On Stević-Sharma operator from the Zygmund space to the Bloch-Orlicz space

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Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\varphi$ an analytic self-map of $\mathbb{D}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. In order to unify the products of composition, multiplication, and differentiation operators, Stević and Sharma introduced the following so-called Stević-Sharma operator:

$$T_{\psi_1, \psi_2, \varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$. Here we characterize the boundedness and compactness of the operator $T_{\psi_1, \psi_2, \varphi}$ from the Zygmund space to the Bloch-Orlicz space.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi, \psi}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi, \psi} f(z) = \psi(z) f(\varphi(z)), \quad z \in \mathbb{D}.$$ 

If $\psi \equiv 1$, it becomes the composition operator, usually denoted by $C_{\varphi}$. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by $M_{\psi}$. Hence, since $W_{\varphi, \psi} = M_{\psi} C_{\varphi}$, it is a product-type operator. A standard problem is to provide function theoretic characterizations when $\varphi$ and $\psi$ induce a bounded or compact weighted composition operator (see, e.g., [1–5] and the references therein).

A systematic study of other product-type operators started by Stević et al. since the publication of papers [6] and [7]. Before that there were a few papers in the topic, e.g., [8]. The differentiation operator on $H(\mathbb{D})$ is defined by

$$D f(z) = f'(z), \quad z \in \mathbb{D}.$$ 

The next two product-type operators $D C_{\varphi}$ and $C_{\varphi} D$, attracted some attention first (see, e.g., [9–12] and the references therein). The publication of [7] attracted some attention in product-type operators involving integral-type ones (see, e.g., [13–17] and the references therein). Since that time there has been a great interest in various product-type operators
on spaces of holomorphic functions. For example, the six product-type operators from Bergman spaces to Bloch type spaces

\[ M_\psi C_\psi D, \quad M_\psi D C_\psi, \quad C_\psi M_\psi D, \quad C_\psi D M_\psi, \quad D C_\psi M_\psi, \quad D M_\psi C_\psi \] (1)

were studied by Sharma in [18]. The next product-type operators \( W_\psi \psi' D \) and \( D W_\psi \psi' \), which were considered in [19] and [20], are included in (1) as the first and sixth operators, respectively. For some other product-type operators, see, e.g., [14, 21–29] and the references therein.

In order to treat operators in (1) in a unified manner, Stević and Sharma introduced the following so-called Stević-Sharma operator:

\[ T_{\psi_1, \psi_2} f(z) = \psi_1 f(\psi(z)) + \psi_2(z)f'(\psi(z)), \quad f \in H(D). \] (2)

For example, in [30] and [31] the operator was studied on the weighted Bergman space.

By using Stević-Sharma operator all six possible products of composition, multiplication, and differentiation operators can be obtained. More specifically we have

\[ M_\psi C_\psi D = T_{0, \psi, \psi}, \quad M_\psi D C_\psi = T_{0, \psi', \psi}, \quad C_\psi M_\psi D = T_{0, \psi, \psi'}, \]

\[ C_\psi D M_\psi = T_{\psi', \psi, \psi'}, \quad D M_\psi C_\psi = T_{\psi', \psi, \psi'}, \quad D C_\psi M_\psi = T_{\psi', \psi, \psi'}. \]

Furthermore, by using this operator all possible difference operators of product-type operators in (1) can also be obtained. For example

\[ M_\psi \phi_1 C_\psi D - M_\psi \phi_2 D C_\psi = T_{\phi_1 \psi - \phi_2 \psi', \phi}, \quad M_\psi \phi_1 C_\psi D - C_\psi M_\phi_2 D = T_{\phi_1 \psi, \phi_2 \psi'}, \]

\[ M_\psi \phi_1 C_\psi D - C_\psi D M_\phi_2 = T_{\phi_1 \psi' - \phi_2 \psi', \phi}, \quad M_\psi \phi_1 C_\psi D - D M_\phi_2 C_\psi = T_{\phi_1 \psi' \phi_2 \psi, \phi'}, \]

\[ M_\psi \phi_1 C_\psi D - D C_\psi M_\phi_2 = T_{\phi_1 \psi' \phi_2 \psi, \phi'}, \quad M_\psi \phi_1 D C_\psi - C_\psi M_\phi_2 D = T_{\phi_1 \psi', \phi_2 \psi}, \]

\[ M_\psi \phi_1 D C_\psi - C_\psi D M_\phi_2 = T_{\phi_1 \psi, \phi_2 \psi'}, \quad M_\psi \phi_1 D C_\psi - D M_\phi_2 C_\psi = T_{\phi_1 \psi', \phi_2 \psi'}, \]

\[ C_\psi M_\phi_1 D - C_\psi D M_\phi_2 = T_{\phi_1 \psi' \phi_2 \psi, \phi'}, \quad C_\psi M_\phi_1 D - D M_\phi_2 C_\psi = T_{\phi_1 \psi' \phi_2 \psi, \phi'}, \]

\[ C_\psi M_\phi_1 D - D C_\psi M_\phi_2 = T_{\phi_1 \psi' \phi_2 \psi, \phi'}, \quad C_\psi D M_\phi_1 - C_\psi M_\phi_2 C_\psi = T_{\phi_1 \psi', \phi_2 \psi'}, \]

\[ C_\psi D M_\phi_1 - D M_\phi_2 C_\psi = T_{\phi_1 \psi', \phi_2 \psi'}, \quad D M_\phi_1 C_\psi - D C_\psi M_\phi_2 = T_{\phi_1 \psi', \phi_2 \psi'}, \]

etc., where \( \psi_1, \psi_2 \in H(D) \). In this paper we characterize the boundedness and compactness of the Stević-Sharma operator from the Zygmund space to the Bloch-Orlicz space. As the applications of our main results, readers can obtain some characterizations for the boundedness and compactness for all six product-type operators in (1), as well as above mentioned differences operators from the Zygmund space to the Bloch-Orlicz space.
Now we present the needed spaces and some facts. For $\alpha > 0$, the weighted Zygmund space $Z_\alpha$ consists of all $f \in H(D)$ such that
\[
\sup_{z \in D} (1 - |z|^2)^\alpha |f'''(z)| < \infty.
\]

It is a Banach space with the norm
\[
\|f\|_{Z_\alpha} = |f(0)| + |f'(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f''(z)|.
\]

When $\alpha = 1$, this space is the Zygmund space and is denoted by $Z$ [32]. From Zygmund's theorem (see Theorem 5.3 in [33]), we know that $f \in Z$ if and only if $f$ is continuous on $\overline{D}$ and
\[
\sup_{h > 0, \theta \in \mathbb{R}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.
\]

For some results on Zygmund-type spaces and some concrete operators on them, see, for example, [15, 23, 32] and the references therein.

Recently, the Bloch-Orlicz space was introduced in [4] by Ramos Fernández. More precisely, let $\Psi$ be a strictly increasing convex function such that $\Psi(0) = 0$. From these conditions it follows that $\lim_{t \to +\infty} \Psi(t) = +\infty$. The Bloch-Orlicz space associated with the function $\Psi$, denoted by $B^\Psi$, is the class of all $f \in H(D)$ such that
\[
\sup_{z \in D} (1 - |z|^2) \Psi(\lambda |f'(z)|) < \infty
\]
for some $\lambda > 0$ depending on $f$. The Minkowski functional
\[
\|f\|_\Psi = \inf \left\{ k > 0 : S_\Psi \left( \frac{f'}{k} \right) \leq 1 \right\}
\]
defines a seminorm for $B^\Psi$, where
\[
S_\Psi(f) = \sup_{z \in D} (1 - |z|^2) \Psi(|f(z)|).
\]

Moreover, $B^\Psi$ is a Banach space with the norm
\[
\|f\|_{B^\Psi} = |f(0)| + \|f\|_\Psi.
\]

In fact, Ramos Fernández in [4] proved that $B^\Psi$ is isometrically equal to $\mu_\Psi$-Bloch space, where
\[
\mu_\Psi(z) = \frac{1}{\Psi^{-1} \left( \frac{1}{1 - |z|^2} \right)}, \quad z \in \mathbb{D}.
\]

Thus, for $f \in B^\Psi$ it follows that
\[
\|f\|_{B^\Psi} = |f(0)| + \sup_{z \in D} \mu_\Psi(z) |f'(z)|.$$
This equivalent norm is useful to us for the study of operator $T_{\psi_1, \psi_2, \phi}$ from the Zygmond space to the Bloch-Orlicz space. It is obvious to see that if $\Psi(t) = t^p$ with $p > 0$, then the space $B^\psi$ coincides with the weighted Bloch space $B^\alpha$, where $\alpha = 1/p$. Also, if $\Psi(t) = t \log(1 + t)$, then $B^\psi$ coincides with the Log-Bloch space (see [34]). For the generalization of the Log-Bloch spaces, see, for example, [35, 36].

Let $X$ and $Y$ be Banach spaces. It is said that a linear operator $L : X \to Y$ is bounded if there exists a positive constant $K$ such that

$$\|Lf\|_Y \leq K\|f\|_X$$

for all $f \in X$. The operator $L : X \to Y$ is said to be compact if it maps bounded sets into relatively compact sets. It is well known that the norm of operator $L : Z \to B^\psi$ is defined by

$$\|L\|_{Z \to B^\psi} = \sup_{\|f\|_Z \leq 1} \|Lf\|_{B^\psi}$$

and written by $\|L\|$.

Throughout this paper, a positive constant $C$ may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \simeq b$.

### 2 Main results and proofs

In order to characterize the compactness, we need the following result, which is proved in a standard way [5]. So, the proof is omitted.

**Lemma 1** Let $\phi$ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in H(\mathbb{D})$. Then the bounded operator $T_{\psi_1, \psi_2, \phi} : Z \to B^\psi$ is compact if and only if for every bounded sequence $\{f_j\}_{j \in \mathbb{N}}$ in $Z$ such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that

$$\lim_{j \to \infty} \|T_{\psi_1, \psi_2, \phi}f_j\|_{B^\psi} = 0.$$  

We state the following useful result whose first estimate was essentially proved in [37], while the second essentially follows from the point evaluation estimate for the Bloch functions (see, e.g., [38]). See also [2].

**Lemma 2** For each $f \in Z$ and $z \in \mathbb{D}$, it follows that

$$|f(z)| \leq \|f\|_Z \quad \text{and} \quad |f'(z)| \leq \log \frac{e}{1 - |z|^2} \|f\|_Z.$$

The following lemma was proved in [37], Lemma 2.5.

**Lemma 3** Let $\{f_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $Z$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$. Then

$$\lim_{j \to \infty} \sup_{z \in \mathbb{D}} |f_j(z)| = 0.$$
For \( w \in \mathbb{D} \) and \( 1/2 < |w| < 1 \), we define the function

\[
    f_w(z) = \left( z - \frac{1}{w} \right) \left[ \left( 1 + \log \frac{e}{1 - wz} \right)^2 + 1 \right].
\]

By using this function, the test functions in the Zygmund space can be obtained as follows:

\[
    g_w(z) = f_w(z) \left( \log \frac{e}{1 - |w|^2} \right)^{-1}, \quad h_w(z) = f_w(z) \left( \log \frac{e}{1 - |w|^2} \right)^{-1} - \int_0^z \log \frac{e}{1 - w\lambda} \, d\lambda.
\]

From [9] we have the next result on the functions \( g_w \) and \( h_w \).

**Lemma 4** Let \( w \in \mathbb{D} \) and \( 1/2 < |w| < 1 \). Then

\[
    g'_w(w) = \log \frac{e}{1 - |w|^2}, \quad g''_w(w) = \frac{2w}{1 - |w|^2}, \quad h''_w(w) = \frac{w}{1 - |w|^2}.
\]

Moreover,

\[
    \sup_{1/2 < |w| < 1} \|g_w\| z \lesssim 1, \quad \sup_{1/2 < |w| < 1} \|h_w\| z \lesssim 1.
\]

Now we characterize the boundedness of the operator \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to \mathcal{B}^{\psi} \).

**Theorem 1** Let \( \psi \) be an analytic self-map of \( \mathbb{D} \) and \( \psi_1, \psi_2 \in \mathcal{H}(\mathbb{D}) \). Then the following statements are equivalent.

(i) The operator \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to \mathcal{B}^{\psi} \) is bounded.

(ii) The functions \( \psi_1, \psi_2, \) and \( \psi \) satisfy the following conditions:

\[
    M_1 := \sup_{z \in \mathbb{D}} \mu_{\psi}(z) |\psi_1(z)| < \infty,
\]

\[
    M_2 := \sup_{z \in \mathbb{D}} \mu_{\psi}(z) |\psi_1(z)| \psi_2(z) |\log \frac{e}{1 - |\psi(z)|^2} | < \infty,
\]

and

\[
    M_3 := \sup_{z \in \mathbb{D}} \mu_{\psi}(z) |\psi_2(z)| |\psi'(z)| \frac{1}{1 - |\psi(z)|^2} < \infty.
\]

Moreover, if the operator \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to \mathcal{B}^{\psi} \) is nonzero and bounded, then

\[
    \|T_{\psi_1, \psi_2, \psi}\| \simeq 1 + M_1 + M_2 + M_3.
\]

**Proof** (i) \( \Rightarrow \) (ii). Suppose that \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to \mathcal{B}^{\psi} \) is bounded. For a fixed \( w \in \mathbb{D} \) and \( |\psi(w)| > 1/2 \), let \( f(z) = h_{\psi(w)}(z) - c_1 + c_2 \), where

\[
    c_1 = g_{\psi(w)}(\psi(w)) = f_{\psi(w)}(\psi(w)) \left( \log \frac{e}{1 - |\psi(w)|^2} \right)^{-1}, \quad c_2 = \int_0^{\psi(w)} \log \frac{e}{1 - \psi(w)\lambda} \, d\lambda.
\]
Then by Lemma 4

\[ f'(\varphi(w)) = f'(\varphi(w)) = 0, \quad f''(\varphi(w)) = k''(\varphi(w)) = \frac{\varphi(w)}{1 - |\varphi(w)|^2}. \]

By using the boundedness of \( T_{\psi_1,\psi_2,\varphi} : Z \to B^\varphi \) to the function \( f \), we have

\[ M_3(w) := \frac{\mu_\varphi(w)|\psi_2(w)||\psi'(w)|}{1 - |\psi(w)|^2} = \mu_\varphi(w)|(T_{\psi_1,\psi_2,\varphi}f)'(w)| \leq C\|T_{\psi_1,\psi_2,\varphi}\|, \quad (3) \]

from which we get

\[ \sup_{|\varphi(z)| \geq 1/2} M_3(z) \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (4) \]

From (4) it follows that

\[ \sup_{|\varphi(z)| \geq 1/2} \frac{\mu_\varphi(z)|\psi_2(z)||\psi'(z)|}{1 - |\psi(z)|^2} \leq 2 \sup_{|\varphi(z)| \geq 1/2} M_3(z) \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (5) \]

Let \( h_0(z) \equiv 1 \in Z \). Then by the boundedness of \( T_{\psi_1,\psi_2,\varphi} : Z \to B^\varphi \), we obtain

\[ M_1 = \sup_{z \in D} \mu_\varphi(z)|\psi_1(z)| \leq \|T_{\psi_1,\psi_2,\varphi}h_0\| \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (6) \]

Considering \( h_1(z) = z \in Z \), by the boundedness of \( T_{\psi_1,\psi_2,\varphi} : Z \to B^\varphi \) we have

\[ \sup_{z \in D} \mu_\varphi(z)|\psi_1(z)\varphi(z) + \psi_1(z)\varphi(z) + \psi_2(z)| \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (7) \]

From (6), (7), the boundedness of \( \varphi \), and the triangle inequality, we obtain

\[ L_1 := \sup_{z \in D} \mu_\varphi(z)|\psi_1(z)\varphi(z) + \psi_2(z)| \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (8) \]

Considering \( h_2(z) = z^2 \in Z \), we have

\[ \sup_{z \in D} \mu_\varphi(z)|\psi_1(z)(\varphi(z))^2 + 2(\psi_1(z)\varphi'(z) + \psi_2(z)\varphi(z) + 2\psi_2(z)\varphi'(z)| \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (9) \]

From (6), (8), (9), the boundedness of \( \varphi^2 \), and the triangle inequality, we get

\[ L_2 := \sup_{z \in D} \mu_\varphi(z)|\psi_2(z)||\varphi'(z)| \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (10) \]

Then from (10) we have

\[ \sup_{|\varphi(z)| \geq 1/2} \frac{\mu_\varphi(z)|\psi_2(z)||\varphi'(z)|}{1 - |\varphi(z)|^2} \leq C\|T_{\psi_1,\psi_2,\varphi}\|. \quad (11) \]

From (5) and (11) we finally have \( M_3 < \infty \).
Now we prove that $M_2 < \infty$. For a fixed $w \in \mathbb{D}$ and $|\psi(w)| > 1/2$, let $g(z) = g_\psi(w) - c_1$.

Then

$$g(\psi(w)) = 0, \quad g'(\psi(w)) = \log \frac{e}{1 - |\psi(w)|^2}, \quad g''(\psi(w)) = \frac{2\psi(w)}{1 - |\psi(w)|^2}.$$

By using the boundedness of $T_{\psi_1,\psi_2,\psi} : \mathbb{Z} \to \mathcal{B}^\psi$, we have

$$\mu_\psi(w) \left( |\psi_1(w)\psi'(w) + \psi'_2(w)|\log \frac{e}{1 - |\psi(w)|^2} + 2 \frac{\psi(w)\psi_2(w)\psi'(w)}{1 - |\psi(w)|^2} \right) = \mu_\psi(w) \left( |T_{\psi_1,\psi_2,\psi}\psi'\psi(\psi_2,w)| \right) \leq C \| T_{\psi_1,\psi_2,\psi} \|.$$

(12)

From (4), (12), and the triangle inequality, it follows that

$$\mu_\psi(w) |\psi_1(w)\psi'(w) + \psi'_2(w)| \log \frac{e}{1 - |\psi(w)|^2} \leq 2M_3(w) + C \| T_{\psi_1,\psi_2,\psi} \|

\leq C \| T_{\psi_1,\psi_2,\psi} \|, \quad (13)$$

and then

$$\sup_{|\psi(z)| < 1/2} \mu_\psi(z) |\psi_1(z)\psi'(z) + \psi'_2(z)| \log \frac{e}{1 - |\psi(z)|^2} \leq C \| T_{\psi_1,\psi_2,\psi} \|. \quad (14)$$

From (8), we obtain

$$\sup_{|\psi(z)| < 1/2} \mu_\psi(z) |\psi_1(z)\psi'(z) + \psi'_2(z)| \log \frac{e}{1 - |\psi(z)|^2} \leq L_1 \log \frac{4e}{3} \leq C \| T_{\psi_1,\psi_2,\psi} \|. \quad (15)$$

Hence, from (14) and (15) we have $M_2 < \infty$.

(ii) $\Rightarrow$ (i). By Lemma 2, for all $f \in \mathbb{Z}$ we have

$$\mu_\psi(z) |T_{\psi_1,\psi_2,\psi} f'(z)| = \mu_\psi(z) \left( |\psi_1(z)\psi'(z) + \psi'_2(z)| \right) \leq \mu_\psi(z) \left( |\psi_1(z)\psi'(z)| + |\psi_2(z)\psi'(z)| \right) \leq (M_1 + M_2 + M_3) \| f \|_\mathbb{Z}. \quad (16)$$

It is clear that

$$|T_{\psi_1,\psi_2,\psi} f(0)| \leq C \| f \|_\mathbb{Z}. \quad (17)$$

Hence from (16) and (17) it follows that $T_{\psi_1,\psi_2,\psi} : \mathbb{Z} \to \mathcal{B}^\psi$ is bounded.

Suppose that the operator $T_{\psi_1,\psi_2,\psi} : \mathbb{Z} \to \mathcal{B}^\psi$ is nonzero and bounded. Then from the proof of (i) $\Rightarrow$ (ii) it is not hard to see that

$$M_1 + M_2 + M_3 \leq \| T_{\psi_1,\psi_2,\psi} \|. \quad (18)$$
Since the operator $T_{\psi_1,\psi_2,\varphi} : \mathcal{Z} \to \mathcal{B}^\psi$ is nonzero, we have $\|T_{\psi_1,\psi_2,\varphi}\| > 0$. From this we can find a positive constant $C$ such that $1 \leq C\|T_{\psi_1,\psi_2,\varphi}\|$, which means that

$$1 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|. \quad (19)$$

Then combining (18) and (19) gives

$$1 + M_1 + M_2 + M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|. \quad (20)$$

It is clear from (16) and (17) that

$$\|T_{\psi_1,\psi_2,\varphi}\| \lesssim 1 + M_1 + M_2 + M_3. \quad (21)$$

Hence from (20) and (21) the asymptotic expression of $\|T_{\psi_1,\psi_2,\varphi}\|$ follows. The proof is finished. \(\square\)

Next we characterize the compactness of operator $T_{\psi_1,\psi_2,\varphi} : \mathcal{Z} \to \mathcal{B}^\psi$.

**Theorem 2** Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in H(\mathbb{D})$. Then the following statements are equivalent.

(i) The operator $T_{\psi_1,\psi_2,\varphi} : \mathcal{Z} \to \mathcal{B}^\psi$ is compact.

(ii) The functions $\psi_1, \psi_2$, and $\varphi$ satisfy the following conditions:

$$M_1 := \sup_{z \in \mathbb{D}} \mu_\varphi(z) \left| \psi'_1(z) \right| < \infty,$$

$$L_1 := \sup_{z \in \mathbb{D}} \mu_\varphi(z) \left| \psi_1(z)\psi'(z) + \psi'_2(z) \right| < \infty,$$

$$L_2 := \sup_{z \in \mathbb{D}} \mu_\varphi(z) \left| \psi_2(z) \right| \left| \psi'(z) \right| < \infty,$$

$$\lim_{|\varphi(z)| \to 1^-} \mu_\varphi(z) \left| \psi_1(z)\psi'(z) + \psi'_2(z) \right| \log \frac{e}{1 - |\varphi(z)|^2} = 0,$$

and

$$\lim_{|\varphi(z)| \to 1^-} \frac{\mu_\varphi(z) \left| \psi_2(z) \right| \left| \psi'(z) \right|}{1 - |\varphi(z)|^2} = 0.$$

**Proof** (i) $\Rightarrow$ (ii). Suppose that (i) holds. Then it is clear that the operator $T_{\psi_1,\psi_2,\varphi} : \mathcal{Z} \to \mathcal{B}^\psi$ is bounded. In the proof of Theorem 1, we have shown that $M_1 < \infty$, $L_1 < \infty$ and $L_2 < \infty$. Consider a sequence $\{\varphi(z_i)\}_{i \in \mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1^-$ as $i \to \infty$. If such a sequence does not exist, then the last two conditions (ii) obviously hold. We may suppose, without loss of generality, that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. Using this sequence, we define the function sequence

$$f_i(z) = f_{\varphi(z_i)}(z) \left( \log \frac{e}{1 - |\varphi(z_i)|^2} \right)^{-1} \left( \log \frac{e}{1 - |\varphi(z_i)|^2} \right)^{-2} \int_0^z \log^3 \frac{e}{1 - \varphi(z_i)w} dw.$$
Then from a calculation we see that \( \sup_{i \in \mathbb{N}} \|f_i\| \leq C \) and \( f_i \to 0 \) uniformly on every compact subset of \( \mathbb{D} \) as \( i \to \infty \). So by Lemma 1

\[
\lim_{i \to \infty} \| T_{\psi_1, \psi_2, \psi} f_i \|_{B^\psi} = 0.
\]

Moreover, we have

\[ f'_i(z_i) = 0, \quad f''_i(z_i) = -\frac{\psi(z_i)}{1 - |\psi(z_i)|^2}. \]

Hence we get

\[
\left| \frac{\mu \psi(z_i) \psi_2(z_i) |\psi'(z_i)||\psi(z_i)|}{1 - |\psi(z_i)|^2} - \mu \psi(z_i) \psi_1'(z_i) |f_i(z_i)| \right| \leq \| T_{\psi_1, \psi_2, \psi} f_i \|_{B^\psi}.
\]

From this, Lemmas 1 and 3, and since \( M_1 \) is finite, we obtain

\[
\lim_{i \to \infty} \left| \frac{\mu \psi(z_i) \psi_2(z_i) |\psi'(z_i)||\psi(z_i)|}{1 - |\psi(z_i)|^2} \right| = 0.
\]

On the other hand, take the sequence \( g_i(z) = g_{\psi(z_i)}(z) - c_i, \ i \in \mathbb{N} \), where \( c_i = g_{\psi(z_i)}(\psi(z_i)) \). Then \( \sup_{i \in \mathbb{N}} \|g_i\| \leq C \),

\[ g_i(\psi(z_i)) = 0, \quad g'_i(\psi(z_i)) = \log \frac{e}{1 - |\psi(z_i)|^2}, \quad g''_i(z_i) = \frac{\psi(z_i)}{1 - |\psi(z_i)|^2}. \]

Hence we have

\[
\left| \mu \psi(z_i) \left( \psi_1(\psi(z_i)) \psi'(z_i) + \psi_2(z_i) \right) \log \frac{e}{1 - |\psi(z_i)|^2} + \frac{\psi(z_i)}{1 - |\psi(z_i)|^2} \right| \leq \| T_{\psi_1, \psi_2, \psi} g_i \|_{B^\psi}.
\]

By the compactness \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to B^\psi \), Lemma 1 and (22), we get

\[
\lim_{i \to \infty} \mu \psi(z_i) \left( \psi_1(\psi(z_i)) \psi'(z_i) + \psi_2(z_i) \right) \log \frac{e}{1 - |\psi(z_i)|^2} = 0.
\]

(ii) \( \Rightarrow \) (i). We first prove that \( T_{\psi_1, \psi_2, \psi} : \mathcal{Z} \to B^\psi \) is bounded. We observe that the conditions in (ii) imply that for every \( \varepsilon > 0 \), there is an \( \eta \in (0, 1) \), such that for any \( z \in K = \{ z \in \mathbb{D} : |\psi(z)| > \eta \} \)

\[
R_1(z) := \mu \psi(z) \left| \psi_1(\psi(z)) \psi'(z) + \psi_2'(z) \right| \log \frac{e}{1 - |\psi(z)|^2} < \varepsilon
\]

and

\[
R_2(z) := \mu \psi(z) \left| \psi_1(z) \psi'(z) + \psi_2(z) \right| \log \frac{e}{1 - |\psi(z)|^2} < \varepsilon.
\]

From the fact \( L_1 < \infty \) and (23), we obtain

\[
M_2 = \sup_{z \in \mathbb{D}} \mu \psi(z) \left| \psi_1(z) \psi'(z) + \psi_2(z) \right| \log \frac{e}{1 - |\psi(z)|^2} \leq \varepsilon + L_1 \log \frac{e}{1 - \eta^2}.
\]
From the fact $L_2 < \infty$ and (24), we also obtain
\[
M_3 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|\psi(z)||\psi'(z)|}{1 - |\psi(z)|^2} \leq \varepsilon + \frac{L_2}{1 - \eta^2}.
\]
Hence from Theorem 1 it follows that the operator $T_{\psi_1, \psi_2, \phi_i} : \mathcal{Z} \to \mathcal{B}^\psi$ is bounded.

In order to prove that the operator $T_{\psi_1, \psi_2, \phi_i} : \mathcal{Z} \to \mathcal{B}^\psi$ is compact, by Lemma 1 we just need to prove that, if $(f_i)_{i \in \mathbb{N}}$ is a sequence in $\mathcal{Z}$ such that $\sup_{i \in \mathbb{N}} \|f_i\| \leq M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then
\[
\lim_{i \to \infty} \|T_{\psi_1, \psi_2, \phi_i}f_i\|_{\mathcal{B}^\psi} = 0.
\]

For such a chosen $\varepsilon$ and $\eta$, by using (23), (24), and Lemma 2 we have
\[
\mu(z)|\psi'(z)||T_{\psi_1, \psi_2, \phi_i}f_i'(z)| = \mu(z)|\psi'(z)||f_i'(\psi(z)) + (\psi_1(z)\psi'(z) + \psi_2'(z))f_i'(\psi(z)) + \psi'(z)\psi_2(z)f_i''(\psi(z))|
\leq \mu(z)(|\psi_1'(z)||f_i'(\psi(z))| + |\psi_1(z)\psi'(z) + \psi_2'(z)||f_i'(\psi(z))|
\leq M_1 \sup_{z \in \mathbb{D}} |f_i(z)| + \left(\sup_{z \in \mathcal{K}} \sup_{z \in \mathbb{D} \setminus \mathcal{K}} \mu(z)|\psi'(z)||\psi_2(z)||f_i''(\psi(z))\right)
\leq 2\varepsilon + M_1 \sup_{z \in \mathbb{D}} |f_i(z)| + L_1 \sup_{|z| \leq \eta} |f_i'(z)| + L_2 \sup_{|z| \leq \eta} |f_i''(z)|.
\]
(25)

Since $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, from (25) and Lemma 3 we get
\[
\lim_{i \to \infty} \sup_{z \in \mathbb{D}} \mu(z)|\psi'(z)||T_{\psi_1, \psi_2, \phi_i}f_i'(z)| = 0.
\]

It is clear that
\[
\lim_{i \to \infty} |T_{\psi_1, \psi_2, \phi_i}f_i(0)| = 0.
\]
(26)

From (25) and (26) we obtain
\[
\lim_{i \to \infty} \|T_{\psi_1, \psi_2, \phi_i}f_i\|_{\mathcal{B}^\psi} = 0.
\]
(27)

Hence from (27) and Lemma 1, we see that $T_{\psi_1, \psi_2, \phi_i} : \mathcal{Z} \to \mathcal{B}^\psi$ is compact. The proof is finished.

Competing interests
The author declares that they have no competing interests.

Author’s contributions
The author performed all tasks of this research: drafting, thinking of the study, writing and revision of paper.
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