A FULLY NONLINEAR SOBOLEV TRACE INEQUALITY

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Abstract. The k-Hessian operator \( \sigma_k \) is the k-th elementary symmetric function of the eigenvalues of the Hessian. It is known that the k-Hessian equation \( \sigma_k(D^2u) = f \) with Dirichlet boundary condition \( u = 0 \) is variational; indeed, this problem can be studied by means of the k-Hessian energy \( -\int u \sigma_k(D^2u) \).

We construct a natural boundary functional which, when added to the k-Hessian energy, yields as its critical points solutions of k-Hessian equations with general non-vanishing boundary data. As a consequence, we prove a sharp Sobolev trace inequality for k-admissible functions \( u \) which estimates the k-Hessian energy in terms of the boundary values of \( u \).

1. Introduction

Let \( X \subset \mathbb{R}^n \) be a bounded smooth domain with boundary \( M = \partial X \). The usual sharp Sobolev trace inequality states that

\[
-\int_X u \Delta u \, dx + \oint_M f u_n \, d\mu \geq \oint_M f (u_f)_n \, d\mu
\]

for all \( f \in C^\infty(M) \) and all \( u \in C^\infty(\overline{X}) \) such that \( u|_M = f \), where \( u_n \) denotes the derivative of \( u \) with respect to the outward-pointing normal along \( M \), \( u_f \) is the harmonic function in \( X \) such that \( u_f|_M = f \), and \( dx, d\mu \) are the volume forms on \( X \) and \( M \), respectively. A standard density argument implies that the trace \( u \mapsto u|_M =: \text{tr} u \) extends to a bounded linear operator \( \text{tr} : W^{1,2}(X) \to W^{1/2,2}(M) \), while the extension \( f \mapsto u_f =: E(f) \) extends to a bounded linear operator \( E : W^{1/2,2}(M) \to W^{1/2}(X) \) such that \( \text{tr} \circ E \) is the identity.

The sharp Sobolev trace inequality (1.1) is a useful tool in many analytic and geometric problems. For example, the Dirichlet-to-Neumann map \( f \mapsto (u_f)_n \) is a pseudodifferential operator with principle symbol \( (-\Delta)^{1/2} \); indeed, it is the operator \( (-\Delta)^{1/2} \) when \( \Omega = \mathbb{R}_+^n \) is the upper half-plane. Thus (1.1) relates the energy of the local operator \( -\Delta \) to the energy of the nonlocal Dirichlet-to-Neumann operator, providing a useful tool for establishing estimates for PDEs stated in terms of the latter operator. This strategy provides a key motivation for the approach of Caffarelli and Silvestre [CS07] for studying fractional powers of the Laplacian. As another example, Escobar [Esc88, Esc90] proved an analogue of (1.1) on compact manifolds with boundary for which both sides of the inequality are conformally invariant. In particular, this recovers (1.1) when \( X = \mathbb{R}_+^n \). Using conformal invariance, he also proved a sharp Sobolev trace inequality which yields the continuous embedding \( W^{1,2}(\mathbb{R}_+^n) \subset L^{\frac{2(n+1)}{n+1}}(\mathbb{R}^{n-1}) \) when \( n \geq 3 \). This work has important implications for the Yamabe Problem on manifolds with boundary [Esc92].
considering weights or higher-order operators, analogues of \((1.1)\) have been established with implications for the energies of fractional powers of the Laplacian of all non-integral orders \([\text{CS07, Yan13}]\) as well as for the energies of conformally 

covariant fractional powers of the Laplacian \([\text{Cas15, CC16, CG11, CY15}]\) and the fractional Yamabe problem \([\text{GQ13}]\).

The purpose of this article is to establish an analogue of \((1.1)\) in terms of the 

\(k\)-Hessian energy \(\sigma_k(D^2u)\). Here \(D^2u\) denotes the Hessian of \(u\) and the \(k\)-th elementary symmetric function \(\sigma_k(A)\) of a symmetric matrix \(A\) is defined by

\[
\sigma_k(A) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}
\]

for \(\lambda_1, \ldots, \lambda_n\) the eigenvalues of \(A\). The Dirichlet problem

\[
\begin{aligned}
\sigma_k(D^2u) &= F(x, u), & \text{in } X, \\
u &= f(x), & \text{on } M
\end{aligned}
\]

(1.2)

has been well-studied for functions \(u\) in the elliptic \(k\)-cone

\[
(1.3) \quad \Gamma_k^+ := \{u \in C^\infty(X) \mid \sigma_j(D^2u) > 0, 1 \leq j \leq k\};
\]

e.g. \([\text{CNS85, TW04, Urb90, Wan94, Wan09}]\). Note that the existence of a solution to \((1.2)\) requires that \(M\) be \((k-1)\)-convex \([\text{CNS85}]\); i.e. the second fundamental form \(L\) of \(M\) must satisfy \(\sigma_j(L) > 0\) for \(1 \leq j \leq k-1\). Indeed, provided \(M\) is \((k-1)\)-convex, X.-J. Wang proved \([\text{Wan94}]\) the fully nonlinear Sobolev inequality

\[
\int_{X} -u\sigma_k(D^2u) dx \geq C(X) \left( \int_{X} |u|^{\frac{n(k+1)}{n-2k}} dx \right)^{\frac{n-2k}{n}}
\]

(1.4)

for all \(u \in \Gamma_k^+\) such that \(u|_M = 0\). In a sense, the Sobolev inequality \((1.4)\) is dual to the desired fully nonlinear analogue of \((1.1)\): in \((1.4)\) the extremal functions are “flat” on the boundary, in the sense \(u|_M = 0\), while in \((1.1)\) the extremal functions are “flat” in the interior, in that \(\Delta u = 0\).

To establish a fully nonlinear analogue of \((1.1)\) requires us to both know that the purported minimizers of the inequality exist and to identify what boundary terms to add to the interior term \(\int u\sigma_k(D^2u) dx\). The first problem is settled: existence and uniqueness of a solution \(u \in \Gamma_k^+\) of the degenerate Dirichlet problem \((1.2)\) with \(F = 0\) is known \([\text{TW04, WX14}]\); here \(\Gamma_k^+\) is the closure of the elliptic \(k\)-cone \((1.3)\) with respect to the \(C^{1,1}\)-norm in \(X\). The second problem is addressed in this article. This is accomplished via the following proposition.

**Proposition 1.1.** Let \(X \subset \mathbb{R}^n\) be a bounded smooth domain with boundary 

\(M = \partial X\) and let \(k \in \mathbb{N}\). Then there is a multilinear differential operator

\[
B_k : (C^1(X) \cap C^2(M))^k \to C^0(M)
\]

(1.5)

such that the multilinear form \(Q_k : (C^2(X) \cap C^2(M) \cap C^1(X))^k \to \mathbb{R}\) defined by

\[
Q_k(u, w^1, \ldots, w^k) := -\int_X u \sigma_k(D^2w^1, \ldots, D^2w^k) dx + \int_M u B_k(w^1, \ldots, w^k) d\mu
\]

(1.6)

is symmetric, where \(\sigma_k(D^2w^1, \ldots, D^2w^k)\) is the polarization of the \(k\)-linear map 

\(w \mapsto \sigma_k(D^2w)\).
Remark 1.2. The notation $(1.5)$ specifies that the operators $B_k$ depend on at most second-order tangential derivatives and at most first-order transverse derivatives of their inputs along the boundary $M$.

An explicit formula for such operators $B_k$ can be deduced from Section 3 and Section 4. From $(1.1)$ we see that $B_1(u) = u_n$ satisfies the conclusions of Proposition 1.1. The following result gives a boundary operator which satisfies the conclusions of Proposition 1.1 when $k = 1$.

Proposition 1.3. Let $X \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $M = \partial X$. Define $B_2: \left( C^1(\overline{X}) \cap C^2(M) \right)^2 \to C^0(M)$ by

\begin{equation}
B_2(v, w) = \frac{1}{2} \left( v_n \Delta w + w_n \Delta v + L(\nabla v, \nabla w) + H v_n w_n \right).
\end{equation}

Then the multilinear form $Q_2: (C^2(X))^3 \to \mathbb{R}$ given by

\begin{equation}
Q_2(u, v, w) = -\int_X u \sigma_2(D^2v, D^2w) dx + \int_M u B_2(v, w) d\mu
\end{equation}

is symmetric.

Here $\Delta$ and $\nabla$ denote the tangential Laplacian and tangential gradient, respectively; i.e. the Laplacian and the gradient defined with respect to the induced metric on the boundary $M$.

Denote by $E_k(u) := Q_k(u, \ldots, u)$ the energy associated to $Q_k$ as in Proposition 1.1. The fact that $(1.6)$ defines a symmetric $(k+1)$-linear form implies that if $v \in C^\infty(\overline{X})$ is such that $v|_M = 0$, then

\begin{equation}
\left. \frac{d}{dt} \right|_{t=0} E_k(u + tv) = -\frac{(k+1)!}{(k+1-j)!} \int_X v \sigma_k(D^2v, \ldots, \underbrace{D^2v, \ldots, D^2u, \ldots, D^2u}_{j-1\text{-times}}) dx
\end{equation}

for all $1 \leq j \leq k+1$. That is, within a class $C_f := \{ u \in C^\infty(\overline{X}) \mid u|_M = f \}$ of functions with fixed trace $f \in C^\infty(M)$, the derivatives of the energies $E_k$ depend only on the interior integrals. In particular, it is straightforward to identify the critical points of $E_k$ and deduce the convexity of $E_k$ within the positive cone $\Gamma^+_k$.

This leads to the following family of fully nonlinear Sobolev trace inequalities.

Theorem 1.4. Fix $k \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be a bounded $(k-1)$-convex domain with boundary $M = \partial X$. Let $B_k$ be as in Proposition 1.1. Given $f \in C^\infty(M)$, let

\begin{equation}
C_{f,k} := \{ u \in C_f \mid D^2u \in \Gamma^+_k \}.
\end{equation}

Then it holds that

\begin{equation}
E_k(u) \geq E_k(u_f)
\end{equation}

for all $u \in \overline{C_{f,k}}$, where $u_f$ is the unique solution to the Dirichlet problem

\begin{equation}
\begin{cases}
\sigma_k(D^2u) = 0, & \text{in } X, \\
u = f, & \text{on } M,
\end{cases}
\end{equation}

and $\overline{C_{f,k}}$ is the closure of $C_{f,k}$ with respect to the $C^{1,1}$-norm in $\overline{X}$.
Note that \( \mathcal{E}_k(u_f) = \oint f B_k(u_f, \ldots, u_f) d\mu \), so that Proposition 1.1 implies that the right-hand side of (1.8) depends only on \( f \), the tangential gradient \( \nabla f \), the tangential Hessian \( \bar{D}^2 f \), and the normal derivative \( (u_f)_n \) of the extension \( u_f \). This is consistent with the expected regularity \( u_f \in C^{1,1}(\overline{X}) \). One may regard (1.8) as a norm inequality for part of the trace embedding \( W^{2k+1, k+1}(X) \subset W^{2k-1, k+1}(M) \).

We conclude this introduction with a few additional comments on the boundary operators \( B_k \) of Proposition 1.1. Given \( f \in C^\infty(M) \) and \( k \in \mathbb{N} \), define

\[
B_k(f) := B_k(u_f, \ldots, u_f)
\]

for \( u_f \) the solution to (1.9). The specification (1.5) of the domain of the boundary operators \( B_k \) implies that \( B_k \) is a well-defined function; it should be regarded as a fully nonlinear analogue of the Dirichlet-to-Neumann map. Theorem 1.4 yields a relationship between the energy of \( B_k \) and the energy associated to the \( \sigma_k \)-curvature.

Motivated by the similar relationship between the energies associated to fractional order operators and the Laplacian induced by (1.1), we propose the study of the operators \( B_k \) as an interesting family of fully nonlinear pseudodifferential operators. In particular, it seems interesting to ask if there exists a constant \( C(M) > 0 \) such that

\[
A(M) \oint_M f B_k(f) d\mu + B(M) \oint_M |f|^{k+1} d\mu \geq \left( \oint_M |f|^{(k+1)(n-2k)} d\mu \right)^{\frac{n-2k}{n-k}}.
\]

If true, this would provide a fully nonlinear analogue of the sharp Sobolev inequality of X.-J. Wang [Wan09]. Note that this is already known in the case \( k = 1 \); cf. [LZ97].

The conditions of Proposition 1.1 do not uniquely determine the boundary operators \( B_k \) of Proposition 1.1; indeed, the operators are not unique even if we require additionally that the operators \( B_k \) commute with diffeomorphisms, as do the operators constructed in the proof of Proposition 1.1. A trivial source of nonuniqueness comes from the freedom to add symmetric zeroth-order terms to \( B_k \). For example, if \( B_k \) satisfies the conclusions of Proposition 1.1, so too does the operator

\[
(w^1, \ldots, w^k) \mapsto B_k(w^1, \ldots, w^k) + cH w^1 \cdots w^k
\]

for any \( c \in \mathbb{R} \), where \( H \) is the mean curvature of the boundary \( M \). More generally, one may add to the boundary operators \( B_k \) any symmetric multilinear operator which is also symmetric upon pairing with integration. For example, consider the operator \( D: (C^1(\overline{X}))^2 \to C^\infty(M) \) defined by

\[
D(v, w) = \delta(L(\nabla(vw))) - L(\nabla v, \nabla w).
\]

It is readily verified that \( (u, v, w) \mapsto \oint u D(v, w) d\mu \) is a symmetric trilinear form, and thus \( D \) can be added to the operator (1.7) to yield another operator \( \tilde{B}_2 \) which satisfies the conclusions of Proposition 1.1.

This article is organized as follows. In Section 2 we collect some useful facts involving the \( k \)-Hessian and the elliptic cones. In Section 3 and Section 4 we prove Proposition 1.1 by explicitly constructing a suitable boundary operator. In Section 5 we prove Theorem 1.4. In Section 6 we discuss in more detail the case \( k = 2 \).
2. Preliminaries

2.1. The $\Gamma_k^+$-cone. In this subsection, we describe some properties of the elementary symmetric functions and their associated convex cones.

**Definition 2.1.** The $k$-th elementary symmetric function for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is

$$
\sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.
$$

The elementary symmetric functions are special cases of hyperbolic polynomials \cite{Gar59}. As such, they enjoy many nice properties in their associated positive cones.

**Definition 2.2.** The positive $k$-cone is the connected component of $\{ \lambda \mid \sigma_k(\lambda) > 0 \}$ which contains $(1, \ldots, 1)$. Equivalently,

$$
\Gamma_k^+ = \{ \lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}.
$$

For example, the positive $n$-cone is

$$
\Gamma_n^+ = \{ \lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0 \}
$$

and the positive 1-cone is the half-space

$$
\Gamma_1^+ = \{ \lambda \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n > 0 \}.
$$

Note that $\Gamma_k^+$ is an open convex cone and that

$$
\Gamma_n^+ \subset \Gamma_{n-1}^+ \cdots \subset \Gamma_1^+.
$$

Applying Gårding’s theory of hyperbolic polynomials \cite{Gar59}, one concludes that $\sigma_k^+$ is a concave function in $\Gamma_k^+$.

**Definition 2.3.** A symmetric matrix $A$ is in the $\tilde{\Gamma}_k^+$ cone if its eigenvalues

$$
\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)) \in \Gamma_k^+.
$$

Suppose $f$ is a function on $\Gamma_k^+$. Denote by $F = f(\lambda(A))$ the function on $\tilde{\Gamma}_k^+$ induced by $f$. It is known \cite{CNSS85} that if $f$ is concave in $\Gamma_k^+$, then the induced function $F$ is concave in $\tilde{\Gamma}_k^+$. For this reason, we shall denote $\tilde{\Gamma}_k^+$ by $\Gamma_k^+$ and $\sigma_k(\lambda(A))$ by $\sigma_k(A)$ when there is no possibility of confusion.

Notice that $\sigma_n(A) = \det(A)$. An equivalent definition of $\det(A)$ is

$$
\det A := \frac{1}{n!} \delta_{i_1 \cdots i_n}^{j_1 \cdots j_n} A_{i_1 j_1} \cdots A_{i_n j_n},
$$

where $\delta_{i_1 \cdots i_n}^{j_1 \cdots j_n}$ is the generalized Kronecker delta: it is zero if $\{i_1, \ldots, i_n\} \neq \{j_1, \ldots, j_n\}$ and equals 1 (resp. equals $-1$) if $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ differ by an even (resp. odd) permutation. Similarly, an equivalent definition of $\sigma_k(A)$ is

$$
\sigma_k(A) := \frac{1}{k!} \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} A_{i_1 j_1} \cdots A_{i_k j_k}.
$$

The Newton transformation tensor is defined as

$$
T_k(A)_{ij} := \frac{1}{k!} \delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} (A)_{i_1 j_1} \cdots (A)_{i_k j_k}.
$$

**Definition 2.4.** The polarization of $\sigma_k$ is

$$
\sigma_k(A_1, \ldots, A_k) := \frac{1}{k!} \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}.
$$
It is called the polarization of $\sigma_k$ because $\sigma_k(A_1, \ldots, A_k)$ is the symmetric multilinear form such that $\sigma_k(A) = \sigma_k(A, \ldots, A)$.

**Definition 2.5.** The polarized Newton transformation tensor is

$$T_k(A_1, \ldots, A_k)_{ij} := \frac{1}{k!} \sigma^{ij_1 \cdots i_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}.$$

When some components in the polarizations are the same, we adopt the notational conventions

$$\sigma_k(B, \ldots, B, C, \ldots, C) := \sigma_k(B, \ldots, B, C, \ldots, C),$$

$$T_k(B, \ldots, B, C, \ldots, C)_{ij} := T_k(B, \ldots, B, C, \ldots, C)_{ij}.$$  

Some useful relations between the Newton transformation tensor $T_k$ and $\sigma_k$ are as follows. For any symmetric matrix $A$, if we denote the trace by $\text{Tr}$, then

$$\sigma_k(A) = \frac{1}{n-k} \text{Tr}(T_k(A)_{ij}),$$

$$\sigma_{k+1}(A) = \frac{1}{k+1} \text{Tr}(T_k(A)_{im} A_{mj}).$$

Many useful algebraic inequalities for elements of $\Gamma^+_k$ can be deduced from Gårding’s theory of hyperbolic polynomials [Gar59]. For us, the important such inequality is the fact that if $A_1, \ldots, A_k \in \Gamma^+_k$, then $T_k(A_1, \ldots, A_k)_{ij}$ is a nonnegative matrix.

3. Construction of the polarized functional

We begin our construction of the boundary integrals of Proposition 1.1. Define

$$S_0(u, w^1, \ldots, w^k) := -2 \sum_p \int_X u_i w^p T_{k-1}(D^2 w^p)_{ij} dx$$

$$- \sum_{p \neq q} \int_X w_i w^p T_{k-1}(D^2 w, D^2 w^p, q)_{ij} dx. \tag{3.1}$$

where $D^2 w^p$ denotes the list $(D^2 w^1, \ldots, D^2 w^{p-1}, D^2 w^{p+1}, \ldots, D^2 w^k)$ obtained from $(D^2 w^1, \ldots, D^2 w^k)$ by removing the entry $D^2 w^p$, and likewise $D^2 w^p, q$ denotes the list obtained from $(D^2 w^1, \ldots, D^2 w^k)$ by removing the entries $D^2 w^p$ and $D^2 w^q$. Similar notation will be used to remove more elements from the list. Using integration by parts to rewrite (3.1) as a sum of an interior and a boundary integral, both of which have integrands which factor through $u$, yields the following first step towards proving Proposition 1.1.

**Proposition 3.1.** There exists a symmetric $\mathbb{R}$-multilinear function $A_k: C^\infty(\overline{X}) \to C^\infty(M)$ such that

$$L(u, w^1, \ldots, w^k) := \int_X u \sigma_k(D^2 w^1, \ldots, D^2 w^k) dx + \oint_M u A_k(w^1, \ldots, w^k) d\mu$$

is symmetric in $u, w^1, \ldots, w^k$.

**Remark 3.2.** The operators $A_k$ constructed by our proof depend on at most 4 derivatives of their inputs.
Proof. Note that $S_0$ is symmetric. Our objective is to rewrite \( \text{(3.1)} \) in the desired form \( \text{(3.2)} \). To that end, writing \( \text{(3.1)} \) as a sum over pairs $p \neq q$ and then integrating by parts in $X$ yields

$$S_0 = \sum_{p \neq q} \left[ -\frac{2}{k-1} \int_X u_i w_j^p T_{k-1}(D^2 w^{\wedge p})_{ij} dx - \int_X w_i^p w_j^q T_{k-1}(D^2 u, D^2 w^{\wedge p,q})_{ij} dx \right]$$

$$= \sum_{p \neq q} \left[ \frac{2}{k-1} \int_X u_i w_j^p T_{k-1}(D^2 w^{\wedge p})_{ij} dx + \int_X w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} dx \right.$$

$$- \left. \frac{2}{k-1} \int_M w_i^p w_j^q T_{k-1}(D^2 u, D^2 w^{\wedge p,q})_{ij} d\mu - \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu \right].$$

Integrating by parts in $X$ once more yields

$$S_0 = \sum_{p \neq q} \left[ \frac{k+1}{k-1} \int_X u_i w_j^p T_{k-1}(D^2 w^{\wedge p})_{ij} dx - \frac{k+1}{k-1} \int_M u_i w_j^p T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu \right.$$

$$- \int_M w_i^p w_j^q T_{k-1}(D^2 u, D^2 w^{\wedge p,q})_{ij} d\mu + \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu \right].$$

Denote the boundary integral by $T$:

$$T = \sum_{p \neq q} \left[ \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu - \int_M w_i^p w_j^q T_{k-1}(D^2 u, D^2 w^{\wedge p,q})_{ij} d\mu \right.$$

$$- (k+1) \int_M w_i^p w_j^q T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu.]$$

Thus

$$S_0 = k^2(k+1) \int_X u \sigma_k(D^2 w^1, \ldots, D^2 w^k) dx + T.$$

We aim to write $T$ as the sum of a symmetric term and a boundary integral of the form $\int_M u B(w^1, \ldots, w^k) d\mu$. To that end, consider the symmetrization of the second term of \( \text{(3.3)} \):

$$S_1 := \sum_{p \neq q} \left[ -\int_M w_i^p w_j^q T_{k-1}(D^2 u, D^2 w^{\wedge p,q})_{ij} d\mu \right.$$

$$- \frac{1}{k-1} \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu - \frac{1}{k-1} \int_M u_i w_j^p T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu \right].$$

Note that $S_1$ is symmetric with respect to $u, w^1, \ldots, w^k$. Combining \( \text{(3.3)} \) and \( \text{(3.4)} \) yields

$$T = S_1 - k \sum_p \int_M w_i^p T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu + k \sum_p \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu.$$

We define

$$U_1 := -k \sum_p \int_M w_i^p T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu,$$

$$Q := k \sum_p \int_M w_i^p u_j T_{k-1}(D^2 w^{\wedge p})_{ij} d\mu.$$
so that
\[ T = U_1 + S_1 + Q. \]

\( U_1 \) is of the correct form \( \int uB(w^1, \ldots, w^p) \, d\mu \). We continue with the term \( Q \). Observe that
\[ Q = k \sum_p \left[ \int_M w^p u_\alpha T_{k-1}(D^2 w^p)_{\alpha\beta} \, d\mu + \int_M w^p u_n T_{k-1}(D^2 w^p)_{nn} \, d\mu \right]. \]

where Greek indices \( \alpha, \beta \in \{1, \ldots, n-1\} \) denote tangential directions and \( n \) denotes the outward-pointing normal along \( M \). By the definition of Newton tensor, \( T_{k-1}(D^2 w^p)_{nn} = \sigma_{k-1}(D^2 w^p|_{TM}) \), where \( D^2 w^p|_{TM} \) denotes the list of the restrictions \( D^2 w^1|_{TM}, \ldots, D^2 w^n|_{TM} \) with the \( p \)-th element removed. Thus
\[ Q = k \sum_p \left[ \int_M w^p u_\alpha T_{k-1}(D^2 w^p)_{\alpha\beta} \, d\mu + \int_M w^p u_n \sigma_{k-1}(D^2 w^p|_{TM}) \, d\mu \right]. \]

Define
\[ U_2 := k \sum_p \int_M w^p u_\alpha T_{k-1}(D^2 w^p)_{\alpha\beta} \, d\mu, \]
\[ Q_1 := k \sum_p \int_M w^p u_n \sigma_{k-1}(D^2 w^p|_{TM}) \, d\mu, \]
so that
\[ Q = U_2 + Q_1. \]

Integrating by parts along \( M \) shows that
\[ U_2 = -k \sum_p \int_M u(w^p T_{k-1}(D^2 w^p)_{\alpha\beta}) \, d\mu. \]

Thus \( U_2 \) is of the correct form \( \int uB(w^1, \ldots, w^p) \, d\mu \). Therefore we need only consider \( Q_1 \).

Consider the symmetrization of \( Q_1 \):
\[ S_2 := \sum_{p \neq q} \left[ \frac{k}{k-1} \int_M w^p u_n \sigma_{k-1}(D^2 w^q|_{TM}) \, d\mu + \frac{k}{k-1} \int_M u w^p_\alpha \sigma_{k-1}(D^2 w^q|_{TM}) \, d\mu \right. \]
\[ + k \int_M w^p w^q_\alpha \sigma_{k-1}(D^2 u|_{TM}, D^2 w^q|_{TM}) \, d\mu \right]. \]

Note that \( S_2 \) is symmetric with respect to \( u, w^1, \ldots, w^k \). Moreover,
\[ Q_1 = S_2 - \frac{k}{k-1} \sum_{p \neq q} \int_M u w^p_\alpha \sigma_{k-1}(D^2 w^q|_{TM}) \, d\mu \]
\[ - k \sum_{p \neq q} \int_M w^p w^q_\alpha \sigma_{k-1}(D^2 u|_{TM}, D^2 w^q|_{TM}) \, d\mu. \]

Denote by \( \bar{D^2} \) the Hessian with respect to the induced metric of \( M \) and by \( L_{\alpha\beta} \) the second fundamental form of \( M \). Given \( v \in C^\infty(\overline{\mathcal{M}}) \), it holds that
\[ D^2 v|_{TM} = \bar{D^2} v + v_n L \]
along $M$. Define

$$U_3 := -\frac{k}{k-1} \sum_{p \neq q} \int_M u w_p u_{n,k-1} (D^2 w |_{TM}) d\mu,$$

$$U_4 := -k \sum_{p \neq q} \int_M w_p u_{n,k-1} (\bar{D}^2 u, D^2 w |_{TM}) d\mu.$$

Integrating by parts along $M$ yields

$$U_4 = -\frac{k}{k-1} \sum_{p \neq q} \int_M u (w_p w_q u_{n,k-1} (L, D^2 w |_{TM})_{\alpha,\beta})_{\alpha,\beta} d\mu,$$

where the bars on $\alpha$ and $\beta$ denote covariant derivatives with respect to the induced metric on $M$. In particular, both $U_3$ and $U_4$ are of the form $\int_M u B(w^1, \ldots, w^k) d\mu$.

Define

$$Q_2 := -k \sum_{p \neq q} \int_M w_p u_{n,k-1} (L, D^2 w |_{TM}) d\mu.$$

It follows from (3.5), (3.6) and the definitions of $U_3, U_4, Q_2$ that

$$Q_1 = S_2 + U_3 + U_4 + Q_2.$$

Now we want to write $Q_2$ in the desired form. To that end, consider the symmetrization of $Q_2$:

$$S_3 := -k \sum_{p \neq q \neq r} \left[ \frac{1}{k-2} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu \right. + \frac{1}{2!(k-2)} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu$$

$$\left. + \frac{1}{2!} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu \right].$$

(3.7)

Note that $S_3$ is symmetric with respect to $u, w^1, \ldots, w^k$. Define

$$U_5 := \frac{k}{2!(k-2)} \sum_{p \neq q \neq r} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu,$$

$$U_6 := \frac{k}{2!} \sum_{p \neq q \neq r} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu.$$

As above, integration by parts along $M$ implies that both $U_5$ and $U_6$ are of the form $\int_M u B(w^1, \ldots, w^k) d\mu$.

Define

$$Q_3 := \frac{k}{2!} \sum_{p \neq q \neq r} \int_M w_p w_q w_r u_{n,k-1} (L, D^2 w |_{TM}) d\mu.$$

From (3.7) and the definitions of $Q_2, U_5, U_6$ and $Q_3$ we deduce that

$$Q_2 = S_3 + U_5 + U_6 + Q_3.$$
Proceeding in this way, for all \( 2 \leq i \leq k \) we make the following definitions. First, define

\[
S_i := (-1)^i k \sum_{p_1 \neq \cdots \neq p_i} \left[ \frac{1}{(i-2)!(k+1-i)} \oint_M w^{p_1}w^{p_2} \cdots w^{p_{i-1}}w_n \right.
\]
\[
\times \sigma_{k-1}(L, \ldots, L, D^2 w|_{TM})^{p_i} \cdots (p_i-1) d\mu
\]
\[
+ \frac{1}{(i-1)!(k+1-i)} \oint_M u w^{p_1} \cdots w^{p_{i-1}} \sigma_{k-1}(L, \ldots, L, D^2 w|_{TM})^{p_i} \cdots (p_i-1) d\mu,
\]
\[
+ \frac{1}{(i-1)!} \oint_M w^{p_1}w^{p_2} \cdots w^{p_i} \sigma_{k-1}(L, \ldots, L, D^2 u|_{TM}, D^2 w|_{TM})^{p_i} \cdots (p_i) d\mu].
\]

Note that \( S_i \) is symmetric with respect to \( u, w^1, \ldots, w^k \). Next, define

\[
U_{2i-1} := \frac{(-1)^{i+1}k}{(i-1)!(k+1-i)} \sum_{p_1 \neq \cdots \neq p_i} \oint_M u w^{p_1} \cdots w^{p_{i-1}}
\]
\[
\times \sigma_{k-1}(L, \ldots, L, D^2 w|_{TM})^{p_i} \cdots (p_i-1) d\mu,
\]
\[
U_{2i} := \frac{(-1)^{i+1}k}{(i-1)!} \sum_{p_1 \neq \cdots \neq p_i} \oint_M w^{p_1}w^{p_2} \cdots w^{p_i} \sigma_{k-1}(L, \ldots, L, D^2 u, D^2 w|_{TM})^{p_i} \cdots (p_i) d\mu.
\]

Integration by parts along \( M \) implies that both \( U_{2i-1} \) and \( U_{2i} \) are of the form \( \oint u B(w^1, \ldots, w^k) d\mu \). Then

\[
Q_i := \frac{(-1)^{i+1}k}{(i-1)!} \sum_{p_1 \neq \cdots \neq p_i} \oint_M w^{p_1}w^{p_2} \cdots w^{p_i} \sigma_{k-1}(L, \ldots, L, D^2 w|_{TM})^{p_i} \cdots (p_i) d\mu
\]

is such that

\[
Q_{i-1} = S_i + U_{2i-1} + U_{2i} + Q_i.
\]

It remains to write \( Q_k \) as the sum of a symmetric integral and a boundary integral whose integrand factors through \( u \). To that end, define

\[
S_{k+1} := \frac{(-1)^{k+1}k}{(k-1)!} \sum_{p_1 \neq \cdots \neq p_k} \left[ \oint_M w^{p_1}w^{p_2} \cdots w^{p_k} u_n \sigma_{k-1}(L) d\mu \right.
\]
\[
+ \frac{1}{k} \oint_M u w^{p_1} \cdots w^{p_k} \sigma_{k-1}(L) d\mu.
\]

Note that \( S_{k+1} \) is symmetric with respect to \( u, w^1, \ldots, w^k \). Also define

\[
U_{2k+1} := \frac{(-1)^k}{(k-1)!} \sum_{p_1 \neq \cdots \neq p_k} \oint_M u w^{p_1} \cdots w^{p_k} \sigma_{k-1}(L) d\mu.
\]

Note that \( U_{2k+1} \) is of the form \( \oint u B(w^1, \ldots, w^k) d\mu \) and that

\[
Q_k = S_{k+1} + U_{2k+1}.
\]
In summary, we have shown that
\[
S_0 - \sum_{i=1}^{k+1} S_i = k^2(k+1) \int_X u \sigma_k(D^2w_1, \ldots, D^2w_k)dx + \sum_{i=1}^{2k+1} U_i
\]
and observed that the left-hand side is symmetric in \( u, w_1, \ldots, w_k \) while the right-hand side is of the form \( \oint uB(w_1, \ldots, w_k)d\mu \). Dividing (3.8) through by \( k^2(k+1) \) yields (3.2). □

4. Adjusted polarized functional

The difference between Proposition 1.1 and Proposition 3.1 is that in the latter result, we only ask that the boundary integrals making up the polarized functional are such that their integrands factor through \( u \). In particular, it is not clear that from the proof of Proposition 3.1 that the functions \( A_k \) depend only on at most second-order tangential derivatives and at most first-order transverse derivatives along \( M \). This arises in two ways. First, the integral \( U_1 \) depends on the second-order derivative \( w_{\alpha n} \). Second, when written in the form \( \oint uB(w_1, \ldots, w_k)d\mu \), the integrals \( U_{2i}, 1 \leq i \leq k \), depend also on third- and fourth-order derivatives of \( w^p \).

By more carefully considering the integration by parts along \( M \) invoked in the proof of Proposition 3.1, we show that the combination \( \sum U_i \) only depends on at most second-order tangential derivatives and at most first-order transverse derivatives of \( w^p \). This proves Proposition 1.1. To that end, we first require a few facts.

**Lemma 4.1.** Let \( X \subset \mathbb{R}^n \) be a bounded smooth domain with boundary \( M = \partial X \). Let \( w_1, \ldots, w_k \in C^\infty(X) \). Then
\[
w_{\beta n} = w_{n\beta} - L_{\alpha\beta}w_\alpha,
\]
(4.1)
\[
T_k(D^2w_1, \ldots, D^2w_k)_{\alpha n} = -\frac{1}{k} \sum_{p=1}^{k} T_{k-1}(D^2w_{1}^{p})_{\alpha\beta}w_{\beta n}^{p},
\]
(4.2)
where \( \alpha, \beta \in \{1, \ldots, n-1\} \) denote tangential directions, \( n \) denotes the outward-pointing normal along the boundary, and \( w_{n\beta} \) denotes the tangential gradient of \( w_n \). Moreover,
\[
T_k(L_1, \ldots, L_i, D^2w_T^{p_1, \ldots, p_i})_{\alpha\beta} = \sum_{p \neq p_1, \ldots, p_i} T_k(L_1, \ldots, L_i, D^2w_T^{p_1, p_2, \ldots, p_i})_{\alpha\beta}w_{\beta n}^{p},
\]
(4.3)
where the left-hand side denotes the divergence with respect to the induced metric on \( M \).

**Proof.** (4.1) follows immediately from the definition of the second fundamental form \( L \) and (4.2) follows immediately from the definitions of the Newton tensors. To prove (4.3), first recall that the Newton tensors are divergence-free with respect to the flat metric in \( X \). From the definition of the second fundamental form, we have that
\[
w_{\alpha\beta, \gamma} = w_{\alpha\beta, \gamma} + L_{\alpha\gamma}w_{\beta n} + L_{\beta\gamma}w_{\alpha n}.
\]
Inserting this into the definition of the Newton tensors yields the result (cf. [Che09, Lemma 11]). □
Lemma [4.1] allows us to carefully perform the integration by parts argument as described above.

**Proof of Proposition [4.1]** Denote \( C := C^1(\mathcal{X}) \cap C^2(M) \). Define

\[
\begin{align*}
\hat{U}_1 &= -k \sum_p \oint_M u w^n_p \sigma_{k-1}(D^2 w|_{TM}) d\mu, \\
\hat{U}_1 &= -k \sum_p \oint_M u w^n_p T_{k-1}(D^2 w) d\mu.
\end{align*}
\]

It follows from (3.6) that \( \hat{U}_1 \) is well-defined on \( C \); i.e. \( \hat{U}_1 \) depends on at most second-order tangential derivatives and first-order transverse derivatives of \( u^1, \ldots, u^k \) on \( M \). Furthermore, we have that

\[
U_1 = \hat{U}_1 + \hat{U}_1.
\]

Consider now \( U_1 + U_2 + U_4 \). Define

\[
\begin{align*}
W_1 &= -k \sum_{p \neq q} \oint_M u w^n_p T_{k-2}(D^2 w|_{TM}) \alpha_\beta \gamma \omega_{\gamma} d\mu, \\
W_2 &= -k \sum_{p \neq q} \oint_M u w^n_p T_{k-2}(D^2 w|_{TM}) \alpha_\beta \omega_{\gamma} d\mu, \\
W_3 &= \frac{k}{k-1} \sum_{p \neq q \neq r} \oint_M u w^n_p w^q_r T_{k-2}(L, D^2 w|_{TM}) \alpha_\beta \gamma \omega_{\gamma} d\mu.
\end{align*}
\]

It follows from (3.6) that \( W_1, W_2, W_3 \) are well-defined on \( C \). Define also

\[
\begin{align*}
V_1 &= \frac{k}{k-1} \sum_{p \neq q} \oint_M u^p u_q T_{k-2}(D^2 w|_{TM}) \alpha_\beta \gamma \omega_{\gamma} d\mu, \\
V_2 &= -\frac{k}{k-1} \sum_{p \neq q \neq r} \oint_M u^p u_q T_{k-2}(L, D^2 w|_{TM}) \alpha_\beta \gamma \omega_{\gamma} d\mu,
\end{align*}
\]

Note that \( V_1 \) and \( V_2 \) still involve derivatives of \( u \); this issue will be dealt with later. Integrating by parts along \( M \) and using Lemma [4.1] yields

\[
\begin{align*}
\hat{U}_1 + U_2 + U_4 &= W_1 + V_1 + \frac{k}{k-1} \sum_{p \neq q} \oint_M u w^n_p T_{k-2}(D^2 w|_{TM}) \alpha_\beta \omega_{n_\beta} d\mu \\
&- \frac{k}{k-1} \sum_{p \neq q} \oint_M w^p T_{k-2}(D^2 w|_{TM}) \alpha_\beta (u_n \omega_n) \beta d\mu \\
&= W_1 + V_1 + V_2 + \frac{k}{k-1} \sum_{p \neq q} \oint_M w^n_p T_{k-2}(D^2 w|_{TM}) \alpha_\beta (u_n \omega_n) \beta d\mu \\
&+ \frac{k}{k-1} \sum_{p \neq q \neq r} \oint_M w^p w^n_p u_q T_{k-2}(L, D^2 w|_{TM}) \alpha_\beta \omega_{n_\beta} d\mu \\
&= W_1 + W_2 + W_3 + V_1 + V_2 + \hat{U}_2 + \hat{U}_3,
\end{align*}
\]
where

\[
\hat{U}_2 := -\frac{k}{k-1} \sum_{p \neq q \neq r} \oint_M w_n^p w_n^q T_{k-2}(L, D^2 w_{TM}^{\langle p, q, r \rangle})_{\alpha\beta} w_n^r d\mu,
\]

\[
\hat{U}_3 := \frac{k}{k-1} \sum_{p \neq q \neq r} \oint_M w_n^p w_n^q u_{\alpha} T_{k-2}(L, D^2 w_{TM}^{\langle p, q, r \rangle})_{\alpha\beta} w_n^r d\mu.
\]

We continue this process by considering \(\hat{U}_2 + \hat{U}_3 + U_6\). More generally, given \(1 \leq i \leq k - 1\), we make the following definitions. First, define

\[
W_{2i-1} := (-1)^i \frac{k}{k-1} \sum_{p_0 \neq \cdots \neq p_i} \frac{1}{(i-1)!} \oint_M w_n^{p_0} w_n^{p_1} \cdots w_n^{p_{i-1}} \times T_{k-2}(L, \ldots, L, D^2 w_{TM}^{\langle p_0, \ldots, p_i \rangle})_{\alpha\beta} L_{\beta\gamma} w_n^p d\mu,
\]

\[
W_{2i} := (-1)^i \frac{k}{k-1} \sum_{p_0 \neq \cdots \neq p_i} \frac{1}{i!} \oint_M w_n^{p_0} \cdots w_n^{p_{i-1}} \times T_{k-2}(L, \ldots, L, D^2 w_{TM}^{\langle p_0, \ldots, p_i \rangle})_{\alpha\beta} L_{\beta\gamma} w_n^p d\mu.
\]

It follows from \([3.3]\) that \(W_{2i-1}\) and \(W_{2i}\) are well-defined on \(\mathcal{C}\). Next, define

\[
V_i := (-1)^{i+1} \frac{k}{k-1} \sum_{p_0 \neq \cdots \neq p_i} \frac{1}{(i-1)!} \oint_M u_{\alpha} w_n^{p_0} w_n^{p_1} \cdots w_n^{p_{i-1}} \times T_{k-2}(L, \ldots, L, D^2 w_{TM}^{\langle p_0, \ldots, p_i \rangle})_{\alpha\beta} L_{\beta\gamma} w_n^p d\mu.
\]

Note that \(V_i\) still involves derivatives of \(u\); this issue will be dealt with later. Finally, define

\[
\hat{U}_{2i} := (-1)^i \frac{k}{k-1} \sum_{p_0 \neq \cdots \neq p_{i+1}} \frac{1}{i!} \oint_M w_n^{p_0} w_n^{p_1} \cdots w_n^{p_i} \times T_{k-2}(L, \ldots, L, D^2 w_{TM}^{\langle p_0, \ldots, p_{i+1} \rangle})_{\alpha\beta} w_n^{p_{i+1}} d\mu,
\]

\[
\hat{U}_{2i+1} := (-1)^{i+1} \frac{k}{k-1} \sum_{p_0 \neq \cdots \neq p_{i+1}} \frac{1}{i!} \oint_M u_{\alpha} w_n^{p_0} w_n^{p_1} \cdots w_n^{p_i} \times T_{k-2}(L, \ldots, L, D^2 w_{TM}^{\langle p_0, \ldots, p_{i+1} \rangle})_{\alpha\beta} w_n^{p_{i+1}} d\mu;
\]

note that \(\hat{U}_{2k-2} = \hat{U}_{2k-1} = 0\). Integrating by parts along \(M\) and using Lemma \([4.1]\) yields

\[
\hat{U}_{2i} + \hat{U}_{2i+1} + U_{2i+4} = V_{i+2} + W_{2i+2} + W_{2i+3} + \hat{U}_{2i+2} + \hat{U}_{2i+3}.
\]

In particular, it follows that

\[
\sum_{i=1}^{2k+1} U_i = \hat{U}_1 + \sum_{i=1}^{k} U_{2i+1} + \sum_{i=1}^{2k-2} W_i + \sum_{i=1}^{k-1} V_i.
\]
Note that $\sum U_{2i+1}$ and $\sum W_i$ are all well-defined on $C$. It remains to check that, after integration by parts, $\sum V_i$ can be written as a boundary integral with integrand the product of $u$ with a function which is well-defined on $C$.

Given $1 \leq i \leq k-1$, define

$$A_i := (-1)^i \frac{k}{(i-1)!(k-1)} \sum_{p_0 \neq \ldots \neq p_{i+1}} \int_M u w_{p_0} w_{p_1} \ldots w_{p_{i-1}} w_{p_i} w_{p_{i+1}}$$

$$\times T_{k-2}(L, \ldots, L, D^2 w_{\alpha \gamma \ldots \beta}) \alpha \gamma \beta d\mu,$$

$$B_i := (-1)^{i+1} \frac{k}{(i-1)!(k-1)} \sum_{p_0 \neq \ldots \neq p_{i}} \int_M u w_{p_0} w_{p_1} \ldots w_{p_{i-2}} w_{p_i}$$

$$\times T_{k-2}(L, \ldots, L, D^2 w_{\alpha \gamma \ldots \beta}) \alpha \gamma \beta d\mu,$$

$$C_i := (-1)^i \frac{k}{(i-1)!(k-1)} \sum_{p_0 \neq \ldots \neq p_i} \int_M u w_{p_0} \ldots w_{p_{i-2}}$$

$$\times T_{k-2}(L, \ldots, L, D^2 w_{\alpha \gamma \ldots \beta}) \alpha \gamma \beta d\mu.$$

Note that $B_i$ and $C_i$ are well-defined on $C$. Moreover, integration by parts along $M$ readily yields

$$V_i = A_i - A_{i-1} + B_i + C_i,$$

where we interpret $A_0 = 0$. Since $A_{k-1} = 0$, it follows that

$$\sum_{i=1}^{k-1} V_i = \sum_{i=1}^{k-1} (B_i + C_i).$$

Combining (4.4) and (4.5) yields the desired result.

5. The First and Second Variation

It is straightforward to compute the first and second variations of the energy functional

$$\mathcal{E}_k(u) := Q_k(u, \ldots, u)$$

associated to the symmetric multilinear form constructed by Proposition 1.1.

**Proposition 5.1.** Let $X \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $M = \partial X$. Let $u, v \in C^\infty(X)$ and suppose that $v|_M = 0$. Then

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_k(u + tv) = -(k+1) \int_X v \sigma_k(D^2 u, \ldots, D^2 u) dx.$$

**Proof.** Since $Q_k$ is symmetric, we compute that

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_k(u + tv) = (k+1) Q_k(v, u, \ldots, u).$$

Since $v|_M = 0$, we see that the boundary integral in (1.6) vanishes. This yields (5.1).
Proposition 5.2. Let $X \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $M = \partial X$. Let $u, v \in C^\infty(X)$ and suppose that $v|_M = 0$. Then
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} E_k(u + tv) = (k + 1) \int_X v_i v_j T_{k-1}(D^2u)_{ij} dx.
\]
In particular, if $u \in \Gamma^+_k$, then
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} E_k(u + tv) \geq 0
\]
for all $v \in C^\infty(X)$ such that $v|_M = 0$.

Proof. Since $Q_k$ is symmetric, we compute that
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} E_k(u + tv) = k(k + 1) Q_k(v, v, u, \ldots, u).
\]
Since $v|_M = 0$, it follows that
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} E_k(u + tv) = -(k + 1) \int_X v T_{k-1}(D^2u)_{ij} v_{ij} dx
\]
\[
= -(k + 1) \int_X v_i v_j T_{k-1}(D^2u)_{ij} dx.
\]
The last conclusion follows from the fact that if $u \in \Gamma^+_k$, then $T_{k-1}(D^2u)_{ij}$ is nonnegative. $\square$

We are now ready to prove Theorem 1.4, which we restate here for convenience.

Theorem 5.3. Let $X \subset \mathbb{R}^n$ be a bounded smooth domain with $(k-1)$-convex boundary $M = \partial X$. Fix $f \in C^\infty(M)$ and denote $C_{f,k} = \{u \in \Gamma^+_k | u|_M = f\}$.

Then
\[
\mathcal{E}_k(u) \geq \mathcal{E}_k(u_f)
\]
for all $u \in \overline{C_{f,k}}$, where $u_f \in \overline{C_{f,k}}$ is the solution to the Dirichlet problem
\[
\begin{cases}
\sigma_k(u_f) = 0, & \text{in } X, \\
u_f = f, & \text{on } M.
\end{cases}
\]

Proof. By Proposition 5.1, the solution $u_f$ to (5.2) is a critical point of the functional $\mathcal{E}_k: C^{1,1}(\overline{X}) \rightarrow \mathbb{R}$. By Proposition 5.2, the restriction $\mathcal{E}_k: \overline{C_{f,k}} \rightarrow \mathbb{R}$ is a convex functional. Since $\overline{C_{f,k}}$ is convex, $u_f$ realizes the infimum of $\mathcal{E}_k: \overline{C_{f,k}} \rightarrow \mathbb{R}$. Indeed, if not, then there is a $u \in \overline{C_{f,k}}$ such that $\mathcal{E}_k(u) < \mathcal{E}_k(u_f)$. Since $\overline{C_{f,k}}$ is convex, it follows that $u + t(1-t)u_f \in \overline{C_{f,k}}$ for all $t \in [0, 1]$. Denote $\mathcal{E}_k(t) := \mathcal{E}_k(tu + (1-t)u_f)$. Since $\mathcal{E}_k(u) < \mathcal{E}_k(u_f)$, there exists a $t^* \in [0, 1]$ such that $\mathcal{E}_k''(t^*) < 0$. This contradicts the facts that $\mathcal{E}_k'(0) = 0$ and $\mathcal{E}_k''(t) \geq 0$ for all $t \in [0, 1]$. $\square$
We conclude this article by considering the specific case \( k = 2 \); the case \( k = 1 \) is covered by \([1.1]\). First, a suitable boundary operator as in Proposition \([1.1]\) is given by Proposition \([1.3]\) which we restate here for convenience.

**Proposition 6.1.** Let \( X \subset \mathbb{R}^n \) be a bounded smooth domain with boundary \( M = \partial X \). Define \( B : (C^1(\overline{X}) \cap C^2(M))^2 \to C^0(M) \) by

\[
B_2(v, w) = \frac{1}{2} (v_n \Delta w + w_n \Delta v + L(\nabla v, \nabla w) + H v_n w_n).
\]

Then the multilinear form \( Q_2 : (C^2(X))^3 \to \mathbb{R} \) given by

\[
Q_2(u, v, w) = -\int_X u \sigma_2(D^2u, D^2w) dx + \oint_M u B_2(v, w) d\mu
\]

is symmetric.

**Proof.** Following the proof of Proposition \([1.1]\) we see that a suitable choice of boundary operator is

\[
\tilde{B}_2(v, w) := \frac{1}{2} (v_n \Delta w + w_n \Delta v + L(\nabla v, \nabla w) + H v_n w_n)
\]

\[+ \frac{1}{6} (A(\nabla v, \nabla w) + v(A, \tilde{D}^2 w) + w(A, \tilde{D}^2 v) + v(\nabla H, \nabla w) + w(\nabla H, \nabla v)).
\]

A straightforward computation yields

\[
\delta (vA(\nabla w)) + \delta (wA(\nabla v)) - A(\nabla v, \nabla w)
\]

\[= A(\nabla v, \nabla w) + v(A, \tilde{D}^2 w) + w(A, \tilde{D}^2 v) + v(\nabla H, \nabla w) + w(\nabla H, \nabla v)
\]

On the other hand,

\[
\oint_M u \left[ \delta (vA(\nabla w)) + \delta (wA(\nabla v)) - A(\nabla v, \nabla w) \right] d\mu
\]

\[= -\oint_M [uA(\nabla v, \nabla w) + vA(\nabla w, \nabla u) + wA(\nabla u, \nabla v)] d\mu
\]

is symmetric in \( u, v, w \). Thus \( B_2 - \tilde{B}_2 \), and hence \( Q_2 \), is symmetric in \( u, v, w \). \( \square \)

Applying this boundary operator in Theorem \([1.4]\) yields the following sharp Sobolev trace inequality.

**Theorem 6.2.** Let \( X \subset \mathbb{R}^n \) be a bounded smooth mean-convex domain with boundary \( M = \partial X \). Given \( f \in C^\infty(M) \), set

\[
C_f = \{ u \in \Gamma_2^+ \mid u|_M = f \}
\]

Then it holds that

\[-\int_X u \sigma_2(D^2u) dx + \oint_M u B_2(u, u) d\mu \geq \oint_M f B_2(u_f, u_f) d\mu
\]

for all \( u \in C_f \), where \( B_2 \) is the operator \([6.1]\) and \( u_f \in C^{1,1}(\overline{X}) \cap \Gamma_2^+ \) is the unique solution to the Dirichlet problem

\[
\begin{cases}
\sigma_2(D^2 u_f) = 0, & \text{in } X, \\
u = f, & \text{on } M.
\end{cases}
\]
SOBOLEV TRACE INEQUALITY

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