The dual geometry of Boolean semirings

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Abstract. It is well known that the variety of Boolean semirings, which is generated by the three element semiring $S$, is dual to the category of partially Stone spaces. We place this duality in the context of natural dualities. We begin by introducing a topological structure $S$ and obtain an optimal natural duality between the quasi-variety $ISP(S)$ and the category $IS_{c}P^{+}(S)$. Then we construct an optimal and very small structure $S_{os}$ that yields a strong duality. The geometry of some of the partially Stone spaces that take part in these dualities is presented, and we call them “hairy cubes”, as they are $n$-dimensional cubes with unique incomparable covers for each element of the cube. We also obtain a polynomial representation for the elements of the hairy cube.

1. Introduction

Extensions of the concept of a Boolean ring to include semirings has been done in several different directions. One source of diversity is the different definitions of semiring. The other is how they get connected to Boolean rings. We will use the concept of semiring commonly used in formal languages and automata theory, that is, the only thing missing in order to be a ring is the existence of additive inverses (see [3] and [6]). As in Guzman [4], we will denote by $BSR$ the variety generated by the two 2-element semirings, and will call it the variety of Boolean semirings. It turns out that this variety is also generated by a 3-element semiring with carrying set $S = \{0, h, 1\}$, that we denote $S$. The semiring $S$ will play a crucial role in this paper.

In [4], following the ideas of Stone [7] in his now famous “Stone representation theorem”, a duality is established between the category $BSR$ of Boolean semirings and the category $PSS$ of partially Stone spaces. On the other hand, Clark and Davey [1] present a thorough study of natural dualities between algebraic and topological quasi-varieties. It is the goal of this paper to place the duality from [4] in the much richer context of [1].

A structured topological space consists of

$$X = \langle X; G, H, R, T \rangle$$

where $\langle X, T \rangle$ is a topological space, $G$ is a set of finitary (total) operations on $X$, $H$ is a set of finitary partial operations on $X$, and $R$ is a set of finitary

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relations on \( X \). The arities of the operations, partial operations, and relations define the type of \( X \). Given a finite discrete structured topological space \( X \), we denote by \( IS_c P^+(\mathcal{X}) \) the category of closed substructures of non-empty products of copies of \( X \).

On one side of the duality we will have the quasi-variety \( \mathcal{A} = ISP(\mathcal{S}) \) generated by \( S \). On the other side, we will have the category of structured topological spaces \( \mathcal{X} = IS_c P^+(\mathcal{S}) \), generated by some appropriate structure \( \mathcal{S} = (S; G, H, R, T) \) having \( S = \{0, h, 1\} \) as underlying set and \( T \) the discrete topology.

We begin by naming three binary relations on \( S \): \( r_1 \), \( r_2 \), and \( r_3 \), and prove the following result.

**Theorem 1.1.** The structure \( \mathcal{S} = (S; \{r_1, r_2, r_3\}, T) \) yields an optimal natural duality on \( \mathcal{A} \).

The proof is in Section 2. The duality is optimal in the sense that if any one of the relations were to be deleted from the structure of \( \mathcal{S} \), duality would be lost.

In the duality \( BSR \rightleftharpoons PSS \), a Boolean semiring \( A \) is mapped to the set of prime filters of \( A \) (recall that a Boolean semiring can be viewed as as partially complemented distributive lattice, see [4]). For finite \( A \), every prime filter is the upset of a join-irreducible element. When \( \mathcal{X} \) is a closed substructure of a finite power of \( \mathcal{S} \), we would like to describe the join-irreducible elements of \( \mathcal{X}(\mathcal{X}, \mathcal{S}) \). We denote the set of all of them by \( \mathcal{X}(\mathcal{X}, \mathcal{S})_J \). In Theorem 2.11, an inductive characterization of join-irreducible elements of \( \mathcal{X}(\mathcal{S}^n, \mathcal{S}) \) is given, see Section 2. In [2] a description of \( \mathcal{X}(\mathcal{X}, \mathcal{S})_J \) is given for any \( \mathcal{X} \in \mathcal{X} \) that is a closed substructure of a finite power of \( \mathcal{S} \). This will appear in a subsequent paper. Here we lay the foundation for those results, a description of the meet semilattice \( \mathcal{X}(\mathcal{S}^n, \mathcal{S})_J \), that we call the “hairy cube”. It consists of an \( n \)-dimensional cube covered by “hairs”. More precisely,

**Theorem 1.2.** The poset \( \mathcal{X}(\mathcal{S}^n, \mathcal{S})_J \) consists of two parts: the “base” \( Y^n \) which is an \( n \)-cube and the “hairs” \( \mathcal{X}(\mathcal{S}^n, \mathcal{S})_J \setminus Y^n \), which are pairwise incomparable. Each element of the base is covered by a unique hair. Each hair covers a unique base element.

In Theorem 3.7 it is shown that these partial order properties of \( \mathcal{X}(\mathcal{S}^n, \mathcal{S})_J \) completely determine it as a partially Stone space. We close Section 3 with a polynomial representation of the join-irreducible elements of \( \mathcal{X}(\mathcal{S}^n, \mathcal{S}) \), see Theorem 3.8.

In Section 4 we first establish that the duality in Theorem 2.6 is neither a full nor a strong duality. Then we discuss why this is true and how that duality can easily be upgraded to a strong duality, following some of the ideas of [1]. Then we show how to construct an optimal and very small structure \( \mathcal{S}_{\text{os}} \) that yields a strong duality on \( \mathcal{A} \). Despite the fact that \( \mathcal{S} \) is not subdirectly