Area estimates
for two-dimensional immersions of mean curvature type
in Euclidean spaces of higher codimension

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Abstract
We establish area bounds for two-dimensional immersions in \( \mathbb{R}^3 \) and \( \mathbb{R}^n \). Namely, for \( \mu \)-stable immersions in \( \mathbb{R}^3 \) (\( \mathbb{R}^n \)), for graphs in \( \mathbb{R}^3 \) which solve quasilinear equations in divergence form, and for graphs which are critical for Fermat-type variational problems in \( \mathbb{R}^n \).

MCS 2000: 35J60, 53A07, 53A10

Keywords: Twodimensional immersions, higher codimension, area estimates

1 Introduction

1.1 The results
In this paper we prove area bounds for the following types of surfaces:

1. \( \mu \)-stable immersions of prescribed mean curvature-type in \( \mathbb{R}^3 \) (\( \mathbb{R}^n \)) in terms of a suitable stability constant and of curvature terms (chapter 2, section 2.3, 2.5, 2.6);

2. graphs \((x, y, \zeta(x, y))\) of mean curvature type in \( \mathbb{R}^3 \), which are solutions of non-homogeneous divergence form equations (chapter 3, section 3.2);

3. graphs \((x, y, \zeta_1(x, y), \ldots, \zeta_{n-2}(x, y))\) in \( \mathbb{R}^n \), \( n \geq 3 \), which are critical for Fermat-type variational problems (chapter 4, section 4.3).

For example, area bounds are crucial for compactness results (see e.g. [7] for such results concerning weighted minimal surfaces, see section 2.2 below), but also for various gradient and curvature estimates for nonlinear differential systems: Here, we mention to [3], where such a bound for Fermat-type graphs in \( \mathbb{R}^n \) was left unproved, further [12]-[16], [20], [21], where area bounds are used for various curvature estimates, and, finally, [17] for curvature estimates of \( n \)-manifolds in \( \mathbb{R}^n \) using techniques from [9], [23].

We will not discuss isoperimetric inequalities. But we mention that such inequalities are established e.g. in [7] for immersed critical points of elliptic variational problems (2.8) using Fourier series methods, or in [9] (and references therein) for mean curvature immersions using a generalized Sobolev inequality. We want to extend this later method in a further paper to prove other area bounds for immersions of mean-curvature type (see the discussion in section 2.2).

2 \( \mu \)-stable geodesic discs of mean curvature-type

In the first part of this note we consider immersions

\[ X = X(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v)) \in C^3(B, \mathbb{R}^3) \]  (2.1)
on the closed unit disc $B := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$ such that rank $\partial X = 2$ in $B$ for its Jacobian $\partial X \in \mathbb{R}^{3 \times 2}$. The unit normal vector of $X$ is defined as

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} \quad \text{in } B$$

(2.2)

with the partial derivatives $X_u$ and $X_v$ of $X$, and $\times$ means the usual vector product in $\mathbb{R}^3$.

In the next two sections we specify the class of immersion we will deal with.

2.1 First step: Introduction of weighted metrics

We equip the immersions with a weighted metric of Finsler type: Let us given a symmetric and positive definite weight matrix $G(X,Z) \in C^2(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}, \mathbb{R}^{3 \times 3})$ (2.3)

with the properties: For all $(X,Z) \in \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}$ there hold

(G1) $G(X,Z) = G(X,\lambda Z)$ for all real $\lambda > 0$;

(G2) $G(X,Z) \circ Z^t = Z^t$, where the upper $t$ denotes transposition;

(G3) $(1 + g_0)^{-1} |\xi|^2 \leq \xi \circ G(X,Z) \circ \xi^t \leq (1 + g_0) |\xi|^2$ for all $\xi \in \mathbb{R}^3$, with real $g_0 \in [0, +\infty]$;

(G4) $\det G(X,Z) = 1$.

Now, let $X$ be an immersion with Gauss map $N$, and let $G(X,Z)$ be a weight matrix as above. We define the components $h_{ij}$ of the weighted first fundamental form of $X$ and the associated line element $ds_g$ as (use the summation convention, and set $u^1 \equiv u$, $u^2 \equiv v$)

$$h_{ij} := X_{u^i} \circ G(X,N) \circ X_{u^j}^t, \quad ds_g^2 = h_{ij} du^i du^j.$$ (2.4)

2.2 Second step: Definition $\mu$-stability

We want to prove an area estimate for so-called $\mu$-stable immersions in $\mathbb{R}^3$:

**Definition.** The immersion $X$ is called $\mu$-stable with real $\mu > 0$ and a function $q \in C^1(B, \mathbb{R})$ iff

$$\int_B \nabla_{ds_g^2}(\varphi, \varphi) W \, du dv \geq \mu \int_B (q - K) W \varphi^2 \, du dv \quad \text{for all } \varphi \in C_0^\infty(B, \mathbb{R})$$ (2.5)

with a weighted metric $ds_g^2$ from (2.4), where $q - K \geq 0$ in $B$ with the Gaussian curvature $K$ of the surface, and where

$$\nabla_{ds_g^2}(\varphi, \psi) := h^{ij} \varphi_{u^i} \psi_{u^j}, \quad h_{ij} h^{jk} = \delta^k_i,$$ (2.6)

is the Beltrami operator w.r.t. $ds_g^2$, $\delta^k_i$ is the Kronecker symbol.

**Examples.**

In the calculus of variations we are faced with “weighted” and “unweighted” problems:

1. A conformally parametrized surface of constant mean curvature $h_0 \in \mathbb{R}$ (as a critical point of the area functional with a suitable volume constraint) is stable iff

$$\int_B |\nabla \varphi|^2 \, du dv \geq 2 \int_B (2h_0^2 - K) W \varphi^2 \, du dv \quad \text{for all } \varphi \in C_0^\infty(B, \mathbb{R}),$$ (2.7)
that is, it is $\mu$-stable with $\mu = 2$ and $q \equiv 2h_0^2$. We have $q \equiv 0$ for minimal surfaces. Furthermore, in this case $G(X, Z) \equiv E^3$ with the three-dimensional unit matrix $E^3 \subset \mathbb{R}^{3 \times 3}$, such that $\nabla ds^2(\varphi, \varphi) = \frac{1}{\mu} |\nabla \varphi|^2 = \frac{1}{\mu} (\varphi_\mu^2 + \varphi_v^2)$ using conformal parameters, and where $ds^2$ stands for the non-weighted line element.

2. Critical points $X$ of variational problems

$$\iint_B F(X, X_u \times X_v) \, du dv \longrightarrow \text{extr!} \quad (2.8)$$

are immersions of mean-curvature type, that is, they solve

$$\nabla ds^2_g(X, N) = -2H_g(X, N) = -\frac{\text{trace} \, F_{XZ}(X, N)}{\sqrt{\det F_{ZZ}(X, N)}}, \quad (2.9)$$

where $F_{XZ} = (F_{x^i z^j})_{i,j=1,2,3} \in \mathbb{R}^{3 \times 3}$ etc., with the weighted mean curvature $H_g(X, Z)$ w.r.t.

$$G(X, Z) = \left( \frac{F_{ZZ}(X, Z)}{\sqrt{\det F_{ZZ}(X, Z)}} + (z^i z^j)_{i,j=1,2,3} \right)^{-1}, \quad (2.10)$$

and $ds^2_g$ chosen as in (2.4). This weight matrix was first introduced in [20]. For example, assume that the integrand in (2.8) has the form $F = F(Z)$. If for a critical point (a so-called G-minimal surface, $H_g(X, N) \equiv 0$) the second variation is non-negative, then it can be shown (see [12])

$$\iint_B \nabla ds^2_g(\varphi, \varphi)W \, du dv \geq \mu \iint_B (-K)W \varphi^2 \, du dv \quad \text{for all } \varphi \in C^\infty_0(B, \mathbb{R}), \quad (2.11)$$

that is, a critical point is $\mu$-stable with $q \equiv 0$ and a suitable $\mu > 0$.

Independent of the theory of the second variation, various stability criteria were developed by analysing spherical properties of the immersions:

3. For example, stability for minimal surfaces [1], for surfaces of prescribed constant mean curvature [19], [13], for F-minimal surfaces [6], or for weighted minimal surfaces [12].
4. In [2] the reader can find stability criteria for minimal surfaces in the three-sphere $S^3$, in the hyperbolic space $H^3$, and in the Euclidean space $\mathbb{R}^n$. This last result was improved for minimal graphs with flat normal bundle in [16].

2.3 An estimate for the area growth of geodesic discs

Using methods which go back to [18] and [20] we prove the following area bound:

**Theorem.** Let the immersion $X$ be $\mu$-stable in the sense of (2.5), such that

$$\mu > \frac{1 + g_0}{2} \quad \text{and} \quad q \geq 0 \quad \text{in } B. \quad (2.12)$$

Let it represent a geodesic disc $\mathcal{B}_r(X_0)$ of radius $r > 0$ and center $X_0 \in \mathbb{R}^3$. Then

$$\mathcal{A}[X] \leq \frac{2 \pi \mu}{2 \mu - (1 + g_0)} r^2 \quad (2.13)$$

for the area $\mathcal{A}[X]$ of the immersion.
Remarks. 1. The assumption \( q \geq 0 \) in \( B \) is needed in the estimate (2.21). If it is not fulfilled, we could proceed in (2.21) with \( q^-(u,v) := \min \{ q(u,v), 0 \} \). It would follow

\[
\mathcal{A}[X] \leq \frac{2\mu}{2\mu - (1 + g_0)} r^2 - \frac{\mu r^2}{2\mu - (1 + g_0)} \int_B q^-(u,v) W(u,v) dudv.
\]  

(2.14)

2. The proof of the theorem uses intrinsic methods. Therefore, it could be extended to \( \mu \)-stable immersions in Euclidean spaces \( \mathbb{R}^n \) for \( n \geq 3 \). But up to now we are not able to transform critical points of general elliptic variational problems in spaces of higher codimension into a weighted form as given in (2.9), (2.10) (see also the remarks in section 4.1).

3. The smallest value for \( \mu \), such that a growth estimate of this form is true, is not known; see the discussion in [11].

For the proof we need the following result (see [12]):

**Lemma.** Let the immersion \( X \) be given. We denote by \( ds^2 \) its non-weighted line element, and by \( ds_\beta^2 \) its weighted element w.r.t. a weight matrix \( G(X,Z) \). Then there hold

\[
(1 + g_0)^{-1} \int_B \nabla_{ds^2} \langle \varphi, \varphi \rangle W \ dudv \leq \int_B \nabla_{ds_{\beta}^2} \langle \varphi, \varphi \rangle W \ dudv \leq (1 + g_0) \int_B \nabla_{ds^2} \langle \varphi, \varphi \rangle W \ dudv
\]

(2.15)

for all \( \varphi \in C^1_0(B, \mathbb{R}) \) with the Beltrami operators \( \nabla_{ds^2} \) and \( \nabla_{ds_{\beta}^2} \) from (2.6).

**Proof of the Theorem.** 1. Due to the Lemma, the \( \mu \)-stability (2.5) yields

\[
\int_B \nabla_{ds^2} \langle \varphi, \varphi \rangle W \ dudv \geq \frac{\mu}{1 + g_0} \int_B (q - K) W \varphi^2 \ dudv.
\]

(2.16)

2. Introduce geodesic polar coordinates \((\varrho, \varphi) \in [0, r] \times [0, 2\pi]\). For curves \( \varrho = \text{const} \) on the surface, the integral formula of Bonnet and Gauss reads

\[
\int_0^{2\pi} \kappa_\varrho(\varrho, \varphi) \sqrt{P(\varrho, \varphi)} \ d\varphi + \int_0^\varrho \int_0^{2\pi} K(\tau, \varphi) \sqrt{P(\tau, \varphi)} \ d\tau d\varphi = 2\pi
\]

(2.17)

with the geodesic curvature \( \kappa_\varrho \). For the area element \( P \) there hold \( P(\varrho, \varphi) > 0 \) for all \((0, r) \times [0, 2\pi]\), as well as

\[
\lim_{\varrho \to 0^+} P(\varrho, \varphi) = 0, \quad \lim_{\varrho \to 0^+} \frac{\partial}{\partial \varrho} \sqrt{P(\varrho, \varphi)} = 1 \quad \text{for all } \varphi \in [0, 2\pi].
\]

(2.18)

Following [5], §81, for such curves it holds \( \kappa_\varrho \sqrt{P} = \frac{\partial}{\partial \varrho} \sqrt{P} \) for \((\varrho, \varphi) \in (0, r) \times [0, 2\pi]\), thus

\[
\frac{\partial}{\partial \varrho} \int_0^{2\pi} \sqrt{P(\varrho, \varphi)} \ d\varphi = \int_0^{2\pi} \kappa_\varrho(\varrho, \varphi) \sqrt{P(\varrho, \varphi)} \ d\varphi
\]

(2.19)

\[
= 2\pi - \int_0^\varrho \int_0^{2\pi} K(\tau, \varphi) \sqrt{P(\tau, \varphi)} \ d\tau d\varphi.
\]
3. Define the function $L(\varrho) := \int_0^\varrho \sqrt{P(\varrho, \varphi)} \, d\varphi$, $0 < \varrho \leq r$, with the derivatives

$$L'(\varrho) = 2\pi \int_0^{2\pi} K(\tau, \varphi) \sqrt{P(\tau, \varphi)} \, d\tau \, d\varphi, \quad L''(\varrho) = -\int_0^{2\pi} K(\varrho, \varphi) \sqrt{P(\varrho, \varphi)} \, d\varphi. \quad (2.20)$$

4. Consider the test function $\Phi(\varrho) := 1 - \frac{\varrho}{r}$, $0 < \varrho \leq r$. It holds $\nabla_{ds^2} (\Phi, \Phi) = \Phi'(\varrho)^2$ with the line element $ds^2$. Using $q \geq 0$ we estimate as follows:

$$\int_0^r \Phi'(\varrho)^2 L(\varrho) \, d\varrho = \int_0^r \Phi'(\varrho)^2 \sqrt{P(\varrho, \varphi)} \, d\varrho \, d\varphi \geq \frac{\mu}{1 + g_0} \int_0^r q(\varrho, \varphi) \Phi(\varrho)^2 \, d\varrho \, d\varphi + \frac{\mu}{1 + g_0} \int_0^r L''(\varrho) \Phi(\varrho)^2 \, d\varrho \quad (2.21)$$

5. Together with (2.18), integration by parts yields

$$\int_0^r L''(\varrho) \Phi(\varrho)^2 \, d\varrho = L'(\varrho) \Phi(\varrho)^2 \bigg|_{\varrho=0}^{\varrho=r} - 2 \int_0^r L'(\varrho) \Phi(\varrho) \Phi'(\varrho) \, d\varrho \quad (2.22)$$

Thus, $\int_0^r L(\varrho) \Phi'(\varrho)^2 \, d\varrho \leq \frac{2\pi \mu}{2\mu - (1 + g_0)},$ and the statement follows with $\Phi'^2 = \frac{1}{r^2}$.

**Example.** Let the immersion $X$ with prescribed constant mean curvature $h_0$ be $\mu$-stable with real $\mu > \frac{1}{2}$ and $q \equiv 2h_0^2$ (compare with (2.7)). Furthermore, let it represent a geodesic disc $\mathcal{B}_r(X_0)$ with geodesic radius $r > 0$ and center $X_0$. Then it holds

$$A[X] \leq \frac{2\pi \mu}{2\mu - 1} r^2 \quad (2.23)$$

due to $G(X, Z) \equiv \mathbb{E}^3$, that is, $g_0 = 0$.

**2.4 Remark: Area bounds for minimizers via outer balls**

In [24] we find area bounds in terms of outer balls enclosing embedded minimizers for the general variational problem (2.8). Namely, denote by $\nu: \mathcal{M} \to S^2$ its unit normal. Intersect the surface
with the closed ball $K_\rho(X_0)$ of radius $\rho > 0$ and center $X_0 \in \mathcal{M}$. Assume that $\mathcal{M} \cap K_\rho(X_0)$ is simply connected. The greater of the two “caps” of the boundary $\partial K_\rho(X_0)$, which are generated by this intersection, is denoted by $\mathcal{K}$. Now, assume that

$$m_1|Z| \leq \tilde{F}(X,Z) \leq m_2|Z| \quad \text{for all } (X,Z) \in \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}$$

for the composition $\tilde{F} = F \circ X$, where $0 < m_1 \leq m_2 < +\infty$. Due to the minimality of $X$ we estimate

$$A[\mathcal{M} \cap K_\rho(X_0)] = \int_{\mathcal{M} \cap K_\rho(X_0)} d\mathcal{M} \leq \frac{1}{m_1} \int_{\mathcal{M} \cap K_\rho(X_0)} \tilde{F}(X,\nu) d\mathcal{M} \leq \frac{1}{m_1} \int_{\mathcal{K}} \tilde{F}(X,\nu) d\mathcal{K} \leq \frac{m_2}{m_1} \int_{\mathcal{K}} d\mathcal{K} < \frac{4m_2\pi}{m_1} \rho^2.$$  

(2.25)

**2.5 An estimate in terms of the curvatura integra**

The proof of our theorem allows the next result (see also [20]):

**Proposition.** Let the immersion $X$ represent a geodesic disc $\mathcal{B}_r(X_0)$ of radius $r > 0$ and center $X_0$. Let its Gaussian curvature satisfy

$$K(\rho,\varphi) \leq K_0 \quad \text{for all } (\rho,\varphi) \in [0,r] \times [0,2\pi]$$

(2.26)

with a real constant $K_0 \in [0, +\infty)$. Then it holds

$$A[X] \leq r^2 \left\{ \pi + \frac{1}{2} \int_0^r \int_0^{2\pi} \left\{ K_0 - K(\rho,\varphi) \right\} \sqrt{P(\rho,\varphi)} d\rho d\varphi \right\}.$$  

(2.27)

**Example.** For minimal surfaces we have $K_0 = 0$.

**Proof of the Proposition.** For curves $\rho = \text{const}$ we conclude from the second line in (2.19)

$$\frac{\partial}{\partial \rho} \int_0^{2\pi} \sqrt{P(\rho,\varphi)} d\varphi \leq 2\pi + \int_0^r \int_0^{2\pi} \left\{ K_0 - K(\tau,\varphi) \right\} \sqrt{P(\tau,\varphi)} d\tau d\varphi.$$  

(2.28)

A first integrating w.r.t. the radius coordinate $\rho$, and then a further integration w.r.t. to $\rho = 0 \ldots r$ proves the statement.

**2.6 An estimate in terms of the boundary curvature**

Given the immersion $X$ with its $C^2$-regular boundary curve. Denote by $\kappa_g$ and $\kappa_n$ its geodesic curvature and normal curvature, resp. It holds $\kappa = \sqrt{\kappa_g^2 + \kappa_n^2} \geq |\kappa_g|$ for the non-negative curvature of the boundary, and due to Bonnet-Gauß we conclude

$$\iint_B (-K) W dudv = \int_{\partial B} \kappa_g(s) ds - 2\pi \leq \int_{\partial B} \kappa(s) ds - 2\pi.$$  

(2.29)

Inserting into (2.27) proves the
**Corollary.** Under the above assumptions it holds

\[ A[X] \leq \frac{K_0}{2} A[X] r^2 + \frac{r^2}{2} \int_{\partial B} \kappa(s) \, ds \quad (2.30) \]

with the constant \( K_0 \in [0, +\infty) \) from (2.26). In particular, if \( K_0 = 0 \) then it holds

\[ A[X] \leq \frac{r^2}{2} \int_{\partial B} \kappa(s) \, ds. \quad (2.31) \]

### 3 Graphs in \( \mathbb{R}^3 \)

#### 3.1 Introductory remarks

In this chapter we want to prove an upper area bound for graphs which solve non-homogeneous quasilinear equations. First, let us give some examples:

1. Critical points of variational problems

\[
\iint_{\Omega} F(x,y,\zeta,\zeta_x,\zeta_y) \, dxdy \rightarrow \text{extr!} \quad (3.1)
\]

have non-homogeneous divergence form (see section 4.2).

2. An equation of the form

\[
A(\zeta_x,\zeta_y)\zeta_{xx} + 2B(\zeta_x,\zeta_y)\zeta_{xy} + C(\zeta_x,\zeta_y)\zeta_{yy} = 0 \quad (3.2)
\]

with smooth coefficients can always be transformed into divergence form (see [4]).

3. By introducing a suitable weight matrix, solutions of

\[
A(x,y,\zeta,\zeta_x,\zeta_y)\zeta_{xx} + 2B(x,y,\zeta,\zeta_x,\zeta_y)\zeta_{xy} + C(x,y,\zeta,\zeta_x,\zeta_y)\zeta_{yy} = R(x,y,\zeta,\zeta_x,\zeta_y) \quad (3.3)
\]

can be transformed into the Beltrami form [2.9] (see i.e. [20], [21] for \( R \equiv 0 \)). This would make the results of chapter 2 applicable to the objects of study in this part (see also the examples discussed in section 2.2).

#### 3.2 An estimate in the general case

The next result follows ideas from [10], where area estimates for homogeneous divergence form equations are established, but where an explicit form as below is not needed. Therefore, we want to demonstrate all the essential steps.

**Theorem.** Let \( \zeta \in C^2(\Omega, \mathbb{R}) \cap C^1(\overline{\Omega}, \mathbb{R}) \), \( \Omega \subset \mathbb{R}^2 \) bounded and simply connected and with \( C^1 \)-regular boundary, solve the elliptic Dirichlet boundary value problem

\[
\frac{d}{dx} F_p(x,y,\zeta,\zeta_x,\zeta_y) + \frac{d}{dy} F_q(x,y,\zeta,\zeta_x,\zeta_y) = R(x,y,\zeta,\zeta_x,\zeta_y) \quad \text{in } \Omega, \\
\zeta(x,y) = \varphi(x,y) \quad \text{on } \partial \Omega, \quad (3.4)
\]

where \( \varphi \in C^1(\mathbb{R}^2, \mathbb{R}) \). Assume that
(A1) for all \((x, y, z, p, q) \in \mathbb{R}^5\)
\[
F_p(x, y, z, p, q)^2 + F_q(x, y, z, p, q)^2 \leq k_0^2
\]  
with a real constant \(k_0 \in [0, +\infty)\);

(A2) with a further real constant \(m_1 \in (0, +\infty)\)
\[
m_1|\xi|^2 \leq (\xi_1, \xi_2) \circ \begin{pmatrix} F_{pp}(x, y, z, \tilde{p}, \tilde{q}) & F_{pq}(x, y, z, \tilde{p}, \tilde{q}) \\ F_{qp}(x, y, z, \tilde{p}, \tilde{q}) & F_{qq}(x, y, z, \tilde{p}, \tilde{q}) \end{pmatrix} \circ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
\]  
for all \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\) and all \(\tilde{p}, \tilde{q} \in \mathbb{R}\) such that \(\tilde{p}^2 + \tilde{q}^2 \leq 1\);

(A3) finally
\[
F_p(x, y, z, 0, 0) = 0, \quad F_q(x, y, z, 0, 0) = 0.
\]

Then it holds
\[
\mathcal{A}[\xi] \leq \left(1 + \frac{||\xi||_{0, \Omega}||R||_{0, \Omega}}{m_1}\right) \mathcal{A}[\Omega] + \frac{||\xi||_{0, \partial \Omega}k_0}{m_1} \mathcal{L}[\partial \Omega]
\]  
with the area \(\mathcal{A}[\Omega]\) of \(\Omega \subset \mathbb{R}^2\) and the length \(\mathcal{L}[\partial \Omega]\) of its boundary curve \(\partial \Omega\), and the usual Schauder norms \(|| \cdot ||_{0, \Omega}\) etc.

**Proof of the Theorem.**  
1. Consider the function
\[
\mu(t) := pF_p(x, y, z, tp, tq) + qF_q(x, y, z, tp, tq), \quad t \in [0, 1].
\]
Assumption (A3) implies \(\mu(0) = 0\) and \(\mu(1) = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q)\).

2. For real \(t \in [0, 1]\) we introduce a real number \(m_1^*(t) = m_1^*(t) \in (0, +\infty)\) such that
\[
m_1^*(t)|\xi|^2 \leq (\xi_1, \xi_2) \circ \begin{pmatrix} F_{pp}(x, y, z, tp, tq) & F_{pq}(x, y, z, tp, tq) \\ F_{qp}(x, y, z, tp, tq) & F_{qq}(x, y, z, tp, tq) \end{pmatrix} \circ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
\]  
for all \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\). Namely, due to (A2) we demand

(a) if \(p^2 + q^2 \geq 1\), then \(m_1^*(t) \geq m_1\) for \(t \leq \frac{1}{\sqrt{p^2 + q^2}}\);

(\beta) if \(p^2 + q^2 \leq 1\), then \(m_1^*(t) \geq m_1\) for \(t \leq 1\).

Note that in both cases \(t^2p^2 + t^2q^2 \leq 1\).

3. Differentiating \(\mu = \mu(t)\) yields
\[
\mu'(t) = F_{pp}(x, y, z, tp, tq)p^2 + 2F_{pq}(x, y, z, tp, tq)pq + F_{qq}(x, y, z, tp, tq)q^2
\]  
and by definition of \(m_1^*(t)\) we have \(m_1^*(t)(p^2 + q^2) \leq \mu'(t)\). It follows that
\[
\mu(1) = \int_0^1 \mu'(t) \, dt \geq (p^2 + q^2) \int_0^1 m_1^*(t) \, dt.
\]  
4. Now, note that
(γ) if \( p^2 + q^2 \geq 1 \), then due to (α)
\[
\mu(1) \geq (p^2 + q^2)^{-\frac{1}{2}} \int_0^1 m_1^*(t) \, dt \geq (p^2 + q^2)^{-\frac{1}{2}} \int_0^1 m_1 \, dt = m_1 \sqrt{p^2 + q^2}; \quad (3.13)
\]

(δ) if \( p^2 + q^2 \leq 1 \), then due to (β)
\[
\mu(1) \geq (p^2 + q^2)^{-\frac{1}{2}} \int_0^1 m_1 \, dt = m_1(p^2 + q^2). \quad (3.14)
\]

Summarising we arrive at (cp. [10], Lemma 4)
\[
p_{F_p}(x, y, z, p, q) + q_{F_q}(x, y, z, p, q) \geq \begin{cases} m_1(p^2 + q^2), & \text{if } p^2 + q^2 \leq 1 \\ m_1 \sqrt{p^2 + q^2}, & \text{if } p^2 + q^2 \geq 1 \end{cases}. \quad (3.15)
\]

5. Making use of the divergence structur of our Dirichlet problem we infer
\[
\text{div} (\zeta_{F_p}, \zeta_{F_q}) = p_{F_p} + q_{F_q} + \zeta \left( \frac{d}{dx} F_p + \frac{d}{dy} F_q \right) = p_{F_p} + q_{F_q} + \zeta R, \quad (3.16)
\]
and integration by parts yields (cp. [10], Lemma 5)
\[
\iint_{\Omega} (p_{F_p} + q_{F_q}) \, dxdy = \iint_{\Omega} \text{div} (\zeta_{F_p}, \zeta_{F_q}) \, dxdy - \iint_{\Omega} \zeta R \, dxdy = \int_{\partial \Omega} \zeta (F_p, F_q) \cdot \nu \, ds + \iint_{\Omega} \zeta R \, dxdy \quad (3.17)
\]
\[
\leq \| \zeta \|_{0, \partial \Omega} \int_{\partial \Omega} \sqrt{F_p^2 + F_q^2} \, ds + \| \zeta \|_{0, \Omega} \| R \|_{0, \Omega} A \| \Omega \|
\]
\[
\leq \| \zeta \|_{0, \partial \Omega} k_0 L[\partial \Omega] + \| \zeta \|_{0, \Omega} \| R \|_{0, \Omega} A \| \Omega \|
\]
with \( \nu = \nu(s) \) normal to the boundary \( \partial \Omega \subset \mathbb{R}^2 \).

6. Taking \( \sqrt{1 + p^2 + q^2} \leq 1 + \sqrt{p^2 + q^2} \) for \( p^2 + q^2 \geq 1 \), and \( \sqrt{1 + p^2 + q^2} \leq 1 + p^2 + q^2 \) for \( p^2 + q^2 \leq 1 \) into account, we calculate for \( p^2 + q^2 \leq 1 \) and \( p^2 + q^2 \geq 1 \) (cp. [10], Proof of Theorem III)
\[
\iint_{\Omega} \sqrt{1 + p^2 + q^2} \, dxdy \leq A \| \Omega \| + \frac{1}{m_1} \iint_{\Omega} (p_{F_p} + q_{F_q}) \, dxdy. \quad (3.18)
\]
The statement follows.

**Remark.** Various variations of the proof are possible: For example, we could alter (3.17) to obtain
\[
\iint_{\Omega} (p_{F_p} + q_{F_q}) \, dxdy \leq \| \zeta \|_{0, \partial \Omega} k_0 L[\partial \Omega] + \| \zeta \|_{0, \Omega} \| R \|_{L^1(\Omega)}
\]
with the \( L^1 \)-norm on \( \Omega \). Then (3.18) would change according to this new estimate.
3.3 Homogeneous divergence equations

**Corollary.** In the homogeneous case $R \equiv 0$ we conclude from the Theorem

$$ A[\zeta] \leq A[\Omega] + \frac{\|\zeta\|_{0,\partial\Omega} k_0}{m_1} L[\partial\Omega]. \quad (3.19) $$

For example, let us consider minimal graphs with $F(p,q) = \sqrt{1 + p^2 + q^2}$ such that

$$ F_p(p,q) = \frac{p}{\sqrt{1 + p^2 + q^2}}, \quad F_q(p,q) = \frac{q}{\sqrt{1 + p^2 + q^2}}. \quad (3.20) $$

For (A1), we calculate

$$ F_p^2 + F_q^2 = \frac{p^2 + q^2}{1 + p^2 + q^2} \leq 1 =: k_0. \quad (3.21) $$

Furthermore, we set $m_1 := \frac{1}{\sqrt{8}}$ due to

$$ \frac{1}{\sqrt{8}} \cdot |\xi|^2 \leq (\xi_1, \xi_2) \circ \left( \begin{array}{cc} 1 + q^2 & -pq \\ (1 + p^2 + q^2)^{3/2} & (1 + p^2 + q^2)^{3/2} \\ -pq & 1 + p^2 \\ (1 + p^2 + q^2)^{3/2} & (1 + p^2 + q^2)^{3/2} \end{array} \right) \circ \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) $$

as well as $\lambda_1 = \frac{1}{(1 + p^2 + q^2)^{3/2}} \geq \frac{1}{\sqrt{8}}$, $p^2 + q^2 \leq 1$, for the “restricted” smallest eigenvalue from (A2).

**Corollary.** For minimal graphs it holds

$$ A[\zeta] \leq A[\Omega] + \sqrt{8} \|\zeta\|_{0,\partial\Omega} L[\partial\Omega]. \quad (3.22) $$

For the inhomogeneous divergence equation

$$ \text{div} \frac{(p,q)}{\sqrt{1 + p^2 + q^2}} = 2H(x,y,z) \quad (3.23) $$

with prescribed mean curvature $H$ such that $h_0 = \|H\|_{0,\Omega}$, we conclude

**Corollary.** In this case of prescribed mean curvature it holds

$$ A[\zeta] \leq \left\{ 1 + 2\sqrt{8} h_0 \|\zeta\|_{0,\Omega} \right\} A[\Omega] + \sqrt{8} \|\zeta\|_{0,\partial\Omega} L[\partial\Omega]. \quad (3.24) $$

**Remark.** This result is not sharp. From the estimate of the next chapter we will conclude

$$ A[\zeta] \leq \left\{ 1 + 2h_0 \|\zeta\|_{0,\Omega} \right\} \text{Area}[\Omega] + \|\zeta\|_{0,\partial\Omega} L[\partial\Omega]. \quad (3.25) $$

3.4 An interior estimate

The next result is motivated from [8] where sharp bounds for mean-curvature-graphs are proved.

**Proposition.** For real $\nu > 0$ we define the interior set

$$ \Omega_\nu := \left\{ (x,y) \in \Omega : \text{dist}((x,y), \partial\Omega) > \nu \right\}. \quad (3.26) $$

Then, under the conditions of the above Theorem and the additional assumption

$$ pF_p(x,y,z,p,q) + qF_q(x,y,z,p,q) \geq 0 \quad \text{for all } (x,y,z,p,q) \in \mathbb{R}^5 \quad (3.27) $$

(compare with (3.20)) it holds

$$ \int\int_{\Omega_\nu} \sqrt{1 + \zeta_x^2 + \zeta_y^2} \, dx dy \leq A[\Omega] + \frac{1}{m_1} \left( \frac{2k_0}{\nu} + \|R\|_{0,\Omega} \right) \|\zeta\|_{0,\Omega} A[\Omega]. \quad (3.28) $$
Proof. Choose a test function $\varphi \in C^\infty_0(\Omega, \mathbb{R})$ such that

$$\varphi(u, v) = 1 \quad \text{in } \Omega \nu, \quad |\nabla \varphi(u, v)| \leq \frac{2}{\nu} \quad \text{in } \Omega.$$  \hfill (3.29)

We compute $\text{div}(\varphi \zeta F_p, \varphi \zeta F_q) = \zeta \nabla \varphi \cdot (F_p, F_q)^t + \varphi(pF_p + qF_q) + \varphi \zeta R$. Integrating the divergence term would give no contribution due to $\varphi = 0$ on $\partial \Omega$. Therefore,

$$\iint\limits_{\Omega \nu} (pF_p + qF_q) \, dxdy \leq \iint\limits_{\Omega} \varphi(pF_p + qF_q) \, dxdy$$

$$= - \iint\limits_{\Omega} \zeta \nabla \varphi \cdot (F_p, F_q)^t \, dxdy - \iint\limits_{\Omega} \varphi \zeta R \, dxdy$$ \hfill (3.30)

$$\leq \left( \frac{2k_0}{\nu} + \|R\|_{0, \Omega} \right) \|\zeta\|_{0, \Omega} \mathcal{A}[\Omega].$$

We proceed as in point 6, i.e. (3.15) and (3.18), of the proof of our theorem. \hfill \square

4 Fermat-type graphs in $\mathbb{R}^n$

4.1 Introductory remarks

In this final chapter we establish an area bound for graphs of Fermat-type in divergence form which are critical for the variational problem $(X = (x, y, \zeta_1, \ldots, \zeta_{n-2}))$

$$\iint\limits_{\Omega} \Gamma(X) W \, dxdy \rightarrow \text{extr!}$$ \hfill (4.1)

1. For $\Gamma(X) \equiv 1$ we have the usual area functional.

2. In contrast to the case of $n = 3$, the following area bounds depend additionally on the derivatives of the graphs on the boundary. For example, the area of the conformally parametrized minimal graph $(z, z^n)$, $z = x + iy \in B$ and $n \in \mathbb{N}$, depends on the maximum norm of the mapping (which does not depend on $n$) and the exponent $n$.

3. It remains open how to transform the Euler-Lagrange system of (4.1) into a Beltrami form by means of a suitable weight matrix (see the remarks in section 3.1).

4.2 The Euler-Lagrange equations

Let us start with the general functional

$$\mathcal{F}[\zeta_1, \ldots, \zeta_{n-2}] = \iint\limits_{\Omega} F(x, y, \zeta_1, \ldots, \zeta_{n-2}, \nabla \zeta_1, \ldots, \nabla \zeta_{n-2}) \, dxdy.$$ \hfill (4.2)

We set $\zeta = (\zeta_1, \ldots, \zeta_{n-2})$, $p_\sigma = \zeta_{\sigma, x}$, $q_\sigma = \zeta_{\sigma, y}$ etc.

Proposition. The $n - 2$ Euler-Lagrange equations of $\mathcal{F}[\zeta_1, \ldots, \zeta_{n-2}]$ are

$$\frac{dF_{p_\sigma}(x, y, \zeta, \nabla \zeta)}{dx} + \frac{dF_{q_\sigma}(x, y, \zeta, \nabla \zeta)}{dy} = F_{z_\sigma}(x, y, \zeta, \nabla \zeta) \quad \text{for } \sigma = 1, \ldots, n - 2.$$ \hfill (4.3)
Corollary. The non-parametric minimal surface system is
\[
\text{div} \left( \frac{p_\sigma q_\sigma}{W} \right) = - \text{div} \left( \frac{p_\sigma \sum_{\theta=1}^{n-2} q_\theta^2 - q_\sigma \sum_{\theta=1}^{n-2} p_\sigma q_\theta, q_\sigma \sum_{\theta=1}^{n-2} p_\theta^2 - p_\sigma \sum_{\theta=1}^{n-2} p_\theta q_\theta}{W} \right)
\]
(4.4)
for \( \sigma = 1, \ldots, n - 2 \).

Proof of the Corollary. From \( X_x = (1,0,\zeta_{1,x}, \ldots, \zeta_{n-2,x}) \), \( X_y = (0,1,\zeta_{1,y}, \ldots, \zeta_{n-2,y}) \) it follows
\[
h_{11} = 1 + \sum_{\sigma=1}^{n-2} \zeta_{\sigma,x}^2 = 1 + p^2, \quad h_{12} = \sum_{\sigma=1}^{n-2} \zeta_{\sigma,x} \zeta_{\sigma,y} = p \cdot q^t, \quad h_{22} = 1 + \sum_{\sigma=1}^{n-2} \zeta_{\sigma,y}^2 = 1 + q^2 \]
(4.5)
setting \( p = (p_1, \ldots, p_{n-2}) \), \( q = (q_1, \ldots, q_{n-2}) \). Therefore, we have (let \( W = F(p,q) \))
\[
A[\zeta] = \iint_{\Omega} F(p,q) \, dxdy \equiv \iint_{\Omega} \sqrt{1 + p^2 + q^2 + p^2q^2 - (p \cdot q^t)^2} \, dxdy.
\]
(4.6)
Differentiation shows
\[
F_{p_\sigma} = \frac{p_\sigma + p_\sigma q^t - q_\sigma (p \cdot q^t)}{W}, \quad F_{q_\sigma} = \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W},
\]
(4.7)
together with \( F_{z_\sigma}(p,q) \equiv 0 \) for \( \sigma = 1, \ldots, n - 2 \). The statement follows. \( \Box \)

Remark. For \( n = 3 \), the minimal surface system reduces to \( \text{div} \sqrt{\zeta} = 0 \) in \( \Omega \).

Corollary. Critical points \((x,y,\zeta)\) of Fermat’s functional with the integrand
\[
F(x,y,z,p,q) = \Gamma(x,y,z) \sqrt{1 + p^2 + q^2 + p^2q^2 - (p \cdot q^t)^2}
\]
(4.8)
solve the Euler-Lagrange system
\[
\text{div} \left( \frac{p_\sigma q_\sigma}{W} \right) = 2H(X, \tilde{N}_\sigma) \sqrt{1 + p_\sigma^2 + q_\sigma^2}
\]
\[
+ \frac{1}{\Gamma W} \left\{ \left[ p^2 + q^2 + p^2q^2 - (p \cdot q^t)^2 \right] \Gamma_{z_\sigma} - \sum_{\omega=1}^{n-2} (p_\sigma p_\omega + q_\sigma q_\omega) \Gamma_{z_\omega} \right\}
\]
\[
- \frac{1}{\Gamma} \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \Gamma, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \Gamma \right)
\]
(4.9)
for \( \sigma = 1, \ldots, n - 2 \) with the mean curvature field
\[
H(X, \tilde{N}_\sigma) = \frac{\Gamma(X) \cdot \tilde{N}_\sigma^t}{2\Gamma(X)W}, \quad X = (x,y,\zeta),
\]
(4.10)
\( w.r.t. \) to the non-orthogonally unit normal field
\[
\tilde{N}_\sigma = \frac{1}{\sqrt{1 + |\nabla \zeta_{\sigma}|^2}} (-\zeta_{\sigma,x}, -\zeta_{\sigma,y}, 0, \ldots, 0, 1, 0, \ldots, 0), \quad \sigma = 1, \ldots, n - 2.
\]
(4.11)
Proof of the Corollary. We compute

\[
F_{p_\sigma} = \frac{p_\sigma + p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \Gamma, \quad F_{q_\sigma} = \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \Gamma, \quad F_{z_\sigma} = \Gamma z_\sigma \Gamma W
\]  

(4.12)
as well as

\[
\frac{dF_{p_\sigma}}{dx} = \Gamma \frac{d}{dx} \frac{p_\sigma + p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \Gamma + \frac{d}{dx} \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} + \frac{p_\sigma + p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dx},
\]

\[
\frac{dF_{q_\sigma}}{dy} = \Gamma \frac{d}{dy} \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \Gamma + \frac{d}{dy} \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} + \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dy}.
\]

Thus, (4.3) takes the form

\[
\text{div} \left( \frac{p_\sigma, q_\sigma}{W} \right) = \frac{\Gamma z_\sigma W}{\Gamma} - \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W}, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \right)
\]

\[
- \frac{p_\sigma + p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dx} - \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dy}.
\]

(4.14)

Performing the differentiation gives \( \frac{\partial}{\partial x} = \Gamma_x + \Gamma_{z_\omega} p_\omega \) etc.

\[
\text{div} \left( \frac{p_\sigma, q_\sigma}{W} \right) = \frac{1}{\Gamma W} \left\{ \Gamma z_\sigma - p_\sigma \Gamma_x - q_\sigma \Gamma_y \right\}
\]

\[
+ \frac{1}{\Gamma W} \left\{ \left[ p^2 + q^2 + p^2 q^2 - (p \cdot q^t)^2 \right] \Gamma z_\sigma - \sum_{\omega=1}^{n-2} (p_\sigma p_\omega + q_\sigma q_\omega) \Gamma z_\omega \right\}
\]

\[
- \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dx} - \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \frac{d\Gamma}{dy}
\]

\[
- \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W}, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \right).
\]

(4.15)

With \( \Gamma_X = (\Gamma_x, \Gamma_y, \Gamma_{z_1}, \ldots, \Gamma_{z_{n-2}}) \), the first row can be transformed into

\[
\frac{1}{\Gamma W} \left\{ \Gamma z_\sigma - p_\sigma \Gamma_x - q_\sigma \Gamma_y \right\} = \frac{1}{\Gamma W} \mathbf{\tilde{N}}_\sigma \cdot \Gamma_X \sqrt{1 + |\nabla \zeta_\sigma|^2} = 2H(X, \mathbf{\tilde{N}}_\sigma) \sqrt{1 + |\nabla \zeta_\sigma|^2},
\]

(4.16)

and the statement follows.

\[ \square \]

Remark. For \( n = 3 \), the Euler-Lagrange system reduces to

\[
\text{div} \left( \nabla \zeta \right) = 2H(X, \mathbf{\tilde{N}}) \quad \text{in} \ \Omega.
\]

(4.17)

4.3 An area estimate

The main result of this chapter is the following

Theorem. Let \( \zeta \in C^1(\overline{\Omega}, \mathbb{R}^{n-2}) \cap C^2(\Omega, \mathbb{R}^{n-2}) \) solve (4.1) where \( \Gamma = \Gamma(x, y) \in C^1(\overline{\Omega}, \mathbb{R}) \) such that with real constants \( \Gamma_0, \Gamma_1, \) and \( \Gamma_2 \) it holds

\[
0 < \Gamma_0 \leq \Gamma(X) \leq \Gamma_1 < +\infty, \quad \Gamma_2 := \|\Gamma\|_{1, \Omega}, \quad h_0 := \sup_{(X, Z) \in \mathbb{R}^3 \times S^1} |H(X, Z)|.
\]

(4.18)
We require the smallness condition
\[ \Lambda := 1 - \frac{\sqrt{2} (n - 2)^2 \Gamma_2}{\Gamma_0} \max_{\sigma=1,\ldots,n-2} \| \zeta_{0,\Omega} \| > 0. \] (4.19)

Then it holds
\[ \Lambda \cdot \mathcal{A}[\zeta] \leq \mathcal{A}[\Omega] + (n - 2) \mathcal{L}[\partial \Omega] \max_{\sigma=1,\ldots,n-2} \| \zeta_{\sigma} \|_{0,\partial \Omega} \]
\[ + 2(n - 2) h_0 \mathcal{A}[\Omega] \max_{\sigma=1,\ldots,n-2} \| \zeta_{\sigma} \|_{0,\Omega} \]
\[ + (n - 2)^2 \mathcal{L}[\partial \Omega] \max_{\sigma=1,\ldots,n-2} \| \zeta_{\sigma} \|_{0,\partial \Omega} ||D^T \zeta_{\sigma}||_{0,\partial \Omega} \] (4.20)

with the tangential derivative \( D^T \zeta_{\sigma} = (q_{\sigma}, -p_{\sigma}) \cdot \nu \), \( \nu \) unit normal along \( \partial \Omega \), \( \sigma = 1, \ldots, n-2 \).

Remarks.
1. The third line in (4.20) does not appear if \( n = 3 \). Furthermore, in this case we set \( \Lambda := 1 \). Furthermore, \( \Gamma_2 = 0 \) implies \( \Lambda = 1 \).
2. If we prescribe boundary values \( \zeta_{\sigma,R}, \), we can replace the tangential derivatives \( D^T \zeta_{\sigma} \) by the derivatives of \( \zeta_{\sigma,R} \).

Proof of the Theorem. 1. We add the \( n - 2 \) identities \( \zeta_{\sigma} \) \( \text{div} \frac{\nabla \zeta_{\sigma}}{W} = \text{div} \frac{\zeta_{\sigma} \nabla \zeta_{\sigma}}{W} - \frac{||\nabla \zeta_{\sigma}||^2}{W} \):
\[ \sum_{\sigma=1}^{n-2} \zeta_{\sigma} \text{div} \frac{\nabla \zeta_{\sigma}}{W} = \sum_{\sigma=1}^{n-2} \text{div} \frac{\zeta_{\sigma} \nabla \zeta_{\sigma}}{W} - \sum_{\sigma=1}^{n-2} \frac{p_{\sigma,0}^2 + q_{\sigma,0}^2}{W} = \sum_{\sigma=1}^{n-2} \text{div} \frac{\zeta_{\sigma} \nabla \zeta_{\sigma}}{W} - \frac{p^2 + q^2}{W}. \] (4.21)

For the area element we have
\[ \frac{1}{W} - W = \frac{1}{W} - \frac{1}{W} - \frac{[1 + p^2 + q^2 + 2p^2q^2 - (p \cdot q)^2]}{W} = - \frac{p^2 + q^2}{W} - \frac{p^2q^2 - (p \cdot q)^2}{W}, \] (4.22)

therefore,
\[ W = \frac{1}{W} + \sum_{\sigma=1}^{n-2} \zeta_{\sigma} \text{div} \frac{\nabla \zeta_{\sigma}}{W} - \sum_{\sigma=1}^{n-2} \zeta_{\sigma} \text{div} \frac{\nabla \zeta_{\sigma}}{W} + \frac{p^2q^2 - (p \cdot q)^2}{W}. \] (4.23)

2. Multiply the Euler-Lagrange equations (4.9) by \( \zeta_{\sigma} \). Summation gives \((\Gamma_{\zeta_{\sigma}} \equiv 0)\)
\[ \sum_{\sigma=1}^{n-2} \zeta_{\sigma} \text{div} \frac{\nabla \zeta_{\sigma}}{W} = 2 \sum_{\sigma=1}^{n-2} H(X, \vec{N}_\sigma) \zeta_{\sigma} \sqrt{1 + p_{\sigma}^2 + q_{\sigma}^2} \]
\[ - \sum_{\sigma=1}^{n-2} \zeta_{\sigma} \text{div} \left( \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^i)}{W}, \frac{q_{\sigma}p^2 - p_{\sigma}(p \cdot q^i)}{W} \right) \]
\[ - \sum_{\sigma=1}^{n-2} \left\{ \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^i)}{W} \zeta_{\sigma} \Gamma_x + \frac{q_{\sigma}p^2 - p_{\sigma}(p \cdot q^i)}{W} \zeta_{\sigma} \Gamma_y \right\}. \] (4.24)

Note that in the second line
\[ \text{div} \left( \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^i)}{W}, \frac{q_{\sigma}p^2 - p_{\sigma}(p \cdot q^i)}{W} \right) \zeta_{\sigma} \]
\[ = \zeta_{\sigma} \text{div} \left( \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^i)}{W}, \frac{q_{\sigma}p^2 - p_{\sigma}(p \cdot q^i)}{W} \right) + \frac{p_{\sigma}^2q^2 - p_{\sigma}q_{\sigma}(p \cdot q^i)}{W} + \frac{q_{\sigma}^2p^2 - q_{\sigma}p_{\sigma}(p \cdot q^i)}{W}, \] (4.25)
and adding up brings
\[
\sum_{\sigma = 1}^{n-2} \zeta_\sigma \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W}, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \right) = \sum_{\sigma = 1}^{n-2} \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \zeta_\sigma \right) - \frac{2}{W} \left\{ p^2 q^2 - (p \cdot q^t)^2 \right\}.
\]

(4.26)

Now, (4.23) can be written as
\[
W = \frac{1}{W} + \sum_{\sigma = 1}^{n-2} \zeta_\sigma \nabla \zeta_\sigma \frac{\zeta_\sigma}{W} - \frac{p^2 q^2 - (p \cdot q^t)^2}{W} - 2 \sum_{\sigma = 1}^{n-2} H(X, \tilde{N}_\sigma) \zeta_\sigma \sqrt{1 + p_\sigma^2 + q_\sigma^2} W
\]
\[
+ \sum_{\sigma = 1}^{n-2} \text{div} \left( \frac{p_\sigma q^2 - q_\sigma (p \cdot q^t)}{W} \zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma (p \cdot q^t)}{W} \zeta_\sigma \right)
\]
\[
+ \sum_{\sigma = 1}^{n-2} \left\{ p_\sigma q^2 - q_\sigma (p \cdot q^t) \zeta_\sigma \zeta_x + q_\sigma p^2 - p_\sigma (p \cdot q^t) \zeta_\sigma \zeta_y \right\}.
\]

(4.27)

3. This last identity will be integrated by parts.

(i) First, observe that due to \( \frac{1}{W} \leq 1 \), if holds
\[
\int \int_{\Omega} \frac{1}{W} \ dxdy \leq A[\Omega].
\]

(ii) The second is evaluated as
\[
\sum_{\sigma = 1}^{n-2} \int_{\Omega} \text{div} \left( \frac{\zeta_\sigma \nabla \zeta_\sigma}{W} \right) \ dx\ dy \leq \sum_{\sigma = 1}^{n-2} \int_{\partial \Omega} |\nabla \zeta_\sigma \cdot \nu| \ ds \leq (n-2) L[\partial \Omega] \max_\sigma \left\| \zeta_\sigma \right\|_{0, \partial \Omega}
\]

\[
\leq (n-2) \sqrt{1 + p_\sigma^2 + q_\sigma^2} W \]

(4.28)

(iii) The third term is non-negative by Hölder’s inequality.

(iv) Analogously, we have
\[
2 \sum_{\sigma = 1}^{n-2} \int_{\Omega} H(X, \tilde{N}_\sigma) \zeta_\sigma \sqrt{1 + p_\sigma^2 + q_\sigma^2} \ dxdy \leq 2(n-2) h_0 A[\Omega] \max_\sigma \left\| \zeta_\sigma \right\|_{0, \Omega}.
\]

(4.29)

(v) We consider the second line in (4.27): First, note that
\[
p_\sigma q^2 - q_\sigma (p \cdot q^t) = \sum_{\theta = 1}^{n-2} (p_\sigma q_\theta - q_\sigma q_\theta) q_\theta, \quad q_\sigma p^2 - p_\sigma (p \cdot q^t) = \sum_{\theta = 1}^{n-2} (q_\sigma q_\theta - q_\theta q_\sigma) p_\theta,
\]

(4.30)

and, therefore, multiplication by \( p_\sigma \) resp. \( q_\sigma \), and summation brings
\[
p^2 q^2 - (p \cdot q^t)^2 = \frac{1}{2} \sum_{\sigma, \theta = 1}^{n-2} (p_\sigma q_\theta - q_\sigma q_\theta)^2.
\]

(4.31)
Integration yields
\[
\sum_{\sigma=1}^{n-2} \iint_{\Omega} \text{div} \left( \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^t)}{W} \zeta_{\sigma} + \frac{q_{\sigma}p^2 - q_{\sigma}(p \cdot q^t)}{W} \zeta_{\sigma} \right) \, dx dy
\]

\[
= \sum_{\sigma, \omega=1}^{n-2} \iint_{\Omega} \text{div} \left( \frac{(p_{\sigma}q_{\omega} - p_{\sigma}q_{\sigma})q_{\omega}}{W} \zeta_{\sigma} - \frac{(p_{\sigma}q_{\omega} - p_{\sigma}q_{\sigma})p_{\omega}}{W} \zeta_{\sigma} \right) \, dx dy
\]

\[
= \sum_{\sigma, \omega=1}^{n-2} \int_{\partial \Omega} \left( \frac{p_{\sigma}q_{\omega} - p_{\sigma}q_{\sigma})}{W} \zeta_{\sigma} \right) (q_{\omega}, -p_{\omega}) \cdot v \, ds
\]

\[
\leq \sum_{\sigma, \omega=1}^{n-2} \int |\zeta_{\sigma}| |D^{\top} \zeta_{\sigma}| \, ds
\]

\[
\leq (n - 2)^2 L[\partial \Omega] \max_{\sigma} \|\zeta_{\sigma}\|_{0, \partial \Omega} \|D^{\top} \zeta_{\sigma}\|_{0, \partial \Omega}.
\]

(vi) To control the last term in (4.27) we use again (4.29), (4.30), and \((p_{\sigma}q_{\theta} - p_{\sigma}q_{\theta})^2 \leq 2[p^2 q^2 - (p \cdot q^t)^2]\) from (4.31)

\[
\left| \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^t)}{W} \right| \leq \sum_{\theta=1}^{n-2} \left| \frac{p_{\sigma}q_{\theta} - q_{\sigma}p_{\theta}}{W} \right| \leq \sqrt{2} \sum_{\theta=1}^{n-2} \sqrt{\frac{p^2 q^2 - (p \cdot q^t)^2}{W}} |q_{\theta}|,
\]

and analogously for \(|q_{\sigma}^2 - q_{\sigma}(p \cdot q^t)|\), we obtain

\[
\frac{|p_{\sigma}q^2 - q_{\sigma}(p \cdot q^t)|}{GW} \leq \sqrt{2} (n - 2) \Gamma_0^{-1} \|q\|, \quad \frac{|q_{\sigma}p^2 - q_{\sigma}(p \cdot q^t)|}{GW} \leq \sqrt{2} (n - 2) \Gamma_0^{-1} |p|.
\]

Then we may estimate
\[
\sum_{\sigma=1}^{n-2} \iint_{\Omega} \left\{ \frac{p_{\sigma}q^2 - q_{\sigma}(p \cdot q^t)}{W} \zeta_{\sigma} \Gamma_x + \frac{q_{\sigma}p^2 - q_{\sigma}(p \cdot q^t)}{W} \zeta_{\sigma} \Gamma_y \right\} \, dx dy
\]

\[
\leq \sum_{\sigma=1}^{n-2} \frac{\Gamma_2}{\Gamma_0} \iint_{\Omega} \left| \zeta_{\sigma} \right| \left\{ \frac{|p_{\sigma}q^2 - q_{\sigma}(p \cdot q^t)|}{W} + \frac{|q_{\sigma}p^2 - q_{\sigma}(p \cdot q^t)|}{W} \right\} \, dx dy
\]

\[
\leq \frac{\sqrt{2} (n - 2)^2 \Gamma_2}{\Gamma_0} \max_{\sigma} \|\zeta_{\sigma}\|_{0, \Omega} \iint_{\Omega} \sqrt{1 + p^2 + q^2 + p^2 q^2 - (p \cdot q^t)^2} \, dx dy.
\]

6. Taking our results together, we arrive at
\[
\mathcal{A}[\cdot] \leq \mathcal{A}[\Omega] + (n-2) L[\partial \Omega] \max_{\sigma} \|\zeta_{\sigma}\|_{0, \partial \Omega} + 2(n-2) h_0 \mathcal{A}[\Omega] \max_{\sigma} \|\zeta_{\sigma}\|_{C^0(\Omega)}
\]

\[
+ (n-2)^2 L[\partial \Omega] \max_{\sigma} \|\zeta_{\sigma}\|_{0, \partial \Omega} \|D^{\top} \zeta_{\sigma}\|_{0, \Omega}
\]

\[
+ \frac{\sqrt{2} (n-2)^2 \Gamma_2}{\Gamma_0} \max_{\sigma} \|\zeta_{\sigma}\|_{0, \Omega} \mathcal{A}[\cdot].
\]

Rearranging proves the statement.

\[ \square \]

Remark. In (4.35) we need the special form \(\Gamma = \Gamma(x, y)\). Otherwise, due to (4.34) there would remain terms quadratically in \(p_{\theta}, q_{\theta}\) in the integrand.
4.4 Minimal surfaces

We consider the special case \( \Gamma \equiv 1 \) (that is, \( h_0 = 0 \)):

**Corollary.** Let \( \zeta \in C^2(\Omega, \mathbb{R}^{n-2}) \cap C^1(\overline{\Omega}, \mathbb{R}^{n-2}) \) solve the minimal surface system \((4.4)\). Then

\[
A[\zeta] \leq A[\Omega] + (n - 2)\mathcal{L}[\partial \Omega] \max_{\sigma=1,\ldots,n-2} \|\zeta_\sigma\|_{0,\partial \Omega}^0
\]

\[
+ (n - 2)^2\mathcal{L}[\partial \Omega] \max_{\sigma=1,\ldots,n-2} \|\zeta_\sigma\|_{0,\partial \Omega}^0 \|D^{\top} \zeta_\sigma\|_{0,\partial \Omega}^0.
\]

(4.37)

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