An inverse problem for a three-dimensional heat equation in thermal imaging and the enclosure method

Masaru IKEHATA∗and Mishio KAWASHITA†

Abstract

This paper studies a prototype of inverse initial boundary value problems whose governing equation is the heat equation in three dimensions. An unknown discontinuity embedded in a three-dimensional heat conductive body is considered. A single set of the temperature and heat flux on the lateral boundary for a fixed observation time is given as an observation datum. It is shown that this datum yields the minimum length of broken paths that start at a given point outside the body, go to a point on the boundary of the unknown discontinuity and return to a point on the boundary of the body under some conditions on the input heat flux, the unknown discontinuity and the body. This is new information obtained by using enclosure method.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^3$ with $C^{2,\alpha_0}$ boundary and $0 < \alpha_0 \leq 1$. Let $D$ be an open subset of Ω with $C^{2,\alpha_0}$ boundary and satisfy that: $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ is connected. We denote by $\nu_x$, $\nu_y$ the unit outward normal vectors at $x \in \partial D$, $y \in \partial \Omega$ on $\partial D$, $\partial \Omega$, respectively. Let $T$ be an arbitrary fixed positive number and $\rho = \rho(x) \in C^{0,\alpha_0}$. Given $f \in L^1(0,T;H^{-1/2}((\partial \Omega)))$ let $u = u(x,t)$ be the weak solution of

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= 0 \text{ in } (\Omega \setminus \overline{D}) \times ]0, T[,
\frac{\partial u}{\partial \nu} + \rho u &= 0 \text{ on } \partial D \times ]0, T[,
\frac{\partial u}{\partial \nu} &= f \text{ on } \partial \Omega \times ]0, T[,
u(x,0) &= 0 \text{ in } \Omega \setminus \overline{D}.
\end{aligned}
$$

(1.1)
For detailed information about the weak solution which follows [4], see subsection 1.5 in this paper.

This paper is concerned with the following problem.

**Inverse Problem.** Fix $T > 0$. Assume that both $D$ and $\rho$ are unknown. Extract information about the location and shape of $D$ from the temperature $u$ on $\partial \Omega$ over finite time interval $[0, T]$ with a fixed known heat flux $f$.

This is a prototype of several inverse problems related to thermal imaging, dynamical remote sensing and very important one. $D$ is a mathematical model of unknown discontinuity embedded in a three-dimensional heat conductive body. There are extensive mathematical studies of uniqueness and stability issues of Inverse Problem. In particular, it is known that the observation data uniquely determine general $D$ itself under a suitable condition on the heat flux on $\partial \Omega$ in the case when $\rho \equiv 0$. See Bryan-Caudill [1], Canuto-Rosset-Vessella [3], Vessella [15] and his survey paper [16] together with references therein for more information about these issues.

1.1 An interpretation of previous one-space dimensional result

In [7] Ikehata started a study that seeks an analytical and constructive approach for the inverse problem. He considered a one-space dimensional version of the problem and related ones. The method used therein is called the *enclosure method* which was introduced by himself in [5, 6]. The enclosure method aims at extracting a domain that encloses an unknown discontinuity, such as inclusion, cavity or crack in a known background medium by observing a signal propagating inside the medium on the boundary of the surface surrounding the medium. Then the Dirichle-to-Neumann map associated with the governing equation of the used signal appears as an idealized mathematical model of the observed data. The enclosure method constructs the so-called the *indicator function* by using the Dirichlet-to-Neumann map or its partial knowledge combined with the *complex geometrical optics solution* of the governing equation. The indicator function has an independent variable which is contained in the complex geometrical optics solution as a large parameter. The complex geometrical optics solution changes its *growing* and *decaying* property as the parameter goes to infinity bordering on, for example, a plane in three dimensions. The behaviour of the indicator function as the independent variable goes to infinity depends on the relative position of the plane to unknown discontinuity and enables us to obtain an enclosing domain. In this sense this original enclosure method can be considered as a method of using the complex geometrical optics solutions. However, note that the way of using this growing and decaying character positively differs from the well known method which goes back to Calderón [2] and Sylvester-Uhlmann [14] since their method is based on the oscillating character of the complex geometrical optics solutions about the parameter.

Now let us describe one of the problems considered in [7]. Let $u = u(x, t)$ with $u_x(0, t) \in$
Let $L^2(0, T)$ be a solution of the problem
\[
\begin{aligned}
&\begin{cases}
u_t = \nu_{xx}\ 	ext{in}\ [0, a[\times]0, T[,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
&\begin{cases}
u_x(a, t) + \rho \nu(a, t) = 0 \quad \text{for} \ t \in ]0, T[, \\
u(x, 0) = 0 \quad \text{in} \ [0, a[.
\end{cases}
\end{aligned}
\]  \tag{1.2}

It is assumed that both constants $a > 0$ and $\rho \in \mathbb{R}$ in (1.2) are unknown. He considered the problem: extract $a$ from $u(0, t)$ and $u_x(0, t)$ for $0 < t < T$. This inverse problem is the one dimensional version of our inverse problem for (1.1). In (1.2), sets $]0, \infty[\times]0, T]$ and $\{a\}$ correspond to $\Omega$, $D$ and $\partial D$ respectively.

In [7], to extract $a$ from $u(0, t)$ and $u_x(0, t)$ ($0 < t < T$), he introduced an indicator function $I(\tau)$ of independent variable $\tau > 0$ given by the integral
\[
I(\tau) = \int_0^T (v_x(0, t) u(0, t) + u_x(0, t) v(0, t)) \, dt,
\]
where $v = v(x, t)$ is a solution of the one-dimensional backward heat equation $v_t + v_{xx} = 0$ of the following form:
\[
v(x, t) = e^{-\tau^2 t} e^{-\tau x}.
\]
For this indicator function $I(\tau)$, he showed that the formula
\[
\lim_{\tau \to -\infty} \frac{1}{\tau} \log |I(\tau)| = -2a
\]  \tag{1.3}
is valid under the condition on $u_x(0, t)$: there exists a constant $\beta_0 \in \mathbb{R}$ such that
\[
\liminf_{\tau \to -\infty} \tau \beta_0 \left| \int_0^T u_x(0, t) e^{-\tau^2 t} \, dt \right| > 0.
\]  \tag{1.4}

Formula (1.3) means that the exact location of the unknown boundary $\{a\}$ of the inside cavity $]a, \infty[$ can be detected by a single set of $u(0, t)$ and $u_x(0, t)$ for $t \in ]0, T]$ provided $u_x(0, t)$ satisfies (1.4). Note that there are other choices of $v$ to define $I(\tau)$ which is useful for detecting the unknown boundary $\partial D = \{a\}$ (for the detail, see [7]).

Our aim is to seek formulae which enable us to extract information about the unknown boundary $\partial D$ for the three-dimensional case. To be our problem clear, we rewrite (1.3) by using another solution of the backward heat equation.

Given $y \in \mathbb{R}^1$ the function
\[
\tilde{E}_\tau(x, y) = \frac{1}{\tau} e^{-\tau|x-y|}
\]
satisfies the equation
\[
\tilde{E}_\tau''(x) - \tau^2 \tilde{E}_\tau(x) + 2\delta(x-y) = 0
\]
in the whole line. Note that $-2^{-1} \tilde{E}_\tau(x, y)$ is a fundamental solution of the operator $\partial^2_x - \tau^2$.

Let $p$ be an arbitrary fixed point in $]-\infty, 0[$. Then $v(x, t) = e^{-\tau^2 t} \tilde{E}_\tau(x, p)$ also satisfies the backward heat equation $v_t + v_{xx} = 0$ for $(x, t) \in ]0, a[\times]0, T]$. Using this function, we define another indicator function
\[
\tilde{I}(\tau, p) = \int_0^T (v_x(0, t) u(0, t) + u_x(0, t) v(0, t)) \, dt.
\]
Since \( v(x,t) = e^{\tau p} e^{-\tau^2 t} e^{-\tau x} / \tau \) on \([0, a]\), we have \( \tilde{I}(\tau, p) = e^{\tau p} I(\tau) / \tau \). From this and (1.3) we obtain another formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |\tilde{I}(\tau, p)| = p - 2a. \tag{1.5}
\]

The point is the interpretation of this right-hand side of (1.5). Since \( p < 0 \), one can write \( p - 2a = -(|p| + 2a) \). Hence we can see that the quantity \(|p| + 2a\) in formula (1.5) coincides with the length of the broken path that starts at \( x = p \), goes to \( a \) (the point of the boundary \( \{a\} \) of the cavity) and returns to \( x = 0 \) (the point of the boundary of medium).

In this paper we establish a three-dimensional analogue of formula (1.5) (which is equivalent to (1.3) as mentioned above).

### 1.2 Description of the main result

First we introduce a three dimensional analogue of \( \tilde{I}(\tau, p) \).

**Definition 1.1** Let \( p \) be an arbitrary point outside \( \Omega \). Define the indicator function for the solution \( u_f(x,t) \) of (1.1) with a fixed \( f \in L^2(\partial\Omega \times ]0, T[) \) by the formula

\[
I(\tau, p) = \int_{\partial\Omega} \int_0^T \left( \frac{\partial v}{\partial \nu}(y,t) u_f(y,t) - f(y,t) v(y,t) \right) dt dS_y,
\]

where

\[
v(x,t) = e^{-\tau^2 t} E_\tau(x,p)
\]

and

\[
E_\tau(x,y) = \frac{e^{-\tau|x-y|}}{2\pi|x-y|}, \quad x \neq y, \quad \tau > 0.
\]

Note that \( E(x) = E_\tau(x,y) \) satisfies the equation \((\Delta - \tau^2) E(x) + 2\delta(x-y) = 0 \) in \( \mathbb{R}^3 \) in the sense of distribution. Thus if \( y \in \mathbb{R}^3 \setminus \overline{\Omega} \), then \( E(x) = E_\tau(x,y) \) satisfies the equation

\[
(\Delta - \tau^2) E(x) = 0 \quad \text{in} \quad \Omega. \tag{1.6}
\]

Hence, the indicator function \( I(\tau, p) \) in definition 1.1 is suited to treat three dimensional analogue of formula (1.5).

Throughout this paper, we always assume that the heat flux \( f(y,t) \) belongs to the space \( L^2(\partial\Omega \times ]0, T[) \). Since the weak solution \( u_f \) of (1.1) uniquely exists, the indicator function \( I(\tau, p) \) is well-defined. Our purpose in this paper is to clarify what information can be obtained from this indicator function. To describe them, we need to introduce the following notations:

**Definition 1.2** Given \( p \) outside \( \Omega \) define

\[
l(p,D) = \inf_{(x,y) \in \partial D \times \partial \Omega} l_p(x,y),
\]

where

\[
l_p(x,y) = |p - x| + |x - y|, \quad (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3.
\]
The quantity \( l(p, D) \) can be interpreted as the minimum length of broken paths that start at \( p \), go to a point on \( \partial D \) and return to a point on \( \partial \Omega \).

We also introduce some sets of pair of points on \( \partial D \) and \( \partial \Omega \) related to \( l(p, D) \).

**Definition 1.3** Given \( z \) outside \( D \) define

\[
\mathcal{G}(z) = \{ x \in \partial D \mid \nu_x \cdot (z - x) = 0 \},
\]

\[
\mathcal{G}^\pm(z) = \{ x \in \partial D \mid \pm \nu_x \cdot (z - x) > 0 \}.
\]

Let \( p \) be an arbitrary point outside \( \Omega \). Define

\[
\mathcal{M}(p) = \{ (x, y) \in \partial D \times \partial \Omega \mid l(p, D) = l_p(x, y) \},
\]

\[
\mathcal{M}_1(p) = \{ (x, y) \in \mathcal{M}(p) \mid x \in \mathcal{G}^+(p) \cap \mathcal{G}^+(y) \},
\]

\[
\mathcal{M}_2^+(p) = \{ (x, y) \in \mathcal{M}(p) \mid x \in \mathcal{G}^+(p) \cap \mathcal{G}^+(y) \},
\]

\[
\mathcal{M}_g(p) = \{ (x, y) \in \mathcal{M}(p) \mid x \in \mathcal{G}(p) \}.
\]

Now we state what the indicator function \( I(\tau, p) \) gives. We put

\[
g(y, \tau) = \int_0^T e^{-\tau^2 t} f(y, t) dt \quad (y \in \partial \Omega, \tau > 0).
\]

**Theorem 1.1** Assume that \( f \in L^2(\partial \Omega \times ]0, T[) \) and there exists a constant \( \mu \in \mathbb{R} \) such that the function \( g(y, \tau) \) defined by (1.7) belongs to \( C^{\alpha_0}(\partial \Omega) \) for all large \( \tau > 0 \) and satisfies

\[
0 < \inf_{y \in \partial \Omega} \liminf_{\tau \to \infty} \tau^\mu \text{Re} g(y, \tau) \leq \limsup_{\tau \to \infty} \tau^\mu \| g(\cdot, \tau) \|_{C^{\alpha_0}(\partial \Omega)} < \infty. \tag{1.8}
\]

Then, the formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I(\tau, p)| = -l(p, D), \tag{1.9}
\]

holds if \( \partial D \) and \( \partial \Omega \) satisfy the following four conditions:

\( (I.1) \) \( \partial D \) is strictly convex, \( (I.2) \) \( \mathcal{M}_g(p) = \emptyset \), \( (I.3) \) \( \mathcal{M}_2^-(p) = \emptyset \), \( (I.4) \) every point \( (x_0, y_0) \in \partial D \times \partial \Omega \) attaining \( l(p, D) \) is non-degenerate critical point of \( l_p(x, y) \).

**Remark 1.1** There exist many \( f \in L^2(\partial \Omega \times ]0, T[) \) satisfying (1.8). For example, (1.8) with \( \mu = 2 \) holds for \( f \in C^1([0, T]; C^{\alpha_0}(\partial \Omega)) \) with \( \inf_{y \in \partial \Omega} f(0, y) > 0 \). Indeed, integration by parts implies that

\[
\| \tau^2 g(\cdot, \tau) - f(0, \cdot) \|_{C^{\alpha_0}(\partial \Omega)} \leq \tau^{-2} \max_{0 \leq t \leq T} \| \partial_t f(t, \cdot) \|_{C^{\alpha_0}(\partial \Omega)} \quad (0 \leq \beta \leq \alpha_0).
\]
Formula (1.9) for the three-dimensional problem (1.1) can be interpreted as the analogous formula of (1.5) for the one-dimensional case (1.2). Note that in the one-dimensional case, $\Omega = [0, \infty]$ and $\partial D = \{ a \}$. Hence the length $l(p, D)$ for a point $p \notin [0, \infty](= \Omega)$ is just $2a - p$ as appeared in (1.5). Thus, from formula (1.5) we can find the unknown boundary $\partial D = \{ a \}$.

In section 2, we prove theorem 1.1. We briefly introduce the decomposition of $I(\tau, p)$ into the main part $I_0(\tau, p)$ and remainder term. This decomposition enables us to reduce the problem to the study of the asymptotic behaviour of $I_0(\tau, p)$, which is stated as theorem 2.1. Sections 3 to 5 are devoted to the proof of theorem 2.1. In the last part of section 2, we explain the necessity of the succeeding sections for the proof of theorem 2.1.

### 1.3 Other previous results using the enclosure method

To obtain other information about $D$ one may think about replacing $v$ in $I(\tau, p)$ with other special solutions of the backward heat equation $(\partial_t + \Delta)v = 0$ in $\Omega$.

In three-space dimensional case, define the indicator function $J_v(\tau)$ by

$$ J_v = \int_{\partial \Omega} \int_0^T \left( \frac{\partial}{\partial \nu}(y, t)u_f(y, t) - f(y, t)v(y, t) \right) dt dS, $$

(1.10)

where $u_f$ is the solution of (1.1), $v(x, t)$ is a solution of the backward heat equation $(\partial_t + \Delta)v = 0$ in $[0, T] \times \Omega$ having the form $v = e^{-\tau^2 t}q(x, \tau)$ and thus $(\Delta - \tau^2)q = 0$ in $\Omega$.

Note that there are several possibilities of the choice of $v$ and $f$ in (1.10).

**Case ($\infty$):** This is an ideal case. It is assumed that one can obtain $u_f$ on $\partial \Omega \times ]0, T[$ corresponding to infinitely many $f$. In this case, we can design input heat flux $f$ to obtain information of $D$. In what follows, for integer $k$, we denote by $H^k(\Omega)$ the $L^2$-Sobolev space defined by $H^2(\Omega) = \{ u \in L^2(\Omega) | \partial_x^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leq 2 \}$, where the derivative $\partial_x^\alpha u$ is in distribution sense. For an appropriate $\varphi \in L^2(0, T)$ and a function $q(x, \tau)$ satisfying $(\Delta - \tau^2)q = 0$ in $\Omega$ with $\|q(\cdot, \tau)\|_{H^2(\Omega)} = O(e^{C\tau})$ ($\tau \to \infty$) for some fixed constant $C > 0$, we input heat flux $f(x, t; \tau)$ depending on $\tau \geq 1$ as

$$ f(x, t; \tau) = \varphi(t) \frac{\partial q}{\partial \nu}(x, \tau) \quad \text{on} \quad \partial \Omega \times ]0, T[. $$

For each $\tau \geq 1$, we put $v(x, t; \tau) = e^{-\tau^2 t}q(x, \tau)$. Since $f \in L^2(\partial \Omega \times ]0, T[)$, from the definition of the weak solutions for (1.1) and $v \in C^1([0, T]; H^2(\Omega))$, using (1.10), we can define $I_q(\tau) = J_v$. As is in [9, 11, 10], from elliptic estimates, it follows that there exists a constant $C > 0$ such that

$$ C^{-1}\|\nabla_x q(\cdot, \tau)\|^2_{L^2(D)} \leq |I_q(\tau)| \leq C\{\|\nabla_x q(\cdot, \tau)\|^2_{L^2(D)} + \tau^2\|q(\cdot, \tau)\|^2_{L^2(D)}\} \quad (\tau \geq 1). $$

(1.11)

From (1.11) and the asymptotic behaviour of $q(x, \tau)$ on $D$ as $\tau \to \infty$, one can extract several quantities such as $h_D(\omega) = \sup_{x \in D} x \cdot \omega$, $d_D(p) = \inf_{x \in D} |x - p|$ and $R_D(y) = \sup_{x \in D} |x - y|$ when $q$ is chosen appropriately. Note also that [9] covers the case where the background conductivity is isotropic, inhomogeneous and known. It makes use of a complex geometrical optics solution constructed by using a Faddeev-type Green function for the modified Helmholtz equation.
Case (I): On the contrary to Case \((\infty)\), let us consider the case where we can only use one set of data \((f, u_f)\) on \(\partial\Omega \times [0, T]\) as the measurement. In this case, we cannot design the indicator function like as Case \((\infty)\). However, as is in [11], we can extract \(dist(\partial\Omega, D) = \inf_{y \in \partial\Omega, x \in D} |x - y|\) from \(u_f\) on \(\partial\Omega \times [0, T]\) for a fixed \(f\). More precisely, we introduce the function \(g(y, \tau)\) defined by (1.7). Taking a function \(\varphi(x, \tau)\) as the weak solution to
\[
\begin{align*}
\left\{ \begin{array}{ll}
(\Delta - \tau^2)q(x, \tau) = 0 & \text{in } \Omega, \\
\frac{\partial q}{\partial \nu}(x, \tau) = g(x, \tau), & \text{on } \partial\Omega,
\end{array} \right.
\end{align*}
\]
and putting \(v(t, x; \tau) = e^{-\tau^2 t}q(x, \tau)\) in (1.10), we define \(I_q(\tau) = J_v\) as the indicator function. The point is: \(v\) depends on \(f\). This idea comes from [8] in which an inverse obstacle scattering problem in the time domain has been considered. For this indicator function, estimate (1.11) can be also shown similarly to Case \((\infty)\). Hence, we can extract \(dist(\partial\Omega, D)\) from the indicator function by studying the asymptotic behaviour of \(q(x, \tau)\) on \(D\) as \(\tau \to \infty\). Note that in the last step we employ the potential theoretic construction of the solution of (1.12)(cf. [11]).

In both cases, the limit
\[
\lim_{\tau \to \infty} -\frac{1}{2\tau} \log |I_q(\tau)| = d_0
\]
gives various quantities related to \(D\), as described above.

The results are listed as follows:

| Case | \(f(x, t)\) | \(q(x, \tau)\) in \(v(x, t; \tau) = e^{-\tau^2}q(x, \tau)\) | \(d_0\) |
|------|-------------|-----------------|-------|
| \((\infty)\) \(\varphi(t)\frac{\partial q}{\partial \nu_x}(x, \tau)\) | \begin{align*}
q &= e^{\tau \omega \cdot x} & \text{with } \omega \in S^2 \\
q &= \frac{e^{-\tau|x-p|}}{|x-p|} & \text{with } p \in \mathbb{R}^3 \setminus \overline{\Omega} \\
q &= \frac{e^{\tau|x-y|} - e^{-\tau|x-y|}}{|x-y|}, & \text{with } y \in \mathbb{R}^3 \\
q &= \frac{2\tau}{|x-y|}, & \text{with } y \in \mathbb{R}^3 \\
q &= e^{c\tau^2(\omega+\lambda\tau\omega^\perp)\cdot x} & \text{with } c\tau > 1, \\
\omega \cdot \omega^\perp &= 0, & \lambda_\tau = \sqrt{1 - \frac{1}{c^2\tau^4}} \omega, & \omega^\perp \in S^2
\end{align*} | \begin{align*}
h_D(\omega) \\
d_D(p) \\
R_D(y) \\
h_D(\omega)
\end{align*} |
| (I) | A fixed \(f\) | \begin{align*}
&\text{the solution to (1.12)} \\
&\text{for } g(y, \tau) \text{ given by (1.7)}
\end{align*} | \begin{align*}
dist(\partial\Omega, D)
\end{align*} |

Note that we can also apply the idea in Case (I) to one-space dimensional case (1.2) and obtain \(dist(\partial\Omega, D) = a\). However, this is different from formula (1.3) (and (1.5)) since in this formula, \(v(x, t)\) does not have any relation with the heat flux \(f(0, t)\) ! Hence, for treating three-space dimensional analogue of formula (1.5) (or (1.3)), we need to choose \(v(x, t)\) in (1.10) being independent of \(f(x, t)\).

In the following table, our result in this paper is described using \(l(p, D)\). However, there are places with question marks. Those indicate that we do not know what kind of information about \(D\) can be extracted from the corresponding indicator function. To fill the places with suitable quantities we need further investigation in future.
Anyway, it seems that the result and proof of this paper suggest us the difficulty of the reconstruction problem using a single set of data. It will be interesting to find a simpler proof of the result.

### 1.4 What is a difference from one-space dimensional case?

It may be suspicious that too many assumption on $f$, $\partial \Omega$ and $\partial D$ appears in theorem 1.1. In this subsection, we will explain why those assumption is required for the proof of (1.9).

In one-space dimensional case, we have formula (1.5) provided the input heat flux at $t = 0$ on the boundary $\{0\}$ satisfies (1.4) for some $\beta_0$. This condition on the heat flux ensures the strength of the input heat flux at $t = 0$ from below implicitly. In three-space dimensional case, assumption (1.8) in theorem 1.1 corresponds to this condition. Moreover, theorem 2.1 in section 2 tells us that we do not need to input the heat flux at $t = 0$ on the whole boundary $\partial \Omega$. If we know, in advance, the set of all points $y \in \partial \Omega$ such that there exists a point $x \in \partial D$ with $(x, y) \in \mathcal{M}_1(p) \cup \mathcal{M}_2^\perp(p)$, then it suffices to input heat flux at $t = 0$ supplied only on such special points $y \in \partial \Omega$. Thus (1.8) can be replaced with weaker one if this is the case, however, it is not practical to assume such a priori information.

In three-space dimensional case, there are several type of the points $(x_0, y_0) \in \partial D \times \partial \Omega$ that attain the minimum length $l(p, D)$ (i.e. $(x_0, y_0) \in \mathcal{M}(p)$). One type consists of broken rays of geometrical optics passing through $y_0$, $x_0$ and $p$ in this order. The pairs of such points $(x_0, y_0)$ consist of the set $\mathcal{M}_1(p)$. Note that in a special case, there may exist a point $(x_0, y_0) \in \mathcal{M}_1(p)$ such that $y_0$ is contained in the line segment $px_0$. This case just corresponds to one-space dimensional case.

In three-space dimensional case, there may also exist points $(x_0, y_0) \in \mathcal{M}(p)$ such that $x_0$ is on the line segment $py_0$. These points belong to one of the three types of disjoint sets $\mathcal{M}^+_2(p)$, $\mathcal{M}_2^\perp(p)$ and $\mathcal{M}_9(p)$. As it can be seen in the proof of theorem 1.1, it is not easy to measure the contribution of points in $\mathcal{M}_9(p)$ to the asymptotic behavior of $I(\lambda, p)$. We can also see that the contribution of points in $\mathcal{M}_2^\perp(p)$ to the asymptotic behavior of $I(\tau, p)$ may cancel the one of the points belonging to $\mathcal{M}_1(p)$ (cf. theorem 2.1). In theorem 1.1, to avoid these cancelations, we assume $\mathcal{M}_9(p) \cup \mathcal{M}_2^\perp(p) = \emptyset$ (i.e. (I.2) and (I.3)).
Thus, in three-space dimensional case, the structure of $\mathcal{M}(p)$ becomes complicated. This is one of the different points from one-space dimensional case and makes the problem for three-space dimensional case harder. However, we can give a condition on $\partial \Omega$ that ensures $\mathcal{M}_g(p) \cup \mathcal{M}_2^p = \emptyset$ (cf. proposition 6.2). And also, in propositions 6.1 and 6.3 a condition to ensure that a point $(x_0, y_0) \in \mathcal{M}(p) \setminus \mathcal{M}_g(p)$ is a non-degenerate critical point of $l_\rho$ on $\partial D \times \partial \Omega$ (cf. propositions 6.1 and 6.3), is given. Using these sufficient conditions, we can give examples covered by theorem 1.1.

As the next step it would be interesting to know what kind of information can be extracted from $l(p,D)$ given at all or some $p \in \mathbb{R}^3 \setminus \overline{\Omega}$. To our best knowledge, the complete answer to the question is unknown. However, in section 6.5 we show that $l(p,D)$ yields some information about an upper bound of the location of $D$.

In theorem 1.1, we also assume that $\partial D$ is strictly convex. It seems that this assumption is too strong for the applications to practical inverse problems. However, at the present time, technically, to treat the case of “one measurement”, we need such kind of a priori information on the unknown object $\partial D$. We can also show a similar result to the case that $D$ consists of several disjoint strictly convex domains. However, to treat this case, we need to repeat the argument which was used in the proof of theorem 1.1. Hence to keep this paper in an appropriate length, we restrict ourselves within introducing theorem 1.1.

### 1.5 A remark on the solution class

Before closing this section, following [4], we describe the class of solutions of the initial boundary value problem for the heat equation (1.1).

For $T > 0$ and a Hilbert space $H$, $L^2(0,T; H)$ denotes the space of $H$-valued $L^2$ functions in $t \in [0,T]$. For two Hilbert spaces $H$ and $V$ with $V \subset H \subset V'$, we also introduce the space $W(0,T; V, V') = \{ u \mid u \in L^2(0,T; V), u' \in L^2(0,T; V') \}$, where $V'$ is the dual space of the Hilbert space $V$, and $u'$ means the (weak) derivative in $t \in [0,T]$.

As is in [4], for any $f \in L^2(0,T; H^{-1/2}(\partial \Omega))$, we call $u \in W(0,T; H^1(\Omega \setminus \overline{D}), (H^1(\Omega \setminus \overline{D}))')$ satisfies

$$
\begin{cases}
  u_t - \Delta u = 0 & \text{in } (\Omega \setminus \overline{D}) \times ]0, T[,
  \\
  \frac{\partial u}{\partial \nu} + \rho u = 0 & \text{on } \partial D \times ]0, T[,
  \\
  \frac{\partial u}{\partial \nu} = f & \text{on } \partial \Omega \times ]0, T[,
\end{cases}
$$

(1.13)

in the weak sense if $u$ satisfies

$$
\langle u'(t), \varphi \rangle_{H^1(\Omega \setminus \overline{D})} + \int_{\Omega \setminus \overline{D}} \nabla u(x,t) \cdot \nabla \varphi(x) dx
- \langle \rho u(t), \varphi|_{\partial D} \rangle_{H^{1/2}(\partial D)} = \langle f(t), \varphi|_{\partial \Omega} \rangle_{H^{-1/2}(\partial \Omega)} \text{ in } ]0, T[.
$$

in the sense of distribution on $]0, T[$ for all $\varphi \in H^1(\Omega \setminus \overline{D})$ and a.e. $t \in ]0, T[$. In the above, the bracket $\langle \cdot, \cdot \rangle_V$ is the pairing between a Hilbert space $V$ and $V'$.

We see that every $u \in W(0,T; H^1(\Omega \setminus \overline{D}),(H^1(\Omega \setminus \overline{D}))')$ is almost everywhere equal to a continuous function of $[0,T]$ in $L^2(\Omega \setminus \overline{D})$ (Theorem 1 on p.473 in [4]). Further, we
have the following inclusion:

\[ W(0, T; H^1(\Omega \setminus \overline{D}), (H^1(\Omega \setminus \overline{D}))') \hookrightarrow C^0([0, T]; L^2(\Omega \setminus \overline{D})), \]

where the space \( C^0([0, T]; L^2(\Omega \setminus \overline{D})) \) is equipped with the norm of uniform convergence. Thus one can consider \( u(t) \) \((0 \leq t \leq T)\) as elements of \( L^2(\Omega \setminus \overline{D}) \). Then we see that for any given \( f \in L^2(0, T; H^{-1/2}(\partial \Omega)) \) and \( u_0 \in L^2(\Omega \setminus \overline{D}) \), there exists a unique \( u \in W(0, T; H^1(\Omega \setminus \overline{D})), (H^1(\Omega \setminus \overline{D}))' \) satisfying (1.13) in the weak sense and the initial condition \( u(0) = u_0 \) (Theorems 1 and 2 on p.512 in [4]). We denote by \( u_f \) the weak solution of (1.13) with \( u(0) = 0 \) and this is the meaning of the weak solution of (1.1).

2 Proof of theorem 1.1.

We begin with choosing the main term \( I_0(\tau, p) \) of \( I(\tau, p) \). Define

\[ w(x, \tau) = \int_0^T e^{-\tau^2 t} u_f(x, t) dt, \quad x \in \Omega \setminus \overline{D}, \quad \tau > 0. \]

Since \( f \in L^2(\partial \Omega \times ]0, T[) \), \( u \in W(0, T; H^1(\Omega \setminus \overline{D})), (H^1(\Omega \setminus \overline{D}))' \) is the weak solution of (1.13). From these facts, we can see that \( w(\cdot, \tau) \in H^1(\Omega \setminus \overline{D}) \) is the unique solution of the following elliptic boundary value problem in the weak sense:

\[
\begin{cases}
(\Delta - \tau^2) w = u(x, T)e^{-\tau^2 T} & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w}{\partial \nu} + \rho(x) w = 0 & \text{on } \partial D, \\
\frac{\partial w}{\partial \nu} = g(y, \tau) & \text{on } \partial \Omega,
\end{cases}
\]  

(2.1)

where \( g(y, \tau) \) is the function defined by (1.7). Using \( w(x, \tau) \), we obtain the expression

\[ I(\tau, p) = \int_{\partial \Omega} \left( \frac{\partial E_\tau(y, p)}{\partial \nu} w(y, \tau) - \frac{\partial w(y, \tau)}{\partial \nu} E_\tau(y, p) \right) dS_y. \]

Let us consider the solution \( w_0(x; \tau) \) of the following elliptic boundary value problem:

\[
\begin{cases}
(\Delta - \tau^2) w_0 = 0 & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w_0}{\partial \nu} + \rho(x) w_0 = 0 & \text{on } \partial D, \\
\frac{\partial w_0}{\partial \nu} = g(y, \tau) & \text{on } \partial \Omega.
\end{cases}
\]  

(2.2)

Note that \( g(\cdot, \tau) \in L^2(\partial \Omega) \) for \( f \in L^2(\partial \Omega \times ]0, T[) \). Hence usual elliptic theory implies that for any \( \tau > 0 \), there exists the unique solution \( w_0(\cdot, \tau) \in H^1(\Omega \setminus \overline{D}) \) of (2.2) in the weak sense. Thus, for \( \tau > 0 \), we can introduce

\[ I_0(\tau, p) = \int_{\partial \Omega} \left( \frac{\partial E_\tau(y, p)}{\partial \nu} w_0(y, \tau) - \frac{\partial w_0(y, \tau)}{\partial \nu} E_\tau(y, p) \right) dS_y. \]

We can show that there exist constants \( C > 0 \) and \( \mu_0 > 0 \) depending on \( \partial D \), \( f \) and \( \rho \) such that

\[ |I(\tau, p) - I_0(\tau, p)| \leq C\tau^{-1/2} e^{-\tau^2 T} \quad (\tau \geq \mu_0). \]
In what follows, when the above estimate holds, we merely write
\[ I(\tau, p) = I_0(\tau, p) + O(\tau^{-1/2}e^{-\tau^2 T}) \quad \tau \to \infty. \] (2.3)

This reduction is well known (cf. section 2 in [10]), however, for this paper to be self-contained, we show it in Appendix C.

Now we state the asymptotic behavior of \( I_0(\lambda, p) \) being the essential part of this paper.

**Theorem 2.1** Assume that \( f \in L^2(\partial \Omega \times [0, T]) \), and \( \partial D \) and \( \partial \Omega \) satisfy (I.1), (I.2) and (I.4) in theorem 1.1. Then the set \( \mathcal{M}(p) \) is finite. Moreover, we have
\[ I_0(\tau, p) = \frac{1}{\tau} e^{-\tau l(\cdot, D)} \{ A(\tau, p)g + \|g(\cdot, \tau)\|_{C^0,0(\partial \Omega)}O(\tau^{-\alpha_0/2}) \} \] (2.4)
as \( \tau \to \infty \), where
\[ A(\tau, p)g = \sum_{(x_0, y_0) \in \mathcal{M}_1(p)} C(x_0, y_0) H^+(x_0, y_0, p)g(y_0, \tau) \]
\[ + \sum_{(x_0, y_0) \in \mathcal{M}_2(p)} C(x_0, y_0) H^-(x_0, y_0, p)g(y_0, \tau). \] (2.5)

In (2.5), the coefficients \( C(x, y) \) for each \( (x, y) \in \mathcal{M}_1(p) \cup \mathcal{M}_2(p) \) is a positive constant independent of \( g \) (i.e \( f \)) and
\[ H^\pm(x, y, p) = \frac{1}{|x-p||x-y|} \nu_x \cdot \left\{ \frac{p-x}{|p-x|} \pm \frac{y-x}{|y-x|} \right\}, (x, y) \in \partial D \times \partial \Omega. \]

**Remark 2.1** We have \( H^+(x_0, y_0, p) > 0 \) for \( (x_0, y_0) \in \mathcal{M}_1(p) \) and \( H^-(x_0, y_0, p) < 0 \) for \( (x_0, y_0) \in \mathcal{M}_2(p) \) (cf. (2) of proposition 3.2 and (3.33) in subsection 3.3).

Note that all the points of the set \( \mathcal{M}(p) \) are critical points of \( l_p \) on \( \partial D \times \partial \Omega \) and for each \( (x, y) \in \mathcal{M}(p) \) the Hessian at \( (x, y) \) of any local representation of \( l_p \) in a neighbourhood of \( (x, y) \) has no negative eigenvalues. Thus a point \( (x, y) \in \mathcal{M}(p) \) is a non-degenerate critical point of \( l_p \) on \( \partial D \times \partial \Omega \) if and only if the Hessian at \( (x, y) \) of a local representation of \( l_p \) in a neighbourhood of \( (x, y) \) is positive definite. Thus the conclusion of the finiteness of \( \mathcal{M}(p) \) in theorem 2.1 is trivial.

Using theorem 2.1, we can obtain theorem 1.1. Here, we continue the proof of theorem 1.1 assuming theorem 2.1 holds.

**Proof of theorem 1.1.** Since we consider the case \( \mathcal{M}_0(p) \cup \mathcal{M}_2(p) = \emptyset \), from (2.5), (1.8) and remark 2.1, it follows that there exist constant \( C > 0 \) and \( \mu_0 > 0 \) such that
\[ C^{-1} \leq \tau^\mu \Re A(\tau, p)g \leq \tau^\mu |A(\tau, p)g| \leq C \quad (\tau \geq \mu_0). \]
Combining this estimate with (2.4) and (1.8), we obtain
\[ C^{-1}_1 \leq \tau^{\mu+1} |e^{\tau l(p,D)} I_0(\tau, p)| \leq C_1 \quad (\tau \geq \mu_1) \] (2.6)
for some constants \( C_1 > 0 \) and \( \mu_1 > 0 \) independent of \( \tau \). From the above estimate and (2.3), it follows that
\[ C^{-1}_2 \leq \tau^{\mu+1} e^{\tau l(p,D)} |I(\tau, p)| \leq C_2 \quad (\tau \geq \mu_2) \]
for some constants \( C_2 > 0 \) and \( \mu_2 > 0 \) independent of \( \tau \). This estimate shows theorem 1.1 holds. 

\[ \square \]

From the above proof of theorem 1.1, we can see that formula (1.9) in theorem 1.1 is given by (2.6). Using \( A(\tau, p)g \) in (2.5), we can give sufficient conditions for getting (2.6) (i.e. (1.9)).

**Corollary 2.1** Assume that there exists a positive number \( \mu \) such that the function \( g(y, \tau) \) defined by (1.7) belongs to \( g(\cdot, \tau) \in C^{0,\alpha_0}(\partial \Omega) \) for all \( \tau > 0 \) large enough and satisfies
\[ \liminf_{\tau \to \infty} \tau^{\mu} |A(\tau, p)g| > 0 \] (2.7)
and
\[ \lim_{\tau \to \infty} \frac{\tau^{\mu} \|g(\cdot, \tau)\|_{C^{0,\alpha_0}(\partial \Omega)}}{\tau^{\alpha_0/2}} = 0. \] (2.8)

Then formula (1.9), that is,
\[ \lim_{\tau \to \infty} \frac{1}{\tau} \log |I(\tau, p)| = -l(p, D) \]
is valid.

Note that in theorem 1.1, we assume (1.8) to ensure (2.7) holds. However, (1.8) is too strong. We do not need to input the heat flux \( f \) at \( t = 0 \) on the whole boundary \( \partial \Omega \). From the form (2.5) of \( A(\tau, p)g \), we can see that it is enough to supply \( f \) at \( t = 0 \) only on the set of all points \( y \in \partial \Omega \) such that there exists a point \( x \in \partial D \) with \((x, y) \in M_1(p) \cup M_2^{-}(p)\). Note that condition (2.7) also gives a lower bound estimate for the strength of the input heat flux \( f \) at \( t = 0 \). If both \( M_1(p) \) and \( M_2^{-}(p) \) are not empty, a cancelation in \( A(\tau, p)g \) may occur (see remark 2.1) and thus it is delicate whether (2.7) holds or not. Another condition (2.8) is not a serious one. For example, if \( f = 1 \) on \( \partial \Omega \times [0, T] \), then (2.8) is satisfied with \( \mu = 2 \). Note that in this case, (2.7) also holds with \( \mu = 2 \) if \( A(\tau, p)g \) does not vanish.

It is crucial to represent the main term \( I_0(\tau, p) \) by using Laplace type integrals (cf. proposition 3.1) for the proof of theorem 2.1. This is done in subsection 3.1. We construct the solution \( w_0(x, \tau) \) of (2.2) by single layer potentials on \( \partial D \) and \( \partial \Omega \) in potential theory. Using this expression, we decompose the main term into some parts. Each term can be reduced to a Laplace type integral over \( \partial \Omega \times \partial D \) with a large parameter \( \tau \).
In each integral, the exponential terms are just given by \( e^{-\tau l_p(x,y)} \). Thus the points \((x_0, y_0) \in \partial D \times \partial \Omega\) attaining the minimum \( l(p, D) \) of \( l_p(x, y) \), (i.e., \((x_0, y_0) \in \mathcal{M}(p)\)) determine the asymptotic behaviour of \( I_0(\tau, p) \). In subsection 3.3 of section 3.1, we study the structure of the set \( \mathcal{M}(p) \).

In section 4, we give a proof of theorem 2.1 using the Laplace method. Here, we need to have asymptotic behaviour of the amplitude functions in the Laplace integrals. These key facts are described in lemma 4.1. In section 5 the proof of lemma 4.1 is given.

Since the amplitude functions contain terms defined by using the inverse of an integral operator on \( \partial D \), the problem is eventually reduced to obtaining some estimates of the kernel \( K_{\tau}^\infty(x, y) \) of an operator of the form \( K_{\tau} (I - K_{\tau})^{-1} \), where \( K_{\tau} \) is an integral operator on \( \partial D \) with the kernel \( K_{\tau}(x, y) \) estimated by

\[
|K_{\tau}(x, y)| \leq C\tau e^{-\tau|x-y|} \quad (x, y \in \partial D, \tau > 0). \tag{2.9}
\]

We need to show that kernel \( K_{\tau}^\infty(x, y) \) can be estimated by the same exponential term \( e^{-\tau|x-y|} \) as in the estimate (2.9). Therefore we need more precise argument than that of usual classical potential theory although we study the kernels of the repeated integral operators \( K_{\tau}^n \ (n = 1, 2, \ldots) \) according to the classical approach. The needed estimates of the integral kernels are given in [12]. Here, only the result used in this paper is summarized in subsection 3.2 (cf. theorem 3.1).

The Laplace method requires the non-degenerateness of \( l_p(x, y) \) at \((x, y) \in \mathcal{M}(p)\). In section 6, sufficient conditions of non-degenerateness of \( l_p(x, y) \) are given. Using these conditions, we can give examples covered by theorems 1.1, 2.1 and corollary 2.1.

To make this paper self-contained we add two appendixes A and B. In Appendix A, we give a proof of one version of the Laplace method used to show the main result. Appendix B is devoted to a computation of Weingarten map for ellipsoids, which is used to treat the examples in section 6.

### 3 Preliminaries

#### 3.1 the decompostion of \( I_0(\tau, p) \)

We employ the layer potential approach for the construction of \( w_0 \).

Given \( g \in C(\partial \Omega) \) and \( h \in C(\partial D) \) define

\[
V_{\Omega}(\tau)g(x) = \int_{\partial \Omega} E_{\tau}(x, y)g(y)dS_y, \quad x \in \mathbb{R}^3 \setminus \partial \Omega,
\]

\[
V_{\Omega}(\tau)h(x) = \int_{\partial D} E_{\tau}(x, z)h(z)dS_z, \quad x \in \mathbb{R}^3 \setminus \partial D.
\]

We construct \( w_0 \) in the form

\[
w_0(x, \tau) = V_{\Omega}(\tau)\varphi(x, \tau) + V_{\Omega}(\tau)\psi(x, \tau), \tag{3.1}
\]
where \(\varphi(\cdot, \tau) \in C(\partial \Omega)\) and \(\psi(\cdot, \tau) \in C(\partial D)\) are unknown functions to be determined.

Here we cite some well known facts for \(V_\Omega(\tau)\) and \(V_D(\tau)\) from potential theory (cf. [13]).

- \(V_\Omega(\tau)g\) satisfies \((\Delta - \tau^2)V_\Omega(\tau)g = 0\) in \(\mathbb{R}^3 \setminus \partial \Omega\).
- \(V_D(\tau)h\) satisfies \((\Delta - \tau^2)V_D(\tau)h = 0\) in \(\mathbb{R}^3 \setminus \partial D\).

These yield that \(w_0\) having the form (3.1) satisfies the equation \((\Delta - \tau^2)w_0 = 0\) in \(\Omega \setminus \overline{D}\).

In what follows, we denote by \(B(X, Y)\) the space consisting of continuous linear operators from a normed space \(X\) to a Fréchet space \(Y\). Note that \(B(X, Y)\) is the space consisting of all bounded linear operators when \(X\) and \(Y\) are Banach spaces. We also put \(B(X) = B(X, X)\).

- \(V_\Omega(\tau) \in B(C(\partial \Omega), C^\infty(\mathbb{R}^3 \setminus \partial \Omega) \cap C(\mathbb{R}^3))\) and the Neumann derivative for \(V_\Omega(\tau)g\) at \(x \in \partial \Omega\)

\[
\frac{\partial}{\partial \nu_x} V_\Omega(\tau)g|_{\partial \Omega}(x) = \lim_{\epsilon \downarrow 0} \sum_{j=1}^{3} (\nu_x)_j \left( \frac{\partial}{\partial x_j} V_\Omega(\tau)g \right)(x - \epsilon \nu(x))
\]

exists and is given by the formula

\[
\frac{\partial}{\partial \nu_x} V_\Omega(\tau)g|_{\partial \Omega}(x) = g(x) + S_{\partial \Omega}(\tau)g(x),
\]

where

\[
S_{\partial \Omega}(\tau)g(x) = \int_{\partial \Omega} \frac{\partial}{\partial \nu_x} E_\tau(x, y)g(y)\,dS_y, \quad x \in \partial \Omega.
\]

- \(V_D(\tau) \in B(C(\partial D), C^\infty(\mathbb{R}^3 \setminus \partial D) \cap C(\mathbb{R}^3))\) and the Neumann derivative for \(V_D(\tau)h\) at \(x \in \partial D\)

\[
\frac{\partial}{\partial \nu_x} V_D(\tau)h|_{\partial D}(x) = \lim_{\epsilon \downarrow 0} \sum_{j=1}^{3} (\nu_x)_j \left( \frac{\partial}{\partial x_j} V_D(\tau)h \right)(x + \epsilon \nu(x))
\]

exists and is given by the formula

\[
\frac{\partial}{\partial \nu_x} V_D(\tau)h|_{\partial D}(x) = -h(x) + S_{\partial D}(\tau)h(x),
\]

where

\[
S_{\partial D}(\tau)h(x) = \int_{\partial D} \frac{\partial}{\partial \nu_x} E_\tau(x, z)h(z)\,dS_z, \quad x \in \partial D.
\]

- For \(\tau > 0\), \(S_{\partial \Omega}(\tau) \in B(C(\partial \Omega))\) and \(S_{\partial D}(\tau) \in B(C(\partial D))\). Moreover there exists a positive constant \(C\) such that these operator norms are bounded by \(C\tau^{-1}\).

Using these properties, we can show that \(w_0\) having the form (3.1) satisfies the boundary conditions in (2.2) if and only if \(\varphi\) and \(\psi\) satisfies the system of integral equations on \(\partial \Omega \cup \partial D\):

\[
\varphi(x, \tau) + S_{\partial \Omega}(\tau)\varphi(x, \tau) + X_{\partial \Omega}(\tau)\psi(x, \tau) = g(x, \tau) \text{ on } \partial \Omega,
\]

\[
\psi(x, \tau) - (X_{\partial D}(\tau) + \rho(x)V_\Omega(\tau))\varphi(x, \tau) - (S_{\partial D}(\tau) + \rho(x)V_D(\tau))\psi(x, \tau) = 0 \text{ on } \partial D,
\]

(3.2)
where

\[ X_{\partial \Omega}(\tau) \psi(x, \tau) = \int_{\partial D} \frac{\partial}{\partial \nu_x} E_\tau(x, z) \psi(z, \tau) dS_z \text{ on } \partial \Omega, \]

\[ X_{\partial D}(\tau) \varphi(x, \tau) = \int_{\partial \Omega} \frac{\partial}{\partial \nu_y} E_\tau(x, y) \varphi(y, \tau) dS_y \text{ on } \partial D. \]

For the concise expression of \( \varphi \) and \( \psi \) we introduce the 2 \times 2 matrix operator acting on \( C(\partial \Omega) \times C(\partial D) \)

\[ Y(\tau) = (Y_{ij}(\tau)) = \begin{pmatrix} -S_{\partial \Omega}(\tau) & -X_{\partial \Omega}(\tau) \\ X_{\partial D}(\tau) + \rho(x) V_\Omega(\tau) & S_{\partial D}(\tau) + \rho(x) V_D(\tau) \end{pmatrix}. \]

Using \( Y(\tau) \), we can write the equations (3.2) as

\[(I - Y(\tau)) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}.\]

Using a similar argument for the proof of the boundedness for \( S_{\partial \Omega}(\tau) \) and \( S_{\partial D}(\tau) \), we know that: if \( \tau > 0 \), then \( X_{\partial \Omega}(\tau) \in B(C(\partial D), C(\partial \Omega)) \), \( X_{\partial D}(\tau) \in B(C(\partial \Omega), C(\partial D)) \), and there exists a positive constant \( C \) such that these operator norms are bounded by \( C/\tau \).

For \( V_\Omega(\tau) \) and \( V_D(\tau) \), we can show that \( V_\Omega(\tau) \in B(C(\partial \Omega), C(\partial D)) \), \( V_D(\tau) \in B(C(\partial D)) \) and they have similar estimates.

Therefore we conclude that there exists a positive constant \( C \) such that, for all \( \tau > 0 \)

\[ \|Y(\tau)\|_{B(C(\partial \Omega) \times C(\partial D))} \leq C \tau^{-1}. \]

This ensures that if \( \tau \) is large enough, then the Neumann series \( \sum_{n=0}^{\infty} Y(\tau)^n \) is absolutely convergent with the operator norm and coincides with \( (I - Y(\tau))^{-1} \). \( \varphi \) and \( \psi \) are given by

\[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = (I - Y(\tau))^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix}. \]

(3.3)

This completes the construction of \( w_0 \).

Next, we write \( I_0(\tau, p) \) in terms of only \( \varphi \) given by (3.3). For the definition of \( I_0(\tau, p) \), it follows that

\[ I_0(\tau, p) = \int_{\partial D} \left( \frac{\partial E_\tau}{\partial \nu} + \rho E_\tau \right) (y, p) w_0(y, \tau) dS_y. \]

Indeed, integration by parts implies that

\[ I_0(\tau, p) = \int_{\partial \Omega} \left( \frac{\partial E_\tau}{\partial \nu} w_0 - \frac{\partial w_0}{\partial \nu} E_\tau \right) dS_y = \int_{\Omega} \left( (\Delta - \tau^2) E_\tau \right) w_0 - \int_{\partial D} \frac{\partial E_\tau}{\partial \nu} w_0 + \frac{\partial w_0}{\partial \nu} E_\tau \right) dS_y = \int_{\partial D} \left( \frac{\partial E_\tau}{\partial \nu} w_0 + \rho w_0 E_\tau \right) dS_y = \int_{\partial D} \left( \frac{\partial E_\tau}{\partial \nu} + \rho E_\tau \right) w_0 dS_y. \]
Using the above equality and (3.1), one has the decomposition

\[ I_0(\tau, p) = J_1(\tau, p) + J_2(\tau, p) \]
\[ = \int_{\partial D} \left( \frac{\partial}{\partial \nu} + \rho \right) E_\tau(x, p) V_\Omega(\tau) \varphi(x, \tau) dS_x \]
\[ + \int_{\partial D} \left( \frac{\partial}{\partial \nu} + \rho \right) E_\tau(x, p) V_D(\tau) \psi(x, \tau) dS_x. \] (3.4)

A direct computation gives

\[ \left( \frac{\partial}{\partial \nu} + \rho(x) \right) E_\tau(x, y) = \frac{1}{2\pi} e^{-|x-y|} H(x, y, \tau) \quad (x \in \partial D \cup \partial \Omega, y \in \mathbb{R}^3, x \neq y), \] (3.5)

where

\[ H(x, y, \tau) = \frac{\nu_x \cdot (y-x)}{|x-y|} \left( -\frac{\tau}{|x-y|} + \frac{1}{|x-y|^2} \right) + \frac{\rho(x)}{|x-y|}. \]

This yields

\[ J_1(\tau, p) = \left( \frac{1}{2\pi} \right)^2 \int_{\partial \Omega} dS_y \varphi(y, \tau) \int_{\partial D} \frac{H(x, p, \tau)}{|x-y|} e^{-\tau t_p(x, y)} dS_x. \] (3.6)

Set \( w_2(x, \tau) = V_D(\tau) \psi(x, \tau) \) and write

\[ J_2(\tau, p) = \int_{\partial D} \frac{\partial}{\partial \nu} E_\tau(x, p) w_2(x, \tau) dS_x + \int_{\partial D} \rho E_\tau(x, p) w_2(x, \tau) dS_x. \]

Note that \( w_2 \in C^\infty(\mathbb{R}^3 \setminus \overline{D}) \) satisfies the equation \((\Delta - \tau^2)w_2 = 0\) in \( \mathbb{R}^3 \setminus \overline{D} \). For sufficiently large \( R > 0 \), this function belongs to \( H^2 \) for \( |x| > R \) and \( \lim_{h \to 0} \frac{\partial w_2}{\partial \nu}(x + h\nu_x) \) exists in \( C(\partial \Omega) \). Since \( E_\tau(x, p) \) satisfies \((\Delta_x - \tau^2)E_\tau(x, p) + 2\delta(x-p) = 0\), integration by parts and the property of \( w_2 \) mentioned above yield

\[ \int_{\partial D} \frac{\partial}{\partial \nu} E_\tau(x, p) w_2(x, \tau) dS_x = 2w_2(p, \tau) + \int_{\partial D} E_\tau(x, p) \frac{\partial w_2}{\partial \nu}(x, \tau) dS_x. \]

From the property of \( V_D(\tau) \) and the second equation in (3.2) we obtain

\[ \left( \frac{\partial}{\partial \nu} + \rho \right) w_2(x, \tau) = -Y_{21}(\tau) \varphi(x, \tau) \text{ on } \partial D. \]

Therefore we have

\[ J_2(\tau, p) = 2w_2(p, \tau) - \int_{\partial D} E_\tau(x, p) Y_{21}(\tau) \varphi(x, \tau) dS_x. \] (3.7)

From (3.5) we know that

\[ Y_{21}(\tau) \varphi(x, \tau) = \frac{1}{2\pi} \int_{\partial \Omega} e^{-\tau|x-y|} H(x, y, \tau) \varphi(y, \tau) dS_y. \] (3.8)
This yields
\[ \int_{\partial D} E_\tau(x,p)Y_{21}(\tau)\varphi(x,\tau)dS_x \]
\[ = \left( \frac{1}{2\pi} \right)^2 \int_{\partial \Omega} dS_y \varphi(y,\tau) \int_{\partial D} H(x,y,\tau) e^{-\tau l_p(x,y)}dS_x. \]  
(3.9)

Note also that
\[ \psi(x,\tau) = (I - Y_{22}(\tau))^{-1}Y_{21}(\tau)\varphi(x,\tau), \tau \gg 1. \]

In what follows we denote by \( t Y_{22}(\tau) \) the formal adjoint operator defined by
\[ \int_{\partial D} (t Y_{22}(\tau)f)(x)h(x)dx = \int_{\partial D} f(x)(Y_{22}(\tau)h)(x)dx \quad (f,h \in C(\partial D)). \]

From the definition of \( t Y_{22}(\tau) \), it follows that
\[ t((I - Y_{22}(\tau))^{-1}Y_{21}(\tau)\varphi(x,\tau) = (I - t Y_{22}(\tau))^{-1}. \]
From these facts, it holds that
\[ w_2(p,\tau) = V_D(\tau)\psi(p,\tau) \]
\[ = \int_{\partial D} E_\tau(p,x)(I - Y_{22}(\tau))^{-1}Y_{21}(\tau)\varphi(x,\tau)dS_x \]
\[ = \frac{1}{2\pi} \int_{\partial D} e^{-\tau|x-p|} (I - Y_{22}(\tau))^{-1}Y_{21}(\tau)\varphi(x,\tau)dS_x \]  
(3.10)
\[ = \frac{1}{2\pi} \int_{\partial D} Y_{21}(\tau)\varphi(x,\tau) \left( (I - t Y_{22}(\tau))^{-1} \frac{e^{-\tau|\cdot-p|}}{|\cdot-p|} \right) dS_x \]
\[ = \frac{1}{2\pi} \int_{\partial D} e^{-\tau|x-p|} Y_{21}(\tau)\varphi(x,\tau) \cdot e^{\tau|x-p|} \left( (I - t Y_{22}(\tau))^{-1} \frac{e^{-\tau|\cdot-p|}}{|\cdot-p|} \right) dS_x. \]

Define
\[ F(x,p,\tau) = e^{\tau|x-p|} \left( (I - t Y_{22}(\tau))^{-1} \frac{e^{-\tau|\cdot-p|}}{|\cdot-p|} \right)(x). \]  
(3.11)

A combination of (3.8) and (3.10) gives
\[ w_2(p,\tau) = \left( \frac{1}{2\pi} \right)^2 \int_{\partial \Omega} dS_y \varphi(y,\tau) \int_{\partial D} e^{-\tau l_p(x,y)}H(x,y,\tau)F(x,p,\tau)dS_x. \]  
(3.12)

Finally from (3.4), (3.6), (3.7), (3.9), (3.12), we obtain the representation formula of \( I_0(\tau,p) \):
\[ (2\pi)^2 I_0(\tau,p) = \int_{\partial \Omega} dS_y \varphi(y,\tau) \]
\[ \times \int_{\partial D} e^{-\tau l_p(x,y)} \left\{ \frac{H(x,p,\tau)}{|x-y|} - \frac{H(x,y,\tau)}{|x-p|} + 2H(x,y,\tau)F(x,p,\tau) \right\} dS_x. \]  
(3.13)
Lemma 3.1

\[ tY_{22}(\tau)h(z) = \frac{1}{2\pi} \int_{\partial D} e^{-\tau|x-z|}H(x, z, \tau)h(x)dS_x, \ h \in C(\partial D), z \in \partial D. \]

Proof. Let \( f, h \in C(\partial D). \) Since \( Y_{22}(\tau) = S_{\partial D}(\tau) + \rho(x)V_{D}(\tau), \) we have

\[
\int_{\partial D} tY_{22}(\tau)h(z) \cdot f(z)dS_z = \int_{\partial D} h(x) \cdot Y_{22}(\tau)f(x)dS_x
\]

\[
= \int_{\partial D} dS_z h(x) \int_{\partial D} \left\{ \frac{\partial}{\partial \nu_x}E_{\tau}(x, z) + \rho(x)E_{\tau}(x, z) \right\} f(z)dS_z
\]

\[
= \int_{\partial D} dS_z f(z) \int_{\partial D} \left\{ \frac{\partial}{\partial \nu_x}E_{\tau}(x, z) + \rho(x)E_{\tau}(x, z) \right\} h(x)dS_x.
\]

This yields

\[
tY_{22}(\tau)h(z) = \int_{\partial D} \left\{ \frac{\partial}{\partial \nu_x}E_{\tau}(x, z) + \rho(x)E_{\tau}(x, z) \right\} h(x)dS_x, \ z \in \partial D.
\]

From this and (3.5) we obtain the desired formula.

\[ \square \]

Define

\[ M(\tau) = tY_{22}(\tau)(I - tY_{22}(\tau))^{-1}. \]

One can write

\[
(I - tY_{22}(\tau))^{-1} = I + tY_{22}(\tau) + (tY_{22}(\tau))^2(I - tY_{22}(\tau))^{-1}
\]

\[
= I + tY_{22}(\tau) + tY_{22}(\tau)M(\tau).
\]

(3.14)

Define, for an arbitrary \( z \neq x \) and \( x \in \partial D \)

\[ H_0(x, z) = \frac{\nu_x \cdot (z - x)}{|x - z|^2}, \]

\[ H_1(x, z) = \frac{1}{|x - z|} \left( \frac{\nu_x \cdot (z - x)}{|x - z|^2} + \rho(x) \right). \]

Since

\[ H(x, z, \tau) = \tau H_0(x, z) + H_1(x, z), \]

from lemma 3.1 we have

\[ tY_{22}(\tau) = M_0(\tau) + \tilde{M}(\tau), \]

(3.15)

(3.16)

where

\[
M_0(\tau)h(z) = \frac{\tau}{2\pi} \int_{\partial D} e^{-\tau|x-z|}H_0(x, z)h(x)dS_x,
\]

\[
\tilde{M}(\tau)h(z) = \frac{1}{2\pi} \int_{\partial D} e^{-\tau|x-z|}H_1(x, z)h(x)dS_x.
\]

(3.17)
Now set
\[ M_1(\tau) = \tilde{M}(\tau) + iY_{22}(\tau)M(\tau) \]  
and
\[ F_j(x, p, \tau) = e^{\tau|x-z|} \left( M_j(\tau) \left( \frac{e^{-\tau|z-p|}}{|z-p|} \right)(x) \right), \quad j = 0, 1. \]

From (3.14), (3.16) and (3.18) we have \((I - iY_{22}(\tau))^{-1} = I + M_0(\tau) + M_1(\tau)\) and thus (3.11) can be rewritten as
\[ F(x, p, \tau) = \frac{1}{|p - x|} + F_0(x, p, \tau) + F_1(x, p, \tau). \]

Substituting this and (3.15) into (3.13), we obtain

**Proposition 3.1** The decomposition
\[ I_0(\tau, p) = \tau I_{00}(\tau, p) + I_{01}(\tau, p), \]
is valid, where
\[ G_0(x, y, p, \tau) = H^+(x, y, p) + 2H_0(x, y)(F_0(x, p, \tau) + F_1(x, p, \tau)), \]
\[ G_1(x, y, p, \tau) = \frac{H_1(x, p)}{|x - y|} + \frac{H_1(x, y)}{|x - p|} + 2H_1(x, y)(F_0(x, p, \tau) + F_1(x, p, \tau)) \]
and
\[ I_{0j}(\tau, p) = \left( \frac{1}{2\pi} \right)^2 \int_{\partial\Omega} dS_y \varphi(y, \tau) \int_{\partial D} e^{-\tau l_p(x, y)} G_j(x, y, p, \tau) dS_x, \quad j = 0, 1. \]

### 3.2 Basic estimates of integral kernels

We introduce basic estimates of the integral kernels of the operators \(M_0(\tau)\) and \(M_1(\tau)\) introduced in (3.17) and (3.18). To obtain the asymptotic behaviour of \(I_0(\tau, p)\), these estimates of the kernels are essentially needed in our proof. In this subsection we always assume that \(D\) is a bounded domain with the boundary \(\partial D\) of class \(C^{2,\alpha_0}\) with \(0 < \alpha_0 \leq 1\).

It is well known that there exists a positive constant \(C\) such that for all \(x, z \in \partial D\)
\[ |\nu_x - \nu_z| \leq C|x - z|, \quad |\nu_x \cdot (x - z)| \leq C|x - z|^2. \]

From (3.17) and (3.20), we see that the integral kernel \(M_0(x, z, \tau)\) of the operator \(M_0(\tau)\) is given by
\[ M_0(x, z, \tau) = \frac{\tau}{2\pi} e^{-\tau|x-z|} \frac{\nu_z \cdot (x - z)}{|x - z|^2} \]
and has the estimate
\[ |M_0(x, z, \tau)| \leq C_0 \tau e^{-\tau|x-z|}, \quad x, z \in \partial D, \quad \tau > 0. \]

For \(M_1(\tau)\) we can obtain the following result:
Theorem 3.1 Assume that ∂D is strictly convex. Then there exist positive constants $C$ and $\mu_0 \geq 1$ such that: for all $\tau \geq \mu_0$ the operator $M_1(\tau)$ has an integral kernel $M_1(x, z, \tau)$ which is measurable for $(x, z) \in \partial D \times \partial D$, continuous for $x \neq z$ and has the estimate

$$|M_1(x, z, \tau)| \leq Ce^{-\tau|x-z|} \left(1 + \frac{1}{|x-z|} + \min \left\{\tau(\tau|x-z|^3)^{1/2}, \frac{1}{|x-z|^3}\right\}\right). \quad (3.23)$$

Remark 3.1 Since $\min \{\sqrt{a}, a^{-1}\} \leq 1$ for all $a > 0$, from (3.23) we get

$$|M_1(x, z, \tau)| \leq C \left(\tau + \frac{1}{|x-z|}\right)e^{-\tau|x-z|}. \quad (3.24)$$

These estimates are essential to obtain theorem 2.1. As is described in section 2, for a proof of theorem 3.1 is given in [12].

3.3 The structure of $\mathcal{M}(p)$

The last of the preliminaries, we study the structure of the set $\mathcal{M}(p)$.

Proposition 3.2 Assume that $\partial D$ is of class $C^2$. Then it holds that:

(1) if $(x_0, y_0) \in \mathcal{M}(p)$, then $\nu_{y_0} = (y_0 - x_0)/|y_0 - x_0|$;

(2) if $(x_0, y_0) \in \mathcal{M}_1(p)$, then $\nu_{x_0}$ has to be on the plane determined by the three points $p$, $x_0$, $y_0$ and the angle between $p - x_0$ and $\nu_{x_0}$ coincides with the angle between $y_0 - x_0$ and $\nu_{x_0}$;

(3) the set $\mathcal{M}(p)$ has the decomposition

$$\mathcal{M}(p) = \mathcal{M}_1(p) \cup \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p);$$

(4) if $(x_0, y_0) \in \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p)$, then there exists $t \in [0, 1]$ such that $x_0 = (1-t)p + ty_0$.

Further assume that $D$ is strictly convex. Then it holds that:

(5) if $(x_0, y_0) \in \mathcal{M}_2^-(p)$, then there exists a unique $x_0^* \in \mathcal{G}^+(p)$ such that $(x_0^*, y_0) \in \mathcal{M}_2^+(p)$;

(6) if $(x_0, y_0) \in \mathcal{M}_2^+(p)$, then there exists a unique $x_0^* \in \mathcal{G}^-(p)$ such that $(x_0^*, y_0) \in \mathcal{M}_2^-(p)$.

Proof. Let $(x_0, y_0) \in \mathcal{M}(p)$. Choose a system of local coordinates $x = x(\sigma)$, $\sigma = (\sigma_1, \sigma_2)$ with $x_0 = x(0)$ in a neighbourhood of $x_0 \in \partial D$. Similarly choose a system of local coordinates $y = y(\theta)$, $\theta = (\theta_1, \theta_2)$ with $y_0 = y(0)$ in a neighbourhood of $y_0 \in \partial \Omega$. Then the function $\tilde{l}_p(\sigma, \theta) = l_p(x(\sigma), y(\theta))$ takes the local minimum at $(\sigma, \theta) = (0, 0)$. Thus we have, for all $j = 1, 2$

$$\frac{\partial}{\partial \sigma_j} \tilde{l}_p(0, 0) = 0, \quad \frac{\partial}{\partial \theta_j} \tilde{l}_p(0, 0) = 0.$$

Since

$$\frac{\partial}{\partial \sigma_j} \tilde{l}_p(\sigma, \theta) = \left(\frac{x - p}{|x-p|} + \frac{x - y}{|x-y|}\right) \frac{\partial x}{\partial \sigma_j} \quad (3.25)$$

$$\frac{\partial}{\partial \theta_j} \tilde{l}_p(\sigma, \theta) = \left(\frac{y - p}{|y-p|} + \frac{y - x}{|y-x|}\right) \frac{\partial y}{\partial \theta_j} \quad (3.26)$$
and
\[ \frac{\partial}{\partial \sigma_j} \mathcal{l}_p(\sigma, \theta) = -\frac{x - y}{|x - y|} \cdot \frac{\partial y}{\partial \theta_j}, \]
we get
\[ \left( \frac{x_0 - p}{|x_0 - p|} + \frac{x_0 - y_0}{|x_0 - y_0|} \right) \cdot \frac{\partial x}{\partial \sigma_j}(0, 0) = 0 \] (3.27)
and
\[ \frac{x_0 - y_0}{|x_0 - y_0|} \frac{\partial y}{\partial \theta_j}(0, 0) = 0. \] (3.28)

This last equality yields that \( \nu_{y_0} \) and \( (y_0 - x_0)/|y_0 - x_0| \) have to be parallel. Assume that \( \nu_{y_0} = -(y_0 - x_0)/|y_0 - x_0| \). Then one can find a point \( y'_0 \) outside \( \Omega \) that is located on the segment \( x_0y_0 \). Since \( x_0 \in \Omega \), one can find a point \( y''_0 \in \partial \Omega \) on the segment \( x_0y'_0 \). Then we have \( l_p(x_0; y''_0) < l_p(x_0; y_0) \). This is against \( (x_0, y_0) \in \mathcal{M}(p) \). Therefore (1) has to be true.

Write
\[ \frac{p - x_0}{|p - x_0|} = \alpha \nu_{x_0} + \beta \frac{\partial x}{\partial \sigma_1}(0, 0) + \gamma \frac{\partial x}{\partial \sigma_2}(0, 0) \]
and
\[ \frac{y_0 - x_0}{|y_0 - x_0|} = \alpha' \nu_{x_0} + \beta' \frac{\partial x}{\partial \sigma_1}(0, 0) + \gamma' \frac{\partial x}{\partial \sigma_2}(0, 0). \]

Since \( \nu_{x_0} \cdot \partial x/\partial \sigma_j(0, 0) = 0 \), we have
\[ \alpha = \frac{p - x_0}{|p - x_0|} \cdot \nu_{x_0}, \quad \alpha' = \frac{y_0 - x_0}{|y_0 - x_0|} \cdot \nu_{x_0}. \]

From (3.27) we get the system of the equations for \( \beta + \beta' \) and \( \gamma + \gamma' \):
\[ \begin{pmatrix} \frac{\partial x}{\partial \sigma_1}(0, 0) \\ \frac{\partial x}{\partial \sigma_2}(0, 0) \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \sigma_1}(0, 0) \& \frac{\partial x}{\partial \sigma_2}(0, 0) \end{pmatrix} \begin{pmatrix} \beta + \beta' \\ \gamma + \gamma' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (3.29)

Since the vectors \( \partial x/\partial \sigma_j(0, 0), j = 1, 2 \) are linearly independent, the coefficients matrix of (3.29) is invertible and one gets \( \beta + \beta' = 0 \) and \( \gamma + \gamma' = 0 \). This yields
\[ \frac{p - x_0}{|p - x_0|} + \frac{y_0 - x_0}{|y_0 - x_0|} = (\alpha + \alpha') \nu_{x_0}. \] (3.30)

Moreover since the vectors \( (p - x_0)/|p - x_0| \) and \( (y_0 - x_0)/|y_0 - x_0| \) have the unit length, \( \beta^2 = \beta'^2, \gamma^2 = \gamma'^2 \) and \( \beta \gamma = \beta' \gamma' \), we get
\[ |\alpha| = |\alpha'|. \] (3.31)

If \( (x_0, y_0) \in \mathcal{M}_1(p) \), then both \( \alpha \) and \( \alpha' \) are positive and from (3.30) and (3.31) we obtain \( \alpha = \alpha' \) and
\[ \frac{p - x_0}{|p - x_0|} + \frac{y_0 - x_0}{|y_0 - x_0|} = 2\alpha \nu_{x_0}. \] (3.32)

This coincides with the law of reflection of the light and yields (2).
For the proof of (3) it suffices to prove that the set \( M(p) \) is contained in \( M_1(p) \cup M_2^+(p) \cup M_2^-(p) \cup M_\theta(p) \). We employ a contradiction argument. Assume that there exists a \((x_0, y_0) \in M(p) \setminus (M_1(p) \cup M_2^+(p) \cup M_2^-(p) \cup M_\theta(p))\). Since \((x_0, y_0)\) does not belong to \( M_\theta(p) \), we get \( x_0 \in G^+(p) \) or \( x_0 \in G^-(p) \). Consider the case when \( x_0 \in G^+(p) \).

Since \((x_0, y_0)\) does not belong to \( M_1(p) \cup M_2^+(p) \), we have \((y_0 - x_0) \cdot \nu_{x_0} = 0\). Then from (3.31) we have \((p - x_0) \cdot \nu_{x_0} = 0\). Contradiction. Next consider the case when \( x_0 \in G^-(p) \). Since \((x_0, y_0)\) does not belong to \( M_2^-(p) \), we have \((y_0 - x_0) \cdot \nu_{x_0} \leq 0\).

(3.31) yields \((y_0 - x_0) \cdot \nu_{x_0} < 0\). Since \( \partial D \) is \( C^2 \) at \( x_0 \), one can find an open ball \( B \) contained in \( D \) such that \( \partial B \cap \partial D = \{x_0\} \). Therefore the set of all numbers \( t \in [0, 1] \) such that \((1-s)x_0 + sy_0 \in D \) for all \( 0 < s < t \), is not empty. Denote by \( t^* \) the least upper bound of the set. It is easy to see that \( 0 < t^* < 1 \) and the point \( x_0' = (1 - t^*)x_0 + t^*y_0 \in \partial D \) and \( x_0 \neq x_0' \). If the points \( x_0', p \) and \( x_0 \) form a triangle, then by the triangle inequality we have \(|p - x_0| + |x_0 - y_0| > |p - x_0'| + |x_0' - y_0|\). If they do not form a triangle, then \( y_0 \) has to be on the segment \( px_0 \) since \((p - x_0) \cdot \nu_{x_0} < 0 \) and \((y_0 - x_0) \cdot \nu_{x_0} < 0 \). Since \(|p - x_0| = |p - x_0'| + |x_0 - x_0'| \) and \(|x_0 - y_0| = |x_0 - x_0'| + |x_0' - y_0|\), we get \(|p - x_0| + |x_0 - y_0| = |p - x_0'| + |x_0' - y_0| + 2|x_0 - x_0'| > |p - x_0'| + |x_0' - y_0|\). This against \((x_0, y_0) \in M(p) \). Contradiction. This completes the proof of (3).

The proof of (4) starts with the simple fact: if \((x_0, y_0) \in M_2^+(p) \cup M_2^-(p) \), then the numbers \((p - x_0) \cdot \nu_{x_0}\) and \((y_0 - x_0) \cdot \nu_{x_0}\) have different signature. This together with (3.31) yields \( \alpha + \alpha' = 0 \) in (3.30). If \((x_0, y_0) \in M_\theta(p) \), then \( \alpha = 0 \) and (3.31) gives again \( \alpha + \alpha' = 0 \) in (3.30). In any case we get

\[
\frac{p - x_0}{|p - x_0|} + \frac{y_0 - x_0}{|y_0 - x_0|} = 0. \tag{3.33}
\]

Therefore \( t = |p - x_0|/l_p(x_0, y_0) \) gives the desired conclusion. (5) and (6) are trivial.

\( \square \)

4 Proof of theorem 2.1

Given \( \delta > 0 \) define

\[
G_\delta(p) = \{x \in \partial D \mid \text{dist}(x, G(p)) \geq \delta\}, \ G_\delta^\pm(p) = G_\delta(p) \cap G^\pm(p).
\]

In this section first we state two crucial lemmas needed for establishing the asymptotic formula for \( l_0(\tau, p) \).

The first lemma is concerned with the asymptotic behaviour of the amplitudes of the integrals in proposition 3.1 and the proof is given in section 5.

**Lemma 4.1** There exists a positive constant \( \mu_0 \) such that the following assertions are true.

1. There exists a positive constant \( C \) such that if \( x \in \partial D \) and \( \tau \geq \mu_0 \), then

\[
|F_j(x, p, \tau)| \leq C\tau, \ j = 0, 1.
\]
(2) Given \( \delta > 0 \) there exists a positive constant \( C_\delta \) such that if \( x \in G^{+}_\delta(p) \) and \( \tau \geq \mu_0 \), then
\[
|F_j(x, p, \tau)| \leq C_\delta \tau^{-1}, \quad j = 0, 1.
\]

(3) Given \( \delta > 0 \) there exists a positive constant \( C_\delta \) such that if \( x \in G^{-}_\delta(p) \), and \( \tau \geq \mu_0 \), then
\[
|F_1(x, p, \tau)| \leq C_\delta \tau^{-1}.
\]

(4) Given \( \delta > 0 \) there exists a positive constant \( C_\delta \) such that if \( x \in G^{-}_\delta(p) \) and \( \tau \geq \mu_0 \), then
\[
|F_0(x, p, \tau) + 1/(|x - p|) \leq C_\delta \tau^{-\alpha_0/2}.
\]

The following lemma gives the asymptotic behaviour of an integral with an exponential weight and the idea behind the derivation is called the Laplace method.

**Lemma 4.2** Let \( U \) be an arbitrary open set of \( \mathbb{R}^n \). Let \( f \in C^{2, \alpha_0}(U) \) and satisfy at a point \( x_0 \in U \), for all \( x \in U \setminus \{x_0\} \) \( f(x) > f(x_0) \) and \( \det(\text{Hess}(f)(x_0)) > 0 \). Then given \( \varphi \in C^{0, \alpha_0}(\overline{U}) \) it holds that
\[
\int_U e^{-\tau f(x)} \varphi(x) dx = \frac{e^{-\tau f(x_0)}}{\sqrt{\det(\text{Hess}(f)(x_0))}} \left( \frac{2\pi}{\tau} \right)^{n/2} \left( \varphi(x_0) + \|\varphi\|_{C^{0, \alpha_0}(\overline{U})} O(\tau^{-\alpha_0/2}) \right).
\]

Moreover there exists a positive constant \( C \) such that, for all \( \tau \geq 1 \)
\[
\left| \int_U e^{-\tau f(x)} \varphi(x) dx \right| \leq \frac{C e^{-\tau f(x_0)}}{\tau^{n/2}} \|\varphi\|_{C(\overline{U})}.
\]

The proof of this lemma is given in Appendix A. We now give a proof of theorem 2.1. Since \( \mathcal{M}(p) \) is a finite set, one can write
\[
\mathcal{M}(p) = \{(x^{(j)}, y^{(j)}) \mid j = 1, 2, \ldots, N\}.
\]

However, by (5) and (6) of proposition 3.2 the counting number of the set \( \mathcal{M}_2^+(p) \) coincides with that of \( \mathcal{M}_2^-(p) \). Then (3) of proposition 3.2 yields that the counting number of the set \( \mathcal{M}(p) \setminus \mathcal{M}_1(p) \) has to be an even number. Hence one can write
\[
\mathcal{M}_1(p) = \{(x^{(j)}, y^{(j)}) \mid j = 1, 2, \ldots, N_1\},
\]
\[
\mathcal{M}_2^+(p) = \{(x^{(j)}, y^{(j)}) \mid j = N_1 + 1, \ldots, N_1 + N_2\},
\]
\[
\mathcal{M}_2^- (p) = \{(x^{(j)}, y^{(j)}) \mid j = N_1 + N_2 + 1, \ldots, N_1 + 2N_2\},
\]
where \( x^{(j)} = (x^{(j+N_2)})^*, \quad j = N_1 + 1, \ldots, N_1 + N_2 \) and \( N = N_1 + 2N_2 \).

From the second equation in (3.2) we have
\[
\psi(x, \tau) = (I - Y_{22}(\tau))^{-1}Y_{21}(\tau)\varphi(x, \tau).
\]
Then from the first equation of (3.2) we obtain the equation of $\varphi$ only:
\[
\{ I - Y_{11}(\tau) - Y_{12}(\tau)(I - Y_{22}(\tau))^{-1}Y_{21}(\tau) \} \varphi(x, \tau) = g(x, \tau).
\]
Since $\|Y_{ij}(\tau)\| = O(\tau^{-1})$ as $\tau \to \infty$, it follows from the equation that
\[
\varphi(y, \tau) = g(y, \tau) + O(\tau^{-1})\|g(\cdot, \tau)\|_{C(\partial \Omega)} \quad (4.1)
\]
as $\tau \to \infty$ uniformly for $y \in \partial \Omega$.

Given $\delta > 0$ set
\[
U_\delta(x^{(j)}) = \{ x \in \partial D \mid |x - x^{(j)}| < \delta \}, \quad V_\delta(y^{(j)}) = \{ y \in \partial \Omega \mid |y - y^{(j)}| < \delta \}.
\]
One can choose a sufficiently small $\delta > 0$ such that, for $j = 1, \ldots, N_1 + 2N_2$ $U_{2\delta}(x^{(j)}) \cap \mathcal{G}(p) = \emptyset$ and $(U_{2\delta}(x^{(j)}) \times V_{2\delta}(y^{(j)})) \cap \mathcal{M}(p) = \{(x^{(j)}, y^{(j)})\}$. Moreover since $l_p(x, y) > l(p, D)$ for all $(x, y)$ in the compact set $(\partial D \times \partial \Omega) \setminus \left(U_{j=1+2N_2}U_{\delta/3}(x^{(j)}) \times V_{\delta/3}(y^{(j)})\right)$, one can find a positive constant $c_0$ such that
\[
l_p(x, y) \geq l(p, D) + c_0 \quad \text{for} \quad (x, y) \in (\partial D \times \partial \Omega) \setminus \left(U_{j=1+2N_2}U_{\delta/3}(x^{(j)}) \times V_{\delta/3}(y^{(j)})\right).
\]
From this, (1) of lemma 4.1, (4.1) and proposition 3.1 one gets, for $k = 0, 1$
\[
I_{0k}(\tau, p) = \left(\frac{1}{2\pi}\right)^{2N_1+2N_2} \sum_{j=1}^{N_1+2N_2} I_{0kj}(\tau, p) + e^{-\tau l(p, D)}O(e^{-c_0\tau/2})\|g\|_{C(\partial \Omega)}.
\]

Here, for $j = 1, \ldots, N_1 + 2N_2$
\[
I_{0kj}(\tau, p) = \int_{V_\delta(x^{(j)})} dS_y \varphi(y, \tau) \int_{U_\delta(x^{(j)})} e^{-\tau l_p(x, y)} \Psi_j(x, y) G_k(x, y, p, \tau) dS_x
\]
and $\Psi_j \in C^2(\bar{U_\delta}(x^{(j)}) \times \bar{V_\delta}(y^{(j)}))$ is a cut-off function with $\Psi_j(x, y) = 1$ in $U_{\delta/2}(x^{(j)}) \times V_{\delta/2}(y^{(j)})$ and $\Psi_j(x, y) = 0$ in $(U_{2\delta/3}(x^{(j)}) \times V_{2\delta/3}(y^{(j)}))^c$.

We study the asymptotic behaviour of $I_{0kj}(\tau, p)$. Choose local coordinate systems $x = s^{(j)}(\sigma)$ with $x^{(j)} = s^{(j)}(0)$ for $U_\delta(x^{(j)})$ and $y = \tilde{s}^{(j)}(\tilde{\sigma})$ with $y^{(j)} = \tilde{s}^{(j)}(0)$ for $V_\delta(y^{(j)})$. Set $\Psi_j(\sigma, \tilde{\sigma}) = \Psi_j(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}))$,
\[
J_j(\sigma, \tilde{\sigma}) = \sqrt{\det \left( \frac{\partial s^{(j)}}{\partial \sigma_p} (\sigma) \cdot \frac{\partial s^{(j)}}{\partial \sigma_q} (\sigma) \right) \det \left( \frac{\partial \tilde{s}^{(j)}}{\partial \tilde{\sigma}_p} (\tilde{\sigma}) \cdot \frac{\partial \tilde{s}^{(j)}}{\partial \tilde{\sigma}_q} (\tilde{\sigma}) \right)}
\]
and $\tilde{l}_p^{(j)}(\sigma, \tilde{\sigma}) = l_p(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}))$. A change of variables gives the expression
\[
I_{0kj}(\tau, p) = \int_{\mathbb{R}^4} e^{-\tau \tilde{l}_p^{(j)}(\sigma, \tilde{\sigma})} \varphi(\tilde{s}^{(j)}(\tilde{\sigma}), \tau) \Psi_j(\sigma, \tilde{\sigma}) G_k(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}), \tau, p, \tau) J_j(\sigma, \tilde{\sigma}) d\sigma d\tilde{\sigma}.
\]

Since the function $x \mapsto (p - x) \cdot \nu_x$ is continuous and $\overline{U_\delta(x^{(j)}) \cap \mathcal{G}(p)} = \emptyset$, we have $U_\delta(x^{(j)}) \subset \mathcal{G}^+(p)$ for $j = 1, \ldots, N_1 + N_2$; $U_\delta(x^{(j)}) \subset \mathcal{G}^-(p)$ for $j = N_1 + N_2 + 1, \ldots, N_1 + 2N_2$. 

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Consider the case when \( j = 1, \cdots, N_1 + N_2 \). It follows from (2) of lemma 4.1
\[
G_0(x, y, p, \tau) = H^+(x, y, p) + O(\tau^{-1}), \quad G_1(x, y, p, \tau) = O(1)
\]
as \( \tau \to \infty \) uniformly for \( (x, y) \in \overline{U_\delta(x^{(j)})} \times \partial \Omega \). Since we have \( \text{Hess} \left( \tilde{I}_p^{(j)}(0, 0) \right) > 0 \), from these estimate, (4.1) and lemma 4.2, we obtain
\[
I_{01j}(\tau, p) = e^{-\tau l(p,D)}\|g(\cdot, \tau)\|_{C(\partial \Omega)} O(\tau^{-2}),
\]
(4.3)
and
\[
I_{00j}(\tau, p) = \frac{J_j(0,0)e^{-\tau l(p,D)}}{\sqrt{\det (\text{Hess} \left( \tilde{I}_p^{(j)}(0, 0) \right))}} \left( \frac{2\pi}{\tau} \right)^2
\times (g(y^{(j)}, \tau)H^+(x^{(j)}, y^{(j)}, p) + O(\tau^{-\alpha/2})\|g(\cdot, \tau)\|_{C^{\alpha,0}(\partial \Omega)}).
\]
(4.4)
Next consider the case when \( j = N_1 + N_2, \cdots, N_1 + 2N_2 \). From (3) and (4) of lemma 4.1 we get
\[
G_0(x, y, p, \tau) = H^-(x, y, p) + O(\tau^{-\alpha/2}), \quad G_1(x, y, p, \tau) = O(1)
\]
as \( \tau \to \infty \) uniformly for \( (x, y) \in \overline{U_\delta(x^{(j)})} \times \partial \Omega \). From these estimates, (4.1) and lemma 4.2 we obtain
\[
I_{01j}(\tau, p) = e^{-\tau l(p,D)}\|g(\cdot, \tau)\|_{C(\partial \Omega)} O(\tau^{-2})
\]
(4.5)
and
\[
I_{00j}(\tau, p) = \frac{J_j(0,0)e^{-\tau l(p,D)}}{\sqrt{\det (\text{Hess} \left( \tilde{I}_p^{(j)}(0, 0) \right))}} \left( \frac{2\pi}{\tau} \right)^2
\times (g(y^{(j)}, \tau)H^-(x^{(j)}, y^{(j)}, p) + O(\tau^{-\alpha/2})\|g(\cdot, \tau)\|_{C^{\alpha,0}(\partial \Omega)}).
\]
(4.6)
From proposition 3.1, (4.2) to (4.6) and the fact that \( H^+(x^{(j)}, y^{(j)}, p) = 0 \) for \( j = N_1 + 1, \cdots, N_1 + N_2 \) (see (3.33)), we obtain the desired asymptotic formula (2.4) for \( I_0(\tau, p) \). The coefficients \( C(x_0, y_0) \) in (2.5) for \( (x_0, y_0) = (x^{(j)}, y^{(j)}) \) are given by
\[
\frac{J_j(0,0)}{\sqrt{\det (\text{Hess} \left( \tilde{I}_p^{(j)}(0, 0) \right))}}
\]
and thus positive. This completes the proof of theorem 2.1.

5 Asymptotic behaviour of \( F_j(x, p, \tau) \)

In this section, we prove lemma 4.1. In the first two subsections, we prepare properties of the broken path and estimates of boundary integrals used to show lemma 4.1. The last subsection, we give a proof of lemma 4.1 using the estimates of the integral kernels of \( M_0(\tau) \) and \( M_1(\tau) \) given in (3.22) and theorem 3.1, respectively.

Throughout this section, we always assume that \( \partial D \) is of class \( C^{2,\alpha_0} \) with \( 0 < \alpha_0 \leq 1 \). We denote by \( B(x, r) \) the open ball centered at \( x \) with radius \( r \).
5.1 Properties of the broken path

The aim of this subsection is to study the behaviour of the function

\[ l_{(p,x)}(z) \equiv |p - z| + |z - x| \]

with the independent variable \( z \in \partial D \), and given \( p \in \mathbb{R}^3 \setminus \overline{\Omega} \) and \( x \in \partial D \).

We start with describing the following well known facts.

**Lemma 5.1** There exists \( 0 < r_0 \) such that, for all \( x \in \partial D \), \( \partial D \cap B(x,2r_0) \) can be represented as a graph of a function on the tangent plane of \( \partial D \) at \( x \), that is, there exist an open neighbourhood \( U_x \) of \((0,0)\) in \( \mathbb{R}^2 \) and a function \( g = g_x \in C^{2,\infty}(\mathbb{R}^2) \) with \( g(0,0) = 0 \) and \( \nabla g(0,0) = 0 \) such that the map

\[ U_x \ni \sigma = (\sigma_1, \sigma_2) \mapsto x + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma_1, \sigma_2) \nu_x \in \partial D \cap B(x,2r_0) \]

gives a system of local coordinates around \( x \), where \( \{e_1, e_2\} \) is an orthogonal basis for \( T_x(\partial D) \). Moreover the norm \( ||g||_{C^{2,\infty}(\mathbb{R}^2)} \) has an upper bound independent of \( x \in \partial D \).

In this paper we call this system of coordinates the standard system of local coordinates around \( x \).

The following lemma plays an important role in the proof of lemma 4.1.

**Lemma 5.2** Assume that \( \partial D \) is strictly convex. If \( x \in \mathcal{G}^+(p) \cup \mathcal{G}(p) \), then the function \( l_{(p,x)}(z), z \in \partial D \) attains the minimum only at \( z = x \). If \( x \in \mathcal{G}^-(p) \), then the points on \( \partial D \) that attain the minimum are given by only two points \( z = x, x^* \). Moreover the following statements are true.

(i) Given \( \delta > 0 \) there exists a positive constant \( C_\delta \) such that if \( x \in \mathcal{G}^+(p) \), then for all \( z \in \partial D \) we have

\[ l_{(p,x)}(z) \geq |p - x| + C_\delta |z - x|. \]

(ii) Given \( \delta > 0 \) there exists a constant \( 0 < \delta' \leq \delta \) such that if \( x \in \mathcal{G}^-(p) \), then \( |x - x^*| \geq 2\delta' \). Further, for any \( 0 < \delta' \leq \delta_0 \), there exists a positive constant \( C_{\delta'} \) such that, for all \( x \in \mathcal{G}^-(p) \) and \( z \in \partial D \setminus B(x^*, \delta') \) we have

\[ l_{(p,x)}(z) \geq |p - x| + C_{\delta'} |z - x|. \]

(iii) Given \( \delta > 0 \) there exist positive constants \( C_\delta \) and \( C_\delta' \) such that, if \( 0 < \delta' \leq C_\delta' \), then for all \( x \in \mathcal{G}^-(p) \) and \( z \in \partial D \cap \overline{B(x^*, \delta')}, \)

\[ l_{(p,x)}(z) \geq |p - x| + C_\delta |z - x^*|^2. \]

**Proof.** It is clear that \( \min_{z \in \partial D} l_{(p,x)}(z) = l_{(p,x)}(x) \). Let \( z \in \partial D \) be a point such that \( l_{(p,x)}(z) = l_{(p,x)}(x) \).

Consider the case \( x \in \mathcal{G}^+(p) \cup \mathcal{G}(p) \). Assume that \( z \neq x \). Since \( |p - z| + |z - x| = |p - x| \), \( z \) has to be on the line segment determined by \( p \) and \( x \). Since \( (p - x) \cdot \nu_x \geq 0 \), we have
\[(z - x) \cdot \nu_x \geq 0. \] On the other hand, since \(\partial D\) is strictly convex and \(z \neq x\), one gets \((z - x) \cdot \nu_x < 0\). This is a contradiction. Thus \(z = x\).

Next consider the case \(x \in \mathcal{G}^{-}(p)\). Assume that \(z \neq x\). Similarly to above one knows that \(z\) is located on the line segment determined by \(p\) and \(x\) and thus gets \(z = x^*\). Therefore the set of all points \(z\) that attain the minimum of \(l_{(p,x)}(\cdot)\) is contained in the set \(\{x, x^*\}\). However since \(l_{(p,x)}(x^*) = l_{(p,x)}(x)\), the function \(l_{(p,x)}(\cdot)\) really attains the minimum at \(z = x, x^*\).

Now we give a proof of (i). Let \(z \neq x\). We have
\[
|p - z|^2 = |p - x|^2 + |z - x|^2 - 2(p - x) \cdot (z - x)
\]
\[
= \left\{|p - x| - |z - x| \left(\frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|}\right)\right|^2 + |z - x|^2 \left\{1 - \left(\frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|}\right)^2\right\}.\]
This yields
\[
|p - z| \geq |p - x| - |z - x| \left(\frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|}\right).
\]
From this we obtain the estimate
\[
l_{(p,x)}(z) \geq |p - x| + |z - x| \left(1 - \frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|}\right). \tag{5.1}
\]
Let \(z'\) be the orthogonal projection of \(z\) onto \(T_x(\partial D)\). We see that \((z - z') \cdot (p - x) \leq 0\) since \((z - z') \cdot \nu_x \leq 0\), \((p - x) \cdot \nu_x \geq 0\) and \(z - z'\) is parallel to \(\nu_x\). It follows from this that
\[
\frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|} = \frac{z - z'}{|z - x|} \cdot \frac{p - x}{|p - x|} + \frac{z' - x}{|z - x|} \cdot \frac{p - x}{|p - x|} \leq \frac{z' - x}{|z - x|} \cdot \frac{p - x}{|p - x|}. \tag{5.2}
\]
First consider the case \((z' - x) \cdot (p - x) \geq 0\). Since \(|z - x| \geq |z' - x|\), from (5.2) we have
\[
\frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|} \leq \frac{z' - x}{|z' - x|} \cdot \frac{p - x}{|p - x|}. \tag{5.3}
\]
Let \(p'\) be the orthogonal projection of \(p\) onto \(T_x(\partial D)\). Since \(z' - x\) and \(p' - x\) are parallel to \(T_x(\partial D)\), we have
\[
\left|\frac{p - x}{|p - x|} - \frac{z' - x}{|z' - x|}\right| \geq \left|\frac{p - x}{|p - x|} - \frac{p' - x}{|p - x|}\right| = \frac{(p - x) \cdot \nu_x}{|p - x|}. \tag{5.4}
\]
Set
\[
A_\delta \equiv \inf_{x \in \mathcal{G}^{-}(p)} \frac{(p - x) \cdot \nu_x}{|p - x|} > 0.
\]
From (5.4) we have
\[
\frac{z' - x}{|z' - x|} \cdot \frac{p - x}{|p - x|} = 1 - \frac{1}{2} \left|\frac{p - x}{|p - x|} - \frac{z' - x}{|z' - x|}\right|^2 \leq 1 - \frac{1}{2} A_\delta^2. \tag{5.5}
\]
Now from (5.1), (5.3) and (5.5) we obtain
\[ l_{(p, x)}(z) \geq |p - x| + \frac{A_0^2}{2} |z - x| \]
provided \((z' - x) \cdot (p - x) \geq 0\). If \((z' - x) \cdot (p - x) < 0\), (5.2) gives
\[ \frac{z - x}{|z - x|} \cdot \frac{p - x}{|p - x|} < 0. \]
Then from (5.1) we have
\[ l_{(p, x)}(z) \geq |p - x| + |z - x|. \]
Therefore (i) holds for \(C_0 = \min \{A_0^2, 2\}/2\).

Next we give a proof of (ii). It is clear that the map: \(G^{-1}(p) \ni x \mapsto x^* \in G^+(p)\) is continuous. Since the set \(G^{-1}_0(p)\) is compact and \(|x - x^*| > 0\) for all \(x \in G^{-1}_0(p)\), we have
\[ B_0 = \inf_{x \in G^{-1}_0(p)} |x - x^*| > 0. \]
Then \(\delta'_0 = \min \{B_0/2, \delta\}\) satisfies the desired condition. Next we prove that
\[ D_{\delta'} = \sup_{x \in G^{-1}_0(p)} \sup_{z \in (\partial D \setminus \{x\}) \cap B(x^*, \delta')} \frac{|x - x^*|}{|z - x|} < 1. \]  
(5.6)

If this is not true, then the compactness of \(G^{-1}_0(p)\) and \(\partial D\) yields the existence of points \(x_0 \in G^{-1}_0(p)\) and \(z_0 \in \partial D\) and sequences \(\{x_n\}\) with \(x_n \in G^{-1}_0(p)\) and \(\{z_n\}\) with \(z_n \in (\partial D \setminus \{x_n\}) \cap B(x^*_n, \delta')\) such that, as \(n \to \infty\), \(x_n \to x_0, z_n \to z_0\) and
\[ \frac{z_n - x_n}{|z_n - x_n|} = \frac{x^*_n - x_n}{|x^*_n - x_n|} \to 1. \]  
(5.7)
Moreover, one may assume that the unit vectors \((z_n - x_n)/|z_n - x_n|\) converges to a unit vector \(\vartheta\). Since \(|x_n - x^*_n| \geq 2\delta'_0\), from the continuity of the map \(G^{-1}_0(p) \ni x \mapsto x^* \in \partial D\) we have \(x_0 \neq x^*_0\). Thus from (5.7) we obtain
\[ \vartheta \cdot \frac{x^*_0 - x_0}{|x^*_0 - x_0|} = 1. \]
This gives \(\vartheta = (x^*_0 - x_0)/|x^*_0 - x_0|\) and since \(\partial D\) is strictly convex, we obtain \(\vartheta \cdot \nu_{x_0} < 0\).

Consider the case when \(z_0 = x_0\). From (3.20) we obtain \(\vartheta \cdot \nu_{x_0} = 0\). This is a contradiction.

Next consider the case when \(z_0 \neq x_0\). In this case we obtain
\[ \frac{z_0 - x_0}{|z_0 - x_0|} = \frac{x^*_0 - x_0}{|x^*_0 - x_0|}. \]
This yields that \(z_0\) is located on the line determined by \(x_0\) and \(x^*_0\). Since \(\partial D\) is strictly convex, we have \(z_0 = x^*_0\). However, we have also \(|z_0 - x^*_0| \geq \delta'\). Contradiction.
Therefore (5.6) is valid. Since \( (x^* - x)/|x^* - x| = (p - x)/|p - x| \), from (5.1) we have

\[
l_{(p, x)}(z) \geq |p - x| + |z - x| \left( 1 - \frac{z - x}{|z - x|} \cdot \frac{x^* - x}{|x^* - x|} \right).
\]

Now the final conclusion of (ii) is true for \( C_{y'} = 1 - D_{y'} \).

Finally we give a proof of (iii). Since \( |p - x| = |p - x^*| + |x^* - x| \), we have

\[
l_{(p, x)}(z) - |p - x| = (|p - z| - |p - x^*|) + (|z - x| - |x^* - x|).
\]

(5.8)

Set \( \xi = z - x^* \). We have

\[
|p - z| - |p - x^*| = \frac{\xi^2 - 2(p - x^*) \cdot \xi}{|p - z| + |p - x^*|},
\]

(5.9)

\[
|z - x| - |x^* - x| = \frac{\xi^2 + 2(x^* - x) \cdot \xi}{|z - x| + |x^* - x|}.
\]

From (5.8) and (5.9) we have the expression

\[
l_{(p, x)}(z) - |p - x| = \left( \frac{1}{|p - z| + |p - x^*|} + \frac{1}{|z - x| + |x^* - x|} \right) \xi^2 + 2R(z) \cdot \xi,
\]

(5.10)

where

\[
R(z) = \frac{x^* - p}{|p - z| + |p - x^*|} + \frac{x^* - x}{|z - x| + |x^* - x|}.
\]

Since

\[
\frac{x^* - p}{|p - z| + |p - x^*|} - \frac{x^* - p}{2|p - x^*|} = \frac{(|p - z| - |p - x^*|)(x^* - p)}{2(|p - z| + |p - x^*|)|p - x^*|},
\]

\[
\frac{x^* - x}{|z - x| + |x^* - x|} - \frac{x^* - x}{2|x^* - x|} = \frac{(|z - x| - |x^* - x|)(x^* - x)}{2(|z - x| + |x^* - x|)|x^* - x|},
\]

from (5.9) one gets

\[
\frac{x^* - p}{|p - z| + |p - x^*|} = \frac{1}{2} \frac{x^* - p}{|p - x^*|}
\]

\[
+ \frac{\{x^* \cdot \xi\}(x^* - p)}{(|p - z| + |p - x^*|)^2|p - x^*|} = \frac{1}{2} \frac{|\xi|^2(x^* - p)}{(|p - z| + |p - x^*|)^2|p - x^*|}
\]

and

\[
\frac{x^* - x}{|z - x| + |x^* - x|} = \frac{1}{2} \frac{x^* - x}{|x^* - x|}
\]

\[
- \frac{\{x^* \cdot \xi\}(x^* - x)}{(|z - x| + |x^* - x|)^2|x^* - x|} = \frac{1}{2} \frac{|\xi|^2(x^* - x)}{(|z - x| + |x^* - x|)^2|x^* - x|}.
\]

Since

\[
\frac{x^* - x}{|x^* - x|} + \frac{x^* - p}{|x^* - p|} = 0,
\]

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it follows that
\[
R(z) \cdot \xi = -\frac{\{(x^* - p) \cdot \xi\}^2}{(|p - z| + |p - x^*|)^2|p - x^*|} - \frac{\{(x^* - x) \cdot \xi\}^2}{(|z - x| + |x^* - x|)^2|x^* - x|}
\]

\[
-\frac{1}{2} \frac{|\xi|^2 (x^* - p) \cdot \xi}{(|p - z| + |p - x^*|)^2|p - x^*|} - \frac{1}{2} \frac{|\xi|^2 (x^* - x) \cdot \xi}{(|z - x| + |x^* - x|)^2|x^* - x|}.
\]

Using the facts
\[
\inf_{(x, z) \in G^\delta(p) \times \partial D} |p - z| + |p - x^*| > 0, \quad \inf_{(x, z) \in G^\delta(p) \times \partial D} |z - x| + |x^* - x| > 0, \quad (5.11)
\]

from (5.10) we obtain
\[
l_{(p, x)}(z) = |p - x| + K(z) \xi \cdot \xi + O(|\xi|^3) \quad (5.12)
\]

uniformly for \(x \in G^\delta_p(p)\), where
\[
K(z) = \left( \frac{1}{|p - z| + |p - x^*|} + \frac{1}{|z - x| + |x^* - x|} \right) I_3
\]

\[
-\frac{2(x^* - p) \otimes (x^* - p)}{(|p - z| + |p - x^*|)^2|p - x^*|} - \frac{2(x^* - x) \otimes (x^* - x)}{(|z - x| + |x^* - x|)^2|x^* - x|}.
\]

Set
\[
\vartheta = \frac{x^* - p}{|x^* - p|}.
\]

Then
\[
\frac{x^* - x}{|x^* - x|} = -\vartheta
\]

and we have
\[
K(z) = \left( \frac{1}{|p - z| + |p - x^*|} + \frac{1}{|z - x| + |x^* - x|} \right) I_3
\]

\[
-2 \left( \frac{|x^* - p|}{(|p - z| + |p - x^*|)^2} + \frac{|x^* - x|}{(|z - x| + |x^* - x|)^2} \right) \vartheta \otimes \vartheta.
\]

Let \(z = z(\sigma) = x^* + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma) \nu_2^*\) be the standard local coordinate system around \(x^*\) for \(z \in \partial D \cap B(x^*, 2r_0)\). From lemma 5.1 we know that, for a suitable constant \(C\) depending only on \(\partial D\) we have \(|\sigma| \leq |\xi| \leq C|\sigma|\). Using (5.11) together with the following facts
\[
\inf_{x \in G^\delta(p)} |p - x^*| > 0, \quad \inf_{x \in G^\delta(p)} |x^* - x| > 0,
\]

we have
\[
\frac{1}{|p - z| + |p - x^*|} = \frac{1}{2|p - x^*|} + O(|\xi|), \quad \frac{1}{|z - x| + |x^* - x|} = \frac{1}{2|x^* - x|} + O(|\xi|)
\]

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uniformly for $x \in G_\delta^-(p)$. These yield

$$K(z) = \frac{1}{2} \left( \frac{1}{|p-x|} + \frac{1}{|x^*-x|} \right) (I_3 - \vartheta \otimes \vartheta) + O(|\xi|).$$

Since $\xi = \sigma_1 e_1 + \sigma_2 e_2 + O(|\xi|^2)$, we obtain

$$K(z) \xi \cdot \xi = \frac{1}{2} \left( \frac{1}{|p-x|} + \frac{1}{|x^*-x|} \right) (I_2 - \vartheta' \otimes \vartheta') \sigma \cdot \vartheta + O(|\xi|^3)$$

where $\vartheta' = (\vartheta \cdot e_1, \vartheta \cdot e_2)^T$. Here we note that the eigenvalues of the $2 \times 2$-matrix $I_2 - \vartheta' \otimes \vartheta'$ are given by $1$ and $1 - |\vartheta \cdot e_1|^2 - |\vartheta \cdot e_2|^2 = |\vartheta \cdot \nu_x|^2$. Therefore we conclude that

$$K(z) \xi \cdot \xi \geq \frac{1}{2} \left( \frac{1}{|p-x|} + \frac{1}{|x^*-x|} \right) |\vartheta \cdot \nu_x|^2|\sigma|^2 + O(|\xi|^3).$$

Since

$$\inf_{x \in G_\delta^-(p)} \frac{p-x^*}{|p-x^*|} \cdot \nu_x^* > 0,$$

from (5.12) we obtain the desired conclusion.

\[\square\]

**Remark 5.1** From the proof of (iii) we obtain the expression

$$l_{(p,x)}(z) - |p-x| = K(z) \xi \cdot \xi - \frac{|\xi|^2(x^*-p) \cdot \xi}{(|p-z| + |p-x^*|)^2 |p-x^*|} - \frac{|\xi|^2(x^*-x) \cdot \xi}{(|z-x| + |x^*-x|)^2 |x^*-x|}$$

with $\xi = z-x^*$. To show theorem 2.1, we need this equality.

### 5.2 Estimates of integrals on the boundary $\partial D$

To show lemma 4.1, we need the following estimates:

**Lemma 5.3** Let $r_0$ be the same as that of lemma 5.1. There exists a positive constant $C$ depending only on $\partial D$ such that

(i) for all $x \in \partial D$, $0 < \rho'_0 \leq r_0$, $\tau > 0$, $0 \leq k < 2$

$$\int_{B(x,\rho'_0) \cap \partial D} \frac{e^{-\tau|x-z|}}{|x-z|^k} dS_z \leq \frac{C}{2-k} \min \{ \tau^{-2+k}, (\rho'_0)^{2-k} \};$$

(ii) for all $x \in \partial D$, $\tau > 0$, $0 \leq k < 2$

$$\int_{\partial D} \frac{e^{-\tau|x-z|}}{|x-z|^k} dS_z \leq \frac{C}{2-k} \tau^{-(2-k)} \left( 1 + \frac{\tau^{2-k} e^{-\tau r_0}}{\rho'_0^k} \right).$$
Proof. Let \( z = s(\sigma) \) be the standard system of local coordinates around \( x \) with \(|\sigma|^2 + g(\sigma)^2 < (2r_0)^2\). We have

\[
\int_{B(x,\rho_0')\cap \partial D} \frac{e^{-\tau|x-z|}}{|x-z|^k} \, dS_z = \int_{|\sigma|^2 + g(\sigma)^2 < (\rho_0')^2} \frac{e^{-\tau\sqrt{|\sigma|^2 + g(\sigma)^2}}}{(|\sigma|^2 + g(\sigma)^2)^{k/2}} \sqrt{1 + |\nabla g(\sigma)|^2} \, d\sigma
\]

\[
\leq C \int_0^{\rho_0'} \int_0^{2\pi} \frac{e^{-\tau r}}{r^{k-1}} r \, dr \, d\theta \leq 2\pi C \int_0^{\rho_0'} e^{-\tau r^{1-k}} \, dr.
\]

Here note that

\[
\int_0^{\rho_0'} e^{-\tau r^{1-k}} \, dr \leq \int_0^{\rho_0'} r^{1-k} \, dr = \frac{(\rho_0')^{2-k}}{2-k}
\]

and

\[
\int_0^{\rho_0'} e^{-\tau r^{1-k}} \, dr = \tau^{k-2} \int_0^{\tau \rho_0'} e^{-\tau r^{1-k}} \, dr
\]

\[
\leq \tau^{k-2} \int_0^{\infty} e^{-\tau r^{1-k}} \, dr = \tau^{k-2} \left( 1 + \frac{1}{2-k} \right) \leq \frac{3}{2-k} \tau^{k-2}.
\]

This proves (i). To verify (ii) we compute

\[
\int_{\partial D \setminus B(x,\rho_0)} \frac{e^{-\tau|x-z|}}{|x-z|^k} \, dS_z \leq e^{-\tau \rho_0'} \int_{\partial D} \frac{1}{r_0^k} \, dS_z \leq C \frac{e^{-\tau \rho_0'}}{r_0^k}.
\]

From this and (i) for \( \rho_0' = r_0 \) we obtain (ii). This completes the proof of lemma 5.3.

\[\square\]

### 5.3 Proof of lemma 4.1

We start with the expression for \( F_j(x,p,\tau) \) for \( j = 0, 1 \) (see (3.19)):

\[
F_j(x,p,\tau) = e^{\tau|x-p|} \int_{\partial D} M_j(x,z,\tau) \frac{e^{-\tau|z-p|}}{|z-p|} \, dS_z, \quad x \in \partial D.
\]

For the case \( j = 0 \) \( M_0(x,y,\tau) \) is given by (3.21) and the case \( j = 1 \) is a consequence of theorem 3.1.

First we prove (1) of lemma 4.1. From (3.22) and (3.24) we get

\[
|F_j(x,p,\tau)| \leq C e^{\tau|x-p|} \int_{\partial D} |M_j(x,z,\tau)| \frac{e^{-\tau|z-p|}}{|z-p|} \, dS_z
\]

\[
\leq C \int_{\partial D} \left( \tau + \frac{1}{|x-z|} \right) \frac{e^{-\tau(|x-z|+|z-p|-|x-p|)}}{|z-p|} \, dS_z.
\]

(5.13)

Since \(|x-z| + |z-p| \geq |x-p|\), the right-hand side of (5.13) has the bound

\[
\frac{C}{\text{dist}(p,\partial D)} \int_{\partial D} \left( \tau + \frac{1}{|x-z|} \right) \, dS_z.
\]
Applying the argument for the proof of (ii) in lemma 5.3 to the integral above, we see that
\[ \sup_{x \in \partial D} \int_{\partial D} \frac{dS_z}{|x - z|} < \infty. \]
Thus one concludes that (1) is true.

Second we prove (2) of lemma 4.1. Consider the case when \( x \in G^+(p) \). One can apply (i) of lemma 5.2 to the integrand in the right-hand side of (5.13) and get
\[
\int_{\partial D} \left( \tau + \frac{1}{|x - z|} \right) e^{-\tau|z - p|} \frac{dS_z}{|z - p|} \leq \int_{\partial D} \left( \tau + \frac{1}{|x - z|} \right) e^{-\tau C_\delta |z - x|} dS_z.
\]
Applying (ii) of lemma 5.3 to the integral of the right-hand side above, one gets
\[
\int_{\partial D} \left( \tau + \frac{1}{|x - z|} \right) e^{-\tau C_\delta |z - x|} dS_z \leq C(\tau \cdot \tau^{-2} + \tau^{-1}).
\]
Thus this together with (5.13) yields that (2) is true.

Third we prove (3) of lemma 4.1. By (ii) and (iii) of lemma 5.2, one can find \( C_\delta > 0 \) and \( \delta' > 0 \) such that, for all \( x \in G^-\delta(p) \)
\[
|x - x^*| \geq 2\delta', \tag{5.14}
\]
\[
|x - z| + |z - p| \geq |p - x| + C_\delta |z - x|, \quad z \in \partial D \setminus B(x^*, \delta'), \tag{5.15}
\]
and
\[
|x - z| + |z - p| \geq |p - x| + C_\delta |z - x^*|^2, \quad z \in \partial D \cap B(x^*, \delta'). \tag{5.16}
\]
We decompose \( \partial D \) into two parts \( \partial D \cap B(x^*, \delta') \) and \( \partial D \setminus B(x^*, \delta') \). Then we have
\[
|F_1(x, p, \tau)| \leq I + II
\]
where
\[
I = e^{\tau|x - p|} \int_{\partial D \cap B(x^*, \delta')} |M_1(x, z, \tau)| \frac{e^{-\tau|z - p|}}{|z - p|} dS_z,
\]
\[
II = e^{\tau|x - p|} \int_{\partial D \setminus B(x^*, \delta')} |M_1(x, z, \tau)| \frac{e^{-\tau|z - p|}}{|z - p|} dS_z.
\]
From (5.14) we see that if \( z \in \partial D \cap B(x^*, \delta') \), then \( |z - x| \geq \delta' \). This together with (3.23) gives
\[
|M_1(x, z, \tau)| \leq Ce^{-\tau|x - x^*|} \left( 1 + \frac{1}{\delta'} + \frac{1}{\delta^2} \right).
\]
It follows from this and (5.16) that
\[
I \leq \frac{C}{\operatorname{dist}(p, \partial D)} \int_{\partial D \cap B(x^*, \delta')} e^{-\tau C_\delta |z - x^*|^2} dS_z.
\]
Note that \( \delta' \) can be arbitrary small and thus one may assume that \( \delta' < 2r_0 \), where \( r_0 \) is given in lemma 5.1. Using the standard local coordinates around \( x^* \), one obtains

\[
\int_{\partial D \cap B(x^*, \delta')} e^{-\tau C_{\delta} |z - x^*|^2} dS_z \leq C \int_0^{\delta'} \int_0^{2\pi} e^{-\tau C_{\delta} r^2} r dr d\theta \leq C' \tau^{-1}
\]

and this thus yields \( I \leq C\tau^{-1} \). For the estimation of \( II \) we make use of (3.24) and (5.15). This gives

\[
II \leq \int_{\partial D \setminus B(x^*, \delta')} \left( \tau + \frac{1}{|x - z|} \right) e^{-\tau C_{\delta} |x - z|} \frac{dS_z}{|z - p|} \leq \frac{C}{\text{dist}(p, \partial D)} \int_{\partial D} \left( \tau + \frac{1}{|x - z|} \right) e^{-\tau C_{\delta} |x - z|} dS_z \leq C\tau^{-1}.
\]

Therefore \( II \leq C\tau^{-1} \) and this completes the proof of (3).

Finally we prove (4) of lemma 4.1. From (3.21) one gets

\[
F_0(x, p, \tau) = \frac{\tau}{2\pi} \int_{\partial D \setminus B(x^*, \delta')} \frac{\nu_z \cdot (x - z)}{|x - z|^2} \frac{e^{-\tau(|x - z| + |z - p| - |x - p|)}}{|z - p|} dS_z + R(\tau),
\]

where

\[
R(\tau) = \frac{\tau}{2\pi} \int_{\partial D \setminus B(x^*, \delta')} \frac{\nu_z \cdot (x - z)}{|x - z|^2} \frac{e^{-\tau(|x - z| + |z - p| - |x - p|)}}{|z - p|} dS_z.
\]

For \( r_0 \) in lemma 5.1, one can choose \( \delta' \) above in such a way that \( \delta' \leq 2r_0 \) and (5.14)-(5.16) are also satisfied. A combination of the second inequality of (3.20) and (5.15) yields

\[
|R(\tau)| \leq \frac{C \tau}{\text{dist}(p, \partial D)} \int_{\partial D} e^{-\tau C_{\delta} |x - z|} dS_z = O(\tau^{-1}).
\]

Denote by \( I(\tau) \) the first integral of the right-hand side of (5.17). Let \( z = s(\sigma) = x^* + \sigma_1 e_1 + \sigma_2 e_2 - g(\nu) \nu_x^* \) be the standard local coordinates around \( x^* \). Choose a function \( \chi \in C^\infty(\mathbb{R}^2) \) such that \( \chi = 1 \) near \( |s(\sigma) - x^*| \leq \delta'/8 \) and \( \chi = 0 \) for \( |s(\sigma) - x^*| \geq \delta'/4 \). Set

\[
L(\sigma) = |p - s(\sigma)| + |s(\sigma) - x^*| - |x - p|.
\]

It follows from (5.16) that if \( \delta'/4 \geq |s(\sigma) - x^*| \geq \delta'/8 \), then \( L(\sigma) \geq C_{\delta}(\delta'/8)^2 \). This yields

\[
I(\tau) = \int_{\mathbb{R}^2} \chi(\sigma) \frac{\nu_z(s(\sigma) \cdot (x - s(\sigma)))}{|x - s(\sigma)|^2} \frac{e^{-\tau L(\sigma)}}{|s(\sigma) - p|} \sqrt{1 + \nabla g(\sigma)^2} d\sigma + O \left( e^{-C_{\delta}(\delta'/8)^2 \tau} \right).
\]

Here we compute \( \det(\text{Hess}(L)(0)) \). From remark 5.1, one can easily obtain

\[
\frac{\partial^2}{\partial\sigma_i \partial\sigma_j} L(0) = 2K(x^*) e_i \cdot e_j = \left( \frac{1}{|p - x^*|} + \frac{1}{|x - x^*|} \right) (I_3 - \vartheta \otimes \vartheta) e_i \cdot e_j
\]

\[
= \left( \frac{1}{|p - x^*|} + \frac{1}{|x - x^*|} \right) (\delta_{ij} - (\vartheta \cdot e_i)(\vartheta \cdot e_j))
\]

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where \( \vartheta = (p - x^*)/|p - x^*| \). This together with the equation \( |p - x^*| + |x - x^*| = |p - x| \) gives

\[
\det (\text{Hess}(L)(0)) = \left( \frac{1}{|p - x^*|} + \frac{1}{|x - x^*|} \right)^2 |\vartheta \cdot \nu_x|^2 = \left( \frac{|p - x|}{|p - x^*| |x - x^*|} \right)^2 |\vartheta \cdot \nu_x|^2.
\]

Since \( \vartheta \cdot \nu_x > 0 \), we obtain

\[
\sqrt{\det (\text{Hess}(L)(0))} = \frac{|p - x|}{|p - x^*| |x - x^*|} \vartheta \cdot \nu_x.
\]

Set

\[
\Phi(\sigma) = \chi(\sigma) \frac{\nu_{s(\sigma)} \cdot (x - s(\sigma))}{|x - s(\sigma)|^2} \frac{e^{-\tau L(\sigma)}}{|s(\sigma) - p|} \sqrt{1 + \nabla g(\sigma)^2}.
\]

Since \( (p - x^*)/|p - x^*| = -(x^* - x)/|x - x^*| \), \( \Phi(0) \) has the form

\[
\Phi(0) = -\frac{\vartheta \cdot \nu_x}{|p - x^*| |x - x^*|}.
\]

This yields

\[
\frac{\Phi(0)}{\sqrt{\det (\text{Hess}(L)(0))}} = -\frac{1}{|p - x|}.
\]

Now we are ready to apply lemma 4.2 to the integral \( I(\tau) \). The result is

\[
I(\tau) = \frac{e^{-\tau L(0)}}{\sqrt{\det (\text{Hess}(L)(0))}} \left( \frac{2\pi}{\tau} \right)^{2/2} (\Phi(0) + O(\tau^{-\sigma_0/2})) + O(\tau^{-\infty})
\]

\[
= -\frac{2\pi}{\tau} \frac{1}{|p - x|} + O(\tau^{-\sigma_0/2 - 1}).
\]

From this together with (5.17) and (5.18) we obtain the desired conclusion.

\( \square \)

## 6 Sufficient conditions and examples

It is curious to know when assumptions of theorems 1.1 and 2.1 are satisfied. We can give sufficient conditions to ensure that a point \((x_0, y_0) \in \mathcal{M}(p) \setminus \mathcal{M}_g(p)\) is a non-degenerate critical point of \( l_p \) on \( \partial D \times \partial \Omega \). The conditions are given by using the Weingarten map of \( C^2 \) surfaces \( S \subset \mathbb{R}^3 \). Assume that \( S \) is the \( C^2 \) boundary of a bounded open set like \( \partial \Omega \) and \( \partial D \). Let \( \nu_x \) be the unit outer normal of \( S \) at \( x \in S \). For a tangential vector field \( v \in T_x(S) \) to \( S \) at \( x \in S \), the Weingarten map \( \mathcal{A}_{S,x}(v) \) is defined by \( \mathcal{A}_{S,x}(v) = D_v \nu_x \). By the original local coordinate of \( \mathbb{R}^3 \), \( \nu_x \) is given as \( \nu_x = (\nu_1(x), \nu_2(x), \nu_3(x)) \), and \( \mathcal{A}_{S,x} \) is expressed as follows:

\[
\mathcal{A}_{S,x}(v) = (v_x(\nu_1), v_x(\nu_2), v_x(\nu_3)) \quad (v \in T_x(S)).
\]

From the definition, we can show that \( \mathcal{A}_{S,x} \) is a linear map on the tangent space \( T_x(S) \), and \( \mathcal{A}_{S,x} = 1/R \) when \( S \) is a ball with radius \( R > 0 \).
Proposition 6.1 Let \( (x_0, y_0) \in \mathcal{M}(p) \setminus \mathcal{M}_g(p) \). Assume that the Weingarten map \( A_{\partial \Omega, y_0} \) of \( \partial \Omega \) at \( y_0 \) satisfies

\[
A_{\partial \Omega, y_0} < \frac{1}{l(p, D)} I.
\]

Then \( (x_0, y_0) \) should be a non-degenerate critical point of \( l_p \) on \( \partial D \times \partial \Omega \).

Remark 6.1 For every point \( (x_0, y_0) \in \mathcal{M}_1(p) \), a sufficient condition (6.1) for non-degenerateness can be relaxed (cf. proposition 6.3).

We need to check \( \mathcal{M}_g(p) = \emptyset \) to apply theorems 1.1, 2.1 and proposition 6.1 About this, we introduce a sufficient condition to satisfy \( \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p) = \emptyset \).

Proposition 6.2 If the set \( \mathcal{L}(p) = \{ y \in \partial \Omega \mid (y - p)/|y - p| \cdot \nu_y = 1 \} \) consists of a single point, then \( \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p) = \emptyset \).

Note that \( \mathcal{L}(p) \neq \emptyset \) since every point \( y_0 \in \partial \Omega \) attaining local maximum of the function \( \partial \Omega \ni y \to |y - p| \) belongs to \( \mathcal{L}(p) \). If \( \Omega \) is a ball, it is clear the assumption of proposition 6.2 is satisfied. However, even if \( \Omega \) is convex, \( \mathcal{L}(p) \) does not always consist of a single point. For example, consider the case that \( \partial \Omega \subset \Omega \) centered at the origin. Assume that \( \partial \Omega \) contains a part of the sphere with the center \( p \) and the radius \( r = \max_{y \in \partial \Omega} |y - p| \).

From propositions 6.1 and 6.2 we can give examples for corollary 2.1 deduced by theorems 1.1 and 2.1 in sections 1 and 2. We begin with introducing the following corollary:

Corollary 6.1 Let \( \Omega \) be the open ball with radius \( R \) centered at the origin. Assume that \( \partial D \) is strictly convex and there exists a \( \eta > 0 \) such that \( D \) contains the open ball with radius \( R/2 + \eta \) centered at the origin. Let \( p \in \mathbb{R}^3 \setminus \overline{\Omega} \) satisfy dist\( (p, \partial \Omega) < 2\eta \). Let \( f(y, t) \) be the function of \( (y, t) \in \partial \Omega \times [0, T] \) having the form \( f(y)\varphi(t) \), where \( f \in C^{0, \alpha}(\partial \Omega) \) with \( f(y) \neq 0 \) for all \( y \in \partial \Omega; \varphi \in L^2(0, T) \) satisfying the following condition: there exists a \( \tau > 0 \) such that

\[
0 < \lim_{\tau \to \infty} \tau \left| \int_0^T e^{-\tau^2 t} \varphi(t) dt \right| < \infty.
\]

Then the formula (1.9) is valid.

Proof. Set \( \epsilon = \text{dist}(p, \partial \Omega) \). Since \( \text{dist}(p, \partial D) \leq \text{dist}(p, \partial B_{R/2+\eta}) \) and

\[
\text{dist}(p, \partial B_{R/2+\eta}) = \frac{R}{2} - \eta + \epsilon,
\]

we have

\[
2\text{dist}(p, \partial D) - \text{dist}(p, \partial \Omega) \leq R - (2\eta - \epsilon).
\]

Since \( \epsilon < 2\eta \), this together with inequality \( 2\text{dist}(p, \partial D) - \text{dist}(p, \partial \Omega) \geq l(p, D) \) which can be easily verified yields \( l(p, D) < R \). Now from this, propositions 6.1, 6.2 and corollary 2.1 we obtain the desired conclusion.

\( \square \)

Now, we give a simple example of a pair of \( \Omega \) and \( D \) in which the minimum length \( l(p, D) \) can be obtained by the indicator function \( I(\tau, p) \).
Example 6.1 Let $R > r > 0$. Let $\Omega$ and $D$ be the open balls with radius $R$ and $r$, respectively centered at a common point $q$. Let $p$ be an arbitrary point outside $\Omega$ with $\text{dist}(p, \partial \Omega) = h > 0$. Then by proposition 6.2 one knows that $\mathcal{M}_p(\nu) = \mathcal{M}_2^\nu(p) = \mathcal{M}_2(p) = \emptyset$ and $\mathcal{M}(p) = \mathcal{M}_1(p)$. Let $(x_0, y_0) \in \mathcal{M}_1(p)$. Since $l(p, D)$ is the minimum of $l_p$, the function $y \mapsto |y - x_0|$ takes a local minimum at $y_0$. This implies that $y_0 - x_0$ and $\partial \Omega$ are parallel (precisely, we have $\nu_{y_0} = |y_0 - x_0|^{-1}(y_0 - x_0)$ as in (1) of proposition 3.2). Since $D$ and $\Omega$ are spheres having the common center, the point $x_0$ has to be on the line determined by $p$ and $q$. Then one gets $\mathcal{M}_1(\nu) = \{(q + r(p - q)/|p - q|, q + R(p - q)/|p - q|)\}$ and $l(p, D) = h + 2(R - r)$. Assume that we know $r_0 \in |R/2, R|$ such that $r > r_0$. This $r_0$ can be considered as an a-priori information about unknown $r$. Choose $p$ in such a way that $h/2 < r_0 - R/2$. Since $\mathcal{A}_{\partial \Omega, y_0} = (1/R)I$, the condition (6.1) is satisfied. In this case, from (2.5) we see that the condition (2.7) becomes $\lim_{\tau \to \infty} \tau |g(q + R(p - q)/|p - q|, \tau)| > 0$. Therefore if only this condition and corresponding one to (2.8) are satisfied, one can extract the quantity $h + 2(R - r)$ from (1.9).

As is mentioned in remark 6.1, assumption (6.1) in proposition 6.1 can be relaxed. Using this fact, we can cover other example containing example 6.1, which also justifies the fact that theorem 1.1 is considered as a three-dimensional analogue of (1.5) (for this example, see subsection 6.3).

6.1 Proof of proposition 6.2

We give a proof of proposition 6.2 in here.

Step 1. We prove: if $y' \in \partial \Omega$ attains the maximum of the function $f(y) = |y - p|$, $y \in \partial \Omega$, then $y' \in \mathcal{L}(p)$.

Using a local coordinate at $y'$, we see that the vectors $(y' - p)/|y' - p|$ and $\nu_{y'}$ are parallel each other. Assume that $(y' - p)/|y' - p| = -\nu_{y'}$. Since $\partial \Omega$ is $C^2$, one can find a point $y''$ on $\partial \Omega$ in such a way that $y'' = y' - \tau \nu_{y'}$ with a $\tau > 0$. Then $f(y'') = \tau f(y') > f(y')$. This is a contradiction. Therefore it must hold that $(y' - p)/|y' - p| = \nu_{y'}$. This is nothing but $y' \in \mathcal{L}(p)$. Since the existence of $y'$ is clear, this implies $\mathcal{L}(p) \neq \emptyset$.

Step 2. We prove: if $(x_0, y_0) \in \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p)$, then $y_0 \in \mathcal{L}(p)$.

By (1) of proposition 3.2, $(y_0 - x_0)/|y_0 - x_0| = \nu_{y_0}$. On the other hand from (4) of proposition 3.2 we have $(y_0 - x_0)/|y_0 - x_0| = (y_0 - p)/|y_0 - p|$. This gives $(y_0 - p)/|y_0 - p| \cdot \nu_{y_0} = 1$, that is, $y_0 \in \mathcal{L}(p)$.

Now assume that $\mathcal{L}(p)$ consists of a single point. Let $(x_0, y_0) \in \mathcal{M}_2^+(p) \cup \mathcal{M}_2^-(p) \cup \mathcal{M}_g(p)$. From the second step we have $y_0 \in \mathcal{L}(p)$. Then the first step implies that $f(y)$ attains the maximum at $y_0$ only. Using the assumption that $\partial D$ is $C^2$ at $x_0$ and choosing a suitable half line that starts at $p$, one can conclude the existence of points $x_1 \in \partial D$, $y_1 \in \partial \Omega$ with $y_1 \neq y_0$ such that $x_1$ is on the line determined by $p$ and $y_1$. Hence $f(y_1) = l_p(x_1, y_1)$. By (4) of proposition 3.2, we have $f(y_0) = l_p(x_0, y_0)$. Since $f(y_1) < f(y_0)$, we obtain $l_p(x_0, y_0) > l_p(x_1, y_1)$. This is against $(x_0, y_0) \in \mathcal{M}(p)$. Therefore one gets the desired conclusion.
6.2 Positive definiteness of the Hessian of \( \tilde{l}_p(\sigma, \theta) \) at \((\sigma, \theta) = (0, 0)\)

In this subsection, we show proposition 6.1. Throughout this subsection, we always assume that \( D \) is of class \( C^2 \) and strictly convex.

As in the proof of proposition 3.2, we choose systems of local coordinates \( x = x(\sigma), \sigma = (\sigma_1, \sigma_2) \) with \( x_0 = x(0) \) and \( y = y(\theta), \theta = (\theta_1, \theta_2) \) with \( y_0 = y(0) \) in a neighbourhood of \( x_0 \in \partial D \) and \( y_0 \in \partial \Omega \) respectively. It suffices to prove that the Hessian of \( \tilde{l}_p(\sigma, \theta) = l_p(x(\sigma), y(\theta)) \) at \((\sigma, \theta) = (0, 0)\) for \((x_0, y_0) \in \mathcal{M}(p) \setminus \mathcal{M}_g(p)\) is positive definite under the constraint (6.1) on the Weingarten map for \( A_{\partial \Omega, y_0} \). This is equivalent to the statement: the quadratic form on \( \mathbb{R}^2 \times \mathbb{R}^2 \)

\[
\sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \sigma_k}(0,0) \xi_j \xi_k + \sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \theta_k}(0,0) \xi_j \eta_k + \sum_{j,k=1}^{2} (\tilde{l}_p)_{\theta_j \theta_k}(0,0) \eta_j \eta_k, \quad (\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2
\]

is positive definite.

First we give an expression for the form by using the Weingarten maps for surfaces. Using (3.25)-(3.28), one gets

\[
(\tilde{l}_p)_{\sigma_j \sigma_k}(0,0) = \nabla_x l_p(x_0, y_0) \cdot \frac{\partial^2 x}{\partial \sigma_j \partial \sigma_k}(0) + \nabla_x \nabla_x l_p(x_0, y_0) \frac{\partial x}{\partial \sigma_k}(0) \cdot \frac{\partial x}{\partial \sigma_j}(0) \tag{6.2}
\]

\[
(\tilde{l}_p)_{\sigma_j \theta_k}(0,0) = -\frac{1}{|x_0 - y_0|} \frac{\partial x}{\partial \sigma_j}(0) \cdot \frac{\partial y}{\partial \theta_k}(0) \tag{6.3}
\]

and

\[
(\tilde{l}_p)_{\theta_j \theta_k}(0,0) = \nabla_y l_p(x_0, y_0) \cdot \frac{\partial^2 y}{\partial \theta_j \partial \theta_k}(0) + \frac{1}{|x_0 - y_0|} \frac{\partial y}{\partial \theta_j}(0) \cdot \frac{\partial y}{\partial \theta_k}(0). \tag{6.4}
\]

First we consider

**Case 1.** \((x_0, y_0) \in \mathcal{M}_1(p)\).

Let \( S \) and \( \tilde{S} \) be an spheroid and a sphere defined by

\[
S = \{ x \in \mathbb{R}^3 \mid l_p(x, y_0) = l_p(x_0, y_0) \} \quad \text{and} \quad \tilde{S} = \{ y \in \mathbb{R}^3 \mid |y - x_0| = |y_0 - x_0| \}.
\]

We denote by \( \hat{\nu}_x, \hat{\nu}_y \) the unit outward normal vectors at \( x \in S \), \( y \in \tilde{S} \) on \( S, \tilde{S} \), respectively. Note that \( x_0 \in \partial D \cap S \) and \( y_0 \in \partial \Omega \cap \tilde{S} \).

Here we claim:

\[
|\nabla_x l_p(x_0, y_0)| = \frac{2(p - x_0) \cdot \nu_{x_0}}{|x_0 - p|} \neq 0, \quad \nu_{x_0} = -\frac{1}{|\nabla_x l_p(x_0, y_0)|} \nabla_x l_p(x_0, y_0) = -\hat{\nu}_x, \tag{6.5}
\]

\[
\nu_{y_0} = \hat{\nu}_y = \nabla_y l_p(x_0, y_0).
\]

The first and second equations come from (3.32) in the proof of (2) of proposition 3.2. The third equation is nothing but (1) of proposition 3.2. (6.5) yields that \( T_{x_0}(S) = T_{x_0}(\partial D) \) and \( T_{y_0}(\tilde{S}) = T_{y_0}(\partial \Omega) \). Then one can choose a local coordinate system \( x = x(\varphi) \) with \( x_0 = x(0) \) of \( S \) in such a way that

\[
\frac{\partial x}{\partial \sigma_j}(0) = \frac{\partial x}{\partial \varphi_j}(0), \quad j = 1, 2.
\]
Since \( l_p(x(\varphi), y_0) = l_p(x_0, y_0) \), we have \((\partial_{\varphi_j} \partial_{\varphi_k}) l_p(x(\varphi), y_0) = 0\). That is,

\[
\begin{align*}
-\nabla_x l_p(x_0, y_0) \cdot \frac{\partial^2 x}{\partial \varphi_j \partial \varphi_k}(0) &= \nabla_x \nabla_x l_p(x_0, y_0) \frac{\partial x}{\partial \varphi_j}(0) \cdot \frac{\partial x}{\partial \varphi_k}(0) \\
&= \nabla_x \nabla_x l_p(x_0, y_0) \frac{\partial x}{\partial \sigma_j}(0) \cdot \frac{\partial x}{\partial \sigma_k}(0).
\end{align*}
\]

(6.6)

Then from (6.2), the second equation in (6.5) and (6.6) we obtain

\[
(l_p)_{\sigma_j \sigma_k}(0, 0) = -|\nabla_x l_p(x_0, y_0)| \left\{ \nu_{x_0} \cdot \frac{\partial^2 x}{\partial \sigma_j \partial \sigma_k}(0) + \hat{\nu}_{x_0} \cdot \frac{\partial^2 x}{\partial \varphi_j \partial \varphi_k}(0) \right\}.
\]

(6.7)

Given \( \xi \in \mathbb{R}^2 \) set

\[
v(\xi) = \sum_{j=1}^2 \xi_j \frac{\partial x}{\partial \sigma_j}(0).
\]

This vector in \( \mathbb{R}^3 \) belongs to \( T_{x_0}(S) = T_{x_0}(\partial D) \). Since we have

\[
\mathcal{A}_{\partial D, x_0} v(\xi) \cdot v(\xi) = -\sum_{j,k=1}^2 \nu_{x_0} \cdot \frac{\partial^2 x}{\partial \sigma_j \partial \sigma_k}(0) \xi_j \xi_k,
\]

\[
\mathcal{A}_{S, x_0} v(\xi) \cdot v(\xi) = -\sum_{j,k=1}^2 \hat{\nu}_{x_0} \cdot \frac{\partial^2 x}{\partial \varphi_j \partial \varphi_k}(0) \xi_j \xi_k,
\]

(6.7) gives

\[
\sum_{j,k=1}^2 (l_p)_{\sigma_j \sigma_k}(0, 0) \xi_j \xi_k = |\nabla_x l_p(x_0, y_0)| \left( \mathcal{A}_{\partial D, x_0} v(\xi) \cdot v(\xi) + \mathcal{A}_{S, x_0} v(\xi) \cdot v(\xi) \right).
\]

(6.8)

Given \( \eta \in \mathbb{R}^2 \) set

\[
\tilde{v}(\eta) = \sum_{j=1}^2 \eta_j \frac{\partial y}{\partial \theta_j}(0).
\]

This belongs to \( T_{y_0}(\partial \Omega) = T_{y_0}(\tilde{S}) \). From (6.4), the third equation in (6.5) and a similar computation we obtain

\[
\sum_{j,k=1}^2 (l_p)_{\sigma_j \theta_k}(0, 0) \eta_j \eta_k = -\mathcal{A}_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) + \mathcal{A}_{S, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta).
\]

(6.9)

And also (6.3) gives

\[
\sum_{j,k=1}^2 (l_p)_{\sigma_j \theta_k}(0, 0) \xi_j \eta_k = -\frac{1}{|x_0 - y_0|} v(\xi) \cdot \tilde{v}(\eta).
\]

(6.10)
Summing (6.8), (6.9) and (6.10) up, we obtain the formula
\[
\sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \sigma_k} (0,0) \xi_j \xi_k + 2 \sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \eta_k} (0,0) \xi_j \eta_k + \sum_{j,k=1}^{2} (\tilde{l}_p)_{\eta_j \eta_k} (0,0) \eta_j \eta_k
\]
\[
= |\nabla x_{l_p} (x_0, y_0)| (A_{\partial D, x_0} v(\xi) \cdot v(\xi) + A_{S, x_0} v(\xi) \cdot v(x))
\]
\[
- \frac{2}{|x_0 - y_0|} v(\xi) \cdot \tilde{v}(\eta) + A_{S', y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) - A_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta).
\]  

(6.11)

In order to prove the positive definiteness of the right hand side of (6.11) first we consider the case when \( \partial D \) is flat and \( \partial \Omega \) is replaced with a sphere in part.

**Lemma 6.1** Let \( S' \) be the sphere centered at \( \tilde{p} = y_0 - l_p (x_0, y_0) \tilde{v}_{y_0} \) with radius \( l_p (x_0, y_0) \). Then, for all \((\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \) we have
\[
|\nabla x_{l_p} (x_0, y_0)| A_{S, x_0} v(\xi) \cdot v(\xi) - \frac{2}{|x_0 - y_0|} v(\xi) \cdot \tilde{v}(\eta)
\]
\[
+ A_{S', y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) - A_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) \geq 0.
\]

**Proof.** Denote by \( \Pi \) the set of all points \( x \) such that \( (x - x_0) \cdot \nu_{x_0} = 0 \). Since \((x_0, y_0) \in M_1(p)\), from (2) of proposition 3.2 one knows that the points \( p \) and \( y_0 \) are in the half space \((x - x_0) \cdot \nu_{x_0} > 0 \). Choose a small neighbourhood \( V \) of \( y_0 \). Given \( x \in \Pi \) and \( y \in S' \cap V \) we have \(|p - x| = |\tilde{p} - x|\) and \(|\tilde{p} - y| = l_p (x_0, y_0)\). The triangle inequality gives
\[
l_p (x, y) = |p - x| + |x - y| = |\tilde{p} - x| + |x - y| \geq |\tilde{p} - y| = l_p (x_0, y_0).
\]

This yields that the function \( l_p (x, y) \) on \( \Pi \times (S' \cap V) \) attains the minimum value. Therefore the Hessian of the local representation of the function on \( \Pi \times (S' \cap V) \) has to be non-negative at \((x_0, y_0)\). This is nothing but the statement of lemma 6.1 since \( A_{\Pi, x_0} = 0 \).

\( \square \)

A combination of lemma 6.1 and (6.11) gives
\[
\sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \sigma_k} (0,0) \xi_j \xi_k + 2 \sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma_j \eta_k} (0,0) \xi_j \eta_k + \sum_{j,k=1}^{2} (\tilde{l}_p)_{\eta_j \eta_k} (0,0) \eta_j \eta_k
\]
\[
\geq |\nabla x_{l_p} (x_0, y_0)| A_{\partial D, x_0} v(\xi) \cdot v(\xi) + \frac{1}{l_p (x_0, y_0)} |\tilde{v}(\eta)|^2 - A_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta).
\]  

(6.12)

Note that we made use of the fact \( A_{S', y_0} = (1/l_p (x_0, y_0)) I \). Then assumption (6.1) on \( A_{\partial \Omega, y_0} \) and strict convexity of \( \partial D \) yield that the right hand side of (6.12) is positive definite. This completes the proof of proposition 6.1 in the case when \((x_0, y_0) \in M_1(p)\).

To complete the proof, from (3) of proposition 3.2, it suffices to consider the following case:
\textbf{Case 2.} \((x_0, y_0) \in \mathcal{M}_2^+(p) \cup \mathcal{M}_2(p)\).

In this case (3.33) in the proof of proposition 3.2 holds. This gives \(\nabla_x l_p(x_0, y_0) = 0\). Then from (6.2), (6.3) and \(\nu_{y_0} = (y_0 - x_0)/|y_0 - x_0|\) we get

\[
\sum_{j,k=1}^2 (\hat{l}_p)_{\sigma_j \sigma_k} (0, 0) \xi_j \xi_k = \frac{l_p(x_0, y_0)}{|p - x_0||y_0 - x_0|} \left\{ v(\xi) \cdot v(\xi) - (v(\xi) \cdot \nu_{y_0})^2 \right\},
\]

\[
\sum_{j,k=1}^2 (\hat{l}_p)_{\sigma_j \sigma_k} (0, 0) \xi_j \eta_k = -\frac{1}{|x_0 - y_0|} v(\xi) \cdot \tilde{v}(\eta)
\]

and we still have (6.9). Summing those up, we obtain

\[
\sum_{j,k=1}^2 (\hat{l}_p)_{\sigma_j \sigma_k} (0, 0) \xi_j \xi_k + 2 \sum_{j,k=1}^2 (\hat{l}_p)_{\sigma_j \sigma_k} (0, 0) \xi_j \eta_k + \sum_{j,k=1}^2 (\hat{l}_p)_{\sigma_j \sigma_k} (0, 0) \eta_j \eta_k
\]

\[
= \frac{l_p(x_0, y_0)}{|p - x_0||x_0 - y_0|} |w(\xi)|^2 - \frac{2}{|x_0 - y_0|} w(\xi) \cdot \tilde{v}(\eta) + \frac{2}{|x_0 - y_0|} \tilde{v}(\eta) \cdot \tilde{v}(\eta) + \mathcal{A}_{\tilde{s}, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) - \mathcal{A}_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta),
\]

where \(w(\xi) = v(\xi) - (v(\xi) \cdot \nu_{y_0})\nu_{y_0}\). Here we note that

\[
\frac{l_p(x_0, y_0)}{|p - x_0||x_0 - y_0|} |w(\xi)|^2 - \frac{2}{|x_0 - y_0|} w(\xi) \cdot \tilde{v}(\eta) + \mathcal{A}_{\tilde{s}, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) - \frac{1}{l_p(x_0, y_0)} \tilde{v}(\eta) \cdot \tilde{v}(\eta)
\]

\[
= \frac{l_p(x_0, y_0)}{|p - x_0||x_0 - y_0|} |w(\xi)|^2 - \frac{2}{|x_0 - y_0|} w(\xi) \cdot \tilde{v}(\eta) + \left( \frac{1}{|x_0 - y_0|} - \frac{1}{l_p(x_0, y_0)} \right) \tilde{v}(\eta) \cdot \tilde{v}(\eta)
\]

\[
= \frac{l_p(x_0, y_0)}{|p - x_0||x_0 - y_0|} |w(\xi) - \frac{|p - x_0|}{l_p(x_0, y_0)} \tilde{v}(\eta)|^2 \geq 0.
\]

Thus the right hand side of (6.13) becomes

\[
\frac{l_p(x_0, y_0)}{|p - x_0||x_0 - y_0|} \left| w(\xi) - \frac{|p - x_0|}{l_p(x_0, y_0)} \tilde{v}(\eta) \right|^2 + \frac{1}{l_p(x_0, y_0)} \tilde{v}(\eta) \cdot \tilde{v}(\eta) - \mathcal{A}_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta).
\]

By the assumption on \(\mathcal{A}_{\partial \Omega, y_0}\), for the positive definiteness of (6.13) it suffices to prove that if

\[
w(\xi) - \frac{|p - x_0|}{l_p(x_0, y_0)} \tilde{v}(\eta) = 0, \quad \tilde{v}(\eta) = 0,
\]

then \(v(\xi) = 0\). First \(\tilde{v}(\eta) = 0\) yields \(w(\xi) = 0\), that is, \(v(\xi) = (v(\xi) \cdot \nu_{y_0})\nu_{y_0}\). Since \(v(\xi) \cdot \nu_{x_0} = 0\), this yields that \(v(\xi) \cdot \nu_{y_0} = 0\) or \(\nu_{x_0} \cdot \nu_{y_0} = 0\). However, if \(\nu_{x_0} \cdot \nu_{y_0} = 0\), then (3.33) gives \((p - x_0) \cdot \nu_{x_0} = 0\). This is against \((x_0, y_0) \notin \mathcal{M}_g(p)\). Therefore it must hold that \(v(\xi) \cdot \nu_{y_0} = 0\) and thus this yields \(v(\xi) = 0\). This completes the proof of proposition 6.1.

\(\square\)
6.3 A sufficient condition of positive definiteness of $l_p$ at $(x_0, y_0) \in \mathcal{M}_1(p)$

For $(x_0, y_0) \in \mathcal{M}_1(p)$ one can relax condition (6.1).

**Proposition 6.3** Let $(x_0, y_0) \in \mathcal{M}_1(p)$. Assume that: there exists a constant $R > d_0 \equiv |x_0 - y_0|$ such that the one of the following holds.

\[
\mathcal{A}_{\partial \Omega, y_0} \leq \frac{1}{R} |\nabla_x l_p(x_0, y_0)| (\mathcal{A}_{\partial D, x_0} + \mathcal{A}_{S, x_0}) > \frac{R}{(R - d_0)d_0}, \quad (6.14)
\]
\[
\mathcal{A}_{\partial \Omega, y_0} < \frac{1}{R} |\nabla_x l_p(x_0, y_0)| (\mathcal{A}_{\partial D, x_0} + \mathcal{A}_{S, x_0}) \geq \frac{R}{(R - d_0)d_0}. \quad (6.15)
\]

Then we have the same conclusion as proposition 2.1.

**Proof.** We start with rewriting (6.11):

\[
\sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma j, \sigma k}(0, 0) \xi_j \xi_k + 2 \sum_{j,k=1}^{2} (\tilde{l}_p)_{\sigma j, \theta k}(0, 0) \xi_j \eta_k + \sum_{j,k=1}^{2} (\tilde{l}_p)_{\theta j, \theta k}(0, 0) \eta_j \eta_k
\]
\[
= |\nabla_x l_p(x_0, y_0)| (\mathcal{A}_{\partial D, x_0} v(\xi) \cdot v(\xi) + \mathcal{A}_{S, x_0} v(\xi) \cdot v(\xi)) + \frac{1}{R} |\tilde{v}(\eta)|^2 - \mathcal{A}_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) + I(\xi, \eta), \quad (6.16)
\]

where

\[
I(\xi, \eta) = -\frac{1}{R} |\tilde{v}(\eta)|^2 - \frac{2}{d_0} v(\xi) \cdot \tilde{v}(\eta) + \mathcal{A}_{\tilde{S}, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta).
\]

From the equation

\[
\mathcal{A}_{\tilde{S}, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) = d_0^{-1} |\tilde{v}(\eta)|^2
\]

it follows that

\[
I(\xi, \eta) = \frac{R - d_0}{Rd_0} |\tilde{v}(\eta)|^2 - \frac{2}{d_0} v(\xi) \cdot \tilde{v}(\eta)
\]
\[
= \frac{R - d_0}{Rd_0} \left| \tilde{v}(\eta) - \frac{R}{R - d_0} v(\xi) \right|^2 - \frac{R}{(R - d_0)d_0} |v(\xi)|^2.
\]

Thus the right-hand side of (6.16) becomes

\[
|\nabla_x l_p(x_0, y_0)| (\mathcal{A}_{\partial D, x_0} v(\xi) \cdot v(\xi) + \mathcal{A}_{S, x_0} v(\xi) \cdot v(\xi)) - \frac{R}{(R - d_0)d_0} |v(\xi)|^2
\]
\[
+ \frac{1}{R} |\tilde{v}(\eta)|^2 - \mathcal{A}_{\partial \Omega, y_0} \tilde{v}(\eta) \cdot \tilde{v}(\eta) + \frac{R - d_0}{Rd_0} \left| \tilde{v}(\eta) - \frac{R}{R - d_0} v(\xi) \right|^2.
\]

(6.17)

Now it is easy to see that (6.14) or (6.15) ensure the positive definiteness of (6.17).

\[\square\]
6.4 An example covered by proposition 6.3

Using propositions 6.2 and 6.3, we can give another example including example 6.1.

Let \( \Omega \) be an open ball centered at the origin \( O \) with radius \( R \) and \( p \) be a point outside \( \Omega \). Let \( D \) be an open ball centered at \( q \) with radius \( r \). Assume that: \( \overline{D} \subset \Omega \), that is, \(|q| + r < R\); the line determined by two points \( p \) and \( q \) passes the origin and \(|p - q| \leq |p|\). By proposition 6.2 one knows that \( \mathcal{M}_q(p) = \mathcal{M}_q^2(p) = \mathcal{M}_q^2(p) = \emptyset \) and \( \mathcal{M}(p) = \mathcal{M}_1(p) \). Let \((x_0, y_0) \in \mathcal{M}_1(p)\). By (1) of proposition 4.1 one knows that \( y_0 - x_0 \) and \( \nu_{x_0} \) are parallel. This yields that \( x_0 \) has to be on the line determined by \( y_0 \) and the origin \( O \). By (2) of proposition 3.2 one knows that the angle between \( p - x_0 \) and \( \nu_{x_0} \) coincides with the one between \( y_0 - x_0 \) and \( \nu_{x_0} \). This yields that \( x_0 \) has to be on the line determined by \( p \) and \( q \). Then one gets \( \mathcal{M}_1(p) = \{(q + r(p - q)/|p - q|, q + (R - |q|)(p - q)/|p - q|)\} \) and \( l(p, D) = |p - x_0| + |x_0 - y_0| \). We point out that the condition

\[
|\nabla_{x_0} l_p(x_0, y_0)| (\mathcal{A}_{\partial D, x_0} + \mathcal{A}_{S, x_0}) > \frac{R}{(R - d_0)d_0},
\]  

(6.18)
is satisfied. Since \( \mathcal{A}_{\partial \Omega, y_0} = (1/R)I \), we conclude that (6.14) is satisfied.

The condition (6.18) for this example is checked as follows. For the ellipsoid \( S \) with the focal points \( p \) and \( y_0 \), it follows that

\[
\mathcal{A}_{S, x_0} = \frac{l_0}{2(l_0 - d_0)d_0} I, \quad l_0 = l_p(x_0, y_0)\]  

(6.19)
The proof of (6.19) is given in Appendix B. Using (6.19), the equations \( \mathcal{A}_{\partial D, x_0} = (1/r)I \) and \( |\nabla_{x_0} l_p(x_0, y_0)| = 2 \) we know that (6.18) is equivalent to the condition

\[
\frac{1}{r} + \frac{l_0}{2(l_0 - d_0)d_0} > \frac{R}{2(R - d_0)d_0}.
\]  

(6.20)

This condition itself is checked by a direct computation, however, we present here the detail for the convenience of the reader. Set \( h = |p - y_0| \). We have \( l_0 = h + 2d_0 \). Noting that \( d_0 = R - (r + |q|) \), one gets

\[
\frac{1}{r} + \frac{l_0}{2(l_0 - d_0)d_0} - \frac{R}{2(R - d_0)d_0} = \frac{2(l_0 - d_0)(R - d_0)d_0 + rl_0(R - d_0) - r(l_0 - d_0)R}{2r(d_0)(l_0 - d_0)(R - d_0)}.
\]  

The numerator of the right-hand side is written as

\[
2(l_0 - d_0)(R - d_0)d_0 + rl_0(R - d_0) - r(l_0 - d_0)R = (l_0 - d_0)(2|q| + r)d_0 + rR(R - d_0) > 0
\]

Therefore (6.20) is valid.

6.5 Upper bound of the location of \( D \)

It should be pointed out that knowing \( l(p, D) \), one can obtain an upper bound of the location of \( D \). For \( p \notin \overline{\Omega} \), and \( y \in \partial \Omega \), we put \( \mathcal{E}_p(y) = \{x \in \mathbb{R}^3 \mid |p - x| + |x - y| \geq l(p, D)\} \), \( \mathcal{E}_p = \bigcap_{y \in \partial \Omega} \mathcal{E}_p(y) \), \( \mathcal{R}_p = \{x \in \mathbb{R}^3 \mid |p - x| \geq 2^{-1}(d_{\partial \Omega}(p) + l(p, D))\} \), \( d_{\partial \Omega}(p) = \inf\{|y - p| \mid y \in \partial \Omega\} \) and \( d_{\mathcal{E}_p \cap \Omega}(p) = \inf\{|x - p| \mid x \in \mathcal{E}_p \cap \Omega\} \).
Proposition 6.4 It holds that: (i) $\overline{D} \subset \mathcal{E}_p \cap \Omega$; (ii) $\mathcal{E}_p \cap \Omega \subset \mathcal{R}_p$ and $2d_{\partial \mathcal{E}_p \cap \Omega}(p) \geq l(p, D) + d_{\partial \Omega}(p)$.

Proof. For $x \in D$, one can find $x_0 \in \partial D$ such that $x_0$ is on the segment connecting $x$ with $p$. The definition of $l(p, D)$ implies that $l(p, D) \leq |p - x_0| + |x_0 - y|$ for any $y \in \partial \Omega$. If $x_0$, $x$ and $y$ are not on a line, from triangle inequality we have $l(p, D) < |p - x_0| + |x_0 - x| + |x - y| = |p - x| + |x - y|$. If $x_0$, $x$ and $y$ are on a line, then this line should be the line passing points $x$ and $p$. If $y$ is located on the segment $xp$, then we have $|p - x_0| < |p - x|$ and $|x_0 - y| < |y - x|$, which also implies that $l(p, D) < |p - x| + |x - y|$. If $y$ is outside of segment $px$, then we have $l(p, D) \leq |p - x_0| + |x_0 - y| = |p - x| + |x - y|$. Hence $D \subset \mathcal{E}_p(y)$. Since $\mathcal{E}_p(y)$ is closed and $\overline{D} \subset \Omega$, we obtain (i).

To show (ii), assume that $x \in \mathcal{E}_p \cap \Omega$. From $x \in \Omega$ and $p \notin \overline{\Omega}$, there exists $t > 0$ such that $y_t = p + t(x - p)/|x - p| \in \partial \Omega$. Since $x \in \mathcal{E}_p(y_t)$ we have $l(p, D) \leq |p - x| + |x - y_t| = 2|p - x| - t = 2|p - x| - |p - y_t|$, which implies that $2|p - x| \geq l(p, D) + |p - y_t| \geq l(p, D) + d_{\partial \Omega}(p)$. This yields (ii).

Hence if we have a set $\Lambda \subset \mathbb{R}^3 \setminus \overline{\Omega}$ such that for every $p \in \Lambda$, $l(p, D)$ can be calculated by formula (1.9), then for an arbitrary set $\Gamma \subset \partial \Omega$ we have $\overline{D} \subset \cap_{(p, y) \in \Lambda \times \Gamma} \mathcal{E}_p(y) \cap \Omega$.

In example 6.1, we also note that if we put $\Lambda = \{p \in \mathbb{R}^3 \setminus \overline{\Omega} \mid h/2 < r_0 - R/2 \}$ we have $l(p, D) = h + 2(R - r)$ for $p \in \Lambda$. Hence proposition 6.4 implies that

$$
\overline{D} \subset \cap_{p \in \Lambda} \{x \in \Omega \mid |p - x| \geq h + (R - r) \} = \{x \in \Omega \mid |q - x| \leq r \} = \overline{D}.
$$

Thus the estimate given in proposition 6.4 is optimal. This can be extended as follows:

Assume that $\Omega$ is convex and consider the case that there exists a point $(x_0, y_0) \in \mathcal{M}_1(p)$ corresponding to the one-dimensional case (i.e. $y_0$ is on the line segment $px_0$). In this case, the argument showing (ii) of proposition 6.4 implies that $l(p, D) = |p - x_0| + |x_0 - y_0| = 2|p - x_0| - |y_0 - y|$. Note that from convexity of $\Omega$, we can characterize $y_0$ as the unique point $y_{\min}(p) \in \partial \Omega$ as the point attaining the minimum $\min_{y \in \partial \Omega} |y - p| = d_{\partial \Omega}(p)$, and thus we have $|p - y_0| = d_{\partial \Omega}(p)$. Indeed, from (1) of proposition 3.2, one can know that $\nu_{y_0} = (y_0 - x_0)^{-1}(y_0 - x_0)$. Hence we have $\nu_{y_0} = |p - y_0|^{-1}(p - y_0)$. The convexity of $\Omega$ implies that the point $y_1 \in \partial \Omega$ satisfying $\nu_{y_1} = |p - y_1|^{-1}(p - y_1)$ should be coincide with $y_{\min}(p)$. Thus the point $x_0$ is determined by $x_0 = p + 2^{-1}(d_{\partial \Omega}(p) + l(p, D))\omega(p)$, where $\omega(p) = (y_{\min}(p) - p)/|y_{\min}(p) - p|$. Note also that we have

$$
h_D(-\omega(p)) = -p \cdot \omega(p) - 2^{-1}(d_{\partial \Omega}(p) + l(p, D)).
$$

Thus in the case corresponding to the one-dimensional case we can find the value of the support function in the direction $-\omega(p)$ like as is in the original enclosure method.

Unfortunately, this equality for $h_D(-\omega(p))$ does not always holds. Even the estimate $h_D(-\omega(p)) \geq -p \cdot \omega(p) - 2^{-1}(d_{\partial \Omega}(p) + l(p, D))$ may not always be true. Note also that even the set $\cap_{(p, y) \in \Lambda \times \partial \Omega} \mathcal{E}_p(y)$ does not always coincide with $D$. However, instead of lines $\omega(p) \cdot x = t$, if we use $\mathcal{E}_p(y)$ for $p \notin \overline{\Omega}$ and $y \in \partial \Omega$, from proposition 6.4, we can give estimates of $D$. 

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A Proof of lemma 4.2

The Taylor theorem gives

\[ f(x) = f(x_0) + (A(x)(x - x_0), x - x_0)_{\mathbb{R}^n}, \]

where

\[ A(x) = (A_{ij}(x)), \quad A_{ij}(x) = \int_0^1 (1 - \theta) f_{x_{i}x_{j}}(x_0 + \theta(x - x_0))d\theta. \]

Since \( f \in C^{2,\alpha_0}(\overline{U}) \), we have, for a positive constant \( C > 0 \)

\[ |A_{ij}(x) - A_{ij}(x_0)| \leq C|x - x_0|^{\alpha_0} \quad (x \in \overline{U}, i, j = 1, 2, \ldots, n). \]

By the assumption, \( A(x_0) = \frac{1}{2}(\text{Hess}(f)(x_0)) > 0 \). Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0 \) be the eigenvalues of \( A(x_0) \). Then there exists an orthogonal matrix \( P \) such that \( {}^tPA(x_0)P = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n) \). Define \( y = {}^tP(x - x_0) \) and set \( B(y) = A(x) - A(x_0), \varphi(y) = \varphi(x) \).

Then there exist constants \( \delta_1 > 0, C > 0 \) such that

\[ |B_{ij}(y)| \leq C|y|^{\alpha_0} \quad (|y| \leq \delta_1, i, j = 1, 2, \ldots, n). \]

Thus we have \( |\Phi_1(y)| < \frac{1}{2}\Phi_0(y) (|y| \leq \delta_1) \), where \( \Phi_0(y) = \frac{1}{2} \sum_{j=1}^{n} \mu_j y_j^2 \) and \( \Phi_1(y) = (B(y)y, y)_{\mathbb{R}^n} \).

Let \( \delta > 0 \). Then, one can choose a constant \( c_1 > 0 \) such that \( f(x) \geq f(x_0) + c_1 \) (\( x \in \overline{U}, |x - x_0| \geq \delta \)), which yields

\[ \left| \int_{\{x \in U \mid |x - x_0| \geq \delta\}} e^{-\tau f(x)} \varphi(x)dx \right| \leq C e^{-\tau f(x_0)} e^{-c_1 \tau} \max_{x \in \overline{U}} |\varphi(x)|. \]
Therefore choosing a suitable function $\psi \in C^\infty_0(\mathbb{R}^n)$ with $\psi(y) = 1 \,(|y| \leq \delta_1/3)$, $\psi(y) = 0 \,(|y| \geq 2\delta_1/3)$, we get

$$
\int_U e^{-\tau f(x)} \varphi(x) \,dx = e^{-\tau f(x_0)} \left\{ \int_{\mathbb{R}^n} e^{-\tau \Phi_0(y) + \Phi_1(y)} \tilde{\varphi}(y) \psi(y) \,dy + O(e^{-c_1\tau} \|\varphi\|_{C(U)}) \right\}
$$
as $\tau \to \infty$.

Here since $|e^X - 1| = |X| \int_0^1 e^{\theta X} \,d\theta \leq |X| |\text{Re} X|$, we have

$$
|e^{-\tau \Phi_0(y)} (e^{-\tau \Phi_1(y)} - 1)| \leq C |\tau||y|^{2+\alpha_0} e^{-\tau \Phi_0(y)} e^\tau |\Phi_1(y)| \leq C |\tau||y|^{2+\alpha_0} e^{-\tau \Phi_0(y)/2}.
$$

This gives

$$
\left| \int_{\mathbb{R}^n} \{ e^{-\tau (\Phi_0(y) + \Phi_1(y))} - e^{-\tau \Phi_0(y)} \} \tilde{\varphi}(y) \psi(y) \,dy \right|
\leq C |\tau| \int_{\mathbb{R}^n} |y|^{2+\alpha_0} e^{-\tau \Phi_0(y)/2} \tilde{\varphi}(y) \psi(y) \,dy
\leq C |\tau| \|\varphi\|_{C(U)} \int_{\mathbb{R}^n} |y|^{2+\alpha_0} e^{-\tau \Phi_0(y)/2} \,dy
= C |\tau| \|\varphi\|_{C(U)} \int_{\mathbb{R}^n} |y|^\alpha \,dy \tau^{-(n+\alpha_0)/2} \leq C\Phi_0^{\alpha_0} \|\varphi\|_{C(U)} \tau^{-(n+\alpha_0)/2}.
$$

Summing up, we obtain

$$
\int_U e^{-\tau f(x)} \varphi(x) \,dx = e^{-\tau f(x_0)} \left\{ \int_{\mathbb{R}^n} e^{-\tau \Phi_0(y)} \tilde{\varphi}(y) \psi(y) \,dy + O(\tau^{-(n+\alpha_0)/2}) \|\varphi\|_{C(U)} \right\}
$$
as $\tau \to \infty$.

Since $\varphi \in C^{0,\alpha_0}(U)$, we have $|\tilde{\varphi}(y) - \varphi(0)| \leq \|\varphi\|_{C^{0,\alpha_0}(U)} |y|^\alpha \,(|y| \leq \delta_1)$. Using a similar argument, one can replace $\tilde{\varphi}(y)$ in the integrand above with $\varphi(0) = \varphi(x_0)$ and obtain

$$
\int_U e^{-\tau f(x)} \varphi(x) \,dx = e^{-\tau f(x_0)} \left\{ \varphi(x_0) \int_{\mathbb{R}^n} e^{-\tau \Phi_0(y)} \psi(y) \,dy + O(\tau^{-(n+\alpha_0)/2}) \|\varphi\|_{C^{0,\alpha_0}(U)} \right\}
$$
as $\tau \to \infty$. Using the asymptotics

$$
\int_{\mathbb{R}^n} e^{-\tau \Phi_0(y)} \psi(y) \,dy = \int_{\mathbb{R}^n} e^{-\tau \Phi_0(y)} \,dy + \int_{\mathbb{R}^n} e^{-\tau \Phi_0(y)} (\psi(y) - 1) \,dy
= \frac{1}{\sqrt{\mu_1 \mu_2 \cdots \mu_n}} \left( \frac{2\pi}{\tau} \right)^{n/2} \left\{ 1 + O(e^{-\tau \mu_0 \delta_1^2 / 36}) \right\}
$$
and the equality $\det(\text{Hess}(f)(x_0)) = \mu_1 \mu_2 \cdots \mu_n$, we obtain the desired asymptotic formula.

For the estimation choose a cut-off function $\psi \in C^\infty_0(U)$ such that $\psi = 1$ near $\{x_0\}$, $\psi \geq 0$ in $U$. Then, there exists a constant $c_2 > 0$ such that $f(x) \geq f(x_0) + c_2 \,(x \in$
If \( \operatorname{supp}(1 - \psi) \). This gives
\[
\left| \int_U e^{-\tau f(x)} \varphi(x) dx \right| \leq \left| \int_U e^{-\tau f(x)} \varphi(x) \psi(x) dx \right| + \left| \int_U e^{-\tau f(x)} (1 - \psi(x)) \varphi(x) dx \right|
\]
\[
\leq \| \varphi \|_{C(\overline{U})} \left\{ \int_U e^{-\tau f(x)} \psi(x) dx + C e^{-\tau f(x_0 + \epsilon_2)} \right\} \quad (\tau \in C_{\delta_0}).
\]
Applying the asymptotic formula established above, we have
\[
\int_U e^{-\tau f(x)} \psi(x) dx = \frac{e^{-\tau f(x_0)}}{\sqrt{\det(Hess(f)(x_0))}} \left( \frac{2\pi}{\tau} \right)^{n/2} \left( \psi(x_0) + \| \psi \|_{C^{0,1}(\overline{U})} O(\tau^{-1/2}) \right) \quad (A.1)
\]
as \( \tau \to \infty \) uniformly in \( \tau \in C_{\delta_0} \). Then a combination of \( (A.1) \) and \( (A.2) \) yields the desired estimate.

\[ \square \]

**B  Proof of (6.19)**

Set
\[ p_0 = 2^{-1}(p + y_0), \]
\[ e_1 = (y_0 - p)/|y_0 - p|, \quad e' = (x_0 - p_0) - ((x_0 - p_0) \cdot e_1)e_1. \]

If \( e' \neq 0 \), then set \( e_2 = |e'|^{-1} e' \) and choose a unit vector \( e_3 \) in such a way that \( e_1, e_2 \) and \( e_3 \) form orthogonal bases of \( \mathbb{R}^3 \); if \( e' = 0 \), then choose unit vectors \( e_2 \) and \( e_3 \) in such a way that \( e_1, e_2 \) and \( e_3 \) form orthogonal bases of \( \mathbb{R}^3 \). Then one can write the equation for the ellipsoid \( S \) as
\[
x = s(\sigma_1, \sigma_2) = p_0 + f(\sigma_1)e_1 + g(\sigma_1)(\cos \sigma_2)e_2 + g(\sigma_1)(\sin \sigma_2)e_3
\]
where
\[ f(\sigma_1) = a \cos \sigma_1, \quad g(\sigma_1) = b \sin \sigma_1, \]
\[ a = l_0/2, \quad b = \sqrt{(l_0/2)^2 - (|p - y_0|/2)^2}. \]

We have
\[
\frac{\partial s}{\partial \sigma_1} = f'(\sigma_1)e_1 + g'(\sigma_1) \cos(\sigma_2)e_2 + g'(\sigma_1) \sin(\sigma_2)e_3,
\]
\[
\frac{\partial s}{\partial \sigma_2} = -g(\sigma_1) \sin(\sigma_2)e_2 + g(\sigma_1) \cos(\sigma_2)e_3,
\]
\[
\frac{\partial^2 s}{\partial \sigma_1^2} = f''(\sigma_1)e_1 + g''(\sigma_1) \cos(\sigma_2)e_2 + g''(\sigma_1) \sin(\sigma_2)e_3,
\]
\[
\frac{\partial^2 s}{\partial \sigma_2^2} = -g(\sigma_1) \cos(\sigma_2)e_2 - g(\sigma_1) \sin(\sigma_2)e_3,
\]
\[
\frac{\partial^2 s}{\partial \sigma_1 \partial \sigma_2} = -g'(\sigma_1) \sin(\sigma_2)e_2 + g'(\sigma_1) \cos(\sigma_2)e_3.
\]
Denoting by \( \hat{\nu}(\sigma) \) the unit normal vector at \( s(\sigma) \) outward to \( S \), one gets
\[
\hat{\nu}(\sigma) = \frac{1}{F(\sigma)}(g'(\sigma_1)e_1 - f'(\sigma_1) \cos(\sigma_2)e_2 - f'(\sigma_1) \sin(\sigma_2)e_3)
\]
where
\[
F(\sigma) = \sqrt{f'(\sigma_1)^2 + g'(\sigma_1)^2}.
\]
A direct computation gives
\[
\hat{\nu}(\sigma) \cdot \frac{\partial^2 s}{\partial \sigma_1^2} = \frac{f''(\sigma_1)g'(\sigma_1) - f'(\sigma_1)g''(\sigma_1)}{F(\sigma)}, \quad \hat{\nu}(\sigma) \cdot \frac{\partial^2 s}{\partial \sigma_1 \partial \sigma_2} = 0,
\]
\[
\hat{\nu}(\sigma) \cdot \frac{\partial^2 s}{\partial \sigma_2^2} = \frac{f'(\sigma_1)g'(\sigma_1)}{F(\sigma)}.
\]
Given \( \xi = (\xi_1, \xi_2) \) set
\[
v(\xi) = \xi_1 \frac{\partial s}{\partial \sigma_1} + \xi_2 \frac{\partial s}{\partial \sigma_2} \in T_x S.
\]
We have
\[
A_{S,x_0}(v(\xi)) \cdot v(\xi) = - \sum_{j,k=1}^2 \hat{\nu}(\sigma) \cdot \frac{\partial^2 s}{\partial \sigma_j \partial \sigma_k}(\sigma) \xi_j \xi_k
\]
\[
= (A\xi, \xi)_{\mathbb{R}^2}
\]
where
\[
A = \frac{1}{F(\sigma)} \begin{pmatrix}
    f'(\sigma_1)g''(\sigma_1) - f''(\sigma_1)g'(\sigma_1) & 0 \\
    0 & -f'(\sigma_1)g'(\sigma_1)
\end{pmatrix}
\]
\[
= \frac{ab}{\sqrt{a^2 \sin^2 \sigma_1 + b^2 \cos^2 \sigma_1}} \begin{pmatrix}
    1 & 0 \\
    0 & \sin^2 \sigma_1
\end{pmatrix}.
\]
It is easy to see that
\[
v(\xi) \cdot v(\xi') = (G\xi, \xi')_{\mathbb{R}^2}, \quad \xi, \xi' \in \mathbb{R}^2
\]
where
\[
G = \begin{pmatrix}
    a^2 \sin^2 \sigma_1 + b^2 \cos^2 \sigma_1 & 0 \\
    0 & b^2 \sin^2 \sigma_1
\end{pmatrix}.
\]
A combination of (A.3) and (A.4) gives
\[
A_{S,x_0} v(\xi) = v(G^{-1}A\xi).
\]
This means that the representation matrix of \( A_{S,x_0} \) with respect to the basis \( \partial s/\partial \sigma_1 \) and \( \partial s/\partial \sigma_2 \) is given by \( G^{-1}A \). Now set \( \sigma = (0,0) \). Since \( x_0 = p_0 + ae_1 \), \( l_0 = 2d_0 + |p - y_0| \) and \( G^{-1}A = (a/b^2)I \), we obtain (6.19).
\[\square\]
C Proof of (2.3)

We put $\varepsilon_f(x, \tau) = w(x, \tau) - w_0(x, \tau)$. From (2.1) and (2.2), it follows that $\varepsilon_f(\cdot, \tau) \in H^1(\Omega \setminus D)$ satisfies

$$
\begin{cases}
(\Delta - \tau^2)\varepsilon_f = u(x, T)e^{-\tau^2 T} & \text{in } \Omega \setminus D, \\
\frac{\partial \varepsilon_f}{\partial \nu} + \rho(x)\varepsilon_f = 0 & \text{on } \partial D, \\
\frac{\partial \varepsilon_f}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
$$

in the weak sense. Integration by parts implies that

$$
\int_{\Omega \setminus D} \left\{ | \nabla x \varepsilon_f |^2 + \tau^2 | \varepsilon_f |^2 \right\} dx = -e^{-\tau^2 T} \int_{\Omega \setminus D} u(x, T) \varepsilon_f(x) dx + \int_{\partial D} \rho | \varepsilon_f |^2 dS(x).
$$

This equality yields

$$
\| \nabla x \varepsilon_f \|^2 + \tau^2 \| \varepsilon_f \|^2 \leq e^{-\tau^2 T} \| u(\cdot, T) \| \| \varepsilon_f \| + \| \rho \|_{C^0(\partial D)} \| \varepsilon_f \|_{L^2(\partial D)}^2 (= Q).
$$

(A.5)

Note that there exists a constant $C > 0$ depends only on $\partial D$ and $\partial \Omega$ such that

$$
\| \varepsilon_f \|_{L^2(\partial \Omega)}^2 + \| \varepsilon_f \|_{L^2(\partial D)}^2 \leq C \left\{ \varepsilon \| \nabla x \varepsilon_f \|^2 + \frac{1}{\varepsilon} \| \varepsilon_f \|^2 \right\} \quad (0 < \varepsilon < 1).
$$

For $\tau > 1$, taking $\varepsilon = \tau^{-1}$, we obtain

$$
\| \varepsilon_f \|_{L^2(\partial \Omega)}^2 + \| \varepsilon_f \|_{L^2(\partial D)}^2 \leq C \tau^{-1} \left\{ \| \nabla x \varepsilon_f \|^2 + \tau^2 \| \varepsilon_f \|^2 \right\} \quad (\tau > 1),
$$

(A.6)

which yields

$$
Q \leq \tau^{-2} e^{-2\tau^2 T} \| u(\cdot, T) \|^2 + 4^{-1} \tau^2 \| \varepsilon_f \|^2 + C \| \rho \|_{C^0(\partial D)} \tau^{-1} \left\{ \| \nabla x \varepsilon_f \|^2 + \tau^2 \| \varepsilon_f \|^2 \right\}.
$$

From the above estimate and (A.5), it follows that there exist constants $C > 0$ and $\mu_0 > 1$ depending only on $\partial D$ and $\rho$ such that

$$
\| \nabla x \varepsilon_f \|^2 + \tau^2 \| \varepsilon_f \|^2 \leq C \tau^{-2} e^{-2\tau^2 T} \| u(\cdot, T) \|^2 \quad (\tau \geq \mu_0).
$$

Combining the above estimate with (A.6), we obtain

$$
\| \varepsilon_f(\cdot, \tau) \|_{L^2(\partial \Omega)} \leq C \tau^{-3/2} e^{-\tau^2 T} \quad (\tau \geq \mu_0).
$$

Hence, we get (2.3) since

$$
I(\tau, p) - I_0(\tau, p) = \int_{\partial \Omega} \left( \frac{\partial E_\tau(y, p)}{\partial \nu} \varepsilon_f(y, \tau) - \frac{\partial \varepsilon_f(y, \tau)}{\partial \nu} E_\tau(y, p) \right) dS(y)
$$

and

$$
\left\| \frac{\partial E_\tau(\cdot, p)}{\partial \nu} \right\|_{L^2(\partial \Omega)} \leq C \tau \quad (\tau > 0).
$$