Fifty Years of Yang-Mills Theory and my Contribution to it.

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MIT-CTP-3484

Abstract

On the fiftieth anniversary of Yang-Mills theory, I review the contribution to its understanding by my collaborators and me.

Contents: 1. Gauge Theories and Quantum Anomalies; 2. Mathematical Connections; 3. Gauge Field Dynamics other than Yang-Mills; 4. Gauge Formalism for General Relativity Variables; A. Christoffel connection as a gauge potential, B. Gravitational Chern-Simons term from gauge theory Chern-Simons term, C. Coordinate transformations in general relativity and gauge theory, (i) Response to changes in coordinates (ii) Invariant fields and constants of motion. References.

To be published by World Scientific.

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Following some precursors (Klein, Pauli, Shaw), Yang and Mills invented non-Abelian gauge theory half a century ago [1]. Governed by the Yang-Mills Lagrange density

\[ \mathcal{L}_{YM} = \frac{1}{2} \left< F^{\mu\nu} F_{\mu\nu} \right> , \]

where \( F^{\mu\nu} \) is the Lie algebra/matrix valued gauge field strength constructed from a gauge potential \( A_\mu \)

\[ F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] , \]

the model generalizes in a natural and elegant fashion Maxwell electrodynamics, to which it also reduces in the Abelian case. (Brackets \( <> \) denote matrix trace.)

Twenty years passed before physicists learned how to quantize, renormalize and put the theory to phenomenological use describing the dynamics of fundamental elementary particles – a length of time comparable to the interval between the invention of quantum physics by Planck and its final formulation by Heisenberg and Schrödinger. In the early seventies, the work of our editor ‘tHooft [2] and his teacher Veltman [2] made it possible to perform calculations for strong, weak and electromagnetic interactions, based on well-defined, non-Abelian gauge theory models, which became subsumed in the “standard model” for elementary particles. While important investigations unraveled the novel dynamics (confinement, renormalization group and asymptotic freedom, large-N limit), my collaborative research focused on the kinematical properties of non-Abelian gauge fields; properties that become visible when close attention is paid to the gauge theory’s mathematical (geometrical, topological) structures, which nevertheless affect physical content. In this celebratory review, I shall first summarize some of our work, and then present remarks on gauge theoretic aspects of gravity theory.

### 1 Gauge Theory and Quantum Anomalies

A precursor to much mathematical analysis of gauge theories is the chiral anomaly, discussed by Bell and me [3] (also Adler and, earlier, others [4]) before non-Abelian theories entered center-stage. We showed that a chiral symmetry of classical dynamics does not in
In the presence of fermions, the continuity equation for the classically conserved, but quantum mechanically non-conserved chiral current, $J^\mu_A$, becomes proportional to the “anomaly” after quantization.

$$\partial_\mu J^\mu_A \propto \langle F_{\mu\nu}^* F_{\mu\nu} \rangle$$

($F_{\mu\nu}^*$ is the dual of $F_{\mu\nu} : F_{\mu\nu}^* \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$.) Physicists later recognized the quantity on the right as the Chern-Pontryagin topological density. The anomaly allowed evading a chiral symmetry-based, model-independent no-go theorem prohibiting the neutral pion from decaying into two photons – a process seen in Nature. Evidently quantum effects destroy the apparent (classical) chiral symmetry and negate the physically unacceptable prohibition. This was very much appreciated by Bell’s good friend Veltman, who (with Sutherland) established the no-go result \[5\]. His pupil 'tHooft later made important connections between the chiral anomaly and properties of the standard model (see below).

The first application of the chiral anomaly to the standard model came with the observation by Gross and me \[6\] (also Bouchiat, Iliopoulos and Meyer \[7\]) that the ‘tHooft-Veltman argument for renormalizability of gauge theories remains valid only if fermion content is arranged so that gauge fields couple to currents that are free of anomalies, in which case the Yang-Mills equation with sources $J^\mu$ is self-consistent.

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \equiv \mathcal{D}_\mu F_{\mu\nu} = J^\nu$$

This requirement together with the strength of the anomaly, fixed experimentally by the $\pi^0 \rightarrow 2\gamma$ decay amplitude, provides up to now one of the few principles for determining the color and family structure of elementary fermions (quarks, leptons) in Nature.

Nevertheless, the subject lay fallow until instantons were found by Belavin, Polyakov, Schwartz and Tyupkin \[8\] (who called them “pseudoparticles”). Instantons are classical gauge field configurations in imaginary time (Euclidean space-time) that are self-dual or anti self-dual.

$$*F_{\mu\nu} = \pm F_{\mu\nu}$$
Therefore, they satisfy the classical, imaginary-time, sourceless Yang-Mills equation,
\[ D_\mu F^{\mu\nu} = 0, \]  
(1.4)
by virtue of the Bianchi identity for the dual field strength.
\[ D_\mu * F^{\mu\nu} = 0 \]  
(1.5)
Moreover, the 4-dimensional integral of the Chern-Pontryagin density evaluated on instanton configurations takes values fixed by an integer that labels the homotopy class to which the gauge field belongs. The non-vanishing value for this integral came as a surprise, because the Chern-Pontryagin density is itself a total divergence, so its integral is a surface term frequently ignored by physicists.
\[ \frac{1}{4} < * F^{\mu\nu} F_{\mu\nu} > = \partial_\mu K^\mu \]
(1.6)
\[ K^\mu \equiv \varepsilon^{\mu\alpha\beta\gamma} < \frac{1}{2} A_\alpha \partial_\beta A_\gamma + \frac{1}{3} A_\alpha A_\beta A_\gamma > \]
\[ \frac{1}{4} < * F^{\mu\nu} F_{\mu\nu} > = \int dtd^3x \partial_\mu K^\mu \]
(1.8)
\[ K^\mu \] is called the Chern-Simons or anomaly current, whose time component \( K^0 \) contains only spatial quantities. \( K^0 \) is also called the Chern-Simons density, about which I shall have more to say later.
\[ CS(A) \equiv K^0 = \varepsilon^{ijk} < \frac{1}{2} A_i \partial_j A_k + \frac{1}{3} A_i A_j A_k > \]
(1.7)
The surprising relevance of all this to quantum physics in Minkowski space-time comes about for the following reasons. Since the baryon number current in the standard model does not couple to a gauge field, it can remain anomalous, even though it is a vector quantity with no chiral component. This possibility was appreciated already in the original paper with Bell [3], but no physical consequence was drawn, because the total accumulated change of the anomalous charge, \( Q_A = \int d^3x J_A^0 \), is a surface term of no apparent importance before the advent of instantons.
\[ \Delta Q_A \equiv \int_{-\infty}^{\infty} dt \frac{d}{dt} \int d^3x J_A^0 \propto \frac{1}{4} \int dt d^3x < * F^{\mu\nu} F_{\mu\nu} > = \int dt d^3x \partial_\mu K^\mu \]
(1.8)
However, ‘tHooft realized that this reasoning is inadequate [9]. He observed that instantons can be used to evaluate approximately the Yang-Mills functional integral continued to imaginary time. Since also the instanton-dominated integral of the Chern-Pontryagin density
is non-vanishing, ’tHooft concluded that baryon number is not conserved in the standard model. By evaluating the Euclidean functional integral in a Gaussian approximation around the instanton solution of Belavin \textit{et al.}, he calculated the baryon lifetime. Fortunately it is exponentially small, but diamonds in principle are not forever.

There remained much to be done in extending ’tHooft’s results. One wished to identify the physical mechanism in Minkowski space-time behind the approximate evaluation of an Euclidean functional integral. Also ’tHooft found that his answer depends on an angle $\theta$, which is not seen in gauge field dynamics; again a physical explanation was needed.

By recalling known procedures in condensed matter physics and quantum chemistry Rebbi and I \cite{10} (also Callan, Dashen and Gross \cite{11}) explained that classical paths in imaginary time signal quantum tunneling, whose probability amplitude in the semi-classical approximations is given by the Euclidean functional integral in Gaussian approximation around the imaginary time path. But where does the tunneling take place? To answer this, and also to understand the $\theta$-angle, we examined the gauge theory in the Schrödinger representation, where quantum states are described by wave functionals $\Psi(A)$ defined on configuration space variables – here the spatial vector potentials $A$. (Although the reality of a space of gauge field variables is obscure, it is no more obscure than the 3 N-dimensional space on which N-body, $N > 1$, quantum mechanical wave functions are defined.)

We then drew the following qualitative but exact picture of the quantum field theory \cite{10}. The Gauss law and conventional gauge fixing (of the kind described by Faddeev in this volume) ensure that $\Psi(A)$ is unchanged when $A$ undergoes a gauge transformation that is deformable to the identity; so called a “small” gauge transformation. However, in non-Abelian groups there are gauge transformations that are homotopically non-trivial and cannot be connected to the identity. For these “large” transformations, labeled by an integer $n$ that indexes the homotopy class to which the gauge transformation belongs, $\Psi(A)$ changes by a phase. This phase is the $\theta$-angle encountered by ’tHooft in his calculation.

In the classical ground state, the magnetic field $B^i \equiv \star F^{i0}$ vanishes. This is achieved not only
with vanishing $A$, but also with pure-gauge vector potentials. Since large gauge functions fall into classes labeled by the integers, the potential energy profile takes on a periodic structure, reminiscent of a Bloch crystal and depicted in the Figure below.

The zero-energy troughs correspond to pure gauge vector potentials constructed from gauge functions belonging to the $n^{th}$ homotopy class. Thus the classical ground state is (infinitely) degenerate. Quantum mechanical tunneling lifts this degeneracy and creates energy bands. Usually in quantum field theory tunneling is suppressed by infinite energy barriers between degenerate vacua – this leads to spontaneous symmetry breaking. However, in Yang-Mills theory there are imaginary time paths in field space, which avoid infinite barriers and interpolate between the vacua. Semi-classically these are the instantons., as can be verified by examining their large (imaginary) time asymptotes. The final picture is portrayed schematically in the above Figure.

To go beyond qualitative considerations we constructed an explicit functional of $A$, which is invariant against small gauge transformations but not large ones. This is just the spatial integral of the Chern-Simons density, introduced in (1.7). One readily checks that

$$W(A) = \frac{1}{4\pi^2} \int d^3x CS(A) = \frac{1}{4\pi^2} \int d^3x \varepsilon^{ijk} \left< \frac{1}{2} A_i \partial_j A_k + \frac{1}{3} A_i A_j A_k \right>,$$

satisfies the Gauss law by virtue of

$$\delta W(A) = \frac{1}{4\pi^2} \int d^3x \delta A \cdot B,$$

$$D \cdot B = 0.$$
But when $A$ is gauge transformed by a large gauge function in the $n^{th}$ homotopy class, $W(A)$ shifts by the “winding” number $n$.

\[
W(A^g) = W(A) - \frac{1}{24\pi^2} \int d^3x \varepsilon^{ijk} < g^{-1} \partial_i g \ g^{-1} \partial_j g \ g^{-1} \partial_k g > \quad (1.11)
\]

\[
A^g_i \equiv g^{-1} A_i \ g + g^{-1} \partial_i \ g \quad (1.12)
\]

The last term in (1.11) evaluates the winding number of $g$ (for well-behaved $g$).

Every Yang-Mills wave functional can be presented as

\[
\Psi(A) = e^{i\theta W(A)} \Psi_{inv}(A), \quad (1.13)
\]

where $\Psi_{inv}(A)$, is invariant against all gauge transformations, small and large, while the non-invariance of $\Psi(A)$ is contained in the universal phase involving $W(A)$.

\[
\Psi(A) \rightarrow e^{i\theta} \Psi(A) \quad (1.14)
\]

But in quantum theory a universal phase of wave functions may be removed at the expense of adding the time derivative of the phase to the theory’s Lagrangian. When this procedure is carried out with the help of (1.6) and (1.7) for the problem at hand, the Yang-Mills quantum theory becomes equivalently described by completely gauge invariant states, $\Psi_{inv}(A)$, while the Yang-Mills Lagrangian acquires the addition $\theta \int d^3x \ast F_{\mu \nu} F_{\mu \nu}$, which does not contribute to equations of motion because it is a total time derivative. This shows that the $\theta$-angle is associated with Lorentz invariant but CP (or T) violating phenomena, which through our analysis are understood in exact terms, while instantons provide a semi-classical description. We are left with a puzzle: What fixes the magnitude of $\theta$, whose experimental consequences (e.g. neutron electric dipole moment) have never been seen? (An analogy with the cosmological constant puzzle is apparent.)

## 2 Mathematical Connections

The anomaly-based instanton investigation of the standard model did not produce any useful numbers for experimentalists to measure. But it affected deeply our understanding of the
theory. Also it suggested a wealth of interesting mathematical problems to which Rebbi and I found solutions by methods drawn from analysis, geometry and topology.

We proved that the Belavin et al. instanton preserves an $SO(5)$ symmetry subgroup of the $SO(5,1)$ conformal invariance group for Euclidean Yang-Mills theory \cite{12}. This allows a group theoretical classification of motions in the presence of the instanton. Furthermore, use of $SO(5)$ covariant coordinates yields simple and elegant formulas, so that evaluating the functional integral in a Gaussian approximation around the instanton becomes transparent. We \cite{13} (also Schwartz, as well as Atiyah, Hitchin and Singer \cite{14}) showed that the most general instanton configuration with Chern-Pontryagin index ($\equiv \int d^4x \ < F^{\mu\nu} F_{\mu\nu} >$) $\propto n$ can be viewed as a non-linear superposition of $|n|$ individual instantons, which depends on $8|n|$ parameters for the $SU(2)$ group: $4|n|$ positions in 4-dimensional space, $|n|$ instanton sizes, and $(k^2 - 1)|n|$ group variables of $SU(2)$. Also following a suggestion by ‘tHooft, we exhibited the most general, explicit multi-instanton formula. \cite{15} Our expression is closed under conformal transformations and maximizes the parameter count for $|n| = 1$ and 2. (For $|n| > 2$, no explicit formula for the general solution is known, but a procedure for constructing it at given $|n|$ has been found by Atiyah, Drinfeld, Hitchin and Manin. \cite{16})

Much of this analysis involves zero eigenvalue solutions to 4-dimensional elliptic differential equations in the presence of instantons. The number of these zero modes is determined by the Chern-Pontryagin index of the background gauge field. This is the statement of the celebrated Atiyah-Singer index theorem which appeared in physics for the first time because of these considerations. \cite{17} Indeed the anomaly equation (1.1) may be viewed as a local version of the theorem.

The zero modes arise not only with 4-dimensional instantons, but also in the presence of extended, topologically interesting field configurations in other dimensions. For example, ‘tHooft and Polyakov \cite{18} found that magnetic monopole configurations are present as classical static solutions to some gauge theories based on semi-simple groups, like $SU(2)$. Rebbi and I \cite{19} (also Hansenfratz and ‘tHooft \cite{20}) then showed that spinless boson fields, carrying half-integer isospin, bind to the monopole with zero energy and form a spin $\frac{1}{2}$ excitation in
the quantum theory, even though all “constituents” carry integer spin. While it is uncertain whether magnetic monopoles and the associated spin fractionization have a physical presence in Nature (they are not features of the standard model) analogous effects can arise in condensed matter systems that are available in a laboratory. For example, 1-dimensional topological kinks form domain walls in lineal polymers like polyactelene. Electrons propagating across these kinks experience fractionization of their quantum numbers – an effect that has been observed experimentally. [21]

The zero modes associated with monopoles and kinks arise in elliptic differential equations in odd dimensions, where the Atiyah-Singer index cannot be used because it is restricted to even dimensions. Callias, a student at that time, provided the necessary extension, and the “Callias index” is now used for counting zero modes in odd-dimensional spaces. [22]

The mathematical activity surrounding instantons and other extended objects, like monopoles, vortices and kinks, seeded an interaction between physics and mathematics, which is still flourishing. At an American Physical Society meeting, where I summarized the above results [23], Singer declaimed a poetic pean to physics – mathematics collaboration:

\[
\begin{align*}
\text{In this day and age} & \quad \text{Mathematicians so blind} & \quad \text{But gauges have flaws} \\
\text{The physicist sage} & \quad \text{Follow slowly behind} & \quad \text{God hems and haws} \\
\text{Writes page after page} & \quad \text{With their clever minds} & \quad \text{As the curtain He draws} \\
\text{On the current rage} & \quad \text{A theorem they’ll find} & \quad \text{O’er His physical laws} \\
\text{The gauge} & \quad \text{Only written and signed} & \quad \text{It may be a lost cause} \\
\end{align*}
\]

I. Singer

The interaction with mathematics became fueled anew when physicists called attention to further gauge theoretic invariants.
3 Gauge Field Dynamics other than Yang-Mills

Exploration of Yang-Mills theory brought physicists’ attention to gauge theoretic invariants, other than the Yang-Mills Lagrange density \( \frac{1}{2} < F_{\mu\nu} F_{\mu\nu} > \). I have already discussed the anomaly-determined Chern-Pontryagin density \( < *F_{\mu\nu} F_{\mu\nu} > \) (1.1). It does not contribute to equations of motion because it is a total derivative, but it controls the \( \theta \)-angle. Further there is the Chern-Simons current \( K^\mu \) (1.6) and density \( K^0 \equiv CS(A) \) (1.7) – “anti-derivatives” of the Chern-Pontryagin density. Defined on 3-space, \( CS(A) \) is not gauge invariant. But its properly normalized integral over 3-space – the Chern-Simons term \( W(A) \) (1.9) – responds only to large gauge transformatives by the integer winding number of the gauge function.

\( W(A) \) first entered physics as the phase of Yang-Mills quantum states, (1.13), where it is responsible for the \( \theta \)-angle.

Another remarkable but futile role for \( W(A) \) is that \( e^{\pm 4\pi^2 W(A)} \) solves the Yang-Mills functional Schrödinger equation with zero eigenvalue by virtue of (1.10a): \( H_{YM}(\frac{1}{i}, A) e^{\pm 4\pi^2 W(A)} = 0 \), where \( H_{YM} \) is the Yang-Mills Hamiltonian. It is astonishing that such a non-linear, integro-differential, functional equation possesses a simple and explicit solution. Unfortunately, \( e^{\pm 4\pi^2 W(A)} \) cannot describe a physical state because \( |e^{\pm 4\pi^2 W(A)}| \) is not gauge invariant against large gauge transformations, and grows exponentially for large (functional) argument. [This “solution” is a great gauge theory teaser, but in fact it possesses a quantum mechanical analog: The zero-eigenvalue Schrödinger equation for a 2-dimensional, \((x, y)\), isotropic harmonic oscillator (unit frequency) is solved by \( e^{\pm xy} \), which diverges for large \( x \) or \( y \), and has no place in the quantal Hilbert space for this system.]

The integer valued gauge non-invariance of \( W(A) \) does not prevent using that 3-dimensional quantity in the action for a gauge theory on a \((2+1)\)-dimensional space-time. Because its variation (1.10a) is gauge covariant, \( W(A) \) contributes a gauge covariant quantity to the equation of motion. Also in a quantum theory only the phase exponential of the action need be gauge invariant. With this in mind, Deser and I \([21]\) (together with students and postdoctoral fellows) proposed that \( mW(A) \) be included in the action for a novel \((2+1)\)-
dimensional gauge theory, where \( m \) is strength of the new term. [Here \( A_\mu \) is a \((2 + 1)\)-dimensional space-time covariant vector potential, not a 3-dimensional spatial vector \( \mathbf{A} \).]

In order to preserve gauge invariance of the phase exponential of the action for a non-Abelian theory, \( m \) must be quantitized to be an integral multiple of \( 2\pi \). Then \( e^{imW(A)} \) remains invariant even against large gauge transformations, which shift \( W(A) \) by an integer. This coupling constant quantization is the precise field theoretic analog of Dirac’s celebrated monopole strength quantization. Of course in an Abelian theory, there are no large gauge transformations; \( W(A) \) is gauge invariant; quantization of the interaction strength is not needed.

The modified but gauge covariant equation of motion, with a covariantly conserved source, reads

\[
\mathcal{D}_\mu F^{\mu\nu} + \frac{m}{8\pi^2} \varepsilon^{\alpha\beta} F_{\alpha\beta} = J^\nu, \tag{3.1}
\]

\[
-\varepsilon^{\nu\alpha\beta} \mathcal{D}_\alpha *F_{\beta} + \frac{m}{4\pi^2} *F^\nu = J^\nu. \tag{3.2}
\]

The non Yang-Mills term on the left comes from varying \( mW(A) \), see (1.10a). Eq. (3.2) exhibits the dual field \((*F^\mu \equiv \frac{1}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta})\), which in 3 dimensions is a vector, obeying the Bianchi identity.

\[
\mathcal{D}_\mu *F^\mu = 0 \tag{3.3}
\]

Note that (3.2) is consistent with (3.3).

The dimension of \( m \) is mass (in units of the gauge coupling constant) and one sees either from the linear portion of (3.1) or from the linear Abelian case that gauge excitations in this theory are massive, while retaining gauge invariance! This provides yet another example where gauge invariance does not enforce masslessness for gauge field excitations. Note that \( P \) and \( T \) are violated by the mass term.

Because the mass term in (3.1) has one fewer derivative than the usual Yang-Mills kinetic term, it dominates at low energies and large distances. In the absence of the Yang-Mills
term, equation (3.1) reduces to a field-current identity.

\[ \frac{m}{8\pi^2} \varepsilon^{\alpha\nu\beta} F_{\alpha\beta} = J^{\nu} \]  
\[ (3.4) \]

This is especially interesting in the Abelian case – planar electrodynamics – where the components of (3.4) read

\[ B = -\frac{4\pi^2}{m} \rho, \]  
\[ (3.5a) \]

\[ E^i = \frac{8\pi^2}{m} \varepsilon^{ij} J^j. \]  
\[ (3.5b) \]

Here \( E^i \) is a planar electric field; \( B \), a magnetic field perpendicular to the plane; \( \rho \) is a charge density and \( J^i \) a planar current. A further consequence of (3.5a), which also follows from the time component of (3.1) or (3.2), is the integrated statement

\[ N = -\frac{4\pi^2}{m} Q, \]  
\[ (3.6) \]

where \( N \) is the magnetic flux through the plane and \( Q \) is the charge.

Relations like (3.5a)–(3.6) arise in descriptions of the quantum Hall regime, and the Chern-Simons term has been widely used to model this planar effect. More recently high temperature superconductivity gave impetus to applications of Chern-Simons structures in speculative descriptions of that phenomenon. Also physics returned the Chern-Simons term to mathematics when Witten used it in a functional integral formula for knot invariants.

My investigations of Chern-Simons based gauge theories mostly concern the truncated equations (3.4) with source currents constructed from specific field theoretic or point particle variables. The dynamics of the sources is also self consistently included.

For sources made from relativistic scalar fields with precisely tuned self interactions, E. Weinberg and I \[ 25 \] (also Hong, Kim and Pac \[ 26 \]) found static vortex solutions, similar to Ginsburg-Landau (Nielsen-Olesen) vortices at the boundary between type I and type II superconductors. Although equations are only partially integrable, we showed that in the Abelian case both topological and non-topological vortices are present. Furthermore, Pi and I \[ 27 \] considered non-relativistic scalar field dynamics, more specifically the non-relativistic
limit of the above mentioned model. In that case the static equations are completely inte-
grable, providing explicit profiles for the vortices.

It is known that conventional vortex models, without the Chern-Simons term, support only
charge-neutral vortices. On the other hand, the Chern-Simons vortices are charged, by virtue
of (3.6). Also generically, they carry arbitrary, unquantized angular momentum. This is a
consequence of planar dynamics where rotations are Abelian and angular momentum need
not be quantized. [28]

It is important that the Chern-Simons term is not merely an ad hoc $P$ and $T$ violating ad-
dition to planar gauge theories, which could be included or omitted at will. Even when it is
absent in a bare Lagrangian containing fermions, it arises from radiative corrections. Massless
fermions in (2 + 1)-dimensional space-time preserve planar parity invariance. Nevertheless
they induce a parity violating Chern-Simons term. This is the so called “parity anomaly”,
discovered by Redlich [29], a student during that research period. It is the odd-dimensional
analog of the even-dimensional chiral anomaly. (The fermion determinant, which is respon-
sible for this effect, can also be evaluated in a heat bath environment at finite temperature;
there its response to large gauge transformations is especially intricate.)

Both chiral and parity anomalies enforce quantum mechanical symmetry breaking in the-
ories that before quantization possess symmetries associated with masslessness. One more
instance of this phenomenon is known: anomalous quantum mechanical breaking of scale and
conformal invariance. Generically these do not survive in non-trivial quantum field theories,
not even in quantum mechanics. [30] Although not necessarily confined to gauge theories,
nor possessing any significant topological aspects, the scale/conformal anomalies share with
the previous two the property of destroying an enhancement of symmetry that masslessness
would entail. This common feature prepared Coleman and me to understand and explain
why the relevant currents are not conserved, with anomalous divergences governed by the
anomalously non-vanishing trace of the energy-momentum tensor. [31] It appears that Na-
ture abhors masslessness, but we do not know why.
Finally we note that the Chern-Simons term $mW(A)$, viewed as a quantity defined on three spatial dimensions, can be inserted into the action for a Lorentz invariant theory in $(3 + 1)$ dimensional space-time. Due to the dimensional mismatch, this acts as a source of Lorentz and CTP violation. The idea has been developed for electrodynamics by Carroll, Field and me.\[32\] There it produces a modification of the Maxwell equations only in Ampère's law, which in the Lorentz violating theory reads

$$-\frac{\partial E}{\partial t} + \nabla \times B = J + \frac{m}{4\pi^2} B.$$ \hspace{1cm} (3.7)

(A source current containing a contribution from a magnetic field is familiar in magnetohydrodynamics.) The physical consequence of (3.7) is that the vacuum becomes birefringent, and propagating light waves undergo a Faraday-like rotation. Light from distant galaxies provides an experimental measure of this effect. Available observational data indicates that it does not occur in Nature; $m = 0$.

## 4 Gauge Formalism for General Relativity Variables

General relativity with its diffeomorphism invariance embodies the symmetry of local translations. Therefore, one should try presenting the theory in a formalism similar to that of a gauge theory. This has been achieved in lower dimensions.

For 3-dimensional Einstein theory in the Dreibein/spin connection formulation, field variables are gauge potentials for the local Poincaré or deSitter groups: local translations are gauged by Dreibeine, local Lorentz rotations, by spin connections. Dynamics is not of the Yang-Mills form; rather the Lagrange density is a Chern-Simons term based on the Poincaré or deSitter groups. Einstein theory does not exist in two dimensions, but various “dilaton” gravities can be formulated as gauge theories based on $SO(2.1)$ or (extended) Poincaré groups, with dynamics governed by a “BF” Lagrange density. In neither dimensionality do these models support propagating, dynamical degrees of freedom.

The above successes do not extend to four dimensions, where no complete gauge theoretic
A formulation has been found for Einstein’s theory. The closest we have again uses Vierbeine/spin connections, with the latter gauging the local Lorentz group, while the Vierbeine remain as covariant, additional variables.

However, if we put aside the issue of gravitational dynamics and focus only on the gravitational field variables, we can find many (notational) analogies to gauge fields. These analogies are useful for motivating and constructing gravitational counterparts to gauge theoretical entities. Correspondingly, aspects of general relativity can inform topics in gauge theory.

Here I shall provide a dictionary between gauge theoretic and general relativistic variables, and then use the relationship between them for further constructions both in general relativity and gauge theory. It is likely that the gravity-gauge theory connection described here is familiar to some (for example Bardeen and Zumino [33]) but I know of no textbook discussion. Thus I hope that my presentation will lead to wider appreciation of these useful formulas.

**A. Christoffel connection as a gauge potential**

Consider the Christoffel connection $\Gamma^\mu_{\alpha\nu}$ (in any dimension $d$) and view it as the $(\mu, \nu)$ component of a gauge potential matrix.

$$\Gamma^\mu_{\alpha\nu} = (A^\mu_\alpha)_\nu \quad (4.1)$$

All quantities are functions of the d-dimensional coordinate $x^\mu$. Next consider a new coordinate system $\bar{x}^\mu(x)$. The conventional formula relating the connection in the new coordinate system, $\bar{\Gamma}^\mu_{\alpha\nu}$, to that in the original coordinates is of course familiar. [31] For our purposes, observe that the transformation formula can be presented with the definition (4.1) as

$$(\bar{A}_\alpha(\bar{x}))^\mu_\nu = \left\{ (U^{-1})^\rho_\mu (A_\beta(x))^\sigma_\rho (U)^\sigma_\nu + (U^{-1})^\mu_\sigma \frac{\partial}{\partial x^\beta} (U)^\sigma_\nu \right\} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \quad (4.2)$$

where

$$(U)^\mu_\nu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \ , \ (U^{-1})^\nu_\mu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \quad (4.3)$$
Thus we see that the matrix field $\mathcal{A}_\beta(x)$ transforms by $\frac{\partial \mathcal{A}_\beta}{\partial \bar{x}^\alpha}$ in its vector index, and also undergoes a gauge transformation by the gauge function $\mathcal{U}$ in its matrix indices. In matrix notation

$$\tilde{\mathcal{A}}_\alpha(\bar{x}) = \left\{ \mathcal{U}^{-1} \mathcal{A}_\beta(x) \mathcal{U} + \mathcal{U}^{-1} \frac{\partial}{\partial \bar{x}^\beta} \mathcal{U} \right\} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \quad (4.A.4)$$

If $\mathcal{U}$ were arbitrary, we would conclude from (4.A.4) that $\mathcal{A}_\alpha$ is a connection for the $GL(d, \mathbb{R})$ group. But the requirement that Christoffel connections be symmetric in their lower indices (no torsion) is met only for $\mathcal{U}$ in the form (4.A.3). (Here and below gauge equivalents to geometrical objects are denoted by scripted letters. Coordinate covariant derivatives acting on geometrical quantities are denoted by $D$; in corresponding formulas where the derivative acts on gauge theory variables we use as before $\mathcal{D}$.)

The Christoffel connection’s notational analogy to a gauge potential continues for covariant derivatives. A contravariant vector behaves as a left-transforming group object.

$$D_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu} V^\nu = \partial_\alpha V^\mu + (\mathcal{A}_\alpha)^\mu_{\nu} V^\nu \sim \mathcal{D}_\alpha V_L \quad (4.A.5a)$$

while a covariant vector is right-transforming.

$$D_\alpha V_\mu = \partial_\alpha V_\mu - \Gamma^{\nu}_{\alpha\mu} V_\nu = \partial_\alpha V_\mu - V_\nu (\mathcal{A}_\alpha)^{\nu}_{\mu} \sim \mathcal{D}_\alpha V_R \quad (4.A.5b)$$

Thus a mixed, second rank tensor is a matrix transforming in the adjoint representation.

$$D_\alpha V^\mu_{\nu} = \partial_\alpha V^\mu_{\nu} + \Gamma^\mu_{\alpha\sigma} V^\sigma_{\nu} - \Gamma^\sigma_{\alpha\nu} V^\mu_{\sigma} = \partial_\alpha V^\mu_{\nu} + (\mathcal{A}_\alpha)^\mu_{\sigma} V^{\nu}_{\sigma} - V^\mu_{\sigma} (\mathcal{A}_\alpha)^{\sigma}_{\nu} \sim \partial_\alpha V_{\text{ADJ}} + [\mathcal{A}_\alpha, V_{\text{ADJ}}] \sim \mathcal{D}_\alpha V_{\text{ADJ}} \quad (4.A.5c)$$

The conventional formula in terms of Christoffel connections for the Riemann curvature tensor translates to the gauge field strength.

$$R^\mu_{\nu\alpha\beta} (\Gamma) = (\mathcal{F}_{\alpha\beta})^\mu_{\nu} = \partial_\alpha (\mathcal{A}_\beta)^\mu_{\nu} - \partial_\beta (\mathcal{A}_\alpha)^\mu_{\nu} + [\mathcal{A}_\alpha, \mathcal{A}_\beta]^\mu_{\nu} \quad (4.A.6)$$

The Bianchi identity for the curvature and commutators of covariant derivatives also translate freely.
Another useful formula relates the above to the Vielbeine \( e^a_\mu \) and spin connections \( \omega^a_{\alpha b} \). Beginning with

\[
D_\alpha e^a_\mu = \partial_\alpha e^a_\mu - \Gamma^\nu_{\alpha \mu} e^a_\nu + \omega^a_{\alpha b} e^b_\mu = 0, \tag{4.A.7a}
\]

we contract with the Vielbein inverse \( e^a_a \) to find

\[
\Gamma^\sigma_{\alpha \mu} = e^\sigma_a \omega^a_{\alpha b} e^b_\mu + e^\sigma_a \partial_\alpha e^a_\mu. \tag{4.A.7b}
\]

Thus upon defining a matrix \( \mathcal{E} \) with indices \((a, \mu)\)

\[
(\mathcal{E})^\mu_a = e^a_\mu, \quad (\mathcal{E}^{-1})^\sigma_a = e^\sigma_a,
\]

and viewing \( \omega^a_{\alpha b} \) as a matrix gauge potential with indices \((a, b)\)

\[
\omega^a_{\alpha b} = (\mathfrak{A}_a)^a_{\ b}, \tag{4.A.8}
\]

Eq. (4.A.7) reads, in matrix notation,

\[
\mathcal{A}_\alpha = \mathcal{E}^{-1} \mathfrak{A}_a \mathcal{E} + \mathcal{E}^{-1} \partial_\alpha \mathcal{E}. \tag{4.A.10}
\]

We see that the spin connection and the Christoffel connection are gauge equivalent, with the Vielbein taking the role of the gauge transformation function. This has the consequence that (gauge) covariant quantities (like the Riemann tensor) can be constructed either with Christoffel or spin connections, and the two expressions are related by the covariant gauge transformation \((\mathcal{E}^{-1}...\mathcal{E})\) built from the Vielbein. Thus for example,

\[
R^\mu_{\nu\alpha\beta}(\Gamma) = e^a_\mu R^a_{\nu\alpha\beta}(\omega)e^b_\nu \tag{4.A.11}
\]

where \( R^a_{\nu\alpha\beta}(\omega) \) is the Riemann curvature constructed in a familiar way from the spin connection.

I have not been able to extend the gauge analogy any further. Quantities arising in gravitational dynamics – the Ricci tensor and scalar – involve contracting “space-time” indices with “gauge” indices. This requires using the metric tensor or the Vielbeine \((\text{recall } g_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab})\). But the former is not present in the gauge formalism, while the latter plays the role of “gauge” transformation functions, see \((4.A.8)-(4.A.11)\). Nevertheless, even the limited analogy can be put to good use.
B. Gravitational Chern-Simons term from gauge theory Chern-Simons term

We know how to construct a gauge theoretic Chern-Simons term in three dimensions. Using the gravity-gauge theory dictionary, specifically (1.9) and (4.A.1), leads immediately to the formula for the gravitational Chern-Simons term appropriate either to (2+1)-dimensional space-time or to 3-dimensional space. [24]

\[
W(\Gamma) = \frac{1}{4\pi^2} \int d^3x \varepsilon^{\alpha\beta\gamma} \left( \frac{1}{2} \Gamma^\mu_{\alpha\nu} \partial_\beta \Gamma^\nu_{\gamma\mu} + \frac{1}{3} \Gamma^\mu_{\alpha\nu} \Gamma^\nu_{\beta\omega} \Gamma^\omega_{\gamma\mu} \right) \quad (4.B.1)
\]

Under coordinate transformations, \( \Gamma(\sim A) \) transforms according to (4.A.2)/(4.A.4). Therefore from (1.11) and (1.12) it follows that

\[
W(\bar{\Gamma}) = W(\Gamma) - \frac{1}{24\pi^2} \int d^3x \varepsilon^{\alpha\beta\gamma} < U^{-1} \partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U >, \quad (4.B.2)
\]

where \( U \) is the coordinate transformation matrix (4.A.3) and the last term in (4.B.2) is its winding number. We can restrict these transformations to be sufficiently well-behaved so that there is no winding. Then \( W(\Gamma) \) is a coordinate invariant.

The Christoffel connection is constructed from the metric tensor \( g_{\mu\nu} \). An interesting geometrical quantity emerges when \( W(\Gamma) \) is varied with respect to \( g_{\mu\nu} \). This is carried out in two steps: first vary (4.B.1) with respect to \( \Gamma \), and use (1.10a) (tensor rotation) as well as the dictionary (4.A.1) and (4.A.6). Then vary \( \Gamma \) according to

\[
\delta \Gamma^\mu_{\alpha\nu} = \frac{g^{\mu\sigma}}{2} \left( D_\alpha \delta g_{\sigma\nu} + D_\nu \delta g_{\sigma\alpha} - D_\sigma \delta g_{\alpha\nu} \right). \quad (4.B.3)
\]

Partial integration and the Bianchi identity leave

\[
\delta W(\Gamma) \equiv -\frac{1}{4\pi^2} \int d^3x \delta g_{\mu\nu} \sqrt{g} C^{\mu\nu}, \quad (4.B.4)
\]

where \( C^{\mu\nu} \), called the Cotton tensor, reads

\[
C^{\mu\nu} = \frac{1}{2\sqrt{g}} \left( \varepsilon^{\mu\alpha\beta} D_\alpha R^\nu_{\beta\beta} + \varepsilon^{\nu\alpha\beta} D_\alpha R^\mu_{\beta\beta} \right). \quad (4.B.5)
\]

\( C^{\mu\nu} \) is like a covariant curl of the Ricci tensor \( R^\nu_{\beta\beta} = R^\mu_{\mu\nu} \). [Instead of \( R^\nu_{\beta\beta} \) one can equivalently write in (4.B.5) the Einstein tensor \( G^\nu_{\beta} \equiv R^\nu_{\beta} - \frac{1}{2} \delta^\nu_{\beta} R_{\mu\nu} \).]
$C^{\mu\nu}$ is obviously symmetric and traceless. Because it arises from the variation of the invariant $W(\Gamma)$, $C^{\mu\nu}$ is also covariantly conserved, as can be verified explicitly.

The role of $C^{\mu\nu}$ in 3-dimensional geometry is the following. In four or more dimensions there exists the Weyl tensor, which functions in two ways. It is the non-Ricci part of the Riemann tensor; also it serves as a template for conformal flatness: vanishing if and only if the space is conformally flat. In three dimensions the Riemann tensor only has a Ricci part; the Weyl tensor is absent. [That is why in 3-dimensional Einstein theory there is no curvature external to matter sources, and there are no propagating excitations.]

In the absence of the 3-dimensional Weyl tensor, $C^{\mu\nu}$ replaces it as the conformal template, vanishing if and only if space-time is conformally flat. While these geometric properties of $C^{\mu\nu}$ are ancient knowledge, the fact that it arises from varying the gravitational Chern-Simons term was a new discovery, made possible by the gauge theory-gravity connection. [24]

The gauge theory Chern-Simons term can be added to a 3-dimensional Yang-Mills theory, giving rise to massive but gauge invariant excitations. So also the gravitational Chern-Simons term can supplement 3-dimensional Einstein theory, where it has even more profound consequences. It converts a theory with no propagating excitations into one with a massive propagating mode, all the time maintaining diffeomorphism invariance! The equation of motion with an energy-momentum tensor source reads

$$G^{\mu\nu} + \frac{1}{4\pi^2 m} C^{\mu\nu} = -8\pi G T^{\mu\nu}. \quad (4.B.6)$$

Since $C^{\mu\nu}$ is of one higher derivative order than $G^{\mu\nu}$, the strength parameter $m$ has dimension of mass, and $m$ is also the mass of the propagating degree of freedom in the linearized theory. Thus in the Einstein limit, corresponding to $m \to \infty$, the massive excitation decouples leaving non-propagating degrees of freedom. The quantum theory is well behaved, in spite of the higher derivatives in $C^{\mu\nu}$. Unlike in the gauge theory, coupling constant quantization is not needed, because the gravitational Chern-Simons term is invariant against (well behaved) coordinate transformations.
The gravitational Chern-Simons term $W(\Gamma)$ may also be presented in terms of the spin connection $W(\omega)$. From (1.11), (4.A.7b)/(4.A.10) and (4.B.1) follows

$$W(\Gamma) = W(\omega) - \frac{1}{24\pi^2} \int d^3x \epsilon^{\alpha\beta\gamma} < \mathcal{E}^{-1} \partial_\alpha \mathcal{E} \mathcal{E}^{-1} \partial_\beta \mathcal{E} \mathcal{E}^{-1} \partial_\gamma \mathcal{E} > .$$  \hspace{1cm} (4.B.7)

Although the second term, being the winding number of $\mathcal{E}$, does not contribute to the equations of motion, it should not be dropped; otherwise confusion arises: $W(\omega)$ in Minkowski space is a Chern-Simons term based on the local Lorentz group SO(2.1), which does not support gauge transformations with non-zero winding; $W(\omega)$ is SO(2.1) invariant. However, it is usually believed that field theory may be analytically continued to Euclidean space. In that case $W(\omega)$ is the Chern-Simons term for SO(3) and this does possess non-trivial windings. So it is difficult to understand how $W(\omega)$ could function in a quantum theory: with Lorentzian signature its coefficient can be arbitrary, with Euclidean signature it must be quantized! Which should it be?

Fortunately we need not answer this question, because $W(\omega)$ does not stand alone: It is supplemented by the last term in (4.B.7) whose response to Euclidean rotations must cancel any non-invariance of $W(\omega)$, because together they equal $W(\Gamma)$, which does not feel local rotations.

Similarly to the work by Carroll et al. on Chern-Simons extended electromagnetism, Pi and I added the gravitational Chern-Simons term to 4-dimensional Einstein relativity. Linearized analysis indicates that, contrary to the electromagnetic case, wave propagation is not affected. Another noteworthy feature is that vacuum space-times, which satisfy the modified equations, necessarily possess vanishing gravitational Chern-Pontryagin density, $\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} R^\mu_{\nu\alpha\beta} R_{\mu\nu\gamma\delta} = 0$. Correspondingly, the Schwarzschild space-time remains a solution, but Kerr space-time becomes modified.
C. Coordinate transformations in general relativity and gauge theory

(i) Response to changes in coordinates

In general, fields respond to an infinitesimal coordinate transformation, generated by the vector $f$, through the Lie derivative $L_f$ with respect to $f$.

$$\delta_f x^\mu = -f^\mu(x)$$  \hspace{1cm} (4.C.1)

$$\delta_f (\text{field}) = L_f (\text{field})$$  \hspace{1cm} (4.C.2)

$L_f$ involves ordinary derivatives; for example for a covariant vector the action of $L_f$ is

$$\delta_f V_\alpha = L_f V_\alpha = f^\mu \partial_\mu V_\alpha + \partial_\alpha f^\mu V_\mu$$  \hspace{1cm} (4.C.3a)

while a contravariant vector responds by

$$\delta_f V^\alpha = L_f V^\alpha = f^\mu \partial_\mu V^\alpha - \partial_\mu f^\alpha V^\mu$$  \hspace{1cm} (4.C.3b)

Moreover, when the space possesses a metric structure Lie derivatives of covariant objects (scalars, vectors, tensors etc.) remain covariant; i.e replacing ordinary derivatives by coordinate covariant derivatives produces no change in the formulas.

This is not true in gauge theories. For example the Lie derivative of the field strength

$$L_f F_{\alpha\beta} = f^\mu \partial_\mu F_{\alpha\beta} + \partial_\alpha f^\mu F_{\mu\beta} + \partial_\beta f^\mu F_{\alpha\mu}$$  \hspace{1cm} (4.C.4)

is not gauge covariant because the derivative acting on $F_{\alpha\beta}$ is not gauge covariant. Similarly, the Lie derivative of a vector potential, which follows formulas (4.C.3) ($V \rightarrow A$) is not gauge covariant. Consequently coordinate transformations in a gauge theory, implemented by Lie derivatives, loose gauge covariance.

We now ask: Is it possible to modify the implementation of coordinate transformations in gauge theories so that gauge covariance is preserved? The positive answer that I gave draws on notational analogies between gauge and gravity fields.
Let us record the gravity formulas. Under (4.C.1) and (4.C.2) the metric tensor transforms as

\[ \delta_f g_{\mu\nu} = L_f g_{\mu\nu} = f^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu f^\alpha g_{\alpha\nu} + \partial_\nu f^\alpha g_{\mu\alpha} = D_\mu f_\nu + D_\nu f_\mu \]  

(4.C.5)

The last equality follows from the previous when ordinary derivatives are replaced by covariant derivatives, which also annihilate \( g_{\mu\nu} \). With (4.C.5) we find the response of the Christoffel connection from (4.B.3),

\[ \delta_f \Gamma^\mu_{\alpha\nu} = f^\beta R^\mu_{\nu\beta\alpha} + D_\alpha (D_\nu f^\mu) \]  

(4.C.6)

and furthermore

\[ \delta_f R^\mu_{\nu\alpha\beta} = D_\alpha \delta \Gamma^\mu_{\beta\nu} - D_\beta \delta \Gamma^\mu_{\alpha\nu} = L_f R^\mu_{\nu\alpha\beta}. \]  

(4.C.7)

Both (4.C.6) and (4) exhibit a coordinate covariant response.

When we consider the gauge theoretic analog to (4.C.6), we expect to find that the coordinate transformation of the covariant vector potential, \( i.e. \) its Lie derivative, can be presented analogously to (4.C.6) as the sum of two terms: a projection on the field strength and a total derivative. This is indeed the case, as is readily established by adding and subtracting suitable terms in (4.C.3a) \( (V_\alpha \to A_\alpha) \).

\[ \delta_f A_\alpha = f^\mu (\partial_\mu A_\alpha - \partial_\alpha A_\mu + [A_\mu, A_\alpha]) \]

\[ + f^\mu (\partial_\alpha A_\mu - [A_\mu, A_\alpha]) + \partial_\alpha f^\mu A_\mu \]

\[ = f^\mu F_{\mu\alpha} + D_\alpha (f^\mu A_\mu) \]  

(4.C.8)

It is the last term in (4.C.8) that spoils gauge covariance. We recognize it as an infinitesimal gauge transformation with gauge function of \( f^\mu A_\mu \). But in a gauge theory, gauge transformations can be performed at will. We use this freedom to redefine the response of gauge variables by supplementing the Lie derivatives with the gauge transformation that removes the last term in (4.C.8). Thus the modified, but gauge equivalent response reads

\[ \delta_f A_\alpha = f^\beta F_{\beta\alpha} \]  

(4.C.9)

and (4.C.9) has the consequence that

\[ \delta_f F_{\alpha\beta} = D_\alpha \delta_f A_\beta - D_\beta \delta_f A_\alpha \]

\[ = f^\mu D_\mu F_{\alpha\beta} + \partial_\alpha f^\mu F_{\mu\beta} + \partial_\beta f^\mu F_{\alpha\mu} \]  

(4.C.10)
This is gauge covariant and differs from the usual formula (4.C.4) by a gauge transformation of $F_{\alpha\beta}$ generated by $f^\mu A_\mu$. We may view (4.C.10) as defining a gauge covariant Lie derivative.

We thus achieve the goal of describing coordinate transformation of gauge fields in a gauge covariant manner, with a gauge covariant Lie derivative. But there is a price to pay: The gauge covariant Lie derivatives follow a closure rule that differs from conventional Lie derivatives. The commutator of two gauge covariant Lie derivatives, with respect to two vectors, $f$ and $g$, closes on the gauge covariant Lie derivative with respect to the Lie bracket of $f$ and $g$ plus a gauge transformation generated by $f^\alpha g^\beta F_{\alpha\beta}$.

(ii) **Invariant fields and constants of motion**

To determine whether a generic field is invariant against a coordinate transformation generated by $f$, we check whether its Lie derivative annihilates $\phi$.

\[ L_f \phi = 0 \Rightarrow \text{(invariant field } \phi) \quad (4.C.11) \]

For the metric tensor this is the statement of the Killing equation, see (4.C.5).

\[ \delta f g_{\mu\nu} = L_f g_{\mu\nu} = D_\mu f_\nu + D_\nu f_\mu = 0 \quad (4.C.12) \]

From (4.C.6) and (4.C.7) immediately follow well known conditions on Killing vectors. [37]

\[ f^\beta R^\mu_{\nu\beta\alpha} = -D_\alpha(D_\nu f^\mu) \quad (4.C.13) \]

\[ L_f R^\mu_{\nu\alpha\beta} = 0 \quad (4.C.14) \]

But in a gauge theory a condition weaker than (4.C.11) is appropriate: A coordinate invariant configuration in a gauge theory need not be annihilated by the Lie derivative, rather a gauge transformation may survive. In other words, a gauge field configuration should still be considered as invariant, if any non invariance can be compensated by a gauge transformation. Applying this condition to the gauge covariant transformation law (4.C.9), there emerges a gauge covariant criterion for an invariant gauge field configurations.

\[ f^\beta F_{\beta\alpha} = D_\alpha \Phi_f \Rightarrow \text{invariant gauge field } F_{\alpha\beta} \quad (4.C.15) \]
Here $\Phi_f$ is an unspecified quantity, linear in $f$. This is the gauge theoretic analog to (4.C.13), except that in the gravity formula the quantity corresponding to $\Phi_f$ is specified explicitly as $-D_\nu f^\mu$.

Manton and I showed that $\Phi_f$ generates the gauge transformation needed to compensate any coordinate asymmetry in an invariant gauge field configuration. Therefore $\Phi_f$ also contributes to the conserved constant of motion, which characterizes motion in the presence of such an invariant gauge field. [38]

This is the origin of the celebrated addition to the angular momentum in the field of a Dirac magnetic monopole: The conserved angular momentum comprises the kinematical angular momentum supplemented by the radial unit vector, which is also present in (4.C.15), when $F_{\beta\alpha}$ is the magnetic monopole field and $f$ generates spatial rotations around the axis $a : f^\mu = (0, \mathbf{r} \times a)$, then $\Phi_f = \mathbf{r} \cdot a$ Similarly, the 'tHooft-Polyakov magnetic monopole is spherically symmetric up to an isospin gauge transformation. The angular momentum constant of motion therefore acquires an isospin component $T$ in addition to the kinematical angular momentum $L$.

$$J = L + T$$  \hspace{1cm} (4.C.16)

This is the origin of the previously mentioned conversion of isospin to spin.

References

[1] C.-N. Yang and R.L. Mills, “Conservation of Isotopic Spin and Isotopic Gauge Invariance” Phys. Rev. 96, 191 (1954).

[2] G. ‘tHooft, “The Renormalization Procedure for Yang-Mills Fields”, Ph.D Thesis, Utrecht University (1972); G. ‘tHooft and M. Veltman, “Regularization and Renormalization of Gauge Fields” Nucl Phys. B44, 189 (1972).

[3] J.S. Bell and R. Jackiw, “A PCAC Puzzle: $\pi^0 \to 2\gamma$ in the $\sigma$-model” Nuovo Cim. A60, 47 (1969).
[4] H. Fukuda and Y Miyamoto, “On the $\gamma$-Decay of Neutral Meson” Prog. Theoret. Phys. 4, 347 (1949); J. Steinberger, “On the Use of Subtraction Fields and the Lifetimes of some Types of Meson Decay” Phys. Rev. 76, 1180 (1949); J. Schwinger, “Gauge Invariance and Vacuum Polarization” Phys. Rev. 82, 664 (1951); S. Adler, “Axial Vector Vertex in Spinor Electrodynamics” Phys. Rev. 177, 2426 (1969).

[5] D. Sutherland, “Current Algebra and some Nonstrong Mesonic Decays” Nucl. Phys. B2, 433 (1967); M. Veltman, “Theoretical Aspects of High Energy Neutrino Interactions” Proc. Roy. Soc. London A 301, 107 (1967).

[6] D. Gross and R. Jackiw, “Effect of Anomalies on Quasirenormalizable Theories” Phys. Rev. D 6, 477 (1972).

[7] C. Bouchiat, J. Iliopoulos and P. Meyer, “An Anomaly free Version of Weinberg’s Model” Phys. Lett. B38, 519 (1972).

[8] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, “Pseudoparticle Solutions of the Yang-Mills Equations” Phys. Lett. B59, 85 (1975).

[9] G. 'tHooft, “Symmetry Breaking Through Bell-Jackiw Anomalies” Phys. Rev. Lett. 37, 8 (1976).

[10] R. Jackiw and C. Rebbi, “Vacuum Periodicity in a Yang-Mills Quantum Theory” Phys. Rev. Lett. 37, 172 (1976).

[11] C. Callan, R. Dashen and D. Gross, “The Structure of the Gauge Theory Vacuum” Phys. Lett. B63, 334 (1976).

[12] R. Jackiw and C. Rebbi, “Conformal Properties of a Yang-Mills Pseudoparticle” Phys. Rev D 14, 517 (1976).

[13] R. Jackiw and C. Rebbi, “Degrees of Freedom in Pseudoparticle Systems” Phys. Lett. 67, 189 (1977).
[14] A. Schwartz, “On Regular Solutions of Euclidean Yang-Mills Equations” Phys. Lett. 67B, 172 (1977); M. Atiyah, N. Hitchin and I. Singer, “Deformations of Instantons” Proc. Nat. Acad. Sci. 74, 2662 (1977).

[15] R. Jackiw, C. Nohl and C. Rebbi, “Conformal Properties of Pseudoparticle Configurations” Phys. Rev. D 15, 1642 (1977).

[16] M. Atiyah, N. Hitchen, V. Drinfeld and Y. Manin, “Construction of Instantons” Phys. Lett. A65, 185 (1978).

[17] R. Jackiw and C. Rebbi, “Spinor Analysis of Yang-Mills Theory” Phys. Rev. D 16, 1052 (1977).

[18] G. ’tHooft, “Magnetic Monopoles in Unified Gauge Theories” Nucl. Phys. 79, 276 (1974); A. Polyakov, “Particle Spectrum in the Quantum Field Theory” Zh. Eksp. Teor. Fiz. Pis’ma Red. 20, 430 (1974) [English translation: JETP Lett. 20, 430 (1974)].

[19] R. Jackiw and C. Rebbi, “Spin from Isospin in a Gauge Theory” Phys. Rev. Lett. 36, 1116 (1976).

[20] P. Hasenfratz and G. ’tHooft, “A Fermion-Boson Puzzle in a Gauge Theory” Phys. Rev. Lett. 36, 1119 (1976).

[21] R. Jackiw and C. Rebbi, “Solitons with Fermion Number $\frac{1}{2}$” Phys. Rev. D 13, 3398 (1976); R. Jackiw and J. R. Schrieffer, “Solitons with Fermion Number $\frac{1}{2}$ in Condensed Matter and Relativistic Field Theories” Nucl. Phys. B190, 253 (1981).

[22] C. Callias, “Index Theorems on Open Spaces” Commun. Math. Phys. 62, 213 (1978); R. Bott and R. Seeley, “Some Remarks on the Paper of Callias” Commun. Math. Phys. 62, 245 (1978).

[23] R. Jackiw, “The Yang-Mills Vacuum as a Bloch Wave” APS Spring Meeting, Washington D.C. (1977); reprinted in R. Jackiw, Diverse Topics in Theoretical and Mathematical Physics (World Scientific, Singapore, 1995).
[24] S. Deser, R. Jackiw and S. Templeton, “Three Dimensional Massive Gauge Theories” Phys. Rev. Lett. 48, 975 (1982); “Topologically Massive Gauge Theories” Ann. Phys. 140, 372 (1982), (E) 185, 406 (1988).

[25] R. Jackiw and E Weinberg, “Self-Dual Chern-Simons Vortices” Phys. Rev. Lett. 64, 2234 (1990).

[26] J. Hong, Y. Kim and P.-Y. Pac, “Multivortex Solitons of the Abelian Chern-Simons-Higgs Theory” Phys. Rev. Lett. 64 2330 (1990).

[27] R. Jackiw and S.-Y. Pi, “Soliton Solutions to the Gauged Nonlinear Schrödinger Equation on the Plane” Phys. Rev. Lett. 64, 2969 (1990).

[28] For reviews see R. Jackiw and S.-Y. Pi, “Self-Dual Chern-Simons Solitons” Prog. Theor. Phys. (Kyoto) Suppl. 107, 1 (1992); G. Dunne “Self-Dual Chern-Simons Theories” Lecture Notes in Physics m36 (Springer, Berlin, 1995).

[29] N. Redlich, “Gauge Noninvariance and Parity Violation of Three-Dimensional Fermions” Phys. Rev. Lett. 52, 18 (1984).

[30] B. Holstein, “Anomalies for Pedestrians” Am. J. Phys. 61, 142 (1993).

[31] S. Coleman and R. Jackiw, “Why Dilatation Generators do not Generate Dilatations?” Ann. Phys. 67, 552 (1971).

[32] S. Carroll, G. Field and R. Jackiw, “Limits on a Lorentz and Parity Violating Modification of Electrodynamics” Phys. Rev. D 41, 1231 (1990); S. Carroll and G. Field, “Is there Evidence for Cosmic Anisotropy in the Polarization of Distant Galaxies?” Phys. Rev. Lett. 79, 2394 (1997).

[33] W. Bardeen and B. Zumino, “Consistent and Covariant Anomalies in Gauge and Gravitational Theories” Nucl. Phys. B244, 421 (1984).

[34] Our definitions and conventions for geometrical entities follow S. Weinberg, *Gravitation and Cosmology* (Wiley, New York NY, 1972), except that our Riemann tensor is the negative of his, as is our Lorentzian metric tensor.
[35] R. Jackiw and S.-Y. Pi, “Chern-Simons Modification of Gravity” Phys. Rev. D 68, 104012 (2003).

[36] R. Jackiw, “Gauge Covariant Conformal Transformations” Phys. Rev. Lett. 41, 1635 (1978); “Invariance, Symmetry and Periodicity in Gauge Theories” Acta Phys. Austr. Suppl. XXII, 383 (1980), reprinted in R. Jackiw, Diverse Topics in Theoretical and Mathematical Physics (World Scientific, Singapore 1995).

[37] Formula (4.C.14) is equivalent to the inchoate (13.1.12) in Weinberg [34].

[38] R. Jackiw and N. Manton, “Symmetries and Conservation Laws in Gauge Theories” Ann. Phys. 127, 257 (1980).