ABSTRACT

The problem of worst case edge deletion from a network is considered. Suppose that you have a communication network and you can delete a single edge. Which edge deletion causes the largest disruption? More formally, given a graph, which edge after deletion disconnects the maximum number of pairs of vertices, where ties for number of pairs disconnected are broken by finding an edge that increases the average shortest path length the maximum amount. This problem is interesting both practically and theoretically. We call it the single edge deletion problem.

Our contributions include formally defining the single edge deletion problem and providing motivations from network analysis. Also, we give an algorithm that solves the problem much faster than a naive solution. The algorithm incorporates sophisticated and novel techniques, and generalises to the problem of computing the all-pairs shortest paths table after deleting each edge individually. This means the algorithm has deep theoretical interest as well as the potential for even wider applications than those we present here.

1 Introduction

What is that worst that could happen if a link gets removed from a network? This is the focal question for the work presented here. In the lexicon of graph theory, we consider what the impact on connectivity and shortest paths is if any single edge is deleted and seek to find the edge that, once deleted, has the largest impact. To this end, we define the novel single edge deletion problem (Section 1.1), explain how it relates to the existing literature (Sections 1.2 and 1.3), and provide a solution to it that is significantly better than the naive solution (Section 2). Further, the algorithm we give that solves the single edge deletion problem uses interesting and novel algorithmic techniques, and generalises to the problem of computing the all-pairs shortest path table after deleting each edge individually. This means the algorithm has deep theoretical interest as well as the potential for even wider applications than those we present here. Finally, we benchmark the practical performance our algorithms and apply them to real network topologies from [1] for illustration (Section 3).
1.1 The Single Edge Deletion Problem

The single edge deletion problem is defined on a graph $G = (V, E)$ comprising a set of vertices $V$ and a set of edges $E$. The most general form of a graph is considered, in which edges are directed and weighted with arbitrary, real weights. Multiple edges between a single ordered pair of vertices are disallowed without loss of generality. In such cases, the minimum weight edge can be kept while all other duplicate edges are removed. This procedure has no effect on the solution to the single edge deletion problem. Likewise, self edges (edges from a vertex to itself) can be removed from the graph, so we assume they do not exist in $E$.

We want to find a single edge $e \in E$ such that the deletion of $e$ from $E$ decreases the total number of connected pairs in the graph the most. In the case where there are multiple edges whose deletion leaves a minimum number of connected pairs, tie-break on the largest average shortest path length after deletion. If there are still ties, they may be broken arbitrarily.

Let $\rho(u, v)$ be true when there is a directed path from vertex $u$ to vertex $v$ in $G$. Using $\rho(u, v)$, define $P(G) = \{(u, v) \mid u, v \in V, \rho(u, v), u \neq v\}$. The set $P(G)$ can be interpreted as the set of all valid connected pairs of vertices in $G$.

Similarly, let $\psi(u, v)$ be the distance of the shortest path between vertices $u$ and $v$ in $G$. Using $\psi(u, v)$, define $\Psi(G) = \sum_{(u,v) \in P(G)} \psi(u, v)$. The value of $\Psi(G)$ can be interpreted as the sum of shortest path lengths for all valid connected vertex pairs in $G$. For a definition of shortest path, see [2]. It is assumed that there are no negative weight cycles in $G$, otherwise the shortest path is not well defined.

We want to find a single edge $e \in E$ such that $|P((V, E \setminus \{e\}))|$ is minimized. Any ties should be broken by maximization of $\Psi((V, E \setminus \{e\}))$. If there are still remaining ties for the choice of $e$, pick one arbitrarily.

Note that maximization of $\Psi$ in the case of ties is equivalent to maximization of average shortest path length. The average is defined as $\frac{\Psi((V, E \setminus \{e\}))}{|P((V, E \setminus \{e\}))|}$ and the denominator is fixed when considering ties on $|P((V, E \setminus \{e\}))|$.

This defines the the single edge deletion problem. It is intended to capture the idea of finding the worst edge to delete in a communication network: an edge that disconnects the most pairs of vertices, tie breaking on highest average latency.

It is possible to generalise the definition of the single edge deletion problem further. The all-pairs shortest paths table single edge deletion problem is to compute the all-pairs shortest paths table $|E|$ times, once for each $G' = (V, E \setminus \{e\})$ such that $e \in E$. We note two things. First, this can be used to compute many metrics for network topology robustness, including finding the worst case edge deletion as is done in the single edge deletion problem described above. Further, the algorithms we present in Section 2 can solve the all-pairs shortest paths table single edge deletion problem with trivial modifications. We present the worst case edge deletion version to show how our algorithms can be practically applied.

1.2 Network Analysis

Chaos engineering is a technique championed by Netflix and widely adopted by Big Tech companies to build resilience to failure into their systems [3]. Random, real-world failures are injected into production systems to enforce failure mitigation into system design. These could include taking down some servers, making certain requests fail, removing links between data centers, or any other failure mode the system may have. Inspired by this, we explore the question of how badly things can go wrong in a network when a single link is removed.

Similar problems can, and have [4, 5], been viewed through an adversarial lens. Decker and Colbert [4] devised techniques for problems similar to those we consider. These include finding and analyzing graph topologies that are resilient to node deletion. They note that such tools are important both for designing military communication networks, and some civilian networks, such as power networks, that must be resilient to attacks that could remove nodes (or edges). They considered connectedness in unweighted graphs. We explore the related, but distinct, problem of finding the worst edge deletions with respect to shortest path lengths as well as connectivity.

More recent work has applied similar network robustness analysis tools to data center networks [6]. Interestingly, average path length is used as a metric in [6], however, it was not used for larger topologies, since the time requirement was too large. Similarly, Ellens and Kooiji [7] use shortest path length as a measure of network robustness. We generalize these approaches by determining the shortest path lengths after edge deletion using new, efficient algorithms.
1.3 Related Algorithmic Problems

The single edge deletion problem, particularly the generalised all-pairs shortest paths table variant, is a close relation of some well studied problems in graph theory. These include the decremental shortest path problem, and the dynamic all-pairs shortest paths problem.

The decremental shortest path problem was originally studied in the context of efficiently handling single-source shortest path queries while edges are deleted [8, 9, 10]. All-pairs shortest path variants of the problem have also been studied [11, 12]. The single edge deletion problem differs from decremental shortest path problems because it considers each edge deletion in isolation, rather than deleting edges in a progressive manner.

The dynamic all-pairs shortest paths problem is to efficiently answer shortest path queries while handling vertex and/or edge modifications in a graph. These modifications typically mean changing the weights of edges in a graph. Efficient algorithms for this problem have been proposed [13, 14]. These algorithms could be used to solve the single edge deletion problem. Each edge in turn could be removed, the all-pairs shortest paths table examined, then the edge could be added back. While doing this would work, the algorithm we present in Section 2.2 is much more efficient and simpler than reducing the single edge deletion problem to the dynamic all-pairs shortest paths problem.

Recently, a fast solution to the problem of finding a single edge whose addition minimizes the average shortest path length was published [15]. The work presented here tackles the inverse problem. That is, we consider edge deletion, rather than addition. While these do superficially appear to be similar problems, the algorithmic techniques used and applications are wholly different. It is a quintessential case where the inverse of a problem requires a very different approach.

2 Algorithms

We describe two algorithms. First, we introduce a naive algorithm for the single edge deletion problem. This is the simpler algorithm to understand, and is likely to be the first approach an algorithm designer explores. It is used as a baseline for comparison as well as a starting point for describing our faster algorithm, which is subtler to derive. The description of this faster algorithm follows.

2.1 Naive Algorithm

The naive approach is essentially brute force. Try deleting each edge in turn, and check the result by recomputing the all-pairs shortest path table.

Pseudocode is given in Algorithm 1. The bulk of the work is done by the subroutine APSP($V, E$), which computes the all-pairs shortest path table for a graph. There are many algorithms that can be used to compute this table [16, 17]. For the sake of simplicity, let us assume that the Floyd-Warshall algorithm [2, 17] is used. In the worst case, when the graph is dense, the Floyd-Warshall algorithm is the best known algorithm, and it requires $\Theta(|V|^3)$ time.

Since the APSP subroutine is called for each edge in $E$, the time complexity is $\Theta(|E||V|^3)$. If we assume the graph is dense ($E \in O(|V|^2)$), then the time complexity is $\Theta(|V|^5)$.

2.2 Faster Algorithm

We begin by assuming that the graph contains no negative weight edges. This is without loss of generality. A graph with negatively weighted edges (and no negative cycles) can be reweighted into a graph containing the same shortest path information but with only non-negative edge weights in $O(|V||E|)$ time [2, 16]. Also, it seems unlikely that a communication network would have negative weight edges, so this step is likely not needed.

Consider a single-source version of the problem. The values of $|P(G)|$ and $\Psi(G)$ are computed considering only paths from a single source vertex for each possible edge deletion. If this can be solved efficiently, the results can be used to solve the full problem at the cost of running the single source algorithm once for each vertex and then examining the total values of $|P(G)|$ and $\Psi(G)$ over all source vertices for each edge deletion.

Dijkstra’s algorithm [18, 2] finds a shortest path from a single source vertex to every other vertex in a graph. These shortest paths, when taken together, form a tree rooted at the source vertex. Call this the shortest path tree. There may be multiple shortest path trees. In such a case, assume one is chosen arbitrarily.

Observe that if an edge is deleted from a graph and it is not in the shortest path tree, then no vertex could be disconnected by the edge deletion and no shortest path distance could have been affected. The reason for this is that the tree comprises shortest paths to every reachable vertex, and none of these paths could have been affected by the edge deletion. In
The worst case for our faster algorithm and naive algorithm is when the graph is dense. The algorithms require $O(|V|^2 |E| + |V|^3 \log |V|)$ time respectively. Our algorithm is a full factor of $|V|^2$ faster.

For sparse graphs, the complexity depends on the choice of all-pairs shortest path algorithm for the naive algorithm. If Johnson’s algorithm \[16\], which is the most efficient available up to polynomial factors, is used, then the complexity of both algorithms is $O(|V|^2 |E| + |V|^3 \log |V|)$. Note that Algorithm 2 can be modified to return every edge that is tied as worst to delete. Further, it can even be modified to find the “worst” edge under many functions of the shortest path table in general. For example, one could...

### Algorithm 1: Naive algorithm for single edge deletion

1: **procedure** \textsc{NaiveSED}(G = (V, E))
2: $P_{\text{best}} \leftarrow \infty$
3: $\Psi_{\text{best}} \leftarrow -\infty$
4: $e_{\text{best}} \leftarrow \emptyset$
5: for $e \in E$ do
6: \quad $A \leftarrow \text{APSP}(V, E \setminus \{e\})$
7: \quad $P \leftarrow 0$
8: \quad $\Psi \leftarrow 0$
9: \quad for $u \in V$ do
10: \quad \quad for $v \in V$ do
11: \quad \quad \quad if $\text{PathExists}(A, u, v)$ then
12: \quad \quad \quad \quad $P \leftarrow P + 1$
13: \quad \quad \quad \quad $\Psi \leftarrow \Psi + \text{DISTANCE}(A, u, v)$
14: \quad \quad \quad end if
15: \quad end for
16: \quad if $P < P_{\text{best}} \lor P = P_{\text{best}} \land \Psi > \Psi_{\text{best}}$ then
17: \quad \quad $P_{\text{best}} \leftarrow P$
18: \quad \quad $\Psi_{\text{best}} \leftarrow \Psi$
19: \quad \quad $e_{\text{best}} \leftarrow e$
20: \quad end if
21: end for
22: return $e_{\text{best}}$
23: **end procedure**
Algorithm 2: Faster algorithm for single edge deletion

1: procedure FastSED\((G = (V, E))\)
2: Initialise \(P_e\) and \(\Psi_e\), which map between each edge and a running total.
3: \(P_e \leftarrow 0\) for each edge \(e \in E\)
4: \(\Psi_e \leftarrow 0\) for each edge \(e \in E\)
5: \(\text{for source} \in V \text{ do}\)
6: \(T \leftarrow \text{Dijkstra}(G, \text{source})\)
7: \(\text{for } e \in \text{Edges}(T) \text{ do}\)
8: \(T' \leftarrow \text{Dijkstra}((V, E \setminus \{e\}), \text{source})\)
9: \(\text{for } v \in \text{Vertices}(T') \text{ do}\)
10: \(P_e \leftarrow P_e + 1\)
11: \(\Psi_e \leftarrow \Psi_e + \text{Distance}(T', \text{source}, v)\)
12: \(\text{end for}\)
13: \(\text{end for}\)
14: \(D \leftarrow \sum_{v \in \text{Vertices}(T)} \text{Distance}(T, \text{source}, v)\)
15: \(\text{for } e \notin \text{Edges}(T) \text{ do}\)
16: \(P_e \leftarrow P_e + |\text{Vertices}(T)|\)
17: \(\Psi_e \leftarrow \Psi_e + D\)
18: \(\text{end for}\)
19: \(\text{end for}\)
20: \(P_{\text{best}} \leftarrow \infty\)
21: \(\Psi_{\text{best}} \leftarrow -\infty\)
22: \(e_{\text{best}} \leftarrow \emptyset\)
23: \(\text{for } e \in E \text{ do}\)
24: \(\text{if } P_e < P_{\text{best}} \lor P_e = P_{\text{best}} \land \Psi_e > \Psi_{\text{best}} \text{ then}\)
25: \(P_{\text{best}} \leftarrow P_e\)
26: \(\Psi_{\text{best}} \leftarrow \Psi_e\)
27: \(e_{\text{best}} \leftarrow e\)
28: \(\text{end if}\)
29: \(\text{end for}\)
30: \(\text{return } e_{\text{best}}\)
31: \(\text{end procedure}\)

find the “best” edge to delete instead: an edge that disconnects the \textit{minimum} number of pairs of vertices, tie-breaking on \textit{minimum} average shortest path lengths.

In the most general terms, our algorithm allows efficient computation of the all-pairs shortest paths table after deleting each edge individually. Note that the Distance function for each source vertex constitutes the all-pairs shortest paths table.

3 Results

3.1 Empirical Run Time Analysis

The complexity analysis was validated empirically. The naive and faster algorithms were implemented and run on an Intel i7-10750H. The code can be found at [https://github.com/maxwardg/single_edge_deletion](https://github.com/maxwardg/single_edge_deletion). Graphs were generated randomly using a Gilbert-like model [20]. Each edge had a 40\% probability of being included in the graph. The probability of an edge existing is independent of the other edges. Edge weights were sampled uniformly at random from the real range \([0,1)\).

The edge probability of 40\% was chosen since, for our choices of \(|V|\), it is likely to see both graphs where the chosen edge deletion disconnects a pair, and where no pair in disconnected. It is also dense enough to see the worst case run time of each algorithm clearly.

The results are given in Figure [1]. As expected, the time needed by both appears to grow as a polynomial function of the number of vertices. Further, the faster algorithm quickly becomes much faster relative to the naive algorithm as
Figure 1: Empirical run time results for both algorithms. The area of the lines denote that minimum and maximum run time over a sample of 5 runs.

Figure 2: The “Abilene” network, taken from [1]. Network vertices and edges are in blue. The edge to delete as selected by our algorithm is in red.

the number of vertices increases. At 80 vertices, the faster algorithm requires about 0.123 seconds while the naive algorithm requires about 25 seconds.

3.2 Example Network Results

Our algorithm was run on selected graphs from the Internet Topology Zoo [1]. We present them here to illustrate a practical use of our algorithm on real-world network topologies. The three examples in Figures 2 to 4 represent typical cases. They include a case where removal of an link disconnects some pairs, where removal of a link impacts the average path length significantly, and where removal of a link has only a small impact on average path length.

The networks were parsed in GML format. Duplicate edges and vertices were removed. It was assumed that all links were bidirectional. All vertex locations were derived from their latitude and longitude. Edges were weighted by the great-circle distance between vertex locations.

In Figure 2, the selected edge increases shortest path lengths, particularly between the locations on the far left and the far right. This shows a case where the worst case edge removal does not disconnect any pair of vertices, but instead increases the average shortest path length significantly. This might translate to a large impact in network latency.

Figure 3 shows a similar situation. The worst edge deletion does not disconnect any pair of vertices. However, the impact on average path length is less severe than in Figure 2. This example shows a more robust network.

A different type of situation is depicted in Figure 4. The worst edge deletion disconnects the part of the graph above the edge from the rest of the graph. This is a case where the number of connected pairs is impacted.
4 Conclusions

We have defined the single edge deletion problem. This problem has applications to analyzing network robustness through identification of vulnerable edges. We have also described an algorithm to solve the problem that is significantly faster than the naive approach. Further, the speed of the algorithm has been analysed experimentally, and the algorithm was used to analyze real-world networks to illustrate how it can be used in practice. We also point out that the algorithm generalises to computing the all-pairs shortest paths table after deleting each edge individually. This gives the general algorithmic techniques we present broad theoretical and potentially practical interest.

While the algorithm we have presented is much faster than the naive algorithm, it is unclear what the lower bound on the problem is. For example, the edge addition version of the problem has been solved in $O(|V|^3 \log |V|)$ time \cite{15}. It is possible that either a faster algorithm is found or a hardness result proves the lower bound on the problem.

We observe some avenues to speed up the algorithm. These do not affect the worst case, but instead may help in the expected case, particularly for random graphs. Our algorithm examines every edge in the shortest path tree in Algorithm 2. However, only the critical edges must be examined. We believe that a modified single-source shortest path algorithm can be used to find these critical edges efficiently when the graph has only positive edge weights. In the worst case, all the edges of the shortest path tree are critical, so the worst case is the same. Also, finding critical edges may be much harder when some edges are non-positive.

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