Patterson–Sullivan distributions for rank one symmetric spaces of the noncompact type

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Abstract

There is a remarkable relation between two kinds of phase space distributions associated to eigenfunctions of the Laplacian of a compact hyperbolic manifold: It was observed in [1] that for compact hyperbolic surfaces $X_T = \Gamma \setminus \mathbb{H}$ Wigner distributions $\int_{S^* X_T} a dW r_j = \langle \text{Op}(a) \phi r_j, \phi r_j \rangle_{L^2(X_T)}$ and Patterson–Sullivan distributions $PS r_j$ are asymptotically equivalent as $r_j \to \infty$. We generalize the definitions of these distributions to all rank one symmetric spaces of noncompact type and introduce off-diagonal elements $PS_{\lambda, \lambda}$. Further, we give explicit relations between off-diagonal Patterson–Sullivan distributions and off-diagonal Wigner distributions and describe the asymptotic relation between these distributions.

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1 Introduction

In this paper we generalize an interesting link between two kinds of phase space distributions which was observed in [1] for hyperbolic surfaces to all rank one symmetric spaces.
Riemannian symmetric spaces of the noncompact type. The distributions of interest arise in the study of quantum ergodicity. To put our results in a general context, we follow [20] to briefly recall some relevant notions of the framework of quantum ergodicity.

If \((X,g)\) is an \(n\)-dimensional compact Riemannian manifold with Laplace operator \(\Delta\), then \(L^2(X) = \oplus_{\lambda} \mathcal{H}_{\lambda}\), where \(\Delta = -\lambda_k\) on the eigenspaces \(\mathcal{H}_{\lambda_k}\) and \(\dim(\mathcal{H}_{\lambda_k}) < \infty\). We fix ordered orthonormal bases \(\{\varphi_k : 1 \leq i \leq \dim(\mathcal{H}_{\lambda_k})\}\) for each \(\mathcal{H}_{\lambda_k}\) to obtain a sequence \(\{\varphi_k : k = 1, 2, 3, \ldots, 1 \leq i \leq \dim(\mathcal{H}_{\lambda_k})\}\) of orthonormal eigenfunctions. Given a calculus of pseudodifferential operators on \(X\), i.e. an assignment \(\text{Op} : C^\infty(S^*X) \to \mathcal{B}(L^2(X))\) of bounded operators \(\text{Op}(a)\) to smooth zero order symbols \(a\), satisfying the usual requirements \[\text{[21]},\] we associate to a given eigenfunction \(\varphi_k\) a distribution \(W_k\), called the \textit{Wigner distribution} for \(\varphi_k\), defined by \(W_k(a) := \langle \text{Op}(a)\varphi_k, \varphi_k \rangle_{L^2(X)}\). A distribution \(\mu \in D'(S^*X)\) is called \textit{weak*-limit point} of the \(\{W_k\}\) if there is a subsequence \(S \subseteq \{\lambda_k\}\) such that \(\lim_{S} W_{k_j}(a) = \mu(a)\) for all \(a\). One of the problems in the framework of quantum ergodicity is the question: What are the weak*-limit points of the \(W_k\)? All such limit distributions are invariant measures for the geodesic flow on \(S^*X\). It is not known which limit points arise and how they depend on the choice of the \(\{\varphi_k\}\).

It was observed in [1] that for compact hyperbolic surfaces \(X = \Gamma\backslash \mathbb{H}\) Wigner distributions are asymptotically equivalent (and hence equivalent for the study of quantum ergodicity) to \textit{Patterson–Sullivan distributions} \(\hat{\mathcal{FS}}_{irr}\), which are also associated to the sequence \(\{\varphi_k\}\) of eigenfunctions. An interesting property of these Patterson–Sullivan distributions is that they are invariant under the geodesic flow, so one might hope that the study of these invariant distributions combined with the relations to Wigner distributions yield more insight into the questions of quantum ergodicity for symmetric spaces.

Before we state our results we have to make a few remarks about the special \(\Psi DO\)-calculus we use in this paper: S. Zelditch ([19]) introduced a natural quantization for \(G/K\), when \(G = PSU(1,1)\), \(K = PSO(2)\). It is in fact possible to generalize this calculus two all rank one symmetric spaces \(X := G/K\), where \(G\) is a connected semisimple Lie group with finite center and \(K\) a maximal compact subgroup of \(G\). The basic definitions and properties of this calculus are given in Section [4]. Full details with all computations concerning this calculus will appear in [12]. An advantage of this calculus is its \(G\)-equivariance: Fix a co-compact and torsion free discrete subgroup \(\Gamma\) of \(G\) and let \(SX\) denote the unit tangent bundle of \(X = G/K\). If \(a \in C^\infty(SX)\) is \(\Gamma\)-invariant (under the natural action of \(G\) on \(SX\), see Section [4]), then it yields a pseudodifferential operator on the quotient \(X_\Gamma := \Gamma\backslash G/K\).

Our setting is as follows: Let \(X = G/K\) denote a general rank one symmetric space of the noncompact type, where \(G\) is a connected semisimple Lie group with finite center and \(K\) a maximal compact subgroup of \(G\). Let \(G = KAN\) be a corresponding Iwasawa decomposition of \(G\) and let \(M\) denote the centralizer of \(A\) in \(K\). The geodesic boundary of \(X\) can be identified with the flag manifold \(B := K/M\). Let \(o := K \in G/K\) denote the \textit{origin} of the symmetric space \(X\). Further, let \(\Delta\), resp. \(\Delta_\Gamma\), denote the Laplace operator of \(X\), resp. \(X_\Gamma\). We consider the following automorphic eigenvalue problem on \(X = G/K\):

\[
\Delta \varphi = -c \varphi \\
\varphi(\gamma z) = \varphi(z) \text{ for all } \gamma \in \Gamma \text{ and for all } z \in X.
\]
In other words, we study the eigenfunctions of the Laplacian on the compact manifold $X_T = \Gamma \backslash X$. If the eigenfunctions $\varphi$ are real-valued, the eigenvalues $-c \in \mathbb{R}$, $-c \leq -(\rho, \rho)$, of $\Delta$ are of the form $-c := -c_\lambda := -((\lambda, \lambda) + (\rho, \rho))$, where $\lambda \in \mathfrak{a}^*$, the real dual of the Lie algebra $\mathfrak{a}$ of $A$, and where $(\cdot, \cdot)$ denotes the inner product on $\mathfrak{a}^*$ induced by the Killing form (see Section 2). We fix a complete $L^2(X_T)$-orthonormal basis $\{\varphi_\lambda\}$ of real-valued and $\Gamma$-invariant eigenfunctions, where the eigenvalues are repeated according to their multiplicity. We hence obtain a corresponding sequence of eigenvalue parameters $\lambda_j \in \mathfrak{a}^*$. Then

$$\Delta \varphi_{\lambda_j} = -((\lambda_j, \lambda_j) + (\rho, \rho))\varphi_{\lambda_j} \text{ for all } j,$$

$$\varphi_{\lambda_j}(\gamma z) = \varphi_{\lambda_j}(z) \text{ for all } \gamma \in \Gamma, z \in X, j \in \mathbb{N}_0.$$

If $Y$ is a manifold, $u$ a distribution or hyperfunction on $Y$ and $\varphi$ a test function, then we denote the pairing $\langle \varphi, u \rangle_Y$ by $\int_Y \varphi(y)u(dy)$.

For each eigenfunction $\varphi_{\lambda_j}$ (with exponential growth, see Section 3) of the negative Laplacian $-\Delta$ with corresponding eigenvalue $c_j = (\lambda_j, \lambda_j) + (\rho, \rho)$ there is a unique distribution boundary value (also described in Section 3 $T_{\lambda_j} \in D'(B)$) such that

$$\varphi_{\lambda_j}(x) = \int_B e^{i(\lambda_j + \rho)(x, b)}T_{\lambda_j}(db).$$

Here $\langle x, b \rangle$ denotes the horocyclic bracket defined in (2.13) below. Given $a \in C^\infty(SX)$, the Wigner distributions are defined by

$$W_{\lambda_j, \lambda_k}(a) := \langle \text{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle_{L^2(X_T)}.$$

In the special case when $j = k$, we write $W_{\lambda_j}(a) := W_{\lambda_j, \lambda_j}(a)$.

Let $B^{(2)} = (B \times B) \setminus \Delta = \{(b, b') \in B \times B : b \neq b'\}$ denote the set of pairs of distinct boundary points ($\Delta$ denotes the diagonal of $B \times B$). We will describe the geodesic boundary in Section 2. Each geodesic of $X$ has a unique forward limit point and a unique backward limit point in $B$. In particular, we identify $B^{(2)}$ with the space of geodesics. We will see in Section 3 that in the case of $\Gamma$-invariant eigenfunctions the boundary values $T_{\lambda_j}$ satisfy the following equivariance property:

$$T_{\lambda_j}(d^\gamma b) = e^{-i(\lambda_j + \rho)(\gamma a, \gamma b)}T_{\lambda_j}(db), \quad \gamma \in \Gamma. \quad (1.1)$$

It is then possible to introduce (see Section 5 for details) functions $d_{\lambda_j}$ on $B^{(2)}$ and a Radon transform $\mathcal{R} : C_c^\infty(SX) \to C_c^\infty(B^{(2)})$ such that the expression

$$\langle a, PS_{\lambda_j} \rangle_{SX} := \int_{B^{(2)}} d_{\lambda_j}(b, b') \mathcal{R}(a)(b, b')T_{\lambda_j}(db)T_{\lambda_j}(db') \quad (1.2)$$

defines a $\Gamma$-invariant distribution on $SX$. We call these distributions the Patterson-Sullivan distributions associated to the $\{\varphi_{\lambda_j}\}$. The $PS_{\lambda_j}$ are invariant under the geodesic flow and under time reversal (see Section 5 for details). The weight functions $d_{\lambda_j}$ will be called intermediate values because of (1.5), which generalizes the intermediate value formula (5.1) for hyperbolic surfaces.

Let $H : KAN \to a$ denote the Iwasawa projection (see Section 2) and let $w$ denote the non-trivial Weyl group element (see Section 2). Given $j \in \mathbb{N}_0$, define

$$L_{\lambda_j}a(g) := \int_N e^{-i(\lambda_j + \rho)(H(nw))}a(gn)dn, \quad a \in C(G), \quad (1.3)$$

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whenever the integral exists.

Following [1] we use a cutoff \( \chi \in C_c^\infty(X) \), which is a smooth replacement for the characteristic function of a fundamental domain \( F \) for \( \Gamma \) (cf. Section 3). A concrete relation between the \( W_{\lambda_j} \) and the \( PS_{\lambda_j} \) is given by the operators \( L_{\lambda_j} \) and it generalizes the "exact formula" in Theorem 1.1 of [1]:

**Theorem 1.1.** Let \( a \in C^\infty(SX_\Gamma) \). Then

\[
(\text{Op}(a) \varphi_{\lambda_j}, \varphi_{\lambda_j})_{L^2(X_\Gamma)} = (L_{\lambda_j}(\chi a), PS_{\lambda_j})_{SX}.
\]

(1.4)

Still following [1], we also define normalized Patterson–Sullivan distributions

\[
\widehat{PS}_{\lambda_j} := \frac{1}{\langle 1, PS_{\lambda_j} \rangle_{SX}} PS_{\lambda_j},
\]

(1.5)

which satisfy the same normalization condition \( \langle 1, \widehat{PS}_{\lambda_j} \rangle_{SX_\Gamma} = 1 \) as the \( W_{\lambda_j} \) on the quotient \( SX_\Gamma \).

As was pointed out in the introduction of [1] it is of interest to also have analogous results for off-diagonal matrix entries. To this end we introduce (in Section 6) off-diagonal Patterson–Sullivan distributions \( PS_{\lambda_j, \lambda_k} \) such that \( PS_{\lambda_j, \lambda_j} = PS_{\lambda_j} \) for all \( j \in \mathbb{N}_0 \). We then prove the off-diagonal analog of Theorem 1.1:

**Theorem 1.2.** Let \( a \in C^\infty(SX_\Gamma) \). Then

\[
(\text{Op}(a) \varphi_{\lambda_j}, \varphi_{\lambda_k})_{L^2(X_\Gamma)} = (L_{\lambda_k}(\chi a), PS_{\lambda_j, \lambda_k}).
\]

(1.6)

Theorem 1.1 is an immediate consequence of Theorem 1.2 but we intentionally separated the definitions and the results. One reason is that the definitions are based on quite different ideas and that the \( PS_{\lambda_j} \) have nicer invariance properties than the \( PS_{\lambda_j, \lambda_k} \). Another reason is that the normalization of the \( PS_{\lambda_j} \) motivates the normalization of the \( PS_{\lambda_j, \lambda_k} \) (see Definition 6.8).

Finally, we generalize the "asymptotic formula" in Theorem 1.1 of [1] to off-diagonal elements:

**Theorem 1.3.** Let \( a \in C^\infty(SX_\Gamma) \). Assume that \( \lambda_{jn}, \lambda_{kn} \to \infty \) are sequences of spectral parameters such that \( |\lambda_{jn} - \lambda_{kn}| \leq \tau \) for some \( \tau > 0 \). Then

\[
(\text{Op}(a) \varphi_{\lambda_{jn}}, \varphi_{\lambda_{kn}})_{L^2(X_\Gamma)} = \langle a, \widehat{PS}_{\lambda_j, \lambda_k} \rangle_{SX_\Gamma} + O(1/|\lambda_{kn}|) \quad (n \to \infty).
\]

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## 2 Preliminaries

In this section we collect a number of geometric definitions and facts needed to formulate our main results.
### Semisimple Lie Groups

Let $G$ be a non-compact connected semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of $G$, and $\langle \cdot , \cdot \rangle$ the Killing form of $\mathfrak{g}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$ such that the form $(X,Y) \mapsto -\langle X, \theta Y \rangle$ is positive definite on $\mathfrak{g} \times \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of $\theta$ and $K$ the analytic subgroup of $G$ with Lie algebra $\mathfrak{k}$. We choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ its dual and $\mathfrak{a}_c^*$ the complexification of $\mathfrak{a}^*$. At this point we do not yet make the assumption that the rank of $X = G/K$, i.e. $\dim \mathfrak{a}$, is one. Later, however, it will be indispensable. Let $A = \exp \mathfrak{a}$ denote the corresponding analytic subgroup of $G$ and let $\log$ denote the inverse of the map $\exp : \mathfrak{a} \to A$.

Given $\lambda \in \mathfrak{a}^*$, put $\mathfrak{g}_\lambda = \{ X \in \mathfrak{g} : [H,X] = \lambda(H)X \ \forall H \in \mathfrak{a} \}$. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then $\lambda$ is called a (restricted) root and $m_\lambda = \dim(\mathfrak{g}_\lambda)$ is called its multiplicity. Let $\mathfrak{g}_C$ denote the complexification of $\mathfrak{g}$ and if $\mathfrak{s}$ is any subspace of $\mathfrak{g}$ let $\mathfrak{s}_C$ denote the complex subspace of $\mathfrak{g}_C$ spanned by $\mathfrak{s}$.

For $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$. Since $\langle \cdot , \cdot \rangle$ is positive definite on $\mathfrak{p} \times \mathfrak{p}$ we put $|\lambda| := (\lambda, \lambda)^{1/2}$ for $\lambda \in \mathfrak{a}^*$ and $|X| := (X,X)^{1/2}$ for $X \in \mathfrak{p}$. The $\mathbb{C}$-bilinear extension of $\langle \cdot , \cdot \rangle$ to $\mathfrak{a}_c^*$ will be denoted by the same symbol.

Let $\mathfrak{a}^*$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. The components of $\mathfrak{a}^*$ are called Weyl chambers. We fix a Weyl chamber $\mathfrak{a}^+$ and call a root $\alpha$ positive ($>0$) if it is positive on $\mathfrak{a}^+$. Let $\mathfrak{a}_c^+$ denote the corresponding Weyl chamber in $\mathfrak{a}_c^*$, that is the preimage of $\mathfrak{a}^+$ under the mapping $\lambda \mapsto H_\lambda$. Let $\Sigma$ denote the set of restricted roots, $\Sigma^+$ the set of positive roots and $\Sigma^-$ the set of negative roots.

Let $\Sigma_0 = \{ \alpha \in \Sigma : \frac{1}{2} \alpha \notin \Sigma \}$, and put $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$, $\Sigma_0^- = \Sigma^- \cap \Sigma_0$. We set $\rho := 2^{-1} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ and let $N$ denote the analytic subgroup of $G$ with Lie algebra $\mathfrak{n} := \Sigma_{\alpha \geq 0} \mathfrak{g}_\alpha$. Then $\Pi = \theta(\mathfrak{n}) = \Sigma_{\alpha < 0} \mathfrak{g}_\alpha$. The involutive automorphism $\theta$ of $\mathfrak{g}$ extends to an analytic involutive automorphism of $G$, also denoted by $\theta$, whose differential at the identity $e \in G$ is the original $\theta$. It thus makes sense to define $\overline{N} = \theta N$. The Lie algebra of $\overline{N}$ is $\theta(\mathfrak{n})$.

Let $M$ denote the centralizer of $A$ in $K$ and let $M'$ denote the normalization of $A$ in $K$. Let $W$ denote the (finite) Weyl group $M'/M$. The group $W$ acts as a group of linear transformations of $\mathfrak{a}$ and also on $\mathfrak{a}_c^*$ by $(s\lambda)(H) := \lambda(s^{-1}H)$ for $s \in W$, $H \in \mathfrak{a}$ and $\lambda \in \mathfrak{a}^*_c$. Let $r$ denote the order of $W$ and let $w_1, \ldots, w_r$ be a complete set of representatives in $M'$. Let $A^+ := \exp(\mathfrak{a}^+)$, $B := K/M$, $P := MAN$. Then we have the decompositions

1. $G = KAN$ (Iwasawa decomposition),
2. $G = \bigcup_{j=1}^r Pw_j P$ (Bruhat decomposition).

Here (1) means that each $g \in G$ can be uniquely written in the form

$$g = k(g) \exp H(g)n(g),$$

where $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$. The functions $k, H, n$ are called Iwasawa projections. In (2), the union is disjoint. Let $w^*$ denote the Weyl group element mapping $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Exactly one of the summands in (2), namely $Pw^*P$, is open in $G$. Thus the set $\overline{N}M\overline{A}N$ is open in $G$.

We call $\dim(\mathfrak{a})$ the real rank of $G$ and the rank of the symmetric space $X = G/K$. Let $\phi := K \in G/K$ denote the origin of $X$. If $X$ has rank one, the Weyl group has only two elements. In this case we denote the nontrivial Weyl group element by $w$ and pick an element $m' \in M'$ such that $m'M = w \in W$. 


By abuse of notation we write $w$ for $m'$. Then we have the important formula

$$waw^{-1} = a^{-1} \quad \forall a \in A. \quad (2.2)$$

If $Y$ is a manifold satisfying the second countability axiom we write $\mathcal{D}(Y)$ for the space of $C^\infty$ functions on $Y$ of compact support. $\mathcal{D}'(Y)$ denotes the dual space of distributions on $Y$. The space $\mathcal{E}(Y)$ denotes the space of $C^\infty$ functions on $Y$ and $\mathcal{E}'(Y)$ denotes the dual space of distributions on $Y$ of compact support.

**Normalization of Measures**

We briefly recall some normalizations of the measures on the homogeneous spaces we work with. We follow [8]. The Killing form induces euclidean measures on $A$, $a$ and $a^*$. For $l = \dim(A)$ we multiply these measures by $(2\pi)^{-l/2}$ and obtain invariant measures $da, dH$ and $d\lambda$ on $A, a$ and $a^*$. This normalization has the advantage that the euclidean Fourier transform of $A$ is inverted without a multiplicative constant. We normalize the Haar measures $dk$ and $dm$ on the compact groups $K$ and $M$ such that the total measure is 1. If $U$ is a Lie group and $P$ a closed subgroup, with left invariant measures $du$ and $dp$, the $U$-invariant measure $du_P = d(uP)$ on $U/P$ (if it exists) will be normalized by

$$\int_U f(u) du = \int_{U/P} \left( \int_P f(up) dp \right) du_P. \quad (2.3)$$

This measure exists in particular if $P$ is a compact subgroup of $U$. In particular, we have a $K$-invariant measure $dk_M = d(kM)$ on $K/M$ of total measure 1. We also have a $G$-invariant measure $dx = dg_K = d(gK)$ on $X = G/K$. By uniqueness, $dx$ is a constant multiple of the measure on $X$ induced by the Riemannian structure on $X$ given by the Killing form. The Haar measures $dn$ and $d\overline{m}$ on the nilpotent groups $N$ and $\overline{N}$ are normalized such that

$$\theta(dn) = d\overline{m}, \quad \int e^{-2\rho(H(\overline{m}))} d\overline{m} = 1. \quad (2.4)$$

The Haar measure on $G$ can ([8], Ch. I, §5) then be normalized such that

$$\int_G f(g) dg = \int_{KAN} f(kan) e^{2\rho(\log a)} dk da dn \quad (2.5)$$

for all $f \in C_c(G)$. Let $f_1 \in C_c(AN), f_2 \in C_c(G), a \in A$. Then ([8], pp. 182)

$$\int_N f_1(na) dn = e^{2\rho(\log a)} \int_N f_1(ann) dn \quad (2.7)$$

and

$$\int_G f_2(g) dg = \int_{KNA} f_2(kna) dk dn da = \int_{ANK} f_2(ank) da dn dk. \quad (2.8)$$

Let $f_3 \in C_c(X)$. It follows from (2.8) that

$$\int_X f_3(x) dx = \int_{AN} f_3(ann \cdot a) da dn. \quad (2.9)$$
For any (restricted) root $\alpha$ we write $\alpha_0 := \alpha/(\alpha,\alpha)$. We will need Harish-Chandra’s $c$-functions ([7], p. 163, the rank of $X$ is arbitrary)

$$e_s(\lambda) = \prod_{\alpha \in \Sigma^+} \Gamma \left( \frac{m_\alpha}{4} + \frac{1}{2} + \frac{(i\lambda,\alpha_0)}{2} \right) \Gamma \left( \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{(i\lambda,\alpha_0)}{2} \right), \quad (2.10)$$

where $s \in W$, $\Sigma^+ = \Sigma^+_0 \cap s^{-1}\Sigma^-_0$ and where $\Gamma$ denotes the classical Gamma-function. Let $X$ have rank one. Now, the set $\Sigma$ of (restricted) roots contains at most two positive elements: $\alpha$ and possibly $2\alpha$. We adopt the usual convention that $m_{2\alpha} = 0$ if $2\alpha$ is not a root. Harish-Chandra’s $c$-function ([5], Ch. IV, §6) is the meromorphic function

$$c(\lambda) = c_0 \frac{2^{-\langle i\lambda,\alpha_0 \rangle}}{\Gamma \left( \frac{m_\alpha}{4} + \frac{1}{2} + \frac{(i\lambda,\alpha_0)}{2} \right) \Gamma \left( \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{(i\lambda,\alpha_0)}{2} \right)}, \quad \lambda \in a_+^*, \quad (2.11)$$

where $c_0 = 2^{\frac{1}{2}m_\alpha + m_{2\alpha}} \Gamma \left( \frac{m_\alpha}{2} + \frac{m_{2\alpha}}{2} + \frac{1}{2} \right)$.

**Geodesics, Boundary and the Unit Tangent Bundle**

$G$ acts on $G/P$ via $g \cdot xP = gxP$ and $K/M \to G/P$, $kM \to kP$ is a diffeomorphism ([9], p. 407) inverted by $gP \mapsto k(g)M$, where $g = k(g) \exp H(g)\alpha(g)$. Hence this map intertwines the $G$-action on $G/P$ with the action on $K/M$ defined by $g \cdot kM = k(gk)M$. These spaces are thus equivalent for the study of $B = K/M = G/P$. Although the following remarks are basically trivial, we write them down for later reference: With respect to the actions described above, the stabilizer of $M \in K/M$ is the subgroup $P = MAN$. The action of the groups $AN$ and $P$ on $G/K$ are transitive. For the remainder of this section, let $X = G/K$ be of rank one. As above, let $w \in M'$ denote a representative of the nontrivial Weyl group element.

**Lemma 2.1.** $P = MAN$ acts transitively on $G/P \setminus \{P\}$.

**Proof.** This follows from the Bruhat decomposition

$$G = P \cup PwP \quad \text{(disjoint union)}.$$

In fact, let $gP \in G/P \setminus \{P\}$. Then $g \notin P$, so $g$ is of the form $p_1wp_2$, where $p_1, p_2 \in P$. Hence $gP = p_1wp = p_1 \cdot wP$ and we have proven that each $gP \neq P \in G/P$ lies in the $P$-orbit of $wP$. \hfill $\Box$

**Remark 2.2.** Let $H_0$ denote the unique unit vector (with respect to the norm on $a$ induced by the Killing form) in $a^+$. It is well known ([8]) that $K \cdot H_0 = S(p)$, i.e., the group $K$ acts transitively on the set $S(p)$ of unit vectors in the tangent space $T_oX = p$. The subgroup $AN = NA$ of $G$ acts transitively on $G/K$, so $G = NAK$ acts transitively on the the unit tangent bundle $SX$ of $X = G/K$. The group $M$ is the stabilizer in $K$ of $H_0 \in S(p)$. Hence the unit tangent bundle of $X$ can be identified $G$-equivariantly with the homogeneous space $G/M$. We will from now on write $SX = G/M$ (for $X$ of real rank one). The geodesic flow on $G/M$ reads as the action of $A$ by right translations on $G/M$. 

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As in the introduction, consider the space $B \times B$ and its diagonal $\Delta$. We let $B^{(2)} = (B \times B) \setminus \Delta$ denote the set of distinct boundary points. We may now describe the space of geodesics and the geodesic connections in the rank one case. We describe the map that assigns to a geodesic its forward and backward limit points.

We call $\gamma_{H_0}(t) = e^{tH_0} \cdot o$ the \textit{standard geodesic}. If we write $B = K/M$, the forward limit point $b_\infty$ of the standard geodesic identifies with $M \in K/M$ (that is $P \in G/P$) and (since $\text{Ad}_G(w)$ is $-\text{id}$ on $a$) its backward limit point $b_{-\infty}$ identifies with $wM \in K/M$ (that is $wP \in G/P$). Since $wM \neq M$ in $K/M$, the point $(M,wM)$ is an element of $B^{(2)}$ and the standard geodesic is the unique (up to parameter translation and time reversal) geodesic of $X$ that joins the boundary points $M$ and $wM$ at infinity. We also write $a_t := e^{tH_0} \in A$. Then the standard geodesic is the curve $t \mapsto a_t \cdot o$.

We consider the action of $G$ on $B^{(2)}$, given by

$$G \times B^{(2)} \to B^{(2)}, \quad g \cdot (b_1,b_2) = (g \cdot b_1,g \cdot b_2).$$

(2.12)

\textbf{Lemma 2.3.} $G$ acts transitively on $B^{(2)}$. The stabilizer of $(b_\infty,b_{-\infty}) \in B^{(2)}$ is the subgroup $MA$ of $G$.

\textbf{Proposition 2.4.} $B^{(2)} = G/MA$ as homogeneous spaces.

\textit{Proof.} Let $b_1 \neq b_2$ be points in $B$. Since $K$ acts transitively on $B$, we find a $k \in K$ such that $k \cdot b_1 = b_\infty$. Since $P$ acts transitively on $B \setminus \{b_\infty\}$, we also find a $p \in P$ such that $p \cdot k \cdot b_2 = b_{-\infty}$. Let $g = pk$. Then $g \cdot (b_1,b_2) = (b_\infty,b_{-\infty})$.

It remains to show $g \cdot (b_\infty,b_{-\infty}) = (b_\infty,b_{-\infty})$ if and only if $g \in MA$. Let $ma \in MA$. Then $ma \cdot b_\infty = b_\infty$ and $ma \cdot b_{-\infty} = m \cdot a \cdot wP = w\tilde{m}aP = wP$, since $M'$ normalizes $A$ and $M$. Hence $MA$ acts trivially on $(b_\infty,b_{-\infty}) \in B^{(2)}$. For the converse assume $g \cdot (b_\infty,b_{-\infty}) = (b_\infty,b_{-\infty})$. Then $g \cdot b_\infty$, so $g = ma \in MAN$.

It suffices to prove $n = e$. For $b \in B$ let $G_b$ denote the subgroup of $G$ fixing $b$. Then $n \in G_{b_\infty} \cap G_{b_{-\infty}} = MAN \cap wMANw^{-1}$, so $n \in wMANw^{-1} = MA \cdot wNw^{-1} = MAN$. Hence there exists an element $\overrightarrow{m} \in N$ such that $n\overrightarrow{m} = ma \in N\overline{N} \cap MA = \{e\}$ (cf. [9], Ch. VI, Exercise B2. See also [7], Lemma 1.6 on page 79.). But $N \cap \overline{N} = \{e\}$, since $g$ is the direct vector space sum of the root-spaces $g_a$. Hence $n = e$ as desired.

\textbf{Definition 2.5.} We will from now on always write $g(b,b')MA \in G/MA$ for the unique coset corresponding to $(b,b') \in B^{(2)}$. The representative $g(b,b')$ is uniquely determined modulo $MA$.

Remark (2.2) yields a $G$-equivariant identification $G/M = SX$. Another identification is $G/M = X \times B$: It is clear that with respect to the diagonal action of $G$ on $X \times B$ the group $M$ is the stabilizer of $(K,P) \in G/K \times G/P$. Using the Iwasawa decomposition (see [12] for details) we also see that $G$ acts transitively on the space $X \times B$. This induces the following $G$-equivariant identification $SX = X \times B$: If $(z,b) \in X \times B$, then let $v(z,b)$ denote the unit vector in $SX$ tangential to the geodesic through $z$ with forward endpoint $b$. This geodesic exists since $X$ has rank one (see [5], [11] and also [12] for details).

\textbf{Horocycle bracket and Iwasawa Projection}

In this subsection, we describe the so-called \textit{horocycle bracket} on $X \times B$, because we need some formulae corresponding to this inner product. For details on the
geometric interpretation of this horocycle bracket see \cite{7}, Ch. II (and \cite{12}).

Let $\langle \cdot, \cdot \rangle : X \times B \to a, (x, b) \mapsto \langle x, b \rangle$ be defined by

$$
\langle x, b \rangle = \langle gK, kM \rangle = -H(g^{-1}k). \tag{2.13}
$$

Each $(x, b)$ is of the form $(gK, kM)$ and it is easy to see that \autoref{lem:invariant} is well-defined. We remark that the use of $\langle \cdot, \cdot \rangle$ is a direct computation. The second part of Proof. (i)

For $h$ and by (2.14) this equals $\langle g \cdot k, k \cdot M \rangle$ which equals $\langle g \cdot k, k \cdot M \rangle$. Hence

Since $A$ normalizes $N$ this yields $\log(a') + \log(a)$.

\begin{lemma}
Let $g_1, g_2 \in G, k \in K$. Then $H(g_1 g_2 k) = H(g_1 k(g_2 k)) + H(g_2 k)$.
\end{lemma}

Proof. Decompose $g_2 k = k' a n$ and $g_1 k = k' a n'$. Then

$$
H(g_1 g_2 k) = H(k' a n' a n) = H(a' n' a).
$$

Since $A$ normalizes $N$ this equals $\log(a') + \log(a)$.

\begin{lemma}
Let $x = hK \in G/K$, $b = kM \in K/M$, $g \in G$. Then

$$
\langle g \cdot x, g \cdot b \rangle = \langle x, b \rangle + \langle g \cdot o, g \cdot b \rangle. \tag{2.14}
$$

Proof. By definition, $(g \cdot x, g \cdot b) = -H(h^{-1}g^{-1}k(gk))$. Then by \autoref{lem:invariant} applied to $g_1 = h^{-1}g^{-1}$ and $g_2 = g$ this equals

$$
-H(h^{-1}g^{-1}k) + H(gk) = -H(h^{-1}k) + H(gk).
$$

For $h = e$ we obtain $\langle g \cdot o, g \cdot b \rangle = -H(k) + H(gk) = H(gk)$. Hence

$$
\langle g \cdot x, g \cdot b \rangle - \langle g \cdot o, g \cdot b \rangle = [-H(h^{-1}k) + H(gk)] - [-H(k) + H(gk)],
$$

which equals $-H(h^{-1}k) = \langle hK, kM \rangle = \langle x, b \rangle$.

\begin{lemma}
Let $\gamma, g \in G$ and $w \in K$. Then

(i) $\langle g \cdot o, g \cdot M \rangle = H(g)$ and $\langle g \cdot o, g \cdot wM \rangle = H(gw)$.

(ii) $H(\gamma g) = H(g) + \langle \gamma \cdot o, \gamma g \cdot M \rangle$ and $H(\gamma gw) = H(gw) + \langle \gamma \cdot o, \gamma g \cdot wM \rangle$.

Proof. (i) is a direct computation. The second part of (ii) follows from the first part applied to cw instead of $g$. For this assertion, let $z = g \cdot o$. Then by (i)

$$
H(\gamma g) = \langle \gamma g \cdot o, \gamma g \cdot M \rangle = \langle \gamma \cdot z, \gamma g \cdot M \rangle
$$

and by (2.14) this equals $\langle z, g \cdot M \rangle + \langle \gamma \cdot o, \gamma g \cdot M \rangle = H(g) + \langle \gamma \cdot o, \gamma g \cdot M \rangle$.

\section{Helgason Boundary Values}

In this section we recall the Poisson transform, which plays a key role in the proofs of our results, and use it to prove the estimate \autoref{lem:boundary} which will allow us to define the Patterson–Sullivan distributions. Even though part of what we describe here could be done in greater generality we restrict ourselves to the case of rank one spaces.
**Eigenfunctions and Poisson Transform**

We fix a co-compact, torsion free discrete subgroup $\Gamma$ of $G$ and choose a $G$-invariant measure $\nu$ on $\Gamma \backslash G$ such that

$$\int_G f(x) dx = \int_{\Gamma \backslash G} \left( \sum_{\gamma} f(\gamma x) \right) d\nu(x)$$

for $f \in C_c(G)$. We will denote the Hilbert space $L^2(\Gamma \backslash G, \nu)$ simply by $L^2(\Gamma \backslash G)$.

The $G$-invariance of $\nu$ implies that the equation

$$(R_\Gamma(g)f)(\Gamma x) = f(\Gamma g)$$

($g, x \in G, f \in L^2(\Gamma \backslash G)$) defines a unitary representation $R_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$, which is called the *right-regular representation* of $G$ on $\Gamma \backslash G$.

As before, let $\Delta$ denote the Laplace operator of $X$. The eigenspaces corresponding to eigenvalues $-c \leq -\langle \rho, \rho \rangle$ of $\Delta$ are (loc. cit.) the spaces $E_\lambda(X) = \{ f \in E(X) : \Delta f = -\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle f \}$, where $\lambda \in a^*$ and where $\langle , \rangle$ denotes the inner product on $a^*$ induced by the Killing form as described in Section 2.

We fix a $\Gamma$-invariant eigenfunction $\varphi \in E_\lambda(X)$ and assume that $\varphi$ is normalized with respect to the $L^2(X_{\Gamma})$-norm. Then $\Delta \varphi = -\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle \varphi$.

Let $\mathcal{A}(B)$ denote the vector space of analytic functions on $B = K/M$, topologized as in [6], Section 5. The *analytic functionals* are (loc. cit.) the functionals in the dual space $\mathcal{A}'(B)$ of $\mathcal{A}(B)$. Fix $\lambda \in a^*$ and recall the following fundamental result (cf. [7], p. 508):

**Theorem 3.1.** The Poisson–Helgason transform $P_{\lambda} : \mathcal{A}'(B) \to E_\lambda(X)$ given by

$$P_{\lambda}(T)(z) := \int_B e^{(\lambda + \rho)(z,b)} T(db)$$

(3.1)

is a bijection of the dual space $\mathcal{A}'(B)$ onto the eigenspace $E_\lambda(X)$.

For an eigenfunction $f \in E_\lambda(X)$ of the Laplacian we call the unique functional $T_f$ with $f = P_{\lambda}(T_f)$, given by Theorem 3.1, the *boundary values* of $f$.

We will now consider a special class of these eigenfunctions that have distributional boundary values: Let $d_X$ denote the distance function on $X$ and define the space $E^*(X)$ of smooth functions of exponential growth by

$$E^*(X) := \left\{ f \in E(X) \mid \exists C > 0 : |f(x)| \leq C e^{C d_X(x, x)} \forall x \in X \right\}.$$  (3.2)

We put $E^*_\lambda(X) := E^*(X) \cap E_\lambda(X)$ and recall (2.10). Then (cf. [7], p. 508):

**Theorem 3.2.** Let $\lambda \in a^*_C$ be such that $e_{\omega}(\lambda) \neq 0$. Then $P_{\lambda}(\mathcal{D}'(B)) = E^*_\lambda(X)$.

$G$ acts on $B$, hence on $\mathcal{D}'(B)$ by push-forward: Given $T \in \mathcal{D}'(B)$, a test function $f \in \mathcal{E}(B)$ and $g \in G$, the action is $(gT)(f) = T(f \circ g^{-1})$. When we denote the pairing between distributions and test functions by an integral, we also write $T(d\gamma b)$ for $(\gamma T)(db)$. Consider a $\Gamma$-invariant eigenfunction $\varphi$
with boundary values $T_\varphi$. Then $\varphi(\gamma z) = \varphi(z)$ for all $\gamma$ and $z$ implies (recall $(g \cdot x, g \cdot b) = (x, b) + (g \cdot a, g \cdot b)$ from equation (2.14))

$$
\varphi(z) = \int_B e^{i(\lambda + \rho)(\gamma z, b)} T_\varphi(db) = \int_B e^{i(\lambda + \rho)(\gamma z, b)} T_\varphi(d\gamma b) = \int_B e^{i(\lambda + \rho)(\gamma z, b)} e^{i(\lambda + \rho)(\gamma a, \gamma b)} T_\varphi(d\gamma b).
$$

By uniqueness of the Poisson–Helgason transform (Theorem 3.1) we obtain

$$
T_\varphi(d\gamma b) = e^{-(i\lambda + \rho)(\gamma a, \gamma b)} T_\varphi(db). \tag{3.3}
$$

**Spherical Principal Series**

We recall some facts concerning the *principal series* representations of $G$. Following [7] and [18], let $\lambda \in \mathfrak{a}$ and consider the representation $\sigma_\lambda(\text{man}) = e^{(i\lambda + \rho) \log(a)}$ of $P = MAN$ on $\mathbb{C}$. We denote the *induced representation* on $G$ by $\pi_\lambda = \text{Ind}^G_P(\sigma_\lambda)$. The *induced picture* of this representation is constructed as follows: A dense subspace of the representation space is

$$
H_\lambda^\infty := \left\{ f \in C^\infty(G) : f(g\text{man}) = e^{-(i\lambda + \rho) \log(a)} f(g) \right\}
$$

with inner product

$$
(f_1, f_2) = \int_{K/M} f_1(k) \overline{f_2(k)} dk = \langle f_1|_{K}, f_2|_{K} \rangle_{L^2(K/M)}
$$

and corresponding norm $||f||^2 = \int_{K/M} |f(k)|^2 dk$. The group action of $G$ is given by $(\pi_\lambda(g)f)(x) = f(g^{-1}x)$. The actual Hilbert space, which we denote by $H_\lambda$, and the representation on $H_\lambda$, which we also denote by $\pi_\lambda$, is obtained by completion (cf. [18], Ch. 9). The representations $\pi_\lambda (\lambda \in \mathfrak{a})$ form the *spherical principal series* of $G$. $(\pi_\lambda, H_\lambda)$ is a unitary ([7], p. 528) and irreducible (loc. cit. p. 530) Hilbert space representation.

Given $f \in C^\infty(K/M)$ we may extend it to a function on $G$ by $\tilde{f}(g) = e^{-(i\lambda + \rho) H(g)} f(k(g))$. A direct computation shows that $\tilde{f} \in H_\lambda^\infty$. On the other hand, if $f \in H_\lambda^\infty$, then the restriction $f|_K$ of $f$ to $K$ is an element of $C^\infty(K/M)$. Moreover, if $f \in C^\infty(K/M)$ and if $\tilde{f}$ as above, then $\tilde{f}|_K = f$. The mapping $f \mapsto \tilde{f}$ described above is isometric with respect to the $L^2(K/M)$-norm. We may hence identify $C^\infty(K/M) \cong H_\lambda^\infty$. The advantage is that the representation space is independent of $\lambda$. The group action on $C^\infty(K/M)$ is realized by

$$
(\pi_\lambda(g)f)(kM) = f(k(g^{-1}k)M)e^{-(i\lambda + \rho) H(g^{-1}k)}. \tag{3.4}
$$

This is called the *compact picture* of the (spherical) principal series. Notice that for $g \in K$ the group action simplifies to the left-regular representation of the compact group $K$ on $K/M$.

Let $\lambda \in \mathfrak{a}$. It follows from

$$
(\pi_\lambda(g)1)(k) = e^{-(i\lambda + \rho) H(g^{-1}k)} = e^{i(\lambda + \rho) (gK, kM)} \tag{3.5}
$$

that the Poisson transform $P_\lambda(T) : G/K \to \mathbb{C}$ of $T \in \mathcal{D}'(B)$ is given by

$$
P_\lambda(T)(gK) = T(\pi_\lambda(g) \cdot 1). \tag{3.6}
$$
Let $\varphi$ denote a $\Gamma$-invariant eigenfunction of the Laplace operator with boundary values $T_{\varphi} \in \mathcal{D}'(B)$ such that $\varphi = P_\lambda(T_{\varphi})$. Let $\pi_\lambda$ denote the dual representation on $\mathcal{D}'(B)$ corresponding to $\pi_\lambda$. Since $\varphi$ is invariant, it follows from [17,9] and the uniqueness of the boundary values that $T_{\varphi}$ is invariant under the actions $\pi_\lambda(\gamma)$, $\gamma \in \Gamma$.

**Regularity of Distribution Boundary Values**

In this subsection we prove a regularity statement for distribution boundary values corresponding to Laplace eigenfunctions with eigenvalue parameter $\lambda \in \mathfrak{a}^*$ on a compact quotient $X_\Gamma$. These estimates may not be the sharpest possible, but they are sufficient for our purposes.

Let $T_{\varphi} \in \mathcal{D}'(K/M)$ be the (unique) preimage (under the Poisson transform) of a normalized $L^2(X_\Gamma)$-eigenfunction $\varphi$ (with exponential growth). Under the identification $H^\infty_\lambda \equiv C^\infty(K/M)$ we view $T_{\varphi}$ as a functional on $H^\infty_\lambda$: For $f \in H^\infty_\lambda$ let $T_{\varphi}(f)$ be defined by $T_{\varphi}(f_{|K})$. Then $T_{\varphi}$ is a continuous linear functional on $H^\infty_\lambda$, invariant under $\pi_\lambda(\gamma)$. As proven in [17], Theorem A.1.4, if $f$ is a smooth vector for the principal series representation, then $f \in H^\infty_\lambda$ is a smooth function on $G$. We consider the mapping

$$\Phi_{\varphi} : H^\infty_\lambda \to C^\infty(\Gamma \backslash G), \quad \Phi_{\varphi}(f)(\Gamma g) = T_{\varphi}(\pi_\lambda(g) f).$$

**Lemma 3.3.** $\Phi_{\varphi}$ is an isometry w.r.t. the norms of $L^2(K/M)$ and $L^2(\Gamma \backslash G)$.

**Proof.** The operator $\Phi_{\varphi}$ is equivariant with respect to the actions $\pi_\lambda$ on $H^\infty_\lambda$ and the right regular representation of $G$ on $L^2(\Gamma \backslash G)$. We pull-back the $L^2(\Gamma \backslash G)$ inner product onto the $(\mathfrak{g}, K)$-module $H^\infty_{\lambda,K}$ of $K$-finite and smooth vectors (which is dense in $H^\infty_\lambda$, [17], p. 81):

$$\langle f_1, f_2 \rangle_{L^2(\Gamma \backslash G)} := \langle \Phi_{\varphi}(f_1), \Phi_{\varphi}(f_2) \rangle_{L^2(\Gamma \backslash G)}.$$

Let $f_1 \in H^\infty_{\lambda,K}$. Then $A_{f_1} : H^\infty_{\lambda,K} \to \mathbb{C}$, $f_2 \mapsto \langle f_1, f_2 \rangle_{L^2(K/M)}$ is a conjugate-linear, $K$-finite functional on the $(\mathfrak{g}, K)$-module $H^\infty_{\lambda,K}$. This module is irreducible and admissible, since $H_\lambda$ is unitary and irreducible ([17], theorems 3.4.10 and 3.4.11). As $A_{f_1}$ is $K$-finite it is nonzero on at most finitely many $K$-isotypic components. It follows that there is a linear map $A : H^\infty_{\lambda,K} \to H^\infty_{\lambda,K}$ such that for each $f_1 \in H^\infty_{\lambda,K}$ the functional $A_{f_1}$ equals $f_2 \mapsto \langle A_{f_1}, f_2 \rangle_{L^2(K/M)}$. The equivariance of $\Phi_{\varphi}$ and the unitarity of $\pi_\lambda$ imply that $A$ is $(\mathfrak{g}, K)$-equivariant. Using Schur’s lemma for irreducible $(\mathfrak{g}, K)$-modules ([17], p. 80), we deduce that $A$ is a constant multiple of the identity and hence $\langle \cdot, \cdot \rangle_2$ is a constant multiple of the original $L^2(K/M)$-inner product on $H^\infty_{\lambda,K}$. This constant is 1:

First, $\Phi_{\varphi}(1) = P_\lambda(T_{\varphi}) = \varphi$ is the $K$-invariant lift of $\varphi$ to $L^2(\Gamma \backslash G)$. Then $\|\Phi_{\varphi}(1)\|_{L^2(\Gamma \backslash G)} = 1 = \|1\|_{L^2(K/M)}$. \(\square\)

Let $(y_j)$ and $(x_i)$ be bases for $\mathfrak{t}$ and $\mathfrak{p}$, respectively, such that $\langle y_j, y_i \rangle = -\delta_{ij}$, $\langle x_j, x_i \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes the Killing form. The Casimir operator of $\mathfrak{t}$ is $\Omega_t = \sum_j y_j^2$ and the Casimir operator of $\mathfrak{g}$ is

$$\Omega_g = -\sum_j x_j^2 + \Omega_t \in \mathcal{Z}(\mathfrak{g}),$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. 

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It follows from $T_\varphi(f) = \Phi_\varphi(f)(\Gamma e)$ that

$$|T_\varphi(f)| \leq \|\Phi_\varphi(f)\|_\infty. \quad (3.7)$$

We may now estimate this by a convenient Sobolev norm on $L^2(\Gamma \setminus G)$. Let $\tilde{\Delta}$ denote the Laplace operator of $\Gamma \setminus G$. Then we have

$$\tilde{\Delta} = -\Omega_g + 2\Omega_t,$$

where $\Omega_g$ and $\Omega_t$ are the Casimir operators on $G$ and $K$, respectively.

**Definition 3.4.** Let $s \in \mathbb{R}$. The Sobolev space $W^{2,s}(\Gamma \setminus G)$ is (cf. [14], p. 22) the space of functions $f$ on $\Gamma \setminus G$ satisfying $(1 + \tilde{\Delta})^{s/2} f \in L^2(\Gamma \setminus G)$ with norm

$$\|f\|_{W^{2,s}(\Gamma \setminus G)} = \|(1 + \tilde{\Delta})^{s/2} f\|_{L^2(\Gamma \setminus G)}.$$

Let $m = \dim(\Gamma \setminus G) = \dim(G)$, and let $s > m/2$. The Sobolev imbedding theorem for the compact space $\Gamma \setminus G$ ([14], p. 19) states that the identity $W^{2,s}(\Gamma \setminus G) \hookrightarrow C^{0}(\Gamma \setminus G)$ is a continuous inclusion ($C^{0}(\Gamma \setminus G)$ is equipped with the usual sup-norm $\| \cdot \|_\infty$). It follows that there exists a $C > 0$ such that

$$\|\Phi_\varphi(f)\|_\infty \leq C\|\Phi_\varphi(f)\|_{W^{2,s}(\Gamma \setminus G)} \quad \forall f \in C^{\infty}(K/M). \quad (3.8)$$

Now we derive the announced regularity estimate for the boundary values: First, by increasing the Sobolev order, we may assume $s/2 \in \mathbb{N}$, so

$$(1 + \tilde{\Delta})^{s/2} = (1 - \Omega_g + 2\Omega_t)^{s/2} \in \mathcal{U}(\mathfrak{g}).$$

Hence $(1 + \tilde{\Delta})^{s/2}$ commutes with each $G$-equivariant mapping. Let $f \in H^\infty_K$. Then

$$\|\Phi_\varphi(f)\|_{W^{2,s}(\Gamma \setminus G)} = \left\| (1 + \tilde{\Delta})^{s/2} \Phi_\varphi(f) \right\|_{L^2(\Gamma \setminus G)} = \left\| \Phi_\varphi((1 - \Omega_g + 2\Omega_t)^{s/2}(f)) \right\|_{L^2(\Gamma \setminus G)} = \left\| (1 - \Omega_g + 2\Omega_t)^{s/2}(f) \right\|_{L^2(K/M)}.$$ \quad (3.9)

Recall $\pi_\lambda(\Omega_t) = \Delta_{K/M}$ and $\Omega_g \in \mathcal{Z}(\mathfrak{g})$. Then (3.9) equals

$$\left\| \sum_{k=0}^{s/2} \binom{s/2}{k} (1 + 2\Delta_{K/M})^k (-\Omega_g)^{s/2-k}(f) \right\|_{L^2(K/M)} \leq \sum_{k=0}^{s/2} \binom{s/2}{k} \left\| (1 + 2\Delta_{K/M})^k (-\Omega_g)^{s/2-k}(f) \right\|_{L^2(K/M)}. \quad (3.10)$$

Assume $f \in H^\infty_{\lambda,K}$ and recall that $\Omega_g$ acts on the irreducible $\mathcal{U}(\mathfrak{g})$-module $H^\infty_{\lambda,K}$ by multiplication with the scalar $-(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)$ (cf. [18], p. 163), that is

$$\Omega_g | H^\infty_{\lambda,K} = - (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle) \text{id}_{H^\infty_{\lambda,K}}.$$
Then (3.10) equals
\[
\sum_{k=0}^{s/2} \binom{s/2}{k} \left(1 + 2\Delta_{K/M}\right)^k (|\lambda|^2 + |\rho|^2)^{s/2-k} \|f\|_{L^2(K/M)}.
\]

(3.11)

But \((|\lambda|^2 + |\rho|^2)^{-k} \leq 1 + |\rho|^{-s} =: C' (0 \leq k \leq s/2),\) so the term in (3.11) is bounded by
\[
C' \left(|\lambda|^2 + |\rho|^2\right)^{s/2} \sum_{k=0}^{s/2} \binom{s/2}{k} \left(1 + 2\Delta_{K/M}\right)^k \|f\|_{L^2(K/M)}.
\]

(3.12)

Since \(H^\infty_{\lambda,K}\) is dense in \(H^\infty_{\lambda}\), this bound holds for all \(f \in H^\infty_{\lambda}\). Using (3.7)-(3.12) we get
\[
|T_\varphi(f)| \leq C' \left(|\lambda|^2 + |\rho|^2\right)^{s/2} \sum_{k=0}^{s/2} \binom{s/2}{k} \left(1 + 2\Delta_{K/M}\right)^k \|f\|_{L^2(K/M)}.
\]

(3.13)

for all \(f \in H^\infty_{\lambda}\) and hence for all \(f \in C^\infty(K/M)\). We estimate (3.13) by a continuous \(C^\infty(K/M)\)-seminorm \(\|\cdot\|\) (independent of \(\varphi\)) and obtain:

**Proposition 3.5.** Let \(2s > \dim(G)\) such that \(s/2 \in \mathbb{N}\). There exists a continuous \(C^\infty(B)\)-seminorm \(\|\cdot\|\) such that
\[
|T_\varphi(f)| \leq (1 + |\lambda|)^s \|f\| \quad \forall f \in C^\infty(K/M)
\]

(3.14)

for the distribution boundary values \(T_\varphi\) corresponding to a real-valued and \(L^2(X_G)\)-normalized eigenfunction \(\varphi\) of \(\Delta_{\Gamma'}\) with eigenvalue \(-(|\lambda|^2 + |\rho|^2)\).

Each \(f \in C^\infty(B) \otimes C^\infty(B)\) has the form \(f = \sum_{i,j} c_{i,j} f_i \otimes f_j\). We define a cross-norm \(\|\cdot\|''\) on 
\[ C^\infty(B) \otimes C^\infty(B) \]

by
\[
\|f\|'' = \inf \left\{ \sum_{i,j} |c_{i,j}| \|f_i\|'' \|f_j\|'' : f = \sum_{i,j} c_{i,j} f_i \otimes f_j \right\}.
\]

This norm induces a continuous seminorm on the projective tensor product 
\[ C^\infty(B) \otimes_{\pi} C^\infty(B) \] (cf. [13], p. 435). Let \(\psi\) denote another normalized eigenfunction with distribution boundary values \(T_\psi \in \mathcal{D}'(B)\) and eigenvalue parameter \(\mu \in \mathfrak{a}^*\). Given \(f = \sum_{i,j} c_{i,j} f_i \otimes f_j \in C^\infty(B) \otimes C^\infty(B)\) we obtain
\[
|T_\varphi(f)| \leq \sum_{i,j} |c_{i,j}| \cdot |T_\varphi(f_i)| \cdot |T_\psi(f_j)|
\]

\[
\leq (1 + |\lambda|)^s (1 + |\mu|)^s \sum_{i,j} |c_{i,j}| \cdot \|f_i\|'' \cdot \|f_j\|'',
\]

(3.15)

which implies (by taking the infimum)
\[
|T_\varphi \otimes T_\psi(f)| \leq (1 + |\lambda|)^s (1 + |\mu|)^s \|f\|''
\]

(3.16)

for all \(f \in C^\infty(B) \otimes C^\infty(B)\). But 
\[ C^\infty(B \times B) \cong C^\infty(B) \otimes_{\pi} C^\infty(B) \] (cf. [13], p. 530) implies that (3.16) holds for all \(f \in C^\infty(B \times B)\).
4 Non-Euclidean Pseudodifferential Operators

We use a special $G$-equivariant $Ψ DO$-calculus that generalizes the Zelditch quantization from [12]. In this section we state some basic definitions and results we need. Full details will appear in [12]. For the moment, we may drop the rank one assumption. Fix a co-compact and torsion free discrete subgroup $Γ$ of $G$. Using the identification $X × B = G/M$ we identify functions $a(z, λ, b) = a(gK, λ, g·M)$ on $X × a^* × B$ with functions $a(gM, λ)$ on $G/M × a^*$. Let $n = \dim G$ and \{\(X_1, \ldots, X_n\)\} be a basis for $\mathfrak{g}$ (the elements are acting on functions on $G/M$ as left-invariant differential operators). A $Ψ DO$ of order 0 is a properly supported operator $A : C_0^∞(X) → C_0^∞(X)$ defined by

\[
Au(z) = ∫_{a^*_+} ∫_B e^{(iλ + ρ)(z,b)}a(z, λ, b)\tilde{u}(\lambda, b)\, db\, dλ, \tag{4.1}
\]

where:

(i) $\tilde{u}(\lambda, b) = ∫_X u(x)e^{-(iλ + ρ)(z,b)}dx$ is Helgason’s non-euclidean Fourier transform of $u$ ([7], p. 223).

(ii) $dλ = \frac{1}{2π} |e(λ)|^{-2} dλ$, where $|W|$ is the order of the Weyl group.

We call $a(z, λ, b)$ the complete symbol of $A$, which is equivalently given by

\[
(Ae_{λ,b})(z) = a(z, λ, b)e_{λ,b}(z), \tag{4.2}
\]

where for $λ ∈ a^*$ and $b ∈ B$ the functions $e_{λ,b} : X → C$, $z → e^{i(λ + ρ)(z,b)}$ are called non-Euclidean plane waves.

Let now $X$ have rank one and denote by $|·|$ the norm on $a^*$ induced by the Killing form. We identify $a = \mathbb{R} = a^*$: Define $λ_0 ∈ a^*_+$ by $λ_0(X) = (X, H_0)$ $(X ∈ a)$. We always assume that $a(z, λ, b)$ is a classical symbol of order 0, i.e. it has an asymptotic expansion of homogeneous symbols of decreasing order:

\[
a(z, λ, b) \sim ∑_{j=0}^∞ λ^{-j} a_{−j}(z, b). \tag{4.3}
\]

Asymptotics here means that $a(z, b, λ) = ∑_{j=0}^R a_j(z, b)λ^{-j+m} ∈ S^{m−R−1}$, where $a ∈ C^∞(X × a^* × B) = C^∞(G/M × a^*)$ is a symbol of order $m ∈ \mathbb{R}$ ($a ∈ S_m$) if for all $β ∈ \mathbb{N}_0$, $α ∈ \mathbb{N}_0^n$ and for each compact subset $C ⊂ G/M$ it satisfies

\[
∥∂_X^β X_1^{α_1} \cdots X_n^{α_n} a(gM, λ)∥ ≤ C(β)(1 + |λ|)^{m−β}. \tag{4.4}
\]

We call $σ_A := a_0$ the principal symbol of $\text{Op}(a) = A$. Theorems 10, 17, 18 only concern principal symbols, so we often assume that $a$ is independent of $λ$.

By $S_m^0$ we denote symbols of order $m$ which are invariant under the diagonal action of $Γ$ on $X × B$:

\[
a(γ · z, λ, γ · b) = a(z, λ, b), \quad γ ∈ Γ. \tag{4.5}
\]

Let $L_0^∞$ be the space of operators associated with such symbols. If $(T_γ u)(z) = u(g · z)$ denotes the translation of functions on $X$ we find (see [12] for details):

**Proposition 4.1.** Let $a ∈ S^0$. Then $\text{Op}(a) : L_2^2(X) → L_2^2(X)$ is continuous. Moreover, $A ∈ L_0^∞$ if and only if $A$ commutes with each $T_γ$, $γ ∈ Γ$. 


Recall from Section 3 that if \( \varphi \) is an eigenfunction of the Laplace operator with eigenvalue \( -(\lambda, \lambda) + \langle \rho, \rho \rangle \) \( \lambda \in \mathfrak{a}^* \) and boundary values \( T \in \mathcal{D}'(B) \), then

\[
\varphi(z) = \int_B e^{i(\lambda + \rho)(z,b)} T(db).
\]  

(4.6)

Let \( \{ \varphi_{\lambda_i} \} \) denote the eigenfunctions of \( \Delta_T \) with corresponding boundary values \( T_{\lambda_i} \in \mathcal{D}'(B) \). Then \( a \in S^0_1 \) induces a bounded operator on \( L^2(X_T) \) by

\[
\text{Op}(a) \varphi_{\lambda_j}(z) = \int_B a(z,b) e^{i(\lambda_j + \rho)(z,b)} T_{\lambda_j}(db),
\]

(4.7)

where we used the formula \( \text{Op}(a) e^{i(\lambda + \rho)(z,b)} = a(z,b) e^{i(\lambda + \rho)(z,b)} \) (cf. (4.2)) and pulled the operator under the integral sign in (1.6).

5 Patterson–Sullivan Distributions

In this section we introduce the central concepts we need to formulate our results: Intermediate values, the Radon transform, which really is a time average in our context, and the Patterson–Sullivan distributions.

Intermediate Values

To motivate the concept of intermediate values, consider the case where \( G = PSU(1,1)/PSO(2) \) is the open unit disk \( \mathbb{D} \) with boundary \( B = \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( \gamma \in G, b, b' \in B \). One has the intermediate value formula (cf. [10], p. 8)

\[
|\gamma(b) - \gamma(b')|^2 = |\gamma'(b)| \cdot |\gamma'(b')| \cdot |b - b'|^2.
\]  

(5.1)

It follows from [5], p. 197, that \( \frac{d(\gamma \cdot b)}{db} = e^{-2\rho(\gamma \cdot o, \gamma \cdot b)} \), where \( \rho = \frac{1}{2} \). Then

\[
|\gamma(b) - \gamma(b')|^2 = e^{-(\gamma \cdot o, \gamma \cdot b)} e^{-(\gamma \cdot o, \gamma \cdot b')} |b - b'|^2.
\]  

(5.2)

To generalize this we construct certain functions \( d_\lambda : G/MA \to \mathbb{C} \), which we call intermediate values, and which satisfy a certain equivariance property generalizing (5.2) (cf. (5.3)). This property then leads to invariance properties of the Patterson–Sullivan distributions.

**Definition 5.1.** By time reversal we mean the involution \( \iota(x, \xi) = (x, -\xi) \) on the unit cosphere bundle \( S^* X \). Under \( \Gamma G/MA = S^* X_T \) the time reversal map takes the form \( \Gamma g \mapsto \Gamma gw \). We say that a distribution \( T \) is time-reversible if \( \iota^* T = T \). Recall that each \( (b, b') \in B^{(2)} \) is of the form \( (g \cdot M, g \cdot wM) \in B^{(2)} \), where \( gMA \in G/MA \) is unique. Since \( w^2 \in M \), time reversal means

\[
(b, b') = (g \cdot M, g \cdot wM) \mapsto (gw \cdot M, g \cdot w^2 M) = (b', b),
\]

which is given by \( (b, b') \mapsto (b', b) \).

**Definition 5.2.** Given \( \lambda \in \mathfrak{a}^* \), we define \( d_\lambda : G/MA \to \mathbb{C} \) by

\[
d_\lambda(gMA) := e^{i(\lambda + \rho)(H(g) + H(gw))}.
\]  

(5.3)

Recall \( w^{-1} aw = a^{-1} \) \( (a \in A) \), which implies that \( d_\lambda \) is well-defined and time reversal invariant. We call the functions \( d_\lambda \) intermediate values.
Lemma 5.3. Let $\gamma, g \in G$. Then
\[
d_{\lambda}(\gamma g) = e^{(i\lambda + \rho)((\gamma \cdot g \cdot M) + (\gamma \cdot g \cdot wM))}d_{\lambda}(g).
\] (5.4)

Proof. This follows from Lemma 2.9.

Note that by Lemma 2.3 we may interpret $d_{\lambda}$ as a function on $B^{(2)}$, that is
\[
d_{\lambda}(b, b') = d_{\lambda}(g \cdot M, g \cdot wM) = e^{(i\lambda + \rho)(H(g) + H(gw))}
\] for $g = g(b, b')$.

Proposition 5.4. $d_{\lambda}(g \cdot M, g \cdot wM) = e^{(i\lambda + \rho)((g \cdot g \cdot M) + (g \cdot g \cdot wM))}$.

Proposition 5.5. Let $(b, b') \in B^{(2)}$ and $\gamma \in G$. Then
\[
(d_{\lambda} \circ \gamma)(b, b') = d_{\lambda}(\gamma \cdot b, \gamma \cdot b') = e^{(i\lambda + \rho)((\gamma \cdot \gamma \cdot b) + \gamma \cdot \gamma \cdot b')}d_{\lambda}(b, b').
\] (5.5)

Proof. Let $g \in G$ such that $(b, b') = (g \cdot M, g \cdot wM)$. Then $d_{\lambda}(\gamma \cdot g, \gamma \cdot g) = d_{\lambda}(g)$, so the assertion follows from Lemma 5.3.

Invariance Properties

As in the introduction, let $c_0 \leq c_1 \leq c_2 \leq \ldots \rightarrow \infty$ denote the spectrum of $-\Delta_F$ and $\{\varphi_{\lambda_j}\}$ a fixed $L^2(X_\Gamma)$-orthonormal basis of real valued eigenfunctions with eigenvalues $c_j = \lambda_j \lambda_j + (\rho, \rho) \in \mathbb{R}$. Then $\lambda_j \in a^* \cup ia^*$ and since $c_j \rightarrow \infty$ there are only finitely many $\lambda_j \in a^*$, so we may assume $\lambda_j \in a^*$ for all $j \in \mathbb{N}_0$. We only consider eigenfunctions with exponential growth and denote the corresponding sequence of distributional boundary values by $\{T_{\lambda_j}\}$.

Definition 5.6. The Patterson–Sullivan distribution $ps_{\lambda_j}$ associated to $\varphi_{\lambda_j}$ is the distribution
\[
ps_{\lambda_j}(db, db') := d_{\lambda_j}(b, b')T_{\lambda_j}(db)T_{\lambda_j}(db').
\] (5.6)

on $C^\infty_c(B^{(2)})$. The same definition [5.6] extends $ps_{\lambda_j}$ to a bounded linear functional on the larger space $d_{\lambda}(b, b')^{-1} \cdot C^\infty(B \times B)$.

Proposition 5.7. Suppose that $\varphi_{\lambda_j}$ is a $\Gamma$-invariant eigenfunction of the Laplacian. Let $T_{\lambda_j}$ denote its boundary values. Then the distribution $ps_{\lambda_j}(db, db')$ is $\Gamma$-invariant and time reversal invariant.

Proof. Time reversibility is obvious. Given a test function $f$ and $\gamma \in \Gamma$ we have
\[
ps_{\lambda_j}(f \circ \gamma^{-1}) = (T_{\lambda_j} \otimes T_{\lambda_j})(d_{\lambda_j} \cdot (f \circ \gamma^{-1})) = (\gamma T_{\lambda_j} \otimes \gamma T_{\lambda_j})((d_{\lambda_j} \circ \gamma) \cdot f).
\]
It follows from (3.3) that
\[
T_{\lambda_j}(d\gamma b)T_{\lambda_j}(d\gamma b') = e^{-(i\lambda_j + \rho)(\gamma \cdot \gamma \cdot b)}e^{-(i\lambda_j + \rho)(\gamma \cdot \gamma \cdot b')}T_{\lambda_j}(db)T_{\lambda_j}(db').
\]
Multiplying with (5.5) completes the proof of $\Gamma$-invariance.

Recall our notation from 2.5. Let $g(b, b')MA \in G/MA$ denote the coset corresponding to $(b, b') \in B^{(2)}$. 

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Definition 5.8. The Radon transform on $SX = G/M$ is given by
\[ Rf(b, b') := \int_A f(g(b, b')aM)da, \]
whenever the integral exists. [3], p. 91, applied to the subgroup $MA$, yields:

Lemma 5.9. $R : C_c(SX) \rightarrow C_c(B(2))$.

Definition 5.10. Let $F$ denote a bounded fundamental domain for $\Gamma$ in $X$. Following [1], pp. 380-381, we say that $\chi \in C_\infty^c(X)$ is a smooth fundamental domain cutoff function if it satisfies
\[ \sum_{\gamma \in \Gamma} \chi(\gamma z) = 1 \quad \forall z \in X. \quad (5.7) \]
Such a function can for example be constructed by taking $\nu \in C_\infty^c(X)$, $\nu = 1$ on $F$, and setting $\chi(z) = \nu(z) \cdot (\sum_{\gamma \in \Gamma} \nu(\gamma z))^{-1}$. If $\chi$ satisfies (5.7), then
\[ \int_F f \, dz = \int_X \chi f \, dz, \quad f \in C(X_\Gamma). \quad (5.8) \]

The following property of these cutoffs is proven in [1], Lemma 3.5:

Proposition 5.11. Let $T \in D'(SX)$ be a $\Gamma$-invariant distribution. Let $a$ be a $\Gamma$-invariant smooth function on $SX$. Then for any $a_1, a_2 \in D(SX)$ such that $\sum_{\gamma \in \Gamma} a_j(\gamma \cdot (z, b)) = a(z, b)$ ($j = 1, 2$) we have $\langle a_1, T \rangle_{SX} = \langle a_2, T \rangle_{SX}$.

Given $T$ and $a$ as in Proposition 5.11 and if moreover $\chi_j$ ($j = 1, 2$) are smooth fundamental domain cutoffs, then $a_j = \chi_j a$ satisfy the assumptions of the proposition. Hence $\langle a, T \rangle_{SX_\Gamma} := \langle \chi a, T \rangle_{SX}$ defines a distribution on the quotient $SX_\Gamma$ and this definition is independent of the choice of $\chi$.

Definition 5.12. (1) The Patterson–Sullivan distributions $PS_{\lambda_j}$ on $SX$ are defined by
\[ \langle a, PS_{\lambda_j} \rangle_{SX} := \int_{B \times B \setminus \Delta} (Ra)(b, b') \, ps_{\lambda_j}(db, db'). \]
(2) On $SX_\Gamma = \Gamma \setminus SX$ we define the Patterson–Sullivan distributions by
\[ \langle a, PS_{\lambda_j} \rangle_{SX_\Gamma} := \langle \chi a, PS_{\lambda_j} \rangle_{SX}, \]
where $\chi$ is a smooth fundamental domain cutoff.
(3) We define normalized Patterson–Sullivan distributions
\[ \widehat{PS}_{\lambda_j} = \frac{1}{\langle 1, PS_{\lambda_j} \rangle_{SX_\Gamma}} PS_{\lambda_j}, \quad (5.9) \]
which satisfy the normalization condition $\langle 1, \widehat{PS}_{\lambda_j} \rangle_{SX_\Gamma} = 1$. Note that $1 = \langle 1, W_{\lambda_j} \rangle_{SX_\Gamma}$.

In view of Proposition 5.11 the definitions made in 5.12 do not depend on $\chi$. Consider the expression
\[ PS_{\lambda_j}(a) = \langle a, PS_{\lambda_j} \rangle = \int_{B(z)} d_{\lambda_j}(b, b') \, R(a)(b, b') \, T_{\lambda_j}(db) \, T_{\lambda_j}(db'). \]
$PS_{\lambda_j}(a)$ is defined if $d_{\lambda_j} R(a) \in C^\infty(B \times B)$, which is the case for $a \in C_c^\infty(SX)$, since then $Ra \in C_c^\infty(B^{(2)})$, which in turn implies $d_{\lambda_j} R(a) \in C_c^\infty(B^{(2)}) \subset C_c^\infty(B \times B) = C^\infty(B \times B)$.

As an immediate consequence of Proposition 5.11 we obtain:

**Proposition 5.13.** Each $PS_{\lambda_j}$ is a geodesic flow invariant and $\Gamma$-invariant distribution on $G/M = SX$. On the quotient $SX_{\Gamma}$, $PS_{\lambda_j}$ still is invariant under the geodesic flow.

**Proof of Theorem 1.1**

**Lemma 5.14.** $L_{\lambda_j} : C_c^\infty(G) \to C^\infty(G)$.

**Proof.** It is well-known (cf. [8], Ch. IV, Lemma 5.14) that

$$\rho(H(\pi)) \geq 0 \quad \forall \pi \in \overline{\Gamma}.$$  

(5.10)

Hence the weight $|e^{-(i\lambda_j + \rho)H(\nu_{a})}| \leq C$ is bounded by a constant. The assertion follows from [8], p. 91, applied to the closed subgroup $N$ of $G$. $\square$

The following formula is the key tool in the proof of Theorem 1.1

**Lemma 5.15.** Let $a \in C^\infty(SX)$, $(b, b') \in B^{(2)}$. Then

$$\int_X \chi(a(z, b)e^{(i\lambda_j + \rho)((z, b) + (z, b'))} \, dz = d_{\lambda_j}(b, b') R(L_{\lambda_j} \chi a)(b, b').$$  

(5.11)

In view of (6.6), (5.11) is the special case $\lambda_j = \lambda_k$ of the more general formula in Lemma 6.3 and hence we do not give a proof here. Recall that the $\varphi_{\lambda_j}$ are real-valued. Let $a \in C^\infty(\Gamma \setminus G/M)$. Then (5.7) yields

$$\langle \text{Op}(a) \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle = \int_{B^{(2)}} \left( \int_X \chi(a(z, b)e^{(i\lambda_j + \rho)((z, b) + (z, b'))} \, dz \right) T_{\lambda_j}(db) T_{\lambda_j}(db').$$

It follows from Lemma 5.15 that $d_{\lambda_j} R(L_{\lambda_j} \chi a)$ has removable singularities in each $(b, b) \in B \times B$. Hence by the same lemma $\langle \text{Op}(a) \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle$ equals

$$\langle d_{\lambda_j} R(L_{\lambda_j} \chi a), T_{\lambda_j} \otimes T_{\lambda_j} \rangle = \langle R(L_{\lambda_j} \chi a), ps_{\lambda_j} \rangle = \langle L_{\lambda_j}(\chi a), PS_{\lambda_j} \rangle,$$

which proves Theorem 1.1.

6 Off-diagonal Patterson–Sullivan Distributions

In this section we generalize the results of Section 5 to the off-diagonal case and thus prove Theorem 1.2.

**Off-diagonal Intermediate Values**

The construction of $PS_{\lambda_j, \lambda_k}$ is different from the construction of $PS_{\lambda_j}$. We will see in this section why it is impossible to define functionals $ps_{\lambda_j, \lambda_k}$ ($\lambda_j \neq \lambda_k$).

**Definition 6.1.** Given $\lambda, \mu \in \mathfrak{a}$, define $d_{\lambda, \mu} : G/M \to \mathbb{C}$ by

$$d_{\lambda, \mu}(g) = e^{(i\lambda + \rho)H(g)} e^{(i\mu + \rho)H(gw)}.$$

(6.1)
This is well-defined, since the Iwasawa projection is $M$-invariant and $M'$ normalizes $M$. What we really need is a geodesic flow invariant function on $G/M$, that is $d_{\lambda,\mu}$ should be invariant under the right action of $A$. In other words, we would wish to have $d_{\lambda,\mu}$ well-defined on $G/MA$. But for $g \in G$, $m \in M$ and $a \in A$ a direct computation shows

$$d_{\lambda,\mu}(gam) = d_{\lambda,\mu}(g)e^{i(\lambda-\mu)\log(a)}.$$  

(6.2)

It follows that $d_{\lambda,\mu}$ is not a function on $G/MA$. This implies that for $\lambda \neq \mu$ we cannot define a more general function $d_{\lambda,\mu}(b,b')$ in analogy with (5.1). We will see in (6.3) how to circumvent this problem. Exactly as in Lemma 5.5 we have:

**Lemma 6.2.** Let $\gamma, g \in G$. Then

$$d_{\lambda,\mu}(\gamma g) = e^{i(\lambda+\gamma)(\gamma^{-1}g-\gamma g)}e^{(\mu+\rho)(\gamma^{-1}g-\gamma g)}d_{\lambda,\mu}(g).$$  

(6.3)

**Invariance Properties**

Let $f$ be a function on $G/M$ and let $\lambda, \mu \in \mathfrak{a}^*$. The *weighted Radon transform* on $G$ is defined by

$$(R_{\lambda,\mu}f)(g) := \int_A d_{\lambda,\mu}(ga)f(ga)da,$$  

(6.4)

whenever the integral exists. As in Lemma 5.9 we deduce:

**Remark 6.3.** Let $f \in C_c^\infty(G/M)$. Then $R_{\lambda,\mu}(f) \in C_c^\infty(G/MA)$ is invariant under the geodesic flow of $G/M = SX$. Hence $R_{\lambda,\mu}(f)$ is defined on $G/MA$ (see (6.2) and its subsequent remark).

**Definition 6.4.** As before, let $g_\lambda(b,b') \in G$ be a representative for the element in $G/MA$ that corresponds to $(b,b') \in B^{(2)}$. Given $f \in C_c^\infty(G/M)$, we define

$$R_{\lambda,\mu}(f)(b,b') := R_{\lambda,\mu}(f)(g_\lambda(b,b')).$$  

(6.5)

Then $R_{\lambda,\mu}f \in C_c^\infty(B^{(2)})$. This definition is independent of the choice of representative $g(b,b')$, since $R_{\lambda,\mu}(f)$ is invariant.

Let $f \in C_c^\infty(G/M)$. The values $|d_{\lambda,\mu}(g)|$ are independent of $\lambda, \mu$ and all derivatives of $d_{\lambda,\mu}$ have polynomial growth in $\lambda, \mu$. It follows that given a continuous seminorm $\| \cdot \|_1$ on $C_c^\infty(B \times B)$ there exist $K_1 > 0$ and a continuous seminorm $\| \cdot \|_2$ on $C_c^\infty(G/M)$ such that

$$\|R_{\lambda,\mu}(f)\|_1 \leq (1 + |\lambda|)^{K_1}(1 + |\mu|)^{K_1}\|f\|_2.$$  

(6.6)

**Definition 6.5.** The *off-diagonal Patterson–Sullivan distribution associated to $\varphi_{\lambda_j}$ and $\varphi_{\lambda_k}$* is the distribution on $SX = G/M$ defined by

$$PS_{\lambda_j,\lambda_k}(f) := \langle f, PS_{\lambda_j,\lambda_k} \rangle := \int_{B^{(2)}} (R_{\lambda_j,\lambda_k}f)(b,b') T_{\lambda_j}(db) T_{\lambda_k}(db').$$  

(6.7)

Assume $R_{\lambda_j,\lambda_k}(f) \in C_c^\infty(B \times B)$. Then $PS_{\lambda_j,\lambda_k}(f)$ is well-defined. A simple example is when $f \in C_c^\infty(SX) = C_c^\infty(G/M)$. In this case, it follows from (5.16), (6.0) and (6.7) that there exist $K > 0$ and a continuous seminorm $\| \cdot \|_2$ on $C_c^\infty(G/M)$ such that

$$|PS_{\lambda_j,\lambda_k}(f)| \leq (1 + |\lambda_j|)^K(1 + |\lambda_k|)^K\|f\|_2.$$  

(6.8)
Remark 6.6. Let \((b, b') \in B^{(2)}\) and \(g = g(b, b')\). Then
\[
\mathcal{R}_{\lambda_j, \lambda_j}(f)(g) = \int_A d_{\lambda_j, \lambda_j}(ga) f(ga) \, da = d_{\lambda_j}(g(b, b'))(\mathcal{R}f)(b, b'),
\]
which implies \(PS_{\lambda_j, \lambda_j} = PS_{\lambda_j}\).

Proposition 6.7. Suppose that \(\varphi_{\lambda_j}\) and \(\varphi_{\lambda_k}\) are \(\Gamma\)-invariant eigenfunctions. Then the distribution \(PS_{\lambda_j, \lambda_k}\) on \(SX = G/M\) is \(\Gamma\)-invariant.

Proof. Let \(f \in C_c^\infty(G/M)\) and let \(f_\gamma\) denote the translation \(f \circ \gamma^{-1}\). Then
\[
\langle f_\gamma, PS_{\lambda_j, \lambda_k} \rangle = \int_{B^{(2)}} \int_A d_{\lambda_j, \lambda_k}(g(b, b')a) f(\gamma^{-1}g(b, b')a) \, da \, T_{\lambda_j}(db) \, T_{\lambda_k}(db'),
\]
where \((b, b') = (g \cdot M, g \cdot wM)\) for \(g = g(b, b')\). By (3.3) this equals
\[
\int_{B^{(2)}} \int_A d_{\lambda_j, \lambda_k}(g(\gamma \cdot (b, b'))) \, f(\gamma^{-1}g(b, b')a) e^{-(i\lambda_j + \rho)(\gamma \cdot o, \gamma \cdot b)} \, d_{\lambda_j, \lambda_k}(ga).
\]
Recall that \(a \in A\) acts trivially on \((M, wM)\). Using this and (6.3) we observe
\[
d_{\lambda_j, \lambda_k}(\gamma ga) = e^{(i\lambda_j + \rho)(\gamma \cdot o, \gamma \cdot b)} e^{(i\lambda_k + \rho)(\gamma \cdot o, \gamma \cdot b')} d_{\lambda_j, \lambda_k}(ga).
\]
We also have \(g(\gamma \cdot (b, b')) = \gamma g(b, b')\), since \((b, b') \mapsto g(b, b') \in G/MA\) is \(G\)-equivariant. Hence \(\gamma^{-1}g(\gamma \cdot (b, b')) = g(b, b')\). Thus we have
\[
\langle f_\gamma, PS_{\lambda_j, \lambda_k} \rangle = \int_{B^{(2)}} \int_A d_{\lambda_j, \lambda_k}(b, b') \, f(b, b') \, da \, T_{\lambda_j}(db) \, T_{\lambda_k}(db') = \langle f, PS_{\lambda_j, \lambda_k} \rangle,
\]
and the proposition follows.

In view of Proposition 6.7 the definition of \(PS_{\lambda_j, \lambda_k}\) descends to \(SX_G = \Gamma \backslash SX\):

Definition 6.8. (1) The off-diagonal Patterson–Sullivan distributions on \(SX_G\) are defined by (\(\chi\) is a smooth fundamental domain cutoff)
\[
\langle a, PS_{\lambda_j, \lambda_k} \rangle_{SX_G} := \langle \chi a, PS_{\lambda_j, \lambda_k} \rangle_{SX_G}.
\]
(2) We normalize these distributions by
\[
PS_{\lambda_j, \lambda_k} = \frac{1}{\langle 1, PS_{\lambda_k, \lambda_k} \rangle_{SX_G}} PS_{\lambda_j, \lambda_k}.
\]

Proof of Theorem 1.2

The following lemma is the off-diagonal analog of Lemma 5.19.

Lemma 6.9. Let \(a \in C_c^\infty(SX_G)\), \((b, b') \in B^{(2)}\). Then
\[
\int_X \chi(z, b) e^{(i\lambda_j + \rho)(z, b)} e^{(i\lambda_k + \rho)(z, b')} \, dz = \mathcal{R}_{\lambda_j, \lambda_k}(L_{\lambda_k}(\chi a))(b, b').
\]

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Proof. Select $g \in G$ such that $(b, b') = (g \cdot M, g \cdot wM)$. The following manipulations do not depend on the choice of $g$. By $G$-invariance of $dz$, the left hand side of (6.13) equals

$$ \int_X \chi_a(g \cdot z, b)e^{(i\lambda_j + \rho)(g \cdot z, b)}e^{(i\lambda_k + \rho)(g \cdot z, b')}dz. \quad (6.14) $$

Identify $\chi_a$ with a function on $G/M$: Then since $b = g \cdot o$ we have

$$ \chi_a(gan \cdot o, b) = \chi_a(gan \cdot o, g \cdot M) = \chi_a(gan \cdot o, gan \cdot M) = \chi_a(ganM) $$

(recall that $P = MAN$ fixes $M \in K/M$, in particular $an \in AN$ fixes $M = b_\infty$). From the integral formula (2.9) we obtain that (6.14) equals

$$ \int_{AN} \chi_a(ganM)e^{(i\lambda_j + \rho)(gan \cdot o, g \cdot M)}e^{(i\lambda_k + \rho)(gan \cdot o, g \cdot wM)}dn \, da. \quad (6.15) $$

But $(gan \cdot o, g \cdot M) = (gan \cdot o, gan \cdot M) = H(gan) = H(ga)$ and $H(n^{-1}w) = H(nw)$ (which is equivalent to $H(\pi) = H(\pi^{-1})$ and thus follows from [8, p. 436 (8)]). Then (2.13) and (2.14) yield

$$ \langle gan \cdot o, g \cdot wM \rangle = -H(n^{-1}a^{-1}w) + H(gw), $$

which by (2.2) equals $-H(nw) + H(gaw)$. Hence (6.15) becomes

$$ \int_{AN} \chi_a(ganM)e^{(i\lambda_j + \rho)H(ga)}e^{(i\lambda_k + \rho)H(gaw)}e^{-(i\lambda_k + \rho)H(nw)}dn \, da $$

$$ = \int_A d\lambda, \lambda e^{(i\lambda_k + \rho)H(ga)} \int_{AN} \chi_a(ganM)e^{-(i\lambda_k + \rho)H(nw)}dn \, da $$

$$ = \int_A d\lambda, \lambda e^{(i\lambda_k + \rho)(gan \cdot o, g \cdot M)}(ganM) \, da = R_{\lambda, \lambda}(\chi_a)(b, b'). $$

The independence of the representative $g(b, b')$ follows from the unimodularity of $A$ and because the mapping $N \ni n \mapsto \tilde{m}^{-1}m \in N \ni m \in M$ preserves the measure $dn$ (since $M$ is compact).

As in Section 3 we may now integrate (6.13) against $T_{\lambda_j}(db)T_{\lambda_k}(db')$, which completes the proof of Theorem 1.2.

7 Proof of Theorem 1.3

Given a phase function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $\text{Im}(\psi) \geq 0$ and an amplitude $\alpha \in C^\infty_c(\mathbb{R}^n)$ and $\tau > 0$, consider the integral

$$ I(\tau) := \int e^{i\tau \psi(x)}\alpha(x)dx. $$

It is well known ([15], p. 195) that if $\psi' \neq 0$ on the support of $a$, then $I(\tau) = O(\tau^{-\infty})$ as $\tau \rightarrow \infty$. Assume that $0 \in \mathbb{R}^n$ is the only critical point of $\psi$ and let $H = \psi''(0)$ be nonsingular at 0. Also assume $\psi(0) = 0$. Then

$$ \psi(x) = \langle Hx, x \rangle/2 + O(|x|^3) \quad \text{as } x \rightarrow 0 $$

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where $R_k = ((H^{-1} D, D)/2i)^k$ is a differential operator on $\mathbb{R}^n$ of order $2k$ with $D = (D_1, \ldots, D_n)$, where $D_j = -i\partial_j$ and $C = |\det H|^{-1/2}e^{\pi i \text{sign}(H)/4}$ is a constant depending on $\psi$. The phase function $\psi$ is defined as in (6.8). Theorem 1.2 implies

$$L_{\lambda}(\chi(a)(g)) = C \cdot (2\pi/\lambda)^{s/2} \sum_n \lambda^{-n} R_{2n}(\chi(a)(g)), \quad (7.2)$$

where $R_{2n}$ is a differential operator on $SX$ of order $2n$ and $R_0$ is the identity. Although we consider off-diagonal elements, the proof in [1] applies with almost no change: Let $K$ be defined as in [8]. Theorem 1.2 implies

$$\langle \text{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{S X_T} = \langle L_{\lambda_k}(\chi(a)), PS_{\lambda_j, \lambda_k}\rangle_{S X} = C \cdot (2\pi/\lambda_k)^{s/2} \sum_{n=0}^{N} \lambda_k^{-n} (R_{2n}(\chi(a)), PS_{\lambda_j, \lambda_k}) + O(\lambda_k^{-N-1+2K}).$$

We choose $N > 2K$. Since $R_0$ is the identity, the operator $L_{\lambda}^{(N)} = \sum_n \lambda^{-n} R_{2n}$ can be inverted up to $O(\lambda^{-N-1})$, i.e. one finds differential operators $M_{\lambda}^{(N)} = \sum_{n=0}^{N} \lambda^{-n} M_{2n}$, where $M_0 = \text{id}$, and $R_{\lambda}^{(N)}$ such that

$$L_{\lambda}^{(N)} M_{\lambda}^{(N)} = \text{id} + \lambda^{-N-1} R_{\lambda}^{(N)}.$$

An application of Theorem 1.2 to $M_{\lambda_k}^{(N)} a$ yields

$$\langle \text{Op}(M_{\lambda_k}^{(N)} a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{S X_T} = \langle L_{\lambda_k}^{(N)} M_{\lambda_k}^{(N)} a, PS_{\lambda_j, \lambda_k}\rangle_{S X} + O(\lambda_k^{-N-1+2K}) = \langle L_{\lambda_k}^{(N)} M_{\lambda_k}^{(N)} a, PS_{\lambda_j, \lambda_k}\rangle_{S X} + O(\lambda_k^{-N-1+2K}) = \langle a, PS_{\lambda_j, \lambda_k}\rangle_{S X_T} + O(\lambda_k^{-N-1+2K}).$$

The second line is a consequence of Proposition 5.11 But

$$M_{\lambda}^{(N)}(a) = a + \lambda^{-1} (M_2 + \ldots + \lambda^{-N+1} M_{2N})(a), \quad (7.3)$$
so the $L^2$-continuity of zero order pseudodifferential operators implies

$$\langle \text{Op}(M^{(N)}_{\lambda_k})(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_T)} = \langle \text{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_T)} + O(1/\lambda_k),$$

(7.4)

which proves

$$C \cdot (2\pi/\lambda_k)^{s/2} \langle a, \overline{PS}_{\lambda_j, \lambda_k}\rangle_{S X_T} = \langle \text{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{S X_T} + O(1/\lambda_k).$$

(7.5)

We put $\langle a, PS_{\lambda_j, \lambda_k}\rangle = \langle 1, PS_{\lambda_k, \lambda_k}\rangle \langle a, \hat{PS}_{\lambda_j, \lambda_k}\rangle$ into (7.5) and obtain

$$C \cdot (2\pi/\lambda_k)^{s/2} \cdot \langle 1, PS_{\lambda_k, \lambda_k}\rangle \cdot \langle a, \overline{PS}_{\lambda_j, \lambda_k}\rangle = \langle a, W_{\lambda_j, \lambda_k}\rangle + O(1/\lambda_k).$$

(7.6)

In particular, for $a = 1$, we get

$$C \cdot (2\pi/\lambda_k)^{s/2} \cdot \langle 1, PS_{\lambda_k, \lambda_k}\rangle_{S X_T} = 1 + O(1/\lambda_k).$$

(7.7)

Together with (7.6) this yields

$$(1 + O(1/\lambda_k)) \cdot \langle a, \overline{PS}_{\lambda_j, \lambda_k}\rangle = \langle a, W_{\lambda_j, \lambda_k}\rangle + O(1/\lambda_k).$$

(7.8)

The Wigner distributions and hence by (7.8) the $\langle a, \overline{PS}_{\lambda_j, \lambda_k}\rangle$ are uniformly bounded. It follows that the left side of (7.8) is asymptotically the same as $\langle a, W_{\lambda_j, \lambda_k}\rangle$. This completes the proof of Theorem 1.3.

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