A multiscale stabilization of the streamfunction form of the steady state Navier-Stokes equations

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Abstract.
In this document, the derivation of a multiscale stabilized finite element method for the streamfunction formulation of the two-dimensional, incompressible Navier-Stokes equations is motivated and outlined. A linearized model problem is developed and analyzed through a variational multiscale approach to determine the form of the stabilized terms.

1. Introduction
While stabilized finite element methods are in their third decade of development, they continue to be a lively research topic. Starting with the Streamline/Upwind Petrov-Galerkin Method (SUPG) [1, 2], whose name reflects the somewhat heuristic construction, the methods evolved and morphed through improved analysis to become the underpinnings of more recent efforts in multiscale analysis and error estimation/indication. In the process, the stabilization became more general as well. The variational multiscale method developed by Hughes et al. [3, 4] has provided a general framework for stabilized methods. By recognizing the impact of small scales on larger scales, the multiscale methodology provides a much clearer physical intuition into the mathematics of such technologies.

While stabilized finite element methods for second-order problems have been well-established, no stabilized methods for fourth-order problems have been developed as of yet. Fourth-order problems arise in fluid dynamics, for example, by condensing the Navier-Stokes equations and continuity. By introducing a streamfunction for the solenoidal velocity field, one may reduce the steady, two-dimensional Navier-Stokes equations from a set of three second-order partial differential equations to a single fourth-order partial differential equation in terms of a streamfunction. This fourth-order partial differential equation can then be solved using \( C^1 \) finite elements.

The popularity of streamfunction formulations has been damped by two limitations. First of all, there is an inherent difficulty in extending streamfunction formulations to three dimensions. Second, \( C^1 \) finite elements must be employed, which are not trivial to develop or implement. Magnetohydrodynamics (MHD) has caused a renewed interest in the streamfunction form of the Navier-Stokes equations as (1) the geometries in consideration can be decomposed into two-dimensional elements in the plane and spectral elements in the toroidal plane and (2) other operators representing MHD physics also benefit from \( C^1 \) continuity [5, 6].
The streamfunction form suffers from the same stability issues as its traditional counterpart. As the Peclet number, which represents the ratio between the advection and diffusion terms, approaches infinity, the numerical solution obtained from the traditional Galerkin methodology suffers wiggles and oscillations. In order to counter this, a stabilization procedure is presented, the first of its kind for fourth-order problems. This allows one to use a coarse spatial discretization in the numerical solution of even the toughest problems in fluid dynamics and still expect to obtain a solution with some approximating power. Then, one may employ adaptivity to progressively improve the solution at the optimal rate of convergence. This stabilization procedure is very similar in nature to those for second-order problems, but some changes must be made in order to account for higher-order operators and continuously differentiable basis functions.

2. The streamfunction formulation and a linearized model problem

In order to properly motivate a stabilized methodology for the streamfunction form of the Navier-Stokes equations, it is important to recognize the physical meaning of such a formulation. Thus, in this section, the streamfunction formulation is derived from the Navier-Stokes equations initially given in terms of two second-order partial differential equations. Consider the steady, two-dimensional, incompressible Navier-Stokes equations in conservative form. These equations may be written as follows:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p. \]

The streamfunction, \( \psi \), is defined as follows:

\[ \mathbf{u} = \left( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right). \]

One notes that if the velocity field satisfies (3), the velocity field satisfies (1) automatically. By taking the curl of both sides of (2) and substituting our expression for the streamfunction, one obtains the streamfunction formulation of the steady Navier-Stokes equations:

\[ \nabla^\perp \psi \cdot \nabla \Delta \psi = \nu \Delta^2 \psi, \]

where

\[ \nabla^\perp = \left( -\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right). \]

One notes that the pure streamfunction formulation is an advection-diffusion equation for the quantity \( \Delta \psi \), the vorticity of the velocity field. The velocity field for this advection-diffusion equation is precisely the velocity field \( \mathbf{u} \). Thus, the streamfunction form simply dictates the transport of vorticity in the domain of interest. One also notes that since the pure streamfunction formulation is fourth-order in nature, traditional \( C^0 \) finite elements cannot be utilized. There are a number of options. One includes introducing auxiliary variables to reduce the system to second order. However, this will necessarily introduce more equations into the system and inflate the size of the system. An alternative approach is to utilize \( C^1 \) finite elements. There have been many difficulties in using \( C^1 \) finite elements in the past. The primary difficulty is in their implementation. \( C^1 \) finite elements are often presented in implicit formulation. That is, given a particular element in a finite element mesh, the basis functions associated with said element are not explicitly known. A linear system must be solved in order to determine these basis functions. However, explicit representations of \( C^1 \) finite element basis functions are possible. These basis functions differ from \( C^0 \) finite elements in that they are not just functions of parent coordinates but also of the element geometry. The authors of this paper are in the process of developing
explicit and efficient representations of $C^1$ finite elements for fluid dynamics in order to combat this traditional limitation [7].

Stabilized finite element methods are inherently dependent on the ability to analyze the stability and convergence properties of a numerical method for a particular set of partial differential equations. Since the analysis of numerical methods for nonlinear equations is in many cases not possible, linearized model problems are developed in order to carry out a full stability and convergence analysis. Replacing $\nabla \cdot \psi$ with a solenoidal velocity field $\vec{u}$ (and thereby making $\psi$ no longer the stream function), one obtains an appropriate linearized model problem for the streamfunction form of the Navier-Stokes equations:

$$\vec{u} \cdot \nabla \Delta \psi = \nu \Delta^2 \psi.$$ (6)

One notes that (6) exhibits the same qualities as (4). In particular, it is also a advection-diffusion equation for the vorticity.

3. Boundary conditions

Boundary conditions for the streamfunction form and its linearized model problem can be a concern. For simplicity, only Dirichlet boundary conditions along the entire boundary will be considered here. This may occur, for example, when no-slip boundary conditions exist for the entire boundary. These boundary conditions are of the form:

$$\vec{u}(x, y) = \vec{f}(x, y) \text{ for } (x, y) \in \partial \Omega.$$ (7)

In order to develop equivalent boundary conditions for the streamfunction $\psi$, one considers the following two relations:

$$\frac{\partial \psi}{\partial \vec{s}} = \vec{u} \cdot n,$$ (8)

$$\frac{\partial \psi}{\partial \vec{n}} = -\vec{u} \cdot s,$$ (9)

where $\vec{n}$ and $\vec{s}$ are respectively the normal and tangent to the boundary. Equations (8) and (9) can be used along the entire boundary to define appropriate boundary conditions. To arrive at more familiar boundary conditions, $\psi$ can be set arbitrarily to 0 along some point at the boundary and (8) can be integrated along the entire boundary to arrive at a Dirichlet boundary condition. Then, one arrives at the following two boundary conditions, one Dirichlet and one Neumann, for the streamfunction form (and similarly for the linearized model problem):

$$\psi(x, y) = f(x, y) \text{ for } (x, y) \in \partial \Omega,$$ (10)

$$\psi_n(x, y) = g(x, y) \text{ for } (x, y) \in \partial \Omega.$$ (11)

4. Variational multiscale method and stabilization

The strong form of the linearized model problem is as follows:

**Strong Form**

Find $\psi : \Omega \to \mathcal{R}$ such that

$$\mathcal{L}\psi(x, y) = 0 \text{ for } (x, y) \in \Omega,$$ (12)

$$\psi(x, y) = f(x, y) \text{ for } (x, y) \in \partial \Omega,$$ (13)

$$\psi_n(x, y) = g(x, y) \text{ for } (x, y) \in \partial \Omega.$$ (14)
where \( f : \partial \Omega \to \mathcal{R}, g : \partial \Omega \to \mathcal{R}, \) and \( \mathcal{L} = \vec{u} \cdot \nabla \Delta - \nu \Delta^2. \)

To develop the variational form of (12)-(14), one introduces the following functional spaces:

\[
S = \left\{ \psi | \psi \in H^2(\Omega), \psi = f, \psi_n = g \text{ on } \partial \Omega \right\},
\]
\[
W = \left\{ w | w \in H^2(\Omega), w = 0, w_n = 0 \text{ on } \partial \Omega \right\},
\]

where \( H^2(\Omega) \) is the Sobolev space of functions with square-integrable functions with square-integrable first and second derivatives. With these spaces defined, the variational form is:

**Variational Form**

Find \( \psi \in S \) such that \( \forall w \in W \)

\[
(w, \mathcal{L}\psi) = \int_{\Omega} w \mathcal{L}\psi d\Omega = 0.
\]

The variational multiscale method consists of a coarse-fine scale decomposition. To continue, one defines:

\[
\psi = \bar{\psi} + \psi',
\]
\[
w = \bar{w} + w',
\]

where \( \bar{\psi} \) and \( \bar{w} \) represent the coarse scales and \( \psi' \) and \( w' \) represent the fine scales. The coarse scales satisfy the Dirichlet and Neumann boundary conditions while the fine scales satisfy the homogeneous counterparts of these boundary conditions. One also defines \( S = \bar{S} \oplus S' \) and \( W = \bar{W} \oplus W' \) where the barred quantities represent the coarse scale spaces and the prime quantities the fine scale spaces. One assumes that the coarse scale and fine scale spaces are linearly independent. Substituting the expressions above for \( \psi \) and \( w \), using the fact that the coarse and fine scale spaces are linearly independent, and using integration by parts formulas on the fine scale terms, one arrives at the following problem:

Find \( \bar{\psi} \in \bar{S} \) and \( \psi' \in S' \) such that \( \forall \bar{w} \in \bar{W}, w' \in W' \)

\[
(\bar{w}, \mathcal{L}\bar{\psi}) + (\mathcal{L}^* \bar{w}, \psi') = 0,
\]
\[
(w', \mathcal{L}\psi') + (\mathcal{L}^* w', \psi') = 0,
\]

where \( \mathcal{L}^* = -\vec{u} \cdot \nabla \Delta - \nu \Delta^2 \) is the adjoint of the linear differential operator.

Multiscale methodologies typically try to solve for an approximation of \( \psi' \) analytically and then insert this result into (20). Through an analysis of (21) utilizing Green’s functions, one finds that \( \psi' = -M' \mathcal{L}\psi \) where \( M' \) is an integral operator involving the Green’s function of the dual problem of (21). As the Green’s function is very difficult and often impossible to determine, the expression for \( M' \) will be replaced by an algebraic approximation \( \tau \). This algebraic approximation, often denoted as the stabilization parameter, is determined by a stability and convergence analysis. This convergence analysis is very similar to that employed for Streamline Upwind/Petrov-Galerkin (SUPG) and Galerkin Least Squares (GLS) [8] methodologies. For an overview on how to conduct this stability and convergence analysis, see [9]. After the stabilization parameter \( \tau \) is defined, one may expand the abstract operators and integrate by parts on the coarse scale equation to balance derivatives. The resulting problem becomes:
Multiscale stabilized formulation of the linearized model problem

Find $\bar{\psi} \in \bar{S}$ such that $\forall \bar{w} \in \bar{W}$

$$\int_{\Omega} \left( \nabla \bar{w} \cdot \bar{u} \Delta \bar{\psi} + \Delta \bar{w} \Delta \bar{\psi} \right) d\Omega + \int_{\Omega} \tau \left( \bar{u} \cdot \nabla \Delta \bar{w} + \nu \Delta^2 \bar{w} \right) \left( \bar{u} \cdot \nabla \Delta \bar{\psi} - \nu \Delta^2 \bar{\psi} \right) d\Omega = 0.$$  \hspace{1cm} (22)

One notes that if $\bar{u}$ is replaced by $\nabla^\perp \bar{\psi}$ in the above equation and the resulting stabilization parameter $\tau$, one obtains a multiscale stabilization for the pure streamfunction formulation of the steady, two-dimensional, incompressible Navier-Stokes equations.

5. Conclusions and future work

In this document, a multiscale stabilization of the streamfunction form of the steady Navier-Stokes equations has been performed. A linearized model problem was developed through a variational multiscale approach to determine the precise form of the stabilized terms. The authors are currently working on the implementation and testing of this multiscale stabilization with Argyris, Bell, and Clough-Tocher elements and the extension of this current work for the streamfunction formulation of the transient Navier-Stokes equations. Exploration of other issues such as frozen coefficient analysis, efficient linearization, time accuracy, error estimation, and boundary condition is planned for the future. We believe that the stabilization will enable much more robust adaptivity for difficult MHD applications.

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