On a type of Static Equation on Certain Contact Metric Manifolds

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Abstract. This paper deals with the investigation of $K$-contact and $(\kappa, \mu)$-contact manifolds admitting a positive smooth function $f$ satisfying the equation:

$$ f\tilde{Ric} = \tilde{\nabla}^2 f $$

where $\tilde{Ric}$, $\tilde{\nabla}^2$ are traceless Ricci tensor and Hessian tensor respectively. We proved that if a complete and simply connected $K$-contact manifold admits such a smooth function $f$, then it is isometric to the unit sphere $S^{2n+1}$. Next, we showed that if a non-Sasakian $(\kappa, \mu)$-contact metric manifold admit such a smooth function $f$, then it is locally flat for $n = 1$ and for $n > 1$ is locally isometric to the product space $E^{n+1} \times S^n$.

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1. Introduction

Static space-times are special and important global solutions to Einstein equations which show the interplay between matter and space-time in general relativity. A static space-time metric $\tilde{g} = -f^2 dt^2 + g$ satisfying the Einstein equation

$$ Ric_{\tilde{g}} - \frac{R_{\tilde{g}}}{2} = T, $$

for the energy-momentum stress tensor $T = -\mu f^2 dt^2 - \rho g$ of a perfect fluid, where the smooth functions $\mu$ and $\rho$ are the energy density and pressure of the perfect fluid respectively. Moreover, $Ric_{\tilde{g}}$ and $R_{\tilde{g}}$ stand for Ricci tensor and the scalar curvature for the metric $\tilde{g}$. For details see [27, 15, 17, 15]. The Friedmann-Lemaitre-Robertson-Walker solutions for the Einstein equation
with perfect fluid as a matter fluid represents a homogeneous, fluid filled universe that is undergoing accelerated expansion [23]. Therefore static perfect fluid space-times acts as a generalizations of the static vacuum spaces and hence is an important topic of study both for physicians and mathematicians.

We now recall the definition of static perfect fluid space-time (see [9, 17, 19]).

**Definition 1.1.** A Riemannian manifold \((M^n, g)\) is said to be the spatial factor of a static perfect fluid space-time if there exist smooth functions \(f > 0\) and \(\rho\) on \(M^n\) satisfying the perfect fluid equations:

\[
f \dot{Ric} = \nabla^2 f \tag{1.2}
\]

and

\[
\Delta f = \left( \frac{n-2}{2(n-1)} R + \frac{n}{n-1} \rho \right) f, \tag{1.3}
\]

where \(Ric\) and \(\nabla^2\) stand for the traceless Ricci and Hessian tensors, respectively. When \(M^n\) has boundary \(\partial M\), we assume in addition that \(f^{-1}(0) = \partial M\).

Coutinho et al. [9] obtained a sharp boundary estimate for static perfect fluid space-time on an \(n\)-dimensional compact manifold with boundary. Kobayashi and Obata [17] gave a classification of conformally flat Riemannian manifold in which \((g, f)\) satisfies (1.2). Recently, Leandro and Solórzano [20] investigate the static perfect fluid space-time \(M^4 \times f \mathbb{R}\) such that \((M^4, g)\) is a half conformally flat Riemannian manifold and proved that \((M^4, g)\) is locally isometric to a warped product manifold \(I \times_\varphi N^3\) where \(I \subset \mathbb{R}\) and \(N^3\) is a space form. As illustrated by Coutinho et al. [9] Schwarzchild space, which serves as a model for a static black hole, is an exact solution of the above equations.

Let \((M^n, g)\) be a compact, oriented, connected Riemannnian manifold with dimension \(n\) at least three, \(\mathcal{M}\) be the set of Riemannian metrics on \(M^n\) of unitary volume, \(\mathcal{C} \subset \mathcal{M}\) be the set of Riemannian metrics with constant scalar curvature. Define the total scalar curvature functional \(\mathcal{R} : \mathcal{M} \to \mathbb{R}\) as

\[
\mathcal{R}(g) = \int_{M^n} R_g dM_g,
\]

where \(R_g\) is the scalar curvature on \(M^n\). It is well-known that the formal \(L^2\)-adjoint of the linearization of the scalar curvature operator \(\mathcal{L}_g\) at \(g\) is defined as

\[
\mathcal{L}^*_g(f) := - (\Delta_g f) g + Hess_g f - fRic_g,
\]

where \(f\) is a smooth function on \(M^n\), and \(\Delta_g, Hess_g\) and \(Ric_g\) stand for the Laplacian, the Hessian form and the Ricci curvature tensor on \(M^n\), respectively. As in [2, 8] we say that \(g\) is a \(V\)-static metric if there exists a smooth function \(f\) on \(M^n\) and a constant \(\kappa\) satisfying

\[
\mathcal{L}^*_g(f) = \kappa g. \tag{1.4}
\]
Coutinho et al. [9] consider Riemannian manifolds satisfying (1.2) and obtained several classification on the compact case. Moreover, if the scalar curvature is constant, then \((M^n, g, f)\) satisfies \(V\)-static equation (Proposition 6, [9]). Also, the Euler-Lagragian equation of Hilbert-Einstein action on the space of Riemannian metric with unit volume and constant scalar curvature is

\[
Ric - \frac{1}{n}Rg = \nabla^2 f - \left( Ric - \frac{1}{n-1}Rg \right) f.
\]

(1.5)

A Riemannian manifold \((M^n, g)(n \geq 3)\) of constant scalar curvature is called CPE if it admits a smooth solution \(f\) satisfying (1.5). It is interesting to note that equations (1.2) and (1.5) are somewhat related. When \(\kappa = 0\) in equation (1.4) together with smooth boundary \(\partial M\) such that \(f^{-1}(0) = \partial M\) is known as vacuum static space (Fishech-Marsden equation [11]) and when \(\kappa = 1\) in (1.4) is known as Miao-Tam critical metric [21]. The significance of equation (1.2) is that this rearrangement enclosed a large group of metrics, such as static metric, CPE metric and Miao-Tam critical metric.

Several authors started investigating the above metrics on different contact structures. In [12], Ghosh and Patra investigated CPE in the framework of \(K\)-contact manifold and \((\kappa, \mu)\)-contact manifold. They showed that the complete \(K\)-contact metric satisfying CPE is Einstein and is isometric to unit sphere \(S^{2n+1}\). The same authors also considered Miao-Tam critical metric on contact geometry [13]. In [24], they investigated CPE in Kenmotsu and Almost Kenmotsu manifolds. Recently, Kumara et al. [18] investigated the characteristics of static perfect fluid space-time metrics on almost Kenmotsu manifolds.

In this paper, we would be investigating certain contact manifolds \((M^n, g), n \geq 3\) admitting a smooth non-trivial function \(f\) satisfying

\[
fRic = \nabla^2 f.
\]

(1.6)

It is clear that (1.6) is similar to the first part of (1.2) without perfect fluid matter. A compact manifold admitting smooth function \(f\) satisfying (1.6) was considered by Coutinho et al. [9]. The advantage of this arrangement, is that it encloses a large class of metric, for instance, the static perfect fluid space-time metric eq. (1.2) and (1.3) [9, 17, 19], the critical point equation eq. (1.5) [3], critical metrics of the volume functional [2, 11, 21, 10] and static spaces [1]. We proved that if a complete and simply connected \(K\)-contact manifold admitting smooth function \(f\) satisfying (1.6) then it is isometric to the unit sphere \(S^{2n+1}\) and verified this by constructing an example. Moreover, a non-Sasakian \((\kappa, \mu)\)-contact metric manifold admitting such function \(f\) satisfying (1.6) is considered.
2. Preliminaries

A 2n+1-dimensional smooth manifold \( M \) is said to have a contact structure if it admits a \((1,1)\)-tensor field \( \varphi \), a vector field \( \xi \) called the characteristic vector field such that \( d\eta(\xi, X) = 0 \) for every vector field \( X \) on \( M \), a 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \) everywhere and an associate metric \( g \) called Riemannian metric satisfying the following conditions:

\[
\varphi^2 = -I + \eta \otimes \xi, \quad d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad (2.1)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)
\]

for any vectors field \( X, Y \) on \( M \). Moreover, if \( \nabla \) denotes the Riemannian connection of \( g \), then the following relation holds:

\[
\nabla_X \xi = -\varphi X - \varphi hX. \quad (2.3)
\]

From the definition, it persues that \( \varphi\xi = 0 \) and \( \eta \circ \varphi = 0 \). Then, the manifold \( M(\varphi, \xi, \eta, g) \) equipped with such a structure is called a contact metric manifold \([4, 6]\).

Given a contact metric manifold \( M \) we define a symmetric \((1,1)\)-tensor field \( h \) and self adjoint operator \( l \) by \( h = \frac{1}{2}L_\xi\varphi \) and \( l = R(., \xi)\xi \), where \( L \) denotes Lie differentiation. Then, \( h\varphi = -\varphi h \), \( Trh = Tr\varphi h = 0 \), \( h\xi = 0 \). Also from \([6]\),

\[
\text{Ric}(\xi, \xi) = g(Q\xi, \xi) = Trl = 2n - |h|^2. \quad (2.4)
\]

\[
\nabla_\xi h = \varphi - \varphi h^2 - \varphi l. \quad (2.5)
\]

A contact metric structure on \( M \) is said to be normal if the almost complex structure on \( M \times \mathbb{R} \) defined by \( J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}) \), where \( f \) is a real function on \( M \times \mathbb{R} \), is integrable. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

\[
(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.6)
\]

for any \( X, Y \in TM \). The vector field \( \xi \) is a killing vector with respect to \( g \) if and only if \( h = 0 \). A contact metric manifold \( M(\varphi, \xi, \eta, g) \) for which \( \xi \) is killing (equivalently \( h = 0 \) or \( Trl = 2n \)) is said to be a K-contact metric manifold. On a K-contact manifold, the following formulas are known \([6]\)

\[
\nabla_X \xi = -\varphi X, \quad (2.7)
\]

\[
Q\xi = 2n\xi, \quad (2.8)
\]

\[
R(\xi, X)Y = (\nabla_X \varphi)Y, \quad (2.9)
\]

where \( \nabla \) is the operator of covarient differentiation of \( g \), \( \text{Ric} \) is the Ricci tensor of type \((0,2)\) such that \( \text{Ric}(X, Y) = g(QX, Y) \), where \( Q \) is Ricci operator and \( R \) is the Riemann curvature tensor of \( g \). A Sasakian manifold is \( K \)-contact.
and the converse is not true except in dimension 3. The following formula also holds for a $K$-contact manifold

$$(\nabla_Y \varphi)X + (\nabla_{\varphi Y} \varphi)X = 2g(Y, X)\xi - \eta(X)(Y + \eta(Y)\xi). \quad (2.10)$$

As a generalization of the Sasakian case, Blair et al. [5] introduced $(\kappa, \mu)$-nullity distribution on a contact metric manifold and gave several reasons for studying it. A full classification of $(\kappa, \mu)$-spaces was given by Boeckx [7].

The $(\kappa, \mu)$-nullity distribution of a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is a distribution

$$N(\kappa, \mu) : p \rightarrow N_p(\kappa, \mu) = \{Z \in T_pM : R(X, Y)Z = \kappa\{g(Y, Z)X - g(X, Z)Y\} + \mu\{g(Y, Z)hX - g(X, Z)hY\}\},$$

for any $X, Y, Z \in T_pM$ and real numbers $\kappa$ and $\mu$. A contact metric manifold $M^{2n+1}$ with $\xi \in N(\kappa, \mu)$ is called a $(\kappa, \mu)$-contact metric manifold. In particular, if $\mu = 0$, then the notion of $(\kappa, \mu)$-nullity distribution reduces to the notion of $\kappa$-nullity distribution, introduced by Tanno [28]. If $\kappa = 1$, the structure is Sasakian, and if $\kappa < 1$, the $(\kappa, \mu)$-nullity condition determines the curvature of the manifold completely.

In a $(\kappa, \mu)$-contact metric manifold the following relations hold [5, 26]

$$h^2 = (k - 1)\varphi^2, \quad k \leq 1, \quad (2.11)$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad (2.12)$$

$$QX = [2(n - 1) - n\mu]X + [2(n - 1) + \mu]hX$$

$$+ [2(1 - n) + n(2k + \mu)]\eta(X)\xi, \quad (2.13)$$

$$R = 2n(2n - 2 + k - n\mu). \quad (2.14)$$

Here, $R$ is the scalar curvature of the manifold.

3. Main results

In this section we characterized $K$-contact manifold and $(\kappa, \mu)$-contact manifold in which $f$ a smooth function satisfies [16]. First, we prove the following result which will be used later on.

**Lemma 3.1.** If a Riemannian manifold $(M^{2n+1}, g)$ admits a smooth non-trivial function $f$ satisfying (1.6), then the curvature tensor $R$ can be expressed as

$$R(X, Y)DF = (Xf)QY - (Yf)QX + f[(\nabla_X Q)Y$$

$$- (\nabla_Y Q)X] + (X\psi)Y - (Y\psi)X,$$

for any vector fields $X, Y$ on $M^{2n+1}$ and a smooth function $\psi = \frac{\Delta f - Rf}{2n+1}$. 


Proof. Suppose a Riemannian manifold \((M^{2n+1}, g)\) admits a smooth non-trivial function \(f\) satisfying (1.6), then (1.6) can be rewritten as
\[
\nabla_X Df = fQX + \psi_X,
\]
(3.1)
Taking the covariant derivative of (3.1) along arbitrary vector field \(Y\) we obtained
\[
\nabla_Y \nabla_X Df = (Yf)QX + f(\nabla_Y Q)X
+ fQ(\nabla_Y X) + (Y\psi)X + \psi(\nabla_Y X).
\]
(3.2)
Inserting repeatedly (3.2) in the expression for curvature tensor,
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]
we get the required result. □

Theorem 3.2. Let \(M^{2n+1}(\varphi, \eta, \xi, g)\) be a complete and simply connected K-contact manifold and \(f\) a non-trivial smooth function satisfying (1.6), then it is compact, Einstein and isometric to the unit sphere \(S^{2n+1}\).

Proof. Taking the covariant derivative of (2.8) along arbitrary vector field \(X\) and using (2.7) gives
\[
(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X,
\]
(3.3)
for any vector field \(X\) on \(M\). As \(\xi\) is Killing, we have \(L_\xi Q = 0\). Making use of (2.8) and (3.3) in the last expression yields
\[
\nabla_\xi Q = Q\varphi - \varphi Q.
\]
(3.4)
Replacing \(X\) by \(\xi\) in Lemma 3.1 and using (3.3), (3.4) and (2.8) we get
\[
R(\xi, X)Df = (\xi f)QX - 2n(Xf)\xi
+ f[2n\varphi X - \varphi QX] + (\xi\psi)X - (X\psi)\xi.
\]
(3.5)
Taking an inner product of (3.5) with vector field \(Y\), then using (2.9) after taking an inner product of (2.9) with \(Df\), we obtain:
\[
g((\nabla_X \varphi)Y, Df) + (\xi f)Ric(X, Y) - 2n(Xf)\eta(Y) + f\{2ng(\varphi X, Y)
- g(\varphi QX, Y)\} + ((\xi\psi)g(X, Y) - (X\psi)\eta(Y) = 0.
\]
(3.6)
Replacing \(X\) by \(\varphi X\) and \(Y\) by \(\varphi Y\) in (3.6) yields
\[
g((\nabla_\varphi X \varphi)Y, Df) + (\xi\psi)Ric(\varphi X, \varphi Y)
- 2nf g(X, \varphi Y) - f g(Q\varphi X, Y) + (\xi\psi)g(\varphi X, \varphi Y) = 0.
\]
(3.7)
Making use of (2.10) and (3.6) in (3.7) we get
\[
(2n - 1)\{(Xf)\eta(Y) - (Yf)\eta(X)\} + \{(Y\psi)\eta(X)
- (X\psi)\eta(Y)\} + 8ng g(\varphi X, Y) + 2fg((Q\varphi + \varphi Q)Y, X) = 0.
\]
(3.8)
Taking \(X = \varphi X\) and \(Y = \varphi Y\) in (3.8) and using the fact that \(f \neq 0\), we get
\[
(Q\varphi + \varphi Q)X = 4n\varphi X,
\]
(3.9)
for any vector field \(X\) on \(M\).
Let \(\{e_i, \varphi e_i, \xi\}, i = 1, 2, \ldots n\) be a \(\varphi\)-basis of \(M\) such that \(Qe_i = \lambda_i e_i\). Then we
have \( \varphi Qe_i = \lambda_i \varphi e_i \). Making use of this in (3.9), we get \( Q \varphi e_i = (4n - \lambda_i) \varphi e_i \). Therefore, the scalar curvature is given as

\[
R = g(Q \xi, \xi) + \sum_{i=1}^{n} g(Qe_i, e_i) + g(Q \varphi e_i, \varphi e_i) = 2n(2n + 1).
\]

Taking an inner product of (3.5) with \( Df \), then using (2.8) gives

\[
(\xi f)_Df - 2n(Df)(\xi f) + f(Q \varphi Df \varphi Df \varphi Df) \varphi Df - (D \psi)(\xi f) = 0.
\]

Again taking an inner product of (3.5) with \( \xi \) then using \( R(\xi, X) \xi = \eta(X) \xi - X \), we obtain:

\[
(2n + 1)[Df - (\xi f) \xi] = [(\xi \psi) - D \psi].
\]

Taking the covariant derivative of (3.11) along arbitrary vector field \( X \), then taking an inner product of forgoing equation with vector field \( Y \), we get

\[
2(2n + 1)(\xi f)g(\varphi X, Y) + 2(\xi \psi)g(\varphi X, Y) = X[(2n + 1)(\xi f) + (\xi \psi)]\eta(Y) - Y[(2n + 1)(\xi f) + (\xi \psi)]\eta(X),
\]

where we make use of \( g(\nabla_X Df, Y) = g(\nabla_Y Df, X) \). Choosing \( X, Y \perp \xi \) and noticing that \( d\eta \neq 0 \) in (3.12) gives \( (2n + 1)(\xi f) + (\xi \psi) = 0 \), hence (3.11) becomes

\[
(2n + 1)Df + D \psi = 0.
\]

Making use of (3.9) and (3.13) in (3.10) gives

\[
(\xi f)[Q Df - 2n Df] - f[Q \varphi Df - 2n \varphi Df] = 0.
\]

Operating (3.14) by \( \varphi \) yields

\[
(\xi f)[\varphi Q Df - 2n \varphi Df] - f[\varphi^2 Q Df - 2n \varphi^2 Df] = 0.
\]

Combining (3.15), (3.14) and using (2.1) we obtain

\[
((\xi f)^2 + f^2)(Q Df - 2n Df) = 0.
\]

As \( f \neq 0 \) we must have \( Q Df = 2n Df \). Taking the covariant derivative of last equation and using (3.1) gives

\[
(\nabla_X Q)Df + fQ^2 X + (\psi - 2nf)Q X - 2n \psi X = 0.
\]

Contracting the obtained equation implies \( |Q|^2 = 2nR \), as \( R = 2n(2n + 1) \) is constant. Then as a consequence, we get,

\[
|Q - \frac{R}{2n} I|^2 = |Q|^2 - 2 \frac{R^2}{2n + 1} + \frac{R^2}{2n + 1} = 0.
\]

Therefore, we must have \( Q = \frac{R}{2n + 1} I = 2n I \), that is, \( M \) is Einstein. As \( M \) is complete, by Myer’s theorem [22] it is compact. Integrating (3.13), we get \( \psi = -(2n + 1)f + k \), where \( k \) is some constant. In consequence of this, and \( Q = 2n I \) in (3.1) yields

\[
\nabla_X Df = (-f + k)X.
\]
We now apply Tashiro’s theorem \cite{29} to conclude that \( M \) is isometric to the unit sphere \( S^{2n+1} \). This completes the proof. \( \square \)

On a Sasakian manifold, it is well-known that the Ricci operator commute with \( \varphi \), that is, \( Q\varphi = \varphi Q \). Making use of this in \eqref{3.9} gives \( Q\varphi X = 2n\varphi X \). Then replacing \( X \) by \( \varphi X \) in the last expression yield \( QX = 2nX \), which implies that \( M \) is Einstein. Therefore proceeding similarly as in Theorem \ref{3.2} we can state the following:

**Corollary 3.3.** Let \( M^{2n+1}(\varphi, \eta, \xi, g) \) be a complete and simply connected Sasakian manifold and \( f \) a non-trivial smooth function satisfying \eqref{1.6}, then it is compact, Einstein and isometric to the unit sphere \( S^{2n+1} \).

It is known that if a Riemannian manifold \( (M^n, g)(n \geq 3) \) with constant scalar curvature and \( f \) a smooth function on \( M^n \) satisfying \eqref{1.2}. Then \( (M^n, g, f) \) satisfies the V-static equation \eqref{1.4} for some constant \( k \) (Proposition 6, \cite{9}). Further, we see that the scalar curvature of a \( K \)-contact manifold is constant from Theorem \ref{3.2} and hence we are in a position to state the following:

**Corollary 3.4.** Let \( M^{2n+1}(\varphi, \eta, \xi, g) \) be a complete and simply connected \( K \)-contact manifold without boundary and \( f \) a non-trivial smooth-function satisfying V-static equation \eqref{1.4}, then it is compact, Einstein and isometric to the unit sphere \( S^{2n+1} \).

**Remark 3.5.** In particular, the V-static equation reduces to Fishcher-Marsden equation \cite{11} for \( k = 0 \) and Miao-Tam critical metric \cite{21} for \( k = 1 \). In \cite{13}, authors studied Miao-Tam critical metric and in \cite{25} the Fishcher-Marsden equation without boundary in certain contact metric manifolds. Therefore, Theorem 3.3 \cite{25} and Theorem 3.2 \cite{13} are particular cases of Corollary 3.4.

**Theorem 3.6.** Let \( M^{2n+1}(\varphi, \eta, \xi, g) \) be a non-Sasakian \((\kappa, \mu)\)-contact metric manifold and \( f \) a non-trivial smooth function satisfying \eqref{1.6}, then \( M \) is locally flat for \( n = 1 \) and for \( n > 1 \) is locally isometric to the product space \( E^{n+1} \times S^n(4) \).

**Proof.** Replacing \( Y \) by \( \xi \) in \eqref{2.12}, then using it in \eqref{2.5} gives

\[
\nabla_\xi h = \mu h \varphi.
\]

Taking the covariant derivative of \eqref{2.13} along \( \xi \) and using \eqref{3.17} we get

\[
(\nabla_\xi Q)X = \mu (2(n - 1) + \mu) h \varphi X,
\]

for any vector field \( X \) on \( M \). Moreover, from \eqref{2.12} we have \( Q\xi = 2n\kappa \xi \). Differentiating this along arbitrary vector field \( X \) and using \eqref{2.1} yields

\[
(\nabla_X Q)\xi = Q(\varphi + \varphi h)X - 2n\kappa(\varphi + \varphi h)X.
\]
Taking an inner product of Lemma 3.1 with $\xi$ and using (3.18) and (3.19), we obtain

$$g(R(X, Y)Df, \xi) = 2n\kappa \{(Xf)\eta(Y) - (Yf)\eta(X)\} + f\{g(Q\varphi X + \varphi QX, Y) + g(Q\varphi hX + h\varphi QX, Y)\} - 4n\kappa f g(\varphi X, Y) + (X\psi)\eta(Y) - (Y\psi)\eta(X).$$  \hspace{1cm} (3.20)

Contracting Lemma 3.1 along $X$ and making use of $Q\xi = 2n\kappa \xi$ yields

$$RDf + 2nD\psi = 0.$$  \hspace{1cm} (3.21)

On the other hand, replacing $X$ by $\xi$ in (3.20) gives

$$g(R(\xi, Y)Df, \xi) = 2n\kappa \{(\xi f)\eta(Y) - (Yf)\eta(Y)\} + (\xi\psi)\eta(Y) - (Y\psi).$$  \hspace{1cm} (3.22)

In consequence of this, Eq. (3.22) becomes

$$(2n + 1)\kappa \{(\xi f)\xi - Df\} + \{(\xi\psi)\xi - D\psi\} - \mu hDf = 0.$$  \hspace{1cm} (3.23)

Replacing $X$ by $\xi$ in (3.21) and using $Q\xi = 2n\kappa \xi$ infer

$$\nabla_\xi Df = (2n\kappa f + \psi)\xi.$$  \hspace{1cm} (3.24)

Combining (3.21) and (3.23), then taking its covariant derivative along $\xi$ and using (3.17) and (3.24), we get

$$[R - 2n(2n + 1)\kappa][2n\kappa f\xi + \psi\xi - (\xi f)\xi] - 2n\mu^2 h\varphi Df = 0.$$  \hspace{1cm} (3.25)

Operating (3.25) by $\varphi$ gives $2n\mu^2 hDf = 0$. Then, operating this by $h$ and using (2.11), we get $2n\mu^2(\kappa - 1)\varphi^2 Df = 0$. Since $M$ is non-Sasakian, we have either (i) $\mu = 0$ or (ii) $\varphi^2 Df = 0$.

Case (i): Replacing $X$ by $\varphi X$ and $Y$ by $\varphi Y$ in (3.20) and noting that $f \neq 0$, we get

$$Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa \varphi X = 0.$$  \hspace{1cm} (3.26)

Replacing $X$ by $\varphi X$ in (2.13), we get

$$Q\varphi X = [2(n - 1) - n\mu]\varphi X + [2(n - 1) + \mu]h\varphi X.$$  \hspace{1cm} Then, operating this by $h$ and using (2.11) infer

$$hQ\varphi X = [2(n - 1) - n\mu]h\varphi X - (\kappa - 1)[2(n - 1) + \mu]\varphi X.$$  Again operating (2.13) by $\varphi$ yields

$$\varphi QX = [2(n - 1) - n\mu]\varphi X + [2(n - 1) + \mu]\varphi hX.$$  \hspace{1cm} Replacing $X$ by $hX$ in the last expression gives

$$\varphi QhX = [2(n - 1) - n\mu]\varphi hX - (\kappa - 1)(n - 1) + \mu]\varphi X.$$  Making use of the last four equations in (3.26), we obtain

$$\mu(n + 1) = \kappa(\mu - 2).$$  \hspace{1cm} (3.27)
By our assumption, $\mu = 0$ implies from above relation $\kappa = 0$. Therefore, (2.12) becomes $R(X,Y)\xi = 0$. Hence we can conclude that $M$ is locally flat for $n = 1$ and for $n > 1$ is locally isometric to the product space of $E^{n+1} \times S^n(4)$ (see [6]).

Case (ii): $\varphi^2 Df = 0$ implies $Df = (\xi f)\xi$. Taking the covariant derivative of this along vector field $X$ and using (3.1) and (2.3), we get

$$\nabla_X Df = X(\xi f)\xi - (\xi f)(\varphi X + \varphi hX).$$

Anti-symmetrizing the last expression yields

$$X(\xi f)\eta(Y) - Y(\xi f)\eta(X) + (\xi f)d\eta(X,Y) = 0.$$

Choosing $X, Y \perp \xi$ and noting that $d\eta \neq 0$, we obtain $\xi f = 0$ implies $Df = 0$, i.e., $f$ is constant on $M$. Also, from (3.21) we see that $\psi$ is constant. In consequence, (3.1) implies $M$ is Einstein, i.e., $QX = 2n\kappa X$. Contracting this we get the scalar curvature as $R = 2n\kappa(2n+1)$. Comparing this with (2.13) it follows that $n\mu = 2(n-1) - 2n\kappa$. Making use of the last relation and $QX = 2n\kappa X$ in (2.13), we obtain $(2(n-1) + \mu)h = 0$. Since $M$ is non-Sasakian we must have $2(n-1) + \mu = 0$. Obviously for $n = 1$, $\mu = 0 = \kappa$ and hence $R(X,Y)\xi = 0$. On the other hand, for $n > 1$, inserting $2(n-1) + \mu = 0$ in (3.27) we obtain $\kappa = n - \frac{1}{n} > 1$, a contradiction. This completes the proof. $\square$

By similar arguments as in Corollary 3.4 we can state the following:

**Corollary 3.7.** Let $M^{2n+1}(\varphi, \eta, \xi, g)$ be a non-Sasakian $(\kappa, \mu)$-contact metric manifold without boundary and $f$ is a non-constant solution of $V$-static equation (1.4), then it is locally flat for $n = 1$ and for $n > 1$ is locally isometric to the product space $E^{n+1} \times S^n(4)$.

Next, we construct an example of $(\kappa, \mu)$-contact metric manifold and $K$-contact manifold in which a non-trivial smooth function $f$ satisfies (1.6).

**Example.** In [14], the authors constructed a 3-dimensional $(1 - \lambda^2, 0)$-contact metric manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$ in the Euclidean space $\mathbb{R}^3$, where $\lambda$ is a real number. Take $f(x, y, z) = x^2$, then one can easily see that the equations (3.1) holds for $\psi = 2(1 - x^2)$ on $M$. In particular, taking $\lambda = 0$ in the above example give a 3-dimensional $K$-contact manifold. Moreover, for $f(x, y, z) = x^2$ and $\psi = 2(1 - x^2)$, the 3-dimensional $K$-contact manifold satisfies (1.6). The expression for Ricci operator is obtained as $QX = 2X$, that is, $M$ is Einstein (see [14]) with constant scalar curvature $R = 6$. Thus Theorem (3.2) is verified.

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