ON THE REMODELING CONJECTURE FOR TORIC CALABI-YAU 3-ORBIFOLDS

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ABSTRACT. The Remodeling Conjecture proposed by Bouchard-Klemm-Mariño-Pasquetti (BKMP) relates the A-model open and closed topological string amplitudes (the all genus open and closed Gromov-Witten invariants) of a semi-projective toric Calabi-Yau 3-manifold/3-orbifold to the Eynard-Orantin invariants of its mirror curve. It is an all genus open-closed mirror symmetry for toric Calabi-Yau 3-manifolds/3-orbifolds. In this paper, we present a proof of the BKMP Remodeling Conjecture for all genus open-closed orbifold Gromov-Witten invariants of an arbitrary semi-projective toric Calabi-Yau 3-orbifold relative to an outer framed Aganagic-Vafa Lagrangian brane. We also prove the conjecture in the closed string sector at all genera.

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1. Introduction

1.1. Background and motivation. Mirror symmetry is a duality from string theory originally discovered by physicists. It says two dual string theories – type IIA and type IIB – on different Calabi-Yau 3-folds give rise to the same physics. Mathematicians became interested in this relationship around 1990 when Candelas, de la Ossa, Green, and Parks \[19\] obtained a conjectural formula of the number of rational curves of arbitrary degree in the quintic 3-fold by relating it to period integrals of the quintic mirror. By late 1990s mathematicians had established the foundation of Gromov-Witten (GW) theory as a mathematical theory of A-model topological closed strings. In this context, the genus $g$ free energy of the topological A-model on a Calabi-Yau 3-fold $\mathcal{X}$ is defined as a generating function $F^g_{\mathcal{X}}$ of genus $g$ Gromov-Witten invariants of $\mathcal{X}$, which is a function on a (formal) neighborhood around the large radius limit in the complexified Kähler moduli of $\mathcal{X}$. The genus $g$ free energy of the topological B-model on the mirror Calabi-Yau 3-fold $\mathcal{X}$ is a section of $\mathcal{V}^{2g}$, where $\mathcal{V}$ is the Hodge line bundle over the complex moduli $\mathcal{M}$.
of $\tilde{X}$, whose fiber over $\tilde{X}$ is $H^0(\tilde{X}, \Omega^3_{\tilde{X}})$. Locally it is a function $\tilde{F}_g^{\tilde{X}}$ near the large radius limit on the complex moduli $\tilde{X}$. Mirror symmetry predicts $F_g^{\tilde{X}} = \tilde{F}_g^{\tilde{X}}$ under the mirror map as a mathematical conjecture. The mirror map and $\tilde{F}_0^{\tilde{X}}$ are determined by period integrals of a holomorphic 3-form on $\tilde{X}$. Period integrals on toric manifolds and complete intersections in them can be expressed by explicit hypergeometric functions. This conjecture has been proved in many cases to various degrees. Roughly speaking, our result is about (a much more generalized version of) this conjecture when $X$ is a toric Calabi-Yau 3-orbifold.

1.1.1. Mirror symmetry for compact Calabi-Yau manifolds. Givental [50] and Lian-Liu-Yau [64] independently proved the genus 0 mirror formula for the quintic. Later they extended their results to Calabi-Yau complete intersections in projective toric manifolds [52, 65, 66]. The genus-one mirror formula $F_1^Q$ for the quintic 3-fold $Q$ was conjectured by Bershadsky-Cecotti-Ooguri-Vafa (BCOV) [9] and proved by Zinger [91]. Combining the techniques of BCOV, results of Yamaguchi-Yau [85], and boundary conditions, Huang-Klemm-Quacken [57] proposed a mirror conjecture on $F_g^Q$ up to $g = 51$. The mirror conjecture on $F_g^Q$ is open for $g \geq 2$. One difficulty is that mathematical theory of higher genus B-model on a general compact Calabi-Yau manifold has not been developed until very recently. In 2012, Costello and Li initiated a mathematical analysis of the BCOV theory [32] based on the effective renormalization method developed by Costello [31]. One essential idea in their construction [33] is to introduce open topological strings on the B-model. The higher genus B-model potentials are then uniquely determined by the genus-zero open B-model potentials. We will see later that this phenomenon also arises in the BKMP Remodeling Conjecture, where the higher genus B-model potentials are determined by the genus-zero open B-model potentials via the Eynard-Orantin recursion.

1.1.2. Gromov-Witten invariants of toric Calabi-Yau 3-manifolds/orbifolds. The technique of virtual localization [55] reduces all genus Gromov-Witten invariants of toric orbifolds to Hodge integrals. When the toric orbifold $X$ is a smooth Calabi-Yau 3-fold, the Topological Vertex [5, 63, 72] provides an efficient algorithm to compute these integrals, and thus to compute Gromov-Witten invariants of $X$ as well as open Gromov-Witten invariants of $X$ relative to an Aganagic-Vafa Lagrangian brane $\mathcal{L}$ (defined in [60, 66, 68, 63] in several ways), in all genera and degrees. The algorithm of the topological vertex is equivalent to the Gromov-Witten/Donaldson-Thomas correspondence for smooth toric Calabi-Yau threefolds [71]; it provides a combinatorial formula for a generating function of all genus Gromov-Witten invariants of a fixed degree. Recently, this effective algorithm has been generalized to toric Calabi-Yau 3-orbifolds with transverse $A_n$ singularities [92, 77, 78, 79], but not for more general toric Calabi-Yau 3-orbifolds.

1.1.3. Mirror symmetry for toric Calabi-Yau 3-manifolds/orbifolds. The topological B-model for the mirror $\hat{X}$ of a semi-projective toric Calabi-Yau 3-manifold/orbifold $X$ can be reduced to a theory on the mirror curve of $\hat{X}$ [50]. Under mirror symmetry, $F_0^X$ corresponds to integrals of 1-forms on the mirror curve along loops, whereas the generating function $F^{X, \mathcal{L}}_{0,1}$ of genus-zero open Gromov-Witten invariants (counting holomorphic disks in $X$ bounded by $\mathcal{L}$) corresponds to integrals of 1-forms on the mirror curve along paths [71, 6]. Based on the work of Eynard-Orantin
and Mariño [70], Bouchard-Klemm-Mariño-Pasquetti [15] [16] proposed a new formalism of the topological B-model on \( X \) in terms of the Eynard-Orantin invariants \( \omega_{g,n} \) of the mirror curve, and conjectured a precise correspondence, known as the BKMP Remodeling Conjecture, between \( \omega_{g,n} \) (where \( n > 0 \)) and the generating function \( F^X,\mathcal{L} \) of open Gromov-Witten invariants counting holomorphic maps from bordered Riemann surfaces with \( g \) handles and \( n \) holes to \( X \) with boundaries in \( \mathcal{L} \). They also proposed a conjectural statement on closed Gromov-Witten invariants \( F^X_g \). These conjectures, known as the BKMP Remodeling Conjecture in both open string and closed string sectors, are all genus open-closed mirror symmetry, and provide an effective algorithm of computing \( F^X,\mathcal{L} \) and \( F^X_g \) recursively, for general semi-projective toric Calabi-Yau 3-orbifolds.

The open string part of the Remodeling Conjecture for \( \mathbb{C}^3 \) was proved independently by L. Chen [24] and J. Zhou [87]. The free energy part of the Remodeling Conjecture for \( \mathbb{C}^3 \) was proved independently by Bouchard-Catuneanu-Marchal-Sulkowski [11] and S. Zhu [90]. Eynard and Orantin provided a proof of the Remodeling Conjecture for general smooth semi-projective toric Calabi-Yau 3-folds in [43]. The authors proved the Remodeling Conjecture for all semi-projective affine toric Calabi-Yau 3-orbifolds [\( \mathbb{C}^3/G \) [46].

1.2. Statement of the main result and outline of the proof. In this paper, we prove the BKMP Remodeling Conjecture for a general semi-projective toric Calabi-Yau 3-orbifold \( X \), in both the open string sector and the closed string sector. We consider a framed Aganagic-Vafa brane \((\mathcal{L},f)\) on an outer leg of \( X \), where \( \mathcal{L} \equiv [(S^1 \times \mathbb{C})/\mu_m] \) for a finite abelian group \( \mu_m \equiv \mathbb{Z}_m \) and \( f \in \mathbb{Z} \). This outer leg may be gerby with a non-trivial isotropy group \( \mu_m \). We define generating functions of open-closed Gromov-Witten invariants:

\[
F^X,\mathcal{L},f_{g,n}(\tau_1,\ldots,\tau_p,\tilde{X}_1,\ldots,\tilde{X}_n)
\]

as \( H^*_{CR}(B\mu_m;\mathbb{C}) \otimes \mathbb{C}^n \)-valued formal power series in A-model closed string coordinates \( \tau = (\tau_1,\ldots,\tau_p) \) and A-model open string coordinates \( \tilde{X}_1,\ldots,\tilde{X}_n \); here \( H^*_{CR}(B\mu_m;\mathbb{C}) \equiv \mathbb{C}^m \) is the Chen-Ruan orbifold cohomology of the classifying space \( B\mu_m \) of \( \mu_m \). (The precise definition of \( F^X,\mathcal{L},f_{g,n} \) is given in Section 3.12.)

On the other hand, we use the Eynard-Orantin invariants \( \omega_{g,n} \) of the framed mirror curve to define B-model potentials

\[
\hat{F}_{g,n}(\underline{q};\hat{X}_1,\ldots,\hat{X}_n)
\]

as \( H^*_{CR}(B\mu_m;\mathbb{C}) \otimes \mathbb{C}^n \)-valued functions in B-model closed string coordinates (complex parameters) \( \underline{q} = (q_1,\ldots,q_p) \) and B-model open string coordinates \( \hat{X}_1,\ldots,\hat{X}_n \). (The precise definitions of \( \omega_{g,n} \) and \( \hat{F}_{g,n} \) are given in Section 6.3 and Section 6.4, respectively.) They are analytic in an open neighborhood of the origin in \( \mathbb{C}^p \times \mathbb{C}^n \). The closed mirror map relates the flat coordinates \( (\tau_1,\ldots,\tau_p) \) to the complex parameters \( (q_1,\ldots,q_p) \) of the mirror curve and the open mirror map relates the A-model open string coordinates \( \tilde{X}_1,\ldots,\tilde{X}_n \) to the B-model open string coordinates \( \hat{X}_1,\ldots,\hat{X}_n \).

Our main results are the following theorems. We have the BKMP Remodeling Conjecture for the open string sector:
Theorem 7.5 (BKMP Remodeling Conjecture: open string sector). For any $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$, under the open-closed mirror map $\tau = \tau(q)$ and $\tilde{X} = \tilde{X}(q, \hat{X})$,

$$F_{g,n}(q; \hat{X}_1, \ldots, \hat{X}_n) = (-1)^{g-1+n|G_0|} F_{g,n}(\mathcal{L}, f)(\tau; \tilde{X}_1, \ldots, \tilde{X}_n).$$

This is more general than the original conjecture in [16], which covers the $m = 1$ case, i.e. when $\mathcal{L}$ is on an effective leg.

In the closed string sector, we also have the BKMP Remodeling Conjecture for free energies. We have the following theorems under the closed mirror map $\tau = \tau(q)$.

Theorem 7.6 When $g > 1$, we have,

$$F_g^X(\tau) = (-1)^{g-1} \tilde{F}_g(q).$$

Theorem 7.9 When $g = 1$, we have,

$$dF_1^X(\tau) = d\tilde{F}_1(q).$$

Theorem 7.10 For any $i, j, k \in \{1, \ldots, p\}$, we have,

$$\frac{\partial^3 F_0^X}{\partial \tau^i \partial \tau^j \partial \tau^k}(\tau) = - \frac{\partial^3 \tilde{F}_0}{\partial \tau^i \partial \tau^j \partial \tau^k}(q).$$

The key idea in the proof of the BKMP Remodeling Conjecture is that we can realize the A-model and B-model higher genus potentials as quantizations on two isomorphic semi-simple Frobenius structures. On the A-model side, we use the Givental quantization formula to express the higher genus GW potential of $X$ in terms of the Frobenius structure of the quantum cohomology of $X$ (genus-zero data). On the B-model side, the Eynard-Orantin recursion determines the higher genus B-model potential by the genus-zero initial data. The bridge connecting these two formalisms on A-model and B-model is the graph sum formula. The quantization formula on the A-model is a formula involving the exponential of a quadratic differential operator. By the classical Wick formula, it can be rewritten as a graph sum formula:

$$F_{g,n}^{X,(\mathcal{L}, f)} = \sum_{\Gamma \in \Gamma_{g,n}(X)} w_A^O(\tilde{\Gamma}) / |\text{Aut}(\tilde{\Gamma})|$$

where $\Gamma_{g,n}(X)$ is certain set of decorated graphs, $\text{Aut}(\tilde{\Gamma})$ is the automorphism group of the decorated graph $\tilde{\Gamma}$, and $w_A^O(\tilde{\Gamma})$ is the A-model weight of the decorated graph $\tilde{\Gamma}$.

On the B-model side, by the result in [38], the Eynard-Orantin recursion is equivalent to a graph sum formula. So the B-model potential $\tilde{F}_{g,n}$ can also be expressed as a graph sum:

$$\tilde{F}_{g,n} = \sum_{\Gamma \in \Gamma_{g,n}(X)} w_B^O(\tilde{\Gamma}) / |\text{Aut}(\tilde{\Gamma})|$$
where $w_B^O(\vec{\Gamma})$ is the B-model weight of the decorated graph $\vec{\Gamma}$. Then we reduce the BKMP remodeling conjecture to

$$w_A^O(\vec{\Gamma}) = w_B^O(\vec{\Gamma}).$$

The weights $w_A^O(\vec{\Gamma})$ and $w_B^O(\vec{\Gamma})$ are determined by the A-model and B-model $R$–matrices (information extracted from the Frobenius structures) together with the A-model and B-model disk potentials. The disk mirror theorem in [45] is precisely what we need to match the disk potentials. The genus-zero mirror theorem [27] identifies the equivariant small quantum cohomology ring of $X$ with the equivariant Jacobian ring of its Landau-Ginzburg B-model. In particular, the quantum differential equations on the A-model and on the Landau-Ginzburg B-model are identified. By the dimensional reduction, we can show that the B-model $R$–matrix is indeed the $R$–matrix in the fundamental solution of the B-model quantum differential equation. The fundamental solution of the quantum differential equation is unique up to a constant matrix. We identify the A-model and B-model $R$–matrices by matching them in degree zero. Putting these pieces together, we have

$$w_A^O(\vec{\Gamma}) = w_B^O(\vec{\Gamma}).$$

1.3. Some remarks. We have the following remarks about our proof:

- Our proof does not rely on the equivalence of the orbifold Gromov-Witten vertex (a generating function of Hurwitz Hodge integrals [76]) and the orbifold Donaldson-Thomas vertex (a generating function of colored 3d partitions). As mentioned in Section 1.1.2 above, the equivalence is known for toric Calabi-Yau 3-orbifolds with transverse $A_n$-singularities [92, 77, 78, 79]. It is not clear how to formulate the equivalence for toric Calabi-Yau 3-orbifolds which do not satisfy the Hard Lefschetz condition. Moreover, the structure of the algorithm for the Topological Vertex is very different from the structure of the Eynard-Orantin topological recursion on the B-model. Roughly speaking, the vertex algorithm comes from degeneration of the target, whereas the topological recursion comes from degeneration of the domain. It seems very difficult, if not impossible, to derive the Remodeling Conjecture from the Topological Vertex.

- Instead, we study the GW theory of $X$ by Givental’s quantization formula, which expresses the higher genus GW potential of $X$ in terms of the abstract Frobenius structure of the quantum cohomology. In [93], the Givental quantization formula for general GKM orbifolds is proved. So we can apply the result in [93] to the case of toric Calabi-Yau 3-orbifolds. It turns out that the Givental quantization formula on the A-model matches the Eynard-Orantin recursion on the B-model perfectly. In particular, we provide new proofs of the BKMP Remodeling Conjecture in the smooth case and the affine case.

- The Remodeling Conjecture provides a very effective recursive algorithm to compute closed and open-closed Gromov-Witten invariants of all semi-projective toric Calabi-Yau 3-orbifolds at all genera (see [15] [16] for the numerical computation). Before the introduction of this algorithm, these invariants were very difficult to compute in the orbifold cases where the Topological Vertex was not applicable.

- One key ingredient in Eynard-Orantin’s recursive algorithm is the open topological string. Only by including the open topological strings can we
determine the higher genus topological strings from the genus zero data by the Eynard-Orantin recursion. This philosophy is in line with the method of Costello-Li [32, 33], and may be enlightening for further study of mirror symmetry.

1.4. Future work. The BKMP Remodeling Conjecture has many interesting applications. We discuss two of them: all genus open-closed Crepant Transformation Conjecture for toric Calabi-Yau 3-orbifolds and modularity for all genus open-closed GW potentials of toric Calabi-Yau 3-orbifolds.

1.4.1. The all genus open-closed Crepant Resolution Conjecture for toric Calabi-Yau 3-orbifolds. One of the most important conjectures in the orbifold GW theory is the Crepant Resolution Conjecture (CRC), first proposed by Ruan [80, 81] and later generalized to various situations by Bryan-Graber [14], Coates-Corti-Iritani-Tseng [27], Coates-Ruan [30], etc. The CRC relates the orbifold GW theory of a Gorenstein orbifold to the GW theory of its crepant resolution.

In the past few years there has been a lot of progress on open CRC for toric Calabi-Yau 3-orbifolds relative to Aganagic-Vafa branes. Cavalieri-Ross [20] proved a disk CRC relating disk invariants of \([\mathbb{C}^2/\mathbb{Z}_2] \times \mathbb{C}\) and its crepant resolution \(K_{\mathbb{P}^1} \times \mathbb{C}\). Brini-Cavaleri-Ross [12] formulated a disk CRC for all toric Calabi-Yau 3-orbifolds and a quantized (all-genus) open CRC for Hard-Lefshetz toric Calabi-Yau 3-orbifolds in terms of winding neutral open potentials; these conjectures have been proved for \([\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}\) by Brini-Cavaleri-Ross [12] and for \([\mathbb{C}^4/(\mathbb{Z}_2 \times \mathbb{Z}_2)]\) by Brini-Cavaleri [13]. Brini-Cavaleri also proved the disk CRC for \([\mathbb{C}^3/\mathbb{Z}_{n+2}]\) and its partial crepant resolution \(K_{\mathbb{P}^3(n,1,1)}\), which are not Hard-Lefshetz. Ke-Zhou [62] proved an open CRC for disk potentials for all semi-projective toric Calabi-Yau orbifolds relative to an effective outer brane using the open mirror theorem proved in [45]. In a forthcoming paper [47], the authors will use Theorem 7.5 to formulate and prove an all-genus open CRC for all semi-projective toric Calabi-Yau 3-orbifolds in terms of the open potentials defined in Section 3.12.

1.4.2. Modularity for all genus open-closed GW potentials of toric Calabi-Yau 3-orbifolds. The modularity of the GW potentials of Calabi-Yau 3-folds has been studied in [1, 89]. In these works, the modularity of the GW potentials provides a powerful tool to construct higher genus B-models. It also produces closed formulae for some GW potentials in terms of quasi-modular forms [89]. The mathematical proof of the modularity for GW potentials remains a difficult problem in general. For toric Calabi-Yau 3-orbifolds, the remodeling conjecture relates the GW potential to the Eynard-Orantin invariants of the mirror curve. Eynard and Orantin studied the modularity of the Eynard-Orantin invariants of any spectral curves [42]. This modularity follows from the modularity of the fundamental differential of the spectral curve. Therefore, the Remodeling Conjecture should imply the modularity for all genus open-closed GW potentials of toric Calabi-Yau 3-orbifolds.

1.5. Overview of the paper. In Section 2, we fix the notation of toric varieties and orbifolds. We also discuss the geometry of toric Calabi-Yau 3-orbifolds and Aganagic-Vafa branes in them.

In Section 3, we introduce the equivariant GW invariants, as well as open-closed GW invariants relative to Aganagic-Vafa branes. Section 3.3 to 3.8 are on the
quantization of the Frobenius manifolds from big equivariant quantum cohomology; the graph sum formula from [93] expressing all genus descendant potential for toric orbifolds is stated in Section 3.8. In Section 3.9 to Section 3.12, we consider restriction to the small phase space. We recall the genus zero mirror theorem from [27] in Section 3.9 and define A-model open potentials \( F_{g,n}(L,f) \) in Section 3.12. They give rise to the same genus-zero theory.

In Section 4, we define three different mirror B-models to a toric Calabi-Yau 3-orbifold. They give rise to the same genus-zero theory.

In Section 5, we study geometry and topology of mirror curves. In particular, we construct a family of mirror curves near the limit point in the B-model moduli space and introduce the flat coordinates in this moduli space in Section 5.4 and 5.5.

In Section 6, we recall the Eynard-Orantin’s topological recursion [42], and the graph sum formula of Eynard-Orantin invariants \( \omega_{g,n} \) derived in [38]. Using the disk mirror theorem [45], we expand this graph sum formula around suitable puncture(s) on the mirror curve and obtain a graph sum formula of the B-model potential \( \mathcal{F}_{g,n} \).

In Section 7, we finish the proof of the Remodeling Conjecture by comparing the A-model and B-model graph sums.

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2. A-model Geometry and Topology

We work over \( \mathbb{C} \). In this section, we give a brief review of semi-projective toric Calabi-Yau 3-orbifolds. We refer to [49, 35] for the theory of general toric varieties. We refer to [10] [48] for the theory of general smooth toric Deligne-Mumford (DM) stacks. In Section 2.3 and Section 2.5, we specialize the definitions in [58, Section 3.1] to toric Calabi-Yau 3-orbifolds.

2.1. The simplicial toric variety and the fan. Let \( N \cong \mathbb{Z}^3 \) be a lattice of rank 3. Let \( X_\Sigma \) be a 3-dimensional simplicial toric variety defined by a (finite) simplicial fan \( \Sigma \) in \( N_R := N \otimes \mathbb{R} \). Then \( X_\Sigma \) contains the algebraic torus \( T = N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^3 \) as an open dense subset, and the action of \( T \) on itself extends to \( X_\Sigma \). We further assume that:

(i) \( X_\Sigma \) is Calabi-Yau: the canonical divisor of \( X_\Sigma \) is trivial;
(ii) \( X_\Sigma \) is semi-projective: the \( T \)-action on \( X_\Sigma \) has at least one fixed point, and the morphism from \( X_\Sigma \) to its affinization \( X_0 = \text{Spec} \mathcal{H}^0(X_\Sigma, \mathcal{O}_{X_\Sigma}) \) is projective.

We introduce some notation:

- Let \( \Sigma(d) \) be the set of \( d \)-dimensional cones in \( \Sigma \).
- Let \( \Sigma(1) = \{ \rho_1, \ldots, \rho_{3+p'} \} \) be the set of 1-dimensional cones in \( \Sigma \), where \( p' \in \mathbb{Z}_{\geq 0} \), and let \( b_i \in N \) be characterized by \( \rho_i \cap N = \mathbb{Z}_{\geq 0} b_i \).

The lattice \( N \) can be canonically identified with \( \text{Hom}(\mathbb{C}^*, T) \), the cocharacter lattice of \( T \); the dual lattice \( M = \text{Hom}(N, \mathbb{Z}) \) can be canonically identified with
Hom(\(\mathbb{T}, \mathbb{C}^*\)), the character lattice of \(\mathbb{T}\). Given \(m \in M\), let \(\chi^m \in \text{Hom}(\mathbb{T}, \mathbb{C}^*)\) denote the corresponding character of \(\mathbb{T}\). Let \(M_\mathbb{R} := M \otimes \mathbb{R}\) be the dual real vector space of \(N_\mathbb{R}\). The Calabi-Yau condition (i) implies that, there exists a vector \(e_i^* \in M\) such that \(\langle e_i^*, b_i \rangle = 1\) for \(i = 1, \ldots, 3+p'\). We may choose \(e_1^*, e_2^*, e_3^*\) such that \(\{e_1^*, e_2^*, e_3^*\}\) is a \(\mathbb{Z}\)-basis of \(M\). Let \(\{e_1, e_2, e_3\}\) be the dual \(\mathbb{Z}\)-basis of \(N\), which defines an isomorphism \(N \cong \mathbb{Z}^3\) given by \(n_1e_1 + n_2e_2 + n_3e_3 \mapsto (n_1, n_2, n_3)\); under this isomorphism, \(b_i = (m_i, n_i, 1)\) for some \((m_i, n_i) \in \mathbb{Z}^2\). We define the Calabi-Yau subtorus of \(\mathbb{T}\) to be \(\mathbb{T}' := \text{Ker}(e_i^* : \mathbb{T} \to \mathbb{C}^*) \cong (\mathbb{C}^*)^3\). Then \(N' := \text{Ker}(e_i^* : N \to \mathbb{Z}) \cong \mathbb{Z}^2\) can be canonically identified with \(\text{Hom}(\mathbb{C}^*, \mathbb{T}')\), the cocharacter lattice of the Calabi-Yau torus \(\mathbb{T}'\). Let \(P \subset N'_\mathbb{R} := N' \otimes \mathbb{R} \cong \mathbb{R}^2\) be the convex hull of \(\{(m_i, n_i) : i = 1, \ldots, 3+p'\}\), and let \(\sigma_0 \subset N_\mathbb{R}\) be the cone over \(P \times \{1\} \subset N'_\mathbb{R} \times \mathbb{R} = N_\mathbb{R}\). Then

\[
\sigma_0 = \bigcup_{\sigma \in \Sigma(3)} \sigma
\]
is a 3-dimensional strongly convex polyhedral cone. We have

\[
H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}) = \mathbb{C}[M \cap \sigma_0^\vee]
\]
where \(\sigma_0^\vee \subset M_\mathbb{R}\) is the dual cone of \(\sigma_0 \subset N_\mathbb{R}\). Therefore, the affine toric variety defined by the cone \(\sigma_0\) is the affinization \(X_0\) of \(X_\Sigma\).

There is a group homomorphism

\[
\phi' : \tilde{N'} := \bigoplus_{i=1}^{3+p'} \tilde{\mathbb{Z}}b_i \to N, \quad \tilde{b}_i \mapsto b_i
\]
with finite cokernel. Applying \(- \otimes_{\mathbb{Z}} \mathbb{C}^*\), we obtain an exact sequence of abelian groups

\[
1 \to G_\Sigma \to \tilde{\mathcal{T}}' \to \mathbb{T} \to 1,
\]
where \(\tilde{\mathcal{T}}' = \tilde{N}' \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{3+p'}\), and \(G_\Sigma\) can be disconnected. The action of \(\tilde{\mathcal{T}}'\) on itself extends to \((\mathbb{C}^*)^{3+p'} = \text{Spec}\mathbb{C}[Z_1, \ldots, Z_{3+p'}]\). Let \(Z_\Sigma := \prod_{\rho \in \Sigma} Z_i\), and let \(Z(\Sigma) \subset (\mathbb{C}^*)^{3+p'}\) be the closed subvariety defined by the ideal generated by \(\{Z_\Sigma : \sigma \in \Sigma\}\). Then \(\tilde{\mathcal{T}}'\) acts on \(U_\Sigma := (\mathbb{C}^*)^{3+p'} - Z(\Sigma)\), and the simplicial toric variety \(X_\Sigma\) is the geometric quotient

\[
X_\Sigma = U_\Sigma / G_\Sigma.
\]

### 2.2. The toric orbifold and the stacky fan

In general, a smooth toric DM stack is defined by a stacky fan \(\Sigma = (N, \Sigma, \beta)\) \cite{Lee}, and a toric orbifold is a smooth toric DM stack with trivial generic stabilizer \cite{Friedman} Section 5). The canonical stacky fan associated to the simplicial fan \(\Sigma\) in Section 2.1 is

\[
\Sigma^{\text{can}} = (N, \Sigma, \beta^{\text{can}} = (b_1, \ldots, b_{3+p'})).
\]

The toric orbifold \(X\) defined by \(\Sigma^{\text{can}}\) is the stacky quotient

\[
X = [U_\Sigma / G_\Sigma] \cong [U_\Sigma / \Sigma^{\text{can}}].
\]

In this paper, we consider semi-projective toric Calabi-Yau 3-orbifolds \(X\) constructed as above.

We will also need an alternative description of \(X\) in terms of an extended stacky fan introduced by Y. Jiang \cite{Jiang}. For toric Calabi-Yau 3-orbifold, there is a canonical extended stacky fan

\[
\Sigma^{\text{ext}} = (N, \Sigma, \beta^{\text{ext}} = (b_1, \ldots, b_{3+p})).
\]
where \( b_i = (m_i, n_i, 1) \) and
\[
\{(m_i, n_i) : i = 1, \ldots, 3 + p\} = P \cap \mathbb{Z}^2.
\]

There is a surjective group homomorphism
\[
\phi : \tilde{N} := \bigoplus_{i=1}^{3+p} \mathbb{Z}\tilde{b}_i \to N, \quad \tilde{b}_i \mapsto b_i
\]

Let \( L = \text{Ker}(\phi) \cong \mathbb{Z}^p \). Then we have a short exact sequence of free \( \mathbb{Z} \)-modules:
\[
0 \to L \xrightarrow{\psi} \tilde{N} \xrightarrow{\phi} N \to 0.
\]

Applying \(- \otimes \mathbb{C}^*\), we obtain an exact sequence of abelian Lie groups:
\[
1 \to G \to \tilde{T} \to T \to 1,
\]
where \( \tilde{T} = \tilde{N} \otimes \mathbb{C}^* \cong \mathbb{C}^{3+p} \), and \( G = \mathbb{L} \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^p \). The action of \( \tilde{T} \) on itself extends to \( \mathbb{C}^{3+p} = \text{Spec} \mathbb{C}[Z_1, \ldots, Z_{3+p}] \) and preserves the Zariski open dense subset \( U_{\Sigma^{\text{ext}}} = U_{\Sigma} \times (\mathbb{C}^*)^{p-p} \subset \mathbb{C}^{3+p} \). Then
\[
X_{\Sigma} = U_{\Sigma^{\text{ext}}}/G, \quad X = [U_{\Sigma^{\text{ext}}}/G].
\]

2.3. Character lattices and integral second cohomology groups. Let \( \tilde{M} := \text{Hom}(\tilde{N}, \mathbb{Z}) \) be the dual lattice of \( \tilde{N} \), which can be canonically identified with the character lattice \( \text{Hom}(\tilde{T}, \mathbb{C}^*) \) of \( \tilde{T} \). The \( \tilde{T} \)-equivariant inclusion \( U_{\Sigma} \to \mathbb{C}^{3+p} \) induces a surjective group homomorphism
\[
\kappa_{\Sigma} : \tilde{M} \cong H^2_\tilde{T}(\mathbb{C}^{3+p}; \mathbb{Z}) \cong H^2_\tilde{T}([\mathbb{C}^{3+p}/G]; \mathbb{Z}) \to H^2_{\tilde{T}}(U_{\Sigma}; \mathbb{Z}) \cong H^2(\mathcal{X}; \mathbb{Z}).
\]

Let \( D_i^\Sigma \in H^2_{\tilde{T}}(\mathbb{C}^{3+p}; \mathbb{Z}) \) be the \( \tilde{T} \)-equivariant Poincaré dual of the divisor \( \{Z_i = 0\} \subset \mathbb{C}^{3+p} \). Then \( \{D_i^\Sigma : i = 1, \ldots, 3+p\} \) is the \( \mathbb{Z} \)-basis of \( \tilde{M} \cong H^2_{\tilde{T}}(\mathbb{C}^{3+p}; \mathbb{Z}) \) which is dual to the \( \mathbb{Z} \)-basis \( \{\tilde{b}_i : i = 1, \ldots, 3+p\} \) of \( \tilde{N} \). We have
\[
\text{Ker}(\kappa_{\Sigma}) = \bigoplus_{i=4+p'}^{3+p} \mathbb{Z}D_i^\Sigma.
\]

Let \( L^\vee = \text{Hom}(L, \mathbb{Z}) \) be the dual lattice of \( L \), which can be canonically identified with the character lattice \( \text{Hom}(G, \mathbb{C}^*) \) of \( G \). The \( G \)-equivariant inclusion \( U_{\Sigma} \to \mathbb{C}^{3+p} \) induces a surjective group homomorphism
\[
\kappa : L^\vee \cong H^2_G(\mathbb{C}^{3+p}; \mathbb{Z}) \cong H^2([\mathbb{C}^{3+p}/G]; \mathbb{Z}) \to H^2_G(U_{\Sigma}; \mathbb{Z}) \cong H^2(\mathcal{X}; \mathbb{Z}).
\]

We have
\[
\text{Ker}(\kappa) = \bigoplus_{i=4+p'}^{3+p} \mathbb{Z}D_i
\]
where \( D_i \in H^2_G(\mathbb{C}^{3+p}; \mathbb{Z}) \) is the \( G \)-equivariant Poincaré dual of the divisor \( \{Z_i = 0\} \subset \mathbb{C}^{3+p} \).

Applying \( \text{Hom}(-; \mathbb{Z}) \) to (5), we obtain the following short exact sequence of free \( \mathbb{Z} \)-modules:
\[
0 \to M \xrightarrow{\phi^\vee} \tilde{M} \xrightarrow{\psi^\vee} L^\vee \to 0.
\]
To summarize, we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 \\
\downarrow & & & & \\
0 & \to & \bigoplus_{i=1+p'} \mathbb{Z} D_i^T & \xrightarrow{\kappa} & \bigoplus_{i=1+p'} \mathbb{Z} D_i & \to 0 \\
\downarrow & & & & \\
0 & \to & M & \to & L' & \to 0 \\
\downarrow \kappa & & \downarrow \kappa & & \downarrow \kappa & \\
0 & \to & H^2(T; \mathbb{Z}) & \to & H^2(X; \mathbb{Z}) & \to 0 \\
\end{array}
\]

(8)

In the above diagram, the rows and columns are short exact sequences of abelian groups. The map \( \psi' \) sends \( D^T_i \) to \( D_i \). For \( i = 1, \ldots, 3 + p \), we define

\[
\bar{D}^T_i := \kappa_T(D^T_i) \in H^2(T; \mathbb{Z}), \quad \bar{D}_i := \kappa(D_i) \in H^2(X; \mathbb{Z}).
\]

Then \( \bar{\psi}'(\bar{D}^T_i) = \bar{D}_i \), and \( \bar{D}^T_i = \bar{D}_i = 0 \) for \( 4 + p' \leq i \leq 3 + p \).

Finally, we have the following isomorphisms

\[
c_1 : \text{Pic}(X) \xrightarrow{\kappa} H^2(X; \mathbb{Z}), \quad c_1^T : \text{Pic}_T(X) \xrightarrow{\kappa} H^2(T; \mathbb{Z}),
\]

where \( c_1 \) and \( c_1^T \) are the first Chern class and the \( \mathbb{T} \)-equivariant first Chern class, respectively.

2.4. Torus invariant closed substacks. Given \( \sigma \in \Sigma \), define

\[
I'_\sigma := \{ i \in \{1, \ldots, 3 + p\} : \rho_i \in \sigma \}, \quad I_\sigma = \{1, \ldots, 3 + p\} \setminus I'_\sigma.
\]

Then \( I'_\sigma \subset \{1, \ldots, 3 + p'\} \) and \( \{4 + p', \ldots, 3 + p\} \subset I_\sigma \subset \{1, \ldots, 3 + p\} \). If \( \sigma \in \Sigma(d) \) then \( |I'_\sigma| = d \) and \( |I_\sigma| = 3 + p - d \).

Let \( \bar{V}(\sigma) \subset U_{\Sigma_{\text{ext}}} \) be the closed subvariety defined by the ideal of \( \mathbb{C}[Z_1, \ldots, Z_{3+p}] \) generated by \( \{ Z_i = 0 \mid i \in I'_\sigma \} \). Then \( \mathcal{V}(\sigma) := \{ \bar{V}(\sigma)/G \} \) is a codimension \( d \) \( \mathbb{T} \)-invariant closed substack of \( X = [U_{\Sigma_{\text{ext}}}/G] \).

The generic stabilizer of \( \mathcal{V}(\sigma) \) is \( G_\sigma = \{ g \in G \mid g \cdot z = z \mbox{ for all } z \in \bar{V}(\sigma) \} \), which is a finite subgroup of \( G \cong (\mathbb{C}^*)^p \). If \( \tau \subset \sigma \) then \( \mathcal{V}(\tau) \subset \mathcal{V}(\sigma) \) so \( G_\tau \) is a subgroup of \( G_\sigma \).

2.5. Extended nef, Kähler, and Mori cones. We first introduce some notation. Given a lattice \( \Lambda \cong \mathbb{Z}^n \) and \( F = \mathbb{Q}, \mathbb{R}, \mathbb{C} \), let \( \Lambda_F \) denote the \( F \) vector space \( \Lambda \otimes \mathbb{Z} F \cong \mathbb{F}^n \).

Given a maximal cone \( \sigma \in \Sigma(3) \), we define

\[
\mathcal{K}_\sigma := \bigoplus_{i \in I_\sigma} \mathbb{Z} D_i
\]

which is a sublattice of \( \mathbb{L}' \) of finite index, and define the extended \( \sigma \)-nef cone to be

\[
\overline{\text{Nef}}_\sigma := \sum_{i \in I_\sigma} \mathcal{K}_\sigma D_i.
\]
which is a top dimensional cone in $\mathbb{L}_Q$. The \textit{extended nef cone} of the extended stacky fan $\Sigma^{\text{ext}}$ is

\[ \text{Nef}(\Sigma^{\text{ext}}) := \cap_{\sigma \in \Sigma(3)} \overline{\text{Nef}}_{\sigma}. \]

The \textit{extended $\sigma$-Kähler cone} $\overline{C}_{\sigma}$ is defined to be the interior of $\overline{\text{Nef}}_{\sigma}$; the \textit{extended Kähler cone} $C(\Sigma^{\text{ext}})$ of $\Sigma^{\text{ext}}$ is defined to be the interior of the extended nef cone $\text{Nef}(\Sigma^{\text{ext}})$. We have an exact sequence of $\mathbb{R}$-vector spaces:

\[ 0 \to \bigoplus_{i=1+p'} \mathbb{R}D_i \to \mathbb{L}_Q^\vee \xrightarrow{\kappa} H^2(\mathcal{X}; \mathbb{R}) = H^2(X_{\Sigma}; \mathbb{R}) \to 0 \]

The \textit{Kähler cone} of $\mathcal{X}$ is $C(\Sigma) = \kappa(C(\Sigma^{\text{ext}})) \subset H^2(\mathcal{X}; \mathbb{R})$. We have

\[ C(\Sigma^{\text{ext}}) = C(\Sigma) \times \prod_{i=1+p'} \mathbb{R}_{\geq 0} D_i. \]

Let $K_{\sigma}$ be the dual lattice of $K_{\sigma}'$; it can be viewed as an additive subgroup of $\mathbb{L}_Q$:

\[ K_{\sigma} = \{ \beta \in \mathbb{L}_Q | (D, \beta) \in \mathbb{Z} \forall D \in K_{\sigma}' \}, \]

where $\langle -, - \rangle$ is the natural pairing between $\mathbb{L}_Q^\vee$ and $\mathbb{L}_Q$. Define

\[ K := \bigcup_{\sigma \in \Sigma(3)} K_{\sigma}. \]

Then $K$ is a subset (which is not necessarily a subgroup) of $\mathbb{L}_Q$, and $\mathbb{L} \subset K$.

We define the \textit{extended $\sigma$-Mori cone} $\overline{\text{NE}}_{\sigma} \subset \mathbb{L}_R$ to be the dual cone of $\overline{\text{Nef}}_{\sigma} \subset \mathbb{L}_R^\vee$:

\[ \overline{\text{NE}}_{\sigma} = \{ \beta \in \mathbb{L}_R | (D, \beta) \geq 0 \forall D \in \overline{\text{Nef}}_{\sigma} \}. \]

The \textit{extended Mori cone} of the extended stacky fan $\Sigma^{\text{ext}}$ is

\[ \text{NE}(\Sigma^{\text{ext}}) := \bigcup_{\sigma \in \Sigma(3)} \overline{\text{NE}}_{\sigma}. \]

We have

\[ \text{NE}(\Sigma^{\text{ext}}) = \text{NE}(\Sigma) \times \prod_{i=1+p'} \mathbb{R}_{\geq 0} b_i, \]

where $\text{NE}(\Sigma) \subset H_2(\mathcal{X}; \mathbb{R})$ is the Mori cone of $\mathcal{X}$.

We define

\[ K_{\text{eff}, \sigma} := K_{\sigma} \cap \overline{\text{NE}}_{\sigma}, \quad K_{\text{eff}} := K \cap \text{NE}(\Sigma^{\text{ext}}) = \bigcup_{\sigma \in \Sigma(3)} K_{\text{eff}, \sigma}. \]

2.6. Anticones and stability. There is an alternative way to define the toric variety $\mathcal{X}$ (see [28, Section 3.1]). Given $D_i \in \mathbb{L}_Q^\vee \cong \mathbb{Z}^p$, for $i = 1, \ldots, p+3$, one choose a stability vector $\eta \in \mathbb{L}_Q^\vee$. Define the anticone

\[ \mathcal{A}_\eta = \{ I \subset \{ 1, \ldots, p+3 \} : \eta \in \sum_{i \in I} \mathbb{R}_{\geq 0} D_i \}. \]

Then the associated toric orbifold is $[(\mathbb{C}^{p+3} \setminus \bigcup_{i \in \mathcal{A}_\eta} C_i')/G]$, where $G = \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^*$. The definition is equivalent to the one in Section 2.2. For the stacky fan $\Sigma^{\text{ext}}$, one can always choose such $\eta$ – for any $\eta$ in $C(\Sigma^{\text{ext}})$ this construction will produce $\mathcal{X}$.
2.7. The inertia stack and the Chen-Ruan orbifold cohomology group. Given $\sigma \in \Sigma$, define $I_{\sigma}$ and $I'_{\sigma}$ as in Section \textbf{2.4} and define

$$\text{Box}(\sigma) := \{ v \in N : v = \sum_{i \in I'_{\sigma}} c_i b_i, \quad 0 \leq c_i < 1 \}.$$

If $\tau \subset \sigma$ then $I'_{\tau} \subset I'_{\sigma}$, so $\text{Box}(\tau) \subset \text{Box}(\sigma)$.

Let $\sigma \in \Sigma(3)$ be a maximal cone in $\Sigma$. We have a short exact sequence of abelian groups

$$0 \rightarrow \mathbb{K}_{\sigma}/\mathbb{L} \rightarrow \mathbb{L}_{\mathbb{L}}/\mathbb{L} \rightarrow \mathbb{L}_{\mathbb{L}}/\mathbb{K}_{\sigma} \rightarrow 0,$$

which can be identified with the following short exact sequence of multiplicative abelian groups

$$1 \rightarrow \mathbb{G}_{\sigma} \rightarrow \mathbb{G}_{\mathbb{L}} \rightarrow (\mathbb{G}_{\mathbb{L}}_{\mathbb{G}_{\sigma}})^{\mathbb{L}} \rightarrow 1$$

where $(\mathbb{G}_{\mathbb{L}}_{\mathbb{G}_{\sigma}})^{\mathbb{L}} \equiv U(1)^p$ is the maximal compact subgroup of $(\mathbb{G}_{\mathbb{L}}_{\mathbb{G}_{\sigma}}) \equiv (\mathbb{C}^*)^p$.

Given a real number $x$, we recall some standard notation: $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$, $\lceil x \rceil$ is the least integer greater or equal to $x$, and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x$. Define $v : \mathbb{K}_{\sigma} \rightarrow N$ by

$$v(\beta) = \sum_{i=1}^{3+p} \lfloor (D_i, \beta) \rfloor b_i,$$

Then

$$v(\beta) = \sum_{i \in I'_{\sigma}} \{-(D_i, \beta)\} b_i,$$

so $v(\beta) \in \text{Box}(\sigma)$. Indeed, $v$ induces a bijection $\mathbb{K}_{\sigma}/\mathbb{L} \cong \text{Box}(\sigma)$.

For any $\tau \in \Sigma$ there exists $\sigma \in \Sigma(3)$ such that $\tau \subset \sigma$. The bijection $\mathbb{G}_{\tau} \rightarrow \text{Box}(\tau)$ restricts to a bijection $\mathbb{G}_{\tau} \rightarrow \text{Box}(\tau)$.

Define

$$\text{Box}(\Sigma^\text{can}) := \bigcup_{\sigma \in \Sigma} \text{Box}(\sigma) = \bigcup_{\sigma \in \Sigma(3)} \text{Box}(\sigma).$$

Then there is a bijection $\mathbb{K}/\mathbb{L} \rightarrow \text{Box}(\Sigma^\text{can})$.

Given $v \in \text{Box}(\sigma)$, where $\sigma \in \Sigma(d)$, define $c_i(v) \in [0,1) \cap \mathbb{Q}$ by

$$v = \sum_{i \in I'_{\sigma}} c_i(v) b_i.$$

Suppose that $k \in \mathbb{G}_{\sigma}$ corresponds to $v \in \text{Box}(\sigma)$ under the bijection $\mathbb{G}_{\sigma} \cong \text{Box}(\sigma)$, then $k$ acts on $(Z_1, \ldots, Z_{3+p}) \in \mathbb{C}^{3+p}$ by

$$k \cdot Z_i = \begin{cases} Z_i, & i \in I_{\sigma}, \\ e^{2\pi \sqrt{-1}c_i(v)} Z_i, & i \in I'_{\sigma}. \end{cases}$$

Define

$$\text{age}(k) = \text{age}(v) = \sum_{i \in I_{\sigma}} c_i(v).$$

Let $IU = \{(z, k) \in U_{\Sigma^\text{ext}} \times G \mid k \cdot z = z\}$, and let $G$ act on $IU$ by $h \cdot (z, k) = (h \cdot z, k)$. The inertia stack $\mathcal{I}X$ of $\mathcal{X}$ is defined to be the quotient stack

$$\mathcal{I}X := [IU/G].$$

Note that $(z = (Z_1, \ldots, Z_{3+p}), k) \in IU$ if and only if

$$k \in \bigcup_{\sigma \in \Sigma} G_{\sigma} \text{ and } Z_i = 0 \text{ whenever } \chi_i(k) \neq 1.$$
orbifold cohomology with coefficient $C$

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Equivariant Chen-Ruan cohomology.

2.8. By slight abuse of notation, the Chen-Ruan orbifold cup product in the equivariant setting is also denoted by $\star_X$.

2.8. Equivariant Chen-Ruan Cohomology. Let $R_T := H^*_\beta(pt) = H^*(BT)$, and let $S_T$ be the fractional field of $R_T$. Then

$$R_T = \mathbb{C}[u_1, u_2, u_3], \quad S_T = \mathbb{C}(u_1, u_2, u_3).$$

As a graded $\mathbb{C}$ vector space and an $R_T$-module, the $T$-equivariant Chen-Ruan orbifold cohomology with coefficient $\mathbb{C}$ is defined to be

$$H^*_\text{CR, T}(\mathcal{X}) = \bigoplus_{v \in \Box(\Sigma^{\text{can}})} H^*_\text{CR}(\mathcal{X}_v)[2\text{age}(v)].$$

By slight abuse of notation, the Chen-Ruan orbifold cup product in the equivariant setting is also denoted by $\star_X$.

Given $\sigma \in \Sigma(3)$, let $\mathcal{X}_\sigma = [\mathbb{C}^3/G_\sigma]$ be the affine toric Calabi-Yau 3-orbifold defined by the cone $\sigma$; its coarse moduli is the affine simplicial toric variety $X_\sigma$ defined by
\( \sigma: X_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}^3/G_\sigma. \) The \( T \)-equivariant orbifold cup product and the \( T \)-equivariant Poincaré pairing define the structure of a Frobenius algebra on \( H^*_\text{CR,T}(X_\sigma) \otimes_{R_\sigma} S_T \) (see e.g. \cite[Section 4.2]{?} for an explicit description). There exists a field \( \bar{S}_T \), which is a finite extension of \( S_T, \) such that \( H^*_\text{CR,T}(X_\sigma) \otimes_{R_\sigma} \bar{S}_T \) is a semisimple \( \bar{S}_T \)-algebra; we choose \( \bar{S}_T \) to be the minimal among such extensions. For a fixed \( X_\sigma \cong [\mathbb{C}^3/G_\sigma], \) we define
\[
\{ 1_h : h \in G_\sigma \}, \quad \{ \bar{1}_h : h \in G_\sigma \}, \quad \{ \bar{\phi}_\gamma : \gamma \in G_\sigma^* \}
\]as in \cite[Section 4.2]{?}. Then \( \{ \bar{\phi}_\gamma : \gamma \in G_\sigma^* \} \) is a canonical basis of \( H^*_\text{CR,T}(X_\sigma) \otimes_{R_\sigma} \bar{S}_T. \)

The Frobenius algebra \( H^*_\text{CR,T}(X) \otimes_{R_T} \bar{S}_T, \) equipped with the \( T \)-equivariant orbifold cup product and the \( T \)-equivariant Poincaré pairing, is isomorphic to a direct sum of Frobenius algebras:
\[
\bigoplus_{\sigma \in \Sigma(3)} \iota^*_\sigma: H^*_\text{CR,T}(X; \mathbb{C}) \otimes_{R_T} \bar{S}_T \xrightarrow{\sim} \bigoplus_{\sigma \in \Sigma(3)} H^*_\text{CR,T}(X_\sigma; \mathbb{C}) \otimes_{R_\sigma} \bar{S}_T,
\]
where \( \iota^*_\sigma: H^*_\text{CR,T}(X) \otimes_{R_T} \bar{S}_T \to H^*_\text{CR,T}(X_\sigma) \otimes_{R_\sigma} \bar{S}_T \) is induced by the \( T \)-equivariant open embedding \( \iota_\sigma: X_\sigma \to X. \) There exists a unique \( \phi_{\sigma,\gamma} \in H^*_\text{CR,T}(X) \otimes_{R_T} \bar{S}_T \) such that \( \phi_{\sigma,\gamma}|_{X_\sigma} = \bar{\phi}_\gamma \) and \( \phi_{\sigma,\gamma}|_{p_{\sigma'}} = 0 \) for \( \sigma' \in \Sigma(3), \sigma' \neq \sigma, \) where \( p_{\sigma'} \) is the \( T \)-fixed point corresponding to \( \sigma'. \) Define \( I_\Sigma := \{ (\sigma, \gamma) : \sigma \in \Sigma(3), \gamma \in G_\sigma^* \}. \) Then
\[
\{ \phi_{\sigma,\gamma} : (\sigma, \gamma) \in I_\Sigma \}
\]
is a canonical basis of the semisimple \( \bar{S}_T \)-algebra \( H^*_\text{CR,T}(X; \mathbb{C}) \otimes_{R_T} \bar{S}_T: \)
\[
\phi_{\sigma,\gamma} \ast_X \phi_{\sigma',\gamma'} = \delta_{\sigma,\sigma'} \delta_{\gamma,\gamma'} \phi_{\sigma,\gamma}.
\]
We have
\[
\sum_{(\sigma, \gamma) \in I_\Sigma} \phi_{\sigma,\gamma} = 1.
\]
The \( T \)-equivariant Poincaré pairing is given by
\[
(\phi_{\sigma,\gamma}, \phi_{\sigma',\gamma'})_{X; \mathbb{T}} = \frac{\delta_{\sigma,\sigma'} \delta_{\gamma,\gamma'}}{\Delta_{\gamma,\gamma}} = |G_\sigma|^2 e(\sigma),
\]
where \( e(\sigma) = w_1(\sigma)w_2(\sigma)w_3(\sigma) \in H^6_\text{cr}(p_\sigma; \mathbb{C}) \cong H^6(BT; \mathbb{C}) \) is the \( T \)-equivariant Euler class of the tangent space \( T_{p_\sigma}X. \)

In the rest of this paper, we sometimes use the bold letter \( \sigma \) for the pair \( (\sigma, \gamma) \) for simplicity. Define
\[
\hat{\phi}_\sigma = \sqrt{\Delta^\sigma} \phi_\sigma, \quad \sigma \in I_\Sigma.
\]
(By a finite field extension, we may assume \( \sqrt{\Delta^\sigma} \in \bar{S}_T \) for all \( \sigma \in I_\Sigma. \)) Then
\[
\phi_{\sigma} \ast_X \hat{\phi}_{\sigma'} = \delta_{\sigma,\sigma'} \sqrt{\Delta^\sigma} \hat{\phi}_\sigma, \quad (\hat{\phi}_\sigma, \hat{\phi}_{\sigma'})_{X; \mathbb{T}} = \delta_{\sigma,\sigma'}.
\]
We call \( \{ \hat{\phi}_\sigma : \sigma \in I_\Sigma \} \) the classical normalized canonical basis. It is a normalized canonical basis of the \( T \)-equivariant Chen-Ruan orbifold cohomology ring \( H^*_\text{CR,T}(X) \otimes_{R_T} \bar{S}_T. \)
2.9. The symplectic quotient and the toric graph. Let \( \{ e_a : a = 1, \ldots, p \} \) be a \( \mathbb{Z} \)-basis of the lattice \( L \) and let \( \{ e^*_a : a = 1, \ldots, p \} \) be the dual \( \mathbb{Z} \)-basis of the dual lattice \( L^\vee \). Then \( \psi : L \to N \) and \( \psi^\vee : M \to L^\vee \) are given by
\[
\psi(e_a) = \sum_{i=1}^{3+p} l_i^{(a)} n_i, \quad \psi^\vee(D^T_i) = \sum_{a=1}^p l_i^{(a)} e^*_a,
\]
for some \( l_i^{(a)} \in \mathbb{Z} \), where \( 1 \leq a \leq p \) and \( 1 \leq i \leq 3 + p \). The \( p \) vectors \( l_i^{(a)} := (l_1^{(a)}, \ldots, l_{3+p}^{(a)}) \in \mathbb{Z}^{3+p} \) are known as the charge vectors in the physics literature.

Let \( G_\mathbb{R} \cong U(1)^p \) be the maximal compact torus of and \( G \cong (\mathbb{C}^*)^p \). Then \( L_\mathbb{R}^\vee \) can be canonically identified with the dual of the Lie algebra of \( G_\mathbb{R} \). The \( G \)-action on \( \mathbb{C}^{3+p} \) restrict to a Hamiltonian \( G_\mathbb{R} \)-action on the symplectic manifold \((\mathbb{C}^{3+p}, \sqrt{-1} \sum_{i=1}^{3+p} dZ_i \wedge d\bar{Z}_i)\). A moment map of this Hamiltonian \( G_\mathbb{R} \)-action is given by
\[
\bar{\mu} : \mathbb{C}^{3+p} \longrightarrow \mathbb{L}^\vee_\mathbb{R}, \quad \bar{\mu}(Z_1, \ldots, Z_{3+p}) = \sum_{a=1}^p \left( \sum_{i=1}^{3+p} l_i^{(a)} |Z_i|^2 \right) e^*_a.
\]
Given a point \( \eta \) in the extended Kähler cone \( C(\Sigma^{\text{ext}}) \subset \mathbb{L}^\vee_\mathbb{R} \), the symplectic quotient \([\bar{\mu}^{-1}(\eta)/G_\mathbb{R}]\) is a Kähler orbifold which is isomorphic to \( X \) as a complex orbifold. The symplectic structure \( \omega(\eta) \) depends on \( \eta \). The map
\[
\kappa_{C(\Sigma^{\text{ext}})} : C(\Sigma^{\text{ext}}) \longrightarrow C(\Sigma) \subset H^2(X_\Sigma; \mathbb{R})
\]
can be identified with \( k \mapsto [\omega(\eta)] \), where \( [\omega(\eta)] \) is the Kähler class of the Kähler form \( \omega(\eta) \).

Let \( T_\mathbb{R} \cong U(1)^3 \) (resp. \( T_\mathbb{R}' \cong U(1)^2 \)) be the maximal compact torus of \( T \cong (\mathbb{C}^*)^3 \) (resp. \( T' \cong (\mathbb{C}^*)^2 \)). Then \( M_\mathbb{R} \) (resp. \( M'_\mathbb{R} \)) is canonically identified with the dual of the Lie algebra of \( T_\mathbb{R} \) (resp. \( T'_\mathbb{R} \)). The \( T \)-action on \( X \) restricts to a Hamiltonian \( T_\mathbb{R} \)-action on the Kähler orbifold \((X, \omega(\eta))\). The map \( \kappa(\eta) \) determines a moment map \( \mu_{T_\mathbb{R}} : X \longrightarrow M_\mathbb{R} \) up to translation by a vector in \( M_\mathbb{R} \). The image \( \mu_{T_\mathbb{R}}(X) \) is a convex polyhedron. The moment map \( \mu_{T'_\mathbb{R}} : X \longrightarrow M'_\mathbb{R} \) is the composition \( \pi \circ \mu_{T_\mathbb{R}} \), where \( \pi : M_\mathbb{R} \cong \mathbb{R}^3 \rightarrow M'_\mathbb{R} \cong \mathbb{R}^2 \) is the projection. The map \( \mu_{T'_\mathbb{R}} \) is surjective. Let \( X^1 \subset X \) be the union of 0-dimensional and 1-dimensional \( T \)-orbits in \( X \). The toric graph is defined by \( \Gamma = \mu_{T'_\mathbb{R}}(X^1) \subset M'_\mathbb{R} \cong \mathbb{R}^2 \), and is determined by \( \kappa(\eta) \) up to translation by a vector in \( M'_\mathbb{R} \). The vertices (resp. edges) of \( \Gamma \) are in one-to-one correspondence to 3-dimensional (resp. 2-dimensional) cones in \( \Sigma \).

2.10. Aganagic-Vafa branes. An Aganagic-Vafa brane in \( X = [\bar{\mu}^{-1}(\eta)/G_\mathbb{R}] \) is a Lagrangian suborbifold of the form \( \mathcal{L} = [\bar{L}/G_\mathbb{R}] \), where
\[
\bar{L} = \{(Z_1, \ldots, Z_r) \in \bar{\mu}^{-1}(\eta) : \sum_{i=1}^{3+p} l_i^1 |Z_i|^2 = c_1, \quad \sum_{i=1}^{3+p} l_i^2 |Z_i|^2 = c_2, \quad \arg(\prod_{i=1}^{3+p} Z_i) = c_3 \}.
\]
for some \( l_1^1, l_2^2 \in \mathbb{Z}^{3+p} \) satisfying \( \sum_{i=1}^{3+p} l_1^1 = \sum_{i=1}^{3+p} l_2^2 = 0 \) and \( c_1, c_2, c_3 \in \mathbb{R} \). The constants \( c_1, c_2 \) are chosen such that \( \mu_{T_\mathbb{R}}(\mathcal{L}) \) is a ray ending at a point in the interior of an edge of the moment polytope. Then \( \mu_{T_\mathbb{R}}(\mathcal{L}) \) is a point in \( \Gamma \) which is not a vertex.

Given a 2-dimensional cone \( \tau \in \Sigma(2) \) such that \( I' = \{ i, j \} \), where \( 1 \leq i < j \leq 3 + p' \), let \( \ell_i \subset X \) be defined by \( Z_i = Z_j = 0 \), and let \( \ell_\tau \) be the coarse moduli space of \( \ell_\tau \). There are two cases:

1. \( \tau \) is the intersection of two 3-dimensional cones and \( \ell_\tau \cong \mathbb{P}^1 \).
2. There is a unique 3-dimensional cone \( \sigma \) containing \( \tau \) and \( \ell_\tau \cong \mathbb{C} \).
An Aganagic-Vafa brane $\mathcal{L}$ intersects a unique 1-dimensional $\mathbb{T}$-orbit closure $I_\tau$, where $\tau \in \Sigma(2)$. We say $\mathcal{L}$ is an inner (resp. outer) brane if $\ell_\tau \cong \mathbb{P}^1$ (resp. $\ell_\tau \cong \mathbb{C}$). In this paper we will consider a fixed $0 \in \Sigma(2)$ such that $\ell_0 \cong \mathbb{C}$, and consider an Aganagic-Vafa (outer) brane intersecting $I_\tau$. Let $\sigma_0$ be the unique 3-dimensional cone containing $\tau_0$. By permuting $b_1, \ldots, b_{3+p'}$ we may assume $\sigma_0$ is spanned by $b_1, b_2, b_3$ and $\tau_0$ is spanned by $b_2, b_3$. We have a short exact sequence of finite abelian groups

$$1 \to G_{\tau_0} \cong \mu_m \to G_{\sigma_0} \to \mu_r \to 1,$$

where $m$ and $r$ are positive integers and $\mu_m$ and $\mu_r$ are finite cyclic groups of orders $m$ and $r$ respectively. We say $I_{\sigma_0}$ is an effective leg (resp. a gerby leg) if $m = 1$ (resp. $m > 1$). By choosing suitable $\mathbb{Z}$-basis $(e_1, e_2, e_3)$ of $N$ we may assume

$$b_1 = re_1 - s e_2 + e_3, \quad b_2 = me_2 + e_3, \quad b_3 = e_3,$$

where $s \in \{0, 1, \ldots, r - 1\}$.

3. A-model Topological Strings

In Section 2.5.1, we consider the 3-dimensional torus $\mathbb{T}$ and the big phase space. In Section 3.8 we specialize to the small phase space. After that, we further specialize to the Calabi-Yau torus $\mathbb{T}'$.  

3.1. Equivariant Gromov-Witten invariants. Given nonnegative integers $g, n$ and an effective curve class $d \in H_2(X; \mathbb{Z})$, let $\overline{\mathcal{M}}_{g,n}(X; d)$ be the moduli stack of genus $g$, $n$-pointed, degree $d$ twisted stable maps to $X$ ([3,11]. [83 Section 2.4]). Let $ev_i : \overline{\mathcal{M}}_{g,n}(X; d) \to \mathcal{I}X$ be the evaluation at the $i$-th marked point. The $\mathbb{T}$-action on $X$ induces $\mathbb{T}$-actions on $\overline{\mathcal{M}}_{g,n}(X; d)$ and on the inertia stack $\mathcal{I}X$, and the evaluation map $ev_i$ is $\mathbb{T}$-equivariant.

Let $\overline{\mathcal{M}}_{g,n}(X; d)$ be the moduli stack of genus $g$, $n$-pointed, degree $d$ stable maps to the coarse moduli $X$ of $X$. Let $\pi : \overline{M}_{g,n+1}(X; d) \to \overline{\mathcal{M}}_{g,n}(X; d)$ be the universal curve, and let $\omega_n$ be the relative dualizing sheaf. Let $s_j : \overline{\mathcal{M}}_{g,n}(X; d) \to \overline{\mathcal{M}}_{g,n+1}(X; d)$ be the section associated to the $j$-th marked point. Then

$$L_j := s_j^* \omega_n$$

is the line bundle over $\overline{\mathcal{M}}_{g,n}(X; d)$ formed by the cotangent line at the $j$-th marked point. The descendant classes on $\overline{\mathcal{M}}_{g,n}(X; d)$ are defined by

$$\psi_j := c_1(L_j) \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q}), \quad j = 1, \ldots, n.$$

The $\mathbb{T}$-action on $X$ induces an $\mathbb{T}$-action on $\overline{\mathcal{M}}_{g,n}(X; d)$, and we choose a $\mathbb{T}$-equivariant lift $\psi_j^T \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q})$ of $\psi_j \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q})$ as in [69 Section 5.1].

The map $p : \overline{\mathcal{M}}_{g,n}(X; d) \to \overline{\mathcal{M}}_{g,n}(X; d)$ is $\mathbb{T}$-equivariant. Following [83 Section 2.5.1], we define

$$\hat{\psi}_j := p^* \psi_j \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q})$$

to be the pullback of $\psi_j \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q})$. (Note that $\hat{\psi}_j$ is denoted $\hat{\psi}_j$ in [27] and $\hat{\psi}_j$ in [93].) Then

$$\hat{\psi}_j^T := p^* \psi_j^T \in H^2(\overline{\mathcal{M}}_{g,n}(X; d); \mathbb{Q})$$

is a $\mathbb{T}$-equivariant lift of $\hat{\psi}_j$. 


Since $X_{\Sigma}$ is not projective, the moduli stack $\mathcal{M}_{g,n}(\mathcal{X}, d)$ is not proper in general, but the $\mathbb{T}$-fixed substack $\mathcal{M}_{g,n}(\mathcal{X}, d)^\mathbb{T}$ is. Given $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ and $\gamma_1, \ldots, \gamma_n \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}; \mathbb{Q})$, we define $\mathbb{T}$-equivariant genus $g$ degree $d$ Gromov-Witten invariants of $\mathcal{X}$ by
\[
\langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n}\rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \int_{[\mathcal{M}_{g,n}(\mathcal{X}, d)^\mathbb{T}]/\mathbb{T}} \frac{\iota^* \left( \prod_{j=1}^n \text{ev}_j^*(\gamma_j) (\hat{\psi}_j^{a_j}) \right)}{e_T(N^\text{vir})} \in S_T = \mathbb{Q}(u_1, u_2, u_3).
\]
where the weighted virtual fundamental class $[\mathcal{M}_{g,n}(\mathcal{X}, d)^\mathbb{T}]^{w,T}$ is induced by the inclusion map $\iota^* : H^*_T(\mathcal{M}_{g,n}(\mathcal{X}, d); \mathbb{Q}) \to H^*_T(\mathcal{M}_{g,n}(\mathcal{X}, d)^\mathbb{T}; \mathbb{Q})$ is induced by the inclusion map $\iota^* : \mathcal{M}_{g,n}(\mathcal{X}, d)^\mathbb{T} \to \mathcal{M}_{g,n}(\mathcal{X}, d)$. More generally, if we insert $\gamma_1, \ldots, \gamma_n \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}) \otimes_{R_T} \mathbb{T}$ then we obtain
\[
\langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n}\rangle_{g,n}^{\mathcal{X}, \mathbb{T}} \in S_T.
\]

3.2. Generating functions. Let $\text{NE}(\Sigma) \subset H_2(\mathcal{X}; \mathbb{R}) = H_2(X_{\Sigma}; \mathbb{R})$ be the Mori cone generated by effective curve classes in $X_{\Sigma}$ (see Section 2.5). Let $E(\mathcal{X})$ denote the semigroup $\text{NE}(\Sigma) \cap H_2(X_{\Sigma}; \mathbb{Z})$. The Novikov ring $\Lambda_{\text{nov}}$ of $\mathcal{X}$ is defined to be the completion of the semigroup ring $\mathbb{C}[E(\mathcal{X})]$.

\[
\Lambda_{\text{nov}} := \mathbb{C}[E(\mathcal{X})] = \left\{ \sum_{d \in E(\mathcal{X})} c_d Q^d : c_d \in \mathbb{C} \right\}.
\]

Given $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $\gamma_1, \ldots, \gamma_n \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}) \otimes_{R_T} \mathbb{T}$, we define the following double correlator with primary insertions:
\[
\langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n}\rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \sum_{m=0}^{\infty} \sum_{d \in E(\mathcal{X})} \frac{Q^d}{m!} \langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n}, t^m \rangle_{g,n+m,d}^{\mathcal{X}, \mathbb{T}}
\]
where $Q^d \in \mathbb{C}[E(\mathcal{X})] \subset \Lambda_{\text{nov}}$ is the Novikov variable corresponding to $d \in E(\mathcal{X})$, and $t \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}) \otimes_{R_T} \mathbb{T}$.

We introduce two types of generating functions of genus $g$, $n$-point $\mathbb{T}$-equivariant Gromov-Witten invariants of $\mathcal{X}$.

1. For $j = 1, \ldots, n$, introduce formal variables $u_j = u_j(z) = \sum_{a \geq 0} (u_{aj})_a z^a$

where $(u_{aj})_a \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}) \otimes_{R_T} \mathbb{T}$. Define
\[
\langle u_1, \ldots, u_n \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \langle u_1(\hat{\psi}), \ldots, u_n(\hat{\psi}) \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \sum_{a_1, \ldots, a_n \geq 0} \langle (u_{a_1})_1 \hat{\psi}^{a_1}, \ldots, (u_{a_n})_n \hat{\psi}^{a_n} \rangle_{g,n}^{\mathcal{X}, \mathbb{T}}.
\]

2. Let $z_1, \ldots, z_n$ be formal variables. Given $\gamma_1, \ldots, \gamma_n \in H^*_\text{CR,}\mathbb{T}(\mathcal{X}) \otimes_{R_T} \mathbb{T}$, define
\[
\langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n} \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \sum_{a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}} \langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n} \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} \prod_{j=1}^n z_j^{-a_j-1}.
\]

The above two generating functions are related by
\[
\langle \gamma_1 \hat{\psi}^{a_1}, \ldots, \gamma_n \hat{\psi}^{a_n} \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} = \langle u_1, \ldots, u_n \rangle_{g,n}^{\mathcal{X}, \mathbb{T}} \big|_{u_j(z) = \frac{\gamma_j}{z_j^{a_j}}}.
\]
3.3. The equivariant big quantum cohomology. Let
\[\chi = \dim_C H^*_{\text{CR}}(X) = \dim_{\bar{S}} H^*_{\text{CR,T}}(X; \bar{S}_T).\]
We choose a \(\bar{S}_T\)-basis of \(H^*_{\text{CR,T}}(X; \bar{S}_T)\) \(\{T_i : i = 0, 1, \ldots, \chi - 1\}\) such that
\[T_0 = 1, \quad T_a = \bar{D}_a^T \text{ for } a = 1, \ldots, p, \quad T_a = 1_{b_{j_a}} \text{ for } a = p + 1, \ldots, p,\]
and for \(i = p + 1, \ldots, \chi - 1, T_i\) is of the form \(T_aT_b\) for some \(a, b \in \{1, \ldots, p\}\). Write \(t = \sum_{a=0}^{\chi-1} \tau^a T_a\), and let \(\tau' = (\tau_1, \ldots, \tau_p), \tau'' = (\tau_0, \tau_{p+1}, \ldots, \tau_{\chi-1})\). By the divisor equation,
\[\langle T_i, T_j, T_k \rangle_{0,3}^{X,T} \in S_T[[\bar{Q}, \tau'', \tau]], \quad \langle \phi_\sigma, \phi_\sigma', \phi_\sigma'' \rangle_{0,3}^{X,T} \in \bar{S}_T[[\bar{Q}, \tau'']],\]
where \(\bar{Q}^d = Q^d \exp(\sum_{a=1}^p \tau_a(T_a, d))\). Let \(S := \bar{S}_T[[\bar{Q}, \tau'']]\). Given \(a, b \in H^*_{\text{CR,T}}(X; \bar{S}_T)\), define the quantum product
\[a \ast b := \sum_{\sigma \in I_\Sigma} \langle a, b, \phi_\sigma \rangle \phi_\sigma \in H^*_{\text{CR,T}}(X; \bar{S}_T) \otimes_{\bar{S}} S.\]
Then \(A := H^*_{\text{CR,T}}(X; \bar{S}_T) \otimes_{\bar{S}} S\) is a free \(S\)-module of rank \(\chi\), and \((A, \ast_t)\) is a commutative, associative algebra over \(S\). Let \(I \subset S\) be the ideal generated by \(\bar{Q}, \tau''\), and define
\[S_n := S/I^n, \quad A_n := A \otimes_S S_n\]
for \(n \in \mathbb{Z}_{\geq 0}\). Then \(A_n\) is a free \(S_n\)-module of rank \(\chi\), and the ring structure \(\ast_t\) on \(A\) induces a ring structure \(\ast_{\underline{\chi}}\) on \(A_n\). In particular,
\[S_1 = \bar{S}_T, \quad A_1 = H^*_{\text{CR,T}}(X; \bar{S}_T),\]
and \(\ast_{\underline{\chi}} = \ast_{\chi}\) is the orbifold cup product. So
\[\{\phi_\sigma^{(1)} := \phi_\sigma : \sigma \in I_\Sigma\}\]
is an idempotent basis of \((A_1, \ast_{\underline{\chi}})\). For \(n \geq 1\), let \(\{\phi_\sigma^{(n+1)} : \sigma \in I_\Sigma\}\) be the unique idempotent basis of \((A_{n+1}, \ast_{n+1})\) which is the lift of the idempotent basis \(\{\phi_\sigma^{(n)} : \sigma \in I_\Sigma\}\) of \((A_n, \ast_{\underline{\chi}})\) [67] Lemma 16]. Then
\[\{\phi_\sigma(t) := \lim \phi_\sigma^{(n)} : \sigma \in I_\Sigma\}\]
is an idempotent basis of \((A, \ast_t)\). The ring \((A, \ast_t)\) is called the equivariant big quantum cohomology ring.
Set
\[\bar{\Lambda}_{\text{nov}}^T := \bar{S}_T \otimes_C \Lambda_{\text{nov}} = \bar{S}_T[[E(X)]]\]
Then \(H := H^*_{\text{CR,T}}(X; \bar{\Lambda}_{\text{nov}}^T)\) is a free \(\bar{\Lambda}_{\text{nov}}^T\)-module of rank \(\chi\). Any point \(t \in H\) can be written as \(t = \sum_{\sigma \in I_\Sigma} t^\sigma \phi_\sigma\). We have
\[H = \text{Spec}(\bar{\Lambda}_{\text{nov}}^T[t^\sigma : \sigma \in I_\Sigma]).\]
Let \(\hat{H}\) be the formal completion of \(H\) along the origin:
\[\hat{H} := \text{Spec}(\bar{\Lambda}_{\text{nov}}^T [[t^\sigma : \sigma \in I_\Sigma]]).\]
Let \(\mathcal{O}_{\hat{H}}\) be the structure sheaf on \(\hat{H}\), and let \(\mathcal{T}_{\hat{H}}\) be the tangent sheaf on \(\hat{H}\). Then \(\mathcal{T}_{\hat{H}}\) is a sheaf of free \(\mathcal{O}_{\hat{H}}\)-modules of rank \(\chi\). Given an open set in \(\hat{H}\),
\[\mathcal{T}_{\hat{H}}(U) \cong \bigoplus_{\sigma \in I_\Sigma} \mathcal{O}_{\hat{H}}(U) \frac{\partial}{\partial t^\sigma}.\]
The big quantum product and the $\mathbb{T}$-equivariant Poincaré pairing defines the structure of a formal Frobenius manifold on $\hat{H}$:

$$\frac{\partial}{\partial \sigma} \hat{\phi}_{\sigma} = \sum_{\rho \in I_S} \langle \hat{\phi}_{\sigma}, \hat{\phi}_{\sigma'}, \hat{\phi}_{\rho} \rangle_{\chi, \tau} \frac{\partial}{\partial u_\rho} \hat{\chi}(\hat{H}; \mathcal{T}_H).$$

Then $\hat{\phi}_{\sigma}$ is normalized canonical basis:

$$\left( \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \tau} \right)_{\chi, \tau} = \delta_{\sigma, \sigma'}.$$  

3.4. The $A$-model canonical coordinates and the $\Psi$-matrix. The canonical coordinates $\{u^\sigma = u^\sigma(t) : \sigma \in I_\Sigma\}$ on the formal Frobenius manifold $\hat{H}$ are characterized by

$$\frac{\partial}{\partial u_\sigma} = \delta_{\sigma, \tau}(t).$$

up to additive constants in $\Lambda_{\text{can}}$. We choose canonical coordinates such that they lie in $\tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$ and vanish when $\tilde{Q} = 0$, $\tilde{\tau}' = \tilde{\tau}'' = 0$. Then $u^\sigma - \sqrt{\Delta^\sigma(t)} \in \tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$ and vanish when $\tilde{Q} = 0$, $\tilde{\tau}' = \tilde{\tau}'' = 0$.

We define $\Delta^\sigma(t) \in \tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$ by the following equation:

$$(\hat{\phi}_{\sigma}(t), \hat{\phi}_{\sigma'}(t))_{\chi, \tau} = \frac{\delta_{\sigma, \sigma'}}{\Delta^\sigma(t)}.$$  

Then $\Delta^\sigma(t) \rightarrow \Delta^\sigma$ in the large radius limit $\tilde{Q}, \tilde{\tau}'' \rightarrow 0$. The normalized canonical basis of $(\hat{H}, *_I)$ is

$$\{\hat{\phi}_{\sigma}(t) := \sqrt{\Delta^\sigma(t)} \hat{\phi}_{\sigma}(t) : \sigma \in I_\Sigma\}.$$  

They satisfy

$$\hat{\phi}_I(t) *_I \hat{\phi}_{\sigma'}(t) = \delta_{\sigma, \sigma'} \sqrt{\Delta^\sigma(t)} \hat{\phi}_{\sigma}(t), \quad (\hat{\phi}_{\sigma}(t), \hat{\phi}_{\sigma'}(t))_{\chi, \tau} = \delta_{\sigma, \sigma'}.$$  

(Note that $\sqrt{\Delta^\sigma(t)} = \sqrt{\Delta^\sigma} \sqrt{\Delta^\sigma(t)}$, where $\sqrt{\Delta^\sigma} \in \tilde{S}_\Sigma$ and $\sqrt{\Delta^\sigma(t)} \in \tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$, so $\sqrt{\Delta^\sigma(t)} \in \tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$.) We call $\{\hat{\phi}_{\sigma}(t) : t \in I_\Sigma\}$ the quantum normalized canonical basis to distinguish it from the classical normalized canonical basis $\{\hat{\phi}_{\sigma} : \sigma \in I_\Sigma\}$. The quantum canonical basis tends to the classical canonical basis in the large radius limit: $\hat{\phi}_{\sigma}(t) \rightarrow \hat{\phi}_{\sigma}$ as $\tilde{Q}, \tilde{\tau}'' \rightarrow 0$.

Let $\Psi = (\Psi_{\sigma, \sigma'})$ be the transition matrix between the classical and quantum normalized canonical bases:

$$\hat{\phi}_{\sigma'} = \sum_{\sigma \in I_\Sigma} \Psi_{\sigma, \sigma'} \hat{\phi}_{\sigma}(t).$$  

Then $\Psi$ is an $\chi \times \chi$ matrix with entries in $\tilde{S}_\Sigma[[\tilde{Q}, \tilde{\tau}'', \tilde{\tau}''']]$, and $\Psi \rightarrow 1$ (the identity matrix) in the large radius limit $\tilde{Q}, \tilde{\tau}'' \rightarrow 0$. Both the classical and quantum normalized canonical bases are orthonormal with respect to the $\mathbb{T}$-equivariant Poincaré pairing $(, )_{\chi, \tau}$, so $\Psi^T \Psi = \Psi \Psi^T = 1$, where $\Psi^T$ is the transpose of $\Psi$, or equivalently

$$\sum_{\rho \in I_S} \Psi_{\rho, \sigma} \Psi_{\rho, \sigma'} = \delta_{\sigma, \sigma'}.$$  

Equation (13) can be rewritten as

$$\frac{\partial}{\partial u^\sigma} = \sum_{\sigma \in I_\Sigma} \Psi_{\sigma, \sigma'} \sqrt{\Delta^\sigma(t)} \frac{\partial}{\partial u^\sigma}.$$
which is equivalent to

\[ \frac{du^\sigma}{\sqrt{\Delta^\sigma(t)}} = \sum_{\sigma' \in I_\Sigma} dt^{\sigma'} \Psi_{\sigma' \sigma}. \]

3.5. The equivariant big quantum differential equation. We consider the Dubrovin connection \( \nabla^z \), which is a family of connections parametrized by \( z \in \mathbb{C} \cup \{ \infty \} \), on the tangent bundle \( T_{\hat{H}} \) of the formal Frobenius manifold \( \hat{H} \):

\[ \nabla^z = \frac{\partial}{\partial t^\sigma} - \frac{1}{z} \hat{\omega}^\sigma \ast t \]

The commutativity (resp. associativity) of \( \ast \) implies that \( \nabla^z \) is a torsion free (resp. flat) connection on \( T_{\hat{H}} \) for all \( z \). The equation

\[ \nabla^z \mu = 0 \]

for a section \( \mu \in \Gamma(\hat{H}, T_{\hat{H}}) \) is called the \( \mathbb{T} \)-equivariant big quantum differential equation (\( \mathbb{T} \)-equivariant big QDE). Let

\[ T_{\hat{H}}^{f,z} \subset T_{\hat{H}} \]

be the subsheaf of flat sections with respect to the connection \( \nabla^z \). For each \( z \), \( T_{\hat{H}}^{f,z} \) is a sheaf of \( \bar{\Lambda}_{nov}^\mathbb{T} \)-modules of rank \( \chi \).

A section \( L \in \text{End}(T_{\hat{H}}) = \Gamma(\hat{H}, T_{\hat{H}}^* \otimes T_{\hat{H}}) \) defines an \( \mathcal{O}_{\hat{H}}(\hat{H}) \)-linear map

\[ L : \Gamma(\hat{H}, T_{\hat{H}}) = \bigoplus_{\sigma \in I_\Sigma} \mathcal{O}_{\hat{H}}(\hat{H}) \frac{\partial}{\partial t^\sigma} \to \Gamma(\hat{H}, T_{\hat{H}}) \]

from the free \( \mathcal{O}_{\hat{H}}(\hat{H}) \)-module \( \Gamma(\hat{H}, T_{\hat{H}}) \) to itself. Let \( L(z) \in \text{End}(T_{\hat{H}}) \) be a family of endomorphisms of the tangent bundle \( T_{\hat{H}} \) parametrized by \( z \). \( L(z) \) is called a fundamental solution to the \( \mathbb{T} \)-equivariant QDE if the \( \mathcal{O}_{\hat{H}}(\hat{H}) \)-linear map

\[ L(z) : \Gamma(\hat{H}, T_{\hat{H}}) \to \Gamma(\hat{H}, T_{\hat{H}}) \]

restricts to a \( \bar{\Lambda}_{nov}^\mathbb{T} \)-linear isomorphism

\[ L(z) : \Gamma(\hat{H}, T_{\hat{H}}^{f,\infty}) = \bigoplus_{\sigma \in I_\Sigma} \bar{\Lambda}_{nov}^\mathbb{T} \frac{\partial}{\partial t^\sigma} \to \Gamma(\hat{H}, T_{\hat{H}}^{f,z}). \]

between rank \( \chi \) free \( \bar{\Lambda}_{nov}^\mathbb{T} \)-modules.

3.6. The \( S \)-operator. The \( S \)-operator is defined as follows. For any cohomology classes \( a, b \in H_{CR, T}^\ast(\mathcal{X}; \hat{S}_T) \),

\[ (a, S(b))_{\mathcal{X}, T} = (a, b)_{\mathcal{X}, T} + \langle a, \frac{b}{z - \psi} \rangle_{0,2} \mathcal{X}_{\psi} \]

where

\[ \frac{b}{z - \psi} = \sum_{i=0}^{\infty} b_i \psi^i z^{-i-1}. \]

The \( S \)-operator can be viewed as an element in \( \text{End}(T_{\hat{H}}) \) and is a fundamental solution to the \( \mathbb{T} \)-equivariant big QDE \([\text{15}]\). The proof for \( S \) being a fundamental solution can be found in \([\text{34}]\) for the smooth case and in \([\text{58}]\) for the orbifold case which is a direct generalization of the smooth case.
Remark 3.1. One may notice that since there is a formal variable \( z \) in the definition of the \( \mathbb{T} \)-equivariant big QDE [15], one can consider its solution space over different rings. Here the operator \( S = 1 + S_1/z + S_2/z^2 + \cdots \) is viewed as a formal power series in \( 1/z \) with operator-valued coefficients.

Remark 3.2. Given \( \tau \in H^2_{CR,T}(\mathcal{X}) \otimes_{R_z} \bar{S}_T \), let \( \tau = \tau' + \tau'' \) where \( \tau' \in H^2_{CR,T}(\mathcal{X}) \otimes_{R_z} \bar{S}_T \) and \( \tau'' \) is a linear combination of elements in \( H^2_{CR,T}(\mathcal{X}) \otimes_{R_z} \bar{S}_T \) and elements in degree 2 twisted sectors. Then by divisor equation, we have

\[
(a, b)_{\mathcal{X}, \tau + \langle a, \frac{b}{z - \psi} \rangle_{0, 2}} = (a, b e^{i\psi} \tau)_{\mathcal{X}, \tau} + \sum_{m=0}^{\infty} \sum_{d \geq 0, m} \int_0^1 \frac{Q^d e^{i\psi} \tau'}{m!} (a, \frac{be^{i\psi}}{z - \psi}, (t'')^m)_{\mathcal{X}, \tau}.
\]

In the above expression, if we fix the power of \( z^{-1} \), then only finitely many terms in the expansion of \( e^{i\psi} \tau' \) contribute. Therefore, the factor \( e^{i\psi} \tau' \) can play the role of \( Q^d \) and hence the restriction \( \langle a, \frac{b}{z - \psi} \rangle_{0, 2} |_{Q=1} \) is well-defined. So the operator \( S|_{Q=1} \) is well-defined.

Definition 3.3 (\( \mathbb{T} \)-equivariant big J-function). The \( \mathbb{T} \)-equivariant big J-function \( J^\text{big}_T(z) \) is characterized by

\[
(J^\text{big}_T(z), a)_{\mathcal{X}, \tau} = (1, S(a))_{\mathcal{X}, \tau}
\]

for any \( a \in H^0_0(\mathcal{X}; \bar{S}_T) \). Equivalently,

\[
J^\text{big}_T(z) = 1 + \sum_{\sigma \in I_{\Sigma}} \langle 1, \frac{\tilde{\phi}_{\sigma}}{z - \psi} \rangle_{0, 2} \tilde{\phi}_{\sigma}.
\]

We consider several different (flat) bases for \( H^*_0(\mathcal{X}; \bar{S}_T) \):

(1) The classical canonical basis \( \{ \phi_{\sigma} : \sigma \in I_{\Sigma} \} \) defined in Section 2.8.

(2) The basis dual to the classical canonical basis with respect to the \( \mathbb{T} \)-equivariant Poincare pairing: \( \{ \phi^\sigma = \Delta^\sigma \phi_{\sigma} : \sigma \in I_{\Sigma} \} \).

(3) The classical normalized canonical basis \( \{ \phi_{\sigma} = \sqrt{\Delta^\sigma} \phi_{\sigma} : \sigma \in I_{\Sigma} \} \) which is self-dual: \( \tilde{\phi}^\sigma = \phi_{\sigma} : \sigma \in I_{\Sigma} \).

For \( \sigma, \sigma' \in I_{\Sigma} \), define

\[
S_{\sigma'}^{\sigma'}(z) := \langle \phi^\sigma, S(\phi_{\sigma'}) \rangle.
\]

Then \( (S_{\sigma}^{\sigma'}(z)) \) is the matrix of the \( S \)-operator with respect to the canonical basis \( \{ \phi_{\sigma} : \sigma \in I_{\Sigma} \} \):

\[
S(\phi_{\sigma}) = \sum_{\sigma' \in I_{\Sigma}} \phi_{\sigma'} S_{\sigma'}^{\sigma'}(z).
\]

For \( \sigma, \sigma' \in I_{\Sigma} \), define

\[
S_{\sigma'}^{\sigma}(z) := \langle \phi_{\sigma'}, S(\phi^\sigma) \rangle.
\]

Then \( (S_{\sigma}^{\sigma'}(z)) \) is the matrix of the \( S \)-operator with respect to the bases \( \{ \phi^\sigma : \sigma \in I_{\Sigma} \} \) and \( \{ \phi_{\sigma} : \sigma \in I_{\Sigma} \} \):

\[
S(\phi^\sigma) = \sum_{\sigma' \in I_{\Sigma}} \phi_{\sigma'} S_{\sigma'}^{\sigma}(z).
\]

Introduce

\[
S_z(a, b) = (a, S(b))_{\mathcal{X}, \tau},
\]

\[
V_{z_1, z_2}(a, b) = \langle \frac{a}{z_1 + z_2}, \frac{b}{z_1 - \psi_1}, \frac{b}{z_2 - \psi_2} \rangle_{0, 2}.
\]
A well-known WDVV-like argument says

\begin{equation}
V_{z_1, z_2}(a, b) = \frac{1}{z_1 + z_2} \sum_i S_{z_1}(T_i, a) S_{z_2}(T_i, b),
\end{equation}

where $T_i$ is any basis of $H^*_C(T; \mathcal{X})$ and $T^i$ is its dual basis. In particular,

\[
V_{z_1, z_2}(a, b) = \frac{1}{z_1 + z_2} \sum_{\sigma \in \Sigma} S_{z_1}(\hat{\phi}_\sigma, a) S_{z_2}(\hat{\phi}_\sigma, b).
\]

3.7. The A-model R-matrix. Let $U$ denote the diagonal matrix whose diagonal entries are the canonical coordinates. The results in [53] and [93] imply the following statement.

**Theorem 3.4.** There exists a unique matrix power series $R(z) = 1 + R_1 z + R_2 z^2 + \ldots$ satisfying the following properties.

1. The entries of $R_1$ lie in $S_2[\mathcal{Q}, \tau]$. 
2. $\tilde{S} = \Psi R(z) e^{U/z}$ is a fundamental solution to the $T$-equivariant big QDE (15).
3. $R$ satisfies the unitary condition $R^T(-z) R(z) = 1$. 
4. \[
\lim_{\tilde{Q}, \tau'' \to 0} R_{\rho, \delta, \gamma}^\sigma(z) = \delta_{\rho, \sigma} \chi_{\rho}(h) \chi_{\rho}(h^{-1}) \prod_{i=1}^3 \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} B_{m+1}(c_i(h)) \left( \frac{z}{w_i(\sigma)} \right)^m \right)
\]

Each matrix in (2) of Theorem 3.4 represents an operator with respect to the classical canonical basis $\{ \hat{\phi}_\sigma : \sigma \in \Sigma \}$. So $R^T$ is the adjoint of $R$ with respect to the $T$-equivariant Poincaré pairing $(\ , \chi).$ The matrix $(\tilde{S}^\rho_{\sigma})^\sigma(z)$ is of the form

\[
\tilde{S}^\rho_{\sigma}(z) = \sum_{\rho \in \Sigma} \Psi^\rho_{\sigma} R_{\rho, \delta, \gamma}^\sigma(z) e^{\omega/\tau} = (\Psi R(z))_{\sigma} e^{\omega/\tau}
\]

where $R(z) = (R_{\rho, \delta, \gamma}^\sigma(z)) = 1 + \sum_{k=1}^{\infty} R_k z^k$.

We call the unique $R(z)$ in Theorem 3.4 the A-model R-matrix. The A-model R-matrix plays a central role in the quantization formula of the descendant potential of $T$-equivariant Gromov-Witten theory of $\mathcal{X}$. We will state this formula in terms of graph sum in the the next subsection.

3.8. The A-model graph sum. In [93], the third author generalizes Givental’s formula for the total descendant potential of equivariant Gromov-Witten theory of GKM manifolds to GKM orbifolds. In order to state this formula, we need to introduce some definitions.

- We define \[
S^\sigma_{\sigma'}(z) := (\hat{\phi}_{\sigma'}(t) , S(\hat{\phi}_\sigma(t))).
\]

Then $\left( S^\sigma_{\sigma'}(z) \right)$ is the matrix of the $S$-operator with respect to the normalized canonical basis $\{ \hat{\phi}_\sigma(t) : \sigma \in \Sigma \}$:

\[
S(\hat{\phi}_\sigma(t)) = \sum_{\sigma' \in \Sigma} \hat{\phi}_\sigma(t) S^\sigma_{\sigma'}(z).
\]
• We define

\[ S_{\vec{G}}(z) := (\hat{\phi}_{\vec{G}}(t), S(\phi_{\vec{G}})). \]

Then \((S_{\vec{G}}(z))\) is the matrix of the \(S\)-operator with respect to the basis \(\{\phi_{\vec{G}} : \vec{G} \in I_\Sigma\}\) and \(\{\hat{\phi}_{\vec{G}}(t) : \vec{G} \in I_\Sigma\\}:

\[ S(\phi_{\vec{G}}) = \sum_{\vec{G} \in I_\Sigma} \hat{\phi}_{\vec{G}}(t) S_{\vec{G}}(z). \]

Given a connected graph \(\Gamma\), we introduce the following notation.

1. \(V(\Gamma)\) is the set of vertices in \(\Gamma\).
2. \(E(\Gamma)\) is the set of edges in \(\Gamma\).
3. \(H(\Gamma)\) is the set of half edges in \(\Gamma\).
4. \(L^0(\Gamma)\) is the set of ordinary leaves in \(\Gamma\). The ordinary leaves are ordered:
\[ L^0(\Gamma) = \{l_1, \ldots, l_n\} \]
where \(n\) is the number of ordinary leaves.
5. \(L^1(\Gamma)\) is the set of dilaton leaves in \(\Gamma\). The dilaton leaves are unordered.

With the above notation, we introduce the following labels:

1. (genus) \(g : V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}\).
2. (marking) \(\sigma : V(\Gamma) \rightarrow I_\Sigma\). This induces \(\sigma : L(\Gamma) = L^0(\Gamma) \cup L^1(\Gamma) \rightarrow I_\Sigma\), as follows: if \(l \in L(\Gamma)\) is a leaf attached to a vertex \(v \in V(\Gamma)\), define \(\sigma(l) = \sigma(v)\).
3. (height) \(k : H(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}\).

Given an edge \(e\), let \(h_1(e), h_2(e)\) be the two half edges associated to \(e\). The order of the two half edges does not affect the graph sum formula in this paper. Given a vertex \(v \in V(\Gamma)\), let \(H(v)\) denote the set of half edges emanating from \(v\). The valency of the vertex \(v\) is equal to the cardinality of the set \(H(v) : \text{val}(v) = |H(v)|\).

A labeled graph \(\vec{\Gamma} = (\Gamma, g, \sigma, k)\) is stable if

\[ 2g(v) - 2 + \text{val}(v) > 0 \]
for all \(v \in V(\Gamma)\).

Let \(\Gamma(\mathcal{X})\) denote the set of all stable labeled graphs \(\vec{\Gamma} = (\Gamma, g, \sigma, k)\). The genus of a stable labeled graph \(\vec{\Gamma}\) is defined to be

\[ g(\vec{\Gamma}) := \sum_{v \in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v \in V(\Gamma)} (g(v) - 1) + (\sum_{e \in E(\Gamma)} 1) + 1. \]

Define

\[ \Gamma_{g,n}(\mathcal{X}) = \{\vec{\Gamma} = (\Gamma, g, \sigma, k) \in \Gamma(\mathcal{X}) : g(\vec{\Gamma}) = g, |L^0(\Gamma)| = n\}. \]

We assign weights to leaves, edges, and vertices of a labeled graph \(\vec{\Gamma} \in \Gamma(\mathcal{X})\) as follows.

1. **Ordinary leaves.** To each ordinary leaf \(l_j \in L^0(\Gamma)\) with \(\sigma(l_j) = \sigma \in I_\Sigma\) and \(k(l) = k \in \mathbb{Z}_{\geq 0}\), we assign the following descendant weight:

\[ (L^0)^k_j(l_j) = [z^k](\sum_{\sigma, m \in I_\Sigma} \frac{u^m(z)}{\sqrt{\Delta^m(z)}} S^m_{\sigma}(z) R(-z)_{\rho, \sigma}), \]

where \([z^k]z\) means taking the nonnegative powers of \(z\).

2. **Dilaton leaves.** To each dilaton leaf \(l \in L^1(\Gamma)\) with \(\sigma(l) = \sigma \in I_\Sigma\) and \(2 \leq k(l) = k \in \mathbb{Z}_{\geq 0}\), we assign

\[ (L^1)^k_l := [z^{k-1}](\sum_{\sigma, m \in I_\Sigma} \frac{1}{\sqrt{\Delta^m(z)}} R_{\sigma, \sigma}(-z)). \]
(3) **Edges.** To an edge connected a vertex marked by $\sigma \in I_\Sigma$ to a vertex marked by $\sigma' \in I_\Sigma$ and with heights $k$ and $l$ at the corresponding half-edges, we assign

$$w_{k,l}^{\sigma,\sigma'} := \left[ z^k u^l \right] \left( \frac{1}{z + w} \delta_{\sigma,\sigma'} - \sum_{\rho \in I_\Sigma} R_\rho \sigma(-z) R_\rho \sigma'(-w) \right).$$

(4) **Vertices.** To a vertex $v$ with genus $g(v) = g \in \mathbb{Z}_{\geq 0}$ and with marking $\sigma(v) = \sigma$, with $n$ ordinary leaves and half-edges attached to it with heights $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and $m$ more dilaton leaves with heights $k_{n+1}, \ldots, k_{n+m} \in \mathbb{Z}_{\geq 0}$, we assign

$$\left( \sqrt{\Delta_{\sigma}(t)} \right)^{2g(v) - 2 + \text{val}(v)} \int_{\mathbb{M}_{g,n+m}} \psi_1^{k_1} \cdots \psi_{n+m}^{k_{n+m}}.$$

We define the weight of a labeled graph $\Gamma \in \Gamma(X)$ to be

$$w_A^{\sigma}(\Gamma) = \prod_{v \in V(\Gamma)} \left( \sqrt{\Delta_{\sigma}(v)}(t) \right)^{2g(v) - 2 + \text{val}(v)} \prod_{h \in H(v)} \tau_{k(h)}(v) \prod_{e \in E(\Gamma)} w_{\rho(h_1(e)), \rho(h_2(e))}^{\sigma(h_1(e))}.$$

With the above definition of the weight of a labeled graph, we have the following theorem which expresses the $\mathbb{T}$-equivariant descendent Gromov-Witten potential of $X$ in terms of graph sum.

**Theorem 3.5** (Zong [93]). Suppose that $2g - 2 + n > 0$. Then

$$\langle u_1, \ldots, u_n \rangle_{g,n}^{X,\mathbb{T}} = \sum_{\Gamma \in \Gamma_{g,n}(X)} \frac{w_A^{\sigma}(\Gamma)}{|\text{Aut}(\Gamma)|}.$$

**Remark 3.6.** In the above graph sum formula, we know that the restriction $S_{\sigma}^\mathbb{T}(z)|_{q=1}$ is well-defined by Remark 3.2. Meanwhile by (1) in Theorem 3.3, we know that the restriction $R(z)|_{q=1}$ is also well-defined. Therefore by Theorem 3.2 we have $\langle u_1, \ldots, u_n \rangle_{g,n}^{X,\mathbb{T}}|_{q=1}$ is well-defined.

We make the following observation.

**Lemma 3.7.**

$$\left. (\mathcal{L}^u)^\sigma_k(l_j) \right|_{u_i(z)=1} = - \left. (\mathcal{L}^l)^\sigma_k(l_j) \right|_{u_i(z)=t}$$

**Proof.** Let $(\ , \ )$ denote the $\mathbb{T}$-equivariant Poincaré pairing $(\ , \ )_{X,\mathbb{T}}$. Given $u \in H^\ast_{CR,T}(X) \otimes R_\psi \otimes S_{\mathbb{T}}$, we define $S_{u}^\mathbb{T}(z) := (\hat{\phi}_\rho, S(u))$. Then

$$\left. (\mathcal{L}^u)^\sigma_k(l_j) \right|_{u_i(z)=1} = \left[ z^k \left( \sum_{\rho \in I_\Sigma} (z S_{-1}^\mathbb{T})_t(z) \right) R(-z) \rho \sigma \right]$$

$$\left. (\mathcal{L}^u)^\sigma_k(l_j) \right|_{u_i(z)=t} = \left[ z^k \left( \sum_{\rho \in I_\Sigma} (S_{-1}^\mathbb{T})_t(z) \right) R(-z) \rho \sigma \right].$$

where

$$\left( z S_{-1}^\mathbb{T}(z) \right)_+ = \frac{z \hat{\phi}_\rho(t), 1 + \langle \hat{\phi}_\rho(t), 1 \rangle_{0,2}}{\sqrt{\Delta_{\rho}(t)}} + \left( \hat{\phi}_\rho(t), t \right)$$

$$\left( \mathcal{L}^u_{-1} \right)_+ = \left( \hat{\phi}_\rho(t), t \right)$$
So
\[
(L^u)^\sigma_k(l_j)\bigg|_{u_j(z)=1} = [z^z] \left( \sum_{\rho \in \Gamma} \frac{z}{\sqrt{\Delta_{\rho}(t)}} R_{\rho}^\sigma(-z) \right) + (L^u)^\sigma_k(l_j)\bigg|_{u_j(z)=t} = -(L^1)^\sigma_k + (L^u)^\sigma_k(l_j)\bigg|_{u_j(z)=t}.
\]

As a special case of Theorem 3.5, if \( g > 1 \) then
\[
\langle \hat{\psi} \rangle_{g,0} = \sum_{\Gamma \in \Gamma_{g,0}(X)} \frac{w_A^n(\hat{\Gamma})}{|\text{Aut}(\hat{\Gamma})|}.
\]

We end this subsection with the following alternative graph sum formula for \( \langle \hat{\psi} \rangle_{g,1} \).

**Proposition 3.8.** If \( g > 1 \) then
\[
\langle \hat{\psi} \rangle_{g,1} = \frac{1}{2g-2} \sum_{\Gamma \in \Gamma_{g,1}(X)} \frac{w_A^n(\hat{\Gamma})}{(L^u)^\sigma_k(l_1)=(L^1)^\sigma_k} + \langle t \rangle_{g,1}.
\]

**Proof.** Theorem 3.5 and Lemma 3.7 imply
\[
\langle \hat{\psi} \rangle_{g,1} = \sum_{\Gamma \in \Gamma_{g,1}(X)} \frac{w_A^n(\hat{\Gamma})}{(L^u)^\sigma_k(l_1)=(L^1)^\sigma_k} + \langle t \rangle_{g,1}.
\]

On the other hand,
\[
\langle \hat{\psi} \rangle_{g,1} = \sum_{m=0}^{\infty} \sum_{d \in E(X)} \frac{Q^d}{m!} \langle \hat{\psi} \rangle_{g,1+m,d} = \sum_{m=0}^{\infty} \sum_{d \in E(X)} \frac{Q^d}{m!} (2g-2+m) \langle t^m \rangle_{g,m,d} = (2g-2) \sum_{m=0}^{\infty} \sum_{d \in E(X)} \frac{Q^d}{m!} \langle t^m \rangle_{g,m,d} + \sum_{m=1}^{\infty} \sum_{d \in E(X)} \frac{Q^d}{(m-1)!} \langle t^m \rangle_{g,m,d} = (2g-2) \langle \hat{\psi} \rangle_{g,0} + \langle t \rangle_{g,1}.
\]

where the second equality follows from the dilaton equation. Therefore,
\[
\langle \hat{\psi} \rangle_{g,1} = (2g-2) \langle \hat{\psi} \rangle_{g,0} + \langle t \rangle_{g,1}.
\]

The proposition follows from Equation (23) and Equation (24). \( \square \)

### 3.9. Genus zero mirror theorem over the small phase space.

In [27], Coates-Corti-Iritani-Tseng proved a genus-zero mirror theorem for toric Deligne-Mumford stacks. This theorem takes a particularly simple form when the extended stacky fan satisfies the weak Fano condition [58, Section 4.1]. We state this theorem for toric Calabi-Yau 3-folds over the small phase space.
3.9.1. The small phase space. So far we work with the big phase space $H^*_\text{CR,\Sigma}(X; S_\Sigma) \cong S_\Sigma^\otimes X$. In this paper, we define the small phase space to be $H^2_{\text{CR,\Sigma}}(X) \cong \mathbb{C}^{3+p}$.

Following [58, Section 3.1], we choose $H_1, \ldots, H_p \in L^\vee \cap \text{Nef}(\Sigma^{\text{ext}})$ (where $H_\nu$ corresponds to the symbol $p_\nu$ in [58]) such that

- $\{H_1, \ldots, H_p\}$ is a $\mathbb{Q}$-basis of $L^\vee_\Sigma$.
- $\{\bar{H}_1, \ldots, \bar{H}_{p'}\}$ is a $\mathbb{Q}$-basis of $H^2(X; \mathbb{Q}) = H^2(X_\Sigma; \mathbb{Q})$.
- $H_a = D_{3+p_a}$ for $a = p' + 1, \ldots, p$.

Then $H_1, \ldots, H_p$ is a $\mathbb{C}$-basis of $H^2_{\text{CR,\Sigma}}(X) \cong \mathbb{C}^p$, and $\bar{H}_1, \ldots, \bar{H}_{p'}$ lie in the Kahler cone $C(\Sigma)$.

For $a = 1, \ldots, p'$ let $\bar{H}_a^\mathbb{C} \in H^2_\mathbb{C}(X)$ be the unique $T$-equivariant lifting of $H_a \in H^2(X)$ such that $H_a^\mathbb{C}|_{p=0} = 0$. Then

$$H^2_{\text{CR,\Sigma}}(X) = \bigoplus_{i=1}^{3+p} \mathbb{C}D_i^\mathbb{C} = \mathbb{C}u_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_3 \oplus \bigoplus_{a=1}^{p'} \mathbb{C}H_a^\mathbb{C} \oplus \bigoplus_{a=p'+1}^{p} \mathbb{C}1_{b_{a+3}}.$$

Any $\tau \in H^2_{\text{CR,\Sigma}}(X, \mathbb{C})$ can be written as

$$\tau = \tau_0 + \sum_{a=1}^{p'} \tau_a \bar{H}_a^\mathbb{C} \oplus \sum_{a=p'+1}^{p} \tau_a 1_{b_{a+3}},$$

where $\tau_0 \in H^2(BT) = \mathbb{C}u_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_3$ and $\tau_1, \ldots, \tau_p \in \mathbb{C}$. We write $\tau = \tau' + \tau''$, where

$$\tau' \in H^2_\mathbb{C}(X) = H^2_\mathbb{C}(X_\Sigma), \quad \tau'' = \sum_{a=p'+1}^{p} \tau_a 1_{b_{a+3}}.$$

3.9.2. The small equivariant quantum cohomology ring. In section 3.3 we defined the big equivariant quantum cohomology ring, where the quantum product $\ast$ depends on the point $t$ in the big phase space $H^*_\text{CR,\Sigma}(X; S_\Sigma)$. The small equivariant quantum cohomology ring is defined by restricting $t$ to the small phase space $H^2_{\text{CR,\Sigma}}(X)$.

More concretely, let $QH^*_\text{CR,\Sigma}(X) := H^*_\text{CR,\Sigma}(X; S_\Sigma) \otimes S_\Sigma[[\bar{Q}, \bar{\tau}]]_{a=p'+1, \ldots, p}$. Then $QH^*_\text{CR,\Sigma}(X)$ is a free $S_\Sigma[[\bar{Q}, \bar{\tau}]]_{a=p'+1, \ldots, p}$-module of rank $X$. Define the small quantum product $\ast$ to be $\ast_{\tau} := \ast|_{t=\tau}$, where $\tau$ is in the small phase space $H^2_{\text{CR,\Sigma}}(X)$. The pair $(QH^*_\text{CR,\Sigma}(X), \ast)$ is called the small equivariant quantum cohomology ring of $X$.

The small equivariant quantum cohomology ring $(QH^*_\text{CR,\Sigma}(X), \ast)$ is still semisimple. In fact, let $\phi_{\sigma}(\tau) := \phi_{\sigma}(t)|_{t=\tau}$ be the restriction of $\phi_{\sigma}(t)$ to the small phase space. Then

$$\{\phi_{\sigma}(\tau) : \sigma \in I_\Sigma\}$$

is a canonical basis of $(QH^*_\text{CR,\Sigma}(X), \ast)$. 

3.9.3. The equivariant small $J$-function. The $T$-equivariant small $J$-function $J_T(\tau, z)$ is the restriction of the $T$-equivariant big $J$-function to the small phase space. Given $\tau \in H^2_{\text{CR,\Sigma}}(X)$

$$J_T(\tau, z) := J^\text{big}_T(z)|_{t=\tau, q=1},$$
where \( J^\log(z) \) is defined in Definition 3.3. The restriction to \( Q = 1 \) is well-defined by Remark 3.2. Therefore,

\[
J_T(\tau, z) = 1 + \sum_{m=0}^{\infty} \sum_{d \in \mathbb{Z}} \sum_{\sigma \in \mathbb{Z}} \frac{1}{m!} (1, \frac{\hat{\sigma}}{z - \psi} \tau^m_{0,2,m,d})^{X, T} \chi \phi(1, \frac{\hat{\sigma}}{z - \psi}, (\tau^m_{0,2,m,d})^{X, T} \chi \phi)
\]

3.9.4. The equivariant small I-function. We define charges \( m_i^{(a)} \in \mathbb{Q} \) by

\[
D_i = \sum_{a=1}^{p} m_i^{(a)} H_a.
\]

Let \( t_0, q = (q_1, \ldots, q_{p'}) \) be formal variables, and define \( q^{\beta} = q_1^{(H_1, \beta)} \cdots q_{p'}^{(H_{p'}, \beta)} \) for \( \beta \in \mathbb{K} \). The limit point \( q \to 0 \) is a B-model large radius/orbifold mixed-type limit point. When \( X \) is a smooth toric variety, we have \( p' = p \), and this point is a large radius limit point. Following [58, Definition 4.1], and [27, Definition 28], we define the equivariant small I-function as follows.

**Definition 3.9.**

\[
I_T(t_0, q, z) = e^{t_0 + \sum_{i=1}^{3+p'} \tilde{H}_i^z \log q_i} \prod_{\beta \in \mathbb{K}_{eff}} \frac{q^\beta \prod_{i=1}^{3+p'} \prod_{m=0}^{\infty} (D_i, \beta - m) z}{\prod_{m=0}^{\infty} (D_i^z + (D_i, \beta - m))} 1_{\epsilon(\beta)}.
\]

Note that \( (H_a, \beta) \geq 0 \) for \( \beta \in \mathbb{K}_{eff} \).

**Remark 3.10.** \( D_i^z \in H^2_\beta(X) \) in this paper corresponds to \( u_i \) in [27, Definition 28], and \( H_a \) (resp. \( \bar{H}_a \)) in this paper corresponds to \( p_a \) (resp. \( \bar{p}_a \)) in [58]. The I-function in [27, Definition 28] depends on variables \( t_1, \ldots, t_{3+p'} \), which are related to the variables \( t_0, \log q_1, \ldots, \log q_{p'} \) in Definition 3.9 by

\[
\sum_{i=1}^{3+p'} t_i D_i^z = t_0 + \sum_{a=1}^{p'} \log q_i \tilde{H}_a^z.
\]

Equivalently,

\[
t_0 = \sum_{i=1}^{3} t_i w_i, \quad \log q_i = \sum_{a=1}^{3+p'} m_i^{(a)} t_i,
\]

where \( w_i \in \mathbb{C} u_1 \oplus \mathbb{C} u_2 \oplus \mathbb{C} u_3 \) is the restriction of \( D_i^z \) to the fixed point \( p_{a_0} \).

We now study the expansion of \( I_T(t_0, q, z) \) in powers of \( z^{-1} \). It can be rewritten as

\[
I_T(t_0, q, z) = e^{(t_0 + \sum_{a=1}^{3+p'} \tilde{H}_a^z \log q_i) / z} \prod_{\beta \in \mathbb{K}_{eff}} \frac{q^\beta \prod_{i=1}^{3+p'} \prod_{m=0}^{\infty} (D_i, \beta - m) z}{\prod_{m=0}^{\infty} (D_i^z + (D_i, \beta - m))} 1_{\epsilon(\beta)}
\]

where \( \hat{\beta} = D_1 + \cdots + D_{3+p} \in C(\Sigma^{ext}) \).
For \( i = 1, \ldots, 3 + p \), we will define \( \Omega_i \subset \mathbb{K}_{\text{eff}} - \{0\} \) and \( A_i(q) \) supported on \( \Omega_i \).

We observe that, if \( \beta \in \mathbb{K}_{\text{eff}} \) and \( v(\beta) = 0 \) then \( \langle D_i, \beta \rangle \in \mathbb{Z} \) for \( i = 1, \ldots, 3 + p \).

- For \( i = 1, \ldots, 3 + p' \), let

\[
\Omega_i = \{ \beta \in \mathbb{K}_{\text{eff}} : v(\beta) = 0, \langle D_i, \beta \rangle < 0 \text{ and } \langle D_j, \beta \rangle \geq 0 \text{ for } j \in \{1, \ldots, 3 + p\} - \{i\} \}.
\]

Then \( \Omega_i \subset \{ \beta \in \mathbb{K}_{\text{eff}} : v(\beta) = 0, \beta \neq 0 \} \). We define

\[
A_i(q) = \sum_{\beta \in \Omega_i} q^\beta \frac{(-1)^{-\langle D_i, \beta \rangle - 1}(-\langle D_i, \beta \rangle - 1)!}{\prod_{j \in \{1, \ldots, 3 + p\} - \{i\}} \langle D_j, \beta \rangle!}
\]

- For \( i = 4 + p', \ldots, 3 + p \), let

\[
\Omega_i = \{ \beta \in \mathbb{K}_{\text{eff}} : v(\beta) = b_i, \langle D_j, \beta \rangle \notin \mathbb{Z}_{>0} \text{ for } j = 1, \ldots, 3 + p \},
\]

and define

\[
A_i(q) = \sum_{\beta \in \Omega_i} q^\beta \frac{\prod_{j=1}^{3+p} \langle (\langle D_j, \beta \rangle - m) \rangle^{\langle D_j, \beta \rangle - m}}{\prod_{m=0}^{\infty} \langle (\langle D_j, \beta \rangle - m) \rangle}.
\]

Let \( \sigma \) be the smallest cone containing \( b_i \). Then

\[
b_i = \sum_{j \in I'_\sigma} c_j(b_i) b_j,
\]

where \( c_j(b_i) \in (0, 1) \) and \( \sum_{j \in I'_\sigma} c_j(b_i) = 1 \). There exists a unique \( D'_i \in \mathbb{L}_\mathbb{Q} \) such that

\[
\langle D_j, D'_i \rangle = \begin{cases} 1, & j = i, \\
-c_j(b_i), & j \in I'_\sigma, \\
0, & j \in I_\sigma - \{i\}.
\end{cases}
\]

Then

\[
A_i(q) = q^{D'_i} + \text{higher order terms}
\]

\[
I(t_0, q, z) = 1 + \frac{1}{z} \left( t_0 + \sum_{a=1}^{p'} \log(q_a) \bar{H}_a + \sum_{i=1}^{3+p'} A_i(q) \bar{D}_i + \sum_{i=4+p'} A_i(q) 1_{b_i} \right) + o(z^{-1}).
\]

where \( o(z^{-1}) \) involves \( z^{-k}, k \geq 2 \) We have

\[
\bar{D}_i = \begin{cases} \sum_{a=1}^{p'} m^{(a)}_i H_a + w_i, & i = 1, 2, 3, \\
\sum_{a=1}^{p'} m^{(a)}_i H_a, & 4 \leq i \leq 3 + p'.
\end{cases}
\]

Let \( S_\sigma(q) := \sum_{i=1}^{3+p'} m^{(a)}_i A_i(q) \). Then

\[
I'(t_0, q, z) = 1 + \frac{1}{z} \left( (t_0 + \sum_{i=1}^{3} w_i A_i(q)) + \sum_{a=1}^{p'} (\log(q_a) + S_\sigma(q)) \bar{H}_a + \sum_{a=4+p'} A_i(q) 1_{b_i} \right) + o(z^{-1}).
\]

3.9.5. The mirror theorem. The main result in [27] implies the following \( T \)-equivariant mirror theorem:

**Theorem 3.11** (Coates-Corti-Iritani-Tseng).

\[
J_T(\tau, z) = I_T(t_0, q, z),
\]

where the equivariant closed mirror map \( (t_0, q) \mapsto \tau(t_0, q) \) is determined by the first-order term in the asymptotic expansion of the \( I \)-function

\[
I(t_0, q, z) = 1 + \frac{\tau(t_0, q)}{z} + o(z^{-1}).
\]
More explicitly, the equivariant closed mirror map is given by
\[
\tau = \tau_0(t_0, q) + \sum_{a=1}^{2} \tau_a(q) \hat H^a + \sum_{a'=3}^{p} \tau_a(q) 1_{b_{a+3}}
\]
where
\[
\tau_0(t_0, q) = t_0 + \sum_{i=1}^{3} w_i A_i(q),
\]
\[
\tau_a(q) = \begin{cases} 
\log(q_a) + S_a(q), & 1 \leq a \leq p', \\
A_{a+3}(q), & p' + 1 \leq a \leq p.
\end{cases}
\]  
(25)

Under this mirror map, the B-model large radius/orbifold mixed-type limit $g \to 0$ corresponds to the A-model large radius/orbifold mixed type limit $\tilde Q \to 0, \tau'' \to 0$.

3.10. **Non-equivariant small $I$-function.** Choose a basis \(\{e_1, \ldots, e_p\}\) of \(H^2_{\text{cr}}(X)\) such that \(\{e_1, \ldots, e_g\}\) is a basis of \(H^2_{\text{cr}, c}(X)\). We choose a basis \(\{e'_1, \ldots, e'_{q}\}\) of \(H^1_{\text{cr}}(X)\) which is dual to \(\{e_1, \ldots, e_g\}\) under the perfect pairing \(H^2_{\text{cr}, c}(X) \times H^1_{\text{cr}}(X) \to \mathbb{C}\). Then
\[
I(q, z) := I_\tau(t_0, q, z) |_{t_0=0, w_i=0} = 1 + \frac{1}{z} \sum_{a=1}^{p} T^a(q) e_a + \frac{1}{z^2} \sum_{b=1}^{q} W_b(q) e_b
\]
for some generating functions \(T^a(q), W_b(q)\) of \(q\). Given \(\psi \in H^2_{\text{cr}, c}(X)\),
\[
\langle \psi \rangle_{0,1} \in \mathbb{C}
\]
is defined in the non-equivariant setting. The mirror theorem implies that, under the mirror map,
\[
W_b(q) = \frac{\partial F^X_0}{\partial T^b} = \langle e_b \rangle_{0,1}^X |_{t=\tau, Q=1}, \quad b = 1, \ldots, q,
\]
where \(F^X_0 = \langle \psi \rangle_{0,1}^X |_{t=\tau, Q=1}\). (See e.g. [34, Proposition 10.3.4].) A basis of the space of solutions to the non-equivariant Picard-Fuchs system are given by
\[
\{1, T^1(q), \ldots, T^p(q), W_1(q), \ldots, W_q(q)\}.
\]
When the coarse moduli space \(X_\Sigma\) of \(X\) is a smooth toric variety (so that \(X_\Sigma = X\), \(T^1, \ldots, T^p\) have logarithm singularities and \(W_1, \ldots, W_q\) have double logarithm singularities.

3.11. **Restriction to the Calabi-Yau torus.** The inclusion \(T' \to T\) induces a surjective ring homomorphism
\[
R_\tau = H^*_\tau(pt) = \mathbb{C}[u_1, u_2, u_3] \twoheadrightarrow R_{\tau'} = H^*_\tau(dt) = \mathbb{C}[u_1, u_2]
\]
given by \(u_1 \mapsto u_1, u_2 \mapsto u_2, u_3 \mapsto 0\). The image of \(\{\phi_\sigma : \sigma \in I_\Sigma\}\) under the surjective ring homomorphism
\[
H^*_{\text{cr}, \tau}(X) \otimes_{R_\tau} \mathcal{S}_\tau \twoheadrightarrow H^*_{\text{cr}, \tau'}(X) \otimes_{R_{\tau'}} \mathcal{S}_{\tau'}
\]
is a canonical basis of the semisimple \(\mathcal{S}_{\tau'}\)-algebra \(H^*_{\text{cr}, \tau}(X) \otimes_{R_\tau} \mathcal{S}_{\tau'}\). In the rest of this paper, we will often consider \(T'\)-equivariant cohomology and. Let the symbol \(\phi'_\sigma\) denote the image of \(\phi_\sigma\) under the ring homomorphism in (26).
3.12. **Open-closed Gromov-Witten invariants.** Let \( \mathcal{L} \subset \mathcal{X} \) be an outer Aganagic-Vafa Lagrangian brane. Let \( G_0 := G_{p_0} \) be the stabilizer of the stacky point \( p_{\eta_0} \). Our notation is similar to that in [?, Section 5]. In particular, the integer \( f \) is a framing, and \( \mathbb{T}_f := \text{Ker}(u_2 - fu_1) \). The morphism \( H^\ast(B\mathbb{T}; \mathbb{Q}) = \mathbb{Q}[u_1, u_2] \to H^\ast(B\mathbb{T}_f; \mathbb{Q}) = \mathbb{Q}[v] \) is given by \( u_1 \mapsto v, u_2 \mapsto fv \). The weights of \( \mathbb{T}_f \)-action on \( T_{p_{\eta_0}} \mathcal{X} \) are

\[
\begin{align*}
    w_1 &= \frac{1}{r} u_1, \quad w_2 = \frac{s}{m} u_1 + \frac{1}{m} u_2, \quad w_3 = -\frac{s + m}{m} u_1 - \frac{1}{m} u_2.
\end{align*}
\]

so the weights of \( \mathbb{T}_f \)-action on \( T_{p_{\eta_0}} \mathcal{X} \) are \( w_1, w_2, w_3 \), where

\[
\begin{align*}
    w_1 &= \frac{1}{r}, \quad w_2 = \frac{s + tf}{rm}, \quad w_3 = -\frac{m - s - tf}{rm}.
\end{align*}
\]

Let the correlator \( \langle \tau \rangle_{g,d,(\mu_1, k_1), \ldots, (\mu_n, k_n)}^{X, (\mathcal{L}, f)} \) denote the equivariant open-closed Gromov-Witten invariant defined in [45, Section 3]. Define the open-closed Gromov-Witten potential

\[
\tilde{F}_{g,n}^{X, (\mathcal{L}, f)}(\tau, Q; \bar{X}_1, \ldots, \bar{X}_n) = \sum_{d \in \text{Eff}(X)} \sum_{\mu_1, \ldots, \mu_n > 0} \sum_{k_1, \ldots, k_n = 0} \sum_{\ell \geq 0} \frac{\langle \tau \rangle_{g,d,(\mu_1, k_1), \ldots, (\mu_n, k_n)}^{X, (\mathcal{L}, f)}}{\ell!} \cdot Q^d \cdot \prod_{j=1}^{\ell} (\bar{X}_j)^{\mu_j} \cdot (-1)^{\frac{1}{m}} \sum_{\nu} \prod_{j=1}^{\ell} (-1)^{\frac{d_{\nu} - 2}{m}} \Gamma'(\frac{d_{\nu}}{m}),
\]

which is an \( H_{CR}^\ast(B\mu_m; \mathbb{C}) \otimes \mathbb{C} \)-valued function, where

\[
H_{CR}^\ast(B\mu_m; \mathbb{C}) = \bigoplus_{k=0}^{m-1} \mathbb{C} 1_k'.
\]

We introduce some notation.

1. Given \( d_0 \in \mathbb{Z} \) and \( k \in \mathbb{Z}_m \), let \( D'(d_0, k) \) be the disk factor defined by Equation (13) in [?], and define

\[
h(d_0, k) := (e^{2\pi \sqrt{-1} d_0 w_1}, e^{2\pi \sqrt{-1} (d_0 w_2 - \frac{1}{m})}, e^{2\pi \sqrt{-1} (d_0 w_3 + \frac{1}{m})}) \in G_0 \subset \mathbb{T} = (\mathbb{C}^\ast)^3.
\]

2. Given \( h \in G_0 \), define

\[
\Phi_h^b(\bar{X}) := \frac{1}{|G_0|} \sum_{(d_0, k) \in \mathbb{Z}_0 \times \mathbb{Z}_m \atop h(d_0, k) = h} D'(d_0, k) \cdot \bar{X}^{d_0} \cdot (-1)^{-k/m} 1_k' \frac{1}{h}.
\]

Then \( \Phi_h(\bar{X}) \) takes values in \( \bigoplus_{k=0}^{m-1} \mathbb{C}v^{a_\text{age}(v) - 2} 1_k' \).

For \( a \in \mathbb{Z} \) and \( h \in G_0 \), we define

\[
\Phi_a^h(\bar{X}) := \frac{1}{|G_0|} \sum_{(d_0, k) \neq 0 \atop h(d_0, k) = h} D'(d_0, k) \cdot (\frac{d_0}{v})^a \cdot \bar{X}^{d_0} \cdot (-1)^{-k/m} 1_k' \frac{1}{h}.
\]

Then \( \Phi_a^h(\bar{X}) \) takes values in \( \bigoplus_{k=0}^{m-1} \mathbb{C}v^{a_\text{age}(v) - 2 - a} 1_k' \), and

\[
\Phi_{a+1}(\bar{X}) = (\frac{d}{v}) \Phi_a^b(\bar{X}).
\]
Proposition 3.12.  

(1) (disk invariants)

\[ \tilde{F}_{0,1}^{X,\vec{h}}(\tau, Q; \tilde{X}) = \Phi_{\tau}^{h_1}(\tilde{X}) + \sum_{a_1} \sum_{a_2} \chi_{a_1}(h_1) \chi_{a_2}(h_2) \Phi_{a_1}(\tilde{X}_1) \Phi_{a_2}(\tilde{X}_2) \]

(2) (annulus invariants)

\[ \tilde{F}_{0,2}^{X,\vec{h}}(\tau, Q; \tilde{X}_1, \tilde{X}_2) = \Phi_{\tau}^{h_1}(\tilde{X}_1) \Phi_{\tau}^{h_2}(\tilde{X}_2) \]

where

\[ (\tilde{X}_1 \frac{\partial}{\partial \tilde{X}_1} + \tilde{X}_2 \frac{\partial}{\partial \tilde{X}_2}) \tilde{F}_{0,2}^{X,\vec{h}}(0; \tilde{X}_1, \tilde{X}_2) \]

(27) 

So we have

\[ \tilde{F}_{0,2}^{X,\vec{h}}(\tau, Q; \tilde{X}_1, \tilde{X}_2) = [z_1^{-1} z_2^{-1}] \sum_{\alpha_1, \alpha_2} V_{\alpha_1 \alpha_2}(\phi_{\sigma_0, \alpha_1}, \phi_{\sigma_0, \alpha_2}) \tilde{\xi}^{\alpha_1}(z_1, \tilde{X}_1) \tilde{\xi}^{\alpha_2}(z_2, \tilde{X}_2). \]
Corollary 3.16. By Remark 3.6 and Proposition 3.12, 

\[
\sum_{a_1, \ldots, a_n \in \mathbb{Z}} \left( \prod_{f \in G_0} \tilde{\xi}_{\sigma}(z, \tilde{X}) \right) \prod_{j=1}^{n} \tilde{\xi}_{\gamma_j}(z_j, \tilde{X}_j).
\]

Remark 3.13. \( \tilde{F}^{X, (L, f)}_{g,n}(0; \tilde{X}_1, \tilde{X}_2) \) is an \( H^*(B\mu_m; \mathbb{C})^{\otimes 2} \)-valued power series in \( \tilde{X}_1, \tilde{X}_2 \) which vanishes at \( (\tilde{X}_1, \tilde{X}_2) = (0, 0) \), so it is determined by \( [\nabla] \).

We now combine Section 3.8 and the above Proposition 3.12 to obtain a graph sum formula for \( \tilde{F}^{X, (L, f)}_{g,n} \). We use the notation in Section 3.8 and introduce the notation

\[
\tilde{\xi}_{\sigma}(z, \tilde{X}) = \begin{cases} 
\tilde{\xi}_{\sigma}(z, \tilde{X}), & \text{if } \sigma = (\sigma, \alpha), \\
0, & \text{if } \sigma = (\sigma, \alpha) \text{ and } \sigma \neq \sigma_0.
\end{cases}
\]

• Given a labelled graph \( \tilde{\Gamma} \in \Gamma_{g,n}(\mathcal{X}) \), to each ordinary leaf \( l_j \in L_\sigma(\gamma) \) with \( \sigma(l_j) = \sigma \in I_\Sigma \) and \( k(l_j) \in \mathbb{Z}_{\geq 0} \) we assign the following weight (open leaf)

\[
(\mathcal{L}_O)_{\tilde{\Gamma}}(l_j) = \left[ z^j \right] \sum_{\rho, \sigma \in I_\Sigma} \left( \tilde{\xi}_{\sigma}(z, \tilde{X}_j) \right)_{\mathbb{Z}}^{\sigma} \left( R(-z)_{\rho} \right)_{\tilde{\Gamma}}(l_j).
\]

• Given a labelled graph \( \Gamma_{g,n}(\mathcal{X}) \), we define a weight

\[
\tilde{w}_A^O(\tilde{\Gamma}) = \prod_{v \in V(\Gamma)} \left( \sqrt{\Delta_{(v)}(t)} \right)_{\tilde{\xi}}^{2g(v)} \prod_{h \in H(v, \sigma)} \tau_{k(h_1)}(\sigma(v)) \prod_{v \in V(\Gamma)} \left( \mathcal{L}_O \right)_{k(h_2)}(\sigma(v)).
\]

Then we have the following graph sum formula for \( \tilde{F}^{X, (L, f)}_{g,n} \).

Theorem 3.14.

\[
\tilde{F}^{X, (L, f)}_{g,n} = \sum_{\tilde{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{\tilde{w}_A^O(\tilde{\Gamma})}{|\text{Aut}(\tilde{\Gamma})|}.
\]

Proof. This follows from Theorem 3.5 and Proposition 3.12.

\( \square \)

Definition 3.15 (Restriction to \( Q = 1 \)). We define

\[
F^{X, (L, f)}_{g,n}(\tau; \tilde{X}_1, \ldots, \tilde{X}_n) := \tilde{F}^{X, (L, f)}_{g,n}(\tau, 1; \tilde{X}_1, \ldots, \tilde{X}_n).
\]

By Remark 3.6 and Proposition 3.12 \( F^{X, (L, f)}_{g,n} \) is well-defined. Theorem 3.14 implies

Corollary 3.16.

\[
F^{X, (L, f)}_{g,n} = \sum_{\tilde{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_A^O(\tilde{\Gamma})}{|\text{Aut}(\tilde{\Gamma})|}.
\]

where \( w_A^O(\tilde{\Gamma}) = \tilde{w}_A^O(\tilde{\Gamma}) |_{Q=1} \).
4. Hori-Vafa mirror, Landau-Ginzburg mirror, and the mirror curve

4.1. The mirror curve and its compactification. We first introduce some notation. Let $P \subset \mathbb{N}'_N \cong \mathbb{R}^2$ be the convex lattice polytope defined in Section 2.1.

We introduce complex variables $q = (q_1, \ldots, q_p)$. Given $(m, n) \in P \cap \mathbb{N}'$, so that $(m, n, 1) = b_i$ for some $i \in \{1, \ldots, 3 + p\}$, we define $a_{m,n}(q)$ as follows.

(1) If $1 \leq i \leq 3 + p'$, we define

$$a_{m,i}(q) := \begin{cases} 1, & 1 \leq i \leq 3, \\ q_{i-3}, & 4 \leq i \leq 3 + p' \end{cases}$$

(2) If $4 + p' \leq i \leq 3 + p$ let $\sigma$ be the minimal cone containing $b_i$. Then $\sigma \in \Sigma(2) \cup \Sigma(3)$, and

$$b_i = \sum_{j \in I_\sigma} c_j(b_i)b_j.$$

We define

$$a_{m,i}(q) := q_{i-3} \prod_{j \in I_\sigma} a_{m_j,n_j}(q)^{c_j(b_i)}$$

We have the relations

$$\prod_{a=1}^p (a_{m+a,3+a}(q))^{(D_{a+3})} = q^\beta, \quad \forall \beta \in \mathbb{K}_{\text{eff}}.$$  \tag{29}

In particular,

$$q_{a+3} = q_{a+3}^{D_{a+3}}.$$  \tag{30}

Under the relations (29) and (30), the parameter $q = q(q)$ depends on the B-model complex parameter $q$. For each $a = 1, \ldots, p$, $q_a$ is a monomial in $q_1, \ldots, q_p$.

Define

$$H(X,Y,q) := \sum_{(m,n) \in P \cap \mathbb{N}'} a_{m,n}(q)X^mY^n$$

$$= X^2Y^2 + Y^{m+1} + \sum_{a=1}^{p'} q_aX^{m_{3+a}}Y^{n_{3+a}} + \sum_{a=p'+1}^{p} a_{m_{3+a},n_{3+a}}(q)X^{m_{3+a}}Y^{n_{3+a}}.$$  

For fixed $q = (q_1, \ldots, q_p) \in \mathbb{C}^p$, $H(X,Y,q)$ is a Laurent polynomial in $\mathbb{C}[X, X^{-1}, Y, Y^{-1}]$; we choose $q$ such that the Newton polytope of $H(X,Y,q)$ is $P$.

The mirror curve of $X$ is

$$C_q = \{(X,Y) \in (\mathbb{C}^*)^2 : H(X,Y,q) = 0\}.$$  

For fixed $q \in \mathbb{C}^p$, $C_q$ is an affine curve in $(\mathbb{C}^*)^2$. The polytope $P$ determines a toric surface $S_P$ with a polarization $L_P$, and $H(X,Y,q)$ extends to a section $s_q \in H^0(S_P, L_P)$. The compactified mirror curve $\overline{C}_q \subset S_P$ is the zero locus of $s_q$.

For generic $q \in \mathbb{C}^p$, $\overline{C}_q$ is a compact Riemann surface of genus $g$ and $\overline{C}_q$ intersects the anti-canonical divisor $\partial S_P = S_P \setminus (\mathbb{C}^*)^2$ transversally at $n$ points, where $g$ and $n$ are the number of lattice points in the interior and the boundary of $P$, respectively.
4.2. **Three mirror families.** The symplectic toric orbifold \((X, \omega(\eta))\) has three mirror families:

- The Hori-Vafa mirror \((\tilde{X}_q, \Omega_q)\), where
  \[
  \tilde{X}_q = \{(u, v, X, Y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* : uv = H(X, Y, q)\}
  \]
  is a non-compact Calabi-Yau 3-fold, and
  \[
  \Omega_q := \text{Res}_{\tilde{x}_q} \left( \frac{1}{H(X, Y, q) - uv} du \wedge dv \wedge \frac{dX}{X} \wedge \frac{dY}{Y} \right)
  \]
  is a holomorphic 3-form on \(\tilde{X}_q\).

- The \(T\)-equivariant Landau-Ginzburg mirror \((\mathbb{C}^*)^3, W_q^{T^*})\), where
  \[
  W_q^{T^*}(X, Y, Z) = H(X, Y, q)Z - u_1 \log X - u_2 \log Y
  \]
  is the \(T\)-equivariant superpotential.

  One obtains this superpotential from the non-equivariant superpotential
  \[
  W = \sum_{i=1}^{p+3} \tilde{X}_i,
  \]
  where
  \[
  \prod_{i=1}^{p+3} \tilde{X}_i^{(D_i, \beta)} = q^\beta, \beta \in \mathbb{Z}_{\text{eff}}.
  \]

  For any top dimensional cone \(\sigma\) with its three 1-cones \(b_{\sigma,1}, b_{\sigma,2}, b_{\sigma,3}\), we can write the superpotential \(W\) in the following form
  \[
  W = \tilde{H}_\sigma(\tilde{X}_\sigma, \tilde{Y}_\sigma) \tilde{Z}_\sigma, \quad \tilde{X}_\sigma \tilde{Y}_\sigma^{-s} \tilde{Z}_\sigma = X_{b_{\sigma,1}}, \quad \tilde{Y}_\sigma \tilde{Z}_\sigma = X_{b_{\sigma,2}}, \quad \tilde{Z}_\sigma = X_{b_{\sigma,3}},
  \]
  \[
  \tilde{H}_\sigma(\tilde{X}, \tilde{Y}) = \tilde{X}_\sigma \tilde{Y}_\sigma^{-s} + \tilde{Y}_\sigma + 1 + \sum_{a=1}^p \tilde{q}_{\sigma,a} \tilde{X}_\sigma^n \tilde{Y}_\sigma^m.
  \]

  For any \(\sigma\), each \(\tilde{q}_{\sigma,a}\) is a monomial of \(q_{\sigma,a}'\), \(a' = 1, \ldots, p\), and \(\lim_{q \to 0} \tilde{q}_{\sigma} = 0\). The group \(G_\sigma\) fits into the following short exact sequence of finite abelian groups:
  \[
  0 \to \mu_{r_\sigma} \to G_\sigma \to \mu_{m_\sigma} \to 0.
  \]

  In Section 2.10 by choosing the Lagrangian \(\mathcal{L}\), we fixed a preferred 3-cone \(\sigma_0\). Writing
  \[
  X = \tilde{X}_{\sigma_0}, \quad Y = \tilde{Y}_{\sigma_0}, \quad Z = \tilde{Z}_{\sigma_0}, \quad \tilde{q}_{\sigma_0,a} = a_{m_{3+a'}, n_{3+a'}}(q),
  \]
  \[
  H(X, Y) = \tilde{H}_{\sigma_0}(\tilde{X}_{\sigma_0}, \tilde{Y}_{\sigma_0}).
  \]

  Let \(e^{-x} = X, e^{-y} = Y, e^{-x_\sigma} = \tilde{X}_\sigma, e^{-y_\sigma} = \tilde{Y}_\sigma, \quad \tilde{x} = u_1 x + u_2 y\). The \(T\)-equivariant superpotential is given by
  \[
  W_q^{T^*} = W + \tilde{x}.
  \]

- The mirror curve \(C_q = \{(X, Y) \in (\mathbb{C}^*)^2 : H(X, Y, q) = 0\}\) and its compactification \(\overline{C}_q\).

  In Section 4.3 (resp. Section 15) below, we will reduce the genus zero B-model on the Hori-Vafa mirror (resp. equivariant Landau-Ginzburg mirror) to a theory on the mirror curve.
4.3. Dimensional reduction of the Hori-Vafa mirror. In this section, we will relate period integrals of the holomorphic 3-form \( \Omega_q \) along 3-cycles in the Hori-Vafa mirror \( \mathcal{X}_q \) to integrals of a 1-form along 1-cycles in the mirror curve \( \mathcal{C}_q \). The references of this subsection are \([37]\) and \([21]\); see also \([61]\).

The inclusion \( J : C_q \to \overline{C}_q \) induces a surjective homomorphism

\[
J_* : H_1(C_q; \mathbb{Z}) \cong \mathbb{Z}^{2g+n-1} \to H_1(\overline{C}_q; \mathbb{Z}) \cong \mathbb{Z}^{2g}.
\]

Let \( J_1(C_q; \mathbb{Z}) \) denote the kernel of the above map. Then \( J_1(C_q; \mathbb{Z}) \cong \mathbb{Z}^{g(n-1)} \) is generated by \( \delta_1, \ldots, \delta_n \), where \( \delta_i \in H_1(C_q; \mathbb{Z}) \) is the class of a small loop around the puncture \( \tilde{p}_i \). They satisfy

\[
\delta_1 + \cdots + \delta_n = 0.
\]

The inclusion \( I : C_q \to (\mathbb{C}^*)^2 \) induces a homomorphism

\[
I_* : H_1(C_q; \mathbb{Z}) \cong \mathbb{Z}^{2g+n-1} \to H_1((\mathbb{C}^*)^2; \mathbb{Z}) = \mathbb{Z}^2
\]

whose cokernel is finite (but not necessarily trivial). Let \( K_1(C_q; \mathbb{Z}) \cong \mathbb{Z}^{2g+n-3} \) denote the kernel of the above map.

Let \( J_1(C_q; \mathbb{Q}) : = J_1(C_q; \mathbb{Z}) \otimes \mathbb{Q} \) and \( K_1(C_q; \mathbb{Q}) : = K_1(C_q; \mathbb{Z}) \otimes \mathbb{Q} \). Then we have the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & J_1(C_q; \mathbb{Q}) & \rightarrow & H_1(C_q; \mathbb{Q}) & \rightarrow & J_* & H_1(\overline{C}_q; \mathbb{Q}) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & K_1(C_q; \mathbb{Q}) & & H_1((\mathbb{C}^*)^2; \mathbb{Q}) & & 0 & & 0 \\
\end{array}
\]

In the above diagram, the row and the column are short exact sequences of vector spaces over \( \mathbb{Q} \). Let \( \overline{C}_q \) be the fiber product of the inclusion \( I : C_q \to (\mathbb{C}^*)^2 \) and the universal cover \( \mathbb{C}^2 \to (\mathbb{C}^*)^2 \). Then \( p : \overline{C}_q \to C_q \) is a regular covering with fiber \( \mathbb{Z}^2 \), and there is an injective group homomorphism

\[
p_* : H_1(\overline{C}_q; \mathbb{Q}) \to H_1(C_q; \mathbb{Q})
\]

whose image is \( K_1(C_q; \mathbb{Q}) \).

The long exact sequence of relative homology for the pair \( ((\mathbb{C}^*)^2, C_q) \) is

\[
\cdots \rightarrow H_2(C_q; \mathbb{Z}) \rightarrow H_2((\mathbb{C}^*)^2; \mathbb{Z}) \rightarrow H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \rightarrow H_1(C_q; \mathbb{Z}) \rightarrow H_1((\mathbb{C}^*)^2; \mathbb{Z}) \rightarrow H_1((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \rightarrow \cdots
\]

where \( H_2(C_q; \mathbb{Z}) = 0 \). So we have a short exact sequence

\[
0 \rightarrow H_2((\mathbb{C}^*)^2; \mathbb{Z}) \rightarrow H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \xrightarrow{\partial} K_1(C_q; \mathbb{Z}) \rightarrow 0.
\]
Lemma 4.1 (\cite{37} Section 5.1 \cite{21} Section 4.2). The map \( \Lambda \mapsto C \) to the non-equivariant Picard-Fuchs system. Let \( \omega \) is well-defined up to addition of an integral multiple of \( \omega_\gamma \). The inclusion \( T^2 = (S^1)^2 \subset (\mathbb{C}^*)^2 \) is a homotopy equivalence, so \( H_2((\mathbb{C}^*)^2; \mathbb{Z}) \cong H_2(T^2; \mathbb{Z}) = \mathbb{Z}[T^2] \). We have

\[
\int_{[T^2]} \omega = (2\pi \sqrt{-1})^2.
\]

Define \( \mu : \tilde{X}_q \to \mathbb{R} \) by \( (u, v, X, Y) \mapsto |u|^2 - |v|^2 \) and \( \pi : \tilde{X}_q \to (\mathbb{C}^*)^2 \) by \( (u, v, X, Y) \mapsto (X, Y) \). Then \( \pi : \mu^{-1}(0) \to (\mathbb{C}^*)^2 \) is a circle fibration which degenerates along \( C_q \subset (\mathbb{C}^*)^2 \). Given a relative 2-cycle \( \Lambda \in Z_2((\mathbb{C}^*)^2, C_q) \), \( \pi^{-1}(\Lambda) \) is a 3-cycle in \( \tilde{X}_q \). The map \( \Lambda \mapsto \pi^{-1}(\Lambda) \) induces a group homomorphism \( M : H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \to H_3(\tilde{X}_q; \mathbb{Z}) \).

Let \( x = -\log(X) \), \( y = -\log(Y) \). Then

\[
\omega = dx \wedge dy = \frac{dX}{X} \wedge \frac{dY}{Y}
\]

is the standard holomorphic symplectic form on \((\mathbb{C}^*)^2\). Note that \( \omega|_{C_q} = 0 \), so \( \omega \) represents a class in \( H^2((\mathbb{C}^*)^2, C_q; \mathbb{C}) \). Therefore,

\[
\int_{\Lambda} \omega
\]

is well-defined for any \( \Lambda \in H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \cong \mathbb{Z}^{2g-2+n} \). The inclusion \( T^2 = (S^1)^2 \subset (\mathbb{C}^*)^2 \) is a homotopy equivalence, so \( H_2((\mathbb{C}^*)^2; \mathbb{Z}) \cong H_2(T^2; \mathbb{Z}) = \mathbb{Z}[T^2] \). We have

\[
\int_{[T^2]} \omega = (2\pi \sqrt{-1})^2.
\]

Define \( \mu : \tilde{X}_q \to \mathbb{R} \) by \( (u, v, X, Y) \mapsto |u|^2 - |v|^2 \) and \( \pi : \tilde{X}_q \to (\mathbb{C}^*)^2 \) by \( (u, v, X, Y) \mapsto (X, Y) \). Then \( \pi : \mu^{-1}(0) \to (\mathbb{C}^*)^2 \) is a circle fibration which degenerates along \( C_q \subset (\mathbb{C}^*)^2 \). Given a relative 2-cycle \( \Lambda \in Z_2((\mathbb{C}^*)^2, C_q) \), \( \pi^{-1}(\Lambda) \) is a 3-cycle in \( \tilde{X}_q \). The map \( \Lambda \mapsto \pi^{-1}(\Lambda) \) induces a group homomorphism \( M : H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \to H_3(\tilde{X}_q; \mathbb{Z}) \).

Lemma 4.1 (\cite{37} Section 5.1 \cite{21} Section 4.2). The map \( M : H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \to H_3(\tilde{X}_q; \mathbb{Z}) \) is an isomorphism, and

\[
\int_{M(\Lambda)} \Omega_q = \int_{M(\Lambda)} \frac{du}{u} \wedge \frac{dX}{X} \wedge \frac{dY}{Y} = 2\pi \sqrt{-1} \int_{\Lambda} \omega
\]

for any \( \Lambda \in H_2((\mathbb{C}^*)^2; C_q; \mathbb{Z}) \). In particular,

\[
\int_{M([T^2])} \Omega_q = (2\pi \sqrt{-1})^3.
\]

If \( \gamma \in K_1(C_q; \mathbb{Z}) \) then

\[
\int_{\gamma} ydx = \frac{1}{(2\pi \sqrt{-1})^3} \int_{\tilde{X}_q} \Omega_q = 1.
\]
There exists $\tilde{A}_1, \ldots, \tilde{A}_p, \tilde{B}_1, \ldots, \tilde{B}_q$ which are flat sections of $H_3(X_q; \mathbb{C})$ such that

$$\frac{1}{(2\pi\sqrt{-1})^3} \int_{\tilde{A}_a} \Omega_q = T^a(q), \quad a = 1, \ldots, p,$$

$$\frac{1}{2\pi\sqrt{-1}} \int_{\tilde{B}_i} \Omega_q = W_i(q), \quad i = 1, \ldots, q.$$

4.4. The equivariant small quantum cohomology. Let $H(X, Y, q)$ be defined as in Section 4.1. The $T$-equivariant Landau-Ginzburg mirror of $X$ is $((\mathbb{C}^*)^3, W_q^T)$, where

$$W_q^T(X, Y, Z) = H(X, Y, q)Z - u_1 \log X - u_2 \log Y - u_3 \log Z$$

Consider the universal superpotential $W^T(X, Y, Z, q) = W_q^T(X, Y, Z)$ defined on $(\mathbb{C}^*)^3 \times (\mathbb{C}^*)^p$. Then

$$\text{Jac}(W^T) := \frac{\tilde{S}_T[q_a, q_a^{-1}, X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]}{\left(\frac{\partial W^T}{\partial X}, \frac{\partial W^T}{\partial Y}, \frac{\partial W^T}{\partial Z}\right)}$$

is an algebra over $\tilde{S}_T[q_a, q_a^{-1}]$. For each fixed $q = (q_1, \ldots, q_p) \in \mathbb{C}^p$, we obtain

$$\text{Jac}(W_q^T) := \frac{\tilde{S}_T[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]}{\left(\frac{\partial W_q^T}{\partial X}, \frac{\partial W_q^T}{\partial Y}, \frac{\partial W_q^T}{\partial Z}\right)}$$

which is an algebra over $\tilde{S}_T$. By the argument in [83], Theorem 3.11 (T-equivariant mirror theorem) implies the following isomorphism of $\tilde{S}_T$-algebras:

$$QH^*_\text{CR, T}(X)|_{T=\tau(q), q=1} \cong \text{Jac}(W_q^T) := \frac{\tilde{S}_T[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]}{\left(\frac{\partial W_q^T}{\partial X}, \frac{\partial W_q^T}{\partial Y}, \frac{\partial W_q^T}{\partial Z}\right)}$$

where $QH^*_\text{CR, T}(X)$ is the small $T$-equivariant quantum cohomology of $X$, and $\tau(q)$ is the closed mirror map.

Under this isomorphism, the $T$-equivariant Poincaré pairing on $QH^*_\text{CR, T}(X)|_{T=\tau(q), q=1}$ corresponds to the residue pairing on $\text{Jac}(W_q^T)$. More precisely, for generic $q, u_1, u_2, u_3$, $W_q^T$ is (locally) a holomorphic Morse function, i.e., the holomorphic 1-form $\omega_q := (\mathbb{C}^*)^3 \rightarrow T^*(\mathbb{C}^*)^3$ intersects the zero section transversally. Then there is a bijection between the zeros of $dW_q^T$ and the set $I_\Sigma$. Let $p_\sigma \in (\mathbb{C}^*)^3$ be the zero of $dW_q^T$ associated to $\sigma \in I_\Sigma$. Then

$$(f, g) := \frac{1}{(2\pi\sqrt{-1})^3} \int_{dW_q^T} f \frac{g dx \wedge dy \wedge dz}{\partial W_q^T \partial \sigma_x \partial \sigma_y \partial \sigma_z} = \sum_{\sigma \in I_\Sigma} \frac{f(p_\sigma)g(p_\sigma)}{\det(\text{Hess}(W_q^T)(p_\sigma))}$$

where

$$\text{Hess}(W_q^T) = \begin{pmatrix}
(W_q^T)_{xx} & (W_q^T)_{xy} & (W_q^T)_{xz} \\
(W_q^T)_{yx} & (W_q^T)_{yy} & (W_q^T)_{yz} \\
(W_q^T)_{zx} & (W_q^T)_{zy} & (W_q^T)_{zz}
\end{pmatrix}$$

To summarize, there is an isomorphisms of Frobenius algebras

$$QH^*_\text{CR, T}(X)|_{T=\tau(q), q=1} \cong \text{Jac}(W_q^T)$$

with

$$\Delta^\sigma(\tau)|_{T=\tau(q), q=1} = \det(\text{Hess}(W_q^T)(p_\sigma))$$
under the closed mirror map \((25)\). Setting \(u_3 = 0\), we have
\[
QH^\ast_{CR,\tau}(\mathcal{X})\big|_{\tau = \tau(q), q = 1} \cong \text{Jac}(W^\tau_q).
\]

4.5. Dimensional reduction of the equivariant Landau-Ginzburg model. The \(T'\)-equivariant mirror of \(\mathcal{X}\) is a Landau-Ginzburg model \(((\mathbb{C}^\ast)^3, W^\tau_q)\), where \(W^\tau_q : (\mathbb{C}^\ast)^3 \to \mathbb{C}\) is the \(T'\)-equivariant superpotential
\[
W^\tau_q = H(X, Y, q)Z - u_1 \log X - u_2 \log Y
\]
which is multi-valued. Here we view \(u_1\) and \(u_2\) as complex parameters. The differential
\[
dW^\tau_q = \frac{\partial W^\tau_q}{\partial X} dX + \frac{\partial W^\tau_q}{\partial Y} dY + \frac{\partial W^\tau_q}{\partial Z} dZ = Z dH + H dZ - u_1 \frac{dX}{X} - u_2 \frac{dX}{Y}
\]
is a well-defined holomorphic 1-form on \((\mathbb{C}^\ast)^3\).

We have
\[
\frac{\partial W^\tau_q}{\partial X}(X, Y, Z) = Z \frac{\partial H}{\partial X}(X, Y, q) - \frac{u_1}{X}
\]
\[
\frac{\partial W^\tau_q}{\partial Y}(X, Y, Z) = Z \frac{\partial H}{\partial Y}(X, Y, q) - \frac{u_2}{Y}
\]
\[
\frac{\partial W^\tau_q}{\partial Z}(X, Y, Z) = H(X, Y, q)
\]
Therefore,
\[
\frac{\partial W^\tau_q}{\partial X} = 0, \quad \frac{\partial W^\tau_q}{\partial Y} = 0, \quad \frac{\partial W^\tau_q}{\partial Z} = 0
\]
are equivalent to
\[
H(X, Y, q) = 0, \quad \frac{\partial H}{\partial X}(X, Y, q) = -\frac{1}{Z} \frac{\partial \hat{x}}{\partial X}, \quad \frac{\partial H}{\partial Y}(X, Y) = -\frac{1}{Z} \frac{\partial \hat{x}}{\partial Y},
\]
where \(\hat{x} = u_1 x + u_2 y\). Therefore, the critical points of \(W^\tau_q(X, Y, Z)\), which are zeros of the holomorphic differential \(dW^\tau_q\) on \((\mathbb{C}^\ast)^3\), can be identified with the critical points of \(\hat{x}\), which are zeros of the holomorphic differential
\[
d\hat{x} = -u_1 \frac{dX}{X} - u_2 \frac{dY}{Y} = -u_1 \left(\frac{dX}{X} + \frac{u_2}{u_1} \frac{dY}{Y}\right).
\]
on the mirror curve
\[
C_q = \{(X, Y) \in (\mathbb{C}^\ast)^2 : H(X, Y, q) = 0\}.
\]
For generic choice of \(q\), \(C_q\) is a smooth Riemann surface of genus \(g\) with \(n\) punctures. For fixed \(q\), the zeros of \(d\hat{x}\) depends only on \(f = u_2/u_1\). For generic \(f \in \mathbb{C}\), the section \(d\hat{x} : C_q \to T^*C_q\) intersects the zero section transversally at \(2g - 2 + n\) points. We conclude that

**Lemma 4.2.**

\[
\text{Jac}(W^\tau_q) \cong H_B,
\]
where

\[
\text{Jac}(W_q^{\gamma}) := \frac{\delta^q [X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]}{(\frac{\partial W_q^{\gamma}}{\partial X}, \frac{\partial W_q^{\gamma}}{\partial Y}, \frac{\partial W_q^{\gamma}}{\partial Z})},
\]

\[
H_B := \frac{\delta^q [X, X^{-1}, Y, Y^{-1}]}{(H(X, Y), u_2 X \frac{H}{\partial X}(X, Y) - u_1 Y \frac{\partial H}{\partial Y}(X, Y))}.
\]

Therefore, there is an equivalence of triangulated categories of B-branes in the two Landau-Ginzburg models:

\[\text{DSing}(\mathbb{C}^3, W_q^{\gamma}) \cong \text{DSing}(C_q, \hat{x})\]

where \text{DSing} is defined in [74].

It is straightforward to check that, \((X_0, Y_0)\) is a solution to

\[H(X, Y, q) = u_2 X \frac{\partial H}{\partial X}(X, Y, q) - u_1 Y \frac{\partial H}{\partial Y}(X, Y, q) = 0\]

if and only of \((X_0, Y_0, \frac{u_1}{X_0 \frac{\partial H}{\partial X}}(X_0, Y_0))\) is a solution to

\[\frac{\partial W_q^{\gamma}}{\partial X} = \frac{\partial W_q^{\gamma}}{\partial Y} = \frac{\partial W_q^{\gamma}}{\partial Z} = 0.\]

Moreover, \(\hat{y} := y/u_1\) is a local holomorphic coordinate near \((X_0, Y_0)\), and

Lemma 4.3.

\[\det \text{Hess}(W_q^{\gamma})(X_0, Y_0, \frac{u_1}{\frac{\partial H}{\partial x}}(X_0, Y_0, q)) = -\frac{d^2 \hat{x}}{dy^2}.\]

Proof.

\[W_q^{\gamma}(X, Y, Z) = H(X, Y, q)Z - u_1 \log X - u_2 \log Y\]

Then

\[\text{Hess}(W_q^{\gamma}) := \begin{pmatrix}
(W_q^{\gamma})_{xx} & (W_q^{\gamma})_{xy} & (W_q^{\gamma})_{xz} \\
(W_q^{\gamma})_{yx} & (W_q^{\gamma})_{yy} & (W_q^{\gamma})_{yz} \\
(W_q^{\gamma})_{zx} & (W_q^{\gamma})_{zy} & (W_q^{\gamma})_{zz}
\end{pmatrix}
= \begin{pmatrix}
H_{xx}Z & H_{xy}Z & -H_xZ \\
H_{yx}Z & H_{yy}Z & -H_yZ \\
-H_xZ & -H_yZ & HZ
\end{pmatrix}
\]

\[\det(\text{Hess}(W_q^{\gamma})) = Z^3 \det \begin{pmatrix}
H_{xx} & H_{xy} & -H_x \\
H_{yx} & H_{yy} & -H_y \\
-H_x & -H_y & H
\end{pmatrix}
= H(X, Y, q) = 0.
\]

\[H_x \frac{dx}{dy} + H_y = 0 \Rightarrow \frac{dx}{dy} = -\frac{H_y}{H_x}.
\]

\[H_{xx}(\frac{dx}{dy})^2 + H_{xy} \frac{dx}{dy} + H_x \frac{d^2 x}{dy^2} + H_{yx} \frac{dx}{dy} + H_{yy} = 0
\]

\[H_{xx} \frac{H_y^2}{H_x^2} - 2H_{xy} \frac{H_y}{H_x} + H_x \frac{d^2 x}{dy^2} + H_{yy} = 0
\]

\[\frac{d^2 x}{dy^2} = \frac{2H_{xy}H_xH_y - H_{xx}H_y^2 - H_{yy}H_x^2}{H_x^3} = \frac{1}{H_x^3} \det \begin{pmatrix}
H_{xx} & H_{xy} & -H_x \\
H_{yx} & H_{yy} & -H_y \\
-H_x & -H_y & 0
\end{pmatrix}
\]
\[
\frac{d^2 \hat{x}}{dy^2} = \left( \frac{u_1}{H_x} \right)^3 \det \begin{pmatrix}
H_{xx} & H_{xy} & -H_x \\
H_{yx} & H_{yy} & -H_y \\
-H_x & -H_y & 0
\end{pmatrix}
\]
Recall that \((X_0, Y_0) = (e^{-x_0}, e^{-y_0})\) satisfies
\[
H(X_0, Y_0) = 0, \quad -u_2 H_x(X_0, Y_0) + u_1 H_y(X_0, Y_0) = 0.
\]
So
\[
\det(\text{Hess}(W_q^\gamma))(X_0, Y_0, \frac{-u_1}{H_x(X_0, Y_0)}) = \left( \frac{-u_1}{H_x(X_0, Y_0)} \right)^3 \det \begin{pmatrix}
H_{xx} & H_{xy} & -H_x \\
H_{yx} & H_{yy} & -H_y \\
-H_x & -H_y & H
\end{pmatrix}(X_0, Y_0)
\]
\[
= -\frac{d^2 \hat{x}}{dy^2} |_{y = 0}
\]
\[
\Box
\]
Denote \(\hat{X} = e^{-\hat{x}}\) and \(\hat{Y} = e^{-\hat{y}}\). Define \(I_\Sigma := \{ (\sigma, \alpha) : \sigma \in \Sigma(3), \alpha \in G^*_\sigma \}\). Then there is a bijection between the zeros of \(dW_{q^\gamma}\) (and also \(d\hat{x}\)) and the set \(I_\Sigma\), and we denote the corresponding critical point by \(p_\sigma(\tau) = (X_\sigma(\tau), Y_\sigma(\tau), Z_\sigma(\tau))\) and \(p_\sigma(\tau) = (\hat{u}^\sigma(\tau), \hat{v}^\sigma(\tau))\). Around \(p_\sigma\) we have
\[
\dot{x} = \hat{u}^\sigma + c_1^2
\]
\[
\dot{y} = \hat{v}^\sigma + \sum_{d=1}^{\infty} h_d^\sigma c_d^2
\]
where
\[
h_1^\sigma = \sqrt{\frac{2}{d^2y^2(\hat{v}^\sigma)}}
\]
Let \(\Gamma_\sigma\) be the Lefschetz thimble of the superpotential \(\hat{x} : C_q \to \mathbb{C}\) such that \(\hat{x}(\Gamma_\sigma) = \hat{u}^\sigma + \mathbb{R}_{\geq 0}\). Then \(\{\Gamma_\sigma : \sigma \in I_\Sigma\}\) is a basis of the relative homology group \(H_1(C_q, \{\hat{x} \gg 0\})\).

**Lemma 4.4.** Given \(\sigma = (\sigma, \gamma) \in I_\Sigma\), there exists \(\tilde{\Gamma}_\sigma \in H_3((\mathbb{C}^*)^3, \{\partial(W_{q^\gamma} / z) \gg 0\}; \mathbb{Z})\) such that
\[
I_\sigma := \int_{\tilde{\Gamma}_\sigma} e^{-\frac{W_{q^\gamma}}{z}} \Omega = 2\pi \sqrt{-1} \int_{\Gamma_\sigma} e^{-\frac{\zeta}{z}} \Phi
\]
where \(\Phi = \hat{y} d\hat{x}\) and \(\Omega = \frac{dX}{\hat{X}} \frac{dY}{\hat{Y}} \frac{dZ}{\hat{Z}}\).

**Proof.** For any 3-cone \(\sigma\), we write
\[
W_{q^\gamma} = \tilde{H}_\sigma(\tilde{X}_\sigma, \tilde{Y}_\sigma)Z_\sigma - \tilde{u}_{\sigma, 1} \log \tilde{X}_\sigma - \tilde{u}_{\sigma, 2} \log \tilde{Y}_\sigma - \log \tilde{q}_{\sigma, 0},
\]
where
\[
\tilde{X}_\sigma = \tilde{X}_\sigma(X, Y, q), \quad \tilde{Y}_\sigma = \tilde{Y}_\sigma(X, Y, q), \quad \tilde{Z}_\sigma = \tilde{Z}_\sigma(X, Y, Z, q), \quad \tilde{q}_{\sigma, 0} = \tilde{q}_{\sigma, 0}(q).
\]
We denote
\[
\dot{x}_\sigma = \tilde{u}_{\sigma, 1} \dot{x}_{\sigma} + \tilde{u}_{\sigma, 2} \dot{y}_{\sigma}.
\]
Then
\[
\dot{x}_\sigma = \dot{x} + \tilde{u}_{\sigma, 0} \log \tilde{q}_{\sigma, 0}, \quad \tilde{u}_{\sigma, 1} = u_1, \quad \tilde{u}_{\sigma, 2} = u_2, \quad \log \tilde{q}_{\sigma, 0} = 0.
\]
In the decomposition $(\tilde{u}_\sigma, 1, \tilde{u}_{\sigma,2},0) = w_{\sigma,1}(r_{\sigma}, -s_{\sigma}, 1) + w_{\sigma,2}(0, m_{\sigma}, 1) + w_{\sigma,3}(0, 0, 1)$, assume that $w_{\sigma,1} > 0$, $w_{\sigma,2} > 0$, $w_{\sigma,3} < 0$. Notice that one can always rearrange the order of the three 1-cones spanning $\sigma$ such that $w_{\sigma,1} > w_{\sigma,2} > w_{\sigma,3}$. We also have $w_{\sigma,1} + w_{\sigma,2} + w_{\sigma,3} = 0$. The case when $w_{\sigma,1} > 0$, $w_{\sigma,2} < 0$, $w_{\sigma,3} < 0$ is similar. We further assume that $u_1, u_2$ are chosen generically such that for any 3-cone $\sigma$ and $j = 1, 2, 3$, there is no $w_{\sigma,j} = 0$.

At $q = 0$, the mirror curve equation $\tilde{H}_\sigma = 0$ becomes

$$\tilde{X}_\sigma r_{\sigma} + \tilde{Y}_\sigma = 1 = 0,$$

which is the equation of the mirror curve $C_q$ of the affine toric Calabi-Yau 3-orbifold $X_\sigma$ defined by the 3-cone $\sigma$. There are $r_{\sigma}, m_{\sigma}$ critical points of the function $\tilde{u}_\sigma, 1, \tilde{u}_{\sigma,2} \tilde{g}_\sigma = \hat{x} - \log \tilde{q}_\sigma, 0$ on $C_q$ which can be holomorphically extended to $C_\sigma$ when $q = 0$—they are all critical points on $C_\sigma$ (see Section 5.4). By direct computation (see e.g. [?, Section 6.5]), for $\gamma \in G^*_\sigma$, each critical point $(\tilde{X}_{\sigma,\gamma}, \tilde{Y}_{\sigma,\gamma})$ at $q = 0$ satisfies

$$-1 < \tilde{X}_{\tau_{\sigma,\gamma}}(0)^{r_{\sigma}} \tilde{Y}_{\tau_{\sigma,\gamma}}(0)^{r_{\sigma}} = -w_{\sigma,1}/w_{\sigma,1} + w_{\sigma,2},$$

$$-1 < \tilde{Y}_{\tau_{\sigma,\gamma}}(0)^{m_{\sigma}} = w_{\sigma,2}/w_{\sigma,1} + w_{\sigma,2} < 0.$$

For each $\sigma = (\tau, \gamma)$, we define a relative cycle

$$\tilde{\Gamma}_{\sigma, q} = \{ \tilde{X}_\sigma \in \mathbb{R}_+ \tilde{X}_\sigma(q), \tilde{Y}_\sigma \in \mathbb{R}_+ \tilde{Y}_\sigma(q) \}.$$

When $q = 0$, we have a disjoint union of connected components

$$\{ \tilde{X}_{\tau_{\sigma,\gamma}} \tilde{Y}_{\tau_{\sigma,\gamma}} \in \mathbb{R}_-, \tilde{Y}_{\tau_{\sigma,\gamma}} \in \mathbb{R}_- \} = \bigsqcup_{\gamma \in G^*_\sigma} \tilde{\Gamma}_{\sigma, q, 0},$$

where $\tilde{\Gamma}_{\sigma, q, 0}$ is the connected component passing through $(\tilde{X}_{\tau_{\sigma,\gamma}}(0), \tilde{Y}_{\tau_{\sigma,\gamma}}(0))$. We define $\tilde{\Gamma}_{\sigma} = \tilde{\Gamma}_{\sigma, q} \times C$, where $C = \{ \tilde{Z}_\sigma \in -1 + \sqrt{-1} \mathbb{R} \}$. As in [?, Section 7.2], we can compute

$$e^{-\log \tilde{q}_\sigma} I_\sigma = \int_{\tilde{\Gamma}_{\sigma, q}} e^{-\log \tilde{q}_\sigma} \cdot e^{\frac{w_{\sigma,1}}{2} \Omega} = \frac{2\pi \sqrt{-1}}{|G_\sigma|} \sum_{\tilde{h} = (r_1, \ldots, r_p), r \in \mathbb{Z}^{G_\sigma}} e^{\frac{1}{2} c_3(\tilde{h})} \chi(\tilde{h}) \prod_{a=1}^p \left( -\tilde{q}_{\sigma, a} r_a \right)^{r_a} \left( \Gamma\left( -\frac{w_{\sigma,3}}{2} - c_3(\tilde{h}) + 1 \right) \right).$$

We define

$$L_\sigma := \{ \tilde{q}_\sigma = (\tilde{q}_{\sigma,1}, \ldots, \tilde{q}_{\sigma,p}) \in \mathbb{C}^p : \tilde{q}_\sigma, \tilde{X}_\sigma(0)^{m_{\sigma}} \tilde{Y}_\sigma(0)^{n_{\sigma}} \in \mathbb{R} \}$$

which is a totally real linear subspace of $\mathbb{C}^p$. When $|q|$ is small (then $|\tilde{q}_\sigma|$ is small), we restrict $\tilde{q}_\sigma$ to $L_\sigma \cong \mathbb{R}^p$, then $\tilde{\Gamma}_{\sigma, q} = \tilde{\Gamma}_{\sigma, 0}$ for $\tilde{q}_\sigma$ in a sufficiently small neighborhood of $0 \in L_\sigma$. Set

$$v^+, v^- \in \mathbb{C}, \quad \tilde{\Gamma}_{\sigma, q} = \tilde{\Gamma}_{\sigma, q} \times \{ v^+ = v^- \}, \quad \tilde{\Gamma}_{\sigma, q} = \tilde{\Gamma}_{\sigma, q} \times \{ v^+ = v^- \},$$

$$\Omega' = \frac{d \tilde{X}_\sigma d \tilde{Y}_\sigma dv^-}{X Y v^-} = \frac{d \tilde{X}_\sigma d \tilde{Y}_\sigma dv^-}{X Y v^-}.$$
Let $\tilde{q}_\sigma \in L_\sigma$. The dimensional reduction is [?, Section 7.2]

$$e^{-\log \pi \sigma_0} I_\sigma = - \int_{\Gamma_\sigma} e^{-\frac{z}{\sigma_0}} \Omega' = 2\pi \sqrt{-1} \int_{\Gamma_\sigma} \frac{\omega}{\pi \sigma_0} \Omega' e^{-\frac{z}{\sigma_0}} \hat{y} \hat{d} \hat{x}.$$ 

The cycle $\Gamma_\sigma \subset \Sigma_\sigma$ satisfies $\tilde{x}_\sigma(\Gamma_\sigma) = [\tilde{x}_\sigma, \gamma, +\infty)$. So when $\tilde{q}_\sigma \in I_\sigma$,

$$I_\sigma = 2\pi \sqrt{-1} \int_{\Gamma_\sigma} e^{-\frac{z}{\sigma_0}} \hat{y} \hat{d} \hat{x}.$$ 

Since both sides of the above equation are analytic in $\tilde{q}_\sigma$, it holds for all $\tilde{q}_\sigma$ in a small neighborhood of $0 \in \mathbb{C}^g$. □

5. GEOMETRY OF THE MIRROR CURVE

5.1. Riemann Surfaces. In this subsection, we recall some classical results on Riemann surfaces. The main reference of this subsection is [44].

Let $C$ be a non-singular complex projective curve, which can also be viewed as a compact Riemann surface. Let $g \in \mathbb{Z}_{\geq 0}$ be the genus of $C$. Let $\cap$ denote the intersection pairing $H_1(C; \mathbb{Z}) \times H_1(C; \mathbb{Z}) \rightarrow \mathbb{Z}$. We choose a symplectic basis $\{A_i, B_j : i = 1, \ldots, g\}$ of $(H_1(C; \mathbb{Z}), \cap)$:

$$A_i \cap A_j = B_i \cap B_j = 0, \quad A_i \cap B_j = -B_j \cap A_i = \delta_{ij}, \quad i, j \in \{1, \ldots, g\}.$$ 

Recall that on a Riemann surface, a differential of the first kind is a holomorphic 1-form; a differential of the second kind on $C$ is a meromorphic 1-form whose residue at any of its poles is zero; a differential of the third kind on $C$ is a meromorphic 1-form with only simple poles. If $\omega$ is a differential of the first or second kind then $\int_A \omega$ is well-defined for $A \in H_1(C; \mathbb{Z})$ (or more generally, $A \in H_1(C; \mathbb{C})$).

The fundamental differential of the second kind on $C$ normalized by $A_1, \ldots, A_g$ is a bilinear symmetric meromorphic differential $B(p_1, p_2)$ characterized by

- $B(p_1, p_2)$ is holomorphic everywhere except for a double pole along the diagonal $p_1 = p_2$, where, if $z_1, z_2$ are local coordinates on $C \times C$ near $(p, p)$ then
  $$B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + f(z_1, z_2)\right)dz_1dz_2.$$ 
  
  where $f(z_1, z_2)$ is holomorphic and $f(z_1, z_2) = f(z_2, z_1)$.

- $\int_{p_i \in A_i} B(p_1, p_2) = 0$, $i = 1, \ldots, g$.

Let $\omega_i \in H^0(C, \omega_C)$ be the unique holomorphic 1-form on $C$ such that

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_i} \omega_i = \delta_{ij}.$$ 

Then $\{\omega_1, \ldots, \omega_g\}$ is a basis of $H^0(C, \omega_C) \cong \mathbb{C}^g$, the space of holomorphic 1-form on $C$, and

$$\int_{p' \in B_i} B(p, p') = \omega_i(p).$$ 

More generally, for any $\gamma \in H_1(C; \mathbb{Z})$,

$$\omega_\gamma(p) := \int_{p' \in \gamma} B(p, p')$$ 

is a holomorphic 1-form on $C$ and

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_j} \omega_\gamma = A_j \cap \gamma.$$
We may extend the intersection pairing to a skew-symmetry \( \mathbb{C} \)-bilinear map \( \cap : H_1(C; \mathbb{C}) \times H_1(C; \mathbb{C}) \to \mathbb{C} \). The above discussion remains valid if we choose a symplectic basis \( \{ A_i, B_i : i = 1, \ldots, g \} \) of \( (H_1(C; \mathbb{C}), \cap) \) instead of \( (H_1(C; \mathbb{Z}), \cap) \).

Let \( \gamma \) be a path connecting two distinct points \( p_1, p_2 \in C \), oriented such that \( \partial \gamma = p_1 - p_2 \). Then

\[
\omega_\gamma(p) := \int_{p \in \gamma} B(p, p')
\]

is a meromorphic 1-form on \( C \) which is holomorphic on \( C \setminus \{ p_1, p_2 \} \) and has simple poles at \( p_1, p_2 \). The residues of \( \omega_\gamma \) at \( p_1, p_2 \) are

\[
\text{Res}_{p \to p_1} \omega_\gamma(p) = 1, \quad \text{Res}_{p \to p_2} \omega_\gamma(p) = -1.
\]

5.2. The Liouville form. Let

\[
\hat{x} = u_1 x + u_2 y, \quad \hat{y} = \frac{y}{u_1}, \quad f = \frac{u_2}{u_1}
\]

as before. Define

\[
\lambda := \hat{y} d\hat{x} = y d(x + f y), \quad \Phi := \lambda|_{C_q}.
\]

Then \( \Phi \) is a multi-valued holomorphic 1-form on \( C_q \). Recall that there is a regular covering map \( p : \overline{C}_q \to C_q \) with fiber \( \mathbb{Z}^2 \) which is the restriction of \( \mathbb{C}^2 \to (\mathbb{C}^*)^2 \) given by \( (x, y) \mapsto (e^{-x}, e^{-y}) \). Then \( p^* \Phi \) is a holomorphic 1-form on \( \overline{C}_q \).

5.3. Differentials of the first kind and the third kind. For any integers \( m, n \),

\[
\omega_{m,n} := \text{Res}_{H(X,Y) = 0} \frac{X^m Y^n}{H(X,Y,q)} \frac{dX}{X} \wedge \frac{dY}{Y} = -X^m Y^n \frac{d\hat{x}}{\partial y} \frac{H(X,Y,q)}{\partial H(X,Y,q)}
\]

is a holomorphic 1-form on the mirror curve \( C_q \) and a meromorphic 1-form on the compactified mirror curve \( \overline{C}_q \).

By results in \( \text{[8]} \), \( \omega_{m,n} \) is holomorphic on \( \overline{C}_q \) iff \( (m, n) \in \text{Int}(P) \cap N' \). Recall that

\[
p = |P \cap N'| - 3, \quad g = |\text{Int}(P) \cap N'|, \quad n = |\partial P \cap N'|.
\]

For generic \( q \), \( \overline{C}_q \) is a compact Riemann surface of genus \( g \), intersecting the anti-canonical divisor \( \partial S_P := S_P \setminus (\mathbb{C}^*)^2 \) transversally at \( n \) points \( \tilde{p}_1, \ldots, \tilde{p}_n \), so \( C_q \) a Riemann surface of genus \( g \) with \( n \) punctures. The space of holomorphic 1-forms on \( \overline{C}_q \), \( H^0(\overline{C}_q, \omega_{\overline{C}_q}) \), is \( g \)-dimensional, where \( g \) can be zero. A basis of \( H^0(\overline{C}_q, \omega_{\overline{C}_q}) \) is given by

\[
\{ \omega_{m,n} : (m, n) \in \text{Int}(P) \cap N' \}.
\]

Let \( D_q^\infty := \overline{C}_q \cap \partial S_P = \tilde{p}_1 + \cdots + \tilde{p}_n \). The space of meromorphic 1-forms on \( \overline{C}_q \) with at most simple poles at \( \tilde{p}_1, \ldots, \tilde{p}_n \), \( H^0(\overline{C}_q, \omega_{\overline{C}_q}(D_q^\infty)) \), is \( g + n - 1 \) dimensional. It is spanned by the \( g + n \) 1-forms

\[
\{ \omega_{m,n} : (m, n) \in P \cap N' \}
\]

with a single relation

\[
\sum_{(m,n) \in P \cap N'} a_{m,n}(q) \omega_{m,n} = 0.
\]
5.4. Toric degeneration. The main reference of this subsection is [75, Section 3].

For a generic choice of \( \eta \in \mathbb{L}_Q^{\vee} \),

\[
\Theta_\eta = \bigcap_{i \in A_n} \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i \subset \mathbb{L}_Q^{\vee}
\]

is a top dimensional convex cone in \( \mathbb{L}_Q^{\vee} \cong \mathbb{Q}^p \). Given a semi-projective toric Calabi-Yau 3-orbifold \( \mathcal{X} \), there is always a choice of such \( \eta \) such that \( \Theta_\eta \) is the Kähler cone \( C(\Sigma^{ext}) \), and the construction in Section 2.4 also defines \( \mathcal{X} \). Any \( \eta \) in this cone give rise to the same \( \Theta_\eta \). The cone \( \Theta_\eta \) together with its faces is a fan in \( \mathbb{L}_Q^{\vee} \) (still denoted by \( \Theta_\eta \)), and determines a \( p \)-dimensional affine toric variety \( X_{\Theta_\eta} \).

Consider the exact sequence

\[
0 \to M' \overset{\phi'}{\to} \tilde{M}' \overset{\psi'}{\to} \mathbb{L}^{\vee} \to 0
\]

where \( M' = M/\langle e_3 \rangle \) and \( \tilde{M}' = \tilde{M}/\langle \psi'(e_3) \rangle \). Let \( D_i^{\psi'} \) be the image of \( D_i^\psi \) when passing to \( \tilde{M}' \).

For any proper subset \( I \subset \{ 1, \ldots, r \} \), define

\[
\bar{\Theta}_I = \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i^{\psi'}, \quad \bar{\Theta}_{I, \eta} = (\psi')^{-1}(\Theta_\eta) \cap \bar{\Theta}_I.
\]

Define a fan

\[
\bar{\Theta}_\eta = \{ \bar{\Theta}_{I, \eta} | I \notin \{ 1, \ldots, 3 + p \} \}.
\]

This fan determines a toric variety \( X_{\bar{\Theta}_\eta} \). There is a fan morphism \( \rho' : \bar{\Theta}_\eta \to \Theta_\eta \), which induces a flat family of toric surfaces \( \rho : X_{\bar{\Theta}_\eta} \to X_{\Theta_\eta} \).

Let \( \Theta_H \subset \mathbb{L}_Q^{\vee} \) be the cone spanned by \( H_1, \ldots, H_p \). Let \( \mathbb{L}_H := \mathbb{Q}^{p+1}_a \mathbb{Z}H_a \) and let \( \mathbb{L}_H \) be the dual lattice. Then \( \mathbb{L}_H^{\vee} \) is a sublattice of \( \mathbb{L}^{\vee} \) of finite index, and \( \mathbb{L} \) is a sublattice of \( \mathbb{L}_H^{\vee} \) of finite index. Let \( \Theta_H \subset \mathbb{L}_Q^{\vee} \) be the top dimensional cone spanned by the vectors \( H_1, \ldots, H_p \) chosen Section 3.9.

Let \( \Theta_\eta^{\vee} \) and \( \Theta_H^{\vee} \) be the dual cones of \( \Theta_\eta \) and \( \Theta_H \), respectively. We have inclusions

\[
\Theta_\eta \subset \Theta_H \subset \Theta_\eta^{\vee} \subset \Theta_H^{\vee} \subset \mathbb{L}_Q^{\vee}.
\]

Note that \( \Theta_\eta \cap \mathbb{L} \) is a subset of \( \Theta_H \cap \mathbb{L}_H \), so we have an injective ring homomorphism

\[
\mathbb{C}[\Theta_\eta \cap \mathbb{L}] \to \mathbb{C}[\Theta_H \cap \mathbb{L}_H] = \mathbb{C}[q_1, \ldots, q_p]
\]

where \( q_1, \ldots, q_p \) are the variables in Section 3.9. Taking the spectrum, we obtain a morphism

\[
\mathbb{A}^p = \text{Spec} \left( \mathbb{C}[q_1, \ldots, q_p] \right) \longrightarrow X_{\Theta_\eta} = \text{Spec} \left( \mathbb{C}[\Theta_\eta^\vee \cap \mathbb{L}] \right).
\]

and a cartesian diagram

\[
\begin{array}{ccc}
\mathbb{X} & \overset{\nu}{\longrightarrow} & X_{\Theta_\eta} \\
\phi \downarrow & & \rho \downarrow \\
\mathbb{A}^p & \overset{\nu}{\longrightarrow} & X_{\Theta_\eta}
\end{array}
\]

where \( \phi : \mathbb{X} \to \mathbb{A}^p \) is a flat family of toric surfaces.

Given \( \sigma \in \Sigma(1) \cup \Sigma(2) \cup \Sigma(3) \), let \( P_{\sigma} \) be the convex hull of \( \{(m_i, n_i) : b_i \in \sigma \} \). This gives a triangulation \( T \) of \( P \) with vertices \( \{ P_\sigma : \sigma \in \Sigma(1) \} \), edges \( \{ P_\sigma : \sigma \in \Sigma(2) \} \), and faces \( \{ P_\sigma : \sigma \in \Sigma(3) \} \).
We choose a Kähler class $[\omega(\eta)] \in H^2(X_\Sigma; \mathbb{Z})$ associated to a lattice point $\eta \in L^\vee$; $[\omega(\eta)]$ is the first Chern class of some ample line bundle over $X_\Sigma$. Then it determines a toric graph $\Gamma \in \mathbb{R}^2$ up to translation by an element in $M' \cong \mathbb{Z}^2$ (see Section 2.2.3). The toric graph gives a polyhedral decomposition of $M_\mathbb{Q}$ in the sense of [75] Section 3. It is a covering $\mathcal{P}$ of $M_\mathbb{Q}$ by strongly convex lattice polyhedra. The asymptotic fan of $\mathcal{P}$ is defined to be
\[ \Sigma_\mathcal{P} := \{ \lim_{\alpha \to 0} a \Xi \subset M'_\mathbb{Q} : \Xi \in \mathcal{P} \}. \]

The fan $\Sigma_\mathcal{P} = \widehat{\Theta}_\eta \cap \rho'^{-1}(0)$ defines the toric surface $\mathbb{S}_\mathcal{P}$, i.e., $X_{\Sigma_\mathcal{P}} = \mathbb{S}_\mathcal{P}$. For each $\Xi \in \mathcal{P}$, let $C(\Xi) \subset M'_\mathbb{Q} \times \mathbb{Q}_{\geq 0}$ be the closure of the cone over $\Xi \times \{ 1 \}$ in $M'_\mathbb{Q} \times \mathbb{Q}$. Then
\[ \Sigma_\mathcal{P} := \{ \sigma \text{ is a face of } C(\Xi) : \Xi \in \mathcal{P} \} = \widehat{\Theta}_\eta \cap \rho'^{-1}(\mathbb{Q}_{\geq 0}) \]
is a fan in $M'_\mathbb{Q} \times \mathbb{Q}$ with support $|\Sigma_\mathcal{P}| = M'_\mathbb{Q} \times \mathbb{Q}_{\geq 0}$. The projection $\pi' : M'_\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ to the second factor defines a map from the fan $\Sigma_\mathcal{P}$ to the fan $\{ 0, \mathbb{Q}_{\geq 0} \}$. This map of fans determines a flat toric morphism $\pi : X_{\Sigma_\mathcal{P}} \to \mathbb{A}^1$, where $X_{\Sigma_\mathcal{P}}$ is the toric 3-fold defined by the fan $\Sigma_\mathcal{P}$. Let $t$ be a closed point in $\mathbb{A}^1$, and let $X_t$ denote the fiber of $\pi$ over $t$. Then $X_t \cong \mathbb{S}_\mathcal{P}$ for $t \neq 0$. As shown in [75], when $t = 0$, we have a union of irreducible components
\[ X_0 = \bigcup_{\sigma \in \Sigma(3)} \mathbb{S}_{\mathcal{P}_\sigma}. \]
If $\tau \in \Sigma(2)$, $\sigma \in \Sigma(3)$, and $\tau \subset \sigma$, then $P_\tau$ is an edge of the triangle $P_\sigma$, and it corresponds to a torus invariant divisor $D_\tau \subset \mathbb{S}_{\mathcal{P}_\sigma}$.

The polytope $\text{Hull}(\bar{b}_1, \ldots, \bar{b}_{p+3}) \subset \bar{N}$ lies on the hyperplane $\langle \phi^\vee(e_3^\vee), \bullet \rangle = 1$. It determines a polytope on $\bar{N}' = \{ \langle \phi^\vee(e_3^\vee), \bullet \rangle = 0 \}$ up to a translation. The associated line bundle $\mathcal{L}$ on $X_{\Sigma_\mathcal{P}}$ has sections $s_i$, $i = 1, \ldots, p+3$ associated to each integer point in this polytope. Define
\[ s = \sum_{i=1}^{p+3} s_i, \quad \mathcal{C} = s^{-1}(0). \]

The divisor $\mathcal{C} \subset X_{\Sigma_\mathcal{P}}$ forms a flat family of curves of arithmetic genus $g$ over $X_{\Sigma_\mathcal{P}}$. Let $\mathcal{C}' := \delta^{-1}(\mathcal{C}) \subset \mathcal{C}$ be the pullback divisor under the morphism $\delta : \mathcal{C} \to X_{\Sigma_\mathcal{P}}$. Then $\mathcal{C}' \to \mathbb{A}_\mathbb{P}$ is a flat family of curves of arithmetic genus $g$ over $\mathbb{A}_\mathbb{P}$.

For $q \neq 0$, $\mathcal{C}_q \subset X_{\mathbb{P}} = \rho^{-1}(q)$ can be identified with the zero locus of
\[ H(X, Y) = \sum_{(m,n) \in \mathbb{P} \cap N} a_{m,n}(q(q)) X^m Y^n. \]
Thus $\mathcal{C}_q = \overline{C}(q(q))$ defined in Section 4.1. When $q = 0$, we have a union of irreducible components
\[ \mathcal{C}_0 = \bigcup_{\sigma \in \Sigma(3)} \overline{C}_\sigma, \]
where $\overline{C}_\sigma \subset \mathbb{S}_{\mathcal{P}_\sigma}$ is the zero locus of
\[ H_\sigma(X_\sigma, Y_\sigma) = \sum_{(m,n) \in \mathbb{P}_\sigma \cap N'} X_\sigma^m Y_\sigma^n. \]
where $X_\sigma, Y_\sigma$ are affine coordinates in $\overline{C}_\sigma$. Let $C_\sigma = \overline{C}_\sigma \cap (\mathbb{S}_{\mathcal{P}_\sigma} \setminus \partial \mathbb{S}_{\mathcal{P}_\sigma})$, where $\partial \mathbb{S}_{\mathcal{P}_\sigma} = \bigcup_{\tau \subset \sigma, \tau \in \Sigma(2)} D_\tau$. Given $\tau \in \Sigma(2)$, let $n_\tau := |G_\tau| = |P_\tau \cap N'| - 1$. Then $\mathcal{C}_\sigma \cap D_\tau$ consists of $n_\tau$ points $\bar{p}_1, \ldots, \bar{p}_{n_\tau}$. When $q = 0$, the group of $n_\tau$-th roots of unity,
Let \( \mu_{n_\tau} \cong \mathbb{Z}/n_\tau \mathbb{Z} \), acts freely and transitively on \( \bar{p}_1^\tau, \ldots, \bar{p}_{n_\tau}^\tau \); we label these points such that \( \mu_{n_\tau} \) acts by cyclic permutation.

We have

\[
\bar{C}_\sigma = C_\sigma \cup \bigcup_{\tau \in \Sigma(2), \tau \subset \sigma} \{ \bar{p}_1^\tau, \ldots, \bar{p}_{n_\tau}^\tau \}.
\]

Let \( \sigma_1, \sigma_2 \in \Sigma(3) \) be two distinct 3-cones in \( \Sigma \). The intersection of \( \bar{C}_{\sigma_1} \) and \( \bar{C}_{\sigma_2} \) is non-empty if and only if \( \sigma_1 \cap \sigma_2 = \tau \) for some 2-dimensional cone \( \tau \in \Sigma(2) \). In this case, \( \bar{C}_{\sigma_1} \) and \( \bar{C}_{\sigma_2} \) intersect at \( n_\tau \) nodes \( \bar{p}_1^\tau, \ldots, \bar{p}_{n_\tau}^\tau \).

The genus of \( \bar{C}_\sigma \) is

\[
g_\sigma = |\text{Int}(P_\sigma) \cap N'|.
\]

Let

\[
n_\sigma = |\partial P_\sigma \cap N'| = \sum_{\tau \in \Sigma(2), \tau \subset \sigma} n_\tau.
\]

Then \( C_\sigma \) is a genus \( g_\sigma \) Riemann surface with \( n_\sigma \) punctures

\[
\{ \bar{p}_i^\tau : \tau \in \Sigma(2), \tau \subset \sigma, 1 \leq i \leq n_\tau \}.
\]

Let \( \Gamma_{\varepsilon_0} \) be the dual graph of the nodal curve \( C_0 \). Then

\[
g = \sum_{\sigma \in \Sigma(3)} g_\sigma + b_1(\Gamma_{\varepsilon_0}).
\]

We let

\[
g' := \sum_{\sigma \in \Sigma(3)} g_\sigma, \quad g'' := b_1(\Gamma_{\varepsilon_0}).
\]

Define

\[
U = \{ (q_1, \ldots, q_{\bar{\epsilon}}) \in \mathbb{C}^p \times \mathbb{C}^{p'-p'} : C_2 \text{ is smooth and intersects } \partial S_P \text{ transversally at } n \text{ points} \}.
\]

Then \( U \) is a dense open subset of \( \mathbb{C}^p \). We define the following cycles.

- At \( q = 0 \), for any \( \sigma \in \Sigma(3) \), we choose a symplectic basis \( A_1^\sigma, B_1^\sigma, \ldots, A_{g_\sigma}^\sigma, B_{g_\sigma}^\sigma \) of \( H_1(\bar{C}_\sigma; \mathbb{C}) \), as in \([2, \text{ Section } 6]\). These cycles can be extended to \( H_1(C_\sigma; \mathbb{C}) \) when \( q \in U_\varepsilon = \{ q \in U_\varepsilon : |q| < \epsilon \} \) when \( \epsilon \) is small.
- For any \( \tau \in \Sigma_c(2) := \{ \tau \in \Sigma(2) : \ell_\tau = \mathbb{P}^1 \} \), let \( \delta_1^\tau, \ldots, \delta_{n_\tau}^\tau \in H_1(C_\sigma; \mathbb{C}) \) be the vanishing cycles associated to the nodes \( \bar{p}_1^\tau, \ldots, \bar{p}_{n_\tau}^\tau \); each \( \delta_i^\tau \) is defined up to a sign. The vanishing cycles are well defined when \( q \in U \) for small \( \epsilon \).
- For any \( \tau \in \Sigma(2) \setminus \Sigma_c(2) \), let \( \delta_j^\tau \in H_1(C_\sigma; \mathbb{C}) \) be the class of a small circle around the puncture \( \bar{p}_j^\tau \). We have

\[
n = \sum_{\tau \in \Sigma_c(2) \setminus \Sigma_c(2)} n_\tau.
\]

We fix \( \epsilon \) as discussed above, and have the following facts on \( U_\varepsilon \).

1. The cycles \( \{ A_i^\sigma : \sigma \in \Sigma(3), 1 \leq i \leq g_\sigma \} \) are linearly independent and span a \( g' \)-dimensional subspace \( V' \) of \( H_1(C_2; \mathbb{C}) \cong \mathbb{C}^{2g_2} \).
2. The vanishing cycles \( \{ \delta_j^\sigma : \tau \in \Sigma_c(2), 1 \leq j \leq n_\tau \} \) are linearly dependent and span a \( g'' \)-dimensional subspace \( V'' \) of \( H_1(C_\sigma; \mathbb{C}) \cong \mathbb{C}^{2g_\sigma} \).
3. We have \( V' \cap V'' = \{ 0 \} \), so that \( V := V' \oplus V'' \cong \mathbb{C}^g \). The bilinear map

\[
V \times H^0(C_2, \omega_{C_2}) \rightarrow \mathbb{C} \text{ given by } (\gamma, \theta) \mapsto \int_\gamma \theta \text{ is a perfect pairing.}
\]
By parallel transport with respect to the Gauss-Manin connection, a rank $n$ complex vector bundle over $\Sigma_c(2)$ extends to a rank $n-1$ flat subbundle of $\mathcal{H}$ over $U_c$. We have a short exact sequence of flat complex vector bundles over $U_c$:

$$0 \to \mathcal{W} \to \mathcal{H} \to \mathcal{H}^{\prime} \to 0.$$ 

5.5. **B-model flat coordinates.** Given $\tau \in \Sigma(2)$ and $i \in \{1, \ldots, n_\tau\}$, define

$$A^\tau_i := \frac{(-1)^{\tau_i}}{n_\tau} \sum_{j=1}^{n_\tau} \left( e^{2\pi \sqrt{-1} \frac{\tau_j}{n_\tau}} \delta_j^\tau \right).$$

In particular,

$$A^\tau_{n_\tau} = \frac{(-1)^{\tau_{n_\tau}}}{n_\tau} (\delta_1^\tau + \cdots + \delta_{n_\tau}^\tau).$$

If $\tau \in \Sigma_c(2)$, then $A^\tau_i$ are flat sections of $\mathcal{V}''$. Otherwise they are flat sections of $\mathcal{W}$. We assume $q \in U_c$. If $a \in \{p'+1, \ldots, p\}$ then there are two cases:

1. There exists a unique $\sigma \in \Sigma(3)$ such that $(m_a, n_a) \in \text{Int}(P_\sigma)$, and there exists a unique $A^\sigma_i \in H_1(C_{q}(\mathbb{C})$), where $1 \leq i \leq g_\sigma$, such that

$$\frac{1}{2\pi \sqrt{-1}} \int_{A^\sigma_i} (\nabla \frac{\partial}{\partial q^j} \Phi)|_{q=0} = \delta_{aj}, \quad \forall b_j \in \text{Int}(P).$$

Then we have a lift $A_a \in K_1(C_{q}(\mathbb{C})$ such that $J_\ast(A_a) = A^\sigma_i$, and

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_a} (\nabla \frac{\partial}{\partial q^j} \Phi)|_{q=0} = \delta_{ab},$$

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_a} \Phi = \tau_a(q) + \text{const} = \text{const} + q_a + O(|q|^2).$$

2. There exists a unique $\tau \in \Sigma_c(2)$ such that $(m_a, n_a) \in \text{Int}(P_\tau)$, and there exists a unique $A^\tau_i \in H_1(C_{q}(\mathbb{C})$, where $1 \leq i \leq n_\tau - 1$, such that

$$\frac{1}{2\pi \sqrt{-1}} \int_{A^\tau_i} (\nabla \frac{\partial}{\partial q^j} \Phi)|_{q=0} = \delta_{aj}, \quad \forall b_j \in \text{Int}(P).$$

Then we have a lift $A_a \in K_1(C_{q}(\mathbb{C})$ such that $J_\ast(A_a) = A^\tau_i$, and

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_a} (\nabla \frac{\partial}{\partial q^j} \Phi)|_{q=0} = \delta_{ab},$$

$$\frac{1}{2\pi \sqrt{-1}} \int_{A_a} \Phi = \tau_a(q) + \text{const} = \text{const} + q_a + O(|q|^2).$$
There exists a unique \( \tau \in \Sigma(2) \setminus \Sigma_s(2) \) such that \( (m_a, n_a) \in \text{Int}(P_s) \), and there exists a unique \( A_\tau^\ast \in H_1(C_{q(\tau)}; \mathbb{C}) \), where \( 1 \leq i \leq n_\tau - 1 \), such that

\[
\frac{1}{2\pi\sqrt{-1}} \int_{A_\tau^\ast} (\nabla \cdot \Phi) |_{\partial = 0} = \delta_{ab}.
\]

Then we have an \( A_a \in K_1(C_{q(\tau)}; \mathbb{C}) \) such that \( A_a - A_\tau^\ast \in J_1(C_{q(\tau)}; \mathbb{C}) \), and

\[
\frac{1}{2\pi\sqrt{-1}} \int_{A_a} \Phi = \tau_a(\mathcal{Q}) + \text{const} = \text{const} + q_a + O(|q|^2).
\]

Suppose that \( \tau \in \Sigma(2) \). Then it is spanned by \( b_i, b_j \), where \( i, j \in \{1, \ldots, 3 + p'\} \). Let \( q_0 = q_{-1} = q_{-2} = 1 \). Then there exists an \( A_\tau'' \in K_1(C_{q(\tau)}; \mathbb{C}) \) such that \( A_\tau'' - A_\tau^\ast \in J_1(C_{q(\tau)}; \mathbb{C}) \)

\[
\int_{A_\tau''} \Phi = \log q_{j-3} - \log q_{i-3} + A_j(q) - A_i(q) + \text{const}.
\]

If \( q_a = \prod_{i=1}^{p'} q_{ab} \), then \( A_a = \sum_{i=1}^{p'} a_{ab} A_b(q) \). For each \( a \in \{1, \ldots, p'\} \), there exists a linear combination \( A_a \) of \( \{A''_\tau \in \Sigma(2)\} \) such that

\[
\int_{\Sigma(2)} A_a \Phi = \tau_a(q) + \text{const}.
\]

Define the \( B \)-model flat coordinate by

\[
\tilde{\tau}_a(q) := \frac{1}{2\pi\sqrt{-1}} \int_{A_a} \Phi, \quad a = 1, \ldots, p = g + n - 3.
\]

Then

\[
\tilde{\tau}_a(q) = \tau_a(q) + c_a
\]

for some constant \( c_a \in \mathbb{C} \). Therefore,

\[
\frac{\partial}{\partial \tau_a} = \frac{\partial}{\partial \tilde{\tau}_a}.
\]

By Lefschetz duality, there is a perfect pairing

\[
\cap : H_1(C_{q}; \mathbb{C}) \times H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C}) \to \mathbb{C},
\]

where \( \dim_{\mathbb{C}} H_1(C_{q}; \mathbb{C}) = \dim_{\mathbb{C}} H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C}) = 2g + n - 1 \). This gives an isomorphism

\[
H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C}) \cong H_1(C_{q}; \mathbb{C}).
\]

We also have an intersection pairing

\[
\cap : H_1(\overline{C}_{q}; \mathbb{C}) \times H_1(\overline{C}_{q}; \mathbb{C}) \to \mathbb{C}.
\]

This gives an isomorphism \( H_1(\overline{C}_{q}; \mathbb{C}) \to H_1(\overline{C}_{q}; \mathbb{C})^\vee \). Under the above isomorphisms, the injective map \( H_1(\overline{C}_{q}; \mathbb{C}) \to H_1(\overline{C}_{q}; D_q^{\infty}) \) can be identified with the dual of the surjective linear map \( J_s : H_1(C_{q}; \mathbb{C}) \to H_1(\overline{C}_{q}; \mathbb{C}) \). Let \( K_1(C_{q}; \mathbb{C})^1 \subset H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C}) \) be the subspace of annihilators of \( K_1(C_{q}; \mathbb{C}) \). Then \( K_1(C_{q}; \mathbb{C})^1 \) is 2-dimensional, and there is a perfect pairing

\[
\cap : K_1(C_{q}; \mathbb{C}) \times H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C})/K_1(C_{q}; \mathbb{C})^1 \to \mathbb{C}.
\]

By permuting \( A_1, \ldots, A_p \) if necessary, we may choose \( B_1, \ldots, B_p \in H_1(\overline{C}_{q}; D_q^{\infty}; \mathbb{C})/K_1(C_{q}; \mathbb{C})^1 \) such that

1. \( \{A_1, \ldots, A_p, B_1, \ldots, B_p\} \) is a symplectic basis of \( H_1(\overline{C}_{q}; \mathbb{C}) \), and \( \{B_1, \ldots, B_p, -A_1, \ldots, -A_p\} \) is the dual basis of \( H_1(\overline{C}_{q}; \mathbb{C}) \).
(2) \( \{A_1, \ldots, A_p, B_1, \ldots, B_q\} \) is a basis of \( K_1(C_q; \mathbb{C}) \), and \( \{B_1, \ldots, B_p, -A_1, \ldots, -A_q\} \) is the dual basis of \( H_1(\overline{C_q}, D^\infty_q; \mathbb{C}) \).

5.6. Differentials of the second kind. Let \( B(p_1, p_2) \) be the fundamental differential of the second kind normalized by \( A_1, \ldots, A_q \). It is also called the Bergman kernel in [42, 43]. Then

\[
\frac{\partial \Phi}{\partial \tau_a}(p) = \int_{p' \in E_a} B(p, p'). \quad a = 1, \ldots, p.
\]

Following [40, 43], given any \( \sigma \in I_\Sigma \) and \( d \in \mathbb{Z}_{\geq 0} \), define

\[
\theta^d_\sigma(p) := (2d - 1)!! 2^{-d} \text{Res}_{p' \to p_\sigma} B(p, p') \zeta_{\sigma}^{-2d-1}.
\]

Then \( \theta^d_\sigma \) satisfies the following properties.

1. \( \theta^d_\sigma \) is a meromorphic 1-form on \( \overline{C_q} \) with a single pole of order \( 2d + 2 \) at \( p_\sigma \).
2. In local coordinate \( \zeta_\sigma = \sqrt{z - \bar{w}_\sigma} \) near \( p_\sigma \),

\[
\theta^d_\sigma = \left( \frac{- (2d + 1)!!}{2^d \zeta_\sigma^{-2d+2}} + f(\zeta_\sigma) \right) d\zeta_\sigma,
\]

where \( f(\zeta_\sigma) \) is analytic around \( p_\sigma \). The residue of \( \theta^d_\sigma \) at \( p_\sigma \) is zero, so \( \theta^d_\sigma \) is a differential of the second kind.

The meromorphic 1-form \( \theta^d_\sigma \) is characterized by the above properties; \( \theta^d_\sigma \) can be viewed as a section in \( H^0(\overline{C_q}, \omega_{\overline{C_q}}((2d + 2)p_\sigma)) \). At \( q = 0 \), \( \theta^d_{\sigma' \mid \overline{C_q}} = 0 \) for \( \sigma' \neq \sigma \).

6. B-model Topological Strings

6.1. Canonical basis in the B-model: \( \theta^d_\sigma \) and \( [V_\sigma(\tau)] \). For any Laurent polynomial \( f \in \tilde{S}_\Sigma[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}] \), denote \( [f] \in \text{Jac}(W^\tau_q) \). For \( \sigma \in I_\Sigma \), let \( V_\sigma(\tau) \) be a Laurent polynomial in \( X, Y, Z \) such that \( V_\sigma(\tau)(P_\sigma) = \delta_{\sigma, \sigma'} \). The collection \( \{[V_\sigma(\tau)]\}_{\sigma \in I_\Sigma} \) is a canonical basis of the semisimple Frobenius algebra \( \text{Jac}(W^\tau_q) \).

Let \( \tilde{H}_a = \left[ \frac{\partial W_\sigma^\tau}{\partial \tau_a} \right] \). Since \( \tilde{H}_1, \ldots, \tilde{H}_p \) multiplicatively generate \( \text{Jac}(W^\tau_q) \), we may choose a basis of \( \text{Jac}(W^\tau_q) \) of the form

\[
1, \tilde{H}_1, \ldots, \tilde{H}_p, E_1, \ldots, E_q,
\]

where \( E_i = \tilde{H}_{a_i} \tilde{H}_{b_i} \) for some \( a_i, b_i \in \{1, \ldots, p\} \). We can write \( [f] \) in the following decomposition

\[
[f] = \sum_{i=1}^q A_i(q) \tilde{H}_{a_i} \cdot \tilde{H}_{b_i} + \sum_{a=1}^p B_a(q) \tilde{H}_a + C(q) 1.
\]

Let \( D_a f = \frac{\partial W_\sigma^\tau}{\partial \tau_a} f - z \frac{\partial f}{\partial \tau_a} \). Define the standard form of \([f]\) to be

\[
\tilde{f} = \sum_{i=1}^q A_i(q) D_a d b_i + \sum_{a=1}^p B_a(q) D_a 1 + C(q) = \sum_{d=0}^{2} z^d f_d,
\]

and define the oscillating integral of \([f]\) to be

\[
\int_{\Gamma} e \frac{W_\sigma^\tau}{\Omega} \tilde{f} \Omega.
\]

We know that \([f] = [\tilde{f}]\). Direct calculation shows the following.
Lemma 6.1. We have the following identities in the Jacobian ring \( \text{Jac}(W_q^T) \).

\[
1 = \sum_{\sigma \in \Sigma} [V_{\sigma}(\tau)],
\]

\[
\tilde{H}_a = -\sum_{\sigma \in \Sigma} B_a^{\sigma}(q)[V_{\sigma}(\tau)],
\]

\[
\hat{H}_a \cdot \hat{H}_b = \sum_{\sigma \in \Sigma} C_{ab}^{\sigma}(q)[V_{\sigma}(\tau)],
\]

in which the coefficients are

\[
B_a^{\sigma}(q) = \left. \frac{\partial H_a}{\partial x_{\sigma}^0} \right|_{(X,Y) = (X_\sigma(q), Y_\sigma(q))},
\]

\[
C_{ab}^{\sigma}(q) = \left. \frac{\partial H_a \partial H_b}{\partial x_{\sigma}^0 \partial x_{\sigma}^0} \right|_{(X,Y) = (X_\sigma(q), Y_\sigma(q))}.
\]

Proof. See the proofs of Proposition 6.3 and Proposition 6.4 of [?]. \( \square \)

By Lemma 6.1 and Lemma 6.2 we conclude this subsection with the following proposition.

Proposition 6.3. If

\[
[V_{\sigma}(\tau)] = \sum_{i=1}^{g} A_i^{\sigma}(q) \tilde{H}_a_i \cdot \tilde{H}_b_i - \sum_{a=1}^{p} B_a^{\sigma}(q) \hat{H}_a + C_{\sigma}(q) 1,
\]

then

\[
\frac{h_1^1}{2} \theta_0^a = \sum_{i=1}^{\tilde{g}} A_i^{\sigma}(q) \left. \frac{\partial^2 \Phi}{\partial \tau_a \partial \tau_b} \right|_{(\sigma) \in \Sigma} + \sum_{a=1}^{p} B_a^{\sigma}(q) d(\frac{\partial \Phi}{\partial x_{\sigma}^0}) + C_{\sigma}(q) d(\frac{d\tilde{y}}{dx}).
\]

If the genus \( g \) of the compactified mirror curve \( \overline{C}_q \) is zero then \( \theta_0^a \sigma \), or more generally any differential of the second kind on \( \overline{C}_q \), is exact.

6.2. Oscillating integrals and the B-model R-matrix. Let \( V_{\sigma} = V_{\sigma}(0) \) be the flat basis in \( \text{Jac}(W_q^T) \) such that when \( q = 0 \), \( V_{\sigma}(P_{\sigma'}) = \delta_{\sigma,\sigma'} \). Then define

\[
\tilde{S}_{\sigma'}^{\sigma}(z) = \int_{\tilde{F}_{\sigma}} e^{-\frac{z}{2} V_{\sigma'}} \Omega.
\]

The matrix \( \tilde{S} \) plays the role of the fundamental solution of the B-model quantum differential equation. The matrix \( \tilde{S}_{\sigma'}^{\sigma}(z) \) has the following asymptotic expansion.
Proposition 6.4.

\[ \tilde{S}_{\sigma'}(z) \sim (-2\pi z)^{\frac{1}{2}} \sum_{\sigma'' \in I_\Sigma} \Psi_{\sigma'} \tilde{\phi}'' \tilde{R}_{\sigma''} e^{-\tilde{u}''}, \]

where \( \Psi_{\sigma'} \) is the matrix such that \( \phi_{\sigma'} = \sum_{\sigma''} \Psi_{\sigma'} \tilde{\phi}{\sigma''} \) as in Equation (14), and \( \tilde{u}'' \) is the critical value of \( W_{\sigma''}^q \) at \( P_{\sigma'} \). Furthermore, the matrix \( \tilde{R}_{\sigma''} = \delta_{\sigma''} + O(z) \).

Proof. By the stationary phase expansion,

\begin{align*}
\tilde{S}_{\sigma'}(z) &= \sum_{\sigma'' \in I_\Sigma} \Psi_{\sigma'} \tilde{\phi}'' \sqrt{-\text{Hess}_{W_{\sigma''}^q}(P_{\sigma''})} \cdot \int_{\Gamma_{\sigma''}} e^{-\frac{1}{2} W_{\sigma''}^q(\tau)} \Omega \\
&\sim (-2\pi z)^{\frac{1}{2}} \sum_{\sigma'' \in I_\Sigma} \Psi_{\sigma'} \tilde{\phi}'' \cdot e^{-\frac{1}{2} \Omega} (\delta_{\sigma''} + O(z)).
\end{align*}

So the matrix \( \tilde{S} \) can be asymptotically expanded in the desired form, and the matrix \( \tilde{R} \) is

\[ \tilde{R}_{\sigma''}(z) = \delta_{\sigma''} + O(z). \]

Given any \( \sigma \in I_\Sigma \), define

\[ \xi_k = (-1)^k \left( \frac{d}{dx} \right)^{k-1} \theta_0, \quad k \in \mathbb{Z}_{\geq 1}, \]

\[ \theta_k = d^k \xi_k, \quad k \geq 1, \quad \theta_0 = \delta_0, \]

\[ \theta_0(z) = \sum_{k=0}^{\infty} \theta_k z^k, \quad \hat{\theta}(z) = \sum_{k=0}^{\infty} \hat{\theta}_k z^k. \]

Notice that \( \hat{\theta}_k \) is a meromorphic function on \( \overline{C_q} \). Following Eynard-Orantin [13], let

\[ f_{\sigma'}(u) = \frac{e^{u \hat{\theta}^0_0}}{2 \sqrt{\pi u}} \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} u \hat{\theta}_0} \] .

Assume

\[ [V_{\sigma'}(\tau)] = \sum_{i=1}^{g} A_{i\sigma'}(\tau) \hat{H}_{a_i} - \sum_{a=1}^{p} B_{a\sigma'}(\tau) \hat{H}_b + C_{\sigma'}(\tau) \]

then by Proposition 6.3 we have

\[ \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} W_{\sigma'}^q(\tau)} \Omega = (z^2 \sum_{i=1}^{\theta_0} A_{i\sigma'}(\tau) \frac{\partial^2}{\partial \tau_{a_i} \partial \tau_{b_i}} + z \sum_{a=1}^{p} B_{a\sigma'}(\tau) \frac{\partial}{\partial \tau_a} + C_{\sigma'}(\tau)) \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} \Omega} \]

\[ = (z^2 \sum_{i=1}^{\theta_0} A_{i\sigma'}(\tau) \frac{\partial^2}{\partial \tau_{a_i} \partial \tau_{b_i}} + z \sum_{a=1}^{p} B_{a\sigma'}(\tau) \frac{\partial}{\partial \tau_a} + C_{\sigma'}(\tau)) 2\pi \sqrt{-1} \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} \Phi} \]

\[ = 2\pi \sqrt{-1} z^2 \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} \phi} \sum_{i=1}^{\theta_0} A_{i\sigma'}(\tau) \frac{\partial^2}{\partial \tau_{a_i} \partial \tau_{b_i}} + \sum_{a=1}^{p} B_{a\sigma'}(\tau) d(\frac{\partial y}{\partial \tau_a}) + C_{\sigma'}(\tau) d(\frac{\partial y}{\partial \tau_a}) \]

\[ = 2\pi \sqrt{-1} z^2 \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} \phi} \frac{\partial^2}{\partial \tau_{a_i} \partial \tau_{b_i}}. \]

From Equation 37,

\[ \int_{\Gamma_{\sigma'}} e^{-\frac{1}{2} W_{\sigma'}^q(\tau)} \Omega \sim \frac{(-2\pi z)^{\frac{1}{2}} e^{-\frac{1}{2} \Omega}}{\sqrt{-\text{Hess}_{W_{\sigma''}^q}(P_{\sigma''})}} \tilde{R}_{\sigma''}, \]
and $h_{\tau}^2 = \frac{2}{\text{radical.al⟪3}}$, it is easy to see
\[
\tilde{R}_{\sigma'}(z) = f_{\sigma'}(\frac{1}{z}).
\]

Following Eynard [40], define Laplace transform of the Bergman kernel
\[
(38)
\]
where $\sigma, \sigma' \in I_{\Sigma}$. By [40, Equation (B.9)],
\[
(39)
\]
Setting $u = -v$, we conclude that $(\tilde{R}^*(\frac{1}{u})\tilde{R}(-\frac{1}{u}))^{\sigma} = \{\sum_{\sigma' \in I_{\Sigma}} \tilde{R}_{\sigma'}(\frac{1}{u})\tilde{R}_{\sigma'}(\frac{1}{v})\} = \delta^{\sigma\sigma'}$. This shows $\tilde{R}$ is unitary.

The following proposition is a consequence of Lemma 6.5 in [7] and Equation (39).

**Proposition 6.5.**
\[
\theta_{\sigma}(z) = \sum_{\sigma' \in I_{\Sigma}} \tilde{R}_{\sigma'}(z)\tilde{\theta}_{\sigma'}(z).
\]

The following proposition is due to Dubrovin [39] and Givental [51, 53], but we only consider the small phase space $H^{2}_{\text{CR}, \tau_{i}}(\mathcal{X}; \mathbb{C})$.

**Proposition 6.6.** Assume functions $\tilde{A}_{\tilde{\sigma}}(\tau_{\leq 2}) = \sum_{\sigma' \in I_{\Sigma}} \tilde{A}_{\sigma'}(z)(\tau_{\leq 2})\phi^{\sigma'}$ allow the following asymptotic expansions
\[
\tilde{A}_{\tilde{\sigma}}(z) \sim \sum_{\sigma' \in I_{\Sigma}} \hat{\Psi}_{\sigma'} \tilde{B}_{\sigma'}(z)e^{-\frac{2z}{\tau_i}},
\]
such that
- $\Psi$ is the transition matrix defined in Equation (14);
- The matrix function $\tilde{B}_{\sigma'}(z) = \delta_{\sigma\sigma'} + O(z)$ is unitary
  \[
  \sum_{\sigma' \in I_{\Sigma}} \tilde{B}_{\sigma'}(z)\tilde{B}_{\sigma'}(-z) = \delta_{\sigma\sigma'};
  \]
- The functions $\tilde{\sigma}$ and the canonical coordinates $u^{\sigma}$ differ by constants, i.e.
  \[
  \frac{\partial \tilde{u}^{\sigma}}{\partial \tau_{i}} = \frac{\partial u^{\sigma}}{\partial \tau_{i}} \text{ for } i = 1, \ldots, p.
  \]

If each function $\tilde{A}_{\tilde{\sigma}}$ satisfies the quantum differential equations for $1 \leq i \leq p$
\[
-z \frac{\partial}{\partial \tau_{i}} \tilde{A}_{\tilde{\sigma}} = \tilde{H}_{i} \cdot \tilde{A}_{\tilde{\sigma}},
\]
then $\tilde{B}$ is unique up to a right multiplication of $\exp(\sum_{i=1}^{\infty} a_{i}z^{2i-1})$, where $a_{i}$ is a constant diagonal matrix.

**Proof.** This proof is essentially the same as the Proposition in [53, Section 1.3 (p1269)]. The only minor difference is that we only considers the small phase space. Let $\tilde{s}$ be the diagonal matrix with the diagonal elements $\{s^{\sigma}\}_{\sigma \in \Sigma}$. Notice that substituting the series into the quantum differential equations gives
\[
\left(\frac{\partial}{\partial \tau_{i}} + \Psi \frac{\partial \Psi}{\partial \tau_{i}}\right)\tilde{B}_{k-1} = -\left[\frac{\partial \tilde{s}}{\partial \tau_{k}}\right].
\]
This gives a recursion which determines $\hat{B}$. The off-diagonal terms in $\hat{B}_k$ are directly expressed in $\hat{B}_{k-1}$ algebraically, and the diagonal terms could be solved by integration, noting that $\sum_k \hat{B}_k$ has vanishing diagonal.

Let $\mathcal{P}^{\sigma,\sigma'}(z) = \sum_{k \geq 0} (\mathcal{P}_k)^{\sigma,\sigma'} z^k = \sum_{\sigma' \in I_{\Sigma}} \hat{B}_{\sigma}^{\sigma'}(z)\hat{B}^{\sigma'}(z)$, then the quantum differential equations produce

\[-\left[ \frac{\partial}{\partial \bar{r}_i} , \mathcal{P}_k \right] = d\mathcal{P}_{k-1} + \left[ \Psi \frac{\partial \Psi}{\partial \bar{r}_{k-1}} , \mathcal{P}_{k-1} \right].\]

Note that $\hat{B}$ is unitary, i.e. $\mathcal{P}_k = 0$ for $k \geq 1$ and $\mathcal{P}_0 = I$. For $k$ odd, the equation above ensures that $\mathcal{P}_k = 0$ from $\mathcal{P}_{k-1} = 0$ (or $I$ when $k = 1$) since $\mathcal{P}_k$ is ant-symmetric. For even $k$, the ambiguity of the integrating constants in determining the diagonal terms of $B_k$ in the process above is fixed by

\[0 = (\hat{B}_k)^{\sigma,\sigma'} + (\hat{B}_k)^{\sigma',\sigma} + \text{terms involving } B_i, i = 1, \ldots, k - 1.\]

We see that this is equivalent to a right multiplication of $\exp(\sum_{i=1}^{\infty} a_i z^{2i-1})$. □

Since $\tilde{u}^{\sigma}$ is a critical value,

\[\frac{\partial \tilde{u}^{\sigma}}{\partial \bar{r}_i} = \frac{dW_q^{\sigma'}(P_\sigma(\tau), \tau)}{d \bar{r}_i} = \frac{\partial W_q^{\sigma'}(P_\sigma(\tau))}{\partial \bar{r}_i}.\]

The Jacobian ring element $\hat{H}_i = \{ \frac{\partial W_q^{\sigma'}}{\partial \bar{r}_i} \}$ corresponds to $H_i$ in the quantum cohomology. Then by the following identity

\[\left[ \frac{\partial W_q^{\sigma'}}{\partial \bar{r}_i} \right] = \sum_{\sigma' \in I_{\Sigma}} \frac{\partial W_q^{\sigma'}}{\partial \bar{r}_i} (P_\sigma(\tau))[V_\sigma(\tau)],\]

we have

\[\frac{\partial u^{\sigma}}{\partial \bar{r}_i} = \frac{\partial \tilde{u}^{\sigma}}{\partial \bar{r}_i},\]

which implies the critical values are canonical coordinates. The function $\tilde{S}^{\sigma} = \sum_{\sigma' \in I_{\Sigma}} \tilde{S}^{\sigma'} \phi^{\sigma'}$ is a solution to the quantum differential equation

\[-\tilde{\tau} \frac{\partial}{\partial \bar{r}_i} \tilde{S}^{\sigma} = \left( \frac{\partial W_q^{\sigma'}}{\partial \bar{r}_i} \right) \tilde{S}^{\sigma},\]

For all $\sigma \in I_{\Sigma}$, $\tilde{S}^{\sigma}$ satisfy the condition of Proposition 6.6.

6.3. The Eynard-Orantin topological recursion and the B-model graph sum. Let $\omega_{g,n}$ be defined recursively by the Eynard-Orantin topological recursion [42]:

\[\omega_{0,1} = 0, \quad \omega_{0,2} = B(p_1, p_2).\]

When $2g - 2 + n > 0$,

\[\omega_{g,n}(p_1, \ldots, p_n) = \sum_{(\sigma, \tau) \in I_{\Sigma}} \text{Res}_{p = p_0}^{\mathcal{E} \in \bar{E}} \frac{\hat{B}(p, \xi)}{2(\Phi(p) - \Phi(\bar{p}))} \left( \omega_{g-1,n+1}(p, \bar{p}, p_1, \ldots, p_{n-1}) + \sum_{g_1 + g_2 = g} \sum_{I \cup J = \{1, \ldots, n-1\}} \omega_{g_1, |I|+1}(p, p_I) \omega_{g_2, |J|+1}(\bar{p}, p_J) \right)\]

Following [33], the B-model invariants $\omega_{g,n}$ are expressed in terms of graph sums. We first introduce some notation.
For any $\sigma \in I_{\Sigma}$, we define

\[ h^\sigma_k := \frac{(2k-1)!!}{2^{k-1}} h^\sigma_{2k-1}. \]

Then

\[ \hat{h}^\sigma_k = \frac{u^{1-k}}{\sqrt{\pi}} e^{u\sigma^k} h^\sigma_k. \]

For any $\sigma, \sigma' \in I_{\Sigma}$, we expand

\[ B(p_1, p_2) = \left( \frac{\delta_{\sigma, \sigma'}}{(\zeta_{\sigma} - \zeta_{\sigma'})^2} + \sum_{k, l \in \mathbb{Z}_{>0}} B_{k, l}^{\sigma, \sigma'} \zeta_{\sigma}^k \zeta_{\sigma'}^l \right) d\zeta_{\sigma} d\zeta_{\sigma'}, \]

near $p_1 = p_\sigma$ and $p_2 = p_{\sigma'}$, and define

\[ \tilde{B}_{k, l}^{\sigma, \sigma'} := \frac{(2k-1)!!(2l-1)!!}{2^{k+l+1}} B_{2k, 2l}^{\sigma, \sigma'}. \]

Then

\[ \tilde{B}_{k, l}^{\sigma, \sigma'} = \left[ -u^{k-l} \right] \left( \frac{u}{u + v} \right)^k \left( \frac{1}{z + w} \right)^l \left( \delta_{\sigma, \sigma'} - \sum_{\gamma \in I_{\Sigma}} f_{\gamma}^\sigma (u) f_{\gamma}^{\sigma'} (v) \right) \]

Given a labeled graph $\tilde{\Gamma} \in \Gamma_{g,n}(\mathcal{X})$ with $L^\sigma(\Gamma) = \{ l_1, \ldots, l_n \}$, and $\bullet = u$ or $O$, we define its weight to be

\[ u^\bullet_B (\tilde{\Gamma}) = (-1)^{g(\tilde{\Gamma})-1} \prod_{v \in V(\tilde{\Gamma})} \left( \frac{h^\sigma_{\gamma(v)}}{\sqrt{\gamma}} \right)^{2g(v) - \text{val}(v)} \left( \prod_{k \in H(v)} \tau_{k(v)} \right) \prod_{e \in E(\tilde{\Gamma})} \tilde{B}_{k(e), l(e)}^{\sigma(v_1(e)), \sigma(v_2(e))} \]

\[ \cdot \prod_{l \in L^1(\tilde{\Gamma})} \left( L^1 \right)_{k(l)} \prod_{j=1}^n \left( L^\bullet \right)_{k(l_j)}^{\sigma(l_j)} (l_j) \]

where

- (dilaton leaf)
  \[ (L^1)_{k}^\sigma = -\frac{1}{\sqrt{-2}} \hat{h}^\sigma. \]

- (descendant leaf)
  \[ (L^u)_{k}^\sigma (l_j) = -\frac{1}{\sqrt{-2}} \theta_{\sigma}^k (p_j) \]

- (open leaf)
  \[ (L^O)_{k}^\sigma (l_j) = \frac{1}{\sqrt{-2}} \sum_{l \in Z_{>0}} \int_{0}^\chi \rho_l^\sigma (\theta_{\sigma}^k) \psi_\ell. \]

In our notation [38], Theorem 3.7 is equivalent to:

**Theorem 6.7** (Dunin-Barkowski–Orantin–Shadrin–Spitz [38]). For $2g - 2 + n > 0$,

\[ \omega_{g,n} = \sum_{\Gamma \in \Gamma_{g,n}(\mathcal{X})} \frac{w^u_B (\tilde{\Gamma})}{|\text{Aut}(\tilde{\Gamma})|}. \]
We now consider the unstable case \((g,n) = (0,2)\). Recall that \(dx = -\frac{dX}{X}\) is a meromorphic 1-form on \(\bar{C}_q\), and \(\frac{d}{dx} = -X\frac{d}{dX}\) is a meromorphic vector field on \(\bar{C}_q\).

Define

\[
(42) \quad C(p_1, p_2) := (-\frac{\partial}{\partial x(p_1)}) - \frac{\partial}{\partial x(p_2)}(\frac{\omega_{0,2}}{dx(p_1) dx(p_2)})(p_1, p_2) d(x(p_1))(d(x(p_2))).
\]

Then \(C(p_1, p_2)\) is meromorphic on \((\bar{C}_q)\) and is holomorphic on \((\bar{C}_q \setminus \{ p_\sigma : \sigma \in I_\Sigma \})^2\).

**Lemma 6.8.**

\[
C(p_1, p_2) = \frac{1}{2} \sum_{\sigma \in I_\Sigma} \theta_{\sigma}^0(p_1) \theta_{\sigma}^0(p_2).
\]

**Proof.** For any \(\sigma, \sigma' \in I_\Sigma\), we compute their Laplace transforms

\[
\int_{p_1 \in \Gamma_\sigma} \int_{p_2 \in \Gamma_{\sigma'}} e^{-\frac{x(p_1)-\sigma}{i_1} - \frac{x(p_2)-\sigma'}{i_2}} C(p_1, p_2)
= -\frac{z_1 + z_2}{z_1 z_2} \int_{p_1 \in \Gamma_\sigma} \int_{p_2 \in \Gamma_{\sigma'}} e^{-\frac{x(p_1)-\sigma}{i_1} - \frac{x(p_2)-\sigma'}{i_2}} \omega_{0,2}
\]

\[
= \frac{2\pi}{\sqrt{z_1 z_2}} \sum_{\sigma'' \in I_\Sigma} \tilde{R}_{\sigma''}(z_1) \tilde{R}_{\sigma''}(z_2)
= \frac{1}{2} \sum_{\sigma'' \in I_\Sigma} \int_{p_1 \in \Gamma_{\sigma''}} \int_{p_2 \in \Gamma_{\sigma''}} e^{\frac{x(p_1)-\sigma''}{i_1} - \frac{x(p_2)-\sigma''}{i_2}} \theta_{\sigma''}^0(p_1) \theta_{\sigma''}^0(p_2).
\]

Define

\[
\omega = C(p_1, p_2) - \frac{1}{2} \sum_{\sigma \in I_\Sigma} \theta_{\sigma}^0(p_1) \theta_{\sigma}^0(p_2).
\]

Since for \(i = 1, \ldots, g\), \(\int_{p_2 \in A_i} \omega_{0,2}(p_1, p_2) = 0\), \(\int_{A_i} \theta_{\sigma}^0 = 0\), we have \(\int_{p_2 \in A_i} \omega = 0\), and the following residue 1-form has

\[
\int_{p_2 \in A_i} \text{Res}_{p_1 \rightarrow p_\sigma} \zeta_{\sigma}(p_1) \omega(p_1, p_2) = 0,
\]

for all \(i = 1, \ldots, g\). Notice that the 1-form \(\text{Res}_{p_1 \rightarrow p_\sigma} \zeta_{\sigma}(p_1) \omega(p_1, p_2)\) has no poles, otherwise a possible double pole at \(p_\sigma\) implies non-zero Laplace transform of \(\omega\) at \(\Gamma_{\sigma} \times \Gamma_{\sigma'}\). It follows from the vanishing A-cycles integrals that

\[
\text{Res}_{p_1 \rightarrow p_\sigma} \zeta_{\sigma}(p_1) \omega(p_1, p_2) = 0,
\]

and then \(\omega\) does not have any poles. Therefore by the vanishing A-periods of \(\omega\) we know \(\omega = 0\). \(\square\)

### 6.4. B-model open potentials

In this section, we fix \(u_1 = 1\) and \(u_2 = f\). Choose \(\delta > 0\), \(\epsilon > 0\) sufficiently small, such that for \(|q| < \epsilon\), the meromorphic function \(\hat{X}: \bar{C}_q \rightarrow \mathbb{C} \cup \{\infty\}\) restricts to an isomorphism

\[
\hat{X}_q^\ell : D^\ell_q \rightarrow D_\delta = \{ \hat{X} \in \mathbb{C} : |X| < \delta \},
\]

where \(D^\ell_q\) is an open neighborhood of \(\hat{p}_\ell \in \hat{X}^{-1}(0)\), \(\ell = 0, \ldots, m - 1\). Define

\[
\rho^\ell_q := (\hat{X}_q^\ell)^{-1} \times \cdots \times (\hat{X}_q^\ell)^{-1} : (D_\delta) \times \cdots \times (D_\delta) \rightarrow D^\ell_q \times \cdots \times D^\ell_q \subset (\bar{C}_q)^n.
\]
(1) (disk invariants) At \( g = 0 \), \( \hat{Y}(\hat{p}_\ell) = -1 \) for \( \ell = 0, \ldots, 1 \). When \( \epsilon \) and \( \delta \) are sufficiently small, \( \hat{Y}(\rho_q^g(\hat{X})) \in \mathbb{C} \setminus [0, \infty) \). Choose a branch of logarithm \( \log : \mathbb{C} \setminus [0, \infty) \to (0, 2\pi) \), and define
\[
\hat{y}_q^g(\hat{X}) = -\log \hat{Y}(\rho_q^g(\hat{X})).
\]
The function \( \hat{y}_q^g(X) \) depends on the choice of logarithm, but \( \hat{y}_q^g(X) - \hat{y}_q^g(0) \) does not. \( d\hat{x} = -d\hat{X}/\hat{X} \) is a meromorphic 1-form on \( \mathbb{C} \) with a simple pole at \( \hat{X} = 0 \), and
\[
(\hat{y}_q^g(X) - \hat{y}_q^g(0))d\hat{x}
\]
is a holomorphic 1-form on \( D_\delta \). Define the B-model disk potential by
\[
\hat{F}_{0,1}(q; \hat{X}) := \sum_{\ell \in \mathbb{Z}_m} \int_0^{\hat{X}} (\hat{y}_q^g(X') - \hat{y}_q^g(0))(-\frac{dX'}{X'}) \cdot \psi_\ell,
\]
which takes values in \( H^*(\mathcal{B} \mu_m; \mathbb{C}) \).

(2) (annulus invariants)
\[
(\rho_q^{g, \ell_1, \ell_2})^* \omega_{0,2} - \frac{d\hat{X}_1 d\hat{X}_2}{(X_1 - X_2)^2}
\]
is holomorphic on \( D_\delta \times D_\delta \). Define the B-model annulus potential by
\[
\hat{F}_{0,2}(q; \hat{X}_1, \hat{X}_2) := \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{\hat{X}_1} \int_0^{\hat{X}_2} ((\rho_q^{g, \ell_1, \ell_2})^* \omega_{0,2} - \frac{dX_1' dX_2'}{(X_1' - X_2')^2}) \cdot \psi_{\ell_1} \otimes \psi_{\ell_2},
\]
which takes values in \( H^*(\mathcal{B} \mu_m; \mathbb{C}) \otimes \mathbb{C}^2 \).

(3) For \( 2g - 2 + n > 0 \), \( (\rho_q^{g, \ell_1, \ldots, \ell_n})^* \omega_{g,n} \) is holomorphic on \( (D_\delta)^n \). Define
\[
\hat{F}_{g,n}(q; \hat{X}_1, \ldots, \hat{X}_n) := \sum_{\ell_1, \ldots, \ell_n \in \mathbb{Z}_m} \int_0^{\hat{X}_1} \cdots \int_0^{\hat{X}_n} (\rho_q^{g, \ell_1, \ldots, \ell_n})^* \omega_{g,n} \cdot \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n},
\]
which takes values in \( H^*(\mathcal{B} \mu_m; \mathbb{C}) \otimes \mathbb{C}^n \).

For \( g \in \mathbb{Z}_{\geq 0} \) and \( n \in \mathbb{Z}_{\geq 0} \), \( \hat{F}_{g,n}(q; \hat{X}_1, \ldots, \hat{X}_n) \) is holomorphic on \( B \times (D_\delta)^n \) when \( \epsilon, \delta > 0 \) are sufficiently small. By construction, the power series expansion of \( \hat{F}_{g,n}(q; \hat{X}_1, \ldots, \hat{X}_n) \) only involves positive powers of \( \hat{X}_i \).

For \( k \in \mathbb{Z}_{\geq 0} \), define
\[
\xi^k_\sigma(\hat{X}) := \sum_{\ell \in \mathbb{Z}_m} \int_0^{\hat{X}} (\rho_q^g)^* \theta^k_\sigma \psi_\ell, \quad \xi_\sigma(z, \hat{X}) := \sum_{\ell \in \mathbb{Z}_m} \int_0^{\hat{X}} (\rho_q^g)^* \hat{\theta}_\sigma(z) \psi_\ell.
\]

6.5. B-model free energies. In this section, \( g > 1 \) is an integer.

**Definition 6.9** (cf. [16] Definition 2.7). The B-model genus \( g \) free energy is defined to be
\[
\hat{F}_g := \frac{1}{2g - 2} \sum_{\sigma \in \Sigma_g} \text{Res}_{p_\sigma} \omega_{g,1}(p) \tilde{\Phi}_\sigma(p).
\]
where \( \tilde{\Phi}_\sigma \) is a function defined on an open neighborhood of \( p_\sigma \) in \( \Sigma_g \) such that \( d\tilde{\Phi}_\sigma = \Phi \).

Notice that the definition does not depend on the choice of \( \tilde{\Phi}_\sigma \).
Proposition 6.10.

\[ \tilde{F}_g = \frac{1}{2 - 2g} \sum_{\Gamma \in \Gamma_g, \mathcal{L}} w_B^\mu(\tilde{\Gamma})_{(\tilde{\mathcal{L}}^\nu)_{(l_1) = (\tilde{\mathcal{L}}^\nu)^\tau}} |\text{Aut}(\tilde{\Gamma})|. \]

Proof. Recall that

\[ (\tilde{\mathcal{L}}^\nu)^\tau = \frac{1}{\sqrt{-2}} \theta^k(p_1) \]

\[ (\tilde{\mathcal{L}}^\nu)^\tau = \frac{1}{\sqrt{-2}} \delta^\tau. \]

By the graph sum formula of \( \omega_{g,1} \) (Theorem 6.7) and the definition of \( \tilde{F}_g \) (Definition 6.9), it suffices to show that

\[ \text{Res}_{p \to p} \theta^k(p) \tilde{\Phi}(p) = -\delta^\tau. \]

Near \( p_\tau \), we have

\[ \theta^k = \left( \frac{(2k + 1)!!}{2^{2k} \zeta(2k + 2)} + f(\zeta) \right) d\zeta, \]

where \( f(\zeta) \) is analytic around \( p_\tau \), and

\[ d\tilde{\Phi} = \tilde{y} \tilde{x} = (\tilde{v} + \sum_{d=1}^\infty h_d \zeta^d)(2\zeta d\zeta), \]

so up to a constant,

\[ \tilde{\Phi} = \tilde{v} + \sum_{d=1}^\infty \frac{2h_d}{d + 2} \zeta^{d+2}. \]

Therefore,

\[ \text{Res}_{p \to p} \theta^k(p) \tilde{\Phi}(p) = -\frac{(2k - 1)!!}{2^{k-1}} h_{2k-1}^\tau = -\delta^\tau. \]

\[ \square \]

7. All Genus Mirror Symmetry

7.1. Identification of A-model and B-model R-matrices. In [27], the genus 0 mirror theorem of any semi-projective toric orbifolds is proved. It implies that the \( T^k \)-equivariant quantum cohomology ring of \( X \) is isomorphic to \( \text{Jac}(W^\tau_q) \) as Frobenius algebras. This isomorphism is of course under the closed mirror map between the extended complexified Kähler parameters \( \tau_1, \cdots, \tau_p \) and complex parameters \( q_1, \cdots, q_p \). Under this isomorphism \( h^\sigma_1 = \sqrt{\frac{\zeta_1}{\zeta(\nu)}} \) is identified with \( \sqrt{\frac{-1}{2\zeta(\tau)}} \).

We are working with non-conformal Frobenius manifolds, and the solution of the quantum differential equation is not unique. The ambiguity is fixed by the following theorem.

Theorem 7.1. For any \( \sigma = (\sigma, \alpha) \) and \( \sigma' = (\sigma', \alpha') \),

\[ R_{\sigma'\sigma}(z) \big|_{\tau, Q = 1} = \tilde{R}_{\sigma'\sigma}(-z). \]
Proof. Both $\bar{S}_\sigma^\sigma(z)$ and $S_\sigma^\sigma(-z)|_{t=\tau,Q=1}$ both satisfy the conditions of Proposition 6.6. We only need to show they match when $q = 0$. Recall from Section 5.2 that when $q = 0$, the compactified mirror curve $\overline{C}_q(q)$ degenerates into a nodal curve $\overline{C}_0 = \overline{\cup_{\sigma\in\Sigma(3)}}\overline{C}_\sigma$, where the irreducible component $\overline{C}_\sigma$ can be identified with the compactified mirror curve of the affine toric Calabi-Yau 3-orbifold $X_{\sigma}$ defined by the 3-cone $\sigma$. The 1-form $\theta^\sigma_{\sigma,\alpha}(0)|_{C_\sigma}$ vanishes when $\sigma' \neq \sigma$, and has only double pole at $p_{\sigma,0}(0)$ if $\sigma' = \sigma$. As computed in [?, Theorem 7.5]

$$R^\sigma_{\sigma'}(z)|_{q=0} = \sum_{\alpha>0} \chi(h) \chi_{\sigma'}(h^{-1}) \exp \left( \sum_{m \geq 1} \frac{(-1)^m}{m} \sum_{i=1}^3 B_{m+1}(c_i(h))(\frac{z}{w_i(\sigma)})^m \right),$$

which is precisely $R^\sigma_{\sigma'}(z)|_{q=0}$ given in Equation 19. 

7.2. Identification of graph sums. In this subsection, we identify the graph sums on A-model and B-model.

For $l = 1, \ldots, n$ and $\sigma \in I_{\Sigma}$, let $u^\sigma_l(z) = \sum_{\alpha > 0} (\bar{u}_l)^\sigma_\alpha z^\alpha := \sum_{\sigma' \in \Sigma} \left( \frac{u^\sigma_l^\prime(z)}{\sqrt{\Delta^\sigma}} \frac{S^\sigma_{\sigma'}(z)}{\Delta^\sigma} \right)$. The identification $R(z)|_{t=\tau,Q=1} = \hat{R}(-z)$ implies the following theorem:

Theorem 7.2. For any $\hat{\Gamma} \in \Gamma_{g,n}(X)$,

$$w_B^u(\hat{\Gamma})_{\bar{\theta}_\sigma^\rho(p_l), (\bar{u}_l)_l} = (-1)^g(\hat{\Gamma})^{-1+n} w_A^u(\hat{\Gamma})_{t=\tau,Q=1},$$

under the closed mirror map.

Proof: (1) Vertex. By the discussion in Section 4.4 and Section 4.5, $h^\sigma = \sqrt{\frac{2}{3\sigma_\tau(\tau)}}$ for any $\sigma \in I_{\Sigma}$. So in the B-model vertex term, $\frac{h^\sigma}{\sqrt{-2}} = \sqrt{\frac{2}{3\sigma_\tau(\tau)}}$. Therefore the B-model vertex matches the A-model vertex.

(2) Edge. By the property for $\tilde{B}_{a,b}^\sigma$, $\tilde{B}_{a,b}^\sigma = \left[ u^{-a}v^{-b} \right] \left( \frac{uv}{u+v} \left( \delta_{\sigma,\rho} - \sum_{\gamma \in \Sigma} f_\gamma^\sigma(u)f_\gamma^\rho(v) \right) \right) = \left[ z^aw^b \right] \left( \frac{1}{z+w} \left( \delta_{\sigma,\rho} - \sum_{\gamma \in \Sigma} f_\gamma^\sigma \frac{1}{z} f_\gamma^\rho \frac{1}{w} \right) \right)$. Therefore, the identification $R(z)|_{t=\tau,Q=1} = \hat{R}(-z) = (f_\rho^\sigma(-\frac{i}{2}))$ gives us $\tilde{B}_{a,b}^\sigma = c_{a,b}^\rho|_{t=\tau,Q=1}$.

(3) Ordinary leaf. By Proposition 6.5, we have the following expression for $\theta^\sigma_\rho$:

$$\theta^\sigma_\rho = \sum_{c=0}^\infty \sum_{p \in I_{\Sigma}} ([z^aw^b]) (\tilde{R}_\rho^\sigma(z)) \hat{\theta}_\rho^c.$$ 

Notice that $\hat{R}(z) = R(-z)|_{t=\tau,Q=1}$. So $$(C^u_k)^\sigma = \sum_{\gamma \in \Sigma} (\bar{u}_l)^\sigma_\gamma (p_l) = -(C^u_k)^\sigma(l_j)|_{t=\tau,Q=1}. $$
Dilaton leaf. We have the following relation between $h^\sigma_\tau$ and $f^i_j(u)$ (see [?])

$$h^\sigma_\tau = [u^{1-k}] \sum_{\rho \in \Sigma} h^\rho_\tau f^\rho_\sigma(u).$$

By the relation

$$R^\sigma_\rho(z)_{t=\tau,q=1} = f^\rho_\sigma(-\frac{1}{z})$$

and the fact $h^\tau_\tau = \sqrt{\sum_{\rho \in \Sigma}}$, it is easy to see that the B-model dilaton leaf matches the A-model dilaton leaf.

7.3. BKMP Remodeling Conjecture: the open string sector. In this subsection, we fix $u_1 = 1$ and $u_2 = q$. We compare A and B-model open leafs. The disk potential with respect to the Aganagic-Vafa brane $L$ is given by localization, as in Proposition 6.12 (computed in [13]).

$$\left(\frac{d}{dx}\right)^2 F_{0,1}^{X,(\ell,f)}(\sigma, \tilde{X}) = \sum_{\sigma \in \Sigma} \tilde{\xi}^\sigma(z, \tilde{X}) S(1, \phi_\sigma)|_{t=\tau, q=1}. $$

The following theorem is proved by Tseng and the first two authors [45].

**Theorem 7.3** (Genus zero open-closed mirror symmetry). Under the closed mirror map given by Equation (25) and the open map give by

$$\log \tilde{X} = \log X + \sum_{m=1}^3 w_i A_i(q),$$

we have

$$|G_0| F_{0,1}^{X,(\ell,f)}(\sigma, \tilde{X}) = \hat{F}_{0,1}(q; \tilde{X}) = \sum_{t \in \tau_m} \left( \int_0^X \rho_t^\ell(y(X') - y(0)) \left( -\frac{dX'}{X'} \right) \psi_\ell. \right)$$

This theorem, together with Equation (43), implies that under the open-closed mirror map, as power series in $\tilde{X}$,

$$U(z)(\sigma, \tilde{X}) := \sum_{\sigma \in \Sigma} \tilde{\xi}^\sigma(z, \tilde{X}) S(1, \phi_\sigma)|_{t=\tau, q=1} = -\sum_{n \geq 0} z^n \left( -\frac{d}{dx} \right)^n \frac{1}{|G_0|} \int_0^X \rho_t^\ell(y) \psi_\ell.$$ 

Notice that from Proposition 6.3 if

$$\hat{\phi}_\sigma(\tau(q)) = \sum_{i=1}^6 \hat{A}_i^\sigma(q) H_{a_i} * \tau H_{b_i} + \sum_{a=1}^p \hat{B}_a(\sigma(q)) H_a + \hat{C}_\sigma(q) \mathbf{1},$$

then

$$\frac{\theta^\sigma}{\sqrt{2}} = \sum_{i=1}^6 \hat{A}_i^\sigma(q) \frac{\partial^2 \Phi}{\partial \tau_{a_i} \partial \tau_{b_i}} + \sum_{a=1}^p \hat{B}_a(\phi_\sigma(q)) \frac{\partial \phi_\sigma}{\partial \tau_a} + \hat{C}_\sigma(q) \frac{d \hat{\phi}_\sigma}{d \tilde{X}}.$$ 

Therefore

$$\sum_{\sigma \in \Sigma} \tilde{\xi}^\sigma(z, \tilde{X}) S(\hat{\phi}_\sigma(\tau), \phi_\sigma)|_{t=\tau, q=1}$$

$$= \sum_{i=1}^6 z^2 \hat{A}_i^\sigma(q) \frac{\partial^2 U}{\partial \tau_{a_i} \partial \tau_{b_i}} + \sum_{a=1}^p z \hat{B}_a(\sigma(q)) \frac{\partial U}{\partial \tau_a} + \hat{C}_\sigma(q) U.$$ 

By Equation (45) and (46), under the open-closed mirror map (47)
\[ z^2 |G_0| \sum_{\sigma' \in \Sigma} \sum_{t, t' \in \mathbb{T}_m} \tilde{\xi}'(z, \tilde{X}) S(\phi_\sigma(\tau), \phi_{\sigma'}) |_{t=t', Q=1} = -\sum_{t, t' \in \mathbb{T}_m} \int_0^{\tilde{X}} \rho_\sigma^* \theta_\sigma(z) \psi_{t'} = -\xi_\sigma(z, \tilde{X}). \]

**Proposition 7.4** (Annulus open-closed mirror symmetry). Under the open-closed mirror map,
\[ \hat{F}_{0,2}(q; \hat{X}_1, \hat{X}_2) = -|G_0|^2 F_0^{X, (L, f)}(\tau; \hat{X}_1, \hat{X}_2). \]

**Proof.** The symmetric meromorphic 2-form \( C(p_1, p_2) \) is defined by (42). Then
\[
(\hat{X}_1 \frac{\partial}{\partial \hat{X}_1} + \hat{X}_2 \frac{\partial}{\partial \hat{X}_2}) \hat{F}_{0,2}(q; \hat{X}_1, \hat{X}_2)
= \sum_{t, t' \in \mathbb{T}_m, \sigma \in \Sigma} \int_0^{\hat{X}_1} \int_0^{\hat{X}_2} (\rho_q^* \psi_t) \otimes \psi_{t'}
= \frac{1}{2} \sum_{t, t' \in \mathbb{T}_m, \sigma \in \Sigma} \int_0^{\hat{X}_1} \int_0^{\hat{X}_2} (\rho_q^* \psi_t) \otimes \psi_{t'}
= \frac{1}{2} \sum_{\sigma, \sigma'} \xi'_\sigma(\hat{X}_1) \xi'_\sigma(\hat{X}_2)
= -|G_0|^2 [z_1 z_2] \sum_{\sigma, \sigma', \sigma'' \in \Sigma} \tilde{\xi}_\sigma''(z_1, \tilde{X}_1) \tilde{\xi}_\sigma''(z_2, \tilde{X}_2) S(\phi_\sigma(\tau), \phi_{\sigma''}) |_{t, Q=1} S(\phi_\sigma(\tau), \phi_{\sigma}) |_{t, Q=1}
= -|G_0|^2 [z_1 z_2] \sum_{\sigma, \sigma'' \in \Sigma} V(\phi_\sigma, \phi_{\sigma''}) |_{t, Q=1} \tilde{\xi}_\sigma''(z_1, \tilde{X}_1) \tilde{\xi}_\sigma''(z_2, \tilde{X}_2)
= -|G_0|^2 [z_1 z_2] \frac{\partial}{\partial \hat{X}_1} \frac{\partial}{\partial \hat{X}_2} F_0^{X, (L, f)}(\tau; \hat{X}_1, \hat{X}_2).
\]

where the second equality follows from Lemma 6.8, the fourth equality is WDVV (Equation 18), and the last equality follows from (27). Both \( W_{0,2}(q; \hat{X}_1, \hat{X}_2) \) and \( F_0^{X, (L, f)}(\tau; \hat{X}_1, \hat{X}_2) \) are \( H_{CM}(B \mu_m; \mathbb{C}) \otimes \mathbb{C} \)-valued power series in \( \hat{X}_1, \hat{X}_2 \) which vanish at \((\hat{X}_1, \hat{X}_2) = (0, 0)\), so
\[ \hat{F}_{0,2}(q; \hat{X}_1, \hat{X}_2) = -|G_0|^2 F_0^{X, (L, f)}(\tau; \hat{X}_1, \hat{X}_2). \]

\[ \square \]

**Theorem 7.5** (All genus open-closed mirror symmetry, a.k.a. BKMP remodeling conjecture). Under the open and closed mirror maps,
\[ \hat{F}_{g,n}(q, \hat{X}_1, \ldots, \hat{X}_n) = (-1)^{g-1+n} |G_0|^n F_{g,n}^{X, (L, f)}(\tau; \hat{X}_1, \ldots, \hat{X}_n). \]

**Proof.** For the unstable cases \((g, n) = (0, 1)\) and \((0, 2)\), this theorem is Theorem 7.3 and Proposition 7.4 respectively.

For stable cases \(2g - 2 + n > 0\), the graph sums are matched in Theorem 7.2 except for leafs. We match them here.

The A-model open leaf is
\[ (L^O)^{\sigma} = -\frac{1}{|G_0|} [z^k] \int_0^{\tilde{X}} \sum_{\sigma' \in \Sigma} \int_0^{\tilde{X}} \left( R_{\sigma'}(z) \right) \rho_\sigma^* \theta_\sigma(z) \psi_{t}. \]
By \( \theta_\sigma(z) = \sum_{\sigma' \in I_\Sigma} \left( R_{\sigma', \sigma}(-z)|_{t_3=\tau, Q=1} \right) \hat{\theta}_{\sigma'}(z) \) (Proposition [6.3]), the B-model open leaf is
\[
(L^O)^\sigma_k = [z^k] \frac{1}{\sqrt{-1}} \sum_\ell \int_0^\infty \sum_{\sigma' \in I_\Sigma} \left( R_{\sigma', \sigma}(-z)|_{t_3=\tau, Q=1} \right) \rho_\ell \hat{\theta}_{\sigma'}(z) \psi_\ell.
\]
Then \(|G_0|(L^O)^\sigma_k = -(\check{L}^O)^\sigma_k\), and this proves the BKMP Remodeling Conjecture.

\[\square\]

7.4. BKMP Remodeling Conjecture: the free energies.

7.4.1. The case \( g > 1 \). \( \| \mathcal{X}_{g,0} \) is a power series in \( \bar{Q} = Q e^{\tau'} \) and \( \tau'' \), so
\[
F_g^\mathcal{X}(\tau) := \langle \| \mathcal{X}_{g,0} \rangle_{t_3=1} \tau'
\]
is well-defined.

**Theorem 7.6.** When \( g > 1 \), we have
\[
F_g^\mathcal{X}(\tau) = (-1)^{g-1} F_3(q).
\]

**Proof.** From the proof of Theorem [7.2]
\[
w^A_{\mathcal{X}}(\bar{\mathcal{F}}) \left|_{(L^A)^\sigma_k=(L^1)^\sigma_k} \right. = (-1)^{g-1} w^A_{\mathcal{X}}(\bar{\mathcal{F}}) \left|_{(L^A)^\sigma_k=(L^1)^\sigma_k} \right. \tau, Q=1
\]
for any labelled graph \( \bar{\mathcal{F}} \in \Gamma_{g,1}(\mathcal{X}) \). Theorem 7.6 follows from Proposition [6.8] Proposition [6.10] and Equation [48].

Theorem 7.6 was proved in the special case \( \mathcal{X} = \mathbb{C}^3 \) in [11].

7.4.2. The case \( g = 1 \). The genus-one free energy has a different formula on both A-model and B-model. On A-model side, since the graph sum formula is for the case when \( 2g-2+n > 0 \), we need to find a different formula for \( F_1^\mathcal{X} \). In [93], the third author proved a formula for the genus-one Gromov-Witten potential of any GKM orbifolds. It expresses \( F_1^\mathcal{X} \) in terms of the Frobenius structures. In our case, we have the following theorem:

**Theorem 7.7 (Zong [93]).** The following formula holds for the genus one Gromov-Witten potential \( F_1^\mathcal{X}(\tau) \):
\[
dF_1^\mathcal{X}(\tau) = \frac{1}{48} d \log \Delta^\sigma(\tau) + \sum_{\sigma' \in I_\Sigma} \frac{1}{2} (R_1)_{\sigma', \sigma} d u^\sigma.
\]

On B-model side, the genus-one free energy is defined in the following way (see [42]):

**Definition 7.8 (genus-one B-model free energy).** The genus-one B-model free energy \( \check{F}_1 \) is defined as
\[
\check{F}_1 = -\frac{1}{2} \log \tau_B - \frac{1}{24} \sum_{\sigma' \in I_\Sigma} \log h_1^\sigma
\]
where \( \tau_B \) is the Bergmann \( \tau \)-function determined by
\[
d(\log \tau_B) = \sum_{\sigma} \text{Res}_{p=0} \left| \frac{B(p, \bar{p})}{d \bar{z}(p)} \right| d u^\sigma.
\]

The Bergmann \( \tau \)-function is defined up to a constant and so is \( \check{F}_1 \). The mirror symmetry for the genus-one free energy is the following theorem:
Theorem 7.9 (mirror symmetry for genus-one free energy). Under the closed mirror map,
\[ dF_1^X(\tau) = d\hat{F}_1(q) \]

Proof. First by the identification \( h_\tau^\sigma = \sqrt{\frac{-2}{\Delta^\sigma(\tau)}} \), we have
\[ -\frac{1}{24} \sum_{\sigma \in I_\Sigma} d\log h_1^\sigma = -\frac{1}{24} \sum_{\sigma \in I_\Sigma} d\log \sqrt{\frac{-2}{\Delta^\sigma(\tau)}} = \frac{1}{48} \sum_{\sigma \in I_\Sigma} d\log \Delta^\sigma(\tau). \]
So in order to prove the theorem, we only need to show that
\[ -\frac{1}{2} d\log \tau_B = \sum_{\sigma \in I_\Sigma} \left( \frac{1}{2} (R_1)_\sigma^\sigma \right) du^\sigma |_{\tau, Q = 1}. \]
Note that since \( \{ \bar{u}^\sigma \}_{\sigma \in I_\Sigma} \) is the set of B-model canonical coordinates and so \( du^\sigma |_{\tau, Q = 1} = d\bar{u}^\sigma \) for any \( \sigma \in I_\Sigma \). Therefore
\[ -\frac{1}{2} d\log \tau_B = \frac{1}{2} \sum_{\sigma \in I_\Sigma} \text{Res}_{p \to \bar{p}} B(p, \bar{p}) \frac{\partial B(p, \bar{p})}{\partial x(p)} du^\sigma |_{\tau, Q = 1}. \]

By the local expansions of \( \hat{x} \) and \( B(p, \bar{p}) \) near \( p_\sigma \), we have
\[ \hat{x} = \bar{u}^\sigma + \zeta_\sigma \]
\[ B(p, \bar{p}) = \left( \frac{1}{2(2\zeta_\sigma)^2} + \sum_{k, l \geq 0} B_{k, l} \zeta_\sigma^k (\zeta_\sigma)^l \right) d\zeta_\sigma d(-\zeta_\sigma). \]
Recall that
\[ \tilde{B}_{k, l} = \frac{(2k - 1)!!(2l - 1)!!}{2^{k+l+1}} B_{2k, 2l}. \]
Substituting the local expansions of \( \hat{x} \) and \( B(p, \bar{p}) \) into \( \text{Res}_{p \to \bar{p}} \frac{B(p, \bar{p})}{\partial x(p)} \), we have
\[ \text{Res}_{p \to \bar{p}} \frac{B(p, \bar{p})}{\partial x(p)} = -\tilde{B}_{0, 0} \]
\[ = -[z^0 w^0] \left( \frac{1}{z + w} (\delta_{\sigma, \sigma} - \sum_{\gamma \in I_\Sigma} f_\gamma (\frac{1}{z}) f_\gamma (\frac{1}{w})) \right) \]
\[ = -(R_1)_\sigma^\sigma \]
\[ = -(R_1)_\sigma^\sigma |_{\tau, Q = 1}. \]
Therefore
\[ -\frac{1}{2} d\log \tau_B = -\frac{1}{2} \sum_{\sigma \in I_\Sigma} \text{Res}_{p \to \bar{p}} \frac{B(p, \bar{p})}{\partial x(p)} du^\sigma |_{\tau, Q = 1} = \sum_{\sigma \in I_\Sigma} \frac{1}{2} (R_1)_\sigma^\sigma du^\sigma |_{\tau, Q = 1} \]
which finishes the proof. \( \square \)

7.4.3. The case \( g = 0 \). Another special case is the genus-zero free energy. In this case, instead of giving the definition of \( \tilde{F}_0 \) directly, we will use the special geometry property to build the mirror symmetry. Recall that we have the following special geometry property (see [32]):
\[ \frac{\partial \omega_{g,n}}{\partial \tau} (p_1, \ldots, p_n) = \int_{p_{n+1} \in B_i} \omega_{g,n+1}(p_1, \ldots, p_{n+1}), \quad i = 1, \ldots, p, \quad (g, n) \neq (0,0), (0,1). \]
Here when $n = 0$, the invariant $\omega_{g,0}$ is just the free energy $\tilde{F}_g$. We will use the special geometry property to show the following theorem:

**Theorem 7.10** (mirror symmetry for genus-zero free energy). For any $i, j, k \in \{1, \ldots, p\}$, we have

$$\frac{\partial^3 F_0^X}{\partial \tau^i \partial \tau^j \partial \tau^k} = \frac{\partial^3 \tilde{F}_0}{\partial \tau^i \partial \tau^j \partial \tau^k}(q)$$

under the closed mirror map.

**Proof.** Let $\{\gamma_1, \ldots, \gamma_p\}$ be the set of basis of $H^2_{\text{CH}, \mathcal{Y}}(X)$ corresponding to the coordinates $\{\tau^1, \ldots, \tau^p\}$. Then

$$\frac{\partial^3 F_0^X}{\partial \tau^i \partial \tau^j \partial \tau^k} = \langle \gamma_i, \gamma_j, \gamma_k \rangle_{0,3} \mid_{l = \tau, q = 1}.$$

By the graph sum formula described in Section 5.5, we know that $\langle \gamma_i, \gamma_j, \gamma_k \rangle_{0,3}$ has the same graph sum formula with that of $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,3}$ except that the ordinary leaves are replaced by

$$(52) \quad [z^0](\sum_{\rho \in I_0} \Psi_1^\rho R(-z)_\rho)$$

with $l = i, j, k$ respectively. Here $\Psi_1^\rho$ is defined as $\gamma_l = \sum_{\rho \in I_0} \Psi_1^\rho \partial_\rho(\tau)$.

Now let us consider $\frac{\partial^3 F_0}{\partial \tau^i \partial \tau^j \partial \tau^k}$. By the special geometry property (49) (50) (51), we have

$$\frac{\partial^3 F_0}{\partial \tau^i \partial \tau^j \partial \tau^k} = \frac{\partial^2}{\partial \tau^i \partial \tau^j} \int_{p \in B_k} \Phi(p) = \frac{\partial}{\partial \tau^i} \int_{p \in B_k} \frac{\partial \Phi}{\partial \tau^j}(p_1)$$

$$= \int_{p \in B_k} \int_{p \in B_k} \omega_{0,2}(p_1, p_2) = \int_{p \in B_k} \int_{p \in B_k} \omega_{0,3}(p_1, p_2, p_3).$$

By the graph sum formula for $\omega_{0,3}$, we know that $\int_{p \in B_k} \int_{p \in B_k} \omega_{0,3}(p_1, p_2, p_3)$ has the same graph sum formula with that of $\omega_{0,3}$ except that the ordinary leaves are replaced by

$$\frac{1}{\sqrt{-2}} \int_{p \in B_k} \theta^0_\rho(p)$$

with $l = k, j, i$ respectively. It is easy to see that

$$\theta^0_\rho(p) = [z^0]\left(\frac{-e^{\frac{\rho}{8}\tau}}{\sqrt{\pi}z} \int_{p' \in \Gamma_\rho} B(p, p') e^{-\frac{\rho}{8}\tau(p')}\right).$$
Therefore
\[
\frac{1}{\sqrt{-2}} \int_{p \in B_{\ell}} \theta^0_{\sigma}(p) = \frac{1}{\sqrt{-2}} \int_{p \in B_{\ell}} [z^0] \left( \frac{-e^{\sigma}}{\sqrt{\pi \tau}} \int_{p' \in \Gamma_{\tau}} B(p, p') e^{-\tau(e_{z^0})} \right)
\]
\[
= [z^0] \left( \frac{-1}{\sqrt{-2}} e^{\sigma} \int_{p' \in \Gamma_{\tau}} \frac{\partial \Phi}{\partial \tau} e^{-\tau(e_{z^0})} \right)
\]
\[
= [z^0] \left( \frac{-e^{\alpha}}{\sqrt{-2} \pi z} S_{\ell} \sigma \right)
\]
\[
= [z^0] \left( - \sum_{\rho \in I_{\Sigma}} \Psi_1^\rho R(z)^\rho \right)
\]
\[
= [z^0] \left( - \sum_{\rho \in I_{\Sigma}} \Psi_1^\rho R(-z)^\rho \right) \mid_{\ell = \tau, Q = 1}.
\]

Comparing with (52), we see that the three new ordinary leaves on A-model differ those on B-model by a minus sign. So by Theorem 7.2 for \((g, n) = (0, 3)\), we conclude that
\[
\frac{\partial^3 F^X}{\partial \tau^i \partial \tau^j \partial \tau^k} = -\frac{\partial^3 \tilde{F}_0}{\partial \tau^i \partial \tau^j \partial \tau^k}.
\]

\[\square\]

**Remark 7.11.** The proof of Theorem 7.10 can be directly generalized to show that the first derivatives of \(F^X_g\) match the first derivatives of \(\tilde{F}_g\) for any \(g \geq 1\) by replacing \([z^0]\) by \([z^k]\) for any \(k \in \mathbb{Z}_{\geq 0}\) in the computation of new ordinary leaves. In particular, this gives another proof of Theorem 7.9.

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