ELLiptic CURves With everywhere Good Reduction

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Abstract. We consider the question of which quadratic fields have elliptic curves with everywhere good reduction. By revisiting work of Setzer, we expand on congruence conditions that determine the real and imaginary quadratic fields with elliptic curves of everywhere good reduction and rational $j$-invariant. Using this, we determine the density of such real and imaginary quadratic fields. If $R(X)$ denotes the number of real quadratic fields $K = \mathbb{Q}[\sqrt{m}]$ such that $|\Delta_K| < X$ and for which there exists an elliptic curve $E/K$ with rational $j$-invariant that has everywhere good reduction, then $R(X) \gg \frac{X}{\sqrt{\log(X)}}$. We also obtain a similar result for imaginary quadratic fields. To obtain these estimates we explicitly construct quadratic fields over which we can construct elliptic curves with everywhere good reduction. The estimates then follow from elementary multiplicative number theory. In addition, we obtain infinite families of real and imaginary quadratic fields such that there are no elliptic curves with everywhere good reduction over these fields.

1. Introduction

It is a well-known result that over $\mathbb{Q}$ there are no elliptic curves $E$ with everywhere good reduction. However, the same is not true over general number fields. For example, let $K = \mathbb{Q}(\sqrt{29})$ and $a = \frac{5 + \sqrt{29}}{2}$. Then the elliptic curve

$$E : y^2 + xy + a^2y = x^3$$

has unit discriminant, and hence has everywhere good reduction over $K$.

This leads to the natural question: Over which number fields do there exist elliptic curves with everywhere good reduction? This question has often been approached by studying $E/K$ with everywhere good reduction which satisfy additional properties, such as those which have a $K$-rational torsion point, admit a global minimal model, or have rational $j$-invariant. We say that an elliptic curve $E/K$ has $EGR(K)$ if it has everywhere good reduction over $K$, and that an elliptic curve $E/K$ has $EGR_{\mathbb{Q}}(K)$ if it additionally has $\mathbb{Q}$-rational $j$-invariant. Similarly, we say a quadratic field has $EGR$ if there exists a $EGR(K)$ elliptic curve and a quadratic field has $EGR_{\mathbb{Q}}$ if there exists a $EGR_{\mathbb{Q}}(K)$ elliptic curve.

For many real and imaginary quadratic fields $K$ of small discriminant, explicit examples of elliptic curves $E/K$ with everywhere good reduction can be found in the literature, such as [8] and [6]. There are also many known examples of such fields for which there do not exist any elliptic curves $E/K$ with everywhere good reduction; see [8], [11], [7] for example.

For example, Kida [8] showed that if $K$ satisfies certain hypotheses, every $E/K$ with $EGR$ has a $K$-rational point of order two. This condition led to a series of non-existence results

2010 Mathematics Subject Classification. 11G05.

Key words and phrases. elliptic curves; quadratic fields; diophantine equations; everywhere good reduction.

The second author thanks the NSF for its support.
for particular real quadratic fields with small discriminant. In [14], Setzer classified elliptic curves with $EGR_\mathbb{Q}$ over real quadratic number fields. Kida extended Setzer’s approach by giving a more general method suitable for computing elliptic curves with $EGR$ over certain real quadratic fields with rational or singular $j$-invariants in [9]. Comalada [1] showed that there exists $E/K$ with $EGR$, a global minimal model, and a $K$-rational point of order two if and only if one of his sets of diophantine equations has a solution. Ishii supplemented this theorem by studying $k$-rational 2 division points in [6] to demonstrate specific real quadratic fields without $EGR$ elliptic curves. Later Kida and Kagawa in [11] generalized Ishii’s result to obtain non-existence results for $\mathbb{Q}(\sqrt{17})$, $\mathbb{Q}(\sqrt{73})$ and $\mathbb{Q}(\sqrt{97})$. Yu Zhao determined criteria for real quadratic fields to have elliptic curves with $EGR$ and a non-trivial 3-division point. In [16], he provides a table for all such fields with discriminant less than 10,000.

For imaginary quadratic fields, Stroeker [15] showed that no $E/K$ with $EGR$ admits a global minimal model. In [13], Setzer showed that there exist elliptic curves with $EGR$ and a $K$-rational point of order two if and only if $K = \mathbb{Q}(\sqrt{-m})$ with $m$ satisfying certain congruence conditions. Comalada and Nart provided criteria to determine when elliptic curves have $EGR$ in [2]. Kida combined this result with a method of computing the Mordell-Weil group in [10] to prove there are no elliptic curves with $EGR$ over the fields $\mathbb{Q}(\sqrt{-35}), \mathbb{Q}(\sqrt{-37}), \mathbb{Q}(\sqrt{-51})$ and $\mathbb{Q}(\sqrt{-91})$. There are no elliptic curves with $EGR_\mathbb{Q}(K)$ for $-37 < m < -1$. However, there are elliptic curves with small discriminant and $EGR_\mathbb{Q}(K)$ for real quadratic fields $K$.

Table 1 shows what is known for $K = \mathbb{Q}(\sqrt{m})$ with square-free positive integers $m \leq 47$. We stop at 47 because to the best of our knowledge, the $m = 51$ case is still unknown.

A combination of the above results gives many methods to prove that a particular quadratic number field has an $EGR$ elliptic curve. Cremona and Lingham [3] described an algorithm for finding all elliptic curves over any number field $K$ with good reduction outside a given set of primes. However, this procedure relies on finding integral points on certain elliptic curves over $K$, which can limit its practical implementation. As a consequence of Setzer’s result regarding the classification of elliptic curves over both real and imaginary quadratic number fields with rational $j$-invariant, it is known that there infinitely many quadratic fields which have an $EGR$ elliptic curve. However, there is no conjectured density result for the proportion of quadratic fields over which there exist elliptic curves $E$ with everywhere good reduction.

Let $R(X)$ be the number of real quadratic number fields $K$ with discriminant at most $X$ and an $EGR_\mathbb{Q}(K)$ elliptic curve. By revisiting the results of Setzer, we prove the following.

**Theorem 1.1.** With $R(X)$ as above, we have that

$$R(X) \gg \frac{X}{\sqrt{\log(X)}}.$$ 

If $I(X)$ is the number of imaginary quadratic number fields $K$ with $|\Delta_K| < X$ and an $EGR_\mathbb{Q}(K)$ elliptic curve, we also obtain the result below.

**Theorem 1.2.** With $I(X)$ as above, we have that

$$I(X) \gg \frac{X}{\sqrt{\log(X)}}.$$
To prove Theorem 1.1, we first show that all real quadratic fields of the form described below in Theorem 1.3 have EGR \( \mathbb{Q} \), and then count these fields.

**Theorem 1.3.** Let \( m = 2q \), where \( q = q_1 \cdots q_n \equiv 3 \pmod{8} \) with \( q_j \equiv 1, 3 \pmod{8} \) distinct primes. Then the real quadratic field \( K = \mathbb{Q}(\sqrt{m}) \) has EGR \( \mathbb{Q} \).

**Remark 1.** If \( m \) is as described in Theorem 1.3, there exists \( E/K \) with EGR \( \mathbb{Q} \) and \( j(E) = 20^3 \) as shown by Setzer in 2.1.

Similarly, to prove Theorem 1.2, we show all imaginary quadratic fields found below in Theorem 1.4 have EGR \( \mathbb{Q} \).

**Theorem 1.4.** Let \( m = 37q \), where \( q = -q_1 \cdots q_n \equiv 1 \pmod{8} \) with \( q_j \) distinct primes such that \( \left( \frac{q}{37} \right) = 1 \). Then the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{m}) \) has EGR \( \mathbb{Q} \).

**Remark 2.** If \( m \) is as described in Theorem 1.4, there exists \( E/K \) with EGR \( \mathbb{Q} \) and \( j(E) = 16^3 \) as shown by Setzer in 2.1.

We can achieve results like Theorem 1.3 and 1.4 for integers other than 2 and 37; these two cases are all is required to prove Theorem 1.1 and 1.2.

To obtain a density result for \( m = qD \), where \( D \) is fixed and \( q \) varies, we define certain ‘good’ \( D \). We say \( D \) is good if it is the square free part of \( A^3 - 1728 \), where \( A \) satisfies certain congruence conditions modulo powers of 2 and 3. Both \( D = 2 \) and \( D = 37 \) are examples of ‘good’ values of \( D \). These congruence conditions will be described explicitly in Section 2.1.
2. If \( D \) is good, then \( K = \mathbb{Q}(\sqrt{Dq}) \) has EGR\(_Q\) whenever \( D \) and \( q \) satisfy certain explicit conditions, see Section 2. For any square-free \( D \), define

\[
\epsilon_D = \begin{cases} 
1 & D \equiv 1 \pmod{4} \\
-1 & \text{otherwise}
\end{cases}
\]

When \( \text{sign}(D) = -\epsilon_D \), we get real quadratic fields \( \mathbb{Q}(\sqrt{qD}) \), and when \( \text{sign}(D) = \epsilon_D \), we get imaginary quadratic fields.

Using this, we show that \( R_D(X) \), the number of \( q \leq X \) such that \( \mathbb{Q}(\sqrt{Dq}) \) is a real EGR\(_Q\) quadratic number field, satisfies the following lower bound:

**Theorem 1.5.** Let \( D \) be good with \( r \) distinct prime factors and \( R_D(X) \), the number of EGR\(_Q\) real quadratic number fields \( \mathbb{Q}(\sqrt{Dq}) \) with \( q \leq X \). Assume that \( \text{sign}(D) = -\epsilon_D \). Then

\[
R_D(X) \gg \frac{X}{\log^{1-1/2^r} X}.
\]

We obtain a similar result to show that \( I_D(X) \), the number of EGR\(_Q\) imaginary quadratic number fields \( \mathbb{Q}(\sqrt{Dq}) \) satisfies the following lower bound.

**Theorem 1.6.** Let \( D \) be good with \( r \) distinct prime factors and \( I_D(X) \), the number of EGR\(_Q\) imaginary quadratic number fields \( \mathbb{Q}(\sqrt{Dq}) \) with \( q \leq X \). Assume that \( \text{sign}(D) = \epsilon_D \). Then

\[
I_D(X) \gg \frac{X}{\log^{1-1/2^r} X}.
\]

**Remark 3.** While we have only looked at curves with rational \( j \)-invariant, Noam Elkies’ computations [4] suggest that very few \( E/K \) with EGR have \( j(E) \notin \mathbb{Q} \) and unit discriminant. Therefore, the theorem below, which to the best of our knowledge has not previously appeared in the literature, suggests that most fields of the form \( K = \mathbb{Q}(\sqrt{\pm p}) \) for primes \( p \equiv 3 \pmod{8} \) are not EGR. This is consistent with Elkies’ data.

Using this approach we were also able to determine nonexistence of EGR\(_Q\) quadratic fields.

**Theorem 1.7.** Let \( p \equiv 3 \pmod{8} \) be prime.

(1) Let \( K = \mathbb{Q}(\sqrt{p}) \). Then there are no \( E/K \) with EGR\(_Q\).

(2) let \( K = \mathbb{Q}(\sqrt{-p}) \). Then there are no \( E/K \) with EGR\(_Q\).

**Remark 4.** In [7], Kagawa showed that if \( p \) is a prime number such that \( p \equiv 3(4) \) and \( p \neq 3,11 \), then there are no elliptic curves with EGR over \( E \) which discriminant is a cube in \( K \). Since all EGR(K) curves have cubic discriminant as shown in Setzer [14], this gives a result similar to Theorem 1.7.

In Section 2, we describe conditions arising from Setzer to define when we have EGR\(_Q\) quadratic fields. In Section 3, we use these conditions to find a lower bound based on an example of Serre. In Section 4, we will give examples of EGR\(_Q\) real quadratic fields and EGR\(_Q\) imaginary quadratic fields.

2. **Constructing EGR\(_Q\) Quadratic Fields**

In [14], given a rational \( j \)-invariant, Setzer determines whether there exists an elliptic curve and number field over which this curve has everywhere good reduction. Following his
notation, we make the following definitions. Let $\mathcal{R}$ be the following set:

$$\mathcal{R} = \{ A \in \mathbb{Z} : 2|A \Rightarrow 16|A \text{ or } 16|A - 4, \text{ and } 3|A \Rightarrow 27|A - 12 \}.$$ 

Note that by the Chinese Remainder Theorem, $\mathcal{R}$ is then the union of the following congruence classes:

- 1, 5 (mod 6)
- 4, 16, 20, 32 (mod 48)
- 39 (mod 54)
- 228, 336 (mod 432)

We say that $D$ is good if it is in the following set:

$$\{ D : Dt^2 = A^3 - 1728, D \text{ square-free}, A \in \mathcal{R}, t \in \mathbb{Z} \}.$$ 

For example, the good $D$ with $|D| < 100$ are exactly

$$-91, -67, -43, -26, -19, -11, -7, 2, 7, 37, 65, 79.$$ 

**Remark 5.** We note that $\pm 1$ are not good, as the elliptic curves $Y^2 = X^3 - 1728, -Y^2 = X^3 - 1728$ have no integral points with $Y \neq 0$.

By Setzer [14], the only candidates for elliptic curves $E$ with $\text{EGR}_K(K)$ over a quadratic field $K$ have $j(E) = A^3$ with $A \in \mathcal{R}$.

**Theorem 2.1** ([14]). Let $K = \mathbb{Q}(\sqrt{m})$ be a quadratic field with $m$ square-free. Then there exists an elliptic curve $E/K$ with $\text{EGR}_K$ if and only if the following conditions are satisfied for some good $D \mid \Delta_K$.

1. $\epsilon_D D$ is a rational norm from $K$.
2. If $D \equiv \pm 3 \pmod{8}$, then $m \equiv 1 \pmod{4}$.
3. If $D$ is even then $m \equiv 4 + D \pmod{16}$.

To prove the theorem, Setzer shows that given a pair $(m, D)$ satisfying the conditions of the theorem, there exists $u \in K^\times$ such that

$$E_{u,A} : y^2 = x^3 - 3A(A^3 - 1728)u^2x - 2(A^3 - 1728)^2u^3$$

has $j$-invariant $A^3$ and $\text{EGR}_K$ over $K$.

**Remark 6.** We correct a mistake in Condition (2) of this theorem as written in [14].

We note that if $u \equiv v \pmod{4\mathcal{O}_K}$ and $m \equiv 2, 3, \pmod{4}$, then we must have that $N(u) \equiv N(v) \pmod{8}$. However, if $m \equiv 1 \pmod{4}$, we only know that $N(u) \equiv N(v) \pmod{4}$. Moreover, we can pick $w \in 4\mathcal{O}_K$ such that $N(u + w) \equiv N(u) + 4 \pmod{8}$.

Condition (2) as written in Setzer’s paper states that if $D \equiv \pm 3 \pmod{8}$, then $m \equiv 5 \pmod{8}$.

$D \equiv \pm 3 \pmod{8}$ implies that a certain element $u \in \mathcal{O}_K$ has $N(u) \equiv 5 \pmod{8}$.

But for the curve to have good reduction at primes dividing 2, it is necessary that $u$ is congruent to a square modulo $4\mathcal{O}_K$. For $m \equiv 2, 3 \pmod{4}$ this is not possible, as no squares can have norm equivalent to 5 modulo 8. However, if $m \equiv 1 \pmod{4}$, the condition that $N(u) \equiv 5 \pmod{8}$ is not an obstacle, as $u$ is congruent modulo $4\mathcal{O}_K$ to elements of norm 1 modulo 8. Setzer mistakenly assumes that this can only happen when $m \equiv 5 \pmod{8}$.

In proving that fields do and do not have elliptic curves with $\text{EGR}_K$, the following equivalent version of Setzer’s theorem will be useful.
Theorem 2.2. Fix $D$ good, and $m = qD$ square-free. $K = \mathbb{Q}(\sqrt{m})$ has EGR$_{\mathbb{Q}}$ if and only if the following conditions are satisfied:

(a) $(-\epsilon_D/q_i) = 1$ for all odd primes $p_i$ dividing $D$;
(b) $(\epsilon_D D/q_j) = 1$ for all odd primes $q_j$ dividing $q$;
(c) $m > 0$ if $\epsilon_D D < 0$;
(d) If $D \equiv \pm 3 \pmod{8}$ then $q \equiv D \pmod{4}$;
(e) If $D$ is even then $q \equiv D + 1 \pmod{8}$.

Proof of Theorem 2.2. We need to show that the conditions in Theorem 2.1 are equivalent to those in Theorem 2.2.

Assume that $K = \mathbb{Q}(\sqrt{m})$ where $m$ is square-free.

Clearly if $m = qD$, $D$ divides $\Delta_K$. We need to show that if $D \mid \Delta_K$ then $D \mid m$. This is trivial for $m \equiv 1 \pmod{4}$, as then $\Delta_K = m$. If $m \equiv 3 \pmod{4}$, then $D$ cannot be even because of condition (3) of Theorem 2.1, so $D \mid m$. If $m \equiv 2 \pmod{4}$, then $D$ must be square-free, so $D \mid m$.

Now, $\epsilon_D D$ is a rational norm from $K$ if and only if there exists a rational solution to $\epsilon_D D = a^2 - b^2 Dq$. Since $D \mid a$, the above is equivalent to the existence of a nontrivial integer solution to $\epsilon_D x^2 - Dy^2 + qz^2 = 0$. By Legendre’s Theorem [5], this equation has a nontrivial integral solution if and only if the following hold:

(i) $\epsilon_D, -D$, and $q$ do not all have the same sign, which is equivalent to condition (c).
(ii) $\epsilon_D D$ is a square modulo $|q|$, which is equivalent to condition (b).
(iii) $-\epsilon_D q$ is a square modulo $|D|$, which is equivalent to condition (a).
(iv) $-Dq$ is a square modulo $|\epsilon_D|$, which is always the case.

Lastly, conditions (d) and (e) are directly equivalent to (2) and (3).

To prove Theorem 1.1, the lower bound for $R_D(X)$ and Theorem 1.2, the lower bound for $I_D(X)$, we require Theorem 1.3 (which considers the case $D = 2$) and Theorem 1.4 (which considers the case $D = 37$). Below, we prove both those theorems using the result above.

Proof of Theorem 1.3. Let $A = 20 \in \mathcal{R}$. This shows that $D = 2$ is good. For $m = 2q$ with $q = q_1 \cdots q_n \equiv 3 \pmod{8}$ and $q_j \equiv 1, 3 \pmod{8}$ distinct primes, all of the conditions in Theorem 2.2 are satisfied, and so $K = \mathbb{Q}(\sqrt{m})$ has EGR$_{\mathbb{Q}}$.

Proof of Theorem 1.4. Let $A = 16 \in \mathcal{R}$. This that shows that $D = 37$ is good. For $m = 37q$ with $q = -q_1 \cdots q_n \equiv 1 \pmod{8}$ and $q_j$ distinct primes such that $\left(\frac{q_j}{37}\right) = 1$, all of the conditions in Theorem 2.2 are satisfied, and so $K = \mathbb{Q}(\sqrt{m})$ has EGR$_{\mathbb{Q}}$.

We also can use Theorem 2.2 to prove nonexistence results about EGR$_{\mathbb{Q}}$ quadratic fields.

Proof of Theorem 1.7. Let $p \equiv 3 \pmod{8}$ be prime.

To show that there are no $E/\mathbb{Q}(\sqrt{p})$ with EGR$_{\mathbb{Q}}$, we must show that neither of the pairs $(D, q) = (p, 1), (-p, -1)$ satisfy the conditions of Theorem 2.2. We note that since $p = D \equiv \pm 3 \pmod{8}$, condition (d) implies that $q \equiv 5D \equiv \pm 1 \pmod{8}$, which is a contradiction.
Similarly, to show that there are no EGR\(_Q(\sqrt{-p})\), we have to show that neither of the
pairs \((D, q) = (p, -1), (-p, 1)\) satisfy the conditions of the theorem. We note that in both
cases, condition (a) implies that \(\left( \frac{-1}{p} \right) = 1\), which is a contradiction. \(\square\)

3. Finding Lower Bounds

To prove the lower bounds, we use an example of Serre [12] as a reference. Let \(K/Q\) be a
Galois extension and \(C \subset \text{Gal}(K/Q)\) be a conjugacy class. Let \(\pi(K/Q, C)\) denote the set of
primes \(p\) that are unramified in \(K/Q\) which Frobenius conjugacy class \(C\).

**Definition 1.** We call a set of primes a Chebotarev set if there are finitely many finite
Galois extensions \(K_i/Q\) and conjugacy classes \(C_i \subset \text{Gal}(K_i/Q)\) such that up to finite sets,
\(P = \cup_i \pi(K_i/Q, C_i)\).

**Definition 2.** We define a set \(E \subset \mathbb{N} > 0\) to be multiplicative if for all pairs \(n_1, n_2\) relatively
prime, we have that \(n_1n_2 \in E\) if and only if \(n_1 \in E\) or \(n_2 \in E\).

Given a multiplicative set \(E\), let \(P(E)\) be the set of primes \(p\) in \(E\). Let \(\bar{E} := \mathbb{N} > 0 - E\),
and \(\bar{E}(X) := \{m \in \bar{E}, m \leq X\}\).

**Theorem 3.1 ([12]).** Suppose that \(E\) is multiplicative and \(P(E)\) is a Chebotarev set with
density \(0 < \alpha < 1\). Then
\[
\bar{E}(X) \sim \frac{cX}{\log \alpha X}
\]
for some \(c > 0\).

We will use the theorem above to prove Theorem 1.5 and Theorem 1.6. As shown in
Section 2, the special cases with \(D = 2, 37\) will then imply Theorem 1.1 and 1.2.

**Proof of Theorem 1.5 and Theorem 1.6.** Let \(D\) be good. Let \(D'\) be the odd part of \(D\), and
\(\delta = \varepsilon_D \varepsilon_{D'} D/D'\). Note that if \(D\) is odd, then \(\delta = 1\).

Also define
\[
\bar{E}_D := \left\{ q_1^{a_1} \cdots q_n^{a_n} : q_j \text{ is prime, } a_j \geq 0, \left( \frac{q_j}{p} \right) = 1 \text{ for all odd primes } p \mid D, \left( \frac{\delta}{q_j} \right) = 1 \right\}.
\]

The compliment \(E_D = \mathbb{N} - \bar{E}_D\) is then multiplicative and \(P(E_D)\) has Chebotarev density
\(\alpha = 1 - 1/2^r\), where \(r\) is the number of prime factors of \(D\). Therefore, by Theorem 3.1, we have
\[
\bar{E}_D(X) \sim \frac{cX}{\log \alpha X}.
\]

Now, we have to relate \(\bar{E}_D(X)\) to \(R_D(X)\) and \(I_D(X)\). We do this by showing that if
\(\pm q \in \bar{E}(X)\) is squarefree and satisfies congruence conditions coming from (d) and (e) of
Theorem 2.2, then \(m = qD\) has EGR\(_Q\).

Let \(C_D\) be the set of \(q \in \mathbb{Z}\) that satisfy the congruence conditions (d) and (e) of Theorem
2.2, so that
\[
C_D = \begin{cases}
\{ q \in \mathbb{Z} : q \equiv D \pmod{4} \} & \text{if } D \equiv \pm 3 \pmod{8} \\
\{ q \in \mathbb{Z} : q \equiv D + 1 \pmod{8} \} & \text{if } D \equiv 0 \pmod{2} \\
\{ q \in \mathbb{Z} \} & \text{otherwise}
\end{cases}
\]
We define
\[ R_D^E(X) := \{ Dq : \text{sgn}(D)q \in \overline{E}_D(X/D), q \text{ squarefree, } q \in C_D \} \]
\[ I_D^E(X) := \{ Dq : -\text{sgn}(D)q \in \overline{E}_D(X/D), q \text{ squarefree, } q \in C_D \} \]

**Lemma 3.2.** For good \( D \), \( R_D^E(X) \subset R_D(X) \) if \( \epsilon_D = -\text{sgn}(D) \) and \( I_D^E(X) \subset I_D(X) \) if \( \epsilon_D = \text{sgn}(D) \).

**Proof.** We need to check (a) and (b) of Theorem 2.2. They follows from the properties of the Jacobi Symbol. Let \( D \) be good. If either \( \epsilon_D = -\text{sgn}(D) \) with \( 0 < qD \) or \( \epsilon_D = \text{sgn}(D) \) with \( 0 > qD \), we have that \( 0 < -\epsilon_Dq = \prod q_j \), so

\[
\left( \frac{-\epsilon_Dq}{p} \right) = \prod \left( \frac{q_j}{p} \right) = 1.
\]

Note that then we always have that \( \epsilon_{D'}D' \equiv 1 \pmod{4} \) and \( \epsilon_DD = \delta D' \epsilon_{D'} \). So

\[
\left( \frac{\epsilon_DD}{q_j} \right) = \left( \frac{\delta}{q_j} \right) \left( \frac{\epsilon_{D'}D'}{q_j} \right) = \left( \frac{q_j}{\epsilon_{D'}D'} \right) = \prod_{p|D \text{ odd}} \left( \frac{q_j}{p} \right) = 1.
\]

Since a positive proportion of \( \pm q \in \overline{E}_D(X/D) \) satisfy the extra conditions of being square-free and in \( C_D \), we have that

\[ R_D^E(X), I_D^E(X) \gg \frac{X}{\log^a X}, \]

and hence the same is true of the bigger sets \( R_D(X), I_D(X) \). □

**Proof of Theorem 1.1.** The theorem follows immediately from Theorem 1.5 and Theorem 1.3. Theorem 1.3 shows \( D = 2 \) is good with \( r = 1 \) distinct factors and the real quadratic field \( K = \mathbb{Q}(\sqrt{qD}) \) has EGR\( _Q \). If \( R(X) \) is the number of these fields, Theorem 1.5 shows

\[ R(X) \gg \frac{X}{\sqrt{\log(X)}}. \]

□

**Proof of Theorem 1.2.** The theorem follows immediately from Theorem 1.6 and Theorem 1.4. Theorem 1.4 shows \( D = 37 \) is good with \( r = 1 \) distinct factors and the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{qD}) \) has EGR\( _Q \). If \( I(X) \) is the number of these fields, Theorem 1.6 shows

\[ I(X) \gg \frac{X}{\sqrt{\log(X)}}. \]

□
In this section, we explain how to find elliptic curves with EGR over \(\mathbb{Q}\) when the conditions of Theorem 2.2 are satisfied, and give examples of elliptic curves with EGR. The results in this section are based on Setzer’s construction in 2.1.

We start with a quadratic field \(K = \mathbb{Q}(\sqrt{m})\) and a factorization \(m = Dq\) with \(D\) good which satisfies the conditions of Theorem 2.2. We want to find \(u\) such that
\[E_{u,A} : y^2 = x^3 - 3A(A^3 - 1728)u^2x - 2(A^3 - 1728)^2u^3\]
has EGR\(_K\). Let \(\alpha \in K\) have norm \(\epsilon_D D\), and pick \(n\) odd such that \(\beta := n\alpha = a + b\sqrt{m} \in \mathcal{O}_K\). Let \(A \in \mathcal{R}\) be such that \(D = d_1^2d_2^2\) with \(d_1\) square-free. If \(m \equiv 1, 2 \pmod{4}\), then one of \(u = \pm \beta d_1\) both work or \(u = \pm \beta d_1\rho\) both work, where \(\rho = \frac{1}{2}(m + 1) + \sqrt{m}\).

The table below has some examples.

| \(A\) | \(D\) | \(d_1\) | \(q\) | \(\alpha\) | \(u\) |
|------|------|------|------|--------|------|
| 20   | 2    | 42   | 3    | \(2 + \sqrt{6}\) | \(-d_1\alpha = -84 - 42\sqrt{6}\) |
| -15  | -7   | 1    | -11  | 35 + 4\sqrt{77} | \(-d_1\alpha = -35 - 4\sqrt{77}\) |
| -32  | -11  | 42   | -15  | 77 + 6\sqrt{165} | \(d_1\alpha = 3234 + 252\sqrt{165}\) |
| -32  | -11  | 42   | -3   | 11 + 2\sqrt{33} | \(-d_1\alpha = -462 - 84\sqrt{33}\) |
| 39   | 79   | 1    | 5    | 79 + 4\sqrt{395} | \(\pm d_1\alpha\rho = \pm(17222 + 871\sqrt{395})\) |
| 16   | 37   | 6    | -7   | 37 + 6\sqrt{-259} | \(\pm d_1\alpha = \pm(222 + 36\sqrt{-259})\) |

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