$B \to \eta^{(l)}(\ell^-\bar{\nu}_\ell, \ell^+\ell^-, K, K^*)$ decays in the quark-flavor mixing scheme

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Abstract

In the quark-flavor mixing scheme, $\eta$ and $\eta'$ are linear combinations of flavor states $\eta_q = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_s = s\bar{s}$ with the masses of $m_{qq}$ and $m_{ss}$, respectively. Phenomenologically, $m_{ss}$ is strictly fixed to be around 0.69, which is close to $\sqrt{2m_K^2 - m_\pi^2}$ by the approximate flavor symmetry, while $m_{qq}$ is found to be 0.18 ± 0.08 GeV. For a large allowed value of $m_{qq}$, we show that the BRs for $B \to \eta^{(l)}X$ decays with $X = (\ell^-\bar{\nu}_\ell, \ell^+\ell^-)$ are enhanced. We also illustrate that $BR(B \to \eta X) > BR(B \to \eta' X)$ in the mechanism without the flavor-singlet contribution. Moreover, we demonstrate that the decay branching ratios (BRs) for $B \to \eta^{(l)}K^{[s]}$ are consistent with the data. In particular, the puzzle of the large $BR(B \to \eta'K)$ can be solved. In addition, we find that the CP asymmetry for $B^\pm \to \eta K^\pm$ can be as large as $-30\%$, which agrees well with the data. However, we cannot accommodate the CP asymmetries of $B \to \eta K^*$ in our analysis, which could indicate the existence of some new CP violating sources.
I. INTRODUCTION

The branching ratio (BR) of $B^0 \to \eta' K^0$ was first observed by the CLEO collaboration with $(89^{+18}_{-16} \pm 9) \times 10^{-6}$ [1], which is much larger than $(20 - 40) \times 10^{-6}$ estimated by the factorization ansatz [2]. With more data accumulated, this incomprehensible value becomes a real puzzle now that the measurements from BELLE and BABAR depart from the theoretical estimations, where the former has observed $BR(B^+ \to \eta K^+) = (1.9 \pm 0.3^{+0.2}_{-0.1}) \times 10^{-6}$ [3], $BR(B^+ \to \eta' K^+) = (69.2 \pm 2.2 \pm 3.7) \times 10^{-6}$ and $BR(B^0 \to \eta' K^0) = (58.9^{+3.6}_{-3.5} \pm 4.3) \times 10^{-6}$ [4], while the latter has measured $BR(B^+ \to \eta K^+) = (3.3 \pm 0.6 \pm 0.3) \times 10^{-6}$ [5], $BR(B^+ \to \eta' K^+) = (68.9 \pm 2.0 \pm 3.2) \times 10^{-6}$ and $BR(B^0 \to \eta' K^0) = (67.4 \pm 3.3 \pm 3.2) \times 10^{-6}$ [6]. To unravel the mystery, many solutions have been proposed, such as the intrinsic charm in $\eta'$ [7], the gluonium state [8], the spectator hard scattering mechanism [9] and the flavor-singlet component in $\eta'$ [10]. Nevertheless, there are still no conclusive solutions yet.

Recently, the BaBar Collaboration [11] has also measured the semileptonic decays with the data as follows:

\[ BR(B^+ \to \eta \ell^+ \nu_\ell) = (0.84 \pm 0.27 \pm 0.21) \times 10^{-4} < 1.4 \times 10^{-4} (90\% \text{ C.L.}) , \]
\[ BR(B^+ \to \eta' \ell^+ \nu_\ell) = (0.33 \pm 0.60 \pm 0.30) \times 10^{-4} < 1.3 \times 10^{-4} (90\% \text{ C.L.}) . \] (1)

Although the significance of the former in Eq. (1) is 2.55σ, the central value is a factor of 2 larger than $0.4 \times 10^{-4}$ calculated by the light-cone sum rules (LCSRs) [12]. Due to these results, we speculate that the mechanism to enhance the BRs of $B \to \eta' K$ may also affect the semileptonic decays of $B^- \to \eta^{(')} \ell \bar{\nu}_\ell$. After surveying various proposed mechanisms, one finds that only the flavor-singlet mechanism (FSM) [10] could have direct influence on the BRs of semileptonic decays [12, 13]. In this paper, inspired by the measurements of the semileptonic decays, we would like to propose another possible mechanism within the quark-flavor mixing scheme to study the decays of $B \to \eta^{(')} (\ell^+ \ell^-, \ell^- \ell^+, K, K^*)$. We will also compare our results with those in the FSM [10, 12, 13, 14, 15] and explore the differences between the two mechanisms, which could be tested in future B experiments.

The paper is organized as follows. In Sec. II, we review the quark-flavor mixing scheme. In Sec. III, we carry out a general analysis for the decay amplitudes and form factors. Numerical results and discussions are presented in Sec. IV. Our conclusions are given in Sec. V.
II. THE QUARK-FLAVOR MIXING SCHEME

It is known that the physical states $\eta$ and $\eta'$ are composed of the flavor octet $\eta_8$ and singlet $\eta_1$, in which the flavor wave functions are denoted as $\eta_8 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$ and $\eta_1 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$, respectively. Due to the $U_A(1)$ anomaly, it is understood that the mass of $\eta'$ is much larger than that of $\eta$. To satisfy the current experimental data, usually one needs to introduce two angles to the mixing matrix, defined by $\eta = \cos \theta_8 \eta_8 - \sin \theta_1 \eta_1$ and $\eta' = \sin \theta_8 \eta_8 + \cos \theta_1 \eta_1$, to describe the connection between physical and flavor states. However, it is known that by using the two-angle scheme, we will encounter a divergent problem in some $B$ decays, such as $B \to \eta'K$. To illustrate this problem, we notice that in these decays, the factorized parts are associated with the matrix element $\langle 0|\bar{s}i\gamma_5 s|\eta_1\rangle$. From the equation of motion, one has $\langle 0|\partial^\mu \bar{s}\gamma_\mu \eta_8|s\rangle = \langle 0|2m_s \bar{s}i\gamma_5 s|\eta_1\rangle = m_{\eta_1}^2 f_{\eta_1}$, leading to $\langle 0|\bar{s}i\gamma_5 s|\eta_1\rangle = m_{\eta_1}^2 f_{\eta_1}/2m_s$, where $f_{\eta_1}(m_{\eta_1})$ is the decay constant (mass) of $\eta_1$. In the chiral limit of $m_s \to 0$, the matrix element diverges because $m_{\eta_1} \neq 0$. To explicitly display the chiral limit, it is better to use the quark-flavor scheme, defined by $\eta = \cos \theta_8 \eta_8 - \sin \theta_1 \eta_1$ and $\eta' = \sin \theta_8 \eta_8 + \cos \theta_1 \eta_1$.

Clearly, in terms of the quark-flavor basis, $m_{qq}$ and $m_{ss}$ are zero in the chiral limit. We note that $m_{qq}$ and $m_{ss}$ are unknown parameters and their values can be obtained by fitting with the data, such as the masses of $\eta^{(i)}$ and the decay rates of some relevant $B$ decays. Note that $m_{qq,ss}$ are related to $m_{qq,ss}^0$ by $m_{\eta_8}^0 = m_{qq}^2/(m_u + m_d)$, $m_{\eta_s}^0 = m_{ss}^2/2m_s$ and $m_K^0 = m_K^2/(m_s + m_q)$. From the divergences of the axial vector currents

$$\partial^\mu q' \gamma_\mu q' = \frac{G_F}{4\pi} 2m_{qq}^2 q' \gamma_5 q'$$

(4)
where \( G = G^{\mu\nu} \) are the gluonic field-strength and \( \tilde{G} = \tilde{G}^{\mu\nu} = \epsilon^{\mu\alpha\beta} G^{\alpha\beta}_{\mu\nu} \), one obtains the \( \eta_{q,s} \) masses as

\[
\begin{pmatrix}
M^2_{qq} & M^2_{qs} \\
M^2_{sq} & M^2_{ss}
\end{pmatrix} = 
\begin{pmatrix}
\langle 0|\partial^\mu J^\mu_{\mu\delta}|\eta_q\rangle/f_q & \langle 0|\partial^\mu J^\mu_{\mu\delta}|\eta_s\rangle/f_s \\
\langle 0|\partial^\mu J^\mu_{\mu\delta}|\eta_s\rangle/f_q & \langle 0|\partial^\mu J^\mu_{\mu\delta}|\eta_s\rangle/f_s
\end{pmatrix} = 
\begin{pmatrix}
m^2_{qq} + 2a^2 & \sqrt{2}ya^2 \\
\sqrt{2}ya^2 & m^2_{ss} + y^2a^2
\end{pmatrix}
\]

(5)

with \( a^2 = \langle 0|\alpha_s G\tilde{G}|\eta_q\rangle/(4\sqrt{2}\pi f_q) \) and \( y = f_q/f_s \). Furthermore, by using the mixing matrix introduced in Eq. (2), we have

\[
\sin \phi = \left[ \frac{(m^2_{qq} - m^2_{ss})(m^2_{ss} - m^2_{qq})}{(m^2_{qq} - m^2_{ss})(m^2_{ss} - m^2_{qq})} \right]^{1/2},
\]

\[
y = \frac{1}{2} \frac{(m^2_{qq} - m^2_{ss})(m^2_{ss} - m^2_{qq})}{m^2_{ss} - m^2_{qq}},
\]

\[
a^2 = \frac{1}{2} \frac{(m^2_{qq} - m^2_{ss})(m^2_{ss} - m^2_{qq})}{m^2_{ss} - m^2_{qq}},
\]

(6)

where \( m_{q(q)} \) is the mass of \( \eta^{(q)} \).

According to the relations in Eq. (3), it is interesting to see that the parameter \( m_{qq(ss)} \) is involved in the distribution amplitude of the \( \eta_{q(s)} \) state, which is defined by

\[
\langle 0|q''(0)j_q''(z)|\eta_{q'}(p)\rangle = \frac{i}{\sqrt{2N_c}} \int^1_0 dx e^{-ixp_z} \left[ (\gamma_5)_{jk} \phi_{q''_{q'}}(x) + (\gamma_5)_{jk} m^0_{q''_{q'}} \phi^{(a)}_{q''_{q'}}(x) + m^0_{q''_{q'}} [\not{n} - \not{n}_+ - 1]_{jk} \phi^{(b)}_{q''_{q'}}(x) \right],
\]

(7)

where \( q'' = u, d \) and \( s_0 \), \( q' = q \) and \( s_1 \), \( \phi_{q''_{q'}}(x) \) and \( \phi^{(a)}_{q''_{q'}}(x) \) denote the twist-2 and twist-3 wave functions of the \( \eta_{q'} \) state, respectively, \( x \) is the momentum fraction, \( m^0_{q''_{q'}} \) stands for the chiral symmetry breaking parameter, and \( n_+ = (1, 0, 0) \) and \( n_- = (0, 1, 0) \) are defined in the light-cone coordinates. On substituting Eq. (7) into Eq. (3), we obtain \( m^0_{q''_{q'}} = m^2_{qq}/(m_u + m_d) \) and \( m^0_{q''_{q'}} = m^2_{ss}/2m_s \). In the next section, it will be clear that the value of \( m_{qq} \) is crucial for the determination of the \( B \to \eta^{(q)} \) transition form factors, which play important roles in the decay branching ratios of \( B \to \eta^{(q)}X \) with \( X = (\ell^- \bar{\nu}_\ell, \ell^+ \ell^- , K) \).
III. DECAY AMPLITUDES AND FORM FACTORS

We first study the semileptonic decays of $B^- \to \eta^{(*)}\ell^- \nu_\ell$ and $\bar{B} \to \eta^{(*)}\ell^+ \ell^-$ by writing the effective Hamiltonians at quark level in the SM as

$$\mathcal{H}_I = \frac{G_F V_{ub}}{\sqrt{2}} \bar{u} \gamma_\mu (1 - \gamma_5) b \bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell, \quad (8)$$

$$\mathcal{H}_{II} = \frac{G_F \alpha_{em} \lambda_9'}{\sqrt{2\pi}} \left[ H_{1\mu} L^\mu + H_{2\mu} L_t^{\mu} \right], \quad (9)$$

respectively, with

$$H_{1\mu} = C_{9\mu}^{\text{eff}} (\mu) \bar{q} \gamma_\mu P_L b - \frac{2m_b}{q^2} C_7 (\mu) \bar{q} i \sigma_{\mu\nu} q' P_R b,$$

$$H_{2\mu} = C_{10} \bar{q} \gamma_\mu P_L b,$$

$$L^\mu = \bar{\ell} \gamma_\mu \ell, \quad L_t^{\mu} = \bar{\ell} \gamma_\mu \gamma_5 \ell,$$  

(10)

where $\alpha_{em}$ is the fine structure constant, $V_{ij}$ denote the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $\lambda_9' = V_{tb} V_{tb}^*$, $C_i$ are the Wilson coefficients (WCs) with their explicit expressions given in Ref. [22], $m_b$ is the current b-quark mass, $q$ is the momentum transfer and $P_{L(R)} = (1 \mp \gamma_5)/2$. Note that the long-distance effects of $c\bar{c}$ bound states have been included in $C_{9\mu}^{\text{eff}}$, given by [22]

$$C_{9\mu}^{\text{eff}} (\mu) = C_9 (\mu) + (3C_1 (\mu) + C_2 (\mu)) \left( h(z, s) - \frac{3}{\alpha_{em}^2} \sum_{V=\Psi, \eta} k_V \frac{\pi \Gamma (V \to \ell^+ \ell^-) M_V}{M_V^2 - q^2 - i M_V \Gamma_V} \right),$$

$$h(z, s) = -\frac{8}{9} \ln \frac{m_b}{\mu} - \frac{8}{9} \ln z + \frac{8}{27} + \frac{4}{9} x - \frac{2}{9} (2 + x) |1 - x|^{1/2}$$

$$\times \left\{ \begin{array}{ll}
\ln \left| \frac{\sqrt{4 - x + 1}}{\sqrt{4 - x - 1}} \right| - i \pi, & \text{for } x \equiv 4z^2/s < 1, \\
2 \arctan \frac{1}{\sqrt{x-1}}, & \text{for } x \equiv 4z^2/s > 1,
\end{array} \right. (11)$$

where $h(z, s)$ describes the one-loop matrix elements of operators $O_1 = \bar{s}_a \gamma_\mu P_L b \bar{c}_\beta \gamma_\mu P_L c_a$ and $O_2 = \bar{s} \gamma_\mu P_L b \bar{c} \gamma_\mu P_L c$ [22] with $z = m_c/m_b$ and $s = q^2/m_b^2$, $M_V$ ($\Gamma_V$) are the masses (widths) of intermediate states. The hadronic matrix elements for the $B \to P$ transition are parametrized as

$$\langle P(p_P) | \bar{q} \gamma_\mu b | \bar{B}(p_B) \rangle = f_+^P (q^2) \left( P^\mu - \frac{P \cdot q}{q^2} q^\mu \right) + f_0^P (q^2) \frac{P \cdot q}{q^2} q'_\mu,$$

$$\langle P(p_P) | \bar{q} i \sigma_{\mu\nu} q' b | \bar{B}(p_B) \rangle = \frac{f_T^P (q^2)}{m_B + m_P} [P \cdot q q'_\mu - q^2 P_\mu],$$

(12)
with $P$ representing the pseudoscalar, $P_\mu = (p_B + p_P)_\mu$, $q_\mu = (p_B - p_P)_\mu$ and $f^P_\alpha(q^2)$ are form factors. Consequently, the transition amplitudes associated with the interactions in Eqs. (8) and (9) can be expressed as

$$M_I = \frac{\sqrt{2}G_FV_{ub}f^P_+(q^2)\bar{\ell} P\ell}{\pi},$$

$$M_{II} = \frac{G_F\alpha_{em}N_C}{\sqrt{2}\pi} [\bar{m}_{\gamma\ell}\bar{\ell} P\ell + \bar{m}_{10}\bar{\ell} P\ell\gamma_5\ell]$$

for $\bar{B} \to P\ell^-\bar{\nu}_\ell$ and $\bar{B} \to P\ell^+\ell^-$, respectively, where

$$\bar{m}_{97} = C_{9}f^P_+(q^2) + \frac{2m_b}{m_B + m_P}C_{7}f^P_+(q^2), \quad \bar{m}_{10} = C_{10}f^P_+(q^2).$$

The differential decay rates for $B^- \to P\ell^-\bar{\nu}_\ell$ and $B_d \to P\ell^+\ell^-$ as functions of $q^2$ are given by [12]

$$\frac{d\Gamma_I}{dq^2} = \frac{C_{9}^2|V_{ub}|^2m_B^2}{3 \cdot 2^6\pi^3} \sqrt{(1 - s + \bar{m}_P^2)^2 - 4\bar{m}_P^2} \left(f^P_+(q^2)\bar{P}_P\right)^2,$$

$$\frac{d\Gamma_{II}}{dq^2} = \frac{C_{9}^2\alpha_{em}m_B^3}{3 \cdot 2^9\pi^5} |\lambda'|^2 \sqrt{(1 - s + \bar{m}_P^2)^2 - 4\bar{m}_P^2\bar{P}_P^2} \left(|\bar{m}_{97}|^2 + |\bar{m}_{10}|^2\right),$$

respectively, with $\bar{P}_P = 2\sqrt{s|\bar{q}_P|}/m_B = \sqrt{(1 - s - \bar{m}_P^2)^2 - 4s\bar{m}_P^2}$. Since we concentrate on the production of the light leptons, we have neglected the terms explicitly related to the lepton mass. We note that due to $C_9 >> C_7$, the effect associated with the form factor of $f^P_+(q^2)$ in Eq. (15) is small. From Eqs. (16) and (17), we see clearly that the semileptonic decays are only sensitive to the form factor $f^P_+(q^2)$. By this property, we can use the data of $B^- \to \eta\ell^-\bar{\nu}_\ell$ to constrain the unknown parameters in the calculations of the form factors. The constrained parameters could make some predictions for the decays $B \to \eta(0)\ell^+\ell^-$ and $B \to \eta(0)K$.

In the large recoil region, i.e. $q^2 \to 0$, the form factors can be evaluated by the perturbative QCD (PQCD) [24, 25] approach, in which the transverse momenta of valence quarks are included to remove the end-point singularities when $x \to 0$. Hence, in terms of Eq. (7) and the flavor diagrams shown in Fig. 1 the form factors $f^P_+(q^2)$, $f^P_-(q^2)$, and $f^P_0(q^2)$ for $B \to P$ can be formulated as [23]

$$f^P_+(q^2) = f^P_1(q^2) + f^P_2(q^2),$$

$$f^P_0(q^2) = f^P_1(q^2)\left(1 + \frac{q^2}{m_B^2}\right) + f^P_2(q^2)\left(1 - \frac{q^2}{m_B^2}\right),$$

$$f^P_-(q^2) = f^P_1(q^2)\left(1 - \frac{q^2}{m_B^2}\right) + f^P_2(q^2)\left(1 + \frac{q^2}{m_B^2}\right).$$
where

\[
\begin{align*}
    f_1^P(q^2) &= 8\pi C_F m_B^2 r_P \int_0^1 [dx] \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \left[ \phi_P^+(x_2) - \phi_P^t(x_2) \right] \\
    &\quad \times E(t^1) h(x_1, x_2, b_1, b_2), \\
    f_2^P(q^2) &= 8\pi C_F m_B^2 \int_0^1 [dx] \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \\
    &\quad \times \left\{ \left[ (1 + x_2\xi) \phi_P(x_2) + 2r_P \left( \left\{ \frac{1 - \xi}{\xi} - x_2 \right\} \phi_P^t(x_2) - x_2 \phi_P^+(x_2) \right) \right] \\
    &\quad \times E(t^1) h(x_1, x_2, b_1, b_2) + 2r_P \phi_P^+(x_2) E(t^2) h(x_2, x_1, b_1, b_2) \right\} , \quad (19)
\end{align*}
\]

and

\[
\begin{align*}
    f_T^P(q^2) &= 8\pi C_F m_B^2 (1 + m_P/m_B) \int_0^1 [dx] \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \\
    &\quad \times \left\{ \left[ \phi_P(x_2) - r_P x_2 \phi_P^+(x_2) + r_P \left( \frac{2}{\xi} + x_2 \right) \phi_P^t(x_2) \right] E(t^1) h(x_1, x_2, b_1, b_2) \\
    &\quad + 2r_P \phi_P^t(x_2) E(t^2) h(x_2, x_1, b_1, b_2) \right\} , \quad (20)
\end{align*}
\]

with \( C_F = 4/3 \), \( \xi = 1 - q^2/m_B^2 \) and \( r_P = m_P^0/m_B \). From Eq. (18), we find that \( f_+(0) = f_0(0) \). The evolution factor is given by \( E(t) = \alpha_s(t) \exp(-S_B(t) - S_P(t)) \) where the Sudakov exponents \( S_{B,P} \) can be found in Ref. [26]. The hard function \( h \) is written as

\[
\begin{align*}
    h(x_1, x_2, b_1, b_2) &= S_t(x_2) K_0(\sqrt{x_1 x_2} m_B b_1) \\
    &\quad \times \left[ \theta(b_1 - b_2) K_0(\sqrt{x_2} m_B b_1) I_0(\sqrt{x_2} m_B b_2) \\
    &\quad + \theta(b_2 - b_1) K_0(\sqrt{x_2} m_B b_2) I_0(\sqrt{x_2} m_B b_1) \right], \quad (21)
\end{align*}
\]

where the threshold resummation effect is described by \( S_t(x) = 2^{1+2c}\Gamma(\frac{3}{2} + c)[x(1 - x)]^c/\sqrt{\pi}\Gamma(1 + c) \) with \( c = 0.3 \) [25]. The hard scales \( t^{(1,2)} \) are chosen to be [27]

\[
\begin{align*}
    t^{(1)} &= \max(\sqrt{m_B^2 x_2^2}, 1/b_1, 1/b_2, \Lambda), \\
    t^{(2)} &= \max(\sqrt{m_B^2 x_1^2}, 1/b_1, 1/b_2, \Lambda),
\end{align*}
\]

FIG. 1: Flavor diagrams for the \( B \to P \) transition with \( \Gamma^\mu = (\gamma^\mu, i\sigma^{\mu\nu} q_\nu) \).
where $\bar{\Lambda}$ is used to exclude the effects from nonperturbative contributions. To get the BRs for the three-body semileptonic decays, besides the values of the form factors at $q^2 = 0$, we also need to know their $q^2$ dependences. To obtain them, we adopt the fitting results calculated by the light-cone sum rules (LCSR) \cite{28}, given by

$$f^P_{+(T)}(q^2) = \frac{f^P_{+(T)}(0)}{(1 - q^2/m_{B^*}^2)(1 - \alpha_{+(T)}q^2/m_{B^*}^2)}$$ \tag{22}

with $\alpha_{+(T)} = 0.52(0.84)$ and $m_{B^*} = 5.32$ GeV. In terms of the quark-flavor mixing scheme, we will calculate the $B \to \eta_{q,s}$ form factors, which are related to those of $B \to \eta^{(l)}$ by

$$f^\eta_{+(T)}(q^2) = \frac{\cos \phi}{\sqrt{2}} f^{\eta q}_{+(T)}(q^2),$$

$$f^{\eta^*}_{+(T)}(q^2) = \frac{\sin \phi}{\sqrt{2}} f^{\eta q}_{+(T)}(q^2).$$ \tag{23}

For the nonleptonic decays of $B \to \eta^{(l)}K$, we will assume the color-transparency \cite{29}, i.e., no rescattering effects in $B$ decays. The effective interaction for the $b \to s$ transition at the quark level is given by \cite{22}

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{q=u,c} V_q \left[ C_1(\mu)O_{1(q)}^{(0)}(\mu) + C_2(\mu)O_{2(q)}^{(0)}(\mu) + \sum_{i=3}^{10} C_i(\mu) O_i(\mu) \right],$$ \tag{24}

where $V_q = V_{qs}^* V_{qb}$ are the CKM matrix elements and the operators $O_1$-$O_{10}$ are defined as

$$O_1^{(q)} = (\bar{q}_a^q q_\beta)_{V-A} (\bar{q}_\beta b_\alpha)_{V-A}, \quad O_2^{(q)} = (\bar{q}_a^q q_\alpha)_{V-A} (\bar{q}_\beta b_\beta)_{V-A},$$

$$O_3 = (\bar{q}_a^q b_\alpha)_{V-A} \sum_q (\bar{q}_\beta q_\beta)_{V-A}, \quad O_4 = (\bar{q}_a^q b_\beta)_{V-A} \sum_q (\bar{q}_\beta q_\alpha)_{V-A},$$

$$O_5 = (\bar{q}_a^q b_\alpha)_{V-A} \sum_q (\bar{q}_\beta q_\beta)_{V+A}, \quad O_6 = (\bar{q}_a^q b_\beta)_{V-A} \sum_q (\bar{q}_\beta q_\alpha)_{V+A},$$

$$O_7 = \frac{3}{2} (\bar{q}_a^q a_\alpha)_{V-A} \sum_q e_q(\bar{q}_\beta q_\beta)_{V+A}, \quad O_8 = \frac{3}{2} (\bar{q}_a^q b_\beta)_{V-A} \sum_q e_q(\bar{q}_\beta q_\alpha)_{V+A},$$

$$O_9 = \frac{3}{2} (\bar{q}_a^q b_\alpha)_{V-A} \sum_q e_q(\bar{q}_\beta q_\beta)_{V-A}, \quad O_{10} = \frac{3}{2} (\bar{q}_a^q b_\beta)_{V-A} \sum_q e_q(\bar{q}_\beta q_\alpha)_{V-A},$$ \tag{25}

with $\alpha$ and $\beta$ being the color indices. In Eq. \cite{24}, $O_1$-$O_2$ are from the tree level of weak interactions, $O_3$-$O_6$ are the so-called gluon penguin operators and $O_7$-$O_{10}$ are the electroweak penguin operators, while $C_i \ (i = 1, 2, \cdots, 10)$ are the corresponding WCs. Using the unitarity condition, the CKM matrix elements for the penguin operators $O_3$-$O_{10}$ can also be expressed as $V_u + V_c = -V_t$. To study the nonleptonic decays, we will encounter the transition matrix elements such as $\langle P_1 P_2 | H_{\text{eff}} | B \rangle = \langle P_1 P_2 | V_q C_i O_i | B \rangle$. To describe the B decay...
amplitudes, we have to know not only the relevant effective weak interactions but also all possible topologies for the specific process. In Fig. 2 we display the flavor diagrams for $B_d \rightarrow \eta_{q(s)} K$ decays, in which (a)-(c), (d)-(e) and (f) illustrate penguin emission, penguin annihilation and tree emission topologies, respectively. Since the $b$-quark is dictated by the weak charged current, its chirality is always left-handed. However, the chiralities for $q\bar{q}$ pairs, produced by gluon, Z-boson and photon penguins, could be both left and right-handed, resulting in processes containing both $V - A$ and $V + A$ currents. In Fig. 2 we have explicitly labeled the associated type of currents except the diagram (f) which is from the tree and only has the left-handed interaction. Note that although we use the states $\eta_{q,s}$ as our basis, the physical states can be easily obtained by using Eq. (2). For the charged $B$ decays, besides the flavor diagrams displayed in Fig. 2 three more diagrams arising from tree emission and annihilation topologies need to be included as shown in Fig. 3. From

FIG. 2: Flavor diagrams for $B_d \rightarrow \eta^{(0)} K^0$ decays: (a)-(e) stand for the penguin contributions while (f) is the tree contribution, where $V \mp A$ denote the left-hand and right-handed currents, respectively.

FIG. 3: Flavor diagrams arising from tree emission and annihilation for charged $B$ decays.
Figs. 2 and 3, the decay amplitudes for $B^{0,+} \rightarrow \eta_q K^{(*)0,+}$ and $B \rightarrow \eta_q K^{(*)0,+}$ are given by

\begin{align}
A_q^0 &= V_t \left( F_{Pa}^0 + N_{Pa}^0 + F_{Pc}^0 + N_{Pc}^0 + F_{Pd}^0 + N_{Pd}^0 \right) - V_u \left( F_{Tf}^0 + N_{Tf}^0 \right), \\
A_s^0 &= V_t \left( F_{P(b+c)}^0 + N_{P(b+c)}^0 + F_{Pe}^0 + N_{Pe}^0 \right),
\end{align}

(26)

and

\begin{align}
A_q^+ &= V_t \left( F_{Pa}^+ + N_{Pa}^+ + F_{Pc}^+ + N_{Pc}^+ + F_{Pd}^+ + N_{Pd}^+ \right) \\
&\quad - V_u \left( F_{T(f+g)}^+ + N_{T(f+g)}^+ + F_{Th}^+ + N_{Th}^+ \right), \\
A_s^+ &= V_t \left( F_{P(b+c)}^+ + N_{P(b+c)}^+ + F_{Pe}^+ + N_{Pe}^+ \right) - V_u \left( F_{Tt}^+ + N_{Tt}^+ \right),
\end{align}

(27)

where $V_t = V_{tb}V_{ts}^* = -\lambda^2$ and $V_u = V_{ub}V_{us}^* = \lambda^4 R_b e^{-i\phi_3}$. $F_{P(k)}^{0,+}$ and $N_{P(k)}^{0,+}$ represent the penguin factorized and nonfactorized contributions for the topology $k$, and $F_{T(k)}^{0,+}$ and $N_{T(k)}^{0,+}$ are the tree factorized and nonfactorized effects, respectively. The lengthy formulas for various factorizable and nonfactorizable parts can be found in Refs. [26, 30]. We note that for simplicity we have used the same notations for the $\eta_{q,s}K$ and $\eta_{q,s}K^*$ modes. Furthermore, from Eq. (2) the physical decays can be written as

\begin{align}
A(B^{0,+} \rightarrow \eta K^{(*)}) &= \frac{\cos \phi}{\sqrt{2}} A_q^{0,+} - \frac{\sin \phi}{\sqrt{2}} A_s^{0,+}, \\
A(B^{0,+} \rightarrow \eta' K^{(*)}) &= \frac{\sin \phi}{\sqrt{2}} A_q^{0,+} + \frac{\cos \phi}{\sqrt{2}} A_s^{0,+}.
\end{align}

(28)

The decay BRs and CP asymmetries (CPAs) are given by

\begin{align}
BR(B^{0,+} \rightarrow \eta^{(*)} K^{(*)}) &= \frac{G_F^2 |\bar{p}| m_B^2 \tau_B^{B^{0,+}}}{16\pi} |A(B^{0,+} \rightarrow \eta^{(*)} K^{(*)})|^2, \\
A_{CP}(B \rightarrow \eta^{(*)} K^{(*)}) &= \frac{BR(B \rightarrow \eta^{(*)} K^{(*)}) - BR(B \rightarrow \eta^{(*)} K^{[*]})}{BR(B \rightarrow \eta^{(*)} K^{[*]}) + BR(B \rightarrow \eta^{(*)} K^{[*]})},
\end{align}

(29)

(30)

which can be evaluated in terms of Eqs. (26), (27) and (28), where $|\bar{p}| = \sqrt{E_K^2 - m_K^2}$ and $E_K = (m_B^2 - m_{\eta_q}^2 + m_{\eta_q}^2)/2m_B$.

**IV. Numerical Results and Discussions**

In the PQCD approach, if we regard the meson wave functions as known objects, the remaining unknown theoretical quantities are the chiral symmetry breaking parameters of states $\eta_{q,s}$ and $K$, denoted by $m_{\eta_{q,s},K}^0$, and the meson decay constants $f_{B,\eta_{q,s},K}$. It is known
that $f_K$ has been determined quite precisely to be around 0.16 GeV by experiment, while the lattice QCD calculations give $f_B = 0.216 \pm 0.022$ GeV [31], which is consistent with the extracted value from the decay $B^- \rightarrow \tau \bar{\nu}_\tau$ measured by Belle [32]. By low-energy experiments, the decay constants of $\eta_{q,s}$ are found to be $f_{\eta_q} = (1.07 \pm 0.02)f_\pi$ and $f_{\eta_s} = (1.34 \pm 0.06)f_\pi$ [19], respectively. Basically, the undetermined parameters in our considerations are the parameters $m_{qq}$ and $m_{ss}$. To obtain the allowed range for $m_{qq,ss}$ in a model-independent way, we adopt the phenomenological approach. The parameters in Eq. (6) are limited to be $\phi = 39.3^\circ \pm 1.0^\circ$, $y = 0.81 \pm 0.03$ and $a^2 = 0.265 \pm 0.010$ [19]. With these values, the allowed ranges for $m_{qq}$ and $m_{ss}$ are presented in Fig. 4. From the figure, we find that $m_{ss}$ has a narrow allowed window around 0.69 GeV, which can be understood in terms of the flavor symmetry, given by $m_{ss} = \sqrt{2m_K^2 - m_\pi^2}$ [20]. However, $m_{qq}$ is relatively broader, given by $0.18 \pm 0.08$ GeV. To do the numerical estimations, we take $f_B = 0.19$ GeV, $f_{\eta_q} = 0.14$ GeV and $\phi = 39.3^\circ$ as the input values. For the nonperturbative wave functions, we use the results derived by the LCSR for the light mesons [28], while for $B$ meson wave function, we use

$$\phi_B(x) = N_B x^2(1 - x)^2 \exp \left[ -\frac{m_B^2 x^2}{2\omega_B^2} \right] \exp \left[ -\frac{\omega_B^2 x^2}{2} \right]$$ (31)

with $N_B = 111.2$ and $\omega_B = 0.38$ [26]. Accordingly, we get the $B \rightarrow K$ form factor of $f_+^K(0)$, defined in Eq. (12), to be 0.36. From Eq. (23), we show the form factors $f_{+T}^{\eta_q}(0)$ in Table I. From the table, we see clearly that they will be enhanced with increasing $m_{qq}$. In addition, it is easy to understand that the behavior $f_{+T}^{\eta_q}(0) \geq f_{+T}^{\eta}(0)$ is always satisfied as seen from Eq. (23) due to $\cos \phi > \sin \phi$ with $\phi \sim 39.3^\circ$. This property is different from that in the

FIG. 4: The allowed ranges for $m_{qq}$ and $m_{ss}$.
TABLE I: \( f_+^{\eta}(0) \) and \( f_+^{\eta}(0) \) with three allowed values of \( m_{qq} \).

| \( m_{qq} \) (GeV) | \( f_+^{\eta}(0) \) | \( f_+^{\eta}(0) \) | \( f_+^{\eta}(0) \) | \( f_+^{\eta}(0) \) |
|------------------|------------------|------------------|------------------|------------------|
| 0.14             | 0.14             | 0.14             | 0.12             | 0.11             |
| 0.18             | 0.21             | 0.20             | 0.18             | 0.17             |
| 0.22             | 0.29             | 0.29             | 0.24             | 0.24             |

FSM, given by \[10\]

\[
\begin{align*}
\eta^0(0) & = \cos \phi \frac{f_q}{f_\pi} f_\pi^\eta(0) + \frac{1}{\sqrt{3}} \left( \sqrt{2} \cos \phi \frac{f_q}{f_\pi} - \sin \phi \frac{f_s}{f_\pi} \right) f_i^{\text{sing}}(0), \\
\eta'(0) & = \sin \phi \frac{f_q}{f_\pi} f_\pi^\eta(0) + \frac{1}{\sqrt{3}} \left( \sqrt{2} \sin \phi \frac{f_q}{f_\pi} + \cos \phi \frac{f_s}{f_\pi} \right) f_i^{\text{sing}}(0),
\end{align*}
\]

where \( f_i^{\text{sing}}(0) \) (\( i = +, T \)) correspond to the new form factors due to the flavor singlet state. Based on \( f_+^{\pi}(0) \approx f_+^{T}(0) \approx 0.26 \) calculated by the LCSRs \[28\], we present the numerical results of Eq. (32) in Table II. From the table, we see that \( f_+^{\eta}(0) < f_+^{\eta}(0) \) in the FSM. Furthermore, by using \(|V_{ub}| = 3.5 \times 10^{-3}, |V_{td}| = 8.1 \times 10^{-3}\) \[33\], Eqs. (16), (17) and (22) and the values in Tables I and II, we show the semileptonic decay BRs in Table III. From the table, we find that the results in both approaches could be consistent with the data of \( B^- \to \eta(0)\ell\nu_\ell \). On the other hand, in our approach, we always predict \( BR(B^- \to \eta^{'+}\ell^- + \nu_\ell) > BR(B^- \to \eta^{'}\ell^+\ell^-) \), whereas the inequality is reversed in the FSM. Similar conclusion can be also drawn for the processes of \( B_d \to \eta(0)\ell^+\ell^- \). We note that the BRs are insensitive to the parametrizations displayed in Eq. (22) \[12\].

We now give our numerical analysis for the nonleptonic decays \( B \to \eta(0)K^{(*)} \). By using the PQCD approach, the values of factorized and nonfactorized contributions for the \( B \) decays are shown in Table IV. Based on these values and \( V_{ts} = -0.041 \) and \( V_{ub} = 4.6 \times 10^{-3}e^{-i\phi_3} \)
TABLE III: BRs of $B^{-} \rightarrow \eta^{(0)} \ell \bar{\nu}_{\ell}$ (in units of $10^{-4}$) and $\bar{B}_{d} \rightarrow \eta^{(0)} \ell^{+} \ell^{-}$ (in units of $10^{-7}$) with $m_{qq} = 0.14, 0.18$ and $0.22$ GeV in our mechanism and $f_{+}^{\text{sing}}(0) = 0.0, 0.1$ and $0.2$ in the FSM. 

$\phi = 39.3^\circ$, the predictions for $\text{BR}(B \rightarrow \eta^{(0)} K^{[s]})$ and $A_{CP}(B \rightarrow \eta^{(0)} K^{[s]})$ are given in Table V and Table VI, respectively. Our results can be summarized as follows:

- From Table V, we see clearly that with $m_{qq} = 0.22$ GeV, the BRs for $B \rightarrow \eta^{(0)} K^{[s]}$ are consistent with the WA data. It is interesting to note that by increasing $m_{qq}$, $\text{BR}(B \rightarrow \eta^{(0)} K)$ tend to be small (large), while $\text{BR}(B \rightarrow \eta^{(0)} K^{*})$ to be large (small), favored by the experiments.

- As seen from Table V with the same value of $m_{qq}$, $\text{BR}(B \rightarrow \eta K) < O(10^{-1})\text{BR}(B \rightarrow \eta' K)$, while $\text{BR}(B \rightarrow \eta K^{*}) > \text{BR}(B \rightarrow \eta' K^{*})$. The phenomena could be ascribed to the signs in the amplitudes of $B \rightarrow (\eta_{q}, \eta_{s}) K^{(*)}$ by comparing Eqs. (26), (27) and (28) with the specific values of $F_{Pa}^{0,+}$ and $F_{P(b+c)}^{0,+}$ in Table IV.

- From Table VI we find that for $m_{qq} = 0.22$ GeV $A_{CP}(B_{u} \rightarrow \eta K^{+})$ is as large as $-30\%$, which agrees well with the data, whereas the other two sets of $m_{qq}$ lead to positive and small asymmetries. In addition, our prediction for $A_{CP}(B_{d} \rightarrow \eta K^{*0})$ is too small, while that of $A_{CP}(B_{u} \rightarrow \eta K^{*+})$ is too large, in comparison with the data. If future experiments display the current tendencies for these CPAs, such phenomena will become new puzzles.

Finally, we remark that in the quark-flavor scheme, as the errors in the decay constants of $f_{q}$ and $f_{s}$ are only $2\%$ and $4\%$, respectively, their effects on BRs and CPAs are mild.
TABLE IV: Factorizable and nonfactorizable parts for the decays $B \rightarrow \eta_{q,s}K^*$ with $m_{qq} = 0.22$ GeV, where the values in the square brackets are for $B \rightarrow \eta_{q,s}K^*_{ss}$.

|            | $F^0_{Pa}10^2$ | $N^0_{Pa}10^5$ | $F^0_{P(b+c)}10^2$ | $N^0_{P(b+c)}10^4$ | $F^0_{Pc}10^2$ | $N^0_{Pc}10^4$ |
|------------|----------------|----------------|-------------------|-------------------|----------------|----------------|
| $\phi$     | -1.10          | 6.36 - $i2.06$ | -0.55             | -0.33 + $i1.22$   | 0.26            | -7.17 + $i3.39$ |
| $\xi$      | [-0.42]        | [3.89 - $i2.37$] | [0.45]            | [-0.89 + $i1.74$] | [0.19]         | [-7.88 + $i2.56$] |

|            | $F^0_{Pd}10^3$ | $N^0_{Pd}10^5$ | $F^0_{Pc}10^3$ | $N^0_{Pc}10^5$ | $F^0_{Pf}10^2$ | $N^0_{Pf}10^4$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\phi$     | -0.61 + $i2.43$ | -5.77 - $i9.62$ | -0.44 + $i1.25$ | -0.51 - $i4.62$ | -0.61          | 3.61 - $i1.57$  |
| $\xi$      | [-0.19 + $i2.37$] | [-5.05 - $i3.36$] | [0.30 - $i1.86$] | [-5.02 - $i9.06$] | [0.41]         | [4.00 - $i1.29$] |

|            | $F^+_{Pa}10^2$ | $N^+_{Pa}10^5$ | $F^+_{P(b+c)}10^2$ | $N^+_{P(b+c)}10^4$ | $F^+_{Pc}10^2$ | $N^+_{Pc}10^4$ |
|------------|----------------|----------------|-------------------|-------------------|----------------|----------------|
| $\phi$     | -1.05          | 3.55 - $i0.27$ | -0.54             | -3.56 + $i1.71$   | 0.21            | -6.24 + $i1.70$ |
| $\xi$      | [-0.43]        | [-1.86 - $i2.61$] | [0.45]            | [-0.89 + $i1.74$] | [0.188]         | [-7.64 + $i3.53$] |

|            | $F^+_{Pd}10^3$ | $N^+_{Pd}10^5$ | $F^+_{Pc}10^3$ | $N^+_{Pc}10^5$ | $F^+_{Pf}10^2$ | $N^+_{Pf}10^4$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\phi$     | -0.63 + $i2.20$ | -2.86 - $i4.46$ | -0.50 + $i1.60$ | -1.59 - $i2.98$ | -0.45          | 3.27 - $i0.90$  |
| $\xi$      | [-0.09 + $i2.37$] | [-2.21 - $i0.72$] | [0.29 - $i1.83$] | [-2.83 - $i4.07$] | [-0.41]         | [3.85 - $i1.74$] |

|            | $F^+_{Tg}10^2$ | $N^+_{Tg}10^3$ | $F^+_{Th}10^3$ | $N^+_{Th}10^3$ | $F^+_{Tf}10^3$ | $N^+_{Tf}10^3$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\phi$     | 10.03          | -1.16 + $i0.18$ | 2.38 + $i0.02$  | 0.99 + $i1.38$  | -1.02 - $i0.02$ | 0.27 + $i1.13$  |
| $\xi$      | [11.60]        | [-1.55 + $i0.06$] | [-2.19 - $i1.14$] | [1.09 + $i0.70$] | [1.61 + $i0.95$] | [0.89 + $i1.55$] |

However, the influence from the mixing angle $\phi$ could be larger. We present the results with the error of $\phi$ in Table VII.

V. CONCLUSIONS

Due to the current experimental limits on the mixing parameters of the $\eta$ and $\eta'$ mesons, we have studied the phenomenologically allowed ranges for $m_{ss}$ and $m_{qq}$. Explicitly, we have found that $m_{ss}$ is around 0.69 GeV and $m_{qq} = 0.18 \pm 0.08$ GeV. We have shown that the semileptonic decays of $B^- \rightarrow \eta^{(0)}\ell \bar{\nu}_\ell$ are sensitive to $m_{qq}$ and thus they can provide strong constraints on its value. In addition, our mechanism based on the quark-flavor mixing scheme naturally leads to $f^\eta_+(0) < f^\eta_0(0)$ as well as $BR(B^- \rightarrow \eta\ell^{-}\bar{\nu}_\ell) > BR(B^- \rightarrow \eta'\ell^{-}\bar{\nu}_\ell)$, in contrast with the reversed inequalities in the FSM due to the flavor-singlet contribution.
TABLE V: BR(B → η(ℓ⁺)K[++]π−) (in units of 10⁻⁶) with m_{qq} = 0.14, 0.18 and 0.22 GeV as well as the world average (WA) values [34].

| m_{qq} | B_d → ηK^0 | B_d → η'K^0 | B_u → ηK^+ | B_u → η'K^+ |
|--------|-------------|-------------|------------|------------|
| 0.14   | 3.01        | 31.44       | 5.66       | 34.60      |
| 0.18   | 0.28        | 44.01       | 1.26       | 47.36      |
| 0.22   | 1.43        | 62.69       | 1.52       | 65.04      |
| WA     | < 1.9       | 64.9 ± 3.5  | 2.2 ± 0.3  | 69.7 ± 2.8 |

TABLE VI: A_{CP}(B → η(ℓ⁺)K[++]π⁻) (in unit of 10⁻²) with m_{qq} = 0.14, 0.18 and 0.22 GeV as well as the world average (WA) values [34].

| m_{qq} | B_d → ηK^0 | B_d → η'K^0 | B_u → ηK^+ | B_u → η'K^+ |
|--------|-------------|-------------|------------|------------|
| 0.14   | -2.10       | 0.69        | 5.62       | -5.28      |
| 0.18   | -2.47       | 0.57        | 5.88       | -6.19      |
| 0.22   | 4.41        | 0.48        | -30.64     | -6.88      |
| WA     | --          | --          | -29 ± 11   | 3.1 ± 2.1  |

| m_{qq} | B_d → ηK^0 | B_d → η'K^0 | B_u → ηK^+ | B_u → η'K^+ |
|--------|-------------|-------------|------------|------------|
| 0.14   | 0.79        | -0.82       | -15.79     | 8.39       |
| 0.18   | 0.67        | -0.98       | -20.51     | 8.83       |
| 0.22   | 0.57        | -1.30       | -24.57     | 4.60       |
| WA     | 19 ± 5      | -8 ± 25     | 2 ± 6      | 30^{+33}_{-37} |

[12, 13]. Similar conclusions can also be drawn for the decays B_d → η(ℓ⁺)ℓ⁻. It is interesting to note that the future measurements on BR(B^− → η(ℓ⁺)ℓ⁻) and BR(B_d → η(ℓ⁺)ℓ⁻) can be used to distinguish the two flavor mechanisms. Moreover, we have shown that
TABLE VII: BRs (in units of $10^{-6}$) and CPAs (in units of $10^{-2}$) for $B \to \eta^{(i)}K^{[s]}$ decays with $m_{qq} = 0.22$ GeV and $\phi = 39.3^\circ \pm 1.0^\circ$.

| Obs. | $B_d \to \eta K^0$ | $B_d \to \eta' K^0$ | $B_u \to \eta K^+$ | $B_u \to \eta' K^+$ |
|------|-------------------|-------------------|-------------------|-------------------|
| BR   | $1.43^{+0.34}_{-0.31}$ | $62.69^{+0.30}_{-0.34}$ | $1.52^{+0.16}_{-0.13}$ | $65.04^{+0.12}_{-0.15}$ |
| $A_{CP}$ | $4.41^{+0.57}_{-0.44}$ | $0.48 \pm 0.009$ | $-30.64^{+4.12}_{-2.87}$ | $-6.88^{+0.13}_{-0.12}$ |

| Obs. | $B_d \to \eta K^{*0}$ | $B_d \to \eta' K^{*0}$ | $B_u \to \eta K^{*+}$ | $B_u \to \eta' K^{*+}$ |
|------|-------------------|-------------------|-------------------|-------------------|
| BR   | $22.31^{+0.28}_{-0.29}$ | $3.35^{+0.29}_{-0.27}$ | $22.13^{+0.26}_{-0.27}$ | $6.38 \pm 0.26$ |
| $A_{CP}$ | $0.57 \pm 0.011$ | $-1.30 \pm 0.08$ | $-24.57^{+0.72}_{-0.27}$ | $4.60^{+1.16}_{-1.32}$ |

$BR(B \to \eta^{(i)}X)$ with $X = (\ell^-\bar{\nu}_\ell, \ell^+\ell^-)$ are enhanced and in particular, the puzzle of the large $BR(B \to \eta'K)$ can be solved with a reasonable large value of $m_{qq}$. We have also demonstrated that $A_{CP}(B^{\pm} \to \eta K^{\pm})$ can be as large as $-30\%$ and $BR(B \to \eta^{(i)}K^{*})$ are consistent with the current data. Finally, we remark that our results for $A_{CP}(B \to \eta K^{*})$ do not agree with the experimental values. According to our analysis, currently, they are the most incomprehensible phenomena. Other mechanisms as well as more precise measurements are needed for a complete description of all the above decays.

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