Non-linear Reconstruction of the Large Scale Structure

V. Bistolas\textsuperscript{1} and Y. Hoffman\textsuperscript{2}

Racah Institute of Physics, The Hebrew University
Jerusalem, Israel

\textbf{ABSTRACT}

The linear algorithm of the Wiener filter and constrained realizations (CRs) of Gaussian random fields is extended here to perform non-linear CRs. The procedure consists of: (1) Using linear CR of low resolution data to construct a high resolution underlying field, as if the linear theory is valid; (2) Taking the linear CR backwards in time, by the linear theory, to set initial conditions for an N-body simulation; (3) Forwarding the field in time by an N-body code. An intermediate step might be introduced to ‘linearize’ the low resolution data.

The non-linear CR can be applied to any observational data set that is linearly related to the underlying field. Here it is applied to the IRAS 1.2Jy catalog using 843 data points within a sphere of $6000\,Km/s$, to reconstruct the full non-linear large scale structure of our ‘local’ universe.

\textsuperscript{1} E-mail: bisto@vms.huji.ac.il
\textsuperscript{2} E-mail: hoffman@vms.huji.ac.il
I. Introduction

In the standard model of cosmology galaxies and the large scale structure of the universe form out of a random perturbation field via gravitational instability. It is assumed that the primordial perturbation field constitutes a random homogenous and isotropic Gaussian field and that on relevant scales its amplitude is small, hence its evolution is described by the linear theory of gravitational instability (cf. Peebles 1980). The theoretical study of structure formation has been a major effort of modern cosmology (cf. Padmanabhan 1993). On the observational side, the large scale structure has been studied mostly by means of red-shift surveys (cf. Strauss and Willick 1995) and peculiar velocities (cf. Dekel 1994). A method for the reconstruction of the underlying dynamical (density and velocity) fields from a given observational data base is presented here.

The problem of recovering the underlying field from given observations, which by their nature are incomplete and have a finite accuracy and resolution, is one often encountered in many branches of physics and astronomy. It has been shown that for a random Gaussian field an optimal estimator of the underlying field is given by a minimal variance solution (Zaroubi, Hoffman, Fisher and Lahav 1996; ZHFL), known also as the Wiener filter (hereafter WF, Wiener 1949, Press et al. 1986). This approach is based on the a priori knowledge of the statistical nature of the field, the so-called prior. Within the framework of Gaussian fields the WF coincides with the Bayesian posterior and the maximum entropy estimations (ZHFL). Indeed, in the cosmological case on large enough scales where linear theory applies and the (over) density and velocity fields are Gaussian the WF is the optimal tool for the reconstruction of the large scale structure. This is further complemented by the algorithm of constrained realizations (CRs) of Gaussian fields (Hoffman and Ribak 1991) to create Monte Carlo simulations of the residual from this optimal estimation. This combined WF/CR approach has been applied recently to a variety of cosmological data bases in an effort to reconstruct the large scale structure. This includes the analysis of the
COBE/DMR data (Bunn et al. 1994), the analysis of the velocity potential (Ganon and Hoffman 1993), the reconstruction of the density field (Hoffman 1993, 1994, Lahav 1993, 1994, Lahav et al. 1994) and the peculiar velocity field (Fisher et al. 1995) from the IRAS redshift survey (Fisher et al. 1993).

A major limitation of the WF/CR approach is that it applies only in the linear regime. Yet, on small scales the perturbations are not small and the full non-linear gravitational instability theory has to be used. Here the WF/CR method is extended to the non-linear regime, and a new algorithm of non-linear constrained realizations (NLCRs) is presented. The general method is presented in §II and its application to the IRAS 1.2Jy catalog is given in §III. The results are presented in §IV and a short discussion (§V) concludes this Letter.

II. Non Linear Constrained Realizations

The general WF/CR method has been fully described in ZHFL and only a very short outline of it is presented here. Consider the case of a set of observations performed on an underlying random field (with $N$ degrees of freedom) $s = \{s_1, ..., s_N\}$ yielding $M$ data points, $d = \{d_1, ..., d_M\}$. Here, only measurements that can be modeled as linear convolution or mapping on the field are considered. The act of observation is represented by

$$d = Rs + \epsilon,$$

where $R$ is a linear operator which represents the point spread function and $\epsilon = \{\epsilon_1, ..., \epsilon_M\}$ gives the statistical errors. Here the notion of a point spread function is extended to include any linear operation that relates the measurements to the underlying field. The WF estimator is:

$$s_{\text{WF}} = \langle s d^\dagger \rangle \langle d d^\dagger \rangle^{-1} d.$$

Here, $\langle ... \rangle$ represents an ensemble average and $\langle s d^\dagger \rangle$ is the cross-correlation matrix of
the data and the underlying field. The data auto correlation matrix is

$$\langle d d^\dagger \rangle = R \langle s s^\dagger \rangle R + \langle \epsilon \epsilon^\dagger \rangle,$$

where the second term represents the statistical errors, \(i.e.\) shot noise. (See ZHFL for a general treatment of the error covariance matrix.)

In the case of a random Gaussian field, the WF estimator coincides with the conditional mean field given the data. A CR of the random residual from the mean is obtained by creating an unconstrained realization of the underlying field (\(\tilde{s}\)) and the errors (\(\tilde{\epsilon}\)), and ‘observing’ it the same way the actual universe is observed. Namely, a mock data base is created by:

$$\tilde{d} = R\tilde{s} + \tilde{\epsilon},$$

A CR is then obtained simply by (Hoffman and Ribak 1991):

$$s^{CR} = \tilde{s} + \left( \langle s d^\dagger \rangle \langle d d^\dagger \rangle \right)^{-1} (d - \tilde{d}).$$

The WF/CR is used now to uncover the finite resolution used to obtain the data. Thus, low resolution data is used to make high resolution CRs. Here, for simplicity the finite resolution is modeled by Gaussian smoothing, however any other kernel can be used as well. The low resolution is modeled by smoothing on scale \(R_L\) and the high resolution by \(R_S\), where \(R_L > R_S\). Often, the low resolution data correspond to observables in, or close to, the linear regime, while the high resolution field lies deep in the non-linear regime. However, the reconstruction is done within the framework of the linear theory. This might be a reasonable assumption for the low resolution data, but it is certainly inconsistent with the high resolution field. The following procedure is suggested here for the NLCR: I. Use low resolution data to construct a high resolution CR, as if linear theory holds on these small scales; II. Take this CR backwards to an early enough epoch by the linear theory; III. Use this CR, which now constitutes a given realization of the initial conditions of our ‘local’ universe constrained by observational data, as an input to an N-body code to evolve
it to the present time. These three steps provide one with a NLCR given the observed data and the assumed prior model.

The three steps proposed here are all consistent with the general framework of the standard model, namely Gaussian primordial perturbation field and gravitational instability. Now, in the case where the smoothing indeed transforms dynamical variables to the linear regime, the constructed field provides one with a particular realization which is fully consistent with the assumed model. The quality of the reconstruction depends on the accuracy of the data and its sampling and on the nature of the prior, namely the ‘strength’ of the correlations. However, often it happens that smoothing does not take the data all the way to the linear regime. In such a case an intermediate step of mapping these quasi-linear variables to the linear ones has to be introduced. Such a mapping cannot be usually rigorously formulated and one should recourse to some approximations. These should be checked against N-body simulations and mock catalogs, to find a mapping suitable to the problem at hand.

III. Application: The IRAS 1.2Jy Catalog

The WF/CR and the NLCR presented here can be used with any data base whose relation to the underlying field can be modeled by Eq. 1. Thus, observations of the velocity field can be used to reconstruct the density field and vice versa. The concrete case studied here is the construction of the density and velocity fields from the IRAS 1.2Jy redshift survey. At present red-shift distortions are ignored, however the formalism can be easily extended to account for these as well (Zaroubi and Hoffman 1995). The sample is defined by its selection function, φ(r), and the boundaries of the survey are defined by a mask of galactic latitude |b| < 5°. The prior assumed here is a CDM power spectrum with a shape parameter Γ = 0.2 and a normalization of σ8 = 0.7 (cf. Strauss and Willick 1995). For simplicity no biasing and a flat universe are assumed.

The underlying density field is evaluated on a Cartesian grid with a sampling rate
of 1000 $Km/s$ (here distances are given in velocity units) within a sphere of 6000 $Km/s$, excluding IRAS’ zone of avoidance. The discrete galaxy distribution is smoothed on a scale $R_L = 1000 $Km/s. This yields $M = 834$ data points:

$$
\Delta_\alpha = \Delta(\mathbf{r}_\alpha) = \left[ \sum_{gal} \frac{1}{\phi(r_{gal})} \exp\left(-\frac{(\mathbf{r}_\alpha - \mathbf{r}_{gal})^2}{2R_L^2}\right) - \bar{n} \right]/\bar{n} \tag{6}
$$

where $\bar{n}$ is the mean number density of the IRAS galaxies. The data autocorrelation function is written as $\langle \Delta_\alpha \Delta_\beta \rangle = \xi_{\alpha\beta} + \sigma_{\alpha\beta}$. The first term is just the autocorrelation function of the smoothed field ($\xi^s(r)$),

$$
\xi_{\alpha\beta} = \xi^s(|\mathbf{r}_\alpha - \mathbf{r}_\beta|) = \frac{1}{(2\pi)^3} \int P(k) \exp\left(-kR_S^2\right) \exp(ik \cdot (\mathbf{r}_\alpha - \mathbf{r}_\beta)) d^3k, \tag{7}
$$

and the shot noise covariance matrix is:

$$
\sigma_{\alpha\beta} = \frac{1}{\bar{n}(2\pi R_L^2)^{3/2}} \int \frac{1}{\phi(x)} \exp\left(-\frac{(\mathbf{r}_\alpha - \mathbf{x})^2 + (\mathbf{r}_\beta - \mathbf{x})^2}{2R_L^2}\right) d^3x \tag{8}
$$

Note that the kernel introduces off-diagonal terms in the error covariance matrix (ZHFL).

The cross-correlation of the high resolution field and the low resolution data is:

$$
\xi_{\alpha}(\mathbf{r}_i) = \frac{1}{(2\pi)^3} \int P(k) \exp\left(-\frac{(kR_S)^2 + (kR_L)^2}{2}\right) \exp(ik \cdot (\mathbf{r}_i - \mathbf{r}_\alpha)) d^3k \tag{9}
$$

Defining the WF operator $W_{i\beta}$,

$$
W_{i\beta} = \xi_{\alpha}(\mathbf{r}_i) \left( \xi_{\alpha\beta} + \sigma_{\alpha\beta} \right)^{-1}, \tag{10}
$$

and a linear high resolution is thus obtained by

$$
\delta(\mathbf{r}_i) = \tilde{\delta}(\mathbf{r}_i) + W_{i\beta} \left( \Delta_\beta - \tilde{\Delta}_\beta \right). \tag{11}
$$

The choice of $R_L$ plays a crucial role in the NLCR algorithm and it involves conflicting considerations. On the one hand a small $R_L$ is desired, so as to keep high resolution information, but on the other one a large $R_L$ would minimize the shot noise errors and
would result in a ‘more’ linear estimator. Here a value of $R_L = 1000Km/s$ is chosen. To check the linearization of this smoothing a non-linear unconstrained realization of the assumed prior has been calculated by a PM N-body code. Now, the smoothed (scale $R_L$) (over)density is evaluated in two ways. One is done by smoothing the initial conditions and propagating it in time by the linear theory, yielding $\delta^L$. The other, $\delta^{NL}$, is obtained by smoothing the full non-linear density field. It is known that even on the 1000Km/s smoothing scale there are systematic deviation and scatter from the desired $\delta^{NL} = \delta^L$ relation (Nusser et al. 1991). One finds that on both ends of high amplitudes positive and negative $\delta$’s, $\delta^{NL}$ is larger than $\delta^L$. Note that this is a pure dynamical phenomenon and the statistical shot noise does not affect it. A minimal variance fitting formula is calculated here, $\delta^L = f(\delta^{NL})\delta^{NL}$, and this is used to recover the linear field. Note that $f(\delta^{NL})$ depends on the assumed model and the smoothing kernel. A consistency check on this simple mapping is the evaluation of the 1-point distribution function of $f(\delta^{NL})\delta^{NL}$. Indeed it is found to be very close to that of $\delta^L$, namely a normal distribution.

Various algorithms have been proposed to trace back non-linear perturbation field to the linear regime (cf. Strauss and Willick 1995). All of these ‘time machines’ recover the initial linear field in the case where the quasi-linear field is known exactly, with no statistical uncertainty. The case of real observational data where the shot noise increases with distance, poses a much more difficult problem. As one goes further away the data becomes more dominant by the shot noise and in the mean the amplitude of the measured field increases with distance. Thus, before applying any ‘time machine’ the signal has to be first cleaned from the noise and only then it can be traced back to the linear regime.

The phenomenological fix to the ‘non-linearity’ of the smoothed data which is used here consists of two steps. First, to account for the scatter in the $(\delta^{NL}, \delta^L)$ relation a new term is introduced to the data auto-covariance matrix, $\sigma^{NL}$. Dealing with the scatter by statistical means is a manifestation of our inability to invert the exact non-local non-linear mapping from the linear to the quasi-linear regime. Here we go to the extreme
simplification and take $\sigma^{NL}_{\alpha\beta} = const.\delta_{\alpha\beta}$. The value of the constant term is determined by the requirement that $\chi^2/d.o.f. = 1$, where the $\chi^2$ takes into account the cosmic variance, shot noise and $\sigma^{NL}$. A WF estimator of the $R_L$-smoothed field is obtained by applying a WF on the data, where $R_S$ is replaced by $R_L$ to obtain low resolution,

$$\delta^{WF,QL}(r_i) = \left[W_{i\alpha}\right]_{R_S=R_L} \Delta_\alpha.$$  \hspace{1cm} (12)

The estimation of the quasi-linear correction is given by $(1 - f(\delta^{WF,QL})\delta^{WF,QL}$. This correction is evaluated at grid points $r_\alpha$ and is used to correct the data points:

$$\Delta^L_\alpha = \Delta_\alpha - (1 - f(\delta^{WF,QL})\delta^{WF,QL}.$$ \hspace{1cm} (13)

The modified (‘linearized’) $\Delta^L_\alpha$’s are now substituted in Eq. 10 to obtain a high resolution CR of the underlying linear field, given the actual data.

The linearization procedure presented here behaves as follows. In the limit of distant data points, where the data is dominated by shot noise, the WF attenuates the estimated field towards zero amplitude. Substituting the resulting estimator in Eq. 12 would hardly change its value. The WF/CR is therefore dominated by the random residual, and consequently the resulting realization lies in the linear regime. For nearby data points where the shot noise is negligible, the WF leaves the signal almost untouched $\delta^{WF,QL} \approx \Delta$. In such a case the fitting formula would linearize the data, as has been checked against the N-body simulations.

**IV. Volume Limited IRAS Catalog**

The IRAS 1.2Jy catalog consists of 5321 galaxies These are used to evaluate the smoothed density field on a Cartesian grid of 1000$Km/s$ spacing within a 6000$Km/s$, excluding the zone of avoidance, yielding 844 data points. NLCRs are created on a finer $64^3$ grid of 250$Km/s$ spacing. A PM N-body code that is used here is based on an FFT Poisson solver. For a CDM-like power spectrum the structure within the 6000$Km/s$ would
be hardly affected by the periodicity on the ±8000 Km/s box. A comprehensive analysis of NLCR, including detailed comparison of reconstruction of mock catalogs and sensitivity to the assumed prior has been conducted and will be presented in a forthcoming paper (Bistolas and Hoffman, 1995a).

The IRAS galaxy distribution is presented in Fig. 1, where the projected galaxy distribution (within ±1000 Km/s) on the nine planes of SGX, SGY, SGZ = ±3000, 0 Km/s is given. The full N-body distribution of the NLCR is presented in Fig. 2 in a manner similar to that of Fig. 1. Note that this consists a realization of a ‘volume limited’ IRAS catalog. Finally, the non-linear reconstructed field at 500 Km/s smoothing is presented in Fig. 3. A full analysis of the cosmography revealed by the NLCR will be given in a forthcoming publication (Bistolas and Hoffman 1995b). Here we just point to the seemingly filamentary structure of the reconstructed galaxy distribution. A closer inspection shows that the generic feature here is more planar (2D) rather than a filamentary (1D) structure, and the apparent filaments are the intersection of the sheets with the planes defined by the plots. Different NLCRs have been performed to study the variance implied by the prior and the data and relatively small scatter is found between the different realizations. In particular, the existence and location of peaks and troughs is a very robust feature of the realization with small scatter in their amplitudes. Also the sheets and filaments remain invariant under the different realizations, however their ‘sharpness’ varies somewhat. A comparison of two such realizations is given in Fig. 4, where the ‘volume limited’ galaxy distribution and 500 Km/s smoothed δ-field at the supergalactic (SGZ = 0) plane are plotted.

V. Discussion

The NLCR algorithm presented here enables one to perform controlled Monte Carlo N-body simulations of the formation of our ‘local’ universe. These are designed to recover the actual observational data, used to constrain them, within the statistical uncertainties
of the data. The new ingredient introduced here is the reconstruction of the non-linear regime, *i.e.* the extrapolation in Fourier space from small to large wavenumbers that are deep in the non-linear regime.

The NLCR introduced here can serve as a tool for studying and analyzing the large scale structure of the universe. Some of the obvious problems where NLCRs are expected to be very useful are: (1) The reconstruction of the velocity field from redshift catalogs; (2) Mapping the zone of avoidance and extrapolating the dynamical fields into unobserved regions; (3) Studying the dynamics of actually observed rich clusters with the actual initial and boundary conditions; (4) Analysis of filaments and pancakes as probes of the initial conditions and the cosmological model; (5) The NLCR can serve as a probe of the biasing mechanism. The main virtue here lies in the fact that different data sets, which in principal can represent different biasing of the underlying dynamical field, can be used to simultaneously set constraints on the realizations. Given all these and the technical simplicity of the algorithm we expect it to be a standard tool of N-body and gas dynamical simulations.

At the time this *Letter* has been written Kolatt *et al.* (1995) have reported on a similar project of NLCR of the IRAS 1.2Jy catalog. Their procedure differs from the present one mainly in not distinguishing between the low resolution (data) and high resolution (realizations). The input data is smoothed on the 500$Km/s$ scale and is heavily dominated by the noise, which is ‘removed’ by a power preserving modified WF. The modified filter is designed to preserve the power, regardless of the noise level. The resulting estimator is therefore more dominated by the noise and less by the *prior* model compared to our method. Yet, both methods seem to yield similar results and are equally efficient. Detailed comparisons against N-body simulations are needed to judge the merits of each method.
Acknowledgments

The members of the IRAS collaboration are gratefully acknowledged for their help with the IRAS data base. We have benefited from many stimulating discussions with L. da Costa, A. Dekel, O. Lahav and S. Zaroubi. This research has been supported in part by The Hebrew University Internal Funds (grant 53/94) and by the Israel Science Foundation administrated by the Israeli Academy of Sciences and Humanities (grant 590/94).
References

Bistolas, V. and Hoffman, Y., 1995a (in preparation).
Bistolas, V. and Hoffman, Y., 1995b (in preparation).
Bunn, E., Fisher, K.B., Hoffman, Y., Lahav, O., Silk, J., & Zaroubi, S. 1994, Ap. J. Lett., 432, L75.
Dekel, A., 1994, Ann. Rev. Astron. Astrophys., 32, 371.
Fisher, K.B., Lahav, O., Hoffman, Y., Lynden-Bell, D. & Zaroubi, S. 1994, M.N.R.A.S., in press.
Ganon, G. and Hoffman, Y., 1993, Ap. J. Lett., 415, L5.
Hoffman, Y. 1993, Proc. of the 9th IAP Conference on Cosmic Velocity Fields, eds. F. Bouchet and M. Lachiéze-Rey, (Gif-sur-Yvette Cedex: Editions Frontières), p. 357
Hoffman, Y. 1994, in ‘Unveiling Large Scale Structures Behind the Milky-Way’, eds. C. Balkowski and R.C. Kraan-Korteweg, PASP conference series.
Hoffman, Y. & Ribak, E. 1991, Ap. J. Lett., 380, L5.
Kolatt, T., Dekel, A., Ganon, G., and Willick, J.A., 1995, Ap. J.(submit,).
Lahav, O. 1993, Proc. of the 9th IAP Conference on Cosmic Velocity Fields, eds. F. Bouchet and M. Lachiéze-Rey, (Gif-sur-Yvette Cedex: Editions Frontières) p. 205
Lahav, O. 1994, in ‘Unveiling Large Scale Structures Behind the Milky-Way’, eds. C. Balkowski and R.C. Kraan-Korteweg, PASP conference series.
Lahav, O., Fisher, K.B., Hoffman, Y., Scharf, C.A., & Zaroubi, S. 1994, Ap. J. Lett., 423, L93.
Nusser, A., Dekel, A., Bertschinger, E., and Blumethal, G.R., 1991, Ap. J., 379, 6.
Padmanabhan, T., 1993, Structure Formation in the Universe, (Cambridge: Cambridge University Ppress).
Peebles, P.J.E. 1980, The Large-Scale Structure of the Universe, (Princeton: Princeton University Press).
Press, W.H., Teukolsky, S.A., Vetterling, W.T., & Flannery, B.P. 1992, Numerical Recipes (Second Edition) (Cambridge: Cambridge University Press).

Strauss, M.A. and Willick, J.A., 1995, Physics Report (in press).

Wiener, N. 1949, in Extrapolation and Smoothing of Stationary Time Series, (New York: Wiley)

Zaroubi, S., Hoffman, Y., Fisher, K.B., and S. Lahav, O., 1995, Ap. J., (in press; ZHFL).

Zaroubi, S., and Hoffman, Y., 1995, Ap. J., (submit.).
Figure Captions

Fig. 1: Raw data: The IRAS 1.2Jy galaxies. The galaxy distribution is presented in 9 planar slabs of thickness of ±10h⁻¹Mpc. (Supergalactic coordinates are used and distances are given in h⁻¹Mpc, where h is Hubble’s constant in units of 100Km/s/Mpc.)

Fig. 2. ‘Volume limited’ IRAS catalog: A non-linear constrained realization based on the IRAS galaxy distribution. The full N-body particle distribution has been diluted to the mean IRAS galaxies mean number density.

Fig. 3. Gaussian smoothing: The non-linear constrained realization shown in Fig. 2 is Gaussian smoothed on a scale R = 500Km/s. Contour spacing is 0.2 and the dashed lines correspond to negative values of δ.

Fig. 4. Different realizations: A comparison of different non-linear constrained realizations of the same input data and prior is presented here by the ‘galaxy’ distribution and the contour plots. Upper and lower raws correspond to two different realizations. The galaxy distribution and contour plots are presented in the same way as in Figs. 2 and 3.
Non-linear Reconstruction of the Large Scale Structure

V. Bistolas$^1$ and Y. Hoffman$^2$

Racah Institute of Physics, The Hebrew University
Jerusalem, Israel

ABSTRACT

The linear algorithm of the Wiener filter and constrained realizations (CRs) of Gaussian random fields is extended here to perform non-linear CRs. The procedure consists of: (1) Using linear CR of low resolution data to construct a high resolution underlying field, as if the linear theory is valid; (2) Taking the linear CR backwards in time, by the linear theory, to set initial conditions for an N-body simulation; (3) Forwarding the field in time by an N-body code. An intermediate step might be introduced to ‘linearize’ the low resolution data.

The non-linear CR can be applied to any observational data set that is linearly related to the underlying field. Here it is applied to the IRAS 1.2Jy catalog using 843 data points within a sphere of $6000Km/s$, to reconstruct the full non-linear large scale structure of our ‘local’ universe.

---

$^1$ E-mail: bisto@vms.huji.ac.il
$^2$ E-mail: hoffman@vms.huji.ac.il
I. Introduction

In the standard model of cosmology galaxies and the large scale structure of the universe form out of a random perturbation field via gravitational instability. It is assumed that the primordial perturbation field constitutes a random homogenous and isotropic Gaussian field and that on relevant scales its amplitude is small, hence its evolution is described by the linear theory of gravitational instability (cf. Peebles 1980). The theoretical study of structure formation has been a major effort of modern cosmology (cf. Padmanabhan 1993). On the observational side, the large scale structure has been studied mostly by means of red-shift surveys (cf. Strauss and Willick 1995) and peculiar velocities (cf. Dekel 1994). A method for the reconstruction of the underlying dynamical (density and velocity) fields from a given observational data base is presented here.

The problem of recovering the underlying field from given observations, which by their nature are incomplete and have a finite accuracy and resolution, is one often encountered in many branches of physics and astronomy. It has been shown that for a random Gaussian field an optimal estimator of the underlying field is given by a minimal variance solution (Zaroubi, Hoffman, Fisher and Lahav 1996; ZHFL), known also as the Wiener filter (hereafter WF, Wiener 1949, Press et al. 1986). This approach is based on the a priori knowledge of the statistical nature of the field, the so-called prior. Within the framework of Gaussian fields the WF coincides with the Bayesian posterior and the maximum entropy estimations (ZHFL). Indeed, in the cosmological case on large enough scales where linear theory applies and the (over) density and velocity fields are Gaussian the WF is the optimal tool for the reconstruction of the large scale structure. This is further complemented by the algorithm of constrained realizations (CRs) of Gaussian fields (Hoffman and Ribak 1991) to create Monte Carlo simulations of the residual from this optimal estimation. This combined WF/CR approach has been applied recently to a variety of cosmological data bases in an effort to reconstruct the large scale structure. This includes the analysis of the
COBE/DMR data (Bunn et al. 1994), the analysis of the velocity potential (Ganon and Hoffman 1993), the reconstruction of the density field (Hoffman 1993, 1994, Lahav 1993, 1994, Lahav et al. 1994) and the peculiar velocity field (Fisher et al. 1995) from the IRAS redshift survey (Fisher et al. 1993).

A major limitation of the WF/CR approach is that it applies only in the linear regime. Yet, on small scales the perturbations are not small and the full non-linear gravitational instability theory has to be used. Here the WF/CR method is extended to the non-linear regime, and a new algorithm of non-linear constrained realizations (NLCRs) is presented. The general method is presented in §II and its application to the IRAS 1.2Jy catalog is given in §III. The results are presented in §IV and a short discussion (§V) concludes this Letter.

II. Non Linear Constrained Realizations

The general WF/CR method has been fully described in ZHFL and only a very short outline of it is presented here. Consider the case of a set of observations performed on an underlying random field (with \( N \) degrees of freedom) \( \mathbf{s} = \{s_1, ..., s_N\} \) yielding \( M \) data points, \( \mathbf{d} = \{d_1, ..., d_M\} \). Here, only measurements that can be modeled as linear convolution or mapping on the field are considered. The act of observation is represented by

\[
\mathbf{d} = \mathbf{R}\mathbf{s} + \mathbf{\epsilon},
\]

where \( \mathbf{R} \) is a linear operator which represents the point spread function and \( \mathbf{\epsilon} = \{\epsilon_1, ..., \epsilon_M\} \) gives the statistical errors. Here the notion of a point spread function is extended to include any linear operation that relates the measurements to the underlying field. The WF estimator is:

\[
\mathbf{s}^{WF} = \langle \mathbf{s} \mathbf{d}^\dagger \rangle \langle \mathbf{d} \mathbf{d}^\dagger \rangle^{-1} \mathbf{d}.
\]

Here, \( \langle ... \rangle \) represents an ensemble average and \( \langle \mathbf{s} \mathbf{d}^\dagger \rangle \) is the cross-correlation matrix of \( \mathbf{s} \) and \( \mathbf{d} \).
the data and the underlying field. The data auto correlation matrix is

$$\langle \mathbf{d} \mathbf{d}^\dagger \rangle = \mathbf{R} \langle \mathbf{s} \mathbf{s}^\dagger \rangle \mathbf{R} + \langle \mathbf{\epsilon} \mathbf{\epsilon}^\dagger \rangle,$$

where the second term represents the statistical errors, i.e. shot noise. (See ZHFL for a general treatment of the error covariance matrix.)

In the case of a random Gaussian field, the WF estimator coincides with the conditional mean field given the data. A CR of the random residual from the mean is obtained by creating an unconstrained realization of the underlying field ($\tilde{\mathbf{s}}$) and the errors ($\tilde{\mathbf{\epsilon}}$), and ‘observing’ it the same way the actual universe is observed. Namely, a mock data base is created by:

$$\tilde{\mathbf{d}} = \mathbf{R} \tilde{\mathbf{s}} + \tilde{\mathbf{\epsilon}},$$

A CR is then obtained simply by (Hoffman and Ribak 1991):

$$\mathbf{s}^{\text{CR}} = \tilde{\mathbf{s}} + \langle \mathbf{s} \mathbf{d}^\dagger \rangle \langle \mathbf{d} \mathbf{d}^\dagger \rangle^{-1} (\mathbf{d} - \tilde{\mathbf{d}}).$$

The WF/CR is used now to uncover the finite resolution used to obtain the data. Thus, low resolution data is used to make high resolution CRs. Here, for simplicity the finite resolution is modeled by Gaussian smoothing, however any other kernel can be used as well. The low resolution is modeled by smoothing on scale $R_L$ and the high resolution by $R_S$, where $R_L > R_S$. Often, the low resolution data correspond to observables in, or close to, the linear regime, while the high resolution field lies deep in the non-linear regime. However, the reconstruction is done within the framework of the linear theory. This might be a reasonable assumption for the low resolution data, but it is certainly inconsistent with the high resolution field. The following procedure is suggested here for the NLCR: I. Use low resolution data to construct a high resolution CR, as if linear theory holds on these small scales; II. Take this CR backwards to an early enough epoch by the linear theory; III. Use this CR, which now constitutes a given realization of the initial conditions of our ‘local’ universe constrained by observational data, as an input to an N-body code to evolve
it to the present time. These three steps provides one with a NLCR given the observed data and the assumed *prior* model.

The three steps proposed here are all consistent with the general framework of the standard model, namely Gaussian primordial perturbation field and gravitational instability. Now, in the case where the smoothing indeed transforms dynamical variables to the linear regime, the constructed field provides one with a particular realization which is fully consistent with the assumed model. The quality of the reconstruction depends on the accuracy of the data and its sampling and on the nature of the *prior*, namely the ‘strength’ of the correlations. However, often it happens that smoothing does not take the data all the way to the linear regime. In such a case an intermediate step of mapping these quasi-linear variables to the linear ones has to be introduced. Such a mapping cannot be usually rigorously formulated and one should recourse to some approximations. These should be checked against N-body simulations and mock catalogs, to find a mapping suitable to the problem at hand.

**III. Application: The IRAS 1.2Jy Catalog**

The WF/CR and the NLCR presented here can be used with any data base whose relation to the underlying field can be modeled by Eq. 1. Thus, observations of the velocity field can be used to reconstruct the density field and *vice versa*. The concrete case studied here is the construction of the density and velocity fields from the IRAS 1.2Jy redshift survey. At present red-shift distortions are ignored, however the formalism can be easily extended to account for these as well (Zaroubi and Hoffman 1995). The sample is defined by its selection function, \( \phi(r) \), and the boundaries of the survey are defined by a mask of galactic latitude \(|b| < 5^\circ\). The *prior* assumed here is a CDM power spectrum with a shape parameter \( \Gamma = 0.2 \) and a normalization of \( \sigma_8 = 0.7 \) (cf. Strauss and Willick 1995). For simplicity no biasing and a flat universe are assumed.

The underlying density field is evaluated on a Cartesian grid with a sampling rate
of 1000\(Km/s\) (here distances are given in velocity units) within a sphere of 6000\(Km/s\), excluding IRAS' zone of avoidance. The discrete galaxy distribution is smoothed on a scale \(R_L = 1000Km/s\). This yields \(M = 834\) data points:

\[
\Delta_\alpha = \Delta(r_\alpha) = \left[ \sum_{gal} \frac{1}{\phi(r_{gal})} \exp\left( -\frac{(r_\alpha - r_{gal})^2}{2R_L^2} \right) - \bar{n} \right] / \bar{n}
\]

(6)

where \(\bar{n}\) is the mean number density of the IRAS galaxies. The data autocorrelation function is written as \(\langle \Delta_\alpha \Delta_\beta \rangle = \xi_{\alpha \beta} + \sigma_{\alpha \beta}\). The first term is just the autocorrelation function of the smoothed field (\(\xi^s(r)\)),

\[
\xi_{\alpha \beta} = \xi^s(|r_\alpha - r_\beta|) = \frac{1}{(2\pi)^3} \int P(k) \exp(- (kR_L)^2) \exp(i k \cdot (r_\alpha - r_\beta)) d^3k,
\]

(7)

and the shot noise covariance matrix is:

\[
\sigma_{\alpha \beta} = \frac{1}{\bar{n}(2\pi R_L^2)^3/2} \int \frac{1}{\phi(x)} \exp\left( -\frac{(r_\alpha - x)^2 + (r_\beta - x)^2}{2R_L^2} \right) d^3x
\]

(8)

Note that the kernel introduces off-diagonal terms in the error covariance matrix (ZHFL). The cross-correlation of the high resolution field and the low resolution data is:

\[
\xi_{\alpha}(r_i) = \frac{1}{(2\pi)^3} \int P(k) \exp\left( -\frac{(kR_S)^2}{2} + \frac{(kR_L)^2}{2} \right) \exp(i k \cdot (r_i - r_\alpha)) d^3k
\]

(9)

Defining the WF operator \(W_{i\beta}\),

\[
W_{i\beta} = \xi_{\alpha}(r_i) \left( \xi_{\alpha \beta} + \sigma_{\alpha \beta} \right)^{-1},
\]

(10)

and a linear high resolution is thus obtained by

\[
\delta(r_i) = \tilde{\delta}(r_i) + W_{i\beta} \left( \Delta_\beta - \tilde{\Delta}_\beta \right).
\]

(11)

The choice of \(R_L\) plays a crucial role in the NLCR algorithm and it involves conflicting considerations. On the one hand a small \(R_L\) is desired, so as to keep high resolution information, but on the other one a large \(R_L\) would minimize the shot noise errors and
would result in a ‘more’ linear estimator. Here a value of $R_L = 1000\text{Km/s}$ is chosen. To check the linearization of this smoothing a non-linear unconstrained realization of the assumed prior has been calculated by a PM N-body code. Now, the smoothed (scale $R_L$) (over)density is evaluated in two ways. One is done by smoothing the initial conditions and propagating it in time by the linear theory, yielding $\delta^L$. The other, $\delta^{NL}$, is obtained by smoothing the full non-linear density field. It is known that even on the $1000\text{Km/s}$ smoothing scale there are systematic deviation and scatter from the desired $\delta^{NL} = \delta^L$ relation (Nusser et al. 1991). One finds that on both ends of high amplitudes positive and negative $\delta$’s, $\delta^{NL}$ is larger than $\delta^L$. Note that this is a pure dynamical phenomenon and the statistical shot noise does not affect it. A minimal variance fitting formula is calculated here, $\delta^L = f(\delta^{NL})\delta^{NL}$, and this is used to recover the linear field. Note that $f(\delta^{NL})$ depends on the assumed model and the smoothing kernel. A consistency check on this simple mapping is the evaluation of the 1-point distribution function of $f(\delta^{NL})\delta^{NL}$. Indeed it is found to be very close to that of $\delta^L$, namely a normal distribution.

Various algorithms have been proposed to trace back non-linear perturbation field to the linear regime (cf. Strauss and Willick 1995). All of these ‘time machines’ recover the initial linear field in the case where the quasi-linear field is known exactly, with no statistical uncertainty. The case of real observational data where the shot noise increases with distance, poses a much more difficult problem. As one goes further away the data becomes more dominant by the shot noise and in the mean the amplitude of the measured field increases with distance. Thus, before applying any ‘time machine’ the signal has to be first cleaned from the noise and only then it can be traced back to the linear regime.

The phenomenological fix to the ‘non-linearity’ of the smoothed data which is used here consists of two steps. First, to account for the scatter in the $(\delta^{NL}, \delta^L)$ relation a new term is introduced to the data auto-covariance matrix, $\sigma^{NL}$. Dealing with the scatter by statistical means is a manifestation of our inability to invert the exact non-local non-linear mapping from the linear to the quasi-linear regime. Here we go to the extreme
simplification and take $\sigma^{NL}_{\alpha\beta} = \text{const.} \delta_{\alpha\beta}$. The value of the constant term is determined by the requirement that $\chi^2/d.o.f. = 1$, where the $\chi^2$ takes into account the cosmic variance, shot noise and $\sigma^{NL}$. A WF estimator of the $R_L$-smoothed field is obtained by applying a WF on the data, where $R_S$ is replaced by $R_L$ to obtain low resolution,

$$\delta^{WF,QL}(r_i) = \left[ W_{i\alpha} \right]_{R_S=R_L} \Delta_{\alpha}. \quad (12)$$

The estimation of the quasi-linear correction is given by $(1 - f(\delta^{WF,QL})\delta^{WF,QL}$. This correction is evaluated at grid points $r_\alpha$ and is used to correct the data points:

$$\Delta^L_{\alpha} = \Delta_{\alpha} - (1 - f(\delta^{WF,QL})\delta^{WF,QL}. \quad (13)$$

The modified (‘linearized’) $\Delta^L_{\alpha}$’s are now substituted in Eq. 10 to obtain a high resolution CR of the underlying linear field, given the actual data.

The linearization procedure presented here behaves as follows. In the limit of distant data points, where the data is dominated by shot noise, the WF attenuates the estimated field towards zero amplitude. Substituting the resulting estimator in Eq. 12 would hardly change its value. The WF/CR is therefore dominated by the random residual, and consequently the resulting realization lies in the linear regime. For nearby data points where the shot noise is negligible, the WF leaves the signal almost untouched $\delta^{WF,QL} \approx \Delta$. In such a case the fitting formula would linearized the data, as has been checked against the N-body simulations.

**IV. Volume Limited IRAS Catalog**

The IRAS 1.2Jy catalog consists of 5321 galaxies. These are used to evaluate the smoothed density field on a Cartesian grid of 1000$Km/s$ spacing within a 6000$Km/s$, excluding the zone of avoidance, yielding 844 data points. NLCRs are created on a finer $64^3$ grid of 250$Km/s$ spacing. A PM N-body code that is used here is based on an FFT Poisson solver. For a CDM-like power spectrum the structure within the 6000$Km/s$ would
be hardly affected by the periodicity on the ±8000Km/s box. A comprehensive analysis of NLCR, including detailed comparison of reconstruction of mock catalogs and sensitivity to the assumed prior has been conducted and will be presented in a forthcoming paper (Bistolas and Hoffman, 1995a).

The IRAS galaxy distribution is presented in Fig. 1, where the projected galaxy distribution (within ±1000Km/s) on the nine planes of $SGX, SGY, SGZ = ±3000, 0Km/s$ is given. The full N-body distribution of the NLCR is presented in Fig. 2 in a manner similar to that of Fig. 1. Note that this consists a realization of a ‘volume limited’ IRAS catalog. Finally, the non-linear reconstructed field at 500Km/s smoothing is presented in Fig. 3. A full analysis of the cosmography revealed by the NLCR will be given in a forthcoming publication (Bistolas and Hoffman 1995b). Here we just point to the seemingly filamentry structure of the reconstructed galaxy distribution. A closer inspection shows that the generic feature here is more planar (2D) rather than a filamentry (1D) structure, and the apparent filaments are the intersection of the sheets with the planes defined by the plots. Different NLCRs have been performed to study the variance implied by the prior and the data and relatively small scatter is found between the different realizations. In particular, the existence and location of peaks and troughs is a very robust feature of the realization with small scatter in their amplitudes. Also the sheets and filaments remain invariant under the different realizations, however their ‘sharpness’ varies somewhat. A comparison of two such realizations is given in Fig. 4, where the ‘volume limited’ galaxy distribution and 500Km/s smoothed $\delta$-field at the supergalactic ($SGZ = 0$) plane are plotted.

V. Discussion

The NLCR algorithm presented here enables one to perform controlled Monte Carlo N-body simulations of the formation of our ‘local’ universe. These are designed to recover the actual observational data, used to constrain them, within the statistical uncertainties
of the data. The new ingredient introduced here is the reconstruction of the non-linear regime, i.e. the extrapolation in Fourier space from small to large wavenumbers that are deep in the non-linear regime.

The NLCR introduced here can serve as a tool for studying and analyzing the large scale structure of the universe. Some of the obvious problems where NLCRs are expected to be very useful are: (1) The reconstruction of the velocity field from redshift catalogs; (2) Mapping the zone of avoidance and extrapolating the dynamical fields into unobserved regions; (3) Studying the dynamics of actually observed rich clusters with the actual initial and boundary conditions; (4) Analysis of filaments and pancakes as probes of the initial conditions and the cosmological model; (5) The NLCR can serve as a probe of the biasing mechanism. The main virtue here lies in the fact that different data sets, which in principal can represent different biasing of the underlying dynamical field, can be used to simultaneously set constraints on the realizations. Given all these and the technical simplicity of the algorithm we expect it to be a standard tool of N-body and gas dynamical simulations.

At the time this Letter has been written Kolatt et al. (1995) have reported on a similar project of NLCR of the IRAS 1.2Jy catalog. Their procedure differs from the present one mainly in not distinguishing between the low resolution (data) and high resolution (realizations). The input data is smoothed on the 500Km/s scale and is heavily dominated by the noise, which is ‘removed’ by a power preserving modified WF. The modified filter is designed to preserve the power, regardless of the noise level. The resulting estimator is therefore more dominated by the noise and less by the prior model compared to our method. Yet, both methods seem to yield similar results and are equally efficient. Detailed comparisons against N-body simulations are needed to judge the merits of each method.
Acknowledgments

The members of the IRAS collaboration are gratefully acknowledged for their help with the IRAS data base. We have benefited from many stimulating discussions with L. da Costa, A. Dekel, O. Lahav and S. Zaroubi. This research has been supported in part by The Hebrew University Internal Funds (grant 53/94) and by the Israel Science Foundation administrated by the Israeli Academy of Sciences and Humanities (grant 590/94).
References

Bistolas, V. and Hoffman, Y., 1995a (in preparation).
Bistolas, V. and Hoffman, Y., 1995b (in preparation).
Bunn, E., Fisher, K.B., Hoffman, Y., Lahav, O., Silk, J., & Zaroubi, S. 1994, Ap. J. Lett., 432, L75.
Dekel, A., 1994, Ann. Rev. Astron. Astrophys., 32, 371.
Fisher, K.B., Lahav, O., Hoffman, Y., Lynden-Bell, D. & Zaroubi, S. 1994, M.N.R.A.S., in press.
Ganon, G. and Hoffman, Y., 1993, Ap. J. Lett., 415, L 5.
Hoffman, Y. 1993, Proc. of the 9th IAP Conference on Cosmic Velocity Fields, eds. F.
Bouchet and M. Lachiéze-Rey, (Gif-sur-Yvette Cedex: Editions Frontières), p. 357
Hoffman, Y. 1994, in ‘Unveiling Large Scale Structures Behind the Milky-Way’, eds. C.
Balkowski and R.C. Kraan-Korteweg, PASP conference series.
Hoffman, Y. & Ribak, E. 1991, Ap. J. Lett., 380, L5.
Kolatt, T., Dekel, A., Ganon, G., and Willick, J.A., 1995, Ap. J.(submit.).
Lahav, O. 1993, Proc. of the 9th IAP Conference on Cosmic Velocity Fields, eds. F.
Bouchet and M. Lachiéze-Rey,(Gif-sur-Yvette Cedex: Editions Frontières) p. 205
Lahav, O. 1994, in ‘Unveiling Large Scale Structures Behind the Milky-Way’, eds. C.
Balkowski and R.C. Kraan-Korteweg, PASP conference series.
Lahav, O., Fisher, K.B., Hoffman, Y., Scharf, C.A., & Zaroubi, S. 1994, Ap. J. Lett., 423, L93.
Nusser, A., Dekel, A., Bertschinger, E., and Blumethal, G.R., 1991, Ap. J., 379, 6.
Padmanabhan, T., 1993, Structure Formation in the Universe, (Cambridge: Cambridge
University Ppress).
Peebles, P.J.E. 1980, The Large-Scale Structure of the Universe, (Princeton: Princeton
University Press).
Press, W.H., Teukolsky, S.A., Vetterling, W.T., & Flannery, B.P. 1992, Numerical Recipes (Second Edition) (Cambridge: Cambridge University Press).

Strauss, M.A. and Willick, J.A., 1995, Physics Report (in press).

Wiener, N. 1949, in Extrapolation and Smoothing of Stationary Time Series, (New York: Wiley)

Zaroubi, S., Hoffman, Y., Fisher, K.B., and S. Lahav, O., 1995, Ap. J., (in press; ZHFL).

Zaroubi, S., and Hoffman, Y., 1995, Ap. J., (submit.).
Figure Captions

Fig. 1: Raw data: The IRAS 1.2Jy galaxies. The galaxy distribution is presented in 9 planar slabs of thickness of $\pm 10h^{-1}Mpc$. (Supergalactic coordinates are used and distances are given in $h^{-1}Mpc$, where $h$ is Hubble’s constant in units of 100Km/s/Mpc.)

Fig. 2. ‘Volume limited’ IRAS catalog: A non-linear constrained realization based on the IRAS galaxy distribution. The full N-body particle distribution has been diluted to the mean IRAS galaxies mean number density.

Fig. 3. Gaussian smoothing: The non-linear constrained realization shown in Fig. 2 is Gaussian smoothed on a scale $R = 500Km/s$. Contour spacing is 0.2 and the dashed lines correspond to negative values of $\delta$.

Fig. 4. Different realizations: A comparison of different non-linear constrained realizations of the same input data and prior is presented here by the ‘galaxy’ distribution and the contour plots. Upper and lower raws correspond to two different realizations. The galaxy distribution and contour plots are presented in the same way as in Figs. 2 and 3.