Threefolds near the Noether line - Part 1
joint work with S. Coughlan [C], Y. Hu [H], T. Zhang [Z]

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Overview

1. Volume, genus and Noether inequalities
2. Fibrations
3. Threefolds on the Noether line
4. Their moduli spaces
Volume and genus

Let $X$ be a complex projective variety whose singularities are at worst canonical. Let $n$ be the dimension of $X$.

Definition

The genus of $X$ is $p_g(X) := h^0(X, K_X)$.

The volume of $X$ is

$$\text{vol}(X) := n! \limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^n}$$

A variety is of general type if and only if its volume is positive. $X$ is a canonical model if the singularities of $X$ are canonical and $K_X$ is ample.

Both genus and volume are birational invariants. If $X$ is a canonical model then

$$\text{vol}(X) = K^n_X.$$
Volume and genus: Noether inequality

If $X$ is a curve ($n = 1$) then $2\mathbb{N} \ni \text{vol}(X) = 2p_g(X) - 2$.
If $X$ is a surface ($n = 2$) we have the Noether inequality

$$\mathbb{N} \ni \text{vol}(X) \geq 2p_g(X) - 4$$

If $X$ is a threefold ($n = 3$) we have the Noether inequality\(^1\)

**Theorem 1.** Let $X$ be a projective $3$-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 11$. Then

$$\text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$ 

[M. Chen, Y. Hu, C. Jiang 2024]: The inequality holds also for $p_g = 5$. Moreover $\text{vol}(X) \in \mathbb{Q}$ may be not integral if the canonical model of $X$ is not Gorenstein.

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\(^1\)Chen, Jungkai A.; Chen, Meng; Jiang, Chen; *The Noether inequality for algebraic $3$-folds*. With an appendix by János Kollár, Duke Math. J. **169** (2020), no.9, 1603–1645.
The canonical surfaces $X$ with $K_X^2 = 2p_g(X) - 4$ are usually referred to as Horikawa surfaces because Horikawa described\(^2\) their moduli space.

Horikawa writes

> In Sections 3 and 5 we shall prove that minimal algebraic surfaces with given $p_g$ and $c_1^2$ satisfying $c_1^2 = 2p_g - 4$ and $p_g \geq 3$ have one and the same deformation type provided that $c_1^2$ is not divisible by 8.

> In Section 7 we shall study the case in which $c_1^2$ is divisible by 8. If we fix $c_1^2$, these surfaces are divided into two deformation types. They are homo-

By this classification, if $p_g \geq 7$, then there is a fibration $f : X \to \mathbb{P}^1$ with fibres of genus 2.

\(^2\)Horikawa, Eiji; *Algebraic surfaces of general type with small $C_1^2$. I.* Ann. of Math. (2) 104 (1976), no.2, 357–387.
Genus 2 fibrations

Let \( f : X \rightarrow B \) be a genus 2 fibration, that is a surjective morphism onto a projective curve \( B \) whose general fibre is a curve of genus 2. Let \( F \) be a fibre.

By the theory of genus 2 fibrations, the canonical ring of \( F \), that is the ring \( \bigoplus_d H^0(F, dK_F) \) is of one of the following two forms:

| If \( F \) is 2–connected | If \( F = C_1 + C_2 \) with \( C_1 C_2 = 1 \) |
|---------------------------|-----------------------------------------------|
| \( \frac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)} \) | \( \frac{\mathbb{C}[x_0, x_1, y, z]}{f_2(x_0, x_1), f_6(x_0, x_1, y, z)} \) |
| \( \deg x_j = 1 \) \quad \deg z = 3 \quad \deg f_6 = 6 \) | \( \deg x_j = 1 \) \quad \deg y = 2 \quad \deg z = 3 \quad \deg f_2 = 2 \quad \deg f_6 = 6 \) |
Invariants of genus 2 fibrations

If $X$ is moreover a canonical surface, then\(^3\)

$$\text{vol}(X) = 2p_g(X) - 4 + 6b + \deg \tau - 2h^1(X, \mathcal{O}_X)$$

where $b$ is the genus of the base curve $B$, and $\tau$ is an effective divisor on $B$ supported on the image of the 2-disconnected fibres, those of the form $C_1 + C_2$ with $C_1 C_2 = 1$.

By an inequality of Debarre, if $K_X^2 < 2p_g$, then $h^1(X, \mathcal{O}_X) = 0$, which in turn implies $b = 0$.

In particular

$$K_X^2 = 2p_g(X) - 4 \iff h^1(X, \mathcal{O}_X) = 0 \text{ and } \tau = 0.$$  

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\(^3\)Catanese, F., Pignatelli, R.; *Fibrations of low genus, I.* Annales Scientifiques de l’École Normale Supérieure, **39**, 6 (2006), 1011–1049
Simple genus 2 fibrations

We can reformulate more precisely the previous remark as follows:

**Definition**

A simple genus 2 fibration is a morphism $f : X \to B$ between projective varieties of respective dimension 2 and 1 such that

1. $B$ is smooth;
2. all singularities of $X$ are canonical;
3. $K_X$ is $f$–ample;
4. for all $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form $\mathbb{C}[x_0, x_1, z]/f_6(x_0, x_1, z)$, with $\deg x_i = 1$, $\deg z = 3$, $\deg f_6 = 6$.

**Theorem**

Let $X$ be a canonical surface with $p_g \geq 7$. Then $K_X^2 = 2p_g - 4$ if and only if $X$ is a regular\textsuperscript{a} simple genus 2 fibration.

\textsuperscript{a}regular means $h^1(X, \mathcal{O}_X) = 0$
Fibrations in (1,2)-surfaces

The proof of the Noether inequality for threefolds shows that, if the volume is not far from $\frac{4}{3}p_g - \frac{10}{3}$ then there exists a fibration

$$f : X \rightarrow \mathbb{P}^1$$

such that the general fibre of $f$ is a (1, 2)–surface.

**Definition**

A (1, 2)-surface is a canonical surface with $K^2 = 1$ and $p_g = 2$. They are the hypersurfaces of degree 10 in $\mathbb{P}(1, 1, 2, 5)$ with at worst canonical singularities.
Simple fibrations in \((1, 2)\)-surfaces

However, the previous analysis suggests the following definition

**Definition (CP 2023\(^a\))**

\(^a\)Coughlan, S.; Pignatelli, R.; *Simple fibrations in (1,2)-surfaces*, Forum of Mathematics, Sigma, Volume 11, 2023, e43

A simple fibration in \((1, 2)\)–surfaces is a morphism \(f : X \to B\) between projective varieties of respective dimension 3 and 1 such that

1. \(B\) is smooth;
2. all singularities of \(X\) are canonical;
3. \(K_X\) is \(f\)–ample;
4. for all \(p \in B\), the canonical ring of the fibre \(X_p = f^{-1}p\) is of the form

\[
\frac{\mathbb{C}[x_0, x_1, y, z]}{f_{10}(x_0, x_1, y, z)}
\]

\(^a\)deg \(x_i = 1\), deg \(y = 2\), deg \(z = 5\), deg \(f_{10} = 10\)
A simple fibration in \((1, 2)\)-surfaces is **regular** if and only if \(h^1(X, \mathcal{O}_X) = 0\). In fact,
\[
h^1(X, \mathcal{O}_X) = 0 \iff B \cong \mathbb{P}^1
\]
Regular and/or Gorenstein simple fibrations in (1, 2)-surfaces

A simple fibration in (1, 2)-surfaces is **regular** if and only if $h^1(X, \mathcal{O}_X) = 0$. In fact

$$h^1(X, \mathcal{O}_X) = 0 \iff B \cong \mathbb{P}^1$$

A simple fibration in (1, 2)-surfaces is **Gorenstein** if and only if $K_X$ is Cartier. Recall that each fibre is a double cover of a quadric cone $\mathbb{P}(1, 1, 2)$ branched on the vertex $v$ and a curve $\Gamma$. Then

$$K_X \text{ is Cartier} \iff \text{on each fibre of } f, \ v \notin \Gamma$$

In other words, $X$ is Gorenstein if and only if any fibre $F$ does not contain the point $(0, 0, 1, 0)$ of $\mathbb{P}(1, 1, 2, 5)$.
Simple fibrations vs Noether inequality

Theorem (CP 2023)

Let $f : X \to B$ be a simple fibration in $(1, 2)$–surfaces. Suppose that $X$ is Gorenstein and regular. Then $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.

Conversely

Theorem (CHPZ 2024)

Coughlan, S.; Hu, Y.; Pignatelli, R.; Zhang, T.; Threefolds on the Noether line and their moduli spaces, arXiv:2409.17847

Suppose that $X$ is a canonical threefold with $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.

1. If $p_g \geq 23$ then $X$ is a Gorenstein regular simple fibration in $(1, 2)$–surfaces.

2. If $p_g \geq 11$ then there is a Gorenstein regular simple fibration in $(1, 2)$–surfaces $f : Y \to \mathbb{P}^1$ with canonical model $X$ and either the map $Y \to X$ is an isomorphism or it contracts a section of $f$ to a point.
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**Theorem (CP 2023)**

Let $f : X \to B$ be a simple fibration in $(1, 2)$–surfaces. Suppose that $X$ is **Gorenstein** and **regular**. Then $K_X^3 = \frac{4}{3} p_g - \frac{10}{3}$.

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Strategy of the proof of the last theorem (CHPZ 2024)

Suppose that $X$ is a canonical threefold with $K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3}$. Assume that $p_g(X) \geq 11$.

[HZ2022]⁴: We can choose a minimal model $X_1$ of $X$ so that $X_1$ admits a fibration $X_1 \to \mathbb{P}^1$ whose general fiber is a smooth $(1,2)$-surface. We know that $X_1$ is Gorenstein.

⁴Hu, Y.; Zhang, T.: *Algebraic threefolds of general type with small volume* Mathematische Annalen, to appear, arXiv: 2204.02222.
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Now replace $X_1 \rightarrow \mathbb{P}^1$ with its relative canonical model $X_0 \rightarrow \mathbb{P}^1$. Let $F_p$ be the fibre over any point $p$. Then $F_p$ is Gorenstein, and by the use of standard exact sequences it has $p_g = 2$, $K^2 = 1$.

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If $F_p$ is stable, by a Theorem of Franciosi, Pardini and Rollenske, $F_p$ is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$.

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4 Hu, Y.; Zhang, T.: *Algebraic threefolds of general type with small volume* Mathematische Annalen, to appear, arXiv: 2204.02222.

Roberto Pignatelli (Trento)  Threefolds near the Noether line I  SCMS, September 29th, 2024  13 / 23
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Suppose that $X$ is a canonical threefold with $K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3}$. Assume that $p_g(X) \geq 11$.

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Not all of our fibres are stable. However we could apply their argument: key steps are showing that $h^1(F_p, nK_{F_p}) = 0$ for all $n \in \mathbb{N}$ and that there is an integral curve $C \in |K_{F_p}|$.

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Now replace $X_1 \to \mathbb{P}^1$ with its relative canonical model $X_0 \to \mathbb{P}^1$. Let $F_p$ be the fibre over any point $p$. Then $F_p$ is Gorenstein, and by the use of standard exact sequences it has $p_g = 2$, $K^2 = 1$.

If $F_p$ is stable, by a Theorem of Franciosi, Pardini and Rollenske, $F_p$ is a hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$.

Not all of our fibres are stable. However we could apply their argument: key steps are showing that $h^1(F_p, nK_{F_p}) = 0$ for all $n \in \mathbb{N}$ and that there is an integral curve $C \in |K_{F_p}|$.

Then $X_0$ is a simple fibration in $(1, 2)$-surfaces and we can apply the results on them in [CP2023].

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4 Hu, Y.; Zhang, T.: *Algebraic threefolds of general type with small volume* Mathematische Annalen, to appear, arXiv: 2204.02222.
$\mathbb{P}(1, 1, 2, 5)$-bundles

Every simple fibration in $(1, 2)$-surfaces $X \to B$ is naturally embedded as a divisor in a $4$–fold $\mathbb{F}$ having a morphism on $B$ whose fibres are all weighted projective spaces $\mathbb{P}(1, 1, 2, 5)$. $X$ has relative degree $10$.

Some of these $\mathbb{F}$ are toric varieties.
Consider $\mathbb{C}^6$ with coordinates $t_0, t_1, x_0, x_1, y, z$.
Consider the toric $4$-fold $\mathbb{F} = \mathbb{C}^6 // (\mathbb{C}^*)^2$ defined by the weight matrix

$$
\begin{pmatrix}
  t_0 & t_1 & x_0 & x_1 & y & z \\
  1 & 1 & a_0 & a_1 & b & c \\
  0 & 0 & 1 & 1 & 2 & 5
\end{pmatrix}
$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$.
Then $(t_0, t_1)$ defines a bundle $f : \mathbb{F} \to \mathbb{P}^1$ in weighted projective spaces $\mathbb{P}(1, 1, 2, 5)$. 
Divisors in $\mathbb{P}(1, 1, 2, 5)$-bundles

**Theorem (CP 2023)**

Let $f : X \rightarrow \mathbb{P}^1$ be a Gorenstein regular simple fibration in $(1, 2)$–surfaces. Then $X =: X(d; d_0)$ is a divisor in the toric 4–fold $\mathbb{C}^6 / (\mathbb{C}^*)^2$ with weight matrix

$$
\begin{pmatrix}
t_0 & t_1 & x_0 & x_1 & y & z \\
1 & 1 & d - d_0 & d_0 - 2d & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 5
\end{pmatrix}
$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$, given by a bihomogeneous (of bidegree $(0, 10)$) equation of the form

$$z^2 = y^5 + \ldots$$

So we know that all canonical 3-folds on the Noether line, at least for $p_g \geq 23$ (or $p_g \geq 11$ if we allow a small modification), are $X(d; d_0)$ for some integer $d, d_0 \in \mathbb{Z}$. 
Fix $d, d_0$. The threefolds $X(d; d_0)$ form an unirational family of threefolds. Direct computations show:

1. The general $X =: X(d; d_0)$ has only canonical singularities iff

\[
\frac{1}{4} d \leq d_0 \leq \frac{3}{2} d
\]

2. If $\min(d, d_0) \geq 1$, then

\[
p_g(X) = 3d - 2 \quad h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0 \quad K_X^3 = 4d - 6
\]

3. $K_X$ is big and nef iff $\min(d; d_0) \geq 2$ and ample iff $\min(d; d_0) \geq 3$

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5. Else $d = d_0 = 0$ and $X = S \times \mathbb{P}^1$.
6. Note that this shows that $d$ is a deformation invariant.
7. The cases with $d_0 = 2$ are those where we need the small contraction.
The canonical image of $X(d; d_0)$

The image $\Sigma$ of the canonical map of $X(d; d_0)$, a subvariety of $\mathbb{P}^{3d-1}$, depends from $d_0$.

In fact, a basis from $H^0(X, K_X)$ is given by the monomials

$$t_0^{d_0-2}x_0, t_0^{d_0-3}t_1x_0, \ldots, t_1^{d_0-2}x_0, t_0^{3d-d_0-2}x_1, t_0^{3d-d_0-3}t_1x_1, \ldots, t_1^{3d-d_0-2}x_1$$

So $\Sigma$ is

- a rational normal curve if $d_0 = 1$.
- the cone over a rational normal curve if $d_0 = 2$
- the Hirzebruch surface $\mathbb{F}_e$ with $e = 3d - 2d_0$ if $d_0 \geq 3$
The singularities of the general $X(d; d_0)$

Recall that the general $X(d; d_0)$ has at worst canonical singularities iff
\[
\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d \iff \frac{5}{2}d \geq e \geq 0.
\]
More precisely the singular locus of the general $X(d; d_0)$ is

1. Empty if
\[
d \leq d_0 \leq \frac{3}{2}d \iff d \geq e \geq 0
\]

2. $8d_0 - 7d$ terminal singularities\(^8\) if
\[
\frac{7}{8}d < d_0 < d \iff \frac{5}{4}d > e > d
\]

3. Empty if
\[
d_0 = \frac{7}{8}d \iff e = \frac{5}{4}d
\]

4. A section of $f: X \to \mathbb{P}^1$ if
\[
\frac{1}{4}d \leq d_0 < \frac{7}{8}d \iff \frac{5}{2}d \leq e < \frac{5}{4}d
\]

\(^8\)locally of the form $z^2 + y^5 + x_1t = 0$
Components of the moduli space containing smooth 3-folds

Theorem (CP 2023)

Assume $d \geq 3$ (equiv. $p_g \geq 7$).

The threefolds $X(d; d_0)$ with $d \leq d_0 \leq \frac{3}{2}d$ belongs all to the same irreducible component of the moduli space of canonical threefolds that I call the KCH\(^{a}\) component.

Moreover $X \left( d; \left\lfloor \frac{3}{2}d \right\rfloor \right)$ is an open subset of the KCH component.

The threefolds $X(d; d_0)$ with $d_0 = \frac{7}{8}d$ belong to a different irreducible component.

\(^{a}\)Threefolds in this component, say the KCH component, had been found first by Y. Chen and Y. Hu, generalizing work of Kobayashi.

The second component shows up only when $d$ is divisible by 8. This is a 3-dimensional version of the ”second component” in the moduli space of the Horikawa surfaces, showing up only when $K^2$ is divisible by 8.
The explicit algebraic description of these threefolds allowed us to prove:

**Theorem (CP 2023)**

*The general element of the KCH component is a Mori Dream Space.*

In fact we proved that the general $X(d; d_0)$ is a Mori Dream Space for all $d \leq d_0 \leq \frac{3}{2}d$.

We still do not know if an analogous statement holds for $d_0 < d$. 
Components of the moduli space not containing smooth 3-folds

Theorem (CHPZ 2024)

Assume $\frac{d}{4} \leq d_0 \leq \frac{25d-3}{26}$. Then the threefolds $X(d; d_0)$ form an irreducible component of the moduli space of threefolds of general type.
Components of the moduli space not containing smooth 3-folds

Theorem (CHPZ 2024)

Assume $\frac{d}{4} \leq d_0 \leq \frac{25d-3}{26}$. Then the threefolds $X(d; d_0)$ form an irreducible component of the moduli space of threefolds of general type.

The idea of the proof is that, if a threefold $X(d, d_0)$ is a degeneration of a family of threefolds $X(d, d'_0)$, then their canonical image $\mathbb{F}_{3d-2d_0}$ is a degeneration of a family of surfaces $\mathbb{F}_{3d-2d'_0}$, which implies $d_0 \leq d'_0$. A direct computation shows that, when $d_0 < d$, the modular dimension of the family of the threefolds $X(d; d_0)$ is strictly decreasing as a function of $d_0$, so the general $X(d; d_0)$ cannot be specialization of any $X(d; d'_0)$ with $d'_0 < d$.

If $d_0 \leq \frac{25d-3}{26}$ this dimension is also bigger than the dimension of the KCH component, completing the proof.
The moduli space

Even if we are not able to determine if the threefolds $X(d; d_0)$ with $\frac{25d-3}{26} < d_0 < d$ form an irreducible component of the moduli space or are specialisation of threefolds in the KCH component, we obtain

**Corollary (CHPZ 2024)**

The number of irreducible components of the moduli space of canonical threefolds with $p_g \geq 11$, $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ is at most $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ and at least $\left\lfloor \frac{p_g+8}{78} \right\rfloor$.

This in contrast with Horikawa's result in dimension 2. However, it is analogous to the huge number$^9$ of irreducible components of the moduli space of stable surfaces with $K^2 = 2p_g - 4$.

$^9$Rana, J., Rollenske, S. *Standard stable Horikawa surfaces*, Algebraic Geometry 11 (4), 2024, 569–592
The case \( \min(d, d_0) = 1 \)

We also studied the threefolds \( X(d, d_0) \) with \( \min(d, d_0) = 1 \),
\[
\frac{1}{4} d \leq d_0 \leq \frac{3}{2} d.
\]
So \( d_0 = 1, 1 \leq d \leq 4 \).

We have found that \( X(1; 1) \) has Kodaira dimension zero, whereas \( X(2; 1), X(3; 1) \) and \( X(4; 1) \) are of general type.

Set then \( X^+(d; 1) \) for the canonical model of \( X(d; 1), d = 2, 3, 4 \). Then

| \( d \) | \( p_g \) | \( K^3_{X^+} \) | Singularities of the general \( X^+(d; 1) \) |
|-----|-----|----------------|------------------------------------------------|
| 2   | 4   | \( 2 + \frac{1}{4} \) | \( 2 \times \frac{1}{2}(1, 1, 1), \frac{1}{4}(1, 3, 3) \) |
| 3   | 7   | \( 6 + \frac{1}{14} \) | \( \frac{1}{2}(1, 1, 1), \frac{1}{7}(3, 4, 6) \) |
| 4   | 10  | \( 10 + \frac{1}{30} \) | \( \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 2, 2), \frac{1}{5}(1, 4, 4) \) |

The threefolds \( X^+(3; 1) \) and \( X^+(4; 1) \) were already in literature \(^{10}\).

\(^{10}\) Chen, M., Jiang, C., Li, B.; \textit{On minimal varieties growing from quasi-smooth weighted hypersurfaces}. J. Differential Geom. \textbf{127} (1), 2024, 35–76