2$ and $k_0 > 1$
the set $S^*_{+}(S_0) \setminus \text{Trap}(\Omega_K)$ is disconnected, and then the argument in [NS1] does not work. Here $\text{Trap}(\Omega_K)$ is the set of all trapped points in $S^*(\Omega_K)$, i.e. points $x = (q, v) \in S^*(\Omega_K)$ such that either the forward billiard trajectory 
$$\gamma_K^+(x) = \{\text{pr}_1(F^{(K)}_t(x)) : t \geq 0\}$$
or the backward trajectory $\gamma_K^-(q, v) = \gamma_K^+(q, -v)$ is infinitely long.

Here we prove the following.

**Theorem 1** Let $K$ and $L$ be obstacles in $\mathbb{R}^2$ such that each of them is a finite disjoint union of strictly convex compact domains with $C^3$ boundaries. If $K$ and $L$ have almost the same travelling times, then $K = L$.

The argument we use is completely different from that in [NS1]. As in [NS1], our argument also proves a similar result in the case when $K$ and $L$ have almost the same scattering length spectrum. A result similar to that in [NS1] concerning non-trapping obstacles$^3$ satisfying certain non-degeneracy conditions was proved recently in [St3].

The set of trapped points plays a rather important role in various inverse problems in scattering by obstacles, and also in problems on metric rigidity in Riemannian geometry. It is known that $\text{Trap}(\Omega_K) \cap S^*_+(S_0)$ has Lebesgue measure zero in $S^*_+(S_0)$. However, as an example of M. Livshits shows (see Ch. 5 in [M] or Figure 1 in [NS1]), in general the set of trapped points $x \in S^*(\Omega_K)$ may contain a non-trivial open set. In the latter case the obstacle cannot be recovered from travelling times. Similar examples in higher dimensions are given in [NS3].

**Definition 1** Let $K, L$ be two obstacles in $\mathbb{R}^d$. We will say that $\Omega_K$ and $\Omega_L$ have conjugate flows if there exists a homeomorphism 
$$\Phi : S^*(\Omega_K) \setminus \text{Trap}(\Omega_K) \longrightarrow S^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$$
which is $C^1$ on an open dense subset of $S^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$, satisfies 
$$F^{(L)}_t \circ \Phi = \Phi \circ F^{(K)}_t, \quad t \in \mathbb{R},$$
and $\Phi = \text{id}$ on $S^*(\mathbb{R}^d \setminus M) \setminus \text{Trap}(\Omega_K) = S^*(\mathbb{R}^d \setminus M) \setminus \text{Trap}(\Omega_L)$.

For $K, L$ in a generic class of obstacles in $\mathbb{R}^d$ ($d \geq 2$), which includes the type of obstacles considered here, it is known that if $K$ and $L$ have almost the same SLS or almost the same travelling times, then $\Omega_K$ and $\Omega_L$ have conjugate flows ([St2] and [NS2]). Thus Theorem 1 is an immediate consequence of the following.

**Theorem 2** Let each of the obstacles $K$ and $L$ be a finite disjoint union of strictly convex compact domains in $\mathbb{R}^2$ with $C^3$ boundaries. If $\Omega_K$ and $\Omega_L$ have conjugate flows, then $K = L$.

We prove Theorem 2 in Sect. 4 below. In Sect. 2 we state some useful results from [St1] and [St2]. It turns out that billiard trajectories with tangent points to the boundary of the obstacle play an important role in the two-dimensional case considered here. We discuss some particular aspects of these in Sect. 3, which are significantly used in Sect. 4. One of the lemmas stated in Sect. 3 is proved in the Appendix.

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$^3$That is, obstacles $K$ with $\text{Trap}(\Omega_K) = \emptyset$. 
2 Preliminaries

Next, we describe some propositions from [St1] and [St2] that are needed in the proof of Theorem 2. We state them in the general case $d \geq 2$, although later on we will use them in the special case $d = 2$.

**Standing Assumption.** $K$ and $L$ are finite disjoint unions of strictly convex domains in $\mathbb{R}^d$ ($d \geq 2$) with $C^3$ boundaries, and with conjugate flows $F^{(K)}_t$ and $F^{(L)}_t$.

**Proposition 1** ([St1]) (a) There exists a countable family $\{M_i\} = \{M_i^K\}$ of codimension 1 submanifolds of $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that every $\sigma \in S_+^*(S_0) \setminus (\cup_i M_i)$ generates a simply reflecting ray in $\Omega_K$. Moreover the family $\{M_i\}$ is locally finite, that is any compact subset of $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ has common points with only finitely many of the submanifolds $M_i$.

(b) There exists a countable family $\{R_i\}$ of codimension 2 smooth submanifolds of $S_+^*(S_0)$ such that for any $\sigma \in S_+^*(S_0) \setminus (\cup_i R_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to $\partial K$.

(c) There exists a countable family $\{Q_i\}$ of codimension 2 smooth submanifolds of $S_0^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$ such that for any $\sigma \in S_0^*(\partial K) \setminus (\cup_i Q_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to $\partial K$.

It follows from the conjugacy of flows and Proposition 4.3 in [St2] that the submanifolds $M_i$ are the same for $K$ and $L$, i.e. $M_i^K = M_i^L$ for all $i$.

The following is Lemma 5.2 in [St1]. In fact the lemma in [St1] assumes $C^\infty$ smoothness for the submanifold $X$, however its proof only requires $C^3$ smoothness.

**Proposition 2** Let $X$ be a $C^3$ smooth submanifold of codimension 1 in $\mathbb{R}^d$, and let $x_0 \in X$ and $\xi_0 \in T_{x_0}X$, $\|\xi_0\| = 1$, be such that the normal curvature of $X$ at $x_0$ in the direction $\xi_0$ is non-zero. Then for every $\epsilon > 0$ there exist an open neighbourhood $V$ of $x_0$ in $X$, a smooth map $V \ni x \mapsto \xi(x) \in T_xX$ and a smooth positive function $t(x) \in [\delta, \epsilon]$ on $V$ for some $\delta > 0$ such that $Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$ is a smooth strictly convex surface with an unit normal field $\mu(y(x)) = \xi(x)$, $x \in V$. That is, the normal field of $Y$ consists of vectors tangent to $X$ at the corresponding points of $V$. (See Figure 2.)

![Figure 2](image-url)

Finally we state one consequence of Corollary 1.2 in [St2] and the fact that for obstacles $K$ considered in this paper the set Trap$(\Omega_K)$ of trapped points has Lebesgue measure zero in $S^*(\Omega_K)$. The latter follows easily e.g. using a small part of the argument of the proof of Proposition 5.1 in [St1] (see the last four paragraphs of Sect. 5 in [St1]).
Proposition 3 Let each of the obstacles $K$ and $L$ be a finite disjoint union of strictly convex compact domains in $\mathbb{R}^2$ with $C^3$ boundaries. If $\Omega_K$ and $\Omega_L$ have conjugate flows, then $\text{Vol}(K) = \text{Vol}(L)$.

3 Tangential singularities

Assume that the obstacles $K$ and $L$ in $\mathbb{R}^2$ satisfy the assumptions of Theorem 2.

As in [NS1], a point $y \in \partial K$ (or $y \in \partial L$) will be called regular if $\partial K = \partial L$ in an open neighbourhood of $y$ in $\partial K$ (resp. $\partial L$). Otherwise $y$ will be called irregular.

Lemma 1 Assume that a connected component $\partial K_i$ of $\partial K$ contains a regular point. Then $\partial K_i$ coincides with some connected component of $\partial L$.

In the proof of the above we will use the following technical lemma.

Lemma 2 Let $f, g \in C^2[-a, a]$ for some $a > 0$ be such that $f(0) = g(0) = 0$, $f'(0) = g'(0) = 0$, $f''(0) = g''(0)$, $f(x) > 0$ and $g(x) > 0$ for $x \neq 0$, and there exists a constant $c > 0$ such that $f''(x) \geq c$ and $g''(x) \geq c$ for all $x \in [-a, a]$. Let $\Gamma$ be the set of those $x \in (0, a)$ for which there exists $y(x) \in (0, a)$ so that the tangent line to

$$\text{graph}(f) = \{(t, f(t)) : t \in [-a, a]\}$$

at $(x, f(x))$ is also tangent to $\text{graph}(g)$ at $(y(x), g(y(x)))$. Assume that for some $\delta \in (0, a]$ the set $\Gamma \cap [0, \delta]$ is everywhere dense in $[0, \delta]$. Then there exists $\epsilon \in (0, \delta]$ such that $f(x) = g(x)$ on $[0, \epsilon]$.

We prove Lemma 2 in the Appendix. We can now prove Lemma 1.

Proof of Lemma 1. Assume that the connected component $\partial K_i$ of $\partial K$ contains a regular point, however $\partial K_i$ is not entirely contained in $\partial L$. Let $V$ be the set of regular points on $\partial K_i$. Then $V \subset \partial K_i \cap \partial L_j$ for some $j$, and clearly $V$ is a proper closed subset of $\partial K_i$ which is a union of non-trivial closed arcs. Let $x_0 \in V$ be a boundary point of $V$ in $\partial K_i$; then there exist points $y \in \partial K_i \setminus \partial L_j$ arbitrarily close to $x_0$. It now follows from Lemma 1 that for every sufficiently small open arc $W$ on $\partial K_i$ with an end point $x_0$ and such that $W$ is not entirely in $\partial L_j$, $W$ contains non-trivial open sub-arcs $W'$ arbitrarily close to $x_0$ such that for all $x \in W'$ the tangent line to $\partial K_i$ at $x$ is not tangent to $\partial L_j$.

Let $\xi_0 \in S^1$ be one of the unit vectors tangent to $\partial K$ at $x_0$. (Later we may have to replace $\xi_0$ by $-\xi_0$.) It follows from Proposition 2 above that there exists a small $\epsilon_0 > 0$, an open neighbourhood $V_0$ of $x_0$ in $\partial K_i$, a $C^2$ map $V_0 \ni x \mapsto \xi(x) \in S^*_x(\partial K)$ and a $C^2$ positive function $t(x) \in [\delta, \epsilon_0]$ on $V_0$ for some $\delta \in (0, \epsilon_0)$ such that

$$\Sigma = \{y(x) = x + t(x)\xi(x) : x \in V_0\}$$

is a $C^2$ strictly convex curve with unit normal field $\nu_{\Sigma}(y(x)) = \xi(x)$, $x \in V_0$. So, for any $x \in V_0$ the straight line through $y(x)$ with direction $\xi(x)$ is tangent to $\partial K$ at $x$. Set $y_0 = x_0 + \epsilon_0\xi_0 \in \Sigma$. 

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It follows from Proposition 1 above that for all but countably many \( y \in \Sigma \) the trajectories \( \gamma_K(y, \nu_\Sigma(y)) \) and \( \gamma_L(y, \nu_\Sigma(y)) \) have at most one tangency to \( \partial K \) and \( \partial L \), respectively. For such \( y \), since \( \gamma_K(y, \nu_\Sigma(y)) \) has a tangent point to \( \partial K \), it must have exactly one tangent point to \( \partial K \). Since the flows \( F_t^{(K)} \) and \( F_t^{(L)} \) are conjugate by assumption, \( \gamma_L(y, \nu_\Sigma(y)) \) also must have exactly one tangent point \( z(y) \) to \( \partial L \). As we noted earlier as a consequence of Lemma 1, taking the curve \( \Sigma \) sufficiently small about \( y_0 \), there exists an open arc \( \Sigma_+ \) on \( \Sigma \), with an end point \( y_0 \), and a subset \( \Sigma'_+ \) of \( \Sigma \) containing non-trivial open sub-arcs arbitrarily close to \( y_0 \) such that for all \( y \in \Sigma'_+ \) the straight line through \( y \) with direction \( \nu_\Sigma(y) \) is not tangent to \( \partial L_j \). This implies that \( \gamma_L(y, \nu_\Sigma(y)) \) is tangent to a connected component of \( \partial L \) different from \( \partial L_j \). That is, \( z(y) \notin \partial L_j \) for all \( y \in \Sigma'_+ \).

Assume for a moment that for every \( z \in \partial L \setminus \partial L_j \) there exists an open neighbourhood \( W_z \) of \( z \) in \( \partial L \) such that \( W_z \cap \{ z(y) : y \in \Sigma'_+ \} \) has Lebesgue measure zero in \( \partial L \). Covering \( \partial L \setminus \partial L_j \) with a finite number of neighbourhoods \( W_z \), it follows that \( \Sigma'_+ \) has Lebesgue measure zero - a contradiction. Thus, there exists \( z_0 \in \partial L \setminus \partial L_j \) and an open neighbourhood \( W_0 \) of \( z_0 \) in \( \partial L \) such that \( W_0 \cap \{ z(y) : y \in \Sigma'_+ \} \) has positive Lebesgue measure in \( \partial L \) and \( z_0 \) is a Lebesgue density point for this set. We have \( z_0 = y'_0 + t_0 \nu_\Sigma(y'_0) \) for some \( y'_0 \in \Sigma'_0 \) and \( t_0 \in \mathbb{R} \) (with \( |t_0| \) larger than the minimal distance between distinct connected components of \( \partial L \), assuming that \( \epsilon_0 > 0 \) is chosen sufficiently small).

We will assume that \( t_0 > 0 \); otherwise we just have to replace \( \xi_0 \) by \( -\xi_0 \) and the surface \( \Sigma \) by \( \{ x - t(x)\xi(x) : x \in V_0 \} \). Let \( F_{t_0}^{(L)}(y_0, \nu_\Sigma(y_0)) = (z_0, -\zeta_0) \). Using again Proposition 2 above, assuming \( \epsilon_0 > 0 \) is sufficiently small and shrinking the open neighbourhood \( W_0 \) of \( z_0 \) in \( \partial L \) if necessary, there exist a \( C^2 \) map \( W_0 \ni z \mapsto \zeta(z) \in \mathbb{S}^+(\partial L) \) and a \( C^2 \) positive function \( s(x) \in [\delta, \epsilon_0] \) on \( W_0 \) for some \( \delta \in (0, \epsilon_0) \) such that \( \zeta(z_0) = \zeta_0 \) and

\[
X = \{ p(z) = z + s(x)\zeta(z) : z \in W_0 \}
\]

is a \( C^2 \) strictly convex curve with unit normal field \( \nu_X(p(z)) = \zeta(z), z \in W_0 \). So, for any \( z \in W_0 \) the straight line through \( p(z) \) with direction \( \zeta(z) \) is tangent to \( \partial L \) at \( z \). Set \( p_0 = z_0 + \epsilon_0 \zeta_0 \in X \). Now as in the proof of Lemma 5.4 in [St1] we derive that the set of those \( p \in X \) for which the billiard trajectory \( \gamma_L^+(p, \nu_X(p)) \) intersects \( \Sigma \) perpendicularly near \( y'_0 \) after time close to \( t_0 \) is countable (see Figure 3). This means that in a small neighbourhood of \( y'_0 \) there is only a countable set of points \( y \in \Sigma'_0 \) such that the billiard trajectory in \( \Omega_L \) issued from \( y \) in direction \( \nu_\Sigma(y) \) is tangent to \( \partial L \) near \( z_0 \). This is a contradiction with the choice of \( z_0 \).

![Figure 3](image-url)

This proves that the connected component \( \partial K_i \) of \( \partial K \) must be entirely contained in \( \partial L \), i.e. it coincides with one of the connected components of \( \partial L \). \( \blacksquare \)
4 Proof of Theorem 2

Assume again that the obstacles $K$ and $L$ in $\mathbb{R}^2$ satisfy the assumptions of Theorem 2. For any $x \in \partial K$ set

$$d_K(x) = \inf\{t > 0 : \exists \sigma \in \mathbb{S}^*(S_0) \text{ such that } \text{pr}_1(\mathcal{F}^{(K)}_t(\sigma)) = x\},$$

which is well-defined as we prove below. For $y \in \partial L$ define $d_L(y)$ similarly. Clearly,

$$d_K(x) = \inf\{t > 0 : \exists \xi \in \mathbb{S}^1 \text{ such that } \langle \xi, \nu_K(x) \rangle \geq 0 \text{ and } \text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi)) \in S_0\}.$$

1) First, notice that $d_K(x)$ is well-defined for all $x \in \partial K$. Indeed, given $x \in \partial K$ consider the 1-dimensional manifold $\tilde{X} = \{\xi \in \mathbb{S}^1 : \langle \xi, \nu_K(x) \rangle > 0\}$. As in the proof of Proposition 5.5 in [St1] (as a consequence of Lemmas 5.4 and 5.2 in [St1]) we derive that the set of those $\xi \in \tilde{X}$ so that $\gamma^+_K(x, \xi)$ has a tangent point to $\partial K$ is finite or countable. Then as in the proof of Proposition 5.1 in [St1] (see the last four paragraphs of Sect. 5 in [St1]), a simple topological argument implies that the set $\text{Trap}(\Omega_K) \cap \tilde{X}$ has topological dimension zero. Thus, there exist (many) directions $\xi \in \tilde{X}$ so that $(x, \xi) \notin \text{Trap}(\Omega_K)$, so $\text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi)) \in S_0$ for some $t > 0$.

2) We have

$$d_K(x) = \min\{t > 0 : \exists \xi \in \mathbb{S}^1 \text{ such that } \langle \xi, \nu_K(x) \rangle \geq 0 \text{ and } \text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi)) \in S_0\}.$$

Indeed, let $x \in \partial K$, and let $\{t_n \searrow t = d_K(x)\}$ as $n \to \infty$, where $\text{pr}_1(\mathcal{F}^{(K)}_{t_n}(x, \xi_n)) \in S_0$ for some sequence $\{\xi_n\} \subset \mathbb{S}^1$ with $\langle \xi_n, \nu_K(x) \rangle \geq 0$ for all $n \geq 1$. Choosing an appropriate subsequence, we may assume $\xi_n \to \xi \in \mathbb{S}^1$ as $n \to \infty$. Then the continuity of the flow $\mathcal{F}^{(K)}_t$ (see e.g. [MS1], or [CFS] for the particular situation considered in this paper) implies that $\text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi)) \in S_0$.

3) The function $d_K : \partial K \to \mathbb{R}$ is continuous, and therefore bounded. Indeed, let $x \in \partial K$ and let $\{x_n\} \subset \partial K$ be a sequence with $x_n \to x$ as $n \to \infty$. It follows from 2) above that for $t = d_K(x)$ we have $\text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi)) \in S_0$ for some $\xi \in \mathbb{S}^1$ with $\langle \xi, \nu_K(x) \rangle \geq 0$. For each $n$, take $\xi_n \in \mathbb{S}^1$ so that $\langle \xi_n, \nu_K(x_n) \rangle \geq 0$ and $\xi_n \to \xi$ as $n \to \infty$. Then the continuity of the flow implies $\text{pr}_1(\mathcal{F}^{(K)}_t(x_n, \xi_n)) \to \text{pr}_1(\mathcal{F}^{(K)}_t(x, \xi))$ in $S_0$ as $n \to \infty$, so the times $t_n$ with $\text{pr}_1(\mathcal{F}^{(K)}_{t_n}(x_n, \xi_n)) \in S_0$ satisfy $t_n \to t$ as $n \to \infty$. Thus, for the sequence $s_n = d_K(x_n)$ we have $s_n \leq t_n$ for all $n$, so $\limsup n \to \infty s_n = s < t$. By 2) above, for each $n$ there exists $\eta_n \in \mathbb{S}^1$ with $\langle \eta_n, \nu_K(x_n) \rangle \geq 0$ such that $\text{pr}_1(\mathcal{F}^{(K)}_{s_n}(x_n, \eta_n)) \in S_0$. Choosing an appropriate subsequence, we may assume that $\eta_n \to \eta \in \mathbb{S}^1$ and $s_n \to s$ as $n \to \infty$. Then $\langle \eta, \nu_K(x) \rangle \geq 0$ and the continuity of the flow yields $\text{pr}_1(\mathcal{F}^{(K)}_{s}(x, \eta)) \in S_0$, a contradiction with $s < t = d_K(x)$ and 2). Hence we must have $s = t$ which means that there exists $\lim_{n \to \infty} d_K(x_n) = d_K(x)$. This proves the continuity of the function $d_K$.

Next, set

$$d_0^{(K)} = \sup\{d > 0 : \text{every point } x \in \partial K \text{ with } d_K(x) < d \text{ is regular }\}. \quad (4.1)$$
In a similar way define $d_0^{(L)}$. We will also need the numbers

$$
\tilde{d}_K = \max_{x \in \partial K} d_K(x), \quad \tilde{d}_L = \max_{y \in \partial L} d_L(y),
$$

which exist by the continuity of the functions $d_K$ and $d_L$. Clearly, $d_0^{(K)} \leq \tilde{d}_K$ and $d_0^{(L)} \leq \tilde{d}_L$.

In what follows without loss of generality we will assume that

$$
d_0^{(K)} \leq d_0^{(L)}. \tag{4.2}
$$

4) There exists $x \in \partial K$ with $d_K(x) = d_0^{(K)}$. To prove this, first notice that if for some $\epsilon > 0$ the interval $(d_0^{(K)}, d_0^{(K)} + \epsilon)$ does not contain any $d_K(y)$ with an irregular point $y$, then the right-hand-side of (4.1) would be $\geq d_0^{(K)} + \epsilon$, a contradiction with the definition of $d_0^{(K)}$. This implies that there exists a sequence $\{x_n\} \subset \partial K$ of irregular points with $d_K(x_n) \searrow d_0^{(K)}$ as $n \to \infty$. Taking a subsequence, we may assume $x_n \to x \in \partial K$ as $n \to \infty$, and then the continuity of $d_K$ implies $d_K(x) = d_0^{(K)}$.

5) Let $x \in \partial K$ be such that $d_K(x) = d_0^{(K)}$. We claim that $x$ is a regular point. There are two cases to consider here.

Case 1. There exists an open neighbourhood $V$ of $x$ in $\partial K$ so that $d_K(y) \geq d_k(x) = d_0^{(K)}$ for all $y \in V$. Setting $t = d_K(x)$, it follows from 2) that there exists $\xi \in S^1$ with $\langle \xi, \nu_K(x) \rangle \geq 0$ and $pr_1(\mathcal{F}_s^{(K)}(x, \xi)) \in S_0$. Since the function $d_K$ has a minimum at $x$ on $V$, the continuity of the flow now implies that $\xi = \nu_K(x)$, i.e. $\gamma_K(x, \xi)$ is a back-scatter trajectory perpendicular to $\partial K$ at $x$. Let $x_1, \ldots, x_k$ be the reflection points of the billiard trajectory $\gamma = \{pr_1(\mathcal{F}_s^{(K)}(x, \xi)) : 0 < s < t\}$, if any. Then $d_K(x_i) < d_K(x) = d_0^{(K)}$, so each $x_i$ is a regular point, i.e. $\partial K = \partial L$ on a neighbourhood of $x_i$. Apart from that, (4.2) implies that $\gamma$ cannot have any common point with $\partial L$ different from $x_1, \ldots, x_k$. Thus, we must have $\gamma = \{pr_1(\mathcal{F}_s^{(L)}(x, \xi)) : 0 < s < t\}$ as well, which show that $x \in \partial L$ and $\nu_K(x) = \nu_L(x)$. Similarly, if $x' \in \partial K$ is sufficiently close to $x$, setting $\xi' = \nu_k(x')$, there exists $t'$ close to $t$ such that $pr_1(\mathcal{F}_s^{(K)}(x', \xi')) \in S_0$. Moreover, the billiard trajectory $\gamma' = \{pr_1(\mathcal{F}_s^{(K)}(x', \xi')) : 0 < s < t'\}$ can only have common points with $\partial K$ near the points $x_i$, and so $\gamma' = \{pr_1(\mathcal{F}_s^{(L)}(x', \xi')) : 0 < s < t'\}$ as well. This implies $x' \in \partial L$.

Thus, $\partial K = \partial L$ on an open neighbourhood of $x$ in $\partial K$, i.e. $x$ is a regular point.

Case 2. There exists a sequence $\{x_n\} \subset \partial K$ with $x_n \to x$ and $d_K(x_n) \nearrow d_0^{(K)} = d_K(x)$ as $n \to \infty$. It follows from the definition of $d_0^{(K)}$ that $x_n$ is a regular point, i.e. $\partial K = \partial L$ on an open neighbourhood of $x_n$ in $\partial K$. If $x \in \partial K_i$, then $x_n \in \partial K_i$ for sufficiently large $n$, so $\partial K_i$ contains regular points. Now Lemma 1 implies that $\partial K_i$ coincides with a connected component of $\partial L$, so in particular $x$ is a regular point.

6) We claim that $d_0^{(K)} = \tilde{d}_K$. Assume for a moment that $d_0^{(K)} < \tilde{d}_K$. As we observed in 4), there exists a sequence $\{x_n\} \subset \partial K$ of irregular points such that $d_K(x_n) \searrow d_0^{(K)}$.

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4The easy formal proof of this requires the assumption (which we can make without loss of generality) that the radius of the circle $S_0$ is sufficiently large, so that the curvature of $S_0$ is much smaller then the minimal curvatures of $\partial K$ and $\partial L$. We leave the details of this argument to the reader.

5If $\gamma$ has a tangency to $\partial K$ at some $x_i$, then $\gamma'$ may not have a common point with $\partial K$ near $x_i$. 

8
Taking a subsequence, we may assume \( x_n \to x \in \partial K \) as \( n \to \infty \), and then the continuity of \( d_K \) implies \( d_K(x) = d_K^0(K) \). Now 5) implies that \( x \) is a regular point, and so \( x_n \) is regular for any sufficiently large \( n \). This is a contradiction which shows that we must have \( d_K^0(K) = d_K \).

Finally, it follows from 6) and the definition of \( d_K^0(K) \) that we must have \( \partial K \subset \partial L \), i.e. \( K \subset L \). Now, since \( \text{Trap}(\Omega_K) \) and \( \text{Trap}(\Omega_L) \) both have Lebesgue measure zero in the corresponding phase spaces, it follows from Proposition 3 above that \( \text{Vol}(K) = \text{Vol}(L) \). This implies \( K = L \).

5 Appendix: Proof of Lemma 2

Set \( g(x) = f(x) + \varphi(x) \), \( x \in [-a,a] \).

Then \( \varphi(0) = \varphi'(0) = \varphi''(0) = 0 \). Replacing \( a \) by a smaller number, we may assume that

\[
|\varphi''(x)| \leq c/2 \quad , \quad x \in [-a,a].
\]

We want to find a function \( z \in C^1(-\delta, \delta) \) for some \( \delta \in (0,a) \) such that \( g'(z) = f'(x) \), i.e.

\[
f'(z(x)) + \varphi'(z(x)) = f'(x) \quad , \quad x \in (-\delta, \delta),
\]

and \( z(0) = 0 \). Consider the equation

\[
u(x,z) = f'(z) + \varphi'(z) - f'(x) = 0.
\]

Since \( u(0,0) = 0 \) and

\[
\frac{\partial u}{\partial z}(x,z) = f''(z) + \varphi''(z) \geq c - c/2 = c/2 > 0,
\]

it follows from the Implicit Function Theorem that there exist \( \delta > 0 \) and a function \( z \in C^1(-\delta, \delta) \) with \( z(0) = 0 \) such that (5.2) holds. Take the constant \( \epsilon > 0 \) so that \( z(x) \in (-\delta, \delta) \) for all \( x \in (-\epsilon, \epsilon) \).

If \( x \in \Gamma_\epsilon = \Gamma \cap (0, \epsilon) \), then for some \( y(x) \in (0,a) \) we have that the tangent line to \( \text{graph}(f) \) at \( (x,f(x)) \) is tangent to \( \text{graph}(g) \) at \( (y(x),g(y(x))) \). In particular, \( g'(y(x)) = f'(x) \), i.e.

\[
f'(y(x)) + \varphi'(y(x)) = f'(x) = f'(z(x)) + \varphi'(z(x)).
\]

However the function \( \psi(t) = f'(t) + \varphi'(t) \) is strictly increasing on \([-a,a] \), since \( \psi'(t) = f''(t) + \varphi''(t) \geq c/2 \), as observed above. Thus, (5.4) implies \( y(x) = z(x) \). In other words, \((z(x),g(z(x)))\) is the only point where the tangent line to \( \text{graph}(f) \) at \((x,f(x))\) could be tangent to \( \text{graph}(g) \).

Next, notice that if \( x \in \Gamma_\epsilon \) and \( x \neq y(x) \), then

\[
g'(y(x)) = f'(x) = \frac{f(x) - g(y(x))}{x-y(x)} = \frac{f(x) - f(y(x)) + \varphi(y(x))}{x-y(x)},
\]
which yields (using \(y(x) = z(x)\) as well)
\[
\Phi(x) = f'(x)(x - z(x)) + f(z(x)) + \varphi(z(x)) - f(x) = 0. \tag{5.5}
\]
The function \(\Phi\) is in \(C^1(-\epsilon, \epsilon)\) and, using (5.2), we get
\[
\Phi'(x) = f''(x)(x - z(x)) + f'(x)(1 - z'(x)) + (f'(z(x)) + \varphi'(z(x))) z'(x) - f'(x)
\]
\[
= f''(x)(x - z(x)) + f'(x)(1 - z'(x)) + f'(x) z'(x) - f'(x) = f''(x)(x - z(x)).
\]

Assuming that \(\Gamma_\epsilon\) is everywhere dense in \((0, \epsilon)\), it follows from (5.5) and Rolle’s Theorem that \(\Phi'(x) = 0\) for all \(x \in \Gamma_\epsilon\). Indeed, if \(x \in \Gamma_\epsilon\) and \(x \neq z(x)\), then for \(x' \in \Gamma_\epsilon\) close to \(x\) we have \(x \neq z(x')\), so the above argument shows that \(\Phi(x) = \Phi(x') = 0\). Then Rolle’s Theorem shows that \(\Phi'(x'') = 0\) for some \(x''\) between \(x\) and \(x'\). Letting \(x' \to x\), gives \(\Phi'(x) = 0\), so \(f''(x)(x - z(x)) = 0\). If \(x = z(x)\) for some \(x \in \Gamma_\epsilon\), then \(f''(x)(x - z(x)) = 0\) holds again. Thus, \(f''(x)(x - z(x)) = 0\) for all \(x \in \Gamma_\epsilon\), and therefore for all \(x \in (0, \epsilon)\). Since \(f''(x) > 0\), this gives \(x = z(x)\) on \((0, \epsilon)\), and then (5.2) implies \(g'(x) = f'(x)\) on \((0, \epsilon)\). Finally, \(g(0) = f(0)\) yields \(g(x) = f(x)\) on \([0, \epsilon]\). ■

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