Uniqueness of stochastic entropy solutions for stochastic balance laws with Lipschitz fluxes

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Abstract

In this paper, we consider a stochastic balance law with a Lipschitz flux and gain the uniqueness for stochastic entropy solutions. The argument is supported by the stochastic kinetic formulation, the Itô formula and the regularization techniques. Furthermore, as an application, we derive the uniqueness of stochastic entropy solutions for stochastic porous media type equations.

Keywords: Uniqueness; Stochastic entropy solution; Stochastic kinetic formula; Itô formula

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1 Introduction

We are interested in the uniqueness of stochastic entropy solutions for the following stochastic balance law:

\[ \partial_t \rho(t, x) + \text{div}_x (B(\rho)) + \partial_{x_i} B_{i,j}(t, \rho) \circ \dot{W}_j(t) = A(t, x, \rho), \]

in \( \Omega \times (0, \infty) \times \mathbb{R}^d \), with given initial condition:

\[ \rho(t, x)|_{t=0} = \rho_0(x) \quad \text{in} \quad \Omega \times \mathbb{R}^d, \]

where \( W(t) = (W_1(t), W_2(t), \ldots, W_d(t))^T \) is an \( d \)-dimensional standard Wiener process on the classical Wiener space \( (\Omega, \mathcal{F}, P, (\mathcal{F}_s)_{s\geq 0}) \), i.e. \( \Omega \) is the space of all continuous functions from \( [0, \infty) \) to \( \mathbb{R}^d \) with locally uniform convergence topology, \( \mathcal{F} \) is the Borel \( \sigma \)-field, \( P \) is the Wiener measure, \( (\mathcal{F}_s)_{s\geq 0} \) is the natural filtration generated by the coordinate process \( W(t, \omega) = \omega(t) \). \( \circ \) is the Stratonovich convention and

\[
\begin{align*}
A(t, x, v) &\in L^1_{\text{loc}}([0, \infty); L^1(\mathbb{R}^d; W^{1,1}_{\text{loc}}(\mathbb{R}_v))) + L^1_{\text{loc}}([0, \infty); L^\infty(\mathbb{R}^d; W^{1,1}_{\text{loc}}(\mathbb{R}_v))), \\
A(t, x, 0) &= 0, \quad B \in W^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^d), \quad B_{i,j} \in L^2_{\text{loc}}([0, \infty); W^{1,2}_{\text{loc}}(\mathbb{R})) \quad 1 \leq i, j \leq d. 
\end{align*}
\]

The initial function is assumed to be non-random and

\[ \rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \]
Here the use of the Stratonovich differential stems from the fact that ordinary differential equations with time dependent converging Brownian motion give rise stochastic differential equations of Stratonovich’s.

When $B_{i,j} = 0$ $(1 \leq i, j \leq d)$, (1.1) reduces to a deterministic partial differential equation known as the balance law

$$\partial_t \rho(t, x) + \text{div}_x(B(\rho)) = A(t, x, \rho), \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d. \quad (1.5)$$

The first pioneering result on the well-posedness is due to Kružkov [1]. Under the Lipschitz assumption on $B$ and $A$, he obtains the existence in company with uniqueness of admissible entropy solutions for (1.6).

When $A = 0$, $B = 0$ and $(B_{i,j}(t, \rho)) = \text{diag}(A_1(\rho), A_2(\rho), \ldots, A_d(\rho))$, Lions, Perthame and Souganidis [2] also study the Cauchy problem (1.1), (1.2). Under the presumption that $A \in C^2$, they develop a pathwise theory for scalar conservation laws with quasilinear multiplicative rough path dependence.

Besides the stochastic quasilinear dependence, recently Feng and Nualart [3], Debussche and Vovelle [4], Chen, Ding and Karlsen [5] (also see Hofmanová [6]) put forward a theory of stochastic entropy solutions of scalar conservation laws with Itô-type, which in our setting take the form

$$\begin{cases}
\partial_t \rho(t, x) + \text{div}_x(B(\rho)) = A(x, \rho) W_t, & \text{in} \quad \Omega \times (0, \infty) \times \mathbb{R}^d, \\
\rho(t, x)|_{t=0} = \rho_0(x), & \text{in} \quad \Omega \times \mathbb{R}^d,
\end{cases} \quad (1.6)$$

where $B \in C^2$, $A$ is a Lipschitz continuous function, $W$ ia a one dimensional Wiener process. In their papers, they gain the uniqueness of stochastic entropy solutions as well as the existence.

It is remarked that all above mentioned works are concentrate their attention of stochastic entropy solutions for stochastic balance laws on $C^2$-fluxes. There are relatively few papers concerned with $C^1$ or Lipschitz fluxes. The aim of the present paper is raising the state of the art of the theory of uniqueness for stochastic entropy solutions of stochastic balance laws with coefficients in Lipschitz spaces to the level of the Kružkov theory for those with balance laws.

Our motivation stems from two facts. The first one comes from fluid dynamics. It is well known that the fundamental fluid dynamics models are based on Navier-Stokes equations and Euler equations. However, abundant experimental observations suggest that stochastic Navier-Stokes or Euler equations seem to be more viable models, and the stochastic balance law with the form of (1.1) can be viewed as a simple caricature of the stochastic Euler equations. In addition, stochastic balance laws like (1.1) arises as models in the theory of mean field games developed by Lasry and Lions [7-9].

The main idea is the stochastic kinetic formulation (see [2,10] for example). In fact, if $\rho \in L^\infty(\Omega; L^\infty_{\text{loc}}([0, \infty); L^\infty(\mathbb{R}^d))) \cap C([0, \infty); L^1(\mathbb{R}^d \times \Omega))$ is a stochastic entropy solution of (1.1), (1.2), with $B$ and $A$ are smooth, then for any $v \in \mathbb{R}$, by the kinetic formulation, $u(t, x, v)$, defined by

$$u(t, x, v) = \chi_{\rho(t,x)}(v) = \begin{cases}
1, & \text{when } 0 < v < \rho, \\
-1, & \text{when } \rho < v < 0, \\
0, & \text{otherwise},
\end{cases} \quad (1.7)$$

is a stochastic weak solution of the stochastic transport equation

$$\partial_t u(t, x, v) + b(v) \cdot \nabla_x u + \partial_{x,v} u \circ \dot{M}_i(t, v) + A(t, x, v) \partial_v u(t, x, v) = \partial_v m, \quad (1.8)$$
in $\Omega \times (0, \infty) \times \mathbb{R}^{d+1}$, supplied with

$$u(t, x, v)|_{t=0} = \chi_{\rho_0}(v) \text{ in } \mathbb{R}^{d+1},$$

(1.9)

where $m$ is a nonnegative measure, and

$$M_i(t, v) = \int_0^t \sigma_{i,j}(s, v) dW_j(s).$$

(1.10)

Thus the uniqueness of stochastic weak solutions for (1.8), (1.9) may lead to the uniqueness of stochastic entropy solutions for (1.1), (1.2), which is of particular mathematical interest and here we give a positive answer to the uniqueness of stochastic entropy solutions for (1.1), (1.2) on Lipschitz fluxes.

This paper is organized as follows. In Section 2, we give a stochastic kinetic formula for stochastic balance laws with rough coefficients. Section 3 is devoted to the uniqueness on stochastic entropy solutions for (1.1), (1.2).

As usual, the notation here is mostly standard. $C(T)$ denotes a positive constant depending only on $T$, whose value may change in different places. The summation convention is enforced throughout this article, wherein summation is understood with respect to repeated indices. $D(\mathbb{R}^d)$ stands for the set of all smooth functions in $\mathbb{R}^d$ and supported in a compact subset in $\mathbb{R}^d$, $D'(\mathbb{R}^d)$ represents its dual space. Correspondingly, $D_+(\mathbb{R}^d)$ is the non-negative elements in $D(\mathbb{R}^d)$. $\mathbb{N}$ denotes the set consisting of all natural numbers. a.s. is the abbreviation of "almost surely".

## 2 A stochastic kinetic formula

In this section, we are interested in the Cauchy problem (1.1), (1.2). For notional simplicity, we denote by $b = B'$, $\sigma_{i,j}(t, v) = \partial_v B_{i,j}(t, v)$, and we assume $\sigma_{i,j} = \sigma_{j,i}$. Moreover, we set

$$a = (a_{i,j})_{1 \leq i, j \leq d} = \frac{1}{2}(\sigma_{i,j})_{1 \leq i, j \leq d}(\sigma_{i,j})_{1 \leq i, j \leq d}.$$  

(2.1)

Initially, we need give some notions.

**Definition 2.1** $\rho \in L^\infty(\Omega; L^\infty_{loc}([0, \infty); L^\infty(\mathbb{R}^d))) \cap C([0, \infty); L^1(\mathbb{R}^d \times \Omega))$ is a stochastic weak solution of (1.1) and (1.2), if for any $\varphi \in D(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho(t, x)\varphi(x) dx$ is an $\mathcal{F}_t$ semi-martingale and with probability one, the below identity

$$\int_{\mathbb{R}^d} \varphi(x)\rho(t, x) dx - \int_0^t \int_{\mathbb{R}^d} B(\rho) \cdot \nabla_x \varphi(x) dx ds - \int_0^t dW_j(s) \int_{\mathbb{R}^d} \partial_x \varphi(x) B_{i,j}(s, \rho) dx + \int_{\mathbb{R}^d} \varphi(x) \rho_0(x) dx + \int_0^t \int_{\mathbb{R}^d} A(s, x, \rho) \varphi(x) dx ds,$$  

(2.2)

holds true, for all $t \in [0, \infty)$.

**Remark 2.1** Obviously, (2.2) admits an equivalent representation:

$$\int_0^\infty \int_{\mathbb{R}^d} \partial_t \psi(t, x)\rho(t, x) dx dt + \int_{\mathbb{R}^d} \rho_0(x)\psi(0, x) dx + \int_0^\infty \int_{\mathbb{R}^d} B(\rho) \cdot \nabla \psi(t, x) dx dt$$
Definition 2.2 A stochastic weak solution of (1.1), (1.2) is a stochastic entropy solution, if for any \( \eta \in \Xi \),
\[
\partial_t \eta(x) + \text{div}(Q(x)) + \partial_x Q_{i,j}(x) \circ \dot{W}_j(t) \leq h(t,x,\rho), \quad \text{a.s. } \omega \in \Omega, \tag{2.3}
\]
in the sense of distributions, i.e., for any \( \psi(t,x) \in \mathcal{D}([0,\infty) \times \mathbb{R}^d) \) and for almost all \( \omega \in \Omega \)
\[
\int_0^\infty dt \int_{\mathbb{R}^d} \partial_t \psi(x) dx + \int_0^\infty dx \int_0^\infty Q(x) \cdot \nabla_x \psi(x) dx + \int_0^T dW_j(t) \int_{\mathbb{R}^d} \partial_x \psi(x) Q_{i,j}(x) dx \geq -\int_0^T \psi(0,x) \eta(0) dx - \int_0^T h(t,x,\rho) \psi(t,x) dx dt,
\]
where
\[
Q(x) = \int_0^\rho \eta'(v)b(v) dv, \quad Q_{i,j}(x) = \int_0^\rho \eta'(v)\sigma_{i,j}(v) dv, \quad h(t,x,\rho) = A(t,x,\rho)\eta'(\rho), \tag{2.4}
\]
and
\[
\Xi = \{c_0 \rho + \sum_{k=1}^n c_k |\rho - \rho_k|, \quad c_0, c_k, \rho_k \in \mathbb{R} \text{ are constants}\}.
\]

Remark 2.2 (i) \( \eta \in \Xi \), so \( \eta \) is convex and thus \( \eta'(v) \) is legitimate for almost all \( v \in \mathbb{R} \). But to make \( A(t,x,\rho)\eta'(\rho) \) well-defined, we need to choose a particular modification for \( \eta'(\rho) \). Here in our mind, we take \( \partial_\rho \rho - \rho_k = \text{sign}(\rho - \rho_k) \).

(ii) If \( \rho, B, A \) and \( B_{i,j} \) are smooth, for any convex function \( \eta \), then
\[
\partial_t \eta(x) + \text{div}_x(Q(x)) + \partial_x Q_{i,j}(x) \circ \dot{W}_j(t) = h(t,x,\rho),
\]
with \( Q, Q_{i,j} \) and \( h \) given by (2.4). In general, (1.1) should serve as the \( \varepsilon \to 0 \) limit of the equation
\[
\partial_t \rho(\varepsilon,t,x) + \text{div}_x(B(\rho(\varepsilon))) + \partial_x B_{i,j}(t,\rho(\varepsilon)) \circ \dot{W}_j(t) - \varepsilon \Delta \rho(\varepsilon,t,x) = A(t,x,\rho(\varepsilon)).
\]

Observe that now
\[
\partial_t \eta(\rho(\varepsilon)) + \text{div}_x(Q(\rho(\varepsilon))) + \partial_x Q_{i,j}(t,\rho(\varepsilon)) \circ \dot{W}_j(t) - \varepsilon \Delta \eta(\rho(\varepsilon)) \leq h(t,x,\rho(\varepsilon)),
\]
so the vanishing viscosity limit should make above inequality to conserve the same sigma. Thus Definition 2.2 is reasonable.

(iii) When \( \eta(\rho, \bar{\rho}) = |\rho - \bar{\rho}| \), then
\[
Q(x, \rho) = \text{sign}(\rho - \bar{\rho})[B(\rho) - B(\bar{\rho})], \quad Q_{i,j}(t,\rho,\bar{\rho}) = \text{sign}(\rho - \bar{\rho})[B_{i,j}(t,\rho) - B_{i,j}(t,\bar{\rho})],
\]
where \( \bar{\rho} \) is a parameter taking values in \( \mathbb{R} \). 

4
Before founding the uniqueness of stochastic entropy solutions, we need establish the equivalence between stochastic entropy solutions of (1.1), (1.2) and stochastic weak solutions for (1.8), (1.9).

**Theorem 2.1 (Stochastic kinetic formula)** Assume that $A$ and $B_{i,j}$ fulfill the assumption (1.3).

(i) Let $\rho$ be a stochastic entropy solution of (1.1), (1.2) and set $u(t, x, v) = \chi_{\rho(t,x)}(v)$. Then

$$u \in L^\infty(\Omega; L^\infty([0, \infty); L^\infty(\mathbb{R}^d_t; L^1(\mathbb{R}^d_v)))) \cap C([0, \infty); L^1(\mathbb{R}^d_t \times \mathbb{R}^d_v \times \Omega)),$$

and it is a stochastic weak solution of the linear stochastic transport problem (1.8), (1.9) with $m$ weak solution of (1.8), (1.9), with $t$.

Proof. Clearly, it suffices to show: for any $(1.8)$ has the following equivalent representation: (Lemma 2.1)

Before founding the uniqueness of stochastic entropy solutions, we need establish the equivalence (1.3).

(ii) Let $\rho(t, x, v) \in L^\infty(\Omega; L^\infty([0, \infty); L^\infty(\mathbb{R}^d_t; L^1(\mathbb{R}^d_v)))) \cap C([0, \infty); L^1(\mathbb{R}^d_t \times \mathbb{R}^d_v \times \Omega))$ be a stochastic weak solution of (1.8), (1.9), with $m$ meeting above properties. Then

$$m([0, s] \times \mathbb{R}^{d+1}) \rightarrow m([0, t] \times \mathbb{R}^{d+1}), \text{ as } s \rightarrow t.$$  

Here $u$ is called a stochastic weak solution of (1.8), (1.9), if for any $\phi \in D(\mathbb{R}^{d+1})$, with probability one,

$$\int_{\Omega} \phi(x, v)u(t, x, v)dxdv - \int_{0}^{t} \int_{\mathbb{R}^{d+1}} b(v) \cdot \nabla \phi(x, v)dxdvdv = \int_{\Omega} \phi(x, v)u_0(x, v)dxdv + \int_{0}^{t} \int_{\mathbb{R}^{d+1}} M_i(\cdot ds, dv) \int_{\mathbb{R}^{d+1}} \partial_{x_i} \phi(x, v)u(s, x, v)dx$$

$$+ \int_{0}^{t} \int_{\mathbb{R}^{d+1}} \partial_{v}[A(s, x, v)\phi(x, v)]u(s, x, v)dxdvdv - \int_{0}^{t} \int_{\mathbb{R}^{d+1}} \partial_{v}[\phi(x, v)m(dx, dv, ds)]$$

is legitimate, for all $t \in [0, \infty)$.

**Lemma 2.1** (1.8) has the following equivalent representation:

$$\partial_t u(t, x, v) + b(v) \cdot \nabla u + \partial_{x_i} uM_i(t, v) - a_{i,j}(t, v)\partial^2_{x_i,x_j} u + A(t, x, v)\partial_v u = \partial_v m,$$  

(2.8)

Proof. Clearly, it suffices to show: for any $\phi \in D(\mathbb{R}^{d+1})$, and for all $t \in [0, \infty)$,

$$\int_{0}^{t} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \phi(x, v)u(s, x, v)dxdm_i(\cdot ds, dv)$$

$$= \int_{0}^{t} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \phi(x, v)u(s, x, v)dxdm_i(ds, dv) + \int_{0}^{t} ds \int_{\mathbb{R}^{d+1}} a_{i,j}(s, v)\partial^2_{x_i,x_j} \phiudxdv.$$

But on the other hand,

$$\int_{0}^{t} \int_{\mathbb{R}^{d+1}} \partial_{x_i} \phi(x, v)u(s, x, v)dxdm_i(\cdot ds, dv)$$
\[
= \int_0^t \int_{\mathbb{R}^d+1} \partial_{x_i} \phi(x,v)u(s,x,v)dx M_i(ds, dv) + \frac{1}{2} \iint_\mathbb{R}^d \partial_{x_i} \phi(x,v)u(\cdot,x,v)dx M_i(\cdot,v)_t dv,
\]
where \([\cdot, \cdot]_t \] denotes the joint quadratic variation, thus it is sufficient to demonstrate
\[
\int_\mathbb{R} \left[ \int_\mathbb{R} \partial_{x_i} \phi(x,v)u(\cdot,x,v)dx M_i(\cdot,v)_t dv \right] dv = 2 \int_0^t ds \int_{\mathbb{R}^d+1} a_{i,j}(s,v) \partial^2_{x_i,x_j} \phi u dx dv.
\]

Note that whichever (1.8) or (2.8) holds, then for any \(\phi \in \mathcal{D}(\mathbb{R}^d+1)\), and for all \(t \in [0, \infty)\), the martingale part of \(\int_{\mathbb{R}^d} \partial_{x_i} \phi(x,v)u(t,x,v)dx\) is given by
\[
\int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i,x_j} \phi(x,v)u(s,x,v)dx M_j(ds, v).
\]
Therefore
\[
\int_\mathbb{R} \left[ \int_\mathbb{R} \partial_{x_i} \phi(x,v)u(\cdot,x,v)dx M_i(\cdot,v)_t dv \right] dv
\]
\[
= \int_\mathbb{R} M_j(ds, v) \int_\mathbb{R} \partial^2_{x_i,x_j} \phi(x,v)u(s,x,v)dx M_i(\cdot,v)_t dv
\]
\[
= \int_\mathbb{R} dv \int_0^t ds \int_\mathbb{R} \partial^2_{x_i,x_j} \phi(x,v)u(s,x,v)\sigma_{i,k}(s,v)\sigma_{j,k}(s,v)dx
\]
\[
= 2 \int_0^t ds \int_{\mathbb{R}^d+1} a_{i,j}(s,v) \partial^2_{x_i,x_j} \phi(x,v)u(s,x,v)dx dv.
\]

Lemma 2.2 For any \(p \in [1, \infty]\), we have the following embedding:
\[
L^p(\mathbb{R}^d; W^{1,1}_{loc}(\mathbb{R})) \hookrightarrow L^p(\mathbb{R}^d; C(\mathbb{R})�).
\]

Proof. Clearly, \(W^{1,1}_{loc}(\mathbb{R}) \hookrightarrow C(\mathbb{R})\), so for any \(f \in L^p(\mathbb{R}^d; W^{1,1}_{loc}(\mathbb{R}))\), \(f(x, \cdot) \in C(\mathbb{R})\) for a.s. \(x \in \mathbb{R}^d\). Let \(-\infty < a < b < \infty\) be two real numbers, then
when \(p < \infty\),
\[
\|f\|_{L^p(\mathbb{R}^d; L^\infty(a,b))}^p = \int_{\mathbb{R}^d} \|f(x, \cdot)\|_{L^\infty(a,b)}^p dx
\]
\[
= \int_{\mathbb{R}^d} \left\| \int_a^b \partial_v f(x,v)dv + f(x,a) \right\|_{L^\infty(a,b)}^p dx
\]
\[
\leq 2^{p-1} \left[ \int_{\mathbb{R}^d} \left[ \int_a^b |\partial_v f(x,v)| dv \right] dx + \int_{\mathbb{R}^d} |f(x,a)|^p dx \right]
\]
\[
< \infty,
\]
when \(p = \infty\),
\[
|f(x,v)| < \int_a^v |\partial_y f(x,y)| dy + |f(x,a)|
\]
\[
\leq \int_a^v |\partial_y f(x,y)| dy + |f(x,a)|
\]
\[
\leq \int_a^b |\partial_y f(x, y)|dy + |f(x, a)|
< \infty
\]
for almost all \(x \in \mathbb{R}^d\), and all \(v \in [a, b]\), which hints
\[
L^p(\mathbb{R}^d; W^{1,1}_{\text{loc}}(\mathbb{R})) \hookrightarrow L^p(\mathbb{R}^d; L^1_{\text{loc}}(\mathbb{R})).
\]
Thus the desired result follows.

**Proof of Theorem 2.1.** For any \(\alpha_1, \alpha_2 \in \mathbb{R}\), notice that
\[
\int_{\mathbb{R}} |\chi_{\alpha_1}(v) - \chi_{\alpha_2}(v)|dv = |\alpha_1 - \alpha_2|,
\]
so (2.7) implies (2.5), and the vice versa is clear, we need to survey the rest of (i) and (ii).

(i) Suppose that \(\rho\) is a stochastic entropy solution of (1.1), (1.2) fulfilling the statement in (i), for any \(v \in \mathbb{R}\), it renders that
\[
\partial_t \eta(\rho, v) + \text{div}_x Q(\rho, v) + \partial_x Q_i(t, \rho, v) \circ \dot{W}_i(t) = \text{sign}(\rho - v)A(t, x, \rho) - 2m,
\]
where
\[
\eta(\rho, v) = |\rho - v| - |v|,
Q(\rho, v) = \text{sign}(\rho - v)[B(\rho) - B(v)] - \text{sign}vB(v),
Q_i(t, \rho, v) = \text{sign}(\rho - v)[B_i(t, \rho) - B_i(t, v)] - \text{sign}vB_{i}(t, v),
m \text{is a nonnegative measure on } [0, \infty) \times \mathbb{R}^d \times \mathbb{R},
\]
for almost all \(\omega \in \Omega\).

On account of (1.3), if one differentiates (2.9) in \(v\) in distributional sense, then
\[
\begin{cases}
\partial_v \eta(\rho, v) = -2u(t, x, v), \\
\partial_v Q(\rho, v) = -2b(v)u(t, x, v), \\
\partial_v Q_i(t, \rho, v) = -2\sigma_{i}(t, v)u(t, x, v), \\
\partial_v \text{sign}(\rho - v)A(t, x, \rho) = 2\partial_v u(t, x, v)A(t, x, v).
\end{cases}
\]
Thus one derives the identity (1.8) in the sense of distributions for almost all \(\omega \in \Omega\).

In fact, if one takes the last identity in (2.11) for an example, then for any \(\phi_1 \in \mathcal{D}(\mathbb{R})\),
\[
\langle \partial_v \text{sign}(\rho - v)A(t, x, \rho), \, \phi_1 \rangle = -2\phi_1(\rho)A(t, x, \rho).
\]
On the other hand
\[
\langle \partial_v u(t, x, v)A(t, x, v), \, \phi_1 \rangle = -\langle u(t, x, v), \, \partial_v (\phi_1(v)A(t, x, v)) \rangle.
\]
Note that
\[
\int_{\mathbb{R}} g'(v)u(t, x, v)dv = g(\rho(t, x)) - g(0), \quad \text{for any } g \in W^{1,1}_{\text{loc}}(\mathbb{R}),
\]
and \(A(t, x, 0) = 0\), it follows that
\[
\langle \partial_v u(t, x, v)A(t, x, v), \, \phi_1 \rangle = -\phi_1(\rho)A(t, x, \rho),
\]
thus \( \partial_t \text{sign}(\rho - v)A(t, x, \rho) = 2\partial_t u(t, x, v)A(t, x, v) \).

In view of that \( \rho \) is bounded local-in-time, from (2.9) and (2.10), for any fixed \( T > 0 \), and almost all \( \omega \in \Omega \), \( m \) is supported in \([0, T] \times \mathbb{R}^d \times [-K, K] \), with \( K = \|\rho\|_{L^\infty((0, T) \times \mathbb{R}^d \times \Omega)} \). Accordingly, it remains to examine that \( m \) is bounded and continuous in \( t \).

Since \( m \geq 0 \) and it is supported in a compact subset in \( v \), we obtain

\[
0 \leq \langle m, \psi \otimes 1 \rangle_{t,x,v} = -\langle \partial_t u + b(v) \cdot \nabla_x u + A\partial_v u + \partial_x u \circ \dot{M}_i(t, v), \psi \otimes v \rangle_{t,x,v},
\]

for any \( \psi \in \mathcal{D}_+(\mathbb{R}) \), and for almost all \( \omega \in \Omega \).

With the aid of Lemma 2.1, then

\[
0 \leq \langle m, \psi \otimes 1 \rangle_{t,x,v} = -\langle \partial_t u + b(v) \cdot \nabla_x u + A\partial_v u + \sigma_{i,j}(t, v) \partial_x W_j(t) - a_{i,j}(t, v) \partial^2_{x_i,x_j} u, \psi \otimes v \rangle_{t,x,v}.
\]

(2.13)

Thanks to (2.12), one computes from (2.13) that

\[
-\langle \partial_t u + b(v) \cdot \nabla_x u + A\partial_v u + \sigma_{i,j}(t, v) \partial_x W_j(t) - a_{i,j}(t, v) \partial^2_{x_i,x_j} u, \psi \otimes v \rangle_{t,x,v}
\]

\[
= \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \partial_t \psi(t, x) \rho^2 dx dt + \frac{1}{2} \int_{\mathbb{R}^d} \psi(0, x) \rho_0^2(x) dx + \int_0^T \int_{\mathbb{R}^d} \rho A(t, x, \rho) \psi(t, x) dx dt + \int_0^T \int_{\mathbb{R}^d} \rho \partial_v u \psi(t, x) dx dt + \int_0^T \int_{\mathbb{R}^d} \partial_t \psi(t, x) \rho dx dt + \int_0^T \int_{\mathbb{R}^d} \rho \partial_v u \psi(t, x) dx dt
\]

\[
\quad + \int_0^T \int_{\mathbb{R}^d} \left[ \sigma_{i,j}(t, \rho(t, x)) \rho(t, x) - \int_0^{\rho(t, x)} A_{i,j}(t, v) dv \right] \partial^2_{x_i,x_j} \psi(t, x) dx dt + \int_0^T \int_{\mathbb{R}^d} \left[ B_{i,j}(t, \rho(t, x)) - \int_0^{\rho(t, x)} B_{i,j}(t, v) dv \right] \partial_t \psi(t, x) dx dt,
\]

for any \( T > 0 \) and \( \psi \in \mathcal{D}_+(\mathbb{R}) \), where \( \partial_v A_{i,j}(t, v) = a_{i,j}(t, v) \).

On account of hypotheses (1.3), in view of Lemma 2.2, this leads to

\[
-\langle \partial_t u + b(v) \cdot \nabla_x u + A\partial_v u + \sigma_{i,j}(t, v) \partial_x W_j(t) - a_{i,j}(t, v) \partial^2_{x_i,x_j} u, \psi \otimes v \rangle_{t,x,v}
\]

\[
\leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \partial_t \psi(t, x) \rho^2 dx dt + \frac{1}{2} \int_{\mathbb{R}^d} \psi(0, x) \rho_0^2(x) dx + C(T) \int_0^T \int_{\mathbb{R}^d} \tilde{a}(t, x) \rho^2 \psi(t, x) dx dt + \int_0^T \int_{\mathbb{R}^d} \rho(t, x) |\nabla_x \psi(t, x)| dx dt + \int_0^T \int_{\mathbb{R}^d} \partial_t \psi(t, x) \partial^2_{x_i,x_j} \psi(t, x) dx dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left[ B_{i,j}(t, \rho) - \int_0^{\rho(t, x)} B_{i,j}(t, v) dv \right] \partial_t \psi(t, x) dx dt,
\]

\[ P - a.s. \quad \omega \in \Omega, \quad (2.14) \]

where

\[
\tilde{a}(t, x) = \sup_{v \in [-K, K]} |A(t, x, v)| \in L^1_{loc}(0, \infty; L^1(\mathbb{R}^d)) + L^1_{loc}(0, \infty; L^\infty(\mathbb{R}^d)),
\]
\[ \tilde{a}_{i,j}(t) = \sup_{v \in [-K,K]} |A_{i,j}(t,v)| \in L^1\text{loc}([0,\infty)). \]

Obviously, (2.14) holds ad hoc for \( \psi(t,x) = \psi_1(t)\theta_n(x) \), where \( \psi_1 \in \mathcal{D}_+(\mathbb{R}^d) \), \( \theta \in \mathcal{D}_+(\mathbb{R}^d) \),

\[ \theta_n(x) = \frac{\theta(x)}{n}, \quad \theta(x) = \begin{cases} 1, & \text{when } |x| \leq 1, \\ 0, & \text{when } |x| > 2, \end{cases} \quad (2.15) \]

and for this fixed \( n \), by an approximation demonstration, one can fetch

\[ \psi_1(t) = \begin{cases} 1, & t \in [0,T - \frac{1}{n}], \\ -n(t-T), & t \in (T - \frac{1}{n},T], \\ 0, & t \in (T,\infty). \end{cases} \]

Observing that the Itô isometry,

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \left[ |B_{i,j}(t,\rho)| \rho - \int_0^{\rho(t,x)} B_{i,j}(t,v)dv \right] \partial_x \psi(t,x) dx dW_j(t) \right]^2
\]

\[ = \int_0^T \mathbb{E} \left[ \int_{\mathbb{R}^d} \left( |B_{i,j}(t,\rho)| \rho - \int_0^{\rho(t,x)} B_{i,j}(t,v)dv \right) \partial_x \psi(t,x) dx \right]^2 dt \]

\[ \leq C(T) \mathbb{E} \int_0^T \tilde{b}_{i,j}^2(t) \left[ \int_{\mathbb{R}^d} |\rho(t,x)||\partial_x \psi(t,x)| dx \right]^2 dt, \]

where

\[ \tilde{b}_{i,j}(t) = \sup_{v \in [-K,K]} |B_{i,j}(t,v)| \in L^2\text{loc}([0,\infty)). \]

Thus we gain from (2.13) and (2.14) by letting \( n \to \infty \), that

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(dt,dx, dv) 
\leq \frac{1}{2} \left[ \int_{\mathbb{R}^d} \rho^2 dx - \int_{\mathbb{R}^d} \rho^2(T,x) dx \right] + C(T) \int_0^T \int_{\mathbb{R}^d} \tilde{a}(t,x) \rho^2(t,x) dx dt, \quad (2.16) 
\]

for \( P - a.s. \omega \in \Omega \), which suggests that \( m \) is bounded on \([0,T] \times \mathbb{R}^d \times \mathbb{R} \), for any given \( T > 0 \), and

\( m \in L^1(\Omega; \mathcal{D}'([0,\infty) \times \mathbb{R}^{d+1})). \)

Specially, when \( T \to 0 \), we obtain

\[ \lim_{T \to 0} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(dt,dx, dv) = 0, \quad P - a.s. \omega \in \Omega. \]

The arguments employed above for 0 and \( T \) adapted to any \( 0 \leq s, t < \infty \) now, yields that

\[ \lim_{t \to s} \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(dr,dx, dv) = 0, \]

which hints \( m \) is continuous in \( t \). By Remark 2.1, \( u \) is a stochastic weak solution of (1.8), (1.9).

(ii) Let us show the reverse fact. Given \( \epsilon > 0 \) and \( \bar{\rho} \in \mathbb{R} \), set

\[ \eta_{\epsilon}(t,\bar{\rho}) = (\sqrt{(t-\bar{\rho})^2 + \epsilon^2} - \epsilon) - |\bar{\rho}| \in C^2(\mathbb{R}), \]
then \( \eta \) is convex, \( \eta'(t, \bar{\rho}) \in C_b(\mathbb{R}) \), and
\[
\eta_k(t, \bar{\rho}) \longrightarrow |t - \bar{\rho}| - |\bar{\rho}| \text{ as } \epsilon \longrightarrow 0.
\]

In a consequence of \( u(t, x, v) \) solving (1.8) and (1.9), by Remark 2.1, it follows that
\[
\langle \partial_t m, \psi \eta'(v, \bar{\rho}) \xi_k(v) \rangle_{t,x,v} = \langle \partial_t u + \partial_{x,i} u \circ \tilde{M}_i(t, v), \psi \eta'(v, \bar{\rho}) \xi_k(v) \rangle_{t,x,v}
+ \langle b(v) \cdot \nabla_x u + A \partial_v u, \psi \eta'(v, \bar{\rho}) \xi_k(v) \rangle_{t,x,v}
= \langle \partial_t u + \sigma_i j(t, v) \partial_{x,i} u \circ \tilde{W}_j(t), \psi \eta'(v, \bar{\rho}) \xi_k(v) \rangle_{t,x,v}
+ \langle b(v) \cdot \nabla_x u + A \partial_v u, \psi \eta'(v, \bar{\rho}) \xi_k(v) \rangle_{t,x,v},
\tag{2.17}
\]
for any \( \psi \in D_+([0, \infty) \times \mathbb{R}^d) \), \( \xi \in D_+(\mathbb{R}) \), where
\[
\xi_k(v) = \xi\left(\frac{v}{k}\right), \quad 0 \leq \xi \leq 1, \quad \xi(v) = \begin{cases} 1, & \text{when } |v| \leq 1, \\ 0, & \text{when } |v| \geq 2. \end{cases}
\tag{2.18}
\]
Applying the partial integration, one deduces
\[
\lim_{k \to \infty} \langle \partial_t m, \psi \eta'(v, \bar{\rho}) \xi_k \rangle_{t,x,v}
= - \lim_{k \to \infty} \langle m, \psi \eta''(v, \bar{\rho}) \xi_k + \eta'(v, \bar{\rho}) \xi'_k \rangle_{t,x,v}
\leq 0, \quad P - a.s. \omega \in \Omega,
\tag{2.19}
\]
for \( m \) yields the properties stated in Theorem 2.1 (i).

Upon using (2.12) and (2.19), from (2.17), we derive
\[
\int_0^\infty dt \int_{\mathbb{R}^d} \partial_t \psi(t, x)[\eta_k(\rho, \bar{\rho}) - \eta_k(0, \bar{\rho})]dx + \int_0^\infty dt \int_{\mathbb{R}^d} Q_\epsilon(\rho, \bar{\rho}) \cdot \nabla_x \psi dx
\geq - \int_{\mathbb{R}^d} \psi(0, x)[\eta_k(\rho_0, \bar{\rho}) - \eta_k(0, \bar{\rho})]dx - \int_0^\infty dt \int_{\mathbb{R}^d} \eta'_k(\rho, \bar{\rho}) A(t, x, \rho) \psi(t, x) dx
- \int_{\mathbb{R}^d} \circ \hat{W}_j \int_{\mathbb{R}^d} \partial_{x,i} \psi Q_{i,j}^\epsilon(t, \rho, \bar{\rho}) dx,
\tag{2.20}
\]
by taking \( k \) to infinity, here
\[
Q_\epsilon(\rho, \bar{\rho}) = \int_{\mathbb{R}} b(v) \eta'(v, \bar{\rho}) u(t, x, v) dv, \quad Q_{i,j}^\epsilon(t, \rho, \bar{\rho}) = \int_{\mathbb{R}} \sigma_i j(t, v) \eta'(v, \bar{\rho}) u(t, x, v) dv.
\]
On the other hand
\[
\lim_{\epsilon \to 0} \eta'(v, \bar{\rho}) = \text{sign}(v - \bar{\rho})
\]
and
\[
\lim_{\epsilon \to 0} Q_\epsilon(\rho, \bar{\rho}) = \text{sign}(\rho - \bar{\rho})[B(\rho) - B(\bar{\rho})] - \text{sign}\rho[B(\rho) - B(0)],
\]
\[
\lim_{\epsilon \to 0} Q_{i,j}^\epsilon(t, \rho, \bar{\rho}) = \text{sign}(\rho - \bar{\rho})[B_{i,j}(t, \rho) - B_{i,j}(t, \bar{\rho})] - \text{sign}\rho[B_{i,j}(t, \rho) - B_{i,j}(t, 0)],
\]
for a.s. \((\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}^d\).

If one lets \( \epsilon \) approach to zero in (2.20), we attain the inequality (2.3), thus \( \rho \) is a stochastic entropy solution.
Remark 2.3  (i) Our proof for Theorem 2.1 is inspired by Theorem 1 in [10], but the demonstration here appears to be finer, and for more details, one can see [10] and the references cited there.

(ii) Until now we are not clear enough how to extend the present result to more general vector-valued function $\rho$, for the proof here relying upon very much ‘scalar features’. But for results on the deterministic $2 \times 2$ hyperbolic system of isentropic gas dynamics in both Eulerian or Lagrangian variables, one can see [11].

We are now in a position to give our main result on uniqueness for stochastic entropy solutions to (1.1), (1.2).

3 Uniqueness of stochastic entropy solutions

Before founding the uniqueness, we need two lemmas below, the first one follows from DiPerna and Lions [12], and the proof is analogue, we only give the details for the second one.

Lemma 3.1 Let $E \in L^p(\Omega; L^p(0, T; W^{1,\infty}_{loc}(\mathbb{R}^n; \mathbb{R}^n)))$, $F \in L^{q_1}(\Omega; L^{q_2}(0, T; L^{\beta}_{loc}(\mathbb{R}^n)))$, with $1 \leq p_1, p_2, q_1, q_2, \alpha, \beta \leq \infty$, $T \in (0, \infty)$. Then

$$(E \cdot \nabla F) \ast \tilde{\varrho}_{\varepsilon_1} - E \cdot \nabla (F \ast \tilde{\varrho}_{\varepsilon_1}) \longrightarrow 0 \quad \text{in} \quad L^{r_1}(\Omega; L^{r_2}(0, T; L^{\gamma}_{loc}(\mathbb{R}^n)))$$

as $\varepsilon_1 \to 0$, where $1 \leq r_1, r_2 < \infty$, satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{1}{\gamma}, \quad \frac{1}{p_1} + \frac{1}{q_1} \leq \frac{1}{r_1}, \quad \frac{1}{p_2} + \frac{1}{q_2} \leq \frac{1}{r_2}.$$ 

$n \in \mathbb{N}$, and

$$\tilde{\varrho}_{\varepsilon_1} = \frac{1}{\varepsilon_1^n} \tilde{\varrho}(\frac{\cdot}{\varepsilon_1}) \quad \text{with} \quad \tilde{\varrho} \in D_+(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \tilde{\varrho}(y)dy = 1, \quad \varepsilon_1 > 0.$$ 

And when $n = d$, we set $\tilde{\varrho}$ by $\varrho_1$.

Lemma 3.2 Let $f \in L^2(\Omega; L^2_{loc}([0, \infty)))$, then

$$\left[\int_0^t f(s)dW_s \ast \varrho_{2, \varepsilon_2}\right](t) \longrightarrow \int_0^t f(s)dW_s, \quad \text{in} \quad L^2(\Omega; L^2_{loc}([0, \infty))) \quad \text{as} \quad \varepsilon_2 \to 0,$$

where $W_t$ is a one dimensional standard Wiener process, and

$$\varrho_{2, \varepsilon_2} = \frac{1}{\varepsilon_2} \varrho_{2}(\frac{\cdot}{\varepsilon_2}) \quad \varrho_2 \in D_+(\mathbb{R}), \quad \int_{\mathbb{R}} \varrho_2(t)dt = 1, \quad \text{supp} \varrho_2 \subset (-1, 0).$$

Proof. In fact, for any $T \in (0, \infty)$, then

$$\mathbb{E} \int_0^T \left| \left[ \int_0^t f(s)dW_s \ast \varrho_{2, \varepsilon_2}\right](t) - \int_0^t f(s)dW_s \right|^2 dt$$

$$= \mathbb{E} \int_0^T \left| \int_{\mathbb{R}} \varrho_{2, \varepsilon_2}(s)ds \int_0^{t-s} f(r)dW_r - \int_0^t f(r)dW_r \right|^2 dt$$

$$= \mathbb{E} \int_0^T \left| \int_{-1}^0 \varrho_2(s)ds \int_0^{t-s} f(r)dW_r - \int_0^t f(r)dW_r \right|^2 dt.$$
For \( f \in L^2(\Omega; L^2_{loc}([0, \infty))) \), thus the stochastic process \( \{ \int_0^t f(r) dW_r, \ t \geq 0 \} \) is a martingale. With the help of Doob’s inequality and the Itô isometry, from (3.1), one obtains

\[
\mathbb{E} \int_0^T \left| \int_0^t f(s) dW_s \right|^2 dt \leq 4 \int_0^T \sup_{0 \leq s \leq t} \mathbb{E} \left| \int_t^{t+\varepsilon_2 s} f(r) dW_r \right|^2 dt.
\]  

(3.2)

Then we finish the proof if one lets \( \varepsilon_2 \) tend to 0 in (3.2).

After above preparation, we give our main result now.

**Theorem 3.1 (Uniqueness of Stochastic Entropy Solutions)** Let \( A, B \) and \( B_{i,j} \) be described as in (1.3), that

\[
[A, B]_+ \in L^1_{loc}([0, \infty); L^\infty(\mathbb{R}^{d+1})),
\]

(3.3)

\[
\frac{A}{1 + |v|} \in L^1_{loc}([0, \infty); L^1(\mathbb{R}^{d+1}) + L^\infty(\mathbb{R}^{d+1}) + L^1(\mathbb{R}^d; L^\infty(\mathbb{R}^d))).
\]

(3.4)

Further, if one presumes that

\[
B \in L^1_{loc}([0, \infty); W^{1,\infty}_{loc}(\mathbb{R}^d, \mathbb{R}^d)), \quad B_{i,j} \in L^2_{loc}([0, \infty); W^{1,\infty}_{loc}(\mathbb{R})),
\]

(3.5)

then the stochastic entropy solutions of (1.1), (1.2) is unique.

**Proof.** Let \( \rho_1 \) and \( \rho_2 \) be two stochastic entropy solutions of (1.1), with initial values \( \rho_{0,1} \) and \( \rho_{0,2} \), respectively. Then \( u_1 = \chi_{\rho_1} \) and \( u_2 = \chi_{\rho_2} \) defined by (1.7) are stochastic weak solutions of (1.8) with nonhomogeneous terms \( \partial_v m_1 \) and \( \partial_v m_2 \), initial data \( u_{0,1} = \chi_{\rho_{0,1}} \) and \( u_{0,2} = \chi_{\rho_{0,2}} \), respectively.

Let \( \varrho_3 \) be another regularization kernel in variables \( v \), i.e.

\[
\varrho_3 \in \mathcal{D}_+(\mathbb{R}), \quad \int_{\mathbb{R}} \varrho_3(v) dv = 1.
\]

For \( \varepsilon_1, \varepsilon_2, \varepsilon > 0 \), set

\[
\varrho_{1,\varepsilon_1}(x) = \frac{1}{\varepsilon_1} \varrho_1(\frac{x}{\varepsilon_1}), \quad \varrho_{2,\varepsilon_2}(t) = \frac{1}{\varepsilon_2} \varrho_2(\frac{t}{\varepsilon_2}), \quad \varrho_{3,\varepsilon}(v) = \frac{1}{\varepsilon} \varrho_3(\frac{v}{\varepsilon}),
\]

then \( u_{i,\varepsilon} := u_i \ast \varrho_{1,\varepsilon_1} \ast \varrho_{2,\varepsilon_2} \ast \varrho_{3,\varepsilon} \) \( (i = 1, 2) \) meets

\[
\begin{cases}
\partial_t u_{i,\varepsilon} + b(v) \cdot \nabla u_{i,\varepsilon} + A(t, x, v) \partial_v u_{i,\varepsilon} + \partial_x u_{i,\varepsilon} \circ M_i(t, v) = \partial_v m_{i,\varepsilon} + R_{i,\varepsilon}, \\
u_{i,\varepsilon}(t, x, v)|_{t=0} = \chi_{\rho_{0,i}} \ast \varrho_{1,\varepsilon_1} \ast \varrho_{3,\varepsilon}(x, v),
\end{cases}
\]

(3.6)
here $R_{i}^{\varepsilon,\kappa} = R_{i,1}^{\varepsilon,\kappa} + R_{i,2}^{\varepsilon,\kappa} + R_{i,3}^{\varepsilon,\kappa}$, and

\[
\begin{align*}
R_{i,1}^{\varepsilon,\kappa} &= b(v) \cdot \nabla x u_{i}^{\kappa,\varepsilon} - [b(v) \cdot \nabla x u_{i}^{\kappa}]
\varepsilon,\kappa,
R_{i,2}^{\varepsilon,\kappa} &= A(t, x, v) \partial_{v} u_{i}^{\kappa,\varepsilon} - [A(t, x, v) \partial_{v} u_{i}^{\kappa}]
\varepsilon,\kappa,
R_{i,3}^{\varepsilon,\kappa} &= \partial_{x_{i}} u_{i}^{\kappa,\varepsilon} \circ \dot{M}_{i}(t, v) - \left[\partial_{x_{i}} u_{i} \circ \dot{M}_{i}(t, v)\right]^{\varepsilon,\kappa}.
\end{align*}
\tag{3.7}
\]

From (3.6), for $i = 1, 2$, it follows that

\[
\frac{d}{dt} \int_{R_{d+1}} |u_{i}^{\varepsilon,\kappa}| \xi_{k}(v) \theta_{n}(x) dxdv
= \int_{R_{d+1}} |u_{i}^{\varepsilon,\kappa}| \xi_{k}(v) b(v) \cdot \nabla x \theta_{n}(x) dxdv + \int_{R_{d+1}} |u_{i}^{\varepsilon,\kappa}| \partial_{v} [\xi_{k}(v) A(t, x, v)] \theta_{n}(x) dxdv
+ \int_{R_{d+1}} |u_{i}^{\varepsilon,\kappa}| \partial_{x_{i}} \theta_{n}(x) \xi_{k}(v) \circ \dot{M}_{i}(t, v) dxdv + \int_{R_{d+1}} \text{sign} u_{i}^{\varepsilon,\kappa} \xi_{k}(v) \theta_{n}(x) R_{i}^{\varepsilon,\kappa}(t, x, v) dxdv
+ \int_{R_{d+1}} \xi_{k}(v) \text{sign} u_{i}^{\varepsilon,\kappa} \theta_{n}(x) \partial_{\varepsilon,\kappa} m_{i}(t, x, v) dxdv,
\tag{3.8}
\]

where $M_{i}$, $\theta_{n}$ and $\xi_{k}$ are given by (1.10), (2.15) and (2.18) respectively.

An analogue calculation also yields that

\[
\frac{d}{dt} \int_{R_{d+1}} u_{1}^{\varepsilon,\kappa}(t, x, v) u_{2}^{\varepsilon,\kappa}(t, x, v) \xi_{k}(v) \theta_{n}(x) dxdv
= \int_{R_{d+1}} u_{1}^{\varepsilon,\kappa} u_{2}^{\varepsilon,\kappa} \xi_{k}(v) b(v) \cdot \nabla x \theta_{n}(x) dxdv + \int_{R_{d+1}} u_{1}^{\varepsilon,\kappa} u_{2}^{\varepsilon,\kappa} \partial_{v} [\xi_{k}(v) A(t, x, v)] \theta_{n}(x) dxdv
+ \int_{R_{d+1}} u_{1}^{\varepsilon,\kappa} u_{2}^{\varepsilon,\kappa} \partial_{x_{1}} \theta_{n}(x) \xi_{k}(v) \circ \dot{M}_{1}(t, v) dxdv + \int_{R_{d+1}} \xi_{k}(v) \theta_{n}[u_{1}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{2}^{\varepsilon,\kappa} + u_{2}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{1}^{\varepsilon,\kappa}] dxdv
+ \int_{R_{d+1}} \xi_{k}(v) \theta_{n}(x) [R_{1}^{\varepsilon,\kappa}(t, x, v) u_{2}^{\varepsilon,\kappa} + R_{2}^{\varepsilon,\kappa}(t, x, v) u_{1}^{\varepsilon,\kappa}] dxdv.
\tag{3.9}
\]

From (3.8) and (3.9), one concludes

\[
\frac{d}{dt} \int_{R_{d+1}} |u_{i}^{\varepsilon,\kappa}| \xi_{k}(v) \theta_{n}(x) dxdv + \frac{d}{dt} \int_{R_{d+1}} |u_{j}^{\varepsilon,\kappa}| \xi_{k} \theta_{n} dxdv - 2 \frac{d}{dt} \int_{R_{d+1}} u_{i}^{\varepsilon,\kappa} u_{j}^{\varepsilon,\kappa} \xi_{k} \theta_{n} dxdv
= \int_{R_{d+1}} [|u_{i}^{\varepsilon,\kappa}| + |u_{j}^{\varepsilon,\kappa}| - 2 |u_{i}^{\varepsilon,\kappa} u_{j}^{\varepsilon,\kappa}|] \xi_{k}(v) b(v) \cdot \nabla x \theta_{n}(x) dxdv
+ \int_{R_{d+1}} [|u_{i}^{\varepsilon,\kappa}| + |u_{j}^{\varepsilon,\kappa}| - 2 |u_{i}^{\varepsilon,\kappa} u_{j}^{\varepsilon,\kappa}|] \partial_{v} [\xi_{k} A(t, x, v)] \theta_{n} dxdv
+ \int_{R_{d+1}} \xi_{k} \theta_{n} [\text{sign} u_{i}^{\varepsilon,\kappa} R_{1}^{\varepsilon,\kappa} + \text{sign} u_{j}^{\varepsilon,\kappa} R_{2}^{\varepsilon,\kappa}] dxdv - 2 \int_{R_{d+1}} \xi_{k} \theta_{n} [R_{1}^{\varepsilon,\kappa} u_{2}^{\varepsilon,\kappa} + R_{2}^{\varepsilon,\kappa} u_{1}^{\varepsilon,\kappa}] dxdv
+ \int_{R_{d+1}} [|u_{i}^{\varepsilon,\kappa}| + |u_{j}^{\varepsilon,\kappa}| - 2 |u_{i}^{\varepsilon,\kappa} u_{j}^{\varepsilon,\kappa}|] \partial_{x_{i}} \theta_{n}(x) \xi_{k}(v) \circ \dot{M}_{i}(t, v) dxdv + I,
\tag{3.10}
\]

where

\[
I = \int_{R_{d+1}} \xi_{k} \theta_{n} [\text{sign} u_{i}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{2}^{\varepsilon,\kappa} + \text{sign} u_{j}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{1}^{\varepsilon,\kappa}] dxdv
- 2 \int_{R_{d+1}} \xi_{k} \theta_{n} [u_{i}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{2}^{\varepsilon,\kappa} + u_{j}^{\varepsilon,\kappa} \partial_{\varepsilon,\kappa} m_{1}^{\varepsilon,\kappa}] dxdv
\]
On the other hand, for fixed \( \varepsilon \), from (3) in (2.11), therefore for large \( k \) sufficiently large,

\[
\int_{\mathbb{R}^{d+1}} \partial_{v} \xi \theta \left[ \text{sign} \ u_{1}^{\varepsilon} m_{1}^{\varepsilon} + \text{sign} \ u_{2}^{\varepsilon} m_{2}^{\varepsilon} \right] dxdv = 0,
\]
\[
\int_{\mathbb{R}^{d+1}} \partial_{v} \xi \theta \left[ u_{1}^{\varepsilon} m_{2}^{\varepsilon} + u_{2}^{\varepsilon} m_{1}^{\varepsilon} \right] dxdv = 0. \tag{3.11}
\]

From (3.6) and (1.7), with the aid of assumption (1.3) and Lemma 2.2, \( m_{i}^{\varepsilon} (i = 1, 2) \) is continuous in \( v \) in a neighborhood of zero. Besides, note that

\[
\text{sign} u_{i}^{\varepsilon} \to \text{sign} u_{i} = \text{sign} v, \quad \text{as} \quad \varepsilon \to 0,
\]

therefore for large \( k \),

\[
\lim_{\varepsilon \to 0} I_{1} = -2 \int_{\mathbb{R}^{d}} \theta_{n}(x)[m_{1}^{\varepsilon}(t, x, 0) + m_{2}^{\varepsilon}(t, x, 0)]dx \tag{3.12}
\]

Moreover, due to (2.12) and the fact \( m_{i} \geq 0 (i = 1, 2) \), so for \( k \) large enough,

\[
I_{2} = \int_{\mathbb{R}^{d+1}} \theta_{1, \varepsilon_{1}}(y) \theta_{2, \varepsilon_{2}}(s) dy ds \int_{\mathbb{R}^{d+2}} \xi_{k}(\rho_{1}(t - s, x - y) + \tau)m_{2}^{\varepsilon}(t, x, \rho_{1}(t - s, x - y) + \tau)
\]
\[
+ \xi_{k}(\rho_{2}(t - s, x - y) + \tau)m_{1}^{\varepsilon}(t, x, \rho_{2}(t - s, x - y) + \tau)] \theta_{3, \varepsilon}(\tau) \theta_{n}(x) dxdvd\tau
\]
\[
- \int_{\mathbb{R}^{d+1}} \xi_{k}(v) \theta_{n}(x)[m_{1}^{\varepsilon}(t, x, v) + m_{2}^{\varepsilon}(t, x, v)] \theta_{3, \varepsilon}(v) dxdv
\]
\[
\geq - \int_{\mathbb{R}^{d+1}} \xi_{k}(v) \theta_{n}(x)[m_{1}^{\varepsilon}(t, x, v) + m_{2}^{\varepsilon}(t, x, v)] \theta_{3, \varepsilon}(v) dxdv
\]
\[
\to - \int_{\mathbb{R}^{d}} \theta_{n}(x)[m_{1}^{\varepsilon}(t, x, 0) + m_{2}^{\varepsilon}(t, x, 0)]dx, \quad \text{as} \quad \varepsilon \to 0, \tag{3.13}
\]

On account of (3.7), thanks to Lemma 3.1, then,

\[
\lim_{\varepsilon_{1} \to 0} \lim_{\varepsilon_{2} \to 0} \lim_{\varepsilon \to 0} R_{1, 2}^{\varepsilon, \varepsilon} = 0, \quad \text{in} \quad L^{1}(\Omega; L^{1}(0, T; L_{\text{loc}}^{1}(\mathbb{R}^{d+1}))), \quad \text{for} \quad i = 1, 2. \tag{3.14}
\]

On the other hand, for fixed \( \varepsilon_{1} \), we have

\[
R_{1, 3}^{\varepsilon, \varepsilon} = \partial_{x_{i}} u_{i}^{\varepsilon, \varepsilon} \circ \dot{M}_{i}(t, v) - [\partial_{x_{i}} u_{i} \circ \dot{M}_{i}(t, v)]^{\varepsilon, \varepsilon} := I_{1, 3}^{\varepsilon, \varepsilon} - \frac{1}{2} J_{1, 3}^{\varepsilon, \varepsilon}
\]

where

\[
I_{1, 3}^{\varepsilon, \varepsilon} = \partial_{x_{i}} u_{i}^{\varepsilon, \varepsilon} \dot{M}_{i}(t, v) - [\partial_{x_{i}} u_{i}^{\varepsilon, \varepsilon} \circ \dot{M}_{i}(t, v)]^{\varepsilon, \varepsilon}
\]
\[
J_{1, 3}^{\varepsilon, \varepsilon} = \partial_{x_{i}}^{2} u_{i}^{\varepsilon, \varepsilon} \sigma_{i}^{k} \sigma_{j}^{k}(t, v) - [\partial_{x_{i}}^{2} u_{i}^{\varepsilon, \varepsilon} \sigma_{i}^{k} \sigma_{j}^{k}(t, v)]^{\varepsilon, \varepsilon}
\]
Due to Lemma 3.2 and (3.5),
\[
\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \lim_{\varepsilon \to 0} J_{t, \varepsilon, \theta}^{\varepsilon, \theta} = 0, \quad \text{in} \quad L^2(\Omega; L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^{d+1}))),
\]
and by virtue of Lemma 3.1,
\[
\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \lim_{\varepsilon \to 0} J_{t, \varepsilon, \theta}^{\varepsilon, \theta} = 0, \quad \text{in} \quad L^1(\Omega; L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^{d+1}))),
\]
for \( t = 1, 2. \)

For \( n \) and \( k \) (\( k \) is big enough) be fixed, if one let \( \varepsilon \) tend to zero first, \( \varepsilon_2 \) approach to zero next, \( \varepsilon_1 \) incline to zero last, with the aid of (3.11)–(3.16), from (3.10), it leads to
\[
\frac{d}{dt} \int_{\mathbb{R}^{d+1}} |u_1| |\xi_k(v)\theta_n(x)| dx dv + \frac{d}{dt} \int_{\mathbb{R}^{d+1}} |u_2| |\xi_k\theta_n dx dv - 2 \frac{d}{dt} \int_{\mathbb{R}^{d+1}} u_1 u_2 \xi_k \theta_n dx dv
\leq \int_{\mathbb{R}^{d+1}} [u_1 + u_2 - 2u_1 u_2] |\xi_k(v)b(v) \cdot \nabla_x \theta_n(x)| dx dv
+ \int_{\mathbb{R}^{d+1}} [u_1 + u_2 - 2u_1 u_2] \partial_v [\xi_k A(t, x, v)] \theta_n dx dv
+ \int_{\mathbb{R}^{d+1}} [u_1 + u_2 - 2u_1 u_2] \partial_{x_1} \theta_n(x) \xi_k(v) \circ \dot{M}_1(t, v) dx dv.
\]
Because of the fact \(|u_1 - u_2|^2 = |u_1 - u_2|\), from (3.17), one derives
\[
\frac{d}{dt} \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| |\xi_k(v)\theta_n(x)| dx dv
\leq \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| |\xi_k(v)b(v) \cdot \nabla_x \theta_n(x)| dx dv
+ \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| \partial_v [\xi_k(v) A(t, x, v)] \theta_n(x) dx dv
+ \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| \partial_{x_1} \theta_n(x) \xi_k(v) \circ \dot{M}_1(t, v) dx dv.
\]
By taking \( n \) to infinity first, \( k \) to infinity second, with the help of (3.4), (3.5), then
\[
\frac{d}{dt} \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| dx dv \leq \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| \partial_v A(t, x, v) dx dv
\leq \|\partial_v A(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^{d+1})} \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1 - u_2| dx dv,
\]
where in the last inequality, we have used the assumption (3.3).

Thus
\[
\mathbb{E} \int_{\mathbb{R}^d} |\rho_1(t) - \rho_2(t)| dx
= \mathbb{E} \int_{\mathbb{R}^{d+1}} |u_1(t) - u_2(t)| dx dv
\leq \int_{\mathbb{R}^{d+1}} |u_{0, 1} - u_{0, 2}| dx dv \exp(\int_0^t \|\partial_v A(s, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^{d+1})} ds)\]
which implies
\[ \rho_{0,1}(x) - \rho_{0,2}(x) \mid dx \exp\left( \int_0^t \| \partial_v A(s, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^{d+1})} ds \right) \]
(3.19)

From (3.19), we complete the proof.

From Theorem 3.1, one clearly has the below comparison result.

**Corollary 3.1 (Comparison Principle)** Let \( \rho_1 \) and \( \rho_2 \) be two stochastic entropy solutions of (1.1), with initial values \( \rho_{0,1} \) and \( \rho_{0,2} \), if \( \rho_{0,1} \leq \rho_{0,2} \), then with probability 1, \( \rho_1 \leq \rho_2 \).

**Proof.** Clearly, mimicking above calculation, we have

\[
\frac{d}{dt} \mathbb{E} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] dx dv
\]

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**Proof.** Clearly, mimicking above calculation, we have

\[
\frac{d}{dt} \mathbb{E} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] dx dv
\]

Observing that

\[
[u_1(t, x, v) - u_2(t, x, v)] = \frac{|u_1 - u_2| - (u_1 - u_2)}{2},
\]

hence

\[
\frac{d}{dt} \mathbb{E} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] dx dv
\]

\[
= \frac{1}{2} \mathbb{E} \frac{d}{dt} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] - \frac{1}{2} \mathbb{E} \frac{d}{dt} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)]
\]

\[
\leq \frac{1}{2} \mathbb{E} \int_{\mathbb{R}^{d+1}} \left[ |u_1(t, x, v) - u_2(t, x, v)| - u_1(t, x, v) + u_2(t, x, v) \right] \partial_v A(t, x, v) dx dv
\]

\[
= \mathbb{E} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] dx dv \| \partial_v A(t, v, \cdot) \|^2_{L^\infty(\mathbb{R}^{d+1})}.
\]

Then the Grönwall inequality applies, one concludes

\[
\mathbb{E} \int_{\mathbb{R}^{d}} [\rho_1(t, x) - \rho_2(t, x)] dx
\]

\[
= \mathbb{E} \int_{\mathbb{R}^{d+1}} [u_1(t, x, v) - u_2(t, x, v)] dx dv
\]

\[
\leq \int_{\mathbb{R}^{d+1}} [u_0,1(x, v) - u_0,2(x, v)] dx dv \exp\left( \int_0^t \| [\partial_v A(s, \cdot, \cdot)] \|^2_{L^\infty(\mathbb{R}^{d+1})} ds \right)
\]

\[
= \int_{\mathbb{R}^{d+1}} [\rho_0,1(x) - \rho_0,2(x)] dx \exp\left( \int_0^t \| [\partial_v A(s, \cdot, \cdot)] \|^2_{L^\infty(\mathbb{R}^{d+1})} ds \right)
\]

\[
= 0,
\]

which implies \( \rho_1 \leq \rho_2 \), \( P - a.s. \).
Remark 3.1 (i) As a special case, one confirms the uniqueness of stochastic entropy solutions for
\[
\begin{aligned}
\partial_t \rho(t,x) + \text{div}_x(B(\rho)) + \nabla \rho(t,x) \cdot \mathcal{W}(t) = A(t,x,\rho), \quad &\text{in } \Omega \times (0, \infty) \times \mathbb{R}^d, \\
\rho(t,x)|_{t=0} = \rho_0(x), \quad &\text{in } \Omega \times \mathbb{R}^d,
\end{aligned}
\]
when $B \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R}^d)$. However, we cannot give an affirm answer on the problem whether the weak solutions is unique or not, when $B$ is non-regular (such as $B \in L^{\infty}(\mathbb{R}; \mathbb{R}^d)$).

(ii) Our proof originates from [13], but the calculation here seems to be finer and the demonstration is more difficult, for more details one can pay his attention on [13] and the references cited up there.

To make our discussion on uniqueness more clear, we exhibit a representative example here.

Example 3.1 A porous medium equation (see [14]) with a nonlinear source, a nonlinear convection term and a stochastic perturbation reads
\[
\begin{aligned}
\partial_t \rho + \text{div}_x(\zeta|\rho|^{-\alpha}) + \vartheta(t)\text{div}_x(|\rho|^{\beta/2}) \circ \mathcal{W}(t) = A(t,\rho), \quad &\text{in } \Omega \times (0, \infty) \times \mathbb{R}^d, \\
\rho(t,x)|_{t=0} = \rho_0(x), \quad &\text{in } \Omega \times \mathbb{R}^d,
\end{aligned}
\]  
(3.20)
where $\zeta \in \mathbb{R}^d$ is a fixed vector, $\alpha, \beta \geq 0$ are constants,
\[
\vartheta(t) \in L^1_{\text{loc}}([0, \infty)), \quad A(t,\rho) = \frac{\lambda(t)\rho^2}{1+\rho^2}, \quad \lambda(t) \in L^1_{\text{loc}}([0, \infty)).
\]

From Theorem 3.1 and Corollary 3.1, we have

Proposition 3.1 Let $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then the stochastic entropy solutions of (3.20) is unique. In addition, if $\rho_0 \geq 0$, the unique stochastic entropy solution $\rho \geq 0$.

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