Persistence of periodic orbits under state-dependent delayed perturbations: computer-assisted proofs

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Abstract

A computer-assisted argument is given, which provides existence proofs for periodic orbits in state-dependent delayed perturbations of ordinary differential equations (ODEs). Assuming that the unperturbed ODE has an isolated periodic orbit, we introduce a set of polynomial inequalities whose successful verification leads to the existence of periodic orbits in the perturbed delay equation. We present a general algorithm, which describes a way of computing the coefficients of the polynomials and optimizing their variables so that the polynomial inequalities are satisfied. The algorithm uses the tools of validated numerics together with Chebyshev series expansion to obtain the periodic orbit of the ODE as well as the solution of the variational equations, which are both used to compute rigorously the coefficients of the polynomials. We apply our algorithm to prove the existence of periodic orbits in a state-dependent delayed perturbation of the van der Pol equation.

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Key words. State-dependent delay equations, periodic orbits, validated numerics, variational equations, Chebyshev series

1 Introduction

The goal of the present work is to develop computer assisted arguments for proving the existence of periodic orbits in state-dependent delay differential equations (SDDEs) arising as perturbations of ordinary differential equations (ODEs). After fixing bounds on the norm of the perturbation, the delay, and their derivatives, our method determines (a) values of the perturbation parameter ε so that the periodic orbit of the ODE persists into the SDDE, and (b) provides explicit bounds on the distance, in an appropriate norm, between the perturbed and unperturbed solutions. Note that, while an ODE generates a finite dimensional dynamical system, an SDDE –if it generates a semi-flow at all– has phase space as an infinite dimensional Banach manifold [Wal03b, Wal21, Wal16, DvGVLW95, Wal03a]. This makes the perturbation arguments fairly delicate, and one novelty of our method is that it does not require to consider the Cauchy problem for the SDDE.

Regularity issues are also quite subtle. For example, if the vector field generating the ODE is real analytic then any periodic orbits are real analytic as well. Not so for SDDEs. It has been shown that periodic solutions for SDDEs may be analytic in the neighborhood of a certain point, and only $C^8$ on other portions of the orbit [MPN14, KW17, MPN19]. Moreover, it has been

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conjectured that this state of affairs is generic. Regularity questions have practical implications on the set-up of the perturbative argument – for example we cannot employ analytic norms.

To manage these difficulties, we employ an approach based on the parameterization method \cite{CFdlL03a, CFdlL03b, CFdlL05, HdlL06a, HdlL06b, HdlL07}. The parameterization method is a functional analytic framework for studying invariant manifolds, which exploits the fact that recurrent enough solutions often have much nicer properties than solutions with arbitrary initial conditions. The idea of the method is to formulate a chart or covering map for the invariant object as the solution of an invariance equation, and studying such an equation allows the problem to be attacked using all the tools of nonlinear analysis and computational mathematics. A much more thorough description of the parameterization method with many applications is in \cite{HCF16}.

Recently a number of authors have made substantial progress using the parameterization method to study invariant manifolds in ill posed problems \cite{dLS19, CdlL20, WdlL20, CGL18}, and in particular the method has been used successfully to study periodic and quasi-periodic solutions of SDDEs and their attached stable/unstable manifolds \cite{HdlL16, HdlL17, CCdlL20, YGdlL21, YGdlL}. We think of this as an application of the Poincaré program in problems where the semi-flow theory is underdeveloped or otherwise problematic. That is, one builds up an understanding of the dynamics one invariant object at a time, putting aside the fact that the dynamics in a full neighborhood of the invariant sets may or may not make sense at all.

The paper \cite{YGdlL} just cited develops a-posteriori theorems for state dependent perturbations of periodic orbits in ODEs. In the present work we implement a procedure sufficient for verifying the hypotheses of \cite{YGdlL} in concrete examples. Here we have to balance two competing considerations. On the one hand, our arguments require a great deal of quantitative information about the perturbing functions and periodic orbit of the unperturbed system. On the other hand, we do not want to restrict our attention to ODEs where explicit formulas for periodic orbits are known analytically. Indeed, our goal is to describe an approach which works in principle for any ODE with an isolated periodic orbit.

In the present work these constraints are simultaneously satisfied using computer-assisted methods of proof for the ODE. This is a very active area of research, and many viable options exist for studying periodic solutions. A thorough review of the literature is a task beyond the scope of this modest introduction, and we refer the interested reader to the review articles \cite{Rum10, vdBL15, KMWZ21} and books \cite{Tuc11, NPW19}. In the present work, for reasons that will be elucidated throughout the manuscript, we employ a computer-assisted method of proof wherein the ODE is projected into a Banach space of rapidly decaying Chebyshev coefficients. The truncated problem is solved numerically using Newton’s method, and the existence of a true solution with Chebyshev series coefficients near our numerical approximation is proven using a Newton-Kantorovich argument. The approach is adapted from \cite{HLMJ16}, using techniques also from \cite{LR14, Les18}. A readable introduction to these ideas is found in the first three chapters of \cite{vdBL18}. Combining the analytical results from \cite{YGdlL} with the mathematically rigorous computational methods just discussed, we prove results for a number of state-dependent delayed perturbations of a van der Pol equation. As far as we are aware, the present paper and \cite{Chu21} - which deals with the problem of rigorous integration – are the only two papers thus far in the literature dealing with computer assisted proofs for SDDEs.

Remark 1.1 (SDDE in electrodynamics). DDEs arise naturally when modeling distributed systems where communication lags between the various subsystems cannot be ignored. When the lags themselves depend on the state of the system, we have SDDEs. Examples are common in biology, control theory, and epidemiology, and we refer to \cite{HKWW06} for an expansive discussion of relevant applications. We single out for further discussion an interesting collection of problems where the perturbative arguments developed in the present work could, under suitable modifications, be very useful.

While the classical Newtonian theory of $N$-body interactions treats gravitational distur-
bances as acting instantaneously, it would be more realistic to incorporate the finite propagation speed of light, perhaps as a delay. For example, since the magnitude of the gravitational force due to one body acting on another is inversely proportional to the square of distance between them, the delay would depend on the distance divided by the speed of light. For systems of particles where this fraction is small, this would result in a state dependent perturbation of the classical equations of motion.

A much more explicit example is found in the 1949 paper of Feynmann and Wheeler [WF49], where they put forward a theory of direct interparticle action for a system of point charges in electromagnetic interaction, and discuss its potential advantages. Subsequent work in this direction is found in Driver’s Ph.D. dissertation on the electromagnetic two body problem [Dri60], and in several of his subsequent works [Dri63a, Dri63b, Dri69, HD90].

For a theory based on SDDEs to be symmetric under the reversal of time, it is necessary to include an “advanced” delay which restores the symmetry of the equations. This induces a counter intuitive dependence in the equations of motion on the future state of the system, in addition to its past [Dri79a, Dri79b]. While dependence on future state is an unusual feature of a physical model, there continues to be much interest in this approach [DLGHP10, DLHR12, DSDL15, DL16].

We mention these works mainly as an opportunity to stress that the a-posteriori framework employed in the present work allows for the inclusion of advanced delays, and could in principle be applied to the electromagnetic theories above. On the other hand, in their current formulation, the results of [YGdlL] require that the periodic orbit of the ODE is isolated, and this never occurs in the Hamiltonian ODEs of classical dynamics. Development of an a-posteriori framework which generalizes the approach of [YGdlL] to systems with continuous symmetries is the subject of an upcoming work.

Remark 1.2 (Regularity of the results). The arguments in the present work are formulated in $C^{1+Lip}$ spaces and, though we can bootstrap to obtain more regularity, we do not obtain bounds on higher derivatives. Bounds on higher derivatives can be obtained using the methods of [YGdlL], but this requires putting more work into the estimates.

Remark 1.3 (Spectral Bases: Chebyshev versus Taylor). The computer-assisted proofs in this paper are formulated using a spectral representation—namely Chebyshev series—for the periodic orbit. For the purposes of this paper it is quite valuable to have a representation of the unperturbed periodic solution as an object in $C^k([0, T], \mathbb{R}^d)$ (with $k \in \mathbb{N}$, or $k = \infty$, or even real analytic), where $T$ is the period and $\mathbb{R}^d$ is the state space for the ODE (rather than for example a representation of the orbit as a fixed or periodic point in a Poincaré section). This is because our method requires bounds on the size of the orbit and its derivatives, and these bounds are easily recovered from the spectral representation.

We note that it is also very natural to study periodic solutions using Fourier series, and to formulate computer-assisted proofs on Banach spaces of rapidly decaying Fourier coefficients as in [HLMJ16]. However, the a-posteriori theory of [YGdlL] requires bounds on solutions of some variational equations associated with the periodic orbit, and solutions of the variational equation are not periodic. This problem could be addressed in Fourier space using the Floquet methods developed [CLMJ15]. However, in the present work we found it expedient to solve the variational equations directly using a Chebyshev scheme. This requires a Chebyshev representation of the periodic orbit, which is why we use Chebyshev series throughout.

The paper is organized as follows. In Section 2 we recall some background from [YGdlL] and introduce a set of polynomial inequalities whose successful verification leads to the existence of periodic orbits in the perturbed SDDE. In Section 3 we introduce our main algorithm, which provides a way of computing the coefficients of the polynomials and optimizing their variables so that the polynomial inequalities are satisfied. In Section 4 we apply our algorithm to prove the existence of periodic orbits in a state-dependent delayed perturbation of the Van der Pol equation. We conclude the paper in Section 5.
2 Formulation of the Problem and the Polynomial Inequalities

In this section, we recall some results from [YGdlL] and describe a set of polynomial inequalities whose successful verification leads to the existence of the periodic orbits for the perturbed SDDE. Consider a smooth ODE on \( \mathbb{R}^d \)

\[
\dot{x}(t) = f(x(t)),
\]

(1)

and assume it has a periodic orbit \( \Gamma \), which is parameterized by \( K_0: \mathbb{R} \to \mathbb{R}^d \) where \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). In other words, \( \Gamma = \{ K_0(t) : t \in [0, 1] \} \). We consider a perturbation involving a state-dependent (forward or backward) delay term, that is

\[
\dot{x}(t) = f(x(t)) + \varepsilon P\left(x(t - r(x(t)))\right),
\]

(2)

where \( r: \mathbb{R}^d \to \mathbb{R} \) is not restricted to be positive.

When all but one Floquet multipliers of \( \Gamma \) are different from 1, the result from [YGdlL] ensures that the perturbed equation (2) also has a periodic orbit for small enough \( \varepsilon \). In order to find values of \( \varepsilon \) so that the periodic orbit persists, we first summarize the proof in [YGdlL]. The proof there is based on the parameterization method, first introduced in [CFdlL05, CFdlL03b, CFdlL03a].

The parameterization \( K_0: \mathbb{R} \to \mathbb{R}^d \) of the periodic orbit for (1) with frequency \( \omega_0 \) satisfies

\[
\omega_0 D K_0(\theta) = f(K_0(\theta)).
\]

(3)

Consider perturbations \( \hat{K} \) and \( \hat{\omega} \), of \( K_0 \) and \( \omega_0 \) respectively, so that \( K \overset{\text{def}}{=} K_0 + \hat{K} \) parameterizes the periodic orbit of the perturbed equation (2) and \( \omega \overset{\text{def}}{=} \omega_0 + \hat{\omega} \) is the new frequency. Then \( \hat{K} \) and \( \hat{\omega} \) satisfy

\[
\omega_0 D \hat{K}(\theta) - D f(K_0(\theta)) \hat{K}(\theta) = B^\varepsilon(\theta, \hat{\omega}, \hat{K}) - \hat{\omega} D K_0(\theta),
\]

(4)

where

\[
B^\varepsilon(\theta, \hat{\omega}, \hat{K}) \overset{\text{def}}{=} N(\theta, \hat{K}) + \varepsilon P(\hat{K}(\theta)) - \hat{\omega} D \hat{K}(\theta),
\]

\[
N(\theta, \hat{K}) \overset{\text{def}}{=} f(K_0(\theta) + \hat{K}(\theta)) - f(K_0(\theta)) - D f(K_0(\theta)) \hat{K}(\theta),
\]

and the term \( \hat{K} \) coming from the delay is given by

\[
\hat{K}(\theta) \overset{\text{def}}{=} K(\theta - \omega r(K(\theta))).
\]

To solve equation (4), we consider the variational equation. For any fixed \( \theta_0 \in \mathbb{T} \), let \( \Phi(\theta; \theta_0) \) be such that

\[
\omega_0 \frac{d}{d\theta} \Phi(\theta; \theta_0) = D f(K_0(\theta)) \Phi(\theta; \theta_0), \quad \Phi(\theta_0; \theta_0) = \text{Id}.
\]

(5)

where \( \text{Id} \) is the identity matrix in \( \mathbb{R}^d \). Then the condition that \( \Gamma \) has all but one Floquet multiplier different from 1 is equivalent to the following nondegenerate assumption [H] on \( \Phi(\theta_0 + 1; \theta_0) \):

(H) \( \Phi(\theta_0 + 1; \theta_0) \) has a simple eigenvalue 1 whose eigenspace is generated by \( D K_0(\theta_0) \).

At the point \( K_0(\theta_0) \) on the periodic orbit, the tangent space has a spectral splitting,

\[
T_{K_0(\theta_0)} \mathbb{R}^d = E_{\theta_0} \oplus \text{Span}\{ D K_0(\theta_0) \}.
\]

(6)

We denote the projections onto \( \text{Span}\{ D K_0(\theta_0) \} \) and \( E_{\theta_0} \) as \( \Pi^\top_{\theta_0} \) and \( \Pi^\bot_{\theta_0} \), respectively.
Now, equation (4) can be solved by a fixed point approach using the variation of constants method. The fixed point of the following operator $\Gamma^\varepsilon$ solves (4), where

$$\Gamma^\varepsilon(\hat{\omega}, \hat{K}) = \begin{pmatrix} \Gamma_1^\varepsilon(\hat{\omega}, \hat{K}) \\ \Gamma_2^\varepsilon(\hat{\omega}, \hat{K}) \end{pmatrix},$$

with

$$\Gamma_1^\varepsilon(\hat{\omega}, \hat{K}) \overset{\text{def}}{=} \left\langle \int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0} \Phi(\theta_0 + 1; s, \hat{\omega}, \hat{K}) \right\rangle ds, DK_0(\theta_0) \right\rangle,$$

$$\Gamma_2^\varepsilon(\hat{\omega}, \hat{K})(\theta) \overset{\text{def}}{=} \Phi(\theta; \theta_0)u_0 + \frac{1}{\omega_0} \int_{\theta_0}^{\theta} \Phi(\theta; s)B^\varepsilon(s, \hat{\omega}, \hat{K}) - \Gamma_1^\varepsilon(\hat{\omega}, \hat{K})DK_0(s)) ds$$

where $u_0 \in E_{\theta_0}$ satisfies

$$[Id - \Phi(\theta_0 + 1; \theta_0)]u_0 = \frac{1}{\omega_0} \int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0} \Phi(\theta_0 + 1; s)B^\varepsilon(s, \hat{\omega}, \hat{K}) ds.$$ 

Basically, $\Gamma_2^\varepsilon$ provides the updated $\hat{K}$ with variation of constants formula. The initial condition $u_0$ is chosen such that $\Gamma_2^\varepsilon$ is periodic, and $\Gamma_1^\varepsilon$ makes sure that there exists such $u_0$.

Let $I_a \overset{\text{def}}{=} [-a, a]$, and

$$\mathcal{E}_\beta \overset{\text{def}}{=} \left\{ g: \mathbb{T} \to \mathbb{R}^d \mid g \text{ is } C^2, \left\| \frac{d^i}{d\theta^i} g(\theta) \right\| \leq \beta_i, \ i = 0, 1, 2 \right\}, \quad (7)$$

where $\| \cdot \|$ is the $C^0$ norm. Note that we could choose any norm in $\mathbb{R}^d$ (e.g. Euclidean norm), but a different choice would lead to a slight change in the estimates. Define the fixed point operator $\Gamma^\varepsilon$ on $I_a \times \mathcal{E}_\beta$. For small given $\varepsilon$, if one can show that: (i) $\Gamma^\varepsilon$ maps $I_a \times \mathcal{E}_\beta$ into itself, and (ii) $\Gamma^\varepsilon$ is a contraction in $C^0$ distance, then there is a fixed point $(\hat{\omega}^*, \hat{K}^*)$ of $\Gamma^\varepsilon$. Notice that $\hat{\omega}^* \in I_a$ and $\hat{K}^*$ is $C^{1+Lip}$, that is differentiable with Lipschitz derivative, using the fact that the closure of $\mathcal{E}_\beta$ is a subset of $C^{1+Lip}$ functions. Therefore, equation (4) is solved. Indeed, (i) ensures the existence of a fixed point, and (ii) guarantees the uniqueness of the fixed point. As demonstrated in [YGdL], in order to verify (i) and (ii), it is sufficient to verify the six following inequalities.

$$|\Gamma_1^\varepsilon(\hat{\omega}, \hat{K})| \leq a \quad \iff \quad Q(\varepsilon, a, \beta_0) \leq a$$

$$\|\Gamma_2^\varepsilon(\hat{\omega}, \hat{K})\| \leq \beta_0 \quad \iff \quad P_0(\varepsilon, a, \beta_0) \leq \beta_0$$

$$\left\| \frac{d}{d\theta} \Gamma_2^\varepsilon(\hat{\omega}, \hat{K}) \right\| \leq \beta_1 \quad \iff \quad P_1(\varepsilon, a, \beta_0, \beta_1) \leq \beta_1$$

$$\left\| \frac{d^2}{d\theta^2} \Gamma_2^\varepsilon(\hat{\omega}, \hat{K}) \right\| \leq \beta_2 \quad \iff \quad P_2(\varepsilon, a, \beta_0, \beta_1, \beta_2) \leq \beta_2$$

$\Gamma^\varepsilon$ is a contraction in $C^0$ \quad \iff \quad \begin{cases} \mu_1(\varepsilon, a, \beta_0, \beta_1) < 1 \\ \mu_2(\varepsilon, a, \beta_0, \beta_1) < 1 \end{cases}$

As we shall see now in the next section, $Q$, $P_0$, $P_1$, $P_2$, $\mu_1$, and $\mu_2$ are polynomials with computable coefficients which are determined by the unperturbed equation, the perturbation term $P$, and the delay term $r$. 

5
2.1 The explicit construction of the polynomials $Q$, $P_0$, $P_1$, $P_2$, $\mu_1$ and $\mu_2$

For a given norm $| \cdot |$ on $\mathbb{R}^d$, denote $|A|$ as the operator norm of matrix $A$. Define constants

$$
C_{1,1} \overset{\text{def}}{=} |\Phi(\theta_0 + 1; s)| = \max_{s \in [\theta_0, \theta_0 + 1]} |\Phi(\theta_0 + 1; s)|,
$$

$$
C_{1,2} \overset{\text{def}}{=} |\Phi(\theta; s)| = \max_{\theta \in [\theta_0, \theta_0 + 1], \ s \in [\theta_0, \theta]} |\Phi(\theta; s)|,
$$

$$
C_{1,3} \overset{\text{def}}{=} |\Phi(\theta; \theta_0)| = \max_{\theta \in [\theta_0, \theta_0 + 1]} |\Phi(\theta; \theta_0)|.
$$

Let

$$
\left\| \frac{d}{ds} \Phi(\theta_0 + 1; s) \right\| = \max_{s \in [\theta_0, \theta_0 + 1]} \left\| \frac{d}{ds} \Phi(\theta_0 + 1; s) \right\|,
$$

and

$$
\left\| \frac{d}{ds} \Phi(\theta) \right\| = \max_{\theta \in [\theta_0, \theta_0 + 1], \ s \in [\theta_0, \theta]} \left\| \frac{d}{ds} \Phi(\theta; s) \right\|,
$$

and

$$
C_{2,1} \overset{\text{def}}{=} 1 + |\Phi(\theta_0 + 1; \theta_0)| + \left\| \frac{d}{ds} \Phi(\theta_0 + 1; s) \right\|,
$$

$$
C_{2,2} \overset{\text{def}}{=} 1 + |\Phi(\theta; \theta_0)| + \left\| \frac{d}{ds} \Phi(\theta; s) \right\|.
$$

Recall the spectral splitting in (6), let $\left\| [I - \Phi(\theta_0 + 1; \theta_0)]^{-1}\right\|_{E_{\theta_0}}$ be the norm of the operator $[I - \Phi(\theta_0 + 1; \theta_0)]^{-1}$ defined in the space $E_{\theta_0}$, let $\|\Pi_{\theta_0}^\perp\| = \|\Pi_{\theta_0}^\perp\|_E$ be the norm of the projections $\Pi_{\theta_0}^\perp$ and $\Pi_{\theta_0}^\parallel$, and let

$$
M \overset{\text{def}}{=} \frac{\left\| [I - \Phi(\theta_0 + 1; \theta_0)]^{-1}\right\|_{E_{\theta_0}} \|\Pi_{\theta_0}^\perp\|}{\omega}.
$$

Now we are ready to provide the polynomials explicitly,

$$
Q(\varepsilon, \alpha, \beta_0) = \frac{\|\Pi_{\theta_0}^\perp\|}{\|DK_0(\theta_0)\|} \left( \varepsilon C_{1,1} \|P\| + C_{2,1} \alpha \beta_0 + \frac{C_{1,3}}{2} \|D^2 f\| \beta_0^2 \right),
$$

where $\|P\|$ is the $C^0$ norm of $P$ in a neighborhood of the periodic orbit of the unperturbed equation (1) (containing the periodic orbit for the perturbed equation (2)), i.e., a small neighborhood of size $2\beta_0$ around the unperturbed periodic orbit, and $\|D^2 f\|$ is the supremum of the norm of the bilinear operator $D^2 f$ in the aforementioned neighborhood of the unperturbed periodic orbit, see more details in section [3.2.1]. In the following, all norms without specification mean the supremum norm.

$$
P_0(\varepsilon, \alpha, \beta_0) = \varepsilon \left( M C_{1,3} C_{1,1} + \frac{C_{1,2}}{\omega_0} \right) \|P\| + \frac{C_{1,2}}{\omega_0} \|DK_0\| \alpha + \left( M C_{1,3} C_{2,1} + \frac{C_{2,2}}{\omega_0} \right) \alpha \beta_0
$$

$$
+ \frac{\|D^2 f\|}{2} \left( M C_{1,3} C_{1,1} + \frac{C_{1,2}}{\omega_0} \right) \beta_0^2,
$$

$$
P_1(\varepsilon, \alpha, \beta_0, \beta_1) = \frac{1}{\omega_0} \left( \varepsilon \|P\| + \|DK_0\| \alpha + \|Df \circ K_0\| \beta_0 + \alpha \beta_1 + \frac{1}{2} \|D^2 f\| \beta_0^2 \right),
$$

$$
P_2(\varepsilon, \alpha, \beta_0, \beta_1, \beta_2) = \frac{1}{\omega_0} \left( \|D^2 K_0\| \alpha + \|D^2 f \circ K_0\| \|DK_0\| \beta_0 + \|Df \circ K_0\| \beta_1 + \left\| \frac{d}{d\theta} B^\perp \right\| \right),
$$

where

$$
\left\| \frac{d}{d\theta} B^\perp \right\| \leq \varepsilon \|DP\| \left( \|DK_0\| + \beta_1 \right) \left[ 1 + (\omega_0 + \alpha) \|Dr\| \left( \|DK_0\| + \beta_1 \right) \right] + \alpha \beta_2
$$

$$
+ \|D^3 f\| \left( \|DK_0\| + \beta_1 \right) \beta_0^2 + 2 \|D^2 f\| \beta_0 \beta_1.
Let
\[ \xi \overset{\text{def}}{=} \varepsilon \|DP\| + \varepsilon \|DP\| (\|DK_0\| + \beta_1) (\|Dr\| (\omega_0 + a) + \|r\|) + \|D^2 f\| \beta_0. \]

Then
\[
\mu_1(\varepsilon, a, \beta_0, \beta_1) = \frac{\|\Pi_{\theta_0}^T C_{2,1}\|}{|DK_0(\theta_0)|} (\beta_0 + a) + \frac{\|\Pi_{\theta_0}^T C_{1,1}\|}{|DK_0(\theta_0)|} \xi, \tag{15}
\]
\[
\mu_2(\varepsilon, a, \beta_0, \beta_1) = \left( MC_{1,3} C_{2,1} + \frac{C_{1,2}\|\Pi_{\theta_0}^T C_{2,1}\|}{\omega_0 |DK_0(\theta_0)|} r \right) (\beta_0 + a) + \left( MC_{1,3} C_{1,1} + \frac{C_{1,2}\|\Pi_{\theta_0}^T C_{1,1}\|}{\omega_0 |DK_0(\theta_0)|} + \frac{C_{1,2}}{\omega_0} \right) \xi. \tag{16}
\]

Note that similar to \(\|P\|\) and \(\|D^2 f\|\), \(|r|\), \(|Dr|\), \(|DP|\), and \(\|D^3 f\|\) are the supremum norms in the same neighborhood of the unperturbed periodic orbit mentioned before.

In the next section, we introduce an algorithm which provides an efficient and automated way of (a) computing the coefficients of the polynomials \(Q, P_0, P_1, P_2\), \(\mu_1\) and \(\mu_2\); and (b) optimizing their variables \(\varepsilon, a, \beta_0, \beta_1\) and \(\beta_2\) so that the polynomial inequalities are satisfied, hence providing rigorous and constructive proofs of existence of periodic orbits for the perturbed SDDE.

## 3 Algorithm

In the section, we first introduce our main algorithm. We assume that the \(C^1\) norms of the perturbation \(P\) and the forward or backward delay \(r\) are given as inputs.

After that, we provide details for each step in the algorithm. In particular, the computations of the bounds and the constants appearing in the inequalities are elaborated. In the algorithm, the word compute means that we use computer-assisted proofs to obtain rigorous enclosures of the quantities we are computing.

**Algorithm 3.1.**

* **Input**: The model \(\dot{x}(t) = f(x(t))\) as in (1), \(\|P\|\), \(\|DP\|\), \(|r|\), and \(\|Dr\|\) for \(P\) and \(r\) in (2).

* **Output**: Positive constants \(\varepsilon, a, \beta_0, \beta_1, \beta_2\) such that the inequalities (11), (12), (13), (14), (15), and (16) are satisfied.

1. Compute a parameterization of the periodic orbit of (1), \(K_0 : \mathbb{T} \to \mathbb{R}^d\), and the frequency \(\omega_0 > 0\).

2. For a fixed \(\theta_0 \in \mathbb{T}\), compute the solution of the forward variational equation (5) and also the solution of the backward variational equation, \(\Phi(\theta_0; \theta) = (\Phi(\theta; \theta_0))^{-1}\), which verifies
\[
\omega_0 \frac{d}{d\theta} \Phi(\theta_0; \theta) = -\Phi(\theta_0; \theta) Df(K_0(\theta)), \quad \Phi(\theta_0; \theta_0) = I_d. \tag{17}
\]

3. Compute the eigenvectors of \(\Phi(\theta_0 + 1; \theta_0)\).

4. Compute \(\|\Pi_{\theta_0}^T\|\), \(\|\Pi_{\theta_0}^T\|\), and \(\|[Id - \Phi(\theta_0 + 1; \theta_0)]^{-1}\|E_{\theta_0}\|\).

5. Compute the constants \(C_{1,1}, C_{1,2}, C_{1,3}\) given in (8), \(C_{2,1}, C_{2,2}\) given in (9), and \(M\) given in (10).
6. Optimize $\varepsilon$ for $a$, $\beta_0$, $\beta_1$, $\beta_2$ in certain ranges so that the following inequalities are verified.

$$
\begin{align*}
Q(\varepsilon, a, \beta_0) - a &\leq 0 \\
P_0(\varepsilon, a, \beta_0) - \beta_0 &\leq 0 \\
P_1(\varepsilon, a, \beta_0, \beta_1) - \beta_1 &\leq 0 \\
P_2(\varepsilon, a, \beta_0, \beta_1, \beta_2) - \beta_2 &\leq 0 \\
\mu_1(\varepsilon, a, \beta_0, \beta_1) - 1 &< 0 \\
\mu_2(\varepsilon, a, \beta_0, \beta_1) - 1 &< 0.
\end{align*}
$$

(18)

**Remark 3.1.** Note that Algorithm 3.1 works for a class of perturbations $P$ and delays $r$, one only needs the $C^1$ norms of them in a neighborhood of the unperturbed periodic orbit.

**Remark 3.2.** Note that if $\varepsilon_0$ verifies the inequalities in (18) given the constants, so will any $0 \leq \varepsilon \leq \varepsilon_0$.

**Remark 3.3.** There are different ways of optimizing, for example, we can also view $\|P\|$, $\|DP\|$, $\|r\|$, and $\|Dr\|$ as variables of the polynomials and modify the optimization step in Algorithm 3.1 to maximize $\|P\|$, $\|DP\|$, $\|r\|$, and $\|Dr\|$ along with $\varepsilon$. The optimization process consists, in general, in local searches and thus the initial guesses play an important role. The choice of the objective function depends on the goals of the problems, which we will specify in Section 4.

### 3.1 Computer-Assisted Proofs for the Unperturbed System

The first two steps of Algorithm 3.1 require computing solutions of ODEs, namely a periodic orbit of (1) and solutions of the forward and backward variational equations about the periodic orbit. These steps are achieved with the tools of rigorously validated numerics. Using a computer to produce constructive proofs of existence of solutions of differential equations is by now well-established, and we refer the interested reader to the survey papers [Nak01, Rum10, KSW96, vdBL15, GS18, KMWZ21] and to the books [NPW19, Tuc11, vdBL18] for more details. Our approach to compute rigorously the ingredients of Steps 1 and 2 of Algorithm 3.1 uses Chebyshev series expansion and a Newton–Kantorovich type theorem (i.e. the radii polynomial approach), as presented in [LR14, HLMJ16, vdBS21]. More precisely, we compute Chebyshev series expansions of the unperturbed periodic orbit $K_0$ of (3), the solution $\Phi$ of the forward variational equation (5), and the solution $\Phi$ of the backward variational equation (17). For each of the three problems, a zero-finding problem is formulated for the Chebyshev coefficients of the solution of the ODE, which lies in the product of weighed $\ell^1$ spaces that we denote $\ell^1_\nu$ (for some geometric decay rate $\nu \geq 1$), see definition in Appendix A.1.

One way to interpret the results is that we have $\tilde{x}$, which is an approximate finite part of the Chebyshev series, and $\tilde{x}$, which is the tail part, such that

$$
x = \tilde{x} + \tilde{x},$$

although we do not know what is exactly $\tilde{x}$, we have the bound

$$
\|\tilde{x}\|_{\ell^1_\nu} \leq R,
$$

for some explicitly given $R > 0$, typically quite small.

In order to use Chebyshev series to represent the solutions, it is standard to rescale the problem and consider solutions defined on the interval $[-1, 1]$, see Appendix A.1.1. Therefore, we define the scaling parameter $L$, which is related with the period, the rescaled periodic orbit $O(s)$, the forward variational flow $F(s)$, and the backward variational flow $B(s)$ as

$$
L = \frac{1}{2\omega_0},
$$

$$
O(s) = K_0\left(\frac{1}{2}(s + 1)\right),
$$

$$
F(s) = \Phi\left(\frac{1}{2}(s + 1); 0\right),
$$

$$
B(s) = \Phi(0; \frac{1}{2}(s + 1)),
$$

8
for all $s$ in $[-1, 1]$, where we note that we fixed $\theta_0 = 0$. In particular, for all $-1 \leq s, t \leq 1$,

$$F(t)B(s) = \Phi(\frac{1}{2}(t + 1); 0)\Phi(0; \frac{1}{2}(s + 1)) = \Phi(\frac{1}{2}(t + 1); \frac{1}{2}(s + 1)),$$

and with this, we can compute $C_{1, 2}$ in (8) and $C_{2, 2}$ in (9).

To solve (3), (5), and (17), we look for Chebyshev series expansions of $O(s)$, $F(s)$ and $B(s)$, and each solution is computed by applying the radii polynomial approach to a specific zero-finding problem. Assume that this has been achieved, we have the numerical approximations and estimations of their tails, that is

$$L = L + \bar{L}, \quad |\bar{L}| \leq R,$$

$$O(s) = \bar{O}(s) + \tilde{O}(s), \quad \|\tilde{O}\| \leq R,$$

$$F(s) = \bar{F}(s) + \tilde{F}(s), \quad \|\tilde{F}\| \leq R,$$

$$B(s) = \bar{B}(s) + \tilde{B}(s), \quad \|\tilde{B}\| \leq R,$$

(19)

in certain norms associated with the norm in $\ell^*_1$.

### 3.2 Computation of the bounds

The bounds in the Algorithm 3.1 require to manage the information from the Section 3.1 and specify the norms. From now on, we will stick to Euclidean norm on $\mathbb{R}^d$, then, the supremum norms of vector fields are straightforward. For the derivatives of the vector fields, we are going to use the Fröbenius norm (i.e. Euclidean norm of the vectorization) as an upper bound of their operator norms.

Note that $O: [-1, 1] \to \mathbb{R}^d$ and $F, B: [-1, 1] \to \mathbb{R}^{d \times d}$, and the norms appeared in the coefficients of the polynomials in Algorithm 3.1 are, in essence, the supremum norms. Since when $\nu = 1$, the $\ell^*_1$ norm is an upper bound of the supremum norm, we then have

$$\|O\| \leq \sqrt{\sum_{i=1}^{d} \|O_i\|_{\ell^*_1}^2} \leq \sqrt{\sum_{i=1}^{d} (\|\tilde{O}_i\|_{\ell^*_1} + R)^2},$$

$$\|F\| \leq \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} \|F_{i,j}\|_{\ell^*_1}^2} \leq \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} (\|\tilde{F}_{i,j}\|_{\ell^*_1} + R)^2},$$

similarly for $B$.

Some of the coefficients in the polynomials involved in Algorithm 3.1 are now straightforward. However, there are still a few bounds requiring more computational effort.

### 3.2.1 Bounds on Neighborhoods of the Periodic Orbit

Some quantities in Algorithm 3.1 namely $\|Df\|$ appearing in (13), $\|D^2f\|$ appearing in (12), (13) and (14), and $\|D^3f\|$ appearing in (14), need to be bounded in a neighborhood of the periodic $K_0$. While these derivatives are defined everywhere, it is however enough to consider their bounds in a neighborhood of the unperturbed periodic orbit.

Therefore, we will rigorously provide a neighborhood enclosing the unperturbed periodic orbit. Since we use interval arithmetic, the enclosure will be provided in terms of a hypercube, which admits an easy computer encoding using interval arithmetic.

Let $O(s)$ be the rigorously proved periodic orbit as in (19). If $O = (O_1, \ldots, O_d)$ component-wise, then for each $O_i$, we consider the optimization problems of minimization and maximization on $s \in [-1, 1]$.

Note that to obtain initial approximations of these optimizations, we consider a non-interval optimization problem with the numerical approximations $\tilde{O}_i$, then we verify it using, e.g.,
verifyconstraintglobalmin in intlab, see [Rum18]. The rigorous verification provides an interval where the min/max is located, we then evaluate $\hat{O}_t$ on this interval. Now taking into account the errors $\bar{O}_i$, we are ready to provide the infimum and the supremum as boundaries of the hypercube containing the periodic orbit. Here, one can consider some safety factors to make the enclosure a little bit bigger although the process described here already provides a rigorous enclosure. We then enlarge the hypercube by size $\beta_0$ on the upper and lower bounds, which makes sure that the periodic orbit of the perturbed equation \[2\] lies in the enlarged neighborhood.

The outputs of these bounds will be intervals containing the exact bounds, to prevent a wrapping effect, the hypercube mesh must be adjusted until those intervals have a small radius. This adjustment will be model-dependent and often will be an ad-hoc process.

Once the hypercube is determined, we evaluate upper bounds of $\|Df\|$, $\|D^2f\|$, and $\|D^3f\|$ on the hypercube (possibly with a hypercube mesh) and return the maximum of these evaluations as the bounds. More precisely, for $f = (f_1, \ldots, f_d)$, using $\partial$ derivative notation, we have that on the hypercube,

$$
\|Df\| \leq \sqrt{\sum_{i,j=1}^{d} \|\partial_i f_j\|_{\ell_1}^2}, \quad \|D^2f\| \leq \sqrt{\sum_{i,j,k=1}^{d} \|\partial_{i,j} f_k\|_{\ell_1}^2}, \quad \|D^3f\| \leq \sqrt{\sum_{i,j,k,l=1}^{d} \|\partial_{i,j,k} f_l\|_{\ell_1}^2},
$$

where $\partial_i f_j$ means the partial derivative of $f_j$ with respect to $x_i$, other expressions are similar. Note that for the first inequality above, we used the fact the Fr"obenius norm is an upper bound of the operator norm of a matrix under Euclidean norm on $\mathbb{R}^d$. The second and third inequalities can be derived from this fact.

### 3.2.2 Bounds on the Convolutions

We have to consider some products involved with the ODE to get several coefficients for the polynomials in Algorithm 3.1. With Chebyshev representations, the products become convolutions. Since the $\ell_{\nu}^1$ space is a Banach algebra (see Section A.1), we have $\|a * b\|_{\ell_{\nu}^1} \leq \|a\|_{\ell_{\nu}^1} \|b\|_{\ell_{\nu}^1}$ for all $a, b \in \ell_{\nu}^1$, where $*: \ell_{\nu}^1 \times \ell_{\nu}^1 \to \ell_{\nu}^1$ denotes the discrete convolution.

However, the above inequality is likely to provide overestimated bounds which will affect the size of $\varepsilon$ in the optimization step of the inequalities (18). To get better results, we should avoid using the inequality as much as possible.

More precisely, let $a = \bar{a} + \bar{a}$ and $b = \bar{b} + \bar{b}$ be $\ell_{\nu}^1$ elements with exact truncated parts and the tail parts. If the tail parts are bounded by $R > 0$, then

$$
\|a * b\|_{\ell_{\nu}^2} \leq \|ar{a} * \bar{b}\|_{\ell_{\nu}^2} + (\|\bar{a}\|_{\ell_{\nu}^1} + \|\bar{b}\|_{\ell_{\nu}^1} + R)R. \quad (20)
$$

Numerically we keep $\bar{a} * \bar{b}$ and the bound of the tail ($\|\bar{a}\|_{\ell_{\nu}^1} + \|\bar{b}\|_{\ell_{\nu}^1} + R)R$. Thus, we can consider a class that encodes the truncated Chebyshev series and a bound of its tail. In that class we overload different operations and make elemental operations, such as sums, products, norms, easily computable.

Note that the bound in (20) becomes more complicated as we increase the number of the convolutions to bound, i.e. for cubic, quartic, quintic, etc. convolutions. More precisely, we can keep the numerical parts and let the tail parts be variables of a polynomial, e.g. $p(s_1, s_2) = (\bar{a} + s_1)(\bar{b} + s_2)$, we then expand everything in monomials and take into account the bounds of the tails. If the tails are bounded by $R$, then we let $s_1 = s_2 = R$ in the expansion.

### 3.2.3 Bounds on the Derivatives

Expressions like $C_{2,1}$ in (9) can be bounded by using the ODE systems. That is,

$$
\frac{d}{ds} F(1)B(s) = -F(1)Lf(O(s))B(s).
$$

10
The norm of the righthand side is now easily computable by convolutions, taking care of the numerical and tail parts of $F(1)$, $L$, $O(s)$, and $B(s)$.

Another possible way to bound the norm of the derivative of a function is to use estimates similar to Cauchy bounds.

### 3.2.4 Bounds on Triangle Meshes

The terms $C_{1,2}$ in [8] and $C_{2,2}$ in [9] are computationally expensive because they require considering a triangular mesh.

Indeed, the terms $F(t)B(s)$ in $C_{1,2}$ and $-F(t)Lf(O(s))B(s)$ in $C_{2,2}$ can be bounded by taking an interval mesh for the triangle $-1 \leq s \leq t \leq 1$. That is, for a $m(m+1)/2$ mesh size, we define the intervals

$$
t_k = -1 + 2[k-1,k]/m, \quad s_j = -1 + 2[j-1,j]/m,
$$

for integers $k = 1, \ldots, m$ and $j = 1, \ldots, k$. We evaluate the expressions in these intervals (adding the radius $R$), computing the norms, and returning the maximum. The value of $m$ is chosen in such a way that the maximum stagnates with respect to larger $m$.

### 3.2.5 Bounds on the Projections

The projections $\Pi_{\theta_0}^t$ and $\Pi_{\theta_0}^\perp$ have the same norm. To bound them, we first consider the case $d = 2$. The higher dimensional case is similar.

In the two-dimensional case, the monodromy matrix has two eigenpairs $(\lambda_i, u_i)$, $i = 1, 2$, without loss of generality, we assume that $\lambda_1 = 1$, $\lambda_2 \neq 1$. An arbitrary vector $u$ in the plane is given by $u = a_1u_1 + a_2u_2$ in the basis $\{u_1, u_2\}$. Let $\alpha$ be the smaller angle between these two eigenvectors. Then by the trigonometric relations (law of sine) (see Figure 1)

$$
\frac{|u_1|}{\sin \alpha} = \frac{|u_1 u_1|}{\sin \beta} = \frac{|u_2 u_2|}{\sin(\alpha - \beta)},
$$

then

$$
|a_1 u_1| = \frac{\sin \beta}{\sin \alpha} |u| \quad \text{and} \quad |a_2 u_2| = \frac{\sin(\alpha - \beta)}{\sin \alpha} |u|.
$$

As a consequence, the norm of both projections can be bounded as $\|\Pi_{\theta_0}^t\|, \|\Pi_{\theta_0}^\perp\| \leq \frac{1}{\sin \alpha}$. In practice, the angle $\alpha$ can be computed by the inner product properties, that is

$$
\cos \alpha = \frac{u_1 \cdot u_2}{|u_1||u_2|}.
$$

Note that the eigenvectors $u_1$ and $u_2$ are those of the matrix $F(1)$, which consists of the numerical part and the tail part. These eigenvectors need to be verified. We can use for example verifyeig in INTLAB for the rigorous verification (see [Rum01]).

In the $d$-dimensional case, by assumption [H], the monodromy matrix $F(1)$ has a simple eigenvalue 1 with eigenvector $u_1$. All the other eigenvectors of eigenvalues $\neq 1$ generate a hyperplane. Let $\alpha$ be the acute angle between $u_1$ and the hyperplane, then similar to the $d = 2$ case, the norms are bounded by $\frac{1}{\sin \alpha}$. If we consider a normal vector $\vec{n}$ to the hyperplane, the angle $\gamma$ between $u_1$ and $\vec{n}$ satisfies $\sin \alpha = \frac{1}{|\cos \gamma|}$, see Figure 1.

### 3.2.6 Bounds in the complement of tangential directions

Now we consider $\| [I_d - \Phi(\theta_0 + 1; \theta_0)]^{-1} E_{\theta_0} \|$ in Algorithm 3.1. Due to the hypothesis [H], we know that restricted to the subspace $E_{\theta_0}$ of [6], the matrix $I_d - \Phi(\theta_0 + 1; \theta_0)$ is invertible. In
Figure 1: On the left, the angles in the two-dimensional case \( d = 2 \), a vector \( u \) in the span of \( u_1 \) and \( u_2 \). On the right, the general case, \( u_1 \) is the eigenvector of the eigenvalue 1 and \( \vec{n} \) is a normal vector of the hyperplane generated by the other eigenvectors.

Our experiments, see Section 4, since \( E_{\theta_0} \) is 1-dimensional, it is easy to invert \( Id - \Phi(\theta_0 + 1; \theta_0) \) and compute its norm in \( E_{\theta_0} \). In general, one could consider the nonzero singular values of \( Id - \Phi(\theta_0 + 1; \theta_0) \). Since under Euclidean norm, the matrix operator norm is the largest singular value, the norm \( \| (Id - \Phi(\theta_0 + 1; \theta_0))^{-1} \|_{E_{\theta_0}} \) we want is the reciprocal of the smallest modulus of the nonzero singular values of \( Id - \Phi(\theta_0 + 1; \theta_0) \).

### 3.3 Solving the Inequalities

The last step in Algorithm 3.1 consists in optimizing the inequalities (18) such that, for instance, the perturbative parameter \( \varepsilon \) is maximized. The variables of that optimization are \( \varepsilon, a, \beta_0, \beta_1, \) and \( \beta_2 \). All of them must be strictly positive, that is more than the epsilon machine, e.g. \( 2^{-52} \). Moreover, the result in [YGdlL] says that \( a \) and \( \beta_0 \) are of the same order as \( \varepsilon \) asymptotically.

The optimization problem that maximizes the \( \varepsilon \) is computed numerically. That is, we take the upper bounds of the coefficient intervals of the polynomials in (18), we apply the numerical maximization problem, and then we check the result with the inequalities having the interval coefficients.

A similar process is applied when we try to optimize the class of perturbations and forward/backward delays, namely \( \| P \|, \| DP \|, \| r \|, \) and \( \| Dr \| \). In this situation, one needs to play more with the objective function of interest, which depends on the goals of the proof itself.

To prevent a loss of information, it is convenient to expand the polynomial in (18) in monomials. Thus the coefficients in the numerical search is as sharp as possible. Appendix C shows the explicit monomial expressions for the two different optimizations proposed here.

In the experiments we use the Optimization Toolbox in Matlab to get an approximation of the maximizations and INTLAB to certify that the validity of the inequalities with the intervals.

### 4 The Van der Pol Example

For the unperturbed ODE, we consider the Van der Pol equation with parameter \( \mu > 0 \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1.
\end{align*}
\]

(21)

Recall that Algorithm 3.1 only requires the \( C^1 \) bounds of \( P \) and \( r \) on a bounded set. Hence, we will not fix \( P \) and \( r \), but rather work with a class of \( P \)'s and \( r \)'s.

### 4.1 Results

The main results consists of three theorems, namely Theorem 4.1, Theorem 4.2 and Theorem 4.3. The first theorem computes and proves the existence of the periodic orbit, the forward, and
the backward variational flows of the ODE in (21). It uses the radii polynomial approach, see Appendix A which provides and explicit distance between the numerical approximation and the exact solution. The second theorem proves the existence of periodic orbits in state-dependent delay perturbations of (21) (in the form of (2)) for some values of the perturbation parameter $\varepsilon$, given the norms of the perturbations and the forward or backward delays. Moreover, we have estimations on the differences between the frequencies and the periodic orbits before and after the perturbation. The third theorem establishes that the perturbation and delay terms can be more general.

**Theorem 4.1.** Fix a parameter value $\mu \in \{j/10 : j = 1, \ldots, 10\}$. Let $\tilde{O} : [-1, 1] \to \mathbb{R}^2$ be a numerical solution of the periodic orbit of (21) (passing close to the point $(2, 0) \in \mathbb{R}^2$) consisting of 200 Chebyshev coefficients (coordinatewise), and $2L$ be the numerical approximate period. Then the true periodic orbit $O : [-1, 1] \to \mathbb{R}^2$ and its period $2L$ satisfy

$$\|O - \tilde{O}\| \leq r_0, \quad |L - \tilde{L}| \leq r_0$$

where $r_0$ depends on $\mu$ and is given in Table 1. Moreover, let $\tilde{F}, \tilde{B} : [-1, 1] \to \mathbb{R}^{2 \times 2}$ be numerical approximations of the forward and backward variational flows of $O$ represented by 200 Chebyshev coefficients (entrywise). Then the true solutions $F, B : [-1, 1] \to \mathbb{R}^{2 \times 2}$ satisfy

$$\|F - \tilde{F}\| \leq r_1, \quad \|B - \tilde{B}\| \leq r_2$$

where $r_1$ and $r_2$ are given in Table 1.

**Remark 4.1.** The norm of the difference between $O$ and $\tilde{O}$ in above theorem should be interpreted as the maximum of the componentwise $\ell^1_\nu$ norm for $\nu = 1.01$, where each component of $\tilde{O}$ (200 Chebyshev coefficients) is viewed as an element in $\ell^1_\nu$ with zero tail. Similar for $\|F - \tilde{F}\|$ and $\|B - \tilde{B}\|$.

**Remark 4.2.** The number 200 of Chebyshev coefficients in Theorem 4.1 are determined by plotting the coefficients and truncating before the stagnation in the tail (due to the precision of the arithmetic as $2^{-52}$ in double-precision.)

Now we are ready to provide our first persistence result.

**Theorem 4.2.** Consider a perturbation of the form (2) to equation (21), assume that the functions $P$ and $r$ satisfy $\|P\|, \|DP\|, \|r\|, \text{ and } |Dr| \leq 1$, then for values of the parameter $\mu$ and constants $a_\alpha$, $\beta_0$, $\beta_1$, and $\beta_2$ in the Table 2 below, if $\varepsilon \leq \varepsilon_0$ in the Table 2 the inequalities in (18) are satisfied. Hence, there exists a $C^{1+\varepsilon_0}$ periodic orbit for the perturbed system (2). The differences of the unperturbed and perturbed periodic orbits and their frequencies lie in the $C^0$ closure of the space $I_a \times \delta_\beta$ as in (7).

**Remark 4.3.** Theorem 1.2 and bootstrapping techniques lead to infinite regularity provided that the unperturbed system and the functions $P$ and $r$ are all smooth.
Table 2: Values of \(a\) and \(\beta\'), and admissible \(\varepsilon_0\) so that the periodic orbit persists under perturbation.

The other parameters admits several options of optimization depending on the aims. Notice that the norm \(\|P\|\) can be normalized to 1 since \(P\) is multiplied by the perturbative parameter \(\varepsilon\). Our second perturbative result, Theorem 4.3, optimizes the threshold \(\varepsilon_0\) and also the class of perturbation \(\|DP\|\) and of the delay \(\|r\|, \|Dr\|\). Then for all \(\varepsilon \leq \varepsilon_0\), \(\|r\| \leq c_r\), \(\|Dr\| \leq c_{Dr}\), and \(\|DP\| \leq c_{DP}\) with \(\varepsilon_0, c_r, c_{Dr},\) and \(c_{DP}\) in Table 4, the inequalities in (18) are satisfied, and so there is a \(C^{1+Lip}\) periodic orbit of the perturbed system.

**Theorem 4.3.** Let (21) be the unperturbed ODE and consider a perturbation of the form (2). Given \(a, \beta_0, \beta_1,\) and \(\beta_2\) in Table 3. Then for all \(\varepsilon \leq \varepsilon_0\), \(\|r\| \leq c_r\), \(\|Dr\| \leq c_{Dr}\), and \(\|DP\| \leq c_{DP}\) with \(\varepsilon_0, c_r, c_{Dr},\) and \(c_{DP}\) in Table 4, the inequalities in (18) are satisfied, and so there exists a \(C^{1+Lip}\) periodic orbit for the perturbed system.

Table 3: Values of the space \(I_a \times \delta_{\beta}'\) in (7) for which the perturbed periodic orbit exists.

Table 4: Optimized perturbative parameters \(\varepsilon_0, c_r, c_{Dr},\) and \(c_{DP}\).

**Remark 4.4.** The quantities in Table 4 have been obtained by an optimization process of the inequalities (18) with the objective function \(\varepsilon^2 \|r\| \|Dr\| \|DP\|^2\) for each of the \(\mu\) values.

### 4.2 The Zero Finding-problems for the Unperturbed System

In order to calculate the required coefficients for this example rigorously, we follow the radii polynomial approach, see [14] and a summary in Appendix A and we use the INTLAB package in Matlab, [4]. We solve a boundary value problem to get the periodic orbit, and initial value problems to get the solutions of the forward and backward variational equations. In the following subsections, each of these problems will be formulated as a zero-finding problem.

In Algorithm 3.1 we need to solve for the periodic orbit \(O(s) = (a_1(s), a_2(s))\) and the scaling parameter \(L\) that verifies
\[
\frac{d}{ds} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = L \begin{pmatrix} \mu(1 - a_1^2) a_2 - a_1 \\ a_1 \end{pmatrix}, \quad O(-1) = O(1), \quad \text{and} \quad a_1(-1) = 0,
\]
(22)
the forward system \( F(s) = (v_{11} \ v_{12}) (s) \) that verifies
\[
\frac{d}{ds} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} = L \begin{pmatrix} 0 & 1 \\ \vartheta & \varphi \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} \quad \text{and} \quad F(-1) = Id_2,
\]
(23)
with \( \vartheta(s) = -2\mu a_1 a_2 - 1 \) and \( \varphi(s) = \mu(1 - a_1^2) \), and the backward system \( B(s) = (u_{11} \ u_{12}) (s) \) that verifies
\[
\frac{d}{ds} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = -L \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \vartheta & \varphi \end{pmatrix} \quad \text{and} \quad B(-1) = Id_2.
\]

We utilize Chebyshev discretization to represent the periodic orbit and its variational (forward and backward) flows. Figure 2 shows the (numerical) solutions for some values of the parameter \( \mu \). Note that for \( \mu \geq 1 \) the backward flow grows rapidly since the periodic orbit attracts strongly as we can see in its (numerical) real non-trivial eigenvalue \( \lambda_{\mu} \) of the monodromy matrix
\[
\begin{align*}
\lambda_{0.5} &= 3.917692025927352 \times 10^{-2}, \\
\lambda_1 &= 8.596950636046152 \times 10^{-4}, \\
\lambda_{1.5} &= 6.466756568013210 \times 10^{-6}.
\end{align*}
\]

**Remark** 4.5. Note that in (22) we considered the spatial section \( O_1(-1) = 0 \). Thanks to hypothesis (H), we can solve the problem successfully.

### 4.2.1 Zero-finding problem for the periodic orbit

The rigorous proof concerning \( L > 0 \) and the periodic orbit \( O: [-1, 1] \to \mathbb{R}^2 \), with sequence of Chebyshev coefficients \((a_1, a_2)\), is done in the Banach space \( X_\nu \overset{\text{def}}{=} (\ell_1^1) \times \mathbb{R} \). The space is endowed with the product norm
\[
\| (a_1, a_2, L) \| \overset{\text{def}}{=} \max\{\|a_1\|_{\ell_1^1}, \|a_2\|_{\ell_1^1}, |L|\},
\]
for \((a_1, a_2, L) \in X_\nu\).

The proof is obtained by solving a boundary value problem, see Appendix A.3 with boundary conditions \( a_1(-1) = 0 \) (the phase condition) and \( a_i(-1) = a_i(1) \), for \( i = 1, 2 \). To define the equivalent zero-finding map \( \mathcal{O}: X_\nu \to X_\nu^*, \) with \( 1 < \nu' \leq \nu \), of (22), we first consider linear operators \( M, T: \ell^1_\nu \to \ell^1_\nu \) and \( \Lambda: \ell^1_{\nu'} \to \ell^1_{\nu'} \) defined as
\[
M \overset{\text{def}}{=} \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{pmatrix},
\]
\[
T \overset{\text{def}}{=} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-1 & 0 & 1 & 0 & 0 & \cdots \\
0 & -1 & 0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]
(24)
Figure 2: On the left $\mu = 0.5$, on the middle $\mu = 1$, and on the left $\mu = 1.5$ with $\mu$ parameter in [21]. From top to bottom the orbit of the periodic orbit passing close to the point $(2,0)$, its forward and backward variational flows.

and

$$\Lambda \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \end{pmatrix} \quad (25)$$

Then the zero-finding map for the periodic orbit is given by $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$, where

$$\mathcal{O}_i(a_1, a_2, L) \overset{\text{def}}{=} (M + \Lambda) a_i + L \cdot T_{f_i}(a), \quad i \in \{1, 2\},$$

$$\mathcal{O}_3(a_1, a_2, L) \overset{\text{def}}{=} (a_1) + 2 \sum_{k \geq 1} (a_1) k (-1)^k,$$

with

$$f_1(a) \overset{\text{def}}{=} a_2,$$

$$f_2(a) \overset{\text{def}}{=} \mu a_2 - \mu (a_1 \cdot a_1 \cdot a_2) - a_1,$$
with \( * \) being the convolution in \( \ell^1 \), see Appendix A.4.

Following the radii polynomial approach, we first compute a numerical solution \((\bar{a}_1, \bar{a}_2, \bar{L}) \in X_\nu \) such that \( \mathcal{O}(\bar{a}_1, \bar{a}_2, \bar{L}) \approx 0 \), that is it is zero up to a given tolerance, for instance \( 10^{-12} \).

Then computing the radii polynomial as in Lemma A.1, there is an exact solution \((a_1, a_2, L) \in X_\nu \) within distance \( r_0 > 0 \) of the numerical solution such that \( \mathcal{O}(a_1, a_2, L) = 0 \).

### 4.2.2 Zero-finding problem for the solution of the forward variational equation

To compute the solution of (23), we first assume that one computed rigorously the values of \( L \) and the periodic orbit \((a_1, a_2)\) verifying (22). We split these information as the sum of the numerical parts and the tail parts, that is

\[
a_i = \bar{a}_i + \hat{a}_i, \quad \text{for } i \in \{1, 2\}, \quad L = \bar{L} + \hat{L}.
\]

From the proof, we know that \( \| \hat{a}_i \|_{\ell^1} \leq r_0 \) and \( |\bar{L}| \leq r_0 \).

The \( \hat{\vartheta} \) and \( \varrho \) in (23) depend on the periodic orbit, and are represented as Chebyshev series. We split them by their numerical and tail parts

\[
\hat{\vartheta} \overset{\text{def}}{=} \hat{\vartheta} + \hat{\vartheta}, \quad \varrho \overset{\text{def}}{=} \varrho + \hat{\varrho}.
\]

Explicitly, the numerical parts are given by

\[
\hat{\vartheta}_k \overset{\text{def}}{=} -2\mu (\hat{a}_1 * \hat{a}_2)_k - \delta_{0,k}, \quad \hat{\varrho}_k \overset{\text{def}}{=} \mu \delta_{0,k} - \mu (\hat{a}_1 * \hat{a}_1)_k,
\]

and the tails contain the crossing terms from the convolutions.

Let \( S: \ell^1_\nu \to \ell^1_\nu \) be the linear operator defined by

\[
S \overset{\text{def}}{=} \begin{pmatrix} -1 & 2 & -2 & 2 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \ldots \end{pmatrix}
\]

and let \( \ell_\alpha: \ell^1_\nu \to \ell^1_\nu \) be the linear operator of convolution multiplication defined, given \( \alpha \in \ell_1 \), by \( \ell_\alpha: b \mapsto \alpha * b \).

The zero-finding problem equivalent to (23) consists in two proofs of initial value problems, see Appendix A.2. Each of them corresponds to a column of \( F(s) \), and they are given by the map \( \mathcal{F}_p: (\ell^1_\nu)^2 \to (\ell^1_\nu)^2 \), with \( 1 \leq \nu' < \nu \) defined as

\[
\mathcal{F}_p(v) \overset{\text{def}}{=} \mathcal{F}_\vartheta(v) + \mathcal{F}_\varrho(v),
\]

\[
\mathcal{F}_\vartheta(v) \overset{\text{def}}{=} p + \begin{pmatrix} S + \Lambda & 0 \\ 0 & S + \Lambda \end{pmatrix} v + \bar{L} \begin{pmatrix} 0 & T \\ T \ell_\vartheta & T \ell_\vartheta \end{pmatrix} v,
\]

\[
\mathcal{F}_\varrho(v) \overset{\text{def}}{=} \bar{L} \begin{pmatrix} 0 & T \\ T \ell_\varrho & T \ell_\varrho \end{pmatrix} v + (\bar{L} + \bar{\bar{L}}) \begin{pmatrix} 0 & 0 \\ T \ell_\varrho & T \ell_\varrho \end{pmatrix} v,
\]

where the operators \( T \) and \( \Lambda \) are defined in (24) and (25) respectively.

The element \( p \in (\ell^1_\nu)^2 \) in (26) is related to the initial condition of the chosen column, when \( v \) corresponds to the first column of \( F(s) \), then \( p \) is the vector of Chebyshev coefficients of the constant vector \( (1) \); otherwise, \( p \) is the vector of Chebyshev coefficients of the constant vector \( (\hat{1}) \).

Now computing the elements in Lemma A.1 for the initial conditions, we prove the existence of the solution \( F(s) = \bar{F}(s) + \bar{F}(s) \) with \( \| \bar{F} \| = \max_{i,j \in \{1, 2\}} \| \tilde{v}_{ij} \|_{\ell_\nu} \leq r_1 \).
4.2.3 Zero-finding problem for the solution of the backward variational equation

Similarly to the forward flow in (26), we consider the map $B_p: (\ell^1_\nu)^2 \rightarrow (\ell^1_\nu)^2$ defined by

\[
B_p(u) \overset{\text{def}}{=} \tilde{B}_p(u) + \tilde{B}_p(u),
\]

\[
\tilde{B}_p(u) \overset{\text{def}}{=} p + \begin{pmatrix} S + \Lambda & 0 \\ 0 & S + \Lambda \end{pmatrix} u - \tilde{L} \begin{pmatrix} 0 & T\ell_{\tilde{\varrho}} \\ T & T\ell_{\tilde{\varrho}} \end{pmatrix} u,
\]

(27)

Now when $p \in (\ell^1_\nu)^2$ in (27) corresponds to $\left( \frac{1}{0} \right)$, we obtain the first row of $B(s)$; otherwise, when $p$ corresponds to $\left( \frac{0}{1} \right)$, we get the second row.

As in the forward flow case, computing the elements in Lemma A.1, we end up proving the existence of the solution $B(s) = \tilde{B}(s) + \tilde{B}(s)$ with $\|\tilde{B}\| = \max_{i,j \in \{1,2\}} \|u_{ij}\|_{\ell^1_\nu} \leq r_2$.

4.3 Details of the Computer-Assisted Proofs for the Unperturbed Systems

We present the results of three proofs. The first one for the periodic orbit and the period is a nonlinear problem with a cubic term in the case of the Van der Pol (21). The two other proofs, (26) and (27), are linear and they depend on the results from the first proof.

We used the radii polynomial approach, see Appendix A, for all the three computer-assisted proofs and a common $\nu = 1.01$ for the $\ell^1_\nu$ space. When we encounter products, like in $\vartheta$ and $\varrho$, which are convolutions in Chebyshev spaces, we keep track of all the terms of the numerical and tail parts, see Section 3.2.2. In particular, we have to manage quintic convolution to get the norm of $D^2K_0$ in the polynomials.

Figure 3 shows the final radii values and the computational times of each proof, in particular, we observed that the backward variational flow is computationally harder for larger $\mu$ because the periodic orbit becomes more attractive, see Figure 2.

Figure 3: Radii of the computer-assisted proofs of the orbit, the forward, and the backward of (21) for different values of the parameter $\mu$.

4.4 Bounds Computation and Optimization Steps

Following Section 3.2 we compute the bounds required in Algorithm 3.1. We use different radii of the computer-assisted proofs, that is, $r_0$ for the period and the periodic orbit, $r_1$ for the forward flow, and $r_2$ for the backward flow. Their values depend on the parameter $\mu$ in (21).
CAPs of Persistence of Periodic orbit

For the triangle meshes we use the size of 5000. Overall the computation required around 3 days for each parameter \( \mu \) of the ODE (21).

We have done two optimization processes. One in Theorem 4.2 and another one in Theorem 4.3. In both cases, the variables \( a \) and \( \beta = (\beta_0, \beta_1, \beta_2) \) for the space \( I_a \times \mathcal{E}_\beta \) in (7) have been restricted to the domains

\[
\begin{align*}
    a, \beta_0 &\in (0, 0.1], \\
    \beta_1, \beta_2 &\in (0.5], \\
    \epsilon &\in (0, +\infty),
\end{align*}
\]

and initial guesses \( a = \beta_0 = 10^{-2}, \beta_1 = \beta_2 = 0.5, \) and \( \epsilon = 10^{-2} \).

In Theorem 4.2 the objective function was just \( \epsilon \) with \( \|P\|, \|DP\|, \|r\|, \) and \( \|Dr\| \) equal to 1. In Figure 4 we used this procedure to illustrate, for different values of the parameter \( \mu \) in (21), how sensitive the numerical threshold \( \epsilon_0 \) is when either \( \|r\|, \|Dr\|, \) or \( \|DP\| \) ranges in the x-axes of the plot and the other two variables are set to 1. Thus, from Figure 4 we observe that \( \|DP\| \) and \( \|Dr\| \) have similar, strong effects on the \( \epsilon_0 \). On the other hand, \( \|r\| \) presents less influence in the \( \epsilon_0 \). In any case, that sensitivity will always depend on the model itself and the inputs of the Algorithm 3.1.

![Figure 4: Value of the \( \epsilon_0 \) in (18) for the Van der Pol equation (21), moving either \( \|r\|, \|Dr\|, \) or \( \|DP\| \) and setting the other two quantities equal to 1.](image)

In Theorem 4.3 the objective function is \( \epsilon^2 \|r\| \|Dr\| \|DP\|^2 \) with \( \|r\|, \|Dr\|, \) and \( \|DP\| \) in the domain \([1, +\infty)\). Figure 5 shows the values \( a \) and \( \beta = (\beta_0, \beta_1, \beta_2) \) for the space \( I_a \times \mathcal{E}_\beta \) in (7) and the numerical thresholds \( \epsilon_0, c_r, c_Dr, \) and \( c_Dr \) in Theorem 4.3. Heuristically, the objective function was chosen considering that different operands have different scales. For instance, \( \epsilon \) will
be, in general, small and the rest will be large. Thus, we use multiplication instead of addition. We are putting $\varepsilon^2$ and $\|DP\|^2$ because we want them to be dominant during the optimization. The reason is that the size of $\varepsilon$ corresponds to the size of the perturbation, and it is not very interesting if the perturbation term $P$ is close to a constant vector.

We note that in our result, it is possible that none of $\varepsilon, \|r\|, \|Dr\|, \|DP\|$ is maximized, since the optimization processes are subject to various tolerances, the initial guess, etc. Our goal is to provide some values of the parameters close to optimized values so that the inequalities (18) are satisfied.

Finally, after each of the numerical optimizations, we verify the inequalities with the interval arithmetic polynomials we got before the optimization to prove rigorously that they verify the inequalities (18).

![Figure 5: Values of the proof optimizing $\varepsilon, \|r\|, \|Dr\|, \|DP\|$ for different values of the parameter $\mu$ in (21) in Theorem 4.3.](image)

### 5 Conclusion

The theorems established here exploit well-known (but fairly state-of-the-art) computer-assisted methods of proofs for ODEs, however the use of these methods in the present work is fairly novel, as the validated numerical computations are used to verify the hypotheses of a very singular perturbation theorem. The statement of the theorem hypothesizes the existence and other more quantitative properties of an isolated periodic orbit in a nonlinear system of ODEs. These hypotheses are notoriously difficult to verify in nonlinear ODEs, especially if the ODEs are far from any perturbative or asymptotic regime. The validated numerical methods allow us to pass from good numerical computations, to mathematically rigorous statements about the desired periodic solution. Moreover, the high order spectral methods used in the present work provide enough control of the orbits that we can obtain, a-posteriori bounds for all the constants appearing in the hypotheses of the perturbation theorem. Indeed, all of this can be made fairly automatic.

The result given here can be generalized easily to higher dimensional systems of ODEs, to the case when there are multiple forward or backward delays, either state-dependent, distributed, or of other types. Of course the polynomials considered here will need some modifications in other cases, but the constructions are not fundamentally different.

Using the same framework, but longer expressions for the polynomials, one should be able to study the case where small delays are present as in electrodynamics, see [YGdlL]. Indeed, applying the arguments developed here to relativistic perturbations of electrodynamics would
be a fascinating future project. Our work used the INTLAB package \cite{Rum99} but other packages providing rigorous bounds can be applied as well, such as, ARB in \cite{Joh17} and the CAPD library \cite{KMWZ21}.

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A **Radii Polynomial Approach**

There are many good references for the radii polynomial approach, see for example, \cite{HLMJ16,LR14,GAL20}, we cite the one in \cite{GAL20}, which suits our problem the best.

**Lemma A.1.** Let \( \bar{x} \in X \) and \( r > 0 \), assume that \( G : X \to Y \) is Fréchet differentiable on the ball \( B_r(\bar{x}) \). Consider bounded linear operators \( A^\dag \in B(X,Y) \) (approximation of \( D G(\bar{x}) \)) and \( A \in B(Y,X) \) (approximate inverse of \( D G(\bar{x}) \)). Assume that \( A \) is injective. Let \( Y_0, Z_0, Z_1, Z_2 \geq 0 \) be bounds satisfying

\[
\begin{align*}
\|A G(\bar{x})\|_X & \leq Y_0 \\
\|I - A A^\dag\|_{B(X)} & \leq Z_0 \\
\|A (D G(\bar{x}) - A^\dag)\|_{B(X)} & \leq Z_1 \\
\|A (D G(\bar{x} + z) - D G(\bar{x}))\|_{B(X)} & \leq Z_2 r, \quad \forall z \in B_r(0).
\end{align*}
\]

Define the radii polynomial

\[
p(r) = Z_2 r^2 + (Z_0 + Z_1 - 1) r + Y_0. \tag{28}
\]

If there exists \( 0 < r_0 \leq r \), s.t. \( p(r_0) < 0 \), then there exists a unique \( x \in B_{r_0}(\bar{x}) \) s.t. \( G(x) = 0 \).

Following \cite{LR14}, we briefly summarize the process to address initial and boundary value problems. In both cases, we use a suitable Banach space which is a Banach algebra under discrete convolutions.

**A.1 The Banach algebra space**

Let \( \omega_k \) be the weights defined in terms of a parameter \( \nu > 1 \) and the Kronecker delta function \( \delta_{i,j} \) as \( \omega_k = (2 - \delta_{0,k}) \nu^k \). The “ell one nu” space is the normed-space defined by

\[
\ell_1^\nu := \left\{ b_k \in \mathbb{R} : \|b\|_{\ell_1^\nu} := \sum_{k \geq 0} \omega_k |b_k| < \infty \right\}. \tag{29}
\]

In this space, we define a product (or convolution) as

\[
(a \ast b)_k = \sum_{|k_1| + |k_2| = k} a_{|k_1|} b_{|k_1|} \tag{30}
\]

for all \( a = \{a_k\}_{k \geq 0} \) and \( b = \{b_k\}_{k \geq 0} \) in \( \ell_1^\nu \). The space is a Banach algebra under convolution (e.g. see \cite{Les18}),

\[
\|a \ast b\|_{\ell_1^\nu} \leq \|a\|_{\ell_1^\nu} \|b\|_{\ell_1^\nu}.
\]
A.1.1 The space of Chebyshev series

The Chebyshev series of a function \( x \) defined on \([-1, 1]\) is an expression of the form

\[
x(t) = a_0 + 2 \sum_{k=1}^\infty a_k T_k(t), \quad a_0, a_k \in \mathbb{R} \text{ and } t \in [-1, 1]
\]

where \( T_k(t) = \cos(k \cos^{-1}(t)) \in [-1, 1] \) are the Chebyshev polynomials, \([Tre13]\).

We consider the sequence of coefficients of (31) in \( \ell^1_\nu \) for a parameter \( \nu \geq 1 \). Thus, the \( \ell^1_\nu \)-norm of the sequence is an upper bound of the \( C^0 \)-norm of the function, i.e.

\[
\|x\|_{C^0} \leq \|x\|_{\ell^1_\nu} \leq \|x\|_{\ell^1_\nu}.
\]

The product of two Chebyshev series corresponds to the convolution (30) of their coefficients. Note that the convolution operator by \( a \) in \( \ell^1_\nu \), i.e. \( b \mapsto a * b \), can be written as a combination of Toeplitz and Hankel (infinite) matrices.

From the numerical point of view, the Chebyshev series (30) is truncated up to a finite number of coefficients and its tail is estimated separately. Because of the aliasing phenomenon, one needs to consider a zero-padding on the inputs to obtain a correct output of convolution.

The Matlab code in Listing 1 codifies an arbitrary number of convolutions using the Toeplitz and Hankel matrices. Note that the code also works with the intlab package for interval arithmetic. There are other approaches to compute rigorously the convolution of Chebyshev series using the FFT, e.g. \([Les18]\). This other approach is, in general, more flexible when the memory resources is a constraint.

```matlab
function a = convo(varargin)
    m = nargin - 1;
    nn = arrayfun(@(x) size(x{1},1), varargin);
    maxn = max(nn);
    a = [varargin{end}; zeros(maxn*(m+1)-nn(end),1)];
    for k = 1:m
        a = mat([varargin{k}; zeros(maxn*(m+1)-nn(k),1)])*a;
    end
end

function A = mat(a)
    m = length(a);
    A = toeplitz(a) + [zeros(m-1,1) hankel(a(2:end)); zeros(1,m)];
end
```

Listing 1: Chebyshev convolutions using matrices.

Another important part for truncated Chebyshev series is the evaluation at a given \( t \in [-1, 1] \), there are various efficient algorithms to compute it, such as the Clenshaw-Curtis \([CC60]\) or Laurent-Horner methods \([AH20]\). Using intlab we found that the direct computation of the Chebyshev polynomials \( T_k(t) = \cos(k \cos^{-1}(t)) \) and an explicit computation of the linear combination in (31) to be more accurate.

A.2 Zero-finding problem for Initial Value Problems

The first step for a rigorous proof of an initial value problem (IVP) in \( \mathbb{R}^d \), see \([LR14]\), like

\[
\dot{x}(t) = Lf(x), \quad x(-1) = p_0,
\]

consists in fixing the final time of integration that, by a temporal change, encoded in the (known) parameter \( L \), ensures \( t \in [-1, 1] \). Note that (32) is equivalent to

\[
x(t) = x(-1) + \int_{-1}^t f(x(s)) \, ds, \quad x(-1) = p_0.
\]
If the orbit \( x(t) \) is represented with Chebyshev coefficients \( \{a_k\}_{k \geq 0} \), as in [31], and we assume that the map \( f \) admits a Chebyshev series representation, possibly by polynomialization techniques [Hen21] [LMJR16], then \( y(t) = f(x(t)) \) has computable coefficients \( \{f_k\}_{k \geq 0} \) in terms of \( \{a_k\} \). Thus, the equation \( \mathscr{G}(a_k) = 0 \) for the Lemma A.1 corresponds to finding zero of the expression

\[
\begin{align*}
    p_0 - a_0 - 2 \sum_{j \geq 1} (-1)^j a_j & \quad k = 0, \\
    2ka_k + Lf_{k+1} - Lf_{k-1} & \quad k \geq 1.
\end{align*}
\]

A.3 Zero-finding problem for the Boundary Value Problems

Following [LR14], a periodic boundary value problem (BVP) of an ODE consists in integrating an IVP, see Appendix A.2, taken into consideration of boundary conditions given by the equation

\[ \mathcal{H}(x(-1), x(1)) = 0 \]

where \( \mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}^m \) and \( p_1 = x(1) \) depend on parameters \( \theta \in \mathbb{R}^m \), possibly including \( L \) in (32) now as an unknown.

Assuming that \( \mathcal{H} \) admits a Chebyshev representation series, say \( H(a_k, \theta) \), then the equation \( \mathscr{G}(a_k, \theta) = 0 \) for Lemma A.1 corresponds to making zero the expression

\[
\begin{align*}
    H(a_k, \theta) & \quad k = -1, \\
    p_1 - a_0 - 2 \sum_{j \geq 1} a_j & \quad k = 0, \\
    2ka_k + Lf_{k+1} - Lf_{k-1} & \quad k \geq 1.
\end{align*}
\]

B The radii polynomial approach applied to the Van der Pol system

The explicit bounds in Lemma A.1 depends on the model itself. Here we give one-by-one each of those bounds for the periodic orbit of the Van der Pol case, [21]. The forward and backward variational flows are simpler since they are linear problems and, in particular, \( Z_2 = 0 \).

First we need to consider the truncation up to Chebyshev coefficient order \( n \), and bound the tails. After that, we can consider operator \( A \) and \( A^\dagger \) and we provide bounds for \( Y_0, Z_0, Z_1, \) and \( Z_2 \).

B.1 Projections and inclusions

Fixed \( n \geq 0 \). Define projection \( \pi^n : \ell^1_\nu \to \mathbb{R}^{n+1} \) by \( (a_k)_{k \geq 0} \mapsto (a_k)_{k=0}^n \). And, similarly, define \( \Pi^{(n)} : X_\nu \to \mathbb{R}^{2n+3} \),

\[ \Pi^{(n)}(a, b, L) \overset{\text{def}}{=} (\pi^n a, \pi^n b, L), \quad \forall (a, b, L) \in X_\nu. \]

Let \( \iota^n : \mathbb{R}^{n+1} \hookrightarrow \ell^1_\nu \) be the inclusion defined by

\[ (a_k)_{k=0}^n \mapsto (a_0, a_1, \ldots, a_n, 0, 0, \ldots). \]

Then similarly as in the projection case, we denote by \( \iota^{(n)} : \mathbb{R}^{2n+3} \to X_\nu \) the mapping

\[ \iota^{(n)}(a, b, L) \overset{\text{def}}{=} (\iota^n a, \iota^n b, L), \quad \forall (a, b, L) \in (\mathbb{R}^{n+1})^2 \times \mathbb{R}. \]

For the periodic orbit, define a finite dimensional projection \( F^{(n)} : \mathbb{R}^{2n+3} \to \mathbb{R}^{2n+3} \) of \( F \):

\[ F^{(n)}(x) \overset{\text{def}}{=} \Pi^{(n)} F(\iota^{(n)} x). \]
B.2 Choices of $A^\dagger$ and $A$ for the periodic orbit

$A^\dagger$ is an approximation of the derivative of the zero-finding function (at the numerical solution). We basically choose it as the derivative of the truncated finite-dimensional zero-finding function plus only the diagonal part for the tail.

More precisely,

$$A^\dagger = \begin{pmatrix} A_{1,1}^{\dagger} & A_{1,2}^{\dagger} & A_{1,3}^{\dagger} \\ A_{2,1}^{\dagger} & A_{2,2}^{\dagger} & A_{2,3}^{\dagger} \\ A_{3,1}^{\dagger} & A_{3,2}^{\dagger} & A_{3,3}^{\dagger} \end{pmatrix} = \begin{pmatrix} A_{1,1}^{\dagger} & 0 & A_{1,3}^{\dagger} \\ 0 & A_{2,2}^{\dagger} & 0 \\ A_{3,1}^{\dagger} & A_{3,2}^{\dagger} & 0 \end{pmatrix}.$$  

For $b = (b_1, b_2, b_3) \in X_\nu$,

$$(A^\dagger b)_i = A_{i,1}^{\dagger} b_1 + A_{i,2}^{\dagger} b_2 + A_{i,3}^{\dagger} b_3 \in \ell^1_\nu, \quad i = 1, 2, 3.$$  

Note that for $i = 1, 2$, $(A_{i,1}^{\dagger} b_3)_k = (\hat{c}_L F_i^{(n)}(\bar{x}) b_3)_k$ for $k \leq n$, $(A_{i,1}^{\dagger} b_3)_k = 0$ otherwise. For $i = 1, 2, 3$, $j = 1, 2$:

$$(A_{i,j}^{\dagger} b_j)_k \overset{\text{def}}{=} \begin{cases} (D_j F_i^{(n)}(\bar{x}))_k & k \leq n, \\
2 k \delta_{ij} (b_j)_k & k > n,
\end{cases}$$  

where $\delta_{ij}$ is the Kronecker delta.

For $A$, we take numerical inverse of the derivative for the truncated finite-dimensional zero-finding function and add the diagonal part for the tail, which is smoothing. Let $A^{(n)}$ be numerical inverse of $DF^{(n)}(\bar{x})$,

$$A^{(n)} \overset{\text{def}}{=} \begin{pmatrix} A_{1,a_1}^{(n)} & A_{1,a_2}^{(n)} & A_{1,L}^{(n)} \\ A_{2,a_1}^{(n)} & A_{2,a_2}^{(n)} & A_{2,L}^{(n)} \\ A_{3,a_1}^{(n)} & A_{3,a_2}^{(n)} & A_{3,L}^{(n)} \end{pmatrix} \in \mathbb{R}^{(2n+3) \times (2n+3)}.$$  

Define approximate inverse $A$ of $DF(\bar{x})$ by

$$A = \begin{pmatrix} A_{1,a_1} & A_{1,a_2} & A_{1,L} \\ A_{2,a_1} & A_{2,a_2} & A_{2,L} \\ A_{3,a_1} & A_{3,a_2} & A_{3,L} \end{pmatrix} = \begin{pmatrix} A_{1,a_1}^{(n)} & A_{1,a_2}^{(n)} & A_{1,L}^{(n)} \\ A_{2,a_1}^{(n)} & A_{2,a_2}^{(n)} & A_{2,L}^{(n)} \\ A_{3,a_1}^{(n)} & A_{3,a_2}^{(n)} & A_{3,L}^{(n)} \end{pmatrix}.$$  

Where we abused the notation $A_{i,L}^{(n)}$ $(i = 1, 2)$ for it to denote an element in $\ell^1_\nu$ by adding a zero tail. For $i = 1, 2, 3$, $j = 1, 2$, and $b = (b_1, b_2, b_3) \in X_\nu$,

$$(A_{i,j} b_j)_k = \begin{cases} (A_{i,j}^{(n)} b_j^{(n)})_k & k \leq n, \\
\frac{1}{2k} \delta_{ij} (b_j)_k & k > n.
\end{cases}$$  

B.3 $Y$ bounds for periodic orbits

With the zero-finding function and our choice of $A$, we are ready to calculate the $Y$ bounds component-wise.

$$Y_i^{(0)} = \left[ \sum_{i=1}^{2} (A_{1,a_1}^{(n)} F_i^{(n)}(\bar{x}))_0 + (A_{1,a_2}^{(n)} F_i^{(n)}(\bar{x}))_0 \right] + 2 \sum_{k \leq n} \left[ \sum_{i=1}^{2} (A_{1,a_1}^{(n)} F_i^{(n)}(\bar{x}))_k + (A_{1,a_2}^{(n)} F_i^{(n)}(\bar{x}))_k \right] \nu^k + \frac{L}{n+1} |(a_2)_n| \nu^{n+1}. $$
\[ Y_2^{(0)} = \sum_{i=1}^{2} \left( A_{2,ai}^{(n)} P_i^{(n)}(\bar{x})_i \right) + (A_{2,L} P_3^{(n)}(\bar{x}))_0 \]
\[ + 2 \sum_{k=1}^{2} \sum_{i=1}^{2} \left( A_{2,ai}^{(n)} P_i^{(n)}(\bar{x})_i \right) + (A_{2,L} P_3^{(n)}(\bar{x}))_k \nu^k \]
\[ + \sum_{k=n+1}^{3n+1} \frac{L^n}{k} \sum_{|k_1+k_2+k_3=k+1} (a_1)|_{k_1} (a_1)|_{k_2} (a_2)|_{k_3} \]
\[ - \sum_{k=n+1}^{3n+1} (a_1)|_{k_1} (a_1)|_{k_2} (a_2)|_{k_3} \nu^k \]
\[ + \frac{L}{n+1} |a(a_2)_n -(a_1)_n| \nu^{n+1}, \]
\[ Y_3^{(0)} = \left| \sum_{j=1}^{2} A_{3,ai}^{(n)} P_i^{(n)}(\bar{x})_i + A_{3,L} P_3^{(n)}(\bar{x})_j \right|. \]

**B.4 Z bounds for periodic orbits**

**B.4.1 Z^0 bounds**

Let \( B = I - AA^\dagger \), block-wise:

\[
B \overset{\text{def}}{=} \begin{pmatrix} B_{1,a1} & B_{1,a2} & B_{1,L} \\ B_{2,a1} & B_{2,a2} & B_{2,L} \\ B_{3,a1} & B_{3,a2} & B_{3,L} \end{pmatrix}.
\]

Define the norm

\[
\| B_{j,a_i} \|_{B(\ell^1, \ell^1)}^{*} \overset{\text{def}}{=} \sup_{m \in \mathbb{N}} \frac{1}{\omega_m} \sum_{k \in \mathbb{N}} |(B_{j,a_i})_{k,m}| \omega_k
\]
\[ = \max_{0 \leq m \leq n} \frac{1}{\omega_m} \sum_{0 \leq k \leq n} |(B_{j,a_i})_{k,m}| \omega_k, \quad \forall i, j \in \{1, 2\}. \]

Then, we can take

\[
Z_j^{(0)} \overset{\text{def}}{=} \| B_{j,a_1} \|_{B(\ell^1, \ell^1)}^{*} + \| B_{j,a_2} \|_{B(\ell^1, \ell^1)}^{*} + \| B_{j,L} \| \nu, \quad \forall j \in \{1, 2\},
\]
\[ Z_3^{(0)} \overset{\text{def}}{=} \| B_{3,a_1} \|_{\ell^\infty} + \| B_{3,a_2} \|_{\ell^\infty} + |B_{3,L}|. \]

**B.4.2 Z^1 bounds**

Let \( z_j = ([DF(\bar{x}) - A^\dagger]d)_j \). Let \( P^n = \ell^n \circ \pi^n : \ell^1 \nu \rightarrow \ell^1 \nu, \)
\[ P^n : (a_k)_{k \geq 0} \mapsto (a_0, a_1, \ldots, a_n, 0, 0, \ldots), \]
and \( P^I = Id - P^n : \ell^1 \nu \rightarrow \ell^1 \nu, \)
\[ P^I : (a_k)_{k \geq 0} \mapsto (0, 0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots). \]

Define operator \( T^I \overset{\text{def}}{=} T - P^n TP^n. \)

\[
z_1 = MP^I d_1 + L \cdot T^I d_2 + d_3 \cdot P^I T d_2.
\]

Notice that

\[
Dc_2(a)(d_1, d_2) = \mu d_2 - 2\mu (d_1 * a_1 * a_2) - \mu (a_1 * a_1 * d_2) - d_1,
\]

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Lemma B.1. Let $P$ be a function of $x$ where $x = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N, 0, 0, \ldots) \in \ell^1$. For $0 \leq k \leq n + 1$, define $\ell^k_{\alpha} \in \ell^\infty$ by

$$
\ell^k_{\alpha}(h) \overset{\text{def}}{=} (\alpha * P^I h)_{k} = \sum_{k_1 + k_2 = k} \alpha_{[k_1]}(P^I h)_{[k_2]}.
$$

Then,

$$
\|\ell^k_{\alpha}\|_{\ell^\infty} \leq \psi_{k}(\bar{\alpha}) \overset{\text{def}}{=} \max_{n-j \leq k \leq N} \left( \frac{\|\bar{\alpha}_{[k-j]} + \bar{\alpha}_{k+j}\|}{2\nu} \right).
$$

In our case, we will let $N = 2n$.

$$
\|A_{1, a_2} z_2\|_{\nu} \leq \frac{1}{\omega_{n+1}} \left( \|A_{1, a_2}^{(n)}\|_{0, \nu} + (\mu + 1)\bar{L}\|A_{1, a_2}^{(n)}\|_{n, \nu}\right) r
$$

$$
+ 2\mu\bar{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{1, a_2}^{(n)})_{i,j}| (\Psi_{j-1}(\bar{a}_1 * \bar{a}_2) + \Psi_{j+1}(\bar{a}_1 * \bar{a}_2)) \omega_i r
$$

$$
+ \mu\bar{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{1, a_2}^{(n)})_{i,j}| (\Psi_{j-1}(\bar{a}_1 * \bar{a}_1) + \Psi_{j+1}(\bar{a}_1 * \bar{a}_1)) \omega_i r,
$$

$$
\|A_{2, a_2} z_2\|_{\nu} \leq \frac{1}{\omega_{n+1}} \left( \|A_{2, a_2}^{(n)}\|_{0, \nu} + (\mu + 1)\bar{L}\|A_{2, a_2}^{(n)}\|_{n, \nu}\right) r
$$

$$
+ 2\mu\bar{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{2, a_2}^{(n)})_{i,j}| (\Psi_{j-1}(\bar{a}_1 * \bar{a}_2) + \Psi_{j+1}(\bar{a}_1 * \bar{a}_2)) \omega_i r
$$

$$
+ \mu\bar{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{2, a_2}^{(n)})_{i,j}| (\Psi_{j-1}(\bar{a}_1 * \bar{a}_1) + \Psi_{j+1}(\bar{a}_1 * \bar{a}_1)) \omega_i r
$$

$$
+ \frac{1}{2(n+1)} \left( (\mu\bar{a}_2 - \bar{a}_1)^{n} \omega_{n+1} + L(\mu + 1)\frac{1}{\nu} + \nu \right) r
$$

$$
+ \frac{1}{2(n+1)} \frac{1}{\nu} + \nu \left( (\bar{a}_1)^{\nu} [\bar{a}_2]^\nu + \bar{L}([\bar{a}_2]^\nu + |\bar{a}_1|^\nu)\right) r.
$$
Now we let

\[ \|A_{3, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{1, L}^n\|_\nu r, \]

\[ \|A_{2, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{2, L}^n\|_\nu r, \]

\[ \|A_{3, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{3, L}^n\|_\nu r. \]

Since \( |z_3| \leq \frac{2r}{\omega_{n+1}} \),

\[ \|A_{1, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{1, L}^n\|_\nu r, \]

\[ \|A_{2, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{2, L}^n\|_\nu r, \]

\[ \|A_{3, L} z_3\|_\nu \leq \frac{2}{\omega_{n+1}} \|A_{3, L}^n\|_\nu r. \]

Now we let

\[ Z_1^{(1)} \overset{\text{def}}{=} \frac{\sum_{k=1}^{n} \|A_{1, a_k}\|_n + \tilde{L} (\|A_{1, a_1}\|_n + (\mu + 1) \|A_{1, a_2}\|_n) + 2 |A_{1, L}^n|_\nu}{\omega_{n+1}} \]

\[ + \frac{1}{2(n + 1)} \left( \tilde{L} (\frac{1}{\nu} + \nu) + \|A_{2, a_2}\|_n \right) \]

\[ + 2 \mu \tilde{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{1, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_1 \ast \tilde{a}_2) + \Psi_{j+1}(\tilde{a}_1 \ast \tilde{a}_2)) \omega_i \]

\[ + \mu \tilde{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{2, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_2 \ast \tilde{a}_1) + \Psi_{j+1}(\tilde{a}_2 \ast \tilde{a}_1)) \omega_i \]

\[ + \frac{1}{2(n + 1)} (\|a\ast \tilde{a}_2 - \tilde{a}_1\|_n + \tilde{L}(\mu + 1) \left( \frac{1}{\nu} + \nu \right) \]

\[ + \frac{\mu}{2(n + 1)} \left( \frac{1}{\nu} + \nu \right) (\|\tilde{a}_2 \ast \tilde{a}_1 \ast \tilde{a}_2\|_\nu + \tilde{L}(2\tilde{a}_1 \ast \tilde{a}_2)_\nu + \|\tilde{a}_1 \ast \tilde{a}_1\|_\nu). \]

\[ Z_2^{(1)} \overset{\text{def}}{=} \frac{\sum_{k=1}^{n} \|A_{2, a_k}\|_n + \tilde{L} (\|A_{2, a_1}\|_n + (\mu + 1) \|A_{2, a_2}\|_n) + 2 |A_{2, L}^n|_\nu}{\omega_{n+1}} \]

\[ + 2 \mu \tilde{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{2, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_1 \ast \tilde{a}_2) + \Psi_{j+1}(\tilde{a}_1 \ast \tilde{a}_2)) \omega_i \]

\[ + \mu \tilde{L} \sum_{i=0}^{n} \sum_{j=1}^{n} |(A_{3, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_2 \ast \tilde{a}_1) + \Psi_{j+1}(\tilde{a}_2 \ast \tilde{a}_1)) \omega_i \]

\[ + \frac{1}{2(n + 1)} \sum_{k=1}^{n} (A_{3, a_2})_{k, 0} + \tilde{L} (\|A_{3, a_1}\|_n + (\mu + 1) \|A_{3, a_2}\|_n) + 2 |A_{3, L}^n|_\nu \]

\[ + 2 \mu \tilde{L} \sum_{i=0}^{n} |(A_{3, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_1 \ast \tilde{a}_2) + \Psi_{j+1}(\tilde{a}_1 \ast \tilde{a}_2)) \omega_i \]

\[ + \mu \tilde{L} \sum_{i=0}^{n} |(A_{3, a_2})_{i, j}| (\Psi_{j-1}(\tilde{a}_2 \ast \tilde{a}_1) + \Psi_{j+1}(\tilde{a}_2 \ast \tilde{a}_1)) \omega_i. \]

B.4.3 \( Z^{(2)} \) bound

Now let \( \zeta_j = [(DF(\bar{x} + b) - DF(\bar{x}))d]_j. \) Then, \( \langle \zeta_i \rangle_0 = 0 \) for \( i = 1, 2, \) and \( \zeta_3 = 0. \)

For \( k \geq 1, \)

\[ \zeta_1 = d_3 \cdot T b_2 + b_3 \cdot T d_2, \]

\[ \zeta_2 = [(\tilde{L} + b_3) \cdot T D c_2 (\tilde{a} + (b_1, b_2)) - \tilde{L} \cdot T D c_2 (\tilde{a})] (d_1, d_2) \]

\[ + d_3 \cdot T [c_2 (\tilde{a} + (b_1, b_2)) - c_2 (\tilde{a})]. \]
Since 
\[ \|T\|_{B(t^*_1,t^*_2)} \leq \frac{1}{\nu} + \nu, \]
then
\[ \|\zeta_1\|_{\nu} \leq 2\left(\frac{1}{\nu} + \nu\right) \]
\[ \|\zeta_2\|_{\nu} \leq \left(\frac{1}{\nu} + \nu\right) \left((\mu + 1) + 2\mu(\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu}) + \mu(\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu} + 2\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu})\right)^2 \]
\[ + \mu \bar{L} \left(2\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu} + 2\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu} + 2\mu(\|\bar{a}_1\|_{\nu} + \|\bar{a}_2\|_{\nu} + 3\|\bar{a}_2\|_{\nu} + 4r^2) \right)^2 \]
\[ + \mu \bar{L} \left(4\|\bar{a}_1\|_{\nu} + 2\|\bar{a}_2\|_{\nu} + 3r\right)^2. \]

Lemma B.2. For \( i, j = 1, 2, \)
\[ \|A_{i,a_j}\|_{B(t^*_1,t^*_2)}^* \leq \max \left\{ \|A_{i,a_j}^{(n)}\|^*, \frac{1}{2(n+1)} \delta_{i,j}\right\}, \]
where \( \|A_{i,a_j}^{(n)}\|^* \overset{\text{def}}{=} \max_{0 \leq m \leq n} \frac{1}{\omega_m} \sum_{0 \leq k \leq n} |(A_{i,a_j}^{(n)})_{k,m}| \omega_k. \)

Therefore we have
\[ \|(A\zeta)_i\|_{\nu} \leq \sum_{j=1}^{2} \|A_{i,a_j}\|_{B(t^*_1,t^*_2)}^* \|\zeta_j\|_{\nu}, \quad |(A\zeta)_3| \leq \sum_{j=1}^{2} \|A_{i,a_j}\|_{B(t^*_1,t^*_2)}^* \|\zeta_j\|_{\nu}. \]

Define
\[ \Xi \overset{\text{def}}{=} \mu \left(4\|\bar{a}_1 \ast \bar{a}_2\|_{\nu} + 2\|\bar{a}_1 \ast \bar{a}_1\|_{\nu} + 6\|\bar{a}_1\|_{\nu} r + 3\|\bar{a}_2\|_{\nu} r + 4r^2\right) \]
\[ + \mu \bar{L} \left(4\|\bar{a}_1\|_{\nu} + 2\|\bar{a}_2\|_{\nu} + 3r\right)^2. \]

Then
\[ Z_1^{(2)} = \left(\frac{1}{\nu} + \nu\right) \|A_{1,a_2}^{(n)}\|^*_\Xi + 2\left(\frac{1}{\nu} + \nu\right) \max \left\{ \|A_{1,a_1}^{(n)}\|^*, \frac{1}{2(n+1)} \right\}, \]
\[ Z_2^{(2)} = \left(\frac{1}{\nu} + \nu\right) \max \left\{ \|A_{2,a_2}^{(n)}\|^*, \frac{1}{2(n+1)} \right\} \Xi + 2\left(\frac{1}{\nu} + \nu\right) \|A_{2,a_1}^{(n)}\|^*, \]
\[ Z_3^{(2)} = \left(\frac{1}{\nu} + \nu\right) \|A_{3,a_2}^{(n)}\|^*_\Xi + 2\left(\frac{1}{\nu} + \nu\right) \|A_{3,a_1}^{(n)}\|^*. \]

C Expansion of the polynomials in monomials

The polynomials in Section 3.1 are used in the optimization step of the Algorithm 3.1. However, during the optimization it is better to expand them in monomials to provide sharper bounds.

The polynomials (11), (12), and (13) are already expanded in monomials of the optimization variables. On the other hand, The other polynomials (14), (15), and (16) have different monomial expansions depending on the optimization variables.
C.1 Monomial expansion for fixed perturbations

In this case, the variables to be optimize are \( \varepsilon, a, \beta_0, \beta_1, \) and \( \beta_2. \) Then

\[
P_2(\varepsilon, a, \beta_0, \beta_1, \beta_2) = \frac{1}{\omega_0} \varepsilon \alpha_2^2 \| DP \| \| Dr \| + \frac{2}{\omega_0} \| D K_0 \| \| DP \| \| Dr \| \varepsilon \alpha_1 + \frac{1}{\omega_0} \varepsilon \beta_0^2 \| DP \| \| Dr \| + \frac{1}{\omega_0} \varepsilon \beta_0 \| DP \| \| Dr \| \varepsilon \beta_0^2 + \frac{1}{\omega_0} \varepsilon \beta_0 \| Dr \| + \frac{1}{\omega_0} \varepsilon \beta_1 \| Dr \| \varepsilon \beta_0^2.
\]

Let us define

\[
\mu_1(\varepsilon, a, \beta_0, \beta_1, \beta_2) = \frac{\| \Pi_0^1 \| C_{1,1}}{\| D K_0(\theta_0) \|} \left( \frac{\| DP \| \| Dr \| \varepsilon \alpha_1}{\omega_0} + \frac{\| D K_0 \| \| DP \| \| Dr \| \varepsilon \beta_0}{\omega_0} + \frac{\| D f \circ K_0 \| \| Dr \| \varepsilon \beta_0}{\omega_0} \right).
\]

\[
\gamma \overset{\text{def}}{=} \frac{C_{1,2} C_{1,1}}{\| D K_0(\theta_0) \|} \frac{\Pi_0^1}{\| D K_0(\theta_0) \|} + \frac{C_{1,2}}{\omega_0} + C_{1,1} C_{1,3} M
\]

\[
\delta \overset{\text{def}}{=} \frac{C_{1,2} C_{1,1}}{\| D K_0(\theta_0) \|} \frac{\Pi_0^1}{\| D K_0(\theta_0) \|} + C_{1,3} C_{2,1} M + \frac{C_{2,2}}{\omega_0}
\]

then

\[
\mu_2(\varepsilon, a, \beta_0, \beta_1, \beta_2) = \frac{\| DP \| \| Dr \| \varepsilon \alpha_1}{\omega_0} + \frac{\| D K_0 \| \| DP \| \| Dr \| \varepsilon \beta_0^2}{\omega_0} + \frac{\| D f \circ K_0 \| \| Dr \| \varepsilon \beta_0^2}{\omega_0} + \frac{\| D f \| \| Dr \| \varepsilon \beta_0^2}{\omega_0}.
\]

C.2 Monomial expansion for class of perturbations

In this case, the variables are \( \varepsilon, a, \beta_0, \beta_1, \beta_2, \| DP \|, \| Dr \| \) and \( \| r \| \). Using (33) for (16),

\[
P_2(\varepsilon, a, \beta_0, \beta_1, \beta_2, \| DP \|, \| Dr \|) = \frac{1}{\omega_0} \varepsilon \alpha_2^2 \| DP \| \| Dr \| + \frac{2}{\omega_0} \| D K_0 \| \| DP \| \| Dr \| \varepsilon \alpha_1 + \frac{1}{\omega_0} \varepsilon \beta_0^2 \| DP \| \| Dr \| + \frac{1}{\omega_0} \varepsilon \beta_0 \| DP \| \| Dr \| \varepsilon \beta_0^2 + \frac{1}{\omega_0} \varepsilon \beta_1 \| DP \| \| Dr \| \varepsilon \beta_0^2 + \frac{1}{\omega_0} \varepsilon \beta_1 \| DP \| \| Dr \| \varepsilon \beta_0^2 + \frac{1}{\omega_0} \varepsilon \beta_1 \| DP \| \| Dr \| \varepsilon \beta_0^2.
\]
\[
\begin{align*}
\mu_1(\varepsilon, a, \beta_0, \beta_1, \beta_2, \|DP\|, \|Dr\|, \|r\|) &= \frac{\|\Pi_{\beta_0}^\perp C_{1,1}\|}{|DK_0(\theta_0)|} \left( \varepsilon a \beta_1 \|DP\| \|Dr\| + \|DK_0\| \varepsilon a \|DP\| \|Dr\| + \varepsilon \beta_1 \|DP\| \|r\| + \|DK_0\| \varepsilon a \|DP\| \|r\| + \varepsilon \|DP\| \right) + \\
&\quad + \frac{\|\Pi_{\beta_0}^\perp C_{2,1}\|}{|DK_0(\theta_0)|} \alpha + \frac{\|\Pi_{\beta_0}^\perp (C_{1,1} \|D^2f\| + C_{2,1})\|}{|DK_0(\theta_0)|} \beta_0,
\end{align*}
\]

and
\[
\mu_2(\varepsilon, a, \beta_0, \beta_1, \beta_2, \|DP\|, \|Dr\|, \|r\|) = \gamma \varepsilon a \beta_2 \|DP\| \|Dr\| + \\
\frac{\|DK_0\| \gamma \varepsilon a \|DP\| \|Dr\| + \omega_0 \gamma \varepsilon \beta_2 \|DP\| \|Dr\| + \gamma \varepsilon \beta_2 \|DP\| \|r\| + \|DK_0\| \omega_0 \gamma \varepsilon \|DP\| \|Dr\| + \gamma \varepsilon \|DP\| \|r\| + \gamma \varepsilon \|DP\| + \delta a + (\|D^2f\| \gamma + \delta) \beta_0.
\]

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