WAVE MECHANICS: BEHAVIOR OF A DISTRIBUTED ELECTRON CHARGE IN AN ATOM

ANDREY VASILYEV

Retired from State Optical Institute, Saint Petersburg, Russia
e-mail: andrey@wavemach.org
Abstract

A hypothesis is forwarded of the electron charge in an atom existing in a distributed form. To check it by methods of electrodynamics and mechanics (without invoking the formalism of quantum mechanics and the concepts of the wave function and of the operators), the potential, kinetic, and total energies were calculated for three states of the hydrogen atom, which were found to agree closely with the available experimental data. The second Chapter of the Paper offers additional assumptions concerning various scenarios of motion of elements of the distributed electron charge which obey fully the laws of theoretical mechanics. The angular momentum of the ground-state hydrogen atom calculated in the frame of theoretical mechanics is shown to coincide with the spin which is $\hbar/2$.

Key words: quantum mechanics, electrodynamics, charge, potential, electric field, energy, angular momentum.
Contents

1 Spherical atom
  1.1 Introduction .................................................. 2
  1.2 Equation for the shape of a distributed electron charge in an atom 3
  1.3 Potential energy of distributed electron charge in a hydrogen atom 6
  1.4 Fields and potentials of a distributed electron charge ............ 8
  1.5 Kinetic and total energies of a distributed electron charge in an atom ........................................... 11

2 Non-spherical atom
  2.1 Shape of a non-spherical charge ................................. 17
  2.2 Potentials of a non-spherical charge distribution ............... 20
  2.3 Analysis of possible patterns of motion of a distributed charge .. 25
  2.4 Angular momentum of the LNS state ........................... 29
  2.5 Comment on the absence of emission from a stationary orbit .... 38
  2.6 Conclusion ..................................................... 39
  2.7 Appendix ....................................................... 41

Bibliography .......................................................... 41
Chapter 1

Spherical atom

1.1 Introduction

Quantum mechanics permits one to calculate with a high accuracy all parameters of an electron in an atom, such as its energy, squared angular momentum, projection of the angular momentum on the Z axis and others. The square of the wave function determines the probability for an electron to be at a certain point in space. It is possible, however, to study the behavior an electron in an atom in a more detailed way. One may invoke for this purpose methods developed in electrodynamics. Significantly, one can calculate many fundamental parameters of the electron in an atom without resorting to such strictly quantum-mechanical concepts as “wave function” or “operator”. It turns out also that such an approach provides in some cases a more detailed description of the behavior of an electron than it would be possible within the frame of quantum mechanics. This opens a way to understanding this behavior more closely. Accordingly, our basic goal here will be not determination of the parameters of an electron in an atom, which are actually well known, but rather an attempt at describing the electron behavior in greater detail. Now if the various parameters of the electron calculated in the context of electrodynamics are found to agree with experimental data and values derived from quantum mechanics, this may be considered as supporting the assumptions formulated below on electron behavior (or, in the case of disagreement, as refuting these ideas).

The first Chapter calculates the potential, kinetic, and total energies of the hydrogen atom for three states based on the above assumptions and invoking only the concepts of electrodynamics and theoretical mechanics.

The second Chapter calculates, likewise drawing only from the concepts of electrodynamics and theoretical mechanics, the angular momentum of the hydrogen atom in the ground state, which turned out to be $\hbar/2$, in excellent agreement with experimental data. We are going to conduct the relevant reasoning and calculations in the nonrelativistic approximation.
1.2 Equation for the shape of a distributed electron charge in an atom

Quantum mechanics assumes the electron to be a point charge of magnitude \((-e)\). A point charge at rest can be presented in the form of an expansion in a Fourier integral in scalar spatial harmonics of the charge:

\[
q(r) = (-e)\delta(r) = (-e) \int_{-\infty}^{\infty} f(k)e^{ikr}dk,
\]  

where \(dk = dk_xdk_ydk_z\) denotes integration over the three wave vectors \(k_x, k_y, k_z\) of Cartesian coordinates. Here \(k = 2\pi/\lambda\), where \(\lambda\) is the wavelength of the corresponding harmonic of charge, is the wave number, and \(\delta(r)\) is Dirac’s delta function. The spectral density \(f(k)\) of expansion into the Fourier integral is defined in the following way:

\[
f(k) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \delta(r)e^{-ikr}dr,
\]

where \(dr = dx dy dz\) denotes integration over the three Cartesian axes.

Each harmonic \(f(k_x)e^{ik_xx}\), \(f(k_y)e^{ik_yy}\), \(f(k_z)e^{ik_zz}\) in Eq. (1.1) is essentially a wave at rest with wave numbers \(k_x, k_y, k_z\). These are, however, not electromagnetic but rather charge waves.

If a point charge propagates with a constant velocity \(v_x\) along the \(x\) axis, it can be presented in the form of an expansion in moving harmonics:

\[
q(r, t) = (-e)\delta(r - vt) = (-e) \int_{-\infty}^{\infty} f(k)e^{i(kr - wt)}dk.
\]  

Each harmonic \(f(k_x)e^{i(k_xx-\omega t)}\) is a plane wave moving with a velocity \(v_x\) along the \(x\) axis. The frequency \(\omega\) of all these waves is coupled to the wave number \(k_x\) through the relation \(k_x = \omega/v_x\). The \(f(k_y)e^{ik_yy}\) and \(f(k_z)e^{ik_zz}\) harmonics are, as before, waves at rest with wave numbers \(k_y\) and \(k_z\).

It should be stressed that the above decompositions are in no way expansions in de Broglie waves. These are nothing more than conventional expansions in spatial or spatial-temporal harmonics.

Significantly, expansion of an object in harmonics is not just a mathematical abstraction. Each harmonic is in actual fact a real physical component of a given process. For instance, when sound or electromagnetic waves are decomposed into a spectrum each harmonic corresponding to a harmonic of the mathematical expansion is a real sound or electromagnetic wave exhibiting all the properties of a wave of this nature. The validity of this statement is not questioned any longer. Nevertheless, one could refer in this connection to the book by A. A. Kharkevich [1] where this point is considered at length. In our case, we meet with the same situation in expanding a charge in separate harmonics of the kind of Eq. (1.3), which represent in this particular case charge waves.

Consider now a point electron that enters an atom and starts rotating about the nucleus. When an electron rotates about the nucleus, each charge harmonic
adding up to the charge of the electron must satisfy the periodicity conditions. Because the lengths of the charge waves making up the electron charge are different, the shape of the point charge of the electron should somehow change. For this reason, the electron, while being point-like in free space, will have to “spread out” in rotation about the nucleus, i.e., assume another shape, different from point-like. Significantly, this will be a real “spread-out”, a real change in the shape of the point charge, rather than the probability of finding a point charge at a certain point in space. Said otherwise, we have here what can be called distribution of the electron charge in space.

Now what shape can assume a charge rotating about a nucleus? In other words, what will be the electron charge distribution in an atom? The Schrödinger equation permits one to calculate with a high precision the various parameters describing the state of an electron in an atom. This licenses us to assume that the charge of an electron rotating about a nucleus should take on the shape coinciding with that of an eigenfunction of Schrödinger’s equation, or of a function related in some way to eigenfunctions of the Schrödinger equation. The set of eigenfunctions of the Schrödinger equation is usually expressed in a spherical reference frame and is well known. Therefore, we are going to use in what follows the spherical coordinate system \( r, \vartheta, \varphi \). The Cartesian system was used above only to make our reasoning more revealing.

What are the eigenfunctions of the Schrödinger equation among which one could look for functions describing the shape of a rotating charge? We note immediately that the shape of the distribution of this charge can be looked for only by invoking \( S \) states of solutions of the Schrödinger equation. Indeed, charge conservation dictates that the total charge of a distribution rotating about a nucleus must be equal to the electron charge. Integration over the whole space of the \( S \) states only yields a nonzero result. The integral over all other states gives zero. Therefore, only the \( S \) state is capable of describing the presence of an electron in an atom. While the atom can contain, besides \( S \), other charge states as well, they will be able only to modify the charge shape.

Hence, in all situations the \( S \) charge state is present in an atom. The \( S \) charge state must be present in an atom always.

Besides, it is only the \( S \) states of the charge that can contribute to the electron potential energy. The potential energy of an electron is mediated by the Coulomb interaction of the electron with the nucleus. We identify the nucleus with a geometric point placed at the origin of the spherical reference frame. The electron charge is distributed somehow in space. In this case, the potential energy of interaction of this charge with the nucleus can be written as

\[
E_{\text{Pot}} = \int e\delta(r)\Phi(r)dV, \tag{1.4}
\]

where \( e\delta(r) \) is the nuclear charge, and \( \Phi(r) \) is the potential generated by the electron charge distribution. Examining Eq. (1.4), we see that the energy \( E_{\text{Pot}} \) is nonzero for the \( S \) states only, because it is solely for the \( S \) states that the potential \( \Phi(0) \) is nonzero. While besides the \( S \) states an atom may contain other charge states as well, they will not be able to contribute to the potential energy of electron interaction with the nucleus. This contribution can come from the \( S \) states of charge only.
Hence, in all situations the S charge state is present in an atom. The S charge state must be present in an atom always.

To find the charge shape assumed by an electron rotating about a nucleus, we write the corresponding equation in the form

\[ \nabla \rho(r) + \frac{4m_e}{\hbar^2}(\mathcal{E} - \mathcal{E}_{\text{Pot}})\rho(r) = 0. \]  

(1.5)

In this equation, \( \rho(r) \) is the density of the electron charge distribution we are looking for, \( m_e \) is the electron mass, \( \mathcal{E} \) is the eigenvalue of energy, \( \mathcal{E}_{\text{Pot}} \) is the electron potential energy, and \( \hbar \) is the Dirac constant: \( \hbar = h/2\pi \), where \( h \) is the Planck constant. Equation (1.5) differs in form from the Schrödinger equation in that it is written not for the wave function \( \Psi \) but rather for the density of electron charge distribution \( \rho(r) \), and in the numerical coefficient 4 in the second term of the equation (in the Schrödinger equation, the coefficient is 2). The meaning of this difference will become clear later. Equation (1.5) may be called “the equation of wave mechanics”. This name stresses that an electron, while being point-like in free space and, hence, obeying the laws of mechanics, changes its shape in an atom as a result of manifestation of its wave properties when expanded in charge waves.

We are going to find the shape of the electron charge distribution by means of Eq. (1.5). Equation (1.5) contains the energy \( \mathcal{E}_{\text{Pot}} \). This is the potential energy of an electron in the nuclear field. We do not know, however, the shape of the electron charge distribution and, hence, do not know the energy \( \mathcal{E}_{\text{Pot}} \). The only thing we can do is to substitute for \( \mathcal{E}_{\text{Pot}} \) the energy of interaction of a point nucleus with a point electron. In doing this, we have got ourselves into an ambiguity: indeed, we are looking for the shape of a distributed electron charge but substitute into Eq. (1.5) the energy of a point electron. One should therefore verify that this will not give rise to an inconsistency. We shall check this later in the particular example of the hydrogen atom.

Introducing the parameter \( \mu_e = 2m_e \), Eq. (1.5) can be recast in the form

\[ \nabla \rho_{\mu}(r) + \frac{2\mu_e}{\hbar^2}(\mathcal{E}_\mu - \mathcal{E}_{\text{Pot}})\rho_{\mu}(r) = 0. \]  

(1.6)

This equation coinciding in form with the Schrödinger equation, one can use the results of its solution, keeping in mind all the time that Eq. (1.6) contains in place of the mass \( m_e \) the parameter \( \mu_e \), which is not the electron mass. Therefore in what follows we are going to label all the results derived from Eq. (1.6) with index \( \mu \). The quantities labeled by \( \mu \) may judiciously be called “modified” to stress that they are related not to the real electron mass \( m_e \) but rather to a parameter \( \mu_e \). Accordingly, Eq. (1.6) may be referred to as a “modified” equation of wave mechanics.

It appears pertinent to specify now the relations connecting the commonly used quantities with the corresponding “modified” quantities.

Modified mass

\[ \mu_e = 2m_e. \]

Modified Bohr radius

\[ a_\mu = \frac{\hbar^2}{\mu_e e^2} = \frac{\hbar^2}{2m_e e^2} = \frac{a}{2}, \]  

(1.7)
where \( a = \frac{\hbar^2}{m_e e^2} \) is the Bohr radius.

Modified reduced radius

\[
\tau_\mu = \frac{r}{a_\mu} = 2\tau,
\]

where \( r \) is the radial coordinate of a spherical reference system, and \( \tau = r/a \) is the reduced radius. We thus obtain \( \tau_\mu a_\mu = \tau a = r \).

1.3 Potential energy of distributed electron charge in a hydrogen atom

Let us find the potential energy of interaction of an electron with the nucleus in a hydrogen atom. One should first find for this purpose the shape of the distributed charge of an electron in the potential field of the nucleus and the energy of interaction of this distributed charge with the nucleus by methods of electrodynamics. To calculate the charge distribution we are looking for, one has to substitute into the equation of wave mechanics (1.5), or Eq. (1.6), the potential energy of electron interaction with the nucleus. As pointed out in Sec. 1.2, the only possibility open for us here is to treat this energy as interaction of a point nucleus with a point electron. The potential energy of interaction of a point nucleus with a point electron can be written as

\[
E_{\text{Pot}} = -\frac{e^2}{r},
\]

where \( r \) is the distance between the nucleus and the electron. We assume the nucleus to be at the origin of the spherical coordinate system. In this case \( r \) is nothing else but the radial component of the spherical reference frame. Substituting Eq. (1.8) into Eq. (1.5), we come to

\[
\Delta \rho + \frac{4m_e}{\hbar^2} \left( E + \frac{e^2}{r} \right) \rho = 0.
\]

One has now to solve this equation and find the charge distribution \( \rho \). In place of solving Eq. (1.9), however, we can write the “modified” equation by substituting Eq. (1.8) into Eq. (1.6), to obtain

\[
\Delta \rho_\mu + \frac{2\mu_e}{\hbar^2} \left( E_\mu + \frac{e^2}{r} \right) \rho_\mu = 0.
\]

Recall that it is not the real electron mass \( m_e \) but rather parameter \( \mu_e = 2m_e \) that enters this equation. Equation (1.10) coincides in form with the Schrödinger equation for the hydrogen atom. The solutions to this equation are well known. In the form of most relevance to us they are presented in monograph [2]. As pointed out in Sec. 1.2, of all the solutions we are interested in the spherical symmetric \( S \) states only. We are writing out these solutions for the quantum numbers \( n = 1, 2, 3 \) below (they are normalized against unity):

\[
\rho_{\mu 1S} = 2e^{-\tau_\mu},
\]
The variable in relations (1.11) is the modified radius \( \tau_\mu = r/a_\mu \), where \( r \) is the radial component of the spherical coordinate system, and \( a_\mu \) is the modified Bohr radius (see Eqs. (1.7)). Equations (1.7) can be used to eliminate the \( \mu \) index, with the solutions acquiring the form

\[
\rho_{1S} = A_1 Se^{-2\tau}, \\
\rho_{2S} = A_2 Se^{-2\tau}(1 - \tau), \\
\rho_{3S} = A_3 Se^{-2\tau}(27 - 36\tau^2 + 8\tau^4). 
\]

We have introduced here normalization factors \( A_nS \), so that there is no sense anymore in retaining the fractional coefficients in the parentheses. The variable in Eqs. (1.12) is the reduced radius \( \tau = r/a \). We are going to use besides \( \tau \) in what follows the \( r \) variable too. The solutions \( \rho_{nS} \) are the densities of electron charge distribution in the atom. The total charge should be equal to the electron charge: \(-e\);

\[
\int \rho_{nS}dV = -e. \tag{1.13}
\]

\( dV \) is here an element of volume in the spherical coordinate system:

\[
dV = r^2 \sin \theta d\theta d\varphi dr = a^2 \tau^2 \sin \theta d\theta d\varphi d\tau = a^2 \tau^2 \sin \theta d\theta d\varphi d\tau. \tag{1.14}
\]

Integrating Eq. (1.13), we come to the following values for the \( A_nS \) coefficients:

\[
A_{1S} = -\frac{e}{\pi a^3}, \\
A_{2S} = \frac{+e}{4\pi a^3} \cdot \frac{1}{4}, \\
A_{3S} = -\frac{e}{4\pi a^3} \cdot \frac{4}{27 \cdot 27 \cdot 3}. \tag{1.15}
\]

Having obtained the solutions (1.12) for the electron charge distribution in an atom, we can find the energy of interaction of this charge with the nucleus:

\[
E_{Pot,nS} = \int \Phi_N \rho_{nS}dV. \tag{1.16}
\]

Here \( \Phi_N = e/r = e/a\tau \) is the nuclear potential. Substituting now the expression for the nuclear potential and equations (1.12) into Eq. (1.16), and going over to a common variable \( \tau \), we obtain after some straightforward calculations

\[
E_{Pot,1S} = \frac{e^2}{a}, \quad E_{Pot,2S} = -\frac{e^2}{a} \cdot \frac{1}{4}, \quad E_{Pot,3S} = -\frac{e^2}{a} \cdot \frac{1}{9}. \tag{1.17}
\]
We have used in the integration the well known formula (valid for an integer \( n \))
\[
\int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}.
\]
(1.18)

The expressions for the energy (1.17) coincide with the eigenvalues of Eq. (1.9)
although they were derived by another method. The expressions (1.17) can be
combined in one formula
\[
\mathcal{E}_{\text{Pot},nS} = -\frac{e^2}{a} \cdot \frac{1}{n^2}.
\]
(1.19)

Equation (1.19) coincides fully with the expression for the potential energy of
interaction between a nucleus and an electron which is well known in quantum
mechanics (it could be obtained, for instance, from the expression \( \mathcal{E}_{\text{Pot},nS} = \int \hat{\Psi}_{nS}^* \hat{H} \Psi_{nS} \, dV \), where \( \hat{H} = -\frac{\nabla^2}{2} \) is the potential energy operator).

Thus, by normalizing the charge distribution against the electron charge and
using standard methods of electrostatics, we have arrived at correct values of the
energies of interaction of a distributed charge with the nucleus, which coincide
with those known from quantum mechanics.

1.4 Fields and potentials of a distributed electron charge

Knowing the distribution of electron charge in an atom, we can readily calculate
the electric fields and potentials generated by these charges. As pointed out in
Sec. 1.2, of all the solutions of the wave mechanics equation we are interested
in spherically symmetric states of charge only. This simplifies greatly our task.
Let us find now the potentials and fields of a distributed electron charge. The
electric field can be derived from the Gauss’ theorem (see, e.g., Ref. [3], p. 109):
\[
\oint_S \mathbf{E}(r) \, dS = 4\pi \int_V \rho(r') \, dV' = 4\pi Q',
\]
(1.20)

where for the surface \( S \) we can take a sphere of radius \( r \), \( Q' \) is the charge inside
this sphere, \( \rho(r') \) is the charge density, and \( V \) is the volume inside the sphere \( S \).
Because by symmetry of the system the field on the surface of such a sphere is
constant (and has only one radial component \( E_r(r) \)), the Gauss’ theorem takes
on the form
\[
E(r) \cdot 4\pi r^2 = 4\pi \int_V \rho(r') \, dV',
\]
(1.21)

whence
\[
E(r) = \frac{1}{r^2} \int_V \rho(r') \, dV' = \frac{4\pi}{r^2} \int_0^r \rho(r') r'^2 \, dr'.
\]
(1.22)

The potential \( \Phi(r) \) can be calculated from the formula
\[
\Phi(r) = -\int_0^r E(r') \, dr'.
\]
(1.23)
Substituting Eq. (1.22) for the field $E(r)$ in this equality and integrating by parts, we come to the following expression for the potential

$$
\Phi(r) = \frac{4\pi}{r} \int_0^r \rho(r')r'^2 dr' + 4\pi \int_r^\infty \rho(r')r'dr'.
$$

(1.24)

Equations (1.22) and (1.24) can be found in monographs [3] (pp. 113, 175) and [4] (pp. 27, 228).

The expressions for distributed charge (1.12) involve the variable $\tau = r/a$.

Equation (1.24) expressed in these variables assumes the form

$$
\Phi(\tau) = \frac{4\pi a^2}{\tau} \int_0^\tau \rho(\tau')\tau'^2 d\tau' + 4\pi a^2 \int_\tau^\infty \rho(\tau')\tau'd\tau'.
$$

(1.25)

Substituting the expressions for distributed charge (1.12) in Eq. (1.25), we come to the following relations for the potential of the distributed electron charge

$$
\Phi_1S = -\frac{e}{a} + \frac{e}{a} e^{-2\tau}(1 + \tau),
$$

$$
\Phi_2S = -\frac{e}{a} + \frac{e}{a} e^{-2\tau} \cdot \frac{1}{4}(\tau^2 + 3\tau + 4),
$$

(1.26)

$$
\Phi_3S = -\frac{e}{a} + \frac{e}{a} e^{-2\tau} \cdot \frac{1}{27 \cdot 9}(8\tau^3 + 36\tau^2 + 27 \cdot 5\tau + 27 \cdot 9).
$$

Examining expressions (1.27), we see immediately that the potentials of distributed electron charge in a hydrogen atom are actually a sum of the potential of a point charge $(-e)$ placed at the origin and the potential of a point charge $(+e)$ but with an exponential factor, likewise located at the origin.

Consider now the limiting cases of the behavior of the potential. For $r \to \infty$, the second term in expressions (1.27) vanishes by virtue of the exponential factor, with the potentials reducing to a potential of a point charge $(-e)$ placed at the origin. For $r \to 0$, the exponentials can be expanded in a series, with two terms retained. This leads us to

$$
\Phi_{1S}(0) = -\frac{e}{a}, \quad \Phi_{2S}(0) = -\frac{e}{a} \cdot \frac{1}{4}, \quad \Phi_{3S}(0) = -\frac{e}{a} \cdot \frac{1}{9}.
$$

(1.28)

Because we assume the nucleus to be point-like, located at the origin, and having a charge $q_N = (+e)$, the potential energy of interaction of the nucleus with a distributed electron charge can be derived without integration

$$
\mathcal{E}_{nS} = q_N\Phi_{nS}(0) = e\Phi_{nS}(0).
$$

(1.29)
This yields for the interaction energy
\[ E_{1S} = -\frac{e^2}{a}, \quad E_{2S} = -\frac{e^2}{a} \cdot \frac{1}{4}, \quad E_{3S} = -\frac{e^2}{a} \cdot \frac{1}{9} \] (1.30)

These expressions coincide naturally with Eqs. (1.17) calculated from Eq. (1.16).

Calculate now the fields corresponding to the charge distributions obtained. By virtue of the spherical symmetry of the charge distribution, the fields will have only one radial component \( E_r \):
\[ E_r = -\frac{1}{a} \cdot \frac{1}{\tau} \Phi \]
Taking a derivative of expressions (1.26), we come to
\[ E_{1S} = -\frac{e}{a^2 \tau^2} + \frac{e}{a^2 \tau^2} e^{-2\tau} (2\tau^2 + 2\tau + 1), \]
\[ E_{2S} = -\frac{e}{a^2 \tau^2} + \frac{e}{a^2 \tau^2} e^{-\tau} \cdot \frac{1}{4} (\tau^3 + 2\tau^2 + 4\tau + 4), \]
\[ E_{3S} = -\frac{e}{a^2 \tau^2} + \frac{e}{a^2 \tau^2} e^{-2\tau} \cdot \frac{1}{4} (16\tau^4 + 24\tau^3 + 9 \cdot 18\tau^2 + 27 \cdot 18\tau + 27 \cdot 27). \] (1.31)

Just as in the case with the potential, we rewrite Eqs. (1.31) in a more revealing way
\[ E_{1S} = -\frac{e}{r^2} + \frac{e}{r^2} e^{-2\tau} (2\tau^2 + 2\tau + 1), \]
\[ E_{2S} = -\frac{e}{r^2} + \frac{e}{r^2} e^{-\tau} \cdot \frac{1}{4} (\tau^3 + 2\tau^2 + 4\tau + 4), \]
\[ E_{3S} = -\frac{e}{r^2} + \frac{e}{r^2} e^{-2\tau} \cdot \frac{1}{4} (16\tau^4 + 24\tau^3 + 9 \cdot 18\tau^2 + 27 \cdot 18\tau + 27 \cdot 27). \] (1.32)

Examining Eqs. (1.32), we see that in all cases the field of a distributed electron charge may be considered as a sum of two fields, more specifically, of a field of a point charge \((-e)\) located at the origin plus that of a point charge \((+e)\) but with an exponential factor, which is likewise placed at the origin.

Consider now the extreme cases. For \( r \to \infty \), the second term in all expressions vanishes by virtue of the exponential factor, to leave the field of a point charge \((-e)\) at the origin. To study the behavior of the field for \( r \to 0 \), one can expand the exponential to third order. Substituting this expansion in Eqs. (1.31), we readily see that all fields in this case vanish.

The expressions for the potential \( \Phi_{1S} \) and field \( E_{1S} \) of the \( 1S \) state can be found in Refs. [3] (pp. 113, 176) and [4] (pp. 27 and 228).

The field acting in the atom is actually a sum of the field of the nucleus \( E_N = e/r^2 \) and of that created by the distributed charge (1.31), (1.32). At a certain distance from the nucleus this total field will tend to zero by virtue of the exponential factor. As one approaches the nucleus, the total field resembles the field of the nucleus \( e/r^2 \).

Similarly, the potential acting in the atom is a sum of that created by the nucleus \( \Phi_N = e/r \) and of the distributed charge potential (1.26), (1.27). At some distance from the nucleus, this total potential tends to zero by virtue of the exponential factor. As one comes closer to the nucleus, the total potential approaches in form the nuclear potential \( e/r \).
CHAPTER 1. SPHERICAL ATOM

1.5 Kinetic and total energies of a distributed electron charge in an atom

We have seen that the charge which was point-like in free space spreads out in an atom to become a distributed rather than point charge. In free space, however, electron has not only a point charge but a point mass as well. On entering an atom, this point mass has to spread out just as this was done by the point charge. The shape of the mass distribution should be analogous to that of the charge distribution, because it forms by summation of absolutely identical harmonics both in the first and the second cases. Indeed, a point electron possesses both a point charge and a point mass. The point charge and mass can be expanded in terms of harmonics as this was done in series (1.1) or (1.3); a difference may appear only in the common coefficient and the common sign of the harmonics. The velocities and directions of motion of the harmonics also coincide, because both the charge and the mass belong to the same point electron. Therefore, as the electron moves in its circular trajectory, summation of all harmonics in which the mass was expanded should produce the same distribution of mass as that of the charge (to within the common coefficient and sign). In other words, we deal here not with a distribution of charge or that of mass but rather with a distribution of the charge/mass object. Therefore, we can write an equation for the distribution of mass similar to Eq. (1.5):

\[ \Delta m(r) + \frac{4m_e}{\hbar^2}(E - E_{pot})m(r) = 0. \]  

(1.33)

For the hydrogen atom, this equation can be recast to the form of Eq. (1.9):

\[ \Delta m + \frac{4m_e}{\hbar^2} \left( E + \frac{e^2}{r} \right) m = 0. \]

We write the solutions to this equation in a form similar to that of Eqs. (1.12):

\[ m_{1S} = B_{1S}e^{-2\tau}, \]

\[ m_{2S} = B_{2S}e^{-\frac{2\tau}{3}}(1 - \tau), \]

\[ m_{3S} = B_{3S}e^{-\frac{2\tau}{3}}(27 - 36\tau + 8\tau^2). \]

(1.34)

To find the coefficients \( B_{nS} \), one has first to normalize the \( m_{nS} \) functions against the electron mass \( m_e \):

\[ \int m_{nS}dV = m_e. \]

(1.35)

Integration yields the following expressions for the \( B_{nS} \) coefficients

\[ B_{1S} = \frac{m_e}{\pi a^3}, \]

\[ B_{2S} = -\frac{m_e}{4\pi a^3} \cdot \frac{1}{4}, \]

\[ B_{3S} = \frac{m_e}{4\pi a^3} \cdot \frac{4}{27 \cdot 27 \cdot 3}. \]

(1.36)

Thus, we have found the shape of the mass distribution. To be precise, this is not the shape of the mass distribution but rather that of the charge/mass object.
What’s more, examining Eqs. (1.15) and (1.36) we see that the distributions of charge and mass are related through

\[ m(r) = -\frac{me}{e} \rho(r). \]  

(1.37)

In what follows, in all cases where the term “charge” or “mass” appears it will be assumed that in actual fact this is the charge/mass object.

A comment is appropriate here. The mass distributions (1.34) are normalized against the electron mass, i.e., they are always positive. Equations (1.34) contain, however, alternating polynomials, with the result that within some intervals the mass may acquire a negative sign. There is nothing particular in this from the mathematical side of view, but theoretical mechanics does not operate with such a notion as negative mass. Most probably, the “negativeness” of the mass may physically come up in combination with some other parameters, for instance, in the momentum or angular momentum. The momentum and angular momentum may have either sign, and the negative sign means only motion or rotation in the opposite direction.

Besides the shape of the mass distribution, one can envisage some other parameters as well which are connected with mass. Indeed, electron in an atom possesses kinetic energy. Hence, its mass should move somehow. The only internal motion allowed for an atom is rotation. In other words, we have to find the form of this rotation. But neither Eq. (1.33) nor Eq. (1.5) can yield the form of rotation, because these relations do not contain any parameters of motion at all. But some form of rotation has to be present in an atom. There is, however, another consideration.

By the theorem of Earnshaw, a stable static configuration of electric charges cannot exist without involvement of any other forces of other than electric origin. For this reason, static charge distributions (1.12) are intrinsically unstable. But the atom is stable. Only rotation can impart stability to the atom. Hence, the charges described by Eqs. (1.12) should rotate.

Motion (considered in nonrelativistic approximation) cannot change the potential energy of a distributed charge. The potential energy of a rotating charge distribution can be readily found assuming the charge being at rest. Indeed, any charge which has departed from a given point is replaced immediately by an identical charge at this point. Potential energy depends only on the magnitude of the charge at a given point and is independent of whether this is the charge that has just left or the second (that has arrived).

Let us look now for the overall pattern of rotation of a distributed charge and a distributed mass. What are the basic considerations we should start from? First, we know from quantum mechanics the expression for the kinetic energy of a rotating electron \( E_{\text{kin}} \) (derived, for instance, from the relation \( E_{\text{kin},nS} = \int \Psi_{nS}^* \hat{H} \Psi_{nS} dV \), where \( \hat{H} = \frac{\hat{p}^2}{2m} \) is the kinetic energy operator, and \( \hat{p} = -i\hbar \nabla \) is the momentum operator):

\[ E_{\text{kin},nS} = \frac{e^2}{2a} \cdot \frac{1}{n^2}. \]  

(1.38)

Second, quantum mechanics offers the following expression for the angular momentum of an electron in the \( S \) states, \( M_{nS} \):

\[ M_{nS} = \text{const} = \hbar/2, \]  

(1.39)
CHAPTER 1. SPHERICAL ATOM

because an electron has in the $S$ states only the spin moment $\hbar/2$. (Recall
that nonrelativistic quantum mechanics required that the angular momenta of
$S$ states be zero, while experiment showed them to be $\hbar/2$ rather than zero.)
Thus, rotation should be such as to satisfy at least these two conditions, (1.38)
and (1.39).

The simplest assumption that comes immediately to mind is as follows. The
distributions of mass and charge rotate as a whole, i.e., as a continuous body.
Let us assume that rotation occurs around a vertical axis ($\vartheta = 0$). Let us call
it for convenience the $Z$ axis, and the plane $\vartheta = \pi/2$, the equatorial plane. In
this case, the velocity $v_\varphi$ of motion of any point is proportional to the radius
and the sine of the angle $\vartheta$:

$$v_\varphi = Ar \sin \vartheta = Aa \tau \sin \vartheta,$$

(1.40)

where $A$ is a constant. The above suggests that as one moves away from the
axis of rotation, $v_\varphi$ increases, and as one approaches this axis, it decreases.

Let us verify that this assumption meets conditions (1.38) and (1.39). Sub-
stituting the expressions for the mass (1.34) in the relation

$$\mathcal{E}_{\text{Kin},nS} = \int \frac{m_n S v_\varphi^2}{2} dV,$$

(1.41)

where the element of volume $dV$ is defined by Eq. (1.14) we come to

$$\mathcal{E}_{\text{Kin},1S} = m_e A^2 a^2 \cdot 1, \quad \mathcal{E}_{\text{Kin},2S} = m_e A^2 a^2 \cdot 8, \quad \mathcal{E}_{\text{Kin},3S} = m_e A^2 a^2 \cdot 57.$$  (1.42)

We calculate now the angular momentum around the $Z$ axis ($\vartheta = 0$)

$$M_{nS} = \int m_n S v_\varphi r \sin \vartheta dV.$$  (1.43)

For the momenta $M_{nS}$ we obtain

$$M_{1S} = m_e A a^2 \cdot 2, \quad M_{2S} = m_e A a^2 \cdot 16, \quad M_{3S} = m_e A a^2 \cdot 114.$$  (1.44)

Examining now Eqs. (1.42) and (1.44), we see that the energies $\mathcal{E}_{\text{Kin},nS}$ do
not scale as $1/n^2$, and that the angular momenta $M_{nS}$ are in no way constant.
Said otherwise, the conditions (1.38) and (1.39) are not met. But this means
that the distributions of charge and mass cannot rotate as a whole, i.e., as a solid body.

To establish the pattern of rotation of the charge and mass that can exist in
an atom, the following reasoning appears to be appropriate. Consider circular
rotation of an element of mass $dm$ with a negative charge $dq$ in the equatorial
plane about a nucleus with charge $(+e)$ as about the center.

Rotation of an element of mass $dm$ along a circle of radius $R$ is driven by
the action on the element of mass of a centripetal force $d\mathbf{F}$ toward the center:

$$d\mathbf{F} = -\frac{dm v_\varphi^2}{R} \mathbf{n},$$

where $v_\varphi$ is the velocity of the element of mass, and $\mathbf{n}$ is the
unit vector in the direction of the radius $R$. The negative sign of the centripetal
force $d\mathbf{F}$ signifies that the force is directed oppositely to the unit vector $\mathbf{n}$. In
our particular case, the centripetal force is essentially the force of attraction
between the charges, \( \mathbf{dF} = \frac{edq}{R^2} \mathbf{n} \). Equating these two expressions, canceling \( R \) and dividing by two, we finally obtain

\[
\frac{dmv^2}{2} = \frac{1}{2} \left( -\frac{edq}{R} \right).
\] (1.45)

We see on the left the kinetic energy of the element of mass, and on the right, one half of the potential energy taken with the opposite sign. This means that circular rotation of an element of mass \( dm \) obeys the relation

\[
\mathcal{E}_{\text{Kin}} = \frac{1}{2} (\mathcal{E}_{\text{Pot}}).
\] (1.46)

The equality (1.46) formulates the Clausius theorem on the virial of forces mediating circular motion of an element of mass in a Coulomb potential well (see, e.g., [5], p. 76). Because the potential, kinetic, and total energies of an element of mass moving circularly are constant, there is no need in averaging the energies, as this would be required by the theorem of virial in a general form.

Let us write now the theorem of virial for an element or sum of elements of mass/charge moving in a Coulomb potential well in a general form (see, e.g., [5]):

\[
\overline{\mathcal{E}_{\text{Kin}}} = \frac{1}{2} \overline{(-\mathcal{E}_{\text{Pot}})},
\] (1.47)

where the bar on top signifies averaging over time.

Equation (1.45) yields the velocity of circular motion of an element \( dm \) (at the equator)

\[
v_\varphi = \sqrt{-\frac{e}{R} \frac{dq}{dm}},
\] (1.48)

and, combined with Eq. (1.37), we finally have

\[
v_\varphi = \frac{e}{\sqrt{m_e}} \frac{1}{\sqrt{R}},
\] (1.49)

i.e., the velocity of an element of mass is inversely proportional to the square root of the distance between the nucleus and the element. Thus, as one approaches the axis of rotation, the velocity increases, and as one moves away from it, the velocity decreases.

Because at the equator the radius \( R \) of the circle along which the element of mass moves coincides with the coordinate \( r \) of the spherical reference frame, Eq. (1.49) can be recast in the form

\[
v_\varphi = \frac{\alpha c}{\sqrt{r}}.
\] (1.50)

In this expression, the radius of the circle \( R = r = a\tau \), where \( a \) is, as before, the Bohr radius: \( a = \frac{\hbar^2}{m_e c^2} \), \( \alpha \) is the fine structure constant: \( \alpha = \frac{e^2}{\hbar c} \), and \( c \) is the velocity of light.

Finding the velocity of an element of mass not lying in the equatorial plane meets with some difficulties. In this case, the centripetal force does not coincide in direction with the force of attraction to the nucleus, and this makes the above reasoning invalid here.
To describe the rotation of a distributed charge as a whole, we can make two assumptions: 1. Rotation occurs in such a way that linear velocity $v_\phi$ of each element depends only on its distance from the nucleus; 2. Rotation occurs such that elements located on the sphere of radius $r$ have the same angular velocity. In the first case, the relation for the velocity of motion of elements of a distributed charge can be written in the way similar to Eq. (1.50)

$$v_\phi = \frac{k' \alpha c}{\sqrt{\tau}}.$$  \hspace{1cm} (1.51)

In the second case, the equation assumes the form

$$v_\phi = \frac{k'' \alpha c}{\sqrt{\tau}} \sin \vartheta.$$  \hspace{1cm} (1.52)

Here $k'$ and $k''$ are some coefficients. In both cases, the charge rotates in a layered pattern depending on the radius $r$, with the velocity of rotation being the higher, the closer is the charge element to the nucleus.

Thus, of the two versions of the velocity, (1.51) and (1.52), we have to choose the right one.

Substituting the velocity from Eq. (1.51) in Eq. (1.41), we obtain

$$E_{\text{Kin},1S} = \frac{e^2}{2a} \cdot \frac{1}{1}, \quad E_{\text{Kin},2S} = \frac{e^2}{2a} \cdot \frac{1}{4}, \quad E_{\text{Kin},3S} = \frac{e^2}{2a} \cdot \frac{1}{9}.$$  \hspace{1cm} (1.53)

Substituting now the velocity from Eq. (1.52) in Eq. (1.41), we come to

$$E_{\text{Kin},1S} = \frac{e^2}{2a} \cdot \frac{2}{3} \cdot \frac{1}{1}, \quad E_{\text{Kin},2S} = \frac{e^2}{2a} \cdot \frac{2}{3} \cdot \frac{1}{4}, \quad E_{\text{Kin},3S} = \frac{e^2}{2a} \cdot \frac{2}{3} \cdot \frac{1}{9}.$$  \hspace{1cm} (1.54)

An analysis of Eqs. (1.53) and (1.54) suggests that the energies $E_{\text{Kin},nS}$ scale as $1/n^2$, as required by Eq. (1.38).

The virial theorem (1.47) permits us now to find the $k'$ and $k''$ coefficients. We come eventually to $k' = 1$, $k'' = \sqrt{3}/2$.

For the kinetic energy we obtain in the two cases the following expressions

$$E_{\text{Kin},1S} = \frac{e^2}{2a} \cdot 1, \quad E_{\text{Kin},2S} = \frac{e^2}{2a} \cdot \frac{1}{4}, \quad E_{\text{Kin},3S} = \frac{e^2}{2a} \cdot \frac{1}{9}.$$  \hspace{1cm} (1.55)

Equations (1.55) can be combined to yield

$$E_{\text{Kin},nS} = \frac{e^2}{2a} \cdot \frac{1}{n^2}.$$  \hspace{1cm} (1.56)

Equation (1.56) coincides exactly with Eq. (1.38) for the kinetic energy of an electron, which is well known from quantum mechanics.

Let us determine now the total energy of the electron. By adding the potential energy (1.17) just found with the kinetic energy (1.55), we arrive at the total electron energy

$$E_{1S} = \frac{e^2}{2a} \cdot 1, \quad E_{2S} = -\frac{e^2}{2a} \cdot \frac{1}{4}, \quad E_{3S} = -\frac{e^2}{2a} \cdot \frac{1}{9}.$$  \hspace{1cm} (1.57)
Expressions (1.57) can now be combined

\[ E_{nS} = -\frac{e^2}{2a} \cdot \frac{1}{n^2}, \]  

(1.58)

in an expression for the total electron energy, likewise well known from quantum mechanics.

Thus, basing on the formulas and methods of electrodynamics and mechanics and applying the above approach, we have come to absolutely correct values of the potential, kinetic, and total energies of an electron in a hydrogen atom. Significantly, in so doing we have not invoked such purely quantum-mechanical concepts as the wave function and the operator.

Let us calculate now the angular momentum of a distributed mass about a vertical axis \( Z \) (\( \vartheta = 0 \)).

Substituting the velocity \( v_\varphi \) from Eqs. (1.51) and (1.52) into the expression for the angular momentum (1.43) we come, respectively, to

\[ M_{1S} = 0.92\hbar, \quad M_{2S} = 1.63\hbar, \quad M_{3S} = 2.40\hbar, \]  

(1.59)

and

\[ M_{1S} = 0.96\hbar, \quad M_{2S} = 1.70\hbar, \quad M_{3S} = 2.49\hbar. \]  

(1.60)

To calculate the radial integrals, we have to use now, in place of Eq. (1.18) valid for an integer exponent \( n \), a more general formula

\[ \int_0^\infty e^{-ax}x^n \, dx = \frac{\Gamma(n+1)}{a^{n+1}} \quad (\text{при } a > 0 \quad n > -1), \]  

(1.61)

because the exponent of \( x \) assumes not integer but rather half-integer values. \( \Gamma(n+1) \) is the gamma function.

Examining now Eqs. (1.55) and (1.56), we see that we have obtained correct values for the kinetic energy of the electron and a correct dependence on number \( n \). As for the angular momenta, they are not equal to \( \hbar/2 \), and, more than that, they are not equal to a constant value at all.

Hence, the assumptions concerning the velocity of rotation of a distributed charge are not accurate, and this problem requires a further study.

In Chapter 2 of this Paper, the velocity of rotation of a distributed charge is analyzed in more detail, and it is demonstrated that the angular momentum of the ground state is \( \hbar/2 \), in full agreement with experimental observations.
Chapter 2
Non-spherical atom

2.1 Shape of a non-spherical charge

As shown in Chapter 1 of the present Paper, if the electron charge distribution is assumed to be spherically symmetric, one cannot obtain a correct value of the projection of angular momentum on the $Z$ axis. One may therefore suggest that the distribution of the charge/mass does depend somehow on the angle $\vartheta$. We have not as yet, however, an equation more accurate than Eq. (1.5) of wave mechanics and, therefore, we cannot know a more accurate solution. This implies that the assumption concerning the actual form of the dependence of the distribution on the angle $\vartheta$ will have to be chosen intuitively. This also means that we can no longer use Eq. (1.9). We are going to employ, however, the main conclusions derived by means of this equation. We shall apply, in particular, the radial dependence of charge density which was derived for a spherically symmetric charge distribution.

Consider the hydrogen atom in ground state.

The simplest assumptions that appear reasonable in this case consist in that the distributed charge scales with $\vartheta$ as $\sin \vartheta$ or $\sin^2 \vartheta$. In these conditions, the charge density can be written as follows

\[ \rho'_{1\text{NS}} = A'_{1\text{NS}} \cdot e^{-2\tau} \cdot \sin \vartheta, \quad (2.1) \]

or

\[ \rho''_{1\text{NS}} = A''_{1\text{NS}} \cdot e^{-2\tau} \cdot \sin^2 \vartheta. \quad (2.2) \]

Here the subscript $1\text{NS}$ identifies the state which is similar to $1S$ but not spherically symmetric.

The coefficients $A'_{1\text{NS}}$ and $A''_{1\text{NS}}$ are found by normalization against the electron charge ($-e$)

\[ \int \rho'_{1\text{NS}} dV = -e, \quad \int \rho''_{1\text{NS}} dV = -e. \quad (2.3) \]

We finally arrive at

\[ A'_{1\text{NS}} = -\frac{4e}{\pi a^2}, \quad A''_{1\text{NS}} = -\frac{3e}{2\pi a^2}. \quad (2.4) \]
One may as reasonably assume that the angular dependence can be a combination of several spherical functions $Y_{lm}$. As shown in Chapter 1, the $l = 0$ spherical function must be present in an atom always, because it is only this function that permits description of the charge in an atom. All the other functions are capable of affecting the charge shape only, without adding or subtracting any charge.

What other functions could be used to describe a distributed charge in an atom? These functions should be symmetric with respect to the angle $\vartheta = \pi/2$; indeed, there are no grounds to assume that an atom can be asymmetric relative to the equator, unless some additional external fields interfere. Besides, we can select for description of the charge only functions with $m = 0$. Functions with $m \neq 0$ have no axial symmetry relative to the $Z$ axis; therefore, any rotation of such an asymmetric charge should give rise to emission of radiation, which in actual fact does not happen.

The simplest function satisfying these requirements is $Y_{20}$. Therefore, we write the angular part of the formula for the charge distribution in the following form:

$$L = D(Y_{00} + D_{20}Y_{20}),$$  \hspace{1cm} (2.5)

where $Y_{00} = \frac{1}{\sqrt{4\pi}}$, $Y_{20} = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \vartheta - \frac{1}{2} \right)$, (see, for instance, Ref. [2]).

Coefficient $D_{20}$ can be found from the condition that function $L$ for $\vartheta = 0$ and $\vartheta = \pi$ (i.e., at the $Z$ axis) be zero. Indeed, if a charge rotates about the $Z$ axis, all elements of charge not on the $Z$ axis are acted upon by both attraction to the nucleus and the centrifugal force. On the $Z$ axis, the centrifugal force is zero, therefore the only possibility for existence of a distributed charge lies in the absence of charge on the $Z$ axis.

Coefficient $D$ will be found from the condition that function $L$ for $\vartheta = \pi/2$ (i.e., at the equator) be unity. We obtain from these conditions

$$D = \frac{2\sqrt{4\pi}}{3}, \quad D_{20} = -\frac{1}{\sqrt{5}}.$$

Thus, function $L$ acquires the form

$$L = \frac{2\sqrt{4\pi}}{3} \left[ \frac{1}{\sqrt{4\pi}} - \frac{1}{\sqrt{5}} \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \vartheta - \frac{1}{2} \right) \right].$$

One can easily verify that this function exactly coincides with the function $\sin^2 \vartheta$. In other words, we obtain in this case two forms for the density of distributed charge:

$$\rho''_1^{\text{NS}} = A''_1^{\text{NS}} e^{-2\tau \sin^2 \vartheta},$$  \hspace{1cm} (2.6)

or

$$\rho''_1^{\text{NS}} = A''_1^{\text{NS}} e^{-2\tau} D(Y_{00} + D_{20}Y_{20}).$$  \hspace{1cm} (2.7)

One can chose conveniently the form most suitable for the actual conditions.

Let us see whether we obtain a correct potential energy of interaction of the nucleus with these forms of distributed charge. The potential produced by a distributed charge at the nucleus, i.e., at the origin, can be written as

$$\Phi'(0) = \int \frac{\rho''_1^{\text{NS}}}{r} dr = \int \frac{\rho''_1^{\text{NS}}}{a\tau} dr, \quad \Phi''(0) = \int \frac{\rho''_1^{\text{NS}}}{r} dr = \int \frac{\rho''_1^{\text{NS}}}{a\tau} dr.$$  \hspace{1cm} (2.8)
We consider the nucleus to be a point at the origin, with the charge \((+e)\). Therefore, for the energy of interaction of the nucleus with the distributed electron charge can be written as

\[ E'_{\text{Pot}} = e\Phi'(0), \quad E''_{\text{Pot}} = e\Phi''(0). \tag{2.9} \]

One can easily verify that in both cases we arrive at the same result:

\[ E'_{\text{Pot}} = E''_{\text{Pot}} = \frac{-e^2}{a}, \tag{2.10} \]

which coincides with the energy of the spherically symmetric 1S state \((1.17)\), as well as with the results known from quantum mechanics.

Let us calculate now the kinetic energy of a rotating charge. On repeating the arguments concerning the mass distribution formulated for a spherically symmetric charge (Sec. 1.5), we come to the conclusion that the mass distribution corresponding to the distribution of charge (2.1) has the form

\[ m'_{1\text{NS}} = B'_{1\text{NS}} e^{-2\tau \sin \vartheta}, \tag{2.11} \]

and that the distribution of mass corresponding to the charge distribution (2.6) or (2.7) reads as

\[ m''_{1\text{NS}} = B''_{1\text{NS}} e^{-2\tau \sin^2 \vartheta}, \tag{2.12} \]

or

\[ m''_{1\text{NS}} = B''_{1\text{NS}} D(Y_{00} + D_{20} Y_{20}). \tag{2.13} \]

The coefficients \(B'_{1\text{NS}}\) and \(B''_{1\text{NS}}\) are found by normalizing the mass density by the electron mass \(m_e\). In this way we obtain for the \(B'_{1\text{NS}}\) and \(B''_{1\text{NS}}\) coefficients

\[ B'_{1\text{NS}} = \frac{4m_e}{a^3 \pi^2}, \quad B''_{1\text{NS}} = \frac{3m_e}{2\pi a^3}. \tag{2.14} \]

This distribution rotates about the \(Z\) axis.

Let us find now the rotation velocity \(v_{\phi}\).

Two versions of the rotation velocity, \((1.51)\) and \((1.52)\), were proposed for the spherical charge distribution. If a charge distribution has no spherical symmetry (Eqs. (2.1) or (2.2)), there is no charge near the \(Z\) axis. This means that we may remove now the assumption of the angular velocity for charges on a sphere of radius \(r\) being constant (as this was done in Chapter 1 of the Paper) and retain only the dependence of the velocity on radius. The dependence of the velocity on radius only can be assigned to the fact that the only source of the force propelling the motion of a distributed charge is the nucleus. Therefore (in contrast to Eq. \((1.52)\)), we write the expression for the velocity of a distributed charge, similar to Eq. \((1.51)\), for \(k' = 1\) in the following form

\[ v_{\phi} = \frac{\alpha e}{\sqrt{\tau}}, \tag{2.15} \]

where \(\alpha\) is the fine structure constant, \(c\) is the velocity of light, and \(\tau = r/a\) is the reduced radius.

Equation \((2.15)\) shows that the velocity \(v_{\phi}\) increases as one approaches the nucleus. The charge rotates, as before, in a stratified fashion, but now it is the
linear velocity $v_\phi$ along the $\phi$ coordinate rather than the angular velocity that depends on radius. We meet, however, as before, with a difficulty of identifying the factor that causes the motion of charges along the $\phi$ coordinate if the force center does not lie in the $\phi$ orbital plane (except for the equator). We still have not got, however, any other pattern of rotation to compare.

Calculate now the kinetic energy of a rotating charge. The kinetic energy $E_{Kin}$ can be written as

$$E_{Kin} = \int \frac{mv^2 \phi^2}{2} dV.$$  (2.16)

Substituting Eqs. (2.11), (2.12), and (2.15) in Eq. (2.16), we come to

$$E'_{Kin} = E''_{Kin} = \frac{1}{2} \cdot \frac{e^2}{a},$$  (2.17)

which coincides with the formula derived for a spherically symmetric $1S$ charge/mass distribution (1.55) and with the expression known from quantum mechanics.

The total energy $E = E_{Pot} + E_{Kin}$ also turns out to be identical for both charge distribution patterns (see Eqs. (2.10) and (2.17)):

$$E' = E'' = -\frac{1}{2} \cdot \frac{e^2}{a},$$  (2.18)

which coincides with the results derived for a spherically symmetric $1S$ charge distribution (1.57) and the appropriate formulas of quantum mechanics.

Calculate now the angular momenta for these two versions of charge distribution. We assume the charge to rotate about the $Z$ axis, with the velocity determined by Eq. (2.15). The angular momentum is given by the formula

$$M_Z = \int m v_\phi R dV,$$  (2.19)

where $R = r \sin \vartheta = a \tau \sin \vartheta$ is the distance from the element of mass to the $Z$ axis. Formula (2.19) yields

$$M'_Z = h \cdot 0.998, \quad M''_Z = h \cdot 1.039.$$  (2.20)

The integrals (2.19) are expressed in terms of the gamma-function with a half-integer index, unlike the integrals of energy (2.16) containing the gamma-function with an integer index.

Experiments suggest that the angular momentum of the $S$ state is $h \cdot 0.5$. This means that the values specified by Eq. (2.20) disagree with the value known from experiment. One should therefore reconsider the process of charge/mass rotation in more detail.

### 2.2 Potentials of a non-spherical charge distribution

In Sec. 2.1, the energies of the $1NS$ state of the hydrogen atom are calculated under the assumption that a distributed charge does not have spherical symmetry. Experiment suggests, however, that the atom is spherically symmetric. It appears now reasonable to study what consequences would ensue from the
assumption of the hydrogen atom being not spherically symmetric, and how such an atom would look to an observer. To do this, we have to calculate the potentials of distributed charges and compare them with the Coulomb potential of the nucleus. The potentials of a distributed charge can be calculated from an expansion in spherical harmonics. The formulas pertaining to a search of a potential with the use of such an expansion can be found, for instance, in Ref. [4]. As demonstrated in Sec. 2.1 of the present Paper, the distribution of the kind of Eq. 2.2 can be expressed through two spherical harmonics, \( Y_{00} \) and \( Y_{20} \); therefore, the series will be truncated with only four terms left (two terms for \( r > r' \) and two terms with \( r < r' \)). The function \( \sin \vartheta \) being not a member of the \( Y_{lm} \) system of spherical functions, the series for the potential is not truncated. As shown in Sec. 2.1, a charge distribution can be fully described by functions with \( m = 0 \) only. Apart from this, these functions should be symmetric with respect to the equator. The second condition suggests that a spherical function can have only even order \( l \). The functions \( Y_{00} \) representing essentially Legendre polynomials were taken from Ref. [6]. Five harmonics were taken for calculation of the potential: \( Y_{00}, Y_{20}, Y_{40}, Y_{60}, Y_{80} \). In this case, the part of the multipole moment which depends on the \( Y_{80} \) function amounts to 0.01 of the moment depending on \( Y_{00} \). The formulas employed in the calculation are specified in Appendix.

The main results of the calculations can be visualized in Figs. 2.1–2.6. The notation accepted is as follows. The radius of the spherical reference frame \( r \) is given in units of the Bohr radius \( a \), i.e., \( r \) in the graphs is actually the parameter \( \tau \) in all of the above formulas. The potentials \( U \) are expressed in units of \( e/a \). The potential \( U(r, \vartheta) \) is the total potential deriving from the whole set of the harmonics involved. \( U(0) \) is the potential deriving only from the spherically symmetric harmonic \( Y_{00} \). Expressed in this notation, the Coulomb potential of a nucleus \( N(r) \) reads as \( 1/r \).

Figure 2.1 plots the potential of a distributed charge, Eq. 2.1, vs. distance \( r \) from the nucleus for three angles: \( \vartheta = 0, \vartheta = \pi/4, \vartheta = \pi/2 \). Shown in Fig. 2.2 is the same graph for the (2.2) distribution. We readily see that the potentials do not depend on the angle \( \varphi \) at all.

Figure 2.3 illustrates for the charge distribution 2.1 the dependence on distance \( r \) from the nucleus of both the potential \( U(0) \) formed by the spherically symmetric harmonic \( Y_{00} \) only and of the total potential \( U(r, \vartheta) \) involving all the harmonics included for two angles, \( \vartheta = 0 \) and \( \vartheta = \pi/2 \). Figure 2.4 shows the same plot constructed for the charge distribution 2.2. Examining these graphs we see that the potentials are formed in both cases mostly by the spherically symmetric harmonic, the contribution of the other harmonics being very small. This is immediately evident though from Figs. 2.1 and 2.2: indeed, if the potential is not small and the angular dependence is very weak, the only conclusion can be that the potential derives primarily from the spherically symmetric harmonic.

Thus, in spite of the charge distributions 2.1 and 2.2 being different from spherical, the potentials produced by these charges deviate very little from the spherical pattern. Therefore, such an atom would superficially look as spherically symmetric.

Consider now the extent to which the spherically symmetric harmonic of the potential of a distributed charge \( U(0) \) affects the potential of the atom. An
Figure 2.1:

Figure 2.2:
CHAPTER 2. NON-SPHERICAL ATOM

Figure 2.3:

Figure 2.4:
CHAPTER 2. NON-SPHERICAL ATOM

Figure 2.5:

Figure 2.6:
atom carries the total potential, i.e., the potential of the nucleus plus that of the distributed charge. Because the distributed charge potential primarily derives from the spherically symmetric harmonic \(U_0(r)\), it is only this harmonic that we shall consider in this sum.

Figure 2.5 plots the total potential of a spherically symmetric harmonic of charge (2.1) plus the Coulomb potential of the nucleus \(N(r)\). The dashed curve shows only the Coulomb potential of the nucleus \(N(r)\). We immediately see that for \(r > 2\) the total potential is very small (which certainly derives from the fact that the field vanishes at some distance from the atom), but it approaches the Coulomb potential of the nucleus as one comes closer to the latter. Figure 2.6 plots the same graph for the charge distribution (2.2). Significantly, the potential generated by the spherically symmetric harmonic \(U_0(r)\) coincides with the potential \(\Phi_{1S}\) of the spherical charge distribution (see Eqs. (1.26) and (1.27)).

Thus, despite the absence of spherical symmetry in the charge distribution, the atom mostly preserves the main features of spherical symmetry; indeed, the deviation of the potential from the spherically symmetric pattern is very small, and as one comes closer to the nucleus, it approaches the Coulomb potential.

### 2.3 Analysis of possible patterns of motion of a distributed charge

As can be seen from Eqs. (2.20), we have not obtained the correct value for the angular momentum of the hydrogen atom in ground state. This stresses the need for reconsidering the process of mass/charge rotation in more detail.

Equation (2.15) shows essentially that each element of charge rotates circularly about the \(Z\) axis. But why should each element of charge rotate in a circle only? Each element of the distributed charge is confined to the Coulomb potential well of the nucleus (we are disregarding as yet the additional potential of the distributed charge itself; as follows from Sec. 2.2, the difference of the total from the Coulomb potential in an atom is small). The behavior of a charge in a Coulomb potential well is known. Monograph [7] could be best suited for our purposes.

Consider the situation in two stages. We shall first be interested in the motion of a charge at the equator.

Recall some well established facts.

In a Coulomb potential well, a constant element of charge \(dq\) with a mass \(dm\) can move along a circle or an ellipse, and the ellipse can degenerate into a straight line. The energy of an element of charge depends on the semimajor axis of the ellipse (or on the radius of the circle). The actual shape of the ellipse (i.e., its semiminor axis) depends on the angular momentum of the particle. Thus, all ellipses with the same semimajor axis but different semiminor axes have the same energies but different angular momenta, down to the zero momentum (in which case the ellipse degenerates into a straight line). Said otherwise, mass/charge elements of the same energy can move in a Coulomb potential well along trajectories which differ in the value of the angular momentum.

Because we have a distributed electron charge in an atom, it appears only logical to assume that each element of charge can move along any allowed trajec-
tory (by an allowed trajectory we understand here any trajectory satisfying the laws of mechanics). Note, however, that different trajectories (different ellipses) intersect. Therefore, when introducing the assumption that elements of charge can move along different allowed trajectories, we have to accept another one as well, namely, that each element of mass/charge can move along its trajectory regardless of those of other charges. In other words, charges may pass through one another without an attendant change of the trajectory. Said otherwise, each element of mass/charge moves in the force field independently of other elements of mass/charge. It goes without saying, that all elements surrounding the element under consideration contribute to the force field.

This would seem to be contrary to the observation that charges interact with one another. But the statement that each mass/charge element moves in a force field independently of other mass/charge elements is just a consequence of the fact that we consider interaction of charges not directly with one another but rather through the field; indeed, each charge interacts with the field generated by another charge or by all the other charges. In other words, we consider charge motion in a force field created by other charges, putting the existence of these charges apart.

Like charges repel one another. Therefore, the assumption that likely charged elements of charge can pass through one another may seem a far-fetched idea. Indeed, the point charges one usually considers have an infinite density at the point where the charge is located. Therefore, like point charges cannot pass through one another; more than that, they cannot even approach one another close enough. The distributed charges treated by us here have a finite charge density. Such charges can penetrate into one another, depending on what external forces act on these charges and what are the forces created by these charges.

In actual fact, the statement that charges can pass through one another does not carry anything supernatural in it. For instance, electromagnetic fields can penetrate one into or through the other without at the same time affecting one another—this is nothing but the standard principle of superposition. Two radar beams can cross without interaction; this is just penetration of ac fields through one another. Superposition of one dc field on another (the principle of superposition) may be regarded as penetration of one field into another. Significantly, in this process the fields do not act in any way on one another.

As for the charges, no statements concerning passage of one charge through another without direct action on one another (interaction of charges is taken into account through the fields created by these charges) have thus far been made, although the principle of superposition is valid for charges as well. This statement should, however, be made. If electromagnetic fields do pass through one another, there would appear nothing strange in admitting that charges likewise can do it. The difference between these statements lies in that ac electromagnetic fields propagate along rectilinear trajectories (trajectories (beams) are straight lines (in vacuum)), while charges move along their trajectories in a potential field. The actual shape of the trajectory is determined by the potential field in which this element moves, as well as by the parameters of this element. A trajectory can be calculated in the frame of theoretical mechanics. In our case, the trajectories along which an element of distributed charge/mass moves in the field of the nucleus are closed curves rather than straight lines.

Consider in more detail the motion of an element of charge in an atom.

Assume an element of charge $dq$ located in the equatorial plane. Consider
the trajectories in moving along which the charge $dq$ has the same total energies (in this case all elliptical trajectories have the same semimajor axes). This element can move in a circular or an elliptical trajectory in the equatorial plane, with the total energy of this element in any trajectory being the same, and only angular momenta different (see, e.g., Ref. [7]). Each angular momentum can be identified with its own elliptical trajectory. Because in all trajectories the element of charge $dq$ has the same energy, this element of charge can move along any trajectory. Moreover, this element of charge can move in all trajectories at the same time. This can be visualized in the following way. Divide element of charge $dq$ in $k$ parts. Then one element of charge $dq' = dq/k$ can move along one elliptical trajectory, another charge $dq'$, along another trajectory, and so on. As $k$ tends to infinity, all the trajectories will criss-cross all of the allowed region containing trajectories of the elements of charge $dq'$ of the same energy but with different angular momenta. Generally speaking, this process may be considered not as motion of elements of charge along trajectories but rather as motion of a continuous medium, of a charge wave.

Let us analyze the various trajectories along which an element of charge $dq$ with a mass $dm$ can move in the case where the total energy of the element in each trajectory is the same.

Figure 2.7 illustrates several such orbits of all possible ones: a circular orbit $a$, eight elliptical orbits $(b - k)$ with different eccentricities (and, hence, different angular momenta), and a linear orbit $l$ into which the ellipse degenerates at an eccentricity of unity. This orbit passes through the nucleus of the atom. All the orbits are characterized by identical semimajor axes (if the energies of the elements are equal, the semimajor axes of the ellipses should likewise be equal). All orbits lie in the same plane. The elements of charge in all orbits rotate in the same direction.

All orbits focus at the same point. In this focus (in our figure, this is the
center $O$ of the circle) the nucleus of the atom is located. Using the focal properties of ellipses, one can readily show that each elliptic trajectory intersects a circular orbit at the point where this ellipse intersects its semiminor axis. The dashed lines confine the region of allowed trajectories along which an element $dm$ can move.

As already mentioned, an element of charge $dq$ with a mass $dm$ can move along all of the above trajectories simultaneously to form not propagation of single particles but rather a wave motion. For this to become possible, the element of charge $dq$ has to split into a multitude of parts $dq'$, each of them moving in its own trajectory.

An element of mass/charge residing in a Coulomb potential well moves in one plane. We have considered motion in the equatorial plane. Note, however, that through a line connecting the element under study with the nucleus one can pass an infinite number of planes. The trajectory of a given element may lie in any of these planes, because Coulomb field possesses spherical symmetry. Moreover, this element can be divided into a multitude of parts, and each part of the element can move in a trajectory in its plane. It thus appears that the element of charge under consideration can move along all of the allowed trajectories in all planes simultaneously. It would apparently be more appropriate to speak here not of the motion of a set of mass/charge elements but rather of that of a wave propagating within a certain solid angle. The solid angle is defined by the set of all allowed trajectories.

This reasoning can be repeated for any element of charge. Each element can be divided into parts, and these parts will propagate in a certain solid angle. What we will have actually is propagation within a certain solid angle of a charge wave, or, to express it more properly, a mass/charge wave.

This reasoning resembles in a large measure the Huygens–Fresnel principle. By the Huygens–Fresnel principle, each point of a propagating wave acts as a source of a secondary wave, and the front of the propagating wave may be visualized as an envelope of all the secondary waves.

The difference of the consideration offered here from the Huygens–Fresnel principle lies essentially in that by the latter principle a wave propagates along a straight line, i.e., the rays of all secondary waves are straight lines. In our case, the trajectories associated with the propagation of charge waves, rather than being straight lines, are mediated instead by the potential of the field in which the element under consideration moves and by the parameters of the element itself. In the cases of interest here, these trajectories are closed curves.

The similarity with the Huygens–Fresnel principle lies in that any point of the distributed charge is a center from which a mass/charge wave propagates within a certain solid angle.

The above pattern may be considered as an attempt to find common features between the corpuscular and wave patterns of behavior. There are even grounds to suggest that the corpuscular and wave concepts actually merge. The grounds underlying this statement may be seen in that the behavior of a distributed charge can be studied in two different contexts. The behavior of each single mass/charge element obeys all laws of theoretical mechanics. Its motion can be calculated with the use of equations derived in theoretical mechanics. On the other hand, when one considers the motion of all elements taken together, it is the motion of a wave. This wave should obey certain partial differential equations involving the effect of potentials on the motion of the “charge waves”.
These equations have not thus far been constructed. But it is with partial differential equations that the motion of “charge waves” is most appropriate to analyze, and this stresses the need for constructing such equations. When such equations allowing for the effect of potentials on “charge wave” motion are obtained, there will be strong grounds to call the field of science described by these equations wave mechanics, because these equations should take into proper account both the wave properties of objects (originating from specific boundary or periodic conditions) and all the characteristics described by theoretical mechanics. Until this is done, the term “wave mechanics” announced in the title of the Paper should be treated rather as an expression of wishful thinking on the part of the author.

One should also attempt to apply the enormously vast amounts of knowledge amassed in theoretical mechanics for point objects to description of the behavior of a continuous medium. We have to admit, however, that the behavior of a continuous medium could be described more adequately by the “wave” formalism, i.e., through the use of partial differential equations.

2.4 Angular momentum of the 1NS state

We turn now to calculating the angular momentum of the ground state of a hydrogen atom. We start by dividing the volume of the distributed mass into elements of magnitude $dV$ with a mass $dm = ndV$. Next we calculate the angular momenta of each element separately and add them subsequently. We have to keep in mind that each element of mass $dm$ can move along different elliptical orbits. Therefore, we have to consider in the beginning the angular momentum of one element of mass.

A Coulomb potential well is spherically symmetric. The presence of a distributed charge, which is anything but spherically symmetric, distorts the symmetry of this potential. As shown in Sec. 2.2, however, the presence of this charge affects very little the potential. This gives us grounds to assume in what follows that elements of charge move in a Coulomb potential field.

Calculate the angular momentum of a charge/mass element subject to the condition that the element moves along different orbits in the same plane but that in all these orbits the energy of the element is the same. We again use the data given in Ref. [7]. The angular momentum $dM$ of a constant element of mass $dm$ in a Coulomb potential well can be written as

$$dM = \frac{2dmf}{T}, \quad (2.21)$$

where $f$ is the area of the orbit, and $T$ is the period of revolution of an element of mass in this orbit. Recall that the period $T$ depends only on the energy of this element with the mass $dm$. Because we consider here orbits of the same energy, the value of $T$ for all the orbits of interest will be the same. In a Coulomb potential well, orbits are actually ellipses; therefore, we obtain $f = \pi xy$, where $x$ is the semimajor, and $y$, the semiminor axes of the ellipse. Because elements of mass in all the orbits of interest to us here have the same energy, the semimajor axes $x$ of all the orbits are identical (the semimajor axis depends on energy only), and the semiminor ones, $y$, are different.
An element placed in a Coulomb potential well moves in one plane only. Consider the motion of an element in one of such planes.

If an element \( dm \) rotates in a circular orbit of radius \( R = x \), for the angular momentum of this element we can write

\[
\frac{dM}{R} = \frac{2dm\pi x^2}{T} = \frac{2dm\pi R^2}{T}.
\]

Calculate now the angular momentum for the case where an element of mass \( dm \) moves in this plane along all orbits at the same time. To do this, divide the element of mass \( dm \) into \( k \) parts: \( dm' = dm/k \) (see Fig. 2.7).

Each element of mass \( dm' \) will move along its elliptical trajectory. For its angular momentum we can write (recall that \( x = \text{const} = R \) for this energy):

\[
\frac{dM'}{T} = \frac{2dm'\pi xy}{T} = \frac{2dm'\pi Ry}{T}.
\]

To calculate the total angular momentum \( dM_{dm} \) of an element \( dm \) moving in all trajectories in the plane under consideration simultaneously, we have to sum all the momenta \( dM' \) (see Eq. (2.23)). Significantly, the parameter \( y \) varies in the process from zero (the case in which the ellipse degenerates into a straight line) to \( R \) (where the ellipse transforms into a circle of radius \( R \)). The element of mass \( dm' \) can be prudently recast to the form \( dm' = dm\frac{dy}{R} \), because the quantity \( R/dy \) is nothing else but the number of parts \( k \) into which the mass \( dm \) was divided. But then the total angular momentum becomes

\[
\frac{dM_{dm}}{R} = \frac{1}{T} \int_0^R 2\pi ydmR \frac{dy}{R} = \frac{1}{T} 2\pi dm \int_0^R ydy = \frac{1}{T} 2\pi dm\pi R^2 \cdot \frac{1}{2}.
\]

Examining Eqs. (2.22) and (2.24), we see that the angular momentum of an element of mass \( dm \), in the case where it moves in all trajectories simultaneously, is only one half that of the element of mass \( dm \) moving in a circular orbit as a whole (the energies of the \( dm \) elements are in both cases the same). In other words, calculation of the angular momentum of an element \( dm \) moving along all allowed elliptical orbits may be replaced by calculation of the angular momentum of the same element but moving along a circular orbit. The necessary condition for this to be valid is that the total energies of the element in the circular and elliptical orbits should be equal (indeed, in both orbits we have the same element). For this to be valid, the semimajor axes of all the ellipses considered should be equal to one another and to the radius of the circle. It may be appropriate to recall that we are speaking here about trajectories confined to one plane. Thus, Eq. (2.24) can be recast in the form

\[
\frac{dM_{dm}}{R} = \frac{dM_R}{2}.
\]

Here \( dM_R \) is the angular momentum of the element \( dm \) rotating along the circle of radius \( R \) in the given plane. \( dM_{dm} \) is the angular momentum of the element \( dm \) moving in all allowed trajectories, likewise in the same plane, with the elements in both cases having equal energies.

We calculated earlier the angular momentum of the distribution of charges rotating about the \( Z \) axis (Eqs. (2.19) and (2.20)). In this case, each element of
charge was rotating in circular orbits whose planes were parallel to the equatorial plane. One just could not conceive at the time of any other pattern of rotation for a charge distribution. In this version of rotation, it was difficult to identify, however, the mechanism accounting for rotation of charges lying outside the equatorial plane, because the plane of their orbits does not pass through the center of force, i.e., the nucleus.

Having allowed for the possibility of charges interpenetrating one another, we could construct a different pattern of rotation, which would appear more natural while not contradicting any laws of mechanics. An element of charge, acted upon by the force of attraction, moves in the Coulomb potential well of the nucleus. This element rotates in actual fact about the nucleus rather than about the $Z$ axis. The trajectories of motion lie in the plane crossing the point where the nucleus is located, i.e., the origin of the coordinate frame. In this case, the trajectories, rather than being parallel to the equatorial plane, can make any angle with it.

A Coulomb potential well is spherically symmetric. As a consequence, the orbit of an element $dm$ may lie in different planes. These planes can be visualized by rotating the original plane about the line connecting the position of the element $dm$ with that of the nucleus. The orbit of an element $dm$ may lie in any of these planes. Moreover, because the energies of the element in each trajectory are equal, the element $dm$ may move along elliptical trajectories in all these planes at the same time. It appears only natural that, as already mentioned, in actual fact one should treat this pattern as motion of a charge wave within a certain solid angle rather than as that of elements.

Consider this situation in more detail. We take first the trajectories lying in
the plane passing through the vectors $Z$ and $r$ (vector $r$ specifies the direction to the element chosen). We will call it plane $C$ (Fig. 2.8). Let the element be located at point $A$. Figure 2.8 displays a “fan” of velocities $v$, i.e., directions along which an element $dm$ can move in the $C$ plane (compare with Fig. 2.7). Recall that all elements rotate in one sense. The direction of the velocities of element $dm$ at point $A$ is defined by that of the tangents to the elliptical trajectories at point $A$. One of such trajectories, i.e., one of the ellipses is identified in Fig. 2.8 with a dashed line. The direction of motion of an element can be described by the angle $\eta$, which we will reckon from the $r$ axis. The angle $\eta$ defines the direction of motion of an element along the trajectory, i.e., along the ellipse. This angle varies from 0 to $\pi$, which corresponds to variation of the ellipse shape from a straight line to a circle and again back to the straight line (see Fig. 2.7).

Because the velocity fan lies in the $C$ plane, the resultant velocity formed by summation of the velocities of the element $dm$, which propagates along different trajectories in the $C$ plane, at point $A$ lies in the same $C$ plane, while the angular momentum $dM_C$ of the element $dm$ is perpendicular to this plane (see Fig. 2.8).

The trajectories of the element $dm$ may lie not only in plane $C$. The other planes to which the trajectories may be confined can be obtained by rotating plane $C$ about the axis connecting the origin $O$ with the position of the element $dm$, i.e., point $A$ (Fig. 2.9). We denote the angle of turn of this plane by $\xi$, and will reckon it from the original position of the $C$ plane. The reason for which this angle is reckoned from this plane will become clear later.

The angular momentum of an element $dm$ moving in any turned plane is perpendicular to this plane. Therefore, as the planes are turned, the tips of the momentum vectors (we will denote them by $dM_\xi$) will trace an arc. This arc will lie in a plane perpendicular to vector $r$ (Fig. 2.9).

Consider the limits within which the angle $\xi$ can vary. Variation of the angle $\xi$ will initiate formation of new planes over which the element $dm$ can move. We have to keep in mind that the element can move not only in these, newly
formed, planes, but in all planes simultaneously. We readily see that if the angle $\xi$ is larger than $\pi$, new planes coinciding with some of the original planes will appear. In these coinciding planes, all allowed trajectories will coincide as well. Significantly, motion over these coinciding trajectories will occur simultaneously in opposite senses, i.e., there will be no motion on these planes. (This can be readily seen from Fig. 2.8 if we turn mentally plane $C$ through $\pi$, or from Fig. 2.9.)

Thus, angle $\xi$ can vary from $0 - \pi$.

As a plane turns through $\pi$, vectors $dM_\xi$ likewise turn through $\pi$.

Consider now why the angle $\xi$ should be reckoned from the $C$ plane.

The overall rotation of the charge in an atom (by overall rotation we understand here rotation of the charge as a whole rather than that of individual elements of charge) occurs in one sense, which accounts for the atom having an angular momentum. We conventionally directed the angular momentum of the atom along the $Z$ axis. Accordingly, the resultant velocity of rotation is directed along the $\varphi$ axis.

Now if the trajectories of an element lie in the $C$ plane, all components of the velocity lie in the same plane, with no velocity components left along the $\varphi$ axis. The angular momentum $dM_C$ of this element will in this case have no components along the $Z$ axis (see Fig. 2.8). If we rotate the $C$ plane by increasing the angle $\xi$, we will detect formation in these turned planes velocity components directed along the $\varphi$ axis, with the $dM_\xi$ momentum acquiring a component along the $Z$ axis (see Fig. 2.9). Both quantities reach a maximum for $\xi = \pi/2$. As the angle $\xi$ increases still further, both quantities will decrease, to vanish eventually at $\xi = \pi$.

As the angle $\xi$ grows still more (as does the corresponding turn of the plane), the elements will rotate in these planes in the opposite sense, with a negative component of the velocity along the $\varphi$ axis, and of the angular momentum along the $Z$ axis, appearing, although they should not exist by our original condition.

Thus, reckoning the angle $\xi$ from the $C$ plane and variation of $\xi$ within the $0 - \pi$ limits provides overall rotation of the charge in one sense and formation of an angular momentum along the $Z$ axis. Significantly, no negative components of the angular momentum appear along the $Z$ axis.

The motion of elements over certain planes we have just considered is only some approximation to reality, because strictly confined to one plane is only motion of an element of a constant magnitude. Now in a real atom an element of mass/charge propagates within a certain solid angle. We will have to calculate now the angular momentum of an element $dm$ in the case of its propagation within a solid angle. We start with constructing a local frame of spherical coordinates $r'$, $\eta$, $\xi$ centered on the element $dm$, i.e., at point $A$. The $\eta = 0$ axis will be directed along the $r$ axis and will be called the $Z'$ axis, and the angle $\xi$ will be reckoned from plane $C$ (Fig. 2.9). We see immediately that the angles $\eta$ and $\xi$ of this coordinate system coincide with the angles $\eta$ and $\xi$ considered above.

If an element propagates into a solid angle, this means actually that its trajectory, rather than being confined to a certain plane, occupies a sector instead. An analog of element motion in a plane will be motion within a small solid angle $d\xi$, the orientation of this solid angle $d\xi$ (an analog of the position of the plane) being determined by the angle $\xi$. As already demonstrated, the angle $\eta$ varies within the $0 - \pi$ limits, and the angle $\xi$ varies within the same limits, $0 - \pi$. 

Thus, the element propagates into a hemisphere; accordingly, the solid angle into which the charge propagates as a wave is $2\pi$.

Our problem lies in finding the resultant angular momentum of the element $dm$ which propagates into a solid angle as a wave, for which purpose one will have to sum all components of the angular momentum oriented in different directions. Significantly, any direction of velocity at a given point (and within the allowed solid angle) is equally probable. This conclusion is valid because an element at a given point which propagates in different directions has the same energies, i.e., all these trajectories are equally probable. Moreover, the magnitude of the velocity should not be dependent on the direction of motion of a given element. This conclusion can be substantiated in the following way. We consider ellipses of the same energy, i.e., the total energy of an element on any ellipse is the same (although the relative magnitudes of the kinetic and potential energies change as the element moves along the ellipse). The potential energy depends on the position of the element only. Hence, at a given point (for instance, at point $A$) the potential energies of an element moving over any ellipse are the same. But if the total energies are equal, and the potential energies at a given point are equal too, then the kinetic energies at this point will be equal as well. For this reason, the velocities are equal irrespective of their direction. To calculate the angular momentum of the element $dm$ propagating into a solid angle $2\pi$, we divide the element $dm$ into parts

$$
 dm' = dm \frac{d\Omega}{2\pi}.
$$

(2.26)

Each element $dm'$ propagates into a solid angle $d\Omega = \sin \eta d\eta d\xi$.

We shall approach this problem in steps. Isolate a sector $d\xi$. Find the projection of the angular momenta of the elements propagating into the $d\xi$ sector onto a plane perpendicular to the $r$ axis (plane $D$ in Fig. 2.9). To do this, we will have to sum the angular momenta over the coordinate $\eta$. Because at a given point the velocities in any direction are equal in magnitude, the distribution of the angular momenta in the $d\xi$ sector is symmetric relative to the $D$ plane. Denoting this projection by $dp_\xi$, we come to

$$
 dp_\xi = \int v \sin \eta dm' = \int v \sin \eta dm \frac{d\Omega}{2\pi} = \int_0^{\pi} \frac{dm}{2\pi} \sin \eta d\xi \sin \eta d\eta = \frac{dmv}{2\pi} \cdot \frac{\pi}{2} \cdot d\xi.
$$

(2.27)

The angular momentum $dM_\xi$ corresponding to the momentum $dp_\xi$ is shown in Fig. 2.9.

We turn now to summation of the vectors $dp_\xi$ obtained over the coordinate $\xi$. We first find the projection of vectors $dp_\xi$ on the plane formed by turning plane $C$ through an angle $\xi = \pi/2$. Because at a given point the velocities in any direction are equal, the distribution of the momenta $dp_\xi$ will be symmetric relative to this plane. The momenta $dp_\xi$ lying in plane $D$ perpendicular to vector $r$, the vector we have obtained (denote it by $dp_\phi$) will coincide in direction with the coordinate $\phi$:

$$
 dp_\phi = \int \sin \xi dp_\xi = \frac{dmv}{2\pi} \cdot \frac{\pi}{2} \cdot \int_0^{\pi} \sin \xi d\xi = \frac{dmv}{2},
$$

(2.28)

or

$$
 dp = \frac{dmv}{2} n_\phi.
$$

(2.29)
where \( \mathbf{n}_\varphi \) is the unit vector along the \( \varphi \) axis.

Thus, the momentum of the element \( dm \) propagating as a wave into a solid angle of \( 2\pi \) is oriented along the \( \varphi \) axis. The momentum has no other components. This momentum is equal in magnitude to one half of the momentum the same element would have if it moved as a whole in one direction.

We can now calculate trivially the angular momentum of the element \( dm \). Using the conventional expression for determination of the angular momentum \( \mathbf{M} = [\mathbf{r} \times \mathbf{p}] \), and bearing in mind that the resultant momentum of the element \( dm \) propagating as a wave is directed along the \( \varphi \) axis, i.e., perpendicular to the \( C \) plane, we arrive at the following expression for the resultant angular momentum of the element \( dm \):

\[
dM_{\Omega} = \frac{dmu_r}{2} [\mathbf{n}_r \times \mathbf{n}_\varphi],
\]

(2.30)

where \( \mathbf{n}_r \) is the unit vector along the \( r \) axis. We are not using here the relation for the angular momentum in its conventional form \( d\mathbf{M} = dm[\mathbf{r} \times \mathbf{v}] \), because the velocity \( \mathbf{v} \) of an element propagating into a hemisphere has not specific direction. As evident from Eq. (2.30), the angular momentum \( dM_{\Omega} \) lies in the plane \( C \) (see Fig. 2.10).

Compare now Eq. (2.30) with the standard expression for the angular momentum of an element moving as a whole along a circular trajectory of radius \( r \) and see what orbit could be identified with motion of the element as a wave. To avoid confusion, we shall denote the element of mass \( dm \) by \( dm_b \) (\( dm_{\text{body}} \)) in the case where we shall believe the element to move as a whole. It should be stressed, however, that \( dm \) and \( dm_b \) are one and the same element. Because the angular momentum of an element in a Coulomb potential is conserved, one
can calculate the momentum at any point we choose. Let us calculate the angular momentum of the element \( dm_b \) at the point of the circular orbit in which this element is farthest from the equatorial plane. At this point, the velocity of the element is directed along the \( \varphi \) axis. The angular momentum \( dM_b \) of this element can be written as

\[
dM_b = dm_b [r \times v] = dm_b vr [n_r \times n_\varphi].
\] (2.31)

A cursory inspection of Eqs. (2.30) and (2.31) reveals that the angular momenta have the same orientation, while in magnitude the angular momentum of the element propagating into the hemisphere is one half only of that of the element moving along a circular orbit. Significantly, \( r \) for the element propagating into the hemisphere is the distance from the element to the nucleus, while for the element in circular motion, \( r \) is not only the distance from the element to the nucleus but the radius of the circular orbit as well, with the orbit located such that the element under consideration is at the point of the orbit farthest from the equatorial plane. Thus, to calculate the angular momentum of the element propagating into a solid angle, one can restrict oneself to finding that of the element rotating in the corresponding orbit and taking one half of it. We shall use subsequently this observation.

As seen from Eqs. (2.30) and (2.31), both the \( dM_\Omega \) and \( dM_b \) angular momenta lie in the \( C \) plane (note that each element has its own \( C \) plane).

Both in Eq. (2.30) and (2.31), velocity \( v \) is the velocity of motion of an individual element \( dm \). Significantly, the velocity of motion of an element does not coincide with that of the wave process (in propagation of the same element into a solid angle). Examining the expression for the momentum (2.29), we see that the velocity of the wave process at a given point is one half the velocity of motion of elements at the same point, and it is directed along the \( \varphi \) axis, whereas the velocities of the elements have the same magnitude but are differently directed within the solid angle of \( 2\pi \). That an element moves not as a whole but propagates rather into a solid angle is accounted for in the expression for the angular momentum through the coefficient \( 1/2 \) (compare the Eqs. (2.30) and (2.31) for circular motion of the element).

Expand the angular momentum of the element \( dm \) into two components, along and perpendicular to the \( Z \) axis (Fig. 2.10). The axial \( Z \) component can be written as

\[
dM_Z = dM_\Omega \cdot \sin \vartheta = \frac{dmvr}{2} \sin \vartheta,
\] (2.32)

where \( \vartheta \) is the coordinate of point \( A \) (see Figs. 2.8 and 2.10). For the component perpendicular to the \( Z \) axis we obtain

\[
dM_\perp = dM_\Omega \cdot \cos \vartheta = \frac{dmvr}{2} \cos \vartheta.
\] (2.33)

Let us calculate the angular momentum of the atom as a whole. This can be done by summing up the angular momenta of all the elements. By virtue of the axial symmetry of the system, the \( M_\perp \) component vanishes to leave the \( M_Z \) one only:

\[
M_Z = \int \frac{1}{2} vr \sin \vartheta dm = \int \frac{1}{2} mvr \sin \vartheta dV.
\] (2.34)
In this expression, one has to substitute for velocity \( v \) the velocity of the element \( dm \) moving circularly over the circle, i.e., actually \( dm_b \). This was substantiated by us earlier in a comparison of Eqs. (2.30) and (2.31).

The velocity of a circularly rotating element can be taken from Eq. (1.50), because this equation describes the motion of an element \( dm \) over a circle of radius \( r = a \tau \). In this formula, however, one has to drop index \( \varphi \), because circular motion of an element with which we compare propagation of an element into the hemisphere can occur in any plane passing through the nucleus.

As already mentioned, the circular orbit which is opposed to propagation of an element into a solid angle should be positioned such that the element under consideration is at the point farthest from the equatorial plane (it is in this case that the angular momentum \( dM_b \) will be confined to the \( C \) plane). This provides a mutual one-to-one correspondence between the position of an element and of the orbit under consideration (excluding the elements at the equator). Indeed, a given element can be crossed by a multitude of other orbits, having different angles of tilt. In all of these orbits, however, the point farthest from the equatorial plane will not coincide with the element we are considering. Therefore, these orbits will contribute to other elements. This leaves only one orbit for the given element. Hence, in such a consideration each element will be identified with one and the only orbit, and in integration over elements we will not be plagued by the danger of taking some orbits more than once into account.

As for an element being capable of moving along any tilted orbit, this has already been taken into account in deriving the expression (2.30) for the angular momentum \( dM_b \), where the coefficient \( 1/2 \) was obtained.

Equation (2.34) has a fairly simple structure. It can be revealed readily by considering an equivalent, circularly rotating element \( dm_b \). The factor \( \sin \theta \) accounts for the tilt of the trajectory of the equivalent circularly rotating element, and coefficient \( 1/2 \), for the real element propagating into a solid angle rather than rotating circularly. The other terms of Eq. (2.34) make up the standard expression for the angular momentum.

Substituting the expressions for velocity (1.50) and for the density of mass, (2.11) or (2.12), into Eq. (2.34), we come to

\[
M'_Z = \frac{1}{2} \int_0^\infty \frac{4m_e e^{-2\tau} \alpha c}{a^3 \pi^2} \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\varphi, \\
M''_Z = \frac{1}{2} \int_0^\infty \frac{3m_e e^{-2\tau} \alpha c}{a^3 \pi^2} \int_0^\pi \sin^4 \theta d\theta \int_0^{2\pi} d\varphi.
\]

(2.35)

(2.36)

Our calculations finally yield

\[
M'_Z = \hbar \cdot 0.499, \quad M''_Z = \hbar \cdot 0.519.
\]

(2.37)

The first value fits better the available experimental data. At the same time, we will not yet reject the second value which corresponds to the charge distribution expressed in terms of spherical functions.

We have to bear in mind the following points. First, these values were obtained in nonrelativistic approximation. Second, by nonrelativistic quantum
mechanics, the moment of the ground state is zero, which is in conflict with experiment.

We are turning now to the virial theorem (1.47). Strictly speaking, we had no grounds for applying this equation in the first Chapter of the Paper. The virial theorem in the form of Eq. (1.47) was used to describe the motion of an element of mass/charge in a Coulomb potential well. In the first part of the Paper, however, an additional statement was tacitly introduced that the velocity of an element $v_\varphi$ is directed along the $\varphi$ axis only. This is in clear conflict with the conditions under which the theorem can be applied. The above reasoning and the assumption that charges can interpenetrate lift this additional statement. Charges can move in a Coulomb field along any allowed orbit. In this case, the virial theorem in the form of Eq. (1.47) is certainly applicable to charges in a Coulomb potential well.

On these grounds one can forward one more comment. In the analysis of Eq. (1.5) it was pointed out, in particular, that it resembles in form the Schrödinger equation, the only difference being that it contains a coefficient 4, whereas in the Schrödinger equation the coefficient is 2. We use Eq. (1.5) to derive the charge distribution and, hence, the potential energy of an electron. This potential energy coincides with the eigenvalue of Eq. (1.5). If we substituted coefficient 2 in place of 4 in Eq. (1.5) and calculated the energy as eigenvalues of this new Eq. (1.5), we would have obtained the numerical value of the total electron energy. A decrease of the coefficient by a factor two brings about a corresponding decrease of the calculated energy to one half. In actual fact, this is simply the result of our having tacitly added to Eq. (1.5) the virial theorem, because by this theorem the total energy of the electron is one half of its potential energy. While this is acceptable if our goal is to numerically calculate the values we are interested in, straightforward logic suggests that Eq. (1.5) should have the coefficient 4, because it is the electron potential energy that we calculate with this equation.

### 2.5 Comment on the absence of emission from a stationary orbit

This is a short comment, but we believe it to be important enough to be presented in a separate paragraph. By quantum mechanics, a point charge rotates around a nucleus. (More precisely, an electron is in a state having a definite energy and a definite projection of its angular momentum on the $Z$ axis). In the frame of electrodynamics, a rotating electron should emit radiation, lose energy and eventually fall on the nucleus. In actual fact this just does not happen. To provide a proper explanation for this, it was assumed that the electron does not radiate when in a stationary orbit. Actually, this is one of the postulates of quantum mechanics. The stationary orbit (more precisely, the steady state) is calculated by solving the Schrödinger equation. Considered in the frame of mathematics, there is nothing that could be questioned; indeed, if, by Schrödinger equation, there is a solution within which an electron is in steady state with a certain energy, hence, this energy does not change, and, hence, the electron will not radiate. Viewed in the physical context, however, it is not clear in what does a stationary electron orbit differ from the non-stationary one. Why
an electron residing in one orbit does radiate, and in another one, does not?

The above assumption of the existence of a distributed electron charge permits one to lift this postulate. Indeed, each element of mass/charge moves in accordance with the laws of theoretical mechanics and electrodynamics. Each individual element rotating about a nucleus is involved in periodic motion and, thus, has to radiate. But the elements make up a distributed charge. Now motion of a distributed charge is no longer periodic. Although each element moves along its own separate trajectory, the motion of the distributed charge as a whole is actually a common circular motion of the total charge. Radiation of variable fields by one element is canceled by that of the other elements.

Significantly, overall motion of the distributed charge as a whole (which now is no longer periodic) generates a magnetic field. This is reflected in the atom having a magnetic moment. An analog of such motion could be a set of closed currents which are known not to radiate periodic fields while having a constant magnetic field.

2.6 Conclusion

The time has come for summing up the outcome of our reasoning.

In the first part of the Paper, we have put forward an assumption that electron in an atom, rather than being a point object, is a distributed charge. Taken as a whole, this charge should be equal to that of the electron; therefore, in all cases there should exist in the atom the $S$-state of the distributed charge. We forwarded the equation of wave mechanics, i.e., the equation which a charge distribution should obey. The solutions of this equation derived for the Coulomb potential of the nucleus identified the shape of the charge distribution in an atom. These solutions were normalized against the electron charge (not by unity). We further invoked standard methods in use in electrostatics to derive the values of the potential energy of interaction of a distributed charge with the nucleus for three states of the hydrogen atom, which were found to coincide with those well known from quantum mechanics. More than that, these values coincide with the eigenvalues of the equation of wave mechanics. Standard methods employed in electrostatics were used to find the fields and potentials of distributed charges and to calculate again the energies of interaction of distributed charges with a nucleus.

This has led to another assumption that the electron represents actually not only a distributed charge but a distributed mass as well. The distributed mass assumes a distribution of the same shape as the distributed charge, and its motion coincides with that of a distributed charge. It thus turns out that the electron is a distributed charge/mass object. The distributed mass should be normalized by the electron mass (rather than by unity). An analysis of various versions of motion of distributed charge and mass led to the conclusion that the velocity of the elements of charge/mass should increase as they approach the nucleus (if the charge of an electron behaved as a solid body, the velocity of the motion of its elements should be increasing with distance from the nucleus). An assumption was made concerning the velocity distribution for the charge/mass distributions found, thus making it possible to calculate the kinetic and total energies for three states of the hydrogen atom. These values were demonstrated to coincide with those known from quantum mechanics.
Next, the same velocity distributions were used to calculate the angular momenta for the same states of the hydrogen atom. It was found that the angular momenta do not coincide with the values known from experiment. This suggested that the atom does not possibly possess spherical symmetry.

In the second part of the Paper, two versions of a non-spherical charge distribution in an atom were advanced. It was shown that while the charge is certainly non-spherical, the potentials of these charges are close to spherical. Said otherwise, such an atom would look as spherical to an observer. Moreover, as one comes closer to the nucleus, this potential approaches ever more nearly the Coulomb potential. These distributions were used to calculate the potential, kinetic, and total energies for three states of the hydrogen atom. All these values were demonstrated to coincide with those well known from quantum mechanics. The angular momenta did not, however, equate to reality. This initiated a deeper analysis of the behavior of elements of charge/mass in a Coulomb potential well.

By the laws of theoretical mechanics, elements of mass in a Coulomb potential well can move along not only circular but elliptical trajectories as well. In order for such motion to become realistic, however, two more suggestions had to be made. First: charges can interpenetrate one another (as electromagnetic waves penetrate one through another). Second: if there are orbits in which an element possesses the same energies, this element can move in any of these orbits, and, more than that, simultaneously along all these orbits. In other words, this is no longer the motion of individual elements; it is rather the motion of a wave. This motion resembles the Huygens–Fresnel principle, the only difference being that the trajectories of the elements obey the laws of theoretical mechanics and are in effect closed curves.

These assumptions formed a basis on which the angular momentum of the hydrogen atom in ground state was calculated, and was found to be equal to $\hbar/2$, in excellent agreement with the experimental data. The experimental observation that the angular momentum of $S$ states is $\hbar/2$ was introduced into theoretical quantum mechanics as a postulate, as an intrinsic angular momentum of the electron (spin). (It may be reminded that in non-relativistic quantum mechanics the angular momentum of the $S$ states is zero.) Using this mechanism, it was found possible to calculate the angular momentum of the ground state of the hydrogen atom drawing solely from the laws of theoretical mechanics.

Each element in an atom undergoes periodic motion along a circle or ellipse. The motion of all elements as a whole is, however, no longer periodic, and represents rather circular motion of the charge as a whole. There being no common periodic motion of the charges, the electron (distributed charge) residing in steady state should not radiate. The only thing that exists is a constant magnetic field generated by a common circular motion of charges. For quantum mechanics, the statement that the electron in a stationary orbit does not radiate is essentially a postulate. Thus, this postulate can now be lifted.

Now how could one visualize an atom containing an electron in the form of a distributed charge? The most pictorial way would possibly be to compare it with a drop of a liquid. Elements of the liquid within the drop move in different directions but, when summed, produce rotation of the drop as a whole. This rotation could be detected only by labeling somehow an element of the liquid. If the element is not labeled, rotation of the drop cannot be detected. In other
words, an observer would believe this drop to be at rest. One should bear in mind, however, that the density of the drop increases toward the center, as does also the velocity of motion of the elements of the liquid.

2.7 Appendix

Equations for calculation of the potential (the notation used is that of Sec. 2.2). Point of observation - \( r \), point of integration - \( r' \).

For \( r > r' \):

\[
U_Q(r, \vartheta) = \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \cdot \frac{Q_l(r)Y_l(\vartheta)}{r'^{l+1}},
\]

where the multipole moment \( Q_l(r) \):

\[
Q_l(r) = \sqrt{\frac{4\pi}{2l+1}} \int_0^r \rho(r', \vartheta')r'^lY_l^*(\vartheta')dV'.
\]

For \( r < r' \):

\[
U_G(r, \vartheta) = \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \cdot r^lG_l(r)Y_l(\vartheta),
\]

where the multipole moment \( G_l(r) \):

\[
G_l(r) = \sqrt{\frac{4\pi}{2l+1}} \int_r^{\infty} \rho(r', \vartheta')r'^{l+1}Y_l^*(\vartheta')dV'.
\]

Functions \( Q_l(r) \) and \( G_l(r) \) depend on \( r \) in the upper and lower limits of integration as on a parameter.

Spherical functions:

\[
Y_l(\vartheta) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \vartheta)
\]

Legendre polynomials:

\[
P_0 = 1
\]
\[
P_2(\cos \vartheta) = \frac{1}{2} \cdot (3 \cos^2 \vartheta - 1)
\]
\[
P_4(\cos \vartheta) = \frac{1}{8} \cdot (35 \cos^4 \vartheta - 30 \cos^2 \vartheta + 3)
\]
\[
P_6(\cos \vartheta) = \frac{1}{48} \cdot (63 \cdot 11 \cdot 3 \cos^6 \vartheta - 15 \cdot 63 \cdot 3 \cos^4 \vartheta + 15 \cdot 21 \cdot \cos^2 \vartheta - 15)
\]
\[
P_8(\cos \vartheta) = \frac{1}{48 \cdot 3} \cdot (99 \cdot 13 \cdot 28 \cdot 13 \cdot 3 \cos^8 \vartheta - 99 \cdot 30 \cdot 7 \cdot 3 \cos^6 \vartheta + 99 \cdot 30 \cdot 7 \cdot 3 \cos^4 \vartheta - 63 \cdot 15 \cdot 4 \cdot \cos^2 \vartheta + 15 \cdot 7)
\]

Shape of distributed charge in this notation:

\[
\rho_{\text{NS}} = -\frac{4}{\pi^2} e^{-2r} \sin \vartheta, \quad \rho_{\text{NS}}'' = -\frac{3}{2\pi} e^{-2r} \sin^2 \vartheta.
\]

The potential at a point \( r \) is a sum of potentials calculated for \( r > r' \) and \( r < r' \):

\[
U(r, \vartheta) = U_Q(r, \vartheta) + U_G(r, \vartheta).
\]
Bibliography

[1] A. A. Kharkevich, Spectra and analysis (In Russian), Publ. GTTI, Moscow, Leningrad, 1952.

[2] H. A. Bethe, E. E. Salpeter, Quantum Mechanics of One- and Two- Electron Atoms, Springer-Verlag, Berlin, Gottingen, Heidelberg, 1957.

[3] V. V. Batygin and I. N. Toptygin, Modern Electrodynamics (In Russian), Pt. 1. Microscopic Theory, Publ. Institute of Computer Studies, Moscow, Izhevsk, 2003.

[4] V. V. Batygin and I. N. Toptygin, Problems in electrodynamics, Academic Press, Second Edition, (Translation from the Russian), 1978.

[5] I. I. Ol'khovskii, Theoretical Mechanics for Physicists (In Russian), Second Edition, Publ. Moscow University, Moscow, 1974.

[6] H. B. Dwight, Tables of Integrals and other Mathematical Data, fourth edition, the Macmillan Company, New York, 1961.

[7] L. D. Landau, E. M. Lifshitz, Mechanics, third edition, Butterworth-Heinemann, (Translation from the Russian), 1976.