I discuss the relation between harmonic polynomials and invariant theory and show that homogeneous, harmonic polynomials correspond to ternary forms that are apolar to a base conic (the absolute). The calculation of Schlesinger that replaces such a form by a polarised binary form is reviewed.

It is suggested that Sylvester’s theorem on the uniqueness of Maxwell’s pole expression for harmonics is renamed the Clebsch–Sylvester theorem.

The relation between certain constructs in invariant theory and angular momentum theory is enlarged upon and I resurrect the Joos–Weinberg matrices.

Hilbert’s projection operators are considered and their generalisations by Story and Elliott are related to similar, more recent constructions in group theory and quantum mechanics, the ternary case being equivalent to SU(3).
1. Introduction.

There has recently been increased activity in some aspects of classical spherical harmonic theory partly in response to the very accurate measurements of the cosmic microwave background and the need to analyse these in a significant fashion. One approach has lead to the rediscovery of Maxwell’s way of picturing spherical harmonics through their ‘poles’, (e.g. [1–5]), the uniqueness of which description is the content of Sylvester’s theorem.

Because of the relation between this topic, and angular momentum theory in general, and invariant theory, I thought it might be helpful to set down a personal resumé of some of these things in a more historical context, if only to draw attention to sometimes forgotten, but relevant, work. I therefore try to draw together mostly existing material. I also include some peripheral constructions that I feel are worth exposing again, such as the Joos–Weinberg matrices.

The heyday of classical, constructive invariant theory was the second half of the 19th century and the first quarter of the 20th. However, it never really went away and has recently undergone a resurgence of interest (e.g. Olver, [6]) finding applications in coding, computer aided design and pattern recognition.\(^2\)

It is possible to find parallels (and equivalences) between many constructions and processes in classical invariant theory and those in ‘modern’ applied mathematics. Although my pointing out some of these is probably more of interest than utility, I hope to provide something of value.

2. Spherical harmonics.

Discussions of spherical harmonics are many. The term itself seems to date to the treatise of 1867 by Thomson and Tait, [8], where fairly general definitions can be found, but the subject had its beginning in a work of Legendre of 1782 followed by Laplace’s paper of 1782 on potential theory (as we now call it). Later fundamental papers are those of Green (1828) and Gauss (1841). Maxwell, [9], lists some standard technical references of his time.

As might be expected, there are a number of approaches. Here I just try to give some relevant facts without bothering too much about logical ordering.

For this reason, I begin with the addition theorem for (solid) spherical harmonics which can be regarded as ‘classic’, being derived, effectively, by Legendre.

\(^2\) Olver’s introduction in [6] and Sturmfels’ in Hilbert, [7], supply useful historical comments.
It is,

\[(r' \cdot r)^L P_L(\cos \gamma) = C^L_M(r') \cdot C^M_L(r)\]  

(1)

where \(\gamma\) is the angle between \(r\) and \(r'\), \(rr' \cdot \cos \gamma = r \cdot r'\).

I give here, because it’s convenient, the useful composition law for these standard solid harmonics, \(C^M_L(r)\), in my conventions,

\[C^{M_1}_{L_1}(r) C^{M_2}_{L_2}(r) = -\sum_{L_3} (ir)^{L_1+L_2-L_3} \left( \begin{array}{ccc} L_1 & L_2 & L_3 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} M_1 & M_2 & L_3 \\ L_1 & L_2 & M_3 \end{array} \right) C^{M_3}_{L_3}(r),\]  

(2)

and the ‘inverse’,

\[C^{M_1}_{L_1}(r) C^{M_2}_{L_2}(r) \left( \begin{array}{ccc} L_1 & L_2 & M_3 \\ M_1 & M_2 & L_3 \end{array} \right) = -(ir)^{L_1+L_2-L_3} \left( \begin{array}{ccc} L_1 & L_2 & L_3 \\ 0 & 0 & 0 \end{array} \right) C^{M_3}_{L_3}(r).\]  

(3)

The left–hand side of (1) is a homogeneous bipolynomial, of degree \(2L\), in \(r\) and \(r'\) and is homogeneous of degree \(L\) in \(r\) and of degree \(L\) in \(r'\). It is symmetrical in \(r\) and \(r'\) and is a solid spherical harmonic in either set being sometimes referred to as the bi axial harmonic of \(r\) and \(r'\), \(\{8\}^3, [10]\). \(P_L(\cos \gamma)\) is a Laplace coefficient \((cf \ MacMillan, [11] p.377^4)\) and the explicit form of the left–hand side of (1) can be found from the classic series for \(P_L\) as, \((cf \ [11] p.383 eqn.(2))\),

\[(r' \cdot r)^L P_L(\cos \gamma) \equiv H_L(r, r')
\]

\[= \frac{1}{2L} \sum_K (-1)^K \left( \begin{array}{c} L \\ K \end{array} \right) \left( \begin{array}{c} 2L - 2K \\ L \end{array} \right) r'^{2K} r^{2K}(r, r')^{L-2K}.\]  

(4)

Conversely one could derive this expression from first principles, using rotational invariance and the harmonic condition, and then deduce the standard series expansion of \(P_L\), \(cf \ [12]\).

The vectors \(r\) and \(r'\) can be taken complex, subject to complex orthogonal transformations (complex rotations or, equivalently, Lorentz transformations). An important case is when one of them, say \(r'\), is isotropic, or null, \(r' = 0\), which can be achieved by treating the spin–one quantity \(r'\), \(\equiv a\), as composed from a 2–spinor, \(\psi = (\xi \eta) \in \mathbb{C}^2\),

\[a_1 = -\frac{a_x - ia_y}{i\sqrt{2}} = \xi^2, \quad a_{-1} = \frac{a_x + ia_y}{i\sqrt{2}} = \eta^2, \quad a_0 = -ia_z = \sqrt{2} \xi \eta.\]  

(5)

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3 Thomson and Tait, [8] p.159, derive (1) by means of Taylor’s theorem. See also Hobson, [10] §90.

4 As noted by MacMillan on p.300, (1) can be thought of as a transformation from a harmonic with its pole on the z–axis to one whose pole is on the line \((x', y', z')\).
sometimes called the Cartan map, [12].

\((a_x, a_y, a_z)\) are the Cartesian components of \(a\) and I have chosen a normalisation in keeping with the standard/contrastandard definitions. This accounts for the differences with [12]. I should therefore say that my convention is that upstairs indices correspond to contrastandard, and downstairs ones to standard behaviour, in the terminology of Fano and Racah [13], to whom I generally adhere for signs, factors of \(i\) etc. However my \(3j\)-symbols are the more usual ones of Wigner and my angle and rotation conventions follow those of Brink and Satchler, [14]. The relation between their surface harmonics and my solid ones is

\[ r^L C_{LM}^{(BS)}(\theta, \phi) = (-i)^L C_M^L(\mathbf{r}). \]

I also give my raising and lowering convention, which is the transpose of Wigner’s and is the same as that adopted by Williams, [15]. Specifically, for a spin–\(j\) object, \(\phi\), I set

\[ \phi^m = (-1)^{j+m} \phi_{-m}. \]  

Using (5), (1) reduces to,

\[ \frac{(2L)!}{2L^L} (a \cdot r)^L = C_M^L(a) C_L^M(\mathbf{r}). \]  

\(C_M^L(a)\) has the familiar, monomial expression (e.g. Weyl, [16], van der Waerden, [17], Wigner, [18], Bargmann, [19], Schwinger, [20]),

\[ C_M^L(a) = \frac{(2L)!) \xi_{L+M} \eta_{L-M}}{2^{L/2} L! [(L+M)!(L-M)!]^{1/2}} \]

\[ = \frac{\sqrt{(2L)!}}{2^{L/2} L!} \xi_M^{(L)}, \]

where \(\xi_m^{(j)}\) is (for \(j\) half–integral) what I have termed, [21], a null \((2j+1)\)-spinor,

\[ \xi_m^{(j)} = \binom{2j}{j-m}^{1/2} \xi^{j+m} \eta^{j-m}. \]  

The factors here\(^5\) are chosen so that the scalar product of \(\xi_m^{(j)}\) with another null spinor,

\[ \alpha_m^{(j)} = \binom{2j}{j-m}^{1/2} \alpha^{j+m} \beta^{j-m}, \]

\(^5\) Landau and Lifschitz, a very useful book, make the same choice, [22] eqn.(97.4). See also Fano and Racah, [13], App.F.
is the simple power of a bracket, (cf [12] (6.150)),

$$\alpha^m_m \xi^{(j)} = (\eta \alpha - \xi \beta)^{2j} = \begin{vmatrix} \alpha & \beta \\ \xi & \eta \end{vmatrix}^{2j}. \quad (10)$$

In terms of this notation, (7) reads,

$$\sqrt{\frac{(2L)!}{2L/2L!}} (\mathbf{a} \cdot \mathbf{r})^L = \xi_M^L C_M^L (\mathbf{r}), \quad (11)$$

quoted, with conventional phases, by Schwinger [20]; see [12] 6.149. The left-hand side can be considered to be a generating function for the spherical harmonics (cf Erdélyi et al, [23], §11.5.1). Likewise, and rather trivially, the right-hand side of (10) can be taken as a generating function for the null spinor $\xi^{(j)}$. Just set $t \equiv \alpha/\beta$ and compare with [23] equn 11.7(10).

If $\mathbf{r}$ is also a null vector, $\mathbf{r} = \mathbf{b}$, constructed from the 2-spinor $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ by the Cartan map, then, from (10) and (11), or directly from (5),

$$\left(\mathbf{a} \cdot \mathbf{b}\right)^L = (\eta \alpha - \xi \beta)^{2L}. \quad (12)$$

As an interlude, I now rederive (11) in order to introduce some related material that I feel deserves revisiting. I start from the equation for the product of $2j$ null 3-vectors, (see [21] equn.(9)),

$$a^{i_1} \ldots a^{i_2j} = \rho_j \xi^m \xi^{(j)} \xi^m_{n} \equiv \rho_j \xi \xi^{(j)} \xi, \quad (13)$$

where the $t^{i_1 \ldots i_{2j}}$ are the spatial components of the Joos–Weinberg matrices, $t^{(\mu)} = t^{\mu_1 \ldots \mu_{2j}}$ which are higher-spin analogues of the space–time Pauli matrices, $\sigma^\mu$. They are symmetric and traceless on $\mu$. $\rho_j$ is the normalisation,

$$\rho_j = 2^{-2j} e^{-i\pi j} (4j)!^{1/2}. \quad (14)$$

They form a complete set (another one!) of $(2j + 1) \times (2j + 1)$ hermitian matrices and can be expressed as symmetrized powers of the angular momentum matrices, examples being, (beware: I use ‘$j$’ in two roles),

$$t^{0 \ldots 0} = 1; \quad t^{i_0 \ldots 0} = \frac{1}{j} J^i; \quad t^{ij 0 \ldots 0} = \frac{1}{2j - 1} \left( \frac{1}{j} [J^i, J^j]_+ + \delta^{ij} \right). \quad (15)$$

The $t^{(\mu)}$, when sandwiched between $(2j + 1)$–spinors, transform like 4–tensors. Equation (13) is a higher-spin version of the Cartan map (5). The simplest case is $j = 1/2$, identical to (5),

$$a_i = \frac{1}{i\sqrt{2}} \bar{\psi} \sigma_i \psi, \quad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \bar{\psi} = (-\eta, \xi), \quad (16)$$
where $\sigma_i$ are the standard spatial Pauli matrices.

I now contract (13) with rs to give,

$$(a \cdot r)^{2j} = \rho_j \bar{\xi} (r_i r_j \ldots t^{ij\ldots}) \xi.$$ (17)

From the tensor operator (adjoint) transformation of the $t^{ij\ldots}$ one can easily show that, [24],

$$r_i r_j \ldots t^{ij\ldots} = e^{i\pi j} r^{2j} e^{-i\pi r} J.$$ (18)

Further, the important multiplication law for null spinors is,

$$\xi^{(j_1)} \xi^{(j_2)} = (-1)^{2j_1 (2j_1 + 2j_2 + 1)} \left( \begin{array}{ccc} j_1 & j_2 & m_3 \\ m_1 & m_2 & j_1 + j_2 \end{array} \right) \xi^{(j_1+j_2)},$$ (19)

essentially by Wigner–Eckart. This means, in particular, that,

$$\xi^{(j_1)} \xi^{(j_2)} \left( \begin{array}{ccc} m_1 & m_2 & j_3 \\ j_1 & j_2 & m_3 \end{array} \right) = 0,$$ (20)

unless $j_3 = j_1 + j_2$.

So, from (19), (17) reads,

$$(a \cdot r)^{2j} = \rho_j (4j + 1)^{1/2} e^{i\pi j} (-r)^{2j} \xi^{(2j)} \left. \text{Tr} \left( u^{m_j}_{2j}(j) D^{(j)}(\pi r) \right) \right|,$$ (21)

where $u^M_L(J)$ are the matrices discussed by Racah and have matrix elements,

$$\left[ u^M_L(J) \right]_{m'}^m = \langle J m | u^M_L(J) | J m' \rangle = \left( \begin{array}{ccc} J & M & m' \\ m & L & J \end{array} \right).$$ (22)

The trace on the right–hand side of (21) is a hyperspherical harmonic,\(^6\) which is a spin–$j$ SU(2) representation matrix expressed in polar coordinates, $(\chi, \xi, \eta)$, on the 3–sphere. It is evaluated on the equator of $S^3$, $\chi = \pi/2$, and is just a surface harmonic, $C$. Generally,

$$\text{Tr} \left( u^M_L D^{(j)}(g) \right) = i^L H_{j,L}(\chi) \xi^{M}_{L}(\xi, \eta),$$ (23)

where the radial function is related to Gegenbauer polynomials,

$$H_{j,L}(\chi) = L! \left( \frac{(2J - L)!}{(2J + L + 1)!} \right)^{1/2} (2i \sin \chi)^L C_{2J-L}^{L+1}(\cos \chi).$$ (24)

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\(^6\) These are standard objects and will form the basis of a further work. I refer here now only to Talman, [25].
In (21), \( \chi = \pi/2, J = j \) and \( L = 2j \) so one needs,

\[
H_{j, 2j}(\pi/2) = (2i)^{2j} (2j)! \frac{1}{(4j + 1)!^{1/2}}.
\]  

(25)

Therefore, from (23), using (25) and (14),

\[
(a \cdot r)^{2j} = \frac{\rho_j}{(4j)!^{1/2}} e^{i\pi j} (2i)^{2j} (-r)^{2j} C_{2j}^m(\hat{r}) \xi_m^{(2j)}
\]

\[
= 2^{2j} \frac{\rho_j}{(4j)!^{1/2}} C_{2j}^m(r) \xi_m^{(2j)}
\]

\[
= e^{-i\pi j} C_{2j}^m(r) \xi_m^{(2j)}.
\]

I have thus regained (11), as promised, with integer \( M = m \).

I wish to prove something that might not be immediately apparent i.e. that the quantity on the right–hand side of (13) is traceless, as it ought to be since the \( a \) are null. The spatial \( t^{i_1...j} \) themselves are not traceless but the null nature of \( \xi \) saves the day as I now show. Firstly, the space–time traceless property gives the spatial trace,

\[
t_i^{i_1...i_{2j-2}} = t^{00...i_{2j-2}},
\]

which contains at most a product of \( (2j - 2) \) angular momentum matrices and the claim, therefore, is that the quantity,

\[
\xi^{(j)} J^{i_1} ... J^{i_n} \xi^{(j)}, \quad n \leq 2j - 2,
\]

(26)

vanishes. Writing the \( J^i \) as \( 3j \) symbols, the products can be recombined in turn using \( 6j \) symbols until the \( \xi \)s are coupled by the same \( 3j \) symbol and the null theorem (20) can be applied. Because \( j_3 \) is never greater than \( 2j - 2 \), the desired result follows. Alternatively, (19) can be applied directly to (26) to give,

\[
\xi \cdot \text{Tr} (u_{2j}(J) J^{i_1} ... J^{i_n}) ,
\]

in terms of the Racah \( u \) matrix, (22), which can be shown to vanish. The \( J \) are spin–\( j \) matrices. I leave the details as an exercise.

I give some further details about the Joos–Weinberg matrices (Weinberg [26]) in particular on the relation (18). In fact this relation, or rather the one from which I shall shortly derive it, was used by Weinberg to compute the matrices as products of angular momentum matrices. A direct algebraic method, which is easily implemented is the following. Setting,

\[
e^{-i\theta} n \cdot J = \sum_{k=0}^{2j} c_k(\theta) (n \cdot J)^k ,
\]

(27)
for $\theta = \pi$, only even powers occur for integral $j$ and odd powers for $j$ half-odd integral and one has to invert the set of equations, obtained by diagonalisation,

$$( -1)^m = \sum_{k=0}^{j} m^{2k} c_{2k}(\pi), \quad m = 1, \ldots, j, \quad j \text{ integral}$$

$$i(-1)^{m-1/2} = \sum_{k=0}^{j-1/2} m^{2k+1} c_{2k+1}(\pi), \quad m = 1/2, \ldots, j, \quad j \text{ half-odd integral},$$

with $c_0(\pi) = 1$. Torruella, [27], provides an analytic inversion as a function of $j$ but it is easier (and equivalent), for any particular $j$, to use the tabled formulae,

$$( -1)^{m} = \sum_{n=0}^{j} (-1)^n 2^n m^2 (m^2 - 1^2) (m^2 - 2^2) \ldots (m^2 - (n-1)^2) \over (2n)!$$

$$( -1)^{m-1/2} = \sum_{n=0}^{j-1/2} (-1)^n 2^{n+1} m (m^2 - (1/2)^2) (m^2 - (3/2)^2) \ldots (m^2 - (n-1/2)^2) \over (2n+1)!$$

For example,

$$j = 1, \quad c_2(\pi) = -2$$

$$j = 3/2, \quad c_1(\pi) = 7i/3, \quad c_3(\pi) = -4i/3$$

$$j = 2, \quad c_2(\pi) = -8/3, \quad c_4(\pi) = 2/3.$$

From the computed expansion, using (18), the spatial Joos–Weinberg matrices can be extracted as sums of symmetrised products of the $\mathbf{J}^i$, if desired. I will not do so here but refer to the calculation in Weinberg’s paper, of which the above is a particular case. (Just set $q_0 = 0$ in that reference.)

The $\ell^{(\mu)}$ are actually geared to the relativistic situation. They transform under Lorentz transformations, regarded conveniently as complex 3-rotations, as

$$e^{(\alpha - i\beta) \cdot \mathbf{J}} \ell^{\mu} \ldots e^{(\alpha + i\beta) \cdot \mathbf{J}} = \ell^{\nu} \ldots \Lambda_\nu^\mu \ldots$$

where $\Lambda$ is the 4-vector Lorentz transformation. The first thing to note is that $t^{00 \ldots 0}$ is a scalar under the spatial rotation subgroup, $\alpha = 0$, and so can be set equal to 1 by Schur’s lemma. With this in mind, the next step, [26], is to saturate the $\mu$ indices with the four-vector, $q_\mu$, and choose $\Lambda$ to be a pure Lorentz boost, $\beta = 0$, that kills the 3-vector part, i.e. $q_\mu \rightarrow q'_\mu = (q, 0)$, where $q^2 = q_0^2 - |\mathbf{q}|^2$. Then (27) becomes,

$$q_\mu \ldots t^{\mu \ldots} = q^{2j} e^{-2\alpha \cdot \hat{q}} \cdot \mathbf{J},$$

(28)
where
\[
\hat{q} = \frac{q}{|q|}, \quad \sinh \alpha = \frac{|q|}{q}.
\]

Expansion of the right–hand side of (28), as a polynomial in \(q_\mu\), allows the \(t^{(\mu)}\) to be read off, as mentioned before, [26].

To obtain (18) from (28), formally continue \(q_0\) to zero. Then \(\sinh \alpha = i\) so \(\alpha = i\pi/2\) and we are back to ordinary rotations (through \(\pi!\)).

Before leaving this topic, I briefly mention two uses of the \(t^{(\mu)}\). Firstly, the massless equations (Weyl, Maxwell, linearised vacuum gravity, Dirac’s higher spin) can all be written in the form, [28],
\[
t^{\mu \ldots \nu} \partial_\nu \phi(x) = 0,
\]
and, secondly, the quantity \(\phi^{(j)}_{(j)} t^{(\mu)} \phi_{(j)}\) is a higher–spin analogue of the Bel–Robinson tensor, [24].

3. Invariant Theory.

If \(\xi\) and \(\eta\) are viewed as variables, and \(\alpha\) and \(\beta\) as coefficients, the bracket power in (10) is the symbolic form of a binary \(2j\)–ic, traditionally written \((\alpha_1 x_1 + \alpha_2 x_2)^n\), \(n = 2j\) (e.g. Grace and Young [29]). For example, for the general quadratic (spin one), \(ax_1^2 + 2b x_1 x_2 + cx_2^2\), one has \(a = \alpha_1^2\), \(b = \alpha_1 \alpha_2\) and \(c = \alpha_2^2\), but \(ac \neq b^2\) and the coefficients, \((a, \sqrt{2}b, c)\), do not constitute the (spherical) components of a null 3–vector (3–spinor).

Comparing with (5), it cannot escape notice that the Cartan map had already appeared algebraically in the 1850’s, when Aronhold introduced the symbolic method into invariant theory. Since this theory plays a part in what I wish to say, I present some expository material. I will not avail myself of the various renderings of classical invariant theory into modern algebraic terms, preferring the earlier language.

The general \(2j\)–ic, \(\Phi_{2j}\), formed with the coefficients \(\phi^{m}_{(j)}\), is written,
\[
\Phi_{2j}(\phi, \xi, \eta) = \phi^{m}_{(j)} \xi_m \equiv \phi \xi
\]
\[= (-1)^{2j} \xi^m \phi^{(j)} \equiv (-1)^{2j} \xi \phi. \tag{29}\]

In the case that the coefficients form a null \((2j + 1)\)–spinor, say \(\phi_{(j)} = \alpha_{(j)}\), \(\Phi_{2j}\) takes the simple power form (10) and the \(2j\) roots of \(\Phi_{2j} = 0\), for \(t \equiv \xi/\eta\), are
all equal to $\alpha/\beta$. The necessary and sufficient condition for all roots to be equal, is that the Hessian of the quantic vanish identically. The Hessian for the quantic (29) is given in my notation, by

$$H = \left(\begin{array}{ccc} 2j - 2 & j & j \\ l & m & n \end{array}\right) \xi^l_{(2j-2)} \phi^m_l \phi^n_l,$$

and so one has the theorem, [21], that if,

$$\left(\begin{array}{ccc} 2j - 2 & j & j \\ l & m & n \end{array}\right) \phi^m_l \phi^n_l = 0,$$

then $\phi_{(j)}$ is null. The converse is contained in (20).

It is fundamental that subjecting the two spinor, $\psi = (\xi \eta)$, to a transformation belonging to $SL(2,\mathbb{C})$, and requiring the binary $n$–ic, $\Phi_n$, (29), to be invariant, induces a (particular) Lorentz transformation on the coefficients, $\phi_m$. Invariant theory treatments can be found in Turnbull, [30], Chap.VIII §8, Glenn, [31] §1.2.4, [29], §16. I repeat some standard details here out of interest.

The basic transformation $\psi \rightarrow \psi'$, expressed as the inverse for convenience, is,

$$\left(\begin{array}{c} \xi \\ \eta \end{array}\right) = \left(\begin{array}{cc} l_1 & m_1 \\ l_2 & m_2 \end{array}\right) \left(\begin{array}{c} \xi' \\ \eta' \end{array}\right).$$

The imposed invariance of $\Phi$ implies, setting $\xi'/\eta' = t$,

$$\Phi_n(\bar{\phi}, \xi, \eta) = \Phi_n(\bar{\phi}', \xi', \eta') = \eta^m \Phi_n(\bar{\phi}', t, 1)$$

$$= \eta^m \Phi_n(\bar{\phi}', l_1 t + m_1, l_2 t + m_2)$$

$$= \eta^m e^{t \mathbf{l} \nabla_m} \Phi_n(\bar{\phi}, m_1, m_2)$$

where $\mathbf{l} \nabla_m$ is the polarisation operator (or translation operator or directed derivative),

$$\mathbf{l} \nabla_m = l_1 \frac{\partial}{\partial m_1} + l_2 \frac{\partial}{\partial m_2} \equiv \left( l \frac{\partial}{\partial m} \right).$$

This is an extreme example of a nullform, as defined by Hilbert to be a form all of whose invariants vanish. For this one needs only a root of multiplicity $[n/2] + 1$, $n$ being the form order.

The group in invariant theory does not have to be unimodular. For those requiring the basics of invariant theory in short, but expert, compass I recommend Dickson’s book, [32], which treats both symbolic and non–symbolic methods and employs group theory concepts.
I interject a small, necessary calculational point. The normalisation of the coefficients, \( \phi^m \), differs from that of the coefficients as usually defined in invariant theory. The two conventions are exhibited in the forms,

\[
\Phi^2_j(\phi, \xi, \eta) = \sum_{m=-j}^{j} \phi^m \left( \frac{2j}{j-m} \right)^{1/2} \xi^{j+m} \eta^{j-m}
\]

(33)

with \( n = 2j \) and \( r = j - m \), so that,

\[
\phi^{j-r} = \binom{n}{r}^{1/2} a_r,
\]

(34)

connects the two sets of coefficients.

The normalisation for the \( \phi \)s corresponds to the use of the \( 1-j \) symbol as the raising and lowering metric in weight space, (6), as in (29), and elsewhere, and this is an appropriate place to make some elementary remarks on duality which I will need later.

The spinor \( \overline{\psi} = (-\eta, \xi) \) is dual to \( \psi = (\xi, \eta) \). I can make this projective by setting \( -\eta \to u \) and \( \xi \to v \), and then refer to \( u, v \) as line, or tangential, coordinates. \( \xi \) and \( \eta \) are ‘point coordinates’.

The point binary form (33) can be written in line coordinates as follows,

\[
\Phi^2_j(\phi, \xi, \eta) = \left( \frac{2j}{j-m} \right)^{1/2} (-1)^{j-m} \phi^m \xi^{j+m} (-\eta)^{j-m}
\]

(35)

corresponding to (29). The sign factor could be absorbed into the coefficients of the line form.

Putting (33) into the invariance (31) yields

\[
\sum_{r=0}^{n} \binom{n}{r} a_r t^{n-r} = e^{t^1 \nabla m} \Phi_n(a, m_1, m_2)
\]
and expansion gives the relation between the old and new coefficients,

\[ a'_r = \frac{r!}{n!} (1. \nabla_m)^{n-r} \Phi_n(a, m_1, m_2), \]  

(36)
i.e.

\[ \phi^m = \sqrt{\frac{(j-m)!}{(2j)!(j+m)!}} (1. \nabla_m)^{j+m} \Phi_{2j}(\bar{\phi}, m_1, m_2). \]  

(37)

If one rather uses \( \eta' = t \xi' \), then the alternative formula,

\[ a'_r = \frac{(n-r)!}{n!} (m \cdot \nabla_1)^r \Phi_n(a, l_1, l_2), \]  

(38)
i.e.

\[ \phi^m = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} (m \cdot \nabla_1)^{j-m} \Phi_{2j}(\bar{\phi}, l_1, l_2), \]  

(39)
arises ([30], [31]).

Infinitesimal behaviour and projection operators.

As mentioned, the induced transformation on the coefficients is that of the spin–\( j \) representation of the rotation group (when \( \text{SL}(2, \mathbb{C}) \) is reduced, inessentially, to \( \text{SU}(2) \)) and these relations are more or less identical to a standard method of finding the representation matrices, \( D^j \), of the rotation group, e.g. [33–35, 19]. It is worth expanding on this in the following way.

A binary covariant, \( K \), is an \( \text{SL}(2, \mathbb{C}) \), and hence \( \text{SU}(2) \), invariant, i.e. \( (cf (31)) \),

\[ K(a; \xi, \eta) = K(a'; \xi', \eta'). \]

Useful information follows from the infinitesimal expression of this invariance.

The binary (spin–half) realisation of the generating angular momentum operators is, (see Sharp, [36], Bargmann, [19]),

\[ J_+ \rightarrow -\xi \frac{\partial}{\partial \eta}, \quad J_- \rightarrow -\eta \frac{\partial}{\partial \xi}, \]

\[ J_z \rightarrow \frac{1}{2} \left( \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right), \]  

(40)
acting on functions, \( f(\xi, \eta) \). A commutation relation is

\[ [J_+, J_-] = 2J_z. \]  

(41)
For ease of comparison, I have reverted to the notation $J_\pm = J_x \pm iJ_y$. These generators serve also for the binary invariant group, SL(2).

The total angular momentum (Casimir operator) is

$$J^2 = \frac{1}{4} \left( \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \right) \left( \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + 2 \right),$$

and the simultaneous eigenfunctions, $f^j_m$, of $J_z$ and $J^2$ are easily determined by a standard procedure. From (42), they must be homogeneous of degree $2j$, where $j(j+1)$ is the eigenvalue of $J^2$, with $j$ a half–integer. The eigenvalue of $J_z$ equals $m$, with $-j \leq m \leq j$. As usual, the $f^j_m$ can be determined, using the lowering $J_-$, from the ‘highest weight’ function $f^j_j$, which itself satisfies $J_+ f^j_j = 0$ and so is fixed to be proportional to the power, $\xi^j$. Then, lowering by induction, yields, see (9),

$$f^j_m = \xi_m^{(j)},$$

and we see that the $\xi_m^{(j)}$ are function space variants of the quantum vectors, $|jm\rangle$.

The matrix elements of $J_\pm$ in this basis are the usual ones. I do not give them because they are contained in the expressions for the transformations induced by (40) on the coefficients of the quantic, (29). These already occur in early invariant theory in the guise of the Cayley–Sylvester–Aronhold operators, $\Omega$ and $O$.

 Acting on a binary covariant, $K$, of the ground form, (29), (33), the operators $J_\pm$ are equivalent to the annihilators, $\Omega$ and $O$, i.e.,

$$\Omega'K \equiv \left( \Omega - \eta \frac{\partial}{\partial \xi} \right)K = 0$$

$$O'K \equiv \left( O - \xi \frac{\partial}{\partial \eta} \right)K = 0$$

(43)

with, in classic notation,

$$\Omega \equiv \sum_{k=0}^{n-1} (k+1) a_k \frac{\partial}{\partial a_{k+1}}, \quad O \equiv \sum_{k=0}^{n-1} (n-k) a_{k+1} \frac{\partial}{\partial a_k}. \quad (44)$$

This theorem was proved by invoking the invariance of the covariant under the infinitesimal shears, $\xi \to \xi + \lambda \eta$, $\eta \to \eta$ and $\xi \to \xi$, $\eta \to \lambda \xi + \eta$, generated by $J_-$ and $J_+$, respectively. (See e.g., Salmon, [37] §148. Dickson, [32,38], gives a succinct treatment and Elliott, [39] Chaps.VI, VII a more lengthy one.) The operator $J_z$ generates the scalings that maintain $\xi \eta$. Olver, in his nice, modern reworking of classical invariant theory, [6], discusses the Lie algebra aspects of $\Omega$ and $O$. 

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Transcribed into SU(2) notation via (34), equation (44) reads, \((n = 2j)\),

\[
\Omega = \sum_{m=1-j}^{j} ((j - m + 1)(j + m))^{1/2} \phi^m \frac{\partial}{\partial \phi^{m-1}}
\]

\[
O = \sum_{m=1-j}^{j} ((j - m + 1)(j + m))^{1/2} \phi^{m-1} \frac{\partial}{\partial \phi^{m}}
\]

\[
= - \sum_{m=-j}^{j-1} ((j + m + 1)(j - m))^{1/2} \phi^{m+1} \frac{\partial}{\partial \phi_{m+1}},
\]

in which the angular momentum matrix elements can be recognised using the relation

\[
X_a = \varphi^i (G^j_a) \frac{\partial}{\partial \varphi^j}
\]

between the Lie derivative operators, \(X_a\), and the matrix generators, \(G_a\), in the (matrix) representation in the carrier space of which \(\varphi^i\) is a vector. Thus

\[
\varphi^j \rightarrow \ '\varphi^j = e^{q^a X_a} \varphi^j = \varphi^j [e^{q^a G_a}]^i \varphi^j.
\]

The general structure of the SU(2) covariant, \(K\), of degree \(g\) and order \(\varpi = 2s\) is a Clebsch–Gordan coupling of \(g\) spin–\(j\) objects, \(\phi\), to a spin–\(s\) quantity (the set of coefficients) which is then coupled to a spin–\(s\) null spinor (the variables) to a zero spin resultant. The invariance conditions (43) are then equivalent to the standard invariance of the 3–\(j\) symbols under SU(2).

The operators \(\Omega\) and \(O\) perform important functions in the construction of concomitants and thence in the proof of the Hilbert finite basis theorem. The algebra of \(\Omega\) and \(O\), developed for this purpose, can be translated into angular momentum terms using (45) and (46).

Corresponding to \(J_z\), (41), there is the commutator (‘alternant’)

\[
\frac{1}{2} [O, \Omega] = \frac{1}{2} \sum_{k=0}^{n} (2k - n) a_k \frac{\partial}{\partial a_k} = - \sum_{m=-j}^{j} m \phi^m \frac{\partial}{\partial \phi^m}
\]

originally constructed by Cayley, [40]. Therefore, in total, \(\Omega'\), \(O'\) and \(\frac{1}{2} [O', \Omega']\) correspond to \(\mathbf{J} + \frac{1}{2} \mathbf{\sigma}\).

I reckon weight following Cayley’s first scheme, [40], in which \(a_r\) has weight \(r - n/2\), and \(\xi\) and \(\eta\) have weights \(\pm 1/2\). This is more in keeping with the group representation value than the one usual in classical texts, being the eigenvalue of
and has the advantage that the commutator \( \frac{1}{2}[O', \Omega'] \) is the weight, or scaling, operator since \( \phi^m \) has weight \(-m\).

A basic result, essentially due to Hilbert, is that the quantity,

\[
K(a; \xi, \eta) = \left( 1 - \frac{O'O'}{1.2} + \frac{O'^2\Omega'^2}{1.2^2.3} - \frac{O^3\Omega^3}{1.2^2.3^2.4} + \ldots \right) F(a; \xi, \eta),
\] (48)

where \( F \) is a rational, integral, isobaric \(^9\) function of the \( a_r \) and \( \xi, \eta \) of zero total weight, is a covariant of the ground form (33), or zero.\(^{10}\)

The operator, \( H_+ \), in (48), and its partner,

\[
H_- \equiv \left( 1 - \frac{\Omega'O'}{1.2} + \frac{\Omega'^2O'^2}{1.2^2.3} - \frac{\Omega^3O^3}{1.2^2.3^2.4} + \ldots \right),
\] (49)

take the forms of general projection operators constructed from raising and lowering operators so that one could refer to \( K \) as the covariant projection of \( F \). It is a theorem that every covariant can be obtained in this way.

The operator equivalent of this projection, customarily couched in angular momentum terms, is that the terminating polynomial,

\[
\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(1+r)!} J_r^+ J_r^-
\] (50)

is equivalent, using (41), to

\[
\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(1+r)!} J_r^+ J_r^-
\] (51)

when acting on an eigenstate of \( J_z \) with zero eigenvalue, \( m = 0 \), and projects onto the identity, or trivial, representation, \( i.e. \) to a scalar. This last statement accords with the scalar quality of the covariant, \( K \). The \( m = 0 \) condition is just the zero weight one on \( F \).

The operator, (50), is a special case of a projector derived by Löwdin, [41], equn.(32) and rederived by Shapiro, [42]. This more general operator is equivalent, for quantics, to \( (\alpha)_n \) is the Pochhammer symbol,

\[
\left( 1 - \frac{O\Omega}{1!(2m+2)_1} + \frac{O'^2\Omega'^2}{2!(2m+3)_2} - \frac{O^3\Omega^3}{3!(2m+4)_3} + \ldots \right),
\] (52)

\(^9\) Isobaric means that each separate term of an expression has the same weight. Olver defines it to mean invariance under the scaling subgroup, which is more restrictive.

\(^{10}\) See Elliott, [39], §182.
where, for simplicity, I consider actions on functions of just the coefficients, e.g. on homogeneous, isobaric functions i.e. gradients, $G(a)$, of excess $2m$, a number which determines the scaling of $G$ via

$$\frac{1}{2} [O, \Omega] G = mG.$$  

For these terms, consult Elliott, [39].

The sufficiency of dealing with just the projection onto the identity has been exploited by Shapiro, [43], for arbitrary groups. The basic notion is one commonly used in selection rule calculations that computing how much of a given irrep there is in, say, a product of two irreps, is equivalent to coupling the three irreps to the trivial one. This corresponds to a simple group average, e.g. [35]. It is the same as the choice one has in invariant theory of dealing either with the form as a whole (a scalar) or with just its coefficients (a carrier space vector).

As an example of a higher group, Noz and Shapiro, [44], give the identity projection for SU(3) as a polynomial in the generators. In the invariant theory setting this involves an extension to the ternary domain, geometrically more interesting, but more complicated. The most systematic and elegant treatment of projection operators is given by Story, [45], and a summary in [46]. As an example, the group GL(3) is generated by 6 shears (determinant one), and 3 expansions. In Elliott, [39], Chap.XVI, the shears are the annihilators, denoted $\Omega'_{ij}$ with $i, j = x, y, z; i \neq j$, and the expansions are the non–zero commutators, $H'_1, H'_2, H'_3$ with $\sum H'_i = 0$. Elliott effectively works out the structure constants of $\mathfrak{sl}(3)$ in the Cartan basis, the $H'_i$ being the Cartan subalgebra in precisely the modern notation, e.g. Racah, [47], equn.(72). For example, in terms of the Gell–Mann matrices $\lambda_3 \sim H'_3$ and $\lambda_8 \sim H'_1 - H'_2$ in the fundamental, quark representation, $(x, y, z)$. $H'_1, H'_2$ and $H'_3$ generate scalings of the pairs $(y, z), (z, x)$ and $(x, y)$ respectively, corresponding to the three SU(2) ‘binary’ subgroups which give V–spin, U–spin and I–spin. These scalings leave the products, $yz, zx$ and $xy$ invariant, each pair as in the binary case, leading onto the definition of weights used by Story.  

The projection operators are assembled out of the $\Omega'_{ij}$, which are the raising and lowering operators, usually denoted by $E_\alpha$ in Lie algebra theory. A comparison of the projectors derived by Story, [46], equns.(8), (9) and Elliott, with those of Noz

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11 Elaboration of this will form part of a further communication.

12 For those who wish to quickly refresh their knowledge on the Lie algebra of GL(n)⊃ SL(n), I suggest Racah, [47], and Gourdin, [48], esp. Chap.3.
and Shapiro, [44], eqns.(2.4), (2.17), shows agreement.\(^\text{13}\) See also Asherova and Smirnov, [49]. All methods involve chains of subgroups e.g. SU(3)⊃SU(2).

There is a close connection between the construction of Lie group/algebra representations and that of covariants from semi-covariants,\(^\text{14}\) in particular from one, given, term of the covariant (its ‘source’) by repeated action of \(\Omega’\) or \(O’\), as in Roberts’ theorem in the binary domain. (I recommend [32], [39] and [7]). The various matrix elements will emerge as in (45) for \(\mathfrak{sl}(2)\). The projection operator method for simple Lie groups is given in more detail by Ašerova et al, [50].

4. Polars and angular momentum addition.

The polar operators \(\mathbf{1.}\nabla_m\), (32), and their obvious extensions to higher domains, play important roles in invariant theory.

The introduction of \(u\) and \(v\) is not necessary in the binary domain and so is not usually made in the classical works. One can easily work directly with \((-\eta,\xi)\). Line coordinates are needed however for ternary and higher forms and their introduction in the binary case allows a more unified treatment. \(cf\) Todd, [51],§§1.6, 2.12. For example I now bring in the famous ‘Omega process’ used in the composition of invariants, and covariants, to give others. To be a little more systematic, I denote point variables by \(x_i\) \((i = 1,\ldots,p)\) and the dual line coordinates by \(u^i\). A \(p\)–ary \(n\)–ic is written symbolically as,

\[
\begin{align*}
  f : & \quad a^i_x = (\mathbf{a.x})^n = (a^i x_i)^n = (a^1 x_1 + a^2 x_2 + \ldots + a^p x_p)^n \\
  F : & \quad u^i_\alpha = (\mathbf{u.}\alpha)^n = (u^i \alpha_i)^n = (u^1 \alpha_1 + u^2 \alpha_2 + \ldots + u^p \alpha_p)^n
\end{align*}
\]

in point and line coordinates respectively. I am most concerned with \(p = 2\) and \(p = 3\).

The geometrical interpretation of forms becomes more involved the higher \(p\) is. One interpretation\(^\text{15}\) follows on taking the \(x_i\) as homogeneous point coordinates in a \((p−1)\)–dimensional space and the vanishing of a form yields a co–dimension 1 variety (a hypersurface) in this space. In the binary case this is a range of points, in the ternary case, a curve and quaternary, a surface. This interpretation provides a useful language and visualisation, at least for low \(p\).

\(^{13}\)This will be considered elsewhere.

\(^{14}\)These are the analogues of the extreme weight states, \(|jj\rangle\) and \(|j,−j\rangle\).

\(^{15}\)Not the only possibility.
Consider two distinct \( p \)-forms of generally different orders, \( f = a^\alpha_n \equiv u^\alpha_n = F \) and \( g = b^\beta_m \equiv u^\beta_m = G \), where I have indicated both the point and line forms.

Following Clifford, [52], define the generalised polar of \( G \) in respect of \( f \),

\[
h = \frac{(n-m)!}{m!n!} (\nabla_u \cdot \nabla_x)^m u^\beta_m a^\alpha_n \equiv \frac{(n-m)!}{m!n!} \Omega^m u^\beta_m a^\alpha_n \tag{53}
\]

\[
= \frac{(n-m)!}{n!} (\beta \cdot \nabla_x)^m a^\alpha_n
\]

\[
= (a \cdot \beta)^m a^{n-m} = a^m a^{n-m}, \quad n \geq m.
\]

The first line is in Grace and Young (for the ternary case), [29], §241, the second is Clifford’s definition (also in the ternary case). The last line is the symbolic expression. In the ternary case, \( h \) gives the polar locus, of order \((n - m)\), of \( G \) in respect of \( f \).

\( h \) is zero if \( n < m \). However in this case, we can reverse the roles of \( f \) and \( G \) and compute the generalised polar of \( f \) in respect of \( G \),

\[
h' = \frac{(n-m)!}{m!n!} (\nabla_x \cdot \nabla_u)^n u^\alpha_n a^\beta_m \equiv \frac{(n-m)!}{m!n!} \Omega^n u^\alpha_n a^\beta_m
\]

\[
= \frac{(n-m)!}{m!} (a \cdot \nabla_u)^n u^\beta_m
\]

\[
= (a \cdot \beta)^n a^{m-n} = a^n u^{m-n}, \quad m \geq n,
\]

which represents the polar envelope, of class \((m - n)\), of \( f \) in respect of \( G \).\(^{16}\)

For binary forms, the situation is summarised, explicitly in point coordinates, by Sturm, [59], who also describes the geometry in terms of ranges of points on the complex projective line in a standard way. For this purpose, it is expressive to rewrite \( h \), say, using the factored form of \( u^\beta_m \), which is, reverting to the original binary notation,

\[
u^\beta_m = \prod_{k=1}^m (v_k u - u_k v),
\]

so that setting \( v_k \to \xi_k \) and \( u_k \to -\eta_k \) and also, according to the definition (53), \( u \to \partial_\xi, v \to \partial_\eta \),

\[
h = \frac{(n-m)!}{n!} \prod_{k=1}^m \left( \xi_k \frac{\partial}{\partial \xi} + \eta_k \frac{\partial}{\partial \eta} \right) f , \tag{55}
\]

\(^{16}\) A class curve is often called an envelope or a tangential curve and an order curve is a ‘locus’.

A ‘curve’ can usually be expressed in either of these dual ways. The best known example is the conic, which is of order 2, or equivalently, class 2. Simple descriptions, in English, can be found in school textbooks such as Robson, [53], Milne, [54], Sommerville, [55]. Possibly the most attractive is Askwith, [56]. More advanced is Salmon, [37]. The rather rare book by Scott, [57], should be mentioned and the more modern Todd, [51], and Semple and Kneebone, [58].
which shows \( h \) as a *mixed* polar of \( f \) with respect to the points (roots) \((\xi_k, \eta_k)\).

Writing \( h \) as the form \((c \cdot x)^{n-m}\), the coefficients, \( c_r \), can be determined in terms of \( a_r \) and \( b_r \). They are given in Sturm, for example, [59], and would allow one to calculate the \( 3j \) symbols \( \binom{n/2}{m/2} \binom{(n-m)/2}{*} \).\(^{17}\)

In the binary case, the quantity \( \Omega \) is the operator introduced by Cayley and which, in point coordinates, is expressed as a determinant\(^ {18}\). Use of line coordinates allows it to be written as a contraction, \( \Omega = \nabla_u \cdot \nabla_x \) (see Todd, [51], §2.12.3). Two binary forms, \( f \) and \( g \), can be combined into a third using \( \Omega \) to give their \( r \)th transvectant,

\[
(f,g)^r = \frac{(n-r)!(m-r)!}{n!m!} \Omega^r(fG).
\]

In particular the polars are the ‘end values’,

\[
h = (f,g)^m, \quad h' = (f,g)^n,
\]

and generally \((f,g)^r\) is a form of order \((m+n-2r)\) on making the replacement \((u_1, u_2) \rightarrow (-x_2, x_1)\) (or a form of class \((m+n-2r)\) on making the dual replacement \((x_1, x_2) \rightarrow (u_2, -u_1)\). The order (class) runs from \((m+n)\) to \(|m-n|\) and transvection corresponds to addition of angular momenta via Clebsch–Gordan, or \(3j\), symbols. As a well known example, the second transvectant of a form with itself is proportional to its *Hessian*, \( H \), (30).

I can now introduce the important notion of *apolarity*, originally developed in the binary case but later extended to higher domains.\(^{19}\)

The straight analytical definition is that two forms, \( f \) and \( G \), are said to be *apolar* if their polar, \( h \), defined in (53), vanishes. I take \( n \geq m \) for ease.

An important fact is that the form \( f \) is apolar to any form that contains \( G \) as a factor and whose class does not exceed \( n \). The proofs, which are simple, are given in Grace and Young, [29] pp.225, 304. In the binary case this result corresponds to elementary angular momentum addition rules, as I now explain. \( f \) and \( g \) correspond, respectively, to spins \( j_1 = n/2 \) and \( j_2 = m/2 \) as explained earlier.

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\(^{17}\) As might be expected, these \(3j\) symbols have simple explicit forms, *e.g.* Hamermesh, [35], p.375 eqn.(9-123). They occur later.

\(^{18}\) In the ternary case the operator \( \Omega \), as defined here, occurs on p.296 of [29] as \( Q \).

\(^{19}\) The notion appears to be due to Reye. I will not dilate on its general significance but just refer to Coolidge, [60] pp.368, 410, Grace and Young, [29], Semple and Kneebone, [58], Todd, [51].

\(^{20}\) Whether we say apolar to \( G \) or to its dual, \( g \), is, metrically, immaterial and I will be somewhat slack in making this distinction.
The polar $h$ corresponds to spin $j_1 - j_2 = (n - m)/2$. If it vanishes, so that $f$ and $G$ are apolar, one has the $3j$ equivalent

$$f^{m_1} g^{m_2} \left( \begin{array}{ccc} j_1 & j_2 & j_1 - j_2 \\ m_1 & m_2 & \ast \end{array} \right) = 0,$$

(57)

i.e. $2(j_1 - j_2) + 1 = n - m + 1$ conditions.

The angular momentum analogue of the product of two forms, say $g \phi$, can be found from the multiplication rule for null spinors, (19). Thus,

$$g \phi = g^{m_2} \phi^{m_3} \xi^{(j_2)}_{m_2} \xi^{(j_3)}_{m_3}$$

$$= (-1)^{2j_2} (2j_2 + 2j_3 + 1)^{1/2} g^{m_2} \phi^{m_3} \left( \begin{array}{ccc} j_2 & j_3 & m_4 \\ m_2 & m_3 & j_2 + j_3 \end{array} \right) \xi^{(j_2+j_3)}_{m_4}$$

(58)

$$\equiv \psi^{m_4} \xi^{(j_2+j_3)}_{m_4}$$

$$= \psi.$$

The statement is that $\psi = g \phi$, is apolar to $f$ if $g$ is, i.e., that

$$0 = f^{m_1} \psi^{m_4} \left( \begin{array}{ccc} j_1 & j_2 + j_3 (j_1 - j_2 - j_3) \\ m_1 & m_4 & \ast \end{array} \right)$$

$$\propto f^{m_1} g^{m_2} \phi^{m_3} \left( \begin{array}{ccc} j_2 & j_3 & m_4 \\ m_2 & m_3 & j_2 + j_3 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 + j_3 (j_1 - j_2 - j_3) \\ m_1 & m_4 & \ast \end{array} \right)$$

is true if (57) holds. To show this, the $3j$ symbols are recoupled, using $6j$ symbols, so that $j_1$ is coupled to $j_2$ and $j_3$ to $j_1 - j_2 - j_3$. Without writing out the full expression, these couplings quickly show that the smallest intermediate angular momentum is $j_1 - j_2$ and the largest one is also $j_1 - j_2$. This allows (57) to come into play and the result follows.

Two apolar forms of the same order (class), $n$, are sometimes called *conjugate*.\(^{21}\)

From (57), this means that the joint invariant,

$$f^{m_1} g_{m_1} = (a \cdot \beta)^n,$$

(59)

vanishes.

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\(^{21}\) The terminology seems to originate with Rosanes, [61]. Reye uses the term apolarity as do Grace and Young, [29]. The classical expression for the joint invariant (or *lineal* linear invariant) can be seen in many places e.g. Elliott, [39], §49, Grace and Young, [29] §177, Glenn, [31], equin (71), Coolidge, [60] p.369, Rosanes, [61]. The relation between (36) and (38) is on p.58 of Elliott. Following Sylvester, Elliott considers polars as examples of *emanants*.
A form, \( \phi \), of order \( n \), is conjugate to the \( n \)th power (cf (10)) of one of its factors, essentially by definition of the roots of \( \phi = 0 \). These powers are null forms, on my previous definition, and their coefficients are the components of what I termed the \emph{principal null} \( 2j + 1 \)-spinors of \( \phi \), [21]. \textit{In general} there are \( n = 2j \) of them and any linear combination is obviously conjugate to \( \phi \). Conversely, any form conjugate to \( \phi \) can be written as a linear combination of the principal null forms of \( \phi \), essentially by a counting argument (Rosanes, [61]).

Since odd order forms (half–odd–integer spin) are self–conjugate, they can always be expanded as sums of \( n \)th powers.

An easy theorem, which has sometimes been used as the definition of apolarity, is that \( f \) is apolar to \( g \) if, and only if, it is conjugate to the product \( g\phi \) for \emph{all} factors \( \phi \). Thus, conjugacy reads, using (58) for the product of forms,

\[
f^{m_1}g^{m_2}\phi^{m_3}\begin{pmatrix} j_2 + j_3 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0.
\]

If true for all \( \phi \) then \( \phi \) can be removed and, setting \( j_1 \equiv j_2 + j_3 \), there results

\[
f^{m_1}g^{m_2}\begin{pmatrix} j_1 & j_2 & j_1 - j_2 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0.
\]

This proves necessity. Sufficiency has been shown just above.

A specific example, that combines these lemmas, follows from the groupings of the equation,

\[
a^{m_1}b^{m_2}c^{m_3}\begin{pmatrix} j & j - \frac{1}{2} & \frac{1}{2} \\ m_1 & m_2 & m_3 \end{pmatrix} = 0,
\]

which say, in form language, that, if \( a^n_x \) and \( b^{n-1}_x c_x \) are conjugate \( n \)–ics, then \( (ac)a^{n-1}_x \) and \( b^{n-1}_x \) are conjugate \( (n - 1) \)–ics. The particular \( 3j \) symbol here is the same as the matrix–spinor \( u_A(j) \) \( (A = \pm 1/2) \) employed by Dirac in his higher spin equations (see Corson, [62]).

As another illustration, in terms of \( 3j \) symbols the necessary and sufficient condition for there to be at least \( e \) equal roots of the binary quantic equation \( \Phi_{2j} = 0 \), is that the apolarity,

\[
\left( \begin{array}{ccc} m'' & j' & j \\ j - j' & m' & m \end{array} \right) \phi^m \xi^{m'}_{(j')} = 0, \quad e = 2(j - j') + 1,
\]

should have solutions for \( \xi \) corresponding to the equal root. This implies

\[
\left( \begin{array}{ccc} m'' & j' - \frac{1}{2} & j \\ j - j' + \frac{1}{2} & m' & m \end{array} \right) \phi^m \xi^{m'}_{(j'-\frac{1}{2})} = \mu \xi^{m''}_{(j-j'+\frac{1}{2})}.
\]
In particular, as a trivial check, when \( e = 2j \), (62) implies \( \phi \propto \xi(j) \), a null spinor. At the other extreme, if \( e = 1 \) (i.e. general position), (61) is just the polynomial, \( \Phi_{2j} = 0 \), each of the \( 2j \) roots qualifying as an ‘equal’ root.

In general, (61) is a set of \( e \) simultaneous \( (2j + 1 - e) \)-ics with at least one common root, the equal root. If there are two common roots, there are two sets of equal roots of the original quantic, and so on.

As intimated earlier, for calculational purposes there is no need to evaluate the \( 3j \)-symbols in (61) as they are coded in the polar, the derivative form of which, (53) or (55), shows that the \( e \) polynomials are just derivatives applied to the form \( \Phi_{2j} \).

Explicitly (cf Sturm, [59]), putting in a bookkeeping null spinor, \( \alpha_{(j-j')} \),

\[
\alpha_{(j-j')}^{m''} \varepsilon_{(j')}^{m'} \left( j - j' j' j \right) \phi^{m} = (-1)^{2j'} \left( \frac{2j + 1}{(2j - 2j')!} \right)^{1/2} \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{2(j-j')} \Phi_{2j}(\phi; \xi, \eta). \tag{63}
\]

I also remark that the purely 2–spinor standpoint, (Penrose, [63]), is equally as effective.

To summarise the correspondances so far between angular momentum and binary invariant theory: Forms of order (or class) \( n \) correspond to spin–(\( n/2 \)) spinors. The polar of one form with respect to another corresponds to the addition of the angular momenta to give the minimum total value. The product of two forms does the same thing but to the maximum value, while the transvectant interpolates and gives all allowed values.\(^{22}\)

For the ternary case things are not so complete or straightforward but they are geometrically more interesting. I will return to this topic later and now take up the theory of harmonic polynomials, for which it is pertinent.

\(^{22}\) The connection of invariant theory and angular momentum theory is, of course, well known but does not seem to be specifically used, to any degree. Any necessary calculations tend to be done again. However, for interesting modern developments, with an algebraic–geometric slant, see Abdesselam and Chipalkatti, [64], and references given there.
5. Harmonic projection. Poles.

Historically, the Laplace coefficient \( P_n(\cos \gamma) \) arose in the Taylor expansion \(^{23}\) of the ‘displaced’ \(1/r\) potential, or static Green function,

\[
\frac{1}{|r - r'|} = \sum_{n=0}^{\infty} \frac{(rr')^n P_n(\cos \gamma)}{r^{2n+1}} = \sum_{n=0}^{\infty} \frac{H_n(r, r')}{r^{2n+1}}
\]

\[
= e^{-r' \cdot \nabla} \frac{1}{r} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (r' \cdot \nabla)^n \frac{1}{r}
\]

yielding

\[
H_L(r, r') = \frac{(-1)^L}{L!} r^{2L+1} (r' \cdot \nabla)^L \frac{1}{r},
\]

given by Thomson and Tait, [8] p.157 eqn.(54). Symmetry produces the equivalent expression,

\[
H_L(r, r') = \frac{(-1)^L}{L!} r'^{2L+1} (r \cdot \nabla')^L \frac{1}{r'}.
\]

A form which displays the symmetry can be found by remarking that, as we know (7), the left–hand side of (7), \((a_x x + a_y y + a_z z)^L\), is a spherical harmonic, of degree \(L\). This can be checked by direct differentiation, assuming the null condition, \(a \cdot a = 0\). Hence, in particular,

\[
(r \cdot \nabla')^L \frac{1}{r'}
\]

is a spherical harmonic in \(r\), in agreement with (66) of course. The next step involves an application to (67) of the useful differentiation theorem, due to Niven, [65], see Hobson, [10], and [12] (6.169),

\[
Y_n(r) = (-1)^n \frac{2^n n!}{(2n)!} r^{2n+1} Y_n(\nabla) \frac{1}{r},
\]

where \(Y_n\) is any solid spherical harmonic. There results Niven’s interesting form,

\[
H_L(r, r') = \frac{2^L (rr')^{2L+1}}{(2L)!} (\nabla \cdot \nabla')^L \frac{1}{rr'}.
\]

I note the similarity of the equivalent expressions, (69) and (65), with the polar forms in (54).

\(^{23}\) I do not bother with conditions or regions of convergence.
The expression of spherical harmonics as derivatives, normal or fractional, of $1/r$ was taken as fundamental by Thomson and Tait, [8], and also was adopted and extended by Maxwell, [9].

In this connection, it is best to begin historically with what might be called Gauss’ harmonic expansion. Gauss proved constructively, in a brief ‘Nachlass’, that a rational, integral polynomial function, of degree $n$, $f_n(r)$, can be represented by a finite sum of solid spherical harmonics, $Y_m(r)$, as,

$$f_n = Y_n + r^2 Y_{n-2} + r^4 Y_{n-4} + \ldots$$

$$= \sum_{s=0}^{[n/2]} r^{2s} Y_{n-2s},$$

(70)

By repeatedly hitting the left–hand side with the Laplacian until zero is obtained and then solving the resulting equations from the bottom up, all the harmonics, $Y_m$ can be found in terms of $f_n$. The expansion of a given $f_n$ is, therefore, unique.

To find an explicit expression for the $Y_m$ in terms of $f_n$, which would be desirable, one can start by obtaining the harmonic, $Y_n$, from $f_n$. This can be found in the work by Clebsch, [67]. The relevant theorem, in the way he states it, can be motivated, in our terms, by writing (70) as,

$$f_n = Y_n + r^2 (Y_{n-2} + r^2 Y_{n-4} + \ldots)$$

$$= Y_n + r^2 f_{n-2},$$

(71)

where the first line is the iteration of the second. This formula expresses the standard decomposition of (the space of) homogeneous polynomials and is usually derived first (e.g. Vilenkin, [33], Berger et al, [68]) with the expansion (70) as a consequence.

One refers to $Y_n$ as the harmonic projection of $f_n$, $Y_n = H(f_n)$, and Gauss’ expansion can be construed as a (finite) series of harmonic projections,

$$f_n = H(f_n) + r^2 H(f_{n-2}) + r^4 H(f_{n-4}) + \ldots$$

(72)

Clebsch, [67] eqn.3, gives the harmonic projection, although he does not use

\[\text{A specific example is worked out by MacMillan, [11], §207. See also Heine, [66] vol.1, pp.324–325.}\]
this terminology, in the form, \(^{25}\)

\[
Y_m = H(f_m) \equiv \left[ 1 - \frac{r^2 \Delta}{2(2m-1)} + \frac{r^4 \Delta^2}{2.4(2m-1)(2m-3)} - \cdots \right] f_m . \quad (73)
\]

Exhibiting all the projections, \(H(f_m)\), in terms of the starting out function, \(f_n\), will complete the evaluation. To do this, I follow the Gauss procedure, by writing out the series again,

\[
f_n = Y_n + r^2 Y_{n-2} + \ldots + r^{2s} Y_{n-2s} + r^{2s+2} Y_{n-2s-2} + \ldots ,
\]

and then acting on it with \(\Delta^s\), which kills the first \(s\) terms. This leaves

\[
\Delta^s f_n = A_s(n) Y_{n-2s} + Br^2 Y_{n-2s-2} + \ldots,
\]

where \(A_s(n)\) is the constant,

\[
A_s(n) = 2.4 \ldots 2s (2n - 2s + 1)(2n - 2s - 1) \ldots (2n - 4s + 3) \quad (74)
\]

and \(B\) is a constant that I do need because the harmonic (first) part of the equation gives,

\[
Y_{n-2s} \equiv H(f_{n-2s}) = \frac{1}{A_s(n)} H(\Delta^s f_n) . \quad (75)
\]

In this way, all the terms in (72) can be considered calculated. Equation (75) with (74), was obtained by Prasad, [71], and later by Dougall, [72] \(^{26}\).

Prasad goes on to express the expansion in a compact way using the more general differentiation theorem of Hobson, [73], [10], §80,

\[
H(f_n) = (-1)^n \frac{2^n n!}{(2n)!} r^{2n+1} f_n(\nabla) \frac{1}{r} , \quad (76)
\]

which is an extension of Clebsch’s result, (73). (The more specific result (68) follows from (76).) Easy substitution yields the final expansion,

\[
f_n(r) = (-1)^n \sum_{s=0}^{[n/2]} \frac{(2n - 4s + 1)}{A_s(n)} r^{2n-2s+1} \left[ \Delta^s \rho f_n(\rho) \right] \frac{1}{r} ,
\]

\(^{25}\) A more sophisticated treatment is given in Vilenkin, [33] Chap.XI. Also the quantities \(r^2\) and \(\Delta\) are reciprocal, in the sense of Salmon, and the projection operator takes a typical form. Compare with the Hilbert projectors, (48) (49) (52). \(r^2\) is a raising, and \(\Delta\) a lowering, operator, \(\text{cf}\) Elliott, [69]. The Lie algebra aspect of spherical harmonics is outlined by Howe, [70], as an example of a bigger scheme. His other examples are also instructive.

\(^{26}\) There is an ellipsis of an ellipsis in the expression for \(C_{2p}\), in Hobson [10], §96.
where $\rho$ is the vector of gradients, $\rho = (\alpha, \beta, \gamma) = (\partial_x, \partial_y, \partial_z)$, and

$$\Delta_\rho = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \gamma^2}.$$  

As noted by Clebsch, specific choices of $f_n$, produce known harmonics. For example

$$f_n(r) = C (r.r')^n, \quad C = 2^{-2n} \frac{(2n)!}{n!n!} = \frac{1.3.\ldots(2n-1)}{n!}$$  

yields the usual Laplace coefficients, $P_n(\cos \gamma)$, as follows immediately from (76) and (65). Hobson, [10], uses (76) to develop the properties of the zonal and tesseral spherical harmonics following Maxwell, and says everything that needs saying.

Now $(r.r')^n$ is a rather special $n$th order polynomial (in $r$) which suggests a natural extension to the general product of $n$ factors,

$$f_n(r) = C \prod_{i=1}^n (p_{(i)} \cdot r), \quad p_{(i)} = (x_{(i)}, y_{(i)}, z_{(i)})$$  

whose harmonic projection, with $C$ as in (77), gives the solid spherical harmonic

$$Y_n(r, \hat{p}_{(i)}) = \frac{(-1)^n}{n!} r^{2n+1} \prod_{i=1}^n (p_{(i)} \cdot \nabla) \frac{1}{r} = \frac{(-1)^n}{n!} r^{2n+1} \prod_{i=1}^n \nabla_{p_{(i)}} \frac{1}{r},$$  

where $\nabla_{p_{(i)}}$ is a directed derivative.

Maxwell refers to the points where the rays, determined by the vectors $p_{(i)}$, intersect the unit 2-sphere as the poles of the harmonic, $Y_n$, and considers the harmonic to depend on them. I have indicated this by the dependence on the unit vectors $\hat{p}_{(i)}$. As noted by Maxwell, Gauss, in the same Nachlass referred to earlier, sketchily outlines a geometrical meaning of ‘Sphere functions’ referring to the poles as ‘bestimmte Punkte’.

Conversely, by a non–rigorous counting method, Maxwell shows that any rational integral harmonic, $Y_n$ of order $n$ can be represented as a multipole, (79). This is an important point and is the content of Sylvester’s theorem.

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27 Usually the vectors $p_{(i)}$ are taken to be unit ones.
6. Clebsch–Sylvester theorem.

Proceeding a little more carefully, cf [10], [73], [74], [75] §51, the question is, given a specific harmonic, $Y_n$, can one find a general homogeneous polynomial, $f_n$, of product form

$$f_n(r) = C \prod_{i=1}^{n} (p_i \cdot r),$$

of which $Y_n$ is the harmonic projection?

The relation between $Y_n$ and $f_n$ is the basic Gauss expansion, (71). We see that this does not determine $f_n$ uniquely because of the final term, which can be chosen arbitrarily, given just $Y_n$. It is this ‘gauge’ freedom that can be used to work $f_n$ into the product form, (80). Saying it again, we are looking to write the harmonic $Y_n$ is the form,

$$Y_n(r) = C \prod_{i=1}^{n} (p_i \cdot r) + r^2 G_{n-2}(r),$$

by appropriate choice of the general homogeneous polynomial of order $(n-2)$, $G_{n-2}$. This is a purely algebraic problem, first raised, and solved, by Clebsch and, later clearly, but more cursorily, by Sylvester. The question is independent of the notion of poles.

Interpreting $(x, y, z)$ as projective (homogeneous) coordinates in the plane, the terms in (81), correspond to lines $h_i$, $p_i \cdot r = 0$, a curve, $C_n$, of order $n$, $Y_n(r) = 0$, a curve, $C_{n-2}$, of order $(n-2)$, $G_{n-2} = 0$, and an absolute null conic, $C_2$, $r^2 = r \cdot r = x^2 + y^2 + z^2 = 0$. Geometrically (81) says that the $n$ lines $h_i$ meet the curve $C_n$ in the $2n$ points where the conic $C_2$ cuts that curve. These points occur in $n$ complex conjugate pairs and it is only lines joining such pairs that are real lines, and so can be identified with $h_i$. There is therefore only one way of doing this, so that the decomposition into a real product is unique. This is commonly referred to as Sylvester’s theorem but there is a very clear and explicit statement of it in Clebsch, [67] p.350, and so I suggest it be called the Clebsch–Sylvester theorem. The problem may be that Clebsch uses a rather complicated elimination procedure.

A further piece of old fashioned geometric terminology is to say that the line, $h$, is the polar, with respect to the null conic, of the point (the pole) whose homogeneous coordinates are the line coordinates (direction cosines) of $h$.\footnote{There seems to be a curious congruence of terms as the word pole is used both in its projective geometric sense and in its physical sense, although both derive from the same root meaning ‘axis’.}
The determination of the intersections, \textit{i.e.} of the poles, by elimination, is a separate algebraic matter best approached by first using the Cartan map to parametrise, this time, the variables, \((x, y, z)\), rather than the coefficients, of the ternary \(n\)-ic, \(Y_n\),

\[
Y_n(x, y, z) = \sum_{p+q+r = n} \frac{n!}{p!q!r!} a_{pqr} x^p y^q z^r;
\]

so that the null conic, \(C_2\), is automatically satisfied.

In my previous notation I denoted the \textit{null} \(r\) by \(b\), with the Cartan map, or \textit{uniformisation} in algebraic geometry,

\[
\begin{align*}
\xi_1 &= -\frac{b_x - ib_y}{i\sqrt{2}} = \xi^2, \\
\xi_{-1} &= \frac{b_x + ib_y}{i\sqrt{2}} = \eta^2, \\
\eta_0 &= -ib_z = \sqrt{2}\xi\eta,
\end{align*}
\]  

which differs, only by conventions, from existing choices, \cite{74}, \cite{75}.

Before pursuing the required elimination, I wish to make some comments related to (83). Substitution of (83) into the ternary \(n\)-ic, (82), yields a \textit{binary} \(2n\)-ic,

\[
Y_n(b_x, b_y, b_z) = Y_{2n}(\xi, \eta) = \sum_{p+q = 2n} \frac{(2n)!}{p!q!} a_{pq} \xi^p \eta^q; \quad p + q = 2n.
\]  

Equation (12) can be considered as a formal expression of this equivalence\footnote{The equivalence between the invariant theory of a binary quantic and an orthogonal ternary quantic, which is what we really have here, has been utilised by Littlewood \cite{76}. He makes no mention of the work of Cartan on spinors nor of the relevant considerations of Burnside and Salmon.} through the equality of the symbolic forms of the two types of quantics. In fact it is helpful, in the present situation, to draw attention to the similarity between the true power \((r' \cdot r)^n\) of (77), whose harmonic projection gives \(P_n\), and the symbolic form \((a \cdot r)^n\) of a general ternary quantic.\footnote{See Problem 9, \cite{10}, p.176. \textit{cf} \cite{67}.} Then, still symbolically, the harmonic projection yields

\[
Y_n(r, a) = \frac{(-1)^n}{n!} r^{2n+1} (a \cdot \nabla)^n \frac{1}{r}
\]

\[
= \frac{(-1)^n}{n!} r^{2n+1} \nabla a \frac{1}{r}.
\]
7. Computation of the poles.

The elimination, to find the factors, \textit{i.e.} the poles, is reduced to solving the \(2n\)-ic,

\[
Y_{2n}(\xi, \eta) = 0, \tag{86}
\]

for the ratio \(t = \xi / \eta\). From reality, for every root, \(t\), there is another equal to \(-1/t^*\). This means that the solutions for the corresponding vector \(b\) (projectively a point) are complex conjugates, as mentioned above. The line, \(h\), joining two such points is,

\[
\begin{vmatrix}
  x & y & z \\
  b_x & b_y & b_z \\
  b_x^* & b_y^* & b_z^*
\end{vmatrix} = 0, \tag{87}
\]

or

\[ h : \quad r.(b \times b^*) = 0, \]

so that the real pole vectors, \(p_{(i)}\), in (81) are,

\[ p_{(i)} = \pm i \frac{b \times b^*}{b \cdot b^*}, \tag{88} \]

for every pair of null solutions \(b, b^*\). The direction cosines (line coordinates) are

\[ \hat{p_{(i)}} = \pm i \frac{b \times b^*}{b \cdot b^*}. \]

The spherical harmonics, (79) now look like,

\[
Y_n(r, \{b\}) = C r^{2n+1} \prod_b \left( b^* \cdot (b \times i \nabla) \right) \frac{1}{r} \]

\[ = C r^{2n+1} \prod_b \begin{vmatrix}
  i\partial_x & i\partial_y & i\partial_z \\
  b_x & b_y & b_z \\
  b_x^* & b_y^* & b_z^*
\end{vmatrix} \frac{1}{r}. \tag{89} \]

As an example take the real harmonic polynomial,

\[ H(r) = -3x^4 - 3y^4 - 8z^4 - 6x^2y^2 + 24y^2z^2 + 24x^2z^2 - 60\sqrt{2}x^2yz + 20\sqrt{2}y^3z. \]

Replace \(r\) by \(b\) and then use the Cartan map to get the octavic (without an 8th power),

\[ H(\xi, \eta) = 20\xi\eta(2i\sqrt{2}(\xi^6 + \eta^6) - 7\eta^3\xi^3) = 0, \]
whose roots are easily obtained. I give only one,
\[ \frac{\xi}{\eta} = -\frac{1}{2} \sqrt{\frac{3}{2}} + \frac{i}{2\sqrt{2}}. \]
Substitution back into the Cartan map yields the null vector,
\[ b = \left( -\frac{\sqrt{6}}{8} + \frac{3\sqrt{2}i}{8}, \frac{5\sqrt{2}}{8} - \frac{\sqrt{6}i}{8}, -\frac{1}{2} - \frac{\sqrt{3}i}{2} \right), \]
and evaluation of the direction cosines (88) gives,
\[ \hat{p}(1) = \left( -\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -\frac{1}{3} \right). \]
In fact, in this case, the poles are the corners of a regular tetrahedron, centred on the origin with one corner on the \( z \)-axis, one lying in the \( xy \)-plane and one edge parallel to the \( x \)-axis.\(^{31}\)

The computation of the null vectors can be bypassed by going back essentially to the factorisations and the basic equality (84) which is,
\[ \prod_{i=1}^{n} (b \cdot p_{(i)}) = \eta^{2n} \prod_{i=1}^{n} \left( t - t_i \right) \left( t + \frac{1}{t_i^*} \right). \] (90)
On rewriting both sides, the left one best in spherical coordinates, \( b_1 = \xi^2 = \eta^2 t^2 \), \( b_{-1} = \eta^2 \), \( b_0 = \sqrt{2}\xi\eta = \sqrt{2}\eta^2 t \), one finds,
\[ \prod_{i=1}^{n} \left( b_m p_{m(i)}^n \right) = \eta^{2n} \prod_{i=1}^{n} \left( t^2 p_{(i)}^1 + \sqrt{2} t p_{(i)}^0 + p_{(i)}^{-1} \right) \]
\[ = \eta^{2n} \prod_{i=1}^{n} \left( t^2 + \frac{1 - |t|^2}{|t_i^2|} t_i^* t - \frac{t_i^2}{|t_i^2|} \right). \] (91)
From this identity the values of the spherical (contrastandard) components of the pole vectors can be read off. When normalised, I find
\[ \hat{p}(t) = -\frac{\sqrt{2}i}{1 + |t|^2} \left( t^*, \frac{1}{\sqrt{2}} (1 - |t|^2), t \right), \] (92)
where I have labelled the pole by the root, \( t \) (dropping the index ‘\( i \)’). The transformation \( t \rightarrow -1/t^* \) just reverses the direction of the ray, giving nothing new.

\(^{31}\) The converse constructive calculation of \( H(r) \) is given in MacMillan, [11], p.397.
For convenience, I give the Cartesian components as well,
\[
\mathbf{P}(t) = \frac{2}{1 + |t|^2} \left( \text{Re } t, -\text{Im } t, -\frac{1}{2} (1 - |t|^2) \right).
\]  
(93)

The use of the Cartan map to solve the elimination problem, leading to the result (93), is probably the most efficient technique of finding the poles. It occurs in Backus, [77], and Baerheim, [78]. The articles of Katz and Weeks, [2] and Weeks, [3] contain a handy independent reworking of the basics and some relevant numerics.

After several steps of elimination, not directly involving the Cartan map, Clebsch, [67], also provides a means of finding the poles by first factorising a polynomial constructed from the harmonic, giving what I have denoted by \( b \). The poles are then obtained from (87).

More explicitly, in his notation, \( \Omega \) is a homogeneous harmonic polynomial of order \( n \), and Clebsch proves that the (finite) expression, of order \( 2n \),
\[
\Upsilon \equiv \Omega^2 - \frac{n + 2}{n!} \Delta \Omega^2 + \frac{n(n + 1)}{n(n - 1)!} \Delta^2 \Omega^2 - \ldots,
\]  
(94)
is always factorisable into \( 2n \) linear factors.\(^{32}\) These are the \( b \)s.

As an example I choose the simple, \( n = 2 \) harmonic, \( \Omega = xy + yz + zx \), and find,
\[
\Upsilon = x^4 - 2x^3y - 2x^3z + 3x^2y^2 + 3x^2z^2 - 2xy^3 - 2xz^3 + y^4 - 2y^3z + 3y^2z^2 - 2yz^3 + z^4
\]
\[
= (x^2 + y^2 + z^2 - xy - yz - zx)^2
\]
\[
= \left( x - \frac{y + z}{2} + \frac{\sqrt{3}i|y - z|}{2} \right)^2 \left( x - \frac{y + z}{2} - \frac{\sqrt{3}i|y - z|}{2} \right)^2,
\]  
(95)
giving the null vector (in Cartesians),
\[
b \sim \left( 1, -\frac{1 - \sqrt{3}i}{2}, -\frac{1 + \sqrt{3}i}{2} \right),
\]
and its complex conjugate, twice. Computing the poles via (88) gives \( \frac{1}{\sqrt{3}}(1, 1, 1) \), twice. For comparison, the alternative method can be seen later, in connection with another matter.

In the general situation, the linear factors can be found by first setting \( z = 1 \) and then, in crudest fashion, substituting \( y = mx + c \) into the identity, \( \Upsilon = 0 \) to give \( \Upsilon(x) = 0, \forall x \). In fact, one needs only two terms, say \( \Upsilon(0) = 0 \) and \( \Upsilon'(0) = 0 \). The former determines \( c \) as the solution of an ordinary polynomial of degree \( 2n \) and the latter fixes \( m \) linearly in terms of \( c \). This method is, perhaps, less efficient than the one based on the binary form because of the redundant information.

\(^{32}\) There is a misprint on p.349 in [67], \( n \) factors being stated.
8. Normal forms.

In many cases, the harmonic polynomial will be presented as a combination of the standard set of spherical harmonics, $C_{LM}^L(r)$. This is a ‘normal form’ of the harmonic, [10] §87. Starting from this combination allows a re-expression of the factorisation theorem.

For a specific order $L$, let the harmonic,

$$\Phi_L(r) = \phi_L^M C_{LM}^L(r), \quad (96)$$

be the real data under consideration. (I called this $Y_n(r)$ before.) Using

$$\left(C_{LM}^L(r)\right)^* = C_{LM}^L(r), \quad (97)$$

reality implies the condition,

$$\phi_L^* = (-1)^{L-M} \phi_L^M = \phi_L^M, \quad (98)$$

familiar from mode decompositions into spherical harmonics in many physical situations such as hydrodynamics. In the terminology of [21], the raising and lowering operator is a charge conjugation operator.

Replacing $r$ by a null vector, $b \in \mathbb{C}^3$, the real harmonic, (96), turns into,

$$\Phi_L(b) = \phi_L^M C_{LM}^L(b), \quad (99)$$

still with the reality condition (98) on the data. Quite generally, one has $\Phi_L(b)^* = \Phi_L(b^*)$, which shows that if $b$ is a solution, i.e. if

$$\Phi_L(b) = 0,$$

then so is $\pm b^*$, as Clebsch and Sylvester said.\(^{33}\)

It is a useful exercise to confirm this, since the replacement $r \to b$, modifies the reality properties, (97), of the harmonics. One can use either $b$ or $(\xi, \eta)$ of (83) to express this. We also have the explicit formulae (8) and (9) to check things out.

Taking the complex conjugate of $\Phi_L$ gives,

$$\Phi_L(b)^* = \phi_L^* M \left(C_{LM}^L(b)\right)^* = (-1)^{L+M} \phi_L^M \left(C_{LM}^L(b)\right)^*$$

$$= (-1)^{L-M} \phi_L^M \left(C_{-M}^L(b)\right)^*$$

$$= \phi_L^M C_{LM}^L(b^*)$$

$$= \Phi_L(b^*).$$

\(^{33}\) For clarity, $b^*$ here refers to the vector with complex conjugated Cartesian components.
\( \Phi_L \) is an easily found binary \(^{34} \) 2L-ic in the variables \( \xi \) and \( \eta \) (cf equations (29) and (84) in a different notation) whose roots correspond to the Maxwell poles.\(^{35} \) Although the algebra and the properties of the polynomial have been given before, I rederive them in a not really different way.

Looking at the polynomial equation (equivalent to (86)) as one in \( t = \xi / \eta \), the replacement \( b \rightarrow b^* \) corresponds to charge conjugation, \( \xi \rightarrow -\eta^* \), \( \eta \rightarrow \xi^* \), and so, if \( t \) is a root, then so is \(-1/t^*\), as I said before. I have thus obtained the factorisation displayed on the right-hand side of (90). To determine the poles, this 2–spinor factorisation is simply rewritten in terms of vectors, as on the left–hand side of (90).

The spinor \( \psi^\dagger = (-\eta^*, \xi^*) \) is the spin conjugate spinor to \( \psi = (\xi / \eta) \) and one can refer to the complex conjugate null 3–vector \( b^* \) as the conjugate vector \( b^\dagger = b^* \).\(^{36} \)

I remark that this analysis depends on the fact that the harmonic data is real. I also remark that pairing other roots produces other, complex, vector factors. The non-triviality of the result lies in the fact that the factors are real (if \( \Phi \) is). Any binary quantic can be factorised and hence so can \( \Phi(b) \), but with complex factors, in general. Some further remarks on this can be found in \[82\].

If \( r \) is reinstated, the factorisation statement, (81), is that the harmonic polynomial \( \Phi_L(r) \) can be decomposed as

\[
\Phi_L(r) = C \prod_{i=1}^{L} (r \cdot p_{(i)}) + r^2 G_{L-2}(r).
\]

Both the polynomial, \( G_{L-2} \), and the constant, \( C \), up to scaling of the \( p_{(i)} \), are uniquely determined by \( \Phi \).

The constant \( C \) can be found by substituting a value for \( r \) that lies on the conic and then \( G \) follows by subtraction. It can also be constructed from the poles, as will be shown later.

I also make the important point that the factors are real, if \( f_n \) is.

For completeness, the constant \( C \) can also be determined as follows, (e.g. \[75\],

\[\text{It is of course also a ternary form as mentioned earlier.}\]

\[\text{To determine the poles, it is not necessary to convert this to a polynomial in (} x, y, z \text{) as in}\]

\[\text{Weeks, [3].}\]

\[\text{A detailed discussion of the notion of spinor and null vector in connection with rotations in}\]

\[\text{3–space can be found in Kramers’ nice book on quantum mechanics, [79], based on his earlier}\]

\[\text{papers, [80]. The definitions mean that his null vectors are i times mine. He also usefully}\]

\[\text{extends to the relativistic case. His approach can be found more fully in Brinkman, [81].}\]
§51, [74]). Consider the pencil of curves,

$$\Phi_L(r) - \lambda \prod_{i=1}^{L} (r \cdot p_{(i)}) = 0,$$

(101)

for varying $\lambda$. Each member of this pencil goes through the $2L$ intersections (of $C_2$ and $C_L$). Select a point, not one of these intersections, on the conic, $C_2$, and adjust $\lambda$ so that the corresponding curve of the pencil goes through it, which is possible because none of the lines $r \cdot p_{(i)} = 0$ vanishes at this point. Since this curve of the pencil now has $2L + 1$ points in common with the absolute conic, $C_2$, it must completely contain it, and so degenerate into the conic and a curve of the $(n - 2)$nd order, $C_{n-2}$. This implies that the left-hand side of (101) must have $r^2$ as a factor, the sufficiency of which is obvious. (A proof of the necessity was given by Ostrowski, [83], [74].) The so determined $\lambda$ is the sought for constant, $C$, the value being independent of the initial point selected.

9. Apolarity and harmonic polynomials.

This development can be given an older analytical-geometric terminology and, to this end, I return to invariant theory. The relevant statement is that if a homogeneous polynomial in $(x, y, z)$ (a ternary form) is harmonic, it is apolar to the fundamental (null) conic, $C_2$, and conversely. This is easily shown from the definition (53) which says that the polar is obtained by replacing the line coordinates, $u$, by the derivative, $\nabla$, in the envelope equation of the conic, and letting it act on the ternary form (curve). In my conventions, the point locus equation of $S = C_2$ reads

$$s^2 = x^2 + y^2 + z^2 = r_1 r^1 + r_{-1} r^{-1} + r_0 r^0 = 2r_1 r_{-1} - r_0^2 = 0$$

and its envelope equation, $\Sigma = K_2$,

$$u_{ij}^2 = u^2 + v^2 + w^2 = u_1 u^1 + u_{-1} u^{-1} + u_0 u^0 = 2u_1 u_{-1} - u_0^2 = 0$$

where I have given both the Cartesian and the spherical forms. The latter is the preferred (canonical) form in analytical geometry\(^{37}\) and what we have referred to as spherical coordinates are trilinear coordinates with respect to a specially chosen

\(^{37}\) e.g. Salmon, [37], Sommerville, [55], Todd, [51] §3.3, Meyer, [84], Grace and Young, [29], Semple and Kneebone, [58].
triangle. For my normalisations, the locus and envelope equations have exactly the same form. The replacement \( u \rightarrow \nabla \) can be made in any coordinate system and we have for the polar,

\[
h = \frac{1}{n(n-1)} \Delta a_x^n \equiv \Delta Y_n(r) = 0,
\]
since \( Y_n \) is assumed harmonic. This is the announced result whose significance is that the construction of curves apolar to a conic is a known topic in invariant theory.

Using suitable parametric coordinates, the analysis of such forms can be reduced to that of binary forms. The equation of the curve, \( C_n \), factorises into two binary forms of order \( n \), Grace and Young, [29] §§ 241-244, Schlesinger, [86]. I give a discussion since the material is not generally familiar.

The parametric coordinates of a (complex) point, \( \kappa \), on the fundamental conic, \( S \), are just the Cartan map,

\[
\begin{align*}
r_1 &= \kappa_1^2, \quad r_0 = \sqrt{2} \kappa_1 \kappa_2, \quad r_{-1} = \kappa_2^2. 
\end{align*}
\]

(I use the notation of [29] and [86]. To compare with (5), \( \xi = \kappa_1 \) and \( \eta = \kappa_2 \).)

If the straight line \( u^i r_i = 0 \) cuts the conic in the points \( \lambda \) and \( \mu \) it is easily shown that,

\[
\begin{align*}
u^1 &= \lambda_2 \mu_2, \quad v^0 = -\frac{1}{\sqrt{2}} (\lambda_1 \mu_2 + \lambda_2 \mu_1), \quad v^{-1} = \lambda_1 \mu_1. 
\end{align*}
\]

The tangents at \( \lambda \) and \( \mu \) meet at the pole point,

\[
\begin{align*}
r_1 &= \lambda_1 \mu_1, \quad r_0 = \frac{1}{\sqrt{2}} (\lambda_1 \mu_2 + \lambda_2 \mu_1), \quad r_{-1} = \lambda_2 \mu_2. 
\end{align*}
\]

Using raising and lowering, one might dispense with the \( u \) symbol, since \( u^i = r^i \), the dual with respect to the conic.\(^{39}\)

\(^{38}\) It corresponds to the choice of \( k = 2 \) in [58], Chap.V case 4. A treatment that uses the general form for the base conic is given by Lindemann, [85]. It is interesting historically to note the polite way these authors claim priority. This reflects the importance of invariant theory at that time.

\(^{39}\) This standard way of representing a point and a line by a pole and its polar with respect to a fixed conic seems to date back to Hesse in the 1850’s following Poncelet, and others. It was used by Darboux and there exists a a well known higher dimensional (spin) generalisation due to Clifford, involving rational norm curves.
Equation (104) can be written as in (16) but this time with two two-spinors, \( \lambda \) and \( \mu \),

\[
 r_i = \frac{1}{i\sqrt{2}} \overline{\lambda} \sigma_i \mu .
\]

If one requires the Cartesian coordinates to be real, then \( \lambda_1 = -\mu_2^* \) and \( \lambda_2 = \mu_1^* \) which means that \( \overline{\lambda} = \mu^* \), i.e. the complex conjugate transpose, which is what is normally referred to as the ‘adjoint’, or hermitian conjugate, \( \mu^\dagger \).

Schlesinger gives an instructive way of finding the equation of an envelope of class \( n \) apolar to the base conic, which, as I have said, corresponds to finding a harmonic homogeneous form. This condition is equivalent to the envelope being conjugate to all curves of order \( n \) that contain \( S \) as a factor.

Let \( c_x^n \) be such a curve, then the necessary and sufficient condition that it contain \( S \) as a part is that, from (102),

\[
 (c^1 \kappa^2_1 + c^0 \sqrt{2} \kappa_1 \kappa_2 + c^{-1} \kappa^2_2)^n = 0 \tag{105}
\]

for each \( \kappa_1 \) and \( \kappa_2 \). The trick now is to think of the symbolic coefficients, \( c \), as variables, \( u \), and \( \kappa_1 \) and \( \kappa_2 \) as fixed coefficient quantities, in a way to be explained shortly. The resulting class curve (envelope) will be conjugate to \( c_x^n \) by virtue of (105), with \( c \rightarrow u \), and hence will be apolar to \( S \).

Thus we substitute, from (103),

\[
 c^1 = \lambda_2 \mu_2 \equiv (\lambda \mu)_1 \\
 c^0 = -\frac{1}{\sqrt{2}} (\lambda_1 \mu_2 + \lambda_2 \mu_1) \equiv (\lambda \mu)_0 \\
 c^{-1} = \lambda_1 \mu_1 \equiv (\lambda \mu)_{-1}
\]

into (105) which factorises into two,

\[
 (\lambda_2 \kappa_1 - \lambda_1 \kappa_2)^n (\mu_2 \kappa_1 - \mu_1 \kappa_2)^n \tag{106}
\]

with \( \lambda \) and \( \mu \) as variables.

The important point is that, since the \( \kappa \) are fixed, arbitrary parameters, one obtains an apolar curve if one replaces the different products of the \( \kappa \) by any quantities. In other words, \( \kappa_1 \) and \( \kappa_2 \) can be replaced by the symbolic coefficients of a binary form so that the above factors become (part) binary forms. Putting, precisely, \( \kappa_1 = a_2 \) and \( \kappa_2 = -a_1 \) produces the polarised binary form,

\[
 a^n_\lambda a^n_\mu = 0, \tag{107}
\]

35
as the envelope equation of a curve of the $n$th class apolar to the fundamental conic, $S$, as soon as one sets $u^i = (\lambda \mu)_i$, using the symbolic,

$$a_\lambda^n a_\mu^n = [a_1^2(\lambda \mu)_{-1} - \sqrt{2} a_1 a_2(\lambda \mu)_0 + a_2^2(\lambda \mu)_1]^n. \quad (108)$$

Precisely the same equation holds for the locus of the apolar curve, but now the $\lambda$ and $\mu$ are translated into point coordinates by the dual (104).

When $\lambda = \mu = \kappa$, the equation, $a_\kappa^{2n} = 0$, gives the points of contact of the $2n$ common tangents of the apolar curve and the conic. On the other hand, the locus equation gives the points where the curve intersects the conic. The expression in (107) is the $n$th polar with respect to $\lambda$ of $a_\mu^{2n}$.

What this result says is that a homogeneous harmonic polynomial in $x, y, z$ can be reconstituted from its evaluation on the absolute conic $S : r^2 = 0$. I will check this by performing a round trip from the simple harmonic polynomial, $xy + yz + zx$, through its evaluation on the absolute to give a binary quartic, thence by polarisation to a product of two binary quadratics and finally back to $xy + yz + zx$.

For the purposes of solution, it is useful to introduce the non–homogeneous ratios (the Darboux coordinates),

$$\mu_1 : \mu_2 = t : 1, \quad \lambda_1 : \lambda_2 = s : 1$$

so that, e.g.,

$$r_1 : r_0 : r_{-1} = s t : \frac{1}{\sqrt{2}} (s + t) : 1. \quad (109)$$

On the absolute, $S$, $s = t$ and (102) holds. Therefore, on $S$,

$$xy = \frac{(t^2 + 1)(t^2 - 1)}{2i}, \quad yz = it(t^2 + 1), \quad zx = t(t^2 - 1)$$

and so

$$(xy + yz + zx)|_S = \frac{1}{2i} t^4 + (1 + i)t^3 - (1 - i)t - \frac{1}{2i}, \quad (110)$$

which is a binary quartic. Polarising this according to (107) one finds, up to a factor,

$$t^2 s^2 - (1 - i) s t (s + t) - (1 + i) (t + s) - 1,$$

and substituting (109) one regains $xy + yz + zx$.

The solutions for the points on $S$ follow by rewriting (110);

$$(2\zeta^2 - 2\zeta - 1)^2 = 0, \quad \text{with} \quad t = (1 - i)\zeta,$$
so that
\[
\zeta = \frac{1}{2} \left( 1 \pm \sqrt{3} \right),
\]
indicating two double roots which are switched under \( t \to -1/t^* \). The corresponding Maxwell poles are proportional to \((1, 1, 1)\), as we know, which could have been deduced immediately from the trivial identity,
\[
(xy + yz + zx) + \frac{1}{2} (x^2 + y^2 + z^2) = \frac{1}{2} (x + y + z)^2.
\]

There are three resolutions of
\[
(xy + yz + zx) − \lambda (x^2 + y^2 + z^2)
\]
into two linear factors, \( \lambda \) being a solution of the resolving cubic, \(4\lambda^3 - I\lambda + J = 0\), of the binary quartic, (110), corresponding (in this case\(^{40}\)) to \( xy + yz + zx \) via the Cartan map. \( I \) and \( J \) are the standard invariants of this quartic, here, \( I = 3, J = -1 \). The real resolution occurs for \( \lambda = -1/2 \), (twice since \( I^3 = 27J^2 \)). The complex one, for \( \lambda = 1 \), is exhibited in (94).

Schlesinger calls the unique apolar curve determined by the binary form, the curve associated with the binary form. Thus, given an arbitrary curve of the \( n \)th order, \( f^n_x = 0 \), there is associated with it, via its \( 2n \) intersections with the conic, \( S \), a unique \( n \)th order curve, \( a^n_\lambda a^n_\mu = 0 \), apolar to \( S \). This gives a geometrical interpretation of harmonic projection and of its uniqueness. Furthermore, there are, in addition to the \( 2n \) points of intersection, on \( S \), of \( a^n_\lambda a^n_\mu = 0 \) and \( f^n_x = 0 \), another \( n(n-2) \) through which it is known\(^{41}\) that a curve, \( m^{n-2}_x = 0 \), of order \( n(n-2) \) can be drawn, that is,
\[
f^n_x = \rho a^n_\lambda a^n_\mu + s^2_x m^{n-2}_x,
\]
which is the equation expressing the harmonic projection \( f \to a a \) (with (104) to relate \( x \) and \( \lambda, \mu \)), equivalent to Gauss’ formula, (71).

A system of \( n \) lines, \( L^n_x \), through the \( 2n \) points on \( S \) can be considered to be a curve of the \( n \)th order (an \( n \)-side) and could be used as a specific \( f^n_x \) in (112). The reverse question is whether, given a form apolar to \( S \), one can find a \( L^n_x \) such that
\[
a^n_\lambda a^n_\mu = \rho L^n_x + s^2_x m^{n-2}_x,
\]

\(^{40}\) The general quartic is discussed by Burnside, [87], who gives some geometrical interpretation.
\(^{41}\) This is Noether’s intersection theorem. Roughly speaking, a curve, \( F \), of order \( n \), that goes through the intersections of curve, \( C \), of order \( n \), and conic, \( S \), of order \( 2 \), is \( F = \rho C + MS \), where \( M \) is a curve of order \( n - 2 \) which goes through the remaining intersections of \( F \) and \( C \).
but this would seem to be obvious because the apolar ternary form, $a^n_\lambda a^n_\mu$, determines the binary form $a^{2n}_\kappa$ (and vice versa) which fixes the $2n$ points on $S$ from which the $n$–side, $L^n_x$, can be constructed. Then (112) can be employed giving (113). Of course, this is exactly Sylvester’s approach to Maxwell’s poles, which is equivalent to Clebsch’s earlier theory.

There are many theorems regarding this set up. For example, it is easy to show that if two binary forms are conjugate, then the associated curve, i.e. harmonic ternary form, of one of them is conjugate to any curve, i.e. ternary form, passing through the $2n$ points (on $S$) corresponding to the other. Conversely, if, through $2n$ points on $S$, one can draw a curve, $b^n_x = 0$, that is conjugate to the apolar curve, $a^n_\lambda a^n_\mu = 0$, then every $n$th order curve that can be drawn through these points has the same property and the binary form of these $2n$ points is conjugate to $a^{2n}_\kappa$.

For, if $c^n_x = 0$ is a second curve which goes through the $2n$ points, then

$$c^n_x = \rho b^n_x + s^2_x m^{n-2}_x.$$

However, both curves on the right–hand side are conjugate to $a^n_\lambda a^n_\mu = 0$, the first by assumption and the second because $a^n_\lambda a^n_\mu = 0$ is apolar to $s^2_x = 0$ and so, therefore, is $c^n_x$. Thus a system of $n$ lines containing the $2n$ points represents a conjugate $n$–side of $a^n_\lambda a^n_\mu = 0$ which cuts the conic in $2n$ points whose binary form is conjugate to $a^{2n}_\kappa$.

In terms of $s$ and $t$, the conjugate transformation $\lambda_1 \rightarrow -\mu_2^*, \lambda_2 \rightarrow \mu_1^*$ is $t \rightarrow -1/s^*$. The point coordinates are real (in Cartesians) if $t s^* = -1$ and there are 2 real degrees of freedom, as required for the real plane.

For applications, one has to impose the condition that the curve be a real one, i.e. that the form, (108), that gives it is real.

**10. More angular momentum theory.**

After this brief foray into the realms of invariant theory and simple analytical geometry, I return to angular momentum theory by remarking that in this, one concentrates on the coefficients, $\phi^m$, of the forms, see (33), as exemplified in my earlier discussion of apolarity using $3j$–symbols. Writing the harmonic as (96) allows one, sometimes, to exploit the considerable amount of existing angular momentum algebra.

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42 This way of replacing a binary form by two ternary ones is outlined in Salmon, [37], §190. See also Burnside [87].
In dealing with higher–spin quantities, which is what Φ is, there is always the option of using a (multi) two–spinor description. Since two–spinors are the defining representation there are few basic algebraic operations, which is a big advantage, but calculations can become longwinded. Penrose’s, [88], treatment of the algebraic structure of the Riemann curvature is, however, a good example of the elegance that can be achieved (Pirani, [89]). A well known alternative, for integer spins, is to use ‘bivector’, or 3–spinor, space, in which higher spins are represented by symmetric, traceless tensors, e.g. \( \phi_{m_1...m_j} \), \((m_i = 1, 0, -1)\).\(^{43}\)

In this connection, I firstly note that the null 2\(j+1\)–spinor, (9), which comprises the variables of the binary form, can be written using 3\(j\)–symbols as, [21];

\[
\xi^{(j)}_m = \frac{1}{[(2j)!]^{1/2}} u^{m_1}(j) ... u^{m_{2j}}(\frac{1}{2}) \xi^{\frac{1}{2}}_{m_1} ... \xi^{\frac{1}{2}}_{m_{2j}}, \tag{114}
\]

where \(u^m\) is the rectangular matrix–spinor exhibited, up to a factor, as a 3\(j\)–symbol in (60) and \(\xi^\frac{1}{2} = (\xi^\eta)\). The combination of the \(u\)s is the generalised 3\(j\) symbol that uniquely relates the symmetrised \(\frac{1}{2} \otimes ... \otimes \frac{1}{2}\) representation to the spin–\(j\) one (e.g. Fierz, [91], Williams, [15], Ansari, [92], [12], pp.421+.). Assuming \(j\) integral, the two–spinors can be combined in pairs to give the same spin–1 null vector, \(a\), ((5), (16)), and, after some 3\(j\) recombinations, one encounters the familiar polynomial expression for the spherical harmonics,

\[
C^L_M(a) = N \left( \begin{array}{ccc} L & m_1 & m \\ M & 1 & L - 1 \end{array} \right) \left( \begin{array}{ccc} L - 1 & m_2 & m' \\ m & 1 & L - 2 \end{array} \right) ... \left( \begin{array}{ccc} 1 & m_L & 0 \\ m' & 1 & 0 \end{array} \right) a_{m_1} a_{m_2} ... a_{m_L} \tag{115}
\]

where

\[
N = \prod_{k=1}^{L} \left[ \frac{4k^2 - 1}{k} \right]^{1/2},
\]

or, in matrix form,

\[
C^L(a) = N A^{m_1}(L) A^{m_2}(L - 1) ... A^{m_L}(1) a_{m_1} a_{m_2} ... a_{m_L} \tag{116}
\]

with the definition,

\[
[A^{m_1}(L)]_{Mm} = \left( \begin{array}{ccc} L & m_1 & m \\ M & 1 & L - 1 \end{array} \right).
\]

\(^{43}\) This is employed by Debever, Synge and others in the classification of the Riemann tensor, \(j = 2\). (See [90] for comments and a (2\(j + 1\)) treatment with the binary quartic concomitant basis written in 3\(j\) form.)
This expansion is valid for any 3-vector (a 3-spinor) as it follows from the general recursion,

\[ C^L_M(r) = \left[ \frac{4L^2 - 1}{L} \right]^{1/2} \left( \begin{array}{cc} L & m_1 \\ M & 1 \end{array} \right) C^L_{m_2} - 1(r) C^1_{m_1} (r). \] (117)

which yields,

\[ C^L(r) = N A^{m_1}(L) A^{m_2}(L - 1) \cdots A^{m_L}(1) r_{m_1} r_{m_2} \cdots r_{m_L}. \] (118)

However the ‘inverse’ formula,

\[ a_{m_1} a_{m_2} \cdots a_{m_L} = C^L(a) \cdot A_{m_1}(L) A_{m_2}(L - 1) \cdots A_{m_L}(1), \] (119)

is not valid for any 3–vector, \( r \), differing by terms having \( r^2 \) as a factor.

The complete harmonic, (96), is

\[ \Phi_L(r) = \phi^L \cdot C^L(r) \\
= N \phi^L \cdot A^{m_1}(L) A^{m_2}(L - 1) \cdots A^{m_L}(1) r_{m_1} r_{m_2} \cdots r_{m_L} \] (120)

which defines the symmetric, traceless components, \( \phi^{m_1 \cdots m_L} \), of the harmonic in textbook fashion. It is easily checked, using angular momentum theory, that the product of matrix–spinors, \( A^{m_1} \cdots A^{m_L} \), is traceless on the \( m \)s.

In this approach, Maxwell’s poles are introduced by noting that any \((2L + 1)\)–spinor can be expressed in terms of a symmetrised product of, generally different, 3–spinors, \( \psi^i \),

\[ \phi^L = A^{m_1}(L) A^{m_2}(L - 1) \cdots A^{m_L}(1) \psi^1_{m_1} \psi^2_{m_2} \cdots \psi^L_{m_L}, \] (121)

corresponding to the irreducible bivector (tensor) equation,

\[ \phi_{m_1 \cdots m_j} = \psi^1_{m_1} \psi^2_{m_2} \cdots \psi^L_{m_L} - \text{trace terms}. \] (122)

(There are no ‘trace terms’ for 2–spinors.)

The reality condition (98) can be achieved, courtesy of 3j properties, by making all the ‘factors’, \( \psi^i_m \), obey the same requirement, \( i.e. \) they correspond to real 3–vectors.
The factors, \( \psi^i \), are the Maxwell poles, \( p_i \), as can be seen by reconstructing the harmonic (96), using (120) and (122),

\[
\Phi_L(r) = \phi^L \cdot C^L(r) = C \prod_{i=1}^{L}(r \cdot \psi^i) + r^2 G_{L-2}(r),
\]

where \( G \) arises from the trace terms and can be written out explicitly in terms of the \( \psi^i = p_i \).

We thus regain the decomposition structure, (100), in a slightly heavy-handed, but interesting, way.

A numerical example is always helpful and I return to the harmonic polynomial used at the end of §5. The four poles can be found in MacMillan, [11]. They are, in Cartesians,

\[
p_{(1)} = (0, 0, 1) , \quad p_{(2)} = \frac{1}{3} (0, 2\sqrt{2}, -1) \\
p_{(3)} = -\frac{1}{3} (\sqrt{6}, \sqrt{2}, 1) , \quad p_{(4)} = \frac{1}{3} (\sqrt{6}, -\sqrt{2}, -1).
\]

Then

\[
135 \prod_{i=1}^{4}(r \cdot p_{(i)}) = -5z(12\sqrt{2}x^2y - 6x^2z - 4\sqrt{2}y^3 - 6y^2z + z^3),
\]

where the constant, 135, follows by substituting in values satisfying \( x^2 + y^2 + z^2 = 0 \).

As mentioned before, the last term, \( G \), in (123) could be found by subtraction but now I wish to obtain it directly from the poles via the trace terms in (122) which I contract with \( rs \) to get

\[
r^{m_1 \ldots m_4}\phi_{m_1 \ldots m_4} = \prod_{i=1}^{4}(r \cdot p^{(i)}) - \text{trace terms}.
\]

Using the standard expression for the traceless part of a fourth rank tensor\(^{45}\) (\textit{e.g.} Jarić, [95]), or from a short direct calculation, it follows that,

\[
\text{trace terms} = \frac{r^2}{n+4} \Sigma_6 - \frac{(r^2)^2}{(n+4)(n+2)} \Sigma_3 , \quad n = 3 ,
\]

\(^{44}\) The tensor approach to Maxwell’s poles has been discussed by Zou and Zheng, [93]. See also Applequist, [94].

\(^{45}\) Applied either to \( \phi \) or to the product of \( rs \). A general expression is available, but I will not write it out.
where \( \Sigma_6 \) and \( \Sigma_3 \) are the sums of the 6 pair traces and of the 3 double pair traces, respectively,

\[
\Sigma_6 = r^2 (p_{(3)} \cdot p_{(4)}) + \ldots, \quad \Sigma_3 = (p_{(1)} \cdot p_{(2)}) (p_{(3)} \cdot p_{(4)}) + \ldots.
\]

For my example, arithmetic gives \( \Sigma_6 = 2r^2 / 9, \Sigma_3 = 1 / 3 \) and, simply,

\[
G_2 = \frac{1}{3} r^2,
\]

agreeing with the result of subtraction.

The fact that the remainder term in (123) can be expressed in terms of the poles also follows by noting that there are \( L(L - 1)/2 \) constants in \( G_{L-2} \) which could be found by substituting the \( L(L - 1)/2 \) cross products, \( p_{(i)} \times p_{(j)} \), for \( r \) in (123), when the first term on the right-hand side vanishes.

The standard spherical harmonic, \( C^L_M(r) \), can be expressed in terms of the null \((2j + 1)\)-spinors constructed from the \( \lambda \) and \( \mu \) introduced earlier, (104),

\[
C^{2j}_M(r) = (-1)^{2j} \frac{[(4j + 1)!]^{1/2}}{2j(2j)!} \left( \begin{array}{c} M \\ j \end{array} \right) \lambda^{(j)}_{m_1} \mu^{(j)}_{m_2}, \quad 2j = L,
\]

with \( \lambda^{(1/2)} = (\lambda_1, \lambda_2) \) and \( \mu^{(1/2)} = (\mu_1, \mu_2) \). Also,

\[
(a \cdot r)^{2j} = (\bar{X}^{(j)}_\xi \xi^{(j)}_\mu) = ((\lambda_2 \xi_1 - \lambda_1 \xi_2) (\mu_2 \xi_1 - \mu_1 \xi_2))^{2j},
\]

(cf (106)).

A two-spinor approach to Maxwell’s poles and Sylvester’s theorem is given by Dennis, [4], who makes some useful comments on the relation to other techniques.

11. Discussion.

Maxwell’s motivation for introducing poles was to provide a coordinate independent geometrical characterisation of spherical harmonics in 3-space; in his own words (in the first edition of his book 46) “emancipating our ideas from the thrall-dom of systems of coordinates”. This aspect, no doubt, accounts for the recent rediscovery of the procedure when analysing astronomical data.

The geometrical viewpoint can also be used advantageously when finding the harmonics invariant under some finite subgroup of the rotation group. Maxwell’s

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46 It is interesting to compare the treatments of this topic in the first two editions.
theory of poles has been employed in this regard by Poole, [96], and Laporte, [97], although it is not required of course. Polyhedral harmonics had earlier been introduced by Pockels in his classic book, [98].

It is not clear to me whether the equivalence of apolarity and harmonicity discussed in §9 meets Sylvester’s requirement of a geometrical or algebraical definition of a (spherical) harmonic. In fact it is not obvious that one needs such a definition in view of Maxwell’s rebuttal on pp.201–202 of his second edition (without mentioning names) of Sylvester’s objections.

The limitation to integral polynomials can be removed. Gauss’ harmonic expansion then becomes an infinite series, as noted by Hobson, [10], p.128. In invariant theory this can be covered by using Sylvester’s perpetuant construction, which allows the order to become infinite.

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