Spectrometer Calibration by Expectation Maximization Method
L. Yuan

aDepartment of Physics, Hampton University, Hampton, VA 23668

Expectation Maximization (EM) algorithm is a parameter estimation method from incomplete observations. In this paper, an implementation of this method to the calibration of HKS spectrometer at Jefferson Lab is described. We show that the application of EM method is able to calibrate the spectrometer properly in the presence of high background noise, while the traditional nonlinear Least Square method fail. The preliminary results of HKS spectrometer calibration is presented.

1. Introduction

Expectation Maximization (EM) algorithm is a statistical method for parameter estimation from incomplete observations. It is an extension of the Maximum Likelihood (ML) method well known to physicists. This method was first proposed by A. Dempster etc in [1]. Since its introduction, this method has been used in a wide varieties of applications such as signal processing, medical image processing and genetics, to name a few [2].

The EM method is well suited to handle problems with observations diluted by large amount of noise, since it is not known a priori a observation is signal or noise. That is why it is introduced into High Energy physics for track reconstruction in the ATLAS detector at LHC [3], [4]. In the presence of track noise, the EM based tracking algorithm can obtain a track resolution more than two order of magnitude better than traditional Least Square tracking method.

We have used the EM method in the spectrometer calibration for HKS experiment at Jefferson Lab. Jefferson Lab HKS experiment aims at obtaining high resolution hypernuclear spectroscopy by (e,e’K) reaction. To achieve this goal, it is essential to perform a proper spectrometer calibration to optimize the reconstruction resolution of the momentum and angles of scattering electrons and Kaons [5]. The only high precision calibration method is to make use of the known masses of Λ,Σ^0 hyperons produced from hydrogen in CH_2 target and the narrow width of Λ_B hypernuclear ground state from ^12C target [6]. These masses can be produced at the same spectrometer kinematics as the production of hypernuclei.

2. Nonlinear Least Square Method

Let \{\pi\} denote the set of parameters which defines the reconstruction function. For example, the \{\pi\} can be a set of polynomial coefficients in the polynomial expansion of
reconstruction function. The task of calibration now is to find the best set of parameters \( \{ \pi \} \) to optimize the reconstruction resolution. The missing mass \( E_m \) of \((e,e'K)\) reaction can be calculated from the focal plane measurement \( X_i \), \( i = 1, \ldots, N \), \( i \) denotes each event number, and the reconstruction parameters \( \{ \pi \} \):

\[
E_m^i = f(X_i, \{ \pi \}),
\]

where \( f \) is a nonlinear function. The format of \( f \) can be derived from the kinematics equations. \( X_i \) represent the trajectories of the particles at spectrometer focal plane. Let \( \Delta M_i \) be the difference between the calculated mass and the known mass value from Particle Data Book \( M^{PDB} \),

\[
\Delta M_i = E_m^i - M^{PDB}.
\]

Finally, we define a Chisquare as the weighted sum of squared \( \Delta M_i \) over all events:

\[
\chi^2 = \sum_{i=1}^{N} w_i \Delta M_i^2 = \sum_{i=1}^{N} w_i (f(X_i, \{ \pi \}) - M^{PDB})^2,
\]

where \( w_i \) is the relative weights of \( \Lambda, \Sigma \) and \( ^{12}_\Lambda B \) GS events.

The set of parameters \( \{ \pi \} \) which minimize the Chisquare function will define our optimized reconstruction function. This is a typical nonlinear Least Square (NLS) problem. The Chisquare function is still a complex nonlinear function and have to be minimized by numerical method. It is carried out by using CERNLIB Fortran program package LEAMAX ([7]).

In case we have a clean signal of hyperons and hypernuclear bound states (The signal to noise \((S/N)\) ratio better than 6:1), the NLS method works well, as is shown for simulated HKS data in our Arxiv paper ([5]). However, the HKS spectrometer setup detects very forward angle \( e' \) and Kaons in order to increase hypernuclear yield. Thus in real experiment, we see high accidental background between Kaon arm and electrons produced by Bremsstrahlung photons. For the real data, The \( S/N \) ratio in the missing mass spectrum is almost 1:1 (fig.1). Applying the NLS method to the data results in wrong calibration. As shown in fig.2 the accidental background under the \( ^{12}_\Lambda B \) gs peak, which should be flat, now forms an artificial “bump” as a result of the NLS calibration. Clearly, the EM algorithm which is robust to noise observations should be used for the spectrometer calibration of HKS.

3. Expectation Maximization method

We have two considerations in order to implement the EM method:

1. EM method is used for parameter estimation with incomplete observation. Taking advantage of this feature, we can define a variable \( S_i \), \( i = 1, \ldots, N \), which denotes whether event \( i \) is a real signal (real coincidence) : \( S_i = 1 \) or noise (accidental coincidence): \( S_i = 0 \), although \( S_i \) can not be observed by the spectrometer. Thus instead of minimize the chisquare in eq.3 we will minimize an energy function defined as

\[
E(\{S_i\}, \{ \pi \}) = \sum_i [S_i \Delta M_i^2 + \lambda (S_i - 1)^2],
\]
Figure 1. $^{12}_\Lambda$B excitation energy spectrum. The shaded region is accidental background.
Figure 2. \(^{12}\text{B}\) missing mass spectrum used in the improper calibration of HKS spectrometer by nonlinear Least Square method. The shaded region is accidental background.
where $\lambda$ is a cut off parameter.

2. There is a large number of parameters in the parameter set $\{\pi\}$, to avoid the calibration process ending up in a local minimum, we introduce an annealing process \((\mathbb{H})\). One requires each configuration of the system with energy $E$ obey the Boltzmann distribution at temperature $T$. One then minimizes the expectation value of the energy function at successively lower temperatures until final result at $T \to 0$.

According to Boltzmann distribution, the probability for the system to have configuration $\{S_i, \Delta M_i, i = 1, \ldots N\}$, is:

$$P({\{S_i\}, \{\pi\}}) = e^{-\beta E({\{S_i\}, \{\pi\}})}/Z, \quad (5)$$

where $\beta$ is the inverse temperature $\beta = 1/T$ and $Z$ is the partition function:

$$Z = \sum_{\{S_i\}} \int d\{\pi\} e^{-\beta E({\{S_i\}, \{\pi\}})} \quad (6)$$

The EM algorithm can be divided into the expectation step and the maximization step. In the expectation step, the expectation value of the energy function over the unobserved variable is calculated:

$$Q({\{\pi\}} | \{\pi^f\}) = \sum_{\{S_i\}} E({\{S_i\}, \{\pi\}})P({\{S_i\}} | \{\pi^f\})$$

$$= \sum_{\{S_i\}} E({\{S_i\}, \{\pi\}}) \cdot \frac{P({\{S_i\}, \{\pi^f\}})} {P_M({\{\pi^f\}})} \quad (7)$$

where $P({\{S_i\}} | \{\pi^f\})$ is the probability function of assignment variables $S_i$ conditioned on the parameter set $\{\pi^f\}$, $P_M$ is the marginal probability function:

$$P_M({\{\pi^f\}}) = \sum_{\{S_i\}} P({\{S_i\}, \{\pi^f\}}) = e^{-\beta E_{eff}}/Z, \quad (8)$$

Effective energy

$$E_{eff} = -\frac{1}{\beta} \sum_{i=1}^{N} \log(e^{-\beta \lambda} + e^{-\beta \Delta M_i^2}). \quad (9)$$

Substitute equations 5 and 8 into equation 7, we can write up the expectation value as:

$$Q({\{\pi\}} | \{\pi^f\}) = \sum_{i=1}^{N} [\Delta M_i^2 \frac{e^{-\beta \Delta M_i^2}}{e^{-\beta \lambda} + e^{-\beta \Delta M_i^2} + \lambda \frac{e^{-\beta \lambda}}{e^{-\beta \lambda} + e^{-\beta \Delta M_i^2}}]} + \lambda p_0^f]. \quad (10)$$
\( p_i \) can be interpreted as the probability that event \( i \) is a real signal. The minimization step is then to minimize the \( Q \) function with respect to the parameter set \( \{ \pi \} \). Because the second term is independent of \( \{ \pi \} \), in the maximization step, we will minimize function:

\[
\mathcal{g}(\{ \pi \} | \{ \pi' \}) = \sum_{i=1}^{N} w_i \Delta M_i^2 p_{i|t} = \sum_{i=1}^{N} w_i (f(X_i, \{ \pi \}) - M^{PDB})^2 p_{i|t}
\]

with respect to \( \{ \pi \} \). \( p_{i|t} \) is defined by equation (10). Again, we have added the relative weights \( w_i \) to adjust for the effect of \( \Lambda, \Sigma \) and \( \Lambda^0 \) BS events in the calibration. The new values of the parameter is used to update the probabilities \( p_{i|t} \), and the \( g \) function is again minimized. Comparing with eq.3, we can see that the EM algorithm in this case is nothing but an iteratively reweighted least-square procedure. The weights or probabilities are not constants now, but functions of \( \Delta M_i \) or \( \{ \pi \} \). Example probability functions calculated for the HKS spectrometer calibration is shown in fig.3.

The minimization of function \( g \) is also carried out by CERNLIB Fortran program package LEAMAX. We have obtained preliminary reconstruction functions by the EM method described above. The preliminary missing mass spectra of \( \Lambda, \Sigma^0 \) from CH\(_2\) target and hypernucleus \( \Lambda^0 B \) from C\(_{12}\) are shown in fig.1 and fig.4 overlayed with background.

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Figure 3. The dependences of calculated probability functions $p_i$ for $\Lambda, \Sigma$ and $^{12}\Lambda B$ on mass differences $\Delta M_i$. 
Figure 4. $\Lambda$ and $\Sigma$ missing mass distribution produced by $p(e,e'K)$ reaction from CH$_2$ target. The shaded region is accidental background.