NONCLASSICAL EIGENVALUE ASYMPTOTICS
FOR OPERATORS OF SCHröDINGER TYPE

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We consider operators in the form

$$A = -\nabla \cdot \rho \nabla + V(x)$$

on $\mathbb{R}^n$, where metric $\rho = (\rho_{ij}(x)) \geq 0$ and potential $V(x) \geq 0$. The classical Weyl principle for asymptotic distribution of large eigenvalues of $A$ states that the counting function

$$N(\lambda) = \# \{\lambda_j \leq \lambda \} \sim \text{Vol}\{ (x; \xi) | \rho \xi \cdot \xi + V(x) \leq \lambda \} \quad \text{as} \ \lambda \to \infty.$$  

(See for instance [Gu].) Integrating out variable $\xi$ we can rewrite it as

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int (\lambda - V)^{n/2} \frac{dx}{\sqrt{\det \rho}}.$$ 

If potential $V$ and metric $\rho$ are assumed to be homogeneous in $x$, $V(x) = |x|^\alpha V(x')$; $\rho_{ij}(x) = |x|^\beta \rho_{ij}(x')$, $x' = x/|x|$, then (1) reduces to

$$N(\lambda) \sim C \lambda^{n/2 + (1-\beta/2)n/\alpha} \int V^{-n/\alpha}(1-\beta/2) dS;$$

integration over the unit sphere $S$ with constant

$$C = \frac{\omega_n}{(2\pi)^n \alpha} B \left( \frac{n}{2} + 1; \frac{n}{\alpha} (1 - \beta/2) \right),$$

which depends on the volume $\omega_n$ of the unit sphere in $\mathbb{R}^n$ and the beta function.

Assuming $\beta < 2$ we see that integral (2) becomes divergent if $V(x')$ vanishes to a sufficiently high order. The simplest such potential is $V(x,y) = |x|^\alpha |y|^\beta$ on $\mathbb{R}^n + \mathbb{R}^m$.

The Weyl (volume counting) principle, when applied to the corresponding Schrödinger operator $-\Delta + V(x)$, fails to predict discrete spectrum below any energy level $\lambda > 0$. However, as was shown by D. Robert [Ro] and B. Simon [Si], $A$ has purely discrete spectrum $\{\lambda_j\} \to +\infty$ (for qualitative explanation of this phenomenon see [Fe]). Moreover, the "nonclassical" asymptotics of $N(\lambda)$ was derived for such $A$.

Recently M. Solomyak [So] studied a general class of Schrödinger operators

$$-\Delta + V(x)$$

with homogeneous potentials $V$ subject to the following constraint:

(A) zeros of $V$, $\{x : V(x) = 0\}$ form a smooth cone $\Sigma$ in $\mathbb{R}^n$ of dimension $m$, and $V$ vanishes on $\Sigma$ "uniformly" to order $b > 0$.

Introducing variables $x \in \Sigma$ and $y \in N_x$ (the normal to $\Sigma$ at $\{x\}$), hypothesis (A) means that there exists

$$\lim_{t \to 0} t^{-b} V(x + ty) = V_0(x,y).$$
It is easy to see that $V_0(x,y)$ has mixed homogeneity
\[ V_0(x,y) = |x|^a |y|^b V_0(x',y'); \quad a + b = \alpha \]
and $V_0$ approximates $V$ in a small conical neighborhood $\Sigma_\varepsilon$ of $\Sigma$:
\[ \Sigma_\varepsilon = \{ x + y | x \in \Sigma; |y| < \varepsilon |x| \}. \]

Under hypothesis (A) M. Solomyak [So] derived asymptotics of $N(\lambda)$ for such operators $A = -\Delta + V(x)$ in terms of eigenvalues $\{ \lambda_j(x) \}_{1}^{\infty}$ of an auxiliary family of Schrödinger operators $\{ L(x) = -\Delta_y + V_0(x,y) \}_{x \in \Sigma}$. Namely,
\[ N(\lambda) \sim C \lambda^{\frac{m}{2}(1 + \frac{2 + b}{a})} \int_{\Sigma'} \sum_{j=1}^{\infty} \lambda_j(x')^{-m(2+b)/2a} dS, \]
the integral is over $\Sigma' = \Sigma \cap S$ (unit sphere).

Notice that each operator $L(x)$ has "classical type," so Weyl's principle (2) applies to $\{ \lambda_j(x) \}_{1}^{\infty}$.
\[ \#\{ \lambda_j(x) \leq \lambda \} \sim c(\lambda)^{(n-m)(1/2 + 1/b)}. \]

Let us also observe that a polynomial asymptotics of $N(x) \sim c \lambda^p$ implies convergence of the series
\[ \sum_{j=1}^{\infty} \lambda_j^{-q} < \infty, \text{ with any } q > p. \]

Hence by (5) the sum in (4) converges provided
\[ q = m(2 + b)/2a > p = (n - m)(1/2 + 1/b). \]

Condition (6) is sufficient for validity of (4). In the critical case $q = p$ an additional log $\lambda$ factor appears in (4).

The method of [So] was based on the variational formulation of the problem and certain eigenvalue estimates for Schrödinger operators in conical regions obtained in [Ros].

In the present paper we shall outline a different approach based on pseudodifferential calculus with operator-valued symbols in the spirit of [Ro]. This method allows us to recover Solomyak's result (4) and to extend it in various directions, including operators of the form $-\nabla \cdot \rho \nabla + V(x)$.

We propose the following principle, which governs nonclassical asymptotics: the main contribution to $N(\lambda)$ comes from the degeneracy set $\Sigma$ (critical set) of $V$.

According to this principle we want to "localize" $A$ to a small (conical) neighborhood of $\Sigma$. Precisely, let us introduce the "model" operator
\[ A_0 = -\Delta_\Sigma + L(x) = -\Delta_\Sigma + [-\Delta_N - 2 \nabla_x \cdot \rho' \nabla y + V_0(x,y)] \]
on the manifold $N(\Sigma) = \bigcup_{x \in \Sigma} N_x$, normal bundle to $\Sigma$, where $\Delta_{\Sigma}, \Delta_N$ are the Laplace-Beltrami operators on $\Sigma$ and the normal space, $N = N_x$, with respect to the metrics induced by $\rho_{ij}$ and $\rho'$ is the "off diagonal" part of $\rho$.

Writing $A = -\nabla \cdot \rho \nabla + V$ in normal coordinates $(x,y)$ one can show that $A = A_0 + " \text{small perturbation}"$ in a conical neighborhood $\Sigma_\varepsilon$ of $\Sigma$. So we expect $N(\lambda; A) \sim N(\lambda; A_0)$, as $\lambda \to \infty$. 

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To study the eigenvalue distribution one usually works with certain integral “transforms” of \( N(\lambda) \), like \( \text{tr} e^{-tA} = \int_{-\infty}^{\infty} e^{-\lambda t} dN(\lambda) \) or \( \text{tr}(\zeta + A)^{-l} = \int_{-\infty}^{\infty} (\zeta + \lambda)^{-l} dN(\lambda) \).

We prefer to work with the latter. Once the asymptotics
\[
(8) \quad \text{tr}(\zeta + A)^{-l} \sim c_0 \zeta^{-l+p} \quad \text{as} \quad \zeta \to \infty
\]
is established for \( \text{tr} R_\lambda^l \) one can go back to the asymptotics of \( N(\lambda) \sim c \lambda^p \), as \( \lambda \to \infty \), by the Tauberian Theorem of M. V. Keldysh (see [Ro]). The relation between the two constants is \( c = c_0/pB(p; l - p) \).

So we need to establish (8).

Operator \( A \) can be thought of as a differential operator on \( \Sigma \) with operator-valued symbol \( \sum g^{ij} \xi_i \xi_j + L(x) \), where metric \( g = \rho_\Sigma - \rho^* \rho_N^{-1} \rho^* \) on \( \Sigma \) is constructed from the tangent \( \rho_\Sigma \) and normal \( \rho_N \) components of \( \rho \). Then the parametrix (approximate inverse) of \( (\zeta + A_0)^{-l} \) can be constructed as an operator-valued \( \Psi \text{DO} \ K = K^{(l)}_\zeta \) with symbol
\[
\sigma_K = \left[ \zeta + \sum g^{ij} \xi_i \xi_j + L(x) \right]^{-l}.
\]

According to our principle we want to localize kernels \( R_\zeta^l = (\zeta + A)^{-l} \); \( \tilde{R}_\zeta^l = (\zeta + A_0)^{-l} \) and \( K^{(l)}_\zeta \) to a small conical neighborhood \( \Sigma_\epsilon \) of \( \Sigma \). Let us introduce a cut-off function
\[
\chi_\epsilon = \begin{cases} 1 & \text{on } \Sigma_\epsilon, \\ 0 & \text{outside,} \end{cases}
\]
and define an orthogonal projection \( P_\epsilon u = \chi_\epsilon u \) from \( L^2(\mathbb{R}^n) \) onto \( L^2(\Sigma_\epsilon) \).

The following lemma plays the central role in the localization procedure.

**Lemma.** All traces below are equivalent as \( \zeta \to \infty \).

(i) \( \text{tr}(\zeta + A)^{-l} \sim \text{tr} P(\zeta + A)^{-l} P \),

(ii) \( \text{tr}(\zeta + A_0)^{-l} \sim \text{tr} P(\zeta + A_0)^{-l} P \),

(iii) \( \text{tr} K^{(l)}_\zeta \sim \text{tr} PK^{(l)}_\zeta P \),

(iv) traces of “truncated” operators: \( P(\zeta + A)^{-l} P, P(\zeta + A_0)^{-l} P, \) and \( PK^{(l)}_\zeta P \) are all equivalent.

From the lemma follows
\[
(9) \quad \text{tr}(\zeta + A)^{-l} \sim \text{tr} K^{(l)}_\zeta \quad \text{as} \quad \zeta \to \infty.
\]

Now it remains to compute the trace of an operator-valued \( \Psi \text{DO} \ K^{(l)}_\zeta \)
\[
(10) \quad \text{tr} K^{(l)}_\zeta = \int \int \sum_{k=1}^{\infty} \left[ \zeta + \sum g^{ij} \xi_i \xi_j + \lambda_k(x) \right]^{-l} d\xi dx.
\]

Integrating out variables \( \xi \), using homogeneity of \( \lambda_j(x) \) and \( \rho(x) \), and introducing “polar coordinates” on \( \Sigma \) to reduce integration over the cone \( \Sigma \) to a subset \( \Sigma' = \Sigma \cap S \), we get
\[
(11) \quad \text{tr} K^{(l)}_\zeta = C_0 \zeta^{-l+m(1/2+\theta)} \int_{\Sigma} \sum_{j=1}^{\infty} \lambda_j(x')^{-m\theta} \frac{dx'}{\sqrt{\det g^{ij}(x')}}
\]
with constants

\[ s = \frac{\beta b + 2a}{2 + b}; \quad \theta = \frac{1}{s} (1 - \beta/2); \quad C_0 = \int_0^\infty r^{m(1-\beta/2)}(1 - r^s)^{m/2-1} \, dr. \]

Remembering that \{\lambda_j(x')\} obey the classical asymptotics (5) with exponent \( p = (n-m)(2+b)/2b \), we obtain a sufficient condition of convergence of series (11)

\[ m\theta = \frac{m}{s}(1 - \beta/2) > p = \frac{(n-m)(2+b)}{2b} \quad \text{or} \quad \frac{m(2-\beta)}{b+2a} < \frac{n-m}{b}. \]

Thus we have established the following

**Theorem.** If operator \( A = -\nabla \cdot \rho \nabla + V \) with homogeneous potential \( V(x) = |x|\alpha V(x') \geq 0 \) and nondegenerate metric \( \rho_{ij}(x) = |x|\beta \rho_{ij}(x') > 0 \) satisfies hypothesis (A), then spectral function \( N(\lambda) \) of \( A \) admits the nonclassical asymptotics

\[ N(\lambda) \sim C\lambda^{m(1/2+\theta)} \int_{\Sigma} \sum_{1}^{\infty} \lambda_j(x')^{-m\theta} \frac{dx'}{\sqrt{\det g^{ij}(x')}}, \]

provided sufficient condition (13) holds. The metric \((g^{ij})\) on \( \Sigma \) is obtained from components of metric \( \rho \).

**Remarks.** (i) Formula (14) includes both the classical formula (2) with \( \beta = 0 \) and \( s = a \) (i.e., \( b = 0 \)) and all previously studied nonclassical asymptotics [Ro, Si, So] (the latter corresponds to \( \beta = 0 \)).

(ii) In the critical case (equality \( m\theta = p \) in (13)) an additional log \( \lambda \) factor appears in (16). The argument requires some modification: Before passing to the limit in the sum \( \sum_{1}^{\infty} \lambda_j^{-m\theta} \) and integration over \( \Sigma \) one has to “localize” \( K_{ij} \) to a compact region in \( \Sigma \).

We shall illustrate our theorem and conditions by the following

**Example.** Take scalar metric \((\rho_{ij}) = \rho = (t^2 + |x|^2)^{\beta/2} I_{n \times n}\) and potential \( V = (t^2 - |x|^2)^{\beta/2} \) in the space \( \mathbb{R}^n = \{(t, x): t \in \mathbb{R}; x \in \mathbb{R}^{n-1}\} \). The degeneracy set of \( V \) is the standard cone \( \Sigma = \{(t, x): t = \pm |x|\} \) in \( \mathbb{R}^n \).

Direct calculation shows: \( a = b = \alpha/2 \) and \( V_0(x, y) = |x|^{\alpha/2}|y|^{\alpha/2} \).

Condition (15) for convergence of the series of eigenvalues \( \sum_{j} \lambda_j^{-(n-1)\theta} \) of the operator \( L(x) = -d^2/dy^2 + |y|^{\alpha/2} \) on \( \mathbb{R} \) becomes

\[ \frac{\beta + 2}{2 - \beta} < n - 1 \quad \text{or} \quad \beta < \frac{2(n-2)}{n}, \]

and the eigenvalue asymptotics takes a form

\[ N(\lambda) \sim C\lambda^{(n-1)(1/2+\theta)} \sum_{1}^{\infty} \lambda_j^{-(n-1)\theta} \quad \text{with} \quad \theta = \frac{4 + \alpha(2 - \beta)}{2\alpha(\beta + 2)}. \]

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