A note on the asymptotics of random density matrices

M Kornyik

Eotvos Loránd University Department of Probability Theory and Statistics Pázmány Péter sétány 1/C., H-1117, Budapest, Hungary
Wigner Research Centre for Physics Quantum Optics and Quantum Information Group Konkoly-Thege Miklós út 29-33., H-1121, Budapest, Hungary
E-mail: koma@cs.elte.hu

Keywords: random matrix theory, Marchenko-Pastur law, Tracy-Widom law, density matrix, quantum information theory, von Neumann entropy

Abstract

We show in this note that the asymptotic spectral distribution, location and distribution of the largest eigenvalue of a large class of random density matrices coincide with that of Wishart-type random matrices using proper scaling. As an application, we show that the asymptotic entropy production rate is logarithmic. These results are generalizations of those of Nechita, and Sommers and Życzkowski.

1. Introduction

Density matrices are fundamental tools of quantum mechanics and quantum information theory for describing the state of a quantum system [1–3]. While the theory of deterministic density matrices is well developed, random density matrices have not been considered by many before [4–6]. These matrices are particularly useful when the state of the system is either unknown or just partially known. Random (density) matrices also appear in tomography [7], when the matrix elements come from measurements, although in this case the semi-definiteness can happen to fail. Due to the randomness quantities like the entropy or entanglement cannot be computed exactly, they have to be estimated, and in order to do this a probability measure has to be introduced on the set of density matrices. While there is a uniquely defined uniform distribution on the set of pure states, since these are the rays of a Hilbert space, for density matrices, i.e. mixed states, there is no candidate for a canonical measure.

There are two main classes of probability measures on the set of density matrices (described in more details in [6]). The first class consists of metric measures, which are generated by metrics on the set of density matrices, e.g. the Bures distance defined by the metric $d(\rho, \sigma) = \arccos \text{Tr} (\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}^\dagger})^{1/2}$. The second class consists of the induced measures, where density matrices are obtained by partially tracing a random pure state of a larger system. The topic of this note is confined to random density matrices of the second class.

In order to describe the second class assume a quantum system is in some pure state $|X\rangle \in \mathcal{H} \otimes \mathcal{K}$, where $\mathcal{H}$ is a $p$ dimensional Hilbert space of the observer, and $\mathcal{K}$ is an $n$ dimensional Hilbert space representing the (unknown) environment. The state of the system in the observer’s space is given by $\rho = \text{Tr}_\mathcal{K} |X\rangle \langle X|$, i.e. the partial trace of $|X\rangle$ with respect to $\mathcal{K}$. It can be shown that $\rho$ has the form $XX^\dagger / \text{Tr}(XX^\dagger) = XX^\dagger / \|X\|_{HS}^2$, where $X$ is a $p \times n$ matrix, $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm, and $X^\dagger$ denotes the conjugated transpose of $X$ (for more details on the tensor analytic and matrix algebraic description see [8]). Due to the fact that $\mathcal{K}$ is unknown, there is a degree of freedom in choosing the distribution of $|X\rangle$. In case the distribution of $|X\rangle$ is invariant under unitary conjugation, it can be shown that the elements of the matrix $X$ are independent and normally distributed complex random variables. By analyzing the asymptotic behavior of density matrices we can get a useful insight on large quantum systems, i.e. when $\mathcal{H} \otimes \mathcal{K}$ is of large dimension. Since the aforementioned random density matrices are functions of the generating $X$, or more precisely, functions of $XX^\dagger$, it is reasonable to analyze the spectral asymptotics of $XX^\dagger$.

The theory of the asymptotics of positive semi-definite matrices of the form $XX^\dagger$ is well established. It is known that after the proper scaling the limit of the spectral distribution is given by the compactly supported Marchenko-Pastur law and the limit distribution of the largest eigenvalue is governed by the Tracy-Widom law.

© 2018 The Author(s). Published by IOP Publishing Ltd
under quite general conditions (see eg. [9–15]). Furthermore, after proper scaling the largest and smallest eigenvalues converge to the respective edge of the support with probability one [16]. Some results from this theory can be translated to the case of random density matrices of the previously mentioned type. Nechita showed in [4] that after proper scaling and under the assumption that the elements of X are independent with standard normal distribution the limit laws coincide with those of XX†. In section 2 we will generalize these results for a larger class of random density matrices, while section 3 consists of the proofs of the generalized theorems.

Sommers and Życzkowski computed the two point correlation functions of a random density matrix of the second type from the invariant ensemble. They have also shown asymptotic results for the mean of the von Neumann entropy for a special class of random density matrices [5]. Section 4 generalizes their results.

2. Spectral asymptotics of general random density matrices

First, for the sake of completeness, let us introduce some notation and definitions.

**Definition 1.** The matrix \( \rho \in \mathbb{C}^{n \times n} \) is a density matrix, if it is positive semi-definite (denoted by \( \rho \succeq 0 \)) and \( \text{Tr} \rho = 1 \).

Given any \( x \in \mathbb{R} \), denote by \( \delta_x \) the Dirac measure concentrated at \( x \).

**Definition 2.** Let \( c > 0 \) and denote by \( \nu_c \) the Marchenko-Pastur law of parameter \( c \), i.e. let

\[
\nu_c = (1 - 1/c) \mathbb{I}_{[1,\infty)}(c) \delta_0 + \kappa_c
\]

and

\[
d\kappa_c(x) = \frac{1}{2\pi \sqrt{c}} \sqrt{(c_+ - x)(x - c_-)} \cdot \mathbb{I}_{[x_-,x_+]}(x) \, dx
\]

with \( x_{\pm} = (\sqrt{c} \pm 1)^2 \). For a given set \( A \subset \mathbb{R} \) we will denote its indicator function by \( \mathbb{I}_A(x) \).

Given any Hermitian matrix \( A \) we will denote its \( j \)th largest eigenvalue by \( \lambda_j(A) \).

We note that the density matrix is in close relation with the density operator, i.e. a linear, bounded operator of a Hilbert space with trace equal to 1. It can be shown that in finite dimensional Hilbert spaces there is an equivalence between these two objects.

Now let us recall Nechita’s results [4] about the asymptotics of random density matrices. Let \( X = (x_{ij})_{1 \leq i,j \leq d} \) be a family of independent, identically distributed (from this point on abbreviated as IID) random variables with standard complex Gaussian distribution \( \mathcal{N}_C(0,1) \). Assume \( p = p(n) \) is such that \( \lim_n (p(n)/n) = c \) and consider the empirical distribution

\[
\mu_n = \frac{1}{p} \sum_{j=1}^p \delta_{\lambda_j(\rho_n)}
\]

where \( \rho_n = X_n X_n^\dagger / \text{Tr}(X_n X_n^\dagger) \) and \( X_n = (x_{ij})_{1 \leq i,j \leq d} \). Then

\[
\mathbb{P}(\mu_n \rightarrow \nu_c \text{ in distribution}) = 1,
\]

where \( \nu_c \) is the Marchenko-Pastur distribution with parameter \( c \) (defined rigorously in the next section).

Furthermore we also have

\[
\lim_{n} c n \lambda_j(\rho_n) = (\sqrt{c} + 1)^2 \quad \text{with probability one},
\]

and

\[
n^{2/3} \frac{c n \lambda_j(\rho_n)}{(\sqrt{c} + 1)(1/\sqrt{c} + 1)^{2/3}} \longrightarrow_{n \rightarrow \infty} F_2 \quad \text{in distribution},
\]

where \( F_2 \) denotes the Tracy-Widom distribution with parameter 2. (For more details on the Tracy-Widom law see e.g. [17].) While working with density matrices it is also interesting to analyze the asymptotic behavior of the entropy. As an application of the main results we show in section 4 that the asymptotic entropy rate is logarithmic.

In the following we will state the main results.

**Theorem 1.** Assume that \( \{x_{ij}, i, j = 1, 2, \ldots\} \) is a collection of complex IID random variables, with \( \mathbb{E}[x_{ij}] = 0, \mathbb{E}[x_{ij}^2] = 0 \) and \( \mathbb{E}[|x_{ij}|^2] = 1 \), and assume that \( \lim_n p(n)/n = c \). Then
\[ \mathbb{P}(\mu_n \rightarrow \nu_\xi \text{ in distribution}) = 1 \]

where \( \mu_n \) denotes the same measure as in equation (1) and \( \mathbb{E} \) denotes the expectation functional.

**Theorem 2.** Assume \( c \in (0, \infty) \) and \( \lim_n p(n)/n = c \). Consider the sequence of random density matrices 
\[ \rho_n = X_n X_n^\dagger / \text{Tr}(X_n X_n^\dagger) \] 
where \( X_n = (x_{kl})_{k,l} \) with \( (x_{kl}) \) as in the previous theorem, and suppose 
\[ \mathbb{E}[|x_{11}|^4] < \infty. \] Then
\[ \lim_{n \to \infty} c n \lambda_n(\rho_n) = (\sqrt{c} + 1)^2 \] with probability one,
\[ \lim_{n \to \infty} c n \lambda_n(\rho_n) = (\sqrt{c} - 1)^2 \] with probability one,
\[ n^{2/3} \left( \frac{c n \lambda_n(\rho_n)}{1 + \sqrt{\frac{p(n)}{n}}} \right)^2 \left( 1 + 1/\sqrt{\frac{p(n)}{n}} \right)^{1/3} \to F_2 \text{ in distribution.} \]

Furthermore
\[ n^{2/3} \left( \frac{c n \lambda_n(\rho_n)}{1 - \sqrt{\frac{p(n)}{n}}} \right)^2 \left( \sqrt{\frac{p(n)}{n}} - 1 \right)^{1/3} \to F_2 \text{ in distribution,} \]
if \( \exists \gamma_1, \gamma_2 > 0 \) constants such that \( \gamma_1 < p(n)/n < \gamma_2 \) and the \( (x_{kl}) \)'s have subexponential decay, i.e., \( \exists \tau_0, T > 0 \) independent of \( k, l, n \), such that
\[ \mathbb{P}(|x_{kl}| > t) \leq \tau_0^{-1} \exp(-t^{\tau_0}) \text{ for } t \geq T, \]
and \( F_2 \) denotes the Tracy-Widom law of parameter 2.

Note that when determining the asymptotic distribution of \( \lambda_n(\rho_n) \) the quantity \( \sqrt{p/n} \) cannot be replaced by \( \sqrt{c} \) as the convergence of \( p/n \) can be arbitrarily slow. Figures 1(a) and (b) show numerical evidence for theorems 1 and 2. Both simulations were done using matrices with IID elements uniformly distributed on the set \( \{ \pm 1/\sqrt{2} \} \). The other parameters were chosen as \( n = 2000, c = 1/2 \) and the sample size was 5000. The density function of the Tracy-Widom law of parameter 2 was computed with the routine \texttt{dtw(x, beta=2)} of the R package called 'RMTstat', while the eigenvalue statistics were computed in Julia.

**3. Proof of theorem 1 and theorem 2**

Before the proof let us evoke the well known theorem of Marchenko and Pastur.
Theorem 3 (Marchenko-Pastur [13]). Suppose $\{x_{kl} \mid 1 \leq k, l \}$ is a family of IID complex random variables such that $E[x_{kl}] = 0$, $E|x_{kl}|^2 = 1$, furthermore suppose $p = p(n)$ and $\lim_{n} p(n)/n = c > 0$. Let $X_n = (x_{kl})_{1 \leq k, l \leq n}$ and $W_n := \frac{1}{n} X_n X_n^\dagger$ and $\mu' := \frac{1}{p(n)} \sum_{j=1}^{np(n)} \delta_{\lambda'(W_{nj})}$. Then we have
\begin{equation}
\mathbb{P}(\mu'_n \to \nu_c \text{ in distribution}) = 1,
\end{equation}
where $\nu_c$ denotes the Marchenko-Pastur distribution with parameter $c$.

Note that this is a more general and more concisely phrased version of the original theorem. A proof can be found in [13, 18].

Given a measure $\mu$ supported on $\mathbb{R}^+$, introduce the notation
\[ S(\varepsilon, \mu) := \int \frac{1}{x + \varepsilon} d\mu(x) \quad \varepsilon > 0. \]
We will also need the following lemma (for a proof see the appendix).

Lemma 1. [19] Let $\mu_n, (\mu_n, n \in \mathbb{N})$ be random probability measures with support in $[0, \infty)$. Then $\mu_n$ converges weakly to $\mu$ with probability one if and only if $S(\varepsilon, \mu_n)$ converges to $S(\varepsilon, \mu)$ with probability one $\forall \varepsilon > 0$.

Proof of theorem 1. According to lemma 1, equation (2) is equivalent to $\mathbb{P}(S(\varepsilon, \mu_n) \to S(\varepsilon, \nu_c)) = 1$ for all $\varepsilon > 0$, hence it is sufficient to prove that
\begin{equation}
\mathbb{P}(S(\varepsilon, \mu_n) - S(\varepsilon, \mu'_n) \to 0) = 1 \quad \forall \varepsilon > 0.
\end{equation}

It can be easily checked that
\[ S(\varepsilon, \mu_n) = \frac{\text{Tr} X_n X_n^\dagger}{np(n)} \cdot S\left(\frac{\text{Tr}(X_n X_n^\dagger)}{np(n)}, \mu'_n\right). \]

By assumption the elements of $X$ are IID, thus $\text{Tr} X_n X_n^\dagger / (np(n)) \to 1$ with probability one, since the Strong Law of Large Numbers (SLLN) is applicable and $\lim_{n} p(n)/n = c$. Also, because of $\mu$ being a finite measure, $S(\varepsilon, \mu)$ is a continuous function of $\varepsilon$, which implies (3).

Proof of theorem 2. The first part of the theorem is quite obvious. Geman showed in [16] that under the assumption of the present theorem we have
\[ \frac{\lambda_1(X_n X_n^\dagger)}{n} \to (1 + \sqrt{c})^2 \quad \text{with probability one}, \]
\[ \frac{\lambda_n(X_n X_n^\dagger)}{n} \to (1 - \sqrt{c})^2 \quad \text{with probability one}, \]
and hence
\[ cn \lambda_1(\rho_n) = \frac{cn^2}{\text{Tr}(X_n X_n^\dagger)} \lambda_1(X_n X_n^\dagger) \to (1 + \sqrt{c})^2 \quad \text{with probability one}, \]
\[ cn \lambda_n(\rho_n) = \frac{cn^2}{\text{Tr}(X_n X_n^\dagger)} \lambda_n(X_n X_n^\dagger) \to (1 - \sqrt{c})^2 \quad \text{with probability one}, \]
according to the SLLN.

To justify the second part we have to compare the largest eigenvalue of $X_n X_n^\dagger$ with that of $\rho_n$. According to the results of Bao et al in [10] we have
\[ \frac{n^{2/3}}{\lambda_1(X_n X_n^\dagger)/n} - \left(1 + \frac{\sqrt{p(n)}}{n}\right)^2 \to F_2 \quad \text{in distribution}, \]
which means that it is sufficient to show that $n^{2/3} (\lambda_1(X_n X_n^\dagger)/n - cn \lambda_1(\rho_n)) \to 0$ in distribution. Since
\[ \frac{\lambda_1(X_n X_n^\dagger)}{n} - cn \lambda_1(\rho_n) = \frac{\lambda_1(X_n X_n^\dagger)}{n} \left(1 - \frac{cn^2}{\text{Tr}(X_n X_n^\dagger)}\right), \]
and \( \lambda_i(X_n X_n^\dagger) / n \to (1 + \sqrt{\varepsilon})^2 \) with probability one, it remains to prove that
\[
n^{2/3} \left(1 - \frac{cn^2}{\text{Tr}(X_n X_n^\dagger)}\right) \to 0 \quad \text{in probability.} \tag{5}
\]

For arbitrary, but fixed \( p \) and \( n \) let \( S_{p,n} := \sum_{i<j} \psi_i \psi_j \), then \( \mathbb{E}[S_{p,n}] = np \), \( \text{Var}(S_{p,n}) = np(\mathbb{E}[|\psi_i|^4] - 1) \) and \( S_{p,n} / np \to 1 \) with probability 1 if \( p = p(n) \) and \( n \to \infty \). Since \( \gamma_1 < p(n) / n < \gamma_2 \) it is enough to show that
\[
n^{2/3} \left(1 - \frac{np(n)}{S_{p(n),n}}\right) \to 0 \quad \text{in probability.}
\]

Furthermore let \( A_{p,n} := \{ S_{p,n} \in [np - \beta, np + \beta] \} \) for some small \( \beta > 0 \) and for any fixed \( \varepsilon > 0 \), and let
\[
B_{\varepsilon,p,n} := \left\{ n^{2/3} \left(1 - \frac{np}{S_{p,n}}\right) < \varepsilon \right\},
\]
then
\[
\mathbb{P}(B_{\varepsilon,p,n}) \leq \mathbb{P}(B_{\varepsilon,p,n} \cap A_{p,n}) + \mathbb{P}(\Omega \setminus A_{p,n}),
\]
which proves that
\[
\mathbb{P}(\Omega \setminus A_{p(n),n}) \to 0 \quad \text{as} \quad n \to \infty.
\]

Fix \( p \) and let \( n \to \infty \), then
\[
\limsup_n \mathbb{P}(\Omega \setminus A_{p(n),n}) \leq \limsup_n \mathbb{P}\left(\frac{S_{p,n}}{np} - 1 \to 0\right) \leq \limsup_n \mathbb{P}\left(\frac{S_{p,n}}{np} - 1 \to 0\right) = 0,
\]
where in the second inequality we used Fatou’s lemma and the last equality is due to the SLN. Switching the role of \( p \) and \( n \) we obtain the same for fixed \( n \) and \( p \to \infty \). This proves that \( \mathbb{P}(\Omega \setminus A_{p(n),n}) \to 0 \) as \( n \to \infty \).

On the event \( A_{p,n} \) we have
\[
n^{2/3} \frac{S_{p,n} - np}{S_{p,n}} \leq n^{2/3} \frac{\beta}{np - \beta} = O(n^{1/3} p^{-1}),
\]
and the quantity on the right hand side tends to 0 whenever \( p = p(n) \) and \( n \to \infty \). This proves \( \mathbb{P}(B_{\varepsilon,p(n),n} \cap A_{p(n),n}) \to 0 \) as \( n \to \infty \) for any fixed \( \varepsilon > 0 \), therefore (5) holds true. Since convergence in probability implies convergence in distribution the proof is completed.

In the case of the smallest eigenvalue Feldheim and Sodin showed in [20] that
\[
\frac{\lambda_i(X_n X_n^\dagger)}{n} - \left(\frac{p(n)}{n} - 1\right)^2 \to F_2 \quad \text{in distribution.}
\]

The proof is essentially the same as for the previous case, thus it is left to the reader. \( \square \)

### 4. Application: Asymptotic entropy

In this section we are going to investigate the von Neumann entropy of random density matrices of the previously discussed type. We will prove that it exhibits a Strong Law of Large Numbers type of behavior for large systems. As it is meant to characterize the chaos present in a system, the results of this section show how much disorder is to be expected in the observed system \( \mathcal{H} \) after tracing out the environment \( \mathcal{K} \). We will also see that, not surprisingly, the asymptotic entropy depends on the ratio of the size of the observation space \( \mathcal{H} \) and the environment \( \mathcal{K} \). First let us define the von Neumann entropy of a density matrix.

**Definition 3.** Let \( \rho \) denote a density matrix on a finite dimensional Hilbert space \( \mathcal{H} \). Denote by
\[
H(\rho) = -\text{Tr}(\rho \log \rho) = -\sum_j \lambda_j(\rho) \log \lambda_j(\rho)
\]
the so-called von Neumann (also known as Shannon) entropy of \( \rho \). In case 0 is an eigenvalue define \( 0 \log 0 := 0 \).

Sommers and Życzkowski computed asymptotic results for the mean von Neumann entropy in [5] by showing that
\[
EH(\rho_n) = \ln n - \frac{1}{2} + O\left(\frac{\ln n}{n}\right),
\]

if \(\rho_n = XX^\dagger / \text{Tr}(XX^\dagger)\) is such that \(X\) is an \(n \times n\) Gaussian random matrix with IID elements. Our next proposition generalizes their result.

**Proposition 1.** Let \((\rho_n)\) be a sequence of random density matrices of type introduced in theorem 2. Then the following relations hold with probability one:

\[
\lim_{n} (H(\rho_n) - \log n) = \log c - \frac{1}{2\pi c} \int_{x_-}^{x_+} \log x \cdot \sqrt{(x_+ - x)(x - x_-)} \, dx,
\]

where \(x_{\pm} = (1 \pm \sqrt{c})^2\), and as a consequence

\[
\lim_{n} \frac{H(\rho_n)}{\log n} = 1.
\]

**Proof.** The proof is a series of rather simple calculations. For the sake of simplicity we will assume that \(p = p(n)\) and write only \(p\) throughout the proof. Using the definition of \(H\) and applying algebraic transformations yield

\[
H(\rho_n) = -\sum_{j=1}^{p} \lambda_j(\rho_n) \log \lambda_j(\rho_n)
\]

\[
= -\frac{p}{cn} \cdot \frac{1}{p} \sum_{j=1}^{p} c n \lambda_j(\rho_n) \log(c n \lambda_j(\rho_n)) + \log cn.
\]

Notice that \(\frac{1}{p} \sum_{j=1}^{p} c n \lambda_j(\rho_n) \log(c n \lambda_j(\rho_n)) = \int \lambda \log \lambda \, d\mu_\lambda(\lambda)\), where \(\mu_\lambda = \frac{1}{p} \sum_{j=1}^{p} \delta_{c n \lambda(j)}\), furthermore that \(\eta : x \mapsto -x \log x\) with \(\eta(0) := 0\) is a continuous function for \(x \geq 0\). Let \(K > 0\) and \(\eta_K(x) := \max\{\eta(x), K\}\) for \(x \geq 0\), then

\[
\left| \int \eta(x) d\mu_n(x) - \int \eta(x) d\nu(x) \right| \\
\leq \left| \int (\eta(x) - \eta_K(x)) d\mu_n(x) \right| \\
+ \left| \int \eta_K(x) (d\mu_n(x) - d\nu(x)) \right| \\
+ \left| \int (\eta(x) - \eta_K(x)) d\nu(x) \right| \\
= I_1^{(n)} + I_2^{(n)} + I_3^{(n)}.
\]

Now let \(\varepsilon > 0\) be arbitrary. Since \(\mu_n\) converges to \(\nu\) weakly with probability one, and \(\eta_K\) is a bounded, continuous function, \(I_2^{(n)} < \varepsilon/3\) if \(n\) is large enough with probability one.

Due to the definition of \(\eta_K\) we have \(\lim_{K \to \infty} \eta_K(x) = \eta(x)\) and \(|\eta_K(x)| \leq |\eta(x)|\) for every \(x \geq 0\). Moreover, \(\supp \nu \) being compact implies \(\int |\eta(x)| d\nu(x) < \infty\). According to the dominated convergence theorem \(I_1^{(n)} < \varepsilon/3\) if \(K\) is large enough.

It can be easily checked that \(|\eta(x) - \eta_K(x)| \leq x^2\) for \(x \geq K\) and \(K \geq 1\), which means that \(I_3^{(n)} \leq \int_{K}^{\infty} x^2 d\mu_n(x)\). Weak convergence of \(\mu_n\) with probability one implies \(\int x^2 d\mu_n(x) \to \int x^2 d\nu(x)\) with probability one. By writing \(x^2 = \min\{K, x^2\} + (x^2 - K)I_{[K, \infty)}(x) = f_1(x) + f_2(x)\), the function \(f_1(x)\) is continuous and bounded, hence \(\int f_1(x) d\mu_n(x) \to \int f_1(x) d\nu(x)\) with probability one if \(K\) is such that \(K > x_+^2\), therefore \(\int f_2(x) d\mu_n(x) = \int_{K}^{\infty} x^2 d\mu_n(x) - K \mu_n((K, \infty)) \to 0\) with probability one. Weak convergence also implies \(\mu_n((K, \infty)) \to 0\) for any fixed \(K > 0\) with probability one, meaning that we have \(\int_{K}^{\infty} x^2 d\mu_n(x) \to 0\), implying \(\int |\eta(x) - \eta_K(x)| d\mu_n(x) \leq \int_{K}^{\infty} x^2 d\mu_n(x) \to 0\), thus \(I_3^{(n)} < \varepsilon/3\) with probability one if \(n\) is large enough.

Summarizing the above arguments yields that the quantity in (8) is less than \(\varepsilon\). After subtracting \(\log n\) from \(H(\rho_n)\) and taking the limit \(n \to \infty\) we obtain

\[
\lim_{n} (H(\rho_n) - \log n) = \log c - \lim_{n} \frac{p}{cn} \int x \log x \, d\mu_n
\]

\[
= \log c - \int x \log x \, d\nu(x)
\]

\[
= \log c - \frac{1}{2\pi c} \int_{x_-}^{x_+} \log x \sqrt{(x_+ - x)(x - x_-)} \, dx \quad \text{with prob. 1},
\]

due to the assumption \(\lim_{n} p/n = c\). The second part of the proposition is a consequence of the first part and can be easily proved using equation (7). \(\square\)
Remark 1. Usually the entropy rate of a stochastic process \( \{X_n, n \in \mathbb{N}\} \) is defined as \( \lim_{n \to \infty} \frac{H_n}{n} \), with

\[
H_n = - \sum_{x_1, \ldots, x_n} p(x_1, \ldots, x_n) \log p(x_1, \ldots, x_n) \text{ for continuous, and}
\]

\[
H_n = \sum_{x_1, \ldots, x_n} P(X_1 = x_1, \ldots, X_n = x_n) \log P(X_1 = x_1, \ldots, X_n = x_n) \text{ for discrete random variables } X_1, X_2, \ldots.
\]

In the case of this paper there is no trivial way, if any, to define a stochastic process \( \{X_n, n \in \mathbb{N}\} \) such that \( H(X_1, \ldots, X_n) = H(\mu_k) \). The most natural way would be to define \( \langle X_1, \ldots, X_n \rangle \) so that they follow the same distribution as \( \langle \lambda_1(\mu_k), \ldots, \lambda_n(\mu_k) \rangle \). If \( E(f, x_1, \ldots, x_n) \) denotes the distribution function of \( (X_1, \ldots, X_n) \) for any \( n \geq 1 \), then the following strong compatibility condition has to be satisfied

\[
\forall k \geq 0 \quad \forall x_1, \ldots, x_n \int dE_{n+k}(x_1, \ldots, x_n, dx_{n+1}, \ldots, dx_{n+k}) = E(x_1, \ldots, x_n).
\]

It can be checked that this fails to happen even in the case of (Gaussian) Wishart matrices.

5. Conclusion

Nechita showed in [4] that the spectral asymptotics of random density matrices of the form \( XX^\dagger / Tr(XX^\dagger) \) coincide with that of \( XX^\dagger \) after proper scaling, where the elements of the \( X \) are independent and their distribution is standard complex Gaussian. In this paper, the previously mentioned results are generalized for the same type of random density matrices, but for the case when \( X \) comes from a larger class of random matrices.

Since using the formula \( XX^\dagger / Tr(XX^\dagger) \) is a very simple way of simulating random density matrices, these results can be used to approximate properties like the spectral distribution, the location and distribution of the largest eigenvalue, and the von Neumann entropy of large dimensional random density matrices.

In the application section we have generalized results of Sommers and Życzkowski by showing that random density matrices generate infinite entropy in the limit, but the production rate is logarithmic and surprisingly independent of the parameter \( c \).

An interesting further generalization of these results would be to consider random density matrices, where the columns of the generating \( X \) matrix are independent, but the elements of a column are not. Yaskov showed in [19], that assuming \( X \) consists of independent copies of the isotropic \( p \) dimensional (real) vector \( x_p \), the Marchenko-Pastur theorem is equivalent to a concentration of the quadratic form of the resolvent of \( \frac{1}{n} XX^\dagger \). In light of the aforementioned result, it would be interesting to show whether or not the IID condition could be relaxed in theorems 1 and 2.

Acknowledgments

This work was supported by the Hungarian National Research, Development and Innovation Office (Project Nos. K124351 and 2017–1.2.1–NKPI–2017–00001).

The author thanks O Kálmán, G Michaletzky, T Kiss and T Clark for their useful comments and observations, and G Tóth for bringing empirical density matrices to his attention.

Appendix

Proof of lemma 1. Suppose \( \mu_n \) converges to \( \mu \) weakly with probability one, fix \( \varepsilon > 0 \) and let \( f(x) = \frac{1}{x + \varepsilon} \). Then \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a continuous bounded function, thus

\[
S(\varepsilon, \mu_n) = \int_0^\infty \frac{\mu_n(dx)}{x + \varepsilon} - \int_0^\infty \frac{\mu(dx)}{x + \varepsilon} = S(\varepsilon, \mu) \text{ with probability one.}
\]

Now suppose \( \mathbb{P}(S(\varepsilon, \mu_n) \to S(\varepsilon, \mu)) = 1 \) for all \( \varepsilon > 0 \), then

\[
\mathbb{P}(S(\varepsilon, \mu_n) \to S(\varepsilon, \mu) \quad \forall \varepsilon \in \mathbb{Q} \cap (0, \infty)) = 1.
\]

Let \( \nu \) be an arbitrary finite measure supported on \( \mathbb{R}_{\geq 0} \) and fix \( \delta > \eta > 0 \), then

\[
|S(\eta, \nu) - S(\delta, \nu)| \leq |\eta - \delta| \cdot (\delta \eta)^{1/4} \nu(\mathbb{R}_{\geq 0}).
\]

Note that \( S(\varepsilon, \mu_n) \to S(\varepsilon, \mu) \) can be written as

\[
\forall m \in \mathbb{N} \quad \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \quad |S(\varepsilon, \mu_n) - S(\varepsilon, \mu)| < \frac{1}{3m}.
\]
Now for each $m$ and $\varepsilon > 0$ there is a $q_{m,\varepsilon} \in \mathbb{Q}$, such that $|q_{m,\varepsilon} - \varepsilon| < \frac{1}{3m^{2}q_{m,\varepsilon}}$, hence

$$|S(\varepsilon, \mu_{n}) - S(\varepsilon, \mu)| \leq |S(q_{m,\varepsilon}, \mu_{n}) - S(q_{m,\varepsilon}, \mu_{n})| + |S(q_{m,\varepsilon}, \mu_{n}) - S(q_{m,\varepsilon}, \mu)|$$

and this yields

$$\mathbb{P}(S(\varepsilon, \mu_{n}) \to S(\varepsilon, \mu) \quad \forall \varepsilon > 0) = 1,$$

implying (Theorem 2.2 and remark 2.3 in [21]) that $\mu_{n}$ converges to $\mu$ vaguely on every compact subset of $[0, \infty]$ with probability one. For a finite measure $\nu$ on $\mathbb{R}_{>0}$, the measure $\nu(B)$ is defined as

$$\nu(B) = \int_{0}^{\infty} \frac{1}{x + 1} \nu(dx) \quad \text{where } B \text{ is a Borel set of } \mathbb{R}_{>0}.$$

The function $f_{\varepsilon}(x) = \frac{x + 1}{x - \varepsilon}$ with $f(\infty) = 1$ is continuous on $[0, \infty]$ for all $z \in \mathbb{C}$ with $\text{Re} z > 0$, hence

$$S(\varepsilon, \mu_{n}) = \int_{0}^{\infty} \frac{1}{x - \varepsilon} \mu_{n}(dx) = \int_{0}^{\infty} f_{\varepsilon}(x) \mu_{n}(dx) \to \int_{0}^{\infty} f_{\varepsilon}(x) \mu(dx) = S(\varepsilon, \mu).$$

By the standard Stieltjes continuity theorem (Theorem B.9 on page 515 in [18]) this implies that $\mu_{n}$ converges to $\mu$ vaguely. For probability measures vague convergence is equivalent to weak convergence. \hfill \Box

**ORCID iDs**

M Kornyik 🌐 https://orcid.org/0000-0002-8685-3383

**References**

[1] Alicki R and Fannes M 2001 *Quantum Dynamical Systems* (Oxford: Oxford University Press)
[2] Nielsen M A and Chuang I 2002 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[3] Wilde M M 2013 *Quantum Information Theory* (Cambridge: Cambridge University Press)
[4] Nechita I 2007 Asymptotics of random density matrices *Annales Henri Poincaré* vol 8 (Berlin: Springer) pp 1521–38
[5] Sommers H-J and Zyczkowski K 2004 random density matrices *Journal of Physics A: Mathematical and General* 37 8457
[6] Zyczkowski K, Penson K A, Nechita I and Collins B 2011 *Generating random density matrices* *J. Math. Phys.* 52 062201
[7] Knips L, Schwemmer C, Klein N, Reuter J, Tóth G and Weinfurter H 2015 How long does it take to obtain a physical density matrix? arXiv:1512.06866v1
[8] Joshi A W 1995 *Matrices and tensors in Physics* (New Delhi: New Age International)
[9] Bai Z D and Silverstein J W 2004 CLT for linear spectral statistics of large-dimensional sample covariance matrices *The Annals of Probability* 32 553–605
[10] Bao Z, Pan G and Zhou W 2015 Universality for the largest eigenvalue of sample covariance matrices with general population *The Annals of Statistics* 43 382–421
[11] Forrester P J 1993 *The spectrum edge of random matrix ensembles* Nucl. Phys. B 402 709–728
[12] Karoui N E 2007 Tracy-Widom limit for the largest eigenvalue of a large class of complex sample covariance matrices *The Annals of Probability* 35 663–714
[13] Marchenko V A and Pastur L A 1967 Distribution of eigenvalues for some sets of random matrices *Matematicheski Sbornik* 114 507–36
[14] Nagao T and Forrester P J 1995 Asymptotic correlations at the spectrum edge of random matrices *Nucl. Phys. B* 435 401–20
[15] Pastur L, Shcherbina M and Shcherbina M 2011 *Eigenvalue distribution of large random matrices* vol 171 (Providence, RI: American Mathematical Society)
[16] Germán S 1980 A limit theorem for the norm of random matrices *The Annals of Probability* 8 52–261
[17] Tracy C A and Widom H 2002 Distribution functions for largest eigenvalues and their applications arXiv:math-ph/0210034
[18] Bai Z D and Silverstein J W 2010 *Spectral Analysis of Large Dimensional Random Matrices* vol 20 (New York: Springer)
[19] Yaskov P 2016 Necessary and sufficient conditions for the Marchenko–Pastur theorem *Electronic Communications in Probability* 21 8
[20] Feldheim O N and Sodin S 2010 A universality result for the smallest eigenvalues of certain sample covariance matrices *Geometric and Functional Analysis* 20 88–123
[21] Schilling R L, Song R and Vondracek Z 2012 *Bernstein Functions: Theory and Applications* vol 37 (Göttingen: Walter de Gruyter)