AS-configurations and skew-translation generalised quadrangles

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Dedicated to the memory of Ákos Seress.

Abstract. The only known skew-translation generalised quadrangles (STGQ) having order $(q, q)$, with $q$ even, are translation generalised quadrangles. Equivalently, the only known groups $G$ of order $q^3$, $q$ even, admitting an Ahrens-Szekeres (AS-)configuration are elementary abelian. In this paper we prove results in the theory of STGQ giving (i) new structural information for a group $G$ admitting an AS-configuration, (ii) a classification of the STGQ of order $(8, 8)$, and (iii) a classification of the STGQ of order $(q, q)$ for odd $q$ (using work of Ghinelli and Yoshiara).

1. Introduction

The point-line incidence structures known as generalised polygons were introduced by Jacques Tits [19] in 1959, and they have since played an important role in the theory of buildings and incidence geometry. A finite generalised $d$-gon is a point-line geometry whose bipartite incidence graph has diameter $d$ and girth $2d$. We will focus on generalised quadrangles where $d = 4$. These have the property that if $\ell$ is a line and $P$ is a point not on $\ell$, then there is a unique point on $\ell$ collinear with $P$. Hence this geometry contains no triangles. It follows that there exist constants $s$ and $t$ such that every line contains $s + 1$ points, and every point is incident with $t + 1$ lines (provided there are more than two lines through any point and more than two points on any line). We will refer to such a generalised quadrangle as having order $(s, t)$. Interchanging the points and lines of a generalised quadrangle of order $(s, t)$, gives the dual generalised quadrangle whose order is $(t, s)$. Up to point-line duality, the only known examples are the Hermitian quadrangles of order $(q^2, q^3)$, the Payne derived quadrangles, and the skew-translation generalised quadrangles (STGQ).

The precise definition of an STGQ will be given in the next section, but it suffices at this point to regard them as a large class of the known generalised quadrangles. To the authors’ knowledge, the known STGQ are as follows:

(i) the symplectic generalised quadrangles $W(3, q)$ of order $(q, q)$;
(ii) the parabolic quadric generalised quadrangles $Q(4, q)$ of order $(q, q)$;
(iii) various examples of order $(q^2, q)$ including the flock generalised quadrangles, the duals of the examples $T_3(O)$ of Tits, or the Roman generalised quadrangles (see [11]).

We will be interested in STGQ of order $(q, q)$. Upon reading the recent work of Ghinelli [7], the authors realised that her main result could be combined with Yoshiara’s earlier
work \[20\] to prove the following breakthrough in the theory of skew-translation generalised quadrangles.

**Theorem 1.1.** Any skew-translation generalised quadrangle of order \((q, q)\), \(q\) odd, is isomorphic to the symplectic generalised quadrangle \(W(3, q)\).

While Theorem 1.1 has an apparently very short proof (see Corollary 3.4), it should be noted that there is an error in the original work of Ghinelli, which Professors Ghinelli and Ott graciously corrected. We have included this correction in the Appendix.

For \(q\) even, the situation is still unclear. The only known examples arise as Tits’ construction \(T_2(\mathcal{O})\) where \(\mathcal{O}\) is an oval of the Desarguesian projective plane \(\text{PG}(2, q)\) (see [14, §3.1.2]), and so every known example of a generalised quadrangle of order \((q, q)\), \(q\) even, is a translation generalised quadrangle. If \(q \leq 4\), then the STGQ are known (see [14, §6]), and the smallest unknown case is when \(q = 8\). We can express the problem of classifying STGQ of order \((q, q)\) in perhaps a more accessible way in the language of elementary group theory. Following [7, §1], an AS-configuration\(^1\) is a group \(G\) of order \(q^3\) together with a family of subgroups \(U_0, U_1, \ldots, U_{q+1}\) of \(G\), each of order \(q\), satisfying

\[
\begin{align*}
(AS1) & \quad U_0 \text{ is normal in } G, \\
(AS2) & \quad U_i U_j \cap U_k = \{1\} \text{ for all distinct } i, j, k \in \{0, 1, \ldots, q + 1\}.
\end{align*}
\]

Throughout this paper we will refer to the family of subgroups \(U_0, U_1, \ldots, U_{q+1}\) as the AS-configuration for a group \(G\), and in this case we will say that \(G\) admits an AS-configuration. It should be noted that an AS-configuration is a special example of a 4-gonal partition of a group of order \(q^3\) where one of the groups is a normal subgroup of \(G\), see [14, §10.2]. It is further shown in [14, §10.2] that a group of order \(q^3\) admitting an AS-configuration gives rise to an STGQ of order \((q, q)\), and vice versa. (See Section 2 for more details.) A single group \(G\) could admit more than one AS-configuration, and indeed, for elementary abelian 2-groups, an AS-configuration is simply a pseudo-hyperoval of projective space (see [15]). The only known groups of even order admitting an AS-configuration are elementary abelian 2-groups, and indeed, Payne conjectured that there are no other examples [12, p. 498].

In this paper, we give structural information on a group \(G\) admitting an AS-configuration, and we classify the STGQ of order \((8, 8)\).

**Theorem 1.2.** A group of order 512 admitting an AS-configuration must be an elementary abelian 2-group. In particular, a skew-translation generalised quadrangle of order \((8, 8)\) is a translation generalised quadrangle.

We note that the translation generalised quadrangles of order \((8, 8)\) were classified by Payne [13], so our result can be viewed as an extension of Payne’s classification.

The structure of this paper is as follows. Section 2 describes the relationship between Kantor families and AS-configurations, and Section 3 describes the relationship between partial difference sets and AS-configurations and proves Theorem 1.1, establishing the classification of the STGQ of order \((q, q)\) with \(q\) odd. In Section 4 we deduce new structural constraints satisfied by a group \(G\) with an AS-configuration. Our goal in Section 5 is to prove Theorem 1.2 and to show that of the 10 494 213 groups of order \(2^9\), only one group (the elementary abelian group) can have an AS-configuration. This is a remarkable testimony to the power of the triple condition (AS2) above. Finally, in the Appendix we correct an oversight in Ghinelli’s groundbreaking paper [7].

\(^1\)The term AS-configuration was introduced by Ghinelli as they give rise to the so-called Ahrens-Szekeres generalised quadrangles.
2. Background theory

In this section we give the necessary background on skew-translation generalised quadrangles and elation generalised quadrangles. The skew-translation generalised quadrangles are a subfamily of elation generalised quadrangles, which can all be constructed by Kantor’s coset geometry construction [10]. Provided a generalised quadrangle has sufficient local symmetry properties, we can model it completely within a particular group of automorphisms known as an elation group. Given a point \( P \) of a generalised quadrangle \( Q \), an elation about \( P \) is an automorphism \( \theta \) that is either the identity, or it fixes \( P \) and each line incident with \( P \), and no point not collinear with \( P \). If there exists a group \( G \) of elations of \( Q \) about a point \( P \) such that \( G \) acts regularly on the points not collinear with \( P \), then we say that \( G \) is an elation group and that \( Q \) is an elation generalised quadrangle (EGQ).

The number of points of a generalised quadrangle of order \((s, t)\) is equal to \((s + 1)(st + 1)\), and there are \(s(t + 1)\) points collinear with a given point. Therefore, there are \(s^2t\) points not collinear with a given point and hence \( G \) must necessarily have order \(s^2t\). Kantor [10] showed that there was a remarkable connection between elation generalised quadrangles and certain configurations of subgroups in finite groups. Let \( G \) be a group of order \(s^2t\) (where \(s, t > 1\)) and suppose that \( G \) contains a collection \( F \) of \( t + 1 \) subgroups of order \(s\), and a collection \( F^* \) of \( t + 1 \) subgroups of order \(st\), such that

- (K1) for every element \( A^* \) of \( F^* \), there is a unique element \( A \) of \( F \) contained in \( A^* \);
- (K2) \( A^* \cap B = \{1\} \) holds for \( A^* \in F^*, B \in F, \) and \( B \not\subset A^* \);
- (K3) \( AB \cap C = \{1\} \) holds for distinct elements \( A, B, C \in F \).

Nowadays, the pair \((F, F^*)\) is called a Kantor family or 4-gonal family for \( G \). Every elation generalised quadrangle produces a 4-gonal family for its elation group, and conversely, a 4-gonal family of a group of order \(s^2t\) (as above) gives rise to an elation generalised quadrangle of order \((s, t)\) (see [10, Theorem 2]).

Let \( S \) be a generalised quadrangle of order \((s, t)\) and let \( P \) be a point of \( S \). A symmetry about \( P \) is an elation about \( P \) which fixes each point collinear with \( P \). The symmetries about \( P \) form a group with order dividing \( t \) (see [14, pp. 165]). If \( G \) contains the full group of \( t \) symmetries about \( P \), then we say that \( S \) is a skew-translation generalised quadrangle. In this situation, we have that \( t \) is no greater than \( s \), and both parameters are powers of the same prime [8, Corollary 2.6]. The Kantor family \((F, F^*)\) of \( G \) gives rise to a skew-translation generalised quadrangle if and only \( \cap F^* \) is normal in \( G \) of order \( t \) [14, 8.2.2]. For the case that \( s = t \), there is an alternative characterisation due to K. Thas [18]: an elation generalised quadrangle \( Q \) of order \((s, s)\) is a skew-translation generalised quadrangle if and only if \( Q \) has a regular point, i.e., a point \( P \) such that

\[
\left| \{P, R\}^{s+1} \right| = t + 1 \text{ for all points } R \text{ not collinear with } P.
\]

As an example of such a situation, it turns out that all points of the symplectic generalised quadrangle \( W(3, q) \) are regular.

A large class of elation generalised quadrangles are the translation generalised quadrangles (TGQ) whereby the given elation group \( G \) contains a full group of \( s \) symmetries about each line on the base point \( P \). In fact, an EGQ is a TGQ if and only if the given elation group \( G \) is abelian [14, 8.2.3, 8.3.1]. In this instance, \( G \) is actually elementary abelian [14, 8.5.2]. The only known groups admitting AS-configurations are elementary abelian 2-groups, or Heisenberg groups of odd order \(q^3\) with a centre of order \( q \). Hence, it is an open problem (Payne’s Conjecture) whether there exists an AS-configuration of a nonabelian group of even order.

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2For a set \( S \) of points, \( S^\perp \) denotes the set of points collinear with each element of \( S \).
The construction of *Payne derived quadrangles*, which include the Ahrens-Szekeres quadrangles, has as input a generalised quadrangle \( Q \) of order \((s, s)\) and a regular point \( P \) of \( Q \). We can construct a new generalised quadrangle \( Q^P \) whose points are the points of \( Q \) not collinear with \( P \). There are two types of lines: (i) the lines of \( Q \) not incident with \( P \), and (ii) the sets \( \{P, R\}^\perp \) where \( R \) is not collinear with \( P \). The incidence relation is inherited from \( Q \). We can also construct these generalised quadrangles from an AS-configuration \( U_0, U_1, \ldots, U_{q+1} \) of a group \( G \) of order \( q^3 \). We take the points to be the elements of \( G \), and we let the lines be the right cosets of the of \( U_i \), \( i > 0 \), and we obtain a generalised quadrangle of order \((q - 1, q + 1)\).

**Lemma 2.1.** [14, 10.2.1]. A skew-translation generalised quadrangle of order \((q, q)\) gives rise to an AS-configuration of a group \( G \) of order \( q^3 \), and conversely.

One can produce a Kantor family for \( G \) given an AS-configuration as follows:

**Lemma 2.2.** [12, III.4]. An AS-configuration \( U_0, U_1, \ldots, U_{q+1} \) of \( G \) yields a Kantor family \((F, F^*)\) of \( G \) by defining \( F := \{U_i : i = 1, \ldots, q + 1\} \) and \( F^* := \{U_iU_j : i = 1, \ldots, q + 1\} \).

This construction yields all skew-translation generalised quadrangles of order \((q, q)\). So if one can find a novel AS-configuration, it is likely that two new generalised quadrangles would arise.

### 3. Partial difference sets and AS-configurations of symplectic type

A *partial difference set* \( \Delta \) of a group \( G \) is an inverse-closed set of nontrivial elements of \( G \) such that there are two constants \( \lambda \) and \( \mu \), so that every element \( g \in G \setminus \{1\} \) has exactly \( \lambda \) (resp. \( \mu \)) representations of the form \( g = s_is_j^{-1} \) for \( g \in \Delta \) (resp. \( g \not\in \Delta \)) where \( s_i, s_j \in \Delta \). We know that the right-multiplication action of \( G \) yields collineations of the generalised quadrangle arising from an AS-configuration, and this action is regular on the points of the generalised quadrangle. So we fix a point to be the identity element of \( G \), and since the collinearity graph of a generalised quadrangle is strongly regular, we see that the neighbourhood of 1 is a partial difference set. In particular, for an AS-configuration, we have the following:

**Lemma 3.1.** [7, proof of Theorem 1]. Let \( U_0, U_1, \ldots, U_{q+1} \) be an AS-configuration of \( G \). Then

\[
\Delta := \bigcup_{i=0}^{q+1} (U_i \setminus \{1\}) = \left( \bigcup_{i=0}^{q+1} U_i \right) \setminus \{1\}
\]

is a partial difference set of \( G \) with parameters \( \lambda = q - 2 \) and \( \mu = q + 2 \).

Ghinelli showed that the known family of examples for groups of odd order could be characterised by a feature of their partial difference set, and it turns out that this is enough for a complete classification.

**Theorem 3.2.** [7, Theorem 6]. Suppose \( U_0, U_1, \ldots, U_{q+1} \) is an AS-configuration of \( G \) such that each nontrivial conjugacy class of \( G \) intersects the partial difference set \( \Delta \) defined in (1). Then \( G \) is the 3-dimensional Heisenberg group of order \( q^3 \) and the AS-configuration is the known example arising from the symplectic generalised quadrangle \( W(3, q) \).

There is an error in a supporting result within Ghinelli’s paper [7, Lemma 6]. Professors Ghinelli and Ott, in correspondence with the authors, generously solved the error in
the proof. It is with their permission that we present their argument in the Appendix to this paper.

Ghinelli’s result, together with an older result of Yoshiara, provide a classification of AS-configurations of groups of odd order.

**Theorem 3.3.** [20, Lemma 6]. Suppose $G$ acts regularly on a generalised quadrangle $Q$ of order $(s,t)$ with $s > 1$ and $t > 1$, and let $\Delta$ be the associated partial difference set. If $\gcd(s,t) > 1$, then $\Delta$ intersects every nontrivial conjugacy class of $G$.

Together with Ghinelli’s result (Theorem 3.2), we have the following, which by Lemma 2.1, is an equivalent statement of Theorem 1.1.

**Corollary 3.4.** Suppose $U_0, U_1, \ldots, U_{q+1}$ is an AS-configuration of $G$ where $q$ is odd. Then $G$ is the 3-dimensional Heisenberg group of order $q^3$ and the AS-configuration is the known example arising from the symplectic generalised quadrangle $W(3,q)$.

**Proof.** By [7, Theorem 1] the AS-configuration gives rise to a generalised quadrangle with parameters $(q-1, q+1)$. Since $q$ is odd, we have $\gcd(q-1, q+1) = 2$. The result now follows from Theorems 3.2 and 3.3. □

### 4. The structure of groups admitting an AS-configuration

Throughout this section we assume that $U_0, U_1, \ldots, U_{q+1}$ is an AS-configuration of a group $G$ of order $q^3$. We will use standard notation for certain characteristic subgroup constructions that can found in such texts as [16]. Throughout, we will use $\Delta$ for the partial difference set arising from the AS-configuration as defined in 1, and $\Delta'$ will denote the complement of $\Delta$ in $G$. Ghinelli established the following facts about $G$:

**Lemma 4.1.** [7].

(i) $\Phi(G) \leq U_0$,

(ii) $U_i$ is elementary abelian for all $i > 0$, and $G$ is a $p$-group,

(iii) $U_i^g \leq U_0 U_i$ for all $g \in G$,

(iv) $\bigcup_{i=1}^{q+1} U_0 U_i = G$,

(v) if $g \in U_0 U_j$ and $g \not\in U_0 \cup U_j$, then for each $1 \leq i \leq q+1$ with $i \neq j$ there exists exactly one factorisation $g = u_i u_k$, where $u_i \in U_i$, $k$ is unique and depends on $i$, and $1 \neq u_k \in U_k$.

(vi) $G = U_i U_j U_k$ for distinct $i,j,k$.

**Proof.** Parts (i) and (ii) follow from [7, Corollary 1]. Since $\Phi(G) \leq U_0 U_i \triangleleft G$, part (iii) is true. See [7, Equation (4)] for part (iv), and see [7, Equation (6)] for part (v). Part (vi) is true since $|U_i U_j| = q^2$ and $U_i U_j \cap U_k = \{1\}$ implies $|U_i U_j U_k| = q^3 = |G|$. □

**Lemma 4.2.** [12, III.5]. Suppose $U_0, U_1, \ldots, U_{q+1}$ is an AS-configuration of $G$. If one of the $U_1, \ldots, U_{q+1}$ is normal in $G$, then $q$ is even and $G$ is an elementary abelian 2-group.

**Lemma 4.3.** Suppose $x \in G$ and $i > 0$. Then $x$ centralises $U_i$.

**Proof.** Note that the following is the same argument given in the proof of [7, Lemma 5]. Let $u \in U_i$ such that $u \not\in U_j$. Note that $[u, x] = u^{-1} u^x \in U_i$. Since $G$ is nilpotent, we know that $[G, G] \leq \Phi(G)$ and hence $[u, x] \in U_i \cap [G, G] \leq U_i \cap U_0 = \{1\}$.

Therefore, $x$ centralises $u$. □

**Lemma 4.4.** For all $i > 0$, $N_G(U_i) = C_G(U_i)$. 

Proof. Let \( x \in N_G(U_i) \). Then \( U^x_i \cap U_i = U_i \), and the result follows from Lemma 4.3.

An element \( x \) in a group \( G \) is called real if \( x^g = x^{-1} \) for some \( g \in G \). If \( \chi \) is a character of a complex representation of \( G \), then \( \chi(x) = \chi(x^g) = \chi(x^{-1}) = \chi(x) \), so \( \chi(x) \) is a real number.

Lemma 4.5. If \( u, v \) are involutions in a group \( G \) and \( x = uv \) has order \( m \), then \( \langle u, v \rangle \) is a dihedral group of order \( 2m \) and \( x^u = x^v = x^{-1} \).

Proof. Since \( u^2 = v^2 = 1 \), we see \( (uv)^{-1} = v^{-1}u^{-1} = vu \). Thus \( x^u = (uv)^u = vu = x^{-1} \) and similarly \( x^v = x^{-1} \). Now \( \langle u, v \rangle = \langle u, x \rangle \) is a quotient group of the dihedral group \( \langle u, x \mid u^2 = x^m = 1, x^u = x^{-1} \rangle \) of order \( 2m \), and so \( \langle u, v \rangle \) is this group if and only if \( u \not\in \langle x \rangle \). However, \( u \in \langle x \rangle \) implies \( x = x^u = x^{-1} \) and \( m = 1, 2 \). Thus if \( m > 2 \), then \( \langle u, v \rangle \) is dihedral of order \( 2m \). If \( m = 1, 2 \), then \( \langle u, v \rangle \) is dihedral and abelian of order \( 2m \).

Lemma 4.6. Assume that \( q \) is even and \( G/\Phi(U_0) \) is nonabelian. Then the following hold:

(i) If \( x^2 \not\in \Phi(U_0) \), then there exists \( a \in U_i \), with \( i > 0 \), such that \( x^u = x^{-1} \) (mod \( \Phi(U_0) \)).

(ii) \( [G, U_0] \not\subseteq \Phi(U_0) \).

(iii) \( Z(G/\Phi(U_0)) \) is an elementary abelian subgroup of \( G/\Phi(U_0) \).

(iv) \( G/\Phi(U_0) \) has exponent 4.

Proof. The subgroup \( N := \Phi(U_0) \) is characteristic in \( U_0 \), and hence normal in \( G \). Let \( \overline{G} \) denote the map \( G \to G/N : g \mapsto Ng \). Since \( G \) is nonabelian, it is not elementary abelian and so must contain an element \( \overline{x} := Nx \) of order divisible by 4. Since \( U_0 \) is elementary abelian, we have \( \overline{x} \not\in U_0 \). By Lemma 4.1(ii), \( \overline{x} \not\in U_i \) for any \( i > 0 \). As \( G = \bigcup_{i=1}^{\infty} U_0 U_j \) by Lemma 4.1(iv), there exists \( j > 0 \) such that \( x \in U_0 U_j \). Let \( \overline{x} = \overline{u} \overline{v} \), where \( u \in U_0 \), \( v \in U_j \), and \( \overline{u}, \overline{v} \) are involutions in \( \overline{G} \). Thus \( x^u \equiv x^{-1} \) (mod \( \Phi(U_0) \)) by Lemma 4.5, showing (i). Since \( x^2 \not\in N \), \( x^u \equiv x^{-1} \) (mod \( N \)), and thus \( \overline{x} \not\in Z(\overline{G}) \). Hence \( U_0 \) is not contained in \( Z(\overline{G}) \), showing (ii). By (i) every element of order 4 in \( \overline{G} \) is real, so no element of order 4 in \( \overline{G} \) lies in \( Z(\overline{G}) \). Thus \( Z(\overline{G}) \) has exponent 2 and is elementary abelian, showing (iii). Finally, given \( g \in G \), \( g^2 \in U_0 \) by Lemma 4.1(i). Since \( g^2 \in U_0 \) and \( U_0 = \{ \overline{1} \} \), we have \( U^2 = \{ \overline{T} \} \). We know that \( \overline{G} \) contains at least one element of order 4, and so \( \overline{G} \) has exponent 4, showing (iv).

Proposition 4.7. If \( q \) is even, \( G \) is nonabelian, and \( U_0 \not\subseteq Z(G) \), then \( U_0 \) is not elementary abelian and \( G \) has exponent 4.

Proof. By [4, Lemma 10], \( U_0 \) has exponent at most 4. Since \( U_0 \not\subseteq Z(G) \), we have that \( U_0^4 = \{ 1 \} \). If \( U_0^2 = \{ 1 \} \), then \( \Phi(U_0) = \{ 1 \} \) and Lemma 4.6(ii) implies that \( [G, U_0] \not\subseteq \{ 1 \} \) or \( U_0 \not\subseteq Z(G) \), a contradiction. Thus \( U_0^4 = \{ 1 \} \) and \( U_0^2 \not\subseteq \{ 1 \} \), so \( U_0 \) has exponent precisely 4. Let \( g \in G \). Then for some \( i > 0 \), \( g = uv \), where \( u \in U_0 \) and \( v \in U_i \). Since \( U_0 \not\subseteq Z(G) \) and \( U_i \) is elementary abelian, \( g^2 = uvuv = v^2u^2 = u^2 \). Therefore, \( G \) has exponent 4, as desired.

Lemma 4.8. Suppose \( q \) is even and \( G \) is nonabelian.

(i) If \( U_0^m = \{ 1 \} \), then \( G^{2m} = \{ 1 \} \).

(ii) If \( \Phi(U_0)^k = \{ 1 \} \), then \( G^{2k} = \{ 1 \} \).

(iii) If \( G \) has class 2, \( m \) is even, and \( U_0^m = (G^m)^{m/2} = \{ 1 \} \), then \( G^m = \{ 1 \} \).

Proof. Since \( G/U_0 \) is elementary abelian and \( U_0^m = \{ 1 \} \), it follows that \( G^{2m} = \{ 1 \} \) proving (i). The proof of (ii) is similar because \( G^4 \not\subseteq \Phi(U_0) \) by Lemma 4.6. Consider part
(iii) and assume that $G$ has class 2, $m$ is even, and $U_0^m = (G^m)^{m/2} = \{1\}$. An element $g \in U_0$ certainly satisfies $g^m = 1$. Suppose $g \notin U_0$. Then $g = u_0 u_i$ for a unique $i > 0$ where $u_0 \in U_0$ and $u_i \in U_i$. But $g^m = u_0^m u_i^m u_0 u_i$ as $G$ has class 2. However, $u_0^m = u_i^m = 1$ and $[u_1, u_0]^2 = 1$ as $\binom{m}{2}$ is divisible by $m/2$ and $(G^m)^{m/2} = \{1\}$. Thus $g^m = 1$ and so $G^m = \{1\}$.

Lemma 4.9. If $q$ is even and $Z(G) \subseteq \Delta \cup \{1\}$, then $Z(G) \leq U_0$.

Proof. Assume by way of contradiction that $Z(G) \subseteq \Delta \cup \{1\}$ and there exists a $z \in Z(G) \setminus U_0$. Then $1 \neq z \in \Delta \setminus U_0$ and $z \in U_i$ for some $i > 0$. On the other hand, $\{1\} \neq U_0 \triangleleft G$, and so there exists $1 \neq u \in U_0 \cap Z(G)$. However, this means that $1 \neq uz \in Z(G) \setminus \Delta$, contrary to our assumption.

Proposition 4.10. If $q$ is even and $Z(G)^2 \neq \{1\}$, then $Z(G) \leq U_0$.

Proof. Suppose $Z(G)^2 \neq \{1\}$, and let $z \in Z(G)$ such $z^2 \neq 1$. Assume first that $z \notin U_0$. Since for $i > 0$ the $U_i$ are elementary abelian by Lemma 4.1(ii), we have that $z \notin \Delta$. By Lemma 4.1(v), without a loss of generality up to reordering the $U_i$ we know that $z$ can be written as

$$z = u_1 u_2$$

for some $u_1 \in U_1$ and $u_2 \in U_2$. Since $z$ is central, and $u_2$ is an involution, we have

$$[u_1, u_2] = [zu_2, u_2] = [u_2, u_2] = 1.$$ 

Therefore, $u_1$ and $u_2$ commute. So

$$z^2 = u_1 u_2 u_1 u_2 = u_1^2 u_2^2 = 1$$

as $u_1$ and $u_2$ are involutions, a contradiction. Thus every $z \in Z(G)$ that has order at least 4 is in $U_0$. Since $Z(G)^2 \neq \{1\}$ and every finite abelian group is generated by elements of maximal order, we have that $Z(G) \leq U_0$, as desired.

Corollary 4.11. If $q$ is even, $G$ is nonabelian, and $U_0 \leq Z(G)$, then $U_0 = Z(G)$.

Proof. If $\Phi(U_0) = \{1\}$, then Lemma 4.6(ii) implies that $[G, U_0] \neq \{1\}$ contrary to our assumption that $U_0 \leq Z(G)$. Thus $U_0$ is not elementary abelian and neither is the supergroup $Z(G)$. Hence $Z(G)^2 \neq \{1\}$ and Proposition 4.10 shows the reverse containment $Z(G) \leq U_0$. Consequently $U_0 = Z(G)$, as desired.

Lemma 4.12. If $q$ is even and $U_0 \leq Z(G)$, then $U_0^2 = G^2 = \Phi(G)$, and therefore, $\Phi(G)$ is a proper subgroup of $U_0$.

Proof. Let $g \in G$. By [7, Equation (4)], $G$ is a union of the $U_0 U_i$ and hence there exists $i \in \{1, 2, \ldots, q + 1\}$ such that $g \in U_0 U_i$. So there exists $u_0 \in U_0$ and $u_i \in U_i$ such that $g = u_0 u_i$. Now

$$g^2 = (u_0 u_i)(u_0 u_i) = u_0^2 u_i^2 = u_0^2$$

as $U_0$ is central. Therefore, $g \in U_0^2$, and so it follows that $G^2 = U_0^2$. By [9, III.3.14(b)], we know for a 2-group $G$ that $\Phi(G) = G^2$, and hence $\Phi(G)$ is elementary abelian, since we know from Proposition 4.7 that $G$ has exponent 4. Hence $\Phi(G) < U_0$ (by Proposition 4.7).

Theorem 4.13. If $U_0 = \Phi(G)$, then $Z(G) \leq U_0$.

Proof. This is true for $q$ odd by [7, Corollary 2], so we assume that $q$ is even. Assume that $Z(G)$ is not a subgroup of $U_0$. By Lemma 4.9, this implies that there is $1 \neq z \in Z(G) \cap \Delta^c$. Without a loss of generality up to reindexing, by Lemma 4.1(v), we may assume that $z = u_0 u_1 = u_2 u_3 = u_4 u_5 = \cdots = u_q u_{q+1}$, where $u_i \in U_i$ for all $i$. 

Note that for $j$ even, if $x \in U_{j+1}$, then $z^2 = z$, and so $u_j u_{j+1} = u_j^2 u_{j+1}$, i.e., $u_j = u_j^2$, and $u_j \in C_G(U_{j+1})$. Similarly, $u_{j+1} \in C_G(U_j)$ when $j > 0$ is even. Note that this means that the $q + 2$ ways of writing $z$ as a product of elements from $\Delta$ are precisely $z = u_0 u_1 = u_1 u_0 = u_2 u_3 = \cdots = u_q u_{q+1} = u_{q+1} u_q$, since $u_j$ commutes with $u_{j+1}$ for $j$ even.

On the other hand, since $G = U_0 U_j U_{j+1}$ and $U_0 = \Phi(G)$, we have $G = \langle U_j, U_{j+1} \rangle$ for $j > 0$, and thus for all $i > 1$, $u_i \in Z(G)$. Now, if $u_1 \notin Z(G)$, we may look at all the products $u_2 u_j$ for $j \geq 3$. Each of these is in some $V_k := U_0 U_k$ by Lemma 4.1(iv); if for some $j$, $u_2 u_j \in V_k$ for $k \neq 1$, then proceeding as above, we find an element in $U_1 \cap Z(G)$; otherwise, each $u_2 u_j \in V_1$. In this case, $u_2 u_j$ is factored uniquely as a nontrivial element of $U_0$ multiplied on the right by a nontrivial element of $U_1$, and so we let $u_2 u_j = x_{0,j} x_{1,j}$, where $x_{i,j} \in U_i$ (and note that $u_0 = x_{0,3}$, $u_1 = x_{1,3}$ since $z = u_0 u_1 = u_2 u_3$). Suppose that $x_{0,i} = x_{0,j}$ for $i \neq j \geq 3$. Then

$$1 \neq u_i u_j = (u_i u_2)(u_2 u_j) = (x_{0,i}^{-1} x_{0,i}^{-1})(x_{0,j} x_{1,j}) = x_{0,i}^{-1} x_{1,j} \in U_1,$$

a contradiction. Thus each $x_{0,j}$ must be different, and since no $u_2 u_j$ is in $U_1$, each $x_{0,j}$ is nontrivial. However, there are $q - 1$ such $x_{0,j}$, and $Z(G) \cap U_0$ has at least one nonidentity element. Thus some $x_{0,j}$ is central, and then so is $x_{1,j} = x_{0,j}^{-1} u_2 u_j$. In any case, there is a nontrivial element of the centre in each $U_i$.

Choose elements $v_i \in U_i \cap Z(G)$ for each $i > 0$ and define $K := \langle v_1, v_2, \ldots, v_q \rangle$. For each $i$, define $K_i := KV_i / V_i$. For a particular $i$ and any $j \neq i$, the image of $V_i v_j$, which will be denoted by $\overline{v_j}$, is nontrivial in $\overline{K_i}$ since $U_j \cap U_0 U_i = \{1\}$ for all $i \neq j > 0$. Now, if $v_k = \overline{v_j}$, then $v_j v_k \in V_i$. Since $U_j U_k \cap U_0 = \{1\}$, each $v_j v_k$ is contained in a unique $V_m$. On the other hand, there are exactly $q(q+1)/2$ pairs and $(q+1)$ different subgroups $V_1, \ldots, V_q$, so without a loss of generality there are at most $q/2$ products $v_j v_k$ in $V_1$.

This means that $\overline{K} := \overline{K_1}$ has at least $q/2$ nonidentity elements, and, since $|\overline{K}|$ is a power of 2, $|\overline{K}| \geq q$. Hence $|KV_i| = |\overline{K}| \cdot |V_i| \geq q^3$, and so $G = KV_i$.

Let $u \in U_1$. For any $g \in G$, we may write $g = z x_1 x_0$, where $z \in K$ and $x_i \in U_i$ for each $i$, and so $u^g = u^{z x_1 x_0} = u^{x_0}$, which means that $u^G = u^{U_0}$ for any $u \in U_1$. For any $j > 1$, $U_0 \leq V_j$, and so $u^{V_j} = u^G$ as well. Since $|V_j| = q^2$ and $|U_0| = q$, this means that

$$q^2 = |C_{V_j}(u)| \cdot |u^{V_j}| \quad \text{and} \quad q = |C_{U_0}(u)| \cdot |u^{U_0}|.$$

Putting these two equations together, we see that $|C_{V_j}(u)| = q|C_{U_0}(u)|$. On the other hand,

$$|C_{V_j}(u)| U_0 = |U_0| \cdot |C_{V_j}(u)| U_0 / U_0 = |U_0| \cdot |C_{V_j}(u)| / C_{U_0}(u) = q^2,$$

and so it must be that $V_j = C_{V_j}(u)/U_0$. This is true for any $j > 1$, and so

$$G = V_2 V_3 = C_{V_2}(u) U_0 C_{V_3}(u) U_0 = \langle C_{V_2}(u), C_{V_3}(u) \rangle,$$

since $U_0 = \Phi(G)$. This means that $u$ commutes with every element of $G$, and so $u \in Z(G)$. On the other hand, $u$ was arbitrary, and so $U_1 \leq Z(G)$. Hence $G = KV_1 = Z(G) U_0$, and, since $U_0 = \Phi(G)$, $G$ is generated by central elements, a contradiction as $G$ is nonabelian. Therefore, if $U_0 = \Phi(G)$, then $Z(G) \leq U_0$, as desired.

**Corollary 4.14.** If $q$ is even and $U_0 = \Phi(G)$, then $Z(G) < U_0$ and $q > 2$.

**Proof.** By Theorem 4.13, $Z(G) \leq U_0$, and by Lemma 4.12, $Z(G) \neq U_0$. Thus $Z(G) < U_0$ and $q > 2$ (otherwise $|Z(G)| = |U_0| = 2$).

**Lemma 4.15.** If $G$ has an extraspecial epimorphic image of size 8 or 32, then $G$ does not admit an AS-configuration.

**Proof.** We use the conventional notation $D_8$, $Q_8$, $2^{1+4}$, $2^{1+4}$ for the extraspecial groups of order 8 or 32. Assume, by way of contradiction, that a group $G$ admits an
As-configuration $U_0, U_1, \ldots, U_{q+1}$ where $|G| = q^3$, and has a quotient group $G/N$ isomorphic to $D_8$, $Q_8$, $2^{1+4}_+$, or $2^{1+4}_-$. Consider the natural epimorphism $G \to G/N$ with $g \mapsto \overline{g} := Ng$, and set $\overline{G} := G/N$. Then $|\overline{G}| = 8$ implies $q \geq 2$, and $|\overline{G}| = 32$ implies $q \geq 8$.

Case $\overline{G} \cong Q_8$. For $i > 0$, each $U_i$ is elementary abelian, and so each $\overline{U_i}$ is elementary abelian. On the other hand, only one element $Q_8$ has order 2; hence $\overline{U_i} \leq \mathbb{Z}(\overline{G})$ for each $i > 0$. However, $G = U_1U_2U_3$ implies that $\overline{G} = \overline{U_1U_2U_3} \leq \mathbb{Z}(\overline{G})$. This is a contradiction as $\overline{G} \cong Q_8$.

Case $\overline{G} \cong D_8$. There are two maximal elementary abelian subgroups of $\overline{G}$ which we denote $H_1$ and $H_2$. Observe that $H_1 \cong H_2 \cong C_2 \times C_2$, and both $H_j$ are generated by the involution of $\mathbb{Z}(\overline{G})$ and a non-central involution. Hence each $\overline{U_i}$ is isomorphic to a subgroup of one $H_j$. Since $q + 1 \geq 5$, the Pigeonhole Principle implies that at least three of $\overline{U_1}$, $\overline{U_2}$, $\overline{U_3}$, $\overline{U_4}$, and $\overline{U_5}$ must be subgroups of the same elementary abelian subgroup, say of $H_1$. Without a loss of generality, assume that $\overline{U_1}, \overline{U_2}$, and $\overline{U_3}$ are subgroups of $H_1$. However, $G = U_1U_2U_3$, and so $\overline{G} = \overline{U_1U_2U_3} \leq H_1 < \overline{G}$, a contradiction. Thus this case also does not arise.

Suppose now that $\overline{G}$ is extraspecial of order 32. The map $Q : \overline{G}/\mathbb{Z}(\overline{G}) \to \mathbb{Z}(\overline{G})$ given by $Q(\mathbb{Z}(\overline{G})\overline{g}) := \overline{g}$ is a non-singular quadratic form on $\overline{G}/\mathbb{Z}(\overline{G})$, of plus or minus type. Note that the ambient vector space is 4-dimensional and so the totally singular subspaces comprise an elliptic or hyperbolic quadric of $PG(3,2)$. If $A$ is an elementary abelian subgroup of $\overline{G}$, then $(\mathbb{Z}(\overline{G}), A)$ is elementary abelian. Hence the maximal elementary abelian subgroups of $\overline{G}$ are normal and contain $\mathbb{Z}(\overline{G})$. Moreover, they correspond to maximal totally singular subspaces in $\overline{G}/\mathbb{Z}(\overline{G})$; for the minus-type case, we obtain the non-singular elliptic quadric of five points, whereas in the plus-type case, we obtain the non-singular hyperbolic quadric having nine points and six lines.

Case $\overline{G} \cong 2^{1+4}_+$. There are six maximal elementary abelian subgroups $H_1, \ldots, H_6$ of $\overline{G}$, and they each have order 8. By renumbering if necessary, we may assume that any two subgroups from $\{H_1, H_2, H_3\}$ and any two from $\{H_4, H_5, H_6\}$ generate $\overline{G}$, and that $\langle H_i, H_j \rangle < \overline{G}$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. Each of $U_1, \ldots, U_{q+1}$ is elementary abelian, and so each of $\overline{U_1}, \ldots, \overline{U_{q+1}}$ is a subgroup of (at least) one of the six elementary abelian subgroups $H_1, \ldots, H_6$. Since $q + 1 \geq 9$ the Pigeonhole Principle implies that at least one subgroup contains $\overline{U_i}$ and $\overline{U_j}$ for distinct $i, j \in \{1, \ldots, q+1\}$. Without a loss of generality, assume that $H_1$ contains $\overline{U_1}$ and $\overline{U_2}$. Suppose first that for some $k \in \{3, \ldots, 9\}$ and some $j \in \{4, 5, 6\}$ we have $\overline{U_k} \leq H_j$. Then

$$\overline{G} = \overline{U_1U_2U_k} \leq \langle H_1, H_j \rangle < \overline{G},$$

a contradiction. Suppose now that for all $k \in \{3, \ldots, 9\}$ we have $\overline{U_k}$ is a subgroup of $H_1$, $H_2$, or $H_3$. The Pigeonhole Principle now implies that some $H_j$ contains $\overline{U_k}, \overline{U_{k'}}, \overline{U_{k''}}$ for distinct $k, k', k'' \in \{3, \ldots, 9\}$. Then $\overline{G} = \overline{U_kU_{k'}U_{k''}} \leq \langle H_j \rangle < \overline{G}$, again a contradiction. Therefore, no such AS-configuration can exist, as desired.

Case $\overline{G} \cong 2^{1+4}_-$. There are five maximal elementary abelian subgroups of $\overline{G}$, and they each have order 4. Since $q + 1 \geq 9$, the Pigeonhole Principle implies that at least one of these subgroups contains two different $\overline{U_i}$'s with $i > 0$. Thus, there exist (at most) two different subgroups of order 4 that contain three distinct $\overline{U_i}$'s. However, no two of these subgroups of order 4 generate $\overline{G}$; a contradiction. Therefore, $G$ cannot have an AS-configuration. □
5. Skew-translation generalised quadrangles of order \((8,8)\)

A skew-translation generalised quadrangle has an elation group of order 512, and we use the theory of Section 4, plus the known catalogue of finite groups of order 512 available in the computational algebra systems \textsf{GAP} \cite{GAP} and \textsf{Magma} \cite{Magma}, to show that such an elation group is elementary abelian. The 10,494,213 groups \(G\) of order \(2^9\) are numbered in the same order by both \textsf{GAP} and \textsf{Magma}, and we say that the \(k\)th group has Id-number \(k\). The structural constraints on \(G\) found in the previous section radically reduce the number of feasible groups to just four examples. These groups are described in Table 1. We write \(2_+^{1+2m}\) and \(2_+^{1-2m}\) for the extraspecial group of order \(2^{1+2m}\) of plus-type and minus-type, respectively. The central products \(2_+^{1+2m} \circ C_4\) and \(2_+^{1-2m} \circ C_4\), with central involutions amalgamated, are isomorphic.

**Lemma 5.1.** There are at most four nonabelian groups \(G\) of order 512 admitting an AS-configuration, and they are described in Table 1.

| IdNumber | \(Z(G)\) | \(\Phi(G)\) | \(\text{Aut}(G)\) | Name/Description |
|----------|----------|-------------|------------------|-----------------|
| 10,494,208 | \(C_2^3\) | \(C_2\) | \(2^{6+14}.(\text{GO}^+(6,2) \times S_3)\) | \(2_+^{1+6} \times C_2^2\) |
| 10,494,210 | \(C_4 \times C_2\) | \(C_2\) | \(2^{7+8}.\text{Sp}(6,2)\) | \((2_+^{1+6} \circ C_4) \times C_2\) |
| 10,494,211 | \(C_2\) | \(C_2\) | \(2^8 \cdot \text{GO}^+(8,2)\) | \(2_+^{1+8}\) |
| 10,494,212 | \(C_2\) | \(C_2\) | \(2^8 \cdot \text{GO}^-(8,2)\) | \(2_+^{1-8}\) |

**Table 1.** Four remaining groups

**Proof.** Suppose \(G\) is a nonabelian group of order 512, and assume that \(G\) admits an AS-configuration \(U_0, U_1, \ldots, U_9\). Recall that \(\Phi(G) \leq U_0\) by Lemma 4.1(i). Now, from the structural information of the last section, \(G\) must satisfy the following conditions:

(i) \(U_0 = \Phi(G) \implies Z(G) \leq U_0\) (see Theorem 4.13);
(ii) \(U_0 \leq Z(G) \implies Z(G) = U_0 > U_0^2 = G^2 = \Phi(G) > G^4 = \{1\}\) (see Proposition 4.7, Corollary 4.11, Lemma 4.12);
(iii) \(\exp(Z(G)) > 2 \implies Z(G) \leq U_0\) (see Proposition 4.10);
(iv) \(\exp(U_0) = 2 \implies \exp(G) = 4\) and \(\exp(Z(G)) = 2\) and \(U_0 \not\leq Z(G)\) (see Proposition 4.7, Lemma 4.8, Proposition 4.10);
(v) \(G\) has no extraspecial quotient groups of order 8 or 32 (see Lemma 4.15).

Of the 10,494,213 groups \(G\) of order \(2^9\) only 9367 satisfy conditions (i) – (iv) above. This number reduces to 552 upon applying condition (v). The smallest Id-number for \(G\) is 10,493,076, and the largest is 10,494,212. By Lemma 4.2, for each \(i > 0\), we must have \(U_i\) not normal in \(G\). Suppose \(i > j > 0\) and \(U_i = U_j^g\) for some \(g \in G\). By Lemma 4.1(iii), \(U_i \leq U_0 U_i \cap U_0 U_j = U_0\), a contradiction. Therefore, the \(U_i\) are pairwise non-conjugate. Of these remaining 552 groups, only 283 have at least nine non-normal subgroups of order 8 that are pairwise not conjugate and trivially intersect the Frattini subgroup of the given group. Of these 283 groups, only four have a clique of size 9 in a graph whose vertices are subgroups of order 8 and \((H, K)\) is an edge when

\[
\Phi(G) \cap HK = \Phi(G) \cap KH = K \cap \Phi(G) H = H \cap \Phi(G) K = \{1\}.
\]

The four remaining groups are listed in Table 1. 

\[\Box\]

The \textsf{GAP} \cite{GAP} code for all of our computational work is provided in the \texttt{arXiv}-version of this paper \cite{GAP}. 

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We now show that none of the four groups $G$ in Table 1 admits an AS-configuration. Note that $G$ is a finite 2-group of exponent 4, with a Frattini subgroup $\Phi(G)$ of order 2, and centre $Z(G)$ satisfying $\Phi(G) \leq Z(G)$. Let $\overline{g}$ be the element $\Phi(G)g$ of the Frattini quotient $G/\Phi(G)$. As is customary, we identify the multiplicative groups $\Phi(G) = G^2$ and $G/\Phi(G)$ with the additive groups of the field $F_2$ and the vector space $V = F_2^3$, respectively. The square map induces a well-defined quadratic form $Q: V \to F_2$ on $V$ given by:

$$Q(\overline{g}) := g^2.$$ 

The map $B: V \times V \to F_2$ given by $B(\overline{g}, \overline{h}) = [g, h] = g^{-1}h^{-1}gh$ is a well-defined alternating bilinear form. Indeed, $B$ is the bilinear form associated with $Q$ because

$$B(\overline{g}, \overline{h}) = Q(\overline{gh})Q(\overline{g})Q(\overline{h}) = (gh)^2g^2h^2 = g^2h^2[h, g]g^2h^2 = [h, g] = [g, h].$$

Suppose $H \leq G$. We use the following facts and definitions in the following:

1. the subspace $\overline{\mathbf{t}}$ of $V$ is totally singular if and only if $H^2 = 1$;
2. the subspace $\overline{\mathbf{H}}$ of $V$ is totally isotropic if and only if $H' = 1$ (i.e. $H$ is abelian);
3. the (bilinear) radical of $B$, denoted $\text{Rad}(B)$ or $\text{Rad}(Q)$, equals $Z(G)/\Phi(G)$;
4. the singular radical of $Q$, denoted $\text{SRad}(Q)$, equals $Z(G)^2\Phi(G)/\Phi(G)$;
5. the bilinear form $B$ gives us a null polarity: $\langle \overline{g} \rangle = C_G(\overline{g}) = \{ \overline{h} : h \in C_G(\overline{g}) \}$.

It is straightforward to prove that an AS-configuration $U_0, U_1, \ldots, U_9$ of $G$ gives rise to nine 3-dimensional subspaces $U_1, \ldots, U_9$ of $V$, satisfying the following generalised definition of a singular pseudo-arc. Let $Q: V \to F$ be a (possibly degenerate) quadratic form on a vector space $V = F^d$, and let $n$ be an integer satisfying $d/3 \leq n \leq d/2$. A set $\{W_1, \ldots, W_m\}$ of subspaces of $V$ satisfying the following conditions is called a singular pseudo-arc:

- (SP1) each $W_i$ is totally singular with respect to $Q$, i.e. $Q(W_i) = 0$ for $1 \leq i \leq m$;
- (SP2) $\dim(W_1) = \cdots = \dim(W_m) = n$;
- (SP3) $V = W_i + W_j + W_k$ for all distinct $i, j, k \in \{1, \ldots, m\}$.

We call the set $\{W_1, \ldots, W_m\}$ a singular $(m, n)$-pseudo-arc of $V$.

Observe that (SP1), (SP2), and Witt’s theorem implies that $n \leq d/2$, and if $m \geq 3$ then property (SP3) implies $d/3 \leq n$. Thus $d/3 \leq n \leq d/2$ is guaranteed to hold when $m \geq 3$. The standard definition of a pseudo-arc has $d = 3n$, and in this case the sum in (SP3) is direct. The groups in Table 1 have $|G| = 2^9$, $|\Phi(G)| = 2$, and $V = G/\Phi(G) \cong F_2^8$. Since 8 is not divisible by 3, our subsequent deliberations involve the generalised definition of a pseudo-arc.

**Lemma 5.2.** The group $2^{4+6} \times C_2^2$ in Table 1 does not admit an AS-configuration.

**Proof.** Set $G = 2^{4+6} \times C_2^2$. Assume by way of contradiction that $G$ has an AS-configuration $U_0, U_1, \ldots, U_9$. Then as remarked above $W_i, \ldots, W_9$ is a singular $(9, 3)$-pseudo-arc of $V := G/\Phi(G)$ relative to the squaring quadratic form $Q: V \to F_2$ where $W_i = U_i$. In this case, the (bilinear) radical $\text{Rad}(B)$ equals the singular radical $\text{SRad}(Q)$. Up to isometry, we may suppose that

$$Q \left( \sum_{i=1}^8 x_i e_i \right) = x_1x_2 + x_3x_4 + x_5x_6 \quad \text{where } V = \langle e_1, \ldots, e_8 \rangle = F_2^8.$$

Note that $\text{Rad}(B) = \text{SRad}(Q) = \langle e_7, e_8 \rangle$.

We used a computer to search for all singular $(9, 3)$-pseudo-arcs in $V$, and we outline in detail how this computation proceeded. First, we found all singular $(6, 3)$-pseudo-arcs of $V$, up to symmetry in the isometry group of $Q$ (which is isomorphic to $2^{12} : (GO^+(6, 2) \times S_3)$). This involved using the command **SmallestImageSet** in the GAP package **Gape** [17]. We then used depth-first backtrack search to find the singular $(9, 3)$-pseudo-arcs extending...
the singular $(6,3)$-pseudo-arcs. In total, we found eight singular $(9,3)$-pseudo-arcs up to equivalence. We then take the preimage of each singular $(9,3)$-pseudo-arc, yielding a set of nine subgroups of $G$, each of size 16. We then take all complements of the Frattini subgroup in each of our set of subgroups of order 16, giving us 72 subgroups of size 8 that meet $\Phi(G)$ trivially (as $\Phi(G) \cap U_i \subseteq U_0 \cap U_i = \{1\}$ for $i > 0$). We then perform eight depth-first searches on eight sets of 72 subgroups to find an AS-configuration for $G$. None of the eight searches found an AS-configuration.

\begin{lemma}
The group $(2_+^{1+6} \cdot C_4) \times C_2$ in Table 1 does not admit an AS-configuration.
\end{lemma}

\begin{proof}
Set $G = (2_+^{1+6} \cdot C_4) \times C_2$. In this case, $U_0 = Z(G) \cong C_4 \times C_2$. Mimicking the proof of Lemma 5.2, assume that $G$ has an AS-configuration $U_0, U_1, \ldots, U_9$, and define $\tilde{B}: \tilde{V} \times \tilde{V} \to \mathbb{F}_2$ by $\tilde{B}(\tilde{g}, \tilde{h}) = [g, h]$ where $\tilde{V} = G/Z(G)$. The outer automorphism group of $2_+^{1+6} \cdot C_4$ equals $\text{Sp}(\tilde{B})$, and $\text{Sp}(6,2)$ acts transitively on the maximal totally isotropic subspaces of $\tilde{V}$. A computer calculation shows that $\text{Aut}(G)$ acts transitively on elementary abelian subgroups of $G$ of order $2^6$. Indeed, it also shows that $\text{Aut}(G)$ acts transitively on the elementary abelian subgroups that have order 8 and meet $Z(G)$ trivially. So we can fix $U_1$ for a putative AS-configuration. Upon stabilising $U_1$ in $\text{Aut}(G)$ and looking at the elementary abelian subgroups meeting $U_0 U_1$ trivially, we see that we can also fix a third element $U_2$ of a putative AS-configuration. Now we take the stabiliser $\text{Aut}(G)_{U_1,U_2}$. There are 784 elementary abelian subgroups of order 8 that meet each of $U_0 U_1, U_0 U_2$, and $U_1 U_2$ trivially, and $\text{Aut}(G)_{U_1,U_2}$ has precisely two orbits on this set of subgroups: one of size 672 and one of size 112. If we choose $U_3$ from the small orbit (of size 112), then there are no elementary abelian subgroups of order 8 that meet trivially each $U_i U_j$ where $i$ and $j$ are distinct elements of $\{0, 1, 2, 3\}$. So $U_3$ can be chosen from the large orbit (of size 672). The set of elementary abelian subgroups of order 8 that meet trivially $U_i U_j$, where $i$ and $j$ are distinct elements of $\{0, 1, 2, 3\}$, has size 48. Upon using a depth-first backtrack computer search, and we discovered that there are no partial AS-configurations of size 6 on this set of subgroups. This contradiction shows that $G$ does not have an AS-configuration.
\end{proof}

\begin{remark}
We cannot use geometry on the Frattini quotient $V \cong \mathbb{F}^3_2$ to rule out $(2_+^{1+6} \cdot C_4) \times C_2$. There does exist a singular $(9,3)$-pseudo-arc of $V$ with each element intersecting the radical $\pi_0$ trivially. To construct an example, consider the conic defined by the quadratic $xy = z^2$ of the Desarguesian projective plane on the 3-dimensional vector space $\mathbb{F}^3_2$. (The conic has nine projective points corresponding to the pairwise disjoint subspaces $\{(1,y,y^2)\}, y \in \mathbb{F}_2$, and $\{(0,0,1)\}$ which is the radical of this conic.) We now apply field reduction from $\mathbb{F}^3_8$ to $\mathbb{F}^3_2$. Write $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$ where $\alpha^3 + \alpha + 1 = 0$, and let $T: \mathbb{F}_8 \to \mathbb{F}_2$ be the trace map $\beta \mapsto \beta + \beta^2 + \beta^4$. Every $\mathbb{F}_2$-linear map $\mathbb{F}_8 \to \mathbb{F}_2$ has the form $\beta \mapsto T(\gamma \beta)$ for some $\gamma \in \mathbb{F}_8$. Identify $(x, y, z) \in \mathbb{F}^3_8$ with $(x_0 + x_1 \alpha + x_2 \alpha^2, y_0 + y_1 \alpha + y_2 \alpha^2, z_0 + z_1 \alpha + z_2 \alpha^2) \in \mathbb{F}^3_2$. This gives a quadratic form $Q_\gamma: \mathbb{F}^3_2 \to \mathbb{F}^2_2$ defined by

$$Q_\gamma(x_0 + x_1 \alpha + x_2 \alpha^2, y_0 + y_1 \alpha + y_2 \alpha^2, z_0 + z_1 \alpha + z_2 \alpha^2) = T(\gamma(x y + z^2)).$$

As $T(\alpha) = T(\alpha^2) = T(\alpha^4) = 0$ and $T(\alpha^3) = T(\alpha^5) = T(\alpha^6) = 1$, we see that $Q := Q_1$ equals

$$Q(x_0 + x_1 \alpha + x_2 \alpha^2, y_0 + y_1 \alpha + y_2 \alpha^2, z_0 + z_1 \alpha + z_2 \alpha^2) = x_0 y_0 + x_1 y_2 + x_2 y_1 + z_0^2.$$ 

A short calculation shows that each quadratic form $Q_\gamma$ with $\gamma \in \mathbb{F}_8, \gamma \neq 0$, is equivalent to $Q$. Thus we obtain pairwise disjoint singular planes lying in the singular points of the quadratic form $x_0 y_0 + x_1 y_2 + x_2 y_1 + z_0^2$. The radical of the conic is mapped to the radical of this form, which is the plane $"z = 0"$ of $\text{PG}(8,2)$. We then take a point $P$ of the singular radical $\text{SRad}(Q)$ and quotient by $P$ to $\text{PG}(7,2)$. The image of our conic is then sent to disjoint planes of $\text{PG}(7,2)$, and we see that these planes are singular with respect to the
induced form $Q$ and $\text{Rad}(Q)/P$ is the radical $\pi_0$ for this form. It turns out that we obtain the $W_i$ realising a singular pseudo-arc.

We will need the following result in order to show that the extraspecial group $2^{1+8}$ has no AS-configuration.

**Lemma 5.5.** Suppose for some $i > 0$ that $C_{U_j}(U_i) = 1$ for all $i \neq j > 0$. Then $C_G(U_i) \leq U_0 U_i$.

**Proof.** Suppose that $C_{U_j}(U_i) = 1$ for all $i > 1$. Assume that $x \in C_G(U_i)$ but that $x \notin U_0 U_1$. We know that $x \in U_j U_0$ for a unique $j > 1$. Since $x \notin U_j$ and $x \notin U_0 U_1$ by assumption, we have that $x = u_j u_0$, where $1 \neq u_j \in U_j$ and $1 \neq u_0 \in U_0$. Since $x \notin U_0 \cup U_j$, this means that $x$ is not in the partial difference set $\Delta := \bigcup_{k=0}^{+1} (U_k \setminus \{1\})$. By [7, Equation (6)], there is a unique factorisation $x = u_1 u_k$ where $k > 1$, the element $u_1$ is a nontrivial element of $U_1$, and $1 \neq u_k \in U_k$. However, this implies that $1 \neq u_k \in U_k \cap C_G(U_1) = C_{U_k}(U_1)$, a contradiction, and the result is proved.

**Lemma 5.6.** The extraspecial group $2^{1+8}$ in Table 1 does not admit an AS-configuration.

**Proof.** Set $G = 2^{1+8}$. Then squaring defines a non-degenerate form $Q$ of minus-type on $V := G/\Phi(G)$. Since the Witt index of this form is 3, we know that each $U_i$ for $i > 0$ maps to a maximal totally isotropic subspace $\overline{U}_i$ of $(V, Q)$. If $0 < i < j$, then $\overline{U}_i \cap \overline{U}_j = \{0\}$ is equivalent to $\overline{U}_i^\perp \cap \overline{U}_j = \{0\}$. So by Lemma 5.5, we know from their sizes that $\overline{U}_i^\perp = \overline{U}_0 U_j$, or in other words, $C_G(U_i) = U_0 U_i$ (for each $i > 0$). Therefore, $U_0 \leq C_G(U_1) \cap C_G(U_2) \cap C_G(U_3)$, which is a contradiction as the right-hand side is simply the centre of $G$ (as $G = U_1 U_2 U_3$). Therefore, $G$ does not have an AS-configuration.

**Lemma 5.7.** The extraspecial group $2^{1+8}$ in Table 1 does not admit an AS-configuration.

**Proof.** Set $G = 2^{1+8}$. Here, squaring defines a non-degenerate quadratic form $Q$ of plus-type on $V := G/\Phi(G)$. By Witt’s theorem, the stabiliser in $\text{GO}^+(8, 2)$ of a singular point acts transitively on the set of singular 3-spaces. We used a computer to search for all singular $(9, 3)$-pseudo-arcs in $V$ with respect to the form

$$Q(x) := x_1 x_2 + x_3 x_4 + x_5 x_6 + x_7 x_8.$$  

We outline in detail how this computation proceeded. First, we found all singular $(6, 3)$-pseudo-arcs of $V$, up to symmetry in $\text{GO}^+(8, 2)$. This was aided by using the command $\text{SmallestImageSet}$ in the GAP [6] package Grape [17]. We then used depth-first backtrack search to find all extensions of singular $(6, 3)$-pseudo-arcs to singular $(9, 3)$-pseudo-arcs. From the 1402 possible singular $(6, 3)$-pseudo-arcs, none extended to a $(9, 3)$-pseudo-arc.

We can now prove the second main theorem of the Introduction.

**Proof of Theorem 1.2.** By Lemma 5.1, the only possible nonabelian groups of order 512 that can admit an AS-configuration are listed in Table 1. In Lemmas 5.2, 5.3, 5.6, and 5.7, we progressively ruled out these four groups. Thus the only possible groups $G$ of order 512 that can admit an AS-configuration are abelian, and by Lemma 4.2, $G$ must then be elementary abelian. Hence an STGQ of order 8 is a translation generalised quadrangle, and these were classified by Payne [13] in 1976.

**Corollary 5.8.** Every skew-translation generalised quadrangle of order $(q, q)$ with $q \leq 8$ is isomorphic to the classical generalised quadrangle $\mathcal{W}(3, q)$, or to $T_2(\mathcal{O})$ where $\mathcal{O}$ is a non-classical translation oval of $\text{PG}(2, 8)$. 

\[\square\]
PROOF. As noted in the introduction, the generalised quadrangles of order \((q, q)\) are classified for \(q \leq 4\) (see [14, §6]), and they are classical generalised quadrangles. Similarly, a generalised quadrangle of order \((q, q)\) with \(q\) prime is classical [2, Proposition 3]. This leaves \(q = 8\) as we know that \(q\) is a prime power by [5, Theorem 1]. Let \(\mathcal{Q}\) be a skew-transformation generalised quadrangle of order \((q, q)\) with \(q = 8\). By Theorem 1.2, \(\mathcal{Q}\) is a translation generalised quadrangle. There are precisely two TGQ of order \((8, 8)\) according to Payne’s classification [13]: (i) the classical example \(\mathcal{W}(3, 8) \cong \mathcal{Q}(4, 8)\), and (ii) a generalised quadrangle of the form \(T_2(O)\) where \(O\) is a non-classical translation oval of \(\text{PG}(2, 8)\).

It is natural to consider groups of order \(q^3\) when \(q = 2^m > 8\). There are roughly \(2^{2m^3}\) such groups. As \(m\) increases, more and more groups will satisfy the numerous structural constraints in Section 4. It is not obvious that examples of AS-configurations will not arise for 2-groups that are central products of extraspecial groups and abelian groups, nor is it particularly obvious how one might construct a nonabelian 2-group with an AS-configuration.

One reason to believe that examples may arise in larger 2-groups would be the fact that non-classical translation planes of order \(q\) only come into existence when we consider \(q \geq 16\). If \(G\) is a group of order \(q^3\) admitting an AS-configuration \(U_0, U_1, \ldots, U_{q+1}\), then in the quotient \(G/U_0\), the subgroups \(U_iU_0/U_0\), \(i > 0\), form a spread of the group \(G/U_0\). This implies that \(G/U_0\) is elementary abelian and we obtain a translation plane by the famous construction of André. For \(q \leq 8\), we must obtain the Desarguesian plane \(\text{AG}(2, q)\), whereas for larger values of \(q\), there are more exotic examples of translation planes. The fact that we have not discovered a non-classical skew translation generalised quadrangle of order \((q, q)\) might be a by-product of a conjecture: the skew-translation generalised quadrangle arising from an AS-configuration is classical if and only if the associated spread yields a Desarguesian translation plane.

Appendix A. A work-around for Lemma 6 of Ghinelli (2012)

The first line of the proof of Lemma 6 of [7] contains an oversight where “\(hg^{-1}\) normalises \(U\) given that \(U^{gh^{-1}} \cap U\) is nontrivial” does not necessarily hold from the assumptions provided. Lemma 6 can be avoided in the proof of her main result (Theorem 6) by using the following lemma and then reading on from Corollary 4 hence. We are very grateful for the contribution of Professors Ghinelli and Ott who collaborated in the lemma below, via correspondence with the authors on 2013.10.01.

**Lemma A.1.** Let \(G\) be a group of order \(q^3\), \(q\) odd, and suppose \(U_0, U_1, \ldots, U_{q+1}\) is an AS-configuration for \(G\). Then

(i) for each \(i > 0\), if \(1 \neq u \in U_i\), then \(C_G(u) = N_G(U_i) = C_G(U_i) = U_0U_i\), and

(ii) \(U_0 = \mathbb{Z}(G)\).

**Proof.** We may suppose without loss of generality that \(i = 1\). By way of contradiction, let \(u \in U_1\) and suppose that \(C_{U_0}(u)\) is strictly smaller than \(U_0\). Then by [7, Lemma 4], we have \(u^{U_0} = U_0u\), and \(|C_G(u)| = q^2\). Let \(2 \leq j \leq q + 1\). Since \(U_1\) centralises \(u\) (as \(U_1\) is abelian), we have \(G = U_1U_0U_j \leq C_G(U_1)U_0U_j\) and hence \(G = C_G(u)U_0U_j\). Therefore, \(U_0U_j\) acts transitively on \(uG\), and in particular,

\[q^2 = |U_0U_j| = |u^{U_0U_j}|C_{U_0U_j}(u)| = q|C_{U_0U_j}(u)|\]

and so \(|C_{U_0U_j}(u)| = q\). We claim that there exists an index \(2 \leq k \leq q + 1\) such that \(C_{U_k}(u) = \{1\}\). Suppose not. Then for each \(2 \leq k \leq q + 1\) there is a nontrivial element...
$v_k \in U_k$ that commutes with $u$. Then the cosets

$$U_1, U_1v_2, \ldots, U_1v_{q+1}$$

each centralise $u$. Now these cosets are distinct, since if $U_1v_s = U_1v_t$, and $v_s \notin U_1$, then $v_sv_t^{-1} \in U_1 \cap U_2U_t = \{1\}$. Thus we conclude that

$$|C_G(u)| \geq (q + 1)q > q^2$$

a contradiction. Therefore, without loss of generality, we may suppose that $U_2 \cap C_{U_0}v_2(u) = U_2 \cap C_G(u) = \{1\}$; that is, $U_2$ acts regularly by conjugation on $u^G$.

Now by assumption, $U_0 \not\supseteq C_{U_0}v_2(u)$, and we know from above that $|C_{U_0}v_2(u)| = q$, so there exists $g \in C_{U_0}v_2(u) \setminus U_0$. Write $g = u_0u_2$, where $u_0 \in U_0$ and $u_2 \in U_2$, and note that $u_0 \neq 1$ and $u_2 \neq 1$. Now $u_0u_2 = u_2^h$ for some $h \in G$ (as $u_2^G = U_0u_2$) and hence $u_2 \in C_G(u)^h = C_G(u^h)$. Therefore, $u_2$ is a nontrivial element of $U_2$ fixing an element of $u^G$, contradicting the fact that $U_2$ acts regularly on $u^G$. Therefore, $C_G(u) = U_0U_1$ for each nontrivial $u \in U_1$. Moreover, by Corollaries 2 and 3 of [7], we have that $U_0 = Z(G)$. □

Acknowledgements. We would like to thank Professors Ghinelli and Ott for their help with the proof of Lemma A.1. We also thank Dr. Gabriel Verret for his comments on an earlier draft. The first author acknowledges the support of the Australian Research Council Future Fellowship FT120100036. The second author acknowledges the support of the Australian Research Council Discovery Grants DP130100106 and DP1401000416. The third author acknowledges the support of the Australian Research Council Discovery Grant DP120101336.

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Appendix: GAP code

Here we present the GAP code used for the classification of groups of order 512 having an AS-configuration.

```gap
LoadPackage("fining");
LoadPackage("autpgrp");
stream := OutputTextFile("AScomputation", true);
AppendTo(stream, "You will be able to see the progress of the computation here\n");

############################################
# The following function tests to see if a group G satisfies the
# conclusions of Theorem 4.13, Proposition 4.7, Corollary 4.11,
# Lemma 4.12, Lemma 4.8, and Proposition 4.10.
############################################

SufficientConditionCheck := function(G)
  local z, frat, exp, flag1, flag2, flag3, flag4, hom, fratquo, n, subs, u0, i;
  if IsAbelian(G) then
    return false;
  fi;
  frat := FrattiniSubgroup(G);
  n := Root(Size(G), 3);
  if Size(frat) > n then
    return false;
  fi;
  if Size(frat) = n then
    u0 := frat;
    z := Centre(G);
    exp := Exponent(G);
    # Z(G) < U0 (Corollary 4.14)
    flag1 := IsSubgroup(u0, z) and z <> u0;
    # U0 elt. ab. => Exponent 4 (Lemma 4.8)
    flag2 := flag1 and (not IsElementaryAbelian(u0) or exp = 4);
    return flag2;
  fi;
  hom := NaturalHomomorphismByNormalSubgroup(G, frat);
  fratquo := Image(hom);
  # All possible subgroups U0 / Frat(G) of G / Frat(G)
  subs := InvariantSubgroupsElementaryAbelianGroup(fratquo, [], [ Log(n/Size(frat),2) ]);;
  z := Centre(G);
  exp := Exponent(G);
  for i in [1..Size(subs)] do
    u0 := PreImage(hom, subs[i]);
    # U0 <= Z(G) => Exponent 4 and U0 = Z(G)
    flag1 := not IsSubgroup(z, u0) or (exp = 4 and u0 = z);
    # Z(G) not elt. ab. => Z(G) <= U0
    flag2 := flag1 and (IsElementaryAbelian(z) or IsSubgroup(u0, z));
    # U0 elt. ab. => Exponent 4 and Z(G) elt.ab. and U0 not contained in Z(G)
    flag3 := flag2 and (not IsElementaryAbelian(u0) or (exp = 4 and IsElementaryAbelian(z) and not IsSubgroup(z, u0)));
    # U0 <= Z(G) => U0^2 = G^2 = Phi(G)
    flag4 := flag3 and (not IsSubgroup(z, u0) or (Agemo(u0, 2) = Agemo(G,2) and Agemo(u0, 2) = frat));
    if flag4 then
      return true;
    fi;
  od;
  return flag4;
end;
```
The following function tests Lemma 4.15 of the paper: does there exist a normal subgroup $N$ of $G$ of index 8 or 32 such that $G/N$ is extraspecial? Such a group cannot admit an $AS^{-}$ configuration.

Note that we are using an iterator for the $LowIndexSubgroupsFpGroup$ since in many cases a false answer returns quite soon. It would be even quicker if there was a routine for just the low index normal subgroups.

Here is the code for the function:

```plaintext
Lemma415 := function ( G )
local d, ns, iso, iter, im, n;
isomorphism := IsomorphismFpGroup(G);
im := Image(iso);
d := DerivedSubgroup( im );
Print("Doing normal subgroups of index 8 ... ");
it = LowIndexSubgroupsFpGroupIterator(im, 8);
repeat
  n := NextIterator(iter);
  if IsNormal(im, n) and not IsSubgroup(n, d) then
    Print("false
    return false;
  fi;
until IsDoneIterator(iter);
Print("Doing normal subgroups of index 32 ... ");
it = LowIndexSubgroupsFpGroupIterator(im, 32);
repeat
  n := NextIterator(iter);
  if IsNormal(im, n) and IdGroup(im / n) in [[32,49], [32,50]] then
    Print("false
    return false;
  fi;
until IsDoneIterator(iter);
Print("true
return true;
end;
```

The following function tests to see if we have at least $n+1$ conjugacy classes of non-normal subgroups of order $n$ in a group $G$ of order $n^3$, that meet the Frattini subgroup trivially. This is a necessary condition for $G$ to admit an $AS^{-}$ configuration.

```
EnoughSubgroupsOfSize_n := function (G, n)
local subs, frat;
sub := Filtered(SubgroupsSolvableGroup( G,
  rec( consider := ExactSizeConsiderFunction(n) ) ), t -> Size(t) = n);
frat := FrattiniSubgroup(G);
sub := Filtered(sub, t -> IsTrivial(Intersection(frat,t)) and not IsNormal(G,t));
return Size(subs) >= n + 1;
end;
```

The following function simply gives us all of the non-normal subgroups of order $n$ in a group $G$ of order $n^3$, that intersect the Frattini subgroup trivially.

```
GoodSubgroupsOfSize_n := function (G, n)
local subs, frat;
sub := Filtered(SubgroupsSolvableGroup( G,
  rec( consider := ExactSizeConsiderFunction(n) ) ), t -> Size(t) = n);
frat := FrattiniSubgroup(G);
sub := Filtered(sub, t -> IsTrivial(Intersection(frat,t)) and not IsNormal(G,t));
sub := Union(List(subs,x -> AsSet(ConjugacyClassSubgroups(G,x))));
```
return subs;
end;

# The following function asks whether a group G of order n^3 has a set
# of n+1 subgroups of order n, intersecting the FrattiniSubgroup trivially,
# such that they pairwise satisfy the symmetric
# relation Phi(G) \cap AB = Phi(G) \cap BA = 1.
# To make this even quicker, we could also ask that
# A and B are not conjugate in G.
HasCliqueOfSize9 := function( G )
local n, subs, frat, graph, clique, aut, test_triple;
  n := Root(Size(G), 3);
subs := GoodSubgroupsOfSize_n(G, n);
  if Size(subs) < n + 1 then return [] fi;
  frat := FrattiniSubgroup(G);
  aut := AutomorphismGroup(G);
  test_triple := function(a,b,c)
    local x, blist, flag;
    blist := AsList(b); flag := false;
    for x in AsList(c) do
      if not IsOne(x) then
        flag := ForAny(blist, t -> x*t in a);
      fi;
    od;
    return true;
  end;
  graph := Graph(aut, subs, 
    function(x, t) return Image(t,x) end,
    function(i,j) return i<>j and 
    test_triple(frat, i, j) and test_triple(frat, j, i) and 
    test_triple(i, frat, j) and test_triple(j, frat, i) end);
  clique := CompleteSubgraphsOfGivenSize(graph, n + 1, 0, false);
  return Size(clique) > 0;
end;

# Determining groups of order 512 that do not admit an AS-configuration.
# There is repetition in the code, so it could be made quicker. However,
# we are interested in how many groups survive at each step.

range := [1..NrSmallGroups(512)];
repeat
  midpoint := Int(Size(range)/2) + Minimum(range);
  G := SmallGroup(512, midpoint);
  phi := FrattiniSubgroup(G);
  if Size(phi) > 8 then
    range := [midpoint+1..Maximum(range)];
  else
    range := [Minimum(range)..midpoint];
  fi;
  Print(range, "\n");
until Size(range) = 2;
min := First(range, t -> Size( FrattiniSubgroup(SmallGroup(512,t)) ) <= 8);
AppendTo(stream, Concatenation("...the smallest index for which SmallGroup(512, i) has a Frattini subgroup of order at most 8 is", String(min), "\n") );
# It should turn out that the smallest index is 7532393.

# We use the function SufficientConditionCheck
# to find all the groups of order 512 which satisfy
# the displayed conditions in the proof of Lemma 5.1.
AppendTo(stream, "Now checking to see which groups satisfy (i)−(iv) of the proof of Lemma 5.1.\n") ;

leftover := [];
for i in [min..NrSmallGroups(512)] do
  G := SmallGroup(512, i);
  if SufficientConditionCheck( G ) then
    Add(leftover, i);
  fi;
  if (i − min + 1) mod 1000 = 0 then
    AppendTo(stream, Concatenation("...progress: ", String(i − min + 1), ", done out of ", String(NrSmallGroups(512)−min), "\n") );
  fi;
od;

AppendTo(stream, Concatenation("We now have ", String(Size(leftover)), " groups.\n") );

# Now checking to see which of these groups have an extraspecial quotient
# of order at most 32.
AppendTo(stream, "Now checking to see which groups satisfy (v) of the proof of Lemma 5.1.\n") ;

leftover2 := [];
for i in leftover do
  G := SmallGroup(512, i);
  if Lemma415( G ) then
    Add(leftover2, i);
  fi;
  if Position(leftover, i) mod 10 = 0 then
    AppendTo(stream, Concatenation("...progress: ", String(Position(leftover,i)), ", done out of ", String(Size(leftover2)), "\n") );
  fi;
od;

AppendTo(stream, Concatenation("We now have ", String(Size(leftover2)), " groups.\n") );

# Are there enough subgroups? (The U_i, i > 0)
AppendTo(stream, "Now checking to see which groups have enough conjugacy classes of non−normal subgroups of order 8.\n") ;

leftover3 := [];
for i in leftover2 do
  G := SmallGroup(512, i);
  if EnoughSubgroupsOfSize_n( G, 8 ) then
    Add(leftover3, i);
  fi;
  if Position(leftover2, i) mod 10 = 0 then
    AppendTo(stream, Concatenation("...progress: ", String(Position(leftover2,i)), ", done out of ", String(Size(leftover2)), "\n") );
  fi;
AppendTo(stream, Concatenation("We now have ", String(Size(leftover3)), ".groups\n") );

# Now we will see if there are enough subgroups which satisfy the relation  
# *U_0\cap U_i U_j = \{1\} or U_0\cap U_j U_i = \{1\} for a fixed U_0.  
AppendTo(stream, "Now checking to see which groups have a clique of size 9.\n") ;

leftover4 := [];  
for i in leftover3 do  
    G := SmallGroup(512, i);  
    if HasCliqueOfSize9( G ) then  
        Print(i, ".\n");  
        Add(leftover4, i);  
    fi;  
    if Position(leftover3, i) mod 10 = 0 then  
        AppendTo(stream, Concatenation("... progress: ",  
            String(Position(leftover3,i)), ".done out of ", String(Size(leftover3)), ".\n");  
    fi;  
od;  
AppendTo(stream, Concatenation("We now have ", String(Size(leftover4)), ".groups\n") );

BackTracker := function( size, domain, seeds, ispartialsolution )  
# "size" is the ultimate size of the sets we want to find  
# "domain" is the search space  
# "seeds" is a set which all of our sets are forced to contain  
# "ispartialsolution" is a function that takes as input 'x' which is the  
# current partial solution, the node we are visiting, and 'new' is the  
# extra element that we are adding to 'x'.  
local results, node_visit, children_of_l, isleaf, count;

def children_of_l( domain, l )  
local pos;  
if IsEmpty(l) then  
    return domain;  
else  
    pos := Position(domain, l[Size(l)]);  
    return domain{[pos+1..Size(domain)]};  
fi;  
end;

def results := [];

def count := 1;

node_visit := function( x, new )  
local child, pos;  
# just checks progress  
if not IsEmpty(x) then  
    pos := Position(domain,x[1]);  
    if pos > count then count := pos; Print("count, ".\n"); fi;  
fi;

def BackTracker( size, domain, seeds, ispartialsolution )  
# 'size' is the ultimate size of the sets we want to find  
# 'domain' is the search space  
# 'seeds' is a set which all of our sets are forced to contain  
# 'ispartialsolution' is a function that takes as input 'x' which is the  
# current partial solution, the node we are visiting, and 'new' is the  
# extra element that we are adding to 'x'.  
local results, node_visit, children_of_l, isleaf, count;

def children_of_l( domain, l )  
local pos;  
if IsEmpty(l) then  
    return domain;  
else  
    pos := Position(domain, l[Size(l)]);  
    return domain{[pos+1..Size(domain)]};  
fi;  
end;

def results := [];

def count := 1;

node_visit := function( x, new )  
local child, pos;  
# just checks progress  
if not IsEmpty(x) then  
    pos := Position(domain,x[1]);  
    if pos > count then count := pos; Print("count, ".\n"); fi;  
fi;

def BackTracker( size, domain, seeds, ispartialsolution )  
# 'size' is the ultimate size of the sets we want to find  
# 'domain' is the search space  
# 'seeds' is a set which all of our sets are forced to contain  
# 'ispartialsolution' is a function that takes as input 'x' which is the  
# current partial solution, the node we are visiting, and 'new' is the  
# extra element that we are adding to 'x'.  
local results, node_visit, children_of_l, isleaf, count;

IsPartialPseudoArc := function( f )
  # f is a set of subspaces of a projective space
  # returns: boolean, true or false
  local flag, a, b, c, tplus1, d;
  d := ProjectiveDimension( AmbientSpace( f[1] ) );
  tplus1 := Size(f);
  flag := true;
  for a in [1..tplus1−2] do
    for b in [a+1..tplus1−1] do
      for c in [b+1..tplus1] do
        flag := ProjectiveDimension( Span( f{a,b,c} ) ) = d and
                ForAll(Combinations([a,b,c], 2), t −> ProjectiveDimension( Meet( f{t} ) ) = −1);
        if not flag then
          return flag;
        fi;
      od;
    od;
  od;
  return flag;
end;

OrbitRepsOnPutativeTuples := function( g, s, max, property )
  # Takes a permutation group g, starting set s (can be empty),
  # and finds all tuples of size max containing s up to equivalence in g.
  # The variable property only keeps tuples that fulfil a hereditary property
  # (like IsPartialPseudoArc)
  local tuplefinder, tuples;
  tuples := [];
  tuplefinder := function(g, s, max)
    local stab, sx, ssx, orb, orbs, rest, x;
    if (Length(s) = max) then
      Print(s,"\n"); Add(tuples, s);
      return;
    fi;
    if IsEmpty(s) then stab := g;
    else stab := Stabilizer(g, s, OnSets);
    fi;
    orbs := Orbits(stab,[1..DegreeAction(g)]);
    for orb in orbs do
      x := Minimum(orb);
      if (IsEmpty(s) or x > Maximum(s)) and property(Concatenation(s,[x])) then
        sx := Union(s, [x ]);
        ssx := SmallestImageSet(g, sx);
        if ssx = sx then
          tuplefinder(g, sx, max);
        fi;
      fi;
    od;
    tuplefinder(g,s,max);
  return tuples;
end;
FindASConfigsViaPseudoArcs := function(perm, omega, seedsize, group, stream)
# This code is for groups of order 512 that have a Frattini subgroup
# of order 2. We look to the Frattini factor, where we have a quadratic
# form on PG(7,2), and we first want to find (if they exist) all
# singular (9,3)−pseudoarcs. We then pull these back to the group
# where we use backtracking to find AS−configurations.
# We have included an extra variable "stream" since we want more
# verbose printing in this function.
local tuples, t, rest, eggs, basket, frat, hom, quot, gens, preimages, s,
comps, asconfigs, ispartialsolution, ispartialsolution2, test_triple, allASconfigs;
# Find some partial pseudo−arcs of an intermediate size, up to symmetry
AppendTo(stream, Concatenation("Finding all partial singular pseudo−arcs of size", "+", String(seedsize), "\(n\)"));
tuples := OrbitRepsOnPutativeTuples(perm, [], x > IsPartialPseudoArc(omega{Concatenation([x], t)}));
# extend to full pseudo−arc
eggs := BackTracker(9, Concatenation(t, Difference(rest, t)), t, ispartialsolution);
Append(basket, eggs);
end;

ispartialsolution := function( x, new )
local a, b;
if IsEmpty(x) or IsEmpty(new) then
    return true;
fi;
if not ForAll(x, t −> ProjectiveDimension( Meet( omega[t], omega[new[1]] ) ) ) = -1) then
    return false;
fi;
for a in [1..Size(x)−1] do
    for b in [a+1..Size(x)] do
        if not (ProjectiveDimension( Span( omega{[x[a], x[b], new[1]]} ) ) ) = 7) then
            return false;
        fi;
    od;
return true;
end;

basket := [];
for t in tuples do
    AppendTo(stream, Concatenation("*** Doing :\.

rest := Filtered([1..Size(omega)], i −> IsPartialPseudoArc(omega{Concatenation([i], t)}));
# extend to full pseudo−arc
eggs := BackTracker(9, Concatenation(t, Difference(rest, t)), t, ispartialsolution);
Append(basket, eggs);
od;

AppendTo(stream, Concatenation("We have found \(n\)\) singular (9,3)−pseudoarcs in the Frattini factor\(n\)\));

test_triple := function(a,b,c)
local x, blist, flag;
## exists c <> 1 such that cb in A
blist := AsList(b); flag := false;
for x in AsList(c) do
    if not IsOne(x) then
        flag := ForAny(blist, t −> x*t in a);
    fi;
    if flag then return false; fi;
od;
return true;
end;
ispartialsolution2 := function( x, new )
local flag, a, b;
if IsEmpty(x) then
    return true;
fi;
flag := true;
if not IsEmpty(new) then
  for a in \[1..\text{Size}(x)-1\] do
    for b in \[a+1..\text{Size}(x)\] do
      flag := test_triple(x[a], x[b], new[1]);
      if not flag then
        return flag;
      fi;
    od;
  fi;
return flag;
end;

allASconfigs:=[];
if not IsEmpty(basket) then
  AppendTo(stream, "Sorting out equivalence of pseudo-arcs in the group...
"
);
  basket := Set(basket, t -> \text{SmallestImageSet}(\text{perm}, \text{Set}(t)));
  AppendTo(stream, "Done."
);
  basket := List(basket, t -> \text{omega}(t));
  AppendTo(stream, "We have found \(9,3\)-pseudo-arcs up to equivalence in the full isometry group.
"
);
# Now taking preimages into the 2-group
frat := FrattiniSubgroup(group);
hom := NaturalHomomorphismByNormalSubgroup(group, frat);
quot := Image(hom);
gens := GeneratorsOfGroup(quot);
preimages := List(basket, t -> List(t, x -> \text{Group(List(x, j -> Product(gens[j])))}));
preimages := List(preimages, t -> List(t, x -> PreImage(hom, x)));
AppendTo(stream, "Found preimages, now looking for complements to the Frattini subgroup.
"
);
for s in preimages do
  AppendTo(stream, Concatenation("Preimage: ", String(Position(preimages, s)), "\n"));
  comps := Union(List(s, t -> \text{ComplementClassesRepresentatives}(t, frat)));
  asconfigs := BackTracker(9, comps, [], ispartialsolution2);
  AppendTo(stream, Concatenation("\# Number of examples: ", String(Number(asconfigs, t -> not IsEmpty(t))), "\n") );
  Append(allASconfigs, asconfigs);
  od;
else
  AppendTo(stream, "No pseudo-arcs found\n");
  return [];
fi;
return allASconfigs;
end;

# Initial setup
pg := PG(7,2);
pgl := ProjectivityGroup(pg);
points := AsList(Points(pg));
planes := AsList(Planes(pg));
r := PolynomialRing(GF(2), 8);
# Tests for 10 494 208

AppendTo(stream, "****_Ruling_out_SmallGroup(512,10494208)_****\n");

form := QuadraticFormByPolynomial( r.1*r.2+r.3*r.4+r.5*r.6, r );
singpts := Filtered(points, t -> IsZero((t^_)^form));;

gens := [ CollineationOfProjectiveSpace([1, 0, 1, 1, 1, 1, 0, 0], [1, 1, 1, 1, 1, 0, 0, 0], [1, 0, 1, 1, 0, 1, 0, 0], [0, 0, 1, 1, 0, 1, 0, 0], [0, 0, 0, 0, 1, 1, 1, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0]], Z(2), IdentityMapping( GF(2) ), GF(2))]; stab := Subgroup(pgl, gens);

# pre–computed stabiliser of the degenerate quadric

orbits_planes := FiningOrbits(stab, planes, OnProjSubspaces);;
singplanes := Concatenation(Filtered(orbits_planes, t > Number(singpts, u > u * t[1])=7));;
act := ActionHomomorphism(stab, singplanes, OnProjSubspaces);
perm := Image(act);
omega := HomeEnumerator(UnderlyingExternalSet(act));;
asconfigs := FindASConfigsViaPseudoArcs(perm, omega, 6, SmallGroup(512,10494208), stream);

AppendTo(stream, Concatenation("For_SmallGroup(512,10494208), we found ", String(Size(asconfigs)), ", _AS–configurations,\n"));

# Tests for 10 494 210

AppendTo(stream, "****_Ruling_out_SmallGroup(512,10494210)_****\n");

form := QuadraticFormByPolynomial( r.1*r.2+r.3*r.4+r.5*r.6+r.7^2, r );
singpts := Filtered(points, t -> IsZero((t^_)^form));;

gens := [ CollineationOfProjectiveSpace([0, 0, 0, 1, 1, 1, 1, 0], [0, 1, 0, 0, 0, 0, 0, 1], [1, 0, 1, 0, 1, 1, 0, 1], [0, 0, 1, 0, 1, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0]], Z(2), IdentityMapping( GF(2) ), GF(2))]; stab := Subgroup(pgl, gens);

orbits_planes := FiningOrbits(stab, planes, OnProjSubspaces);;
singplanes := Concatenation(Filtered(orbits_planes, t > Number(singpts, u > u * t[1])=7));;
act := ActionHomomorphism(stab, singplanes, OnProjSubspaces);
perm := Image(act);
omega := HomeEnumerator(UnderlyingExternalSet(act));;
asconfigs := FindASConfigsViaPseudoArcs(perm, omega, 6, SmallGroup(512,10494210), stream);
AppendTo(stream, Concatenation("For SmallGroup(512, 10494210), we found ", String(Size(asconfigs)), ", AS-configurations.\n"));

########################################################################
# Tests for 10 494 212
########################################################################
AppendTo(stream, "*** Ruling out SmallGroup(512, 10494212) ***");
form := QuadraticFormByPolynomial( r.1*r.2+r.3*r.4+r.5*r.6+r.7*r.8, r );
singpts := Filtered(points, t -> IsZero((t^_)^form));;
stab := IsometryGroup( PolarSpace(form) );
orbits_planes := FiningOrbits(stab, planes, OnProjSubspaces);
singplanes := Concatenation(Filtered(orbits_planes, t -> Number(singpts, u -> u * t[1])=7));;
act := ActionHomomorphism(stab, singplanes, OnProjSubspaces);
perm := Image(act);
omega := HomeEnumerator(UnderlyingExternalSet(act));;
asconfigs := FindASConfigsViaPseudoArcs(perm, omega, 6, SmallGroup(512,10494212), stream);
AppendTo(stream, Concatenation("For SmallGroup(512, 10494212), we found ", String(Size(asconfigs)), ", AS-configurations.\n"));
CloseStream(stream);

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