Small vibrations of mechanical systems in the case of degenerate eigenvalues

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Abstract. Free vibrations of mechanical and hydrodynamic systems are considered. Some well-known works in this field are summarized and some aspects of computational nature are discussed. The eigenvalue boundary value problem is solved in the case of multiple roots for a system with $n$ degrees of freedom. The case of the zero root of a semidefinite system is considered. The question of orthogonality of forms of natural oscillations is considered. It is shown that in some cases, the proximity of eigenvalues occurs in problems of dynamics of an aircraft containing elastic containers with a liquid. On the example of a mechanical analog of transverse vibrations of a cylindrical tank with a liquid, the possibility of rotation of the free surface of the liquid is shown.

1. Introduction
The problem of free vibrations of mechanical and hydrodynamic systems with a finite and infinite number of degrees of freedom in the case of multiple and close eigenvalues is a complex mathematical problem. There are known statements and solutions of such problems in works on the theory of vibrations by Strelkov S. P. [1], Kolesnikov K. S. [2] and other authors [3-8]. In this paper, based on these sources, studies were conducted that allowed us to obtain some clarifying additional results. For example, in comparison with [9], the case of multiple roots and the case of the zero root of a semidefinite system are considered in detail. The question of orthogonality of forms of natural oscillations is also considered. The results obtained may be of interest to specialists dealing with vibrations of various systems.

The boundary value problem for eigenvalues in a special case - in the case of multiple solutions - is considered. Since the solution of the boundary value problem is reduced mainly to the problem of determining the eigenvalues of a matrix of finite dimension, this paper deals with the problem of close (multiple) eigenvalues for a mechanical system with $n$ degrees of freedom. In General, the structure of a matrix, whose eigenvalues and eigenvectors are found, is quite complex [10-13]. Therefore, this problem is a fairly serious mathematical problem of linear algebra and is not considered here in such a General statement. This problem is narrowed down in the article. Some well-known works in this field [2-6,8,14,15] are summarized and some aspects of a computational nature that can be useful in practice from a computational point of view are discussed. In addition, it is shown that in some cases (examples are given), the proximity of eigenvalues occurs in problems of the dynamics of an aircraft containing elastic containers with a liquid. Hydrodynamic problems with multiple eigenvalues are also formulated.

Generalizing some known data, the authors come to the conclusion that the method of half division (and related methods that are widely used in solving problems using the finite element method) in the
case of close (multiple) eigenvalues of the matrix becomes ineffective. Due to the large dimension of the matrix in the method of finite element and due to the above circumstances, in this case it is advisable to use, if possible, the method of eigenfunctions or projection methods.

Using the theory of vibrations of a rigid cylindrical tank with a liquid in a linear setting, it is shown that under certain initial conditions, the movement of the system is reduced to the rotation of the mirror of the free surface of the liquid relative to the axis of symmetry of the vessel. This fact was established earlier by other researchers based on the analysis of nonlinear equations of motion of a tank with a liquid in the presence of a free liquid surface.

2. The case of multiple eigenvalues of a boundary value problem

Let the differential equation of the boundary value problem have the form \[1\]:
\[
M[y] = \lambda \cdot N[y].
\]

Here \(M[y]\) and \(N[y]\) are some linear differential operators for functions of several independent variables \(y_1, y_2, \ldots\).

Boundary conditions are added to this equation:
\[
u_{\mu}[y] = 0, \quad \mu = 1, 2, \ldots, k.
\]

Here \(\lambda\) is the eigenvalue of the boundary value problem, \(y\) is an unknown function, \(u_{\mu}[y]\) is a linear homogeneous differential expressions relative to \(y\), taken on some given boundary curve (or boundary surface) of the \(\tau\) area. Let's call this area \(\tau\) main. The numbers \(\lambda\) for which there are non-zero solutions \(y \neq 0\) of equation \(1\) that satisfy the boundary conditions \(2\) (eigenfunctions) are the eigenvalues of the problem.

It is assumed [1] that the eigenvalue \(\lambda\) is \(r\)-fold degenerate or has a multiplicity of \(r\) if there are exactly \(r\) independent eigenfunctions \(y_1, y_2, \ldots, y_r\), corresponding to this eigenvalue [10]:
\[
M[y_i] = \lambda \cdot N[y_i], \quad u_{\mu}[y_i] = 0, \quad i = 1, 2, \ldots, r.
\]

Here \(u_{\mu}[y_i] = 0\) is a symbolic recording of boundary conditions.

If the problem has degenerate eigenvalues, you can use the results as if they are not present. However, the evidence becomes more complex and requires more extensive support tools [10, 13].

Let the linearly independent functions \(y_1, y_2, \ldots, y_r\), chosen as eigenfunctions of the boundary value problem \(1\)-(2) correspond to \(r\)-fold degenerate eigenvalue of \(\lambda\). Then the functions
\[
\psi_g = \sum_{j=1}^{r} a_{gj} y_j \quad (g = 1, 2, \ldots, r)
\]
for any constants \(a_{gj}\), they are also eigenfunctions for the eigenvalue \(\lambda\). Besides that, constants \(a_{gj}\) can be defined so that the functions \(\psi_g\) will be orthogonal in a generalized sense:
\[
\int_a^b \psi_g \cdot N[y_j] \, dx = 0 \quad \text{when} \quad g \neq j.
\]

And constant multipliers for \(\psi_g\) can be found up to the sign, so that \(\psi_g\) will be "normalized", that is:
\[
\int_a^b \psi_g \cdot N[y_j] \, dx = 1 \quad \text{when} \quad g = j.
\]
The proof can be found in the work [10].

Thus, we can assume that

$$\int_{a}^{b} \psi_i \cdot N[\psi_k] dx = \begin{cases} 0 & \text{npu } i \neq k \\ 1 & \text{npu } i = k \end{cases}. \quad (4)$$

The following theorem is valid.

In the case of a self-adjoint fully defined eigenvalue boundary value problem (1)-(2), there is a normalized in the generalized sense orthogonal system of eigenfunctions $\psi_i$. The ratio (4) is true for it.

2.1. Semi-defined eigenvalue problems

In this case we have the equation:

$$M[y] = \lambda \cdot N[y], \quad u_\mu[y] = 0.$$ 

Among other values of $\lambda$, $\lambda = 0$ appears.

For example in the problem of bending vibrations of a homogeneous free at both ends rod with a length of $l$:

$$y IV = \lambda y, \quad y IV (0) = y IV (l) = 0$$

there is a eigenvalue $\lambda = 0$, that is doubly degenerate with eigenfunctions of $y = 1$ and $y = x$. In this case, the system moves as a solid whole.

2.2. An example of a doubly degenerate eigenvalue from hydrodynamics

Consider the boundary value problem of free vibrations of the free surface of an ideal incompressible liquid in a rigid concentric vertical tank [2-6,8]. The scheme of the mechanical system is shown in figure 1.

Figure 1. Scheme of a part of a concentric tank with the liquid (top view).

Figure 2. Graphic representation of the cross section of the tank with the liquid by the vertical radial plane.

Vibrations are assumed to be small. The cross sections of the liquid move along the $s$ axis according to the law of the $\xi(s,t)$ (figure 2). And their height gets a deviation $\eta(s,t)$ from the height $h$ of the free surface of the liquid (figure 3). The rounding radius of the channel $R$ is much larger than the size of its cross-section $b$ (figure 2).
Figure 3. Diagram of deflection of particles of the free surface of the liquid in the tank.

The pressure in the liquid depending on the height of the cross section during fluctuations is determined by the hydrostatic pressure formula:  

\[ p = \rho gy. \]

Using the D'alembert principle for a fluid element with length \( ds \), we obtain the equation of motion of the system:

\[
\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \eta}{\partial s}. \tag{5}
\]

Express the deviation \( \eta(s,t) \) in terms of the displacement \( \xi(s,t) \) using the continuity equation:

\[ \eta = -h \frac{\partial \xi}{\partial s}. \]

As a result, the differential equation (5) will take the form:

\[
a^2 \frac{\partial^2 \xi}{\partial s^2} = \frac{\partial^2 \xi}{\partial t^2}. \]

The boundary conditions of the problem will be as follows:

\[ \xi(0,t) = \xi(l,t), \quad \frac{\partial \xi}{\partial s}(0,t) = \frac{\partial \xi}{\partial s}(l,t), \]

where \( l = 2\pi R \).

Let \( \xi(s,t) = f(s)T(t) \). Then \( \lambda^2 = \omega^2 g h \) - eigenvalue of the task:

\[
\frac{\partial^2 f}{\partial s^2} + \lambda^2 f = 0.
\]

The solution of this differential equation has the form

\[ f(s) = A \cos \lambda s + B \sin \lambda s. \]

From the boundary conditions (6), we obtain that \( \lambda_i = \frac{2\pi i}{l} \) and for each \( i = 1,2,3,... \), there are simultaneously two linearly independent functions that satisfy the differential equation and boundary conditions:

\[ f_{1i} = \cos \frac{2\pi i}{l} s, \quad f_{2i} = \sin \frac{2\pi i}{l} s. \]

These functions satisfy orthogonality conditions. The validity of the orthogonality conditions for these functions is checked directly.
3. The case of multiple eigenvalues in the problem of oscillations of systems with a finite number of degrees of freedom

Let a mechanical system have two degrees of freedom. Then its equations of motion have the form [1,9]:

\[
\begin{align*}
q_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + c_{11}q_1 + c_{12}q_2 &= 0, \\
q_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + c_{12}q_1 + c_{22}q_2 &= 0.
\end{align*}
\]  

(7)

An oscillating system will have multiple frequencies \( \omega_1 = \omega_2 = \omega \), if

\[
\frac{c_{11}}{a_{11}} = \frac{c_{22}}{a_{22}} = \frac{c_{12}}{a_{12}} = \omega^2.
\]

(8)

Here \( q_i \) (\( i = 1,2 \)) are the generalized coordinates, \( c_{ij} \) are the system's reduced stiffness coefficients, and \( a_{ij} \) are the system's reduced inertia coefficients. The system (7) with consideration for (8) can be converted to the form:

\[
\begin{align*}
q_{11}X + a_{12}Y &= 0, \\
a_{12}X + a_{22}Y &= 0.
\end{align*}
\]

(9)

where \( X = \ddot{q}_1 + \omega^2 q_1 \), \( Y = \ddot{q}_2 + \omega^2 q_2 \).

Since \( \Delta = a_{11}a_{22} - a_{12}^2 \neq 0 \) and \( \Delta > 0 \), the solution of system (9) will be as follows [11]:

\( X = 0, \ Y = 0. \)

Hence we have two equations

\[
\begin{align*}
\ddot{q}_1 + \omega^2 q_1 &= 0, \\
\ddot{q}_2 + \omega^2 q_2 &= 0.
\end{align*}
\]

The solution of the last two equations has the form:

\( q_1 = c_1 \cos(\omega t + \alpha), \quad q_2 = c_2 \cos(\omega t + \beta). \)

Thus, the eigenvectors \( \vec{\eta}_i \) (\( i = 1,2 \)) of this problem are:

\( \vec{\eta}_1 = [1, 2], \quad \vec{\eta}_2 = [0, 1]. \)

They are obviously orthogonal.

Examples of such a problem are a mathematical pendulum with a spherical suspension that performs small movements in a uniform field of gravity (figure 4), and a weightless elastic pinched rod with a load at the end that performs bending vibrations (figure 5).

Note that the above reasoning and results are correct at the physical level of rigor. Strictly mathematically, the rotation of the liquid mirror is proved in the nonlinear formulation [6].

Figure 4. Mathematical pendulum with a spherical suspension.

Figure 5. Weightless elastic pinched rod with a load at the end.
4. Conclusion
The article considers the free vibrations of a mechanical and hydro-mechanical systems based on classical works [2,6] in the case of close eigenfrequencies and in the case when one eigenfrequency is zero (the case of a semi-definite system). The orthogonality of the forms of natural vibrations of the systems is revealed. The possibility of the existence of rotation of the liquid mirror, proved earlier in Druzhinin’s work [6], is shown by the example of a mechanical analog of transverse vibrations of a concentric cylindrical tank with a liquid.

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