Relativistic Temperature Transformation Revisited, One hundred years after Relativity Theory

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An attempt has been made to find a consistent and logical form for relativistic temperature transformation. Other works in this area have been discussed. Our approach is based on the kinetic theory of ideal gases.

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Introduction

According to Galilean Transformation (GT), the speed of light is different in two inertial frames, moving with constant velocity with respect to each other. In the late 18th and early 19th century, there was considerable evidence supporting the idea that the speed of light is an invariant quantity with respect to all inertial observers. In 1905, Einstein\textsuperscript{1} made the ground breaking discovery that when transforming physical quantities from one inertial observer’s frame to another, the Lorentz transformations must be used so that the speed of light remains an invariant quantity. The principle of relativity modified the Newton’s laws of motion and in a sense unified concepts like space and time. Since then, much effort has been devoted to the search for the relativistic forms of other laws of physics. Many authors have tried to modify other fields, such as thermodynamics, to render it compatible with the relativity theory.

Generally, quantities are transformed in any field of physics by presuming invariance of some basic quantities or covariance of the form of laws in that discipline under Lorentz transformation. For example electric and magnetic fields are transformed by assuming covariance of Maxwell equations. In our efforts to find the relativistic temperature transformation, we have conducted a fairly ex-
tensive survey of the existent literature on the subject. This included books such as "Feynman Lectures on Physics" [2] as well as numerous other papers published in this area since the birth of relativity by Planck, Einstein, Tolman, Ott and many others. (See the reference section for citations).

Generally, it can be seen that the results in the current literature, has been obtained by postulating the covariance of the first and/or second law(s) of thermodynamics under Lorentz transformation. It is fair to say that at this moment and to the best of our knowledge, lack of distinct experimental evidence, has prevented a consensus to be reached among experts on the subject. As we have seen the principle of the invariance of the speed of light has modified the laws of physics. In the light of this principle we should take a more careful approach when we are about to make fundamental assumptions. To avoid ambiguity, the authors of this paper have chosen a method which involves using equations and concepts that are verified by experiment and theory in the relativistic domain, such as relativistic form of energy. We have sought the clearest and simplest set of assumption.

In what follows, we begin with a rather short historical review (for complete reviews see the references section), and then we try to derive our transformation from a new point of view.

Our notation is as follows: symbols $T$, $P$, $Q$, $S$, $U$ and $W$ are used for temperature, pressure, heat, entropy, internal energy and work, respectively.

### I. HISTORICAL REVIEW

Einstein and Planck [3, 4] were the first who proposed a transformation for temperature. They applied principle of the least action to a moving black-body cavity and assumed the first and the second laws of thermodynamics [11], [2] to be covariant:

\[ dU = dQ + dW \]  

\[ dQ = T dS, \]  

and they argued for the following set of transformations:

\[ T = T'/\gamma, \quad Q = Q'/\gamma \]  

\[ S = S', \quad P = P', \]
where primed quantities denote, the ones measured in the proper frame of reference that is moving with constant velocity $\vec{v}$ with respect to a stationary reference frame and $\gamma$ is:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$  \hspace{1cm} (5)

Several years later, a new transformation of temperature was derived by Eddington [5] and Blanusa [6] based on the covariance of the second law of thermodynamics [2]. In 1963, Ott [7] provided a critical overview of the subject. He made the assertion that the transformations, derived by Eddington and Blanusa [5, 6] are the correct transformations for the thermodynamic quantities. These are:

$$T = \gamma T' \hspace{1cm} Q = \gamma Q' \hspace{1cm} (6)$$

$$S = S' \hspace{1cm} P = P' \hspace{1cm} (7)$$

However, Landsberg [8, 9] argued that the thermodynamic quantities that are statistical in nature, namely $T, S, U$ should not be expected to change for an observer who judges the center of mass to be undergoing a uniform motion. This approach, leads to the conclusion that some thermodynamic relationships such as the second law [2] are not covariant and results in the following equations:

$$\gamma Q = T dS \hspace{1cm} (8)$$

$$Q = Q'/\gamma \hspace{1cm} (9)$$

$$P = P' \hspace{1cm} (10)$$

The above discussions can be summarized as the following:

$$T = \gamma^a T' \hspace{1cm} (11)$$

where $a = -1, 0, 1$ comes from Planck/Einstein, Landsberg and Ott views, respectively.

The mentioned proposed transformations for temperature have been criticized by many authors. Here we briefly review some of the critique.
In 1967, Moller [14] some years after publishing his book on theory of relativity, criticized the issue in a rather detailed paper and called Ott’s transformation as the correct formula, and reminisced accepting of Einstein derivation wrongly as correct formula for half a century as strange and unique incident in history of physics.

Landsberg pointed out that Ott’s view as expressed in [14] requires force to be redefined, a requirement which is difficult for many to accept [15].

Shizhi stated that the different formulations are describing the same physical quantities and are therefore compatible. He believes that the difference between Einstein’s and Planck’s opinions from one hand and Ott’s from the other, depends on the selection of the frame of reference [16].

The kinetic theory of ideal gases has been the basis of some additional work on the subject in which other relationships have been proposed [10, 11, 12, 13].

II. DERIVATION

The authors of this paper take the appropriate definition of the relativistic temperature to be one that is based on the kinetic theory of ideal gases. Before deriving our transformation for temperature let us try to calculate the relativistic equipartition theorem. According to Boltzmann probability distribution for a classical (classic vs. quantum) ideal gas in a thermal bath, average energy is related to reservoir temperature, $T$, as below:

$$\langle E \rangle = \bar{E} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E e^{-\beta E} \frac{d^3 \vec{p}}{h^3}}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta E} \frac{d^3 \vec{p}}{h^3}},$$

where $h$ is a normalization constant with dimensions of action, $\beta = 1/(kT)$ and $k$ is the Boltzmann constant. In most thermodynamics textbooks the above relationship is used to calculate $\langle E \rangle$. By using the non-relativistic expression for kinetic energy, $p^2/2m$, it is shown that $\langle E \rangle = \frac{3}{2} kT$, where the upper limit for the integral over momentum goes to infinity which requires us to use the relativistic form of energy.

Here, we use the relativistic form of energy-momentum relationship [13] and (A1), (A2) and (A3) from Appendix A and invoke the change of variable [14] to evaluate $\langle E \rangle$ as below:

$$E^2 = (pc)^2 + (mc^2)^2$$

$$p = mc \sinh(\chi)$$
\[ \langle E \rangle = mc^2 \left( \frac{K_1(u)}{K_2(u)} + \frac{3}{u} \right), \]  

(15)

where \( u = \beta mc^2 \) and \( K's \) are modified Bessel functions.

Now, as a consistency check it is useful to obtain classical \((u \gg 1)\) limit of the above relation. From (15) and (A4) in appendix we have:

\[ K_1(u) \simeq \left( \frac{\pi}{2u} \right)^{1/2} e^{-u} \left[ 1 + \frac{3}{8u} \right] \]  

(16)

\[ K_2(u) \simeq \left( \frac{\pi}{2u} \right)^{1/2} e^{-u} \left[ 1 + \frac{15}{8u} \right], \]  

(17)

by using (15), (16) and (17):  

\[ \langle E \rangle - mc^2 = \langle E_k \rangle = \frac{3}{2} kT, \]  

(18)

\( E_k \) denotes kinetic energy.

Now, to obtain the relativistic transformation of temperature, an important question is: What should be kept invariant under Lorentz transformation?! We presume average energy in any frame of reference to be related to its temperature by Boltzmann probability distribution. For clarifying the issue let us explain our purpose by a useful example. Let the thermodynamic system be, say, a bottle of helium gas. Strap the bottle onto the seat of a rocket. As measured in the rocket frame, the temperature of the helium inside the bottle is \( T' \). Now, Let the Rocket be moving with velocity \( \vec{v} \) relative to the ground-based lab frame. What is the helium temperature \( T \) as measured in the lab frame compare to \( T' \)? From 4-vector momentum transformation for each particle in the bottle we have:

\[ E = \gamma (E' + \vec{v} \cdot \vec{p}'), \]  

(19)

and for the whole bottle of helium gas, we have: (index \( b \) refers to the bottle)

\[ \langle E \rangle_b = \gamma \left( \langle E' \rangle_b + \vec{v} \cdot \langle \vec{p} \rangle \right), \]  

(20)

where \( \langle \vec{p} \rangle = 0 \) in the rocket frame.

Now, substitute average energies by their related temperatures:

\[ \frac{K_1(u)}{K_2(u)} + \frac{3}{u} = \gamma \left( \frac{K_1(u')}{K_2(u')} + \frac{3}{u'} \right), \]  

(21)
where $u = \frac{mc^2}{kT}$ and $u' = \frac{mc^2}{kT'}$. The relation $u = mc^2/kT$ as can be seen in the FIG. 11, is a continuous and monotonically decreasing function. The fact that it is a single valued function, implies, in eq. (21), existence of one and just one $T$ corresponding to $T'$ where $T \geq T'$ because $\gamma \geq 1$.

In both classical ($u \gg 1$) and relativistic ($u \ll 1$) order above equation, (21), reduces to Ott transformation (6). Asymptotic aspects of the above relation are:

\[
\gamma \rightarrow 1 \Rightarrow T = T' 
\]

\[
\gamma \rightarrow \infty \Rightarrow T = \infty.
\]

III. CONCLUSION

The transformation derived here using the kinetic theory, applies to an ideal gas. This result is true for all potentials which depend only on position (see eq. (12)), and predicts that moving objects appear hotter to stationary observers. This is in agreement with Ott’s view.

At the present time no temperature transformation has been agreed upon. To reach consensus, it seems necessary that firm experimental evidence is obtained.

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APPENDIX A: MODIFIED BESSEL FUNCTIONS

In this appendix we give some basic mathematical formulae related to modified Bessel functions $K_n(u)$. The expressions are taken from [17]:
\[
K_n(u) = \frac{\sqrt{\pi}}{\Gamma(n + \frac{1}{2})} \left(\frac{u}{2}\right)^n \int_0^\infty e^{-u \cosh(\chi)} \sinh^{2n}(\chi) \, d\chi
\]  
(A1)

\[uK'_n(u) - nK_n(u) = -uK_{n+1}(u)\]  
(A2)

\[uK'_n(u) + nK_n(u) = uK_{n-1}(u)\]  
(A3)

\[
K_n(u) = \left(\frac{\pi}{2u}\right)^{1/2} e^{-u} \left[ 1 + \frac{4n^2 - 1}{1!8u} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8u)^2} + \ldots \right]
\]  
(A4)

[1] Einstein A., *Ann. Phys. (Liepzig)*, 17, (1905).
[2] *The Feynman lectures on physics*. R. P. Feynman, Robert B. Leighton, Matthew L. LinkSands, Addison-Wesley, (1989).
[3] Einstein A., *Jahrb. Radioakt. Elektron.* 4 (1907) 411.
[4] Planck M., *Ann. Phys. (Liepzig)*, 26 (1908) 1.
[5] Eddington A. S., *The Mathematical Theory of Relativity*, Cambridge, Univ. Press (1923) p.34;
[6] Blanusa D., Kolokvij 17.12.1947, *Proceedings: Prvi kongres mat. i fiz. FNRJ*, Bled (8-12 November 1949)
[7] Ott H., *Z.Phys.*, 175 (1963) 70.
[8] Landsberg P. T., In a critical review of thermodynamics, edited by Stuart E. B., Brainard A.J. and GAL-OR B. (*Mono Books, Baltimore*) 1970,p.253.
[9] Landsberg P. T., *Nature*, 213 (1966)571; 214 (1967) 903.
[10] Tolman R. C., *relativity, thermodynamics and cosmology*, (Claredon Press, Oxford, 1934)
[11] Shaffer J., *Nuovo Cimento* B, 103 (1989) 259.
[12] Bevelacqua J. J., *Phys. Essays*, 2 (1989) 230.
[13] Trout K. P. and Greiner J., *Nuovo Cimento* Vol.113 B, N.
[14] Moller C., *Dan. Vid. Selsk. Math-Phys. Meddel. 36* (1967) 1
[15] Landsberg P. T.,*Eur. J. Phys.*, 2 (1981) 203.
[16] Cui Shizhi, *ACTA Scientiarum Naturalism Universities Sunyateseni*,no. 3 (1987) 60.
[17] Arfken G., *Mathematical methods for physicist*, 3rd ed., p.614-618.
FIG. 1: Average energy vs $u = \frac{mc^2}{kT}$