Large Deviations in Renewal Theory and Renewal Models of Statistical Mechanics

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Abstract We present and establish large deviations principles for general multivariate renewal-reward processes associated with a classical discrete-time renewal process. A renewal-reward process describes a cumulative reward over time, supposing that a broad-sense multivariate reward is obtained at each occurrence of the event that is renewed under the renewal process. We consider both the standard model and a constrained model that is constructed conditioning on the event that one of the renewals occurs at a predetermined time. With a different interpretation of the time coordinate, the constrained renewal model includes several important models of statistical mechanics, such as the model of polymer pinning, the Poland-Scheraga model of DNA denaturation, the Wako-Saitô-Muñoz-Eaton model of protein folding, and the Tokar-Dreyssé model of strained epitaxy. We attack the problem of large deviations in constrained renewal models by an argument based on convexity and super-additivity. Then, we transfer results to standard renewal processes by resorting to conditioning. In the context of constrained renewal models, we also propose an explicit application of the general theory to deterministic rewards that grow no faster than the time elapsed between two successive occurrences of the renewed event. This type of rewards codifies the extensive variables of statistical mechanics.

Keywords Large deviations · Renewal processes · Renewal-reward processes · Statistical mechanics · Polymer pinning models · DNA denaturation · Protein folding · Critical behavior

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1 Introduction

Renewal processes are models of stochastic phenomena where an event is renewed continuously over time. Renewal processes play an important role in different areas...
of applied mathematics including insurance [1], finance [2], and queuing systems [3] among others. They also appear in statistical mechanics since several models of this field can be recast in terms of renewal processes under the condition that one of the renewals occurs at a predetermined time [4]. This condition reflects a different interpretation of the time coordinate and gives rise to constrained renewal models. The original standard processes that are free from this condition can be referred as free renewal models. Two well-known examples of constrained renewal models are the model of polymer pinning [4] and the Poland-Scheraga model of DNA denaturation [4, 5, 6], the former being formulated as a free renewal model as well. Lesser-known models that can be mapped in constrained renewal models are the model for protein folding introduced independently by Wako and Saitô first [7, 8] and Muñoz, Eaton, and co-workers later [9, 10, 11] and the model of strained epitaxy proposed by Tokar and Dreyssé [12, 13, 14]. Renewal models of statistical mechanics are exactly solvable and able to account for phase transitions of any integer order [4]. This way, they have attracted the interest of many researchers often encouraging generalizations, as in the case of the Wako-Saitô-Muñoz-Eaton model [15, 16, 17, 18, 19, 20, 21, 22] familiar to the author. The Wako-Saitô-Muñoz-Eaton model and the Tokar-Dreyssé model are customarily presented as lattice-spin models with binary degrees of freedom. The link with stochastic processes stems from the fact that these binary variables can be regarded as indicators of hypothetical renewals, constituting the so-called regenerative phenomenon associated by Kingman with a renewal process [23].

The ubiquity of renewal processes in the context of stochastic processes has stimulated the quest for many large deviations principles within free renewal models [24, 25, 26, 27, 28, 29, 30, 31]. On the contrary, in spite of the large amount of work done on renewal models of statistical mechanics, no large deviations principle has been investigated so far within this class of models to the best of our knowledge. The present paper is intended to lay the foundation for the use of the large deviations formalism in constrained renewal models. This formalism has been successfully applied to other lattice-spin models of statistical mechanics, such as the Curie-Weiss model [32], the Curie-Weiss-Potts model [33], the mean-field Blume-Emery-Griffiths model [34], and the Ising model [35, 36, 37, 38].

In this work we introduce and prove large deviations principles for general multivariate renewal-reward processes associated with constrained renewal models and, as a by-product, with also free renewal models. A renewal-reward process is defined as a cumulative reward over time, imagining that a broad-sense multivariate reward is obtained at each occurrence of the event that is renewed under the renewal process. Rewards that are a deterministic function of the time elapsed between two successive renewals are of special interest and are called deterministic rewards in this paper. From the perspective of statistical mechanics, renewal-reward processes corresponding to deterministic rewards are macroscopic variables. This way, our results are concerned with the (possible joint) fluctuations of the observables involved in the thermodynamic description of the system. From the perspective of stochastic processes, our results extend existing large deviations principles for renewal-reward processes with deterministic and univariate rewards [27, 29] to cases where rewards are any multivariate random variables. We attack the problem of large deviations in constrained renewal models by an argument based on convexity and super-additivity, and then we transfer results to free renewal models by resorting to conditioning. This argument is a suitable
generalization to the present context of the method used by Bahadur and Zabell [39] to derive the modern form of the Cramér’s Theorem [40]. In turn, this method can be traced back to the approach of Ruelle [41] and Lanford [42] for proving the existence of various thermodynamic limits.

The connection between critical phenomena and the non-analytic behavior of rate functions in large deviations principles has been gaining considerable interest in the physics community as demonstrated by several, although often heuristic, recent works [43,44,45,46,47,48,49,50,51,52,53,54]. We point out that two of these works deal with renewal processes [52,53]. Renewal models are able to account for different kinds of phase transitions and provide at the same time a framework where explicit results are feasible, in contrast to most models. This way, renewal processes ultimately supply a perfect framework to probe the above connection. In this respect, we supplement our general theory with an application to deterministic rewards in the context of constrained renewal models, which are the most relevant for statistical physics. In particular, we characterize the rate function corresponding to deterministic rewards that increase linearly or sublinearly (in magnitude) with the time elapsed between two successive occurrences of the renewed event. Due to their property, this type of rewards exactly codify the typical extensive variables of statistical mechanics.

The paper is organized as follows. In Sect. 2 we introduce the mathematical context of renewal processes and present our results on large deviations in constrained renewal models and free renewal models. Such results are proven in Sect. 3 for the constrained case and in Sect. 4 for the free case. The general theory is then applied in Sect. 5 to deterministic rewards in constrained renewal models.

### 2 Models and Main Results

In this section we summarize the main results of the work as follows. Fundamentals on renewal processes are first introduced in Sect. 2.1. Then, we expose our large deviations principles in Sect. 2.2. Last, we outline the connection between regenerative phenomena and statistical mechanics in Sect. 2.3. With this purpose, we briefly review the Poland-Scheraga model, the Wako-Saitô-Muñoz-Eaton model, and the Tokar-Dreyssé model providing the mapping with renewal processes. To the best of our knowledge, this mapping was not known and has never been shown before for the latter two models.

#### 2.1 Basics on Renewal Theory

Let on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) be given a sequence \(\{S_i\}_{i \geq 1}\) of independent and identically distributed random variables taking values in \(\{1, 2, \ldots\} \cup \{\infty\}\).

The sequence \(\{T_i\}_{i \geq 0}\) defined by \(T_0 := 0\) and \(T_i := S_1 + \cdots + S_i\) for all \(i \geq 1\) is called a **discrete-time renewal process**. The variable \(T_i\) is the \(i^{th}\) **renewal time** and is usually thought of as the instant at which a specific event occurs for the \(i^{th}\) time. In this respect, the number of occurrences of the event by the time \(t \geq 0\) is \(N_t := \sup\{i \geq 0 : T_i \leq t\}\). The variable \(S_i\) is the \(i^{th}\) **waiting time** and represents the waiting time for the \(i^{th}\) occurrence after the previous occurrence. The common distribution \(p\) of waiting times is specified by \(p(s) := \mathbb{P}[S_1 = s]\) for every \(s \geq 1\). The
renewal process is called recurrent if \( P[S_1 < \infty] = \sum_{i \geq 1} p(s) = 1 \) and transient if \( P[S_1 < \infty] < 1 \). The renewal process is aperiodic if there does not exist an integer \( \lambda > 1 \) with the property that \( p(s) = 0 \) except when \( s \) is a multiple of \( \lambda \). Any renewal process is aperiodic on a proper time scale so that aperiodic renewal processes cover all interesting situations. The probability that a renewal occurs at the time \( t \geq 0 \) is \( P[X_t = 0] \), \( X_t \) being the indicator variable that takes value 0 if \( t \in \{T_i\}_{i \geq 0} \) and value 1 otherwise. This probability satisfies the initial condition \( P[X_0 = 0] = 1 \) and the renewal equation \( P[X_t = 0] = \sum_{i=1}^t p(s) P[X_{t-s} = 0] \) for each \( t \geq 1 \). This equation comes from conditioning on the first renewal time \( T_1 = S_1 \) and then using the fact that the renewal process starts over at every renewal time. For a presentation of discrete-time renewal processes we refer, for example, to [55].

Denoting by \( S \) the support \( \{s \geq 1 : p(s) > 0\} \) of the waiting time distribution \( p \) and by \( \gcd\{s_1, \ldots, s_n\} \) the greatest common divisor of \( n \) integers \( s_1, \ldots, s_n \), hereafter we assume that for some \( n \geq 1 \) the set \( S \) contains at least \( n \) numbers \( s_1, \ldots, s_n \) such that \( \gcd\{s_1, \ldots, s_n\} = 1 \). This hypothesis is tantamount to say that the renewal process is aperiodic and entails that any sufficiently large time \( t \) can be a renewal instant with positive probability. Indeed, the straight bound \( P[X_{t+s} = 0] \geq P[S_1 = s \land X_{t+s} = 0] = p(s) P[X_t = 0] \) valid for each \( t \geq 0 \) and \( s \geq 1 \) yields after iteration that \( P[X_{\sum_{i=1}^n \nu_{s_i}} = 0] \geq \prod_{i=1}^n p(s_i)^{\nu_i} > 0 \) for all non-negative integers \( \nu_1, \ldots, \nu_n \) with \( s_1, \ldots, s_n \) as above. On the other hand, since \( \gcd\{s_1, \ldots, s_n\} = 1 \) there exists a largest integer that cannot be expressed in the form \( \sum_{i=1}^n \nu_i s_i \) with non-negative integer coefficients \( \nu_1, \ldots, \nu_n \). This integer number is the Frobenius number associated with \( s_1, \ldots, s_n \) and every \( t \) larger than it satisfies \( P[X_t = 0] > 0 \).

Suppose now that a sequence \( \{R_i\}_{i \geq 1} \) of random variables taking values in \( \mathbb{R}^d \) is given on \( (\Omega, \mathcal{F}, P) \) in such a way that \( \{(S_i, R_i)\}_{i \geq 1} \) is a sequence of independent and identically distributed random vectors. We think of \( R_i \) as a broad-sense reward associated with the waiting time \( S_i \). The variables \( R_i \) and \( S_i \) are allowed to be dependent in general. The total reward by the time \( t \geq 0 \) is

\[
W_t := \sum_{i=1}^{N_t} R_i,
\]

with the convention that \( \sum_{i=1}^n R_i := 0 \) if \( n = 0 \). The sequence \( \{W_t\}_{t \geq 0} \) is called a renewal-reward process. The aim of this work is to establish a large deviations principle for the variable \( W_t \) with respect to both free renewal models and constrained renewal models. The free renewal model is represented by the original probability space \( (\Omega, \mathcal{F}, P) \). In addition to \( P \), on the measurable space \( (\Omega, \mathcal{F}) \) we consider the probability measure \( \tilde{P} \) whose support comprises only those samples where a renewal occurs at the predetermined time \( t \), namely where \( X_t = 0 \). We refer to the probability space \( (\Omega, \mathcal{F}, \tilde{P}) \) as the constrained renewal model and for each \( t \geq 0 \) we precisely define it in terms of the Radon-Nikodym derivative

\[
\frac{d\tilde{P}}{dP} := \frac{1 - X_t}{P[X_t = 0]}
\]

We stress that the constrained renewal model is well-defined at least for any sufficiently large \( t \), as we have shown that \( P[X_t = 0] > 0 \) holds in such case.
2.2 Main Results

Let $E$ denote expectation with respect to the probability measure $P$ and let $u \cdot v$ denote the dot product of two vectors $u$ and $v$ in $\mathbb{R}^d$. Let $\|u\|_2 := \sqrt{u \cdot u}$ be the Euclidean norm of $u$. Our main result on constrained renewal models requires at first the introduction of two functions. The former is the function $\zeta$ that maps each point $k \in \mathbb{R}^d$ in the extended real number $\zeta(k)$ defined by

$$\zeta(k) := \inf \left\{ z \in \mathbb{R} : E\left[ \exp (k \cdot R_1 - zS_1) \mathbb{1}(S_1 < \infty) \right] \leq 1 \right\},$$

(1)

where the infimum over the empty set is customarily interpreted as $\infty$. The latter is the function $I$ built from $\zeta$ that associates every vector $w \in \mathbb{R}^d$ with the extended real number $I(w)$ given by

$$I(w) := \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k - \zeta(k) + \zeta(0) \right\}.$$

(2)

A part from the constant $\zeta(0)$, the function $I$ is the convex conjugate of $\zeta$. We point out that $\zeta(0)$ is finite. Indeed, on the one hand we have that $\zeta(0) \leq 0$ because $E[\exp(-zS_1)\mathbb{1}(S_1 < \infty)] \leq 1$ for all $z \geq 0$ and on the other hand we have that $\zeta(0) > -\infty$ since the support $S$ of the waiting time distribution $p$ is nonempty by hypothesis so that $E[\exp(-zS_1)\mathbb{1}(S_1 < \infty)] = \sum_{s \in S} \exp(-zs)p(s) > 1$ for each number $z$ sufficiently negative. If the renewal process is recurrent, then $\zeta(0) = 0$ due to the fact that $E[\exp(-zS_1)\mathbb{1}(S_1 < \infty)] > \sum_{s \in S} p(s) = 1$ for any $z < 0$ in such case.

The following theorem is concerned with the fluctuations of the scaled total reward $W_t/t$ at large times $t$ with respect to the constrained renewal model and represents our first main result. It is proven in Sect. 3.

**Theorem 1** The following conclusions hold:

(a) the function $I$ is lower semicontinuous and proper convex. If in addition $\zeta$ is finite in an open neighborhood of the origin, then $I$ has compact level sets\(^1\);

(b) if $G$ is any open set in $\mathbb{R}^d$, then

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t \left[ \frac{W_t}{t} \in G \right] \geq - \inf_{w \in G} \{ I(w) \}.$$

If $G$ is convex as well as open, then the limit exists and

$$\lim_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t \left[ \frac{W_t}{t} \in G \right] = - \inf_{w \in G} \{ I(w) \};$$

(c) if $F$ is any compact set or any closed convex set in $\mathbb{R}^d$, then

$$\limsup_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t \left[ \frac{W_t}{t} \in F \right] \leq - \inf_{w \in F} \{ I(w) \}.$$

If $\zeta$ is finite in an open neighborhood of the origin, then this bound is valid more generally for each closed set $F$ in $\mathbb{R}^d$.

\(^1\) A level set of the function $I$ is a set of the form $\{ w \in \mathbb{R}^d : I(w) \leq \lambda \}$ with an arbitrarily given real number $\lambda$. Level sets of lower semicontinuous functions are always closed.
The lower bound in Part (b) of Theorem 1 and the upper bound in Part (c) are usually called large deviations lower bound and large deviations upper bound [40], respectively. When a lower semicontinuous function $I$ exists in such a way that the large deviations lower bound holds for each open set $G$ and the large deviations upper bound holds for each compact set $K$, then $W_t$ is said to satisfy a weak large deviations principle with rate function $I$ [40]. If the large deviations upper bound holds more generally for every closed set $F$, then $W_t$ is said to satisfy a full large deviations principle [40]. A rate function $I$ is a good rate function if it has compact level sets [40]. Theorem 1 states that, within the constrained renewal model, the total reward $W_t$ satisfies a weak large deviations principle with rate function $I$ given by (2) under the only innocuous assumption of aperiodicity. If in addition the function $\zeta$ is finite in an open neighborhood of the origin, then $W_t$ satisfies a full large deviations principle with good rate function $I$. An interesting case that gives rise to a function $\zeta$ that is finite in an open neighborhood of the origin and everywhere is when the reward $R_1$ does not grow faster than the waiting time $S_1$ in the sense that a positive constant $M < \infty$ exists so that $\| R_1 \|_2 \leq MS_1$. In this case, for each point $k \in \mathbb{R}^d$ and number $z \geq M\|k\|_2$ we have that $k \cdot R_1 - zS_1 \leq \|k\|_2\|R_1\| - zS_1 \leq (M\|k\|_2 - z)S_1 \leq M\|k\|_2 - z$. Consequently, $\mathbb{E}(\exp(k \cdot R_1 - zS_1) \mathbb{1}(S_1 < \infty)) \leq \exp(M\|k\|_2 - z) \leq 1$ for any fixed $k$ and every sufficiently large $z$, thus implying that $\zeta(k) < \infty$ according to definition (1).

The study of the fluctuations of the scaled total reward with respect to the free renewal model takes advantage of the analysis of constrained renewal models and brings us to our second main result. In order to present this result, we set $\xi_* := \liminf_{t \uparrow \infty} (1/t) \ln \mathbb{P}[S_1 > t]$ and we introduce the function $I_*$ built again from $\zeta$ that maps each $w \in \mathbb{R}^d$ in the extended real number $I_*(w)$ given by

$$I_*(w) := \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k - \max \{ \zeta(k), \xi_* \} \right\}. \tag{3}$$

Similarly, we set $\xi^* := \limsup_{t \uparrow \infty} (1/t) \ln \mathbb{P}[S_1 > t] \geq \xi_*$ and we define $I^*$ to be the function that associates every $w \in \mathbb{R}^d$ with

$$I^*(w) := \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k - \max \{ \zeta(k), \xi^* \} \right\}. \tag{4}$$

The following theorem quantifies the fluctuations of $W_t/t$ at large $t$ with respect to the free renewal model. The proof is given in Sect. 4.

**Theorem 2** The following conclusions hold:

(a) the functions $I_*$ and $I^*$ are both lower semicontinuous and proper convex. If in addition $\zeta$ is finite in an open neighborhood of the origin, then $I_*$ and $I^*$ have compact level sets;

(b) if $G$ is any open set in $\mathbb{R}^d$, then

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq \inf_{w \in G} \left\{ I_*(w) \right\};$$

(c) if $K$ is any compact set in $\mathbb{R}^d$, then

$$\limsup_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in K \right] \leq \inf_{w \in K} \left\{ I^*(w) \right\}.$$
If $\zeta$ is finite in an open neighborhood of the origin, then this bound is valid more generally for each closed set $I$ in $\mathbb{R}^d$ in place of $K$.

Theorem 2 states that, within the free renewal model, the total reward $W_t$ satisfies a weak large deviations principle with rate function $I_*$ in all those cases where $I_* = I^*$. Finiteness of $\zeta$ in an open neighborhood of the origin gives a full large deviations principle with good rate function $I_*$ when $I_* = I^*$. We find that $I_* = I^*$ if $\xi^* = -\infty$ or, more generally, $\xi_* = \xi^*$ as expected in all real applications. We have that $I_* = I^*$ even if $\xi_* < \xi^*$ provided that the condition $\zeta(k) \geq \xi^*$ is met for all $k \in \mathbb{R}^d$. Such condition is not trivial, as $\xi^* > -\infty$ if $\xi^* > \xi_*$, but is verified in the interesting case of a recurrent renewal process with a reward $R_i$ that grows more slowly than the waiting time $S_1$ in the sense that there exists a function $\varphi$ from $\{1, 2, \ldots\} \cup \{\infty\}$ to $\mathbb{R}$ such that $\|R_1\|_2 \leq \varphi(S_1)$ and $\lim_{s \to \infty} \varphi(s)/s = 0$. In this case $I_* = I^* = I$ because $\zeta(0) = 0$ due to the recurrence property. To show that $\zeta(k) \geq \xi^* > -\infty$ for any fixed point $k$, we pick an arbitrary real number $z < \xi^*$ and then a number $\varepsilon > 0$ and an integer $t_o \geq 0$ such that $\varepsilon\|k\|_2 + z < \xi^* \leq 0$ and $\|R_1\|_2 \leq \varepsilon S_1$ if $S_1 > t_o$. The integer $t_o$ exists due to the hypotheses that $\|R_1\|_2 \leq \varphi(S_1)$ and $\lim_{s \to \infty} \varphi(s)/s = 0$. If $S_1 > t_o$, then we have that $k \cdot R_1 - z S_1 \geq -\varepsilon\|k\|_2 + z + z) \geq 0$ which shows that $\mathbb{E}[\exp(k \cdot R_1 - z S_1) \geq \mathbb{E}[\exp(k \cdot R_1 - z S_1) \|S_1 > t_o\|$ for all $t \geq t_o$. Sending $t$ to infinity we realize that $\mathbb{E}[\exp(k \cdot R_1 - z S_1) = \infty$ since $\varepsilon\|k\|_2 + z < \xi^*$ and $\zeta^* = \lim sup_{s \to \infty}(1/t)\ln \mathbb{P}[S_1 > t]$. At this point, recurrence yields that $\mathbb{E}[\exp(k \cdot R_1 - z S_1) \|S_1 < \infty\] = \mathbb{E}[\exp(k \cdot R_1 - z S_1)] = \infty$ and the arbitrariness of $z < \xi^*$ allows us to conclude that $\zeta(k) \geq \xi^*$ according to (1).

The large deviations principles stated by Theorems 1 and 2 describe the fluctuations of the random vector $W_t/t$ at large $t$ for very general rewards. However, in applications and especially in statistical physics a major role is played by deterministic rewards. We say that rewards are deterministic if there exists a function $f$ from $\{1, 2, \ldots\} \cup \{\infty\}$ to $\mathbb{R}^d$ in such a way that $R_i = f(S_i)$ for all $i \geq 1$. Previous large deviations principles exist for deterministic rewards in the context of stochastic processes and are recovered by Theorem 2. In particular, the case where the renewal process is recurrent and the total reward $W_t$ is the number $N_t$ of renewals by the time $t$, corresponding to the choice $f(s) := 1$ for all $s \geq 1$, was first studied in [27]. Then, recurrent renewal processes and deterministic rewards with an univariate and bounded function $f$ were considered in [29]. Following this trend, in Sect. 5 we shall analyze in detail the case of constrained renewal models and deterministic rewards with a possible unbounded function $f$ that is supposed to fulfill the property that the limit $\lim_{s \to \infty} f(s)/s =: \ell$ exists in $\mathbb{R}^d$. The interest in this case is twofold: on the one hand it is loose enough to include the models of statistical mechanics with most extensive variables; on the other hand it is definite enough to make feasible explicit calculations by methods of convex analysis. In particular, in Sect. 5 we will demonstrate how to compute the rate function $I$, determining its effective domain and the family of subdifferentials of the function $\zeta$, which is shown to be proper convex in Sect. 3. The analytic behavior of $I$ depends on the differential properties of $\zeta$ and, for instance, the graph of $I$ exhibits affine stretches if $\zeta$ is not differentiable at some point.

In Sect. 5 we will also prove that for constrained renewal models and deterministic rewards that grow no faster than waiting times the scaled total reward $W_t/t$ always converges in probability to a constant vector $w_o$ as $t$ is sent to-
finity, meaning that \( \lim_{t \to \infty} P_t[|W_t/t - w_o|_2 \geq \delta] = 0 \) for each \( \delta > 0 \). This convergence is exponential apart from peculiar constrained renewal models where it results slower than exponential due to persistent fluctuations. We name this special constrained renewal models \textit{critical} in the spirit of statistical mechanics.

To characterize critical renewal models we need to introduce the number \( \xi := \limsup_{s \to \infty} (1/s) \ln p(s) \leq \xi^* \), which is exactly equal to \( \xi^* \) when the renewal process is recurrent. Critical renewal models will be shown to be constrained renewal models corresponding to waiting time distributions \( p \) that satisfy \( \xi > -\infty \), \( \sum_{s \geq 1} \exp(-\xi s) p(s) = 1 \), and \( \sum_{s \geq 1} s \exp(-\xi s) p(s) < \infty \). For this class of models, the sum \( \sum_{s \geq 1} f(s) \exp(-\xi s) p(s) \) exists when \( f(s)/s \) is bounded at large \( s \) and the constant vector \( w_o \) is found to be

\[
w_o := \frac{\sum_{s \geq 1} f(s) \exp(-\xi s) p(s)}{\sum_{s \geq 1} s \exp(-\xi s) p(s)}.
\]

From the perspective of large deviations principles, we stress that the rate function has a unique minimum point (where it takes value zero) in normal situations. In contrast, the rate function \( I \) corresponding to critical renewal models and deterministic rewards for which the limit \( \lim_{s \to \infty} f(s)/s =: \ell \) exists has an entire segment of minimum points. This segment is the segment connecting \( w_o \) to \( \ell \) and can be called \textit{phase transition segment} according to the literature on large deviations principles in statistical mechanics [35,43]. As the phase transition segment is the set of the zeros of the rate function \( I \), Theorem 1 loses effectiveness to describe the fluctuations of the random vector \( W_t/t \) at large \( t \) over measurable sets that intersect the phase transition segment. These fluctuations are exactly the persistent fluctuations that prevent the convergence in probability to \( w_o \) from being exponential and require further studies to be characterized. The nature of such fluctuations is currently under investigation for certain classes of waiting time distributions and will be the subject of a future paper [56].

### 2.3 Regenerative Phenomena and Statistical Mechanics

The binary process \( \{X_t\}_{t \geq 0} \) of (non-)renewal indicators plays an important role in connection with regenerative phenomena and statistical mechanics. We recall that \( X_t := 0 \) if \( t \in \{T_n\}_{n \geq 1} \) and \( X_t := 1 \) otherwise for each \( t \geq 0 \). According to Kingman [23], the binary process \( \{X_t\}_{t \geq 0} \) is a \textit{discrete-time regenerative phenomenon}\(^2\) if the following property holds for any \( n \geq 1 \) instants \( 0 < \tau_1 < \cdots < \tau_n \):

\[
P[X_{\tau_1} = \cdots = X_{\tau_n} = 0] = \prod_{l=1}^{n} P[X_{\tau_l - \tau_{l-1}} = 0].
\]  

(5)

As a matter of fact, this property holds. Indeed, some renewals (not necessarily consecutive) occur at the \( n \) instants \( 0 < \tau_1 < \cdots < \tau_n \) if and only if positive integers \( k_1, \ldots, k_n \) exist so that \( T_{k_1 + \cdots + k_l} = \tau_l \) for every \( l \leq n \). Setting \( k_0 := 0 \) and \( \tau_0 := 0 \), we can equivalently state that renewals occur at the instants \( \tau_1, \ldots, \tau_n \).

\(^2\) Actually, what is a discrete time regenerative phenomenon according to Kingman is the binary process \( \{1 - X_t\}_{t \geq 0} \) that associates the value one instead of zero to renewals but the present setting is more convenient for connections to statistical mechanics.
provided that $\sum_{i=1}^{k_l} S_{k_0+\cdots+k_{l-1}+i} = T_{k_0+\cdots+k_l} - T_{k_0+\cdots+k_{l-1}} = \tau_l - \tau_{l-1}$ for all $l \leq n$. As the $S_i$'s are independent of the $S_j$'s when $i$ and $j$ run over values satisfying $k_0+\cdots+k_{l-1} < i \leq k_0+\cdots+k_l$ and $k_0+\cdots+k_{l-1} < j \leq k_0+\cdots+k_m$ with $l \neq m$, we get that

$$
P[X_{\tau_1} = \cdots = X_{\tau_n} = 0] = \sum_{k_1 \geq 1} \cdots \sum_{k_n \geq 1} P \left[ \sum_{i=1}^{k_l} S_{k_0+\cdots+k_{l-1}+i} = \tau_l - \tau_{l-1} \text{ for } l = 1, \ldots, n \right]
$$

$$
= \sum_{k_1 \geq 1} \cdots \sum_{k_n \geq 1} \prod_{l=1}^{n} P \left[ \sum_{i=1}^{k_l} S_i = \tau_l - \tau_{l-1} \right]
$$

$$
= \prod_{l=1}^{n} \sum_{k_l \geq 1} P \left[ \sum_{i=1}^{k_l} S_i = \tau_l - \tau_{l-1} \right].
$$

This identity is (5) because the case with $n = 1$ shows that $\sum_{k \geq 1} P[\sum_{i=1}^{k} S_i = \tau]$ is equal to $P[X = 0]$ for all $\tau \geq 1$.

The finite-dimensional marginals of the present discrete-time regenerative phenomenon $\{X_t\}_{t \geq 0}$ constitute a bridge towards some models of statistical mechanics and can be determined through the following argument. Fix a time $t \geq 1$ and binary numbers $0 = x_0, x_1, \ldots, x_t$ that contain $n + 1 := \sum_{i=0}^{t-1} (1 - x_i)$ zeros in certain positions, say $0 = t_0 < t_1 < \cdots < t_n \leq t$. The frequency $\#_{s|t}(x_0, \ldots, x_t)$ by which the distance between consecutive zeros is $s \leq t$ is equal to zero by convention if $n = 0$ and equal to $\sum_{s_i=1}^{t} \mathbb{1}(t_i - t_{i-1} = s)$ if $n > 0$. In both cases $\#_{s|t}(x_0, \ldots, x_t)$ can be written as

$$
\#_{s|t}(x_0, \ldots, x_t) := \sum_{\tau=1}^{t-s+1} (1 - x_{\tau-1}) \prod_{k=\tau}^{\tau+s-2} x_k (1 - x_{\tau+s-1}),
$$

where the intermediate factor is not present when $s = 1$. The distance between the position $t_n$ of the last zero and $t$ is $t - t_n = \sum_{\tau=1}^{t} x_k$. We want to obtain a formula for the probability $P[X_0 = x_0 \wedge \ldots \wedge X_t = x_t]$. To this aim, we observe that the condition $X_\tau = x_\tau$ for each $\tau \leq t$ is the same as the condition $T_\tau = t_\tau$ for every $i \leq n$ and $T_{n+1} > t$. In turn, according to whether $n = 0$ or $n > 0$, the latter is tantamount to $S_1 = T_1 > t - t_0$ or to $S_{n+1} = T_{n+1} - T_n > t - t_n$. It follows that $P[X_0 = x_0 \wedge \ldots \wedge X_t = x_t]$ is equal to $P[S_1 > t - t_0]$ if $n = 0$ and equal to $\prod_{s=1}^{n} p(t_i - t_{i-1}) P[S_1 > t - t_n] = \prod_{s=1}^{n} p(s) \sum_{s=1}^{t} \mathbb{1}(t_i - t_{i-1} = s) P[S_1 > t - t_n]$ when $n > 0$. Explicitly $\sum_{s=1}^{t} \mathbb{1}(t_i - t_{i-1} = s)$ and $t - t_n$ in terms of $x_0, \ldots, x_t$ as above, we find in both cases that

$$
P[X_0 = x_0 \wedge \ldots \wedge X_t = x_t] = (1 - x_0) \prod_{s=1}^{t} p(s)^{\#_{s|t}(x_0, \ldots, x_t)} P[S_1 > \sum_{\tau=1}^{t} x_{\tau+s-1}].
$$

This formula is valid for every integer $t \geq 1$ and binary numbers $x_0, \ldots, x_t$, even if $x_0 = 0$ because $X_0 = 0$, and expresses the finite-dimensional marginals of the process $\{X_t\}_{t \geq 0}$ with respect to the free renewal model. As far as the constrained renewal model is concerned, adding the condition $X_t = 0$ we get that

$$
P_t[X_0 = x_0 \wedge \ldots \wedge X_t = x_t] = \frac{(1 - x_0)(1 - x_t)}{P[X_t = 0]} \prod_{s=1}^{t} p(s)^{\#_{s|t}(x_0, \ldots, x_t)}. 
$$
The corresponding probability distribution for waiting times is found by noticing that \( n \) renewals occur by the time \( t \) and a renewal exactly occurs at the time \( t \) if and only if \( T_n = \sum_{i=1}^{n} S_i = t \). Thus, we have that for every positive integers \( n \) and \( s_1, \ldots, s_n \)

\[
P_t[S_1 = s_1 \land \ldots \land S_n = s_n \land N_t = n] = \frac{1}{P[X_t = 0]} \prod_{i=1}^{n} p(s_i). \tag{8}
\]

The probability distribution (7) is precisely the finite-volume Gibbs state associated with the homogeneous Wako-Saitō-Muñoz-Eaton model and the Tokar-Dreyssé model, whereas the probability distribution (8) is the finite-volume Gibbs state corresponding to the homogeneous model of polymer pinning and to the Poland-Scheraga model. In these cases, \( t \) is the number of microscopic degrees of freedom of the system and measures the volume rather than the time. The renewal equation allows us to compute the normalization factor \( P[X_t = 0] \), which is the partition function in the language of statistical mechanics. Below, we briefly review the Poland-Scheraga model, the Wako-Saitō-Muñoz-Eaton model, and the Tokar-Dreyssé model drawing the mapping with renewal processes. We refer to [4] for polymer pinning models, which are commonly introduced from the beginning in terms of renewal processes.

2.3.1 The Model by Poland and Scheraga for Melting of DNA

Most DNA molecules consist of two strands made up of nucleotide monomers. Monomers on one strand can be either bound or unbound to a specific matching monomer on the other strand. Thermal denaturation of double-stranded DNA is the process by which the two strands unbind upon heating. The Poland-Scheraga model for melting of DNA considers the DNA molecule as being composed of an alternating sequence of bound and denaturated segments that do not interact with one another [5, 6]. A bound segment of length \( l \geq 1 \) is favored by the energetic gain \( \epsilon l \), the binding energy \( \epsilon < 0 \) being taken to be the same for all matching monomers [6]. We suppose that \( \epsilon \) is measured in unit of \( k_B T \), where \( k_B \) is the Boltzmann constant and \( T \) is the temperature. A denaturated segment of length \( l \geq 1 \) is favored by an entropic gain \( \sigma l > 0 \) due to the added configurations arising from a loop of length \( 2l \) [6]. The construction of the Poland-Scheraga model with \( t \) monomers per strand assumes that there is a certain number, say \( n \), of consecutive stretches of positive lengths \( s_1, \ldots, s_n \) that span a chain of \( t \) monomers: \( n \leq t \) and \( \sum_{i=1}^{n} s_i = t \). Set \( t_0 := 0 \) and \( t_i := s_1 + \cdots + s_i \) for \( i \geq 1 \). The \( i^{th} \) stretch starts at position \( t_{i-1} \) and is imagined to consist of just one bound monomer if \( s_i = 1 \) and one bound monomer followed by one denaturated segment of length \( s_i - 1 \) if \( s_i > 1 \). We notice that the first monomer in position \( t_0 \) is, without loss of generality, always bound and that there is actually a bound segment of length \( l \geq 2 \) starting at \( t_{i-1} \) if \( s_i = \cdots = s_{i+l-2} = 1 \) and \( s_{i+l-1} > 1 \). With the convention that \( \sigma_0 := 0 \), the statistical weight of the Poland-Scheraga model reads

\[
\prod_{i=1}^{n} \exp \left( -\epsilon + \sigma_{s_i-1} - \eta s_i \right). \tag{9}
\]

The real number \( \eta \) can be any without altering the physical description of the system because of the constraint \( \sum_{i=1}^{n} s_i = t \).
The statistical weight of the configuration with only one large denatured segment is \( \exp(-\epsilon + \sigma_{s-1} - \eta t) \), showing that a necessary condition for the existence of a thermodynamic limit is that \( \eta_o := \limsup_{t \to \infty} \sigma_t/t < \infty \). If this is the case, then the Cauchy-Hadamard Theorem yields that \( \eta \geq \eta_o \) can be found so that \( \sum_{s \geq 1} \exp(-\epsilon + \sigma_{s-1} - \eta s) \leq 1 \). This way, setting \( p(s) := \exp(-\epsilon + \sigma_{s-1} - \eta s) \) for each \( s \geq 1 \), a renewal process with waiting time distribution \( p \) can be constructed.

For this renewal process, the probability \( P_t[S_1 = s_1 \land \ldots \land S_n = s_n \land N_t = n] \) is proportional to the statistical weight defined by (9) whenever \( \sum_{i=1}^n s_i = t \), as confirmed by (8). Thus, the Poland-Scheraga model can be regarded as a constrained renewal model. In this context, renewal times correspond to bound monomers. We point out that if \( \sum_{s \geq 1} \exp(-\epsilon + \sigma_{s-1} - \eta_s) \geq 1 \), then \( \eta \) can be chosen in such a way that \( \sum_{s \geq 1} \exp(-\epsilon + \sigma_{s-1} - \eta s) = 1 \) and the renewal process associated with \( p \) is recurrent. If on the contrary \( \sum_{s \geq 1} \exp(-\epsilon + \sigma_{s-1} - \eta_s) < 1 \), then any renewal process associated with Poland-Scheraga model is necessarily transient.

As far as the total reward \( W_t \) is concerned, we notice that \( W_t \) counts the number \( N_t \) of bound monomers per strand when \( R_t := 1 \) for each \( i \geq 1 \). If instead \( R_i := \sigma_{S_{i-1}} \) for every \( i \geq 1 \), then \( W_t \) measures the total loop entropy. The joint fluctuations of the number of bound monomers and the total loop entropy can be investigated by taking \( R_t := (1, \sigma_{S_{i-1}}) \) for any \( i \geq 1 \). These are few examples of deterministic rewards that give rise to total rewards corresponding to extensive thermodynamic variables.

2.3.2 The Model by Wako, Saitô, Muñoz, and Eaton for Protein Folding

Most proteins consist of a long chain of amino acid monomers held together by peptide bonds. Protein folding is the process by which a polypeptide chain folds into its functional shape from random coil. The model by Wako and Saitô [7,8] and Muñoz and Eaton [9,10,11] describes a protein made up of \( t + 1 \) monomers as a sequence of \( t \) peptide bonds. A configuration of the protein is identified by associating the \( i^{th} \) bond with a binary variable \( x_i \) taking value one if the bond is in its functional conformation and value zero otherwise. We assume that \( i \) runs from \( 0 \) to \( t-1 \) and that, without loss of generality, \( x_0 := 0 \). Bonds \( i \) and \( j > i \) are supposed to interact only if they are contained in a string of consecutive functional-like bonds and only if they are in contact in the global three-dimensional functional structure of the protein. For homogeneous systems like homopolymers [7], their interaction involves the energetic gain \( \epsilon_{j-i} \leq 0 \) that we express in unit of \( k_B T \), with \( \epsilon_{j-i} := 0 \) if the two bonds are not in contact in the protein architecture. The model also takes into account the entropic loss \( \sigma > 0 \) of fixing one peptide unit in the functional conformation. The statistical weight of the Wako-Saitô-Muñoz-Eaton model reads

\[
\exp \left[ - \sum_{i=0}^{t-2} \sum_{j=i+1}^{t-1} \epsilon_{j-i} \prod_{k=i}^{j} x_k + \sigma \sum_{i=0}^{t-1} (1 - x_i) + \eta t \right],
\]

where \( \eta \) is any real number. This statistical weight can be conveniently manipulated introducing the frequency \( \#_{i}(x_0,\ldots,x_{t-1},0) \) defined by (6). To this aim, we set \( u_0 := 0 \) and \( u_l := \sum_{s=1}^{l} (l-s) \epsilon_s \) for each \( l \geq 1 \). We notice that \( u_l \) is the energetic contribution of a stretch of \( l \) consecutive functional-like bonds. Then,
simple algebra in combination with the assumption that \(x_0 := 0\) shows that (10) can be rewritten in the form
\[
\exp \left[ \sum_{s=1}^{t} (\sigma - u_{s-1} + \eta s) \#_{s|t}(x_0, \ldots, x_{t-1}, 0) \right].
\] (11)

The statistical weight of the configuration with all the peptide bonds in their functional conformation is \(\exp(\sigma - u_{t-1} + \eta t)\), so that a necessary condition for the existence of a thermodynamic limit is that \(\eta_t := \liminf_{t \to \infty} u_t / t > -\infty\). If this is the case, then a number \(\eta \leq \eta_t\) exists with the property that \(\sum_{s \geq 1} p(s) \leq 1\) with \(p(s) := \exp(\sigma - u_{s-1} + \eta s)\) for each \(s \geq 1\). For the renewal process with waiting time distribution \(p\), the probability \(P_t[X_0 = x_0 \land \ldots \land X_t = x_t]\) in (7) is proportional to (11) when \(x_0 = 0\) and \(x_t = 0\). Thus, the Wako-Saitô-Muñoz-Eaton model can be mapped in a constrained renewal model where renewal times correspond to peptide bonds that do not assume their functional conformation, namely, to positions \(\tau\) where \(X_\tau = 0\). These bonds are \(N_t\) in number and mark the beginning of functional-like stretches, the \(i^{th}\) of which has length \(S_i - 1\) and provides the energetic contribution \(u_{S_i-1}\). The total energy is the total reward \(W_t\) associated with the deterministic rewards \(R_i := u_{S_i-1}\) for every \(i \geq 1\). The renewal process corresponding to the Wako-Saitô-Muñoz-Eaton model can be chosen to be recurrent if \(\sum_{s \geq 1} \exp(\sigma - u_{s-1} + \eta_t s) \geq 1\), otherwise it is necessarily transient.

### 2.3.3 The Model by Tokar and Dreyssé for Strained Epitaxy

Epitaxy is the growth process of a crystal film on a crystalline substrate used in nanotechnology and in semiconductor fabrication. In most cases where the film material is different from the substrate material, the strain of the crystal film to accommodate the lattice geometry of the substrate leads to the self-assembly of coherent nanostructures. The model by Tokar and Dreyssé [12,13,14] for strained epitaxy aims at describing the size distribution of these atomic structures assuming that atoms interact only when they belong to the same cluster. In the one-dimensional case, clusters of contiguous atoms are defined unambiguously by their size and the energetic gain \(u_l \leq 0\) completely characterizes the one made up of \(l \geq 1\) atoms [12,13,14]. As before, parameters are expressed in unit of \(k_B T\). If atoms are arranged on \(t\) lattice sites and if the occupation variable \(x_i\) is associated with site \(i\) running from 0 to \(t - 1\), then the model by Tokar and Dreyssé with chemical potential \(\mu\) is defined by the statistical weight
\[
\exp \left[ -\sum_{s=2}^{t} u_{s-1} \#_{s|t}(x_0, \ldots, x_{t-1}, 0) + \mu \sum_{i=0}^{t-1} x_i + (\eta - \mu) t \right],
\] (12)

where \(\eta\) is again any real number. The condition \(x_0 := 0\) is imposed here in such a way that \(\#_{s|t}(x_0, \ldots, x_{t-1}, 0)\) defined by (6) counts the number of clusters with \(s \geq 1\) atoms in the lattice configuration identified by \(x_0, \ldots, x_{t-1}\). With the convention that \(u_0 := 0\), this condition also entails that (12) is the same as
\[
\exp \left[ \sum_{s=1}^{t} (\mu - u_{s-1} + \eta s) \#_{s|t}(x_0, \ldots, x_{t-1}, 0) \right].
\] (13)
Large Deviations in Renewal Theory and Renewal Models of Statistical Mechanics

As for the Wako-Saitô-Muñoz-Eaton model, \( \eta_0 := \liminf_{t \to \infty} u_t / t > -\infty \) is a necessary condition for the existence of a thermodynamic limit and guarantees that a number \( \eta \) exists so that \( \sum_{s \geq 1} \exp(-\mu - u_{s-1} + \eta s) \leq 1 \). The renewal process with waiting time distribution \( p \) defined by \( p(s) := \exp(-\mu - u_{s-1} + \eta s) \) for each \( s \geq 1 \) yields that \( \sum_{s \geq 1} \exp(-\mu - u_{s-1} + \eta_0 s) \geq 1 \), otherwise it is transient.

3 Large Deviations in Constrained Renewal Models

This section addresses the proof of Theorem 1 as follows. In Sect. 3.1 we show the existence of a weak large deviations principle, discussing the possible extension to a full large deviations principle. The corresponding rate function is then related with the function \( \zeta \) in Sect. 3.2, where the generalized renewal equation formalism is introduced for explicit calculations. In order to not interrupt the flow of the presentation, we postpone the most technical proofs in Sect. 3.3. We recall that the support \( S := \{ s \geq 1 : p(s) > 0 \} \) of the waiting time distribution \( p \) is supposed to fulfill the property that for some \( n \geq 1 \) there are numbers \( s_1, \ldots, s_n \) in \( S \) such that \( \gcd(s_1, \ldots, s_n) = 1 \). This hypothesis guarantees the existence of an integer \( t_0 \) so that \( \prod_{l=1}^{t_0} \{ X_{s_l} \} = \{ s \geq 1 : p(s) > 0 \} \) for each \( t \geq t_0 \) as we have seen in Sect. 2.1. No other assumption on the distribution \( p \) is made and, in particular, the transient case \( \sum_{s \geq 1} p(s) < 1 \) is important for applications to statistical mechanics is allowed as well as the recurrent case \( \sum_{s \geq 1} p(s) = 1 \).

Our theory of large deviations takes advantage of a certain conditional independence of the increments of the renewal-reward process \( \{ W_t \}_{t \geq 0} \). This feature comes from the following factorization.

**Lemma 1** Let 0 :\( \tau_0 < \tau_1 < \cdots < \tau_n \) be any integer times and let \( A_1, \ldots, A_n \) be any Borel sets in \( \mathbb{R}^d \). Set \( \sigma_l := \tau_l - \tau_{l-1} \) for each \( l \). Then

\[
P \left[ W_{\tau_l} - W_{\tau_{l-1}} \in A_l \land X_{\tau_l} = 0 \text{ for } l = 1, \ldots, n \right] = \prod_{l=1}^{n} P \left[ W_{\sigma_l} \in A_l \land X_{\sigma_l} = 0 \right].
\]

If \( \prod_{l=1}^{n} P \left[ X_{\tau_l} = \cdots = X_{\tau_n} = 0 \right] > 0 \), then Lemma 1 states that the increments \( W_{\tau_1} - W_{\tau_0}, \ldots, W_{\tau_n} - W_{\tau_{n-1}} \) of \( \{ W_t \}_{t \geq 0} \) are independent conditional on the event that some renewals (not necessarily consecutive) occur at the \( n \) instants \( \tau_1, \ldots, \tau_n \). In addition, Lemma 1 shows that for each \( l \leq n \) and measurable set \( A \subseteq \mathbb{R}^d \) we have that

\[
P \left[ W_{\tau_l} - W_{\tau_{l-1}} \in A \land X_{\tau_l} = \cdots = X_{\tau_n} = 0 \right] = P \left[ W_{\sigma_l} \in A \land X_{\sigma_l} = 0 \right].
\]

3.1 Weak and Full Large Deviations Principles

We leave normalization aside for the moment to focus on the measure \( \mu_t \) over the Borel \( \sigma \)-field \( \mathcal{B}(\mathbb{R}^d) \) defined for each time \( t \geq 1 \) and measurable set \( A \) by

\[
\mu_t(A) := P \left[ \frac{W_t}{t} \in A \land X_t = 0 \right].
\]
We notice that $\mu_t(\mathbb{R}^d) = \mathbb{P}[X_t = 0]$, so that $\mu_t(\mathbb{R}^d) > 0$ if $t \geq t_0$, and that $\mu_t(A)$ is indeed $\mathbb{P}_t[W_t/t \in A]$ a part from the normalization $\mathbb{P}[X_t = 0]$. When $C$ is a measurable convex set in $\mathbb{R}^d$, then the sequence over time of the $\mu_t(C)$'s satisfies the fundamental super-multiplicativity property

$$
\mu_{t+\tau}(C) \geq \mu_t(C) \cdot \mu_\tau(C)
$$

(14)

for all integers $\tau \geq 1$ and $t \geq 1$. This property is not fulfilled by the $\mathbb{P}_t[W_t/t \in C]$'s precisely because of the normalization. In order to verify (14), we observe that $W_{t+\tau}/(\tau+t) \in C$ whenever $W_t/\tau \in C$ and $(W_{t+\tau} - W_t)/t \in C$ since $C$ is convex and $W_{t+\tau}/(\tau+t) = AW_t/\tau + (1-\lambda)(W_{t+\tau} - W_t)/t$ with $\lambda := \tau/(\tau+t)$. It follows that

$$
\mu_{t+\tau}(C) \geq \mathbb{P} \left[ \frac{W_t}{\tau} \in C \land \frac{W_{t+\tau} - W_t}{t} \in C \land X_{\tau+t} = 0 \right] \\
\geq \mathbb{P} \left[ \frac{W_t}{\tau} \in C \land \frac{W_{t+\tau} - W_t}{t} \in C \land X_\tau = X_{\tau+t} = 0 \right],
$$

where the breakthrough conditioning on $X_\tau = 0$ has been added to obtain the last minorization. This conditioning allows the last probability to factorize as specified by Lemma 1 with the consequence that

$$
\mu_{t+\tau}(C) \geq \mathbb{P} \left[ \frac{W_t}{\tau} \in C \right] \cdot \mathbb{P} \left[ \frac{W_t}{\tau} \in C \right] = \mu_t(C) \cdot \mu_\tau(C).
$$

Super-multiplicativity, which becomes super-additivity once logarithms are taken, makes possible to describe in general terms the exponential decay with $t$ of the measure $\mu_t$. To this purpose, we denote by $\mathcal{L}(A)$ the extended real number associated with any $A \in \mathcal{B}(\mathbb{R}^d)$ by the formula

$$
\mathcal{L}(A) := \sup_{t \geq 1} \left\{ \frac{1}{t} \ln \mu_t(A) \right\}.
$$

If $C \in \mathcal{B}(\mathbb{R}^d)$ is convex, then (14) entails that $\ln \mu_{k\tau}(C) \geq k \ln \mu_\tau(C)$ for all integers $k \geq 1$ and $\tau \geq 1$. Consequently, $\limsup_{t \uparrow \infty} (1/t) \ln \mu_t(C) \geq (1/\tau) \ln \mu_\tau(C)$ for arbitrary $\tau$, giving $\limsup_{t \uparrow \infty} (1/t) \ln \mu_t(C) \geq \mathcal{L}(C)$. As the opposite inequality is obvious, we get that $\limsup_{t \uparrow \infty} (1/t) \ln \mu_t(C) = \mathcal{L}(C)$. The following lemma improves this result when $C$ is open as well as convex.

**Lemma 2** If $C$ is an open convex set in $\mathbb{R}^d$, then the limit $\lim_{t \uparrow \infty} (1/t) \ln \mu_t(C)$ exists as an extended real number and is equal to $\mathcal{L}(C)$.

With the aim of identifying a rate function for $\mu_t$, we introduce the function $J$ that maps any $w \in \mathbb{R}^d$ in the extended real number $J(w)$ defined by

$$
J(w) := - \inf_{\delta > 0} \left\{ \mathcal{L}(B_{w,\delta}) \right\},
$$

where $B_{w,\delta} := \{ v \in \mathbb{R}^d : \|v - w\|_2 < \delta \}$ is the Euclidean open ball of center $w$ and radius $\delta$. We point out that $J(w) \geq 0$ because $\mu_t(A) \leq \mathbb{P}[X_t = 0] \leq 1$ for all $t \geq 1$ and $A \in \mathcal{B}(\mathbb{R}^d)$. The first important properties of $J$ are presented in the following lemma.
Lemma 3 The function $J$ is lower semicontinuous and convex.

The function $J$ allows us to control the measure decay of open and compact sets as follows. Let $G$ be an open set in $\mathbb{R}^d$ and let $w$ be a given point in $G$. Since there exists $\delta > 0$ such that $B_{w, \delta} \subseteq G$ and since $B_{w, \delta}$ is open and convex, we get that  \[ \liminf_{t \to \infty} (1/t) \ln \mu_t(G) \geq \lim_{t \to \infty} (1/t) \ln \mu_t(B_{w, \delta}) = \mathcal{L}(B_{w, \delta}) \geq -J(w) \]  from Lemma 2. The arbitrariness of $w$ yields  \[ \liminf_{t \to \infty} (1/t) \ln \mu_t(G) \geq -\inf_{w \in G} \{J(w)\}. \]  Let now $K$ be a compact set in $\mathbb{R}^d$ and let $\lambda < \inf_{w \in K} \{J(w)\}$ be a real number. Since there exists $\eta > \lambda$ such that $\inf_{\delta > 0} \{\mathcal{L}(B_{w, \delta})\} = -J(w) \leq -\eta$ for all $w \in K$, a number $\delta_w > 0$ can be found for each $w \in K$ in such a way that $\mathcal{L}(B_{w, \delta_w}) \leq -\lambda$. Then, Lemma 2 gives that  \[ \lim_{t \to \infty} (1/t) \ln \mu_t(B_{w, \delta_w}) \leq -\lambda \]  for such $\delta_w$. Due to the compactness of $K$, there exist finitely many points $w_1, \ldots, w_n$ in $K$ such that  \[ K \subseteq \bigcup_{i=1}^n B_{w_i, \delta_{w_i}}. \]  It follows that  \[ \mu_t(K) \leq \sum_{i=1}^n \mu_t(B_{w_i, \delta_{w_i}}), \]  which in turn yields  \[ \limsup_{t \to \infty} (1/t) \ln \mu_t(K) \leq -\lambda. \]  This way, sending $\lambda$ to $\inf_{w \in K} \{J(w)\}$ we reach the bound  \[ \limsup_{t \to \infty} (1/t) \ln \mu_t(K) \leq -\inf_{w \in K} \{J(w)\}. \]  In conclusion, we have proven the following results.

Proposition 1 It holds that:

(a)  \[ \liminf_{t \to \infty} \frac{1}{t} \ln \mu_t(G) \geq -\inf_{w \in G} \{J(w)\} \quad \text{for each open set } G \subseteq \mathbb{R}^d; \]

(b)  \[ \limsup_{t \to \infty} \frac{1}{t} \ln \mu_t(K) \leq -\inf_{w \in K} \{J(w)\} \quad \text{for each compact set } K \subseteq \mathbb{R}^d. \]

These results can be strengthened for convex sets as follows.

Proposition 2 It holds that:

(a)  \[ \lim_{t \to \infty} \frac{1}{t} \ln \mu_t(G) = -\inf_{w \in G} \{J(w)\} \quad \text{for each open convex set } G \subseteq \mathbb{R}^d; \]

(b)  \[ \limsup_{t \to \infty} \frac{1}{t} \ln \mu_t(F) \leq -\inf_{w \in F} \{J(w)\} \quad \text{for each closed convex set } F \subseteq \mathbb{R}^d. \]

The measure $\mu_t$ is exponentially tight if for any real number $\lambda > 0$ there exists a compact set $K$ in $\mathbb{R}^d$ with the property that  \[ \limsup_{t \to \infty} (1/t) \ln \mu_t(\mathbb{R}^d \setminus K) \leq -\lambda. \]  The upper bound (b) in Propositions 1 and 2 can be extended to any closed set when $\mu_t$ is exponentially tight. Indeed, if $F$ is a closed set in $\mathbb{R}^d$ and $K$ is the compact set associated with $\lambda$ by exponential tightness, then the set $F \cap K$ is compact and Proposition 1 applies showing that  \[ \limsup_{t \to \infty} (1/t) \ln \mu_t(F \cap K) \leq -\inf_{w \in F \cap K} \{J(w)\}. \]  On the other hand, for each $t \geq 0$ we have that  \[ \mu_t(F) \leq \mu_t(F \cap K) + \mu_t(\mathbb{R}^d \setminus K). \]  Thus, taking logarithms, dividing by $t$, and sending $t$ to infinity, we get at the result  \[ \limsup_{t \to \infty} (1/t) \ln \mu_t(F) \leq -\min \{\inf_{w \in F} \{J(w)\}, \lambda\}. \]  This bound proves the following proposition once $\lambda$ is sent to infinity.

Proposition 3 If $\mu_t$ is exponentially tight, then for each closed set $F \subseteq \mathbb{R}^d$

\[ \limsup_{t \to \infty} \frac{1}{t} \ln \mu_t(F) \leq -\inf_{w \in F} \{J(w)\}. \]

3.2 The Rate Function

In this section we show that the rate function $J$ is the convex conjugate of the function $\zeta$ defined by (1) and then we verify Theorem 1 point by point. In order
to establish a connection between $J$ and $\zeta$, explicit calculations are needed and entrusted to the formalism of the generalized renewal equation that is introduced at first.

### 3.2.1 Expectations and Generalized Renewal Equation

Let $\{(S_t, V_t)\}_{t \geq 1}$ be a sequence of independent and identically distributed random vectors on $(\Omega, \mathcal{F}, \mathbb{P})$, the $V_t$'s taking values in $[0, \infty)$, and for each time $t \geq 1$ denote by $Z_t$ the expected value

$$Z_t := \mathbb{E} \left[ (1 - X_t) \prod_{i=1}^{N_t} V_i \right].$$

We notice that the conditions $t \geq 1$ and $X_t = 0$ imply that $N_t \geq 1$. Here, we determine the asymptotic exponential rate of growth of $Z_t$ with respect to $t$. The solution to this problem comes from the fact that the number $Z_t$ satisfies the generalized renewal equation

$$Z_t = \sum_{s=1}^{t} a_s Z_{t-s}$$

(16)

with initial condition $Z_0 := 1$, where $a_s := \mathbb{E}[V_1 \mathbb{I}(S_1 = s)]$ is a non-negative extended real number. This equation is deduced conditioning on the first renewal time $T_1 = S_1$ and then using the property that the renewal process starts over at every renewal time. We exclude the trivial situation where $Z_t = 0$ for all $t \geq 1$, so that an integer $s_0 \geq 1$ satisfying $a_{s_0} > 0$ must exist.

For each $z \in \mathbb{R}$, the expected value $A(z) := \mathbb{E}[V_1 \exp(-zS_1) \mathbb{I}(S_1 < \infty)]$ exists as an extended real number because the variable $V_1$ is non-negative and defines a lower semicontinuous function $A$ that maps $z$ in $A(z)$. Introducing the sequence $\{a_s\}_{s \geq 1}$, we can write that $A(z) = \sum_{s \geq 1} a_s \exp(-zs)$. The extended real number $\zeta_o$ given by

$$\zeta_o := \inf \left\{ z \in \mathbb{R} : A(z) \leq 1 \right\},$$

(17)

where the infimum over the empty set is customarily interpreted as $\infty$, is going to play a major role. The level set $\{z \in \mathbb{R} : A(z) \leq 1\}$ is bounded from below as $A(z) \geq a_{s_0} \exp(-zs_0) > 1$ for all $z$ sufficiently negative and closed due to lower semicontinuity. Consequently, $\zeta_o > -\infty$ and $A(\zeta_o) \leq 1$ if $\zeta_o < \infty$. These properties yield that $Z_t \leq \exp(\zeta_o t)$ for all $t \geq 1$. While this is trivial when $\zeta_o = \infty$, it follows by induction from (16) in the case $\zeta_o < \infty$. Indeed, if $\zeta_o < \infty$ and $Z_s \leq \exp(\zeta_o s)$ for $s = 0, \ldots, t-1$, which is certainly true when $s = 0$ since $Z_0 = 1$, then $Z_t \leq \exp(\zeta_o t) \sum_{s=1}^{t} a_s \exp(-\zeta_o s) \leq \exp(\zeta_o t) A(\zeta_o) \leq \exp(\zeta_o t)$. Under the assumption of aperiodicity, the number $\zeta_o$ exactly is the asymptotic exponential rate of growth of $Z_t$ with respect to $t$, as stated by the following proposition. The sequence $\{a_s\}_{s \geq 1}$ is said to be aperiodic if $n \geq 1$ positive integers $s_1, \ldots, s_n$ can be found in such a way that $a_{s_l} > 0$ for all $l \leq n$ and and gcd$\{s_1, \ldots, s_n\} = 1$.

**Proposition 4** The bound $Z_t \leq \exp(\zeta_o t)$ holds for all $t \geq 1$. If in addition the sequence $\{a_s\}_{s \geq 1}$ is aperiodic, then $\lim_{t \to \infty} (1/t) \ln Z_t$ exists and is equal to $\zeta_o$. 

For a first application of this result we consider the case where \( V_i := 1 \) for every \( i \geq 1 \). In this case, \( Z_t = \mathbb{P}[X_t = 0] \) for all \( t \geq 0 \) and \( a_s = p(s) \) for all \( s \geq 1 \). The sequence \( \{a_s\}_{s \geq 1} \) inherits aperiodicity from the waiting time distribution and a direct comparison with (1) shows that the number \( \zeta_o \) associated with \( V_1 \) by (17) is exactly \( \zeta(0) \). Then, Proposition 4 gives that \( \lim_{t \to \infty} (1/t) \ln \mathbb{P}[X_t = 0] = \zeta(0) \).

### 3.2.2 Connection with the Function \( \zeta \)

We finally prove that the rate function \( J \) is the convex conjugate of \( \zeta \), namely that \( J(w) = \sup \{ w \cdot k - \zeta(k) \} \) for all \( w \in \mathbb{R}^d \), and that the measure \( \mu_2 \) is exponentially tight whenever \( \zeta \) is finite in an open neighborhood of the origin. To show that \( J \) is the convex conjugate of \( \zeta \) we exploit Proposition 4 as follows. Pick a point \( k \in \mathbb{R}^d \) and set \( V_i := \exp(k \cdot R_i) \) for each \( i \geq 1 \). In this case, the number \( \zeta_o \) associated with \( V_i \) by (17) is exactly the number \( \zeta(k) \) defined by (1). Then, bearing in mind that \( W_t := \sum_{i=1}^{N_t} R_i \), we get from Proposition 4 that for all \( t \geq 1 \)

\[
\mathbb{E} \left[ (1-X_t) \exp(k \cdot W_t) \right] = \mathbb{E} \left[ (1-X_t) \prod_{i=1}^{N_t} V_i \right] \leq \exp \left[ \zeta(k) t \right]. \tag{18}
\]

At the same time, given a point \( w \in \mathbb{R}^d \) and a real number \( \delta > 0 \), the fact that \( k \cdot W_t \geq tk \cdot w - \|W_t - tw\|_2 \|k\|_2 \geq tk \cdot w - t\delta \|k\|_2 \) when \( W_t/t \in B_{w,\delta} \) entails that for every \( t \geq 1 \)

\[
\mathbb{E} \left[ (1-X_t) \exp(k \cdot W_t) \right] \geq \mathbb{E} \left[ \exp(k \cdot W_t) 1 \left( \frac{W_t}{t} \in B_{w,\delta} \land X_t = 0 \right) \right] \\
\geq \exp \left( tk \cdot w - t\delta \|k\|_2 \right) \mathbb{P} \left[ \frac{W_t}{t} \in B_{w,\delta} \land X_t = 0 \right] \\
= \exp \left( tk \cdot w - t\delta \|k\|_2 \right) \mu(B_{w,\delta}). \tag{19}
\]

This way, combining (18) with (19), taking logarithms, dividing by \( t \), and sending \( t \) to infinity, we get that \( \zeta(k) \geq k \cdot w - \delta \|k\|_2 + \mathcal{L}(B_{w,\delta}) \geq k \cdot w - J(w) + \delta \|k\|_2 \) thanks to Lemma 2. Thus, sending \( \delta \) to zero first and appealing to the arbitrariness of \( w \) later we reach the bound \( \zeta(k) \geq \sup_{w \in \mathbb{R}^d} \{ k \cdot w - J(w) \} \). Actually, a more sophisticated use of Proposition 4 leads to equality, as stated by the following proposition.

**Proposition 5** \( \zeta(k) = \sup_{w \in \mathbb{R}^d} \{ k \cdot w - J(w) \} \) for all \( k \in \mathbb{R}^d \).

Proposition 5 shows that \( \zeta \) is the convex conjugate of \( J \), which is the opposite of what we wanted. However, as the function \( J \) is lower semicontinuous and convex, the relationship obtained between \( \zeta \) and \( J \) can be inverted (see [57], Theorem 12.2), thus proving that \( J(w) = \sup_{k \in \mathbb{R}^d} \{ w \cdot k - \zeta(k) \} \) for all \( w \in \mathbb{R}^d \). Interestingly, Proposition 5 entails that \( \zeta \) is lower semicontinuous and convex as any convex conjugate (see [57], Theorem 12.2). The function \( \zeta \) is proper convex thanks to the fact that \( \zeta(k) > -\infty \) for each \( k \in \mathbb{R}^d \) and, in turn, \( J \) results proper convex (see [57], Theorem 12.2). In order to show that \( \zeta(k) > -\infty \) for any fixed point \( k \), we notice
that there exists a number \( r \) large enough such that \( \mathbb{P}(\|R_1\|_2 \leq r \land S_1 = s) > 0 \) for at least one \( s \) in the support \( S \) of \( p \). It follows that for every sufficiently negative \( z \)

\[
\mathbb{E} \left[ \exp \left( k \cdot R_1 - zS_1 \right) 1(S_1 < \infty) \right] \geq \mathbb{E} \left[ \exp \left( k \cdot R_1 - zS_1 \right) 1(\|R_1\|_2 \leq r \land S_1 < \infty) \right] \\
\geq \sum_{s \in S} \exp \left( -r\|k\|_2 - s \right) \mathbb{P}(\|R_1\|_2 \leq r \land S_1 = s) \\
> 1.
\]

This yields that \( \zeta(k) > -\infty \) according to definition (1).

Part (a) of Theorem 1 is proven observing that \( I = J + \zeta(0) \), so that the function \( I \) inherits the property to be lower semicontinuous and proper convex from \( J \). It follows in particular that the level set \( \{w \in \mathbb{R}^d : I(w) \leq \lambda \} \) is closed for any fixed number \( \lambda \). On the other hand, if \( \zeta \) is finite in an open neighborhood of the origin, then there exist constants \( \delta > 0 \) and \( M < \infty \) such that \( \zeta(k) \leq M \) for all \( k \) satisfying \( \|k\|_2 \leq \delta \). Consequently, if \( w \neq 0 \) is a vector such that \( I(w) \leq \lambda \) and if \( k \) denotes \( \delta w/\|w\|_2 \), then \( \delta \|w\|_2 = w \cdot k \leq w \cdot -\zeta(k) + M \leq J(w) + M \leq \lambda + M - \zeta(0) \). This shows that the set \( \{w \in \mathbb{R}^d : I(w) \leq \lambda \} \) is bounded, and hence compact, when \( \zeta \) is finite in an open neighborhood of the origin.

Part (b) of Theorem 1 and Part (c) of Theorem 1 for compact sets and closed convex sets are obtained as a corollary of Propositions 1 and 2 once normalization is restored because \( \ln \mathbb{P}[W_i/t \in A] = \mu(A) - \ln \mathbb{P}[X_i = 0] \) for each \( t \geq t_0 \) and \( A \in \mathcal{B}(\mathbb{R}^d) \) on the one hand and \( \lim_{t \to \infty} (1/t) \ln \mathbb{P}[X_i = 0] = \zeta(0) \) on the other hand. Part (c) for general closed sets becomes a consequence of Proposition 3 if we show that the finiteness of the function \( \zeta \) in an open neighborhood of the origin entails exponential tightness for the measure \( \mu_t \). This can be demonstrated by appealing again to the upper bound (18) as we now show. Suppose that \( \zeta \) is finite in an open neighborhood of the origin and denote by \( \{e_1, \ldots, e_d\} \) the canonical basis of \( \mathbb{R}^d \). Then, there exist two constants \( \delta > 0 \) and \( M < \infty \) with the property that both \( \zeta(-\delta e_i) \leq M \) and \( \zeta(\delta e_i) \leq M \) for each \( i \leq d \). Combining the Chernoff bound with (18) we get that for every point \( k \in \mathbb{R}^d \), number \( \rho > 0 \), and positive integer \( t \)

\[
\mathbb{P}[k \cdot W_t > \rho t \land X_t = 0] \leq \mathbb{E} \left[ (1 - X_t) \exp \left( \delta k \cdot W_t - \delta \rho t \right) \right] \\
\leq \exp \left[ \zeta(\delta k)t - \delta \rho t \right].
\]

(20)

This way, taking any real number \( \lambda > 0 \), choosing \( \rho > 0 \) in such a way that \( \delta \rho - M \geq \lambda \), and denoting by \( K \) the compact set \([-\rho, \rho]^d \), we find that

\[
\mu_t(\mathbb{R}^d \setminus K) = \mathbb{P}\left[ -e_i \cdot W_t > \rho t \lor e_i \cdot W_t > \rho t \text{ for some } i \land X_t = 0 \right] \\
\leq \sum_{i=1}^d \mathbb{P}\left[ -e_i \cdot W_t > \rho t \land X_t = 0 \right] + \sum_{i=1}^d \mathbb{P}\left[ e_i \cdot W_t > \rho t \land X_t = 0 \right] \\
\leq \sum_{i=1}^d \exp \left[ \zeta(-\delta e_i)t - \delta \rho t \right] + \sum_{i=1}^d \exp \left[ \zeta(\delta e_i)t - \delta \rho t \right] \\
\leq 2d \exp \left( Mt - \delta \rho t \right) \leq 2d \exp(-\lambda t).
\]

This bound proves that \( \mu_t \) is exponentially tight.
3.3 Proofs

3.3.1 Proof of Lemma 1

Some renewals occur at the \( n \) instants \( 0 < \tau_1 < \cdots < \tau_n \) if and only if positive integers \( k_1, \ldots, k_n \) can be found in such a way that \( T_{k_1} + \cdots + k_l = \tau_l \) for every \( l \leq n \). Setting \( k_0 := 0 \) and \( \tau_0 := 0 \), we can also say that renewals occur at \( \tau_1, \ldots, \tau_n \) if and only if \( \sum_{i=1}^{k_i} S_{k_0 + \cdots + k_{i-1} + i} = T_{k_0 + \cdots + k_i} - T_{k_0 + \cdots + k_{i-1}} = \tau_i - \tau_{i-1} = \sigma_l \) for all \( l \leq n \). We observe that the event \( T_n = t \) entails that \( N_t = n \) in general, so that we have that \( N_{\tau_1} = k_0 + \cdots + k_1 \) for every \( l \leq n \). This fact shows that \( W_{\tau_1} - W_{\tau_1} = \sum_{i=1}^{N_{\tau_1}} R_i - \sum_{i=1}^{N_{\tau_1}-1} R_i = \sum_{i=1}^{k_i} R_{k_0 + \cdots + k_{i-1} + i} \) for each \( l \leq n \). We also point out that the \( (S_i, R_i)'s \) are independent of the \( (S_j, R_j)'s \) when the indices \( i \) and \( j \) run over values satisfying \( k_0 + \cdots + k_{i-1} < i \leq k_0 + \cdots + k_l \) and \( k_0 + \cdots + k_{m-1} < j \leq k_0 + \cdots + k_m \) with \( l \neq m \). It follows that for any Borel sets \( A_1, \ldots, A_n \) in \( \mathbb{R}^d \)

\[
P[W_{\tau_1} - W_{\tau_1} \in A_l \cap X_{\tau_1} = 0 \text{ for } l = 1, \ldots, n]
\]

\[
\begin{align*}
&= \sum_{k_1 \geq 1} \cdots \sum_{k_n \geq 1} \prod_{i=1}^{k_i} \left[ R_{k_0 + \cdots + k_{i-1} + i} \in A_l \cap \sum_{i=1}^{k_i} S_{k_0 + \cdots + k_{i-1} + i} = \sigma_l \text{ for } l = 1, \ldots, n \right] \\
&= \sum_{k_1 \geq 1} \cdots \sum_{k_n \geq 1} \prod_{i=1}^{n} \left[ R_i \in A_l \cap \sum_{i=1}^{k_i} S_i = \sigma_l \right] \\
&= \prod_{i=1}^{n} \sum_{k_i \geq 1} \left[ R_i \in A_l \cap \sum_{i=1}^{k_i} S_i = \sigma_l \right].
\end{align*}
\]

This identity proves Lemma 1 because the case with \( n = 1 \) shows that the sum \( \sum_{i=1}^{k_i} R_i \in A \cap \sum_{i=1}^{k_i} S_i = \tau_l \) is equal to \( P[W_r \in A \cap X_r = 0] \) for any \( r \geq 1 \) and measurable set \( A \).

3.3.2 Proof of Lemma 2

Let \( C \) be an open convex set in \( \mathbb{R}^d \). The content of Lemma 2 is trivial when \( \mu_t(C) = 0 \) for all \( t \geq 1 \), as it is evident that \( \lim_{t \to \infty} (1/t) \ln \mu_t(C) \) exists and is equal to \( \mathcal{L}(C) = -\infty \) in such case. Then, we assume that \( \mu, \sigma(C) > 0 \) for some integer \( \tau_0 \). We shall prove later that the hypothesis that \( C \) is open entails in this case that \( \tau \geq \tau_0 \) exists with the property that \( \mu_t(C) > 0 \) for all \( t \geq \tau \). Now we use this fact to show that for every \( s \geq 1 \)

\[
\liminf_{t \to \infty} \frac{1}{t} \ln \mu_t(C) \geq \frac{1}{s} \ln \mu_s(C).
\]  

(21)

This bound implies that \( \liminf_{t \to \infty} (1/t) \ln \mu_t(C) \geq \mathcal{L}(C) \), with the consequence that the limit \( \lim_{t \to \infty} \ln \mu_t(C) \) exists and is equal to \( \mathcal{L}(C) \) as we already know that \( \limsup_{t \to \infty} (1/t) \ln \mu_t(C) = \mathcal{L}(C) \). In order to verify (21), pick \( s \geq \tau \) at first. Then, there exists a constant \( M > -\infty \) such that \( \ln \mu_t(C) \geq M \) for each \( r \) satisfying \( s \leq r < 2s \). Writing a certain integer \( t \geq 2s \) as \( t = q_s + r \) with integers \( q \geq 1 \) and \( r \in [s, 2s) \), we find that \( \ln \mu_t(C) \geq q \ln \mu_s(C) \).
thanks to (14). Thus, dividing by $t$ and sending $t$ to infinity, we get (21) under the restriction $s \geq \tau$. To conclude, pick any $s \geq 1$ and take an integer $k$ sufficiently large so that $ks \geq \tau$. Then, the bound (21) applies with $ks$ in place of $s$ and gives\[ \liminf_{t \to \infty} \left(\frac{1}{t}\right) \ln \mu_t(C) \geq (1/ks) \ln \mu_{ks}(C) \geq (1/s) \ln \mu_s(C), \]

where (14) has been exploited again to obtain the last inequality.

It remains to show that if $\mu_{\tau}(C) > 0$ for some $\tau > 0$, then there exists $\tau \geq \tau_0$ such that $\mu_t(C) > 0$ for all $t \geq \tau$. As $\mu_{\tau}(C) \geq \mu_t(C) > 0$ for all integers $k \geq \tau$ thanks to (14), we can assume that $\tau_0 > t_0$ changing $\tau_0$ with one of its multiple if necessary. We recall that the integer $t_0 \geq 0$ satisfies $\mathbb{P}[X_t = 0] > 0$ for all $t \geq t_0$. To begin with, suppose that for every $w \in C$ there exists a number $\delta_w > 0$ such that $\mu_{\tau}(B_{w,\delta}) = 0$, $B_{w,\delta} := \{v \in \mathbb{R}^d : \|v - w\|_2 < \delta\}$ being the Euclidean open ball of center $w$ and radius $\delta$. Because $\mathbb{R}^d$ is separable, the open covering $\{B_{w,\delta_w} : w \in C\}$ of $C$ contains a countable subcollection covering $C$. This yields $\mu_{\tau}(C) = 0$, contradicting the assumption that $\mu_{\tau}(C) > 0$. Consequently, there exists a point $w_0 \in C$ such that $\mu_{\tau}(B_{w_0,\delta}) > 0$ for all $\delta > 0$. In particular, we have that $\mu_{\tau}(B_{w_0,\delta}) > 0$ with $\delta > 0$ chosen in such a way that $B_{w_0,2\delta} \subseteq C$. The number $\delta$ exists since $C$ is open. Setting $B_k := B_{w_0,k\delta}$ for shortness, we can write that $\mu_{\tau}(B_1) > 0$ and that $B_2 \subseteq C$.

Since $\lim_{t \to \infty} \mu_t(B_k) = \mu_k(\mathbb{R}^d) = \mathbb{P}[X_t = 0] > 0$ for all $t \geq t_0$ and $\tau_0 > t_0$, there exists an integer $k_0 \geq 1$ such that $\mu_{\tau}(B_{k_0}) > 0$ if $r$ satisfies $\tau_0 \leq r < 2\tau_0$. Set $\tau := 2k_0\tau_0$ and pick any $t \geq \tau$. We are going to show that $\mu_t(C) > 0$. Because $t \geq \tau$ and $t \geq 2\tau_0$, we can write $t = q\tau_0 + r$ with integers $q \geq 1$ and $r \in [\tau_0, 2\tau_0)$. This way, if $\|W_t - q\tau_0w_0\|_2 < \delta_0q\tau_0$ and $\|W_t - q\tau_0w_0\|_2 < \delta_0q\tau_0 + k_0\delta_0r$, then the facts that $q\tau_0 < t$ and $r < 2\tau_0$ give that
\[
\|W_t - tw_0\|_2 \leq \|W_{q\tau_0} - q\tau_0w_0\|_2 + \|W_t - W_{q\tau_0} - rw_0\|_2
\leq \delta_0q\tau_0 + k_0\delta_0r < \delta_0t + 2k_0\delta_0\tau_0 = \delta_0t + \delta_0\tau \leq 2\delta_0t.
\]

In other words, this means that if $W_{q\tau_0}/(q\tau_0) \in B_1$ and $(W_t - W_{q\tau_0})/r \in B_{k_0}$, then $W_t/t \in B_2 \subseteq C$. It follows that
\[
\mu_t(C) \geq \mathbb{P}\left[ \frac{W_t}{t} \in B_2 \land X_t = 0 \right] \geq \mathbb{P}\left[ \frac{W_{q\tau_0}}{q\tau_0} \in B_1 \land \frac{W_t - W_{q\tau_0}}{r} \in B_{k_0} \land X_t = 0 \right]
\geq \mathbb{P}\left[ \frac{W_{q\tau_0}}{q\tau_0} \in B_1 \land \frac{W_t - W_{q\tau_0}}{r} \in B_{k_0} \land X_{q\tau_0} = X_t = 0 \right].
\]

The last probability factorizes as stated by Lemma 1, which allows us to obtain the result
\[
\mu_t(C) \geq \mathbb{P}\left[ \frac{W_{q\tau_0}}{q\tau_0} \in B_1 \land X_{q\tau_0} = 0 \right] \cdot \mathbb{P}\left[ \frac{W_t}{r} \in B_{k_0} \land X_r = 0 \right] = \mu_{q\tau_0}(B_1) \cdot \mu_r(B_{k_0}).
\]

Finally, invoking (14) in combination with the fact that $B_1$ is a convex set we get that $\mu_t(C) \geq q^2 \mu_t(B_1) \cdot \mu_r(B_{k_0})$. Consequently, $\mu_t(C) > 0$ as both $\mu_{\tau_0}(B_1) > 0$ and $\mu_r(B_{k_0}) > 0$ by construction.
3.3.3 Proof of Lemma 3

Lower semicontinuity descends from the fact that for each given point \( w \in \mathbb{R}^d \) and real number \( \lambda < J(w) \) there exists \( \delta_0 > 0 \) so that \( J(v) \geq \lambda \) for all \( v \in B_{w,\delta_0} \). Indeed, a number \( \delta > 0 \) can be found in such a way that \( L(B_{w,\delta}) \leq -\lambda \) since \( \inf_{\delta > 0} \{ L(B_{w,\delta}) \} = -J(w) < -\lambda \). If \( v \in B_{w,\delta_0} \), then we have that \( B_{v,\delta} \subseteq B_{w,\delta_0} \) and \( -J(v) = \inf_{\delta > 0} \{ L(B_{v,\delta}) \} \leq L(B_{v,\delta_0}) \leq -\lambda \) follows.

As far as the convexity of \( J \) is concerned, due to the lower semicontinuity it is sufficient to verify that for each \( v \) and \( w \) in \( \mathbb{R}^d \)

\[
J\left(\frac{v + w}{2}\right) \leq \frac{J(v) + J(w)}{2}.
\]  

(22)

To this aim, we notice that for each real number \( \delta > 0 \) and integer \( t \geq 1 \) the conditions \( W_t / t \in B_{v,\delta} \) and \( (W_{2t} - W_t) / t \in B_{w,\delta} \) imply that \( W_{2t} / (2t) \in B_{(v+w)/2,\delta} \) as \( \|W_{2t} - (v + w) t\|_2 \leq \|W_t - vt\|_2 + \|W_{2t} - W_t - vt\|_2 \leq 2\delta t \). It follows that

\[
\mu_{2t}(B_{\frac{v+w}{2},\delta}) = \mathbb{P}\left[ \frac{W_{2t}}{2t} \in B_{\frac{v+w}{2},\delta} \land X_{2t} = 0 \right] \\
\geq \mathbb{P}\left[ \frac{W_t}{t} \in B_{v,\delta} \land \frac{W_{2t} - W_t}{t} \in B_{w,\delta} \land X_{2t} = 0 \right] \\
\geq \mathbb{P}\left[ \frac{W_t}{t} \in B_{v,\delta} \land \frac{W_{2t} - W_t}{t} \in B_{w,\delta} \land X_t = X_{2t} = 0 \right].
\]

The last probability factorizes as specified by Lemma 1, so that we have

\[
\mu_{2t}(B_{\frac{v+w}{2},\delta}) \geq \mathbb{P}\left[ \frac{W_t}{t} \in B_{v,\delta} \land X_t = 0 \right] \cdot \mathbb{P}\left[ \frac{W_t}{t} \in B_{w,\delta} \land X_t = 0 \right] = \mu_{2t}(B_{v,\delta}) \cdot \mu_{2t}(B_{w,\delta}).
\]

This way, taking logarithms, dividing by \( 2t \), and sending \( t \) to infinity, we get that \( L(B_{(v+w)/2,\delta}) \geq (1/2)L(B_{v,\delta}) + (1/2)L(B_{w,\delta}) \geq -(1/2)J(v) - (1/2)J(w) \) thanks to Lemma 2 because open balls are open convex sets. Inequality (22) follows from here by the arbitrariness of \( \delta \).

3.3.4 Proof of Proposition 2

As we already know that \( \liminf_{t \to \infty} (1/t) \ln \mu_t(G) \geq - \inf_{w \in \mathcal{B}} \{ J(w) \} \) for each open set \( G \) and that \( \limsup_{t \to \infty} (1/t) \ln \mu_t(C) = L(C) \) for every \( C \in \mathcal{B}(\mathbb{R}^d) \) convex, it suffices to show that

\[
L(C) \leq - \inf_{w \in \mathcal{B}} \{ J(w) \}
\]

(23)

when \( C \) is either a given open convex set or a given closed convex set. This is trivial if \( L(C) = -\infty \), so that we assume that \( L(C) > -\infty \).

Pick \( \epsilon > 0 \). Since \( \limsup_{t \to \infty} (1/t) \ln \mu_t(C) = L(C) \), there exists an integer \( \tau \geq 1 \) such that \( L(C) \leq (1/\tau) \ln \mu_\tau(C) + \epsilon \). As \( \mu_\tau(A) \leq \mathbb{P}[X_t = 0] \leq 1 \) for all \( A \in \mathcal{B}(\mathbb{R}^d) \), the measure \( \mu_\tau \) is regular (see [58], Theorem 2.18) and, consequently, a compact set \( K_0 \subseteq C \) can be found so that \( \mu_\tau(C) \leq \mu_\tau(K_0) + [1 - \exp(-\epsilon)] \mu_\tau(C) \). Thus, \( \mu_\tau(C) \leq \exp(\epsilon) \mu_\tau(K_0) \) and \( L(C) \leq (1/\tau) \ln \mu_\tau(K_0) + 2\epsilon \) follows. Let \( K \) be the convex hull of \( K_0 \), which is the smallest convex set containing \( K_0 \).
The set $K$ inherits compactness from $K_o$ (see [57], Theorem 17.2) and satisfies $K_o \subseteq K \subseteq C$ since $C$ is convex. Then, using the fact that $K_o \subseteq K$ we reach the further bound $L(C) \leq (1/\tau) \ln \mu_k(K) + 2\varepsilon \leq L(K) + 2\varepsilon$. At this point, we notice that on the one hand $L(K) = \sup_{t \uparrow \infty} (1/t) \ln \mu_k(K)$ since $K$ is convex and on the other hand $\lim \sup_{t \uparrow \infty} (1/t) \ln \mu_k(K) \leq - \inf_{w \in K} \{J(w)\}$ since $K$ is compact. Thus, $L(C) \leq - \inf_{w \in K} \{J(w)\} + 2\varepsilon \leq - \inf_{w \in C} \{J(w)\} + 2\varepsilon$ because $K \subseteq C$ and (23) follows from the arbitrariness of $\varepsilon$.

3.3.5 Proof of Proposition 4

As $Z_t \leq \exp(\zeta_o t)$ for all $t \geq 1$, we have that $\lim \inf_{t \uparrow \infty} (1/t) \ln Z_t \leq \zeta_o$ and in order to prove the proposition it remains to show that under aperiodicity

$$
\lim \inf_{t \uparrow \infty} \frac{1}{t} \ln Z_t \geq \zeta_o.
$$

(24)

Suppose that $n \geq 1$ integers $s_1, \ldots, s_n$ can be found with the property that $a_{s_l} > 0$ for every $l \leq n$ and $\gcd\{s_1, \ldots, s_n\} = 1$. Then, there exists $\tau \geq 0$ such that each integer $t \geq \tau$ can be written in the form $t = \sum_{l=1}^n t_l s_l$ with non-negative integer coefficients $t_1, \ldots, t_n$ as discussed in Sect. 2.1. At the same time, the generalized renewal equation (16) shows that $Z_{t+s} \geq a_s Z_t$ for any $t \geq 0$ and $s \geq 1$. Iterating this bound, we get that $Z_{\sum_{l=1}^n t_l s_l} \geq \prod_{l=1}^n (a_{s_l})^{t_l} > 0$ for all non-negative integers $t_1, \ldots, t_n$. This shows that $Z_t > 0$ for every $t \geq \tau$. This property allows us to prove (24) as follows. Pick a real number $z < \zeta_o$ and notice that there exists an integer $\tau_o \geq 1$ so that $\sum_{l=1}^{\tau_o} a_s \exp(-zs) \geq 1$. On the contrary we would have that $A(z) \leq 1$, which contradicts the hypothesis that $z < \zeta_o$. Since $Z_t > 0$ for all $t \geq \tau$, we can find a constant $M > -\infty$ such that $\ln Z_t \geq M + zt$ for every $t$ satisfying $\tau \leq t < \tau + \tau_o$. As a matter of fact, this bound is valid for all $t \geq \tau$. Indeed, an argument by induction based on the generalized renewal equation (16) yields that if $t \geq \tau + \tau_o$ and $\ln Z_{t-s} \geq M + z(t-s)$ for $s = 1, \ldots, \tau_o$, then

$$
Z_t = \sum_{s=1}^t \sum_{s=1}^{\tau_o} a_s Z_{t-s} \geq \sum_{s=1}^\tau \sum_{s=1}^{\tau_o} a_s Z_{t-s} \geq \exp(M + zt) \sum_{s=1}^{\tau_o} a_s \exp(-zs) \geq \exp(M + zt).
$$

This way, we reach the result $\lim \inf_{t \uparrow \infty} (1/t) \ln Z_t \geq z$, which provides (24) once $z$ is sent to $\zeta_o$.

3.3.6 Proof of Proposition 5

Given a point $k \in \mathbb{R}^d$, in order to prove the proposition it is enough to show that $\zeta(k) \leq \sup_{w \in \mathbb{R}^d} \{k \cdot w - J(w)\}$. We split the proof in two steps. At first we verify that for each compact set $K$ in $\mathbb{R}^d$

$$
\lim \sup_{t \uparrow \infty} \mathbb{E} \left[ \exp(k \cdot W_t) \mathbb{I} \left( \frac{W_t}{t} \in K \cap X_t = 0 \right) \right] \leq \sup_{w \in K} \{k \cdot w - J(w)\} \quad (25)
$$

and

$$
\lim \sup_{t \uparrow \infty} \mathbb{E} \left[ \exp(k \cdot W_t) \mathbb{I} \left( \frac{W_t}{t} \in \mathbb{R}^d \setminus K \cap X_t = 0 \right) \right] \leq \sup_{w \in \mathbb{R}^d} \{k \cdot w - J(w)\}. \quad (26)
$$
Then, we demonstrate that for each real number $z < \zeta(k)$ there exists a compact set $K \subset \mathbb{R}^d$ with the property that

$$z \leq \lim_{t \uparrow \infty} \sup \mathbb{E} \left[ \exp(k \cdot W_t) 1 \left( \frac{W_t}{t} \in K \wedge X_t = 0 \right) \right].$$

The proposition follows combining (27) with (26) first and sending $z$ to $\zeta(k)$ later.

Pick a compact set $K$ in $\mathbb{R}^d$ and observe that $\sup_{w \in K} \{ k \cdot w - J(w) \} < \infty$ as $J$ is non-negative. To prove (25), let $\lambda > \sup_{w \in K} \{ k \cdot w - J(w) \}$ and $\epsilon > 0$ be real numbers. Since there exists $\eta < \lambda$ such that $\inf_{\delta > \eta} \{ L(B_{w, \delta}) \} = -J(w) \leq \eta - k \cdot w$ for all $w \in K$, for each $w \in K$ a number $\delta_w > 0$ can be found in such a way that $\delta_w \|k\|_2 < \epsilon$ and $L(B_{w, \delta_w}) \leq \lambda - k \cdot w$. For such $\delta_w$ Lemma 2 gives that $\lim_{t \uparrow \infty} (1/t) \ln \mu_t(B_{w, \delta_w}) \leq \lambda - k \cdot w$. Furthermore, if for a certain $t \geq 1$ we have that $W_t/t \in B_{w, \delta_w}$, then $k \cdot W_t - tk \cdot w \leq \|W_t - tw\|_2 \|k\|_2 \leq t\delta_w \|k\|_2 < t\epsilon$. From the compactness of $K$, there exist finitely many points $w_1, \ldots, w_n$ in $K$ so that $K \subseteq \bigcup_{i=1}^n B_{w_i, \delta_{w_i}}$. It follows that for all $t \geq 1$

$$\mathbb{E} \left[ \exp(k \cdot W_t) 1 \left( \frac{W_t}{t} \in K \wedge X_t = 0 \right) \right] \leq \sum_{i=1}^n \mathbb{E} \left[ \exp(k \cdot W_t) 1 \left( \frac{W_t}{t} \in B_{w_i, \delta_{w_i}} \wedge X_t = 0 \right) \right] \leq \sum_{i=1}^n \mu_t(B_{w_i, \delta_{w_i}}) \exp(tk \cdot w_i + t\epsilon).$$

Combining this bound with the fact that $\lim_{t \uparrow \infty} (1/t) \ln \mu_t(B_{w_1, \delta_{w_1}}) \leq \lambda - k \cdot w_1$ for each $l \leq n$, we find that

$$\lim_{t \uparrow \infty} \sup \mathbb{E} \left[ \exp(k \cdot W_t) 1 \left( \frac{W_t}{t} \in K \wedge X_t = 0 \right) \right] \leq \lambda + \epsilon.$$

This way, sending $\lambda$ to $\inf_{w \in K} \{ k \cdot w - J(w) \}$ and $\epsilon$ to 0 we reach (25).

We now verify (27). Recall that $n \geq 1$ numbers $s_1, \ldots, s_n$ with the property that $\gcd\{s_1, \ldots, s_n\} = 1$ can be found in the support $S$ of the waiting time distribution $p$. Pick a real number $z < \zeta(k)$ and observe that necessarily $\mathbb{E}[\exp(k \cdot R_1 - zS_1) 1(S_1 < \infty)] > 1$ by definition of $\zeta(k)$. Then, there exists a sufficiently large number $r$ such that both

$$\mathbb{E} \left[ \exp(k \cdot R_1 - zS_1) 1(\|R_1\|_2 \leq rS_1 < \infty) \right] \geq 1$$

and $\mathbb{P}(\|R_1\|_2 \leq rS_1 \wedge S_1 = s) \geq (1/2)\mathbb{P}(S_1 = s) = (1/2)p(s)$ for each $l \leq n$. The latter condition yields in particular that for any $l \leq n$

$$\mathbb{E} \left[ \exp(k \cdot R_1) 1(\|R_1\|_2 \leq rS_1 \wedge S_1 = s) \right] \geq \exp(-rs_1\|k\|_2) \mathbb{P}(\|R_1\|_2 \leq rS_1 \wedge S_1 = s) \geq \frac{\exp(-rs_1\|k\|_2)}{2} p(s).$$

This way, setting $V_l := \exp(k \cdot R_l) 1(\|R_l\|_2 \leq rS_l)$ for all $i \geq 1$ and introducing the real number $\zeta_o$ defined by

$$\zeta_o := \inf \left\{ y \in \mathbb{R} : \mathbb{E} \left[ V_1 \exp(-yS_1) 1(S_1 < \infty) \right] \leq 1 \right\},$$
we have that \( z \leq \zeta_0 \) from (28). At the same time, (29) shows that the sequence \( \{a_s\}_{s \geq 1} \) with \( a_s := \mathbb{E}[V_1 \mathbb{1}(S_1 = s)] \) for each \( s \geq 1 \) is aperiodic. At this point, applying Proposition 4 with the present \( V_i \)'s we realize that

\[
z \leq \zeta_0 = \lim_{t \uparrow \infty} -\frac{1}{t} \ln \mathbb{E} \left[ (1 - X_t) \prod_{i=1}^{N_t} V_i \right] = \lim_{t \uparrow \infty} -\frac{1}{t} \ln \mathbb{E} \left[ (1 - X_t) \exp(k \cdot W_t) \prod_{i=1}^{N_t} \mathbb{1}(|R_i|_2 \leq r S_i) \right].
\]

On the other hand, the conditions \( X_0 = 0 \) and \( |R_i|_2 \leq r S_i \) for \( i = 1, \ldots, N_t \) entail that \( \sum_{i=1}^{N_t} S_i = t \) and that \( |W_t|_2 \leq r \sum_{i=1}^{N_t} S_i = rt \), namely that \( W_t / t \) belongs to the compact set \( K := \{ w \in \mathbb{R}^d : \|w\|_2 \leq r \} \). It follows that

\[
z \leq \lim_{t \uparrow \infty} -\frac{1}{t} \ln \mathbb{E} \left[ (1 - X_t) \exp(k \cdot W_t) \prod_{i=1}^{N_t} \mathbb{1}(|R_i|_2 \leq r S_i) \right] \leq \limsup_{t \uparrow \infty} -\frac{1}{t} \ln \mathbb{E} \left[ \exp(k \cdot W_t) \mathbb{1} \left( \frac{W_t}{t} \in K \cap X_t = 0 \right) \right].
\]

\section{Large Deviations in Free Renewal Models}

In this section we prove Theorem 2. Large deviations principles in free renewals models can be made a consequence of the corresponding principles in constrained renewal models exploiting conditioning as follows. Pick an integer time \( t \geq 0 \) and a set \( A \in \mathcal{B}(\mathbb{R}^d) \). For each non-negative \( n \leq t \) the event \( N_t = n \) is tantamount to \( T_n \leq t \) and \( T_{n+1} > t \). Thus, bearing in mind that \( T_{n+1} = T_n + S_{n+1} \) and that \( S_{n+1} \) is independent of both \( T_n \) and \( R_1, \ldots, R_n \), we can write that

\[
\mathbb{P} \left[ \frac{W_t}{t} \in A \right] = \sum_{n=0}^{t} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{n} R_i \in A \cap T_n \leq t \wedge T_{n+1} > t \right] = \sum_{n=0}^{t} \sum_{\tau=0}^{t} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{n} R_i \in A \cap T_n = \tau \wedge S_{n+1} > t - \tau \right] = \sum_{\tau=0}^{t} \sum_{n=0}^{\tau} \mathbb{P} \left[ \frac{1}{t} \sum_{i=1}^{n} R_i \in A \cap T_n = \tau \right] \cdot \mathbb{P} \left[ S_1 > t - \tau \right].
\]

On the other hand, the event \( X_\tau = 0 \) takes place if and only if there exists \( n \leq \tau \) such that \( T_n = \tau \), and in such case \( N_\tau = n \). This fact results in the identity \( \mathbb{P}[W_t / t \in A \cap X_\tau = 0] = \sum_{n=0}^{\tau} \mathbb{P}[(1/t) \sum_{i=1}^{n} R_i \in A \cap T_n = \tau] \), giving

\[
\mathbb{P} \left[ \frac{W_t}{t} \in A \right] = \sum_{\tau=0}^{t} \mathbb{P} \left[ \frac{W_t}{t} \in A \cap X_\tau = 0 \right] \cdot \mathbb{P} \left[ S_1 > t - \tau \right]. \tag{30}
\]

This expression connects free renewal models with constrained renewal models and represents the starting point of the present theory.

In Sect. 4.1 we use (30) to derive Part (b) of Theorem 2 and in Sect. 4.1 we exploit (30) to get at Part (c). The supposed aperiodicity of the renewal process
makes possible to invoke the results of the previous section when needed. The function $I_\star$, defined by (3) is the convex conjugate of the proper convex function that maps $k \in \mathbb{R}^d$ in $\max\{\zeta(k), \xi_\star\}$. For this reason $I_\star$ is lower semicontinuous and proper convex (see [57], Theorem 12.2). Similarly to the constrained case, $I_\star$ has compact level sets if there exist constants $\delta > 0$ and $M < \infty$ such that $\max\{\zeta(k), \xi_\star\} \leq M$ for all $k$ that satisfy $\|k\|_2 \leq \delta$. This occurs only if $\zeta$ is finite in an open neighborhood of the origin. The same argument applies to $I^\star$ and, this way, Part (a) of Theorem 2 is proven.

4.1 The Lower Large Deviations Bound

Here we prove Part (b) of Theorem 2 for a given open set $G$ in $\mathbb{R}^d$. The proof is immediate when $\xi_\star = -\infty$. Indeed, keeping only the term corresponding to $\tau = t$ in the r.h.s. of (30), we get that $\mathbb{P}[W_t/t \in G] \geq \mu_t(G)$ for all $t \geq 1$. This way, Proposition 1 yields that

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq - \inf_{w \in G} \{ J(w) \}. \tag{32}$$

On the other hand, definition (3) directly shows that $I_\star = J$ when $\xi_\star = -\infty$, so that the proof of the lower large deviations bound is already concluded in this case.

The proof of Part (b) of Theorem 2 is much more laborious when $\xi_\star > -\infty$. Assume that $\xi_\star > -\infty$ from now on. In order to demonstrate the lower large deviations bound, it is sufficient to verify that for each point $w \in G$

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq -I_\star(w). \tag{31}$$

Pick a point $w \in G$. As $G$ is open, there exists a number $\delta > 0$ such that $B_{w, 3\delta} \subseteq G$. The first step to verify (31) is to prove that for each number $\beta \in (0, 1]$

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq - \inf_{v \in D_{w, \delta}} \sup_{k \in \mathbb{R}^d} \left\{ v \cdot k - \beta \zeta(k) - (1 - \beta)\xi_\star \right\}, \tag{32}$$

where $D_{w, \delta} := \{ v \in \mathbb{R}^d : \|v - w\|_2 \leq \delta \}$ is the Euclidean closed ball of center $w$ and radius $\delta$.

Keeping only the term corresponding to $\tau = t$ in the r.h.s. of (30), we get that $\mathbb{P}[W_t/t \in G] \geq \mathbb{P}[W_t/t \in B_{w, 3\delta}] \geq \mu_t(B_{w, 3\delta})$ for all $t \geq 1$. Then, Proposition 2 provides the bound

$$\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in G \right] \geq - \inf_{v \in B_{w, 3\delta}} \{ J(v) \} \geq - \inf_{v \in D_{w, \delta}} \sup_{k \in \mathbb{R}^d} \left\{ v \cdot k - \zeta(k) \right\}.$$

This bound is (32) when $\beta = 1$. Next, fix $\beta \in (0, 1)$ and let $t$ be an integer sufficiently large in such a way that $\|w\|_2 < \beta t$. Denoting by $\lfloor t \rfloor$ the greatest integer that is less than or equal to $\beta t$, we have that the event $W_{\lfloor t \rfloor}/\lfloor t \rfloor \in B_{w, 3\delta}/\beta$ implies $W_t/t \in B_{w, 3\delta}$. Indeed, as $0 \leq t - \lfloor t \rfloor / \beta < 1 / \beta$ and $\|w\|_2 < \beta t$, it holds
that \( ||W_{i_\ell} - tw||_2 \leq ||W_{i_\ell} - (i_\ell/\beta)w||_2 + (t - i_\ell/\beta)||w||_2 < ||W_{i_\ell} - (i_\ell/\beta)w||_2 + \delta t.\) It follows that if \( ||W_{i_\ell} - (i_\ell/\beta)w||_2 < 2\delta t/\beta \leq 2\delta t,\) then \( ||W_{i_\ell} - tw||_2 < 3\delta t.\) This way, keeping only the term corresponding to \( \tau = i_\ell \) in the r.h.s. of (30), we find that

\[
\mathbb{P}\left[ \frac{W_{\ell}}{t} \in G \right] \geq \mathbb{P}\left[ \frac{W_{\ell}}{t} \in B_{w,3\delta} \right] \geq \mathbb{P}\left[ \frac{W_{\ell}}{t} \in B_{w,3\delta} \wedge X_{i_\ell} = 0 \right] \cdot \mathbb{P}[S_{1} > t - i_\ell] \\
= \mu_{i_\ell} \left( B_{w,3\delta} \right) \cdot \mathbb{P}[S_{1} > t - i_\ell].
\]

Then, combining the fact that \( 1/t \leq \beta/i_\ell \) with \( \ln \mu_{i_\ell}(B_{w,3\delta/\beta}) \leq 0 \) and the fact that \( (t - i_\ell)/t \leq 1 + 1/t - \beta \) with \( \ln \mathbb{P}[S_{1} > t - i_\ell] \leq 0, \) we reach the result

\[
\frac{1}{t} \ln \mathbb{P}\left[ \frac{W_{\ell}}{t} \in G \right] \geq \frac{1}{t} \ln \mu_{i_\ell} \left( B_{w,3\delta/\beta} \right) + \frac{1}{t} \ln \mathbb{P}[S_{1} > t - i_\ell] \\
\geq \frac{\beta}{i_\ell} \ln \mu_{i_\ell} \left( B_{w,3\delta/\beta} \right) + \frac{1 + 1/t - \beta}{t - i_\ell} \ln \mathbb{P}[S_{1} > t - i_\ell].
\]

This result is valid for any \( t \) with the property that \( ||w||_2 < \beta \delta t.\) Sending \( t \) to infinity and observing that both \( \lim_{t \uparrow \infty} i_\ell = \infty \) and \( \lim_{t \uparrow \infty} (t - i_\ell) = \infty \) because \( 0 < \beta < 1, \) we obtain (32) when \( \beta \in (0, 1) \) thanks to Proposition 2 on the one hand and the limit \( \lim \inf_{t \uparrow \infty} (1/t) \ln \mathbb{P}[S_{1} > t] = \xi_{*} \) on the other hand:

\[
\lim \inf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_{\ell}}{t} \in G \right] \geq \frac{\beta}{i_\ell} \inf_{v \in B_{w,3\delta/\beta}} \left\{ J(v) \right\} + (1 - \beta)\xi_{*} \\
= - \beta \inf_{v \in B_{w,3\delta/\beta}} \sup_{k \in \mathbb{R}^{d}} \left\{ v \cdot k - \zeta(k) \right\} + (1 - \beta)\xi_{*} \\
= - \inf_{v \in B_{w,3\delta/\beta}} \sup_{k \in \mathbb{R}^{d}} \left\{ v \cdot k - \beta \zeta(k) - (1 - \beta)\xi_{*} \right\} \\
\geq - \inf_{v \in D_{w,\delta}} \sup_{k \in \mathbb{R}^{d}} \left\{ v \cdot k - \beta \zeta(k) - (1 - \beta)\xi_{*} \right\}.
\]

Now that the bound (32) has been demonstrated, we point out that the infimum and the supremum in the r.h.s. of (32) can be exchanged. Indeed, for any given \( \beta \in (0, 1], \) the function that maps \( (k, v) \in \mathbb{R}^{d} \times D_{w,\delta} \) to \( v \cdot k - \beta \zeta(k) \) is concave and upper semicontinuous with respect to \( k \) for each fixed \( v \in D_{w,\delta} \) and convex and continuous with respect to \( v \) for each fixed \( k \in \mathbb{R}^{d}. \) It is a proper saddle-function since \( \zeta \) is a proper convex function. Then, the compactness of \( D_{w,\delta} \) entails that (see [57], Corollary 37.3.1)

\[
\inf_{v \in D_{w,\delta}} \sup_{k \in \mathbb{R}^{d}} \left\{ v \cdot k - \beta \zeta(k) \right\} = \sup_{k \in \mathbb{R}^{d}} \inf_{v \in D_{w,\delta}} \left\{ v \cdot k - \beta \zeta(k) \right\} \\
= \sup_{k \in \mathbb{R}^{d}} \left\{ w \cdot k - \beta \zeta(k) - \delta \|k\|_2 \right\}.
\]

Consequently, (32) can be recast for each \( \beta \in (0, 1] \) in the form

\[
\lim \inf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_{\ell}}{t} \in G \right] \geq - \sup_{k \in \mathbb{R}^{d}} \left\{ w \cdot k - \beta \zeta(k) - (1 - \beta)\xi_{*} - \delta \|k\|_2 \right\}. \quad (33)
\]
The second step to verify (31) is to optimize the bound (32), or (33), with respect to \( \beta \). This way, we find that for each number \( \eta \in (0,1] \)

\[
\liminf_{t \uparrow \infty} \frac{1}{t} \ln P \left( \frac{W_t}{t} \in G \right) \geq \inf_{\beta \in [\eta,1]} \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k - \beta \zeta(k) - (1 - \beta) \xi_* - \delta \|k\|_2 \right\}.
\]

This bound can be conveniently manipulated similarly to what we did to go from (32) to (33). Take a number \( \eta \in (0,1] \) and recall that any norm is convex. The function that maps \((k, \beta) \in \mathbb{R}^d \times [\eta,1]\) in \( w \cdot k - \beta \zeta(k) - (1 - \beta) \xi_* - \delta \|k\|_2 \) is concave and upper semicontinuous with respect to \( k \) for each fixed \( \beta \in [\eta,1] \) and convex and continuous with respect to \( \beta \) for each fixed \( k \in \mathbb{R}^d \). Thus, the compactness of \([\eta,1]\) offers the possibility to exchange an infimum over \( \beta \) with a supremum over \( k \) (see [57], Corollary 37.3.1) to obtain that

\[
\liminf_{t \uparrow \infty} \frac{1}{t} \ln P \left( \frac{W_t}{t} \in G \right) \geq \sup_{k \in \mathbb{R}^d} \inf_{\beta \in [\eta,1]} \left\{ w \cdot k - \beta \zeta(k) - (1 - \beta) \xi_* - \delta \|k\|_2 \right\}.
\]

At this point, we observe that there exist two constants \( a \geq 0 \) and \( b \geq 0 \) with the property that \( \zeta(k) - \xi_* \geq -a \|k\|_2 - b \) for all \( k \in \mathbb{R}^d \) because \( \zeta \) is proper convex (see [57], Corollary 12.1.2). Then, for each \( \eta \in (0,1] \) sufficiently small to satisfy \( a\eta \leq \delta \) and every \( k \in \mathbb{R}^d \) we have that

\[
\max \left\{ \zeta(k), \eta \zeta(k) + (1 - \eta) \xi_* \right\} + \delta \|k\|_2 \geq \max \left\{ \zeta(k), \xi_* \right\} + \eta \|k\|_2 \geq \max \left\{ \zeta(k), \xi_* \right\} + \eta \left( a \|k\|_2 + b \right) + \delta \|k\|_2 \geq \max \left\{ \zeta(k), \xi_* \right\} - \eta b.
\]

Consequently, we realize that for every positive \( \eta \) small enough

\[
\liminf_{t \uparrow \infty} \frac{1}{t} \ln P \left( \frac{W_t}{t} \in G \right) \geq - \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k - \max \left\{ \zeta(k), \xi_* \right\} + \eta b \right\}
= - I_\star(w) - \eta b.
\]

We get (31) from here sending \( \eta \) to zero.

4.2 The Upper Large Deviations Bound

We conclude the proof of Theorem 2 verifying Part (c). We prove the upper large deviations bound for compact sets first. Then, we discuss exponential tightness to extend this bound to any closed set. Let \( K \) be a compact set in \( \mathbb{R}^d \) and let \( \lambda < \inf_{w \in K} \{ I_\star(w) \} \) and \( \epsilon > 0 \) be real numbers. As there exists \( \eta > \lambda \) such that \( I_\star(w) \geq \eta \) for all \( w \in K \), a point \( k_w \in \mathbb{R}^d \) can be found for each \( w \in K \) with the property that \( w \cdot \delta \cdot k_w - \max \{ \zeta(k_w), \xi_* \} \geq \lambda \). It is manifest that \( \zeta(k_w) < \infty \) for such \( k_w \). Let \( \delta_w > 0 \) be a number that satisfies \( \delta_w \|k_w\|_2 \leq \epsilon \). For every integers \( t \geq 1 \) and \( \tau \leq t \), the condition \( W_\tau/t \in B_w, \delta_w \) entails that \( \delta_w \cdot W_\tau/t \geq \delta \) as
\( k_w \cdot W_t - tk_w \cdot w \geq -\|W_t - tw\|_2 \|k_w\|_2 \). Thus, recalling the bound (18) we get that for each \( w \in K \), \( t \geq 1 \), and \( \tau \leq t \)
\[
\mathbb{P} \left[ \frac{W_t}{t} \in B_{w, \delta w} \land X_t = 0 \right] \leq \mathbb{E} \left[ (1 - X_t) \exp \left( k_w \cdot W_t - tk_w \cdot w + te \right) \right]
\leq \exp \left[ \zeta(k_w)\tau - tk_w \cdot w + te \right]
\leq \exp \left[ \tau \max \{ \zeta(k_w), \xi^* \} - tk_w \cdot w + te \right].
\]

(34)

Due to the compactness of \( K \), there exist finitely many points \( w_1, \ldots, w_n \) in \( K \) such that \( K \subseteq \bigcup_{i=1}^n B_{w_i, \delta w_i} \). Consequently, the identity (30) combined with (34) shows that
\[
\mathbb{P} \left[ \frac{W_t}{t} \in K \right] \leq \sum_{i=1}^n \mathbb{P} \left[ \frac{W_t}{t} \in B_{w_i, \delta w_i} \right]
= \sum_{i=1}^n \sum_{t=0}^{\infty} \mathbb{P} \left[ \frac{W_t}{t} \in B_{w_i, \delta w_i} \land X_t = 0 \right] \cdot \mathbb{P} \left[ S_1 > t - \tau \right]
\leq \sum_{i=1}^n \sum_{t=0}^{\infty} \exp \left[ \tau \max \{ \zeta(k_{w_i}), \xi^* \} - tk_{w_i} \cdot w_i + te \right] \cdot \mathbb{P} \left[ S_1 > t - \tau \right].
\]

At the same time, the facts that \( \limsup_{s \to \infty} (1/s) \ln \mathbb{P} [S_1 > s] = \xi^* \) and that both \( \max \{ \zeta(k_{w_i}), \xi^* \} \geq \xi^* \) and \( \max \{ \zeta(k_{w_i}), \xi^* \} > -\infty \) for all \( l \leq n \) ensure the existence of a constant \( M > 0 \) such that \( \mathbb{P} [S_1 > s] \leq M \exp \left[ s \max \{ \zeta(k_{w_i}), \xi^* \} + se \right] \) for each \( s \geq 0 \) and \( l \leq n \). It follows from the previous bound that for every \( t \geq 1 \)
\[
\mathbb{P} \left[ \frac{W_t}{t} \in K \right] \leq M \sum_{i=1}^n \sum_{t=0}^{\infty} \exp \left[ t \max \{ \zeta(k_{w_i}), \xi^* \} - tk_{w_i} \cdot w_t + (2t - \tau) \epsilon \right]
\leq Mn(t + 1) \exp \left( -t\lambda + 2t\epsilon \right),
\]

which in turn yields
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{W_t}{t} \in K \right] \leq -\lambda + 2\epsilon.
\]

From here, we obtain Part (c) of Theorem 2 for the compact set \( K \) sending \( \epsilon \) to zero and \( \lambda \) to \( \inf_{w \in K} \{ \Pi^t(w) \} \).

The upper large deviations bound can be extended from compact sets to closed sets with the same arguments that gave rise to Proposition 3 if the measure that associates any \( A \in \mathcal{B}(\mathbb{R}^d) \) with \( \mathbb{P} [W_t/t \in A] \) is exponentially tight. Exponential tightness means that for each real number \( \lambda > 0 \) there exists a compact set \( K \) in \( \mathbb{R}^d \) with the property that \( \limsup_{t \to \infty} (1/t) \ln \mathbb{P} [W_t/t \notin K] \leq -\lambda \). Assume that the function \( \zeta \) is finite in an open neighborhood of the origin and denote by \( \{ \epsilon_1, \ldots, \epsilon_d \} \) the canonical basis of \( \mathbb{R}^d \). Then, two constants \( \delta > 0 \) and \( M < \infty \) can be found so that both \( \max \{ \zeta(-\delta \epsilon_i), 0 \} \leq M \) and \( \max \{ \zeta(\delta \epsilon_i), 0 \} \leq M \) for any \( i \leq d \). The bound (20) shows that for every point \( k \in \mathbb{R}^d \), number \( \rho > 0 \), and integers \( t \geq 1 \) and \( \tau \leq t \)
\[
\mathbb{P} \left[ k \cdot W_t > \rho t \land X_t = 0 \right] \leq \exp \left[ \zeta(\delta k)\tau - \delta \rho t \right]
\leq \exp \left[ \max \{ \zeta(\delta k), 0 \} t - \delta \rho t \right].
\]
This way, (30) gives that for all $k \in \mathbb{R}^d$, $\rho > 0$, and $t \geq 1$

$$
\mathbb{P}[k \cdot W_t > \rho t] = \sum_{\tau=0}^{t} \mathbb{P}[k \cdot W_\tau > \rho t \land X_\tau = 0] \cdot \mathbb{P}[S_1 > t - \tau] \\
\leq (t + 1) \exp \left[ \max \{\zeta(\delta k), 0\} t - \delta \rho t \right].
$$

At this point, taking any real number $\lambda > 0$, choosing $\rho > 0$ so that $\delta \rho - M \geq \lambda$, and denoting by $K$ the compact set $[-\rho, \rho]^d$, we obtain the result

$$
\mathbb{P} \left[ \frac{W_s}{t} \in \mathbb{R}^d \setminus K \right] = \mathbb{P} \left[ -e_i \cdot W_t > \rho t \text{ or } e_i \cdot W_t > \rho t \text{ for some } i \right] \\
\leq \sum_{i=1}^{d} \mathbb{P}[-e_i \cdot W_t > \rho t] + \sum_{i=1}^{d} \mathbb{P}[e_i \cdot W_t > \rho t] \\
\leq (t + 1) \sum_{i=1}^{d} \exp \left[ \max \{\zeta(\delta e_i), 0\} t - \delta \rho t \right] \\
+ (t + 1) \sum_{i=1}^{d} \exp \left[ \max \{\zeta(\delta e_i), 0\} t - \delta \rho t \right] \\
\leq 2d(t + 1) \exp (Mt - \delta \rho t) \leq 2d(t + 1) \exp(-\lambda t).
$$

It follows from here that the measure defined on $\mathcal{B}(\mathbb{R}^d)$ by $\mathbb{P}[W_t / t \in A]$ for each measurable set $A$ is exponentially tight.

#### 5 Deterministic Rewards in Constrained Renewal Models

In this section we consider constrained renewal models and apply our theory of large deviations to deterministic rewards $R_i =: f(S_i)$ for each $i \geq 1$ with $f$ a function from $\{1, 2, \ldots \} \cup \{\infty\}$ to $\mathbb{R}^d$. In order to obtain explicit results, we assume that $\lim_{s \downarrow \infty} f(s) / s =: \ell$ exists in $\mathbb{R}^d$. Most extensive variables in the context of statistical mechanics are total rewards associated with this type of deterministic rewards, such as the number of bound monomers within the Poland-Scheraga model, the total loop entropy within the same model if the limit $\lim_{s \downarrow \infty} \sigma_s / s$ exists finite, and the total energy within the Wako-Saitô-Muñoz-Eaton model and the Tokar-Dreyssé model whenever $\lim_{s \downarrow \infty} u_s / s$ exists and is finite.

Hereafter, we denote by $\text{dom} \zeta := \{k \in \mathbb{R}^d : \zeta(k) < \infty\}$ the effective domain of $\zeta$ and by $\partial \zeta(k) := \{g \in \mathbb{R}^d : g \text{ is a subgradient of } \zeta \text{ at } k\}$ the subdifferential of $\zeta$ at the point $k \in \mathbb{R}^d$. Similarly, $\text{dom} J$ and $\partial J(w)$ are the effective domain of $J$ and the subdifferential of $J$ at $w$. Given a set $A$ in $\mathbb{R}^d$, we denote by $\text{int} A$, the interior and closure of $A$, respectively, and by $\text{bd} A := \text{cl} A \setminus \text{int} A$ its boundary. The interior which results when $A$ is regarded as a subset of its affine hull is the relative interior, denoted by $\text{ri} A$. The set $\text{ri} A$ coincides with $\text{int} A$ when $A$ is full dimensional. We refer to [57] for detailed definitions. In Sect. 3.2.2 we have shown that the function $\zeta$ is proper convex and lower semicontinuous, as well as the function $J$. Lower semicontinuity has the nice consequence that the condition $k \in \partial J(w)$ is tantamount to $w \in \partial \zeta(k)$ for all $k$ and $w$ in $\mathbb{R}^d$ (see [57], Theorem 23.5). This way, since at least a subgradient $k$ of $J$ at $w$ exists if $w \in \text{ri}(\text{dom} J)$
(see [57], Theorem 23.4), we get that for each \( w \in \text{ri}(\text{dom } J) \) there exists a point \( k \) such that \( w \in \partial \zeta(k) \). Due to the fact that \( J(w) = w \cdot k - \zeta(k) \) whenever \( w \in \partial \zeta(k) \) (see [57], Theorem 23.5), it follows that \( J \) can be easily computed on \( \text{ri}(\text{dom } J) \) once the effective domain of \( J \) and the subdifferentials of \( \zeta \) have been identified. Importantly, the function \( J \) is completely determined by its values on \( \text{ri}(\text{dom } J) \) again by lower semicontinuity. Indeed, if \( w \in \text{cl}(\text{dom } J) \) and \( v \in \text{ri}(\text{dom } J) \), then \( \lambda w + (1 - \lambda)v \in \text{ri}(\text{dom } J) \) for every \( \lambda \) such that \( 0 \leq \lambda < 1 \) (see [57], Theorem 6.1) and \( J(w) = \lim_{\lambda \uparrow 1} J(\lambda w + (1 - \lambda)v) \) (see [57], Corollary 7.5.1). We have thus proven the following general result.

**Lemma 4** Let \( w \) be any vector in \( \text{cl}(\text{dom } J) \). The following conclusions hold:

(a) if \( w \in \text{ri}(\text{dom } J) \), then a point \( k \) in \( \mathbb{R}^d \) exists with the property that \( w \in \partial \zeta(k) \) and \( J(w) = w \cdot k - \zeta(k) \) for any \( k \) with such property;

(b) for every \( v \in \text{ri}(\text{dom } J) \) and \( \lambda \in [0, 1) \) the vector \( \lambda w + (1 - \lambda)v \) belongs to \( \text{ri}(\text{dom } J) \) and \( J(w) = \lim_{\lambda \uparrow 1} J(\lambda w + (1 - \lambda)v) \).

A remark is in order. The function \( \zeta \) is differentiable at a certain point \( k \in \text{dom } \zeta \) if and only if \( \partial \zeta(k) \) is a singleton (see [57], Theorem 25.1). On the other hand, we have seen above that \( J(w) = w \cdot k - \zeta(k) \) when \( w \in \partial \zeta(k) \). It follows that the graph of \( J \) exhibits affine stretches if \( \zeta \) is not differentiable at some point.

Motivated by Lemma 4, we devote Sect. 5.1 to characterize the effective domain of the rate function \( J \) and the subdifferentials of \( \zeta \) for deterministic rewards such that \( \lim_{s \uparrow \infty} f(s)/s =: \zeta \) exists. We recall that the rate functions \( I \) and \( J \) differ for the constant \( \zeta(0) \), so that studying \( J \) means studying \( I \). In Sect. 5.2 we give the conditions for a constrained renewal model to produce persistent fluctuations for scaled total rewards, thus introducing the notion of critical renewal models. The most technical proofs are postponed in Sect. 5.3 to not interrupt the flow of the presentation.

5.1 The Rate Function in Practice

The first step to compute the rate function \( J \) is to find its effective domain. The set \( \text{dom } J \) can be described in general terms under the assumption that \( R_i := f(S_i) \) for all \( i \geq 1 \), without the need to specify the features of the function \( f \). In such case, definition (1) explicitly reads for each \( k \in \mathbb{R}^d \)

\[
\zeta(k) := \inf \left\{ z \in \mathbb{R} : \sum_{s \geq 1} \exp \left[ k \cdot f(s) - zs \right] p(s) \leq 1 \right\} \quad (35)
\]

and we have that \( \sum_{s \geq 1} \exp[k \cdot f(s) - \zeta(k)s]p(s) \leq 1 \) if \( \zeta(k) < \infty \). Denoting again by \( S \) the support \( \{s \geq 1 : p(s) > 0\} \) of the waiting time distribution \( p \), in order to characterize \( \text{dom } J \) we focus for a moment on the vectors \( f(s)/s \) with \( s \in S \). The smallest convex set \( C \) containing such vectors is the convex hull of the set \( \{f(s)/s\}_{s \in S} \) and results in all the convex combinations of the elements of this set (see [57], Theorem 2.3). This way, \( w \) belongs to \( C \) if and only if there exist integers \( s_1, \ldots, s_n \) in \( S \) and positive real numbers \( \lambda_1, \ldots, \lambda_n \) such that \( w = \sum_{i=1}^n \lambda_i f(s_i)/s_i \) and \( \sum_{i=1}^n \lambda_i = 1 \). The interest in the set \( C \) stems from the fact that the effective domain of \( J \) differs very little from \( C \), as stated by the next lemma. In particular, the lemma shows that \( \text{cl}(\text{dom } J) = \text{cl}C \), so that
Lemma 5 Let $C$ be the convex hull of $\{f(s)/s\}_{s \in S}$. Then, $C \subseteq \text{dom } J \subseteq \text{cl } C$.

The next step towards $J$ is represented by the study of the function $\xi$ with particular attention to differential properties. In contrast to the problem of determining dom $J$, this task demands some assumption on the asymptotic behavior of $f$. The hypothesis that $\lim_{s \to \infty} f(s)/s =: \xi$ exists in $\mathbb{R}^d$ is made from now on. Set $\xi := \limsup_{s \to \infty} (1/s) \ln p(s) \leq \xi^*$. Invoking the Cauchy-Hadamard Theorem we find that the series $\sum_{s \geq 1} \exp[k \cdot f(s) - zs] p(s)$ either diverges or converges according to $z < k \cdot \ell + \xi$ or $z > k \cdot \ell + \xi$. This fact immediately results in the lower bound $\zeta(k) \geq k \cdot \ell + \xi$ valid for all $k \in \mathbb{R}^d$. The form of the function $\zeta$ crucially depends on the behavior of $\sum_{s \geq 1} \exp[k \cdot f(s) - zs] p(s)$ at $z = k \cdot \ell + \xi$, so that it is convenient to introduce the extended real number $\theta(k)$ defined by

$$\theta(k) := \sum_{s \geq 1} \exp[k \cdot f(s) - (k \cdot \ell + \xi)s] p(s).$$

Clearly, $\theta(k) = \infty$ for all $k$ if $\xi = -\infty$. The function $\theta$ that maps each $k \in \mathbb{R}^d$ in $\theta(k)$ is convex and lower semicontinuous because it is the sum of positive convex functions. As a consequence, the possible empty level set $\Theta := \{k \in \mathbb{R}^d : \theta(k) \leq 1\}$ is convex and closed. If $k \in \Theta$, then $\sum_{s \geq 1} \exp[k \cdot f(s) - zs] p(s) \leq \theta(k) \leq 1$ for all $z \geq k \cdot \ell + \xi$. This yields that $\zeta(k) \leq k \cdot \ell + \xi$ if $k \in \Theta$, which shows that $\zeta(k) = k \cdot \ell + \xi$ when $k \in \Theta$ once combined with the above general lower bound. If instead $k \notin \Theta$, then $\sum_{s \geq 1} \exp[k \cdot f(s) - zs] p(s)$ as a function of the variable $z > k \cdot \ell + \xi$ is finite, continuous, strictly decreasing to zero as $z$ goes to infinity, and satisfies $\lim_{z \to k \cdot \ell + \xi} \sum_{s \geq 1} \exp[k \cdot f(s) - zs] p(s) = \theta(k) > 1$. It follows that for each $k \notin \Theta$ there exists a unique real number $z_0 > k \cdot \ell + \xi$ solving the equation $\sum_{s \geq 1} \exp[k \cdot f(s) - z_0s] p(s) = 1$. The value of $\zeta$ at $k \notin \Theta$ is exactly such number $z_0$. In conclusion, we have that $\zeta(k) = k \cdot \ell + \xi$ if $\theta(k) \leq 1$, whereas $\zeta(k)$ is that unique number $z_0 > k \cdot \ell + \xi$ satisfying $\sum_{s \geq 1} \exp[k \cdot f(s) - z_0s] p(s) = 1$ if $\theta(k) > 1$. The function $\zeta$ is finite everywhere and hence continuous on the whole space due to convexity (see [57], Theorem 10.1).

The following lemma supplies a complete description of the differential properties of $\zeta$. We point out that when $k$ is a point such that $\theta(k) > 1$, then the fact that $\zeta(k) > k \cdot \ell + \xi$ entails that $\sum_{s \geq 1} s^\gamma \exp[k \cdot f(s) - \zeta(k)s] p(s) < \infty$ for all number $\gamma$. In particular, it follows from here that the vector $\sum_{s \geq 1} f(s) \exp[k \cdot f(s) - \zeta(k)s] p(s)$ exists if $\theta(k) > 1$ because $f(s)/s$ is bounded for large $s$ due to the hypothesis that the limit $\lim_{s \to \infty} f(s)/s$ exists.

Lemma 6 Let $k$ be any point in $\mathbb{R}^d$. The following conclusions hold:

(a) if $\theta(k) < 1$, then $\zeta$ is differentiable at $k$ and its gradient $\nabla \zeta(k)$ at $k$ equals $\ell$;
(b) if $\theta(k) = 1$, then $\partial \zeta(k) = \{\ell\}$ or

$$\partial \zeta(k) = \left\{ (1 - \alpha)\ell + \frac{\sum_{s \geq 1} f(s) \exp[k \cdot f(s) - \zeta(k)s] p(s)}{\sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s)} : \alpha \in [0, 1] \right\}$$

according to the series $\sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s)$ diverges or converges;
(c) if \( \theta(k) > 1 \), then \( \zeta \) is differentiable at \( k \) and
\[
\nabla \zeta(k) = \frac{\sum_{s \geq 1} f(s) \exp[k \cdot f(s) - \zeta(k)s]}{\sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s]} p(s).
\]

5.2 Critical Renewal Models

The last issue we discuss in the context of constrained renewal models is if the scaled total reward \( W_t/t \) converges in probability to some constant vector \( w_o \in \mathbb{R}^d \) as \( t \) is sent to infinity. The answer to this question will allow us to single out situations where the fluctuations of the scaled total reward corresponding to deterministic rewards are more persistent than expected. Convergence in probability to a constant vector holds for at least the class of constrained renewal models identified by the following proposition. Here rewards are any (not necessarily deterministic) random variable.

**Proposition 6** Assume that there exists a real number \( z_o \) satisfying the following conditions: \( \mathbb{E}[\exp(-z_o S_1) \mathbb{1}(S_1 < \infty)] = 1 \), \( \mathbb{E}[S_1 \exp(-z_o S_1) \mathbb{1}(S_1 < \infty)] < \infty \), and \( \mathbb{E}[\|R_1\|^2 \exp(-z_o S_1) \mathbb{1}(S_1 < \infty)] < \infty \). Then, \( \lim_{t \uparrow \infty} \mathbb{P}(\|W_t/t - w_o\|_2 \geq \delta) = 0 \) for any \( \delta > 0 \) with
\[
w_o := \frac{\mathbb{E}[R_1 \exp(-z_o S_1) \mathbb{1}(S_1 < \infty)]}{\mathbb{E}[S_1 \exp(-z_o S_1) \mathbb{1}(S_1 < \infty)]}.
\]

The convergence in probability is expected to be the stronger exponential convergence in normal situations. According to Ellis [32], we say that \( W_t/t \) converges exponentially to a constant vector \( w_o \) if for any \( \delta > 0 \) there exists a number \( \lambda > 0 \) such that for all sufficiently large \( t \)
\[
\mathbb{P}(\|W_t/t - w_o\|_2 \geq \delta) \leq \exp(-\lambda t).
\] (36)

The quest for exponential convergence leads us to consider the large deviations principle and, successively, to focus on the set \( Z := \{w \in \mathbb{R}^d : I(w) = 0\} \) of the zeros of the rate function \( I \). The set \( Z \) is nothing but \( \partial \zeta(0) \), which is nonempty when \( 0 \in \text{ri} \text{ (dom } \zeta) \) (see [57], Theorem 23.4). Indeed, if \( w \in Z \), then \( w \) is a subgradient of \( \zeta \) at the origin because \( w \cdot k - \zeta(k) + \zeta(0) \leq I(w) = 0 \) for all \( k \in \mathbb{R}^d \). Conversely, if \( w \in \partial \zeta(0) \), then \( J(w) = -\zeta(0) \) (see [57], Theorem 23.5) and \( I(w) = J(w) + \zeta(0) = 0 \) follows. We point out that the set \( Z \) contains a unique element \( w_o \) if and only if the function \( \zeta \) is differentiable at the origin and \( \nabla \zeta(0) = w_o \) (see [57], Theorem 25.1). Our interest in the set \( Z \) stems from the fact that if \( \zeta \) is finite in an open neighborhood of the origin, then \( Z \) is nonempty and \( W_t/t \) converges exponentially to a certain \( w_o \), if and only if \( Z \) is constituted by the only \( w_o \). To prove this fact, suppose that \( \zeta \) is finite in an open neighborhood of the origin, so that the rate function \( I \) has compact level sets, and assume first that \( Z \) is made up of the only \( w_o \). Given a number \( \delta > 0 \), the possible empty set \( K \) of those vectors \( w \) that satisfy \( \|w - w_o\|_2 \geq \delta \) and \( I(w) \leq 1 \) is compact. If \( K \) is empty, then \( I(w) \geq 1 - 2\lambda \) whenever \( \|w - w_o\|_2 \geq \delta \). If \( K \) is nonempty, then \( I \) attains a minimum over \( K \) due to compactness and lower semicontinuity, meaning that there exists \( v \in K \) such that \( I(w) \geq \min \{I(v), 1\} = 2\lambda \) whenever \( \|w - w_o\|_2 \geq \delta \).
In both cases, we have that \( \lambda > 0 \) since \( v \neq w_o \) when \( v \in K \). Finiteness of \( \zeta \) in an open neighborhood of the origin allows us to apply the large deviations upper bound to the closed set of vectors \( w \) such that \( \|w - w_o\|_2 \geq \delta \). This way, we get that
\[
\limsup_{t \to \infty} (1/t) \ln \mathbb{P}[^{\|W_t - w_o\|_2 \geq \delta}] \leq -2\lambda,
\]
which implies (36). Conversely, if (36) holds for a fixed \( \delta > 0 \) and the corresponding \( \lambda > 0 \), then the large deviations lower bound shows that
\[
-I(w) \leq \liminf_{t \to \infty} (1/t) \ln \mathbb{P}[^{\|W_t - w_o\|_2 > \delta}] \leq -\lambda
\]
whenever \( \|w - w_o\|_2 > \delta \). This entails that if \( w \in Z \), then \( \|w - w_o\| \leq \delta \) and the arbitrariness of \( \delta \) gives that \( w = w_o \). In conclusion, we have proven the following result.

**Proposition 7** Assume that \( \zeta \) is finite in an open neighborhood of the origin. Then, \( W_t / t \) converges exponentially to a constant vector \( w_o \) if and only if \( I(w) = 0 \) only for \( w = w_o \).

When the function \( \zeta \) is finite in an open neighborhood of the origin but the set \( Z \) is not a singleton, then the scaled total reward exhibits a complex behavior where either convergence in probability to a constant vector is slower than exponential or it is not possible at all. For the deterministic rewards \( R_i = f(S_i) \) for each \( i \geq 1 \) with a function \( f \) satisfying the property that \( \lim_{s \to \infty} f(s) / s =: \ell \) exists, \( \zeta \) is finite everywhere and Lemma 6 allows us to identify the models where \( Z = \partial \zeta(0) \) is not a singleton. Precisely, a necessary condition for a certain waiting time distribution \( p \) to give rise to a non-singleton \( Z \) is that \( \theta(0) = 1 \), which explicitly reads \( \xi > -\infty \) and \( \sum_{s \geq 1} \exp(-\xi s) p(s) = 1 \). In addition to this, noticing that \( \zeta(0) = \xi \) when \( \theta(0) = 1 \), it is required that \( \sum_{s \geq 1} s \exp(-\xi s) p(s) < \infty \). Under these conditions on \( p \) the set \( Z \) results in
\[
Z = \left\{ (1 - \alpha)\ell + \alpha \frac{\sum_{s \geq 1} f(s) \exp(-\xi s) p(s)}{\sum_{s \geq 1} s \exp(-\xi s) p(s)} : \alpha \in [0, 1) \right\}. \tag{37}
\]
This set is not a singleton provided that peculiar rewards with the trait that \( \sum_{s \geq 1} f(s) \exp(-\xi s) p(s) \) is exactly equal to \( \ell \sum_{s \geq 1} s \exp(-\xi s) p(s) \) are not encountered. In conclusion, we find that most scaled total rewards associated with deterministic rewards that grow no faster than waiting times cannot converge exponentially to any constant vector if \( \xi > -\infty \), \( \sum_{s \geq 1} \exp(-\xi s) p(s) = 1 \), and \( \sum_{s \geq 1} s \exp(-\xi s) p(s) < \infty \). Referring to statistical mechanics, we are led to name critical a constrained renewal model corresponding to a waiting time distribution \( p \) that fulfills such conditions. It is important to stress that a critical renewal model is also a constrained renewal model where the hypotheses of Proposition 6 are verified with \( z_o := \xi \). Thus, since \( \sum_{s \geq 1} \|f(s)\|_2 \exp(-\xi s) p(s) < \infty \) when \( \sum_{s \geq 1} s \exp(-\xi s) p(s) < \infty \) and \( f(s) / s \) is bounded at large \( s \), we have that in critical renewal models the scaled total reward \( W_t / t \) converges in probability to \( w_o \) as \( t \) is sent to infinity with
\[
w_o := \frac{\sum_{s \geq 1} f(s) \exp(-\xi s) p(s)}{\sum_{s \geq 1} s \exp(-\xi s) p(s)}.
\]
The convergence in probability is necessarily slower than exponential and we can informally say that for deterministic rewards that grow no faster than waiting times in critical renewal models the fluctuations of the scaled total reward around a constant vector are more persistent than normal. The set \( Z \) where persistent
fluctuations concentrate results from (37) in the segment that connects \( w_o \) to \( \ell \). This segment can be called phase transition segment according to the literature on large deviations principles in statistical mechanics [35, 43].

5.3 Proofs

5.3.1 Proof of Lemma 5

Let \( \mathcal{H} \) be the closed convex set in \( \mathbb{R}^d \) defined by

\[
\mathcal{H} := \left\{ w \in \mathbb{R}^d : k \cdot w \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot \frac{f(s)}{s} \right\} \text{ for all } k \in \mathbb{R}^d \right\}.
\]

We prove in the order that \( \mathcal{C} \subseteq \text{dom } J \), that \( \text{cl } \mathcal{C} = \mathcal{H} \), and that \( \text{dom } J \subseteq \mathcal{H} \).

If \( w \in \mathcal{C} \), then there exist integers \( s_1, \ldots, s_n \) in the support \( \mathcal{S} \) of \( p \) and positive real numbers \( \lambda_1, \ldots, \lambda_n \) such that \( w = \sum_{l=1}^{n} \lambda_l f(s_l)/s_l \) and \( \sum_{l=1}^{n} \lambda_l = 1 \). Let \( M < \infty \) be a constant satisfying \( -\ln(p(s_l)) \leq M s_l \) for all \( l \leq n \). We have that \( k \cdot f(s_l)/s_l - \zeta(k) \leq M \) for all \( k \in \mathbb{R}^d \) and \( l \leq n \). Indeed, while this is obvious if \( \zeta(k) = \infty \), it is due to the fact that \( k \cdot f(s_l) - \zeta(k)s_l + \ln(p(s_l)) \leq 0 \) for all \( l \leq n \) when \( \zeta(k) < \infty \) because \( \sum_{s \geq 1} \exp[k \cdot f(s) - \zeta(k)s]p(s) \leq 1 \) in such case. It follows that

\[
k \cdot w - \zeta(k) = \sum_{l=1}^{n} \lambda_l \left[ k \cdot \frac{f(s_l)}{s_l} \right] - \zeta(k) = \sum_{l=1}^{n} \lambda_l \left[ k \cdot \frac{f(s_l)}{s_l} - \zeta(k) \right] \leq M
\]

for all \( k \in \mathbb{R}^d \) as \( \sum_{l=1}^{n} \lambda_l = 1 \) and \( \lambda_l > 0 \) for every \( l \). Consequently, we realize that \( J(w) \leq M < \infty \) and hence \( w \in \text{dom } J \). The arbitrariness of \( w \) shows that \( \mathcal{C} \subseteq \text{dom } J \).

If \( w = \sum_{l=1}^{n} \lambda_l f(s_l)/s_l \) is as above with \( \lambda_l > 0 \) for each \( l \) and \( \sum_{l=1}^{n} \lambda_l = 1 \), then we get for all \( k \in \mathbb{R}^d \) that

\[
k \cdot w = \sum_{l=1}^{n} \lambda_l \left[ k \cdot \frac{f(s_l)}{s_l} \right] \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot \frac{f(s)}{s} \right\}.
\]

This gives \( w \in \mathcal{H} \). Thus, \( \mathcal{C} \subseteq \mathcal{H} \) is deduced from the arbitrariness of \( w \) and \( \text{cl } \mathcal{C} \subseteq \mathcal{H} \) follows since \( \mathcal{H} \) is closed. In order to show that \( \text{cl } \mathcal{C} = \mathcal{H} \) it remains to prove that \( \mathcal{H} \subseteq \text{cl } \mathcal{C} \). By contradiction, if there exists \( v \in \mathcal{H} \) that is not contained in \( \text{cl } \mathcal{C} \), then we can find a point \( h \) and a number \( \epsilon > 0 \) such that \( h \cdot w + \epsilon \leq h \cdot v \) for all \( w \in \text{cl } \mathcal{C} \) (see [57], Corollary 11.4.2). In particular, as \( f(s)/s \in \text{cl } \mathcal{C} \) for all \( s \in \mathcal{S} \), we obtain that \( h \cdot f(s)/s + \epsilon \leq h \cdot v \) for each \( s \in \mathcal{S} \). This contradicts the fact that \( k \cdot v \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot f(s)/s \right\} \) for every \( k \in \mathbb{R}^d \).

To conclude, we prove that \( \text{dom } J \subseteq \mathcal{H} \). To begin with, we point out that the bound \( \zeta(k) \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot f(s)/s \right\} \) is valid for all \( k \in \mathbb{R}^d \). Indeed, if \( k \) is a given point in \( \mathbb{R}^d \) and \( z < \zeta(k) \) is a real number, then \( \sum_{s \geq 1} \exp[k \cdot f(s) - zs]p(s) > 1 \) by definition of \( \zeta(k) \). Consequently, an integer \( t \in \mathcal{S} \) exists with the property that \( k \cdot f(t) - zt > 0 \). This way, we find that \( z < k \cdot f(t)/t \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot f(s)/s \right\} \) and \( \zeta(k) \leq \sup_{s \in \mathcal{S}} \left\{ k \cdot f(s)/s \right\} \) follows by sending \( z \) to \( \zeta(k) \).
The fact that \( \zeta(k) \leq \sup_{s \in S} \{ k \cdot f(s) / s \} \) for each \( k \) entails that for all vectors \( w \) and \( k \) in \( \mathbb{R}^d \) and all real numbers \( \rho > 0 \)

\[
\rho k \cdot w \leq J(w) + \zeta(pk) \leq J(w) + \rho \sup_{s \in S} \left\{ k \cdot \frac{f(s)}{s} \right\}.
\]

Thus, if \( J(w) < \infty \), then dividing first by \( \rho \) and sending \( \rho \) to infinity later we obtain that \( k \cdot w \leq \sup_{s \in S} \{ k \cdot f(s) / s \} \) for any \( k \). This shows that \( w \in H \) whenever \( w \in \text{dom} \; J \).

5.3.2 Proof of Lemma 6

A practical way to determine subgradients of convex functions relies on directional derivatives. The one-sided directional derivative \( \zeta'(k; u) \) of \( \zeta \) at \( k \in \text{dom} \; \zeta = \mathbb{R}^d \) with respect to the vector \( u \in \mathbb{R}^d \) is defined by

\[
\zeta'(k; u) := \lim_{\epsilon \downarrow 0} \frac{\zeta(k + \epsilon u) - \zeta(k)}{\epsilon}.
\]

It exists as an extended real number and \( \zeta(k + \epsilon u) \geq \zeta(k) + \zeta'(k; u) \epsilon \) for all positive \( \epsilon \) because the difference quotient in this definition is a non-decreasing function of the parameter \( \epsilon > 0 \) (see [57], Theorem 23.1). The vector \( g \) is a subgradient of \( \zeta \) at \( k \) if and only if \( g \cdot u \leq \zeta'(k; u) \) for all \( u \in \mathbb{R}^d \) (see [57], Theorem 23.2). The function \( \zeta \) is differentiable at \( k \) if and only if a vector \( v \) (necessarily unique) exists so that \( \zeta'(k; u) = v \cdot u \) for every \( u \in \mathbb{R}^d \) (see [57], Theorem 25.2). If such a \( v \) exists, then the gradient \( \nabla \zeta(k) \) of \( \zeta \) at \( k \) is equal to \( v \).

With reference to Part (b) of the lemma, we examine what happens when a vector \( v \neq \ell \) exists so that \( \zeta'(k; u) = \max \{ \ell \cdot u, v \cdot u \} \) for all \( u \in \mathbb{R}^d \). This is manifest. Conversely, if \( g = (1 - \alpha)\ell + \alpha v \) with \( \alpha \in [0, 1] \) satisfies \( g \cdot u \leq \zeta'(k; u) \) for each \( u \), as it is.

Thus, \( g = (1 - \alpha)\ell + \alpha v \) and taking \( u = (v - \ell) \) first and \( u = v - \ell \) later in \( g \cdot u \leq \max \{ \ell \cdot u, v \cdot u \} \) we find that \( \alpha \geq 0 \) and that \( \alpha \leq 1 \), respectively. In conclusion, if \( \zeta'(k; u) = \max \{ \ell \cdot u, v \cdot u \} \) for any \( u \in \mathbb{R}^d \), then \( g \) is a subgradient of \( \zeta \) at \( k \) if and only if there exists \( \alpha \in [0, 1] \) such that \( g = (1 - \alpha)\ell + \alpha v \). This is true even in the case where \( v = \ell \), to which a function \( \zeta \) differentiable at \( k \) corresponds.

Pick \( k \) and \( u \) in \( \mathbb{R}^d \) and if \( \sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s) < \infty \), then set

\[
v := \frac{\sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s)}{\sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s)}.
\]

The above arguments show that to prove the lemma it suffices to check that:

(a) if \( \theta(k) < 1 \), then \( \zeta'(k; u) = \ell \cdot u \);

(b) if \( \theta(k) = 1 \), then \( \zeta'(k; u) = \ell \cdot u \) or \( \zeta'(k; u) = \max \{ \ell \cdot u, v \cdot u \} \) according to the

series \( \sum_{s \geq 1} s \exp[k \cdot f(s) - \zeta(k)s] p(s) \) diverges or converges;

(c) if \( \theta(k) > 1 \), then \( \zeta'(k; u) = v \cdot u \).
We shall do this addressing Part (c) first, then Part (b), and lastly Part (a). In order to simplify formulas, we set $c_ε(s) := \exp[(k + eu) \cdot f(s) - \zeta(k + eu)s]p(s)$ and $\Delta_ε(s) := \epsilon s \cdot f(s) - \zeta(k + eu)s + \zeta(k)s$ for each $s \geq 1$ and $\epsilon \geq 0$. We notice that $c_ε(s) = \exp[\Delta_ε(s)]c_0(s)$ for all $s$ and $\epsilon$, that $\sum_{s \geq 1} c_ε(s) \leq 1$ for every $\epsilon$ as $\zeta$ is always finite, and that $\sum_{s \geq 1} c_ε(s) = 1$ if $k + eu \notin \Theta$. In addition, $\theta(k) = \sum_{s \geq 1} c_0(s)$ if $k \in \Theta$ because $\zeta(k) = k : \ell + \xi$ in such case. As the function $\zeta$ is finite everywhere and convex, $\zeta$ is *Lipschitz continuous* relative to any compact set (see [57], Theorem 10.4). It follows that there exists a constant $b > 0$ such that $|\zeta(k + eu) - \zeta(k)| \leq b \epsilon$ for all non-negative $\epsilon \leq 1$. Furthermore, since $f(s)/s$ is bounded for large $s$, the constant $b$ can be chosen in such a way that it also holds that $|\Delta_ε(s)| \leq b \epsilon s$ for each $s \geq 1$ and non-negative $\epsilon \leq 1$.

*Part (c):* if $\theta(k) > 1$, then $k \notin \Theta$ and $\zeta(k) > k : \ell + \xi$. Consequently, a number $\eta > 0$ can be found so that $\zeta(k) - \eta > k : \ell + \xi$. For such $\eta$ it holds that $\sum_{s \geq 1} s^2 \exp[k \cdot f(s) - \zeta(k)s + \eta s]p(s) = \sum_{s \geq 1} s^2 \exp(\eta s)c_0(s) < \infty$. As $\Theta$ is closed and $k \notin \Theta$, there exists $\epsilon_0 \leq 1$ with the properties that $k + eu \notin \Theta$ and that $eb \leq \eta$ for all non-negative $\epsilon < \epsilon_0$. It follows that $\sum_{s \geq 1} c_ε(s) = 1$ and that $|\Delta_ε(s)| \leq b \epsilon s \leq \eta s$ for all such $\epsilon$ and $s \geq 1$. In particular, for every $\epsilon \in (0, \epsilon_0)$, the former relationship yields that $\sum_{s \geq 1} c_ε(s) = \sum_{s \geq 1} c_0(s)$, whereas the bound $|\exp(z) - 1 - z^2| \leq 2^2 |\exp(|z|)|$ valid for each real number $z$ combined with the latter gives that $|\exp[\Delta_ε(s)] - 1 - \Delta_ε(s)| \leq (b \epsilon s)^2 \exp(ns)$ for all $s \geq 1$. This way, we find that for any $\epsilon \in (0, \epsilon_0)$

$$\Gamma(k + eu) - \zeta(k) - \epsilon v \cdot u = \left| \frac{\sum_{s \geq 1} \Delta_ε(s)c_0(s)}{\sum_{s \geq 1} s c_0(s)} \right| \leq \frac{\sum_{s \geq 1} |\exp[\Delta_ε(s)] - 1 - \Delta_ε(s)|c_0(s)}{\sum_{s \geq 1} s c_0(s)} \leq (eb)^2 \frac{\sum_{s \geq 1} s^2 \exp(\eta s)c_0(s)}{\sum_{s \geq 1} s c_0(s)}.$$ 

From here, we conclude that $\zeta'(k; u) = v \cdot u$ dividing by $\epsilon$ and sending $\epsilon$ to zero.

*Part (b):* if $\theta(k) = 1$, then $k \in \Theta$ and hence $\zeta(k) = k : \ell + \xi$. The fact that $\zeta(h) \geq h : \ell + \xi$ for all $h \in \mathbb{R}^d$ gives that $\zeta(k + hu) - \zeta(k) \geq \ell : u$ for any $\epsilon$, showing that $\zeta'(k; u) \geq \ell : u$. Furthermore, $k \in \Theta$ implies that $\sum_{s \geq 1} c_0(s) = \theta(k) = 1$. If follows from the bound $\exp(z) \geq 1 + z$ valid for every real number $z$ that if $\sum_{s \geq 1} s c_0(s) < \infty$, then the series $\sum_{s \geq 1} \Delta_ε(s)c_0(s)$ exists for each $\epsilon > 0$ and

$$1 \geq \sum_{s \geq 1} c_ε(s) = \sum_{s \geq 1} \exp[\Delta_ε(s)]c_0(s) \geq \sum_{s \geq 1} c_0(s) + \sum_{s \geq 1} \Delta_ε(s)c_0(s) = 1 + \sum_{s \geq 1} \Delta_ε(s)c_0(s).$$

This yields that $(1/\epsilon)\sum_{s \geq 1} \Delta_ε(s)c_0(s) \geq 0$ and sending $\epsilon$ to zero we obtain that $\zeta'(k; u) \\geq v \cdot u$ whenever $\sum_{s \geq 1} s c_0(s) < \infty$. As we have shown that $\zeta'(k; u) \geq \ell : u$ in any case, we can conclude that $\zeta'(k; u) \geq \ell : u$ or $\zeta'(k; u) \geq \max\{\ell : u, v \cdot u\}$ according to $\sum_{s \geq 1} s c_0(s) = \infty$ or $\sum_{s \geq 1} s c_0(s) < \infty$.  


Now we deduce the opposite bounds $\zeta'(k; u) \leq \ell \cdot u$ if $\sum_{s \geq 1} s c_0(s) = \infty$ and $\zeta'(k; u) \leq \max\{\ell \cdot u, v \cdot u\}$ if $\sum_{s \geq 1} s c_0(s) < \infty$. These bounds complete the proof of Part (b). Pick a number $\eta > 0$ and observe that the assumption that $\lim_{s \to \infty} f(s)/s =: \ell$ exists ensures us that a positive integer $\tau_0$ can be found with the property that $[f(s) - \ell s] \cdot u \leq \eta s$ for each $s > \tau_0$. Then, fix a number $\epsilon > 0$ and suppose for a moment that $\epsilon$ satisfies $k + \epsilon u \notin \Theta$. This way, $\zeta$ is differentiable at $k + \epsilon u$ as stated by Part (c) of the present lemma and it follows from convexity that for all $\tau \geq \tau_0$

$$\frac{\zeta(k + \epsilon u) - \zeta(k)}{\epsilon} \leq \nabla \zeta(k + \epsilon u) \cdot u = \sum_{s \geq 1} f(s) \cdot u c_s(s) \sum_{s \geq 1} s c_s(s)$$

$$\leq \ell \cdot u + \sum_{s \geq 1} [f(s) - \ell s] \cdot u c_s(s) \sum_{s \geq 1} s c_s(s)$$

$$\leq \ell \cdot u + \sum_{s \geq 1} [f(s) - \ell s] \cdot u c_s(s) + \eta.$$  

We get from here that for any integer $t \geq 1$

$$\frac{\zeta(k + \epsilon u) - \zeta(k)}{\epsilon} \leq \ell \cdot u + \max\left\{0, \sum_{s = 0}^{\ell} [f(s) - \ell s] \cdot u c_s(s) \right\} + \eta$$

$$\leq \ell \cdot u + \max\left\{0, \sum_{s = 0}^{t} [f(s) - \ell s] \cdot u c_s(s) \right\} + \eta.$$  

This inequality trivially holds even if $k + \epsilon u \in \Theta$, as $\zeta(k + \epsilon u) - \zeta(k) = \epsilon \ell \cdot u$ in such case due to the fact that $k \in \Theta$. Thus, it holds whatever $\epsilon > 0$ is. Sending $\epsilon$ to zero and observing that $\lim_{s \to \infty} c_s(s) = \lim_{s \to \infty} \exp\{\Delta_k(s)c_0(s)\} = c_0(s)$ for each $s \geq 1$, we find that for all $\tau \geq \tau_0$ and $t \geq 1$

$$\zeta'(k; u) \leq \ell \cdot u + \max\left\{0, \sum_{s = 0}^{t} [f(s) - \ell s] \cdot u c_0(s) \right\} + \eta.$$  

At this point, sending first $t$ to infinity, then $\tau$ to infinity, and finally $\eta$ to zero, we get that $\zeta'(k; u) \leq \ell \cdot u$ or $\zeta'(k; u) \leq \max\{\ell \cdot u, v \cdot u\}$ according to $\sum_{s \geq 1} s c_0(s)$ diverges or converges.

Part (a): if $k$ satisfies $\Theta(k) < 1$, then $k \in \Theta$ and hence $\zeta(k) = k \cdot \ell + \xi$. If there exists a number $\delta > 0$ with the property that $k + \delta u \in \Theta$, then $k + \epsilon u \in \Theta$ for every non-negative $\epsilon \leq \delta$ since $k + \epsilon u = (1 - \epsilon/\delta)k + (\epsilon/\delta)(k + \delta u)$ and both $k$ and $k + \delta u$ belong to the convex set $\Theta$. Consequently, $\zeta(k + \epsilon u) = (k + \epsilon u) \cdot \ell + \xi$ for all non-negative $\epsilon \leq \delta$ and $\zeta'(k; u) = \ell \cdot u$ follows immediately. The non trivial case is when the above number $\delta$ does not exist, namely when $k \in \partial \Theta$ and $k + \epsilon u \notin \Theta$ for all $\epsilon > 0$. In this case, we will prove that for each $t \geq 1$ there exists an integer $s > t$ with the property that $\zeta'(k; u)s < u \cdot f(s)$. This fact yields

$$\zeta'(k; u) \leq \limsup_{s \to \infty} \frac{u \cdot f(s)}{s} = \ell \cdot u.$$  

On the other hand, since $\zeta(h) \geq h \cdot \ell + \xi$ for all $h \in \mathbb{R}^d$ and $\zeta(k) = k \cdot \ell + \xi$, we also have that $\zeta'(k; u) \geq \ell \cdot u$. This shows that $\zeta'(k; u) = \ell \cdot u$ in any case.
Assume that $k \in \text{bd } \Theta$ and that $k + \epsilon u \notin \Theta$ for all $\epsilon > 0$. We prove that for each $t \geq 1$ there exists $s > t$ such that $\zeta'(k; u)s < u \cdot f(s)$ by contradiction. If an integer $t \geq 1$ with the property that $\zeta'(k; u)s \geq u \cdot f(s)$ for all $s > t$ exists, then $\zeta(k + \epsilon u) - \zeta(k) \geq \zeta'(k; u)\epsilon \geq u \cdot f(s)/s$ for each $s > t$ and $\epsilon > 0$ by convexity. This means that $\Delta_s(s) \leq 0$ for all $s > t$ and $\epsilon > 0$. At the same time, we have that $\sum_{s \geq 1} c_s(s) = 1$ for all $\epsilon > 0$ as $k + \epsilon u \notin \Theta$ by hypothesis. It follows that $1 = \sum_{s \geq 1} c_s(s) \leq \sum_{s=1}^\infty \exp[\Delta_s(s)]c_0(s) + \sum_{s=t+1}^\infty c_0(s)$ for every $\epsilon > 0$. This way, sending $\epsilon$ to zero and observing that $\sum_{s \geq 1} c_s(s) = \theta(k)$ because $k \in \Theta$, we get that $\theta(k) \geq 1$. This contradicts the fact that $\theta(k) < 1$.

5.3.3 Proof of Proposition 6

To begin with, we consider a new probability space $(\Omega, F, P_0)$ where a sequence \{(S_i, R_i)\}_{i \geq 1} of independent and identically distributed pairs of waiting times and rewards is defined in such a way that $P_0[R_i = A \wedge S_i = s]$ coincides with $\exp(-z_s)P[R_i = A \wedge S_i = s]$ for each $A \in \mathcal{B}(\mathbb{R}^d)$ and $s \geq 1$. We introduce the new waiting time distribution $p_0$ given by $p_0(s) := P_0[S_i = s] = \exp(-z_s)p(s)$ for all $s \geq 1$ and we denote by $E_0$ expectation with respect to the probability measure $P_0$. The hypothesis that $E[\exp(-z_s)1_{S_i < \infty}] = 1$ entails that $P_0[S_i < \infty] = 1$. Then, the hypotheses that $E[S_i \exp(-z_s)1_{S_i < \infty}] < \infty$ and that $E[\|R_i\|_2 \exp(-z_s)1_{S_i < \infty}] < \infty$ combined with the present fact that $P_0[S_i = \infty] = 0$ imply that $E_0[S_i < \infty]$ and $E_0[\|R_i\|_2] < \infty$.

The probability space $(\Omega, F, P_0)$ fulfills the following properties, in contrast to $(\Omega, F, P)$. First of all, we have that $\lim_{t \to \infty} P_0[X_t = 0] = 1/E_0[S_1]$. Indeed, since $\sum_{s \geq 1} p_0(s) = P_0[S_i < \infty] = 1$, this limit is obtained by applying the Renewal Theorem to the renewal equation $P_0[X_t = 0] = \sum_{s=1}^\infty p_0(s)P_0[X_{t-s} = 0]$ that is valid for every $t \geq 1$ (see [55], Theorem 1 in Chapter XIII.10). Second, we have that $N_t$ goes $P_0$-almost surely to infinity when $t$ is sent to infinity because the event where one of the waiting times is infinite has probability zero with respect to the probability measure $P_0$. Last, the facts that $E_0[S_i < \infty]$ and that $E_0[\|R_i\|_2] < \infty$ make possible to invoke the strong law of large numbers to conclude that both $(1/n)\sum_{i=1}^n S_i$ goes $P_0$-almost surely to $E_0[S_i]$ and $(1/n)\sum_{i=1}^n R_i$ goes $P_0$-almost surely to $E_0[R_i]$ when $n$ is sent to infinity. It follows that $\sum_{i=1}^{N_t} R_i / \sum_{i=1}^{N_t} S_i$ goes $P_0$-almost surely to $E_0[R_i]/E_0[S_i]$ when $t$ is sent to infinity. The constant vector $E_0[R_i]/E_0[S_i]$ is nothing but $w_0$. This way, since almost sure convergence implies converge in probability, we get that for any $\delta > 0$

$$\lim_{t \to \infty} P_0 \left[ \frac{\sum_{i=1}^{N_t} f(S_i)}{\sum_{i=1}^{N_t} S_i} \notin B_{w_0, \delta} \right] = 0. \quad (38)$$

Now we come back to the original probability measure $P$, relating this measure to $P_0$. Recall that the event $X_t = 0$ with $t \geq 1$ is tantamount to the condition that an integer $n \geq 1$ exists so that $S_1 + \cdots + S_n = t$, which in particular yields that $N_t = n$. Then, noticing that $\prod_{i=1}^n p(s_i) = \exp(z_0t)\prod_{i=1}^n p_0(s_i)$ if $s_1 + \cdots + s_n = t$
we have that for every $A \in \mathcal{B}([0,\infty))$

$$\mathbb{P}\left[\frac{W_t}{t} \in A \wedge X_t = 0\right] = \sum_{n \geq 1} \mathbb{P}\left[\frac{1}{t} \sum_{i=1}^{n} f(S_i) \in A \wedge s_1 + \cdots + s_n = t\right]$$

$$= \sum_{n \geq 1} \sum_{s_1 \geq 1} \cdots \sum_{s_n \geq 1} \mathbb{P}\left[\frac{1}{t} \sum_{i=1}^{n} f(s_i) \in A \wedge s_1 + \cdots + s_n = t\right] \prod_{i=1}^{n} p(s_i)$$

$$= \exp(z_0t) \sum_{n \geq 1} \sum_{s_1 \geq 1} \cdots \sum_{s_n \geq 1} \mathbb{P}\left[\frac{1}{t} \sum_{i=1}^{n} f(s_i) \in A \wedge s_1 + \cdots + s_n = t\right] \prod_{i=1}^{n} p_o(s_i)$$

$$= \exp(z_0t) \mathbb{P}_o\left[\sum_{i=1}^{N_t} f(S_i) \in A \wedge X_t = 0\right]$$

$$\leq \exp(z_0t) \mathbb{P}_o\left[\sum_{i=1}^{N_t} f(S_i) \in A\right].$$

The identity (39) with $A = \mathbb{R}^d$ yields that $\mathbb{P}[X_t = 0] = \exp(z_0t) \mathbb{P}_o[X_t = 0]$. We notice that $\mathbb{P}_o[X_t = 0] \geq 1/(2\mathbb{E}_o[S_1])$ for all sufficiently large $t$ as we have shown that $\lim_{t \to \infty} \mathbb{P}_o[X_t = 0] = 1/\mathbb{E}_o[S_1]$ with $\mathbb{E}_o[S_1] < \infty$. Thus, dividing the bound (40) by $\mathbb{P}[X_t = 0]$ we find that for each $A \in \mathcal{B}(\mathbb{R}^d)$ and all sufficiently large $t$

$$\mathbb{P}_t\left[\frac{W_t}{t} \in A\right] \leq \mathbb{P}_o\left[\frac{\sum_{i=1}^{N_t} f(S_i)}{\sum_{i=1}^{N_t} S_i} \in A\right]/(2\mathbb{E}_o[S_1]).$$

This bound and the properties of $(\Omega,\mathcal{F},\mathbb{P}_o)$ prove the proposition. Fixing a number $\delta > 0$ and taking $A := \mathbb{R}^d \setminus B_{w_o,\delta}$ we get that $\lim_{t \to \infty} \mathbb{P}_t[\|W_t \cdot t - w_o\|_2 \geq \delta] = 0$ from (41) thanks to (38).

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