Abstract Shot noise processes have been extensively studied due to their mathematical properties and their importance in single neuron modelling, where they are used to model membrane voltages, synaptic input currents and conductances. Since neurons receive a large amount of inputs arriving at high frequency and small jump amplitudes, it is of interest to study if, and under which conditions, a sequence of shot noise admits a diffusion process as limit. Here, we first show that the obtained limit process is a Lévy-driven Ornstein-Uhlenbeck (OU), whose features depend on the underlying jump distributions of the shot noise. Then, we derive the necessary conditions guaranteeing the diffusion limit to a Gaussian OU process, show that they are not met unless allowing for negative jumps happening with probability going to zero, and quantify the error occurred when replacing the shot noise with the OU process. The results offer a new class of models to be used instead of the commonly applied Gaussian OU processes to approximate synaptic input currents, membrane voltages or conductances modelled by shot noise.

Keywords Diffusion processes · Weak convergence · Lévy processes · Lévy-driven Ornstein-Uhlenbeck · Non-Gaussian Ornstein-Uhlenbeck · diffusion approximation

1 Introduction

Shot noise processes, initially proposed to model shot noise in vacuum tubes [3], play a crucial role in single-neuron modelling, where they are known as Stein neuronal model [50] without inhibition and were initially proposed to describe membrane voltage in the framework of Leaky Integrate-and-Fire models [12] [55]. More recently, shot noise processes have been used to describe synaptic input currents in Integrate-and-Fire neurons [15] and Leaky Integrate-and-Fire models [1] [52] [40], as well as conductances in conductance-based models [45]. Since the mathematical formulation of the shot noise process is the same, the results derived in this paper do not depend on the underlying modelling framework, and can therefore be applied to all considered scenarios.

Suppose that events (e.g. jumps representing the excitatory postsynaptic potentials) occur according to a homogeneous Poisson process $N$ with constant rate $\lambda > 0$. Associated with the $k$th event is a positive random variable $J[k]$, quantifying the nonnegative random jump amplitude of $k$th input. Denote by $\tau_k$ the time of the $k$th jump. Assume jumps to be identically distributed, independent of each other and of the point process. Then, the Poissonian shot noise process
\{X(t), t \geq 0\}, or simply shot noise process, is the resulting superposition

\[ X(t) = x_0 e^{-\alpha t} + \sum_{k=1}^{N(t)} j[k] e^{-\alpha (t-t_k)}, \quad X(0) = x_0 \geq 0, \]  

where \( \alpha > 0 \) is a constant determining the exponential decay rate, e.g., the inverse of the membrane time constant when \( X(t) \) models the membrane voltage of a neuron. In general, when the distribution of \( J \) assumes real values, the model is known as Stein model \([50]\) in the neuroscience literature.

Limiting results have been proposed for the drifted, rescaled in time and normalized shot noise process or its generalised versions, see e.g. \([18, 4, 17]\). Here, instead, we are interested in deriving limiting results for the original nonnegative shot noise process \((1)\). From a mathematical point of view, this is commonly done by investigating a sequence of jump processes \((X_n)_{n \in \mathbb{N}} = (\{X_n(t), t \geq 0\})_{n \in \mathbb{N}}\) for which the distribution of the trajectories gets close to that of a limiting process \(Y\).

Depending on how space and time are rescaled, i.e., on the assumptions on the frequencies and on the distributions of the jumps of the underlying stochastic point process, the limit process can be either deterministic, obtained, e.g., as a solution of systems of ordinary/partial differential equations \([19, 20, 21, 22, 23]\), or stochastic \([24, 26, 27]\). Limits of the first type are called fluid limits, thermodynamic limits or hydrodynamic limits, and give rise to what is also known as Kurtz approximation \([20]\), see e.g. \([29]\) for a review.

In this paper, we focus on limits of the second type, dealing with weak convergence of stochastic processes \([24, 26, 31]\), illustrated, e.g., in \([54, 46, 30]\) in the neuronal context. The necessary conditions for the convergence of dimensional processes to diffusion processes are that the limits of the first two infinitesimal moments (also known as Kramers-Moyal coefficients) of the jump process converge to those of the diffusion process, and that the fourth infinitesimal moment goes to zero \([31, 33, 34]\). The vanishing of the fourth infinitesimal moment guarantees that the limit process is continuous \([35]\), a necessary condition for obtaining diffusion processes. In mathematical neuroscience, such approach was often studied by Ricciardi and his coworkers \([25, 47, 28]\), and has been referred to as diffusion approximation \([27, 42]\). A problem arises for the use of several notions of diffusion approximations. For example, an alternative concept of diffusion approximation, called usual approximation or truncated Kramers-Moyal expansion, was for discontinuous models with relatively small jumps and states being far from the boundaries in \([52]\). This method replaces the discontinuous process \(X\) with a diffusion process \(Y\) having the same first two infinitesimal moments. Another notion, which we will refer to as Gaussian approximation or matched synaptic distributions \([30, 39]\), consists in replacing the discontinuous process \(X\) with a Gaussian process \(Y\) having the same first two infinitesimal moments. Since these two notions involve neither limiting results nor infinitesimal moments of order higher than two, the limit process may also be discontinuous, yielding thus a low approximation accuracy. Over the years, the rigorous approach of diffusion approximation by Ricciardi, mathematically supported by the findings of Gikhman and Skorokhod \([31]\), has been unconsciously and unintentionally replaced by the usual and Gaussian approximations in the mathematical and computational neuroscience community, see e.g. \([43, 45, 53]\).

The goal of this paper is to consider a sequence of shot noise processes \((1)\) and investigate if, and under which conditions for the amplitudes and frequencies of the jumps, a sequence of them admits a diffusion process as limit. By studying its weak convergence, we show that the obtained limit process is a (discontinuous) non-Gaussian OU process, also known as Lévy-driven OU or OU Lévy process \([37, 38]\), i.e., an OU process having a non-Gaussian Lévy process as driving noise, as the initial shot noise process. Then, we characterise the limiting Lévy measures which depend on the jump amplitude distributions of the shot noise. We refer to \([39, 41]\) as standard references on Lévy processes. Moreover, we derive the necessary conditions to perform a diffusion approximation and show that these are not simultaneously met. Hence, the Gaussian OU process cannot be obtained as a diffusion approximation of the shot noise process, but only as a usual or Gaussian approximation \([33, 35, 44]\). However, we show that allowing for jumps with negative
amplitudes happening with probability going to 0 is enough to guarantee the weak convergence to a Gaussian OU process as diffusion limit.

On one hand, our results show how the different limit approaches and notions of diffusion approximation may lead to different approximating models. In particular, ignoring the vanishing of the fourth infinitesimal moment results in assuming the limit process to be continuous, while in fact it is not. On the other hand, they may be used to improve existing results on single neuronal models and their corresponding firing statistics, by replacing the membrane voltage, the synaptic input currents or the conductances modelled by the shot noise with the obtained Lévy-driven (and not Gaussian) OU processes. Our findings are not specific for the shot noise process, but can be directly applied to all those models where a diffusion approximation involving sequences of nonnegative and/or nonpositive random variables is commonly used, e.g. neuronal models with synaptic reversal potentials [46], as previously observed in [56].

2 Poissonian shot noise

Throughout, we consider a shot noise process (1) whose random jump amplitudes \( J[k], k \in \mathbb{N} \) are independent and identically distributed random variables, independent on the Poisson process \( N(t) \). We denote by \( J \) the distribution of the jumps, with cumulative distribution function (cdf) \( F_J \). The shot noise process (1) is obtained as the solution of the stochastic differential equation

\[
dX(t) = -\alpha X(t)dt + JdN(t), \quad X(0) = x_0 \geq 0,
\]

and has state space \([0, \infty)\). Denote by

\[
L(t) = \sum_{k=1}^{N(t)} J[k],
\]

a compound Poisson process with \( L(0) \equiv 0 \), i.e. a Lévy process with finite (bounded) Lévy measure \( \lambda F_J(dx) \), drift 0 and no diffusion component (the Gaussian part). The characteristic function of \( L(t) \) becomes

\[
\mathbb{E}[e^{iuL(t)}] = \exp \left( t\lambda \int (e^{iux} - 1) F_J(dx) \right), \tag{3}
\]

while that of \( X_t \) is

\[
\mathbb{E}[e^{isX(t)}] = \exp \left( \lambda \int_0^t (e^{isxe^{-\alpha y}} - 1)dy F_J(dx) + isx_0e^{-\alpha t} \right). \tag{4}
\]

The mean, covariance and variance of the shot noise process are given by [36]

\[
\mathbb{E}[X(t)] = \frac{\lambda}{\alpha} \mathbb{E}[J](1 - e^{-\alpha t}) + x_0e^{-\alpha t}, \tag{5}
\]

\[
\text{Cov}(X(t), X(t+s)) = \frac{\lambda}{2\alpha} e^{-\alpha s} \mathbb{E}[J^2](1 - e^{-2\alpha t}), \tag{6}
\]

\[
\text{Var}(X(t)) = \frac{\lambda}{2\alpha} \mathbb{E}[J^2](1 - e^{-2\alpha t}). \tag{7}
\]

The fourth moment of \( X \) can be obtained from the characteristic function, yielding

\[
\mathbb{E}[X^4(t)] = \mathbb{E}^4[X(t)] + 6\mathbb{E}^2[X(t)]\text{Var}(X(t)) + 3\text{Var}^2(X(t)) + \frac{4\lambda}{3\alpha} \mathbb{E}[X(t)]\mathbb{E}[J^3](1 - e^{-3\alpha t}) + \frac{\lambda}{4\alpha} \mathbb{E}[J^4](1 - e^{-4\alpha t}).
\]
3 Lévy-driven OU process as limit model of the shot noise

Let us consider a sequence of compound Poisson processes \((L_n)_{n \in \mathbb{N}}\) with PSPs having jump distribution \(F_{J_n}\) under the assumptions that

\[
\lim_{n \to \infty} \lambda_n = \infty \quad (8)
\]

and

\[
\lambda_n F_{J_n}(dx) \xrightarrow{\mathcal{L}} \nu(dx), \quad (9)
\]

where \(\nu\) is an unbounded Lévy measure and the convergence in law \(\mathcal{L}\) in (9) is defined such that

\[
\lambda_n \int h(x) F_{J_n}(dx) \to \int h(x) \nu(dx), \quad n \to \infty,
\]

for all bounded and continuous functions \(h\), differentiable at \(x = 0\) with \(h(0) = 0\). Then, we have the following

**Theorem 1** Under conditions (8) and (9), the sequence of shot noise processes \((X_n)_{n \in \mathbb{N}}\) converges weakly to a nonnegative Lévy-driven OU process \(Y = \{Y(t), t \geq 0\}\) given as the solution of the stochastic differential equation

\[
dY(t) = -\alpha Y(t) dt + L_\infty(t), \quad Y(0) = x_0^n \quad (10)
\]

where \((L_n)_{n \in \mathbb{N}}\) converge weakly to \(L_\infty(t)\), i.e., \(L_n \xrightarrow{\mathcal{L}} L_\infty\).

**Proof** The proof is reported in Appendix A.

Hence, in the presence of jumps happening at high frequency with amplitudes going to zero, the shot noise process (modeling e.g. the membrane voltage, the synaptic current and the conductance) may be replaced by a Lévy-driven OU process. The limiting Lévy measures obtained under different jump distributions are reported in Table 1. If the jumps \(J_n\) are Bernoulli or Poisson distributed, the resulting Lévy measure is that of a Poisson process (PP); if the jumps are \(\chi^2\) or gamma distributed, the resulting Lévy measure is that of a gamma process (GP); if the jumps are inverse Gaussian distributed, the resulting Lévy measure is that of an inverse Gaussian process (IGP), while beta distributed jumps with \(\mu = 1\) yield the Lévy measure of a Beta process. In particular, we obtain

1. Poisson process (PP) with Lévy measure \(\nu(x) = \mu \delta(x - 1)\), where \(\delta\) denotes the Dirac delta function. This is a PP with jump 1 and intensity \(\mu\).
2. Gamma process (GP) with Lévy measure \(\nu(x) = \beta x^{-1} e^{-\gamma x}\) concentrated on \((0, \infty)\) and Gamma distributed increments with scale \(\gamma\) and shape \(\beta\).
3. Inverse Gaussian process (IGP) with \(\nu(x) = s e^{-v^2 x^2/2} / \sqrt{2\pi x^3}\), concentrated on \((0, \infty)\) and Inverse Gaussian distributed increments with mean \(s/b\) and shape \(s^2\).
4. Beta process (BP) with Lévy measure \(\nu(x) = \beta x^{-1} (1 - x)^{\beta - 1}\).

If \(J_n\) are \(\chi\), generalised gamma, beta and beta prime distributed, the derived Lévy measures do not correspond to a known process (results not shown). The fact that \(\chi^2\) distributed jump amplitudes lead to the Lévy measure of a Gamma process may be explained with the fact that the \(\chi^2\) distribution is a special case of a gamma distribution.

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1. If the original and the limit process do not start from the same position, we require that \(\lim_{n \to \infty} x_0^n = y_0\) and \(Y(0) = y_0\), as done, for example, in [30].
4 Diffusion approximation of the shot noise process

The shot noise process would converge weakly to a Gaussian limit if \( L_n \xrightarrow{L} B \), a standard Brownian motion, which would require

\[ \mathbb{E}[e^{isL_n(t)}] \rightarrow e^{-ts^2/2}. \]

Alternatively, we may rely on the conditions for the weak convergence of one dimensional jump processes to diffusion processes provided by [27, 33, 31]. For a sequence of processes \((X_n)_{n \in \mathbb{N}}\), denote by \( \Delta X_n(t) \) the increment in \((t, t+\Delta)\), i.e. \( \Delta X_n(t) := X_n(t+\Delta) - X_n(t) \). The \( k \)th infinitesimal moment of \( X_n(t) \), denoted by \( M_{k,X_n}(x) \), \( k \in \mathbb{N} \), is defined by

\[ M_{k,X_n}(x) := \lim_{\Delta t \to 0^+} \frac{\mathbb{E}[(\Delta X_n(t))^k|X_n(t) = x]}{\Delta t}. \]

An analogous definition holds for \( \Delta Y(t) \) and \( M_{k,Y}(x) \), for the process \( Y \). The conditions for the weak convergence to a diffusion process are those proposed in [27, 33, 31], that we now report for simplicity.

**Theorem 2 (Theorem from [31])** A diffusion process \( Y \) starting in \( Y(0) = x_0 \) is the diffusion approximation of a sequence of jump processes \((X_n)_{n \in \mathbb{N}}\) starting in \( x_{0n} = x_0 \) if the following conditions are met

\[
\lim_{n \rightarrow \infty} M_{1,X_n}(x) \rightarrow M_{1,Y}(x) < \infty. \tag{12}
\]
\[
\lim_{n \rightarrow \infty} M_{2,X_n}(x) \rightarrow M_{2,Y}(x) > 0. \tag{13}
\]
\[
\lim_{n \rightarrow \infty} M_{4,X_n}(x) \rightarrow 0, \tag{14}
\]

for all \( x \) in the state space of \( X_n \).

It is well known that diffusion processes have all infinitesimal moments of order higher than two null [34]. At the same time, Pawula theorem [35] states that if the infinitesimal moments \( M_k \) of a stochastic process exist for all \( k \in \mathbb{N} \), the vanishing of any even order infinitesimal moment larger than two implies \( M_k = 0 \) for \( k \geq 3 \). This has two key consequences. First, if a process has a finite number of nonzero infinitesimal moments, this number is at most two. Second, if the fourth infinitesimal moment goes to zero, then all other \( M_k, k \geq 3 \) goes to zero, motivating condition (14).

4.1 Conditions for the diffusion approximation of the shot noise

Throughout, we assume that \( x_{0n} = x_0 = y_0 \). Replacing \( X \) with \( X_n \) in (2), we get

\[ \Delta X_n(t) = -\alpha X_n(t) \Delta t + J_n \Delta N_n(t), \tag{15} \]

where \( \Delta N_n(t) \) is the increment of the Poisson process in \((t, t+\Delta)\). The first, second and fourth infinitesimal moments of \( X_n(t) \) are given by

\[
M_{1,X_n}(x) = -\alpha x + \lambda_n \mathbb{E}[J_n], \tag{16}
\]
\[
M_{2,X_n}(x) = \lambda \mathbb{E}[J_n^2], \tag{17}
\]
\[
M_{4,X_n}(x) = \lambda_n \mathbb{E}[J_n^4], \tag{18}
\]

where we used the fact that \( \mathbb{E}[(\Delta N_n(t))^4] = \lambda_n \Delta t [1 + 7\lambda_n \Delta t + 6(\lambda_n \Delta t)^2 + (\lambda_n \Delta t)^3] \) in the calculation of \( M_{4,X_n} \). The diffusion regime requires

\[
\lim_{n \rightarrow \infty} \mathbb{E}[J_n] = 0, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty. \tag{19}
\]
For the shot noise process, using the infinitesimal moments of \( X_n \) given by (16)–(18), we see that the conditions (12)–(14) guaranteeing the diffusion approximation are given by

\[
\lim_{n \to \infty} \lambda_n E[J_n] = \mu < \infty, \tag{20}
\]

\[
\lim_{n \to \infty} \lambda_n E[J_n^2] = \sigma^2, \quad \sigma > 0, \tag{21}
\]

\[
\lim_{n \to \infty} \lambda_n E[J_n^4] = 0, \tag{22}
\]

which are equivalent to

\[
E[J_n] = O(\lambda_n^{-1}), \quad E[J_n^2] = O(\lambda_n^{-1}), \quad E[J_n^4] = O(\lambda_n^{-1-\eta}), \eta > 0, \tag{23}
\]

where \( a_n \in O(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = l > 0 \). That is, the mean and the second moment of the jump amplitudes \( J_n \) should go to zero as fast as \( \lambda_n \) goes to infinity, while the fourth moment should go to zero faster than \( 1/\lambda_n \).

**Remark 1** As required in (21), the limit of the second infinitesimal moment should not be zero, otherwise the limit process will be deterministic.

**Remark 2** Conditions (20), (21), (22) under (19) are the same as (3.9), (3.11), (3.18) under (2.10) for the diffusion approximation of a jump process with synaptic reversal potential in [46].

**Remark 3** If the conditions are fulfilled, the diffusion limit of the sequence of shot noise processes is a Gaussian OU process starting in \( y_0 = x_0 = x_0 \) with mean, covariance and variance given by

\[
E[Y(t)] = \frac{\mu}{\alpha}(1 - e^{-\alpha t}) + y_0 e^{-\alpha t}, \tag{24}
\]

\[
\text{Cov}(Y(t), Y(t+s)) = \frac{\sigma^2}{2\alpha} e^{-\alpha t}(1 - e^{-2\alpha t}), \tag{25}
\]

\[
\text{Var}(Y(t)) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \tag{26}
\]

This diffusion process, if existing, would coincide with that obtained by the usual and the Gaussian approximations.

**Conjecture 1** A sequence of nonnegative random variables \((J_n)_{n \in \mathbb{N}}\) satisfying the conditions (20)–(22) under (19) does not exist.

Hence, unless the jump amplitudes do not depend on \( n \) [13], the OU process, obtained via the usual and Gaussian approximations, cannot be obtained as diffusion approximation of the shot noise process. In particular, when (19), (20), (21) are satisfied, (22) is not fulfilled, i.e. the fourth infinitesimal moment does not vanish, unless \( J_n \) assumes negative values, as shown in the following.

### 4.2 Examples

We now consider different families of jump distributions, discuss the consequences of violating the vanishing of the fourth infinitesimal moment and investigate the errors when replacing the shot noise with the Gaussian OU process.

#### 4.2.1 The exponential distribution

Conditions (20) and (21) cannot be violated, otherwise the limit process would either have a mean going to infinity or be a deterministic process, respectively. The latter is what happens if \( J_n \) is exponentially distributed with mean \( \theta_n \). Indeed, if \( \lambda_n E[J_n] = \lambda_n \theta_n \to \mu \), then \( \lambda_n E[J_n^2] = 2\lambda_n \theta_n^2 \to 0 \) as \( n \to \infty \). Hence, a shot noise process with exponential distributed jumps yields a deterministic and not a diffusion process, differently to what is stated in [43] [45] [48]. This may explain why the considered diffusion approximation misrepresents the subthreshold voltage distribution for certain types of synaptic drive, see, e.g. [45], or the lack of fit of the derived firing statistics [43] [48]. Similar results hold when considering jumps to be lognormal distributed (results not shown).
Table 1 List of the nonnegative jump amplitude distributions yielding non-null first two infinitesimal moments, i.e., satisfying conditions (20), (21) under (19). For all jump distributions, \( \lim_{n \to \infty} \lambda_n \mathbb{E}[J_n^2] = \lambda_n \mathbb{E}[J_n^2] = M_2(\gamma)(x) \) and \( \lim_{n \to \infty} \lambda_n \mathbb{E}[J_n^2] = \lambda_n \mathbb{E}[J_n^2] = M_4(\gamma)(x) \).

| Distribution | Parameters | \( \lim_{n \to \infty} \lambda_n \mathbb{E}[J_n^2] = M_2(\gamma)(x) \) | \( \lim_{n \to \infty} \lambda_n \mathbb{E}[J_n^2] = \lambda_n \mathbb{E}[J_n^2] = M_4(\gamma)(x) \) | \( \lambda_n F_{J_n}(dx) \xrightarrow{\mu} \nu(dx) \) |
|-------------|------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| Bernoulli   | \( p = \frac{1}{2} \) | \( \mu \) | \( \mu \) | \( \mu \) (PP) |
| Poisson     | \( \mu \) | \( \mu \) | \( \mu \) | \( \mu \) (PP) |
| \( \chi^2(\mu) \) | \( \mu \) | \( 2\mu \) | \( 48\mu \) | \( \frac{1}{2} x^{-1} e^{-x^2/2} (GP) \) |
| \( \chi^2(\mu, \lambda) \) | \( \beta = \frac{\mu^2}{\lambda^2} \) | \( \sigma^2 \) | \( 6\sigma^2 \) | \( \frac{x^2 e^{-x^2/\sigma^2}}{2\pi \sigma^2} \) (IGP) |
| Inverse Gaussian | mean = \( \frac{\mu}{\lambda} \) | \( \sigma^2 \) | \( \frac{\mu^2}{\lambda^2} \) | \( \sqrt{\pi \sigma^2} \) |
| Beta (0,1) support \( (\alpha, \beta) \) | \( \alpha = \frac{\mu}{\beta} \) | \( \frac{\mu^2}{\beta^2} \) | \( \frac{6\mu}{(\beta + 1)(\beta + 2)(\beta + 3)} \) | \( \frac{\beta x^{-1}(1 - x)^{\beta - 1}}{\nu} \) (BP if \( \mu = 1 \)) |

4.2.2 Results when the fourth infinitesimal moment does not vanish

In Table 1, we report a list of discrete and continuous nonnegative random variables \( J_n \) satisfying conditions (20), (21) under (19). Other random variables fulfilling these requirements are the \( \chi \), the generalised gamma and the beta prime distribution (results not shown). We provide both the distribution parameters and the resulting first, second and fourth infinitesimal moments. Since none of the fourth infinitesimal moment vanishes, condition (22) is not fulfilled, meaning that the limit process is not a diffusion. However, the fourth infinitesimal moment can be made arbitrarily small by letting \( \mu \to \infty \) if the jumps are gamma, inverse Gaussian or beta (by setting, e.g., \( \beta = \mu/\sigma^2 \)) distributed. Intuitively, if \( \mu \) increases, the probability that the OU process at time \( t \) assumes negative values decreases, reducing thus the discrepancy between the state space of the original and the limit process, improving thus the quality of the approximation. For the other considered jump distributions, the fourth infinitesimal moment cannot vanish, otherwise all infinitesimal moments would vanish, yielding a degenerate limit in a point. Finally, if \( J_n \) is beta distributed, one of its underlying parameter can be arbitrarily chosen in a way such that, for fixed \( \mu \) and \( \sigma^2 \), it yields the smallest fourth infinitesimal moment among the considered distributions.

4.2.3 Error caused by replacing the shot noise with the Ornstein-Uhlenbeck process

To investigate the role played by the jump amplitude distributions on the quality of the approximation of the shot noise with the OU process, we proceed as follows. If the jumps are Bernoulli, Poisson, \( \chi^2 \), gamma or inverse Gaussian distributed, using the package CharFun in the computing environment R [19], we numerically invert the characteristic function of \( X_n(t) \), to obtain both its probability density function (pdf) and cdf. The results agree with those obtained performing Monte Carlo simulations of \( N = 10^6 \) values of \( X_n(t) \), see Figure 1. Since the proposed numerical approach is less computationally expensive than running simulations, we will focus on it. If \( J_n \) are \( \chi \), generalised gamma, beta or beta prime distributed, their characteristic function, and thus that of \( X_n(t) \), either involves special functions (as for the \( \chi \), beta or beta prime distributions) or it is not available in close form, making the proposed numerical approach either not reliable or not possible. In that case, we will rely on Monte Carlo simulations. A R-package called shotnoise will be made publicly available on github at \( \text{https://github.com/massimilianotamborrino/shotnoise} \) upon publication. The package provides the numerical evaluation of the pdf and cdf of \( X_n(t) \) and the codes for running Monte Carlo simulations for all considered jump distributions.

To measure the error when approximating the shot noise process \( X_n(t) \) with the OU process \( Y(t) \) (normally distributed with mean and variance given by (24) and (26), respectively), we

\footnote{The choice of \( \beta \) guarantees that \( M_2(\gamma) \) converges to \( \sigma^2 \) as \( \mu \to \infty \).}
Fig. 1 Density of $X_n(t)$ obtained through $10^6$ Monte Carlo simulations (black lines) and by numerically inverting its characteristic function (grey lines) if $J_n$ is Bernoulli (left panel), Poisson (second left panel), $\chi^2$ (mid panel), gamma (second right panel) or inverse Gaussian distributed (right panel), with parameters given in Table 1 for $t = 5, \mu = 10, \sigma^2 = 3, \Lambda = 1000, \alpha = 1, x_0 = 0$.

Consider the integrated absolute error (IAE) defined as

$$\text{IAE}(f_{X_n(t)}, f_{Y(t)}) := \int_0^\infty |f_{X_n(t)}(x) - f_{Y(t)}(x)| dx.$$ 

In Figure 2, we report this error as a function of $t$ (left figure) and $\mu$ (right figure) for different jump distributions and $x_0 = x_0 = y_0 = \mu/\alpha$. The quality of the approximation improves if either $\mu$ or $t$ increase, while it decreases if $\sigma^2$ increases and $J_n$ depends on it (figure not shown). This can be explained as follows. While the shot noise is always nonnegative, the probability of the OU of being nonnegative, knowing that its state space is $\mathbb{R}$, is increasing in $t$ and $\mu$ and decreasing in $\sigma^2$, being given by

$$P(Y(t) \geq 0) = 1 - \Phi\left(\frac{\mu(1 - e^{-\alpha t})}{\sqrt{\sigma^2\alpha(1 - e^{-2\alpha t})}}\right),$$

where $\Phi(x)$ denotes the cdf of a standard normal distribution. Among the considered jump distributions of $J_n$, the Bernoulli and the Poisson yield similar IAEs, which are smaller than those from the other distributions unless $t$ is small and $\mu$ is large (cf. Figure 2, right panel), when the gamma distribution yields the lowest IAE. Hence, choosing a jump distribution yielding a fourth infinitesimal moment lower than another, does not necessarily guarantee a smaller IAE, as it can be observed by comparing, for example, the results from the inverse Gaussian and the Bernoulli distributions (cf. Table 1). Finally, note that, despite the yielded fourth infinitesimal moment is $4$, the IAEs of the Bernoulli and the Poisson distributions may differ, see for example Figure 2 right panel for $t = 2$ and $\mu \in (0, 4)$.

4.3 Results for negative jump amplitudes

Throughout this section, we relax the assumption of having nonnegative jumps, allowing $J_n$ to take negative values, but with probability going to 0 as $n \to \infty$. Let $J_n$ be defined by

$$J_n = \begin{cases} J_n^+ & \text{with } P(J_n = J_n^+) = 1 - f_n; \\ J_n^- & \text{with } P(J_n = J_n^-) = f_n, \end{cases}$$

where $J_n^+, J_n^-$ are a nonnegative and a nonpositive random variables, respectively, and $f_n$ is a function in $(0, 1)$ such that $f_n \to 0$ as $n \to \infty$ in the sense of pointwise convergence of a function. Nonnegative jump distributions can be immediately recovered by setting $f_n = 0$. If $f_n \in (0, 1)$, negative jumps happen with probability going to 0 as $n \to \infty$. Nevertheless, this is enough to guarantee that the $J_n$ defined by (27) fulfils conditions (19), (22), yielding thus the OU process with mean (24) and variance (26) as the limit process of $X_n(t)$. We prove this result for specific $J_n$ and $f_n$, but the theorem and the limit OU process hold for other suitable choices of $J_n$ and $f_n$. 

Fig. 2 IAE\(I_{\text{AE}}(f_X(t), f_Y(t))\) as a function of \(t\) (left figure) and \(\mu\) (right figure) when \(\alpha = 1, \sigma^2 = 3\) and the jump amplitude \(J_n\) is Bernoulli (red lines), Poisson (green lines), \(\chi^2\) (blue lines), gamma (light blue lines) or IG distributed with parameters as in Table 1 and \(\lambda = 100\) (dashed lines) or 100000 (solid lines). Left figure: \(\mu = 10\). Right figure: \(t = 2\).

**Theorem 3** Let \(J_n^-\) be a univariate degenerate distribution assuming only the value \(\theta_n < 0\), with pdf, cdf, \(k\)-moment, \(k \in \mathbb{N}\) and variance given by

\[
 f_{J_n^-}(x) = \delta(x - \theta_n), \quad F_{J_n^-}(x) = \mathbb{I}_{(x \geq \theta_n)}, \quad \mathbb{E}[(J_n^-)^k] = \theta_n^k \quad \text{Var}(J_n^-) = 0,
\]

where \(\mathbb{I}_A\) denotes the indicator function of the set \(A\). Let \(J_n^+\) be gamma distributed, with shape parameter \(\sigma^2 \lambda_n^{-1+2\gamma}\) and scale parameter \(\lambda_n^{-\gamma}\) for \(\sigma > 0, \gamma \in (0, 1/4)\). Consider \(f_n = \lambda_n^{-1+3\gamma}\) and \(\theta_n = \lambda_n^{-2\gamma}(-\sigma^2 + \mu \lambda_n^{-\gamma})\). Then, conditions (20), (21) and (22) are fulfilled.

**Proof** The proof is reported in Appendix B.

This result can be explained as follows. Since \(J_n\) can assume negative values, we could rewrite the stochastic differential equation (2) as

\[
 dX_n(t) = -\alpha X_n(t)dt + E_n dN_n^+(t) + I_n dN_n^-(t),
\]

where \(E_n = J_n^+, I_n = J_n^- = \theta_n < 0\) and \(N_n^+(t)\) and \(N_n^-(t)\) are Poisson processes with intensities \(\lambda_n(1 - f_n)\) and \(\lambda_n f_n\), respectively. This process corresponds to a shot noise process with real valued shot effects, known as generalised Stein model [50], with random jump amplitudes \(E_n\) and \(I_n\) modelling the excitatory and the inhibitory components, respectively, and intensities

\[
 \lambda_n(1 - f_n) \to \lambda_n \to \infty \quad \lambda_n f_n = \lambda_n^{3\gamma} \to \infty, \quad \text{as } n \to \infty.
\]

Therefore, despite the probability of having negative jumps vanishes since \(f_n \to 0\) as \(n \to \infty\), the frequency of the negative jumps explodes, \(\lambda_n f_n \to \infty\), enabling the diffusion limit. The weak convergence of a sequence of Stein processes with constant jump amplitudes to an OU process have been already shown in [51], and it can be seen as a direct consequence of [2] for a shot effect assuming real values.
5 Discussion

Poissonian shot noise processes are used to model single neuronal membrane potential, synaptic input currents impinging on the neuron and conductances in single neuron modelling. When the positive jumps (excitatory inputs) impinge on the neuron with frequency going to infinity and jump amplitudes going to zero, diffusion approximations are commonly considered to replace them with Gaussian OU processes [43, 44, 45, 48], even when the jump amplitudes are exponential distributed.

If a sequence of jump processes converges weakly to a diffusion process, the notions of weak convergence, diffusion approximation and usual approximation yield the same limit diffusion process, which coincides with that from the Gaussian approximation if the diffusion process is Gaussian. However, if the limit process obtained via the weak convergence approach is not continuous, as it happens here for the shot noise, only the notion of diffusion approximation detects this via a non-vanishing fourth infinitesimal moment, yielding a warning in the choice of approximating the initial process by a diffusion. While the conditions guaranteeing the weak convergence of stochastic process may be too impractical or technical to be investigated, when dealing with one-dimensional processes, checking the convergence of the first two infinitesimal moments and the vanishing of the fourth [27], represents an analogous [31], intuitive and powerful tool which could be adopted when performing diffusion approximations, improving the reliability and quality of the approximation.

Here, we prove that the Poissonian shot noise converges weakly to a Lévy-driven non-Gaussian OU process. From a modelling point of view, the derived process could then be used to replace the shot noise process modelling membrane voltages, synaptic input currents or conductances, improving the existing results on single-neuron modelling and their firing statistics based on the Gaussian OU process as a result of the usual and Gaussian approximations. A qualitative study of the introduced improvement needs to be carried over.

The lack of a limiting diffusion process is not specific for the shot noise process. Similar results hold for all models involving only nonnegative and/or nonpositive random variables for the jumps, e.g. neuronal models with synaptic reversal potentials, see e.g. [56]. For example, the conditions guaranteeing the diffusion approximation of the jump model in [46] (cf. Theorem 1) are not met, meaning that the provided diffusion process cannot be obtained via a diffusion approximation, but only as a usual approximation.

Several generalisations of [1] have been proposed in the literature, e.g. non-Poisson inputs, non-renewal dynamics, non-stationary dynamics, general shot effect \( g(t) \) instead of the considered \( g(t) = J[k]e^{-\alpha t} \), all under the name of generalised shot noise process [2], or the recently-proposed random process with immigration [4]. All are characterised by being piecewise-deterministic Markov processes, also known as stochastic hybrid systems, i.e. processes with deterministic behavior between jumps [5]. Besides neuroscience, these processes have been used to model various phenomena in several areas of applications, e.g. anomalous diffusion in physics, earthquakes occurrences in geology, rainfall modelling in meteorology, network traffic in computer sciences, insurance, finance; see [4] and references therein for an exhaustive overview. Depending on the underlying conditions for the generalised shot noise processes (e.g. the shot effects , and thus the shot noise themselves, are commonly assumed to assume values in \( \mathbb{R} \), see e.g. [2, 11, 17]) and on how they are drifted, rescaled in time and normalized, functional central limit theorems have been proved, which have Gaussian processes [2, 3, 7, 8], self-similar Gaussian processes [9], infinite-variance stable processes [10], fractional Brownian motion [11], stable (non-Gaussian) processes [4, 12, 13, 15], stationary [16] or non-stationary [17] stochastic processes as limits. Our result contributes to this discussion, under the assumption of a nonnegative and non drifted shot noise process.

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Appendix

A Proof of Theorem 1

Under conditions (5) and (6), the characteristic function of \( L_n \) computed from (4) converges weakly to

\[
\mathbb{E}[e^{itL_n(t)}] = \exp \left( t \int_0^\infty (e^{i\alpha x} - 1) \nu(dx) \right).
\]

Hence, \( L_\infty \) is a Lévy process with initial value 0, drift 0, no Gaussian part and Lévy measure \( \nu \). Since \( X_n \) is a continuous functional of \( L_n \), see (2), the weak convergence of \( L_n \) implies the weak convergence of \( X_n \) for the continuous mapping theorem. In particular,

\[
\mathbb{E}[e^{itX_n(t)}] = \exp \left( \lambda_n \int_0^\infty (e^{i\alpha x} - 1) dF_{J_n}(dx) + isx_0 e^{-at} \right)
\]

\[
\to \exp \left( \int_0^\infty (e^{i\alpha x} - 1) d\nu(dx) + isx_0 e^{-at} \right)
\]

guarantees that the limit process of \( X_n \) is \( Y \) given by (10), as \( n \to \infty \).

B Proof of Theorem 3

Since \( J_n^+ \) is gamma distributed with shape parameter \( \alpha = \sigma^2 \lambda_n^{-1+2\gamma} \) and scale parameter \( \beta = \lambda_n^{-\gamma} \), with \( \sigma > 0, \gamma \in (0,1/4) \), its moments are given by

\[
\mathbb{E}[J_n^+] = \sigma^2 \lambda_n^{-1+\gamma},
\]

\[
\mathbb{E}[(J_n^+)^2] = \sigma^2 \lambda_n^{-1}(1 + \sigma^2 \lambda_n^{-1+2\gamma}),
\]

\[
\mathbb{E}[(J_n^+)^3] = \sigma^2 \lambda_n^{-1-3\gamma}(1 + \sigma^2 \lambda_n^{-1+2\gamma})(2 + \sigma^2 \lambda_n^{-1+2\gamma})(3 + \sigma^2 \lambda_n^{-1+2\gamma}).
\]

Set \( J_n^- = \theta_n = \lambda_n^{-2\gamma}(1 - \sigma^2 + \mu \lambda_n^{-\gamma}) \) and \( f_n = \lambda_n^{-1+3\gamma} \) and consider \( J_n \) given by (27). Then

\[
\lambda_n \mathbb{E}[J_n] = \mu - \sigma^2 \lambda_n^{-1+4\gamma} = \mu + o(\lambda_n^{-1+4\gamma}),
\]

\[
\lambda_n \mathbb{E}[(J_n^+)^2] = \sigma^2 + o(\lambda_n^{-1+3\gamma}) + o(\lambda^{-\gamma}),
\]

\[
\lambda_n \mathbb{E}[(J_n^+)^3] = o(\lambda_n^{-3\gamma}),
\]

where \( o(\lambda_n^{-\alpha}) \) denotes the terms of order less than \( \lambda_n^{-\alpha} \) as \( n \to \infty \). Since \( \lambda_n \to \infty \) as \( n \to \infty \), having \( \gamma \in (0,1/4) \) guarantees that conditions (28), (29) are satisfied.

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