Lame Generalized Problem for Viscoelastic Materials

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Abstract. Article presents the Lame generalized problem of the thick-walled pipe deformation when the pipe material has viscoelastic properties. The model was calculated using fractional numerical methods. The results obtained are well compatible with the classic solution.

1. Introduction

A lot of modern polymeric materials have viscoelastic properties. As [1-4] shows, in the case one-dimensional computation, the stress-strain state simulation requires only the following:

\[ \sigma(t) + bD^\theta \sigma(t) = E_1\varepsilon(t) + E_1D^\beta \varepsilon(t), \]  

(1)

which, in the absence of instantaneous elasticity, which is typical of most polymers, is reduced to the following simpler ratio:

\[ \sigma(t) = E_1D^\beta \varepsilon(t), \]  

(2)

wherein \( \sigma(t) \) is stress, and \( \varepsilon(t) \) is deformation, \( E_1 \) and \( 0 < \beta < 1 \) are parameters of the material. Operator here \( D^\beta \) is a fractional differentiation operator in the Caputo definition [5]

\[ D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x (x-\tau)^{-\beta} f'(\tau) d\tau. \]  

(3)

To analyze possibilities of the numerical methods in two-dimensional cases, let us consider the classical problem of strength of a thick-walled cylinder exposed to internal and external pressures (Figure 1). A hollow round cylinder with a wall of constant thickness \( t \) is exposed to internal and external pressures. Due to symmetry of the cylinder and the loads, the resulting deformations and stresses will also be symmetrical along the axis. In this case, the cylinder is considered to be thick-walled, if \( t \geq 0.1 \cdot D \) (wherein \( D \) is the outer diameter). Solution of such problems was proposed by the French mechanic Lame in 1828.
2. Computational model for the generalized Lame problem

Let us assume that the pipe material has viscoelastic properties. In this case, its stress-strain state is described using fractional derivatives, according to (2). There are the following ratios between circumferential and radial stresses, and strains:

\[
\begin{align*}
\sigma_r &= \frac{E_i}{1-\mu} D^\alpha (\varepsilon_r + \mu \varepsilon_\theta), \\
\sigma_\theta &= \frac{E_i}{1-\mu} D^\alpha (\varepsilon_\theta + \mu \varepsilon_r).
\end{align*}
\]  

(4)

Following the classical problem (Figure 1), we denote the offset \(u(r,t)\). Taking into account the Cauchy ratio \(\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}\), we derive from (4)

\[
\begin{align*}
\sigma_r &= \frac{E_i}{1-\mu} D^\alpha \left( \frac{\partial u}{\partial r} + \mu \frac{u}{r} \right), \\
\sigma_\theta &= \frac{E_i}{1-\mu} D^\alpha \left( \frac{u}{r} + \mu \frac{\partial u}{\partial r} \right).
\end{align*}
\]  

(5)

Substituting (5) into the classical equilibrium equation

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0,
\]  

(6)

will get us

\[
\frac{r}{1-\mu} \frac{\partial}{\partial r} \left( D^\alpha \frac{\partial u}{\partial r} + \frac{\mu}{r} D^\alpha u \right) + D^\alpha \frac{\partial u}{\partial r} - \frac{1}{r} D^\alpha u = 0
\]  

(7)
This generalized problem is a non-stationary one, i.e. the displacement function \( u(r, t) \) depends on time. Let us formulate boundary conditions: at the initial moment the pipe is not deformed, radial stresses on the outer surface are equal to the external pressure, radial stresses on the inner surface are equal to the internal pressure

\[
\begin{align*}
  u(r, 0) &= 0 \\
  \sigma_r(R_1, t) &= -p_1 \\
  \sigma_r(R_2, t) &= -p_2
\end{align*}
\]  

To solve the problem (7) – (8), let us determine a two-dimensional grid in the \([R_1, R_2] \times [0, T]\) area and draw up a difference scheme on it (Figure 2).

**Figure 2.** Two-dimensional grid for the difference scheme of a generalized problem

Grid parameters:

\[
\begin{align*}
  N_1 h_1 &= R_2 - R_1, \\
  N_2 h_2 &= T, \\
  r_i &= R_1 + i h_1, \\
  t_j &= j h_2, \\
  i &= 0 \ldots N_1, j = 0 \ldots N_2.
\end{align*}
\]  

Wherein \( R_1, R_2 \) are internal and external radii, \( T \) is an arbitrary specified period of time, \( h_1, h_2 \) are grid steps. Grid points, \((N_1 + 1)(N_2 + 1)\) total.

Difference analogues of partial regular and fractional derivatives at the grid points can be written as [6]

\[
\begin{align*}
  \frac{\partial u}{\partial r}(i, j) &= \frac{\Delta u_r(i, j)}{h_1}, \\
  D^\alpha u(i, j) &= \frac{h_2^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} \Delta u_r(i, k) \Delta S^\alpha_{j-k},
\end{align*}
\]  

wherein

\[
\begin{align*}
  \Delta u_r(i, j) &= u_{ij} - u_{ij-1}, \\
  \Delta u_r(i, j) &= u_{ij} - u_{i-1,j}, \\
  \Delta S^\alpha_{j-k} &= t^\alpha_{j-k+1} - t^\alpha_{j-k}.
\end{align*}
\]  

To write the equilibrium equation (10) in a difference form, we introduce the following grid functions:
\[
\Delta t \frac{\partial u(i,j)}{\partial t} = \frac{\partial u(i,j) - \partial u(i,j-1)}{h^2} = \frac{u_{ij} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}}{h^2};
\]
\[
D^\alpha \frac{\partial u(i,j)}{\partial r} = \frac{h^2 - a}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} \Delta t \frac{\partial u(i,k)}{\partial r} \Delta S^\alpha_{j-k};
\]
\[
g_{ij} = D^\alpha \frac{\partial u(i,j)}{\partial r} + \frac{\mu}{r_i} D^\alpha u(i,j) =
\]
\[
= \frac{h^2 - a}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} \left( \frac{\Delta t}{\partial r} (i,k) \Delta S^\alpha_{j-k} + \frac{\mu}{r_i} h^2 - a \Gamma(2 - \alpha) \sum_{k=1}^{j} \Delta u_t(i,k) \Delta S^\alpha_{j-k} =
\]
\[
= \frac{h^2 - a}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} \left( \frac{\Delta t}{\partial r} (i,k) + \frac{\mu}{r_i} \Delta u_t(i,k) \right) \Delta S^\alpha_{j-k}.
\]

As a result, we get an equilibrium equation for the grid internal points
\[
\frac{r_i}{1 - \mu} \frac{g_{ij} - g_{i-1,j}}{h^2} + \frac{h^2 - a}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} \left( \frac{\Delta t}{\partial r} (i,k) - \frac{1}{r_i} \Delta u_t(i,k) \right) \Delta S^\alpha_{j-k} = 0
\]
\[
i = 1 \ldots N_1 - 1, \quad j = 1 \ldots N_2 - 1.
\]

We formulate the boundary conditions, based on (8)

\[
u_{i0} = 0.
\]

As a result, the system should have \( N_1(N_2 - 1) \) equations for unknown values of the desired grid function.
3. Results

(12), (15), (16) system is calculated using the following parameters:

\[
\begin{align*}
\alpha &= 0.5; \quad \mu = 0.3; \quad E_1 = 200\text{MPa} \cdot s^\alpha; \\
R_1 &= 1\text{m}; \quad R_2 = 2\text{m}; \quad p_1 = 10\text{ MPa}; \quad p_2 = 1\text{ MPa}; \\
N_1 &= 4; \quad N_2 = 3; \quad h_1 = 0.25\text{m}; \quad h_2 = 1\text{s}.
\end{align*}
\]

PTC Mathcad 15 software package is used for the calculation. Table 1 includes the following values obtained for the displacement grid function.

Table 1. (12) – (16) difference scheme calculation results

| \(i\) | \(j = 0\) | \(j = 1\) | \(j = 2\) | \(j = 3\) |
|------|----------|----------|----------|----------|
| \(i = 0, R_1\) | 0 | 0.008 | 0.029 | 0.061 |
| \(i = 1\) | 0 | 0.006 | 0.020 | 0.049 |
| \(i = 1\) | 0 | 0.005 | 0.015 | 0.039 |
| \(i = 1\) | 0 | 0.004 | 0.013 | 0.032 |
| \(i = 4, R_2\) | 0 | 0.003 | 0.011 | 0.025 |

Diagrams of radial and circumferential stresses are created (Figure 3 and Figure 4).

4. Conclusions

Results of the calculations correspond to the theoretical expected results and are an extension of the classical model. In fact, the classical model is a special case of the absolutely elastic material with \(\alpha = 1\). In case of a viscoelastic material, stresses inside the pipe proved to be lower, which is quite logical and proves adequacy of the model. Therefore, the model presented and the difference scheme can be used to calculate other non-stationary problems and structures by using viscoelastic materials.
Figure 4. Circumferential stress diagrams:
1 – Classical elastic problem (α = 0); t₂, t₃ – stress in grid points at α = 0.5 and t = t₂, t₃

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