Open group transformations within the Sp(2)-formalism

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Abstract

Previously we have shown that open groups whose generators are in arbitrary involutions may be quantized within a ghost extended framework in terms of the nilpotent BFV-BRST charge operator. Here we show that they may also be quantized within an Sp(2)-frame in which there are two odd anticommuting operators called Sp(2)-charges. Previous results for finite open group transformations are generalized to the Sp(2)-formalism. We show that in order to define open group transformations on the whole ghost extended space we need Sp(2)-charges in the nonminimal sector which contains dynamical Lagrange multipliers. We give an Sp(2)-version of the quantum master equation with extended Sp(2)-charges and a master charge of a more involved form, which is proposed to represent the integrability conditions of defining operators of connection operators and which therefore should encode the generalized quantum Maurer-Cartan equations for arbitrary open groups. General solutions of this master equation are given in explicit form. A further extended Sp(2)-formalism is proposed in which the group parameters are quadrupled to a supersymmetric set and from which all results may be derived.

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1 Introduction

Open groups in which the generators are in arbitrary involutions constitute a very general class of continuous groups. The only framework in which one may hope to treat them systematically in quantum theory is within a ghost extended BRST frame. This is because the quantum generators and their algebra may always be represented in terms of a BRST charge constructed according to the BFV-prescription \[1\]. The quantization is then consistent if the corresponding BRST charge operator is nilpotent, which always is possible to achieve for finite number of degrees of freedom \[2\]. In two recent papers we have developed the technique to integrate these arbitrary involutions and to construct finite group transformations within the operator BFV-BRST frame \[3, 4\]. The most basic new equation we found was a quantum master equation involving an odd extended nilpotent BFV-BRST charge and an even master charge \[3\]. This master equation is naturally defined in terms of the quantum antibrackets defined in \[3, 6\]. It encodes all integrability conditions for defining operators of the quantum connections, which in turn were shown to encode generalized Maurer-Cartan equations. In \[4\] the formalism was further developed. In particular we found an extended framework with more ghosts from which all properties could be extracted. Furthermore, we gave an explicit form of the master charge that satisfies the master equation.

Since it is possible to embed generators in arbitrary involutions also into two anticommuting nilpotent odd operators, the BRST and the antiBRST charges \[7\], there should exist a generalization of the above results to this case. In gauge theories the BRST and the antiBRST formulation has been very useful and powerful. (There is also a corresponding Lagrangian formulation of gauge theories which have been treated in many papers \[8\].) Here we shall consider an Sp(2)-version in which the BRST and antiBRST charges are cast into a charge with an Sp(2)-index. A first attempt to generalize the above results for open group transformations to the Sp(2)-formalism was given in \[9\]. There we showed that there is an Sp(2)-valued connection operator whose integrability conditions are determined by an Sp(2)-valued quantum master equation in direct analogy with the one of \[3\]. This master equation was also shown to encode generalized Maurer-Cartan equations. However, the resulting finite group transformations were not defined on the whole ghost extended space in this construction. In this paper we propose a new more genuine Sp(2)-version of \[3\] in which the open group transformations are defined on the entire extended space. As we shall see this construction requires the Sp(2)-charges to be given in the nonminimal sector which also contains dynamical Lagrange multipliers. The results of the present paper allow us then to generalize all properties given in \[3\] to the Sp(2)-formalism. We give a general Sp(2)-invariant solution of the master equation, and we find an extended Sp(2)-frame from which all results may be derived.

2 Brief review of previous results.

Let \(\theta_\alpha\) with Grassmann parities \(\varepsilon_\alpha(=0,1)\) be quantum generators of open group transformations. They satisfy the commutator algebra
\[
[\theta_\alpha, \theta_\beta] = i\hbar U_{\alpha\beta}^{\gamma} \theta_\gamma,
\]
(1)
where $U_{\alpha\beta}^\gamma$ are operators in general. Here and in the following we will only use graded commutators defined by

$$[f,g] \equiv fg - gf(-1)^{\varepsilon_f\varepsilon_g}. \quad (2)$$

By means of additional ghost operators the algebra (1) may be embedded into one hermitian odd operator $\Omega$ satisfying

$$\frac{1}{2}[\Omega, \Omega] = \Omega^2 = 0. \quad (3)$$

$\Omega$ is the BFV-BRST charge [1] and this embedding is always possible to achieve for finite number of degrees of freedom [2]. Explicitly $\Omega$ is given by

$$\Omega = \sum C^\alpha \theta^\alpha \gamma + \frac{1}{2} C^\beta C^\alpha U_{\alpha\beta}^\gamma \mathcal{P}_\gamma (-1)^{\varepsilon_{\beta\gamma} + \varepsilon_{\gamma}} + \cdots, \quad (4)$$

where $C^\alpha$ are ghost operators and $\mathcal{P}_\alpha$ their conjugate momenta with Grassmann parities $\varepsilon(C^\alpha) = \varepsilon(\mathcal{P}_\alpha) = \varepsilon_{\alpha} + 1$, satisfying the properties

$$[C^\alpha, \mathcal{P}_\beta] = i\hbar \delta_{\alpha\beta}, \quad (C^\alpha)^\dagger = C^{\dagger \alpha}, \quad \mathcal{P}_\alpha^\dagger = -(\cdot)^{\varepsilon_{\alpha}} \mathcal{P}_\alpha. \quad (5)$$

$C^\alpha$ and $\Omega$ have ghost number one and $\mathcal{P}_\alpha$ ghost number minus one. The $\Omega$ in (3) is given in a $\mathbb{CP}$-ordered form and the dots denote terms of higher powers in $C^\alpha$ and $\mathcal{P}_\alpha$ which are determined by (3). Notice that the hermiticity of $\Omega$ implies peculiar hermiticity properties of $\theta^\alpha$ in general.

In [3, 4] we presented some new techniques to analyze finite open group transformations within the above BRST framework. The results of this analysis showed that open group transformations on the ghost extended space may be expressed in terms of unitary operators of the form

$$U(\phi) = \exp \{-i\hbar^{-2}[\Omega, \rho(\phi)]\}, \quad (6)$$

where $\phi^\alpha$, $\varepsilon(\phi^\alpha) = \varepsilon_{\alpha}$, are group parameters. $\rho(\phi)$ is an hermitian odd operator with ghost number minus one, which was required to satisfy $[\Omega, \rho(0)] = 0$ which in turn yields $U(0) = 1$. A natural choice is $\rho(\phi) \propto \mathcal{P}_\alpha \phi^\alpha$ since $[\Omega, \mathcal{P}_\alpha]$ represent the group generators in the extended BRST frame. The latter satisfy a closed algebra if the rank of the theory is zero or one. For rank two and higher they only satisfy a closed algebra together with $\mathcal{P}_\alpha$. (The rank is equal to the maximal power of $\mathcal{P}_\alpha$ in $\Omega$.)

In terms of $U(\phi)$ we may define open group transformed operators $A(\phi)$ by $A(\phi) \equiv U(\phi) A U^{-1}(\phi)$. It follows that $A(\phi)$ satisfies the Lie equations ($\partial_{\alpha} \equiv \partial/\partial \phi^\alpha$)

$$A(\phi) \stackrel{\triangleleft}{\partial_{\alpha}} A(\phi) \stackrel{\triangleleft}{\partial_{\alpha}} - i\hbar^{-1} [A(\phi), Y_\alpha(\phi)] = 0, \quad (7)$$

where the quantum connection operator $Y_\alpha(\phi)$ is given by

$$Y_\alpha(\phi) \equiv i\hbar U(\phi) \left(U^{-1}(\phi) \stackrel{\triangleleft}{\partial_{\alpha}} \right), \quad (8)$$

The existence of a unitary operator $U(\phi)$ requires $Y_\alpha$ to satisfy the integrability conditions

$$Y_\alpha \stackrel{\triangleleft}{\partial_{\beta}} - Y_\beta \stackrel{\triangleleft}{\partial_{\alpha}} (-1)^{\varepsilon_{\alpha\beta}} = (i\hbar)^{-1} [Y_\alpha, Y_\beta], \quad (9)$$
which also follow from the Lie equations (8). In order to have a representation of the form (9) the quantum connection $Y_\alpha$ should from (8) be of the form

$$Y_\alpha(\phi) = (i\hbar)^{-1}[\Omega, \Omega_\alpha(\phi)], \quad \varepsilon(\Omega_\alpha) = \varepsilon_\alpha + 1. \quad (10)$$

This was the starting point of ref. [3]. $\Omega_\alpha(\phi)$ has obviously ghost number minus one. From (9) we found then that the operators $\Omega_\alpha(\phi)$ must satisfy the integrability conditions

$$\Omega_\alpha \overset{\partial_\beta}{\rightarrow} - \Omega_\beta \overset{\partial_\alpha}{\rightarrow} (1 - \varepsilon_\alpha\varepsilon_\beta) - (i\hbar)^{-2}(\Omega_\alpha, \Omega_\beta)\Omega + \frac{1}{2}(i\hbar)^{-1}[\Omega_{\alpha\beta}, \Omega] = 0, \quad (11)$$

where $\Omega_{\alpha\beta}$ in general is a $\phi^\alpha$-dependent operator with ghost number minus two and Grassmann parity $\varepsilon(\Omega_{\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta$. The quantum antibracket in (11) is defined by (5) (for their properties see [3, 4])

$$(f, g)_\Omega \equiv \frac{1}{2} ([f, [\Omega, g]] - [g, [\Omega, f]](-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}). \quad (12)$$

From (11) one may then derive integrability conditions for $\Omega_{\alpha\beta}$ which in turn introduces an operator $\Omega_{\alpha\beta\gamma}$ with ghost number minus three together with higher quantum antibrackets when the $\Omega$-commutator is divided out. Thus, $Y_\alpha$ is replaced by a whole set of operators, and the integrability conditions (9) for $Y_\alpha$ is replaced by a whole set of integrability conditions for these operators. These integrability conditions may be viewed as generalized Maurer-Cartan equations. The explicit form of $Y_\alpha$ and $\Omega_\alpha$ are

$$Y_\alpha(\phi) = \lambda^\beta_\alpha(\phi)\theta_\beta(-1)^{\varepsilon_\alpha+\varepsilon_\beta} + \{\text{possible ghost dependent terms}\}$$
$$\Omega_\alpha(\phi) = \lambda^\beta_\alpha(\phi)\mathcal{P}_\beta + \{\text{possible ghost dependent terms}\}, \quad \lambda^\beta_\alpha(0) = \delta^\beta_\alpha, \quad (13)$$

where $\lambda^\beta_\alpha(\phi)$ are operators in general. For Lie group theories $\lambda^\beta_\alpha(\phi)$ are pure functions and we may choose $\Omega_\alpha(\phi) = \lambda^\beta_\alpha(\phi)\mathcal{P}_\beta$ and $\Omega_{\alpha\beta} = 0$ in which case (11) reduces to the standard Maurer-Cartan equations

$$\partial_\alpha \lambda^\gamma_\beta - \partial_\beta \lambda^\gamma_\alpha(-1)^{\varepsilon_\alpha\varepsilon_\beta} = \lambda^\gamma_\alpha \lambda^\beta_\beta \mathcal{U}_{\beta\gamma}(1 - (-1)^{\varepsilon_\theta + \varepsilon_\sigma + \varepsilon_\delta}). \quad (14)$$

The crucial new discovery in [3] was that the set of integrability conditions for $\Omega_\alpha$, $\Omega_{\alpha\beta\gamma}$, ..., are encoded in a simple quantum master equation involving an extended BRST charge $\Delta$ and a master charge $S$. (This was proved to third order in $\eta^\alpha$.) It is

$$(S, S)_\Delta = i\hbar[S, \Delta], \quad (15)$$

where the left-hand side is the antibracket (12) with $\Omega$ replaced by $\Delta$. The odd operator $\Delta$ is defined by

$$\Delta \equiv \Omega + \eta^\alpha \pi_\alpha(-1)^{\varepsilon_\alpha}, \quad \Delta^2 = 0, \quad (16)$$

where $\pi_\alpha$ is the conjugate momentum operator to $\phi^\alpha$ now turned into an operator $([\phi^\alpha, \pi_\beta] = i\hbar\delta^\beta_\alpha)$, and where $\eta^\alpha$, $\varepsilon(\eta^\alpha) = \varepsilon_\alpha + 1$, are new parameters which may be viewed as superpartners to $\phi^\alpha$. The even operator $S$ is given by the following power expansion in $\eta^\alpha$

$$S(\phi, \eta) \equiv G + \eta^\alpha \Omega_\alpha(\phi) + \frac{1}{2}\eta^\alpha\eta^\beta \Omega_{\alpha\beta}(-1)^{\varepsilon_\beta} +$$
$$+ \frac{1}{6}\eta^\gamma\eta^\beta\eta^\alpha \Omega_{\alpha\beta\gamma}(-1)^{\varepsilon_\beta + \varepsilon_\alpha\varepsilon_\gamma} + \text{terms of higher powers in } \eta^\alpha, \quad (17)$$

where $G$ is the ghost charge operator. For further details of this construction see [3, 4].
3 Previous Sp(2)-version

In the Sp(2)-version of the BRST-theory the quantum generators $\theta_\alpha$ in (1) are embedded into two odd, hermitian charge operators $\Omega^a$ where $a(=1,2)$ is an Sp(2)-index. $\Omega^a$ have ghost number one and satisfy

$$[\Omega^a, \Omega^b] = \Omega^{(a}\Omega^b) = 0. \quad (18)$$

They have the following $CP$-ordered form

$$\Omega^a = C^{\alpha a} \theta_\alpha + \frac{1}{2} C^{\beta a} C^{\alpha \alpha} U_{\alpha \beta}^\gamma \mathcal{P}_{\gamma b}(-1)^{\xi_\beta + \xi_\gamma} + \cdots, \quad (19)$$

where $C^{\alpha a}$ ($\xi(C^{\alpha a}) = \xi_\alpha + 1$) are Sp(2)-valued ghost operators and $\mathcal{P}_{aa}$ their conjugate momenta satisfying

$$[C^{\alpha a}, \mathcal{P}_{\beta b}] = i\hbar \delta_\alpha^\alpha \delta_\beta^b, \quad (C^{\alpha a})^\dagger = C^{\alpha a}, \quad \mathcal{P}^\dagger_{aa} = -(1)^{\xi_a} \mathcal{P}_{aa}. \quad (20)$$

The dots in (19) denote terms of higher orders in the ghost variables which are determined by (18).

In [9] we proposed in analogy with (7) and (10) that group transformed operators $A(\phi)$ satisfy the Sp(2)-valued Lie equations

$$A(\phi) \overset{\sim}{\nabla}^{ab}_{\alpha a} = A(\phi) \partial_\alpha \delta_\alpha^b - (i\hbar)^{-1}[A(\phi), Y_{aa}^b(\phi)] = 0, \quad (21)$$

where the Sp(2)-valued connection operators are of the form

$$Y_{aa}^b(\phi) = (i\hbar)^{-1}[\Omega^b, \Omega_{aa}(\phi)], \quad \xi(\Omega_{aa}) = \xi_\alpha + 1. \quad (22)$$

The analysis of these connections led to the result that $Y_a \equiv \frac{1}{2} Y_{aa}^a$ may be viewed as a conventional connection operator like (8) while

$$T_{ab}^a(\phi) \equiv \xi_{acc} Y_{aa}^b(\phi) \quad (23)$$

are constraint operators. The unitary operator corresponding to $Y_a$ only acts within a ghost restricted subset of operators $A$ satisfying $[A, T_{ab}^a(\phi)] = 0$. In a way this is natural since the Sp(2)-formulation contains more ghost variables then the conventional BRST frame. By restricting the ghost sector we essentially reduce the Sp(2)-formalism to the standard one. Anyway within this formalism we found that the integrability conditions of $Y_{aa}^a$ led to a whole set of integrability conditions for $\Omega_{aa}$ and higher operators $\Omega_{\alpha\beta ab}$, $\ldots$, which were symmetric in lower Sp(2)-indices and which was shown to constitute generalized Maurer-Cartan equations. This set of equations were then shown to be encoded in the Sp(2)-valued quantum master equation (proved to third order in $\eta^a$)

$$(S, S)^a_{\Delta} = i\hbar[\Delta^a, S], \quad (24)$$

where the Sp(2)-valued quantum antibracket is defined by [5] (their properties are given in [3, 3])

$$(f, g)^a_{\Delta} \equiv \frac{1}{2} \left( [f, [\Delta^a, g]] - [g, [\Delta^a, f]](-1)^{(\xi_f + 1)(\xi_{g} + 1)} \right). \quad (25)$$
The right-hand side is equal to (12) with $\Omega$ replaced by $\Delta^a$. The two odd operators $\Delta^a$ were defined by (cf (16))

$$\Delta^a \equiv \Omega^a + j^a \eta^\alpha \pi_\alpha (-1)^{\varepsilon_\alpha}, \quad [\Delta^a, \Delta^b] = 0,$$

(26)

where again $\pi_\alpha$ is the conjugate momentum to $\phi^\alpha$. $j^a$ are even $\text{Sp}(2)$-valued parameters. The even master charge $S$ was in turn given by the following power expansion in $\eta^{\alpha a}$

$$S(\phi, \eta, j) \equiv G + j^a \eta^\alpha \Omega_{\alpha a}(\phi) + \frac{1}{4} j^b j^a \eta^\beta \eta^\alpha (-1)^{\varepsilon_\beta} \Omega_{\alpha \beta ab}(\phi) +$$

$$+ \frac{1}{36} j^c j^b j^a \eta^\gamma \eta^\beta \eta^\alpha (-1)^{\varepsilon_\beta + \varepsilon_\alpha + \varepsilon_\gamma} \Omega_{\alpha \beta \gamma abc}(\phi) + \ldots .$$

(27)

The parameters $j^a$ provide for the necessary $\text{Sp}(2)$-symmetrization in the generalized Maurer-Cartan equations. For further details see [9].

4 The new $\text{Sp}(2)$-proposal

In this paper our object is to construct open group transformations within the $\text{Sp}(2)$-formalism which are defined on the complete extended space. We expect then that the group transformed operators $A(\phi)$ satisfy the Lie equations (7) and that the quantum connections $Y_\alpha$ are given by (8) in terms of a unitary group element $U(\phi)$. A natural generalization of (6) is the following $\text{Sp}(2)$-invariant expression

$$U(\phi) = \exp \{- i \hbar^{-2} \varepsilon_{ab} [\Omega^b, [\Omega^a, R(\phi)]]\},$$

(28)

where $R(\phi)$ is an even operator with ghost number minus two and such that $U(0) = 1$. $\varepsilon_{ab}$ is the $\text{Sp}(2)$-metric ($\varepsilon_{ab} = -\varepsilon_{ba}, \varepsilon_{ab} \varepsilon_{bc} = \delta_c^a, \varepsilon^{12} = 1$). We notice that commutators of operators of the form $\varepsilon_{ab} [\Delta^b, [\Delta^a, A]]$ yield operators of the same form, which means that $\varepsilon_{ab} [\Delta^b, [\Delta^a, A]]$ is a natural expression for a group generator within the $\text{Sp}(2)$-formalism. If $U(\phi)$ is of the form (28) then the quantum connection (8) may always be written in the following $\text{Sp}(2)$-invariant form

$$Y_\alpha = (i \hbar)^{-2} \frac{1}{2} \varepsilon_{ab} [\Omega^b, [\Omega^a, X_\alpha(\phi)]],$$

(29)

where $X_\alpha$ has Grassmann parity $\varepsilon_\alpha$, the same as $Y_\alpha$. In this way we avoid the $\text{Sp}(2)$-valued connections (22) considered in [8]. However, it remains to establish the existence of quantum connections of this form. We know that $Y_\alpha$ should satisfy the boundary conditions (13). Now this expression cannot be obtained from (29) for any choice of $X_\alpha$ if $\Omega^a$ are given in the minimal sector as in (19). However, there is a nonminimal sector involving dynamical Lagrange multipliers which does allow for a solution of the form (29). In this sector the $\Omega^a$-operators have the form

$$\Omega^a = C^{\alpha a} \theta_\alpha + \frac{1}{2} C^{b \beta} C^{\alpha a} U_{\alpha \beta} \gamma P_{\beta b} (-1)^{\varepsilon_\beta + \varepsilon_\gamma} +$$

$$+ \varepsilon^{ab} P_{\beta b} \lambda^\beta + \frac{1}{2} \lambda^\beta C^{\alpha a} U_{\alpha \beta} \gamma \zeta_\gamma + \cdots ,$$

(30)

where $\lambda^\alpha, \varepsilon(\lambda^\alpha) = \varepsilon_\alpha$, are the Lagrange multipliers and $\zeta_\alpha$ their conjugate momenta ($[\lambda^\alpha, \zeta_\beta] = i \hbar \delta^\alpha_\beta$). $\lambda^\alpha$ has ghost number two and $\zeta_\alpha$ minus two. The expressions (30) are
both $\mathcal{CP}$- and $\lambda\zeta$-ordered, and the higher order terms are determined by (38). In the nonminimal sector it is easily seen that the boundary conditions (13) for $Y_\alpha$ is reproduced by (29) with $X_\alpha$ given by

$$X_\alpha(\phi) = -\lambda_\alpha^\beta(\phi)\zeta_\beta(-1)^{\varepsilon_\alpha+\varepsilon_\beta} + \{\text{possible ghost dependent terms}\}.$$  

Thus, the quantum connections $Y_\alpha$ may have the $\text{Sp}(2)$-invariant form (29) provided $\Omega^a$ are given in the nonminimal sector with the form (30). The integrability conditions (4) for $Y_\alpha$ lead through (29) to the following integrability conditions for $X_\alpha$

$$X_\alpha \dot{\partial}_\beta - X_\beta \dot{\partial}_\alpha (-1)^{\varepsilon_\alpha+\varepsilon_\beta} + (i\hbar)^{-3}\frac{1}{2}\{X_\alpha, X_\beta\} \Omega = (i\hbar)^{-1}[X_{\alpha\beta a}, \Omega^a],$$

where the operator $X_{\alpha\beta a}$ has ghost number minus three and Grassmann parity $\varepsilon(X_{\alpha\beta a}) = \varepsilon_\alpha + \varepsilon_\beta + 1$, and is in general $\phi^a$-dependent. In (32) we have introduced a new quantum bracket defined by

$$\{f, g\} \Omega \equiv \left[f, \Omega^a\right], \varepsilon_{ab}\left[\Omega^b, g\right].$$

It has similar properties to the graded commutator (2) (see appendix A). For Lie group theories we may choose $X_\alpha(\phi) = -\lambda_\alpha^\beta(\phi)\zeta_\beta(-1)^{\varepsilon_\alpha+\varepsilon_\beta}$ where $\lambda_\alpha^\beta(\phi)$ are functions in which case (22) with $X_{\alpha\beta a} = 0$ leads to the Maurer-Cartan equations (14).

The integrability conditions for $X_{\alpha\beta a}$ in (32) lead to higher operators whose integrability conditions lead to still higher operators etc. It is obviously a nontrivial task to invent a master equation which encodes all these operators and their integrability conditions. However, in the next section we will propose such a master equation. To make it easier to understand this proposal it should be helpful to view the above construction from the to understand this proposal it should be helpful to view the above construction from the

In (13) the quantum connections $Y_\alpha$ were given in terms of the $\text{Sp}(2)$-valued connections (22) by

$$Y_\alpha(\phi) = \frac{1}{2}Y_{aa}^a(\phi) = \frac{1}{2}(i\hbar)^{-1}[\Omega^a, \Omega_{aa}(\phi)].$$

Obviously we may reproduce the expression (29) if we choose

$$\Omega_{aa}(\phi) = \varepsilon_{ba}(i\hbar)^{-1}[\Omega^b, X_\alpha(\phi)].$$

We notice then that such a relation only is possible in the nonminimal sector. It is easily seen that (35) with (30) and (31) yields

$$\Omega_{aa}(\phi) = \lambda_\alpha^\beta(\phi)P_{\beta a} + \{\text{possible ghost dependent terms}\},$$

which are the boundary conditions imposed in (9). If we insert the expression (34) into the $\text{Sp}(2)$-valued connections (22) we find

$$Y_{aa}^b(\phi) = \varepsilon_{ca}(i\hbar)^{-2}[\Omega^b, [\Omega^c, X_\alpha(\phi)]].$$

By means of the Jacobi identities and (18) we find then that

$$T_{ab}^{bc}(\phi) \equiv \varepsilon^{bac}Y_{ac}^b(\phi) = -(i\hbar)^{-2}[\Omega^b, [\Omega^a, X_\alpha]] = 0.$$  

This implies that the choice (35) makes the $\text{Sp}(2)$-treatment in (3) completely unconstrained and equivalent to our new proposal. Notice that (36) means that $Y_{ab}^c$ have only one independent nonzero element. In fact, it implies

$$Y_{a2} = Y_{a1} = Y_{a1} - Y_{a2} = 0 \Rightarrow Y_\alpha = Y_{a1} = Y_{a2}. $$

However, it is a nontrivial task to implement the constraints (35) into the master equation.
5 The master equation

We propose that the master equation

\[ (S, S)^\alpha_\Delta = i\hbar [\Delta^\alpha, S] , \quad (40) \]

encodes all integrability conditions starting with \([42]\) for the operator \(X_\alpha\) in our new proposal \([23]\) of the quantum connection operator \(Y_\alpha\). Although \([41]\) is of the same form as \([24]\) used in \([3]\) the operators \(\Delta^\alpha\) and \(S\) are here different from the ones used there, which were \([26]\) and \([27]\). The two odd operators \(\Delta^\alpha\) we propose to have the following new form

\[ \Delta^\alpha \equiv \Omega^\alpha + \eta^{\alpha 0} \pi_\alpha (-1)^{e_\alpha} + \rho^\alpha \xi_{ab} e^{ab} (-1)^{e_\alpha}, \]

\[ [\Delta^\alpha, \Delta^\beta] = \Delta^{(a \Delta^b)} = 0, \quad \tag{41} \]

where \(\Omega^\alpha\) are the hermitian \(\mathrm{Sp}(2)\)-charges in the nonminimal sector given by \([11]\). \(\eta^{\alpha 0}\), \((\varepsilon (\eta^{\alpha 0}) = e_\alpha + 1)\), are new operators and \(\xi_{ab}\) their conjugate momenta \([\eta_{\alpha 0}, \xi_{\beta b}] = i\hbar \delta^{\alpha}_{\beta} \delta^0_b\). \(\rho^\alpha\), \((\varepsilon (\rho^\alpha) = e_\alpha)\), are new parameters. \(\phi^0\), \(\rho^\alpha\) and \(\eta^{\alpha 0}\) may be viewed as a superset of new coordinates. The even master charge \(S\) has also a new form. Here we propose it to be given by a general power expansion in \(\eta^{\alpha 0}\) and \(\rho^\alpha\) (cf.\([27]\)),

\[ S(\phi, \rho, \eta) = G + \eta^{\alpha 0} \Omega_{\alpha 0}(\phi) + \rho^0 \Omega_\alpha(\phi) + \frac{1}{2} \eta^{\alpha \beta} \eta^{\alpha 0} (-1)^{\varepsilon_\beta} \Omega_{\alpha \beta 0}(\phi) + \frac{1}{2} \rho^\alpha \rho^\beta \Omega_{\alpha \beta 0}(\phi) + \rho^0 \eta^{\alpha 0} \Omega_{\alpha 0 0}(\phi) + \text{terms of higher powers in } \rho^\alpha \text{ and } \eta^{\alpha 0}, \quad (42) \]

where \(G\) now is the ghost charge operator including the dynamical Lagrange multipliers, i.e.

\[ G \equiv -\frac{1}{2} \left( \mathcal{P}_{\alpha 0} \mathcal{C}^{\alpha 0} - \mathcal{C}^{\alpha 0} \mathcal{P}_{\alpha 0} (-1)^{e_\alpha} \right) - \left( \zeta_\alpha \lambda^\alpha + \lambda^\alpha \zeta_\alpha (-1)^{e_\alpha} \right). \quad \tag{43} \]

The Grassmann parities of the coefficient operators in \([42]\) are determined by the ones of \(\rho^\alpha\), \(\eta^{\alpha 0}\) and that \(S\) is even. Their symmetry properties are

\[ \Omega_{\alpha \beta 0} = \Omega_{\beta \alpha 0} (-1)^{\varepsilon_\alpha \varepsilon_\beta}, \quad \Omega_{0 \beta 0} = -\Omega_{0 \beta 0} (-1)^{\varepsilon_\alpha \varepsilon_\beta}, \quad \ldots \quad \tag{44} \]

all determined by the \(m\eta\)-monomials the coefficients are multiplied by in \(S\). Their ghost numbers are determined by the following conditions: \(S\) is required to have total ghost number zero and \(\Delta^\alpha\) total ghost number one. The latter requires \(\eta^{\alpha 0}\) to have ghost number one which implies that \(\xi_{ab}\) has ghost number minus one which in turn requires \(\rho^\alpha\) to have ghost number two. One may notice that \([11]\) and \([42]\) essentially reduce to \([26]\) and \([27]\) if we set \(\rho^\alpha = 0\) and \(\eta^{\alpha 0} = j^a \eta^a\). However, if we first calculate \([40]\) and then take this limit then we will not get the same equations as in \([3]\). The expressions \([11]\) and \([42]\) yield

\[ [S, \Delta^\alpha] = i\hbar \Omega^\alpha + \eta^{\beta \delta} \Omega_{\beta \delta 0} (\Omega^\alpha + \rho^\alpha \left( -i\hbar \Omega_{ab} e^{ab} + [\Omega_\alpha, \Omega^\alpha] \right) + \]

\[ + \frac{1}{2} \eta^{\gamma \delta} (-1)^{\varepsilon_\gamma} \left( i\hbar \Omega_{\gamma \delta} \Omega_{\gamma c} \delta^0_c - i\hbar \Omega_{\gamma c} \Omega_{\gamma \delta} \delta^0_c (-1)^{\varepsilon_\beta \varepsilon_\gamma} + [\Omega_{\gamma \beta 0}, \Omega^\gamma] \right) + \]

\[ + \frac{1}{2} \rho^\beta \left( -i\hbar \Omega_{\beta c} e^{ab} (-1)^{\varepsilon_\gamma} - i\hbar \Omega_{\beta 0} e^{ab} (-1)^{\varepsilon_\gamma} + [\Omega_{\beta \gamma 0}, \Omega^\gamma] \right) + \]

\[ + \rho^\gamma \eta^{\beta \delta} \left( i\hbar \Omega_{\gamma \delta} \delta^0_c (-1)^{\varepsilon_\gamma} - i\hbar \Omega_{\gamma \beta 0} e^{ac} (-1)^{\varepsilon_\gamma} + [\Omega_{\beta \gamma 0}, \Omega^\gamma] \right) + \]

\[ + \text{terms of higher powers in } \rho^\alpha \text{ and } \eta^{\alpha 0}. \quad \tag{45} \]
Since the master equation (40) may equivalently be written as
\[
[S, [S, \Delta^a]] = i\hbar[S, \Delta^a]
\] (46)
it follows from (41) that
\[
[[S, \Delta^a], [S, \Delta^b]] = 0.
\] (47)
From (46) it is also straightforward to calculate (40) order by order in \(\eta^{\alpha a}\) and \(\rho^\alpha\). To the zeroth order and to order \(\eta^{\alpha a}\) the master equation (40) is identically satisfied. Up to second order we find the following nontrivial equations:

(To order \(\eta^{\alpha a}\eta^{\beta b}\):)
\[
\Omega^{\alpha a}(-1)^{\varepsilon_{\beta}} + \Omega^{\beta a}(-1)^{\varepsilon_{\alpha}(\varepsilon_{\beta}+1)} - \frac{1}{4}(i\hbar)^{-2}(\Omega^{\alpha a}, \Omega^{\beta b})_{\Omega} - \frac{1}{2}(i\hbar)^{-1}[\Omega^{\gamma c\delta b}, \Omega^{\alpha a}].
\] (49)

(To order \(\rho^\alpha\rho^\beta\):)
\[
\Omega^{\alpha a}(-1)^{\varepsilon_{\beta}} + \Omega^{\beta a}(-1)^{\varepsilon_{\alpha}(\varepsilon_{\beta}+1)} - \frac{1}{4}(i\hbar)^{-2}(\Omega^{\alpha a}, \Omega^{\beta b})_{\Omega} - \frac{1}{2}(i\hbar)^{-1}\varepsilon_{ab}[\Omega^{\alpha a}, \Omega^{\beta b}] = 0.
\] (50)

(To order \(\rho^\alpha\eta^{\beta b}\):)
\[
\Omega^{\alpha a}(-1)^{\varepsilon_{\beta}} + \Omega^{\beta a}(-1)^{\varepsilon_{\alpha}(\varepsilon_{\beta}+1)} - \frac{1}{4}(i\hbar)^{-2}(\Omega^{\alpha a}, \Omega^{\beta b})_{\Omega} - \frac{1}{2}(i\hbar)^{-1}\varepsilon_{ab}[\Omega^{\alpha a}, \Omega^{\beta b}] = 0.
\] (51)

where
\[
Z^{\alpha a}_{\beta b} \equiv \frac{1}{3}(i\hbar)^{-1}[\Omega^{\alpha a}, \Omega^{\beta b}](-1)^{\varepsilon_{\beta}} - \frac{2}{3}(i\hbar)^{-2}(\Omega^{\alpha a}, \Omega^{\beta b})_{\Omega} \varepsilon_{cb} - \frac{2}{3}(i\hbar)^{-1}[\Omega^{\alpha a}, \Omega^{\beta b}]\varepsilon_{cb}(-1)^{\varepsilon_{\beta}},
\] (52)
where in turn the quantum Sp(2)-antibrackets are given by (25) with \(\Delta^a\) replaced by the nonminimal \(\Omega^a\).

Let us now analyze these equations. Eq.(48) determines \(\Omega^{\alpha a}\) in terms of \(\Omega^\alpha\). When compared to (35) it suggests that
\[
\Omega^\alpha = 2X^\alpha(-1)^a
\] (53)
provided \(\Omega^{\alpha a}\) in \(S\) may be identified with \(\Omega^{\alpha a}\) in (22). This identification is possible since the \(bc\)-symmetric parts of (49) agree exactly with the integrability conditions for \(\Omega^{\alpha a}\) in (3). Now (49) seems to imply stronger conditions on \(\Omega^{\alpha a}\), since the \(bc\)-antisymmetric part of (49) is the nontrivial equation
\[
\varepsilon^{ab}\left(\Omega^{\beta b} \partial^{\bar{a}} + \Omega^{\gamma c} \partial^{\bar{b}} (-1)^{\varepsilon_{\beta} \varepsilon_{\gamma}}\right) = (i\hbar)^{-2}\varepsilon^{cb}(\Omega^{\beta b}, \Omega^{\gamma c})_{\Omega} - \frac{1}{2}(i\hbar)^{-1}[\varepsilon^{cb}\Omega^{\gamma bc}, \Omega^a].
\] (54)
However, we notice then first that (51) together with (44) imply
\[ \Omega_{\alpha\beta ab} = -\frac{1}{2}\varepsilon_{ab}\left(\partial_{\alpha}\Omega_{\beta} + \partial_{\beta}\Omega_{\alpha}(-1)^{\varepsilon_{a}\varepsilon_{b}}\right) + \frac{1}{2}\left(Z_{\alpha\beta ab} - Z_{\beta\alpha ba}(-1)^{\varepsilon_{a}\varepsilon_{b}}\right), \]  
(55)
which means that \(\Omega_{\alpha\beta ab}\) are completely expressed in terms of \(\Omega\)-operators with less indices. By means of (55), (48) and (50) we find then that (44) is identically satisfied. This is related to the identical vanishing of \(T_{\alpha}^{ab}(\phi)\) in (48). Thus, only the bc-symmetric parts of (48) are nontrivial.

Eq. (51) together with (44) imply
\[ \partial_{\alpha}\Omega_{\beta} - \partial_{\beta}\Omega_{\alpha}(-1)^{\varepsilon_{a}\varepsilon_{b}} = \frac{1}{2}\varepsilon_{ba}\left(Z_{\alpha\beta ab} + Z_{\beta\alpha ba}(-1)^{\varepsilon_{a}\varepsilon_{b}}\right), \]
(56)
where the right-hand side is straightforward to calculate from (52). We find by means of (58) that it reduces to
\[
\partial_{\alpha}\Omega_{\beta} - \partial_{\beta}\Omega_{\alpha}(-1)^{\varepsilon_{a}\varepsilon_{b}} = \frac{1}{4}(ih)^{-3}\{\Omega_{\alpha}, \Omega_{\beta}\}\Omega - \frac{1}{12}(ih)^{-3}\varepsilon_{ab}\left[\Omega^{b}, \{\Omega^{a}, [\Omega_{\alpha}, \Omega_{\beta}]\}\right] - \\
-\frac{1}{3}(ih)^{-1}\left[\Omega_{\alpha b a}(-1)^{\varepsilon_{b}} - \Omega_{\beta a a}(-1)^{\varepsilon_{a}(\varepsilon_{b}+1)}, \Omega^{a}\right].
\]  
(57)
These equations are identical to the integrability conditions (32) after the identifications (33) and
\[ X_{\alpha\beta a} \equiv \frac{1}{6}\left(\Omega_{\alpha b a}(-1)^{\varepsilon_{a}} - \Omega_{\beta a a}(-1)^{\varepsilon_{a}(\varepsilon_{b}+1)} + (ih)^{-2}\varepsilon_{ab}\left[[X_{\alpha}, X_{\beta}], \Omega^{b}\right]\right). \]
(58)
Notice that the combination of the \(\Omega_{\alpha\beta a}\)-operators entering \(X_{\alpha\beta a}\) are not determined by (50). Eq. (57) yields through (48) the bc-symmetric parts of (48).

To conclude we have found that the master equation (41) with the ansatz (11) and

To conclude we have found that the master equation (41) with the ansatz (11) and
(23) for \(\Delta^{a}\) and \(S\) yields up to second order in the new variables \(\eta^{aa}\) and \(\rho^{a}\) exactly the integrability conditions (32) of \(X_{\alpha}\) in our basic ansatz (29) for the quantum connection operator \(Y_{\alpha}\). This convinces us that we will find the integrability conditions of \(X_{\alpha\beta a}\) in (58) at the third order in \(\eta^{aa}\) and \(\rho^{a}\) which in turn will involve new \(X\)-operators whose integrability conditions are obtained at the fourth order and so on exactly as we had in

6 Formal properties of the quantum master equation

Consider the operators \(S(\alpha)\) and \(\Delta^{a}(\alpha)\) defined by
\[ S(\alpha) \equiv e^{\xi^{a}F}Se^{-\xi^{a}F}, \quad \Delta^{a}(\alpha) \equiv e^{\xi^{a}F}\Delta^{a}e^{-\xi^{a}F}, \]
(59)
where \(\alpha\) is a real parameter and \(F\) an arbitrary even operator. Obviously
\[ [\Delta^{a}(\alpha), \Delta^{b}(\alpha)] = \Delta^{(\alpha)}(\Delta^{b}\alpha) = 0 \]
(60)
and
\[ (S(\alpha), S(\alpha))_{\Delta(\alpha)}^{\Delta(\alpha)} = i\hbar[\Delta^{a}(\alpha), S(\alpha)], \]
(61)
provided $S$ satisfies the original master equation (40), and provided $\Delta^a$ is given by (41). For $F = S$ we have in particular

$$S(\alpha) = S, \quad \Delta^a(\alpha) = \Delta^a + (i\hbar)^{-1}[\Delta^a, S](1 - e^{-\alpha}).$$  

(62)

Thus, the master equation is satisfied if we replace $\Delta^a$ by $\Delta^a + \beta(i\hbar)^{-1}[\Delta^a, S]$ for any real parameter $\beta$.

If $F$ in (59) satisfies

$$[\Delta^a, F] = 0,$$  

(63)

then $\Delta^a(\alpha) = \Delta^a$. In this case $S(\alpha)$ is another solution of the master equation (40) if $S$ is a given solution. In order for $S(\alpha)$ to have total ghost number zero like $S, F$ must have total ghost number zero, and in order for $S(\alpha)$ to have the same form (12) as $S, F$ should not depend on $\pi_\alpha$ and $\xi_{\alpha 0}$. If we assume that $F(\phi, \eta, \rho)$ may be expanded in powers of $\phi^a, \eta^a$ and $\rho^a$ then the solutions of (63) may be written as (the proof is given in appendix B)

$$F(\phi, \eta, \rho) = F(0, 0, 0) + \frac{1}{2}\varepsilon_{ab}(i\hbar)^{-2}[\Delta^b, \Delta^a, \Phi(\phi, \eta, \rho)],$$  

(64)

where $\Phi$ is an even operator with total ghost number minus two which does not depend on $\pi_\alpha$ and $\xi_{\alpha 0}$ and which satisfies

$$\varepsilon_{ab}[\Delta^b, \Delta^a, \Phi(\phi, \eta, \rho)]|_{\phi=\eta=\rho=0} = 0.$$  

(65)

($\Phi$ has the form (B.11) in appendix B.) $F(0, 0, 0)$ satisfies $[\Omega^a, F(0, 0, 0)] = 0$. It is both natural and consistent to impose the restriction $F(0, 0, 0) = 0$ in which case we find the following class of invariance transformations of the master equation (40)

$$S \rightarrow S' \equiv \exp \left\{ -(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]] \right\} S \exp \left\{ (i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]] \right\},$$  

(66)

which leave the $\phi^a=\eta^a=\rho^a=0$ component of $S$ invariant. This was proposed to be the natural automorphisms of the master equation in [9]. For the corresponding infinitesimal transformations we have

$$\delta S = (i\hbar)^{-3}[S, \frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]]],$$

$$\delta_{21} S \equiv (\delta_2 \delta_1 - \delta_1 \delta_2) S = (i\hbar)^{-3}[S, \frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi_{21}]]],$$

$$\Phi_{21} = (i\hbar)^{-3}\frac{1}{2}\{\Phi_1, \Phi_2\} \Delta.$$  

(67)

We conjecture that the general solution of the master equation (40) satisfying the boundary condition $S|_{\phi^a=\eta^a=\rho^a=0} = G$ is given by

$$S = \exp \left\{ -(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]] \right\} G \exp \left\{ (i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]] \right\},$$  

(68)

where $\Phi(\phi, \eta, \rho)$ has the same form as in (44). Notice that $S = G$ is a trivial solution of (40). The transformations (66) act transitively on (68).
7 Open group transformations within an extended Sp(2)-formalism

The invariance transformations of the quantum master equation (40) which follow from the transformation formulas (59) together with (63) suggest that we could define extended group transformations within an extended Sp(2)-element (69) which means that it satisfies the Lie equation

\[ U(\phi, \eta, \rho) \equiv \exp \left\{ \frac{\hbar}{i} F(\phi, \eta, \rho) \right\}, \quad [\Delta^a, F] = 0, \quad [\tilde{G}, F] = 0, \quad F(0, 0, 0) = 0, \quad (69) \]

where \( \tilde{G} \) is the extended ghost charge

\[ \tilde{G} \equiv G - \frac{1}{2} \left( \xi_{\alpha a} \eta^{\alpha a} - \eta^{\alpha a} \xi_{\alpha a} (-1)^{\varepsilon_{\alpha}} \right) - \left( \sigma_{\alpha} \rho^\alpha + \rho^\alpha \sigma_{\alpha} (-1)^{\varepsilon_{\alpha}} \right), \quad (70) \]

where \( G \) is given by (33). \( \sigma_{\alpha} \) are the conjugate momenta to \( \rho^\alpha \) which now are turned into operators \( [\rho^\alpha, \sigma_{\beta}] = i\hbar \delta_{\beta}^\alpha \). Obviously

\[ [\tilde{G}, S] = 0, \quad [\tilde{G}, \Delta^a] = i\hbar \Delta^a. \quad (71) \]

\( F \) in (69) is according to appendix B given by (cf (44))

\[ F(\phi, \eta, \rho) = \frac{1}{2} \varepsilon_{ab}(i\hbar)^{-2}[\Delta^b, [\Delta^a, \Phi(\phi, \eta, \rho)]]. \quad (72) \]

We notice that the \( \eta^{\alpha a} = \rho^\alpha = 0 \) component of (39) agrees with (28) if we make the identification \( R(\phi) = \Phi(\phi, 0, 0) \).

By means of (29) we may define extended group transformed operators by

\[ \tilde{A}(\phi, \eta, \rho) \equiv A U(\phi, \eta, \rho) A^{-1}(\phi, \eta, \rho), \quad (73) \]

where \( A \) does not depend on \( \phi^\alpha, \eta^{\alpha a}, \rho^\alpha \) and their conjugate momenta. At \( \eta^{\alpha a} = \rho^\alpha = 0 \) \( \tilde{A} \) is the group transformed operators \( A(\phi) \) satisfying the Lie equations (6) with the operator connections (28). The operators \( \tilde{A} \) in (28) satisfy the extended Lie equation

\[ \tilde{A}(\phi, \eta, \rho) \tilde{\nabla}_a \equiv \tilde{A}(\phi, \eta, \rho) \tilde{\partial}_a - (i\hbar)^{-1}[\tilde{A}(\phi, \eta, \rho), \tilde{Y}_a(\phi, \eta, \rho)] = 0, \quad (74) \]

where

\[ \tilde{Y}_a(\phi, \eta, \rho) = i\hbar U(\phi, \eta, \rho) \left( U^{-1}(\phi, \eta, \rho) \tilde{\nabla}_a \right). \quad (75) \]

The expression (69) for \( U(\phi, \eta, \rho) \) implies that \( \tilde{Y}_a \) has the form

\[ \tilde{Y}_a(\phi, \eta, \rho) = (i\hbar)^{-2} \frac{1}{2} \varepsilon_{ab}[\Delta^b, [\Delta^a, \tilde{X}_a(\phi, \eta, \rho)]]. \quad (76) \]

From our conjecture that (68) is the general solution of the master charge \( S \) we see that the master charge \( S \) itself is a group transformed ghost charge under the extended group element (69) which means that it satisfies the Lie equation

\[ S(\phi, \eta, \rho) \tilde{\nabla}_a \equiv S(\phi, \eta, \rho) \tilde{\partial}_a - (i\hbar)^{-1}[S(\phi, \eta, \rho), \tilde{Y}_a(\phi, \eta, \rho)] = 0. \quad (77) \]
This equation together with (76) may be used to resolve \( \tilde{X}_\alpha(\phi, \eta, \rho) \) in terms of \( S \) (cf the corresponding treatment in section 6 of [3]).

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Appendix A

Properties of the new quantum bracket (33).

In (33) we introduced a new quantum bracket which in a general algebraic framework is defined by

\[
\{f, g\}_Q \equiv \left[ [f, Q^a], \varepsilon_{ab} [Q^b, g] \right], \quad [Q^a, Q^b] = 0,
\]

(A.1)

where \( f \) and \( g \) are any operators with Grassmann parities \( \varepsilon(f) \equiv \varepsilon_f \) and \( \varepsilon(g) \equiv \varepsilon_g \) respectively. \( Q^a, a = 1, 2, \) are two odd anticommuting operators. The commutators on the right-hand side are graded commutators defined by (2). The new quantum bracket (A.1) has similar properties to the graded commutator (2). Its properties are

1) Grassmann parity

\[
\varepsilon(\{f, g\}_Q) = \varepsilon_f + \varepsilon_g.
\]

(A.2)

2) Antisymmetry

\[
\{f, g\}_Q = -\{g, f\}_Q (-1)^{\varepsilon_f \varepsilon_g}.
\]

(A.3)

3) Linearity

\[
\{f + g, h\}_Q = \{f, h\}_Q + \{g, h\}_Q, \quad \text{(for } \varepsilon_f = \varepsilon_g).\]

(A.4)

4) If one entry is an odd/even parameter \( \lambda \) we have

\[
\{f, \lambda\}_Q = 0 \quad \text{for any operator } f.
\]

(A.5)

5) The generalized Jacobi identities

\[
\{f, \{g, h\}_Q\}_Q (-1)^{\varepsilon_f \varepsilon_h} + \text{cycle}(f, g, h) =
\]

\[
= \left( 2([f, g]_Q, \tilde{h}) + \{[f, \tilde{g}] + [f, g], \tilde{h}\}_Q \right) (-1)^{\varepsilon_f \varepsilon_h} + \text{cycle}(f, g, h),
\]

(A.6)

where the tilde operators are defined by

\[
\tilde{f} \equiv -\frac{1}{2} \varepsilon_{ab} [Q^b, [Q^a, f]].
\]

(A.7)
6) The generalized Leibniz rule

\[
\{f, gh\}_Q - \{f, g\}_Q h - g\{f, h\}_Q (-1)^{\varepsilon_f \varepsilon_g} = \\
= \varepsilon_{ab}[[f, Q^a], g][Q^b, h](-1)^{\varepsilon_g} + \varepsilon_{ab}[Q^b, g][[f, Q^a], h](-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)} = \\
= -[f, [g, Q^a]_Q][Q^b, h] + [[f, g], Q^a]_Q[Q^b, h] + [g, Q^a]_Q[Q^b, [f, h]](-1)^{\varepsilon_f \varepsilon_g}.
\] (A.8)

The properties 1)-4) agree exactly with the corresponding properties of the graded commutator \([f, g]\) for arbitrary operators \(f\) and \(g\). However, the graded commutator satisfies 5) and 6) with zero on the right-hand sides.

The new quantum bracket (A.1) may also be expressed in terms of the quantum Sp(2)-antibrackets. We find

\[
\{f, g\}_Q = \frac{1}{2}\varepsilon_{ab}\left( (f, Q^a)_Q^b - (f, [Q^a, g])_Q^b \right) + \frac{1}{4}\varepsilon_{ab}[Q^b, [Q^a, [f, g]]],
\] (A.9)

where the antibrackets on the right-hand side are defined by (25) with \(\Delta^b\) replaced by \(Q^b\).

**Appendix B**

**Proof of (64).**

Let \(F(\phi, \eta, \rho)\) be an operator which does not depend on the conjugate momenta to \(\phi^a\), \(\eta^{\alpha a}\) and \(\rho^a\), i.e. \(\pi_\alpha\), \(\xi_{\alpha a}\) and \(\sigma_\alpha\). Let it furthermore be expandable in powers of \(\phi^a\), \(\eta^{\alpha a}\) and \(\rho^a\). We want to solve the condition (63), i.e.

\[
[\Delta^a, F(\phi, \eta, \rho)] = 0.
\] (B.1)

To this purpose we introduce the operators

\[
\Lambda^a \equiv \eta^{\alpha a}\pi_\alpha(-1)^{\varepsilon_\alpha} + \rho^a\xi_{\alpha a}\varepsilon^{ab}(-1)^{\varepsilon_\alpha}, \\
\tilde{\Lambda}_a \equiv \xi_{\alpha a}\phi^\alpha(-1)^{\varepsilon_\alpha} + \sigma_\alpha\eta^{ab}\varepsilon_{ba}(-1)^{\varepsilon_\alpha},
\] (B.2)

with the properties

\[
\varepsilon(\Lambda^a) = \varepsilon(\tilde{\Lambda}_a) = 1, \quad [\Lambda^a, \Lambda^b] = [\tilde{\Lambda}_a, \tilde{\Lambda}_b] = 0.
\] (B.3)

\(\Lambda^a\) has total ghost charge plus one and \(\tilde{\Lambda}_a\) minus one. We have then

\[
\Delta^a = \Omega^a + \Lambda^a, \quad [\Delta^a, \tilde{\Lambda}_b] = i\hbar \delta^a_bN, \\
N = \pi_\alpha\phi^\alpha + \xi_{\alpha a}\eta^{\alpha a} + \sigma_\alpha\rho^a, \quad [N, \Lambda^a] = [N, \tilde{\Lambda}_a] = 0.
\] (B.4)

By commuting (B.1) with \(\tilde{\Lambda}_a\) and taking (B.4) into account we get

\[
i\hbar[N, F] = -\frac{1}{2}[\Delta^a, [F, \tilde{\Lambda}_a]].
\] (B.5)

This in turn implies

\[
(i\hbar)^2[N, [N, F]] = \frac{1}{2}[\Delta^a, [\Delta^b, [[F, \tilde{\Lambda}_b], \tilde{\Lambda}_a]]] = \frac{1}{4}\varepsilon_{ba}[\Delta^a, [\Delta^b, [[F, \tilde{\Lambda}_d], \tilde{\Lambda}_c]]] \varepsilon^{cd},
\] (B.6)
where the last equality follows from $[\Delta^a, \Delta^b] = 0$. Now
\begin{equation}
(i\hbar)^{-1}[N, F(\phi, \eta, \rho)] = -\left(\phi^a \frac{\partial}{\partial \phi^a} + \eta^a \frac{\partial}{\partial \eta^a} + \rho^a \frac{\partial}{\partial \rho^a}\right)F(\phi, \eta, \rho)
\end{equation}
which equivalently may be written as
\begin{equation}
(i\hbar)^{-1}[N, F(\alpha\phi, \alpha\eta, \alpha\rho)] = -\alpha \frac{d}{d\alpha} F(\alpha\phi, \alpha\eta, \alpha\rho)
\end{equation}
where $\alpha$ is a real parameter. From this expression it follows that the solution of (B.6) may be written as
\begin{equation}
F(\phi, \eta, \rho) = F(0, 0, 0) + \frac{1}{4}(i\hbar)^{-2} \int_0^1 \frac{d\alpha}{\alpha} \int_0^\alpha \frac{d\beta}{\beta} \varepsilon_{ba}[\Delta^a, [\Delta^b, [[F(\beta\phi, \beta\eta, \beta\rho), \tilde{\Lambda}_d], \tilde{\Lambda}_c]]] \varepsilon^{cd} \equiv F(0, 0, 0) + (i\hbar)^{-1}[\Delta^a, K_a(\phi, \eta, \rho)],
\end{equation}
where $F(0, 0, 0)$ satisfies
\begin{equation}
[\Delta^a, F(0, 0, 0)] = [\Omega^a, F(0, 0, 0)] = 0.
\end{equation}
Eq. (B.9) is the assertion in [10]. Notice that $\Phi$ has the explicit form
\begin{equation}
\Phi(\phi, \eta, \rho) = \frac{1}{2}(i\hbar)^{-2} \int_0^1 \frac{d\alpha}{\alpha} \int_0^\alpha \frac{d\beta}{\beta} [[F(\beta\phi, \beta\eta, \beta\rho), \tilde{\Lambda}_d], \tilde{\Lambda}_c] \varepsilon^{cd} + \alpha \varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]],
\end{equation}
where $K_a(\phi, \eta, \rho)$ are arbitrary odd operators with total ghost number minus three which do not depend on $\pi_\alpha, \xi_{aa}$ and $\sigma_\alpha$.

A similar analysis was performed within the Sp(2)-version of the BV-quantization in [10].

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