Conformal mapping and impedance tomography

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Abstract. Over the last decade Akduman, Haddar and Kress [1, 3, 5] have employed a conformal mapping technique for the inverse problem to recover a perfectly conducting or a non-conducting inclusion in a homogeneous background medium from Cauchy data on the accessible exterior boundary. More recently, Haddar and Kress [4] proposed an extension of this approach to two-dimensional inverse electrical impedance tomography with piecewise constant conductivities. A main ingredient of this extension is the incorporation of the transmission condition on the unknown interior boundary via a nonlocal boundary condition in terms of an integral equation. We present an outline of the foundations of this new method.

1. Introduction

In a new numerical scheme for solving inverse boundary value problems for the Laplace equation in a doubly connected two-dimensional domain $D$ via a conformal mapping technique introduced by Akduman, Haddar and Kress [1, 3, 5] the reconstruction of the non-accessible interior boundary curve $\Gamma_0$ from over determined Cauchy data on the accessible exterior boundary curve $\Gamma_1$ is based on a conformal map $\Psi : B \to D$ that takes an annulus $B$ bounded by two concentric circles $C_0$ and $C_1$ onto $D$. The Cauchy–Riemann equations provide a nonlocal and nonlinear ordinary differential equation for the boundary values $\Psi|_{C_1}$ on the exterior circle that can be solved by successive approximations. Then a Cauchy problem for the holomorphic function $\Psi$ has to be solved by a regularized Laurent expansion to retrieve the unknown interior boundary curve via $\Gamma_0 = \Psi(C_0)$. For the reconstruction of a perfectly conducting or a non-conducting inclusion, i.e., the inverse problem with a homogeneous Dirichlet or Neumann condition on $\Gamma_0$ this conformal mapping method separates the inverse problem into the nonlinear well-posed problem for the ordinary differential equation and the linear ill-posed Cauchy problem.

The inverse electrical impedance problem to reconstruct the shape of a conducting inclusion with a constant conductivity that is different from the constant background conductivity of $D$ corresponds to an inverse transmission problem. For this case, when applying the conformal mapping idea, in principle, two conformal maps are required. In addition to the mapping $\Psi : B \to D$ also a map taking the interior of $C_0$ onto the interior of $\Gamma_0$ is needed. Furthermore, the homogeneous transmission condition on $\Gamma_0$ transforms into a more complicated transmission condition on $C_0$ containing the traces of the two conformal maps at different locations for both sides of $C_0$.

Restricting themselves to the case where the two conformal maps are extensions of each other, and consequently have to coincide with a Moebius transform, in a first attempt Dambrine and Kateb [2] were able to extend parts of the above methods to the inverse transmission problem. In a recent paper, Haddar and Kress [4] proposed a different approach that uses only
the conformal map for the annulus and incorporates the transformed transmission condition on $C_0$ by a nonlocal boundary condition in terms of a boundary integral equation for the trace of the solution to the transmission problem on $\Gamma_0$.

2. The inverse algorithm

Let $D_0$ and $D_1$ be two simply connected bounded domains in $\mathbb{R}^2$ with $C^2$ smooth boundaries $\Gamma_0$ and $\Gamma_1$ such that $D_0 \subset D_1$. Let $D := D_1 \setminus \overline{D_0}$ and assume the unit normal $\nu$ to $\Gamma_0$ and $\Gamma_1$ to be directed into the exterior of $D_0$ and $D_1$, respectively. For a given function $f \in H^{1/2}(\Gamma_1)$ and given positive constants $\sigma_0$ and $\sigma_1$ we consider the transmission problem for the Laplace equation

$$\Delta u_0 = 0 \quad \text{in } D_0, \quad \Delta u = 0 \quad \text{in } D \quad (2.1)$$

for $u_0 \in H^1(D_0)$ and $u \in H^1(D)$ with boundary condition

$$u = f \quad \text{on } \Gamma_1 \quad (2.2)$$

and transmission conditions

$$u_0 = u, \quad \sigma_0 \frac{\partial u_0}{\partial \nu} = \sigma_1 \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_0. \quad (2.3)$$

After denoting the normal derivative of $u$ on $\Gamma_1$ by

$$g := \frac{\partial u}{\partial \nu}|_{\Gamma_1}$$

the inverse problem under consideration is to determine the shape of the interior boundary curve $\Gamma_0$ from pairs of Cauchy data $(f, g)$.

In the sequel we will identify $\mathbb{R}^2$ and $\mathbb{C}$. We introduce the annulus $B$ bounded by the concentric circles $C_0$ with radius $\rho$ and $C_1$ with radius one centered at the origin. By the conformal mapping theorem there exists a uniquely determined radius $\rho$ and a holomorphic function $\Psi$ that maps $B$ bijectively onto $D$ such that the boundaries $C_0$ and $C_1$ are mapped onto $\Gamma_0$ and $\Gamma_1$, respectively. Assuming for simplicity that the total length of $\Gamma_1$ is $2\pi$, we denote by $\gamma : [0, 2\pi] \rightarrow \Gamma_1$ the parameterization of $\Gamma_1$ in terms of arc length.

We define a function $\varphi : [0, 2\pi] \rightarrow [0, 2\pi]$ by setting

$$\varphi(t) := \gamma^{-1}(\Psi(e^{it})). \quad (2.4)$$

Roughly speaking, $\varphi$ describes how $\Psi$ maps arc length on $C_1$ onto arc length on $\Gamma_1$. Clearly, the boundary values $\varphi$ uniquely determine $\Psi$ as the solution to the Cauchy problem with $\Psi$ on $C_1$ given through $\Psi(e^{it}) = \gamma(\varphi(t))$. Hence, the operator

$$N_\rho : \varphi \mapsto \chi \quad (2.5)$$

where $\chi(t) := \Psi(\rho e^{it})$ is well defined. The function $\chi : [0, 2\pi] \rightarrow \mathbb{C}$ parameterizes the interior boundary curve $\Gamma_0$ and determining $\chi$ solves the inverse transmission problem.

We denote by $A_\rho : H^{1/2}[0, 2\pi] \times H^{1/2}[0, 2\pi] \rightarrow H^{-1/2}[0, 2\pi]$ the Dirichlet-to-Neumann operator for the annulus $B$ that maps function pairs $(F_1, F_2)$ onto the normal derivative

$$(A_\rho(F_1, F_2))(t) := \frac{\partial v}{\partial \nu}(e^{it}) - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial \nu}(e^{it}) \, dt, \quad t \in [0, 2\pi],$$
of the harmonic function \( v \in H^1(B) \) with boundary values on \( C_1 \) and \( C_0 \) given by
\[
v(e^{it}) = F_1(t) \quad \text{and} \quad v(\rho e^{it}) = F_2(t) \quad \text{for} \quad t \in [0, 2\pi].
\]

In terms of this operator \( A \) from the Cauchy–Riemann equations for \( u \) and \( v = u \circ \Psi \) and their corresponding harmonic conjugates one can deduce the nonlocal differential equation
\[
\frac{d\varphi}{dt} = A_\rho(f \circ \gamma \circ \varphi, u \circ N_\rho \varphi)
g \circ \gamma \circ \varphi
\tag{2.6}
\]
for the boundary map \( \varphi \).

In order to eliminate the unknown trace of \( u \) on the interior boundary curve from (2.6) we introduce the double-layer operator \( K : H^{1/2}(\Gamma_0) \to H^{1/2}(\Gamma_0) \) by
\[
(K\beta)(x) := 2\int_{\Gamma_0} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \beta(y) \, ds(y), \quad x \in \Gamma_0,
\]
with the fundamental solution
\[
\Phi(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x - y|}
\]
to Laplace’s equation in \( \mathbb{R}^2 \). In terms of the Cauchy data \((f, g)\) we define the combined single- and double-layer potential
\[
w(x) := \int_{\Gamma_1} \left\{ g(y)\Phi(x, y) - f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} \, ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_1.
\tag{2.7}
\]
Then, as a consequence of Green’s representation theorem, the trace \( \beta := u|_{\Gamma_0} \) is given as the unique solution to the integral equation
\[
(1 + \mu)\beta + (1 - \mu)K\beta = 2\mu w|_{\Gamma_0}
\]
where
\[
\mu := \frac{\sigma_1}{\sigma_0}.
\]

Via \( \eta := u \circ \chi \) we transform this into
\[
(1 + \mu)\eta + (1 - \mu)H_\chi \eta = 2\mu w \circ \chi
\tag{2.8}
\]
with the parameterized double-layer operator \( H_\chi \) given by \( H_\chi(\beta \circ \chi) := (K\beta) \circ \chi \). Since the double-layer operator is compact and has its spectrum contained in \([-1, 1]\), the operator on the left-hand side of (2.8) is bijective with bounded inverse
\[
M_\chi := 2\mu [(1 + \mu)I + (1 - \mu)H_\chi]^{-1} : H^{1/2}[0, 2\pi] \to H^{1/2}[0, 2\pi].
\]

Now we can write
\[
u \circ N_\rho \varphi = M_{N_\rho \varphi}(w \circ N_\rho \varphi)
\]
and, finally, the nonlocal differential equation for \( \varphi \) assumes the form
\[
\frac{d\varphi}{dt} = A_\rho(f \circ \gamma \circ \varphi, M_{N_\rho \varphi}(w \circ N_\rho \varphi))
g \circ \gamma \circ \varphi
\tag{2.9}
\]
The differential equation (2.9) has to be complemented by the boundary conditions \( \varphi(0) = 0 \) and \( \varphi(2\pi) = 2\pi \).
To avoid difficulties in solving (2.9) arising from zeros of the function $g$ occurring in the denominator, two pairs of Cauchy data can be used. If $(f_1, g_1)$ and $(f_2, g_2)$ are two pairs of Cauchy data on $\Gamma_1$, then

$$
\frac{d\varphi}{dt} = \sum_{j=1}^{2} (g_j \circ \gamma \circ \varphi) A_{\rho}(f_j \circ \gamma \circ \varphi, M_{N_{\rho \varphi}}(w_j \circ N_{\rho \varphi}))
$$

where $w_1$ and $w_2$ denote the combined single- and double-layer potential (2.7) associated with the real-valued Cauchy pairs $(f_1, g_1)$ and $(f_2, g_2)$, respectively. To condense the notation, after introducing the complex valued functions

$$
f = f_1 + if_2, \quad g = g_1 + ig_2 \quad \text{and} \quad w = w_1 + iw_2
$$

we rewrite (2.10) in the shorter form

$$
\frac{d\varphi}{dt} = \Re \left( \left( g \circ \gamma \circ \varphi \right) A_{\rho}(f \circ \gamma \circ \varphi, M_{N_{\rho \varphi}}(w \circ N_{\rho \varphi})) \right) / |g \circ \gamma \circ \varphi|^2.
$$

After defining Fourier coefficients depending on the data $(f, g)$ and on $\varphi$ by setting

$$
a_m(\varphi) := \int_0^{2\pi} f(\varphi(t)) e^{-imt} dt,
\quad b_m(\varphi) := \int_0^{2\pi} g(\varphi(t)) \varphi'(t) e^{-imt} dt,
\quad c_m(\varphi, \rho) := \int_0^{2\pi} (M_{N_{\rho \varphi}}(w \circ N_{\rho \varphi}))(t) e^{-imt} dt,
$$

a straightforward application of Green's integral theorem yields

$$
|m| a_m(\varphi) + b_m(\varphi) |\rho|^{2|m|} + |m| a_m(\varphi) - b_m(\varphi) = 2|m|\rho|m|c_m(\varphi, \rho).
$$

By this equation the radius is given in terms of $\varphi$ and the data $(f, g)$. Under appropriate assumptions, (2.12) can be solved iteratively via

$$
\rho_{j+1} = \left| \frac{b_m(\varphi) - |m| a_m(\varphi) + 2|m|\rho_j|m|c_m(\varphi, \rho_j)}{b_m(\varphi) + |m| a_m(\varphi)} \right|^\frac{1}{|\rho_j|}.
$$

Finally we define an operator $V$ by setting $(V\psi)(t) = t + \psi(t)$ and introduce the operator $T : H_0^1[0, 2\pi] \to H_0^1[0, 2\pi]$ by

$$
(T\psi)(t) := \int_0^t \left( U\psi - \frac{1}{2\pi} \int_0^{2\pi} U\psi \, d\theta \right) \, dr, \quad t \in [0, 2\pi],
$$

where

$$
U\psi := \Re \left( \left( g \circ \gamma \circ V\psi \right) A_{\rho}(V\psi)(f \circ \gamma \circ V\psi, M_{N_{\rho \varphi}}(w \circ N_{\rho \varphi} V\psi)) \right) / |g \circ \gamma \circ V\psi|^2,
$$

and $\rho(V\psi)$ indicates the solution of (2.12) for $\varphi = V\psi$ and an appropriately chosen $m \in \mathbb{N}$. Then we can summarize the above results into the following theorem.
**Theorem 2.1** Let \((f,g)\) be a pair of Cauchy data of the form (2.11) for the transmission problem. Then, in terms of the holomorphic map \(\Psi : B \to D\) and its boundary values \(\varphi\) the function \(\psi = V^{-1}\varphi\) is a fixed point of \(T\).

Theorem 2.1 suggests the following iteration scheme: Given a current approximation \(\psi_0\), we update it in two steps.

(i) For an appropriate choice of \(m\) we solve (2.12) with the boundary map \(\varphi_0 = V\psi_0\) via iterations as indicated in (2.13) to obtain a radius \(\rho_0\).

(ii) In view of Theorem 2.1 we update the boundary map by \(\psi_1 = T(\psi_0)\) using the radius \(\rho_0\) and a regularized version of the operator \(N_\rho\) for the Cauchy problem.

Of course, the whole scheme then consists in repeating these two steps iteratively. For a convergence result on this iteration scheme for sufficiently small transmission coefficients \(\mu\) and numerical examples exhibiting the feasibility of the method we refer to [4].

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