Lower Bound on the Localization Error in Infinite Networks with Random Sensor Locations

Itsik Bergel and Yair Noam

Abstract

We present novel lower bounds on the mean square error (MSE) of the location estimation of an emitting source via a network where the sensors are deployed randomly. The sensor locations are modeled as a homogenous Poisson point process. In contrast to previous bounds which are a function of the specific locations of all the sensors, we present CRB-type bounds on the expected mean square error; that is, we first derive the CRB on the MSE as a function of the sensors’ location, and then take expectation with respect to the distribution of the sensors’ location. Thus, these bounds are not a function of a particular sensor configuration, but rather of the sensor statistics. Hence, these novel bounds can be evaluated prior to sensor deployment and provide insights into design issues such as the necessary sensor density, the effect of the channel model, the effect of the signal power, and others. The derived bounds are simple to evaluate and provide a good prediction of the actual network performance.

I. INTRODUCTION

Localization via sensor networks plays an important role in many applications such as personal security and safety, location based billing, machinery monitoring, radio-resource management, intelligent transportation systems, etc. (e.g., [2]–[8]). To determine the source location, each sensor takes measurements about the source; then, the location is estimated by a joint processing of the measurements of all sensors. The sensors typically measure time-of-arrival (TOA), time-difference-of-arrival (TDOA), angle-of-arrival (AOA), or received-signal-strength (RSS).
There has been extensive research over the years to improve localization accuracy and robustness under various network conditions (see, for example, [9]–[13] and references therein). A significant part of this research has to do with localization performance analysis, which is carried out primarily by deriving the Cramér-Rao bound (CRB) on the mean square error (MSE) of the location estimate in a variety of network setups (e.g., [14]–[19]). In all these works, the derived CRBs are functions of the sensor locations, making it less useful for designing and analyzing networks with arbitrary sensor deployment; e.g., if the sensors are dropped randomly from the air, or if the designer is not familiar with the terrain and/or the constraints under which the sensors will actually be placed. In such cases, a random network analysis is more suitable, where the sensors’ location is modeled as a realization of a random field. This approach has been applied to different aspects of ad-hoc wireless networks, and has provided many important answers and insights, including capacity scaling-laws (e.g., [20]–[22]), closed-form expressions for various performance metrics (e.g., [23]–[27]), throughput bounds (e.g., [21],[28]–[30]) and etc.

In many respects, the problems of ad-hoc networks and localization are related; e.g., both share similar channel models, network structures and system parameters. Therefore, applying stochastic geometry to the analysis of source localization via sensor networks can lead to novel insights. Numerical studies of the localization performance of networks with random sensor placement (e.g.; [31]–[34]) have yielded interesting results about the behavior of random networks. An important approach involves modeling the sensor locations as Poisson Point Processes (PPP). Lazos and Poovendran [35] used PPP modeling in a numerical study of localization robustness. Aldalahmeh et al. [36] evaluated the probability of target detection by proximity sensors, and [37] [38] further determined the scaling law of the localization error as a function of the sensor-density. Nevertheless, to date, no closed-form expression for the localization error in random networks has been put forward.

In this paper, using stochastic geometry, we consider the sensor locations as a realization of a homogenous PPP, and derive CRB-type bounds on the achievable localization error. First, we derive the CRB as a function of sensor locations, in the case of an infinitely countable number of sensors, each measuring a continuous time signal. While such a bound has been derived for a finite number of sensors, it has never been properly extended to the case of an infinite countable number. This step, which also includes the derivation of a likelihood function, is crucial to take the expectation with respect to the PPP distribution, and obtain a bound which is not a function.
of the sensors’ location, but rather a function of the sensor statistics. We present a bound on the MSE in a network that utilizes both TOA and RSS measurements, and show that in the case of joint TOA and RSS measurements, the bound does not satisfy the assumptions of [37] and does not scale as \( \lambda^a \) for any \( a > 0 \), where \( \lambda \) is the sensor density. We also provide closed-form asymptotic expressions for the bound in the wideband and narrowband cases. Note that the narrowband asymptote is equivalent to a network that only uses RSS measurements, while the wideband asymptote is equivalent to a network that only uses TOA measurements. Unlike the general bound, the two asymptotic bounds do scale as \( \lambda^a \) (such a scaling was observed in [37], using simplifying assumptions on the measurement model). Note that in this work we provide closed-form bounds, while the analysis of [37] required a numerical evaluation of the coefficients of this proportionality.

The remainder of the paper is organized as follows. In Section II we present the system model, and in Section III we extend the known CRB for the case of a finite number of sensors to the case of an infinite number of sensors. In Section IV we present our novel lower bound on the localization error, and also give simple expressions for the bound in specific cases of interest. In Section V we present some numerical examples and Section VI concludes.

Notation: Vectors and matrices are denoted by boldface lower-case and upper-case letters respectively. \( \| \cdot \| \) denotes the Euclidian norm. \( \mathbb{R}_+ \) denotes the positive part of the real line.

II. System Model

Consider a two-dimensional source localization problem with an infinitely countable number of sensors, where sensor \( m \) is located at the point \( \psi_m = (a_m, b_m) \in \mathbb{R}^2 \), and the unknown source location is \((x, y)\). We assume that the location of all sensors, \( \psi = \{\psi_m\}_{m=1}^{\infty} \), is a realization of a \( \lambda \)-density homogenous PPP, \( \Psi \), defined on the probability space \((\Omega, \mathcal{F}, P)\), and that after the sensors have been randomly dispersed in the plane, \( \psi \) becomes known to the fusion center. The received signal at the \( m \)-th sensor is given by

\[
r_m(t) = k_0 D_m^{-\gamma/2} s(t - \tau_m) + v_m(t), \quad t \in \mathbb{R}_+, \ m \in \mathbb{N},
\]

where \( s(t) \) is a known signal waveform satisfying

\[
\left| \frac{ds(t)}{dt} \right|, \left| \frac{d^2 s(t)}{dt^2} \right|, \int_0^{\infty} |s(t)| dt, \int_0^{\infty} |ds(t)/dt| < \infty;
\]

\(^1\) For simplicity, we limit the discussion to real signals.

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Fig. 1. System model. The black circles denote the sensors; the star denotes the source.

$k_0$ is a constant that depends on the transmission power, antenna gains and carrier frequency. Furthermore, $\gamma$ is the path-loss exponent, $\tau_m = D_m/c$ is the time delay, $c$ is the speed of light, 

$$D_m(\theta) = \sqrt{(x-a_m)^2 + (y-b_m)^2}$$

(3)

is the distance between the source and the $m$-th sensor, and $v_m(t)$ is a white Gaussian noise with spectral density $N_0/2$.

The fusion center receives the signals from all of the sensors, from which it estimates $\hat{\theta} = [\hat{X}, \hat{Y}]$. We denote the estimation error by $\Delta \theta \triangleq \hat{\theta} - \theta$, and the mean square estimation error (MSE) for a given sensor locations is $E\{\|\Delta \theta\|^2 | \Psi \} = E\{ (\hat{X} - x)^2 + (\hat{Y} - y)^2 | \Psi \}$. At the system design stage, the sensor locations are not yet known. Thus, in this work we focus on the expected MSE with respect to the distribution of the sensor locations: $E\{\|\Delta \theta\|^2 \}$.

### III. Derivation of the CRB and Average CRB

In this section we derive the average CRB by taking the expectation of the CRB over all possible sensor placements. To do so, we must first calculate the CRB for a given sensor placement $\psi$. This CRB, which involves an infinitely countable number of sensors has never

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2In this work we take the standard approach of assuming that the path loss exponent is known. See [39] for a discussion on the estimation of the path loss exponent in random networks.
been rigorously derived. This derivation is mandatory if we want to take the expectation of the CRB with respect to the distribution of \( \Psi \) (see Appendix A for further details).

**Theorem 1:** Let \( \psi = \{ \psi_m \}_{m=1}^{\infty} \in \mathbb{R}^{2 \times \infty} \) be an infinite sequence of points in \( \mathbb{R}^2 \), where point \( \psi_m = (a_m, b_m) \) denotes the location of the \( m \)th sensor, whose received signal is given in (1).

For \( \Psi = \psi \), let \( \hat{\theta} = [\hat{X}, \hat{Y}] \) be an unbiased estimate of the source location \( \theta = [x, y] \in \mathbb{R}^2 \); i.e., \( E\{\hat{\theta} - \theta | \psi\} = 0 \). If \( \psi \) satisfies

\[
\sum_{m=1}^{\infty} D_m^{-\gamma} < \infty, 
\]

for every \( \theta \), where \( D_m \) is defined in (3), then, the CRB on the mean square error is:

\[
E\{\|\hat{\theta} - \theta\|^2 | \psi\} \geq \text{CRB}(\theta, \psi),
\]

where

\[
\text{CRB}(\theta, \psi) = \frac{1}{\rho} \sum_{m=1}^{\infty} \sum_{j>m} g(D_m) g(D_j) \sin^2 (\phi_m - \phi_j),
\]

\[
g(D) = D^{-\gamma - 2} \left( \gamma^2 + \frac{4W_e c^2}{D^2} \right),
\]

\( E_s = k_0^2 \int_{-\infty}^{\infty} s^2(t) dt \) is the received signal power at a unit distance, \( \rho = E_s/2N_0 \), and \( W_e = k_e^2 \int_{-\infty}^{\infty} (ds(t)/dt)^2 dt \) is the effective bandwidth; \( \phi_m \) is the angle between the source and the \( m \)th sensor; i.e., \( \cos (\phi_m) = (x - a_m) / D_m \) and \( \sin (\phi_m) = (y - b_m) / D_m \), as depicted in Fig. 1.

**Proof:** The CRB for the model (1) is already known in the case of a finite number of sensors in deterministic locations [18]. However, extending this result to the infinite case is not trivial and requires careful treatment. The full proof is given in Appendix A.

The bound (6) is a function of the sensor locations, \( \psi \) (due to the fact that \( \{\phi_m\}_{m=1}^{\infty} \) and \( \{D_m\}_{m=1}^{\infty} \) are functions of \( \psi \)), which is assumed to be deterministic. Next, by treating the sensor locations as PPP \( \Psi \) and taking the expectation of \( \text{CRB}(\theta, \Psi) \) with respect to the distribution of \( \Psi \), we obtain the average CRB. This is a more general tool that depends on the sensor density, rather than a specific sensor locations. Because \( \text{CRB}(\theta, \psi) \) is defined solely under the constraint (4), one must show that \( \text{CRB}(\theta, \Psi) \) is a well-defined random variable.

**Theorem 2:** Let \( \Psi \) be a PPP, then \( \text{CRB}(\theta, \Psi) \) is a well-defined random variable; i.e., (4) is satisfied with probability one.

**Proof:** see Appendix B
Now that it is possible to take the expectation, the average CRB is given by

\[ \overline{\text{CRB}}(\theta) = \mathbb{E}\{\text{CRB}(\theta, \Psi)\} = \frac{1}{\rho} \mathbb{E} \left[ \frac{\sum_m g(D_m)}{\sum_m \sum_{j > m} g(D_m) g(D_j) \sin^2(\phi_j - \phi_j)} \right]. \]  

(8)

It is important to stress that unlike \( \text{CRB}(\theta, \Psi) \), \( \overline{\text{CRB}}(\theta) \) is not a bound on the estimation error that a particular network (with a particular \( \psi \)) achieves, but rather a bound on the expected error over all possible networks whose sensors’ placement is a realization of a PPP.

We conclude this section by a discussion of the relationship between (8) and other CRB-type bounds as well as the underlying unbiased-condition required by each bound. The bound \( \overline{\text{CRB}}(\theta) \) looks similar to other CRB-type bounds, such as the Hybrid CRB [40], the modified CRB [41–43] and the Miller-Chang bound [44], which were designed for hybrid estimation problems; i.e., problems that include unknown deterministic parameters (for example, \( \theta \)) and unknown random (for example \( \psi \)) parameters. Similar to \( \overline{\text{CRB}}(\theta) \), these bounds are obtained by first deriving the CRB (or the FIM in the modified CRB and the HCRB) for a given value of some random parameter, and then taking the expectation with respect to that random parameter. However, the bound in (8) is completely different, since the argument of the expectation here is \( \text{CRB}(\theta, \varphi) \), which is derived under the assumption that \( \varphi \) is known to the estimator; i.e., it is not an unknown parameter, or even an unknown nuisance parameter. With respect to the unbiased condition, \( \overline{\text{CRB}}(\theta) \) requires that the estimate of \( \theta \) be unbiased for every value of \( \varphi \), except possibly for sets of probability zero. This condition is similar (but not identical since in our case, \( \varphi \) is perfectly known) to that of Miller-Chang bound [44], and is stronger than the condition of the Hybrid and the Modified CRBs, which require unbiasedness only on the average. For further reference on CRB-type bounds, the reader is referred to [45].

IV. LOWER BOUND ON THE LOCALIZATION ERROR

Theorem 2 presents the average CRB on the localization error. This CRB can be primarily evaluated by Monte Carlo simulations (while truncating the sum over \( m \) to a large enough number of sensors). The following theorem presents a simple lower bound on \( \overline{\text{CRB}}(\theta) \), expressed in terms of one-dimensional integrals.

**Theorem 3:** Consider a sensor network whose sensor locations is a realization of a homogenous PPP with \( \lambda \) sensors per unit area. Then, the average CRB in (8) is lower bounded by:

\[ \overline{\text{CRB}}(\theta) \geq \text{CRB}_{LB} = \frac{4}{\rho} \int_0^\infty \exp\left\{ -2\pi \lambda Z(s) \right\} ds, \]

(9)
where
\[
Z(s) = \int_0^\infty \left[ 1 - e^{-sg(r)} \right] r dr. \tag{10}
\]

Although presented in an integral form, the bound \(\text{CRB}_{LB}\) is very informative. As expected, the bound on the average CRB, \(\text{CRB}_{LB}\), scales linearly with the noise power and inverse linearly with the transmission power. Moreover, because \(Z(s)\) is always positive, \(\text{CRB}_{LB}\) is a monotonically decreasing function of the sensor density, \(\lambda\).

**Proof:** From (8)
\[
\text{CRB}(\theta) = \frac{1}{\rho} \mathbb{E} \left[ \sum_m g(D_m) \sum_{m \neq j} g(D_j) \sin^2(\phi_m - \phi_j) \right] \{D_m\}_{m=1}^\infty \geq \frac{1}{\rho} \mathbb{E} \left[ \sum_m g(D_m) \frac{1}{2} \sum_m \sum_{m \neq j} g(D_m) g(D_j) \mathbb{E}[\sin^2(\phi_m - \phi_j)] \{D_m\}_{m=1}^\infty \right], \tag{11}
\]
\[
= \frac{4}{\rho} \mathbb{E} \left[ \frac{\sum_m g(D_m)}{\sum_m \sum_{m \neq j} g(D_m) g(D_j)} \right], \tag{12}
\]

where (11) follows from Jensen’s inequality for conditional expectations, and (12) is due to the uniform distribution of the angles. Next, by adding the missing diagonal terms of the double sum to the denominator, one obtains
\[
\text{CRB}(\theta) \geq \frac{4}{\rho} \mathbb{E} \left[ \frac{\sum_m g(D_m)}{\sum_m \sum_{m \neq j} g(D_m) g(D_j) + \sum_m g^2(D_m)} \right] \tag{13}
\]
\[
= \frac{4}{\rho} \mathbb{E} \left[ \frac{1}{\sum_m g(D_m)} \right]. \tag{14}
\]

Considering the denominator in (14), we define
\[
G = \sum_m g(D_m); \tag{15}
\]
a quantity whose characteristic function is known in stochastic geometry [46-48],
\[
\varphi_G(s) = \mathbb{E}\{e^{-sG}\} = e^{-2\pi\lambda Z(s)}. \tag{16}
\]

The average CRB can be evaluated from \(\varphi_G(s)\) via the formula (see for example [28]):
\[
\mathbb{E}\{G^{-1}\} = \int_0^\infty \varphi_G(s) ds. \tag{17}
\]
Substituting (16) and (17) into (14) establishes the desired result.

As expected, $\text{CRB}_{\text{LB}}$, scales linearly with the noise power and inverse linearly with the transmission power. Moreover, because $Z(s)$ is always positive, $\text{CRB}_{\text{LB}}$ is a monotonically decreasing function of the sensor density, $\lambda$. However, obtaining further insights is more difficult because the integrals in (9) and (10) have no closed-form expression in general. To this end, we derive closed-form expressions in two extreme cases.

1) Wideband extreme: In many cases, particularly in the case of wideband sources, the TOA is much more informative on the source’s location than the RSS is. Keeping that in mind, we now characterize the CRB in the wideband extreme case. The intuition is that for large enough $W_e$, $g(D) \approx 4W_eD^{-\gamma}/c^2$, implying that the RSS is negligible. Substituting the latter approximation of $g(D)$ into (9) and (10) yields:

$$\text{CRB}_{\text{LB},W} = \frac{c^2}{\rho W_e} \left( \frac{\pi \lambda}{2} \right)^{-\frac{2}{\gamma}} \left( 1 - \frac{2}{\gamma} \right) \Gamma \left( 1 + \frac{\gamma}{2} \right).$$

(18)

The following corollary formalizes the approximation:

**Corollary 4 (Wideband localization):** Consider $\text{CRB}_{\text{LB}}$ of Theorem 3 and $\text{CRB}_{\text{LB},W}$ of (18), then

$$\left( 1 - \frac{\pi \lambda c^2 \gamma}{2W_e} \right) \text{CRB}_{\text{LB},W} < \text{CRB}_{\text{LB}} < \text{CRB}_{\text{LB},W};$$

(19)

hence, $\text{CRB}_{\text{LB}}/\text{CRB}_{\text{LB},W} \to 1$ as $W_e \to \infty$ with a convergence rate of $O(1/W_e)$.

**Proof:** We first bound the integrand (10) from below and above, using

$$1 - \left( 1 - \gamma^2 s \gamma - \gamma^2 \right) e^{-4sW_e r^{-\gamma}/c^2}$$

$$> 1 - e^{-sg(r)} > 1 - e^{-4sW_e r^{-\gamma}/c^2}.$$ 

(20)

where for the right hand side (RHS) of (20) we used the inequality $x + 1 < e^x$ and the left hand side (LHS) is because $e^{-\gamma^2 s \gamma - (\gamma^2 + 2)} < 1$. Let

$$U(s) \triangleq \int_0^\infty (1 - e^{-4sW_e r^{-\gamma}/c^2}) r dr$$

(21)

Then, from (20)

$$U(s) < Z(s) < \int_0^\infty (1 - (1 - \gamma^2 s \gamma - \gamma^2) e^{-4sW_e r^{-\gamma}/c^2}) r dr$$

$$= U(s) + \frac{c^2 \gamma}{4W_e}.$$ 

(22)

(23)

where for the RHS we used the identity

$$\int_0^\infty e^{-bx-a}x^{-1-c}dx = \frac{b^{-\frac{a}{c}}}{a} \Gamma \left( \frac{b}{a} \right).$$

(24)
From (22)

\[
\text{CRB}_{LB,W} = \frac{4}{\rho} \int_0^\infty e^{-2\pi U(s)} ds < \text{CRB}_{LB} \tag{25}
\]

\[
< \frac{4}{\rho} \int_0^\infty e^{-2\pi \left(U(s) + \frac{c^2}{4W_e}\right)} ds < \left(1 - \frac{\pi \lambda c^2}{2W_e}\right) \text{CRB}_{LB,W}
\]

where we used \(\exp\{-x\} > 1 - x\). This establishes the proof.

The inverse linear relation between \(\text{CRB}_{LB}\) in (18) and the effective bandwidth is, again, quite expected. However, this bound also indicates the effect of the sensor-density and the path-loss exponent; i.e., it decreases as \(\lambda^{-\gamma/2}\). Thus, for \(\gamma\) which are close to 2, the bound is inversely proportional to the sensor-density. In this extreme case, the sensor density \(\lambda\) has a similar effect as the number of sensors in the more traditional setup of sensors with independent observations. On the other hand, in the more typical setups where \(\gamma > 2\), the density \(\lambda\) has a stronger effect than the number of sensors. This may be explained by the fact that large values of \(\lambda\) indicate a high probability of having sensors in close proximity to the target.

2) Narrowband extreme: Here, the TOA is not informative enough and the localization relies primarily on the RSS information. One practical example (that is not covered by this model) is the case of a dense multipath, where it is very difficult to determine the location from TOA information. Another example, which is easily obtained from our model, is the narrowband extreme case, where \(W_e\) is small enough such that the TOA data is insignificant; i.e., \(g(D) \approx \gamma^2 D^{-\gamma-2}\). Again, similar to (18) the approximate bound is given by

\[
\text{CRB}_{LB,N} = \frac{4}{\rho \gamma^2} \left(\pi \lambda \Gamma \left(\frac{\gamma}{\gamma+2}\right)\right)^{-\gamma/2-1} \Gamma \left(2 + \frac{\gamma}{2}\right). \tag{26}
\]

The exact formulation is established in the following corollary.

**Corollary 5 (Narrowband localization):** The CRB bound of Theorem 3 satisfies

\[
|\text{CRB}_{LB,N} - \text{CRB}_{LB}| < O(W_e), \tag{27}
\]

where \(\text{CRB}_{LB,N}\) is given in (26). Hence, \(\text{CRB}_{LB}\) converges to \(\text{CRB}_{LB,N}\) as the bandwidth decreases.

**Proof:** Again, we bound the integrand (10) from below and above,

\[
1 - e^{-s\gamma^2 r - \gamma^2} \left(1 - \frac{4sW_e r - \gamma^2}{c^2 r}\right) = 1 - e^{-s g(r)} > 1 - e^{-s \gamma^2 r - \gamma^2}. \tag{28}
\]
Similar to (20), integrating the RHS of (28) yields \( \text{CRB}_{LB} < \text{CRB}_{LB,N} \), which establishes the RHS of (27). To show the LHS of (27), we multiply the LHS of (28) by \( r \), integrate and by using (24) we obtain

\[
Z(s) < \int_0^\infty \left( 1 - e^{-s\gamma^2 r^{-\gamma^2 - 2}} \left( 1 - \frac{4sW_e\gamma - 1}{e^2 r^{-\gamma^2 - 2}} \right) \right) rdr = V(s) + Q(s)
\]

where

\[
V(s) \triangleq \int_0^\infty \left( 1 - e^{-s\gamma^2 r^{-\gamma^2 - 2}} \right) rdr
\]

\[
Q(s) \triangleq \frac{4sW_e(s\gamma^2)^{\frac{1}{\gamma^2 + 2}}}{e^{2(\gamma^2 + 2)}} \Gamma \left( \frac{\gamma - 1}{\gamma + 2} \right).
\]

Hence

\[
\text{CRB}_{LB,N} = \frac{4}{\rho} \int_0^\infty \exp \left\{ -2\pi \lambda V(s) \right\} ds
\]

\[
> \text{CRB}_{LB} = \frac{4}{\rho} \int_0^\infty \exp \left\{ -2\pi \lambda Z(s) \right\} ds
\]

\[
> \frac{4}{\rho} \int_0^\infty \exp \left\{ -2\pi \lambda (V(s) + Q(s)) \right\} ds
\]

and therefore

\[
|\text{CRB}_{LB} - \text{CRB}_{LB,N}| < \frac{4}{\rho} \int_0^\infty (e^{-2\pi \lambda V(s)} - e^{-2\pi \lambda (V(s) + Q(s))}) ds
\]

\[
< \frac{4}{\rho} \int_0^\infty e^{-2\pi \lambda V(s)} \left( 1 - e^{-2\pi \lambda Q(s)} \right) ds < \frac{4}{\rho} \int_0^\infty e^{-2\pi \lambda V(s)} 2\pi \lambda Q(s) ds
\]

\[
< \frac{16\pi e^{-\frac{1}{2}(\gamma + 3)} \lambda W_e}{\gamma^4 \rho} \Gamma \left( \frac{\gamma}{\gamma + 2} \right)^{-\frac{5+5}{\gamma^2 + 2}} \Gamma \left( \frac{\gamma - 1}{\gamma + 2} \right) \Gamma \left( \frac{\gamma + 5}{2} \right)
\]

where in the third line we used \( e^{-x} > 1 - x \) for \( x > -1 \). This establishes the proof.

The performance in the narrowband case is typically not as good as in the wideband case, due to the effective bandwidth factor, \( W_e \), in (18) which significantly reduces the bound. Corollary 5 shows that in the narrowband regime, the bound decreases as \( \lambda^{-\gamma/2 - 1} \), which is a faster decay rate than in the wideband case. This dependence shows that RSS localization depends heavily on the probability of having sensors very close to the target.

V. NUMERICAL EXAMPLES

In this section we use the closed-form bounds, and show their usefulness in predicting the average localization error. All simulations were carried out using the pulse

\[
s(t) = \sqrt{\frac{2E_s}{3Tk_0^2}} \cdot (1 - \cos(2\pi t/T)), \quad t \in [0, T],
\]

\[\text{(34)}\]
which satisfies $W_e = \frac{4\pi^2}{3\gamma^2}$. We begin by demonstrating the behavior of the bound as a function of the SNR. Fig. 2 depicts the MSE of the maximum likelihood localizer, the average CRB, $\text{CRB}(\theta)$ from (8), which was approximated by averaging the $\text{CRB}(\theta, \Psi)$ in (6) over 2000 Monte Carlo trials with $10^3$ sensors (instead of infinity) deployed randomly according to a PPP distribution at each trial. The figure also depicts $\text{CRB}_{\text{LB}}$ from (9). The sensor-density is one per 100m$^2$, the path-loss exponent is $\gamma = 4$ and $T$ is set to $10^{-6}$ or $10^{-8}$ seconds. As expected, both bounds are inversely proportional to the SNR. The closed-form lower bound $\text{CRB}_{\text{LB}}$ is, indeed, not as tight as the average CRB. Nevertheless, it exhibits exactly the same behavior, and can serve as a good indication of the actual achievable performance. The ML localizer demonstrates the well-known threshold effect, where for SNR values of more than 54dB (for 1 meter) the localizer achieves the average CRB bound, while for lower SNRs the localization error is significantly larger.

Fig. 3 shows the localization error as a function of the path-loss exponent for the same sensor density and an SNR of $\rho = 50$ dB at one meter away from source. Interestingly, the figure shows that for $\gamma \leq 3.5$, this SNR is sufficient for the ML localizer to converge to the average CRB, while for higher values of $\gamma$ the latter SNR is insufficient. As expected from (7), for small enough values, the effective bandwidth does not affect the localization performance. In this scenario, this occurs for $T < 10^{-6}$, whereas below it, the contribution of the time-of-arrival information becomes negligible, and the localization relies solely on the received power information. In addition, the slope of the $\log_{10} \text{MSE}$ behaves differently for wideband and narrowband signals, where in the narrowband case ($T > 10^{-6}$) this slope is $d \log_{10} \text{MSE}/d\gamma = 0.8$ while in the
Fig. 3. Source localization MSE of the ML estimator against the average CRB $\text{CRB}(\theta)$ and the lower bound $\text{CRB}_{\text{LB}}$ as a function of the path-loss exponent, $\lambda = 0.01 \text{m}^2$ and $\rho = 50 \text{dB}$.

For the wideband case ($T < 10^{-8}$), this slope is $d \log_{10} \text{MSE}/d \gamma = 1.2$.

Finally, Fig. 4 presents the localization performance as a function of the sensor density, $\lambda$, for $\gamma = 4$ and $\rho = 50 \text{dB}$. Again, the closed-form lower bound provides a very good characterization of the average CRB. This figure also exhibits a difference between the slope of the MSE for narrowband and wideband signals. For narrowband signals ($T > 10^{-6}$) this slope is $d \log_{10} \text{MSE}/d \log_{10} \lambda = -1.5$ while for wideband signals ($T > 10^{-8}$) this slope is $d \log_{10} \text{MSE}/d \log_{10} \lambda = -1$. Fig. 4 also presents a unique threshold effect as a function of the sensor density. This threshold appears in densities between 0.01 and 0.03 sensors per square meter and in different bandwidths. Interestingly, this type of threshold is different than the commonly observed phenomenon, which occurs when increasing the signal to noise ratio or signal snapshot (which is completely equivalent to a higher signal to noise ratio) or with the number of sensors, which increases the number of observations. This case is different, because the number of sensors is infinite for all values of $\lambda$, but the statistics of the received signals vary with the density. Thus, at a higher density, there are typically more sensors close to the target, leading to a threshold $\lambda$ below which the ML estimator is characterized by large errors while above this threshold, it approaches the average CRB.

VI. CONCLUSION

We presented novel bounds on the localization MSE using stochastic geometry. Obtaining these bounds required the derivation of the CRB in the case of an infinitely countable number of
sensors, each measuring a continuous-time signal, for which we provided a rigorous derivation, which includes an expression for the likelihood function \(49\). The latter likelihood can be used in the future to derive other type of bounds (see e.g. \[49\]) or to analyze similar estimation problems. After we derived the CRB for a particular sensor locations, by taking the expectation with respect to a PPP distribution, we obtained the average CRB which bounds the expected MSE over all possible sensor locations. This bound depends solely on the network statistics and is not a function of a particular sensor deployment. In addition to the average CRB, we derived several bounds in closed-form. Both bounds provide a good characterization of the network performance, and can be used in the network planning stage to determine the performance for a given sensor density. The bounds exhibit different behavior for wideband and narrowband signals. Further research is required to better characterize these differences.

The comparison of the ML estimate’s MSE and the derived bounds demonstrates a new and unique threshold effect, where in the low sensor-density regime, the MSE is far above the average CRB whereas in the high sensor density regime, the MSE approaches the average CRB. This type of threshold is different from the known threshold effect with respect to signal energy and/or the number of sensors. In this scenario, the number of sensors is infinite for all sensor densities. However, the statistics of the received signals change with the density. Thus, at a higher density, there is greater chance of having sensors in close proximity to the target, which allows the ML error to converge to the average CRB. Further research is also required to characterize and quantify this new threshold effect.

Fig. 4. Source localization MSE of the ML estimator against the averaged CRB \(\text{CRB}(\theta)\) and the lower bound \(\text{CRB}_{LB}\) as a function of the sensor density, \(\gamma = 4\) and \(\rho = 50\)dB.
APPENDIX A
DERIVATION OF THE CRB

Before providing a rigorous proof for Theorem 1, we briefly discuss the main issues that must be addressed when deriving the CRB for an infinitely countable number of sensors. As stated in Section III, the CRB for the model (1) is already known for the case of a finite number of sensors [18]. However, extending this result to the infinite case is not straightforward and requires justification and proper definitions, even if the limit does exist of the CRB as the number of sensors goes to infinity. In other words, let $M < \infty$ be the number of sensors located at $\psi^M = \{(a_m, b_m)\}_{m=1}^M$ and $\text{CRB}(\theta, \psi^M)$ be the corresponding CRB. Next, let $\psi^M$ be an increasing sequence; i.e., $\psi^1 \subset \psi^2 \subset \psi^3 \cdots$. It is easy to show that $\lim_{M \to \infty} \text{CRB}(\theta, \psi^M)$ is indeed a lower bound for the infinite case, where the existence of the limit is guaranteed since $\text{CRB}(\theta, \psi^M)$ is non-negative and monotonically decreasing in $M$. However, it is not trivial to show that the limit as $M$ approaches infinity is indeed the CRB bound, $\text{CRB}(\theta, \psi)$, which is defined with respect to the likelihood of the measurements taken from the infinite number of sensors. In other words, if one could calculate the CRB directly, rather than first calculating $\text{CRB}(\theta, \psi^M)$, and then take the limit as $M \to \infty$, the results would not necessarily be equal and therefore, $\lim_{M \to \infty} \text{CRB}(\theta, \psi^M)$ is not necessarily the actual CRB. Obtaining the actual CRB bound is beneficial, because the CRB has been studied extensively, and possesses many desired characteristics; in particular, tightness in the high SNR regime, where it is achieved by the ML estimator.

In more concrete terms, let $p_{\theta, \psi^M}$ and $p_{\theta, \psi}$ be the likelihoods for estimating $\theta$ in the case of a finite and an infinite number of sensors, respectively. The CRB is the inverse of the Fisher information matrix (FIM), which is given respectively for the finite and the infinite case by

$$I_{\psi^M}(\theta) = \mathbb{E}\left\{\frac{\partial \log p_{\theta, \psi^M}}{\partial \theta} \frac{\partial \log p_{\theta, \psi^M}}{\partial \theta} \mid \psi^M \right\},$$

$$I_{\psi}(\theta) = \mathbb{E}\left\{\frac{\partial \log p_{\theta, \psi}}{\partial \theta} \frac{\partial \log p_{\theta, \psi}}{\partial \theta} \mid \psi \right\};$$

By definition, to derive the CRB for the infinite case, one has to calculate $I_{\psi}(\theta)$ directly according to (36); i.e., to define the infinite likelihood, $p_{\theta, \psi}$, and substitute it in (36). Another alternative would be to calculate (36) by taking the limit of (35) as $M \to \infty$; i.e.,

$$\lim_{M \to \infty} I_{\psi^M}(\theta) = I_{\psi}(\theta).$$
However, for $\text{(37)}$ to be true, the likelihood $p_{\theta,\psi,M}$ must converge to $p_{\theta,\psi}$ as $M \to \infty$, and similarly for the derivative of the log-likelihood with respect to $\theta$, as well as the product of the derivatives. If these conditions are satisfied, it is possible to calculate the CRB via $\text{(37)}$.

This proof has several parts; some of which are cumbersome. Hence, to make it tractable, we begin with a high level description of each part. In Part A we define the probability space on which the observations from a finite number of sensors are defined as random elements. We then define the likelihood in this case. In Part B, we extend the probability space to the case of an infinite number of sensors; i.e., we construct a probability space such that the observations from all the sensors constitute a sample point. This is necessary to obtain a well defined likelihood function, which is used to estimate $\theta$. In Part C we derive, in closed-form, the likelihood in the case of an infinite number of sensors $p_{\theta,\psi}$, which is then used in Part D (Lemma 7) to derive a formula for the FIM in the case of an infinite number of sensors with additive white Gaussian noise. It turns out, from Lemma 7, that the infinite FIM can be calculated via $\text{(37)}$. In Part E we substitute the model, $\text{(1)}$, into the formula for the FIM and calculate the CRB.

**Proof of Theorem 1**

**Part A. Definition of the probability space and the likelihood function in the case of a finite number of sensors:**

We start with some definitions and known results, which will later be used in the derivation. We begin with defining a suitable probability space in which the likelihoods are defined. Because white Gaussian noise does not exist as an ordinary mathematical stochastic process, we consider the integral of $\text{(1)}$ and replace the model of $\text{(1)}$ with the equivalent model (see e.g. [50], Sections VI.C.2 and VII.B)

$$R_m(t) = k_0 D_m^{-\gamma/2} \int_0^t s(u - \tau_m) \, du + W_m(t), \quad t \in \mathbb{R}_+,$$

where $W_m(t)$ is a Wiener process with covariance $\mathbb{E}(W_m(t)W_m(u)) = N_0/2 \min\{t, u\}$. Because the Wiener process is continuous a.s., for each $m$, $W_m(\cdot)$ can be seen as an element in $S$, the space of continuous functions on $\mathbb{R}$; i.e., $S = C(\mathbb{R})$. Furthermore, because $s(t - \tau)$ is integrable for every $\tau$, each $R_m(\cdot)$ is also an element in $S$. The space $S$ is a complete separable metric space, dubbed Polish space (see e.g. [51], Sec. 2.4), with respect to the distance $d(f_1, f_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{t \in [0,n]} |f_1(t) - f_2(t)|$. Next, consider the measurable space $(S, \mathcal{B}(S), \mu)$, where $\mathcal{B}(S)$ is the Borel $\sigma-$algebra and $\mu$ is the Wiener measure. Given a node location, $\psi_m$, each observation
The probability space and likelihood function in the case of an infinite number of sensors: The probability space has been defined, to define a likelihood, one needs a measure with respect to which \( P_{\theta, \psi_m} \) is absolutely continuous. Such a measure is \( \mu \) if \( \int_0^\infty s^2(t - \tau_m)dt < \infty \), and the likelihood function for estimating \( \theta \) from the observation \( R_m \), given \( \psi_m \) can be written as \([52]\)

\[
p_{\theta, \psi_m}(R_m(t)) = \frac{dP_{\theta, \psi_m}}{d\mu}(R_m(t)) = \exp \left\{ \frac{2}{N_0} \int_0^\infty s_m(t; \theta)dR_m(t) - \frac{1}{N_0} \int_0^\infty s_m^2(t; \theta)dt \right\} \tag{39}
\]

where \( dP_{\theta, \psi_m}/d\mu \) denotes the Radon-Nikodym derivative, and

\[
s_m(t; \theta) = k_0 D_m^{-\gamma/2} s(t - \tau_m). \tag{40}
\]

Furthermore, consider the observation \( R_M \triangleq \{R_m(t)\}_{m \in M} \in \mathbb{S}^{\left| M \right|} \) where \( M = \{\alpha_1, \alpha_2, \ldots, \alpha_M\} \subset \mathbb{N} \). Because \( W_m(t), W_n(t) \) are independent processes for every \( m \neq n \), \( R_M \) is naturally represented as a random element in \( (\mathbb{S}^{\left| M \right|}, \mathcal{B}(\mathbb{S}^{\left| M \right|}), P_{\theta, \psi_M}) \), where \( P_{\theta, \psi_M} \) is the product measure of \( \{P_{\theta, \psi_m}\}_{m \in M} \), with the product defined in the usual way (see e.g. \([53]\)). The likelihood of the \( R_M \) given \( \psi_M = \{\psi_m : m \in M\} \) is

\[
p_{\theta, \psi_M}(R_M) = \exp \left\{ \frac{2}{N_0} \sum_{m \in M} \int_0^\infty s_m(t; \theta)dR_m(t) - \frac{1}{N_0} \sum_{m \in M} \int_0^\infty s_m^2(t; \theta)dt \right\} \tag{41}
\]

\[
- \prod_{m \in M} p_{\theta, \psi_m}(R_m(t)).
\]

**Part B.** The probability space and likelihood function in the case of an infinite number of sensors:

Let \( R \triangleq \{R_m(t)\}_{m=1}^\infty \); to derive the likelihood for an infinite number of sensors we need a probability space \((\Omega_\psi, \mathcal{F}_\psi, P_{\theta, \psi})\) on which \( R \) is a sample point. This can be accomplished by using Kolmogorov’s consistency theorem (see e.g., \([54]\), Theorem 6.3.1 and Remark 6.3.4), but first we need the following definition.

**Definition 1:** Let \( M = \{\alpha_1, \alpha_2, \ldots, \alpha_M\} \subset \mathbb{N} \), \( M < \infty \), with \( \alpha_i \neq \alpha_j \), \( \forall i \neq j \). A finite dimensional cylinder is a set \( C_M \subset \mathbb{S}^\infty \) where there exists \( B \in \mathcal{B}(\mathbb{S}^M) \) such that every \((x_1, x_2, x_3, \ldots) \in C_M\) satisfies \( x_M \triangleq (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_M}) \in B \). The set \( B \) is called the basis of the cylinder \( C_M \).

Next, from Kolmogorov’s consistency theorem it follows that for every \( \psi \) there exists a probability space \((\Omega_\psi, \mathcal{F}_\psi, P_{\theta, \psi})\), where \( \Omega_\psi = \mathbb{S}^\infty \), \( \mathcal{F}_\psi \) is the sigma algebra generated by the

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\(^3\)In our case \( P_{\theta, \psi_m}(R_m(t) \in B) = \mu(B - s(t - \tau_m)) \) where \( B - s(t - \tau_m) = \{x(t) \in \mathbb{S} : x(t) - s(t - \tau_m) \in B\} \).
set of finite dimensional cylindrical sets and for every finite dimensional cylinder \( C \in \mathcal{F}_\psi \) with basis \( B \) the following is satisfied

\[
P_{\theta,\psi}(C_M) = P_{\theta,\psi,M}(B).
\] (42)

Similarly, there exists a measure \( P_0 \) on \((\Omega_\psi, \mathcal{F}_\psi)\) such that

\[
P_0(R_{\psi,M} \in B) = \mu^{|M|}(B).
\] (43)

Because \( P_{\theta,\psi,M} \ll \mu^{|M|} \) for every \( M \subset \mathbb{N} \) with \( |M| < \infty \), and since both \( P_0 \) and \( P_{\theta,\psi} \) are defined using the Kolmogorov consistency theorem, it follows that \( P_{\theta,\psi} \ll P_0 \).

Now that we have a well defined probability space \((\Omega_\psi, \mathcal{F}_\psi, P_{\theta,\psi})\), and a measure \( P_0 \) which dominates \( P_{\theta,\psi} \), the likelihood of \( \mathcal{R} \)

\[
p_{\theta,\psi}(\mathcal{R}) = \frac{dP_{\theta,\psi}}{dP_0}(\mathcal{R}),
\] (44)

is well defined and the CRB is given by \( CRB(\psi) = I^{-1}(\theta, \psi) \), where,

\[
I(\theta, \psi) = \mathbb{E} \left\{ \frac{\partial \log p_{\theta,\psi}(\mathcal{R})}{\partial \theta} \frac{\partial \log p_{\theta,\psi}(\mathcal{R})}{\partial \theta} \right\}.
\] (45)

**Part C. Derivation of the likelihood \( p_{\theta,\psi} \) in closed-form:**

We now derive the likelihood \( p_{\theta,\psi}(\mathcal{R}) \) in closed-form by showing that the marginal likelihood \( p_{\theta,\psi,M} \) converges w.p. 1 to \( p_{\theta,\psi} \). If the observations \( \{R_m(t)\}_{m=1}^{\infty} \) were random variables (i.e., for each \( m \), \( R_m \) was a Borel measurable function from \( \Omega \) to \( \mathbb{R} \)), the convergence of the marginal likelihood to the infinite one would follow immediately from Grenander’s theorem (See [55], Chapter 3, Corollary 1). In our problem, however, \( R_m(\cdot) \) is a random element on \((\mathbb{S}, \mathcal{B}(\mathbb{S}), \mu)\); thus, it is necessary to extend Grenander’s theorem to the case at hand.

**Theorem 6:** Let \( \{\mathcal{M}_\nu\}_{\nu=1}^{\infty} \subset \mathbb{N} \) be an increasing sequence of sets (that is \( \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \)), and for each \( \nu \), define \( p_{\theta,\psi,M_\nu} \) as in (41). Consider \( p_{\theta,\psi} \) defined in (44), then,

\[
p_{\theta,\psi}(\mathcal{R}) = \lim_{\nu \to \infty} p_{\theta,\psi,M_\nu}(\mathcal{R}_{M_\nu}), \text{ w.p. } 1.
\] (46)

**Proof:** For each \( \mathcal{M}_\nu \), let \( C_\nu \subset \mathbb{S}^\infty \) be a finite dimensional cylinder with basis \( (\mathcal{M}_\nu, B_\nu) \), where \( B_\nu \in \mathcal{B}(\mathbb{S}^{|\mathcal{M}_\nu|}) \). Then, by the Radon-Nykodym theorem

\[
P_{\theta,\psi}(C_\nu) = \int_{C_\nu} p_{\theta,\psi}(\mathbf{x})dP_0(\mathbf{x}) = \int_{B_\nu} p_{\theta,\psi,M_\nu}(\mathbf{x}_{M_\nu})d\mu^{|\mathcal{M}_\nu|}(\mathbf{x}_{M_\nu}).
\] (47)

\(^4\text{This is because every } C \in \mathcal{B}(\mathbb{S}^\infty) \text{ with } P_0(C) = 0 \text{ can be written as } C = \bigcup_{n=1}^{\infty} C_n \text{ where } C_n \text{ are cylindrical sets with bases } B_n \in \mathcal{B}(\mathbb{S}^{k_n}), k_n < \infty, \forall n. \text{ Now, by subadditivity, } P_0(C_n) = 0, \text{ thus } \mu^{k_n}(B_n) = 0 \Rightarrow P_{\theta,\psi,A_n}(B_n) = 0 \text{ for every } A_n \subset \mathbb{N}, |A_n| = k_n, \text{ and therefore, by subadditivity } P_{\theta,\psi}(C) = 0.\n\)
Since this holds for every $B_\nu \in \mathcal{B}(\mathcal{S}^{\lfloor M_\nu \rfloor})$, it follows that

$$p_{\theta, \psi M_\nu}(x) = \mathbb{E}_{P_0} \{ p_{\theta, \psi}(x) | \mathcal{F}_{M_\nu} \},$$

(48)

where $\mathcal{F}_{M_\nu} = \sigma(C_N)$, where $C_N$ is the collection of all finite dimensional cylindrical sets, with basis $(\mathcal{N}, B)$, such that $\mathcal{N} \subset \mathbb{N}$ and $B \in \mathcal{B}(\mathcal{S}^{\lfloor N \rfloor})$. Note that $\{ \mathcal{F}_{M_i} \}_{i=1}^\infty$ is a filtration since $M_1 \subseteq M_2 \subseteq \cdots$. Thus, $p_{\theta, \psi M}$ is an abstract Doob Martingale (see [56], Theorem 4) for every $M_1 \subseteq M_2 \subseteq \cdots$, and therefore converges to $p_{\theta, \psi}$ a.s.-$P_0$, which establishes the desired result (recall that $P_0$ dominates $P_\theta, \psi$).

To summarize, Theorem 6 provides a close-form expression for the likelihood

$$p_{\theta, \psi}(R) = e^{-\sum_{m=1}^\infty \left( 2 \int_0^\infty s_m(t; \theta) dR_m(t) - \int_0^\infty s_m^2(t; \theta) dt \right)},$$

(49)

**Part D. Formula for the Fisher Information matrix in the case of an infinite number of sensors:**

Now that we have the likelihood in closed-form, we can derive the FIM, and show that (37) is satisfied. To this end we need the following lemma, which extends the well known formula for the FIM in the case where the observation is a finite set of continuous signals, each with additive Gaussian white noise, to the case of infinitely countable continuous signals.

**Lemma 7:** If $s(t)$ is bounded, differentiable, with a bounded derivative, then for every $\psi$ which satisfies (4), the $i,j$ entry of the $2 \times 2$ FIM is given by

$$[I(\theta)]_{i,j} = \frac{2}{N_0} \sum_{m=1}^\infty \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_i} \frac{\partial s_m(t; \theta)}{\partial \theta_j} dt$$

(50)

**Proof:** To prove the lemma, one must prove that (45) reduces to (50). In order to evaluate (45), we need the following proposition:

**Proposition 8:** Consider the likelihood $p_{\theta, \psi}(R)$ in (49), then $\log p_{\theta, \psi}(R) \in L_2(\Omega_\psi, \mathcal{F}_\psi, P_\psi)$ and

$$\frac{\partial \log p_{\theta, \psi}}{\partial \theta_j} = \sum_{m=1}^\infty \frac{\partial \log p_{\theta, \psi_m}(R_m(t))}{\partial \theta_j} \in L_2(\Omega_\psi, \mathcal{F}_\psi, P_\psi)$$

(51)

where the equality is in the mean-square sense; i.e., with respect to the norm of $L_2(\Omega_\psi, \mathcal{F}_\psi, P_\psi)$.

**Proof:** From (49),

$$\log p_{\theta, \psi}(R) = \sum_{m=1}^\infty \log(p_{\theta, \psi_m}(R_m(t))),$$

(52)

and from (39)

$$\log(p_{\theta, \psi_m}(R_m(t))) = \frac{2}{N_0} \int_0^\infty k_0 D_m^{-\gamma/2} s(t - \tau_m(\theta)) dR_m(t) - \frac{1}{N_0} \int_0^\infty k_0 D_m^{-\gamma} s^2(t - \tau_m(\theta)) dt$$

(53)
Because $s(t)$ is bounded and square integrable, it follows that
\[ Z_m \triangleq \int_0^\infty s(t - \tau_m(\theta)) dR_m(t) \in L_2, \tag{54} \]
and
\[ \xi_m \triangleq \int_0^\infty s^2(t - \tau_m(\theta)) < \infty. \tag{55} \]
Thus, $\log(p_{\theta,\psi_m}(R_m)) \in L_2$. Furthermore, because $\int_0^\infty s^2(t) dt < \infty$, there exists a constant, $u$, (which is not a function of $\theta$ and $m$) such that $E\{Z_m^2\} \leq u$, and since $\sum_{m=1}^\infty D_m^{-\gamma} < \infty$, it follows that
\[ \log p_{\theta,\psi}(R_m) = \sum_{m=1}^\infty \frac{1}{N_0}(2k_0 D_m^{-\gamma/2} Z_m + D_m^{-\gamma} \xi_m) \in L_2 \tag{56} \]
This establishes the first part of the proposition. For the second part, we note $s'(t) \triangleq ds(t)/dt$ exists bounded, square integrable with a bounded derivative, and hence also $\partial s_m(t; \theta)/\partial \theta_j$. Thus, the order of the derivative and the integration in (53) can be exchanged:\footnote{This is possible due to the dominant convergence theorem for stochastic integrals (see e.g., \cite{57} Theorem 1.1.19), and the regularity conditions, \cite{2}. It is then possible to justify it similar to \cite{53}, theorem 2.27.}
\[ \frac{\partial \log p_{\theta,\psi_m}(R_m)}{\partial \theta_j} = \frac{1}{N_0} \left( 2 \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_j} dR_m(t) - \int_0^\infty \frac{\partial s_m^2(t; \theta)}{\partial \theta_j} dt \right), \tag{57} \]
which can be written as (see e.g., \cite{50} sec VII.B)
\[ \frac{\partial \log p_{\theta,\psi_m}(R_m)}{\partial \theta_j} = \frac{2}{N_0} \left( \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_j} dW_m(t) \right). \tag{58} \]
Note that $|\partial D_m/\partial \theta_j| \leq 1$; thus, $|\partial D_m^{-\gamma/2}/\partial \theta_j| = |D_m^{-\gamma/2-1} \partial D_m/\partial \theta_j| \leq D_m^{-\gamma/2-1}$. Denote $\partial D_m/\partial \theta_j = D_m^{',j}$, then
\[ \frac{\partial \log p_{\theta,\psi_m}(R_m)}{\partial \theta_j} = \frac{2}{N_0} \left( \int_0^\infty \left[ D_m^{-\gamma/2-1} \frac{\partial D_m}{\partial \theta_j} s(t - \tau_m(\theta)) + \frac{D_m^{-\gamma/2}}{c} \frac{\partial D_m}{\partial \theta_j} s'(t - \tau_m(\theta)) \right] dW_m(t) \right) \]
\[ = (O(D_m^{-\gamma/2})(S_m(\theta) + S'_m(\theta))), \tag{59} \]
where
\[ S_m(\theta) \triangleq \left( \int_0^\infty s(t - \tau(\theta)) dW_m(t) \right) \in L_2 \tag{60} \]
\[ S'_m(\theta) \triangleq \left( \int_0^\infty s'(t - \tau(\theta)) dW_m(t) \right) \in L_2. \tag{61} \]
Hence
\[ \frac{\partial \log p_{\theta,\psi_m}(R_m)}{\partial \theta_j} \in L_2 \tag{62} \]
Recalling that $\sum_{m=1}^{\infty} D_{m}^{-\gamma} < \infty$ and using (52), it follows that

$$\sum_{m=1}^{\infty} \partial \log p_{\theta, \psi_{m}}(R_{m}(t)) \in L_{2}(\Omega, F, P_{\psi})$$

(63)

It remains to show that $p_{\theta, \psi}(\mathbb{R})$ is differentiable and that its derivative is given by (63). Using (56) and $\sum_{m=1}^{\infty} D_{m}^{-\gamma} < \infty$, (52) can be written as

$$\log p_{\theta, \psi}(\mathbb{R}) = \sum_{m=1}^{M} \log(p_{\theta, \psi_{m}}(R_{m})) + \tilde{U}_{M}(\theta),$$

(64)

where

$$\tilde{U}_{M} = \sum_{m=M+1}^{\infty} \frac{k_{0}D_{m}^{-\gamma/2}}{N_{0}} (2Z_{m} + \xi_{m}) \in L_{2}. \quad (65)$$

Thus,

$$\frac{\Delta_{j}(\log p_{\theta, \psi}(\mathbb{R}))}{\Delta \theta_{j}} = \sum_{m=1}^{M} \frac{\Delta_{j}(\log(p_{\theta, \psi_{m}}(R_{m}))}{\Delta \theta_{j}} + \frac{\Delta_{j}(\tilde{U}_{M}(\theta))}{\Delta \theta_{j}} \quad (66)$$

where $\Delta_{j}$ is an operator defined on functions from $\mathbb{R}^{2}$ to $\mathbb{R}$, such that for a given $g : \mathbb{R}^{2} \to \mathbb{R}$, $\Delta_{1}(g(\theta_{1}, \theta_{2})) = g(\theta_{1} + \Delta \theta_{1}, \theta_{2})$ and $\Delta_{2}(g(\theta_{1}, \theta_{2})) = g(\theta_{1}, \theta_{2} + \Delta \theta_{2})$. We now show that $\Delta_{j}(\tilde{U}_{M}(\theta))/\Delta \theta_{j}$ converges to zero as $M \to \infty$ and $\Delta \theta_{j} \to 0$. From (54), (55) and $\sum_{m=1}^{\infty} D_{m}^{-\gamma} < \infty$, it follows that for every $\epsilon > 0$ there exists $M_{0}(\epsilon)$, such that $\forall M \geq M_{0}(\epsilon), E\{\tilde{U}_{M}^{2}(\theta)\} < \epsilon$ for every $\theta$. Thus, by letting $\epsilon = o(\Delta \theta_{j})^{2}$ we have

$$\frac{\Delta_{j}(\log p_{\theta, \psi}(\mathbb{R}))}{\Delta \theta_{j}} = \sum_{m=1}^{M_{0}} \frac{\Delta_{j}(\log(p_{\theta, \psi_{m}}(R_{m}))}{\Delta \theta_{j}} + o_{ms, \Delta \theta_{j}}(1), \quad (67)$$

where $o_{ms, \Delta \theta_{j}}(1)$ denotes a random function $Q(\Delta \theta_{j})$ which converges to zero in the mean square sense as $\Delta \theta_{j} \to 0$. Denote

$$W_{M_{1}, \Delta \theta_{j}} = \sum_{m=1}^{M_{1}} \frac{\Delta_{j}(\log(p_{\theta, \psi_{m}}(R_{m}))}{\Delta \theta_{j}} - \sum_{m=1}^{M_{1}} \frac{\partial \log(p_{\theta, \psi_{m}}(R_{m}))}{\partial \theta_{j}}. \quad (68)$$

Then, substituting into (67), we obtain

$$\frac{\Delta_{j}(\log p_{\theta, \psi}(\mathbb{R}))}{\Delta \theta_{j}} = \sum_{m=1}^{M_{1}} \frac{\partial \log(p_{\theta, \psi_{m}}(R_{m}))}{\partial \theta_{j}} + W_{M_{1}, \Delta \theta_{j}} + o_{ms, \Delta \theta_{j}}(1), \quad (69)$$

Next, by the mean value theorem, for every $\Delta \theta_{j}$ there exists $\theta_{j,m}$ between $\theta_{j}$ and $\theta_{j} + \Delta \theta_{j}$ such that

$$V_{M_{1}, \Delta \theta_{j}} = \sum_{m=M_{1}}^{M_{0}} \frac{\Delta_{j}(\log(p_{\theta, \psi_{m}}(R_{m}))}{\Delta \theta_{j}} - \sum_{m=M_{1}}^{M_{0}} \frac{\partial \log(p_{\theta, \psi_{m}}(R_{m}))}{\partial \theta_{j}} \bigg|_{\theta_{j} = \theta_{j,m}} \quad (70)$$

Thus,

$$\frac{\Delta_{j}(\log p_{\theta, \psi}(\mathbb{R}))}{\Delta \theta_{j}} = \sum_{m=1}^{M_{1}} \frac{\partial \log(p_{\theta, \psi_{m}}(R_{m}))}{\partial \theta_{j}} + W_{M_{1}, \Delta \theta_{j}} + V_{M_{1}, \Delta \theta_{j}} + o_{ms, \Delta \theta_{j}}(1), \quad (71)$$
Thus, \( m \), where the last inequality follows from the fact that for sufficiently large
into (45), it follows that
which establishes (51), thus establishing Proposition 8.

Exaining the last equality, we observe that \( \lim_{M_1 \to \infty} \lim_{\Delta \theta_j \to \infty} W_{M_1, \Delta \theta_j} = 0 \) in the m.s. As
for \( V_{M_1, \Delta \theta_j} \), using (59), it can be written as
\[
V_{M_1, \Delta \theta_j} = \sum_{m=M_1}^{M_0} \left( (S_m(\theta) + S_m'(\theta))|_{\theta_j=\theta_j,m} \right) O((D_m(\theta))^{-\gamma/2})|_{\theta_j=\theta_j,m}. \tag{73}
\]
Because \( S_m(\theta) + S_m'(\theta) \in L_2 \) for every \( \theta \), in order to show that \( \lim_{M_1 \to \infty} \lim_{\Delta \theta_j \to \infty} V_{M_1, \Delta \theta_j} = 0 \)
in the m.s., it is sufficient to show that \( \sum_{m=1}^{M_0} (D_m(\theta))^{-\gamma} |_{\theta_j=\theta_j,m} < \infty \) uniformly over \( \Delta \theta_j \).
Recalling that \( \theta = [x, y] \), and assuming without loss of generality that \( \theta_j = x \), \( \Delta \theta_j = \Delta x \), we obtain
\[
D_m^{-\gamma/2}(\theta)|_{\theta_j=\theta_j,m} = D_m^{-\gamma/2}(x + \Delta x_m, y) = \left( \frac{1}{\sqrt{(x+\Delta x_m-a_m)^2+(y-b_m)^2}} \right)^{\gamma/2}, \tag{74}
\]
where \( |\Delta x_m| \leq |\Delta x| \). After some algebraic manipulation and using the Bernoulli inequality\(^6\)
\[
D_m^{-1}(x + \Delta x_m, y) \leq \frac{|\Delta x_m + 2a_m \Delta x_m|}{2(a_m^2+b_m^2)^{3/2}} + \frac{1}{\sqrt{a_m^2+b_m^2}} \leq \frac{5/2}{\sqrt{a_m^2+b_m^2}}, \tag{75}
\]
where the last inequality follows from the fact that for sufficiently large \( m \), \( \sqrt{a_m^2+b_m^2} > |\Delta x_m| \).
Since (75) implies that \( D_m^{-1}(x + \Delta x_m, y) \leq 5D_m^{-1}(x, y)/2 \), uniformly on \( \Delta \theta_j \), and because
\( \sum_{m=1}^{\infty} (D_m(\theta))^{-\gamma} \leq \sum_{m=1}^{M_0} (D_m(\theta))^{-\gamma} |_{\theta_j=\theta_j,m} \leq \sum_{m=1}^{\infty} (D_m(\theta))^{-\gamma} |_{\theta_j=\theta_j,m} \leq \infty \) uniformly on \( \Delta \theta_j \). Thus, \( \lim_{M_1 \to \infty} \lim_{\Delta \theta_j \to \infty} V_{M_1, \Delta \theta_j} = 0 \) in the m.s. It follows that
\[
\lim_{\Delta \theta_j \to 0} \frac{\Delta_j(\log p_{\theta, \psi}(R))}{\Delta \theta_j} = \sum_{m=1}^{\infty} \frac{\partial \log (p_{\theta, \psi}(R_m))}{\partial \theta_j}, \text{ m.s.} \tag{76}
\]
which establishes (51), thus establishing Proposition 8.

Back to the proof of Lemma 7 substituting (58), into (51) and then substituting the result into (45), it follows that
\[
[I(\theta)]_{ij} = E \left\{ \sum_{m,n=1}^{\infty} \frac{2}{N_0} \int_0^{t} \frac{\partial s_m(t; \theta)}{\partial \theta_i} dW_m(t) \frac{2}{N_0} \int_0^{t} \frac{\partial s_n(t; \theta)}{\partial \theta_j} dW_n(t) \right\} \tag{77}
\]
where the infinite sum and the expectation are interchangeable (see e.g. [33], Proposition 5.21).
Thus,
\[
[I(\theta)]_{ij} = \sum_{m,n=1}^{\infty} \delta_{m,n} \times [i_m(\theta)]_{ij}, \tag{78}
\]
(1 + x)^r \leq 1 + rx \) for \( x > -1 \) and \( r \in [0, 1] \).

\(^6\)
where $\delta_{m,n}$ is the Kronecker delta function, and

$$[\hat{i}(\theta)]_{i,j} = E \left\{ \frac{2}{N_0} \left( \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_j} dW_m(t) \right) \frac{2}{N_0} \left( \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_i} dW_m(t) \right) \right\}$$

(79)

$$= \frac{2}{N_0} \int_0^\infty \frac{\partial s_m(t; \theta)}{\partial \theta_j} \frac{\partial s_m(t; \theta)}{\partial \theta_i} dt,$$

(80)

where the second equality follows from the properties of the stochastic integral (see e.g., [50], Proposition VI.D.1). This establishes (50), thus establishing Lemma 7.

**Part E. Derivation of the CRB for the model in (1)**

Now, using Proposition 7, it is possible to derive $I(\theta)$. We begin with $\partial s_m(t; \theta)/\partial \theta_i$ for $i = 1$ (recall that $\theta_1 = x$)

$$\frac{\partial s_m(t)}{\partial x} = \frac{\partial}{\partial x} \left( k_0 D_m^{-\frac{3}{2}} s \left( t - \frac{D_m}{c} \right) \right) = -\frac{k_0}{2c} D_m^{-\frac{3}{2} - 2} (x - \alpha_m) \left( 2 D_m s' \left( t - \frac{D_m}{c} \right) + c \gamma s \left( t - \frac{D_m}{c} \right) \right)$$

(81)

where we used $\partial D_m/\partial x = (x - \alpha_m)/D_m$. The expression for $\partial s_m(t)/\partial y$ can be obtained by substituting $y$ and $\beta_m$ in $\partial s_m(t)/\partial x$, for $x$ and $\alpha_m$, respectively. Substituting (81) into (80), one obtains

$$[\hat{i}_m(\theta)]_{1,1} = \frac{E_0}{N_0} D_m^{-2} (x - \alpha_m)^2 g(D_m),$$

(82)

where we used the assumption $\int_0^\infty s(t) \frac{d W(t)}{dt} dt = 0$. The rest of the entries of $\hat{i}_m(\theta)$ can be derived similarly to obtain

$$\hat{i}_m(\theta) = \rho g(D_m) \begin{bmatrix} \cos^2 \phi_m & \sin \phi_m \cos \phi_m \\ \sin \phi_m \cos \phi_m & \sin^2 \phi_m \end{bmatrix},$$

(83)

where $\cos(\phi_m) = (x - \alpha_m)/D_m$ and $\sin(\phi_m) = (y - \beta_m)/D_m$. Thus, the FIM is given by

$$I(\theta)$$

(84)

$$= \rho \sum_{i=1}^M \left[ \begin{array}{cc} \cos^2 (\phi_i) g(D_i) & \sin (\phi_i) \cos (\phi_i) g(D_i) \\ \sin (\phi_i) \cos (\phi_i) g(D_i) & \sin^2 (\phi_i) g(D_i) \end{array} \right]$$

The error $E\{\|\hat{\Theta} - \Theta\|^2\} = E\{(\hat{x} - x)^2 + (\hat{y} - y)^2\}$ is bounded by

$$E\{\|\hat{\Theta} - \Theta\|^2\} \geq \text{Tr}(I^{-1}(\Theta)) = \frac{[I(\theta)]_{11} + [I(\theta)]_{22}}{\det(I(\theta))}$$

(85)

To obtain the bound, we first derive $\det(I(\theta))$,

$$\det(I(\theta)) = \sum_{m=1}^{M} \sum_{n=m+1}^{M} \left[ \hat{i}_m(\theta) \right]_{1,1} \left[ \hat{i}_j(\theta) \right]_{2,2} - \left[ \hat{i}_m(\theta) \right]_{2,1} \left[ \hat{i}_j(\theta) \right]_{1,2}$$

$$= \rho^2 \sum_{m=1}^{M} \sum_{n=m+1}^{M} g(D_m) g(D_n) \sin^2 (\phi_m - \phi_n)$$

(86)
Substituting the latter into (85) establishes the desired result.

APPENDIX B
PROOF OF THEOREM 2

We begin with some notation and definitions. Recall that sensor-locations \( \psi = \{(a_m, b_m)\}_{m=1}^{\infty} \) is a realization of a homogeneous PPP \( \Psi \) defined on \((\Omega, \mathcal{F}, P)\); i.e., \( \psi = \Psi(\omega) \). Denote by \( c_A(\psi) \), the number of points in the intersection of \( \psi \) with \( A \subseteq \mathcal{B}(\mathbb{R}^2) \); i.e., \( c_A(\psi) = \sum_{m=1}^{\infty} \chi(\psi_m \in A) \), where \( \chi_A(\cdot) \) is the indicator function. Let \( C_A : \Omega \rightarrow \mathbb{R} \) be the random variable \( C_A(\omega) = c_A(\Psi(\omega)) \).

Now to the proof. If (4) is satisfied, the measurability of \( CRB(\theta, \Psi) \) follows immediately from the fact that its numerator and denominator, being a limit of measurable functions, are measurable functions of \( \psi \). It remains to show that (4) is satisfied with probability 1. Denote \( B_m = \{z \in \mathbb{R}^2 : m \leq \|z - \theta\| < m + 1 \} \) for every \( m = 0, 1, 2, ... \) and \( M_0 = C_{B_0} \). Then, for every \( M \in \mathbb{N} \)

\[
\sum_{m=1}^{M} D_m^{-\gamma} \leq \sum_{m=1}^{M_0} D_m^{-\gamma} + \sum_{m=1}^{M} m^{-\gamma} C_{B_m};
\]

(87)

where \( \sum_{m=M_1}^{M_2} z_m = 0 \) if \( M_1 > M_2 \). Thus, to show that \( \sum_{m=1}^{\infty} D_m < \infty \) it is sufficient to show that each of the series \( Q_{M_0} \) and \( K_N \) converges \( P \)-a.s. to some random variable\(^7\) Because in homogenous PPP \( C_{B_m} < \infty \) \( P \)-a.s. for every \( A \subseteq \mathbb{R}^2 \) with a finite Lebesgue measure, it follows that \( M_0 < \infty \) \( P \)-a.s. Thus, \( Q_{M_0} \) is a finite sum of random variables \( D_1^{-\gamma}, ..., D_{M_0}^{-\gamma} \), where each \( D_m, m = 1, ..., M_0 \) is a continuous random variable \( D_m : \Omega \rightarrow [0, 1) \). Because \( P(D_m = 0) = 0 \), it is possible to define a continuous random variable \( Z_m : \Omega \rightarrow \mathbb{R}_+ \) as \( Z_m = D_m^{-\gamma} \) if \( D_m \neq 0 \).

\(^7\) The bound \( CRB(\theta, \Psi) \) is a well defined random variable if it is \((\Omega, \mathcal{F}) - (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) measurable. Because the process \( \Psi \) is a function \( \Psi : \Omega \rightarrow \Lambda \) which is \((\Omega, \mathcal{F}) - (\Lambda, \mathcal{L}) \) measurable, where \( \Lambda \) is the set of all locally finite sets in \( \mathbb{R}^2 \); i.e., \( \psi \in \Lambda \), if for every \( B \subseteq \mathbb{R}(\mathbb{R}^2) \) with a finite Lebesgue-measure \( c_B(\psi) < \infty \) and \( \mathcal{L} \) is the minimal sigma-algebra of sets in \( \Lambda \) such that \( \psi \mapsto c_B(\psi) \) is measurable for every \( B \subseteq \mathbb{R}(\mathbb{R}^2) \) with finite Lebesgue measure. Hence, to show that \( CRB(\theta, \Psi) \) is a well defined random variable, it is sufficient to show that it is \((\Lambda, \mathcal{L}) - (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) measurable.

\(^8\) We use the standard definition for a random variable; i.e., a Borel-measurable function \( X : \Omega \rightarrow \mathbb{R} \), where \( \mathbb{R} \) does not include \( \pm \infty \); therefore, \( |X(\omega)| < \infty \) for every \( \omega \). Under this definition, if the limit diverges; i.e., \( \sum_{m=1}^{\infty} D_m = \infty \), is not a random variable.
and \( Z_m = 0 \) for \( D_m = 0 \). While \( \mathbb{E}\{Z_m\} \) does not exist\(^9\), \( Z_m \) is finite w.p. 1 and because \( M_0 \) is finite w.p. 1, it follows that \( Q_{M_0} \) is a well defined random variable, and therefore finite.

Next we show that \( K_M \) converges \( P- \) a.s. To this end, we use the Khinchine Kolmogorov 1-series theorem (see e.g. [54] Theorem 8.3.4)

**Theorem 9 (Khinchine-Kolmogorov’s 1-series theorem):** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of independent random variables on a probability space \( (\Omega, \mathcal{F}, P) \) such that \( \mathbb{E}\{X_n\} = 0 \) and \( \sum_{n=1}^{\infty} \mathbb{E}\{X_n^2\} < \infty \). Then, \( I_n \triangleq \sum_{j=1}^{n} X_j \) converges in the mean square and \( P- \) a.s. as \( n \to \infty \).

Note that \( K_M \) is a weighted sum of Poisson random variables \( C_{B_m} \) with mean and variance equal to \( v_L(B_m) \lambda \), where \( v_L \) is the Lebesgue measure on \( \mathbb{R}^2 \). Therefore, \( \mathbb{E}\{C_{B_m}\} = \text{Var}(C_{B_m}) = \lambda \pi (2m + 1) \) and because

\[
\sum_{m=1}^{\infty} \mathbb{E}\{m^{-\gamma}C_{B_m} - \mathbb{E}\{m^{-\gamma}C_{B_m}\}\}^2 = \sum_{m=1}^{\infty} \text{Var}(m^{-\gamma}C_{B_m}) = \sum_{m=1}^{\infty} m^{-2\gamma} \lambda \pi (2m + 1) < \infty, \tag{88}
\]

it follows that

\[
K'_M = \sum_{m=1}^{M} \left( m^{-\gamma}C_{B_m} - m^{-\gamma}\mathbb{E}\{C_{B_m}\} \right)
\]

converges \( P- \) a.s. Furthermore, note that \( \lim_{M \to \infty} \mathbb{E}\{K_M\} = \sum_{m=1}^{\infty} m^{-\gamma} \mathbb{E}\{C_{B_m}\} = \pi \lambda \sum_{m=1}^{\infty} m^{-\gamma}(2m+1) < \infty \) where we replaced the order of the expectation with the infinite sum since \( C_{B_m} \) is non-negative for every \( m \) (see e.g., [53] Theorem 2.15). Hence \( \lim_{M \to \infty} K'_M + \lim_{M \to \infty} \mathbb{E}\{K_M\} = \lim_{M \to \infty} (K'_M + \mathbb{E}\{K_M\}) = \lim_{M \to \infty} K_M' \); i.e., \( K_M \) converges w.p. 1 to a random variable, and therefore, \( \lim_{M \to \infty} K_M < \infty \) \( P- \) a.s.

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\(^9\)The integral diverges to infinity for every \( m \); this is denoted by \( \mathbb{E}\{Z_m\} = \infty \). If one considers the extended real line, \( \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \) the expectation is well defined. In this paper we do not use \( \bar{\mathbb{R}} \).
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