Global existence for Schrödinger-Debye system for initial data with infinite mass

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Abstract

We obtain global existence results for the Cauchy problem associated to the Schrödinger-Debye system for a class of data with infinite mass ($L^2$-norm). A smallness condition on data is assumed. Our results include data such as singular-homogeneous functions and some types of data blowing up at finitely many points. We also study the asymptotic stability of the solutions. Our analysis is performed in the framework of weak-$L^p$ spaces.

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1 Introduction

We consider the Cauchy problem for the Schrödinger-Debye (S-D) system, which consists of the following coupled equations:

\[
\begin{cases}
    i\partial_t u + \frac{1}{2}\Delta_x u = uv, & t \geq 0, \ x \in \mathbb{R}^n, \\
    \mu\partial_t v + v = \lambda|u|^p, & \mu > 0, \ \lambda = \pm 1, \\
    u(x, 0) = u_0(x), \ v(x, 0) = v_0(x),
\end{cases}
\]

(1.1)

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where $u$ is a complex-valued function and $v$ is a real-valued function. For the case $p = 2$, these equations appear in the context of nonlinear optics modeling interactions of an electromagnetic wave with a non resonant medium, where the material response time $\mu$ is relevant. More precisely, $u$ denotes the envelope of a light wave that goes through a media whose response is non resonant and $v$ is a change induced in its refraction index with a slight delay $\mu$. Similar to the physical theory of the nonlinear Schrödinger equation, for the system (1.1), the parameter $\lambda = -1$ and $\lambda = 1$ model focusing and defocusing situations, respectively. See Newell and Moloney [12] for a more complete discussion of this model. For the sake of simplicity, throughout this paper, $v$ and $v_0$ will be abusively named as delay and initial delay.

The mass of $u$ is invariant for the system (1.1), namely

$$\int_{\mathbb{R}^n} |u(x,t)|^2 dx = \int_{\mathbb{R}^n} |u_0(x)|^2 dx. \quad (1.2)$$

It is not known the existence of other conservation law for this system, but, in the physical case ($p = 2$) we have the following pseudo-Hamiltonian:

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left\{ |\nabla u|^2 + \lambda |u|^4 - \lambda \mu^2 |\partial_t v|^2 \right\} dx = 2\lambda \mu \int_{-\infty}^{\infty} |\partial_t v|^2 dx. \quad (1.3)$$

Local and global-in-time well-posedness for the Cauchy problem (1.1) in the continuous ($x \in \mathbb{R}^n$) and periodic ($x \in \mathbb{T}$) context have been developed in the works [1, 4, 5, 7, 8] with initial data $(u_0, v_0)$ in classical Sobolev spaces $H^s_1 \times H^{s_2}$. All results from these references only consider the dimensions $n = 1, 2$ or $3$. Specifically, in [8], the authors studied the case $n = 1$ and $p = 2$ and they have obtained global-well posedness in the Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, with $-\frac{3}{14} < s < 0$, (1.4)

which contain data with infinite mass. This result shows that, under point of view of global-in-time existence, the system (1.1) does not have the same behavior than the associated nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2} \Delta_x u = \lambda |u|^p, \quad (1.5)$$

because, for $n = 1$ and $p = 2$, the flux of (1.5) is not uniformly continuous on bounded sets of Sobolev spaces with negative indices. More recently, in [9], for $n = 2$ and $p = 2$, the authors obtained global solutions in the energy space $H^1 \times L^2$ for any data and this is another difference with the associated cubic nonlinear Schrödinger equation, where global results are given only for small data.

In this paper we focus on data without finite mass (infinite $L^2$-norm) for the system (1.1). Our motivation are the results of [3, 6] and [10] in spaces with infinite $L^2$-norm for nonlinear Schrödinger and generalized Boussinesq equations, respectively. In these works, by employing scaling techniques, the authors have obtained global existence results in time-dependent spaces based on Lebesgue spaces $L^r$ (with $r \neq 2$) and weak-$L^r$ spaces.
In the context of the system (1.1), we are interested in understanding how the delay \( v \) influences the behavior of (1.1). The system (1.1) is closely related to the Schrödinger equation, which is obtained by formally substituting \( v = |u|^p \) in the first equation of (1.1). This relationship leads us to investigate existence of small global-in-time solutions for (1.1) in the classes

\[
\sup_{t>0} t^\alpha \| u \|_{L^{(p+2, \infty)}} < \infty \quad \text{and} \quad \sup_{t>0} t^\beta \| v \|_{L^{(\frac{p+2}{p}, \infty)}} < \infty,
\]

(1.6)

where the exponents \( \alpha \) and \( \beta \) are chosen so that the norms in (1.6) are invariant by the “intrinsical scaling” of (1.1), see Theorem 2.1. A result about \( L^p \)-regularity of solutions is also proved. The initial condition \((u_0, v_0)\) is taken in such a way that

\[
\sup_{t>0} t^\alpha \| S(t)u_0 \|_{L^{(p+2, \infty)}} < \infty \quad \text{and} \quad v_0 \in L^{(\frac{p+2}{p}, \infty)}.
\]

(1.7)

This class allows us to consider singular-homogeneous initial conditions, which do not belong to any \( L^r(\mathbb{R}^n) \), and contain data blowing up at finitely many points. For \( p \) in a given interval, these initial data do not belong to \( L_{loc}^2(\mathbb{R}^n) \times L_{loc}^2(\mathbb{R}^n) \), that is, they do not have finite local mass, see item (a) in Remark 2.1. Moreover, there is no any inclusion relation between the classes (1.4) and (1.7), even when \( p = 2 \).

Here we work in the range \( p_0 < p < \frac{4}{n-2} \) (\( p_0 < p < \infty \) if \( n = 1, 2 \)) and consider high dimensions \( 1 \leq n < 6 \), where \( p_0 \) is the positive root of \( np^2 + (n-2)p - 4 = 0 \). In comparison with (1.5), a feature is that the presence of delay \( v \) imposes a further restriction on dimension \( n < 6 \), see item (b) in Remark 2.1. On the other hand, comparing with the mathematical literature of (1.1), our results provide global existence of solutions for some high dimensions of the domain \( \mathbb{R}^n \) which do not have been covered previously.

We also study the asymptotic stability of these global solutions and prove that some types of initial perturbations vanish as \( t \to \infty \) (see Theorem 2.2). This result shows a distinct influence of the initial envelope of light \( u_0 \) and initial delay \( v_0 \) on the long-time behavior of the solutions, see item (d) in Remark 2.2.

The paper is organized as follows. In Section 2 we review some basic properties about weak-\( L^p \) spaces and state our results, which are proved in Section 3.

2 Functional setting and results

In this section we prove the existence of global solutions for system (1.1) including homogeneous data. We start by recalling that \( f \in L^{(r, \infty)}(\text{weak-}L^r) \) if only if \( \| f \|_{(r, \infty)} = \sup_{t>0} t^{\lambda_f(t)}(t) < \infty \), where

\[
\lambda_f(t) = |\{ x \in \mathbb{R}^n : |f(x)| > t \}|
\]

is the distribution function of \( f \) and \( |A| \) denotes the Lebesgue measure of a set \( A \). The quantity \( \| \cdot \|_{(r, \infty)} \) is not a norm, but, for \( 1 < r \leq \infty \), \( L^{(r, \infty)} \) endowed with the norm
\[ \| \cdot \|_{(r, \infty)} = \sup_{t>0} t^\frac{1}{r} f^{**}(t) \] is a Banach space, where \( f^{**} \) is defined by

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad \text{and} \quad f^*(t) = \inf \{ s > 0 : \lambda_f(s) \leq t \} . \]

Also, \( \| \cdot \|_{(r, \infty)} \leq \| \cdot \|_{(r, \infty)} \leq \frac{1}{r-1} \| \cdot \|_{(r, \infty)} \) which imply that \( \| \cdot \|_{(r, \infty)} \) and \( \| \cdot \|_{(r, \infty)} \) induce the same topology on \( L^{(r, \infty)} \). Hölder inequality works well in the framework of \( L^{(r, \infty)} \)-spaces, namely

\[ \| f g \|_{(r, \infty)} \leq \frac{r}{r-1} \| f \|_{(r_1, \infty)} \| g \|_{(r_2, \infty)} , \quad (2.1) \]

for \( 1 < r_1, r_2 < \infty \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \). For a deeper discussion about weak-\( L^p \) we refer the reader to [2].

Let \( \alpha = \frac{1}{p} - \frac{n}{2(p+2)} \) and \( \beta = p\alpha \). We define \( \mathcal{M}_p^0 \) as the space of all pairs \((u_0, v_0) \in (S'(\mathbb{R}^n), L^{(\frac{p+2}{p}, \infty)})\) such that the norm

\[ \|(u_0, v_0)\|_p^0 = \max \left\{ \left( \sup_{t>0} t^{\alpha} \|S(t)u_0\|_{(p+2, \infty)} \right), \|v_0\|_{(\frac{p+2}{p}, \infty)} \right\} < \infty. \]

Also, \( \mathcal{M}_p \) denotes the space of Bochner measurable functions \((u, v) : \mathbb{R} \rightarrow L^{(p+2, \infty)} \times L^{(\frac{p+2}{p}, \infty)}\) such that the norm \( \|(u, v)\|_{\mathcal{M}_p} = \max \{ \|u\|_{(\infty, \alpha)}, \|v\|_{(\infty, \beta)} \} \) is finite, where

\[ \|u\|_{(\infty, \alpha)} = \sup_{t>0} t^{\alpha} \|u(\cdot, t)\|_{(p+2, \infty)} \quad \text{and} \quad \|v\|_{(\infty, \beta)} = \sup_{t>0} t^\beta \|v(\cdot, t)\|_{(\frac{p+2}{p}, \infty)}. \]

Recall that \( p_0 \) denotes the positive root of the equation \( np^2 + (n-2)p - 4 = 0 \). In what follows, we state our global-in-time existence theorem with weak-data.

**Theorem 2.1.** Let \( \max\{p_0, 1\} < p < \frac{4}{n-2} \) and \((u_0, v_0) \in \mathcal{M}_p^0\).

(i) (Existence) There exists \( \varepsilon > 0 \) such that if

\[ \sup_{t>0} t^{\alpha} \|S(t)u_0\|_{(p+2, \infty)} \quad \text{and} \quad \|v_0\|_{(\frac{p+2}{p}, \infty)} \leq \varepsilon \]

(2.2)

then the system (1.7) has a global solution \((u, v) \in \mathcal{M}_p\).

(ii) (\( L^p \)-regularity) Moreover, if, in addition, \( \sup_{t>0} t^{\alpha} \|S(t)u_0\|_{p+2} \) and \( \|v_0\|_{p+2} \) are small enough, then the previous solution \((u, v)\) satisfies

\[ \sup_{t>0} t^{\alpha} \|u(\cdot, t)\|_{p+2} < \infty \quad \text{and} \quad \sup_{t>0} t^\beta \|v(\cdot, t)\|_{\frac{p+2}{p}} < \infty. \]

(2.3)

**Remark 2.1.** Now we stand out two points about the Theorem 2.1.
(a) Homogeneous and singular data. Theorem 2.1 allows us consider homogeneous and singular initial data. Let $P_m(x)$ and $Q_m(x)$ be homogeneous polynomials of degree $m$, with $Q_m$, harmonic. A simple computation shows that $v_0 = \varepsilon_1 P_m(x) |x|^{-\frac{pm}{p+2}-m} \in L^\infty$ and, by [8] Prop. 3.7-3.9, $u_0 = \varepsilon_2 Q_m(x) |x|^{-\frac{pm}{p+2}-m}$ satisfies
\[ \sup_{t>0} t^\alpha \| S(t)u_0 \|_{(p+2,\infty)} = \| S(1)u_0 \|_{(p+2,\infty)} \leq \varepsilon_2 \| S(1) \left( Q_m(x) |x|^{-\frac{2m}{p+2}} \right) \|_{p+2} < \infty. \]
Notice that, in despite of $u_0, v_0 \to 0$ as $|x| \to \infty$, $u_0, v_0 \notin L^r(\mathbb{R}^n)$ for all $1 \leq r \leq \infty$.
We also can take $(u_0, v_0)$ blowing up at finitely many points. For instance
\[ u_0 = \sum_{j=1}^{N_1} \gamma_j P_{m,j}(x-x_j) |x-x_j|^{-\frac{p}{p+2}-m} \quad \text{and} \quad v_0 = \sum_{j=1}^{N_2} \eta_j Q_{m,j}(x-x_j) |x-x_j|^{-\frac{pm}{p+2}-m}, \]
where $x_j, \tilde{x}_j \in \mathbb{R}^n$, $P_{m,j}$ is as $P_m$, $Q_{m,j}$ is as $Q_m$, and $\gamma_j, \eta_j$ are small enough. Observe that if either $p \leq \frac{4}{n-2}$ or $p \geq 2$ then $(u_0, v_0) \notin L^p_{\text{loc}}(\mathbb{R}^n) \times L^p_{\text{loc}}(\mathbb{R}^n)$, that is, the data does not have finite local mass.

(b) Delay influence on dimension $n$. Making a comparison with the Schrödinger equation (1.3), the presence of the delay $v$ imposes a further restriction on $n$, namely $1 < \frac{n-2}{n-2} \iff n < 6$, which is a necessary condition for the term $|u|^p$ be locally Lipschitz continuous, fact that was used to prove the contraction property for the map $\Phi(u,v)$ defined by (3.1)-(3.4) (see (3.10)). We observe that for (1.3) this restriction is not necessary because the nonlinearity $u |u|^p$ is locally Lipschitz continuous when $p > 0$.

In view of the definition of the space $M_p$, the solution $(u, v)$ provided by item $(i)$ of Theorem 2.1 satisfies $\| u(\cdot, t) \|_{(p+2,\infty)} = O(t^{-\alpha})$ and $\| v(\cdot, t) \|_{(p+2,\infty)} = O(t^{-\beta})$ as $t \to \infty$.
Among others, the next result proves that certain perturbations of $(u, v)$ are negligible, for large value of $t$, with respect to quantity $t^{\alpha+h} \| \cdot \|_{(p+2,\infty)}$ and $t^{\beta+h} \| \cdot \|_{(p+2,\infty)}$, for $0 \leq h < 1 - p\alpha$.

**Theorem 2.2 (Asymptotic stability).** Under the hypotheses of Theorem 2.1. Let $0 \leq h < 1 - p\alpha$ and $(u, v)$ and $(\tilde{u}, \tilde{v})$ be two global solutions of system (1.1) obtained through Theorem 2.1, with the respective initial conditions $(u_0, v_0)$ and $(\tilde{u}_0, \tilde{v}_0)$, small enough in $M_p^0$. Then
\[ \lim_{t \to \infty} t^{\alpha+h} \| S(t)(u_0 - \tilde{u}_0) \|_{(p+2,\infty)} = 0 \]
if and only if
\[ \lim_{t \to \infty} t^{\alpha+h} \| u(\cdot, t) - \tilde{u}(\cdot, t) \|_{(p+2,\infty)} = \lim_{t \to \infty} t^{\beta+h} \| v(\cdot, t) - \tilde{v}(\cdot, t) \|_{(p+2,\infty)} = 0. \]
Moreover, the same conclusion holds if we change the norm $\| \cdot \|_{(r,\infty)}$ by its strong version $\| \cdot \|_r$. 

Remark 2.2. Now we give some observations about the Theorem 2.2.

(a) The step 1 of the proof of Theorem 2.2 actually gives us more, namely
\[
\sup_{t>0} t^{\alpha+h} \|u(\cdot, t)\|_{(p+2, \infty)} < \infty \text{ and } \sup_{t>0} t^{\beta+h} \|v(\cdot, t)\|_{(\frac{p+2}{p}, \infty)} < \infty, \tag{2.6}
\]
provided that \(\| (u_0, v_0) \|^p_p \) is small enough and \( \limsup_{t \to \infty} t^{\alpha+h} \|S(t)u_0\|_{(p+2, \infty)} < \infty \).

Since \( h \geq 0 \), these two last constraints imply
\[
\sup_{t>0} t^{\alpha+h} \|S(t)u_0\|_{(p+2, \infty)} < \infty. \tag{2.7}
\]

In fact the proof of (2.6) requires only (2.7) and a smallness assumption on \( \| (u_0, v_0) \|^p_p \). Any smallness condition on \( \sup_{t>0} t^{\alpha+h} \|S(t)u_0\|_{(p+2, \infty)} \) is not needed.

(b) By taking \( \bar{u}_0 = 0 \) and \( \bar{v} = 0 \), a consequence of Theorem 2.2 is that \( \| u(\cdot, t) \|_{(p+2, \infty)} = o(t^{-(\alpha+h)}) \) and \( \| v(\cdot, t) \|_{(\frac{p+2}{p}, \infty)} = o(t^{-(\alpha+h)}) \) provided \( \|S(t)u_0\|_{(p+2, \infty)} = o(t^{-(\alpha+h)}) \), as \( t \to \infty \).

Moreover, note that \( \frac{1}{p} - \frac{n(p+1)}{2(p+2)} < 0 \) when \( p > p_0 \). Thus, if \( u_0 - \bar{u}_0 \in L^{\frac{p+2}{p+1}, \infty} \) and \( 0 \leq h < \frac{n(p+1)}{2(p+2)} - \frac{1}{p} \), then (2.4) holds; in fact
\[
0 \leq \lim_{t \to \infty} t^{\alpha+h} \|S(t)(u_0 - \bar{u}_0)\|_{(p+2, \infty)} \\
\leq C_S \left( \lim_{t \to \infty} t^{\frac{1}{p} - \frac{n(p+1)}{2(p+2)} + h} \right) \|u_0 - \bar{u}_0\|_{(\frac{p+2}{p+1}, \infty)} = 0.
\]

(c) Basin of attraction. Let \((u, v)\) be the solution obtained through Theorem 2.2 with initial data \((u_0, v_0) \in \mathcal{M}^0_p\). Take \( \psi_0 = u_0 + \phi \) and \( \omega_0 = v_0 + \phi \in L^{\frac{p+2}{p+1}, \infty} \) small enough with \( \phi \in \mathcal{S}(\mathbb{R}^n) \), and consider the solution \((\psi, \omega)\) corresponding to \((\psi_0, \omega_0) \in \mathcal{M}^0_p\). Since \( \psi_0 - u_0 = \omega_0 = \phi \) satisfies (2.4) then
\[
\lim_{t \to \infty} t^\alpha \|\psi(\cdot, t) - u(\cdot, t)\|_{(p+2, \infty)} = \lim_{t \to \infty} t^\beta \|\omega(\cdot, t) - v(\cdot, t)\|_{(\frac{p+2}{p}, \infty)} = 0. \tag{2.8}
\]

Therefore, considering smooth perturbations, one obtains a basin of attraction, in sense of (2.8) around each initial data \((u_0, v_0) \in \mathcal{M}^0_p\). In fact, this basin is characterized by all perturbations \( \phi \) that satisfy (2.7).

(d) Distinct influence of \( u_0 \) and \( v_0 \). For \( \phi = \frac{\varepsilon}{|x|^{2/p}} (\notin \mathcal{S}(\mathbb{R}^n)) \)
\[
\lim_{t \to \infty} t^\alpha \left\| S(t) \frac{\varepsilon}{|x|^{2/p}} \right\|_{(p+2, \infty)} = \left\| S(1) \frac{\varepsilon}{|x|^{2/p}} \right\|_{(p+2, \infty)} \neq 0
\]
and then, by Theorem 2.2 (with \( \phi = 0 \) or not),
\[
\lim_{t \to \infty} t^\alpha \|\psi(\cdot, t) - u(\cdot, t)\|_{(p+2, \infty)} \neq 0.
\]
In other words, the disturbance \( \psi(\cdot, t) - u(\cdot, t) \) persists at long times. Next notice that in Theorem 2.2 we have not assumed any restriction of type (2.4) on \( \phi = \omega_0 - v_0 \). So, in case \( \phi = 0 \), the effect of any initial-delay small perturbation \( \phi \in L_{p+\frac{1}{p}}(\infty) \) vanishes as \( t \to \infty \). This reveals us a distinct influence of initial conditions \( u_0 \) and \( v_0 \) on the long-time behavior of solutions.

3 Proofs of the main results

3.1 Proof of Theorem 2.1

The proof of our results is based on the following basic properties of the Schrödinger group (see [6, 10]):

\[
\|S(t)\phi\|_{(p+2, \infty)} \leq C_S |t|^{-\frac{np}{p+2}} \|\phi\|_{(p+2, \infty)},
\]

(3.1)

\[
\|S(t)\phi\|_{p+2} \leq \tilde{C}_S |t|^{-\frac{np}{p+2}} \|\phi\|_{p+2},
\]

(3.2)

where \( \| \cdot \|_r \) denotes the norm of the Lebesgue space \( L^r \) and \( C_S, \tilde{C}_S \) are positive constants that depend only on \( p \). Moreover, the following elementary result will be useful in the proofs.

**Lemma 3.1** *(Beta function).* Let \( a \) and \( b \) positive real numbers. Then, the integral

\[
B(1 - a, 1 - b) = \int_0^1 \frac{dx}{(1 - x)^a x^b}
\]

is finite if, and only if, \( \max\{a, b\} < 1 \).

We begin by proving the existence of solutions.

**Existence.** Consider the ball

\[
B_p(a) = \{(u, v) \in \mathcal{M}_p; \|u\|_{\infty, \alpha} \leq a \text{ and } \|v\|_{\infty, \beta} \leq a \}
\]

and the metric \( d[\cdot, \cdot] \) in \( \mathcal{M}_p 
\]

\[
d[(u, v), (\bar{u}, \bar{v})] = \max (\|u - \bar{u}\|_{\infty, \alpha}, \|v - \bar{v}\|_{\infty, \beta}).
\]

In order to apply a fixed point argument, we will show that the mapping \( \Phi(u, v) := (\Phi_1(u, v), \Phi_2(u, v)), \) defined by

\[
\Phi_1(u, v) := S(t)u_0 - i \int_0^t S(t-s)u(\cdot, s)v(\cdot, s)ds
\]

(3.3)

\[
\Phi_2(u, v) := e^{-\frac{1}{\mu}t}v_0 + \frac{\lambda}{p} \int_0^t e^{-\frac{1}{\mu}s} |u(\cdot, s)|^p ds,
\]

(3.4)

is a contraction on the metric space \( (B_p(a), d) \), for a suitable value of \( a \).
Computing \( \| \cdot \|_{\infty, \alpha} \) and \( \| \cdot \|_{\infty, \beta} \) in (3.3) and (3.4) we obtain

\[
\| \Phi_1(u, v) \|_{\infty, \alpha} \leq \sup_{t > 0} t^\alpha \| S(t) u_0 \|_{(p+2, \infty)} + \sup_{t > 0} t^\alpha \int_0^t \| S(t - s) uv(\cdot, s) \|_{(p+2, \infty)} \, ds \quad (3.5)
\]

\[
\| \Phi_2(u, v) \|_{\infty, \beta} \leq \sup_{t > 0} t^\beta \| e^{- \frac{t}{\nu}} v_0 \|_{(p+2, \infty)} + \frac{1}{\mu} \sup_{t > 0} t^\beta \int_0^t \| e^{- \frac{t-s}{\nu}} |u(\cdot, s)|^p \|_{(p+2, \infty)} \, ds \quad (3.6)
\]

In the following we estimate the nonlinearities:

\[
N_1(u, v) = \int_0^t \| S(t - s) uv(\cdot, s) \|_{(p+2, \infty)} \, ds
\]

\[
N_2(u, v) = \frac{1}{\mu} \int_0^t \| e^{- \frac{t-s}{\nu}} |u(\cdot, s)|^p \|_{(p+2, \infty)} \, ds.
\]

By using (3.1) and afterwards Hölder’s inequality (2.1) one has

\[
N_1(u, v) \leq C \int_0^t (t - s)^{- \frac{np}{2(p+2)}} \| uv \|_{(p+2, \infty)} \, ds
\]

\[
\leq C \int_0^t (t - s)^{- \frac{np}{2(p+2)}} \| u \|_{(p+2, \infty)} \| v \|_{(p+2, \infty)} \, ds
\]

\[
\leq \| u \|_{\infty, \alpha} \| v \|_{\infty, \beta} C \int_0^t (t - s)^{- \frac{np}{2(p+2)}} s^{-(\alpha + \beta)} \, ds
\]

\[
\leq C t^{-(\alpha + \beta) + 1} \int_0^1 (1 - s)^{- \frac{np}{2(p+2)}} s^{-(\alpha + \beta)} \, ds = K_1 t^{-\alpha} \| u \|_{\infty, \alpha} \| v \|_{\infty, \beta}, \quad (3.7)
\]

and also

\[
N_2(u, v) \leq \frac{1}{\mu} \int_0^t \| e^{- \frac{t-s}{\nu}} |u(\cdot, s)|^p \|_{(p+2, \infty)} \, ds \leq \frac{1}{\mu} \int_0^t \| e^{- \frac{t-s}{\nu}} |u(\cdot, s)|^p \|_{(p+2, \infty)} \, ds
\]

\[
\leq \| u \|_{\infty, \alpha} \frac{1}{\mu} \left( \int_0^{t/2} e^{- \frac{t-s}{\nu}} s^{-\rho \alpha} \, ds + \int_{t/2}^t e^{- \frac{t-s}{\nu}} s^{-\rho \alpha} \, ds \right)
\]

\[
\leq \| u \|_{\infty, \alpha} \frac{1}{\mu} \left( e^{- \frac{t}{2\nu}} \int_0^{t/2} s^{-\beta} \, ds + 2 \beta t^{-\beta} \int_{t/2}^t e^{- \frac{t-s}{\nu}} s^{-\rho \alpha} \, ds \right)
\]

\[
= \| u \|_{\infty, \alpha} \left( \frac{1}{1 - \beta \rho} e^{- \frac{t}{2\nu}} t^{-\beta} + 2 \beta t^{-\beta} (1 - e^{- \frac{t}{2\nu}}) \right)
\]

\[
\leq K_2 t^{-\beta} \| u \|_{\infty, \alpha}^p. \quad (3.8)
\]

Next assume that \((u, v) \in B_p(2\varepsilon)\) with \(0 < 4K_1 \varepsilon < 1\) and \(0 < 2^p K_2 p \varepsilon^{p-1} < 1\). Then

\[
d[\Phi_1(u, v), \Phi_2(u, v), (0, 0)] \leq \max (\| \Phi_1(u, v) \|_{\infty, \alpha}, \| \Phi_1(u, v) \|_{\infty, \beta})
\]

\[
\leq \max (\varepsilon + 4K_1 \varepsilon, \varepsilon + 2^p K_2 \varepsilon^{p-1} \varepsilon)
\]

\[
\leq \max (\varepsilon + \varepsilon, \varepsilon + \varepsilon) \leq 2\varepsilon,
\]

(3.9)
and so $\Phi(B_p(2\varepsilon)) \subset B_p(2\varepsilon)$. Also, by recalling the inequality

$$||u|^p - |\bar{u}|^p| \leq p |u - \bar{u}|(|u|^{p-1} + |\bar{u}|^{p-1}) \quad \text{for} \quad p > 1,$$

(3.10)
a similar argument yields

$$d[\Phi(u, v) - \Phi(\bar{u}, \bar{v})] \leq \max(4K_1\varepsilon, 2^pK_2p\varepsilon^{p-1})d[(u, v), (\bar{u}, \bar{v})],$$

and therefore $\Phi$ is a contraction in $B_p(2\varepsilon)$. Now, an application of Banach fixed point theorem concludes the proof of the existence statement.

**Regularity.** From point fixed argument, we know that the previous solution is the limit in $B_p(2\varepsilon)$ of the following Picard sequence $\{(u_m, v_m)\}_{m \geq 1}$:

$$u_1(\cdot, t) = S(t)u_0 \quad \text{and} \quad u_{m+1}(\cdot, t) = S(t)u_0 - i \int_0^t S(t - s)u_m(\cdot, s)v_m(\cdot, s)ds \quad (3.11)$$

$$v_1(\cdot, t) = e^{-\frac{t}{p}}v_0 \quad \text{and} \quad v_{m+1}(\cdot, t) = e^{-\frac{t}{p}}v_0 + \frac{1}{p} \int_0^t e^{-\frac{s}{p}}|u_m(\cdot, s)|^pds. \quad (3.12)$$

In order to prove the regularity (2.3), it is sufficient to show

$$\sup_{t > 0} t^\alpha \|u_m(\cdot, t)\|_{p+2} \leq C \quad \text{and} \quad \sup_{t > 0} t^\beta \|v_m(\cdot, t)\|_{p+\frac{2}{p}} \leq C, \quad \text{for all} \quad m \geq 1. \quad (3.13)$$

To this end, we proceed as in the proof of the inequalities (3.5)–(3.9); but in this time, we employ (3.2) and Hölder inequality in $L^r$-spaces, instead of (3.1) and (2.1), in order to bound (3.11)–(3.12) as

$$\sup_{t > 0} t^\alpha \|u_{m+1}(\cdot, t)\|_{p+2} \leq \sup_{t > 0} t^\alpha \|S(t)u_0\|_{p+2} + \tilde{K}_1 \sup_{t > 0} t^\alpha \|u_m(\cdot, t)\|_{p+2} \sup_{t > 0} t^\beta \|v_m(\cdot, t)\|_{p+\frac{2}{p}} \quad (3.14)$$

$$\sup_{t > 0} t^\beta \|v_{m+1}(\cdot, t)\|_{p+\frac{2}{p}} \leq \|v_0\|_{p+\frac{2}{p}} + \tilde{K}_2 \left(\sup_{t > 0} t^\beta \|u_m(\cdot, t)\|_{p+\frac{2}{p}}\right)^p \quad (3.15)$$

Let us denote $U_m = \sup_{t > 0} t^\alpha \|u_m(\cdot, t)\|_{p+2}$, and $V_m = \sup_{t > 0} t^\beta \|v_m(\cdot, t)\|_{p+\frac{2}{p}}$, for $m \geq 2$, with $U_1 = \sup_{t > 0} t^\alpha \|S(t)u_0\|_{p+2}$ and $V_1 = \|v_0\|_{p+\frac{2}{p}}$. Notice that (3.14)–(3.15) can be rewrite as the following system of recurrence inequalities:

$$U_{m+1} \leq U_1 + \tilde{K}_1U_mV_m \quad \text{and} \quad V_{m+1} \leq V_1 + \tilde{K}_2(U_m)^p, \quad m \geq 1. \quad (3.16)$$

It is known that systems of inequalities as (3.16) can be solved provided that $U_1, V_1$ are small enough (see, for instance, [11]). More precisely, if $U_1, V_1 \leq \tilde{\varepsilon}$, with $\tilde{\varepsilon}$ chosen in a such way that $4\tilde{K}_1\tilde{\varepsilon} < 1$ and $2^p\tilde{K}_2\tilde{\varepsilon}^{p-1} < 1$, then one can find $C > 0$ such that $U_m, V_m \leq C$, for all $m \geq 1$. ■
3.2 Proof of Theorem 2.2

The proof is divided in three steps.

First step. We start by showing that

\[
\sup_{t>0} t^\alpha u_{m+1}(t) \leq \sup_{t>0} t^\alpha u(0) < \infty \quad \text{and} \quad \sup_{t>0} t^\alpha v_{m+1}(t) \leq \sup_{t>0} t^\alpha v(0) < \infty.
\]

Let \( \{ (u_m, v_m) \}_{m \geq 1} \) and \( \{ (\tilde{u}_m, \tilde{v}_m) \}_{m \geq 1} \) be the Picard sequences converging to \((u, v)\) and \((\tilde{u}, \tilde{v})\), respectively (see (3.11)-(3.12)). We estimate

\[
t^{\alpha + h} \| u_{m+1}(t) - \tilde{u}_{m+1}(t) \|_{(p+2,\infty)} \leq \sup_{t>0} t^{\alpha + h} \| S(t)(u_0 - \tilde{u}_0) \|_{(p+2,\infty)} + \left\{ C_s \int_0^1 (1-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} ds \right\} \times \left\{ \| v_m \|_{\infty,\beta} \sup_{t>0} t^{\alpha + h} \| u_m - \tilde{u}_m \|_{(p+2,\infty)} + \| v_m \|_{(p+2,\infty)} \sup_{t>0} t^{\beta + h} \| v_m - \tilde{v}_m \|_{(p+2,\infty)} \right\}
\]

and

\[
t^{\beta + h} \| v_m(t) - \tilde{v}_m(t) \|_{(p+2,\infty)} \leq \sup_{t>0} t^{\beta + h} \| v(0) - \tilde{v}_0 \|_{(p+2,\infty)} + \frac{\nu p}{2} \int_0^1 e^{-\frac{t-s}{p}} \| u_m - \tilde{u}_m \|_{(p+2,\infty)} (\| u_m \|_{(p+2,\infty)}^{p-1} + \| \tilde{u}_m \|_{(p+2,\infty)}^{p-1}) ds \leq \sup_{t>0} t^{\beta + h} \| v(0) - \tilde{v}_0 \|_{(p+2,\infty)} + \frac{\nu p}{2} \int_0^1 e^{-\frac{t-s}{p}} s^{-\alpha-h} ds \sup_{t>0} t^{\alpha + h} \| u_m - \tilde{u}_m \|_{(p+2,\infty)}.
\]

Denote \( U_m = \sup_{t>0} t^{\alpha + h} \| u_m - \tilde{u}_m \|_{(p+2,\infty)} \) and \( V_m = \sup_{t>0} t^{\beta + h} \| v_m - \tilde{v}_m \|_{(p+2,\infty)} \), for all \( m \geq 2 \). Observe that (3.18)-(3.19) imply the following recursive inequality

\[
U_{m+1} + V_{m+1} \leq \{ U_1 + V_1 \} + \max \{ \varepsilon K_1^*, \varepsilon^{p-1} K_2^* \} \{ U_m + V_m \},
\]

where \( U_1 + V_1 = \sup_{t>0} t^{\alpha + h} \| S(t)(u_0 - \tilde{u}_0) \|_{(p+2,\infty)} + \| v(0) - \tilde{v}_0 \|_{(p+2,\infty)} < \infty \) (by hypotheses) and \( K_1^*, K_2^* \) depend only on \( h, \alpha, p, \mu, n \). By taking \( \varepsilon > 0 \) small enough such that \( L_\varepsilon = \max \{ \varepsilon K_1^*, \varepsilon^{p-1} K_2^* \} \leq 1 \), we obtain that the sequences \( U_m, V_m \) are bounded and, by standard arguments, (3.17) holds.

Second step. We first prove that (2.4) implies (2.5). To this end, we subtract the equations satisfied by \((u, v)\) and \((\tilde{u}, \tilde{v})\), and afterwards take the norms \( t^{\alpha + h} \| u \|_{(p+2,\infty)} \) and \( t^{\beta + h} \| v \|_{(p+2,\infty)} \) to obtain the following inequalities:
\[ t^{\alpha + h} \| u(t) - \bar{u}(t) \|_{(p+2, \infty)} \leq \sup_{t > 0} t^{\alpha + h} \| S(t)(u_0 - \bar{u}_0) \|_{(p+2, \infty)} + t^{\alpha + h} \left\| \int_0^t S(t-s)[(u - \bar{u})v]ds \right\|_{(p+2, \infty)} + t^{\alpha + h} \left\| \int_0^t S(t-s)[\bar{u}(v - \bar{v})]ds \right\|_{(p+2, \infty)} := I_0(t) + I_1(t) + I_2(t) \] (3.20)

and

\[ t^{\beta + h} \| v(t) - \bar{v}(t) \|_{(p+2, \infty)} \leq \left\| e^{-\frac{t}{\mu}} (v_0 - \bar{v}_0) \right\|_{(p+2, \infty)} + \frac{\lambda}{\mu} \left\| \int_0^t e^{-\frac{ts}{\mu}} (|u(s)|^p - |\bar{u}(s)|^p) ds \right\|_{(p+2, \infty)} := J_0(t) + J_1(t) \] (3.21)

Since \((u, v) \in B_p(2\varepsilon)\), we employ the change of variable \( s \mapsto ts \) and estimate \( I_1 \) as

\[ I_1(t) \leq t^{\alpha + h} C_S \int_0^1 (t-s)^{\frac{np}{p+2}-\alpha} (ts)^{\alpha + h} \| v(s) \|_{(p+2, \infty)} ds \leq 2\varepsilon C_S \int_0^1 (1-s)^{\frac{np}{p+2}-\alpha} (ts)^{\alpha + h} \| v(ts) - \bar{v}(ts) \|_{(p+2, \infty)} ds. \] (3.22)

For \( I_2 \), a similar computation yields

\[ I_2(t) \leq 2\varepsilon C_S \int_0^1 (1-s)^{\frac{np}{p+2}-\alpha} (ts)^{\beta + h} \| v(ts) - \bar{v}(ts) \|_{(p+2, \infty)} ds. \] (3.23)

Before dealing with \( J_1(t) \), we remember that \( \sup_{t>0} te^{-at} = \frac{1}{a} e^{-1} \).

\[ J_1(t) \leq \frac{\lambda}{\mu} p t^{\beta + h} \int_0^t e^{-\frac{ts}{\mu}} \left\| u - \bar{u} \right\|_{(p+2, \infty)} ds \leq \frac{\lambda}{\mu} p t^{\beta + h} \int_0^t e^{-\frac{ts}{\mu}} s^{-\beta h} [s^{\alpha(p-1)} (|u|^{p-1} + |\bar{u}|^{p-1}) + s^{\alpha+1} |u - \bar{u}|] ds := H(t), \] (3.24)

where we have used the Hölder inequality in last estimate. Next we split the integral within \( H(t) \) into two parts and estimate

\[ \leq \frac{\lambda}{\mu} p \int_0^{t/2} e^{-\frac{(t-s)}{\mu}} s^{-\beta h} ds \sup_{t>0} t^{\alpha + h} \| u(t) - \bar{u}(t) \|_{(p+2, \infty)} + t^{\beta + h} \int_{t/2}^t e^{-\frac{(t-s)}{\mu}} s^{-\beta h} s^{\alpha + h} \| u(s) - \bar{u}(s) \|_{(p+2, \infty)} ds \]
\[
\leq \lambda p^p \epsilon^{p-1} \left( \frac{1}{\mu} t e^{-\left(\frac{1}{2p}\right)s} \cdot \frac{(1/2)^{1-\beta-h}}{(1-\beta-h)} \cdot \sup_{t>0} t^{\alpha+h} \|u(t) - \bar{u}(t)\|_{(p+2,\infty)} + 2^{\beta+h} \int_{1/2}^{1} \frac{1}{\mu} t e^{-\left(\frac{1}{2p}\right)s} ds \sup_{t/2<s<t} s^{\alpha+h} \|u(s) - \bar{u}(s)\|_{(p+2,\infty)} \right) \\
:= H_1(t) + H_2(t). \quad (3.25)
\]

By (3.17) and elementary properties of \(\limsup_{t \to \infty}\) we get
\[
\limsup_{t \to \infty} H_1(t) \leq C \limsup_{t \to \infty} te^{-\frac{1}{2p}t} = 0 \quad (3.26)
\]
\[
\limsup_{t \to \infty} H_2(t) \leq \lambda p^p \epsilon^{p-1} 2^{\beta+h} \limsup_{t \to \infty} \left\{ (1 - e^{-\frac{1}{2p}}) \sup_{t/2<s<t} s^{\alpha+h} \|u(s) - \bar{u}(s)\|_{(p+2,\infty)} \right\} \leq \lambda p^p \epsilon^{p-1} 2^{\beta+h} \limsup_{t \to \infty} t^{\alpha+h} \|u(t) - \bar{u}(t)\|_{(p+2,\infty)}. \quad (3.27)
\]

Now, define
\[
A := \limsup_{t \to \infty} t^{\alpha+h} \|u(t) - \bar{u}(t)\|_{(p+2,\infty)} + \limsup_{t \to \infty} t^{\beta+h} \|v(t) - \bar{v}(t)\|_{(p+2,\infty)} < \infty.
\]

So, taking \(\limsup_{t \to \infty}\) in (3.20) and (3.21) and using the bounds (3.22)-(3.27), we obtain
\[
A \leq \limsup_{t \to \infty} (I_0(t) + J_0(t)) + \max \left\{ \limsup_{t \to \infty} [I_1(t) + I_2(t)] , \limsup_{t \to \infty} [H_1(t) + H_2(t)] \right\} \leq 0 + \max \left\{ 2\epsilon CS \int_0^1 (1 - s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} ds, \lambda p^p \epsilon^{p-1} 2^{\beta+h} \right\} A \\
:= \Gamma(\epsilon) A. \quad (3.28)
\]

Choosing \(\epsilon > 0\) small enough such that \(\Gamma(\epsilon) < 1\), we get that \(A = 0\) which implies (2.5).

**Third step.** In order to prove the reciprocal assertion, we proceed as in the proof above, by changing the place of \(t^{\alpha+h} \|u(t) - \bar{u}(t)\|_{(p+2,\infty)}\) and \(t^{\alpha+h} \|S(t)(u_0 - \bar{u}_0)\|_{(p+2,\infty)}\) by the other one. In this case, instead of (3.28) we get
\[
\limsup_{t \to \infty} \left\{ \limsup_{t \to \infty} t^{\alpha+h} \|u(t) - \bar{u}(t)\|_{(p+2,\infty)} \right\} \\
\leq 0 + \max \left\{ 2\epsilon CS \int_0^1 (1 - s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} ds, \lambda p^p \epsilon^{p-1} 2^{\beta+h} \right\} A \\
= 0,
\]

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because $A = 0$ by hypothesis.

The strong version of the results follows by an entirely parallel way to previous proof by taking the quantities $t^{\alpha+h} \| \cdot \|_{p+2}^p$ and $t^{\beta+h} \| \cdot \|_{p+2}^{p+2}$ in place of $t^{\alpha+h} \| \cdot \|_{(p+2,\infty)}^p$ and $t^{\beta+h} \| \cdot \|_{(p+2,\infty)}^{p+2}$, respectively. The proof is finished.

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