Quantitative Helly-Type Theorems via Sparse Approximation

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Received: 24 August 2021 / Revised: 14 August 2022 / Accepted: 8 September 2022 / Published online: 10 November 2022
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Abstract
We prove the following sparse approximation result for polytopes. Assume that \(Q\) is a polytope in John’s position. Then there exist at most \(2^d\) vertices of \(Q\) whose convex hull \(Q'\) satisfies \(Q \subseteq -2^d Q'\). As a consequence, we retrieve the best bound for the quantitative Helly-type result for the volume, achieved by Brazitikos, and improve on the strongest bound for the quantitative Helly-type theorem for the diameter, shown by Ivanov and Naszódi: We prove that given a finite family \(F\) of convex bodies in \(\mathbb{R}^d\) with intersection \(K\), we may select at most \(2^d\) members of \(F\) such that their intersection has volume at most \((cd)^{3d/2} \text{vol } K\), and it has diameter at most \(2d^2 \text{diam } K\), for some absolute constant \(c > 0\).

Editor in Charge: János Pach

The research of Gergely Ambrus was supported by NKFIH grant KKP-133819; by the EFOP-3.6.1-16-2016-00008 Project, which in turn has been supported by the European Union, co-financed by the European Social Fund; and by the national project TKP2021-NVA-09. Project no. TKP2021-NVA-09 has been implemented with the support provided by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, financed under the TKP2021-NVA funding scheme.

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1 History and Results

Helly’s theorem, dated from 1923 [13], is a cornerstone result in convex geometry. Its finitary version states that the intersection of a finite family of convex sets is empty if and only if there exists a subfamily of \( d + 1 \) sets such that its intersection is empty. In 1982, Bárány et al. [4] introduced the following quantitative versions of Helly’s theorem: there exist positive constants \( v(d) \) and \( \delta(d) \) such that for a finite family \( \mathcal{F} \) of convex bodies (that is, compact convex sets with non-empty interior) in \( \mathbb{R}^d \), one may select \( 2d \) members such that their intersection has volume at most \( v(d) \text{vol}(\bigcap \mathcal{F}) \), or has diameter at most \( \delta(d) \text{diam}(\bigcap \mathcal{F}) \).

The problem of finding the optimal values of \( \delta(d) \) and \( v(d) \) has enjoyed special interest in recent years (see e.g. the excellent survey article [3]). In [4] (see also [5]) the authors proved that \( v(d) \leq d^{2d^2} \) and \( \delta(d) \leq d^{2d} \), and they conjectured that \( v(d) \approx d^{c_1d} \) and \( \delta(d) \approx c_2d^{1/2} \) for some positive constants \( c_1, c_2 > 0 \).

For the volume problem, in a breakthrough paper, Naszódi [17] proved that \( v(d) \leq e^{d+1}d^{2d+1/2} \), while \( v(d) \geq d^{d/2} \) must hold. Improving upon his ideas, Brazitikos [6] found the current best upper bound for volume: \( v(d) \leq (cd)^{3d/2} \) for a constant \( c > 0 \).

For the diameter question, Brazitikos [8] proved the first polynomial bound on \( \delta(d) \) by showing that \( \delta(d) \leq cd^{11/2} \) for some \( c > 0 \). In 2021, Ivanov and Naszódi [14] found the best known upper bound, \( \delta(d) \leq (2d)^3 \), and also proved that \( \delta(d) \geq cd^{1/2} \). Thus, the value conjectured in [4] for \( \delta(d) \) would be asymptotically sharp.

In the present note, we show that given a finite family \( \mathcal{F} \) of closed convex sets, one can select at most \( 2d \) members such that their intersection sits inside a scaled version of \( \bigcap \mathcal{F} \) for a suitable location of the origin. Clearly, it suffices to prove this statement for the special case when \( \mathcal{F} \) consists of closed halfspaces intersecting in a convex body. As an application, we obtain an improvement on the diameter bound, \( \delta(d) \leq 2d^2 \), and retrieve the best known bound for \( v(d) \). The crux of the argument is the following sparse approximation result for polytopes, which is a strengthening of [14, Thm. 2].

**Theorem 1.1** Let \( \lambda > 0 \) and \( Q \subset \mathbb{R}^d \) be a convex polytope such that \( Q \subseteq -\lambda Q \). Then there exist at most \( 2d \) vertices of \( Q \) whose convex hull \( Q' \) satisfies

\[
Q \subseteq -(\lambda + 2)d Q'.
\]

We immediately obtain the following corollary.

**Corollary 1.2** Assume that \( Q = -Q \) is a symmetric convex polytope in \( \mathbb{R}^d \). Then we may select a set of at most \( 2d \) vertices of \( Q \) with convex hull \( Q' \) such that

\[
Q \subseteq 3d Q'.
\]
As usual, let $B^d$ denote the unit ball of $\mathbb{R}^d$ and let $S^{d-1}$ be the unit sphere of $\mathbb{R}^d$. A standard consequence of Fritz John’s theorem [16] states that if $K \subset \mathbb{R}^d$ is a convex body in John’s position, that is, the largest volume ellipsoid inscribed in $K$ is $B^d$, then $B^d \subseteq K \subseteq dB^d \subseteq -dK$ (see e.g. [2]). Thus, we derive

**Corollary 1.3** Assume that $Q \subset \mathbb{R}^d$ is a convex polytope in John’s position. Then there exists a subset of at most $2d$ vertices of $Q$ whose convex hull $Q'$ satisfies

$$Q \subseteq -2d^2 Q'.$$

For $n \in \mathbb{N}^+$, we will use the notation $[n] = \{1, \ldots, n\}$. For a family of sets $\{K_1, \ldots, K_n\} \subset \mathbb{R}^d$ and for a subset $\sigma \subset [n]$, let

$$K_\sigma = \bigcap_{i \in \sigma} K_i,$$

as in [14]. Also, let $\binom{[n]}{\leq k}$ stand for the set of all subsets of $[n]$ with cardinality at most $k$. Using this terminology, we are ready to state our quantitative Helly-type result.

**Theorem 1.4** Let $\{K_1, \ldots, K_n\}$ be a family of closed convex sets in $\mathbb{R}^d$ with $d \geq 2$ such that their intersection $K = K_1 \cap \cdots \cap K_n$ is a convex body. Then there exists a $\sigma \in \binom{[n]}{\leq 2d}$ such that

$$\text{vol}_d K_\sigma \leq (cd)^{3d/2} \text{vol}_d K \quad \text{and} \quad \text{diam } K_\sigma \leq 2d^2 \text{diam } K$$

for some constant $c > 0$.

To conclude the section we formulate the following conjecture, which was essentially stated already in [4]. This would imply the asymptotically sharp bound for $v(d)$, see the remark after the proof of Theorem 1.4.

**Conjecture 1.5** Assume that $\{u_1, \ldots, u_n\} \subset S^{d-1}$ is a set of unit vectors satisfying the conditions of Fritz John’s theorem. That is, there exist positive numbers $\alpha_1, \ldots, \alpha_n$ for which $\sum_{i=1}^n \alpha_i u_i = 0$ and $\sum_{i=1}^n \alpha_i u_i \otimes u_i = I_d$, the identity operator on $\mathbb{R}^d$. Then there exists a subset $\sigma \subset [n]$ with cardinality at most $2d$ so that

$$B^d \subset cd \text{ conv } \{u_i : i \in \sigma\}$$

with an absolute constant $c > 0$.

That the above estimate would be asymptotically sharp is shown by taking $n = d + 1$ and letting $\{u_1, \ldots, u_n\}$ to be the set of vertices of a regular simplex inscribed in $S^{d-1}$.

Note that we study quantitative Helly-type questions that require selecting at most $2d$ sets, which is the smallest cardinality for which such estimates may hold. Versions obtained by relaxing this cardinality bound have been studied e.g. by Brazitikos [7], Dillon and Soberón [9], and Ivanov and Naszódi [14]. In particular, an estimate which matches Theorem 1.1 asymptotically was given in [14] when selecting $2d + 1$ vertices of the polytope, and an asymptotically sharp estimate for the quantitative Helly-type
theorem for the diameter was proved in [9] for sufficiently large sub-families. Further quantitative Helly-type results have been studied in [15] (for log-concave functions) and [11] (continuous versions).

2 Proofs

Proof of Theorem 1.1 The condition \( Q \subseteq -\lambda Q \) ensures that 0 \( \in \text{int} \ Q \). Among all simplices with \( d \) vertices from the set of vertices of \( Q \) and one vertex at the origin, consider a simplex \( S = \text{conv} \{0, v_1, \ldots, v_d\} \) with maximal volume. We write \( S \) in the form

\[
S = \left\{ x \in \mathbb{R}^d : x = \alpha_1 v_1 + \cdots + \alpha_d v_d \text{ for } \alpha_i \geq 0 \text{ and } \sum_{i=1}^d \alpha_i \leq 1 \right\}.
\] (1)

For every \( i \in [d] \), let \( H_i \) be the hyperplane spanned by \( \{0, v_1, \ldots, v_d\} \setminus \{v_i\} \), and let \( L_i \) be the closed strip between the hyperplanes \( v_i + H_i \) and \( -v_i + H_i \). Define \( P = \bigcap_{i \in [d]} L_i \) (see Fig. 1). Note that

\[
P = \{ x \in \mathbb{R}^d : \text{vol}_d(\text{conv}(\{0, x, v_1, \ldots, v_d\} \setminus \{v_i\})) \leq \text{vol}_d(S) \text{ for all } i \in [d] \}.
\] (2)

This follows from the volume formula

\[
\text{vol}_d(\text{conv}(\{0, w_1, \ldots, w_d\})) = \frac{1}{d!} |\det(w_1 w_2 \ldots w_d)|
\]

for arbitrary \( w_1, \ldots, w_d \in \mathbb{R}^d \), which implies that for all \( x \in \mathbb{R}^d \) of the form \( x = cv_i + w \) with \( w \in H_i, i \in [d] \),

\[
\text{vol}_d(\text{conv}(\{0, x, v_1, \ldots, v_d\} \setminus \{v_i\})) = |c| \text{vol}_d(S).
\]

Next, we show that

\[
P = \{ x \in \mathbb{R}^d : x = \beta_1 v_1 + \cdots + \beta_d v_d \text{ for } \beta_i \in [-1, 1] \}.
\] (3)

Indeed, since \( v_1, \ldots, v_d \) are linearly independent, we may consider the linear transformation \( A \) with \( A(v_i) = e_i \) for \( i \in [d] \). Note that

\[
A(P) = \bigcap_{i \in [d]} A(L_i) = \{ x \in \mathbb{R}^d : x = \beta_1 e_1 + \cdots + \beta_d e_d \text{ for } \beta_i \in [-1, 1] \}.
\]

Thus, (3) holds. Since \( S \) is chosen maximally, (2) shows that for any vertex \( w \) of \( Q \), \( w \in P \). By convexity,

\[
Q \subseteq P.
\] (4)
Let $S' = -2dS + (v_1 + \cdots + v_d)$. By (1),

$$S' = \left\{ x \in \mathbb{R}^d : x = \gamma_1 v_1 + \cdots + \gamma_d v_d \text{ for } \gamma_i \leq 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geq -d \right\}.$$  \hfill (5)

Then, from (3) and (5),

$$P \subseteq S'.$$  \hfill (6)

Let $u = (1/d)(v_1 + \cdots + v_d)$ be the centroid of the facet $\text{conv}\{v_1, \ldots, v_d\}$ of $S$. Let $y$ be the intersection of the ray from 0 through $-u$ and the boundary of $Q$. By Carathéodory’s theorem, we can choose $k \leq d$ vertices $\{v'_1, \ldots, v'_k\}$ of $Q$ such that $y \in \text{conv}\{v'_1, \ldots, v'_k\}$. Set $Q' = \text{conv}\{v_1, \ldots, v_d, v'_1, \ldots, v'_k\}$.

Note that $[y, u] \subseteq Q'$, which implies $0 \in Q'$. Thus,

$$S \subseteq Q'.$$  \hfill (7)

Since $Q \subseteq -\lambda Q$, we have that $-u \in \lambda Q$. Since $\lambda y$ is on the boundary of $\lambda Q$, we also have that $-u \in [0, \lambda y]$. We know that $0, \lambda y \in \lambda Q'$, so

$$u \in -\lambda Q'.$$  \hfill (8)

Combining (4), (6), (7), and (8):

$$Q \subseteq P \subseteq S' = -2dS + du \subseteq -2dQ' - \lambda dQ' = -(\lambda + 2)dQ'.$$  \hfill (9)
Proof of Theorem 1.4. As shown in [4], we may assume that the family \( \{ K_1, \ldots, K_n \} \) consists of closed halfspaces such that \( K = \bigcap K_i \) is a \( d \)-dimensional polytope. Let \( T \) be the affine transformation sending \( K \) to John’s position. Let \( \tilde{K}_i = T K_i \) for \( i \in [n] \), \( \tilde{K} = TK \), and for some \( \sigma \subset [n] \), let \( \tilde{K}_\sigma = \bigcap_{i \in \sigma} \tilde{K}_i \). We claim that there exists \( \sigma \in \binom{[n]}{\leq 2d} \) such that the following two properties hold:

\[
\tilde{K}_\sigma \subseteq -2d^2 \tilde{K},
\]

\[
\text{vol}_d \tilde{K}_\sigma \leq (cd)^{3d/2} \text{vol}_d \tilde{K}
\]

for some constant \( c > 0 \). Estimates (10) and (11) are affine invariant, so this will suffice to prove Theorem 1.4.

Recall that since \( \tilde{K} \) is in John’s position, \( B^d \subseteq \tilde{K} \subseteq dB^d \) (see [2] or [12, Thm. 5.1]). Setting \( Q = (\tilde{K})^\circ \), this yields that \( (1/d)B^d \subseteq Q \subseteq B^d \) (here and later on, \( K^\circ \) stands for the polar set: \( K^\circ = \{ x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in K \} \)). In particular, \( Q \subseteq -dQ \). Hence, we may apply Theorem 1.1 to \( Q \) with \( \lambda = d \), we obtain a subset of at most \( 2d \) vertices of \( Q \) such that their convex hull \( Q' \) satisfies

\[
Q \subseteq -(d + 2)d Q' \subseteq -2d^2 Q'.
\]

The family of closed halfspaces supporting the facets of \( (Q')^\circ \) is a subset of \( \{ \tilde{K}_1, \ldots, \tilde{K}_n \} \) with at most \( 2d \) elements. Thus, we can choose \( \sigma \in \binom{[n]}{\leq 2d} \) such that \( \tilde{K}_\sigma = (Q')^\circ \). Taking the polar of (12), we obtain

\[
\tilde{K}_\sigma \subseteq -(d + 2)d \tilde{K} \subseteq -2d^2 \tilde{K},
\]

which shows (10).

Let \( P \) be the parallelotope enclosing \( Q \) from the proof of Theorem 1.1 and set \( P' = -(1/(2d^2)) P \). Statement (9) implies

\[
Q' \supseteq P'.
\]

Since \( S \) is chosen maximally, the volume of \( S \) is at least the volume of the simplex obtained from the Dvoretzky–Rogers lemma [10] (see also [17, Lem. 1.4]):

\[
\text{vol}_d(S) \geq \frac{1}{\sqrt{d!}d^{d/2}}.
\]

Using (13),

\[
\text{vol}_d(P') = (2d^2)^{-d} \text{vol}_d(P) = (2d^2)^{-d} \cdot 2^d d! \text{vol}_d(S) \geq d^{-5d/2}(d!)^{1/2}.
\]
Note that $P'$ is centrally symmetric, so we can use the Blaschke–Santaló inequality (see [1, Thm. 1.5.10]) for $P'$:

$$\text{vol}_d(P') \cdot \text{vol}_d((P')^\circ) \leq \text{vol}_d(B_d^d)^2. \quad (15)$$

Using the inclusions $\tilde{K} \supseteq B_d^d$ and $\tilde{K}_\sigma = (Q')^\circ \subseteq (P')^\circ$, as well as (14) and (15):

$$\frac{\text{vol}_d(\tilde{K}_\sigma)}{\text{vol}_d(\tilde{K})} \leq \frac{\text{vol}_d((P')^\circ)}{\text{vol}_d(B_d^d)} \leq \frac{\pi^{d/2}d^{5d/2}(d!)^{-1/2}}{\Gamma((d/2) + 1)} \leq (cd)^{3d/2} \quad (16)$$

for some absolute constant $c > 0$. This shows (11) and concludes the proof. \(\square\)

**Remark** We briefly explain how Conjecture 1.5 would imply the asymptotically optimal bound on $v(d)$. First note that the estimate (12) would hold with the factor $cd$ instead of $2d^2$. Then, in the rest of the proof of Theorem 1.4, we could replace all instances of the factor $2d^2$ with $cd$. In particular, one would get the linear upper bound $\delta(d) \leq cd$ from the improvement of (10), while the rest of the calculations would show that the final quotient in (16) is at most $(c'd)^{d/2}$ for some absolute constant $c' > 0$.

**Acknowledgements** This research was done under the auspices of the Budapest Semesters in Mathematics program. We are grateful to the anonymous referees for their valuable comments on the article.

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