Viscosity Solutions to Second Order Path-Dependent Hamilton-Jacobi-Bellman Equations and Applications *

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Abstract
In this article, a notion of viscosity solutions is introduced for second order path-dependent Hamilton-Jacobi-Bellman (PHJB) equations associated with optimal control problems for path-dependent stochastic differential equations. We identify the value functional of optimal control problems as unique viscosity solution to the associated PHJB equations. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property. Applications to backward stochastic Hamilton-Jacobi-Bellman equations are also given.

Key Words: Path-dependent Hamilton-Jacobi-Bellman equations; Viscosity solutions; Optimal control; Path-dependent stochastic differential equations; Backward stochastic Hamilton-Jacobi-Bellman equations

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1 Introduction
The notion of viscosity solutions for Hamilton-Jacobi-Bellman (HJB) equations, first introduced in 1983 by Crandall and Lions [9], has become an indispensable tool in optimal control theory and numerous subjects related to it. We refer to the survey paper of Crandall, Ishii and Lions [8] and the monographs of Fleming and Soner [18] and Yong and Zhou [41] for a detailed account for the theory of viscosity solutions. For viscosity solutions in infinite dimensional Hilbert spaces, we refer to Fabbri, Gozzi and Świąč [17], Gozzi, Rouy and Świąč [19], Lions [22, 23, 24] and Świąč [37].

Dupire in his work [10] introduced horizontal and vertical derivatives for functionals defined in the space of càdlàg paths and provided a functional Itô formula (see Cont and Fournié [4, 5] for a more general and systematic research). Soon after, Dupire’s functional Itô formula was applied to second order path-dependent HJB (PHJB) equations. Peng [30] made the first attempt to extend Crandall-Lions framework to path-dependent case, by focusing on the uniqueness part. Tang and Zhang [38] proposed a different notion of viscosity solutions for path-dependent Bellman equations. They verified that the value functional of the corresponding optimal control problem is a viscosity solution, but did not investigate the comparison principle.

At the same time, Ekren, Keller, Touzi and Zhang [12] introduced a new notion of viscosity solutions for semi-linear path-dependent partial differential equations (PPDEs) in the space of...
continuous paths in terms of a nonlinear expectation. In the subsequent works, Ekren [11], Ekren, Touzi and Zhang [13, 14] and Ren [35] extended the notion to fully nonlinear case when the Hamilton function \( H \) is uniformly nondegenerate. Ren, Touzi and Zhang [36] considered the degenerate case and established the comparison principle when the nonlinearity \( H \) is \( d_p \)-uniformly continuous in the path. We refer to Barrasso and Russo [1], Cosso and Russo [6], Leao, Ohashi and Simas [21], Peng and Song [31] and Peng and Wang [32] for other notions of solutions to path-dependent semi-linear equations, and to Viens and Zhang [39] and Wang, Yong and Zhang [40] for some more general PPDEs. We also mention that Lukoyanov [25] developed a theory of viscosity solutions to semi-linear equations, and to Viens and Zhang [39] and Wang, Yong and Zhang [40] for some more general PPDEs. We refer to Barrasso and Russo [1], Cosso and Russo [6], Leao, Ohashi and Simas [21], Peng and Song [31] and Peng and Wang [32] for other notions of solutions to path-dependent semi-linear first order Hamilton-Jacobi equations when Hamilton function \( H \) is \( d_p \)-locally Lipschitz continuous with respect to the path.

In this paper, we consider the following controlled path-dependent stochastic differential equation (PSDE):

\[
\begin{align*}
    dX^{\gamma, u}(s) &= b(X^{\gamma, u}_s, u(s))ds + \sigma(X^{\gamma, u}_s, u(s))dW(s), \quad s \in [t, T], \\
    X^{\gamma, u}_t &= \gamma_t \in \Lambda_t.
\end{align*}
\]

In the above equation, \( \Lambda_t \) denotes the set of all continuous \( \mathbb{R}^d \)-valued functions \( \gamma \) defined over \([0, t]\), and let \( \Lambda^* = \bigcup_{s \in [s, T]} \Lambda_t \) and \( \Lambda \) denote \( \Lambda^0 \); the unknown \( X^{\gamma, u}(s) \), representing the state of the system, is an \( \mathbb{R}^d \)-valued process; \( X^{\gamma, u}_s \) is the whole history of \( X^{\gamma, u}(\cdot) \) from time 0 to \( s \); \( \{W(t), t \geq 0\} \) is an \( n \)-dimensional standard Wiener process; \( u(\cdot) = (u(s))_{s \in [t, T]} \) is progressively measurable with respect to the Wiener filtration and takes values in some Polish space \((U, d_1)\) (we will say that \( u(\cdot) \in U[t, T] \)). We define a norm on \( \Lambda_t \) and a metric on \( \Lambda \) as follows: for any \((t, \gamma_t), (s, \eta_s) \in [0, T] \times \Lambda_t,\)

\[
\|\gamma_t\|_0 := \sup_{0 \leq t \leq T} |\gamma_t(l)|, \quad d_\infty(\gamma_t, \eta_s) := |t - s| + \sup_{0 \leq t \leq T} |\gamma_t(l \wedge t) - \eta_s(l \wedge s)|.
\]

We assume that the coefficients \( b : \Lambda \times U \to \mathbb{R}^d \) and \( \sigma : \Lambda \times U \to \mathbb{R}^{d \times n} \) satisfy Lipschitz condition under \( \| \cdot \|_0 \) with respect to the path.

We aim at maximizing a cost functional of the form:

\[
J(\gamma_t, u(\cdot)) := Y^{\gamma, u}(t), \quad (t, \gamma_t) \in [0, T] \times \Lambda,
\]

over \( U[t, T] \), where the process \( Y^{\gamma, u} \) is defined via solution of backward stochastic differential equation (BSDE):

\[
Y^{\gamma, u}(s) = \phi(X^{\gamma, u}_T) + \int_s^T q(X^{\gamma, u}_l, Y^{\gamma, u}(l), Z^{\gamma, u}(l), u(l))dl - \int_s^T Z^{\gamma, u}(l)dW(l), \quad a.s., \quad \forall s \in [t, T].
\]

Here \( q \) and \( \phi \) are given real functionals on \( \Lambda \times \mathbb{R} \times \mathbb{R}^n \) and \( \Lambda_T \), respectively, and satisfy Lipschitz condition under \( \| \cdot \|_0 \) with respect to the path. We define the value functional of the optimal control problem as follows:

\[
V(\gamma_t) := \text{esssup}_{u(\cdot) \in U[t, T]} Y^{\gamma, u}(t), \quad (t, \gamma_t) \in [0, T] \times \Lambda.
\]

We characterize this value functional \( V \) with the following PHJB equation:

\[
\begin{align*}
    \mathcal{L}V(\gamma_t) := \partial_t V(\gamma_t) + H(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) &= 0, \quad (t, \gamma_t) \in [0, T] \times \Lambda, \\
    V(\gamma_T) &= \phi(\gamma_T), \quad \gamma_T \in \Lambda_T;
\end{align*}
\]
where

\[ H(\gamma_t, r, p, \ell) = \sup_{u \in U} \left[ \langle p, b(\gamma_t, u) \rangle + \frac{1}{2} \text{tr} \left[ \sigma(\gamma_t, u) \sigma^\top(\gamma_t, u) \right] + q(\gamma_t, r, \sigma^\top(\gamma_t, u)p, u) \right], \quad (t, \gamma_t, r, p, \ell) \in [0, T] \times \Lambda \times \mathbb{R} \times \mathbb{R}^d \times S(\mathbb{R}^d). \]

Here, \(\sigma^\top\) is the transpose of the matrix \(\sigma\), \(S(\mathbb{R}^d)\) the set of all \((d \times d)\) symmetric matrices, \(\langle \cdot, \cdot \rangle\) the scalar product of \(\mathbb{R}^d\), and \(\partial_t, \partial_x\) and \(\partial_{xx}\) the so-called pathwise (or functional or Dupire; see [10, 14, 15]) derivatives, where \(\partial_t\) is known as the horizontal derivative, while \(\partial_x\) and \(\partial_{xx}\) are the first and second order vertical derivatives, respectively.

The primary objective of this article is to develop a concept of viscosity solutions to PHJB equations on the space of continuous paths (see Definition 4.2 for details). We shall show that the value functional \(V\) defined in (1.4) is unique viscosity solution to the PHJB equation (1.5) when the coefficients \((b, \sigma, q, \phi)\) are uniformly Lipschitz in the path under \(|| \cdot \||_0\).

The lack of smoothness of \(|| \cdot \||_0\) makes it more difficult to define the viscosity solutions and to prove its uniqueness.

In this paper we want to extend the theory of viscosity solutions to the second order path-dependent case. We adopt the natural generalization of the well-known Crandall-Lions definition in terms of test functions. Since we assume the coefficients \((b, \sigma, q, \phi)\) only satisfy \(|| \cdot \||_0\)-Lipschitz conditions with respect to the path and do not impose uniformly nondegenerate requirement on the coefficients, none of these results in Ekren [11], Ekren, Touzi and Zhang [13, 14], Lukoyanov [25], Peng [30], Ren [35] and Ren, Touzi and Zhang [36] are directly applicable to our case.

The main contribution of this paper is the introduction of an appropriate notion of viscosity solutions and the proof of uniqueness. The uniqueness property is derived from the comparison theorem. For the proof of the comparison theorem, we generalize the classical methodology in Crandall, Ishii and Lions [8] to the path-dependent case. We introduce functional \(\Upsilon: \Lambda \times \Lambda \rightarrow \mathbb{R}\) defined by

\[ \Upsilon(\gamma_t, \eta_s) = S(\gamma_t, \eta_s) + 3|\gamma_t(t) - \eta_s(s)|^6 \]

and

\[ S(\gamma_t, \eta_s) = \begin{cases} \frac{(||\gamma_t - \eta_s||_0^6 - ||\gamma_t - \eta_s||_0^6)\ell}{||\gamma_t - \eta_s||_0^2}, & ||\gamma_t - \eta_s||_0 \neq 0; \\ 0, & ||\gamma_t - \eta_s||_0 = 0 \end{cases} \]

for \(\gamma_t, \eta_s \in \Lambda\). Here \(||\gamma_t - \eta_s||_0 = \sup_{t \in [0, T]} ||\gamma_t(t) - \eta_s(t)||\).

This key functional is the starting point for the proof of comparison theorem. First, for every fixed \((\ell, a_{\ell}) \in [0, T] \times \Lambda\), define \(f: \Lambda^{\ell} \rightarrow \mathbb{R}\) by

\[ f(\gamma_{\ell}) := \Upsilon(\gamma_{\ell}, a_{\ell}), \quad \gamma_{\ell} \in \Lambda^{\ell}. \]

We show that it is equivalent to \(|| \cdot \||_0^6\) and study its regularity in the sense of horizontal/vertical derivatives. Then the test function in our definition of viscosity solutions and the auxiliary function \(\Psi\) in the proof of comparison theorem can include \(f\) (see Step 1 in the proof of Theorem 6.1) as
we show that it satisfies a functional Itô formula. By this, the comparison theorem is established when the coefficients only satisfy Lipschitz assumption under $\| \cdot \|_0$.

Second, we use $\Upsilon$ to define a smooth gauge-type function $\Upsilon : \Lambda \times \Lambda \rightarrow \mathbb{R}$ by

$$\Upsilon(\gamma_t, \eta_s) := \Upsilon(\gamma_t, \eta_s) + |s - t|^2, \quad (t, \gamma_t), (s, \eta_s) \in [0, T] \times \Lambda.$$ 

Then we can apply a modification of Borwein-Preiss variational principle (see Borwein and Zhu [2, Theorem 2.5.2]) to get a maximum of a perturbation of the auxiliary function $\Psi$.

Unfortunately, the second spatial derivative $\partial_{xx} S(\cdot, a_t)$ is not equal to $0$ (see Lemma 3.1), in order to apply Crandall-Ishii lemma (see Theorem 5.3) to prove comparison theorem (see Theorem 6.1), a stronger convergence property of auxiliary functional is needed. Thanks to the Step 2 in the proof of Theorem 6.1, we can find the expected convergence property of auxiliary functional and prove the comparison theorem.

Regarding existence, we prove that the value functional $V$ defined in (1.4) is a viscosity solution to the PHJB equation given in (1.5) under our definition by functional Itô formula and dynamic programming principle.

Cosso and Russo [7] define a smooth functional and apply the Borwein-Preiss variational principle to study the comparison theorem (which imply the uniqueness) of viscosity solutions for the path-dependent heat equation. The construction of their smooth functional seems to be more complicated than ours, and they only study the path-dependent heat equation.

Backward stochastic partial differential equation (BSPDE) is another interesting topic. Peng [28] obtained the existence and uniqueness theorem for the solution to backward stochastic HJB (BSHJB) equations in a triple. The relationship between forward backward stochastic differential equations and a class of semi-linear BSPDEs was established in Ma and Yong [26]. For viscosity solutions of BSPDEs, we refer to Ekren and Zhang [15] and Qiu [33, 34]. As an application of our results, we give a definition of viscosity solutions to BSHJB equations, and characterize the value functional of the optimal stochastic control problem as unique viscosity solution to the associated BSHJB equation.

The outline of this article is as follows. In the following section, we introduce the framework of [5] and [10], preliminary results on path-dependent stochastic optimal control problems, and a modification of Borwein-Preiss variational principle. In Section 3, we present the smooth functionals $\Upsilon_{m,M}$ which are the key to proving the stability and uniqueness results of viscosity solutions. In Section 4, we define classical and viscosity solutions to our PHJB equations and prove that the value functional $V$ defined by (1.4) is a viscosity solution to PHJB equation (1.5). We also show the consistency with the notion of classical solutions and the stability result. A Crandall-Ishii lemma for path-dependent case is given in Section 5. The uniqueness of viscosity solutions for (1.5) is proved in Section 6 and Section 7 is devoted to applications to BSHJB equations.

2 Preliminaries

2.1. Pathwise derivatives. For the vectors $x, y \in \mathbb{R}^d$, the scalar product is denoted by $\langle x, y \rangle$ and the Euclidean norm $\| x \|_2$ is denoted by $|x|$ (we use the same symbol $| \cdot |$ to denote the Euclidean norm on $\mathbb{R}^k$, for any $k \in \mathbb{N}^+$. If $A$ is a vector or matrix, its transpose is denoted by $A^\top$; For a matrix $A$, denote its operator norm and Hilbert-Schmidt norm by $|A|$ and $|A|_2$, respectively. Denote by $\mathcal{S}(\mathbb{R}^d)$ the set of all $(d \times d)$ symmetric matrices. Let $T > 0$ be a fixed number. For each $t \in [0, T]$, define $\Lambda_t := D([0, t]; \mathbb{R}^d)$ as the set of càdlàg $\mathbb{R}^d$-valued functions on $[0, t]$. We denote $\Lambda^t = \bigcup_{s \in [t, T]} \Lambda_s$ and let $\hat{\Lambda}$ denote $\Lambda^0$. 


A very important remark on the notations: as in [10], we will denote elements of \( \hat{\Lambda} \) by lower case letters and often the final time of its domain will be subscripted, e.g. \( \gamma \in \Lambda_t \subset \hat{\Lambda} \) will be denoted by \( \gamma_t \). Note that, for any \( \gamma \in \hat{\Lambda} \), there exists only one \( t \) such that \( \gamma \in \Lambda_t \). For any \( 0 \leq s \leq t \), the value of \( \gamma_t \) at time \( s \) will be denoted by \( \gamma_t(s) \). Moreover, if a path \( \gamma_t \) is fixed, the path \( \gamma_t|_{[0,s]} \), for \( 0 \leq s \leq t \), will denote the restriction of the path \( \gamma_t \) to the interval \([0,s]\).

For convenience, define for \( x \in \mathbb{R}^d \), \( \gamma_t \in \hat{\Lambda} \), \( 0 \leq t \leq \bar{t} \leq T \),

\[
\gamma^x_t(s) := \gamma_t(s)\mathbf{1}_{[0,t]}(s) + (\gamma_t(t) + x)\mathbf{1}_{[t,\bar{t}]}(s), \quad s \in [0,\bar{t}]; \\
\gamma_{t,f}(s) := \gamma_t(s)\mathbf{1}_{[0,t]}(s) + \gamma_t(t)\mathbf{1}_{[t,\bar{t}]}(s), \quad s \in [0,\bar{t}].
\]

We define a norm on \( \hat{\Lambda}_t \) and a metric on \( \hat{\Lambda} \) as follows: for any \( 0 \leq t \leq \bar{t} \leq T \) and \( \gamma_t, \tilde{\gamma}_t \in \hat{\Lambda} \),

\[
||\gamma_t||_0 := \sup_{0 \leq s \leq t} |\gamma_t(s)|, \\
d_\infty(\gamma_t, \tilde{\gamma}_t) = d_\infty(\bar{\gamma}_t, \gamma_t) := |t - \bar{t}| + ||\gamma_t - \tilde{\gamma}_t||_0.
\]

In the sequel, for notational simplicity, we use \( ||\gamma_t - \tilde{\gamma}_t||_0 \) to denote \( ||\gamma_t - \tilde{\gamma}_t||_0 \). Then \((\hat{\Lambda}_t, || \cdot ||_0)\) is a Banach space and \((\hat{\Lambda}^t, d_\infty)\) is a complete metric space. Following Dupire [10], we define spatial derivatives of \( f : \hat{\Lambda} \to \mathbb{R} \), if exist, in the standard sense: for the basis \( e_i \) of \( \mathbb{R}^d \), \( i, j = 1, \ldots, d \),

\[
\partial_{x_i} f(g_s) := \lim_{h \to 0} \frac{1}{h} \left[f(\gamma_s^{h e_i}) - f(g_s)\right], \\
\partial_{x_i,x_j} f := \partial_{x_i}(\partial_{x_j} f), \quad (s, g_s) \in [0, T] \times \hat{\Lambda},
\]

and the right time-derivative of \( f \), if exists, as:

\[
\partial_t f(g_s) := \lim_{h \to 0, h > 0} \frac{1}{h} \left[f(\gamma_{s+h}) - f(g_s)\right], \quad (s, g_s) \in [0, T] \times \hat{\Lambda}.
\]

For the final time \( T \), we define

\[
\partial_t f(\gamma_T) := \lim_{\bar{t} \downarrow T} \partial_t f(\gamma_T |_{[0,\bar{t}]}) \quad \gamma_T \in \hat{\Lambda}.
\]

We take the convention that \( \gamma_s \) and \( \partial_x f \) denote column vector and \( \partial_{xx} f \) denotes \( d \times d \)-matrix.

**Definition 2.1.** Let \( t \in [0, T) \) and \( f : \hat{\Lambda}^t \to \mathbb{R} \) be given.

(i) We say \( f \in C^0(\hat{\Lambda}^t) \) if \( f \) is continuous in \( \gamma_s \) on \( \hat{\Lambda}^t \) under \( d_\infty \).

(ii) We say \( f \in C^{1,2}(\hat{\Lambda}^t) \subset C^0(\hat{\Lambda}^t) \) if \( \partial_t f, \partial_{x_i} f, \partial_{x_i x_j} f \) exist in \( \hat{\Lambda}^t \) and are in \( C^0(\hat{\Lambda}^t) \) for all \( i, j = 1, 2, \ldots, d \).

(iii) We say \( f \in C^{1,2}_p(\hat{\Lambda}^t) \subset C^{1,2}(\hat{\Lambda}^t) \) if \( f \) and all of its derivaties grow in a polynomial way.

For each \( t \in [0, T] \), let \( \Lambda_t := C([0, t], \mathbb{R}^d) \) be the set of all continuous \( \mathbb{R}^d \)-valued functions defined over \([0, t]\). We denote \( \Lambda^t := \bigcup_{s \in [0, t]} \Lambda_s \) and let \( \Lambda \) denote \( \Lambda^0 \). Clearly, \( \Lambda := \bigcup_{t \in [0, T]} \Lambda_t \subset \hat{\Lambda} \), and each \( \gamma \in \Lambda \) can also be viewed as an element of \( \hat{\Lambda} \). \( (\Lambda_t, || \cdot ||_0) \) is a Banach space, and \((\Lambda^t, d_\infty)\) is a complete metric space. \( f : \Lambda^t \to \mathbb{R} \) and \( \hat{f} : \hat{\Lambda}^t \to \mathbb{R} \) are called consistent on \( \Lambda^t \) if \( f \) is the restriction of \( \hat{f} \) on \( \Lambda^t \).

**Definition 2.2.** Let \( t \in [0, T) \) and \( f : \Lambda^t \to \mathbb{R} \) be given.

(i) We say \( f \in C^0(\Lambda^t) \) (resp., \( f \in USC^0(\Lambda^t) \), \( f \in LSC^0(\Lambda^t) \)) if \( f \) is continuous (resp., upper semicontinuous, lower semicontinuous) in \( \gamma_s \) on \( \Lambda^t \) under \( d_\infty \).
(ii) We say \( f \in C_p^{1,2}(\Lambda^t) \) if there exists \( \hat{f} \in C_p^{1,2}(\hat{\Lambda}^t) \) which is consistent with \( f \) on \( \Lambda^t \).

By Dupire [10] and Cont and Fournie [5, Theorem 4.1], we have the following functional Itô formula.

**Theorem 2.3.** Suppose \( X \) is a continuous semi-martingale and \( f \in C_p^{1,2}(\Lambda^t) \) for some fixed \( i \in [0,T] \). Then for any \( t \in [i,T] \):

\[
f(X_t) = f(X_i) + \int_i^t \partial_t f(X_s)ds + \frac{1}{2} \int_i^t \partial_{xx} f(X_s)ds + \int_i^t \partial_x f(X_s)dX(s), \quad P\text{-a.s.}
\]

(2.4)

Here and in the following, for every \( s \in [0,T] \), \( X(s) \) denotes the value of \( X \) at time \( s \), and \( X_s \) the whole history path of \( X \) from time 0 to \( s \).

**Proof.** Define a family of functionals \( F := (F_t)_{t \in [0,T]} \) where

\[
F_t(\gamma_t, \xi_t) = f(\gamma_t), \quad (\gamma_t, \xi_t) \in \hat{\Lambda}_t \times D([0,t], S_d^+),
\]

and \( D([0,t], S_d^+) \) denotes the space of càdlàg functions on \([0,t]\), taking values in the set \( S_d^+ \) of positive \( d \times d \) matrices. It is clear that \( F \) satisfies conditions (10) and (14) in Cont and Fournie [5].

By \( f \in C_p^{1,2}(\Lambda^t) \), we have \( F \in C^{1,2}((i,T)) \) (see Definition 3.6 in Cont and Fournie [5]). Thus \( F \) satisfies conditions in Theorem 4.1 of Cont and Fournie [5]. Then, by functional Itô formula (30) in Theorem 4.1 of Cont and Fournie [5], we get (2.4) holds true. \( \square \)

By the above Theorem, we have the following important result.

**Lemma 2.4.** Let \( f \in C_p^{1,2}(\Lambda^t) \) and \( \hat{f} \in C_p^{1,2}(\hat{\Lambda}^t) \) such that \( \hat{f} \) is consistent with \( f \) on \( \Lambda^t \), then the following definition

\[
\partial_t f := \partial_t \hat{f}, \quad \partial_x f := \partial_x \hat{f}, \quad \partial_{xx} f := \partial_{xx} \hat{f}
\]

is independent of the choice of \( \hat{f} \). Namely, if there is another \( \hat{f}' \in C_p^{1,2}(\hat{\Lambda}^t) \) such that \( \hat{f}' \) is consistent with \( f \) on \( \Lambda^t \), then the derivatives of \( \hat{f}' \) coincide with those of \( \hat{f} \) on \( \Lambda^t \).

**Proof.** By the definition of the horizontal derivative, it is clear that \( \partial_t \hat{f}(\gamma_t) = \partial_t \hat{f}'(\gamma_t) \) for every \( \gamma_t \in \Lambda^t \). Next, for every \( l \in [0,T] \), define

\[
\Theta_l := \{ \gamma_t \in \Lambda_t : \text{there exists a constant } L > 0 \text{ such that } \sup_{r_1, r_2 \in [0,l]} |\gamma_t(r_1) - \gamma_t(r_2)| \leq L|r_1 - r_2| \}.
\]

Then \( \Theta_l \) is dense in \((\Lambda_t, || \cdot ||_0)\). For every \((l, \gamma_l) \in [l,T] \times \Theta_l \) and \( h \in \mathbb{R}^d \), let

\[
X(r) = \gamma_l(r \wedge l) + (r \vee l - l)h, \quad r \in [l,T].
\]

It is clear that \( X \) is a continuous semi-martingale. Then by Lemma 2.3

\[
\int_l^s \langle \partial_x \hat{f}(X_r), h \rangle dr = \int_l^s \langle \partial_x \hat{f}'(X_r), h \rangle dr, \quad s \in [l,T].
\]

By the continuity of \( \partial_x \hat{f} \) and \( \partial_x \hat{f}' \) and the arbitrariness of \( h \in \mathbb{R}^d \), we have \( \partial_x \hat{f}(\gamma_t) = \partial_x \hat{f}'(\gamma_t) \).

Notice that \( \Theta_l \) is dense in \((\Lambda_t, || \cdot ||_0)\), by also the continuity of \( \partial_x f \) and \( \partial_x f' \), we obtain that \( \partial_x \hat{f}(\gamma_t) = \partial_x \hat{f}'(\gamma_t) \) for every \( \gamma_t \in \Lambda^t \). Finally, for every \((l, \gamma_l) \in [l,T] \times \Theta_l \) and \( a \in \mathbb{R}^{d \times n} \), let

\[
X(r) = \gamma_l(r \wedge l) + a(W(r \vee l) - W(l)), \quad r \in [0,T].
\]
It is clear that $X$ is a continuous semi-martingale. Then by Lemma 2.3,

$$
\int_t^s \text{tr}(\partial_{xx} f(X_r)aa^\top)dr = \int_t^s \text{tr}(\partial_{xx} f'(X_r)aa^\top)dr, \ s \in [t,T].
$$

By the continuity of $\partial_{xx} f$ and $\partial_{xx} f'$, the arbitrariness of $a \in \mathbb{R}^{d \times n}$ and the denseness of $\Theta_t$, we also have $\partial_{xx} f(\gamma_t) = \partial_{xx} f'(\gamma_t)$ for every $\gamma_t \in \Lambda_t$. $\square$

2.2. Value functional. Let $\Omega := \{ \omega \in C([0,T], \mathbb{R}^n) : \omega(0) = 0 \}$, the set of continuous functions with initial value 0, $W$ the Wiener measure, $P$ the Wiener measure, $\mathcal{F}$ the Borel $\sigma$-field over $(\Omega, || \cdot ||_0)$, completed with respect to the Wiener measure $P$ on this space. Then $(\Omega, \mathcal{F}, P)$ is a complete space. Here and in the sequel, for notational simplicity, we use 0 to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. By $\{ \mathcal{F}_t \}_{0 \leq t \leq T}$ we denote the filtration generated by $\{ W(t), 0 \leq t \leq T \}$, augmented with the family $\mathcal{N}$ of $P$-null of $\mathcal{F}$. The filtration $\{ \mathcal{F}_t \}_{0 \leq t \leq T}$ satisfies the usual conditions.

We introduce the admissible control. Let $t, s$ be two deterministic times, $0 \leq t \leq s \leq T$.

**Definition 2.5.** An admissible control process $u(\cdot) = \{ u(r), r \in [t,s] \}$ on $[t,s]$ is an $\{ \mathcal{F}_t \}_{t \leq r \leq s}$ progressively measurable process taking values in some Polish space $(\Omega, d_1)$. The set of all admissible controls on $[t,s]$ is denoted by $\mathcal{U}[t,s]$. We identify two processes $u(\cdot)$ and $\tilde{u}(\cdot)$ in $\mathcal{U}[t,s]$ and write $u(\cdot) \equiv \tilde{u}(\cdot)$ on $[t,s]$, if $P(u(\cdot) = \tilde{u}(\cdot)$ a.e. in $[t,s]) = 1$.

Now we consider the controlled state equation (1.1) and cost equation (1.3). First we make the following assumption.

**Hypothesis 2.6.** $b : \Lambda \times U \rightarrow \mathbb{R}^d$, $\sigma : \Lambda \times U \rightarrow \mathbb{R}^{d \times n}$, $q : \Lambda \times \mathbb{R} \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and $\phi : \Lambda_T \rightarrow \mathbb{R}$ are continuous, and there exists $L > 0$ such that, for all $(t, \gamma_t, \eta_T, y, z, u)$, $(t, \gamma_t', \eta_T', y', z', u) \in [0,T] \times \Lambda \times \Lambda_T \times \mathbb{R} \times \mathbb{R}^n \times U$,

$$
\begin{align*}
&|b(\gamma_t, u)|^2 \vee |\sigma(\gamma_t, u)|^2 \leq L^2(1 + ||\gamma_t||_0^2), \\
&|b(\gamma_t, u) - b(\gamma_t', u)| \vee |\sigma(\gamma_t, u) - \sigma(\gamma_t', u)| \leq L||\gamma_t - \gamma_t'||_0, \\
&q(\gamma_t, y, z, u) \leq L(1 + ||\gamma_t||_0 + |y| + |z|), \\
&q(\gamma_t, y, z, u) - q(\gamma_t', y', z', u) \leq L(||\gamma_t - \gamma_t'||_0 + |y - y'| + |z - z'|), \\
&|\phi(\eta_T) - \phi(\eta_T')| \leq L||\eta_T - \eta_T'||_0.
\end{align*}
$$

The following lemma is standard; see, for example, [38, Lemmas 3.1 and 3.2] (see also El Karoui, Peng and Quenez [13, Theorem 2.1 and Proposition 2.1] and Karatzas and Shreve [20, Theorem 3.28 in Chapter 3 and Theorem 2.9 in Chapter 5] for details).

**Lemma 2.7.** Assume that Hypothesis 2.6 holds. Then for every $u(\cdot) \in \mathcal{U}[0,T]$, $(t, \gamma_t) \in [0,T] \times \Lambda$ and $p \geq 2$, PSDE (L.1) admits a unique strong solution $X^{\gamma_t,u}$, and BSDE (L.3) admits a unique pair of solutions $(Y^{\gamma_t,u}, Z^{\gamma_t,u})$. Furthermore, let $X^{\gamma_t,u}$ and $(Y^{\gamma_t,u}, Z^{\gamma_t,u})$ be the solutions of PSDE (L.1) and BSDE (L.3) corresponding $(t, \gamma_t) \in [0,T] \times \Lambda$ and $v(\cdot) \in \mathcal{U}[0,T]$. Then the following estimates hold:

$$
\begin{align*}
\mathbb{E}[||X_T^{\gamma_t,u} - X_T^{\gamma_t',u}||_p^p | \mathcal{F}_t] \leq C_p ||\gamma_t - \gamma_t'||_0^p + C_p \int_t^T \mathbb{E}[|b(X_t^{\gamma_{t'}u}, u(l)) - b(X_t^{\gamma_{t'}u}, v(l))|^p | \mathcal{F}_t]dl \\
+ C_p \int_t^T \mathbb{E}[|\sigma(X_t^{\gamma_{t'}u}, u(l)) - \sigma(X_t^{\gamma_{t'}u}, v(l))|^p | \mathcal{F}_t]dl;
\end{align*}
$$

(2.5)
Theorem 2.10. (see [38, Theorem 3.4]) Assume Hypothesis 2.6 holds true, the value functional
\[ \mathbb{E}[||X_T^{\gamma_{t,u}}||_0^p|\mathcal{F}_t] \leq C_p(1 + ||\gamma_t||_0^p); \] (2.6)
\[ \mathbb{E}[||X_T^{\gamma_{t,u}} - \gamma_t||_0^p|\mathcal{F}_t] \leq C_p(1 + ||\gamma_t||_0^p)(r - t)^{\frac{p}{2}}, \quad r \in [t, T]; \] (2.7)
and
\[ \mathbb{E}[||Y_T^{\gamma_{t,u}} - Y_T^{\gamma_{t,v}}||_0^p|\mathcal{F}_t] \]
\[ \leq C_p||\gamma_t - \gamma_t^\prime||_0^p + C_p \int_t^T \mathbb{E}[|b(X_t^{\gamma_{t,u}}, u(l)) - b(X_t^{\gamma_{t,v}}, v(l))|^p \]
\[ + |\sigma(X_t^{\gamma_{t,u}}, u(l)) - \sigma(X_t^{\gamma_{t,v}}, v(l))|_2^p \]
\[ + |q(X_t^{\gamma_{t,u}}, Y_t^{\gamma_{t,u}}(l), Z_t^{\gamma_{t,u}}(l), u(l)) - q(X_t^{\gamma_{t,v}}, Y_t^{\gamma_{t,v}}(l), Z_t^{\gamma_{t,v}}(l), v(l))|^p|\mathcal{F}_t|dl; \] (2.8)
\[ \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_t^{\gamma_{t,u}}(s)|^p |\mathcal{F}_t \right] + \mathbb{E} \left[ \left( \int_t^T |Z_t^{\gamma_{t,u}}(l)|^2 dl \right)^{\frac{p}{2}} |\mathcal{F}_t \right] \leq C_p(1 + ||\gamma_t||_0^p). \] (2.9)

The constant \( C_p \) depending only on \( p, T \) and \( L \).

Formally, under Hypothesis 2.6 the value functional \( V(\gamma_t) \) defined by (1.4) is \( \mathcal{F}_t \)-measurable. However, according to the similar proof procedure of Proposition 3.3 in Buckdahn and Li [3], we can prove the following.

**Theorem 2.8.** Suppose the Hypothesis 2.6 holds true. Then \( V \) is a deterministic functional.

The following property of the value functional \( V \) is an immediate consequence of Lemma 2.7.

**Lemma 2.9.** Assume that Hypothesis 2.6 holds, then \( V \in C^0(\Lambda) \) and there exists a constant \( C > 0 \) such that, for all \( (t, \gamma_t, \gamma_t') \in [0, T] \times \Lambda \times \Lambda \),
\[ |V(\gamma_t) - V(\gamma_t^\prime)| \leq C||\gamma_t - \gamma_t^\prime||_0; \quad |V(\gamma_t)| \leq C(1 + ||\gamma_t||_0). \] (2.10)

We now discuss a dynamic programming principle (DPP) for the optimal control problem (1.1), (1.3) and (1.4). For this purpose, we define the family of backward semigroups associated with BSDE (1.3), following the idea of Peng [29].

Given the initial condition \( (t, \gamma_t) \in [0, T] \times \Lambda \), a positive number \( \delta \leq T - t \), an admissible control \( u(\cdot) \in \mathcal{U}[t, t + \delta] \) and a real-valued random variable \( \eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R}) \), we put
\[ G_{s,t+\delta}^{\gamma_{t,u}}[\eta] := \hat{Y}_{s,t+\delta}^{\gamma_{t,u}}(s), \quad s \in [t, t + \delta], \] (2.11)
where \( (\hat{Y}_{s,t+\delta}^{\gamma_{t,u}}(s), \hat{Z}_{s,t+\delta}^{\gamma_{t,u}}(s))_{t \leq s \leq t + \delta} \) is the solution of the following BSDE with the time horizon \( t + \delta \):
\[ \begin{aligned}
    dY_{s,t}^{\gamma_{t,u}}(s) &= -q(X_s^{\gamma_{t,u}}, \hat{Y}_{s,t}^{\gamma_{t,u}}(s), \hat{Z}_{s,t}^{\gamma_{t,u}}(s), u(s))ds + \hat{Z}_{s,t}^{\gamma_{t,u}}(s)dW(s), \\
    \hat{Y}_{t,t+\delta}^{\gamma_{t,u}}(t+\delta) &= \eta,
\end{aligned} \] (2.12)
and \( X_{s,t}^{\gamma_{t,u}}(\cdot) \) is the solution of PSDE (1.1).

**Theorem 2.10.** (see [38, Theorem 3.4]) Assume Hypothesis 2.6 holds true, the value functional \( V \) obeys the following DPP: for any \( (t, \gamma_t) \in [0, T] \times \Lambda \) and \( 0 < \delta \leq T - t \),
\[ V(\gamma_t) = \operatorname{esssup}_{u(\cdot) \in \mathcal{U}[t, t + \delta]} G_{t,t+\delta}^{\gamma_{t,u}}[V(X_{t+\delta}^{\gamma_{t,u}})]. \] (2.13)
Lemma 2.10 says that the value functional $V$ is Lipschitz continuous in $\Lambda_t$. From Theorem 2.10 we have

**Theorem 2.11.** (see [25, Theorem 3.7]) Under Hypothesis 2.6 there is a constant $C > 0$ such that, for every $0 \leq t \leq t' \leq T$ and $\gamma_{t}, \gamma_{t'} \in \Lambda$,

$$|V(\gamma_{t}) - V(\gamma_{t'})| \leq C[|\gamma_{t} - \gamma_{t'}|_0 + (1 + |\gamma_{t}|_0)(t' - t)^{\frac{1}{2}}].$$ (2.14)

2.3. **Borwein-Preiss variational principle.** In this subsection, we introduce a modification of Borwein-Preiss variational principle (see Borwein and Zhu [2, Theorem 2.5.2]) which plays a crucial role in the proof of the comparison Theorem. We firstly recall the definition of gauge-type function for compete metric space $(H, d)$.

**Definition 2.12.** Let $(H, d)$ be a compete metric space. We say that a continuous functional $\rho : H \times H \rightarrow [0, \infty)$ is a gauge-type function on the compete metric space $(H, d)$ provided that:

(i) $\rho(x, x) = 0$ for all $x \in H$,

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in H$, we have $\rho(x, y) \leq \delta$ implies that $d(x, y) < \varepsilon$.

**Lemma 2.13.** Let $t \in [0, T]$ be fixed and let $f : \Lambda^t \rightarrow \mathbb{R}$ be an upper semicontinuous functional bounded from above. Suppose that $\rho$ is a gauge-type function on $(\Lambda^t, d_{\infty})$ and $\{\delta_i\}_{i \geq 0}$ is a sequence of positive number, and suppose that $\varepsilon > 0$ and $(t_0, \gamma_{t_0}) \in [t, T] \times \Lambda^t$ satisfy

$$f(\gamma_{t_0}) \geq \sup_{(s, \gamma_s) \in [t, T] \times \Lambda^t} f(\gamma_s) - \varepsilon.$$

Then there exist $(\tilde{t}, \gamma_{\tilde{t}}) \in [t, T] \times \Lambda^t$ and a sequence $\{(t_i, \gamma_{t_i})\}_{i \geq 1} \subset [t_0, T] \times \Lambda^t$ such that

(i) $\rho(\gamma_{t_i_0}, \gamma_{\tilde{t}}) \leq \frac{\varepsilon}{\delta_0}$, $\rho(\gamma_{t_i}, \gamma_{\tilde{t}}) \leq \frac{\varepsilon}{\delta_i}$ and $t_i \uparrow \tilde{t}$ as $i \rightarrow \infty$,

(ii) $f(\gamma_{\tilde{t}}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_{t_i}, \gamma_{\tilde{t}}) \geq f(\gamma_{t_0})$,

(iii) $f(\gamma_s) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_{t_i}, \gamma_s) < f(\gamma_{\tilde{t}}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_{t_i}, \gamma_{\tilde{t}})$ for all $(s, \gamma_s) \in [\tilde{t}, T] \times \Lambda^t \setminus \{(\tilde{t}, \gamma_{\tilde{t}})\}$.

This lemma is similar to Borwein and Zhu [2, Theorem 2.5.2], the only difference is that here contains one more result than the latter, that is, the time series $\{t_i\}_{i \geq 1}$ is monotonically increasing. For the convenience of readers, we give its proof in Appendix A.

For every $t \in [0, T]$, define $\Lambda^t \otimes \Lambda^t := \{(\gamma_s, \eta_s) | \gamma_s, \eta_s \in \Lambda^t\}$. It is clear that $(\Lambda^t \otimes \Lambda^t, d_{1, \infty})$ is a compete metric space, where $d_{1, \infty}((\gamma_1, \eta_1), (\gamma_2, \eta_2)) = d_{\infty}(\gamma_1, \eta_1) + d_{\infty}(\gamma_2, \eta_2)$ for all $(t, (\gamma_1, \eta_1)), (s, (\eta_s, \eta_s^2)) \in [0, T] \times (\Lambda \otimes \Lambda)$. Similar to Lemma 2.13 we have the following.

**Lemma 2.14.** Let $t \in [0, T]$ be fixed and let $f : \Lambda^t \otimes \Lambda^t \rightarrow \mathbb{R}$ be an upper semicontinuous functional bounded from above. Suppose that $\rho$ is a gauge-type function on $(\Lambda^t \otimes \Lambda^t, d_{1, \infty})$ and $\{\delta_i\}_{i \geq 0}$ is a sequence of positive number, and suppose that $\varepsilon > 0$ and $(t_0, (\gamma_{t_0}, \eta_{t_0})) \in [t, T] \times (\Lambda^t \otimes \Lambda^t)$ satisfy

$$f(\gamma_{t_0}, \eta_{t_0}) \geq \sup_{(s, (\gamma_s, \eta_s)) \in [t, T] \times (\Lambda^t \otimes \Lambda^t)} f(\gamma_s, \eta_s) - \varepsilon.$$

Then there exist $(\tilde{t}, (\gamma_{\tilde{t}}, \eta_{\tilde{t}})) \in [t, T] \times (\Lambda^t \otimes \Lambda^t)$ and a sequence $\{(t_i, (\gamma_{t_i}, \eta_{t_i}))\}_{i \geq 1} \subset [t_0, T] \times (\Lambda^t \otimes \Lambda^t)$ such that
(i) \(\rho((\gamma^0_i, \eta^0_i), (\hat{\gamma}_i, \hat{\eta}_i)) \leq \frac{\varepsilon}{\delta} \), \(\rho((\gamma^i, \eta^i), (\hat{\gamma}_i, \hat{\eta}_i)) \leq \frac{\varepsilon}{\delta} \), and \(t_i \uparrow \hat{t} \) as \(i \to \infty\).

(ii) \(f(\gamma^i_i, \eta^i_i) - \sum_{i=0}^{\infty} \delta_i \rho((\gamma^i_i, \eta^i_i), (\hat{\gamma}_i, \hat{\eta}_i)) \geq f(\gamma^0_i, \eta^0_i)\), and

(iii) \(f(\gamma_s, \eta_s) - \sum_{i=0}^{\infty} \delta_i \rho((\gamma^i_i, \eta^i_i), (\gamma_s, \eta_s)) < f(\gamma^i_i, \eta^i_i) - \sum_{i=0}^{\infty} \delta_i \rho((\gamma^i_i, \eta^i_i), (\hat{\gamma}_i, \hat{\eta}_i))\) for all \((s, (\gamma_s, \eta_s)) \in [\hat{t}, T] \times (\Lambda^f \otimes \Lambda^i) \setminus \{(\hat{t}, (\hat{\gamma}_i, \hat{\eta}_i))\}.

3 Smooth gauge-type functions.

In this section we introduce the functionals \(\Upsilon^{m,M}\), which are the key to proving the uniqueness and stability of viscosity solutions.

For every \(m \in \mathbb{N}^+\), define \(S_m : \hat{A} \times \hat{A} \to \mathbb{R}\) by, for every \((t, \gamma_t), (s, \eta_s) \in [0, T] \times \hat{A}\),

\[
S_m(\gamma_t, \eta_s) = \begin{cases} \frac{\||\gamma_t - \eta_s\|^{2m} - |\gamma(t) - \eta(s)|^{2m}}{\||\gamma_t - \eta_s\|^{2m}}, & \||\gamma_t - \eta_s\| \neq 0; \\ 0, & \||\gamma_t - \eta_s\| = 0. \end{cases} \quad (3.1)
\]

We recall that \(\||\gamma_t - \eta_s\| = 0 = ||\gamma_t, \eta_s - \eta_s, \eta_s||\). For every \(m \in \mathbb{N}^+\) and \(M \in \mathbb{R}\), define \(\Upsilon^{m,M}\) and \(\Upsilon^{m,M}\) by

\[
\Upsilon^{m,M}(\gamma_t, \eta_s) := S_m(\gamma_t, \eta_s) + M|\gamma(t) - \eta(s)|^{2m}, \quad (t, \gamma_t), (s, \eta_s) \in [0, T] \times \hat{A}, \quad (3.2)
\]

and

\[
\Upsilon^{m,M}(\gamma_t, \eta_s) := \Upsilon^{m,M}(\gamma_t, \eta_s) + |s - t|^2, \quad (t, \gamma_t), (s, \eta_s) \in [0, T] \times \hat{A}. \quad (3.3)
\]

For simplicity, we let \(\Upsilon^{m,M}(\gamma_t, \eta_s)\) denote \(\Upsilon^{m,M}(\gamma_t, \eta_s)\) when \(\eta_s(l) \equiv \mathbf{0}\) for all \(l \in [0, t]\). It is clear that \(\Upsilon^{m,M}(\gamma_t, \eta_s) = \Upsilon^{m,M}(\gamma_t - \eta_s)\) for all \(\gamma_t, \eta_s \in \hat{A}\). We also let \(S\), \(\Upsilon\) and \(\Upsilon\) denote \(S_3\), \(\Upsilon^{3,3}\) and \(\Upsilon^{3,3}\), respectively.

Now we study the regularity of \(\Upsilon^{m,M}\) in the sense of horizontal/vertical derivatives.

**Lemma 3.1.** For every fixed \(m \in \mathbb{N}^+\), \(M \in \mathbb{R}\) and \((\hat{t}, \hat{a}_i) \in [0, T] \times \hat{A}\), define \(\Upsilon^{m,M}_{\hat{a}_i} : \hat{A} \to \mathbb{R}\) by

\[
\Upsilon^{m,M}_{\hat{a}_i}(\gamma_t) := \Upsilon^{m,M}(\gamma_t, \hat{a}_i), \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{A}. \quad (3.4)
\]

Then, for all \((t, \gamma_t) \in [\hat{t}, T] \times \hat{A}\), \(\partial_t \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t), \partial_x \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t), \partial_{xx} \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t)\) exist and

\[
\partial_t \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t) = 0; \quad (3.4)
\]

\[
|\partial_x \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t)| \leq 2m(3 + |M - 3|)|\gamma(t) - \hat{a}_i(t)|^{2m-1}; \quad (3.5)
\]

\[
|\partial_{xx} \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t)| \leq 2m[3(6m - 1) + (2m - 1)|M - 3|]|\gamma(t) - \hat{a}_i(t)|^{2m-2}. \quad (3.6)
\]

If we also assume \(m \geq 2\), \(\Upsilon^{m,M}_{\hat{a}_i}(\cdot) \in C^{1,2}_p(\hat{A})\).

**Proof.** First, by the definition of \(\Upsilon^{m,M}_{\hat{a}_i}\), it is clear that \(\Upsilon^{m,M}_{\hat{a}_i} \in C^0(\hat{A})\) and \(\partial_t \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t) = 0\) for \((t, \gamma_t) \in [\hat{t}, T] \times \hat{A}\). Second, we consider \(\partial_x \Upsilon^{m,M}_{\hat{a}_i}(\gamma_t)\). Define \(S^{\hat{a}_i}_m : \hat{A} \to \mathbb{R}\) by

\[
S^{\hat{a}_i}_m(\gamma_t) := S_m(\gamma_t, \hat{a}_i), \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{A},
\]
and define \( g : \hat{\Lambda}^i \to \mathbb{R} \)

\[
g(\gamma_t) := |\gamma_t(t) - a_i(\hat{t})|^{2m}, \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}^i.
\]

It is clear that

\[
\Upsilon_{a_i}^{M}(\gamma_t) = S_m^{a_i}(\gamma_t) + Mg(\gamma_t), \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}^i.
\]

Clearly, if \( \hat{t} = 0 \),

\[
\partial_x S_m^{a_i}(\gamma_0) = 0, \quad \gamma_0 \in \hat{\Lambda}_0.
\]  

(3.7)

For every \( (t, \gamma_t) \in (0, T] \times \hat{\Lambda} \), let \( ||\gamma_t||_{\hat{t}}^{2m} = \sup_{0 \leq s < t} |\gamma(t)|^{2m} \) and \( (\gamma_t)_i(t) = \langle \gamma(t), e_i \rangle, \quad i = 1, 2, \ldots, d \). Then, for every \( t \in [\hat{t}, T] \) and \( t > 0 \), if \( |\gamma_t(t) - a_i(\hat{t})| < ||\gamma_t - a_i||_{\hat{t}}^{0} \),

\[
\partial_x S_m^{a_i}(\gamma_t) = \lim_{h \to 0} \frac{S_m^{a_i}(\gamma_t) - S_m^{a_i}(\gamma_t)}{h} = \lim_{h \to 0} \frac{(||\gamma_t - a_i||_{\hat{t}}^{2m} - |\gamma_t(t) + he_i - a_i(\hat{t})|^{2m})^3 - (||\gamma_t - a_i||_{\hat{t}}^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^3}{h ||\gamma_t - a_i||_{\hat{t}}^{4m}}
\]

\[
= -\frac{6m(||\gamma_t - a_i||_{\hat{t}}^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^2|\gamma_t(t) - a_i(\hat{t})|^{2m - 2}((\gamma_t)_i(t) - (a_i)_i(\hat{t}))}{||\gamma_t - a_i||_{\hat{t}}^{4m}};
\]  

(3.8)

if \( |\gamma_t(t) - a_i(\hat{t})| > ||\gamma_t - a_i||_{\hat{t}}^{0} \),

\[
\partial_x S_m^{a_i}(\gamma_t) = 0; \tag{3.9}
\]

if \( |\gamma_t(t) - a_i(\hat{t})| = ||\gamma_t - a_i||_{\hat{t}}^{0} \neq 0 \), since

\[
||\gamma_t^{he_i} - a_i||_{\hat{t}}^{2m} - |\gamma_t(t) + he_i - a_i(\hat{t})|^{2m} = \begin{cases} 0, & |\gamma_t(t) + he_i - a_i(\hat{t})| \geq |\gamma_t(t) - a_i(\hat{t})|; \\ |\gamma_t(t) - a_i(\hat{t})|^{2m} - |\gamma_t(t) + he_i - a_i(\hat{t})|^{2m}, & |\gamma_t(t) + he_i - a_i(\hat{t})| < |\gamma_t(t) - a_i(\hat{t})|, \end{cases}
\]  

(3.10)

we have

\[
0 \leq \lim_{h \to 0} \frac{|S_m^{a_i}(\gamma_t^{he_i}) - S_m^{a_i}(\gamma_t)|}{|h|} \leq \lim_{h \to 0} \frac{||\gamma_t(t) - a_i(\hat{t})||^{2m - 2} - |\gamma_t(t) + he_i - a_i(\hat{t})||^{2m - 3} |\gamma(t) - a_i(\hat{t})|^{2m - 2}}{|h| \times ||\gamma_t^{he_i} - a_i||_{\hat{t}}^{4m}} = 0; \tag{3.11}
\]

if \( |\gamma_t(t) - a_i(\hat{t})| = ||\gamma_t - a_i||_{\hat{t}}^{0} = 0 \),

\[
\partial_x S_m^{a_i}(\gamma_t) = 0. \tag{3.12}
\]

Notice that

\[
\partial_x g(\gamma_t) = 2m|\gamma_t(t) - a_i(\hat{t})|^{2m - 2}((\gamma_t)_i(t) - (a_i)_i(\hat{t})), \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}^i. \tag{3.13}
\]

From (3.7), (3.8), (3.9), (3.11), (3.12) and (3.13) we obtain that, for all \((t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}^i\),

\[
\partial_x \Upsilon_{a_i}^{M}(\gamma_t) = \partial_x S_m^{a_i}(\gamma_t) + Mg(\gamma_t)
\]

\[
= 6m ||\gamma_t - a_i||_{\hat{t}}^{4m} - (||\gamma_t - a_i||_{\hat{t}}^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^2 |\gamma_t(t) - a_i(\hat{t})|^{2m - 2}
\]
From (3.14) it follows that

\[ |\partial_x \Upsilon_{a_i}^{m,M}(\gamma_t)| \leq 6m|\gamma_t(t) - a_i(\hat{t})|^{2m-1} + 2m|M - 3||\gamma_t(t) - a_i(\hat{t})|^{2m-1} = 2m(3 + |M - 3|)|\gamma_t(t) - a_i(\hat{t})|^{2m-1}. \]

That is (3.5).

We now consider \( \partial_{xx} \Upsilon_{a_i}^{m,M} \). Clearly, if \( \hat{t} = 0 \),

\[ \partial_{xx} S_m^a(\gamma_0) = 0, \quad \gamma_0 \in \Lambda_0. \] (3.15)

For every \((t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda} \) and \( t > 0 \), if \(|\gamma_t(t) - a_i(\hat{t})| < ||\gamma_t - a_i||_0^{-} \),

\[
\partial_{xx} S_m^a(\gamma_t) = \lim_{h \to 0} \left[ \frac{-6m(||\gamma_t - a_i||_0^{2m} - |\gamma_t(t) + he_j - a_i(\hat{t})|^{2m})^2|\gamma_t(t) + he_j - a_i(\hat{t})|^{2m-2}}{h||\gamma_t - a_i||_0^{4m}} \right. \\
\times ((\gamma_t)_i(t) - (a_i)_i(\hat{t}) + h1_{\{i=j\}}) + \frac{6m(||\gamma_t - a_i||_0^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^2|\gamma_t(t) - a_i(\hat{t})|^{2m-2}((\gamma_t)_i(t) - (a_i)_i(\hat{t}))}{h||\gamma_t - a_i||_0^{4m}} \\
\left. \times (\gamma_t)_j(t) - (a_j)_j(\hat{t}) - \frac{12m(m - 1)(||\gamma_t - a_i||_0^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^2|\gamma_t(t) - a_i(\hat{t})|^{2m-4}}{||\gamma_t - a_i||_0^{4m}} \\
\times ((\gamma_t)_i(t) - (a_i)_i(\hat{t}))((\gamma_t)_j(t) - (a_j)_j(\hat{t}))1_{\{m>1\}} - \frac{6m(||\gamma_t - a_i||_0^{2m} - |\gamma_t(t) - a_i(\hat{t})|^{2m})^2|\gamma_t(t) - a_i(\hat{t})|^{2m-2}1_{\{i=j\}}}{||\gamma_t - a_i||_0^{4m}} \right].
\] (3.16)

if \(|\gamma_t(t) - a_i(\hat{t})| > ||\gamma_t - a_i||_0^{-} \),

\[ \partial_{xx} S_m^a(\gamma_t) = 0; \] (3.17)

if \(|\gamma_t(t) - a_i(\hat{t})| = ||\gamma_t - a_i||_0^{-} \neq 0 \), by (3.10), we have

\[ 0 \leq \lim_{h \to 0} \frac{|\partial_{xx} S_m^a(\gamma_t, he_j) - \partial_{xx} S_m^a(\gamma_t)|}{h} \leq \lim_{h \to 0} \frac{6m(|\gamma_t(t) - a_i(\hat{t})|^{2m} - |\gamma_t(t) - a_i(\hat{t}) + he_j|^{2m})^2|\gamma_t(t) - a_i(\hat{t}) + he_j|^{2m-2}}{h \times ||\gamma_t - a_i||_0^{4m}} \\
\times ((\gamma_t)_i(t) - (a_i)_i(\hat{t}) + h1_{\{i=j\}} = 0; \] (3.18)

if \(|\gamma_t(t) - a_i(\hat{t})| = ||\gamma_t - a_i||_0^{-} = 0 \),

\[ \partial_{xx} S_m^a(\gamma_t) = 0. \] (3.19)

Notice that

\[ \partial_{xx} g(\gamma_t) = 4m(m - 1)|\gamma_t(t) - a_i(\hat{t})|^{2m-4}(|\gamma_t(t) - a_i(\hat{t})|(|\gamma_t(t) - a_i(\hat{t})|)|_{\{m>1\}} \]
Then, from (3.4), (3.5) and (3.6) we have
\[ \partial \frac{\partial Y_{m}^{M} (\gamma_t)}{\partial x} (\gamma_t) = \partial \frac{\partial S_{m}^{M} (\gamma_t)}{\partial x} (\gamma_t) + M \partial \frac{\partial g (\gamma_t)}{\partial x} (\gamma_t) \]

and
\[ \frac{\partial}{\partial t} Y_{m}^{M} (\gamma_t) = \frac{\partial}{\partial t} S_{m}^{M} (\gamma_t) + \frac{\partial}{\partial t} g (\gamma_t) \]

Combining (3.15), (3.16), (3.17), (3.18), (3.19) and (3.20), we obtain, for all \((t, \gamma_t) \in \hat{t}, T \times \hat{A}^i\),
\[ \partial \frac{\partial Y_{m}^{M} (\gamma_t)}{\partial x} (\gamma_t) - a_i (\hat{t}) |2m - 2| I, \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{A}^i. \] (3.20)

From (3.21) it follows that
\[ |\partial \frac{\partial Y_{m}^{M} (\gamma_t)}{\partial x} (\gamma_t)| \leq 2(2m^2 + 12m(m - 1) + 6) |\gamma_t| (t, \gamma_t) - a_i (\hat{t}) |2m - 2| I \]
\[ + 2m |M - 3| (1 + 2(m - 1)) |\gamma_t| (t, \gamma_t) - a_i (\hat{t}) |2m - 2| I \]
\[ = 2m [6(6m - 1) + (2m - 1)|M - 3| |\gamma_t| (t, \gamma_t) - a_i (\hat{t}) |2m - 2| I. \] (3.22)

That is (3.6). By (3.14), it is clear that \(\partial x, Y_{m}^{M} (\gamma_t) \in C^0 (\hat{A}^i)\) for all \(i = 1, 2, \ldots, d\). By (3.21), it is clear that \(\partial x, Y_{m}^{M} (\gamma_t) \in C^0 (\hat{A}^i)\) for all \(i, j = 1, 2, \ldots, d\) and \(m \geq 2\). Notice that
\[ |\gamma_t| (t, \gamma_t) \in [\hat{t}, T] \times \hat{A}^i. \] (3.23)

Then, from (3.4), (3.5) and (3.6) we have \(\gamma_{m, M} (\cdot) \in C_p^{1,2} (\hat{A}^i)\) for \(m \geq 2\). The proof is now complete. \(\square\)

For every \(m \in \mathbb{N}^+\) and \(M \in \mathbb{R}\), define \(\gamma_{m, M} (\cdot) : (\Lambda \otimes \Lambda)^2 \rightarrow \mathbb{R}\) as follows: for every \((\gamma_t (\cdot), (\eta_s, \eta_s')) \in \Lambda \otimes \Lambda^i\),
\[ \gamma_{m, M} (\gamma_t (\cdot), (\eta_s, \eta_s')) := \gamma_{m, M} (\gamma_t, \eta_s) + \gamma_{m, M} (\gamma_t, \eta_s') + |t - s|, \]
We now collect some properties of \(\gamma_{m, M}\), \(\gamma_{m, M} (\cdot)\), and \(\gamma_{m, M} (\cdot)\).

**Lemma 3.2.** For every \(m \in \mathbb{N}^+\), \(M \in \mathbb{R}\),
\[ |\gamma_t| (t, \gamma_t) \in [0, T] \times \hat{A}^i. \] (3.23)

If we also assume \(M \geq 3\), then, for every \(t \in [0, T]\), \(\gamma_{m, M} (\cdot)\) (resp., \(\gamma_{m, M} (\cdot)\)) is a gauge-type function on complete metric space \((\Lambda^t, d_{\infty})\) (resp., \((\Lambda^t \otimes \Lambda^i, d_{\infty})\)).

**Proof.** If \(|\gamma_t| = 0\), it is clear that (3.23) holds. Then we may assume that \(|\gamma_t| = 0\). Letting \(\alpha := |\gamma_t| (t, \gamma_t) (2m\) and \(M = 3\), we have
\[ \gamma_{m, M} (\gamma_t) = \frac{(|\gamma_t| (t, \gamma_t) (2m\}^3 + 3|\gamma_t| (t, \gamma_t) (2m := f (\alpha) = \left(\frac{|\gamma_t| (t, \gamma_t) (2m - \alpha\}^3 + 3\alpha. \]

By
\[ f'(\alpha) = -3 \frac{||\gamma_t||_0^{2m} - \alpha^2}{||\gamma_t||_0^{4m}} + 3 \geq 0, \quad \text{for all } 0 \leq \alpha \leq ||\gamma_t||_0^{2m}, \] (3.24)
we get that
\[ ||\gamma_t||_0^{2m} = f(0) \leq \Upsilon^{m,3}(\gamma_t) = f(\alpha) \leq f(||\gamma_t||_0^{2m}) = 3||\gamma_t||_0^{2m}, \quad (t, \gamma_t) \in [0, T] \times \Lambda. \]

Notice that \( \Upsilon^{m,3}(\gamma_t) = \Upsilon^{m,3}(\gamma_t) + (M - 3)||\gamma_t(t)||^{2m} \), we get (3.23).

Let \( M \geq 3 \). It follows from (3.23) that, for every \((t, (\gamma_t, \gamma'_t)), (s, (\eta_s, \eta'_s)) \in [0, T] \times (\Lambda \otimes \Lambda), \)
\[ \Upsilon^{m,M}(\gamma_t, \eta_s) = \Upsilon^{m,M}(\gamma_t, \eta_s - \eta_s, \eta_s) + |s - t|^2 \geq ||\gamma_t - \eta_s||_0^{2m} + |s - t|^2, \] (3.25)
\[ \Upsilon^{m,M,2}(\gamma_t, \gamma'_t, (\eta_s, \eta'_s)) = \Upsilon^{m,M}(\gamma_t, \eta_s) + \Upsilon^{m,M}(\gamma'_t, \eta'_s) + |t - s|^2 \geq ||\gamma_t - \eta_s||_0^{2m} + ||\gamma'_t - \eta'_s||_0^{2m} + |s - t|^2. \] (3.26)

Recalling that \( d_{\infty}(\gamma_t, \eta_s) = |t - s| + ||\gamma_t - \eta_s||_0 \) and \( d_{1,\infty}((\gamma_t, \gamma'_t), (\eta_s, \eta'_s)) = d_{\infty}(\gamma_t, \eta_s) + d_{\infty}(\gamma'_t, \eta'_s), \)
we get that \( \Upsilon^{m,M} \) (resp., \( \Upsilon^{m,M,2} \)) is a gauge-type function on complete metric space \((\Lambda^t, d_{\infty})\) (resp., \((\Lambda^t \otimes \Lambda^t, d_{1,\infty})\)). □

**Remark 3.3.** For every fixed \((\hat{t}, a_t) \in [0, T] \times \hat{\Lambda}_t, \) since \( ||\gamma_t - a_t||_0^6 \) does not belong to \( C_p^{1,2}(\hat{\Lambda}^t) \), it cannot appear as an auxiliary functional in the proof of the uniqueness and stability of viscosity solutions. However, by the above Lemmas 2.7 and 2.8 we can replace \( ||\gamma_t - a_t||_0^6 \) with the equivalent functional \( \Upsilon(\gamma_t, a_t). \)

In the proof of uniqueness of viscosity solutions, we also need the following lemma.

**Lemma 3.4.** For \( m \in \mathbb{N}^+ \) and \( M \geq 3, \) we have
\[ (\Upsilon^{m,M}(\gamma_t + \eta_t))^\frac{1}{2m} \leq (\Upsilon^{m,M}(\gamma_t))^\frac{1}{2m} + (\Upsilon^{m,M}(\eta_t))^\frac{1}{2m}, \quad (t, \gamma_t, \eta_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda}. \] (3.27)

**Proof.** If one of \( ||\gamma_t||_0, ||\eta_t||_0 \) and \( ||\gamma_t + \eta_t||_0 \) is equal to 0, it is clear that (3.27) holds. Then we may assume that all of \( ||\gamma_t||_0, ||\eta_t||_0 \) and \( ||\gamma_t + \eta_t||_0 \) are not equal to 0. By the definition of \( \Upsilon^{m,M} \), we get, for every \((t, \gamma_t, \eta_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda}, \)
\[ \Upsilon^{m,M}(\gamma_t + \eta_t) = \frac{||\gamma_t + \eta_t||_0^{2m} - ||\gamma_t(t) + \eta_t(t)||_0^{2m}^2}{||\gamma_t + \eta_t||_0^{4m}} + M|\gamma_t(t) + \eta_t(t)|^{2m} \]
\[ = ||\gamma_t + \eta_t||_0^{2m} - \frac{||\gamma_t(t) + \eta_t(t)||_0^{6m}}{||\gamma_t + \eta_t||_0^{4m}} + 3 \frac{||\gamma_t(t) + \eta_t(t)||_0^{4m}}{||\gamma_t + \eta_t||_0^{2m}} + (M - 3)|\gamma_t(t) + \eta_t(t)|^{2m}. \]

Letting \( x := ||\gamma_t + \eta_t||_0^{2m} \) and \( y := ||\gamma_t(t) + \eta_t(t)||_0^{2m}, \) we have
\[ \Upsilon^{m,M}(\gamma_t + \eta_t) = f(x, y) := x - \frac{y^3}{x^2} + 3 \frac{y^2}{x} + (M - 3)y. \]

By
\[ f_x(x, y) = 1 + 2 \left( \frac{y}{x} \right)^3 - 3 \left( \frac{y}{x} \right)^2 = \left( \frac{2y}{x} + 1 \right) \left( \frac{y}{x} - 1 \right)^2 \geq 0, \quad 0 \leq y \leq x, \quad x > 0, \]
\[ f_y(x, y) = -3 \frac{y^2}{x^2} + 6 \frac{y}{x} + (M - 3) \geq 0, \quad 0 \leq y \leq x, \quad x > 0, \]
For the following Lemma 3.5, we have that
\[ g(\gamma_t + \eta_t) = \frac{1}{m} - (\frac{a\alpha + b\beta}{\alpha + \beta}) - \alpha g(a) - \beta g(b) \]

where
\[ g(x) = \left(1 - x^6m + 3x^4m + (M - 3)x^{2m}\right)^{\frac{1}{2m}}, \quad x \in [0, 1]. \] (3.28)

By the following Lemma 3.5, we have that \( g \) is a convex function on \([0, 1]\), then
\[
\left(\Upsilon^m(x) + \eta_t\right) \frac{1}{m} - \left(\Upsilon^m(x) + \eta_t\right) \frac{1}{m} - \left(\Upsilon^m(x) + \eta_t\right) \frac{1}{m} \leq (\alpha + \beta)\left(g\left(\frac{a\alpha + b\beta}{\alpha + \beta}\right) - \alpha g(a) - \beta g(b)\right)
\]

To complete the proof, it remains to state and prove the following lemma.

**Lemma 3.5.** For \( m \in \mathbb{N}^+ \) and \( M \geq 3 \), the function \( g \) defined by (3.28) is a convex function on \([0, 1]\).

**Proof.** By the definition of \( g \), for all \( x \in [0, 1] \),
\[ g'(x) = \frac{1}{2m}g^{1-2m}(x)(-6mx^6m^{-1} + 12mx^{4m-1} + 2m(M - 3)x^{2m-1}), \]
and
\[ g''(x) = \frac{1}{2m}g^{1-2m}(x)(-6m(6m - 1)x^{6m-2} + 12m(4m - 1)x^{4m-2} + 2m(2m - 1)(M - 3)x^{2m-2}) - \frac{1}{2m}(1 - \frac{1}{2m})g^{1-4m}(x)(-6mx^{6m-1} + 12ma^{4m-1} + 2m(M - 3)x^{2m-1})^2 \]

\[ = 3g^{1-4m}(x)x^{4m-2}(2x^6m - (2m + 7)x^{6m} + 6x^{4m} - (6m - 1)x^{2m} + 8m - 2) + g^{1-4m}(x)x^{2m-2}(M - 3)(-8m + 2)x^{6m} + (6m + 3)x^{4m} + 2m - 1). \]
Notice that
\[
2x^{8m} - (2m + 7)x^{6m} + 6x^{4m} - (6m - 1)x^{2m} + 8m - 2 \\
\geq 2x^{8m} - (2m + 7)x^{6m} + 6x^{4m} - (6m - 1)x^{2m} + (2m - 1)x^{4m} + 6m - 1 \\
= x^{4m}(1 - x^{2m})(2m + 5 - 2x^{2m}) + (6m - 1)(1 - x^{2m}) \geq 0, \quad x \in [0, 1],
\]
and
\[
-(8m + 2)x^{6m} + (6m + 3)x^{4m} + 2m - 1 \geq -(8m + 2)x^{6m} + (6m + 3)x^{4m} + (2m - 1)x^{4m} \\
= (8m + 2)x^{4m}(1 - x^{2m}) \geq 0, \quad x \in [0, 1],
\]
we get that \(g''(x) \geq 0\) for all \(x \in [0, 1]\) and the function \(g\) is a convex function on \([0, 1]\). The proof is now complete. \(\square\)

4 Viscosity solutions to PHJB equations: Existence theorem.

In this section, we consider the second order path-dependent Hamilton-Jacobi-Bellman (PHJB) equation \([1.5]\). As usual, we start with classical solutions.

**Definition 4.1.** (Classical solution) A functional \(v \in C_p^{1,2}(\Lambda)\) is called a classical solution (resp., subsolution, supersolution) to the PHJB equation \([1.5]\) if the terminal condition, \(v(\gamma_T) = (\text{resp.}, \leq, \geq)\phi(\gamma_T)\) for all \(\gamma_T \in \Lambda_T\) is satisfied, and \(Lv(\gamma_t) = (\text{resp.}, \geq, \leq) 0\) for all \((t, \gamma_t) \in [0, T] \times \Lambda\).

We will prove that the value functional \(V\) defined by \([1.4]\) is a viscosity solution of PHJB equation \([1.5]\). We give the following definition for the viscosity solutions. For every \((t, \gamma_t) \in [0, T] \times \Lambda\) and \(w \in USC^0(\Lambda)\), define
\[
\mathcal{A}^+(\gamma_t, w) := \left\{ \varphi \in C_p^{1,2}(\Lambda^t) : 0 = (w - \varphi)(\gamma_t) = \sup_{(s, \eta_s) \in [t, T] \times \Lambda} (w - \varphi)(\eta_s) \right\},
\]
and, for every \((t, \gamma_t) \in [0, T] \times \Lambda\) and \(w \in LSC^0(\Lambda)\), define
\[
\mathcal{A}^-(\gamma_t, w) := \left\{ \varphi \in C_p^{1,2}(\Lambda^t) : 0 = (w - \varphi)(\gamma_t) = \inf_{(s, \eta_s) \in [t, T] \times \Lambda} (w - \varphi)(\eta_s) \right\}.
\]

**Definition 4.2.** \(w \in USC^0(\Lambda)\) (resp., \(w \in LSC^0(\Lambda)\)) is called a viscosity subsolution (resp., supersolution) to \([1.5]\) if the terminal condition, \(w(\gamma_T) \leq \phi(\gamma_T)\) (resp., \(w(\gamma_T) \geq \phi(\gamma_T)\)) for all \(\gamma_T \in \Lambda_T\) is satisfied, and whenever \(\varphi \in \mathcal{A}^+(\gamma_s, w)\) (resp., \(\varphi \in \mathcal{A}^-(\gamma_s, w)\)) with \((s, \gamma_s) \in [0, T] \times \Lambda\), we have
\[
L\varphi(\gamma_s) \geq 0 \quad (\text{resp.}, \ L\varphi(\gamma_s) \leq 0).
\]
\(w \in C^0(\Lambda)\) is said to be a viscosity solution to equation \([1.5]\) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 4.3.** Assume that the coefficients \(b(\gamma_t, u) = \overline{b}(t, \gamma_t(t), u), \sigma(\gamma_t, u) = \overline{\sigma}(t, \gamma_t(t), u), q(\gamma_t, y, z, u) = \overline{q}(t, \gamma_t(t), y, z, u), \phi(\eta_T) = \overline{\phi}(\eta_T)\) for all \((t, \gamma_t, y, z, u) \in [0, T] \times \Lambda \times \mathbb{R} \times \mathbb{R}^n \times U\) and \(\eta_T \in \Lambda_T\). Then there exists a function \(V : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) such that \(V(\gamma_t) = \overline{V}(t, \gamma_t(t))\) for all \((t, \gamma_t) \in [0, T] \times \Lambda\), and PHJB equation \([1.5]\) reduces to the following HJB equation:
\[
\begin{cases}
\mathcal{V}_{t} + \mathcal{H}(t, x, \nabla V(t, x), \nabla_x^2 V(t, x)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d, \\
\nabla V(T, x) = \overline{\phi}(x), \quad x \in \mathbb{R}^d;
\end{cases}
\]
where
\[
\Pi(t, x, r, p, \iota) = \sup_{u \in U} [p \cdot \bar{b}(t, x, u)] + \frac{1}{2} \text{tr} [\sigma(t, x, u)\sigma^T(t, x, u)]
+ \varpi(t, x, r, \sigma^T(t, x, u)p, \iota), \quad (t, x, r, p, \iota) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d).
\]
Here and in the sequel, \(\nabla_x\) and \(\nabla_x^2\) denote the standard first and second order derivatives with respect to \(x\). However, slightly different from the HJB literature, \(\nabla_t^+\) denotes the right time-derivative of \(\nabla\).

The following theorem show that our definition of viscosity solutions to PHJB equation (1.5) is a natural extension of classical viscosity solution to HJB equation (4.1).

**Theorem 4.4.** Consider the setting in Remark 4.3. Assume that \(V\) is a viscosity solution (resp., subsolution, supersolution) of PHJB equation (1.5) in the sense of Definition 4.2. Then \(\nabla\) is a viscosity subsolution of equation (1.5), it follows that, for every \(x \in \mathbb{R}^d\),
\[
V(T, x) = V(\gamma_T) \leq \phi(\gamma_T) = \phi(x),
\]
where \(\gamma_T \in \Lambda\) with \(\gamma_T(T) = x\).
Next, let \(\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)\) and \((t, x) \in [0, T] \times \mathbb{R}^d\) such that
\[
0 = (\nabla - \varphi)(t, x) = \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} (\nabla - \varphi)(s, y).
\]
We can modify \(\varphi\) such that \(\varphi, \varphi_t, \nabla_x \varphi, \nabla_x^2 \varphi\) and \(\nabla_x^2 \varphi\) grow in a polynomial way. Here and in the sequel, \(\varphi_t\) denotes the time-derivative of \(\varphi\). Define \(\varphi : \Lambda \to \mathbb{R}\) by
\[
\varphi(\gamma_s) = \varphi(s, \gamma_s(s)), \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda},
\]
and define \(\gamma_t \in \Lambda_t\) by
\[
\gamma_t(s) = x, \quad s \in [0, t].
\]
It is clear that,
\[
\partial_x \varphi(\gamma_s) = \nabla_x \varphi(s, \gamma_s(s)), \quad \partial_{xx} \varphi(\gamma_s) = \nabla_x^2 \varphi(s, \gamma_s(s)), \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda},
\]
\[
\partial_t \varphi(\gamma_s) = \varphi_t(s, \gamma_s(s)), \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda},
\]
and
\[
\partial_t \varphi(\gamma_T) = \lim_{s<T,\gamma_T|_{[0,s]}\in \mathbb{R}^d} \partial_t \varphi(\gamma_T|_{[0,s]}) = \lim_{s<T,\gamma_T|_{[0,s]}\in \mathbb{R}^d} \varphi_t(s, \gamma_T(s)) = \varphi_t(s, \gamma_T(s)), \quad \gamma_T \in \hat{\Lambda}_T.
\]
Thus we have \(\varphi \in C^{1,2}_p(\Lambda) \subset C^{1,2}_p(\Lambda')\). Moreover, by the definitions of \(V\) and \(\varphi\),
\[
0 = (V - \varphi)(\gamma_t) = (\nabla - \varphi)(t, x) = \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} (\nabla - \varphi)(s, y) = \sup_{(s, \gamma_s) \in [t, T] \times \Lambda} (V - \varphi)(\gamma_s).
\]
Therefore, \(\varphi \in A^+(\gamma_t, V)\) with \((t, \gamma_t) \in [0, T] \times \Lambda\). Since \(V\) is a viscosity subsolution of PHJB equation (1.5), we have
\[
\mathcal{L} \varphi(\gamma_t) \geq 0.
\]
Thus,
\[ \Psi_{t+}(t, x) + \Pi(t, x, \psi(t, x), \nabla_x \psi(t, x), \nabla_x^2 \psi(t, x)) \geq 0. \]

By the arbitrariness of \( \psi \in C^{1,2}([0, T] \times \mathbb{R}^d) \), we see that \( \psi \) is a viscosity subsolution of HJB equation (4.1), and thus completes the proof. \( \square \)

We note that the viceversa of Theorem 4.4 does not hold true. Indeed, let \( \varphi \in A^+(\hat{\gamma}_t, V) \) with \((t, \hat{\gamma}_t) \in [0, T) \times \Lambda \). Since \( \varphi \) belongs only to \( C^p_{\text{loc}}(\Lambda') \) and not to \( C^p_{\text{loc}}(\Lambda) \), we cannot construct \( \psi \in C^{1,2}([0, T] \times \mathbb{R}^d) \) from \( \varphi \) such that \( \psi_t(t, \hat{\gamma}_t(t)) = \partial_t \varphi(\hat{\gamma}_t), \nabla_x \psi(t, \hat{\gamma}_t(t)) = \partial_x \varphi(\hat{\gamma}_t), \nabla_x^2 \psi(t, \hat{\gamma}_t(t)) = \partial_{xx} \varphi(\hat{\gamma}_t) \) and

\[ 0 = (\psi - \varphi)(t, \hat{\gamma}_t(t)) = \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} (\psi - \varphi)(s, y). \]

We are now in a position to give the existence and consistency results for the viscosity solutions.

**Theorem 4.5.** Suppose that Hypothesis 2.6 holds. Then the value functional \( V \) defined by (1.3) is a viscosity solution to equation (1.3).

**Theorem 4.6.** Let Hypothesis 2.6 hold true, \( v \in C^{1,2}_{p}(\Lambda) \). Then \( v \) is a classical solution (resp., subsolution, supersolution) of equation (1.3) if and only if it is a viscosity solution (resp., subsolution, supersolution).

The proof of Theorems 4.5 and 4.6 is rather standard. Moreover, note that a viscosity solution in the sense of [13] is a viscosity solution in our sense, then these results can be implied by [13] directly. However, our conditions are weaker than those in [13]. For the sake of the completeness of the article and the convenience of readers, we give their proof in the appendix B.

We conclude this section with the stability of viscosity solutions.

**Theorem 4.7.** Let \( b, \sigma, q, \phi \) satisfy Hypothesis 2.6 and \( v \in C^0(\Lambda) \) (resp., \( v \in \text{USC}^0(\Lambda) \), \( v \in \text{LSC}^0(\Lambda) \)). Assume

(i) for any \( \varepsilon > 0 \), there exist \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon \) and \( v^\varepsilon \in C^0(\Lambda) \) (resp., \( v^\varepsilon \in \text{USC}^0(\Lambda) \), \( v^\varepsilon \in \text{LSC}^0(\Lambda) \)) such that \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon \) satisfy Hypothesis 2.6 and \( v^\varepsilon \) is a viscosity solution (resp., subsolution, supersolution) of PHJB equation (1.3) with generators \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon \); (ii) as \( \varepsilon \to 0 \), \( (b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon, v^\varepsilon) \) converge to \((b, \sigma, q, \phi, v)\) uniformly in the following sense:

\[
\lim_{\varepsilon \to 0} \sup_{(t, \gamma_t, x, y, u) \in [0, T] \times \Lambda \times \mathbb{R}^d \times U} \sup_{\eta_T \in \Lambda_T} [ |(b^\varepsilon - b) + |\sigma^\varepsilon - \sigma|_2| + |q^\varepsilon - q| + |\phi^\varepsilon - \phi|(|\gamma_T) + |v^\varepsilon - v||\gamma_T|] = 0. \tag{4.2}
\]

Then \( v \) is a viscosity solution (resp., subsolution, supersolution) of PHJB equation (1.3) with generators \( b, \sigma, q, \phi \).

**Proof.** Without loss of generality, we shall only prove the viscosity subsolution property. First, from \( v^\varepsilon \) is a viscosity subsolution of equation (1.3) with generators \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon \), it follows that

\[ v^\varepsilon(\gamma_T) \leq \phi^\varepsilon(\gamma_T), \quad \gamma_T \in \Lambda_T. \]

Letting \( \varepsilon \to 0 \), we have

\[ v(\gamma_T) \leq \phi(\gamma_T), \quad \gamma_T \in \Lambda_T. \]

Next, we let \( \varphi \in A^+(\hat{\gamma}_t, v) \) with \((t, \hat{\gamma}_t) \in [0, T) \times \Lambda \). By (1.2), there exists a constant \( \delta > 0 \) such that for all \( \varepsilon \in (0, \delta) \),

\[ \sup_{(t, \gamma_t) \in [t, T] \times \Lambda} (v^\varepsilon(\gamma_t) - \varphi(\gamma_t)) \leq 1. \]
Denote $\varphi_1(\gamma_t) := \varphi(\gamma_t) + \bar{\Upsilon}(\gamma_t, \dot{\gamma}_t)$ for all $(t, \gamma_t) \in [0, T] \times \Lambda$. By Lemma 3.11 we have $\varphi_1 \in C_{p}^{1,2}(\Lambda^\ell)$. For every $\varepsilon \in (0, \delta)$, it is clear that $v^\varepsilon - \varphi_1$ is an upper semicontinuous functional and bounded from above on $\Lambda^\ell$. Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{2^i}$ for all $i \geq 0$. Since $\bar{\Upsilon}(\cdot, \cdot)$ is a gauge-type function on $(\Lambda^\ell, d_{\infty})$, from Lemma 2.13 it follows that, for every $(t_0, \gamma_{t_0}^0) \in [\hat{t}, T] \times \Lambda^\ell$ satisfying
\[(v^\varepsilon - \varphi_1)(\gamma_{t_0}^0) \geq \sup_{(s, \gamma_s) \in [\hat{t}, T] \times \Lambda^\ell} (v^\varepsilon - \varphi_1)(\gamma_s) - \varepsilon, \quad \text{and} \quad (v^\varepsilon - \varphi_1)(\gamma_{t_0}^0) \geq (v^\varepsilon - \varphi_1)(\gamma_{t_0}^0),\]
there exist $(t_\varepsilon, \gamma_{t_\varepsilon}^\varepsilon) \in [\hat{t}, T] \times \Lambda^\ell$ and a sequence $\{(t_i, \gamma_{t_i}^i)\}_{i \geq 1} \subset [t_0, T] \times \Lambda^\ell$ such that
(i) $\bar{\Upsilon}(\gamma_{t_0}^0, \gamma_{t_\varepsilon}^\varepsilon) \leq \varepsilon$, $\bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon) \leq \frac{1}{2^i}$ and $t_i \uparrow t_\varepsilon$ as $i \to \infty$,
(ii) $(v^\varepsilon - \varphi_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon) \geq (v^\varepsilon - \varphi_1)(\gamma_{t_0}^0)$, and
(iii) $(v^\varepsilon - \varphi_1)(\gamma_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_s) < (v^\varepsilon - \varphi_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon)$ for all $(s, \gamma_s) \in [t_\varepsilon, T] \times \Lambda^\ell \setminus \{(t_\varepsilon, \gamma_{t_\varepsilon}^\varepsilon)\}$.

We claim that
\[d_{\infty}(\gamma_{t_\varepsilon}^\varepsilon, \gamma_{t_\varepsilon}^\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.3}\]
Indeed, if not, by (3.25) and the definition of $d_{\infty}$, we can assume that there exists a constant $\nu_0 > 0$ such that
\[\bar{\Upsilon}(\gamma_{t_\varepsilon}^\varepsilon, \gamma_{t_\varepsilon}^\varepsilon) \geq \nu_0.\]
Thus, by the property (ii) of $(t_\varepsilon, \gamma_{t_\varepsilon}^\varepsilon)$, we obtain that
\[0 = (v - \varphi)(\dot{\gamma}_{t_\varepsilon}) = \lim_{\varepsilon \to 0} (v^\varepsilon - \varphi_1)(\dot{\gamma}_{t_\varepsilon}) \leq \lim_{\varepsilon \to 0} \sup_{\varepsilon \to 0} \left[ (v^\varepsilon - \varphi_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon) \right] \]
\[= \lim_{\varepsilon \to 0} \sup_{\varepsilon \to 0} \left[ (v^\varepsilon - \varphi_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon) \right] \leq \lim_{\varepsilon \to 0} \sup_{\varepsilon \to 0} \left[ (v - \varphi)(\gamma_{t_\varepsilon}^\varepsilon) + (v^\varepsilon - v)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i, \gamma_{t_\varepsilon}^\varepsilon) \right] - \nu_0 \leq (v - \varphi)(\dot{\gamma}_{t_\varepsilon}) - \nu_0 = -\nu_0,
\]
contradicting $\nu_0 > 0$. We notice that, by (3.5), (3.6) and the property (i) of $(t_\varepsilon, \gamma_{t_\varepsilon}^\varepsilon)$, there exists a generic constant $C > 0$ such that
\[2 \sum_{i=0}^{\infty} \frac{1}{2^i} (t_\varepsilon - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\varepsilon}{2^i} \right) \frac{1}{2} \leq C \varepsilon \frac{1}{2};\]

\[|\partial_x \bar{\Upsilon}(\gamma_{t_\varepsilon}^\varepsilon - \dot{\gamma}_{t_\varepsilon}, t_\varepsilon)| \leq C|\dot{\gamma}_{t_\varepsilon}(t_\varepsilon) - \dot{\gamma}_{t_\varepsilon}(t_\varepsilon)|^5; \quad |\partial_{xx} \bar{\Upsilon}(\gamma_{t_\varepsilon}^\varepsilon - \dot{\gamma}_{t_\varepsilon}, t_\varepsilon)| \leq C|\dot{\gamma}_{t_\varepsilon}(t_\varepsilon) - \dot{\gamma}_{t_\varepsilon}(t_\varepsilon)|^4;
\]

\[\left| \partial_x \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i - \dot{\gamma}_{t_\varepsilon}, t_\varepsilon) \right| \leq 18 \sum_{i=0}^{\infty} \frac{1}{2^i} |\dot{\gamma}_{t_i}^i(t_i) - \dot{\gamma}_{t_i}^i(t_\varepsilon)|^5 \leq 18 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\varepsilon}{2^i} \right) \frac{1}{2} \frac{1}{2} \leq C \varepsilon \frac{1}{2};\]

\[\text{and}\]

\[\left| \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{\Upsilon}(\gamma_{t_i}^i - \dot{\gamma}_{t_\varepsilon}, t_\varepsilon) \right| \leq 306 \sum_{i=0}^{\infty} \frac{1}{2^i} |\dot{\gamma}_{t_i}^i(t_i) - \dot{\gamma}_{t_i}^i(t_\varepsilon)|^4 \leq 306 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\varepsilon}{2^i} \right) \frac{1}{2} \frac{1}{2} \leq C \varepsilon \frac{1}{2}.\]
Then for any $\rho > 0$, by (1.2) and (1.3), there exists $\varepsilon > 0$ small enough such that

$$\hat{t} \leq t_\varepsilon < T, \quad 2|t_\varepsilon - \hat{t}| + 2\sum_{i=0}^{\infty} \frac{1}{2^i}(t_\varepsilon - t_i) \leq \frac{\rho}{4},$$

and

$$|\partial_t \varphi(\gamma_{t_\varepsilon}) - \partial_t \varphi(\hat{\gamma}_t)| \leq \frac{\rho}{4}, \quad |I| \leq \frac{\rho}{4}, \quad |II| \leq \frac{\rho}{4},$$

where

$$I = H^\varepsilon(\gamma_{t_\varepsilon}, \rho_\varepsilon, \sigma_\varepsilon, \partial_x \varphi_2(\gamma_{t_\varepsilon}), \partial_xx \varphi_2(\gamma_{t_\varepsilon})) - H((\gamma_{t_{\varepsilon}}, v^\varepsilon(\gamma_{t_{\varepsilon}}), \partial_x \varphi_2(\gamma_{t_{\varepsilon}}), \partial_xx \varphi_2(\gamma_{t_{\varepsilon}}))),$$

$$II = H(\gamma_{t_\varepsilon}, v^\varepsilon(\gamma_{t_\varepsilon}), \partial_x \varphi_2(\gamma_{t_\varepsilon}), \partial_xx \varphi_2(\gamma_{t_\varepsilon})), - H(\hat{\gamma}_t, \varphi(\hat{\gamma}_t), \partial_x \varphi(\hat{\gamma}_t), \partial_xx \varphi(\hat{\gamma}_t)), $$

$$\varphi_2(\gamma_{t_\varepsilon}) = \varphi_1(\gamma_{t_\varepsilon}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Pi(\gamma_{t_i}, \gamma_{t_\varepsilon}),$$

and

$$H^\varepsilon(\gamma_t, r, p, 1) = \sup_{u \in U} \{\langle p, b^\varepsilon(\gamma_t, u) \rangle + \frac{1}{2} \text{tr}[\sigma^\varepsilon(\gamma_t, u) \sigma^\varepsilon T(\gamma_t, u)]$$

$$+ \rho^\varepsilon(\gamma_t, r, \sigma^\varepsilon T(\gamma_t, u), p, u), \quad (t, \gamma_t, r, p, 1) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d).$$

Since $v^\varepsilon$ is a viscosity subsolution of PHJB equation (1.5) with generators $b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, \phi^\varepsilon$, we have

$$\partial_t \varphi(\gamma_{t_\varepsilon}) + H^\varepsilon(\gamma_{t_\varepsilon}, v^\varepsilon(\gamma_{t_\varepsilon}), \partial_x \varphi_2(\gamma_{t_\varepsilon}), \partial_xx \varphi_2(\gamma_{t_\varepsilon})) \geq 0.$$ 

Thus

$$0 \leq \partial_t \varphi(\gamma_{t_\varepsilon}) + 2(t_\varepsilon - \hat{t}) + 2\sum_{i=0}^{\infty} \frac{1}{2^i}(t_\varepsilon - t_i) + H(\hat{\gamma}_t, \varphi(\hat{\gamma}_t), \partial_x \varphi(\hat{\gamma}_t), \partial_xx \varphi(\hat{\gamma}_t)) + I + II \leq \mathcal{L}(\varphi(\hat{\gamma}_t)) + \rho.$$ 

Letting $\rho \downarrow 0$, we show that $\mathcal{L}(\varphi(\hat{\gamma}_t)) \geq 0$. Since $\varphi \in C^{1,2}_{p^\varepsilon}(A^\varepsilon)$ is arbitrary, we see that $\varphi$ is a viscosity subsolution of PHJB equation (1.5) with generators $b, \sigma, q, \phi$, and thus completes the proof.  \( \square \)

5 Viscosity solutions to PHJB equations: Crandall-Ishii lemma.

In this section we extend Crandall-Ishii lemma to the path-dependent case. It is the cornerstone of the theory of viscosity solutions, and is the key result in the comparison proof that will be given in the next section.

**Definition 5.1.** Let $(\tilde{t}, \tilde{x}) \in (0, T) \times \mathbb{R}^d$ and $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function bounded from above. We say $f \in \Phi(\tilde{t}, \tilde{x})$ if there is a constant $r > 0$ such that, for every $L > 0$, there is a constant $C_0 \geq 0$ depending only on $L$ such that, for every function $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that $f(s, y) - \varphi(s, y)$ has a maximum over $[0, T] \times \mathbb{R}^d$ at a point $(t, x) \in (0, T) \times \mathbb{R}^d$, if

$$|t - \tilde{t}| + |x - \tilde{x}| < r, \quad |f(t, x)| + |\nabla_x \varphi(t, x)| + |\nabla^2_x \varphi(t, x)| \leq L,$$

then

$$\varphi_t(t, x) \geq -C_0.$$  \( (5.1) \)
Definition 5.2. Let $\hat{t} \in [0, T)$ be fixed and $w : \Lambda \to \mathbb{R}$ be an upper semicontinuous function bounded from above. Define, for $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\hat{w}^t(t, x) := \sup_{\xi \in \Lambda^t, \hat{t} \in \Lambda} w(\xi), \quad t \in [\hat{t}, T]; \quad \hat{w}^t(t, x) := \hat{w}^t(\hat{t}, x) - (\hat{t} - t)^\frac{1}{2}, \quad t \in [0, \hat{t}).$$

Let $\hat{w}^{t,*}$ be the upper semicontinuous envelope of $\hat{w}^t$ (see Definition D.10), i.e.,

$$\hat{w}^{t,*}(t, x) = \limsup_{(s, y) \in [0, \hat{t}] \times \mathbb{R}^d, (s, y) \to (t, x)} \hat{w}^t(s, y).$$

In what follows, by a modulus of continuity, we mean a continuous function $\rho_1 : [0, \infty) \to [0, \infty)$, with $\rho_1(0) = 0$ and subadditivity: $\rho_1(t + s) \leq \rho_1(t) + \rho_1(s)$, for all $t, s > 0$; by a local modulus of continuity, we mean a continuous function $\rho_1 : [0, \infty) \times [0, \infty) \to [0, \infty)$, with the properties that for each $r \geq 0$, $t \to \rho_1(t, r)$ is a modulus of continuity and $\rho_1$ is non-decreasing in the second variable.

Theorem 5.3. (Crandall-Ishii lemma) Let $w_1, w_2 : \Lambda \to \mathbb{R}$ be upper semicontinuous functionals bounded from above and such that

$$\limsup_{||\gamma_t|| \to \infty} \frac{w_1(\gamma_t)}{||\gamma_t||} < 0; \quad \limsup_{||\gamma_t|| \to \infty} \frac{w_2(\gamma_t)}{||\gamma_t||} < 0. \quad (5.2)$$

Recalling $\Lambda^t \otimes \Lambda^t := \{(\gamma_{s, \eta})|\gamma_s, \eta_t \in \Lambda^t\}$ for all $t \in [0, T]$. Let $\varphi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ be such that

$$w_1(\gamma_t) + w_2(\eta_t) - \varphi(\gamma_t(t), \eta_t(t))$$

has a maximum over $\Lambda^t \otimes \Lambda^t$ at a point $(\hat{t}, \hat{\gamma}_t)$ with $\hat{t} \in (0, T)$. Assume, moreover, $\hat{w}_1^{t,*} \in \Phi(\hat{t}, \hat{\gamma}_t(\hat{t}))$ and $\hat{w}_2^{t,*} \in \Phi(\hat{t}, \hat{\eta}_t(\hat{t}))$, and there exists a local modulus of continuity $\rho_1$ such that, for all $\hat{t} \leq t \leq s \leq T$, $\gamma_t \in \Lambda$,

$$w_1(\gamma_t) - w_1(\gamma_{t,s}) \leq \rho_1(||s - t||, ||\gamma_t||, 0), \quad w_2(\gamma_t) - w_2(\gamma_{t,s}) \leq \rho_1(||s - t||, ||\gamma_t||, 0). \quad (5.3)$$

Then for every $\kappa > 0$, there exist sequences $(t_k, \gamma_{t_k}^k, s_k, \eta_{s_k}^k) \in [\hat{t}, T] \times \Lambda^\kappa$ and sequences of functionals $\varphi_k \in C^{1,2}_p(\Lambda^\kappa), \psi_k \in C^{1,2}_p(\Lambda^\kappa)$ bounded from below such that

$$w_1(\gamma_t) - \varphi_k(\gamma_t)$$

has a strict maximum 0 at $\gamma_{t_k}^k$ over $\Lambda^\kappa$,

$$w_2(\eta_t) - \psi_k(\eta_t)$$

has a strict maximum 0 at $\eta_{s_k}^k$ over $\Lambda^\kappa$, and

$$\left( t_k, \gamma_{t_k}^k(t_k), w_1(\gamma_{t_k}^k), \partial_t \varphi_k(\gamma_{t_k}^k), \partial_x \varphi_k(\gamma_{t_k}^k), \partial_{xx} \varphi_k(\gamma_{t_k}^k) \right)$$

$$k \to \infty (\hat{t}, \hat{\gamma}_t, 0, \hat{\gamma}_t, \nabla_x \varphi(\hat{\gamma}_t(\hat{t}), \hat{\eta}_t(\hat{t})), X) \quad (5.4)$$

$$\left( s_k, \eta_{s_k}^k(s_k), w_2(\eta_{s_k}^k), \partial_t \psi_k(\eta_{s_k}^k), \partial_x \psi_k(\eta_{s_k}^k), \partial_{xx} \psi_k(\eta_{s_k}^k) \right)$$

$$k \to \infty (\hat{t}, \hat{\eta}_t, 0, \hat{\eta}_t, \nabla_x \varphi(\hat{\gamma}_t(\hat{t}), \hat{\eta}_t(\hat{t})), Y) \quad (5.5)$$
where \( b_1 + b_2 = 0 \) and \( X, Y \in \mathcal{S}(\mathbb{R}^d) \) satisfy the following inequality:
\[
-\left(\frac{1}{\kappa} + |A|\right)I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \kappa A^2,
\]
and \( A = \nabla^2_x \varphi(\gamma(t), \hat{\eta}(t)) \). Here \( \nabla^2 \varphi \) denotes the standard second order derivative of \( \varphi \) with respect to the variable \( x = (x_1, x_2) \in \mathbb{R}^{2d} \), and \( \nabla_x x \varphi \) and \( \nabla_x \varphi \) denote the standard first order derivative of \( \varphi \) with respect to the first variable and the second variable, respectively.

**Proof.** By the following Lemma 5.5, we have that
\[
\bar{w}_1(t, x) + \bar{w}_2(t, y) - \varphi(x, y) \quad \text{has a maximum over } [0, T] \times \mathbb{R}^d
\]
Moreover, we have \( \bar{w}_1(t, \gamma(t)) = \bar{w}_1(\tilde{\gamma}(t), \tilde{\eta}(t)) \). Then, by \( \tilde{w}_1 \in \Phi(\tilde{\gamma}(t), \tilde{\eta}(t)) \) and Remark 5.4, the Theorem 8.3 in [8] and Lemma 5.4 of Chapter 4 in [41] can be used to obtain sequences of functions \( \varphi_k, \psi_k \in C^{1,2}([0, T] \times \mathbb{R}^d) \) bounded from below such that \( \bar{w}_1(t, x) - \varphi_k(t, x) \) has a strict maximum 0 at some point \( (t_k, x_k) \in (0, T) \times \mathbb{R}^d \) over \( [0, T] \times \mathbb{R}^d \), \( \bar{w}_2(s, y) - \psi_k(s, y) \) has a strict maximum 0 at some point \( (s_k, y_k) \in (0, T) \times \mathbb{R}^d \) over \( [0, T] \times \mathbb{R}^d \), and such that
\[
(t_k, x_k, \bar{w}_1(t_k, x_k), (\varphi_k)_t(t_k, x_k), \nabla_x \varphi_k(t_k, x_k), \nabla_x^2 \varphi_k(t_k, x_k)) \quad \text{where } b_1 + b_2 = 0 \quad \text{and } \quad \text{Lemma 5.6 is satisfied.}
\]
We claim that we can assume the sequences \{\( t_k \)\}_{k \geq 1} \in \hat{\Lambda} \times [\tilde{\gamma}, T) \) and \{\( s_k \)\}_{k \geq 1} \in \hat{\Lambda} \times [\tilde{\gamma}, T) \). Indeed, if not, for example, there exists \( \gamma \) such that \( t_k < \gamma \) for all \( k \geq 1 \). Since \( \bar{w}_1(t, x) - \varphi_k(t, x) \) has a maximum at \( (t_k, x_k) \) on \( [0, T] \times \mathbb{R}^d \), we obtain that
\[
(\tilde{\varphi})_t(t_k, x_k) = \frac{1}{2}(\tilde{\varphi}_k(t_k, x_k) - \frac{1}{2} \tilde{t} - t_k)^{-\frac{1}{2}} \to \infty, \quad \text{as } k \to \infty,
\]
which contradicts that \( (\tilde{\varphi})_t(t_k, x_k) \to b_1 \in \mathbb{R} \).

Now we consider the functionals, for \( (t, \gamma_t), (s, \eta_s) \in \hat{\Lambda} \times \Lambda \),
\[
\Gamma_k^1(\gamma_t) := w_1(\gamma_t) - \varphi_k(t, \gamma_t(t)), \quad \Gamma_k^2(\eta_s) := w_2(\eta_s) - \psi_k(s, \eta_s(s)).
\]
It is clear that \( \Gamma_k^1, \Gamma_k^2 \) are upper semicontinuous functionals bounded from above on \( \Lambda^i \). Define a sequence of positive numbers \{\( \delta_i \)\}_{i \geq 0} by \( \delta_i = \frac{1}{2^i} \) for all \( i \geq 0 \). Since \( \Upsilon(\cdot, \cdot) \) is a gauge-type function on \( (\Lambda^i, d_{\infty}) \), for every \( k \geq 1 \), applying Lemma 2.13, we have that, for every \( (t_0, \gamma_{t_0}, s_0, \eta_{s_0}) \in (\tilde{\gamma}, T) \times \Lambda^i \), there exist \( (t_{k,j}, \gamma_{t_{k,j}}, s_{k,j}, \eta_{s_{k,j}}) \) and two sequences \{\( (\hat{t}_i, \hat{\gamma}_{t_i}) \)\}_{i \geq 1} \subset [\tilde{\gamma}, T] \times \Lambda^i \), \{\( (s_i, \hat{\eta}_{s_i}) \)\}_{i \geq 1} \subset [s_0, T] \times \Lambda^i \) such that
\[
\Gamma_k^1(\gamma_{t_0}) \geq \sup_{(t, \gamma_t) \in [\tilde{\gamma}, T] \times \Lambda^i} \Gamma_k^1(\gamma_t) - \frac{1}{j}, \quad \Gamma_k^2(\eta_{s_0}) \geq \sup_{(s, \eta_s) \in [s_0, T] \times \Lambda^i} \Gamma_k^2(\eta_s) - \frac{1}{j},
\]
(i) \( \Upsilon(\gamma_{i,\ell}^0, \gamma_{\ell,k}^{k}) \setminus \Upsilon(\eta_{\ell_0}, \eta_{\ell,k}^{k}) \leq \frac{1}{2}, \Upsilon(\gamma_{i,\ell}^0, \gamma_{\ell,k}^{k}) \setminus \Upsilon(\eta_{\ell_0}, \eta_{\ell,k}^{k}) \leq \frac{1}{2} \) and \( \tilde{t}_i \uparrow t_{k,j}, \tilde{s}_i \uparrow s_{k,j} \) as \( i \to \infty \),

(ii) \( \Gamma_{k}^{1} (\gamma_{\ell,k}^{k}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\gamma}_{i,\ell}^{i}, \gamma_{\ell,k}^{k}) \geq \Gamma_{k}^{1} (\eta_{\ell_0}, \eta_{\ell,k}^{k}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\eta}_{i,\ell}^{i}, \eta_{\ell,k}^{k}) \geq \Gamma_{k}^{1} (\eta_{0_{k}}) \), and

(iii) for all \((t, \gamma_{t}) \in [t_{k,j}, T] \times \Lambda_{k,j} \setminus \{(t_{k,j}, \gamma_{k,j})\},

\[
\Gamma_{k}^{1}(\gamma_{t}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\gamma}_{i,t}^{i}, \gamma_{t}) \leq \Gamma_{k}^{1}(\gamma_{t}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\gamma}_{i,t}^{i}, \gamma_{t}),
\]

and for all \((s, \eta_{s}) \in [s_{k,j}, T] \times \Lambda_{k,j} \setminus \{(s_{k,j}, \eta_{k,j})\},

\[
\Gamma_{k}^{2}(\eta_{s}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\eta}_{i,s}^{i}, \eta_{s}) \leq \Gamma_{k}^{2}(\eta_{s}) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\tilde{\eta}_{i,s}^{i}, \eta_{s}).
\]

By the following Lemma 5.6 we have

\[
(t_{k,j}, \gamma_{k,j}^{k}(t_{k,j})) \to (t_{k,j}, x^{k}), (s_{k,j}, \eta_{k,j}^{k}(s_{k,j})) \to (s_{k,j}, y^{k}) \quad \text{as} \quad j \to \infty,
\]

\[
\lim_{j \to \infty} \tilde{w}_{1}(t_{k,j}, \gamma_{k,j}^{k}(t_{k,j})) = \tilde{w}_{1}^{*}(t_{k,j}, x^{k}), \quad \tilde{w}_{2}(s_{k,j}, \eta_{k,j}^{k}(s_{k,j})) = \tilde{w}_{2}^{*}(s_{k,j}, y^{k}) \quad \text{as} \quad j \to \infty.
\]

Using these and \(5.8\) and \(5.9\) we can therefore select a subsequence \( j_{k} \) such that

\[
(t_{k,j_{k}}, \gamma_{k,j_{k}}^{k}(t_{k,j_{k}}), w_{1}(\gamma_{k,j_{k}}^{k}), (\varphi_{k}(t), \nabla_{x} \varphi_{k}, \nabla_{x} \varphi_{k}^{2})(t_{k,j_{k}}, \gamma_{k,j_{k}}^{k}(t_{k,j_{k}})))
\]

\[
\to \infty \left( \tilde{t}, \tilde{r}_{i}(\tilde{t}), \tilde{w}_{1}(\tilde{r}_{i}(\tilde{t}, \tilde{r}_{i}(\tilde{t}, \tilde{r}_{i}(\tilde{t}, X)))ight),
\]

\[
(s_{k,j_{k}}, \eta_{k,j_{k}}^{k}(s_{k,j_{k}}), w_{2}(\eta_{k,j_{k}}^{k}), (\tilde{\varphi}_{k}(t), \nabla_{x} \tilde{\varphi}_{k}, \nabla_{x} \tilde{\varphi}_{k}^{2})(s_{k,j_{k}}, \eta_{k,j_{k}}^{k}(s_{k,j_{k}})))
\]

\[
\to \infty \left( \tilde{t}, \tilde{r}_{i}(\tilde{t}), \tilde{w}_{2}(\tilde{r}_{i}(\tilde{t}, \tilde{r}_{i}(\tilde{t}, \tilde{r}_{i}(\tilde{t}, Y)))ight).
\]

We notice that, by \(3.5\), \(3.6\) and the property (i) of \((t_{k,j}, \gamma_{k,j}^{k}(t_{k,j}), s_{k,j}, \eta_{k,j}^{k}(s_{k,j}))\), there exists a generic constant \( C > 0 \) such that

\[
\frac{1}{2} \left\{ (s_{k,j_{k}} - \tilde{s}_{i}) + (t_{k,j_{k}} - \tilde{t}_{i}) \right\} \leq C_{j_{k}}^{- \frac{1}{2}};
\]

\[
\left| \partial_{x} \left[ \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\gamma_{i}, \gamma_{i,t}) - \gamma_{i}^{i} \right] \right| + \left| \partial_{x} \left[ \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\eta_{i}, \eta_{i,k}) - \eta_{i}^{i,k} \right] \right| \leq C_{j_{k}}^{- \frac{1}{2}};
\]

and

\[
\left| \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\gamma_{i}, \gamma_{i,t}) - \gamma_{i}^{i} \right] \right| + \left| \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\eta_{i}, \eta_{i,k}) - \eta_{i}^{i,k} \right] \right| \leq C_{j_{k}}^{- \frac{1}{2}}.
\]

Therefore the lemma holds with \( \varphi_{k}(t) := \varphi_{k}(t, \gamma_{t}(t)) \),

\[
w_{1}(\gamma_{k,j}(s_{k,j})), \quad w_{2}(\eta_{k,j}(s_{k,j})), \quad \psi_{k}(s_{k,j}) := \psi_{k}(s_{k,j}, \eta_{k,j}(s_{k,j})), \quad w_{1}(\gamma_{k,j}(s_{k,j})), \quad w_{2}(\eta_{k,j}(s_{k,j})), \quad \text{and} \quad t_{k} := t_{k,j}, \quad \gamma_{t} := \gamma_{t_{k,j}}, \quad s_{k} := s_{k,j}, \quad \eta_{s} := \eta_{s_{k,j}}.
\]
Remark 5.4. As mentioned in Remark 6.1 in Chapter V of [18], Condition (5.1) is stated with reverse inequality in Theorem 8.3 of [3]. However, we immediately obtain results (5.4)-(5.6) from Theorem 8.3 of [3] by considering the functions \( u_1(t, x) := \tilde{w}_1^{t_1}(T-t, x) \) and \( u_2(t, x) := \tilde{w}_2^{t_1}(T-t, x) \).

To complete the proof of Theorem 5.3, it remains to state and prove the following two lemmas.

Lemma 5.5. Let all the conditions in Theorem 5.3 hold. Recall that \( (\hat{t}, \hat{\gamma}_I, \hat{\eta}_I) \) is given in Theorem 5.3. \( \tilde{w}_1^{\hat{t}} \) and \( \tilde{w}_2^{\hat{t}} \) are defined in Definition 5.2 with respect to \( w_1 \) and \( w_2 \) given in Theorem 5.3, respectively. Then \( \tilde{w}_1^{\hat{t}}(t, x) + \tilde{w}_2^{\hat{t}}(t, y) - \varphi(x, y) \) has a maximum over \([0, T] \times \mathbb{R}^d \times \mathbb{R}^d\) at \( (\hat{t}, \hat{\gamma}_I(\hat{t}), \hat{\eta}_I(\hat{t})) \). Moreover, we have

\[
\tilde{w}_1^{\hat{t}}(\hat{t}, \hat{\gamma}_I(\hat{t})) = w_1(\hat{\gamma}_I), \quad \tilde{w}_2^{\hat{t}}(\hat{t}, \hat{\eta}_I(\hat{t})) = w_2(\hat{\eta}_I). \tag{5.15}
\]

Proof. For every \( \hat{t} \leq t \leq s \leq T \) and \( x \in \mathbb{R}^d \), from the definition of \( \tilde{w}_1^{t} \) it follows that

\[
\tilde{w}_1^{t}(t, x) - \tilde{w}_1^{s}(s, x) = \sup_{\gamma_t \in \Lambda^t, \gamma_t(t) = x} w_1(\gamma_t) - \sup_{\eta_s \in \Lambda^s, \eta_s(s) = x} w_1(\eta_s). \tag{5.16}
\]

By \( \text{(5.2)} \), there exist constants \( M_1 > 0 \) and \( \varepsilon > 0 \) such that

\[
\frac{w_1(\gamma_t)}{||\gamma_t||_0} \leq -\varepsilon, \quad \text{if} \quad ||\gamma_t||_0 \geq M_1, \quad \gamma_t \in \Lambda^t.
\]

Thus,

\[
w_1(\gamma_t) \leq -\varepsilon ||\gamma_t||_0, \quad \text{if} \quad ||\gamma_t||_0 \geq M_1, \quad \gamma_t \in \Lambda^t. \tag{5.17}
\]

For every \( t \in [\hat{t}, T] \), define \( \xi_t \in \Lambda^t \) by

\[
\xi_t(l) = x, \quad l \in [0, t],
\]

then, by \( \text{(5.3)} \),

\[
w_1(\xi_t) - \sup_{l \in [\hat{t}, T]} \rho_1(|l - \hat{t}|, |x|) \leq w_1(\xi_t) \leq \sup_{\xi_t \in \Lambda^t, \xi_t(t) = x} w_1(\xi_t).
\]

Notice that \( w_1(\xi_t) - \sup_{l \in [\hat{t}, T]} \rho_1(|l - \hat{t}|, |x|) \) depends only on \( x \), there exists a constant \( C_x^1 > 0 \) depending only on \( x \) such that, for all \( (t, \gamma_t) \in [\hat{t}, T] \times \Lambda^t \) satisfying \( ||\gamma_t||_0 \geq C_x^1 \),

\[
-\varepsilon ||\gamma_t||_0 < w_1(\xi_t) - \sup_{l \in [\hat{t}, T]} \rho_1(|l - \hat{t}|, |x|) \leq \sup_{\xi_t \in \Lambda^t, \xi_t(t) = x} w_1(\xi_t). \tag{5.18}
\]

Taking \( C_x = M_1 \lor C_x^1 \), by \( \text{(5.17)} \) and \( \text{(5.18)} \),

\[
w_1(\gamma_t) < \sup_{\xi_t \in \Lambda^t, \xi_t(t) = x} w_1(\xi_t), \quad \text{if} \quad ||\gamma_t||_0 \geq C_x, \quad \gamma_t \in \Lambda^t. \tag{5.19}
\]

Combining \( \text{(5.16)} \) and \( \text{(5.19)} \), from \( \text{(5.3)} \) we have

\[
\tilde{w}_1^{\hat{t}}(t, x) - \tilde{w}_1^{s}(s, x) = \sup_{\gamma_t \in \Lambda^t, ||\gamma_t||_0 \leq C_x, \gamma_t(t) = x} w_1(\gamma_t) - \sup_{\eta_s \in \Lambda^s, \eta_s(s) = x} w_1(\eta_s) \leq \sup_{\gamma_t \in \Lambda^t, ||\gamma_t||_0 \leq C_x, \gamma_t(t) = x} [w_1(\gamma_t) - w_1(\gamma_{t,s})]
\]

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We claim that we can assume that there exists a constant $M$. Indeed, if not, for every $n \in \mathbb{A}^i$, $\|\gamma_n\|_0 \leq C_x$, $\gamma_n(t) = x$. Then, by (5.20), we get that

$$
\sup_{\gamma \in \mathbb{A}^i, \|\gamma\|_0 \leq C_x, \gamma(t) = x} \rho_1(|s - t|, ||\gamma||_0) \leq \rho_1(|s - t|, C_x).
$$

Clearly, if $0 \leq t \leq \hat{t}$, we have

$$
\hat{w}^i_1(t, x) - \hat{w}^i_1(s, x) = -(\hat{t} - t)^{\frac{1}{2}} + (\hat{t} - s)^{\frac{1}{2}} \leq 0,
$$

and, if $0 \leq t \leq \hat{t} \leq T$, we have

$$
\hat{w}^i_1(t, x) - \hat{w}^i_1(s, x) \leq \hat{w}^i_1(\hat{t}, x) - \hat{w}^i_1(s, x) \leq \rho_1(|s - \hat{t}|, C_x).
$$

On the other hand, for every $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, by the definitions of $\hat{w}^i_1(t, x)$ and $\hat{w}^i_2(t, y)$, there exist sequences $(l_i, x_i), (\tau_i, y_i) \in [0, T] \times \mathbb{R}^d$ such that $(l_i, x_i) \to (t, x)$ and $(\tau_i, y_i) \to (t, y)$ as $i \to \infty$ and

$$
\hat{w}^i_1(t, x) = \lim_{i \to \infty} \hat{w}^i_1(l_i, x_i), \quad \hat{w}^i_2(t, y) = \lim_{i \to \infty} \hat{w}^i_2(\tau_i, y_i).
$$

Without loss of generality, we may assume $l_i \leq \tau_i$ for all $i > 0$. By (5.20) and (5.22), we have

$$
\hat{w}^i_1(t, x) = \lim_{i \to \infty} \hat{w}^i_1(l_i, x_i) \leq \lim \inf_{i \to \infty} \hat{w}^i_1(\tau_i, x_i) + \rho_1(|\tau_i - l_i|, C_{x_i})].
$$

Here $C_{x_i} > 0$ is the constant that makes the following formula true:

$$
w_1(\gamma_t) < \sup_{\xi \in \mathbb{A}^i, \xi(t) = x_i} w_1(\xi_t), \text{ if } ||\gamma||_0 \geq C_{x_i}, \gamma_t \in \mathbb{A}^i.
$$

We claim that we can assume that there exists a constant $M_2 > 0$ such that $C_{x_i} \leq M_2$ for all $i \geq 1$. Indeed, if not, for every $n$, there exists $i_n$ such that

$$
\hat{w}^i_1(l_{i_n}, x_{i_n}) = \begin{cases} 
\sup_{\gamma_n \in \mathbb{A}^i, ||\gamma_n||_0 > n, \gamma_n(l_{i_n}) = x_{i_n}} [w_1(\gamma_{i_n})], & l_{i_n} \geq \hat{t}; \\
\sup_{\gamma \in \mathbb{A}^i, ||\gamma||_0 > n, \gamma(t) = x_{i_n}} [w_1(\gamma_t)] - (\hat{t} - l_{i_n})^{\frac{1}{2}}, & l_{i_n} < \hat{t}.
\end{cases}
$$

Letting $n \to \infty$, by (5.22), we get that

$$
\hat{w}^i_1(l_{i_n}, x_{i_n}) \to -\infty \text{ as } n \to \infty,
$$

which contradicts the convergence that $\hat{w}^i_1(t, x) \to -\infty$ as $n \to \infty$.

Then, by (5.21),

$$
\hat{w}^i_1(t, x) \leq \lim \inf_{i \to \infty} [\hat{w}^i_1(\tau_i, x_i) + \rho_1(|\tau_i - l_i|, M_2)] = \lim \inf_{i \to \infty} \hat{w}^i_1(\tau_i, x_i).
$$

Therefore, by (5.23) and (5.26) and the definitions of $\hat{w}^i_1$ and $\hat{w}^i_2$,

$$
\hat{w}^i_1(t, x) + \hat{w}^i_2(t, y) - \varphi(x, y) \\
\leq \lim \inf_{i \to \infty} [\hat{w}^i_1(\tau_i, x_i) + \hat{w}^i_2(\tau_i, y_i) - \varphi(x_i, y_i)] \\
\leq \sup_{(l, x_0, y_0) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d} [\hat{w}^i_1(l, x_0) + \hat{w}^i_2(l, y_0) - \varphi(x_0, y_0)] \\
= \sup_{(l, x_0, y_0) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d} [\hat{w}^i_1(l, x_0) + \hat{w}^i_2(l, y_0) - \varphi(x_0, y_0)].
$$
We also have, for \((l, x_0, y_0) \in [\bar{t}, T] \times \mathbb{R}^d \times \mathbb{R}^d\),
\[
\bar{w}_1^\ast(l, x_0) + \bar{w}_2^\ast(l, y_0) - \varphi(x_0, y_0) = \sup_{\gamma, \eta \in \Lambda^l, \gamma(l) = x_0, \eta(l) = y_0} \{w_1(\gamma_l) + w_2(\eta_l) - \varphi(\gamma_l(l), \eta_l(l))\}
\leq w_1(\bar{\gamma}_l) + w_2(\bar{\eta}_l) - \varphi(\bar{\gamma}_l(l), \bar{\eta}_l(l)),
\] (5.28)
where the inequality becomes equality if \(l = \bar{t}\) and \(x_0 = \bar{\gamma}_l(l), y_0 = \bar{\eta}_l(l)\). Combining (5.27) and (5.28), we obtain that
\[
\bar{w}_1^\ast(l, x) + \bar{w}_2^\ast(l, y) - \varphi(x, y) \leq w_1(\bar{\gamma}_l) + w_2(\bar{\eta}_l) - \varphi(\bar{\gamma}_l(l), \bar{\eta}_l(l)).
\] (5.29)

By the definitions of \(\bar{w}_1^\ast\) and \(\bar{w}_2^\ast\), we have \(\bar{w}_1^\ast(l, x) \geq \bar{w}_1(t, x), \bar{w}_2^\ast(t, y) \geq \bar{w}_2(t, y)\). Then by also (5.28) and (5.29), for every \((t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d\),
\[
\bar{w}_1^\ast(l, x) + \bar{w}_2^\ast(l, y) - \varphi(x, y) = \bar{w}_1(\bar{\gamma}_l(l)) + \bar{w}_2(\bar{\eta}_l(l)) - \varphi(\bar{\gamma}_l(l), \bar{\eta}_l(l))
\leq \bar{w}_1^\ast(\bar{\gamma}_l(l)) + \bar{w}_2^\ast(\bar{\eta}_l(l)) - \varphi(\bar{\gamma}_l(l), \bar{\eta}_l(l)).
\] (5.30)
Thus we obtain that (5.15) holds true, and \(\bar{w}_1^\ast(l, x) + \bar{w}_2^\ast(l, y) - \varphi(x, y)\) has a maximum at \((\bar{\gamma}_l(l), \bar{\eta}_l(l))\) on \([0, T] \times \mathbb{R}^d \times \mathbb{R}^d\). The proof is now complete. \(\Box\)

**Lemma 5.6.** Let all the conditions in Theorem 5.5 hold. Recall that \(\Gamma_1^k\) and \(\Gamma_2^k\) are defined in (5.17). Then the maximum points \(\gamma_{k,j}^l\) of \(\Gamma_1^k(\gamma_l) - \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\gamma_{i,k}^l, \gamma_l)\) and the maximum points \(\eta_{k,j}^l\) of \(\Gamma_2^k(\eta_l) - \sum_{i=0}^{\infty} \Upsilon(\eta_{i,k}^l, \eta_l)\) satisfy conditions (5.12), (5.13) and (5.14).

**Proof.** Recall that \(\bar{w}_1^\ast\) and \(\bar{w}_2^\ast\), by the definitions of \(\bar{w}_1^\ast\) and \(\bar{w}_2^\ast\), we get that
\[
\bar{w}_1^\ast(t, k, \gamma_{k,j}^l(t,k,j)) - \bar{\varphi}_k(t, k, \gamma_{k,j}^l(t,k,j)) = w_1(\gamma_{k,j}^l(t,k,j)) - \varphi_k(t, k, \gamma_{k,j}^l(t,k,j)) = \Gamma_1^k(\gamma_{k,j}^l(t,k,j))
\]
and
\[
\bar{w}_2^\ast(s, k, \eta_{k,j}^l(s,k,j)) - \bar{\psi}_k(s, k, \eta_{k,j}^l(s,k,j)) = w_2(\eta_{k,j}^l(s,k,j)) - \varphi_k(s, k, \eta_{k,j}^l(s,k,j)) = \Gamma_2^k(\eta_{k,j}^l(s,k,j))
\]
We notice that, from (5.11), and the property (ii) of \((t, k, j, \gamma_{k,j}^l, s, k, j, \eta_{k,j}^l)\),
\[
\Gamma_1^k(\gamma_{k,j}^l) \geq \Gamma_1^k(\gamma_{0,k}) \geq \sup_{(t, \gamma_l) \in [\bar{t}, T] \times \Lambda^l} \Gamma_1^k(\gamma_l) - \frac{1}{j}, \quad \Gamma_2^k(\eta_{k,j}^l) \geq \Gamma_2^k(\eta_{0,k}) \geq \sup_{(s, \eta_l) \in [\bar{t}, T] \times \Lambda^l} \Gamma_2^k(\eta_l) - \frac{1}{j}
\]
and by the definitions of \(\bar{w}_1^\ast\) and \(\bar{w}_2^\ast\),
\[
\sup_{(t, \gamma_l) \in [\bar{t}, T] \times \Lambda^l} \Gamma_1^k(\gamma_l) \geq \bar{w}_1^\ast(t, k, x) - \bar{\varphi}_k(t, k, x) \geq \sup_{(s, \eta_l) \in [\bar{t}, T] \times \Lambda^l} \Gamma_2^k(\eta_l) \geq \bar{w}_2^\ast(s, k, y) - \bar{\psi}_k(s, k, y)
\]
Therefore,
\[
\bar{w}_1^\ast(t, k, \gamma_{k,j}^l(t,k,j)) - \bar{\varphi}_k(t, k, \gamma_{k,j}^l(t,k,j)) \geq \Gamma_1^k(\gamma_{k,j}^l) \geq \bar{w}_1^\ast(t, k, x) - \bar{\varphi}_k(t, k, x) - \frac{1}{j},
\] (5.31)
\[
\bar{w}_2^\ast(s, k, \eta_{k,j}^l(s,k,j)) - \bar{\psi}_k(s, k, \eta_{k,j}^l(s,k,j)) \geq \Gamma_2^k(\eta_{k,j}^l) \geq \bar{w}_2^\ast(s, k, y) - \bar{\psi}_k(s, k, y) - \frac{1}{j}
\] (5.32)
By (5.2) and that $\hat{\varphi}_k$ and $\tilde{\psi}_k$ are bounded from below, there exists a constant $M_3 > 0$ that is sufficiently large that

$$\Gamma_k^1(\gamma_t) < \sup_{(t,\xi_t) \in [\hat{t}, T] \times \Lambda^t} \Gamma_k^1(\xi_t) - 1, \quad t \in [\hat{t}, T], \quad ||\gamma_t||_0 \geq M_3,$$

and

$$\Gamma_k^2(\eta_s) < \sup_{(r,\zeta_r) \in [\hat{t}, T] \times \Lambda^t} \Gamma_k^2(\zeta_r) - 1, \quad s \in [\hat{t}, T], \quad ||\eta_s||_0 \geq M_3.$$

Thus, we have $||\gamma_{k,j}||_0 \lor ||\eta_{s,k,j}||_0 < M_3$. In particular, $|\gamma_{k,j}(t_{k,j})| \lor |\eta_{s,k,j}(s_{k,j})| < M_3$. We note that $M_3$ is independent of $j$. Then letting $j \to \infty$ in (5.31) and (5.32), we obtain (5.12). Indeed, if not, we may assume that there exist $(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \in [0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d$ and a subsequence of $(t_{k,j}, \gamma_{k,l}(t_{k,j}), s_{k,j}, \eta_{s,k,j}(s_{k,j}))$ still denoted by itself such that

$$(t_{k,j}, \gamma_{k,l}(t_{k,j}), s_{k,j}, \eta_{s,k,j}(s_{k,j})) \to (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \neq (t_k, x^k, s_k, y^k).$$

Letting $j \to \infty$ in (5.31) and (5.32), by the upper semicontinuity of $w_1^{\hat{t},*} + w_2^{\hat{t},*} - \hat{\varphi}_k - \tilde{\psi}_k$, we have

$$w_1^{\hat{t},*}(\hat{t}, \hat{x}) + w_2^{\hat{t},*}(\hat{s}, \hat{y}) - \hat{\varphi}_k(\hat{t}, \hat{x}) - \tilde{\psi}_k(\hat{s}, \hat{y}) \geq w_1^{\hat{t},*}(t_k, x^k) + w_2^{\hat{t},*}(s_k, y^k) - \hat{\varphi}_k(t_k, x^k) - \tilde{\psi}_k(s_k, y^k),$$

which contradicts that $(t_k, x^k, s_k, y^k)$ is the strict maximum point of $w_1^{\hat{t},*}(t, x) + w_2^{\hat{t},*}(s, y) - \hat{\varphi}_k(t, x) - \tilde{\psi}_k(s, y)$ on $[0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d$.

By (5.12), the upper semicontinuity of $w_1^{\hat{t},*}$ and $w_2^{\hat{t},*}$ and the continuity of $\hat{\varphi}_k$ and $\tilde{\psi}_k$, letting $j \to \infty$ in (5.31) and (5.32), we obtain

$$w_1^{\hat{t},*}(t_k, x_k) \geq \lim_{j \to \infty} w_1^{\hat{t},*}(t_{k,j}, \gamma_{k,l}(t_{k,j})) \geq \lim_{j \to \infty} w_1^{\hat{t},*}(t_{k,j}, \gamma_{k,l}(t_{k,j})) \geq \tilde{\psi}_1(t_k, x_k),$$

$$w_2^{\hat{t},*}(s_k, y_k) \geq \lim_{j \to \infty} w_2^{\hat{t},*}(s_{k,j}, \eta_{s,k,j}(t_{k,j})) \geq \lim_{j \to \infty} w_2^{\hat{t},*}(s_{k,j}, \eta_{s,k,j}(t_{k,j})) \geq \tilde{\psi}_2(s_k, y_k).$$

Thus, we get (5.13) holds true. Letting $j \to \infty$ in (5.31) and (5.32), by (5.13) and the definitions of $\Gamma_k^1$ and $\Gamma_k^2$,

$$w_1^{\hat{t},*}(t_k, x_k) = \lim_{j \to \infty} w_1(t_{k,j}, x_k), \quad w_2^{\hat{t},*}(s_k, y_k) = \lim_{j \to \infty} w_2(s_{k,j}, y_k).$$

Thus, we obtain (5.14). The proof is now complete. □

6 Viscosity solution to PHJB equation: Uniqueness theorem.

This section is devoted to a proof of uniqueness of viscosity solutions to (1.5). This result, together with the results from Section 4, will be used to characterize the value functional defined by (1.4).

By Proposition 11.2.13, without loss of generality we assume that there exists a constant $K > 0$ such that, for all $(t, \gamma_t, p, \ell) \in [0, T] \times \Lambda \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d)$ and $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$,

$$H(\gamma_t, r_1, p, \ell) - H(\gamma_t, r_2, p, \ell) \geq K(r_2 - r_1). \quad (6.1)$$

We now state the main result of this section.
Theorem 6.1. Suppose Hypothesis **[2.7]** holds. Let $W_1 \in C^0(\Lambda)$ (resp., $W_2 \in C^0(\Lambda)$) be a viscosity subsolution (resp., supersolution) to equation (1.5) and let there exist constant $L > 0$ and a local modulus of continuity $\rho_2$ such that, for any $(t, \gamma_t), (s, \eta_s) \in [0, T] \times \Lambda$,

$$|W_1(\gamma_t)| \vee |W_2(\gamma_t)| \leq L(1 + ||\gamma_t||_0);$$

$$|W_1(\gamma_t) - W_1(\eta_s)| \vee |W_2(\gamma_t) - W_2(\eta_s)| \leq \rho_2(|s - t|, ||\gamma_t||_0 \vee ||\eta_s||_0) + L||\gamma_t - \eta_s||_0. \quad (6.3)$$

Then $W_1 \leq W_2$.

Theorems [4.5] and [6.1] lead to the result (given below) that the viscosity solution to PHJB equation given in (1.5) corresponds to the value functional $V$ of our optimal control problem given in (1.1), (1.3) and (1.4).

Theorem 6.2. Let Hypothesis **[2.7]** hold. Then the value functional $V$ defined by (1.4) is the unique viscosity solution to (1.5) in the class of functionals satisfying (6.2) and (6.3).

**Proof.** Theorem [4.5] shows that $V$ is a viscosity solution to equation (1.5). Thus, our conclusion follows from Theorems [2.11] and [6.1]. \[ \Box \]

Next, we prove Theorem [6.1]. We note that for $\rho > 0$, the functional defined by $\tilde{W} := W_1 - \frac{\rho}{t(1)^2}$ is a viscosity subsolution for

$$\begin{align*}
\partial_t \tilde{W}(\gamma_t) + H(\gamma_t, \tilde{W}(\gamma_t), \partial_x \tilde{W}(\gamma_t), \partial_{xx} \tilde{W}(\gamma_t)) &= -\frac{\rho}{(t+1)^2}, \quad (t, \gamma_t) \in [0, T) \times \Lambda, \\
\tilde{W}(\gamma_T) &= \phi(\gamma_T), \quad \gamma_T \in \Lambda_T.
\end{align*} \quad (6.4)$$

We mention that $\tilde{W}$ is also a viscosity subsolution of (6.4) if the second argument of $H$ is $W_1(\gamma_t)$ instead of $\tilde{W}(\gamma_t)$. As $W_1 \leq W_2$ follows from $\tilde{W} \leq W_2$ in the limit $\rho \downarrow 0$, it suffices to prove $W_1 \leq W_2$ under the additional assumption given below:

$$\partial_t W_1(\gamma_t) + H(\gamma_t, W_1(\gamma_t), \partial_x W_1(\gamma_t), \partial_{xx} W_1(\gamma_t)) \geq c, \quad c := \frac{\rho}{(T + 1)^2}, \quad (t, \gamma_t) \in [0, T) \times \Lambda. \quad (6.5)$$

**Proof of Theorem [6.1]** The proof of this theorem is rather long. Thus, we split it into several steps.

1. **Step 1.** Definitions of auxiliary functionals.
   We only need to prove that $W_1(\gamma_t) \leq W_2(\gamma_t)$ for all $(t, \gamma_t) \in [T - \bar{a}, T) \times \Lambda$. Here,

$$\bar{a} = \frac{1}{2(342L + 36)L} \wedge T. \quad \text{and} \quad \tilde{\gamma}_T = \gamma_T = \Lambda_T.$$

Then, we can repeat the same procedure for the case $[T - \bar{a}, T - (i - 1)\bar{a})$. Thus, we assume the converse result that $(\bar{t}, \tilde{\gamma}_T) \in (T - \bar{a}, T) \times \Lambda$ exists such that $\bar{m} := W_1(\tilde{\gamma}_T) - W_2(\tilde{\gamma}_T) > 0$.

Consider a small number $\varepsilon > 0$ such that

$$W_1(\bar{\gamma}_T) - W_2(\bar{\gamma}_T) - 2\varepsilon \frac{\nu T - \bar{t}}{\nu T} Y(\bar{\gamma}_T) > \frac{\bar{m}}{2},$$

and

$$\frac{\varepsilon}{\nu T} \leq \frac{c}{4}, \quad \text{and} \quad \nu = 1 + \frac{1}{2T(342L + 36)L}. \quad (6.5)$$
Next, recalling that \( \Lambda^t \otimes \Lambda^t := \{(\gamma_s, \eta_s)|\gamma_s, \eta_s \in \Lambda^t\} \) for all \( t \in [0, T] \), we define for any \( \beta \in (0, \infty) \) and \( (\gamma_t, \eta_t) \in \Lambda^{T-a} \otimes \Lambda^{T-a} \),

\[
\Psi(\gamma_t, \eta_t) = W_1(\gamma_t) - W_2(\eta_t) - \beta \Upsilon(\gamma_t, \eta_t) - \beta \sqrt{t} |\gamma_t(t) - \eta_t(t)|^2 - \frac{\nu T - t}{\nu T} (\Upsilon(\gamma_t) + \Upsilon(\eta_t)).
\]

By (3.23) and (6.2), it is clear that \( \Psi \) is bounded from above on \( \Lambda^{T-a} \otimes \Lambda^{T-a} \). Moreover, by Lemma 3.1, \( \Psi \) is a continuous functional. Define a sequence of positive numbers \( \{\delta_i\}_{i \geq 0} \) by \( \delta_i = \frac{1}{2^i} \) for all \( i \geq 0 \). Since \( \Upsilon^\beta(\cdot, \cdot) \) is a gauge-type function on \( (\Lambda^\tilde{t} \otimes \Lambda^\tilde{t}, d_1, \infty) \), from Lemma 2.14 it follows that, for every \( (\gamma_0^0, \eta_0^0) \in \Lambda^\tilde{t} \otimes \Lambda^\tilde{t} \) satisfying

\[
\Psi(\gamma_0^0, \eta_0^0) \geq \sup_{(s, (\gamma_s, \eta_s)) \in [\tilde{t}, T] \times (\Lambda^\tilde{t} \otimes \Lambda^\tilde{t})} \Psi(\gamma_s, \eta_s) - \frac{1}{\beta}, \quad \text{and} \quad \Psi(\gamma_0^0, \eta_0^0) \geq \Psi(\tilde{\gamma}_i, \tilde{\eta}_i) > \frac{\bar{m}}{2},
\]

there exist \( (\tilde{t}, (\tilde{\gamma}_i, \tilde{\eta}_i)) \in [\tilde{t}, T] \times (\Lambda^\tilde{t} \otimes \Lambda^\tilde{t}) \) and a sequence \( \{(t_i, (\gamma_i^i, \eta_i^i))\}_{i \geq 1} \subset [t_0, T] \times (\Lambda^\tilde{t} \otimes \Lambda^\tilde{t}) \) such that

(i) \( \Upsilon(\gamma_0^0, \tilde{\gamma}_i) + \Upsilon(\eta_0^0, \tilde{\eta}_i) + |\tilde{t} - t_0|^2 \leq \frac{1}{2^i}, \) \( \Upsilon(\gamma_i^i, \tilde{\gamma}_i) + \Upsilon(\eta_i^i, \tilde{\eta}_i) + |\tilde{t} - t_i|^2 \leq \frac{1}{2^i} \) and \( t_i \uparrow \tilde{t} \) as \( i \to \infty \),

(ii) \( \Psi_1(\tilde{\gamma}_i, \tilde{\eta}_i) \geq \Psi(\gamma_0^0, \eta_0^0) \),

(iii) for all \( (s, (\gamma_s, \eta_s)) \in [\tilde{t}, T] \times (\Lambda^\tilde{t} \otimes \Lambda^\tilde{t}) \setminus \{(\tilde{t}, (\tilde{\gamma}_i, \tilde{\eta}_i))\}, \)

\[
\Psi_1(\gamma_s, \eta_s) < \Psi_1(\tilde{\gamma}_i, \tilde{\eta}_i), \quad (6.6)
\]

where we define

\[
\Psi_1(\gamma_t, \eta_t) := \Psi(\gamma_t, \eta_t) - \sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon(\gamma_i^i, \gamma_t) + \Upsilon(\eta_i^i, \eta_t) + |t - t_i|^2], \quad (\gamma_t, \eta_t) \in \Lambda^\tilde{t} \otimes \Lambda^\tilde{t}.
\]

We should note that the point \( (\tilde{t}, (\tilde{\gamma}_i, \tilde{\eta}_i)) \) depends on \( \beta \) and \( \varepsilon \).

**Step 2.** There exists \( M_0 > 0 \) independent of \( \beta \) such that

\[
||\tilde{\gamma}_i||_0 \lor ||\tilde{\eta}_i||_0 < M_0, \quad (6.7)
\]

and the following result holds true:

\[
\beta ||\tilde{\gamma}_i - \tilde{\eta}_i||_0 + \beta |\tilde{\gamma}_i(\tilde{t}) - \tilde{\eta}_i(\tilde{t})|^4 \to 0 \text{ as } \beta \to \infty. \quad (6.8)
\]

Let us show the above. First, noting \( \nu \) is independent of \( \beta \), by the definition of \( \Psi \), there exists a constant \( M_0 > 0 \) independent of \( \beta \) that is sufficiently large that \( \Psi(\gamma_t, \eta_t) < 0 \) for all \( t \in [T - \bar{a}, T] \) and \( ||\gamma_t||_0 \lor ||\eta_t||_0 \geq M_0 \). Thus, we have \( ||\tilde{\gamma}_i||_0 \lor ||\tilde{\eta}_i||_0 \lor ||\gamma_0^0||_0 \lor ||\eta_0^0||_0 < M_0 \). We note that \( M_0 \) depends on \( \varepsilon \).

Second, by (6.6), we have

\[
2\Psi_1(\tilde{\gamma}_i, \tilde{\eta}_i) \geq \Psi_1(\tilde{\gamma}_i, \tilde{\gamma}_i) + \Psi_1(\tilde{\eta}_i, \tilde{\eta}_i). \quad (6.9)
\]

This implies that

\[
2\beta \Upsilon(\tilde{\gamma}_i, \tilde{\eta}_i) + 2\beta |\tilde{\gamma}_i(\tilde{t}) - \tilde{\eta}_i(\tilde{t})|^2 \leq |W_1(\tilde{\gamma}_i) - W_1(\tilde{\eta}_i)| + |W_2(\tilde{\gamma}_i) - W_2(\tilde{\eta}_i)| + \sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon(\eta_i^i, \tilde{\gamma}_i) + \Upsilon(\gamma_i^i, \tilde{\eta}_i)]. \quad (6.10)
\]
On the other hand, notice that
\[ \mathcal{Y}(\gamma_t, \eta_s) = \mathcal{Y}(\gamma_t - \eta_s, t), \quad \gamma_t, \eta_s \in \Lambda, \quad 0 \leq s \leq t \leq T, \]
by Lemma 3.4 and the property (i) of \((\hat{t}, (\hat{\gamma}, \hat{\eta}))\),
\[
\sum_{i=0}^{\infty} \frac{1}{2i} [\mathcal{Y}(\eta_i, \hat{\gamma}_i) + \mathcal{Y}(\gamma_i, \hat{\eta}_i)] \\
\leq 2^5 \sum_{i=0}^{\infty} \frac{1}{2i} [\mathcal{Y}(\eta_i, \hat{\gamma}_i) + \mathcal{Y}(\gamma_i, \hat{\eta}_i) + 2\mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i)] \\
\leq \frac{2^6}{2\beta} + 2^7 \mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i). \tag{6.11}
\]
Combining (6.10) and (6.11), from (6.2) and (6.7) we have
\[
(2\beta - 2^7)\mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i) + 2\beta^\frac{3}{2} |\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^2 \\
\leq |W_1(\hat{\gamma}_i) - W_1(\hat{\eta}_i)| + |W_2(\hat{\gamma}_i) - W_2(\hat{\eta}_i)| + \frac{2^6}{2\beta} \\
\leq 2L(2 + ||\hat{\gamma}_i||_0 + ||\hat{\eta}_i||_0) + \frac{2^6}{2\beta} \leq 4L(1 + M_0) + \frac{2^6}{2\beta}. \tag{6.12}
\]
Letting \(\beta \to \infty\), we have
\[
\mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i) \leq \frac{1}{2\beta - 2^7} \left[ 4L(1 + M_0) + \frac{2^6}{2\beta} \right] \to 0 \quad \text{as} \quad \beta \to \infty.
\]
In view of (3.23), we have
\[
||\hat{\gamma}_i - \hat{\eta}_i||_0 \to 0 \quad \text{as} \quad \beta \to \infty. \tag{6.13}
\]
From (3.23), (6.3), (6.10), (6.11) and (6.13), we conclude that
\[
\beta ||\hat{\gamma}_i - \hat{\eta}_i||_0^6 + \beta^\frac{3}{2} |\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^2 \leq \beta \mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i) + \beta^\frac{3}{2} |\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^2 \\
\leq \frac{1}{2} ||W_1(\hat{\gamma}_i) - W_1(\hat{\eta}_i)|| + ||W_2(\hat{\gamma}_i) - W_2(\hat{\eta}_i)|| + \frac{2^5}{2\beta} + 2^8 \mathcal{Y}(\hat{\gamma}_i, \hat{\eta}_i) \\
\leq L||\hat{\gamma}_i - \hat{\eta}_i||_0^6 + \frac{2^5}{2\beta} + 2^8 ||\hat{\gamma}_i - \hat{\eta}_i||_0^6 \to 0 \quad \text{as} \quad \beta \to \infty. \tag{6.14}
\]
Multiply the leftmost and rightmost sides of inequality (6.14) by \(\beta^\frac{3}{2}\), we obtain that
\[
\beta^\frac{3}{2} |\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^2 \leq L\beta^\frac{3}{2} ||\hat{\gamma}_i - \hat{\eta}_i||_0^6 + \frac{2^5}{2\beta} + 2^8 \beta^\frac{3}{2} ||\hat{\gamma}_i - \hat{\eta}_i||_0^6. \tag{6.15}
\]
By also (6.14), the right side of above inequality converges to 0 as \(\beta \to \infty\). Then we have that
\[
\beta^\frac{3}{2} |\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^2 \to 0 \quad \text{as} \quad \beta \to \infty.
\]
Combining with (6.14), we have (6.8).

\textbf{Step 3}. There exists \(N > 0\) such that \(\hat{t} \in [\hat{t}, T)\) for all \(\beta \geq N\).

By (6.13), we can let \(N > 0\) be a large number such that
\[
L||\hat{\gamma}_i - \hat{\eta}_i||_0 \leq \frac{\tilde{m}}{4},
\]
where \(\tilde{m}\) is a positive number to be chosen later.
for all $\beta \geq N$. Then we have $\hat{t} \in [\hat{t}, T)$ for all $\beta \geq N$. Indeed, if $\hat{t} = T$, we deduce the following contradiction:

$$\frac{\bar{m}}{2} \leq \Psi(\gamma_{\hat{t}}, \eta_{\hat{t}}) = \phi(\hat{\gamma}_{\hat{t}}) \leq \phi(\hat{\eta}_{\hat{t}}) \leq L \|\hat{\gamma}_{\hat{t}} - \hat{\eta}_{\hat{t}}\|_0 \leq \frac{\bar{m}}{4}.$$

**Step 4.** Crandall-Ishii lemma.

From above all, for the fixed $N > 0$ in step 3, we find $(\hat{t}, \hat{\gamma}_{\hat{t}}), (\hat{t}, \hat{\eta}_{\hat{t}}) \in [\hat{t}, T) \times \Lambda^{\hat{t}}$ satisfying $\hat{t} \in [\hat{t}, T)$ for all $\beta \geq N$ such that

$$\Psi_1(\hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}}) \geq \Psi(\hat{\gamma}_{\hat{t}}, \hat{\gamma}_{\hat{t}}) \text{ and } \Psi_1(\hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}}) \geq \Psi_1(\gamma_t, \eta_t), \quad (\gamma_t, \eta_t) \in \Lambda^t \times \Lambda^t.$$

We define, for $(t, \gamma_t, \eta_t) \in [0, T] \times \Lambda \times \Lambda,$

$$w_1(\gamma_t) = W_1(\gamma_t) - 2^5 \beta \Upsilon(\gamma_t, \hat{\xi}_t) - \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\gamma_t) - \varepsilon \bar{\Upsilon}(\gamma_t, \hat{\gamma}_t) - \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \Upsilon_i(\gamma_t), \eta_t), \quad (6.17)$$

$$w_2(\eta_t) = -W_2(\eta_t) - 2^5 \beta \Upsilon(\eta_t, \hat{\xi}_t) - \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\eta_t) - \varepsilon \bar{\Upsilon}(\eta_t, \hat{\eta}_t) - \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \Upsilon_i(\eta_t), \eta_t), \quad (6.18)$$

where $\hat{\xi}_t = \frac{\gamma_t + \eta_t}{2}$. We note that $w_1, w_2$ depend on $\hat{\xi}_t$, and thus on $\beta$ and $\varepsilon$. We also note that the last term in (6.18) is $\sum_{i=0}^{\infty} \frac{1}{2^{2i}} \Upsilon_i(\gamma_t, \eta_t)$ rather than $\sum_{i=0}^{\infty} \frac{1}{2} \Upsilon_i(\eta_t, \eta_t)$. This is because we divide the term $\sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_t, \eta_t) + \Upsilon_i(\eta_t, \eta_t) + |t - t_i|^2$ in $\Psi_1$ into two terms $\sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon_i(\gamma_t, \eta_t)$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon_i(\eta_t, \eta_t)$. Define $\varphi \in C^2(\mathbb{R}^{d} \times \mathbb{R}^d)$ by

$$\varphi(x, y) = \beta^4 |x - y|^2, \quad (x, y) \in \mathbb{R}^{d} \times \mathbb{R}^{d}. \quad (6.19)$$

By the following Lemma $6.15$, $w_1$ and $w_2$ satisfy the conditions of Theorem $5.3$ with $\varphi$ defined by (6.19). Then by Theorem $5.3$, there exist sequences $(l_k, \hat{\gamma}_{l_k}^k), (s_k, \hat{\eta}_{s_k}^k) \in [\hat{t}, T) \times \Lambda^{l_k}$ and sequences of functionals $\varphi_k \in C^{1,2}_p(\Lambda^{l_k}), \psi_k \in C^{1,2}_p(\Lambda^{s_k})$ bounded from below such that

$$w_1(\gamma_t) - \varphi_k(\gamma_t) \quad (6.20)$$

has a strict maximum 0 at $\hat{\gamma}^k_{l_k}$ over $\Lambda^{l_k}$, and

$$w_2(\eta_t) - \psi_k(\eta_t) \quad (6.21)$$

has a strict maximum 0 at $\hat{\eta}^k_{s_k}$ over $\Lambda^{s_k}$, and

$$\left(l_k, \hat{\gamma}^k_{l_k}(l_k), w_1(\hat{\gamma}^k_{l_k}), \partial_k \varphi_k(\hat{\gamma}^k_{l_k}), \partial_x \varphi_k(\hat{\gamma}^k_{l_k}), \partial_{xx} \varphi_k(\hat{\gamma}^k_{l_k}) \right)$$

$k \to \infty \left(l_k, \hat{\gamma}^k_{l_k}(l_k), w_1(\hat{\gamma}^k_{l_k}), b_1, 2 \beta^4 \left(\hat{\gamma}^k_{l_k}(l_k) - \hat{\eta}^k_{l_k}(l_k)\right), X \right), \quad (6.22)$

$$\left(s_k, \hat{\eta}^k_{s_k}(s_k), w_2(\hat{\eta}^k_{s_k}), \partial_k \psi_k(\hat{\eta}^k_{s_k}), \partial_x \psi_k(\hat{\eta}^k_{s_k}), \partial_{xx} \psi_k(\hat{\eta}^k_{s_k}) \right)$$

$k \to \infty \left(s_k, \hat{\eta}^k_{s_k}(s_k), w_2(\hat{\eta}^k_{s_k}), b_2, 2 \beta^4 \left(\hat{\eta}^k_{s_k}(s_k) - \hat{\gamma}^k_{s_k}(s_k)\right), Y \right), \quad (6.23)$

where $b_1 + b_2 = 0$ and $X, Y \in \mathcal{S}(\mathbb{R}^{d})$ satisfy the following inequality:

$$-6 \beta^4 \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & Y \\ 0 & 0 & I \end{array} \right) \leq \left( \begin{array}{ccc} X & 0 & 0 \\ 0 & 0 & Y \\ 0 & 0 & I \end{array} \right) \leq 6 \beta^4 \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & Y \\ 0 & 0 & I \end{array} \right). \quad (6.24)$$
We note that sequence \((\bar{\gamma}^k_{l_k}, \bar{\eta}^k_{s_k}, l_k, s_k, \varphi_k, \psi_k)\) and \(b_1, b_2, X, Y\) depend on \(\beta\) and \(\varepsilon\). We also note that (6.24) follows from (5.6) choosing \(\kappa = \frac{1}{2} \beta^{-\frac{1}{2}}\). In fact, by (6.19),

\[
A = \nabla^2_x \Psi(\bar{\gamma}(0), \bar{\eta}(0)) = 2\beta^{\frac{1}{2}} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},
\]

and thus, if \(\kappa = \frac{1}{2} \beta^{-\frac{1}{2}}\),

\[
A + \kappa A^2 = (1 + 4\kappa \beta^{\frac{1}{2}})A = 3A,
\]

and

\[
-\frac{1}{\kappa} + |A| = -\left(2\beta^{\frac{1}{2}} + 4\beta^{\frac{1}{2}}\right) = -6\beta^{\frac{1}{2}}.
\]

Then from (5.6) it follows that (6.24) holds true. By the following Lemma 6.6, we have

\[
\lim_{k \to \infty} [d_\infty(\bar{\gamma}^k_{l_k}, \bar{\eta}^k) + d_\infty(\bar{\eta}^k_{s_k}, \bar{\eta})] = 0. \tag{6.25}
\]

For every \((t, \gamma_t), (s, \eta_s) \in [T - \bar{a}, T] \times \Lambda^{T - \bar{a}}\), let

\[
\chi^k(\gamma_t) := \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\gamma_t) + \varepsilon \Upsilon(\gamma_\bar{t}) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon(\gamma^i_{l_k}, \gamma_t) + 2^5 \beta \Upsilon(\gamma_t, \hat{\xi}_t) + \varphi_k(\gamma_t),
\]

\[
h^k(\eta_s) := -\varepsilon \frac{\nu T - s}{\nu T} \Upsilon(\eta_s) - \varepsilon \Upsilon(\eta_\bar{t}) - \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon(\eta^i_{s_k}, \eta_s) - 2^5 \beta \Upsilon(\eta_s, \hat{\xi}_t) - \psi_k(\eta_s).
\]

It is clear that \(\chi^k(\cdot) \in C_p^{1,2}(\Lambda^k), h^k(\cdot) \in C_p^{1,2}(\Lambda^k)\). Moreover, by (6.20), (6.21) and definitions of \(w_1\) and \(w_2\),

\[
(W_1 - \chi^k)(\gamma_{l_k}) = \sup_{(t, \gamma_t) \in [l_k, T] \times \Lambda^k} (W_1 - \chi^k)(\gamma_t),
\]

\[
(W_2 - h^k)(\eta_{s_k}) = \inf_{(s, \eta_s) \in [s_k, T] \times \Lambda^k} (W_2 - h^k)(\eta_s).
\]

From \(l_k \to \bar{t}, s_k \to \bar{t}\) as \(k \to \infty\) and \(\bar{t} < T\) for \(\beta > N\), it follows that, for every fixed \(\beta > N\), constant \(K_\beta > 0\) exists such that

\[
l_k \vee s_k < T, \text{ for all } k \geq K_\beta.
\]

Now, for every \(\beta > N\) and \(k > K_\beta\), from the definition of viscosity solutions it follows that

\[
\partial_t \chi^k(\gamma_{l_k}) + \mathbf{H}(\gamma_{l_k}^k, W_1(\gamma_{l_k}^k), \partial_x \chi^k(\gamma_{l_k}^k), \partial_{xx} \chi^k(\gamma_{l_k}^k)) \geq c, \tag{6.26}
\]

and

\[
\partial_t h^k(\eta_{s_k}) + \mathbf{H}(\eta_{s_k}^k, W_2(\eta_{s_k}^k), \partial_x h^k(\eta_{s_k}^k), \partial_{xx} h^k(\eta_{s_k}^k)) \leq 0, \tag{6.27}
\]

where, for every \((t, \gamma_t) \in [l_k, T] \times \Lambda^k\) and \((s, \eta_s) \in [s_k, T] \times \Lambda^s\), from Lemma 3.1

\[
\partial_t \chi^k(\gamma_t) = \partial_t \varphi_k(\gamma_t) - \frac{\varepsilon}{\nu T} \Upsilon(\gamma_t) + 2\varepsilon(t - \bar{t}) + 2\sum_{i=0}^{\infty} \frac{1}{2i}(t - t_i),
\]
where

\[ \partial\chi^k(\gamma_t) = \partial\varphi_k(\gamma_t) + \varepsilon \frac{\nu T - t}{\nu T} \partial\chi(\gamma_t) + \varepsilon \partial\chi(\gamma_t - \hat{\gamma}_{t,i}) + 2^5 \beta \partial x\chi(\gamma_t - \hat{\xi}_{t,i}) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\gamma_t - \gamma_{t,i}) \right], \]

\[ \partial_{xx}\chi^k(\gamma_t) = \partial_{xx}(\varphi_k(\gamma_t)) + \varepsilon \frac{\nu T - t}{\nu T} \partial_{xx}\chi(\gamma_t) + \varepsilon \partial_{xx}\chi(\gamma_t - \hat{\gamma}_{t,i}) + 2^5 \beta \partial_{xx}\chi(\gamma_t - \hat{\xi}_{t,i}) + \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\gamma_t - \gamma_{t,i}) \right], \]

\[ \partial_t h^k(\eta_s) = -\partial_t \psi_k(\eta_s) + \frac{\varepsilon}{\nu T} \nu T \chi(\eta_s) - 2\varepsilon(s - i), \]

\[ \partial_x h^k(\eta_s) = -\partial_x \psi_k(\eta_s) - \varepsilon \frac{\nu T - s}{\nu T} \partial_x \chi(\eta_s) - \varepsilon \partial_x \chi(\eta_s - \hat{\eta}_{t,i}) - 2^5 \beta \partial_x \chi(\eta_s - \hat{\xi}_{t,i}) - \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\eta_s - \eta_{t,i}) \right], \]

\[ \partial_{xx} h^k(\eta_s) = -\partial_{xx} \psi_k(\eta_s) - \varepsilon \frac{\nu T - s}{\nu T} \partial_{xx} \chi(\eta_s) - \varepsilon \partial_{xx} \chi(\eta_s - \hat{\eta}_{t,i}) - 2^5 \beta \partial_{xx} \chi(\eta_s - \hat{\xi}_{t,i}) - \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\eta_s - \eta_{t,i}) \right]. \]

**Step 5. Calculation and completion of the proof.**

We notice that, by \((3.5)\) and \((3.0)\), there exists a generic constant \(C > 0\) such that

\[ |\partial_x \chi(\hat{\gamma}_k) - \hat{\gamma}_{t,i} | + |\partial_x \chi(\hat{\eta}_{s_k} - \hat{\eta}_{s_k}) | \leq C |\hat{\gamma}_k(t) - \hat{\gamma}_k(l_k)|^5 + C |\hat{\eta}_k(t) - \hat{\eta}_{s_k}(s_k)|^5; \]

\[ \partial_{xx} \chi(\hat{\gamma}_k) - \hat{\gamma}_{t,i} | + |\partial_{xx} \chi(\hat{\eta}_{s_k} - \hat{\eta}_{s_k}) | \leq C |\hat{\gamma}_k(t) - \hat{\gamma}_k(l_k)|^4 + C |\hat{\eta}_k(t) - \hat{\eta}_{s_k}(s_k)|^4. \]

Letting \(k \to \infty\) in \((6.26)\) and \((6.27)\), and using \((6.22)\), \((6.23)\) and \((6.25)\), we obtain

\[ b_1 - \frac{\varepsilon}{\nu T} \nu T \chi(\hat{\eta}_i) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (t_i - t_i) + H(\hat{\eta}_i, W_1(\hat{\gamma}_i), \partial_x \chi(\hat{\gamma}_i), \partial_{xx} \chi(\hat{\gamma}_i)) \geq c; \]  

\[ -b_2 + \frac{\varepsilon}{\nu T} \nu T \chi(\hat{\eta}_i) + H(\hat{\eta}_i, W_2(\hat{\gamma}_i), \partial_x h(\hat{\eta}_i), \partial_{xx} h(\hat{\eta}_i)) \leq 0, \]

where

\[ \partial_x \chi(\hat{\gamma}_i) := 2\beta \frac{1}{\nu T} (\hat{\gamma}_i - \hat{\eta}_i) + 2^5 \beta \partial_x \chi(\hat{\gamma}_i - \hat{\xi}_i) + \varepsilon \frac{\nu T - t_i}{\nu T} \partial_x \chi(\hat{\gamma}_i) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\hat{\gamma}_i - \gamma_{t,i}) \right], \]

\[ \partial_{xx} \chi(\hat{\gamma}_i) := X + 2^5 \beta \partial_{xx} \chi(\hat{\gamma}_i - \hat{\xi}_i) + \varepsilon \frac{\nu T - t_i}{\nu T} \partial_{xx} \chi(\hat{\gamma}_i) + \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \chi(\hat{\gamma}_i - \gamma_{t,i}) \right], \]
\[\partial_x h(\hat{\eta}_i) := 2\beta^\frac{1}{2}(\hat{\gamma}_i(t) - \hat{\eta}_i(t)) - 2\beta^\frac{1}{2}\partial_x \Upsilon(\hat{\eta}_i - \xi_i) - \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_x \gamma(\hat{\eta}_i) - \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\hat{\eta}_i - \eta_{i,t,i}^i) \right],\]

and

\[\partial_{xx} h(\hat{\eta}_i) := -Y - 2\beta^\frac{1}{2}\partial_{xx} \Upsilon(\hat{\eta}_i - \xi_i) - \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_{xx} \gamma(\hat{\eta}_i) - \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\hat{\eta}_i - \eta_{i,t,i}^i) \right].\]

Notice that \(b_1 + b_2 = 0\) and \(\hat{\xi}_i = \frac{\hat{\gamma}_i + \hat{\eta}_i}{2}\), combining \((6.28)\) and \((6.29)\), we have

\[c + \frac{\varepsilon}{\nu T}(\Upsilon(\hat{\gamma}_i) + \Upsilon(\hat{\eta}_i)) - 2 \sum_{i=0}^{\infty} \frac{1}{2^i}(\hat{t} - t_i) \leq \mathbf{H}(\hat{\gamma}_i, W_1(\hat{\gamma}_i), \partial_x \chi(\hat{\gamma}_i), \partial_{xx} \chi(\hat{\gamma}_i)) - \mathbf{H}(\hat{\eta}_i, W_2(\hat{\eta}_i), \partial_x h(\hat{\eta}_i), \partial_{xx} h(\hat{\eta}_i)).\] (6.30)

On the other hand, by \((6.1)\) and via a simple calculation we obtain

\[\mathbf{H}(\hat{\gamma}_i, W_1(\hat{\gamma}_i), \partial_x \chi(\hat{\gamma}_i), \partial_{xx} \chi(\hat{\gamma}_i)) - \mathbf{H}(\hat{\eta}_i, W_2(\hat{\eta}_i), \partial_x h(\hat{\eta}_i), \partial_{xx} h(\hat{\eta}_i)) \leq \sup_{u \in U} (J_1 + J_2 + J_3),\] (6.31)

where from Hypothesis \((2.6)\), \((3.5)\), \((3.6)\) and \((6.24)\), we have

\[J_1 = \langle b(\hat{\gamma}_i, u), \partial_x \chi(\hat{\gamma}_i) \rangle - \langle b(\hat{\eta}_i, u), \partial_x h(\hat{\eta}_i) \rangle \]

\[\leq 2\beta^{\frac{1}{2}}L|\hat{\gamma}_i(t) - \hat{\eta}_i(t)| \times ||\hat{\gamma}_i - \hat{\eta}_||_0 + 18\beta|\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^5 L(2 ||\hat{\gamma}_||_0 + ||\hat{\eta}_||_0)\]

\[+ 18L \sum_{i=0}^{\infty} \frac{1}{2^i} [||\gamma_i^i(t_i) - \gamma_i(t)|^5 + ||\eta_i^i(t_i) - \eta_i(t)|^5] (1 + ||\hat{\gamma}_||_0 + ||\hat{\eta}_||_0)\]

\[+ 36\varepsilon \frac{\nu T - \hat{t}}{\nu T} L(1 + ||\hat{\gamma}_||^6_0 + ||\hat{\eta}_||^6_0);\] (6.32)

\[J_2 = \frac{1}{2} \text{tr}[\partial_{xx} \chi(\hat{\gamma}_i) \sigma(\hat{\gamma}_i, u) \sigma^T(\hat{\gamma}_i, u)] - \frac{1}{2} \text{tr}[\partial_{xx} h(\hat{\eta}_i) \sigma(\hat{\eta}_i, u) \sigma^T(\hat{\eta}_i, u)]\]

\[\leq 3\beta^{\frac{1}{2}}||\sigma(\hat{\gamma}_i, u) - \sigma(\hat{\eta}_i, u)||^2 + 306\beta|\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^4 (||\sigma(\hat{\gamma}_i, u)||^2 + ||\sigma(\hat{\eta}_i, u)||^2)\]

\[+ 153 \varepsilon \frac{\nu T - \hat{t}}{\nu T} ||\hat{\gamma}_i(t)||^4 + ||\hat{\eta}_i(t)||^4 ||\sigma(\hat{\gamma}_i, u)||^2 + ||\sigma(\hat{\eta}_i, u)||^2\]

\[+ 153 \sum_{i=0}^{\infty} \frac{1}{2^i} ||\gamma_i^i(t_i) - \hat{\gamma}_i(t)||^4 |\sigma(\hat{\gamma}_i, u)||^2 + \sum_{i=0}^{\infty} \frac{1}{2^i} ||\gamma_i^i(t_i) - \hat{\eta}_i(t)||^4 |\sigma(\hat{\eta}_i, u)||^2\]

\[\leq 3\beta^{\frac{1}{2}}L^2 ||\hat{\gamma}_i - \hat{\eta}_i||^2_0 + 306\beta L^2 |\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^4 (2 + ||\hat{\gamma}_||^2_0 + ||\hat{\eta}_||^2_0)\]

\[+ 36\varepsilon \frac{\nu T - \hat{t}}{\nu T} L^2 (1 + ||\hat{\gamma}_||_0^6 + ||\hat{\eta}_||_0^6)\]

\[+ 153 \left( \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ ||\gamma_i^i(t_i) - \hat{\gamma}_i(t)||^4 + ||\gamma_i^i(t_i) - \hat{\eta}_i(t)||^4 \right] \right) L^2 (1 + ||\hat{\gamma}_||^6_0 + ||\hat{\eta}_||^6_0);\] (6.33)

and

\[J_3 = q(\hat{\gamma}_i, W_2(\hat{\eta}_i), \sigma^T(\hat{\gamma}_i, u) \partial_x \chi(\hat{\gamma}_i), u) - q(\hat{\eta}_i, W_2(\hat{\eta}_i), \sigma(\hat{\eta}_i, u) \partial_x h(\hat{\eta}_i), u)\]
≤ \( L||\hat{\gamma}_i - \hat{\eta}_i||_0 + 2\beta T L^2|\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})| \times ||\hat{\gamma}_i - \hat{\eta}_i||_0 + 18\beta L^2|\hat{\gamma}_i(\hat{t}) - \hat{\eta}_i(\hat{t})|^5 (2 + ||\hat{\gamma}_i||_0 + ||\hat{\eta}_i||_0) \\
+18L^2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |\gamma_i^0(t_i) - \hat{\gamma}_i(\hat{t})|^5 + |\eta_i^0(t_i) - \hat{\eta}_i(\hat{t})|^5 \right] (1 + ||\hat{\gamma}_i||_0 + ||\hat{\eta}_i||_0) \\
+36\varepsilon \frac{\nu T - \hat{t}}{\nu T} L^2 (1 + ||\hat{\gamma}_i||_0^6 + ||\hat{\eta}_i||_0^6). \tag{6.34} \)

We notice that, by the property (i) of \((\hat{t}, \hat{\gamma}_i, \hat{\eta}_i))

\[ 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\hat{t} - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right)^{\frac{3}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{3}{2}}, \]

\[ \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |\gamma_i^0(t_i) - \hat{\gamma}_i(\hat{t})|^5 + |\eta_i^0(t_i) - \hat{\eta}_i(\hat{t})|^5 \right] \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right)^{\frac{3}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{3}{2}}, \]

and

\[ \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |\gamma_i^0(t_i) - \hat{\gamma}_i(\hat{t})|^4 + |\eta_i^0(t_i) - \hat{\eta}_i(\hat{t})|^4 \right] \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right)^{\frac{3}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{3}{2}}. \]

Combining (6.30) with (6.34), then by (6.7) and (6.8) we can let \( \beta > 0 \) be large enough such that

\[ c \leq -\frac{\varepsilon}{\nu T} (\mathcal{Y}(\hat{\gamma}_i) + \mathcal{Y}(\hat{\eta}_i)) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} (342L + 36) L (1 + ||\hat{\gamma}_i||_0^6 + ||\hat{\eta}_i||_0^6) + \frac{c}{4}. \tag{6.35} \]

Recalling \( \nu = 1 + \frac{1}{2T(342L + 36)L} \) and \( \bar{a} = \frac{1}{2T(342L + 36)L} \wedge T \), by (3.23) and (6.5), the following contradiction is induced:

\[ c \leq \frac{\varepsilon}{\nu T} + \frac{c}{4} \leq \frac{c}{2}. \]

The proof is now complete. \( \square \)

To complete the previous proof, it remains to state and prove the following lemmas. In the following Lemmas of this section, let \( \hat{w}_1^i, \hat{w}_1^{i^*} \) and \( \hat{w}_2^i, \hat{w}_2^{i^*} \) be the definitions in Definition 5.2 with respect to \( w_1 \) defined by (6.17) and \( w_2 \) defined by (6.18), respectively.

Lemma 6.3. There exists a local modulus of continuity \( \rho_1 \) such that the functionals \( w_1 \) and \( w_2 \) defined by (6.17) and (6.18) satisfy condition (5.3).

Proof. From (6.3) and the definition of \( w_1 \), we have that, for every \( \hat{t} \leq t \leq s \leq T \) and \( \gamma_i \in \Lambda^i \),

\[ w_1(\gamma_i) - w_1(\gamma_{ts}) = W_1(\gamma_i) - 2^5 \beta T \mathcal{Y}(\gamma_i, \hat{\gamma}_i) - \varepsilon \frac{\nu T - \hat{t}}{\nu T} \mathcal{Y}(\gamma_i, \hat{\gamma}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} \mathcal{Y}(\gamma_i^i, \gamma_{ts}) \\
- W_1(\gamma_{ts}) + 2^5 \beta T \mathcal{Y}(\gamma_{ts}, \hat{\gamma}_i) + \varepsilon \frac{\nu T - s}{\nu T} \mathcal{Y}(\gamma_{ts}, \hat{\gamma}_i) + \sum_{i=0}^{\infty} \frac{1}{2^i} \mathcal{Y}(\gamma_{ts}^i, \gamma_{ts}) \\
= W_1(\gamma_i) - W_1(\gamma_{ts}) + \varepsilon \frac{t - s}{\nu T} \mathcal{Y}(\gamma_i) + \varepsilon ((s - \hat{t})^2 - (t - \hat{t})^2) + \sum_{i=0}^{\infty} \frac{1}{2^i} ((s - t_i)^2 - (t - t_i)^2) \\
\leq \rho_2(|s - t|, ||\gamma_i||_0) + 2T(2 + \varepsilon)|s - t|. \]

Taking \( \rho_1(l, x) = \rho_2(l, x) + 2T(2 + \varepsilon)l, (l, x) \in [0, \infty) \times [0, \infty) \), it is clear that \( \rho_1 \) is a local modulus of continuity and \( w_1 \) satisfies condition (5.3) with it. In a similar way, we show that \( w_2 \) satisfies condition (5.3) with this \( \rho_1 \). The proof is now complete. \( \square \)
Lemma 6.4. \( \hat{w}_1^{i*} \in \Phi(\hat{t}, \hat{\gamma}_t(\hat{t})) \) and \( \hat{w}_2^{i*} \in \Phi(\hat{t}, \hat{\eta}_t(\hat{t})) \).

Proof. We only prove \( \hat{w}_1^{i*} \in \Phi(\hat{t}, \hat{\gamma}_t(\hat{t})) \). \( \hat{w}_2^{i*} \in \Phi(\hat{t}, \hat{\eta}_t(\hat{t})) \) can be obtained by a symmetric way. Set \( r = \frac{1}{\varphi(T - \hat{t} \wedge \hat{t})} \), for given \( L > 0 \), let \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}^d) \) be a function such that \( \hat{w}_1^{i*}(t, x) - \varphi(t, x) \) has a maximum at \( (\hat{t}, \hat{x}) \in (0, T) \times \mathbb{R}^d \), moreover, the following inequalities hold true:

\[
|\hat{t} - \hat{t}| + |\hat{x} - \hat{\gamma}_{\hat{t}}(\hat{t})| < r = \frac{1}{2}(|T - \hat{t} \wedge \hat{t}|),
\]

\[
|\hat{w}_1^{i*}(\hat{t}, \hat{x})| + |\nabla_x \varphi(\hat{t}, \hat{x})| + |\nabla_x^2 \varphi(\hat{t}, \hat{x})| \leq L.
\]

By Lemma 5.4 of Chapter 4 in [11], we can modify \( \varphi \) such that \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}^d) \) bounded from below, \( \varphi, \varphi_t, \nabla_x \varphi \) and \( \nabla_x^2 \varphi \) grow in a polynomial way, \( \hat{w}_1^{i*}(t, x) - \varphi(t, x) \) has a strict maximum 0 at \( (\hat{t}, \hat{x}) \in (0, T) \times \mathbb{R}^d \) on \([0, T] \times \mathbb{R}^d\) and the above two inequalities hold true. If \( \tilde{t} < \hat{t} \), we have \( \varphi(\tilde{t}, \tilde{x}) = \frac{1}{2}(\hat{t} - \tilde{t}) > 0 \). If \( \tilde{t} \geq \hat{t} \), recall that \( w_1 \) is defined in (6.17), we consider the functional

\[
\Gamma(\gamma_t) = w_1(\gamma_t) - \varphi(t, \gamma_t(t)), \quad (t, \gamma_t) \in [\hat{t}, T] \times \Lambda.
\]

Note that \( w_1 \) is a continuous functional bounded from above and \( \varphi \) is a continuous function bounded from below, \( \Gamma \) is a continuous functional bounded from above on \( \Lambda \). Define a sequence of positive numbers \( \{\delta_i\}_{i \geq 0} \) by \( \delta_i = \frac{1}{2^i} \) for all \( i \geq 0 \). For every \( 0 < \delta < 1 \), by Lemma 2.13 we have that, for every \( (\tilde{t}_0, \tilde{\gamma}_{\tilde{t}_0}) \in [\hat{t}, T] \times \Lambda \) satisfying

\[
\Gamma(\gamma_{\tilde{t}_0}) \geq \sup_{(s, \gamma_s) \in [\hat{t}, T] \times \Lambda} \Gamma(\gamma_s) - \delta,
\]

there exist \( (\tilde{t}, \tilde{\gamma}_{\tilde{t}}) \in [\hat{t}, T] \times \Lambda \) and a sequence \( \{(\tilde{t}_i, \tilde{\gamma}_{\tilde{t}_i})\}_{i \geq 1} \subset [\hat{t}_0, T] \times \Lambda \) such that

(i) \( \overline{\gamma}(\tilde{t}_0, \tilde{\gamma}_{\tilde{t}_0}) \leq \delta, \overline{\gamma}(\tilde{t}_i, \tilde{\gamma}_{\tilde{t}_i}) \leq \frac{\delta}{2^i} \) and \( t_i \uparrow \hat{t} \) as \( i \to \infty \),

(ii) \( \Gamma(\tilde{\gamma}_{\tilde{t}_i}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \overline{\gamma}(\tilde{t}_i, \tilde{\gamma}_{\tilde{t}_i}) \geq \Gamma(\tilde{\gamma}_{\tilde{t}_0}) \), and

(iii) for all \( (s, \gamma_s) \in [\hat{t}, T] \times \Lambda \) \( \backslash \{(\tilde{t}, \tilde{\gamma}_{\tilde{t}})\} \),

\[
\Gamma(\gamma_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} \overline{\gamma}(\tilde{t}_i, \gamma_s) < \Gamma(\gamma_{\tilde{t}}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \overline{\gamma}(\tilde{t}_i, \gamma_{\tilde{t}_i}).
\]

We should note that the point \( (\tilde{t}, \tilde{\gamma}_{\tilde{t}}) \) depends on \( \delta \). By the definitions of \( \hat{w}_1^i \) and \( \hat{w}_1^{i*} \), we have

\[
\hat{w}_1^{i*}(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) = \limsup_{s \geq \hat{t}, (s, y) \to (\hat{t}, \hat{x})} (\hat{w}_1^i(s, y) - \varphi(s, y)) = \limsup_{s \geq \hat{t}, (s, y) \to (\hat{t}, \hat{x})} \left( \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} w_1(\xi_s) - \varphi(s, y) \right).
\]

Note that by Lemma 6.3, \( w_1 \) satisfies condition (5.3). Then, for every \( (s, \xi_s) \in [\hat{t}, \hat{t}] \times \Lambda \),

\[
w_1(\xi_s) \leq w_1(\xi_s, \hat{t}) + \rho_1(|\hat{t} - s|, ||\xi_s||_0).
\]
By the definition of $w_1$, there exists a constant $M_4 > 0$ such that
\[
\sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} w_1(\xi_s) = \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y, \|\xi_s\|_0 \leq M_4} w_1(\xi_s). \tag{6.41}
\]
Thus, by (6.40) and (6.41),
\[
\begin{align*}
\limsup_{s \geq \bar{t}, (s, y) \to (\bar{t}, \bar{x})} \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} [w_1(\xi_s) - \varphi(s, y)] & \leq \limsup_{s \geq \bar{t}, (s, y) \to (\bar{t}, \bar{x})} \left( \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} w_1(\xi_s) - \varphi(s, y) \right) \\
& \leq \limsup_{s \geq \bar{t}, (s, y) \to (\bar{t}, \bar{x})} \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y, \|\xi_s\|_0 \leq M_4} [w_1(\xi_s) - \varphi(s, y)] \\
& \leq \limsup_{s \geq \bar{t}, (s, y) \to (\bar{t}, \bar{x})} \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} [w_1(\xi_s) - \varphi(s, y)]. \tag{6.42}
\end{align*}
\]
Therefore, by (6.39) and (6.42),
\[
\tilde{w}_1^t(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) = \limsup_{s \geq \bar{t}, (s, y) \to (\bar{t}, \bar{x})} \sup_{\xi_s \in \Lambda^t, \xi_s(s) = y} [w_1(\xi_s) - \varphi(s, y)] \leq \sup_{(s, \gamma_s) \in [\bar{t}, T] \times \Lambda^t} \Gamma(\gamma_s).
\]
Combining with (6.38),
\[
\Gamma(\tilde{\gamma}_t) \geq \sup_{(s, \gamma_s) \in [\bar{t}, T] \times \Lambda^t} \Gamma(\gamma_s) - \delta \geq \tilde{w}_1^t(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) - \delta. \tag{6.43}
\]
Recall that $\tilde{w}_1^t \geq \tilde{w}_1^i$. Then, by the definition of $\tilde{w}_1^i$, the property (ii) of $(\bar{t}, \tilde{\gamma}_t)$ and (6.43),
\[
\tilde{w}_1^i(\bar{t}, \tilde{\gamma}_t(\bar{t})) - \varphi(\bar{t}, \tilde{\gamma}_t(\bar{t})) \geq \tilde{w}_1^i(\bar{t}, \gamma_t(\bar{t})) - \varphi(\bar{t}, \gamma_t(\bar{t})) \geq w_1(\gamma_t) - \varphi(\bar{t}, \gamma_t(\bar{t})) \geq \Gamma(\tilde{\gamma}_t) \geq \tilde{w}_1^i(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) - \delta = -\delta. \tag{6.44}
\]
Noting that $\nu$ is independent of $\delta$ and $\varphi$ is a continuous function bounded from below, by the definitions of $\Gamma$ and $w_1$ (6.2) and (3.23) there exists a constant $M_5 > 0$ depending only on $\varphi$ that is sufficiently large that $\Gamma(\tilde{\gamma}_t) < \Gamma(\gamma_t) < \Gamma(\gamma_t) - 1$ for all $t \in [\bar{t}, T]$ and $\|\gamma_t\|_0 \geq M_5$. Thus, we have $\|\tilde{\gamma}_t\|_0 \leq M_5$. In particular, $|\tilde{\gamma}_t(t)| < M_5$. Letting $\delta \to 0$, by the similar proof procedure of (5.12) and (5.13), we obtain
\[
\bar{t} \to \bar{t}, \tilde{\gamma}_t(\bar{t}) \to \bar{x}, \tilde{w}_1^i(\bar{t}, \tilde{\gamma}_t(\bar{t})) \to \tilde{w}_1^i(\bar{t}, \bar{x}) \text{ as } \delta \to 0. \tag{6.45}
\]
Noting that $\tilde{w}_1^i(t, x) - \varphi(t, x)$ has a strict maximum 0 at $(\bar{t}, \bar{x}) \in (0, T) \times \mathbb{R}^d$ on $[0, T] \times \mathbb{R}^d$, by (6.37) and (6.45) there exists a constant $0 < \Delta < 1$ such that for all $0 \leq \delta < \Delta$,
\[
\varphi(\bar{t}, \tilde{\gamma}_t(\bar{t})) \geq \tilde{w}_1^i(\bar{t}, \tilde{\gamma}_t(\bar{t})) \geq \tilde{w}_1^i(\bar{t}, \bar{x}) - 1 \geq -(L + 1).
\]
Then, by the definitions of $\Gamma$ and $w_1$, (6.23), (6.2) and (6.44), there exists a constant $M_6 > 0$ depending only on $L$ such that, for all $0 < \delta < \Delta$ and $\|\gamma_t\|_0 \geq M_6$ satisfying $\tilde{\gamma}_t(t) = \gamma_t(\bar{t})$,
\[
\Gamma(\tilde{\gamma}_t) < -2 \leq \Gamma(\tilde{\gamma}_t_0) - 1 \leq \sup_{(s, \gamma_s) \in [\bar{t}, T] \times \Lambda^t} \Gamma(\gamma_s) - 1.
\]
Thus, we have $||\overline{\gamma}_t||_0 < M_6$ for all $0 < \delta < \Delta$. From (6.36), it follows that $\overline{t} < \hat{t} + \frac{|T - \hat{t}|}{2} \leq T$. Then, by (6.45), we have $\overline{t} < T$ provided that $\delta > 0$ is small enough. Thus, the definition of the viscosity subsolution can be used to obtain the following result:

$$\partial_t \mathcal{S}(\gamma_t) + H(\gamma_t, W_1(\gamma_t), \partial_x \mathcal{S}(\gamma_t), \partial_{xx} \mathcal{S}(\gamma_t)) \geq c. \quad (6.46)$$

where, for every $(t, \gamma_t) \in [\overline{t}, T] \times \Lambda,$

$$\mathcal{S}(\gamma_t) := \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\gamma_t) + \varepsilon \Upsilon(\gamma_t, \hat{\gamma}_t) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_{i_t}, \gamma_t) + 2^5 \beta \Upsilon(\gamma_t, \Upsilon_t) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_{i_t}^i, \gamma_t) + \varphi(t, \gamma_t(t)),$$

$$\partial_t \mathcal{S}(\gamma_t) := -\frac{\varepsilon}{\nu T} \Upsilon(\gamma_t) + 2\varepsilon(t - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} [(t - t_i) + (t - \hat{t}_i)] + \varphi(t, \gamma_t(t)),$$

$$\partial_x \mathcal{S}(\gamma_t) := \varepsilon \frac{\nu T - t}{\nu T} \partial_x \Upsilon(\gamma_t) + \varepsilon \partial_x \Upsilon(\gamma_t - \hat{\gamma}_{i,t}) + 2^5 \beta \partial_x \Upsilon(\gamma_t - \Upsilon_t)$$
$$+ \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_t - \gamma_{i,t}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_t - \gamma_{i,t}^i) \right] + \nabla_x \varphi(t, \gamma_t(t)),$$

$$\partial_{xx} \mathcal{S}(\gamma_t) := \varepsilon \frac{\nu T - t}{\nu T} \partial_{xx} \Upsilon(\gamma_t) + \varepsilon \partial_{xx} \Upsilon(\gamma_t - \hat{\gamma}_{i,t}) + 2^5 \beta \partial_{xx} \Upsilon(\gamma_t - \Upsilon_t)$$
$$+ \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_t - \gamma_{i,t}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\gamma_t - \gamma_{i,t}^i) \right] + \nabla^2_x \varphi(t, \gamma_t(t)).$$

We notice that $||\overline{\gamma}_t||_0 < M_6$ for all $0 < \delta < \Delta$ and $M_6$ only depends on $L$. Then letting $\delta \rightarrow 0$ in (6.46), by the definition of $H$, (6.37) and (6.45), it follows that there exists a constant $C_0 \geq 0$ depending only on $L$ such that $\varphi(t, \hat{x}) \geq -C_0$. The proof is now complete. \(\square\)

**Lemma 6.5.** The functionals $w_1$ and $w_2$ defined by (6.17) and (6.18) satisfy the conditions of Theorem 5.3 with $\varphi$, where $\varphi$ is the function defined by (6.19).

**Proof.** From (4.28) and (6.2), $w_1$ and $w_2$ are continuous functionals bounded from above and satisfy (5.2). By Lemmas 6.3 and 6.4, $w_1$ and $w_2$ satisfy condition (5.3), and $w_1^k \Phi(\hat{t}, \hat{\gamma}_t(\hat{t}))$ and $w_2^k \Phi(\hat{t}, \hat{\eta}_t(\hat{t}))$. Moreover, let $\varphi$ be the function defined by (6.19). By Lemma 3.4 and (6.6) we obtain that, for all $(t, (\gamma_t, \eta_t)) \in [\overline{t}, T] \times (\Lambda^i \otimes \Lambda^i)$,

$$w_1(\gamma_t) + w_2(\eta_t) - \varphi(\gamma_t(t), \eta_t(t)) = w_1(\gamma_t) + w_2(\eta_t) - \beta^\frac{1}{2} \gamma(t) - \eta_t(t)^2$$
$$\leq \Psi_1(\gamma_t, \eta_t) \leq \Psi_1(\hat{\gamma}_t, \hat{\eta}_t) = w_1(\hat{\gamma}_t) + w_2(\hat{\eta}_t) - \beta^\frac{1}{2} \gamma(\hat{t}) - \eta_t(\hat{t})$$
$$= w_1(\hat{\gamma}_t) + w_2(\hat{\eta}_t) - \varphi(\hat{\gamma}_t(\hat{t}), \hat{\eta}_t(\hat{t})). \quad (6.47)$$

where the last inequality becomes equality if and only if $t = \hat{t}$, $\gamma_t = \hat{\gamma}_t$, $\eta_t = \hat{\eta}_t$. Then we obtain that $w_1(\gamma_t) + w_2(\eta_t) - \varphi(\gamma(t), \eta(t))$ has a maximum over $\Lambda^i \otimes \Lambda^i$ at a point $(\hat{\gamma}_t, \hat{\eta}_t)$ with $\hat{t} \in (0, T)$. Thus $w_1$ and $w_2$ satisfy the conditions of Theorem 5.3 with $\varphi$ defined by (6.19). \(\square\)

**Lemma 6.6.** The maximum points $(\hat{\gamma}_t^k, \hat{\eta}_t^k)$ satisfy condition (6.27).
Proof. Without loss of generality, we may assume \( s_k \leq l_k \), by (6.21), (6.3), (6.6) and the definitions of \( w_1 \) and \( w_2 \), we have that
\[
\begin{align*}
& w_1(\hat{\gamma}_k^l) + w_2(\hat{\eta}_k^l) - \beta^k \hat{\gamma}_k^l(l_k) - \hat{\eta}_k^l(s_k)^2 \\
\leq & \Psi_1(\hat{\gamma}_k^l, \hat{\eta}_k^l) - W_2(\hat{\eta}_k^l) + W_2(\hat{\eta}_k^l, \hat{\gamma}_k^l) - \varepsilon \left[ \bar{\mathbb{E}}(\hat{\gamma}_k^l, \hat{\gamma}_k^l) + \mathbb{E}(\hat{\eta}_k^l, \hat{\gamma}_k^l) \right] \\
\leq & \Psi_1(\hat{\gamma}_k^l, \hat{\eta}_k^l) + \rho_2(l_k - s_k, ||\hat{\eta}_k^l||_0) - \varepsilon \left[ \bar{\mathbb{E}}(\hat{\gamma}_k^l, \hat{\gamma}_k^l) + \mathbb{E}(\hat{\eta}_k^l, \hat{\gamma}_k^l) \right].
\end{align*}
\]
By (6.23), \( w_2(\hat{\eta}_k^l) \rightarrow w_2(\hat{\eta}_l) \) as \( k \rightarrow \infty \). Then by that \( w_2 \) satisfies condition (6.2), there exists a constant \( M_7 > 0 \) that is sufficiently large that
\[
||\hat{\eta}_k^l||_0 \leq M_7, \text{ for all } k > 0.
\]
Letting \( k \rightarrow \infty \) in (6.48), by (6.22) and (6.23) we have that
\[
\Psi_1(\hat{\gamma}_k^l, \hat{\eta}_k^l) = w_1(\hat{\gamma}_k^l) + w_2(\hat{\eta}_k^l) - \beta^k \hat{\gamma}_k^l(l_k) - \hat{\eta}_k^l(s_k)^2 \leq \Psi_1(\hat{\gamma}_k^l, \hat{\eta}_k^l) - \varepsilon \lim \sup_{k \rightarrow \infty} \left[ \bar{\mathbb{E}}(\hat{\gamma}_k^l, \hat{\gamma}_k^l) + \mathbb{E}(\hat{\eta}_k^l, \hat{\gamma}_k^l) \right].
\]
Thus,
\[
\lim_{k \rightarrow \infty} \left[ \bar{\mathbb{E}}(\hat{\gamma}_k^l, \hat{\gamma}_k^l) + \mathbb{E}(\hat{\eta}_k^l, \hat{\gamma}_k^l) \right] = 0.
\]
Then by (6.23) we get (6.25) holds true. The proof is now complete. \( \square \)

Remark 6.7. The continuity condition (6.3) is crucial in our current approach. In fact, condition (6.3) is used to prove (6.3) and Lemmas 6.3 and 6.6. In particular, since \( \partial_{xx} S_u(\cdot) \) is not equal to 0 (see Lemma 7.1), the convergence property (6.8) is the key to prove comparison theorem. It will be very interesting to structure a new smooth function to see if we can apply it to prove comparison theorem without condition (6.3).

7 Application to BSHJB equations.

In this section, we show that our PHJB equations includes backward stochastic HJB (BSHJB) equations as a special case (see also Example 4.5). In the following, we let \( n = d \).

We consider the controlled state equation:
\[
\begin{align*}
\bar{X}^{t,x,u}(s) &= x + \int_t^s b(W_l, \bar{X}^{t,x,u}(l), u(l))dl + \int_t^s \sigma(W_l, \bar{X}^{t,x,u}(l), u(l))dW(l), \ s \in [t, T],
\end{align*}
\]
and the associated BSDE:
\[
\begin{align*}
\bar{Y}^{t,x,u}(s) &= \bar{\phi}(W_T, \bar{X}^{t,x,u}(T)) + \int_s^T \bar{q}(W_l, \bar{X}^{t,x,u}(l), \bar{Y}^{t,x,u}(l), \bar{Z}^{t,x,u}(l), u(l))dl \\
& \quad - \int_s^T \bar{Z}^{t,x,u}(l)dW(l), \ s \in [t, T],
\end{align*}
\]
with \( \bar{b} : \Lambda \times \mathbb{R}^m \times U \rightarrow \mathbb{R}^m, \bar{\sigma} : \Lambda \times \mathbb{R}^m \times U \rightarrow \mathbb{R}^{m \times d}, \bar{q} : \Lambda \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \times U \rightarrow \mathbb{R} \) and \( \bar{\phi} : \Lambda_T \times \mathbb{R}^m \rightarrow \mathbb{R} \). The value functional of the optimal control is defined by
\[
\bar{V}(t, x) := \text{esssup}_{u(\cdot) \in \mathcal{U}[t,T]} \bar{Y}^{t,x,u}(t), \ (t, x) \in [0, T] \times \mathbb{R}^m.
\]
This problem is path dependent on \( \omega_l \) and state dependent on \( \bar{X}(t) \). Now we transform this problem into the path-dependent case.
In this section, for each \( t \in [0, T] \), define \( \Lambda^{d+m}_t \) as the set of continuous \( \mathbb{R}^{d+m} \)-valued functions on \([0, t]\). We denote \( \Lambda^{d+m} = \bigcup_{t \in [0, T]} \Lambda^{d+m}_t \). For any \((\omega_t, \xi_t), (\omega_T, \xi_T) \in \Lambda^{d+m}, (y, z) \in \mathbb{R} \times \mathbb{R}^d\) and \( u \in U \), we define \( b : \Lambda^{d+m} \times U \rightarrow \mathbb{R}^{d+m}, \sigma : \Lambda^{d+m} \times U \rightarrow \mathbb{R}^{(d+m) \times d}, q : \Lambda^{d+m} \times \mathbb{R} \times \mathbb{R}^d \times U \rightarrow \mathbb{R} \) and \( \phi : \Lambda^{d+m}_T \rightarrow \mathbb{R} \) as

\[
\begin{align*}
  b((\omega_t, \xi_t), u) &:= \begin{pmatrix} 0 \\ \tilde{b}(\omega_t, \xi_t(t), u) \end{pmatrix}, &
  \sigma((\omega_t, \xi_t), u) &:= \begin{pmatrix} I \\ \tilde{\sigma}(\omega_t, \xi_t(t), u) \end{pmatrix}, \\
  q((\omega_t, \xi_t), y, z, u) &:= \tilde{q}(\omega_t, \xi_t(t), y, z, u), &
  \phi(\omega_T, \xi_T) &:= \tilde{\phi}(\omega_T, \xi_T(T)).
\end{align*}
\]

We assume \( b, \sigma, q, \phi \) satisfy Hypothesis [2.6] then following [1.1], [1.3] and [1.4], for any \((\omega_t, \xi_t) \in \Lambda^{d+m} \) and \( u(\cdot) \in U[t, T] \) we can define \( X^{(\omega_t, \xi_t), u}, Y^{(\omega_t, \xi_t), u} \) and \( V^{(\omega_t, \xi_t)} := \text{esssup}_{u(\cdot) \in U[t, T]} Y^{(\omega_t, \xi_t), u}(t) \).

Noting \( V^{(\omega_t, \xi_t)} \) only depends on the state \( x = \xi_t(t) \) of the path \( \xi_t \) at time \( t \), we can rewrite \( X^{(\omega_t, \xi_t), u}, Y^{(\omega_t, \xi_t), u} \) and \((\omega_t, \xi_t)\) into \( X^{\omega_t, x, u}, Y^{\omega_t, x, u} \) and \( V^{(\omega_t, x)} \), respectively. Then, in view of Theorem [6.2] \( V^{(\omega_t, x)} \) is a unique viscosity solution to the PHJB equation:

\[
\begin{align*}
  \{ \partial_t V^{(\omega_t, x)} + \sup_{u \in U} \left[ (\nabla_x V^{(\omega_t, x)}, \tilde{b}(\omega_t, x, u)) + \frac{1}{2} \text{tr}(\nabla_x^2 V^{(\omega_t, x)} \tilde{\sigma}(\omega_t, x, u) \tilde{\sigma}^\top(\omega_t, x, u)) \right] \\
  + \frac{1}{2} \text{tr}(\nabla^2 V^{(\omega_t, x)} \tilde{\sigma}(\omega_t, x, u) \tilde{\sigma}^\top(\omega_t, x, u)) + \frac{1}{2} \text{tr}(\nabla^2 W^{(\omega_t, x)} \tilde{\sigma}(\omega_t, x, u) \tilde{\sigma}^\top(\omega_t, x, u)) \\
  + \phi(\omega_T, x, u) \right] = 0, \quad (t, x, \omega) \in [0, T] \times \mathbb{R}^m \times \Omega.
\end{align*}
\]

Here, \( \partial_t \) and \( \partial_{\gamma} \) are the spatial derivatives in \( \gamma_t \in \Lambda \), and \( \nabla_x \) and \( \nabla_x^2 \) are the classical partial derivatives in the state variable \( x \).

If \( V^{(\omega_t, x)} \) is smooth enough, applying functional Itô formula to \( V(W_t, x) \), we obtain

\[
dV(W_t, x) = [\partial_t V(W_t, x) + \frac{1}{2} \text{tr} \partial_{\gamma} V(W_t, x)]dt + \partial_{\gamma} V(W_t, x)dW(t), \quad \text{P-a.s.}, \quad (t, x) \in [0, T] \times \mathbb{R}^m.
\]

Define the pair of \( \mathcal{F}_t \)-adapted processes

\[
\begin{align*}
  (\tilde{V}(t, x), p(t, x)) := (V(W_t, x), \partial_{\gamma} V(W_t, x)), \quad (t, x) \in [0, T] \times \mathbb{R}^m.
\end{align*}
\]

and combine with [7.4], we have

\[
\begin{align*}
  d\tilde{V}(t, x) &= -\sup_{u \in U} \left[ (\nabla_x \tilde{V}(t, x), \tilde{b}(W_t, x, u)) + \frac{1}{2} \text{tr}(\nabla_x^2 \tilde{V}(t, x) \tilde{\sigma}(W_t, x, u) \tilde{\sigma}^\top(W_t, x, u)) \right] \\
  &\quad + \text{tr}(\tilde{\sigma}^\top(W_t, x, u) \nabla_x p(t, x)) + \tilde{q}(W_t, x, \tilde{V}(t, x), p(t, x)) \\
  &\quad + \tilde{\phi}(W_t, x, u) \nabla_x \tilde{V}(t, x, u)] + p(t, x)dW(t), \quad (t, x) \in [0, T] \times \mathbb{R}^m, \quad \text{P-a.s.,}
\end{align*}
\]

Thus we obtain that

**Theorem 7.1.** If the value functional \( V \in C^{1,2}_{p, \ell}(\Lambda^{d+m}) \), then the pair of \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \((\tilde{V}(t, x), p(t, x))\) defined by [7.5] is a classical solution to [7.6].

Notice that \((\tilde{V}(t, x), p(t, x))\) is only dependent on \( V \) which is a unique viscosity solution to PHJB equation [7.4], then we can give the definition of viscosity solutions to BSHJ equation [7.6].

**Definition 7.2.** If \( V \in C^{0}(\Lambda^{d+m}) \) is a viscosity solution to PHJB [7.3], then we call \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted process \( \tilde{V}(t, x) := V(W_t, x) \) defined by [7.3] is a viscosity solution to BSHJ equation [7.6].
By Theorem 6.2, we obtain that

**Theorem 7.3.** Let $b, \sigma, q, \phi$ satisfy Hypothesis 2.6. Then the $\mathcal{F}_t$-adapted process $\bar{V}(t,x) := V(W_t,x)$ defined by (7.5) is a unique viscosity solution to BSHJB equation (7.6).

**Remark 7.4.** If the coefficients in (7.6) are independent of $x$ and $u$, the BSHJB equation (7.6) reduces to a BSDE:

$$
\begin{cases}
    d\bar{V}(t) = -\bar{q}(W_t, \bar{V}(t), p(t))dt + p(t)dW(t), \quad t \in [0,T), \\
    \bar{V}(T) = \bar{\phi}(W_T),
\end{cases}
$$

(7.7)

We refer to the seminal paper by Pardoux and Peng [27] for the wellposedness of such BSDEs. Moreover, for any $(t, \gamma_t) \in [0,T] \times \Lambda$, by [27] the following BSDE on $[t,T]$ has a unique solution:

$$
\begin{cases}
    d\bar{V}^\gamma(s) = -\bar{q}(W^\gamma_s, \bar{V}^\gamma(s), p^\gamma(s))ds + p^\gamma(s)dW(s), \quad s \in [0,T), \\
    \bar{V}^\gamma(T) = \bar{\phi}(W^\gamma_T),
\end{cases}
$$

(7.8)

where

$$
W^\gamma_s(l) = \begin{cases}
    \gamma_t(l), & l \in [0,t], \\
    \gamma_t(l) + W(l) - W(t), & l \in (t,T].
\end{cases}
$$

Define $V(\gamma_t) := \bar{V}^\gamma(t)$, in the similar (even easier) process of the proof of Theorem 4.5, we show that $V(W_t)$ is a viscosity solution to BSDE (7.7) in our definition. On the other hand, it is easy to show that, for any $t \in [0,T]

\[ V(W_t) = \bar{V}(t), \quad \text{P-a.s.} \]

Thus, the viscosity solution to BSDE (7.7) coincides with its classical solution. Therefore, our definition of viscosity solution to BSHJB equation (7.6) is a natural extension of classical solution to BSDE (7.7).

**Appendix A** Borwein-Preiss variational principle

**Proof (of Lemma 2.13).** Define sequences $\{(t_1, \gamma_{t_1}^i)\}_{i \geq 1}$ and $\{B_i\}_{i \geq 1}$ inductively starting with

\[ B_0 := \{(s, \gamma_s) \in [t_0,T] \times \Lambda^0 | f(\gamma_s) - \delta_0 \rho(\gamma_s, \gamma_{t_0}^0) \geq f(\gamma_{t_0}^0)\}. \tag{A.1} \]

Since $(t_0, \gamma_{t_0}^0) \in B_0$, $B_0$ is nonempty. Moreover it is closed because both $f$ and $-\rho(\cdot, \gamma_{t_0}^0)$ are upper semicontinuous functionals. We also have that, for all $(s, \gamma_s) \in B_0$,

\[ \delta_0 \rho(\gamma_s, \gamma_{t_0}^0) \leq f(\gamma_s) - f(\gamma_{t_0}^0) \leq \sup_{(s, \gamma_s) \in [t,T] \times \Lambda^0} f(\gamma_s) - f(\gamma_{t_0}^0) \leq \varepsilon. \tag{A.2} \]

Take $(t_1, \gamma_{t_1}^1) \in B_0$ such that

\[ f(\gamma_{t_1}^1) - \delta_0 \rho(\gamma_{t_1}^1, \gamma_{t_0}^0) \geq \sup_{(s, \gamma_s) \in B_0} [f(\gamma_s) - \delta_0 \rho(\gamma_s, \gamma_{t_0}^0)] - \frac{\varepsilon}{2\delta_0}, \tag{A.3} \]

and define similarly

\[ B_1 := \left\{ (s, \gamma_s) \in B_0 \cap [t_1,T] \times \Lambda^{t_1} \left| f(\gamma_s) - \sum_{k=0}^{1} \delta_k \rho(\gamma_s, \gamma_{t_k}^k) \geq f(\gamma_{t_1}^1) - \delta_0 \rho(\gamma_{t_1}^1, \gamma_{t_0}^0) \right\} \right. \tag{A.4} \]
In general, suppose that we have defined \((t_j, \gamma_{t_j}^i)\), \(B_j\) for \(j = 1, 2, \ldots, i - 1\) satisfying

\[
f(\gamma_{t_j}^i) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_{t_j}^i, \gamma_{t_j}^k) \geq \sup_{(s, \gamma_s) \in B_{j-1}} \left[ f(\gamma_s) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_s, \gamma_{t_j}^k) \right] - \frac{\delta_j \varepsilon}{2^j \delta_0},
\]

(A.5)

and

\[
B_j := \left\{ (s, \gamma_s) \in B_{j-1} \cap [t_j, T] \times \Lambda^{i_j} \mid f(\gamma_s) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_s, \gamma_{t_j}^k) \geq f(\gamma_{t_j}^i) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_{t_j}^i, \gamma_{t_j}^k) \right\}.
\]

(A.6)

We choose \((t_i, \gamma_{t_i}^i) \in B_{i-1}\) such that

\[
f(\gamma_{t_i}^i) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_{t_i}^i, \gamma_{t_i}^k) \geq \sup_{(s, \gamma_s) \in B_{i-1}} \left[ f(\gamma_s) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_s, \gamma_{t_i}^k) \right] - \frac{\delta_i \varepsilon}{2^i \delta_0},
\]

(A.7)

and we define

\[
B_i := \left\{ (s, \gamma_s) \in B_{i-1} \cap [t_i, T] \times \Lambda^{i_i} \mid f(\gamma_s) - \sum_{k=0}^{i} \delta_k \rho(\gamma_s, \gamma_{t_i}^k) \geq f(\gamma_{t_i}^i) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_{t_i}^i, \gamma_{t_i}^k) \right\}.
\]

(A.8)

We can see that for every \(i = 1, 2, \ldots\), \(B_i\) is a closed and nonempty set. It follows from (A.7) and (A.8) that, for all \((s, \gamma_s) \in B_i\),

\[
\delta_i \rho(\gamma_s, \gamma_{t_i}^i) \leq \left[ f(\gamma_s) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_s, \gamma_{t_i}^k) \right] - \left[ f(\gamma_{t_i}^i) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_{t_i}^i, \gamma_{t_i}^k) \right]
\]

\[
\leq \sup_{(s, \gamma_s) \in B_{i-1}} \left[ f(\gamma_s) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_s, \gamma_{t_i}^k) \right] - \left[ f(\gamma_{t_i}^i) - \sum_{k=0}^{i-1} \delta_k \rho(\gamma_{t_i}^i, \gamma_{t_i}^k) \right] \leq \frac{\delta_i \varepsilon}{2^i \delta_0},
\]

which implies that

\[
\rho(\gamma_s, \gamma_{t_i}^i) \leq \frac{\varepsilon}{2^i \delta_0}, \text{ for all } (s, \gamma_s) \in B_i.
\]

(A.9)

Since \(\rho\) is a gauge-type function, inequality (A.9) implies that \(\sup_{(s, \gamma_s) \in B_i} d_\infty(\gamma_s, \gamma_{t_i}^i) \to 0\) as \(i \to \infty\), and therefore, \(\sup_{(s, \gamma_s) \in B_i} d_\infty(\gamma_s, \eta_i) \to 0\) as \(i \to \infty\). Since \([t, T] \times \Lambda^i\) is complete, by Cantor’s intersection theorem there exists a unique \((\hat{t}, \hat{\gamma}_{\hat{t}}^i) \in \bigcap_{i=0}^{\infty} B_i\). Obviously, we have \(d_\infty(\gamma_{t_i}^i, \hat{\gamma}_{\hat{t}}^i) \to 0\) and \(t_i \uparrow \hat{t}\) as \(i \to \infty\). Then \((\hat{t}, \hat{\gamma}_{\hat{t}}^i)\) satisfies (i) by (A.2) and (A.9). For any \((s, \gamma_s) \in [\hat{t}, T] \times \Lambda^\hat{t}\) and \((s, \gamma_s) \not= (\hat{t}, \hat{\gamma}_{\hat{t}}^i)\), we have \((s, \gamma_s) \not\in \bigcap_{i=0}^{\infty} B_i\), and therefore, for some \(j\),

\[
f(\gamma_s) - \sum_{k=0}^{\infty} \delta_k \rho(\gamma_s, \gamma_{t_k}^k) \leq f(\gamma_s) - \sum_{k=0}^{j} \delta_k \rho(\gamma_s, \gamma_{t_k}^k) \leq f(\gamma_{t_j}^i) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_{t_j}^i, \gamma_{t_k}^k).
\]

(A.10)

On the other hand, it follows from (A.1), (A.8) and \((\hat{t}, \hat{\gamma}_{\hat{t}}^i) \in \bigcap_{i=0}^{\infty} B_i\) that, for any \(q \geq j\),

\[
f(\gamma_0^\hat{t}) \leq f(\gamma_{t_j}^i) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma_{t_j}^i, \gamma_{t_k}^k) \leq f(\gamma_{t_j}^i) - \sum_{k=0}^{q-1} \delta_k \rho(\gamma_{t_j}^i, \gamma_{t_k}^k)
\]

\[
\leq f(\gamma_{\hat{t}}^i) - \sum_{k=0}^{q-1} \delta_k \rho(\gamma_{\hat{t}}^i, \gamma_{t_k}^k).
\]

(A.11)
Letting $q \to \infty$ in (A.11), we obtain
\[ f(\gamma^0_j) \leq f(\gamma^j_j) - \sum_{k=0}^{j-1} \delta_k \rho(\gamma^j_j, \gamma^k_k) \leq f(\gamma^j_j) - \sum_{k=0}^{\infty} \delta_k \rho(\gamma^j_j, \gamma^k_k), \] (A.12)
which verifies (ii). Combining (A.10) and (A.12) yields (iii). □

Appendix B  Existence and consistency for viscosity solutions.

Proof (of Theorem 4.5). We let $\varphi \in A^+(\tilde{\gamma}, V)$ with $(\tilde{t}, \tilde{\gamma}) \in [0, T) \times \Lambda$. For $0 < \delta \leq T - \hat{t}$, we have $\tilde{t} < \hat{t} + \delta \leq T$, then by the DPP (Theorem 2.10), we obtain the following result:
\[ 0 = V(\tilde{\gamma}) - \varphi(\tilde{\gamma}) = \operatorname{esssup}_{u(\cdot) \in U[\tilde{t}, \tilde{t} + \delta]} G^{\tilde{\gamma}, u^\varepsilon}_{\tilde{t}, \tilde{t} + \delta}[V(X^{\tilde{\gamma}, u^\varepsilon})] - \varphi(\tilde{\gamma}). \] (B.1)
Then, for any $\varepsilon > 0$ and $0 < \delta \leq T - \hat{t}$, we can find a control $u^\varepsilon(\cdot) \equiv u^\varepsilon, \delta(\cdot) \in U[\tilde{t}, \tilde{t} + \delta]$ such that the following result holds:
\[ -\varepsilon \delta \leq G^{\tilde{\gamma}, u^\varepsilon}_{\tilde{t}, \tilde{t} + \delta}[V(X^{\tilde{\gamma}, u^\varepsilon})] - \varphi(\tilde{\gamma}). \] (B.2)
We note that $G^{\tilde{\gamma}, u^\varepsilon}_{s, \tilde{t} + \delta}[V(X^{\tilde{\gamma}, u^\varepsilon})]$ is defined in terms of the solution of the BSDE:
\[ \begin{cases} 
\, dY^{\tilde{\gamma}, u^\varepsilon}(s) = -q(X^{\tilde{\gamma}, u^\varepsilon}(s), Z^{\tilde{\gamma}, u^\varepsilon}(s), u^\varepsilon(s))ds + Z^{\tilde{\gamma}, u^\varepsilon}(s)dW(s), & s \in [\tilde{t}, \tilde{t} + \delta], \\
\, Y^{\tilde{\gamma}, u^\varepsilon}(\tilde{t} + \delta) = V(X^{\tilde{\gamma}, u^\varepsilon}), 
\end{cases} \] (B.3)
by the following formula:
\[ G^{\tilde{\gamma}, u^\varepsilon}_{s, \tilde{t} + \delta}[V(X^{\tilde{\gamma}, u^\varepsilon})] = Y^{\tilde{\gamma}, u^\varepsilon}(s), \quad s \in [\tilde{t}, \tilde{t} + \delta]. \]
Applying functional Itô formula (2.4) to $\varphi(X^{\tilde{\gamma}, u^\varepsilon})$, we get that
\[ \varphi(X^{\tilde{\gamma}, u^\varepsilon}) = \varphi(\tilde{\gamma}) + \int_{\hat{t}}^{\tilde{t}} (\hat{L}\varphi)(X^{\tilde{\gamma}, u^\varepsilon}(l))dl - \int_{\hat{t}}^{\tilde{t}} q(X^{\tilde{\gamma}, u^\varepsilon}, \varphi(X^{\tilde{\gamma}, u^\varepsilon}), \sigma(X^{\tilde{\gamma}, u^\varepsilon}, u^\varepsilon(l))) \\
\times \partial_x \varphi(X^{\tilde{\gamma}, u^\varepsilon})dl + \int_{\hat{t}}^{\tilde{t}} \sigma(X^{\tilde{\gamma}, u^\varepsilon}, u^\varepsilon(l))\partial_x \varphi(X^{\tilde{\gamma}, u^\varepsilon})dW(l), \] (B.4)
where
\[ (\hat{L}\varphi)(\gamma_l, u) = \partial_t \varphi(\gamma_l) + (\partial \varphi(\gamma_l), b(\gamma_l, u)) + \frac{1}{2} \text{tr} [\partial_{xx} \varphi(\gamma_l) \sigma(\gamma_l, u)\sigma^*(\gamma_l, u)] + q(\gamma_l, \varphi(\gamma_l), \sigma^*(\gamma_l, u))\partial_x \varphi(\gamma_l, u), \quad (t, \gamma_l, u) \in [0, T] \times \Lambda \times U. \]
Set
\[ Y^{2, \tilde{\gamma}, u^\varepsilon}(s) := \varphi(X^{\tilde{\gamma}, u^\varepsilon}) - Y^{\tilde{\gamma}, u^\varepsilon}(s), \quad s \in [\tilde{t}, \tilde{t} + \delta], \]
\[ Z^{2, \tilde{\gamma}, u^\varepsilon}(s) := \sigma^*(X^{\tilde{\gamma}, u^\varepsilon}, u^\varepsilon(s))\partial_x \varphi(X^{\tilde{\gamma}, u^\varepsilon}) - Z^{\tilde{\gamma}, u^\varepsilon}(s), \quad s \in [\tilde{t}, \tilde{t} + \delta]. \]
Comparing (B.3) and (B.4), we have, P-a.s.,
\[ dY^{2, \tilde{\gamma}, u^\varepsilon}(s) \]
Obviously, \( \tilde{\varphi} \) where

\[
\phi \in \Lambda^i, \quad \varphi(\gamma_t^i, V_t) \leq C(1 + \| \gamma_t \|_0)^p.
\]

In view of Lemma 2.7, we also have

\[
\sup_{\gamma_t \in U} \mathbb{E} \left[ \sup_{\gamma_t \in U} |\tilde{\varphi}(\gamma_t, u)| \right] \leq C\delta.
\]

Thus, by \( \varphi \in A^+(\tilde{\gamma}_t, V) \),

\[
I = -\frac{1}{\delta} \mathbb{E} \left[ \left( \varphi(X_{\tilde{\gamma}_t+i}^i, u^\varepsilon(s)) - Y_{\tilde{\gamma}_t+i}^i, u^\varepsilon(s) \right) \right] \Gamma_t(i + \delta)
\]

\[
= -\frac{1}{\delta} \mathbb{E} \left[ \left( V(X_{\tilde{\gamma}_t+i}^i, u^\varepsilon(s)) - \varphi(X_{\tilde{\gamma}_t+i}^i) \right) \right] \Gamma_t(i + \delta) \leq 0;
\]

\[
II \leq \frac{1}{\delta} \mathbb{E} \left[ \int_{\tilde{\gamma}_t}^{\tilde{\gamma}_t+i} \sup_{u \in U} |\tilde{\varphi}(\tilde{\gamma}_t, u)| dl \right] = \mathcal{L}_\varphi(\tilde{\gamma}_t).
\]

Now we estimate higher order terms III and IV. By (B.7) and the dominated convergence theorem,

\[
\lim_{\delta \to 0} \mathbb{E} \sup_{\gamma_t \leq i \leq i + \delta} |\tilde{\varphi}(\gamma_t^i, u^\varepsilon(s)) - (\tilde{\varphi})(\gamma_t^i, u^\varepsilon(s))| = 0,
\]

where \( |A| \) is the value of the linear conditions, combining the regularity of \( \varphi \in \mathcal{C}_p^{1,2}(A) \), there exist an integer \( \bar{\gamma}_t \) and a constant \( C \geq 0 \) independent of \( u \in U \) such that, for all \( (t, \gamma_t, u) \in \mathbb{R} \times \Lambda \times U \),
then
\[ \lim_{\delta \to 0} |III| = \lim_{\delta \to 0} \sup_{t \leq t \leq t + \delta} \mathbb{E}[(\tilde{L}\varphi)(X^\xi_{t+\delta}, \beta) - (\tilde{L}\varphi)(\xi_t, \beta)] = 0; \]  
(B.10)
and, for some finite constant \( C > 0 \),
\[ |IV| \leq \frac{1}{\delta} \int_{t}^{t+\delta} \mathbb{E}\left[ |\varphi| - 1 \right] \mathbb{E}((\tilde{L}\varphi)(X^\xi_t, u_t)(l)|dl\right) \]
\[ \leq \frac{1}{\delta} \int_{t}^{t+\delta} \mathbb{E}(\Gamma(l)|l^2 \mathbb{E}((\tilde{L}\varphi)(X^\xi_t, u_t)(l)|\right) \frac{1}{2} dl \]
\[ \leq C(1 + \|\gamma_t\|_0)^2 \delta^{\frac{1}{2}}. \]  
(B.11)
Substituting (B.8), (B.9) and (B.11) into (B.6), we have
\[ -\varepsilon \leq \mathcal{L}\varphi(\gamma_t) + III + C(1 + \|\gamma_t\|_0)^2 \delta^{\frac{1}{2}}. \]  
(B.12)
Sending \( \delta \) to 0, by (B.10), we have
\[ -\varepsilon \leq \mathcal{L}\varphi(\gamma_t). \]
By the arbitrariness of \( \varepsilon \), we show \( \mathcal{V} \) is a viscosity subsolution to (1.3).

In a symmetric (even easier) way, we show that \( \mathcal{V} \) is also a viscosity supersolution to equation (1.3). This step completes the proof. \( \square \)

**Proof (of Theorem 4.6).** We prove the subsolution property only. Assume \( v \) is a viscosity subsolution. It is clear that \( v(\gamma_T) \leq \phi(\gamma_T) \) for all \( \gamma_T \in \Lambda_T \). For any \( (t, \gamma_t) \in [0, T] \times \Lambda \), since \( v \in C^{1,2}_{p}(\Lambda) \), by definition of viscosity subsolutions we see that \( \mathcal{L}v(\gamma_t) \geq 0 \).

On the other hand, assume \( v \) is a classical subsolution. Let \( \varphi \in \mathcal{A}^\tau(\gamma_t, v) \) with \( t \in [0, T) \). For every \( \alpha \in \mathbb{R}^d \) and \( \beta \in \mathbb{R}^{d \times n} \), let
\[ X(s) = \gamma_t(s) + \int_t^s \alpha dl + \int_t^s \beta dW(l), \quad s \in [t, T], \]
and \( X(s) = \gamma_t(s), \quad s \in [0, t] \). Then \( X(\cdot) \) is a continuous semi-martingale on \([t, T] \). Applying functional Itô formula (2.3) to \( \varphi \) and noticing that \( (v - \varphi)(\gamma_t) = 0 \), we have, for every \( 0 < \delta \leq T - t \),
\[ 0 \leq \mathbb{E}(\varphi - v)(X_{t+\delta}) \]
\[ = \mathbb{E} \int_t^{t+\delta} [\partial_t(\varphi - v)(X_t) + \langle \partial_x(\varphi - v)(X_t), \alpha \rangle] dl + \frac{1}{2} \mathbb{E} \int_t^{t+\delta} \text{tr}((\partial_{xx}(\varphi - v)(X_t))\beta\beta^T) dl \]
\[ = \mathbb{E} \int_t^{t+\delta} \tilde{\mathcal{H}}(X_t) dl, \]  
(B.13)
where
\[ \tilde{\mathcal{H}}(\eta) = \partial_t(\varphi - v)(\eta_t) + \langle \partial_x(\varphi - v)(\eta_t), \alpha \rangle + \frac{1}{2} \text{tr}((\partial_{xx}(\varphi - v)(\eta_t))\beta\beta^T), \quad (s, \eta_s) \in [0, T] \times \Lambda. \]
Letting \( \delta \to 0 \) in (B.13),
\[ \tilde{\mathcal{H}}(\gamma_t) \geq 0. \]  
(B.14)
Let \( \beta = 0 \), by the arbitrariness of \( \alpha \),
\[ \partial_t \varphi(\gamma_t) \geq \partial_t v(\gamma_t), \quad \partial_x \varphi(\gamma_t) = \partial_x v(\gamma_t). \]
Then, for every \( u \in U \), let \( \beta = \sigma(\gamma_t, u) \) in (B.14). Noting that \( \phi(\gamma_t) = v(\gamma_t) \), we have

\[
\frac{\partial_t \phi(\gamma_t)}{\partial t} + \langle \partial_x \phi(\gamma_t), b(\gamma_t, u) \rangle + \frac{1}{2} \text{tr} (\partial_{xx} \phi(\gamma_t) \sigma(\gamma_t, u) \sigma^\top(\gamma_t, u)) \\
+ q(\gamma_t, \phi(\gamma_t), \sigma^\top(\gamma_t, u) \partial_x \phi(\gamma_t), u)
\geq \frac{\partial_t v(\gamma_t)}{\partial t} + \langle \partial_x v(\gamma_t), b(\gamma_t, u) \rangle + \frac{1}{2} \text{tr} (\partial_{xx} v(\gamma_t) \sigma(\gamma_t, u) \sigma^\top(\gamma_t, u)) \\
+ q(\gamma_t, v(\gamma_t), \sigma^\top(\gamma_t, u) \partial_x v(\gamma_t), u).
\]

Note that \( L v(\gamma_t) \geq 0 \), taking the supremum over \( u \in U \), we see that

\[
L \phi(\gamma_t) \geq L v(\gamma_t) \geq 0.
\]

Thus, we have that \( v \) is a viscosity subsolution of equation (1.5). □

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