On the integrability of the discrete nonlinear Schrödinger equation

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Abstract – In this letter we present an analytic evidence of the nonintegrability of the discrete nonlinear Schrödinger equation, a well-known discrete evolution equation which has been obtained in various contexts of physics and biology. We use a reductive perturbation technique to show an obstruction to its integrability.

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Introduction. – The nonlinear Schrödinger (NLS) equation

\[ i\partial_t f + \partial_{xx} f = \sigma |f|^2 f, \quad f = f(x,t), \quad \sigma = \pm 1, \]

(1)
is a universal nonlinear integrable partial differential equation (PDE) for models with weak nonlinear effects [1]. It has been central for almost forty years in a large variety of areas in sciences and it appears in many physical contexts, see for instance [2–6]. From the integrability of this PDE, there follows the existence of infinitely many symmetries and conservation laws, and the possibility of solving its Cauchy problem, once the initial data are prescribed. In correspondence with its symmetries, one finds an infinite number of exact solutions, the solitons, which, up to a phase, emerge unperturbed from the interaction among themselves.

Many physical and biological applications involve lattice systems. In literature, one may find a few discrete forms of the NLS equation. The most relevant one is the discrete NLS (DNLS) equation

\[ i\partial_t f_n + \frac{1}{2h^2}(f_{n+1} - 2f_n + f_{n-1}) = \sigma |f_n|^2 f_n, \]

(2)

where \( n \in \mathbb{Z} \) is a discrete index, \( t \) is a real variable, \( f_n \) is a complex function and \( h \) is a real parameter related to the space discretization. Its continuous limit (\( h \to 0, n \to \infty, x = nh \) finite) goes into the integrable NLS equation (1). Equation (2) is one of the most studied lattice models (see for instance [4,7–12] and references therein). Among the many recent applications of eq. (2) let us just mention the theory of Bose-Einstein condensates in optical lattices [12] and semiconductors [11]. The DNLS equation possesses exact discrete breather solutions, where the bumps are spatially localized and periodic in time [10,13]. However, just a few number of conserved quantities is known and thus the DNLS equation (2) is supposed to be nonintegrable [14]. Numerical schemes have been used to exhibit its chaotic behavior [15]. As far as we know, no proof of its nonintegrability is known and a few articles can be found on this subject [16].

An integrable discretization of the NLS equation has been found by Ablowitz and Ladik [17]. It reads

\[ i\partial_t f_n + \frac{1}{2h^2}(f_{n+1} - 2f_n + f_{n-1}) = \sigma |f_n|^2 f_n, \]

(3)

where \( \sigma \) is a real parameter. This equation has an infinite number of explicit standing as well as travelling soliton solutions [4]. In the limit as \( h \to 0 \), also eq. (3) goes into the NLS equation (1). Equations (2) and (3) exhibit very different responses to the same initial data. In the case of an infinite lattice with rapidly decaying boundary conditions, eq. (3) has soliton solutions while eq. (2) does not [15,18].

Multiscale perturbation techniques [19] have proved to be important tools for finding approximate solutions to many physical problems by reducing a given PDE to a simpler equation, which can be integrable [1]. These multiscale expansions are structurally strong and can be applied to both integrable and nonintegrable systems. Zakharov and Kuznetsov [20] have shown that, starting from an
integrable PDE and performing a proper multiscale expansion, one may obtain other integrable systems. In particular, they showed that the slow-varying amplitudes of a dispersive wave solution of eq. (1) satisfies the Korteweg-de Vries (KdV) equation (1) and vice versa. Calogero et al. [21] have used the multiscale perturbation technique as a tool to give necessary conditions for the integrability of large classes of PDEs both in 1+1 and 2+1 dimensions. In particular the nonintegrability of the resulting multiscale reduction is a consequence of the nonintegrability of the ancestor system. Multiscale techniques have been used also in the papers [22–24] to prove integrability of several PDEs.

Some attempts to extend this approach to discrete equations have been proposed [25–33]. In [30,31,33] one can find a multiscale expansion technique on the lattice which, starting from dispersive integrable $\mathbb{Z}^2$-lattice equations, provides other $\mathbb{Z}^2$-lattice equations. To do so, one had to introduce a slow-varying condition on the amplitudes by requiring

$$\left(\Delta_n\right)^{p+1} f_n = 0,$$

(4) $p$ being a positive integer and $\Delta_n f_n = f_{n+1} - f_n$. As a consequence, the resulting reduced equation turned out to be nonintegrable even if the ancestor equation was integrable. However, as shown in [27], if $p = \infty$ the reduced equations become formally continuous and their integrability may be preserved by the discrete reductive perturbation procedure. In this way the multiscale expansions easily fit with both difference-difference and differential-difference equations. The illustrative example considered in [27] has been the lattice potential KdV equation, a dispersive nonlinear $\mathbb{Z}^2$-lattice equation obtained from the superposition formula for the soliton solutions of the KdV equation. There one performed the multiscale expansion of the weak plane-wave solutions of the discrete dispersive linear system. A proper representation of the discrete shift operators in terms of differential ones provides the integrable NLS equation (1) as the lowest-order secularity condition from the multiscale expansion. Further examples have been considered in [29]. They confirmed a discrete analog of the Zakharov-Kuznetsov’s claim [20]: “if a nonlinear dispersive discrete equation is integrable then its lowest order multiscale reduction is an integrable NLS equation”.

In the present letter we consider the multiscale perturbation analysis of eq. (2). Multiscale analysis of the DNLS equation (2) has been already considered in [26], giving a differential-difference system which does not fulfill any integrability criterium. However this result did not prove the nonintegrability of the DNLS equation, as similar results have been obtained in the case of the integrable equation (3) [26,32]. Here, extending to lattice equations the approach used in [20,22–24] and computing the higher-order terms in the reductive perturbation expansion, we are able to provide an analytical evidence of the nonintegrability of eq. (2). In fact, even if its lowest-order multiscale reduction is an integrable KdV-type equation, the higher-orders reductions exhibit nonintegrable behaviors.

**Discrete multiscale analysis of the DNLS equation.** – By the position $f_\nu(t) = \left|\nu_\nu(t)\right|^{1/2}\exp[i\nu_\nu(t)]$ the DNLS equation (2) may be written as the following system of real differential-difference equations:

$$\partial_t \nu_\nu + \frac{1}{h^2} \left(\sqrt{\alpha^+_n} \sin \beta^+_n + \sqrt{\alpha^-_n} \sin \beta^-_n\right) = 0,$$

(5)

$$\partial_t \phi_\nu + \frac{1}{h^2} \left(\sqrt{\gamma^+_n} \cos \beta^+_n + \sqrt{\gamma^-_n} \cos \beta^-_n\right) + \sigma \nu_\nu = 0,$$

(6)

where $\alpha^+_n(t) = \nu_n \nu_{n+1}$, $\beta^+_n(t) = \phi_n \nu_{n+1} - \phi_n$, and $\gamma^+_n(t) = \nu_{n+1} \nu_{n+1}$. By analogy with the continuous case, see [20], we expand the real fields $\nu_\nu(t)$ and $\phi_\nu(t)$ around the constant solution $f_\nu(t) = \exp(-i\sigma t)$ of the DNLS

$$\nu_\nu(t) = 1 + \sum_{i=1}^{\infty} \epsilon^i \nu^{(i)}(\kappa, \{t_m\}_{m \geq 1}),$$

(7)

$$\phi_\nu(t) = -\sigma t + \sum_{i=1}^{\infty} \epsilon^{2i-1} \phi^{(i)}(\kappa, \{t_m\}_{m \geq 1}),$$

(8)

where $\epsilon$, with $0 \leq |\epsilon| \ll 1$, is the perturbation parameter. The fields $\nu^{(i)}$ and $\phi^{(i)}$ in eqs. (7), (8) depend on the slow-space variable $\kappa = \epsilon \zeta_\nu, \zeta \in \mathbb{R}$, and the slow-time variables $t_m = \epsilon^{2m-1} t, m \geq 1$. The free parameter $\zeta$ can be fixed so as to obtain the proper continuous limit.

Given a function $u_n(t) = \nu(\kappa, \{t_m\}_{m \geq 1})$, we expand $u_{n\pm 1}(t)$ and $\partial_t u_n(t)$ in terms of the slow variables $\kappa$ and $\{t_m\}_{m \geq 1}$. Introducing the shift operator $T_\kappa$ such that $T_\kappa u_n = u_{n\pm 1}$, we have

$$u_{n\pm 1}(t) = (T^{\pm 1}_\kappa)^{\zeta} v(\kappa, \{t_m\}_{m \geq 1}) = \sum_{i=0}^{\infty} \epsilon^i A_i^\pm v(\kappa, \{t_m\}_{m \geq 1}),$$

(9)

$$\partial_t u_n(t) = \sum_{i=1}^{\infty} \epsilon^{2i-1} \partial_t v(\kappa, \{t_m\}_{m \geq 1}),$$

(10)

where, defining $\Delta_i^\pm = (T^{\pm 1}_\kappa - 1)^i$ to be the $i$-th order difference operator, one gets

$$A_i^\pm = \frac{(\pm \zeta \delta_\kappa)^i}{i!}, \quad \delta_\kappa = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_i^\kappa.$$

(11)

If $u_n$ is a slow-varying function of order $p$, see eq. (4), we can truncate the infinite series in eq. (11). In such a case the $\delta_\kappa$-operators reduce to polynomials in the $\Delta_\kappa$-operators of order at most $p$. Hereafter we shall assume $p = \infty$, and the $\delta_\kappa$-operators are formal differential operators.
Taking into account the expansions (7), (8) and eq. (9), we have the following formulas for the shifts of the functions \( \nu_n(t) \) and \( \phi_n(t) \):

\[
\nu_{n\pm 1}(t) = 1 + \sum_{j=2}^{\infty} \sum_{i=1}^{[j/2]} e^j A_{j-2i, j} \phi(i)(\kappa, \{ t_m \}_{m \geq 1}),
\]

\[
\phi_{n\pm 1}(t) = -\sigma t + \sum_{j=1}^{\infty} \sum_{i=1}^{[(j+1)/2]} e^j A_{j-2i+1, j} \phi(i)(\kappa, \{ t_m \}_{m \geq 1}),
\]

where \([x]\) denotes the integer part of \(x\).

Let us introduce the multiscale expansions (12), (13), together with eq. (10), into eqs. (5), (6) and require that these equations be satisfied at all orders in \(\epsilon\).

At the lowest nontrivial order \(\epsilon^2\) one finds \(\phi^{(1)} = -\sigma \partial_\kappa \phi^{(1)}\). From now on all results will be presented just for the functions \(\phi^{(1)}\).

At order \(\epsilon^3\) we get \(\partial^2_\xi - c^2 \partial_\xi^2 \phi^{(1)} = 0\), where \(c = \pm (\zeta \sigma^{1/3})/h\). As \(c\) has to be real, then \(\sigma = 1\); moreover we choose \(\zeta = h\) so that \(c\) remains finite as \(\epsilon \to 0\). Therefore the resulting equation at this order is satisfied by \(\phi^{(1)}(\xi, \{ t_m \}_{m \geq 2})\) with \(\xi = \kappa - ct\).

At order \(\epsilon^3\), the no-secular term condition implies \(\partial^2_\xi - c^2 \partial_\xi^2 \phi^{(2)} = 0\), i.e. \(\phi^{(2)} = \phi^{(2)}(\xi, \{ t_m \}_{m \geq 2})\). The evolution equation for \(\phi^{(1)}\) w.r.t. the slow time \(t_2\) reads

\[
\partial_{t_2} \phi^{(1)} = K_2(\phi^{(1)}),
\]

\[
K_2(\phi^{(1)}) = a \left[ \frac{3}{4a} \partial_\xi \phi^{(1)} \right]^2 - \frac{3}{4a} \partial_\xi \phi^{(1)} - f(\phi^{(1)}),
\]

with \(a = c(3 - h^2)/24\). Equation (14) is a potential KdV equation. Therefore, if eq. (2) has to be integrable, then its multiscale reductions should provide the integrable evolution equations \((j \geq 3)\)

\[
\partial_{t_3} \phi^{(1)} = K_3(\phi^{(1)}) = b_j \int \xi y^{j-1} \left[ \partial_\xi^3 \phi^{(1)} \right],
\]

\[
L[f(\xi)] = \partial_\xi^2 f(\xi) - \frac{\partial_{t_3} \phi^{(1)}}{a} f(\xi) - \frac{\partial_{t_3} \phi^{(1)}}{2a} \int y \, df(y),
\]

where \(L\) is the recursive operator associated with the KdV hierarchy and the \(b_j\)'s are free coefficients to be fixed later.

We assign a formal degree to the \(\xi\)-derivatives of the functions \(\phi^{(j)}\), \(\deg(\partial^j_\xi \phi^{(j)}) = \ell + 2j - 1\), \(\ell \geq 0\), and define \(P_n\) as the vector space spanned by the products with total degree \(n\) of all derivatives \(\partial^j_\xi \phi^{(j)}\). We denote by \(P^{(r)}_n \subset P_n\) the subspace spanned by those products of derivatives \(\partial^j_\xi \phi^{(j)}\) with \(j \leq r\). Similar vector spaces have been introduced in [22,23] in the case of the multiscale analysis of PDEs.

After caring for secularities, the order \(\epsilon^7\) yields \(\phi^{(3)} = \phi^{(3)}(\xi, \{ t_m \}_{m \geq 2})\) and the following nonhomogeneous evolution equation for the field \(\phi^{(2)}\) w.r.t. the slow time \(t_2\), depending on \(\phi^{(1)}\) and its derivatives:

\[
\partial_{t_2} \phi^{(2)} = c^3(3 - h^2) \partial^3_\xi \phi^{(2)} + \frac{3}{2} \partial_{t_3} \phi^{(1)} \partial_\xi \phi^{(2)} = -\partial_{t_3} \phi^{(1)} - \frac{5h^2}{64} \left( \partial_\xi^2 \phi^{(1)} \right)^2 + \frac{c^2 h^2}{12} \left( \partial_\xi \phi^{(1)} \right)^3 - \frac{3}{16} \partial_\xi \phi^{(1)} \partial_\xi \phi^{(2)} - \frac{c^2 h^2 - 30h^2 - 15}{1920} \partial_\xi \phi^{(1)}.\]

Substituting eq. (15) into eq. (16) with \(j = 3\) and fixing \(b_3 = -c(h^4 - 30h^2 - 15)/1920\) in order to remove residual secularities, eq. (16) reduces to the following evolution equation for the field \(\phi^{(2)}\) w.r.t. the slow time \(t_2\):

\[
\partial_{t_2} \phi^{(2)} - K_3(\phi^{(1)}) \phi^{(2)} = f(t_2),
\]

where \(K_3(\phi^{(1)}) \phi^{(2)} = (d/d\theta) K_3(\phi^{(1)} + \theta \phi^{(2)})|_{\theta = 0}\) is the Fréchet derivative of \(K_3(\phi^{(1)})\) along the direction of \(\phi^{(2)}\).

In eq. (17) the forcing term \(f(t_2)\) is a well-defined element of \(P_3\), namely a linear combination of three independent differential monomials, with coefficients that are rational functions of \(h\). At this same order, the integrability of eq. (2) implies the existence of the following evolution equation for the field \(\phi^{(2)}\) w.r.t. the slow time \(t_3\):

\[
\partial_{t_3} \phi^{(2)} - K_3(\phi^{(1)}) \phi^{(2)} = f(t_3),
\]

where \(f(t_3)\) is an element of the space \(P_3\), which if eq. (2) has to be integrable, must satisfy the compatibility condition \([\partial_{t_3} - K_3(\phi^{(1)})]f(t_3) = [\partial_{t_3} - K_3(\phi^{(1)})]f(t_2)\). Such a condition allows one to express the coefficients of the polynomial \(f(t_3)\) in terms of those of \(f(t_2)\), and does not impose any further constraint on the coefficients of \(f(t_2)\). As this condition is satisfied, eventual obstructions to the integrability of eq. (2) will appear at higher perturbative orders.

Let us now consider the order \(\epsilon^9\). The resulting equations provide the evolution of the field \(\phi^{(3)}\) w.r.t. the slow time \(t_2\). This is given by an integro-differential equation. Introducing the fields \(\phi^{(3)} = \partial_t \phi^{(3)}\), taking care of secularities and taking into account that \(\phi^{(1)}\) evolves w.r.t. the slow time \(t_4\) according to eq. (15), we get \(\phi^{(4)}(\xi, \{ t_m \}_{m \geq 2})\) and

\[
\partial_{t_4} \varphi^{(3)} - H_4(\varphi^{(1)}) \varphi^{(3)} = g(t_4),
\]

where \(H_4(\varphi^{(1)}) \varphi^{(3)}\) is the Fréchet derivative along \(\varphi^{(3)}\) of the KdV flow \(H_3(\varphi^{(1)}) = \partial_t K_3(\varphi^{(1)})\). Here \(\varphi^{(3)} \in P_3\) is a linear combination of fourteen differential monomials, whose coefficients are well-defined rational functions of \(h\). The evolution equation of \(\varphi^{(3)}\) w.r.t. the slow time \(t_3\) takes the form

\[
\partial_{t_3} \varphi^{(3)} - H_3(\varphi^{(1)}) \varphi^{(3)} = g(t_3),
\]

where \(H_3(\varphi^{(1)}) \varphi^{(3)}\) is the Fréchet derivative along \(\varphi^{(3)}\) of the higher-order KdV flow \(H_3(\varphi^{(1)}) = \partial_t K_3(\varphi^{(1)})\). Here \(g(t_3)\) is an element of the 31-dimensional vector space.
\( \mathcal{P}^{(2)}_{11} \) whose coefficients are determined by requiring the compatibility condition

\[
[\partial_\alpha - H'_{2}(\varphi^{(1)})]g^{(t_2)} = [\partial_\alpha - H'_{2}(\varphi^{(1)})]g^{(t_2)}.
\]

Equation (21) is a necessary condition for the integrability of eq. (2). In this case only nine out of the fourteen coefficients of \( g^{(t_2)} \) are independent. Thus we have five integrability conditions, whose explicit expressions will be published elsewhere in a more detailed paper [34]. The obtained constraints on the polynomial \( g^{(t_2)} \) are not satisfied by the coefficients computed in eq. (19). Then, this incompatibility implies that the DNLS equation (2) cannot be integrable.

**Concluding remarks.** – By performing a discrete multiscale analysis, we have proven that the DNLS equation (2) is not integrable. Although its lowest-order reduction gives a KdV equation, the higher-order terms do not satisfy the required integrability conditions. It is remarkable to note that a similar analysis performed on the integrable discrete NLS equation (3) provides integrable reductions [34].

Moreover, by using the expansions (7), (8), our perturbation technique enables us to construct approximate solutions of eq. (2) starting from the exact solutions of the integrable eq. (14). However these solutions will break down in the far-field region, due to nonintegrability of eq. (2).

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