Long-time dynamics in plate models with strong nonlinear damping

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Abstract

We study long-time dynamics of a class of abstract second order in time evolution equations in a Hilbert space with the damping term depending both on displacement and velocity. This damping represents the nonlinear strong dissipation phenomenon perturbed with relatively compact terms. Our main result states the existence of a compact finite dimensional attractor. We study properties of this attractor. We also establish the existence of a fractal exponential attractor and give the conditions that guarantee the existence of a finite number of determining functionals. In the case when the set of equilibria is finite and hyperbolic we show that every trajectory is attracted by some equilibrium with exponential rate. Our arguments involve a recently developed method based on the “compensated” compactness and quasi-stability estimates. As an application we consider the nonlinear Kirchhoff, Karman and Berger plate models with different types of boundary conditions and strong damping terms. Our results can be also applied to the nonlinear wave equations.

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1 Introduction

We study a class of plate models with the strong nonlinear damping which abstract form is the following Cauchy problem in a separable Hilbert space $H$:

$$
\partial_{tt} u + D(u, \partial_t u) + A u + F(u) = 0, \quad t > 0; \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \quad (1)
$$

We impose the following set of hypotheses:

**Assumption 1.1 (A)** The operator $A$ is a linear self-adjoint positive operator densely defined on a separable Hilbert space $H$ (we denote by $| \cdot |$ and $(\cdot, \cdot)$ the norm and the scalar product in this space). We assume that the resolvent of $A$ is compact in $H$. We also denote by $H_s$

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(with $s > 0$) the domain $D(A^{s/2})$ equipped with the graph norm $\| \cdot \|_s = \| A^{s/2} \cdot \|$. In this case $H_{-s}$ denotes the completion of $H$ with respect to the norm $\| \cdot \|_{-s} = \| A^{-s/2} \|$. Below we denote by $\{e_k\}$ the orthonormal basis in $H$ consisting of the eigenfunctions of the operator $A$:

$$Ae_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty,$$

and by $P_N$ the orthoprojector onto $\text{Span}\{e_k : k = 1, 2, \ldots, N\}$.

(D) For some value $\theta \in (0, 1]$ the damping operator $D$ maps $H_1 \times H_\theta$ into $H_{-\theta}$ and possesses the properties:

(i) For every $\varrho > 0$ there exist $\alpha_\varrho > 0$ and $\beta_\varrho > 0$ such that

$$\langle D(u, v), v \rangle \geq \alpha_\varrho |v|^2 \quad \text{and} \quad |D(u, v)|_{-\theta} \leq \beta_\varrho |v|^\varrho$$

for all $(u; v) \in H_1 \times H_\theta$, $|u|^2 + |v|^2 \leq \varrho^2$ (it is allowed that $\alpha_\varrho \to 0$ and $\beta_\varrho \to \infty$ as $\varrho \to \infty$).

(ii) For every $\varrho > 0$ there exist $\gamma_\varrho > 0$ and $C_\varrho > 0$ such that

$$\langle D(u^1, v^1) - D(u^2, v^2), v^1 - v^2 \rangle \geq \gamma_\varrho |v^1 - v^2|^2_{\varrho} - C_\varrho |u^1 - u^2|^2_{1-\delta}(1 + |v^1|^2_{\varrho} + |v^2|^2_{\varrho})$$

for all $(u^1; v^1) \in H_1 \times H_\theta$, $|u^i|^2 + |v^i|^2 \leq \varrho^2$, and for some $\delta > 0$ (it is allowed that $\gamma_\varrho \to 0$ and $C_\varrho \to \infty$ as $\varrho \to \infty$).

(iii) We assume that for every $u \in L_\infty(0, T; H_1)$ the mapping $v \mapsto D(u, v)$ is weakly continuous from $L_2(0, T; H_\theta)$ into $L_2(0, T; H_{-\theta})$ and

$$|D(u^1, v) - D(u^2, v)|_{-1} \leq C_\varrho |u^1 - u^2|^1_{1-\delta}(1 + |v|^2_{\varrho})$$

for all $|u^i|^1, |v| \leq \varrho$, $u^i \in H_1$, $v \in H_{\theta}$, and for some $\delta > 0$.

(iv) In the case $\theta < 1$ we have

$$|D(u^1, v^1) - D(u^2, v^2)|_{-\theta} \leq C_\varrho \left[|v^1 - v^2|_{\varrho} + |u^1 - u^2|^1(1 + |v|^2_{\varrho} + |v|^2_{\varrho}) \right]$$

for all $(u^i; v^i) \in H_1 \times H_\theta$, $|v^i|^2_{\varrho} + |v^i|^2_{\varrho} \leq \varrho^2$. In the case $\theta = 1$ we assume a stronger (compared to (4) with $\theta = 1$) inequality:

$$|D(u^1, v^1) - D(u^2, v^2)|_{-1} \leq C_\varrho \left[|v^1 - v^2|^1 + |u^1 - u^2|^1_{1-\delta}(1 + |v|^1 + |v|^2_{1}) \right]$$

for all $(u^i; v^i) \in H_1 \times H_1$, $|v^i|^2_{1} + |v^i|^2_{1} \leq \varrho^2$, where $\delta > 0$.

We note that the conditions in (2) follows from (3) and (4) or (5) provided $D(u, 0) \equiv 0$ for every $u \in H_1$.

(F) There exists $\delta > 0$ such that the nonlinear operator $F$ maps $H_{1-\delta}$ into $H_{-\theta}$ and is locally Lipschitz, i.e.,

$$|F(u_1) - F(u_2)|_{-\theta} \leq L(\varrho)|u_1 - u_2|_{1-\delta}, \quad \forall |A^{1/2} u_i| \leq \varrho,$$

We also assume that $F(u) = \Pi'(u)$, where $\Pi(u)$ is a $C^1$-functional on $H_1 = D(A^{1/2})$, and $'$ stands for the Fréchet derivative. We assume that $\Pi(u)$ is locally bounded on $H_1$ and there exist $\eta < 1/2$ and $C \geq 0$ such that

$$\eta |A^{1/2} u|^2 + \Pi(u) + C \geq 0, \quad u \in H_1 = D(A^{1/2})$$
Remark 1.2 Our conditions can be relaxed in different directions. For instance, for the well-posedness in Theorem 2.1 we can assume that $\delta = 0$ in (3) (and (6)), and require $F$ to map $H_{1-\sigma}$ into $H_{-l}$ continuously for some $\sigma > 0$ and $l > 0$. In a similar way, instead of (Diii) we can assume other continuity properties of the damping operator $D$ that will allow us to perform the limit transition in the corresponding Galerkin approximations. To obtain the global attractor existence, we can also relax (3) by including into its latter term the expression $\varepsilon |u^1 - u^2|_1$ with a small parameter $\varepsilon$ (see, e.g., Assumption 3.21 in [10] for a similar requirement in the case of the monotone damping). Moreover, instead of (3) we can assume that

\[
(D(u^1, v^1) - D(u^2, v^2), v^1 - v^2) \geq \gamma_\theta |v^1 - v^2|_\theta^2 - |u^1 - u^2|_1^2 \left[ \varepsilon + C_\theta(\varepsilon) \left( |v^1|_\theta^2 + |v^2|_\theta^2 \right) \right]
\]

for every $\varepsilon > 0$. However, we do not pursue these possible generalizations because our abstract hypotheses are motivated by the plate models described below. We note that in this paper we concentrate on the case when $\theta$ is positive. For some results for the case when $\theta = 0$ and $D(u, u_t)$ is linear with respect to $u_t$, we refer to [8]. It is also worth mentioning that our damping operator $D$ is positive (see the first relation in (2)) but not monotone (see (3)) in general. Thus, we cannot apply here the theory developed in [10].

Our main applications are plate models (with hinged boundary conditions, for definiteness). In this case

- $A = (-\Delta_D)^2$, where $\Delta_D$ is the Laplace operator in a bounded smooth domain $\Omega$ in $\mathbb{R}^2$ with the Dirichlet boundary conditions. We have then that $H = L_2(\Omega)$ and

\[\mathcal{D}(A) = \{ u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega \}.\]

We also have that $H_s = (H^{2s} \cap H_0^1(\Omega))$ for $1/2 \leq s \leq 1$ and $H_\sigma = H_0^{2\sigma}(\Omega)$ for $0 < \sigma < 1/2$. Here above $H^{\sigma}(\Omega)$ is the corresponding $L_2$-based Sobolev space, $H_0^{\sigma}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^{\sigma}(\Omega)$.

- The damping operator $D(u, u_t)$ may have the form

\[
D(u, u_t) = \Delta [\sigma_0(u) \Delta u_t] - \text{div} \left[ \sigma_1(u, \nabla u) \nabla u_t \right] + g(u, u_t),
\]

where $\sigma_0(s_1), \sigma_1(s_1, s_2, s_3)$ and $g(s_1, s_2)$ are locally Lipschitz functions of $s_i \in \mathbb{R}, i = 1, 2, 3$, such that $\sigma_0(s_1) > 0, \sigma_1(s_1, s_2, s_3) \geq 0$ and $g(s_1, s_2) s_2 \geq 0$. Also the functions $\sigma_1$ and $g$ satisfy some growth conditions (for a more detailed discussion of properties of the damping functions we refer to Section 5 below). We note that every term in (9) represents a different type of damping mechanisms. The first one is the so-called viscoelastic Kelvin–Voight damping, the second one represents the structural damping and the term $g(u, u_t)$ is the dynamical friction (or viscous damping). We refer to [20, Chapter 3] and to the references therein for a discussion of stability properties caused by each type of the damping terms in the case of linear systems.

- The nonlinear feedback (elastic) force $F(u)$ may have one of the following forms (which represent different plate models):

  (a) Kirchhoff model: $F(u)$ is the Nemytskii operator

\[
u \mapsto -\kappa \cdot \text{div} \left\{ |\nabla u|^q \nabla u - \mu |\nabla u|^p \nabla u \right\} + \varphi(u) - p(x),
\]

\[
 u \mapsto -\kappa \cdot \text{div} \left\{ |\nabla u|^q \nabla u - \mu |\nabla u|^p \nabla u \right\} + \varphi(u) - p(x),
\]
where $\kappa \geq 0$, $q > r \geq 0$, $\mu \in \mathbb{R}$ are parameters, $p \in L_2(\Omega)$ and $\varphi \in \text{Lip}_{\text{loc}}(\mathbb{R})$ fulfills the condition

$$\liminf_{|s| \to \infty} \frac{\varphi(s)}{s} > -\lambda_1^2,$$

where $\lambda_1$ is the first eigenvalue of the Laplacian with the Dirichlet boundary conditions.

(b) **Von Karman model:** $F(u) = -[u, v(u) + F_0] - p(x)$, where $F_0 \in H^4(\Omega)$ and $p \in L_2(\Omega)$ are given functions, the von Karman bracket $[u, v]$ is given by

$$[u, v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2 \cdot \partial^2_{x_1 x_2} u \cdot \partial^2_{x_1 x_2} v,$$

and the Airy stress function $v(u)$ solves the following elliptic problem

$$\Delta^2 v(u) + [u, u] = 0 \text{ in } \Omega, \quad \frac{\partial v(u)}{\partial n} = v(u) = 0 \text{ on } \partial \Omega.$$  \hfill (12)

Von Karman equations are well known in nonlinear elasticity and constitute a basic model describing nonlinear oscillations of a plate accounting for large deflections, see [21, 11] and the references therein.

(c) **Berger Model:** In this case the feedback force has the form

$$F(u) = -\kappa \int_{\Omega} |\nabla u|^2 dx - \Gamma \Delta u - p(x),$$

where $\kappa > 0$ and $\Gamma \in \mathbb{R}$ are parameters, $p \in L_2(\Omega)$; for some details and references see, e.g., [6, Chapter 4] and [10, Chapter 7].

Long-time dynamics of second order equations with a nonlinear damping was studied by many authors. We refer to [2, 15, 24, 25, 8, 18] for the case of a damping with a nonlinear displacement-dependent coefficient and to [9, 10, 11] and to the references therein for the velocity-dependent damping. Models with different types of a strong (linear) damping in wave equations were considered in [3, 4, 17, 23, 29], see also the literature quoted in these references.

The main novelty of the current paper is the following: (i) we can consider the strong nonlinear displacement- and velocity-dependent damping of a general structure (thus we cannot use analyticity of the corresponding model with a zero source term which takes place in the case when $D(u, u_t) = B u_t$, where $B$ is a self-adjoint operator satisfying (2) with $\theta \in [1/2, 1]$, see, e.g., [20, Chapter 3] and the references therein); (ii) this damping can be perturbed by low order terms.

Our main result (see Theorem 3.1) states the existence of a compact global attractor and describes other asymptotic (long-term) properties of the system generated by (1). To establish this result we use the recently developed approach (see [9] and also [10] and [11, Chapters 7,8]). We first prove that the corresponding system is quasi-stable in the sense of the definition given in [11, Section 7.9], and then we apply general theorems on properties of quasi-stable systems. In the same framework we also establish a result on the rate of stabilization (see Theorem 3.2) which states that under some additional conditions every solution is attracted by an equilibrium with an exponential rate. To obtain this result we rely on some type of the observability inequality and use the same idea as in [10, Section 4.3] (see also [9] and [11]).

The paper is organized as follows. In the preliminary Section 2 we discuss well-posedness of our abstract model and the dynamical system generation. We also recall several notions and
results from the theory of dissipative dynamical systems. Our main results on the global attractor for (1) and on asymptotic properties of individual trajectories are stated in Section 3. The proofs based on the quasi-stability property of the corresponding system are given in Section 4. In Section 5 we discuss some applications.

Below constants denoted by the same symbol may vary from line to line.

2 Preliminaries

In this section we show that problem (1) generates a dynamical system.

2.1 Well-posedness

We first prove the existence and uniqueness of weak solutions to problem (1). We recall that a function $u(t)$ is a weak solution to (1) on an interval $[0, T]$ if

$$u \in L_{\infty}(0, T; \mathcal{D}(A^{1/2})),$$

$$\partial_t u \in L_{\infty}(0, T; H) \cap L_2(0, T; \mathcal{D}(A^{\theta/2}))$$

and (1) is satisfied in the sense of distributions.

The main statement of the section is the following assertion which also contains some auxiliary solution properties needed for the results on asymptotic dynamics.

Theorem 2.1 Let Assumption 1.1 be in force and $(u_0; u_1) \in \mathcal{H} \equiv \mathcal{D}(A^{1/2}) \times H$. Then the following assertions hold.

1. Problem (1) has a unique weak solution $u(t)$ on $\mathbb{R}_+$. This solution belongs to the class

$$\mathcal{W} \equiv C(\mathbb{R}_+; \mathcal{D}(A^{1/2})) \cap C^1(\mathbb{R}_+; H),$$

and the following energy relation

$$\mathcal{E}(u(t), \partial_t u(t)) + \int_0^t (D(\partial_t u(\tau), \partial_t u(\tau)), \partial_t u(\tau))d\tau = \mathcal{E}(u_0, u_1)$$

holds for every $t > 0$, where the energy $\mathcal{E}$ is defined by the formula

$$\mathcal{E}(u_0, u_1) = E(u_0, u_1) + \Pi(u_0) \equiv \frac{1}{2} \left( |u_1|^2 + |A^{1/2}u_0|^2 \right) + \Pi(u_0).$$

Moreover, this solution $u(t)$ satisfies the estimate

$$\sup_{t \geq 0} E(u(t), \partial_t u(t)) + \int_0^{+\infty} |A^{\theta/2}\partial_t u(t)|^2 dt \leq C(R) \text{ if } E(u_0, u_1) \leq R^2. \quad (14)$$

2. If $u^1(t)$ and $u^2(t)$ are two weak solutions such that $E(u^i(0), \partial_t u^i(0)) \leq R^2$, $i = 1, 2$, then their difference $z(t) = u^1(t) - u^2(t)$ satisfies the relation

$$E(z(t), \partial_t z(t)) + a_R \int_0^t |A^{\theta/2}\partial_t z(\tau)|^2 d\tau \leq b_R E(z(0), \partial_t z(0)) e^{c_R t} \quad (15)$$

for some constants $a_R, b_R, c_R > 0$. 

5
Proof. To prove the existence of solutions, we use the standard Galerkin method of seeking for approximations of the form
\[ u_N(t) = \sum_{k=1}^{N} C_k(t)e_k, \quad N = 1, 2, \ldots \]
that solve the finite-dimensional projections of (1). Such solutions exist, and after multiplication of the corresponding projection of (1) by \( \partial_t u_N(t) \) we get that \( u_N(t) \) satisfies the energy relation (13). By (7) we obtain that
\[ c_0 E(u_0, u_1) - c_1 \leq \mathcal{E}(u_0, u_1) \leq C(R) \]
whenever \( E(u_0, u_1) \leq R^2 \). Therefore, by (2) the energy relation for \( u_N(t) \) yields estimate (14) for approximate solutions with the constant \( C(R) \) independent of \( N \). Using the equation for \( u_N(t) \) and also the conditions (2) and (6), it can be shown in the standard way that
\[ \int_{0}^{T} |A^{-1/2} \partial_t u_N(t)|^2 dt \leq C_T(R), \quad N = 1, 2, \ldots, \]
for every \( T > 0 \). These a priori estimates show that \( (u_N; \partial_t u_N; \partial_{tt} u_N) \) is \( \ast \)-weakly compact in
\[ L_\infty(0; T; H_1) \times [L_\infty(0; T; H) \cap L_2(0; T; H_\theta)] \times L_2(0; T; H_{-1}) \]
for every \( T > 0 \). Thus the Aubin-Dubinsky theorem (see [26, Corollary 4]) yields that \( (u_N; \partial_t u_N) \) is compact in \( C(0; T; H_{1-\varepsilon} \times H_{-\varepsilon}) \) for every \( \varepsilon > 0 \). These compactness properties make it possible to show the existence of weak solutions satisfying (14). For the limit transition in the nonlinear terms we use the property (Diii) in Assumption 1.1 and relation (6). It is also clear that \( t \mapsto (u(t); \partial_t u(t)) \) is a weakly continuous function in \( \mathcal{H} = \mathcal{D}(A^{1/2}) \times H \). To obtain the energy relation in (13), we note that the function \( u^n(t) = P_n u(t) \) solves an equation of the form
\[ \partial_t u^n + Au^n = h(t) \]
with some \( h \in L_2(0; T; H) \). This makes it possible to obtain a certain energy relation for \( u^n \) which gives us (13) after the limit transition \( n \to \infty \). This also allows us to obtain the strong continuity properties of \( t \mapsto (u(t); \partial_t u(t)) \) in \( \mathcal{H} \) by the standard method.

To prove (15), we note that \( z(t) = u^1(t) - u^2(t) \) solves the equation
\[ \partial_t z + D(u^1, \partial_t u^1) - D(u^2, \partial_t u^2) + Az + F(u^1) - F(u^2) = 0. \quad (16) \]
Thus, multiplying this equation by \( \partial_t z \) and integrating from \( s \) to \( t \) we have
\[ E_z(t) + \int_{s}^{t} (D(u^1, \partial_t u^1) - D(u^2, \partial_t u^2), \partial_t z) d\tau = E_z(s) - \int_{s}^{t} (F(u^1) - F(u^2), \partial_t z) d\tau \quad (17) \]
for any \( 0 \leq s < t \), where \( E_z(t) = E(z(t), \partial_t z(t)) \). Therefore, using (3), (6) and (14) we obtain that
\[ E_z(t) + \frac{\gamma R}{2} \int_{s}^{t} |A^{3/2} \partial_t z|^2 d\tau \leq E_z(s) + c_R \int_{s}^{t} (1 + |\partial_t u_1|^2 + |\partial_t u_2|^2) |A^{1/2} z|^2 d\tau, \quad s < t, \]
for some \( c_R > 0 \). Now we can apply the Gronwall lemma to obtain (15) which, in particular, implies the uniqueness of weak solutions. \( \square \)
2.2 Generation of a dynamical system

We recall that a dynamical system (see, e.g., [6, 16, 27]) is a pair \((X, S(t))\) of a complete metric space \(X\) and a family of continuous mappings \(S(t) : X \to X, \ t \geq 0\), such that (i) \(t \mapsto S(t)y\) is continuous in \(X\) for every \(y \in X\) and (ii) the semigroup property is satisfied, i.e., \(S(t + \tau) = S(t) \circ S(\tau)\) for any \(t, \tau \geq 0\) and \(S(0)\) is the identity operator.

We also recall that the system \((X, S(t))\) is gradient if it possesses a strict Lyapunov function, i.e., there exists a continuous functional \(\Phi(y)\) on \(X\) such that (i) \(\Phi(S(t)y) \leq \Phi(y)\) for all \(t \geq 0\) and \(y \in X\); (ii) the equality \(\Phi(y) = \Phi(S(t)y)\) may take place for all \(t > 0\) if only \(y\) is a stationary point of \(S(t)\).

Applying Theorem 2.1 we obtain the following assertion.

**Proposition 2.2** Let Assumption 1.1 be in force. Then problem (1) generates a dynamical system in the space \(\mathcal{H} = \mathcal{D}(\mathcal{A}^{1/2}) \times H\) with the evolution operator \(S(t)\) given by

\[
S(t)y = (u(t); \partial_t u(t)), \quad \text{where } y = (u_0; u_1) \text{ and } u(t) \text{ solves (1)}.
\]

This system is gradient with the full energy \(E(u_0; u_1)\) as a strict Lyapunov function (this follows from the energy relation in (13)).

We also recall that a system \((X, S(t))\) is called asymptotically smooth (see [16]) if for any closed bounded set \(B \subset X\) that is positively invariant \((S(t)B \subset B)\) one can find a compact set \(\mathcal{K} = \mathcal{K}(B)\) which uniformly attracts \(B\): \(\sup\{\text{dist}_X(S(t)y, \mathcal{K}) : y \in B\} \to 0\) as \(t \to \infty\). The global attractor (see, e.g., [1, 6, 16, 27]) of a dynamical system \((X, S(t))\) is defined as a bounded closed set \(\mathfrak{A} \subset X\) which is invariant \((S(t)\mathfrak{A} = \mathfrak{A}\) for all \(t > 0\)) and uniformly attracts all other bounded sets:

\[
\lim_{t \to \infty} \sup\{\text{dist}_X(S(t)y, \mathfrak{A}) : y \in B\} = 0 \quad \text{for any bounded set } B \text{ in } X.
\]

In this paper we use the following criterion of the global attractor existence for gradient systems (see, e.g., [28, Theorem 4.6]):

**Theorem 2.3** Let \((X, S(t))\) be an asymptotically smooth gradient system such that for any bounded set \(B \subset X\) there exists \(\tau > 0\) such that \(\gamma_\tau(B) \equiv \bigcup_{t \geq \tau} S(t)B\) is bounded. If the set \(\mathcal{N}\) of stationary points is bounded, then \((X, S(t))\) has a compact global attractor \(\mathfrak{A}\) which coincides with the unstable set \(\mathbb{M}_+(\mathcal{N})\) emanating from \(\mathcal{N}\), i.e., \(\mathfrak{A} = \mathbb{M}_+(\mathcal{N})\).

We recall (see, e.g., [1]) that the unstable set \(\mathbb{M}_+(\mathcal{N})\) emanating from \(\mathcal{N}\) is a subset of \(X\) such that for each \(z \in \mathbb{M}_+(\mathcal{N})\) there exists a full trajectory \(\{y(t) : t \in \mathbb{R}\}\) satisfying \(u(0) = z\) and \(\text{dist}_X(y(t), \mathcal{N}) \to 0\) as \(t \to -\infty\).

**Remark 2.4** We note that we can avoid the hypothesis of Theorem 2.3 that \(\gamma_\tau(B)\) is bounded if the following requirements on the corresponding Lyapunov function \(\Phi(x)\) are added: (i) \(\Phi(x)\) is bounded from above on any bounded set; (ii) the set \(\Phi_R = \{x \in X : \Phi(x) \leq R\}\) is bounded for every \(R\) (see, e.g., [10, Corollary 2.29]).

3 Main results

Our first main result is the following theorem.
Theorem 3.1 Let Assumption 1.1 be in force. Then the dynamical system \((\mathcal{H}, S(t))\) generated by (1) possesses a compact global attractor \(\mathfrak{A}\). Moreover,

1. \(\mathfrak{A} = \mathbb{M}_+(N)\), where \(N = \{(u; 0) \in \mathcal{H} : Au + F(u) = 0\}\) is the set of stationary points and

\[
\text{dist}_\mathcal{H}(S(t)y, N) \equiv \inf \{|S(t)y - e|_\mathcal{H} : e \in N\} \to 0 \quad \text{as} \quad t \to +\infty \quad \text{for every} \quad y \in \mathcal{H}. \quad (18)
\]

2. This attractor has a finite fractal dimension.

3. Any trajectory \(\gamma = \{(u(t); \partial_t u(t)) : t \in \mathbb{R}\}\) from the attractor \(\mathfrak{A}\) possesses the property

\[
(u; \partial_t u; \partial_{tt} u) \in L_\infty(\mathbb{R}; H_{2-\theta} \times H_1 \times H),
\]

and there is \(R > 0\) such that

\[
\sup_{\gamma \subset \mathfrak{A}} \sup_{t \in \mathbb{R}} \left( |u|_{2-\theta}^2 + |\partial_t u|_1^2 + |\partial_{tt} u|_1^2 \right) \leq R^2.
\]

4. The system \((\mathcal{H}, S(t))\) possesses a (generalized) fractal exponential attractor \(\mathfrak{A}_{\text{exp}}\) whose dimension is finite in the space \(\mathcal{H} = H_\theta \times H^{-1}\).

5. Let \(\mathcal{L} = \{l_j : j = 1, ..., N\}\) be a finite set of functionals on \(H_1\) with the completeness defect

\[
\epsilon_\mathcal{L} = \epsilon_\mathcal{L}(H_1, H) \equiv \sup \{ |u| : u \in H_1, l_j(u) = 0, j = 1, ..., N, |u|_1 \leq 1 \}.
\]

Then there exists \(\epsilon_0 > 0\) such that under the condition \(\epsilon_\mathcal{L} \leq \epsilon_0\) the set \(\mathcal{L}\) is (asymptotically) determining in the sense that the property

\[
\lim_{t \to \infty} \max_j \int_t^{t+1} |l_j(u^1(s) - u^2(s))|^2 ds = 0 \quad \text{for two solutions} \quad u^1 \text{ and } u^2
\]

implies that \(\lim_{t \to \infty} |S(t)y_1 - S(t)y_2|_\mathcal{H} = 0\). Here above \(S(t)y_i = (u^i(t); \partial_t u^i(t))\), \(i = 1, 2\).

We recall that the fractal dimension \(\dim^X M\) of a compact set \(M\) in a complete metric space \(X\) is defined as

\[
\dim^X M = \lim_{\varepsilon \to 0} \sup \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},
\]

where \(N(M, \varepsilon)\) is the minimal number of closed sets of diameter \(2\varepsilon\) in \(M\) needed to cover the set \(M\).

We also recall (see, e.g., [13] and also [10, 22] and the references therein) that a compact set \(\mathfrak{A}_{\text{exp}} \subset \mathcal{H}\) is said to be a (generalized) fractal exponential attractor for the dynamical system \((\mathcal{H}, S(t))\) iff \(\mathfrak{A}_{\text{exp}}\) is a positively invariant set of finite fractal dimension (in some extended space \(\mathcal{H}\)) and for every bounded set \(D \subset \mathcal{H}\) there exist positive constants \(t_D, C_D\) and \(\gamma_D\) such that

\[
\text{dist}_\mathcal{H}(S(t)x, \mathfrak{A}_{\text{exp}}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad t \geq t_D.
\]

As for the determining functionals, we mention that this notion goes back to the papers by Foias and Prodi [14] and by Ladyzhenskaya [19] for the 2D Navier-Stokes equations. For the further development of the theory we refer to [12] and to the survey [5] and to the references quoted therein (see also [6, Chap.5]). We note that for the first time determining functionals for second
order (in time) evolution equations with a nonlinear damping was considered in [7], see also a discussion in [11, Section 8.9] We also refer to [5, 6] for a description of sets of functionals with a small completeness defect. Determining modes and nodes are among them.

Using the same idea as in [9, 10, 11] we can establish the following result on convergence of individual solutions to equilibria with an exponential rate.

**Theorem 3.2** In addition to Assumption 1.1 we assume that $F(u)$ is Fréchet differentiable and its derivative $F'(u)$ possesses the properties
\[
|\langle F'(u), w \rangle|_{-1} \leq C_R |w|_1, \quad w \in H_1,
\]
and
\[
|\langle F'(u) - F'(v), w \rangle|_{-1} \leq C_R |u - v|_{1-\delta} \cdot |w|_1, \quad w \in H_1,
\]
for any $u, v \in H_1$ such that $|u|_1 \leq R$ and $|v|_1 \leq R$ with $\delta > 0$. Here $(F'(u), w)$ is the value of $F'(u)$ on the element $w$. Let the set $\mathcal{N}$ be finite and all equilibria be hyperbolic in the sense that the equation $Au + \langle F'(\phi), u \rangle = 0$ has only a trivial solution for each $(\phi; 0) \in \mathcal{N}$. Then for any $y \in \mathcal{H}$ there exists an equilibrium $e = (\phi; 0) \in \mathcal{N}$ and constants $\gamma > 0, C > 0$ such that
\[
|S(t)y - e|_{\mathcal{H}} \leq Ce^{-\gamma t}, \quad t > 0.
\]

We note that this type of stabilization theorems is well-known in literature for different classes of gradient systems, and several approaches to the question on stabilization rates are available (see, e.g., [1] and also [9, 10, 11] and the references therein). The approach presented in [1] relies on the analysis of linearized dynamics near each equilibrium and requires the hyperbolicity condition in a dynamical form. Here we use the method developed in [9, 10] (see also a discussion in [11]), and we need this condition in a weaker form.

## 4 Proofs

As it was already mentioned, the main ingredient of the proof of Theorem 3.1 is a quasi-stability property of the dynamical system $(\mathcal{H}, S(t))$ generated by (1).

### 4.1 Quasi-stability

We show that under the conditions listed in Assumption 1.1 the system $(\mathcal{H}, S(t))$ is quasi-stable in the sense of the definition given in [11, Section 7.9]. Namely, we prove the following proposition.

**Proposition 4.1** Let Assumption 1.1 hold. Assume that $u^i(t), i = 1, 2$ are two weak solutions to problem (1) with initial data $y_i = (u^i_0; u^i_1)$ such that $|A^{1/2}u^i_0|^2 + |u^i_1|^2 \leq R^2$ for some $R > 0$. We denote $S(t)y_i = (u^i(t); \partial_t u^i(t)), i = 1, 2$. Then there exist $C(R), \gamma(R) > 0$ such that
\[
|S(t)y_1 - S(t)y_2|^2_{\mathcal{H}} \leq C(R) \left[ |y_1 - y_2|^2_{\mathcal{H}} e^{-\gamma(R)t} + \int_0^t e^{-\gamma(R)(t-\tau)} |u^1(\tau) - u^2(\tau)|^2 d\tau \right], \quad t > 0.
\]

This type of estimates was originally introduced in [9] and related to a decomposition of the evolution operator $S(t)$ into uniformly exponentially stable and compact parts, see also a discussion in [10] and [11, Section 7.9].

We start with two preliminary lemmas.
Lemma 4.2  Under Assumption 1.1 there exist $T_0 > 0$ and a constant $c > 0$ independent of $T$ such that for any pair $u^1$ and $u^2$ of weak solutions to (1) we have the following relation

$$TE_z(T) + \int_0^T E_z(t)dt \leq c \left\{ \int_0^T |z(t)|^2 dt + \int_0^T |(D(t), z(t))|dt + \int_0^T |(D(t), z)| dt + \Psi_T(u^1, u^2) \right\}$$

for every $T \geq T_0$, where $z(t) = u^1(t) - u^2(t)$, and the functionals $E_z$, $D$, and $\Psi_T$ are defined as

$$E_z(t) = E_0(z(t), z_l(t)) = \frac{1}{2} ((z_l(t), z_l(t)) + (Az(t), z(t))),$$

$$D(t) = D(u^1(t), u^1_l(t)) - D(u^2(t), u^2_l(t)),$$

$$\Psi_T(u^1, u^2) = \left| \int_0^T (G(\tau), z_l(\tau))d\tau \right| + \left| \int_0^T (G(t), z(t)) dt \right| + \left| \int_0^T d\tau \int_t^T (G(\tau), z_l(\tau))d\tau \right|$$

with $G(t) = F(u^1(t)) - F(u^2(t))$.

Proof.  We use the standard arguments involving the multipliers $z_l$ and $z$ for (16). We refer to the proof of Lemma 3.23 in [10] and also [11, Lemma 8.3.1], where this lemma is proved under another set of hypotheses concerning the damping operator. However the corresponding argument does not depend on a structure of the damping operator.  □

Lemma 4.3  Let $u^1$ and $u^2$ be two solutions to (1) with the initial data $(u^1_0, u^1_i)$. We assume that $|u^1_i|^2 + |u^1_0|^2 \leq R^2$. Then

$$\max_{[0,T]} E_z(t) \leq c_0 \left[ E_z(T) + \int_0^T |(D(t), z_l)|dt \right] + C_R \left[ \int_0^T |A^{\theta/2} z_l|^2 dt + \int_0^T |A^{1/2} z|^2 dt \right],$$

with $z = u^1 - u^2$, where $E_z$ and $D(t)$ are the same as in Lemma 4.2.

Proof.  This follows from (17), the Lipschitz property for $F$ and the uniform estimate (14) for $u^i(t)$, $i = 1, 2$. We refer to [10, Lemma 3.25] for a similar assertion.  □

Now we complete the proof of Proposition 4.1. Using (4) with $\theta \in (0,1]$ we obtain that

$$|(D(t), z_l)| \leq C_{R, \varepsilon} |z_l|^2 + \varepsilon |z|^2 (1 + |u^1_t|^2 + |u^1_l|^2)$$

(25)

for any $\varepsilon > 0$. We also have from (4) for $\theta < 1$ and from (5) for $\theta = 1$ that

$$|(D(t), z)| \leq C_{R, \varepsilon} |z_l|^2 + \varepsilon |z|^2 (1 + |u^1_t|^2 + |u^1_l|^2) + C_{R, \varepsilon} |z|^2.$$

The subcritical estimate in (6) yields

$$|\Psi_T| \leq C_R \int_0^T |z_l|^2 dt + \varepsilon \int_0^T |z|^2 dt + C_{R, \varepsilon} \int_0^T |z|^2 dt \quad \text{for every} \quad \varepsilon > 0.$$
Therefore, Lemma 4.2 implies
\[ TE_z(T) + \int_0^T E_z(t) dt \leq C_{R,\varepsilon} \int_0^T |z(t)|^2 dt + \varepsilon \int_0^T |z|^3(T)|u_1^1|_{\partial \Omega}^2 + |u_1^2|_{\partial \Omega}^2 + \varepsilon + C_{R,\varepsilon, T} \int_0^T |z|^2 dt \tag{26} \]
for every \( T \geq T_0 \). By Lemma 4.3 using (14) and (25) we have that
\[ \max_{[0,T]} E_z(t) \leq 2c_0 E_z(T) + C_R \left[ \int_0^T |A^{\theta/2} z_t|^2 dt + \int_0^T |A^{1/2} z|^2 dt \right], \]
where the constants \( c_0 \) and \( C_R \) are independent of \( T \). Therefore, from (26)
\[ TE_z(T) + \int_0^T E_z(t) dt \leq C_{R,\varepsilon} \int_0^T |z(t)|^2 dt + \varepsilon \max_{[0,T]} E_z(t) \int_0^T \left( |u_1^1|_{\partial \Omega}^2 + |u_1^2|_{\partial \Omega}^2 \right) dt + C_{R,\varepsilon, T} \int_0^T |z|^2 dt \]
which, after an appropriate choice of \( \varepsilon \), implies that
\[ TE_z(T) + \int_0^T E_z(t) dt \leq C_R \int_0^T |z_t(t)|^2 dt + C_{R,\varepsilon, T} \int_0^T |z|^2 dt \tag{27} \]
for every \( T \geq T_0 \). Using (3), (6) and (17) we conclude that there exists \( \tilde{\gamma}_R > 0 \) such that
\[ \tilde{\gamma}_R \int_0^T |z_t(s)|_{\partial \Omega}^2 dt \leq E_z(0) - E_z(T) + \varepsilon \int_0^T |z(t)|^2 dt \]
\[ + C_{R,\varepsilon} \int_0^T |z|^3(T)|u_1^1|_{\partial \Omega}^2 + |u_1^2|_{\partial \Omega}^2 + \varepsilon + C_{R,\varepsilon, T} \int_0^T |z|^2 dt. \]
Consequently, choosing \( \varepsilon \) small enough, by (27) we have that
\[ TE_z(T) \leq c_R[E_z(0) - E_z(T)] \]
\[ + C_R \int_0^T E_z(t)(|u_1^1|_{\partial \Omega}^2 + |u_1^2|_{\partial \Omega}^2) dt + C_{R,\varepsilon, T} \int_0^T |z|^2 dt. \]
Thus
\[ E_z(T) \leq \kappa_R E_z(0) + C_R \int_0^T E_z(t)(|u_1^1|_{\partial \Omega}^2 + |u_1^2|_{\partial \Omega}^2) dt + C_{R,\varepsilon, T} \int_0^T |z|^2 dt \]
with \( \kappa_R < 1 \) and \( T \geq T_0 \). Now the standard argument (cf., e.g., [10, p. 62] or [11, p.414]) leads to (24). This concludes the proof of Proposition 4.1.

4.2 Completion of the proof of Theorem 3.1

1. Proposition 4.1 means that the system \((H, S(t))\) is quasi-stable in the sense of Definition 7.9.2 [11]. Therefore, by Proposition 7.9.4 [11] \((H, S(t))\) is asymptotically smooth. By Proposition 2.2 \((H, S(t))\) is a gradient system. Thus, Remark 2.4 and Theorem 2.3 imply that there exists a compact global attractor. By the standard results on gradient systems with compact attractors (see, e.g., [1, 6, 27]) we have that \( A = M_+(V) \) and (18) holds.

2. Since \((H, S(t))\) is quasi-stable, the finiteness of the fractal dimension \( \dim_f \mathfrak{A} \) follows from Theorem 7.9.6 [11].
3. To obtain the result on regularity stated in (19) and (20), we apply Theorem 7.9.8 [11].

4. One can see from (1) and Theorem 2.1 that any weak solution $u(t)$ possesses the property

$$
\int_{t}^{t+1} |\partial_t u(\tau)|^2 d\tau \leq C R \text{ for all } t > 0,
$$

provided $(u_0; u_1) \in B_R = \{ y \in \mathcal{H} : |y|_{\mathcal{H}} \leq R \}$. This implies that $t \mapsto S(t)y$ is a 1/2-Hölder continuous function with values in $\tilde{\mathcal{H}} = H_\theta \times H_{-1}$ for every $y \in B_R$. Therefore, the existence of a fractal exponential attractor follows from Theorem 7.9.9 [11].

5. To prove the statement concerning determining functionals, we use the same idea as in the proof of Theorem 8.9.3 [11], see also Theorem 7.9.11 [11].

4.3 Proof of Theorem 3.2

We use the same idea as in [9, 10, 11].

Since $\mathcal{N}$ is finite by (18) in Theorem 3.1 we have that for any $y \in \mathcal{H}$ there exists an equilibrium $e = (\phi; 0) \in \mathcal{N}$ such that

$$
|S(t)y - e|_{\mathcal{H}} \to 0, \quad t \to \infty. \quad (28)
$$

Thus we need only to prove that $S(t)y$ tends to $e$ with the stated rate.

Let $S(t)y = (u(t); u_t(t))$. We can assume that $\sup_{t \geq 0} |S(t)y|_{\mathcal{H}} \leq R$, for some $R > 0$. The function $z(t) = u(t) - \phi$ satisfies the following equation

$$
z_{tt}(t) + D(\phi + z(t), z_t(t)) + Az(t) + F(\phi + z(t)) - F(\phi) = 0, \quad t > 0. \quad (29)
$$

Let $\tilde{E}(t) = E_z(t) + \Phi(t)$, where $E_z(t)$ is the same as in Lemma 4.2 and

$$
\Phi(t) = \Pi(\phi + z(t)) - \Pi(\phi) - (F(\phi), z) \equiv \int_0^1 (F(\phi + \lambda z) - F(\phi), z) d\lambda.
$$

One can see that

$$
\tilde{E}(t) + \int_0^t (D(\phi + z(\tau), z_t(\tau)), z_t(\tau)) d\tau = \tilde{E}(0). \quad (30)
$$

In particular, we have that $\tilde{E}(t)$ is non-increasing. Moreover, since $(z; z_t) \to 0$ in $H_1 \times H$ as $t \to +\infty$, we have that $\tilde{E}(t) \to 0$ when $t \to +\infty$. Thus $\tilde{E}(t) \geq 0$ for all $t \geq 0$. It is also clear from (6) that

$$
|\tilde{E}(t) - E_z(t)| \leq C_R |z(t)|^2 - \delta |z(t)|_\theta \leq \varepsilon |z(t)|^2 + C_{R, \varepsilon} |z(t)|^2, \quad \forall \varepsilon > 0. \quad (31)
$$

Applying (27) for the case when $u^1(t) = u(t)$ and $u^2(t) = \phi$ we obtain that

$$
TE_z(T) + \int_0^T E_z(t) dt \leq C_R \int_0^T |z_t(t)|^2 dt + C_{R, T} \max_{[0, T]} |z(t)|^2
$$

for $T \geq T_0$ with some $T_0 > 0$. Now we prove the following lemma.

Lemma 4.4 Let $z(t)$ be a weak solution to (29) such that

$$
\int_{T-1}^T E_z(t) dt \leq \delta \quad \text{and} \quad \sup_{t \in \mathbb{R}_+} E_z(t) \leq \varrho \quad (33)
$$

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with some $\delta, \varrho > 0$ and $T > 1$. Then there exists $\delta_0 > 0$ such that
\[
\max_{[0,T]} |z(t)|^2 \leq C \int_0^T |z(t)|_0^2 dt
\]  
(34)

for every $0 < \delta \leq \delta_0$, where the constant $C$ may depend on $\delta$, $\varrho$ and $T$.

**Proof.** Assume that (34) is not true. Then for some $\delta > 0$ small enough there exists a sequence of solutions $\{z^n(t)\}$ satisfying (33) and such that
\[
\lim_{n \to \infty} \left\{ \max_{[0,T]} |z(t)|^2 \left[ \int_0^T |z^n(t)|_0^2 dt \right]^{-1} \right\} = \infty.
\]  
(35)

By (33) $\max_{[0,T]} |z^n(t)|^2 \leq C_\varrho$ for all $n$. Thus (35) implies that
\[
\lim_{n \to \infty} \int_0^T |z^n(t)|_0^2 dt = 0.
\]  
(36)

Therefore we can assume that there exists $z^* \in H_1$ such that
\[
(z^n; z^n) \to (z^*; 0) \text{-weakly in } L_{\infty}(0,T; H_1 \times H).
\]  
(37)

It follows from (2) and (36) that for $u^n(t) = \phi + z^n(t)$ we have the relation
\[
\lim_{n \to \infty} \int_0^T |(D(u^n(t), u^n(t)), \psi(t))| dt = 0 \text{ for any } \psi \in L_2(0,T; H_\varrho).
\]

This allows us to conclude that $u^* = \phi + z^* \in H_1$ solves the problem $Au + F(u) = 0$. From (33) we have that $|A^{1/2}(u^* - \phi)|^2 \leq 2\delta$. If we choose $\delta_0 > 0$ such that $|A^{1/2}(\phi_1 - \phi_2)|^2 > 2\delta$ for every couple $\phi_1$ and $\phi_2$ of stationary solutions (we can do it because the set $\mathcal{N}$ is finite), then we can conclude that $u^* = \phi$ provided $\delta \leq \delta_0$. Thus we have $z^* = 0$ in (37).

Now we normalize the sequence $z^n$ by defining $\tilde{z}^n \equiv c_n^{-1} z^n$ with $c_n = \max_{[0,T]} |z(t)|$, where we account only for a suitable subsequence of nonzero terms in $c_n$. It is clear from (37) with $z^* = 0$ that $c_n \to 0$ as $n \to \infty$. By (35) we also have that
\[
\int_0^T |\tilde{z}^n(t)|_0^2 dt \to 0 \text{ as } n \to \infty.
\]  
(38)

Relations (32) and (38) imply the following uniform estimate
\[
\sup_{t \in [0,T]} \left\{ |\tilde{z}^n(t)|^2 + |A^{1/2}\tilde{z}^n(t)|^2 \right\} \leq C, \quad n = 1, 2, \ldots.
\]

Thus we can suppose that there exists $\tilde{z}^* \in H_1$ such that
\[
(\tilde{z}^n; \tilde{z}^n) \to (\tilde{z}^*; 0) \text{-weakly in } L_{\infty}(0,T; H_1 \times H).
\]  
(39)

The function $\tilde{z}^n$ satisfies the equation
\[
\frac{\tilde{z}^n_t}{c_n} + \frac{1}{c_n} D(\phi + z^n, \tilde{z}^n) + A\tilde{z}^n + \frac{1}{c_n} [F(\phi + z^n) - F(\phi)] = 0.
\]  
(40)
As above, from (2) and (38) we conclude that
\[
\frac{1}{c_n} \int_0^T |\langle D(\phi + z^n, z^n_t), \psi \rangle| dt \to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad \psi \in L_2(0, T; H_\theta).
\]
It also follows from (21) and (22) that
\[
\frac{1}{c_n} [F(\phi + z^n) - F(\phi)] \to \langle F'(\phi), \hat{z}^* \rangle \quad \text{weakly in} \quad L_2(0, T; H_{-1}).
\]
Therefore, after the limit transition in (40) we conclude that \(\hat{z}^*\) satisfies
\[
A\hat{z}^* + \langle F'(\phi), \hat{z}^* \rangle = 0
\]
and, by hyperbolicity of \(\phi\) we conclude that \(\hat{z}^* = 0\). Thus (39) and Aubin-Dubinski theorem [26, Corollary 4] imply that \(\max_{[0, T]} |\hat{z}^n| \to 0 \quad \text{as} \quad n \to \infty\), which is impossible. \(\square\)

**Completion of the proof of Theorem 3.2.** By (28) we choose \(T_0 > 0\) such that (33) holds with \(\delta \leq \delta_0\) and \(T > T_0\). From (31) we have that \(\tilde{E}(T) \leq c_R E_z(T)\). The energy relation in (30) and the lower bound in (2) yield that
\[
\int_0^T |z_t(t)|^2 dt \leq c_R \left[ \tilde{E}(0) - \tilde{E}(T) \right]. \quad (41)
\]
Therefore Lemma 4.4, relation (32) and the energy relation in (30) imply that \(\tilde{E}(T) \leq \gamma_R \tilde{E}(0)\) for some \(0 < \gamma_R < 1\). This implies that \(E_z(mT) \leq \gamma_R^n \tilde{E}(0)\) for \(m = 1, 2, \ldots\). By (31), (34) and (41) we have that
\[
E_z(mT) \leq 2\tilde{E}(mT) + c_R \max_{[mT; (m+1)T]} |\hat{z}^n|^2 \leq c_R \tilde{E}(mT), \quad m = 1, 2, \ldots
\]
Thus \(E_z(mT) \leq c_R \gamma_R^m\) for \(m = 1, 2, \ldots\). Now using (15) we obtain (23). The proof is complete.

**5 Applications**

As it was mentioned in the introduction, our main applications are plate models.

**5.1 Plate models**

For the definiteness, we concentrate on the hinged boundary conditions (the results remain true with other types of self-adjoint boundary conditions). Below \(\| \cdot \|_s\) is the norm in the Sobolev space \(H^s(\Omega)\) of order \(s\).

**Forcing term:** We first check that the forcing term \(F\) satisfies Assumption 1.1(F) for all cases described above.

In the case of the Kirchhoff model, the embeddings
\[
H^s(\Omega) \subset L_{2/(1-s)}(\Omega), \quad L_{2/(1+s)}(\Omega) \subset H^{-s}(\Omega), \quad H^{1+\eta}(\Omega) \subset L_\infty(\Omega)
\]
for \(0 < s < 1\) and \(\eta > 0\) imply that for any \(\theta \in [1/2, 1]\) the force \(F\) given by (10) satisfies (6) with some \(\delta < 1/2\). In the case \(\theta \in (0, 1/2)\) we rely on the inequality
\[
\|f \cdot g\|_s \leq C\|f\|_{s+\sigma}\|g\|_{1-\sigma}, \quad f, g \in H^1(\Omega),
\]
for \(s < 1\).
The potential energy $\Pi$ has the form

$$\Pi(\mathbf{u}) = \int_\Omega \Phi(\mathbf{u}(x)) dx + \frac{\kappa}{q + 2} \int |\nabla \mathbf{u}(x)|^{q+2} dx - \frac{\kappa \mu}{r + 2} \int |\nabla \mathbf{u}(x)|^{r+2} dx - \int_\Omega \mathbf{u}(x) p(x) dx,$$

where $\Phi(s) = \int_0^s \varphi(\xi) d\xi$ is the antiderivative of $\varphi$. It follows from (11) that there exist $\gamma < \lambda_1^2$ and $C \geq 0$ such that $\Phi(s) \geq -\gamma s^2/2 - C$ for all $s \in \mathbb{R}$. This implies (7).

In the case of the von Karman model, we have that the Airy stress function $v(\mathbf{u})$ defined in (12) satisfies the inequality

$$\|v(u_1,v(u_1)) - v(u_2,v(u_2))\|_{-\eta} \leq C(\|u_1\|_2^2 + \|u_2\|_2^2)\|u_1 - u_2\|_{2-\eta}$$

for every $\eta \in [0,1]$ (see Corollary 1.4.5 in [11]). Thus, (6) holds for every $0 < \theta \leq 1$ with $\delta = \theta$. The potential energy $\Pi$ has the form

$$\Pi(\mathbf{u}) = \frac{1}{4} \int_\Omega \left[|v(\mathbf{u})|^2 - 2(|u| - 2p)u\right] dx$$

and possesses the properties listed in Assumption 1.1(F), see, e.g., [11, Chapter 4].

One can also see that the Berger model satisfies Assumption 1.1(F) for every $0 < \theta \leq 1$, and (6) holds with $\delta = \theta$; for some details see [6, Chapter 4] and [10, Chapter 7].

**Damping terms:** Now we consider possible forms of the damping operator. **Case $\theta = 1$:** Our main example is (9) under the following conditions:

- $\sigma_0(s_1), \sigma_1(s_1,s_2,s_3)$ and $g(s_1,s_2)$ are locally Lipschitz functions of $s_i \in \mathbb{R}$, $i = 1,2,3$;
- $\sigma_0(s_1) > 0$, $\sigma_1(s_1,s_2,s_3) \geq 0$ and $g(s_1,s_2)s_2 \geq 0$ for all $s_i$;
- there exists $\varrho \geq 0$ such that

$$|\sigma_1(\xi) - \sigma_1(\xi^*)| \leq C_R|\xi - \xi^*|(1 + |\xi_2|^q + |\xi_3|^q + |\xi_2^*|^q + |\xi_3^*|^q)$$

for all $\xi = (\xi_1,\xi_2,\xi_3), \xi^* = (\xi_1^*,\xi_2^*,\xi_3^*) \in \mathbb{R}^3$ such that $|\xi_1|,|\xi_1^*| \leq R$;

- there exist $q_1 \leq 4$ and $q_2 \leq 2$ such that

$$|g(\xi) - g(\xi^*)| \leq C_R \left[(1 + |\xi_2|^{q_1} + |\xi_2^*|^{q_1})|\xi_1 - \xi_1^*| + (|\xi_2|^{q_2} + |\xi_2^*|^{q_2})|\xi_2 - \xi_2^*| \right]$$

for all $\xi = (\xi_1,\xi_2), \xi^* = (\xi_1^*,\xi_2^*) \in \mathbb{R}^2$ such that $|\xi_1|,|\xi_1^*| \leq R$. 


In particular, we can take
\[ \sigma_1(u, \nabla u) = \sigma_{10}(u) + \sigma_{11}(u)|\nabla u|^r \quad \text{and} \quad g(u, u_t) = g_0(u)u_t + g_1(u)u_t^3, \]
where \( \sigma_{10}, \sigma_{11}, g_0 \text{ and } g_1 \) are nonnegative locally Lipschitz functions, \( r \geq 1 \).

**Case** \( \theta = 1/2 \): We consider the case when \( \sigma_0 \equiv 0 \) and \( \sigma_1 \) is independent of \( \nabla u \), i.e., we consider a damping of the form
\[ D(u, u_t) = -\text{div} [\sigma_1(u)\nabla u_t] + g(u, u_t). \quad (43) \]

One can see that Assumption 1.1(D) holds with \( \theta = 1/2 \) under the following conditions:

- \( \sigma_1(s_1) \) and \( g(s_1, s_2) \) are locally Lipschitz functions of \( s_i \in \mathbb{R}, i = 1, 2; \)
- \( \sigma_1(s_1) > 0 \) and \( g(s_1, s_2)s_2 \geq 0 \) for all \( s_i; \)
- the function \( g \) satisfies (42) with some \( q_1 < 3 \) and \( q_2 < 2 \).

We can also consider more general (anisotropic) damping operators \( D(u, u_t) \) which are defined variationally by the formula
\[ (D(u, u_t), \psi) = \sum_{i,j,k,l} \int_{\Omega} a_{ijkl}(u)u_{x_i,x_j}x_{x_k}x_{x_l} dx + \sum_{i,j} \int_{\Omega} b_{ij}(u)u_{x_i}x_{x_j} dx + \int_{\Omega} c(u, \nabla u, u_t)\psi dx \]
for every \( \psi \in (H^2 \cap H^1_0)(\Omega) \) under appropriate positivity and smoothness hypotheses concerning the coefficients.

### 5.2 Wave equation with strong damping

As an example, we can also consider the following wave equation on a bounded domain \( \Omega \) in \( \mathbb{R}^3 \) with a nonlocal damping coefficient:
\[ u_{tt} - \sigma_0(\|u\|_\eta)\Delta u_t + \sigma_1(u)u_t - \Delta u + \varphi(u) = f(x), \quad u|_{\partial \Omega} = 0. \quad (44) \]

We assume that \( \eta < 1 \) and the following conditions concerning the damping functions \( \sigma_0 \) and \( \sigma_1 \) are valid: (i) \( \sigma_j(s) \) are locally Lipschitz functions of \( s \in \mathbb{R}, j = 0, 1; \) (ii) \( \sigma_0(s) > 0, \sigma_1(s) \geq 0 \) for all \( s \in \mathbb{R}; \) (iii) there exists \( q_1 < 3 \) such that
\[ |\sigma_1(\xi) - \sigma_1(\xi^*)| \leq C(1 + |\xi|^{q_1} + |\xi^*|^{q_1})|\xi - \xi^*|, \quad \xi, \xi^* \in \mathbb{R}. \]

We also assume that the source term \( \varphi \in C^1(\mathbb{R}) \) possesses the properties
\[ \lim_{|s| \to \infty} \{\varphi(s)s^{-1}\} > -\lambda_1, \quad |\varphi'(s)| \leq C(1 + |s|^q), \quad s \in \mathbb{R}, \quad q < 4, \]
where \( \lambda_1 \) is the first eigenvalue of the Laplacian with the Dirichlet boundary conditions.

In this case we can apply Theorem 3.1 for the model in (44) with \( \theta = 1 \). We also note that basing on a requirement like (8) we can also cover the case when \( \eta = 1 \) in (44). As for the case of the critical growth exponents (\( q_1 = 3 \) and \( q = 4 \)) of the damping coefficient \( \sigma_1 \) and the force \( \varphi \), we cannot apply here our abstract approach. This case requires a separate consideration involving a specific structure of the model.

In a similar way, we can also consider the wave model (44) in arbitrary dimension \( d \) and with another structure of the damping operator.
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