Musielak Orlicz bumps and Bloom type estimates for commutators of Calderón Zygmund and fractional integral operators on variable Lebesgue spaces via sparse operators

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Abstract

We obtain Musielak Orlicz bumps conditions on a pair of weights for the boundedness of Calderón Zygmund operators and their commutators between variable Lebesgue spaces with different weights. The symbols of the commutators belong to a wider class of functions.

We also give Bloom type estimates for commutators of Calderón Zygmund and fractional integral operators in the variable Lebesgue context.

The techniques involved in both type of results are related with the theory of sparse domination.

1 Introduction and main results

One of the main purpose of this paper is to obtain sufficient conditions on a pair of weights in order to attain two-weighted norm inequalities for Calderón Zygmund operators (CZO’s), and their commutators, between variable Lebesgue spaces. We give Musielak Orlicz bump conditions on the weights that guarantee these results. The symbols of the commutators belong to a wider class of functions including BMO and Lipschitz spaces.

The main motivation for studying the results above is [25]. In this article the author studied sufficient conditions on a pair of weights in order to obtain boundedness results for potential operators between Lebesgue spaces with different weights. Later in [9], a similar problem was studied for CZO’s and their commutators with BMO symbols, obtaining Orlicz bump inequalities on a pair of weights as sufficient conditions. In that paper, Cruz Uribe and Pérez conjectured that weaker conditions that involve Young functions are sufficient to obtain the desired boundedness. This conjecture have been studied extensively, for a complete history we refer the reader to [8, 7, 6, 17] and [10] for the

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extensive references that they contain. One of our result extend the main theorem in [9] to the context of Musielak Orlicz spaces.

Another goal in this paper is to obtain Bloom type estimates on variable Lebesgue spaces for commutators of CZO’s and fractional integral operators with symbols belonging to other modified Lipschitz class.

In [2], Bloom obtained boundedness results of the type $L^p(\mu) \to L^p(\lambda)$ with $\mu$ and $\lambda \in A_p$, for commutator of the Hilbert transform. The symbol involved belongs to a weighted version of the bounded mean oscillation space, $BMO_\nu$, where $\nu = (\mu/\lambda)^{1/p}$. Later, in [14] and [19], the authors extend the results above to $\omega$-Calderón Zygmund operators, with $\omega(t) = t^\gamma$, $\gamma > 0$, and for general $\omega$, respectively (see also [20] for higher order commutators).

On the other hand, in [15] and [1], a version of the Bloom’s result for the fractional integral operator and their commutators were given.

The principal tools in order to obtain the mentioned results are related with the sparse domination techniques (see section §2.1).

We now introduce the general context where we shall be working with.

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a measurable function. For $A \subset \mathbb{R}^n$ we define

$$p^-_A = \text{ess inf}_{x \in A} p(x) \quad \text{and} \quad p^+_A = \text{ess sup}_{x \in A} p(x).$$

For simplicity we denote $p^- = p^-_{\mathbb{R}^n}$ and $p^+ = p^+_{\mathbb{R}^n}$.

With $p'(\cdot)$ we denote the conjugate exponent of $p(\cdot)$ given by $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$. It is not hard to prove that $(p')^- = (p^+)'$ and $(p')^+ = (p^-)'$.

We say that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if $1 \leq p^- \leq p^+ \leq \infty$ and we denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of the exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfy the following inequalities

$$\frac{1}{p(x)} - \frac{1}{p(y)} \leq \frac{C}{\log(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^n$$

and

$$\frac{1}{p(x)} - \frac{1}{p_\infty} \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n$$

for some positive constants $C$ and $p_\infty$. It is easy to see that the inequality (1.1) implies that $\lim_{|x| \to \infty} 1/p(x) = 1/p_\infty$. The conditions on $1/p(\cdot)$ above are known as local and global log-Hölder conditions, respectively.

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we define the function

$$\varphi_{p(\cdot)}(y, t) = \begin{cases} t^{p(y)}, & 1 \leq p(y) < \infty \\ \infty \cdot \mathcal{X}_{(1,\infty)}(t), & p(y) = \infty, \end{cases}$$

for $t \geq 0$ and $y \in \mathbb{R}^n$, with the convention $\infty \cdot 0 = 0$. Then the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions $f$ defined on $\mathbb{R}^n$ such that, for some positive $\lambda$,
A Luxemburg norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \varphi_{p(\cdot)}(x, |f(x)|/\lambda) \, dx \leq 1 \right\}.$$ 

By $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ we denote the space of the functions $f$ such that $f \in L^{p(\cdot)}(\cdot)$ on $U$ for every compact set $U \subset \mathbb{R}^n$.

A locally integrable function $w$ defined in $\mathbb{R}^n$ which is positive almost everywhere is called a weight. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ we define the weighted variable Lebesgue space $L^{p(\cdot)}_w(\mathbb{R}^n)$ as the set of the measurable functions $f$ defined on $\mathbb{R}^n$ such that $fw \in L^{p(\cdot)(\mathbb{R}^n)}$. (See [5] and [11] for more information about variable Lebesgue spaces).

By a cube $Q$ in $\mathbb{R}^n$ we shall understand a cube with sides parallel to the coordinate axes. By $\chi_Q$ and $f_Q$ we denote the characteristic function of $Q$ and the average of $f$ over $Q$, respectively.

We shall say that $A \lesssim B$ if there exist a positive constant $C$ such that $A \leq CB$.

Throughout this paper, we use $m$ to denote a nonnegative integer.

We now introduce the operators we shall be working with and state the corresponding main results for each one.

Let $\omega : [0, 1] \to [0, \infty)$ a continuous, increasing and subadditive function such that $\omega(0) = 0$. We say that a linear operator $T$ is an $\omega$-Calderón-Zygmund operator on $\mathbb{R}^n$ if $T$ is bounded on $L^2(\mathbb{R}^n)$, and can be represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad x \notin \text{supp} \, f.$$ 

The kernel $K$ satisfies the size condition

$$|K(x, y)| \leq \frac{C_K}{|x - y|^n}, \quad x \neq y,$$

for some positive constant $C_K$, and the smoothness condition given by

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega \left( \frac{|x - x'|}{|x - y|} \right) \frac{1}{|x - y|^n}, \quad |x - y| > 2|x - x'|.$$ 

We denote $T \in \omega$-CZO if $T$ is an $\omega$-Calderón-Zygmund operator with $\omega$ satisfying the Dini condition

$$\int_0^1 \omega(t) \frac{dt}{t} < \infty.$$ 

Given a linear operator $T$ and a locally integrable function $b$, formally define the commutator of $T$ with symbol $b$ by

$$[T, b(f)] = T(bf) - b(Tf).$$
\[ [b, T] f(x) = b(x) T f(x) - T(b f)(x), \quad x \in \mathbb{R}^n. \]

The higher order commutator of order \( m \) of \( T \) is defined by

\[ T^0_b = T, \quad T^m_b = [b, T^{m-1}_b]. \]

We say that the functional \( a \) satisfies the \( T_\infty \) condition (and we denote \( a \in T_\infty \)) if there exists a positive constant \( t_\infty \) such that for every cube \( Q \) and every cube \( Q' \subset Q \),

\[ a(Q') \leq t_\infty a(Q). \tag{1.2} \]

We denote the least constant \( t_\infty \) in (1.2) by \( \|a\|_{t_\infty} \). Clearly, \( \|a\|_{t_\infty} \geq 1 \).

Let \( a \in T_\infty \), we say that a function \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \) belongs to the generalized Lipschitz space \( L_a \) if

\[ \sup_Q \frac{1}{a(Q)|Q|} \int_Q |b(x) - b_Q| \, dx < \infty, \]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \).

We are now in position to state our first result.

**Theorem 1.1.** Let \( T \in \omega \text{-CZO} \) and \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) such that \( 1 < p^- \leq p^+ < \infty \). Let \( m \in \mathbb{N} \cup \{0\} \) and \( b \in \mathcal{L}_a \) with \( a \in T_\infty \). Suppose that \((v, w)\) is any couple of weights such that \( v \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) \) and for some constants \( S > p^+/p^- \) and \( R > (p')^+(p')^- \),

\[ \sup_Q a(Q)^m \frac{\|X_Q w\|_{S'p(\cdot)}}{\|X_Q\|_{S'p(\cdot)}} \frac{\|X_Q v^{-1}\|_{R'p(\cdot)}}{\|X_Q\|_{R'p(\cdot)}} < \infty. \tag{1.3} \]

Then

\[ T^m_b : L^{p(\cdot)}_v(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}_w(\mathbb{R}^n). \]

**Remark 1.2.** Note that the weights \( w \) and \( v = \sup_Q a(Q)^m \|X_Q w\|_{S'p(\cdot)} / \|X_Q\|_{S'p(\cdot)} \) where \( S > p^+/p^- \) satisfies condition (1.3). In fact,

\[ a(Q)^m \frac{\|X_Q w\|_{S'p(\cdot)}}{\|X_Q\|_{S'p(\cdot)}} \frac{\|X_Q v^{-1}\|_{R'p(\cdot)}}{\|X_Q\|_{R'p(\cdot)}} \leq \frac{\|X_Q w\|_{S'p(\cdot)}}{\|X_Q\|_{S'p(\cdot)}} \frac{\|X_Q\|_{S'p(\cdot)}}{\|X_Q w\|_{S'p(\cdot)}} = 1. \]

In the classical Lebesgue spaces, a proof can be found in [9] for the case \( \omega(t) = t^\gamma, \gamma > 0 \) and \( b \in \text{BMO} = \mathcal{L}_a \) with \( a \equiv 1 \).

Let us observe that, if \( a(Q) = |Q|^{\delta/n}, 0 < \delta < 1 \), then \( a \in T_\infty \) and it is known that \( \mathcal{L}_a := \mathbb{L}(\delta) \) coincides with the classical Lipschitz spaces define as the set of functions \( b \) such that

\[ |b(x) - b(y)| \lesssim |x - y|^{\delta}, \quad x, y \in \mathbb{R}^n. \]
On the other hand, if $r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $n/r \leq \alpha$ and $\delta(\cdot)$ is the exponent defined by

$$\frac{\delta(\cdot)}{n} = \frac{\alpha}{n} - \frac{1}{r(\cdot)},$$

(1.4)

the functional $a(Q) = \|X_Q\|_{n/\delta(\cdot)}$ satisfies the $T_\infty$ condition and $L_a = \mathbb{L}(\delta(\cdot))$ is a variable version of the spaces $\mathbb{L}(\delta)$ defined above.

Particularly, if $b \in \mathbb{L}(\delta(\cdot))$, we can improve the theorem above in the sense that we can consider weaker norms on the weights than those in (1.3), involving generalized $\Phi$-functions, denoted by $\Phi$-functions (see section §2.2 for more information about $\Phi$-functions). In order to state the result we need some previous definitions.

The norm associated to a given $\Phi$-function $\Psi$ is define by

$$\|f\|_{\Psi(\cdot,L)} = \inf \left\{ \lambda > 0 : \frac{\int_{\mathbb{R}^n} \Psi \left( x, \left( \frac{|f(x)|}{\lambda} \right) \right) dx}{\lambda} \leq 1 \right\}$$

and we denote by $L^\Psi(\mathbb{R}^n)$ the space of functions $f$ such that $\|f\|_{\Psi(\cdot,L)} < \infty$.

A corresponding maximal operator associated to $\Psi$ is

$$M_{\Psi(\cdot,L)} f(x) = \sup_{Q \ni x} \frac{\|X_Q f\|_{\Psi(\cdot,L)} \|X_Q\|_{\Psi(\cdot,L)}}{\|X_Q\|_{\Psi(\cdot,L)}}, \quad x \in \mathbb{R}^n.$$

For $\beta(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a fractional type version of the maximal defined above is given by

$$M_{\beta(\cdot),\Psi(\cdot,L)} f(x) = \sup_{Q \ni x} \frac{\|X_Q f\|_{\Psi(\cdot,L)} \|X_Q\|_{\Psi(\cdot,L)}}{\|X_Q\|_{\Psi(\cdot,L)}}, \quad x \in \mathbb{R}^n.$$

We say that a 3-tuples of $\Phi$-functions $(A, B, D)$ satisfy condition $F$ if they verify

1.5. $\|X_Q\|_{A(\cdot,L)} \|X_Q\|_{B(\cdot,L)} \lesssim \|X_Q\|_{D(\cdot,L)}$.

1.6. $A^{-1}(x,t)B^{-1}(x,t) \lesssim D^{-1}(x,t)$ where $A^{-1}$ denotes the inverse of $A$ (for the definition of the inverse of a $\Phi$-function see section §2.2).

1.7. $\|X_Q\|_{D(\cdot,L)} \|X_Q\|_{D^*(\cdot,L)} \lesssim |Q|$, where $D^*$ is the conjugate function of $D$ (for its definition see section §2.2).

Necessary conditions on $D$ where given in [11] in order to verify 1.7.

We can now state our result.

**Theorem 1.3.** Let $T \in \omega-CZo$ and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let $0 < \alpha < n$ and $r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $n/\alpha < r$ and $r_\infty = r(\cdot)$, $\delta(\cdot)$ be defined as in (1.4) and $b \in \mathbb{L}(\delta(\cdot))$. Assume that $(A, B, D)$ and $(E, H, J)$ are 3-tuples of $\Phi$-functions satisfying condition $F$ and

$$M_{B(\cdot,L)} : L^p(\mathbb{R}^n) \hookrightarrow L^{p^\prime}(\mathbb{R}^n), \quad M_{H(\cdot,L)} : L^{p^\prime}(\mathbb{R}^n) \hookrightarrow L'^{p}(\mathbb{R}^n).$$

(1.8)
Suppose that \((v, w)\) is any couple of weights such that \(v \in L^p_{\text{loc}}(\mathbb{R}^n)\) and
\[
\sup_Q \|\mathcal{X}_Q\|_{n/\delta(\cdot)}^m \|\mathcal{X}_Q w\|_{E(\cdot, L)} \|\mathcal{X}_Q v^{-1}\|_{A(\cdot, L)} \|\mathcal{X}_Q\|_{A(\cdot, L)} < \infty.
\]
Then
\[
T^m_b : L^p_v(\mathbb{R}^n) \hookrightarrow L^p_w(\mathbb{R}^n).
\]

Let us give some examples of \(G\Phi\)-functions that satisfy the hypothesis of the theorem above. Notice first that, if we consider \(x, t\) \(\Psi(\cdot) = L(\log x, t)\) satisfy condition \((\cdot) = \mathcal{X}_Q v^{-1}\) \(A(\cdot, L)\) with \(1 < p^- \leq p^+ < \infty\), \(\Psi(x, t) = t^p(x)(\log(e + t))^q(x)\) is a \(G\Phi\)-function. In this case, the space \(L^\Psi(\mathbb{R}^n)\) will be denoted by \(L^p(\log L)^q(\mathbb{R}^n)\). In \([21], \text{Proposition 2.5}\) the authors proved that the Hardy-Littlewood maximal operator \(M\) is bounded in this space when \(p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) with \(1 < p^- \leq p^+ < \infty\), and \(q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\). We say that \(q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) if \(q(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}\) with \(p^+ < \infty\) such that, for some positive constant \(C\), it satisfies the following inequality
\[
|q(x) - q(y)| \leq \frac{C}{\log(e + \log(e + 1/|x - y|))}, \text{ for every } x, y \in \mathbb{R}^n.
\]

Let \(p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) with \(1 < p^- \leq p^+ < \infty\) and \(\sigma > (p')^+/(p')^-\). The following \(G\Phi\)-functions satisfy condition \(\mathcal{F}\) and the hypotheses \((\cdot)\) of the Theorem 1.3.

**Example 1.4.** \(A_1(x, t) = t^{\sigma p'(x)}(\log(e + t))^{\sigma p'(x)}, B_1(x, t) = t(\sigma p')'(x)\) and \(D_1(t) = t \log(e + t)\).

**Example 1.5.** If, in addition, \(\mu(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) with \(1 < \mu^- \leq \mu^+ < \infty\) such that
\[
1/\sigma p'(\cdot) - 1/\mu(\cdot) > \epsilon
\]
for some constant \(\epsilon \in (0, 1)\) and \(\nu(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) then, an example is given by
\[
A_2(x, t) = t^{-\mu(x)}(\log(e + t))^{\nu(x)\mu(x)}, B_2(x, t) = t^{\sigma p'(x)}
\]
and \(D_2(x, t) = t^\alpha(x)(\log(e + t))^{\alpha(x)\nu(x)}\) where \(\alpha(\cdot)\) is defined by \(1/\alpha(\cdot) = 1/\mu(\cdot) + 1/(\sigma p')'(\cdot)\).

In \([23]\) we checked above examples.

**Remark 1.6.** Note that the pair of weights \((v, w)\), where \(v\) is defined by
\[
v(x) = \sup_Q \|\mathcal{X}_Q\|_{n/\delta(\cdot)}^m \|\mathcal{X}_Q w\|_{E(\cdot, L)} / \|\mathcal{X}_Q\|_{A(\cdot, L)}
\]
satisfies condition \((\cdot)\). In fact,
\[
\|\mathcal{X}_Q\|_{n/\delta(\cdot)}^m \|\mathcal{X}_Q w\|_{E(\cdot, L)} \|\mathcal{X}_Q v^{-1}\|_{A(\cdot, L)} \|\mathcal{X}_Q\|_{A(\cdot, L)} \leq \|\mathcal{X}_Q w\|_{E(\cdot, L)} \|\mathcal{X}_Q\|_{E(\cdot, L)} = 1.
\]
Another class of symbols we shall consider is related with the Bloom type estimates in the variable Lebesgue spaces.

**Definition 1.7.** Let \( \eta \) be a weight, \( 0 \leq \delta(\cdot) < n \) and \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \). We say that \( b \in BMO^\delta(\cdot) \) if

\[
\|b\|_{BMO^\delta(\cdot)} = \sup_Q \frac{1}{\|\chi_Q\|_{n/\delta(\cdot)} \eta(Q)} \int_Q |b - b_Q| < \infty.
\]

When \( \delta(\cdot) \equiv 0 \), \( BMO^0_\nu = BMO_\nu \), the space introduced in \([2]\). If \( \delta(\cdot) \equiv \delta \), with \( 0 < \delta < 1 \), \( BMO^\delta_\nu \) is a Lipschitz type space defined in \([12]\).

Our first result generalizing Bloom’s theorem in the variable Lebesgue context for CZO is the following. For the definitions of the classes of weights see section \( \S 3. \)

**Theorem 1.8.** Let \( T \in \omega \text{-CZO} \) and \( p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) such that \( 1 < p^- \leq p(\cdot) < q(\cdot) \leq q^+ < \infty \). Let \( \delta(\cdot) \) be the exponent defined by

\[
\frac{m\delta(\cdot)}{n} = \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}.
\]

Let \( \mu, \lambda \in A_{p(\cdot), q(\cdot)} \) and \( \nu = \mu/\lambda \). Assume that \( b \in BMO^\delta(\cdot)_{\mu/\lambda} \), then

\[
T^m_b : L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).
\]

When \( p, q \) are constants, Theorem 1.8 was proved in \([19]\) for the first order commutator and in \([20]\) for higher order.

A similar result in the spirit of theorem above for the higher order commutator of the fractional integral operator is given by the following theorem. Recall first that the fractional integral operator is defined, for \( 0 < \alpha < n \), by

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.
\]

**Theorem 1.9.** Let \( 0 < \alpha < n \) and \( p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) such that \( 1 < p^- \leq p(\cdot) < q(\cdot) \leq q^+ < \infty \). Let \( \delta(\cdot) \) be the exponent defined by

\[
\frac{m\delta(\cdot) + \alpha}{n} = \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}.
\]

Let \( \mu, \lambda \in A_{p(\cdot), q(\cdot)} \) and \( \nu = \mu/\lambda \). Assume that \( b \in BMO^\delta(\cdot)_{\mu/\lambda} \), then

\[
(I_\alpha)_b^m : L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).
\]

When \( p, q \) are constants and \( 1/p - 1/q = \alpha/n \), the result above was proved in \([15]\) for the first order commutator and in \([1]\) for higher order.
2 Preliminaries

In order to prove our results we give some preliminaries definitions and technical lemmas.

2.1 Sparse operators

We now introduce the dyadic structures we will working with. These definitions and a profound treatise on dyadic calculus can be found in [18].

We say that a collection of cubes $D$ in $\mathbb{R}^n$ is a dyadic grid if it satisfies the following properties:

1. If $Q \in D$, then $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
2. If $P, Q \in D$, then $P \cap Q \in \{P, Q, \emptyset\}$.
3. For every $k \in \mathbb{Z}$, the cubes $D_k = \{Q \in D : \ell(Q) = 2^k\}$ form a partition of $\mathbb{R}^n$.

Given a dyadic grid $D$, a set $S \subseteq D$ is sparse if for every $Q \in S$,

$$\left| \bigcup_{P \in S, P \subseteq Q} P \right| \leq \frac{1}{2} |Q|.$$

Equivalently, if we define

$$E(Q) = Q \setminus \bigcup_{P \in S, P \subseteq Q} P,$$

then the sets $E(Q)$ are pairwise disjoint and $|E(Q)| \geq \frac{1}{2} |Q|$.

The classic example of a dyadic grid and sparse family are the standard dyadic grid on $\mathbb{R}^n$ and the Calderón-Zygmund cubes associated with an $L^1$ function.

The following results establish pointwise sparse domination for higher order commutators of $T \in \omega$-CZO and the fractional integral operator $I_\alpha$. For simplicity we introduce the following notation. Let $m, h$ be two integers, $0 \leq \alpha < n$ and $S$ be a sparse family, we denote $A^{m,h}_{S,\alpha}$ the fractional sparse operator given by

$$A^{m,h}_{S,\alpha}(b, f)(x) = \sum_{Q \in S} |b(x) - b_Q|^{m-h}|Q|^{\alpha/n} |(b - b_Q)^h f|_Q \cdot X_Q(x), \quad b, f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

When $\alpha = 0$ we denote $A^{m,h}_{S,0} = A^{m,h}_{S}$ and, if $m = h = \alpha = 0$ we denote $A^{0,0}_{S,0}(b, f) = A_{S}(f)$.

**Theorem 2.1 ([16]).** Let $T \in \omega$-CZO. For every bounded function $f$ with compact support and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exist $3^n$ sparse families $S_j$ such that

$$|T^m_b f(x)| \leq C(m, n, T) \sum_{j=1}^{3^n} \sum_{h=0}^{m} A^{m,h}_{S_j}(b, f)(x), \quad a.e. \ x \in \mathbb{R}^n.$$
A more general version of the statement above was proved in [16].

**Theorem 2.2** ([1]). Let $0 < \alpha < n$. For every bounded function $f$ with compact support and $b \in L^m_{\text{loc}}(\mathbb{R}^n)$, there exist $3^n$ sparse families $\mathcal{S}_j$ such that

$$|(I_\alpha)_{b} f(x)| \leq C(m,n,T) \sum_{j=1}^{3^n} \sum_{h=0}^{m} A_{\mathcal{S}_j,\alpha}(b,f)(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$  

We shall use the following result in the proof of Proposition 3.1.

**Lemma 2.3** ([19]). Let $\mathcal{D}$ be a dyadic grid and let $\mathcal{S} \subset \mathcal{D}$ be a sparse family. Assume that $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then there exists a sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ such that $\mathcal{S} \subset \tilde{\mathcal{S}}$ and for every cube $Q \in \tilde{\mathcal{S}},$

$$|b(x) - b_Q| \lesssim \sum_{R \in \tilde{\mathcal{S}}} \frac{1}{|R|} \int_{R} |b(y) - b_R| \, dy \cdot A_R(x), \quad \text{a.e. } x \in Q.$$  

### 2.2 Generalized $\Phi$-functions

With $\mathcal{M}$ we denote the set of all Lebesgue real valued, measurable functions on $\mathbb{R}^n$.

A convex function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$, $\lim_{t \to 0^+} \psi(t) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$ is called a $\Phi$-function.

A real function $\Psi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is said to be a generalized $\Phi$-function (G$\Phi$-function), and we denote $\Psi \in \Phi(\mathbb{R}^n)$, if $\Psi(x,t)$ is Lebesgue-measurable in $x$ for every $t \geq 0$ and $\Psi(x, \cdot)$ is a $\Phi$-function for every $x \in \mathbb{R}^n$.

If $\Psi \in \Phi(\mathbb{R}^n)$, then for any $x \in \mathbb{R}^n$ we denote by $\Psi^\ast(x, \cdot)$ the conjugate function of $\Psi(x, \cdot)$ which is defined by

$$\Psi^\ast(x,u) = \sup_{t \geq 0} (tu - \Psi(x,t)), \quad u \geq 0.$$  

Also we can define $\Psi^{-1}$, the generalized inverse function of $\Psi$ by
\[ \Psi^{-1}(x,t) = \inf\{u \geq 0 : \Psi(x,u) \geq t\}, \quad x \in \mathbb{R}^n, t \geq 0. \]

The following result is a generalization of the classical Hölder inequality to the Musielak–Orlicz spaces.

**Lemma 2.4.** Let \( \Psi \in \Phi(\mathbb{R}^n) \), then
\[
\int_{\mathbb{R}^n} f(x)g(x) \, dx \lesssim \|f\|_{\Psi(\cdot, L)} \|g\|_{\Psi^*(\cdot, L)} \tag{2.2}
\]
for all \( f \in L^\Psi(\mathbb{R}^n) \) and \( g \in L^{\Psi^*}(\mathbb{R}^n) \).

For the definition of \( \Psi^* \), the following generalization of the Young’s inequality holds in this context,
\[ tu \leq \Psi(\omega, t) + \Psi^*(\omega, u), \quad \omega \in \mathbb{R}^n, t, u \geq 0 \]
for any \( \Psi \in \Phi(\mathbb{R}^n) \). Moreover, it can be proved that if \( \Psi, \Lambda, \Theta \in \Phi(\mathbb{R}^n) \) and \( \Psi^{-1}(x,t)\Lambda^{-1}(x,t) \leq \Theta^{-1}(x,t) \) then
\[ \Theta(x,tu) \leq \Psi(x,t) + \Lambda(x,u). \]

The inequality above allow us to prove the following generalized Hölder type inequality.

**Lemma 2.5 (22).** Let \( \Psi, \Lambda, \Theta \in \Phi(\mathbb{R}^n) \) such that \( \Psi^{-1}(x,t)\Lambda^{-1}(x,t) \leq \Theta^{-1}(x,t) \). Then
\[
\|fg\|_{\Theta(\cdot, L)} \lesssim \|f\|_{\Psi(\cdot, L)} \|g\|_{\Lambda(\cdot, L)} \tag{2.3}
\]
for all \( f \in L^\Psi(\mathbb{R}^n) \) and \( g \in L^\Lambda(\mathbb{R}^n) \).

See [11] and [13] for more information about generalized \( \Phi \)-functions.

### 2.3 Variable Lebesgue spaces

When we deal with variable Lebesgue spaces, we have the following known results that we shall be using along this paper.

**Lemma 2.6 ([11]).** Let \( s(\cdot), p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) be such that \( 1/s(\cdot) = 1/p(\cdot) + 1/q(\cdot) \). Then
\[
\|fg\|_{s(\cdot)} \lesssim \|f\|_{p(\cdot)} \|g\|_{q(\cdot)} . \tag{2.4}
\]
Particularly, if \( s(\cdot) \equiv 1 \), the inequality above gives
\[
\int_{\mathbb{R}^n} |f(y)g(y)| \, dy \lesssim \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \tag{2.5}
\]
which is an extension of the classical Hölder inequality.
Lemma 2.7 ([11]). Let $p(\cdot) \in P(\mathbb{R}^n)$ and $s \geq 1/p^\prime$. Then $\|f^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s$.

Lemma 2.8 ([11]). Let $p(\cdot) \in P^{log}(\mathbb{R}^n)$. Then $\|X_Q\|_{p(\cdot)} \|X_Q\|_{p^\prime(\cdot)} \simeq |Q|$, for every cubes $Q \subset \mathbb{R}^n$.

Moreover, we have the following result.

Lemma 2.9 ([22]). Let $p(\cdot), q(\cdot) \in P^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$. Suppose that $1/p(\cdot) = 1/\beta(\cdot) + 1/q(\cdot)$ then, for every cube $Q \subset \mathbb{R}^n$, $\|X_Q\|_{p(\cdot)} \simeq \|X_Q\|_{\beta(\cdot)} \|X_Q\|_{q(\cdot)}$.

Theorem 2.10 ([11]). Let $p(\cdot), s(\cdot), l(\cdot) \in P^{log}(\mathbb{R}^n)$ such that $p(\cdot) = s(\cdot)l(\cdot)$ with $l^\prime > 1$. Then

$$M_{L^s(\cdot)} : L^p(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n).$$

Theorem 2.11 ([22]). Let $p(\cdot), q(\cdot) \in P^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leq q(\cdot)$. Let $\beta(\cdot)$ and $s(\cdot) \in P^{log}(\mathbb{R}^n)$ be two functions such that $1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)$, $(p/s)^{-} > 1$ and $s^+ < \infty$. Then

$$M_{\beta(\cdot), L^s(\cdot)} : L^p(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n).$$

Let $p \in P(\mathbb{R}^n)$, we say that a weight $w \in A_p(\cdot)$ if there exists a positive constant $C$ such that, for every cube $Q \subset \mathbb{R}^n$,

$$\|X_Qw\|_{p(\cdot)} \|X_Qw^{-1}\|_{p^\prime(\cdot)} \leq C|Q|. \quad (2.6)$$

Lemma 2.12 ([4]). Let $p(\cdot) \in P^{log}(\mathbb{R}^n)$ with $p^\prime > 1$ and $w \in A_{p(\cdot)}$. Then there exist a constant $s \in (1/p^\prime, 1)$ such that $w^{1/s} \in A_{s p(\cdot)}$.

Lemma 2.13 ([23]). Let $k$ be a positive integer and $p(\cdot) \in P^{log}(\mathbb{R}^n)$ with $1 < p^\prime \leq p^+ < \infty$. Let $a \in T_\infty$ and $b \in L_a$. Then, for every cube $Q \subset \mathbb{R}^n$,

$$\|X_Q(b - b_Q)^k\|_{p(\cdot)} \lesssim a(Q)^k \|b\|^k_{L_a}.$$

Lemma 2.14 ([23]). Let $r(\cdot) \in P^{log}(\mathbb{R}^n)$ with $r_\infty \leq r(\cdot)$, $\delta(\cdot)$ be defined as in (1.4) and $b \in L(\delta(\cdot))$. Let $Q$ be a cube in $\mathbb{R}^n$ and $z \in kQ$ for some positive integer $k$. Then

$$|b(z) - b_Q| \lesssim \|X_Q\|_{n/\delta(\cdot)}.$$

3 Some previous results

The following proposition is useful in order to prove the Theorem 1.8.

Proposition 3.1. Let $\eta$ be a weight, $0 \leq \delta(\cdot) < n$ and $k$ a non negative integer. Let $S$ be a sparse family contained in a dyadic grid $\mathcal{D}$. Assume that $b \in BMO_{n}^{\delta(\cdot)}$. Then there exists a sparse family $\tilde{S} \subset \mathcal{D}$ such that $S \subset \tilde{S}$ and for every cube $Q \subset \tilde{S}$,
\[
\int_Q |b(x) - b_Q|^k |f(x)| \, dx \lesssim \|b\|_{BMO_0^b}^k \|\lambda_Q\|_{L^1(\Delta)}^k \int_Q (A_{\tilde{S}})^k \eta \, f(x) \, dx, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),
\]
where \((A_{\tilde{S}})^k \eta \) denotes the operator \((A_{\tilde{S}})^k \eta \) iterated \(k\) times.

When \(\delta(\cdot) \equiv 0\), the result above was proved in [19] in the case \(k = 1\), and in [20] for \(k > 1\).

**Proof of Proposition 3.1.** Let \(\tilde{S}\) be the sparse family provided by Lemma 2.3 and \(Q \in \tilde{S}\). Then, by this lemma, we have

\[
|b(x) - b_Q| \lesssim \sum_{R \in \tilde{S}} \frac{1}{|R|} \int_R |b(y) - b_R| \, dy \cdot \lambda_R(x) \lesssim \|b\|_{BMO_0^b} \|\lambda_Q\|_{L^1(\Delta)} \sum_{R \in \tilde{S}} \frac{\|\lambda_R\|_{L^1(\Delta)}}{|R|} \cdot \lambda_R(x).
\]

Therefore,

\[
\int_Q |b(x) - b_Q|^k |f(x)| \, dx \lesssim \|b\|_{BMO_0^b}^k \|\lambda_Q\|_{L^1(\Delta)}^k \int_Q \left( \sum_{R \in \tilde{S}} \frac{\|\lambda_R\|_{L^1(\Delta)}}{|R|} \cdot \lambda_R(x) \right)^k |f(x)| \, dx.
\]

Since the cubes from \(\tilde{S}\) are dyadic,

\[
\left( \sum_{R \in \tilde{S}} \frac{\|\lambda_R\|_{L^1(\Delta)}}{|R|} \cdot \lambda_R(x) \right)^k = \sum_{\{R_1, R_2, \ldots, R_k\} \subseteq \tilde{S}} \left( \prod_{i=1}^k \frac{\|\lambda_{R_i}\|_{L^1(\Delta)}}{|R_i|} \right) \cdot \lambda_{R_1}(x) \lambda_{R_2}(x) \ldots \lambda_{R_k}(x)
\]

\[
= \sum_{\{R_1, R_2, \ldots, R_k\} \subseteq \tilde{S}} \left( \prod_{i=1}^k \frac{\|\lambda_{R_i}\|_{L^1(\Delta)}}{|R_i|} \right) \cdot \lambda_{R_1 \cap R_2 \cap \ldots \cap R_k}(x)
\]

\[
\leq k! \sum_{\{R_1, R_2, \ldots, R_k\} \subseteq \tilde{S}} \left( \prod_{i=1}^k \frac{\|\lambda_{R_i}\|_{L^1(\Delta)}}{|R_i|} \right) \cdot \lambda_{R_k}(x).
\]

Hence

\[
\int_Q \left( \sum_{R \in \tilde{S}} \frac{\|\lambda_R\|_{L^1(\Delta)}}{|R|} \cdot \lambda_R(x) \right)^k |f(x)| \, dx \lesssim \sum_{\{R_1, R_2, \ldots, R_k\} \subseteq \tilde{S}} \left( \prod_{i=1}^k \frac{\|\lambda_{R_i}\|_{L^1(\Delta)}}{|R_i|} \right) \int_{R_k} |f(x)| \, dx.
\]
\[
\sum_{(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k) \subseteq \mathcal{S}} \left( \frac{\prod_{i=1}^{k} \eta(R_i)}{|R_i|} \right) |f|_{\mathcal{R}_k} \eta(x) \, dx \\
= \sum_{(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k) \subseteq \mathcal{S}} \left( \frac{\prod_{i=1}^{k-1} \eta(R_i)}{|R_i|} \right) \int_{\mathcal{R}_{k-1}} \left( \sum_{\mathcal{R}_k \subseteq \mathcal{S}} \frac{|f|_{\mathcal{R}_k} \cdot \mathcal{X}_{\mathcal{R}_k}(x)}{\mathcal{R}_k} \right) \eta(x) \, dx \\
\leq \sum_{(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k) \subseteq \mathcal{S}} \left( \frac{\prod_{i=1}^{k-1} \eta(R_i)}{|R_i|} \right) \int_{\mathcal{R}_{k-1}} (A_{\mathcal{S}}(f)) \eta f(x) \, dx.
\]

Using this argument \(k\) times we conclude
\[
\int_Q \left( \sum_{\mathcal{R} \subseteq \mathcal{S}} \frac{\eta(R)}{|R|} \cdot \mathcal{X}_R(x) \right)^k |f(x)| \, dx \lesssim \int_Q (A_{\mathcal{S}}^k)^k f(x) \, dx.
\]

We are done. \(\square\)

For \(\beta(\cdot) \in \mathcal{P}(\mathbb{R}^n)\) and \(\mathcal{S}\) a sparse family, we define the variable fractional sparse operator \(T_{\mathcal{S}}^{\beta(\cdot)}\) by
\[
T_{\mathcal{S}}^{\beta(\cdot)} f(x) = \sum_{Q \in \mathcal{S}} \| \mathcal{X}_Q \|_{\beta(\cdot)} f Q \cdot \mathcal{X}_Q(x), \quad f \in L^1_{\text{loc}}(\mathbb{R}^n).
\]

If \(0 < \alpha < n\) and \(\beta(\cdot) \equiv n/\alpha\), this operator was studied in [3] in the classical context of weighted Lebesgue spaces. We are interested in studying the boundedness properties of the operators \(T_{\mathcal{S}}^{\beta(\cdot)}\) on weighted variable Lebesgue spaces.

The classes of weights we will be dealing with are a variable version of the well known \(A_{p,q}\) classes of Muckenhoupt and Wheeden (see [24]). Let \(p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\). We say that a weight \(w \in A_{p(\cdot), q(\cdot)}\) if there exists a positive constant \(C\) such that, for every cube \(Q \subset \mathbb{R}^n\),
\[
\frac{\| \mathcal{X}_Q w \|_{q(\cdot)} \| \mathcal{X}_Q w^{-1} \|_{p(\cdot)}}{\| \mathcal{X}_Q \|_{q(\cdot)} \| \mathcal{X}_Q \|_{p(\cdot)}} \leq C.
\] (3.1)
The smallest of such constants will be denoted by \([w]_{A_{p(\cdot),q(\cdot)}}\). Note that \(w \in A_{p(\cdot),q(\cdot)}\) is equivalent to \(w^{-1} \in A'_{p(\cdot),p'(\cdot)}\). When \(p(\cdot) = q(\cdot)\), we obtain the \(A_{p(\cdot)}\) class given in [4] that characterizes the boundedness of the Hardy–Littlewood maximal operator on \(L^{p(\cdot)}_w(\mathbb{R}^n)\).

We obtain the following boundedness result for \(\mathcal{I}^{\beta(\cdot)}_S\) between variable weighted Lebesgue spaces.

**Proposition 3.2.** Let \(p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) such that \(1 < p^- \leq p(\cdot) \leq q(\cdot) \leq q^+ < \infty\). Let \(\beta(\cdot)\) be the exponent defined by \(1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)\) and \(S\) be a sparse family. Assume \(w \in A_{p(\cdot),q(\cdot)}\), then

\[
\mathcal{I}^{\beta(\cdot)}_S : L^{p(\cdot)}_w(\mathbb{R}^n) \to L^{q(\cdot)}_w(\mathbb{R}^n).
\]

Note that if \(p(\cdot) \equiv q(\cdot)\), \(w \in A_{p(\cdot)}\) and \(S\) be a sparse family, from the proposition above we obtain that

\[
A_S : L^{p(\cdot)}_w(\mathbb{R}^n) \to L^{p(\cdot)}_w(\mathbb{R}^n).
\] \hspace{1cm} (3.2)

This generalizes the well-known result proved in [8] or [18] for the sparse operator \(A_S\) in the classical context.

In order to prove the Proposition 3.2 let see some useful properties of the classes \(A_{p(\cdot),q(\cdot)}\).

Note that if \(p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\), \(p(\cdot) \leq q(\cdot)\), the opposite inequality of (3.1) follows by Hölder’s inequality (2.4) and Lemma 2.9, so we have that

\[
\frac{\|X_Q w\|_{q(\cdot)}}{\|X_Q\|_{q(\cdot)}} \frac{\|X_Q w^{-1}\|_{p'(\cdot)}}{\|X_Q\|_{p'(\cdot)}} \simeq 1
\]

if \(w \in A_{p(\cdot),q(\cdot)}\).

Let \(p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) such that \(p(\cdot) \leq q(\cdot)\), then

\[
\frac{\|X_Q f\|_{p(\cdot)}}{\|X_Q\|_{p(\cdot)}} \lesssim \frac{\|X_Q f\|_{q(\cdot)}}{\|X_Q\|_{q(\cdot)}}, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n).
\] \hspace{1cm} (3.3)

Indeed, let \(\beta(\cdot)\) be defined by \(1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)\). Then \(\beta(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) and, by Hölder’s inequality (2.4) and Lemma 2.9 we obtain (3.3).

**Lemma 3.3.** Let \(p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)\) such that \(p(\cdot) \leq q(\cdot)\) and \(w \in A_{p(\cdot),q(\cdot)}\). Then

(i) \(w \in A_{p(\cdot)} \cap A_{q(\cdot)}\)

(ii) \(w \in A_{q(\cdot),p(\cdot)}\). Moreover,

\[
\frac{\|X_Q w\|_{p(\cdot)}}{\|X_Q\|_{p(\cdot)}} \frac{\|X_Q w^{-1}\|_{q'(\cdot)}}{\|X_Q\|_{q'(\cdot)}} \simeq 1.
\]
Proof. Let us see (i). Since \( p(\cdot) \leq q(\cdot) \), by (3.3) we have
\[
\frac{\|\mathcal{X}^w\|_{p(\cdot)}}{\|\mathcal{X}\|_{p(\cdot)}} \leq \frac{\|\mathcal{X}^w\|_{q(\cdot)}}{\|\mathcal{X}\|_{q(\cdot)}} \leq [w]_{A_{p(\cdot),q(\cdot)}}.
\]
In the same way, since \( q'(\cdot) \leq p'(\cdot) \), we have
\[
\frac{\|\mathcal{X}^w\|_{q(\cdot)}}{\|\mathcal{X}\|_{q(\cdot)}} \leq \frac{\|\mathcal{X}^w\|_{q'(\cdot)}}{\|\mathcal{X}\|_{q'(\cdot)}} \leq [w]_{A_{p(\cdot),q(\cdot)}}.
\]
In order to prove (ii), by applying Hölder’s inequality (2.5) and Lemma 2.9 we have
\[
1 \leq \frac{\|\mathcal{X}^w\|_{p(\cdot)}}{\|\mathcal{X}\|_{p(\cdot)}} \frac{\|\mathcal{X}^w\|_{q'(\cdot)}}{\|\mathcal{X}\|_{q'(\cdot)}} \leq [w]_{A_{p(\cdot),q(\cdot)}}[w]_{A_q(\cdot)}.
\]

The following proposition provides us with an “openness” type property of class \( A_{p(\cdot),q(\cdot)} \).

**Proposition 3.4.** Let \( p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) such that \( 1 < p^- \leq p(\cdot) \leq q(\cdot) \leq q^+ < \infty \) and \( w \in A_{p(\cdot),q(\cdot)} \). Then there exist \( u(\cdot), v(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) such that \( (p/u) > 1, (q'/v') > 1 \) and \( w \in A_{u(\cdot),v(\cdot)} \).

For the case \( p(\cdot) \equiv q(\cdot) \), this proposition was proved in [4].

**Proof of Proposition 3.4.** Since \( w \in A_{p(\cdot),q(\cdot)} \), by Lemma 3.3(i), we have that \( w \in A_{p(\cdot)} \). Similarly, since \( w^{-1} \in A_{q'(\cdot),p'(\cdot)} \), \( w^{-1} \in A_{q'(\cdot)} \). Then by Lemma 2.12, since \( p^- > 1 \) and \( (q')^- > 1 \), there exist two constants \( s \in (1/p^-,1) \) and \( r \in (1/(q')^-,1) \) such that
\[
w^{1/s} \in A_{sp(\cdot)} \quad \text{and} \quad w^{-1/r} \in A_{rq'(\cdot)}.
\]
We denote \( u'(\cdot) = \frac{1}{s}(sp(\cdot))' \) and \( v'(\cdot) = \frac{1}{r}(rq'(\cdot))' \). Note that
\[
\frac{p(\cdot)}{u(\cdot)} = p(\cdot)(1-s) + 1 \geq (p)^-(1-s) + 1
\]
and
\[
\frac{q'(\cdot)}{v'(\cdot)} = q'(\cdot)(1-r) + 1 \geq (q')^-(1-r) + 1.
\]
so that \((p/u)^- > 1\) and \((q'/v')^- > 1\). By (3.4) and Lemma 2.7, we have that
\[
\|X_Q w\|_{p(u)} \|X_Q w^{-1}\|_{u'} \simeq 1 \quad \text{and} \quad \|X_Q w\|_{v(\cdot)} \|X_Q w^{-1}\|_{q'} \simeq 1
\]
respectively. Thus, by Lemma 3.3(ii) we obtain \(w \in A_{u(\cdot),v(\cdot)}\).

We can now proceed with the proof of Proposition 3.2.

Proof of Proposition 3.2. By duality and since \(S\) is sparse we have
\[
\|T_S^{\beta(\cdot)} f\|_{L_w^{\beta(\cdot)}} = \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \int_{\mathbb{R}^n} g(x) T_S^{\beta(\cdot)} f(x) \, dx = \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \sum_{Q \in S} |Q| \|X_Q\|_{\beta(\cdot)} f_Q g_Q
\]
\[
\lesssim \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \sum_{Q \in S} |E(Q)| \|X_Q\|_{\beta(\cdot)} f_Q g_Q.
\]
Let \(u(\cdot), v(\cdot)\) the exponents provided by Proposition 3.4. Then by Hölder’s inequality and Lemma 2.8 we obtain
\[
\|T_S^{\beta(\cdot)} f\|_{L_w^{\beta(\cdot)}} \lesssim \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \sum_{Q \in S} |E(Q)| \|X_Q\|_{\beta(\cdot)} \frac{\|X_Q f w\|_{u(\cdot)} \|X_Q w^{-1}\|_{u'(\cdot)} \|X_Q g w^{-1}\|_{v'(\cdot)} \|X_Q w\|_{v(\cdot)}}{\|X_Q\|_{u(\cdot)} \|X_Q\|_{u'(\cdot)} \|X_Q\|_{v(\cdot)}}
\]
\[
\lesssim \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \int_{\mathbb{R}^n} M_{\beta(\cdot),L^{u(\cdot)}}(f w)(x) M_{L^{v'(\cdot)}}(g w^{-1})(x) \, dx
\]
\[
\lesssim \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \left\| M_{\beta(\cdot),L^{u(\cdot)}}(f w) \right\|_{q(\cdot)} \left\| M_{L^{v'(\cdot)}}(g w^{-1}) \right\|_{q'(\cdot)}
\]
\[
\lesssim \sup_{\|g\|_{L_w^{q'(\cdot)}} \leq 1} \|f w\|_{p(\cdot)} \|g w^{-1}\|_{q'(\cdot)} \leq \|f\|_{L^p_{\nu(\cdot)}},
\]
where we have used that, by Theorem 2.10, \(M_{L^{v'(\cdot)}} : L^{q'(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{q'(\cdot)}(\mathbb{R}^n)\) and by Theorem 2.11, \(M_{\beta(\cdot),L^{u(\cdot)}} : L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)\). \(\square\)

The following lemma will be useful in the proof of the Theorem 1.8.

Lemma 3.5. Let \(p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\), \(\mu, \lambda \in A_{p(\cdot),q(\cdot)}\) and \(\nu = \mu / \lambda\). Then \(\lambda^{\nu} \mu^{\frac{m-h}{m}} \in A_{p(\cdot),q(\cdot)}\) for every \(m \in \mathbb{N}\) and for each \(h = 0, 1, \ldots, m\).
Let $m \in \mathbb{N}$ and $k \in \{1, 2, \ldots, m\}$. Note that if $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p(\cdot) \leq q(\cdot)$, $\mu, \lambda \in \mathcal{A}_{p(\cdot), q(\cdot)}$, $\nu = \mu/\lambda$ and we denote $\eta = \nu^{1/m}$ we have

$$
\left\| \langle A_S \rangle^k \right\|_{L^{q(\cdot)}_{\lambda^k}} \lesssim \|f\|_{L^{q(\cdot)}_{\lambda^m k}}, \quad \forall F \in L^1_{\text{loc}}(\mathbb{R}^n)
$$

and

$$
\left\| \langle A_S \rangle^k \right\|_{L^{p(\cdot)}_{\lambda^m k}} \lesssim \|f\|_{L^{p(\cdot)}_{\lambda^m}}, \quad \forall F \in L^1_{\text{loc}}(\mathbb{R}^n).
$$

Proof of Lemma 3.5. By Hölder's inequality (2.4) and Lemma 2.7, we have

$$
\left\| \mathcal{X}_Q \lambda \nu^{\frac{m-h}{m}} \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \lambda^{-1} \nu^{-\frac{m-h}{m}} \right\|_{p'(\cdot)} \leq \left( \left\| \mathcal{X}_Q \lambda \lambda^{\frac{h}{m}} \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \lambda^{-1} \lambda^{-\frac{h}{m}} \right\|_{p'(\cdot)} + \left\| \mathcal{X}_Q \mu \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \mu^{-1} \right\|_{p'(\cdot)} \right) \left( \left\| \mathcal{X}_Q \lambda \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \lambda^{-1} \right\|_{p'(\cdot)} + \left\| \mathcal{X}_Q \mu \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \mu^{-1} \right\|_{p'(\cdot)} \right) \left( \left\| \mathcal{X}_Q \lambda \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \lambda^{-1} \right\|_{p'(\cdot)} + \left\| \mathcal{X}_Q \mu \right\|_{q(\cdot)} \left\| \mathcal{X}_Q \mu^{-1} \right\|_{p'(\cdot)} \right)
$$

4 Proofs of main results

Proof of Theorem 1.1. Since $v \in L^p_{\text{loc}}(\mathbb{R}^n)$ implies that the set of bounded functions with compact support is dense in $L^p_{\text{loc}}(\mathbb{R}^n)$ and taking into account Theorem 2.1, it is enough to show that for a sparse family $S$,

$$
\left\| A_S^{\mu,h}(b, f) \right\|_{L^p_{\text{loc}}(\mathbb{R}^n)} \lesssim \|f\|_{L_{p'}(\mathbb{R}^n)} \quad h \in \{0, 1, \ldots, m\}
$$

for each nonnegative bounded function with compact support $f$. Let $h \in \{0, 1, \ldots, m\}$, by duality

$$
\left\| A_S^{\mu,h}(b, f) \right\|_{L^p_{\text{loc}}(\mathbb{R}^n)} \lesssim \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} g(x) A_S^{\mu,h}(b, f)(x) \ dx
$$

$$
= \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)} \leq 1} \sum_{Q \in S} \frac{1}{|Q|} \int_Q |b(x) - b_Q|^h f(x) \ dx \frac{1}{|Q|} \int_Q |b(x) - b_Q|^{m-h} g(x) \ dx.
$$

(4.1)
Let us denote \( s(\cdot) = Rp'(\cdot) \) and \( l(\cdot) = Sp(\cdot) \). Since \( (p')^+ < R(p')^- \) and \( p^+ < Sp^- \), \( (s')^+ < p^- \) and \( (l')^+ < (p')^- \) then, we can take two constants \( A, B \) such that

\[
(s')^+ < A < p^- \quad \text{and} \quad (l')^+ < B < (p')^-,
\]

and \( \omega(\cdot), \tau(\cdot) \) defined by

\[
\frac{1}{\omega(\cdot)} = \frac{1}{s(\cdot)} + \frac{1}{A} \quad \text{and} \quad \frac{1}{\tau(\cdot)} = \frac{1}{l(\cdot)} + \frac{1}{B}.
\]

Observe that \( \omega(\cdot), \tau(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) since \( s(\cdot), l(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \).

On the other hand note that, for \( k \) an integer, an exponent \( r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) with \( 1 < r^- \leq r^+ < \infty \) and \( H \in L^{1}_{\text{loc}}(\mathbb{R}^n) \), by Hölder’s inequality (2.5), Lemmas 2.8 and 2.13, we have

\[
\frac{1}{|Q|} \int_{Q} |b(y) - b_Q|^k H(y) \, dy \lesssim \frac{\|X_Q|b - b_Q|^k\|_{r(\cdot)} \|X_Q H\|_{r(\cdot)}}{\|X_Q\|_{r(\cdot)}} \lesssim a(Q)^k \|b\|_{L^a} \|X_Q H\|_{r(\cdot)}.
\]

Thus, by (4.1), we have

\[
\left\| A^m_{S}(b, f) \right\|_{L^0_{w}} \lesssim \left\| b \right\|_{L^\alpha_a} \sup_{g \in L^{p'(\cdot)}_{w} \leq 1} \sum_{Q \in S} |Q(a(Q)^m \|X_Q f\|_{w(\cdot)} \|X_Q g\|_{\tau(\cdot)}|.
\]

Using that \( S \) is a sparse family and Hölder’s inequality (2.4), Lemma 2.8 and the hypothesis on the weights we obtain that

\[
\left\| A^m_{S}(b, f) \right\|_{L^0_{w}} \lesssim \left\| b \right\|_{L^\alpha_a} \sup_{g \in L^{p'(\cdot)}_{w} \leq 1} \sum_{Q \in S} |E(Q)|a(Q)^m \|X_Q f\|_{A(\cdot, L)} \|X_Q g\|_{A(\cdot, L)} \|X_Q w\|_{B(\cdot, L)} \|X_Q w\|_{l(\cdot)}
\]

\[
\lesssim \|M_{L\alpha}(f)\|_{p(\cdot)} \|M_{L\beta}(g w^{-1})\|_{p'(\cdot)}
\]

\[
\lesssim \|f\|_{L^p_{w}} \|g\|_{L^{p'(\cdot)}_{w} \leq 1} \|M_{L\beta}(g w^{-1})\|_{p'(\cdot)}
\]

\[
\lesssim \|f\|_{L^p_{w}} \|g\|_{L^{p'(\cdot)}_{w} \leq 1} \|M_{L\alpha}(f)\|_{p(\cdot)} \|M_{L\beta}(g w^{-1})\|_{p'(\cdot)}
\]

where we have used that by Theorem 2.10, \( M_{L\alpha} \colon L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n) \) since \( A < p^- \) and \( M_{L\beta} \colon L^{p'(\cdot)}(\mathbb{R}^n) \to L^{p'(\cdot)}(\mathbb{R}^n) \) since \( B < (p')^- \).

**Proof of Theorem 1.3.** As before it suffices to provide suitable estimates for

\[
\sum_{Q \in S} |Q|(|b - b_Q|^h f)_{Q}(|b - b_Q|^{m-h} g)_{Q}, \quad h \in \{0, 1, \ldots, m\}
\]
for each nonnegative bounded function with compact support $f$ and each $g$ with $\|g\|_{L^{p'}_{e_{w-1}}} \leq 1$, where $\mathcal{S}$ is a sparse family. Note that, for $k$ an non negative integer and $H \in L^1_{\text{loc}}(\mathbb{R}^n)$, by Lemma 2.14 we have

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q|^k H(x) \, dx \lesssim \|\mathcal{X}_Q\|^k_{n/\delta(\cdot)} H_Q.$$  

Then we have

$$\sum_{Q \in \mathcal{S}} |Q| |(b - b_Q)^h f)_Q (b - b_Q)^{m-h} g)_Q \lesssim \sum_{Q \in \mathcal{S}} |Q| \|\mathcal{X}_Q\|^{m}_{n/\delta(\cdot)} f_Q g_Q. \quad (4.2)$$

By condition $\mathcal{F}$ and Hölder’s inequalities (2.2) and (2.3) we have

$$f_Q \lesssim \frac{\|\mathcal{X}_Q f\|_{D(\cdot, L)}}{\|\mathcal{X}_Q\|_{D^\ast(\cdot, L)}} \|\mathcal{X}_Q\|_{D^\ast(\cdot, L)} \lesssim \frac{\|\mathcal{X}_Q f\|_{B(\cdot, L)}}{\|\mathcal{X}_Q\|_{B(\cdot, L)}} \|\mathcal{X}_Q\|_{A(\cdot, L)}$$

and

$$g_Q \lesssim \frac{\|\mathcal{X}_Q g\|_{J(\cdot, L)}}{\|\mathcal{X}_Q\|_{J^\ast(\cdot, L)}} \|\mathcal{X}_Q\|_{J^\ast(\cdot, L)} \lesssim \frac{\|\mathcal{X}_Q g\|_{H(\cdot, L)}}{\|\mathcal{X}_Q\|_{H(\cdot, L)}} \|\mathcal{X}_Q\|_{E(\cdot, L)}.$$

Then from (4.2), since $\mathcal{S}$ is sparse, by the hypothesis on the weights and (1.8) we have

$$\sum_{Q \in \mathcal{S}} |Q| |(b - b_Q)^h f)_Q (b - b_Q)^{m-h} g)_Q$$

$$\lesssim \sum_{Q \in \mathcal{S}} |E(Q)| \|\mathcal{X}_Q\|^{m}_{n/\delta(\cdot)} \frac{\|\mathcal{X}_Q f\|_{B(\cdot, L)}}{\|\mathcal{X}_Q\|_{B(\cdot, L)}} \|\mathcal{X}_Q\|_{A(\cdot, L)} \|\mathcal{X}_Q g\|_{H(\cdot, L)} \|\mathcal{X}_Q\|_{E(\cdot, L)}$$

$$\lesssim \int_{\mathbb{R}^n} M_{B(\cdot, L)} (f v)(x) M_{H(\cdot, L)} (gw^{-1})(x) \, dx \lesssim \|M_{B(\cdot, L)} (f v)\|_{p(\cdot)} \|M_{H(\cdot, L)} (gw^{-1})\|_{p'(\cdot)}$$

$$\lesssim \|f v\|_{p(\cdot)} \|gw^{-1}\|_{p'(\cdot)} \lesssim \|f\|_{L^p(\cdot)}.$$

**Proof of Theorem 1.8.** Taking into account Theorem 2.1, it suffices to prove the estimate for the sparse operators

$$A^m_{\mathcal{S}} (b, f)(x) = \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-h} (b - b_Q)^h f)_Q \cdot \mathcal{X}_Q(x), \quad h \in \{0, 1, ..., m\},$$

for each nonnegative bounded function with compact support $f$. By duality we have

$$\left\| A^m_{\mathcal{S}} (b, f) \right\|_{L^1(\cdot)} \lesssim \sup_{\|g\|_{p(\cdot)} \leq 1} \int_{\mathbb{R}^n} \lambda(x) g(x) A^m_{\mathcal{S}} (b, f)(x) \, dx$$

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Let $\tilde{S}$ be the sparse family provided by Proposition 3.1 and $Q \in \tilde{S}$. Then, by this proposition and denoting $\eta = v^{1/m}$, we have

$$
\left\| A_{q,\lambda}^{\eta}(b, f) \right\|_{L^q(\cdot)} \lesssim \sup_{\|g\|_{L^q(\cdot)} \leq 1} \left\| b \right\|_{BMO_{\lambda}^{\eta}(\cdot)} \sum_{Q \in \tilde{S}} \frac{\|XQ\|_{\eta}(\cdot)}{|Q|} \int_{\lambda} (A_{S}^{\eta}f(x) \cdot XQ(x)) \lambda(x)g(x) dx.
$$

Noting that $\|XQ\|_{\eta}(\cdot) = \|XQ\|_{\eta/n\delta(\cdot)}$, denoting $\beta(\cdot) = n/m\delta(\cdot)$ and $A_{\beta} = A$, we have

$$
\left\| A_{\beta}^{m,h}(b, f) \right\|_{L^q(\cdot)} \lesssim \sup_{\|g\|_{L^q(\cdot)} \leq 1} \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \sum_{Q \in \tilde{S}} \frac{\|XQ\|_{\beta}(\cdot)}{|Q|} \int_{\lambda} A_{\eta}^{h}f(x) \cdot XQ(x) \lambda(x)g(x) dx.
$$

Using $m - h$ times that $A$ is self-adjoint, we have

$$
\int_{\mathbb{R}^n} T_{\beta}^{\eta}(A_{\eta}^{h}f)(x) \cdot A_{\eta}^{m-h}(\lambda g)(x) dx = \int_{\mathbb{R}^n} A \left( A_{\eta}^{m-h-1}\left[ \left(T_{\beta}^{\eta}(A_{\eta}^{h}f)\right) \lambda(x)g(x) \right] \right)(x) \lambda(x)g(x) dx.
$$

Combining the preceding estimates and Hölder’s inequality, we obtain that

$$
\left\| A_{\beta}^{m,h}(b, f) \right\|_{L^q(\cdot)} \lesssim \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \sup_{\|g\|_{L^q(\cdot)} \leq 1} \int_{\mathbb{R}^n} A \left( A_{\eta}^{m-h-1}\left[ \left(T_{\beta}^{\eta}(A_{\eta}^{h}f)\right) \lambda(x)g(x) \right] \right)(x) \lambda(x)g(x) dx
$$

$$
\lesssim \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \sup_{\|g\|_{L^q(\cdot)} \leq 1} \left\| A \left( A_{\eta}^{m-h-1}\left[ \left(T_{\beta}^{\eta}(A_{\eta}^{h}f)\right) \lambda(x)g(x) \right] \right) \right\|_{L^q(\cdot)}
$$

$$
\lesssim \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \left\| A \left( A_{\eta}^{m-h-1}\left[ \left(T_{\beta}^{\eta}(A_{\eta}^{h}f)\right) \lambda(x)g(x) \right] \right) \right\|_{L^q(\cdot)}
$$

where we have used that $\lambda \in A_{\eta}(\cdot)$ and (3.2). By inequality (3.5) we obtain

$$
\left\| A_{\beta}^{m,h}(b, f) \right\|_{L^q(\cdot)} \lesssim \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \left\| \left(T_{\beta}^{\eta}(A_{\eta}^{h}f)\right) \right\|_{L^q(\cdot)} = \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \left\| T_{\beta}^{\eta}(A_{\eta}^{h}f) \right\|_{L^q(\cdot)}.
$$

Using Lemma 3.5 and Proposition 3.2 we obtain that

$$
\left\| A_{\beta}^{m,h}(b, f) \right\|_{L^p(\cdot)} \lesssim \left\| b \right\|_{BMO_{\lambda}^{\beta}(\cdot)} \left\| A_{\eta}^{h}f \right\|_{L^p(\cdot)}.
$$
and applying (3.6), we conclude that
\[
\left\| A_{S,\alpha}^{m,h}(b,f) \right\|_{L^q_{\lambda}} \lesssim \left\| b \right\|_{BMO_{\eta}^{\delta(\cdot)}}^m \left\| f \right\|_{L^p_{\lambda m,\eta}} = \left\| b \right\|_{BMO_{\eta}^{\delta(\cdot)}}^m \left\| f \right\|_{L^p_{\mu}}.
\]

\[\square\]

**Proof of Theorem 1.9.** We first consider \( b \in L^m_{loc}(\mathbb{R}^n) \). Since \( \mu \in L^p_{loc}(\mathbb{R}^n) \) and taking into account Theorem 2.2, it is enough to show that, for a sparse family \( S \),

\[
\left\| A_{S,\alpha}^{m,h}(b,f) \right\|_{L^q_{\lambda}} \lesssim \left\| f \right\|_{L^p_{\mu}}, \quad h \in \{0,1,\ldots,m\},
\]

holds for each nonnegative bounded function with compact support \( f \). Let \( h \in \{0,1,\ldots,m\} \), by duality

\[
\left\| A_{S,\alpha}^{m,h}(b,f) \right\|_{L^q_{\lambda}} \lesssim \sup_{\|g\|_{q(\cdot)} \leq 1} \int_{\mathbb{R}^n} \lambda(x)g(x)A_{S,\alpha}^{m,h}(b,f)(x) \, dx
\]

\[
= \sup_{\|g\|_{q(\cdot)} \leq 1} \sum_{Q \in S} \left| Q \right|^\alpha/n \left( b - b_Q \right)^h f \int_Q \left| b(x) - b_Q \right|^m \lambda(x)g(x) \, dx.
\]

(4.4)

Let \( \tilde{S} \) be the sparse family provided by Proposition 3.1 and \( Q \in \tilde{S} \). Then, by this proposition and denoting \( \eta = \nu^{1/m} \), we have that \( \left\| A_{S,\alpha}^{m,h}(b,f) \right\|_{L^q_{\lambda}} \) is bounded by a multiple of

\[
\left\| b \right\|_{BMO_{\eta}^{\delta(\cdot)}}^m \sup_{\|g\|_{q(\cdot)} \leq 1} \sum_{Q \in \tilde{S}} \left| Q \right|^\alpha/n \left\| \mathcal{X}_Q \right\|_{n/\delta(\cdot)}^m \int_Q (A_{\tilde{S}}^{m,h})_{\eta} f \, dx \int_Q (A_{\tilde{S}}^{m,h})_{\eta} (\lambda g) \, dx.
\]

Note that, if we denote \( \beta(\cdot) = n/(m\delta(\cdot) + \alpha) \), by Lemma 2.9, \( |Q|^\alpha/n \left\| \mathcal{X}_Q \right\|_{n/\delta(\cdot)}^m \simeq \left\| \mathcal{X}_Q \right\|_{\beta(\cdot)} \). Thus, we get the same inequality as in (4.3). So we can proceed in the same way as in the proof of Theorem 1.8 to get the desired result for the case \( b \in L^m_{loc}(\mathbb{R}^n) \).

In order to complete the proof we must show that if \( b \in BMO_{\eta}^{\delta(\cdot)} \) then \( b \in L^m_{loc}(\mathbb{R}^n) \). In fact, for any compact set \( K \) we choose a cube \( Q \) with \( |Q| > 1 \) such that \( K \subset Q \). Then

\[
\int_K |b|^m \leq \int_Q |b|^m \leq \int_Q |b - b_Q|^m + \left( \int_Q |b| \right)^m.
\]

Since \( b \in L^1_{loc}(\mathbb{R}^n) \), the last term is bounded. For the first term note that, by Lemma 3.1,

\[
\int_Q |b - b_Q|^m \mathcal{X}_Q \lesssim \left\| b \right\|_{BMO_{\eta}^{\delta(\cdot)}}^m \left\| \mathcal{X}_Q \right\|_{n/\delta(\cdot)}^m \int_Q (A_{\tilde{S}})_{\eta}^m \mathcal{X}_Q
\]

\[
\lesssim \left\| b \right\|_{BMO_{\eta}^{\delta(\cdot)}}^m \left\| \mathcal{X}_Q \right\|_{n/\delta(\cdot)}^m \left\| \lambda (A_{\tilde{S}})_{\eta}^m \mathcal{X}_Q \right\|_{q(\cdot)} \left\| \lambda^{-1} \mathcal{X}_Q \right\|_{q(\cdot)}^m.
\]

By Lemma 3.5 and (3.5),

\[
\left\| \lambda (A_{\tilde{S}})_{\eta}^m \mathcal{X}_Q \right\|_{q(\cdot)} = \left\| (A_{\tilde{S}})_{\eta}^m \mathcal{X}_Q \right\|_{L^q_{\lambda}} \lesssim \left\| \mathcal{X}_Q \right\|_{L^q_{\lambda m,\eta}} = \left\| \mathcal{X}_Q \right\|_{L^p_{\mu}}.
\]

So we are done. \(\square\)
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