Chiral Poincaré transformations and their anomalies*

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Abstract

I consider global transformations of a Dirac fermion field, that are generated by the generators of Poincaré transformations, but with a $\gamma_5$ appended. Such chiral translations and chiral Lorentz transformations are usually not symmetries of the Lagrangian, but naively they are symmetries of the fermionic measure. However, by using proper time regularization in Minkowski space, I find that they in general give rise to a nontrivial Jacobian. In this sense they have “anomalies”. I calculate these anomalies in a theory of a massive fermion coupled to an external Abelian vector field. My motivation for considering chiral Poincaré transformations is the possibility that they are relevant to bosonization in four dimensions.

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1 Introduction

Chiral phase rotations have the property that they are, naively, symmetries of the fermionic measure. This is also true for the chiral Poincaré transformations, and they are therefore singled

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out from the set of all possible transformations of a Dirac fermion that are not symmetries of the Lagrangian.

In this paper I investigate chiral Poincaré transformations within a simple Abelian model of a fermion coupled to an external vector field. My motivation for this has to do with bosonization [1]. While bosonization is a well established concept in two dimensions (see e.g. [2] and references therein), the same thing cannot be said about four dimensions (however, see [3]). If we consider the various bosonization schemes in the path integral [2], they all have the property that the bosonic field in the bosonized theory is the chiral phase of the fermion, and the bosonic action $S_{\text{bos}}$ is given by the Jacobian $J = \exp(iS_{\text{bos}})$ of a local chiral transformation of the fermion. But in four dimensions it cannot be expected that just the chiral phase of a fermion is enough to describe the physics. It is therefore necessary to identify further degrees of freedom of the fermion that are relevant for bosonization. I think the degrees of freedom that correspond to chiral Poincaré transformations are good candidates for this.

If chiral Poincaré transformations really is to play a similar role in four dimensional bosonization as chiral phase rotations does in the two-dimensional case, then they must give rise to a Jacobian when a change of variables is performed in the path integral. In this paper I will restrict my attention to this Jacobian and calculate it for infinitesimal chiral Lorentz transformations and translations. A calculation of the Jacobian for finite transformations and an investigation of the bosonization procedure itself will be reported in a separate publication [4].

The Jacobian $J[\beta] = \exp(i \int d^4x \beta A)$ of a chiral rotation with parameter $\beta(x)$ contains the chiral anomaly $A(x)$. It is a quantum correction to the naive expectation that $J$ is unity. Thus, since the same thing happens for chiral Poincaré transformations they also have a kind of “anomaly”. This is a distinguishing property of chiral Poincaré transformations and is logically independent of the fact that they are not symmetries of the Lagrangian. In lack of a better word, I will sometimes refer to the quantity that corresponds to $A$ for chiral Poincaré transformations as ‘anomaly’ as well.

I should also mention a paper by Alvarez [5] which is slightly related to this work. He tried to calculate the fermionic determinant of a fermion that is coupled to a vector field by using a decoupling transformation. His decoupling transformation corresponds to the spin part of a chiral Lorentz transformation. There is also the recent paper [6] where a similar decoupling transformation is used. However, my work is not directly related to investigations of local Lorentz symmetry [7].

The organization of the paper is the following. In sec. 2, I discuss classical properties of the chiral Poincaré transformations. I point out that they make sense if they are regarded as active, transforming the physical system, rather than passive. In sec. 3, I consider the quantum theory which I regularize using the proper time representation of the fermionic determinant in Minkowski space. I prefer this to the usual Euclidean formulation (see e.g. the review by Ball [8]), because better control over the anomalies is achieved. In sec. 4, I calculate the Jacobians, which is a generalization of the calculation that leads to the Adler–Bell–Jackiw (ABJ) anomaly [9]. The new features which we must give attention to is the presence of derivative operators and a $\sigma_{\mu\nu} \lambda$-matrix in the generators. In sec. 5, I summarize and discuss the generalization to non-Abelian theories. I also speculate on a possible application in strong interaction physics.

2 Chiral Poincaré transformations

The model we consider is a Dirac fermion $\psi$ with mass $m$ coupled to an external vector field $A_\mu$. The simplicity of this model allows us to discuss the main ideas without the complications
of non-Abelian structure or couplings to other fields. The Lagrangian is

$$L = \bar{\psi}[i\partial - A - m]\psi.$$  \hspace{1cm} (1)

$A_\mu$ can be a gauge potential or an arbitrary external source.

We first recall the elementary discussion of phase and Poincaré transformations. This is useful for comparison. Since we are considering an Abelian model, the phase transformations are generated by the unit operator. The Poincaré transformations are generated by

$$P_\mu = i\partial_\mu,$$ \hspace{1cm} (2)

the generator of translations, and

$$J_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu} + (x_\mu i\partial_\nu - x_\nu i\partial_\mu) \equiv S_{\mu\nu} + L_{\mu\nu},$$ \hspace{1cm} (3)

the generator of Lorentz transformations.

Our transformations are active. That is, they transform the physical system, the fermion field, in contrast to passive transformations where the coordinate system is transformed. This is essential for the chiral transformations. Thus no transformation law is assigned to the $A_\mu$-field.

The transformations are:

1. Phase transformations:

$$\psi(x) \to e^{i\alpha}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha}$$ \hspace{1cm} (4)

$\alpha$ is a dimensionless constant. The Lagrangian is invariant.

2. Translations:

$$\psi(x) \to e^{ia_\mu P_\mu}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-ia_\mu P_\mu}$$ \hspace{1cm} (5)

$a_\mu$ is a constant vector with the dimension of length. For $a_\mu$ infinitesimal, $L$ transforms into

$$L \to \bar{\psi}[i\partial - A - a_\mu \partial^\nu A_\nu - m]\psi.$$ \hspace{1cm} (6)

If we assign the transformation law

$$A_\mu \to A_\mu - a_\nu \partial^\nu A_\mu$$ \hspace{1cm} (7)

to $A_\mu$ to accompany (5), we will have that $L$ is invariant. The infinitesimal transformation (7) is of course identical to that found from a translation of the coordinate system.

3. Lorentz transformations:

$$\psi(s) \to e^{i\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu}}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}e^{-i\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu}}$$ \hspace{1cm} (8)

$\omega_{\mu\nu}$ is a dimensionless constant antisymmetric tensor. For infinitesimal $\omega_{\mu\nu}$ we have

$$L \to \bar{\psi}[i\partial - A - \omega^{\mu\nu} A_\nu \gamma_\mu - \frac{1}{2}\omega^{\mu\nu}(x_\mu \partial_\nu A_\nu - x_\nu \partial_\mu A_\nu) - m]\psi$$ \hspace{1cm} (9)

from which we can construct the transformation rule

$$A_\mu \to A_\mu + i\frac{1}{2}\omega^{\rho\sigma}(J_{\rho\sigma})_{\mu\nu} A_\nu$$
$$J_{\rho\sigma})_{\mu\nu} \equiv i(g_{\rho\mu} g_{\sigma\nu} - g_{\sigma\mu} g_{\rho\nu}) + (x_\rho i\partial_\sigma - x_\sigma i\partial_\rho)g_{\mu\nu}$$ \hspace{1cm} (10)

which leads to invariance.
The chiral versions of these are then the following.

1. Chiral phase transformations:

\[ \psi(x) \rightarrow e^{i\beta\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\beta\gamma_5} \]  

(11)

\( \beta \) is a dimensionless pseudoscalar constant. These are the usual ones, and lead to the Lagrangian

\[ \mathcal{L} \rightarrow \bar{\psi}[i\not\!\partial - A - m - 2im\beta\gamma_5]\psi. \]  

(12)

2. Chiral translations:

\[ \psi(x) \rightarrow e^{ib_\mu P^\mu\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{ib_\mu P^\mu\gamma_5} \]  

(13)

\( b_\mu \) is a constant axial vector with the dimension of length. This leads to

\[ \mathcal{L} \rightarrow \bar{\psi}[i\not\!\partial - A - b_\mu \partial^\mu A\gamma_5 - m + 2m\gamma_5 b_\mu \partial^\mu]\psi. \]  

(14)

3. Chiral Lorentz transformations:

\[ \psi(x) \rightarrow e^{i\frac{1}{2}j_{\mu\nu}J_{\mu\nu}\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\frac{1}{2}j_{\mu\nu}J_{\mu\nu}\gamma_5} \]  

(15)

\( j_{\mu\nu} \) is a “dual” tensor (i.e. it has properties like \( \tilde{F}_{\mu\nu} \) under \( C, P \) and \( T \)) and is dimensionless. This leads to

\[ \mathcal{L} \rightarrow \bar{\psi}[i\not\!\partial - A - j_{\mu\nu} A_\nu \gamma_5 - \frac{1}{2}j_{\mu\nu}(x_\mu \partial_\nu A - x_\nu \partial_\mu A)\gamma_5 - m - imj_{\mu\nu}J_{\mu\nu}\gamma_5]\psi. \]  

(16)

It is now impossible to find transformation rules for \( A_\mu \) which restores invariance. Chiral transformations includes \( \gamma_5 \) in the infinitesimal generators, and the left- and right-handed parts of the fermion are transformed in an opposite sense, as can be seen from the representation where \( \gamma_5 \) is diagonal:

\[ \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \]  

(17)

In this case there are no interpretations in terms of transformations of the coordinate system, but as long as they are regarded as active the chiral Poincaré transformations make just as much sense as the non-chiral ones. Note also that new derivative operators have been generated in the Dirac operator.

3 Proper time representation and Poincaré invariance

I now turn to the the quantum theory and the calculation of anomalies, or in other words, Jacobians. I use proper time regularization of the fermionic determinant [8], which I will discuss in some detail even if it is a well known scheme. The reasons for this are, first, that I work in Minkowski space and that there are differences from the usual Euclidean formulation. Second, I need to demonstrate that the scheme is both phase and Poincaré invariant. And third, compared to the calculation of the ABJ anomaly, there are additional features coming from the \( \sigma_{\mu\nu} \) and the derivative operators in the generators. The calculations in this section are essentially model independent.
In Minkowski space, ambiguities are resolved by adding a small positive imaginary part to the Dirac operator:

$$\mathcal{L} = \bar{\psi}(D + i\epsilon)\psi, \quad D = i\partial - A - m.$$  \hspace{1cm} (18)

This is the usual $\epsilon$-prescription which will render all momentum integrals well defined, and which gives the fermion mass a small negative imaginary part, i.e. the fermion mass is $m_c \equiv m - i\epsilon$.

The quantum theory is expressed by the path integral $Z$, which defines the fermionic determinant,

$$Z = \int D\psi D\bar{\psi} e^{i\int d^4x \bar{\psi}D\psi} \equiv \text{Det}D,$$  \hspace{1cm} (19)

where $\psi$ and $\bar{\psi}$ are the independent quantities, and the effective action $W (Z \equiv e^{iW})$ is given by

$$W = -i\text{Tr} \ln D.$$  \hspace{1cm} (20)

These quantities are formal and will become well defined when we regularize the theory.

I will calculate the Jacobian of a local infinitesimal change of path integration variables. Let us collectively denote the phase and Poincaré transformations by

$$\psi \to e^{iA} \psi, \quad \bar{\psi} \to \bar{\psi} e^{-iA},$$  \hspace{1cm} (21)

where

$$A = \alpha, \quad \frac{1}{2}(a_\mu P^\mu + P^\mu a_\mu), \quad \text{or} \quad \frac{1}{2}\left(\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} + \frac{1}{2} J^{\mu\nu} \omega_{\mu\nu}\right).$$  \hspace{1cm} (22)

The parameters are now local, $\alpha = \alpha(x)$, etc. We use symmetric products for the local translations and Lorentz transformations, since these are Dirac hermitian. Similarly, their chiral counterparts are

$$\psi \to e^{iB_5} \psi, \quad \bar{\psi} \to \bar{\psi} e^{iB_5},$$  \hspace{1cm} (23)

with

$$B = \beta, \quad \frac{1}{2}(b_\mu P^\mu + P^\mu b_\mu), \quad \text{or} \quad \frac{1}{2}\left(\frac{1}{2} \phi_{\mu\nu} J^{\mu\nu} + \frac{1}{2} J^{\mu\nu} \phi_{\mu\nu}\right).$$  \hspace{1cm} (24)

These transformations induce a change in the Dirac operator:

$$D \to e^{-iA+iB_5} De^{iA+iB_5}$$

$$= D + i(DA - AD) + i(DB_5 + B_5 D)$$

$$\equiv D + \delta D,$$  \hspace{1cm} (25)

which in turn induces a change in $W$:

$$\delta W = -i\text{Tr} \delta D \frac{1}{D}.$$  \hspace{1cm} (26)

The Jacobian $J$ is then determined by the requirement that the path integral $Z$ is unchanged by a change of variables:

$$Z = J e^{iW + i\delta W} \equiv e^{iS_J} e^{iW + i\delta W} = e^{iW},$$  \hspace{1cm} (27)
where we have defined the action \( S_J \equiv -i \ln J \). Therefore \( S_J \) can be found by

\[
S_J = -\delta W = i \text{Tr} \delta D \frac{1}{\bar{D}}.
\]

(28)

This is the quantity we are interested in, and which contains the anomalies.

We now introduce an operator \( \tilde{D} \) and a proper time integral, thereby defining the formal trace:

\[
S_J = i \text{Tr} \delta D \frac{1}{\bar{D}} = i \text{Tr} \delta \tilde{D} \frac{1}{\bar{D}} \tilde{D} = \int_{1/\Lambda^2}^{\infty} ds \text{Tr} \delta D \tilde{D} e^{is(D\tilde{D} + i\epsilon)}. \tag{29}
\]

The operator \( \tilde{D} \) is a priori arbitrary, except that it must be chosen to give the right \( \epsilon \)-prescription, like I have written here. This will then ensure convergence at the upper integration limit. For the lower integration limit the cutoff \( \Lambda \) is introduced, which is to be taken to infinity at the end of the calculation. I will discuss the appropriate choice for \( \tilde{D} \) below. When this choice is made \( S_J \) will be regular and well defined.

We can use the expression for \( \delta D \) (eq. (25)) to perform the proper time integral and write \( S_J \) in a Fujikawa-like form [9] (see also [10]):

\[
S_J = \int_{1/\Lambda^2}^{\infty} ds \text{Tr} i(DA - AD + DB\gamma_5 + B\gamma_5 D) \tilde{D} e^{is(D\tilde{D} + i\epsilon)}
\]

(30)

Here I have used the cyclicity of the trace, the identity \( \tilde{D} e^{is\tilde{D}D} = \tilde{D} e^{is\tilde{D}D} \), and the fact that only the lower limit of the integral contributes due to the implicit presence of the \( \epsilon \).

We must now make an appropriate choice for \( \tilde{D} \), one which preserves phase and Poincaré invariance and leads to the right \( \epsilon \)-prescription. In principle \( \tilde{D} \) can still be completely unrelated to \( D \); for instance we can choose the unit operator, \( \tilde{D} = I \). But the choice I will adopt, to be justified in a moment, is

\[
\tilde{D} = (i\gamma_5)D(i\gamma_5). \tag{31}
\]

For our QED-like theory we get \( \tilde{D} = i\partial - A + m \), with plus in front of the mass, and

\[
\tilde{D}D = D\tilde{D} = -D_{\mu}D^{\mu} - \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu} - m^2, \quad D_{\mu} = \partial_{\mu} + iA_{\mu},
\]

(32)

the familiar “square” of the Dirac operator. \( i\epsilon \) is implied. Using the cyclicity of the trace and the fact that \( \gamma_5 \) commutes with \( A \), we have

\[
S_J = -2\text{Tr} B\gamma_5 e^{iD\tilde{D}/\Lambda^2}.
\]

The terms proportional to \( A \) in eq. (30) have thus cancelled out. The vector current, energy-momentum tensor and angular momentum tensor are then conserved, and both phase and Poincaré invariance are intact.

I have two reasons for the choice (31) of \( \tilde{D} \). First, I want to relate it to \( D \) such that \( \tilde{D}D \) and \( D\tilde{D} \) are second order derivative operators. This simplifies later calculations. Second, it automatically respects both phase and Poincaré invariance. Furthermore, in a non-Abelian model with an external vector \( V_\mu \) and axial vector \( A_\mu \), this choice gives the Bardeen anomaly – the Wess–Zumino consistent anomaly.
4 Jacobians for the chiral transformations

For the calculation of the Jacobians, it is necessary to consider each form of \( B \) separately. As I have already mentioned, there are two further ingredients in the calculation of the chiral Poincaré Jacobians compared to the chiral phase case, namely the occurrence of \( \sigma_{\mu \nu} \) in \( J_{\mu \nu} \) and of derivative operators in \( P_\mu \) and \( J_{\mu \nu} \).

Let us write

\[
\tilde{D} D = -D_\mu D^\mu - Y, \quad Y = \frac{1}{2} \sigma_{\mu \nu} F^{\mu \nu} + m^2.
\]  

(33)

It is useful first to recall the familiar calculation of the ABJ anomaly

\[
S_\beta = -2 \text{Tr} \beta \gamma_5 e^{i \tilde{D} D / \Lambda^2} \equiv \int d^4 x \mathcal{L}_\beta.
\]  

(34)

\( \mathcal{L}_\beta \) is given by

\[
\mathcal{L}_\beta = -2 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} \beta \gamma_5 e^{i(-D^2-Y)/\Lambda^2} e^{-i k x} = -2 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \beta \gamma_5 e^{i(k^2+2i k D-D^2-Y)/\Lambda^2} = -2\Lambda^4 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} \beta \gamma_5 \exp \left( \frac{2i k D}{\Lambda} - \frac{D^2 + Y}{\Lambda^2} \right),
\]  

(35)

where I have scaled \( k_\mu \rightarrow \Lambda k_\mu \). Expanding the exponential and keeping only terms that are not suppressed by the cutoff \( \Lambda \), we get

\[
\mathcal{L}_\beta = -2 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} \beta \gamma_5 \sigma_{\mu \nu} \sigma_{\rho \sigma} \beta (-\frac{1}{8} F^{\mu \nu} F^{\rho \sigma}) = \frac{1}{8\pi^2} \beta F \tilde{F}.
\]  

(36)

In comparison we have for chiral translations

\[
\mathcal{L}_b = -2 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} \left( \frac{1}{2} b_\mu i \partial^\mu + \frac{1}{2} i \partial^\mu b_\mu \right) \gamma_5 e^{i \tilde{D} D / \Lambda^2} e^{-i k x} = -2 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ -\frac{1}{2} (i \partial_\mu b^\mu) + b_\mu k^\mu \right] \gamma_5 e^{i(k^2+2i k D-D^2-Y)/\Lambda^2} = \mathcal{L}_b^{(1)} + \mathcal{L}_b^{(2)}
\]  

(37)

I have used partial integration to bring the first term into the form of the ABJ anomaly, for which we get

\[
\mathcal{L}_b^{(1)} = \frac{1}{8\pi^2} (-i \frac{1}{2} \partial_\mu b^\mu) F \tilde{F}.
\]  

(38)

The second term, after scaling, is

\[
\mathcal{L}_b^{(2)} = -2\Lambda^5 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} b_\mu k^\mu \gamma_5 \exp \left( \frac{2i k D}{\Lambda} - \frac{D^2 + Y}{\Lambda^2} \right)
\]  

\[
= -2 \int \frac{d^4 k}{(2\pi)^4} e^{i k x} \text{tr} b_\mu k^\mu \gamma_5 \sigma_{\rho \sigma} \sigma_{\alpha \beta} \beta
\]  

\[
= \frac{1}{16\pi^2} (i \partial_\mu b^\mu) F \tilde{F} + \frac{1}{8\pi^2} b_\mu A^\mu F \tilde{F}.
\]  

(39)
Summing the two contributions, we get

$$\mathcal{L}_b = \frac{1}{8\pi^2} b_\mu A^\mu F \tilde{F}$$  \hspace{1cm} (40)$$

This expression is not invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu \alpha$ (when $A_\mu$ is a gauge potential). This is slightly surprising, since we were careful about gauge invariance when we regularized our theory. I will discuss this point in the next section.

For the chiral Lorentz transformations, there are again two terms which are handled essentially in this way, but in addition there is now a third term, coming from the $\frac{1}{2} \sigma_{\mu\nu} \gamma_5$ in $J_{\mu\nu} \gamma_5$, which requires special attention. This time the trace leads to only one factor of $F_{\mu\nu}$, and the term computes to

$$\mathcal{L}_\phi^{(3)} = -2 \int \frac{d^4k}{(2\pi)^4} e^{ikx} \text{tr}(\frac{1}{2} \phi_{\mu\nu} \frac{1}{2} \sigma_{\mu\nu} \gamma_5) e^{iD/\Lambda^2} e^{-ikx} = \frac{1}{48\pi^2} \phi_{\mu\nu}(\Box + 6m^2) \tilde{F}^{\mu\nu}. \hspace{1cm} (41)$$

The complete chiral Lorentz anomaly is given by

$$\mathcal{L}_\phi = \frac{1}{48\pi^2} \phi_{\mu\nu}(\Box + 6m^2) \tilde{F}^{\mu\nu} + \frac{1}{8\pi^2} \frac{1}{2} \phi_{\mu\nu}(x^\mu A^\nu - x^\nu A^\mu) F \tilde{F} \hspace{1cm} (42)$$

In addition to being gauge non-invariant, this is also not manifestly Poincaré invariant. However, this is an artifact of writing the anomaly in terms of a Lagrangian since we integrate this expression over spacetime to find the action: $S_\phi = \int d^4x \mathcal{L}_\phi$. The integration then picks out only the Poincaré invariant part of $\mathcal{L}_\phi$, and physics does not depend on the coordinate system.

5 Discussion

To summarize, I used proper time regularization in Minkowski space to regularize a theory of a massive Dirac fermion coupled to an external vector field. When the regularization is chosen to respect phase rotation and Poincaré invariance, this lead to a nontrivial transformation of the fermionic measure under chiral translations and Lorentz transformations in the path integral. I emphasize that the chiral Poincaré transformations are “anomalous transformations” and not “anomalous symmetries”. The “anomalies” depend on external sources and the mass of the fermion, which at the same time break the would-be symmetry explicitly.

The existence of a Jacobian for infinitesimal chiral Poincaré transformations implies of course that finite transformations give rise to a Jacobian as well, which is what we want for the four dimensional bosonization procedure. Such a finite Jacobian is a complicated nonlinear quantity with infinitely many derivatives, and is very hard to calculate. Furthermore, if the bosonic fields $\theta$, $\phi_{\mu\nu}$, and $b_\mu$ are relevant degrees of freedom for bosonization, and if $L_J$ is to be the bosonized Lagrangian it is necessary to make them decouple from the fermion. Work on this problem is in progress [4].

Let us return to the problem of the gauge non-invariance of the anomalies, eqs. (40) and (42). Since we were careful with choosing a gauge invariant regularization of the theory, the full path integral cannot depend on the part of $A_\mu$ that is a gradient. That is, if we make the Hodge decomposition

$$A_\mu = \partial_\mu \eta + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \xi^{\rho\sigma}, \hspace{1cm} (43)$$
it does not depend on $\partial_\mu \eta$. We are therefore allowed to make the replacement

$$A_\mu \rightarrow \Pi_\mu A_\nu = \left( g_\mu^\nu - \frac{\partial_\mu \partial^\nu}{\Box} \right) A_\nu = \frac{1}{\Box} \partial^\nu F_{\nu\mu}$$

in all expressions, where $\Pi_{\mu\nu}$ is a projection operator which removes the gradient from $A_\mu$. Due to our gauge invariant regularization, this can be done without loss of generalization.

Let me also remark on our choice of regularization operator $\tilde{D}$. Another reasonable choice would be $\tilde{D} = -CD^T C^{-1}$, where $C$ is the charge conjugation matrix and the transposition is with respect to the Dirac matrix structure. This is the “charge conjugate” of $D$. It gives identical results for our QED-like theory, but in the non-Abelian case with a vector and an axial vector it leads to the covariant chiral anomaly instead of the Bardeen anomaly, and thus the vector current is not conserved in this case. However, other terms proportional to $A$ in eq. (44) are non-zero as well. The energy-momentum tensor and angular momentum tensor are then also not conserved. In other words, Poincaré invariance is lost!

It is possible to generalize the discussion to non-Abelian fermions. Let us call the internal quantum number ‘flavor’. First of all, since we have taken the active point of view for our transformations, it is possible to “translate” or “Lorentz transform” each flavor of fermion separately. These transformations are generated by $t^a P_\mu$ and $t^a J_{\mu\nu}$, where $t^a$ are generators of rotations in flavor space. Their chiral counterparts are generated by $t^a P_\mu \gamma_5$ and $t^a J_{\mu\nu} \gamma_5$. All of these transformations are naive symmetries of the fermionic measure, but none of them are symmetries of the Lagrangian, except when the theory under consideration is trivial. They are also not the generators of a group, since their algebra does not close. However, the chiral set of generators gives rise to nontrivial Jacobians, and for this reason these may still, together with chiral phase rotations, play a role in four dimensional non-Abelian bosonization.

Finally, I will speculate on a possible application of flavored chiral Poincaré transformations in strong interaction physics. Recall that hadrons can be classified, with reasonable success, in terms of the group $SU(6)$, which is assumed to contain $SU(2)_{\text{spin}} \times SU(3)_{\text{flavor}}$ [11]. According to the $SU(6)$ scheme, the $\rho$’s enter in the same multiplet as the $\pi$’s. But the $\pi$’s can be understood within QCD as Goldstone bosons of spontaneously broken chiral symmetry, which raises the question of whether the $\rho$’s could be Goldstone bosons as well. This question was discussed in a paper by Caldi and Pagels [12]. However, shortly after the discovery of $SU(6)$ in the 60’s, it became clear that a symmetry group that mixes internal and spacetime degrees of freedom in any but a trivial way does not exist for a relativistic theory (see the lecture note and reprint volume by Dyson [13]). This means that the $\rho$’s cannot correspond to the broken generators of any symmetry group. Caldi and Pagels suggested that the $\rho$’s were nevertheless “dormant” Goldstone bosons in a certain static limit.

I claim that the $\rho$’s correspond to chiral Lorentz transformations, and that these are in some sense “broken”. This is not in conflict with the no-go theorems of the 60’s [13] because the chiral Lorentz transformations are neither symmetries nor a group. If we consider eqs. (12) and (16) in flavor space, we can read off from the mass terms the quark wavefunctions that correspond to the transformation parameters. They are

$$\beta^a \sim \bar{q} i t^a \gamma_5 q,$$
$$\phi^a_{\mu\nu} \sim \bar{q} i t^a \frac{1}{2} J_{\mu\nu} \gamma_5 = \bar{q} i t^a \frac{1}{4} \sigma_{\mu\nu} \gamma_5 q + \bar{q} i t^a \frac{1}{2} L_{\mu\nu} \gamma_5 q.$$ (45)

The latter implies

$$\tilde{\phi}^a_{\mu\nu} \sim \bar{q} t^a \frac{1}{4} \sigma_{\mu\nu} q + \text{orbital part.}$$ (46)

The “electric” part of the spin part of $\tilde{\phi}_{\mu\nu}$ is the same wavefunction for the $\rho$-meson as that suggested in ref. [12]. Thus, if the field $\beta^a$ describes the pseudoscalar octet $(\pi, K, \eta)$, and the
field $\phi^{a}_{\mu}$ describes the vector mesons ($\rho, K^{*}, \phi$) and $\omega$ (for $t^{0} \equiv I$), then they correspond to quark wavefunctions with the correct quantum numbers. Perhaps it is possible to test this idea by the methods of chiral perturbation theory.

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