CONTINUOUS DATA ASSIMILATION FOR TWO-PHASE FLOW: ANALYSIS AND SIMULATIONS

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Abstract. We propose, analyze, and test a novel continuous data assimilation two-phase flow algorithm for reservoir simulation. We show that the solutions of the algorithm, constructed using coarse mesh observations, converge at an exponential rate in time to the corresponding exact reference solution of the two-phase model. More precisely, we obtain a stability estimate which illustrates an exponential decay of the residual error between the reference and approximate solution, until the error hits a threshold depending on the order of data resolution. Numerical computations are included to demonstrate the effectiveness of this approach, as well as variants with data on sub-domains. In particular, we demonstrate numerically that synchronization is achieved for data collected from a small fraction of the domain.

1. Introduction

1.1. Data assimilation. Data assimilation (DA) refers to a class of methodologies which combines information from grain coarse observational data with simulation/dynamical model in order to obtain a more accurate forecast. The method has a long history, with applications in weather modeling and environmental forecasting [63], as well as the medical, environmental and biological sciences, [56,58], imaging, traffic control, finance and oil exploration [6]. There are a variety of data assimilation techniques, with which actual measured quantities over time are incorporated in system models. One classical technique, which is based on a linear-quadratic estimation, is known as the Kalman filter. This Bayesian approach gives exact probabilistic predictions, although the underlying system and any corresponding observation models are assumed to be linear. This approach has been modified to cover more general cases in ensemble Kalman filter, extended Kalman filter and the unscented Kalman filter; see [8,31,60].

A major difficulty of applying a physical model to real life applications is that, usually the initial condition cannot be known/measured exactly, and only an approximation over a coarse spatial resolution is known. This imprecision in measuring the initial condition may sometimes cause an exponential growing error in the solution of a nonlinear system. To overcome this difficulty, a promising approach, known as the continuous data assimilation, was proposed and analyzed by Azouani, Olson, and Titi in [10,11] based on techniques coming from control theory. This approach introduces a feedback control term at the PDE level to synchronize the computed solution with the true solution corresponding to the observed data.

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To describe the method, we consider a dynamical system

\[
\frac{du}{dt} = F(u),
\]

with insufficient/inaccurate knowledge of the initial state \( u(0) \), but with a solution \( u(t) \) on a coarse grid that is believed to accurately reflect some aspects of the underlying physical reality. Given observational data of the system at a coarse spatial resolution of size \( H \), i.e. \( \Pi_H(u(t)) \) from some given interpolation operator \( \Pi_H \), the algorithm is to construct an approximate solution \( v(t) \) from the observations that satisfies the auxiliary equation

\[
\frac{dv}{dt} = F(v) - \mu \Pi_H (v - u), \quad v(0) = \text{arbitrary},
\]

where \( \mu > 0 \) is a relaxation (nudging) parameter. The goal is to pick \( \mu > 0 \) and \( H > 0 \) such that

\[ v(t) \to u(t) \]

as \( t \to \infty \) in a suitable spatial space. The above algorithm is designed to work for many nonlinear dissipative dynamical systems of the form (1.1), with their solution well-known to be unstable. Owing to this instability, it is expected that any small error in the initial data could lead to an exponentially growing error in the solutions. In these cases, the dissipative term (only) controls the small scales and instabilities occur at large scales. The feedback term in (1.2) that is newly introduced then aims to stabilize the system and damp the error term at large scales by forcing (nudging) the large spatial scales of the measured solution (of the auxiliary equation) back to the reference solution.

In the context of the incompressible 2D Navier-Stokes equations, the authors in [10] proved that, for large enough \( \mu \) and small enough \( h \), the approximate solution to (1.2), \( v \), converges exponentially fast to the exact solution \( u \). Numerical experiments were carried out successfully to test this algorithm for many nonlinear systems, for instance, the 2D Navier-Stokes equation [17, 43, 44, 50, 59], the Rayleigh-Bénard equations [7, 33], and the Kuramoto-Sivashinsky equations [61]. Continuous data assimilation applied to other physical phenomena and PDEs includes non-Newtonian fluids [21], magnetohydrodynamic equations [15], Leray-\( \alpha \) model of turbulence [52], quasi-geostrophic equation [52], Darcy’s equation [64], KdV equations [51], primitive equations [70], and many others. While the aforementioned works assume noise-free observations, the method is later extended to the case when only noisy data can be obtained, e.g. in [12, 41, 49]. The authors refer the readers to other recent literature on this topic; see, e.g. [9, 16, 32, 34, 36–41, 46, 48, 61, 62, 68, 76].

1.2. Multi-phase flow. Thanks to the development of numerical reservoir simulation, oil-field corporations have deeply benefited from the technology in terms of confidently predicting oil recovery estimates and determining the selection of operations during the deployment of specific recovery technologies. During the screening stage, numerical reservoir model is established and simulations are run to determine the feasibility of many injection/production options. This requires reservoir simulation to be accurate and time efficient. Furthermore, for oil and gas reservoirs that are already in production, it is necessary to determine reservoir parameters (e.g. permeability, porosity) with increasing certainty [1, 65]. As the oil field matures, more effective practices such as production enhancement, chemical treatment or
infill drilling can greatly extend the life of the oil reservoir, thereby increasing the overall recovery rate.

A commonly used model in reservoir simulation is the multi-phase flow model \[18,57,72,74\]. In recent decades, both the mathematical analysis and numerical simulation of the two-phase flows have been a focus of study for many researchers and practitioners, thanks to their important applications in petroleum engineering and hydrology. The system of equations governing two-phase immiscible incompressible flows in porous media consists of a nonlinear elliptic Darcy-type equation for the global pressure and a nonlinear parabolic equation with degenerate diffusion term for the saturation, which are coupled by means of the total velocity, recuperated from Darcy’s equation \([4,23,26]\).

Due to the nonlinear nature of the problem, the velocity of the fluid in different phases highly depends on the saturation and the pressure of the respective phase. So as to obtain an accurate simulation, we are required to have a good initialization of the model parameters as well as an accurate initial condition of the saturation. While uncertainty quantification (UQ) and parameter estimation have been used to predict reservoir parameters \([55,66,67]\), unfortunately it is not feasible to obtain accurate microscope data of saturation and pressure at a particular time slice. Nonetheless, a coarse scale approximate of the saturation and pressure field can be obtained using seismic waves data and well logs data. In this work, we consider a simple two-phase model (3.8) and (3.10), and inject these coarse-scale data directly into a system via our proposed data assimilation algorithm to control the error of the solution without using any microscope initial condition.

1.3. Main result of this paper. While two-phase models have a long history of success on certain problems, they tend to lose accuracy on more complicated problems due to the insufficient and inaccurate knowledge of the initial state. Meanwhile, continuous data assimilation (DA) has recently been used to improve accuracy by incorporating measurement data into the simulation. In this work, we introduce a data assimilation model (4.1) which combines the coarse grid saturation measurement data with the multi-phase flow problem. For illustrative purpose, instead of a general multi-phase flow model, we only focus on an immiscible incompressible two-phase flow model (3.7), (3.8), (3.9) and (3.10). After performing an error analysis for the data assimilation algorithm, we prove an exponentially decaying error bound between the exact and approximate solutions until the error reaches a certain level (Theorem 4.2). More precisely, for a given data resolution \(H\), we find \(t_0\) in terms of \(H\) such that synchronization is guaranteed for all \(t < t_0\) for large enough \(\mu \sim O(1/H)\). In this case, coarser data resolution leads to a smaller \(t_0\) and visa versa.

In addition, we illustrate the efficacy of the algorithm by extensive computational studies. We find that the nudging algorithm achieves synchronization with data that is much more coarse than required by the rigorous estimates in Theorem 4.2. Furthermore, we demonstrate numerically that observation on a small fraction of the domain suffices for global assimilation, which may in practice indeed be the case for data collected on the smaller portion of the whole domain.

Organization of this paper. In section 2, we briefly introduce basic notations and preliminaries used in the analysis. Section 3 provides background on the two-phase model and
revisits the existence and uniqueness argument of the model. Later, in sections 4, after introducing the data assimilation algorithm, we state and prove our main results. We give conditions under which the approximate solutions, obtained by the data assimilation algorithm, converge to the solution of the two-phase model. In section 5, we present numerical results to demonstrate the performance of our proposed data assimilation model.

2. Notation and Preliminaries

Let \( \Omega \subset \mathbb{R}^d \) be a bounded open domain. From what follows, we always write \( C \) as a generic positive constant independent of the model parameters. Let \( p \in [1, \infty] \), and the Lebesgue space \( L^p(\Omega) \) is the space of all measurable functions \( v \) on \( \Omega \) with which

\[
\|v\|_{L^p} := \left( \int_{\Omega} |v(x)|^p \, dx \right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1, \infty),
\]

\[
\|v\|_{L^\infty} := \text{ess-sup}_{x \in \Omega} |v(x)| < \infty, \quad \text{if } p = \infty.
\]

The \( L^2 \) norm and inner product will be denoted by \( \|\cdot\| \) and \( (\cdot, \cdot) \) respectively, while all other norms will be labeled with subscripts. Let \( V \) be a Banach space of functions defined on \( \Omega \) with the associated norm \( \|\cdot\|_V \). We denote by \( L^p(a,b; V) \), \( p \in [1, \infty] \), the space of functions \( v : (a,b) \rightarrow V \) such that

\[
\|v\|_{L^p(a,b; V)} := \left( \int_a^b \|v(t)\|_V^p \, dt \right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1, \infty),
\]

\[
\|v\|_{L^\infty(a,b; V)} := \text{ess-sup}_{t \in (a,b)} \|v(t)\|_V < \infty, \quad \text{if } p = \infty.
\]

From now on, for notational sake, we will denote an integral of a function \( f \) over a domain \( \Omega \) with one of the following three notations if no confusion will arise

\[
\int_{\Omega} f(x) \, dx, \quad \int_{\Omega} f \, dx, \quad \text{and} \quad \int_{\Omega} f.
\]

In addition, we consider

\[
V_0 = \{ v \in H^1(\Omega) | \int_{\Omega} v = 0 \},
\]

\[
V_0^* = \{ v \in H^{-1}(\Omega) | \int_{\Omega} v = 0 \}.
\]

**Definition 2.1.** The bi-linear operator \( a(\cdot, \cdot) \) and semi-norm \( |\cdot|_K \) are given as

\[
a(u,v) = \int_{\Omega} K \nabla u \cdot \nabla v, \quad \forall \ u, v \in H^1(\Omega),
\]

and

\[
|u|_K^2 = a(u,u).
\]

**Remark 2.1.** Note that \( |\cdot|_K \) define a norm for \( V_0 \).

**Definition 2.2.** The Green’s operator \( G : V_0^* \rightarrow V_0 \) is given by

\[
a(G(u), v) = (u, v).
\]

We define the norm of \( V_0 \) and \( V_0^* \) respectively as \( \|v\|_{V_0} = |v|_K \) and \( \|u\|_{V_0^*} := \|G(u)\|_{V_0} \) for all \( u \in V_0^*, v \in V_0 \).
Our data assimilation method requires that the observational measurements $\Pi_H(u)$ be given as linear interpolant observables satisfying $\Pi_H : L^2(\Omega) \to L^2(\Omega)$ such that
\begin{align}
\|\Pi_H \varphi\| &\leq c_I \|\varphi\|, \quad \forall \varphi \in L^2(\Omega), \\
\|\varphi - \Pi_H \varphi\| &\leq c_0 H \|\varphi\|_{H^1(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\end{align}
An example of such interpolation operators can be given by a projection operator onto the Fourier modes with wave numbers $|k| \leq 1/H$. Other physical examples include the volume elements and constant finite element interpolation [53].

3. A simple two-phase flow Model

In this section, we describe an immiscible incompressible two-phase flow model. In this model, the flow of the fluids is governed by the Darcy law of the water and the oil phase. More precisely, the velocity of phase $\alpha$, $\tilde{v}_\alpha$, is described as
\[ \tilde{v}_\alpha = -\frac{k_{ra}}{\mu_\alpha} K(\nabla p_\alpha), \quad \text{in } (0, T) \times \Omega, \quad \alpha = o, w, \]
where $k_{ra}, \mu_\alpha, p_\alpha$ and $\rho_\alpha$ are the relative permeability, viscosity, pressure and density of phase $\alpha$, $K$ is the absolute permeability, $T$ is the final time and $\Omega$ is the reservoir domain. We consider the capillary pressure $p_{cow}$ defined as
\[ p_{cow} := p_o - p_w. \]
The saturation of phase $\alpha$, denoted by $S_\alpha$, is governed by the mass balance equation
\[ \frac{\partial(\phi \rho_\alpha S_\alpha)}{\partial t} + \nabla(\rho_\alpha \tilde{v}_\alpha) = \rho_\alpha \tilde{q}_\alpha, \quad \text{in } (0, T) \times \Omega, \quad \alpha = o, w, \]
where $\tilde{q}_\alpha$ is the volumetric input of phase $\alpha$ and $\phi$ is the porosity of the medium. The saturation $S_\alpha$ then satisfies
\begin{equation}
S_o + s_w = 1.
\end{equation}
To simplify the model, in our work, we only consider the case when the density $\rho_\alpha$, the viscosity $\mu_\alpha$, the absolute permeability $K$ and the porosity $\phi$ are constant functions, and that the relative permeability $k_{ra}$, the capillary pressure $p_{cow}$ are functions only depending on $S_\alpha$. Using (3.1), we have $S_o = 1 - s_w$ and therefore $k_{ra}$ and $p_{cow}$ can be written as a function of $s_w$. We then define the function $\kappa_\alpha$ as
\[ \kappa_\alpha(s_w) = \frac{k_{ra}(s_w)}{\phi \mu_\alpha} K. \]
We furthermore assume $\kappa_\alpha \in L^\infty(0, 1)$, $K$ is a positive tensor and
\[ \kappa \geq (\kappa_w(s_w) + \kappa_o(s_w)) \geq \kappa \geq 0, \]
\[ \kappa \geq \kappa \geq \kappa \geq 0 \quad \forall \xi \in \mathbb{R}^d, \|\xi\| = 1, \]
for some constants $\kappa, \pi, K$ and $\kappa$. Since $\phi, \rho_\alpha$ are constants, the mass balance equations can be simplified as
\begin{equation}
\frac{\partial(S_\alpha)}{\partial t} + \nabla(v_\alpha) = q_\alpha, \quad \text{in } (0, T) \times \Omega, \quad \alpha = o, w,
\end{equation}
where $q_\alpha = \frac{\bar{q}_\alpha}{\phi}$ and

$$v_\alpha = -\kappa_\alpha K(\nabla p_\alpha), \text{ in } (0, T) \times \Omega, \alpha = o, w.$$

We then obtain a pressure equation by summing the equations (3.2) and (3.3)

$$-\nabla \cdot \left( K(\kappa_o(\nabla p_o) + \kappa_w(\nabla p_w)) \right) = \nabla (v_o \pm v_w) = q_o + q_w.$$  \hspace{1cm} (3.3)

With this, the system can be simplified to

$$-\nabla \cdot \left( (\kappa_w + \kappa_o) K(\nabla p_w) \right) = q_t \pm \nabla \cdot (\kappa_o \nabla p_{cow}), \hspace{1cm} (3.4)$$

$$\partial (s_w) \partial t + \nabla \cdot (\kappa_w K(\nabla p_w)) = q_w, \hspace{1cm} (3.5)$$

where $q_t = q_o + q_w$.

**Global pressure.** We recall how to reformulate the equations with the help of the concept of global pressure. The global pressure, $p$, is defined as

$$p(S) = p_o(S) + \int_0^S f_w \beta(\xi)d\xi,$$

where $\beta = -\frac{\partial p_{cow}}{\partial s_w}$, $f_i = \frac{\kappa_i}{\kappa}$ and $\kappa = \sum_{i=o,w} \kappa_i$. We may then check directly that

$$\nabla \left( \int_0^S f_w \beta(\xi)d\xi \right) = f_w \beta \nabla S = -f_w \nabla p_{cow}.$$  \hspace{1cm} (3.6)

Hence, we have $\kappa \nabla p = \kappa \nabla p_o - \kappa_w \nabla p_{cow} = \kappa_o \nabla p_o + \kappa_w \nabla p_w$.

Now define a function $\theta$ as

$$\theta(S) := \int_0^S \frac{\kappa_w(\xi) \kappa_o(\xi)}{\kappa(\xi)} \beta(\xi)d\xi,$$

and obtain

$$\kappa(S) \nabla \theta(S) = \kappa_w(S) \kappa_o(S) \alpha(S) \nabla S = -\kappa_w(S) \kappa_o(S) \nabla p_{cow}(S).$$

We therefore have

$$\nabla \theta + \kappa_w \nabla p = -\frac{\kappa_o \kappa_{w}}{\kappa} \nabla p_{cow} + \frac{\kappa_o \kappa_{w}}{\kappa} \nabla p_o + \frac{\kappa_w^2}{\kappa} \nabla p_w$$

$$= \kappa_w \nabla p_w = -K^{-1}v_w$$

and

$$-\nabla \theta + \kappa_o \nabla p = \frac{\kappa_o \kappa_{w}}{\kappa} \nabla p_{cow} + \frac{\kappa_o^2}{\kappa} \nabla p_o + \frac{\kappa_o \kappa_{w}}{\kappa} \nabla p_w$$

$$= \kappa_o \nabla p_o = -K^{-1}v_o.$$  \hspace{1cm} (3.7)

This leads to the equation

$$-\nabla (\kappa(S) K(S) \nabla p(S, x)) = q_t(x), \quad \forall x \in \Omega,$$

where $q_t = q_o + q_w$. We then define an inverse map $T$ of $\theta$, such that

$$T(\theta(S)) = S.$$
We notice that $T$ is well-defined and $\theta$ is irreversible since $\theta$ is monotone. With the help of the aforementioned notations, the system can be now rewritten as

(3.7) \[-\nabla(\kappa K \nabla p) = q_t,\]
(3.8) \[\partial_t(S) - \nabla \cdot \left(K \nabla \theta(S) + \kappa_w K \nabla p\right) = q_w;\]
(3.9) \[-\nabla(\kappa K \nabla p) = q_t,\]
(3.10) \[\partial_t(T(\theta)) - \nabla \cdot \left(K \nabla \theta + \kappa_w K \nabla p\right) = q_w.

3.1. **Analysis of the two-phase model; Existence and Uniqueness results.** In this subsection, we first revisit some classical results of the two-phase model, and then review the existence and uniqueness results of the weak solution of the problems (3.7) and (3.8) as in [2, 3, 5, 19, 24]. The remaining analysis is proceed with the following common assumptions [24,75].

A1. $\Omega \subset \mathbb{R}^d$ for $d \in \{2, 3\}$ is a connected Lipschitz domain.
A2. $\kappa_\alpha$ are continuous and $\kappa_w(0) = \kappa_o(0) = 0$

- $\kappa_o(S_o) > 0$ if $S_o > 0$,
- $\kappa_w(s_w) > 0$ if $s_w > 0$.
A3. $q_w, q_o \in L^\infty(0, T; H^{-1}(\Omega))$.
A4. $\kappa_w$ and $\kappa_o$ satisfy

\[\kappa_w(S) = C_w S^{1+\xi_w},\]
\[\kappa_o(S) = C_o (1 - S)^{1+\xi_o},\]

for some $\xi_o, \xi_w, C_w, C_o > 0$.
A5. Denote $b(S) = \frac{\kappa_w(S) \kappa_o(S)}{\kappa(S)} \beta(S)$, and then $b(S) \leq C_0$ for all $0 \leq S \leq 1$.
A6. $\beta(S)$ satisfies the bound

\[c_\alpha(S) - \beta_w(1 - S)^{-\beta_o} \leq \beta(S) \leq c_\alpha(S)^{-\beta_w}(1 - S)^{-\beta_o},\]

for some $\beta_o, \beta_w, C_\alpha, c_\alpha > 0$.
A7. $\kappa$ satisfies the inequality

\[|\kappa(S_2) - \kappa(S_1)| \leq C(\theta(S_2) - \theta(S_1), S_2 - S_1)^{\frac{1}{2}},\]

for $S_1, S_2 \in [0, 1]$.
A8. $p \in L^\infty(0, T, W^{1,\infty}(\Omega))$.

**Remark 3.1.** Assumption A8 can also be derived directly from some mild assumptions given in [24,75].

**Proposition 3.2.** With assumptions A4 – A6, there are positive constants $\delta < 1/2$, $\tau_i$ and $C_i > 0$ such that

\[C_1 S^{\tau_w} \leq b(S) \leq C_2 S^{\tau_w}, \quad S \in [0, \delta],\]
\[C_3 \leq b(S) \leq C_4, \quad S \in [\delta, 1 - \delta],\]
\[ C_5 (1 - S)^{\tau_o} \leq b(S) \leq C_6 (1 - S)^{\tau_o}, \quad S \in [1 - \delta, 1], \]

where \( \tau_w = 1 + \xi_w - \beta_w \), and \( \tau_o = 1 + \xi_o - \beta_o \).

With the aforementioned assumptions, we arrive at the following lemma, which will be important for our subsequent analysis.

**Lemma 3.3.** Assuming \( A_4 - A_6 \), for any \( S_1, S_2 \in [0, 1] \), then we have

\[ \tilde{C}(S_2 - S_1)^{2+\tau} \leq (\theta(S_2) - \theta(S_1))(S_2 - S_1) \leq C_0 (S_2 - S_1)^2 \]

where \( \tau = \max(\tau_w, \tau_o) \).

**Proof.** Without loss of generality, assume \( S_2 \geq S_1 \), then by the definition of \( \theta \) in (3.6), we have

\[ (\theta(S_2) - \theta_1(S_1)) = \int_{S_1}^{S_2} b(S) dS \leq C_0 \int_{S_1}^{S_2} dS, \]

\[ (\theta(S_2) - \theta_1(S_1)) \geq c \int_{S_1}^{S_2} (1 - S)^{\tau_o} S^{\tau_w} , \]

and therefore

\[ \tilde{C}(S_2 - S_1)^{1+\tau} \leq |\theta(S_2) - \theta(S_1)| \leq C_0 (S_2 - S_1). \]

\( \square \)

We now state the well-posedness result for the model (3.7) and (3.8), equipped with Neumann boundary conditions, although the results can be extended to more general boundary conditions. The complete proof can be found in the above mentioned references. For the readers’ convenience and in order to make the paper self-contained, we have included in a short sketch of proof of some of the following properties. Readers may refer to [24] for further details.

**Theorem 3.4 (Existence and uniqueness of weak solutions of the two-phase flow model).** Let \( \Omega \subset \mathbb{R}^d \) for \( d \in \{2, 3\} \) is a connected Lipschitz domain. Assume \( s_w(0) \in L^2(\Omega) \) with homogeneous Neumann boundary conditions in (3.7). Problems (3.7) and (3.8) has a unique weak solution on \((0, \infty)\) satisfying for all \( T > 0 \)

\[ s_w = T(\theta), \quad \partial_t s_w \in L^2(0, T; H^{-1}(\Omega)), \quad 0 \leq \theta(x, t) \leq \theta^*. \]

Moreover

\[ \int_{\Omega} \kappa(s_w) K \nabla p(t, \cdot) \nabla w = \int_{\Omega} q_t w, \quad \forall w \in H^1(\Omega), \]

\[ \int_0^T \int_{\Omega} \left( \partial_t s_w v + K(\nabla \theta(s_w) + \kappa_w \nabla p) \cdot \nabla v \right) = \int_0^T \int_{\Omega} q_w v, \quad \forall v \in L^2(0, T; H^1(\Omega)). \]

**Proof.** The proof of existence starts with defining a discrete time solution of the problem. Denote

\[ \partial^n v(t) = \frac{v(t + \eta) - v(t)}{\eta}, \]

for any function \( v(t) \), where \( \eta = T/N \). Next define
Therefore, we have
\[ I_{i,n}(V) = \left\{ v \in L^\infty(0,T; V) : v \text{ is piece-wise polynomial with degree } \right\} \]

\[ i \text{ in time on each sub-interval } J_i \subset J \}, \]

where \( J_i = (t_i, t_{i+1}] \), \( t_i = i\eta \) and \( t_N = T \). We then define the discrete time solution \( p^n \in I_{0,n}(H^1(\Omega)) \) with
\[ \int_\Omega p^n(t, \cdot) = 0, \quad \text{and} \quad \theta^n \in I_{1,n}(H^1), \]
which satisfy
\[ (3.12) \]
\[ \int_0^T \int_\Omega (\partial^n(T(\theta^n))v + \kappa_w \nabla p^n \cdot \nabla v + K \nabla \theta^n \cdot \nabla v) = \int_0^T \int_\Omega q_n v \quad \forall v \in I_{0,h}(H^1(\Omega)). \]

We now need the following tow result, which are stated without proof.

**Lemma 3.5.** The discrete problems (3.12) are well-posed.

**Lemma 3.6.** Let \( d : \mathbb{R} \to \mathbb{R} \) be an increasing function such that \( d(0) = 0 \) and \( \{c_i\} \) be a sequence of real numbers. Then for any number \( m > 0 \),
\[ \sum_{k=1}^m (d(c_k) - d(c_{k-1}))c_k \geq D(c_m) - D(c_0) \geq -D(c_0) \]
where
\[ D(c) = \int_0^c (b(c) - b(\xi))d\xi. \]

Next, we recall the following Lemma, which is important to our subsequent analysis.

**Lemma 3.7.** The discrete solution \( p^n, \theta^n \) satisfy
\[ \| p^n \|_{L^\infty(0,T;H^1(\Omega))} + \| \theta^n \|_{L^2(0,T;H^1(\Omega))} \leq C, \]
where \( C \) is independent of \( \eta \).

**Proof.** Since \( p^n(t, \cdot) \in V_0 \), set \( w = p^n \) in (3.12) to obtain
\[ \kappa(t_{i+1} - t_i)|p^n(t, \cdot)|^2 \leq \int_{t_i}^{t_{i+1}} \int_\Omega \kappa(s_w) K|\nabla p^n(t, \cdot)|^2 = \int_{t_i}^{t_{i+1}} \int_\Omega q_n(t, \cdot)p^n(t, \cdot) \]
\[ \leq C(t_{i+1} - t_i) \| q_n \|_{L^\infty(0,T;H^{-1}(\Omega))} \| p^n \|_{L^\infty(0,T;H^1(\Omega))}. \]

Therefore, we have \( c_K \| p^n \|_{L^\infty(0,T;V_0)} \leq \| p^n \|_{L^\infty(0,T;H^1(\Omega))} \leq C_K \| p^n \|_{L^\infty(0,T;V_0)} \) and
\[ \| p^n \|_{L^\infty(0,T;H^1)} \leq C_K^2 \| q_n \|_{L^\infty(0,T;H^{-1}(\Omega))}. \]

Then consider \( v = \theta^n \) in (3.12) and obtain
\[ \int_0^T \int_\Omega (\partial^n(T(\theta^n))\theta^n + \kappa_w K \nabla p^n \cdot \nabla \theta^n + K |\nabla \theta^n|^2) = \int_0^T \int_\Omega q_n \theta^n. \]
After using Lemma 3.6, we have
\[ \int_0^T \int_\Omega \left( \partial^\alpha (T(\theta^n)) \theta^n \right) \geq - \int_0^{\theta^n_0} (T(c) - T(\xi)) d\xi, \]
and
\[ \int_\Omega T(\theta^n(t, \cdot)) - \int_\Omega T(\theta^n)(0, \cdot) = \int_0^t \int_\Omega q_w. \]
Hence, we obtain
\[ | \int_\Omega \theta^n(t, \cdot)| \leq C \left( | \int_\Omega \theta^n(0, \cdot)| + | \int_0^t \int_\Omega q_w | \right), \]
and therefore
\[ \|\theta^n(t, \cdot)\|_{H^1}^2 \leq C \left( \|\theta^n(t, \cdot)\|_{V_0}^2 + \left| \int_0^t \int_\Omega q_w \right|^2 + \left| \int_\Omega \theta^n(0, \cdot) \right|^2 \right). \]

With the above inequalities, one can arrive at
\begin{align*}
|\theta^n|_{L^2(0,T;H^1)}^2 &\leq C \left( \|q_0\|_{L^2(0,T;H^{1-\alpha}(\Omega))}^2 + \|q_t\|_{L^2(0,T;H^{1-\alpha}(\Omega))}^2 + \int_0^{\theta^n_0} (T(\theta^n) - T(\xi)) d\xi \\
&\quad + \left| \int_0^T \int_\Omega q_w \right|^2 + \left| \int_\Omega \theta^n(0, \cdot) \right|^2 \right). 
\end{align*}
(3.13)

From Lemma 3.6, and since \( q^n \) and \( \theta^n \) remains bounded, we have the following corollary.

**Corollary 3.8.** Let \( q^n \) and \( \theta^n \) be solutions to (3.12), then

1. For any \( 2 \leq r < \infty \), there exist a subsequence \( q^n \to q \) weakly in \( L^r(0,T;H^1(\Omega)) \), and \( \theta^n \to \theta \) weakly in \( L^2(0,T;H^1(\Omega)) \).
2. There is a subsequence \( \theta^n \to \theta \) strongly in \( L^2(0,T;L^2(\Omega)) \).
3. There is a subsequence \( \theta^n \to \theta \) strongly in \( L^2(0,T;H^{1-\alpha}(\Omega)) \) for any \( 0 < \alpha < 1/2 \), and \( s^n \to s \) pointwise a.e. on \( (0,T) \times \Omega \) where \( s^n = T(\theta^n) \).

In the rest of this section, we analyze the stability of the weak solution by bounding the difference between the two solutions \((s_i, p_i), i = 1, 2\) as follows.

**Lemma 3.9.** Let \((s_i, p_i), i = 1, 2\) are two weak solutions to (3.11) given by Theorem 3.4. Then we have
\[ \|\nabla (p_2 - p_1)\|_{L^2} \leq C \left( \|\kappa(s_2) - \kappa(s_1)\|_{L^2} + \|q_{t,2} - q_{t,1}\|_{L^2} \right). \]

**Proof.** After subtracting the two equations, we have
\[ - \int_0^T \int_\Omega \kappa(s_1) K \nabla (p_2 - p_1) \cdot \nabla w = \int_0^T \int_\Omega (\kappa(s_2) - \kappa(s_1)) K \nabla p_2 \cdot \nabla w + \int_0^T \int_\Omega (q_{t,1} - q_{t,2}) w. \]
for all test function $w \in V_0$. Set $w = p_1 - p_2$ to get
\begin{equation}
\int_0^T \| \nabla(p_2 - p_1) \|_{L^2}^2 \leq C \int_0^T \int_\Omega \kappa(s_1) K |\nabla(p_2 - p_1)|^2
\end{equation}
\begin{align*}
= \int_0^T \int_\Omega (\kappa(s_2) - \kappa(s_1)) K \nabla p_2 \cdot \nabla (p_1 - p_2) + \int_0^T \int_\Omega (q_{t,1} - q_{t,2})(p_1 - p_2) \\
\leq C \int_0^T \| \nabla p_2 \|_{L^\infty} \| \kappa(s_2) - \kappa(s_1) \|_{L^2} \| \nabla (p_2 - p_1) \|_{L^2} + \int_0^T \| q_{t,1} - q_{t,2} \|_{L^2} \| p_2 - p_1 \|_{L^2}.
\end{align*}
And since $\| p_2 - p_1 \|_{L^2} \leq C \| \nabla (p_2 - p_1) \|_{L^2}$, we obtain the result
$$
\| \nabla (p_2 - p_1) \|_{L^2} \leq C (\| \kappa(s_2) - \kappa(s_1) \|_{L^2} + \| q_{t,2} - q_{t,1} \|_{L^2}).
$$

With $\pi$ to be the average operator $\pi(u) = \frac{1}{|\Omega|} \int_\Omega u$, and $e = (I - \pi)(s_2 - s_1)$, one can prove the following stability result.

**Lemma 3.10.** Let $(s_i, p_i), i = 1, 2$ are two weak solutions to (3.11) given by Theorem 3.4. With the assumptions A1 – A8, we have
\begin{align*}
\frac{1}{2} \int_0^t \| e \|_{V_0}^2 + \int_0^t (\theta_2 - \theta_1, s_2 - s_1) \\
\leq \frac{\gamma e}{p} \int_0^t \| s_1 - s_2 \|_{L^p}^p + C \frac{q}{q\delta} \left( \| \pi(s_2 - s_1)(0, \cdot) \|_{L^q}^q + \| q_{w,2} - q_{w,1} \|_{L^q(0,T;L^q)}^2 \right) \\
+ C \frac{\delta}{2} \int_0^t \left( \| \kappa(s_2) - \kappa(s_1) \|_{L^2}^2 + \| \nabla(p_2 - p_1) \|_{L^2}^2 + \| q_{w,2} - q_{w,1} \|_{L^2}^2 \right) + C \frac{1}{2\delta} \int_0^t \| \nabla G(e) \|_{L^2}^2.
\end{align*}
where $\frac{1}{p} + \frac{1}{q} = 1$, $\delta, \delta_p > 0$.

**Proof.** After subtracting the two equations, we get
\begin{equation}
\begin{align*}
\int_0^t \int_\Omega (\kappa(s_2) - \kappa(s_1)) K \nabla p_2 \cdot \nabla v + \int_0^t \int_\Omega (q_{w,1} - q_{w,2}, v) \\
= \int_0^t \int_\Omega (\kappa(w(s_1)) - \kappa(w(s_2))) K \nabla p_2 \cdot \nabla v + \int_0^t \int_\Omega (q_{w,1} - q_{w,2}, v),
\end{align*}
\end{equation}
for all test functions $v \in L^2(0,T;H^1(\Omega))$. Using Definition 2.2, we have the following estimates on the first two terms of the above equations
$$\int_\Omega \partial_t(s_2 - s_1)G(e) = \int_\Omega \partial_t(e)G(e) = a(G(\partial_t(e)), G(e)) = \frac{1}{2} \partial_t \| G(e) \|_{V_0}^2 = \frac{1}{2} \partial_t \| e \|_{V_0}^2,$$
and
$$\int_\Omega K \nabla(\theta_2 - \theta_1) \cdot \nabla G(e) = a(\theta_2 - \theta_1, G(e)) = (\theta_2 - \theta_1, e)$$
$$= (\theta_2 - \theta_1, s_2 - s_1) - (\theta_2 - \theta_1, \pi(s_2 - s_1)),$$
By Holder’s inequality and Young’s inequality, we then obtain
\[
\int_{\Omega} K \nabla (\theta_2 - \theta_1) \cdot \nabla G(e) \geq (\theta_2 - \theta_1, s_2 - s_1) - \|\theta_2 - \theta_1\|_{L^p} \|\pi(s_2 - s_1)\|_{L^q}
\]
\[
\geq (\theta_2 - \theta_1, s_2 - s_1) - \frac{\delta_p}{p}\|\theta_2 - \theta_1\|_{L^p} - \frac{1}{q\delta_p} \|\pi(s_2 - s_1)\|_{L^q}^q.
\]
Since \(p_2 \in L^\infty(0,T,W^{1,\infty}(\Omega))\), the term \(\int_0^t \int_{\Omega} (\kappa_w(s_1) - \kappa_w(s_2)) K \nabla p_2 \cdot \nabla G(e)\) in (3.15) can be estimated as
\[
\int_0^t \int_{\Omega} (\kappa_w(s_1) - \kappa_w(s_2)) K \nabla p_2 \cdot \nabla G(e) \leq C \|p_2\|_{L^\infty(0,T,W^{1,\infty})} \int_0^t \int_{\Omega} |(\kappa_w(s_1) - \kappa_w(s_2))| \cdot |\nabla G(e)|
\]
\[
\leq C \int_0^t \|\kappa_w(s_1) - \kappa_w(s_2)\|_{L^2} \|\nabla G(e)\|_{L^2}
\]
\[
\leq C \int_0^t \left(\frac{\delta}{2}\|\kappa(s_2) - \kappa(s_1)\|_{L^2} + \frac{1}{2\delta}\|\nabla G(e)\|_{L^2}^2\right).
\]
Similarly, the other two terms \(\int_0^t (q_{w,1} - q_{w,2}, G(e))\) and \(\int_0^t \int_{\Omega} \kappa_w(s_1) K \nabla (p_2 - p_1) \cdot \nabla G(e)\) in (3.15) can be bounded above as
\[
\int_0^t (q_{w,1} - q_{w,2}, G(e)) \leq \int_0^t \frac{\delta}{2}\|q_{w,1} - q_{w,2}\|_{L^2}^2 + \frac{1}{2\delta}\int_0^t \|G(e)\|_{L^2}^2,
\]
and
\[
\int_0^t \int_{\Omega} \kappa_w(s_1) K \nabla (p_2 - p_1) \cdot \nabla G(e) \leq C \int_0^t \left(\frac{\delta}{2}\|\nabla (p_2 - p_1)\|_{L^2}^2 + \frac{1}{2\delta}\|\nabla G(e)\|_{L^2}^2\right).
\]
After inserting the above estimates into (3.15), we obtain
\[
\frac{1}{2}\partial_t \int_0^t \|e\|_{V_0^p}^2 + \int_0^t (\theta_2 - \theta_1, s_2 - s_1)
\]
\[
\leq \frac{\delta_p}{p} \int_0^t \|\theta_2 - \theta_1\|_{L^p}^p + \frac{1}{q\delta_p} \int_0^t \|\pi(s_2 - s_1)\|_{L^q}^q + C \frac{1}{2\delta} \int_0^t \|\nabla G(e)\|_{L^2}^2 + \|G(e)\|_{L^2}^2
\]
\[
+ C \frac{\delta}{2} \int_0^t \left(\|\kappa(s_2) - \kappa(s_1)\|_{L^2}^2 + \|\nabla (p_2 - p_1)\|_{L^2}^2 + \|q_{w,1} - q_{w,2}\|_{L^2}^2\right),
\]
for some constant \(C > 0\). Since \(|\theta_2 - \theta_1| \leq C|s_2 - s_1|\) and \(\|e\|_{L^2} \leq C\|\nabla G(e)\|_{L^2}\), we have
\[
\frac{1}{2}\partial_t \int_0^t \|e\|_{V_0^p}^2 + \int_0^t (\theta_2 - \theta_1, s_2 - s_1)
\]
\[
\leq \frac{\delta_p}{p} \int_0^t \|s_2 - s_1\|_{L^p}^p + \frac{1}{q\delta_p} \int_0^t \|\pi(s_2 - s_1)\|_{L^q}^q + C \frac{1}{2\delta} \int_0^t \|\nabla G(e)\|_{L^2}^2
\]
\[
+ C \frac{\delta}{2} \int_0^t \left(\|\kappa(s_2) - \kappa(s_1)\|_{L^2}^2 + \|\nabla (p_2 - p_1)\|_{L^2}^2 + \|q_{w,1} - q_{w,2}\|_{L^2}^2\right).
\]
Then consider $v = \pi(s_2 - s_1)$ in (3.15), and obtain
\[ \frac{1}{q} \left( \| \pi(s_2 - s_1)(t, \cdot) \|^q_{L^q} - \| \pi(s_2 - s_1)(0, \cdot) \|^q_{L^q} \right) = \int_0^t \int_{\Omega} \partial_t (s_2 - s_1) (\pi(s_2 - s_1))^{q-1} \]
\[ = \int_0^t (q_{w,2} - q_{w,1}, (\pi(s_2 - s_1))^{q-1}) \].

Therefore, we have
\[ \frac{1}{q} \| \pi(s_2 - s_1) \|^q_{L^q(0,t;L^q)} \leq \| \pi(s_2 - s_1) \|^q_{L^q(0,t;L^q)} \| q_{w,2} - q_{w,1} \|_{L^q(0,t;L^q)} + \frac{1}{q} \| \pi(s_2 - s_1)(0, \cdot) \|^q_{L^q(\Omega)}, \]
and
\[ \| \pi(s_2 - s_1) \|^q_{L^q(0,t;L^q)} \leq C_q \left( \| \pi(s_2 - s_1)(0, \cdot) \|^q_{L^q} + \| q_{w,2} - q_{w,1} \|_{L^q(0,t;L^q)} \right), \]
which proves the lemma.

We will give the stability estimate and thus the uniqueness with the following theorem.

**Theorem 3.11.** Let $(s_i, p_i)$, $i = 1, 2$ are two weak solutions to (3.11) given by Theorem 3.4. With the assumptions A1–A8, we have
\[ \| (I - \pi)(s_2 - s_1) \|_{L^q(0,T,V^q_0)} \leq C e^{C_t} \left( \| \pi(s_2 - s_1)(0, \cdot) \|_{L^q(\Omega)} + \| q_{w,2} - q_{w,1} \|_{L^q(0,T;L^q(\Omega))} \right), \]
\[ + \| (I - \pi)(s_2 - s_1)(0, \cdot) \|_{V^q_0}^2 + \| q_{w,2} - q_{w,1} \|_{L^2(0,T;L^2(\Omega))}^2 + \| q_{t,2} - q_{t,1} \|_{L^2(0,T;L^2(\Omega))}^2), \]
for some $C > 0$, $\tau$ defined as in Lemma 3.3, and $q_0 = \frac{2 + \tau}{1 + \tau}$.

**Proof.** From Lemma 3.9, we have
\[ \| \nabla (p_2 - p_1) \|_{L^2}^2 \leq C \left( \| \kappa(s_2) - \kappa(s_1) \|_{L^2}^2 + \| q_{t,2} - q_{t,1} \|_{L^2}^2 \right). \]
Therefore, we observe that
\[ \frac{1}{2} \partial_t \int_0^t \| e \|^2_{V^q_0} + \frac{c}{p} \int_0^t \| s_1 - s_2 \|^p_{L^p} + \frac{C_q}{q \delta p} \left( \| \pi(s_2 - s_1)(0, \cdot) \|_{L^q}^q + \| q_{w,2} - q_{w,1} \|_{L^q(0,t;L^q)} \right) \]
\[ + \frac{C \delta}{2} \int_0^t \left( \| \kappa(s_2) - \kappa(s_1) \|_{L^2}^2 + \| q_{w,2} - q_{w,1} \|_{L^2}^2 + \| q_{t,2} - q_{t,1} \|_{L^2}^2 \right) + C \int_0^t \| \nabla G(e) \|_{L^2}^2. \]

Since $|\kappa(s_2) - \kappa(s_1)| \leq (\theta_2 - \theta_1)(s_2 - s_1)$, we have
\[ \frac{1}{2} \partial_t \int_0^t \| e \|^2_{V^q_0} + \left( 1 - \frac{C \delta}{2} \right) \int_0^t (\theta_2 - \theta_1, s_2 - s_1) \]
\[ \leq \frac{c}{p} \int_0^t \| s_1 - s_2 \|^p_{L^p} + \frac{C_q}{q \delta p} \left( \| \pi(s_2 - s_1)(0, \cdot) \|_{L^q}^q + \| q_{w,2} - q_{w,1} \|_{L^q(0,t;L^q)} \right) \]
\[ + \frac{C \delta}{2} \int_0^t \left( \| q_{w,2} - q_{w,1} \|_{L^2}^2 + \| q_{t,2} - q_{t,1} \|_{L^2}^2 \right) + C \int_0^t \| \nabla G(e) \|_{L^2}^2. \]
By taking $\delta = C^{-1}$ in the above inequality, we obtain
\[
\partial_t \int_0^t \|e\|^2_{V_0^*} + \int_0^t (\theta_2 - \theta_1, s_2 - s_1) \\
\leq 2 \left( \frac{\delta_p}{p} \int_0^t \|s_1 - s_1\|^2_{L^p} + \frac{C_q}{q \delta_q} \left( \|\pi(s_2 - s_1)(0, \cdot)\|^q_{L^q} + \|q_{w,2} - q_{w,1}\|^q_{L^q(0,T;L^q)} \right) \\
+ \frac{1}{2} \int_0^t \left( \|q_{w,2} - q_{w,1}\|^2_{L^2} + \|q_{t,2} - q_{t,1}\|^2_{L^2} \right) + \frac{C^2}{2} \int_0^t \|\nabla G(e)\|^2_{L^2} \right).
\]
Since $\int_0^t (\theta_2 - \theta_1, s_2 - s_1) \geq C \int_0^t \int_\Omega |s_2 - s_1|^{2+\tau}$, we can choose a $\delta_p > 0$ such that
\[
\partial_t \int_0^t \|e\|^2_{V_0^*} + \frac{1}{2} \int_0^T (\theta_2 - \theta_1, s_2 - s_1) \\
\leq C \left( \|\pi(s_2 - s_1)(0, \cdot)\|^{q_0}_{L^{q_0}} + \|q_{w,2} - q_{w,1}\|^{q_0}_{L^{q_0}(0,T;L^{q_0})} \right) \\
+ C \int_0^t \|\nabla G(e)\|^2_{L^2} + \frac{1}{2} \|q_{w,2} - q_{w,1}\|^2_{L^2(0,T;L^2)} + \frac{1}{2} \|q_{t,2} - q_{t,1}\|^2_{L^2(0,T;L^2)}.
\]
Then denote $E = \|\nabla G(e)\|^2_{L^2}$, and since
\[
\|e\|^2_{V_0^*} = \int_\Omega K|\nabla G(e)|^2 \geq c \|\nabla G(e)\|^2_{L^2} = cE,
\]
and
\[
\|\nabla G(e)\|_{L^{q_0}}^{q_0} \leq C \|\nabla G(e)\|_{L^2}^{q_0},
\]
we can get
\[
E(t) - E(0) + \frac{1}{2} \int_0^t (\theta_2 - \theta_1, s_2 - s_1) \\
\leq C \int_0^t E + C \left( \|\pi(s_2 - s_1)(0, \cdot)\|^{q_0}_{L^{q_0}(\Omega)} + \|q_{w,2} - q_{w,1}\|^{q_0}_{L^{q_0}(0,T;L^{q_0}(\Omega))} \right) \\
+ \frac{1}{2} \left( \|q_{w,2} - q_{w,1}\|^2_{L^2(0,T;L^2(\Omega))} + \|q_{t,2} - q_{t,1}\|^2_{L^2(0,T;L^2(\Omega))} \right).
\]
With that, we at last arrive at
\[
E(t) \leq C e^{Ct} \left( \|\pi(s_2 - s_1)(0, \cdot)\|^{q_0}_{L^{q_0}} + \|q_{w,2} - q_{w,1}\|^{q_0}_{L^{q_0}(0,T;L^{q_0})} + E(0) \\
+ \|q_{w,2} - q_{w,1}\|^2_{L^2(0,T;L^2)} + \|q_{t,2} - q_{t,1}\|^2_{L^2(0,T;L^2)} \right),
\]
where $q_0 = \frac{2 + \tau}{1 + \tau}$. \hfill \qed

4. Data assimilation algorithm for the two-phase flow problem

In this section, we first describe the nudging algorithm for the two-phase flow equations (3.7), (3.8), (3.9) and (3.10). The domain $\Omega$ is partitioned into a coarse partition $T_H$, where the observed data are collected. More precisely, we collect the data, denoted by $\Pi^*_H(s_w)$, of
the averaged water saturation $s_w$ in each coarse element containing $\{x_i\}_{i}^{N} \subset \Omega$, where $x_i$ are the points that physical measurements are performed. The data assimilation algorithm for two-phase flow problem is defined as

$$-
abla \cdot (\kappa K \nabla p) = q_t,$$

(4.1)

$$\frac{\partial (\bar{s}_w)}{\partial t} + \nabla \cdot \left( \frac{\kappa w}{\kappa} K \nabla p \right) + \mu (\Pi_H^* (\bar{s}_w) - \Pi_H^* (s_w)) = q_w,$$

where $\Pi_H : H^1(\Omega) \to L^2(\Omega)$ is a linear interpolant operator satisfying

$$\|\Pi(s) - s\| \leq CH\|s\|_{H^1},$$

(4.2)

$$\|\Pi(s)\| \leq C\|s\|,$$

which can naturally be extended to $L^2(\Omega)$. The dual operator is given by $\Pi_H^* : L^2(\Omega) \to H^{-1}(\Omega)$

$$\int_\Omega \Pi(s) v = \int_\Omega s \Pi^*(v), \quad \forall v \in L^2(\Omega).$$

For instance, $\Pi_H$ may be considered to be the $L^2$ projection operator to the piece-wise constant finite element space (for more details see [53]), namely,

$$\Pi(s_w)(x) = \int_{K_i} s_w, \quad \forall x \in K_i,$$

where $K_i$ is a coarse element in $K_i$, and in this case, we have

(4.3)

$$\Pi_H^* = \Pi_H.$$

**Definition 4.1 (Weak solution to the data assimilation algorithm).** Let $(s, p)$ be the solution to the two-phase problem from Theorem 3.4. The continuous data assimilation equations (4.1) has a unique weak solutions $(\bar{s}, \bar{p})$ that satisfies for all $T > 0$

$$\bar{s}_w = T(\theta), \quad \partial_t \bar{s}_w \in L^2(0, T; H^{-1}(\Omega)), \quad 0 \leq \theta(x, t) \leq \theta^*,$$

and

$$\int_\Omega \kappa(\bar{s}) \nabla \bar{p} \nabla w = \int_\Omega q_t w, \quad \forall w \in H^1(\Omega),$$

(4.4)

$$\int_0^T \int_\Omega \left( \partial_t \bar{s} v + (\nabla \bar{\theta} + \kappa w(\bar{s}) \nabla \bar{p}) \cdot \nabla v + \mu \Pi_H^* (\bar{s} - s)(v) \right) = \int_0^T \int_\Omega q_w v, \quad \forall v \in L^2(0, T; H^1(\Omega)).$$

**Remark 4.1.** Although the well-posedness of (4.4) is not the focus of this work, we speculate that it can be proved by a usual compactness argument [24], or by the fixed-point argument which has recently been suggested in [21] for a similar problem.

In the next theorem, we analyze the residual error coming from the data assimilation algorithm with unknown initial condition, which is the main analytical result of this work.
Theorem 4.2. Let $\Omega \subset \mathbb{R}^d$ for $d \in \{2, 3\}$ is a connected Lipschitz domain. Consider $s$ be a solution of the two-phase equations with initial data $s_w(0) \in L^2(\Omega)$, ensured by Theorem 3.4, and $\Pi_H : L^2(\Omega) \rightarrow L^2(\Omega)$ be a linear map satisfying (2.1). Let $\tilde{s}$ be a solution to the data assimilation algorithm given by (4.1) with homogeneous Neumann boundary conditions. Then for all $H > 0$, if

$$
\mu := \mu(\tilde{\gamma}, C, C_2, H) = 2\tilde{\gamma}\left(\frac{C_2}{C}\right)^{-\frac{1}{2}}H^{-1},
$$

the following bound holds for $t < t_0$

$$
\|(I - \pi)(\tilde{s} - s)\|_{V_0}^2 \leq e^{-\frac{\mu}{2}t}\left(\|e(0, \cdot)\|_{V_0}^2 + 2C_5\|\pi(\tilde{s} - s)(0, \cdot)\|_{L^{2+\tau}}^2\right),
$$

where $t_0 := t_0(\tilde{\gamma}, C, C_2, H)$ is given as

$$
t_0 = \max \left\{0 \leq \zeta < T, \text{ s.t. } c^* H^{2+\tau} \leq \|\tilde{s}(t) - s(t)\|_{L^{2+\tau}}^2, \forall 0 \leq t \leq \zeta \right\}.
$$

Herein, $\tilde{\gamma}, C, C_2, C_5$ are constants depending only on $\Omega$, and $c^* := c^*(\tilde{\gamma}, C, C_2)$ is a constant appearing in (4.16), whereas $\tilde{\gamma}$ is a chosen constant of order 1 with respect to $H$.

Proof. Subtracting (3.11) and (4.4), the difference satisfies the following error equations

$$
\int_{\Omega} \kappa(\tilde{s}) \nabla(\tilde{p} - p) \nabla w = \int_{\Omega} \left(\kappa(s) - \kappa(\tilde{s})\right) \nabla p \cdot \nabla w, \quad \forall w \in H^1,
$$

$$
\int_0^T \int_{\Omega} \left(\partial_t(\tilde{s} - s)v + \nabla(\tilde{\theta} - \theta) \cdot \nabla v + \kappa_w(s) \nabla(\tilde{p} - p) \cdot \nabla v + \mu(\tilde{s} - s)\Pi_H(v)\right)
$$

$$
= \int_0^T \int_{\Omega} \left(\kappa_w(s) - \kappa_w(\tilde{s})\right) \nabla p \cdot \nabla v, \quad \forall v \in L^2(0, T; H^1(\Omega)).
$$

Denote $e = (I - \pi)(\tilde{s} - s)$ and set $v = G(e)$ in (4.9). With that, we obtain

$$
\int_0^T \int_{\Omega} \left(\partial_t(\tilde{s} - s)G(e) + \nabla(\tilde{\theta} - \theta) \cdot \nabla G(e) + \mu(\tilde{s} - s)\Pi_H(G(e))\right)
$$

$$
= \int_0^T \int_{\Omega} \left(\left(\kappa_w(s) - \kappa_w(\tilde{s})\right) \nabla p \cdot \nabla G(e) - \kappa_w(\tilde{s}) \nabla(\tilde{p} - p) \cdot \nabla G(e)\right).
$$

With a similar argument in the proof of Lemma 3.10, we obtain

$$
\int_{\Omega} \partial_t(\tilde{s} - s)G(e) = \frac{1}{2} \partial_t\|G(e)\|_{V^*}^2 = \frac{1}{2} \partial_t\|e\|_{V^*}^2,
$$

and

$$
\int_{\Omega} \nabla(\tilde{\theta} - \theta) \cdot \nabla G(e) = (\tilde{\theta} - \theta, \tilde{s} - s) - (\tilde{\theta} - \theta, \pi(\tilde{s} - s)).
$$

Then choose a test function $v = p(\pi(\tilde{s} - s))^{p-1}$ in (4.9) to get

$$
\|\pi(\tilde{s} - s)(t, \cdot)\|_{L^p(\Omega)}^p + p\mu \int_0^t \|\pi(\tilde{s} - s)(\cdot, \cdot)\|_{L^p(\Omega)}^p = \|\pi(\tilde{s} - s)(0, \cdot)\|_{L^p(\Omega)}^p,
$$

where $\mu := \mu(\tilde{\gamma}, C, C_2, H)$ as defined in (4.5).
and
\[
\|\pi(\bar{s} - s)(t, \cdot)\|_{L^p(\Omega)}^p = e^{-p\mu t} \|\pi(\bar{s} - s)(0, \cdot)\|_{L^p(\Omega)}^p,
\]
for any \( p > 1 \). For any \( u \in V^* \), we have
\[
\int_{\Omega} \nabla G(u) \cdot \nabla G(u) = \int_{\Omega} G(u)(u) = \int_{\Omega} (I - \Pi_H)G(u)(u) + \int_{\Omega} \Pi_H G(u)(u)
\]
\[
\leq \tilde{C}^2 H \|\nabla G(u)\|_{L^2}^2 u + \int_{\Omega} \Pi_H G(u)(u).
\]
We hence obtain
\[
(4.13) \quad \|u\|_{V_0}^2 \leq \tilde{C} H^2 \|u\|_{L^2}^2 + 2 \int_{\Omega} \Pi_H G(u)(u).
\]
Combining (4.11), (4.12) and (4.13) with (4.10), we have
\[
\int_0^t \left( \frac{1}{2} \frac{d}{dt} e_{V_0}^2 + (\tilde{\theta} - \theta, \bar{s} - s) + \frac{\mu}{2} \|\nabla G(e)\|_{L^2}^2 \right)
\]
\[
\leq \int_0^t \left( \frac{1}{2} \frac{d}{dt} e_{V_0}^2 + (\tilde{\theta} - \theta, \bar{s} - s) + \mu \int_{\Omega} \Pi_H G(e) + \frac{\mu \tilde{C}}{2} H^2 \|e\|_{L^2}^2 \right)
\]
\[
= \int_0^T \int_{\Omega} \left( \left( \kappa_w(s) - \kappa_w(\bar{s}) \right) \nabla p \cdot \nabla G(e) - \kappa_w(\bar{s}) \nabla (\bar{p} - p) \cdot \nabla G(e)
\]
\[
+ (\tilde{\theta} - \theta, \pi(\bar{s} - s)) + \frac{\mu \tilde{C}}{2} H^2 \|e\|_{L^2}^2 \right)
\]
\[
\leq C_1 \int_0^t \left( \left( \|s - \bar{s}\|_{L^p} + \|\nabla (\bar{p} - p)\|_{L^p(\Omega)} \right) \|\nabla G(e)\|_{L^2}
\]
\[
+ \|\tilde{\theta} - \theta\|_{L^q} \|\pi(\bar{s} - s)\|_{L^p} + \frac{\mu \tilde{C}}{2} H^2 \|e\|_{L^2}^2 \right),
\]
for any \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and an arbitrary \( \mu > 0 \) which will be determined later. With an argument similar to the proof of Lemma 3.9, we can also obtain
\[
\|\nabla (\bar{p} - p)\|_{L^p}^p \leq \tilde{C}_p \|\bar{s} - s\|_{L^p}^p,
\]
hence, with \( p = 2 = q \), we have
\[
\int_0^t \left( \frac{1}{2} \frac{d}{dt} e_{V_0}^2 + (\tilde{\theta} - \theta, \bar{s} - s) + \frac{\mu}{2} \|\nabla G(e)\|_{L^2}^2 \right)
\]
\[
(4.14) \quad \leq C_1 \int_0^t \left( \left( \|s - \bar{s}\|_{L^2} + \|\nabla (\bar{p} - p)\|_{L^2(\Omega)} \right) \|\nabla G(e)\|_{L^2}
\]
\[
+ \|\tilde{\theta} - \theta\|_{L^{2+\varepsilon}} \|\pi(\bar{s} - s)\|_{L^{2+\varepsilon}} + \frac{\mu \tilde{C}_s}{2} H^2 \|e\|_{L^2}^2 \right),
\]
\[
\int_0^t \left( \frac{1}{2} \partial_t \|e\|_{V_0}^2 + (\tilde{\theta} - \theta, \bar{s} - s) + \frac{\mu}{2} \|\nabla G(e)\|_{L^2}^2 \right) \\
\leq C_1 \int_0^t \left( \|s - \bar{s}\|_{L^2} + \|\nabla (\bar{p} - p)\|_{L^2(\Omega)} \right) \|\nabla G(e)\|_{L^2} \\
+ \|\tilde{\theta} - \theta\|_{L^{2+r}} \|\pi(s - \bar{s})\|_{L^{2+r}} + \frac{\mu \tilde{C}}{2} H^2 \|e\|_{L^2}^2, \tag{4.15}
\]

and

\[
\int_0^t \left( \|s - \bar{s}\|_{L^2} + \|\nabla (\bar{p} - p)\|_{L^2(\Omega)} \right) \|\nabla G(e)\|_{L^2} \leq \frac{\mu}{4C_1} \|\nabla G(e)\|_{L^2}^2 + \frac{2C_2}{\mu} \|s - \bar{s}\|_{L^2}^2,
\]

where \(C_2 := 4C_1(1 + \tilde{C}_2)\).

We next estimate the term \(\|\tilde{\theta} - \theta\|_{L^{2+r}} \|\pi(s - \bar{s})\|_{L^{2+r}}\) as

\[
\|\tilde{\theta} - \theta\|_{L^{2+r}} \|\pi(s - \bar{s})\|_{L^{2+r}} \leq \frac{\tilde{C}}{2C_1C_0} \|\tilde{\theta} - \theta\|_{L^{2+r}}^2 + \frac{2C_1C_0(1 + \tau)}{2C(2 + \tau)^2} \|\pi(s - \bar{s})\|_{L^{2+r}}^2 \\
\leq \frac{\tilde{C}}{2C_1} \|s - \bar{s}\|_{L^{2+r}}^2 + \frac{C_3}{C_1} \|\pi(s - \bar{s})\|_{L^{2+r}}^2,
\]

where \(C_3 = \frac{2C_0^2C_0(1 + \tau)}{C(2 + \tau)^2}\). Now, from \(\int_0^t (\tilde{\theta} - \theta, s - \bar{s}) \geq \tilde{C} \int_0^t \int_\Omega |s - \bar{s}|^{2+r}\), we get that

\[
\int_0^t \left( \frac{1}{2} \partial_t \|e\|_{V_0}^2 + \tilde{C} \right) \int_0^t \int_\Omega |s - \bar{s}|^{2+r} + \frac{\mu}{4} \|\nabla G(e)\|_{L^2}^2 \\
\leq \int_0^t \left( \frac{C_2}{\mu} \|s - \bar{s}\|_{L^2} + C_3 \|\pi(s - \bar{s})\|_{L^{2+r}}^2 + \frac{\mu \tilde{C}}{2} H^2 \|e\|_{L^2}^2, \right.
\]

Then with the help of \(\|e\|_{L^2}^2 \leq \|s - \bar{s}\|_{L^2}^2\) and \(\|\pi(s - \bar{s})\|_{L^{2+r}}^2 \leq e^{-2(\frac{\gamma}{2} + \tilde{\gamma}) t} \|\pi(s - \bar{s})\|_{L^{2+r}}^2\), we obtain

\[
\int_0^t \left( \frac{1}{2} \partial_t \|e\|_{V_0}^2 + \tilde{C} \right) \int_0^t \int_\Omega |s - \bar{s}|^{2+r} + \frac{\mu}{4} \|\nabla G(e)\|_{L^2}^2 \\
\leq \left( \frac{2C_2}{\mu} + \frac{\mu \tilde{C}}{2} H^2 \right) \|s - \bar{s}\|_{L^2}^2 + C_3 \int_0^t e^{-2(\frac{\gamma}{2} + \tilde{\gamma}) t} \|\pi(s - \bar{s})\|_{L^{2+r}}^2.
\]

We can now take \(\mu\) to be the form \(\mu = 2\tilde{\gamma} \left( \frac{C_2}{C} \right)^{-\frac{1}{2}} H^{-1}\) (where \(\tilde{\gamma}\) is a chosen constant) to get

\[
\left( \frac{2C_2}{\mu} + \frac{\mu \tilde{C}}{2} H^2 \right) \|s - \bar{s}\|_{L^2}^2 = H \left( \frac{C_2}{C} \right)^{\frac{1}{2}} \left( \frac{1}{\gamma} + \tilde{\gamma} \right).
\]
With \( \int_0^t \int_\Omega |\bar{s} - s|^2 \leq \tilde{C}_r \int_0^t \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{\frac{2}{2+\tau}} \), we obtain
\[
\int_0^t \left( \frac{1}{2} \partial_t \| e \|^2_{V_0^*} + \tilde{C} \int \int_\Omega |\bar{s} - s|^{2+\tau} + \frac{\mu}{4} \| \nabla G(e) \|^2_{L^2} \right) \\
\leq H \left( \frac{C_2}{C} \right)^{\frac{1}{2}} (\frac{1}{\gamma} + \tilde{\gamma}) \int_0^t \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{\frac{2}{2+\tau}} + C_3 \int_0^t e^{-2(\frac{2+\tau}{1+\tau})t} \| \pi(\bar{s} - s)(0, \cdot) \|^{\frac{2+\tau}{1+\tau}}.
\]

Now consider
\[
I := \left\{ 0 < t < T, \right. \text{ such that } c^* H^{2+\tau} \leq \left( \int_\Omega |\bar{s}(t, \cdot) - s(t, \cdot)|^{2+\tau} \right) \right\},
\]
where
\[
c^* = \left( \frac{C_2}{C} \right)^{\frac{1}{2}} \left( \frac{1}{\gamma} + \tilde{\gamma} \right) \left( \frac{\tilde{C}}{4} \right)^{-2-\tau}.
\]

Now for all \( t \in I \), we have
\[
\left( \frac{C_2}{C} \right)^{\frac{2+\tau}{1+\tau}} (\frac{1}{\gamma} + \tilde{\gamma})^{2+\tau} \left( \frac{\tilde{C}}{4} \right)^{-2-\tau} H^{2+\tau} \leq \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{\frac{2}{2+\tau}},
\]
and
\[
H \left( \frac{C_2}{C} \right)^{\frac{1}{2}} (\frac{1}{\gamma} + \tilde{\gamma}) \int_0^t \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{\frac{2}{2+\tau}} \\
= \left( \frac{C_2}{C} \right)^{\frac{2+\tau}{1+\tau}} (\frac{1}{\gamma} + \tilde{\gamma})^{2+\tau} H^{2+\tau} \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{-\tau} \int_0^t \left( \int_\Omega |\bar{s} - s|^{2+\tau} \right)^{\frac{2}{2+\tau}} \\
\leq \frac{\tilde{C}}{4} \int_0^t \int_\Omega |\bar{s} - s|^{2+\tau}.
\]

Hence, we have
\[
\int_0^t \left( \frac{1}{2} \partial_t \| e \|^2_{V_0^*} + \frac{C}{4} \int \int_\Omega |\bar{s} - s|^{2+\tau} + \frac{\mu}{4} \| \nabla G(e) \|^2_{L^2} \right) \leq C_3 \int_0^t e^{-2(\frac{2+\tau}{1+\tau})t} \| \pi(\bar{s} - s)(0, \cdot) \|^{\frac{2+\tau}{1+\tau}} \leq C_5 \| \pi(\bar{s} - s)(0, \cdot) \|^{\frac{2+\tau}{1+\tau}}.
\]

Which proves the theorem as
\[
\| e(t, \cdot) \|^2_{V_0^*} \leq e^{-\frac{\mu t}{2}} \left( \| e(0, \cdot) \|^2_{V_0^*} + 2C_5 \| \pi(\bar{s} - s)(0, \cdot) \|^{\frac{2+\tau}{1+\tau}} \right).
\]

\[
\text{Remark 4.3.} \text{ We remark that our method can be likewise extended to a general multi-phase model with no obstruction; however, for the sake of simplicity, we will postpone that to a future work.}
\]
5. Computational study

We now present results of two numerical tests that illustrate the theory given in the last section. We consider Algorithm (4.1) for two scenarios with two different permeability profiles. In both tests, the domain is $\Omega = [0,100]^2$. The reference and the approximate solutions are calculated in a fine square mesh with fine mesh size $h = 1/100$ by upwinding finite volume method with time step size $dt = 0.05$. The data is obtained in a coarse square mesh with coarse mesh size $H = 1/10$, while the nudge parameter $\mu = 200$.

Since we do not have access to true (reference) solutions for these problems, we instead use a computed solution. The reference solutions were evolved from a zero initial value, and is run to $t = 325$ using the above setting. For the DA computation, we start from zero initial conditions use the same spatial and temporal discretization parameters as the reference solution, and start assimilation with the $t = 25$ reference solution (i.e., time 0 for DA corresponds to $t = 25$ for the reference).

5.1. Computational study I. In our first experiment, we consider the relative permeability $k_{ra}$ defined as

$$k_{ro} = (1 - s_w)^2, \quad k_{rw} = (s_w)^2,$$

with the injection and production located at the top-left and bottom-right corner of the domain respectively.

In this experiment, we carried out three tests. The first one is with data taken on full domain $\Omega$, and the other two tests are carried out with data only taken on one of the following square sub-domains at the top - left corner:

$$\Omega_1 = [0, 50]^2, \quad \Omega_2 = [0, 25]^2.$$

The plots in Figure 1 shows the saturation error in $L^2$ vs. time, where the observational data are collected from different fractions of the domain. The solution without data assimilation $\mu = 0$ has only negligible drop (in blue) in its residual error. This underscores the significance of a nudged solution synchronizing with the reference solution. Full nudging (in red) indicates synchronization with the reference solution roughly at an exponential rate. We then continue by testing the effect of the size of the sub-domain. Machine precision is reached for data collected over the whole domain. By then, in the case of $\Omega_1$ and $\Omega_2$, the error is within $10^{-10}$ and $10^{-5}$ respectively. Particularly in the case of subdomains $\Omega_1$ and $\Omega_2$, we see from the snapshot plots in Figures 2, 3, 4, and 5 that the main spatial features over the full domain $\Omega$ are nevertheless captured as time evolves.

The convergence of the DA solution to the true solution in time can also be seen in the snapshot plots of the solutions in Figures 6. Here at $t = 0.1$, there is of course a major difference, since the DA simulation starts at 0. The accuracy of DA is seen to increase by $t = 1$ and further by $t = 10$. Finally by $t = 100$ there is no visual difference between DA and reference solution, which we expect since the $L^2$ difference between the solutions at $t = 100$ is seen in Figure 1 to be near $10^{-15}$. More over, the snapshot plots of the true solution starting from zero initial value (bottom plots in Figure 6) indicate the sensitivity of the solution to the initial conditions.
Figure 1. Error comparison $\|s(t) - \tilde{s}(t)\|_{L^2}$. Left: $L_2$ error. Right: $L_2$ error in log-scale.

Figure 2. Snapshots and comparison of the solutions with different data at $T = 0.1$. Left: two-phase solution. Middle-Left: full data. Middle-Right: $\frac{1}{4}$ data. Right: $\frac{1}{16}$ data

Figure 3. Snapshots and comparison of the solutions with different data at $T = 1$. Left: two-phase solution. Middle-Left: full data. Middle-Right: $\frac{1}{4}$ data. Right: $\frac{1}{16}$ data
Figure 4. Snapshots and comparison of the solutions with different data at $T = 10$. Left: two-phase solution. Middle-Left: full data. Middle-Right: $\frac{1}{4}$ data. Right: $\frac{1}{16}$ data

Figure 5. Snapshots and comparison of the solutions with different data at $T = 100$. Left: two-phase solution. Middle-Left: full data. Middle-Right: $\frac{1}{4}$ data. Right: $\frac{1}{16}$ data
5.2. **Computational study II.** We present our second numerical test, with its source terms coinciding with that of the first example. On the other hand, we have a medium that have a totally different permeability profile as shown in figure 7.

The same initialization process as in the first example is considered here, which provides us with the initial condition of the exact solution as $s_w(0) = s_w(25)$. With this new medium profile, we compute an approximate solution without any prior knowledge of our initial value, and apply our data assimilation algorithm with $\mu = 200$ as in Example 1

The convergence history of this test is given in figure 8, and the snapshots of the solutions are now shown in figures 9. We again observe an exponential decay of the residual error in
Figure 7. Absolute permeability for case 2.

Figure 8. Error comparison. Left: data assimilation vs two-phase flow solution. Right: error of data assimilation in model log-scale.

Figure 8, with a clear linear fit in the log scale, This has numerically validated the theoretical result that we have proved in Theorem 4.2, and the effectiveness of our proposed data assimilation algorithm.

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Figure 9. Snapshot of the solutions. Top: Reference solution (exact). Middle-Left: data assimilation solution. Middle-Right: difference between data assimilation solution and reference solution. Bottom-Left: two-phase solution. Bottom-Right: difference between two-phase solution and reference solution.
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