THE WILD MCKAY CORRESPONDENCE FOR CYCLIC GROUPS OF PRIME POWER ORDER

MAHITO TANNO AND TAKEHIKO YASUDA

Abstract. The $v$-function is a key ingredient in the wild McKay correspondence. In this paper, we give a formula to compute it in terms of valuations of Witt vectors, when the given group is a cyclic group of prime power order. We apply it to study singularities of a quotient variety by a cyclic group of prime square order. We give a criterion whether the stringy motive of the quotient variety converges or not. Furthermore, if the given representation is indecomposable, then we also give a simple criterion for the quotient variety being terminal, canonical, log canonical, and not log canonical. With this criterion, we obtain more examples of quotient varieties which are klt but not Cohen–Macaulay.

1. Introduction

The subject of this paper is the wild McKay correspondence for cyclic groups of prime power order. The $v$-function plays an essential role in the wild McKay correspondence; also it is considered as a common generalization of the age invariant in the tame McKay correspondence and of the Artin conductor, an important invariant in the number theory, see Wood–Yasuda [19] for details. In spite of its importance, it is difficult to compute the $v$-functions in a general situation. We give an explicit formula of this function in the case of cyclic group of prime power order, generalizing the one by the second author [20] for the case of prime order. We then apply it to study the discrepancies of singularities of quotient varieties by the cyclic group of prime square order.

The McKay correspondence relates an invariant of a representation $V$ of a finite group $G$ with an invariant of the associated quotient variety $X := V/G$. Depending on which type of invariant one considers, there are different approaches to the McKay correspondence. The one using motivic invariants originates in the works of Batyrev [2] and Denef and Loeser [8] in characteristic zero. The second author [24] generalized their results to arbitrary characteristics, in particular, including the wild case, that is, the case where the finite group in question has order divisible by the characteristic of the base field. In what follows, we denote by $k$ an algebraically closed field of characteristic $p > 0$.

Theorem 1.1 ([24, Corollary 16.3]). Assume that $G$ acts on an affine space $\mathbb{A}^d_k$ linearly and effectively and that $G$ has no pseudo-reflection. Then we have

$$M_{st}(X) = \int_{G\text{-Cov}(D)} L^{d-v}.$$  

Here $M_{st}(X)$ denotes the stringy motive of the quotient variety $X$, $G\text{-Cov}(D)$ denotes the moduli space of $G$-covers of $D := \text{Spec } k[[t]]$, and $v$ is the $v$-function $v: G\text{-Cov}(D) \to \mathbb{Q}$ associated to the $G$-action on $\mathbb{A}^d_k$.

Since stringy motives contain information on singularities of the quotient variety, the above theorem allows us to study singularities of the quotient variety $X$ in terms of the moduli space $G\text{-Cov}(D)$ and the $v$-function on it. For this purpose, it is important to understand the precise structure of the moduli space and compute the $v$-function. The

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second author [20] worked out the case $G = \mathbb{Z}/p\mathbb{Z}$, using the Artin–Schreier theory. We extend it to the case $G = \mathbb{Z}/p^n\mathbb{Z} \ (n > 0)$, using the Artin–Schreier–Witt theory.

We find out that the $\sigma$-function in this case can be computed in terms of the ramification jumps of the field extension corresponding to the given $G$-cover. Let $E$ be a connected $G$-cover of $D$ and $L/k((t))$ the corresponding $G$-extension; the case of connected covers is essential and the case of non-connected covers is reduced to the case of a smaller group. According to the Artin–Schreier–Witt theory, the extension $L/k((t))$ is given by an equation $\varphi(g_0, g_1, \ldots, g_{n-1}) = (f_0, f_1, \ldots, f_{n-1})$, where $(f_0, f_1, \ldots, f_{n-1}) \in W_n(k((t)))$ is a reduced Witt vector. We can decompose the extension $L/k((t))$ into a tower of $p$-cyclic extensions

$$L = K_{n-1} \supset K_{n-2} \supset \cdots \supset K_1 = k((t)),$$

where $K_i = K_{i-1}(g_i)$. Key facts here are first that the value $\nu(E)$ of the $\sigma$-function at $E$ is expressed in terms of ramification jumps of extensions $K_i/K_{i-1}$ (Lemma 3.4) and second that these ramification jumps are determined by orders of $f_i$ (Corollary 3.10). We denote $l_m = -\text{ord } f_m$. Then the $(i+1)$-th upper ramification jump $u_i$ and $(i+1)$-th lower ramification jump $l_i$ are given by

$$u_i = \max \{p^{n-1-m}f_m \mid m = 0, 1, \ldots, i - 1\},$$
$$l_i = u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i,$$

see the proof of Theorem 3.11 for details. Since the $\sigma$-function is additive with respect to the direct sum of representations (it is immediate from Definition 3.3 see [19] Lemma 3.4), the case of indecomposable representations is essential. We also note that for each integer $d \leq p^n$, there exists exactly one indecomposable representation of dimension $d$ modulo isomorphisms (see, for instance, [7] p. 431, (64.2) Lemma); it corresponds to the Jordan block of size $d$ with eigenvalue 1.

**Theorem 1.2 (Theorem 3.11).** Assume that the $G$-representation $V$ is indecomposable of dimension $d$. With the notation as above, we have

$$\sigma(E) = \sum_{0 \leq h_0 + \cdots + h_{n-1} < d} \left[ \frac{l_0p^{n-1} + i_1p^{n-2}l_1 + \cdots + i_{n-1}l_{n-1}}{p^d} \right].$$

In relation to the minimal model program, it is natural to ask: how can we determine representation-theoretically when a quotient variety $V/G$ (with $G$ an arbitrary finite group) is terminal, canonical, log terminal or log canonical? From the wild McKay correspondence with the formula (2), we can give a partial answer to this question. Note that the formula (2) implicitly includes many maxima so that we have to make a case-by-case analysis to compute the integral $\int_{G/\text{Con}(D)} L^{d-n}$. Thus the computation rapidly becomes harder, as the exponent $n$ increases. For this reason, we focus on the case $n = 2$ to evaluate the integral and get some results on singularities. Before stating our results in this direction, we need to introduce some invariants of representations. For an indecomposable representation $V$ of $G = \mathbb{Z}/p^2\mathbb{Z}$ of dimension $d$, writing $d = qp + r \ (0 \leq r < p)$, we define

$$B_V = \frac{qp(q - 1)}{2} + qr,$$
$$C_V = p \left( \frac{qp(p - 1)}{2} + \frac{r(r - 1)}{2} \right) + (p^2 - p + 1) \left( \frac{qp(q - 1)}{2} + qr \right).$$

We generalize them to decomposable representations in the way that they become additive for direct sums.
Theorem 1.3 (Theorem 5.3). Assume that $G = \mathbb{Z}/p^2\mathbb{Z}$. The integral $\int_{G\text{-Cov}(D)} L_d \psi$ converges if and only if the following inequalities hold:

$$B_V \geq p, \quad C_V \geq p^3 - p + 1.$$ 

From the wild McKay correspondence, the convergence of the integral $\int_{G\text{-Cov}(D)} L_d \psi$ is equivalent to that of the stringy motive $M_G(X)$. The latter implies that $X$ is log terminal and the converse holds if the pair has a log resolution. Thus we obtain the following corollary:

Corollary 1.4 (Corollary 5.4). Assume that $G = \mathbb{Z}/p^2\mathbb{Z}$ has no pseudo-reflection. If the inequalities $B_V \geq p$ and $C_V \geq p^3 - p + 1$ hold, then the quotient variety $X = V/G$ is log terminal. Furthermore, if there exists a log resolution of $X$, then the converse is also true.

Furthermore, for an indecomposable $\mathbb{Z}/p^2\mathbb{Z}$-representation $V$, we can estimate the discrepancies/total discrepancy of the quotient variety:

Theorem 1.5 (Theorem 5.16). Assume that $G = \mathbb{Z}/p^2\mathbb{Z}$ and $V$ is an indecomposable $G$-representation of dimension $d$ ($p + 1 < d \leq p^2$). Then,

$$X \begin{cases} \text{terminal,} \\ \text{canonical,} \\ \text{log canonical,} \\ \text{not log canonical} \end{cases} \quad \text{if and only if} \quad \begin{cases} d \geq 2p + 1, \\ d \geq 2p, \\ d \geq 2p - 1, \\ d < 2p - 1. \end{cases}$$

We note that the indecomposable representation of $\mathbb{Z}/p^2\mathbb{Z}$ of dimension $d$ is not effective if $d \leq p$, has pseudo-reflections if $d = p + 1$ and does not have a pseudo-reflection if $d > p + 1$.

Related to the minimal model program in positive characteristics, some singularities which are klt but not Cohen–Macaulay are constructed in recent years (see Kovács [11], Yasuda [20, 23], Cascini–Tanaka [6], Bernasconi [3], Arvidsson–Bernasconi–Lacini [1], and Totaro [18]; see also [22]). The theorem above provides more such examples; for instance, if $V$ is the indecomposable $\mathbb{Z}/4\mathbb{Z}$-representation of dimension 4 in characteristic 2, the quotient variety $V/(\mathbb{Z}/4\mathbb{Z})$ is canonical but not Cohen–Macaulay.

We now give a few comments on the case $G$ has pseudo-reflections. Generally, if a finite group $G$ has a pseudo-reflection, then we can find a $\mathbb{Q}$-Weil divisor $\Delta$ on $X = V/G$ such that $V \rightarrow (X, \Delta)$ is crepant. The wild McKay correspondence theorem holds for log pairs by replacing $M_G(X)$ by $M_G(X, \Delta)$. For a representation of $G = \mathbb{Z}/n\mathbb{Z}$ with general $n$, we determine when there is a pseudo-reflection (Corollary 5.7). Moreover we show that if the given effective $G$-representation has a pseudo-reflection, then the divisor $\Delta$ as above on the quotient variety $X$ is irreducible and has multiplicity $p - 1$ (Proposition 4.8), hence the pair $(X, \Delta)$ is not log canonical unless $p = 2$. If $p = n = 2$, then whether or not the pair is log canonical depends on whether the representation has a direct summand of dimension one (Remark 5.7).

We also note that although we work over an algebraically closed field throughout the paper for the simplicity reason, it is straightforward to generalize our results to any field of characteristic $p > 0$ simply by the base change.

The outline of this paper is as follows. In Section 2 we first recall basic facts about the Artin–Schreier–Witt theory. After that, we describe the moduli space $G\text{-Cov}(D)$ of $G$-covers of $D = \text{Spec } k[[t]]$ and decompose it to strata $G\text{-Cov}(D; j)$. In Section 3 we see that $q$-functions are written by valuations of Witt vectors and by upper/lower ramification jumps of $G$-extensions. In Section 4 we briefly review the wild McKay correspondence and its application to singularities. In Section 5 we discuss the case $G = \mathbb{Z}/p^2\mathbb{Z}$ and give our main results as corollaries of Theorems 5.11.
2.1. The Artin–Schreier–Witt theory. Let us recall some basic facts from the theory of Witt vectors. We denote by $W_m(K)$ the ring of Witt vectors of length $m$ over $K$. We introduce important morphisms. One is the Frobenius morphism

$$\text{Frob}: W_m(K) \to W_n(K), \ (a_0, a_1, \ldots) \mapsto (a_0^p, a_1^p, \ldots).$$

We denote by $\varphi := \text{Frob} - \text{id}$ the Artin–Schreier morphism. The other is the Verschiebung morphism

$$W_m(K) \to W_{m+1}(K), \ (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots).$$

They are homomorphisms of additive groups. Note that the Verschiebung morphism commutes with $\varphi$ and that

$$(a_0, a_1, \ldots, a_{l-1}, a_l, \ldots) = (a_0, a_1, \ldots, a_{l-1}, 0, \ldots) + (0, \ldots, 0, a_l, a_{l+1}, \ldots)$$

holds for every $l \geq 1$.

Let us denote by $K^{\text{sep}}$ the separable closure of $K$, by $K_{p^n}$ the maximal abelian extension of exponent $p^n$ over $K$. As sets, we can describe

$$\text{G-Cov}(D^*) = \text{Hom}_{\text{cont}}(\text{Gal}(K^{\text{sep}}/K), \mathbb{Z}/p^n\mathbb{Z})$$

$$= \text{Hom}_{\text{cont}}(\text{Gal}(K_{p^n}/K), \mathbb{Z}/p^n\mathbb{Z})$$

$$= \text{Hom}_{\text{cont}}(\text{Gal}(K_{p^n}/K), \mathbb{Z}/p^n\mathbb{Z})$$

$$\subseteq \text{Hom}_{\text{cont}}(\text{Gal}(K_{p^n}/K), \mathbb{Z}/p^n\mathbb{Z})$$

$$= W_n(K)/\varphi(W_n(K)).$$

Since every element of $\text{Gal}(K_{p^n}/K)$ has order dividing $p^n$, the image of any morphisms $\text{Gal}(K_{p^n}/K) \to \mathbb{Q}/\mathbb{Z}$ is contained in $(1/p^n)\mathbb{Z}/\mathbb{Z}$ and hence the equality $(\subseteq)$ holds (see [13] pp. 340–341 for details). The equality $(\subseteq)$ is a consequence of [14] Theorem 6.1.9). For a Witt vector $f \in W_n(K)$, we denote by $E_f^p$ the $G$-cover of $D^*$ corresponding to the class of $f$. Note that $E_f^p$ is connected if and only if $f_0 \notin \varphi(K)$. More explicitly, we can see the following (see, for instance, [12] Chapter VI, Exercise 50]).

Proposition 2.1. For a Galois extension $L/K$, it is $p^n$-cyclic if and only if there exists a Witt vector $f = (f_0, f_1, \ldots, f_{n-1}) \in W_n(K)$ with $f_0 \notin \varphi(K)$ such that $L = K(g_0, g_1, \ldots, g_{n-1})$ where the Witt vector $g = (g_0, g_1, \ldots, g_{n-1})$ is a root of an equation $\varphi(g) = f$. Moreover, a generator $\sigma$ of the Galois group $\text{Gal}(L/K)$ is given by $\sigma(g) = g + 1$.

We next find good representatives of elements of $W_n(K)/\varphi(W_n(K))$. 

Notation and convention. Unless otherwise noted, we follow the following notation. We denote by $k$ an algebraically closed field of characteristic $p > 0$ and by $K = k((t))$ the field of formal Laurent power series over $k$. We set $G = \langle \sigma \rangle$ a cyclic group of order $p^n$.

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Notation 2.2. We put \( \mathbb{N}' := \{ j \in \mathbb{Z} \mid j > 0, p \nmid j \} \).

Lemma 2.3. For \( f \in K \), there exists a unique Laurent polynomial of the form
\[
g = \sum_{i \in \mathbb{N}'} g_i t^{-i} \in k[t^{-1}] \subset K
\]
such that \( f - g \in \varphi(K) \).

Proof. See, for instance, [20, Lemma 2.3]. \( \square \)

We call a Laurent polynomial of the above form a representative polynomial. We denote by \( \text{RP}_k \) the set of representative polynomials. We can extend Lemma 2.3 as follows.

Lemma 2.4. For a Witt vector \( f \in W_n(K) \), there exists a unique \( g = (g_0, g_1, \ldots, g_{n-1}) \in W_n(K) \) such that each \( g_i \) is a representative polynomial and \( f - g \in \varphi(W_n(K)) \).

Proof. We prove by induction on \( n \). The case \( n = 1 \) is just Lemma 2.3. Let us denote by \( f_i \) the \( i \)-th component of \( f \). Take \( h_0 \in K \) satisfying \( g_0 = f_0 \) \( \mod \varphi(h_0) \), where \( g_0 \) is the unique representative polynomial. In the Witt ring \( W_n(K) \), we have
\[
(f_0, \ldots) + \varphi(h_0, \ldots) = (f_0 + \varphi(h_0), \ldots)
\]
without loss of generality, we may assume that \( f = (g_0, f_1, \ldots) \). From the induction hypothesis, there exists \( g_1, g_2, \ldots, g_{n-1} \) uniquely such that each \( g_i \) is a representative polynomial and
\[
(f_1, \ldots, f_{n-1}) \equiv (g_1, \ldots, g_{n-1}) \pmod{\varphi(W_{n-1}(K))}
\]
holds. Since the Verschiebung morphism commutes with \( \varphi \), thus we have
\[
(0, f_1, \ldots, f_{n-1}) \equiv (0, g_1, \ldots, g_{n-1}) \pmod{\varphi(W_{n}(K))}.
\]
Then
\[
(g_0, f_1, \ldots, f_{n-1}) = (g_0, 0, \ldots, 0) + (0, f_1, \ldots, f_{n-1})
\]
\[
\equiv (g_0, 0, \ldots, 0) + (0, g_1, \ldots, g_{n-1}) \pmod{\varphi(W_{n}(K))}
\]
\[
= (g_0, g_1, \ldots, g_{n-1}).
\]
The first and last equality follows from the property of the Verschiebung morphism. Therefore, we have proved the existence of \( g \).

Next, we show the uniqueness. [1] shows that the first entry \( g_0 \) is uniquely determined. Suppose that \((g_0, g_1, \ldots, g_{n-1})\) and \((g_0, g'_1, \ldots, g'_{n-1})\) satisfy the condition. Then we have
\[
(0, f_1, \ldots, f_{n-1}) \equiv (0, g_1, \ldots, g_{n-1}) \equiv (0, g'_1, \ldots, g'_{n-1}) \pmod{\varphi(W_{n}(K))}.
\]
Again from the induction hypothesis, this shows that \( g_1, \ldots, g_{n-1} \) are uniquely determined. \( \square \)

We call a Witt vector \( g = (g_0, g_1, \ldots, g_{n-1}) \) consisting of representative polynomials \( g_i \) is a representative Witt vector. More generally, a Witt vector \( f = (f_i)_l \subset W_n(K) \) is called reduced (or standard form) if \( p \nmid \varphi_l(f_i) \) and \( \varphi_l(f_i) < 0 \) for every \( l \), where \( \varphi_l \) denotes the normalized valuation on \( k \).

Corollary 2.5. We have a one-to-one correspondence
\[
G \text{-Cov}(D^*) \leftrightarrow (\text{RP}_k)^n.
\]

Remark 2.6. The corollary shows that \( G \text{-Cov}(D^*) \) is identified with the \( k \)-point set of the ind-scheme \( \mathbb{A}_k^\infty \equiv \varprojlim_{n \in \mathbb{N}} \mathbb{A}_k^n \), where the transition map \( \mathbb{A}_k^n \to \mathbb{A}_k^{n+1} \) is the standard closed embedding. In fact, the coarse moduli space of \( G \text{-Cov}(D^*) \) is the inductive perfection (that is, the inductive limit with respect to Frobenius morphisms) of this space \( \mathbb{A}_k^\infty \), see [9]. To get the fine moduli stack, we further need to take the product of it with the stack \( BG \), see [15].
2.2. Stratification and parameterization. In what follows, we follow the convention that \( \text{ord} 0 = \infty \). For a Witt vector \( f = (f_i)_i \in W_n(K) \), we denote the vector \( \text{ord} f := (\text{ord} f_i)_i \). When \( E^* \) is a \( G \)-cover of \( D^* \) corresponding to the representative Witt vector \( f \), we denote \( \text{ord} E^* = \text{ord} f \).

**Definition 2.7.** For an \( n \)-tuple \( j = (j_i)_i \in (\mathbb{N} \cup \{-\infty\})^n \), set \( -j = (-j_i)_i \). We define \( G^-\text{Cov}(D^*; j) := \{ E^* \in G^-\text{Cov}(D^*) \mid \text{ord} E^* = -j \} \).

\[
\text{RP}_{k,j} := \prod_{i=0}^{n-1} \{ f \in \text{RP}_k \mid \text{ord} f = -j_i \}
\]

For the case \( j = (j_0) \), we write \( \text{RP}_{k,j} \) in stead of \( \text{RP}_{k,(j_0)} \).

**Remark 2.8.** We remark that we consider \( \mathbb{N} \cup \{0\} \) instead of \( \mathbb{N} \cup \{-\infty\} \) in the previous paper [20]. However, our convention in the present paper is more suitable for computation below.

When \( n = 1 \), we have the following one-to-one correspondences (see [20, 10, Proposition 2.11])

\[
G^-\text{Cov}(D^*; j) \leftrightarrow \text{RP}_{k,j} \leftrightarrow k^\times \times k^{j_1 - \lfloor j_1/p \rfloor}.
\]

Here \([\bullet]\) denotes the floor function, which assigns a real number \( a \) to the greatest integer \( \lfloor a \rfloor \) less than or equal to \( a \). When \( j = -\infty \), the space \( G^-\text{Cov}(D^*; -\infty) \) is a point.

The following is straightforward.

**Proposition 2.9.** For \( j = (j_i)_i \in (\mathbb{N} \cup \{-\infty\})^n \), we have one-to-one correspondences

\[
G^-\text{Cov}(D^*; j) \leftrightarrow \text{RP}_{k,j} \leftrightarrow \prod_{j_1 \neq -\infty} \left( k^\times \times k^{j_1 - \lfloor j_1/p \rfloor} \right).
\]

We now regard \( k^\times \times k^n \) as the variety \( G_{m,k} \times A_k^n \). Then the above correspondence gives a structure of variety to \( G^-\text{Cov}(D^*; j) \). Thus, \( G^-\text{Cov}(D^*; j) \) can be thought of as an infinite-dimensional space admitting the stratification

\[
G^-\text{Cov}(D^*) = \bigsqcup_j G^-\text{Cov}(D^*; j)
\]

into countable finite-dimensional strata.

**Remark 2.10.** Varieties \( G_{m,k} \times A_k^n \) are neither fine or coarse moduli spaces of \( G \)-covers (see Remark 2.6). However we can construct families of \( G \)-covers over these spaces in a similar way as in [20, Section 2.4] and get morphisms from these spaces to the corresponding fine moduli stacks which are bijective on geometric points. Thus, as justified in [17], we can use the above varieties as our parameter spaces of \( G \)-covers in our context of motivic integration.

2.3. Explicit description of \( G \)-actions on \( G \)-covers. Let \( f = (f_i)_i \in \text{RP}_{k,j} \) be a representative Witt vector of order \( \text{ord} f = -j \) and \( g = (g_i)_i \) a root of \( \varphi(g) = f \). We assume that the extension \( L = K(g) \) is a \( G \)-extension of \( K \) and that the generator \( \sigma \) of \( G \) acts on \( L \) by \( \sigma(g) = g + 1 \). We can decompose the extension \( L/K \) into a tower of \( p \)-cyclic extensions

\[
L = K_{n-1} \supset K_{n-2} \supset \cdots \supset K_0 \supset K_{-1} = K
\]

where \( K_i = K_{i-1}(g_i) \). Indeed, \( \sigma^{p^i}|_{K_i} \) fixes \( K_{i-1} \) and its order is \( p \). For each extension \( K_i/K_i-1, g_i \) is a root of an equation

\[
\sigma_i^{p_i} - g_i + (\text{polynomial in } g_{0}, g_1, \ldots, g_{i-1}) = f_i.
\]

We denote by \( \vartheta_{K_i} \) the normalized valuation on \( K_i \). For each \( i \), there exists an \( h_i \in K_{i-1} \) such that \( \tilde{f}_i = (g_i + h_i)^p - (g_i + h_i), \) \( p \not| \vartheta_{K_{i-1}}(\tilde{f}_i) \) and \( \vartheta_{K_{i-1}}(\tilde{f}_i) < 0 \). We set \( \tilde{g}_i := g_i + h_i \) and \( \tilde{g} := g_1 \). Since \( g_i^p \quad (0 \leq i < p) \) form a basis of \( K_i/K_i-1, \) thus \( g_0^p, g_1^p, \ldots, g_{i-1}^p \quad (0 \leq i_0, i_1, \ldots, i_{n-1} < p) \) form a basis of \( L/K \).
Notation 2.11. For a $k$-algebra $M$ endowed with a $G$-action, we denote $\delta := \sigma - \id_M$ a $k$-linear operator. For $d \in \mathbb{Z}_{\geq 0}$, we write $M^{\delta = 0} := \text{Ker}(\delta^d : M \to M)$.

For an $n$-tuple $I = (i_0, i_1, \ldots, i_{n-1}) \in \{0, 1, \ldots, p-1\}^n$, we use a multi-index notation $\tilde{g}^I = \tilde{g}_0^{i_0} \tilde{g}_1^{i_1} \cdots \tilde{g}_{n-1}^{i_{n-1}}$. We remark that to give an $n$-tuple $I = (i_1)$ is equivalent to give an integer $a_1 = \sum_{i=0}^{n-1} i p^i$.

Proposition 2.12. For any integer $a_1$ with $1 \leq a_1 < p^n$ and for any $h \in K$, we have $\delta^{a_1}(\tilde{g}^I h) \in k^\times \cdot h$ and $\delta^{a_1+1}(\tilde{g}^I h) = 0$. Therefore, for each integer $d$ with $0 \leq d \leq p^n$, we have

$$L^{\delta^{a_1} = 0} = \bigoplus_{a_i = 0}^{d-1} K \cdot \tilde{g}^I.$$

Proof. The case $n = 1$ is [20, Lemma 2.15]. By direct computation, we get $\delta^m = \sigma^m - \id$ for $0 \leq m \leq n$. The Artin–Schreier–Witt theory says that $\sigma^m$ fixes the subfield $K_{m-1} = K(g_0, g_1, \ldots, g_{m-1})$ and that $\sigma^m(g_m) = g_m + 1$. Furthermore, $\delta^m$ is not only $k$-linear but also $K_{m-1}$-linear. For $1 \leq i_m < p$, $\delta^m(\tilde{g}^m) = (\tilde{g} + 1)^m - \tilde{g}^m = i_m \tilde{g}_{m-1} + \cdots + i_m \tilde{g} + 1$.

Applying $(\delta^m)^{m-1}$ to this, by the induction on $i_m$, we get

$$(\delta^m)^{m-1}(\tilde{g}^m) = i_m \cdot (\delta^m)^{m-1}(\tilde{g}^m-1)$$

and hence $(\delta^m)^{m-1}(\tilde{g}^m) = i_m!$. Then we have

$$\delta^{a_1+m}\cdots i_1 p^n \tilde{g}^I \tilde{g}_0^{i_0} \tilde{g}_1^{i_1} \cdots \tilde{g}_n h = \delta^{a_1+m-i_1 \cdots i_{n-1}} \cdot \tilde{g}^I \tilde{g}_0^{i_0} \tilde{g}_1^{i_1} \cdots \tilde{g}_n h = \delta^{a_1+m-i_1 \cdots i_{n-1}} \cdot \tilde{g}^I \tilde{g}_0^{i_0} \tilde{g}_1^{i_1} \cdots \tilde{g}_n^{i_{n-1}} \cdot \tilde{g}^I h,$$

and the first assertion follows from the induction on $n$.

It is clear that $L^{\delta^{a_1} = 0} = L = \bigoplus_{a_i = 0}^{p^n-1} K \cdot \tilde{g}^I$ holds. Assume $x = \sum_{i} x_i \tilde{g}^I \in L^{\delta^{a_1} = 0}$. Since $L^{\delta^{a_1} = 0} \subset L^{\delta^{a_1} = 0}$, thus we have $x_i = 0$ for $a_i \geq d$ by the induction on $d$. From the first assertion, we have $\delta^{d-1}(x) = \delta^{d-1}(x_j \tilde{g}^I) = 0$, where $j$ is the index satisfying $a_j = d - 1$. Again from the first assertion, this shows $x_j = 0$. Therefore, we have $L^{\delta^{a_1} = 0} \subset \bigoplus_{a_i = 0}^{d-2} K \cdot \tilde{g}^I$. The converse also follows from the first assertion. \hfill $\square$

Corollary 2.13. We denote by $D_L$ the integer ring of $L$ and $v_d$ the normalized valuation on $L$. For an $n$-tuple $I = (i_0, i_1, \ldots, i_{n-1}) \in \{0, 1, \ldots, p-1\}^n$, we put $n_I := \lceil -v_d(\tilde{g}^I) / p^n \rceil$. Here $\lceil \bullet \rceil$ denotes the ceiling function, which assigns a real number $a$ to the least integer $[a]$ greater than or equal to $a$. Then we have

$$D_L = \bigcap_{v_d(\tilde{g}^I t^n) \geq 0} k \cdot \tilde{g}^I t^n.$$

Moreover, for each integer $d$ with $0 \leq d \leq p^n$, we have

$$D_L^{\delta^{a_1} = 0} = \bigcap_{v_d(\tilde{g}^I t^n) \leq 0, 0 \leq a_i < d} k \cdot \tilde{g}^I t^n.$$

Proof. By definition, $v_k(\tilde{g}^I)$ takes distinct values modulo $p$ when $i_l$ runs from $0$ to $p - 1$. Therefore, $v_d(\tilde{g}^I)$ takes distinct values modulo $p^n$ when $a_i$ runs from $0$ to $p^n - 1$. This proves the first assertion. The second assertion follows from Proposition 2.12 and the first assertion. \hfill $\square$
3. \( \nu \)-FUNCTIONS

Suppose that we are given a faithful \( k \)-linear action of \( G \) on \( k^d \). Let \( E \) be a \( G \)-cover of \( D = \text{Spec} \ k[[t]] \), which means the normalization of \( D \) in a \( G \)-cover \( D' = \text{Spec} \ K \). Let \( \Omega_E \) be the coordinate ring of \( E \). Then the direct sum \( \Omega_E^d \) has two \( G \)-actions. One is the diagonal action induced from the \( G \)-action on \( \Omega_E \). The other is given by the composition \( G \to GL(d, k) \hookrightarrow GL(d, \Omega_E) \), where the left map is associated to the \( G \)-action on \( k^d \).

**Definition 3.1.** We define the tuning module \( \Xi_E \subset \Omega_E^d \) to be the submodule of elements on which the two actions above coincide.

**Lemma 3.2** ([21] Proposition 6.3]). The tuning module \( \Xi_E \) is a free \( k[[t]] \)-module of rank \( d \).

**Definition 3.3.** We define the \( \nu \)-function \( \nu : G \text{-Cov}(D) \to \mathbb{Q} ; E \mapsto \nu(E) \) as follows. Let \( x_i = (x_{ij})_{1 \leq j \leq d} \in \Omega_E^d \) \( (1 \leq i \leq d) \) be a \( k[[t]] \)-basis of \( \Xi_E \). Then we define

\[
\nu(E) = \frac{1}{\#G} \text{length} \left( \frac{\Omega_E}{(\det (x_{ij})_{i,j})} \right)
= \frac{1}{\#G} \text{length} \Omega_E^{\#} \cdot \Xi_E.
\]

By abuse of notation, we write \( \Xi_{E'} = \Xi_E \) and \( \nu(E') = \nu(E) \), because \( E \) is the normalization of \( D \) in \( E' \).

The \( \nu \)-function depends on the given \( G \)-representation. We sometimes write the \( \nu \)-function as \( \nu_V \), referring to the representation \( V \) in question. If \( E \) is connected and \( \nu \) denotes the normalized valuation on \( \Omega_E \), then we have

\[
\nu(E) = \frac{1}{\#G} \nu \left( \det (x_{ij})_{i,j} \right).
\]

When \( E \) is not connected and \( E' \) is a connected component with the stabilizer subgroup \( H \subset G \), then we have

\[
\nu_V(E) = \nu_W(E'),
\]
where \( W \) is the restriction of \( V \) to \( H \).

### 3.1. The indecomposable case

Let \( V \) be an indecomposable \( G \)-representation of dimension \( d \). Since \( \nu_{V \oplus W} = \nu_V + \nu_W \) holds, thus the case of indecomposable representations is essential. We denote the coordinate ring of the affine space \( V' \) by \( k[x] = k[x_1, x_2, \ldots, x_d] \). We choose coordinates so that the chosen generator \( \sigma \) of \( G \) acts by

\[
x_i \mapsto \begin{cases} x_i + x_{i+1} & (i \neq d) \\ x_d & (i = d). \end{cases}
\]

It amounts to taking the Jordan standard form of \( \sigma \). We have \( d \leq p^n \), since the order of a Jordan block of size \( m \) with eigenvalue \( 1 \) is the greatest power of \( p \) that does not exceeding \( m \). Let \( E' \) be a \( G \)-cover of \( D' = \text{Spec} \ K \). We also assume that \( E' = \text{Spec} \ L \), where \( L/K \) is a \( G \)-extension. With the notation of Section 2.3, the tuning module \( \Xi_{E'} \) of \( E' \) is written as

\[
\Xi_{E'} = \left\{ (a_1, a_2, \ldots, a_d) \in \mathcal{O}_L^d \mid \sigma(a_i) = a_i + a_{i+1} (1 < i < d), \sigma(a_d) = a_d \right\}
= \left\{ (a, \delta(a), \ldots, \delta^{d-1}(a)) \in \mathcal{O}_L^d \mid a \in \mathcal{O}_L^{p^d=0} \right\}.
\]

**Corollary 2.13** gives us a \( k[[t]] \)-basis of \( \mathcal{O}_L^{p^d=0} \). Then, we now have

**Lemma 3.4.** With the notation above, we have

\[
\nu(E') = \sum_{0 \leq i_0, i_1, \ldots, i_{n-1} < d} \frac{i_0 v_{L}(\tilde{g}_0) + i_1 v_{L}(\tilde{g}_1) + \cdots + i_{n-1} v_{L}(\tilde{g}_{n-1})}{p^n}.
\]
Proof. Let \( n_1 \) be an integer as in Corollary 2.13. By Proposition 2.12, we find that the matrix \( (\delta^m(\tilde{g}^t r^nu))_{1,m} \) is a triangular and that the diagonal components \( \delta^m(\tilde{g}^t r^nu) \) are of the form \( h t^{nu} \) (\( 0 \neq h \in k \)). Then

\[
\sigma(E^*) = \frac{1}{\# G} v_L \left( \det (\delta^m(\tilde{g}^t r^nu))_{1,m} \right)
\]

\[
= \frac{1}{p^n} \sum_{\nu \leq \sigma < d} v_L(t^{nu})
\]

\[
= \sum_{\nu \leq \sigma < d} n_{t,}
\]

which is the desired conclusion. \( \square \)

3.2. Ramification jumps. We next determine the values \( v_L(\tilde{g}_t) \) by studying ramification of \( L/K \). We begin with recalling the notions of lower and upper ramification groups. The basic reference here is [15]. Let \( K \) be a complete discrete valuation field with the perfect residue field of characteristic \( p > 0 \). Consider a finite Galois extension \( L/K \). We denote the valuation ring of \( L \) by \( O_L \) and the prime ideal of \( O_L \) by \( \mathfrak{p}_L \). Put \( G := \text{Gal}(L/K) \).

For each integer \( i \geq -1 \), we set

\[
G_i := \{ \gamma \in G \mid \gamma \text{ acts trivially on } O_L/\mathfrak{p}^{i+1}_L \}
\]

and call it the \( i \)-th lower ramification group of \( L/K \). The lower ramification groups form a descending sequence \( \{G_i\}_i \) of normal subgroups of \( G \), and \( G_i = \{1\} \) for sufficiently large \( i \).

Let us next define upper ramification groups. We put for \( t \in \mathbb{R}_{\geq -1} \)

\[
G_t := G_{\{t\}},
\]

\[
(G_0 : G_t) := \begin{cases} 1 & (t < 0), \\
(G_0 : G_{\{t\}}) & (t \geq 0). \end{cases}
\]

We define the Hasse–Herbrand function \( \varphi = \varphi_{L/K} : \mathbb{R}_{\geq -1} \rightarrow \mathbb{R}_{\geq -1} \) by

\[
\varphi_{L/K}(u) := \int_{0}^{u} \frac{dt}{(G_0 : G_t)}
\]

This function \( \varphi_{L/K} \) is strictly increasing and a self-homeomorphism of \( \mathbb{R}_{\geq -1} \). We denote \( \psi = \varphi_{L/K}^{-1} : \varphi_{L/K}^{-1} \). For \( u \geq -1 \), we call \( G^u := G_{\varphi_{L/K}(u)} \) the \( u \)-th upper ramification group of \( L/K \). As with the lower ramification groups, the upper ramification groups form a descending sequence \( \{G^u\}_u \) of normal subgroups of \( G \), and \( G^u = \{1\} \) for sufficiently large \( u \).

For each subgroup \( H \subset G \) of \( G \) and for each integer \( i \geq -1 \), we have

\[
H_i = G_i \cap H.
\]

Here the filtration \( \{H_i\}_i \) on \( H \) is induced from the \( H \)-extension derived from \( L/K \). Similarly, for each normal subgroup \( H \subset G \) of \( G \) and for each real number \( u \geq -1 \), we have

\[
(G/H)^u = G^u H/H.
\]

We say that \( i \) is a lower ramification jump of \( L/K \) if \( G_i \neq G_{i+1} \). Also, we say that \( u \) is an upper ramification jump of \( L/K \) if \( G^u \neq G^u \) for all \( \epsilon > 0 \).

We now restrict ourselves to the case of our principal interest where \( G = \mathbb{Z}/p^n \mathbb{Z} \) and \( K = k((t)) \). From [15, p. 67, Corollary 2], each graded piece \( G_i/G_{i+1} \) is either 1 or \( \mathbb{Z}/p \mathbb{Z} \). Therefore, there are exactly \( n \) lower ramification jumps and hence there are exactly \( n \) upper ramification jumps.

By definition, we can write

\[
\psi(u) = \int_{0}^{u} (G^0 : G^w)dw.
\]
Let $u_0 < u_1 < \cdots < u_{m-1}$ be the upper ramification jumps. Then, for any real number $u$ with $u_{i-1} < u \leq u_i$, we have $(G^0 : G^u) = p^i$. We remark that $u_0 \geq 0$ because the residue field $k$ is algebraically closed. Therefore, we get

\begin{equation}
\psi(u_i) = \int_0^{u_i} (G^0 : G^w)dw
\end{equation}

\begin{equation}
= \int_0^{u_0} + \int_{u_0}^{u_1} + \cdots + \int_{u_{i-1}}^{u_i} (G^0 : G^w)dw
\end{equation}

\begin{equation}
= u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i.
\end{equation}

Note that $\psi(u_i)$ are the lower ramification jumps of $L/K$ by definition. In particular, when $G = \mathbb{Z}/p\mathbb{Z}$, the unique lower ramification jumps is equal to the unique upper ramification jump; we call it simply the ramification jump.

The following is immediate from the equality $H_i = G_i \cap H$.

**Lemma 3.5.** The highest lower ramification jump of $K_m/K$ is equal to the ramification jump of $K_i/K_{i-1}$.

**Lemma 3.6.** The highest upper jump of $K_m/K$ is equal to the $(m+1)$-th upper jump $u_{m}$ of $L/K$.

**Proof.** Note that the Galois group of $K_m/K$ is the quotient of $G = \mathbb{Z}/p^n\mathbb{Z}$ by the subgroup $p^{m-1}\mathbb{Z}/p^n\mathbb{Z}$. From (2), the upper ramification jump of $K_m/K$ is exactly $u_0, u_1, \ldots, u_m$, those jumps from a subgroup of $G$ to another both of which contain $p^{m-1}\mathbb{Z}/p^n\mathbb{Z}$. The highest one among them is $u_m$. \hfill \square

The following claim follows from the lemmata above.

**Proposition 3.7.** The $(i+1)$-th upper ramification jump of $L/K$ is equal to the ramification jump of $K_i/K_{i-1}$.

**Proof.** The $(i+1)$-th upper ramification jump $u_i$ is equal to the higher upper ramification jump of $K_i/K$, which is equal to the ramification jump of $K_i/K_{i-1}$. \hfill \square

**Lemma 3.8.** The ramification jump of $K_i/K_{i-1}$ is equal to $v_K(\tilde{g}_i)$.

**Proof.** The proof given in [20 Proposition 2.10] works for our situation, because of $p \nmid u_{K_{i-1}}(\tilde{f}_i)$. \hfill \square

From Lemma 3.4, $v(Spec L)$ is expressed in terms of valuations of $\tilde{g}_i$'s, which are in turn related to upper ramification jumps of $L/K$ by the above results. To determine $v(Spec L)$, we now compute the upper ramification jumps in terms of the corresponding representational Witt vectors.

**Theorem 3.9.** Let $L/K$ be a $G$-extension given by an equation $\varphi(g) = f$, where $f$ is reduced. Then, the highest upper ramification jump is given by

$$\max\{-p^{n-1-}v_K(f_i) \mid i = 0, 1, \ldots, n-1\}.$$ 

Here we follow the convention that $v_K(0) = \infty$.

**Proof.** For an integer $m$, we define

$$W_n^{(m)}(K) := \{ (f_0, \ldots, f_{n-1}) \mid p^{n-1-}v_K(f_i) \geq m \}.$$ 

From [4] p. 26, Corollary], for $f \in W_n^{(m)}(K) \setminus W_n^{(m-1)}(K)$, the corresponding extension $L/K$ has Artin conductor $m + 1$ (for the character $\chi : G \to \mathbb{C}$ of any faithful irreducible $G$-representation over $\mathbb{C}$). From [15 Chapter VI, Proposition 5], the highest upper ramification jump is $m$. \hfill \square

This theorem together with Lemma 3.6 shows the following corollary:
We denote by group action, then closed \( \text{field} \)

Let \( \text{Theorem 3.11.} \)

\( m \)

of \( f \)

McKay correspondence, we need also the following relation. For a morphism \( \) of stable subsets.

\( \text{Remark 3.12.} \)

\( n \)

\( \) on each \( \)

The previous theorem in particular shows that the function \( \) is constant on each \( \text{-Cov}(D; j) \).

4. DISCREPANCIES OF SINGULARITIES

In this section, we shall briefly review the wild McKay correspondence proved in \([23]\)

and explain how it relates discrepancies of quotient singularities with the moduli space \( \text{G-Cov}(D) \) and \( \) function on it, following the line of \([20, 23]\).

4.1. Motivic integration. The Grothendieck ring of varieties over \( k \), denoted by \( K_0 \), is the abelian group generated by isomorphic classes \( [Y] \) of varieties over \( k \) subject to the following relation: if \( Z \) is a closed subvariety of \( Y \), then \( [Y] = [Y \setminus Z] + [Z] \). It has a ring structure by defining \( [Y] [Z] = [Y \times Z] \). We denote \( L \) to \( A^1_k \). In application to the McKay correspondence, we need also the following relation. For a morphism \( f : Y \to Z \) of \( k \)-varieties and an integer \( m \geq 0 \), if every geometric fiber of \( f \) say over an algebraic closed field \( L \) is universally homeomorphic to the quotient of \( A^m_k \) by some finite group action, then \( [Y] = L^m [Z] \). We define \( K'_n \) to be the quotient of \( K_0 \) by this relation. We denote by \( M' := K'_n[L^{-1}] \) the localization by \( L \). Subgroups \( F_m := \langle [X] \rangle | \dim X + i \leq -m \) of \( M' \) form a filtration. We define \( \hat{M}' := \lim M'/F_m \), which is again a commutative ring and complete with respect to the induced topology.

For \( n \in \mathbb{N} \), let \( \pi_n : J_n X \to J_{n-1} X \) be the truncation map to \( n \)-jets. We call a subset \( C \subset J_n X \) stable if there exists \( n \in \mathbb{N} \) such that \( \pi_n(C) \) is a constructible subset of \( J_n X \), \( C = \pi_{n-1}^{-1}(\pi_n(C)) \) and the map \( \pi_{n+1}(C) \to \pi_n(C) \) is a piecewise trivial \( A^d_k \) bundle for every \( m \geq n \). We define the measure \( \mu_X(C) \) of a stable subset \( C \subset J_n X \) by

\[
\mu_X(C) := [\pi_n(C)]L^{-md} \in \hat{M}'.
\]

For a more general measurable subset of \( J_n X \), we define its measure as the limit of ones of stable subsets.
Let $C \subset J_a X$ be a measurable subset and $F : C \to \mathbb{Z} \cup \{\infty\}$ a function on it. We say that $F$ is measurable if every fiber of $F$ is measurable. Now we define the integral
\[
\int_C \mathcal{L}^F := \sum_{m \in \mathbb{Z}} \mu_C(F^{-1}(m)){\mathcal{L}}^m \in \hat{M} \cup \{\infty\}.
\]
Note that $\int_C \mathcal{L}^F$ does not necessarily converge.

### Stringy motives

To state the wild McKay correspondence theorem, we shall define the stringy motive. Firstly, we shall recall basic notations concerning singularities. Let $X$ be a normal $k$-variety, $f : Y \to X$ a modification (proper birational morphism) such that $Y$ is a normal $k$-variety. Assume that both the exceptional locus $\text{Exc}(f)$ and the preimage $f^{-1}(X_{\text{sing}})$ of $X_{\text{sing}}$ are pure-dimension $d - 1$. We call such a morphism an admissible modification. Note that the last condition implies $f^{-1}(X_{\text{sing}}) \subset \text{Exc}(f)$. When $X$ is $\mathbb{Q}$-Gorenstein, we can define the relative canonical divisor $K_f$ in the usual way, which is a $\mathbb{Q}$-Weil divisor with a support contained in $\text{Exc}(f)$. Let $\text{Exc}(f) = \bigcup_{i \in E_f} E_i$ and $f^{-1}(X_{\text{sing}}) = \bigcup_{i \in S_f} E_i$ be the decomposition into irreducible components with $S_f \subset E_f$ and write $K_f = \sum a_i E_i$. We call $a_i$ the discrepancy of $E_i$ with respect to $X$ and define
\[
d(X) = \text{discrep}(\text{center} \subset X_{\text{sing}}; X) := \inf_{f \in S_f} a_i.
\]
Here $f$ runs over admissible modifications of $X$. We say that $X$ is terminal (resp. canonical, log terminal, log canonical) if $d(X) > 0$ (resp. $\geq 0$, $> -1$, $\geq -1$). Note that if $d(X) < -1$, then $d(X) = -\infty$.

We also need to consider log pairs as is usual in birational geometry. By a log pair, we mean the pair $(X, \Delta)$ of a normal $\mathbb{Q}$-Gorenstein variety $X$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor $\Delta$ on it. We say that a log pair $(X, \Delta)$ is $klt$ (resp. lc) if for any admissible modification $f : Y \to X$, $K_f - f^* \Delta$ has coefficients $> 1$ (resp. $\geq 1$).

**Remark 4.1.** If $\Delta = 0$, then the notions klt (Kawamata log terminal), plt (pure log terminal) and dlt (divisorial log terminal) coincide. In this case, we say that $X$ is log terminal. Furthermore, if $K_X$ is Cartier, then log terminal implies canonical. For details, see [10, pp. 42–43].

**Definition 4.2.** Let $X$ be a normal variety of pure-dimension $d$. We assume that the canonical sheaf $\omega_X$ is invertible. We define the $\omega$-Jacobian ideal $\mathcal{J}_X$ by
\[
\mathcal{J}_X \omega_X = \text{Im} \left( \bigwedge^d \Omega_{X/k} \to \omega_X \right).
\]
For a log pair $(X, \Delta)$, we define the stringy motive $M_{st}(X, \Delta)$ by
\[
M_{st}(X, \Delta) := \int_{\mathcal{J}_X} \mathcal{L}^{\text{ord} \Delta \text{ord} \mathcal{J}_X}.
\]
Here ord denotes the order function associated to a divisor or an ideal sheaf.

**Remark 4.3.** The invariant ring by a linear action of a $p$-group is a UFD ([5, Theorem 3.8.1]). Especially, in our situation where $G = \mathbb{Z}/p^n \mathbb{Z}$ linearly acts on $\mathbb{A}^d_k$, the quotient variety $X := \mathbb{A}^d_k/G$ is $1$-Gorenstein, that is, $\omega_X$ is invertible.

### The wild McKay correspondence

Let $G$ be a finite group (not necessarily cyclic of prime power order). Generally, we can construct the moduli space $G$-Cov($D$) of $G$-covers of $D = \text{Spec} \ k[[t]]$. Furthermore, we can define a measure on it. For a locally constructible function $F : G$-Cov($D$) $\to \mathbb{Q}$, we can define the integral $\int_{G\text{-Cov}(D)} F$. We remark that the $\nu$-function is locally constructible. For details, see [17].
**Theorem 4.4** ([24 Corollary 16.3]). Let $G$ be a finite group. Assume that $G$ acts on $K^d_k$ linearly and effectively. Put $X := K^d_k/G$ and let $\Delta$ be the $Q$-Weil divisor on $X$ such that $K^d_k \to (X, \Delta)$ is crepant. Then we have

$$M_{\text{ct}}(X, \Delta) = \int_{G-\text{Cov}(D)} L^{d-\alpha}.$$ 

We shall consider the case $G = \mathbb{Z}/p^m\mathbb{Z}$, which is of our principal interest. In this case, we can describe the measure on $G-\text{Cov}(D)$ explicitly as follows. For a constructible subset $C$ of $G-\text{Cov}(D; f)$, then the measure $\nu(C)$ is given by

$$\nu(C) := [C] \in \hat{M}'.$$

Suppose that $F : G-\text{Cov}(D) \to \mathbb{Q}$ is constant on each stratum $G-\text{Cov}(D; f)$. We write $F(f) = F(G-\text{Cov}(D; f)).$ Then we can write

$$\int_{G-\text{Cov}(D)} L^{F} = \sum_{r \in \mathbb{Q}} v(F^{-1}(r))L^r = \sum_{f} v(G-\text{Cov}(D; f))L^{F(f)}.$$ 

Putting $j = (j_0, j_1, \ldots, j_{n-1}) \in (\mathbb{N} \cup \{-\infty\})^n$, we have

$$v(G-\text{Cov}(D; j)) = \prod_{h \neq \infty} (L - 1)L^{h-1-[h/p]}$$

and hence

$$\int_{G-\text{Cov}(D)} L^{F} = \sum_{j \neq \infty} \left( \prod_{h \neq \infty} (L - 1)L^{h-1-[h/p]} \right) L^{F(j)}.$$ 

In Theorem 4.4 if the $G$-action has no pseudo-reflection, then $\Delta = 0$. If the action is indecomposable, we can check easily whether $G$ has pseudo-reflections or not as follows.

**Lemma 4.5.** Let $J$ be a Jordan block of size $d$ with eigenvalue 1 over $k$. We write $d = qp + r (0 \leq r < p)$. Then the Jordan standard form of $J^p$ has $r$ Jordan blocks of size $q + 1$ and $p - r$ Jordan blocks of size $q$; in particular, it has exactly $p$ blocks.

**Proof.** In general, the following holds: Let $A$ be a square matrix of size $d$. We denote by $C_m(\lambda)$ the number of Jordan block of size $m$ with eigenvalue $\lambda$ in the Jordan standard form of $A$. Then, we have

$$C_m(\lambda) = \text{rank } (A - \lambda E)^{m-1} - 2 \text{rank } (A - \lambda E)^m + \text{rank } (A - \lambda E)^{m+1},$$

where $E$ denotes the identity matrix. To prove the formula, we may assume that $A$ is a Jordan block say with eigenvalue $\lambda'$. If $\lambda' \neq \lambda$, the formula is obvious. Let us assume $\lambda' = \lambda$. Then $A - \lambda E$ is nilpotent and $\text{rank } (A - \lambda E)^m = \max\{0, d - m\}$. By direct computation, we get the formula.

Set $A = J^p$. It is easy to see that $\text{rank } (J^p - E)^m = \text{rank } (J - E)^{mp} = \max\{0, d - pm\}$. Especially, we have $\text{rank } (J^p - E)^{q-1} = d - p(q-1) = p+r$, $\text{rank } (J^p - E)^q = d - pq = r$ and $\text{rank } (J^p - E)^{q+1} = \text{rank } (J^p - E)^{q+2} = 0$. Therefore, we get $C_{q+1}(1) = r$ and $C_q(1) = p - r$. The equality $r(q+1) + (p-r)q = qp + r = d$ completes the proof. 

**Lemma 4.6.** Let $J_d$ be the Jordan block of size $d$ with eigenvalue 1 ($1 \leq d \leq p^n$). For $1 \leq m < n$, $J_d^m$ is a pseudo-reflection if and only if $d = p^m + 1$. 

Proof. We shall prove by induction on \( m \). When \( m = 1 \), the claim follows immediately from Lemma 4.5. Let \( m > 1 \). We write \( d = qp + r \) \((0 \leq r < p)\). Then

\[
J_d^m = (J_d^p)^{m-1} \\
\equiv (J_q^p)^{m-1} \\
= (J_q^{p-1})^{p-r} \oplus (J_q^{m-1})^{p-r},
\]

where \( \equiv \) denotes the similarity equivalence. The matrix \( J_d^m \) is a pseudo-reflecting if and only if one of the following holds:

1. \( J_q^{m-1} \) is a pseudo-reflecting, \( r = 1 \) and \( J_q^{p-1} = 1 \),
2. \( J_q^{m-1} \) is a pseudo-reflecting, \( p - r = 1 \) and \( J_q^{p-1} = 1 \).

In the latter case (2), by the induction hypothesis, we get \( q = p^{m-1} + 1 \), which contradicts the equality \( J_q^{m-1} = 1 \). In the former case (1), we get \( q + 1 = p^{m-1} + 1 \) and hence \( d = p^m + 1 \).

Conversely, it is obvious that \( J_d^m \) is a pseudo-reflecting. \( \Box \)

**Corollary 4.7.** Let \( J \) be a matrix of the Jordan normal form with a unique eigenvalue \( 1 \).

1. For a given integer \( m \geq 0 \), the matrix \( J^m \) is a pseudo-reflecting if and only if \( J \) has one Jordan block of size \( p^m + 1 \) and all the other blocks have size \( \leq p^m \).
2. Let \( p^n \) be the order of \( J \). The group \( \langle J \rangle = \mathbb{Z}/p^n\mathbb{Z} \) contains a pseudo-reflecting if and only if \( J \) has one Jordan block of size \( p^{n-1} + 1 \) and all the other blocks have size \( \leq p^{n-1} \). Moreover, if this is the case, the pseudo-reflectings in the group are \( J^{p^{n-1}} \), \( 1 \leq i \leq p - 1 \).

Proof. (1). The "if" part immediately follows from the last lemma. If there are at least two blocks say \( A \) and \( B \) of size \( > p^m \), then neither \( A^p \) or \( B^p \) are the identity matrix. This shows that \( J^m \) is not a pseudo-reflecting. If there is no block of size \( > p^m \), then \( J^m = 1 \), which is not a pseudo-reflecting. Thus, for \( J^m \) being a pseudo-reflecting, \( J \) needs to have one and only one block of size \( > p^m \) whose \( p^m \)-th power is a pseudo-reflecting. Again from the last lemma, this block needs to have size \( p^m + 1 \).

(2). For the group having pseudo-reflectings, the matrix \( J \) needs to be of the form as in (1) for some \( m \). Because of the order, we have \( m = n - 1 \). Conversely, if \( J \) is of this form for \( m = n - 1 \), then the group contains the pseudo-reflecting \( J^{p^{n-1}} \). Thus the first assertion of (2) holds. To show the second assertion, we first note that when two elements \( A, B \) of \( \langle J \rangle \) generate the same subgroup, then \( A \) is a pseudo-reflecting if and only if \( B \) is a pseudo-reflecting. Therefore we only need to consider the \( p \)-powers \( J^{p^m} \). The only pseudo-reflecting among them is the one for \( m = n - 1 \). The second assertion follows. \( \Box \)

**Proposition 4.8.** Suppose that \( G = \mathbb{Z}/p^m\mathbb{Z} \) acts on \( \mathbb{A}^d_k \) linearly and effectively and that there exists a pseudo-reflecting. Let \( H \subset \mathbb{A}^d_k \) be the hyperplane fixed by a pseudo-reflecting in \( G \) (this hyperplane is independent of the pseudo-reflecting from the above corollary). Let \( \overline{H} \) be the image of \( H \) in the quotient variety \( \mathbb{A}^d_k/G \) with the reduced structure. Then the map \( \mathbb{A}^d_k \to (\mathbb{A}^d/G, (p-1)\overline{H}) \) is crepant.

Proof. Let \( V := \mathbb{A}^d_k \) and \( X := V/G \). Let \( v \in H \) be a general \( k \)-point whose stabilizer subgroup \( S \subset G \) has order \( p \) and let \( x \in \overline{H} \) be its image. To compute the right coefficient of the boundary divisor on \( X \), it is enough to consider the morphism \( \text{Spec} \mathcal{O}_{V,v} \to \text{Spec} \mathcal{O}_{X,x} \) between the formal neighborhoods of \( v \) and \( x \). This morphism is isomorphic to the one similarly defined for the quotient morphism \( V \to V/S \) associated to the induced action of \( S = \mathbb{Z}/p\mathbb{Z} \) on \( V \) with pseudo-reflectings. In this case, we know from [24] that the coefficient of the boundary divisor is \( p - 1 \). This shows the proposition. \( \Box \)
4.4. Discrepancies of singularities. We shall recall the relation between stringy motives and discrepancies.

**Proposition 4.9** ([20] Proposition 6.6). If the stringy motive $M_{st}(X, \Delta)$ converges, then the pair $(X, \Delta)$ is klt. Furthermore, if there exists a resolution $f: Y \to X$ such that $K_Y - f^*(K_X + \Delta)$ is a simple normal crossing $\mathbb{Q}$-Cartier divisor, then the converse is also true.

For a measurable subset $U$ of $J_{st}X$, we define

$$\lambda(U) := \dim \left( \int_U L^{\text{ord} J_\mathcal{K}} d\mu_X \right),$$

provided that the integration converges. We say that a measurable subset $U$ of $J_{st}X$ is *small* if the relevant integration converges. We also denote the truncation map by $\pi: J_{st}X \to J_0X = X$.

The following proposition tell us that we can estimate the discrepancies of $A_k^d/G$ by computing the integration $\int_{G-\text{Cov}(D)} L^{d-\sigma}$.

**Proposition 4.10** ([23] Proposition 2.1). Let $C_r \subset \pi^{-1}(X_{\text{sing}})$, $(r \in \mathbb{N})$ be a countable collection of small measurable subset such that $\pi^{-1}(X_{\text{sing}})$ and $\bigcup_{r \in \mathbb{N}} C_r$ coincide outside a measurable subset. Then

$$d(X) = d - 1 - \sup_r \lambda(C_r).$$

We now suppose that $X := A_k^d/G$ the quotient variety associated to an effective linear action of $G = \mathbb{Z}/p^n\mathbb{Z}$. The quotient morphism $A_k^d \to X$ and an arc $D = \text{Spec} K \to X$ induce a $G$-cover of $D$, unless the arc maps into the branch locus of $A_k^d \to X$; the last exceptional case occurs only for arcs in a measure zero subset of $J_{st}X$. For $j \in (\mathbb{N} \cup \{-\infty\})^n$, let $M_j \subset \pi^{-1}(X_{\text{sing}})$ be the locus of arcs including a $G$-cover $E$ with ord $E = -j$. The collection of $M_j$ satisfies the condition of Proposition 4.10. Suppose that $G$ has no pseudo-reflection. As a variant of Theorem 4.4 for $j \neq (-\infty, -\infty, \ldots, -\infty)$, we have

$$\int_{M_j} L^{\text{ord} J_\mathcal{K}} = \int_{G-\text{Cov}(D;j)} L^{d-\sigma} = [G-\text{Cov}(D;j)]L^{d-\sigma(j)}$$

and

$$\lambda(M_j) = \dim v(G-\text{Cov}(D;j)) + d - v(j).$$

The case $j = (-\infty, -\infty, \ldots, -\infty)$ corresponds to the trivial $G$-cover $[D \to D$. We have

$$\int_{M_{(-\infty,-\infty,\ldots,-\infty)}} L^{\text{ord} J_\mathcal{K}} = [R/G] = [B],$$

where $R \subset A_k^d$ and $B \subset X$ are the ramification and the branch loci of $A_k^d \to X$ respectively. In particular,

$$\lambda(M_{(-\infty,-\infty,\ldots,-\infty)}) = \dim R = \dim B.$$

These formulae for $\lambda$ together with Proposition 4.10 enable us to estimate $d(X)$ in terms of the $\sigma$-function in theory. We shall carry it out in the case $n = 2$; computation in this case is already rather complicated.

5. The case $G = \mathbb{Z}/p^2\mathbb{Z}$

As an application of Theorem 5.11 we shall compute $\sigma$-function for the case $G = \mathbb{Z}/p^2\mathbb{Z}$ and give a criterion for convergence the stringy motive $M_{st}(A_k^d/G, \Delta)$. 

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**THE WILD MCKAY CORRESPONDENCE FOR CYCLIC GROUPS OF PRIME POWER ORDER**

15
5.1. Some invariants. For integers \(d, j_0, j_1\) with \(0 < d \leq p^2\), writing \(d = qp + r (0 \leq r < p)\), we define
\[
e^\tilde{p}(j_0, j_1) = e^\tilde{p}(j_0) := \sum_{0 \leq i_0, i_1 < p, 0 \leq i_0 + i_1 + p < d} \left[ \frac{pi_0j_0 + (p^2 - p + 1)i_1j_1}{p^2} \right],
\]
\[
e^\tilde{e}(j_0, j_1) := \sum_{0 \leq i_0, i_1 < p, 0 \leq i_0 + i_1 + p < d} \left[ \frac{pi_0j_0 + (-p - 1)j_0 + p(j_0)i_1}{p^2} \right].
\]

Let \(V\) be a \(d\)-dimensional \(G\)-representation. We do not assume \(0 < d \leq p^n\) here. It uniquely decomposes into indecomposables as
\[
V = \bigoplus_{a=1}^{a} V_{da} \quad (1 \leq d_a \leq p^2, \sum_{a=1}^{a} d_a = d),
\]
where \(V_{da}\) denotes the unique indecomposable \(G\)-representation of dimension \(d_a\). Components \(V_{da}\) correspond to Jordan blocks in the Jordan normal form of a generator of \(G\). For integers \(j_0, j_1\), we define
\[
e^\tilde{e}_V(j_0) := \sum_{a=1}^{a} e^\tilde{e}_{da}(j_0), \quad e^\tilde{e}_V(j_0, j_1) := \sum_{a=1}^{a} e^\tilde{e}_{da}(j_0, j_1).
\]

For a connected \(G\)-cover \(E^*\) of \(D^*\) with order \(-j = (-j_0, -j_1)\), from Theorem 3.11 we have
\[
\nu_V(E^*) = \begin{cases} 
e^\tilde{e}_V(j_0) & \text{if } p j_0 > j_1, \\
\ne^\tilde{e}_V(j_0, j_1) & \text{if } p j_0 < j_1. 
\end{cases}
\]

Actually the functions \(e^\tilde{e}_V\) and \(e^\tilde{e}_V\) are both the sum of a linear function and a periodic function. To describe the linear part, we introduce some invariants. For \(0 < d \leq p^2\), again writing \(d = qp + r (0 \leq r < p)\), we define
\[
A_d := \sum_{0 \leq i_0, i_1 < p, 0 \leq i_0 + i_1 + p < d} i_0 = \frac{qp(p - 1)}{2} + \frac{r(r - 1)}{2},
\]
\[
B_d := \sum_{0 \leq i_0, i_1 < p, 0 \leq i_0 + i_1 + p < d} i_1 = \frac{qp(q - 1)}{2} + qr.
\]

For a \(G\)-representation \(V\) with decomposition as above, we define
\[
A_V := \sum_{a=1}^{a} A_{da}, \quad B_V := \sum_{a=1}^{a} B_{da}
\]
and
\[
C^p_V := pA_V + (p^2 - p + 1)B_V, \quad C^{<0}_V := pA_V - (p - 1)B_V, \quad C^{<1}_V := pB_V.
\]

Note that all these invariants are integers.

**Lemma 5.1.** For integers \(n_i, s_i (i = 0, 1)\), we have
\[
e^\tilde{e}_V(n_0p^2 + s_0) = C^p_V \cdot n_0 + e^\tilde{e}_V(s_0), \quad e^\tilde{e}_V(n_0p^2 + s_0, n_1p^2 + s_1) = C^{<0}_V n_0 + C^{<1}_V n_1 + e^\tilde{e}_V(s_0, s_1).
\]
Proof. Without loss of generality, we may assume that \( V = V_d \). By direct computation, we get

\[
e_{j}^\circ (n_0 p^2 + s_0)
\]

\[
= \sum_{0 \leq i_0 < p, 0 \leq i_1 + i_0 p < d} \left[ p i_0 (n_0 p^2 + s_0) + (p^2 - p + 1) i_1 (n_0 p^2 + s_0) \right]
\]

\[
= \sum_{0 \leq i_0 < p, 0 \leq i_1 + i_0 p < d} \left[ p i_0 n_0 + (p^2 - p + 1) i_1 n_0 + \frac{p i_0 s_0 + (p^2 - p + 1) i_1 s_0}{p^2} \right]
\]

\[
= \left( p \sum_{0 \leq i_0 < p, 0 \leq i_1 + i_0 p < d} i_0 + (p^2 - p + 1) \sum_{0 \leq i_1 < p, 0 \leq i_1 + i_0 p < d} i_1 \right) n_0 + e_{j}^\circ (s_0)
\]

\[
= (p A_d + (p^2 - p + 1) B_d) n_0 + e_{j}^\circ (s_0)
\]

\[
= C_{V_d}^\circ \cdot n_0 + e_{j}^\circ (s_0),
\]

which induces the first equality. Similarly, we get

\[
e_{j}^\circ (n_0 p^2 + s_0, n_1 p^2 + s_1)
\]

\[
= \sum_{0 \leq i_0 < p, 0 \leq i_1 + i_0 p < d} \left[ p i_0 (n_0 p^2 + s_0) + (-p - 1) n_1 p^2 + p (n_1 p^2 + s_1)) i_1 \right]
\]

\[
= \sum_{0 \leq i_0 < p, 0 \leq i_1 + i_0 p < d} \left[ p i_0 - (p - 1) i_1 \right] n_0 + p i_1 n_1 + \frac{p i_0 s_0 + (p - 1) s_0 + p s_1) i_1}{p^2}
\]

\[
= (p A_d - (p - 1) B_d) n_0 + p B_d + e_{j}^\circ (s_0, s_1)
\]

\[
= C_{V_d}^{<0} \cdot n_0 + C_{V_d}^{<1} \cdot n_1 + e_{j}^\circ (s_0, s_1),
\]

which completes the proof. \( \square \)

We will need the following upper and lower bounds of \( A_d \) and \( B_d \) later.

**Lemma 5.2.** With notations as above, we have

\[
\frac{d(p - 1)}{2} - \frac{p^2}{8} \leq A_d \leq \frac{d(p - 1)}{2},
\]

\[
\frac{d(d - p)}{2p} \leq B_d \leq \frac{d(d - p)}{2p} + \frac{p}{8}.
\]

**Proof.** By definition, we have

\[
A_d = \frac{(d - r)(p - 1)}{2} + \frac{r(r - 1)}{2}
\]

\[
= \frac{d(p - 1)}{2} + \frac{r(r - p)}{2}.
\]

From the inequality of arithmetic and geometric means,

\[
0 \geq r(r - p)/2
\]

\[
\geq -(r + (r - p))^2/8
\]

\[
= -p^2/8.
\]

Therefore, we get

\[
\frac{d(p - 1)}{2} - \frac{p^2}{8} \leq A_d \leq \frac{d(p - 1)}{2}.
\]
Similarly, we have
\[ B_d = \frac{(d-r)\left(\frac{d-r}{p} - 1\right)}{2} + \frac{d-r}{p} \]
\[ = \frac{d(d-p)}{2p} - \frac{r(r-p)}{2p}, \]
which completes the proof. \( \square \)

5.2. **A criterion for convergence.** With the notation above, we state the main result of this section as follows.

**Theorem 5.3.** The integral \( \int_{G\text{-Cov}(D)} L^{d-p} \) converges if and only if the following inequalities hold:

\[ B_V \geq p, \]
\[ C_V^G \geq p^3 - p + 1. \]

**Corollary 5.4.** Let \( X := V/G \) be the quotient and \( \Delta \) the \( \mathbb{Q} \)-Weil divisor on \( X \) such that \( V \to (X, \Delta) \) is crepant. If the inequalities \( B_V \geq p \) and \( C_V^G \geq p^3 - p + 1 \) holds, then the pair \( (X, \Delta) \) is klt. Furthermore, if there exists a log resolution of \( (X, \Delta) \), then the converse is also true.

**Proof.** If the inequalities holds, then the integral \( \int_{G\text{-Cov}(D)} L^{d-p} \) converges from Theorem 5.3 and hence the stringy motive \( M_{st}(X, \Lambda) \) also converges from Theorem 4.4. We can prove the claim from Proposition 4.9. \( \square \)

The rest of this subsection is devoted to the proof of Theorem 5.3.

According to the decomposition of \( (\mathbb{N} \cup \{-\infty\})^2 \) into four parts
\[ \{(\infty, -\infty)\}, \{(\infty, j) \mid j \in \mathbb{N}\}, \{(j_0, j_1) \mid p j_0 > j_1\}, \{(j_0, j_1) \mid j_0 \neq -\infty, p j_0 < j_1\}, \]
we divide the integral over \( G\text{-Cov}(D) \) into four parts:
\[ \int_{G\text{-Cov}(D)} L^{d-p} = \mathbb{L}^d \]
\[ + \sum_j v(G\text{-Cov}(D; \infty, j)) L^{d-p(\infty, j)} \]
\[ + \sum_{p j_0 > j_1} v(G\text{-Cov}(D; j_0, j_1)) L^{d-p(j_0, j_1)} \]
\[ + \sum_{p j_0 < j_1} v(G\text{-Cov}(D; j_0, j_1)) L^{d-p(j_0, j_1)}. \]

The integral converges if and only if all the three sums on the right hand side converge.

We will study convergence of these sums in turn.

5.2.1. **Sum (4).** This part corresponds to the \( G \)-covers of \( D \) which have \( p \) connected components. Each component is then an \( H \)-cover with \( H \subset G \) the subgroup of order \( p \). Let \( E \) be such a \( G \)-cover and let \( E' \) be a connected component of it. We have
\[ \nu(E) = \nu'(E'), \]
where \( \nu' \) denotes the \( \nu \)-function \( \nu' : H\text{-Cov}(D) \to \mathbb{Q} \) associated to the induced \( H \)-action on \( A^d_k \).

By Lemma 4.5, we find that the restriction of the indecomposable \( G \)-representation \( V_{\alpha a} \) to \( H \) is isomorphic to
\[ W_{\alpha a} \oplus W_{\alpha a}^{p-r_a}, \]
where \( d_\alpha = q_\alpha p + r_\alpha \) (\( 0 \leq r_\alpha < p \)) and \( W_\varepsilon \) denotes the indecomposable \( H \)-representation of dimension \( \varepsilon \). Therefore, by [20 Proposition 6.9], we find that the infinite sum\[
\sum_{j \in \mathbb{N}} \nu(\text{G-Cov}(D; -\infty, j))L^{d-\alpha(j)} \sum_{j \in \mathbb{N}} \nu(\text{H-Cov}(D; j))L^{d-\alpha'(j)}
\]
converges if and only if the inequality \( D_W \geq p \) holds, where \( D_W \) is the invariant defined in [20 Definition 6.8]. By definition, we have\[
D_W = \sum_{a=1}^{a} \left( \frac{r_\alpha (q_\alpha + 1)q_\alpha}{2} + (p - r_\alpha) \frac{q_\alpha(q_\alpha - 1)}{2} \right)
= \sum_{a=1}^{a} \left( q_\alpha p(q_\alpha - 1) + q_\alpha r_\alpha \right)
= B_V.
\]
Therefore, the infinite sum \( (5) \) converges if and only if the inequality \( (8) \)

\[ B_V \geq p \]

holds.

5.2.2. Sum \( (5) \). Assume \( j_0 \neq -\infty \). Applying Theorem 3.11 we have
\[
\nu(j_0, j_1) = \begin{cases}
\epsilon_V^\kappa(j_0) & \text{if } p j_0 > j_1, \\
\epsilon_V^\kappa(j_0, j_1) & \text{if } p j_0 < j_1.
\end{cases}
\]

Firstly, we shall consider the case \( p j_0 > j_1 \). We shall compute the infinite sum
\[
\sum_{j_0, j_1 \in \mathbb{N}\setminus\{-\infty\}, \ p j_0 > j_1} \nu(\text{G-Cov}(D; j_0, j_1))L^{d-\epsilon_V^\kappa(j_0)} + \sum_{j_0, j_1 \in \mathbb{N}, \ p j_0 > j_1} \nu(\text{G-Cov}(D; j_0, j_1))L^{d-\epsilon_V^\kappa(j_0)}.
\]
Since \( \dim \nu(\text{G-Cov}(D; j_0, -\infty)) < \dim \nu(\text{G-Cov}(D; j_0, j_1)) \) \( (j_1 > -\infty) \), the first sum of the right hand side converges whenever the second sum converges. Thus it is enough to study the convergence of the second sum. We have
\[
\sum_{j_0, j_1 \in \mathbb{N}, \ p j_0 > j_1} \nu(\text{G-Cov}(D; j_0, j_1))L^{d-\epsilon_V^\kappa(j_0)}
= \sum_{j_0, j_1 \in \mathbb{N}, \ p j_0 > j_1} (\mathbb{L} - 1)^2 \mathbb{L}^{-2} \mathbb{L}^{\mathbb{N}_h - \mathbb{N}_p} \mathbb{L}^{d-\epsilon_V^\kappa(j_0)}
= (\mathbb{L} - 1)^2 \mathbb{L}^{-d-2} \sum_{j_0 \in \mathbb{N}} \mathbb{L}^{\mathbb{N}_h - \mathbb{N}_p} \epsilon_V^\kappa(j_0) \sum_{j_1=1}^{p j_0 - 1} \mathbb{L}^{j_1 - \mathbb{N}_p}.
\]
Since \( j_1 - \lfloor j_1/p \rfloor \) is an increasing function in \( j_1 \),
\[
\dim \sum_{j_1=1, \ p j_1}^{p j_0 - 1} \mathbb{L}^{j_1 - \mathbb{N}_p} = (p j_0 - 1) - \lfloor (p j_0 - 1)/p \rfloor = (p - 1) j_0.
\]
Therefore sum \( (5) \) converges if and only if
\[
\sum_{j_0 - \lfloor j_0/p \rfloor - \epsilon_V^\kappa(j_0)} + (p - 1) j_0.
\]
tends to $-\infty$ as $j_0$ tends to $\infty$. From Lemma 5.1 this function in $j_0$ is equivalent to the following one modulo bounded functions:

$$j_0 - j_0/p - C^{c,0}_V j_0/p^2 + (p - 1)j_0 = p^{-2}j_0 (p^3 - p - C^0_V).$$

We conclude:

**Proposition 5.5.** The infinite sum (5) converges if and only if the inequality

$$p^3 - p - C^0_V \leq -1$$

holds.

### 5.2.3. Sum (5)

Sum (5) converges if and only if

$$j_0 - \lfloor j_0/p \rfloor + j_1 - \lfloor j_1/p \rfloor - e_V^c (j_0, j_1) \quad \left(= \dim v(G \text{-Cov}(D; j_0, j_1))L^{d^\sigma(j_0, j_1)} - d \right)$$

tends to $-\infty$ as $j_0 + j_1$ tends to $\infty$. This function in $j_0$ and $j_1$ is equivalent to the following one modulo bounded functions:

$$f(j_0, j_1) := j_0 \left(1 - p^{-1} - C^{c,0}p^{-2}\right) + j_1 \left(1 - p^{-1} - C^{c,1}p^{-2}\right).$$

We observe that $f(j_0, j_1)$ lies between $f(1, j_1)$ and $f(\lfloor j_1/p \rfloor, j_1)$. Thus Sum (5) converges if and only if both $f(1, j_1)$ and $f(\lfloor j_1/p \rfloor, j_1)$ tend to $-\infty$ as $j_1$ tends to $\infty$. Since, modulo bounded functions,

$$f(1, j_1) \equiv j_1 \left(1 - p^{-1} - C^{c,1}p^{-2}\right)$$

and

$$f(\lfloor j_1/p \rfloor, j_1) \equiv j_1 p^{-1} \left(1 - p^{-1} - C^{c,0}p^{-2}\right) + j_1 \left(1 - p^{-1} - C^{c,0}p^{-2}\right)$$

$$\equiv j_1 \left(1 - p^{-2} - C^{c,1}p^{-2} - C^{c,0}p^{-3}\right),$$

we conclude:

**Proposition 5.6.** The infinite sum (5) converges if and only if the following inequalities hold:

(10) $$p^2 - p - C^{c,1}_V \leq -1,$$

(11) $$p^3 - p - C^{c,0}_V - pC^{c,1}_V \leq -1.$$

### 5.2.4. Completing the proof of Theorem 5.3

We found that the integral \( \int_{G \text{-Cov}(D)} L^{d^\sigma} \) converges if and only if inequalities (11) hold. Since $C^c_V = C^{c,0} + pC^{c,1}_V$, (9) holds if and only if (11) does. Similarly, since $C^c_V = pB_V$, (8) implies (10). Thus conditions (10) and (11) are redundant and the theorem follows.

**Remark 5.7.** Consider the $G$-representation $V = V_1^{\boxtimes x} \oplus V_2^{\boxtimes y} \oplus V_3$ with characteristic $p = 2$. Note that when $G = \mathbb{Z}/p^2\mathbb{Z}$, the $G$-representations with a pseudo-reflection are exactly the ones of this form (Corollary 4.7). Let $\Lambda$ be the divisor on $X = V/G$ such that $V \rightarrow (X, \Lambda)$ is crepant. From Proposition 4.8 $\Lambda$ is irreducible and reduced. We see that the pair $(X, \Lambda)$ is log canonical if and only if $y > 0$. Indeed, by direct computation, we get

$$B_V = 1, \quad C^{c}_V = 5 + 2y, \quad C^{c,0}_V = 1 + 2y, \quad C^{c,1}_V = 4.$$

If $y = 0$, then $C^c_V = 5$ and hence $p^3 - p - C^c_V = 1 > 0$. Therefore sum (5) has a term of dimension arbitrarily large. From a variant of Proposition 4.10 for log pairs (see also [24].
Similarly, as for the value of \( \lambda \) from Lemma 4.5, we have
\[
\begin{align*}
p^3 - p - C_V^p &= 1 - 2y \leq -1, \\
p^2 - p - C_V^{p,1} &= -2 \leq -1, \\
p^3 - p - C_V^{p,0} - pC_V^{p,1} &= -3 - 2y \leq -1,
\end{align*}
\]
and hence sums (5) and (6) all converge. From the equality \( B_V = p - 1 \) and from Corollary 1.4], we also see that sum (6) has terms of dimensions bounded above. Again from the variant of Proposition 4.10, \( (X, \Delta) \) is log canonical.

5.3. Evaluation of discrepancies. Using Proposition 4.10 we can evaluate \( d(X) \) of the quotient \( X = \mathbb{A}^d_k / G \). As in the paragraph after Proposition 4.10 let \( M_j \) be a stratum of \( J_0X \) corresponding a stratum \( G - \text{Cov}(D; j) \) of \( G - \text{Cov}(D) \). Let us compute \( \lambda(M_j) = \dim \nu(G - \text{Cov}(D; j_0, j_1)) + d - \sigma(j_0, j_1) \).

5.3.1. The case \( j_0 = -\infty \). From arguments in Section 5.2.1 we have
\[
\begin{align*}
\lambda(M_{(-\infty, j)}) &= \dim \nu(G - \text{Cov}(D; -\infty, j)) + d - \nu(-\infty, j) \\
&= \dim \nu(G - \text{Cov}(D; j)) + d - \sigma(j) \\
&= \lambda(N_j),
\end{align*}
\]
where \( N_j \) is defined in the same way as \( M_j \) for \( \mathbb{A}^d_k / H \). From (3.1), (3.2), if \( p - 1 - B_V \leq 0 \), then we have
\[
\begin{align*}
\sup_{j_0 = -\infty} \lambda(M_j) &= b + \max_{1 \leq i \leq p} \{ l_1 - \text{sht}_W(l_1) \} \\
&= d - B_V + \max_{1 \leq i \leq p} \{ \text{sht}_W(p - l_1) + l_1 \},
\end{align*}
\]
where \( b \) denotes the number of the indecomposable direct summands of the induced \( H \)-representation \( W \) and \( \text{sht}_W \) the function associated to \( W \) defined as follows. For an indecomposable representation \( W_e \) of dimension \( e \), we define
\[
\text{sht}_W(l) := \sum_{i=1}^{e-1} \left| \frac{il}{p} \right|.
\]
In general, for the case \( W = \bigoplus_e W_e \), we define \( \text{sht}_W := \sum_e \text{sht}_W \). As for the value of \( b \), from Lemma 4.3 we have
\[
b = \sum_{d_e < p} d_e + \sum_{d_e \geq p} d_e.
\]
Similarly, as for the value of \( \text{sht}_W(l) \), from (7), we have
\[
\text{sht}_W(l) = \sum_{a=1}^{a} \left( r_a \sum_{i=1}^{q_a} \left| \frac{il}{p} \right| + (p - r_a) \sum_{i=1}^{q_a-1} \left| \frac{il}{p} \right| \right).
\]
If \( p - 1 - B_V > 0 \), then we have \( \sup_{j_0 = -\infty} \lambda(M_j) = \infty \).

5.3.2. The case \( j_0 \neq -\infty \). If \( j_1 = -\infty \), then we have
\[
G - \text{Cov}(D; j_0, -\infty) = \bigoplus_m \times A^h_{-1 \cup \{ j_0 / p \}}.
\]
Since \( \nu(j_0, -\infty) = e^p \) depends only on \( j_0 \), thus \( \lambda(M_{(j_0, -\infty)}) < \lambda(M_{(j_0, j_1)}) \) for \( -\infty < j_1 < p/j_0 \). Therefore, we may assume that \( j_1 \neq -\infty \) to evaluate \( \sup \lambda(M_j) \). Assuming \( j_0, j_1 \neq -\infty \), if we write \( j_1 = n_j p^2 + m_j p + l_j \) \((i = 1, 2, 0 < m_i < p, 1 < l_i < p)\), then we have
\[
\lambda(M_j) = d + j_0 - [j_0 / p] + j_1 - \lfloor j_1 / p \rfloor - \nu(j_0, j_1)
\]
\[
= d + (p^2 - p) n_j + (p - 1) m_j + l_j + (p^2 - p) n_1 + (p - 1) m_1 + l_1 - \nu(j_0, j_1).
\]
Firstly, we consider the case \( p/j_0 > j_1 \).
Lemma 5.8. We have

\[
\sup_{p,j_h > h} \lambda(M_j) = \begin{cases} 
    d + \max \{ (p^2 - 1)m_0 + l_0p - e_V^\ell(m_0p + l_0) \} & \text{if } C_V^c \geq p^3 - p, \\
    \infty & \text{otherwise.}
\end{cases}
\]

Proof. Let \( p_j > j_h \neq -\infty \). The maximum value of \( \lambda(M_j) \) \((p_j > j_h)\) is given when \( j_h = p_j - 1 \) similarly, \( n_1 = n_0p + m_0, m_1 = l_0 - 1 \) and \( l_1 = p - 1 \). By Lemma 5.1, we get

\[ \lambda(M_j) = d + (p^3 - p - C_V^c)n_0 + (p^2 - 1)m_0 + l_0p - e_V^\ell(m_0p + l_0). \]

If \( p^3 - p - C_V^c > 0 \), then

\[ \sup_{p,j_h > h} \lambda(M_j) = \infty. \]

Otherwise, we have

\[ \sup_{p,j_h > h} \lambda(M_j) = d + \max_{0 \leq m_0, m_1 < p, l_1 < p} \{ (p^2 - 1)m_0 + l_0p - e_V^\ell(m_0p + l_0) \}, \]

which completes the proof.

Secondly, we consider the case \( p_j < j_h \).

Lemma 5.9. If \( C_V^{c,0} \geq p^2 - p \) and \( C_V^{c,1} \geq p^3 - p \), then we have

\[
\sup_{p,j_h < h} \lambda(M_j) = d + \max_{0 \leq m_0, m_1 < p, l_1 < p} \{ (p^2 - 1)m_0 + l_0p - e_V^\ell(m_0p + l_0, m_1p + l_1) \}.
\]

If \( C_V^{c,1} < p^2 - p \), then \( \sup_{p,j_h < h} \lambda(M_j) = \infty. \)

Proof. By Lemma 5.1, we get

\[ \lambda(M_j) = d + (p^3 - p - C_V^{c,0})n_0 + (p - 1)m_0 + l_0 \\
+ (p^2 - p - C_V^{c,1})m_1 + l_1 - e_V^\ell(m_0p + l_0, m_1p + l_1). \]

Assuming that \( C_V^{c,0} \geq p^2 - p \) and \( C_V^{c,1} \geq p^3 - p \), we get the first assertion. It is obvious that \( \sup_{p,j_h < h} \lambda(M_j) = \infty \) if \( C_V^{c,1} < p^2 - p \).

Theorem 5.10. If \( B_V \geq p^2 - 1 \) and \( C_V^c \geq p^3 - p \), then we have \( \sup_j \lambda(M_j) < \infty \) and

\[ \sup_j \lambda(M_j) = \max \left\{ \sup_{j_h = -\infty} \lambda(M_j), \sup_{p,j_h > h} \lambda(M_j), \sup_{p,j_h < h} \lambda(M_j) \right\}, \]

where the suprema on the right hand side are given by formulae (12–14). Conversely, if \( B_V < p^2 - 1 \) or \( C_V^c < p^3 - p \), then \( \sup_j \lambda(M_j) = \infty \).

Proof. Since \( C_V^{c,1} = pB_V \), thus \( B_V \geq p^2 - 1 \) implies that \( C_V^{c,1} \geq p^2 - p \). Therefore, it is enough to show that \( C_V^{c,0} \geq p^2 - p \). Assume that \( C_V^{c,0} < p^2 - p \). Then, the lemma below shows that the G-representation \( V \) is of the form \( V_1^{\oplus x} \oplus V_2 \) \((p = 2)\). However, if this is the case, we have \( C_V^c = 5 < 6 = p^3 - p \), which contradicts to the assumption \( C_V^c \geq p^3 - p \).

Consequently, if \( B_V \geq p^2 - 1 \) and \( C_V^c \geq p^3 - p \), then the suprema \( \sup_{p,j_h < h} \lambda(M_j) \) are all finite and they are given by (12–14). The converse is obvious.

Lemma 5.11. Let \( V \) be a faithful G-representation. We have the following.

1. The inequality \( C_V^{c,0} \geq p^2 \) holds if \( p \geq 5 \).
2. The inequality \( C_V^{c,0} \geq p^2 - p \) holds except if the G-representation \( V \) is of the form \( V_1^{\oplus x} \oplus V_2 \) \((p = 2)\).

Proof. (1). Without loss of generality, we may assume \( V \) is indecomposable of dimension \( d \) \((p + 1 \leq d \leq p^2)\). By Lemma 5.2, we get

\[
C_V^{c,0} - p^2 = pA_d - (p - 1)B_d - p^2 \\
\geq \frac{(p - 1)d}{2} - \frac{p^2}{2} - (p - 1)\left( \frac{d(d - p)}{2p} + \frac{p}{8} \right) - p^2.
\]
We denote by $\Phi(d)$ the last expression above. Let us show that $\Phi(d) \geq 0$. By direct computation, we have

$$\Phi(d) = -\frac{p - 1}{2p} d^2 + \frac{p^2 - 1}{2} d - \frac{p^3 + 9p^2 - p}{8},$$

hence $\Phi(d)$ is upward-convex with $d$ regarded as a real variable. It is enough to check the values $\Phi(p)$ are both non-negative. We get

$$\Phi(p) = \frac{p^2(3p(p - 3) - 7) + 4}{8p},$$

$$\Phi(p^3) = \frac{p(3p(p - 5) + 2p + 1)}{8}. $$

It is obvious that $\Phi(p) \geq 0$ and $\Phi(p^3) \geq 0$ if $p \geq 5$. Consequently, we have $C^{c,0}_V - p^2 \geq \Phi(d) \geq 0$.

(2) We may assume $p \leq 3$. Firstly, we consider the case that $V$ is indecomposable of dimension $d \geq p + 1$. When $p = 3$, $d$ varies from 4 to 9. Checking each case by direct computation, we get

$$C^{c,0}_{V_4} = 7, \quad C^{c,0}_{V_5} = 8, \quad C^{c,0}_{V_6} = 12, $$

$$C^{c,0}_{V_7} = 8, \quad C^{c,0}_{V_8} = 7, \quad C^{c,0}_{V_9} = 9, $$

and hence $C^{c,0}_V \geq p^2 - p$ holds.

If $p = 2$, we have

$$C^{c,0}_{V_2} = 0, \quad C^{c,0}_{V_3} = 2, \quad C^{c,0}_{V_4} = 1, \quad C^{c,0}_{V_5} = 2, $$

and hence the proof is completed. □

**Example 5.12.** Let $p = 3$ and $V$ the indecomposable $G = \mathbb{Z}/p^2\mathbb{Z}$-representation of dimension $d$ $(p + 1 < d \leq p^3)$. Then, according to computations with Sage [14], we get the following:

| $d = \text{dim} V$ | $\sup_{j = -\infty} \lambda(M_j)$ | $\sup_{p|j > j_0} \lambda(M_j)$ | $\sup_{p|j < j_1} \lambda(M_j)$ | $d(X)$ |
|-----------------|-------------------------------|-------------------------------|-------------------------------|--------|
| 4               | $\infty$                      | $\infty$                      | $\infty$                      | $-1$   |
| 5               | 5                             | 3                             | 4                             | 2      |
| 6               | 5                             | 2                             | 3                             | 0      |
| 7               | 4                             | 1                             | 4                             | 2      |
| 8               | 4                             | 0                             | 4                             | 3      |
| 9               | 4                             | $-2$                          | 3                             | 4      |

**Table 1.** discrepancies in characteristic 3

### 5.4. Upper and lower bounds

We shall give lower bounds of $e^{c}_V$ and $e^{v}_V$ and apply them to determine when the quotient variety $X$ is terminal, canonical or log canonical under the condition that the given $G$-representation is indecomposable.

**Lemma 5.13.** We have

$$e^{c}_V(j_0) \geq \frac{j_0 C^{c,0}_V}{p^2}, \quad e^{v}_V(j_0, j_1) \geq \frac{j_0 C^{c,0}_V + j_1 C^{c,1}_V}{p^2}. $$
Proof. By definition, we get
\[
\epsilon_{d_a}^\gamma(j_0) = \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} \left[ \frac{p_{b_0j_0} + (p^2 - p + 1)l_1j_0}{p^2} \right]
\]
\[
\geq \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} \frac{p_{b_0j_0} + (p^2 - p + 1)l_1j_0}{p^2}
\]
\[
= \frac{j_0}{p^2} \left( p \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} i_0 + (p^2 - p + 1) \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} i_1 \right)
\]
\[
= \frac{j_0 C_{d_a}^{\gamma}}{p^2}.
\]
Taking sum over \(a\), we get the first inequality.

Similarly, we have
\[
\epsilon_{d_a}^< (j_0, j_1) = \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} \left[ \frac{p_{b_0j_0} - (p - 1)l_1j_0 + p_{l_1j_1}}{p^2} \right]
\]
\[
\geq \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} \frac{p_{b_0j_0} - (p - 1)l_1j_0 + p_{l_1j_1}}{p^2}
\]
\[
= \frac{j_0}{p^2} \left( p \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} i_0 - (p - 1) \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} i_1 \right) + \frac{j_1}{p^2} \sum_{0 \leq b+l+p < d_a, \ 0 \leq b, l < p} i_1
\]
\[
= \frac{j_0 C_{d_a}^{<,0} + j_1 C_{d_a}^{<,1}}{p^2},
\]
and hence we get the second inequality. \(\Box\)

We can now give the upper and bound of \(\sup \lambda(M_j)\). For the \(j_0 = -\infty\) part, from [23 Theorem 1.2], we have the following bounds:

Lemma 5.14 ([23]). We have
\[
(15) \quad \sup_{j_0 = -\infty} \lambda(M_j) \geq d + p - 1 - B_V.
\]
Furthermore, if \(B_V \geq p\), then we have
\[
(16) \quad \sup_{j_0 = -\infty} \lambda(M_j) \leq d + 1 - \frac{2B_V}{p}.
\]

Proposition 5.15. Assume that \(B_V \geq p, C_V^\geq \geq p^3\) and \(C_V^{<,0} \geq p^2\). Then, we have
\[
\sup \lambda(M_j) \leq \max \left\{ d + 1 - \frac{2B_V}{p}, d + p - \frac{C_V^\geq}{p^2}, d + 2 - \frac{C_V^{<,0} + C_V^{<,1}}{p^2} \right\}.
\]

Proof. We shall first consider \(\sup_{p, h > j} \lambda(M_j)\). By Lemma 5.13, we have
\[
(p^2 - 1)m_0 + h_0p - e_{V}^\gamma(m_0p + l_0) \leq \left( p^2 - 1 - \frac{C_V^\geq}{p} \right)m_0 + \left( p - \frac{C_V^\geq}{p^2} \right)l_0.
\]
Since we assume $C_V^0 \geq p^3$, thus the coefficients of $m_0$ and $l_0$ in the right hand side are non-positive. Therefore, the right hand side attains the maximum at $m_0 = 0$ and $l_0 = 1$. From Lemma 5.9, we get

$$\sup_{p,h > l_1} \lambda(M_j) \leq d + p - \frac{C_V^0}{p^2}. \quad (17)$$

Next, we shall consider $\sup_{p,h < l_1} \lambda(M_j)$. By Lemma 5.13, we have

$$(p - 1)m_0 + l_0 + (p - 1)m_1 + l_1 - e_v^\gamma(m_0p + l_0, m_1p + l_1)$$

$$\leq \left(p - 1 - \frac{C_V^0}{p}\right)m_0 + \left(1 - \frac{C_V^0}{p^2}\right)l_0 + \left(p - 1 - \frac{C_V^{<3}}{p}\right)m_1 + \left(1 - \frac{C_V^{<3}}{p^2}\right)l_1.$$ 

Note that the assumption $B_V \geq p$ implies $C_V^{<3} \geq p^2$. Since we have $C_V^0 \geq p^3$ and $C_V^{<3} \geq p^2$, the coefficients in the last expression are non-positive. Therefore, the last expression takes the maximum at $m_0 = m_1 = 0$ and $l_0 = l_1 = 1$. From Lemma 5.9, we get

$$\sup_{p,h < l_1} \lambda(M_j) \leq d + 2 - \frac{C_V^0 + C_V^{<3}}{p^2}. \quad (18)$$

Combining (16–18), we get the claim. \qed

As a conclusion of this section, we get the following.

**Theorem 5.16.** Assume that $V = V_d$ is an indecomposable $G$-representation of dimension $d$ ($p + 1 < d \leq p^2$) (with this assumption, $V$ has no pseudo-reflection and $V \rightarrow X := V/G$ is crepant). Then,

$$X \text{ is terminal, canonical, log canonical, if and only if }$$

$$\begin{align*}
\text{not log canonical} & \quad d > 2p + 1, \\
\text{log canonical} & \quad d \geq 2p, \\
\text{canonical} & \quad d \geq 2p - 1, \\
\text{terminal} & \quad d < 2p - 1.
\end{align*}$$

**Proof.** First, we consider the case $d < 2p - 1$. From the definition of $B_V$, we get $B_V < p - 1$. By Theorem 5.10, we get $\sup \lambda(M_j) = \infty$ and hence $d(X) = -\infty$.

Next, we consider the case $d = 2p - 1$. Since we also assume $d > p + 1$, thus we have $p \geq 3$. By direct computation, we have $B_V = p - 1$ and $C_V^0 = 2p^3 - 4p^2 + 3p - 1 > p^3 - p$ and hence $d(X) > -\infty$ by Theorem 5.10. We remark that by (15) we have

$$\sup \lambda(M_j) \geq \sup_{j_0 = -\infty} \lambda(M_j) \geq d + p - 1 - B_V.$$ 

and hence $d(X) \leq -1$ by Proposition 4.10. Therefore, we get $d(X) = -1$.

Thirdly, we consider the case $d = 2p$. Then, we have

$$B_V = p,$$

$$C_V^0 = 2p^3 - 2p^2 + p \geq p^3 \geq p^3 - p + 1.$$ 

and hence by Corollary 5.4, the quotient $X = V/G$ is klt and $d(X) \geq 0$ (recall that since $\omega_X$ is invertible, thus $d(X) > -1$ implies $d(X) \geq 0$). On the other hand, by (15), we have

$$\sup \lambda(M_j) \geq \sup_{j_0 = -\infty} \lambda(M_j) \geq d + p - 1 - B_V = 2p - 1,$$

and hence $d(X) \leq 0$. Thus we get $d(X) = 0$.

Finally, we consider the case $d \geq 2p + 1$. When $p = 3$, the assertion follows from Example 5.12. We assume that $p \geq 5$. We remark that $A_d$ and $B_d$ are monotonically
increasing function in $d$, so are $C_V^r$ and $C_V^{r,0} + C_V^{r,1}$. From Proposition \ref{prop:5.15}, it is enough to show
\begin{equation}
\max \left\{ d + 1 - \frac{2B_V}{p}, d + p - \frac{C_V^r}{p^2}, d + 2 - \frac{C_V^{r,0} + C_V^{r,1}}{p^2} \right\} < d - 1
\end{equation}
in the case $d = 2p + 1$. In this case, we have
\begin{align*}
B_V &= p + 2, \\
C_V^r &= 2p^3 - p + 2, \\
C_V^{r,0} + C_V^{r,1} &= p^3 - p^2 + p + 2,
\end{align*}
and hence
\begin{align*}
d + 1 - \frac{2B_V}{p} &= d - 1 - \frac{4}{p}, \\
d + p - \frac{C_V^r}{p^2} &= d - 1 + 1 + p - \frac{2p^3 - p + 2}{p^2} \\
&= d - 1 + 1 - p - \frac{p - 2}{p^2}, \\
d + 2 - \frac{C_V^{r,0} + C_V^{r,1}}{p^2} &= d - 1 + 3 - \frac{p^3 - p^2 + p + 2}{p^2} \\
&= d - 1 + 4 - p - \frac{p + 2}{p^2}.
\end{align*}
Thus \eqref{eq:19} holds. \hfill \Box

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, JAPAN
Email address: mahito@presche.me
Email address: u529757k@ecs.osaka-u.ac.jp

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, JAPAN
Email address: takehikoyasuda@math.sci.osaka-u.ac.jp