Lipschitz Equivalence of Self-Similar Sets: Algebraic and Geometric Properties

Hui Rao, Huo-Jun Ruan, and Yang Wang

Abstract. In this paper we provide an up-to-date survey on the study of Lipschitz equivalence of self-similar sets. Lipschitz equivalence is an important property in fractal geometry because it preserves many key properties of fractal sets. A fundamental result by Falconer and Marsh [On the Lipschitz equivalence of Cantor sets, Mathematika, 39 (1992), 223–233] establishes conditions for Lipschitz equivalence based on the algebraic properties of the contraction ratios of the self-similar sets. Recently there has been other substantial progress in the field. This paper is a comprehensive survey of the field. It provides a summary of the important and interesting results in the field. In addition we provide detailed discussions on several important techniques that have been used to prove some of the key results. It is our hope that the paper will provide a good overview of major results and techniques, and a friendly entry point for anyone who is interested in studying problems in this field.

1. Introduction

In the study of fractal geometry a fundamental problem is to find ways that measure the similarity or difference of fractal sets. The concept of dimension, whether it is the Hausdorff dimension or the box counting dimension, is widely used for such a purpose: Two sets having different dimensions are considered to be unalike. However for measuring differences dimension by itself is quite inadequate. Two compact sets, even with the same dimension, may in fact be quite different in many ways. Thus it is natural to seek a suitable quality that would allow us to tell whether two fractal sets are “similar”. Generally, Lipschitz equivalence is thought to be such a quality. In [5] it was pointed out that while topology may be regarded as the study of equivalence classes of sets under homeomorphism, fractal geometry is sometimes thought of as the study of equivalence classes under
bi-Lipschitz mappings. More restrictive maps such as isometry tend to lead to rather uninteresting equivalent classes, while far less restrictive maps such as general continuous maps take us completely out of geometry into the realm of pure topology (see [6]). Bi-Lipschitz maps offer a good balance, which lead to equivalent classes that are interesting and intriguing both geometrically and algebraically.

There has been notable progress on the study of bi-Lipschitz equivalence classes, especially in recent years. Yet much is still unknown, and this progress has led to more unanswered questions. The goal of this paper is to provide a comprehensive survey of the area. It is our hope that the paper will provide a good overview of major results and techniques, and a friendly entry point for anyone who is interested in studying problems in this field.

Let $E, F$ be compact sets in $\mathbb{R}^d$. We say that $E$ and $F$ are Lipschitz equivalent, denoted by $E \sim F$, if there exists a bijection $f : E \to F$ which is bi-Lipschitz, i.e. there exists a constant $C > 0$ such that

$$C^{-1} |x - y| \leq |f(x) - f(y)| \leq C |x - y|$$

for all $x, y \in E$. The general problem we consider is to find conditions under which the two sets $E$ and $F$ are Lipschitz equivalent.

Recall that in general we characterize a self-similar set as the attractor of an iterated function system (IFS). Let $\{\phi_j\}_{j=1}^m$ be an IFS on $\mathbb{R}^d$ where each $\phi_j$ is a contractive similarity with contraction ratio $0 < \rho_j < 1$. The attractor of the IFS is the unique nonempty compact set $F$ satisfying $F = \bigcup_{j=1}^m \phi_j(F)$, see [8]. We say that the attractor $F$ is dust-like, or alternatively, the IFS $\{\phi_j\}$ satisfies the strong separation condition (SSC), if the sets $\{\phi_j(F)\}$ are disjoint. We remark that by definition, “dust-like self-similar set” is not the same as a “totally disconnected self-similar set”. It is well known that if $F$ is dust-like then the Hausdorff dimension $s = \dim_H(F)$ of $F$ satisfies $\sum_{j=1}^m \rho_j^d = 1$.

Now for any $\rho_1, \ldots, \rho_m \in (0, 1)$ with $\sum_{j=1}^m \rho_j^d < 1$, we will call $\rho = (\rho_1, \ldots, \rho_m)$ a contraction vector, and use the notation $D(\rho) = D(\rho_1, \ldots, \rho_m)$ to denote the set of all dust-like self-similar sets that are the attractor of some IFS with contraction ratios $\rho_j, j = 1, \ldots, m$ on $\mathbb{R}^d$. Clearly all sets in $D(\rho)$ have the same Hausdorff dimension, which we denote by $s = \dim_H D(\rho)$. The following property is well known, see e.g. [13].

**Proposition 1.1.** Any two sets in $D(\rho)$ are Lipschitz equivalent.

This result tells us that in the dust-like setting all that matters is the contraction vector. The translations in the similitudes in the IFS do not matter. In fact, all sets in $D(\rho)$ are Lipschitz equivalent to a symbolic space defined by $\rho$. For any $m \geq 1$ let $\Sigma_m$ denote the set of infinite words $w = i_1 i_2 i_3 \cdots$ where each $i_j \in \{1, 2, \ldots, m\}$. For such a $w \in \Sigma_m$ we use the notation $w(k) = i_k$ and $[w]_k = i_1 i_2 \cdots i_k$. For any $\rho = (\rho_1, \rho_2, \ldots, \rho_m), 0 < \rho_j < 1$, we can define a metric $d_\rho(\cdot, \cdot)$ on $\Sigma_m$ as follows: Let $z, w \in \Sigma_m$. If $z(1) \neq w(1)$ then set $d_\rho(z, w) = 1$; otherwise set $d_\rho(z, w) = \rho_{|z|_k}$, where $|z|_k = |w|_k$ but $z(k+1) \neq w(k+1)$, and $\rho_{|z|_k} := \prod_{j=1}^{k} \rho_{z(j)}$. It is well known that $(\Sigma_m, d_\rho)$ is a metric space. The following is easy to prove:

**Proposition 1.2.** Let $\rho = (\rho_1, \ldots, \rho_m)$ be a contraction vector and $E \in D(\rho)$. Then there exists a bi-Lipschitz map from $(\Sigma_m, d_\rho)$ to $E$.

**Remark 1.3.** It was noted in [12] that the proof for Proposition 1.2 leads to the following simple but interesting result: Assume that $D(\rho_1, \ldots, \rho_m)$ and
\( \mathcal{D}(\tau_1, \ldots, \tau_n) \) are Lipschitz equivalent. Let \( s = \dim_H \mathcal{D}(\rho_1, \ldots, \rho_m) \). Then for any \( r > s \), \( \mathcal{D}(\rho_1^r, \ldots, \rho_m^r) \) and \( \mathcal{D}(\tau_1^r, \ldots, \tau_n^r) \) are also Lipschitz equivalent.

Proposition 1.1 gives a “trivial condition” for Lipschitz equivalence. A generalization of this “trivial condition” is when one contraction ratio is derived from another.

Let \( \Sigma_m := \bigcup_{k=1}^{\infty} \{1, 2, \ldots, m\}^k \). For any word \( i = i_1 \cdots i_k \in \Sigma_m \), we denote \( |i| = \{i w : w \in \Sigma_m\} \) and call it a symbol cylinder. A finite set \( \{j_1, \ldots, j_n\} \subset \Sigma_m \) is called a cut set of \( \Sigma_m \) if the symbol cylinders \( [j_1], \ldots, [j_n] \) tile \( \Sigma_m \), i.e., they are disjoint and their union is \( \Sigma_m \).

Let \( \rho = (\rho_1, \ldots, \rho_m) \) and \( \tau = (\tau_1, \ldots, \tau_n) \) be two contraction vectors. We say that \( \tau \) is derived from \( \rho \) if there exists a cut set \( \{j_1, \ldots, j_n\} \) of \( \Sigma_m \) such that \( \tau = (\rho_{j_1}, \ldots, \rho_{j_n}) \), where \( \rho_{j_1} \cdots \rho_{j_n} = \rho_{i_1} \cdots \rho_{i_k} \).

**Definition 1.4.** Let \( \rho \) and \( \tau \) be two contraction vectors. We say \( \rho \) and \( \tau \) are equivalent, denoted by \( \rho \sim \tau \), if there exists a sequence

\[ \rho = \rho_1, \rho_2, \ldots, \rho_N = \tau \]

such that \( \rho_{j+1} \) is derived from \( \rho_j \) or vice versa for \( 1 \leq j < N \).

**Proposition 1.5.** Assume that \( \rho \) is equivalent to \( \tau \). Then \( \mathcal{D}(\rho) \sim \mathcal{D}(\tau) \).

**Proof.** We need only show the conclusion holds when \( \tau \) is derived from \( \rho \). Suppose \( E \in \mathcal{D}(\rho) \) is the attractor of the IFS \( \Phi = \{\phi_j\}_{j=1}^m \), then \( E \) is also the attractor of the IFS \( \{\phi_{j_1}, \ldots, \phi_{j_n}\} \), where \( \phi_{i_1} \cdots \phi_{i_k} := \phi_{i_1} \circ \cdots \circ \phi_{i_k} \). Hence \( \mathcal{D}(\rho) \) and \( \mathcal{D}(\tau) \) have a common element \( E \), and they are equivalent. \( \square \)

The central question in the study of Lipschitz equivalence of dust-like Cantor sets is: Under what conditions are two dust-like Cantor sets Lipschitz equivalent even if they have different contraction vectors? Are there any “nontrivial conditions” that also lead to equivalence?

**Problem 1.6.** Find nontrivial sufficient conditions and necessary conditions on \( \rho \) and \( \tau \) such that \( \mathcal{D}(\rho) \sim \mathcal{D}(\tau) \). In particular, is it possible that \( \mathcal{D}(\rho) \sim \mathcal{D}(\tau) \) but \( \rho \) and \( \tau \) are not equivalent?

As it turns out, among the known results concerning this central question, the algebraic properties of contraction vectors have played a fundamental role. This is a main focus of this survey.

One of the very first and most fundamental results in this area is the following theorem, proved by Falconer and Marsh ([5], Theorem 3.3), that establishes a connection to the algebraic properties of the contraction ratios:

**Theorem 1.7 (Falconer and Marsh).** Let \( \mathcal{D}(\rho) \) and \( \mathcal{D}(\tau) \) be Lipschitz equivalent, where \( \rho = (\rho_1, \ldots, \rho_m) \) and \( \tau = (\tau_1, \ldots, \tau_n) \) are two contraction vectors. Let \( s = \dim_H \mathcal{D}(\rho) = \dim_H \mathcal{D}(\tau) \). Then

1. \( \mathcal{Q}(\rho_1^p, \ldots, \rho_m^p) = \mathcal{Q}(\tau_1^q, \ldots, \tau_n^q) \), where \( \mathcal{Q}(a_1, \ldots, a_m) \) denotes the subfield of \( \mathbb{R} \) generated by \( \mathcal{Q} \) and \( a_1, \ldots, a_m \).
2. There exist positive integers \( p, q \) such that

\[ \text{sgp}(\rho_1^p, \ldots, \rho_m^p) \subseteq \text{sgp}(\tau_1^q, \ldots, \tau_n^q), \]
\[ \text{sgp}(\tau_1^q, \ldots, \tau_n^q) \subseteq \text{sgp}(\rho_1^p, \ldots, \rho_m^p), \]

where \( \text{sgp}(a_1, \ldots, a_m) \) denotes the subsemigroup of \( (\mathbb{R}^+, \times) \) generated by \( a_1, \ldots, a_m \).
Using this theorem, it is trivial to construct dust-like self-similar sets $E$ and $F$ such that $\dim_H E = \dim_H F$ but $E$ and $F$ are not Lipschitz equivalent. For example, let $E$ be the middle-third Cantor set and $F$ be the dust-like Cantor set given by the IFS $\Phi := \{\rho x, \rho x + \frac{1}{2}(1 + \rho), \rho x + 1 - \rho\}$ where $\rho = 3^{-\log_2 3}$. Then $E$ and $F$ have the same dimension. However, they are not Lipschitz equivalent by Theorem 1.7.

Along the direction of the theorem of Falconer and Marsh, several other theorems have been established in recent years. These theorems further establish connections between Lipschitz equivalence and algebraic properties of the contractions. We shall discuss them, along with several key techniques, later in this paper.

Another interesting question on Lipschitz equivalence, in a different direction, considers the geometric structures of self-similar sets. Perhaps the best known problem is the one proposed by David and Semmes ([3], Problem 11.16):

**Problem 1.8.** Let $\phi_i(x) := x/5 + (i - 1)/5$ where $i \in \{1, \ldots, 5\}$. Let $M$ and $M'$ be the attractor of the IFS $\{\phi_1, \phi_3, \phi_5\}$ and the IFS $\{\phi_1, \phi_4, \phi_5\}$, respectively. Are $M$ and $M'$ Lipschitz equivalent?

![Figure 1. Basic intervals of the self-similar sets $M$ and $M'$](image)

We call $M$ the $\{1, 3, 5\}$-set and $M'$ the $\{1, 4, 5\}$-set. The problem is generally known as the $\{1, 3, 5\}-\{1, 4, 5\}$ problem. In this setting, $M$ is dust-like and $M'$ has certain touching structure, see Figure 1. In this problem, the contraction ratios are all identical so the difference lies entirely in the geometry of the two IFSs. David and Semmes conjectured that $M \not\sim M'$. However, by examining graph-directed structures of the attractors and introducing techniques to study Lipschitz equivalence on these structures, Rao, Ruan and Xi [13] proved that in fact $M \sim M'$. Naturally one may ask whether this result extends to the general setting, where we consider the equivalence of two IFSs with the same contraction vector, but one is dust-like while the other has some touching structure. We shall discuss this problem in more details also later in the paper.

In other direction, some recent works are done on the Lipschitz equivalence of $\lambda$-Cantor sets, which are self-similar sets with overlap. We refer the readers to [2, 4].

2. Techniques for Lipschitz Equivalence of Dust-Like Cantor Sets

2.1. Techniques in [5]. In [5] Falconer and Marsh had developed several important techniques to study the Lipschitz equivalence of dust-like self-similar sets. These techniques are now viewed as being fundamental to the area. These techniques had allowed Falconer and Marsh to prove Theorem 1.7.

Let us first introduce some notation. Let $E$ be the attractor of the IFS $\Phi = \{\phi_1, \ldots, \phi_m\}$. For any word $i = i_1 \cdots i_k \in \Sigma_m^*$, we call $k$ the length of the word
i and denote it by \(|i|\). Furthermore, a cylinder \(E_i\) is defined to be \(E_i = \phi_i(E) := \phi_{i_1} \circ \cdots \circ \phi_{i_n}(E)\).

In this section we consider the Lipschitz equivalence of two dust-like self-similar sets \(E\) and \(F\) with the following setup: We assume that \(E \in \mathcal{D}(\rho_1, \ldots, \rho_m)\) is the attractor of \(\Phi = \{\phi_1, \ldots, \phi_m\}\) and \(F \in \mathcal{D}(\tau_1, \ldots, \tau_n)\) is the attractor of \(\Psi = \{\psi_1, \ldots, \psi_n\}\). We also assume in subsections 2.1 and 2.2 that \(s = \dim_H E = \dim_H F\) and \(f : E \to F\) is a bi-Lipschitz map.

An important result is the following lemma:

**Lemma 2.1** ([5]). There exists an integer \(n_0\) such that for any \(i \in \Sigma_m^*\), there exist \(k, j_1, \ldots, j_p \in \Sigma_n^*\) such that \(F_{kj_1}, \ldots, F_{kj_p}\) are disjoint and

\[
 f(E_i) = \bigcup_{r=1}^{p} F_{kj_r} \subset F_k,
\]

where each \(|j_r| \leq n_0\). In particular \(\mathcal{H}^s(f(E_i)) = \mathcal{H}^s(F_k) \sum_{r=1}^{p} (\tau_{j_r})^s\).

**Remark 2.2.** The above lemma implies that a bi-Lipschitz map must behave “nicely” by mapping a cylinder onto a union of cylinders. We can require \(F_k\) to be the smallest cylinder containing \(f(E_i)\). It is clear that we can also require each \(|j_r| = n_0\) in the above lemma. Consequently the set \(\{k, j_1, \ldots, j_p\}\) is uniquely determined by \(i\). We will write \(p_i\) for \(p\) if necessary. We call this unique decomposition to be the maximum decomposition of \(f(E_i)\) with respect to \(F\) and \(n_0\). From now on, we fix \(n_0\) in this section. We remark that \(p\) in (2.1) is bounded since \(p \leq n^{n_0}\).

One of the key techniques in [5] is the introduction of a sequence of functions \(g_k : E \to \mathbb{R}\), given by

\[
 g_k(x) = \frac{\mathcal{H}^s(f(E_i))}{\mathcal{H}^s(E_i)}
\]

for \(x \in E_i\), where \(i \in \{1, \ldots, m\}^k\). This sequence plays a crucial role in studying the Lipschitz equivalence of dust-like Cantor sets. We shall abuse notation by writing \(g_k(E_i) = \frac{\mathcal{H}^s(f(E_i))}{\mathcal{H}^s(E_i)}\). It is easy to show that

\[
 g_k(E_i) = \sum_{i=1}^{m} \frac{\mathcal{H}^s(E_i)}{\mathcal{H}^s(E_i)} g_{k+1}(E_i).
\]

Furthermore, it is not difficult to prove:

**Lemma 2.3** ([5]). The set \(\{\frac{g_{k+1}(x)}{g_k(x)} : x \in E, k \geq 1\}\) is finite.

An important observation is that \(\{g_k\}\) form a martingale with respect to the normalized Hausdorff measure \(\mathcal{H}^s\) and a suitable filtration. By the Martingale Convergence Theorem the sequence of functions \(\{g_k\}\) converges almost everywhere with respect to \(\mathcal{H}^s\). However, note that \(g_k(x)\) take on only finitely many values by Lemma 2.3. It follows that for almost every \(x \in E\), there exists a \(k_0\) such that for \(k \geq k_0\) we must have \(g_k(x) = g_{k_0}(x)\). Using this result, Theorem 1.7 can be proved.

### 2.2. Measure-preserving property

In [1], Cooper and Pignataro studied the order-preserving bi-Lipschitz functions between two dust-like Cantor subsets of \(\mathbb{R}\). They proved that such functions have certain measure preserving property. Xi and Ruan [21] observed that this property also holds in more general case.
LEMMA 2.4 ([11] [21]). There is a cylinder $E_{i_0}$ and a constant $c > 0$ such that $g_k(x) = c$ for all $x \in E_{i_0}$ and $k \geq |i_0|$.  

**Proof.** Set $T = \sup_{k \geq 1} \max_{|i|=k} g_k(E_i)$. Since $f$ is bi-Lipschitz, we have $T < +\infty$.

If $\frac{g_{k+1}(x)}{g_k(x)} = 1$ for all $x \in E$ and all $k \geq 1$, then the lemma clearly holds. Otherwise set $\delta = \min \left( \left\{ \frac{g_{k+1}(x)}{g_k(x)} - 1 : x \in E, k \geq 1 \right\} \setminus \{0\} \right)$. Then $\delta > 0$ by Lemma 2.3. Choose $i_0$ such that (with $\ell = |i_0|$)

\begin{equation}
\frac{g_{\ell}(E_{i_0})}{g_\ell(E_{i_0})} > T/(1 + \delta).
\end{equation}

Then $\frac{g_{\ell+1}(E_{i_0})}{g_{\ell}(E_{i_0})} < 1 + \delta$ for all $j$ and hence $\frac{g_{\ell+1}(E_{i_0})}{g_{\ell}(E_{i_0})} \leq 1$ by the definition of $\delta$.

Now formula (2.4) implies that $\frac{g_{\ell+1}(E_{i_0})}{g_{\ell}(E_{i_0})} = 1$ for all $j$. Hence each $E_{i_0j}$ satisfies (2.4) and we can repeat the same argument with $E_{i_0j}$ in place of $E_{i_0}$. Set $c = g_\ell(E_{i_0})$ and the lemma is proved. \hfill $\square$

This lemma means that the restriction of $f$ on $E_{i_0}$ is measure-preserving up to a constant. More precisely for any Borel set $A \subset E_{i_0}$ we have

\begin{equation}
\frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)} = c = \frac{\mathcal{H}^s(f(E_{i_0}))}{\mathcal{H}^s(E_{i_0})}.
\end{equation}

We shall call any such cylinder $E_{i_0}$ in Lemma 2.4 a stable cylinder with respect to the map $f$. In the rest of this section we fix a stable cylinder $E_{i_0}$. Going back to Lemma 2.3 and Remark 2.2, for any $i \in \Sigma_m^*$, there is a (unique) maximum decomposition of $f(E_{i,1})$ with respect to $F$ and $n_0$:

$$f(E_{i,1}) = \bigcup_{r=1}^{p_{i,0}} F_{k_{j_r}},$$

where $|j_r| = n_0$. This allows us to prove the following observation, which serves as a key result in the development of the matchable condition technique in [12].

**Lemma 2.5 ([12]).** The set $\mathcal{M} = \bigcup_{i \in \Sigma_m^*} \left\{ \frac{\mathcal{H}^s(E_{i,1})}{\mathcal{H}^s(F_{k_{j_r}})} : 1 \leq r \leq p_{i,0} \right\}$ is finite. Consequently, the sets

\begin{equation*}
\mathcal{M}' = \bigcup_{i \in \Sigma_m^*} \left\{ \frac{\text{diam} E_{i,1}}{\text{diam} F_{k_{j_r}}} : 1 \leq r \leq p_{i,0} \right\}
\end{equation*}

and

\begin{equation*}
\mathcal{M}'' = \bigcup_{i \in \Sigma_m^*} \left\{ \frac{p_{i,0}}{r_{k_{j_r}}} : 1 \leq r \leq p_{i,0} \right\}
\end{equation*}

are finite.

**2.3. Pseudo-basis and distance function.** The recent paper [12] introduced several techniques such as pseudo-basis, distance function and matchable relation. These techniques allowed us to prove several theorems that could not be obtained using the classical techniques.

The paper [12] considered the notion of rank for a contraction vector $\mathbf{\rho} = (\rho_1, \ldots, \rho_m)$. Let $\langle \rho_1, \ldots, \rho_m \rangle$ denote the subgroup of $(\mathbb{R}^+, \times)$ generated by $\rho_1, \ldots, \rho_m$, it is a free abelian group. It follows that $\langle \rho_1, \ldots, \rho_m \rangle$ has a nonempty basis and we can define the rank of $\langle \rho_1, \ldots, \rho_m \rangle$, denoted by $\text{rank}(\mathbf{\rho})$, to be the cardinality of the basis. Clearly $1 \leq \text{rank}(\mathbf{\rho}) \leq m$. If $\text{rank}(\mathbf{\rho}) = m$, we say $\mathbf{\rho}$ has full rank. For more about the rank of a free abelian group see e.g. [7].
According to Theorem 1.7 (2), if \( D(\rho) \sim D(\tau) \), then \( \text{rank}(\rho) = \text{rank}(\tau) \), where \( \langle \rho, \tau \rangle := \langle \rho_1, \ldots, \rho_m, \tau_1, \ldots, \tau_n \rangle \).

We call \( w_1, \ldots, w_L \) a pseudo-basis of \( V = \langle \rho, \tau \rangle \) if \( L = \text{rank} V \) and \( V \subseteq \langle w_1, \ldots, w_L \rangle \). It is clear that a basis of \( V \) is natural to be a pseudo-basis. For any \( x_1, x_2 \in V \), we define their distance with respect to the pseudo-basis \( w_1, \ldots, w_L \) by

\[
(2.6) \quad h(x_1, x_2) := \sqrt{\sum_{j=1}^{L} (s_j - t_j)^2},
\]

where \( s_j, t_j \in \mathbb{Z} \) are the unique integers such that \( x_1 = \prod_{j=1}^{L} w_j^{s_j}, x_2 = \prod_{j=1}^{L} w_j^{t_j} \).

**Remark 2.6.** It is easy to show that if \( h_1 \) and \( h_2 \) are distances on \( V \) with respect to two different pseudo-bases, then they are comparable, i.e., there exists a constant \( C \geq 1 \) such that

\[
C^{-1} h_1(x_1, x_2) \leq h_2(x_1, x_2) \leq Ch_1(x_1, x_2), \quad \forall x_1, x_2 \in V.
\]

Let \( \rho_{\max} = \max\{\rho_1, \ldots, \rho_m\} \) and \( \rho_{\min} = \min\{\rho_1, \ldots, \rho_m\} \). For any \( t \in (0, 1) \) let

\[
\mathcal{W}(E, t) := \{i \in \Sigma_n^* : \rho_1 \leq t < \rho_i\},
\]

where \( i^* \) is the word obtained by deleting the last letter of \( i \), i.e., \( i^* = i_1 \cdots i_{k-1} \) if \( i = i_1 \cdots i_k \). We define \( \rho_i = 1 \) if the length of \( i \) equals 1. Similarly, we may define \( \mathcal{W}(F, t) \) with respect to its contraction vector \( \tau \).

Pick some \( i \in \Sigma_n^* \). There is a (unique) maximum decomposition of \( f(E_i) \) with respect to \( F \) and \( n_0 \):

\[
f(E_i) = \bigcup_{r=1}^{p_i} F_{k_j^r},
\]

where \( |j^r| = n_0 \). We define a relation \( \mathcal{R}(i, t, f) \subset \mathcal{W}(E, t) \times \mathcal{W}(F, t) \) by

\[
(2.7) \quad \mathcal{R}(i, t, f) := \left\{ (i', j') \in \mathcal{W}(E, t) \times \mathcal{W}(F, t) : f(E_{i'}) \cap \bigcup_{r=1}^{p_i} F_{k_j^r, j'} \neq \emptyset \right\}.
\]

It is shown in [12] that

**Theorem 2.7 ([12]).** Assume that \( f : E \rightarrow F \) is bi-Lipschitz and let \( E_{i_0} \) be a stable cylinder for some \( i_0 \in \Sigma_n^* \). Let \( h \) be a distance on \( V = \langle \rho, \tau \rangle \) defined by (2.6). Then there exists a constant \( M_0 > 0 \) such that for any \( t \in (0, 1) \) we have

1. For any \( i \in \mathcal{W}(E, t) \),

\[
1 \leq \text{card} \{j : (i, j) \in \mathcal{R}(i_0, t, f)\} \leq M_0.
\]

2. Similarly, for any \( j \in \mathcal{W}(F, t) \),

\[
1 \leq \text{card} \{i : (i, j) \in \mathcal{R}(i_0, t, f)\} \leq M_0.
\]

2.4. **Matchable condition.** One of the most important techniques introduced in [12] is the matchable relation. It is also one of the more technical ones. Let \( E \) and \( F \) be two dust-like self-similar sets with contraction vectors \( \rho \) and \( \tau \) respectively. Let \( h \) be a distance on \( V = \langle \rho, \tau \rangle \) defined by (2.6).

Let \( M_0 \) be a constant. For \( t \in (0, 1) \), a relation \( \mathcal{R} \subset \mathcal{W}(E, t) \times \mathcal{W}(F, t) \) is said to be \( (M_0, h) \)-matchable, or simply \( M_0 \)-matchable when there is no confusion, if

1. \( 1 \leq \text{card} \{j : (i, j) \in \mathcal{R}\} \leq M_0 \) for any \( i \in \mathcal{W}(E, t) \), and \( 1 \leq \text{card} \{i : (i, j) \in \mathcal{R}\} \leq M_0 \) for any \( j \in \mathcal{W}(F, t) \).
(ii) If \((i, j) \in \mathcal{R}\), then \(h(\rho_i, \tau_j) \leq M_0\).

We also say that \(W(E, t)\) and \(W(F, t)\) are \((M_0, h)\)-matchable, or \(M_0\)-matchable when there exists a \((M_0, h)\)-matchable relation \(\mathcal{R} \subset W(E, t) \times W(F, t)\).

Definition 2.8. We shall call two self-similar sets \(E\) and \(F\) are matchable, if there exists a constant \(M_0\) such that for any \(t \in (0, 1)\), \(W(E, t)\) and \(W(F, t)\) are \(M_0\)-matchable.

We remark that the matchable property does not depend on the choice of pseudo-basis of \(\langle \rho, \tau \rangle\). Obviously Theorem 2.7 implies the following result:

Theorem 2.9 ([12]). Let \(E\) and \(F\) be two dust-like self-similar sets. If \(E \sim F\), then \(E\) and \(F\) are matchable.

3. Recent Results on dust-like self-similar sets

The techniques developed in Falconer and Marsh [5] had led to some fundamental theorems on the Lipschitz equivalence of dust-like Cantor sets, such as Theorem 1.7. However, to further advance the field these techniques are clearly not sufficient. As a result there has not been much significant progress until recently, when several new results on the Lipschitz equivalence of dust-like Cantors sets were established in [12, 19, 21]. In particular, the equivalence of several classes have been completely characterized in [12]. These results, which we shall state here, are based on the new techniques outlined in the previous section. As an important observation, a common theme among these results is the link between Lipschitz equivalence and the algebraic properties of the contractions.

One of the new results on the equivalence of two dust-like Cantor sets concerns the special case where one of the contraction vectors has full rank. Lipschitz equivalence in this setting forces strong rigidity on the contraction vectors. The following result is derived by using the distance function and Theorem 2.9.

Theorem 3.1 ([12]). Let \(\rho = (\rho_1, \ldots, \rho_m)\) and \(\tau = (\tau_1, \ldots, \tau_m)\) be two contraction vectors with \(\text{rank}(\rho) = m\). Then \(D(\rho)\) and \(D(\tau)\) are Lipschitz equivalent if and only if \(\tau\) is a permutation of \(\rho\).

If the length of \(\tau\) is not equal to \(m\) then the characterization of \(\tau\) is open. We make the following conjecture:

Conjecture 3.2. Let \(\rho = (\rho_1, \ldots, \rho_m)\) such that \(\text{rank}(\rho) = m\). Assume that \(\tau = (\tau_1, \ldots, \tau_n)\). Then \(D(\rho)\) and \(D(\tau)\) are Lipschitz equivalent if and only if \(\tau\) is derived from \(\rho\).

Another interesting and natural class to consider is when the contraction vectors have two ratios. Namely we may ask under what conditions are \(D(\rho_1, \rho_2) \sim D(\tau_1, \tau_2)\). This question is completely answered in [12].

Theorem 3.3. Let \((\rho_1, \rho_2)\) and \((\tau_1, \tau_2)\) be two contraction vectors with \(\rho_1 \leq \rho_2\), \(\tau_1 \leq \tau_2\). Assume that \(\rho_1 \leq \tau_1\). Then \(D(\rho) \sim D(\tau)\) if and only if one of the two conditions holds:

1. \(\rho_1 = \tau_1\) and \(\rho_2 = \tau_2\).
2. There exists a real number \(0 < \lambda < 1\), such that \((\rho_1, \rho_2) = (\lambda^5, \lambda)\) and \((\tau_1, \tau_2) = (\lambda^3, \lambda^2)\).
We provide a quick sketch of the proof here. First, assume that rank(ρ₁, ρ₂) = 2 or rank(τ₁, τ₂) = 2. Then we must have ρ₁ = τ₁ and ρ₂ = τ₂ by Theorem 3.1. So we now only need to consider the case where rank(ρ₁, ρ₂) = rank(τ₁, τ₂) = 1. By Theorem 1.7 we know there exists a t such that ρ_j = t^{m_j} and τ_j = t^{n_j} where m_j, n_j ∈ ℤ⁺. Set x = t^s where s is the dimension of D(ρ₁, ρ₂). Then
\[ x^{m_1} + x^{m_2} - 1 = 0, \quad x^{n_1} + x^{n_2} - 1 = 0. \]

For the above two polynomials to have a common root they must have a common factor. The irreducibility of trinomials, however, has been classified by Ljunggren [10] (Theorem 3 in the paper). Applying the results in [10] one can show that
\[ (ρ₁, ρ₂) = (λ^5, λ) \quad \text{and} \quad (τ₁, τ₂) = (λ^3, λ^2) \]
for some 0 < λ < 1, which takes on the form λ = t^k for some k ∈ ℤ⁺.

As an application of Theorem 3.3 we can see that the conditions in Theorem 1.7 are necessary but not sufficient via the following example.

**Example 3.4.** Let x, y, 0 < x, y < 1, be the solution of the equations
\[ x^6 + y = 1 \quad \text{and} \quad x^3 + y^4 = 1. \]

One can easily check that the solution indeed exists. Let s be a real number such that 0 < s < 1. Suppose that the contraction vectors of E and F are (x^{6/s}, y^{1/s}) and (x^{3/s}, y^{4/s}), respectively. Then E and F have the same Hausdorff dimension and satisfy the conditions in Theorem 1.7. However, E and F are not Lipschitz equivalent by Theorem 3.3.

Another case where the Lipschitz equivalence of dust-like self-similar sets can be characterized completely is when one of them has uniform contraction ratios.

**Theorem 3.5 ([21]).** Let \( ρ = (ρ₁, \ldots, ρ_m) = (ρ_i) \) and \( τ = (τ₁, \ldots, τ_n) \). Then \( D(ρ) \) and \( D(τ) \) are Lipschitz equivalent if and only if the following conditions hold:

1. \( \dim_H D(τ) = \dim_H D(ρ) = \log m / \log ρ^{-1} \).
2. There exists a q ∈ ℤ⁺ such that m^{1/q} ∈ ℤ and
\[
\log τ_j \in \frac{1}{q} \mathbb{Z} \quad \text{for all} \quad j = 1, 2, \ldots, n.
\]

Note that by Theorem 1.7 all τ_j must be rational powers of ρ. The above theorem shows that one needs more to achieve Lipschitz equivalence.

In other direction, using a measure-preserving property, Xi and Ruan [21] and Xi [19] showed that the graph-directed structure can be used to characterize the Lipschitz equivalence of two dust-like self-similar sets. We remark that the idea of studying graph-directed structures of self-similar sets appeared in [15], where they deal with self-similar sets with overlaps.

We recall the definition of graph-directed sets (see [11]). Let \( G = (V, Γ) \) be a directed graph and \( d \) a positive integer. Suppose for each edge \( e ∈ Γ \), there is a corresponding similarity \( φ_e : \mathbb{R}^d → \mathbb{R}^d \) with ratio \( ρ_e ∈ (0, 1) \). Assume that for each vertex \( i ∈ V \), there exists an edge starting from \( i \). Then there exists a unique family \( \{E_i\}_{i ∈ V} \) of compact subsets of \( \mathbb{R}^d \) such that for any \( i ∈ V \),
\[ E_i = \bigcup_{j ∈ V} \bigcup_{e ∈ E_{ij}} φ_e(E_j), \]
where $\mathcal{E}_{ij}$ is the set of edges starting from $i$ and ending at $j$. In particular, if the union in (3.1) is disjoint for any $i$, we call \( \{E_i\}_{i \in V} \) dust-like graph-directed sets on \((V, \Gamma)\).

Now, let \( \{F_i\}_{i \in V} \) be dust-like graph-directed sets on \((V, \Gamma)\) satisfying
\[
F_i = \bigcup_{j \in V} \bigcup_{e \in \mathcal{E}_{ij}} \psi_e(F_j), \quad i \in V.
\]
If similarities $\phi_e$ and $\psi_e$ have the same ratio for each $e \in \Gamma$, we say that \( \{E_i\}_{i \in V} \) and \( \{F_i\}_{i \in V} \) have the same graph-directed structure.

Recall that $E$ and $F$ are the attractors of the IFSs $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Psi = \{\psi_1, \ldots, \psi_n\}$, respectively. Given a finite subset $\Lambda$ of $\Sigma_\ast^n$ and a positive real number $r$, we call $r \cdot \bigcup_{i \in \Lambda} \psi_i(F)$ a finite copy of $F$. It was proved in [1, 21] that a finite copy of $F$ is always Lipschitz equivalent to $F$.

**Theorem 3.6 ([19, 21]).** Let $E$ and $F$ be two dust-like self-similar subsets of $\mathbb{R}^d$. Then $E \sim F$ if and only if there exist graph-directed sets \( \{E_i\}_{i=1}^\ell \) and \( \{F_i\}_{i=1}^\ell \) such that

1. \( \{E_i\}_{i=1}^\ell \) and \( \{F_i\}_{i=1}^\ell \) have the same graph-directed structures,
2. $E_i = E$ for $i = 1, \ldots, \ell$,
3. $F_i$ is a finite copy of $F$ for $i = 1, \ldots, \ell$.

Notice that the conditions in the above theorem are often difficult to check. We pose the following problem.

**Problem 3.7.** Given two contraction ratios $\rho$ and $\tau$, devise an algorithm to determine in finite steps the Lipschitz equivalence of $D(\rho)$ and $D(\tau)$.

### 4. Touching IFS and Lipschitz equivalence: One dimensional case

So far we have focused almost exclusively on the algebraic properties of contraction ratios. Yet we should not overlook the importance of geometry in the study. One interesting question in Lipschitz equivalence concerns the geometric structures of the generating IFSs of self-similar sets. One such problem is the Lipschitz equivalence of two self-similar sets have the same contraction ratios but one is dust-like while another has some touching structures. The best known example is Problem 1.8 in Section 1, known as the \( \{1, 3, 5\} - \{1, 4, 5\} \) problem proposed by David and Semmes ([3], Problem 11.16). As we mentioned in Section 1, this problem was settled in [13], which proved that the two sets are indeed Lipschitz equivalent. In this section we give a more detailed description of the techniques used in [13] to solve the \( \{1, 3, 5\} - \{1, 4, 5\} \) problem. These techniques have also led to further recent developments ([17, 20] on the Lipschitz equivalence of touching IFSs in more general settings. We shall provide more details on these developments as well.

#### 4.1. The \( \{1, 3, 5\} - \{1, 4, 5\} \) problem and the graph-directed method.

An important technique is the graph-directed method, and here we show how it works by proving the equivalence of the sets $M$ and $M'$. Recall from Section 1, Problem 1.8 that $M$ is the dust-like \( \{1, 3, 5\} \)-set while $M'$ is the \( \{1, 4, 5\} \)-set, which has touching structure, see Figure [1]

**Theorem 4.1 ([13]).** Suppose that dust-like graph-directed sets \( \{E_i\}_{i \in V} \) and \( \{F_i\}_{i \in V} \) have the same graph-directed structure. Then $E_i \sim F_i$ for each $i \in V$.
PROOF. We shall use the notations in (3.1) and (3.2). Since \( \{E_i\}_{i \in V} \) are dust-like, for any \( x \in E_i \), there is a unique infinite path \( e_1 \cdots e_k \cdots \) starting at \( i \) such that
\[
\{x\} = \bigcap_{k=1}^{\infty} \phi_{e_1 \cdots e_k}(E_{i_k})
\]
where the edge \( e_k \) ends at \( i_k \) for every \( k \). We say that \( e_1 e_2 \cdots \) is the coding of \( x \). Hence the mapping \( f : E_i \rightarrow F_i \) defined by
\[
\{f(x)\} = \bigcap_{k=1}^{\infty} \psi_{e_1 \cdots e_k}(F_{i_k}).
\]
is a bijection. It remains to show that \( f \) is bi-Lipschitz.

Suppose \( x, x' \in E_i \). Let \( e_1 e_2 e_3 \cdots \) and \( e'_1 e'_2 e'_3 \cdots \) be the coding of \( x \) and \( x' \), respectively. Let \( m \) be the largest integer such that \( e_1 e_2 \cdots e_m = e'_1 e'_2 \cdots e'_m \). Since both \( x \) and \( x' \) are in the set \( \phi_{e_1 \cdots e_m}(E_{i_m}) \), we have
\[
|x - x'| \leq \min \phi_{e_1 \cdots e_m}(E_{i_m}) = \left( \prod_{i=1}^{m} \rho_{e_i} \right) \text{diam}(E_{i_m}).
\]
On the other hand, by the maximality of \( m \), we have
\[
|x - x'| \geq \min \phi_{e_1 \cdots e_m}(E_{i_{m+1}}), \phi_{e'_1 \cdots e'_{m+1}}(E_{i'_{m+1}}) \geq \left( \prod_{i=1}^{m} \rho_{e_i} \right) \min_{(e, e')} d(\phi_e(E_j), \phi_{e'}(E_j)),
\]
where the minimum is taking over all the pairs \((e, e')\) of distinct edges stemming from a common vertex. For such a pair, let \( j \) and \( j' \) be the end vertices of \( e \) and \( e' \). Since \( e \) and \( e' \) start from a common vertex \( i \), \( \phi_e(E_j) \) and \( \phi_{e'}(E_{j'}) \) are disjoint closed subsets of \( E_i \). Hence the minimum is a positive number.

Therefore, there exists a constant \( c_1 > 0 \) depending only on \( \{E_i\} \) and \( \{\phi_e\} \) such that
\[
c_1^{-1} \left( \prod_{i=1}^{m} \rho_{e_i} \right) \leq |x - x'| \leq c_1 \left( \prod_{i=1}^{m} \rho_{e_i} \right).
\]
Similarly, there exists a constant \( c_2 > 0 \) depending only on \( \{F_i\} \) and \( \{\psi_e\} \) such that
\[
c_2^{-1} \left( \prod_{i=1}^{m} \rho_{e_i} \right) \leq |f(x) - f(x')| \leq c_2 \left( \prod_{i=1}^{m} \rho_{e_i} \right).
\]
It follows that \( c_1^{-1} c_2^{-1} |x - x'| \leq |f(x) - f(x')| \leq c_1 c_2 |x - x'|. \)

Remark 4.2. Theorem 4.1 and its proof are natural extensions of Proposition 1.1.

With the above lemma we can show that the \( \{1, 3, 5\} \)-set and the \( \{1, 4, 5\} \)-set are Lipschitz equivalent.

Proposition 4.3 ([13]). The \( \{1, 3, 5\} \)-set \( M \) and the \( \{1, 4, 5\} \)-set \( M' \) are Lipschitz equivalent.
The touching self-similar set is regular and the contraction vector is ordered. 

\[ T \]  
\[ \rho \]  
\[ \tau \]  
\[ \psi \]  

The answer was given in Xi and Ruan \[ 20 \] dust-like setting, the order of the contractions does make a difference. A complete alization of the \( \{ \psi_i(x) = \rho_i x + t_i \}_{i=1}^n \) be an IFS on \( \mathbb{R} \) satisfying the following three properties: 

1. The subintervals \( \psi_1([0, 1]), \ldots, \psi_n([0, 1]) \) are spaced from left to right without overlapping, i.e. their interiors do not intersect. This means the contraction ratio is ordered. 

2. There exists at least one \( i \in \{1, 2, \ldots, n-1\} \), such that the intervals \( \psi_i([0, 1]) \) and \( \psi_{i+1}([0, 1]) \) are touching, i.e. \( \psi_i(1) = \psi_{i+1}(0) \). 

3. The left endpoint of \( \psi_1[0,1] \) is 0 and the right endpoint of \( \psi_n[0,1] \) is 1. This means the touching is regular. 

Denote by \( T \) the attractor of the IFS \( \Psi \). We call \( T \) a (regular) touching self-similar set with (ordered) contraction vector \( \rho \). In this section, we will always assume that the touching self-similar set is regular and the contraction vector is ordered. 

Denote by \( \mathcal{T}(\rho) \) the family of all touching self-similar sets with contraction vector \( \rho \). We have the following theorem: 

**Theorem 4.4 (13).** Assume that \( \rho = (\rho_1, \ldots, \rho_n) = (\rho, \ldots, \rho) \). Then \( T \sim D \) for every \( T \in \mathcal{T}(\rho) \) and \( D \in \mathcal{D}(\rho) \). 

**4.2. Generalization of the \{1,3,5\} - \{1,4,5\} problem.** A natural generalization of the \{1,3,5\} - \{1,4,5\} problem is when the contraction ratios are no longer uniform. That is, one may consider the Lipschitz equivalence of \( D \in \mathcal{D}(\rho) \) and \( T \in \mathcal{T}(\rho) \), where \( \rho = (\rho_1, \rho_2, \rho_3) \) is a contraction vector in \( \mathbb{R} \). Unlike in the dust-like setting, the order of the contractions does make a difference. A complete answer was given in Xi and Ruan \[ 20 \]. Somewhat surprisingly, it is shown that \( D \) and \( T \) are Lipschitz equivalent if and only if \( \log \rho_1 / \log \rho_3 \) is rational. 

From this result, one naturally asks the following question. 

**Problem 4.5.** Let \( \rho = (\rho_1, \rho_2, \rho_3) \) and \( \tau = (\rho_1, \rho_3, \rho_2) \) be two contraction vectors. Let \( T \in \mathcal{T}(\rho) \) and \( T' \in \mathcal{T}(\tau) \) have initial structure shown as in Figure 3. Under what conditions are \( T \) and \( T' \) Lipschitz equivalent? 

The result in \[ 20 \] is nevertheless a very special case. It is natural to exploit such algebraic and geometric connections further in more general settings. The proof in \[ 20 \] is quite complex, and allying it to the more general setting appears to be very daunting. Recent work by Ruan, Wang and Xi \[ 17 \] has overcome some of the difficulties by introducing a geometric notion called substitutable. It leads to several results that provide insight into the problem. 

Assume that \( \rho = (\rho_1, \ldots, \rho_n) \) is a contraction vector (in \( \mathbb{R} \)) with \( n \geq 3 \). In the rest of this section, we assume that \( D \in \mathcal{D}(\rho) \), \( T \in \mathcal{T}(\rho) \) and \( T \) is the attractor of an IFS \( \Psi = \{ \psi_i(x) = \rho_i x + t_i \}_{i=1}^n \) on \( \mathbb{R} \).
A letter \( i \in \{1, 2, \ldots, n\} \) is a (left) touching letter if \( \psi_i([0, 1]) \) and \( \psi_{i+1}([0, 1]) \) are touching, i.e., \( \psi_i(1) = \psi_{i+1}(0) \). We use \( \Sigma_T \subset \{1, \ldots, n\} \) to denote the set of all (left) touching letters. For simplicity we shall drop the word “left” for \( \Sigma_T \). Let \( \alpha \) be the maximal integer such that \( \bigcup_{i=1}^\alpha \psi_i[0, 1] \) is an interval. Similarly, let \( \beta \) be the maximal integer such that \( \bigcup_{i=n-\beta+1}^n \psi_i[0, 1] \) is an interval.

Given a cylinder \( T_i \) and a nonnegative integer \( k \), we can define respectively the level \((k+1)\) left touching patch and the level \((k+1)\) right touching patch of \( T_i \) to be

\[
L_k(T_i) = \bigcup_{j=1}^\alpha T_{[1]^{\ell}j}, \quad R_k(T_i) = \bigcup_{j=n-\beta+1}^n T_{[n]^{\ell}j},
\]

where \( [\ell]^k \) is defined to be the word \( \ell \ldots \ell \) for any \( \ell \in \{1, \ldots, n\} \), with \( [1]^k j \) as the concatenation of \( i \), \( [1]^k \) and the letter \( j \) (similarly for \( [n]^k j \)). We remark that \( L_0(T_i) = \bigcup_{j=1}^n T_{ij} \) and \( R_0(T_i) = \bigcup_{j=n-\beta+1}^n T_{ij} \).

A letter \( i \in \Sigma_T \) is called left substitutable if there exist \( j \in \Sigma_T^* \) and \( k, k' \in \mathbb{N} \), such that \( \text{diam} \ L_k(T_{i+1}) = \text{diam} \ L_{k'}(T_{ij}) \) and the last letter of \( j \) does not belong to \( \{1\} \cup \Sigma_T + 1 \). Geometrically it simply means that a certain left touching patch of the cylinder \( T_{i+1} \) has the same diameter as that of some left touching patch of a cylinder \( T_{ij} \), and as a result we can substitute one of the left touching patches by the other without disturbing the other neighboring structures in \( T \) because they have the same diameter. Similarly, \( i \in \Sigma_T \) is called right substitutable if there exist \( j \in \Sigma_T^* \) and \( k, k' \in \mathbb{N} \), such that \( \text{diam} \ R_k(T_i) = \text{diam} \ R_{k'}(T_{(i+1)j}) \) and the last letter of \( j \) does not belong to \( \{n\} \cup \Sigma_T \). We say that \( i \in \Sigma_T \) is substitutable if it is left substitutable or right substitutable.

**Remark 4.6.** Both left and right substitutable properties can also be characterized algebraically. By definition, it is easy to check that \( \text{diam} \ L_k(T_{i+1}) = \text{diam} \ L_{k'}(T_{ij}) \) is equivalent to

\[
\rho_{i+1} \rho_i^k = \rho_i \rho_{i+1}^k \rho_j,
\]

while \( \text{diam} \ R_k(T_i) = \text{diam} \ R_{k'}(T_{(i+1)j}) \) is equivalent to

\[
\rho_i \rho_i^k = \rho_{i+1} \rho_i^{k'} \rho_j.
\]

**Example 4.7.** Let \( \rho = (\rho_1, \rho_2, \rho_3) \) with \( \Sigma_T = \{2\} \). Then \( \alpha = 1 \) and \( \beta = 2. \) Assume that \( \log \rho_1 / \log \rho_3 \in \mathbb{Q} \), i.e. there exist \( u, v \in \mathbb{Z}^+ \) such that \( \rho_1^u = \rho_3^v \). Pick \( k = v + 1, k' = 0 \) and \( j = 2[1]^k \). It is easy to check that (4.3) holds for \( i = 2 \) and the last letter of \( j \) is \( \notin \{3\} \cup \Sigma_T \). Thus the touching letter 2 is right substitutable.

Two main results of [17] are listed as follows.
Theorem 4.8 ([17]). Assume that $D \sim T$. Then $\log \rho_1/\log \rho_n \in \mathbb{Q}$.

Theorem 4.9 ([17]). Assume that $\log \rho_1/\log \rho_n \in \mathbb{Q}$. Then, $D \sim T$ if every touching letter for $T$ is substitutable.

Theorem 4.9 allows us to establish a more general corollary. The argument used to show the substitutability in Example 4.7 is easily extended to prove the following corollary:

Corollary 4.10 ([17]). $D \sim T$ if one of the following conditions holds:

1. $\log \rho_i/\log \rho_j \in \mathbb{Q}$ for all $i,j \in \{1, n, \alpha\} \cup (\Sigma_T + 1)$.

2. $\log \rho_i/\log \rho_j \in \mathbb{Q}$ for all $i,j \in \{1, n, n - \beta + 1\} \cup \Sigma_T$.

The following result, which we state as a theorem because of the simplicity of its statement, is a direct corollary of Corollary 4.10.

Theorem 4.11 ([17]). Assume that $\log \rho_i/\log \rho_j \in \mathbb{Q}$ for all $i,j \in \{1, \ldots, n\}$. Then $D \sim T$.

5. Touching IFS and Lipschitz equivalence: Higher dimensional case

Much of the work on Lipschitz equivalence with touching structure is set in $\mathbb{R}$. What about higher dimensions? While many of the results in $\mathbb{R}$ should generalize to higher dimensions, some may not.

Let $Q = [0,1] \times [0,1]$ be the unit square. Given a positive integer $n \geq 3$ and a digit set $\mathcal{D} \subset \{0,1,\ldots,n-1\}^2$, there exists a unique nonempty compact $K \subset Q$ satisfying

$$K = \bigcup_{d \in \mathcal{D}} \frac{1}{n}(K + d).$$

We denote the set $K$ by $K(n, \mathcal{D})$. Xi and Xiong [22] obtained the following result.

Theorem 5.1 ([22]). Assume that $K(n, \mathcal{D}_1)$ and $K(n, \mathcal{D}_2)$ are totally disconnected. Then $K(n, \mathcal{D}_1) \sim K(n, \mathcal{D}_2)$ if and only if $\# \mathcal{D}_1 = \# \mathcal{D}_2$.

Lau and Luo [9], Roinestad [16], and Wen, Zhu and Deng [18] discussed the Lipschitz equivalence of $K(n, \mathcal{D}_1)$ and $K(n, \mathcal{D}_2)$ when at least one of them has touching structure. However, unlike the one dimensional case, $K$ may contain nontrivial connected components which makes the problem much harder.

Problem 5.2. Establish necessary and sufficient conditions for the Lipschitz equivalence of $K(n, \mathcal{D}_1)$ and $K(n, \mathcal{D}_2)$. Clearly we must have $\# \mathcal{D}_1 = \# \mathcal{D}_2$, but in general this is not sufficient. A simple case is $n = 3$ and

$$\mathcal{D}_1 = \{(0,0),(0,1),(0,2),(2,0),(2,2)\},$$

$$\mathcal{D}_2 = \{(0,0),(0,1),(0,2),(2,1),(2,2)\}.$$ See Figure 3. It is not known whether $K(3, \mathcal{D}_1) \sim K(3, \mathcal{D}_2)$.

The sets discussed above are all self-similar. It is natural also to consider the Lipschitz equivalence of self-affine.

Problem 5.3. What happen if the sets are self-affine but not self-similar? For example, when are McMullen carpets Lipschitz equivalent?
Rao, Ruan and Yang [14] defined gap sequences for compact subsets in higher dimensional Euclidean space. [14] also proved that the gap sequence is a Lipschitz invariant. However, we do not know whether gap sequences can be used to prove that two self-similar sets (or self-affine sets) with the same Hausdorff dimension are not Lipschitz equivalent.

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**Department of Mathematics, Hua Zhong Normal University, Wuhan 430079, China**  
*E-mail address*: hrao@mail.ccnu.edu.cn

**Department of Mathematics, Zhejiang University, Hangzhou 310027, China**  
*E-mail address*: ruanhj@zju.edu.cn

**Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA**  
*E-mail address*: ywang@math.msu.edu