Symmetry Realization, Poisson Kernel and the AdS/CFT Correspondence

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Two kinds of realizations of symmetry on classical domains or the Euclidean version of AdS space are used to study AdS/CFT correspondence. Mass of free particles is defined as an AdS group invariant, the Klein-Gordon and Dirac equations for relativistic particles in the AdS space are set up as a simple mimic in the case of Minkowskian space. The bulk-boundary propagator on the AdS space is given by the Poisson kernel. Theorems on the Poisson kernel guarantee the existence and sole of the bulk-boundary propagator. The propagator is used to calculate correlators of the theories that live on the boundary of the AdS space and show conformal invariance, which is desired by the AdS/CFT correspondence.

1. To understand the relations between gauge field theories and string theory is a longstanding problem. The answer to the question will offer us a theory of strong coupling mysteries of QCD. The AdS/CFT correspondence states that string theory in Anti-de Sitter (AdS) spacetime is holographically dual to a conformal field theory (CFT) on the spacetime boundary of the AdS$^{[1]}$. Beyond reasonable doubt, the

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Maldacena dualities between 4-dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory and the type IIB string theory on the background $\text{AdS}_5 \times S^5$ has been established. Equivalence between theories in different dimensions immediately raises questions about how detailed bulk information in one theory can be completely coded in low dimensional degrees of freedom. Although there exist a large amount of evidence, we have not yet any direct translation of the configurations of one theory to the other. Nonperturbative formulations for string and M theory should be set up to investigate such a translation between theories in different dimensional spacetimes.

The Maldacena conjecture is heavily dependent on peculiar properties of the AdS spacetime. It is natural to begin such a program from discussing symmetry realizations of the AdS group in the AdS spacetime. Here we use two kinds of realizations on classical domains, which may be viewed as the Euclidean version of the AdS spacetime, to study some properties of the AdS/CFT correspondence.

2. It is useful to consider the $(n + 1)$-dimensional AdS space $\text{AdS}_{n+1}$ as a submanifold of a pseudo-Euclidean $(n + 2)$-dimensional embedding space with coordinates $(y^\alpha) = (y^0, y^1, \cdots, y^{n+1})$ and metric

$$
\eta_{\alpha\beta} = \text{diag}(+,-,\cdots,-,+) 
$$

with “length squared”

$$(y^0)^2 - \sum_{i=1}^n (y^i)^2 + (y^{n+1})^2 = a^2
$$

preserved by the AdS group $SO(2,n)$ acting as

$$
y^\alpha \rightarrow \tilde{y}^\alpha = \Lambda^\alpha_{\beta} y^\beta, \quad \Lambda^\alpha_{\beta} \in SO(2,n). \quad (1)
$$

The metric of the AdS in the embedding space is given by

$$
ds^2 = -\sum_{i=1}^n (dy^i)^2 + (dy^0)^2 + (dy^{n+1})^2. \quad (2)
$$

Making use of the coordinate $y^\alpha$, we can discuss motions of free relativistic particles in the AdS space. For this aim, we first define the $(n+2)$-dimensional angular momentum
$L^{\alpha\beta}$ as

$$L^{\alpha\beta} = y^{\alpha} \frac{dy^{\beta}}{ds} - y^{\beta} \frac{dy^{\alpha}}{ds}. \quad (3)$$

$L^{\alpha\beta}$ should be conservative for the free particle, so that

$$\frac{dL^{\alpha\beta}}{ds} = 0 \quad (4)$$

should be its equations of motion. On the other hand, $L^{\alpha\beta}$ may be viewed as a set of classical realizations for generators of the AdS group $SO(2, n)$. At this representation, the Casimir-like invariant of the AdS group may be defined as square of mass of the free particle as follows:

$$m_0^2 = \frac{1}{2a^2} L^{\alpha\beta} L_{\alpha\beta}$$

$$= E^2 - P^2 - \frac{L^2}{a^2}, \quad (5)$$

where $E \equiv \frac{1}{a} L^{n+1,0}$ and $P \equiv \frac{1}{a} L^{n+1,i}$.

It is interesting to see that in the projective space realization (see below in 4.), we may identify $P^i$ and $L^{ij}$ as momentum and angular momentum of the free particle in the AdS space, respectively. And Eq.(3) gives rise to the generalization of Einstein’s famous formula $E^2 - P^2 = m_0^2$ in the special relativity.

On the other hand, it is well-known that the operator form of Einstein’s formula gives the Klein-Gordon equation – the relativistic quantum equation of motion for a free scalar particle or the relativistic equation of motion for free scalar field. A simple mimic of quantization at flat spacetime gives the operator counterpart of the classical variables $L_{\alpha\beta}$,

$$L_{\alpha\beta} \rightarrow \hat{L}_{\alpha\beta} = i \left( y_{\alpha} \frac{\partial}{\partial y^{\beta}} - y_{\beta} \frac{\partial}{\partial y^{\alpha}} \right). \quad (6)$$

The operator forms of $L_{\alpha\beta}$ are, in fact, generators of the AdS group. The Corresponding Casimir operator is of the form

$$\hat{Q} = \frac{1}{2a^2} \hat{L}_{\alpha\beta} \hat{L}^{\alpha\beta}. \quad (7)$$

At this stage, we can set up equations of motion in relativistic quantum mechanics of AdS invariant as eigenvalue equation of the Casimir operator $\hat{Q}$ and it is independent
of details of representation of the AdS. The Klein-Gordon equation for relativistic scalar particles in AdS spacetime can be introduced as

\[(\hat{Q} - m_0^2)\Phi = 0.\]  

(8)

In a general frame with coordinates \(x^i\) and metric \(g_{ij}\) \((i, j = 0, 1, \ldots, n)\), the Casimir operator \(\hat{Q}\) of the AdS group is of the form

\[\hat{Q} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ij} \frac{\partial}{\partial x^j} \right).\]  

(9)

Thus, a general form of the Klein-Gordon equation in AdS space is

\[\left[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ij} \frac{\partial}{\partial x^j} \right) - m_0^2 \right] \Phi(x) = 0.\]  

(10)

For spinors, analogue to Dirac operator in the Minkowskian space, we can introduce an AdS invariant operator as

\[\hat{L} = \frac{1}{2} \gamma^\alpha \gamma^\beta \hat{L}_{\alpha\beta},\]  

(11)

where

\[\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}.\]

It is easy to check that

\[\frac{a^2}{2} \hat{L}^{\alpha\beta} \hat{L}_{\alpha\beta} + \frac{9}{4} a^{-2} = \left( \hat{L} - \frac{3}{2} a^{-1} \right)^2.\]  

(12)

Therefore, similar to the case of Minkowskian space, the Dirac operator in AdS space is a square root of the Casimir operator of the AdS group. Then the eigenvalue equation of the Dirac-like operator \(\hat{L}\) may be viewed as the relativistic equation of motion for spinors,

\[[-i\gamma^i(\partial_i - \Gamma_i) + m_0]\Psi(x) = 0,\]  

(13)

where \(\Gamma_i\) is the Ricci rotational coefficient

\[[\Gamma_k, \gamma_i] = \partial_k \gamma_i - \left\{ \begin{array}{l} l \end{array} \right\} \gamma_l.\]
3. In what follows, let us focus on the Euclidean version of the AdS\(_{n+1}\). If we write a real \((n+1)\)-dimensional vector as \(x = (x^0, x^1, \cdots, x^n)\) and \(x'\) is the transport of \(x\), the AdS\(_{n+1}\) can be expressed as

\[
x x' < 1 ,
\]

(14)

with the metric

\[
ds^2 = \frac{4a^2 \sum_{i=0}^{n} (dx^i)^2}{(1 - xx')^2} .
\]

The coordinates \(x^i\) and \(y^a\) are related as

\[
y^0 = a \frac{1 + xx'}{1 - xx'} , \quad iy^{n+1} = a \frac{2x^{n+1}}{1 - xx'} ,
\]

\[
y^i = a \frac{2x^i}{1 - xx'} , \quad (i = 1, 2, \cdots, n) .
\]

(15)

It is not difficult to show that the transformation

\[
\chi = \frac{x - b - xx'b + x(2b'b - bb'I)}{1 - 2bx' + bb'xx'} , \quad bb' < 1 ,
\]

(16)

makes the AdS \((xx' < 1)\) to AdS \((\chi \chi' < 1)\), boundary of the AdS \((xx' = 1)\) to boundary of the AdS \((\chi \chi' = 1)\) and the center \((x = b)\) of the AdS to the center \((\chi = 0)\) of the AdS. Here \(I\) denotes the identity operator in the AdS.

The inverse transformation is of the form

\[
x = \frac{\chi + b + \chi \chi'b + \chi(2b'b - bb'I)}{1 + 2b\chi' + bb'\chi\chi'} , \quad bb' < 1 .
\]

(17)

We see that, besides of the point \(\chi = -\frac{b}{bb'}\), the transformation (16) is 1-1 correspondent.

Besides the above transformations, it is easy to see that

\[
\chi = x \Gamma , \quad \Gamma \Gamma' = I
\]

(18)

also transforms the AdS to AdS but with invariant original point. The transformations (16) and (18) give a kind of realization of the AdS group on the AdS space. This realization may be called hyper-sphere realization.
For this realization of the AdS group, there is an invariant differential of the form on the AdS space
\[
\frac{d\chi d\chi'}{(1 - \chi \chi')^2} = \frac{dx dx'}{(1 - xx')^2}.
\] (19)

And the invariant differential operator is
\[
(1 - \chi \chi')^{n+1} \sum_{i=0}^{n} \frac{\partial}{\partial \chi^i} \left[ (1 - \chi \chi')^{1-n} \frac{\partial \Phi(\chi)}{\partial \chi^i} \right] = 0.
\] (20)

This is just the Laplace equation, i.e. the Klein-Gordon equation, of the massless scalar field.

It is convenient to introduce the spherical coordinates
\[
x = pu, \quad uu' = 1,
\]
\[
dx dx' = d\rho^2 + \rho^2 dudu',
\]
\[
dudu' = d^2 \theta_1 + \sin^2 \theta_1 d^2 \theta_2 + \sin^2 \theta_1 \sin^2 \theta_2 d^2 \theta_3 + \cdots + \sin^2 \theta_1 \cdots \sin \theta_{n-1} d^2 \theta_n, \quad 0 \leq \theta_1, \cdots, \theta_{n-1} \leq \pi, 0 \leq \theta_n \leq 2\pi.
\] (21)

The unit area element on the AdS boundary is as follows:
\[
d\Omega_u = \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \cdots \sin \theta_{n-1} d\theta_1 d\theta_2 \cdots d\theta_n.
\] (22)

From
\[
d\chi d\chi' = \left( \frac{1 - bb'}{1 - 2bu + bb'xx'} \right)^2 dx dx',
\] (23)

we know that on the spherical surface \( x = u, \ \chi = v \), \( uu' = vv' = 1 \) there is a relation
\[
dvdv' = \left( \frac{1 - bb'}{1 - 2bu + bb'uu'} \right)^2 dudu'.
\] (24)

For the \( n \)-dimensional vector \( d\Omega_u \), similarly, we have
\[
d\Omega_v = \left( \frac{1 - bb'}{1 - 2bu + bb'} \right)^n d\Omega_u.
\] (25)

This suggests the Poisson formula
\[
\Phi(x) = \frac{1}{\omega_n} \int_{uu'=1} \cdots \int d\Omega_u G_{B\theta}^E(x, u) \Phi_0(u)
\]
\[
= \frac{1}{\omega_n} \int_{uu'=1} \cdots \int d\Omega_u \left( \frac{1 - xx'}{1 - 2ux' + xx'} \right)^n \Phi_0(u),
\] (26)
where
\[ G_{B\partial}^E(x, u) = \left( \frac{1 - xx'}{1 - 2xu' + xx'} \right)^n \]  
(27)
is the bulk-boundary propagator in the AdS, \( \omega_n = \int_{uu'=1} \cdots \int d\Omega_u = \frac{2\pi^{n+1}}{\Gamma \left( \frac{n+1}{2} \right)} \) is the total areas of the AdS boundary in Euclidean version, \( \Phi(x) \) satisfies Eq.(20) and \( \Phi(x)|_{x \to u} = \Phi_0(u) \). Let \( x = \rho v \), we can rewrite the bulk-boundary propagator as
\[ G_{B\partial}^E(x, u) = \left( \frac{1 - \rho^2}{1 - 2\rho \cos \langle u, v \rangle + \rho^2} \right)^n, \]  
(28)
where \( \langle u, v \rangle \) denotes the angle between the vectors \( u \) and \( v \). The bulk-boundary propagator \( G_{B\partial}^E(x, u) \) has the following properties:

• Because \( 1 - 2\rho \cos \langle u, v \rangle + \rho^2 \geq (1 - \rho)^2 \), we always have \( G_{B\partial}^E(x, u) > 0 \), for \( 0 \leq \rho < 1 \).

•
\[ \lim_{\rho \to 1} G_{B\partial}^E(x, u) = \left\{ \begin{array}{ll} 0, & \text{for } u \neq v, \\ \infty, & \text{for } u = v. \end{array} \right. \]  
(29)

•
\[ \frac{1}{\omega_n} \int_{uu'=1} \cdots \int d\Omega_u G_{B\partial}^E(x, u) = 1. \]  
(30)

• At the case of \( \rho < 1 \), the bulk-boundary propagator satisfies Eq.(20).

These properties guarantee the existence of the Poisson formula Eq.(26). This shows the existence of the bulk-boundary propagator for Klein-Gordon equation. The AdS/CFT correspondence states that\[ Z_{\text{bulk}}[\Phi(\Phi_0)] = \langle e^{-\int_{uu'=1} \cdots \int d\Omega_u \Phi_0(u) O(u)} \rangle_{\text{CFT}}. \]  
(31)
Here \( \ln Z_{\text{bulk}} \) is the effective action for string theory on AdS\( _{n+1} \) considered as a functional of the boundary data \( \Phi_0(u) \) for the fields \( \Phi(x) \). The right side is the generating functional of correlators of the operator \( O(u) \) dual to \( \Phi_0(u) \). Based upon the above
discussion, we know that the free classical solutions of scalar particle may be suggestively written in terms of the boundary value $\Phi_0(u)$ and a bulk-boundary propagator $G_{B\delta}^E(x, u)$.

Then the CFT correlators are given in terms of truncated bulk Green functions as

$$
\langle O(u_1)O(u_2)\cdots O(u_m)\rangle_{\text{CFT}} = \int \prod_{i=1}^m dx_i G_{B\delta}^E(x_i, u_i) \langle \Phi(x_1)\Phi(x_2)\cdots \Phi(x_m) \rangle_T .
$$

(32)

To check our results, some sample calculations will now be carried out, in the approximation of classical supergravity. Consider an AdS theory that contains a massless scalar $\Phi$ with action

$$
I(\Phi) = \frac{1}{2} \int_{xx' < 1} \cdots \int d^{n+1}x \sqrt{g} g^{ij} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^j} .
$$

(33)

Using the bulk-boundary propagator $G_{B\delta}^E(x, u)$, the solution of the Laplace equation in the AdS space with boundary values $\Phi_0$ is

$$
\Phi(x) = \int_{uu' = 1} \cdots \int d\Omega_u \left( \frac{1 - xx'}{1 - 2xu' + xx'} \right)^n \Phi_0(u) .
$$

(34)

It follows that

$$
\frac{\partial \Phi(\rho, u)}{\partial \rho} = -2n(1-\rho^2)^{n-1} \int_{vv' = 1} \cdots \int d\Omega_v \frac{\Phi_0(v)}{[(u-v)(u'-v')]} n - O((1-\rho^2)^{n+1}) , \quad \rho \to 1 .
$$

(35)

By integrating by parts, one can express $I(\Phi)$ as a surface integral,

$$
I(\Phi) = (2a)^{n-1} n \int_{uu' = 1} \cdots \int d\Omega_u \int_{vv' = 1} \cdots \int d\Omega_v \frac{\Phi_0(u)\Phi_0(v)}{|u-v|^{2n}} .
$$

(36)

Hence the two points function of the operator $O$ is proportional to $|u-v|^{-2n}$, as expected for a field $O$ of conformal dimensional $n$.

4. We have discussed the Euclidean version of AdS/CFT correspondence from symmetry realization point of view and in terms of the Poisson formula for the hypersphere realization. It is important to mention that there is another realization of the symmetry transformations of the AdS group, besides of (16). Namely, the real
projective space realization of the AdS space and the transformations of the AdS group on it.

In this realization, the global transformations are given by

$$x \rightarrow \chi = \frac{\sqrt{1 - bb'(x - b)(1 + \lambda b'b)}}{1 - bx'},$$

(37)

where $bb' < 1$ and

$$\lambda = \frac{1 - \sqrt{1 - bb'}}{bb'\sqrt{1 - bb'}}, \quad 1 + \lambda bb' = \frac{1}{\sqrt{1 - bb'}}.$$

The invariant differential metric for this kind of transformations is of the form

$$\frac{d\chi(I - \chi'\chi)^{-1}d\chi'}{1 - \chi\chi'} = \frac{dx(I - x'x)^{-1}dx'}{1 - xx'}.$$

(38)

At the boundary of the AdS space $x = u, \chi = v$, we have $duv' = 0$ and thus

$$dvdv' = \frac{1 - bb'}{(1 - bu')^2}dudu'.$$

(39)

The bulk-boundary propagator for this kind of symmetry is of the form

$$G_{E \partial}^E(x, u) = \frac{(1 - xx')^{\frac{n}{2}}}{(1 - ux')^n}.$$

(40)

The invariant differential of this kind of transformations, which is satisfied by the above bulk-boundary propagator, is as follows:

$$\sum_{i=0}^{n} \frac{\partial^2 \Phi}{\partial x^i \partial x^i} - \sum_{i,j=0}^{n} \frac{x^i x^j}{\partial x^i \partial x^j} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - 2 \sum_{i=0}^{n} x^i \frac{\partial \Phi}{\partial x^i} = 0.$$

(41)

At the boundary of the AdS space, the transformation law for unit area elements is

$$d\Omega_v = \frac{(1 - bb')^{\frac{n}{2}}}{(1 - bu')^n}d\Omega_u.$$

(42)

After added of the transformations

$$\chi = x\Gamma, \quad \Gamma' = I,$$

(43)

which making the original point $(x = 0)$ invariant, we complete this kind of realization of the AdS group. Similar Euclidean version of the AdS/CFT correspondence with transformations (16) can be presented also.
What very interested here is that in this realization $x^i$ are what is called the Beltrami coordinates of the AdS. For the non-Euclidean version, the properties of the Beltrami coordinates of the AdS spacetime have been discussed thoroughly [11, 12]. It has been shown that for the AdS spacetime the Beltrami coordinates form an initial frame in certain sense. And the equations of motion (4) for the classical free particle just correspond to the first integral of the geodesic equation of the Beltrami metric. In other words, the special relativity with AdS symmetry may be set up based on the Beltrami coordinates. Solutions of the Klein-Gordon and Dirac equation in the AdS space can also be given in a more straightforward way at the Beltram frame [13].

Details about these subjects will be presented in a forthcoming paper.

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