Free multiflows in bidirected and skew-symmetric graphs

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Abstract

A graph (digraph) G = (V, E) with a set T ⊆ V of terminals is called inner Eulerian if each nonterminal node v has even degree (resp. the numbers of edges entering and leaving v are equal). Cherkassky [1] and Lovász [15] showed that the maximum number of pairwise edge-disjoint T-paths in an inner Eulerian graph G is equal to \( \frac{1}{2} \sum_{s \in T} \lambda(s) \), where \( \lambda(s) \) is the minimum number of edges whose removal disconnects s and T − {s}. A similar relation for inner Eulerian digraphs was established by Lomonosov [14]. Considering undirected and directed networks with “inner Eulerian” edge capacities, Ibaraki, Karzanov, and Nagamochi [10] showed that the problem of finding a maximum integer multiflow (where partial flows connect arbitrary pairs of distinct terminals) is reduced to \( O(\log T) \) maximum flow computations and to a number of flow decompositions.

In this paper we extend the above max-min relation to inner Eulerian bidirected and skew-symmetric graphs and develop an algorithm of complexity \( O(V E \log T \log(2 + V^2/E)) \) for the corresponding capacitated cases. In particular, this improves the bound in [10] for digraphs. Our algorithm uses a fast procedure for decomposing a flow with \( O(1) \) sources and sinks in a digraph into the sum of one-source-one-sink flows.

KeyWords: bidirected graph, skew-symmetric graph, edge-disjoint paths, multiflow.
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1 Introduction

A graph (digraph) \(G = (V, E)\) with a distinguished subset \(T\) of nodes is said to be inner Eulerian if each node \(v \in V - T\) has even degree (resp. the indegree and outdegree of \(v\) are equal). The nodes in \(T\) and in \(V - T\) are called terminals and inner nodes, respectively. A simple path in \(G\) is called a \(T\)-path if its ends are distinct terminals and the other nodes are inner. There is a nice max-min relation established by Cherkassky [1] and Lovász [15] for graphs, and by Lomonosov [14] for digraphs:

(1) If \((G, T)\) is inner Eulerian, then the maximum number \(\nu_{G,T}\) of pairwise edge-disjoint \(T\)-paths is equal to \(\frac{1}{2} \sum_{s \in T} \lambda_{G,T}(s)\).

Here and later on, for a subset \(X\) of nodes, \(\delta(X) = \delta_G(X)\) denotes the set of edges with one end in \(X\) and the other in \(V - X\), called the cut induced by \(X\). For \(s \in T\), we refer to a subset \(X \subseteq V\) with \(X \cap T = \{s\}\) as an \(s\)-set. Then \(\lambda_{G,T}(s)\) is defined to be the minimum cardinality \(|\delta(X)|\) among the \(s\)-sets \(X\).

The above max-min relation has an obvious extension to the capacitated case. Given a nonnegative integer function \(c : E \to \mathbb{Z}_+\) of edge capacities, let us say that the triple \((\text{network}) (G, T, c)\) is inner Eulerian if for each inner node \(v\), the total capacity of edges incident with \(v\) is even when \(G\) is a graph, and the total capacity of edges entering \(v\) is equal to that of edges leaving \(v\) when \(G\) is a digraph. Then (1) yields the following relation for an inner Eulerian \((G, T, c)\):

(2) \[
\max\{\text{val}(\mathcal{F})\} = \frac{1}{2} \sum_{s \in T} \lambda_{c,T}(s),
\]

where \(\lambda_{c,T}(s)\) denotes the minimum cut capacity \(c(\delta(X))\) among the \(s\)-sets \(X\), the maximum is taken over all collections \(\mathcal{F}\) of \(T\)-paths \(P_1, \ldots, P_k\) along with nonnegative integer weights \(\alpha_1, \ldots, \alpha_k \in \mathbb{Z}_+\) that satisfy the packing condition

(3) \[
\sum (\alpha_i : e \in P_i) \leq c(e) \quad \text{for all } e \in E,
\]

and \(\text{val}(\mathcal{F})\) denotes the total value of \(\mathcal{F}\), defined to be \(\alpha_1 + \ldots + \alpha_k\). (Hereinafter for a function \(g : S \to \mathbb{R}\) and a subset \(S' \subseteq S\), \(g(S')\) stands for \(\sum_{e \in S'} g(e)\).)

A collection \(\mathcal{F}\) consisting of \(T\)-paths \(P_i\) with real weights \(\alpha_i \in \mathbb{R}_+\) that obeys (3) is called a free multiflow (the adjective “free” is used to emphasize that any pair of distinct terminals is allowed to be connected by a path, i.e., the commodity graph in the multiflow maximization problem is complete). A multiflow achieving the equality in (2) is called maximum. Thus, whenever \((G, T, c)\) is inner Eulerian, there exists an integer maximum free multiflow (i.e., having the weights of all paths integral).

Cherkassky [1] showed that such a multiflow in an inner Eulerian undirected network can be found in strongly polynomial time. Subsequently much faster algorithms both for graphs and digraphs have been developed. They apply a “divide-and-conquer” approach in which a current network \((G, T, c)\) with \(|T| \geq 4\) is recursively

\(^1\)Originally relation (2) was stated for fractional multiflows \(\mathcal{F}\) in [12], with a flaw in the proof.
replaced by two networks \((G', T', c')\) and \((G'', T'', c'')\) such that \(|T'|, |T''| \leq \lceil |T|/2 \rceil + 1\). Originally, such an approach was applied in [11] to find, in \(O(\phi(V, E) \log T)\) time, a half-integer maximum multiflow in a graph \(G\) with integer edge capacities (but not guaranteeing integrality in the inner Eulerian case). Hereinafter, in notation involving functions of numerical arguments or time bounds, we indicate sets for their cardinalities, and \(\phi(n, m)\) stands for the complexity of an algorithm for finding a maximum flow in a network with \(n\) nodes and \(m\) edges.

This algorithm was improved and extended in [10] so as to find an integer maximum free multiflow in an inner Eulerian undirected network in the same time \(O(\phi(V, E) \log T)\), and in an inner Eulerian directed network in \(O(\phi(V, E) \log T + V^2 E)\) time.

**Remark 1.** The inner Eulerianness condition is important. Withdrawing it makes the undirected problem more difficult, though still polynomially solvable in the non-capacitated case (a max-min relation is due to Mader [17] and an original polynomial algorithm is due to Lovász [16]), and makes the directed noncapacitated problem NP-hard already for two terminals [4].

The purpose of this paper is to extend the above theoretical and algorithmic results to bidirected graphs. (This sort of nonstandard graphs was introduced by Edmonds and Johnson [2] in connection with one important class of integer linear programs generalizing problems on flows and matchings; for a survey, see also [13, 18].)

Recall that in a bidirected graph \(G = (V, E)\) three types of edges are allowed: (i) a usual directed edge, or an arc, that leaves one node and enters another one; (ii) an edge from both of its ends; or (iii) an edge to both of its ends. When \(u = v\), the edge becomes a loop; in what follows we admit only loops of types (ii) and (iii) (as loops of type (i) do not affect our problem and can be excluded from consideration).

A nonloop edge entering a node \(v\) contributes one unit to the indegree \(\deg^\text{in}(v)\) of \(v\), while a loop of type (iii) at \(v\) contributes two units to \(\deg^\text{in}(v)\); the outdegree \(\deg^\text{out}(v)\) of \(v\) is specified in a similar way. Edges \(e, e'\) connecting nodes \(u, v\) are called parallel if \(e\) enters \(u\) if and only if \(e'\) does so, and similarly for \(v\). If \(G\) has no parallel edges, then \(|E| \leq 2|V|^2\). An instance of bidirected graphs is drawn in Fig. 1.

The notion of inner Eulerianness for a bidirected graph \(G\) with a set \(T\) of terminals is analogous to that for usual digraphs: \(\deg^\text{in}(v) = \deg^\text{out}(v)\) for all inner nodes \(v\). Inner Eulerian triples \((G, T, c)\), where \(c : E \to \mathbb{Z}_+\), are those that turn into inner Eulerian pairs \((G', T)\) when each edge \(e\) is replaced by \(c(e)\) parallel edges.

In order to be able to extend the above results to bidirected graphs, we need to admit \(T\)-paths with restricted self-intersections. (In undirected or directed graphs, when one node is reachable from another one by a path, then it is reachable by a simple path as well, but this need not hold in a bidirected graph.) A walk in a bidirected graph \(G\) is an alternating sequence \(P = (s = v_0, e_1, v_1, \ldots, e_k, v_k = t)\) of nodes and edges such that each edge \(e_i\) connects nodes \(v_{i-1}\) and \(v_i\), and for \(i = 1, \ldots, k - 1\), the edges \(e_i, e_{i+1}\) form a transit pair at \(v_i\), which means that one of \(e_i, e_{i+1}\) enters and
the other leaves $v_i$. Note that $e_1$ may enter $s$ and $e_k$ may leave $t$; nevertheless, we refer to $P$ as a walk from $s$ to $t$, or an $s$–$t$ walk. $P$ is a cycle if $v_0 = v_k$ and the pair $e_1, e_k$ is transit at $v_0$; a cycle is usually considered up to cyclically shifting. Observe that an $s$–$s$ walk is not necessarily a cycle. By a path in a bidirected graph $G$ we mean an edge-simple walk $P$, i.e., a walk with all edges different. Similar to usual graphs/digraphs, a $T$-path (a $T$-walk) is meant to be a path (resp. walk) whose ends are distinct terminals and the other nodes are inner.

Define the number $\lambda_{G,T}(s)$ for $s \in T$ as before. We show the following

**Theorem 1.1** Property (1) remains valid for a bidirected graph $G$ and the set of $T$-paths in it.

**Remark 2.** In this theorem it suffices to consider only minimal $T$-paths, where a path (edge-simple cycle) $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ is called minimal if no part of $P$ from $v_i$ to $v_j$ with $i < j$ (resp. $0 < j - i < k$) forms a cycle. (A minimal path/cycle need not be simple but it passes any node of $G$ at most twice.) Moreover, one can consider only those $T$-paths whose induced bidirected graphs contain no cycle. (In the underlying undirected graph of such a path, each edge belongs to at most one circuit.)

A usual digraph is a special case of bidirected graphs and Theorem 1.1 generalizes the above-mentioned result in [1]. Also there is a natural correspondence between the $T$-paths in an undirected graph $G$ and the minimal $T$-paths in the bidirected graph $G'$ formed from $G$ as follows: direct each edge of $G$ from both of its ends, and for each inner node $v$, assign $\lceil \deg(v)/2 \rceil$ loops entering (twice) $v$. Then a $T$-path $P$ in $G$ is turned into a $T$-path in $G'$ by adding one loop to each intermediate node of $P$. Moreover, $(G', T)$ is inner Eulerian if $(G, T)$ is such. Due to this correspondence, Theorem 1.1 generalizes the above-mentioned Cherkassky–Lovász’ result for undirected graphs as well.

Like the pure graph and digraph cases, one can reformulate Theorem 1.1 in capacitated terms: relation (2) concerning integer free multiflows $F$ remains valid when $G$
is bidirected and \((G, T, c)\) is inner Eulerian. In this case one should consider \(T\)-walks, rather than \(T\)-paths, and refine the packing condition \((3)\) as

\[
\sum_{i=1}^{k} \alpha_i n_i(e) \leq c(e) \quad \text{for all } e \in E,
\]

where \(n_i(e)\) is the number of occurrences of an edge \(e\) in a walk \(P_i\). Thus, the above problem for undirected and directed networks is generalized as:

\[(P) \quad \text{Given an inner Eulerian network } (G, T, c), \text{ where } G \text{ is bidirected, find a collection (free multiflow) } F \text{ of } T\text{-walks } P_1, \ldots, P_k \text{ with weights } \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_+ \text{ that satisfies } (4) \text{ and maximizes the value } \text{val}(F) := \alpha_1 + \ldots + \alpha_k.\]

**Remark 3.** Let \(X\) be an arbitrary subset of nodes of \(G\). One can modify \(G\) as follows: for each node \(v \in X\) and each edge \(e\) incident with \(v\), reverse the direction of \(e\) at \(v\). Also for an arbitrary arc \(e\) incident with a terminal \(s\), one can reverse the direction of \(e\) at \(s\). Both transformations preserve the inner Eulerianness of \((G, T, c)\) and the set of \(T\)-walks. Therefore, problem \((P)\) remains equivalent under such transformations.

Another appealing class of nonstandard graphs was introduced by Tutte [20] who originated a mini-theory, parallel to [2] in a sense, aiming to unify and generalize flow and matching problems. These are so-called skew-symmetric graphs (or antisymmetrical digraphs, in Tutte’s terminology), digraphs with involutions on the nodes and on the arcs which reverse the orientation of each arc (a precise definition is given in Section 3). His and other researchers’ study of structural and optimization problems on skew-symmetric graphs has resulted in a number of interesting theorems, methods and applications.

There is a close relationship between skew-symmetric and bidirected graphs, and typically results on the former can be reformulated for the latter, and vice versa. So is for the problem of our study, too. We take advantage from both representations. The language of bidirected graphs is more preferable for us to work in the non-capacitated case; we prove Theorem 1.1 directly and obtain its analog for skew-symmetric graphs as a corollary. On the other hand, we prefer to deal with skew-symmetric graphs in algorithmic design for the capacitated case. (A serious reason is that a flow in a bidirected network is defined as a packing of \(T\)-paths and we do not see reasonable alternative settings for it, while a flow in a skew-symmetric network can be given in a more compact form, via a function on the arc set.) Some facts about skew-symmetric flows and technical tools elaborated for them help us to devise a fast algorithm for the skew-symmetric analog of problem \((P)\) concerning integer skew-symmetric free multiflows in an inner Eulerian skew-symmetric network. This yields a fast algorithm for \((P)\) as well.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 (which is relatively simple) relying on the fact that an inner Eulerian bidirected graph can be decomposed into cycles and paths with both ends in \(T\). Section 3 explains the
correspondence between bidirected and skew-symmetric graphs, reviews some known results about the latter (in particular, Tutte’s result on symmetric decompositions of skew-symmetric flows) and gives a skew-symmetric analog of Theorem 1.1. Section 4 develops an algorithm for finding a maximum integer skew-symmetric free multiflow in an inner Eulerian skew-symmetric network. It relies on a general approach in [11] and some ingredients from [10] and attracts additional combinatorial ideas and techniques. As a consequence, problem (P) is solved in time \(O(VE \log T \log(2 + V^2/|E|))\) (if the \(O(nm \log(2 + n^2/m))\)-algorithm of Goldberg and Tarjan [9] is applied for finding a maximum flow in a directed network with \(n\) nodes and \(m\) arcs). This improves the bound in [10] for digraphs. To achieve this bound, we use a faster procedure for the particular flow decomposition problem: given an integer flow \(f\) with \(O(1)\) sources and sinks in a digraph with \(n\) nodes and \(m\) arcs, decompose \(f\) into the sum of integer flows, each connecting one source to one sink. The procedure developed in Section 5 solves this problem in \(O(m \log(2 + n^2/m))\) time. In the concluding Section 6, this procedure is extended to symmetric flows in skew-symmetric graphs (it is not used in the algorithm for (P) but may be of interest for other applications).

2 Proof of Theorem 1.1

Let \(G = (V, E)\) be an inner Eulerian bidirected graph with a terminal set \(T\). One may assume that \(G\) has no loops incident with terminals. Since \((G, T)\) is inner Eulerian, for each inner node \(v\), one can choose a set \(\pi(v)\) of transit pairs at \(v\) so that each non-loop edge incident with \(v\) occurs in exactly one pair and each loop at \(v\) (if any) occurs in two pairs. The collection \(\{\pi(v) : v \in V - T\}\) determines a decomposition of \((\text{the edge-set of})\ G\) into a collection \(C\) of edge-simple cycles and a collection \(P\) of paths with both ends at \(T\). More precisely, each edge \(e \in E\) belongs to exactly one member \(P\) of \(C \cup P\) and satisfies the following condition: for each end \(v\) of \(e\), if \(v \in V - T\) and \(\{e, e'\} \in \pi(v)\), then either \(e, v, e'\) or \(e', v, e\) are three consecutive elements in \(P\), while if \(v \in T\), then \(P\) begins with \(v, e\) or ends with \(e, v\). Note that all nodes of any cycle in \(C\) and all intermediate nodes of any path in \(P\) are inner. So each path in \(P\) is a \(T\)-path unless it connects equal terminals. When needed, we may reverse some paths in \(P\).

For \(s \in T\), let \(P_s (Q_s)\) denote the set of paths in \(P\) with exactly one end (resp. with both ends) at \(s\). Since \(|P_s| + 2|Q_s| = \deg(s) \geq \lambda_{G,T}(s)\) (where \(\deg(v)\) is the full degree \(\deg^\text{out}(v) + \deg^\text{in}(v)\) of \(v\)), the theorem becomes trivial when all sets \(Q_s\) are empty. In a general case, we try to transform the decomposition so as to increase the “useful value” \(\eta(P, C) := \sum_{s \in T} |P_s|\), by applying a certain augmenting approach.

Consider \(s \in T\) and assume, w.l.o.g., that all paths in \(P_s\) begin at \(s\). Let \(L = (x_0, x_1, \ldots, x_q)\) be a sequence of distinct nodes such that

\[(5) \quad \text{either } L = \{s\}, \text{ or } x_0 \text{ belongs to a path in } Q_s, \text{ and for } i = 1, \ldots, q, \text{ the nodes } x_i, x_{i-1} \text{ occur in a cycle in } C \text{ or occur in this order in a path in } P_s.\]
We say that \( L \) is augmenting if \( x_q \) belongs to a path in \( \mathcal{P} \) having both ends in \( T - \{s\} \). Consider two cases.

**Case 1.** There is no augmenting sequence for \( s \). Let \( X_s \) be the set of all nodes occurring in sequences \( L \) as in (5). Clearly \( X_s \cap T = \{s\} \). Consider an edge \( e \) of the cut \( \delta(X_s) \); let \( u, v \) be the ends of \( e \) in \( X_s \) and \( V - X_s \), respectively. Observe that \( e \) belongs to neither a cycle in \( \mathcal{C} \) nor a path in \( \mathcal{P} - \mathcal{P}_s \). Also (by (5)) if \( e \) belongs to a path \( P \in \mathcal{P}_s \), then all nodes of \( P \) from \( s \) to (the last occurrence of) \( u \) are contained in \( X_s \), i.e., \( P \) traverses the cut \( \delta(X_s) \) exactly once. This implies \( |\mathcal{P}_s| = |\delta(X_s)| \).

Hence, if none of terminals admits an augmenting sequence as above, then the number of \( T \)-paths in \( \mathcal{P} \) is at least \( \frac{1}{2} \sum_{s \in T} \lambda_{G,T}(s) \), as required.

**Case 2.** An augmenting sequence \( L = (x_0, x_1, \ldots, x_q) \) for \( s \) exists. Let \( L \) be chosen so that no proper subsequence in it is augmenting. Then:

(6) any cycle in \( \mathcal{C} \) meets at most two nodes in \( L \) and these nodes are consecutive in \( L \);

(7) if a path \( P \in \mathcal{P}_s \) contains a node \( x_i \), then the part of \( P \) from \( s \) to (the last occurrence of) \( x_i \) can contain at most one node \( x_j \) with \( j > i \); moreover, if such an \( x_j \) exists then \( j = i + 1 \).

We transform \((\mathcal{P}, \mathcal{C})\) along \( L \), step by step, as follows. Choose a \( Q \in \mathcal{Q}_s \) containing \( x_0 \). At the first step, if (a) \( x_0, x_1 \) belong to a cycle \( C \in \mathcal{C} \), then we combine \( Q \) and \( C \) into one \( s-s \) path. And if (b) \( x_0, x_1 \) belong to a path \( P \in \mathcal{P}_s \) from \( s \) to \( t \), say, and if \( x_1 \) occurs in \( P \) earlier than \( x_0 \), then we replace \( Q \) by the concatenation of the part \( P' \) of \( P \) from \( s \) to (the last occurrence of) \( x_0 \) and the part \( Q' \) of \( Q \) from \( x_0 \) to \( s \), and replace \( P \) by the concatenation of the rest of \( Q \) (from \( s \) to \( x_0 \)) and the rest of \( P \) (from \( x_0 \) to \( t \)). (We assume, w.l.o.g., that the last edge of \( P' \) and the first edge of \( Q' \) form a transit pair at \( x_0 \); otherwise reverse \( Q \).) As a result, we obtain an \( s-s \) path, denoted by \( Q \) as before, that contains \( x_1 \). In case (a), the cycle \( C \) vanishes, and in case (b), the new path \( P \) goes from \( s \) to \( t \) as before, and its part from \( x_0 \) to \( t \) preserves. This together with (6) and (7) implies validity of (5) for the remaining sequence \((x_1, \ldots, x_q)\); moreover, (6) and (7) are maintained as well. At the second step, we consider the pair \( x_1, x_2 \) and act in a similar way, and so on.

Eventually, after \( q \) steps, the current \( s-s \) path \( Q \) contains the node \( x_q \). Since \( L \) is augmenting, \( x_q \) also belongs to some \( t-p \) path \( R \in \mathcal{P} \) with \( t, p \in T - \{s\} \) (possibly \( p = t \)). Now splitting \( Q, R \) at \( x_q \) and concatenating the arising four pieces in another way, we obtain two \( T \)-paths, one connecting \( s \) and \( p \) and the other connecting \( s \) and \( t \). This gives a new decomposition \((\mathcal{P}, \mathcal{C})\) of \( G \) having a larger value of \( \eta \), and the theorem follows. ■

In fact, the above proof is constructive and prompts a polynomial algorithm for finding a maximum number of pairwise edge-disjoint \( T \)-paths in an inner Eulerian bidirected graph. A more efficient and more general algorithm (dealing with the capacitated case) is described in Section 4.
3 Skew-Symmetric Graphs

This section contains terminology and some basic facts concerning skew-symmetric graphs and explains the correspondence between these and bidirected graphs. For a more detailed survey on skew-symmetric graphs, see, e.g., [20] [7] [8].

A skew-symmetric graph is a digraph $G = (V, E)$ endowed with two bijections $\sigma_V, \sigma_E$ such that: $\sigma_V$ is an involution on the nodes (i.e., $\sigma_V(v) \neq v$ and $\sigma_V(\sigma_V(v)) = v$ for each $v \in V$), $\sigma_E$ is an involution on the arcs, and for each arc $e$ from $u$ to $v$, $\sigma_E(e)$ is an arc from $\sigma_V(v)$ to $\sigma_V(u)$. For brevity, we combine the mappings $\sigma_V, \sigma_E$ into one mapping $\sigma$ on $V \cup E$ and call $\sigma$ the symmetry (rather than skew-symmetry) of $G$.

For a node (arc) $x$, its symmetric node (arc) $\sigma(x)$ is also called the mate of $x$, and we will often use notation with primes for mates, denoting $\sigma$. For each $v \in V$ we will often use notation with primes for mates, denoting $\sigma(v)$.

We admit parallel arcs, but not loops, in $G$. Observe that if $G$ contains an arc $e$ from a node $v$ to its mate $v'$, then $e'$ is also an arc from $v$ to $v'$ (so the number of arcs of $G$ from $v$ to $v'$ is even and these parallel arcs are partitioned into pairs of mates).

By a path (circuit) in $G$ we mean a simple directed path (cycle), unless explicitly stated otherwise. The symmetry $\sigma$ is extended in a natural way to paths, subgraphs, and other objects in $G$. In particular, two paths or circuits are symmetric to each other if the elements of one of them are symmetric to those of the other and go in the reverse order: for a path (circuit) $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$, the symmetric path (circuit) $\sigma(P)$ is $(v'_k, e'_k, v'_{k-1}, \ldots, e'_1, v'_0)$. One easily shows that $G$ cannot contain self-symmetric circuits (cf. [8]).

Following terminology in [7], a path or circuit in $G$ is called regular if it contains no pair of symmetric arcs (while symmetric nodes in it are allowed). For a function $h$ on $E$, its symmetric function $h'$ is defined by $h'(e') := h(e)$, $e \in E$, and $h$ is called (self-)symmetric if $h = h'$.

For a function $f : E \to \mathbb{R}$ and a node $v \in V$, define

$$\text{div}_f(v) := \sum (f(e) : e \in \delta^\text{out}(v)) - \sum (f(e) : e \in \delta^\text{in}(v)),$$

(the divergence of $f$ at $v$), where $\delta^\text{out}(v)$ ($\delta^\text{in}(v)$) denotes the set of arcs of $G$ leaving (resp. entering) $v$. Let $f$ be nonnegative, integer-valued and symmetric, and let $S$ be a subset of nodes not intersecting $S' = \sigma(S)$. When $\text{div}_f(v)$ is nonnegative at each $v \in S$ and zero at each $v \in V - (S \cup S')$, $f$ is said to be an IS-flow (integer symmetric flow) from $S$ to $S'$. The value $\text{val}(f)$ of $f$ is $\sum_{s \in S} \text{div}_f(s)$. By a multiterminal version of a theorem due to Tutte [20], an IS-flow $f$ from $S$ to $S'$ has an integer symmetric decomposition. This means that

$$f$$ is representable as $f = \alpha_1 \chi_{P_1} + \alpha_1 \chi_{P_1'} + \ldots + \alpha_k \chi_{P_k} + \alpha_k \chi_{P_k'}$, where for $i = 1, \ldots, k$, $P_i$ is a path from $S$ to $S'$ or a circuit, $P'_i$ is the path (also going from $S$ to $S'$) or circuit symmetric to $P_i$, and $\alpha_i \in \mathbb{Z}_+$. 

(8) $$f$$ is representable as $f = \alpha_1 \chi_{P_1} + \alpha_1 \chi_{P_1'} + \ldots + \alpha_k \chi_{P_k} + \alpha_k \chi_{P_k'}$, where for $i = 1, \ldots, k$, $P_i$ is a path from $S$ to $S'$ or a circuit, $P'_i$ is the path (also going from $S$ to $S'$) or circuit symmetric to $P_i$, and $\alpha_i \in \mathbb{Z}_+$. 

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Here $\chi^P$ denotes the incidence vector of the arc-set of a path/circuit $P$, i.e., for $e \in E$, $\chi^P(e) = 1$ if $e$ belongs to $P$, and 0 otherwise. Note that paths/chains in $[8]$ need not be regular. Considering $\alpha_i$ as the weight of $P_i$ and of $P_i'$, observe that the total weight of paths from $S$ to $S'$ is equal to $\text{val}(f)$. Similar to flow decomposition in usual digraphs, an integer symmetric decomposition of an IS-flow $f$ can be found in $O(VE)$ time.

Let $(G, T)$ be inner Eulerian, where the terminal set $T$ is (self-)symmetric. Take a partition $\{S, S' = \sigma(S)\}$ of $T$ such that $\text{deg}^\text{out}(s) \geq \text{deg}^\text{in}(s)$ for all $s \in S$. Since the all-unit function $f$ on $E$ represents an IS-slow from $S$ to $S'$, [8] implies that

\[(9) \quad \text{there exists a symmetric collection } \mathcal{P} \text{ of circuits and paths from } S \text{ to } S' \text{ in } G \text{ such that the members of } \mathcal{P} \text{ are pairwise arc-disjoint and cover } E, \text{ and each terminal } s \in S \text{ is the beginning of exactly } \text{deg}^\text{out}(s) - \text{deg}^\text{in}(s) \text{ paths in } \mathcal{P}.\]

Moreover, the members of $\mathcal{P}$ are regular (for if some $P \in \mathcal{P}$ contains mates $e, e' \in E$, then $e, e'$ are in $\sigma(P)$ as well, which is impossible).

Next we explain the correspondence between skew-symmetric and bidirected graphs (cf. [8] Sec. 2). For sets $X, A, B$, we may use notation $X = A \cup B$ when $X = A \cup B$ and $A \cap B = \emptyset$. Given a skew-symmetric graph $G = (V, E)$, choose an arbitrary partition $\pi = \{V_1, V_2\}$ of $V$ such that $V_2$ is symmetric to $V_1$. Then $G, \pi$ determine bidirected graph $H$ with node set $V_1$ whose edges correspond to the pairs of symmetric arcs in $G$. More precisely, arc mates $a, a'$ of $G$ generate one edge $e$ of $H$ connecting nodes $u, v \in V_1$ such that: (i) $e$ goes from $u$ to $v$ if one of $a, a'$ goes from $u$ to $v$ (and the other goes from $v'$ to $u'$ in $V_2$); (ii) $e$ leaves both $u, v$ if one of $a, a'$ goes from $u$ to $v'$ (and the other from $v$ to $u'$); (iii) $e$ enters both $u, v$ if one of $a, a'$ goes from $u'$ to $v$ (and the other from $v'$ to $u$). In particular, $e$ is a loop if $a, a'$ connect a pair of symmetric nodes.

Conversely, a bidirected graph $H$ with node set $V_1$, say, determines skew-symmetric graph $G = (V, E)$ with symmetry $\sigma$ as follows. Take a copy $\sigma(v)$ of each element $v$ of $V_1$, forming the sets $V_2 := \{\sigma(v) : v \in V_1\}$ and $V := V_1 \cup V_2$. For each edge $e$ of $H$ connecting nodes $u$ and $v$, assign two “symmetric” arcs $a, a'$ in $G$ so as to satisfy (i)-(iii) above (where $u' = \sigma(u)$ and $v' = \sigma(v)$). An example is depicted in Fig. 2.

**Remark 4.** A bidirected graph generates one skew-symmetric graph, while a skew-symmetric graph generates a number of bidirected ones, depending on the partition $\pi$ of $V$ that we choose in the first construction. The latter bidirected graphs are produced from each other by the edge reversing transformation with respect to a subset of nodes as indicated in Remark 3 in the Introduction, so they are equivalent for us.

A terminal set $S$ in $H$ generates the symmetric terminal set $T := S \cup \sigma(S)$ in $G$, and vice versa. One easily checks that $(H, S)$ is inner Eulerian if and only if $(G, T)$ is such. Also there is a correspondence between the $S$-paths in $H$ and certain $T$-paths.
in $G$. More precisely, let $\tau$ be the natural mapping of $V \cup E$ to $V_1 \cup E(H)$ (where $E(H)$ is the edge set of $H$). Each walk (cycle) $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$ in $G$ induces the sequence $\tau(P) := (\tau(v_0), \tau(a_1), \tau(v_1), \ldots, \tau(a_k), \tau(v_k))$ of nodes and edges in $H$.

Conversely, for a walk (cycle) $Q = (w_0, e_1, w_1, \ldots, e_k, w_k)$ in $H$, form the sequence $\tilde{\tau}(Q) := (v_0, a_1, v_1, \ldots, a_k, v_k)$ of nodes and arcs in $G$ by the following rule:

(R) $v_0 := w_0$ if $e_1$ leaves $w_0$, and $v_0 := \sigma(w_0)$ if $e_1$ enters $w_0$; and for $i = 1, \ldots, k$:
(a) if $e_i$ leaves $w_{i-1}$, then $a_i$ is the arc in $\tau^{-1}(e_i)$ that leaves $w_{i-1}$, and $v_i$ is the head of $a_i$;
(b) if $e_i$ enters $w_{i-1}$, then $a_i$ is the arc in $\tau^{-1}(e_i)$ that leaves $\sigma(w_{i-1})$, and $v_i$ is the head of $a_i$.

(When $e_i$ is a loop, the arcs in $\tau^{-1}(e_i)$ are parallel, and the arc $a_i$ in this set is chosen arbitrarily.) It is not difficult to conclude that (R) provides:

(10) for a walk (cycle) $Q$ in $H$,

(i) $\tilde{\tau}(Q)$ is a walk (cycle) in $G$ and $\tau(\tilde{\tau}(Q)) = Q$;
(ii) if $Q$ is edge-simple and minimal (see Remark 2 in the Introduction), then $\tilde{\tau}(Q)$ is a regular path (circuit).

Also the walk (cycle) reverse to $Q$ determines the walk (cycle) in $G$ symmetric to $\tilde{\tau}(Q)$ (up to the choice of arcs $a_i$ for loops $e_i$). The corresponding converse properties to those in (10) also take place.

Let us say that a $T$-walk $P$ from $s$ to $t$ in $G$ is essential if $t$ is different from $\sigma(s)$. Thus, we have a natural bijection between the essential regular $T$-paths in $G$ (considered up to parallel arc mates) and the minimal $S$-paths in $H$. This gives

(11) $\tilde{\nu}_{G,T} = 2\nu_{H,S},$

where $\tilde{\nu}_{G,T}$ is the maximum cardinality of a symmetric collection of pairwise arc-disjoint essential $T$-paths in $G.$
Given an inner Eulerian network $\delta$ where $\tilde{\lambda}(\lambda)$ terms, if $(\tilde{\lambda}, 12)$ Corollary 3.1 For a skew-symmetric graph $G$ of terminals. By an integer symmetric free multiflow (or, briefly, an IS-multiflow) in the network $(G, T, c)$ we mean a collection $F$ of integer flows $f_{st}$ for the ordered pairs $(s, t)$ of distinct terminals in $S$ such that: (a) $f = f_{st}$ is a flow from $(s, s')$ to $(t, t')$ (i.e., $\text{div}_f(v)$ is nonnegative for $v = s, s'$, nonpositive for $v = t, t'$, and 0 otherwise); (b) each $f_{st}$ is symmetric to $f_{ts}$; and (c) $F$ is $c$-admissible, i.e.,

$$\sum_{st} f_{st}(e) \leq c(e) \quad \text{for each } e \in E.$$

The (total) value $\text{val}(F)$ of $F$ is $\sum_{st} \text{val}(f_{st})$. The problem is:

(PS) Given an inner Eulerian network $(G, T, c)$, where $G$ is a skew-symmetric graph and $T$ and $c$ are symmetric, find a maximum IS-multiflow, i.e., an IS-multiflow $F$ maximizing $\text{val}(F)$.

To see how this problem is related to (P), let $(G, T)$ correspond to $(H, S)$, where $H$ is bidirected. Let $\tilde{c}$ be the corresponding capacity function in $H$, i.e., $\tilde{c}(e) = c(a)$ for an edge $e \in E(H)$ and its images $a, a'$ in $G$. The inner Eulerianness of $(G, T, c)$ implies that of $(H, S, \tilde{c})$, and vice versa. Given an IS-multiflow $F$ in $(G, T, c)$, represent each flow $f_{st}$ in the path packing form:

$$f_{st} = \alpha_1 \chi_{P_1} + \ldots + \alpha_k \chi_{P_k}, \text{ where } \alpha_i = \alpha(P_i) \in \mathbb{Z}_+ \text{ and } P_i \text{ is a circuit or a (simple) path from } \{s, s'\} \text{ to } \{t, t'\}.$$

We assume that the representation of each flow $f_{st}$ is symmetric to that of $f_{ts}$. Then the set $\mathcal{P}$ of (essential) $T$-paths in these representations is symmetric, with $\alpha(P) = \alpha(P')$ for each $P \in \mathcal{P}$, and we have $\text{val}(F) = \sum_{P \in \mathcal{P}} \alpha(P) : P \in \mathcal{P})$. Now each pair $P, P' \in \mathcal{P}$ of path mates determines an $S$-walk $\tilde{P}$ in $H$ (considered up to reversing), and taking together these paths $\tilde{P}$ with weights $\alpha(P)$, we obtain a multiflow $\mathcal{F}$ in $(H, S, \tilde{c})$ satisfying $\text{val}(\mathcal{F}) = \frac{1}{2} \text{val}(F)$.

Conversely, let $\mathcal{F} = (\tilde{P}, \tilde{c})$ be an integer multiflow in $(H, S, \tilde{c})$, where $\tilde{P}$ consists of $S$-walks. One may assume that for each edge $e$ of $H$, no path $\tilde{P} \in \tilde{P}$ traverses $e$ twice.
in the same direction (for otherwise one can remove a cycle from \( \hat{P} \)). Then each \( s-t \) walk \( \hat{P} \) determines an arc-simple directed walk \( P \) from \( \{s, s'\} \) to \( \{t, t'\} \) and its mate \( P' \) from \( \{t, t'\} \) to \( \{s, s'\} \) in \( G \). Assign \( \alpha(P) := \alpha(P') := \hat{\alpha}(\hat{P}) \). Let \( f_{st} \) be the sum of functions \( \alpha v^P \) over the obtained walks \( P \) from \( \{s, s'\} \) to \( \{t, t'\} \). Then \( f_{ts} \) is symmetric to \( f_{st} \). These flows form an IS-multiflow \( F \) in \((G, T, c)\) satisfying \( \text{val}(F) = 2\text{val}(\mathcal{F}) \).

Thus, problems (PS) and (P) (regarding \( H, S, \hat{\alpha} \)) are reduced to each other. In the next section we devise an efficient algorithm for finding an optimal solution to (PS) and then explain that it can be transformed into an optimal solution to the corresponding instance of (P) without increasing the time bound.

4 Algorithm

In this section we describe an algorithm to solve problem (PS) and estimate its complexity. We use terminology and facts from the previous section.

Let \((G = (V, E), T, c)\) be an inner Eulerian skew-symmetric network. As before, we represent the terminal set \( T \) as \( S \sqcup S' \) and associate with \((G, T, c)\) the corresponding bidirected network \((H, S, \hat{\alpha})\). One may assume that no arc in \( G \) connects a pair of terminal mates. Also if \( G \) has an arc entering a terminal \( s \in S \), then replacing its head \( s \) by \( s' \) and symmetrically replacing the tail \( s' \) of the symmetric arc \( e' \) by \( s \) does not affect the problem in essence. So we may assume that \( \deg^{\text{in}}(s) = 0 \) for each terminal \( s \in S \) in \( G \). Then any flow from \( \{s, s'\} \) to \( \{t, t'\} \), where \( s, t \in S \), is essentially a flow from \( s \) to \( t' \), and its symmetric flow is a flow from \( t \) to \( s' \); this property will simplify technical details in our construction. In terms of \( H \), the latter assumption says that each edge incident with a terminal \( s \) in \( H \) leaves \( s \) (cf. Remark 3 in Section 1).

The algorithm uses a recursion analogous to that in [11], and the case \(|S| = 3\) is the base in it. We first consider this special case (which generalizes the case \(|S| = 2\)).

4.1 Case \(|S| = 3\).

The algorithm for this case uses one auxiliary skew-symmetric graph \( G_1 = (V, E_1) \). It is obtained from \( G \) by adding, for each pair \( v, v' \) of inner node mates, four auxiliary arcs connecting \( v \) and \( v' \): two arc mates \( a_v, a'_v \) going from \( v \) to \( v' \) and two arc mates \( a_{v'}, a'_{v'} \) from \( v' \) to \( v \), regardless of the existence of such arcs in \( G \). This \( G_1 \) corresponds to the bidirected graph \( H_1 \) obtained from \( H \) by adding two auxiliary loops at each inner node \( v \), one leaving \( v \) (twice) and the other entering \( v \).

The algorithm consists of three stages. Let \( S = \{s_i, s_2, s_3\} \).

At Stage 1, we apply the algorithm for inner Eulerian graphs from [11] to find a maximum integer free multiflow in the underlying undirected graph \( \overline{H} \) for \( H \) having the same set \( S \) of terminals and the same capacities \( \hat{\alpha} \). It runs in \( O(\phi(V, E)) \) time (since \(|S| = O(1)|\) and outputs (simple) \( S \)-paths \( \overline{P_1}, \ldots, \overline{P_k} \) in \( \overline{H} \) and weights \( \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_+ \) satisfying the packing condition w.r.t. \( \hat{\alpha} \). (Recall that \( \phi \) is a bound
Figure 3: A path \( \overline{P}_i \) and its images in the graphs \( H_1 \) and \( G_1 \) for the applied max flow algorithm; we assume \( \phi = \Omega(VE) \). It also outputs pairwise disjoint \( s_i \)-sets \( X_i, i = 1, 2, 3 \), such that for each \( i \), the sum of weights \( \alpha_j \) of paths \( \overline{P}_j \) connecting \( s_i \) and \( S - \{s_i\} \) is equal to \( \hat{c}(\delta_H(X_i)) \). However, some pairs of consecutive edges in \( \overline{P}_j \) may be non-transit in \( H \), i.e., \( \overline{P}_j \) is not necessarily a path in \( H \).

At Stage 2, we transform \( \overline{P}_1, \ldots, \overline{P}_k \) into paths in the auxiliary bidirected graph \( H_1 \). More precisely, for each \( \overline{P}_i = (v_0, e_1, v_1, \ldots, e_q, v_q) \) and for each non-transit pair \( e_j, e_{j+1} \) in it, if both edges \( e_j, e_{j+1} \) enter (leave) \( v_j \), then the element \( v_j \) of \( \overline{P}_i \) is replaced by the string \( v_j, \ell, v_j \), where \( \ell \) is the auxiliary loop leaving (resp. entering) \( v_j \). This results in minimal \( S \)-paths \( \tilde{P}_1, \ldots, \tilde{P}_k \) in \( H_1 \).

Each path \( \tilde{P}_i \) and its reverse one are then lifted to \( G_1 \) (by the method explained in Section 3), giving regular \( T \)-paths \( P_i, P'_i \) symmetric to each other. (Figure 3 illustrates paths \( \overline{P}_i, \tilde{P}_i, P_i \).) For each pair \( s_j, s_p \) (\( j \neq p \)), the functions \( \alpha_i \chi^{P_i} \) or \( \alpha_i \chi^{P'_i} \) for the paths from \( s_j \) to \( s'_p \) are added up, forming \( s_j - s'_p \) flow \( g_{jp} \). This gives a symmetric collection of six integer flows in \( G_1 \); see Figure 4. The \( \hat{c} \)-admissibility of the above multiflow in \( \overline{H} \) and the fact that each path \( P_i \) is regular imply that the total flow though each arc \( e \) of \( G \) does not exceed \( c(e) \). Also the fact that the cuts \( \delta_H(X_i) \) are saturated implies that

\[
\text{(14)} \quad \text{for } i = 1, 2, 3, \text{ the arcs in } \delta^{\text{out}}(X_i) \text{ are saturated by } g_{i,i-1} + g_{i,i+1}, \text{ and symmetrically, the arcs in } \delta^{\text{in}}(X_i) \text{ are saturated by } g_{i-1,i} + g_{i+1,i},
\]

where \( X_i := X_i \sqcup X'_i \) and the indices are taken modulo 3. So the IS-multiflow consisting of these six flows has maximum value.

At Stage 3, we improve the above flows \( g_{ij} \) in \( G_1 \) by reducing their values on the auxiliary arcs to zero, eventually obtaining the desired multiflow in \( G \). In view
of \( G \), for \( i = 1, 2, 3 \), one may assume that \( g_{i-1,i+1} \) and \( g_{i+1,i-1} \) take zero values on all arcs of the subgraph \( \langle X_i \rangle \) of \( G \) induced by \( X_i \).

Take the residual capacities \( \Delta(e) := c(e) - \sum_{ij} g_{ij}(e) \) of arcs \( e \in E \). The divergency of \( c \) (w.r.t. \( E \)) and of each \( g_{ij} \) (w.r.t. \( E_1 \)) at any inner node is zero, therefore,

\[
\text{div}_\Delta(v) = \sum_{ij} g_{ij}(v, v') \quad \text{for each } v \in V - T,
\]

where \( g(v, v') \) denotes \( g(a_v) + g(a'_v) - g(a'_{v'}) - g(a_{v'}) \) (recall that \( a_v, a'_v \) are the auxiliary arcs from \( v \) to \( v' \)). The function \( \Delta \) on \( E \) is nonnegative, integer-valued and symmetric. Also (15) and

\[
g_{ij}(v, v') = -g_{ij}(v', v) = g_{ji}(v, v') \quad \text{for each } v \in V - T
\]

imply that \( \text{div}_\Delta(v) \) is even for each \( v \in V - T \). Hence we can extend \( \Delta \) to the auxiliary arcs so as to obtain an IS-flow in \( (G_1, T) \). (The extended \( \Delta \) satisfies \( \Delta(v, v') + \sum_{ij} g_{ij}(v, v') = 0 \) for each \( v \in V - T \).)

Notice that \( \Delta(e) = 0 \) for each arc \( e \) in the cut \( \delta(X_i) \), \( i = 1, 2, 3 \), by (14). Therefore, the restriction \( \Delta_i \) of \( \Delta \) to the set \( A_i \) of arcs of the subgraph \( \langle X_i \rangle \) is an IS-flow from \( s_i \) to \( s'_i \). In its turn, the restriction \( \Delta_0 \) of \( \Delta \) to the set \( A_0 \) of arcs with both ends in \( W := V - (X_1 \cup X_2 \cup X_3) \) is an integer symmetric circulation in the subgraph \( (W, A_0) \). (Recall that the sets \( X_1, X_2, X_3 \) are pairwise disjoint.)

We start with getting rid of nonzero arc values of the above flows on the auxiliary arcs within the subgraph \( \langle X_1 \rangle \). To this aim, apply the integer symmetric decomposition procedure to \( \Delta_1 \) (cf. 3) to represent it as the sum of integer \( s_1-s'_1 \)-flows \( h, h' \), where \( h' \) is symmetric to \( h \). Combine \( g := g_{12} + g_{13} + h \) and \( g' := g_{21} + g_{31} + h' \) (where \( h, h' \) are formally extended by zeros on \( E_1 - A_1 \)). Then \( g \) is an integer flow from \( s_i \)

![Figure 4: Flows \( g_{ij} \) in \( G_1 \)](image-url)
to $S'$, and $g'$ is the flow from $S$ to $s'_1$ symmetric to $g$. Also
\[ g(v, v') = g_{12}(v, v') + g_{13}(v, v') + h(v, v') = 0 \quad \text{for each } v \in X_1 - \{s_1, s'_1\}, \]
in view of $h(v, v') = \frac{1}{2}\Delta_1(v, v')$, $\Delta(v, v') + \sum_{ij} g_{ij}(v, v') = 0$ and $g_{23}(e) = g_{32}(e) = 0$ for all $e \in A_1$. So we can reduce $g, g'$ to zero on all auxiliary arcs in $(X_1)$. Now using standard flow decomposition, we represent the new flow $g$ as the sum of three integer flows $f_1, f_2, f_3$, from $s_1$ to $s'_1$, from $s_1$ to $s'_2$, and from $s_1$ to $s'_3$, respectively. Note that $g(e) = 0$ for each $e \in \delta^{in}(X_1)$ implies that $f_1$ is zero on all arcs of the cut $\delta(X_1)$. Update $g_{12} := f_2$ and $g_{13} := f_3$; the flows $g_{21}$ and $g_{31}$ are updated symmetrically. Then the resulting four flows together with the remaining flows $g_{23}, g_{32}$ satisfy (13) as before (thus forming a maximum IS-multiflow) and take zero values on the auxiliary arcs in $(X_1)$, as required. Do similarly for $X_2$ and $X_3$.

The task of improving the flows within the subgraph $(W) = (W, A_0)$ is a bit more involved. First of all we modify $g_{12}$ (and $g_{21}$) so as to get
\[ g_{12}(v, v') + g_{23}(v, v') + g_{31}(v, v') = 0 \quad \text{for each } v \in W \] (this situation is technically simpler). This is done by decomposing the above-mentioned symmetric circulation $\Delta_0$ in $(W)$ into the sum of an integer circulation $\omega$ and its symmetric circulation $\omega'$ and then by updating $g_{12} := g_{12} + \omega$ and $g_{21} := g_{21} + \omega'$ (with $\omega$, $\omega'$ extended by zeros to $E_1 - A_0$). Then the equality $\Delta(v, v') + \sum_{ij} g_{ij}(v, v') = 0$ provides (17).

The process of improving the flows within $(W)$ consists of $O(W)$ iterations (the idea is borrowed from the algorithm for digraphs in [10]). At a current iteration, we choose a node $v \in W$ where some $g_{ij}(v, v')$ is nonzero. W.l.o.g, one may assume that $g_{12}(v, v') > 0$ and $g_{13}(v, v'), g_{23}(v, v') \leq 0$. Let $r_0 := |g_{13}(v, v')|$ and $r_1 := |g_{23}(v, v')|$; then $g_{12}(v, v') = r_0 + r_1$, by (17). Let $B$ be the set of (four) auxiliary arcs connecting $v$ and $v'$.

First of all we represent $g_{12}$ as the sum of two integer $s_1 - s'_2$ flows $g_0, g_1$ such that $g_0(v, v') = r_0$ and $g_1(v, v') = r_1$. To do so, replace $B$ by new terminals $t, t_0, t_1$ and arcs $a = (t, v'), a_0 = (v, t_0)$ and $a_1 = (v, t_1)$, and add an arc $b$ from $s'_2$ to $s_1$. Define $h(a_i) := r_i, i = 0, 1, h(a) := r_0 + r_1, h(b) := \text{val}(g_{12})$ and $h(e) := g_{12}(e)$ for the remaining arcs $e$. This turns $h$ into a flow from $t$ to $\{t_0, t_1\}$, and we decompose it into the sum of integer flows $h_0, h_1$, from $t$ to $t_0$ and from $t$ to $t_1$, respectively. These $h_0, h_1$ determine the desired $g_0, g_1$ in a natural way.

Combine $f := g_0 + g_{13}$. Then $f(v, v') = g_0(v, v') + g_{13}(v, v') = 0$. Update $f(e) := 0$ for each $e \in B$ and decompose the updated $s_1 - \{s'_2, s'_3\}$ flow $f$ into the sum of integer flows $f_0, f_{13}$, from $s_1$ to $s'_2$ and from $s_1$ to $s'_3$, respectively. Then $\text{val}(f_0) = \text{val}(g_0)$ and $\text{val}(f_{13}) = \text{val}(g_{13})$.

Doing similarly for the flow $g_{23}$ and the flow $g'_1$ symmetric to $g_1$ (which have the source $s_2$ in common), we obtain corresponding $s_2 - s'_3$ flow $f_{23}$ and $s_2 - s'_1$ flow $f'_1$. Finally, update $g_{12} := f_0 + f_1$ (where $f_1$ is symmetric to $f'_1$), $g_{13} := f_{13}$ and $g_{23} := f_{23}$.\[14\]
The updated flows $g_{ij}$ together with their symmetric ones satisfy $g_{ij}(e) = 0$ for each arc $e \in B$.

Then we choose a next pair of node mates in $W$, and so on. Upon termination of the process, the resulting flows $g_{ij}$ take zero values on all auxiliary arcs, and it is easily seen from the construction that $\text{val}(g_{ij})$ preserves for all pairs $ij$. So their restrictions to $E$ form a maximum IS-multiflow in $(G, T, c)$, as required.

The above algorithm runs in $O(\phi(V, E))$ time plus the time needed to perform $O(W)$, or $O(V)$, flow decompositions during the iterative process at Stage 3 (the other operations including those in $O(1)$ symmetric decompositions take $O(VE)$ time). Each of these decompositions is applied to a flow with $O(1)$ sources and sinks, and we use the procedure in Section 5 to implement it in $O(E \log(2 + V^2/E))$ time. This gives the bound $O(V E \log(2 + V^2/E))$ for the six (or four) terminal case.

### 4.2 General case.

We now describe the algorithm for an arbitrary $|S| \geq 4$. It is based on a recursive network partition approach.

For a current inner Eulerian skew-symmetric network $N = (G, T, c)$, with $T = S \cup S'$, the network partition procedure partitions $S$ into two sets $S_1, S_2$ such that $|S_1| = \lceil |S|/2 \rceil$ and $|S_2| = \lfloor |S|/2 \rfloor$ and finds a symmetric subset $X \subseteq V$ with $X \cap T = S_1 \cup S'_1$ whose induced cut $\delta(X)$ has minimum capacity $c(\delta(X))$. This is done by finding a minimum capacity cut $\delta(Y)$ with $Y \cap T = S_1 \cup S'_1$ in the underlying undirected network for $(G, c)$, and by making the symmetrization $X := Y \cup Y'$ (relying on $c(\delta(Y \cup Y')) + c(\delta(Y \cap Y')) \leq c(\delta(Y)) + c(\delta(Y')) = 2c(\delta(Y))$).

Next we shrink the subgraph $(V - X)$ of $G$ into two new (extra) terminals $t_1, t'_1$, making each arc in $\delta^\text{out}(X)$ enter $t'_1$, and each arc in $\delta^\text{in}(X)$ leave $t_1$. Similarly, $(X)$ is shrunk into extra terminals $t_2, t'_2$, each arc in $\delta^\text{in}(X)$ becomes entering $t'_2$ and each arc in $\delta^\text{out}(X)$ becomes leaving $t_2$. This produces two smaller inner Eulerian networks $N_i = (G_i = (V_i, E_i), T_i, c_i)$ with $T_i = S_i \cup S'_i \cup \{t_i, t'_i\}$, $i = 1, 2$, satisfying

$$|T_i| \leq 4|T|/5, \quad |V_i| \leq |V|, \quad |E_i| \leq |E|, \quad \text{and} \quad |V_1| + |V_2| = |V| + 4$$

(since $|T_1| = 8$ when $|T| = 10$). Also for $X_i := \{t_i, t'_i\}$, the cut $\delta(X_i)$ of $G_i$ has minimum capacity among the cuts separating $\{t_i, t'_i\}$ and $T_i - \{t_i, t'_i\}$.

One application of the network partition procedure, to a current $N$, takes one minimum cut computation, so it runs in $O(\phi(V, E))$ time.

Let $F_i$ be a (recursively found) maximum free IS-multiflow in $N_i$. The aggregation procedure transforms $F_1, F_2$ into a maximum free IS-multiflow $F$ in $N$. The flows in $F_i$ going from $S_i$ to $t'_i$ are combined into one (multisource) flow $f_i$ from $S_i$ to $t'_i$, and symmetrically, the flows from $t_i$ to $S'_i$ are combined into one flow $f'_i$. By the maximality of $F_i$ and the minimality of $c_i(\delta(X_i))$, $f_i$ saturates $\delta^\text{in}(X_i)$ and $f'_i$ saturates $\delta^\text{out}(X_i)$. We glue together (the images of) $f_1$ and $f'_1$, obtaining $S_1 - S'_2$ flow.
f in \(N\), and do symmetrically for \(f_2, f'_1\), obtaining \(f'\). These \(f, f'\) are decomposed symmetrically into a symmetric collection of integer one-source-one-sink flows. Then the flows formed from \(f, f'\) together with the remaining flows in \(F_1, F_2\) connecting pairs of terminals in \(T_1\) or in \(T_2\) give the desired \(F\). (The maximality of \(F\) follows from the fact that for each \(s \in S_1\), the total value of flows in \(F_1\) leaving \(s\) or entering \(s'\) is equal to the minimum capacity of a cut in \(N_1\) separating \(\{s, s'\}\) and \(T_1 - \{s, s'\}\), and similarly for \(S_2\). The above construction maintains such an equality for \(F\) and each \(s \in S\).

At the bottom level (\(|S| = 3\)), we apply the algorithm described in \([13]\).

One application of the aggregation procedure, to current \(N_1, N_2\), takes \(O(S_1E_1 + S_2E_2)\) time to create the flows \(f_1, f_2\) as above plus \(O(VE)\) time to decompose \(f\), or \(O(VE)\) time in total (in view of (18)).

It remains to explain that the resulting multiflow \(F\) in the initial network \(N\) can be efficiently transformed into a maximum integer free multiflow in the corresponding bidirected network \((H, S, \tilde{c})\). We show that \(O(VE \log T)\) time is sufficient to create from \(F\) a corresponding symmetric collection \((\mathcal{P}, \alpha)\) of weighted \(T\)-paths in \(N\); these paths determine weighted \(T\)-walks in \(H\) forming an optimal solution to problem \((P)\) with \((H, S, \tilde{c})\), by the relationship explained in Section \([3]\). We assume that each flow \(f\) in \(F\) is explicitly given only within its support \(\text{supp}(f) := \{e \in E : f(e) \neq 0\}\).

Let \(T\) be the binary rooted tree formed by all networks arising during the recursion, with the natural ordering on them. The height of \(T\) (or the depth of the recursion) is \(O(\log T)\), in view of the first inequality in (18). For a network \(\tilde{N}\) in \(T\), let \(A(\tilde{N})\) be the set of terminals from the initial \(T\) that are contained in \(\tilde{N}\), and \(F(\tilde{N})\) the set of flows in \(F\) with both terminals in \(A(\tilde{N})\). We use the fact that for incomparable \(N', N''\) in \(T\), the supports of flows in \(F(N')\) are disjoint from those in \(F(N'')\). (Indeed, for the closest common predecessor \(\tilde{N}\) of \(N', N''\), the minimum cut found by the network partition procedure for \(\tilde{N}\) separates \(A(N')\) and \(A(N'')\) and is saturated by the flows not in \(F(N') \cup F(N'')\).)

We proceed as follows. For each non-leaf network \(\tilde{N}\) with children \(N_1, N_2\), combine the flows in \(F\) with the source in \(A(N_1)\) and the sink in \(A(N_2)\) into one multiterminal flow \(\tilde{f}\) (in the initial network), and then decompose \(\tilde{f}\) into a set \(\mathcal{P}(\tilde{N})\) of weighted paths from \(A(N_1)\) to \(A(N_2)\) (the circuits appeared in the decomposition are removed). This takes \(O(V \text{supp}(\tilde{f}))\) time. Taken together, the sets \(\mathcal{P}(\tilde{N})\), their symmetric sets and corresponding paths appeared by decomposing the flows in \(F\) having both terminals in one leaf network, constitute the desired symmetric collection \((\mathcal{P}, \alpha)\). To estimate the complexity, consider the networks \(\tilde{N}\) at height \(i\) in \(T\). They are incomparable, so the supports of flows \(\tilde{f}\) as above in them are pairwise disjoint. Hence to form the sets \(\mathcal{P}(\tilde{N})\) for these \(\tilde{N}\) takes \(O(VE)\) time in total. This gives the bound \(O(VE \log T)\) for the whole procedure, as declared.
4.3 Complexity of the algorithm

We show that the above algorithm runs in \( O(VE \ell(V, E) \log T) \) time, where \( \ell(V, E) := \max\{1, \ln(V^2/E)\} \), assuming \( \phi(n, m) = O(nm \log(2 + n^2/m)) \) (as in Goldberg–Tarjan’s max flow algorithm). We use induction on the height \( h(T) \) of the binary tree \( T \) (it depends only on \( |T| \)). When \( h(T) = 0 \) (i.e. \( |T| = 6 \)), the required time bound was shown in 4.1.

Let \( h(T) \geq 1 \) and let \( N_1, N_2 \) be the children of \( N \) in \( T \). For \( i = 1, 2 \), we have \( h(T_i) \leq h(T) - 1 \), and by induction the time \( \tau_i \) of the algorithm to solve the problem for \( N_i \) is bounded from above as

\[
\tau_i \leq Cn_im_i\ell(n_i, m_i) \log T_i
\]

for some appropriately chosen constant \( C > 0 \) (specified later). Here \( n_i := |V_i| \) and \( m_i := E_i \), keeping notation from 4.2. The network partition and aggregation procedures applied to \( N \) take time \( O(\ell(n, m)) \) and \( O(nm) \), respectively, or \( Dnm\ell(n, m) \) time together, where \( D \) is some constant \( > 0 \), \( n := |V| \) and \( m := |E| \). Therefore, the time \( \tau \) to solve the problem for \( N \) is estimated as

\[
(19) \quad \tau \leq C(n_1m_1\ell(n_1, m_1) \log T_1 + n_2m_2\ell(n_2, m_2) \log T_2) + Dnm\ell(n, m).
\]

We have \( \ell(n_i, m_i) \leq \ell(n, m_i) \) (since \( n_i \leq n \)) and \( m_i\ell(n, m_i) \leq m\ell(n, m) \) (this follows from \( m_i \leq m \) and from \( \ln a > \frac{1}{b} \ln(ab) \) for \( b > 1 \) and \( \ln a > 1 \)). Also \( n_1 + n_2 = n + 4 \) and \( \log T_i \leq \log T - \log \frac{5}{4} \), by (18). Then (19) implies

\[
(20) \quad \tau \leq C \left( nm \log T - nm \log \frac{5}{4} + 4m \log T - 4m \log \frac{5}{4} \right) \ell(n, m) + Dnm\ell(n, m).
\]

Since \( n \geq |T| \) and \( \frac{1}{b}nm \log \frac{5}{4} \) grows faster than \( 4m \log n \), one can choose constants \( n_0 \) and \( C \) (depending on \( D \)) such that the right hand side value in (20) becomes smaller than \( Cn\ell(n, m) \log T \) for any \( n > n_0 \). (For the networks with \( |V| \leq n_0 \), the problem is solved in \( O(E) \) time.) This yields the desired time bound.

**Theorem 4.1** A maximum IS-multiflow (resp. a maximum integer free multiflow) in an inner Eulerian skew-symmetric (resp. bidirected) network \( (G = (V, E), T, c) \) can be found in \( O(VE \log T \log(2 + V^2/E)) \) time.

5 Fast Flow Decomposition

For a fixed \( k \in \mathbb{Z}_+ \), we consider the problem:

\( \text{(D)} \) Given a flow (integer flow) \( f \) from \( S \) to \( T \), with \( |S| + |T| = k \), in a digraph \( G = (V, E) \), find a decomposition \( f = \sum (f_{st} : s \in S, t \in T) \), where each \( f_{st} \) is a flow (resp. integer flow) from \( s \) to \( t \).
and show the following (allowing parallel arcs in $G$ and assuming $|V| = O(E)$).

**Theorem 5.1** (D) can be solved in $O(E \log(2 + V^2/E))$ time.

Note that when $G$ is acyclic, a decomposition (into one-source-one-sink flows or into weighted paths) of any flow in $G$ is carried out in $O(E)$ time by using a topological sorting of the nodes. Sleator and Tarjan [19] showed that any flow $f$ in an arbitrary digraph can be decomposed, in $O(E \log V)$ time, into a circulation and a flow whose support induces an acyclic subgraph of $G$ (so a decomposition of $f$ into one-source-one-sink flows can be found with the same complexity $O(E \log V)$). The algorithm in [19] uses sophisticated computational tools, so-called dynamic trees.

Our approach to solve (D) is based on a node splitting technique and uses only simple data structures. Let $\Pi$ be the set of pairs $st$ with $s \in S$ and $t \in T$.

In the beginning of the algorithm, we delete from $G$ the arcs $e$ with $f(e) = 0$. Also we sort the nodes $v$ by increasing their degrees $\deg(v)$. (This takes $O(E)$ time.) The algorithm applies $|V|$ iterations.

At each iteration, we choose a node $v$ with $\deg(v)$ minimum in the current graph $G = (V, E)$. First of all we scan the arcs incident with $v$ to select parallel arcs among them. Each tuple of parallel arcs is merged into one arc (and the flows on these are added up). The node degrees and the ordering on $V$ are updated accordingly. (This preliminary stage is performed in $O(\deg(v))$ time. As a result, the degree of $v$ becomes less than $2|V|$.) Then we make at most $\deg(v)$ splittings at $v$.

More precisely, at a current step of the iteration, we choose an arc entering $v$ and an arc leaving $v$, say, $e = (u, v)$ and $e' = (v, w)$. If $e$ or $e'$ is a loop, we simply delete it from $G$. Otherwise define $\epsilon := \min\{f(e), f(e')\}$. The splitting-off operation applied to $(e, v, e')$ creates a new arc $e''$ from $u$ to $w$, assigns $f(e'') := \epsilon$, updates $f(e) := f(e) - \epsilon$ and $f(e') := f(e') - \epsilon$, and deletes from $G$ the arc (or arcs) for which the new value becomes zero; it takes $O(1)$ time. The ordering on $V$ is updated accordingly (in $O(1)$ time). Clearly the operation maintains both the divergency at each node and the flow integrality (when the original flow is integer). Also $\text{div}(v)$ decreases and the number of all arcs does not increase.

At the next step of the iteration, the operation is applied to another pair of arcs, one entering and the other leaving $v$, and so on until such pairs no longer exist. After that, if $v \notin S \cup T$, then $v$ is removed from $G$ (as $\text{div}(v) = 0$ implies $\deg(v) = 0$).

At the next iteration, we again choose a vertex where the current degree is minimum, and so on. One can see that after $|V|$ iterations, each arc of the resulting graph $\overline{G}$ goes from a source $s \in S$ to a sink $t \in T$. The decomposition $\overline{D} = \{f_{st} : st \in \Pi\}$ for the resulting $f$ in $\overline{G}$ is trivial: $f_{st}(s, t) := f(s, t)$ and $f_{st}(e) := 0$ for $e \neq (s, t)$ (letting $f_{st} := 0$ if the arc $(s, t)$ does not exist in $\overline{G}$).

Now going in the reverse order and applying the corresponding restoration procedure reverse to the splitting-off one, we transform $\overline{D}$ into the desired decomposition of the initial flow. More precisely, consider a current graph $G$ and the arcs $e = (u, v)$, $e' = (v, w)$, $e'' = (u, w)$ as above, and let $f_{st}$, $st \in \Pi$, be the flows already obtained
for the graph $G'$ formed from $G$ by the splitting-off operation w.r.t. $(e, v, e')$. For each $st \in \Pi$, add $f_{st}(e'')$ to $f_{st}(e)$ and to $f_{st}(e')$ and then delete $e''$. (The backward iteration concerning $v$ finishes with restoring the corresponding tuples of parallel arcs incident with $v$ and assigning, in a due way, the flows $f_{st}$ on these arcs.) Eventually, we obtain the desired decomposition $\{f_{st} : st \in \Pi\}$ of the initial $f$. (Strictly speaking, we have $g := \sum_{st} f_{st} \leq f$ and $\text{div}_{f-g}(v) = 0$ for all $v \in V$; so one should add the circulation $f - g$ to one of the flows $f_{st}$.)

Next we estimate complexity of the above algorithm. Let $v_1, v_2, \ldots, v_{|V|}$ be the sequence of nodes in the splitting-off process. Since $|\Pi| = O(1)$, the restoration process is only $O(1)$ times slower than the splitting-off one. (This is just where we essentially use the condition that $f$ has $O(1)$ terminals.) Using this fact, one can conclude that the algorithm runs in $O(E + \Delta)$ time for the initial $E$, where $\Delta := \deg^*(v_1) + \ldots + \deg^*(v_{|V|})$ and $\deg^*(v)$ denotes the degree of $v$ at the beginning of splitting at $v$. Each iteration $i$ in the former process does not increase the number of arcs of the current graph and decreases the number of nodes by one, unless $v_i \in S \cup T$. So $\deg^*(v_{i+1})$ is at most $2|E|/(|V| - i)$. Summing up the latter numbers over $i$, we obtain $\Delta = O(E \log V)$, which is worse than the time bound in Theorem 5.4.

However, we can estimate $\Delta$ more carefully, by using the inequality $\deg^*(v_i) < 2(|V| - i + k + 1)$ (provided by merging parallel arcs incident with $v_i$). For any integer $1 \leq \lambda \leq |V|$, apply the first bound on $\deg^*(v_i)$ for $i = 1, \ldots, |V| - \lambda$, and the second bound for $i = |V| - \lambda + 1, \ldots, |V|$. This gives

$$\Delta \leq 2|E| \left( \frac{1}{|V|} + \frac{1}{|V| - 1} + \ldots + \frac{1}{\lambda + 1} \right) + 2\lambda(\lambda + k),$$

or $\Delta = O(E \log(V/\lambda) + \lambda^2)$. Now taking $\lambda := \min\{|V|, \lceil \sqrt{|E|} \rceil\}$, we obtain $\Delta = O(E \log(2 + V^2/E))$, and the theorem follows.

6 Fast Skew-Symmetric Flow Decomposition

In this section Theorem 5.1 is extended to (skew-)symmetric flows. For a fixed $k \in \mathbb{Z}_+$, we consider the problem:

(DS) **Given an integer symmetric flow $f$ from $S = \{s_1, \ldots, s_k\}$ to $S' = \sigma(S)$ in a skew-symmetric graph $G = (V, E)$, find a decomposition of $f$ of the form**

$$f = \sum_{1 \leq i \leq j \leq k} (f_{ij} + f_{ij}'),$$

**where each $f_{ij}$ is an integer flow from $s_i$ to $s_j$ and $f_{ij}'$ is symmetric to $f_{ij}$.**

Note that $f_{ij}'$ is a flow from $s_j$ to $s_i'$. So in the above decomposition, for $i, j \in \{1, \ldots, k\}$, $s_i$ and $s_j'$ are connected by the only flow $f_{ij}$ if $i < j$, by only $f_{ji}'$ if $i > j$, and by the two flows $f_{ii}$ and $f_{ii}'$ if $i = j$. We show the following.
Theorem 6.1 (DS) can be solved in $O(E \log(2 + V^2/E))$ time.

This generalizes Theorem 5.1 for integer flows because a digraph $D$ with an integer $S$–$T$ flow $g$ is turned into a skew-symmetric graph with an integer symmetric $(S \cup T')$–$(S' \cup T)$ flow by adding a disjoint copy of the reverse to $D$ with the flow reverse to $g$ in it.

Our algorithm to solve (DS) relies on the following lemma (where, as before, primes are used for the corresponding mate objects).

Lemma 6.2 Let $g$ be a (not necessarily symmetric) half-integer flow from $S$ to $T$ in a skew-symmetric graph $G = (V, E)$ such that $\text{div}_g(v)$ is an integer for each $v \in V$. Let $g + g'$ be integer. Then there exists, and can be found in $O(E)$ time, an integer flow $h$ in $G$ such that $h + h' = g + g'$ and $\text{div}_h(v) = \text{div}_g(v)$ for all $v \in V$.

Proof. Let $E_0$ be the set of arcs $e$ with $g(e) \notin \mathbb{Z}$. The integrality of $g + g'$ implies $E_0' = E_0$, so the subgraph $\langle E_0 \rangle$ induced by $E_0$ is skew-symmetric. Also the half-integrality of $g$ and the integrality of $\text{div}_g$ imply that each node is incident with an even number of arcs in $E_0$. So the underlying undirected graph $H$ of $\langle E_0 \rangle$ is Eulerian.

We grow a (simple) path $P$ in $\langle E_0 \rangle$ such that $P \cap P' = \emptyset$, starting with an arbitrary node $v_0$ and allowing backward arcs in $P$. Let $v$ be the last node of the current $P$, and choose an arc $e \in E_0$ incident with $v$ and different from the last arc of $P$ ($e$ exists as $H$ is Eulerian). Let $u$ be the end of $e$ different from $v$. Three cases are possible. (i) If both $u, u'$ are not in $P$, we increase $P$ by adding $e, u$, and continue the process. (ii) If $u \in P$, we remove the part of $P$ from $u$ to $v$, obtaining the new current path from $v_0$ to $u$, and add $e$ to the removed part, forming circuit $C$ (with possible backward arcs). (iii) If $u' \in P$, we remove the part $Q$ of $P$ from $u'$ to $v$, obtaining the new current path, and add $e, Q'$ and $e'$ to $Q$, forming circuit $C'$ (which is reverse to $C'$).

In case (ii), we update $g$ by pushing half-unit along $C$ (i.e., by setting $g(a) := g(a) + \frac{1}{2}$ for the forward arcs $a$ in $C$, and $g(a) := g(a) - \frac{1}{2}$ for the backward arcs $a$) and by pushing half-unit along the circuit reverse to $C'$. Accordingly, we update $g'$ by pushing half-unit along $C'$ and along the circuit reverse to $C$. And in case (iii), $g$ ($g'$) is updated by pushing half-unit along $C$ (resp. $C'$). In both cases, the new $g'$ is symmetric to the new $g$ and each of the functions $g + g'$ and $\text{div}_g$ preserves. Also $E_0$ decreases by the set of arcs occurring in $C \cup C'$, and the new $H$ is Eulerian. We continue the process with the new $P$.

The final $g, g'$ give the desired $h, h'$. The bound $O(E)$ is obvious.

Return to problem (DS). Add to $G$ new nodes $t, t'$ and arcs $(t, s_i)$ and $(s'_i, t')$, forming skew-symmetric graph $G_1$, and extend $f$ to an IS-flow from $t$ to $t'$ in $G_1$ in a natural way. The fact that $f$ is integer and symmetric implies that $\text{div}_f(t)$ is even.

So we can apply Lemma 6.2 to the flows $g := g' := \frac{1}{2}f$, obtaining corresponding integer flows $h, h'$. The restriction $\overline{h}$ of $h$ to $E$ is an integer flow from $S$ to $S'$, and we apply the $O(E \log(2 + V^2/E))$-algorithm from Section 5 to decompose it as

$$
\overline{h} = \sum_{1 \leq i, j \leq k} h_{ij},
$$
where $h_{ij}$ is an integer flow from $s_i$ to $s_j$. Then the flows $f_{ii} := h_{ii}$ for $i = 1, \ldots, k$, and $f_{ij} := h_{ij} + h'_{ji}$ for $1 \leq i < j \leq k$ are as required, and Theorem 6.1 follows.

**Remark.** The above proof involves the following corollary from Lemma 6.2.

**Corollary 6.3** Let $f$ be an IS-flow from $S$ to $S'$ in a skew-symmetric graph $G = (V, E)$ (where $|S|$ is not fixed), and let $\text{div}_f(v)$ be even for all $v \in V$. Then there exists, and can be found in $O(E)$ time, an integer flow $g$ from $S$ to $S'$ such that $f = g + g'$ and $\text{div}_f(v) = 2\text{div}_g(v)$ for all $v \in V$.

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