Abstract. We study the simple Bershadsky–Polyakov algebra \( W_k = W_k(sl_3, f_\theta) \) at positive integer levels and classify their irreducible modules. In this way, we confirm the conjecture from [9]. Next, we study the case \( k = 1 \). We discover that this vertex algebra has a Kazama–Suzuki-type dual isomorphic to the simple affine vertex superalgebra \( L_{k'}(osp(1|2)) \) for \( k' = -5/4 \). Using the free-field realization of \( L_{k'}(osp(1|2)) \) from [3], we get a free-field realization of \( W_k \) and their highest weight modules. In a sequel, we plan to study fusion rules for \( W_k \).

1. Introduction

In recent years, minimal affine \( W \) algebras have attracted a lot of interest. They are obtained using quantum hamiltonian reduction from affine vertex algebras, and they can be described using generators and relations (cf. [29], [31]).

The Bershadsky–Polyakov algebra \( W_k = W_k(sl_3, f_\theta) \) ([15], [39]) is the simplest minimal affine \( W \)-algebra. T. Arakawa proved in [12] that \( W_k \) is rational for \( k + 3/2 \in \mathbb{Z}_{\geq 0} \), while in other cases it is a nonrational vertex algebra. More recently, for \( k \) admissible and nonintegral, irreducible \( W_k \)-modules were classified in [9] in some special cases, and in [21] in full generality. A realization of \( W_k \), when \( 2k + 3 \notin \mathbb{Z}_{\geq 0} \), and its relaxed modules are presented in [8], which gives a natural generalization of the realization of the affine vertex algebra \( V_k(sl(2)) \) from [3]. Let us now mention certain problems for Bershadsky–Polyakov vertex algebras, which remain unsolved in the papers listed above.

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A. Classification of ordinary irreducible $\mathcal{W}_k$-modules for integer levels $k, k + 2 \in \mathbb{Z}_{\geq 0}$

In [21], authors classified irreducible highest weight $\mathcal{W}_k$-modules for $k$ admissible, nonintegral. They showed that every irreducible highest weight module for $\mathcal{W}_k$ is obtained as an image of the admissible modules for $L_k(sl(3))$ (which are classified by T. Arakawa in [13]). However, when we pass to integral $k$, the methods of [21] are no longer applicable, since in this case quantum hamiltonian reduction sends $L_k(sl(3))$-modules to zero.

In [9], we began the study of the representation theory of $\mathcal{W}_k$ for integer levels $k, k + 2 \in \mathbb{Z}_{\geq 0}$. The starting point was explicit formulas for singular vectors in $\mathcal{W}_k$ (which generalized those of Arakawa in [12]). Using these singular vectors, one concludes that on any weak $\mathcal{W}_k$-module $M$, we should have the relation

$$G^\pm(z)^{k+2} = 0.$$  

This implies that if $M$ is $\mathbb{Z}_{\geq 0}$-graded and irreducible, then the top level $M_{\text{top}}$ is at most a $(k + 2)$-dimensional irreducible module for the Zhu’s algebra $A(\mathcal{W}_k)$, and therefore $M$ is an ordinary module. Therefore, the classification of irreducible ordinary $\mathcal{W}_k$-modules also solves the problem of classification of all irreducible $\mathbb{Z}_{\geq 0}$-graded $\mathcal{W}_k$-modules. In particular, $\mathcal{W}_k$ has no irreducible, relaxed highest weight modules (see also [8, Rem. 6.4]).

We presented in [9] a conjecture on the classification of ordinary irreducible $\mathcal{W}_k$-modules for $k + 2 \in \mathbb{Z}_{\geq 0}$, which we proved in cases $k = -1, 0$ using explicit realizations of $\mathcal{W}_k$. One of the main results in this paper is the proof of this conjecture.

B. Free-field realization of $\mathcal{W}_k$ and their modules for $k + 2 \in \mathbb{Z}_{\geq 0}$

In [8], the Bershadsky–Polyakov algebra $\mathcal{W}_k$ is realized as a vertex subalgebra of $Z_k \otimes \Pi(0)$ (where $Z_k = \mathcal{W}_k(sl(3), f_{\text{prime}})$ is the Zamolodchikov $W$-algebra [43]), for $2k + 3 \notin \mathbb{Z}_{\geq 0}$. A realization of $\mathcal{W}_k$, when $2k + 3 \in \mathbb{Z}_{\geq 0}$, requires a different approach. In [9] we constructed a free-field realization of $\mathcal{W}_0$, but the cases when $k > 0$ were not solved either in [9] or [8].

In the current paper, we continue with our study of the Bershadsky–Polyakov algebra $\mathcal{W}_k = \mathcal{W}_k(sl_3, f_\theta)$ at positive integer levels $k$ and completely solve problem (A) for $k \geq 0$. We also partially solve problem (B) for $k = 1$ and find duality relation of $\mathcal{W}_1$ with the affine vertex superalgebra associated to $osp(1|2)$.

Classification of irreducible representations

In [9], we found a necessary condition for $\mathcal{W}_k$-modules, parametrizing the highest weights as zeroes of certain polynomial functions (cf. Proposition 4.5)

$$h_i(x, y) = \frac{1}{i} (g(x, y) + g(x + 1, y) + ... + g(x + i - 1, y))$$

$$= -i^2 + ki - 3xi + 3i - 3x^2 - k + 2kx + 6x + ky + 3y - 2,$$

and conjectured that this provides the complete list of irreducible modules for the Bershadsky–Polyakov algebra $\mathcal{W}_k$ when $k \in \mathbb{Z}, k \geq -1$. This conjecture was
proved in cases \( k = -1 \) and \( k = 0 \) in [9], using explicit realizations of \( \mathcal{W}_{-1} \) and \( \mathcal{W}_0 \) as the Heisenberg vertex algebra and a subalgebra of lattice vertex algebra, respectively. In this paper, we prove this conjecture for \( k \in \mathbb{Z}, k \geq 1 \), thus obtaining a classification of irreducible modules for \( \mathcal{W}_k \) at positive integer levels.

Let \( L(x, y) \) denote the irreducible highest weight representation of \( \mathcal{W}^k \) generated by a highest weight vector \( v(x, y) \) with highest weight \( (x, y) \) (cf. Section 3 and 4.1).

**Theorem 1.1.** The set \( \{ L(x, y) \mid (x, y) \in S_k \} \) is the set of all irreducible ordinary \( \mathcal{W}_k \)-modules, where

\[
S_k = \left\{ (x, y) \in \mathbb{C}^2 \mid \exists i, \ 1 \leq i \leq k + 2, \ h_i(x, y) = 0 \right\}.
\]

In order to prove Theorem 1.1, we need to show that \( L(x, y), (x, y) \in S_k \), are indeed \( \mathcal{W}_k \)-modules. Idea of the proof is to construct an infinite family of irreducible \( \mathcal{W}_k \)-modules \( L(x, y) \) such that \( h_i(x, y) = 0 \) for arbitrary \( 1 \leq i \leq k + 2 \), using spectral flow construction of \( \mathcal{W}_k \)-modules (cf. Section 4).

- We first consider a family of simple-current \( \mathcal{W}_k \)-modules \( \Psi^n(\mathcal{W}_k), n \in \mathbb{Z}_{\geq 0} \).
- We show that they are highest weight \( \mathcal{W}_k \)-modules satisfying

\[
\Psi^n(\mathcal{W}_k) = L(x_n, y_n) \quad \text{and} \quad h_1(x_{2n}, y_{2n}) = h_{k+2}(x_{2n+1}, y_{2n+1}) = 0.
\]

- Using a certain version of algebraic continuation (based on the fact that highest weights of modules for Zhu’s algebra must be zeros of finitely many curves in \( \mathbb{C}^2 \)), we conclude that \( L(x, y) \) are \( \mathcal{W}_k \)-modules whenever \( h_1(x, y) = 0 \) or \( h_{k+2}(x, y) = 0 \).
- Next, for every \( 2 \leq i \leq k + 1 \), we find special points \( (x^i, y^i) \) such that \( h_i(x^i, y^i) = h_{k+2}(x^i, y^i) = 0 \), and again apply the spectral-flow automorphism \( \Psi^n \). In this way we are able to construct infinitely many highest weight \( \mathcal{W}_k \)-modules \( L(x^i_{2n}, y^i_{2n}) \) such that \( h_i(x^i_{2n}, y^i_{2n}) = 0 \).
- Again using the algebraic continuation, we conclude that \( L(x, y) \) are \( \mathcal{W}_k \)-modules for each point of the curve \( h_i(x, y) = 0 \).

**Realization of \( \mathcal{W}_1 \) and duality with \( L_{-5/4}(osp(1|2)) \)**

Next, we give an indepth study of the case \( k = 1 \). First we show that the Bershadsky–Polyakov vertex algebra \( \mathcal{W}_1 \) can be embedded into the tensor product of the affine vertex superalgebra \( L_{k'}(osp(1|2)) \) at level \( k' = -5/4 \) and the Clifford vertex superalgebra \( F \). The affine vertex algebra \( V^k(osp(1|2)) \) associated to the Lie superalgebra \( osp(1, 2) \) was realized by the first named author in [3]. Using this result, and the fact that at level \( k' = -5/4 \) there is a conformal embedding of \( L_{k'}(sl(2)) \) into \( L_{k'}(osp(1|2)) \) (cf. [7], [19, Sect.10]), we obtain a realization of the Bershadsky–Polyakov algebra \( \mathcal{W}_1 \) (cf. Theorem 5.2).

Let \( F_{-1} \) be the lattice vertex superalgebra associated to the negative definite lattice \( \mathbb{Z}_\sqrt{-1} \). We show that the simple affine vertex superalgebra \( L_{-5/4}(osp(1|2)) \) can be realized as a subalgebra of \( \mathcal{W}_1 \otimes F_{-1} \) (cf. Theorem 5.4). Moreover, there is a duality between \( \mathcal{W}_1 \) and the affine vertex superalgebra \( L_{k'}(osp(1|2)) \) for \( k' = -5/4 \), in the sense that

\[
\mathcal{W}_1 = \text{Com} \left( M_{1,1}(1), L_{k'}(osp(1|2)) \otimes F \right),
\]

\[
L_{k'}(osp(1|2)) = \text{Com} \left( M_{-1}(1), \mathcal{W}_1 \otimes F_{-1} \right).
\]
where $M_0(1)$ and $M_0^\perp(1)$ are Heisenberg vertex algebras defined in Section 5.

In [3] it was proved that $L_{-5/4}(\mathfrak{osp}(1, 2))$ can be realized on the vertex superalgebra $F^{1/2} \otimes \Pi^{1/2}(0)$, where $\Pi^{1/2}(0)$ is a lattice type vertex algebra, and $F^{1/2}$ is a Clifford vertex superalgebra (cf. Subsection 2.2). Using the fact that all irreducible $L_{-5/4}(\mathfrak{osp}(1|2))$-modules can be constructed in this way, we can construct an explicit realization of irreducible $\mathcal{W}_1$-modules.

Consequences of duality and future work

The notion of Kazama–Suzuki dual was first introduced in the context of the duality of the $N = 2$ superconformal algebra and affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ (cf. [22], [1], [2]). Later it was shown that analogous duality relations hold for some other affine vertex algebras and $\mathcal{W}$-algebras (cf. [4], [5], [17], [16]). Our result shows that $L_k(\mathfrak{osp}(1|2))$ is the Kazama–Suzuki dual of $\mathcal{W}_1$. Relaxed modules for $L_k(\mathfrak{osp}(1|2))$ are mapped to the ordinary $\mathcal{W}_1$-modules, for which one expects it is easier to obtain the tensor category structure and calculate the fusion rules. Recent results [10], [16] show compelling evidence that fusion rules and (vertex) tensor category structure can be transferred onto duals. We expect that the duality between $L_k(\mathfrak{osp}(1|2))$ and $\mathcal{W}_1$ could be used to study fusion rules in the category of relaxed modules for $L_k(\mathfrak{osp}(1|2))$ (conjectured in [40]).

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Setup

- The universal Bershadsky–Polyakov algebra of level $k$ will be denoted with $\mathcal{W}_k(\mathfrak{sl}_3, \mathfrak{f}_\theta)$ or $\mathcal{W}_k^k$, and its unique simple quotient with $\mathcal{W}_k(\mathfrak{sl}_3, \mathfrak{f}_\theta)$ or $\mathcal{W}_k$.
- The spectral flow automorphism of $\mathcal{W}_k$ is denoted by $\Psi$.
- Modes of fields with respect to Virasoro vector $\omega$: $J_n$, $L_n$, $G^\pm_n$.
- $L_{x,y}$ denotes the irreducible highest weight representation with highest weight $(x, y)$ with respect to $(J_0, L_0)$ and highest weight vector $v_{x,y}$.
- Modes of fields with respect to Virasoro vector $\overline{\omega} = \omega + \frac{1}{2} DJ$: $J(n) = J_n$, $L(n) = L_n - \frac{n+1}{2} J_n$, $G^+(n) = G^+_n$, $G^-(n) = G^-_{n+1}$.
- $L(x, y)$ denotes the irreducible highest weight representation with highest weight $(x, y)$ with respect to $(J(0), L(0))$ and highest weight vector $v(x, y)$.
- We have $L(x, y) = L_{x,y+x/2}$.
- The Zhu algebra associated to the vertex operator algebra $V$ with the Virasoro vector $\omega$ will be denoted with $A_\omega(V)$.
- The Smith algebra corresponding to the polynomial $g(x, y) \in \mathbb{C}[x, y]$ is denoted with $R(g)$.
- $F_{-1}$ is the lattice vertex superalgebra associated to the lattice $\mathbb{Z}\sqrt{-1}$ defined in Section 2.4.
- $F^{1/2}$ is the Clifford vertex superalgebra, also called the free fermion algebra (cf. Section 2.2). It has an automorphism $\sigma_{F^{1/2}}$ of order two which is lifted from the automorphism $\Phi(r) \mapsto -\Phi(r)$ of the Clifford algebra $\mathbb{C}l^{1/2}$. The $\sigma_{F^{1/2}}$-twisted $F^{1/2}$-modules are denoted by $M_F^\pm$. 
• $F$ is the Clifford vertex superalgebra, also called the charged fermion algebra or the $bc$ system (cf. Section 2.3). It has an automorphism $\sigma_F$ of order two which is lifted from the automorphism $\Psi^+(r) \mapsto -\Psi^+(r), \Psi^-(r) \mapsto -\Psi^-(r)$ of the Clifford algebra $\text{Cl}$. The $\sigma_F$-twisted $F$ module is denoted by $M_{F}^{tw}$.

• $L_k(\text{osp}(1|2))$ is the simple affine vertex superalgebra associated to the Lie superalgebra $\text{osp}(1|2)$ at level $k$. The spectral flow automorphism of $L_k(\text{osp}(1|2))$ is denoted by $\rho$.

2. Preliminaries

In this section, we review certain properties of Clifford vertex superalgebras (cf. [24], [23]) and a construction of twisted modules for vertex superalgebras by H. Li (cf. [35]). Twisted modules for Clifford vertex superalgebras (cf. [24]) will play a key role in the realization of the Bershadsky–Polyakov algebra $\mathcal{W}_1$.

2.1. Twisted modules for vertex superalgebras

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vertex superalgebra (cf. [23], [42]), with the vertex operator structure given by

$$Y : V \to (\text{End } V)[[z, z^{-1}]], \quad Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1},$$

for $v \in V$, $v_n \in \text{End } V$. Then any element in $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is said to be even (resp. odd). For any homogenous element $u$, we define $|u| = 0$ if $u \in V_{\bar{0}}$ and $|u| = 1$ if $u \in V_{\bar{1}}$.

We say that a linear automorphism $\sigma : V \to V$ is a vertex superalgebra automorphism if it holds that

$$\sigma Y(v, z)\sigma^{-1} = Y(\sigma v, z)$$

for $v \in V$. Then $\sigma V_{\alpha} \subset V_{\alpha}$ for $\alpha \in \{\bar{0}, \bar{1}\}$.

Let $V$ be a vertex superalgebra and $\sigma$ an automorphism of $V$ with period $k \in \mathbb{Z}_{\geq 0}$ (that is, $\sigma^k = 1$). Let us now recall a construction of $\sigma$-twisted $V$-modules (cf. [35]).

Let $h \in V$ be an even element such that

$$L(n)h = \delta_{n,0}h, \quad h_n h = \delta_{n,1}\gamma \mathbb{1} \quad \text{for} \quad n \in \mathbb{Z}_{>0},$$

for fixed $\gamma \in \mathbb{Q}$. Assume that $h_0$ acts semisimply on $V$ with rational eigenvalues. It follows that $h_n$ satisfies

$$[L(m), h_n] = -nh_{m+n}, \quad [h_m, h_n] = m\gamma \delta_{m+n,0},$$

for $m, n \in \mathbb{Z}$.

Set

$$\Delta(h, z) = z^{h_0}\exp \left( \sum_{k=1}^{\infty} \frac{h_k}{-k} (-z)^{-k} \right).$$

Note that $e^{2\pi ih_0}$ is an automorphism of $V$. Set $\sigma_h = e^{2\pi ih_0}$ and assume that $\sigma_h$ is of finite order. The following was proved in [35].
Proposition 2.1 ([35]). Let $V$ be a vertex superalgebra and let $h \in V$ be an even element such that (1) holds and $h_0$ acts on $V$ with rational eigenvalues. Let $(M, Y_M(\cdot, z))$ be a $V$-module. Then $(M, Y_M(\Delta(h, z) \cdot, z))$ carries the structure of a $\sigma_h$-twisted $V$-module.

2.2. Clifford vertex superalgebra $F^{1/2}$ and its twisted modules
Let $C^{1/2}$ be the Clifford algebra with generators $\Phi(r), \ r \in \frac{1}{2} + \mathbb{Z}$ and commutation relations
\[
\{ \Phi(r), \Phi(s) \} = \delta_{r+s,0}, \quad r, s \in \frac{1}{2} + \mathbb{Z}.
\]

The fields
\[
\Phi(z) = \sum_{n \in \mathbb{Z}} \Phi(n + \frac{1}{2}) z^{-n-1}
\]
generate on
\[
F^{1/2} = \wedge (\Phi(-n - \frac{1}{2}) \mid n \in \mathbb{Z}_{\geq 0})
\]
a unique structure of a vertex superalgebra with conformal vector
\[
\omega_{F^{1/2}} = \frac{1}{2} \Phi(-\frac{3}{2}) \Phi(-\frac{1}{2}) \mathbb{1},
\]
of central charge $c_{F^{1/2}} = 1/2$ (cf. [26], [28]). Note also that the field $\Phi(z)$ is usually called neutral fermion field, and $F^{1/2}$ is called free-fermion theory in physics literature.

A basis of $F^{1/2}$ is given by
\[
\Phi(-n_1 - \frac{1}{2}) \cdots \Phi(-n_r - \frac{1}{2}),
\]
where $n_1 > \cdots > n_r \geq 0$.

The vertex superalgebra $F^{1/2}$ has the automorphism $\sigma_{F^{1/2}}$ of order two, which is lifted from the automorphism $\Phi(r) \mapsto -\Phi(r)$ of the Clifford algebra. The fixed points of this automorphism is the Virasoro vertex algebra $L^{\text{Vir}}(\frac{1}{2}, 0)$. Moreover, $F^{1/2} = L^{\text{Vir}}(\frac{1}{2}, 0) \oplus L^{\text{Vir}}(\frac{1}{2}, \frac{1}{2})$.

We briefly recall the properties of twisted modules for Clifford vertex superalgebras, while details can be found in [24].

Define the twisted Clifford algebra $C_{tw}^{1/2}$ generated by $\Phi(m), \ m \in \mathbb{Z}$, and relations
\[
\{ \Phi(m), \Phi(n) \} = \delta_{m+n,0}, \quad m, n \in \mathbb{Z}.
\]

Let
\[
M_{F^{1/2}}^\pm = \bigoplus_{n=0}^{\infty} M_{F^{1/2}}^\pm(n)
\]
be the two irreducible modules for the Clifford algebra $C_{tw}^{1/2}$, such that $\Phi(0)$ acts on the one-dimensional top component $M_{F^{1/2}}^\pm(0)$ as $\pm \frac{1}{\sqrt{2}} \text{Id}$.

Let
\[
\Phi_{tw}(z) = \sum_{m \in \mathbb{Z}} \Phi(m) z^{-m-1/2},
\]
and
\[ Y\left( \Phi(-n_1 - \frac{1}{2}) \cdots \Phi(-n_r - \frac{1}{2}), z \right) =: \partial_{n_1} \Phi^{tw}(z_1) \cdots \partial_{n_r} \Phi^{tw}(z_r), \]
and extend by linearity to all of \( F^{1/2} \).

Define the twisted operator
\[ Y^{tw}_{F^{1/2}}(v, z) := Y(e^{\Delta_z} v, z), \]
where
\[ \Delta_z = \frac{1}{2} \sum_{m,n \in \mathbb{Z}_{\ge 0}} C_{m,n} \Phi(m + \frac{1}{2}) \Phi(n + \frac{1}{2}) z^{-m-n-1}, \]
and
\[ C_{m,n} = \frac{1}{2} \frac{m-n}{m+n+1} \begin{pmatrix} -1/2 \\ m \end{pmatrix} \begin{pmatrix} -1/2 \\ n \end{pmatrix}. \]
It holds that (cf. [23], [24])
\[ e^{\Delta_z} \omega_{F^{1/2}} = \omega_{F^{1/2}} + \frac{1}{16} z^{-2} \mathbb{1}. \]

Then \( (M^{\pm}_{F^{1/2}}, Y^{tw}_{F^{1/2}}) \) has the structure of a \( \sigma_{F^{1/2}} \)-twisted module for the vertex superalgebra \( F^{1/2} \).

Recall also that as a \( L^{Vir}(\frac{1}{2}, 0) \)-module, we have (cf. [24])
\[ M^{\pm}_{F^{1/2}} \cong L^{Vir}(\frac{1}{2}, \frac{1}{16}). \]

2.3. Clifford vertex superalgebra \( F \) and its twisted modules
Consider the Clifford algebra \( Cl \) with generators \( \Psi^{\pm}(r), \ r \in \frac{1}{2} + \mathbb{Z} \) and relations
\[ \{ \Psi^{+}(r), \Psi^{-}(s) \} = \delta_{r+s,0}, \quad \{ \Psi^{\pm}(r), \Psi^{\pm}(s) \} = 0, \quad r, s \in \frac{1}{2} + \mathbb{Z}. \]
The fields
\[ \Psi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \Psi^{\pm}(n + \frac{1}{2}) z^{-n-1} \]
generate on
\[ F = \wedge \left( \Psi^{\pm}( - n - 1/2) \mid n \in \mathbb{Z}_{>0} \right) \]
a unique structure of a simple vertex superalgebra. This vertex algebra is sometimes called \( bc \)-system.

Let \( \alpha =: \Psi^{+} \Psi^{-} :. \) Then
\[ \omega_{F} = \frac{1}{2} : \alpha \alpha : \]
is a conformal vector for \( F \) of central charge \( c_{F} = 1 \).

A basis of \( F \) is given by
\[ \Psi^{+}( - n_1 - \frac{1}{2}) \cdots \Psi^{+}( - n_r - \frac{1}{2}) \Psi^{-}( - k_1 - \frac{1}{2}) \cdots \Psi^{-}( - k_s - \frac{1}{2}), \]
where \( n_i, k_i \in \mathbb{Z}_{\geq 0}, n_1 > \cdots > n_r, k_1 > \cdots > k_s \).

Note that by the boson-fermion correspondence, the Clifford vertex superalgebra \( F \) is isomorphic to the lattice vertex superalgebra \( V_{\mathbb{Z}^{\alpha}} \) (cf. [25], [28]), that is,

\[
F \cong V_{\mathbb{Z}^{\alpha}}.
\]

The vertex superalgebra \( F \) has an automorphism \( \sigma_F \) of order two which is lifted from the automorphism \( \Psi^+(r) \mapsto -\Psi^+(r), \Psi^-(r) \mapsto -\Psi^-(r) \) of the Clifford algebra.

Let \( (M_{F, tw}, Y_{F, tw}(-, z)) \) so that \( M_{F, tw} = F \) as a vector space, and the vertex operator is defined by

\[
Y_{F, tw}(v, z) = Y(\Delta(\alpha/2, z)v, z).
\]

By Proposition 2.1, we have that \( (M_{F, tw}, Y_{F, tw}) \) has the structure of a \( \sigma_F \)-twisted module for the vertex superalgebra \( F \).

2.4. Lattice vertex superalgebras \( F_{-1} \)

Consider rank one lattice \( L = \mathbb{Z}\varphi, \langle \varphi, \varphi \rangle = -1 \). Let \( F_{-1} \) be the associated vertex algebra. This vertex superalgebra is used for a construction of the inverse of Kazama–Suzuki functor in the context of duality between affine \( \widehat{sl}(2) \) and \( N = 2 \) superconformal algebra (cf. [22], [1], [2]).

As a vector space \( F_{-1} = \mathbb{C}[L] \otimes M_{\varphi}(1) \), where \( \mathbb{C}[L] \) is a group algebra of \( L \), and \( M_{\varphi}(1) \) the Heisenberg vertex algebra generated by the Heisenberg field \( \varphi(z) = \sum_{n \in \mathbb{Z}} \varphi(n)z^{-n-1} \) such that

\[
[\varphi(n), \varphi(m)] = -n\delta_{n+m,0}.
\]

\( F_{-1} \) is generated by \( e^{\pm \varphi} \). We shall need the relations

\[
\begin{align*}
e_{n}^{\pm \varphi}e_{m}^{\pm \varphi} &= 0 \quad \text{for } n \geq 1, \\
e_{-m}^{\pm \varphi}e_{m}^{\pm \varphi} &= S_m(\pm \varphi)e^{2\varphi} \quad \text{for } m \geq 0, \\
e_{n}^{\varphi}e^{-\varphi} &= 0 \quad \text{for } n \geq -1, \\
e_{-m}^{-\varphi}e^{\varphi} &= S_m(\varphi) \quad \text{for } m \geq 0,
\end{align*}
\]

where \( S_m(\varphi) := S_m(\varphi(-1), \varphi(-2), \ldots) \) is the \( m \)-th Schur polynomial in variables \( \varphi(-1), \varphi(-2), \ldots \).

2.5. Kazama–Suzuki duality

In this subsection, we will define a duality of vertex algebras which is motivated by the duality between \( N = 2 \) superconformal vertex algebra and affine vertex algebra \( L_k(sl(2)) \).

Recall first that if \( S \) is a vertex subalgebra of \( V \), we have the commutant subalgebra of \( V \) (cf. [37])

\[
\text{Com}(S, V) := \{ v \in V \mid a_n v = 0, \forall a \in S, \forall n \in \mathbb{Z}_{\geq 0} \}.
\]
Assume that $U, V$ are vertex superalgebras. We say that $V$ is the Kazama–Suzuki dual of $U$ if there exist injective homomorphisms of vertex superalgebras

$$\varphi_1 : V \rightarrow U \otimes F, \quad \varphi_2 : U \rightarrow V \otimes F_{-1},$$

so that

$$V \cong \text{Com}(H^1, U \otimes F), \quad U \cong \text{Com}(H^2, V \otimes F_{-1}),$$

where $H^1$ (resp. $H^2$) is a rank one Heisenberg vertex subalgebra of $U \otimes F$ (resp. $V \otimes F_{-1}$).

3. Bershadsky–Polyakov algebra $W_k(sl_3, f_\theta)$

Bershadsky–Polyakov vertex algebra $W_k(= W_k^{sl_3, f_\theta})$ is the minimal affine $W$-algebra associated to the minimal nilpotent element $f_\theta$ (cf. [12], [27], [31], [29], [33]). The algebra $W_k$ is generated by four fields $T, J, G^+, G^-$, of conformal weights $2, 1, 3/2, 3/2$ and is a $\frac{1}{2}\mathbb{Z}$-graded VOA.

**Definition 3.1.** Universal Bershadsky–Polyakov vertex algebra $W_k$ is the vertex algebra generated by fields $T, J, G^+, G^-$, which satisfy the following relations:

$$J(x)J(y) \sim \frac{2k + 3}{3}(z - w)^{-2}, \quad G^\pm(z)G^\pm(w) \sim 0,$$

$$J(z)G^\pm(w) \sim \pm G^\pm(w)(z - w)^{-1},$$

$$T(z)T(w) \sim -\frac{c_k}{2}(z - w)^{-4} + 2T(w)(z - w)^{-2} + DT(w)(z - w)^{-1},$$

$$T(z)G^\pm(w) \sim \frac{3}{2}G^\pm(w)(z - w)^{-2} + DG^\pm(w)(z - w)^{-1},$$

$$T(z)J(w) \sim J(w)(z - w)^{-2} + DJ(w)(z - w)^{-1},$$

$$G^+(z)G^-(w) \sim (k + 1)(2k + 3)(z - w)^{-3} + 3(k + 1)J(w)(z - w)^{-2} + (3 : J(w)J(w) : + \frac{3(k + 1)}{2}DJ(w) - (k + 3)T(w))(z - w)^{-1},$$

where $c_k = (3k + 1)(2k + 3)/(k + 3)$.

Vertex algebra $W_k$ is called the universal Bershadsky–Polyakov vertex algebra of level $k$. For $k \neq -3$, $W_k$ has a unique simple quotient which is denoted by $W_k$.

Let

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1},$$

$$G^+(z) = \sum_{n \in \mathbb{Z}} G^+_n z^{-n-1},$$

$$G^-(z) = \sum_{n \in \mathbb{Z}} G^-_n z^{-n-1}.$$
The following commutation relations hold:
\[ [J_m, J_n] = \frac{2k + 3}{3} m \delta_{m+n,0}, \quad [J_m, G^\pm_n] = \pm G^\pm_{m+n}, \]
\[ [L_m, J_n] = -n J_{m+n}, \]
\[ [L_m, G^\pm_n] = \left( \frac{1}{2} m - n + \frac{1}{2} \right) G^\pm_{m+n}, \]
\[ [G^+_m, G^-_n] = \frac{3(J^2)_{m+n-1} + \frac{3}{2}(k+1)(m-n)J_{m+n-1} - (k+3)L_{m+n-1}}{2} \]
\[ + \frac{(k+1)(2k+3)(m-1)}{2} \delta_{m+n,1}. \]

By applying results from [29], we see that for every \((x,y) \in \mathbb{C}^2\) there exists an irreducible representation \(L_{x,y}\) of \(\mathcal{W}^k\) generated by a highest weight vector \(v_{x,y}\) such that
\[ J_0 v_{x,y} = xv_{x,y}, \quad J_n v_{x,y} = 0 \text{ for } n > 0, \]
\[ L_0 v_{x,y} = yv_{x,y}, \quad L_n v_{x,y} = 0 \text{ for } n > 0, \]
\[ G^+_n v_{x,y} = 0 \text{ for } n \geq 1. \]

### 3.1. Structure of the Zhu algebra \(A(\mathcal{W}^k)\)

Let \(A_\omega(V)\) denote the Zhu algebra associated to the VOA \(V\) (cf. [44]) with the Virasoro vector \(\omega\), and let \([v]\) be the image of \(v \in V\) under the mapping \(V \mapsto A_\omega(V)\).

For the Zhu algebra \(A_\omega(\mathcal{W}^k)\), it holds the following.

**Proposition 3.2 ([9, Prop. 3.2.]).** There exists a homomorphism \(\Phi : \mathbb{C}[x, y] \rightarrow A_\omega(\mathcal{W}^k)\) such that
\[ \Phi(x) = [J], \quad \Phi(y) = [T]. \]

It can be shown that the homomorphism \(\Phi : \mathbb{C}[x, y] \rightarrow A_\omega(\mathcal{W}^k)\) is in fact an isomorphism, i.e., that \(A_\omega(\mathcal{W}^k) \cong \mathbb{C}[x, y]\).

If we switch to a shifted Virasoro vector \(\overline{\omega} = \omega + \frac{1}{2} DJ\), the vertex algebras \(\mathcal{W}^k\) and \(\mathcal{W}_k\) become \(\mathbb{Z}_{\geq 0}\)-graded with respect to \(L(0) = \overline{\omega}_1\). In this case, the Zhu algebras are no longer commutative, and they carry more information about the representation theory. The Zhu algebra associated to \(\mathcal{W}^k\) is then realized as a quotient of another associative algebra, the so-called Smith algebra \(R(g)\) (introduced in [41], see also [20]). These algebras were used by T. Arakawa in [12] to prove rationality of \(\mathcal{W}_k(sl_3, f_p)\) for \(k = p/2 - 3, p \geq 3, p \text{ odd}\).

We expand the original definition of Smith algebras \(R(f)\) by adding a central element.

**Definition 3.3.** Let \(g(x, y) \in \mathbb{C}[x, y]\) be an arbitrary polynomial. Associative algebra \(R(g)\) of Smith type is generated by \(\{E, F, X, Y\}\) such that \(Y\) is a central element and the following relations hold:
\[XE - EX = E, \quad XF - FX = -F, \quad EF - FE = g(X,Y).\]

In fact, Zhu algebra associated to \(\mathcal{W}^k\) is a quotient of the Smith-type algebra \(R(g)\) for a certain polynomial \(g(x, y) \in \mathbb{C}[x, y]\).
**Proposition 3.4 ([9, Prop. 4.2.]).** Zhu algebra $A_{\bar{\omega}}(W^k)$ is a quotient of the Smith algebra $R(g)$ for $g(x, y) = -(3x^2 - (2k + 3)x - (k + 3)y)$.

4. Vertex algebra $W_k$ for $k + 2 \in \mathbb{Z}_{\geq 1}$

In this section, we study the representation theory of the Bershadsky–Polyakov algebra $W_k$ at positive integer levels. In [9], we parametrized the highest weights of irreducible $W_k$-modules as zeroes of certain polynomial functions (cf. Proposition 4.5), and conjectured that this provides the complete list of irreducible modules for the Bershadsky–Polyakov algebra $W_k$ when $k \in \mathbb{Z}$, $k \geq -1$. This conjecture was proved in cases $k = -1$ and $k = 0$ in [9]. In this paper, we will prove this conjecture for $k \in \mathbb{Z}$, $k \geq 1$.

4.1. Setup

Let us choose a new Virasoro field

$$L(z) := T(z) + \frac{1}{2} DJ(z).$$

Then $\bar{\omega} = \omega + \frac{1}{2} DJ$ is a conformal vector $\bar{\omega}_{n+1} = L(n)$ with central charge

$$\bar{c}_k = -\frac{4(k+1)(2k+3)}{k+3}.$$

The fields $J, G^+, G^-$ have conformal weights $1, 1, 2$, respectively. Set $J(n) = J_n$, $G^+(n) = G^+_n$, $G^-(n) = G^-_{n+1}$. We have

$$L(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \ G^+(z) = \sum_{n \in \mathbb{Z}} G^+(n) z^{-n-1}, \ G^-(z) = \sum_{n \in \mathbb{Z}} G^-(n) z^{-n-2}.$$

This defines a $\mathbb{Z}_{\geq 0}$-gradation on $W^k$.

Let $L(x, y)$ denote the irreducible highest weight representation with highest weight $(x, y)$ with respect to $(J(0), L(0))$ and highest weight vector $v(x, y)$. We have $L(x, y) = L_{x,y+x/2}$ (cf. [9]).

Define

$$\Delta(-J, z) = z^{-J(0)} \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} J(k)}{k z^k} \right),$$

and let

$$\sum_{n \in \mathbb{Z}} \psi(a_n) z^{-n-1} = Y(\Delta(-J, z)a, z),$$

for $a \in W^k$.

The operator $\Delta(h, z)$ associates to every $V$-module $M$ a new structure of an irreducible $V$-module (cf. [34]). Let us denote this new module (obtained using the mapping $a_n \mapsto \psi(a_n)$) with $\psi(M)$. As the $\Delta$-operator acts bijectively on the set of irreducible modules, there exists an inverse $\psi^{-1}(M)$. 

Remark 4.1. Operators $\psi^m$ are also called the spectral flow automorphisms of $\mathcal{W}_k$ (see [8, Sect. 2] and [21, Subsect. 2.2] for more details).

From the definition of $\Delta(-J,z)$ we have that

$$\psi(J(n)) = J(n) - \frac{2k + 3}{3} \delta_{n,0}, \quad \psi(L(n)) = L(n) - J(n) + \frac{2k + 3}{3} \delta_{n,0},$$

$$\psi(G^+(n)) = G^+(n - 1), \quad \psi(G^-(n)) = G^-(n + 1).$$

Let

$$L(x, y)_{\text{top}} = \{v \in L(x, y) : L(0)v = yv\},$$

and denote

$$\widehat{x}_i = x + i - 1 - \frac{2k + 3}{3}, \quad \widehat{y}_i = y - x - i + 1 + \frac{2k + 3}{3}.$$ 

The following was proved in [12].

Lemma 4.2 ([12, Prop. 2.3]). Let $\dim L(x, y)_{\text{top}} = i$. Then it holds that

$$\psi(L(x, y)) \cong L(\widehat{x}_i, \widehat{y}_i).$$

4.2. Necessary condition for $\mathcal{W}_k$-modules

Starting point in the classification of irreducible $\mathcal{W}_k$-modules for integer levels $k$ is the following formula for singular vectors (cf. [9]). These generalize a construction of a family of singular vectors by T. Arakawa in [12], where he found a similar formula for singular vectors in $\mathcal{W}_k$ at levels $k = p/2 - 3$, $p \geq 3$, $p$ odd.

Lemma 4.3 ([9, Lem. 8.1]). Vectors

$$G^+(-1)^n \mathbb{1}, \quad G^-(-2)^n \mathbb{1}$$

are singular in $\mathcal{W}_k$ for $n = k + 2$, where $k \in \mathbb{Z}$, $k \geq -1$.

Let

$$g(x, y) = -(3x^2 - (2k + 3)x - (k + 3)y) \in \mathbb{C}[x, y]$$

and define polynomials $h_i(x, y)$, for $i \in \mathbb{N}$ (cf. [12]) as

$$h_i(x, y) = \frac{1}{i} (g(x, y) + g(x + 1, y) + \ldots + g(x + i - 1, y))$$

$$= -i^2 + ki - 3xi + 3i - 3x^2 - k + 2kx + 6x + ky + 3y - 2.$$ 

The next technical lemma follows immediately from the definition of $h_i(x, y)$ and $\widehat{x}_i, \widehat{y}_i$.

Lemma 4.4. Assume that $i + j = k + 3$. Then it holds that

$$h_i(x, y) = h_j(\widehat{x}_i, \widehat{y}_i).$$
Proof. We have
\[
\begin{align*}
  h_{k+3-i}(\tilde{x}_i, \tilde{y}_i) &= (-3x^2 - 9x + 12x + 6x - 3x + 3xi - 3kx - 6xi + 4kx + 2kx - kx) \\
  &+ (ky + 3y) + (-i^2 + 2ki + 6i^2 - 6k - 9 + k^2 + 3k - k + 3i^2) \\
  &- 5ki - 15i + 2k^2 + 12k + 18 + 3k + 9 - 3i - 3i^2 + 4ki \\
  &+ 12i - \frac{4}{3}k^2 - 8k - 12k + 2ki - 2k - \frac{4}{3}k^2 + 6i - 6 - 4k \\
  &- 6 - ki + k + \frac{2}{3}k^2 + k - 3i + 3 + 2k + 1) \\
  &= (-3x^2 - 3ix + 2kx + 6x) + (ky + 3y) + (-i^2 + ki + 3i - k - 2) \\
  &= h_i(x, y). \quad \square
\end{align*}
\]

Define the set
\[
S_k = \{(x, y) \in \mathbb{C}^2 \mid \exists i, 1 \leq i \leq k + 2, \ h_i(x, y) = 0\}.
\]

In [9] we proved that in order for \( L(x, y) \) to be an irreducible ordinary \( \mathcal{W}_k \)-module, \((x, y)\) needs to belong to the set \( S_k \).

**Proposition 4.5 ([9]).** Let \( k \in \mathbb{Z}, k \geq -1 \). Then we have:

1. The set of equivalency classes of irreducible ordinary \( \mathcal{W}_k \)-modules is contained in the set
   \[
   \{L(x, y) \mid (x, y) \in S_k\}.
   \]

2. Every irreducible \( \mathcal{W}_k \)-module in the category \( \mathcal{O} \) is an ordinary module.

We stated the following conjecture (and proved it for \( k = -1 \) and \( k = 0 \)).

**Conjecture 4.6 ([9]).** The set \( \{L(x, y) \mid (x, y) \in S_k\} \) is the set of all irreducible ordinary \( \mathcal{W}_k \)-modules.

The proof of Conjecture 4.6 is reduced to showing that \( L(x, y), (x, y) \in S_k \), are indeed \( \mathcal{W}_k \)-modules. In what follows, we shall prove Conjecture 4.6.

### 4.3. Simple current \( \mathcal{W}_k \)-modules

First step in the proof is to construct an infinite family of irreducible \( \mathcal{W}_k \)-modules \( L(x, y) \) satisfying the conditions \( h_1(x, y) = 0 \) or \( h_{k+2}(x, y) = 0 \). We have the following important lemma.

**Lemma 4.7.** Assume that \( n \in \mathbb{Z}_{\geq 0} \). Define
\[
\begin{align*}
  x_{2n} &= -n \frac{k + 3}{3}, \\
  x_{2n+1} &= -n - 1 - \frac{(n + 2)k}{3}, \\
  y_{2n} &= n \frac{3(2k + (k + 3)n)}{3}, \\
  y_{2n+1} &= n + 1 \frac{3(n(k + 3) + 2k + 3)}{3}.
\end{align*}
\]

Then for each \( n \in \mathbb{Z}_{\geq 0} \) we have
\[ L(x_n, y_n) \text{ is irreducible } \mathcal{W}_k\text{-module.} \]
\[ \Psi^n(\mathcal{W}_k) \cong L(x_n, y_n). \]
\[ h_1(x_{2n}, y_{2n}) = h_{k+2}(x_{2n+1}, y_{2n+1}) = 0. \]

**Proof.** By direct calculation we have

\[ h_i(x_{2n}, y_{2n}) = 0 \iff i \in \{1, k + 2 + n(k + 3)\}, \]
\[ h_i(x_{2n+1}, y_{2n+1}) = 0 \iff i \in \{k + 2, 2(k + 2) + n(k + 3)\}. \]

We see that \((x_n, y_n)\) is the unique solution of the following recursive relations:

\[
\begin{align*}
  x_0 &= y_0 = 0, \\
  x_{2n+1} &= x_{2n} - \frac{2k + 3}{3}, \\
  y_{2n+1} &= \hat{y}_{2n} = y_{2n} - x_{2n+1}, \\
  x_{2n+2} &= \hat{x}_{2n+1} = x_{2n+1} + \frac{k}{3}, \\
  y_{2n+2} &= \hat{y}_{2n+1} = y_{2n+1} - x_{2n+2}.
\end{align*}
\]

Then for each \(n \in \mathbb{Z}_{\geq 0}\) we have

\[ L(x_n, y_n) = \Psi^n(L(0, 0)) = \Psi^n(\mathcal{W}_k). \]

So \(L(x_n, y_n)\) is an irreducible \(\mathcal{W}_k\)-module. \(\square\)

**Lemma 4.8.** Assume that \(n \in \mathbb{Z}_{<0}\). Define

\[
\begin{align*}
  x_{2n} &= -n \frac{k + 3}{3}, \\
  x_{2n-1} &= 1 - n - \frac{(n - 2)k}{3}, \\
  y_{2n} &= -\frac{n}{3}(k - (k + 3)n), \\
  y_{2n-1} &= -\frac{n}{3}(2k + 3 - n(k + 3)).
\end{align*}
\]

Then for each \(n \in \mathbb{Z}_{<0}\) we have

\[ L(x_n, y_n) \text{ is irreducible } \mathcal{W}_k\text{-module.} \]
\[ \Psi^n(\mathcal{W}_k) \cong L(x_n, y_n). \]
\[ h_{k+2}(x_{2n}, y_{2n}) = h_1(x_{2n+1}, y_{2n+1}) = 0. \]

**Proof.** By direct calculation we have

\[ h_i(x_{2n}, y_{2n}) = 0 \iff i \in \{k + 2, 1 + n(k + 3)\}, \]
\[ h_i(x_{2n-1}, y_{2n-1}) = 0 \iff i \in \{1, -1 - k + n(k + 3)\}. \]

We see that \((x_n, y_n)\) is the unique solution of the following recursive relations:

\[
\begin{align*}
  x_0 &= y_0 = 0, \\
  x_{2n-1} &= x_{2n} + \frac{2k + 3}{3}, \\
  y_{2n-1} &= y_{2n} + x_{2n}, \\
  x_{2n-2} &= x_{2n-1} - \frac{k}{3}, \\
  y_{2n-2} &= y_{2n-1} + x_{2n-1}.
\end{align*}
\]
We have
\[ \Psi^{-1}(L(0,0)) = \Psi^{-1}(W_k) = L \left( \frac{2k + 3}{3}, 0 \right). \]
Then for each \( n \in \mathbb{Z}_{\leq 0} \) we have
\[ L(x_n, y_n) = \Psi^n(L(0,0)) = \Psi^n(W_k). \]
Hence \( L(x_n, y_n) \) is an irreducible \( W_k \)-module. \( \square \)

By using \([34, \text{Thm. 2.15}] \) (see also \([10, \text{Prop. 3.1}] \)) we get the following.

**Corollary 4.9.** In the fusion algebra of \( W_k \), \( L(x_n, y_n) \) are simple-current modules, i.e., the following fusion rules hold:
\[ \Psi^n(W_k) \times L(x, y) = L(x_n, y_n) \times L(x, y) = \Psi^n(L(x, y)). \]

**4.4. Modules \( L(x, y) \) such that \( h_1(x, y) = 0 \) or \( h_{k+2}(x, y) = 0 \)**

**Theorem 4.10.** Assume that \( h_1(x, y) = 0 \) or \( h_{k+2}(x, y) = 0 \). Then \( L(x, y) \) is an irreducible ordinary \( W_k \)-module.

**Proof.** The solution of the equation \( h_1(x, y) = 0 \) is
\[ y = g_1(x) = \frac{-3x - 2kx + 3x^2}{3 + k} \quad (x \in \mathbb{C}). \]

So we need to prove that \( L(x, g_1(x)) \) is an \( W_k \)-module for every \( x \in \mathbb{C} \). On the other hand, the Zhu’s algebra \( A_\omega(W_k) \) is isomorphic to a certain quotient of \( \mathbb{C}[x, y] \) by an ideal \( I \). Since \( \mathbb{C}[x, y] \) is Noetherian, the ideal \( I \) is finitely generated by finitely many polynomials, say \( P_1, \ldots, P_\ell \). Hence the highest weights \( (x, y) \) of irreducible \( W_k \)-modules are solutions of the equations
\[ P_i(x, y + x/2) = 0, \quad i = 1, \ldots, \ell. \]

It remains to prove that
\[ P_i(x, g_1(x) + x/2) = 0, \quad \forall x \in \mathbb{C}. \]

By Lemma 4.7, we have that \( L(x, g(x)) \) are \( W_k \)-modules for \( x = x_{2n} = -n \frac{k + 3}{3} \). So we have
\[ P_i(x_{2n}, g_1(x_{2n}) + x_{2n}/2) = 0, \quad i = 1, \ldots, \ell. \]

Since each \( P_i(x, g_1(x) + x/2) \) is a polynomial in one variable, and given that it already has infinitely many zeros, it follows that \( P_i(x, g_1(x) + x/2) \equiv 0 \). This proves that \( L(x, g_1(x)) \) is \( W_k \)-module for every \( x \in \mathbb{C} \).

Applying Lemma 4.4, we get that \( L(x, y) \) is a \( W_k \)-module for each solution of the equation \( h_{k+2}(x, y) = 0 \). \( \square \)

**4.5. Proof of Conjecture 4.6**

The proof of the conjecture is reduced to the existence of irreducible \( W_k \) modules \( L(x, y) \) such that \( h_i(x, y) = 0 \) for arbitrary \( 1 \leq i \leq k + 2 \).
Lemma 4.11. Let $1 \leq i \leq k+2$. There exists an irreducible $\mathcal{W}_k$-module $L(x^i, y^i)$ with highest weight
\[
(x^i, y^i) = \left( \frac{1-i}{3}, \frac{(-1+i)(-1+i-k)}{3(3+k)} \right)
\]
such that $\dim L(x, y)_{\text{top}} = i$.

Proof. Note that
\[
h_j(x^i, y^i) = (i-j)(-2+j-k) = 0 \iff j \in \{i, k+2\}.
\]
By Theorem 4.10, we know that $L(x^i, y^i)$ is an $\mathcal{W}_k$-module. But since $h_i(x^i, y^i) = 0$ and $h_j(x^i, y^i) \neq 0$ for $j < i$, we conclude that $\dim L(x^i, y^i)_{\text{top}} = i$. The proof follows.

Now we shall continue as in the previous subsection. We will apply the automorphism $\Psi$ and construct new infinite family of $\mathcal{W}_k$-modules.

Lemma 4.12. For each $n \in \mathbb{Z}_{\geq 0}$, $\Psi^n(L(x^i, y^i)) = L(x_n^i, y_n^i)$, where
\[
x_{2n}^i = \frac{1-i}{3} - \frac{k+3}{3n},
\]
\[
y_{2n}^i = \frac{(i-1)^2+k-ik+12n-3in+10kn-ikn+2k^2n+n^2(k+3)^2}{3(3+k)},
\]
\[
x_{2n+1}^i = \frac{-5+2i-3n-k(2+n)}{3},
\]
\[
y_{2n+1}^i = \frac{(i-4)^2+12k-3ik+2k^2+21n-3in+16kn-ikn+3k^2n+n^2(k+3)^2}{3(3+k)}.
\]

Proof. Using direct calculation, we get
\[
h_j(x_{2n}, y_{2n}) = (i-j)(-2+j-3n-k(1+n)),
\]
\[
h_j(x_{2n+1}, y_{2n+1}) = -(3+i+j-k)(-5+i+j-2k-3n-kn),
\]
which implies that
\[
\dim L(x_{2n}, y_{2n})_{\text{top}} = i, \quad \dim L(x_{2n+1}, y_{2n+1})_{\text{top}} = k+3-i.
\]
We see that $(x_n^i, y_n^i)$ is the unique solution of the following recursive relations:
\[
x_0^i = x^i, \quad y_0^i = y^i,
\]
\[
x_{2n+1}^i = x_{2n}^i + (i-1) - \frac{2k+3}{3}, \quad y_{2n+1}^i = y_{2n}^i - x_{2n+1}^i,
\]
\[
x_{2n+2}^i = x_{2n+1}^i + k + 2 - i - \frac{2k+3}{3}, \quad y_{2n+2} = y_{2n+1}^i - x_{2n+2}^i.
\]
This proves that for each $n \in \mathbb{Z}_{\geq 0}$ we have
\[
L(x_n^i, y_n^i) = \Psi^n(L(x^i, y^i)).
\]
Since $L(x^i, y^i)$ is a $\mathcal{W}_k$-module, we have that $L(x_n^i, y_n^i)$ is an irreducible $\mathcal{W}_k$-module. The proof follows.
Theorem 4.13. Assume that $h_i(x, y) = 0$ for $1 \leq i \leq k + 2$. Then $L(x, y)$ is an irreducible ordinary $\mathcal{W}_k$-module.

Proof. By Lemma 4.12, we have that $L(x_{2n}, y_{2n})$ are $\mathcal{W}_k$-modules for every $n \in \mathbb{Z}_{\geq 0}$. Since $h_i(x_{2n}, y_{2n}) = 0$, we conclude that there is an infinite family of points $(x, y)$ of the curve $h_i(x, y) = 0$ such that $L(x, y)$ is a $\mathcal{W}_k$-module. Applying the same argument as in the proof of Theorem 4.10, we get that $L(x, y)$ is a $\mathcal{W}_k$-module for every $(x, y)$ such that $h_i(x, y) = 0$. □

Theorem 4.13 concludes the proof of Conjecture 4.6.

5. The duality of $\mathcal{W}_1$ and $L_{-5/4}(osp(1|2))$

In this section, we construct an embedding of the Bershadsky–Polyakov vertex algebra $\mathcal{W}_1$ into the tensor product of the affine vertex superalgebra $L_k(osp(1|2))$ at level $k = -5/4$ and the Clifford vertex superalgebra $F$. The affine vertex superalgebra $V^k(osp(1|2))$ and its modules were realized by the first named author in [3]. Moreover, we will show that $L_{-5/4}(osp(1|2))$ is the Kazama–Suzuki dual of $\mathcal{W}_1$ by proving that there is an embedding of $\mathcal{W}_1$ into $F$, where $F$ is the vertex superalgebra associated to the lattice $\mathbb{Z}\sqrt{-1}$.

5.1. Affine vertex superalgebra $V^k(osp(1|2))$

Recall that $g = osp(1, 2)$ is the simple complex Lie superalgebra with basis $\{e, f, h, x, y\}$ such that the even part is $g^0 = \text{span}_\mathbb{C}\{e, f, h\}$ and the odd part is $g^1 = \text{span}_\mathbb{C}\{x, y\}$.

The anti-commutation relations are given by

\begin{align*}
[e, f] &= h, \quad [h, e] = 2e, \quad [h, f] = -2f, \\
[h, x] &= x, \quad [e, x] = 0, \quad [f, x] = -y, \\
[h, y] &= -y, \quad [e, y] = -x, \quad [f, y] = 0, \\
\{x, x\} &= 2e, \quad \{x, y\} = h, \quad \{y, y\} = -2f.
\end{align*}

Choose the nondegenerate super-symmetric bilinear form $(\cdot, \cdot)$ on $g$ such that nontrivial products are given by

\begin{align*}
(e, f) &= (f, e) = 1, \quad (h, h) = 2, \quad (x, y) = -(y, x) = 2.
\end{align*}

Let $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] + CK$ be the associated affine Lie superalgebra, and $V^k(g)$ (resp. $L_k(g)$) the associated universal (resp. simple) affine vertex superalgebra. As usual, we identify $x \in g$ with $x(-1)1$.

The Sugawara conformal vector of $V^k(osp(1, 2))$ is given by

$$\omega_{\text{sug}} = \frac{1}{2K + 3} \left( : ef : + : fe : + \frac{1}{2} : hh : - \frac{1}{2} : xy : + \frac{1}{2} : yx : \right).$$

The notion of Ramond–twisted modules is defined as usual in the case of affine vertex superalgebras [30] (see also [40]).
Recall also the spectral flow automorphism $\rho$ for $\widetilde{osp}(1|2)$:
\[
\rho e(n) = e(n - 2), \quad \rho x(n) = x(n - 1), \quad \rho f(n) = f(n + 2), \\
\rho y(n) = y(n + 1) \quad \rho h(n) = h(n) - 2\delta_{n,0}K, \quad (n \in \mathbb{Z}).
\]

As in the case of the automorphism $\Psi$ of the $\mathcal{W}_k$, one shows that for any $L_k(osp(1|2))$-module $(M, Y_{\rho^n(1)}(\cdot, z))$ and $n \in \mathbb{Z}$, $\rho^n(M)$ is again a $L_k(osp(1|2))$-module with vertex operator structure given by
\[
Y_{\rho^n(M)}(\cdot, z) := Y_M(\Delta(-n h, z)\cdot, z)
\]
(see also [4, Prop. 2.1] for the proof of similar statement for spectral-flow automorphism of $L_k(sl(2))$, and [10, Prop. 3.1] in the case of $\beta - \gamma$ system).

5.2. Embedding of $\mathcal{W}_1$ into $L_{-5/4}(osp(1|2)) \otimes F$

The free field realization of $L_k(osp(1|2))$ is presented in [3]. In what follows, we will show that the simple Bershadsky–Polyakov algebra $\mathcal{W}_1$ can be embedded into $L_{-5/4}(osp(1|2)) \otimes F$, where $F$ is the Clifford vertex superalgebra introduced in Section 2.3.

Set
\[
\tau^+ =: \Psi^+ x :, \quad \tau^- =: \Psi^- y :.
\]

Let $\alpha =: \Psi^+ \Psi^- :$. Then
\[
\omega_F = \frac{1}{2} : \alpha \alpha :.
\]

Denote by $M_{h^\perp}(1)$ the Heisenberg vertex algebra generated by $h^\perp = \alpha - h$. For $s \in \mathbb{C}$, denote by $M_{h^\perp}(1,s)$ the irreducible $M_{h^\perp}(1)$-module on which $h^\perp(0)$ acts as $s$. Note that $h^\perp(1)h^\perp = -\frac{3}{2}$. The Virasoro vector of central charge $c = 1$ in $M_{h^\perp}(1)$ is $\omega^\perp = -\frac{1}{3} h^\perp(-1)^2$. The following lemma can be proved easily using results from [34], [36].

**Lemma 5.1.** Consider the $M_{h^\perp}(1)$-module $M_{h^\perp}(1,s)$ with the vertex operator $Y_s(\cdot, z)$. Then for every $n \in \mathbb{Z}$
\[
(\overline{M_{h^\perp}(1,s)}, \overline{Y_s(\cdot, z)}) := (M_{h^\perp}(1,s), Y_s(\Delta(-\frac{2n}{3} h^\perp, z)\cdot, z))
\]
is an irreducible $M_{h^\perp}(1)$-module isomorphic to $M_{h^\perp}(1, n + s)$.

**Theorem 5.2.**

(i) There is a nontrivial homomorphism of vertex algebras
\[
\Phi : \mathcal{W}_1 \rightarrow L_{-5/4}(osp(1|2)) \otimes F
\]
uniquely determined by

\[ G^+ = 2\tau^+ = 2 : \Psi^+ x : , \]
\[ G^- = 2\tau^- = 2 : \Psi^- y : , \]
\[ J = \frac{5}{3} \alpha - \frac{2}{3} h , \]
\[ T = \omega_{\text{sug}} + \omega_F - \omega^\perp . \]

(ii) \( \text{Im}(\Phi) \) is isomorphic to the simple vertex algebra \( W_1 \).

(iii) As a \( W_1 \otimes M_{h^+}(1) \)-module,

\[
L_{-5/4}(osp(1|2)) \otimes F \cong \bigoplus_{n \in \mathbb{Z}} \Psi^{-n}(W_1) \otimes M_{h^+}(1, n) \\
= \bigoplus_{n \in \mathbb{Z}} L(x_n, y_n) \otimes M_{h^+}(1, -n).
\]

(iv) We have \( \text{Com}(M_{h^+}(1), L_{-5/4}(osp(1|2)) \otimes F) \cong W_1 \).

Proof. Let \( k = 1 \). Using the formula

\[
(\Psi^+_1 x)_n = \sum_{i=0}^{\infty} (\Psi^+_{1-i} x(n + i) - x(n - i - 1) \Psi^+_i)
\]

we obtain

\[
\tau_2^+ \tau^- = -2k' \mathbb{1},
\]
\[
\tau_1^+ \tau^- = -2k' \alpha - h ,
\]
\[
\tau_0^+ \tau^- = -\alpha h : xy : -2k' : D \Psi^+ \Psi^- : .
\]

From the above formulas and OPE relations for \( \mathcal{W}^k \), it follows that for \( k = 1 \), \( k' \)
needs to be equal to \(-5/4\). For level \( k' = -5/4 \), there exists a conformal embedding
of \( L_{k'}(sl(2)) \) into \( L_{k'}(osp(1|2)) \) (cf. [7]), that is,

\[ \omega_{\text{sug}} = \omega_{sl(2)} , \]

where

\[
\omega_{sl(2)} = \frac{1}{2(k' + 2)} \left( : e f : + : f e : + \frac{1}{2} : h h : \right) .
\]

We have

\[
G_2^+ G^- = 4\tau_2^+ \tau^- = 10 \mathbb{1} = (k + 1)(2k + 3) \mathbb{1},
\]
\[
G_1^+ G^- = 4\tau_1^+ \tau^- = -4 \left( -\frac{5}{2} \alpha + h \right) = 10 \alpha - 4h = 6J = 3(k + 1)J,
\]
\[
G_0^+ G^- = 4\tau_0^+ \tau^- = -4 : h \alpha : -4 : xy : +5 : \alpha \alpha : +5D \alpha .
\]
Using the realization in [3] and the fact that there is a conformal embedding of \( L_{-5/4}(sl(2)) \) into \( L_{-5/4}(osp(1|2)) \), it follows that

\[
\omega_{\text{sug}} = \frac{1}{2}(xy : -yx : ) =: xy : -\frac{1}{2} Dh.
\]

This implies that

\[
T =: xy : -\frac{1}{2} Dh + \frac{1}{2} : \alpha \alpha : + \frac{1}{3} (\alpha \alpha : + hh : -2 : \alpha h : )
\]

\[
=: xy : -\frac{1}{2} Dh + \frac{5}{6} : \alpha \alpha : + \frac{1}{3} : hh : -\frac{2}{3} : \alpha h : .
\]

Hence

\[
G^+_0 G^- = -4 : h\alpha : -4 : xy : + 5 : \alpha \alpha : + 5D\alpha
\]

\[
= 3 : J^2 : + 3 : DJ : -4T
\]

\[
= 3 : J^2 : + \frac{3(k + 1)}{2} : DJ : -(k + 3)T.
\]

This proves assertion (i).

Let us prove that \( \overline{W}_1 = \text{Im}(\Phi) \) is simple.

Let \( \overline{W} = \text{Ker}_{L_{-5/4}(osp(1,2)) \otimes Fh^\perp}(0) \). It is clear that \( \overline{W} \) is a simple vertex algebra which contains \( \overline{W}_1 \otimes M_{h^\perp}(1) \).

The simplicity of \( \overline{W}_1 \) follows from the following claim.

**Claim 1.** \( \overline{W} \) is generated by \( G^+, G^-, J, T, h^\perp \).

For completeness, we shall include a proof of Claim 1.

Let \( U \) be the vertex subalgebra of \( \overline{W} \) generated by \( \{G^+, G^-, J, T, h^\perp\} \). Then clearly \( U \cong \overline{W}_1 \otimes M_{h^\perp}(1) \).

Let \( U^{(n)} \) be the \( U \)-module obtained by the simple current construction

\[
(U^{(n)}, Y^{(n)}(\cdot, z)) := (U, Y(\Delta(n\alpha, \cdot), z)).
\]

Note next that \( \alpha = J - \frac{2}{3} h^\perp \), which implies that

\[
\Delta(\alpha, z) = \Delta\left(-\frac{2}{3} h^\perp, z\right) \Delta(J, z).
\]

- For a \( \overline{W}_1 \)-module \( W \), by applying the operator \( \Delta(nJ, z) \), we get the module \( \Psi^{-n}(W) \).
- Using Lemma 5.1, we see that by applying the operator \( \Delta(-\frac{2}{3} n h^\perp, z) \) on \( M_{h^\perp}(1) \), we get the module \( M_{h^\perp}(1) \)-module \( M_{h^\perp}(1, n) \).

We get

\[
U^{(n)} = \Psi^{-n}(\overline{W}_1) \otimes M_{h^\perp}(1, n).
\]
Since by the boson-fermion correspondence $F$ is isomorphic to the lattice vertex superalgebra $V_{2\alpha}$, we get that $U^{(n)}$ is realized inside of $L_{-5/4}(osp(1|2)) \otimes F$:

$$U^{(n)} \cong U.e^{n\alpha}.$$  

Note that $h^\perp(0) \equiv n \text{Id}$ on $U^{(n)}$. Using H. Li construction from [36] (see also [28]), we get that

$$U = \bigoplus U^{(n)}$$  

is a vertex subalgebra of $L_{-5/4}(osp(1|2)) \otimes F$. But one shows that $U$ contains all generators of $L_{-5/4}(osp(1|2)) \otimes F$, so

$$U = L_{-5/4}(osp(1|2)) \otimes F.$$  

This proves that $U = \overline{W} \cong \overline{W}_1 \otimes M_{h^\perp}(1)$. Since $\overline{W}$ is simple, we have that $\overline{W}_1$ is simple, and therefore isomorphic to $W_1$. This proves Claim 1.

The decomposition (iii) follows from relations (3)–(5). The assertion (iv) follows directly from (iii). □

The proof of the following result is similar to the one given in [4, Thm. 6.2] and [10, Thm. 5.1].

**Theorem 5.3.** Assume that $N$ (resp. $N^{tw}$) is an irreducible, untwisted (resp. Ramond twisted) $L_{-5/4}(osp(1|2))$-module such that $h(0)$ acts semisimply on $N$ and $N^{tw}$:

$$N = \bigoplus_{s \in \mathbb{Z} + \Delta} N^s, \quad h(0)|N^s \equiv s \text{Id} \quad (\Delta \in \mathbb{C}),$$

$$N^{tw} = \bigoplus_{s \in \mathbb{Z} + \Delta'} (N^{tw})^s, \quad h(0)|(N^{tw})^s \equiv s \text{Id} \quad (\Delta' \in \mathbb{C}).$$

Then $N \otimes F$ and $N^{tw} \otimes M^{tw}_F$ are completely reducible $W_1$-modules:

$$N \otimes F = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(N), \quad \mathcal{L}_s(N) = \{ v \in N \otimes F \mid h^\perp(0)v = (s + \Delta)v \},$$

$$N^{tw} \otimes M^{tw}_F = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(N^{tw}), \quad \mathcal{L}_s(N^{tw}) = \{ v \in N^{tw} \otimes M^{tw}_F \mid h^\perp(0)v = (s + \Delta')v \},$$

and each $\mathcal{L}_s(N)$ and $\mathcal{L}_s(N^{tw})$ are irreducible $W_1$-modules.

**5.3. Embedding of $L_{-5/4}(osp(1|2))$ into $W_1 \otimes F_{-1}$**

In this section, we consider the tensor product of $W_1$ with the lattice vertex superalgebra $F_{-1}$ (defined in Section 2.4). We will show that the simple affine vertex superalgebra $L_{-5/4}(osp(1|2))$ can be realized as a subalgebra of $W_1 \otimes F_{-1}$.

Let $\overline{h} = J + \varphi$. Then for $n \geq 0$ we have $\overline{h}(n)\overline{h} = \frac{2}{3} \delta_{n,1}1$. Let $M_{\overline{h}}(1)$ be the Heisenberg vertex algebra generated by $\overline{h}$, and $M_{\overline{h}}(1, s)$ the irreducible highest weight $M_{\overline{h}}(1)$-module on which $\overline{h}(0) \equiv s \text{Id}$. 

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There is a homomorphism of vertex algebras $\Phi^{\text{inv}} : L_{-5/4}(osp(1|2)) \to \mathcal{W}_1 \otimes F_{-1}$ uniquely determined by

$$
\begin{align*}
  x &= \frac{1}{2} : G^+ e^{\varphi} :, \\
  y &= -\frac{1}{2} : G^- e^{-\varphi} :, \\
  e &= \frac{1}{8} : (G^+) e^{2\varphi} :, \\
  f &= -\frac{1}{8} : (G^-) e^{-2\varphi} :, \\
  h &= -\frac{3}{2} J - \frac{5}{2} \varphi.
\end{align*}
$$

We have $\mathcal{W}_1 \otimes F_{-1} \cong \bigoplus_{n \in \mathbb{Z}} \rho^n(L_{-5/4}(osp(1|2))) \otimes M_{\mathcal{F}}(1,-n)$.

Direct calculation shows that

$$
\begin{align*}
  x(1)y &= -\frac{1}{4} G_2^+ G = -\frac{(k + 1)(2k + 3)}{4} = -\frac{5}{4} (x,y) \mathbf{1} = -\frac{5}{2} \mathbf{1}, \\
  x(0)y &= -\frac{1}{4} (\varphi(-1)G_2^+ G + G_1^+ G) = -\frac{(k + 1)(2k + 3)}{4} \varphi - \frac{3(k+1)}{4} J \\
  &= -\frac{5}{2} \varphi - \frac{3}{2} J = h, \\
  x(0)x &= 2e, \\
  x(0)e &= 0, \\
  y(0)y &= -2f, \\
  y(0)e &= 0, \\
  y(0)f &= 0 \\
  (\text{note that above we used relations } (G^\pm)^3 = 0), \\
  h(0)x &= x, \\
  h(0)y &= -y, \\
  h(1)h &= \frac{9}{4} J_1 J + \frac{25}{4} \varphi(1) \varphi = \left( \frac{9}{4} \frac{2k+3}{3} - \frac{25}{4} \right) = -\frac{5}{2} \mathbf{1} = -\frac{5}{4} (h,h) \mathbf{1}, \\
  h(0)e &= -\frac{3}{16} J_0 (G^+) e^{2\varphi} - \frac{5}{16} (G^+) \varphi(0) e^{2\varphi} \\
  &= -\frac{3}{8} (G^+) e^{2\varphi} + \frac{5}{8} (G^+) e^{2\varphi} = \frac{1}{4} (G^+) e^{2\varphi} = 2e, \\
  h(0)f &= \frac{3}{16} J_0 (G^-) e^{-2\varphi} + \frac{5}{16} (G^-) \varphi(0) e^{-2\varphi} \\
  &= -\frac{3}{8} (G^-) e^{-2\varphi} + \frac{5}{8} (G^-) e^{-2\varphi} = \frac{1}{4} (G^-) e^{-2\varphi} = -2f.
\end{align*}
$$

Proof. We need to show that the following relations hold for $n \geq 0$:

$$
\begin{align*}
  h(n)x &= \delta_{n,0} x, \\
  e(n)x &= 0, \\
  x(n)f &= \delta_{n,0} y, \\
  h(n)y &= -\delta_{n,0} y, \\
  e(n)y &= -\delta_{n,0} x, \\
  f(n)y &= 0, \\
  x(n)x &= 2\delta_{n,0} e, \\
  y(n)y &= -2\delta_{n,0} f, \\
  x(0)y &= h, \\
  x(1)y &= -\frac{5}{2} \mathbf{1}, \\
  h(0)e &= 2e, \\
  h(0)f &= -2f, \\
  e(0)f &= h.
\end{align*}
$$

Theorem 5.4.
We use the formula (cf. [9, Sect. 8.1])
\[ G_2^+(G^-)^n = 2n(k - (n - 2))(k - (n - 2) + n/2)(G^-)^{n-1}, \]
\[ G_1^+(G^-)^n = 3n(k - (n - 2))J_1(G^-)^{n-1} + n(n - 1)(k - (n - 2))G_{-2}(G^-)^{n-2}, \]
which implies that for \( n = 2 \) and \( k = 1 \) we get
\[ G_2^+(G^-)^2 = (4 \ast 1 \ast 2)G^- = 8G^- .\]

We have
\[ x(0)f = -\frac{1}{16} G_2^+(G^-)^2 e^{-\varphi} = -\frac{1}{16} 8G^- e^{-\varphi} = -\frac{1}{2} G^- e^{-\varphi} = y, \]
\[ e(0)y = -\frac{1}{16} (2G_{-1} G_2^+ G^- e^\varphi - 2G_0^+ G_1^+ G^- e^\varphi) \]
\[ = -\frac{1}{16} (2(k + 1)(2k + 3)G^+ e^\varphi + 6(k + 1)G_0^+ J_1 e^\varphi) \]
\[ = -\frac{1}{16} (20G^+ e^\varphi - 12G^+ e^\varphi) = -\frac{1}{2} G^+ e^\varphi = -x, \]
\[ e(0)f = -\frac{1}{64} (2G_1^+ G_2^+(G^-)^2 + (G_2^+)^2(G^-)^2 2\varphi) = -\frac{1}{64} (16G_1^+ G^- + 16G_2^+ G^- \varphi) \]
\[ = -\frac{1}{64} (16 \cdot 3(k + 1)J + 16(k + 1)(2k + 3)\varphi) = -\frac{3}{2} J - \frac{5}{2} \varphi = h. \]

The operator \( \overline{h}(0) \) acts semisimply on \( W_1 \otimes F_{-1} \) and we have the following decomposition:
\[ W_1 \otimes F_{-1} = \bigoplus_{n \in \mathbb{Z}} Z^{(n)}, \quad Z^{(n)} = \{ v \in W_1 \otimes F_{-1} \mid \overline{h}(0)v = nv \}. \]

We have that \( Z^{(0)} \) is a simple vertex superalgebra and each \( Z^{(n)} \) is a simple \( Z^{(0)} \)-module. The rest of the proof follows from the following claim.

Claim 2. \( Z^{(0)} = \ker_{W_1 \otimes F_{-1}} \overline{h}(0) \) is generated by \( \{ x, y, e, f, h, \overline{h} \} \).

The proof of Claim 2 is completely analogous to the proof of Claim 1. These arguments are also similar to those in [4, Thm. 6.2], [6, Prop. 5.4], [18, Sect. 5] in a slightly different setting.

Claim 2 implies that
\[ Z^{(0)} \cong L_{-5/4}(osp(1|2)) \otimes M_{\overline{h}}(1). \]

As in the proof of Theorem 5.2, using formula
\[ \Delta(n\varphi, z) = \Delta(-nh, z) \Delta\left( -\frac{3n}{2} \overline{h}, z \right), \]
we get
\[ Z^{(-n)} = \rho^n(L_{-5/4}(osp(1|2))) \otimes M_{\overline{h}}(1, -n). \]

The decomposition (ii) follows now from relations (6) and (7). The assertion (iii) is a direct consequence of (ii).

This concludes the proof of the theorem. □
5.4. Parafermion subalgebras
Following [11] we define
\[ C_\ell = \text{Com}(M_J(1), \mathcal{W}_{\ell-3/2}(sl(3), f_{\text{min}})), \]
where \( M_J(1) \) is the Heisenberg subalgebra of \( \mathcal{W}_{\ell-3/2}(sl(3), f_{\text{min}}) \) generated by the Heisenberg field \( J(z) \). For \( g = sl(2) \) or \( g = osp(1|2) \), let
\[ N_k(g) = \text{Com}(M_h(1), L_k(g)) \]
be the parafermion subalgebra of \( L_k(g) \), where \( M_h(1) \) is the Heisenberg subalgebra generated by the field \( h(z) \). Set \( k' = -5/4 \) as before. Using the following decomposition of the conformal embedding from [7],
\[ L_{k'}(osp(1|2)) = L_{k'}(sl(2)) \oplus L_{k'}(\omega_1), \]
we get \( N_{k'}(osp(1|2)) = N_{k'}(sl(2)) \).
The results of our paper give an explicit realization of the isomorphism
\[ N_{k'}(sl(2)) \cong C_{5/2}. \]
Indeed, from Theorem 5.2 we get the embedding
\[ \Phi|_{N_{k'}(sl(2))} : N_{k'}(sl(2)) \hookrightarrow C_{5/2}. \]
Theorem 5.4 gives the opposite embedding
\[ \Phi^{\text{inv}}|_{C_{5/2}} : C_{5/2} \hookrightarrow N_{k'}(sl(2)). \]
The same conclusion can be also obtained from [38]:
- [38, Thm. 10.3] implies that \( N_{k'}(sl(2)) = \mathcal{W}_{k'}(sl(5), f_{\text{princ}}) \),
- [38, Thm. 10.4] implies that \( C_{5/2} = \mathcal{W}_{k'}(sl(5), f_{\text{princ}}) \).

6. From relaxed \( L_{-5/4}(osp(1|2)) \)-modules to ordinary \( \mathcal{W}_1 \)-modules

The free-field realization of the affine vertex superalgebra \( L_{k'}(osp(1|2)) \) and its irreducible modules was obtained in [3]. Specifically for \( k' = -5/4 \), it holds that \( L_{k'}(osp(1|2)) \) can be realized on the vertex superalgebra \( F^{1/2} \otimes \Pi^{1/2}(0) \), where \( \Pi^{1/2}(0) \) is a certain lattice type vertex algebra (cf. [3, Thm. 11.3]). All irreducible \( L_{k'}(osp(1|2)) \)-modules can be constructed using this realization.

In this section, we will construct an explicit realization of irreducible \( \mathcal{W}_1 \)-modules, using the fact that the Bershadsky–Polyakov algebra \( \mathcal{W}_1 \) can be embedded into \( L_{k'}(osp(1|2)) \otimes F \), where \( F \) is a Clifford vertex superalgebra (cf. Theorem 5.2).

6.1. Relaxed \( L_{k'}(osp(1|2)) \)-modules

The vertex algebra \( \Pi^{1/2}(0) \) was introduced in [3], where
\[ \Pi^{1/2}(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}^C_2]. \]
It is closely related to the lattice-type vertex algebra \( \Pi(0) \) from [14]. Here \( c := \frac{2}{k}(\mu - \nu) \), where \( \mu, \nu \) satisfy \( \langle \mu, \mu \rangle = -\langle \nu, \nu \rangle = \frac{k}{2}, \langle \mu, \nu \rangle = \langle \nu, \mu \rangle = 0 \). It is easy to see that \( g = \exp(\pi i \mu(0)) \) is an automorphism of order two for \( \Pi^{1/2}(0) \).

We will need the following fact about \( \Pi^{1/2}(0) \)-modules from [3].
Proposition 6.1 ([3, Prop. 4.1]). Let $\lambda \in \mathbb{C}$ and $g = \exp(\pi i \mu(0))$. Then $\Pi_{-1}^{1/2}(\lambda) := \Pi_{-1}^{1/2}(0)e^{-\mu + \lambda c}$ is an irreducible $g$-twisted $\Pi^{1/2}(0)$-module.

Using the realization of $\mathcal{W}_1$, we have the following.

Lemma 6.2. If $U^{tw}$ is any $g$-twisted $\Pi^{1/2}(0)$-module, then $F \otimes M_{F^{1/2}}^\pm \otimes U^{tw}$ and $M_{F^w}^\pm \otimes F^{1/2} \otimes U^{tw}$ are $\mathcal{W}_1$-modules.

Proof. In [3, Cor. 13.1] it was proved that if $U^{tw}$ is any $g$-twisted $\Pi^{1/2}(0)$-module, then $M_{F^{1/2}}^\pm \otimes U^{tw}$ is an untwisted $L_{-5/4}(osp(1,2))$-module, and $F^{1/2} \otimes U^{tw}$ is a Ramond twisted $L_{-5/4}(osp(1,2))$-module. The claim now follows from the realization of the vertex algebra $\mathcal{W}_1$ in Theorem 5.2. \qed

We will consider the following $F^{1/2} \otimes \Pi(0)$-modules:

- $\sigma_{F^{1/2}} \otimes g$-twisted module $\mathcal{F}_\lambda := M_{F^{1/2}}^\pm \otimes \Pi_{-1}^{1/2}(\lambda)$,
- $g = 1 \otimes g$-twisted module $\mathcal{E}_\lambda := F^{1/2} \otimes \Pi_{-1}^{1/2}(\lambda)$.

First we recall the result from [3].

Proposition 6.3 ([3, Thm. 13.2]). $\mathcal{F}_\lambda$ is an untwisted, relaxed $L_{-5/4}(osp(1,2))$-module. $\mathcal{F}_\lambda$ is irreducible if and only if $\lambda \notin \frac{1}{2} + \frac{1}{2}\mathbb{Z}$.

Using irreducibility of relaxed $L_{-5/4}(sl(2))$-modules we get the following proposition.

Proposition 6.4.

1. $\mathcal{E}_\lambda$ is a Ramond twisted $L_{-5/4}(osp(1,2))$-module.
2. $\mathcal{E}_\lambda = \mathcal{E}_\lambda^0 \oplus \mathcal{E}_\lambda^1$, where as $L_{-5/4}(sl(2))$-modules
   \[
   \mathcal{E}_\lambda^0 = L^{Vir}(\frac{1}{2}, 0) \otimes \Pi_{-1}(\lambda) \bigoplus L^{Vir}(\frac{1}{2}, \frac{1}{2}) \otimes \Pi_{-1}(\lambda + \frac{1}{2}),
   \]
   \[
   \mathcal{E}_\lambda^1 = L^{Vir}(\frac{1}{2}, 0) \otimes \Pi_{-1}(\lambda + \frac{1}{2}) \bigoplus L^{Vir}(\frac{1}{2}, \frac{1}{2}) \otimes \Pi_{-1}(\lambda).
   \]
3. $\mathcal{E}_\lambda^0$ (resp. $\mathcal{E}_\lambda^1$) is irreducible Ramond twisted $L_{-5/4}(osp(1,2))$-modules if $\lambda \notin \mathbb{Z} \cup (-\frac{1}{4} + \frac{1}{2}\mathbb{Z})$ (resp. $\lambda + \frac{1}{2} \notin \mathbb{Z} \cup (-\frac{1}{4} + \frac{1}{2}\mathbb{Z})$).

Proof. Using Proposition 6.1 and Lemma 6.2, we see that $\mathcal{E}_\lambda$ is Ramond twisted $L_{-5/4}(osp(1|2))$-module. This proves (1). Since as a module for the Virasoro vertex algebra $L^{Vir}(\frac{1}{2}, 0)$
\[
F^{1/2} = L^{Vir}(\frac{1}{2}, 0) \oplus L^{Vir}(\frac{1}{2}, \frac{1}{2}),
\]
and as a $\Pi(0)$-module
\[
\Pi_{-1}^{1/2}(\lambda) = \Pi_{-1}(\lambda) \oplus \Pi_{-1}(\lambda + \frac{1}{2}),
\]
we easily get the decomposition in (2). Using the irreducibility results from [3] and [32], we get that as $L_{-5/4}(sl(2))$-modules:

- $L^{Vir}(\frac{1}{2}, 0) \otimes \Pi_{-1}(\lambda)$ is irreducible iff $\lambda \notin -\frac{1 \pm 1}{8} + \mathbb{Z}$,
- $L^{Vir}(\frac{1}{2}, \frac{1}{2}) \otimes \Pi_{-1}(\lambda + 1)$ is irreducible iff $\lambda + \frac{1}{2} \notin -\frac{1 \pm 1}{8} + \mathbb{Z}$,
- $L^{Vir}(\frac{1}{2}, \frac{1}{2}) \otimes \Pi_{-1}(\lambda)$ is irreducible iff $\lambda \notin -\frac{1 \pm 5}{8} + \mathbb{Z}$,
- $L^{Vir}(\frac{1}{2}, \frac{1}{2}) \otimes \Pi_{-1}(\lambda + 1)$ is irreducible iff $\lambda + \frac{1}{2} \notin -\frac{1 \pm 5}{8} + \mathbb{Z}$. 

One easily see that all the modules appearing above are irreducible as \( L_{-5/4}(sl(2)) \)-modules if and only if \( \lambda \notin \frac{1}{4} \mathbb{Z} \). Using the decomposition

\[
L_{-5/4}(osp(1\,|\,2)) = L_{-5/4}(sl(2)) \oplus L_{-5/4}(\omega_1),
\]

we easily see that as (Ramond twisted) \( L_{-5/4}(osp(1\,|\,2)) \)-modules

- \( \mathcal{E}^0_\lambda \) is irreducible iff \( \lambda \notin \mathbb{Z} \cup (-\frac{1}{4} + \mathbb{Z}) \),
- \( \mathcal{E}_\lambda^1 \) is irreducible iff \( \lambda + \frac{1}{2} \notin \mathbb{Z} \cup (-\frac{1}{4} + \mathbb{Z}) \).

The proof follows. \[\square\]

### 6.2. Explicit realization of \( \mathcal{W}_1 \)-modules

From Theorem 4.13 it follows that the set

\[
\{ L(x, y) \mid (x, y) \in \mathbb{C}^2, \ h_i(x, y) = 0, \ 1 \leq i \leq 3 \}
\]

is the set of all irreducible ordinary \( \mathcal{W}_1 \)-modules. Now we will construct explicit realizations of these modules, using results from the previous subsection.

**Lemma 6.5.** The irreducible highest weight \( \mathcal{W}_1 \) modules

\[
T_{(2)} := \{ L(x, y) \mid (x, y) \in \mathbb{C}^2, \ h_2(x, y) = 0 \}
\]

are realized as irreducible quotients of

\[
U_{(2)}(\lambda) = \mathcal{W}_1.E^\lambda_2, \ \ \lambda \in \mathbb{C},
\]

where \( E^\lambda_2 = 1_F \otimes 1_M^{tw}_{F^{1/2}} \otimes e^{-\mu + \lambda c} \) are highest weight vectors for \( \mathcal{W}_1 \) of highest weight

\[
(x_\lambda, y_\lambda) := \left( -\frac{2}{3}(-k + 2\lambda), -\frac{1}{4} + \frac{1}{3}(-k + 2\lambda)^2 + \frac{1}{3}(-k + 2\lambda) \right). \quad (8)
\]

**Proof.** Consider the \( \sigma_{F^{1/2}} \otimes g \)-twisted \( F \otimes F^{1/2} \otimes \Pi^{1/2}(0) \)-module \( \mathcal{F}_{(2)}(\lambda) := F \otimes M_{F^{1/2}}^{\pm} \otimes \Pi^{1/2}_{-1}(\lambda) \). Then \( \mathcal{F}_{(2)}(\lambda) \) is an untwisted \( \mathcal{W}_1 \)-module. It holds that (cf. [3])

\[
h(n)E^\lambda_2 = \delta_{n,0}(-k + 2\lambda)E^\lambda_2,
\]

\[
L_{sug}(n)E^\lambda_2 = -\frac{1}{4} \delta_{n,0}E^\lambda_2, \quad n \in \mathbb{Z}_{\geq 0}.
\]

We have

\[
J(0)E^\lambda_2 = -\frac{2}{3}(-k + 2\lambda)E^\lambda_2,
\]

\[
L(0)E^\lambda_2 = \left( -\frac{1}{4} + \frac{1}{3}(-k + 2\lambda)^2 + \frac{1}{3}(-k + 2\lambda) \right) E^\lambda_2
\]

\[
= \frac{1}{12} (2(-k + 2\lambda) + 3)(2(-k + 2\lambda) - 1) E^\lambda_2.
\]
Set $x_{\lambda} := -\frac{2}{3}(-k + 2\lambda)$ and $y_{\lambda} := \frac{1}{12}(2(-k + 2\lambda) + 3)(2(-k + 2\lambda) - 1)$, so that

$$J(0)E_2^\lambda = x_{\lambda}E_2^\lambda, \quad L(0)E_2^\lambda = y_{\lambda}E_2^\lambda.$$ 

Since $y = \frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{4}$, the pair $(x_{\lambda}, y_{\lambda}) \in \mathbb{C}^2$ satisfies the relation

$$h_2(x, y) = -3x^2 + 2x + 1 + 4y = 0.$$

Hence $W_1$ has a family of highest weight modules $U_{(2)}(\lambda), \lambda \in \mathbb{C}$, with highest weights $(x, y)$. In particular, their irreducible quotients $L(x, y)$ are also modules for $W_1$. □

We have the following irreducibility result.

**Proposition 6.6.** Assume that $\lambda \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$. Then $F_\lambda \otimes F$ is a completely reducible $W_1 \otimes M_{h_+}(1)$-module:

$$F_\lambda \otimes F \cong \bigoplus_{n \in \mathbb{Z}} \Psi^{-n}(L(x_{\lambda}, y_{\lambda})) \otimes M_{h_+}(1, \Delta + n)$$

where $\Delta = k - 2\lambda$ and weights $(x_{\lambda}, y_{\lambda})$ are given by (8). In particular, $U_{(2)}(\lambda)$ is an irreducible $W_1$-module and it holds that

$$U_{(2)}(\lambda) = L(x_{\lambda}, y_{\lambda}).$$

**Proof.** Since $F_\lambda$ is an irreducible $L_{-5/4}(osp(1, 2))$-module for $\lambda \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$, (cf. Proposition 6.3), by applying Theorem 5.3 we see that $F_\lambda \otimes F$ is a completely reducible $W_1 \otimes M_{h_+}(1)$-module:

$$F_\lambda \otimes F \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n(F_\lambda),$$

where

$$
\mathcal{L}_n(F_\lambda) = \{v \in F_\lambda \otimes F \mid h_+^+ (0)v = (n + \Delta)v \}
$$
is an irreducible $W_1 \otimes M_{h_+}(1)$-module. By using Lemma 6.5, we see that $\mathcal{L}_0(F_\lambda)$ must be isomorphic to the irreducible highest weight module $L(x_{\lambda}, y_{\lambda}) \otimes M_{h_+}(1, \Delta)$. Since $\Psi^{-n}(W_1) \otimes M_{h_+}(1, n)$ are simple-current $W_1 \otimes M_{h_+}(1)$-modules, we get that

$$
\mathcal{L}_n(F_\lambda) = (\Psi^{-n}(W_1) \otimes M_{h_+}(1, n)) \times (L(x_{\lambda}, y_{\lambda}) \otimes M_{h_+}(1, \Delta))
= \Psi^{-n}(L(x_{\lambda}, y_{\lambda})) \otimes M_{h_+}(1, \Delta + n)
$$

for every $n \in \mathbb{Z}$. (Here “×” denotes the fusion product in the category of $W_1 \otimes M_{h_+}(1)$-modules).

The proof follows. □

**Lemma 6.7.** $W_1$ has a family of irreducible highest weight modules

$$T_{(1)} := \{L(x, y) \mid (x, y) \in \mathbb{C}^2, \ h_1(x, y) = 0\}$$

which are realized as irreducible quotients of

$$U_{(1)}(\lambda) = W_1.E_1^\lambda, \quad \lambda \in \mathbb{C},$$

where $E_1^\lambda := 1_{F_1^0} \otimes 1_{F_{1/2}} \otimes e^{-\mu + \lambda c}$ are highest weight vectors for $W_1$. 

Proof. Consider the $\sigma_F \otimes g$-twisted $F \otimes F^{1/2} \otimes \Pi^{1/2}(0)$-module $\mathcal{F}(\lambda) := M_{F}^{tw} \otimes F^{1/2} \otimes \Pi^{1/2}(\lambda)$, $\lambda \in \mathbb{C}$. Then $\mathcal{F}(\lambda)$ is an untwisted $\mathcal{W}_1$-module.

Let $Y^{tw}_F(\omega, z) = Y(\Delta(h, z)\omega, z)$ and set $h = \frac{a}{2}$. We have

$$\Delta\left(\frac{\alpha}{2}, z\right)\omega = \omega + \frac{\alpha}{2} z^{-1} + \frac{1}{2} \frac{\alpha}{2} (1) \frac{\alpha}{2} z^{-2} = \omega + \frac{\alpha}{2} z^{-1} + \frac{1}{8} z^{-2}\mathbb{I},$$

hence $L(0)^{tw} = 1/8\mathbb{I}^{tw}$.

Similarly,

$$\Delta\left(\frac{\alpha}{2}, z\right)\alpha = \alpha + \frac{1}{2} \alpha(1)\alpha(-1) z^{-1} = \alpha + \frac{1}{2} z^{-1}\mathbb{I},$$

hence $\alpha(0)^{tw} = 1/2\mathbb{I}^{tw}$.

It holds that

$$h(n)E_1^{\lambda} = \delta_{n, 0}(\delta + 2\lambda)E_1^{\lambda},$$

$$L_{sug}(n)E_1^{\lambda} = -\frac{5}{16} \delta_{n, 0}E_1^{\lambda}, \quad n \in \mathbb{Z}_{\geq 0}.$$

We have

$$J(0)E_1^{\lambda} = \left(\frac{5}{6} - \frac{2}{3}(\delta + 2\lambda)\right)E_1^{\lambda},$$

$$L(0)E_1^{\lambda} = \left(\frac{5}{16} + \frac{1}{8} + \frac{1}{3}(\frac{1}{2} - (\delta + 2\lambda))^2 - \frac{5}{12} + \frac{1}{3}(\delta + 2\lambda)\right)E_1^{\lambda}$$

$$= \frac{1}{48}(4(\delta + 2\lambda) - 5)(4(\delta + 2\lambda) + 5)E_1^{\lambda}.$$

Set

$$x := x_\lambda = \frac{5}{6} - \frac{2}{3}(\delta + 2\lambda),$$

$$y := y_\lambda = \frac{1}{48}(4(\delta + 2\lambda) - 5)(4(\delta + 2\lambda) + 5)$$

so that

$$J(0)E_1^{\lambda} = x_\lambda E_1^{\lambda}, \quad L(0)E_1^{\lambda} = y_\lambda E_1^{\lambda}.$$

Since $y = \frac{3}{4}x^2 - \frac{5}{4}x$, the pair $(x_\lambda, y_\lambda) \in \mathbb{C}^2$ satisfies the relation

$$h_1(x, y) = -3x^2 + 5x + 4y = 0.$$

Hence $\mathcal{W}_1$ has a family of highest weight modules $U_{(\lambda)}$, $\lambda \in \mathbb{C}$ with highest weights $(x, y)$. In particular, their irreducible quotients $L(x, y)$ are also modules for $\mathcal{W}_1$. \qed

Using a twisted variant of Theorem 5.3 and Proposition 6.4, we get the following irreducibility result.
Proposition 6.8. Assume that $\lambda \notin \mathbb{Z} \cup (-\frac{1}{4} + \mathbb{Z})$. Then $E^0_\lambda \otimes M^w_F$ is a completely reducible $W_1 \otimes M_{h_\perp}(1)$-module:

$$E^0_\lambda \otimes M^w_F \cong \bigoplus_{n \in \mathbb{Z}} \Psi^{-n}(L(x_\lambda, y_\lambda)) \otimes M_{h_\perp}(1, \Delta' + n)$$

where $\Delta' = \frac{1}{2} + k - 2\lambda$ and weights $(x_\lambda, y_\lambda)$ are given by (10)-(11). In particular, $U(1)(\lambda)$ is an irreducible $W_1$-module and it holds that

$$U(1)(\lambda) = L(x_\lambda, y_\lambda).$$

Remark 6.9. Theorem 4.13 implies that there exists another family of irreducible highest weight $W_1$-modules $L(x, y)$, for which it holds that $h_3(x, y) = 0$. Indeed, these modules can be obtained from $T(1)$ using the spectral flow automorphism $\psi^{-1}$ as

$$T(3) := \{\psi^{-1}(L(\bar{x}_1, \bar{y}_1)) \mid (x, y) \in \mathbb{C}^2, h_1(x, y) = 0\}.$$  

From Lemma 4.4, it easily follows that $T(3) = \{L(x, y) \mid (x, y) \in \mathbb{C}^2, h_3(x, y) = 0\}$. These modules also appear in the decomposition in Proposition 6.8.

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