Purely affine elementary $su(N)$ fusions

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Abstract

We consider three-point couplings in simple Lie algebras – singlets in triple tensor products of their integrable highest weight representations. A coupling can be expressed as a linear combination of products of finitely many elementary couplings. This carries over to affine fusion, the fusion of Wess-Zumino-Witten conformal field theories, where the expressions are in terms of elementary fusions. In the case of $su(4)$ it has been observed that there is a purely affine elementary fusion, i.e., an elementary fusion that is not an elementary coupling. In this note we show by construction that there is at least one purely affine elementary fusion associated to every $su(N > 3)$.

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1 Introduction

It is well-known that the set of elementary (three-point) couplings of any simple Lie algebra is finite. It is natural to expect that three-point fusions may also be expressed in terms of non-negative linear combinations of products of finitely many elementary fusions (EFs). A fusion may be labelled by the weights \((\lambda, \mu, \nu)\) of the associated coupling, and its threshold level \(t: (\lambda, \mu, \nu)_t\). It has been conjectured \(^1\) that fusion multiplicities are uniquely determined from the tensor product multiplicities \(T_{\lambda, \mu, \nu}\) and the associated multi-set of minimum levels \(\{t\}\) at which the various couplings first appear. Therefore, to the triplet \((\lambda, \mu, \nu)\) there correspond \(T_{\lambda, \mu, \nu}\) distinct couplings, hence \(T_{\lambda, \mu, \nu}\) values of \(t\), one for each distinct coupling. These values are called threshold levels. The threshold levels associated to two different couplings may be identical.

All elementary couplings must be EFs when assigned the associated threshold levels. However, there may in general be more EFs than elementary couplings. The extra EFs will be called purely affine EFs. The only known example is in \(su(4)\) \(^5\), and it reads

\[(\Lambda^1 + \Lambda^3, \Lambda^1 + \Lambda^3, \Lambda^1 + \Lambda^3)_2\]

Here \(\{\Lambda^i\}_{i=1,...,r}\) is the set of fundamental weights, and \(r\) is the rank of the Lie algebra. The tensor product multiplicity is two:

\[T_{\Lambda^1 + \Lambda^3, \Lambda^1 + \Lambda^3, \Lambda^1 + \Lambda^3} = 2\]

and the set of threshold levels is \(\{2, 3\}\).

The objective of the present work is the extension of this result to higher rank. Thus, we shall prove that for every \(su(r + 1)\), \(r \geq 3\),

\[(\Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r)_2\]

is a purely affine EF. As for \(su(4)\), the tensor product multiplicity is two:

\[T_{\Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r} = 2\]

and the set of threshold levels is \(\{2, 3\}\).

A knowledge of EFs is important, for example, when constructing three-point functions in Wess-Zumino-Witten theories along the lines discussed in \(^6\). There and here we use a formalism where couplings are represented by polynomials, and the threshold levels correspond to particular properties of those. We construct the polynomial for \((3)\) in sect. 3 below.

The existence of \((3)\) as a purely affine elementary fusion may also be proved just by counting fusion multiplicities, as we will show. The corresponding polynomial contains more information, however, and in the process of constructing it, we will explain why purely affine elementary fusions can appear. On the other hand, the simpler counting argument is useful in that it can be much more easily adapted to algebras besides \(su(N)\), as we will indicate.

\(^1\)Let us review here the status of this conjecture. Modulo rigour, it follows from the Gepner-Witten depth rule \(^2\). A rigorous proof would follow from a refinement of the depth rule conjectured in \(^3\). A version of this formula was independently proved in \(^4\) in the context of vertex operator algebras (VOAs). The only thing missing at this point is the proof that the fusion rules of affine VOAs, as calculated in \(^5\), are identical to those given by the Verlinde formula.
2 Background

Here we shall provide the necessary background and fix our notation, by discussing certain results on realizations of Lie algebras, their highest weight modules, tensor products, affine fusion, threshold levels, and elementary couplings.

2.1 Differential operators and polynomial realizations

A general and convenient differential operator realization of any simple Lie algebra (in particular $su(r + 1)$) was provided in \cite{9, 10}. It is given in terms of the flag variables (or triangular coordinates) of the Lie algebra – there is one independent parameter $x^\alpha$ for every positive root $\alpha > 0$. Let $\lambda$ be the highest weight of the representation, then according to \cite{9, 10},

\[
E_\alpha(x, \partial) = \sum_{\beta > 0} V^\beta_\alpha(x) \partial_\beta
\]

\[
H_i(x, \partial, \lambda) = \sum_{\beta > 0} V^\beta_i(x) \partial_\beta + \lambda_i
\]

\[
F_\alpha(x, \partial, \lambda) = \sum_{\beta > 0} V^\beta_{-\alpha}(x) \partial_\beta + \sum_{j=1}^{r} P^j_{\alpha}(x) \lambda_j
\]

is a differential operator realization of the Lie algebra. $V$ and $P$ are polynomials in the flag variables $x$ provided in \cite{9, 10}. Here we only need to know explicitly

\[
E_\theta(x, \partial) = \partial_\theta
\]

\[
P^j_{\alpha}(x) = \left[e^{-C(x)}\right]^{-\alpha}_j
\]

where $C^b_\alpha(x) = -\sum_{\beta > 0} x^\beta f_{\beta, a}^b$ with $f_{a,b}^c$ being a structure constant. $\theta$ is the highest root. If $\text{wt}(x^\alpha ... x^\beta) = \alpha + ... + \beta$ and $\text{wt}$ is linear, it follows that $P$ is homogeneous in its weight, and that it is $\text{wt}(P^j_{\alpha}(x)) = \alpha$.

One further utilization of the flag variables is to represent states in highest weight modules $M_\lambda$ \cite{10}, or tensor products thereof \cite{3, 8}, as polynomials. For example, the polynomial (or wave function) associated to the state

\[
F_\alpha \cdots F_\beta |\lambda\rangle
\]

is

\[
b_{\alpha,...,\beta}(x, \lambda) = F_\alpha(x, \partial, \lambda) \cdots F_\beta(x, \partial, \lambda) 1
\]

Likewise, the polynomial associated to

\[
F_{\alpha(1)} \cdots F_{\beta(n)} |\lambda^{(1)}\rangle \otimes \cdots \otimes F_{\alpha(n)} \cdots F_{\beta(n)} |\lambda^{(n)}\rangle
\]

is the following ordinary product of polynomials in the $n$ sets of flag variables $x_1, ..., x_n$:

\[
b_{\alpha(1),...,\beta(1)}(x_1, \lambda^{(1)}) \cdots b_{\alpha(n),...,\beta(n)}(x_n, \lambda^{(n)})
\]
We are interested in the linear combinations of states like (9) that correspond to the singlet, and we shall focus mainly on the case \( n = 3 \). That corresponds to considering three-point couplings (to the singlet).

In [6], we showed how the polynomial description above is ideally suited to the implementation of the Gepner-Witten depth rule [2] of affine fusion. The rule encodes the level-dependence. Incorporating the idea of threshold levels, the key point is the following. A polynomial associated to a (three-point) coupling may be assigned the threshold level given by the highest power of \( x_\theta \), using \( x_\theta^l = x_\theta^l \) for all \( l = 1, 2, 3 \). That follows essentially from the differential operator realization of \( E_\theta \) [3].

Certain relations (or syzygies) among different couplings and their decompositions into elementary couplings, complicate the determination of the multi-set of threshold levels. We will have more to say about that below.

2.2 Elementary couplings and threshold levels

Let \( E \) denote a generic element in the set \( \mathcal{E} \) of elementary (three-point) couplings. The associated elementary polynomial is \( R^E(x_1, x_2, x_3) \). Consider a decomposition of the coupling \((\lambda, \mu, \nu)\) on \( \mathcal{E} \):

\[
(\lambda, \mu, \nu) = \prod_{E \in \mathcal{E}} E^{c_E}
\]

where \( c_E \) are non-negative integers. We indicate a multiplication of couplings by \( \times \), e.g., \( E \times E' \) and \( E^{\times 2} = E \times E \). The polynomial associated to (11) enjoys the factorization

\[
R^{\lambda, \mu, \nu}(x_1, x_2, x_3) = \prod_{E \in \mathcal{E}} \left( R^E(x_1, x_2, x_3) \right)^{c_E}
\]

It follows that we can assign the threshold level

\[
t(R^{\lambda, \mu, \nu}) = \sum_{E \in \mathcal{E}} c_E \ t(R^E)
\]

to (12). Actually, a similar result holds even if the factorization (12) is not on \( \mathcal{E} \).

Decompositions like (11) are in general not unique. That results in certain algebraic relations among the elementary couplings (and so among the elementary polynomials) that must be taken into account. These relations are sometimes called syzygies. They can be implemented by excluding certain combinations of the elementary couplings (or products of the elementary polynomials). The resulting number of allowed decompositions (11) is the associated tensor product multiplicity. Here we do not need any detailed knowledge of that procedure.

Having obtained the \( T_{\lambda, \mu, \nu} \) linearly independent polynomials (12) associated to the triplet \((\lambda, \mu, \nu)\), \( \{R^{\lambda, \mu, \nu}(x_1, x_2, x_3)\}_{s=1, \ldots, T_{\lambda, \mu, \nu}} \), we are still faced with the problem of determining the multi-set of threshold levels. Naively, one could just “measure” the threshold levels of each of the polynomials \( R^{\lambda, \mu, \nu} \), but that would in general produce a multi-set of values slightly greater than the actual multi-set of threshold levels. The reason is found in the fact that \( \{R^{\lambda, \mu, \nu}_s\} \) is merely a convenient basis for the description of the tensor product. When discussing fusion, i.e., when also determining the multi-set of threshold levels, one might need a different basis. Any such basis will consist of linear combinations of the original basis elements \( R^{\lambda, \mu, \nu}_s \). Now, it is obvious that the threshold level of the sum of polynomials in general will be equal

\[ \]
to the maximum value of the individual threshold levels. Nevertheless, it may occur that a linear combination of polynomials will have a smaller maximum power of \( x^\theta \) than any of the participating polynomials. That is precisely what we shall encounter below.

A subset of the elementary three-point couplings is governed by the two-point couplings, since a two-point coupling is a three-point coupling with one of the three weights being zero. The set of elementary two-point couplings is easily described. It consists of the couplings \((\Lambda^i, \Lambda^{i+}, 0), i = 1, ..., r\), and permutations thereof. Thus, the number of independent, elementary two-point couplings is \(3r\).

Two- and three-point functions in Wess-Zumino-Witten theories were discussed in [7] and [8], respectively. They are essentially constructed by working out a basis \(\{R_{\lambda, \mu, \nu}^s\}\) of the associated tensor products. Their level-dependence was considered in [6].

3 Elementary fusions

When discussing fusion above, the procedure has been to decompose into elementary couplings, implement the syzygies, and eventually consider linear combinations of polynomials in order to determine the multi-set of threshold levels. Here we advocate an alternative approach where extra EFs are introduced [5, 6]. That will increase the number of syzygies and the complexity of their implementation, but result in a good basis \(\{R_{s}^{\lambda, \mu, \nu}\}\) for discussing fusion. Thus, the merit is that one may avoid the final step of considering linear combinations of polynomials. The multi-set of threshold levels is simply given by the threshold levels of the basis polynomials: \(\{t\} = \{t(R_{s}^{\lambda, \mu, \nu})\}\). It is natural to conjecture that only a finite number of such extra purely affine EFs will be needed.

Implicit in our discussion has been that all elementary couplings are also elementary fusions. One only needs to assign their threshold levels. However, that is easy to show using our polynomial correspondence. We also note that any coupling \((\lambda, \mu, \nu)\) with \(T_{\lambda, \mu, \nu} = 1\) has a unique (up to normalization) polynomial realization \(R_{s}^{\lambda, \mu, \nu}\). The associated threshold level is therefore given unambiguously by \(t(R_{s}^{\lambda, \mu, \nu})\). We stress, though, that not all elementary couplings correspond to tensor products of unit multiplicity. \(G_2\) provides a simple example [11]: \(T_{2\Lambda_2^2, 2\Lambda_2^2, \Lambda_1^1 + \Lambda_2^2} = 2\), and one of the two couplings (both with \(t = 3\)) is elementary.

It is evident that the only coupling of threshold level 0 is \((0, 0, 0)\). Thus, all other couplings have positive threshold levels.

3.1 \(su(r + 1)\) fusion: polynomials

In the following we will concentrate on \(su(r+1)\). We recall that \(\Lambda^{i+} = \Lambda^{r+1-i}\) for all \(i = 1, ..., r\). Since \(\theta = \Lambda^1 + \Lambda^r\), we shall pay particular attention to the two fundamental highest weight modules \(M_{\Lambda^1}\) and \(M_{\Lambda^r}\). All weights in the modules appear with multiplicity one, so choosing a basis is straightforward. We have made the following simple choice of bases, and listed their associated polynomials:

\[
|\Lambda^1\rangle \leftrightarrow 1, \quad F_{\alpha_1} |\Lambda^1\rangle \leftrightarrow P_{-\alpha_1}^1, \quad \ldots, \quad F_{\alpha_1,j} |\Lambda^1\rangle \leftrightarrow P_{-\alpha_1,j}^1, \quad \ldots, \quad F_\theta |\Lambda^1\rangle \leftrightarrow P_{-\theta}^1 \tag{14}
\]

and

\[
|\Lambda^r\rangle \leftrightarrow 1, \quad F_{\alpha_r} |\Lambda^r\rangle \leftrightarrow P_{-\alpha_r}^r, \quad \ldots, \quad F_{\alpha_r,j} |\Lambda^r\rangle \leftrightarrow P_{-\alpha_r,j}^r, \quad \ldots, \quad F_\theta |\Lambda^r\rangle \leftrightarrow P_{-\theta}^r \tag{15}
\]
Here we have labeled the positive roots as

$$\alpha_{i,j} = \alpha_i + \ldots + \alpha_j , \quad 1 \leq i \leq j \leq r$$

(16)

Using the commutators

$$[E_{\alpha_{i,j}}, E_{\alpha_{k,l}}] = \delta_{k-j,1} E_{\alpha_{i,l}} - \delta_{i-l,1} E_{\alpha_{k,j}}$$

(17)

and the well-known symmetries of the $su(N)$ structure constants such as $f_{-\alpha, -\beta}^{-(\alpha+\beta)} = -f_{\alpha, \beta}^{\alpha+\beta}$ and $f_{\beta, -(\alpha+\beta)}^{-\alpha} = f_{\alpha, \beta}^{\alpha+\beta}$, it is straightforward to verify that the elementary polynomial associated to $(\Lambda^1, \Lambda^r, 0)$ is

$$R^{\Lambda^1, \Lambda^r, 0}(x_1, x_2, x_3) = P^r_\theta(x_2) + \sum_{i=1}^{r-1} P^1_{-\alpha_1,i}(x_1) P^r_{-\alpha_{i+1,r}}(x_2) - P^1_\theta(x_1)$$

$$= x_2^\theta - x_1^\theta + \ldots$$

(18)

The dots indicate a non-vanishing polynomial in $x$ independent of $x^\theta$. One may actually derive the $x^\theta$-dependence (18) without working out the entire two-point polynomial. One simply observes that $\text{wt}(R^{\Lambda^1, \Lambda^r, 0}) = \theta$, and then considers the vanishing action of $E_\theta$ on $R^{\Lambda^1, \Lambda^r, 0}$. The apparent asymmetry in signs in the general expression (18) is due to our choices (14), (15) and (17).

Useful in the following are the tensor products

$$M_{\Lambda^1} \otimes M_{\Lambda^1} = M_{\Lambda^2} \oplus M_{2\Lambda^1}$$

$$M_{\Lambda^1} \otimes M_{\Lambda^r} = M_0 \oplus M_\theta$$

$$M_{\Lambda^r} \otimes M_{\Lambda^r} = M_{\Lambda^r-1} \oplus M_{2\Lambda^r}$$

(19)

which are true for $r \geq 2$.

We are now in a position to consider the triple product $M_\theta \otimes M_\theta \otimes M_\theta$, cf. (8). It is easily seen that the coupling $(\theta, \theta, \theta)$ admits the following two decompositions on the set of two-point couplings:

$$(\theta, \theta, \theta) = (\Lambda^1, \Lambda^r, 0) \times (0, \Lambda^1, \Lambda^r) \times (\Lambda^r, 0, \Lambda^1)$$

$$= (\Lambda^r, \Lambda^1, 0) \times (0, \Lambda^r, \Lambda^1) \times (\Lambda^1, 0, \Lambda^r)$$

(20)

They are only distinct when $r \geq 2$, which we will assume in the following. For $r \geq 3$, (20) exhaust the possible decompositions. That follows from a simple inspection of (15). For $r = 2$, on the other hand, $(\Lambda^1, \Lambda^1, \Lambda^1)$ and $(\Lambda^r, \Lambda^r, \Lambda^r)$ are also elementary couplings, leading to the well-known $su(3)$ syzygy. Summarizing the $su(3)$ case, $T_{\theta, \theta, \theta} = 2$ and the multi-set of threshold levels is \{2, 3\}.

For $r \geq 3$, the situation is radically different. So far we have found that $1 \leq T_{\theta, \theta, \theta} \leq 2$, and the two candidate polynomials are

$$R^\theta_{1, \theta, \theta}(x_1, x_2, x_3) = R^{\Lambda^1, \Lambda^r, 0} R^{0, \Lambda^1, \Lambda^r} R^{\Lambda^r, 0, \Lambda^1}$$

$$= (x_2^\theta - x_1^\theta)(x_3^\theta - x_2^\theta)(x_1^\theta - x_3^\theta) + S_1^{\theta, \theta, \theta}(x_1, x_2, x_3)$$

$$R^\theta_{2, \theta, \theta}(x_1, x_2, x_3) = R^{\Lambda^r, \Lambda^1, 0} R^{0, \Lambda^r, \Lambda^1} R^{\Lambda^1, 0, \Lambda^r}$$

$$= (x_1^\theta - x_2^\theta)(x_2^\theta - x_3^\theta)(x_3^\theta - x_1^\theta) + S_2^{\theta, \theta, \theta}(x_1, x_2, x_3)$$

(21)
According to (33), \( S_1^{0,0,0} \) and \( S_2^{0,0,0} \) are non-vanishing, second-order polynomials in \( x^0 \). Now, if \( T_{\theta,0,0} = 1 \) there would have to be a syzygy relating \( R_1^{0,0,0} \) and \( R_2^{0,0,0} \). Following (31), that would imply that \( R_1^{0,0,0} + R_2^{0,0,0} = 0 \), and in particular \( S_1^{0,0,0} + S_2^{0,0,0} = 0 \). The contributions to \( S_1^{0,0,0} \) and \( S_2^{0,0,0} \) that are quadratic in \( x_3^0 \) and otherwise only depend on \( x_2 \), are

\[
S_1^{0,0,0} : (x_3^0)^2(-P_{-\theta}(x_2) + x_2^0), \quad S_2^{0,0,0} : (x_3^0)^2(P_{-\theta}(x_2) - x_2^0) \tag{22}
\]

So a syzygy would require \( P_{-\theta}(x_2) = P_{-\theta}(x_2) \). However, it is seen (8) that

\[
P_{-\theta}^1(x) = x^0 + \frac{1}{2} x^0 x^{a_2, r} + \ldots
\]

\[
P_{-\theta}^r(x) = x^0 - \frac{1}{2} x^0 x^{a_2, r} + \ldots \tag{23}
\]

Thus, we may conclude that \( T_{\theta,0,0} = 2 \), and that \( S_1^{0,0,0} + S_2^{0,0,0} \neq 0 \).

To determine the threshold levels, we must consider all non-vanishing linear combinations of \( R_1^{0,0,0} \) and \( R_2^{0,0,0} \) to obtain a basis with lowest possible powers of \( x^0 \). That is easily done, and we find

\[
R_+^{0,0,0} = R_1^{0,0,0} + R_2^{0,0,0} \tag{24}
\]

in addition to \( R_1^{0,0,0} \), for example. The threshold levels are

\[
t(R_+^{0,0,0}) = 2, \quad t(R_1^{0,0,0}) = 3 \tag{25}
\]

\( R_+^{0,0,0} \) does not enjoy a factorization in terms of elementary polynomials associated to elementary couplings. Furthermore, since it is the unique polynomial with threshold level 2, a good fusion basis \( \{R_s^{0,0,0}\} \) must contain it. Thus, \( R_+^{0,0,0} \) corresponds to a purely affine EF. This completes the proof of our assertion (3), and the explicit construction of the associated polynomial.

When \( R_+^{0,0,0} \) is added to the elementary couplings in the basis of EFs, we simultaneously introduce the syzygy

\[
R_+^{0,0,0} = R^{\Lambda_1, \Lambda_0, \Lambda_0} R^{0, \Lambda_1, \Lambda_0} R^{0, \Lambda_0, \Lambda_1} + R^{\Lambda_1, \Lambda_0, \Lambda_0} R^{0, \Lambda_0, \Lambda_1} R^{0, \Lambda_1, \Lambda_0} \tag{26}
\]

It may be implemented by excluding the product \( R^{\Lambda_1, \Lambda_0, \Lambda_1} R^{0, \Lambda_0, \Lambda_1} R^{0, \Lambda_1, \Lambda_0} \), for example. That was our choice above (22).

### 3.2 \( su(r + 1) \) fusion: counting

Here we shall deduce the existence of the purely affine elementary coupling (3) without constructing its polynomial (24), simply by calculating certain fusion multiplicities.

A simple analysis (using the Kac-Walton formula, or modified Weyl character method, for example) reveals that for \( r \geq 3 \)

\[
\theta \otimes \theta = (0)_2 \oplus (\Lambda^2 + \Lambda^{-1})_2 \oplus (\Lambda^2 + 2\Lambda^1 + \Lambda^{-1})_3 \oplus (\theta)_2 \oplus (\theta)_3 \oplus (2\theta)_4 \tag{27}
\]

Here we have abbreviated \( M_{\lambda} \) by \( \lambda \). The non-trivial part is the determination of the threshold levels which have been indicated by subscripts: \( (\lambda)_t \) corresponds to \( (\theta, \theta, \lambda^+)_t \), where \( \lambda^+ \) is the
weight conjugate to $\lambda$. So, there exist two couplings $(\theta, \theta, \theta)$, and corresponding fusions with threshold levels 2 and 3.

We need to show that (i) the two couplings $(\theta, \theta, \theta)$ are not elementary as couplings, and (ii) for $r > 2$ the fusion $(\theta, \theta, \theta)_2$ is elementary as a fusion.

It is not hard to show that all couplings corresponding to fusions at level 1 are of the form

$$\begin{align*}
\left\{ (\Lambda^i, \Lambda^j, \Lambda^k)_1 \mid i, j, k \in \{0, 1, 2, \ldots, r\}, \ i + j + k \in \{r + 1, 2(r + 1)\} \right\}
\end{align*}$$

(28)

Here we use $\Lambda^0 = 0$, so that the $3r$ independent elementary two-point couplings are included. Clearly, each elementary coupling in (28) corresponds to an elementary fusion with threshold level 1.

Now, (i) can be demonstrated easily: the two couplings $(\theta, \theta, \theta)$ are expressible as two distinct products of three couplings in (28), as already discussed (20). Part (ii) is also simple: a decomposition of (3)

$$(\theta, \theta, \theta)_2 = (\Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r, \Lambda^1 + \Lambda^r)_2 = (\Lambda^i, \Lambda^j, \Lambda^k)_1 \times (\Lambda^l, \Lambda^m, \Lambda^n)_1$$

(29)

can not be on EFs; for $r > 2$ there is no solution for $\{i, j, k, l, m, n\}$ respecting $i + j + k, l + m + n \in \{r + 1, 2(r + 1)\}$.

We conclude by observing that for all simple Lie algebras besides $A_r \simeq su(r + 1)$, the triple adjoint couplings have tensor product multiplicity 1 and threshold level 3. Because the adjoint representation is a fundamental representation for all simple Lie algebras but $A_r$ and $C_r$, the triple adjoint coupling is both an elementary coupling and an EF, for those algebras. For the case $C_r$, where the highest root is $\theta = 2\Lambda^1$, it is easy to see that the triple adjoint is neither an elementary coupling nor an EF. In conclusion, it is only in the case $A_r$ that we encounter a purely affine EF when considering the triple adjoint coupling.

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