THE BERKOVITS COMPLEX AND SEMI-FREE EXTENSIONS OF KOSZUL ALGEBRAS

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Abstract. In his extension [3] of W. Siegel’s ideas on string quantization, N. Berkovits made several observations of a purely algebraic nature which deserve further study and development. Indeed, interesting accounts of this work have already appeared in the mathematical literature [8, 13] and in a different guise due to Avramov. In this paper we bridge between these three approaches, by providing a complex that is useful in the calculation of some homologies.

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1. Introduction

This paper began with an observation of the importance of Koszul duality theory for commutative algebras in the work of Berkovits [3] on string quantization and string/gauge theory duality. In Berkovits’ paper the commutative algebra in question is the projective coordinate algebra $S$ of the orthogonal Grassmannian $OG(5, 10)$, related to the spinor representation of the group $SO(10, \mathbb{C})$. This is a commutative quadratic algebra, and for such algebras it is an easy consequence of the definition that its Koszul dual is the universal enveloping algebra of a graded Lie superalgebra $L = \bigoplus_{i \geq 1} L_i$.

Berkovits’ first observation, made in the context of string and gauge theories, was that the algebra of syzygies of $S$ is isomorphic to the cohomology $H^*(L_{\geq 2}, \mathbb{C})$ of the graded Lie superalgebra $L_{\geq 2} = \bigoplus_{i \geq 2} L_i$. He proposed a further complex relevant for his quantization procedure, and posed the question of calculating its homology. His construction is an iteration of the Koszul homology of a sequence of elements of an algebra, first applied to the algebra $S$ and then applied to its syzygies. Namely, if $A$ is finitely generated and presented as $\mathbb{C}[a_1, \ldots, a_n]/I$, the Koszul homology with respect to the sequence $\{a_i\}$ is called the algebra of syzygies of $A$. The generators $\Gamma_j$ of the ideal $I$ represent syzygies
in the lowest degree. The Koszul homology of the algebra of syzygies with respect to the sequence \( \{ \Gamma_j \} \) is the algebra studied by Berkovits.

The same construction has appeared in the study of deviations of a local ring (see [2]). They are a sequence of integers that are attached to a local ring, that measure how far it is from being regular or a complete intersection. In calculating these integers, Avramov constructs complexes that are analogous to Berkovits. A consequence of Avramov’s work is that if \( A \) is Koszul, the Berkovits homology can be described explicitly in terms of the graded Lie superalgebra \( L \). Namely, for an arbitrary finitely generated commutative Koszul algebra, the Berkovits homology is isomorphic to the cohomology \( H^*(L_{\geq 3}, \mathbb{C}) \).

This result was also shown in the very interesting paper [13] for the case \( A = S \). Further, Movshev and Schwarz suggested an algebraic technique to study the Berkovits homology of the algebra \( S \) by constructing an equivalent but “smaller” complex. This complex does not seem to appear as an accident, we believe that it is naturally attached to a commutative Koszul algebra. In this paper we are investigating this issue. We construct such a “smaller” complex for Koszul algebras whose algebra of syzygies is also Koszul.

At this point we would like to mention a connection with a conjecture made by Avramov: Conjecture C10 of [1] states that if \( A \) is not a complete intersection, then the appropriate Lie superalgebra \( L \) contains a free nonabelian graded Lie subalgebra. The algebra \( S \) above provides an example confirming the conjecture, according to [10, 13]. Indeed, the “smaller” complex allows one to calculate that \( H^2(L_{\geq 3}, \mathbb{C}) \) is trivial. Our construction of a “smaller” complex allows us to show that the Lie superalgebra \( L_{\geq 3} \) is free for the projective coordinate algebra of \( G(2, 5) \), providing another confirmation of C10, and is not free for \( G(2, N) \) for \( N \geq 6 \).

The algebra of syzygies of \( S \) is in fact non-quadratic, and yet the “smaller” complex still exists. So, we believe that our construction can be extended to a more general case, but requires (to our knowledge) a substantial extension of the notion of Koszulness to non-quadratic algebras.

The paper is organised as follows: in Section 2 we discuss some preliminaries including Koszul algebras and Lie superalgebras; in Section 3.2 we define the Berkovits complex and how it arises as a semi-free extension of the initial algebra. The first main theorem of the paper is Theorem 3.11, which shows that for any commutative Koszul algebra, its Berkovits homology is isomorphic to \( H^*(L_{\geq 3}, \mathbb{C}) \). The second main theorem is in Section 4, where we construct the “small” complex and prove that it calculates the Berkovits homology when we have a Koszul algebra whose algebra of syzygies is quadratic (Corollary 4.3). As an application we consider the Grassmanians \( G(2, N) \) for \( N \geq 5 \) in detail (Theorem 4.7).

The first author was partially supported by Spanish Government grants MTM2010-15831, MTM2010-20692, MTM2012-38122-C03-01 and MTM2013-42178-P and Catalan Government grants SGR1092-2009 and SGR634-2014, and the fourth author by MTM2010-15831 and MTM2013-42178-P. All the authors visited the Max Planck Institute while working on this project and are grateful to the Institute for excellent working conditions. The authors also thank to L.A. Bokut, A. Losev and M. Movshev for very useful discussions and advice, and A. Conca and S. Iyengar for pointing us to Avramov’s commutative algebra constructions in [2]. Of course we also are very grateful to Vadim Schechtman with whom this project originally was started and to whom we dedicate it with our best wishes.

2. Preliminaries on commutative Koszul algebras

We fix our ground field \( k = \mathbb{C} \). Unless otherwise stated, all algebras are graded \( A = \bigoplus_{i \geq 0} A_i \) and locally finite dimensional, i.e. \( \operatorname{dim} A_i < \infty \) for all \( i \geq 0 \). Further,
we assume that $A_0 = \mathbb{C}$, so that $A = \mathbb{C} \oplus A_+$ is augmented, with augmentation ideal $A_+ = \bigoplus_{i \geq 0} A_i$.

2.1. Koszul Algebras.

**Definition 2.1.** Let $A$ be a quadratic algebra defined by a presentation $A = T(V)/Q$, where $V$ is a finite dimensional vector space of generators in degree one, and $Q$ is a two-sided ideal generated by quadratic elements. The Koszul dual algebra $A^!$ is the graded algebra defined as

$$A^! = T(V^*/Q^!),$$

where $V^*$ is concentrated in degree one and $Q^! \subset V^* \otimes V^*$ is the annihilator of $Q$.

Clearly $A^! = A$.

The Koszul complex $(K(A), d_A)$ of a quadratic algebra $A$ is defined by the sequence of right $A \otimes A^!$-modules

$$K_p(A) := A \otimes (A^!)_p \quad (p \geq 0).$$

Here $A$ is a right $A$-module via multiplication and $(A^!)$ is a right $A^!$-module via

$$(\varphi b)(c) := \varphi(bc), \quad (\varphi \in (A^!)) \quad (b, c \in A^!).$$

The differential $d_A$ is the action of the identity $id_V \in \text{Hom}(V, V) = V \otimes V^* \subset A \otimes A^!$. It has degree $-1$, and satisfies $d_A^2 = 0$ since $A$ is quadratic.

Recall that since $A$ is graded, the algebra $\text{Ext}_A(\mathbb{C}, \mathbb{C})$ is bigraded, and decomposes as

$$\text{Ext}_A^i(\mathbb{C}, \mathbb{C}) = \bigoplus_{j \geq 0} \text{Ext}_A^{ij}(\mathbb{C}, \mathbb{C}),$$

where $i$ is the cohomological grading and $j$ is the grading descending from $A$.

**Definition 2.2.** An augmented algebra $A$ is a Koszul algebra if $\text{Ext}_A^{ij}(\mathbb{C}, \mathbb{C}) = 0$ whenever $i \neq j$.

The following theorem gives some equivalent definitions of a Koszul algebra. A proof may be found, for example, in [14, Chapter 2, Theorem 4.1].

**Theorem 2.3.** Let $A$ be a quadratic algebra. Then the following are equivalent:

1. $A$ is Koszul,
2. $A^!$ is Koszul,
3. $\text{Ext}_A(\mathbb{C}, \mathbb{C}) \cong A^!$ as algebras with $\text{Ext}_A^i(\mathbb{C}, \mathbb{C}) = A_i$,
4. $(K(A), d_A)$ is a resolution of $\mathbb{C}$.

**Example 2.4.** Consider the symmetric algebra, $S(V)$. It is a quadratic algebra and the ideal of relations $Q$ inside $T(V)$ is precisely $(u \otimes v - v \otimes u \mid u, v \in V)$. The annihilator of $Q$ inside $T(V^*)$ is generated by tensors of the form $u^* \otimes v^* + v^* \otimes u^*$ and so $S(V)^! = \bigwedge V^*$. Therefore the Koszul complex of $S(V)$ has graded components

$$(1) \quad K_p(S(V)) = S(V) \otimes \bigwedge^p W_1 \quad (p \geq 0)$$

where $W_1$ is a copy of $V$ inside $(S(V)^!)^*$. If $a_1, \ldots, a_n$ and $\xi_1, \ldots, \xi_n$ are bases of $V$ and $W_1$ respectively, then the differential can be written as

$$\sum_i a_i \frac{\partial}{\partial \xi_i}.$$ 

This makes $K(S(V))$ into a DG algebra over $S(V)$, with the obvious product structure. It is a resolution of $\mathbb{C}$ (see [12, Proposition VII.2.1] for a proof), so $S(V)$ and $S(V)^! = \bigwedge V^*$ are Koszul algebras.
2.2. Lie Superalgebras.

Definition 2.5. A Lie superalgebra over \(\mathbb{C}\) is a \(\mathbb{Z}_2\)-graded vector space \(L = L_0 \oplus L_1\) with a map \([\cdot, \cdot] : L \otimes L \to L\) of \(\mathbb{Z}_2\)-graded spaces, satisfying:

1. (anti-symmetry) \([x, y] = -(-1)^{|x||y|}[y, x]\) for all homogeneous \(x, y \in L\),
2. (Jacobi identity) \((-1)^{|x||y|}[x, [y, z]] + (-1)^{|y||z|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0\) for all homogeneous \(x, y, z \in L\).

Here \(|x| = i\) when \(x \in L_i\) for \(i = 0, 1\), and an element \(x\) in \(L_0\) or \(L_1\) is termed even or odd respectively. We recover the familiar definition of a Lie algebra (over \(\mathbb{C}\)) in the case \(L = L_0\).

A graded Lie superalgebra is a Lie superalgebra \(L\) together with a grading compatible with the bracket and supergrading. That is, \(L = \bigoplus_{m \geq 1} L_m\) such that \([L_i, L_j] \subset L_{i+j}\) and \(L_i = \bigoplus_{m \geq 1} L_{2m-i}\) for \(i = 0, 1\).

Remark 2.6. In the literature a graded Lie superalgebra is sometimes called simply a graded Lie algebra. However, this could also refer to an ordinary Lie algebra \(L = L_0\) with a grading compatible with the bracket. In order to avoid this ambiguity we prefer the terminology defined above.

Let \(V\) be a vector space with basis \(a_1, \ldots, a_n\) and consider a quadratic commutative algebra \(A = T(V)/Q\). Then, \(Q = C \oplus I\) where \(C\) is the ideal \((u \otimes v - v \otimes u \mid u, v \in V)\) and \(I = (\Gamma_1, \ldots, \Gamma_m) \subset S^2(V)\) so that we can write \(A = S(V)/I\). Further, \(Q \subset C\) implies that \(Q^\perp \subset C^\perp = S^2(V^*)\) and so \(Q^\perp\) is generated by certain linear combinations of anti-commutators \([a_i^*, a_j^*] = a_i^*a_j^* - a_j^*a_i^*\). We can therefore describe the Koszul dual as the universal envelope of a graded Lie superalgebra,

\[A^! = U(L), \quad L = \bigoplus_{m \geq 1} L_m = \mathbb{L}(V^*)/J,\]

where \(\mathbb{L}\) is the free Lie superalgebra functor, the space of (odd) generators \(V^*\) is concentrated in degree 1, and \(J\) is the Lie ideal with the same generators as \(Q^\perp\) but viewed as linear combinations of supercommutators.

Definition 2.7. For \(k \geq 2\), we define the graded Lie superalgebras

\[L_{\geq k} = \bigoplus_{m \geq k} L_m.\]

The interpretation of the algebras \(L_{\geq k}\) for \(k > 2\) was outlined in [3]. In this paper, we concentrate on the case \(k = 3\).

2.3. Deviations. Consider the Hilbert series of the commutative algebra \(A = S(V)/I\)

\[H_A(t) := \sum_{r=0}^{\infty} \dim(A_r) t^r = \frac{h(t)}{(1-t)^n},\]

where \(h(t)\) is a polynomial such that \(h(0) = 1\). Gauss’ cyclotomic identity [7] allows us to expand this rational function into an infinite product

\[\frac{h(t)}{(1-t)^n} = \prod_{s=1}^{\infty} (1 - t^s)^{(-1)^s \varepsilon_s(A)},\]

where the exponents \(\varepsilon_s(A)\) are integers known as the deviations of \(A\).

When \(A\) is a Koszul algebra, the exponents \(\varepsilon_s(A)\) give the dimensions of the graded components \(L_s\) of the Lie superalgebra \(L\) (see for example [11]), and in particular we have
Indeed, by the Poincaré–Birkhoff–Witt theorem for the universal enveloping algebra of a graded Lie superalgebra, the Hilbert series of \( A^1 = U(L) \) is equal to

\[
H_{A^1}(t) = \prod_{s=1}^{\infty} (1 - (-t)^s)^{-1}^{\dim(L_s)}
\]

and from the relation \( H_A(t) H_{A^1}(-t) = 1 \) it follows that \( \varepsilon_s(A) = \dim(L_s) \) in (4). See [14, Section 2.2] for further details, and also Remark 3.10 below.

Dividing by \( k^{-1} \prod_{s=1}^{k-1} (1 - t^s)^{-1}^{\varepsilon_s(A)} \)

we obtain the Hilbert series of \( L_{\geq k} \) as the following infinite product

\[
\prod_{s=k}^{\infty} (1 - t^s)^{-1}^{\varepsilon_s(A)}.
\]

### 3. Koszul homology and Berkovits complex

#### 3.1. Koszul homology and the algebra of syzygies.

Let \( A = S(V)/I \) be a commutative quadratic algebra with \( \{a_1, \ldots, a_n\} \) a basis of \( V \).

**Definition 3.1.** The Koszul homology of \( A \) with respect to a sequence of elements \( x_1, \ldots, x_k \) in \( A \) is the homology of the complex

\[
A[W_1] := \left( A \otimes \bigwedge W_1, \sum_{i=1}^{k} x_i \frac{\partial}{\partial \theta_i} \right)
\]

where \( A \) has homological degree zero and \( W_1 \) is the vector space spanned by elements \( \theta_1, \ldots, \theta_k \) in homological degree one.

The algebra of syzygies of \( A \) is the Koszul homology of \( A \) with respect to the sequence \( \{a_1, \ldots, a_n\} \).

The algebra of syzygies of \( A \) is a graded finite dimensional algebra by Hilbert’s syzygy theorem (c.f. [6, Theorem 1.13]).

**Remark 3.2.** We make the following trivial remarks:

1. Suppose \( A \) is the symmetric algebra \( S(V) \). Then \( A^1 = \bigwedge V^* \) and the complex \( A[W_1] \) calculating the algebra of syzygies of \( A \) is precisely the Koszul complex, compare Example 2.4.
2. The complex \( A[W_1] \) is a DG algebra. The algebra structure is given by

\[
(ab \otimes \omega) \cdot (b \otimes \eta) = ab \otimes \omega \wedge \eta,
\]

where \( a, b \in A \) and \( \omega, \eta \in \bigwedge W_1 \). Koszul homology, and in particular the algebra of syzygies, therefore inherits a graded algebra structure.
3. When \( A \) is graded, the Koszul homology is bigraded. In the case of Koszul homology with respect to a sequence of generators, the homological grading will be called the order of the syzygies and the sum of the homological grading and the grading on \( A \) will be called the degree of the syzygies.

We observe that \( \text{Tor}^{S(V)}(A, \mathbb{C}) \) is also calculated by the complex \( A[W_1] \), and so coincides with the algebra of syzygies. Indeed, \( K(S(V)) \) is a resolution of \( \mathbb{C} \) by \( S(V) \)-modules, so \( \text{Tor}^{S(V)}(A, \mathbb{C}) \) is the homology of the complex

\[
A \otimes_{S(V)} K(S(V)) = A \otimes_{S(V)} S(V) \otimes_{\mathbb{C}} \bigwedge W_1 = A \otimes_{\mathbb{C}} \bigwedge W_1
\]
Alternatively, suppose we have a minimal free resolution of $A$,

\[ \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0, \]

by graded free $S(V)$-modules

\[ F_p = \bigoplus_q R_{pq} \otimes S(V). \]

Here minimality means that the differential vanishes on tensoring this complex with the trivial $S(V)$-module $\mathbb{C}$, and hence

\[ \text{Tor}_p^{S(V)}(A, \mathbb{C}) = R_p := \bigoplus_q R_{pq}. \]

Thus $R_{pq}$ is the finite dimensional vector space of $p$-th order syzygies of degree $q$ for $A$.

**Example 3.3.** Consider the Plücker embedding of the Grassmannian $G(2, 5)$,

\[ G(2, 5) \longrightarrow \mathbb{P} \left( \bigwedge^2 \mathbb{C}^5 \right), \]

\[ \langle v_1, v_2 \rangle_{\mathbb{C}} \longmapsto [v_1 \wedge v_2]. \]

Let $\{e_1, \ldots, e_5\}$ be a basis of $\mathbb{C}^5$, then $G(2, 5)$ can be written as the intersection of five quadrics $\Gamma_i = 0$ where

\[ \Gamma_1 = -e_{24}e_{35} + e_{23}e_{45} + e_{25}e_{34}, \]

\[ \Gamma_2 = -e_{14}e_{35} + e_{13}e_{45} + e_{15}e_{34}, \]

\[ \Gamma_3 = -e_{14}e_{25} + e_{12}e_{45} + e_{15}e_{24}, \]

\[ \Gamma_4 = -e_{13}e_{25} + e_{12}e_{35} + e_{15}e_{23}, \]

\[ \Gamma_5 = -e_{13}e_{24} + e_{12}e_{34} + e_{14}e_{23}. \]

Here $e_{ij}$ is the coordinate function dual to $e_i \wedge e_j$, etc. The projective coordinate algebra of $G(2, 5)$ is $S(V)/I$ where $V$ is the vector space with basis $\{e_{ij}\}$ and $I = (\Gamma_1, \ldots, \Gamma_5)$.

We now explicitly construct a minimal free resolution by $S(V)$ modules,

\[ 0 \longrightarrow \langle c^* \rangle \longrightarrow \langle \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_5 \rangle \longrightarrow \langle \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_5 \rangle \longrightarrow \langle c \rangle \longrightarrow S(V)/I \longrightarrow 0. \]

The zeroth order syzygy $R_{00}$ is given by a copy of $\mathbb{C}$, generated by $c$. For the first order syzygies we take the free $S(V)$-module $R_{12} = \langle \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_5 \rangle$, with $\partial \tilde{\Gamma}_i = \Gamma_i \cdot c$. The kernel of this boundary map is given by the relations

\[ -e_{12}\Gamma_2 + e_{13}\Gamma_3 - e_{14}\Gamma_4 + e_{15}\Gamma_5 = 0, \]

\[ e_{12}\Gamma_1 = -e_{23}\Gamma_3 + e_{24}\Gamma_4 - e_{25}\Gamma_5 = 0, \]

\[ -e_{13}\Gamma_1 + e_{23}\Gamma_2 - e_{34}\Gamma_4 + e_{35}\Gamma_5 = 0, \]

\[ e_{14}\Gamma_1 - e_{24}\Gamma_2 + e_{34}\Gamma_3 - e_{45}\Gamma_5 = 0, \]

\[ -e_{15}\Gamma_1 + e_{25}\Gamma_2 - e_{35}\Gamma_3 + e_{45}\Gamma_4 = 0. \]

so for the second order syzygies we take the free $S(V)$-module $R_{23} = \langle \tilde{\Gamma}_1^*, \ldots, \tilde{\Gamma}_5^* \rangle$, with

\[ \partial \tilde{\Gamma}_1^* = (0, -e_{12}, e_{13}, -e_{14}, e_{15}), \]

\[ \partial \tilde{\Gamma}_2^* = (e_{12}, 0, -e_{23}, e_{24}, -e_{25}), \]

\[ \partial \tilde{\Gamma}_3^* = (-e_{13}, e_{23}, 0, -e_{34}, e_{35}), \]

\[ \partial \tilde{\Gamma}_4^* = (e_{14}, -e_{24}, e_{34}, 0, -e_{45}), \]

\[ \partial \tilde{\Gamma}_5^* = (-e_{15}, e_{25}, -e_{35}, e_{45}, 0). \]
The third and highest order syzygy \( R_{35} \) has a single generator \( c^* \) with
\[
\partial c^* = \sum_{i=1}^{5} \Gamma_i \tilde{\Gamma}_i^*.
\]

We return to this example in Example 4.6, where we write down the algebra structure on these syzygies.

3.2. The Berkovits complex. Let \( A \) be a commutative quadratic algebra given by \( S(V)/I \), where \( V \) is a vector space with basis \( a_1, \ldots, a_n \), and let \( W_1 \) be the vector space concentrated in degree 1 with basis \( \theta_1, \ldots, \theta_n \).

Lemma 3.4. Suppose the quadratic ideal \( I \) is spanned by
\[
\Gamma_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j, \quad (1 \leq k \leq m).
\]
Then the first order syzygies are given by the following homology classes in \( A[W_1] \),
\[
\tilde{\Gamma}_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j, \quad (1 \leq k \leq m).
\]

This leads to the following definition.

Definition 3.5. The Berkovits complex of a commutative quadratic algebra \( A \) is
\[
A[W_1 \oplus W_2] := \left( A \otimes \bigwedge W_1 \otimes S(W_2), \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \tilde{\Gamma}_k \frac{\partial}{\partial y_k} \right),
\]
where \( W_2 \) is the vector space spanned by elements \( y_1, \ldots, y_m \) in homological degree two.

The Berkovits homology of \( A \) is the homology of this complex.

Definitions 3.1 and 3.5, which calculate Koszul and Berkovits homology respectively, are examples of semi-free extensions of commutative DG algebras, or relative Sullivan algebras (see for example [2, 9, 15, 16]):

Definition 3.6. A semi-free extension of \( A \) is an inclusion of commutative DG algebras
\[
A \to A \otimes S_*(W)
\]
for some positively graded vector space \( W \). We will write \( A[W] \) for \( A \otimes S_*(W) \).

By \( S_* \) we mean the symmetric algebra in the graded sense,
\[
S_*(W) = S(W_{\text{even}}) \otimes \bigwedge W_{\text{odd}}.
\]
For instance Definition 3.5 would read \( A[W_1 \oplus W_2] = A \otimes S_*(W_1 \oplus W_2) \).

3.3. Minimal models for commutative Koszul algebras. Assume that \( A \) is a commutative Koszul algebra with \( A^1 = U(L) \) as above. We construct a resolution of \( A \) in the category of DG algebras from the Chevalley complex of \( L \).

Definition 3.7. The Chevalley complex of \( L \) is the cochain complex with
\[
\text{Ch}^i(L) = \left( \bigwedge^i L \right)^*,
\]
and the differential \( d_C : \text{Ch}^k(L) \to \text{Ch}^{k+1}(L) \) is defined on \( \varphi \in \left( \bigwedge^k L \right)^* \) by
\[
(d_C \varphi)(x_0, \ldots, x_k) = \sum_{i < j} (-1)^{j + \varepsilon(i,j)} \varphi(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_k),
\]
where each \( x_r \) is homogeneous, \( \hat{x}_j \) means omit \( x_j \) and \( \varepsilon(i,j) = |x_j|(|x_{i+1}| + \cdots + |x_{j-1}|) \).
The Chevalley complex is a cochain complex that calculates the cohomology of \(L\) with trivial coefficients, \(H^*(L, \mathbb{C})\). Chevalley and Eilenberg in [5] introduced a shuffle product \(\odot\) with respect to which the differential \(d_C\) is a derivation and gives this complex an algebra structure that descends to the algebra structure on the cohomology of \(L\).

For locally finite dimensional graded Lie superalgebras, one can prove that the shuffle product \(\odot\) is skew-supersymmetric and that there is an algebra isomorphism
\[
\left(\bigwedge L^*, \wedge\right) \cong \left(\left(\bigwedge L\right)^*, \odot\right).
\]

Hence, the Chevalley complex can be alternatively defined as the exterior algebra on \(L^*\), with differential \(d_C\) the extension as a derivation of the map dual to \([\ , \ ] : L \wedge L \to L\).

As well as being a cochain complex whose cohomology is that of \(L\), the Chevalley complex may also be considered a chain complex which defines a resolution of \(A\). The chain complex is given by defining \(\text{Ch}^k(L) = \bigwedge^k L^* \oplus \bigwedge^k L^* \wedge \bigwedge^{k-1} L^*\) for \(k \geq 1\), so that the homological and cohomological gradings together give the total degree in \(\bigwedge L^*\). We illustrate the Chevalley complex as a chain complex with homological grading as follows:

\[
\cdots \xrightarrow{d_C} \text{Ch}_4(L) \xrightarrow{d_C} \text{Ch}_3(L) \xrightarrow{d_C} \text{Ch}_2(L) \xrightarrow{d_C} \text{Ch}_1(L) \xrightarrow{d_C} \text{Ch}_0(L)
\]

The original cohomological grading is seen on the diagonals: \(\text{Ch}^1(L) = L^*\) is the first non-zero diagonal and \(\text{Ch}^2(L)\) is the diagonal containing exterior products of two factors.

**Definition 3.8.** A minimal model of a commutative algebra \(A = S(V)/I\) is a semi-free extension \(S(V) \to S(V)[W]\) such that

1. the canonical quotient \(S(V)[W] \to S(V) \to A\) is an isomorphism in homology,
2. the differential is decomposable, \(\partial W \subseteq S(V) \otimes S^{\geq 2}W\).

This concept arose in rational homotopy theory, see for example [9, Definition 1.10] or [2, Section 7.2].

**Proposition 3.9.** For a commutative Koszul algebra \(A = U(L)\), the Chevalley complex of \(L\) with homological grading is a minimal model of \(A\).

**Proof.** We observe that \(\text{Ch}_0(L) = \bigwedge L^*_1 = S(V)\). If we take \(W_k = L^*_k\) for \(k \geq 1\) then \(\text{Ch}(L) = \bigwedge^k L^*_1\) is the semi-free extension \(S(V)[W]\).

As \(L\) is graded, so is \(\text{Ext}^*_{U(L)}(\mathbb{C}, \mathbb{C})\), and since \(A\) is Koszul
\[
H^i(U(L), \mathbb{C}) \cong \text{Ext}^i_{U(L)}(\mathbb{C}, \mathbb{C}) = \begin{cases} A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},
\]
where \(j\) is the total degree. Using our homological grading of \(\text{Ch}(L)\) we have therefore,
\[
H_i(\text{Ch}(L)) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}
\]
and the Chevalley complex of \(L\) is a resolution of \(A\).

Finally, the differential is decomposable, since \(d_C(L^*_n) \subseteq \sum_{i=1}^{n-2} L^*_i \wedge L^*_{n-i} \).

**Remark 3.10.** It follows that the Hilbert series of \(A\) may also be calculated as the Euler characteristic of the Chevalley complex with homological grading, and we recover (4).
We will now give a proof of the following theorem, based on a more general result of Avramov.

**Theorem 3.11.** The algebra of syzygies of $A$ is isomorphic to $H^*(L_{\geq 2}, \mathbb{C})$ and the Berkovits homology is isomorphic to $H^*(L_{\geq 3}, \mathbb{C})$.

Recall first the following definition from [2, Section 6.3].

**Definition 3.12.** The acyclic closure of a commutative DG algebra $A$ is a semi-free extension $A \rightarrow A[W]$ such that $A[W]$ is a resolution of $\mathbb{C}$ and $\{\partial w \mid w \in W_{n+1}\}$ minimally generates the reduced homology $\tilde{H}_n(A[W_{\leq n}])$ for each $n \geq 0$.

Thus the complexes calculating the Koszul and Berkovits homology are simply the first two steps of the acyclic closure of $A$. The following result of Avramov may be used to relate a general stage of the acyclic closure to the minimal model of $A$.

Recall that two DG algebras $X$ and $X'$ are quasi-isomorphic if there is a sequence of quasi-isomorphisms of DG algebras $X \sim X^1 \sim \cdots \sim X^n \sim X'$ pointing in either direction.

**Theorem 3.13 ([2, Theorem 7.2.6]).** Suppose $A$ is a commutative DG algebra with acyclic closure $A[W]$ and minimal model $S(V)[W']$. Then, for each $n \geq 1$, the DG algebras $A[W_{\leq n}]$ and $S(V)[W'][/W_{\leq n}]$ are quasi-isomorphic.

If $A$ is Koszul with $A' = U(L)$ then the Chevalley complex of $L$ with homological grading is a minimal model $S(V)[W']$ of $A$, where $W_i = L_{i+1}^*$. Since $S(V)[W']/(W_{\leq n}) = \text{Ch}(L_{\geq n+1})$ the theorem gives a quasi-isomorphism

$$A[W_{\leq n}] \sim \text{Ch}(L_{\geq n+1}).$$

**Theorem 3.11** is just the cases $n = 1$ and 2.

4. A SMALLER COMPLEX

Let $A$ be a Koszul commutative algebra, with Koszul dual $A'$ the universal enveloping algebra of a graded Lie superalgebra $L$. We have seen above that the Chevalley complex of $L$ (with homological grading) is a resolution of $A$, and the Chevalley complexes of $L_{\geq 2}$ and $L_{\geq 3}$ calculate the algebra of syzygies $R$ and the Berkovits homology respectively.

The Berkovits complex is not module over $\mathbb{C}[y_1, \ldots, y_m]$, which makes the its homology difficult to calculate in practice. There is a construction for the case of the orthogonal Grassmanian $\text{OG}(5, 10)$ alluded to in the introduction which appears in Movshev–Schwarz [13], where they found an alternative complex quasi-isomorphic to the Berkovits complex, which is a module over $\mathbb{C}[y_1, \ldots, y_m]$. This gives them a more manageable complex and they are able to calculate its homology directly.

Inspired by this construction, we give an alternative complex for calculating the Berkovits homology, under the assumption that the algebra of syzygies is itself quadratic, although this condition may be relaxed (see 4.2 below). We also borrow the name ‘$bv_{\mu}$’ from Movshev–Schwarz.

Suppose the algebra of syzygies $R$ is generated by a finite dimensional vector space $U$, containing in particular the elements $\Gamma_k$ of Lemma 3.4. Choose a basis $\{y_1, \ldots, y_m\}$ of $L_2$ inside the symmetric algebra $S(L_2)$ and consider the complex

$$bv_{\mu} = S(L_2) \otimes_{R'} K(R') \cong S(L_2) \otimes_{R'} R^i \otimes_{\mathbb{C}} R^*.$$

Here $L_2$ is concentrated in degree 0, and as usual for the Koszul complex we have $R^i$ concentrated in degree 0 and $R^*$ has grading induced by taking $U^*$ as concentrated in
degree 1. Consider the degree zero map of quadratic algebras

$$\bigwedge L^*_2 \to R, \quad y_k^* \mapsto \tilde{\Gamma}_k.$$

The dual map $U^* \to L_2$ induces a map $R^i \to S(L_2)$ which gives an action of $R^i$ on $S(L_2)$ by multiplication.

The differential is $1 \otimes d$ where $d$ is the Koszul differential of $K(R!)$.

Taking into account the natural grading on the symmetric algebra we see that $b_{\nu}$ is filtered by the complexes $S^{\geq p}(L_2) \otimes R^*$ and the associated graded is isomorphic to the bigraded vector space

$$\text{Gr}(b_{\nu}) = S(L_2) \otimes R^*.$$  \hspace{1cm} (6)

Choose a basis $\{q_1, \ldots, q_m\}$ of $L_2$ inside $U(L \geq 2)$. Now let

$$E_{\mu} = \left( \bigwedge L_2 \otimes U(L \geq 2), \sum_{k=1}^m y_k \frac{\partial}{\partial q_k} \right)$$

with the standard augmented algebra structure. This has a filtration $\bigwedge^{\geq p} L_2 \otimes U(L \geq 2)$ whose associated graded $\text{Gr}(E_{\mu})$ is the algebra itself but with zero differential. The reduced bar complex $B^+ E_{\mu}$ has an induced filtration, and we observe that the corresponding associated complex is isomorphic to the bar complex of the associated graded,

$$\text{Gr}(B^+ E_{\mu}) = B^+(\text{Gr}(E_{\mu})) = B^+ \left( \bigwedge L_2 \otimes U(L \geq 2) \right).$$ \hspace{1cm} (7)

We show that $b_{\nu}$ calculates the cohomology of $L_{\geq 3}$ by adapting the argument from [10, Section 3] which studies the case where $R^i \cong U(L \geq 2)$ is the Yang-Mills algebra, and which is also inspired by [13].

The idea is to use the complex $B^+ E_{\mu}$ as an intermediary whose homology is the same as $b_{\nu}$ and $\text{Ch}(L_{\geq 3})$:

$$b_{\nu} \sim \sim \to B^+ E_{\mu} \sim \sim \to \text{Ch}(L_{\geq 3}).$$

**Theorem 4.1.** Homologies of the complexes $b_{\nu}$ and $B^+ E_{\mu}$ coincide.

**Proof.** Consider the spectral sequences associated to the filtrations on $b_{\nu}$ and $B^+ E_{\mu}$ defined above, which converge to their respective homologies and which have zero pages the associated graded objects (6), (7).

By Theorem 3.11 we can say that there exists a chain map

$$R^* \cong R \underbrace{\cong \text{Ch}(L \geq 2)} \underbrace{\cong B^+(U(L \geq 2))}$$

that induces isomorphism of homology groups, given by some choice of homology classes in the Chevalley or bar complexes. Using this, together with classical Koszul duality for symmetric and exterior algebras, and the Künneth theorem, we can construct a chain map

$$S(L_2) \otimes R^* \sim \to B^+ \left( \bigwedge L_2 \otimes B^+(U(L \geq 2)) \right) \sim \to B^+ \left( \bigwedge L_2 \otimes U(L \geq 2) \right)$$

inducing isomorphism in homology. In other words, we have a chain map inducing isomorphism between the homologies of the associated graded objects above,

$$\text{Gr}(b_{\nu}) \sim \to \text{Gr}(B^+ E_{\mu}).$$
This is a weak equivalence between the zero pages of the spectral sequences associated to the filtrations on $bv_\mu$ and $B^+E_\mu$. Hence we have an isomorphism between the first pages.

The result follows from the comparison theorem of spectral sequences, as long as we have a map $p : bv_\mu \to B^+E_\mu$ that induces the map on the associated filtrations of $bv_\mu$ and $B^+E_\mu$. The reduced bar complex is constructed as a quasi-isomorphism $E_\mu \to B^+E_\mu$ and so only need to construct a chain map $bv_\mu \to E_\mu$. We can take the map $y_k \mapsto y_k$ and $\Gamma_k \mapsto q_k$. 

For the second quasi-isomorphism we adapt the proof in [10].

**Proposition 4.2.** The algebras $E_\mu$ and $U(L_{\geq 3})$ are quasi-isomorphic, or the homologies of $B^+(E_\mu)$ and $\text{Ch}(L_{\geq 3})$ coincide. The quasi-isomorphism is given by the inclusion $z \in U(L_{\geq 3}) \hookrightarrow 1 \otimes z \in E_\mu$.

**Proof.** We must prove that the cohomology of $E_\mu$ is $U(L_{\geq 3})$. Consider the following filtration of $U(L_{\geq 2})$:

$$F^j := \begin{cases} 0 & j = 0 \\ \{ z \in U(L_{\geq 2}) \mid \frac{\partial}{\partial q_i}(z) \in F^{j-1} \forall i \} & j > 0 \end{cases}$$

Poincaré-Birkhoff-Witt implies that $F^1 = U(L_{\geq 3})$ (see [10, Proposition 3.1] for more details) and by [13, Lemma 28] the filtration is multiplicative, exhaustive, Hausdorff ($\bigcap F^j = 0$) and $q_1, q_2 \in F^2$. We define a filtration of $E_\mu$ as in [10, Proposition 3.7]:

$$F_p E_{\mu,q} := F^{p+m-q} \otimes \bigwedge^{m-q} L_2.$$ 

It is clear that $d(F_p E_{\mu,q}) \subseteq F_{p-2} E_{\mu,q-1}$, where $d$ is the differential on the complex $E_\mu$, and so the differentials on the $E^0$ and $E^1$-pages of the spectral sequence associated to this filtration are zero.

Since [13, Lemma 29] is true in this context, we have that $\text{Gr}_F(U(L_{\geq 2})) \cong U(L_{\geq 3}) \otimes \mathbb{C}[\tilde{q}_1, \ldots, \tilde{q}_m]$, where $\tilde{q}_1, \ldots, \tilde{q}_m$ are the images of $q_1, \ldots, q_m$ in $\text{Gr}_F(U(L_{\geq 2}))$. The $E^2$-page is given by

$$E^2_{pq} = F_p E_{\mu,p+q} / F_{p-1} E_{\mu,p+q} \cong U(L_{\geq 3}) \otimes \mathbb{C}[\tilde{q}_1, \ldots, \tilde{q}_m] \otimes \bigwedge^{m-p-q} L_2.$$ 

The differential on the $E^2$-page is nothing but the Koszul differential for $\bigwedge L_2$ (see [10, Proposition 3.7] for more details), which is a resolution of $\mathbb{C}$. Hence, the spectral sequence collapses on the $E^3$-page to $E^3_{0,m} = U(L_{\geq 3})$. 

We have the following immediate corollary.

**Corollary 4.3.** If the algebra of syzygies $R$ is quadratic, then the complex $bv_\mu$ calculates the Berkovits homology.

4.1. **Application to Grassmanians** $G(2, N)$. Consider the Grassmanian as a homogeneous space $\text{SL}_N(\mathbb{C})/P$, where $P$ is some minimal complex parabolic subgroup that is not a Borel. In the usual way, the Plücker embedding of $G(2, N)$ is given by the orbit of the highest weight vector in a highest weight representation of $\text{SL}_N(\mathbb{C})$. The corresponding projective coordinate algebra $A_{G(2,N)}$ is quadratic and Koszul and moreover, as shown by Gorodentsev et al. in [8], its algebra of syzygies $R_{G(2,N)}$ is also quadratic and Koszul. They in fact give a full set of generators and relations for $R_{G(2,N)}$ in terms of semistandard Young tableaux. We recall some definitions regarding semistandard Young tableaux and state this result here.
Definition 4.4. A partition $\lambda = [\lambda_1, \ldots, \lambda_k]$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, is represented by a Young diagram with rows of lengths $\lambda_1, \ldots, \lambda_k$. The transposed diagram is denoted $\lambda' = [\lambda'_1, \ldots, \lambda'_k]$. A partition $\lambda$ is written in Frobenius notation as

$$(\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p) \quad \lambda_i = \alpha_i + i, \quad \lambda'_i = \beta_i + i.$$ 

For instance, the partitions $[6, 4, 3, 3, 1]$ and $[6, 4, 3, 3, 1]' = [5, 4, 4, 2, 1, 1]$ are depicted

and are written $(5, 2, 0|4, 2, 1)$ and $(4, 2, 1|5, 2, 0)$ in Frobenius notation.

A semistandard Young tableau is a Young diagram in which the boxes are labelled with numbers $1, \ldots, N$ weakly increasing across rows and strictly increasing down columns. For instance,

\begin{align*}
&
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
& 7 & 8 & 9 & 10 \\
& 11 & 12 \\
& & 13 \\
& & & 14 \\
& & & & 15 \\
& & & & & 16
\end{array}
\end{align*}

(8)
The set of all semistandard Young tableaux of shape $(\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)$ is denoted

$$\pi_{(\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)}.$$ 

For each partition $\lambda$ there is an irreducible $GL_k(\mathbb{C})$-module (and in fact $SL_k(\mathbb{C})$-module), also denoted by $\pi_{(\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)}$, obtained as the quotient space

$$\left( \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_p} V \right) / \text{column exchange relations},$$

where $V$ is the standard module for $GL_k(\mathbb{C})$ or $SL_k(\mathbb{C})$. We do not make these relations explicit, except to note that the semistandard Young tableaux in $\pi_{(\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)}$ define basis vectors in this representation. For example the basis vector for (8) is

$$(e_1 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6) \otimes (e_1 \wedge e_3 \wedge e_5 \wedge e_6) \otimes (e_2 \wedge e_4 \wedge e_5 \wedge e_6) \otimes (e_4 \wedge e_6) \otimes e_5 \otimes e_5,$$

where $e_1, \ldots, e_6$ are a basis of the standard module for $GL_6(\mathbb{C})$ or $SL_6(\mathbb{C})$.

One introduces the algebra

$$A(N) = \bigoplus_{p,q} A_{p,q}(N)$$

where

$$A_{p,q}(N) := \bigoplus_{N-2 \geq i_1 \geq \cdots \geq i_p \geq 2} \pi_{((i_1-2), \ldots, (i_p-2)|(i_1+1), \ldots, (i_p+1))}.$$ 

This algebra is generated by

$$A_{1,r} = \pi_{(r-2|r+1)},$$

for $2 \leq r \leq N-2$. The multiplication $A_{1,r_1} \otimes A_{1,r_2} \to A_{2,r_1+r_2}$ is given by the projection:

$$\pi_{(r_1-2|r_1+1)} \otimes \pi_{(r_2-2|r_2+1)} \to \pi_{(r_1-2,r_2-2|r_1+2,r_2+1)}.$$ 

We have the following:

Theorem 4.5. [8, Theorem 4.4.1] The syzygies of $G(2, N)$ form a bigraded supercommutative Frobenius Koszul algebra isomorphic to $A(N)$ with

$$(R_{G(2,5)})_{pq} = A_{q-p,q}(N).$$
Example 4.6. We return to Example 3.3 and consider the algebra of syzygies of $G(2,5)$, in which case

$$A(5) = A_{0,0}(5) \oplus A_{1,2}(5) \oplus A_{1,3}(5) \oplus A_{2,5}(5)$$

$$= \pi_0 \oplus \pi_{(0|3)} \oplus \pi_{(1|4)} \oplus \pi_{(10|43)}$$

$$= (R_{G(2,5)})_{00} \oplus (R_{G(2,5)})_{12} \oplus (R_{G(2,5)})_{23} \oplus (R_{G(2,5)})_{35},$$

where $\pi_0$ denotes the empty semistandard Young tableau and the corresponding generator in $R_{G(2,5)}$ is given by $c$. The full list of other semistandard Young tableaux along with their generators in $R_{G(2,5)}$ are given by

$$\pi_{(0|3)}: \Gamma_1 = \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array}, \quad \Gamma_2 = \begin{array}{c} 1 \end{array} \begin{array}{c} 4 \end{array}, \quad \Gamma_3 = \begin{array}{c} 1 \end{array} \begin{array}{c} 5 \end{array}, \quad \Gamma_4 = \begin{array}{c} 2 \end{array} \begin{array}{c} 5 \end{array}, \quad \Gamma_5 = \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array}$$

$$\pi_{(1|4)}: \tilde{\Gamma}_i = \begin{array}{c} i \end{array} \begin{array}{c} i+1 \end{array}, \quad \pi_{(10|43)}: \ c^* = \begin{array}{c} 1 \end{array} \begin{array}{c} 1 \end{array} \begin{array}{c} 1 \end{array} \begin{array}{c} 1 \end{array} \begin{array}{c} 1 \end{array}$$

so that $|\pi_{(0|3)}| = |\pi_{(1|4)}| = 5$ and $|\pi_{(10|43)}| = 1$. Further, the algebra structure is given by $\tilde{\Gamma}_i \tilde{\Gamma}_j \tilde{\Gamma}_k = \Gamma_j \tilde{\Gamma}_i = (-1)^{i+j} \delta_{ij} c^*$, $c$ is the unit of the algebra and all other multiplications are zero.

Corollary 4.3 and Theorem 4.5 mean that we can use the complex $bv_\mu$ to calculate some cohomologies of $L_{\geq 3}$.

Theorem 4.7. For $N = 5$, we have $H^2(L_{\geq 3}, \mathbb{C}) = 0$ and so $L_{\geq 3}$ is free. On the other hand, for $N > 5$, we have $H^{N-4}(L_{\geq 3}, \mathbb{C}) \neq 0$, and so $L_{\geq 3}$ is not free.

Proof. For $N = 5$, we use a result from [4, Ex II.2.9] that if $L_{\geq 3}$ is positively graded then $L_{\geq 3}$ is a free Lie (super)algebra if and only if $H^2(L_{\geq 3}, \mathbb{C}) = 0$. The differential in the complex $bv_\mu$ is:

$$d(c) = 0, \quad d(\tilde{\Gamma}_i) = y_i c, \quad d(\tilde{\Gamma}_i^*) = 0, \quad d(c^*) = \sum_{i=1}^{5} (-1)^{i+1} y_i \tilde{\Gamma}_i^*.$$  

Since the only contribution to $H^2(L_{\geq 3}, \mathbb{C})$ could be given by $c^*$, and this is not a cycle, we obtain $H^2(L_{\geq 3}, \mathbb{C}) = 0$.

For $N > 5$, we use the fact that if $H^k(L_{\geq 3}, \mathbb{C}) \neq 0$ for some $k \geq 2$ then $L_{\geq 3}$ is not a free Lie superalgebra. We always have a collection of generators of $(R_{G(2,N)})_{12} = A_{1,2}(N)$ which are semistandard Young tableaux of shape $(0|3)$:

$$\begin{array}{c} \setlength{\arraycolsep}{0pt} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \end{array} \begin{array}{c} \setlength{\arraycolsep}{0pt} \begin{array}{c} 1 \end{array} \begin{array}{c} 4 \end{array} \end{array} \begin{array}{c} \setlength{\arraycolsep}{0pt} \begin{array}{c} 1 \end{array} \begin{array}{c} 5 \end{array} \end{array} \begin{array}{c} \setlength{\arraycolsep}{0pt} \begin{array}{c} 2 \end{array} \begin{array}{c} 5 \end{array} \end{array} \begin{array}{c} \setlength{\arraycolsep}{0pt} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \end{array} \end{array}$$

We continue to call these generators $\tilde{\Gamma}$. We also have a unique generator of highest order $N - 3$ given by the rectangular $((N - 3) \times N)$ shape semistandard Young tableau associated to the set

$$A_{N-3,(N(N-3)/2)}(N) = \pi_{(N-4,N-5,...,0,N-1,N-2,...,1)}.$$

We continue to call this generator $c^*$. We now exhibit semistandard Young tableaux which is of order $N - 4$ giving non-trivial homology classes in the complex $bv_\mu$. Namely, we consider the elements $\tilde{\Gamma}^*$ in

$$A_{N-4,(N(N-4)(N+1)/2)}(N) = \pi_{(N-4,N-5,...,1,N-1,N-2,...,4)},$$
with the rectangular shape of $c^*$ but missing the shape $(0|3)$ at the bottom right:

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

In order to have $d(\tilde{\Gamma}^*) \neq 0$, these tableaux would have to factorize into products of tableaux of shape $(0|3)$ and tableaux of some shape $\kappa$. However no such shape $\kappa$ is found in $A(N)$, so $d(\tilde{\Gamma}^*) = 0$ and the $\tilde{\Gamma}^*$ are cycles. Finally we observe that $d(c^*)$ is still a sum of terms of the form $y\tilde{\Gamma}^*$, as in the case $N = 5$, and hence the cycles $\tilde{\Gamma}^*$ are not boundaries.

This means we have exhibited a non-trivial cohomology class in $H^{N-4}(L \geq 3, \mathbb{C})$, proving the claim. □

4.2. Further generalization. We presented a simple construction of a complex $bv_\mu$ quasi-isomorphic to the Berkovits homology and to the Lie algebra $L \geq 3$, which made certain calculations very straightforward. Although it is already quite general, the construction may be further extended.

We suspect that one may enrich the homology isomorphism between the complexes $bv_\mu$ and $B^+ E_\mu$. Namely, one would like quasi-isomorphisms of differential graded algebras: between the cobar construction $\Omega bv_\mu$, the algebra $E_\mu$, and the algebra $U(L \geq 3)$. For this, however, it would be necessary to define an appropriate coalgebra structure on $bv_\mu$. In the case that the algebra of syzygies $R$ is quadratic, $bv_\mu$ will be a ‘strict’ differential graded coalgebra, and other cases it will be an $A_\infty$-coalgebra. An explicit definition of such an $A_\infty$ structure was given in the paper [13] of Movshev–Schwarz for the example they considered.

Observe that we may also construct a complex, termed $bv_0$ in [13], whose cobar $\Omega bv_0$ is quasi-isomorphic as an algebra to $U(L \geq 2)$ and whose homology coincides with the algebra of syzygies $R$. In fact there is a canonical differential $Q$ on the cobar construction of the cohomology $H^*(L \geq 2, \mathbb{C})$ which gives an algebra quasi-isomorphism

\[
(\Omega H^*(L \geq 2, \mathbb{C}), Q) \sim U(L \geq 2).
\]

In other words, $H^*(L \geq 2, \mathbb{C})$, which is just $R$, has an $A_\infty$ coalgebra structure, and we term this structure $bv_0$. Then the $A_\infty$ coalgebra structure on $bv_\mu$ should be a perturbation of that on the tensor product $Gr(bv_\mu) = S(L_2) \otimes R^*$.

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