Fans in the Theory of Real Semigroups
I. Algebraic Theory

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Abstract

In [DP1] we introduced the notion of a real semigroup (RS) as an axiomatic framework to study diagonal quadratic forms with arbitrary entries over (commutative, unitary) semi-real rings. (For the axioms of RS, cf. [17].) Two important classes of RSs were studied at length in [DP2], [DP3]. In this paper we introduce and develop the algebraic theory of RS-fans, a third class of RSs providing a vast generalization of homonymous notions previously existing in field theory and in the theories of abstract order spaces and of reduced special groups; for a background on fans, see paragraph A of the Introduction, below. The contents of this paper are briefly reviewed in paragraph B of the Introduction. The combinatorial theory of the structures dual to RS-fans, called ARS-fans, is the subject of [DP5b], a continuation of the present paper.

Introduction

In [DP1] we introduced the notion of a real semigroup (henceforth abbreviated RS), an axiomatic framework aimed at studying diagonal quadratic forms with arbitrary entries over commutative, unitary rings admitting a minimum of orderability. For ready reference we have included below the axioms defining RSs (1.7) and their underlying structures, the ternary semigroups (abbreviated TS, see 1.1). The basic properties of these structures are proved in [DP1], §§ 1,2, pp. 100-112, and [DP2], § 2, pp. 57-59. We also proved ([DP1], Thm. 4.1, p. 115) that the RSs are categorically dual to the abstract real spectra (ARS), previously introduced in [M], Chs. 6 – 9, with a similar goal.

In [DP2], [DP3] we introduced and studied two outstanding classes of RSs, the Post algebras and the spectral real semigroups, and their dual ARSs. The aim of this paper and its continuation [DP5b] is to present a third natural class of RSs (and their dual ARSs), namely fans, and develop their theory.

A. Background on fans. Fans were introduced by Becker and Köpping [BK] as a distinguished class of preorders in fields, and further investigated by several authors. Chapter 5 of the monograph [La] gives a quite complete picture of the role of fans in the context of fields, and contains many bibliographical references.

A further step was taken by Marshall, see [M], Ch. 3, who generalized the notion of a fan to the context of abstract spaces of orderings (AOS), an axiomatic framework extending the field case. In [Li] (see also [DMI], Ex. 1.7, pp. 8-9, and pp. 89-90) this notion was treated in the framework of reduced special groups (RSG), and its functorial duality with the corresponding notion of fan in the category of AOSs proved.

1 In this paper simply referred to as rings.
2 Namely, having a non-empty real spectrum or, equivalently, that −1 is not a sum of squares.
3 See [ABR], p. 84, and [La], Notes on § 5, p. 48.
Fans surfaced again in [ABR], Chs. 3, 5, under the still more general clothing of *spaces of signs*, a framework equivalent to that of ARSs. This book extensively witnesses the key role that fans play in real algebraic and real analytic geometry; see, e.g., [ABR], Thms. IV.7.3 and V.1.4 (the “generation formulae”), and [AR], pp. 1-7, where further references can be found. However, the notion of a fan used in [ABR] (cf. Def. 3.12, p. 75) is that of an AOS-fan suitably embedded in an ARS; see also [M], p. 162.

A sufficiently general and *intrinsic* notion of fan in the categories of ARSs and of RSGs does not exist at present. The aim of this paper and its continuation [DP5b], is to present and study such a notion. The key leading to this goal consists in bringing into play the (enriched) semigroup structures underlying the real semigroups, namely the *ternary semigroups* ([DP1], Def. 1.1, p. 100; see also 1.1 below). This task is not a straightforward extrapolation of the situation in the categories RSG and AOS, on two accounts. Firstly, owing to the rather complex algebraic and topological structure of the TSs, far more involved than the (trivial) structures underlying the RSGs (namely groups of exponent 2 with a distinguished element $-1$). Secondly, because the natural topology on ARSs is spectral —i.e., has non-trivial specialization— while the corresponding topology on AOSs is Boolean, and therefore has a trivial specialization order. While the first of these factors plays a central role in the present paper, the second will be crucial in its sequel, [DP5b], where we develop the combinatorial theory of ARS-fans.

To motivate the main ideas presented in this paper, we begin by briefly reviewing the definition of a fan in the (dual) categories AOS and RSG (for more details, see [M], Ch. 3; [Li], Ex. 1.1.6, pp. 30-31; [DM], Ex. 1.7, pp. 8-9).

— A fan in the category AOS (henceforth called an AOS-fan) is an abstract space of orders $(X, G)$ where “$X$ is biggest possible”; there are two equivalent ways of making sense of this idea:

1. $X$ consists of all group homomorphisms $h : G \to \{ \pm 1 \}$ such that $h(-1) = -1$.
2. $(X, G)$ is an AOS and $X$ is closed under the product of any three of its members.

— A fan in the category RSG (henceforth an RSG-fan) is a reduced special group $G$ whose binary representation relation is “smallest possible”: there is only one way of making sense of this:

$$ a \in D_G(b, c) \text{ iff either } b = -c \text{ or } (b \neq -c \text{ and } a \in \{b, c\}). $$

**Remarks 0.1**

(a) While condition (1) above implies that $(X, G)$ is an AOS, the last requirement in (2) alone is not sufficient to guarantee that $(X, G)$ is an AOS; in addition, one must require that:

1. $X$ separates points in $G$, i.e., $\bigcap_{\sigma \in X} \ker(\sigma) = \{1\}$.
2. $X$ verifies the following maximality condition (see [M], axiom [AX2] for AOSs, p. 22): for every group homomorphism $\sigma : G \to \{ \pm 1 \}$, if $\sigma(-1) = -1$ and $a, b \in \ker(\sigma)$ then $D_X(a, b) \subseteq \ker(\sigma)$, then $\sigma \in X$.

(b) The definition of binary representation given by condition [RSG-fan] above (together with $1 \neq -1$) implies that $G$ is a RSG ([Li], Prop. 1.1.14, pp. 34-36). Thus, every non-trivial group of exponent 2 is endowed with a structure of RSG. This is not the case for ternary semigroups, where an additional condition (called condition [Z]) ought to be satisfied for a TS to be endowed with a structure of RS; cf. Fact [L3] Proposition [L6] and Theorem 3.3.

We define the notion of a fan in the category ARS of abstract real spectra by postulating the analogs of conditions (1) and (2) above, upon replacing the underlying notion of a group of

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4 The categories of ARSs, RSs, AOSs, RSG’s, under natural morphisms, will be denoted by boldfacing the corresponding acronyms.
exponent 2 with a distinguished element \(-1\) by that of a ternary semigroup and, of course, the target group \(\{ \pm 1 \}\) by the ternary semigroup \(3 = \{-1, 0, 1\}\):

**Definition 0.2** Given a ternary semigroup \(G\) and a non-empty set \(X \subseteq \text{Hom}_{\text{TS}}(G, 3)^5\)

1. \((X, G)\) is a **fan** \(_1\) iff \(X\) consists of all TS-homomorphisms from \(G\) to \(3 = \{-1, 0, 1\}\), i.e., \(X = \text{Hom}_{\text{TS}}(G, 3)\).

2. \((X, G)\) is a **fan** \(_2\) iff it is an ARS and \(X\) is closed under the product of any three of its members.

We shall frequently use in the sequel the following weaker notion to which we give a name:

3. \((X, G)\) is a **q-fan** (quasi-fan) iff \(X\) is closed under the product of any three of its members and \(X\) separates points in \(G\), i.e., for every \(a, b \in G, a \neq b\), there is \(h \in X\) such that \(h(a) \neq h(b)\).

\(\square\)

**Remarks 0.3**

(a) The set \(3 = \{-1, 0, 1\}\) under obvious operations has a unique structure of TS. In fact, endowed with suitable ternary representation and transversal representation relations, see [1.8], it has a **unique** structure of RS. It obviously is a fan \(_1\).

(b) In [0.2](2) we allow products of type \(h^2_1 h_2\); as opposed to the case of special groups, squaring a TS-homomorphism does not produce a map constantly equal to 1. Note also that \(h^3 = h\), and that the product of any three TS-homomorphisms is again a TS-homomorphism.

(c) An obvious example of q-fan over a TS, \(G\), is \((\text{Hom}_{\text{TS}}(G, 3), G):\) \(\text{Hom}_{\text{TS}}(G, 3)\) is closed under product of any three members, and separates points in \(G\) by the separation theorem for TSs, [DP1], Thm. 1.9, pp. 103-104. \(\square\)

**B. Contents of the paper.**

In Section 1 we include for ready reference the axioms defining ternary semigroups [1.1] and real semigroups [1.7], as well as the basic example [1.8]. Other than these, the section reviews briefly some algebraic and topological notions and results used throughout the paper which do not appear in print elsewhere.

In §2 we prove (Theorem 2.3) that the ternary representation relation induced on any TS by a non-empty set of TS-characters separating points, automatically satisfies all axioms for real semigroups with the exception of the strong associativity axiom [RS3], see Definition 1.7.

In §3 we give a purely algebraic characterization of the representation and the transversal representation relations naturally occurring in a q-fan satisfying condition [Z] (Theorem 3.1). We also show that any ternary semigroup endowed with a ternary relation satisfying these algebraic requirements is automatically a real semigroup (Theorem 3.4). A number of important consequences follow from these results; notably

- The identity of the two notions of fan defined in [0.2] above (Proposition 3.5).
- Various properties substantiating the fact that both representation relations in RS-fans are “smallest possible” (Corollaries 3.7 – 3.10).

Section 4 gives a few examples of finite RS- and ARS-fans constructed from ternary semigroups on one and three generators. For each of these examples we draw the graph of the representation partial order of the corresponding RS-fan (cf. Definition [1.10]), and that of the specialization root system of its dual ARS-fan. We prove that, under the representation partial order, each RS-fan is a bounded lattice (not modular, in general), Theorem [1.5].

After a brief presentation of a general notion of congruence in real semigroups, in §5 we prove that the class of RS-fans is closed under the formation of arbitrary quotients (Proposition

\[5\] \(\text{Hom}_{\text{TS}}(G, 3)\) denotes the set of all TS-homomorphisms from the TS \(G\) into the TS \(3\).
5.8), and that quotients of RS-fans by congruences defined by ideals are (upon omitting zero) fans in the category of reduced special groups (Proposition 5.11). Section 6 contains a characterization of RS-fans in terms of specialization and quotients (Theorem 6.1). It follows that certain geometric configurations of the character space of RSs give rise to RS-fans (Corollaries 6.6 and 6.7).

In § 7 we apply the theory previously developed to the real semigroups associated to preordered rings. Proposition 7.8 gives a characterization of the ternary semigroup characters of the RS $G_{A,T}$ associated to a preordered ring $\langle A, T \rangle$ in terms of the algebraic operations of $A$ and the preorder $T$. This yields a natural extension of the original definition of fans by [BK] (cf. [La], Def. 5.1, p. 39) to the context of rings and, hence, a characterization of those RSs $G_{A,T}$ that are fans in terms of $\langle A, T \rangle$ (Proposition 7.10 (2)).

A total preorder $T$ of a ring $A$ is a (proper) preorder such that $T \cup -T = A$. Theorem 7.21 proves that if $T$ is either a total preorder of $A$ or the intersection of two total preorders such that the set of $T$-convex prime ideals of $A$ is totally ordered under inclusion, then $G_{A,T}$ is a RS-fan. This gives a ring-theoretic analog of the notion of a trivial fan, well-known in the field case ([La], Prop. 5.3, p. 39).

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1 Preliminaries

A. Ternary semigroups.

Definition 1.1 A ternary semigroup (abbreviated TS) is a structure $\langle S, \cdot, 1, 0, -1 \rangle$ with individual constants $1, 0, -1$, and a binary operation “·” such that:

[TSS1] $\langle S, \cdot, 1 \rangle$ is a commutative semigroup with unit.

[TSS2] $x^3 = x$ for all $x \in S$.

[TSS3] $-1 \neq 1$ and $(-1)(-1) = 1$.

[TSS4] $x \cdot 0 = 0$ for all $x \in S$.

[TSS5] For all $x \in S, x = -1 \cdot x \Rightarrow x = 0$.

We shall write $-x$ for $-1 \cdot x$. 

In § 1 of [DP1], pp. 100-105, the reader will find an account of basic results on ternary semigroups. In particular, the separation theorem [DP1], Thm. 1.9, pp. 103-104, and notation and results on the spectral and constructible topologies on the set $\text{Hom}_{TS}(T, 3)$ of characters of a TS, $T$, with values in $3 = \{1, -1, 0\}$, cf. [DP1], pp. 104-105, are repeatedly used in this paper.

Warning. Throughout this paper the default topology on all character spaces is the spectral topology. Whenever the associated constructible topology is used, the modifier $(\cdot)_{\text{con}}$ will be attached to the name of the space.

Beyond the results in [DP1], §1, we shall need the following results, which do not appear therein. The next Lemma gives several characterizations of the specialization order of the spectral topology in ternary semigroups.

Lemma 1.2 Let $T$ be a TS, and let $g, h \in X_T$. The following are equivalent:

(1) $g \sim h$ (i.e., $h$ is an specialization of $g$).
(2) $h^{-1}[1] \subseteq g^{-1}[1]$ (equivalently, $h^{-1}[-1] \subseteq g^{-1}[-1]$).
(3) $g^{-1} \{0, 1\} \subseteq h^{-1} \{0, 1\}$.
(4) $Z(g) \subseteq Z(h)$ and $\forall a \in G (a \not\in Z(h) \Rightarrow g(a) = h(a))$.
(5) $h = h^2g$ (equivalently, $h^2 = hg$).

**Proof.** The specialization partial order in any spectral space is defined by:

$$g \leadsto h \iff h \sqsubseteq \bar g \iff \text{for every subbasic open } U, \ h \in U \Rightarrow g \in U.$$ 

Since the subbasic opens of $X_T$ are the sets $\{ h \in X_T \mid h(a) = 1 \}$ for $a \in G$, we get at once the equivalence of (1) and (2).

**(1)/(3) \Rightarrow (4).** For the first assertion, if $g(a) = 0$, (3) gives $h(a) \in \{0, 1\}$, but (2) precludes $h(a) = 1$. For the second, (2) gives $h(a) = 1 \Rightarrow g(a) = 1$; if $h(a) = -1$, just replace $a$ by $-a$.

**(4) \Rightarrow (5).** The identity $h = h^2g$ obviously holds if $h(a) = 0$; if $h(a) \neq 0$, it follows from the second assertion in (4) and $(h(a)) = 1$.

**(5) \Rightarrow (2).** $h = h^2g$ and $h(a) = 1$ clearly imply $g(a) = 1$.

We also register the following algebraic characterizations of inclusion and equality of zero-sets of elements of $X_T$.

**Lemma 1.3** Let $T$ be a TS, and let $u, g, h \in X_T$. Then:

(1) $Z(g) \subseteq Z(h) \iff h = hg^2$.
(2) $Z(g) = Z(h) \iff g^2 = h^2$.
(3) If $u \leadsto g, h$, then $Z(g) \subseteq Z(h)$ if and only if $g \leadsto h$.

**Proof.** (1) and (2) are straightforward.

(3) Lemma [L2](4) proves the implication ($\Rightarrow$) and that $u \leadsto g, h$ implies $Z(u) \subseteq Z(g) \cap Z(h)$.

($\Leftarrow$) Assuming $Z(g) \subseteq Z(h)$, it suffices to verify the second clause of [L2](4). Let $a \not\in Z(h)$. Then, $a \not\in Z(g)$, and the equivalence of items (1) and (4) in [L2] together with $u \leadsto g$ and $u \leadsto h$ yields $u(a) = g(a)$ and $u(a) = h(a)$, respectively. Thus, $g(a) = h(a)$, as required.

**Remark.** Contrary to the case of real semigroups ([M], Prop. 6.4.1, p. 114), the specialization order of the character space of arbitrary ternary semigroups may not be a root system. A counterexample is given in [DP1], Ex. 1.14, p. 105. However, it is a normal space in the usual topological sense of this notion ([DP4], Prop. I.1.21).

**Fact 1.4** Let $G$ be a ternary semigroup, let $X \subseteq \text{Hom}_T(G, 3)$, and assume that $(X, G)$ is a q-fan. A necessary condition for $(X, G)$ to be an ARS is that for all $a, b \in G$, either $Z(a) \subseteq Z(b)$ or $Z(b) \subseteq Z(a)$. Here, $Z(a) = \{ h \in X \mid h(a) = 0 \}$.

**Proof.** Assume $(X, G) \models$ ARS but there are $a, b \in G$ so that $Z(a) \not\subseteq Z(b)$ and $Z(b) \not\subseteq Z(a)$, i.e., $h_1(a) = 0, h_1(b) \neq 0, h_2(b) = 0, h_2(a) \neq 0$, for some $h_1, h_2 \in X$. Since $(X, G)$ is a q-fan, $h_1^2h_2 \in X$. By [M], Prop. 6.1.5, p. 103, $D^2_G(a^2, b^2) = \{ c^2 \}$ for some $c \in G$, and $Z(a) \cap Z(b) = Z(c)$. This entails $h_1(c) = h_1(b) \neq 0$ and $h_2(c) = h_2(a) \neq 0$, whence $h_1^2h_2(c) \neq 0$, contradicting that $h_1^2h_2 \in Z(a) \cap Z(b) = Z(c)$.

**Fact 1.5** Let $T$ be a ternary semigroup and let $a, b \in T$. The following are equivalent:

(1) $Z(a) \subseteq Z(b)$; (2) $a^2b^2 = b^2$; (3) $a \mid b$ (i.e., $a$ divides $b$, i.e., $b = ax$ for some $x \in G$).
(4) $I_b \subseteq I_a$, where $I_c = c \cdot T$ is the ideal of $T$ generated by $c \in T$.

**Proof.** (4) $\iff$ (3) $\Rightarrow$ (1) are clear. For (2) $\Rightarrow$ (3), scaling the equality (2) by $b$ yields $a^2b = b$, i.e., $b = a(ab)$. For the implication (1) $\Rightarrow$ (2) we use the separation theorem for TS’s, [DP1],
Thm. 1.9: if $a^2b^2 \neq b^2$, there is $h \in \text{Hom}_{\text{RS}}(T, 3)$ such that $h(a^2b^2) \neq h(b^2)$; hence $h(b^2) = 1$ and $h(a^2b^2) = h(a^2) = 0$, i.e., $h(a) = 0$ and $h(b) \neq 0$, i.e., $Z(a) \not\subseteq Z(b)$.

Our next result gives alternative characterizations of the necessary condition in Fact 1.3.

**Proposition 1.6** Let $T$ be a ternary semigroup. The following conditions are equivalent:

1. The family $\{Z(a) \mid a \in T\}$ is totally ordered under inclusion.
2. For all $a, b \in T$, either $a^2b^2 = a^2$ or $a^2b^2 = b^2$.
3. For all $a, b \in T$, either $b \mid a$ or $b \mid a$.
4. Every proper ideal of $T$ is prime (i.e., $ab \in I \Rightarrow a \in I$ or $b \in I$).
5. The set of ideals of $T$ is totally ordered under inclusion.

**Proof.** The equivalence of (1) – (3) follows immediately from 1.3.

(2) $\Rightarrow$ (4). Let $I$ be an ideal of $T$, and suppose $ab \in I$; then $a^2b^2 \in I$ and, by (2), $a^2 \in I$ or $b^2 \in I$, which implies $a \in I$ or $b \in I$ (as $x = x^3 = x^2x$).

(4) $\Rightarrow$ (5). If $J_1, J_2$ are incomparable ideals, then $J_1 \cap J_2$ is not prime (if $a \in J_2 \setminus J_1, b \in J_1 \setminus J_2$ then $ab \in J_1 \cap J_2$ but $a \not\in J_1 \cup J_2$).

(5) $\Rightarrow$ (2). Given $a, b \in T$, by (5), either $I_a \subseteq I_b$ or $I_b \subseteq I_a$, and 1.3 yields $b \mid a$ or $a \mid b$.

**B. Real semigroups.** For easy reference we state the axioms defining real semigroups. The language for real semigroups, denoted $L_{\text{RS}}$, is that of ternary semigroups (\{·, 1, 0, −1\}) enriched with a ternary relation $D$. In agreement with standard notation (cf. [M], p. 99 ff.), we write $a \in D(b, c)$ instead of $D(a, b, c)$. We set:

\[
\text{[t-rep]} \quad a \in D^t(b, c) \iff a \in D(b, c) \land b \in D(-a, c) \land -c \in D(b, -a).
\]

The relations $D$ and $D^t$ are called representation and transversal representation, respectively.

**Definition 1.7** A real semigroup (abbreviated RS) is a ternary semigroup together with a ternary relation $D$ satisfying the following axioms:

- **[RS0]** $c \in D(a, b)$ if and only if $c \in D(b, a)$.
- **[RS1]** $a \in D(a, b)$.
- **[RS2]** $a \in D(b, c)$ implies $ad \in D(bd, cd)$.
- **[RS3]** (Strong associativity) If $a \in D^t(b, c)$ and $c \in D^t(d, e)$, then there exists $x \in D^t(b, d)$ such that $a \in D^t(x, e)$.
- **[RS4]** $c \in D(c^2a, d^2b)$ implies $e \in D(a, b)$.
- **[RS5]** If $ad = bd, ae = be$, and $c \in D(d, e)$, then $ac = bc$.
- **[RS6]** $c \in D(a, b)$ implies $c \in D^t(c^2a, c^2b)$.
- **[RS7]** (Reduction) $D^t(a, -b) \cap D^t(b, -a) \neq \emptyset$ implies $a = b$.
- **[RS8]** $a \in D(b, c)$ implies $a^2 \in D(b^2, c^2)$.

For a detailed treatment of real semigroups the reader is referred to [DP1], §§2–4, pp. 106-119, or [DP2], §2, pp. 57-59. Many of the properties of RSs and their duals, the abstract real spectra (abridged ARS) appearing in these and other references (e.g., [M], Chs. 6–8) are used below. For easy reference we state the following fundamental:

**Example 1.8** ([DP1], Corollary 2.4, p. 109) The ternary semigroup $3 = \{1, 0, -1\}$ has a unique structure of real semigroup, with representation given by:

- $D_3(0, 0) = \{0\}$
- $D_3(0, 1) = D_3(1, 0) = D_3(1, 1) = \{0, 1\}$
- $D_3(0, -1) = D_3(-1, 0) = D_3(-1, -1) = \{0, -1\}$
- $D_3(1, -1) = D_3(-1, 1) = 3$
and transversal representation given by:
\[
\begin{align*}
D_3'(0,0) &= \{0\}; & D_3'(0,1) &= D_3'(1,0) = D_3'(1,1) = \{1\}; \\
D_3'(0,-1) &= D_3'(-1,0) = D_3'(-1,-1) = \{-1\}; & D_3'(1,-1) &= D_3'(-1,1) = 3.
\end{align*}
\]

As we will see in §2 below, the crucial axiom in the list above is the strong associativity axiom [RS3]. We register ([M], Prop. 6.1.1, p. 100, and Thm. 6.2.4, pp. 107-108) that [RS3] is equivalent to the conjunction of the weak associativity axiom [RS3a] obtained by replacing transversal representation by ordinary representation in [RS] and

[RS3b] For all \(a, b\), \(D^l(a, b) \neq \emptyset\).

The following equivalent form of [RS3] turns out to be very useful at the time of checking strong associativity in concrete examples:

**Proposition 1.9** In the presence of axiom [RS2], the following is equivalent to axiom [RS3]:

\[
\forall a, b, c, d (D^l(a, b) \cap D^l(c, d) \neq \emptyset) \Rightarrow D^l(a, c) \cap D^l(b, d) \neq \emptyset.
\]

**Proof.** [RS3] \(\Rightarrow\) [RS3′]. Let \(x \in D^l(a, b) \cap D^l(c, d)\); the definition of \(D^l\) yields \(-b \in D^l(a, -x)\) ([DP1], Prop. 2.3 (0), p. 107), and scaling by \(-1\) ([RS2]) gives \(-x \in D^l(-c, -d)\). By [RS3] there is \(y \in D^l(a, -c)\) so that \(-b \in D^l(y, -d)\). Again, the definition of \(D^l\) and [RS2] yield \(-y \in D^l(b, -d)\), and \(y \in D^l(-b, d)\). Hence, \(D^l(a, -c) \cap D^l(-b, d) \neq \emptyset\).

[RS3′] \(\Rightarrow\) [RS3]. Assume [RS3′] and let \(x \in D^l(a, b)\) with \(b \in D^l(c, d)\). By the definition of \(D^l\), \(-b \in D^l(a, -x)\), and by [RS2], \(b \in D^l(-a, x)\), i.e., \(D^l(-a, x) \cap D^l(c, d) \neq \emptyset\). By [RS3′] there is \(y \in D^l(-a, -c) \cap D^l(-x, d)\). By the same manipulation as above, we get \(-y \in D^l(a, c)\) and \(x \in D^l(-y, d)\). So, [RS3] is verified with witness \(-y\). \(\square\)

**Remark.** Note that, while the weak associativity axiom [RS3a] is a non-trivial property (in the sense that it is not a consequence of the remaining axioms), the corresponding weak version of [RS3′], obtained by replacing \(D\) for \(D^l\), does follow from the remaining axioms for RSs, as [RS1] and [RS4] imply \(0 \in D(a, b)\) for all \(a, b\) ([DP1], Prop. 2.3 (1), p. 107). \(\square\)

**The representation partial order on real semigroups.** Recall ([DMI], Cor. 4.4 (c), p. 62) that for a reduced special group, \(G\), the binary relation \(a \leq b \iff a \in D_G(1, b)\) is a partial order for which the operation “multiplication by \(-1\)” is an involution. Further, this relation is induced from the partial order of the Boolean hull of \(G\) ([DMI], Cor. 4.12, p. 69).

In the context of RS’s, none of the binary relations \(a \in D(1, b)\) or \(a \in D^l(1, b)\) defines a partial order for which the operation “\(-\)” (multiplication by \(-1\)) is an involution. However, since every RS, \(G\), is canonically embedded in a Post algebra (seen as a RS, its “Post hull”, [DP2], Prop. 4.1, p. 62), which is a distributive lattice, the latter induces a partial order on \(G\) given by:

**Definition 1.10** ([DP2], Rmk. 2.5, p. 59) Let \(G\) be a RS, and let \(a, b \in G\). We set:

\[
a \leq_G b \quad \text{iff} \quad a \in D_G(1, b) \quad \text{and} \quad -b \in D_G(1, -a).
\]

[Unless necessary we omit the subscript in \(\leq_G\).] This relation is a partial order on \(G\) ([DMI] (1)) called the representation partial order. \(\square\)

When \(G = 3\) this definition gives \(1 <_3 0 <_3 -1\), the opposite of the order of these elements as integers. The binary relation just defined has the following properties:

**Theorem 1.11** Let \(G\) be a RS. For \(a, b, x, y \in G\) we have:

1. The relation \(\leq\) is a partial order on \(G\) such that \(a \leq b \iff -b \leq -a\).
2. For all \(a \in G\), \(1 \leq a \leq -1\).
3. \(a \leq 0 \iff a = a^2 \in \text{Id}(G)\).
0 ≤ a ⇔ a = −a^2 ∈ Id(G).

[Id(G) = \{a^2 | a ∈ G\} is the set of idempotents of G.]

(4) Let \( X_G \) be the character space of \( G \). For \( a, b ∈ G \),
\[
 a ≤_G b ⇔ ∀ h ∈ X_G (h(a) ≤_3 h(b)) ⇔
\]
∀ h ∈ X_G [(h(b) = 1 ⇒ h(a) = 1) ∧ (h(b) = 0 ⇒ h(a) ∈ \{0, 1\})].

(5) The following are equivalent:

(i) \( a^2 ≤ b ≤ −a^2 \);
(ii) \( Z(a) ≤ Z(b) \);
(iii) \( b = a^2b \).

In particular,

(6) \( a^2 ≤ ab ≤ −a^2 \) (hence \( a^2 ≤ ±a ≤ −a^2 \)).

(7) If \( a^2 ≤ b ≤ −a^2 \) and \( b \) is invertible, then \( a \) is invertible.

(8) \( a ≤ x, y ⇒ a ≤ −xy \). Hence, \( x, y ≤ a ⇒ xy ≤ a \).

(9) For all \( a ∈ G \), the infimum and the supremum of \( a \) and \( −a \) for the representation partial order \( ≤ \) exist, and \( a ∧ −a = a^2 \), \( a ∨ −a = −a^2 \). In particular,

(10) \( a ∧ −a ≤ 0 ≤ b ∨ −b \) for all \( a, b ∈ G \).

The proof of Theorem 1.11 appears in [DP4], Propositions I.6.4, I.6.5.

2 From ternary semigroups to real semigroups

In this short section we show that any non-empty set of TS-characters of a ternary semigroup induces ternary relations that satisfy all RS-axioms except, possibly, the strong associativity axiom [RS3] (cf. Definition 1.7).

**Definition 2.1** Given a ternary semigroup, \( G \), and a set \( H ⊆ X_G = \text{Hom}_{TS}(G, 3) \), we define a ternary relation \( D_{G,H} \) on \( G \) — abridged \( D_H \) if \( G \) is clear from context — as follows: for \( a, b, c ∈ G \),
\[
 [D]_H \quad a ∈ D_{G,H}(b, c) ⇔ \text{For all } h ∈ H, h(a) ∈ D_3(h(b), h(c)).
\]

To avoid triviality we assume \( H ≠ \emptyset \); to get best results we also make the rather mild assumption that the set \( H \) separates points in \( G \); given \( a ≠ b \) in \( G \), there is \( h ∈ H \) such that \( h(a) ≠ h(b) \).

In a similar way a corresponding “transversal” relation is defined by:
\[
 [D^t]_H \quad a ∈ D^t_{G,H}(b, c) ⇔ \text{For all } h ∈ H, h(a) ∈ D^t_3(h(b), h(c)).
\]

**Remark 2.2** Given a TS, \( G \), and a set \( H ⊆ G^3 \), in [M], p. 99, Marshall defines representation relations on \( G \), as follows: for \( a, b, c ∈ G \),
\[
 [R] \quad a ∈ D_H(b, c) \iff ∀ h ∈ H [h(a) = 0 ∨ (h(a) ≠ 0 ∧ (h(a) = h(b) ∨ h(a) = h(c)))].
\]
\[
 [TR] \quad a ∈ D^t_H(b, c) \iff ∀ h ∈ H [(h(a) = 0 ∧ h(b) = −h(c)) ∨ (h(a) ≠ 0 ∧
\[∧ (h(a) = h(b) ∨ h(a) = h(c)))]
\]

The representation relation \( D_H \) defined by clause [R] is identical with the relation defined by clause \( [D]_H \) in 2.1. This is obvious by the fact that conditions \( x ∈ D_3(y, z) \) and \( x = 0 ∨ (x ≠ 0 ∧ (x = y ∨ x = z)) \) are equivalent for all \( x, y, z ∈ 3 \); this is straightforward checking using Example 1.8.

Likewise, the transversal representation relation \( D^t_H \) defined by [TR] is identical to the transversal representation relation defined in terms of \( D^t_{G,H} \) by clause [t-rep], Section I since conditions \( x ∈ D^t_3(y, z) \) and \( ((x = 0 ∧ y = −z) ∨ (x ≠ 0 ∧ (x = y ∨ x = z))) \) are equivalent, again by [1.8].

\(^6\) Called the Kleene inequality, cf. [DP2], Rmk. 1.2 (b), p. 55.
Theorem 2.3 Let $G$ be a ternary semigroup and let $\mathcal{H}$ be a non-empty subset of $X_G$ separating points in $G$. The ternary relation $D_\mathcal{H}$ defined in 2.7 satisfies all axioms for real semigroups except, possibly, the axiom [RS3] of strong associativity.

Proof. The verification of axioms [RS0], [RS1], [RS2], [RS4] and [RS8] being straightforward, we deal only with the remaining axioms.

[RS5] Let $a, b, c, d, e \in G$ be such that $ad = bd$, $ae = be$ and $c \in D_\mathcal{H}(d, e)$. Let us prove that $ac = be$. Since $\mathcal{H}$ separates points in $G$, this boils down to proving $h(ac) = h(bc)$ for all $h \in \mathcal{H}$. This is clear if $h(c) = 0$. Let $h(c) \neq 0$. Since $c \in D_\mathcal{H}(d, e)$, either $h(c) = h(d)$ or $h(c) = h(e)$. Since $ad = bd$ and $ae = be$, invoking Definition 2.1 in both cases we get the equality $h(ac) = h(bc)$. By $[D]_{\mathcal{H}}$ once again, we conclude that $ac = be$, as required.

[RS6] Let $a, b, c \in G$ be such that $c \in D_\mathcal{H}(a, b)$, and take $h \in \mathcal{H}$. Then, $h(c) \in D_3(h(a), h(b))$. The real semigroup 3 verifies [RS6], and then $h(c) \in D_3(h(c)^2h(a), h(c)^2h(b))$. From the definition of $D^t$ (cf. §11 [t-rep]), we have the following relations:

(i) $h(c) \in D_3(h(c^2a), h(c^2b))$,  
(ii) $-h(c^2a) \in D_3(-h(c), h(c^2b))$,  
(iii) $-h(c^2b) \in D_3(-h(c), h(c^2a))$.

Since $h$ is arbitrary, from (i), (ii), (iii) we get:

(i') $c \in D_\mathcal{H}(c^2a, c^2b)$,  
(ii') $-c^2a \in D_\mathcal{H}(-c, c^2b)$,  
(iii') $-c^2b \in D_\mathcal{H}(-c, c^2a)$,

which, together, amount to $c \in D^t_{\mathcal{H}/G}(c^2a, c^2b)$.

[RS7] Let $a, b \in G$ be such that $D^t_\mathcal{H}(a, -b) \cap D^t_\mathcal{H}(b, -a) \neq \emptyset$. Take an element $c \in G$ in this intersection. We must prove that $a = b$. By 2.14 $[D]_{\mathcal{H}}$ this boils down to showing that $h(a) = h(b)$ for all $h \in \mathcal{H}$. We consider the following cases:

(i) $h(c) = 0$. If either $h(a) \neq 0$ or $h(b) \neq 0$, from the relations $-a \in D_\mathcal{H}(-c, -b)$ and $-b \in D_\mathcal{H}(-c, -a)$ we obtain $h(-a) = h(-b)$, and then $h(a) = h(b)$. If $h(a) = h(b) = 0$, there is nothing to prove.

(ii) $h(c) \neq 0$. Since $c \in D_\mathcal{H}(a, -b) \cap D_\mathcal{H}(b, -a)$, we have $h(c) = h(a)$ or $h(c) = -h(b)$, and $h(c) = h(b)$ or $h(c) = -h(a)$. If $h(a) \neq h(b)$, these conditions yield either $h(c) = h(a) = -h(a)$ or $h(c) = h(b) = -h(b)$; in both cases we have $h(c) = 0$, a contradiction. Hence, $h(a) = h(b)$.

Remark. Theorem 2.3 reduces the question of checking whether a TS with a ternary relation defined in terms of characters as in 2.4 above is a RS, to checking whether the single axiom [RS3] holds. For example, when the set $\mathcal{H}$ of characters has 2 elements, or has 3 elements with a non-trivial specialization, axiom [RS3] holds. However, there are examples of finite sets of characters $\mathcal{H}$ for which [RS3] does not hold in $G_\mathcal{H}$ (DP2), §1.3.

3 Fans are real semigroups and abstract real spectra

Our main aim in this section is to prove that (ARS-)fans, in any of the two senses considered in Definition 1.2, are abstract real spectra. The first step to achieve this is to work out the explicit form of the representation relations corresponding to the notion of “q-fan” (under the assumption that the necessary condition in 1.4 is verified); this is done in Theorem 3.1. It follows that any TS verifying this necessary condition and endowed with the relations thus obtained is a real semigroup (Theorem 3.4). A number of results 3.5–3.10 which determine to a large extent the structure of RS-fans follow from these theorems.

Theorem 3.1 Let $G$ be a ternary semigroup verifying $[Z] \forall a, b \in G (Z(a) \subseteq Z(b) \text{ or } Z(b) \subseteq Z(a))$. 


Let $X \subseteq \text{Hom}_{TS}(G, 3)$ be such that $(X, G)$ is a q-fan. With $D = D_X$ and $D^t = D_X^t$, denoting the representation relations defined by clauses $[R]$ and $[TR]$ in Definition 3.1, for $a, b \in G$ we have:

$$[D^t] \quad D^t(a, b) = \begin{cases} 
\{a\} & \text{if } Z(a) \subset Z(b) \\
\{b\} & \text{if } Z(a) \subset Z(b) \\
\{a, b\} & \text{if } Z(a) = Z(b) \text{ and } b \neq -a \\
a \cdot G (= b \cdot G) & \text{if } b = -a.
\end{cases}$$

$[D] \quad D(a, b) = a \cdot \text{Id}(G) \cup b \cdot \text{Id}(G) \cup \{x \in G \mid xa = -xb \land x = a^2x\}.

Remark 3.2 The inclusion $\supseteq$ in item $[D]$ of Theorem 3.1 holds for an arbitrary TS, $G$, and any set $X \subseteq \text{Hom}_{TS}(G, 3)$ (indeed, it follows from axioms $[RS1]$ and $[RS4]$): use Definition 2.1 and that $3$ is a real semigroup (Example 1.8).

Theorem 3.3 Let $G$ be a ternary semigroup verifying condition $[Z]$ of Theorem 3.1. Then, conditions $[D]$ and $[D^t]$ in Definition 3.1 are interdefinable in the following sense:

1. Assuming that a ternary relation $D$ on $G$ is defined as in $[D]$ and the corresponding transversal representation is given by the clause
   
   $a \in D^t(b, c) \iff a \in D(b, c) \land -b \in D(-a, c) \land -c \in D(b, -a),$
   
   then $D^t$ verifies condition $[D^t]$ of Definition 3.1.

2. Conversely, if $D^t$ is defined as in $[D^t]$ and the associated ternary representation relation $D$ is defined by the stipulation
   
   $a \in D(b, c) \iff a \in D^t(a^2b, a^2c),$
   
   then $D$ verifies clause $[D]$ of Definition 3.1.

Theorem 3.4 Let $G$ be a ternary semigroup verifying condition $[Z]$ of Theorem 3.1. With the ternary relation $D$ defined as in Definition 3.1, $(G, D)$ is a real semigroup.

This result is a natural extension of [12], Prop. 1.1.14 (see [11] (b)) to the theory of real semigroups.

Before engaging in the proof of these theorems we draw some important consequences of them.

Proposition 3.5 Let $G$ be a TS verifying condition $[Z]$ of Theorem 3.1 and $X \subseteq \text{Hom}_{TS}(G, 3)$. The following are equivalent:

1. $(X, G) = \text{fan}_1$ (i.e., $X = \text{Hom}_{TS}(G, 3)$).

2. i) $(X, G)$ is a q-fan.

   ii) For every subsemigroup $S$ of $G$ such that $S \cup -S = G$ and $S \cap -S$ is a (proper) prime ideal, there is $h \in X$ such that $S = h^{-1}[0, 1]$.

3. $(X, G) = \text{fan}_2$.

Proof. (3) $\Rightarrow$ (2). Assumption (3) implies that $(X, G)$ is an ARS. By axiom $[AX1]$ ([M], p. 99), $X$ separates points of $G$ and is closed under product of any three of its members; so, (2.i) holds. By Theorem 3.1 the representation relation $D$ is given by the equality $[D]$ therein. Since $S \cup -S = G$ we have $S \supseteq \text{Id}(G)$ and, by Corollary 3.10(2), $S$ is saturated for $D$. So, axiom $[AX2]$ in [M], p. 99 holds, and therefore $(X, G)$ verifies (2.ii).

(2.ii) $\Rightarrow$ (1). We must prove that $\text{Hom}_{TS}(G, 3) \subseteq X$. Let $g \in \text{Hom}_{TS}(G, 3)$; set $S := g^{-1}[0, 1]$. Clearly, this set verifies the assumptions in (2.ii). Hence, there is $h \in X$ so that $S = h^{-1}[0, 1]$, whence $g^{-1}[0] = S \cap -S = h^{-1}[0]$. The equalities $g^{-1}[0, 1] = h^{-1}[0, 1]$ and $g^{-1}[0] = h^{-1}[0]$ entail $g = h$, and hence $g \in X$.

(1) $\Rightarrow$ (3). Clearly, $X = \text{Hom}_{TS}(G, 3)$ is closed under product of any three elements. To prove it is an ARS we argue as follows. By the above and the separation theorem for TS,
for RSs ([DP1], Thm. 1.9, pp. 103–104), we conclude that Corollary 3.8. Axiom [RS6] for RSs implies its dual structure, \((X, \overline{G})\), is an ARS; hence, \((X, G)\) is a fan. 

**Definition and Notation 3.6 (Fan)** Henceforth we simply write “fan” (or “ARS-fan”) for either of the equivalent conditions \(\text{fan}_1\) or \(\text{fan}_2\). In using the notation \(\langle X, G \rangle \models \text{fan}\) we implicitly assume that the underlying ternary semigroup \(G\) verifies condition \([Z]\) in Theorem 3.1. This assumption is crucial and, in fact, distinguishes fans from many other classes of ARSs. We shall also say “\(G\) is a fan” (or a “RS-fan”), tacitly assuming that its representation relations are those given in Theorem 3.1.

**Corollary 3.7** Let \(G\) be a TS verifying condition \([Z]\) of Theorem 3.1. Let \(H\) be a real semigroup, and let \(f : G \rightarrow H\) be a homomorphism of ternary semigroups. Then, \(f\) preserves the representation relation \(D\) defined by clause \([D]\) of 3.1 and hence it is a RS-homomorphism from \((G, D)\) into \(H\). In other words, \(\text{Hom}_{RS}((G, D), H) = \text{Hom}_{TS}(G, H)\).

**Proof.** In view of the definition of \(D\), the proof boils down to the following obvious facts:

1. \(f\) preserves products and idempotents, hence the clauses defining the relation \(D\).
2. For \(a, b, c\) in an arbitrary RS, \(H\), we have
   
   (i) \(a \cdot \text{Id}(H) \subseteq D_H(a, b)\), and (ii) \(ca = -cb \land c = a^2c \Rightarrow c \in D_H(a, b)\).

Item \(2.1\) follows from axiom [RS4] of real semigroups. For the proof of \(2.2\), observe that, for \(h \in X_H\), \(h(c) \neq 0\) and \(c = a^2c\) imply \(h(a) \neq 0\). If \(h(c) \neq h(a)\), then \(ca = -cb\) yields \(h(c) = h(b)\). This shows that \(h(c) \in D_3(h(a), h(b))\) for all \(h \in X_H\); by the Separation Theorem for RSs ([DP1], Thm. 4.4, p. 116), we conclude that \(c \in D_H(a, b)\).

In particular we have:

**Corollary 3.8** Let \(G\) be a TS verifying condition \([Z]\) of Theorem 3.1. Then,

1. \(\text{Hom}_{RS}((G, D), 3) = \text{Hom}_{TS}(G, 3)\).

Hence,

2. The ARS dual to the real semigroup \((G, D)\) is \((\text{Hom}_{TS}(G, 3), \overline{G})\).

3. \(\text{Hom}_{TS}(G, 3), \overline{G})\) is a fan (hence an ARS-fan, see 3.6).

**Proof.** \(1\) is 3.7 with \(H = 3\), and \(2\) comes from the definition of the ARS dual to any RS (see proof of the Duality Theorem 4.1, [DP1], p. 117). For \(3\), see 0.2(1) and 3.6.

**Corollary 3.9** Let \((G, D)\) be a RS-fan. Then, the set \(G^\times = \{a \in G \mid a^2 = 1\}\) of invertible elements of \(G\) with representation induced by restriction of \(D\) to \(G^\times\), is a RSG-fan, i.e., a fan in the category of reduced special groups.

**Proof.** Since \(Z(a) = \emptyset\) for \(a \in G^\times\), only the last two clauses in the characterization of \(D_G^t\) given by Theorem 3.1 apply, whenever \(a, b \in G^\times\), and we have:

\[
D_G^t(a, b) = \begin{cases} 
\{ a, b \} & \text{if } b \neq -a, \\
G & \text{if } b = -a.
\end{cases}
\]

But this is exactly the definition of representation in a RSG-fan, cf. [RSG-fan] (Introduction). Axiom [RS6] for RSs implies \(D_G^t(a, b) \cap G^\times = D_G(a, b) \cap G^\times\), proving our contention.

---

7 Recall that \(\overline{G} = \{ \tilde{a} \mid a \in G \}\), where \(\tilde{a} \in 3^{\text{Hom}_{RS}(G, 3)}\) is the map “evaluation at \(a\): for \(\sigma \in \text{Hom}_{RS}(G, 3)\), \(\tilde{a}(\sigma) := \sigma(a)\).
Corollary 3.10 Let $G$ be a TS verifying condition $[Z]$ of Theorem 3.1 and let $D$ be the ternary relation on $G$ defined by clause $[D]$ therein. Then,
(1) Every TS-ideal of $G$ is a saturated prime ideal of the real semigroup $(G, D)$.
(2) A TS-subsemigroup $S$ of $G$ is saturated in $(G, D)$ iff it contains $\text{Id}(G) = \{x^2 \mid x \in G\}$ and $S \cap -S$ is an ideal.

Proof. (1) Straightforward verification, using (1.6).
(2) The implication $(\Rightarrow)$ is obvious. For the converse, write $I = S \cap -S$; $I$ is a prime ideal $\text{Id}(G)$) such that $c \neq a$ and $c \neq b$. Since $X$ separates points, these inequalities, together with $b \neq -a$, give TS-characters $h_1, h_2, h_3 \in X$ whose images at the points $a, b, c$ verify the corresponding inequalities in (3). By assumption, $h = h_1h_2h_3 \in X$, and we prove below that $h$ contradicts $c \in D_X^t(a, b)$; more precisely, $h$ verifies either
(*) $h(c) = 0$ and $h(a) \neq -h(b)$, or
(**) $h(c) \neq 0$, $h(c) \neq h(a)$ and $h(c) \neq h(b)$.

(I) Since $c \neq a$, there is $h_1 \in X$ so that $h_1(c) \neq h_1(a)$. According to the values of $h_1(c) \in \{0, 1, -1\}$, conditions $c \in D_X^t(a, b)$ and $Z(a) \subseteq Z(b)$ yield the following alternatives:

I.a. : $h_1(c) = 0$ and $h_1(a)h_1(b) = -1$ or,
I.b. : $h_1(c) = h_1(b) \neq 0$ and $h_1(a)h_1(b) = -1$.

(II) Assumption $c \neq b$, yields a character $h_2 \in X$ so that $h_2(c) \neq h_2(b)$. An analysis similar to that of (I) narrows the possible values of $h_2$ at the points $a, b, c$ down to:

II.a. : $h_2(c) = 0$ and $h_2(a)h_2(b) = -1$ or,
II.b. : $h_2(c) = h_2(a) \neq 0$ and $h_2(b) \in \{0, -h_2(a)\}$.

(III) The hypothesis $b \neq -a$ gives an $h_3 \in X$ such that $h_3(b) \neq h_3(-a)$. An argument similar to that of (I) and (II), using the assumptions $c \in D_X^t(a, b)$ and $Z(a) \subseteq Z(b)$, shows that $h_3$ can only take the following combination of values at $a, b, c$:

III.a. : $h_3(a) = h_3(b) = h_3(c) \in \{\pm 1\}$ or,
III.b. : $h_3(b) = 0$ and $h_3(a) = h_3(c) \in \{\pm 1\}$. With these data, a long, tedious, but straightforward checking of all possible combinations of values of the characters $h_i (i = 1, 2, 3)$ at the points $a, b, c$, shows that $h = h_1h_2h_3$ has properties (*) and (**), contradicting $c \in D_X^t(a, b)$. This proves item (2).

Next we show:

(3) $Z(a) \subseteq Z(b) \Rightarrow D_X^t(a, b) = \{a\}$.

Proof of (3). $b \in D_X^t(a, b)$ implies $Z(b) \subseteq Z(a)$ (immediate verification); hence $b \notin D_X^t(a, b)$. 

Since \( Z(a) \subset Z(b) \) implies \( b \neq -a \), items (1) and (2) give the conclusion.

The assertions (1) through (3) yield at once:

(4) If \( b \neq -a \), then \( D^t_X(a, b) \equiv \{a, b\} \leftrightarrow \) \( Z(a) = Z(b) \leftrightarrow a^2 = b^2 \).

(5) \( c \in D^t_X(a, -a) \leftrightarrow c = a^2c \leftrightarrow c = a^2x \) for some \( x \in G \).

Proof of (5). The last equivalence is obvious: \( c = a^2x \) implies \( a^2c = a^2(a^2x) = a^2x = c \). As for the first, we have:

\[(\Leftarrow) \text{ Let } h \in X. \text{ Obviously, } h(a) = -h(-a). \text{ The equality } c = a^2c \text{ implies } Z(a) \subseteq Z(c); \text{ hence } h(c) \neq 0 \text{ implies } h(a) \neq 0, \text{ and } h(c) \text{ equals either } h(a) \text{ or } h(-a), \text{ proving } c \in D^t_X(a, -a). \]

\[(\Rightarrow) \text{ For } h \in X, \ c \in D^t_X(a, -a) \text{ and } h(c) \neq 0 \text{ imply } h(c) = h(a) \text{ or } h(c) = -h(a); \text{ hence } h(a) \neq 0. \text{ This shows that } Z(a) \subseteq Z(c), \text{ which implies } a^2c = c^2 \text{ (cf. } 13); \text{ scaling by } c \text{ gives } c = a^2c. \]

This completes the proof of statement (A).

Next we deal with the identity \([D]\). We shall, in fact, prove assertion (2) of Theorem 3.3.

(B) If the ternary relation \( D^t \) is defined as in \([D^t]\) of 3.1 and the equivalence

\[ (\dagger) \ c \in D(a, b) \iff c \in D^t(c^2a, c^2b) \]

holds for all \( a, b, c \in G \), then \( D \) verifies the equality \([D]\).

We write \( \text{Id} \) for \( \text{Id}(G) \).

Proof of (6). Let \( x \in G \). By (\dagger) it suffices to show:

\[ (\dagger\dagger) \ ax^2 \in D^t(ax^2, a^2x^2b). \]

Since \( Z(ax^2) \subseteq Z(a^2x^2b) \), in case \( a^2x^2b \neq -ax^2 \), the first and third clauses of \([D^t]\) give (\dagger\dagger), and in case \( a^2x^2b = -ax^2 \) the last clause in \([D^t]\) proves (\dagger\dagger).

A similar argument gives \( b \cdot \text{Id} \subseteq D(a, b) \).

(7) \( ca = -cb \wedge c = a^2c \Rightarrow c \in D(a, b) \).

Proof of (7). Since \( c^2a = -c^2b \) and \( c = a^2c = (c^2a^2)c \), the last clause in \([D^t]\) yields \( c \in D^t(c^2a, c^2b) \), whence, by (\dagger), \( c \in D(a, b) \).

Items (6) and (7) prove the inclusion \( \supseteq \) in \([D]\). Conversely, assuming \( c \in D(a, b) \), we have \( c \in D^t(c^2a, c^2b) \), by (\dagger). An analysis according to the inclusions of the zero-sets of \( c^2a \) and \( c^2b \) gives:

(8) If \( Z(c^2a) \subseteq Z(c^2b) \) and \( c^2a \neq -c^2b \), then \( c \in D^t(c^2a, c^2b) \subseteq \{c^2a, c^2b\} \), implying \( c \in a \cdot \text{Id} \cup b \cdot \text{Id} \).

(9) If \( c^2a = -c^2b \), scaling by \( c \) gives \( ca = -cb \) and (by the last clause in \([D^t]\)) \( c = c^2a^2x \) for some \( x \in G \); this proves \( a^2c = a^2(c^2a^2x) = c^2a^2x = c \), as required. \( \Box \)

Proof of Theorem 3.3. Item 3.3(2) has just been proved (item (B), proof of 3.1).

Proof of 3.3(1). Assume \( G \) as in the statement, the ternary relation \( D \) defined by clause \([D]\) in 3.1 and \( D^t \) given by:

\[ c \in D^t(a, b) \iff c \in D(a, b) \wedge -a \in D(-c, b) \wedge -b \in D(a, -c). \]

The right-hand side of this equivalence amounts to:

(I) \( c \in a \cdot \text{Id}(G) \cup b \cdot \text{Id}(G) \cup \{x \in G \mid xa = -xb \land x = a^2x\} \).

(II) \( -a \in -c \cdot \text{Id}(G) \cup b \cdot \text{Id}(G) \cup \{x \in G \mid xc = xb \land x = c^2x\} \).

(III) \( -b \in a \cdot \text{Id}(G) \cup -c \cdot \text{Id}(G) \cup \{x \in G \mid xa = xc \land x = a^2x\} \).

The equivalence (\dagger) is readily checked to hold for the relations \( D_X \) and \( D^t_X \), using their definitions [R] and [TR] in 2.2.

\[ \dagger \]The equivalence (\dagger) is readily checked to hold for the relations \( D_X \) and \( D^t_X \), using their definitions [R] and [TR] in 2.2.
As above we write \( \text{Id} \) for \( \text{Id}(G) \). Remark that

\((*)\) \( x \in y \cdot \text{Id} \iff x = xy^2 \), and \((**)\) \( xy = xz \Rightarrow xy^2 = xz^2 (= xyz) \).

We argue by cases, according to the various clauses in \([D^1]\).

1. \( Z(a) \subset Z(b) \Rightarrow c = a \).

The clauses \(-a \in b \cdot \text{Id} \) and \(-a = c^2(-a) = b^2(-a) \) (see \((**)) \) in (II) imply \( Z(b) \subset Z(a) \), and hence are excluded; thus, (II) reduces to \( a \in c \cdot \text{Id} \). The following cases arise from (I) and (II):

(1.i) \( c \in a \cdot \text{Id} \) and \( a \in c \cdot \text{Id} \).

By \((*)\), \( c = ac^2 \) and \( a = ca^2 \). Hence, \( c = ac^2 = (ca^2)c^2 = ca^2 = a \).

(1.ii) \( c \in b \cdot \text{Id} \) and \( a \in c \cdot \text{Id} \).

Then, \( c = bc^2 \) and \( a = ca^2 \), implying \( a = ca^2 = a^2c^2b \); it follows that \( Z(b) \subset Z(a) \), contrary to the assumption in (1).

(1.iii) \( ca = -cb \land c = a^2c = b^2c \land a \in c \cdot \text{Id} \).

The middle equality implies \( Z(b) \subset Z(c) \) and the last \( Z(c) \subset Z(a) \). Hence \( Z(b) \subset Z(a) \), and this case is also excluded.

2. \( Z(b) \subset Z(a) \Rightarrow c = b \).

Same argument as in (1) interchanging \( a \) and \( b \).

3. \( Z(a) = Z(b) \land b \neq -a \Rightarrow c \in \{a, b\} \).

The first assumption gives \( a^2 = b^2 \). Each of the clauses \(-a \in b \cdot \text{Id} \) in (II) and \(-b \in a \cdot \text{Id} \) in (III) yield \(-a = ba^2 = b \) and hence are excluded. From (I) – (III) the following cases arise:

(3.i) \( c \in a \cdot \text{Id} \) and \( a \in c \cdot \text{Id} \).

We have \( c = ac^2 \) and \( a = ca^2 \); as in (1.i) we get \( c = a \).

(3.ii) \( c \in a \cdot \text{Id} \), \( ac = ab \) and \( a = c^2a \).

The first term gives \( c = ac^2 \); hence \( a = c \). The cases

(3.iii) \( c \in b \cdot \text{Id} \) and \(-b \in -c \cdot \text{Id} \), and

(3.iv) \( c \in b \cdot \text{Id} \), \( ab = ac \) and \( b = c^2b \),

are similar to (3.i) and (3.ii) —with \( b \) replacing \( a \)—, and yield \( c = b \).

(3.v) \( ca = -cb \), \( c = a^2c = b^2c \) and \(-a \in -c \cdot \text{Id} \).

As in (3.ii) this gives \( c = a^2c = a \) (see \((*)\)).

(3.vi) \( ca = -cb \), \( c = a^2c = b^2c \) and \(-b \in -c \cdot \text{Id} \).

As in (3.v) we obtain \( c = b \).

(3.vii) \( ca = -cb \), \( c = a^2c = b^2c \), \( ac = ab \), \( a = c^2a \), \( ab = bc \) and \( b = c^2b \) (the last disjunct from (I), (II) and (III)).

We have \( ac = -bc \), \( ac = ab \) and \( ab = bc \); hence \( bc = -bc \). Scaling by \( b \), \( b^2c = -b^2c \), whence \( c = -c \). It follows that \( c = 0 \), which clearly implies \( a = b = 0 \), i.e., \( a, b, c \) are all 0.

4. \( b = -a \Rightarrow c = a^2c = b^2c \).

Each disjunct in (I) implies \( c = a^2c \). The third disjunct contains this condition. If \( c \in a \cdot \text{Id} \), then \( c = ac^2 \) (by \((*)\)); hence \( a^2c = a^2(ac^2) = ac^2 = c \). Likewise, \( c \in b \cdot \text{Id} \), implies \( c = b^2c = a^2c \).

\[ \square \]

**Proof of Theorem 3.4.** We check that the axioms for real semigroups, cf. [17] hold in \((G, D)\).

By Theorem [4,3] we need only prove the associativity axiom [RS3] or, instead, the equivalent statement [RS3'], see Proposition [19]. To abridge we shall write:

\((*)\) \( \forall a, b, c, d (D^i(a, b) \cap D^i(c, d) \neq \emptyset) \), the hypothesis of [RS3'], and

\((**)\) \( D^i(a, -c) \cap D^i(-b, d) \neq \emptyset \), its conclusion.
The characterization of transversal representation in Theorem \ref{thm:y} will be of constant use and will be referred to as "$[D^l]$".

We consider three cases and, for each of them, several subcases according to the mutual inclusions of the zero-sets of $a, b, c, d$.

**Case A.** $a = -b$.

(A.i) $Z(c) \subset Z(a)$.

By the last clause in $[D^l]$ and (*) there is $x \in G$ so that $a^2 x \in D^l(c, d)$. If $Z(c) \subset Z(d)$, we would have $D^l(c, d) = \{c\}$, whence $a^2 x = c$, which implies $Z(a) \subset Z(c)$, contrary to (A.i).

Hence, $Z(d) \subset Z(c) \subset Z(a)$, and this yields $a^2 x \neq c, d$, implying $D^l(c, d) \notin \{c, d\}$. By $[D^l]$ we then have $d = -c$, and it follows that $D^l(a, -c) \cap D^l(-b, d) = D^l(a, -c) = \{-c\} \neq \emptyset$.

(A.ii) $Z(a) \subset Z(c)$.

From $[D^l]$ we have $a \in D^l(a, -c)$, and show that $a \in D^l(-b, d) = D^l(a, d)$. Otherwise, by $[D^l]$ again, we must have $Z(d) \subset Z(a) \subset Z(c)$, and assumption (*) gives an $x \in G$ such that $a^2 x \in D^l(c, d) = \{d\}$, implying $Z(a) \subset Z(d)$, a contradiction.

**Case B.** $c = -d$. Argument similar to that of Case A.

**Case C.** $a \neq -b$ and $c \neq -d$.

The first three clauses of $[D^l]$ show that $D^l(a, b) \subset \{a, b\}$ and $D^l(c, d) \subset \{c, d\}$. We consider the following subcases:

(C.i) $Z(a) \subset Z(b)$ and $Z(c) \subset Z(d)$.

In this case $[D^l]$ and (*) imply $a = c$, and (***) reduces to $D^l(a, -a) \cap D^l(-b, d) \neq \emptyset$. Since $Z(a) = Z(c) \subset Z(b) \cap Z(d)$, we get $b = a^2 b$ and $d = a^2 d$ \[1.5\]. If $b \neq d$, the first three clauses of $[D^l]$ show that one of $-b$ or $d$ is in $D^l(-b, d)$, and the preceding equalities give $a^2 x \in D^l(-b, d)$ for some $x \in G$. By the last clause of $[D^l]$ this also holds if $b = d$, proving $D^l(a, -a) \cap D^l(-b, d) \neq \emptyset$, as required.

(C.ii) $Z(a) \subset Z(b)$ and $Z(d) \subset Z(c)$.

From (*) we have $a \in D^l(c, d)$; then $Z(d) \subset Z(c)$ implies $a = c$ or $a = d$. In either case, $Z(d) \subset Z(b)$, whence $D^l(-b, d) = \{d\}$, and we are reduced to prove $d \in D^l(a, -c)$.

In case $a = c$ we must show that $a^2 x = c^2 x = d$ for some $x \in G$. If $Z(d) = Z(a)$ this holds with $x = d$ by \[1.5\]. If $Z(d) \subset Z(a) = Z(c)$, (*) gives $a \in D^l(c, d) = \{d\}$, leading to $a = c = d$, a contradiction.

Finally, in case $a = d$, since $D^l(-b, d) = \{d\}$, we are reduced to prove $d \in D^l(d, -c)$. Since we may assume $d \neq c$, this follows from $Z(d) \subset Z(c)$, using $[D^l]$.

(C.iii) $Z(a) = Z(b)$ and $Z(c) \subset Z(d)$.

Assumptions (*) and (3) imply $c = a$ or $c = b$, and we have $Z(a) = Z(b) = Z(c)$. Then, the conclusion (***) boils down to $-b \in D^l(a, -c)$. If $c = b$ this follows from $Z(a) = Z(-b)$ by the third clause of $[D^l]$. If $c = a$ then $Z(a) = Z(-b)$ implies $-b = a^2(-b)$, and the conclusion holds as well.

(C.iv) $Z(a) = Z(b)$ and $Z(d) \subset Z(c)$.

If $Z(d) \subset Z(c)$, $[D^l]$ and the assumptions (*) and (3) give $d \in D^l(a, b) = \{a, b\}$. If $d = a$, the desired conclusion boils down to $a \in D^l(-b, a)$, as $D^l(a, -c) = \{a\}$. Since $Z(a) = Z(-b)$, this holds by the last two clauses of $[D^l]$. If $d = b$ (and $a \neq b$), conclusion (***) reduces to $a \in D^l(-b, b)$, since $D^l(a, -c) = \{a\}$. But $Z(a) = Z(b)$ implies $a = b^2 a \in D^l(-b, b)$.

If $Z(d) = Z(c)$, assumptions (*) and (3) give $\{a, b\} \cap \{c, d\} \neq \emptyset$. It follows that $Z(a) = Z(b) = Z(c) = Z(d)$, and then $\{a, -c\} \subseteq D^l(a, -c)$, $\{-b, d\} \subseteq D^l(-b, d)$. If $a = c$, then $Z(a) = Z(c) = Z(d)$ entails $d \in D^l(a, -c)$, and (***) follows. If $d = b$, the same argument
shows $a \in D'(−b,d)$. If $a \neq c$ and $d \neq b$, then $a = d$ and $b = c$; this obviously implies $(a,−c) \cap \{−b,d\} \neq \emptyset$, whence $D'(a,−c) \cap D'(−b,d) \neq \emptyset$.

(C.v) $Z(b) \subset Z(a)$ and $Z(c) \subset Z(d)$. In this case we have $D'(a,b) = \{b\}$ and $D'(c,d) = \{c\}$, whence $b = c$, by (*). Therefore $Z(c) \subset Z(a)$, $Z(b) \subset Z(d)$, which yields $D'(a,−c) = \{−c\}$, $D'(−b,d) = \{−b\}$, and hence (**). (C.vi) $Z(b) \subset Z(a)$ and $Z(d) \subset Z(c)$.

The first clause of $[D']$ gives $D'(a,b) = \{b\}$, and hence $b = c$ or $b = d$, by (*) and (3). In the latter case we must show that $b^2 x \in D'(a,−c)$ for some $x \in G$. If $a \in D'(a,−c)$, since $b^2 a = a$ (as $Z(b) \subset Z(a)$), it suffices to take $x = a$. If $a \notin D'(a,−c)$, then $Z(c) \subset Z(a)$, and the second and fourth clauses of $[D']$ yield $−c \in D'(a,−c)$; since $b^2(−c) = −c$ ($Z(b) \subset Z(c)$), we can take $x = −c$.

Finally, if $b = c$, then $D'(a,−c) = \{−c\}$. If $−c \notin D'(−c,d) = D'(−b,d)$, then $Z(d) \subset Z(c) = Z(b) \subset Z(a)$, and (*) yields $b = d$, a contradiction. Thus, $−c \in D'(−b,d)$, verifying (**)) and completing the proof of Theorem 3.3.

3.11 A digression on q-fans.

The results proved above use in a crucial way the auxiliary—but nonetheless important— notion of a q-fan, introduced in 0.2(3). In Corollary 3.3 we proved that q-fans verifying Marshall’s axiom [AX2] for ARSs are the same thing as fans. The following example shows that this notion is genuinely weaker than that of a fan.

Example 3.12 A q-fan that is not a fan.

Let $C$ be the set of all non-decreasing functions $f : \mathbb{N} \rightarrow \{0,1\}$, where $0 < 1$; thus, $f \in C$ iff $f = 0$ or there is $n \in \mathbb{N}$ such that $f(m) = 0$ if $m < n$ and $f(m) = 1$ if $m \geq n$. Then, $−C = \{(−1) \cdot f | f \in C\}$ is the set of non-increasing maps $g : \mathbb{N} \rightarrow \{0,−1\}$ (with $−1 < 0$). Let $T = C \cup −C$. Straightforward checking shows that, with pointwise defined product, $T$ is a ternary semigroup, having as distinguished elements the constant functions with values 1, 0 and $−1$. In addition, $T$ verifies:

(i) $\text{Id}(T) = C$;

(ii) If $f, g \in T$, then $f \cdot g = f$ or $f \cdot g = g$.

For $n \in \mathbb{N}$, let $\pi_n : T \rightarrow \{0,1,−1\}$ stand for the projection onto the $n$-th coordinate: $\pi_n(f) = f(n)$ ($f \in T$). Straightforward checking, using that the functions in $C$ are non-decreasing and those in $−C$ are non-increasing, proves:

(i) For $k, n, m \in \mathbb{N}$, $k \leq n \leq m \Rightarrow \pi_k \cdot \pi_n \cdot \pi_m = \pi_k$.

In other words, the projections $\pi_n$ ($n \in \mathbb{N}$) form a subset $P$ of $X_T$ closed under the product of any three of its members. It is also clear that $P$ separates points in $T$. Hence, $(P,T)$ is a q-fan.

However, $(P,T)$ is not a fan. To see this, consider the map $h : T \rightarrow \{-1,0,1\}$ defined by:

$$h(f) = \begin{cases} 1 & \text{if } f \in C \setminus \{0\} \\ 0 & \text{if } f = 0 \\ -1 & \text{if } f \in −C \setminus \{0\}. \end{cases}$$

The reader can easily check that $h$ is a TS-character of $T$. However, $h \notin P$; for, if $n \in \mathbb{N}$, the map $f_n$ given by: $f_n(k) = 0$ for $k \leq n$ and $f_n(k) = 1$ for $k > n$, is in $C \setminus \{0\}$, whence $h(f_n) \neq 0$, while $\pi_n(f_n) = 0$.

The following result gives an interesting topological characterization of q-fans.
Theorem 3.13 Let $G$ be a ternary semigroup and let $X \subseteq X_G = \text{Hom}_{TS}(G, 3)$ be a non-empty set of $TS$-characters closed under product of any three of its members. Then, 

$$(X, G) \text{ is a } q\text{-fan } \iff X \text{ is dense for the constructible topology of } X_G.$$  

In particular, if $X$ is proconstructible, i.e., closed in the constructible topology, then $X = X_G$, and hence $(X, G)$ is a fan.

Proof. We shall write $a \equiv_X b$ to mean $g(a) = g(b)$ for all $g \in X$; cf. [5.2](c). Hence, the clause that $X$ separates points in $G$ translates as $a \equiv_X b \iff a = b.$

$(\Rightarrow)$ Since $X$ is assumed to be closed under products of any three of its members, we need only show that $X$ separates points in $G$. Let $a \neq b$. By the separation theorem for $TS$, [DPT], Thm. 1.9, pp. 103–104, the set \{ $h \in X_G \mid h(a) \neq h(b)$ \} is non-empty. It is easily checked that this set is open in the constructible topology of $X_G$, and hence it intersects $X$, i.e., there is $h \in X$ such that $h(a) \neq h(b)$.

$(\Rightarrow)$ By definition the sets

(*) $U = U(a_1) \cap \ldots \cap U(a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k),$

with $a_1, \ldots, a_n, b_1, \ldots, b_k \in G$, form a basis for the constructible topology of $X_G$. We must show: $U \neq \emptyset \Rightarrow U \cap X \neq \emptyset$.

Assume otherwise, and fix $h \in U$. We first show

Claim A. Let $U$ be a basic open containing $h$ such that $U \cap X = \emptyset$, with minimal $n$ having the form (*) for some $b_1, \ldots, b_k \in G$ ($k \geq 0$). Then, $n > 1$.

Proof of Claim A. Case 1. $n = 0$ and $k = 1$. Thus, $Z(b_1) \cap X = \emptyset$; this means $b_1^2 \equiv_X b_1$, hence $b_1^2 = 1$. Since $h \in \text{Hom}_{TS}(G, 3)$, we get $h(b_1^2) = 1$, contradicting $h \in Z(b_1)$.

Case 2. $n = 0$ and $k \geq 2$. Let $k$ be the smallest integer such that there are elements $b_1, \ldots, b_k$ so that $h \in \bigcap_{j=1}^k Z(b_j)$ and $\bigcap_{j=1}^k Z(b_j) \cap X = \emptyset$. Then, $\bigcap_{j=1}^{k-1} Z(b_j) \cap X \neq \emptyset$ and $\bigcap_{j=2}^k Z(b_j) \cap X \neq \emptyset$: let $h_1, h_2$ belong, respectively, to these sets. Since $X$ is closed under products of any three elements, $h_1 h_2 \in X$; clearly, $h_1 h_2 \in \bigcap_{j=1}^k Z(b_j)$, contradicting the choice of $k$.

Case 3. $n = 1$ and $k = 0$. Then $U(a_1) \cap X = \emptyset$: this implies $a_1 \equiv_X -a_1^2$ and hence $a_1 = -a_1^2$, contradicting $h(a_1) = 1$.

Case 4. $n = 1$ and $k = 1$. Then,

(†) $U(a_1) \cap Z(h_1) \cap X = \emptyset$.

Thus, $h(a_1) = h(h_1) = 0$: this yields $a_1^2 \neq a_1 h_1$. Since $X$ separates points, there is $h_1 \in X$, so that $h_1(a_1) = 0$ and $h_1(a_1) \neq 0$. From (†) we get $h_1(a_1) = -1$. On the other hand, the minimality assumption implies $U(a_1) \cap X \neq \emptyset$, i.e., there is $h_2 \in X$ such that $h_2(a_1) = 1$. Then, $h_2 h_1 h_2 \in X$, $h_2 h_2(a_1) = 1$ and $h_2 h_2(a_1) = 0$, i.e., $h_2 h_2 \in U(a_1) \cap Z(b_1) \cap X = \emptyset$, contrary to (†).

Case 5. $n = 1$ and $k \geq 2$. Let $k$ be the least integer such that there are $a_1, b_1, \ldots, b_k \in G$ with $U(a_1) \cap \bigcap_{j=1}^k Z(b_j) \cap X = \emptyset$. Then, there are $h_1, h_2 \in X$ so that $h_1(a_1) = h_2(a_1) = 1$, $h_1(b_1) = \ldots = h_1(b_{k-1}) = 0$ and $h_2(b_2) = \ldots = h_2(b_k) = 0$. Hence, $h_1 h_2 \in X \cap U(a_1) \cap \bigcap_{j=1}^k Z(b_j)$, contradiction.

Next, observe that setting $c = \prod_{i=1}^n a_i^2$ we have $U(c a_i) \subseteq U(a_i)$ for all $i = 1, \ldots, n$, and

$$\bigcap_{i=1}^n U(c a_i) = \bigcap_{i=1}^n U(a_i).$$

Hence, 

$$h \in \bigcap_{i=1}^n U(c a_i) \cap \bigcap_{j=1}^k Z(b_j) \text{ and } \bigcap_{i=1}^n U(c a_i) \cap \bigcap_{j=1}^k Z(b_j) \cap X = \emptyset.$$

Choosing $n$ minimal so that (**) holds for some $a_i, b_j$ ($k \geq 0$), for each index $i \in \{1, \ldots, n\}$ there is $g_i \in X_G$ such that
Lemma 4.1

Let the following supplement to Theorem 1.11, valid in the case of fans.

Examples

4 Examples

4.2 The examples.

Example 4.2. A. Ternary semigroups on one generator.
Call $x$ the generator. We treat first the case where there are no additional relations (“free” case). The corresponding TS is:

$$F_1 = \{1, 0, -1, x, -x, x^2, -x^2\}.$$ 

The necessary condition $[Z]$ is trivially verified. Characters are determined by their value on $x$, and any value 1, 0 and $-1$ is possible; hence the dual ARS, $X_{F_1}$, consists of three characters given by: $h_1(x) = 0$, $h_2(x) = 1$, $h_3(x) = -1$. Clearly, $h_i = h_1 \cdot h_i$, whence $h_i \sim h_1$, for $i = 2, 3$ (Lemma 1.2). So we get the specialization root-system below left.

Specialization root-system of $X_{F_1}$

![Specialization root-system of $X_{F_1}$]

Representation partial order of $F_1$

![Representation partial order of $F_1$]

The representation partial order of the real semigroup $F_1$—illustrated in Figure 1, right—follows straightforwardly from Theorem 1.11 (2)–(4).

Remark. Barring the case where the generator $x$ becomes invertible (i.e., $x^2 = 1$, which gives a four element RSG-fan with an added 0), the only possible additional relation is $x^2 = x$, which eliminates the character $h_3$. Thus, we get the following diagrams for the specialization order (left) and the representation order (right):

![Specialization root-system of $X_{F_1}$]

A more interesting example is:

**Example 4.2 B.** Ternary semigroups on three generators.

Generators: $x$, $y$, $z$. Condition $[Z]$ gives raise to the following possible relations:

1. $x^2 = y^2 = z^2$. ([Z] is automatically verified in this case.)
2. $x^2 = y^2 \neq z^2$ and $x^2z^2 = y^2z^2 \in \{x^2, z^2\}$.

The two identities obtained from the last clause give rise to non-isomorphic cases, and, upon permutation, all cases where two of the three generators have equal squares (i.e., equal zero-sets) are isomorphic to these.

3. $x^2, y^2, z^2$ are pairwise different, and $x^2y^2 \in \{x^2, y^2\}$, $x^2z^2 \in \{x^2, z^2\}$, $y^2z^2 \in \{y^2, z^2\}$.

A case-by-case analysis of all eight combinations of these values shows that, up to isomorphism by permutation, the only surviving case is $x^2y^2 = x^2z^2 = x^2$ and $y^2z^2 = y^2$. As an illustration we analyze the following configuration:

(a) $x^2 = y^2 \neq z^2$ and $x^2z^2 = y^2z^2 = x^2$.

This amounts to $Z(z) \subset Z(x) = Z(y)$ (Lemma 1.3). We focus on two alternatives:

i) No relations other than the above.
Routine checking shows that the following are all possible characters:

- \( h_1 \) sends all three generators to 0;
- \( h_2, h_3 \) send \( x, y \) to 0 and, say, \( h_2(z) = 1, h_3(z) = -1; \)
- \( h_4, \ldots, h_{11} \) assign to the generators all possible combinations of values \( \pm 1 \), with, say, \( h_4, \ldots, h_7 \) sending \( z \) to 1, and \( h_8, \ldots, h_{11} \) sending \( z \) to \(-1\).

Call \( F_2 \) the TS corresponding to this case. Using Lemma 1.2 one sees at once that the specialization root-system of the ARS dual to \( F_2 \) looks as in Figure 3.

![Figure 3. Specialization root-system of \( X_{F_2} \)](image)

Since \( X_{F_2} \) has 11 elements, by [DP5b], Cor. 1.10, we must have \( \text{card}(F_2) = 23 \); the reader is invited to check that:

\[
F_2 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, z^2, -z^2, xy, -xy, xz, -xz, yz, -yz, x^2z, -x^2z, xyz, -xyz\}.
\]

The Hasse diagram of the representation partial order of \( F_2 \) is drawn in Figure 4 below.

![Figure 4. Representation partial order of \( F_2 \)](image)

One may also consider fans arising by adding relations between generators; as an example we describe the fan obtained from the preceding one by adding:

ii) The extra relation \( xz = x \).
Under this relation, each character sending $z$ to $-1$ must also send $x$ to 0. Thus, the characters $h_8, \ldots, h_{11}$ in the preceding example disappear. Specialization among the remaining characters does not change; the order of specialization of the resulting ARS is obtained by omitting these characters in Figure 3. The resulting RS-fan is:

$$F_3 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, z^2, -z^2, xy, -xy\}.$$ 

The diagram of its representation partial order is computed in much the same way as in the preceding example; details are left to the reader.

iii) The relations $xz = x$ and $z^2 = 1$.

The additional relation $z^2 = 1$ makes $z$ invertible and hence excludes the character $h_1$ sending $z$ to 0. This makes the characters $h_2, h_3$ become “disconnected” (cf. [DP51], Def. 2.14(a)); we obtain a two-component root-system:

$$F_4 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, xy, -xy\},$$

with representation partial order:

Figure 5. Specialization root-system of $X_{F_4}$

Figure 6. Representation partial order of $F_4$

Summarizing a common feature of the examples presented above, we shall prove that, under the representation partial order $\leq$, every RS-fan is a bounded lattice.

**Lemma 4.3** Let $F$ be a RS-fan and let $a, b \in F$. Then,

1. If $a < b$, then $a \in \text{Id}(F)$ or $b \in -\text{Id}(F)$.

   In particular,

2. If $a, b \notin \text{Id}(F) \cup -\text{Id}(F) \quad \text{i.e.,} \quad \pm a \text{ and } \pm b \text{ are not squares} \quad \text{— and } a \neq b$, then $a, b$ are incomparable under $\leq$.

**Proof.** (2) follows easily from (1).

1. The assumption entails $a \in D(1, b), -b \in D(1, -a)$ and $a \neq b \ [1.10]$. The definition of representation in a RS-fan (Theorem 3.1) yields

   $$a \in \text{Id}(F) \lor a \in b \cdot \text{Id}(F) \lor a = -ab \quad \text{and} \quad -b \in \text{Id}(F) \lor b \in a \cdot \text{Id}(F) \lor b = ab.$$ 

   Combining these alternatives gives:

   — If any one of the first disjuncts occur, we are done.

   — If both the middle disjuncts occur, i.e., $a = bx^2$ and $b = ay^2$ for some $x, y \in F$, we get $a = ax^2y^2$, $b = bx^2y^2$ and $by^2 = ay^2$, whence $b = by^2x^2 = ay^2x^2 = a$, contradiction.

   — If $a = bx^2$ and $b = ab$, then $a = bx^2 = abx^2 = a^2 \in \text{Id}(F)$.

   — Likewise, $b \in a \cdot \text{Id}(F)$ and $a = -ab$ yield $-b \in \text{Id}(F)$. 


— The two last disjunctions imply \( a = -b \), whence \( a = -ab = a^2 \in \text{Id}(F) \).  

\[\text{Lemma 4.4} \] Let \( F \) be a RS-fan and let \( x, b \in F \). If \( b \notin \text{Id}(F) \) (i.e., \( b \neq b^2 \)), then \( b \leq -x^2 \leq -b^2 \) implies \( b^2 = x^2 \). That is, \(-b^2\) is the smallest \( y \in \text{Id}(F) \) such that \( b \leq -y \). Dually, \( b^2 \) is the largest \( y \in \text{Id}(F) \) such that \( y \leq b \).

\[\text{Proof.} \] Assume \( b \leq -x^2 < -b^2 \). Since \( b^2 \leq 0 \leq -x^2 \leq -b^2 \), we get \( b^2 x^2 = x^2 \) (and \( Z(b) \subseteq Z(x) \)) (Theorem \(1.11\)). On the other hand, \( b \leq -x^2 \) yields \( b \in D(1, -x^2) \), and then \( b \in D'(b^2, -b^2 x^2) = D'(b^2, -x^2) \). If \( Z(b) \subseteq Z(x) \), the first clause in \(3.1\) gives \( b = b^2 \), contrary to assumption. So, \( Z(b) = Z(x) \), and we get \( b^2 = x^2 \). The dual assertion is obvious.  

\[\text{Theorem 4.5} \] Let \( F \) be a RS-fan and let \( \leq \) denote its representation partial order \(1.10\). Then, \( (F, \leq) \) is a lattice with smallest element 1 and largest element \(-1\).

\[\text{Notation.} \] For elements \( a, b \) in a RS, \( G \), we write \( a \perp b \) to mean that \( a \) and \( b \) are incomparable under the representation partial order of \( G \).

\[\text{Proof.} \] We must show that every pair of elements \( a, b \in F \) has a least upper bound, \( \lor \), and a greatest lower bound, \( \land \), for the order \( \leq \). If \( a, b \) are comparable under \( \leq \) there is nothing to prove; so, we may assume \( a \perp b \).

Since \( F \) is a RS-fan, the zero-sets of \( a \) and \( b \) are comparable under inclusion. This, together with \( a \perp b \), implies that one of \( a \) or \( b \) is not in \( \text{Id}(F) \cup -\text{Id}(F) \); indeed:

- If \( Z(a) \subseteq Z(b) \), \(1.11\) yields \( a^2 \leq b \leq -a^2 \); hence, \( a \perp b \) implies \( a \notin \text{Id}(F) \cup -\text{Id}(F) \).
- Likewise, \( Z(b) \subseteq Z(a) \) implies \( b \notin \text{Id}(F) \cup -\text{Id}(F) \).

Since \( -a^2 \) and \( -b^2 \) are \( \leq \)-comparable \(1.11\), we may assume without loss of generality that \( -a^2 \leq -b^2 \). Further, Lemma \(4.4\) shows

\[ (*) \quad -b^2 = \text{least } x \in -\text{Id}(F) \text{ such that } a, b \leq x. \]

\[\text{Claim.} \] \(-b^2 = a \lor b. \)

\[\text{Proof of Claim.} \] By assumption, \( a, b \leq -b^2 \); so we need only prove:

\[ \forall c \in F (c \geq a, b \Rightarrow c \geq -b^2). \]

Note that \( c \geq a, b \) and \( a \perp b \) imply \( c \neq a, b \). If \( c \notin \text{Id}(F) \cup -\text{Id}(F) \), since one of \( a, b \) — say \( a \) — is not in \( \text{Id}(F) \cup -\text{Id}(F) \), then, by Lemma \(4.3\) \( c \neq a \) implies \( c \perp a \), absurd; hence \( c \in \text{Id}(F) \cup -\text{Id}(F) \). If \( c \in \text{Id}(F) \), then \( c \geq a, b \), implies \( a, b \in \text{Id}(F) \), whence \( a, b \) are \( \leq \)-comparable, contradiction. So, \( c \in -\text{Id}(F) \), and \( (*) \) gives \( c \geq -b^2 \), as claimed.

Under the current assumptions \(-a^2 \leq -b^2 \) and \( a \perp b \); upon observing that

\[ b^2 = \text{largest } y \in \text{Id}(F) \text{ such that } y \leq a, b, \]

a similar argument yields \( b^2 = a \lor b. \)

\[\text{Remarks 4.6} \] (a) A closer look at the examples presented above shows that the lattices \((F, \leq)\) are not modular —hence not distributive either— except in very special cases. In fact, most of these lattices contain the configuration

![Diagram](cf. [B], Ch. V, §2, Thm. 2, p. 66). For instance, in Figures 4 and 6 above, the sublattices \( \{z^2 < x^2 < -x^2 < -z^2; z\} \) and \( \{1 < x^2 < -x^2 < 1; z\} \), respectively, form such a

![Diagram](image-url)
pentagon. Example 4.2 A is modular but not distributive. Note that a RSG-fan (i.e., a reduced special group that is a fan, cf. [RSG-fan], Introduction) is a modular lattice under the order $a \leq b \iff a \in D(1, b)$. We also register that in [DP3], Thm. 6.6, pp. 396-397, we proved that the representation partial order in spectral real semigroups is a distributive lattice.

(b) Since $\text{Id}(F) \cup -\text{Id}(F)$ is a totally ordered subset of $(F, \leq)$, the proof of Theorem 4.5 shows that the lattice operations in $(F, \leq)$ satisfy the following identities:

$$a \land b = \begin{cases} \min \{a^2, b^2\} & \text{if } a \perp b \\ \min \{a, b\} & \text{if } a, b \text{ are } \leq\text{-comparable}, \end{cases}$$

and

$$a \lor b = \begin{cases} \max \{-a^2, -b^2\} & \text{if } a \perp b \\ \max \{a, b\} & \text{if } a, b \text{ are } \leq\text{-comparable}. \end{cases}$$

Note that, if $a \perp b$, then $a \land b, a \lor b \in \text{Id}(F) \cup -\text{Id}(F)$.

(c) The operation $x \mapsto -x$ ($x \in F$) is not a complement in the lattice-theoretic sense, but it verifies:

(c1) The Kleene inequality $a \land -a \leq 0 \leq b \lor -b$. (A special case of Theorem 1.11 (10).)

(c2) The De Morgan laws:

(i) $-(a \land b) = -a \lor -b$; (ii) $-(a \lor b) = -a \land -b$.

This is clear if $a$ and $b$ are comparable under $\leq$. If $a \perp b$, assuming, without loss of generality, $-a^2 \leq -b^2$, (i.e., $b^2 \leq a^2$), from (b) we get:

(i) $-(a \land b) = -(a^2 \land b^2) = b^2$, and $-a \lor -b = -(a^2) \lor -(b^2) = -a^2 \lor -b^2 = b^2$.

(ii) $-(a \lor b) = -(a^2 \lor -b^2) = -(-b^2) = b^2$, and $-a \land -b = -(a^2) \land -(b^2) = a^2 \land b^2 = b^2$. □

5 Quotients of fans

A. Preliminaries. Congruences of ternary semigroups and real semigroups.

Definition 5.1 A congruence of ternary semigroups (abbreviated TS-congruence) ternary semigroup (TS)-congruence congruence of ternary semigroup (TS)-congruence of a ternary semigroup $S$ is an equivalence relation $\equiv$ on a TS, $G$, compatible with the semigroup operation and such that the induced quotient structure $G/\equiv$ is a ternary semigroup. [This is equivalent to require $\equiv$ to be proper, i.e. $\equiv \subset G \times G$, and that $x \equiv y \Rightarrow x \equiv 0$.]

Remarks 5.2 (a) The condition that $\equiv$ is proper ensures that $1 \not\equiv 0$, and hence, by the last requirement, $1 \not\equiv -1$.

(b) Since the axioms for TSs are universal, the quotient map $\pi : G \longrightarrow G/\equiv$ is automatically a TS-homomorphism.

(c) For each non-empty set $H \subseteq \text{Hom}_{TS}(G, 3)$, the relation

$$(\dagger)_H \quad a \equiv_H b \iff \text{For all } h \in H, \, h(a) = h(b), \quad (a, b \in G),$$

defines a TS-congruence of $G$ (straightforward checking). We shall write $G/H$ for the quotient TS $G/\equiv_H$.

(d) The set $H \subseteq \text{Hom}_{TS}(G, 3)$ can be identified with a subset $\hat{H} = \{\hat{h} \mid h \in H\}$ of $3^{G/H}$ by the map $h \mapsto \hat{h}$, where $\hat{h} : G/H \longrightarrow 3$ is defined by the functional equation $\hat{h} \circ \pi = h$. By clause (1) above, $\hat{h}$ is well-defined and the map $h \mapsto \hat{h}$ is obviously injective. The reader can easily check that $\text{Hom}_{TS}(G/H, 3) = \hat{H}$. Further, if $\text{Hom}_{TS}(G, 3)$ and $\text{Hom}_{TS}(G/H, 3)$ are endowed with their spectral topologies and $H$ is a proconstructible subset of $\text{Hom}_{TS}(G, 3)$ closed under the product of any three of its members, the map above induces a homeomorphism between $\text{Hom}_{TS}(G/H, 3)$ and $H$ (DP2, Thm. I.1.26).
(e) In [DP4], Thm. I.1.26, it is shown that every TS-congruence of a ternary semigroup \( G \) is of the form \( \equiv_{\mathcal{H}} \) for a suitable set \( \mathcal{H} \) of TS-characters. The set \( \mathcal{H} \) can even be taken to be proconstructible in \( \text{Hom}_{\text{TS}}(G, 3) \).\( \square \)

**Definition 5.3** A (RS-)congruence of a real semigroup \( G \) is an equivalence relation \( \equiv \) satisfying the following requirements:

(i) \( \equiv \) is a congruence of ternary semigroups \([5.1]\).

(ii) There is a ternary relation \( D_{G/\equiv} \) in the quotient ternary semigroup \( (G/\equiv, \cdot, -1, 0, 1) \) so that \( (G/\equiv, \cdot, D_{G/\equiv}, -1, 0, 1) \) is a real semigroup, and the canonical projection \( \pi : G \rightarrow G/\equiv \) is a RS-morphism.

(iii) (Factoring through \( \pi \).) For every RS-morphism \( f : G \rightarrow H \) into a real semigroup \( H \) such that \( a \equiv b \) implies \( f(a) = f(b) \) for all \( a, b \in G \), there exists a RS-morphism (necessarily unique), \( \hat{f} : G/\equiv \rightarrow H \), such that \( \hat{f} \circ \pi = f \), i.e. the following diagram commutes

\[
\begin{array}{ccc}
G & \overset{f}{\rightarrow} & H \\
\pi \downarrow & & \downarrow \hat{f} \\
G/\equiv & &
\end{array}
\]

**Definition 5.4** With notation as in \([5.2]\), if \( G \) is a RS and \( \mathcal{H} \subseteq \text{Hom}_{\text{RS}}(G, 3) = X_G^2 \), we define a ternary relation \( D_{G/\mathcal{H}} \) on \( G/\mathcal{H} \) as follows: for \( a, b, c \in G \),

\[
(\dagger\dagger)_{\mathcal{H}} \quad \pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c)) \iff \text{For all } h \in \mathcal{H}, \ h(a) \in D_3(h(b), h(c)).
\]

Cf. clause \([D]_\mathcal{H}\) in Definition \([2.1]\). Obviously \( D_{G/\mathcal{H}} \) is well-defined and every RS-congruence of real semigroups is obtained in this way. To be precise, we state the following result, a straightforward consequence of the separation theorems for RSs and for TSs ([DPI], Thms. 4.4, pp. 116–117 and 1.9, pp. 103-104):

**Proposition 5.5** Given a real semigroup \( G \) and a RS-congruence \( \equiv \) of \( G \), let

\[ \mathcal{H}_\equiv = \{ p \in X_G | \text{There exists } \sigma \in X_{G/\equiv} \text{ so that } p = \sigma \circ \pi \}. \]

Then,

1. For \( a, b, c \in G \),
   - \( (i) \ \pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c)) \iff \text{For all } p \in \mathcal{H}_\equiv, \ p(a) \in D_3(p(b), p(c)). \)
   - \( (ii) \ a \equiv b \iff \text{For all } p \in \mathcal{H}_\equiv, \ p(a) = p(b). \)

2. \( (G/\equiv, D_{G/\equiv}) \cong (G/\mathcal{H}_\equiv, D_{G/\mathcal{H}_\equiv}) \) as \( \mathcal{L}_{\text{RS}} \)-structures.

Hence,

3. \( (G/\mathcal{H}_\equiv, D_{G/\mathcal{H}_\equiv}) \) is a RS.\( \square \)

**B. Congruences of fans.** We shall now consider the structure of congruences of RS-fans, giving an explicit description of them, and proving, in particular, that arbitrary quotients of fans are fans. We start with some preliminary observations used in the proof.

**Fact 5.6** Let \( G \) be a TS satisfying condition \([Z]\) in \([3.1]\) and let \( \emptyset \neq \mathcal{H} \subseteq \text{Hom}_{\text{TS}}(G, 3) \). Then, the quotient TS, \( G/\mathcal{H} \), also satisfies condition \([Z]\).

**Proof.** Follows from (1) \( \iff \) (2) in Proposition \([3.6]\).\( \square \)

**Fact 5.7** Let \( G \) be a RS and let \( \emptyset \neq \mathcal{H} \subseteq \text{Hom}_{\text{TS}}(G, 3) \). With the relation \( D_{G/\mathcal{H}} \) defined by clause \((\dagger\dagger)\mathcal{H}\) in \([5.4]\) for \( a, b \in G \), we have
\[ \pi(a) \cdot \text{Id}(G/H) \cup \pi(b) \cdot \text{Id}(G/H)) \cup \{\pi(c) | \pi(c)(\pi(a) = -\pi(c)(\pi(b)) \text{ and } \pi(c) = \pi(a^2)\pi(c)\} \subseteq D_{G/H}(\pi(a), \pi(b)). \]

**Proof.** See Remark 3.2. \[\Box\]

**Proposition 5.8** Let \( F \) be a RS-fan and let \( H \) be a non-empty subset of \( X_F \), which is 3-closed (i.e., stable under product of any three of its elements). Then the quotient \( F/H \) is a RS-fan (and \( \equiv_H \) is a RS-congruence).

**Proof.** With \( \pi : F \rightarrow F/H \) denoting the quotient map, we must show, for \( a, b, c \in F \):

\[ \pi(c) \in D_{F/H}(\pi(a), \pi(b)) \iff \pi(c) \in \pi(a) \cdot \text{Id}(F/H) \lor \pi(c) \in \pi(b) \cdot \text{Id}(F/H) \lor \pi(c)(\pi(a) = -\pi(c)(\pi(b)) \text{ and } \pi(c) = \pi(a^2)\pi(c). \]

\((\Rightarrow)\) Assume \( \pi(c) \in D_{G/H}(\pi(a), \pi(b)) \) and \([1]–[3]\) false. By \((\dagger\dagger)_H \) in [5.3] the negation of \([1]\) and \([2]\) yield characters \( h_1, h_2 \in H \) such that \( h_1(c) \neq h_1(a)h_1(c^2) \) and \( h_2(c) \neq h_2(b)h_2(c^2) \). The negation of \([3]\) is equivalent to \([3.i] \lor [3.iii], \)

\[ [3.i] \pi(c)(\pi(a) = -\pi(c)(\pi(b)), \]

\[ [3.iii] \pi(c)(\pi(a) = -\pi(c)(\pi(b)) \text{ and } \pi(c) = \pi(a^2)\pi(c). \]

Case \([3.i]\) yields \( h_3 \in H \) so that \( h_3(c)h_3(a) \neq -h_3(c)h_3(b) \). These inequalities obviously imply \( h_i(c) \neq 0, \text{ i.e., } h_i(c^2) = 1, \) for \( i = 1, 2, 3, \) and \( h_1(c) \neq h_1(a), h_2(c) \neq h_2(b) \).

Let \( h := h_1h_2h_3. \) Then, \( h \in X_F \) (\( F \) is a fan), \( h \in H \) (\( H \) is 3-closed), and \( h(c) \neq 0. \) The representation assumption implies, then, \( h(c) = h(a) \) or \( h(c) = h(b) \); suppose the first equality holds. From \( h_3(c) \neq h_2(b) \) we get \( h_2(c) = h_2(a) \), and hence \([4] h_1(c)h_3(c) = h_1(a)h_3(a). \)

Since \( h_1(c)h_3(c) \neq 0, \) we have \( h_1(a), h_3(a) \neq 0; \) from \( h_1(c) \neq h_1(a) \) comes \( h_1(c) = -h_1(a) \) and, by \([4]\), \( h_3(c) = -h_3(a). \) The representation assumption yields, then, \( h_3(c) = h_3(a), \) wherefrom \( h_3(b) = -h_3(a). \) Scaling by \( h_3(c) \) we get \( h_3(c)h_3(b) = -h_3(c)h_3(a), \) contrary to assumption \([3.i]. \)

The case \( h(c) = h(b) \) is dealt with by a similar argument.

Case \([3.iii]\) yields \( h_3 \in H \) so that \( h_3(c) \neq h_3(a^2)h_3(c). \) Squaring the equality \( \pi(c)(\pi(a) = -\pi(c)(\pi(b)) \) and scaling by \( \pi(c) \) we get \( \pi(c)(\pi(a^2) = \pi(c)(\pi(b^2))), \) whence \( h_3(c)h_3(a^2) = h_3(c)h_3(b^2) \neq h_3(c). \) This implies \( h_3(c) \neq 0, h_3(a) = 0 \) and \( h_3(b) = 0. \) Assumption \( \pi(c) \in D_{G/H}(\pi(a), \pi(b)) \) and \( h_3 \in H \) yield \( h_3(c) \in D_3(h_3(a), h_3(b)) = D_3(0, 0), \) whence \( h_3(c) = 0, \) contradiction.

That \( \equiv_H \) is a RS-congruence follows from the fact, just proved, that \( F/H \) is a RS-fan \[\Box\]

This completes the proof of Proposition 5.8. \[\Box\]

Observe that all RS-congruences of a fan are obtained in the way given by the preceding Proposition:

**Corollary 5.9** Let \( F \) be a RS-fan and let \( \equiv \equiv_H \) be a RS-congruence of \( F \). Then:

\( (a) \equiv = \equiv_H \) for some proconstructible, 3-closed set \( H \subseteq X_F. \) Hence,

\( (b) F/\equiv \) is a RS-fan.

\( (c) \) The correspondence \( H \mapsto \equiv_H \) establishes an inclusion-reversing bijection between proconstructible 3-closed subsets of \( X_F \) and the set of RS-congruences of \( F. \)

**Proof.** (a) The set \( H = \equiv_H \) is given by Proposition 5.3. Item (b) follows from Proposition 5.8 and item (c) is proved in [DP4], Thm. I.1.26. \[\Box\]

**Remark.** We register in passing that quotients of fans have a much stronger property called transversal 2-regularity, introduced and studied in [DP4], Ch. III, §3. The proof of this property appears in [DP4], Thm. VI.11.3. \[\Box\]
C. Quotients modulo ideals. As a last point in this section we address the special case of quotients of RS-fans modulo ideals. Amongst the outstanding cases of congruences of a RS (cf. [DP4], Ch. II, §3, [M], §§6.5, 6.6) one considers those determined by saturated prime ideals.

A saturated prime $I$ ideal of a RS, $G$, determines the set of characters $\mathcal{H}_I = \{ h \in X_G^0 \mid Z(h) = I \}$. The congruence $\sim_I$ induced by $\mathcal{H}_I$ will be denoted by $\sim$, and the corresponding quotient set by $G/I$. In [DP4], Thm. II.3.15, we characterize the congruence $\sim_I$ and both representation relations of $G/I$ solely in terms of the data carried by $G$. We also prove that the representation relation $D_{G/I}$ induces on the set $G_* := (G/I) \setminus \{ \pi(0) \}$, obtained from $G/I$ by omitting zero, the structure of a reduced special group. Including proofs of these results in full generality will take us too far afield. However, in the special case of RS-fans this property follows from Proposition 6.8. The following fact is used in the proof and elsewhere in this paper:

**Lemma 5.10** Let $I$ be an ideal of a RS-fan $F$. Then, for $a, b \in F \setminus I$:

$$a \sim b \iff \exists z \notin I \ (az = bz).$$

**Proof.** The implication ($\Rightarrow$) is clear: if $az = bz$ with $z \notin I$ and $h \in \mathcal{H}_I$, i.e., $Z(h) = I$, then $h(a)h(z) = h(b)h(z)$ entails $h(a) = h(b)$.

($\Leftarrow$) Assume $a \sim b$; then $ab \sim b^2$. Since $b \notin I$, we have $h(b) \neq 0$, i.e., $h(b^2) = 1$, for all $h \in \mathcal{H}_I$, whence $ab \sim b^2 \sim 1$. Set $p := ab$ and assume there is no $z \in F \setminus I$ such that $pz = z$, i.e., $\{ z \in F \mid pz = z \} \subseteq I$. Let $S$ be the semigroup of $F$ generated by $I \cup \text{Id}(F) \cup \{ p \}$. Clearly, $S = I \cup \text{Id}(F) \cup -p \cdot \text{Id}(F)$. Next, observe that $S \cap -S = I$. Indeed, if $x \in S \cap -S$, we have $x^2 \in S$. If $-x^2 \in I$, clearly $x \in I$. If $-x^2 \in \text{Id}(F)$, then $x = 0 \in I$. Finally, if $-x^2 \in -p \cdot \text{Id}(F)$, then $x^2 = pz^2$, whence $pz^2x^2 = z^2x^2$, yielding $z^2x^2 \in I$, and hence (by [1,6]) $z \in I$ or $x \in I$. In either case we conclude $x \in I$, proving $S \cap -S \subseteq I$.

Let $T$ be a subsemigroup of $F$ containing $S$ and maximal for $T \cap -T = I$. By Lemma 1.5, p. 102 of [DP4] there is a character $h \in X_F$ such that $Z(h) = I$ and $T = h^{-1}[0,1]$. Since $-p \in S \subseteq T$ and $p \notin I$, we get $h(p) = -1$, contradicting $p \sim 1$. This proves that $pz = abz = z$ for some $z \in F \setminus I$. Scaling by $b^2$ we get $b(bz) = a(bz)$ with $b \notin F \setminus I$, as required. □

**Proposition 5.11** Let $F$ be a RS-fan. Let $I$ be a proper ideal of $F$. Let $\pi = \pi_I : F \to F/I$ denote the canonical quotient map. Then, $F_* = (F/I) \setminus \{ \pi(0) \}$ is a RSG-fan.

**Proof.** It suffices to prove:

(i) $F_*$ is a group of exponent 2 with $1 \neq -1$.

Indeed, a straightforward computation using clause [D] of 3.1 and (i) proves that $(F_*, D_{F_*})$ satisfies condition [RSG-fan] (Introduction) defining RSG-fans, i.e., for $a, b \in F$ so that $\pi(a), \pi(b) \neq 0$ and $\pi(b) \neq \pi(-1)$,

$$\pi(a) \in D_{F/I}(\pi(1), \pi(b)) \Rightarrow \pi(a) = \pi(1) \lor \pi(a) = \pi(b).$$

**Proof of (i).** We must prove: if $a \in F$ is such that $\pi(a) \neq 0$ (i.e., $a \notin I$), then $\pi(a^2) = 1$ (i.e., $a^2 \sim 1$). By Lemma 5.10 the conclusion is equivalent to $\exists z \notin I \ (a^2z = z)$; take $z = a$. Note also that $\pi(1) \neq \pi(-1)$, i.e., $1 \neq 1$, since $1 \cdot z \neq (-1) \cdot z$ holds for every $z \neq 0$. □

**Remarks 5.12** (a) Quotients of real semigroups produce a considerable amount of collapse. An example is the quotient of the RS-fan $F_3$ of 3.2.B(ii) by the ideal $\{ 0 \}$: since the relation $z \cdot x = x = 1 \cdot x$ holds in $F_3$, by 5.10 the generator $z \in F_3$ collapses to 1 in $F_3/\{ 0 \}$.

(b) There is also collapse “from above”: any ideal $J \supset I$ collapses onto the whole of $F/I$: $\pi[J] = F/I$. 

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9 Quotients of this type have been considered by Marshall in the dual category of abstract real spectra; cf. [M], p. 102 and Cor. 6.6.9.
Clearly, Z(3) This is Lemma 1.5, p. 102 (alternatively, Lemma 3.5, p. 114) in [DP1].

By construction, I(c) (c) It can be proved that the inverse images \( \pi_f^{-1}[\Delta] \) of proper (saturated) subgroups \( \Delta \) of the RSG-fan \( F_1 \) are exactly the saturated subsemigroups \( \Gamma \subseteq F \) such that \( \Gamma \cap I = \emptyset \) and \( \Gamma \supseteq \pi_f^{-1}[1] \). The proof is omitted.

6 Characterizations of fans

The main result of this section is the following characterization of RS-fans:

Theorem 6.1 For a real semigroup \( G \), the following are equivalent:

(1) \( G \) is a RS-fan.

(2) \( G \) satisfies the following conditions:

(i) For every saturated prime ideal \( I \) of \( G \), the quotient reduced special group \( (G_1, D_{G_1}) \) is a RSG-fan.

(ii) For a real semigroup \( G \), and every ideal \( I \) of \( F \), the RSG-fan \( G \subseteq \] of proper (saturated) subgroups \( \Delta \) of the RSG-fan \( F_1 \) are exactly the saturated subsemigroups \( \Gamma \subseteq F \) such that \( \Gamma \cap I = \emptyset \) and \( \Gamma \supseteq \pi_f^{-1}[1] \). The proof is omitted.

Remark. The implication (1) \( \Rightarrow \) (2.iii) is Proposition 5.11 Condition (2.i) is assumption [Z] in the definition of a RS-fan [3,6]; see also Fact 1.4 and Proposition 1.6. Therefore, to complete the proof of 6.1 we must only take care of (1) \( \Rightarrow \) (2.ii) and (2) \( \Rightarrow \) (1), proved, respectively, in Propositions 6.2 (2) and 6.4 below.

Proposition 6.2 Let \( G \) be a RS-fan. Then:

(1) For all elements \( g, h \in X_G \), such that \( g \sim h \) (hence \( Z(g) \subseteq Z(h) \)) and every ideal \( I \) such that \( Z(g) \subseteq I \subseteq Z(h) \) there is \( f \in X_G \) such that \( g \sim f \sim h \) and \( Z(f) = I \).

(2) For every \( g \in X_G \), and every ideal \( I \supseteq Z(g) \) there is a (necessarily unique) \( f \in X_G \) such that \( g \sim f \) and \( Z(f) = I \).

(3) For every ideal \( I \) of \( F \) there is an \( f \in X_G \) such that \( Z(f) = I \).

Proof. Since \( G \) is a RS-fan, every TS-character \( f : G \to \mathbf{3} \) is a RS-homomorphism. Thus, it suffices to construct TS-homomorphisms \( f : G \to \mathbf{3} \) verifying (1) – (3) of the statement.

First we prove (1); the same proof, omitting item (c) below, also proves (2). Let \( f : G \to \mathbf{3} \) be defined by:

\[
f \upharpoonright I = 0 \quad \text{and} \quad f \upharpoonright (G \setminus I) = g \upharpoonright (G \setminus I).
\]

(a) \( Z(f) = I \).

By construction, \( I \subseteq Z(f) \). Since \( Z(g) \subseteq I \), \( f(x) = g(x) \neq 0 \) for \( x \in G \setminus I \), i.e., \( Z(f) \subseteq I \).

(b) \( g \sim f \).

Clear, from (a) and Lemma 1.2 (4),

(c) \( f \sim h \).

Clearly, \( Z(f) \subseteq I \subseteq Z(h) \). If \( h(a) \neq 0 \), then \( a \notin I \); since \( g \sim h \), then \( g(a) = h(a) \). Hence, \( f(a) = g(a) = h(a) \), and we get \( f \sim h \) by Lemma 1.2 (4).

(d) \( f \) is a TS-homomorphism.

Clearly \( f(0) = 0 \) and \( f(\pm 1) = g(\pm 1) = \pm 1 \). Let \( a, b \in G \). If one of \( a, b \) is in \( I \), so is \( ab \), and we have \( f(a)f(b) = 0 = f(ab) \). If \( a, b \notin I \), then \( ab \notin I \), and \( f \) and \( g \) take the same value on \( a, b \) and \( ab \); the result follows from the fact that \( g \) is a TS-character. Since \( G \) is a fan, \( f \in X_G \).

(3) This is Lemma 1.5, p. 102 (alternatively, Lemma 3.5, p. 114) in [DP1].

\[\Box\]
Remark 6.3 The element $f$ such that $g \rightarrow f$ and $Z(f) = I$ in (6.2) can also be obtained by taking any $h \in X_G$ with $Z(h) = I$ (6.2) and setting $f = h^2 g$.

The next Proposition proves the implication (2) $\Rightarrow$ (1) in Theorem 6.1.

**Proposition 6.4** Let $G$ be a real semigroup verifying conditions (2.i) – (2.iii) of Theorem 6.1. Then, $G$ is a RS-fan.

**Proof.** Item (2.i) is condition [Z] of the definition of RS-fan, 3.6. It suffices to show that transversal representation in $G$ satisfies clause $[D]_i$ in 3.1, i.e., for $a, b, c \in G$:

(I) $c \in D^i_G(a, b)$ and $Z(a) \subseteq Z(b)$ imply $c = a$.

(II) $c \in D^i_G(a, b)$, $Z(a) = Z(b)$ and $a \neq -b$ imply $c = a$ or $c = b$.

**Proof of (I).** We first observe that the assumptions of (I) imply $Z(a) = Z(c)$.

Let $h \in X_G$. If $h(a) = 0$, then $h(b) = 0$ (as $Z(a) \subseteq Z(b)$), and $c \in D^i_G(a, b)$ yields $h(c) = 0$; hence, $Z(a) \subseteq Z(c)$.

If $Z(c) \subseteq Z(b)$, then $c \in D^i_G(a, b)$ yields $-a \in D^i_G(-c, b)$, and so $Z(c) \subseteq Z(a)$. If $Z(b) \subseteq Z(c)$, then $-a \in D^i_G(-c, b)$ entails $Z(b) \subseteq Z(a)$, contrary to assumption. Hence, $Z(c) \subseteq Z(a)$, and $Z(a) = Z(c)$.

In order to prove $c = a$, let $h \in X_G$. If $h(b) = 0$, then $h(c) \in D^i_G(h(a), 0) = \{h(a)\}$, whence $h(c) = h(a)$. Henceforth, assume $h(b) \neq 0$. Since $Z(a) \subseteq Z(b)$, there is $g \in X_G$ such that $g(b) = 0$ and $g(a) \neq 0$. Since the set of ideals of $G$ is totally ordered under inclusion, $h(b) \neq 0$ and $g(b) = 0$, we have $Z(h) \subseteq Z(g)$. By (2.ii), there is $g' \in X_G$ such that $Z(g') = Z(g) = h \rightarrow g'$. Then, $g'(b) = 0$; from $c \in D^i_G(a, b)$ and $g(a) \neq 0$ comes $g'(a) = g'(c) \neq 0$. From $h \rightarrow g'$ we infer $h(a) = g'(a)$ and $h(c) = g'(c)$ (Lemma 1.2(4)), and from $g'(a) = g'(c)$ we conclude $h(a) = h(c)$, and hence $a = c$.

**Proof of (II).** Assume $c \in D^i_G(a, b)$, $Z(a) = Z(b)$ and $a \neq -b$; then, there is $g \in X_G$ so that $g(b) = g(a) \neq 0$. First we claim:

**Claim 1.** Under the assumptions of (II), $Z(c) = Z(a) = Z(b)$.

**Proof of Claim 1.** In fact, $c \in D^i_G(a, b)$ yields $Z(a) = Z(b) \subseteq Z(c)$. Assume, towards a contradiction, that there is $h \in X_G$ such that $h(c) = 0$ and $h(a) \neq 0$. From $c \in D^i_G(a, b)$ and $g(b) = g(a)$ we get $g(c) = g(b) = g(a) \neq 0$. Since the set of ideals of $G$ is totally ordered under inclusion, this and $h(c) = 0$ imply $Z(g) \subseteq Z(h)$. By (2.ii), there is $h' \in X_G$ such that $Z(h') = Z(h)$ and $g \rightarrow h'$; it follows that $h'(a) \neq 0$ and, since $Z(a) = Z(b)$, $h'(b) \neq 0$. Invoking Lemma 1.2(4), we get $h'(a) = g(a)$ and $h'(b) = g(b)$; from $g(b) = g(a)$ we obtain $h'(h') = h'(a)$. On the other hand, $c \in D^i_G(a, b)$ and $h'(c) = h(c) = 0$ entail $h'(a) = -h'(b)$, whence $h'(a) = h'(b) = 0$, contradiction. This proves $Z(c) = Z(a) = Z(b)$, as asserted.

If one of $a$ or $b$ is 0, the equality of zero-sets in Claim 1 implies $c = a = b = 0$. So, assume, e.g., $b \neq 0$. Let $I$ be an ideal of $G$ — necessarily prime and saturated — maximal for $b \not\in I$. Let $\sim_I$ be the congruence relation on $G$ determined by $I$, namely, for $x, y \in G$,

$$x \sim_I y \iff h(x) = h(y) \text{ for all } h \in X_G \text{ such that } Z(h) = I.$$  

(Cf. §5.C)

Note that the equality of zero-sets established in Claim 1, together with $b \not\in I$, implies that none of $a, b, c$ is in $I$.

**Claim 2.** $a \not\sim_I b$.

**Proof of Claim 2.** Assume that $a \sim_I b$. Since $g(b) \neq 0$, i.e., $b \not\in Z(g)$, maximality of $I$ entails $Z(g) \subseteq I$. By (2.ii), there is $h \in X_G$ such that $Z(h) = I$ and $g \rightarrow h$. Since $h(b), h(a) \neq 0$, the specialization $g \rightarrow h$ yields $h(a) = g(a)$ and $h(b) = g(b)$ (1.2(4)), which, by $g(a) = g(b)$, entails
Assumption and AOS Proposition 6.4, and of Theorem 6.1.

(Chain length) There is a well-known characterization of fans in the categories Remark 6.5 empty subbasic opens is \( \leq \) valid for ARSs or RSs; an easy computation shows that the RS-fan length also makes sense for ARSs, cf. \[M\], p. 167. However, this characterization is no longer h

The maximality of I

Z

Example 4.2 B.(i), with a finite spectrum; this is easily proved using Theorem 6.1 and see also \[M\], Thm. 8.5.3, p. 167.

The next two corollaries of Theorem 6.1 give stylized (abstract) versions of the notion of a trivial fan, a basic concept in the theory of (pre-)orders on fields (see \[L\], Prop. 5.3, p. 39). Their translation in the case of preordered rings is given in Theorem 7.21 below, where it will be obvious that in the case of fields they boil down to the notion of a trivial fan.

Corollary 6.6 Let G be a real semigroup such that the character space \( X_G \) is totally ordered under specialization. Then, G is a RS-fan.

Proof. We check that conditions (2.i) – (2.iii) of Theorem 6.1 hold.

Since every saturated prime ideal of G is the zero-set of some character (\[DP1\], Lemma 3.5, p. 114) and \( g \sim h \Rightarrow Z(g) \subseteq Z(h) \) for \( g, h \in X_G \), \[12\](4)), the set of saturated prime ideals of G is an inclusion chain, i.e., item (2.i) of 6.1 holds.

Further, every saturated prime ideal is the zero-set of a unique character: if \( h_1, h_2 \in X_G \) are such that \( Z(h_1) = Z(h_2) \), then \( h_1^2 = h_2^2 \) \[12\](1)); if, say, \( h_1 \sim h_2 \), by Lemma \[12\](5), \( h_1 = h_2^2 h_1 = h_2 h_1 \). It follows that, for every saturated prime ideal I the quotient \( G/I \) has a unique character, and hence \( G/I \cong 3 \), which is a RS-fan, showing that condition (2.iii) of 6.1 holds.

Finally, to check item \[6.1\](2.ii), observe that the linearity assumption and the uniqueness proved in the preceding paragraph yield \( Z(g) \subseteq Z(h) \Rightarrow g \sim h \).
Remark. We refer the reader to [M], Prop. 8.8.4, pp. 178-179, where he proves that the RSs in 6.6 are spectral. Conversely, it is an easy exercise to prove that the RSs which are simultaneously spectral and fans are exactly those whose character space is totally ordered by specialization.

Corollary 6.7 Let G be a real semigroup satisfying the following requirements:
(1) Condition [Z] in 3.1
(2) The character space $X_G$ of G is the union of two specialization chains, $C_0, C_1$.
(3) For every saturated prime ideal $I$ of G and for $i = 0, 1$, there is $h_i \in C_i$ such that $Z(h_i) = I$.
Then, G is a RS-fan.

Remark. The specialization chains in item (2) may not be disjoint, and the characters $h_0, h_1$ in (3) may be identical. Using condition (3) (and 6.1 (2.ii)) it can be shown that the chains in (2) are maximal.

Proof. Again, we check that conditions (2.i) – (2.iii) of Theorem 6.1 hold. Condition (1) is item 6.1 (2.i).
(2.ii) Let $g \in X_G$ and let $I$ be a saturated prime ideal of G such that $Z(g) \subseteq I$. Condition (2) implies that either $g \in C_0$ or $g \in C_1$, say the first. By (3) there is $h_0 \in C_0$ so that $Z(h_0) = I$. Since $C_0$ is a specialization chain and $g, h_0 \in C_0$, the inclusion $Z(g) \subseteq I = Z(h_0)$ yields $g \rightarrow h_0$, proving (2.ii).
(2.iii) For every saturated prime ideal $I$ of G, the structure $G_I = (G/I, \{ \pi_I(0) \})$, with representation induced by $D_G$, is a RSG ([DP4], Thm. II.3.15 (d)) and $X_{G_I} = \{ h \in X_G \mid Z(h) = I \}$ which, by assumptions (2) and (3), equals $\{ h_0, h_1 \}$. Since every reduced special group with at most two characters is a RSG-fan, so is $G_I$, as required.

Remarks. (a) There are examples satisfying conditions (2.i) and (2.iii) of Theorem 6.1 but not condition (2.ii).
(b) The real semigroup $G_{C(X)}$ associated to the ring $C(X)$ of continuous, real-valued functions on a topological space $X$ satisfies conditions (2.ii) and (2.iii) of Theorem 6.1 but, in general, not (2.i); cf. [M], 5.2 (6), p. 87.
(c) Even in the presence of conditions (1) and (3) of 6.7, if the character space $X_G$ of G is the union of more than two maximal specialization chains, the situation becomes more involved, as illustrated by the examples in [DP5b], 2.18.

7 Fans and preordered rings
In this section we prove a number of results about, and exhibit some examples of semi-real rings and preordered rings (hereafter p-rings) whose associated real semigroups are fans.

I. Properties of p-rings whose associated real semigroup is a fan.
Throughout this subsection we assume that $\langle A, T \rangle$ is a p-ring.

A. Basic correspondences. Let $\langle A, T \rangle$ be a p-ring and let $G_{A,T}$ denote its associated real semigroup ([DP2], §1, p. 51 or [DP3], 9.1 (A), pp. 406-407). Let Sat($G$) denote the set of all saturated ideals of a real semigroup $G$.
For an ideal $I$ of $A$, let $\overline{I} = \{ a \mid a \in I \}$, and for $J \in$ Sat($G_{A,T}$), set $\widehat{J} := \{ a \in A \mid \overline{a} \in J \}$. The following facts are easily verified or their proofs briefly indicated:

Fact 7.1 With notation as above and $J, J_1, J_2 \in$ Sat($G_{A,T}$), we have:
(i) $\overline{I}$ is a saturated ideal of $G_{A,T}$.
(ii) $\widehat{J}$ is an ideal of $A$.
(iii) $\overline{\widehat{J}} = J$. 

(iv) $J$ prime $\iff \hat{J}$ prime.  
(v) $J_1 \subseteq J_2 \iff \hat{J}_1 \subseteq \hat{J}_2$.  
(vi) The map $J \mapsto \hat{J}$ is injective.

Proof. We only prove saturatedness in (i). Let $a, b \in I$ and $c \in A$ be so that $c \in D_G(\overline{a}, \overline{b})$. By [M], Prop. 5.5.1 (5), p. 95, there are $t_0, t_1, t_2 \in T_1$ so that $t_0 c = t_1 a + t_2 b$ and $\overline{t}_0 c = \overline{t}_1$. From $a, b \in I$ follows $t_0 c \in I$, whence $\overline{c} = \overline{\overline{t}_0 c} \in T_1$. \hfill $\square$

The following notions from real algebra are used in the sequel:

Definition 7.2 Given a (proper) preorder $T$ of $A$, an ideal $I \subseteq A$ is

(i) $T$-radical iff for all $a \in A$ and $t \in T$, $a^2 + t \in I \Rightarrow a \in I$ (and hence $t \in I$).

(ii) $T$-convex iff for $t_1, t_2 \in T$, $t_1 + t_2 \in I \Rightarrow t_1, t_2 \in I$. 

Remark that an ideal is $T$-radical iff it is $T$-convex and radical ([BCR], Prop. 4.2.5, p. 87). We denote by $PConv(A, T)$ the set of all $T$-convex prime (equivalently, $T$-prime) ideals of $A$. For further properties of $T$-convexity (e.g., the definition of the $T$-radical of an ideal, $\sqrt{T}$), the reader is referred to [BCR], §§ 4.2, 4.3. We prove:

Proposition 7.3 Let $J$ be a saturated ideal of $G = G_{A,T}$. Then $\hat{J}$ is a $T$-radical ideal of $A$.

Proof. Assume $a^2 + t \in \hat{J}$, where $a \in A$, $t \in T$; we must show that $a \in \hat{J}$. Write $J = a^2 + t$; then $J \in J$ (definition of $\hat{J}$), and also $J \in D_G^1(\pi^2, \overline{\pi})$ (cf. [M], p. 96). Recall that $X_2 = Sper(A, T)$.

Let $\alpha \in Sper(A, T)$ be such that $\overline{J}(\alpha) = 0$; then, $\pi(\alpha) = -\overline{\pi}(\alpha)$. Since $t \in T \subseteq \alpha$, we have $-\overline{\pi}(\alpha) \in \{0, -1\}$. On the other hand, $\pi^2(\alpha) \in \{0, 1\}$, since $\pi^2$ is a square. Thus, the equality above forces $\overline{\pi}(\alpha) = \overline{\pi}(\alpha) = 0$, proving that $Z(J) \subseteq Z(\overline{\pi}) \cap Z(\overline{\pi}) \subseteq Z(\overline{\pi})$. This inclusion is equivalent to $\overline{\pi}^2 = \pi^2 \cdot \overline{\pi}^2$ (see [4.3](2)). Then, $\overline{\pi}^2 \in J$, whence $\overline{\pi} \in J$, which proves $a \in \hat{J}$. \hfill $\square$

We register the following consequences:

Corollary 7.4 For any ideal $I$ of $A$ we have:

(i) $\overline{\overline{\pi}}$ is the smallest $T$-radical ideal containing $I$, i.e., $\overline{\overline{\pi}} = \sqrt{\sqrt{I}}$.

(ii) $\sqrt{\sqrt{I}} = \overline{\overline{I}} = \overline{I} = T$. \hfill $\square$

Notation. Given a p-ring $(A, T)$, we denote by $Sper(G_{A,T})$ the set of all ideals of the real semigroup $G_{A,T}$. If $G_{A,T}$ is a RS-fan, we know (1.0.5) that $Sper(G_{A,T})$ is totally ordered under inclusion. By (7.4)(v) the set $\{\hat{J} \mid J \in Sper(G_{A,T})\}$ of ideals of $A$ is totally ordered under inclusion as well. \hfill $\square$

Fact 7.5 If $G_{A,T}$ is a fan, then:

(i) Every $T$-radical ideal of $A$ is prime.

(ii) $\{\hat{J} \mid J \in Sper(G_{A,T})\} = PConv(A, T)$.

(iii) The map $J \mapsto \hat{J}$ is an order-preserving bijection from $Sper(G_{A,T})$ onto $PConv(A, T)$.

Proof. (i) Let $I$ be a $T$-radical ideal of $A$. By (7.4)(i), $I = \hat{I}$ and, by (7.4)(i), $\overline{\overline{I}} = \hat{\overline{I}}$, in turn yields $I = \hat{I}$ prime.

(ii) Let $I \in PConv(A, T)$. Then $I$ is a $T$-radical ideal of $A$, and by (7.4)(i) we have $I = \hat{I} (= \sqrt{\sqrt{I}})$.

(iii) Follows from (i) and (7.4)(vi, v). \hfill $\square$

Remarks 7.6 Assume, as above, that $G_{A,T}$ is a RS-fan.

(i) $\overline{\overline{0}}$ is an ideal of $G_{A,T}$ (7.4)(i), hence prime and saturated [1.0.4]; since $\overline{\overline{0}}$ is prime, and by (7.4)(i), $\overline{\overline{0}} = \sqrt{0} = \{a \in A \mid a^{2k} \in T \text{ for some } k \geq 0\}$ is the smallest element of $PConv(A, T)$.

(ii) The maximal element of $Sper(G_{A,T})$ is:

\[ 2M = \text{set of non-invertible elements of } G_{A,T} = \{x \mid x \in A \text{ and } \pi^2 \neq 1\}. \]

Then, the ideal
\[ M = \mathcal{M} = \{ a \in A \mid \pi \in \mathcal{M} \} = \{ a \in A \mid \pi^2 \neq 1 \} = \{ a \in A \mid \exists \alpha \in \text{Sper}(A,T) \text{ such that } \pi(\alpha) = 0 \} = \{ a \in A \mid \exists \alpha \in \text{Sper}(A,T) \text{ such that } a \in \text{supp}(\alpha) \} = \bigcup \{ \text{supp}(\alpha) \mid \alpha \in \text{Sper}(A,T) \}, \]

is the maximal element of \( \text{PConv}(A,T) \).

(iii) **Warning.** Even though the ideal \( M \) is maximal in \( \text{PConv}(A,T) \), it may not be a maximal ideal of \( A \) (e.g., \( (\mathbb{Z}, \sum \mathbb{Z}^2) \)); however, it is maximal in some important cases, e.g., when \( (A,T) \) is a bounded inversion ring; cf. \cite{DM2}, Prop. 7.2, p. 78.

\[ \square \]

**7.7 Ternary semigroup characters of \( G_{A,T} \).** The characterization of ARS-fans given in Proposition 3.5(1), can be restated as follows:

\[ \{ \]

\[ 1 \]

A real semigroup \( G \) is a RS-fan if and only if the set of its prime ideals is totally ordered under inclusion and every character of ternary semigroup \( G \rightarrow 3 \) preserves representation.

In the case \( G = G_{A,T} \), we register, without proof, a description of the TS-characters of \( G_{A,T} \) in terms of the p-ring \( (A,T) \).

**Proposition 7.8** Let \( (A,T) \) be a p-ring. There is a bijective correspondence between the TS-characters of \( G_{A,T} \) and the family of all subsets \( S \subseteq A \) satisfying the following conditions:

1. \( T \subseteq S \).
2. \( S \) is closed under product.
3. \( 1 \notin S \).
4. \( S \cup -S = A \).
5. The set \( S \cap -S \) (not necessarily an ideal!) is prime: for all \( x, y \in A \), \( xy \in S \cap -S \) implies \( x \in S \cap -S \) or \( y \in S \cap -S \).
6. \( S \cap -S \) is \( T \)-convex: for all \( t_1, t_2 \in T \), if \( t_1 + t_2 \in S \cap -S \), then \( t_1, t_2 \in S \cap -S \).

By this characterization, the (necessary and sufficient) condition for \( G_{A,T} \) to be a RS-fan given by \ref{7.7}(†) translates as the conjunction of:

\[ \text{7.9(i)} \]

The set \( \text{PConv}(A,T) \) of \( T \)-convex prime ideals of \( A \) is totally ordered under inclusion, and

\[ \text{7.9(ii)} \]

Every subset \( S \subseteq A \) satisfying conditions (1) – (6) of \ref{7.8} is closed under addition (i.e., is an element of \( \text{Sper}(A,T) \)).

However, we show that in the present case we can dispense with \ref{7.9}(i):

**Proposition 7.10** Let \( (A,T) \) be a p-ring. With notation as above,

1. Condition \ref{7.3} (ii) implies \ref{7.9}(i). Hence,
2. \( G_{A,T} \) is a RS-fan if and only if every subset \( S \subseteq A \) satisfying conditions (1) – (6) of \ref{7.8} is closed under addition.

**Proof.** We need only prove (1). Let \( I, J \in \text{PConv}(A,T) \); let \( \alpha \in \text{Sper}(A,T) \) be such that \( I = \text{supp}(\alpha) \) (cf. \cite{BCR}, Prop. 4.3.8, p. 90). Set \( S = J \cup \alpha \).

We first observe that \( S \) satisfies conditions (1) – (6) of \ref{7.8}. Conditions (1) – (3) are obvious.

4. Since \( -S = J \cup -\alpha \) and \( \alpha \cup -\alpha = A \), we have \( S \cup -S = J \cup \alpha \cup -\alpha = A \).
5. From the previous item we have \( S \cap -S = (J \cup \alpha) \cap (J \cup -\alpha) = J \cup (\alpha \cap -\alpha) = J \cup I \).
   Since both \( I, J \) are prime (ideals), we get \( xy \in S \cap -S \) implies \( x \in S \cap -S \) or \( y \in S \cap -S \).
6. Again, since \( S \cap -S = J \cup I \) and both \( I, J \) are \( T \)-convex, we get the desired conclusion.

By assumption, \( S \) is additively closed. Assume, towards a contradiction, that there are \( a, b \in A \) such that \( a \in I \setminus J \) and \( b \in J \setminus I \). In particular, \( a \in I \subseteq \alpha \subseteq S \) and \( b \in J \subseteq S \), whence \( a + b \in S \). If \( a + b \in J \), since \( -b \in J \) we get \( a = (a + b) + (-b) \) \in J, contradiction. Then, \( a + b \in \alpha \), and from \( a \in I \subseteq -\alpha \), we get \( b = (a + b) + (-a) \in \alpha \). Next, since \( -a \in I \setminus J \) and \( -b \in J \setminus I \), the preceding argument can be carried out with \( a, b \) replaced with \( -a, -b \), respectively, to conclude that \( -b \in \alpha \). Thus, \( b \in \alpha \cap -\alpha = I \), contradiction. \( \square \)
Remark 7.11 Proposition 7.10(2) is the exact ring-theoretic analog of the definition of a fan (as a preorder) in a field, due to BK; namely: A preorder $T$ of a field $F$ is a fan iff for any set $S \supseteq T$ such that $-1 \notin S$ and $S^\times = S \setminus \{0\}$ is a subgroup of index 2 in $F^\times$, then $S$ is closed under addition (La, Def. 5.1, p. 39).

The characterization in 7.10(2) yields a first batch of natural examples of p-rings whose associated real semigroup is a fan.

Corollary 7.12 Let $K$ be a field and $T$ be a preorder of $K$ which is a fan. Let $A$ be a subring of $K$ whose field of fractions is $K$. Then, the real semigroup $G_{A,T \cap A}$ is a fan. In particular, if $A = A_v$ is the valuation ring of a $T$-compatible valuation $v$ of $K$, the real semigroup $G_{A_v,T \cap A_v}$ is a fan.

Proof. According to Proposition 7.10(2) it suffices to check that any set $S \subseteq A$ satisfying conditions (1) – (6) in $\langle A, T \cap T \rangle$ is closed under addition. Let $S' = \{ \frac{a}{b} | a, b \in A, b \neq 0 \text{ and } ab \in S \} \subseteq K$. We first show:

$S' \setminus \{0\}$ is a subgroup of $K^\times$, $T \subseteq S'$ and $-1 \notin S'$.

Clearly, $S \subseteq S'$ and (by (3)) $-1 \notin S'$. Since $K$ is the field of fractions of $A$, any element of $T$ can be written as $\frac{a}{b}$, with $a, b \in A, b \neq 0$. Then, $ab = \frac{a}{b} \cdot b^2 \in T \cap A$. Since $T \cap A \subseteq S$ ((1)), we get $\frac{a}{b} \in S'$, hence $T \subseteq S'$. Condition (2) implies that $S' \setminus \{0\}$ is a subgroup of $K^\times$.

Since, by assumption, $T$ is a fan in the field $K$, the set $S'$ is closed under addition in $K$ (see 7.11), which clearly implies that $S$ is additively closed in $A$.

B. Total preorders and trivial fans in rings.

Notation 7.13 For a p-ring $\langle A, T \rangle$ and a prime ideal $I$ of $A$, we let

- $A_I$ denote the localization of $A$ at $I$,
- $M_I = I \cdot A_I$ denote the maximal ideal of $A_I$, and
- $T_I = T \cdot (A \setminus I)^{-2}$ denote the preorder induced by $T$ on $A_I$.

Fact 7.14 If $I \in \text{PConv} (A, T)$, then $T_I / M_I$ is a proper preorder of the field $A_I / M_I$ (and $T_I$ a proper preorder of $A_I$).

Proof. It suffices to prove the first assertion. Clearly $T_I / M_I$ is a preorder of $A_I / M_I$. To show it is proper, assume, on the contrary, that $-1 \in T_I / M_I$, i.e., $-1 = \left(\frac{t}{x^2}\right) / M_I$, with $t \in T$ and $x \in A \setminus I$; that is, $\frac{t}{x^2} + 1 \in M_I = I \cdot A_I$, i.e., $\frac{t + x^2}{x^2} = \frac{y}{i}$, for some $i \in I$ and $y \in A \setminus I$. Since $I$ is prime, we get $y \cdot (t + x^2) \equiv x^2 i \pmod{I}$, whence $y \cdot (t + x^2) \in I$, and $t + x^2 \in I$. Since $t, x^2 \in T$ and $I$ is $T$-convex and radical, we obtain $x \in I$, contradiction.

Definition 7.15 A total preorder in a ring $A$ is a (proper) preorder $T$ such that $T \cup -T = A$.

Fact 7.16 For a total preorder $T$ of a ring $A$, $T \cap -T$ is a proper $T$-convex ideal of $A$. Any $T$-convex ideal of $A$ contains $T \cap -T$.

Remarks 7.17 (i) The ideal $T \cap -T$ may not be prime (see Example 7.18). When it is, the notion of “total preorder” coincides with “prime cone”, i.e., element of Sper $(A)$.

(ii) When $T \cap -T = \{0\}$ the total preorders are just the total orders of $A$.

Example 7.18 Let $A := \mathbb{R}[X]/(X^2)$; the elements of $A$ are uniquely representable in the form $aX + b$ with $a, b \in \mathbb{R}$. Clearly, the zero ideal of $A$ is not radical, hence not prime either: $X \neq 0$ but $X^2 = 0$. We define a total (pre)order $T$ in $A$ by the stipulation:

$aX + b \in T$ iff $b > 0$ or $(b = 0$ and $a \geq 0$).

Checking that $T$ is a total (pre)order of $A$ is routine, left to the reader. However, the ideal $T \cap -T = \{0\}$ is not prime (not even radical).
Proposition 7.19 shows that total preorders are preserved by localization at, and lifting by convex prime ideals.

**Proposition 7.19** Let \( I \) be a prime ideal of a ring \( A \), \( T \) be a preorder of \( A \), and \( Q \) be a preorder of the localisation \( A_J \). Let \( \iota_J : A \to A_J \) be the canonical map \( a \mapsto a|_I \) \((a \in A)\). Then,

(i) If \( T \) is a total preorder, then so is \( T_J \).

(ii) \( P := \iota_I^{-1}[Q] \) is a preorder of \( A \).

(iii) \( P_I = Q \).

(iv) \( Q \) is total if and only if \( P \) is total.

(v) The maximal ideal \( M_J \) of \( A_J \) is \( Q \)-convex if and only if \( I \) is \( P \)-convex.

**Proof.** (i), (ii) and (iv) are straightforward checking.

(iii) We show:

- \( P_I \subseteq Q \). Let \( z \in P_I \), i.e., \( z = \frac{p}{x} \), with \( p \in P, x \not\in I \). Then, \( \frac{p}{x} \in Q \), \( \frac{x^2}{1} \) is invertible in \( A_J \), and \( \frac{1}{x} \in Q \). It follows that \( z = \frac{p}{x} \cdot \frac{1}{x} \in Q \).

- \( Q \subseteq P_I \). Let \( z \in Q \); then, \( z = \frac{xy}{y} \) with \( x, y \in A, y \not\in I \), which implies \( z = \frac{xy}{y^2} \); this gives \( \frac{y^2}{1} \cdot z = \frac{xy}{1} \). Clearly, \( \frac{y^2}{1} = (\frac{y}{1})^2 \in Q \), whence \( \frac{y^2}{1} \cdot z \in Q \), and \( \frac{xy}{1} \in Q \), which shows that \( xy \in P \). Hence, \( z = \frac{xy}{y^2} \in P \).

(v) \((\Rightarrow)\). Let \( p_1, p_2 \in P \) be such that \( p_1 + p_2 \in I \). Then, \( \frac{p_i}{1} \in Q \) \((i = 1, 2)\), and \( \frac{p_i}{1} + \frac{p_i}{1} \in I \cdot A_J = M_J \). By the convexity assumption, \( \frac{p_1}{1}, \frac{p_2}{1} \in I \cdot A_J \). For \( i = 1, 2 \), we have \( \frac{p_i}{1} = \frac{2}{x} \), with \( j \in I, x \not\in I \). It follows that \( x p_i \not\in I \); since \( x \not\in I \), we get \( p_i \in I \), as required.

\((\Leftarrow)\). Let \( \frac{x_i}{y_i} \in Q \) \((x_i \in A, y_i \not\in I; i = 1, 2)\) be such that \( \frac{x_1}{y_1} + \frac{x_2}{y_2} = \frac{x_1 y_2 + x_2 y_1}{y_1 y_2} \in M_I \). Then, \( \frac{x_1 y_2 + x_2 y_1}{y_1 y_2} = \frac{w}{w} \) with \( z \in I, w \not\in I \). We get \( w(x_2 y_1 + x_2 y_1) = zy_1 y_2 \in I \); since \( w \not\in I \), we have \( x_1 y_2 + x_2 y_1 \in I \). By (iii) we get \( \frac{x_i}{y_i} = \frac{p_i}{s_i} \), with \( p_i \in P, s_i \not\in I \), whence

\((i)\) \( x_i s_i^2 = y_i p_i \) \((i = 1, 2)\).

Scaling \( x_1 s_1^2 = y_1 p_1 \) by \( y_2 s_2^2 \) yields \( x_1 y_2 s_1^2 s_2^2 = y_1 y_2 p_1 s_2^2 \). Likewise, we obtain \( x_2 y_1 s_1^2 s_2^2 = y_1 y_2 p_1 s_2^2 \). Adding these terms gives \( s_1^2 s_2^2(x_1 y_2 + x_2 y_1) = y_1 y_2 (p_1 s_2^2 + p_2 s_1^2) \). Since \( x_1 y_2 + x_2 y_1 \in I \) and \( y_1 y_2 \not\in I \), we get \( p_1 s_2^2 + p_2 s_1^2 \in I \). By \( P \)-convexity of \( I \), \( p_1 s_2^2, p_2 s_1^2 \in I \). From \( s_1, s_2 \not\in I \) comes \( p_1, p_2 \in \) \(\text{I} \); whence, by (i), \( x_i s_i^2 \in I \). Since \( s_i \not\in I \), we get \( x_i \in I \), wherefrom \( \frac{x_i}{y_i} \in I A_J = M_I \). \( \square \)

**Remark 7.20** Even if \( Q \) is a total order of \( A_J \), \( P \) may not be a total order of \( A \). In fact,

\[ P \cap -P = \iota_I^{-1}[Q \cap -Q] = \iota_I^{-1}[0], \]

which, in general is not \( \{0\} \). Note that, for \( x \in A \),

\[ x \in \iota_I^{-1}[0] \iff \iota_J(x) = 0 \quad \text{in } (A_J) \iff \exists z \not\in I (zx = 0); \]

in particular, \( x \) is a zero-divisor. Thus, \( P \) is a total order when \( A \) is an integral domain. \( \square \)

The following result proves two important properties of total preorders in rings:

**Theorem 7.21** (i) Let \( T \) be a total preorder of a ring \( A \). Then, the real semigroup \( G_{A,T} \) is a fan and \( \text{Sper}(A,T) \) is totally ordered by specialization.

(ii) Let \( T_0, T_1 \) be total preorders of a ring \( A \), and let \( T = T_0 \cap T_1 \). Assume that the set \( \text{PConv}(A,T) \) of \( T \)-convex prime ideals of \( A \) is totally ordered under inclusion. Then, the real semigroup \( G_{A,T} \) is a fan.
Remark. In case the ring $A$ is a field, $K$, a total preorder is just a (total) order of $K$. Thus, Theorem 7.21 is a ring-theoretic analog of the well-known fact that the intersection of at most two total orders of a field is a fan, namely the trivial fans, cf. [LA], Prop. 5.3, p. 39. □

Proof. (i) By Corollary 6.10 it suffices to check that $\operatorname{Sper}(A,T)$ is totally ordered under inclusion (= specialization); the proof is identical to that showing that the real spectrum of a ring is a root system: let $\alpha, \beta \in \operatorname{Sper}(A,T)$ and assume that $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$; let $a \in \alpha \setminus \beta$ and $b \in \beta \setminus \alpha$; since the preorder $T$ is total, either $a - b \in T \subseteq \beta$ or $b - a \in T \subseteq \alpha$; hence, $a = b + (a - b) \in \beta$ or $b = a + (b - a) \in \alpha$, absurd.

(ii) We check that assumptions (1) – (3) of 6.7 are verified by $G_{A,T}$.

Since the saturated prime ideals of $G_{A,T}$ are in a bijective, inclusion-preserving correspondence with the $T$-convex prime ideals of $A$ (cf. Fact 7.5), the argument proving Proposition 7.10(2) shows that $G_{A,T}$ verifies condition $[Z]$ in Theorem 3.1, i.e., assumption 6.7(1).

Assumption 6.7(2) follows from the proof of (i) and:

(*) $\operatorname{Sper}(A,T) = \operatorname{Sper}(A,T_0) \cup \operatorname{Sper}(A,T_1)$.

Proof of (*). Clearly, $\operatorname{Sper}(A,T_i) \subseteq \operatorname{Sper}(A,T)$ for $i = 0, 1$. Assume there is $\alpha \in \operatorname{Sper}(A)$ such that $T \subseteq \alpha$ but $T_0 \cap T_1 \not\subseteq \alpha$; for $i = 0, 1$, let $t_i \in T_i \setminus \alpha$. Then, $-t_0 \in \alpha$ and $t_0 \not\in T_1$ (otherwise, $t_0 \in T_0 \cap T_1 \subseteq \alpha$). Since $T_1$ is a total preorder, $t_0 \in -T_1$. Likewise, $-t_1 \in \alpha$ and $t_1 \in -T_0$.

From $t_0 \in T_0$ and $-t_1 \in T_1$ we get $-t_0 t_1 \in T_0$; from $t_1 \in T_1$ and $-t_0 \in T_1$ we get $-t_0 t_1 \in T_1$; hence, $-t_0 t_1 \in T_0 \cap T_1 \subseteq \alpha$. From $-t_0 t_1 \in \alpha$ comes $t_0 t_1 = (-t_0)(-t_1) \in \alpha$. Hence, $t_0 t_1 \in \alpha \cap -\alpha = \operatorname{supp}(\alpha)$. Since this is a prime ideal, $t_i \in \operatorname{supp}(\alpha) \subseteq \alpha$ for $i = 0$ or $i = 1$, contradiction.

In order to prove assumption (3) of 6.7 we first show:

(**) Every $T$-convex prime ideal $I$ of $A$ is both $T_0$-convex and $T_1$-convex.

Proof of (**). From [BCR], Prop. 4.2.8(ii), p. 87, we know that $I$ is either $T_0$-convex or $T_1$-convex. Assume, towards a contradiction, that $I$ is $T_0$-convex but not $T_1$-convex. Then, there are elements $t_0, t_1 \in T_1$ such that $t_0 + t_1 \in I$, but $t_0 t_1 \not\in I$. Since $I$ is $T$-convex, we have $t_0, t_1 \not\in T_0$ and, since $T_0$ is a total preorder, $-t_0, -t_1 \in T_0$. As we have $-(t_0 + t_1) \in I$, $T_0$-convexity yields $t_0 t_1 \in I$, whence $t_0 t_1 \in I$, contradiction.

Now, [BCR], Prop. 4.3.8, p. 90, finishes the proof: for $i = 0, 1$, there is $\alpha_i \in \operatorname{Sper}(A,T_i)$ such that $\operatorname{supp}(\alpha_i) = I$. □

Remark 7.22 The following example shows that the requirement in item (ii) of Theorem 7.21 does not hold automatically. Let $A = C(\mathbb{R})$ be the ring of real-valued continuous functions on the reals. For $i = 0, 1$, let $T_i = \{ f \in A \mid f(i) \geq 0 \}$ and $M_i = \{ f \in A \mid f(i) = 0 \}$. The (maximal) ideal $M_i$ is $T_i$-convex; hence, with $T = T_0 \cap T_1$, both $M_0$ and $M_1$ are $T$-convex; however, $M_0$ and $M_1$ are incomparable under inclusion. □

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