Lax pair, Darboux Transformations and solitonic solutions for a (2+1) dimensional non-linear Schrödinger equation

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Abstract

In this paper the Singular Manifold Method has allowed us to obtain the Lax pair, Darboux transformations and \( \tau \) functions for a non-linear Schrödinger equation in 2+1 dimensions. In this way we can iteratively build different kind of solutions with solitonic behavior.

1. Introduction

The integrability and structure of (2+1) dimensional systems have received considerable attention in the last few years \cite{1}, \cite{10}. Non-linear Schrödinger type equations are a particular case of interest.

These equations were discovered by Calogero \cite{2} and then discussed by Zakharov \cite{31}. Their geometrical properties have been studied in \cite{15}.

The equation under study in this paper is the following non-linear Schrödinger equation in (2+1)

\[
\begin{align*}
  i\psi_t &= \psi_{xy} + r^2 V\psi \\
  V_x &= 2\partial_y |\psi|^2
\end{align*}
\]  

(1.1)

Strachan \cite{27} rederived this equation by dimensionally reducing the self-dual Yang Mills equation \cite{28} and proved it to be integrable from the point of view of geometrical considerations. In \cite{24} its integrability is studied in the sense of having the Painlevé property and exact solutions with solitonic behavior are obtained using Hirota’s bilinear method. One and two-soliton solutions can be found in \cite{26} and dromionic ones are obtained in \cite{25} whereas n-soliton solutions may be found in \cite{17}.

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The geometrical equivalence (called the Lakshmanan equivalence) between spin systems and non-linear Schrödinger equations is studied in [16], [19], [18] and [20].

Equation (1.1) is the Lakshmanan equivalent of the Myrzakulov-I (M-I) equation

\[ \vec{S}_t = (\vec{S} \wedge \vec{S}_y + u\vec{S})_x \]
\[ u_x = -\vec{S} \left( \vec{S}_x \wedge \vec{S}_y \right) \] (1.2)

proposed in [16] as an extension to (2+1) dimensions of Heisenberg’s 1-dimensional spin model [12], [13].

The equivalence between (1.1) and (1.2) is proved in [21] and [22].

If we redefine:

\[ \psi = u \quad \psi^* = \omega \quad V = -2m_y \quad t = it \] (1.3)

and take \( r^2 = -1 \), equation (1.1) becomes:

\[ u_t - u_{xy} - 2m_y u = 0 \]
\[ \omega_t + \omega_{xy} + 2m_y \omega = 0 \]
\[ m_x + u\omega = 0 \] (1.4)

which is the PDE we shall study here.

The plan of this paper is as follows: In section II we shall apply the singular manifold method to equation (1.4) to obtain the singular manifold equations. In section 3 we use the SMM to linearize the singular manifold equations and obtain the Lax pair. Section 4 is devoted to determining the Darboux transformations and \( \tau \)-functions. We apply the results of section 4 to obtain solitonic solutions in section 5. The conclusions are presented in section 6.

2. Singular Manifold Method

Leading term analysis

In order to perform the Painlevé analysis [23] for equation (1.4) we need to expand the fields \( u, \omega \) and \( m \) in a generalized Laurent expansion in terms of an arbitrary singularity manifold \( \chi(x, y, t) = 0 \). Such an expansion should be of the form [30]:

\[ u = \sum_{j=0}^{\infty} u_j(x, y, t) [\chi(x, y, t)]^{j-\alpha} \]
\[ \omega = \sum_{j=0}^{\infty} \omega_j(x, y, t) [\chi(x, y, t)]^{j-\beta} \]
\[ m = \sum_{j=0}^{\infty} m_j(x, y, t) [\chi(x, y, t)]^{j-\gamma} \] (2.1)
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By substituting (2.1) in (1.4), we have for the leading terms:

$$\alpha = \beta = \gamma = 1 \quad m_0 = \chi x \quad u_0 \omega_0 = \chi_x^2$$

from which we see that leading analysis is not able to determine $u_0$ and $\omega_0$ independently and only gives us their product. This suggests that we should write the dominant terms $u_0$ and $\omega_0$ in the more general way as:

$$u_0 = A(x, y, t)\chi_x \quad \omega_0 = \frac{1}{A^2} \chi_x$$

(2.3)

Truncated expansions. Auto-Bäcklund transformations

If we truncate expansions (2.1) at the constant level, as is required by the SMM [29], we can write the solutions in terms of a singular manifold, $\phi$, which is not yet an arbitrary function because it is determined by the truncation condition. We can therefore write the solutions (2.1) to equation (1.4) in the following way:

$$m' = m + \frac{\phi_x}{\phi}$$

$$u' = u + \frac{A\phi_x}{\phi}$$

$$\omega' = \omega + \frac{\phi_x}{A\phi}$$

(2.4)

The set of equations (2.4) are the auto-Bäcklund transformations between two solutions of (1.4).

Expression of the solutions in terms of the Singular Manifold

Substituting equations (2.4) in (1.4), we obtain a polynomial in $\phi$. If we require all the coefficients of this polynomial to be zero we obtain the following expressions after some algebraic manipulations (we used MAPLE V to handle the calculation. The details are in the appendix):

$$u = -\frac{A}{2} \left( v + \left( \frac{A_x}{A} + h \right) \right)$$

(2.5)

$$\omega = -\frac{1}{2A} \left( v - \frac{A_x}{A} + h \right)$$

(2.6)

$$m_x = \frac{1}{4} \left( \left( \frac{A_x}{A} + h \right)^2 - v^2 \right)$$

(2.7)

$$m_y = \frac{1}{2} \left( \frac{A_t}{A} - \frac{A_x A_y}{A^2} - v_y \right)$$

(2.8)

where $v$, $w$ and $q$ are defined as:

$$v = \frac{\phi_{xx}}{\phi_x}$$
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\[
w = \frac{\phi_t}{\phi_x} \quad (2.9)
\]

\[
q = \frac{\phi_y}{\phi_x}
\]

and \( h = h(y,t) \) is an \( x \)-independent function which arises after performing an integration in \( x \) (see appendix).

It is useful to notice that the compatibility conditions between the definitions (2.9) give rise to the following equations:

\[
\phi_{xx} = \phi_{txx} \implies v_t = (w_x + vw)_x
\]

\[
\phi_{xy} = \phi_{yxx} \implies v_y = (q_x + vq)_x
\]

\[
\phi_{yt} = \phi_{ty} \implies q_t = w_y + wq_x - qw_x
\]

**Singular manifold equations**

Furthermore, substitution of (2.4) in (1.4) provides equations to be satisfied by the singular manifold. These equations are (see appendix):

\[
0 = w + hq - \frac{A_y}{A} \quad (2.11)
\]

\[
0 = \left( \frac{A_x A_y}{A^2} - \frac{A_t}{A} \right)_x + \left( v_x - \frac{v^2}{2} + \frac{1}{2} \left( \frac{A_x}{A} + h \right)^2 \right)_y
\]

\[
(2.12)
\]

and the following equation for \( h \)

\[
h_t + hh_y = 0 \quad (2.13)
\]

The set (2.10)-(2.13) are the singular manifold equations.

3. Lax pairs

3.1 Painlevé analysis in singular manifold equations

We can consider the singular manifold equations (2.10)-(2.12) as a system of non-linear coupled PDE’s in \( v, q, w \) and \( A \). It is useful to define

\[
\alpha = \frac{A_x}{A} + h
\]

\[
\beta = \frac{A_y}{A} + h_y x
\]

\[
\gamma = \frac{A_t}{A} + h_t x
\]

\[
(3.1)
\]
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in which case (2.10)-(2.12) can be combined to give:

\[ 0 = v_t + hv_y - [(\beta - h_y)x + (\beta - h_x)v]_x \]
\[ 0 = (\beta\alpha - h\beta - x\alpha h_y - \gamma)_x + \left( v_x - \frac{v^2}{2} + \frac{\alpha^2}{2} \right)_y \]

This allows us to perform the leading terms analysis by setting:

\[ v \sim v_0 \chi^a \]
\[ \alpha \sim \alpha_0 \chi^b, \quad \beta \sim \beta_0 \chi^b; \quad \gamma \sim \gamma_0 \chi^b \]

Substitution of (3.4) in (3.2)-(3.3) yields the leading powers:

\[ a = b = -1 \]

and the leading coefficients:

\[ v_0 = \chi_x \]
\[ \alpha_0 = \pm \chi_x, \quad \beta_0 = \pm \chi_y, \quad \gamma_0 = \pm \chi_t \]

The ± sign tells us that the Painlevé expansion has two branches. The problem of systems with two Painlevé branches has been discussed in [3], [7], [6], [5]. These references suggest that we should consider both branches simultaneously by using two singular manifolds, one for each branch.

### 3.2 Eigenfunctions and the singular manifold

In agreement with the foregoing, we can write the dominant terms of \( v \) and \( A \) as:

\[ v = \frac{\psi^+}{\psi^+} + \frac{\psi^+}{\psi^-}, \quad \alpha = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-}, \quad \beta = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-}, \quad \gamma = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-} \]
\[ \frac{A_x}{A} = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-} - h, \quad \frac{A_y}{A} = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-} - h_y x, \quad \frac{A_t}{A} = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-} - h_t x \]

where \( \psi^+ \) is the singular manifold for the positive branch and \( \psi^- \) for the negative one. We will see later on that \( \psi^+ \) and \( \psi^- \) are the eigenfunctions of the Lax pair.

Integrating (3.7), we obtain the expressions of \( \phi \) and \( A \) in terms of the eigenfunctions:

\[ \phi_x = \psi^+ \psi^-, \quad \phi_t + h \phi_y = \psi^- \psi^+ - \psi^+ \psi^- - h_x \psi^+ \psi^- \]
\[ A = \frac{\psi^+}{\psi^-} e^{-hx} \]
### 3.3 Linearization of the singular manifold equations: the Lax pair

Substitution of (3.8) in (2.5)-(2.8) gives us the following expressions of $u$, $\omega$, $m_x$ and $m_y$ in terms of $\psi^+$ and $\psi^-$:

\[
\begin{align*}
    u &= -\frac{\psi^+}{\psi^-} e^{-hx} \\
    \omega &= -\frac{\psi^-}{\psi^+} e^{hx} \\
    m_x &= -\frac{\psi^+ \psi^-}{\psi^+ \psi^-} \\
    m_y &= \left( \frac{\psi^+ + h\psi^- + u_y e^{hx} \psi^-}{2\psi^+} \right) + \left( \frac{-\psi^- - h\psi^- + w_y e^{-hx} \psi^+}{2\psi^-} \right)
\end{align*}
\]

Equations (3.9) and (3.10) can be considered to be the spatial part of the Lax pair, which written in a more appropriate way reads:

\[
\begin{align*}
    0 &= \psi^+_x + w\psi^- e^{hx} \\
    0 &= \psi^-_x + \omega\psi^+ e^{-hx}
\end{align*}
\]

The temporal part of the Lax pair can be obtained as follows: If we substitute (3.7) in (3.2) and use (3.13), we obtain (after an integration in $x$) the equation:

\[
\left( \frac{\psi^+ + h\psi^- + u_y e^{hx} \psi^-}{-\psi^+} \right) - \left( \frac{-\psi^- - h\psi^- + w_y e^{-hx} \psi^+}{-\psi^-} \right) - h_y = 0
\]

Adding and subtracting (3.12) and (3.14) we obtain the temporal part of the Lax pair:

\[
\begin{align*}
    \psi^+_t - m_y \psi^+ + h\psi^- + u_y \psi^- e^{hx} + \frac{1}{2} h y \psi^+ &= 0 \\
    \psi^-_t + m_y \psi^- + h\psi^+ - \omega_y \psi^+ e^{-hx} + \frac{1}{2} h y \psi^- &= 0
\end{align*}
\]

It is interesting to notice that the compatibility condition between (3.13) and (3.15) is the equation (1.4) together with the condition (2.13). Therefore (3.13) and (3.15) form the Lax pair for (1.4) and $h$ is the spectral parameter although it is non-isospectral.

### 4. Darboux transformations

Summarizing the above results:
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• Let be $u$, $\omega$ and $m$ solutions of (1.4) and $\phi_1$ a singular manifold for them. This singular manifold can be constructed from two eigenfunctions $\psi^+_1$ and $\psi^-_1$ as:

$$\phi_{1x} = \psi^+_1 \psi^-_1$$  \hspace{1cm} (4.1)

where $\psi^+_1$ and $\psi^-_1$ satisfy the Lax pairs:

$$0 = \psi^+_1 x + u \psi^-_1 e^{h_1 x}$$
$$0 = \psi^-_1 x + \omega \psi^+_1 e^{-h_1 x}$$
$$0 = \psi^+_1 t - m_y \psi^-_1 + h_1 \psi^+_1 + u_y \psi^-_1 e^{h_1 x} + \frac{1}{2} h_{1y} \psi^+_1$$
$$0 = \psi^-_1 t + m_y \psi^+_1 + h_1 \psi^-_1 - \omega_y \psi^+_1 e^{-h_1 x} - \frac{1}{2} h_{1y} \psi^-_1$$  \hspace{1cm} (4.2)

and the spectral parameter $h_1$ satisfies:

$$h_{1t} + h_1 h_{1y} = 0$$  \hspace{1cm} (4.3)

• Substituting (3.8) in (2.4), we can define new solutions $u'$, $\omega'$ and $m'$:

$$u' = u + \frac{(\psi^+_1)^2 e^{-h_1 x}}{\phi_1}$$
$$\omega' = \omega + \frac{(\psi^-_1)^2 e^{h_1 x}}{\phi_1}$$
$$m' = m + \frac{\phi_{1x}}{\phi_1}$$  \hspace{1cm} (4.4)

whose Lax pairs will be:

$$0 = \psi'^+_x + u' \psi'^- e^{h_2 x}$$
$$0 = \psi'^-_x + \omega' \psi'^+ e^{-h_2 x}$$
$$0 = \psi'^+_t - m_y \psi'^+ + h_2 \psi'^_y + u'_y \psi'^- e^{h_2 x} + \frac{1}{2} h_{2y} \psi'^+$$
$$0 = \psi'^-_t + m_y \psi'^- + h_2 \psi'^-_y - \omega'_y \psi'^+ e^{-h_2 x} - \frac{1}{2} h_{2y} \psi'^-$$  \hspace{1cm} (4.5)

and we can construct a singular manifold $\phi'$ for the iterated fields $u'$, $\omega'$ and $m'$ through $\psi'^+$ and $\psi'^-$ as:

$$\phi'_x = \psi'^+ \psi'^-$$  \hspace{1cm} (4.6)

Truncated expansion in the Lax pair

We can consider the Lax pair (4.5) as a system of coupled non-linear PDE’s ([6], [11]) in $\psi'^+$, $\psi'^-$, $m'$, $u'$ and $\omega'$. Therefore, the singular manifold method can be applied to the Lax pair itself and truncated expansions for $\psi'^+$ and $\psi'^-$ should be added to the expansions (4.4). Such expansions can be written as:
The seminal solutions \( m, u, \omega, \psi^+ \) and \( \psi^- \) must satisfy the same Lax pair with the spectral parameter \( h_2 \), which means:

\[
\begin{align*}
0 &= \psi^+_2 x + u \psi^-_2 e^{h_2 x} \\
0 &= \psi^+_2 x + \omega \psi^-_2 e^{-h_2 x} \\
0 &= \psi^+_2 - m \psi^-_2 + h_2 \psi^+_2 y + u \psi^-_2 e^{h_2 x} + \frac{1}{2} h_2 \psi^-_2 \\
0 &= \psi^+_2 + m \psi^-_2 + h_2 \psi^+_2 y - \omega \psi^-_2 e^{-h_2 x} + \frac{1}{2} h_2 \psi^-_2 
\end{align*}
\]

Substituting the truncated expansions (4.4) and (4.7) in the Lax pair (4.5) and after some calculation (we used MAPLE V for it) we obtain:

\[
\begin{align*}
\Omega^+ &= \frac{\psi^+_2 \psi^-_2 e^{-(h_1-h_2)x} - \psi^+_2 \psi^-_2}{h_2 - h_1} \\
\Omega^- &= \frac{\psi^+_2 \psi^-_2 - \psi^+_2 \psi^-_2 e^{(h_1-h_2)x}}{h_2 - h_1}
\end{align*}
\]

Summarizing: the set of equations

\[
\begin{align*}
u' &= u + \psi^+_2 \psi^-_2 e^{h_1 x} \\
\omega' &= \omega + \psi^+_2 \psi^-_2 e^{h_1 x} \\
m' &= m + \phi_1 x \\
\psi'^+ &= \psi^+_2 - \frac{\psi^+_2 \Omega^+}{\phi_1} \\
\psi'^- &= \psi^-_2 - \frac{\psi^-_2 \Omega^-}{\phi_1}
\end{align*}
\]

where \( \Omega^+ \) and \( \Omega^- \) are given by (4.9), constitutes a transformation of potentials and eigenfunctions that leaves the Lax pairs invariant. Hence, (4.10) should be considered as a Darboux transformation \[14\].

5. Iteration of the singular manifold: \( \tau \)-functions

The \( \tau \)-functions of Hirota’s bilinear method \[9\] can be built through the singular manifold and Darboux transformations as follows:
Equation (4.6) can be considered as a non-linear equation in \( \phi', \psi'^+ \) and \( \psi'^- \) and it is therefore pertinent to add the following truncated expansion to the set (4.10):
\[
\phi' = \phi_2 + \frac{\Delta}{\phi_1}
\]
(5.1)
where \( \phi_2 \) satisfies:
\[
\phi_{2x} = \psi_2^+ \psi_2^-
\]
(5.2)
Substituting (5.1) and (4.7) in (4.6) one has:
\[
\Delta = -\Omega^+\Omega^-
\]
(5.3)
Since (5.1) defines a singular manifold for \( m' \), can be used to build an iterated solution:
\[
m'' = m' + \frac{\phi'}{\phi}
\]
(5.4)
Substitution of equation (4.4) for \( m' \) in (5.4) gives:
\[
m'' = m + \frac{\tau_x}{\tau}
\]
(5.5)
where
\[
\tau = \phi'\phi_1 = \phi_1\phi_2 - \Omega^+\Omega^-
\]
(5.6)
is the Hirota \( \tau \)-function.

6. Solutions

In this section, we obtain solutions to the system (1.4) in a systematic way using the previous results. The steps followed in this iterative procedure can be summarized as:

- 1) We start from seminal solutions of (1.4) and write the Lax pair for them. The solitonic or dromionic behavior of the iterated solutions will depend on our choice of the seminal ones.
- 2) Solving the Lax pairs, we obtain \( \psi_1^+, \psi_1^- \), \( \psi_2^+ \) and \( \psi_2^- \).
- 3) We use the results of 2) in (4.1), (4.9), and (5.2) to obtain \( \phi_1, \phi_2, \Omega^+ \) and \( \Omega^- \).
- 4) We use (4.4) and (5.4) to obtain the first and second iterations \( m' \) and \( m'' \), respectively.
6.1 Line solitons $m = -abx$, $u = a$, $\omega = b$

If we restrict ourselves to the case in which $h_1$ and $h_2$ are constants, non-trivial solutions of the Lax pairs (4.2) and (4.8) are:

$$
\psi_1^+ = \exp \left[ \alpha_1 x + \beta_1 y + \left( \frac{ab}{\alpha_1} - \alpha_1 \right) \beta_1 t \right] \quad \psi_1^- = -\frac{\alpha_1}{a} \exp \left[ \frac{ab}{\alpha_1} x + \beta_1 y + \left( \frac{ab}{\alpha_1} - \alpha_1 \right) \beta_1 t \right]
$$

$$
\psi_2^+ = \exp \left[ \alpha_2 x + \beta_2 y + \left( \frac{ab}{\alpha_2} - \alpha_2 \right) \beta_2 t \right] \quad \psi_2^- = -\frac{\alpha_2}{a} \exp \left[ \frac{ab}{\alpha_2} x + \beta_2 y + \left( \frac{ab}{\alpha_2} - \alpha_2 \right) \beta_2 t \right]
$$

(6.1)

where $\beta_1, \beta_2$ are arbitrary constants and $\alpha_1$ and $\alpha_2$ are related to the spectral parameter as:

$$
h_i = \alpha_i - \frac{ab}{\alpha_i}
$$

(6.2)

If we define:

$$
P_1 = \exp \left[ \alpha_1 x + \beta_1 y + \left( \frac{ab}{\alpha_1} - \alpha_1 \right) \beta_1 t \right] \quad Q_1 = \exp \left[ \frac{ab}{\alpha_1} x + \beta_1 y + \left( \frac{ab}{\alpha_1} - \alpha_1 \right) \beta_1 t \right]
$$

$$
P_2 = \exp \left[ \alpha_2 x + \beta_2 y + \left( \frac{ab}{\alpha_2} - \alpha_2 \right) \beta_2 t \right] \quad Q_2 = \exp \left[ \frac{ab}{\alpha_2} x + \beta_2 y + \left( \frac{ab}{\alpha_2} - \alpha_2 \right) \beta_2 t \right]
$$

(6.3)

integration of (3.8) yields:

$$
\phi_1 = -\frac{\alpha_1}{a} \frac{1}{\alpha_1 + \frac{ab}{\alpha_1}} (c_1 + P_1 Q_1)
$$

$$
\phi_2 = -\frac{\alpha_2}{a} \frac{1}{\alpha_2 + \frac{ab}{\alpha_2}} (c_2 + P_2 Q_2)
$$

(6.4)

where $c_1$ and $c_2$ are arbitrary constants. Using (4.9) one has:

$$
\Omega^+ = -\frac{\alpha_1 \alpha_2}{a(\alpha_1 \alpha_2 + ab)} P_2 Q_1
$$

$$
\Omega^- = -\frac{\alpha_1 \alpha_2}{a(\alpha_1 \alpha_2 + ab)} P_1 Q_2
$$

(6.5)

The first iteration provides the solution (figure 1):

$$
m'_x = -ab + \partial_{xx}[\ln \phi_1]
$$

(6.6)

and the second (figure 2):

$$
m''_x = -ab + \partial_{xx}[\ln \tau]
$$

(6.7)
where

\[ \phi_i = -\frac{\alpha_i c_i}{a \alpha_i + ab \alpha_i}(1 + F_i) \]  (6.8)

\[ \tau = \phi_1 \phi_2 - \Omega^+ \Omega^- = \frac{\alpha_1 \alpha_2}{a^2} \frac{1}{(\alpha_1 + ab \alpha_1)(\alpha_2 + ab \alpha_2)}[1 + F_1 + F_2 - A_{12} F_1 F_2] \]  (6.9)

and

\[ F_i(x, y, t) = \exp \left[ \left( \alpha_i + \frac{ab}{\alpha_i} \right)x + 2\beta_i y - 2 \left( \alpha_i - \frac{ab}{\alpha_i} \right) \beta_i t + \varphi_i \right] \]  (6.10)

\[ A_{12} = \frac{ab(\alpha_1 - \alpha_2)^2}{(\alpha_1 \alpha_2 + ab)^2} \]  (6.11)

and we have redefined \( c_i \) as: \( c_i = e^{-\varphi_i} \).

A particularly interesting case occurs when \( \alpha_1 = \alpha_2 \) and hence the interaction term \( A_{12} \) vanishes. This case is termed the **resonant state** (figure 3).

**6.2 Dromions** \( m = 0, \ u = 0, \ \omega = b \)

For this seminal solution (similar solutions are obtained for \( u = a, \ \omega = 0 \)), assuming that \( h_1 \) and \( h_2 \) are constants, the easiest non-trivial solutions of (4.2) and (4.8) are:

\[ \psi_{1+} = K_1(y, t) \quad \psi_{1-} = \frac{b}{h_1} e^{-h_1 x} K_1(y, t) \]
\[ \psi_{2+} = K_2(y, t) \quad \psi_{2-} = \frac{b}{h_2} e^{-h_2 x} K_2(y, t) \]  (6.12)

where \( K_i \) are \( x \)-independent functions that satisfy

\[ K_{it} + h_i K_{iy} = 0 \]  (6.13)

Integration of (3.8) yields:

\[ \phi_i = -\frac{b}{h_i^2} \left( R_i(y, t) + K_i^2(y, t) e^{-h_i x} \right) \]  (6.14)

where

\[ R_{it} + h_i R_{iy} = 0 \]  (6.15)

and from (4.9):

\[ \Omega^+ = -\frac{b}{h_1 h_2} K_1 K_2 e^{-h_1 x} \]
\[ \Omega^+ = -\frac{b}{h_1 h_2} K_1 K_2 e^{-h_2 x} \]  (6.16)
The first iteration provides:

\[ m'_y = \partial_{xy} \ln[\phi_1] \quad (6.17) \]

and the second one

\[ m''_y = \partial_{xy} \ln[\tau] \quad (6.18) \]

where

\[ \tau = \phi_1 \phi_2 - \Omega^+ \Omega^- = \frac{b^2}{h_1^2 h_2^2} \left\{ R_1 R_2 + R_2 K_1^2 e^{-h_1 x} + R_1 K_2^2 e^{-h_2 x} \right\} \quad (6.19) \]

Due to the arbitrariness of \( R_i \) and \( K_i \), (6.17) and (6.18) are a rich collection of solutions. Let us consider some particular cases.

**Case 1**

This corresponds to the choice:

\[ R_i = 1 + e^{c_1(y-h_i t)} \]
\[ K_i^2 = 1 + a_i e^{c_1(y-h_i t)} \]

Fig. 4 represents \( m'_y \) for this choice. Figures 5.a-5.c are the second iteration \( m''_y \) for different times.

**Case 2**

Another possibility is:

\[ R_1 = 1 + e^{c_1(y-h_1 t)} + e^{c_2(y-h_1 t)} \]
\[ K_i^2 = 1 + a_1 e^{c_1(y-h_1 t)} + a_2 e^{c_2(y-h_1 t)} \]
\[ R_2 = K_2 = 0 \]

\( m'_y \) is represented in Fig.6 and Fig.7 for different values of the parameters.

7. **Conclusions**

- The real version of a (2+1) dimensional integrable generalization of the non-linear Schrödinger equation, which has been discussed by several authors, is studied from the point of view of Painlevé analysis.
- In section 2 we applied the singular manifold method to equation (1.4), obtaining the leading terms and the singular manifold equations.
- In section 3, the Lax pairs were obtained by performing Painlevé analysis on the singular manifold equations. This allowed us to define the eigenfunctions of the Lax pair.
In section 4 we considered the Lax pairs as system of coupled PDE’s in the fields and eigenfunctions and we obtained the Darboux transformations between two solutions of (1.4). This permits us to determine an iterative procedure for obtaining solutions from already known ones.

Section 6 is devoted to constructing different kinds of solutions with solitonic behavior by using the Darboux transformations with different seminal solutions.

A Appendix

Substitution of the truncated expansion (2.4) in (1.4) provides three polynomials in $\frac{1}{p}$. Setting each coefficient of every polynomial at zero we obtain the following equations:

a) for $m_x + u\omega = 0$

$$0 = vA + u + A^2\omega$$  \hfill (A.1)

b) for $u_t - u_{xy} - 2um_y = 0$

$$0 = A_y + qAx - Aw + vqA + 2uq$$  \hfill (A.2)

$$0 = A_t - A_{xy} - A_x(q_x + qv) - vA_y + A(w_x + wv - v_y - vq_x -qv^2 - 2m_y) - 2u(q_x + vq)$$  \hfill (A.3)

c) for $\omega_t + \omega_{xy} + 2m = 0$

$$0 = A_y + qAx - Aw + vqA - 2\omega A^2 = 0$$  \hfill (A.4)

$$0 = -A_t - A_{xy} + 2A_xA_yA - A_x(q_x + qv) - vA_y + A(w_x + wv + v_y + vq_x + qv^2 + 2m_y) + 2\omega A^2(q_x + \omega q)$$  \hfill (A.5)

From (A.2) and (A.4), we can obtain:

$$u = -\left(\frac{1}{2q}\right)(A_y + qAx + qvA - wA)$$  \hfill (A.6)

$$\omega = \left(\frac{1}{2qA^2}\right)(A_y + qAx - qvA - wA)$$  \hfill (A.7)

(A.6) and (A.7) satisfy (A.1) identically and their substitution in (A.3) and (A.5) gives (after addition and subtraction):

$$-\frac{A_{xy}}{qA} + \frac{q_xA_y}{q^2A} + \frac{A_xA_y}{qA^2} + \frac{w_x}{q} - \frac{q_xw}{q^2} = 0$$  \hfill (A.8)

and

$$A_t - A\omega_y - \frac{A_xA_y}{A} - 2Am_y = 0$$  \hfill (A.9)

(A.8) can be integrated in $x$. The result is:

$$\left(\frac{1}{q}\right)(w - \frac{A_y}{A}) + h(y, t) = 0$$  \hfill (A.10)
where \( h(y,t) \) is the constant with respect to the integration in \( x \). Substitution of (A.10) in (A.6), (A.7) and (A.9) provides:

\[
\begin{align*}
  u &= -\frac{A}{2} \left( v + \left( h + \frac{A_x}{A} \right) \right) \quad (A.11) \\
  \omega &= -\frac{1}{2A} \left( v - \left( h + \frac{A_x}{A} \right) \right) \quad (A.12) \\
  m_y &= \frac{1}{2} \left( \frac{A_t}{A} - \frac{A_x A_y}{A^2} - v_y \right) \quad (A.13)
\end{align*}
\]

Finally, imposing the condition that \( u, \omega \) and \( m \) should be solutions of (1.4), we obtain:

\[
m_x = \frac{1}{4} \left( \left( \frac{A_x}{A} + h \right)^2 - v^2 \right) \quad (A.14)
\]

and

\[
-A_t + \frac{A_x A_t}{A} + \frac{A_x A_{xy}}{A} + \frac{A_y A_{xx}}{A} - 3 \frac{A_x^2 A_y}{A^2} + A(v_{xy} - vv_y) + h_y A_x - h_t A + h(A_{xy} - \frac{A_x A_y}{A}) = 0 \quad (A.15)
\]

Imposing the compatibility condition \( m_{xy} = m_{yx} \) on (A.13) and (A.14), we have:

\[
-A_t + \frac{A_x A_t}{A} + \frac{A_x A_{xy}}{A} + \frac{A_y A_{xx}}{A} - 3 \frac{A_x^2 A_y}{A^2} + A(v_{xy} - vv_y) + h_y A_x + hh_y A + h(A_{xy} - \frac{A_x A_y}{A}) = 0 \quad (A.16)
\]

Adding and subtracting (A.15) and (A.16), the result is:

\[
h_t + hh_y = 0 \quad (A.17)
\]

\[
\left( \frac{A_x A_y}{A^2} - \frac{A_t}{A} \right)_x + \left( v_x - \frac{v^2}{2} + \frac{1}{2} \left( \frac{A_x}{A} + h \right)^2 \right)_y \quad (A.18)
\]

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Fig. 4: One Dromion Solution
Figure 7: Dromion with two positive jumps
Fig. 6: Dromion with one positive and one negative jump
Figure 5.a: Two Dromion Solution. $t<0$
Figure 5.b: Two Dromion Solution. t=0
Figure 5.c: Two Dromion Solution. $t > 0$
Figure 1: Line Soliton
Figure 2: Two Line Soliton
Figure 3: Resonant Soliton