SHELLING-TYPE ORDERINGS OF REGULAR CW-COMPLEXES 
AND ACYCLIC MATCHINGS OF THE SALVETTI COMPLEX

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Abstract. Motivated by the work of Salvetti and Settepanella ([23, Remark 4.5]) we introduce certain total orderings of the faces of any shellable regular CW-complex (called shelling-type orderings) that can be used to explicitly construct maximum acyclic matchings of the poset of cells of the given complex. Building on an application of this method to the classical zonotope shellings (i.e., those arising from linear extensions of the tope poset) we describe a class of maximum acyclic matchings for the Salvetti complex of a linear complexified arrangement. To do this, we introduce and study a new purely combinatorial stratification of the Salvetti complex. For the obtained acyclic matchings we give an explicit description of the critical cells that depends only on the chosen linear extension of the poset of regions. It is always possible to choose the linear extension so that the critical cells can be explicitly constructed from the chambers of the arrangement via the bijection to no-broken-circuit sets defined by Jewell and Orlik [17]. Our method generalizes naturally to abstract oriented matroids.

1. Introduction

The idea of shelling was initially introduced by Bruggesser and Mani [9] as a (geometrically defined) technique to deconstruct polytopes in a ‘controlled way’ allowing an accurate bookkeeping of certain combinatorial data. The required total ordering of the polytope’s facets was obtained from the order in which a general position line in meets the affine hulls of the facets. Much work has been spent on a purely combinatorial characterization of this process, and on a corresponding generalization of the method beyond polytopes. In fact, shellability can be defined for general (possibly nonpure) regular cell complexes [7, 8]. A line of research initiated by Björner [4] studies combinatorial properties of posets that ensure shellability of the associated order complexes. A considerable amount of work was dedicated to this subject (see e.g. [4, 3, 7, 8]). Particular attention was dedicated to the posets of cells of regular CW-complexes: Björner characterized them combinatorially (see [3, Definition 2.1 and Proposition 3.1]), and proved that shelling orders of the facets of the associated CW-complex correspond to recursive coatom orderings of the posets ([3, Proposition 4.2], see also [8, Theorem 13.2]).

Recently, Forman [15] proposed a combinatorial version of Morse theory, called Discrete Morse Theory. The idea is that, given any regular CW-complex, one can define a combinatorial analog of the Morse vector fields (i.e., acyclic matchings on the poset of cells; see Definition [24] and [11, Proposition 3.3]) such that the original complex is homotopy equivalent to a complex having as many cells of dimension $d$ as there are ‘critical points’ (i.e., non-matched cells) of rank $d+1$. Moreover, the attaching maps can be reconstructed from the knowledge of the ‘gradient paths’ (i.e., alternating paths in the poset). Since at the topological core of both shellability and discrete Morse theory lies the idea of collapsing cells (along matched edges or

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along the shelling order), it is natural to study the relation between these concepts: this study was undertaken by various authors, e.g. in [1][11][18]. A comprehensive and careful exposition of the nowadays established combinatorial framework of discrete Morse theory can be found in the book of Kozlov [19].

The motivation for our considerations was given by a joint work of Mario Salvetti with Simona Settepanella [24], where discrete Morse theory is used to explicitly obtain a minimal CW-complex that models the homotopy type of the complement of a complexified arrangement of hyperplanes, thus providing a constructive proof of the minimality result for general arrangements that was obtained independently by Randell [22] and Dimca and Papadima [12]. Another recent study of minimal complexes for complexified arrangements is due to Yoshinaga [26]. For the basic definitions about arrangements of hyperplanes we refer to [21].

The starting point of [24] is the Salvetti complex (introduced in [23] as a combinatorial model for the topology of the complement of complexified arrangements), and the main tool used to construct a maximum acyclic matching of its poset of cells is a certain total order on the faces of the arrangement that is called polar ordering by the authors. The name refers to the fact that this total order is obtained by considering polar coordinates with respect to a generic flag and then ordering the faces according to their smallest point in the lexicographical order of the polar coordinates (for the precise definition see [24, Definition 4.4]). It is explicitly asked for a completely combinatorial formulation of this method [24, Remark 4.5].

In an attempt to answer this question, we keep the idea of constructing acyclic matchings by considering the arrangement from a ‘generic’ point of view, but we try to stay in the context of oriented matroids. These are widely studied combinatorial objects that encode the structure of real arrangements of pseudospheres, and in particular of linear hyperplanes (for an introductory reference see [6, Chapter 1]). Thus, we actually loose the generality of [24], where the results hold also for affine arrangements. However, our method has the advantage that it does not need the choice of a generic flag in the ambient space, and that it holds for general abstract oriented matroids.

One of the ways one can think to look ‘generically’ at an oriented matroid is to consider a shelling of its zonotope. This is a polytope that is classically associated to every oriented matroid and that, if the oriented matroid corresponds to a real arrangement, is combinatorially isomorphic to the polyhedral subdivision of the unit sphere given by the hyperplanes (for a precise account of this subject, see [6, Section 2.2 and Chapter 4]). It is well-known that to every linear extension of the tope poset of the oriented matroid corresponds a (class of) shelling(s) of the associated zonotope: in fact, one can construct recursive coatom orderings of the zonotope’s poset of faces.

We first show a way to construct maximum acyclic matchings of (CW-) posets that admit a recursive coatom ordering. We do this using shelling-type orderings: a class of total orderings of the involved poset that are associated to recursive coatom orderings. Then we apply this construction to the special case of a zonotope.

It turns out that linear extensions of tope posets describe also a nice decomposition of the Salvetti complex that, to the best of our knowledge, has not been considered up to now. The above obtained zonotope shellings give acyclic matchings of every ‘piece’ of this decomposition that can be ‘pasted together’ to give an acyclic matching of the poset of cells of the whole Salvetti complex. To every critical cell correspond canonically a (unique) chamber and a flat of the underlying matroid which codimension equals the dimension of the critical cell. Both are uniquely determined by the chosen linear extension of the tope poset. Maximality
of the matching follows from the fact that the critical cells are in bijection with no-broken circuits, and thus with generators of the homology (by e.g. [17, 20]).

This correspondence can be made more precise and explicit: we show that, for an adequate choice of the ordering of the hyperplanes and of the linear extension of the base poset, the bijection between chambers and no-broken-circuits given by Jewell and Orlik in [17] associates to every chamber a basis of the flat that carries the corresponding critical cell.

The paper is organized as follows. After introducing the main characters, in Section 2 we prove that every recursive coatom ordering of a CW-poset induces a shelling-type total ordering of its faces (Lemma 2.10). From this total ordering, in Proposition 1 we construct an acyclic matching of the given poset that turns out to be ‘optimal’ (for a comparison with known related results of Chari [11] and Babson and Hersh [1] see Remark 2.8). Then, Section 3 introduces oriented matroids, explains the construction of the zonotope shelling associated to a linear extension of the tope poset and compares (in Remark 3.8) the associated shelling-type ordering with the polar orderings of [24]: this is our (kind of) answer to [24, Remark 4.5]. In Section 4 we study the stratification of the Salvetti complex induced by a linear extension of the tope poset (in the context of arrangements also called ‘poset of regions’ and first considered in [13]). First, we prove a general property of tope posets (Theorem 4.15) that, given a linear extension, allows to associate a unique flat $X_C$ to every tope $C$. It turns out that the stratum associated to a tope $C$ corresponds naturally to the oriented matroid obtained by contraction of the flat $X_C$. On the one hand, this allows to construct acyclic matchings for every stratum and to verify acyclicity and maximality of the ‘patchworked’ matching (Proposition 2). On the other hand, in Section 5 we show that for some orderings of the hyperplanes (Definition 5.4) there is a linear extension of the tope poset (Definition 5.12) for which the flat $X_C$ is spanned by the no-broken circuit set that corresponds to $C$ under the bijection described in [17] (Proposition 3).

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2. Shellings and acyclic matchings

2.1. On partially ordered sets. In this work we will deal extensively with partially ordered structures. We outline the basic definitions, pointing to [25, Chapter 3] for a comprehensive reference. A poset is a set (say $P$) endowed with a partial order relation (say $<$), and will be written as a pair $(P, <)$ or, if no misunderstanding about the partial order will be possible, just denoted by $P$. Moreover, the posets we will consider will be locally finite, meaning that for each $p \in P$ there are only finitely many $q$ with $p < q$ or $p > q$. An element $p \in P$ is said to cover $q$ in $P$ whenever $p > q$ and there is no $x \in P$ with $p > x > q$. If $p$ covers $q$ with respect to the order relation $>$ (or $\succ$, $\succ$, ...), then we will write $p \succ q$ (respectively $\succ$, $\succ$).

The set of all elements of $P$ that are covered by $p$ will be called, by slight abuse of notations, the set of coatoms of $p$, and denoted by coat($p$). In fact, for every $q \in P$, the set coat($q$) is the set of coatoms of the poset $P_{\leq q} := \{p \in P \mid p \leq q\}$. This poset is called the principal lower ideal generated by $q$, a lower ideal being in general any subposet of $P$ that can be written as an intersection of principal lower ideals; (principal) upper ideals are defined accordingly. Any subset of the form $P_{\leq q} \cap P_{\geq p}$ is called an interval of $P$. We will write $P_{< q} := P_{\leq q} \setminus \{q\}$. A totally ordered subset
A poset \( P \) is said to be a lattice if every two \( p, q \in P \) have a unique least upper bound (called join and denoted \( p \lor q \)) and a unique greatest lower bound (called meet and denoted by \( p \land q \)).

Remark 2.1. An upper (lower) ideal in a lattice is principal if and only if it is closed under meet (join).

Sometimes we will have to consider different order relations on the same set. If needed, the concerned order relation will be specified in a subscript. Thus, for example, \( \max_{\succ} A \) denotes the maximal element of \( A \) with respect to the order \( \succ \).

A linear extension of a partial order \(<\) is a total order \(\preceq\) such that \( p \prec q \) whenever \( p < q \).

A poset \( P \) is called bounded if it possesses a maximal and a minimal element. Let \( P \) denote the poset \( P \) with a maximal and a minimal element added, if \( P \) has none. The maximal and minimal elements of \( P \) are customarily denoted by \( \hat{1} \) and, respectively, \( \hat{0} \). In a poset with \( \hat{0} \) a principal lower ideal is also called a lower interval.

Given a (possibly nonpure) CW-complex \( K \), we define its poset of faces \( \mathcal{F}(K) \) as the set of (closed) cells of \( K \) ordered by containment, with a minimal element \( \hat{0} \) added (the ‘\(-1\)-dimensional cell’). Note that, for every cell \( k \), every maximal chain in \( \mathcal{F}(K)_{\leq k} \) has the same length. The height of \( k \) is \( h(k) = \ell(\mathcal{F}(K)_{\leq k}) \), the length of the corresponding lower interval. Geometrically, we have \( \dim(k) = h(k) + 1 \) for every cell \( k \).

Figure 2.1. The regular CW-complex \( K_1 \) given by a filled hexagon, and its poset of faces \( \hat{\mathcal{F}}(K_1) \).

2.2. Shellability and Recursive Coatom Orderings. A regular CW complex \( K \) is said to be shellable if its maximal cells can be given an order along which the complex can be ‘rebuilt’ in a very controlled way. For the precise definition we refer to [8, Definition 13.1], where shellability was first extended from simplicial complexes to regular CW-complexes. The complexes that we will consider are given by means of their poset of cells. Therefore we take a point of view that is more tailored to our context: we will define recursive coatom orderings of posets, and then see how they correspond to shellings of regular CW-complexes.

Definition 2.2 (Definition 5.10 of [7]). A bounded poset \( (P, \prec) \) is said to admit a recursive coatom ordering \( \preceq \) if \( \ell(P) = 1 \), or if \( \ell(P) > 1 \) and there is a total ordering \( \preceq = \preceq_1 \) on the set \( \text{coat}(\hat{1}) \) of coatoms of \( P \) such that
(1) for all \( p \in \text{coat}(\hat{1}) \), the interval \([\hat{0}, p]\) admits a recursive coatom ordering \( \prec_p \) in which the coatoms of the intervals \([\hat{0}, q]\) for \( q \prec \hat{1} \) come first.

(2) for all \( p \prec \hat{1} q \), if \( p, q > y \), then there is \( p' \prec \hat{1} q \) and \( z \in \text{coat}(q) \) such that \( p' > z \geq y \).

This definition is one of the criteria introduced by Björner to check shellability of the order complex of a poset. It turned out that, in the context of regular CW-complexes, this property is equivalent to shellability. We state these facts in the next theorem.

**Theorem 2.3** (See [4], [8]). If a poset \( P \) admits a recursive coatom ordering, then \( \Delta(P) \) is shellable. If \( P \) is the poset of faces of a regular CW-complex \( K \), then a total ordering of the maximal faces of \( K \) is a shelling order if and only if it is a recursive coatom ordering of \( \hat{P} \).

![Figure 2.2. A shelling order of the maximal faces of the boundary complex \( K_2 \) of a hexagon, and a corresponding Recursive Coatom Ordering of the poset \( \hat{\mathcal{F}}(K_2) \). The arrows give the R.C.O. of the corresponding lower intervals. Below the VI face, the ordering does not matter.](image)

### 2.3. Matchings and Discrete Morse Theory

We introduce here some basic concepts of Discrete Morse Theory, omitting their proofs. The interested reader will find reference to the publications where the statements first appeared. For a comprehensive exposition of the subject in its entirety we refer to the book of Kozlov [19].

**Definition 2.4** (Compare Proposition 3.3 of [11]). Let \((P, <)\) be any poset. The set of edges of \( P \) is \( \mathcal{E} := \{(p, q) \in P \times P \mid p > q\} \). A subset \( \mathcal{M} \subset \mathcal{E} \) is called a matching of \( P \) if every element of \( P \) appears in at most one matched pair, i.e., a pair \( (p, q) \in \mathcal{M} \). A cycle in a matching \( \mathcal{M} \) is a subset \( \{(p_1, q_1), \ldots, (p_k, q_k)\} \subset \mathcal{M} \) such that

\[
q_1 \prec p_2, \quad q_2 \prec p_3, \ldots, \quad q_k \prec p_1.
\]

The matching \( \mathcal{M} \) is called acyclic if it contains no cycle.

Much of the terminology is borrowed from the theory of graphs, the idea being that \( \mathcal{M} \) is actually a matching of the Hasse diagram of \( P \), i.e., the graph defined by the set of edges \( \mathcal{E} \) on the vertex set \( P \) (informally speaking, this is the graph one usually draws when graphically representing a poset). A matching \( \mathcal{M} \) will be called maximal if there is no matching \( \mathcal{M}' \supseteq \mathcal{M} \). If, in addition, \( \mathcal{M} \) has maximal cardinality among all matchings of \( P \), then it is called a maximum matching. A perfect matching is a matching such that every element of \( P \) is contained in a
matched pair. In general, \( p \in P \) is called **critical** for \( \mathcal{M} \) if it is not contained in any matched pair.

The following result is very useful in dealing with acyclic matchings.

**Lemma 2.5** (Theorem 11.2 of [19]). A matching \( \mathcal{M} \) of a poset \( P \) is acyclic if and only if there is a linear extension \( \prec \) of \( P \) such that \( p \prec q \) whenever \( (p, q) \in \mathcal{M} \).

From a topological point of view, the interest of acyclic matchings of posets is explained in the following (weak) version of the main theorem of Discrete Morse Theory.

**Theorem 2.6** (Theorem 11.13 of [19]. See also [11, 15]). Let \( K \) be a regular CW-complex \( K \) and \( \mathcal{M} \) an acyclic matching of \( \mathcal{F}(K) \setminus \{0\} \). Let \( c_i \) denote the number of critical elements of rank \( i \). Then \( K \) is homotopy equivalent to a CW-complex that has \( c_i \) cells in dimension \( i \).

![Figure 2.3.](image)

**Remark 2.7.** If we consider the whole \( \mathcal{F}(K) \) we can say that if a perfect acyclic matching of \( \mathcal{F}(K) \) exists then \( K \) is contractible. Moreover, if there is an acyclic matching of \( \mathcal{F}(K) \) that has critical elements only in one rank level, say the \( i \)-th, then \( K \) is homotopy equivalent to a wedge of \( i \)-spheres.

### 2.4. From recursive coatom orderings to acyclic maximum matchings.

We now describe a construction of certain acyclic maximum matchings of the poset of cells of every shellable regular CW-complex. The core of the argument is Lemma 2.10 where a convenient linear ordering of all cells is produced.

**Remark 2.8.** It has to be pointed out that our approach via recursive coatom orderings differs from those taken in [9] and [11]. Babson and Hersh [9] consider a certain kind of shellability (i.e. lexicographic) of a particular class of simplicial complexes (order complexes of posets) and, in this case, they construct Morse functions “with a relatively small number of critical cells” ([9, Introduction]). Our argument works for any shelling order of any regular CW-complex \( K \) and gives always a ‘best possible’ matching. In this sense, when \( K \) is the order complex of
a poset, and the the shelling order is the lexicographic one, our result improves [1] Theorem 2.2. After the first version of this paper, we learned that also Chari [11] proved the existence of ‘best possible’ matchings for regular CW-complexes with a generalized shelling (for the precise meaning and the definitions see [11] Page 103 and Corollary 4.3]). Our approach is different, and more constructive. We use the algorithmic language of recursive coatom orderings, and exploit the structure given by the shelling-type linear orderings in the construction of the matching. This structure allows a more accurate understanding of the matchings, and we decided to include it as a stepping stone toward the study of the boundary relations in the minimal complexes produced in Proposition [2] a task that we plan to undertake in future work.

We would like to point out that our shelling-type orderings appear to be a kind of generalized shellings where the bounded faces are exactly the homology facets of the considered CW-complex.

As a first step, we define the class of posets that will be the object of our study. It is clear that these posets include the posets of cells of (possibly nonpure) regular CW-complexes.

**Definition 2.9.** A poset $P$ will be called locally ranked if all its principal lower ideals are ranked. It then possesses a well-defined height function $h$ that assigns to every element the rank of the lower principal ideal it generates. Let $h(P) := \max\{h(p) \mid p \in P\}$. For technical reasons, we will denote by $P_i$ the set of all $p \in P$ with $h(p) = h(P) - i$.

The set of maximal elements of a given locally ranked poset $P$ will be denoted by $M_P$ or simply $M$ if no misunderstanding can occur. If an ordering $\prec$ of $M_P$ is specified, then we can associate to every $p \in P$ a unique element

$$m_p := \max_\prec\{m \in M_P \mid m \geq p\}$$

(informally, the last among the maximal elements that lie above $p$).

We proceed to prove the key technical lemma toward Proposition [4]

**Lemma 2.10.** Let $(P, \prec)$ be a locally ranked poset, and let a recursive coatom ordering $\prec$ be defined on $\bar{P}$. Then it is possible to define a family of total orders $(\{P_i, \sqsubseteq_i\})_{i=0, \ldots, h(P)}$ with the following properties:

- given $p \in P_i$, and writing $Q_p := \bigcup_{p' \sqsubseteq_i p} \mathrm{coat}(p')$,
- (1) the order induced by $\sqsubseteq_{i+1}$ on $D_p := \mathrm{coat}(p) \setminus Q_p$ can be extended to a recursive coatom ordering $\prec_p$ of $\mathrm{coat}(p)$ in which the elements of $Q_p$ come first.
- (2) for all $p' \sqsubseteq_i p$ in $P_i$, if $p', p > z$, then there is $p'' \sqsubseteq_i p$ and $w \in \mathrm{coat}(p)$ such that $p'' > w \geq z$.

**Proof.** The orderings $\sqsubseteq_i$ will be defined recursively for increasing $i$. First, since $P_0 \subseteq M$, it makes sense to let $\sqsubseteq_0$ coincide with the given recursive coatom ordering $\prec$. By hypothesis, for every $p \in P_0$ there is a recursive coatom ordering $\prec_p$ of $P_{\leq p}$ in which the elements of $Q_p$ come first. Therefore we can define $\sqsubseteq_1$ by declaring

$$x \sqsubseteq_1 y \iff \begin{cases} x \prec_p y & \text{if there is } p \in P_0 \text{ with } x, y \in D_p, \\ p \sqsubseteq_0 q & x \in D_p, y \in D_q, \\ m_x \prec_j y & \text{if } y \in M. \end{cases}$$

This ordering is well-defined because by construction $D_p \cap D_q = \emptyset$ if $p \neq q$. Moreover, it clearly satisfies the requirement.

Now let $i > 1$ and suppose that the orderings $\sqsubseteq_j$ are defined for $j \leq i$.

**Definitions:** For $p \in P_i$, let $q_p := \min_{\sqsubseteq_{i-1}}\{q \in P_{i-1} \mid q > p\}$ (and note that this
implies \( p \in D_q \). Moreover, let
\[
N_p := \text{coat}(p) \setminus \{ y \in \text{coat}(p') \mid p' \in \text{coat}(q_p), p' \sqsubseteq_j p \}
\]
and note that by definition \( P_{i+1} = \bigsqcup_{p \in P_i} N_p \). We define also
\[
A_p := \{ y \in \text{coat}(p) \mid y < q \text{ for } q' \sqsubseteq_{i-1} q_p \} \quad \text{and} \quad B_p := Q_p \cap P_{<q_p},
\]
so that \( N_p = \text{coat}(p) \setminus B_p \) (see Figure 2.4).

\[\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.4}
\caption{Figure 2.4.}
\end{figure}\]

**Remark:** For every \( p \in P_i \) we have \( A_p \subseteq B_p \). In fact, given \( p \in P_i \) and \( x \in A_p \), by assumption on \( \sqsubseteq_{i-1} \) there is \( w \in \text{coat}(q_p) \) such that \( w > x \) and \( w \sqsubseteq_i p \).

Because \( \sqsubseteq_i \) induces a recursive coatom ordering on \( \text{coat}(q_p) \), we know that there is a recursive coatom ordering \( \prec_p \) of \( \text{coat}(p) \) such that the elements of \( B_p \) come first.

For \( x, y \in P_{j+1} \) we define:
\[
x \sqsubseteq_{i+1} y \Leftrightarrow \begin{cases} 
  x \prec_p y & \text{if there is } p \text{ such that } x, y \in N_p, \\
  p \sqsubseteq_i p' & \text{if } x \in N_p, y \in N_p', \\
  m_x \prec y & \text{if } y \in M
\end{cases}
\]

At this point it is worth to point out that, given \( p \in P_i \), \( Q_p = \bigcup_{p' \sqsubseteq_i p} N_p \) and \( D_p = N_p \).

We have now to check the conditions. (1) is clear: given \( p \in P_i \) and \( D_p = N_p \), \( \prec_p \) is a recursive coatom ordering of \( \text{coat}(p) \) such that the elements of \( B_p \), and thus every \( x \in \text{coat}(p) \setminus Q_p \), come first. For (2) take \( x, x' \in P_{i+1} \) such that \( x' \sqsubseteq_{i+1} x \) and \( z < x', x \). If \( x' \in N_p' \) and \( x \in N_p \) for \( p \neq p' \), then we have \( p' \sqsubseteq_i p \), and by property (2) of \( \sqsubseteq_i \) there is \( p'' \sqsubseteq_i p \) and \( y \in \text{coat}(p) \) such that \( z \leq y \leq p'' \).

Applying Definition 2.2 (2) to \( \prec_p \) we obtain an \( x'' \prec_p x \) and a \( w \in \text{coat}(x) \) such that \( z \leq w < x'' \). The proof is concluded by the remark that \( x'' \prec_p x \) implies \( x'' \sqsubseteq_{i+1} x \) because \( x \in N_p \).

**Definition 2.11** (Shelling-type orderings). Let \( P \) be a locally ranked poset. We introduce functions \( \pi_i : P_i \to P_{i+1} \) defined by
\[
\pi_i(q) := \max\{ p \in P_{i+1} \mid q > p \},
\]
where the \( \sqsubseteq_i \) are the orderings associated to some shelling via Lemma 2.10.

Then we define a linear extension \( < \) of \( P \) by:
\[
p < q \Leftrightarrow \begin{cases} 
  p \sqsubseteq_i q & \text{if there is } i \text{ such that } p, q \in P_i, \\
  p \sqsubseteq_i \pi_i \pi_i+1 \cdots \pi_{j-1}(q) & \text{if } p \in P_i, q \in P_j \text{ and } i > j.
\end{cases}
\]

The easy check that this is a well-defined linear order is left to the reader. Every linear extension \( < \) of \( P \) that is constructed in this way from a recursive coatom ordering will be called a **shelling-type ordering** of \( P \).
We can now construct an acyclic matching for any shelling-type ordering of a locally ranked poset.

**Lemma 2.12.** Every shelling-type ordering \(<\) of a locally ranked poset \(P\) induces an acyclic matching \(\mathcal{M}\) on \(P\).

**Proof.** For \(i = 1, \ldots, h(P)\) let \(\sqsubset_i\) denote the restriction of \(<\) to \(P_i\). By definition, every \((P_i, \sqsubset_i)\) satisfies the claim of Lemma 2.10. We write \(P_i = \{p_i^1, \ldots, p_i^k\}\), where \(p_j^i \sqsubset_i p_j^{i+1}\) for all \(j = 1, \ldots, k_i\).

**Definition of the matching \(\mathcal{M}\):**

We start with the one-element matching \(\mathcal{M} := \{(p_1^1, \pi_1(p_1^1))\}\). For every \(j = 2, \ldots, k_1\) we add \((p_j^1, \pi_1(p_j^1))\) to \(\mathcal{M}\) if \(\pi_1(p_j^1)\) is not already matched (or, equivalently, if \(\pi_1(p_j^1) \neq \pi_1(p_l^1)\) for all \(l < j\)).

For \(i = 1, \ldots, h(P)\) we further expand \(\mathcal{M}\) as follows: for \(j = 1, \ldots, k_i\), if \(p_j^i\) is not already matched and \(\pi_i(p_j^i) \neq \pi_i(p_l^i)\) for all \(l < j\), then add \((p_j^i, \pi_i(p_j^i))\) to \(\mathcal{M}\).

Since, by construction, \(p \triangleright \pi_i(p)\) whenever \((p, \pi_i(p)) \in \mathcal{M}\), this matching is acyclic by Lemma 2.5. \(\square\)

So far we stayed in the full generality of locally ranked posets. If we restrict ourselves to the case of posets of cells of CW-complexes, we can have even more control on the critical elements. The stepping stone for this is the following easy lemma, that we prove for completeness.

**Lemma 2.13.** Let \(K\) be a regular CW-decomposition of a sphere. Then in every shelling order of \(K\) the only homology facet is the last one.

**Proof.** The argument is by contraposition. Indeed, if the claim would not hold, then there would be a counterexample, say a regular CW-complex \(K\), a homeomorphism \(\phi : K \to S^d\), and a shelling order on the facets of \(K\) such that the last facet, call it \(F\), is not a homology facet. This means that the union \(K'\) of all the facets other than \(F\) is a shellable complex with still a homology facet - in particular, it is not contractible. But on the other hand, this complex has to be homeomorphic to \(S^d \setminus \phi(F \setminus K')\), which is contractible because \(F \setminus K'\) is. A contradiction follows. \(\square\)

**Proposition 1.** Every shelling of a regular CW-complex \(K\) induces an acyclic matching of the poset of faces of \(K\). Moreover, the critical cells of this matching correspond to the homology facets of the given shelling.

**Proof.** It is known that every shelling of a regular CW-complex corresponds to a recursive coatom ordering of its poset of cells (see e.g. [8, Theorem 13.2]) and, by Lemma 2.10, to a family of orderings \((P_i, \sqsubset_i)\) giving rise to a shelling-type ordering \(<\). Via our Lemma 2.12 this \(<\) every shelling order of the facets of \(K\) defines an acyclic matching \(\mathcal{M}\) of the poset of cells \(P := \mathcal{F}(K)\). We have to study the critical cells.

Consider a critical element \(p \in P_i\) such that \(p\) is not maximal in \(P\). Several situations can occur:

(i) **There is \(q > p\) such that \((q, \pi_{i-1}(q)) \in \mathcal{M}\).** Then \(p \sqsubset_i \pi_{i-1}(q)\) and, since \(p\) was not matched, there must be \(\tilde{p} \sqsubset_i p\) such that \(\pi_i(p) = \pi_i(\tilde{p})\). In particular, every element of \(x \in \text{coat}(p)\) is coatom of some \(p' \sqsubset_i p\) by Lemma 2.10(1). We may assume without loss of generality that \(p' \in \text{coat}(q)\), because else by property (2) of Lemma 2.10 we can find \(p'' \in \text{coat}(q)\) such that \(x < p''\). This all means that, in the shelling of \(P_{<q}\) that is induced by \(\sqsubset_i\), the whole boundary of \(p\) is already taken when the turn of \(p\) comes. But since \(p\) is not the last element of this shelling (which is \(\pi_{i-1}(q)\)), using Lemma 2.13 we get a contradiction with the fact that \(P_{<q}\) is a shellable sphere. This case can therefore not enter. \(\diamondsuit\)

(ii) **There is \(q > p\) that is not matched.** If for this \(q\) we have \(p \sqsubset_i \pi_{i-1}(q)\), then
the same reasoning of item (i) applies to get a contradiction. On the other hand, if \( \pi_{i-1}(q) = p \) then our algorithm should have taken the edge \((q, \pi_{i-1}(q) = p)\) into \( \mathcal{M} \) when examining the elements of \( P_{i-1} \): indeed, \( p \) was not already taken as \( \pi_{i-1}(q') \) for any \( q' \subset_{i-1} q \) (and actually it will remain free until the end!). So, this second situation can also not happen. 

(iii) Else: every \( q > p \) is matched ‘from above’, i.e., by an edge \((w, \pi_{i-2}(w) = q)\). In this case, let \( q_1, \ldots, q_k \) be any enumeration of the elements that cover \( p \). We know that no edge \((q_j, p)\) is matched, but we have supposed also that for every \( j = 1, \ldots, k \) there is \( w_j \) such that \((w_j, q_j) \in \mathcal{M}\). Since \( P \) is a CW-poset, we know (e.g. by [3, Proposition 2.2]) that every interval of length 2 has four elements - so that to every \( j \in \{1, \ldots, k\} \) we can associate \( \phi(j) \in \{1, \ldots, k\} \) such that the interval \([p, w_j]\) has elements \( \{p, q_j, q_{\phi(j)}, w_j\} \). In this interval by assumption the edge \((w_j, q_j)\) is matched, and therefore for sure \((w_j, q_{\phi(j)}) \notin \mathcal{M}\). This implies in particular \( w_j \neq w_{\phi(j)} \) for every \( j \). But then the alternating path \( q_j, w_j, q_{\phi(j)}, w_{\phi(j)}, q_{\phi^2(j)}, \ldots \) must be a cycle, because \( \phi \) can take only finite many values (we supposed the CW-complexes to be locally finite). Thus, also this case cannot enter.

It follows that every critical element is a maximal element of \( P \), i.e., by a facet of \( K \). But a maximal element \( m \in P \) is not matched exactly when \( \max_{\subset,\leq} \text{coat}(m) \) is matched by some \( p \subset \leq m \) (and hence, by item (i) above, when all its coatoms are).

In topological words, \( m \) is critical exactly if, when its turn in the shelling comes, its whole boundary was already taken. This means exactly that \( m \) is a homology facet of the given shelling. \( \square \)

Example 2.14. The acyclic matching depicted in Figure [2.3] is induced from the shelling order and the recursive coatom ordering of Figure [2.2] by the following shelling-type ordering:

\[ I \subset a \subset b \subset II \subset f \subset III \subset e \subset IV \subset c \subset V \subset d \subset VI \subset \emptyset. \]

Remark 2.15. Proposition [1] gives a perfect acyclic matching of \( \hat{P} \) whenever \( P \) is the poset of faces of a regular CW-complex that is homotopy equivalent to a sphere. Indeed, in that case the only critical cell of \( P \) can be matched by the added element \( \hat{1} = \hat{P} \setminus P \).

3. Shelling-type orderings of oriented matroids

In this section we apply Proposition [1] to a special situation, as an attempt to answer [24, Remark 4.5] and as a stepping stone to the results of Section [4]. If we consider the fan defined by a set of real linear hyperplanes, we see that the boundary of the associated polar polytope is a shellable (CW-) sphere. The combinatorics of real arrangements of hyperplanes is customarily encoded by oriented matroids. These combinatorial objects are more general than real linear hyperplane arrangements; however, to every oriented matroid corresponds a shellable CW-sphere that, in case the oriented matroid describes an arrangement, is combinatorially isomorphic to the associated polar polytope.

In what follows we state the precise definitions and recall the results that we will need for this paper. The standard reference for a comprehensive overview on oriented matroids is [4].

In Remark [3] we will return to the case where the oriented matroid comes from an arrangement of real hyperplanes to compare our shelling-type orderings to the polar ordering of [24].

Definition 3.1 (Oriented matroid). Given a ground set \( E \), a collection \( V \subset \{+, -, 0\}^E \) is the set of vectors of an oriented matroid \( \mathcal{M} \) if and only if following properties are satisfied:
(1) \((0,0,\ldots,0)\in\mathcal{V},\)
(2) if \(X\in\mathcal{V},\) then \(-X\in\mathcal{V},\)
(3) for all \(X,Y\in\mathcal{V},\) \(X\cap Y\in\mathcal{V},\)
(4) for all \(X,Y\in\mathcal{V},\)
   given \(e,f\in E\) such that \(X_e = -Y_e\) and not both \(X_f, Y_f\) equal 0,
   there is \(Z\in\mathcal{V}\) such that \(Z_e = 0, Z_f \neq 0,\) and if \(Z_i \neq 0\) then \(Z_i\) equals \(X_i\)
or \(Y_i.\)

Let us point out that this is only one of the many ways to characterize oriented
matroids. For a complete account of the many different possible axiomatizations
we refer to Chapter 5 of [6].

**Remark 3.2.** Let \(\mathcal{V}\) be the set of vectors of an oriented matroid \(\mathcal{M}\). Let \(\mathcal{V}^*\)
denote the set of all \(G\in\{+, -, 0\}^E\) such that \(\sum_{e\in E} G_e X_e = 0\) for all \(X\in\mathcal{V}\) (the
multiplication and the sum being performed by thinking of + as +1 and of – as
\(-1).\) Then \(\mathcal{V}^*\) is the set of *covectors* of \(\mathcal{M}.* It is a matter of fact that \(\mathcal{V}^*\)
satisfies the axioms of Definition 3.1: it is the set of vectors of an oriented matroid
that is called dual to \(\mathcal{M}\) (note that \((\mathcal{V}^*)^* = \mathcal{V}).\) For a proof of this see [6, Proposition
3.7.12].

The *support* of a subset \(X\subset\{+, -, 0\}^E\) is \(\text{supp}(X) := \{e\in E \mid X_e \neq 0\}.\) We
define a partial order on \(\mathcal{V}\) by setting

\[
X \leq Y \iff \forall e\in E : \begin{cases} 
\text{supp}(X) \subseteq \text{supp}(Y) \text{ and } \Y_e \neq 0 \Rightarrow X_e \neq -Y_e. 
\end{cases}
\]

**Definition 3.3.** The set \(\mathcal{V}^*\) endowed with this ordering is called the *face poset* of
the oriented matroid \(\mathcal{M}\) and is denoted by \(\mathcal{F}(\mathcal{M}).\) It has a unique minimal element
but in general it possesses many maximal elements, that are called *topes* of the
oriented matroid. the set of topes of an oriented matroid \(\mathcal{M}\) will be denoted by
\(T(\mathcal{M})\) (or just \(T\)).

For \(T\in T\) and \(F\in\mathcal{V}^*\) we define \(T_F\in T\) by \((T_F)_e = T_e\) if \(F_e = 0\) and \((T_F)_e = F_e\)
else (see Remark 3.4 for a geometric interpretation of this operation).

It turns out that also the set \(T\) can be given interesting partial orders. These
were introduced by Edelman [13] in the context of arrangements of hyperplanes
and independently by Edmonds and Mandel [14] for abstract oriented matroids.

**Definition 3.4** (See also Definition 4.2.9 of [13]). Let an oriented matroid \(\mathcal{M}\) be
given and consider its set of topes \(T\). For \(T,T'\in T\) let

\[
S(T,T') := \{e\in E \mid T_e = -T'_e\}.
\]

To every tope \(B\in T\) we can associate a partial order \(\prec_B\) on \(T\) defined by

\[
T_1 \prec_B T_2 \iff S(B,T_1) \subset S(B,T_2).
\]

The set \(T\) endowed with the order relation \(\prec_B\) is called *tope poset of \(\mathcal{M}\) based
at \(B\) and will be denoted by \(T_B(\mathcal{M})\) or simply by \(T_B.\) This poset is ranked by
\(r(T) = |S(B,T)|.\) We will use the symbol \(\dagger\) to indicate total orderings that are
linear extensions of the ordering of a tope poset.

It is a nice fact that, for any oriented matroid \(\mathcal{M}, \mathcal{F}(\mathcal{M})\dagger\) (suitably augmented
by an additional 0-element, if needed) is the poset of faces of a convex polytope
that is called the *zonotope* of \(\mathcal{M}.* The 1-skeleton of its dual polytope is isomorphic, as a
graph, to the Hasse diagram of \(T_B(\mathcal{M})\) for every \(B\in T\). In this sense, specifying
a linear extension of \(T_B\) amounts to somehow ‘specify a direction’ in the ambient
space of the zonotope. Indeed, such a linear ordering is all what one needs to get
a shelling of the zonotope.
Figure 3.1. On the left is the face poset $F(M)$ of an oriented matroid on 3 elements. Its (augmented) dual $F(M)^{op}$ appeared already in Figure 2.1 so that the zonotope of this oriented matroid is the hexagon $K_1$. The dual polytope of $K_1$ is again an hexagon, so that the tope poset $T_{(+,+,+)}(M)$ for this oriented matroid is the poset depicted on the right.

**Theorem 3.5** (Proposition 4.3.2 of [6]). Let $M$ be a simple oriented matroid and $B$ be a tope of $M$. Every linear extension of the tope poset $T_B(M)$ induces a recursive coatom ordering of $F(M)$.

Thus, an application of Proposition 1 gives immediately the following existence result.

**Theorem 3.6.** Let $M$ be a simple oriented matroid and $B$ be a tope of $M$. Every linear extension of the tope poset $T_B(M)$ defines an acyclic matching $M$ of the face poset $F(M)$ such that the only critical element is $-B$, the tope opposite to $B$.

**Example 3.7.** One possible linear extension of the tope poset of Figure 3.1 is given by

$$(+,-,+) \vdash (+,+,-) \vdash (+,-,-) \vdash (-,+,+) \vdash (-,-,+) \vdash (-,-,-).$$

Comparing Figure 2.2 we see that this linear extension corresponds indeed to the shelling order $I, II, \ldots, VI$ of $K_2$ via the correspondence of the posets of faces, and thus induces on the poset $F(M) = F(K_2)$ the acyclic matching indicated in Figure 2.3.

**Remark 3.8** (On polar orderings). As we will explain in detail in the next Section, to every real linear arrangement of hyperplanes is associated an oriented matroid whose covectors correspond to the induced stratification of $\mathbb{R}^n$. These ‘special’ oriented matroids can be therefore also given a polar ordering in the sense of Salvetti and Settepanella [24]. This makes a comparison of the two orderings possible. The outcome is that shelling-type orderings are different from the polar orderings of [24] on linear arrangements (although they can be used for the same scope, as we will see in the next section): indeed, a polar ordering is never a linear extension of the face poset (as can be easily seen comparing Theorem 4 of [24]). Moreover, the order induced on the chambers by a polar ordering is never a linear extension of a poset of regions: otherwise, there would be no other choice for the base chamber $B$ as to take the chamber containing the basepoint of the polar ordering. But then we see that there is a maximal chain in $T_B$ (determined by the general position line $V_1$ of [24]) whose elements form by definition an initial segment in the order of chambers induced by the polar ordering. This is clearly incompatible with being a linear extension of $T_B$. 

\[ \text{Figure 3.1. On the left is the face poset } F(M) \text{ of an oriented matroid on 3 elements. Its (augmented) dual } F(M)^{op} \text{ appeared already in Figure 2.1 so that the zonotope of this oriented matroid is the hexagon } K_1. \text{ The dual polytope of } K_1 \text{ is again an hexagon, so that the tope poset } T_{(+,+,+)}(M) \text{ for this oriented matroid is the poset depicted on the right.} \]
Nevertheless, at a first glance the ordering induced on the chambers by the polar orders seems to be a shelling order for the zonotope. We leave this as an open question.

**Remark 3.9.** The proofs of [9] Proposition 4.3.1 and 4.3.2 are constructive. Hence, by taking a closer look at the arguments used there one can give an explicit description of the shelling-type orderings (and thus of the matchings) that result from our construction. To do this, we need some notation. For every element $e$ of the oriented matroid let $R(e) := \min\{B_F \mid |F| = e\}$, and let $F(e)$ be the unique face with $F(e) \prec R(e)$ and $|F(e)| = e$. Then, for every $R \in T_B$ choose a maximal chain $\omega_R$ in the interval $[B, -R] \subset T_R(M)$. For $i = 0, \ldots, n$ let $\omega_R(i)$ denote the $i$-th element of the chain (counted from the bottom).

For every maximal element $R$ of $F$ we can express $D_R$, $Q_R$ and $\pi_0(R)$ (see Lemma 2.10) as follows:

$$D_R = \{F \in \text{coat}(R) \mid |F| \in S(-R)\} \quad (= \{F \in \text{coat}(R) \mid R_F = T_F\}),$$

$$Q_R = \{F \in \text{coat}(R) \mid |F| \in S(R)\} \quad (= \{F \in \text{coat}(R) \mid R_F \neq T_F\}),$$

$$\pi_0(R) = \omega_R(n).$$

We conclude that the induced ordering $\sqsubseteq_1$ on $F$ can be expressed by

$$F_1 \sqsubseteq_1 F_2 \Leftrightarrow \left\{ \begin{array}{l}
T_{F_1} \nleq T_{F_2} \text{ or } T_{F_1} = T_{F_2} =: R \text{ and } |F_1| \prec_R |F_2|,
\end{array} \right.$$  

where $\prec_R$ is the order in which the elements appear as $S(\omega_R(i), \omega_R(i + 1))$ for increasing $i$.

Moreover, according to the construction of [9] 4.3.1 and 4.3.2, the recursive ordering of $\text{coat}(F)$ is given as above by any maximal chain in $T_F(M/|F|)$ that contains $F'$, where $F'$ is the face where $\omega_R$ crosses $|F|$. In particular, the elements of $D_F$ are ordered according to a maximal chain in the interval $[F', -F] \subset T_F(M/|F|)$, and so on.

4. **Acyclic maximum matchings for the Salvetti complex**

The main motivation of Salvetti and Settepanella for considering polar orderings in [24] was to use these total orderings in the construction of what they call the polar gradient. The polar gradient of [24] is essentially an acyclic maximum matching of the poset of cells of the Salvetti complex - a regular CW complex that was introduced by Mario Salvetti in order to model the homotopy type of the complement of a complexified arrangement of hyperplanes (see Definition 4.1 and [23]).

In this section we want to construct acyclic matchings for the Salvetti complex of linear arrangements using shelling-type orderings. In fact, the outcome is that linear extensions of tope posets give a very nice stratification of the Salvetti complex (see Lemma 4.20) and allow us to paste together different choices of acyclic matchings of the strata.

Let us begin by the definition of the poset of cells of the Salvetti complex for a general oriented matroid. We present it here in his general form and as a formally defined object to underline the fact that it can be defined in purely combinatorial terms. In a second step we will introduce the terminology (and the geometric intuition) of arrangements of hyperplanes.

**Definition 4.1.** Given an oriented matroid $M$, we define a poset $S(M)$ (denoted simply by $S$ if no confusion can arise). The elements of $S(M)$ are all pairs $(F, T)$ where $F \in \mathcal{F}(M)$, $T \in \mathcal{T}(M)$ and $F < T$ in $\mathcal{F}(M)$. The order relation in $S$ will

$$F \sqsubseteq_1 T.$$
be denoted \( \prec_s \) and defined by setting
\[
\langle F, T \rangle \prec_s \langle F', T' \rangle \text{ if } F >_F F' \text{ in } \mathcal{F}(\mathcal{M}) \text{ and } T = T'.
\]
Recall that the poset \( \mathcal{F}(\mathcal{M}) \) has a unique minimal element that we denote by \( P \). For any given tope \( T \) let \( S_T := S(\mathcal{M})_{\leq (P, T)} \). It is clear that \( S_T \) is isomorphic to \( \mathcal{F}(\mathcal{M})^{op} \) as a poset. If no confusion can arise we will write just \( S, F, T \) for \( S(\mathcal{M}), \mathcal{F}(\mathcal{M}), \mathcal{T}(\mathcal{M}) \).

Now fix a “base tope” \( B \in \mathcal{T} \). If a linear extension \( \triangleleft \) of \( T_B \) is given, define, for every \( R \in \mathcal{T} \),
\[
S(R) := \bigcup_{T \triangleleft R} S_T \quad \text{and} \quad N(R) := S(R) \setminus S(R'),
\]
where \( R' \) is the tope that precedes \( R \) in \( \triangleleft \).

**Example 4.2.** The poset of Figure 4.3 is \( S(\mathcal{M}) \) for the (realizable) oriented matroid \( \mathcal{M} \) of Figure 3.1, where for better readability we denoted the co vectors by the corresponding strata in \( \mathbb{R}^2 \) (see Figure 4.1).

A real arrangement of hyperplanes is a set \( \mathcal{A} := \{H_1, \ldots, H_n\} \) where the \( H_i \) are codimension 1 affine subspaces of \( \mathbb{R}^d \). The arrangement \( \mathcal{A} \) is called linear if every \( H_i \) is a linear subspace. The combinatorial data of a real linear arrangement \( \mathcal{A} \) is encoded by the associated oriented matroid \( \mathcal{M}_A \) of the signed linear dependencies among the vectors \( \{v_1, \ldots, v_n\} \) where, for all \( i \), \( v_i \) is normal to \( H_i \). An oriented matroid that is of the form \( \mathcal{M}_A \) for some real linear arrangement \( \mathcal{A} \) is called realizable.

![Figure 4.1](image)

**Figure 4.1.** Our main example, the arrangement of three lines in the plane. On the left the ‘plain’ arrangement, with our choice of normal vectors to build the oriented matroid \( \mathcal{M}_A \). On the right, the cells of the induced stratification of \( \mathbb{R}^2 \).

The relevance of Definition 4.1 comes from the following fundamental result by Mario Salvetti (which actually holds also in a general form for affine arrangements).

**Theorem 4.3** (Theorem 1 of [23]). Let \( \mathcal{A} \) an arrangement of linear real hyperplanes. Then \( S(\mathcal{M}_A) \cup \{\hat{0}\} \) is the poset of cells of a regular CW-complex, called **Salvetti complex**, that is homotopy equivalent to the complement in \( \mathbb{C}^d \) of the complexification of \( \mathcal{A} \).

We see that, although \( S \) can be defined for any oriented matroid, the main topological interest of the construction is in the context of arrangements of hyperplanes. Therefore we will from now sometimes use the more geometrically intuitive language of this setting, that we are going to explain.
If $\mathcal{M}$ is a realizable oriented matroid corresponding to the arrangement $\mathcal{A}$, then the poset $\mathcal{F}(\mathcal{M})$ is the poset of the closed strata determined by $\mathcal{A}$ in $\mathbb{R}^d$, ordered by inclusion of the topological closures.

**Example 4.4.** For $\mathcal{A}$ as in Figure 4.1, $\mathcal{M}_A$ is the oriented matroid $\mathcal{M}$ of Figure 3.1. In particular, we can compare the poset $\mathcal{F}(\mathcal{M})$ of Figure 3.1 with the stratification of $\mathbb{R}^2$ on the right hand side of Figure 4.1. For instance, the covector $(+,0,-)$ represents the stratum of all vectors of $\mathbb{R}^2$ which scalar product with $v_1$ is positive, with $v_2$ equals 0 and with $v_3$ is negative (i.e., the points 'in front of' $H_1$, 'on' $H_2$ and 'behind' $H_3$ with respect to the base chamber $B = (+,+,+)$).

The topes are the maximal strata - i.e., the closure of the connected components of the complement $\mathbb{R}^d \setminus \bigcup A$ of $\mathcal{M}_A$ - and are customarily called chambers (or regions) of $\mathcal{A}$ (given a set $A := \{a_1, a_2, \ldots, a_n\}$, we will write $\bigcap A$ for the set $a_1 \cap a_2 \cap \ldots \cap a_n$ and $\bigcup A$ for $a_1 \cup a_2 \cup \ldots \cup a_n$). Accordingly, $\mathcal{T}_B(\mathcal{A})$ is often referred to as the poset of regions of $\mathcal{A}$ (e.g., in his first appearance in the context of hyperplane arrangements, see [13]). For any two chambers $C_1, C_2$ of $\mathcal{A}$ (topes of $\mathcal{M}_A$), the elements of $S(C_1, C_2)$ correspond to the hyperplanes that separate $C_1$ from $C_2$, i.e., the hyperplanes that are met by any line segment connecting a point in the interior of $C_2$ with a point in the interior of $C_1$. Since the arrangements corresponding to oriented matroids are linear, every chamber is a convex cone. The hyperplanes supporting the facets of the cone determined by the chamber $C$ are called walls of $C$. The set of walls of $C$ is denoted by $W_C$.

**Remark 4.5.** For every wall $H \in W_C$ there is a chamber $K \in \mathcal{T}(\mathcal{A})$ such that $S(C, K) = \{H\}$. In fact, this can be taken as the ‘abstract’ definition in the setting of arbitrary oriented matroids.

**Notation 4.6.** We will denote by $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{M})$ (or just by $\mathcal{L}$) the lattice of flats of the underlying matroid; this is indeed a geometric lattice and we will think of it as of the poset of intersections of the hyperplanes ordered by reverse inclusion (see the top of Figure 4.2 for a picture of $\mathcal{L}(\mathcal{A})$ when $\mathcal{A}$ is the arrangement of three lines through the origin of the plane). For every face $F \in \mathcal{F}(\mathcal{M})$ we write $|F|$ for what corresponds to the “affine span” of $F$, i.e., the flat given by the elements of $\text{supp}(F)$. Given any flat $Y \in \mathcal{L}$, we denote by $\mathcal{A}_Y$ the arrangement given by the hyperplanes that contain $Y$ and set $\mathcal{A}_Y := \text{supp}(Y)$. We write $\mathcal{A}^Y$ for the arrangement $\{H \cap Y \mid H \notin \mathcal{A}_Y\}$ that is determined on $Y$ by the hyperplanes that intersect $Y$ nontrivially. The oriented matroid associated to $\mathcal{A}^Y$ is the contraction $\mathcal{M}(\mathcal{A})/Y$ of the oriented matroid associated to $\mathcal{A}$ (see [13, Section 3.3]). The natural map $\pi(\mathcal{A}) \to \mathcal{T}(\mathcal{A}^Y)$ will be denoted by $\pi_Y$. We will use it to explain the geometric content of the operation described in Definition 3.3.

**Remark 4.7.** Let $\mathcal{M}$ be a realizable oriented matroid and $\mathcal{A}$ the corresponding arrangement. Let $C$ be one of its toposes (chambers) and $F$ be some covector (face) of $\mathcal{M}(\mathcal{A})$. Then the tope $\mathcal{T}_F$ corresponds to the unique chamber that is contained in $\pi_F(\mathcal{A})$ and contains $F$.

**Important Remark 4.8.** In all what follows, unless explicitly stated,

- $\mathcal{A}$ will denote a finite arrangement of $n$ linear hyperplanes in $\mathbb{R}^d$.

Moreover, we fix from now an (arbitrarily chosen) base chamber $B \in \mathcal{T}(\mathcal{A})$ and a (also arbitrary) linear extension $\pi$ of $\mathcal{T}_B(\mathcal{A})$.

---

The use of the word ‘separation’ arose in the literature while considering the chambers to be the open sets that are obtained subtracting $\mathcal{A}$ from $\mathbb{R}^d$, so that any two chambers are really disjoint and ‘separated’ by the hyperplanes in the set $S(C_1, C_2)$. For consistency we let here the chambers be, as any other face, closed. The combinatorics of course works as well, and we will save some cumbersome distinctions in the last section.
Let us also point out that everything we will say can be easily translated into the language of (and thus: holds for) abstract oriented matroids. As the ‘grammar’ and the ‘vocabulary’ for this translation we refer to [6].

**Notation 4.9.** Given $H \in A$, let $A' := A \setminus \{H\}$. Given $C \in T(A)$, we will write $C'$ for the unique chamber of $A'$ that contains $C$. This natural inclusion of chambers induces an order preserving map

$$
\psi : T_B(A') \to T_B(A); \quad C' \mapsto \min \{ C \in T_B(A) \mid C \subseteq C' \}.
$$

Note that if $C' \in T(A')$ contains two chambers $C_1, C_2 \in T(A)$ then, up to renumbering, $C_1 \preceq_B C_2$. So this definition could have been phrased as well in terms of $\preceq$, the partial ordering of $T_B(A)$, instead of $\triangleleft$.

This map is clearly injective, and thus for $C_1', C_2' \in T(A')$ the ordering $\preceq'$ of $T_B(A')$ satisfies

$$
C_1' \preceq' C_2' \iff \psi(C_1') \preceq \psi(C_2').
$$

Given any linear extension $\triangleright$ of $T_B(A)$ we let $\triangleright'$ denote the linear extension of $T_B(A')$ that is in a sense the ‘pullback’ of $\triangleright$ along $\psi$:

$$
C_1' \triangleright' C_2' :\iff \psi(C_1') \triangleright \psi(C_2').
$$

As we will see this construction is canonical.

**Lemma 4.10.** Given two distinct hyperplanes $H_1, H_2 \in A$, for both $i = 1, 2$ write $A_i := A \setminus \{H_i\}$ and let $B_i$ be the unique chamber of $A_i$ containing $B$. Let $\psi_i$ denote the map $T_{B_i}(A_i) \to T_B(A)$ defined in [4.9]. Let then $\hat{B}$ be the unique chamber of $A_1 \cap A_2$ that contains $B_1$ and $B_2$, and write $\psi_1$ for the corresponding map $T_{\hat{B}} \to T_{B_i}(A_i)$. Then the diagram of poset maps

$$
\begin{array}{ccc}
T_B(A_1 \cap A_2) & \xrightarrow{\psi_1} & T_{B_1}(A_1) \\
\psi_2 & \downarrow & \psi_1 \\
T_{\hat{B}}(A_2) & \xrightarrow{\psi_2} & T_B(A)
\end{array}
$$

commutes.

**Proof.** For brevity, let $\hat{A} := A_1 \cap A_2$. Consider $\hat{C} \in T(\hat{A})$. By definition we have

$$
\hat{\psi}_1(\hat{C}) = \min \{ C' \in T_{B_1}(A_i) \mid C' \subset \hat{C} \},
$$

where $\preceq_i$ is the ordering of $T_{B_i}(A_i)$. This, in view of equation [4.1], means

$$
\min \{ C \in T(A) \mid C \subset \hat{\psi}_1(\hat{C}) \} \preceq \min \{ C \in T(A) \mid C \subseteq C' \subseteq \hat{C} \text{ for a } C' \in T_{B_i}(A_i) \}
$$

or, equivalently,

$$
\min \{ C \in T(A) \mid C \subset \hat{\psi}_1(\hat{C}) \} \preceq \min \{ C \in T(A) \mid C \subseteq \hat{C} \}.
$$

Now, because we are taking away from $A$ exactly two hyperplanes, the right side of the last expression takes the minimum over a poset that either has only one element, or is a two-element chain, or has four elements and rank two (depending on whether none, one or both of $H_1$ and $H_2$ cut $\hat{C}$). Thus, in any case the right side above identifies a unique $C \in T_B(A)$, and this is $\psi_1 \hat{\psi}_1(\hat{C})$. Summarizing, we have

$$
\psi_1 \hat{\psi}_1(\hat{C}) = \min \{ C \in T(A) \mid C \subseteq \hat{C} \}.
$$

Since this expression does not depend on $i$, we are done. $\square$

We will need the following corollary.
Corollary 4.11. In the setting of Lemma 4.10, for \( i = 1, 2 \) let \( \bar{\cdot}_i \) be the linear extension induced from \( \bar{\cdot} \) on \( T_{B_i}(A_i) \), and \( \bar{\cdot}_i \) the linear extension of \( T_{B}(A) \) induced from \( \bar{\cdot}_i \). Then

for all \( \hat{C}, \hat{K} \in T_{B}(A_1 \cap A_2) \), \( \hat{C} \bar{\cdot}_1 \hat{K} \iff \hat{C} \bar{\cdot}_2 \hat{K} \)

Proof. For both \( i = 1, 2 \), \( \hat{C} \bar{\cdot}_i \hat{K} \) if and only if \( \psi_i \hat{\psi}_i(\hat{C}) \bar{\cdot} \psi_i \hat{\psi}_i(\hat{K}) \). The claim follows with Lemma 4.10. \( \square \)

Now we can define the object we will study in the next few statements. Recall that we fixed a linear extension \( \bar{\cdot} \) of the tope poset of \( A \).

Definition 4.12. For every \( C \in \mathcal{T}(A) \) we let

\[ \mathcal{J}(C) := \{ X \in \mathcal{L}(A) \mid \text{supp}(X) \cap S(C, K) \neq \emptyset \text{ for every } K \vdash C \} \]

which is easily seen to be an upper ideal in \( \mathcal{L}(A) \).

Notation 4.13. Let \( H \in A \) be given and recall the notation 4.9. We write \( \mathcal{J}'(C') \) for the order ideal of \( \mathcal{L}(A') \) associated to \( C' \) and \( \bar{\cdot}' \) in Definition 4.12. The inclusion \( A' \hookrightarrow A \) induces an order preserving injection

\[ \iota : \mathcal{L}(A') \rightarrow \mathcal{L}(A), \ X \mapsto \bigcap \text{supp}(X). \]

We will identify \( \mathcal{J}'(C') \) with its image under this map.

Lemma 4.14. Let a chamber \( C \in T_{B}(A)_{<1} \) be given, choose \( H \in A \setminus S(B, C) \) (such an hyperplane exists because \( C \neq -B \)) and let \( A' := A \setminus \{ H \} \).

For every \( Y \in \mathcal{J}(C) \) we have

\[ \bigcap (\text{supp}(Y) \setminus \{ H \}) \in \mathcal{J}'(C'). \]

Proof. Fix any \( Y \in \mathcal{L}(A) \). As a first step, observe that

\[ (\ast) \text{ If } H \notin \text{supp}(Y), \text{ then } Y \in \mathcal{J}(C) \iff \bigcap (\text{supp}(Y) \setminus \{ H \}) \in \mathcal{J}'(C'), \]

because in this case \( \bigcap (\text{supp}(Y) \setminus \{ H \}) = Y \), and the conditions for being in \( \mathcal{J}'(C') \) and \( \mathcal{J}(C) \) become equivalent. Therefore suppose from now \( H \in \text{supp}(Y) \).

We want to argue by induction on \( |A| \). If \( |A| = 1 \) the claim is trivial. So suppose \( |A| > 1 \) and that the claim holds for every smaller arrangement. We need to distinguish two cases:

- **Case 1:** \( Y = \bar{1} \in \mathcal{L}(A) \). In this situation

\[ \bigcap (\text{supp}(Y) \setminus \{ H \}) = \bar{1} \in \mathcal{L}(A'). \]

Since both \( \mathcal{J}(C) \) and \( \mathcal{J}'(C') \) are nonempty upper ideals, we have \( Y \in \mathcal{J}(C) \) and \( \bigcap (\text{supp}(Y) \setminus \{ H \}) \in \mathcal{J}'(C') \) and the claim holds.

- **Case 2:** \( Y \neq \bar{1} \in \mathcal{L}(A) \). Thus we can find \( \bar{H} \in A \setminus \text{supp}(Y) \). Since \( H \in \text{supp}(Y) \), in particular \( \bar{H} \neq H \). We need a couple of definitions, in order to apply Lemma 4.10

Let \( \bar{A} := A \setminus \{ \bar{H} \}, \bar{\cdot} \) the induced linear extension, \( \bar{\mathcal{J}}(\bar{C}) \) the corresponding upper ideal (where \( \bar{C} \) is the unique chamber containing \( C \)) and define \( \bar{Y} := \bigcap (\text{supp}(Y) \setminus \{ \bar{H} \}) \). Moreover, let \( \bar{A}' := \bar{A} \setminus \{ H \} = \bar{A} \setminus A' \) and define \( \bar{\cdot}' \), \( \bar{C} \) and \( \bar{\mathcal{J}}'(\bar{C}') \) (noting that by Corollary 4.11 it does not matter to specify whether \( \bar{\cdot} \) is induced by \( \bar{\cdot}' \) or \( \bar{\cdot}' \)). We have the following implications:

- **(I)** \( Y \in \mathcal{J}(C) \Rightarrow \bar{Y} \in \bar{\mathcal{J}}(\bar{C}), \) e.g. by (\( \ast \)).
- **(II)** \( \bar{Y} \in \bar{\mathcal{J}}(\bar{C}) \Rightarrow \bigcap (\text{supp}(\bar{Y}) \setminus \{ \bar{H} \}) \in \bar{\mathcal{J}}'(\bar{C}'), \) by the inductive hypothesis, since \( H \in S(C, -B) \subseteq S(\bar{C}, \bar{B}) \) and \( |\bar{A}| < |A| \) (here \( \bar{\cdot}' \) is viewed as being induced from \( \bar{\cdot} \)).
(III) \( \cap (\text{supp}(\overline{Y}) \setminus \{H\}) \in \mathcal{J}'(C') \Rightarrow \cap (\text{supp}(Y) \setminus \{H\}) \in \mathcal{J}'(C') \) again by (\ast),
where we used Corollary 4.11 in switching point of view and considering \( \overline{C'} \) to be induced from \( C' \).

The lemma follows by chaining up these implications.

\[ \square \]

**Theorem 4.15.** For every \( C \in T_B(A) \), \( \mathcal{J}(C) \subset \mathcal{L}(A) \) is a principal upper ideal.

**Proof.** We will again argue by induction on the size of \( A \), for if \( A \) contains only one hyperplane the claim is trivial. So suppose \(|A| > 1\), and let the claim hold for every arrangement of size at most \(|A| - 1\). Choose chambers \( B, C \in T(A) \) and a linear extension \( \vdash \) of \( T_B(A) \). We will prove that the associated \( \mathcal{J}(C) \) is closed under the join operation (see Remark 2.1).

If \( C = -B \), then clearly \( \mathcal{J}(C) = \{1\} \subset \mathcal{L}(A) \) and the claim holds. If \( C \) is not \( -B \), in particular there is \( H \in S(C, -B) = A \setminus S(B, C) \), and \( A' := A \setminus \{H\} \) satisfies the theorem by induction hypothesis.

By Lemma 4.13 the (order preserving) map
\[
\lambda : \mathcal{L}(A) \to \mathcal{L}(A'), Y \mapsto \cap (\text{supp}(Y) \setminus \{H\})
\]
satisfies \( \lambda(\mathcal{J}(C)) \subseteq \mathcal{J}'(C') \). Note that the inclusion \( \iota \) of \( \mathcal{J}'(C') \) into \( \mathcal{J}(C) \) is well defined because whenever \( K \vdash C \), then \( K' \vdash C' \) and \( S(C', K') \cap \text{supp}(Y) \subset S(C, K) \cap \text{supp}(Y) \): if the former is nonempty, then so is the latter.

If we look at the composition of \( \lambda \) with \( \iota \), we see that \( \iota \lambda(Y) \leq Y \in \mathcal{L}(A) \) for every \( Y \in \mathcal{J}(C) \). Now consider two elements \( Y_1, Y_2 \in \mathcal{J}(C) \): by induction hypothesis \( \lambda(Y_1) \land \lambda(Y_2) \) exists in \( \mathcal{J}'(C') \). In \( \mathcal{J}(C) \) the latter.

This theorem ensures the existence of the object that we are going to define. For a construction of this object one needs some more refined considerations that we will carry out in Section 5.

**Definition 4.16.** Choose, as usual, a base chamber \( B \in T(A) \), let a linear extension \( \vdash \) of \( T_B(A) \) be given, and recall Definition 4.12.

For every \( C \in T(A) \) define
\[
X_C := \min \mathcal{J}(C).
\]

From the arguments stated above we can also obtain

**Corollary 4.17.** With the assumptions and notations of Definition 4.16, if we define \( F_C := X_C \cap C \), we have \(|F_C| = X_C|\).

**Proof.** Let \( A, B \) and \( \vdash \) be given, and consider \( C \in T(A) \). We will show that \( \dim(X_C \cap C) = \dim(X_C) \) whenever \( C \neq -B \) (in the remaining case, there is nothing to show).

Since the claim is trivial when \(|A| = 1\), we will proceed by induction, assuming from now that \(|A| > 1\) and that the claim holds for every arrangement with at most \(|A| - 1\) hyperplanes.

Choose \( H \in W_C \cap S(C, -B) \) (this can be done without loss of generality) and note that then \( C \) is the intersection of \( C' \) with the (closed) halfspace \( H^+ \) bounded by \( H \) and containing \( B \). Thus,
\[
C = C' \cap H^+.
\]

We will write \( X_C = \min \mathcal{J}(C) \) and \( X_C' := \min \mathcal{J}(C') \). By induction hypothesis:
\[
\dim(X_C \cap C') = \dim(X_C').
\]
Recall now the maps defined in the proof of Theorem 4.14. We have
\[ \lambda(X_C) = X'_{C'} \]
by injectivity of \( \iota \).

Therefore, only two cases can happen: either \( \bigcap \operatorname{supp}(X_C) = \bigcap (\operatorname{supp}(X_C) \setminus \{H\}) \), and thus \( X_C = X'_{C'} \), or \( \bigcap \operatorname{supp}(X_C) \neq \bigcap (\operatorname{supp}(X_C) \setminus \{H\}) \), which implies \( X_C = X'_{C'} \cap H \).

If \( X_C = X'_{C'} \), then in particular \( X_C \subset H \) and thus
\[ \dim(C \cap X_C) = \dim(C' \cap H^+ \cap X'_{C'}) = \dim(X'_{C'} \cap H^+) = \dim(X_C). \]

If on the contrary \( X_C = X'_{C'} \cap H \), then
\[ \dim(C \cap X_C) = \dim(C' \cap H^+ \cap X'_{C'} \cap H) = \dim(C' \cap X'_{C'} \cap H) = \dim(X'_{C'} \cap H) = \dim(X_C). \]

\[ \square \]

Question 4.18. It seems likely that the previous arguments can be carried out also for arrangements of affine hyperplanes, at least if \( B \) is assumed to be an unbounded chamber. Since this is not directly relevant for this work, we leave this as a question.

We return to the ‘linear’ case. The following lemma states, for later reference, an easy reformulation of the definition of \( X_C \).

Lemma 4.19. By Definitions 4.12 and 4.16, the flat \( X_C \) is uniquely determined by the following properties:

1. \( S(K, C) \cap \operatorname{supp}(X_C) \neq \emptyset \) for all \( K \vdash C \), and

2. For every \( Y \in \mathcal{L}(A) \) such that \( Y \not\vdash X_C \) there is a chamber \( K \vdash C \) such that \( S(K, C) \cap \operatorname{supp}(Y) = \emptyset \).

Proof. Clear.

The next lemma shows the point of the above definitions: the \( X_C \) actually describe in very compact way the strata \( N(C) \) of Definition 4.1.

Lemma 4.20. Let \( \mathcal{M} \) denote the oriented matroid associated to a real, linear arrangement \( A \), choose a base region \( B \in T(A) \) and a linear extension \( \vdash \) of \( T_B(A) \), and recall Definition 4.7. Then
\[ N(C) \simeq \mathcal{F}(\mathcal{M}/X_C). \]

Proof. By definition \( N(C) = \{ (F, C) \in \mathcal{S}_C \mid C_F \neq K_F \text{ for all } K \vdash C \} \). Since the order is induced by \( \mathcal{S}_C \), we only have to prove equality of sets.

The right-to-left inclusion is easy. Indeed, if \( F \in \mathcal{F}(\mathcal{M}/X_C) \), then \( S(C_F, K) \cap \operatorname{supp}(F) = S(C, K) \cap \operatorname{supp}(F) \) for all \( K \). By Lemma 4.19(1), for all \( K \vdash C \) we have \( S(C, K) \cap \operatorname{supp}(F) \neq \emptyset \), and thus \( C_F \neq K_F \). For the other direction, suppose \( (F; C) \in N(C) \setminus \mathcal{F}(\mathcal{M}/X_C) \),

so that \( F < F' \) in \( \mathcal{F}^{\text{op}} \), hence \( |F'| < X_C \). Then by Lemma 4.19(2) there is \( K \vdash C \) with \( S(C, K) \cap \operatorname{supp}(F) = \emptyset \), and thus \( K_F = C_F \): a contradiction.

Now we can apply the preceding work to construct a family of maximum acyclic matchings of the Salvetti complex.

Proposition 2. Let \( A \) be an arrangement of linear hyperplanes in real space and fix any \( B \in T(A) \). To every linear extension of \( T_B(A) \) corresponds a family of acyclic maximum matchings of the associated Salvetti complex \( S(\mathcal{M}_A) \) which critical cells are in natural bijection with the chambers of \( A \).
Figure 4.2. The Salvetti complex for the arrangement of three lines in the plane, “assembled” by attaching the top cells to the 1-skeleton along the linear extension of the tope poset that was described in Example 3.7 (see also Figure 3.1 and 4.1). The shaded regions represent the “contributions to homotopy” that every top cell gives to the total complex.

Proof. Let $\vdash$ denote a linear extension of the ordering $\prec_B$ of $T_B$ and recall Definition 4.1.

We will prove recursively that every poset $S(C)$ possesses a maximum acyclic matching with as many critical cells as there are chambers $C' \vdash C$. 

For \( S(B) \) this follows from Theorem 3.6, so let the claim hold for a chamber \( C \vdash B \). We have to find an acyclic matching of the ‘new’ part \( N(C) \).

For any chamber \( K \) let
\[
N(C, K) := S_C \setminus S_K = \{ \langle F, C \rangle \in S_C \mid C_F \neq K_F \}.
\]

Clearly \( N(C) = \bigcap_{K \vdash C} N(C, K) \), and thus, with every \( N(C, K) \), also \( N(C) \) is an upper ideal in \( S(C) \). Since by Lemma 4.20 \( N(C) \) is the face poset of an oriented matroid, with Theorem 3.6 we have an acyclic matching of \( N(C) \). These matchings can be pasted together to give a matching of the whole \( S \). The acyclicity of the ‘patchwork-matching’ can be shown with Lemma 2.5 by considering the linear extension of \( S \) given by the concatenation of the linear extensions of the \( N(C) \)s so that an element of \( N(C_1) \) comes after an element of \( N(C_2) \) whenever \( C_1 \vdash C_2 \) (for a precise proof see the more general statement of [19, Theorem 11.10] on ‘patchwork of acyclic matchings’).

By Theorem 3.6, the shelling induced on \( N(C) \) has only one homology cell, and thus the corresponding acyclic matching has exactly one critical element. With the ‘pigeon hole principle’ we now see that the obtained ‘global’ acyclic matchings on \( S \) are in fact maximum acyclic matchings: indeed, the number of critical elements

\[
\langle P, C_1 \rangle \prec \langle P, C_2 \rangle \prec \langle P, C_3 \rangle \prec \langle P, C_4 \rangle \prec \langle P, C_5 \rangle \succ \langle P, C_6 \rangle.
\]

The stratification corresponds to the one of Figure 4.2. Note that the induced shelling-type ordering of Example 2.14 translates into:

\[
C_1 \prec F_1 \prec F_2 \prec C_2 \prec F_6 \prec C_3 \prec F_5 \prec C_4 \prec F_4 \prec C_5 \prec F_3 \prec C_6 \prec P.
\]

On each stratum we depict the associated acyclic matching by thickening the edges of the matching. The resulting critical cells are enclosed into boxes.
and the number of generators in homology both equal the cardinality of the family of the no broken circuit sets (see e.g. [17]).

Remark 4.21. The matchings of the previous proposition are obtained by pasting together acyclic matchings for the different \( N(C) \)s. In principle, any choices of acyclic maximum matchings of the \( N(C) \)s can be pasted together. But since it is easy to see that a shelling-type ordering of a locally ranked poset restricts to a shelling-type ordering of any of its lower ideals, we can construct the whole matching keeping the freedom of choice to a minimum: it is possible to give an explicit description of the critical elements of the matching induced on \( S \) by the choice of a base chamber \( B \), of a linear extension \( \sqsubseteq \) of \( T_B \), and of maximal chains \( \omega_C \) in \( [B, -C] \) for all \( C \in T \); the critical point added with \( N(C) \) is \( \langle F(C), C \rangle \),

\[
F(C) := \max_{\sqsubseteq, t(C)} \{ F' \in F \mid |F'| = X_C \},
\]

where \( \sqsubseteq \) is the shelling-type ordering induced on \( F^{op} \) and \( t(C) \) is the rank (i.e., the codimension) of \( X_C \).

5. No broken circuits and critical elements

In this last section we want to relate our construction to no-broken-circuit sets. It is not easy to track back the origin of these widely studied combinatorial objects that can be defined for every geometric lattice; let us here mention just [10, 5] as ‘early references’. We only recall that they give a basis for the Whitney homology of the associated geometric lattice (see [2, 5]) and, in the context of arrangements of hyperplanes, the no-broken-circuit sets of size \( k \) index a basis of the \( k \)-th degree of the Orlik-Solomon algebra (see e.g. [20, 16] and the textbook [21]), which is known to be isomorphic to the (integral) cohomology algebra of the arrangement’s complement [20]. For a comprehensive and very readable account of these objects, and for more bibliography, see the survey of Yuzvinsky [27].

We will continue our ‘geometric’ treatment of the subject and, as above, leave to the interested reader the translation into the language (and the strength) of abstract oriented matroids.

Definition 5.1. (no-broken-circuit sets) Translating the classical definition for matroids, a circuit of \( A \) is a minimal set \( C \) of hyperplanes such that every \( H \in C \) contains the intersection of the other elements of \( C \). In particular, for every \( H \in C \) the set \( C \setminus \{H\} \) is minimal with the property that the intersection of its hyperplanes equals \( \bigcap \{H\} \). If a linear ordering of the set of hyperplanes is given, a broken circuit is a subset \( B \subset A \) that can be written as \( C \setminus \{H\} \), where \( H \) is the minimal element of \( C \) in the chosen total order.

A no-broken-circuit set, also called simply nbc set, is an independent subset of \( A \) that contains no broken circuit, or the empty set. It is clear that the nbc sets give a simplicial complex, denoted \( \nbc(A) \), on the ground set \( A \). Note that we formally consider also the simplex of dimension \(-1\) given by the empty set - thus, \( \emptyset \in \nbc(A) \) for all \( A \).

Example 5.2. For the arrangement \( A \) of three lines in the plane, with the lattice depicted on the right of Figure 4.2, we have only one circuit, namely \( \{H_1, H_2, H_3\} \), and thus we get

\[
\nbc(A) = \{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \{H_1, H_2\}, \{H_1, H_3\} \}.
\]

A corresponding notion exists for arbitrary geometric lattices (i.e., for arbitrary matroids): the interested reader is referred to [5].
It is important to point out that, for technical reasons, our definitions differ from those of [17] in that our broken circuits fail to contain a minimal (instead of a maximal) element. The other definitions are then adapted to this change.

Before to state the main definitions, let us fix some notation that will accompany us through the remainder of this paper.

**Notation 5.3.** We keep the conventions of the Important Remark 4.8 but now, in addition, we suppose a linear ordering \( \{H_1, \ldots, H_n\} \) to be given on the set of hyperplanes. For the moment no special requirements are made on this ordering.

We will write 
\[ A_j := \{H_1, \ldots, H_j\} \text{ for } 1 \leq j \leq n, \quad A' := A_{n-1}, \quad A'' := A^{H_n}, \]
where \( A^{H_n} = \{H \cap H_n \mid H \in A'\} \), according to the Notation 4.6. Clearly every \( A_j \) inherits the ordering from \( A \). Moreover, there is a canonical ordering of \( A^{H_n} \) obtained by numbering every element \( L \in A'' \) according to the ‘smallest’ hyperplane \( H(L) \in A \) in which it is contained. As above, every \( C \in T(A) \) is contained in exactly one chamber of \( A' \), that we will denote by \( C' \). Thus, \( B' \) is the only chamber of \( A' \) that contains the base chamber \( B \) of \( A \).

For every \( H \in A \) let \( H^+ \) denote the closed halfspace that is bounded by \( H \) and contains \( B \). Clearly \( B = \bigcap_{H \in A} H^+ \) and \( B' = \bigcap_{H \in A'} H^+ \). More generally, there is a canonical choice of a base region \( B_j \) for \( A_j \); we define \( B_j := \bigcap_{i \leq j} H_i^+ \). Thus, \( B' \) is the only chamber of \( A' \) that contains the base chamber \( B \) of \( A \).

The last requirement on \( H_n \) is necessary to ensure that the intersection defining \( B'' \) has indeed maximal dimension inside \( H_n \). It is clear that with this hypothesis
\[ B'' = B' \cap H_n. \]

We will need this property to hold inductively: this is the motivation of the following definition.

**Definition 5.4** (Cut property). A total ordering \( \{H_1, \ldots, H_n\} \) of \( A \) satisfies the cut property with respect to the base chamber \( B \) if, for every \( j = 2, \ldots, n \), \( H_j \) intersects the interior of \( B_{j-1} \) (we will say: \( H_n \) cuts \( B_{j-1} \)).

We need to check that an ordering with this property exists. The next Lemma explains that those orderings correspond to known objects. Namely: maximal chains in the poset of regions.

**Lemma 5.5.** An ordering \( \{H_1, \ldots, H_n\} \) of the hyperplanes of an arrangement \( A \) satisfies the cut property if and only if there is a maximal chain
\[ B = C_0 \prec C_1 \prec \ldots \prec C_n = -B \]
in \( T_B(A) \) such that \( S(C_{i-1}, C_i) = \{H_i\} \) for all \( 1 \leq i \leq n \).

**Proof.** Clear. \( \square \)

We see that every arrangement can be ordered so to satisfy the cut property (for example, the ordering of the hyperplanes in figure 4.1 satisfies the cut property). Indeed, Definition 5.3 turns out to describe the property we were seeking for.

**Remark 5.6.** If the ordering \( A = \{H_1, \ldots, H_n\} \) satisfies the cut property with respect to the chamber \( B \), then for every \( j = 1, \ldots, n \) there is a canonical choice of a base region in \( (A_j)'' \):
\[ B'' := H_j \cap B_{j-1}. \]
Moreover, the induced ordering of \( (A_j)'' \) satisfies the cut property with respect to \( B''_j \).
Definition 5.7. Let $\mathcal{A} := \{H_1, \ldots, H_n\}$ be ordered such that $H_n \in \mathcal{W}_B$. With the Notations of 5.3 we define:

$$T := T_B(A), \quad T' := T_{B'}(A'), \quad T'' := T'_{B''}(A'').$$

Moreover, let $\mathcal{B}'$ (or $\mathcal{B}'(A)$ if specification is needed) denote the set of all chambers of $\mathcal{A}'$ that are 'cut' by $H_n$. Every $C' \in \mathcal{B}'$ contains therefore two chambers $C^1 \prec_B C^1$ of $T$. Define

$$\mathcal{B}^1 := \{C^1 \mid C \in \mathcal{B}\}, \quad \mathcal{B}^1 := \{C^1 \mid C \in \mathcal{B}\},$$

$$\mathcal{U} := T' \setminus \mathcal{B}', \quad \mathcal{B}''' := \{H_n \cap C \mid C \in \mathcal{B}'(A)\}.$$

Remark 5.8. Clearly,

$$T = \mathcal{U} \cup \mathcal{B}^1 \cup \mathcal{B}^1, \quad T' = \mathcal{U} \cup \mathcal{B}', \quad T'' = \mathcal{B}'',$$

with the evident order preserving bijections:

$$\beta' : \mathcal{B}' \rightarrow \mathcal{B}^1, \quad \beta'' : \mathcal{B}^1 \rightarrow \mathcal{B}''.$$

We want to describe a particular linear extension of $T$ that allows us to explicitly index the critical elements of the associated acyclic matchings with the no broken circuit sets of the arrangement. We will make use of an indexing of the chambers of $\mathcal{A}$ by $\text{nbc}$ sets that is inspired by a result of Jewell and Orlik [17].

Definition 5.9 (see Section 3.4 of [17]). Consider an ordering $\mathcal{A} = \{H_1, \ldots, H_n\}$ that satisfies the cut property with respect to the chamber $B$ and keep the notations introduced above. We define a map

$$\eta : T_B(A) \rightarrow \mathcal{P}(\mathcal{A})$$

recursively in the number of elements of $\mathcal{A}$ as follows:

- If $\mathcal{A} = \{H_1\}$, let $\eta_1(H_1^+) := \emptyset$ and $\eta_1(-H_1^+) := \{H_1\}$.
- Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $n > 1$ and suppose we are able to define such functions for every arrangement of cardinality at most $n - 1$. In particular the functions $\eta'$ and $\eta''$ associated to $\mathcal{A}', \mathcal{A}''$ are defined. Then, for $C \in T(\mathcal{A})$ we define

$$\eta(C) := \left\{ \begin{array}{ll}
\eta'(C) & \text{if } C \in \mathcal{U} \cup \mathcal{B}^1, \\
\min\{H \in \mathcal{A} \mid H \cap H_n = L\} \mid L \in \eta''(\beta''(C)) & \text{if } C \in \mathcal{B}^1
\end{array} \right.$$

where we slightly abused notation in implicitly identifying $T'$ with $\mathcal{U} \cup \mathcal{B}^1$ using the bijection $\beta'$ of Definition 5.3.

In particular, for $C \in \mathcal{B}''(A)$ we have $\eta(C^1) = \eta'(C)$ and a natural bijective correspondence between $\eta(C^1)$ and $\eta''(C \cap H_n) \cup \{H_n\}$. The map $\eta$ was introduced in [17] as a bijection between no-broken circuit sets and chambers of the arrangement, as we state in the following lemma.

Lemma 5.10 (see Lemma 3.14 of [17]). The map $\eta$ is a bijection $T(\mathcal{A}) \rightarrow \text{nbc}(\mathcal{A})$ with $\eta(B) = \emptyset$.

As a first step let us prove a technical property that derives from our particular choice of the ordering of the hyperplanes.

Lemma 5.11. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of linear real hyperplanes and $B$ a chamber of $\mathcal{A}$. Suppose that the ordering of the hyperplanes satisfies the cut property with respect to $B$. Then

$$\bigcap \eta'(C) \cap H_n = \bigcap \eta''(C \cap H_n) \quad \forall C \in \mathcal{B}'(A).$$
is easy to see that if $C$ because it holds for $\eta$ that the order induced on $B$ let us denote by $\nu, \nu'$ where $\nu'' = \nu'$.

**Proof.** Again, we argue recursively on the number of hyperplanes of $A$. If $A = \{H_1\}$ there is nothing to prove. So let $A = \{H_1, \ldots, H_n\}$ with $n > 1$ and suppose that the ordering satisfies the cut property with respect to the chamber $B$. Let $A := A \setminus \{H_{n-1}\}$. Clearly the induced ordering on $A$ satisfies the cut property with respect to the chamber $B := \bigcap_{j \neq n-1} H_j^+$ and thus, by induction, the claim holds and ensures
\[
\bigcap \eta''(C) \cap H_n = \bigcap \eta''(C \cap H_n) \quad \forall C \in B'(A).
\]

Also, the induction hypothesis applies to the arrangement $A''$ with respect to the induced order and the chamber $B'' = B \cap H_n$; thus, if we define $L := H_n \cap H_{n-1}$, when there is no $j < n - 1$ with $H_j \supset L$ we can write
\[
\bigcap \nu''(C) \cap L = \bigcap \nu''(C \cap L) \quad \forall C \in B'(A''),
\]
where $\nu, \nu', \nu''$ are the maps obtained by applying Definition 5.9 to $A''$. Finally, let us denote by $\mu, \mu', \mu''$ the maps associated to $A' = \{H_1, \ldots, H_{n-1}\}$. We know that the order induced on $A'$ satisfies the cut property with respect to the unique chamber $B' \supset B$ and thus, by induction,
\[
\bigcap \mu'(C) \cap H_{n-1} = \bigcap \mu''(C \cap H_{n-1}) \quad \forall C \in B'(A').
\]

We would like to point out the following (tautological) relations:
\[
\mu = \eta', \quad \eta' = \mu', \quad \eta'' = \nu', \quad \nu = \eta''.
\]

Now we proceed with the proof. Let $A$ be as above, and choose $C \in B'(A)$. It is easy to see that if $C \subset H_{n-1}^+$ or if $H_{n-1}$ is not a wall of $C$, then the claim holds because it holds for $A$.

So suppose that $H_{n-1}$ is a wall of $C$ and that $C \not\subset H_{n-1}^+$. Then we have
\[
\eta'(C) = \mu(C) = \{H_{n-1}\} \cup \mu''(C \cap H_{n-1})
\]
and
\[
\eta''(C \cap H_n) = \begin{cases} \eta''(C \cap H_n) & \text{if there is } j < n - 1 \text{ with } L \subset H_j, \\ \{L\} \cup \nu''((C \cap H_n) \cap H_{n-1}) & \text{else.} \end{cases}
\]

**Figure 5.1.** The last step in the inductive construction of $\eta$ for the arrangement given on the left of Figure 4.1, where we see that $B^+ = \{C_3, C_5\}$, $B^- = \{C_2, C_4\}$, $U = \{C_1, C_6\}$. For every chamber $C$, the set $\eta(C)$ is written inside $C$ to show the bijective correspondence.
Moreover, we can write
\[
\bigcap \eta'(C) \cap H_n = \bigcap \left[ \{H_{n-1} \} \cup \mu''(C \cap H_{n-1}) \right] \cap H_n = \bigcap \mu'(C) \cap H_{n-1} \cap H_n = \bigcap \eta''(C) \cap H_n \cap H_{n-1} = \bigcap \eta''(C \cap H_n) \cap H_{n-1}.
\]

Since we know that \( H_{n-1} \subset \eta'(C) \), this implies \( \bigcap \eta'(C) \cap H_n = \bigcap \eta''(C \cap H_n) \).

To conclude the proof we distinguish two cases:

Case 1. If there is \( j < n - 1 \) with \( L \subseteq H_j \), the claim follows immediately, because then \( \eta''(C \cap H_n) = \eta''(C \cap H_n) \).

Case 2. If there is no such \( j \), then the induction hypothesis applies to \( \nu \) and gives
\[
\bigcap \eta''(C \cap H_n) \cap H_{n-1} = \bigcap \nu'(C \cap H_n) \cap H_{n-1} = \bigcap \nu''(C \cap H_n \cap H_{n-1}) = \eta''(C),
\]
where the last inequality holds because every element of \( \nu''(C \cap H_n \cap H_{n-1}) \) is contained in \( L \).

Thus, in any case the claim holds.

Now the idea is to consider a linear extension that behaves well under ‘taking \( A' \) and \( A'' \).

**Definition 5.12.** For every \( H \subseteq A \) let \( H^+ \) denote the open halfspace that is bounded by \( H \) and contains the base chamber \( B \). To every \( C \subseteq T \) we associate an array \( \sigma(C) := (\sigma_1(C), \ldots, \sigma_n(C)) \) by setting \( \sigma_i(C) = 0 \) if \( C \subseteq H_i^+ \), and \( \sigma_i(C) = 1 \) else.

We denote by \( \vec{\eta}^f \) (or \( \vec{\eta}_{A,B} \) when specification is needed) the total order on \( T \) induced by the lexicographic ordering of the corresponding arrays.

**Example 5.13.** The linear extension of example 5.7 translates into

\[
(0, 0, 0) \triangleright (0, 0, 1) \triangleright (1, 0, 0) \triangleright (0, 1, 1) \triangleleft (1, 1, 1)
\]

and is therefore \( \vec{\eta}_{A} \) for the arrantement \( A \) of Figure 4.1.

**Remark 5.14.** In the language of oriented matroids the above definition just fixes the acyclic orientation associated with the tope \( B \) and then associates to every tope its signed covector.

**Lemma 5.15.** The ordering \( \vec{\eta}_{A,B}^f \) is a linear extension of \( T_B(A) \), and satisfies:

1. the ordering of \( T' \) induced via the maps \( \delta, \beta', \gamma \) is \( \vec{\eta}_{A',B'}^f \).
2. the ordering of \( T'' \) induced via the map \( \beta'' \) is \( \vec{\eta}_{A'',B''}^f \).

**Proof.** We have to show that if \( C \prec_B C' \), then \( C \prec_{A} C' \). But the former means \( S(B, C) \subseteq S(B, C') \); thus, \( \sigma(C') \) is obtained from \( \sigma(C) \) by switching from 0 to 1 the entries corresponding to the elements of \( S(C, C') \), and \( \vec{\eta}^f \) is therefore a linear extension. Item (1) is easy to see. For (2), recall that every hyperplane of \( A'' \) corresponds to a codimension 2 subspace of \( A \) and gets the number of the smallest \( i < n \) such that \( H_i \) contains the subspace.

The next step will be to prove that the critical cells of the acyclic matching of Proposition 2 are completely determined by the associated chamber, provided that the chosen linear extension is the one associated via Definition 4.16 to an ordering of the hyperplanes that satisfies the cut property.

We will show that, for every base chamber \( B \) and every ordering of \( A \) satisfying the cut property with respect to \( B \), \( \eta(C) \) is a basis of the flat \( X_C \) if the chosen linear extension of \( T_B(A) \) is the one of Definition 5.12.
Theorem 5.16. Let the ordering \( \{H_1, \ldots, H_n\} \) of \( A \) satisfy the cut property with respect to the chamber \( B \) and consider the linear extension \( \gamma^d \) of \( T_B \). We have

\[
X_C = \bigcap \eta(C).
\]

Proof. Again, the claim is trivial if \( A = 1 \). So let \( n := |A| > 1 \) and suppose that the claim holds for every arrangement of at most \( n - 1 \) hyperplanes (and thus, in particular, for \( A' \) and \( A'' \)).

Given \( C \in T(A) \), let

\[
Y_C := \bigcap \eta(C).
\]

We are going to prove that \( Y_C \) satisfies \( \text{Lemma } 4.13 \) (1) and \( \text{Lemma } 4.13 \) (2).

It is easily seen that this is true if \( H_n \in S(B, C) \), because the above properties hold for \( A' \) and depend only on the position of the flat with respect to the union of the chambers \( K \) that come before \( C \). In fact, the chosen linear extension is such that the union of all \( K \vdash^d C \) equals (as a subset of \( \mathbb{R}^d \)) the union of the chambers that come before \( C' \) with respect to the ordering \( \prec^d_{A', B'} \) (recall that \( C' \) is the unique chamber of \( A' \) containing \( C \)).

So let \( C \in B^1 \) and recall that by definition we have

\[
\eta(C) = \{H_n\} \cup \left\{ \min \{H \in A \mid H \cap H_n = L) \mid L \in \eta''(C \cap H_n) \right\}.
\]

We now have to check the properties of Definition 4.19.

\( \text{Lemma } 4.14 \) (1): \( \supp(Y_C) \cap S(C, K) \neq \emptyset \) for all \( K \vdash^d C \).

This assertion is clear if \( H_n \in S(B, K) \), since then \( H_n \in S(C, K) \cap \supp(Y_C) \). On the other hand, if \( H_n \not\in S(B, K) \cdot \emptyset \), then we know that \( S(C, K) \cap \supp(\bigcap \eta'(C')) \neq \emptyset \) by induction hypothesis. But Lemma 5.11 allows us to write

\[
Y_C = \bigcap \eta(C) = \bigcap \eta''(C \cap H_n) = \bigcap \eta'(C) \cap H_n,
\]

whence \( \supp(Y_C) \supseteq \supp(\bigcap \eta'(C)) \), and the claim follows.

\( \text{Lemma } 4.14 \) (2): For every flat \( Z \geq Y_C \) in \( L(A) \) there is a chamber \( K \vdash^d C \) such that \( \supp(Z) \cap S(C, K) = \emptyset \).

Clearly if \( H_n \not\in \supp(Z) \), we are easily done by taking \( K = (C')^1 \) so that \( S(C, K) \) is the unique \( K \vdash^d C \). We are left with the case where \( H_n \in \supp(Z) \). Then \( Z \not\geq \bigcap \eta''(C'') \) in \( L(A') \) - recall Lemma 5.11 - and that \( C'' := C' \cap H_n \). And by induction hypothesis we know that there is \( K'' \vdash^d_{A', B''} C'' \) with no hyperplane of \( A'' \) containing \( Z \) and separating \( K'' \) from \( C'' \). Now let \( K \) be the chamber of \( A \) that is ‘just above’ (or: the preimage with respect to \( \beta'' \) of) \( K'' \) (so that \( K \vdash^d C \) by Lemma 5.11). For every \( H \in S(C, K) \), \( H \cap H_n \) separates \( C'' \) from \( K'' \) in \( A'' \). Thus, if there were \( H \in \supp(Z) \cap S(C, K) \), then there would be \( L := H \cap H_n \in \supp''(Z) \) separating \( C'' \) from \( K'' \) (where \( \supp''(Z) \) is naturally defined as \( \{L \in A'' \mid Z \subset L\} \) - a contradiction.

We can now summarize our results leaving the greatest generality in the attempt to approach the greatest naturality. The proof is an easy combination of Proposition 2.1 Theorem 5.16 Remark 4.21 and Corollary 4.17.

Proposition 3. Let \( A \) denote a real arrangement of linear hyperplanes and choose a chamber \( B \in T(A) \). Every ordering of \( A \) that satisfies the cut property with respect to \( B \) gives rise to a bijection \( \eta \) between chambers and nbc-sets as in Definition 7.9 and to an acyclic matching of the Salvetti complex which critical cells are precisely those of the form

\[
\{ \bigcap \eta(C) \cap C, C \}.
\]

In particular, the resulting CW-complex has one cell of dimension \( |\eta(C)| \) for every \( C \in T(A) \).
Example 5.17. By comparing Figure 4.1 with Figures 4.2, 4.3 and 5.1 one sees immediately the claimed correspondence:

\[ \eta(C_1) = \emptyset, \quad \bigcap \emptyset = \mathbb{R}^d = \emptyset = X_{C_1}, \quad \mathbb{R}^d \cap C_1 = C_1, \quad \langle C_1, C_1 \rangle \text{ is critical}; \]
\[ \eta(C_2) = \{H_3\}, \quad \bigcap \{H_3\} = H_3 = X_{C_2}, \quad H_3 \cap C_2 = F_1, \quad \langle F_1, C_2 \rangle \text{ is critical}; \]
\[ \eta(C_3) = \{H_2\}, \quad \bigcap \{H_2\} = H_2 = X_{C_3}, \quad H_2 \cap C_3 = F_6, \quad \langle F_6, C_3 \rangle \text{ is critical}; \]
\[ \eta(C_4) = \{H_1\}, \quad \bigcap \{H_1\} = H_1 = X_{C_4}, \quad H_1 \cap C_1 = F_2, \quad \langle F_2, C_4 \rangle \text{ is critical}; \]
\[ \eta(C_5) = \{H_1, H_3\}, \quad H_1 \cap H_3 = P = X_{C_5}, \quad P \cap C_5 = P, \quad \langle P, C_5 \rangle \text{ is critical}; \]
\[ \eta(C_6) = \{H_1, H_2\}, \quad H_1 \cap H_2 = P = X_{C_6}, \quad P \cap C_6 = P, \quad \langle P, C_6 \rangle \text{ is critical}; \]

and there are no further critical cells.

Remark 5.18. The importance of the chambers in the above characterization of the critical cells is mainly to give the order along which we decompose the Salvetti complex. It is now natural to ask if such ordering can be defined purely in terms of the no-broken-circuit sets. This would actually allow to describe the situation without referring to the geometry of \( \mathbb{R}^d \). However, this task might be particularly subtle: for instance, compare the arrangement of Coxeter type \( A_2 \) and the coordinate arrangement in \( \mathbb{R}^3 \) (let us call it \( K_3 \)). Up to symmetry, in both cases there is only one linear ordering induced on the families of no-broken-circuit sets:

\[ A_2: \emptyset, \{3\}, \{2\}, \{1\}, \{1, 2\}, \{1, 3\}, \]
\[ K_3: \emptyset, \{3\}, \{2\}, \{2, 3\}, \{1\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\} \]

(where we wrote \( j \) for \( H_j \)) and we see that \( \{1, 2\} \) and \( \{1, 3\} \) are switched in the two orderings. This seems to indicate that one should consider also some `global' property of the lattice, other than just examining the no-broken-circuit sets.

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