RIBBON STRUCTURES OF THE DRINFIELD CENTER

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Abstract. We classify the ribbon structures of the Drinfeld center $Z(\mathcal{C})$ of a finite tensor category $\mathcal{C}$. Our result generalizes Kauffman and Radford’s classification result of the ribbon elements of the Drinfeld double of a finite-dimensional Hopf algebra. As a consequence, we see that $Z(\mathcal{C})$ is a modular tensor category in the sense of Lyubashenko if $\mathcal{C}$ is a spherical finite tensor category in the sense of Douglas, Schommer-Pries and Snyder.

1. Introduction

A braided monoidal category is a monoidal category $\mathcal{B}$ equipped with an isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the hexagon axiom, and a ribbon category is a braided rigid monoidal category $\mathcal{B}$ equipped with a ribbon structure (also called a twist), that is, a natural isomorphism $\theta : \text{id}_B \to \text{id}_B$ satisfying

\[ \theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ \sigma_{Y,X} \circ \sigma_{X,Y}, \]

\[ (\theta_X)^* = \theta_X^* \]

for all $X, Y \in \mathcal{B}$, where $(-)^*$ is the duality functor; see, e.g., [EGNO15]. These notions are used, for example, to formulate and construct several kinds of topological invariants or, more generally, topological quantum field theory.

Given a rigid monoidal category $\mathcal{C}$, we have a braided rigid monoidal category $Z(\mathcal{C})$ called the Drinfeld center of $\mathcal{C}$ (see Subsection 3.1 for our convention). The Drinfeld center does not admit a twist in general. In this paper, we classify the ribbon structures of $Z(\mathcal{C})$ in the case where $\mathcal{C}$ is a finite tensor category in the sense of [EO04]. A typical example of a finite tensor category is the category $H$-mod of finite-dimensional left modules over a finite-dimensional Hopf algebra $H$. As is well-known, the Drinfeld center of $H$-mod is identified with $D(H)$-mod, where $D(H)$ is the Drinfeld double. Our result can be thought of as a categorical generalization of Kauffman and Radford’s classification result of the ribbon elements of the Drinfeld double of a finite-dimensional Hopf algebra [KR93].

Etingof, Nikshych and Ostrik [ENO04] have introduced the distinguished invertible object $\alpha$ of a finite tensor category $\mathcal{C}$. Following [ENO04], there is a natural isomorphism $\delta_X : \alpha \otimes X \to X^{***} \otimes \alpha$ ($X \in \mathcal{C}$). As this theorem generalizes the celebrated Radford $S^4$-formula, we call $\delta$ the Radford isomorphism. Our classification result claims that the ribbon structures of $Z(\mathcal{C})$ are parametrized by ‘square roots’ of the Radford isomorphism (cf. [KR93, Theorem 3]). The precise statement requires a bit big system of notations, so we omit it here.

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Our result yields a new example of ‘non-semisimple’ modular tensor categories in the sense of Lyubashenko [Lyu95a, Lyu95b, Lyu95c, KL01]. If $B$ is a braided finite tensor category, then the coend $F = \int^X \otimes B X \otimes X^*$ has a canonical paring $\omega : F \otimes F \to 1$ defined in terms of the braiding. We say that $B$ is non-degenerate if $\omega$ is. A modular tensor category [KL01] is a non-degenerate ribbon finite tensor category. The braided finite tensor category $Z(\mathcal{C})$ is always non-degenerate by [Shi16a] and [EGNO15, Proposition 8.6.3], but it does not have a ribbon structure in general. Our result determines when $Z(\mathcal{C})$ admits a ribbon structure, and hence is modular. For example, $Z(\mathcal{C})$ is a modular tensor category if $\mathcal{C}$ is spherical in the sense of Douglas, Schommer-Pries and Snyder [DSS13, Definition 4.5.2]. Thus we have obtained an answer to Open Problem (7) of [Mug10, Section 6].

**Organization of this paper.** This paper is organized as follows: In Section 2, we collect some basic results on monoidal categories from [ML98, EGN015] and fix some notations used throughout in this paper.

In Section 3, for two tensor functors $F, G : \mathcal{C} \to \mathcal{D}$ between finite tensor categories $\mathcal{C}$ and $\mathcal{D}$, we introduce the category $Z(F, G)$. An object of this category is a pair $(V, \sigma)$ consisting of an object $V \in \mathcal{D}$ and a natural isomorphism

$$\sigma_X : V \otimes F(X) \to G(X) \otimes V \quad (X \in \mathcal{C})$$

satisfying certain conditions. The Drinfeld center $Z(\mathcal{C})$ is the case where $F$ and $G$ are the identity functor. Unlike $Z(\mathcal{C})$, the category $Z(F, G)$ does not have a tensor product. Though, for three tensor functors $F, G, H : \mathcal{C} \to \mathcal{D}$, one can define the tensor product $\otimes : Z(G, H) \times Z(F, G) \to Z(F, H)$. These categories, as well as this tensor product, are useful to formulate our classification result.

The main result of Section 3 is a monadic description of $Z(F, G)$. Given tensor functors $F, G : \mathcal{C} \to \mathcal{D}$, one can define an algebra $A_{F,G} \in \mathcal{D} \otimes \mathcal{D}^{\text{rev}}$ as a coend of a certain functor. There is a canonical action of $\mathcal{D} \otimes \mathcal{D}^{\text{rev}}$ on $\mathcal{D}$, and hence the algebra $A_{F,G}$ defines a monad on $\mathcal{D}$. We see that the Eilenberg-Moore category of this monad can be identified with $Z(F, G)$. As a consequence, $Z(F, G)$ is a finite abelian category (Theorem 3.4).

The results of Section 3 also allows us to use representation-theoretical techniques to analyze the category of the form $Z(F, G)$. In Section 4, we use this strategy to introduce the Radford object and study its relation to the relative modular object introduced in [Shi17b].

Let $\mathcal{C}$ be a finite tensor category, and let $\text{Hom}$ be the internal Hom functor of the $C \boxtimes C^{\text{rev}}$-module category $\mathcal{C}$. Set $A = \text{Hom}(1, 1)$. Etingof, Nikshych and Ostrik [ENO04] have proved that there is an equivalence

$$K : \mathcal{C} \to (\text{the category of $A$-modules in } C \boxtimes C^{\text{rev}}), \quad V \mapsto (V \boxtimes 1) \otimes A$$

of left $(C \boxtimes C^{\text{rev}})$-module categories. By the results of Section 3, we see that this equivalence induces an equivalence

$$Z(\text{id}_C, S^4_C) \approx (\text{the category of } A^{**}-A\text{-bimodules in } C \boxtimes C^{\text{rev}})$$

of categories, where $S^4_C$ is the duality functor of $C$. The Radford object is defined to be the object $\alpha_C \in Z(\text{id}_C, S^4_C)$ corresponding to $A^*$. This object capsulates the main result of [ENO04]; see Appendix A.

Given a tensor functor $F : \mathcal{C} \to \mathcal{D}$ whose right adjoint is exact, one can define the relative modular object $\mu_F \in \mathcal{D}$ [Shi17b]. As noted in [Shi17b], this object has
a canonical isomorphism \( \gamma_X : \mu_F \otimes F(X) \to F(X) \otimes \mu_F \) (\( X \in \mathcal{C} \)) such that, in our notations, \( \mu_F := (\chi_F, \gamma) \in \mathcal{Z}(F, F) \). Refining the main result of \cite{Shi17b}, we show that there is an isomorphism \( \alpha_{Z(F)} \otimes \mu_F \cong F(\alpha_c) \) in \( \mathcal{Z}(F, S^1_\mathcal{C}F) \); see Subsection 3.3 for the notation.

The main result of this paper is stated and proved in Section 5. We note that the set of natural isomorphisms \( \theta : \text{id}_{\mathcal{Z}(\mathcal{C})} \to \text{id}_{\mathcal{Z}(\mathcal{C})} \) satisfying (1.1) is in bijection with the set of pivotal structures of \( \mathcal{Z}(\mathcal{C}) \). We say that a pivotal structure of \( \mathcal{Z}(\mathcal{C}) \) is ribbon if the corresponding natural isomorphism \( \theta : \text{id}_{\mathcal{Z}(\mathcal{C})} \to \text{id}_{\mathcal{Z}(\mathcal{C})} \) is a ribbon structure. By the results of \cite{ENO04}, the condition for a pivotal structure of \( \mathcal{Z}(\mathcal{C}) \) to be ribbon can be written in terms of the Radford object of \( \mathcal{Z}(\mathcal{C}) \).

For this reason, it is important to know the Radford object of \( \mathcal{Z}(\mathcal{C}) \). By using the result of \cite{Shi16b}, it is described as follows: Given a tensor autoequivalence \( F \) on \( \mathcal{C} \), we denote by \( \tilde{F} \) the braided tensor autoequivalence induced by \( F \). The main result of \cite{Shi16b} claims that there is a bijection
\[
\Theta : \mathcal{Z}(F, G)^{\times} \to \text{Nat}_{\otimes}(\tilde{F}, \tilde{G})
\]
for two tensor autoequivalences \( F \) and \( G \) on \( \mathcal{C} \), where \( \mathcal{Z}(F, G)^{\times} \) is the set of isomorphism classes of invertible objects of \( \mathcal{Z}(F, G) \). By considering the relative modular object of the forgetful functor \( U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \), we have
\[
\alpha_{\mathcal{Z}(\mathcal{C})} = (\text{id}_{\mathcal{Z}(\mathcal{C})}, \Theta(\alpha_c))
\]
as an object of \( \mathcal{Z}(\text{id}_{\mathcal{Z}(\mathcal{C})}, S^1_\mathcal{Z}(\mathcal{C})) \). Finally, by using the functorial property of the map \( \Theta \), we prove that the bijection \( \Theta \) restricts to a bijection between the set
\[
\{ [\beta] \in \mathcal{Z}(\text{id}_{\mathcal{C}}, S^1_\mathcal{C})^{\times} \mid S^1_\mathcal{C}(\beta) \otimes \beta \cong \alpha_c \}
\]
and the set of ribbon pivotal structures of \( \mathcal{Z}(\mathcal{C}) \) (Theorem 5.8). This generalizes a result of Kauffman and Radford to the setting of finite tensor categories \cite{KR93}.

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2. Preliminaries

2.1. Monoidal categories. A monoidal category \cite[VII.1]{ML08} is a category \( \mathcal{C} \) endowed with a functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) (called the tensor product), an object \( \mathbb{1} \in \mathcal{C} \) (called the unit object), and natural isomorphisms
\[
(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \text{and} \quad \mathbb{1} \otimes X \cong X \otimes \mathbb{1} \quad (X, Y, Z \in \mathcal{C})
\]
satisfying the pentagon and the triangle axioms. If these natural isomorphisms are identities, then \( \mathcal{C} \) is said to be strict. In view of the Mac Lane coherence theorem, we may assume that all monoidal categories are strict.

2.2. Monoidal functors. Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories. A (lax) monoidal functor \cite[XI.2]{ML08} from \( \mathcal{C} \) to \( \mathcal{D} \) is a functor \( F : \mathcal{C} \to \mathcal{D} \) endowed with a morphism \( F_0 : \mathbb{1} \to F(\mathbb{1}) \) in \( \mathcal{C} \) and a natural transformation
\[
F_2(X, Y) : F(X) \otimes F(Y) \to F(X \otimes Y) \quad (X, Y \in \mathcal{C})
\]
satisfying certain conditions. A monoidal functor \( F \) is said to be strong if \( F_2 \) and \( F_0 \) are invertible, and said to be strict if \( F_2 \) and \( F_0 \) are identities.

Let \( F \) and \( G \) be monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \). A monoidal natural transformation from \( F \) to \( G \) is a natural transformation \( \xi : F \to G \) between underlying
functors satisfying $\xi_1 \circ F_0 = G_0$ and $\xi_{X \otimes Y} \circ F_2(X, Y) = G_2(X, Y) \circ (\xi_X \otimes \xi_Y)$ for all objects $X, Y \in \mathcal{C}$. If $\mathcal{C}$ is essentially small, then we denote by $\text{Nat}_{\otimes}(F, G)$ the set of all monoidal natural transformations from $F$ to $G$.

2.3. **Rigidity.** We fix our convention for dual objects in a monoidal category. Let $L$ and $R$ be objects of a monoidal category $\mathcal{C}$, and let $\varepsilon : L \otimes R \to \mathbb{1}$ and $\eta : \mathbb{1} \to R \otimes L$ be morphisms of $\mathcal{C}$. We say that $(L, \varepsilon, \eta)$ is a left dual object of $\mathcal{C}$ and $(R, \varepsilon, \eta)$ is a right dual object of $L$ if the equations

$$(\varepsilon \otimes \text{id}_L) \circ (\text{id}_L \otimes \eta) = \text{id}_L \quad \text{and} \quad (\text{id}_R \otimes \varepsilon) \circ (\eta \otimes \text{id}_R) = \text{id}_R$$

are satisfied. The monoidal category $\mathcal{C}$ is rigid if every object of $\mathcal{C}$ has a left dual object and a right dual object. If this is the case, then we denote by

$$(V^*, \text{ev}_V, \text{coev}_V) \quad \text{and} \quad (*V, \text{ev}'_V, \text{coev}'_V)$$

a (fixed) left dual object and a (fixed) right dual object of $V \in \mathcal{C}$, respectively. The assignment $V \mapsto V^*$ extends to a strong monoidal functor $(-)^* : \mathcal{C}^{\text{op}} \to \mathcal{C}^{\text{op}}$, which we call the left duality functor of $\mathcal{C}$. A right duality functor $*(-) : \mathcal{C}$ is defined analogously. We may assume that $(-)^*$ and $*(-)$ are strict monoidal functors and $*(-)$ is the inverse of $(-)^*$.

2.4. **Modules over a monoidal category.** Let $\mathcal{C}$ be a monoidal category. A left $\mathcal{C}$-module category is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (called the action) and natural isomorphisms

$$a_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M) \quad \text{and} \quad \ell_M : \mathbb{1} \otimes M \to M$$

satisfying certain coherence conditions similar to the axioms for monoidal categories. Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-module categories. A lax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F : \mathcal{M} \to \mathcal{N}$ equipped with a natural transformation

$$\xi_{X,M} : X \otimes F(M) \to F(X \otimes M) \quad (X \in \mathcal{C}, M \in \mathcal{M})$$

compatible with the natural isomorphisms $a$ and $\ell$ in the above. We omit the definition of morphisms of lax left $\mathcal{C}$-module functors; see [EGNO15, Chapter 7] for the precise definitions.

We note that $\mathcal{M}^{\text{op}}$ and $\mathcal{N}^{\text{op}}$ are left $\mathcal{C}^{\text{op}}$-module categories. An oplax left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is just a lax left $\mathcal{C}^{\text{op}}$-module functor from $\mathcal{M}^{\text{op}}$ to $\mathcal{N}^{\text{op}}$; see [DSS14, Definition 2.6]. The following lemma is well-known:

**Lemma 2.1** ([DSS14, Lemma 2.11]). Let, as above, $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-module categories. Let $L : \mathcal{M} \to \mathcal{N}$ be a functor, and let $R : \mathcal{N} \to \mathcal{M}$ be a left adjoint of $L$ with unit $\eta : \text{id}_M \to RL$ and counit $\varepsilon : LR \to \text{id}_N$. If $L$ has a structure

$$\xi_{X,M} : L(X \otimes M) \to X \otimes L(M) \quad (X \in \mathcal{C}, M \in \mathcal{M})$$

of an oplax left $\mathcal{C}$-module functor, then $R$ is a lax $\mathcal{C}$-module functor with

$$\xi_{X,M} = \left( X \otimes R(M) \xrightarrow{\eta} RL(X \otimes R(M)) \xrightarrow{\text{id}} R(X \otimes LR(M)) \xrightarrow{R(X \otimes \varepsilon)} R(X \otimes M) \right).$$

This gives a one-to-one correspondence between the structures of oplax left $\mathcal{C}$-module functors on $L$ and the structures of lax left $\mathcal{C}$-module structures on $R$.

We say that an (op)lax $\mathcal{C}$-module functor is strong if its structure morphism is invertible. We also note the following important result:
Lemma 2.2 ([DSS14] Lemma 2.10). Suppose that \( \mathcal{C} \) is rigid. Then every oplax and every lax \( \mathcal{C} \)-module functors are strong.

Thus, when \( \mathcal{C} \) is rigid, lax \( \mathcal{C} \)-module functors and oplax \( \mathcal{C} \)-module functors are simply called \( \mathcal{C} \)-module functors. Lemma 2.1 says that the class of \( \mathcal{C} \)-module functors is closed under taking an adjoint of the underlying functor.

The notions of a right \( \mathcal{C} \)-module category and lax/oplax/strong right \( \mathcal{C} \)-module functors between them are defined analogously. There also are the notion of a \( \mathcal{C} \)-bimodule category and related notions. The same results hold for right module functors and bimodule functors.

2.5. Finite tensor categories. Let \( k \) be an algebraically closed field of arbitrary characteristic. By an algebra over \( k \), we mean an associative and unital algebra over the field \( k \). Given an algebra \( R \) over \( k \), we denote by \( R\text{-mod} \) the category of finite-dimensional left \( R \)-modules.

A finite abelian category over \( k \) is a \( k \)-linear category that is equivalent to \( \mathcal{A}\text{-mod} \) for some finite-dimensional algebra \( \mathcal{A} \) over \( k \). A finite tensor category \( [EO04] \) is a rigid monoidal category \( \mathcal{C} \) such that \( \mathcal{C} \) is a finite abelian category over \( k \), the tensor product functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is \( k \)-linear in each variable, and the unit object \( 1 \in \mathcal{C} \) is a simple object.

We now collect basic notions and results in the theory of finite tensor categories for convenience. Till the end of this subsection, we assume that \( \mathcal{C} \) is a finite tensor category over the field \( k \).

2.5.1. Finite module categories. A finite left \( \mathcal{C} \)-module category is a \( \mathcal{C} \)-module category \( \mathcal{M} \) such that \( \mathcal{M} \) is a finite abelian category over \( k \) and the action of \( \mathcal{C} \) on \( \mathcal{M} \) is \( k \)-linear and right exact in each variable (this condition implies that the action is exact in each variable; see [DSS14]). Finite right \( \mathcal{C} \)-module categories are defined analogously.

Now let \( \mathcal{M} \) be a left \( \mathcal{C} \)-module category. An algebra \( \mathcal{A} \) in \( \mathcal{C} \) (\( = \) a monoid object) defines a monad \( \mathcal{A}\text{id}_\mathcal{M} \) on \( \mathcal{M} \). We define the category \( \mathcal{A}\text{-mod}_\mathcal{M} \) of left \( \mathcal{A} \)-modules in \( \mathcal{M} \) to be the Eilenberg-Moore category of this monad. The following lemma is well-known:

Lemma 2.3. If \( \mathcal{M} \) is a finite left \( \mathcal{C} \)-module category, then the category \( \mathcal{A}\text{-mod}_\mathcal{M} \) is a finite abelian category over \( k \).

If \( \mathcal{N} \) is a right \( \mathcal{C} \)-module category, then the category \( \mathcal{N}\mathcal{A} \) of right \( \mathcal{A} \)-modules in \( \mathcal{N} \) is defined analogously. If \( \mathcal{B} \) is an algebra in \( \mathcal{C} \) and \( \mathcal{L} \) is a \( \mathcal{C} \)-bimodule category, then the category \( \mathcal{A}\mathcal{L}_\mathcal{B} \) of \( \mathcal{A}\mathcal{B} \)-bimodules in \( \mathcal{C} \) is defined. The categories \( \mathcal{N}\mathcal{A} \) and \( \mathcal{A}\mathcal{L}_\mathcal{B} \) are finite abelian categories provided that \( \mathcal{N} \) and \( \mathcal{L} \) are finite.

2.5.2. Internal Hom functors. Let \( \mathcal{M} \) be a finite left \( \mathcal{C} \)-module category, and let \( M \in \mathcal{M} \) be an object. Then the functor \( \text{id}_\mathcal{C} \otimes M \) from \( \mathcal{C} \) to \( \mathcal{M} \) is \( k \)-linear and exact, and hence has a right adjoint. We denote it by \( \text{Hom}(M,-) \). Namely,

\[
\text{Hom}_\mathcal{C}(V, \text{Hom}(M,N)) \cong \text{Hom}_\mathcal{M}(V \otimes M,N) \quad (V \in \mathcal{C}, M, N \in \mathcal{M}).
\]

The assignment \( (M,N) \mapsto \text{Hom}(M,N) \) extends to a functor \( \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{C} \), which we call the internal Hom functor of \( \mathcal{M} \). There are natural isomorphisms

\[
\text{Hom}(M, X \otimes N) \cong X \otimes \text{Hom}(M,N),
\]
\[
\text{Hom}(X \otimes M, N) \cong \text{Hom}(M,N) \otimes X^*.
\]
for \( X \in \mathcal{C} \) and \( M, N \in \mathcal{M} \). We note that the former isomorphism arises from the fact that \( \text{Hom}(M, -) \) is right adjoint to the functor \( \text{id}_C \otimes M : \mathcal{C} \to \mathcal{M} \), which has an obvious structure of a left \( \mathcal{C} \)-module functor.

2.5.3. Tensor functors. By a tensor functor, we mean a \( k \)-linear exact strong monoidal functor between finite tensor categories. Let also \( \mathcal{D} \) be a finite tensor category over \( k \). If \( F : \mathcal{C} \to \mathcal{D} \) is a tensor functor, then there are canonical isomorphisms

\[
F(X^*) \cong F(X)^* \quad \text{and} \quad F(\ast X) \cong \ast F(X)
\]

for \( X \in \mathcal{C} \) [NS07, Section 1]. We say that \( F \) strictly preserves the duality if the isomorphisms (2.3) are the identities. The following lemma will be used in later to avoid some technical difficulties:

**Lemma 2.4.** For every tensor functor \( F : \mathcal{C} \to \mathcal{D} \), there is a finite tensor category \( \mathcal{C}' \) over \( k \), an equivalence \( K : \mathcal{C} \to \mathcal{C}' \) of \( k \)-linear monoidal categories, and a tensor functor \( U : \mathcal{C}' \to \mathcal{D} \) such that \( U \) is strict monoidal, strictly preserves the duality, and \( F = U \circ K \) as tensor functors.

**Proof.** Let \( L \) be a left adjoint of \( F \). Then \( T := FL \) has a canonical structure of a \( k \)-linear right exact Hopf monad on \( \mathcal{D} \). We define \( \mathcal{C}' \) to be the Eilenberg-Moore category of \( T \). Now let \( K : \mathcal{C} \to \mathcal{C}' \) be the comparison functor, and let \( U : \mathcal{C}' \to \mathcal{D} \) be the forgetful functor. By the basic results on Hopf monads, \( K \) and \( U \) satisfies the required conditions; see, e.g., [BN11, Subsection 1.8]. \( \Box \)

2.6. Hom and tensor over an algebra. Let \( A \) and \( B \) be an algebra in a finite tensor category \( \mathcal{C} \). Then the category \( \mathcal{A} \mathcal{C}_B \) of \( A \)-\( B \)-bimodules in \( \mathcal{C} \) is a finite abelian category over \( k \). We note that the duality functor of \( \mathcal{C} \) induces anti-equivalences

\[
(-)^* : \mathcal{A} \mathcal{C}_{B} \to B^* \mathcal{A} \quad \text{and} \quad ^*(-) : \mathcal{A} \mathcal{C}_{B} \to B \mathcal{C}^* A
\]

of \( k \)-linear categories; see, e.g., [DSS14, Lemma 3.4.13].

We denote by \( \text{Hom}_\mathcal{A} \) the internal Hom functors of left \( \mathcal{C} \)-module categories \( \mathcal{C} \) and \( \mathcal{C}_A \), respectively. One has \( \text{Hom}(V, W) = W \otimes V^* \) for \( V, W \in \mathcal{C} \). The following lemma shows that \( \text{Hom}_\mathcal{A} \) is a subfunctor of \( \text{Hom}_- \).

**Lemma 2.5.** Given a right \( A \)-module \( M \) with action \( \iota_M \), we define

\[
\delta_M = (\iota_M)^* \quad \text{and} \quad \delta'_M = (\iota_M \otimes \text{id}_A^*) \circ (\text{id}_M \otimes \text{coev}_A).
\]

Then, for \( M, N \in \mathcal{C}_A \), there is an equalizer diagram

\[
\text{Hom}_\mathcal{A}(M, N) \xrightarrow{\delta_M} \text{Hom}(M, N) \xrightarrow{\delta'_M} N \otimes M^* \xrightarrow{\delta_M \otimes \text{id}} N \otimes A^* \otimes M^*.
\]

**Proof.** Let \( X \in \mathcal{C} \) be an object. Given a morphism \( f : X \otimes M \to N \) in \( \mathcal{C} \), we denote by \( f^\flat \) the morphism corresponding to \( f \) via the canonical isomorphism

\[
\text{Hom}_\mathcal{C}(X \otimes M, N) \cong \text{Hom}_\mathcal{C}(X, \text{Hom}(M, N)) = \text{Hom}(X, N \otimes M^*).
\]

It is routine to check that \( f \) is a morphism of right \( A \)-modules if and only if

\[
(id \otimes \delta'_N) \circ f^\flat = (\delta_M \otimes id) \circ f^\flat.
\]

Thus the above isomorphism restricts

\[
\text{Hom}_\mathcal{A}(X \otimes M, N) \cong \text{Hom}_\mathcal{C}(X, \text{Eq}(id \otimes \delta'_N, \delta_M \otimes id)),
\]

where \( \text{Hom}_\mathcal{A} \) is the Hom functor of \( \mathcal{C}_A \). The proof is done by comparing this result with the definition of the internal Hom functor. \( \Box \)
Unless otherwise noted, we regard $\text{Hom}_A(M, N)$ as a subobject of $\text{Hom}(M, N)$ by this lemma. This convention allows us to assume that the natural isomorphism (2.1) and (2.2) are the identities:

$$\text{Hom}_A(X \otimes M, N) = X \otimes \text{Hom}_A(M, N),$$

$$\text{Hom}_A(M, X \otimes N) = \text{Hom}_A(M, N) \otimes X^*.$$  

Given $L \in \mathcal{C}_A$ and $M \in \mathcal{A}_C$, we denote by $L \otimes_A M$ the tensor product of $L$ and $M$ over $A$. By definition, there is the coequalizer diagram

$$L \otimes A \otimes M \xrightarrow{\text{id}_L \otimes \triangleleft_M} L \otimes M \twoheadrightarrow L \otimes_A M,$$

where $\triangleright_L$ and $\triangleleft_M$ are the actions of $A$ on $L$ and $M$, respectively. If $M \in \mathcal{A}_B$, then we have a left $\mathcal{C}$-module functor $\mathcal{C}_A \rightarrow \mathcal{C}_B$ given by tensoring $M$ over $A$. A right adjoint of this functor is given by $\text{Hom}_B(M, -) : \mathcal{C}_B \rightarrow \mathcal{C}_A$, where $\text{Hom}_B$ is the internal Hom functor of $\mathcal{C}_B$ (the Tensor-Hom adjunction).

Applying the duality functor to the equalizer diagram of Lemma 2.5, we also have the coequalizer diagram

$$M \otimes A \otimes \ast N \xrightarrow{\text{id}_\otimes \triangleright_N} M \otimes \ast N \twoheadrightarrow \ast \text{Hom}_A(M, N)$$

for $M, N \in \mathcal{C}_A$, where $\triangleleft_M : M \otimes A \rightarrow M$ and $\triangleright_N : A \otimes \ast N \rightarrow \ast N$ are the actions of $A$ on $M$ and $\ast N$, respectively. In conclusion, we have the following description of the internal Hom functor (cf. [EO04, Example 3.19]).

**Lemma 2.6.** $\text{Hom}_A(M, N) = (M \otimes_A \ast N)^*.$

Let $R$ be an algebra in $\mathcal{C}$, and let $M$ be a $B$-$A$-bimodule in $\mathcal{C}$. Since $\text{Hom}_A(M, -)$ is a left $\mathcal{C}$-module functor from $\mathcal{C}_A$ to $\mathcal{C}_B$, it induces a functor

$$\text{Hom}_A(M, -) : R\mathcal{C}_A \rightarrow R\mathcal{C}_B.$$

We consider the case where $R = A^{**}$ and compute the image of $A^* \in A^{**} \mathcal{C}_A$ under this functor. By the above lemma, we immediately have:

**Lemma 2.7.** For all $M \in \mathcal{A}_B$, there is an isomorphism

$$\text{Hom}_A(M, A^*) \cong M^*$$

of $A^{**}$-$B$-bimodules.

3. The Drinfeld center and its variants  

3.1. The Drinfeld center and its variants. Let $k$ be an algebraically closed field of arbitrary characteristic. Throughout this section, we fix finite tensor categories $\mathcal{C}$ and $\mathcal{D}$ over $k$. Given two tensor functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, we define the category $\mathcal{Z}(F, G)$ as follows: An object of this category is a pair $(V, \sigma)$ consisting of an object $V \in \mathcal{D}$ and a natural transformation

$$\sigma_X : V \otimes F(X) \rightarrow G(X) \otimes V \quad (X \in \mathcal{C})$$
Lemma 3.1. If $m$ morphism of the corresponding bimodule functor. Thus, by Lemma 2.2, we have:

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For all objects $X, Y \in \mathcal{C}$, we denote by $\mathcal{Z}(F, \sigma)$ for all objects $\mathcal{Z}(F, G)$, then a morphism $f : \mathcal{V} \to \mathcal{W}$ in $\mathcal{Z}(F, G)$ is a morphism $f : V \to W$ in $\mathcal{D}$ satisfying the equation

$$\tau_X \circ (f \otimes \text{id}_{F(X)}) = (\text{id}_{G(X)} \otimes f) \circ \sigma_X$$

for all objects $X \in \mathcal{C}$. The composition of morphisms in $\mathcal{Z}(F, G)$ is defined by the composition as morphisms in $\mathcal{D}$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. The category $\mathcal{Z}(F) := \mathcal{Z}(F, F)$ is often called the centralizer of $F$. The category $\mathcal{Z}(\mathcal{C}) := \mathcal{Z}(\text{id}_C)$ is called the Drinfeld center of $\mathcal{C}$. It is well-known that the former is a monoidal category, and the latter is even a braided monoidal category.

3.2. $\mathcal{Z}(F, G)$ and bimodule functors. Given two tensor functors $F, G : \mathcal{C} \to \mathcal{D}$, we denote by $\langle G \rangle \mathcal{D}(F)$ the category $\mathcal{D}$ regarded as a finite $\mathcal{C}$-bimodule category by the left action $\otimes_G$ and the right action $\otimes_F$ defined by

$$X \otimes_G V = G(X) \otimes V \quad \text{and} \quad V \otimes_F X = V \otimes F(X) \quad (X \in C, V \in D),$$

respectively. If $T : \mathcal{C} \to \langle G \rangle \mathcal{D}(F)$ is a $\mathcal{C}$-bimodule functor, then there are natural isomorphisms

$$T(1) \otimes F(X) = T(1) \otimes_G X \overset{\cong}{\longrightarrow} T(1 \otimes X) = T(X) = T(X \otimes 1)$$

$$\overset{\cong}{\longrightarrow} X \otimes G T(1) = G(X) \otimes T(1)$$

for $X \in \mathcal{C}$. Hence $T(1)$ turns into an object of $\mathcal{Z}(F, G)$. Conversely, given an object $V = (V, \sigma) \in \mathcal{Z}(F, G)$, we have a $\mathcal{C}$-bimodule functor

$$T_V : \mathcal{C} \to \langle G \rangle \mathcal{D}(F), \quad X \mapsto G(X) \otimes V \quad (X \in \mathcal{C})$$

with the structure morphisms given by

$$X \otimes_G T_V(M) = G(X) \otimes G(M) \otimes V \overset{G_2 \otimes \text{id}}{\longrightarrow} G(X \otimes M) \otimes V = T_V(X \otimes M),$$

$$T_V(M) \otimes_F X = G(M) \otimes V \otimes F(X) \overset{\text{id} \otimes \sigma_X}{\longrightarrow} G(M) \otimes G(X) \otimes V \overset{G_2 \otimes \text{id}}{\longrightarrow} G(M \otimes X) \otimes V = T_V(M \otimes X)$$

for $M, X \in \mathcal{C}$. These two constructions are functorial and mutually inverse to each other (up to isomorphisms). We thus have a category equivalence

$$\{\mathcal{C}\text{-bimodule functors} \mathcal{C} \to \langle G \rangle \mathcal{D}(F)\} \approx \mathcal{Z}(F, G), \quad T \mapsto T(1).$$

We note that the structure morphism of an object of $\mathcal{Z}(F, G)$ becomes the structure morphism of the corresponding bimodule functor. Thus, by Lemma 2.2, we have:

**Lemma 3.1.** If $(V, \sigma)$ is an object of $\mathcal{Z}(F, G)$, then $\sigma$ is invertible.
3.3. **Tensor product and dual.** In the case where $F \neq G$, the category $\mathcal{Z}(F, G)$ does not seem to have a natural structure of a monoidal category. Though, for three tensor functors $F, G, H : \mathcal{C} \to \mathcal{D}$, one can define the tensor product

$$\otimes : \mathcal{Z}(G, H) \times \mathcal{Z}(F, G) \to \mathcal{Z}(F, H)$$

by $(V, \sigma) \otimes (W, \tau) = (V \otimes W, \rho)$ for $(V, \sigma) \in \mathcal{Z}(G, H)$ and $(W, \tau) \in \mathcal{Z}(F, G)$, where the natural transformation $\rho$ is defined by

$$\rho_X = \left( V \otimes W \otimes F(X) \xrightarrow{id_V \otimes \tau_X} V \otimes G(X) \otimes W \xrightarrow{\sigma_X \otimes id_W} V \otimes W \otimes H(X) \right)$$

for an object $X \in \mathcal{C}$.

The class of tensor functors from $\mathcal{C}$ to $\mathcal{D}$ form a bicategory with the above tensor product. All 1-cells of this bicategory are dualizable: Given an object $V = (V, \sigma)$ of $\mathcal{Z}(F, G)$, we define the object $V^* \in \mathcal{Z}(G, F)$ by $V^* = (V^*, \sigma^*)$, where

$$\sigma^*_X = \left( V^* \otimes G(X) \xrightarrow{id \otimes \text{coev}^*} V^* \otimes G(X) \otimes V \otimes V^* \xrightarrow{id \otimes \sigma^{-1}_X \otimes id} V^* \otimes V \otimes F(X) \otimes V^* \xrightarrow{\text{ev} \otimes id} F(X) \otimes V^* \right)$$

for $X \in \mathcal{C}$. We also define $^*V \in \mathcal{Z}(G, F)$ by $^*V = (^*V, \sigma^*)$, where

$$\sigma^*_X = \left( ^*V \otimes G(X) \xrightarrow{\cong} ^*V \otimes ^*G(X^*) = ^*(G(X^*) \otimes V) \right.$$

$$\left. \xrightarrow{\cong (\sigma_X^*)} (V \otimes F(X^*)) = F^*(X^*) \otimes V \xrightarrow{\cong} F(X) \otimes ^*V \right)$$

for $X \in \mathcal{C}$. Then it is easy to see that

$$\text{ev}_V : V^* \otimes V \to \mathbb{1}_{\mathcal{Z}(F, F)}$$

and

$$\text{coev}_V^* : \mathbb{1}_{\mathcal{Z}(F, F)} \to V \otimes V^*$$

are morphisms in $\mathcal{Z}(F, F)$, and

$$\text{ev}_V^* : V \otimes V^* \to \mathbb{1}_{\mathcal{Z}(G, G)}$$

and

$$\text{coev}_V^* : \mathbb{1}_{\mathcal{Z}(G, G)} \to V \otimes V^*$$

are morphisms in $\mathcal{Z}(G, G)$. We note that $V = (V, \sigma) \in \mathcal{Z}(F, G)$ is invertible if and only if the underlying object $V \in \mathcal{D}$ is invertible.

3.4. **$\mathcal{Z}(F, G)$ as the Eilenberg-Moore category.** We use $\boxtimes$ to denote the Deligne tensor product of $k$-linear abelian categories [De90]. We consider the coend

$$A = \int_{X \in \mathcal{C}} X^* \boxtimes X$$

in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ with universal dinatural transformation $i_X : X^* \boxtimes X \to A$. There is a unique morphism $m : A \otimes A \to A$ such that the diagram

$$(X^* \boxtimes X) \otimes (Y^* \boxtimes Y) \xrightarrow{i_X \otimes i_Y} A \otimes A \xrightarrow{m} A$$

$$\xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{m} A$$

commutes for all objects $X, Y \in \mathcal{C}$. The coend $A$ is an algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ with multiplication $m$ and unit $i_1$ (cf. [Sh17a] Lemma 4.5).
Remark 3.2. Let \( n \) be an integer. Since the left duality functor \( S := (-)^* \) of \( C \) is an anti-equivalence, the dinatural transformation
\[
i_{S_n(X)} : S^n(X) \boxtimes S^{n-1}(X) \to A \quad (X \in C)
\]
is universal. Hence we may identify \( A \) with the coend
\[
A_n := \int_{X \in C} S^n(X) \boxtimes S^{n-1}(X)
\]
by the universal property. Thus, although the coend \( A_0 \) is mainly used in \( \text{[Shi17a]} \), all the results of \( \text{[Shi17a]} \) can be translated to for our coend \( A = A_1 \).

Given tensor functors \( F, G : C \to D \), we set \( A_{F,G} := (G \boxtimes F^{\text{rev}}) (A) \). This object is an algebra in \( D \boxtimes D^{\text{rev}} \) as the image of the algebra \( A \) under a tensor functor. We note that \( D \) is a finite left \( D \boxtimes D^{\text{rev}} \)-module category by
\[
(3.1) \\
(X \boxtimes Y) \otimes V = X \otimes V \otimes Y \quad (V, X, Y \in D).
\]
Since the functor \( G \boxtimes F^{\text{rev}} \) and the action \( \otimes \) are \( k \)-linear and exact, we have
\[
A_{F,G} \otimes V = \int_{X \in C} G(X^*) \otimes V \otimes F(X) \cong \int_{X \in C} G(X)^* \otimes V \otimes F(X)
\]
for all \( V \in D \). We thus have natural isomorphisms
\[
\text{Hom}_D(A_{F,G} \otimes V, W) \cong \int_{X \in C} \text{Hom}_D(G(X)^* \otimes V \otimes F(X), W)
\]
\[
\cong \int_{X \in C} \text{Hom}_D(V \otimes F(X), G(X) \otimes V)
\]
\[
\cong \text{Nat}(V \otimes F, G \otimes W)
\]
for \( V, W \in D \).

Now let \( \rho : A_{F,G} \otimes V \to V \) be a morphism in \( D \), and let \( \sigma : V \otimes F \to G \otimes W \) be the natural transformation corresponding to \( \rho \) via the above isomorphism. Then the pair \((V, \rho)\) is an \( A_{F,G} \)-modules in \( D \) if and only if the pair \((V, \sigma)\) is an object of \( Z(F, G) \) (cf. Day-Street [DS07] and Bruguières-Virelizier [BV12]). This observation establishes the following lemma:

**Lemma 3.3.** \( Z(F, G) \) is isomorphic to the category of left \( A_{F,G} \)-modules in \( D \).

By Lemmas 2.3 and 4.3 we have:

**Theorem 3.4.** \( Z(F, G) \) is a finite abelian category over \( k \).

3.5. **Functors induced by tensor functors.** For later use, we introduce some notations for functors between the categories of the form \( Z(\_ \_ \_ , \_ \_ \_) \) induced by a tensor functor \( F : C \to D \).

**Notation 3.5.** Let \( B \) be a finite tensor category, and let \( G, G' : B \to C \) be two tensor functors. Given an object \( V = (V, \sigma) \in Z(G, G') \), we define
\[
F(V) = (F(V), F(\sigma)),
\]
where \( F(\sigma) \) is the natural transformation defined by
\[
\begin{align*}
F(\sigma)_X &= (F(V) \otimes FG(X) \xrightarrow{F_2} F(V \otimes G(X)) \\
&\quad \xrightarrow{F(\sigma)_X} F(G'(X) \otimes V) \xrightarrow{(F_2)^{-1}} FG'(X) \otimes F(X)
\end{align*}
\]
for $X \in \mathcal{B}$. The assignment $\mathbf{V} \mapsto F(\mathbf{V})$ extends to a functor
\begin{equation}
F : \mathcal{Z}(G, G') \rightarrow \mathcal{Z}(FG, FG'), \quad \mathbf{V} \mapsto F(\mathbf{V}).
\end{equation}
From the monadic point of view (Subsection 3.4), this functor can be understood as follows: Consider the coends
\[ A_{G,G'} = \int_{X \in \mathcal{B}} G'(X)^* \otimes G(X) \quad \text{and} \quad A_{FG,FG'} = \int_{X \in \mathcal{B}} FG(X)^* \otimes FG'(X). \]
We regard $\mathcal{D}$ as a left $\mathcal{C} \boxtimes \mathcal{E}^{rev}$-module category by
\[ (X \boxtimes Y) \otimes_F V = F(X) \otimes V \otimes F(Y) \quad (X, Y \in \mathcal{C}, V \in \mathcal{D}), \]
Then we have $A_{FG,FG'} \otimes (\cdot) \cong A_{G,G'} \otimes_F (\cdot)$ as monads on $\mathcal{D}$. Since $F : \mathcal{C} \rightarrow \mathcal{D}$ is a $\mathcal{C} \boxtimes \mathcal{E}^{rev}$-module functor, it induces a functor
\[ F : A_{G,G'} \mathcal{C} \rightarrow A_{G,G'} \mathcal{D} \quad (= A_{FG,FG'} \mathcal{D}) \]
between the categories of $A_{G,G'}$-modules. The functor (3.2) corresponds to this functor via the identification by Lemma 3.3.

**Notation 3.6.** Let $\mathcal{E}$ be a finite tensor category, and let $G, G' : \mathcal{D} \rightarrow \mathcal{E}$ be two tensor functors. Given an object $\mathbf{V} = (V, \sigma) \in \mathcal{Z}(G, G')$, we define
\[ \mathbf{V}|_F = (V, \sigma_{F(-)}) \in \mathcal{Z}(GF, G'F). \]
The assignment $\mathbf{V} \mapsto \mathbf{V}|_F$ extends to a functor
\begin{equation}
(\cdot)|_F : \mathcal{Z}(G, G') \rightarrow \mathcal{Z}(GF, G'F), \quad \mathbf{V} \mapsto \mathbf{V}|_F.
\end{equation}
We consider the coends
\[ A_{G,G'} = \int_{X \in \mathcal{D}} G'(X)^* \otimes G(X) \quad \text{and} \quad A_{FG,FG'} = \int_{X \in \mathcal{E}} GF(X)^* \otimes GF(X) \]
with universal dinatural transformations
\[ i_X : G'(X)^* \otimes G(X) \rightarrow A_{G,G'} \quad \text{and} \quad j_X : GF(X)^* \otimes GF(X) \rightarrow A_{FG,FG'}. \]
respectively. By the universal property, there is a unique morphism $\phi : A_{FG,FG'} \rightarrow A_{G,G'}$ in $\mathcal{E}$ such that $\phi \circ j_X = i_{F(X)}$ for all $X \in \mathcal{C}$. It is easy to see that $\phi$ is a morphism of algebras in $\mathcal{E} \boxtimes \mathcal{E}^{rev}$. The functor (3.3) corresponds to the restriction along $\phi$.

4. **Relative modular object**

4.1. **Radford isomorphism.** Let $\mathcal{C}$ be a finite tensor category over an algebraically closed field $k$. The finite tensor category $\mathcal{C} \boxtimes \mathcal{E}^{rev}$ acts on $\mathcal{C}$ by (3.1). Let $\text{Hom}$ denote the internal Hom functor of the $\mathcal{C} \boxtimes \mathcal{E}^{rev}$-module category $\mathcal{C}$. As we have seen in [Shu17a], the algebra $\text{Hom}(\mathbb{1}, \mathbb{1})$ is identical to the algebra
\[ A = \int_{X \in \mathcal{C}} X^* \boxtimes X. \]
By the natural isomorphisms (2.1) and (2.2), we have isomorphisms
\begin{equation}
\tau_V : (\mathbb{1} \boxtimes V) \otimes A \xrightarrow{\cong} \text{Hom}(\mathbb{1}, V) \xrightarrow{\cong} (V \boxtimes \mathbb{1}) \otimes A \quad \text{for } V \in \mathcal{C}.
\end{equation}
By the fundamental theorem for Hopf bimodules \cite[Proposition 2.3]{ENO04}, we have an equivalence
\begin{equation}
K_C : \mathcal{C} \rightarrow (\mathcal{C} \boxtimes \mathcal{C}^{rev})_A \quad V \mapsto (V \otimes \mathbb{1}) \otimes A.
\end{equation}
of \(\mathcal{C} \boxtimes \mathcal{C}^{rev}\)-module categories. This induces an equivalence
\begin{equation}
A^{**} \rightarrow A^{**}(\mathcal{C} \boxtimes \mathcal{C}^{rev})_A, \quad V \mapsto (V \otimes \mathbb{1}) \otimes A_A
\end{equation}
between the categories of \(A^{**}\)-modules. Now let \(S_C = (-)^*\) be the left duality functor of \(\mathcal{C}\). Then, by Remark 3.2, we have isomorphisms
\[ A^{**} \cong \int_{X \in \mathcal{C}} (X^* \boxtimes X)^{**} \cong \int_{X \in \mathcal{C}} X^{**} \boxtimes X \cong \int_{X \in \mathcal{C}} S^d_C(X)^* \boxtimes X = A_{id_C, S^d_C}
\]of algebras in \(\mathcal{C} \boxtimes \mathcal{C}^{rev}\).

**Definition 4.1.** The Radford object \(\alpha_C\) of \(\mathcal{C}\) is the object of \(\mathcal{Z}(id_C, S^A_C)\) corresponding to the \(A^{**}\)-\(A\)-bimodule \(A^*\) via the equivalence
\[
\mathcal{Z}(id_C, S^A_C) \xrightarrow{\text{Lemma 3.3}} A^{**}_C \xrightarrow{4.3} A^{**}(\mathcal{C} \boxtimes \mathcal{C}^{rev})_A.
\]

The distinguished invertible object \cite{ENO04} of \(\mathcal{C}\) is defined as the object \(\alpha \in \mathcal{C}\) such that \(\text{Hom}(\mathbb{1}, \alpha) \cong A^*\) as right \(A\)-modules. Thus there is a natural isomorphism
\[
\delta_X : \alpha \otimes X \rightarrow S^A_C(X) \otimes \alpha \quad (X \in \mathcal{C})
\]such that \(\alpha_C = (\alpha, \delta)\). By tedious computation, we see that the isomorphism \(\delta\) is same as the isomorphism given in \cite[Appendix A]{ENO04}. We will refer to \(\delta\) as the Radford isomorphism of \(\mathcal{C}\).

**4.2. Relative modular object.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be finite tensor categories over the field \(k\), and let \(F : \mathcal{C} \rightarrow \mathcal{D}\) be a tensor functor. Then the category \(\mathcal{D}\) is a finite \(\mathcal{C}\)-bimodule category by the action
\[
X \otimes V \otimes Y = F(X) \otimes V \otimes F(Y) \quad (X, Y \in \mathcal{C}, V \in \mathcal{D}).
\]
Now we suppose that \(F : \mathcal{C} \rightarrow \mathcal{D}\) is a perfect tensor functor (see Subsection 2.5 for the definition). Then a right adjoint \(R\) of \(F\) is exact, and thus \(R\) also has a right adjoint. Let \(G\) be a right adjoint of \(R\). Then, by Lemma 2.1 the functor \(G\) has a canonical structure of a \(\mathcal{C}\)-bimodule functor.

**Definition 4.2.** Let \(F\) and \(G\) be as above. The relative modular object of \(F\) is the object \(\mu_F \in \mathcal{Z}(F)\) corresponding to \(G\) via the category equivalence
\[
\{\mathcal{C}\text{-bimodule functors } \mathcal{C} \rightarrow \langle F, \mathcal{D}(F) \rangle \} \cong \mathcal{Z}(F), \quad T \mapsto T(\mathbb{1})
\]established in Subsection 3.2

As explained in \cite{Shi17b}, the relative modular object \(\mu_F\) is a categorical counterpart of the relative modular function of \cite{FMS97}. Let \(\alpha_C\) and \(\alpha_D\) be the distinguished invertible objects of \(\mathcal{C}\) and \(\mathcal{D}\), respectively. The main result of \cite{Shi17b} shows that, if we write \(\mu_F = (\mu_F, \gamma)\), then there is an isomorphism
\[
\mu_F \cong \alpha_D^* \otimes F(\alpha_C)
\]in \(\mathcal{C}\) (remark that \(\alpha_C\) and \(\mu_F\) in this paper are \(\alpha_C^*\) and \(\mu_F^*\) of \cite{Shi17b}, respectively).
This result is not sufficient for our aim: For the purpose of this paper, we also require a description of the isomorphism \(\gamma\) in terms of the Radford isomorphisms.
4.3. A description of the relative modular object. We now state the main result of this section. Let $F : \mathcal{C} \to \mathcal{D}$ be a perfect tensor functor between finite tensor categories $\mathcal{C}$ and $\mathcal{D}$. With the notations introduced in Subsection 3.5, we have the following two objects:

$$F(\alpha_\mathcal{C}) \in \mathcal{Z}(F, FS^4), \quad \alpha^*_\mathcal{D}|_F \in \mathcal{Z}(S^4F, F).$$

In view of Lemma 2.4, we may assume, for simplicity, that the tensor functor $F$ is strict monoidal and strictly preserves the duality. Then the tensor product of the above two objects makes sense, since $FS^4 = S^4F$. The main result of this section is the following formula of the relative modular object:

**Theorem 4.3.** $\mu_F \cong \alpha^*_\mathcal{D}|_F \otimes F(\alpha_\mathcal{C})$ in $\mathcal{Z}(F, F)$.

This theorem is equivalent to that there is an isomorphism

$$\alpha_\mathcal{D}|_F \otimes \mu_F \cong F(\alpha_\mathcal{C})$$

in $\mathcal{Z}(F, S^4F)$. In other words, if we write

$$\mu_F = (\mu_F, \gamma), \quad \alpha_\mathcal{C} = (\alpha_\mathcal{C}, \delta) \quad \text{and} \quad \alpha_\mathcal{D} = (\alpha_\mathcal{D}, \delta),$$

then there is an isomorphism $j : F(\alpha_\mathcal{C}) \to \mu_F \otimes \alpha_\mathcal{D}$ in $\mathcal{D}$ such that the diagram

$$\begin{array}{ccc}
F(\alpha_\mathcal{C} \otimes X) & \xrightarrow{F(\delta)} & F(\alpha_\mathcal{C}) \otimes F(X) \\
\downarrow & & \downarrow j \otimes \text{id} \\
F(X^{****} \otimes \alpha_\mathcal{C}) & \xrightarrow{j \otimes \text{id}} & F(X)^{****} \otimes F(\alpha_\mathcal{C})
\end{array}$$

commutes for all objects $X \in \mathcal{C}$.

4.4. Monadcity of module functors. To prove Theorem 4.3, we require a monadic description of the category of module functors. Let $\mathcal{M}$ and $\mathcal{N}$ be finite right $\mathcal{C}$-module categories. Then $\mathcal{M}^{op} \boxtimes \mathcal{N}$ is a finite right $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module category by the action determined by

$$(\mathcal{M}^{op} \boxtimes \mathcal{N}) \otimes (X \boxtimes Y) = (M \otimes Y^{**})^{op} \boxtimes (N \otimes X)$$

for $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $X, Y \in \mathcal{C}$, where $\mathcal{M}^{op}$ means the object $M$ regarded as an object of $\mathcal{M}^{op}$.

We denote by $\text{Lex}(\mathcal{M}, \mathcal{N})$ and $\text{LEX}_\mathcal{C}(\mathcal{M}, \mathcal{N})$ the category of $k$-linear left exact functors and the category of $k$-linear left exact $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$, respectively. As noted in Shi17a, there is an equivalence

$$\mathcal{M}^{op} \boxtimes \mathcal{N} \to \text{Lex}(\mathcal{M}, \mathcal{N}), \quad \mathcal{M}^{op} \boxtimes \mathcal{N} \to \text{Hom}_\mathcal{M}(M, -) \otimes_k N$$

of $k$-linear categories, where $\otimes_k$ is the canonical action of the category of finite-dimensional vector spaces over $k$ (see also FSS16 for related results). If we define a right action of $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ on $\text{Lex}(\mathcal{M}, \mathcal{N})$ by

$$(F \otimes (X \boxtimes Y))(M) = F(M \otimes Y^{**}) \otimes X$$

for $F \in \text{Lex}(\mathcal{M}, \mathcal{N})$, $M \in \mathcal{M}$ and $X, Y \in \mathcal{C}$, then the equivalence 4.4 is in fact a module functor. In particular, $\text{Lex}(\mathcal{M}, \mathcal{N})$ is a finite module category.
Lemma 4.4. Let $A$ be the algebra in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ introduced in Subsection 3.4. Then there is an isomorphism $\text{LEX}(\mathcal{M}, \mathcal{N})_A \cong \text{LEX}_e(\mathcal{M}, \mathcal{N})$ of categories commuting with the forgetful functors to $\text{LEX}(\mathcal{M}, \mathcal{N})$.

Proof. The proof is essentially same as [Shi17b, Lemma 3.7]. Let $i_X : X \boxtimes X \to A$ be the universal dinatural transformation. We may identify $A = \int_{X \in \mathcal{C}} ^X \boxtimes X$

by Remark 3.2. Since the action of $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ on $\text{LEX}(\mathcal{M}, \mathcal{N})$ is $k$-linear and exact, we have isomorphisms

$$\text{Nat}(F \otimes A, F) \cong \int_{X \in \mathcal{C}} \text{Hom}_X(F \otimes (*X \boxtimes X), F)$$

for $F \in \text{LEX}(\mathcal{M}, \mathcal{N})$. Let $\rho : F \otimes A \to F$ be a morphism in $\text{LEX}(\mathcal{M}, \mathcal{N})$, and let $\xi_{M,X} : F(M \otimes X) \to F(M) \otimes X$ be the natural transformation corresponding to $\rho$ via the above isomorphisms. Then $(F, \rho)$ is a right $A$-module if and only if $(F, \xi)$ is an oplax $\mathcal{C}$-module functor (cf. the proof of Lemma 3.3). Thus we have the desired isomorphism of categories.

Now we consider the case where $\mathcal{M} = \mathcal{N} = \mathcal{C}$. Then $\text{LEX}(\mathcal{C}) := \text{LEX}(\mathcal{C}, \mathcal{C})$ is not only a right $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module category but also a left $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module category by the action determined by

$$(X \boxtimes Y) \otimes F = X \otimes F(Y \otimes -) \quad (X, Y \in \mathcal{C}, F \in \text{LEX}(\mathcal{C})).$$

There is an equivalence $\mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{C}^{op} \boxtimes \mathcal{C}$ given by $X \boxtimes Y \mapsto Y^* \boxtimes X$. Composing this with the equivalence (1.3), we obtain an equivalence

$$\Phi_e : \mathcal{C} \boxtimes \mathcal{C}^{rev} \to \text{LEX}(\mathcal{C}), \quad X \boxtimes Y \mapsto \text{Hom}_{\mathcal{E}}(Y^*, -) \otimes_k X$$

of $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-bimodule categories. Finally, we introduce the Cayley functor:

$$\Upsilon_e : \mathcal{C} \to \text{LEX}_e(\mathcal{C}) := \text{LEX}_e(\mathcal{C}, \mathcal{C}), \quad V \mapsto V \otimes (-).$$

Lemma 4.5. There is the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Upsilon_e} & \text{LEX}_e(\mathcal{C}) \\
\Downarrow \text{Lemma 1.3} \quad \Phi_e & & \Downarrow \text{Lemma 1.3} \\
\mathcal{C} \boxtimes \mathcal{C}^{rev} & \xrightarrow{\Phi_e} & \text{LEX}(\mathcal{C})
\end{array}$$

commuting up to isomorphisms in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-mod.

Proof. A quasi-inverse of $\Phi_e$ is given by $\text{LEX}(\mathcal{C}) \to \mathcal{C} \boxtimes \mathcal{C}^{rev}, \quad F \mapsto \int_{X \in \mathcal{C}} F(X) \boxtimes X,$

and, in particular, we have $\Phi_e(A) \cong \text{id}_e$ [Shi17a]. Now the claim can be checked directly. □
4.5. Proof of Theorem 4.3. Let \( \mathcal{C} \) and \( \mathcal{D} \) be finite tensor categories over \( k \), and let \( F : \mathcal{C} \to \mathcal{D} \) be a perfect tensor functor. In view of Lemma 2.4, we assume that \( F \) is strict monoidal and strictly preserves the duality.

We first give a convenient realization of a right adjoint of \( F \). We fix a left adjoint functor \( L \) of \( F \). The functor \( F \) is a \( \mathcal{C} \)-bimodule functor if we view \( \mathcal{D} \) as a \( \mathcal{C} \)-bimodule category via \( F \). Hence, by Lemma 2.1, the functor \( L \) is also a \( \mathcal{C} \)-bimodule functor as a left adjoint of \( F \). We note that the structure morphisms of \( L \) are given by

\[
\xi_{X,V}^{(r)} := \varepsilon_X \otimes L(\varepsilon_L) : L(F(X) \otimes V) \to X \otimes L(V),
\]

\[
\xi_{X,V}^{(l)} := \varepsilon_{L(V) \otimes X} : L(V \otimes F(X)) \to L(V) \otimes X
\]

for \( V \in \mathcal{D} \) and \( X \in \mathcal{C} \), where \( \varepsilon : \text{id}_D \to FL \) and \( \varepsilon : LF \to \text{id}_C \) are the unit and the counit of the adjunction \( L \dashv F \), respectively.

Now we define \( R : \mathcal{D} \to \mathcal{C} \) by \( R(V) = \varepsilon(V) \) for \( V \in \mathcal{D} \). As is well-known, \( R \) is a right adjoint of \( F \). The unit \( \eta' \) and the counit \( \varepsilon' \) of \( F \dashv R \) are given by

\[
\eta'_X : X = * \to ^* LF(X) = * L(F(X)) = RF(X),
\]

\[
\varepsilon'_V : FR(V) = F(* R(V)) = * FR(V) \to * V = V
\]

for \( X \in \mathcal{C} \) and \( V \in \mathcal{D} \), respectively. By Lemma 2.1, the functor \( R \) is a \( \mathcal{C} \)-bimodule functor as a right adjoint of \( F \). A straightforward computation shows:

**Lemma 4.6.** By using the structure morphisms of \( L \) given by (4.9) and (4.10), the structure morphisms of \( R \) as a \( \mathcal{C} \)-bimodule functor are expressed as follows:

\[
R(V) \otimes F(X) = \varepsilon(V) \otimes L(F(X)) \xrightarrow{\xi_{X,V}^{(r)}} \varepsilon_{L(F(X))} = L(F(X) \otimes V) = R(V \otimes F(X)),
\]

\[
F(X) \otimes R(V) = \varepsilon_{L(V) \otimes X} \otimes F(X) \xrightarrow{\xi_{X,V}^{(l)}} \varepsilon_X = L(V \otimes F(X)) = R(F(X)) \otimes V).
\]

Since \( F \) and \( L \) are \( k \)-linear and left exact, one can define the functor

\[
\text{LEX}(F,L) : \text{LEX}(\mathcal{C}) \to \text{LEX}(\mathcal{D}), \quad T \mapsto LTF.
\]

In what follows, we regard \( \mathcal{D} \boxtimes \mathcal{D}^{\text{rev}} \) and \( \text{LEX}(\mathcal{D}) \) as finite \( \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \)-bimodule categories via \( F \boxtimes F^{\text{rev}} \). The functor \( \text{LEX}(L,F) \) is a \( \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \)-bimodule functor in an obvious way.

**Lemma 4.7.** \( \Phi_e \circ (L \boxtimes R^{\text{rev}}) = \text{LEX}(F,L) \circ \Phi_e \) as \( \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \)-bimodule functors.

**Proof.** Set \( \Theta = \Phi_e \circ (L \boxtimes R^{\text{rev}}) \) and \( \Psi = \text{LEX}(F,L) \circ \Phi_e \) for simplicity. Then,

\[
\Theta(V \boxtimes W) = \text{Hom}_C(L(W^*), -) \otimes_k L(V),
\]

\[
\Psi(V \boxtimes W) = \text{Hom}_D(W^*, F(-)) \otimes_k L(V).
\]

Thus \( \Theta \cong \Psi \) by the adjunction \( L \dashv F \). One can check that this isomorphism is indeed a morphism of bimodule functors.

We now consider the three coends

\[
A = \int_{X \in \mathcal{C}} X^* \boxtimes X, \quad A_F = \int_{X \in \mathcal{C}} F(X)^* \boxtimes F(X) \quad \text{and} \quad B = \int_{V \in \mathcal{D}} V^* \boxtimes V
\]

with universal dinatural transformations

\[
i_X : X^* \boxtimes X \to A, \quad i_X' : F(X)^* \boxtimes F(X) \to A, \quad \text{and} \quad j_V : V^* \boxtimes V \to B,
\]
respectively. We recall that the coend \( A \) an algebra in \( \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \), and the others are algebras in \( \mathcal{D} \otimes \mathcal{D}^{\text{rev}} \). By the universal property, there is a unique morphism \( \phi : A_F \to B \) in \( \mathcal{D} \otimes \mathcal{D}^{\text{rev}} \) such that \( \phi \circ i'_X = j_{F(X)} \) for all \( X \in \mathcal{C} \). This is in fact a morphism of algebras, and thus we have the restriction-of-scalars functor

\[
\text{Res}_\phi : (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_B \to (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_A
\]

along \( \phi \). By the definition of the algebra \( A_F \), we may identify \( (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_A = (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_B \) and \( \text{Lex}(\mathcal{D})_A = \text{Lex}(\mathcal{D})_B \).

**Lemma 4.8.** The diagram

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{L} & \mathcal{C} \\
\text{Res}_\phi & \downarrow & \downarrow \\
(\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_B & \xrightarrow{(\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_A} & (\mathcal{C} \otimes \mathcal{C}^{\text{rev}})_A
\end{array}
\]

commutes up to isomorphisms in \( \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \)-mod.

**Proof.** We consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\Upsilon} & \text{LEX}_\mathcal{D}(\mathcal{D}) \\
& \downarrow & \downarrow \\
\mathcal{C} & \xrightarrow{\Upsilon} & \text{LEX}_\mathcal{C}(\mathcal{C})
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\text{Res}_F} & \text{LEX}_\mathcal{D}(\mathcal{D}) \\
& \downarrow & \downarrow \\
\mathcal{C} & \xrightarrow{\text{Res}_F} & \text{LEX}_\mathcal{C}(\mathcal{C})
\end{array}
\]

\[
\begin{array}{ccc}
(\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_B & \xrightarrow{\Phi} & (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_A \\
& \downarrow & \downarrow \\
(\mathcal{C} \otimes \mathcal{C}^{\text{rev}})_A & \xrightarrow{\Phi} & (\mathcal{C} \otimes \mathcal{C}^{\text{rev}})_A
\end{array}
\]

Here, \( \Upsilon \)'s are the Cayley functor given by \( (\mathbf{1.8}) \), \( \Phi \)'s are functors induced by a quasi-inverse of the equivalence (\( \mathbf{1.7} \)), and \( \text{Res}_F \) restricts a \( \mathcal{D} \)-module functor along \( F \) to obtain a \( \mathcal{C} \)-module functor. By Lemma \( \mathbf{4.5} \), it is sufficient to show the commutativity of this diagram (up to isomorphisms) to prove this lemma.

It is easy to check that the square labeled (\( \heartsuit \)) is commutative if we use the fact that a quasi-inverse of the Cayley functor \( \Upsilon_\mathcal{C} \) is given by

\[
\Upsilon_\mathcal{C} : \text{LEX}_\mathcal{C}(\mathcal{C}) \to \mathcal{C}, \quad T \mapsto T(\mathbf{1}).
\]

Lemma \( \mathbf{4.7} \) implies that the square labeled (\( \spadesuit \)) is commutative. The commutativity of the other squares are obvious. The proof is done.

Now let \( G \) be a right adjoint of \( R \). Taking right adjoints of the functors in the diagram \( \mathbf{4.11} \), we obtain the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{Hom}(\mathcal{F},-)} & \mathcal{D} \\
& \downarrow & \downarrow \\
(\mathcal{C} \otimes \mathcal{C}^{\text{rev}})_A & \xrightarrow{\text{Hom}(\mathcal{F},-)} & (\mathcal{D} \otimes \mathcal{D}^{\text{rev}})_B
\end{array}
\]

commuting up to isomorphisms in \( \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \)-mod. Let \( k\text{-Cat} \) be the 2-category of \( k \)-linear categories. Applying the 2-functor

\[
\mathcal{C} \otimes \mathcal{C}^{\text{rev}} \rightarrow \text{k-Cat}, \quad \mathcal{M} \mapsto A^* \mathcal{M}
\]
to the above diagram, we obtain the diagram

\[
\begin{array}{ccc}
A^{**}C & \xrightarrow{F} & A^{**}D \\
\downarrow & & \downarrow \\
A^{**}(C \boxtimes C^{rev})_A & \xrightarrow{F \boxtimes G^{rev}} & A^{**}(D \boxtimes D^{rev})_A \\
\end{array}
\]

commuting up to isomorphisms in \(k\text{-}\mathbb{Cat}\). There are category isomorphisms

\[
A^{**}C \cong \mathcal{Z}(\text{id}_C, S_C^2) \quad \text{and} \quad A^{**}D \cong \mathcal{Z}(F, F S_D^2),
\]

where \(S_C\) and \(S_D\) are the left duality functors of \(C\) and \(D\), respectively. Theorem \ref{thm:classification} is now proved by chasing the object \(\alpha_C\) around this diagram.

**Proof of Theorem** \ref{thm:classification}. There is an isomorphism

\[
(4.12) \quad (F(\alpha_C) \boxtimes \mathbb{1}) \otimes B \cong \text{Hom}_{A^f}(B, (F \boxtimes G^{rev})(A^*))
\]

in \(A^{**}(D \boxtimes D^{env})_B\) by the above commutative diagram. The left-hand side corresponds to \(F(\alpha_C) \in \mathcal{Z}(F, F S_D^2)\). We consider the right-hand side of \ref{eq:iso}. By the definition of the relative modular object, there are isomorphisms

\[
\text{Hom}_{A^f}(B, (F \boxtimes G^{rev})(-)) \cong \text{Hom}_{A^f}(B, (1 \boxtimes \mu_F) \otimes (F \boxtimes F^{rev})(-)) \cong (1 \boxtimes \mu_F) \otimes \text{Hom}_{A^f}(B, (F \boxtimes F^{rev})(-))
\]

of left \(C \boxtimes C^{env}\)-module functors. By Lemma \ref{lem:iso} and the definition of the distinguished invertible object, we also have isomorphisms

\[
\text{Hom}_{A^f}(B, (F \boxtimes F^{env})(A^*)) \cong B^* \cong (\alpha_D \boxtimes \mathbb{1}) \otimes B
\]

of \(A^{**}\)-bimodule in \(D \boxtimes D^{env}\). Hence the right-hand side of \ref{eq:iso} is isomorphic to \((\alpha_D \boxtimes \chi_F) \otimes B\) as an \(A^{**}\)-bimodule in \(D \boxtimes D^{env}\). By using \ref{eq:iso}, we finally establish an isomorphism

\[
(\mathbb{1} \otimes 1) \otimes B \cong ((\alpha_D \otimes \mu_F) \boxtimes \mathbb{1}) \otimes B
\]

of \(A^{**}\)-modules in \(D \boxtimes D^{env}\). This implies \ref{thm:classification}, as desired. \(\square\)

5. Classification of the ribbon structures

5.1. **Reformulation of** \texttt{arXiv:1608.05905} **Let** \(\mathcal{C}\) **be a finite tensor category over an algebraically closed field** \(k\). **In this section, we classify the ribbon structures of the Drinfeld center** \(\mathcal{Z}(\mathcal{C})\). **Our result is based on the classification result of the pivotal structures of** \(\mathcal{Z}(\mathcal{C})\) **given in** \texttt{arXiv:1608.05905} [Shi16h]. **We first recall the main result of** [Shi16h] **in a slightly reformulated form.**

Given a tensor autoequivalence \(F\) of \(\mathcal{C}\), we denote by \(\tilde{F} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C})\) the braided tensor autoequivalence of \(\mathcal{Z}(\mathcal{C})\) induced by \(F\). Namely, it is defined by

\[
\tilde{F}(\mathbb{V}) = (F(\mathbb{V}), \sigma^F)
\]

for \(\mathbb{V} = (\mathbb{V}, \sigma) \in \mathcal{Z}(\mathcal{C})\), where \(\sigma_X^F : F(\mathbb{V}) \otimes X \rightarrow X \otimes F(\mathbb{V})\) \((X \in \mathcal{C})\) is the unique natural isomorphism such that the diagram

\[
\begin{array}{ccc}
F(\mathbb{V}) \otimes F(X) & \xrightarrow{F(\mathbb{V}, X)} & F(\mathbb{V} \otimes X) \\
\sigma_F^X \downarrow & & \downarrow F(\sigma_X) \\
F(X) \otimes F(\mathbb{V}) & \xrightarrow{F(\sigma_X)} & F(X \otimes \mathbb{V})
\end{array}
\]
commutes for all \( X \in \mathcal{C} \).

Now let \( F \) and \( G \) be tensor autoequivalences of \( \mathcal{C} \). Given a pair \( \beta = (\beta, j) \) consisting of an invertible object \( \beta \in \mathcal{C} \) and an isomorphism \( j : F^\beta \to G \) of tensor functors, where \( F^\beta(X) = \beta \otimes F(X) \otimes \beta^* \), we define \( \Phi(\beta) : \bar{F} \to \bar{G} \) by

\[
\Phi(\beta)_V = \left( F(V) \xrightarrow{id \otimes \text{coev}} F(V) \otimes \beta \otimes \beta^* \xrightarrow{\sigma^F \otimes \text{id}} \beta \otimes F(V) \otimes \beta^* \xrightarrow{j} G(V) \right)
\]

for an object \( V = (V, \sigma) \in \mathcal{Z}(\mathcal{C}) \) with \( \bar{F}(V) = (F(V), \sigma^F) \). The main result of \cite{Shi16b} claims that the map \( \beta \mapsto \Phi(\beta) \) gives a bijection

\[
(5.1) \quad \bigcup_{n=1}^\infty \left\{ \beta = (\beta_s, j) \mid j \in \text{Nat}_{\otimes}(F^\beta, G) \right\} \xrightarrow{\cong} \text{Nat}_{\otimes}(\bar{F}, \bar{G}),
\]

where \( \{\beta_1, \ldots, \beta_n\} \) is a complete set of representatives of the isomorphism classes of invertible objects of \( \mathcal{C} \).

Given a pair \( \beta = (\beta, j) \) as above, we define

\[
\tau_X : \beta \otimes F(X) \xrightarrow{id \otimes \text{coev}^{-1}} \beta \otimes F(X) \otimes \beta^* \otimes \beta \xrightarrow{j \otimes \text{id}} G(X) \otimes \beta
\]

for \( X \in \mathcal{C} \). Hence we obtain an invertible object \( (\beta, \tau) \in \mathcal{Z}(F, G) \) and, moreover, any invertible object of \( \mathcal{Z}(F, G) \) can be obtained in this way. Now let \( \mathcal{Z}(F, G)^\times \) be the set of the isomorphism classes of invertible objects of \( \mathcal{Z}(F, G) \). The bijection \( (5.1) \) is reformulated as follows:

**Theorem 5.1.** Given an invertible object \( \beta = (\beta, \tau) \) of \( \mathcal{Z}(F, G) \), we define

\[
\Theta(\beta)_V \otimes \text{id}_\beta = \left( F(V) \otimes \beta \xrightarrow{\sigma^F} \beta \otimes F(V) \xrightarrow{\tau_V} G(V) \otimes \beta \right)
\]

for an object \( V = (V, \sigma) \in \mathcal{Z}(\mathcal{C}) \) with \( \bar{F}(V) = (F(V), \sigma^F) \). Then the assignment \( \beta \mapsto \Theta(\beta) \) gives a well-defined bijection

\[
\Theta : \mathcal{Z}(F, G)^\times \xrightarrow{\cong} \text{Nat}_{\otimes}(\bar{F}, \bar{G}).
\]

Suppose that we have three tensor autoequivalences \( F, G \) and \( H \) of \( \mathcal{C} \). We recall that there is the tensor product between \( \mathcal{Z}(G, H) \) and \( \mathcal{Z}(F, G) \). If \( \beta_1 \in \mathcal{Z}(G, H) \) and \( \beta_2 \in \mathcal{Z}(F, G) \) are invertible objects, then

\[
(5.2) \quad \Theta(\beta_1 \otimes \beta_2)_X = \left( \bar{F}(X) \xrightarrow{\Theta(\beta_1)_X} \bar{G}(X) \xrightarrow{\Theta(\beta_2)_X} \bar{H}(X) \right)
\]

for all \( X \in \mathcal{Z}(\mathcal{C}) \). We will use this system to formulate our classification result of the ribbon structures of \( \mathcal{Z}(\mathcal{C}) \).

### 5.2. Braided bimodule categories

Let \( \mathcal{B} \) be a braided finite tensor category over \( k \) with braiding \( \sigma \), and let \( \mathcal{M} \) be a finite \( \mathcal{B} \)-bimodule category with left action \( \otimes \) and right action \( \circ \). We note that \( \mathcal{M} \) has an alternative left \( \mathcal{B} \)-action given by

\[
X \otimes M = M \otimes X \quad (X \in \mathcal{B}, M \in \mathcal{M})
\]

with the associativity isomorphism given by the braiding \( \sigma \). We say that the \( \mathcal{B} \)-bimodule category \( \mathcal{M} \) is *braided* if it is equipped with a natural isomorphism

\[
\sigma^M_{X,M} : X \otimes M \to X \otimes M \quad (X \in \mathcal{B}, M \in \mathcal{M})
\]

such that the pair \( (\text{id}_M, \sigma^M) : (\mathcal{M}, \otimes) \to (\mathcal{M}, \hat{\circ}) \) is an isomorphism of left \( \mathcal{B} \)-module categories.
Suppose that $\mathcal{M}$ and $\mathcal{N}$ are braided finite $\mathcal{B}$-bimodule categories with braiding $\sigma^\mathcal{M}$ and $\sigma^\mathcal{N}$, respectively. We say that a $\mathcal{B}$-bimodule functor $F : \mathcal{M} \to \mathcal{N}$ is **braided** if the diagram

$$
\begin{array}{ccc}
X \otimes F(M) & \xrightarrow{\sigma^\mathcal{N}_{F(M)}} & F(X \otimes M) \\
\sigma^\mathcal{N}_{F(M)} & & F(\sigma^\mathcal{N}_{M}) \\
F(M) \otimes X & \xrightarrow{F(\sigma^\mathcal{N}_{M})} & F(M \otimes X)
\end{array}
$$

commutes for all $X \in \mathcal{B}$ and $M \in \mathcal{M}$, where the horizontal arrows express the structure morphism of $F$ as a $\mathcal{B}$-bimodule functor.

**Lemma 5.2.** Let $\mathcal{M}$ and $\mathcal{N}$ be as above, and let $R : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{B}$-bimodule functor admitting a left adjoint $L$. Then $R$ is braided if and only if $L$ is.

**Proof.** We only prove ‘if’ part, since the other direction can be proved in a similar manner. Let $\eta : \text{id}_\mathcal{N} \to RL$ and $\varepsilon : LR \to \text{id}_\mathcal{M}$ be the unit and the counit of the adjunction $L \dashv R$. We consider the diagram

$$
\begin{array}{c}
X \otimes R(M) \\
\xrightarrow{\eta}
\end{array}
\begin{array}{c}
RL(X \otimes R(M)) \\
\xrightarrow{R(\zeta^{(f)}_{X,M})}
\end{array}
\begin{array}{c}
R(X \otimes M) \\
\xrightarrow{R(\sigma^\mathcal{N})}
\end{array}
\begin{array}{c}
X \otimes M \\
\xrightarrow{\sigma^\mathcal{N}}
\end{array}
\begin{array}{c}
RL(X \otimes M) \\
\xrightarrow{R(\zeta^{(f)}_{X,M})}
\end{array}
\begin{array}{c}
R(X \otimes M) \\
\xrightarrow{R(\sigma^\mathcal{N})}
\end{array}
\begin{array}{c}
R(M \otimes X) \\
\xrightarrow{\varepsilon}
\end{array}
$$

for $X \in \mathcal{B}$ and $M \in \mathcal{M}$, where

$$
\zeta^{(f)}_{X,M} : L(X \otimes M) \to X \otimes L(M) \quad \text{and} \quad \zeta^{(r)}_{M,X} : L(M \otimes X) \to L(M) \otimes X
$$

are the structure morphisms of $L$ as an oplax $\mathcal{B}$-bimodule functor. The central square is commutative by the assumption that $L$ is braided. The left and the right squares are also commutative by the naturality of $\eta$ and $\sigma^\mathcal{N}$, respectively. Thus we have proved that the above diagram commutes. This implies that $R$ is braided. \(\Box\)

Let $\mathcal{B}$ be a braided finite tensor category over $k$, and let $\mathcal{C}$ be a (not necessarily braided) finite tensor category over $k$. We say that a tensor functor $F : \mathcal{B} \to \mathcal{C}$ is **central** [DNO13, Definition 2.3] if there is a braided tensor functor $\tilde{F} : \mathcal{B} \to 2(\mathcal{C})$ such that $F = U \circ \tilde{F}$, where $U$ is the forgetful functor from $2(\mathcal{C})$. We now assume that $F$ is central and fix such a braided tensor functor $\tilde{F}$. Then we can write

$$
\tilde{F}(X) = \left( F(X), \Sigma_X : F(X) \otimes \text{id}_\mathcal{C} \to \text{id}_\mathcal{C} \otimes F(X) \right) \in 2(\mathcal{C})
$$

for some natural isomorphism $\Sigma_X$. The above lemma can be applied to give another description of the relative modular object:

**Theorem 5.3.** Notations are as above. If the tensor functor $F : \mathcal{B} \to \mathcal{C}$ is perfect, then the relative modular object of $F$ is given by $\mu_F = (\mu, \gamma)$, where

$$
\mu = \alpha^{\mathcal{C}}_\mathcal{B} \otimes F(\alpha^{\mathcal{B}}),
$$

and the natural transformation $\gamma$ is given by

$$
\gamma_X : \mu \otimes F(X) \xrightarrow{\Sigma_X(\mu)^{-1}} F(X) \otimes \mu \quad (X \in \mathcal{B}).
$$
Proof. Equation (5.3) follows from Theorem 5.3. We prove (5.4). The category $\mathcal{B}$ is a braided $\mathcal{B}$-bimodule category in an obvious way. The category $\mathcal{C}$ is a braided $\mathcal{B}$-module category with the action

$$X \otimes_F V = F(X) \otimes V \quad \text{and} \quad V \otimes_F V = V \otimes F(X) \quad (V \in \mathcal{C}, X \in \mathcal{B})$$

and the braiding given by

$$X \otimes_F V = F(X) \otimes V \xrightarrow{\Sigma_{X,V}} V \otimes F(X) = V \otimes_F X \quad (V \in \mathcal{C}, X \in \mathcal{B}).$$

The $\mathcal{B}$-bimodule functor $F : \mathcal{B} \to \mathcal{C}$ is braided by the assumption that $\tilde{F}$ is a braided tensor functor. Now let $G$ be a right adjoint of a right adjoint of $F$. By the definition of the relative modular object, we may assume that $\mu = G(1)$. The previous lemma implies that the $\mathcal{B}$-bimodule functor $G$ is braided. Thus we have the commutative diagram

$$\begin{array}{ccc}
\mu \otimes F(X) = G(1) \otimes_F X & \xrightarrow{(\Sigma_X)^{-1}} & X \otimes_F G(1) = F(X) \otimes \mu \\
\downarrow & & \downarrow \\
G(1 \otimes X) & \xrightarrow{\sigma_X^{-1}} & G(X \otimes 1)
\end{array}$$

for $X \in \mathcal{B}$, where the vertical arrows express the structure of $G$ as a $\mathcal{B}$-bimodule functor. Since $\sigma_X$ is the identity, we obtain (5.4). \qed

The forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is perfect [Shi17, Corollary 4.9]. We note that $\mathcal{Z}(\mathcal{C})$ is unimodular, that is, $\alpha_{\mathcal{Z}(\mathcal{C})}$ is isomorphic to the unit object [EGNO15, Proposition 8.10.10]. Applying the above theorem to $U$, we obtain:

**Corollary 5.4.** The relative modular object of $U$ is given by $\mu_U = (\alpha^*, \gamma)$, where $\alpha$ is the distinguished invertible object of $\mathcal{C}$ and $\gamma$ is given by

$$\gamma_X = (\alpha^* \otimes U(X)) \xrightarrow{(\sigma_X)^{-1}} X \otimes \alpha^* = U(X) \otimes \alpha^*$$

for $X = (X, \sigma) \in \mathcal{Z}(\mathcal{C})$.

### 5.3. The Radford isomorphism of the Drinfeld center.

Let $\mathcal{C}$ be a finite tensor category, and let $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. We have two descriptions of the relative modular object: Theorem 4.3 and Corollary 5.4. Using these results, we can determine the Radford isomorphism of $\mathcal{Z}(\mathcal{C})$ as follows:

**Theorem 5.5.** $\alpha_{\mathcal{Z}(\mathcal{C})} = (\mathbb{1}_{\mathcal{Z}(\mathcal{C})}, \Theta(\alpha_{\mathcal{C}}))$.

As we have mentioned, $\mathcal{Z}(\mathcal{C})$ is known to be unimodular. Thus the Radford isomorphism of $\mathcal{Z}(\mathcal{C})$ is of the form $\tilde{\delta}_X : X \to X^{***} \quad (X \in \mathcal{Z}(\mathcal{C}))$. This theorem claims that the isomorphism $\tilde{\delta}$ is given by

$$(5.5) \quad \tilde{\delta}_X \otimes \id_\alpha = \left( X \otimes \alpha \xrightarrow{\sigma_{X,\alpha}} \alpha \otimes X \xrightarrow{\delta_X} X^{***} \otimes \alpha \right)$$

for $X = (X, \sigma_X) \in \mathcal{Z}(\mathcal{C})$, where $\alpha = \alpha_C$ is the distinguished invertible object of $\mathcal{C}$ and $\delta$ is the Radford isomorphism of $\mathcal{C}$.

**Proof.** Let $\tilde{\delta}$ be the Radford isomorphism of $\mathcal{Z}(\mathcal{C})$. Then, by Theorem 4.3, we have an isomorphism $(1, \tilde{\delta}) \cong \alpha_C \otimes \mu_U$ in $\mathcal{Z}(U, S^4U)$. We see that the isomorphism $\delta$ is
rigid monoidal category is ribbon case where $B$.

Now (5.7) is clear. The converse is proved in a similar way. □

5.4. Classification of the ribbon structures. Let $\mathcal{B}$ be a braided rigid monoidal category with braiding $\sigma$. The Drinfeld isomorphism is the natural isomorphism $u: \text{id}_{\mathcal{B}} \rightarrow S^2_{\mathcal{B}}$ defined by

$$u_X = \left( X \xrightarrow{id \otimes \text{coev}} X \otimes X^* \otimes X^{**} \xrightarrow{\sigma \otimes \text{id}} X^* \otimes X \otimes X^{**} \xrightarrow{\text{ev} \otimes \text{id}} X^{**} \right)$$

for $X \in \mathcal{B}$. It is well-known that $u$ satisfies

$$u_{X \otimes Y} = (u_X \otimes u_Y) \sigma_{X,Y}^{-1} \sigma_{Y,X}^{-1}$$

for all objects $X, Y \in \mathcal{B}$ [EGNO15 Proposition 8.9.3]. Thus a natural isomorphism $\theta : \text{id}_{\mathcal{B}} \rightarrow \text{id}_{\mathcal{B}}$ satisfies (1.1) if and only if $j := u \theta$ is a pivotal structure, that is, a monoidal natural transformation from $\text{id}_{\mathcal{B}}$ to $S^2_{\mathcal{B}}$. This observation suggests that the ribbon structures of $\mathcal{B}$ can be identified with a subset of the set of the pivotal structures of $\mathcal{B}$.

**Theorem 5.6.** Let $j : \text{id}_{\mathcal{B}} \rightarrow S^2_{\mathcal{B}}$ be a pivotal structure of the braided rigid monoidal category $\mathcal{B}$. Then $\theta := u^{-1} j$ is a ribbon structure of $\mathcal{B}$ if and only if

$$j_X^{**} \circ j_X = u_X^{**} \circ (u_X^{*})^{-1}$$

for all $X \in \mathcal{B}$.

**Proof.** We note that $j$ satisfies

$$j_X = (j_X^*)^{-1}$$

for all $X \in \mathcal{B}$ [Sch04 Appendix A]. Since $\theta$ satisfies (1.1), $\theta$ is a ribbon structure if and only if it satisfies (1.2), or, equivalently, $(\theta_X)^* = \theta_X$ for all $X \in \mathcal{B}$. We now suppose that $\theta$ is a ribbon structure. Then, by (5.8), we have

$$u_X^{-1} \circ j_X = \theta_X = (\theta_X)^* = (j_X^*)^* \circ (u_X^*)^{-1} = j_X^{-1} \circ (u_X^*)^{-1}$$

for all $X \in \mathcal{B}$. Hence,

$$j_X = u_X \circ j_X^{-1} \circ (u_X^*)^{-1} = j_X^{-1} \circ u_X^* \circ (u_X^*)^{-1} = (j_X^*)^{-1} \circ (u_X^*)^{-1}.$$

Now (5.7) is clear. The converse is proved in a similar way. □

By slight abuse of terminology, we say that a pivotal structure $j$ of a braided rigid monoidal category is ribbon if $\theta = u^{-1} j$ is a ribbon structure. We consider the case where $\mathcal{B}$ is a braided finite tensor category. Let $(\alpha, \delta)$ be the Radford object $\mathcal{B}$, and let $u_X$ be the right-hand side of (5.7). Then the equation

$$\delta_X = \left( X \otimes \alpha \xrightarrow{\sigma_{X,\alpha}} \alpha \otimes X \xrightarrow{\text{id} \otimes u_X} \alpha \otimes X^{**} \right)$$

holds for all $X \in \mathcal{B}$ [EGNO15 Theorem 8.10.7]; see also Appendix A.3. Thus, by the above theorem, we have:
Corollary 5.7. Notations are as above. A pivotal structure $j$ of the braided finite tensor category $\mathcal{B}$ is ribbon if and only if
\[
\delta_X = (j_X^{**} \otimes \text{id}_\alpha) \circ \sigma_{\alpha,X}
\]
holds for all $X \in \mathcal{B}$. Suppose, moreover, that $\mathcal{B}$ is unimodular. Then $j$ is ribbon if and only if $\delta_X = j_X^{**}$ for all $X \in \mathcal{C}$.

Now we give the following classification of the ribbon structures (more precisely, the ribbon pivotal structures) of the Drinfeld center of a finite tensor category.

Theorem 5.8. Let $\mathcal{C}$ be a finite tensor category, and let $\alpha_\mathcal{C}$ be the Radford object of $\mathcal{C}$. Then the bijection $\Theta : \mathcal{Z}(\text{id}_\mathcal{C}, S^2_\mathcal{C}) \times \rightarrow \text{Nat}^{\otimes}(\mathcal{Z}(\text{id}_\mathcal{C}), S^2_\mathcal{Z}(\mathcal{C}))$ given in Theorem 5.1 restricts to a bijection between the set
\[
\{ [\beta] \in \mathcal{Z}(\text{id}_\mathcal{C}, S^2_\mathcal{C}) \times \mid S^2_\mathcal{Z}(\beta) \otimes \beta \cong \alpha_\mathcal{C} \}
\]
and the set of ribbon pivotal structures of $\mathcal{Z}(\mathcal{C})$.

Proof. Let $\beta \in \mathcal{Z}(\text{id}_\mathcal{C}, S^2_\mathcal{C})$ be an invertible object. By Theorem 5.5 and Corollary 5.7, the pivotal structure $\Theta(\beta)$ of $\mathcal{Z}(\mathcal{C})$ is ribbon if and only if
\[
\Theta(\beta)^{**} \circ \Theta(\beta)X = \Theta(\alpha_\mathcal{C})X
\]
for all $X \in \mathcal{Z}(\mathcal{C})$. By the functorial property (5.2), this is equivalent to
\[
\Theta(S^2_\mathcal{C}(\beta) \otimes \beta) = \Theta(\alpha_\mathcal{C}).
\]
Now the claim follows from the bijectivity of $\Theta$. \qed

We give an application of this theorem to not necessarily semisimple modular tensor categories in the sense of Lyubashenko [KL01]. If $\mathcal{B}$ is a braided finite tensor category, then the coend $F = \int_{X \in \mathcal{B}} X \otimes X^*$ has a natural structure of a Hopf algebra in $\mathcal{B}$. The Hopf algebra has a canonical Hopf paring $\omega : F \otimes F \rightarrow \mathbb{1}$. We say that $\mathcal{B}$ is non-degenerate if $\omega$ is. A modular tensor category is then defined as a non-degenerate ribbon finite tensor category. We note that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is always non-degenerate by [Shi16a] and [EGNO15, Proposition 8.6.3], but it does not have a ribbon structure in general. The above theorem completely determines when $\mathcal{Z}(\mathcal{C})$ admits a ribbon structure, and hence is modular.

A spherical pivotal structure [BW99] is a pivotal structure such that the associated left trace and the right trace coincide. Although spherical fusion categories are an important class of tensor categories, such a kind of trace condition is often meaningless in the non-semisimple setting. From the viewpoint of topological quantum field theory, Douglas, Schommer-Pries and Snyder [DSS13, Definition 4.5.2] introduced an alternative notion of the sphericity of finite tensor categories. In our notation, a spherical finite tensor category in their sense is a finite tensor category $\mathcal{C}$ equipped with a pivotal structure $j$ such that
\[
S^2_\mathcal{C}(1, j) \otimes (1, j) \cong \alpha_\mathcal{C}
\]
in $\mathcal{Z}(\text{id}_\mathcal{C}, S^2_\mathcal{C})$. By the above theorem, we have:

Theorem 5.9. The Drinfeld center of a spherical finite tensor category is a modular tensor category.

Thus we have obtained an answer to Problem (7) of [Müg10, Section 6].
Appendix A. Remarks on the Radford isomorphism

A.1. Original definition. Let $\mathcal{C}$ be a finite tensor category over an algebraically closed field $k$. For simplicity, we write $X^{****} = X^{*4}$. In Subsection 4.1, we have introduced an invertible object $\alpha \in \mathcal{C}$ and a natural isomorphism $\delta_X : \alpha \otimes X \to X^{*4} \otimes \alpha$ ($X \in \mathcal{C}$) which we call the Radford isomorphism. The first aim of this appendix is to check that the isomorphism $\delta_X$ is identical to the isomorphism introduced by Etingof, Nikshych and Ostrik [ENO04, Theorem 3.3].

We first recall the definition given in [ENO04]. As in Subsection 4.1, we make $\mathcal{C}$ as a finite left $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module category by $(X \boxtimes Y) \otimes V = X \otimes V \otimes Y$. Let $\text{Hom}$ be the internal Hom functor of the $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module category $\mathcal{C}$. Then $A := \text{Hom}(1, 1)$ is an algebra in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$. As we have recalled, there is an equivalence $K : \mathcal{C} \to (\mathcal{C} \boxtimes \mathcal{C}^{rev})_A$, $V \mapsto \text{Hom}(1, V) \cong (V \boxtimes 1) \otimes A$ of left $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module categories.

The distinguished invertible object of $\mathcal{C}$ [ENO04, Definition 3.1] is the object corresponding to the right $A$-module $A^*$ via the above equivalence. Let $\alpha$ be the distinguished invertible object. Then, by definition, there is an isomorphism $\varphi : (A^{**} \boxtimes 1) \otimes A \to A^*$ of right $A$-modules in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$. This induces an isomorphism of algebras in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$.

$\tau_X : (1 \boxtimes X) \otimes A \to (X \boxtimes 1) \otimes A$ ($X \in \mathcal{C}$) obtained from the structure of $\text{Hom}(1, -)$ as a $\mathcal{C} \boxtimes \mathcal{C}^{rev}$-module functor. We now define the natural isomorphism $\tilde{\tau}_X : (1 \boxtimes X) \otimes (\alpha \boxtimes 1) \otimes A \to (X^{*4} \boxtimes 1) \otimes (\alpha \boxtimes 1) \otimes A$ ($X \in \mathcal{C}$) to be the unique morphism such that the diagram

$$\begin{array}{ccc}
(1 \boxtimes ** Y) \otimes A^{**} & \xrightarrow{id \otimes \varphi} & (1 \boxtimes ** Y) \otimes (\alpha \boxtimes 1) \otimes A \otimes (\alpha^* \boxtimes 1) \\
\tau_Y \downarrow & & \downarrow \tilde{\tau}_X \otimes id \\
(Y^{*} \boxtimes 1) \otimes A^{**} & \xrightarrow{id \otimes \varphi} & (Y^{**} \boxtimes 1) \otimes (\alpha \boxtimes 1) \otimes A \otimes (\alpha^* \boxtimes 1)
\end{array}$$

commutes, where $Y = X^{**}$. For reader’s convenience, we note that $\tau_X$ and $\tilde{\tau}_X$ are the inverses of the natural isomorphisms $\rho_X$ and $\tilde{\rho}_X$ of [ENO04], respectively. Now the isomorphism $\delta_X^{ENO} : \alpha \otimes X \to X^{*4} \otimes \alpha$ ($X \in \mathcal{C}$) given in [ENO04, Theorem 3.3] is described as follows:
Definition A.1. We define $\delta_{\text{ENO}}^X$ to be the isomorphism in $\mathcal{C}$ such that

$$K(\delta_{\text{ENO}}^X) = \left( K(\alpha \otimes X) = (\alpha \boxtimes 1) \otimes (X \boxtimes 1) \otimes A \xrightarrow{id \otimes \tau_X} (\alpha \boxtimes 1) \otimes (1 \boxtimes X) \otimes A \xrightarrow{\tau_X^*} (X^{*4} \boxtimes 1) \otimes (\alpha \boxtimes 1) \otimes A = K(X^{*4} \otimes \alpha) \right).$$

A.2. The algebra $A$ as a coend. We have considered the coend $A' = \int^{X \in \mathcal{C}} X \boxtimes X^*$ in $\mathcal{C} \boxtimes \mathcal{C}^\text{rev}$ in this paper. By the result of [Shi17a], the algebra $A = \text{Hom}(1, 1)$ can be identified with the coend $A'$. To be precise, let $i_X : X \boxtimes X^* \to A'$ be the universal dinatural transformation of the coend. We define $\varepsilon^0 : A' \otimes 1 \to 1$ to be the unique morphism in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc}
(X^* \boxtimes X) \otimes 1 & \xrightarrow{i_X \otimes id_1} & A' \otimes 1 \\
\downarrow \quad & & \downarrow \varepsilon^0 \\
X^* \otimes X & \xrightarrow{ev} & 1
\end{array}$$

commutes for all $X \in \mathcal{C}$. For $M \in \mathcal{C} \boxtimes \mathcal{C}^\text{rev}$ and $V \in \mathcal{C}$, we have the map

$$\text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^\text{rev}}(M, (V \boxtimes 1) \otimes A') \to \text{Hom}_{\mathcal{C}}(M \otimes 1, V)$$

sending a morphism $f : M \to (V \boxtimes 1) \otimes A'$ in $\mathcal{C} \boxtimes \mathcal{C}^\text{rev}$ to

(A.1) $M \otimes 1 \xrightarrow{f \otimes id_1} ((V \boxtimes 1) \otimes A') \otimes 1 = V \otimes (A' \otimes 1) \xrightarrow{id_V \otimes \varepsilon^0} V.$

By the discussion of [Shi17a] Subsection 4.3], the map (A.2) is bijective. Thus, by the definition of the internal Hom functor, we may identify

(A.2) $\text{Hom}(1, V) = (V \boxtimes 1) \otimes A'$

for all $V \in \mathcal{C}$ and, in particular, $A = A'$. It is moreover shown in [Shi17a] that $A$ and $A'$ are identified with algebras in $\mathcal{C} \boxtimes \mathcal{C}^\text{rev}$ if we endow the coend $A'$ with an algebra structure as in Subsection 3.4 of this paper.

In what follows, we identify $\text{Hom}(1, V)$ with $(V \boxtimes 1) \otimes A'$ as in (A.2) and, in particular, $A = A'$. We note that the natural isomorphism $\tau$, which have played an important role in the definition of $\delta_{\text{ENO}}$, is defined in terms of the structure of $\text{Hom}(1, -)$ as a module functor. To express the isomorphism $\tau$ in terms of the universal dinatural transformation, we first give the following descriptions of the unit and the counit of (A.2).

Lemma A.2. The unit of the adjunction (A.2), which we denote by

$$\eta_M : M \to \text{Hom}(M \otimes 1) \quad (M \in \mathcal{C} \boxtimes \mathcal{C}^\text{rev}),$$

finds a pullback:

Diagram (A.5) $\begin{array}{ccc}
\text{Hom}(M \otimes 1, V) & \xrightarrow{\varepsilon} & \text{Hom}(1, V) \\
\downarrow \quad & & \downarrow \varepsilon^0 \\
\text{Hom}(M, V) & \xrightarrow{\tau_M} & \text{Hom}(1, V)
\end{array}$

for all $M \in \mathcal{C} \boxtimes \mathcal{C}^\text{rev}$ and $V \in \mathcal{C}$.
is a unique natural transformation such that
\[ \eta_{X \boxtimes Y} = \left( X \boxtimes Y \overset{\text{id} \otimes \text{coev}}{\longrightarrow} (X \otimes Y \otimes Y^*) \boxtimes Y = (X \otimes Y) \boxtimes (Y^* \boxtimes Y) \right) \]
for all objects \( X, Y \in \mathcal{C} \). The counit of (A.2),
\[ \varepsilon_V : \text{Hom}(\mathbb{1}, V) \otimes \mathbb{1} \rightarrow V \quad (V \in \mathcal{C}), \]
is given by
\[ \varepsilon_V = \left( \text{Hom}(\mathbb{1}, V) \otimes \mathbb{1} = V \otimes (A \otimes \mathbb{1}) \overset{\text{id} \otimes \varepsilon^0}{\longrightarrow} V \otimes \mathbb{1} = V \right). \]

**Proof.** It is obvious from (A.1) that the counit is given as stated. We note that the unit of the adjunction is the morphism corresponding to the identity via
\[ \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}}(M, \text{Hom}(\mathbb{1}, M \otimes \mathbb{1})) \underset{(A.2)}{\longrightarrow} \text{Hom}(M \otimes \mathbb{1}, M \otimes \mathbb{1}). \]
Thus the description of the unit follows from the equation
\[ \text{id}_{X \otimes Y} = \left( X \otimes Y \overset{\eta_{X \boxtimes Y} \otimes 1}{\longrightarrow} X \otimes Y \otimes (A \otimes \mathbb{1}) \overset{\text{id} \otimes \varepsilon^0}{\longrightarrow} X \otimes Y \right), \]
which is easily verified. \(\square\)

By Lemma 2.1 the structure morphism of \( \text{Hom}(\mathbb{1}, -) \) is given by
\[ M \otimes \text{Hom}(\mathbb{1}, V) \underset{\eta}{\longrightarrow} \text{Hom}(\mathbb{1}, (M \otimes \text{Hom}(\mathbb{1}, V)) \otimes \mathbb{1}) = \text{Hom}(\mathbb{1}, M \otimes (\text{Hom}(\mathbb{1}, V) \otimes \mathbb{1})) \]
\[ \underset{\text{Hom}(\mathbb{1}, \text{id} \otimes M \otimes \varepsilon_V)}{\longrightarrow} \text{Hom}(\mathbb{1}, M \otimes V) \]
for \( M \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \) and \( V \in \mathcal{C} \). The morphism \( \tau_X \) for \( X \in \mathcal{C} \) is the case where \( V = \mathbb{1} \) and \( M = \mathbb{1} \boxtimes X \). Now the straightforward computation shows:

**Lemma A.3.** The following diagram commutes for all \( X, Y \in \mathcal{C} \).

\[
\begin{array}{ccc}
(\mathbb{1} \boxtimes X) \otimes (Y^* \boxtimes Y) & \overset{\text{id} \otimes \varepsilon_Y}{\longrightarrow} & (\mathbb{1} \boxtimes X) \otimes A \\
((X \boxtimes X^*) \boxtimes Y) & \underset{\tau_X}{\longrightarrow} & (X \boxtimes Y) \otimes (Y^* \boxtimes X) \\
((X \boxtimes \mathbb{1}) \otimes (Y \otimes X^*) \boxtimes (Y \otimes X)) & \overset{\text{id} \otimes \varepsilon_Y \otimes X}{\longrightarrow} & (X \boxtimes \mathbb{1}) \otimes A
\end{array}
\]

**A.3. Proof of the equivalence.** The algebra \( A^{**} \) acts on \( (\alpha \boxtimes \mathbb{1}) \otimes A \) by
\[ (A.3) \quad \hat{\rho} = \left( A^{**} \otimes (\alpha \boxtimes \mathbb{1}) \otimes A \overset{\text{id} \otimes \phi}{\longrightarrow} A^{**} \otimes A^* \overset{m^1}{\longrightarrow} A^* \overset{\phi^{-1}}{\longrightarrow} (\alpha \boxtimes \mathbb{1}) \otimes A \right). \]
The distinguished invertible object \( \alpha \) is defined by \( (\alpha \boxtimes \mathbb{1}) \otimes A \cong A^* \). Since \( K \) is an equivalence of module categories, \( \alpha \) is an \( A^{**} \)-module in \( \mathcal{C} \). Let \( \rho \) be the action of \( A^{**} \) on \( \alpha \). If we fix a quasi-inverse \( \overline{K} \) of \( K \), then \( \rho \) is given by
\[ \rho = \left( A^{**} \otimes \alpha \overset{\text{tr}}{\longrightarrow} A^{**} \otimes \overline{K}(\alpha) \overset{\text{tr}}{\longrightarrow} \overline{K}(A^{**} \otimes K(\alpha)) \overset{\overline{K}(\rho)}{\longrightarrow} \overline{K}(\alpha) \overset{\text{tr}}{\longrightarrow} \alpha \right), \]
where the second arrow is the structure of \( K \) as a \( \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \)-module functor. The Radford isomorphism \( \delta \), introduced in Subsection 4.1, is now given by

\[
\delta_X = \left( \alpha \otimes X \xrightarrow{\text{coev} \otimes \text{id} \otimes \text{id}} X^* \otimes X^5 \otimes \alpha \otimes X = X^* \otimes (X^{**} \boxtimes X^{**})^* \otimes \alpha \right) \xrightarrow{\text{id} \otimes (i^* \otimes \alpha)} X^* \otimes (A^* \otimes \alpha) \xrightarrow{\text{id} \otimes \rho} X^* \otimes \alpha.
\]

for \( X \in \mathcal{C} \). We now prove:

**Theorem A.4.** \( \delta = \delta^{\text{ENO}} \).

**Proof.** Recall that the multiplication \( m \) of \( A \) is the unique morphism in \( \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \) such that

\[
m \circ (i_X \otimes i_Y) = i_Y \otimes X \text{ for all objects } X, Y \in \mathcal{C}.
\]

Thus we have

(A.4) \[ \tau_X = m \circ (\zeta_X \otimes \text{id}_A) \]

by Lemma A.3, where

\[
\zeta_X = \left( \mathbb{1} \boxtimes X \xrightarrow{\text{coev} \otimes \text{id} \otimes \text{id}} (X \otimes X^*) \boxtimes X = (X \boxtimes \mathbb{1}) \otimes (X^* \boxtimes X) \xrightarrow{\text{id} \otimes i_X} (X \boxtimes \mathbb{1}) \otimes A \right).
\]

Let \( m^\dagger : A^{**} \otimes A^* \rightarrow A^* \) be the action of \( A^{**} \) of \( A^* \). Since \( \phi^* \) is a morphism of right \( A \)-modules, \( \phi^* \) is a morphism of left \( A^{**} \)-modules. Namely,

\[
\phi^* \circ m^{**} = (m^\dagger \otimes (\text{id}_{A^{**}} \boxtimes \text{id}_A)) \circ (\text{id}_{A^{**}} \otimes \phi^*).
\]

Hence, for all \( X \in \mathcal{C} \), the following diagram is commutative:

\[
\begin{array}{ccc}
(1 \boxtimes X^{**}) \otimes A^{**} & \xrightarrow{\text{id} \otimes \phi^*} & (1 \boxtimes X^{**}) \otimes A^* \otimes (\alpha^* \boxtimes \mathbb{1}) \\
\zeta_X \otimes \text{id} & \downarrow & \zeta_X \otimes \text{id} \\
(X^{**} \boxtimes \mathbb{1}) \otimes A^{**} \otimes A^{**} & \xrightarrow{\text{id} \otimes \text{id} \otimes \phi^*} & (X^{**} \boxtimes \mathbb{1}) \otimes A^{**} \otimes A^* \otimes (\alpha^* \boxtimes \mathbb{1}) \\
\text{id} \otimes m^{**} & \downarrow & \text{id} \otimes m^\dagger \otimes \text{id} \\
(X^{**} \boxtimes \mathbb{1}) \otimes A^{**} & \xrightarrow{\text{id} \otimes \phi^*} & (X^{**} \boxtimes \mathbb{1}) \otimes A^* \otimes (\alpha^* \boxtimes \mathbb{1}).
\end{array}
\]

By (A.4), the composition along the first row is \( \tau_X^{**} \). Thus,

\[
\tilde{\tau}_X = \left( (1 \boxtimes X) \otimes (\alpha \boxtimes \mathbb{1}) \otimes A \xrightarrow{\zeta_X \otimes \phi} (X^* \boxtimes \mathbb{1}) \otimes A^{**} \otimes A^* \xrightarrow{\text{id} \otimes m^\dagger} (X^* \boxtimes \mathbb{1}) \otimes A^* \xrightarrow{\phi^{-1}} (X^* \boxtimes \mathbb{1}) \otimes (\alpha \boxtimes \mathbb{1}) \otimes A \right).
\]

Comparing this result with the definition of \( \rho \), we obtain

(A.5) \[ \tilde{\tau}_X = (\text{id}_{1 \boxtimes X^*} \otimes \rho) \circ (\zeta_X^{**} \otimes \text{id}_{\alpha \boxtimes \mathbb{1}} \otimes \text{id}_A). \]
Now we consider the following diagram:

\[
\begin{array}{cccc}
K(\alpha \otimes X) & \cong & (\mathbb{1} \boxtimes X) \otimes K(\alpha) \\
K(\text{coev} \otimes \text{id} \otimes \text{id}) & & (\text{coev} \otimes \text{id}) & \\
K(X^4 \otimes X^5 \otimes \alpha \otimes X) & \cong & (X^4 \boxtimes 1) \otimes (X^5 \boxtimes X) \otimes K(\alpha) \\
K(\text{id} \otimes (\rho^*) \otimes \text{id}) & & \text{id} \otimes \rho^* & \\
K(X^4 \otimes (\rho^* \otimes \alpha)) & \cong & (X^4 \boxtimes 1) \otimes \rho^* \otimes K(\alpha) \\
K(\text{id} \otimes \rho) & & \text{id} & \\
K(X^4 \otimes \alpha) & \rightarrow & (X^4 \boxtimes 1) \otimes K(\alpha),
\end{array}
\]

where the horizontal arrows express the structure of $K$ as a module functor. This diagram is commutative. The composition along the first row is $\tilde{\tau}_X$, while the composition along the second row is $\tilde{\tau}_X$ by (A.5). By the definition of the natural isomorphism $\tau$, the topmost horizontal arrow is

$$K(\alpha \otimes X) = (\alpha \boxtimes \mathbb{1}) \otimes (\mathbb{1} \boxtimes \mathbb{1}) \otimes A$$

$$\xrightarrow{\text{id} \otimes \tau_X} (\alpha \boxtimes \mathbb{1}) \otimes (\mathbb{1} \boxtimes X) \otimes K(\alpha).$$

Hence, by the definition of $\delta^{\text{ENO}}$, we obtain $K(\delta_X) = K(\delta^{\text{ENO}}_X)$. Since $K$ is an equivalence, the equation $\delta_X = \delta^{\text{ENO}}_X$ holds for all objects $X \in \mathcal{C}$. \hfill $\square$

A.4. The braided case. Suppose that $\mathcal{C}$ has a braiding $\sigma$. We define the Drinfeld isomorphism $u : \text{id}_\mathcal{C} \rightarrow S^2_\mathcal{C}$ as in Subsection 5.4 and then define

$$w_X = \left( X \xrightarrow{(u_X)^{-1}} X^{**} \xrightarrow{\sigma_X} X^{****} \right) = \left( X \xrightarrow{u_X} X^{**} \xrightarrow{(u_X)^{-1}} X^{****} \right)$$

for $X \in \mathcal{C}$. Then the Radford isomorphism has the following expression:

**Theorem A.5.** For all $X \in \mathcal{C}$, we have

$$\delta_X = \left( \alpha \otimes X \xrightarrow{\sigma_{X}} X \otimes \alpha \xrightarrow{w_X \otimes \text{id}} X^4 \otimes \alpha \right).$$

This theorem is a generalization of Radford’s result on the distinguished grouplike elements of finite-dimensional quasitriangular Hopf algebras [Rad92] and has been proved in [ENO04] in the unimodular case and in [EGNO15, Theorem 8.10.7] in the general case. We give an alternative proof of this theorem.

To give a proof, we endow $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ with the braiding $\tilde{\sigma}$ determined by

$$\tilde{\sigma}_{X \boxtimes Y, Y \boxtimes Y'} = \sigma_{X, Y} \boxtimes \sigma^{-1}_{X', Y'}, \quad (X, Y, X', Y' \in \mathcal{C}).$$

**Lemma A.6.** The algebra $A$ is commutative.

**Proof.** For all objects $X, Y \in \mathcal{C}$, we have

$$m \circ \tilde{\sigma}_{A, A} \circ (i_X \otimes i_Y) = m \circ (i_Y \otimes i_X) \circ \tilde{\sigma}_{X \boxtimes X, Y \boxtimes Y}$$

$$= i_{X \otimes Y} \circ (\sigma_{X, Y} \boxtimes \sigma^{-1}_{X, Y})$$

$$= i_{X \otimes Y} \circ (\sigma_{X, Y} \boxtimes \sigma^{-1}_{X, Y})$$

$$= i_Y \otimes X \circ (\text{id}_{X \otimes Y} \boxtimes \sigma_{X, Y} \sigma_{X, Y}^{-1}) \quad \text{(by the dinaturality)}$$

$$= i_Y \otimes X = m \circ (i_X \otimes i_Y).$$
Hence \( m = m \circ \tilde{\sigma}_{A,A} \), that is, \( A \) is commutative.

To proceed further, we use the graphical calculus; see, e.g., [Kas95]. Our convention is that a morphism goes from the top to the bottom of the diagram. The evaluation, the coevaluation, the braiding and its inverse are expressed by

\[
\begin{array}{c}
\xymatrix{X^* \ar@/^/[r] & X \ar@/^/[l] & Y \ar@/^/[r] \\
X \ar@/^/[u] & X^* \ar@/^/[u] & Y \ar@/^/[u] \ar@/^/[l] \\
Y \ar@/^/[u] & X \ar@/^/[u] & Y \ar@/^/[u] \ar@/^/[l]
}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\xymatrix{Y \ar@/^/[r] & Y^* \ar@/^/[l] & X \ar@/^/[r] \\
Y \ar@/^/[u] & X \ar@/^/[u] & X^* \ar@/^/[u] \ar@/^/[l]
}
\end{array}
\]

respectively. For example, the Drinfeld isomorphism and its inverse are

\[
u_X = \begin{array}{c}
\xymatrix{X \ar@/^/[r] & X^* \ar@/^/[l]}
\end{array}
\quad \text{and} \quad
\nu_X^{-1} = \begin{array}{c}
\xymatrix{X^* \ar@/^/[r] & X \ar@/^/[l]}
\end{array}
\]

We denote by \( \tilde{\nu} \) the Drinfeld isomorphism of \( \mathcal{C} \boxtimes \mathcal{C}^{rev} \). By the above graphical expression of \( \nu \) and \( \nu^{-1} \), we have:

**Lemma A.7.** \( \tilde{\nu}_{XY} = \nu_X \boxtimes \nu_Y^{-1} \) for all \( X, Y \in \mathcal{C} \), where \( \tilde{\nu}_Y = (\nu^{-1})^{-1} \).

**Lemma A.8.** The left action \( \tilde{\rho} \) of \( A^{**} \) on \( K(A) = (\alpha \boxtimes \mathbb{1}) \otimes A \), defined by (A.3), is equal to the following composition of morphisms:

\[
\tilde{\rho} = 
\begin{array}{c}
\begin{array}{c}
A^{**} \otimes (\alpha \boxtimes \mathbb{1}) \otimes A
\xymatrix{\frac{\nu^{-1}_{A^*} \otimes \text{id}}{\tilde{\sigma}_{A^{**},A^*}}}
A \otimes (\alpha \boxtimes \mathbb{1}) \otimes A
\xymatrix{\text{id} \otimes m}
\end{array}
\end{array}
\]

*Proof.* By the commutativity of the algebra \( A \), we compute:

\[
\begin{array}{c}
\xymatrix{A^* \boxtimes A \ar[r]^{\tilde{\nu}} & A \boxtimes A^* \ar[r]^{m^*} & A \boxtimes A^*}
\end{array}
\]

As before, we denote by \( m^\dagger : A^{**} \otimes A^* \to A^* \) the action of \( A^{**} \) on \( A^* \). By the above computation, we have

\[
m^\dagger \circ (\tilde{\nu}_A \otimes \text{id}_{A^*}) = \begin{array}{c}
\begin{array}{c}
\xymatrix{A \boxtimes A \ar[r] & A \boxtimes A^* \ar[r] & A \boxtimes A^*}
\end{array}
\end{array}
\]

We recall that \( \phi : A^* \to (\alpha \boxtimes \mathbb{1}) \otimes A \) is a morphism of right \( A \)-modules, that is,
The morphism \( \tilde{\rho} \) is computed as follows:

\[
\tilde{\rho} = \phi^{-1} \circ m^+ \circ (\text{id}_{A^{**}} \otimes \phi)
\]

\[
= \phi^{-1} \circ m^+ \circ (\tilde{u}_A \otimes \phi) \circ (\tilde{u}_A^{-1} \otimes \text{id})
\]

\[
= (\text{id}_{\alpha \circ \tilde{u}_1} \otimes m) \circ (\sigma_{A, \alpha \circ \tilde{u}_1} \otimes \text{id}_A) \circ (\tilde{u}_A^{-1} \otimes \text{id}_{\alpha \circ \tilde{u}_1} \otimes \text{id}_A).
\]

The proof is done. \( \Box \)

**Proof of Theorem A.2** By the above lemma, \( \tilde{\rho} \circ (i_X^{**} \otimes \text{id}_{\alpha \circ \tilde{u}_1} \otimes \text{id}_A) \)

\[
= (\text{id}_{\alpha \circ \tilde{u}_1} \otimes m) \circ (\sigma_{A, \alpha \circ \tilde{u}_1} \otimes \text{id}_A) \circ ((\tilde{u}_A^{-1} \otimes i_X^{**}) \otimes \text{id}_{\alpha \circ \tilde{u}_1} \otimes \text{id}_A)
\]

\[
= (\text{id}_{\alpha \circ \tilde{u}_1} \otimes m) \circ (\sigma_{A, \alpha \circ \tilde{u}_1} \otimes \text{id}_A) \circ ((\tilde{u}_A^{-1} \otimes \text{id}_{\alpha \circ \tilde{u}_1} \otimes \text{id}_A) \circ (i_X \otimes \text{id}_{\alpha \circ \tilde{u}_1} \otimes \text{id}_A)
\]

Thus we have the following commutative diagram (cf. the proof of Theorem A.4):

\[
\begin{array}{ccc}
K(\alpha \otimes X^{**}) & \cong & (1 \otimes X^{**}) \otimes (\alpha \otimes 1) \otimes A \\
K(\text{coev} \otimes \text{id}) & & (\text{coev} \otimes \text{id}) \otimes \text{id} \\
K(X^{**} \otimes X^{**} \otimes \alpha \otimes X^{**}) & \cong & (X^{**} \otimes 1) \otimes (X^{**} \otimes X^{**} \otimes \alpha \otimes 1) \otimes A \\
K(\text{id} \otimes u_{X^{**} \otimes \alpha}^{-1}) & & (\text{id} \otimes (u_{X^{**} \otimes \alpha}^{-1} \otimes \text{id} \otimes \text{id}) \\
K(\alpha \otimes X^{**} \otimes \alpha \otimes X^{**}) & \cong & (X^{**} \otimes 1) \otimes (X^{**} \otimes X^{**} \otimes \alpha \otimes 1) \otimes A \\
K(\text{id} \otimes \text{id} \otimes (\text{id} \otimes 1)) & & (\text{id} \otimes (\text{id} \otimes \text{id} \otimes 1)) \\
K(\alpha \otimes (A \otimes 1)) & \cong & (X^{**} \otimes 1) \otimes (\alpha \otimes 1) \otimes A \\
\text{id} \otimes \text{id} \otimes c^0 & & (\text{id} \otimes \text{id} \otimes \text{id}) \\
K(\alpha) & \cong & (X^{**} \otimes 1) \otimes (\alpha \otimes 1) \otimes A
\end{array}
\]
By the definition of the Radford isomorphism, the composition along the first row is \(k(\delta \cdot \alpha)\). Replacing \(X\) with \(X^{\ast\ast}\), we obtain

\[
\delta_X = \begin{array}{ccc}
\alpha & \xrightarrow{u_{X^{\ast\ast}}^{-1}} & X \\
\xrightarrow{\sigma_{\alpha \rightarrow}} & \alpha & \xrightarrow{1} \alpha \\
\end{array}
\]

Since \((u_{X^{\ast\ast}}^{-1})^\ast = (u_{X^{\ast}}^{-1})^\ast\) and \(\tilde{u}_{X^{\ast\ast}}^{-1} = u_X\), we have

\[
\delta_X = \left(\alpha \otimes X \xrightarrow{\sigma_{\alpha \rightarrow X}} X \otimes \alpha \xrightarrow{u_X \otimes \text{id}} X^{\ast\ast} \otimes \alpha\right).
\]

Once we obtain this formula, we prove \(\sigma_{X^{\ast\ast},\alpha}^{-1} = \sigma_{\alpha,X}^{-1}\) by the same way as [EGNO15, Corollary 8.10.8]. Hence the proof is done. \(\square\)

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