Exponential tropical varieties and complex
Monge-Ampere operator*

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Sometimes it is possible to extend some using Newton polyhedra computations in algebraic geometry from polynomials to exponential sums. For this purpose it is useful to consider analogues of tropical varieties in complex space. These analogues are called exponential tropical varieties (ETV). We construct the ring of ETV. Algebraic tropical varieties form the subring of the ring of ETV. In this paper we connect ETV with the complex Monge-Ampere operator action on the space of piecewise linear functions in complex vector space. We show that all ETV arise as results of such operator action. We give some applications of this connection. One of the applications is a criterion for zero value of a mixed Monge-Ampere operator. This criterion is the modification of the criterion for zero value of a mixed volume of convex bodies. The proof is the modification of A. Khovanskii’s unpublished proof of the corresponding theorem on mixed volumes. In the part 1 we give the definition of ETV and detail statements of theorems (without proofs). In the part 2 we prove the theorems on the action of the Monge-Ampere operator.

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1 Definition and basic properties of ETV

1.1 The definition of ETV

Let $X$ be a finite set of closed convex (not necessarily bounded) $k$-dimensional polytopes in a real vector space $E$. We say that $X$ is a $k$-dimensional polyhedral set, if intersection of any two polytopes either is empty or is their common face. Any face of any polytope is called a cell. By default $E$ is the space $\mathbb{C}^n$ considered as a real vector space. Also, we identify the dual space $E^*$ with $\mathbb{C}^n$ using the pairing $(z, z^*) = \Re\langle z, z^* \rangle$. By definition, a chain of degree $m$ on $X$ is the odd function on the set of oriented $k$-dimensional cells taking $\Delta$ to $X_{\Delta} \in \bigwedge_\mathbb{C}^m \mathbb{C}^n$ (the function is called odd, if it’s value at the argument $\Delta$ changes sign with the changing of orientation of $\Delta$). The polyhedral set with a fixed chain we call a framed polyhedral set. The odd form $X_\Delta$ we call the frame of a cell $\Delta$.

Definition 1.1. The union of $k$-dimensional cells $\Delta \in X$ with nonzero frames $X_\Delta$ is denoted supp $X$ and is called the support of the framed polyhedral set. Say that two framed $k$-dimensional polyhedral sets $X, Y$ with the common support are equivalent, if $X_\Delta = Y_\Theta$ for any $k$-dimensional cells $\Lambda \in X, \Theta \in Y$ with $k$-dimensional intersection.

The set of $(k - 1)$-dimensional cells of framed $k$-dimensional polyhedral set $X$ form the $(k - 1)$-dimensional polyhedral set $\partial X$. Make it framed as

$$(\partial X)_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = k} X_\Delta,$$

where the orientations of the cells $\Lambda$ and $\Lambda$ agreed as usual. The framed polyhedral set is called closed if supp $\partial X = \emptyset$.

Corollary 1.1. The framed polyhedral set $\partial X$ is closed.

Let $E_\Delta$ be a tangent space of the cell $\Delta$ and $\mathbb{C}_\Delta$ be a maximal complex subspace of $E_\Delta$.

Definition 1.2. Let $k \geq n$ and $X$ be a closed framed $k$-dimensional polyhedral set with the chain of degree $2n - k$. Say that $X$ is an exponential tropical polyhedral set (ETP), if for any $k$-dimensional cell $\Delta$
(1) the restriction $X_{\Delta;R}$ of the form $X_{\Delta}$ to the space $E_{\Delta}$ is real valued, i.e. $X_{\Delta;R} \in \bigwedge_{R}^{2n-k} E_{\Delta}$;
(2) $X_{\Delta}(\xi_1, \cdots, \xi_k) = 0$ if $\exists i: \xi_i \in C_{\Delta}$.

Remark 1.1. Condition (2) follows from condition (1).

Remark 1.2. In [1] we consider the exponential tropical varieties with polynomial weights. So in [1] for ETP $X$ the frame $X_{\Delta}$ is an exterior form multiplied by a polynomial in the space $E_{\Delta}$. The corresponding construction of tropical geometry see in [4].

Definition 1.3. Say that a real subspace $E$ of $\mathbb{C}^n$ is degenerate, if $\text{codim}_{\mathbb{C}} E < \text{codim} E$, where $E_{\mathbb{C}}$ is the maximal complex subspace of $E$. Say that the cell $\Delta$ is degenerate, if the space $E_{\Delta}$ is degenerate.

If $k = 2n, 2n - 1$, then any $k$-dimensional subspace is nondegenerate. If $k < n$, then any $k$-dimensional subspace is degenerate. If the $k$-dimensional subspace $E$ is nondegenerate, then $\dim E - \dim_{\mathbb{C}} E_{\mathbb{C}} = 2n - k$.

Corollary 1.2. Let $\Delta$ be a $k$-dimensional degenerate cell of $k$-dimensional ETP $X$. Then $X_{\Delta} = 0$.

Definition 1.4. The equivalence class of ETP is called exponential tropical variety (ETV). If the dimension of ETP is $k$, then (by definition) the dimension of ETV is $k - n$.

In what follows the record $X \Rightarrow \mathcal{P}$ means that ETP $X$ lies in the equivalence class ETV $\mathcal{P}$.

Let $X$ be a ETP. We define the current of measure type $\bar{X}$ as

$$\bar{X}(\varphi) = \sum_{\Delta \in X, \dim \Delta = k} \int_{\Delta} X_{\Delta;R} \wedge \varphi. \quad (1.2)$$

Corollary 1.3. ETP $X, Y$ are equivalent, if and only if $\bar{X} = \bar{Y}$.

Corollary 1.4. Let $\mathcal{P}$ be a $d$-dimensional ETV. Then $\bar{\mathcal{P}}$ is a current of bedegree $(d, d)$. If $d > 0$, the the current $\bar{\mathcal{P}}$ is closed.

Example 1.1. Corner loci of piecewise linear functions. The continuous function $h: \mathbb{C}^n \rightarrow \mathbb{R}$ is called piecewise linear, if it is a real polynomial of degree 1 on any $P \in \{P\}$, where $\{P\}$ is a finite set of convex polytopes such that $\bigcup_{P \in \{P\}} P = \mathbb{C}^n$. Let the support of $(2n - 1)$-dimensional polyhedral set $X$ be a corner locus of a piecewise linear function $h$. Then any $(2n - 1)$-dimensional cell $\Delta \in X$ in locally (near any internal point of $\Delta$) is a
common face of two halfspaces $A$ and $B$. Let $h_A = h|_A$ and $h_B = h|_B$. The ordering of the pair $(A, B)$ sets the coorientation of $\Delta$. The standard orientation of $\mathbb{C}^n$ and the coorientation of $\Delta$ together set the orientation of the cell $\Delta$. Using this orientation put $X_{\Delta;R} = d^c h_A - d^c h_B$ (remind that $d^c g(x_i) = dg(ix_i)$). Easy to verify that $\overline{P} = dd^c h$, where $X \Rightarrow P$. For any $(n - 1)$-dimensional ETV $P$ there exists a piecewise linear function $h$ such that $\overline{P} = dd^c h$ (Theorem 2.1).

1.2 Addition of ETV

Let $Q_1, Q_2$ be $(n - k)$-dimensional ETV and $X_i \Rightarrow Q_i$. There is a $k$-dimensional polyhedral set $X$ such that

1. supp $X$ = supp $X_1 \cup$ supp $X_2$

2. if $\Delta \cap \Xi \neq \emptyset$, where $\Delta$ and $\Xi$ are cells of polyhedral sets $X$ and $X_i$, then $\Delta \cap \Xi$ is a cell of polyhedral set $X$.

Let $\Delta \in X$ be a $k$-dimensional cell. Set $X_\Delta$ equal to the sum of (one or two) frames of the cells (one or two) of polyhedral sets $X_i$ containing $\Delta$.

**Corollary 1.5.** The polyhedral set $X$ is ETP and $\bar{X} = \bar{X_1} + \bar{X_2}$.

The equivalence class of $X$ does not depend on the choice of $X_i$ (corallary 1.3). Now define $Q_1 + Q_2 = Q$, where $X \Rightarrow Q$.

**Corollary 1.6.** $m$-dimensional ETV form the commutative group and the map $P \mapsto \overline{P}$ is injection.

Below we consider the formally defined addition of ETV of all dimensions as a graded group. By definition, the degree of $m$-dimensional ETV is $n - m$.

Using the odd volume form $\omega$ of a real vector space we can define the volume of any bounded domain $U$ as $\int_U \omega$. Indeed, this integral does not depend on the choice of orientation. Say that the odd volume form $\omega$ is positive (negative) if $\omega$ give positive (negative) volumes of bounded domains.

**Definition 1.5.** Let $\Delta$ be a $k$-dimensional cell of a $k$-dimensional ETP $X$. Say that the cell $\Delta$ is positive (negative), if the direct image of the form $X_{\Delta;R}$ on the space $E_{\Delta}/\mathbb{C}_\Delta$ is positive (negative) volume form. The construction of direct image of the odd form $X_{\Delta;R}$ requires the coordination of the choice of orientations of the spaces $E_{\Delta}$ and $E_{\Delta}/\mathbb{C}_\Delta$. We do it using the orientation of $\mathbb{C}_\Delta$ as the standard orientation of a complex vector space.

**Definition 1.6.** ETP with all nonnegative cells is called positive. If $X \Rightarrow P$ and ETP $X$ is positive, then ETV $P$ also is called positive.

**Corollary 1.7.** Any ETV is the difference of two positive ETV.
Indeed, let $X \Rightarrow P$. Let $\Delta \in X$ be a $k$-dimensional cell with nonzero frame $X_\Delta$. Consider the single-celled ETP $X^\Delta$ with the cell $E_\Delta$ and its frame $c_\Delta X_\Delta$, where $c_\Delta$ is such a real number, that the cell $E_\Delta$ is positive. Let $X^\Delta \Rightarrow Q^\Delta$ and let $|c_\Delta|$ be sufficiently large. Then the ETV $P + \sum_\Delta Q^\Delta$ is positive and $P = (P + \sum_\Delta Q^\Delta) - (\sum_\Delta Q^\Delta)$.

Say that the $k$-dimensional cells $\Delta, \Lambda$ of $k$-dimensional polyhedral set are neighbor, if $\dim \Delta \cap \Lambda = k - 1$. The set of $k$-dimensional cells is called connected, if for any pair of cells $\Delta, \Lambda$ of this set there exists a sequence of cells $\Delta_1 = \Delta, \Delta_2, \cdots, \Delta_m = \Lambda$ such that $\forall i$ the cells $\Delta_i, \Delta_{i+1}$ are neighbor.

Let $\Xi_1, \cdots, \Xi_q$ be maximal connected subsets of cells of $k$-dimensional polyhedral set $X$. Let $Y_i$ be a polyhedral set formed by cells from the subset $\Xi_i$. Then $\text{supp} X = \bigcup_{1 \leq i \leq q} \text{supp} Y_i$. If $i \neq j$ then $\dim(\text{supp} Y_i \cap \text{supp} Y_j) < k - 1$. If $X$ is ETP, then the polyhedral sets $Y_i$ with inherited chains are ETP also. The latest statement is the direct corollary of definition 1.2.

Say that ETP $Y_i$ is the irreducible component of ETP $X$. If $q = 1$ then ETP $X$ is said to be irreducible. If $X \Rightarrow P$, then the irreducible components of ETV $P$ are well defined.

**Corollary 1.8.** Any ETV is the sum of it’s irreducible components. Irreducible components of positive ETV are positive.

### 1.3 ETV and convex polytops in $\mathbb{C}^n$

For a formulation of theorems 1.1, 1.2 we need some simple geometrical facts and definitions.

**Definition 1.7.** The fan of cones is a polyhedral set such that any cell is a cone with a zero vertex. If the fan is ETP, then we call it a homogeneous ETP. The corresponding ETV also is called homogeneous.

Let $\gamma$ be a convex bounded polytope in $\mathbb{C}^n$. The dual cone $\Delta$ of the face $\delta$ of polytope $\gamma$ is, by definition, the set of points $z \in \mathbb{C}^n$ such that $\max_{z^* \in \gamma} \text{Re}(z, z^*)$ is reached at any $z^* \in \delta$. If $\dim \delta = m$, then $\dim \Delta = 2n - m$. Dual cones of $m$-dimensional faces of $\gamma$ form the $(2n - m)$-dimensional fan of cones $X_{\gamma, 2n-m}$.

**Lemma 1.1.** Let $U$ be an open bounded domain of $m$-dimensional real vector space $E$. Then for any orientation $\alpha$ of $E$ there exists the only multivector $p_U(\alpha) \in \bigwedge^m E$ such that $\int_U \omega = \omega(p_U(\alpha))$ for any volume form $\omega$ of the space $E$.

It’s obviously that $p_U(-\alpha) = -p_U(\alpha)$, where $(-\alpha)$ is the different from $\alpha$ orientation of $E$. I.e. the multivector $p_U$ is odd. The odd multivector $p_U$ is called a volume of domain $U$.  


Let $E_\delta$ be a tangent space of the face $\delta$. The set of $m$-dimensional faces of polytope $\gamma$ form a polyhedral complex. The function $\delta \mapsto p_\delta \in \bigwedge^m E_\Delta \subset \bigwedge^m \mathbb{C}^n$ on the set of $m$-dimensional cells is $m$-cochain of this complex with values in $\bigwedge^m \mathbb{C}^n$.

**Lemma 1.2.** The $m$-cochain $\delta \mapsto p_\delta$ is a cocycle.

The lemma is equivalent to the Pascal conditions for $(m+1)$-dimensional faces of polytope $\gamma$. (The Pascal conditions for $k$-dimensional polytope $\gamma$ is as follows: $\sum_\delta v_\delta = 0$, where $\delta$ is $(k-1)$-dimensional face of $\gamma$ and $e_\delta$ is an external normal of the length equal to the $(k-1)$-dimensional volume of the face $\delta$).

The volume of $(m+1)$-dimensional convex polytope equals to the sum of volumes of its $m$-dimensional faces multiplied by the lengths of corresponding heights and divided by $m+1$. Using cocycle $p_\delta$ we can write it as

$$p_\delta = \frac{1}{m+1} \sum_{\theta \subset \delta, \dim \theta = m} w_\theta \wedge p_\theta,$$

where $w_\theta$ is an arbitrary point of $m$-dimensional face $\theta \subset \delta$.

**Lemma 1.3.** Consider the complex vector space $V$ as the real vector space $E$. There exists the only ring homomorphism $\varphi: \bigwedge^m E \to \bigwedge^m V$ such that the map $\varphi: E \to V$ is the identity.

Let $\delta_\mathbb{C}$ be a minimal complex subspace of $\mathbb{C}^n$ containing the real subspace $E_\delta$. Say that the face $\delta$ of polytope $\gamma$ is degenerate if $\dim \delta > \dim_\mathbb{C} \delta_\mathbb{C}$. Any face of dimension $0, 1$ is nondegenerate. Any face of dimension $> n$ is degenerate.

**Corollary 1.9.** If the face $\delta$ is degenerate, then $\varphi(p_\delta) = 0$.

**Corollary 1.10.** The $m$-cochain $\delta \mapsto \varphi(p_\delta) \in \bigwedge^m \mathbb{C}^n$ is a cocycle. If $m > n$ then this cocycle is zero.

In 2.1 we use the complex variant of formula (1.3).

$$\varphi(p_\delta) = \frac{1}{m+1} \sum_{\theta \subset \delta, \dim \theta = m} w_\theta \wedge \varphi(p_\theta),$$

where (in contrast to the formula (1.3)) we deal with complex multivectors.

For $m \leq n$ we define a homogeneous $(2n - m)$-dimensional ETP, corresponding to a polytope $\gamma \subset \mathbb{C}^n$. The cells of ETP are the cones of the fan $X^\gamma_{2n-m}$. The frame $X^\gamma_{2n-m}$ of cone $\Delta$ is constructed as follows.
Set \( X_{\Delta}^{\gamma,2n-m} = 0 \) if the face \( \delta \) is degenerate. Let the face \( \delta \) be nondegenerate. We consider the multivector \((-i)^m \rho(p_\delta)\) as an exterior \(m\)-form \( W_{\Delta}^{\gamma,2n-m} \) on \( \mathbb{C}^n \). The sign of this form depends on the choice of the face \( \delta \) orientation. To construct the frame \( X_{\Delta}^{\gamma,2n-m} \) from \( W_{\Delta}^{\gamma,2n-m} \) we must establish the correspondence of orientations of the spaces \( E_{\Delta} \) and \( E_{\delta} \).

The bilinear form \( \text{Im} \langle z, z^* \rangle \) give the nodegenerate pairing \( E_\Delta / \mathbb{C}_\Delta \otimes E_\delta \rightarrow \mathbb{R} \). Consider the corresponding symplectic form \( \omega \) on the space \( E_\Delta / \mathbb{C}_\Delta \oplus E_\delta \). Now coordinate the orientations of the spaces \( E_\Delta / \mathbb{C}_\Delta \) and \( E_\delta \) by choosing the orientation of \( \mathbb{C}_\Delta \) as the standard orientation of a complex vector space. Now the orientations of the spaces \( E_\delta \) and \( E_\Delta \) are agreed.

The conditions (1) and (2) of definition \( \text{1.2} \) satisfied by construction. The closedness of the framed polyhedral set \( X_{\Delta}^{\gamma,2n-m} \) is equivalent to the statement of Lemma \( \text{1.10} \). Thus, for \( k \geq n \) the framed fan \( X^{\gamma,k} \) is homogeneous ETP. Corresponding homogeneous ETV also denoted \( X^{\gamma,k} \).

**Theorem 1.1.** For any \( k \)-dimensional homogeneous ETV \( X \) exists a finite set \( \{ \gamma \} \) of convex polytopes in the space \( \mathbb{C}^{n*} \) such that

\[
X = \sum_{\gamma \in \{ \gamma \}} \pm X^{\gamma,k}.
\]

(1.5)

In (1.5) and (1.6) \( tX \) is, by definition, the ETP with the same cells as ETP \( X \), and the cell’s frames multiplied by \( t \).

**Theorem 1.2.** For any \( k \)-dimensional ETV \( X \) exists a finite set \( \{ \gamma \} \) of convex polytopes in the space \( \mathbb{C}^{n*} \) and the set of vectors \( \{ a_\gamma \} \) such that

\[
X = \sum_{\gamma \in \{ \gamma \}} \pm (a_\gamma + X^{\gamma,k}),
\]

(1.6)

where \( a_\gamma + X^{\gamma,k} \) is a translation of ETV \( X^{\gamma,k} \) by the vector \( a_\gamma \).

1.4 Stable intersections of ETP and multiplication of ETV.

Say that polyhedral sets \( X \) and \( Y \) are transversal, if for any pair of cells \( \Delta \in X, \Lambda \in Y \) with nonempty intersection the intersection of spaces \( E_\Delta \) and \( E_\Lambda \) is transversal.
Let $X,Y$ be transversal polyhedral sets of dimensions $p,q$. Then the pairwise intersections of cells of $X$ and $Y$ form a polyhedral set $X \cap Y$. The dimension of the polyhedral set $X \cap Y$ is equal to $p + q - 2n$ (if $X \cap Y \neq \emptyset$, then $p + q \geq 2n$). For transversal ETP $X,Y$ on a polyhedral set $X \cap Y$ with $\dim X \cap Y \geq n$ we can determine the structure of ETP as follows.

Let $X,Y$ be transversal framed polyhedral sets satisfying all the conditions of definition 1.2, except, perhaps, the condition of closedness. Let $\Delta \in X, \Lambda \in Y$ be cells of higher dimensions and let $\Xi = \Delta \cap \Lambda$. Below we define an odd form $X_\Delta \wedge Y_\Lambda$ on the cell $\Xi$. Then the frame of $\Xi$ is defined as $(X \cap Y)_\Xi = X_\Delta \wedge Y_\Lambda$.

Choose the orientations $\alpha, \beta$ of cells $\Delta, \Lambda$ and let $X_\Delta^\alpha, Y_\Lambda^\beta$ be exterior forms corresponding to the chosen orientations. We make an odd form $X_\Delta \wedge Y_\Lambda$, attributing to the form $X_\Delta^\alpha \wedge Y_\Lambda^\beta$ sign depending on the orientation of the cell $\Xi$: make the form of $X_\Delta \wedge Y_\Lambda$ positive (definition 1.5), if forms $X_\Delta, Y_\Lambda$ are both positive or both negative. Otherwise, make the form $X_\Delta \wedge Y_\Lambda$ negative.

**Corollary 1.11.** If the polyhedral sets $X$ and $Y$ of dimension $p$ and $q$ are transversal and closed, then the polyhedral set $(X \cap Y)$, framed as described above, is ETP.

Indeed, let $\Upsilon = \Delta \cap \Gamma$ be a $(p + q - 2n - 1)$-dimensional cell of the polyhedral set $X \cap Y$, where $\Delta$ is a $p$-dimensional cell polyhedral set $X$ and $\Gamma$ is $(q - 1)$-dimensional cell of polyhedral set $Y$. Then, using the definition of the boundary of a polyhedral set (equation (1.1)), get

$$
(\partial (X \cap Y))_\Upsilon = \sum_{Y \supset \Delta \supset \Gamma} X_\Delta \wedge Y_\Lambda = X_\Delta \wedge \sum_{Y \supset \Delta \supset \Gamma} Y_\Lambda = X_\Delta \wedge (\partial Y)_\Gamma = 0.
$$

Thus, the closedness of a polyhedral set $(X \cap Y)$ is a consequence of closedness of polyhedral sets $X,Y$. The remaining conditions of Definition 1.2 are obvious.

**Definition 1.8.** We call ETV $\mathcal{P}$ and $\mathcal{Q}$ transversal if there exist transversal ETP $X,Y$ such that $X \Rightarrow \mathcal{P}$ and $Y \Rightarrow \mathcal{Q}$.

Let $X \Rightarrow \mathcal{P}$ and $Y \Rightarrow \mathcal{Q}$, where ETP $X$ and $Y$ are transversal. Say that $\mathcal{P} \mathcal{Q} = \mathcal{X}$, where $X \cap Y \Rightarrow \mathcal{X}$. The product of transversal ETP is well defined.

Also, if $\dim \mathcal{P} + \dim \mathcal{Q} < n$, then (for any ETV) set $\mathcal{P} \mathcal{Q} = 0$. In any case for the definition of the product of ETV we do the following.

The additive group of the space $\mathbb{C}^n$ acts on the set of polyhedral sets by shifts. For any fixed pair of polyhedral sets $X,Y$ shifted polyhedral sets $X,z + Y$ are transversal for almost all $z \in \mathbb{C}^n$. 8
Theorem 1.3. If ETP $X$ and $z_i + Y$ are transversal, then for $z_i \to 0$ the sequence of currents $X \cap (z_i + Y)$ converges to the limit current $Z$, where $Z$ is ETP of dimension $(p + q - 2n)$. The limit current does not depend on the choice of the sequence $z_i$. If $X \Rightarrow P$ and $Y \Rightarrow Q$, then the equivalence class of ETP $Z$ gives ETV $PQ$ of dimension $(p-n) + (q-n) - n$ independent on the choice of ETP $X, Y$.

The following explains that if ETP $X, Y$ are positive, then ETP $Z$ from the statement of Theorem 1.3 is a stable intersection of ETP $X$ and $Y$.

Definition 1.9. Let $A, B$ be subsets of finite-dimensional vector space $E$. A point $x \in A \cap B$ is called stable if any neighborhood of $x$ contains the points of the set $A \cap (y + B)$ at sufficiently small shifts $y$. The set of stable points of intersection is called a stable intersection and is denoted $A \cap_{st} B$.

Locally any polyhedral set coincides with a fan of cones. Indeed, let $\Theta \in X$ and $e \in \text{Int} \Theta$, where $\text{Int} \Theta$ is the interior of the cell $\Theta$ (assuming that the only point of 0-dimensional cell is internal). The shifted polyhedral set $(X - e)$ in a neighborhood of zero coincides with the fan of cones. Denote this fan by $X_\Theta$ and call it $\Theta$-localization of polyhedral set $X$. The fan $X_\Theta$ is independent of the choice of internal point $e$ of the cell $\Theta$.

Denote by $K_{min}$ the minimal cone of $K$. A minimal cone of a fan is a subspace. Cone $K_{min}$ is contained in all cones of $K$. Minimal cone of $X_\Theta$ is the subspace $E_\Theta$. For any cell $\Theta$ of ETP $X$ the localization $X_\Theta$ is homogeneous ETP, if we equip cells of the polyhedral set $X_\Theta$ with frames inherited from $X$.

Corollary 1.12. Let $\Delta \in X, \Lambda \in Y$ and $x \in \text{Int} \Delta \cap \text{Int} \Lambda$. Then the following conditions are equivalent

1. $x \in \text{supp } X \cap \text{supp } Y$
2. $\Delta \cap \Lambda \subset \text{supp } X \cap \text{supp } Y$
3. $(X_{\Delta})_{\text{min}} \cap (Y_{\Lambda})_{\text{min}} \subset \text{supp } X_{\Delta} \cap \text{supp } Y_{\Lambda}$

Corollary 1.13. $\dim X \cap_{st} Y \leq \dim X + \dim Y - 2n$ (it is assumed that a set of negative dimension is empty).

Let $K, L$ be homogeneous ETP, $\dim K = k, \dim L = l$, $\text{supp } K \cap_{st} L = K_{\text{min}} \cap L_{\text{min}}$ and polyhedral sets $K, z + L$ are transversal. Set $V = K_{\text{min}} \cap L_{\text{min}}$.

Lemma 1.4. If ETP $K, L$ be positive, then $\dim V = k + l - 2n$.

Assume that ETP $K, L$ are positive. Let $\{\Xi_i(z)\}$ be a set of non-empty intersections $\Delta \cap (z + \Lambda)$, where $\Delta \in K$ and $\Lambda \in L$ are the cells of dimension
$k$ and $l$ respectively. Then \( \{\Xi_i(z)\} \) is a set of cells of the highest dimension of ETP $K \cap (z + L)$ (corollary 1.11). Any $\Xi_i(z)$ equals to $V + z_i$ for some $z_i \in \mathbb{C}^n$. So their frames $(K \cap L)\Xi_i(z)$ can be considered as exterior forms on $\mathbb{C}^n$ with odd dependence on the orientation of the subspace $V$. The sum of these forms is denoted by $W_{K,L}(z)$.

**Lemma 1.5.** Let $K, L$ be positive homogeneous ETP. Then the form $W_{K,L}(z)$ is constant (does not depend on $z$).

Let $X, Y$ be positive ETP, $\dim X = k, \dim L = l$. Let $\Theta \subset \supp X \cap \supp Y$, $\dim \Theta = k + l - 2n$ and $\Int \Theta = \Int \Delta \cap \Int \Lambda$, where $\Delta \in X, \Lambda \in Y$. Let $K = X_{\Delta}$ and $L = Y_{\Lambda}$. Assign to $\Theta$ the frame $W_{K,L}(z)$ from Lemma 1.5.

**Theorem 1.4.** Let $X, Y$ be positive ETP, $\dim X = k, \dim L = l$. Then all $(k + l - 2n)$-dimensional cells of polyhedral set $X \cap \supp Y$ with above described frames form the positive ETP $X \cap \supp Y$. For ETP $X \cap \supp Y$ all the statements of Theorem 1.3 for a polyhedral set $Z$ are true.

Now we define the multiplication of ETV as follows. Let $X_1, X_2, Y_1, Y_2$ be positive ETP and let $(X_1 - X_2) \Rightarrow P, (Y_1 - Y_2) \Rightarrow Q$. Then we set

\[
(X_1 \cap \supp Y_1 - X_1 \cap \supp Y_2 - X_2 \cap \supp Y_1 + X_2 \cap \supp Y_2) \Rightarrow P \cdot Q.
\]

It is easy to check that the product is well-defined.

### 1.5 Bergman fans of ETV

Let $\Delta \subset \mathbb{R}^N$ be a $k$-dimensional convex polytope. Let $\Delta^\infty$ be a convex polyhedral cone formed by the limit points of polytopes $t\Delta$ as $t \to +0$. This cone lies in the subspace $E_\Delta$ (recall: the subspace $E_\Delta$ is generated by the differences of points of the polytope $\Delta$). If $\Delta$ is bounded, then $\Delta^\infty = 0$. Else $\dim \Delta^\infty \neq 0$. Any face of the cone $\Delta^\infty$ is a cone $\Lambda^\infty$, where $\Lambda$ is a face of $\Delta$.

For a polyhedral set $X$, we set $S = \cup_{\Delta \in X} \Delta^\infty$. Let $\varphi \in S$ and let $L(\varphi)$ be a set of cells $\Delta \in X$ such that $\varphi \in \Delta^\infty$. The vectors $\varphi, \psi$ of $S$ are called equivalent, if $L(\varphi) = L(\psi)$. The equivalence classes are convex cones of dimension $\leq k$, where $k = \dim X$. Let $S^\infty$ be a closure of the union of equivalence classes, represented by $k$-dimensional cones. If $S^\infty = \emptyset$, then set $X^\infty = 0$. Else there exists a $k$-dimensional fan of cones $X^\infty$ such that

1. $\supp X^\infty = S^\infty$
2. any cone $K \in X^\infty$ is contained in some equivalence class of the set $S$.

Let $X$ be a framed $k$-dimensional polyhedral set in $\mathbb{C}^n$ and $K \in X^\infty$, $\dim K = k$. Define the frame $(X^\infty)_K$ of cone $K$ as $\sum_{\Delta \in L(\varphi)} X_\Delta$, where $\varphi$ is some interior point of the cone $K$. 

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Theorem 1.5. If $X$ is ETP, then $X^\infty$ is a homogeneous ETP. ETV corresponding to ETP $X^\infty$ depends on the equivalence class of ETP $X$ only.

Definition 1.10. Homogeneous ETV $X^\infty$ is called a Bergman fan of ETV $X$.

Theorem 1.6. The map $\beta : X \to X^\infty$ is a homomorphism of the ring of ETV to the ring of homogeneous ETV.

Corollary 1.14. If ETV $X$ is represented as (1.6), then $X^\infty = \sum_{\gamma \in \gamma} \pm X^{\gamma,k}$.

Corollary 1.15. Homomorphism $\beta$ is invariant under the action $z: X \mapsto (z + X)$ of the additive group of $\mathbb{C}^n$ on the ring of ETV. I.e. $(z + X)^\infty = X^\infty$ for any $z \in \mathbb{C}^n$.

Theorem 1.7. If $X$ is a nonzero positive ETV, then also $X^\infty$ is a nonzero positive ETV.

Corollary 1.16. Let $X,Y$ be positive ETV. Thus $XY \neq 0$, if and only if $X^\infty Y^\infty \neq 0$.

Corollary 1.17. Let $X_1, \cdots, X_m$ be positive ETV. Then $X_1 \cdots X_m = 0$, if and only if $(a_1 + X_1) \cdots (a_m + X_m) = 0$ for any $a_i \in \mathbb{C}^n$.

2 Mixed complex Monge-Ampere operator

2.1 Monge-Ampere operator and ETV

The mixed complex Monge-Ampere operator of degree $k$ is (by definition) the map $(h_1, \cdots, h_k) \mapsto dd^c h_1 \wedge \cdots \wedge dd^c h_k$ (remind that $d^c g(x_{\text{tang}}) = dg(ix_{\text{tang}})$, where $g$ is a real function on a complex manifold $M$). Below the map $(h_1, \cdots, h_k) \mapsto dd^c h_1 \wedge \cdots dd^c h_k$ is called the Monge-Ampere operator.

If $h_1, \cdots, h_k$ are continuous convex functions on $\mathbb{C}^n$, then [2] the Monge-Ampere operator value $dd^c h_1 \wedge \cdots \wedge dd^c h_k$ is well defined as a current (that is a functional on the space of smooth compactly supported differential $(2n - 2k)$-forms). This means that if the sequence of smooth convex functions $(f_i)_j$ converges locally uniformly to $h_i$, then the sequence of currents $dd^c (f_1)_j \wedge \cdots dd^c (f_k)_j$ converges to the limit current, independent on the choice of approximation. This limit current is the current of measure type, i.e. it may be continued to a functional on the space of continuous compactly supported forms. It follows that any polynomial in the variables $dd^c g_1, \cdots, dd^c g_q$
with continuous convex functions \( g_i \) is well defined as a current of measure type. It is easy to prove that any piecewise linear function can be written as a difference of two convex piecewise linear functions. It follows that the action of Monge-Ampere operator and above defined currents are well defined for any piecewise linear functions also.

**Remark 2.1.** For piecewise linear (not necessarily convex) functions the current \( dd^c h_1 \wedge \cdots \wedge dd^c h_k \) depends only on the product of functions \( h_1 \cdots h_k \) \[1\]. This property is a very specific for piecewise linear functions. However, it is partially retained for piecewise pluriharmonic functions on a complex manifold \[5\].

Let \( B \) be a ring generated by the currents \( dd^c g \), where \( g \) are piecewise linear functions on \( \mathbb{C}^n \). Attach to the ring \( B \) the current \( \phi \mapsto \int_{\mathbb{C}^n} \phi \), where \( \phi \) is volume form with compact support on \( \mathbb{C}^n \) (the latest current is a unit of the ring \( B \)). The ring \( B \) is graded by degrees of corresponding polynomials in the variables \( dd^c g \).

**Statement 1.** (\[1\]) For any homogeneous element \( b \) of the ring \( B \) there exist \( (n - \deg b) \)-dimensional ETV \( \iota(b) \) such that \( b = \iota(b) \).

**Theorem 2.1.** Let \( B \) be a graded ring of ETV. Then the map \( \iota: B \to B \) is an isomorphism of graded rings.

**Remark 2.2.** The theorem was suggested as a conjecture in \[1\].

**Definition 2.1.** (1) Function \( g: \mathbb{C}^n \to \mathbb{R} \) is said to be positively homogeneous of degree 1, if \( \forall \lambda \geq 0: g(\lambda z) = \lambda g(z) \).

(2) Function \( g: \mathbb{C}^n \to \mathbb{R} \) is said to be \( \mathbb{R} \)-generated, if \( g(z + y) = g(z) \) for any \( y \in \text{Im} \mathbb{C}^n \).

(3) ETV \( X \) is said to be \( \mathbb{R} \)-generated, if \( y + X = X \) for any \( y \in \text{Im} \mathbb{C}^n \) (here \( y + X \) is a shift of ETV by the vector \( y \)).

The following statements follow from Theorem 2.1.

**Theorem 2.2.** Let \( B_h \) be a ring of homogeneous ETV, and \( B_h \) be a ring, generated by currents \( dd^c g \), where \( g \) are positively homogeneous of degree 1 piecewise linear functions on \( \mathbb{C}^n \). Then the map \( \iota|_{B_h}: B_h \to B_h \).

**Theorem 2.3.** (\[3\]) Let \( B_\mathbb{R} \) be a ring of \( \mathbb{R} \)-generated ETV and \( B_\mathbb{R} \) be a ring, generated by currents \( dd^c g \), where \( g \) are \( \mathbb{R} \)-generated piecewise linear functions on \( \mathbb{C}^n \). Then the map \( \iota|_{B_\mathbb{R}}: B_\mathbb{R} \to B_\mathbb{R} \).
The ring $\mathcal{B}_\mathbb{R}$ coincides with the ring of tropical varieties in $\mathbb{R}^n$. 

**Corollary 2.1.** The rings $\mathcal{B}, \mathcal{B}_h, \mathcal{B}_\mathbb{R}$ are generated by the elements of a first degree.

The proof of Theorem 2.1 is based on the following statement ([1], Theorem 2 and Proposition 2).

**Statement 2.** Let $X$ be a $k$-dimensional ETP and $X \Rightarrow \mathcal{P}$. Let $h$ be a piecewise linear function that is linear on any cell of polyhedral set $X$. Construct a new frame of degree $2n - k + 1$ on any $k$-dimensional cell $\Delta \in X$ as

$$Y_\Delta = d^c G_\Delta \wedge X_\Delta,$$

where $G_\Delta$ is any linear function such that $(G_\Delta)|_\Delta = h$. Set $D_c(hX) = \partial Y$. Then

1. $(k - 1)$-dimensional framed polyhedral set $D_c(hX)$ is ETP and does not depend on the choice of functions $G_\Delta$.

2. ETP, corresponding to ETP $D_c(hX)$, depends only on the equivalence class of ETP $X$.

3. $D_c(hX) = dd^c(h\overline{\mathcal{P}})$

It is obvious, that the map $\iota$ is bijective on the elements of degree 0. Using induction on degrees of graded ring $\mathcal{B}$ and applying statement 2, obtain the statement 1 (details are in [1]). Thus, the map $\iota$ is defined as a homomorphism of graded abelian groups. By definition the map $\iota$ is injective. Therefore, the proof of bijectivity of $\iota$ is reduced to the following statement.

**Theorem 2.4.** For any $(k - 1)$-dimensional ETV $\mathcal{P}$ there exist finite sets of

(a) $k$-dimensional ETP $\{\mathcal{Q}\}$

(b) piecewise linear functions $\{h_\mathcal{Q}\}$ on the space $\mathbb{C}^n$

(c) ETP $\{X_\mathcal{Q} \Rightarrow \mathcal{Q}\}$

such that any function $h_\mathcal{Q}$ is linear on each cell of ETP $X_\mathcal{Q}$ and

$$\sum_{\mathcal{Q} \in \{\mathcal{Q}\}} D_c(h_\mathcal{Q} \mathcal{Q}) = \mathcal{P}.$$

Let $h_\gamma$ be a support function of convex polytope $\gamma \subset (\mathbb{C}^n)^*$ (remind: support function is a piecewise linear function $h_\gamma(z) = \max_{z^* \in \gamma} \text{Re}(z, z^*)$ on $\mathbb{C}^n$). From the definition follows that $\forall k$ the function $h_\gamma$ is linear on any cell of ETP $X^{\gamma,k}$.

**Proposition 2.1.** $D_c(h_\gamma X^{\gamma,k}) = (2n - k + 1)X^{\gamma,k-1}$. 

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Proof. Let $\Theta$ be a $k$-dimensional cone dual to a face $\theta$ of polytope $\gamma$. Then the frame $(D_c(h_\gamma X^{\gamma,k}))_\Delta$ of $(k-1)$-dimensional cone $\Delta$ (by definition of the map $D_c$), equals to

$$\sum_{\Theta \supset \Delta} d^c G_\Theta \wedge X^{\gamma,k}_\Theta,$$

where $G_\Theta$ is a linear function such that $(G_\Theta)|_\Theta = h_\gamma$. Set $G_\Theta(z) = \text{Re}(z, w_\theta)$, where $w_\theta$ is an arbitrary point of the face $\theta$. Then $d^c G_\Delta = \text{Re}(dz, -iw_\theta)$.

Let $\delta$ be a face of polytope $\gamma$ dual to cone $\Delta$. Then (by definition) $X^{\gamma,k-1}_\Delta = ((-i)^{2n-k+1}\varrho(p_\delta))$. Applying (1.4), we get

$$X^{\gamma,k-1}_\Delta = (-i)^{2n-k+1}\varrho(p_\delta) = \frac{1}{2n-k+1} \sum_{\Theta \supset \Delta, \dim \Theta = 2n-k} (-iw_\theta) \wedge ((-i)^{2n-k+1}\varrho(p_\delta)) = \frac{1}{2n-k+1} \sum_{\Theta \supset \Delta, \dim \Theta = k} d^c G_\Delta \wedge X^{\gamma,k}_\Theta = \frac{1}{2n-k+1} (D_c(h_\gamma X^{\gamma,k}))_\Delta$$

Proof of Theorem 2.4. Theorem 1.2 ETV $P$ can be represented as

$$P = \sum_{\gamma \in \{\gamma\}} \pm(a_\gamma + X^{\gamma,k-1})$$

Set $Q = \frac{1}{2n-k+1}(a_\gamma + X^{\gamma,k})$ (here $tX$ is, by definition, the ETV $X$ with the frames multiplied by $t$).

Set $h_Q = h_\gamma(z - a_\gamma)$, where $h_\gamma$ be a support function of polytope $\gamma$. The function $h_Q$ is linear on any cell of ETP $X^{\gamma,k}$ by construction. It follows from Proposition 2.1 that

$$a_\gamma + X^{\gamma,k-1} = \frac{1}{2n-k+1} D_c(h_\gamma(a_Q + X^{\gamma,k})) = D_c(h_QQ).$$

Thus $P = \sum_{Q \in \{Q\}} \pm D_c(h_QQ)$. □

Now it remains to show that the map $\iota$ preserves the products of ETV (ie is an isomorphism of rings).

The additive group of space $C^n$ acts by shifts on the set of piecewise linear functions. This action extends to a continuous action on the ring $B$. Similarly (Theorem 1.3), this group acts on the ring $B$. These actions commute with the map $\iota$. Therefore, the statement on the isomorphism of
the rings is reduced to the case of transversal intersections of corner loci (Proposition 2.2).

Let ETP \( X_1, \ldots, X_k \) be corner loci of piecewise linear functions \( h_1, \ldots, h_k \). Say that ETP \( X_1, \ldots, X_k \) are transversal, if for all sets of cells \( \Delta_i \in X_i \) with \( \Delta_1 \cap \cdots \cap \Delta_k \neq \emptyset \) the tangent spaces \( E_{\Delta_i} \) of cells \( \Delta_i \) are transversal.

**Proposition 2.2.** If ETP \( X_1, \ldots, X_k \) are transversal then

\[
\iota(\text{dd}^c h_1 \wedge \cdots \wedge \text{dd}^c h_k) = P_1 \cdots P_k,
\]

where \( X_i \Rightarrow P_i \).

**Proof.** Let \( \Delta_i \in X_i \) be a \((2n-1)\)-dimensional cell. Consider a single-celled ETP \( A_i \) with the cell \( E_{\Delta_i} \) framed as \((X_i)_{\Delta_i} \). Set \( A_i \Rightarrow A_i \).

Locally \( \Delta_i \) looks as a common hyperplane of two halfspaces \( B_1, B_2 \). Let a piecewise linear function \( g_i \) is linear on halfspaces \( B_1, B_2 \) and \( g_i|_{B_j} = h_i|_{B_j} \) near any inner point of the cell \( \Delta_i \).

It follows from transversality condition that the statement of Proposition 2.2 is equivalent to the series of equations

\[
\iota(\text{dd}^c g_1 \wedge \cdots \wedge \text{dd}^c g_k) = A_1 \cdots A_k,
\]

where \( \Delta_1 \in X_1, \ldots, \Delta_k \in X_k \) is any set of \((2n-1)\)-dimensional cells. This last equality is easy to verify directly.

### 2.2 The zero values of mixed Monge-Ampere operator at piecewise linear functions

**Definition 2.2.** Let \( H \) be a complex \( p \)-dimensional subspace of \( \mathbb{C}^n \) and let \( k + p > n \). The set of piecewise linear functions \( g_1, \ldots, g_k \) on \( \mathbb{C}^n \) is called \( H \)-degenerate, if there exist linear functions \( \varphi_i : \mathbb{C}^n \rightarrow \mathbb{R} \) such that \( \varphi_i(z + h) + g_i(z + h) = \varphi_i(z) + g_i(z) \) for any \( z \in \mathbb{C}^n \) and any \( h \in H \) (i.e. functions \( \varphi_i + g_i \) are the pullbacks of some piecewise linear functions on the space \( \mathbb{C}^n/H \) by the projection \( \mathbb{C}^n \rightarrow \mathbb{C}^n/H \)). The set \( g_1, \ldots, g_k \) is called nondegenerate if there is no subspace \( H \) such that the set \( g_1, \ldots, g_k \) is \( H \)-degenerate.

**Theorem 2.5.** Let \( h_1, \ldots, h_k \) be convex piecewise linear functions. Then \( \text{dd}^c h_1 \wedge \cdots \wedge \text{dd}^c h_k \neq 0 \) if and only if the set \( h_1, \ldots, h_k \) is nondegenerate.

**Question 1.** Let \( h_1, \ldots, h_k \) be any convex functions on \( \mathbb{C}^n \). Is it true statement of theorem 2.5?
The theorem 2.5 is a successor of the criterion of a zero value of mixed volume. We explain this more.

**Statement 3.** Let $A_1, \ldots, A_n$ be compact convex bodies in $\mathbb{R}C^n$ and let $h_i: \mathbb{C}^n \to \mathbb{R}$ be their support functions. Consider some Euclidean metric on $\mathbb{R}C^n$, the corresponding Hermitian metric on $\mathbb{C}^n$ and the dual Hermitian metric on $\mathbb{C}^n$. Then the current $dd^c h_1 \wedge \cdots \wedge dd^c h_n$ is a Euclidean measure on the space $\text{Im}\mathbb{C}^n$ multiplied by the mixed volume of convex bodies $A_1, \ldots, A_n$ and by $n!$

The criterion of zero value of mixed volume is formulated as follows.

**Corollary 2.2.** Following statements are equivalent:

1. the mixed volume of convex bodies $A_1, \ldots, A_n$ in $\mathbb{R}^n$ is zero
2. there exist $p \leq n$ and a subset $B_1, \ldots, B_p$ of the set $A_1, \ldots, A_n$ such that for some $a_i \in \mathbb{R}^n$ all shifted bodies $a_1 + B_1, \ldots, a_p + B_p$ are contained in some $(p - 1)$-dimensional subspace of $\mathbb{R}^n$.

Let $A_i$ be a convex polytope. Then the corollary 2.2 follows directly from Theorem 2.5. Indeed, put the sets $A_1, \ldots, A_n$ into the space $\mathbb{R}C^n$ and (using the statement 3) apply Theorem 2.5 to the support functions of polytopes. For any convex bodies $A_i$ corollary follows from the monotonicity of the mixed volume: if $B_i \subseteq A_i$, then $V(B_1, \ldots, B_n) \leq V(A_1, \ldots, A_n)$.

Here are three auxiliary statements, used in the proof of Theorem 2.5. The first is a direct consequence of the definition of ETV, second is obvious. Proof of the third statement is given after the proof of Theorem 2.5.

**Lemma 2.1.** Let $X_1, \ldots, X_k$ be single-celled nonzero $(2n - 1)$-dimensional ETP and let $X_i = P_i$. Let $\Delta_i$ be a $(2n - 1)$-dimensional cells of ETP $X_i$ and let $E_i$ be a tangent spaces of the cells $\Delta_i$. Let $0 \neq \varphi_i \in \mathbb{C}^{n^*}$ be an equation of the hyperplane $E_i$, i.e. $\text{Re}(\langle z, \varphi_i \rangle) = 0$ for any $z \in E_i$. Then $\prod_{i=1,\ldots,k} P_i = 0$, if and only if the vectors $\varphi_1, \ldots, \varphi_k$ are linearly dependent over $\mathbb{C}$.

**Lemma 2.2.** Let $h$ be a piecewise linear function on $\mathbb{R}^N$. Let $v_i \in \mathbb{R}^{N^*}$ be differentials of function $h$ in it’s areas of linearity. Suppose that for any $i, j$ the vectors $v_i - v_j$ are contained in some fixed subspace $H \subset \mathbb{R}^{N^*}$. Then the function $h - v_1$ is a pullback of some piecewise linear function on $\mathbb{R}^N/H^\perp$ by the projection $\mathbb{R}^N \to \mathbb{R}^N/H^\perp$, where the subspace $H^\perp$ is the orthogonal complement of subspace $H$.

Let $A_1, \cdots, A_k$ be nonempty finite sets of $n$-dimensional vector space $E$ (over arbitrary field). If $0 < k \leq n$ and any set of vectors $a_1 \in A_1, \cdots, a_k \in A_k$ is linearly dependent, then the set $A_1, \cdots, A_k$ is called degenerate.
Proposition 2.3. Let a set $A_1, \ldots, A_k$ is degenerate. The there exist
(a) $1 \leq p \leq k$
(b) $(p - 1)$-dimensional subspace $H \subset E$
(c) subset $B_1, \ldots, B_p$ of the set $A_1, \ldots, A_k$
such that $\forall j : B_j \subset H$.

Proof of Theorem 2.5 Let ETP $X_i$ be a corner locus of function $h_i$. Let $A_i \subset \mathbb{C}^n$ be a set of equations of real hyperplanes $E_\Delta$ for all $\Delta$ from the set of $(2n - 1)$-dimensional cells of $X_i$ (see formulation of lemma 2.1).

From the convexity of functions $h_i$ it follows that ETV $X_i$ are positive. Therefore, applying Corollary 1.17 and Lemma 2.1 we get the following: $dd^c h_1 \wedge \cdots \wedge dd^c h_k = 0$, if and only if the set of finite sets $A_1, \ldots, A_k$ is degenerate.

If the set $A_1, \ldots, A_k$ is degenerate, then applying Proposition 2.3 obtain a subset $g_1, \ldots, g_p$ of a set of functions $h_1, \ldots, h_k$ and $(p - 1)$-dimensional complex subspace $H$ of the space $\mathbb{C}^n$ such that each of the functions $g_i$ satisfies the condition of Lemma 2.2. Now theorem 2.5 follows from Lemma 2.2.

□

Proof of Proposition 2.3. Let $\bar{C} = \{C_1, \ldots, C_m\}$ be a maximal nondegenerate subset of the set $\bar{A} = \{A_1, \ldots, A_k\}$. The nondegeneracy of the set $\bar{C}$ implies existence of linearly independent vectors $c_1 \in C_1, \ldots, c_m \in C_m$. Lemma 2.3 follows from the maximality of subset $\bar{C}$.

Lemma 2.3. Let $V$ be an $m$-dimensional subspace, generated by vectors $c_1, \ldots, c_m$, $C \in \bar{A} \setminus \bar{C}$, and $L$ be a subspace, generated by vectors of the set $C$. Then $L \subset V$.

Lemma 2.4. Let $V_i$ be an $(m - 1)$-dimensional subspace of $E$, generated by vectors $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m$. Then, if $C_i \not\subset V$, then $L \subset V_i$.

Proof. Let $\tilde{V}$ be a subspace generated by vectors $c_1, \ldots, \tilde{c}_i, \ldots, c_{m-1}, c_m$, where $C_i \ni \tilde{c}_i \not\in V$. Then $\dim \tilde{V} = m$. So (Lemma 2.3) $L \subset \tilde{V}$. It remains to see that $V_i = V \cap \tilde{V}$.

Sort now the collection of sets $\bar{C} = \{C_1, \ldots, C_m\}$ so that $C_i \subset \bar{V}$ if $i \leq l$ and $C_i \not\subset V$ if $i > l$. If $l = m$, then the subsets $C_1, \ldots, C_m, C$ are contained in $m$-dimensional subspace $V$ as it required in Proposition 2.3.

Lemma 2.5. Let $Q$ be an $l$-dimensional subspace generated by vectors $c_1, \ldots, c_l$. Then $L \subset Q$.

Proof. By Lemma 2.4 $L \subset V_{i+1} \cap \cdots \cap V_m = Q$.

Note that $l > 0$, as otherwise, by Lemma 2.5 $L = 0$, which is impossible.
If $C_i \subset Q$ for $i = 1, \ldots, l$, then the statement of Proposition 2.3 is true for $p = l + 1$, for a subset $C_1, \ldots, C_l, C$ of the set $\bar{A}$ and for the subspace $Q$. Suppose that $\exists i \leq l$ such that $C_i \not\subset Q$.

**Lemma 2.6.** Let $i \leq l$ and $C_i \not\subset Q$. Then $L \subset Q_i$, where $Q_i$ is a $(l - 1)$-dimensional subspace, generated by vectors $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_l$.

**Proof.** Let $C_i \ni \tilde{c}_i \not\subset Q$. As $i \leq l$, then $\tilde{c}_i \in V$. Consider the decomposition

$$\tilde{c}_i = \sum_{j \leq m} \alpha_j c_j$$

(2.1)

respect to the basis $\{c_j\}$ of $V$. There may be cases $\alpha_i \neq 0$ and $\alpha_i = 0$.

In the first case, the vectors

$$c_1, \ldots, c_{i-1}, \tilde{c}_i, c_{i+1}, \ldots, c_m$$

(2.2)

are linearly independent (i.e. form the basis of the space $V$). Then, using Lemma 2.5, we obtain that $L$ belongs to the subspace generated by the vectors $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_l$ and to the subspace generated by the vectors $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_l$. Hence $L$ belongs to their intersection, ie to the subspace $Q_i$, as required.

In the second case (when $\alpha_i = 0$) the vectors (2.2) are linearly dependent. In this case, since $\tilde{c}_i \not\subset Q$, then $\alpha_j \neq 0$ for some $j > l$. Let $V \not\ni \tilde{c}_j \in C_j$. Then the vectors

$$c_1, \ldots, c_{i-1}, \tilde{c}_i, c_{i+1}, \ldots, c_{j-1}, \tilde{c}_j, c_{j+1}, \ldots, c_m$$

(2.3)

are linearly independent. Then (Lemma 2.4), $L$ belongs to subspace $\tilde{Q}$, generated by vectors

$$c_1, \ldots, c_{i-1}, \tilde{c}_i, c_{i+1}, \ldots, c_{j-1}, c_{j+1}, \ldots, c_m.$$ 

Then $c_i \not\in \tilde{Q}$. Indeed, the otherwise we would obtain a decomposition of the form (2.1) with non-zero $\alpha_i$. It follows that $Q \cap \tilde{Q} = Q_i$ and so $L \subset Q_i$. Lemma is proved. \hfill \Box

Now sort the set $C_1, \ldots, C_l$ so that $C_{\leq q} \subset Q$ and $C_{>q} \not\subset Q$. Then the space $L$ belongs to the subspace, generated by vectors $c_1, \ldots, c_q$. This follows from Lemma 2.6, because this last subspace coincides with the subspace $Q_{q+1} \cap \cdots \cap Q_l$. So $q + 1$ sets $C_1, \ldots, C_q, C$ are in $q$-dimensional subspace. Proposition 2.3 is proved.
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