Redundant Components in the $3\alpha$ Faddeev Equation
Using $2\alpha$ RGM Kernel

Yoshikazu Fujiwara, Yasuyuki Suzuki*, Kazuya Miyagawa**
Michio Kohno***, Hidekatsu Nemura****

Department of Physics, Kyoto University, Kyoto 606-8502, Japan
*Department of Physics, Niigata University, Niigata 950-2181, Japan
**Department of Applied Physics, Okayama Science University, Okayama 700-0005, Japan
***Physics Division, Kyushu Dental College, Kitakyushu 803-8580, Japan
****Institute of Particle and Nuclear Physics, KEK, Ibaragi 305-0801, Japan

(Received October 26, 2018)

The $3\alpha$ Faddeev equation using $2\alpha$ RGM kernel involves redundant components whose contribution to the total wave function completely cancels out. We propose a practical method to solve this Faddeev equation, by eliminating the admixture of such redundant components. A complete equivalence between the present Faddeev approach and a variational approach using the translationally invariant harmonic-oscillator basis is numerically shown with respect to the $3\alpha$ bound state corresponding to the ground state of $^{12}\text{C}$.

A first issue for applying realistic quark-model baryon-baryon interactions to few-baryon systems, such as the hypertriton, is to find a basic three-cluster equation which is formulated by using microscopic two-cluster quark-exchange kernel of the resonating-group method (RGM). This issue is non-trivial, not only because the quark-exchange kernel is non-local and energy dependent, but also because RGM equations sometimes involve redundant components due to the effect of the antisymmetrization; i.e., the Pauli-forbidden states. A desirable feature of such a three-cluster equation is that it can be solved in either or both of the variational approach and the Faddeev formalism, yielding completely the same result. In our previous paper[1], which is referred to I hereafter, we have proposed a simple three-cluster equation, which is similar to the orthogonality condition model (OCM), but employs the two-cluster RGM kernel as the interaction potential. The three-cluster Pauli-allowed space is constructed by the orthogonality of the total wave functions to the pairwise Pauli forbidden states. Although this definition of the three-cluster Pauli-allowed space is not exactly equivalent to the standard definition given by the three-cluster normalization kernel, this assumption is essential to find a complete equivalence between the proposed three-cluster equation and the Faddeev equation which employs a singularity-free $T$-matrix derived from the RGM kernel (the RGM $T$-matrix).

More explicitly, the Faddeev equation for systems composed of three identical bosons is expressed as

$$\lambda \varphi = G_0 \tilde{T} S \varphi ,$$

(1)
where $S$ implies the real symmetric ($S^T = S$) matrix for rearrangement of the three-types of the Jacobi-coordinates: $\varphi_\beta + \varphi_\gamma = [(123) + (123)^2] \varphi_\alpha = S \varphi_\alpha$. Note that $1 + S$ is semi-positive definite. In Eq. (1), $\varphi = \varphi_\alpha$ is the $\alpha$-component of the total wave function $\Psi = \varphi_\alpha + \varphi_\beta + \varphi_\gamma = (1 + S) \varphi$, $G_0 = G_0(E) = 1/(E - H_0 + i0)$ is the 3-body free Green function for the negative energy $E$, and $\tilde{T} = \tilde{T}^{(3)}(E, \varepsilon)$ is essentially the two-cluster $T$-matrix derived from the RGM kernel (see Eqs. (2.8), (2.9) and (3.20) of I). This $\tilde{T}$ satisfies the basic relationship (Eq. (2.23) of I)

$$\langle u|1 + G_0 \tilde{T}\rangle = [1 + \tilde{T} G_0]|u\rangle = 0 .$$

(2)

Here we assume only one Pauli forbidden state $|u\rangle$, for simplicity.

In order to find a trivial solution of Eq. (1), we first solve

$$S|uf^\tau\rangle = \tau |uf^\tau\rangle ,$$

(3)

where $|uf\rangle = |u\rangle |f\rangle$ is a product of two functions corresponding to the two momentum Jacobi-coordinate vectors, $k$ and $q$, respectively. Each function of $|u\rangle$ and $|f\rangle$ is assumed to be normalized as $\langle u|u\rangle = \langle f|f\rangle = 1$. In the $3d'$ system discussed in I, there appears no $\tau = -1$ eigen-value, while in the $3\alpha$ system we find two solutions with $\tau = -1$, as is shown below. We find these by solving the eigen-value problem

$$\langle u|S|uf^\tau\rangle = \tau |f^\tau\rangle$$

(4)

with smaller dimensionality than Eq. (3). In fact, if we have a $\tau = -1$ solution $|f^\tau\rangle$ for Eq. (4), the relationship $(1 + S)^2 = [1 + (123) + (123)^2]^2 = 3(1 + S)$ implies that $\langle u|(1 + S)|uf^\tau\rangle = 0$. If we further use $S^T = S$, this leads to $\langle (1 + S)|uf^\tau|(1 + S)|uf^\tau\rangle = 0$, namely, $(1 + S)|uf^\tau\rangle = 0$ for $\tau = -1$. Such a solution with $\tau = -1$ corresponds to the [21] symmetric component of the total wave function with respect to the permutations of the three bosons. Let us assume that we only have one $\tau = -1$ solution for simplicity of discussion. The Faddeev component constructed from this solution

$$\varphi_0^\tau = G_0|uf^\tau\rangle \quad \text{with} \quad \tau = -1$$

(5)

gives vanishing contribution to the total wave function $\Psi$, since $G_0$ is [3] symmetric. On the other hand, it is very easy to show that $\varphi_0^\tau$ is a $\lambda = 1$ solution of Eq. (1):

$$\varphi_0^\tau = G_0\tilde{T} S \varphi_0^\tau \quad \text{for} \quad \tau = -1 .$$

(6)

This is because of the very special nature of $\tilde{T}$, satisfying the basic relationship Eq. (2).

Since our true solution of Eq. (1) with $\lambda = -1$ includes such a component as $\varphi_0^\tau$, a general solution of Eq. (1) with $\lambda = -1$ is expressed as

$$\varphi = \tilde{\varphi} + C \varphi_0^\tau ,$$

(7)

where $\tilde{\varphi}$ is a special solution of Eq. (1) with $\lambda = -1$. An optimum choice of the coefficient $C$ in Eq. (7) is to make the value $\langle \varphi|1 + S|\varphi\rangle/\langle \varphi|\varphi\rangle$ maximum. If we
expand \( \varphi \) by the complete orthonormalized basis of Eq. (3) (including the \( \tau = -1 \) solutions), we find that this condition is expressed as the orthogonality
\[
\langle u f^\tau \mid \varphi \rangle = 0 \quad \text{for} \quad \tau = -1 .
\] (8)

Namely, we only need to solve Eq. (1) under the condition of Eq. (8).

The following procedure to eliminate the redundant components with \( \tau = -1 \) from the Faddeev equation is very similar to the technique used for finding a unique solution of RGM equations. We modify Eq. (1) as
\[
\lambda \varphi = \left[ G_0 \tilde{T} S - G_0 |uf^\tau\rangle \frac{1}{\langle u f^\tau \mid G_0 |uf^\tau\rangle} \langle u f^\tau \mid \varphi \rangle \right] \varphi .
\] (9)

If we multiply \( \langle uf^\tau \rangle \) from the left, \( \langle uf^\tau \mid G_0 \tilde{T} S = \langle uf^\tau \rangle \) yields
\[
\lambda \langle uf^\tau \mid \varphi \rangle = 0 .
\] (10)

Therefore, the solution \( \varphi \) of Eq. (9) satisfies Eq. (8) so long as \( \lambda \neq 0 \). The fact that \( \varphi_0^\tau \) is a \( \lambda = 0 \) solution of Eq. (9) implies that the trivial \( \lambda = 1 \) solution of Eq. (1) is pushed down to the \( \lambda = 0 \) eigen-state in Eq. (9). The solution of Eq. (9) also satisfies Eq. (1) when \( \lambda \neq 0 \), since the second term of Eq. (9) does not contribute because of Eq. (8). If we multiply Eq. (9) only with \( \langle u \rangle \) from the left and leave the degree of freedom for \( q \), we obtain
\[
\lambda \langle u \rangle = - \langle u \mid S \varphi \rangle - \langle u \mid G_0 |uf^\tau\rangle \frac{1}{\langle u f^\tau \mid G_0 |uf^\tau\rangle} \langle u f^\tau \rangle \langle \varphi \} .
\] (11)

Again, the second term of the right hand side does not contribute if \( \lambda \neq 0 \), and we find
\[
\langle u \mid \lambda + S \rangle \varphi \rangle = 0 \quad \text{for} \quad \lambda \neq 0 .
\] (12)

In particular, the \( \lambda = 1 \) solution of Eq. (9) leads to the condition
\[
\langle u \mid 1 + S \rangle \varphi \rangle = 0 \rightarrow \langle u \mid \varphi_\alpha + \varphi_\beta + \varphi_\gamma \rangle = 0 ,
\] (13)

which implies that our total wave function \( \Psi \) does not contain the pairwise redundant component \( \langle u \rangle \varphi \rangle = 0 \). In summary, solving Eq. (9) automatically guarantees the solution
\[
\lambda \varphi = G_0 \tilde{T} S \varphi , \quad \langle uf^\tau \rangle \varphi \rangle = 0 , \quad \langle u \mid \lambda + S \rangle \varphi \rangle = 0 \quad \text{for} \quad \lambda \neq 0 .
\] (14)

Table I. \( |uf\rangle \) SU\(_3\) states for 3 \( \alpha \) system. \( N = N_1 + N_2 \) is the total h.o. quanta of the whole system. Only non-negative \( \lambda \) and \( \mu \) are allowed.

| \( N_1 \) | \( N_2 \) | \( (\lambda \mu) \) |
|---|---|---|
| 0 | \( N \) | \( (N\lambda) \) |
| 2 | \( N - 2 \) | \( (N\lambda), (N - 2, 1), (N - 4, 2) \) |
Table II. [3] and [21] symmetric basis for $|uf\rangle$, classified by the $SU_3$ basis with $(\lambda\mu)$.

| $N$ | $N_1$ | $N_2$ | $(\lambda\mu)$ | [3] | [21] |
|-----|-------|-------|----------------|-----|-----|
| 0   | 0     | 0     | (0 0)         | (00) | -   |
| 2   | 0     | 2     | (2 0)         | (20) | (20) |
| 4   | 0     | 4     | (4 0)         | (40), (02) | (40), (21) |
| 6   | 0     | 6     | (6 0)         | (60), (22) | (41) |
|     | 2     | 4     | (6 0), (41), (22) | (60), (22) | (41) |

Next, let us consider the analytic solution of Eq. (4) in a particular case of the $3\alpha$ system. It is convenient to use the translationally invariant harmonic-oscillator (h.o.) basis used in I. The Pauli forbidden states of the $2\alpha$ system consist of the h.o. states with $N_1 = 0$ (for the relative angular momentum $\ell = 0$) and $N_1 = 2$ (for $\ell = 0$ and 2), where $N_1$ is the total h.o. quanta $(2n + 1)$ for the variable $k$. The odd $N_1$ state is omitted from the very beginning, by assuming only even partial waves for the $2\alpha$ relative motion. We have to solve the problem which $SU_3$ states are classified to the [21] symmetry among the $SU_3$-coupled 2-particle h.o. states $[U_{(N_1)}(k)U_{(N_2)}(q)]_{(\lambda\mu)a}$. For $N_1 = 0$ and 2, these $SU_3$ states are constructed as in Table I. Since we know that all the $SU_3$ state of the $3\alpha$ system is Pauli forbidden for the total h.o. quanta $N = N_1 + N_2 \leq 6$, it is sufficient to consider only the states given in Table II. This result is obtained by enumerating all the [3] symmetric basis states using the Moshinsky’s method. For example, for the two independent (20) states constructed from $N_1 = 2$, $N_2 = 0$ and $N_1 = 0$, $N_2 = 2$ h.o. bases, only one (20) state is [3] symmetric, while the other (20) state belongs to the [21] symmetry. Similarly, only one [3] symmetric (40) state is made for $N = 4$, and another [21] symmetric Pauli forbidden state is constructed for the total angular-momentum $L = 0$ states of the $3\alpha$ system. These [21] symmetric $SU_3$ states are explicitly given by

$$\varphi_{a(21)}^{[21](20)} = \frac{1}{\sqrt{2}} \left[ U_{(20)a}(q) - U_{(20)a}(k) \right],$$

$$\varphi_{a(21)}^{[21](40)} = \sqrt{\frac{2}{5}} U_{(40)a}(q) - \sqrt{\frac{3}{5}} \left[ U_{(20)}(k)U_{(20)}(q) \right]_{(40)a}. \quad (15)$$

Note that the (21) and (41) $SU_3$ states are not possible for $L = 0$. This analysis shows that the two $\tau = -1$ solutions of Eq. (4) for the $L = 0 3\alpha$ system is nothing

---

* Here, again, we can explicitly construct the basis states using the theory of Double Gel’fand polynomials.

** A simple discussion using the tensor components of the 3-dimensional Jacobi-coordinate vectors $k$ and $q$ also leads to this conclusion. Namely, the 21-dimensional representation of the $N = 2$ 2-particle h.o. states, $k_\alpha k_\beta$ (6 dim.), $q_\alpha q_\beta$ (6 dim.), $k_\alpha q_\beta$ (9 dim.), is decomposed into the following $SU_3$ irreducible representations: [3](20): $k_\alpha k_\beta + q_\alpha q_\beta$ (6 dim.), [21](20): $k_\alpha k_\beta - q_\alpha q_\beta$ (6 dim.), [21](20): $k_\alpha q_\beta + q_\alpha k_\beta$ (6 dim.), [111](01): $\epsilon_{\alpha\beta\gamma} k_\beta q_\gamma$ (3 dim.). The [21](20) state in Eq. (15) corresponds to the second $SU_3$ state, which is [2] symmetric with respect to the exchange of the first two particles, but is not totally [3] symmetric.
but the [21](20) and [21](40) SU$_3$ states in Eq. (15). It is also apparent why we have no $\tau = -1$ solution for the 3$d'$ system. In this system, the Pauli forbidden state is only (0s) state for 2$d'$, and there exists no [21] symmetric Pauli forbidden state constructed for $N = 0$.

It should be noted that the existence of the [21] symmetric trivial solutions in the original Faddeev equation (1) is essential to eliminate the three-cluster Pauli forbidden states, which can not be trivially eliminated only through the pairwise orthogonality conditions with respect to the variable $k$. The mechanism to guarantee such a favorable result is furnished by the cooperative role with the exchange symmetry of the boson system and the partial elimination of the functional space $f(q)$ corresponding to the other variable $q$.

It would be legitimate to ask why such [21] symmetric components admixed to the Faddeev component $\varphi_\alpha$ do not play an important role in the usual Faddeev equations. Suppose we have a situation $\varphi_\beta + \varphi_\gamma = S\varphi_\alpha = -\varphi_\alpha$. Then the usual Faddeev equation, $\varphi_\alpha = G_0T_\alpha(\varphi_\beta + \varphi_\gamma)$, becomes $\varphi_\alpha = -G_0T_\alpha\varphi_\alpha$. If one uses the relationship $G_0T_\alpha = G_\alpha V_\alpha$, this equation is reduced to $(E - H_0)\varphi_\alpha = 0$, and $\varphi_\alpha$ turns out to be the plane wave. This implies that we have no square-integrable trivial solutions. Even if we use $\lambda\varphi_\alpha = G_0T_\alpha(\varphi_\beta + \varphi_\gamma)$, we obtain

$$\left[ E - H_0 - \left(1 - \frac{1}{\lambda}\right)V_\alpha \right] = 0,$$

which implies that there are no square-integrable solutions below the 2-body threshold. (Note that we are interested in the situation $\lambda > 0$.) Since the [21] components admixed to $\varphi_\alpha$ do not contribute to $\Psi$ anyway, there is no need to worry about such admixture.

For a practical application of the basic equation Eq. (9) to the 3$\alpha$ system, the self-consistency procedure for determining the energy dependence of the exchange term $\varepsilon K$ in the allowed space, discussed in I, is very important. We therefore need to evaluate the expectation value $\varepsilon$ of the 2$\alpha$ Hamiltonian through

$$\varepsilon = \frac{1}{3}E + \frac{1}{2}\langle \varphi | H_0(1 + S) | \varphi \rangle,$$

where $H_0$ is the 3-body free kinetic-energy operator and the Faddeev component $\varphi$ is normalized as $3\langle \varphi | 1 + S | \varphi \rangle = 1$. Starting from some specific values of $\varepsilon$ and $E$, we solve Eq. (9) and find a negative 3-body energy $E$ such that the eigen-value $\lambda(E)$ becomes 1. The normalized Faddeev component $\varphi$ yields a new value of $\varepsilon$ through Eq. (17). Since it is usually not equal to the starting value, we repeat the process by using the new value. This process of double iteration converges very fast if the starting values of $\varepsilon$ and $E$ are properly chosen. For numerical calculation, we discretize the continuous variables $k$ and $q$, using the Gauss-Legendre $n_1$- and $n_2$-point quadrature formula, respectively, for each of the three intervals of 0 - 1 - 3 - 6 fm$^{-1}$. The small contribution from the intermediate integral over $k$ beyond $k_0 = 6$ fm$^{-1}$ in the 2$\alpha$ $T$-matrix calculation is also taken into account by using the Gauss-Legendre $n_3$-point quadrature formula through the mapping $k = k_0 +$
\[ \tan(\pi/4)(1 + x) \]

The momentum region \( q = 6 \text{ fm}^{-1} - \infty \) is also discretized by the \( n_3 \) point formula just as in the \( k \) discretization case. The values of \( n_1-n_2-n_3 \) are given in Table III. The partial-wave decomposition of the \( 2\alpha \) RGM kernel is carried out numerically using the Gauss-Legendre 20-point quadrature formula. The incorporation of the Coulomb force to the Faddeev formalism is only possible by using the shielded Coulomb potential \( u(r) = (1/r)\theta(R_C - r) \), where \( \theta(x) \) is the Heaviside step function. Here we use a rather small value \( R_C = 6 \text{ fm} \), in order to make the phase-shift calculation of the \( 2\alpha \) system for higher partial-waves numerically stable. The Coulomb exchange kernel for this interaction is explicitly calculated.

Since our Faddeev component \( \varphi \) oscillates due to the orthogonality to the redundant components, we need to include the \( 2\alpha \) partial waves at least up to \( \ell = 4 \). The convergence of the result is confirmed by including the partial waves up to \( \ell \leq 8 \). The numerical inaccuracy of the angular-momentum projection is examined by extending the 20-point quadrature formula to the 30-point formula, and we found that the error is less than 1 keV if numbers of discretization points for \( k \) and \( q \) are large enough. Furthermore, the modified spline interpolation technique developed by Glöckle et al.\(^5\) is employed for constructing the rearrangement matrix \( S \). For the diagonalization of the large non-symmetric matrix, the Arnoldi-Lanczos algorithm recently developed in the ARPACK subroutine package\(^6\) is very useful. For the effective 2-nucleon force for the \( 2\alpha \) RGM kernel, we use the Volkov No. 2 force with \( m = 0 \).\(^59\), following the \( 3\alpha \) RGM calculation by Fukushima and Kamimura\(^7\). The h.o. constant for the \( \alpha \) cluster is assumed to be \( \nu = 0.275 \text{ fm}^{-2} \).

Table III shows the solution of the Faddeev equation for the \( 3\alpha \) system, obtained by solving Eq. (9). We find that the convergence is satisfactory, if the relative \( 2\alpha \) partial waves up to \( \ell_{\text{max}} = 8 \) are taken into account. We compare the best result of the Faddeev calculation with the variational calculation employing the \([3]\) symmetric translationally-invariant h.o. basis. The model is explained in I. The numbers

* These \( n_3 \) points for \( k \) are not included for solving the Faddeev equation (9), since it causes a numerical inaccuracy for the interpolation.

| \( \ell_{\text{max}} \) | \( n_1-n_2-n_3 \) | \( n_{\text{max}} \) | \( \epsilon(2\alpha) \) | \( E(3\alpha) \) | \( c_{(04)} \) |
|---|---|---|---|---|---|
| 4 | 15-10-5 | 4,725 | 8.5826 (9.4542) | -11.2023 (-5.9313) | 0.8262 (0.7905) |
| | 20-10-10 | 7,200 | 8.5825 (9.4541) | -11.2032 (-5.9318) | 0.8262 (0.7905) |
| 6 | 15-10-5 | 6,300 | 8.4490 (9.3511) | -11.4151 (-6.1188) | 0.8212 (0.7861) |
| | 20-10-10 | 9,600 | 8.4488 (9.3509) | -11.4158 (-6.1193) | 0.8212 (0.7861) |
| 8 | 15-10-5 | 7,875 | 8.4467 (9.3492) | -11.4179 (-6.1212) | 0.8211 (0.7860) |
| | 20-10-10 | 12,000 | 8.4465 (9.3490) | -11.4187 (-6.1217) | 0.8211 (0.7860) |
Redundant Components in the 3α Faddeev Equation

without the parentheses in Table IV indicate the results without the Coulomb force, while those in the parentheses when the shielded Coulomb force with $R_C = 6\ \text{fm}^{-1}$ is included. (Note that the 3α energy with the present shielded Coulomb force is slightly (about 150 keV) more attractive in comparison with the that of the full Coulomb calculation in Table I of I.) Table IV also lists the expectation value of the 2α subsystem, $\varepsilon$, and the overlap with the simple shell-model wave function $c_{(04)}$ as well. Here the $SU_3 (04)$ wave function is expressed in the 3α cluster model as

$$\varphi_a^{[3](04)} = \left[ U_{(40)}(k)U_{(40)}(q) \right]_{(04)a} = \frac{8}{15} R_{20}(k,b_1)R_{20}(q,b_2)Y_{(00)0}(\hat{k},\hat{q})$$

$$- \frac{4}{3\sqrt{5}} R_{12}(k,b_1)R_{12}(q,b_2)Y_{(22)0}(\hat{k},\hat{q}) + \frac{3}{5} R_{04}(k,b_1)R_{04}(q,b_2)Y_{(44)0}(\hat{k},\hat{q}),$$

(18)

where $b_1 = 1/4\gamma$ and $b_2 = 3/16\gamma$ with $\gamma = \mu\nu = 2\nu$, $R_{nl}(x,\nu)$, the radial part of the h.o. wave function, and $Y_{(\lambda L)J}(\hat{k},\hat{q})$ the coupled angular-momentum functions. The difference between the Faddeev calculation and the variational calculation is very small. In particular the difference of the 3α energies is less than 1 keV both in the Coulomb on and off cases.

In summary, we have found that the Faddeev equation using 2-cluster RGM kernel, derived in the previous paper, may involve redundant components, if multi Pauli-forbidden states exist for the 2-cluster relative motion as in the 3α system. For systems of three identical bosons, these components do not contribute to the total wave function, since they belong to [21] symmetry with respect to the permutations of the three bosons. We have proposed a simple method to eliminate these redundant components through diagonalization procedure of the rearrangement matrix in the 3-cluster model space with the 2-cluster forbidden components. The equivalence between this modified Faddeev equation and the variational approach using the translationally invariant harmonic-oscillator basis is numerically shown for the ground state of the 3α system.

This work was supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture (No. 12640265).

References

[1] Y. Fujiwara, H. Nemura, Y. Suzuki, K. Miyagawa and M. Kohno, KUNS-1749, nucl-th/0112070, submitted to Prog. Theor. Phys. (2002).
[2] H. Horiuchi, Prog. Theor. Phys. 51 (1974), 1266; 53 (1975), 447.
[3] See, for example, M. Moshinski and Y. F. Smirnov, The Harmonic Oscillator in Modern Physics, Contemporary Concepts in Physics, Vol. 9 (Harwood Academic Publishers, Amsterdam, 1996), Chapter III.17, p. 58.
[4] Y. Fujiwara and H. Horiuchi, Memoirs of the Faculty of Science, Kyoto University, Series A of Physics, Astrophysics, Geophysics and Chemistry, Vol. XXXVI, No. 2, Article 1, (1983), 197.
[5] W. Glöckle, G. Hasberg and A.R. Neghabian, Z. Phys. A - Atoms and Nuclei 305 (1982), 217.
[6] See ARPACK homepage, http://www.caam.rice.edu/software/ARPACK/
[7] Y. Fukushima and M. Kamimura, Proc. Int. Conf. Nuclear Structure, Tokyo, 1977, J. Phys. Soc. Japan 44 (1978) Suppl. p. 225.
Table IV. The $L = 0$ lowest eigen-values for $2\alpha$ and $3\alpha$ systems, obtained by diagonalization using [3] symmetric translationally-invariant h.o. basis. $N$ stands for the maximum total h.o. quanta included in the calculation, and $n_{\text{max}}$ the number of the basis states. The numbers without the parentheses indicate the results without the Coulomb force, while those in the parentheses when the shielded Coulomb force with $R_c = 6$ fm$^{-1}$ is included.

| $N$ | $n_{\text{max}}$ | $E(2\alpha)$ | $\varepsilon$ | $E(3\alpha)$ | $c_{(0\alpha)}$ |
|-----|----------------|-------------|-------------|-------------|----------------|
| 4   | 9.7474 (1.8262) | 20.7705 (23.1969) | 5.4467 (12.6290) | 1           |
| 6   | 5.0088 (6.9205) | 22.7245 (25.1180) | 1           |
| 8   | 2.5254 (4.3239) | 15.9632 (18.0165) | 0.9567 (0.9528) |
| 10  | 1.1182 (2.8321) | 12.9965 (14.8444) | 0.9160 (0.9077) |
| 12  | 0.2649 (1.9122) | 11.3357 (13.0258) | 0.8860 (0.8736) |
| 16  | 0.5010 (1.3140) | 10.3336 (11.8894) | 0.850 (0.8490) |
| 18  | 0.6396 (0.9073) | 9.6897 (11.1403) | 0.8511 (0.8315) |
| 20  | 0.6676 (0.6203) | 9.2852 (10.6295) | 0.8415 (0.8190) |
| 22  | 0.1069 (0.4111) | 9.0119 (10.2742) | 0.8101 (0.8101) |
| 24  | 0.1899 (0.2544) | 8.8295 (10.0235) | 0.8035 (0.8037) |
| 26  | 0.1283 (0.1343) | 8.7071 (9.8446) | 0.7990 (0.7990) |
| 28  | 0.3547 (0.0403) | 8.6246 (9.7157) | 0.7957 (0.7957) |
| 30  | 0.4099 (0.0346) | 8.5686 (9.6222) | 0.7932 (0.7932) |
| 32  | 0.4511 (0.0952) | 8.5306 (9.5538) | 0.7914 (0.7914) |
| 34  | 0.4842 (0.1443) | 8.5047 (9.5034) | 0.7901 (0.7901) |
| 36  | 0.5105 (0.1860) | 8.4869 (9.4660) | 0.7891 (0.7891) |
| 38  | 0.5317 (0.2205) | 8.4746 (9.4381) | 0.7884 (0.7884) |
| 40  | 0.5487 (0.2496) | 8.4661 (9.4172) | 0.7878 (0.7878) |
| 42  | 0.5626 (0.2745) | 8.4603 (9.4014) | 0.7874 (0.7874) |
| 44  | 0.5741 (0.2959) | 8.4562 (9.3805) | 0.7871 (0.7871) |
| 46  | 0.5835 (0.3144) | 8.4533 (9.3804) | 0.7869 (0.7869) |
| 48  | 0.5914 (0.3305) | 8.4513 (9.3734) | 0.7867 (0.7867) |
| 50  | 0.5979 (0.3446) | 8.4499 (9.3680) | 0.7865 (0.7865) |
| 52  | 0.6034 (0.3570) | 8.4489 (9.3639) | 0.7864 (0.7864) |
| 54  | 0.6081 (0.3680) | 8.4482 (9.3607) | 0.7863 (0.7863) |
| 56  | 0.6120 (0.3778) | 8.4476 (9.3582) | 0.7863 (0.7863) |
| 58  | 0.6153 (0.3865) | 8.4473 (9.3563) | 0.7862 (0.7862) |
| 60  | 0.6182 (0.3942) | 8.4470 (9.3547) | 0.7862 (0.7862) |
| 62  | 0.6206 (0.4013) | 8.4468 (9.3535) | 0.7862 (0.7862) |
| 64  | 0.6227 (0.4076) | 8.4467 (9.3526) | 0.7862 (0.7862) |
| 66  | 0.6245 (0.4133) | 8.4466 (9.3519) | 0.7861 (0.7861) |
| 68  | 0.6261 (0.4185) | 8.4465 (9.3513) | 0.7861 (0.7861) |
| 70  | 0.6274 (0.4231) | 8.4464 (9.3508) | 0.7861 (0.7861) |
| 72  | 0.6286 (0.4270) | 8.4464 (9.3504) | 0.7861 (0.7861) |

Faddeev 8.4465 (9.3490) 11.4187 (6.1217) 0.8211 (0.7860)