A spacetime characterization of the Kerr metric

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Abstract

We obtain a characterization of the Kerr metric among stationary, asymptotically flat, vacuum spacetimes, which extends the characterization in terms of the Simon tensor (defined only in the manifold of trajectories) to the whole spacetime. More precisely, we define a three index tensor on any spacetime with a Killing field, which vanishes identically for Kerr and which coincides in the strictly stationary region with the Simon tensor when projected down into the manifold of trajectories. We prove that a stationary asymptotically flat vacuum spacetime with vanishing spacetime Simon tensor is locally isometric to Kerr. A geometrical interpretation of this characterization in terms of the Weyl tensor is also given. Namely, a stationary, asymptotically flat vacuum spacetime such that each principal null direction of the Killing form is a repeated principal null direction of the Weyl tensor is locally isometric to Kerr.

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1 Introduction.

The Kerr metric plays a very prominent rôle in Einstein’s theory of general relativity. Obtained by R. Kerr in 1963 [1], it was the first explicitly known stationary rotating (i.e. non-static) asymptotically flat vacuum spacetime. Although many other stationary and axially symmetric asymptotically flat vacuum solutions are known at present, the Kerr metric remains, somehow, the simplest one. More importantly, the Kerr spacetime has a very special status among stationary vacuum solutions due to the black hole uniqueness theorem, which states, roughly speaking, that the exterior geometry of a stationary, asymptotically flat, vacuum black hole must be Kerr. Despite its importance, the reasons why this geometry plays such a privileged rôle remain somewhat obscure. In comparison, the Schwarzschild metric has nice uniqueness properties; in addition to Birkhoff’s theorem, this metric is well-known to be the only static and asymptotically flat vacuum solution such that the induced metric of the hypersurfaces orthogonal to the static Killing vector are conformally flat.

This leads one to consider whether there exists any property of the Kerr metric which singles it out among the class of stationary, asymptotically flat vacuum metrics. Some characterizations are already known. The first such was found by B. Carter [2] who analyzed stationary and axisymmetric electrovacuum spacetimes such that the Hamilton-Jacobi and Schrödinger equations are separable in some adapted coordinates and found a finite parameter family of solutions containing the Kerr metric. However, this analysis relies on coordinate dependent conditions, which is not appropriate for characterizing a spacetime. Furthermore, axial symmetry is assumed from the outset.

A second characterization was obtained by Z. Perjés [3], who restricted the analysis to the strictly stationary subclass (i.e. to spacetimes with a Killing vector which is timelike everywhere). Working on the manifold of trajectories (i.e. the set of orbits of the isometry group, which is assumed to be a manifold), Perjés showed that the Kerr metric can be found uniquely by demanding the existence of “geodesic eigenrays” of the Killing vector. The third and, probably the most interesting of the known characterizations of Kerr is due to W. Simon [4], who defined a tensor (now called Simon tensor) on the manifold of trajectories which is identically zero for the Kerr metric. Conversely, if the Simon tensor vanishes in an asymptotically flat, vacuum spacetime, then there exists an open neighbourhood of infinity which can be isometrically embedded into the Kerr spacetime. Moreover, the Simon tensor is equivalent to the Cotton tensor (which vanishes in a three-dimensional Riemannian manifold if and only if the metric is locally conformally flat) when the Killing vector is hypersurface orthogonal, so that one of the known characterizations of Schwarzschild is recovered. Despite the clear interest of this characterization it still suffers from two drawbacks. The first one is that the theorem establishes the isometry with Kerr only in some neighbourhood of infinity. Even though any strictly stationary vacuum metric is analytic [5], this does not ensure that the isometry near infinity extends everywhere, because analytic extensions of manifolds
need not be unique. However, this is probably not a serious objection and could presumably be fixed by exploiting the interesting results by Z. Perjés in [3], who found all the possible local forms of the metric in any strictly stationary spacetime with vanishing Simon tensor. In particular, Perjés found that the most general metric with those properties depends only on a few parameters, thus showing that the asymptotically flatness condition in the characterization of Kerr is only necessary in order to fix the value of some constants.

The second objection is more serious and will be the object of this paper. Since the whole construction of the Simon tensor is done on the manifold of trajectories, the characterization in terms of the Simon tensor only works in the strictly stationary region, i.e. outside any ergosphere. However, for most practical purposes (in particular, for the black hole uniqueness theorems) the relevant region of Kerr is the domain of outer communication, which contains a non-empty ergoregion. Therefore the characterization in terms of objects defined in the manifold of trajectories is not sufficient. Thus, it is necessary to find a property of Kerr that distinguishes this metric irrespectively of the norm of the Killing vector.

In order to treat this problem, we should look for a characterization involving only spacetime objects. Due to the relevance of the Simon tensor, the obvious strategy is to try and obtain a spacetime version of the Simon tensor in the region where the Killing vector is timelike and analyze whether or not this spacetime tensor remains regular at points where the Killing vector becomes null. Provided this is the case, it would remain to show that the extended Simon tensor vanishes everywhere for Kerr and prove also that the converse holds, i.e. that an asymptotically flat, vacuum spacetime with vanishing spacetime Simon tensor is locally isometric to Kerr (we cannot expect a global isometry to exist without further global conditions). This is the problem we want to solve in this paper. More specifically, in section 2 we recall the definition of the Simon tensor on the manifold of trajectories and write down the main result proven in [4]. In section 3, we obtain the spacetime counterpart of the Simon tensor in the strictly stationary region and show that it remains regular at points where the Killing vector becomes null. This requires some analysis of the algebraic properties of the Simon tensor. Then, we define a tensor on any spacetime with a Killing vector (with no assumptions on its norm) such that it coincides with the Simon tensor when projected down into the manifold of trajectories in the strictly stationary region, and we obtain the consequences of the vanishing of this tensor in terms of the Weyl tensor. In section 4, we prove that an asymptotically flat vacuum spacetime with a Killing field which is timelike near infinity and such that the corresponding spacetime Simon tensor vanishes, is locally isometric to Kerr at every point, thus extending the results in [4].

It is worth emphasizing that obtaining a spacetime characterization of Kerr is interesting not only in order to understand better what is so special about Kerr, but also for more practical purposes. The problem we have in mind is the black hole uniqueness
Theorem. The existing proofs for this theorem require rather strong hypotheses on the
spacetime, like connectedness of the black hole event horizon (see Weinstein [7], [8] for
interesting progress in the non-connected case), non-existence of closed timelike curves
in the domain of outer communication (see Carter [9] for a discussion) and analyticity
of the metric and the event horizon (see Chruściel [10], [11]). This last point is cru-
cial in order to apply the so-called Hawking rigidity theorem [12], which ensures the
existence of a second Killing field when the black hole is rotating. It is reasonable to
believe that the theorem still holds when some (or perhaps all) of these hypotheses
are significantly relaxed. However, trying to prove those results is a hard problem,
and it is conceivable that a more detailed knowledge of the Kerr geometry, and in
particular having at hand a spacetime characterization of Kerr can prove helpful for
attacking those questions. In this respect, let us mention that the characterization of
Schwarzschild in terms of the conformal flatness of the hypersurfaces orthogonal to the
static Killing vector is an essential tool in all known proofs of the black hole uniqueness
theorems in the non-rotating case.

2 The Simon tensor on the manifold of trajectories

Let us start by fixing our definitions and conventions. Throughout this paper a $C^n$
spacetime denotes a paracompact, Hausdorff, connected, $C^{n+1}$ four-dimensional man-
ifold endowed with a $C^n$ Lorentzian metric $g$ of signature $(-1, 1, 1, 1)$. Smooth means
$C^\infty$. We will also assume that spacetimes are orientable and time-orienta-
able. The Levi-Civita covariant derivative of $g$ is denoted by $\nabla$, the volume form is $\eta^{\alpha\beta\gamma\delta}$ and
our sign conventions of the Riemann and Ricci tensors follow [13].

Throughout this section $(\mathcal{M}, g)$ denotes a $C^2$ spacetime satisfying Einstein’s vac-
uum field equations $R_{\alpha\beta} = 0$ and admitting a Killing vector field $\vec{\xi}$ which is timelike
everywhere. The construction of the Simon tensor is as follows. First, an equivalence
relation $\approx$ is introduced in $\mathcal{M}$ so that two points $p, q \in \mathcal{M}$ are equivalent if and only
if they belong to the same integral line of $\vec{\xi}$. If the spacetime satisfies the chronology
condition (see [12] for the definition) and the Killing field is complete, then the quo-
tient set $\mathcal{N} = \mathcal{M}/\approx$ is a differentiable manifold [14] such that the canonical projection
$\pi : \mathcal{M} \to \mathcal{N}$ is differentiable. Then, there exists a one-to-one correspondence between
tensors on $\mathcal{N}$ and tensors on $\mathcal{M}$ which are completely orthogonal to $\vec{\xi}$ (i.e. orthogonal
to $\vec{\xi}$ with respect to any index) and with vanishing Lie derivative along $\vec{\xi}$.

Then, the so-called norm and twist of the Killing field are defined by $\lambda = -\xi^\alpha \xi_\alpha$
and $\omega_\alpha = \eta_{\alpha\beta\gamma\delta} \xi^\beta \nabla^\gamma \xi^\delta$. Since $\lambda > 0$ in $\mathcal{M}$, the tensors$h_{\alpha\beta} = g_{\alpha\beta} - \lambda^{-1} \xi_\alpha \xi_\beta$ and
$\gamma_{\alpha\beta} = \lambda h_{\alpha\beta}$ are well-defined on $\mathcal{M}$. The corresponding tensors on $\mathcal{N}$ are denoted by
$\lambda$, $\omega_i$, $h_{ij}$ and $\gamma_{ij}$ respectively (tensors on $\mathcal{N}$ carry Latin indices). Both $h_{ij}$ and $\gamma_{ij}$
are symmetric, non-degenerate and positive definite. Denoting by $D$ the Levi-Civita
covariant derivative of \((N, \gamma)\) and introducing the complex one form 

\[
\sigma_j = D_j \lambda - i \omega_j, \tag{1}
\]

the Simon tensor associated with \(\vec{\xi}\) is defined via \[4\]

\[
S_{ijk} = 2 \left( \sigma_k D_j \sigma_i - u_{[k} \gamma_{j]i} \right), \quad u_k = \gamma^{ij} \sigma_{[k} D_{j]} \sigma_i.
\]

For later convenience, let us introduce a tensor \(\hat{Z}_{ij} = D_j \sigma_i\) (which is symmetric by virtue of the vacuum field equations) so that the Simon tensor reads

\[
S_{ijk} = \hat{Z}_{ij} \sigma_k - \hat{Z}_{ik} \sigma_j - \frac{1}{2} \gamma_{ij} \left( \hat{Z} \sigma_k - \sigma^l \hat{Z}_{lk} \right) + \frac{1}{2} \gamma_{ik} \left( \hat{Z} \sigma_j - \sigma^l \hat{Z}_{lj} \right),
\]

where \(\hat{Z} = \gamma^{ij} \hat{Z}_{ij}\) and indices are raised with \(\gamma^{ij}\). One of the main results proven in \[4\] is (the concept of asymptotically flat end is defined later in section 4)

**Theorem 1 (Simon 1984)** Let \((M, g)\) be a \(C^2\) vacuum spacetime with a timelike Killing field \(\vec{\xi}\). Assume that the spacetime contains an asymptotically flat end \(M^\infty\). If the Simon tensor associated with \(\vec{\xi}\) vanishes identically, then there exists an asymptotically flat open submanifold \(M_1 \subset M^\infty\) which is isometrically diffeomorphic to an open submanifold of the Kerr spacetime.

The key idea of the proof is to show that the set of multipole moments defined in the asymptotic end \(M^\infty\) of \((M, g)\) coincides with the set of multipole moments of the Kerr spacetime (for a certain mass and angular momentum). Then, a previous result by Beig and Simon \[13\] stating that two asymptotically flat spacetimes with the same set of multipole moments must be isometric in a neighbourhood of infinity completes the proof.

### 3 The spacetime Simon tensor

In this section we will obtain a tensor defined on any spacetime with a Killing vector, which coincides with the Simon tensor when projected down into the manifold of trajectories (whenever this exists) and such that it vanishes identically for the Kerr spacetime.

Throughout this section \((\mathcal{M}, g)\) denotes a \(C^2\) vacuum spacetime admitting a non-trivial Killing field \(\vec{\xi}\). We do not require that the orbits of this Killing vector are complete (i.e. the Killing vector \(\vec{\xi}\) need not generate a one parameter isometry group). A key object in our analysis is the exact two-form \(F_{\alpha \beta} = \nabla_\alpha \xi_\beta\) (we use boldface characters to denote two-forms). A complex two-form \(\mathcal{B}\) satisfying \(\mathcal{B}^* = -i \mathcal{B}\), where

\[1\]This definition differs by a non-zero factor from the original one in \[4\].
is the Hodge dual operator, is called self-dual (calligraphic letters will be used to denote them). Given any real two-form \( B_{\alpha\beta} \), its self-dual part \( \mathcal{B}_{\alpha\beta} \), is defined by

\[
\mathcal{B}_{\alpha\beta} = B_{\alpha\beta} + i B^*_{\alpha\beta}.
\]

is called Killing form throughout this paper. Since most of the calculations in this section involve two-forms, let us recall some well-known identities (see e.g. \[16\]). Let \( X \) and \( Y \) be arbitrary two-forms on \( M \), then

\[
X_{\mu\sigma} Y_{\nu} - X^*_{\mu\sigma} Y^*_{\nu} = \frac{1}{2} g_{\mu\nu} X_{\alpha\beta} Y^{\alpha\beta}, \quad X_{\mu\sigma} X^*_{\nu} = \frac{1}{4} g_{\mu\nu} X_{\alpha\beta} X^{*\alpha\beta},
\]

which specialized to arbitrary self-dual two-forms, \( \mathcal{X}, \mathcal{Y} \), give

\[
\mathcal{X}_{\mu\sigma} \mathcal{Y}_{\nu} + \mathcal{Y}_{\mu\sigma} \mathcal{X}_{\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{X}_{\alpha\beta} \mathcal{Y}^{\alpha\beta}, \quad \mathcal{X}_{\mu\sigma} \mathcal{X}_{\nu} = \frac{1}{4} g_{\mu\nu} \mathcal{X}_{\alpha\beta} \mathcal{X}^{*\alpha\beta}. \quad (3)
\]

For any self-dual two-form \( \mathcal{Z} \) we have

\[
\mathcal{Z}_{\mu\nu} \mathcal{Z}^*_{\sigma} - \mathcal{Z}_{\nu\sigma} \mathcal{Z}^*_{\mu} = 0,
\]

where \( \mathcal{Z} \) denotes the real part of \( \mathcal{Z} \). From the equation \( \nabla_\alpha \nabla_\beta \xi_\mu = \xi^\sigma C_{\sigma\alpha\beta\mu} \), which follows from the Killing equations in vacuum, we obtain

\[
\nabla_\alpha \mathcal{F}_{\beta\gamma} = \xi^\sigma \mathcal{C}_{\sigma\alpha\beta\gamma}, \quad (5)
\]

where \( \mathcal{C}_{\alpha\beta\gamma\delta} \) is the so-called right self-dual Weyl tensor, defined as \( \mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{2}{3} \eta_{\gamma\delta\rho\sigma} C_{\alpha\beta}^{\rho\sigma} \). This tensor shares all the symmetries with the Weyl tensor, i.e. it is a symmetric double two-form, with vanishing trace, which satisfies the Bianchi identity \( \mathcal{C}_{\alpha[\beta\gamma\delta]} = 0 \). It follows from (5) that \( \mathcal{F} \) satisfies Maxwell’s equations \( d\mathcal{F} = 0 \). The Ernst one-form is defined by

\[
\sigma_\mu \equiv 2 \xi^\alpha \mathcal{F}_{\alpha\mu} = \nabla_\mu \lambda - i \omega_\mu, \quad (6)
\]

which has as an immediate consequence that

\[
\sigma_\mu \sigma^\mu = -\lambda \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}. \quad (7)
\]

The Ernst one-form \( \sigma_\alpha \) is closed, \( \nabla_{[\mu} \sigma_{\nu]} = 2 \mathcal{F}_{\alpha[\nu} \mathcal{F}^\alpha_{\mu]} + \xi^\alpha \nabla_\alpha \mathcal{F}_{\mu\nu} = 0 \), where \( d\mathcal{F} = 0 \) was used in the first equality and (4) and (5) in the second one. Let us assume for the moment that the set \( \mathcal{M}^+ = \{ p \in \mathcal{M}; \lambda_p > 0 \} \) is non-empty and that \( \mathcal{N}^+ = \mathcal{M}^+ / \approx \) is a manifold, so that the Simon tensor can be defined along the lines described in the previous section. We want to translate the Simon tensor into a tensor in \( \mathcal{M}^+ \) and analyze whether it can be extended to all of \( \mathcal{M} \). The spacetime counterpart of \( \sigma_\alpha \) in
(1) is the Ernst one-form $\sigma_\mu$. The symmetric tensor $\hat{Z}_{ij}$ will also have a spacetime counterpart in $\mathcal{M}^+$, which we denote by $\hat{Z}_{\mu\nu}$. As we shall see below, $\hat{Z}_{\mu\nu}$ is singular on the boundary of $\mathcal{M}^+$ (if non-empty). In order to see that the Simon tensor remains nevertheless regular, we need to analyze the structure of the Simon tensor. To that end, let us consider an arbitrary point $p \in \mathcal{M}^+$ and define the following linear map between two-index symmetric, covariant tensors at $p$ which are completely orthogonal to $\vec{\xi}$ and three-index covariant tensors at $p$

$$U(Z)_{\alpha\beta\gamma} = Z_{\alpha\beta\gamma} - Z_{\alpha\gamma\beta} - \frac{1}{2} h_{\alpha\beta} (Z_{\gamma\gamma} - \sigma^\mu Z_{\mu\gamma}) + \frac{1}{2} h_{\alpha\gamma} (Z_{\beta\gamma} - \sigma^\mu Z_{\mu\beta}),$$

(8)

where $Z = g^{\mu\nu} Z_{\mu\nu}$. The algebraic properties of this map are summarized in the following lemma, which is proven by straightforward calculation.

**Lemma 1** At any point $p \in \mathcal{M}^+$ and for any symmetric tensor $Z_{\alpha\beta}|_p$ completely orthogonal to $\vec{\xi}$, the tensor $U_{\alpha\beta\gamma} = U(Z)_{\alpha\beta\gamma}|_p$ defined by (8) satisfies

1) $U_{\alpha\beta\gamma}$ is completely orthogonal to $\vec{\xi}$,
2) $U_{\alpha\beta\gamma} = -U_{\alpha\gamma\beta}$,
3) $U^\alpha_{\alpha\beta} = 0$,
4) $U|_{\alpha\beta\gamma} = 0$,
5) $\sigma^\alpha \sigma^\beta (U_{\alpha\beta\gamma} \sigma_\delta - U_{\alpha\delta\beta} \sigma_\gamma) + \sigma^\mu \sigma^\alpha \sigma^\beta U_{\alpha\gamma\delta} = 0$.

Properties 2), 3) and 4) are standard for Cotton-like tensors. Property 5) is specific for the Simon tensor. A trivial calculation shows that the map $U(Z)$ satisfies

$$U (Z_{\alpha\beta} + s_1 h_{\alpha\beta} + s_2 \sigma^\alpha \sigma^\beta) = U (Z_{\alpha\beta})$$

(9)

where $s_1$ and $s_2$ are arbitrary complex constants. This indicates that the kernel of $U(Z)$ is at least two dimensional. The following lemma shows that at points where $\mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} \neq 0$ the kernel of $U(Z)$ is indeed two-dimensional and that $U(Z)$ is surjective onto the set of tensors satisfying properties 1) to 5) above.

**Lemma 2** Let $U_{\alpha\beta\gamma}$ be a tensor at $p \in \mathcal{M}^+$ satisfying properties 1) to 5) in lemma 1 and assume that $\mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}|_p \neq 0$. Then $\sigma^\mu \sigma^\alpha|_p \neq 0$ and the general solution of the algebraic equation $U(Z^0)_{\alpha\beta\gamma} = U_{\alpha\beta\gamma}$ is given by

$$Z^0_{\alpha\beta} = \frac{\sigma^\gamma (U_{\beta\gamma} + U_{\alpha\gamma})}{2 \sigma^\mu \sigma^\nu} + \frac{3 \sigma^\nu \sigma^\gamma (U_{\nu\alpha\gamma} \sigma_\beta + U_{\nu\beta\gamma} \sigma_\alpha)}{2 (\sigma^\mu \sigma^\nu)^2} + s_1 h_{\alpha\beta} + s_2 \sigma^\alpha \sigma^\beta|_p$$

(10)

where $s_1$ and $s_2$ are arbitrary constants.

**Proof.** Since $\lambda \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}|_p \neq 0$, equation (7) shows $\sigma^\mu \sigma^\alpha|_p \neq 0$. Making use of property (7) the problem can be restricted to the one of finding the general solution of

$$\tilde{Z}_{\alpha\beta\gamma} - \tilde{Z}_{\alpha\gamma\beta} + \frac{1}{2} h_{\alpha\beta} \sigma^\mu \tilde{Z}_{\mu\gamma} - \frac{1}{2} h_{\alpha\gamma} \sigma^\mu \tilde{Z}_{\mu\beta} = U_{\alpha\beta\gamma},$$

(11)
for tensors \( \tilde{Z}_{\alpha\beta} \) satisfying \( \tilde{Z} = \tilde{Z}_{\alpha\beta}\sigma^\alpha\sigma^\beta|_p = 0 \). Contracting (11) with \( \sigma^\gamma \) and \( \sigma^\alpha \) alternatively, it is easy to see that the only possible solution of this equation is given by the right-hand side of (11) with \( s_1 = s_2 = 0 \). It remains to show that (11) is fulfilled. This can be proven by introducing three mutually orthogonal, unit complex vectors \( e^i_\mu \), \( i = 1, 2, 3 \), at \( p \) satisfying \( \xi^\mu e^i_\mu|_p = 0 \) and such that \( \sigma^\mu|_p = (\sigma^\beta\sigma^\beta)^{1/2}|_p e^3_\mu \). Expanding \( U_{\alpha\beta\gamma} \) in terms of these vectors it is easy to obtain the most general form allowed by the algebraic properties 1)-5) in lemma 1. Expanding also \( \tilde{Z}_{\alpha\beta} \) in terms of this basis, it is a matter of simple calculation to check that (11) holds identically. 

Let us return to the tensor \( \hat{Z}_{\nu\mu} \). The following lemma is key to show that the Simon tensor can be extended to all of \( M_+ \).

**Lemma 3** \( \hat{Z}_{\nu\mu} \), i.e. the spacetime counterpart of \( D_m\sigma_n \) on \( M_+ \), can be written as

\[
\hat{Z}_{\nu\mu} = 2\xi^\alpha\xi^\beta C_{\alpha\mu\beta\nu} - \frac{1}{2}\mathcal{F}^2 h_{\mu\nu} - \frac{1}{2\lambda}\sigma_\mu\sigma_\nu,
\]

where \( \mathcal{F}^2 \equiv \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \).

*Proof.* We shall start with the tensor \( \hat{Z}_{mn} \), defined on \( N_+ \) and find its spacetime counterpart. Since the two metrics \( h_{ij} \) and \( \gamma_{ij} \) are conformally related, it follows that

\[
D_m\sigma_l = D^h_m\sigma_l + \frac{1}{2\lambda} \left( h_{lm}\sigma^kD_k\lambda - \sigma_lD_m\lambda - \sigma_mD_l\lambda \right),
\]

where \( D^h \) denotes the Levi-Civita covariant derivative of \( h \), and indices are raised with \( h^{ij} \). The spacetime counterpart of the right-hand side can be written down by recalling that each covariant derivative on \( N_+ \) must be transformed into a covariant derivative on \( M_+ \) and then projected with \( h^\mu_\mu = \delta^\alpha_\alpha + \lambda^{-1}\xi^\alpha\xi^\alpha \). Hence

\[
D_m\sigma_n \leftrightarrow \hat{Z}_{\mu\nu} = h^\mu_\mu h^\nu_\nu\nabla_\mu\sigma_\nu + \frac{1}{2\lambda} \left( h_{\mu\nu}\sigma^\delta\nabla_\delta\lambda - \sigma_\mu\nabla_\nu\lambda - \sigma_\nu\nabla_\mu\lambda \right),
\]

where the arrow stands for the one-to-one correspondence between tensors on \( N_+ \) and tensors on \( M_+ \). Using (6) and (8), we immediately obtain

\[
h^\mu_\mu h^\nu_\nu\nabla_\mu\sigma_\nu = 2h^\mu_\mu h^\nu_\nu F^\alpha_\mu F^\alpha_\nu + 2\xi^\alpha\xi^\beta C_{\alpha\mu\beta\nu}.
\]

Thus, proving the lemma amounts to showing that

\[
2h^\mu_\mu h^\nu_\nu F^\alpha_\mu F^{\alpha\nu} + \frac{1}{2\lambda} \left( h_{\mu\nu}\sigma^\delta\nabla_\delta\lambda - \sigma_\mu\nabla_\nu\lambda - \sigma_\nu\nabla_\mu\lambda \right) + \frac{1}{2}\mathcal{F}^2 h_{\mu\nu} + \frac{1}{2\lambda}\sigma_\mu\sigma_\nu = 0 \quad (12)
\]

holds on \( M_+ \). The imaginary part of this equation is an immediate consequence of the imaginary part of (7) which reads \( \omega^\alpha \nabla_\alpha\lambda = \lambda F_{\alpha\beta}\mathcal{F}^{\alpha\beta} \). Regarding the real part of
The relation follows from the remarkable identity (here comma stands for partial differentiation)

\[
\omega_{\mu\nu} + \lambda_{\mu} \lambda_{\nu} = 2 (\lambda g_{\mu\nu} + \xi_{\mu} \xi_{\nu}) \nabla^\alpha \xi^\beta \nabla_\alpha \xi_\beta + g_{\mu\nu} \lambda_{\alpha} \lambda^\alpha - 4 \lambda \nabla_\nu \xi_\alpha \nabla_\mu \xi_\alpha - 4 \lambda \xi_{(\nu} \nabla_\mu \xi_{\alpha)}
\]

which holds for any Killing vector. This identity can be proven, after a somewhat long but trivial calculation, by expanding \( \eta_{\alpha\beta\gamma\delta} \eta_{\mu\nu\rho\sigma} \) appearing in \( \omega_{\mu\nu} \) on the left-hand side, in terms of products of \( g_{\alpha\beta} \).

This lemma shows that, although \( \hat{Z}_{\alpha\beta} \) becomes singular at points where \( \lambda \) goes to zero, the diverging part belongs to the kernel of the map \( U(\hat{Z}) \). Therefore, the spacetime counterpart of the Simon tensor, which is \( S = U(\hat{Z}) \), remains regular at the boundary of \( \mathcal{M}^+ \). In other words, let us define the symmetric, trace-free tensor

\[
Y_{\mu\nu} = 2 \xi^\alpha \xi^\beta C_{\alpha\mu\beta\nu}
\]

which satisfies, due to (3), \( \sigma^\alpha Y_{\alpha\beta} = -\lambda \xi^\mu C_{\mu\beta\gamma\delta} F^{\gamma\delta} \). Using lemmas 2 and 3, we find that the spacetime counterpart of the Simon tensor at any point \( p \in \mathcal{M}^+ \) reads

\[
S_{\alpha\beta\nu} = U(Y)_{\alpha\beta\nu} = Y_{\alpha\beta} \sigma_\nu - Y_{\alpha\nu} \sigma_\beta - \frac{1}{2} \gamma_{\alpha\beta} \xi^\mu C_{\mu\nu\rho\delta} F^{\rho\delta} + \frac{1}{2} \gamma_{\alpha\nu} \xi^\mu C_{\mu\beta\rho\delta} F^{\rho\delta}.
\]

All the objects in this expression are well-defined in \( \mathcal{M} \). Moreover, this definition makes sense irrespectively of whether the vacuum field equations hold or not. Thus, let us put forward the following definition

**Definition 1** Let \( (\mathcal{M}, g) \) be a \( C^2 \) spacetime with a Killing field \( \vec{\xi} \). Construct \( F_{\mu\nu} \) and \( \sigma_\mu \) through expressions (3) and (6). The spacetime Simon tensor with respect to \( \vec{\xi} \) is defined by

\[
S_{\alpha\beta\nu} = U(Y)_{\alpha\beta\nu} = 2 \xi^\mu \xi^\rho C_{\mu\alpha\beta\nu\rho} - 2 \xi^\mu \xi^\rho C_{\mu\alpha\nu\beta\rho} - \frac{1}{2} \gamma_{\alpha\beta} \xi^\mu C_{\mu\nu\rho\delta} F^{\rho\delta} + \frac{1}{2} \gamma_{\alpha\nu} \xi^\mu C_{\mu\beta\rho\delta} F^{\rho\delta},
\]

where \( \gamma_{\alpha\beta} = \lambda g_{\mu\nu} + \xi_\alpha \xi_\beta \) and \( \lambda = -\xi^\alpha \xi_\beta \).

Let us now consider the implications of the vanishing of the spacetime Simon tensor. So, let \( (V, g) \) be a spacetime with a Killing field \( \vec{\xi} \) such that the corresponding spacetime Simon tensor vanishes everywhere. Consider a point \( p \in V \) where \( \vec{\xi} \) has non-zero norm and \( F^2 \) is non-zero. From lemma 3 and the fact that \( Y_{\alpha\beta} \) is trace-free it follows that

\[
Y_{\alpha\beta} \big|_p = 2 \xi^\mu \xi^\nu C_{\mu\alpha\nu\beta} \big|_p = \frac{Q(p)}{2} \left( \sigma_{\alpha\beta} + \frac{1}{3} \gamma_{\alpha\beta} F^2 \right) \big|_p
\]

where \( Q(p) \) is the quadratic part of the curvature tensor at point \( p \) and \( \sigma_{\alpha\beta} \) is the symmetric part of the curvature tensor.
where \( Q(p) \) is an arbitrary complex constant. \( C_{\alpha\beta\gamma\delta} \) is a double symmetric self-dual two-form which is well-known to be uniquely characterized by its electric and magnetic parts (see e.g. [13]). Thus we have
\[
C_{\alpha\beta\gamma\delta} \mid_p = Q(p) \left( F_{\alpha\beta} F_{\gamma\delta} - \frac{1}{3} \mathcal{I}_{\alpha\beta\gamma\delta} F^2 \right) \mid_p
\]
where \( \mathcal{I}_{\alpha\beta\gamma\delta} \equiv (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} + i \eta_{\alpha\beta\gamma\delta})/4 \) is the metric in the space of self-dual two-forms. The equality above holds because, by virtue of (15), the electric and magnetic parts of both sides coincide.

The following lemma will be important to prove the characterization of Kerr.

**Lemma 4** Let \( (V, g) \) be a \( C^3 \) vacuum, not locally flat, spacetime with a Killing vector \( \xi \) which is non-null on a dense subset of \( V \) and such that the corresponding spacetime Simon tensor vanishes everywhere. Assume also that \( F^2 \neq 0 \) everywhere. Then, the Ernst one-form is exact, i.e. it exists a function \( \sigma \) such that \( \sigma_\alpha = \nabla_\alpha \sigma \) and the Weyl tensor and \( F^2 \) take the form
\[
C_{\alpha\beta\gamma\delta} = -\frac{6}{c - \sigma} \left( F_{\alpha\beta} F_{\gamma\delta} - \frac{1}{3} \mathcal{I}_{\alpha\beta\gamma\delta} F^2 \right), \quad F^2 = A (c - \sigma)^4
\]
where \( c \) and \( A \neq 0 \) are complex constants.

**Proof.** A trivial consequence (5) and (16) is
\[
\nabla_\alpha F^2 = \frac{2}{3} Q F^2 \sigma_\alpha.
\]
Since the spacetime is \( C^3 \) we can use the second Bianchi identities which, in terms of the right self-dual Weyl tensor, read \( \nabla_\alpha C^\alpha_{\beta\gamma\delta} = 0 \). A straightforward calculation shows
\[
0 = 4 \xi^\beta \xi^\gamma \nabla_\alpha C^\alpha_{\beta\gamma\delta} = \frac{\lambda}{9} F^2 Q^2 \sigma_\delta - \frac{F^2}{3} (\lambda \nabla_\delta Q + \xi_\delta \xi^\alpha \nabla_\alpha Q) - \sigma_\delta \sigma^\alpha \nabla_\alpha Q,
\]
which, after contraction with \( \sigma^\delta \) yields \( \sigma^\alpha \nabla_\alpha Q = \frac{1}{6} \lambda F^2 Q^2 \), where we used that \( \lambda \) is zero at most on a set with empty interior and \( \nabla_\alpha Q \) is continuous. Inserting this expression back into (15) we find \( \lambda \left( \nabla_\delta Q + \frac{1}{6} Q^2 \sigma_\delta \right) - \xi_\delta \xi^\alpha \nabla_\alpha Q = 0 \). From \( C_{\alpha\beta\gamma\delta} F^{\alpha\beta}_{\gamma\delta} = 2/3 (F^2)^2 Q \) we have \( \xi^\alpha \nabla_\alpha Q = 0 \) and, therefore,
\[
\nabla_\alpha Q + \frac{1}{6} Q^2 \sigma_\alpha = 0.
\]
The spacetime is not locally flat by assumption, so \( Q \) is not identically zero. Let us define the set \( K = \{ p \in V; Q(p) \neq 0 \} \) and suppose first that \( K \neq V \). Take a point \( q \in \partial K \) (i.e. in the topological boundary of \( K \)). Recall that the Ernst one-form is closed
on all of $V$ and therefore there exists a sufficiently small open neighbourhood $V_q$ of $q$ where the Ernst one-form is exact, i.e. it exists a function $\sigma|_{V_q}$ such that $\sigma|_{V_q} = \nabla_\alpha \sigma|_{V_q}$. On $K \cap V_q$, equation (19) can be integrated to give $\sigma - \frac{6}{Q} = c$ where $c$ is a complex constant. Since $\sigma$ is well-defined in $V_q$ and in particular bounded at $q$ it follows that $Q$ cannot vanish on $q$, against the assumption. Hence $Q$ is non-zero everywhere. Then, equation (19) shows that $\sigma_\alpha$ is exact, i.e. $\sigma_\alpha = \nabla_\alpha \sigma$. The connectedness of $V$ gives $Q = -6/(c - \sigma)$ everywhere. The integration of (17) completes the proof.

$$\blacksquare$$

4 The main Theorem

A straightforward calculation shows that the spacetime Simon tensor $S_{\alpha\beta\mu}$ with respect to the asymptotically timelike Killing vector vanishes identically in the Ker $r$ metric. The aim of this section is to prove that the converse is also true in the asymptotically flat case. More precisely, we prove the following theorem

**Theorem 2** Let $(V, g)$ be a smooth spacetime with the following properties

1. The metric $g$ satisfies the Einstein vacuum field equations,

2. $(V, g)$ admits a smooth Killing field $\vec{\xi}$ such that the spacetime Simon tensor associated to $\vec{\xi}$ vanishes everywhere,

3. $(V, g)$ contains a stationary asymptotically flat four-end $V^\infty$, $\vec{\xi}$ tends to a time translation at infinity in $V^\infty$ and the Komar mass of $\vec{\xi}$ in $V^\infty$ is non-zero.

Then $(V, g)$ is locally isometric to a Kerr spacetime.

**Remark 1.** By stationary asymptotically flat four-end we understand an open submanifold $V^\infty \subset V$ diffeomorphic to $I \times (\mathbb{R}^3 \setminus B(R))$, $I \in \mathbb{R}$ is an open interval and $B(R)$ is a closed ball of radius $R), such that, in the local coordinates defined by the diffeomorphism, the metric satisfies

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r \partial_r g_{\mu\nu}| \leq Cr^{-\alpha}, \quad \partial_r g_{\mu\nu} = 0$$

where $C, \alpha$ are positive constants, $r = \sqrt{\sum(x^i)^2}$ and $\eta_{\mu\nu}$ is the Minkowski metric. Usually, the definition of asymptotically flat four-end requires $I = \mathbb{R}$, but this is not necessary for our purposes. A result by Kennefick and Ó Murchadha [17] (see also Proposition 1.9 in [11]) shows that the Einstein field equations and the existence of a timelike Killing vector force $\alpha \geq 1$. It is then well-known (see e.g. Beig and Simon [18]) that the metric can be brought into the asymptotic form

$$g_{00} = -1 + \frac{2M}{r} + O(r^{-2}), \quad g_{0i} = -\epsilon_{ijk} \frac{4S^j x^k}{r^3} + O(r^{-3}), \quad g_{ij} = \delta_{ij} + O(r^{-1}), \quad (20)$$
where $M$ is the Komar mass $[19]$ of $\xi$ in the asymptotically flat end $V^\infty$ and $\epsilon_{ijl}$ is the alternating Levi-Civita symbol. It is worth noticing that assumption 3 in the theorem is used only in order to prove lemma 6 below, which fixes the value of the two arbitrary constants appearing in lemma 4. Hence, assumption 3 in the theorem could be replaced by any hypothesis under which lemma 6 still holds.

**Remark 2.** By Kerr spacetime we mean, as usual, the maximal analytic extension of the Kerr metric, as described by Boyer and Lindquist $[20]$ and Carter $[21]$. An element of the Kerr family will be denoted by $(V_{M,a}, g_{M,a})$, where $M$ denotes the Komar mass and $a$ the specific angular momentum. In particular, $(V_{M,0}, g_{M,0})$ is the Kruskal extension of the Schwarzschild spacetime.

**Remark 3.** As lemma 6 above shows, the vanishing of the Simon tensor at points where $\mathcal{F}^2 \neq 0$ and $\lambda \neq 0$ implies that the Weyl tensor takes the form $[16]$, which in particular shows that the Petrov type of the Weyl tensor is D. The general solution for vacuum spacetimes of Petrov type D was found by Kinnersley in $[22]$. Hence, we could in principle use his results in order to prove the theorem and indeed we share with his method the fact that we employ the Newman-Penrose formalism to prove the theorem. However, we do not assume analyticity of the metric as is usually done in the field of exact solutions, where the main motivation is obtaining explicit metrics satisfying Einstein field equations. This requires extra care with the choice of tetrads and coordinate systems and it is more convenient to produce a self-contained proof. In particular, we try to use invariantly defined quantities and avoid changing the null tetrad in order to make some spin coefficients zero. By doing this, the proof becomes geometrically more transparent and some insight is gained into the path leading from the vanishing of the Simon tensor up to the Kerr metric.

**Remark 4.** The theorem states that for any point $p \in (V,g)$, there exists an open neighbourhood $U_p$ of $p$ which is isometrically diffeomorphic to an open submanifold of $V_{M,a}$. Since the characterization of Kerr in terms of the spacetime Simon tensor is local and the only global requirement we make is the existence of an asymptotically flat end (which, as pointed out in Remark 1, is only used in order to fix the value of two constants), we should non expect in principle that the local isometry extends to an isometric embedding of $(V,g)$ into $(V_{M,a}, g_{M,a})$. There can be topological obstructions for this global embedding to exist. Analyzing this question in detail would require classifying the spacetimes which are locally isometric to Kerr and which are asymptotically flat (in the sense above, or perhaps under the stronger requirement $I = \mathbb{R}$). It would be necessary, among other things, to determine the discrete isometry groups of $(V_{M,a}, g_{M,a})$ such that the quotient metric still has an asymptotically flat four-end. This is not an easy problem because the global structure of the Kerr spacetime is not particularly simple. In the context of black hole uniqueness theorems one is mainly interested in the domain of outer communication. In this case, it is probably easy to show, after assuming $I = \mathbb{R}$ in the definition of asymptotically flat four-end, that the domain of outer communication of $(V,g)$ is diffeomorphic to the domain of outer com-
munication of \((V_{M,a}, g_{M,a})\). However, the analysis of this problem will be relevant only if the characterization of Kerr in terms of the spacetimeSimon tensor proves useful for extending the black hole uniqueness theorems to the non-analytic case, and we will not consider this question any further here. We should emphasize, however, that theorem 2 is “semi-local” because the existence of the local isometry is shown everywhere.

Throughout this section \((V, g)\) denotes a spacetime fulfilling the requirements of theorem 2. Let us normalize the Killing vector and choose the integration constant in the twist potential \(\omega\) so that \(\sigma \rightarrow 1\) at infinity in \(V^\infty\). A simple calculation using the asymptotic form of the metric (20) in \(V^\infty\) gives \(\mathcal{F}^2 = -4M^2/r^4 + O(r^{-5})\). Since by assumption \(M \neq 0\) we have that the open submanifold \(\hat{V}_f = \{p \in V; \mathcal{F}^2|_p \neq 0\}\) is non-empty. Possibly after redefining \(V^\infty\) as an appropriate asymptotically flat open submanifold of \(V^\infty\), we can assume that \(V^\infty \subset \hat{V}_f\). Let us define the spacetime \((V_f, g_f)\) as the connected component of \(\hat{V}_f\) containing the asymptotically flat region \(V^\infty\), with the induced metric. We have the following lemma

**Lemma 5** Let \((V, g)\) satisfy the hypotheses of the theorem 2 and \((V_f, g_f)\) be defined as above. Then, the Weyl tensor and \(\mathcal{F}^2\) in \((V_f, g_f)\) take the form

\[
\mathcal{C}_{\alpha\beta\gamma\delta} = \frac{-6}{1 - \sigma} \left( \mathcal{F}_{\alpha\beta} \mathcal{F}_{\gamma\delta} - \frac{1}{3} \mathcal{I}_{\alpha\beta\gamma\delta} \mathcal{F}^2 \right), \quad \mathcal{F}^2 = -\frac{1}{4M^2} (1 - \sigma)^4.
\]

**Proof.** In order to apply lemma 4 we must show that the set of points where \(\vec{\xi}\) has non-zero norm is dense in \(V_f\). We use the equation

\[
\nabla^\mu \nabla_\mu \sigma = -\mathcal{F}^2 = -2 \left( F_{\alpha\beta} F^{\alpha\beta} + i F^*_{\alpha\beta} F^{\alpha\beta} \right), \tag{21}
\]

which follows from \(\sigma_\mu = 2\xi^\alpha F_{\alpha\mu}\) and \(\nabla^\alpha F_{\alpha\beta} = 0\). Assume there exists an open set \(U \subset V_f\) where \(\vec{\xi}\) has zero norm. The real part of (21) implies \(F_{\alpha\beta} F^{\alpha\beta}|_U = 0\) and identity (13) provides \(\omega_\mu|_U = 0\). Using (21) again we find \(\mathcal{F}^2|_U = 0\), which is impossible in \(V_f\). Thus, lemma 4 can be applied. The asymptotic form of the metric (21) implies \(\sigma = 1 - 4M/r + O(r^{-2})\) in \(V^\infty\), which combined with the asymptotic behaviour of \(\mathcal{F}^2\) gives \(c = 1\) and \(A = -1/(4M^2)\).

It is convenient to define a function \(P|_{V_f} = (1 - \sigma)^{-1}|_{V_f}\). Equations (7) and (21) read, in terms of \(P\),

\[
\nabla_\mu P \nabla^\mu P = \frac{\lambda}{4M^2}, \quad \nabla^\mu \nabla_\mu P = \frac{1}{4M^2} \frac{2\overline{P} - 1}{P\overline{P}}, \tag{22}
\]

where the bar denotes complex conjugate. Define the real functions \(y\) and \(z\) by \(P = y + i z\). Notice that \(P\) is nowhere zero on \(V_f\) and thus \(y\) and \(z\) cannot vanish simultaneously in \(V_f\). As we shall see, the scalar functions \(y\) and \(z\) are very closely related with the
radial coordinate $r$ and the angular coordinate $\theta$ in the Boyer-Lindquist coordinates of the Kerr metric. They have an intrinsic definition in terms of the Ernst potential and will be essential for proving the existence of the local isometry with the Kerr spacetime. Since $\mathcal{F}_{\alpha\beta}$ is self-dual and $\mathcal{F}^2|_{V_f} = -1/(4M^2P^4)$, there exist two real, smooth, non-zero, null vector fields $\vec{l}_\pm$ satisfying $(\vec{l}_+, \vec{l}_-) = -1$ such that

$$\mathcal{F}_{\alpha\beta} = \frac{1}{4MP^2} \left( -l_{+\alpha}l_{-\beta} + l_{+\beta}l_{-\alpha} - i \eta_{\alpha\beta\gamma\delta} l_+^\gamma l_-^\delta \right),$$

($\vec{l}_\pm$ are the eigenvectors $\mathcal{F}_{\alpha\beta}$, i.e. $\mathcal{F}_{\alpha\beta} l_\alpha^\alpha \propto l_{\pm\beta}$). The contraction of this equation with $\xi^\alpha$ gives, after splitting into the real and imaginary parts,

$$\nabla_\beta y = \frac{1}{2M} \left[ - (\tilde{\xi}, \tilde{l}_+) l_{-\beta} + (\tilde{\xi}, \tilde{l}_-) l_{+\beta} \right] \quad \nabla_\beta z = -\frac{1}{2M} \eta_{\alpha\beta\gamma\delta} \xi^\alpha l_+^\gamma l_-^\delta,$$

(23)

where the parentheses denote the scalar product with respect to the metric $g_f$. Moreover $L_\xi \mathcal{F}_{\alpha\beta} = 0$ provides, after using $(\vec{l}_+, \vec{l}_-) = -1$, the relation

$$[\tilde{\xi}, \tilde{l}_\pm] = \pm C_1 \tilde{l}_\pm \quad \Rightarrow \quad \xi^\alpha \nabla_\alpha (\tilde{\xi}, \tilde{l}_\pm) = \pm C_1 (\tilde{\xi}, \tilde{l}_\pm)$$

(24)

for some function $C_1$. Contracting $\sigma_\beta = 2 \xi^\alpha \mathcal{F}_{\alpha\beta}$ with $\mathcal{F}_{\mu}^{\beta}$ and using (23) we get

$$\xi_\beta = - (\tilde{\xi}, \tilde{l}_+) l_{-\beta} - (\tilde{\xi}, \tilde{l}_-) l_{+\beta} - 2M \eta_{\beta\mu\gamma\delta} \nabla^{\mu} z l_+^\gamma l_-^\delta.$$

(25)

**Proposition 1** The norms of $\nabla_\alpha z$ and $\nabla_\alpha y$ are

$$\nabla_\alpha z \nabla^{\alpha} z|_{V_f} = \frac{B - z^2}{4M^2 (y^2 + z^2)} |_{V_f} \quad \nabla_\alpha y \nabla^{\alpha} y|_{V_f} = \frac{y^2 - y + B}{4M^2 (y^2 + z^2)} |_{V_f}$$

(26)

where $B$ is a non-negative constant. Moreover $z^2|_{V_f} \leq B$.

**Proof.** The real part of the first equation in (23) reads

$$\nabla_\alpha y \nabla^{\alpha} y - \nabla_\alpha z \nabla^{\alpha} z = \frac{1}{4M^2} \left( 1 - \frac{y}{y^2 + z^2} \right)$$

(27)

so that we only need to prove $4M^2 \nabla_\alpha z \nabla^{\alpha} z = (B - z^2)/(y^2 + z^2)$. Let us define a function $H = 4M^2 P \nabla_\alpha z \nabla^{\alpha} z$. Take an arbitrary point $p \in V_f$ and consider a sufficiently small open neighbourhood $U_p \subset V_f$ of $p$ so that there exists a smooth complex vector field $\vec{m}|_{U_p}$ such that $\{\vec{l}_+, \vec{l}_-, \vec{m}, \vec{m}^*\}$ is a null tetrad in $U_p$ with positive orientation, (i.e. satisfying $\eta_{\alpha\beta\gamma\delta} l_+^\gamma l_-^\delta m^\mu m^\nu = -i$). We shall use the Newman-Penrose (NP) notation to denote the Ricci rotation coefficients associated with this null basis.
We shall follow the conventions in [13] except that we use \( \vec{l}_+ \) and \( \vec{l}_- \) instead of \( \vec{k} \) and \( \vec{l} \). Lemma 5 shows that \( \vec{l}_\pm \) are principal null directions of the Weyl tensor and, from the Goldberg-Sachs theorem [23], they define geodesic and shearfree null congruences, which in NP notation means \( \kappa = \sigma = \nu = \lambda = 0 \). The subset of NP equations we shall require are

\[
D\tau = \rho (\tau + \pi) + \tau (\epsilon - \tau), \quad \delta \rho = \rho (\pi + \beta) + \tau (\rho - \pi), \quad (28)
\]

\[
\Delta \pi = -\mu (\pi + \pi) + \pi (\tau - \gamma), \quad \delta \mu = -\pi (\mu - \pi) - \mu (\alpha + \beta). \quad (29)
\]

For later use, let us also quote the commutators between the tetrad vectors, which read

\[
\Delta D - D \Delta = (\gamma + \gamma) D + (\epsilon + \epsilon) \Delta - (\tau + \pi) \delta - (\tau + \pi) \delta, \quad (30)
\]

\[
\delta D - D \delta = (\pi + \beta - \pi) D - (\pi + \beta - \pi) \delta, \quad (31)
\]

\[
\delta \Delta - \Delta \delta = (\pi - \pi - \beta) \Delta + (\mu - \gamma + \gamma) \delta, \quad (32)
\]

\[
\delta \delta - \delta \delta = (\pi - \pi - \mu) D + (\pi - \pi - \mu) \Delta - (\pi - \pi - \mu) \delta - (\pi - \pi - \mu) \delta. \quad (33)
\]

From lemma 5 the only non-zero component of the Weyl spinor in this tetrad is \( \Psi_2 = -1/(8m^2 P^3) \). The Bianchi identities in the NP formalism are \( DP = -\rho P, \Delta P = \mu P, \delta P = -\tau P \) and \( \delta P = \pi P \). Thus, (23) becomes

\[
\nabla_\beta y|_{V_p} = P (\rho l_\beta - \mu l_\beta), \quad \nabla_\beta z|_{V_p} = -i \pi P m_\beta + i \tau P \bar{m}_\beta \quad (34)
\]

which implies \( \rho P = \rho \bar{P}, \mu P = \mu \bar{P} \) and \( \tau P = \pi \bar{P} \). In terms of the spin coefficients, the function \( H \) reads \( H = 8M^2 P^2 \bar{P}^2 \pi \bar{\pi} \geq 0 \) and equation (27) becomes

\[
H = 8M^2 P^3 \bar{P} \rho \mu + y - y^2 - z^2. \quad (27)
\]

A straightforward calculation using equations (28)-(29) gives \( DH = 0, \Delta H = 0, \delta H = -2i \pi \bar{P} z \). Equation (34) implies \( Dz = \Delta z = 0 \) and \( \delta z = i \pi \bar{P} \), from which the constancy of \( H + z^2 \) in \( U_p \) follows trivially. Since this holds for any \( p \in V_f \) and \( V_f \) is connected, we obtain \( H|_{V_f} = B - z^2|_{V_f} \geq 0 \) where \( B \) is a non-negative constant.

Let us now define two open sets \( V_\pm = \{ p \in V_f ; (\vec{\xi}, \vec{l}_\pm)|_p \neq 0 \} \) and introduce the null vector fields

\[
\vec{s}_\pm|_{V_\pm} = \frac{2M}{(\xi, \vec{l}_\pm)} \vec{l}_\pm|_{V_\pm}, \quad \vec{k}_\pm|_{V_\pm} = \frac{(\xi, \vec{l}_\pm)}{2M} \vec{l}_\pm|_{V_\pm}
\]

which satisfy \( (\vec{s}_\pm, \vec{\xi})|_{V_\pm} = 2M, (\vec{s}_\pm, \vec{k}_\pm)|_{V_\pm} = -1 \). In terms of \( \vec{s}_\pm \) and \( \vec{k}_\pm \), equation (23) becomes

\[
\nabla_\alpha y|_{V_\pm} = \mp (k_{\pm\alpha} + W s_{\pm\alpha})|_{V_\pm} = \mp \left( k_{\pm\alpha} + \frac{-y^2 + y - B}{2M^2(y^2 + z^2)} s_{\pm\alpha} \right)|_{V_\pm}. \quad (35)
\]

\[\text{The spin coefficient } \sigma \text{ has nothing to do with the Ernst potential } \sigma \text{ we have been using throughout. This is the only place where it occurs and therefore no confusion should arise.}\]
where we have defined

\[ W|_{V_f} \equiv -\left( \zeta, \bar{\iota}_+ \right) \cdot \left( \zeta, \bar{\iota}_- \right) \frac{4M^2}{\nu_f} \left| -\frac{y^2 + y - B}{8M^2(y^2 + z^2)} \right|_{V_f}, \tag{36} \]

the equality being a consequence of proposition 1.

The case in which the Killing vector \( \xi \) is hypersurface orthogonal requires a separate treatment. This case is characterized by \( \nabla_{\mu} z = 0 \) in some open set \( U \). Then, the imaginary part of the second equation in (22) implies \( z|_{U} = 0 \). Proposition 1 shows \( B = 0 \) and hence that \( z = 0 \) everywhere.

**Proposition 2** Assume that the Killing vector \( \xi \) of \( (V_f, g_f) \) is hypersurface orthogonal. Then, at any point \( p \in V_+ \cup V_- \) there exists a neighbourhood \( U_p \) of \( p \) which can be isometrically embedded into the Kruskal-Schwarzschild spacetime.

**Proof.** For definiteness, we shall give the proof only for \( p \in V_+ \). The proof for \( V_- \) requires only a few sign changes and will be omitted. So, take a point \( p \in V_+ \) and a sufficiently small open neighbourhood \( U_p \subset V_+ \) of \( p \). We shall use the same notation as in the proof of proposition 1. Since \( P = \gamma \) is real, so are \( \mu \) and \( \rho \). Furthermore \( z = 0 \) implies \( \pi = \tau = 0 \). The commutators (30) and (33) show that the two-planes spanned by \( \{ \bar{\iota}_+, \bar{\iota}_- \} \) and by \( \{ \bar{m}, \bar{\iota}_m \} \) are surface-forming. Denote by \( \{ S_{i} \} \) and \( \{ S_{m} \} \) (\( i \) is an index labeling each surface) the corresponding families of surfaces. Thus, \( U_p \) (or some open subset thereof containing \( p \)) is foliated by two families of mutually orthogonal two-surfaces. We want to show that the induced metric in \( S_{m} \) is that of a two-sphere. To that end, we use the Gauss equation to obtain the curvature scalar of the induced metric in \( S_{m} \). Recall that the second fundamental form \( \kappa \) of a surface is defined by

\[ \kappa(X, Y) = \left( \nabla_{X} Y \right)^{\perp}, \]

where \( X, Y \) are tangent vectors to the surface and \( \perp \) denotes the orthogonal projection to the surface. A simple calculation using the definition of the NP coefficients and (22) gives

\[ \kappa(X, Y)_{\parallel} \bigg|_{S_{m}} = -\frac{\nabla_{\beta} y}{y} g_f \left( \bar{\iota}_m, \bar{\iota}_m \right) \bigg|_{S_{m}}. \tag{37} \]

Using the Gauss equation (see e.g. [24]) combined with lemma 3, we obtain \( \hat{R}^c = 1/(2M^2y^2) \big|_{S_{m}} \), where \( \hat{R}^c \) is the Ricci scalar of the surface \( S_{m} \). Since \( \delta y = 0 \) (i.e. \( y \) is constant on \( S_{m} \)) it follows that \( S_{m} \) has positive constant curvature and therefore its metric is locally the round metric of a two-sphere of radius \( r = 2My \). As \( \nabla_{\alpha} y \) is nowhere zero on \( V_+ \cup V_- \), there exist three functions \( x^0, x^2, x^3 \) defined on \( U_p \) so that \( \{ x^0, y, x^2, x^3 \} \) is a coordinate system on \( U_p \) adapted to the foliation, i.e such that \( S_{m} \) is defined by \( x^0 = \text{const}, \ y = \text{const} \) and \( S_{i} \) is defined by \( x^A = \text{const}, \ A = 2, 3 \). Expression (23) shows that \( \xi \) is non-zero on \( V_+ \cup V_- \) and tangent to \( S_{i} \), hence \( \xi|_{U_p} = \xi^0 \partial_{x^0}|_{U_p} \) for some non-zero function \( \xi^0 \). The Killing equations imply \( \partial_{x^0} \xi^o|_{U_p} = 0 \). Regarding \( s_{+}, \)
Let us take an arbitrary point \( p \) at any point on \( V \). The proof will again be given only for \( V_\pm \cap V_1 \). The proof in \( V_- \cap V_1 \) is essentially the same. First, we must prove that \( \{ u, y, x^2, x^3 \} \) is a coordinate system in \( U_p \) in which \( \zeta = \partial_u|_{U_p} \) and \( \bar{s}_+ = \partial_y|_{U_p} \). It remains to determine \( g_{AB} \equiv (\partial_{x^A}, \partial_{x^B})|_{U_p} \). Using equation (37) we easily obtain

\[
g_{AB} = 4M^2 y^2 \bar{g}_{AB}(x^C),
\]

for some functions \( \bar{g}_{AB} \) independent of \( u \) and \( y \). Imposing that the induced metric of \( S^m \) is the metric of a sphere of radius \( r = 2My \) we obtain, after defining \( r = 2My \), that

\[
d s^2\big|_{U_p} = \left( -1 + \frac{2M}{r} \right) du^2 + 2du dr + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)\big|_{U_p},
\]

which proves that \( U_p \) can be isometrically embedded into the Kruskal-Schwarzschild spacetime. \( \square \)

We now return to the general case (i.e. without imposing staticity). Let us define the vector field

\[
\bar{\eta}|_{V_f} \equiv \frac{1}{2M} \left( B + y^2 \right) \zeta + \frac{(y^2 + z^2)}{2M} \left[ (\zeta, \bar{\ell}_-) \bar{\ell}_+ + (\zeta, \bar{\ell}_+) \bar{\ell}_- \right]|_{V_f},
\]

which, in \( V_\pm \), can be rewritten as

\[
\bar{\eta}|_{V_\pm} = \frac{1}{2M} \left( B + y^2 \right) \zeta + \frac{(y^2 + z^2)}{2M} \bar{k}_\pm + \frac{y^2 - y + B}{8M^2} \bar{s}_\pm|_{V_\pm}.
\]

Let us still define another open set \( V_1 = \{ p \in V_f ; z^2 |p < B \} \) and a vector field \( \bar{b}^0|_{V_1} = 4M^2 (y^2 + z^2)(B - z^2)^{-1} \nabla^a z|_{V_1} \).

**Proposition 3** Assume that the Killing vector \( \zeta \) is not hypersurface orthogonal. Then \( \{ \xi, \bar{s}_+, \bar{b}, \bar{\eta} \} \) define a holonomic basis of vector fields on \( V_\pm \cap V_1 \). Furthermore, at any point \( p \) in \( (V_+ \cup V_-) \cap V_1 \) there exists a neighbourhood \( U_p \) of \( p \) that can be isometrically embedded into the Kerr spacetime.

**Proof.** The proof will again be given only for \( V_+ \cap V_1 \). The proof in \( V_- \cap V_1 \) is essentially the same. First, we must prove that \( \{ \xi, \bar{s}_+, \bar{b}, \bar{\eta} \} \) commute and are linearly independent at any point on \( V_\pm \cap V_1 \). An immediate consequence of (24) is \( [\zeta, \bar{s}_+]|_{V_+} = \bar{0}, [\zeta, \bar{k}_+]|_{V_+} = \bar{0} \) and hence \( [\bar{\xi}, \bar{\eta}]|_{V_+} = \bar{0} \). \( [\xi, \bar{b}]|_{V_1} = \bar{0} \) follows easily from the Killing equations. Let us take an arbitrary point \( p \in V_+ \cap V_1 \) and a sufficiently small neighbourhood \( U_p \subset \bar{V}_+ \cap V_1 \) of \( p \) where there exists a complex null field \( \bar{m} \) such that \( \{ \bar{s}_+, \bar{k}_+, \bar{m}, \bar{\eta} \} \) is a positively oriented null tetrad on \( U_p \). All the relations obtained in the proof of proposition 4 are still valid when all NP spin coefficients referred to this new null tetrad. Moreover, relation (35) implies \( DP = 1 \) and \( \Delta P = W \) and therefore \( \rho = -1/P, \mu = W/P \), which can be used to simplify equations (28) and commutators (30)-(33). In particular, the second equation in (28) gives \( \bar{\alpha} + \beta = \pi \) which simplifies the calculations.
considerably. Using (34), \( \vec{b} \) takes the form \( b^{\alpha}|_{U_p} = 4i M^2 (y^2 + z^2) (B - z^2)^{-1}(p^\alpha p^{\beta} - \pi P m^\beta) \). \([\vec{s}_+, \vec{b}]|_{U_p} = [\eta, \vec{b}]|_{U_p} = 0 \) follow after a simple calculation using (28), (34) and (32). In order to prove \([\vec{s}_+, \vec{\eta}]|_{U_p} = 0 \) we need to evaluate \([\vec{s}_+, \vec{k}_+] \). This is done by rewriting (23) in the null tetrad, \( \vec{\xi}|_{U_p} = 2M \left( W \vec{s}_+ - \vec{k}_+ - P \pi \vec{m} - \pi P \vec{m} \right) \) \( \) \( \) \( \). Imposing \([\vec{\xi}, \vec{k}_+] = 0 \), we easily obtain

\[
\left[ \vec{s}_+, \vec{k}_+ \right] = -\frac{2y}{y^2 + z^2} \vec{k}_+ - \frac{y}{M(y^2 + z^2)} \vec{\xi} + \frac{1 - 2y}{8M^2(y^2 + z^2)} \vec{s}_+ \right|_{V_+},
\]

from which \([\vec{s}_+, \vec{\eta}]|_{V_+} = 0 \) is proven by simple calculation. In order to show that \( \vec{\xi}, \vec{s}_+, \vec{\eta}, \vec{b} \) are linearly independent, we evaluate their scalar products on \( V_+ \cap V_1 \), which give

\[
\begin{align*}
(\vec{\xi}, \vec{\xi}) &= -1 + \frac{y}{y^2 + z^2}, \quad \left( \vec{s}_+, \vec{\xi} \right) = 2M, \quad \left( \vec{\xi}, \vec{b} \right) = 0, \quad \left( \vec{\xi}, \vec{\eta} \right) = \frac{y(B - z^2)}{2M(y^2 + z^2)}, \\
(\vec{s}_+, \vec{s}_+) &= 0, \quad \left( \vec{s}_+, \vec{b} \right) = 0, \quad \left( \vec{s}_+, \vec{\eta} \right) = B - z^2, \quad \left( \vec{b}, \vec{b} \right) = \frac{4M^2(y^2 + z^2)}{B - z^2}, \\
(\vec{b}, \vec{\eta}) &= 0, \quad \left( \vec{\eta}, \vec{\eta} \right) = \frac{B - z^2}{4M^2(y^2 + z^2)} \left[ z^2(y^2 - y + B) + y^4 + 2By^2 + B \right].
\end{align*}
\]

The determinant of the matrix formed by these scalar products (in the obvious order) is \(-4M^2(y^2 + z^2) \neq 0 \) which implies the linear independence of the vectors. Let us define a positive constant \( a \) by \( B = a^2/(4M^2) \) and introduce, as before, a function \( r \) by \( r = 2My|_{V_1} \). Since \( z^2|_{V_1} < B \), we can also define a function \( \theta \) on \( V_1 \) by \( z = a \cos \theta/(2M) \) \( \). The holonomicity of \( \{ \vec{\xi}, \vec{s}_+, \vec{b}, \vec{\eta} \} \) implies that there exist two functions \( u|_{U_p} \) and \( \phi|_{U_p} \) such that \( \{u, r, \theta, \phi\} \) is a coordinate system in \( U_p \) satisfying

\[
\vec{\xi} = \frac{\partial}{\partial a}, \quad \vec{s}_+|_{U_p} = 2M \frac{\partial}{\partial r}, \quad \vec{b}|_{U_p} = -\frac{2M}{a \sin \theta} \frac{\partial}{\partial \theta}, \quad \vec{\eta}|_{U_p} = \frac{a}{8M^3} \frac{\partial}{\partial \phi},
\]

which, after using (39), yields

\[
\begin{align*}
ds^2|_{U_p} &= \left(-1 + \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right) du^2 + 2du dr + \frac{4Mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} du d\phi + 2a \sin^2 \theta dr d\phi \\
&\quad + \left(r^2 + a^2 \cos^2 \theta \right) d\theta^2 + \frac{\sin^2 \theta \left[(r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 - 2Mr + a^2)\right]}{r^2 + a^2 \cos^2 \theta} d\phi^2 \right|_{U_p}.
\end{align*}
\]

This is the form of the Kerr metric with mass \( M \) and specific angular momentum \( a \) in the so-called Kerr coordinates. This shows that \( U_p \) is isometrically diffeomorphic to an open submanifold of the Kerr spacetime. \( \square \)

We have shown before that \( \nabla_a z|_{U} = 0 \) for an open set \( U \subset V_f \) implies \( z|_{V_f} = 0 \) and hence that \( \vec{\xi} \) is hypersurface orthogonal everywhere. This shows that \( V_1 \) is either dense
in $V_f$ (when $\bar{\xi}$ is not hypersurface orthogonal) or $V_1$ is empty (when $\bar{\xi}$ is hypersurface orthogonal). The next lemma gives some properties of the complements of $V_1$ and $V_+ \cup V_-$. 

**Lemma 6** The vector field $\bar{\eta}$ defined above is a Killing vector field in $V_f$. The complement of $V_1$ in $V_f$ is $V_f \setminus V_1 = \{ p \in V_f; \bar{\eta}|_p = 0 \}$. Moreover,

- If $B = 0$ then $V_f \setminus (V_+ \cup V_-) = \{ p \in V_f; \bar{\xi}|_p = 0 \}$
- If $0 < B \leq 1/4$ then $V_f \setminus (V_+ \cup V_-) = \{ p \in V_f; (\bar{\eta} - y_+ \bar{\xi})|_p = 0 \text{ or } (\bar{\eta} - y_- \bar{\xi})|_p = 0 \}$, where $y_\pm = (1 \pm (1 - 4B)^{1/2})/4M$.
- If $B > 1/4$ then $V_f \setminus (V_+ \cup V_-)$ is empty.

**Proof.** In the particular case $B = 0$ we have $z|_{U_f} = 0$ and (38) implies $\bar{\eta}|_{U_f} = 0$, so that the first two claims of the lemma become obvious. When $B \neq 0$, proposition 3 shows that $\bar{\eta}$ is a Killing vector on the non-empty set $(V_+ \cup V_-) \cap V_1$, which is linearly independent of $\bar{\xi}$. In the non-hypersurface orthogonal case $V_1$, is dense in $V_f$ and hence $\bar{\eta}$ is a Killing vector in $V_+ \cup V_-$ (where the bar over a set denotes its closure). Let us show that $V_+ \cup V_- = V_f$. Assume that the open set $U_1 = V_f \setminus (V_+ \cup V_-)$ is non-empty. From the definition of $V_\pm$, we have $(\bar{\xi}, \bar{l}_\pm)|_{U_1} = 0$ and (23) implies $\nabla_\alpha y|_{U_1} = 0$. The real part of the second equation in (22) implies then $y|_{U_1} = 1/2$ and proposition 4 fixes $B = 1/4$. The vector field $\bar{\eta} - 1/(4M)\bar{\xi}$ is a non-zero Killing field in $V_+ \cup V_-$ which vanishes identically on the open set $U_1$, but this is impossible due to well-known properties of Killing vectors. Thus, $\bar{\eta}$ is a Killing field everywhere. The second part of the lemma is an obvious consequence of the definition of $\bar{\eta}$ (38). Regarding the statements for $V_f \setminus (V_+ \cup V_-)$, the case $B = 0$ is trivial from equation (23), while the last two statements follow easily from (38) and (25), after noticing that the vanishing of $(\bar{\xi}, \bar{l}_+)$ implies that of $W$.

We can now prove the main theorem.

**Proof of the Theorem.**

Propositions 3 and 5 show that the local isometry exists in a neighbourhood of any point in $(V_+ \cup V_-)$ in the hypersurface orthogonal case and of any point in $(V_+ \cup V_-) \cap V_1$ in the non-hypersurface orthogonal case. It remains to show that the local isometry exists also in the complement of those sets in $V_f$ which, from lemma 3 correspond to the fixed points of certain Killing vectors in $V_f$. Define the open set $W_1 = V_+ \cup V_- \subset V_f$ in the hypersurface orthogonal case and $W_1 = (V_+ \cup V_-) \cap V_1$ in the non-hypersurface orthogonal one. Take an arbitrary point $q \notin W_1$ and a neighbourhood $U_q \subset V_f$ of $q$. Since the set of fixed points of a Killing vector is either an isolated point or a smooth, two-dimensional, totally geodesic surface (25), the set $U_q = U_q \cap W_1$ is connected. Furthermore, any point in $U_q$ has a neighbourhood which can be isometrically embedded in the Kerr spacetime. Combining this two facts and choosing $U_q$ small enough, it is
easy to see that $\tilde{U}_q$ can be isometrically embedded into the Kerr spacetime. This map can be extended to the closure of $\tilde{U}_q$ by continuity. Since $\tilde{U}_q$ is dense in $U_q$, this defines a map $\Psi_q : U_q \to (V_{M,a}, g_{M,a})$. It is now a simple matter to show, using the continuity of the metric, that $\Psi_q$ is an isometric embedding.

In order to complete the proof we need to show that $V_f = V$. Assume, on the contrary, that $V_f$ is a proper subset of $V$ and take a point $q \in \partial V_f$. From continuity of $F^2$ we must have $\lambda|_q = 1$, hence $\xi$ is timelike in some neighbourhood of $q$. Consider a smooth curve $\gamma_p(s)$ defined on some interval $s_0 < s \leq 1$ such that $\gamma_p(s) \in V_f$, $\forall s \in (s_0, 1)$, $\gamma_p(1) = p$ and such that the tangent vector $\dot{\gamma}(s)$ is orthogonal to $\xi$ and of unit length (the existence of such a curve is easy to establish). Define the real function $Y(s) = (y \circ \gamma_p)(s)$. Since $\sigma \to 1$ when we approach $p$ and $z$ remains bounded, we must have $Y(s) \to \infty$ when $s \to 1$. Using $4M \nabla_\alpha y \nabla^\alpha y|_{Y(s)} \to 1$ when $s \to 1$, we can assume that $\nabla_\alpha y|_{Y(s)}$ is spacelike for $s \in (s_0, 1)$. Then

$$\left(\frac{dY}{ds}(s)\right)^2 = \left(\nabla_\alpha y|_{Y(s)} \dot{\gamma}_p^\alpha(s)\right)^2 \leq \nabla_\alpha y \nabla^\alpha y|_{Y(s)},$$

where we have used the fact that $\{\nabla_\alpha y|_{\gamma_p(s)}, \dot{\gamma}_p(s)\}$ define a spacelike two-plane and we have applied the Schwarz inequality. Hence $\frac{dY}{ds}$ stays bounded which contradicts $Y \to \infty$ when $s \to 1$. This completes the proof of the theorem.

Let us remark that the existence of a local isometry between two spacetimes $(M_1, g_1)$ and $(M_2, g_2)$ on a dense subset of $M_1$ does not imply, in general, the existence of the local isometry everywhere on $M_1$. It is easy to construct counterexamples when the manifolds are not analytic. It is very likely that this cannot happen when the target manifold is analytic and inextendible. In any case, this result is not needed here because the local isometry is ensured on an open, dense and connected set, which makes the extension of the local isometry to all of $M_1$ simple.

5 Discussion

The characterization of the Kerr metric given by W.Simon [4] involved a three-index tensor in the quotient manifold which did not have a clear geometrical interpretation (except in the hypersurface orthogonal case, when it is equivalent to the Cotton tensor). In this paper we have translated the Simon tensor into the spacetime and have extended it everywhere. By doing this, we can also obtain a geometrical interpretation for the Simon tensor and hence a nice geometrical property which characterizes the Kerr metric. Any spacetime with a non-trivial Killing field $\xi$ has a privileged two-form $F_{\alpha\beta} = \nabla_\alpha \xi_\beta$, which can be algebraically classified by analyzing the eigenvalue problem associated with its self-dual two-form $F_{\alpha\beta}$. At points where $F_{\alpha\beta}$ is so-called regular (i.e. $F^2 \neq 0$), there exist two principal null directions defined as the eigenvectors of $F_{\alpha\beta}$ (i.e. $F_{\alpha\beta}l^\beta \propto l_\alpha$. At points where $F_{\alpha\beta}$ is singular (i.e $F^2 = 0$ with $F_{\alpha\beta} = 0$)
there exists one principal null direction. Similarly, the Petrov classification of the Weyl tensor is the algebraic classification of the endomorphism of the space of self-dual two-forms defined by

$$C(\mathcal{X})_{\alpha\beta} = C_{\alpha\beta}^{\mu\nu}\mathcal{X}_{\mu\nu}. \quad (40)$$

There exist four principal null directions of the Weyl tensor (which degenerate when the Weyl tensor is algebraically special). A repeated principal null direction of the Weyl tensor is a non-zero null vector field \(\vec{l}\) satisfying

$$C_{\alpha\gamma\delta\beta}l^\beta l^\delta = F l^\alpha l^\gamma$$

for some function \(F\). In section 3 we have shown that, at points where \(\vec{\xi}\) is non-null and \(\mathcal{F}_{\alpha\beta}\) is regular, the vanishing of the Simon tensor is equivalent to the Weyl tensor taking the form \((16)\). Let us state without proof that this also holds at points \(\mathcal{F}^2 = 0\) provided \(\mathcal{F}_{\alpha\beta} \neq 0\). The Petrov classification of \((16)\) is very easy. In this case, the eigenvalues of the endomorphism \((14)\) are \(2/3Q(p)\mathcal{F}^2|_p\) (with eigenvector \(\mathcal{X}_{\alpha\beta}|_p\)) and \(-1/3Q(p)\mathcal{F}^2|_p\) (the eigenspace of which is the set of self-dual two forms \(\mathcal{X}\) satisfying \(\mathcal{X}_{\alpha\beta}\mathcal{F}^\alpha\beta|_p = 0\)). Hence, the Weyl tensor \((16)\) is of Petrov type D whenever \(Q\mathcal{F}^2|_p \neq 0\), type N when \(\mathcal{F}^2|_p = 0, Q(p) \neq 0\) and Type 0 when \(Q(p) = 0\). More importantly, the principal null directions of the Weyl tensor coincide exactly with the principal null directions of \(\mathcal{F}_{\alpha\beta}\) (whenever \(Q(p) \neq 0\)). At points where \(\mathcal{F}_{\alpha\beta}\) is regular, each principal null direction of \(\mathcal{F}_{\alpha\beta}\) is a double principal null direction of the Weyl tensor and at points where \(\mathcal{F}_{\alpha\beta}\) is singular, the principal null direction of \(\mathcal{F}_{\alpha\beta}\) is a quadruple principal null direction of the Weyl tensor. Hence, the Kerr metric is characterized (among asymptotically flat vacuum solutions with a Killing field which tends to a time translation at infinity) by the fact that the principal null directions of the Weyl tensor coincide with the principal null directions of the Killing form \(\mathcal{F}_{\alpha\beta}\). This also shows why the Kerr metric enjoys such a privileged position among asymptotically flat stationary vacuum solutions. The geometrical simplicity of this characterization also indicates that the Kerr metric is, in that sense, the simplest possible asymptotically flat stationary vacuum metric.

A systematic study of the Killing form \(\mathcal{F}_{\alpha\beta}\) for electrovacuum spacetimes admitting a Killing field has been carried out recently by F.Fayos and C.F.Sopuerta [26], who have found a way to determine the principal null directions of \(\mathcal{F}_{\alpha\beta}\) and of the electromagnetic field. Several conditions on the principal null directions can then be imposed, thus restricting the possible geometries. For instance, the condition that the principal null directions of \(\mathcal{F}_{\alpha\beta}\) are aligned with the principal null directions of the electromagnetic field can be imposed. Similarly, conditions like some (or all) principal null directions of \(\mathcal{F}_{\alpha\beta}\) being shear-free and/or geodesic can also be studied. A particular interesting case would be studying which electrovacuum spacetimes admitting a Killing vector field are allowed such that the principal null directions of \(\mathcal{F}_{\alpha\beta}\) and those of the electromagnetic field are geodesic and shear-free. From the Goldberg-Sachs theorem, these null directions must then be repeated principal null directions of the Weyl
tensor, so that the Petrov type is D, N or 0 (depending on the degree of degeneracy). A straightforward calculation shows that the Kerr-Newman metric has Petrov type D with the two repeated principal null directions aligned with the principal null directions of $\mathcal{F}_{\alpha\beta}$ (as in the Kerr metric) and also aligned with the principal null directions of the electromagnetic field. It is a matter for future research to analyze whether the converse is also true (after assuming asymptotic flatness) in order to obtain a characterization of the Kerr-Newman spacetime.

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