Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Periodic solutions and bifurcation in an SIS epidemic model with birth pulses

Guirong Jiang\textsuperscript{a,b,}\textsuperscript{*}, Qigui Yang\textsuperscript{b}

\textsuperscript{a} School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China
\textsuperscript{b} School of Mathematical Sciences, South China University of Technology, Guangzhou, 510640, China

\textbf{A R T I C L E I N F O}

Article history:
Received 11 August 2008
Received in revised form 14 April 2009
Accepted 27 April 2009

Keywords:
SIS epidemic model
Pulse birth
Periodic solution
Flip bifurcation

\textbf{A B S T R A C T}

The dynamical behavior of an SIS epidemic model with birth pulses and a varying population is discussed analytically and numerically. This paper investigates the existence and stability of the infection-free periodic solution and the endemic periodic solution. By using discrete maps, the center manifold theorem, and the bifurcation theorem, the conditions of existence for bifurcation of the positive periodic solution are derived. Moreover, numerical results for phase portraits, periodic solutions, and bifurcation diagrams, which are illustrated with an example, are in good agreement with the theoretical analysis.

\textcopyright 2009 Elsevier Ltd. All rights reserved.

1. Introduction

Infectious diseases have tremendous influence on human and animal population sizes. For example, severe acute respiratory syndrome (SARS) affected China in 2003, myxomatosis caused enormous decreases in the rabbit population in Australia in the 1950s, the Black Death in Europe in the 14th century killed up to a quarter of the human population. Mathematical modelling is of considerable importance in epidemiology because it may provide understanding of the underlying mechanisms which influence the spread of disease and may suggest control strategies. In the case where the infectious lose immunity and become susceptible immediately after recovering, an SIS epidemic model is used to describe the dynamics of the population [1]. Recent years have also seen wide-scope potential applications of the SIS epidemic model in various scientific fields, such as that of complex networks [2].

The study of the SIS epidemic model mainly concerns global asymptotic stability. For example, Rass [3] obtained asymptotic stability for a multi-type SIS model, and Iannelli et al. [4] considered an age-structured epidemic model of SIS type and analyzed the global dynamical behavior for the model when the population density converges uniformly to a steady state. Global analysis of discrete-time SI and SIS epidemic models was given in [5]. Further, the complex dynamical behavior of the SIS epidemic model is discussed by using bifurcation theory. For example, Zhang et al. [6] obtained the conditions of backward bifurcation and the existence of bistable endemic equilibria. Zhang et al. [7] investigated Hopf bifurcation in a delayed SIS epidemic model with stage structure by using the normal form theory and the center manifold argument.

In the above cited papers, ordinary differential equations (ODEs) were used to build an SIS epidemic model. Impulsive differential equations (IDEs) are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change in their values. The study of IDEs mainly concerns the properties of their solutions, such as the existence, uniqueness, stability, boundedness, and periodicity; see [8,9]. Sufficient conditions for the local and global stability of the susceptible pest-eradication periodic solution are

\textsuperscript{*} Corresponding author at: School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China.
E-mail address: grjiang9@163.com (G. Jiang).

0895-7177/$ – see front matter \textcopyright 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2009.04.021
found by means of Floquet theory and comparison methods in [10]. Nieto and Regan [11] present a new approach via variational methods and critical point theory for obtaining the existence of solutions to impulsive problems. The dynamics of an impulsively controlled three-trophic food chain system with general nonlinear functional responses for the intermediate consumer and the top predator are analyzed using the Floquet theory and comparison techniques in [12]. IDEs are widely used in epidemiology. For example, an SIS epidemic model with a pulse vaccination was investigated in [13] and some results are obtained for the global stability of the disease-free periodic solution as well as the existence and stability of the endemic periodic solution are investigated analytically and numerically.

In most models of population dynamics, increases in population due to births are assumed to be time independent. However, many species give birth only during a single period of the year and Caughley [14] termed this growth pattern a birth pulse. Roberts and Kao [15] proposed a model for the dynamics of a fatal infectious disease and discussed the existence and stability of periodic solutions. Tang and Chen [16] obtained an exact periodic solution and the threshold conditions for its stability by using the stroboscopic map in a single-species model. In this paper, the birth pulse is used to build an SIS epidemic model.

The existence and bifurcation of periodic solutions for an epidemic model built from ODEs have been discussed by many authors; see [6,7] for example. However, little is known about the bifurcation theory of an epidemic model built from IDEs. Periodic boundary value problems for non-Lipschitzian impulsive functional differential equations were considered in [17]. Using a projection method, the bifurcation of periodic solution for an impulsively controlled pest management mode was discussed in [18]. The complex dynamics of a Holling type II prey–predator system with state feedback control was investigated in [19]. Zhou and Liu [13] discussed the existence of a disease-free periodic solution and an endemic periodic solution by using the explicit solution. Lakmeche et al. [20] transformed the problem of periodic solution into a fixed-point problem and obtained conditions for the stability of the trivial solution and the existence of a positive period-1 solution. Many authors (for example, see [21]) discussed only the bifurcation of a non-trivial periodic solution of an epidemic model by using the results obtained in [20]. By using the explicit solution, the stroboscopic map is obtained and used to discuss the bifurcation of the periodic solution in [16]. However, the explicit solution is not easy to obtain in an epidemic model and the bifurcation of the endemic periodic solution is difficult to discuss. Some authors investigated the complex dynamics of epidemic models, such as period-doubling bifurcation, chaos and crisis etc., by means of numerical simulations; see [22] for example. Therefore, theoretically analyzing the bifurcation theory of the SIS model is a challenging task.

In this paper, an ordinary differential system is used to build an SIS epidemic model with a birth pulse and a varying population. To study this epidemic model, we construct discrete maps and present analytical results about the complex dynamical behavior. The rest of the paper is organized as follows. In the next section, an SIS epidemic model with a birth pulse and a varying population is introduced. The conditions for the existence and stability of the periodic infection-free solution are derived in Section 3. In Section 4, the existence and stability of the positive periodic solution are discussed. Bifurcation analysis of the SIS epidemic model is given in detail in Section 5. The parameter value at which the epidemic periodic solution bifurcates from the infection-free periodic solution is calculated. The conditions of existence for flip bifurcation are derived by using the center manifold theorem and the bifurcation theorem. The numerical results are presented in Section 6, to verify the theoretical analysis, and the conclusion is presented in Section 7, finally.

2. Model description

A classical SIS epidemic model was introduced by Kermack and McKendrick [1]. On the assumption that recovery from the nonfatal infectious disease does not confer immunity, one particular case of this classical SIS epidemic model is

$$\begin{align*}
\dot{S} &= \mu - \beta SI - \sigma S + \delta I, \\
\dot{I} &= \beta SI - \delta I - \sigma I, \\
\end{align*}$$

(1)

where the overdot denotes differentiation with respect to time, $S(t)$ is the susceptible component of the population, $I(t)$ is the infected component of the population, $\mu$ represents a constant birth rate, $\beta$ is the average number of adequate contacts with susceptibles for an infective individual per unit time, $\sigma$ is the natural death rate and $\delta$ is the rate at which infective individuals lose immunity and return to the susceptible class.

The dynamics of system (1) is simple. There exists a unique positive equilibrium

$$E^* (S^*, I^*) = \left( \frac{\sigma + \delta}{\beta}, \frac{\mu}{\sigma} \right) - \frac{\sigma + \delta}{\beta},$$

in (1) for $\beta \mu > \sigma (\sigma + \delta)$. Nucci and Leach [23] presented an explicit solution of (1) by means of the Painlevé analysis and the Lie theory of transformation groups.

In system (1), $\mu$ represents a constant birth rate, which means that the population is born throughout the year. However, many species give birth only during a single period of the year and Caughley termed this growth pattern a birth pulse. In most cases, the birth pulse is assumed to be the linear birth pulse $\Delta N = pN$ [24], where $N = S + I$. Roberts and Kao [15] considered the birth pulse $\Delta N = (1 + B(N))N$, where $B(N) = b - cN^q$. In paper [16], $B(N) = b \exp(N) - 1$ and $B(N) = \frac{p}{q}N^q - 1$.

In this paper, the birth pulse is taken as $\Delta N = (b + cN^q)N$, where $c = r(b - d)$, $b$ is the maximum birth rate, $d$ is the maximum death rate, $r$ is a parameter reflecting the relative importance of density-dependent population regulation through births and deaths. If $r = 0$ all density dependence acts through the death rate, and if $r = 1$ all density dependence
acts through the birth rate. The newborn population is assumed to be susceptible to disease, that is, \(\Delta S = (b-c(S+I))(S+I)\) and \(\Delta I = 0\). Now, on the basis of the basis of the impulsive differential equations, we develop system (1) by introducing periodic birth pulses and obtain the following epidemic model with birth pulses:

\[
\begin{aligned}
\dot{S} &= -\sigma S - \beta SI + \delta I, \\
\dot{I} &= \beta SI - \delta I - \sigma I, \\
\Delta S &= (b-c(S+I))(S+I), \\
\Delta I &= 0,
\end{aligned}
\]

where \(0 < b < 1, c > 0\), the meanings of parameters \(\beta, \sigma, \) and \(\delta\) are the same as in model (1), \(n \in \mathbb{N}_+\), \(T\) is the time between two consecutive birth pulses, \(\Delta S(t) = S(t^+) - S(t), S(t^+)\) are the quantities of susceptible components of the population after the birth pulse and \(S(t^+) = \lim_{\tau \to 0^+} S(t + \tau), \Delta I(t) = I(t^+) - I(t), I(t^+) = \lim_{\tau \to 0^+} I(t + \tau)\). For more details about impulsive systems see [8,9].

The epidemic model (2) is considered in the region \(\Omega = \{(S, I) | S \geq 0, I \geq 0, S + I < \frac{b}{c} \exp(\sigma T)\}\) in this paper. On the boundary line \(l_1 : S = 0, I > 0, \dot{S} = -\sigma S - \beta SI + \delta I > 0\) and \(\dot{I} = \beta SI - \delta I - \sigma I < 0\) while on the boundary line \(l_2 : S > 0, I = 0, \dot{S} = -\sigma S - \beta SI + \delta I < 0\) and \(\dot{I} = \beta SI - \delta I - \sigma I = 0\). Set the initial point of system (2) as \((S_0, I_0)\), where \(S_0 > 0, I_0 > 0, S_0 + I_0 < \frac{b}{c} \exp(\sigma T)\). It follows from (2) that

\[
\dot{N} = -\sigma N, \quad t \neq nT, \\
\Delta N = (b-cN)N, \quad t = nT,
\]

where \(N = S + I\). Then \(N(T) = (S_0 + I_0) \exp(-\sigma T) < \frac{b}{c}\) and \(\Delta N(T) > 0\). The trajectory originating from this initial point remains in region \(\Omega\) for \(t \in [0, T)\). This trajectory reaches the point \((\tilde{S}_1, \tilde{I}_1)\) at time \(T\), and next jumps to the point \((S_1, I_1)\) with the effect of the birth pulse, where \(S_1 + I_1 = N(T)\), \(S_1 + I_1 = N(T) + \Delta N(T) = (1 + b - cN(T))N(T)\). In view of \(\frac{b}{c} < \frac{(1+b+1)}{2}\) and the function \(y = (1 + b - cx)\) being a strictly monotone increasing function on \((0, \frac{(1+b)}{2})\), \(S_1 + I_1 = (1 + b - cN(T))N(T) < (1 + b - c \cdot \frac{b}{c}) = \frac{b}{c} < \frac{b}{c} \exp(\sigma T)\). Thus \((S_1, I_1) \in \Omega\) and the region \(\Omega = \{(S, I) | S \geq 0, I \geq 0, S + I < \frac{b}{c} \exp(\sigma T)\}\) is invariant for (2).

### 3. Existence and stability of the infection-free periodic solution

In this section, infectious individuals are entirely absent from the population permanently, i.e., \(I(t) = 0, t > 0\). System (2) yields

\[
\begin{aligned}
\dot{S} &= -\sigma S, \\
\Delta S &= (b-cS)S, \\
\end{aligned}
\]

Suppose the value of \(S(t) = S_k\) for \(t = kT\); then the solution for the first equation of system (3) is

\[
S(t) = S_k \exp(-\sigma t) \quad \text{for} \quad kT \leq t < (k+1)T
\]

and

\[
S((k+1)T^+) = S((k+1)T) + (b-cS((k+1)T))S((k+1)T) = (1 + b - cS_k \exp(-\sigma T))S_k \exp(-\sigma T)
\]

for the effect of the impulse. Thus we obtain the one-dimensional discrete map

\[
S_{k+1} = (1 + b - cS_k \exp(-\sigma T))S_k \exp(-\sigma T) = F(S_k).
\]

For each fixed point of the map (4) there is an associated periodic solution of system (3), and vice versa. The fixed points of map (4) are

\[
S_{01} = 0, \quad S_{02} = \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c}.
\]

In the case of the fixed point \(S_{01}\),

\[
F'(S_{01}) = (1 + b) \exp(-\sigma T) - 2c \exp(-2\sigma T)S_k |_{S_k = S_{01}} = (1 + b) \exp(-\sigma T).
\]

For \(T > \frac{1}{\sigma} \ln(1 + b), 0 < F'(S_{01}) < 1\), then the fixed point \(S_{01}\) is stable. Hence the trivial solution of system (2) is stable for \(T > \frac{1}{\sigma} \ln(1 + b)\).

In the case of the fixed point \(S_{02}\),

\[
F'(S_{02}) = (1 + b) \exp(-\sigma T) - 2c \exp(-2\sigma T)S_k |_{S_k = S_{02}} = 2 - (1 + b) \exp(-\sigma T).
\]

For \(\frac{1}{\sigma} \ln \frac{(1+b)}{3} < T < \frac{1}{\sigma} \ln(1 + b)\), the fixed point \(S_{02}\) is stable, that is, the infection-free periodic solution

\[
\begin{aligned}
S(t) &= \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c} \exp(\sigma(t - nT)), \\
I(t) &= 0,
\end{aligned}
\]

of system (2) is stable.
There are three ways in which a fixed point $P$ of a discrete map $f : \mathbb{R}^n \to \mathbb{R}^n$ may fail to be hyperbolic, that is, $Df(P)$ has an eigenvalue $+1$, an eigenvalue $-1$ or a pair of complex eigenvalues $\lambda, \bar{\lambda}$ with $|\lambda| = 1$ [25]. The bifurcation associated with the appearance of eigenvalue $1$ is called a fold (or tangent) bifurcation. This bifurcation is also referred to as a limit point, saddle-node bifurcation, and turning point, among others. The bifurcation associated with the appearance of eigenvalue $-1$ is called a flip (or period-doubling) bifurcation.

From the above, the trivial solution of system (2) is stable for $T > \frac{1}{\sigma} \ln(1 + b)$ and the infection-free periodic solution (6) of system (2) is stable for $\frac{1}{\sigma} \ln(\frac{1+b}{\sigma^3}) < T < \frac{1}{\sigma} \ln(1 + b)$; then a bifurcation occurs at $T = \frac{1}{\sigma} \ln(1 + b)$. Moreover, $F'(S_0) = 1$ and $F'(S_0) = 1$ for $T = \frac{1}{\sigma} \ln(1 + b)$; hence this bifurcation is a fold bifurcation. Then the following result is obtained.

**Proposition 3.1.** A fold bifurcation occurs at $T = \frac{1}{\sigma} \ln(1 + b)$ in system (2). The trivial solution of system (2) is stable for $T > \frac{1}{\sigma} \ln(1 + b)$ while the infection-free periodic solution (6) of system (2) is stable for $\frac{1}{\sigma} \ln(\frac{1+b}{\sigma^3}) < T < \frac{1}{\sigma} \ln(1 + b)$.

### 4. Existence and stability of the positive periodic solution

In this section, we discuss the existence and stability of the positive periodic solution $(S(t), I(t))$, where $S(t) > 0$ and $I(t) > 0$. To obtain the explicit solution of system (2), let

$$N(t) = S(t) + I(t) \quad \text{and} \quad M(t) = \frac{1}{I(t)}$$

It follows from (2) that

$$\begin{cases}
\dot{N} = -\sigma N, \\
\dot{M} = -(\beta N - (\delta + \sigma))M + \beta, \\
\Delta N = (b - cN)N, \\
\Delta M = 0,
\end{cases} \quad t \neq nT, \quad t = nT. \quad (8)$$

Set the initial point of system (8) as $(N_k, M_k)$. The trajectory originating from this initial point reaches the point $(N_{k+1}, M_{k+1})$ at time $T$, and next jumps to the point $(N_{k+1}, M_{k+1})$ with the effect of the impulse. Like (4), it follows from the first equation of system (8) and the birth pulse $\Delta N = (b - cN)N$ at time $T$ that

$$N(t) = N_k \exp(-\sigma t), \quad \text{for} \ 0 \leq t < T$$

and

$$N_{k+1} = (1 + b - cN_k \exp(-\sigma T)) N_k \exp(-\sigma T). \quad (9)$$

It follows from the second equation of (8) and $\Delta M = 0$ that

$$\frac{dM}{dt} = -(\beta N_k \exp(-\sigma t) - (\delta + \sigma))M + \beta$$

and $M_{k+1} = M(T) = M(T^+)$, that is

$$M_{k+1} = \exp\left( -\int_0^T (\beta N_k \exp(-\sigma t) - (\delta + \sigma)) \, dt \right) \times \left( M_k + \beta \int_0^T \exp\left( \int_0^t (\beta N_k \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau \right) \, dt \right). \quad (10)$$

From (9) and (10), we obtain the following discrete map:

$$\begin{cases}
N_{k+1} = (1 + b - cN_k \exp(-\sigma T)) N_k \exp(-\sigma T), \\
M_{k+1} = \exp\left( -\int_0^T (\beta N_k \exp(-\sigma t) - (\delta + \sigma)) \, dt \right) \\
\quad \times \left( M_k + \beta \int_0^T \exp\left( \int_0^t (\beta N_k \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau \right) \, dt \right).
\end{cases} \quad (11)$$

For each fixed point of the map (11) there is an associated periodic solution of system (2), and vice versa. For the facts that $N(t) = S(t) + I(t)$ and $M(t) = \frac{1}{I(t)}$, the discussion of equilibrium $(0, 0)$ of map (11) is meaningless; we omit it here. Now suppose that the positive fixed point of map (11) is $(N_0, M_0)$; then

$$\begin{cases}
N_0 = (1 + b - cN_0 \exp(-\sigma T)) N_0 \exp(-\sigma T), \\
M_0 = \exp\left( -\int_0^T (\beta N_0 \exp(-\sigma t) - (\delta + \sigma)) \, dt \right) \\
\quad \times \left( M_0 + \beta \int_0^T \exp\left( \int_0^t (\beta N_0 \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau \right) \, dt \right),
\end{cases}$$
Suppose the following conditions hold:

\[
\begin{align*}
\beta &> \exp(\sigma T), \\
\int_0^T \left(\beta N_0 \exp(-\sigma t) - (\delta + \sigma)\right) dt &> 0, \\
\exp\left(\int_0^T \left(\beta N_0 \exp(-\sigma t) - (\delta + \sigma)\right) dt - 1\right) &< 1,
\end{align*}
\]

(12)

For \( N_0 > 0 \) and \( M_0 > 0 \), the following conditions are needed:

\[
\begin{align*}
1 + b - \exp(\sigma T) &> 0, \\
\int_0^T (\beta N_0 \exp(-\sigma t) - (\delta + \sigma)) dt &> 0,
\end{align*}
\]

(13)

that is,

\[
\frac{\beta}{c\sigma} (1 + b - \exp(\sigma T))(\exp(\sigma T) - 1) > (\delta + \sigma)T \quad \text{for } T < \frac{1}{\sigma} \ln(1 + b).
\]

(14)

Hence there exists a nonlinear periodic solution in system (8) under condition (14).

Now we discuss the stability of this positive periodic solution. The associated characteristic polynomial of the fixed point \( (N_0, M_0) \) is given by

\[
\begin{vmatrix}
\lambda - (1 + b) \exp(-\sigma T) + 2cN_0 \exp(-2\sigma T) & 0 \\
-\sigma a_{21} & \lambda - \exp\left(-\int_0^T (\beta N_0 \exp(-\sigma t) - (\delta + \sigma)) dt\right)
\end{vmatrix}
\]

and

\[
\lambda_1 = 2 - (1 + b) \exp(-\sigma T), \quad \lambda_2 = \exp\left((\delta + \sigma)T - \frac{\beta}{c\sigma} (1 + b - \exp(\sigma T))(\exp(\sigma T) - 1)\right).
\]

It is easy to calculate that \(-1 < \lambda_1 < 1\) for \(0 < 1 + b - \exp(\sigma T) < 2 \exp(\sigma T)\). Further, it follows from (14) that \(1 + b - \exp(\sigma T) > 0\) and

\[
(\delta + \sigma)T - \frac{\beta}{c\sigma} (1 + b - \exp(\sigma T))(\exp(\sigma T) - 1) < 0,
\]

and then \(0 < \lambda_2 < 1\).

Thus, under condition (14) and \(1 + b - \exp(\sigma T) < 2 \exp(\sigma T)\), that is,

\[
1 + b < 3 \exp(\sigma T),
\]

(15)

\(-1 < \lambda_1 < 1\) and \(0 < \lambda_2 < 1\), which means that the periodic solution of system (8) is stable. Thus the following proposition about the existence and stability of an epidemic periodic solution of system (2) is obtained.

**Proposition 4.1.** Suppose the following conditions hold:

\begin{itemize}
  \item [(H1)] \(\exp(\sigma T) < 1 + b < 3 \exp(\sigma T)\),
  \item [(H2)] \(\frac{\beta}{c\sigma} (1 + b - \exp(\sigma T))(\exp(\sigma T) - 1) > (\delta + \sigma)T\).
\end{itemize}

System (2) has a stable positive periodic solution.

5. **Bifurcation analysis**

5.1. **Bifurcation of the positive periodic solution near the infection-free periodic solution**

In this subsection, we deal with the problem of the bifurcation of the positive periodic solution of system (2) near the infection-free periodic solution. In the following, \(b\) is viewed as a parameter. As shown in Section 3, the infection-free periodic solution of system (2) is

\[
\begin{align*}
\bar{S}(t) &= \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c} \exp(\sigma (t - nT)), \quad nT \leq t < (n + 1)T, \\
\bar{I}(t) &= 0, \quad nT \leq t < (n + 1)T.
\end{align*}
\]

(16)

It follows from (7) and (11) that the positive periodic solution of system (2) is

\[
\begin{align*}
\hat{S}(t) &= \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c} \exp(\sigma (t - nT)) - \bar{I}(t), \quad nT \leq t < (n + 1)T, \\
\hat{I}(t) &= \exp\left(\int_0^{t-nT} \left(\beta \left(1 + b - \exp(\sigma T)\right) \exp(\sigma T) \exp(-\sigma t) - (\delta + \sigma)\right) dt\right) \\
&\quad \times \left(M_0 + \beta \int_0^{t-nT} \exp\left(\int_0^t \beta N_k \exp(-\sigma \tau) - (\delta + \sigma)d\tau\right) dt\right)^{-1}, \quad nT \leq t < (n + 1)T.
\end{align*}
\]

(17)

where \(M_0\) is shown in (12).
Let $N(t) = S(t) + I(t)$; system (2) may be rewritten as
\[
\begin{cases}
\dot{N} = -\sigma N := G_1(N, I), \\
\dot{I} = \beta(N - I)I - \delta I - \sigma I := G_2(N, I), \\
N(0) = (1 + b - cN(0))N(0) := \theta_1(N(0), I(0)), \\
I(0) = \theta_2(N(0), I(0)).
\end{cases}
\tag{18}
\]
which are satisfied with $G_2(N, 0) \equiv 0$, $\theta_2(N, 0) \equiv 0$, $\theta_1 \neq 0$ for $N \neq 0$ and $\theta_2 \neq 0$ for $I \neq 0$.

Let $\psi$ be the flow associated with (18), $U(t) = \psi(t, N_0, I_0)$, $0 < t \leq T$, where $U_0 = U(N_0, I_0), N_0 = N(0), I_0 = I(0)$. The following notation is given in paper [17];
\[
a' = 1 - \left( \frac{\partial \theta_1}{\partial R} \cdot \frac{\partial \psi_1}{\partial t} \right)(T, U_0), \quad b' = 1 - \left( \frac{\partial \theta_2}{\partial I} \cdot \frac{\partial \psi_2}{\partial t} \right)(T, U_0), \\
B = -\frac{\partial^2 \theta_2}{\partial \theta \partial I} \left( \frac{\partial \psi_1(T, U_0)}{\partial t} + \frac{\partial \psi_1(T, U_0)}{\partial R} \frac{\partial \psi_1(T, U_0)}{\partial t} \right) \frac{\partial \psi_2(T, U_0)}{\partial t} - \frac{\partial \theta_2}{\partial I} \frac{\partial \psi_1(T, U_0)}{\partial t} \frac{\partial \psi_2(T, U_0)}{\partial I} - \frac{\partial \theta_2}{\partial I} \frac{\partial \psi_2(T, U_0)}{\partial t} \frac{\partial \psi_2(T, U_0)}{\partial I}, \\
C = -2 \frac{\partial^2 \theta_2}{\partial \theta \partial I} \left( \frac{\partial \psi_1(T, U_0)}{\partial t} \frac{\partial \psi_1(T, U_0)}{\partial I} \right) \frac{\partial \psi_2(T, U_0)}{\partial I} + 2 \frac{\partial \theta_2}{\partial I} \frac{\partial \psi_2(T, U_0)}{\partial t} \frac{\partial \psi_2(T, U_0)}{\partial I} \frac{\partial \psi_2(T, U_0)}{\partial I} - \frac{\partial \theta_2}{\partial I} \frac{\partial \psi_2(T, U_0)}{\partial I} \frac{\partial \psi_2(T, U_0)}{\partial I}.
\]
The result concerning the bifurcation of the positive periodic solution is the following Lemma:

**Lemma 5.1.** If $|1 - a'| < 1$ and $d' = 0$, then we have:
(a) if $BC \neq 0$, then we have a bifurcation. Moreover, we have a bifurcation of a positive periodic solution if $BC < 0$ and a subcritical case if $BC > 0$;
(b) if $BC = 0$, then we have an undetermined case.

For more details see Ref. [17]. The semi-trivial periodic solution and the positive periodic solutions are given in (16) and (17); then the bifurcation of the positive periodic solution does occur in our case. In what follows, we just calculate the value of the parameter $b$ at which the bifurcation occurs. In our case,
\[
N(t) = \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c} \exp(\sigma t), \\
d' = 1 - \left( \frac{\partial \theta_2}{\partial I} \cdot \frac{\partial \psi_2}{\partial t} \right)(T, U_0) = 1 - \exp \left( \beta \int_0^T N(t) dt - (\sigma + \delta)T \right).
\]
The condition $d' = 0$ gives
\[
\beta \int_0^T N(t) dt - (\sigma + \delta)T = \beta \int_0^T \frac{(1 + b - \exp(\sigma T)) \exp(\sigma T)}{c} \exp(-\sigma t) dt - (\sigma + \delta)T = 0,
\]
that is,
\[
b = \frac{c \sigma (\sigma + \delta)T}{\beta \exp(\sigma T)(\exp(\sigma T) - 1)} + \exp(\sigma T) - 1 \equiv b^*.
\tag{19}
\]
Then we get the following result.

**Proposition 5.1.** In system (2), the positive periodic solution (17) bifurcates from the infection-free periodic solution (16) at $b = b^*$, where $b^*$ is shown in (19).

### 5.2. Flip bifurcation

For $b = 3 \exp(\sigma T) - 1$, one of the eigenvalues of the fixed point $(N_0, M_0)$ (12) is $\lambda_1 = 2 - (1 + b) \exp(\sigma T) = -1$. As mentioned in Section 3, the bifurcation associated with the appearance of the eigenvalue $-1$ is called a flip (or period-doubling) bifurcation. Hence $(N_0, M_0, 3 \exp(\sigma T) - 1)$ is a candidate for a flip bifurcation point. Viewing $b$ as a parameter, we discuss flip bifurcation of system (2) by using the map (11) and the following lemma.

**Lemma 5.2.** Let $f_{\mu} : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of a map such that $f_{\mu_0}$ has a fixed point $x_0$ with eigenvalue $-1$. Assume the following conditions:

1. $\left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial \mu} + 2 \frac{\partial^2 f}{\partial x \partial \mu} \right) \neq 0$ at $(x_0, \mu_0)$;
2. $\bar{a} = \frac{1}{2} \left( \frac{\partial f}{\partial x} \right)^2 + \frac{1}{3} \left( \frac{\partial^2 f}{\partial x^2} \right) \neq 0$ at $(x_0, \mu_0)$.
Then there is a smooth curve of fixed points of \( f_b \) passing through \((x_0, \mu_0)\), the stability of which changes at \((x_0, \mu_0)\). There is also a smooth curve \( \gamma \) passing through \((x_0, \mu_0)\) such that \( \gamma \setminus (x_0, \mu_0) \) is a union of hyperbolic period-2 orbits.

For the proof of Lemma 5.2, refer to Ref. [25]. In (F2) the sign of \( \dot{a} \) determines the stability and the direction of bifurcation of the orbits of period 2. If \( \dot{a} \) is positive, the orbits are stable; if \( \dot{a} \) is negative they are unstable.

**Proposition 5.2.** A flip bifurcation occurs at \( b = 3 \exp(\sigma T) - 1 \) in system (2). For some \( \epsilon > 0 \), system (2) has a stable \( 2T \)-periodic solution for \( b \in (3 \exp(\sigma T) - 1, 3 \exp(\sigma T) - 1 + \epsilon) \).

**Proof.** It follows from Section 4 that one of the eigenvalues of the fixed point \((N_0, M_0)\) is \( \lambda_1 = -1 \) at \( b = 3 \exp(\sigma T) - 1 \), and \((N_0, M_0, 3 \exp(\sigma T) - 1) \) is a candidate for a flip bifurcation point. Map (11) may be rewritten as

\[
F_b : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1 + b - cx \exp(-\sigma T))x \exp(-\sigma T) \\ A(x)y + B(x) \end{pmatrix},
\]

where

\[
A(x) = \exp\left(-\int_0^T (bx \exp(-\sigma t) - (\delta + \sigma)) \, dt\right)
\]

\[
B(x) = \beta \exp\left(-\int_0^T (bx \exp(-\sigma t) - (\delta + \sigma)) \, dt\right) \int_0^T \exp\left(\int_0^t (\beta x \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau\right) \, dt.
\]

The map \( f_n : \mathbb{R} \rightarrow \mathbb{R} \) in Lemma 5.2 is one-dimensional while the map (20) is two-dimensional in our case. The center manifold theorem is used here to obtain a one-dimensional map; hence, flip bifurcation is discussed.

For \( b = 3 \exp(\sigma T) - 1 \), it follows from (12) that the fixed point of map (20) is

\[
(x_0, y_0) = \left(\frac{2 \exp(2\sigma T)}{c}, \frac{\beta}{\sigma} \int_0^T \frac{2 \exp(2\sigma T)}{c} \exp(-\sigma t) - (\delta + \sigma) \, dt \right) \int_0^T \exp\left(\int_0^t (\beta x \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau\right) \, dt - 1\right).
\]

Letting \( u = x - x_0, v = y - y_0 \) and \( \bar{b} = b - 3 \exp(\sigma T) + 1 \), we transform the fixed point \((x_0, y_0)\) of map (20) to the origin; then the map (20) becomes

\[
\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} -u + \frac{2 \exp(\sigma T)}{c} \bar{b} + \exp(-\sigma T)\bar{b}u - c \exp(-2\sigma T)u^2 \\ A(u + x_0)v + \bar{B}(u + x_0) \end{pmatrix}.
\]

where \( \bar{B}(u + x_0) = A(u + x_0)y_0 + B(u + x_0) - y_0. \)

On using

\[
y_0 = A(x_0)y_0 + B(x_0),
\]

\[
\exp(ax) = 1 + ax + O(|x|^2),
\]

\[
A(u + x_0) = \exp\left(-\int_0^T (\beta(u + x_0) \exp(-\sigma t) - (\delta + \sigma)) \, dt\right) = \exp\left(\frac{\beta(\exp(\sigma T) - 1)u}{\sigma}\right) A(x_0),
\]

\[
B(u + x_0) = \exp\left(\frac{\beta(\exp(\sigma T) - 1)u}{\sigma}\right) B(x_0) + A_1 u + O(|u|^2),
\]

(23) becomes

\[
\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ A_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u, v, \bar{b}) \\ g_1(u, v, \bar{b}) \end{pmatrix},
\]

where

\[
A_1 = \exp\left(\frac{\beta(\exp(\sigma T) - 1)u}{\sigma}\right) \int_0^T \exp\left(\int_0^t (\beta(x_0) \exp(-\sigma \tau) - (\delta + \sigma)) \, d\tau\right) \, dt,
\]

\[
f_1(u, v, \bar{b}) = \frac{2 \exp(\sigma T)}{c} \bar{b} + \exp(-\sigma T)\bar{b}u - c \exp(-2\sigma T)u^2,
\]

\[
g_1(u, v, \bar{T}) = \frac{\beta(\exp(\sigma T) - 1)}{\sigma} uv + O(|u|^2 + |v|^2).
If we let
\[
J = \begin{pmatrix}
\frac{1}{A_1} & 0 \\
1 + A_0 & \frac{1}{A_0}
\end{pmatrix}, \quad \text{where } A_0 = A(x_0),
\] (25)
and use the translation \( \left( \frac{u}{v} \right) = J \left( \frac{\tilde{x}}{\tilde{y}} \right) \), then map (24) becomes
\[
\left( \frac{\tilde{x}}{\tilde{y}} \right) \mapsto \left( \begin{array}{cc}
-1 & 0 \\
0 & A_0
\end{array} \right) \left( \frac{\tilde{x}}{\tilde{y}} \right) + \left( \begin{array}{c}
f(\tilde{x}, \tilde{y}, \tilde{b}) \\
g(\tilde{x}, \tilde{y}, \tilde{b})
\end{array} \right),
\] (26)
where
\[
f(\tilde{x}, \tilde{y}, \tilde{b}) = f_1 \left( \tilde{x}, -\frac{A_1}{1 + A_0} \tilde{x} + \frac{1}{A_0} \tilde{y}, \tilde{b} \right) = \frac{2 \exp(\sigma T)}{c} \tilde{b} + \exp(-\sigma T) \tilde{b} \tilde{x} - c \exp(-2\sigma T) \tilde{x}^2,
\]
g(\tilde{x}, \tilde{y}, \tilde{b}) = \frac{A_1 A_0}{1 + A_0} f_1 \left( \tilde{x}, -\frac{A_1}{1 + A_0} \tilde{x} + \frac{1}{A_0} \tilde{y}, \tilde{b} \right) + A_0 g_1 \left( \tilde{x}, -\frac{A_1}{1 + A_0} \tilde{x} + \frac{1}{A_0} \tilde{y}, \tilde{b} \right).

Now the center manifold theorem is used to determine the nature of the bifurcation of the fixed point \((\tilde{x}, \tilde{y}) = (0, 0)\) at \( \tilde{b} = 0 \). There exists a center manifold for (26) which can be represented as follows:
\[
W^c(0) = \left\{ \left( \tilde{x}, \tilde{y}, \tilde{T} \right) \in \mathbb{R}^3 | \tilde{y} = h(\tilde{x}, \tilde{b}), h(0, 0) = 0, Dh(0, 0) = 0 \right\}.
\]
In view of the form of \( f(\tilde{x}, \tilde{y}, \tilde{b}) \), it is not necessary to calculate \( h(\tilde{x}, \tilde{b}) \) in our case, and the map restricted to the center manifold is given by
\[
\tilde{f} : \tilde{x} \mapsto -\tilde{x} + \frac{2 \exp(\sigma T)}{c} \tilde{b} + \exp(-\sigma T) \tilde{b} \tilde{x} - c \exp(-2\sigma T) \tilde{x}^2.
\] (27)

Thus
\[
\left( \frac{\partial f}{\partial \tilde{b}} \frac{\partial^2 \tilde{f}}{\partial \tilde{x}^2} + 2 \frac{\partial^2 \tilde{f}}{\partial \tilde{x} \partial \tilde{b}} \right) = \frac{2 \exp(\sigma T)}{c} \cdot (-2 c \exp(-2\sigma T)) + 2 \exp(-\sigma T)
\]
\[
= -2 \exp(-\sigma T) \neq 0
\]
and
\[
\bar{a} = \frac{1}{2} \left( \frac{\partial^2 \tilde{f}}{\partial \tilde{x}^2} \right)^2 + \frac{1}{3} \left( \frac{\partial^3 \tilde{f}}{\partial \tilde{x}^3} \right)^2 = 2c^2 \exp(-4\sigma T) > 0
\]
at \( (\tilde{x}, \tilde{b}) = (0, 0) \). Then conditions (F1) and (F2) hold. So a flip bifurcation occurs in view of Lemma 5.2. A positive 2T-periodic solution bifurcates from the positive 1T-periodic solution at \( \tilde{b} = 0 \), that is,
\[
\tilde{b} = 3 \exp(\sigma T) - 1 := b^{**}.
\] (28)
Since \( \bar{a} > 0 \) in (F2), then the positive 2T-periodic solution is stable. This also means that for some \( \epsilon > 0 \), system (2) has a positive stable 2T-periodic solution for \( b \in (3 \exp(\sigma T) - 1, 3 \exp(\sigma T) - 1 + \epsilon) \). Thus we complete the proof of Proposition 5.2.

6. Numerical results

Now consider the following example:
\[
\begin{align*}
\dot{S} &= -0.3S - 0.2SI + 0.6I, \\
\dot{I} &= 0.2SI - 0.3I - 0.6I, \\
\Delta S &= (b - c(S + I))(S + I), \\
\Delta I &= 0,
\end{align*}
\] (29)
\[
t \neq nT,
\]
\[
t = nT.
\]

In our case, \( \sigma = 0.3, \beta = 0.2, \delta = 0.6 \). In the following, the infection-free periodic solution and epidemic periodic solution, bifurcation diagram, and detailed results about the existence of chaos are given to illustrate the theoretical analysis.

Set \( b = 0.8, c = 0.5, I(t) = 0, \) and the initial point \((0.3, 0, \frac{1}{0.3} \ln(1 + b) = \frac{1}{0.3} \ln(1.8) \approx 1.96. \) The time series of \( S \) of system (29) for \( T = 1.3 \) and \( T = 2.3 \) are shown in Fig. 1. It is seen that the infection-free periodic solution \((S(t), 0)\) is stable for \( T = 1.3 \) and the trivial solution is unstable for \( T = 2.3 \), which verifies Proposition 3.1.
Thus conditions (H1) and (H2) hold. From Proposition 4.1, system (29) has a stable positive periodic solution (see Fig. 2(a)). It is seen from Fig. 2(b) that the solution with the initial point (4, 0.96) of system (29) tends to this positive periodic solution with $t$ increasing.

Now set $c = 0.5, T = 0.5$. From (19) and (27),

$$b^* = \frac{c \alpha (\sigma + \delta) T}{\beta \exp(\sigma T)(\exp(\sigma T) - 1)} + \exp(\sigma T) - 1 \approx 1.9568$$

$$b^{**} = 3 \exp(\sigma T) - 1 \approx 2.4855.$$  

Fig. 3 shows the bifurcation diagram of stable periodic solutions of system (29) with respect to parameter $b$. In Fig. 3(a), the epidemic periodic solution bifurcates from the infection-free periodic solution at $b^* = 1.9568$. For $b \in (1.9568, 2.4855)$, the infection-free periodic solution is unstable and the epidemic periodic solution is stable. At $b^{**} = 2.4855$, flip bifurcation, that is, period-adding bifurcation, occurs. A period-2 solution bifurcates from the epidemic periodic solution. It is seen from Fig. 3(a) that the numerical results are in good agreement with the theoretical results and system (29) possesses rich dynamics including different kinds of bifurcation and periodic windows. Fig. 3(b) shows the period-3 window. The period-3 solution is stable for $b \in (3.448, 3.462)$ and another flip bifurcation occurs at $b = 3.462$.

It is seen from the bifurcation diagram Fig. 3 that there exist periodic solutions. Fig. 4 shows the $1T$-periodic, $2T$-periodic, and $3T$-periodic solutions in system (29) with $b = 2.45$, $b = 2.492$, and $b = 3.46$, respectively. Both theoretical and experimental investigation have revealed that the three main routes to chaos are the route via torus bifurcation, the period-doubling route, and intermittency. From Fig. 3(a), we know that the route to chaos is period-doubling bifurcation in our case. Fig. 4(d) shows that chaos does exist in system (29) for $b = 3.2$.

7. Conclusion

The dynamics of an SIS epidemic model with birth pulses and a varying population was studied in this paper. It was seen that the dynamics of impulsive system (2) is very rich and interesting, although the corresponding system (1) without impulses is very simple. The existence and stability of the infection-free periodic solution ($S(t), 0$) and positive periodic
solution \((S(t), I(t))\) are investigated. The conditions of existence for the bifurcation of the positive periodic solution are derived by virtue of the bifurcation theorem.

It is shown that the trivial solution of the system is stable for \(T > \frac{1}{\sigma} \ln(1+b)\), the infection-free periodic solution is stable for \(\frac{1}{\sigma} \ln \left(\frac{1+b}{3}\right) < T < \frac{1}{\sigma} \ln(1+b)\), and the time period between two consecutive births \(T\) is important for the population \(S(t) + I(t)\). For avoiding extinction, \(T\) should be less than \(\frac{1}{\sigma} \ln(1+b)\). A positive periodic solution (period 1) bifurcates from this infection-free periodic solution at \(b = b^* (19)\) through a supercritical bifurcation; a period-2 solution bifurcates from this positive periodic solution (period 1) at \(b = 3 \exp(\sigma T) - 1\) through flip bifurcation (period-doubling bifurcations). The numerical results show that the chaotic solution is generated via a cascade of period-doubling bifurcations.

Acknowledgements

We would like to thank the anonymous referees very much for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (Nos 10572011 and 10871074) and the Science Foundation of Guangxi Province, China (No. 0832244).
References

[1] W.O. Kermack, A.G. McKendrick, A contribution to the Mathematical theory of epidemics, Proceedings of the Royal Society of London 115 (1927) 700–721.
[2] H. Shi, Z. Duan, G. Chen, An SIS model with infective medium on complex networks, Physica A 387 (2008) 2133–2144.
[3] L. Rass, Asymptotic results for a multi-type contact birth–death process and related SIS epidemic, Mathematical Biosciences 208 (2007) 552–570.
[4] M. Iannelli, M.Y. Kim, E.J. Park, Asymptotic behavior for an SIS epidemic model and its approximation, Nonlinear Analysis 35 (1999) 797–814.
[5] J. Li, Z. Ma, F. Brauer, Global analysis of discrete-time SI and SIS epidemic models, Mathematical Biosciences and Engineering 4 (4) (2007) 699–710.
[6] X. Zhang, X. Liu, Backward bifurcation and global dynamics of an SIS epidemic model with general incidence rate and treatment, Nonlinear Analysis: Real World Applications 10 (2009) 565–575.
[7] T. Zhang, J. Liu, Z. Teng, Bifurcation analysis of a delayed SIS epidemic model with stage structure, Chaos, Solitons and Fractals 38 (2008) 680–690.
[8] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[9] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive Differential Equations and Inclusions, Vol. 2, Hindawi Publishing Corporation, New York, 2006.
[10] P. Georgescu, H. Zhang, An impulsively controlled predator–pest model with disease in the pest, Nonlinear Analysis: Real World Applications doi:10.1016/j.nonrwa.2008.10.060.
[11] J.J. Nieto, D. O’Regan, Variational approach to impulsive differential equations, Nonlinear Analysis: Real World Applications 10 (2009) 680–690.
[12] P. Georgescu, C. Morosanu, Impulsive perturbations of a three-trophic prey-dependent food chain system, Mathematical and Computer Modelling 48 (2008) 975–997.
[13] Y. Zhou, H. Liu, Stability of periodic solution for an SIS model with pulse vaccination, Mathematical and Computer Modelling 38 (2003) 299–308.
[14] G. Caughley, Analysis of Vertebrate Population, John Wiley & Sons, New York, 1977.
[15] M.G. Roberts, R.R. Kao, The dynamics of an infectious disease in a population with pulses, Mathematical Biosciences 149 (1998) 23–36.
[16] S. Tang, L. Chen, Density-dependent birth rate, birth pulses and their population dynamics consequences, Journal of Mathematical Biology 290 (2002) 185–199.
[17] J.J. Nieto, R. Rodriguez Lopez, Periodic boundary value problems for non-Lipschitzian impulsive functional differential equations, Journal of Mathematical Analysis and Applications 318 (2006) 593–610.
[18] P. Georgescu, H. Zhang, L. Chen, Bifurcation of nontrivial periodic solutions for an impulsively controlled pest management mode, Applied Mathematics and Computation 202 (2) (2008) 675–687.
[19] G. Jiang, Q. Lu, L. Qian, Complex dynamics of a Holling type II prey–predator system with state feedback control, Chaos, Solitons and Fractals 31 (2007) 448–461.
[20] A. Lakmeche, O. Arino, Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment, Dynamics of Continuous, Discrete and Impulsive System 7 (2000) 265–287.
[21] Z. Lu, X. Chi, L. Chen, The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission, Mathematical and Computer Modelling 36 (2002) 1039–1057.
[22] S. Gakkhar, K. Negi, Pulse vaccination in SIRS epidemic model with non monotonic incidence rate, Chaos, Solitons and Fractals 35 (2008) 626–638.
[23] M.C. Nucci, P.G.L. Leach, An integrable SIS model, Journal of Mathematical Analysis and Applications 290 (2004) 506–518.
[24] Zhijun Liu, L. Chen, Periodic solution of a two-species competitive system with toxicant and birth pulse, Chaos, Solitons and Fractals 32 (2007) 1703–1712.
[25] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, in: Applied Mathematical Sciences, vol. 42, Springer-Verlag, New York, 1983.