Extracting short-distance physics from $K_{L,S} \to \pi^0 e^+ e^-$ decays

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Abstract

We present a new analysis of the rare decay $K_L \to \pi^0 e^+ e^-$ taking into account important experimental progress that has recently been achieved in measuring $K_L \to \pi^0 \gamma \gamma$ and $K_S \to \pi^0 e^+ e^-$. This includes a brief review of the direct CP-violating component, a calculation of the indirect CP-violating contribution, which is now possible after the measurement of $K_S \to \pi^0 e^+ e^-$, and a re-analysis of the CP conserving part. The latter is shown to be negligible, based on experimental input from $K_L \to \pi^0 \gamma \gamma$, a more general treatment of the form factor entering the dispersive contribution, and on a comparison with the CP violating rate, which can now be estimated reliably. We predict $B(K_L \to \pi^0 e^+ e^-) = (3.2^{+1.2}_{-0.8}) \times 10^{-11}$ in the Standard Model, dominated by CP violation with a sizable contribution ($\sim 40\%$) from the direct effect, largely through interference with the indirect one. Methods to deal with the severe backgrounds for $K_L \to \pi^0 e^+ e^-$ using Dalitz-plot analysis and time-dependent $K_L$–$K_S$ interference are also briefly discussed.
1 Introduction

The flavour-changing neutral-current transition \( K_L \to \pi^0 e^+e^- \) has been recognized since a long time to be one of the most interesting rare kaon decays. It shows an intriguing interplay of short and long distances, leading to a sum of comparable CP-conserving, direct- and indirect-CP-violating contributions \([1]\). Until very recently, it was impossible to estimate all these contributions with good accuracy, or to predict the total \( K_L \to \pi^0 e^+e^- \) rate. As a consequence, it was not clear to which extent this mode could be used as a probe of the Cabibbo-Kobayashi-Maskawa mechanism of CP violation. The situation has now been changed substantially by two new experimental results of the NA48 Collaboration: the observation of the \( K_S \to \pi^0 e^+e^- \) decay \([2]\) and the precise measurement of the \( K_L \to \pi^0 \gamma\gamma \) spectrum at small diphoton-invariant mass \([3]\). The former allows us to evaluate the indirect-CP-violating part of the amplitude, whereas the latter help us to estimate the CP-conserving contribution. The purpose of this paper is a complete reanalysis of the \( K_L \to \pi^0 e^+e^- \) decay in view of this new experimental information.

The use of experimental data to estimate indirect-CP-violating and CP-conserving contributions of \( K_L \to \pi^0 e^+e^- \) is not completely straightforward. The most delicate issue concerns the CP-conserving contribution and, particularly, the dispersive part of the \( K_L \to \pi^0 \gamma^*\gamma^* \to \pi^0 e^+e^- \) amplitude. So far, this part of the amplitude has been estimated employing a specific model-dependent ansatz for the behavior of the \( K_L \to \pi^0 \gamma^*\gamma^* \) vertex with off-shell photons \([4]\). Here we generalize previous analyses using a more general parameterization of the \( K_L \to \pi^0 \gamma^*\gamma^* \) form factor, which satisfies both low- and high-energy constraints and helps us to estimate the theoretical uncertainties in this calculation. Moreover, we show how to extract the information on the on-shell \( K_L \to \pi^0 \gamma\gamma \) amplitude relevant to \( K_L \to \pi^0 e^+e^- \) in a model-independent way, without relying on model-dependent assumptions on the former, such as the vector-meson-dominance parameterization in terms of \( a_V \). As a result, we are able to derive a conservative upper bound on the total CP-conserving contribution of the \( K_L \to \pi^0 e^+e^- \) rate, which turns out to be well below the level of the interesting short-distance component.

As far as the CP-violating amplitude is concerned, the NA48 result on \( B(K_S \to \pi^0 e^+e^-) \) provides us with an unambiguous indication that the indirect-CP-violating contribution is large and cannot be neglected. Here the most delicate issue is the model-dependent sign of the interference between direct- and indirect-CP-violating components of the amplitude. As we shall show, the measured value of \( B(K_S \to \pi^0 e^+e^-) \), together with theoretical arguments both of perturbative and non-perturbative nature, provides a good indication in favour of a positive interference. Following this indication, we predict \( B(K_L \to \pi^0 e^+e^-)_{SM} \approx 3 \times 10^{-11} \), with a negligible CP-conserving component, about 40% due to the clean short-distance direct-CP-violating amplitude (mainly through the interference with the indirect-CP-violating one) and an overall theoretical error that to a large extent scales with the experimental error on \( B(K_S \to \pi^0 e^+e^-) \).

From a purely theoretical perspective we thus conclude that \( K_L \to \pi^0 e^+e^- \) appears as one of the most interesting candidates for precision tests of CP violation in \( \Delta S = 1 \)
transitions. The most serious problem to reach this goal is the so-called Greenlee background, that is, the large irreducible experimental background induced by the decay $K_L \to \gamma\gamma e^+e^-$. The enhancement of the total $K_L \to \pi^0 e^+e^-$ rate due to the positive interference between direct– and indirect-CP-violating amplitudes provides good news in this respect, suggesting that future high-statistics experiments could be able to detect the $K_L \to \pi^0 e^+e^-$ signal over Greenlee’s background. As we shall show, a Dalitz-Plot analysis and, especially, time-dependent measurements could provide additional handles against this problem.

The paper is organized as follows: in Section 2 we briefly review the prediction for the short-distance direct-CP-violating amplitude of $K_L \to \pi^0 e^+e^-$ within the Standard Model (SM). Section 3 contains one of the two main results of this work, namely the new conservative estimate of the CP-conserving branching ratio. The second main result is presented in Section 4, where we analyse the interference between direct- and indirect-CP-violating amplitudes, and the prediction for the total rate. A discussion of possible Dalitz-Plot and time-dependent analyses against the CP-conserving and, especially, Greenlee’s background is presented in Sections 5 and 6. The results are summarized in the Conclusions. The Appendix contains a self-contained model-independent analysis of the ground is presented in Sections 5 and 6. The results are summarized in the Conclusions.

The Appendix contains a self-contained model-independent analysis of the $K_L \to \pi^0 \gamma\gamma$ amplitude in chiral perturbation theory (CHPT) beyond the lowest non-trivial order.

## 2 Short-distance contribution to $K_L \to \pi^0 e^+e^-$

The direct-CP-violating transition $K_2 \to \pi^0(e^+e^-)_{J=1}$, where the lepton pair forms a vector or axial-vector state, is dominated by short-distance dynamics and is calculable with high accuracy in perturbation theory. Following the notation of, the effective Hamiltonian necessary to compute this amplitude at next-to-leading order accuracy can be written as

$$H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left[ \sum_{i=1}^{6,7V} (z_i(\mu) + \tau y_i(\mu)) Q_i(\mu) + \tau y_{7A}(M_W) Q_{7A}(M_W) \right] + \text{h.c.} \quad (1)$$

where $\tau = -(V_{ts}^* V_{td})/(V_{us}^* V_{ud})$ and $V_{ij}$ denote CKM matrix elements. Here $Q_{1,2}$ are the current–current operators, $Q_{3,6}$ the QCD penguin operators, and

$$Q_{7V} = \bar{s} \gamma^\mu(1 - \gamma_5)d \gamma_\mu \gamma_5 \ell, \quad Q_{7A} = \bar{s} \gamma^\mu(1 - \gamma_5)d \gamma_\mu \gamma_5 \ell \quad (2)$$

the leading electroweak operators. Employing the standard CKM phase convention, the overall factor $V_{us}^* V_{ud}$ is real and direct CP violation is induced only by terms proportional to $\Im \tau$. Since $y_1 = y_2 \equiv 0$, only $Q_{7V}, Q_{7A}$ and, in principle, the QCD penguin operators $Q_{3,6}$, are relevant to estimate the $K_2 \to \pi^0(e^+e^-)_{J=1}$ amplitude.

Ignoring for the moment the contribution of $Q_{3,6}$, whose matrix elements vanish at the tree-level, one has

$$A \left(K_L \to \pi^0 e^+e^- \right)_{\text{s.d.}} = -\langle \pi(p_\pi)e^+(k_1)e^-(k_2) | H_{\text{eff}}^{\Delta S=1} | K_2(p) \rangle$$

$$= \frac{G_F}{\sqrt{2}} \Im (V_{ts}^* V_{td}) f_+(z)(p_\pi + p)^\mu [y_{7V} \bar{u}(k_2)\gamma_\mu v(k_1) + y_{7A} \bar{u}(k_2)\gamma_\mu \gamma_5 v(k_1)] . \quad (3)$$

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where $f_+(z = q^2/m_K^2) \approx 1 + z(m_K/m_\rho)^2$ is the form factor of the vector current and $q^2 = (p-p_\pi)^2$. If considered alone, this theoretically clean part of the amplitude leads to

$$B(K_L \to \pi^0 e^+ e^-)_{\text{CPV-dir}} = \frac{\tau(K_L) B(K_{e3}^+)}{\tau(K^+)} \frac{|V_{us}|^2}{|y_{7A}^2 + y_{7V}^2|} |\Im(V_{ts}^* V_{td})|^2,$$

$$= (2.4 \pm 0.2) \times 10^{-12} \left[ \frac{\Im \lambda}{10^{-4}} \right]^2,$$  \hspace{1cm} (4)

where

$$\Im \lambda = \Im(V_{ts}^* V_{td}) \text{ SM} \Rightarrow (1.36 \pm 0.12) \times 10^{-4}$$ \hspace{1cm} (5)

and the numerical coefficient in (4) has been obtained using $V_{us} = 0.2240 \pm 0.0036$, $\alpha(M_Z) = 1/129$, $y_{7A} = -(0.68 \pm 0.03)\alpha(M_Z)$ and $y_{7V} = (0.73 \pm 0.04)\alpha(M_Z)$, corresponding to $m_t = 167 \pm 5$ GeV and a renormalization scale for $y_{7V}$ chosen between 0.8 and 1.2 GeV.

The contribution of the QCD penguin operators cannot be estimated with good accuracy, due to unknown hadronic matrix elements. However, $Q_{3\ldots6}$ are expected to yield a negligible correction to the leading electroweak operators:

$$\sum_{i=3}^{6} y_i(\mu) \langle \pi^0 e^+ e^- | Q_i | K \rangle \ll y_{7V} \langle \pi^0 e^+ e^- | Q_{7V} | K \rangle.$$ \hspace{1cm} (6)

This assumption is strongly supported by i) the corresponding relation for the quark-level matrix elements

$$\sum_{i=3}^{6} y_i(\mu) \langle d e^+ e^- | Q_i | s \rangle \ll y_{7V} \langle d e^+ e^- | Q_{7V} | s \rangle,$$ \hspace{1cm} (7)

which can be checked explicitly in perturbation theory; ii) the following order-of-magnitude estimate for the impact of the $Q_i$ at the hadronic level:

$$\delta B(K_L \to \pi^0 e^+ e^-)_{Q_i} \sim \left( \frac{y_i}{z^2 \Im \tau} \right)^2 \frac{\tau(K_L)}{\tau(K^+)} B(K^+ \to \pi^+ e^+ e^-) \ll 10^{-14}.$$ \hspace{1cm} (8)

Given these considerations, Eqs. (4) and (5) can be considered as precise estimates of the $K_L \to \pi^0 e^+ e^-$ direct-CP-violating amplitude, and its corresponding branching ratio, within the SM.

It is worth to recall that, being dominated by short distances, this part of the $K_L \to \pi^0 e^+ e^-$ amplitude is strongly sensitive to possible non-standard contributions. In particular, axial and vector components of the amplitude could separately or both be enhanced by factors of $O(1)$ with respect to the SM case. In optimistic but still realistic supersymmetric scenarios, the short-distance branching ratio in Eq. (4) could exceed the $10^{-11}$ level.
3 Estimates of the CP-conserving amplitude

The $|\pi^0 e^+ e^-\rangle$ state is not, in general, a CP eigenstate: its transformation under CP depends on the angular momentum of the lepton pair. The short-distance Hamiltonian in Eq. (1) creates the lepton pair in a state of total angular momentum $J = 1$ (and orbital angular momentum $L = 0$) so that $|\pi^0 e^+ e^-\rangle_{s.d.}$ has opposite CP with respect to $|K_2\rangle$. On the contrary, the long-distance process $K_2 \to \pi^0 \gamma \to \pi^0 e^+ e^-$ can lead to final states with even $J$, allowed in the limit of exact CP symmetry \[1, 11, 12, 13, 14, 15, 16\]. The two-photon exchange is not the only source of CP-conserving contributions to $K_L \to \pi^0 e^+ e^-$, in principle also dimension-8 operators generated by $W$-box diagrams can lead to a final state with $J \neq 1$; however, such contributions turn out to be completely negligible \[17\].

In the limit of exact CP symmetry, the most general amplitude describing the $K_L(p) \to \pi^0(p_3)\gamma(q_1, \epsilon_1)\gamma(q_2, \epsilon_2)$ transition, with on-shell or off-shell photons, can be written as

$$A(K_L \to \pi^0 \gamma \gamma) = \frac{G_8\alpha}{4\pi}\epsilon_1\epsilon_2\left[A(z, y; q_1^2, q_2^2)(q_2^\mu q_1^{\mu} - q_1^\mu q_2^{\mu}) + \frac{2B(z, y; q_1^2, q_2^2)}{m_K^2}(p \cdot q_1 q_2^\mu p^\nu + p \cdot q_2 p^\mu q_1^{\nu} - p \cdot q_1 p \cdot q_2 g^{\mu\nu} - q_1^\mu q_2^{\nu} - q_1^\nu q_2^{\mu})\right],$$

where

$$y = \frac{p \cdot (q_1 - q_2)}{m_K^2}, \quad z = \frac{(q_1 + q_2)^2}{m_K^2}, \quad |G_8| = 9.0 \times 10^{-6} \text{ GeV}^{-2}. \quad (10)$$

and $G_8 = G_F|V_{us}V_{ud}|g_8/\sqrt{2}$ with $|g_8| = 5.1$. Due to Bose symmetry the invariant amplitudes $A$ and $B$ are symmetric under the exchange $q_1 \leftrightarrow q_2$.

Restricting the attention to the on-shell case ($q_1^2 = q_2^2 = 0$), the allowed range of the dimensionless variables is

$$0 \leq |y| \leq \frac{1}{2}\lambda^{1/2}(1, r_\pi^2, z), \quad 0 \leq z \leq (1 - r_\pi^2), \quad (11)$$

where

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc), \quad r_\pi = \frac{m_\pi}{m_K}, \quad (12)$$

and the unpolarized differential rate reads

$$\frac{\partial^2 \Gamma}{\partial y \partial z} = \frac{G_8^2\alpha^2 m_K^5}{2^{13}\pi^5} \left[z^2 |A + B|^2 + \left(y^2 - \frac{1}{4}\lambda(1, r_\pi^2, z)\right)^2 |B|^2\right]. \quad (13)$$

In order to obtain the total rate from \[13\], the integration is to be performed over positive values of $y$ only.

Instead of using the $A-B$ basis, one can employ the $S-B$ amplitude basis, where $S \equiv A + B$. As can be seen from Eq. \[13\], in the $S-B$ basis the two amplitudes do not interfere in the on-shell unpolarized rate. This indicates that $S$ and $B$ describe transitions to different final states. Indeed, neglecting the $y$ dependence, $S$ describes the decay into
in Fig. 2, in order to estimate the CP-conserving $K$ amplitude. The low diphoton invariant mass region. Neglecting the kinematical dependence of $M$ (expected to be very mild in both cases below the $B$ threshold) and setting the cut $M_{\gamma\gamma} < 110$ MeV, to make contact with the NA48 analysis [8], we find

$$B(K_L \to \pi^0 \gamma \gamma)_{M_{\gamma\gamma} < 110 \text{ MeV}} = \frac{1}{2.0 \times 10^{-9}} \int_{z < 0.049} dz \left[ z^2 \lambda(1, r_{\pi}^2, z)^{1/2} |S(0)|^2 + \frac{1}{30} \lambda(1, r_{\pi}^2, z) z^{5/2} |B(0)|^2 \right]$$

If $S(0)/B(0) = O(1)$, the experimental ratio

$$R_B = \frac{B(K_L \to \pi^0 \gamma \gamma)_{M_{\gamma\gamma} < 110 \text{ MeV}}}{2.0 \times 10^{-9}}$$

provides an excellent approximation to $|B(0)|^2$; moreover, the inequality $|B(0)|^2 \leq R_B$ holds independently from any assumption on $S(0)/B(0)$. As shown in Fig. 1, the approximation $B(z) \approx B(0)$ works very well for $|B(0)| \gtrsim 1$, which is the case of interest for $K_L \to \pi^0 e^+ e^-$. Thus $R_B$ provides a powerful model-independent tool to estimate (or constrain) the size of $|B(z)|$ in the whole physical range.

Having determined the absolute value of the $B$ amplitude at $q_1^2 = q_2^2 = 0$ from experiments, we can use it as a (real) effective coupling for the $K_L \to \pi^0 \gamma(q_1) \gamma(q_2)$ vertex in Fig. 2 in order to estimate the CP-conserving $K_L \to \pi^0 e^+ e^-$ amplitude. If we ignore
the dependence on $q_1^2, q_2^2$ the two-photon dispersive integral turns out to be logarithmically divergent. To regularize it we employ the following ansatz

$$B(z, y; q_1^2, q_2^2) = B(z) \times f(q_1^2, q_2^2)$$

where the form factor

$$f(q_1^2, q_2^2) = 1 + a \left( \frac{q_1^2}{q_1^2 - m_V^2} + \frac{q_2^2}{q_2^2 - m_V^2} \right) + b \frac{q_1^2 q_2^2}{(q_1^2 - m_V^2)(q_2^2 - m_V^2)},$$

is defined in analogy with the analysis of the $K_L \to \gamma\gamma \to \mu^+\mu^-$ amplitude presented in [22]. This structure is dictated by the assumption that VMD plays a crucial role in the matching between short and long distances (in the numerical analysis $m_V$ is identified with $m_\rho \simeq 770$ MeV). In order to obtain an ultraviolet convergent integral, as it is the case within the full theory, we need to impose the condition

$$1 + 2a + b = 0.$$  

In this way we are left with a single free parameter, that we can vary to estimate the theoretical uncertainties of this approach.\footnote{In principle it could be possible to obtain additional constraints on this form-factor by means of $K_L \to \pi^0\mu^+\mu^-$ data \cite{23}; however, the present experimental information is not accurate enough to extract any significant constraint \cite{24}.} Obviously Eq. (16) does not represent the
most general parameterization of the $B$ amplitude off shell. However, we are not interested in the detailed structure of this amplitude, rather in its weighted integral relevant to $K_L \to \pi^0 e^+ e^-$. In this respect the possibility to vary one parameter, in order to test the stability of the result, represents an important improvement with respect to the choice made in [4]. The latter is recovered as a special case for $b = -a = 1$.

Following [4] we can write the CP-conserving $K_L(p) \to \pi^0(p_\pi)e^+(k_1)e^-(k_2)$ amplitude as

$$M(K_L \to \pi^0 e^+ e^-)_{\text{CPC}} = \frac{G_8 \alpha^2 B(z) G(z)}{16\pi^2 m_K^2} p \cdot (k_1 - k_2)(p + p_\pi)_\mu \pi(k_2) \gamma^\mu v(k_1),$$

(19)

where $z = s/m_K^2 = (k_1 + k_2)^2/m_K^2$ and $G(z)$ is a dimensionless function resulting from the loop integration. Neglecting terms which are suppressed by inverse powers of $m_\pi^2$ and eliminating $b$ by means of Eq. (18), we find:

$$G(z) = \frac{2}{3} \ln \left(\frac{m_\pi^2}{s}\right) - \frac{1}{9} + \frac{4}{3} (1 + a).$$

(20)

The function $B(z)$ is certainly real for $z < 4m_\pi^2/m_K^2$. Above the $\pi\pi$ threshold this is no longer true; however, as discussed above, if $|B(z)| \gtrsim 1$ the local contribution is dominant and, to a good accuracy, $B(z)$ can be approximated by a real constant in the entire phase space. For this reason the product $|B(z)| \times \Re G(z)$ (where $\Re G(z)$ is obtained from (20) by the prescription $\ln(-m_\rho^2) = \ln m_\rho^2 + i\pi$) represents a good approximation the full absorptive contribution of the CP-conserving $K_L \to \pi^0 e^+ e^-$ amplitude.

Our result for this absorptive part is consistent with the one in [13, 14] but not with the result of [4]. We disagree with [4] also in the dispersive part, $|B(z)| \times \Im G(z)$, computed for $a = -1$, where our form factor is identical to the one of [4]. The main difference between our result and the one in [4] is that we do not find any singularity in the limit $m_e \to 0$. The lacking of this kind of singularity was already noticed and discussed in [13]. Indeed it is possible to explicitly check that there are no possible bremsstrahlung contributions that could cancel such a singular behaviour.
Integrating the amplitude \((19)\) over the phase space, at fixed \(z\), leads to
\[
\frac{d}{dz}B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}} = \frac{G_S^2 \alpha^4 m_K^5}{15\pi^2 2^{16} \Gamma_L} |B(z)G(z)|^2 \lambda(1,r^2_\pi,z)^{5/2}
\]
\[
= 8.1 \times 10^{-13} \times |B(z)|^2 \lambda(1,r^2_\pi,z)^{5/2} \left[ 1 + \left( \frac{3}{2\pi} \Re G(z) \right)^2 \right] . \tag{21}
\]

As we shall discuss in more details in Section 5, the kinematical factor \(\lambda^{5/2}\) strongly suppresses the high-\(z\) contributions, strengthening the validity of the approximation \(B(z) \simeq B(0)\). In this limit, integrating over the full phase space we obtain
\[
B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}} = 7.0 \times 10^{-14} \times |B(0)|^2 \times \left\{ 1 + \left[ 1.4 + 1.4(1+a) + 0.4(1+a)^2 \right] \right\} . \tag{22}
\]

The term between square brackets in (22) represents the model-dependent contribution of the dispersive amplitude: we express it as a function of \((1+a)\), since \((a+1) \approx 0\), corresponding to the choice of Ref. [4], denotes the most natural possibility. As can be noticed, for reasonable values of the form-factor parameters, namely for both \(a\) and \(b = -1 - 2a\) of \(\mathcal{O}(1)\), absorptive and dispersive contributions are of the same order. Assuming the term between curly brackets in (22) to be smaller than 10, which we consider a rather conservative hypothesis, and using the inequality \(|B(0)|^2 \leq R_B\) we can finally write
\[
B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}} < 3.5 \times 10^{-4} \times B(K_L \rightarrow \pi^0 \gamma \gamma)_{M,\gamma<110 \text{ MeV}} . \tag{23}
\]

Using the recent experimental input [8] (see the Appendix)
\[
B(K_L \rightarrow \pi^0 \gamma \gamma)_{M,\gamma<110 \text{ MeV}} < 0.9 \times 10^{-8} \quad \text{at} \quad \text{90\%C.L.} , \tag{24}
\]
our present estimate of the maximal CP-conserving contribution to \(K_L \rightarrow \pi^0 e^+e^-\) reads
\[
B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}} < 3 \times 10^{-12} . \tag{25}
\]

We stress that Eq. (25) is a conservative upper bound. A reasonable estimate of \(B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}}\), based on the assumption that real and absorptive contributions are equal, would lead to values below \(10^{-12}\).

To conclude, we recall that Eq. (23) has been obtained under the assumption \(|B(0)| \gtrsim 1\). Thus, according to Eq. (13), it makes sense only if \(B(K_L \rightarrow \pi^0 \gamma \gamma)_{M,\gamma<110 \text{ MeV}} \gtrsim 2 \times 10^{-9}\). If experiments would find that \(B(K_L \rightarrow \pi^0 \gamma \gamma)_{M,\gamma<110 \text{ MeV}}\) is below this figure, other sub-leading amplitudes could become relevant. In particular, we have neglected the contribution induced by \(y^2\)-dependent terms in \(S\). The latter leads to a CP-conserving amplitude
\[
M \left( K_L \rightarrow \pi^0 e^+e^- \right)_{\text{CPC}}^{\text{S-type}} \approx \frac{G_S \alpha^2}{16\pi^2 m_K} \left( \frac{\partial^2 S}{\partial y^2} \right) \frac{z}{12} p \cdot (k_1 - k_2)(p + p_\pi)_{\mu} \vec{n}(k_2) \gamma^\mu \nu(k_1) , \tag{26}
\]
which is not only suppressed by the smallness of \(\partial^2 S/\partial y^2\), but also by extra kinematical and numerical factors (the coefficient \(z/12\) arises from the calculation with a point-like form factor). These contributions become relevant only if \(B(K_L \rightarrow \pi^0 e^+e^-)_{\text{CPC}}\) is in the \(10^{-13}\) range and thus small enough compared to the CP-violating term.
4 Indirect-CP-violating amplitude and total rate

The CP-conserving transitions $K_S \to \pi^0 \ell^+ \ell^-$ and $K^\pm \to \pi^\pm \ell^+ \ell^-$, with the lepton pair in a vector state, are dominated by the long-distance process $K(p) \to \pi \gamma \to \pi(p_{\pi})\ell^+(k_1)\ell^-(k_2)$ \[26\]. The decay amplitudes can in general be written as

$$A\left(K_i \to \pi \ell^+ \ell^-\right) = -\frac{e^2}{m_{K_i}^2(4\pi)^2} W_i(z)(p + p_{\pi})^{\mu} \bar{u}(k_2)\gamma_\mu v(k_1) \, , \quad (27)$$

where $W_i(z)$ are form factors regular at $z = 0$ ($i = \pm, S$). The latter can be decomposed as a sum of a polynomial piece plus a non-analytic term, $W_i^{\pi\pi}(z)$, generated by the $\pi\pi$ loop and completely determined in terms of the physical $K \to 3\pi$ amplitude \[27\]. Expanding the polynomial term up to $O(p^6)$ we can write

$$W_i(z) = G_F m_{K_i}^2 \left(a_i + b_i z\right) + W_i^{\pi\pi}(z) \, , \quad (28)$$

where the real parameters $a_i$ and $b_i$ encode local contributions starting at $O(p^4)$ and $O(p^6)$, respectively. High-precision data on $K^+ \to \pi^+ e^+ e^-$ by BNL-E865 \[28\] have been successfully fitted using Eq. (28) and lead to

$$a_+ = -0.587 \pm 0.010 \quad b_+ = -0.655 \pm 0.044 \, . \quad (29)$$

Unfortunately, chiral symmetry alone does not help to determine the unknown couplings $a_S$ and $b_S$ in terms of $a_+$ and $b_+$. On the other hand, the non-analytic term $W_S^{\pi\pi}(z)$ is known to be very small, due to the $\Delta I = 3/2$ suppression of the CP-conserving $K_S \to 3\pi$ amplitude. As a consequence, the $K_S \to \pi^0 e^+ e^-$ rate turns out to be dominated by local contributions

$$B(K_S \to \pi^0 e^+ e^-) = \left[0.01 - 0.76 a_S - 0.21 b_S + 46.5 a_S^2 + 12.9 a_S b_S + 1.44 b_S^2\right] \times 10^{-10}$$
$$\approx 5.2 \times 10^{-9} \times a_S^2 \, , \quad (30)$$

where the second line follows from the assumption $b_S/a_S = m_{K_S}^2/m_{\rho}^2$, motivated by VMD.

The first experimental evidence of the $K_S \to \pi^0 e^+ e^-$ transition has been announced very recently by the NA48/1 Collaboration at CERN. The observation of 7 events in a clean signal region (with 0.15 expected background events) leads to the preliminary result \[2\]:

$$B(K_S \to \pi^0 e^+ e^-)_{m_{ee} > 165 \text{ MeV}} = \left(3.0^{+1.5}_{-1.2} \pm 0.2\right) \times 10^{-9} \, , \quad (31)$$

which implies

$$|a_S| = 1.08^{+0.26}_{-0.21} \, . \quad (32)$$

This result is in good agreement with the naive chiral counting expectation $a_S = O(1)$ \[27\] and, within this framework, it is quite compatible with the ranges expected from large-$N_C$ \[29\] and resonance-saturation arguments \[30\]. Interestingly, the central value of the NA48/1 result is also in excellent agreement with a very old prediction by Sehgal, based only on VMD and $\Delta I = 1/2$ rule \[31\].
Taking into account the interference between direct- and indirect-CP-violating components, the full CP-violating contribution to \( K_L \to \pi^0 e^+ e^- \) can be written as

\[
B(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = 10^{-12} \times \left[ C_{\text{mix}} \pm C_{\text{int}} \left( \frac{3\lambda_t}{10^{-7}} \right) + C_{\text{dir}} \left( \frac{3\lambda_t}{10^{-7}} \right)^2 \right],
\]

where the \( \pm \) depends on the relative sign between short- and long-distance amplitudes, and

\[
C_{\text{mix}} = 10^{12} |\epsilon|^2 \frac{\tau(K_L)}{\tau(K_S)} B(K_S \to \pi^0 e^+ e^-) = (15.7 \pm 0.3)|a_s|^2,
\]

\[
C_{\text{dir}} = 10^4 \frac{\tau(K_L)}{\tau(K^+)} \frac{B(K_{S\to L}^+)}{|V_{us}|^2} (y_{7L}^2 + y_{7V}^2) = 2.4 \pm 0.2,
\]

\[
C_{\text{int}} = 2 \sin \phi_e \sqrt{C_{\text{mix}} C_{\text{dir}}} \frac{y_{7V}}{\sqrt{y_{7L}^2 + y_{7V}^2}} F = (6.2 \pm 0.3)|a_s|,
\]

with \( \phi_e = 43.7^\circ \). The numerical expressions for \( C_{\text{mix}} \) and \( C_{\text{int}} \) in terms of \( |a_s| \) are computed assuming \( b_S/a_S = m_K^2/m_\rho^2 \) (or the same form factor for direct- and indirect-CP-violating components) and considering only the quadratic terms in (30); the quoted error reflects the impact of the linear terms. On the other hand, the l.h.s. equations in (34) are valid independently of any assumption on \( W_S(z) \), whose possible difference from the short-distance form factor \( f_+(z) \) is taken into account by the factor

\[
F = \frac{\int dz \lambda^{3/2}(1, z, r_s^2) [\Re [W_S^*(z)f_+(z)] - \Im [W_S^*(z)f_+(z)]]}{[\int dz \lambda^{3/2}(1, z, r_s^2)|W_S(z)|^2]^{1/2} \times [\int dz \lambda^{3/2}(1, z, r_s^2)|f_+(z)|^2]^{1/2}},
\]

where \( r_s = m_\pi/m_K \) and \( \lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) \). We also remark that in evaluating \( B(K_S \to \pi^0 e^+ e^-) \), \( \alpha = 1/137 \) has been used for the electromagnetic coupling. For a given branching ratio, a different choice of \( \alpha \) implies, effectively, a different normalization of \( |a_s| \) without changing the estimate of \( C_{\text{mix}} \).

Given the recent result in (32), we can conclude that, within the SM, the indirect-CP-violating amplitude is the largest component of \( B(K_L \to \pi^0 e^+ e^-) \) and the full CP-violating contribution is well above the CP-conserving one.

**Sign of the interference term**

As can be seen from (33), the relative sign between short- and long-distance contributions is a crucial ingredient for estimating the sensitivity of future high-statistics experiments to perform precision SM tests in this mode. A prediction of this sign requires a better understanding of the dynamical origin of the local couplings \( a_i \) and \( b_i \). To this purpose, we first note that the experimental determination of the ratio \( b_+/a_+ \) does not strictly follow naive dimensional analysis, which would predict \( b_+/a_+ = \mathcal{O}(m_\rho^2/(4\pi F_\pi)^2) \sim 0.2 \). Such a large ratio between \( \mathcal{O}(p^4) \) and \( \mathcal{O}(p^6) \) counterterms naturally points to the presence of large VMD contributions in these channels. Under the rather general hypothesis that the
\( b_i \) terms in Eq. (28) are entirely generated by the expansion of a vector-meson propagator, we can re-write the polynomial contribution to \( W_i(s) \) as

\[
G_F m_K^2 \left( \frac{a_i^{\text{VMD}}}{1 - zm_K^2/m_V^2} + a_i^{\text{nVMD}} \right) \approx G_F m_K^2 \left[ (a_i^{\text{VMD}} + a_i^{\text{nVMD}}) + a_i^{\text{VMD}} \frac{m_K^2}{m_V^2} z \right], \tag{36}
\]

where \( a_i^{\text{nVMD}} \) denotes \( z \)-independent non-VMD contributions.\(^2\) A comparison with the experimental results in Eq. (29) then leads to

\[
a_{\text{VMD}}^+ = \frac{m_V^2}{m_K^2} b_{\text{exp}} = -1.6 \pm 0.1, \quad a_{\text{nVMD}}^+ = a_{\text{exp}} - a_{\text{VMD}}^+ = 1.0 \pm 0.1. \tag{37}
\]

The large value of \( a_{\text{nVMD}}^+ \) can be justified, in the charged channel, by the presence of sizable pion-loop contributions. On the contrary, one would expect a pure VMD contribution in the \( K_S \) mode, where the pion term is negligible. We thus expect \( a_{\text{VMD}}^+ = 0 \) and \( a_{\text{VMD}}^+ = a_S \), an assumption that justifies the use of the same form factor for direct- and indirect-CP-violating components adopted to derive (33).

To make a theoretical prediction for \( a_S \) and, particularly, to determine its sign, we need to take an additional step. The contributions to the amplitude (27) generated by the leading short-distance operator in Eq. (1), namely \( Q_{7V} \), satisfy the \( \Delta I = 1/2 \) isospin relation

\[
(a_S)_{Q_{7V}} = -(a_+^{\text{VMD}})_{Q_{7V}} = \frac{m_K^2}{m_V^2} z, \tag{38}
\]

If this relation is obeyed by the full VMD amplitude, we should expect

\[
(a_S^{\text{VMD}})_{Q_{7V}} = -a_{\text{VMD}}^+ = 1.6 \pm 0.1. \tag{39}
\]

Given the various assumptions behind this prediction, the agreement with the experimental result in (32) is rather encouraging.\(^3\) This brings us to the likely hypothesis that the VMD part of the \( W_i(z) \) is dominated by the CP-conserving contribution induced by \( Q_{7V} \), in the limit of a low matching scale \( (m_K \lesssim \mu \lesssim m_V) \).\(^4\) The form factors computed under this assumption from the short-distance Hamiltonian cannot be trusted in detail, since \( z_{7V}^{Q_{7V}}(\mu) \) exhibits a strong scale dependence (that should be matched by the matrix-elements of four-quark operators) and we would like to extrapolate it beyond the validity region of perturbation theory. However, the VMD structure of the form factors ensures we don’t need to push this scale too low. Moreover, we shall employ this procedure only to fix the relative sign between direct- and indirect-CP-violating contributions. Using the

\(^2\) A similar generic decomposition between VMD and non-VMD contributions has been successfully tested in the \( K_L \to \pi^+\pi^-\gamma \) mode \(^{22,33}\).

\(^3\) The \( \Delta I = 1/2 \) relation between \( O(p^4) \) local counterterms of \( K^+ \to \pi^+e^+e^- \) and \( K_S \to \pi^0e^+e^- \) amplitudes had already been adopted in Ref. \(^{26}\) and later on also in Ref. \(^{30}\). However, in these works no distinction is made between VMD and non-VMD contributions to \( a_+^{\text{VMD}} \), which turns out to be a fundamental ingredient to obtain a phenomenologically acceptable prediction for \( a_S \).

\(^4\) Note that, consistently with this hypothesis, the contributions generated by \( Q_{7V} \) satisfy the relation \( (b_+/a_+)(Q_{7V}) = (b_S/a_S)(Q_{7V}) = m_K^2/m_V^2 \), which follows from the presence of the vector form factor \( f_+(z) \) in Eq. (3).
part of the short-distance Hamiltonian to determine the full $K_L \to \pi^0 e^+ e^-$ vector form factor, we find

$$W_L(z) = G_F m_K^2 \frac{4\pi y_{7V}(\mu)}{\sqrt{2\alpha}} f_+(z) \left[ e^i\phi z_{7V}(\mu) y_{7V}(\mu) - i\Im\lambda_t \right]. \quad (40)$$

Since the ratio $z_{7V}(\mu)/y_{7V}(\mu)$ is negative for $\mu < 1$ GeV \cite{7,8}, we conclude that the most likely possibility is a positive interference between direct- and indirect-CP-violating components.

The prediction of a positive interference between direct- and indirect-CP-violating components of $K_L \to \pi^0 e^+ e^-$ had already been reached in Ref. \cite{1}. This conclusion is reinforced by the observation that the perturbative value of $a_S$, computed from $(Q_{7V}(\mu))$, is smaller than the experimental result in (32) for $\mu = 1$ GeV, but it grows, i.e. it goes in the right direction, for smaller values of $\mu$.

The consistency of this prediction can be further confirmed by the following complementary reasoning: if we trust the $\Delta I = 1/2 +$ VMD argument leading to Eq. (39), we have fully established the sign of $a_S$ within chiral perturbation theory. This sign follows from the sign of $a_+$ which, in turn, is determined experimentally by the interference between local and non-local terms in (28). We have thus established the sign of $a_S$ in terms of the sign of $G_8$, the overall coupling of the $(8_L, 1_R)$ non-leptonic weak chiral Lagrangian (see e.g. Ref. \cite{34}). This implies that the $K_L \to \pi^0 e^+ e^-$ vector form factor can be written as

$$W_L(z) = G_F m_K^2 f_+(z) \left[ a_S e^i\phi \text{sgn}(G_8) - i\frac{4\pi y_{7V}(\mu)}{\sqrt{2\alpha}} \Im\lambda_t \right], \quad (41)$$

with a positive $a_S$. The sign of $G_8$ cannot be determined in a model-independent way; however, it can be predicted by the partonic Hamiltonian in (4) employing naive factorization. By doing so we find $\text{sgn}(G_8) < 0$ confirming the positive interference between direct- and indirect-CP-violating components.

**Total rate and present bounds on $\Im\lambda_t$**

Employing the positive sign in (33) we finally arrive at the prediction

$$B(K_L \to \pi^0 e^+ e^-)_{SM} = \left(3.2^{+1.2}_{-0.8}\right) \times 10^{-11} \quad (42)$$

where the error is completely dominated by the uncertainty in $B(K_S \to \pi^0 e^+ e^-)$ and, in view of (25), we have neglected the CP-conserving term. The possible improvements of this prediction with better data on the $K_S$ mode, and the related sensitivity to $\Im\lambda_t$, are illustrated in Fig. 3.

At present, Eq. (42) has to be compared with the new preliminary upper limit by KTeV

$$B(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{at \ 90\% C.L.,} \quad (43)$$

based on the full 1997+1999 data sample \cite{6}. Although we are still far from the possibility of precision SM tests in this mode, the combination of the preliminary results by KTeV...
Figure 3: SM Prediction for $B(K_L \rightarrow \pi^0 e^+ e^-)$ as a function of $\Im \lambda_t$, neglecting CPC contributions and assuming a positive interference between direct- and indirect-CP-violating components (see text). The three curves correspond to a central value $a_S = 1.08$ and no error (central full line); 5% error (dashed blue lines); 10% error (dashed green lines); present error (red dotted lines).

and NA48/1 allows us to derive significant bounds on realistic non-standard scenarios (see e.g. Ref. [10]). Instead of discussing any specific model in detail, we can simply express these bounds as the limits on $\Im \lambda_t$ dictated by $K_L \rightarrow \pi^0 e^+ e^-$. In particular, we find

$$-1.3 \times 10^{-3} < \Im \lambda_t < 1.0 \times 10^{-3} \quad \text{at 90\% C.L.},$$

which would reduce to $|\Im \lambda_t| < 1.3 \times 10^{-3}$ in the absence of any assumption about the interference term in (33). Together with the complementary and comparable constraints on $\lambda_t$ derived by $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ [25], these limits represent the most precise information presently available about the short-distance structure of the FCNC $s \rightarrow d$ amplitude.

5 Dalitz-plot analysis

The most serious problem in the extraction of the $K_L \rightarrow \pi^0 e^+ e^-$ amplitude from a time-independent rate measurement is the large irreducible background generated by the process $K_L \rightarrow \gamma \gamma e^+ e^-$. Imposing the cut $|M_{\gamma \gamma} - m_{\pi^0}| < 5$ MeV on the two-photon invariant mass spectrum of $K_L \rightarrow \gamma \gamma e^+ e^-$, the latter turns out to have a branching ratio...
more than $10^3$ times larger than the signal. As discussed by Greenlee [5], employing additional cuts on various kinematical variables it is possible to reduce this background down to the $10^{-10}$ level [5, 6], but it is hard to reduce it below this figure without drastic reductions of the signal efficiency. We stress, however, that this does not imply that the signal is unmeasurable in a high-statistic experiment, where the physical background can be measured and modelled with high accuracy [35].

An important point to notice is that the kinematical analysis necessary to suppress and control the Greenlee background provides also a powerful tool to discriminate the CP-conserving component of the $K_L \to \pi^0 e^+ e^-$ rate. As discussed in Section 3, theoretical arguments suggest that the CP-conserving component of $K_L \to \pi^0 e^+ e^-$ is very small; however, it is clearly desirable to cross-check this statement $a posteriori$, in a model-independent way, using $K_L \to \pi^0 e^+ e^-$ data.

As discussed by Greenlee [5], the most convenient kinematical variables to describe the decay distribution of $K_L \to \pi^0 e^+ e^-$ are

$$ z = \frac{(k_1 + k_2)^2}{m_K^2}, \quad \bar{y} = \frac{2 p \cdot (k_2 - k_1)}{m_K^2 \lambda^{1/2}(1, r_\pi^2, z)} , \quad (45) $$

whose uncorrelated boundaries (in the limit $m_e/m_K \to 0$) are given by

$$ 0 \leq z \leq (1 - r_\pi)^2 \quad -1 < \bar{y} < 1 . \quad (46) $$

In terms of these variables, CP-conserving and CP-violating distributions assume the following simple factorized structure:

$$ \frac{d^2 \Gamma(K_L \to \pi^0 e^+ e^-)_{\text{CPV}}}{dz \, d\bar{y}} \propto (1 - \bar{y}^2) \lambda^{3/2}(1, z, r_\pi^2) |W(z)|^2 , \quad (47) $$

$$ \frac{d^2 \Gamma(K_L \to \pi^0 e^+ e^-)_{\text{CPC}}}{dz \, d\bar{y}} \propto \bar{y}^2 (1 - \bar{y}^2) \lambda^{5/2}(1, z, r_\pi^2) |B(z)G(z)|^2 . \quad (48) $$

In Fig. 5 we plot the two distributions, as obtained by setting $W(z) = 1 + zm_K^2/m_V^2$ and approximating $B(z)G(z)$ to a constant: the $\bar{y}$ dependence clearly provides a powerful tool to discriminate the two terms.

To better quantify to which extent the two distributions can be distinguished, in Fig. 5 we show the result of a cut on $\bar{y}$ ($|\bar{y}| \leq |\bar{y}|_{\text{max}}$) on the integrated rate. The three curves correspond to the three non-interfering contributions of CP-conserving and CP-violating amplitudes, and Greenlee background (as obtained with a constant $K_L \to \gamma^* \gamma^*$ form factor), all normalized to one. As can be noted, setting the cut $|\bar{y}|_{\text{max}} = 0.5$ the CP-violating rate is reduced only by 30%, while the CP-conserving one drops almost by a factor of 4. The cut on $|\bar{y}|_{\text{max}}$ in not particularly efficient against the Greenlee background, but also in this case it improves the signal/background ratio.

## 6 Time-dependent interferences

Complementary information on the direct CP-violating component of the $K_L \to \pi^0 e^+ e^-$ amplitude can be obtained by studying the time evolution of the $K_{L,S} \to \pi^0 e^+ e^-$ decay.

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Figure 4: Dalitz plot distributions for CP-violating (left) and CP-conserving (right) contributions to $K_L \rightarrow \pi^0 e^+ e^-$ ($\tilde{y} \rightarrow$ vertical axis, $z \rightarrow$ horizontal axis).

Figure 5: CP-violating (full line) and CP-conserving (dashed line) contributions to the $K_L \rightarrow \pi^0 e^+ e^-$ rate (with arbitrary normalization) as a function of $|\tilde{y}|_{max}$. The dot-dashed line denotes Greenlee background.
Figure 6: Time dependent distributions for $\text{Im}(V_{ts}^*V_{td}) = 0$ (green full), $+1.3 \times 10^{-4}$ (red dotted) and $-1.3 \times 10^{-4}$ (blue dashed). Left: probability distribution for a $|K^0\rangle$ state at $t = 0$ to decay into $|\pi^0(\gamma\gamma)e^+e^-\rangle$, Greenlee background included. Right: Probability distribution for $\Phi \rightarrow K_L, K_S \rightarrow \pi^+\pi^-, \pi^0e^+e^-$ events at a $\Phi$ factory (see text).

Although challenging from the experimental point of view, this method has two intrinsic advantages: i) the interference between $K_S$ and $K_L$ amplitudes is only due to the CP-violating part of the latter; ii) the process $K_S \rightarrow \gamma\gamma e^+e^-$ is very suppressed with respect to $K_S \rightarrow \pi^0 e^+e^-$ [$B(K_S \rightarrow \gamma\gamma e^+e^-) \sim \text{few} \times 10^{-12}$], so the background due to the $|\gamma\gamma e^+e^-\rangle$ final state is almost negligible at small times ($t \ll \tau_L$). As representative examples of time-dependent observables, we shall discuss in more detail two specific cases: i) the time evolution of an initial $|K^0\rangle$ state, representative of a possible fixed-target experimental set up; ii) the time evolution of the $K_SK_L$ coherent state produced at a $\Phi$ factory [37].

In the case of a pure $|K^0\rangle$ beam at $t = 0$, with $N_0$ particles, the number of decays into the final state $|\pi^0 e^+e^-\rangle$ or in the background channel $|e^+e^-\gamma\gamma\rangle$ (with $M_{\gamma\gamma} \sim m_{\pi^0}$), as a function of the proper time $t$, can be written as

$$I_0(t) = \tau_S \frac{dN}{dt} = \tau_S N_0 \frac{2}{2} \left\{ |A_S|^2 e^{-t/\tau_S} + 2 \text{Re} \left[ A_{CPV}^L A_{S}^* e^{-i(m_L-m_S)t} \right] e^{-t(\tau_L+\tau_S)/2\tau_L\tau_S} \right. \right.$$  

$$+ \left. \left[ |A_{CPV}^L|^2 + |A_{CPV}^S|^2 + |A_{bkg}|^2 \right] e^{-t/\tau_L} \right\},$$

(49)

where the decay amplitudes $A_S$ and $A_{CPV}^L$ are those in Eqs. [28] and [11], and an overall phase-space integral is understood. The three curves in Fig. 6 have been obtained for $N_0 = 1$, assuming $\tau_S |A_S|^2 = B(K_S \rightarrow \pi^0 e^+e^-) = 6 \times 10^{-9}$ and $\tau_L |A_{bkg}|^2 = B(K_L \rightarrow \gamma\gamma e^+e^-)_{\text{cuts}} = 10^{-10}$, and employing the following three values of $\text{Im}(V_{ts}^*V_{td})$: 0, $\pm 1.3 \times 10^{-4}$. As can be seen, the interference term is quite sensitive to the value of the direct CP-violating amplitude. On a purely statistical level, in this example one could reach a
\[ \approx 15\% \text{ error on } \text{Im}(V_{ts}^*V_{td}) \text{ with an initial flux of } 10^{15}/e_{\pi^0 e e} \ K^0, \text{ where } e_{\pi^0 e e} \text{ denotes the efficiency for decays occurring within the first } 15 \ K_S \text{ decay lengths.} \]

At \( \Phi \) factories we can take advantage of the quantum properties of the \( K_S K_L \) state produced by the \( \Phi \) decay:

\[ A[\phi \to f_1(t_1), f_2(t_2) = \frac{1}{\sqrt{2}} [A_S(f_1, t_1)A_L(f_2, t_2) - A_S(f_2, t_2)A_L(f_1, t_1)] , \quad (50) \]

where \( f_i \) and \( t_i \) denote respectively final states and decay times. Choosing \( f_1 = |\pi^+\pi^-\rangle \), \( f_2 = |\pi^0 e^+ e^-\rangle \) and integrating over \( t_1 + t_2 \) \( 37 \), we obtain for \( t = t_1 - t_2 \)

\[ I_\Phi(t) = N_0 \frac{B(K_S \to \pi^+ \pi^-)B(K_S \to \pi^0 e^+ e^-)}{(1 + \tau_S/\tau_L)} e^{-|t|/(\tau_L + \tau_S)/2\tau_L\tau_S} \]

\[ \times \left\{ \left| \eta_i^+ \right|^2 e^{i(\tau_L - \tau_S)/2\tau_L\tau_S} + \frac{\tau_S}{\tau_L} \frac{B(K_L \to \pi^0 e^+ e^-)}{B(K_S \to \pi^0 e^+ e^-)} e^{-|t|/(\tau_L - \tau_S)/2\tau_L\tau_S} \right\} . \quad (51) \]

Also in this case we plot in Fig. 4 three curves corresponding to \( \text{Im}(V_{ts}^*V_{td}) = 0 \), and \( \pm 1.3 \times 10^{-4} \), obtained for \( \tau_S |A_S|^2 = B(K_S \to \pi^0 e^+ e^-) = 6 \times 10^{-9} \) and \( N_0 = 1 \) but, for simplicity, here we ignore the Greenlee background. The direct-CP-violating effect appears to be remarkably clear, but the statistics necessary to measure it is probably beyond the near-future possibility of existing facilities.

7 Conclusions

Motivated by recent experimental results from NA48 on \( K_L \to \pi^0 \gamma \gamma \) and \( K_S \to \pi^0 e^+ e^- \), which has been observed for the first time, we have presented a re-analysis of the decay mode \( K_L \to \pi^0 e^+ e^- \).

The results on \( K_L \to \pi^0 \gamma \gamma \) together with a more general treatment of the \( K_L \to \pi^0 \gamma^* \gamma^* \) form factor have led to an updated limit on the CP conserving contribution: \( B(K_L \to \pi^0 e^+ e^-)_{\text{CPC}} < 3 \times 10^{-12} \), which we consider as a conservative upper bound. This estimate implies that the CP conserving part is essentially negligible for \( B(K_L \to \pi^0 e^+ e^-) \).

The new measurement of \( B(K_S \to \pi^0 e^+ e^-) \) has enabled us to fix the size of the indirect CP-violating component in \( K_L \to \pi^0 e^+ e^- \). We have also provided arguments that strongly indicate a positive interference with the direct CP-violating amplitude. Combining both contributions we predict

\[ B(K_L \to \pi^0 e^+ e^-) = (3.2^{+1.2}_{-0.8}) \times 10^{-11} \quad (52) \]

in the Standard Model. The largest contribution is indirect CP-violation, but direct CP-violation is also substantial and amounts to \( \sim 40\% \) of the rate, mostly from interference of the two components. The sizable uncertainty which affects at present this prediction reflects the poor experimental knowledge of the \( K_S \to \pi^0 e^+ e^- \) transition. With more
precise data on the latter, the theoretical error on $B(K_L \to \pi^0 e^+ e^-)$ could in principle be reduced below the 10% level. In this perspective, we stress that a model-independent confirmation of the positive interference between direct and indirect CP-violating amplitudes, which could in principle be obtained by means of lattice QCD, would also be very useful.

As we have briefly discussed, the total rate is not the only interesting observable in this channel: the Dalitz plot analysis and the $K_L - K_S$ interference, in time-dependent distributions, could both be very useful in order to extract the direct CP-violating component of the $K_L \to \pi^0 e^+ e^-$ amplitude.

In summary, our analysis demonstrates that the rare decay $K_L \to \pi^0 e^+ e^-$, despite its complexity, can be well described exploiting additional experimental input. It implies, in particular, that its branching fraction is larger than might have been anticipated and has a substantial sensitivity to direct CP violation and New Physics in $\Delta S = 1$ transitions. If the experimental challenges posed by a measurement of $B(K_L \to \pi^0 e^+ e^-)$ can be overcome, this decay will provide us with most valuable information on quark flavour physics.

Acknowledgments

We thank Augusto Ceccucci for interesting discussions and we acknowledge his successful effort in keeping secret the NA48 result on $B(K_S \to \pi^0 e^+ e^-)$ till the public announcement. We are grateful also to Laurie Littenberg, Mara Martini, and Ivan Mikulec for useful conversations. The work of G.D and G.I. is partially supported by IHP-RTN, EC contract No. HPRN-CT-2002-00311 (EURIDICE).

Appendix: $\mathcal{O}(p^6)$ local contributions to $K^0 \to (\pi^0)\gamma\gamma$

Recent precise data by NA48 [3] and KTeV [38], and some recent theoretical papers [39, 40], have raised an interesting discussion about the determination of the $\mathcal{O}(p^6)$ local contributions to $K_L \to \pi^0 \gamma\gamma$ and the related role of vector-meson contributions [41]. As we shall show in the following, a new important piece of information in this respect is provided by the very precise measurement of the $K_S \to \gamma\gamma$ rate [42]. Using chiral symmetry, we can relate counterterm contributions to $K_L \to \pi^0 \gamma\gamma$ and $K_S \to \gamma\gamma$ amplitudes. This leads to a substantial reduction of the allowed parameter space for the $\mathcal{O}(p^6)$ couplings and, as a consequence, to a better understanding of the vector-meson-dominance (VMD) ansatz.

The $(8_L, 1_R)$ weak chiral Lagrangian of $\mathcal{O}(p^6)$ contains a huge number of operators. However, as long as we are interested only in $K_L \to \pi^0 \gamma\gamma$ and $K_S \to \gamma\gamma$ decays, we can restrict the attention only to three independent combinations [16]. In particular, we can parameterize all the contributions in terms of the following simplified effective
Lagrangian\(^5\)

\[
\mathcal{L}_6 = \frac{G_s \alpha}{4\pi} \left( a_1 F_{\mu\nu} F^{\mu\nu} \langle \Delta \chi \rangle + a_2 F_{\mu\nu} F^{\mu\nu} \langle \Delta u^\mu u_\mu \rangle + a_3 F_{\mu\lambda} F^{\mu\sigma} \langle \Delta \{u^\lambda, u_\sigma\} \rangle \right),
\]

(A.1)

where, following the notation of Ref. [31], we define \(\Delta = u^\lambda u^\dagger_\lambda\), \(u^\mu = iu^\dagger D^\mu U u^\dagger\), and \(\chi^+ = u^\dagger M u^\dagger + M u\). The first operator in (A.1) contributes both to \(K_L \to \pi^0 \gamma \gamma\) and \(K_S \to \gamma \gamma\), whereas the other two can contribute only to \(K_L \to \pi^0 \gamma \gamma\).

Historically, the \(O(p^6)\) local contributions to \(A\) and \(B\) amplitudes of \(K_L \to \pi^0 \gamma \gamma\) are parameterized in terms of the three unknown coefficients \(\alpha_1, \alpha_2\) and \(\beta\), defined by [16]

\[
\begin{align*}
A_{CT} &= \alpha_1 (z - r^2_\pi) + \alpha_2, \\
B_{CT} &= \beta.
\end{align*}
\]

(A.2)

Using the Lagrangian (A.1) we find

\[
\begin{align*}
\alpha_1 &= \frac{m_K^2}{F_\pi^2} (4a_2 + 2a_3), \\
\alpha_2 &= \frac{m_K^2}{F_\pi^2} [8a_1 - 4a_2 + 2a_3] - 0.65, \\
\beta &= \frac{m_K^2}{F_\pi^2} (-4a_3) - 0.13,
\end{align*}
\]

(A.3)

where the extra numerical pieces in \(\alpha_2\) and \(\beta\) are residual polynomial parts of the loop amplitudes, renormalized in the minimal subtraction scheme at the scale \(\mu = m_\rho\) [16]. It is then easy to check that the VMD ansatz of Ref. [19] corresponds to the following choice for the \(a_i\):

\[
\begin{align*}
a_1 &= 0, \\
a_2 &= -\frac{m_K^2}{F_\pi^2}, \\
a_3 &= 2a_V.
\end{align*}
\]

(A.4)

The local contributions in (A.2) have to be added to the unitarized loop amplitudes to obtain the full result. In the case of the \(B\) amplitude this leads to [16]

\[
B(z) = \beta + c \times \left\{ \frac{4r^2_\pi}{z} F \left( \frac{z}{r^2_\pi} \right) + \frac{2}{3} \left( 10 - \frac{z}{r^2_\pi} \right) \left[ \frac{1}{6} + R \left( \frac{z}{r^2_\pi} \right) \right] + \frac{2}{3} \log \left( \frac{m_\rho^2}{m_K^2} \right) \right\},
\]

(A.5)

where \(c\) is the coefficient, determined from \(K \to 3\pi\) quadratic slopes, which rules the strength of unitarity corrections (in the numerical analysis we shall employ the value \(c = 1.1\)) and the explicit expression of \(F(z)\) and \(R(z)\) can be found in [16]. Taking into account that \(F(z) \to -z/12\) and \(R(z) \to z/60\) for \(z \to 0\), it follows that \(B(0) = -1.71 + \beta\).

As shown in Eqs. (14)–(15), the magnitude of \(B(0)\) is determined in a model-independent

\[^5\]The strongest simplification arises from the fact that we are dealing only with neutral fields, which commute with the charge matrix \(Q\). For this reason, the latter is not explicitly inserted in the effective operators in (A.1).
way by the measurement of $B(K_L \to \pi^0\gamma\gamma)$ at low diphoton invariant mass. Starting from the NA48 result

$$B(K_L \to \pi^0\gamma\gamma)_{M_{\gamma\gamma} \in [30-110] \text{ MeV}}, \ |p| \in [0-0.2] < 0.6 \times 10^{-8} \text{ at } 90\% \text{C.L.},$$

(A.6)

which under the assumption of a constant $B$ amplitude becomes

$$B(K_L \to \pi^0\gamma\gamma)_{M_{\gamma\gamma} < 110 \text{ MeV}} < 0.9 \times 10^{-8} \text{ at } 90\% \text{C.L.},$$

(A.7)

we can derive a first clear bound on the $O(p^6)$ counterterms, namely

$$-0.4 < \beta < 3.8, \quad -1.0 < \frac{m_K^2}{F^2_\pi} a_3 < 0.07.$$  

(A.8)

A more precise constraint is obtained by means of $B(K_S \to \gamma\gamma)$. In this case most of the unitarity corrections are implicitly taken into account by the overall coupling $G_8$, extracted from the measured $K_S \to \pi^+\pi^-$ rate. Non-trivial $O(p^6)$ terms are expected only from the (non-VMD) local contributions proportional to $a_1$, and from the tiny unitarity corrections associated to the process $K_S \to \pi^0\pi^0 \to \gamma\gamma$ [20]. The recent precise measurement of NA48, $B(K_S \to \gamma\gamma) = (2.78 \pm 0.072) \times 10^{-6}$ [42], is substantially higher than the $O(p^4)$ prediction, $B(K_S \to \gamma\gamma)^{(4)} = 2.1 \times 10^{-6}$ [43], indicating for the first time the need for these $O(p^6)$ terms. Summing the local term generated by $L_6$ to the leading loop amplitude of $O(p^4)$ [43] we can write

$$A(K_S \to \gamma\gamma) = -i \frac{2\alpha G_8 F_\pi}{\pi m_K^2} (M_K^2 - m^2) \left[ F \left( \frac{1}{r^2_\pi} \right) + \frac{2m_K^2}{F^2_\pi} a_1 \right] \epsilon_1 \bar{\epsilon}_2 [q_1^\mu q_2^\nu - (q_1 q_2) g^{\mu\nu}] .$$

(A.9)

where the relative sign between local and non-local terms has been unambiguously fixed by the sign convention adopted in (A.3). Using $a_1$ as a free parameter to fit the experimental branching ratio we finally obtain

$$\frac{8m_K^2}{F^2_\pi} a_1 = (\alpha_1 + \alpha_2 + \beta) + 0.78 = 1.0 \pm 0.3.$$  

(A.10)

The error in (A.10) is not due to the experimental uncertainty, but is a theoretical estimate of possible subleading terms: the $\pm 1\sigma$ value in (A.10) corresponds to the following two cases: i) Eq. (A.9) only; ii) Eq. (A.9) plus the absorptive contribution due to $K_S \to \pi^0\pi^0 \to \gamma\gamma$.

Taking into account the constraints in (A.10) and (A.8) we are basically left with a single free parameter to fit both rate and high diphoton invariant mass spectrum of $K_L \to \pi^0\gamma\gamma$. Apparently, the constraint on $\beta$ in (A.8) is not very stringent; however, this does not represent a serious limitation. The situation becomes much more clear if we adopt the $S-B$ basis for the $K_L \to \pi^0\gamma\gamma$ amplitude ($S = A + B$): in this framework the bound in (A.8) tells us that the $B$ amplitude plays a very marginal role – but for very low diphoton invariant masses (see Section 3) – and that both rate and high diphoton invariant
mass spectrum of $K_L \to \pi^0\gamma\gamma$ are completely dominated by $S$. The latter depend only on two independent counterterm combinations, namely $\alpha_1$ and $(\alpha_2 + \beta)$, whose sum is severely constrained by (A.10). It is then clear that we are left with a single effective coupling, which we chose to be $\alpha_1$. Summing the local contributions to the unitarized loop amplitude and fitting the NA48 result $B(K_L \to \pi^0\gamma\gamma) = (1.36 \pm 0.05) \times 10^{-6}$, we find

$$\alpha_1 = \frac{m_K^2}{F^2_\pi}(4a_2 + 2a_3) = 3.4 \pm 0.4 \text{ ,}$$ (A.11)

where the error reflects mainly the uncertainty in (A.10).

A detailed discussion of the results in (A.8), (A.10) and (A.11) goes beyond the scope of this work and will be presented elsewhere. The quoted figures should also be taken with some care, since they are not based on a complete $\chi^2$ analysis. On the other hand, we can already draw a few interesting conclusions:

- The local structure of $K_S \to \gamma\gamma$ and $K_L \to \pi^0(\gamma\gamma)_{J=0}$ amplitudes is strongly constrained by chiral symmetry. Taking into account this theoretical constraint and the recent precise data by NA48, we are able to fix in a precise way the two independent combinations of counterterms, $a_1$ and $2a_2 + a_3$, which control these amplitudes. Interestingly, these results rule out completely or strongly constrain most of the solutions proposed in [40], taking into account $K_L \to \pi^0\gamma\gamma$ data only.

- As discussed in Section 3, the smallness of the $K_L \to \pi^0(\gamma\gamma)_{J=2}$ amplitude is a direct consequence of the experimental result in (A.6). We thus agree with the conclusion of Ref. [40] that the estimate of the CPC contribution to $K_L \to \pi^0e^+e^-$ does not depend on specific assumptions about $K_L \to \pi^0\gamma\gamma$ counterterms. Since the low diphoton invariant mass analysis of NA48 contradicts previous findings by KTeV [38], an independent confirmation of (A.6) would be very welcome.

- The evidence for a non-vanishing $a_1$ in (A.10) shows that the VMD ansatz (A.4) is not exactly fulfilled. Nonetheless, VMD contributions are likely to provide the dominant $O(\rho^0)$ contribution. This statement is not very quantitative at the moment due to the uncertainty on $a_3$ in (A.8). However, we note that the values for $a_3$ preferred by the low diphoton invariant mass analysis of NA48 are those close to the boundaries of (A.8): $(m_K^2/F_\pi^2)a_3 \approx -0.9$ or 0.1. In other words, the $B$ amplitude is likely to be just below the exclusion limit implied by (A.6). If $(m_K^2/F_\pi^2)a_3 \approx -0.9$,

---

6 The central value in (A.11) has been obtained with the inclusion of the absorptive contribution due to $K_L \to 3\pi^0 \to \pi^0\gamma\gamma$ in the non-local $A$ amplitude [20]: with this choice, we find that the predicted $K_L \to \pi^0\gamma\gamma$ spectrum is in excellent agreement with the observation of NA48 [3]. If the absorptive contribution due to $K_L \to 3\pi^0 \to \pi^0\gamma\gamma$ is not included, the central value of $\alpha_1$ fitted from the rate turns out to be about 20% higher, but the resulting diphoton spectrum is not in very good agreement with NA48 results. We have also explicitly checked that the central value in (A.11) is completely insensitive to the choice of $\beta$, varied within the interval (A.8).
taking into account also the constraint (A.11), we would conclude that

\[
\frac{m_K^2}{F_\pi^2} (a_3 + a_2) \approx +0.4 \quad \text{VMD} \xrightarrow{\rightarrow} 0 ,
\]

\[
\frac{m_K^2}{F_\pi^2} (a_3 - a_2) \approx -2.2 \quad \text{VMD} \xrightarrow{\rightarrow} 4a_V ,
\]

which can be interpreted as a clear manifestation of vector dominance.

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