Integrable Generalized Principal Chiral Models

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Abstract

We study 2D non-linear sigma models on a group manifold with a special form of the metric. We address the question of integrability for this special class of sigma models. We derive two algebraic conditions for the metric on the group manifold. Each solution of these conditions defines an integrable model. Although the algebraic system is overdetermined in general, we give two examples of solutions. We find the Lax field for these models and calculate their Poisson brackets. We also obtain the renormalization group (RG) equations, to first order, for the generic model. We solve the RG equations for the examples we have and show that they are integrable along the RG flow.

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1. Introduction

A lot of efforts have been dedicated in the last decade to the determination, classification and duality relations between 2D non-linear sigma models who are conformal. Since the $\beta$ function of the sigma model appears in a geometrical form [1] and since conformal invariance manifests itself by the vanishing of the $\beta$ function, we end up having a geometrical criterion for the class of conformally invariant non-linear sigma models.

What can be said about non-linear sigma models if we change the condition of conformal invariance to the more general condition of integrability?

In this paper we take a first step toward an answer to this question. We are limiting ourselves to Lie group manifolds and taking a special form for the metric. This class, that we call generalized principal chiral model (GPCM), is a generalization of the diagonal anisotropic principal chiral model [2]. It is also equivalent to the generalized Thirring model studied recently by Bardakçi and Bernardo [3]. While they were looking for conditions on the coupling constants to insure conformal invariance, we will derive, for this class of models, conditions on the metric (or coupling constants) to insure classical integrability. This is done by demanding that a Lax pair formulation for the equation of motion is possible. Once we have the Lax field and the Poisson structure on the phase space, we calculate the Poisson brackets. We find a surprise in the PCM model because a set of operators, which includes the Lax field, not only closes under the Poisson brackets but also forms an algebra, very similar to the two loop affine Lie algebra [4].

Since integrable models are not, in general, at the fixed point of the renormalization group (RG) equations, it is interesting to know whether they remain integrable along the RG flow. In this paper we find the $\beta$ function, to first order, for the generic model in the class that we consider, and solve the RG equations for two examples. For these two cases the models are integrable along the RG flow.

2. Equations of motion

The models that we consider are generalizations of the principal chiral model (PCM) defined on a Lie group manifold $G$. We replace the trace over the algebra by a more general bilinear form. The action is

$$I = -\int d\sigma d\tau L_{ab}(g^{-1}\partial_{\mu}g)^a(g^{-1}\partial_{\mu}g)^b$$

(2.1)
where \( g \in G \) and \( A_\mu A^\mu = A_\mu A_\nu \eta^{\mu\nu} \) with \( \eta^{00} = -\eta^{11} = 1, \eta^{01} = \eta^{10} = 0 \). Here, \( x_0 \equiv \tau \), \( x_1 \equiv \sigma \), and \( L_{ab} \) is taken to be a symmetric and invertible \( \dim G \times \dim G \) matrix.

We introduce a flat connection by the standard definition:

\[
A_\mu = -ig^{-1} \partial_\mu g = A_\mu^a t_a
\]  

(2.2) 

where \( t_a \) are the generators of the Lie algebra \( G = \text{Lie}(G) \):

\[
[t_a, t_b] = i f_{ab}^c t_c
\]  

(2.3) 

The metric on the group manifold is given by \( \kappa_{ab} = Tr(t_a t_b) \). The inverse is denoted by \( \kappa^{ab} \) such that \( \kappa^{ac} \kappa_{cb} = \kappa_{bc} \kappa^{ca} = \delta^a_b \). The Bianchi identity for the connection reads

\[
\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{bc}^a A_\mu^b A_\nu^c = 0
\]  

(2.4) 

Next we derive the equations of motion. Define \( \delta \rho = -ig^{-1} \delta g \), then

\[
\delta A_\nu^a = \partial_\nu \delta \rho^a - f_{bc}^a A_\nu^b \delta \rho^c
\]  

(2.5) 

and

\[
-\frac{\delta I}{\delta \rho^a} = L_{ac} \partial_\mu A_\mu^a + L_{ab} f_{dc}^b A_\mu^a A_\mu^d = 0
\]  

(2.6) 

Since \( L \) is invertible we can write the equations of motion in the form

\[
d_\mu A_\mu^a = 0
\]  

(2.7) 

where

\[
d_{ab}^a = \delta^a_b \partial_\mu + S_{bc}^a A_\mu^c
\]

\[
S_{bc}^a = \frac{1}{2}(F_{bc}^a + F_{cb}^a)
\]

\[
F_{bc}^a = L_{qb} f_{cp}^q (L^{-1})^{pa}
\]  

(2.8) 

We see that \( S_{bc}^a \) is playing a role of a connection. This will also be apparent below when we discuss the general non-linear sigma model on a group manifold.
3. The Lax pair formulation

A basic theorem in classical field theory asserts that a model is integrable if its equation of motion can be represented as a one parametric Lax pair:

\[ [\partial_0 + \mathcal{M}(\lambda), \partial_1 + \mathcal{L}(\lambda)] = 0 \]  

(3.1)

where \( \lambda \) is the spectral parameter. We will now derive conditions on \( L_{ab} \) such that a Lax pair representation is possible.

Because the equations of motion are quadratic in the current s, we take as an ansatz a commutator between linear combination of the currents:

\[ [\partial_0 + N_{0b}^a A_b^a t_a + N_{1b}^a A_1^a t_a, \partial_1 + N_{0b}^a A_b^a t_a + N_{1b}^a A_1^a t_a] = 0 \]  

(3.2)

where \( N_{\mu b}^a \) are two unknown auxiliary dim \( G \times \text{dim} \ G \) matrices. This linear structure is also present in known integrable models of this type.

Computing explicitly the commutator, using (2.3) and the Bianchi identity, we obtain an equation

\[ (N_{0b}^a f_p^b + i N_{\mu p}^a N_{q c}^f f_{ac}^s) A_0^p A_1^q + N_{1b}^a \partial_{\mu} A_{\mu b}^a + i N_{0p}^a N_{1q}^c f_{ac}^s A_{\mu b}^a A^p A^q = 0 \]  

(3.3)

which gives rise upon comparison with the equations of motion to the following conditions:

\[ N_{0b}^a f_p^b + i N_{\mu p}^a N_{q c}^f f_{ac}^s = 0 \]  

(3.4a)

\[ \frac{1}{2} i f_{cd}^a (N_{0c}^d N_{1q}^p + N_{0q}^d N_{1c}^p) = N_{1b}^a S_{pq}^b \]  

(3.4b)

The second equation relates the ansatz parameters to the coupling constant metric \( L_{ab} \) by eq. (2.8).

The solution to these two algebraic equations is an affine variety \( \mathcal{M} \). The model is integrable if \( \text{dim} \mathcal{M} > 0 \). A naive counting shows that we have \( 2(\text{dim} \ G)^3 \) equations for only \( 2(\text{dim} \ G)^2 \) variables. A generic choice of a matrix \( L_{ab} \) is, as expected, not integrable. Nevertheless the set of solutions of (3.4) is not empty.

Before proceeding to the examples we would like to comment on the generalization where \( L_{ab} \) is not a constant function on the group manifold. In this case the auxiliary matrices \( N_{\mu b}^a \) vary on the group manifold. Nevertheless the conditions (3.4) still hold with the only change \( S_{pq}^a \rightarrow \tilde{S}_{pq}^a = S_{pq}^a + \Gamma_{pq}^a \) where \( \Gamma_{pq}^a \) is a connection

\[ \Gamma_{pq}^a = \frac{1}{2} (L^{-1})^{ab} (\partial_p L_{qb} + \partial_q L_{bp} - \partial_b L_{pq}) \]  

(3.5)
with the notation $\partial_a \equiv e^i_a \partial_i$. Note that $S^a_{pq}$ is playing a role of a connection.

We also have, in this case, chirality conditions on $N_\mu$:

$$\partial_+ N_- = \partial_- N_+ = 0$$

(3.6)

where $\partial_\pm = \partial_0 \pm \partial_1$ and similarly $N_\pm = N_0 \pm N_1$. In a light cone formulation the conditions (3.4) read:

$$(N_+ + N_-)^s_{bpq} f^b_s + i(N^{a+}_p N^{c+}_{-q} + N^a_{-p} N^{c+}_{+q} ) f^{s}_{ac} = 0$$

(3.7)

and similarly

$$(N_+ - N_-)^s_{bpq} + i\frac{1}{2}(N^{a+}_p N^{c-}_{-q} - N^a_{-p} N^{c-}_{+q} ) f^{s}_{ac} = 0$$

For a detailed study of the general integrable non linear sigma model on a group manifold see [5].

Next we show how some known models fit into this framework.

**Example 1:** The principal chiral model (PCM) where $L_{ab} = \frac{1}{g^2}\kappa_{ab}$ is known to be integrable [6] and upon taking $N^a_{\mu b} = \lambda_\mu \delta^a_b$ we find that the system (3.4) is reduced to one equation for two variables

$$\lambda_0 + i(\lambda_0^2 - \lambda_1^2) = 0$$

(3.8)

we can write the solution in a parametric way

$$\lambda_0 = -i \sinh^2 \lambda$$

$$\lambda_1 = -i \sinh \lambda \cosh \lambda$$

(3.9)

**Example 2:** The diagonal anisotropic $SU(2)$ PCM with $L_{ab} = J_a \kappa_{ab}$ (no summation) with $J_1 = J_2 \neq J_3$ is also a known integrable model [2]. Taking $N^a_{\mu b} = \lambda_\mu \delta^a_b$ (no summation) and assuming $\lambda^1_\mu = \lambda^2_\mu \neq \lambda^3_\mu$ gives three equations

$$\lambda^1_0 + i(\lambda^0_0 \lambda^3_0 - \lambda^1_1 \lambda^3_1) = 0$$

$$\lambda^3_0 + i((\lambda^0_0)^2 - (\lambda^1_1)^2) = 0$$

(3.10)

$$\lambda^0_0 \lambda^3_1 - \lambda^1_0 \lambda^3_0 = \frac{J_1 - J_3}{J_1} \lambda^1_1$$

for four variables: $(\lambda^1_\mu, \lambda^3_\mu), \quad \mu = 0, 1$. In parametric representation it reads

$$\lambda^1_1 = \frac{1}{m} \sinh \lambda \cosh \lambda$$

$$\lambda^0_0 = \frac{1}{m} \sinh \lambda (m^2 + \cosh^2 \lambda)^{\frac{1}{2}}$$

$$\lambda^1_3 = -i \cosh \lambda (m^2 + \cosh^2 \lambda)^{\frac{1}{2}}$$

$$\lambda^0_3 = -i \sinh^2 \lambda$$

(3.11)

where $k = \frac{J_1 - J_3}{J_1}$ and $m^2 = ik - 1$.

More analysis is needed in order to find new non trivial solutions. Work in this direction is in progress and will be reported elsewhere.
4. The fundamental Poisson brackets

We take in this section a canonical approach and use a technique first used by Bowcock [7]. In order to compute the Poisson brackets \( \{ L^a(\sigma), L^b(\sigma') \} \) it is convenient to choose a local coordinate system \( x^i \). In terms of these coordinates we write

\[
A^a_\mu = -i(g^{-1} \partial_\mu g)^a = e^a_i(x) \partial_\mu x^i \tag{4.1}
\]

where

\[
e^a_i(x) = -i(\frac{1}{g(x)}g_i^a) \partial_\mu x^i \tag{4.2}
\]

are the vielbeins.

The action now is in the general form of a non linear sigma model

\[
I = \int d\tau d\sigma G_{ij}(x) \partial_\mu x^i \partial_\mu x^j \tag{4.3}
\]

where the metric \( G_{ij}(x) \) is given

\[
G_{ij}(x) = L_{ab} e^a_i(x) e^b_j(x) \tag{4.4}
\]

in terms of the vielbeins.

The canonical conjugate momentum is

\[
\pi_i(\sigma) = \frac{\delta I}{\delta \dot{x}^i} = 2G_{ij} \dot{x}^j \tag{4.5}
\]

where \( \dot{x}^i = \frac{\partial}{\partial \tau} x^i \) and we will also use below the notation \( f' = \frac{\partial}{\partial \sigma} f \). The canonical bracket is

\[
\{ x^i(\sigma), \pi_j(\sigma') \} = \delta^i_j \delta(\sigma - \sigma') \tag{4.6}
\]

The “gauge potentials” \( A^a_\mu \) are functions on the phase space given by

\[
A^a_0 = e^a_i \dot{x}^i = \frac{1}{2} (L^{-1})^{ab} e^a_i \pi_i \tag{4.7}
\]

\[
A^a_1 = e^a_i x^i
\]

where \( e^a_i \)

\[
e^a_i e^b_i = \delta^a_b \; ; \; \; e^a_i e^a_i = \delta^b_b \tag{4.8}
\]

is the inverse of \( e^b_i \).
Using the standard identities

\[
\partial_i e^a_j - \partial_j e^a_i = e^b_i e^c_j f_{bc}^a
\]  

(4.9a)

\[
f(\sigma')\delta'(\sigma - \sigma') - f(\sigma)\delta'(\sigma - \sigma') = f'(\sigma)\delta(\sigma - \sigma')
\]  

(4.9b)

\[
\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma') = -\frac{\partial}{\partial \sigma'}\delta(\sigma - \sigma')
\]  

(4.9c)

one can obtain the commutation relations

\[
\{ \tilde{A}^c_0(\sigma), A^d_1(\sigma') \} = f_p^{\ cd} A^b_1 \delta(\sigma - \sigma') + \kappa^{cd} \delta'(\sigma - \sigma')
\]  

\[
\{ \tilde{A}^c_0(\sigma), \tilde{A}^d_0(\sigma') \} = f_p^{\ cd} \tilde{A}^b_0 \delta(\sigma - \sigma')
\]  

(4.10)

where

\[
\tilde{A}^a_0 = 2\kappa^{ab} L_{bc} A^c_0.
\]  

(4.11)

These commutation relations are the building blocks for the calculation of the Poisson bracket for the Lax field. Recall the definition

\[
\mathcal{L}^a(\sigma) = N^a_{0b} A^b_1 + N^a_{1b} A^b_0.
\]  

(4.12)

With the help of (4.10), a straightforward calculation gives

\[
\{ \mathcal{L}^a(\sigma), \mathcal{L}^b(\sigma') \} = \frac{1}{2}(\Gamma^{ab}_{0q} A^q_1 + \Gamma^{ab}_{1q} A^q_0) \delta(\sigma - \sigma') + \frac{1}{2} Q^{ab} \delta'(\sigma - \sigma')
\]  

(4.13)

where

\[
\Gamma^{ab}_{0q} = (N^a_{0c} N^b_{1d} - N^b_{0c} N^a_{1d})(L^{-1})^{ds} f_{sc}^c
\]

\[
\Gamma^{ab}_{1q} = N^a_{1c} N^b_{1d}(L^{-1})^{cs} (L^{-1})^{dr} f_{sr}^p L_{pq}
\]

\[
Q^{ab} = (N^a_{0c} N^b_{1d} + N^b_{0d} N^a_{1c})(L^{-1})^{dc}.
\]  

(4.14)

This computation did not use the integrability conditions which relate the auxiliary $N$ matrices with the coupling constants matrix $L$. Therefore we cannot expect the Poisson bracket to close simply on some algebra. In fact even using (3.4) does not simplify things, and we were not able to identify, for the generic case, a set of fields which include the Lax field and is closed under the Poisson bracket. Although the Poisson bracket $\{ \mathcal{L}^a(\sigma), \mathcal{L}^b(\sigma') \}$ at equal $\tau$ does not close in general, for the case of the PCM (i.e. $L_{ab} = \frac{1}{g^2} \kappa_{ab}$) we have the following:

\[
\mathcal{L}^a = \lambda_0 A^a_1 + \lambda_1 A^a_0 = -i \sinh(\lambda)(\sinh(\lambda) A^a_1 + \cosh(\lambda) A^a_0)
\]  

(4.15)
Let us define the following fields

$$J_n^a = \frac{2}{g^2} (n \tanh(\lambda) A_1^a + A_0^a)$$  \hspace{1cm} (4.16)$$

where $n \in \mathbb{Z}$, and our Lax field is proportional to $J_1^a$:

$$\mathcal{L}^a = -\frac{i}{4} g^2 \sinh 2(\lambda) J_1^a \hspace{1cm} .$$  \hspace{1cm} (4.17)$$

These operators generate an algebra very similar to the two loop affine algebra.

$$\{ J_n^a(\sigma), J_{m}^b(\sigma') \} = f_{abc}^{} J_{n+m}^c(\sigma) \delta(\sigma - \sigma') + \frac{2(n + m)}{g^2} \tanh(\lambda) \kappa^{ab} \delta'(\sigma - \sigma')$$ \hspace{1cm} (4.18)$$

from which it is now straightforward to find the fundamental Poisson brackets:

$$\{ J_n^a(\sigma), J_{m}^b(\sigma') \} = \frac{4(n + m)}{g^2} \tanh(\lambda) r \delta'(\sigma - \sigma') + [r, J_{n+m} \otimes I - I \otimes J_{n+m}] \delta(\sigma - \sigma')$$ \hspace{1cm} (4.19)$$

where

$$\{ J_n^a(\sigma), J_{m}^b(\sigma') \} = \{ J_n^a(\sigma), J_{m}^b(\sigma') \} t_a \otimes t_b$$

here $I$ is the unit matrix and $r$, which is given by

$$r = \frac{1}{2} \kappa^{ab} t_a \otimes t_b$$ \hspace{1cm} (4.20)$$

is the classical $r$-matrix.

5. The beta function

The beta function for the general sigma model is known \[1\]. It is given in terms of the Riemann tensor. To first order

$$\mu \frac{\partial}{\partial \mu} G_{ij} = \frac{1}{4\pi} R_{ij}$$ \hspace{1cm} (5.1)$$

where $R_{ij} = R_{ikj}^i$.

The curvature is given in terms of the Levi-Civita connection

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^r_{lj} \Gamma^i_{kr} - \Gamma^r_{kj} \Gamma^i_{lr}$$ \hspace{1cm} (5.2)$$

where the Levi-Civita connection is given in terms of the metric

$$\Gamma^i_{lj} = \frac{1}{2} G^{is} (\partial_l G_{sj} + \partial_j G_{sl} - \partial_s G_{lj})$$ \hspace{1cm} (5.3)$$
The Ricci tensor in our special case was written explicitly by Halpern and Yamron [8]. We rederive, for completeness sake, their result together with the explicit form of the Riemann tensor which is needed in higher order corrections to the beta function. Since the metric in our special case (as well as in [8]) is given by

\[ G_{ij} = L_{ab} e_i^a e_j^b \]  

the Levi-Civita connection has the form

\[ \Gamma^i_{lj} = \frac{1}{2} e^i_a (\partial_l e^a_j + \partial_j e^a_l - e^a_{lj}) + \frac{1}{2} F^a_{sr} e^i_s e^r_j + e^r_j e^s_i \]  

A tedious but straightforward calculation now gives

\[ R_{ijkl} = \frac{1}{4} U_{rqs}^b e_i^r e_j^q e_k^s \]  

where

\[ U_{rqs}^b = f_{qs}^a f_{ra}^b + 2 \left( S_{rs}^a f_{aq}^b - S_{sr}^a f_{qa}^b + S_{qa}^b f_{rs}^a - S_{sa}^b f_{rq}^a \right) \]

\[ + 4 \left( S_{ar}^b f_{qs}^a + S_{rs}^b f_{qa}^b - S_{qr}^a f_{sa}^b \right) \]  

and \( S_{rs}^a \) is given in (2.8). The Ricci tensor is now given by contracting indices

\[ R_{ij} = R_{ikj}^k = \frac{1}{4} U_{rbs}^b e_i^r e_j^s \]  

where

\[ U_{rbs}^b = (\hat{\kappa}_{rs} + 2 S_{ar}^b f_{bs}^a - 2 S_{bs}^a f_{ra}^b - 4 S_{br}^a S_{sa}^b) \]  

where \( \hat{\kappa} \) is the dual Coxeter number defined by \( f_{br}^a f_{as}^b = -\hat{\kappa}_{rs} \). We also used the fact that \( S_{ba}^b = 0 \). Further simplification can be achieved using the following identities:

\[ F_{ar}^b f_{bs}^a = -F_{as}^b f_{rb}^a \]

\[ F_{ra}^b f_{bs}^a = F_{as}^b F_{rb}^a \]  

\[ F_{br}^a F_{as}^b = f_{br}^a f_{as}^b = -\hat{\kappa}_{rs} \]  

The final result is

\[ U_{rbs}^b = 2 \hat{\kappa}_{rs} - 2 F_{as}^b f_{rb}^a - F_{rb}^a F_{sa}^b \]  

This is in agreement with [8]. The \( \beta \) function equation reads now

\[ \frac{\partial}{\partial \log \mu} L_{rs} = \beta(L_{rs}) = \frac{1}{16\pi} \left( 2 \hat{\kappa}_{rs} - 2 F_{as}^b f_{rb}^a - F_{rb}^a F_{sa}^b \right) \]
This relation holds whether or not the model is integrable. It is interesting to see some examples:

**Example 1:** The PCM $L_{ab} = \frac{1}{g^2} \kappa_{ab}$. In this case

$$F^a_{\;rb} F^b_{\;sa} = -\hat{h} \kappa_{rs}$$

and

$$R_{ij} = \frac{1}{4} \hat{h} \kappa_{rs} e^r_i e^s_j$$

and the renormalization equation is solved simply

$$\frac{1}{g^2(\mu)} = \frac{1}{16\pi} \hat{h} \log(\frac{\mu}{\Lambda})$$

where $\Lambda = \exp(-\frac{16\pi}{g^2(1)})$. Clearly the model is integrable along the flow.

**Example 2:** The diagonal anisotropic PCM: $L_{ab} = J_a \kappa_{ab}$ (no summation). we take $SU(2)$ with $J_1 = J_2 \neq J_3$, and the dual Coxeter number for $SU(2)$ is $\hat{h} = 2$. The Ricci tensor for this case is

$$R_{ij} = \frac{1}{2} (\hat{h} - \lambda_r(J)) \kappa_{rs} e^r_i e^s_j$$

where

$$\lambda_1(J) = \lambda_2(J) = \frac{J_3}{J_1}$$

$$\lambda_3(J) = 2 - \left(\frac{J_3}{J_1}\right)^2$$

After changing variables

$$t = \log \mu, \quad x = 8\pi J_1, \quad y = 8\pi J_3$$

the renormalization equations read

$$\frac{\partial x}{\partial t} = 2 - \frac{y}{x}$$

$$\frac{\partial y}{\partial t} = \left(\frac{y}{x}\right)^2$$

Define $z = \frac{y}{x}$ then we can separate variables:

$$x \frac{\partial z}{\partial x} = \frac{2z(z-1)}{2 - z}$$

which is solved by

$$x^2 = a^2 \frac{z - 1}{z^2}$$
where $a$ is an integration constant. Using the above equations, we finally get

$$\frac{\partial z}{\partial t} = \frac{2}{a}(z - 1)^{\frac{1}{2}}z^2$$

(5.22)

Changing variables one more time $w^2 = z - 1 = \frac{J_3 - J_1}{J_1}$, this equation is readily solved by

$$t + \text{const} = \frac{a}{2}A^2 \log \frac{w - B}{w + B} - a\left(\frac{B}{A}\right)^2 \tan^{-1} \frac{w}{A}$$

(5.23)

where

$$A = \sqrt{\sqrt{2} + 1}$$

$$B = \sqrt{\sqrt{2} - 1}$$

(5.24)

and again it is clear that the model is integrable along the flow.

6. Conclusion

The main result of this paper is the derivation of the algebraic equations (3.4) as conditions for the integrability of the GPCM. As common in the study of integrable models, these equations define an overdetermined system. Naive counting gives $(\dim G)^3$ equations for only $(\dim G)^2$ variables. It is already remarkable if a solution exists, but this is not enough to guarantee integrability. It must also have, as an algebraic variety, a dimension greater than zero to allow for a dependence on a spectral parameter.

We showed by examples that the set of solutions to these conditions is not empty. These two examples: the PCM and the $SU(2)$ diagonal anisotropic PCM are known to be integrable [2] [6] and it is reassuring to see that they actually solve the system and that the spectral parameter naturally arises. However, these are not the only solutions of (3.4). The search for new solutions is under current study and will be reported elsewhere.

Another important subject in classical integrable models is the classical r-matrix. We note that we were not able to find, in general, a set of fields which includes the Lax field and that is closed under the Poisson brackets. Another way to attack this problem is to use the generalized Thirring model which gives the same equations of motion. It has, though, different conditions for integrability, and different Poisson structure and it may be easier to identify and to close an algebra in this setting [9]. I believe nevertheless that the example of the PCM is very revealing and very intriguing. From one hand the appearance of the two loop affine algebra is surprising since the models manifest only $G \times G$ symmetry, on
the other hand we must remember that the fields that we defined are not the conserved currents. Thus one cannot infer, that this algebra is a symmetry of the model.

The directions for further research are numerous. In fact we only have scratched the tip of an iceberg. At the classical level there are the tasks of adding the antisymmetric tensor and dilaton terms and deriving integrability conditions for these models. Another direction may be adding fermions and studying supersymmetric models. There is also of course the problem of finding new solutions to both overdetermined differentio-algebraic systems eqs. (3.4),(3.6-7). It is also important to find higher conservation laws, and to find and understand the classical r-matrix. For almost all of these problems there is a quantum analog. Yet another direction is to look at the possible relation between these models and the irrational conformal field theories (see [10] and references therein). We hope to address some of these questions in the future.

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