Decays of the $X(3872)$ into $J/\psi$ and Light Hadrons

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Abstract

If the $X(3872)$ is a loosely-bound molecule of the charm mesons $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$, it can decay through the decay of a constituent in a hadronic channel with a nearby threshold, such as $J/\psi\omega$ or $J/\psi\rho$. The differential decay rates of the $X$ into $J/\psi\pi^+\pi^-$, $J/\psi\pi^+\pi^-\pi^0$, $J/\psi\pi^0\gamma$, and $J/\psi\gamma$ are calculated in terms of $XJ/\psi\rho$ and $XJ/\psi\omega$ coupling constants using an effective lagrangian that reproduces the decay rates of the $\omega$ and the $\rho$. The dependence of the coupling constants on the binding energy and the total width of the $X$ is determined by a factorization formula. Results from a model by Swanson are used to predict the partial width of $X$ into $J/\psi\pi^+\pi^-\pi^0$ as a function of the binding energy and the total width of the $X$.

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I. INTRODUCTION

The \( X(3872) \) is a narrow resonance near 3872 MeV discovered by the Belle collaboration in electron-positron collisions through the \( B^- \rightarrow BK^- \) followed by the decay \( X \rightarrow J/\psi \pi^+\pi^- \) [1]. Its existence has been confirmed by the CDF and DØ collaborations through its inclusive production in proton-antiproton collisions [2, 3] and by the Babar collaboration through the discovery mode \( B^\pm \rightarrow XK^\pm \) [4]. The combined measurement of the mass of the \( X \) is [5]

\[
m_X = 3871.9 \pm 0.5 \text{ MeV},
\]

which is within 1 MeV of the threshold for the charm mesons \( D^0 \) and \( \bar{D}^*0 \). The presence of the \( J/\psi \) among the decay products of the \( X(3872) \) motivates its interpretation as a charmonium state with constituents \( c\bar{c} \) [5–9]. Two possibilities motivated by the proximity of the mass in Eq. (1) to the \( D^0\bar{D}^*0 \) threshold are a hadronic molecule with constituents \( DD^* \) [10–20] and a “cusp” at the \( D^0\bar{D}^*0 \) threshold associated with strong coupling to \( D^0\bar{D}^*0 \) or \( D^*0\bar{D}^0 \) [21, 22]. Other proposed interpretations include a tetraquark with constituents \( c\bar{c}q\bar{q} \) [23], a “hybrid charmonium” state with constituents \( c\bar{c}g \) [24, 25], a glueball with constituents \( ggg \) [26], and a diquark-antidiquark bound state with constituents \( cu + \bar{c}\bar{u} \) [27]. The interpretation as a \( DD^* \) molecule is particularly predictive because the small binding energy implies that the molecule has universal properties that are completely determined by the binding energy [11, 13, 15, 18]. The small binding energy can be further exploited through factorization formulas for production and decay rates of the \( X \) [19].

Measurements of the decays of the \( X \) can be used to determine its quantum numbers and narrow down the possibilities [5, 28–32]. The upper bound on the decay width of the \( X \) is [1]

\[
\Gamma_X < 2.3 \text{ MeV (90\% C.L.)},
\]

which is much narrower than other charmonium states above the \( D\bar{D} \) threshold. The product of the branching fractions associated with the discovery channel is [1, 4, 5]

\[
\text{Br}[B^+ \rightarrow XK^+] \text{Br}[X \rightarrow J/\psi \pi^+\pi^-] = (1.3 \pm 0.3) \times 10^{-5}.
\]

The invariant mass distribution of the two pions from the decay \( X \rightarrow J/\psi \pi^+\pi^- \) seems to peak near the upper endpoint, which suggests that the pions come from a virtual \( \rho \) resonance. Recently the Belle collaboration has observed the \( X(3872) \) in the decay mode \( X \rightarrow J/\psi \pi^+\pi^-\pi^0 \) [32] with the branching ratio

\[
\frac{\text{Br}[X \rightarrow J/\psi \pi^+\pi^-\pi^0]}{\text{Br}[X \rightarrow J/\psi \pi^+\pi^-]} = 1.0 \pm 0.4_{\text{stat}} \pm 0.3_{\text{syst}}.
\]

The invariant mass distribution of the three pions indicates that they come predominantly from a virtual \( \omega \) resonance [33]. If the decays into \( J/\psi \pi^+\pi^- \) and \( J/\psi \pi^+\pi^-\pi^0 \) are interpreted as \( J/\psi \mathcal{P}^* \) and \( J/\psi \mathcal{W}^* \), the approximate equality of the branching fractions in Eq. (4) implies a large violation of isospin symmetry. The Belle collaboration also reported evidence for the decay \( X \rightarrow J/\psi \gamma \) [32] with the branching ratio

\[
\frac{\text{Br}[X \rightarrow J/\psi \gamma]}{\text{Br}[X \rightarrow J/\psi \pi^+\pi^-]} = 0.14 \pm 0.05.
\]
The observation of the decay into $J/\psi \gamma$ establishes the charge conjugation of the $X$ to be $+$. By analyzing angular distributions in the decay of $X$ into $J/\psi \pi^+\pi^-$, the Belle collaboration has ruled out all $J^{P+}$ assignments for $X$ with $J \leq 2$ other than $1^{++}$ and $2^{++}$ [32]. Upper limits have been placed on the branching fractions for other decay modes of the $X$, including $D^0\bar{D}^0$, $D^+D^-$, $D^0\bar{D}^0\pi^0$ [34], $\chi_{c1}\gamma$, $\chi_{c2}\gamma$, $J/\psi\pi^0\pi^0$ [35], and $J/\psi\eta$ [36]. Upper limits have also been placed on the partial widths for the decay of $X$ into $e^+e^-$ [37, 38] and into $\gamma\gamma$ [37].

The possibility that charm mesons might form molecular states was considered shortly after the discovery of charm [39–42]. In 1993, Tornqvist made a quantitative study of the possibility of molecular states of charm mesons using a one-pion-exchange potential model [43]. He found that the isospin-0 combinations of $DD^*$ and $D^*\bar{D}$ could form weakly-bound states in the S-wave $1^{++}$ channel and in the P-wave $0^{--}$ channel. After the discovery of the $X(3872)$, Tornqvist pointed out that because the binding energy is small compared to the splitting between the $D^+D^{*-}$ and $D^0\bar{D}^{*0}$ thresholds, there will be large violations of isospin symmetry [10]. Swanson considered a potential model that includes both one-pion-exchange and quark exchange and found that the $C = +$ superposition of $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ could form a weakly-bound state in the S-wave $1^{++}$ channel [14]. Swanson’s model included not only the charmed mesons $D^0\bar{D}^{*0}$, $D^{*0}\bar{D}^0$, $D^+D^{*-}$, and $D^{*+}D^-$, but also two other pairs of hadrons with nearby thresholds: $J/\psi \rho$ and $J/\psi \omega$. His prediction that the branching fraction for the decay of $X$ into $J/\psi \pi^+\pi^-\pi^0$ should be comparable to that for decay into $J/\psi \pi^+\pi^-$ was verified by the Belle collaboration [32, 33].

In this paper, we analyze decays of the $X(3872)$ into $J/\psi$ and light hadrons under the assumption that $X$ is a loosely-bound $DD^*$ molecule and that these decays proceed through transitions of $X$ to $J/\psi \rho$ and $J/\psi \omega$. In Section II, we summarize some of the universal results for a system with large scattering length and we give the current constraints on the real and imaginary parts of the large scattering length for the $DD^*$ system. In Section III, we discuss other hadronic states with thresholds near the $D^0\bar{D}^{*0}$ threshold and we summarize results from Swanson’s model for the $X(3872)$. In Section IV, we calculate the differential distributions for decays of $X$ into $J/\psi \pi^+\pi^-$, $J/\psi \pi^+\pi^-\pi^0$, and $J/\psi \pi^0\gamma$ using an effective lagrangian that reproduces decays of the vector mesons. We also use vector meson dominance to calculate the partial width for the decay into $J/\psi \gamma$. The normalizations of the decay rates are determined by unknown coupling constants for the transitions of $X$ to $J/\psi \rho$ and $J/\psi \omega$. The dependence of the coupling constants on the binding energy and the total width of the $X$ is deduced using a factorization formula. In Section V, we use a two-channel scattering model to illustrate how the coupling constants can be related to probabilities for the $J/\psi \rho$ and $J/\psi \omega$ components of the $X$. We use the probability for $J/\psi \omega$ in Swanson’s model to give a quantitative prediction for the partial decay rate of the $X$ into $J/\psi \pi^+\pi^-\pi^0$ as a function of the binding energy and the total width of the $X$. A summary of our results is given in Section VI. An updated determination of the parameters in the effective lagrangian for light pseudoscalar and vector mesons is given in an Appendix.

**II. UNIVERSALITY AND THE $DD^*$ SYSTEM**

The mass of the $X$ is extremely close to the $D^0\bar{D}^{*0}$ threshold: $m_{\rho^0} + m_{D^{*0}} = 3871.3 \pm 1.0$ MeV. From the mass measurement in Eq. (1), the difference is

$$m_X - (m_{\rho^0} + m_{D^{*0}}) = +0.6 \pm 1.1 \text{ MeV}.$$  

\(6\)
Most of the uncertainty comes from the experimental uncertainty in $2m_{D^0}$, because the mass difference $m_{D^{*0}} - m_{D^0}$ has a much smaller uncertainty. If the $X$ were a $D^0D^{*0}/D^{*0}D^0$ molecule, the energy difference in Eq. (6) would have to be negative, corresponding to a positive binding energy defined by

$$E_X = (m_{D^0} + m_{D^{*0}}) - m_X.$$  \hfill (7)

The measurement of the binding energy in Eq. (6) is compatible with a small negative value corresponding to a $DD^*$ molecule. However, the central value of the energy difference in Eq. (6) is positive, corresponding to a resonance in $D^0D^{*0}$ and $D^{*0}D^0$ scattering rather than a bound state. Bugg has referred to this possibility as a “cusp state” \cite{21, 22}, because the line shape of the $X$ in some of its decay modes has a cusp at the $D^{*0}D^0$ threshold.

The energy difference in Eq. (6) is tiny compared to the natural energy scale for binding by the pion exchange interaction: $m^2_\pi/2\mu \approx 10$ MeV, where $\mu$ is the reduced mass of $D^0$ and $\bar{D}^{*0}$.

$$\mu = \frac{m_{D^0}m_{D^{*0}}}{m_{D^0} + m_{D^{*0}}} = 966.5 \pm 0.3 \text{ MeV}. \hfill (8)$$

Whether the energy difference is positive or negative, its unnaturally small value implies that if the $X$ couples to $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$, the S-wave scattering lengths for those channels must be large compared to the natural length scale $1/m_\pi$ associated with the pion exchange interaction. Since the experimental evidence favors the charge conjugation quantum number $C = +$, we assume that there is a large scattering length $a$ in the $C = +$ channel and that the scattering length in the $C = -$ channel is negligible in comparison. In this case, the scattering lengths for elastic $D^0D^{*0}$ scattering and elastic $D^{*0}\bar{D}^0$ scattering are both $a/2$.

Nonrelativistic few-body systems with short-range interactions and a large scattering length have universal properties that depend on the scattering length but are otherwise insensitive to details at distances small compared to $|a|$ \cite{44}. We consider the scattering length to be large if it is much larger than the natural momentum scale associated with low-energy scattering. The universal results are encoded in the truncated connected transition amplitude, which is a function of the total energy $E$ of the two particles in the rest frame of the pair:

$$A(E) = \frac{2\pi/\mu}{-1/a + \sqrt{-2\mu E}}, \hfill (9)$$

where $\mu$ is the reduced mass of the two particles. If $a$ is real and positive, this amplitude has a pole on the physical sheet at $E = -1/(2\mu a^2)$, indicating the existence of a weakly-bound state with the universal binding energy

$$E_X = \frac{1}{2\mu a^2}. \hfill (10)$$

The universal momentum-space wavefunction of this bound state is

$$\psi(p) = \frac{(8\pi/a)^{1/2}}{p^2 + 1/a^2}. \hfill (11)$$

The universal amplitude for transitions from the bound state to a scattering state consisting of two particles with small relative momentum is determined by the residue of the pole in $A(E)$:

$$A_X = \frac{\sqrt{2\pi}}{\mu} a^{-1/2}. \hfill (12)$$
If there is an inelastic scattering channel, the large scattering length $a$ has a negative imaginary part. It is convenient to express the complex scattering length in the form

$$\frac{1}{a} = \gamma_{re} + i\gamma_{im},$$

(13)

where $\gamma_{re}$ and $\gamma_{im}$ are real and $\gamma_{im} \geq 0$. The universal expression for the binding energy in Eq. (10) has an imaginary part $i\Gamma_X/2$, where

$$\Gamma_X = 2\gamma_{re}\gamma_{im}/\mu.$$

(14)

If $\gamma_{im} < \gamma_{re}$, $\Gamma_X$ is the full width at half maximum of a resonance in the inelastic channel [19]. It therefore can be interpreted as the rate for the decay of the bound state into the inelastic channel. The peak of the resonance is below the threshold by the amount

$$E_X = \gamma_{re}^2/(2\mu).$$

(15)

We can therefore interpret this expression as the binding energy of the resonance [19].

The observed decays of the $X$ imply that there are inelastic scattering channels, so $a$ has a negative imaginary part. It can be parameterized in terms of the real and imaginary parts of $1/a$ as in Eq. (13). Our interpretation of $X$ as a bound state requires $\gamma_{re} > 0$. The energy difference in Eq. (6) puts an upper bound on $\gamma_{re}$:

$$\gamma_{re} < 40 \text{ MeV} \quad (90\% \text{ C.L.}).$$

(16)

The upper bound on the width in Eq. (2) puts an upper bound on the product of $\gamma_{re}$ and $\gamma_{im}$:

$$\gamma_{re}\gamma_{im} < (33 \text{ MeV})^2 \quad (90\% \text{ C.L.}).$$

(17)

There is also a lower bound on the width of the $X$ from its decays into $D^0\bar{D}^0\pi^0$ and $D^0\bar{D}^0\gamma$, which both proceed through the decay of a constituent $D^*$. These decays involve interesting interference effects, but the decay rates have smooth limits as the binding energy is tuned to 0 [11]. In this limit, the constructive interference increases the decay rate by a factor of 2, so the partial widths of $X$ into $D^0\bar{D}^0\pi^0$ and $D^0\bar{D}^0\gamma$ add up to $2\Gamma[D^{*0}]$. The width of $D^{*0}$ has not been measured, but it can be deduced from other information about the decays of $D^{*0}$ and $D^{*+}$. Using the total width of the $D^{*+}$, its branching fraction into $D^+\pi^0$, and isospin symmetry, we can deduce the partial width of $D^{*0}$ into $D^0\pi^0$ to be $42 \pm 10$ keV. The total width of the $D^{*0}$ can then be obtained by dividing by its branching fraction into $D^0\pi^0$: $\Gamma[D^{*0}] = 68 \pm 16$ keV. The sum of the partial widths of $X$ into $D^0\bar{D}^0\pi^0$ and $D^0\bar{D}^0\gamma$ is therefore $136 \pm 32$ keV. The resulting lower bound on the product of $\gamma_{re}$ and $\gamma_{im}$ is

$$\gamma_{re}\gamma_{im} > (7 \text{ MeV})^2 \quad (90\% \text{ C.L.}).$$

(18)

By combining this with the upper bound on $\gamma_{re}$ in Eq. (16), we can infer that $\gamma_{im} > 1$ MeV.

### III. Hadronic States with Nearby Thresholds

The state of the $X$ can be written schematically as

$$|X\rangle = \frac{Z^{1/2}}{\sqrt{2}} (|D^0\bar{D}^{*0}\rangle + |D^{*0}\bar{D}^0\rangle) + \sum_H Z_H^{1/2} |H\rangle,$$

(19)
where $Z$ is the probability for the $X$ to be in the $D^0\bar{D}^{*0}/D^{*0}\bar{D}^0$ state and $Z_H$ is the probability for the $X$ to be in another hadronic state $H$. The hadronic states $H$ in (19) could include charmonium states, other charm meson pairs such as $D^{\pm}D^{*\mp}$, states consisting of a charmonium and a light hadron such as $J/\psi\rho$ and $J/\psi\omega$, etc. An expansion of $|X\rangle$ in terms of hadronic states can be valid only if there is an ultraviolet cutoff on the energy difference with respect to the $D^0\bar{D}^{*0}$ threshold. The probabilities $Z_H$ depend on that cutoff. Universality implies that as $a \to \infty$, $Z$ approaches to 1 and $Z_H$ scales as $1/a$.[13].

The small binding energy of $X$ compared to the natural energy scale $m_\pi^2/(2\mu) = 10$ MeV associated with pion exchange implies resonant S-wave interactions in the $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ systems. If there are other hadronic channels whose thresholds differ from the $D^0\bar{D}^{*0}$ threshold by less than 10 MeV, there would be resonant S-wave interactions in those channels as well. In this case, it would be necessary to treat all the resonating channels as a coupled-channel system, with a large elastic scattering length for each channel and a large transition scattering length for each pair of channels. The hadronic states $D^{\pm}D^{*\mp}$, $J/\psi\rho$, and $J/\psi\omega$ have thresholds that are relatively close to the $D^0\bar{D}^{*0}$ threshold. The energy gaps between these other thresholds and the $D^0\bar{D}^{*0}$ threshold are

$$m_{D^{\pm}} + m_{D^{*\mp}} - (m_{\rho} + m_{D^{*0}}) = +8.1 \pm 0.1 \text{ MeV}, \quad (20a)$$
$$m_{J/\psi} + m_{\rho} - (m_{\rho} + m_{D^{*0}}) = +1.4 \pm 1.1 \text{ MeV}, \quad (20b)$$
$$m_{J/\psi} + m_{\omega} - (m_{\rho} + m_{D^{*0}}) = +8.2 \pm 1.0 \text{ MeV}. \quad (20c)$$

The small uncertainty in Eq. (20a) comes from using mass differences between charm mesons to calculate the energy gap. The uncertainties in Eqs. (20b) and (20c) are dominated by the uncertainty in $2m_{\rho}$. The energy gaps in the $D^{\pm}D^{*\mp}$ and $J/\psi\omega$ channels are comparable to the natural energy scale of about 10 MeV associated with pion exchange. The energy gap in Eq. (20b) for the $J/\psi\rho$ channel is much smaller. However, whether any of these channels can have resonant interactions with $D^0\bar{D}^{*0}$ or $D^{*0}\bar{D}^0$ is determined not only by the real parts of the energy gaps, which are given in Eqs. (20), but also by the imaginary parts, which can be obtained by replacing each mass $m$ by $m - i\Gamma/2$, where $\Gamma$ is the width of the particle. If there are large differences between the widths of the various particles, it is necessary only to take into account the largest width among the particles in each channel. The largest width in each of the three channels is

$$\Gamma[D^{*\mp}] = 0.096 \pm 0.022 \text{ MeV}, \quad (21a)$$
$$\Gamma[\rho] = 150.3 \pm 1.6 \text{ MeV}, \quad (21b)$$
$$\Gamma[\omega] = 8.49 \pm 0.08 \text{ MeV}. \quad (21c)$$

For the $D^{\pm}D^{*\mp}$ and $J/\psi\omega$ channels, the magnitude $|\Delta|$ of the complex energy gap is comparable to the natural energy scale 10 MeV associated with pion exchange between $D$ and $D^*$. The large width of the $\rho$ makes $|\Delta|$ for the $J/\psi\rho$ channel much larger than the natural energy scale.

Because the complex energy gap $\Delta$ for the other hadronic channels with nearby thresholds are comparable to or larger than the natural energy scale, these channels need not be taken into account explicitly in calculations of quantities that have nontrivial universal limits as $a \to \pm \infty$. Their dominant effects enter through the complex-valued scattering length $a$. A coupled-channel model that includes other hadronic states with nearby thresholds could still be useful for estimating nonuniversal quantities or for calculating nonuniversal corrections to the universal predictions.
Swanson has constructed a model of the $X$ and the hadronic states with nearby thresholds and used it to predict some of the properties of the $X$ [14, 16]. In particular, he predicted correctly that the branching fraction for $X \to J/\psi \pi^+\pi^-\pi^0$ is comparable to that for $X \to J/\psi \pi^+\pi^-$. In addition to the channel $D^0\bar{D}^{*0} + D^{*0}\bar{D}^0$, Swanson’s model includes $D^+D^{-} + D^{++}D^-$, $J/\psi \rho$, and $J/\psi \omega$. It includes the S-wave and D-wave channels for $DD^*$, but only the S-wave channel for $J/\psi V$, where $V$ is the vector meson $\rho$ or $\omega$. Thus the model has 6 coupled channels. The interactions between the hadrons are modeled by potentials: one-pion-exchange potentials for the S-wave and D-wave $DD^*$ channels and for transitions between those channels and Gaussian potentials for the transitions between the S-wave $DD^*$ channels and the S-wave $J/\psi V$ channels to simulate the effects of quark exchange. The one-pion-exchange potential is singular at short distances and it was regularized by an ultraviolet momentum cutoff $\Lambda$. The nonrelativistic Schrödinger equation for the 6 coupled channels was solved numerically. A bound state with the quantum numbers $J^{PC} = 1^{++}$ of $X$ appeared when the ultraviolet cutoff exceeded the critical value $\Lambda_c = 1.45$ GeV. The binding energy of the $X$ could be adjusted by varying the ultraviolet cutoff.

Swanson solved the coupled channel problem under the assumption that the $\rho$ and $\omega$ are stable hadrons with equal masses $m_\rho = m_\omega = 782.6$ MeV. The reason for using an unphysical value for $m_\rho$ is that the central PDG value from 2002 and earlier, $m_\rho = 771.1$ MeV, is below the $D^0\bar{D}^{*0}$ threshold. If such a value had been used, it would have been necessary to treat $J/\psi \rho$ states as scattering states. This complication was avoided by using a value of $m_\rho$ above the $D^0\bar{D}^{*0}$ threshold. Note that in Eq. (20b), we have taken the updated 2004 PDG value $m_\rho = 775.8 \pm 0.5$ MeV [45], which gives a $J/\psi \rho$ threshold that is a few MeV above the mass of the $X$.

Swanson calculated the probabilities for each component of the wavefunction of $X$ for values of the ultraviolet cutoff that correspond to varying the binding energy $E_X$ from 0.7 MeV to 23.2 MeV [14]. His results for the probabilities $Z_{\psi\omega}$ and $Z_{\psi\rho}$ are shown as dots in Fig. 1. Since the binding energy of the $X$ is known to be less than 1 MeV, only the lowest two values of $E_X$ could be physically relevant. For the lowest value $E_X = 0.7$ MeV, the probabilities were $Z_{\psi\omega} = 9.6\%$, $Z_{D^0\bar{D}^{*0}} = 7.9\%$, and $Z_{\psi\rho} = 0.86\%$. The total probability for the $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ components of the wavefunction is 81.6\%.

In Fig. 1, the dotted lines have the scaling behavior $E_X^{1/2}$ predicted by universality and are normalized so that they pass through the dot at $E_X = 0.7$ MeV. The probability $Z_{\psi\omega}$...
clearly exhibits the universal behavior. The probability $Z_{\psi\rho}$ does not. This may be related to the fact that $Z_{\psi\rho}$ is more than an order of magnitude smaller than $Z_{\psi\omega}$. Because of the weaker coupling of the $X$ to isospin-1 states, the scaling region for isospin-1 states may not set in until a much smaller value of $E_X$.

Swanson estimated the partial widths for the decays of $X$ into $J/\psi h$, where $h$ is the light hadronic state $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $\pi^0\gamma$, or $\pi^+\pi^0\gamma$, using a simple ad hoc recipe. The partial width into $J/\psi h$ was taken to be the sum over the vector mesons $V = \rho, \omega$ of the product of the probability $Z_{\psi V}$ for the $J/\psi V$ component of the wavefunction and the partial width $\Gamma[V \rightarrow h]$ for the decay of the vector meson:

$$\Gamma[X \rightarrow J/\psi h] \approx \sum_V Z_{\psi V} \Gamma[V \rightarrow h].$$

(22)

For the smallest value of the binding energy that was considered, $E_X = 0.7$ MeV, the resulting estimates of the partial widths for decay into $J/\psi h$ were 1290 keV, 720 keV, 70 keV, and 13 keV for $h = \pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $\pi^0\gamma$, and $\pi^+\pi^0\gamma$, respectively. The ratio of the partial widths into $J/\psi \pi^+\pi^-\pi^0$ and $J/\psi \pi^+\pi^-\pi^0$ was predicted to be 0.56 for $E_X = 0.7$ MeV. Remarkably, this prediction agrees with the subsequent measurement by the Belle collaboration given in Eq. (4) to within the experimental errors [35]. The approximately equal branching fractions are a fortuitous result of an amplitude for $X \rightarrow J/\psi \omega$ that is much larger than the amplitude for $X \rightarrow J/\psi \rho$ and an amplitude for $\omega \rightarrow \pi^+\pi^-\pi^0$ that is much smaller than that for $\rho \rightarrow \pi^+\pi^-$. The suppression of the amplitude for $X \rightarrow J/\psi \rho$ is related to the fact that in the isospin symmetric limit in which the mass difference between neutral and charged $D$'s is neglected, there is binding in the isospin-0 channel but not in the isospin-1 channel [43].

Swanson has also used his model to calculate the rates for several other decay modes of the $X$ [16]. The decay rate into $J/\psi \gamma$ has contributions from transitions to $J/\psi \rho$ and $J/\psi \omega$ that can be calculated using vector meson dominance. It also has contributions from the annihilation of the $u$ and $\bar{u}$ from the charm mesons that are the constituents of the $X$. Swanson’s prediction for the partial width into $J/\psi \gamma$ for an $X$ with a binding energy of 1 MeV is 8 keV. Decay modes that receive contributions only from $u\bar{u}$ annihilation, such as $\psi(2S) \gamma$, $KK^*$, and $\pi\rho$, have much smaller partial widths.

IV. DECAYS OF $X$ INTO $J/\psi h$

In this section, we calculate the differential decay rates of the $X$ into $J/\psi h$, where the hadronic system $h$ is $\pi^+\pi^-\pi^0$, $\pi^+\pi^-$, $\pi^0\gamma$, or $\gamma$. We assume that these decays proceed through transitions of $X$ to $J/\psi \rho$ and $J/\psi \omega$. We calculate the differential decay rates in terms of two unknown complex coupling constants using an effective lagrangian that reproduces the decays of the light vector mesons.

A. Vector Meson Decay Amplitudes

We assume that the decay of $X$ into $J/\psi h$, where $h$ is a system of light hadrons, proceeds through transitions to $J/\psi V$, where $V$ is one of the vector mesons $\rho$ or $\omega$, followed by the decay of the vector meson into $h$. Because the mass of the $X$ is so close to the threshold for $J/\psi V$, the vector meson is almost on its mass shell. Any model that reproduces the decays
of the vector mesons should also accurately describe the decay of the virtual vector meson in the \( J/\psi V \) component of \( X \). In Ref. [46], the semileptonic branching fractions for the \( \tau \) lepton were calculated using an effective lagrangian for light pseudoscalar and vector mesons with \( U(3) \times U(3) \) chiral symmetry. All the parameters in the effective lagrangian, aside from the pion decay constant, were determined directly from decays of the vector mesons \( \rho \) and \( \omega \). That same effective lagrangian can be used to calculate the partial widths of \( X \) into \( J/\psi h \). An updated determination of the parameters in that effective lagrangian is given in the Appendix.

The T-matrix element for the decay of a vector meson \( V \) into the light hadronic state \( h \) can be expressed in the form

\[
\mathcal{T}[V \to h] = \epsilon_V^\mu A_\mu[V \to h],
\]

where \( \epsilon_V \) is the polarization vector of the vector meson. The amplitude \( A_\mu \) for the decay \( \rho \to \pi^+\pi^- \) is

\[
A_\mu[\rho \to \pi^+\pi^-] = \frac{1}{2} G_{\nu\pi\pi} (p_+ - p_-)_\mu.
\]

The value of the coupling constant \( G_{\nu\pi\pi} \) is given in Eq. (A1a). The amplitude \( A_\mu \) for the decay \( \omega \to \pi^+\pi^-\pi^0 \) is

\[
A_\mu[\omega \to \pi^+\pi^-\pi^0] = \frac{4\sqrt{3} (\cos \theta_v + \sqrt{2} \sin \theta_v)}{F_\pi^3} \varepsilon_{\mu\nu\alpha\beta} p_+^\nu p_-^\alpha p_0^\beta \times \left( C_{\nu\pi\pi} + \frac{G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2}{m_\pi^2} (1 - \frac{1}{3} [f_\rho(s_{12}) + f_\rho(s_{23}) + f_\rho(s_{31})]) \right),
\]

where the vector meson resonance factor that vanishes at \( s = 0 \). We have denoted the 4-momenta of \( \pi^+ \), \( \pi^- \), and \( \pi^0 \) by \( p_+ \), \( p_- \), and \( p_0 \), respectively. The pion decay constant is \( F_\pi = 93 \) MeV, the values of the parameters \( C_{\nu\pi\pi} \) and \( G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2/m_\pi^2 \) are given by Eqs. (A1b) and (A1c), and the value of the light vector meson mixing angle \( \theta_v \) is given by Eq. (A3). The amplitudes \( A_\mu \) for the radiative decays of the vector mesons are

\[
A_\mu[\rho^+ \to \pi^+\gamma] = \frac{4e}{3F_\pi} \left( C_{\nu\pi\gamma} + \frac{G_{\nu\pi\gamma} C_{\nu\pi\pi} F_\pi^2}{m_\pi^2} \right) \varepsilon_{\mu\nu\alpha\beta} Q^\nu p^\alpha \epsilon_\gamma^\beta,
\]

\[
A_\mu[\omega \to \pi^0\gamma] = \frac{4(\cos \theta_v + \sqrt{2} \sin \theta_v)}{\sqrt{3}F_\pi} e \left( C_{\nu\pi\gamma} + \frac{G_{\nu\pi\gamma} C_{\nu\pi\pi} F_\pi^2}{m_\pi^2} \right) \varepsilon_{\mu\nu\alpha\beta} Q^\nu p^\alpha \epsilon_\gamma^\beta,
\]

where \( Q \) and \( p \) are the 4-momenta of the vector meson and the pion and \( \epsilon_\gamma \) is the polarization vector of the photon. The values of the parameters \( C_{\nu\pi\gamma} \) and \( G_{\nu\pi\gamma} C_{\nu\pi\pi} F_\pi^2/m_\pi^2 \) are given by Eqs. (A2b) and (A2c).

The amplitudes \( A_\nu \) in Eqs. (24), (25), and (27) all satisfy \( Q^\nu A_\nu = 0 \), where \( Q \) is the 4-momentum of the vector meson. This condition is satisfied in any model consistent with vector meson dominance. The assumption of vector meson dominance is that the amplitude for the production of a real photon in a hadronic process can be expressed as the sum of
over vector mesons $V$ of the amplitude for producing $V$ multiplied by a coupling constant for the transition $V \rightarrow \gamma$. The condition $Q^\nu A_\nu = 0$ is required for the gauge invariance of the resulting amplitude for real photon production.

The T-matrix element for $X$ to decay into $J/\psi$ and a light hadronic system $h$ through a virtual vector meson resonance $Q$ can be expressed as

$$\mathcal{T}[X \rightarrow J/\psi h] = A_\mu[X \rightarrow J/\psi V]\frac{-g^{\mu\nu}}{Q^2 - m_V^2 + im_V\Gamma_V}A_\nu[V \rightarrow h], \quad (28)$$

where $Q$ is the total 4-momentum of the hadronic system $h$ or, equivalently, of the virtual vector meson. We have used the condition $Q^\nu A_\nu = 0$ to simplify the numerator of the vector meson propagator. The quantum numbers of the particles, together with Lorentz invariance, constrains the amplitude for $X \rightarrow J/\psi V$ to be the sum of two terms. One of them is

$$A_\mu[X \rightarrow J/\psi V] = G_{X\psi V} \epsilon_{\mu\nu\alpha\beta} Q^\nu \epsilon_X^\alpha \epsilon_\psi^\beta, \quad (29)$$

where $\epsilon_X$ and $\epsilon_\psi$ are the polarization 4-vectors of the $X$ and the $J/\psi$ and $G_{X\psi V}$ is a dimensionless constant. The contraction of this amplitude with the polarization vector $\epsilon_V^\nu$ of the vector meson reduces in the rest frame of the vector meson to $G_{X\psi V}m_V \epsilon_X \cdot (\epsilon_\psi \times \epsilon_V)^*$. The other independent amplitude $A_\mu$ has the Lorentz structure $\epsilon_{\mu\nu\alpha\beta} P^\nu \epsilon_X^\alpha \epsilon_\psi^\beta$. In the rest frame of the $X$, its contraction with $\epsilon_V^\nu$ is $m_X \epsilon_x \cdot (\epsilon_\psi \times \epsilon_V)^*$. Since the mass of the $X$ is so close to the threshold for $J/\psi V$, the rest frames of the $X$ and $V$ are essentially identical. Thus the two independent Lorentz structures are essentially equivalent for decays that are dominated by the vector meson resonance. They give similar predictions for the partial widths for $X$ into $J/\psi h$ for $h = \pi^+\pi^-\pi^0$, $\pi^+\pi^-$, or $\pi^0\gamma$. The amplitude in Eq. (29) has the advantage that it is also consistent with the constraint $Q^\nu A_\nu = 0$ required by vector meson dominance. Thus this amplitude can be used to calculate the decay of $X$ into $J/\psi \gamma$. We therefore take the transition amplitude for $X$ into $J/\psi V$ to be the expression in Eq. (29).

B. Decay into $J/\psi \pi^+\pi^-$

We assume that the decay of $X$ into $J/\psi \pi^+\pi^-$ proceeds through a transition of $X$ to $J/\psi \rho$. The T-matrix element is then given in terms of the unknown coupling constant $G_{X\psi \rho}$ by Eqs. (28) and (29) with $V = \rho$. The expression for the amplitude $A_{\nu}$ for $\rho \rightarrow \pi^+\pi^-$ is given in Eq. (24). We obtain the decay rate by squaring the amplitude, summing over spins, and integrating over phase space. The differential decay rate into $J/\psi \pi^+\pi^-$ as a function of the invariant mass $Q$ of the two pions is

$$\frac{d\Gamma}{dQ}[X \rightarrow J/\psi \pi^+\pi^-] = \frac{|G_{X\psi \rho}|^2 G_{\pi \pi}^2}{9216\pi^2 m_X^2 m_\psi^2} \frac{(Q^2 - 4m_\pi^2)^{3/2}}{(Q^2 - m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2} \times \left[ (m_X^2 + m_\psi^2)(m_X^2 - m_\psi^2)^2 - 2(m_X^4 - 4m_X^2 m_\psi^2 + m_\psi^4)Q^2 + (m_X^2 + m_\psi^2)Q^4 \right], \quad (30)$$

where $\lambda(x, y, z)$ is the triangle function:

$$\lambda(x, y, z) = x^4 + y^4 + z^4 - 2(x^2y^2 + y^2z^2 + z^2x^2). \quad (31)$$

After integrating over the pion invariant mass, the decay rate is

$$\Gamma[X \rightarrow J/\psi \pi^+\pi^-] = |G_{X\psi \rho}|^2 (223 \text{ keV}). \quad (32)$$
The shape of the pion invariant mass distribution for the decay of $X$ into $J/\psi \pi^+\pi^-$ is shown in Fig. 2. Its qualitative features are dominated by the phase space factor $\lambda^{1/2}(m_X, m_\psi, Q)$, which cuts the distribution off at the endpoint $Q = m_X - m_\psi$, and the vector meson resonance factor, which has its maximum at $Q = m_\rho$ just outside the kinematic region. Most of the support for $d\Gamma/dQ$ comes from within $\Gamma_\rho$ of the upper endpoint.

C. Decay into $J/\psi \pi^+\pi^-\pi^0$

We assume that the decay of $X$ into $J/\psi \pi^+\pi^-\pi^0$ proceeds through a transition of $X$ to $J/\psi \omega$. The T-matrix element is then given in terms of the unknown coupling constant $G_{X\psi\omega}$ by Eqs. (28) and (29) with $V = \omega$. The expression for the amplitude for $\omega \rightarrow 3\pi$ is given in Eq. (25). We obtain the decay rate by squaring the amplitude, summing over spins, and integrating over phase space. The differential decay rate into $J/\psi \pi^+\pi^-\pi^0$ as a function of the invariant mass $Q$ of the 3 pions can be reduced to a 2-dimensional integral:

$$
\frac{d\Gamma}{dQ} [X \rightarrow J/\psi \pi^+\pi^-\pi^0] = \frac{|G_{X\psi\omega}|^2(\cos \theta_v + \sqrt{2} \sin \theta_v)^2}{3072 \pi^5 m_X^2 m_{\psi,\pi}^2 F_\pi^2} \frac{\lambda^{1/2}(m_X, m_\psi, Q)}{Q[(Q^2 - m_\omega^2)^2 + m_\omega^2 \Gamma_\omega^2]} \times \left[ (m_X^2 + m_\omega^2)(m_X^2 - m_\omega^2)^2 - 2(m_X^4 - 4m_X^2 m_\psi^2 + m_\psi^4)Q^2 + (m_X^2 + m_\omega^2)Q^4 \right] \\
\times \int ds_{12} \int ds_{23} \left[ s_{12} s_{23} s_{31} - m_\pi^2 (Q^2 - m_\pi^2)^2 \right] \\
\times \left| C_{\nu3\pi} + \frac{G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2}{m_{\nu}^2} (1 - \frac{1}{3} [f_\rho(s_{12}) + f_\rho(s_{23}) + f_\rho(s_{31})]) \right|^2, \tag{33}
$$

where $s_{12}$, $s_{23}$, and $s_{31}$ are the squares of the invariant masses of the three pairs of pions. We have suppressed the limits of integration in the integrals over $s_{12}$ and $s_{23}$. After integrating over the pion invariant masses, the decay rate is

$$
\Gamma[X \rightarrow J/\psi \pi^+\pi^-\pi^0] = |G_{X\psi\omega}|^2(19.4 \text{ keV}). \tag{34}
$$

The shape of the pion invariant mass distributions for the decay of $X$ into $J/\psi \pi^+\pi^-\pi^0$ is shown in Fig. 3. Its qualitative features are dominated by the phase space factor...
\(\lambda^{1/2}(m_X, m_\psi, Q)\), which cuts the distribution off at the endpoint \(Q = m_X - m_\psi\), and the vector meson resonance factor, which has its maximum at \(Q = m_\omega\) just outside the kinematic region. Most of the support for \(d\Gamma/dQ\) comes from within a few widths \(\Gamma_\omega\) of the upper endpoint.

The ratio of the decay rates in Eqs. (32) and (34) is
\[
\frac{\Gamma[X \to J/\psi \pi^+ \pi^- \pi^0]}{\Gamma[X \to J/\psi \pi^+ \pi^-]} = 0.0870 \left| \frac{G_{X\psi\omega}}{G_{X\psi\rho}} \right|^2.
\]
By comparing this to Belle’s result in Eq. (4) for the ratio of the branching fractions, we can obtain an estimate of the ratio of the coupling constants:
\[
\frac{|G_{X\psi\omega}|^2}{|G_{X\psi\rho}|^2} \approx 11.5 \pm 5.7.
\]

D. Decay into \(J/\psi \pi^0\gamma\)

We assume that the decay of \(X\) into \(J/\psi \pi^0\gamma\) proceeds through transitions of \(X\) to \(J/\psi \rho\) and \(J/\psi \omega\). The T-matrix element is then given by Eq. (28) summed over \(V = \rho, \omega\). The amplitudes for \(V \to \pi^0\gamma\) are given in Eqs. (27). The differential decay rate with respect to the invariant mass \(Q\) of the \(\pi^0\gamma\) is
\[
\frac{d\Gamma}{dQ}[X \to J/\psi \pi^0\gamma] = \frac{\alpha_{em} (C_{\psi\gamma} + G_{\psi\gamma} C_{\psi\mu} F_\pi^2/m_\psi^2)^2}{64\pi^2 m_X^2 m_\psi^2 F_\pi^2} \left( Q^2 - m_\psi^2 \right)^3 \lambda^{1/2}(m_X, m_\psi, Q) \frac{Q}{Q^2 - m_\psi^2 - 2(m_X^2 - 4m_X^4 m_\psi^2 + m_\psi^4)Q^2 + (m_X^2 + m_\psi^2)Q^4}
\times \left[ (m_X^2 + m_\psi^2)(m_X^2 - m_\psi^2)^2 - 2(m_X^4 - 4m_X^2 m_\psi^2 + m_\psi^4)Q^2 + (m_X^2 + m_\psi^2)Q^4 \right]
\times \left| \frac{G_{X\psi\rho}}{Q^2 - m_\rho^2 + i m_\rho \Gamma_\rho} + \frac{G_{X\psi\omega} \sqrt{3} (\cos \theta_\omega + \sqrt{2} \sin \theta_\omega)}{Q^2 - m_\omega^2 + i m_\omega \Gamma_\omega} \right|^2.
\]
After integrating over the \(\pi^0\gamma\) invariant mass, the decay rate is
\[
\Gamma[X \to J/\psi \pi^0\gamma] = \left[ |G_{X\psi\omega}|^2 + 0.026 |G_{X\psi\rho}|^2 \right]
+ (0.163 \cos \phi + 0.215 \sin \phi) |G_{X\psi\omega}| |G_{X\psi\rho}| (3.24 \text{ keV}),
\]
where \(\exp(i\phi)\) is the relative phase between \(G_{X\psi\omega}\) and \(G_{X\psi\rho}\). The estimate of the ratio \(|G_{X\psi\omega}|^2/|G_{X\psi\rho}|^2\) in Eq. (36) suggests that the \(|G_{X\psi\omega}|^2\) term in Eq. (38) dominates. If this is the case, the branching fraction for the decay of \(X\) into \(J/\psi\pi^0\gamma\) should be smaller than that for \(J/\psi\pi^+\pi^-\pi^0\) by a factor of about 0.17.

**E. Decay into \(J/\psi\gamma\)**

Having chosen the transition amplitude in Eq. (29) so that it satisfies \(Q^\mu A_{\mu} = 0\), we can use vector meson dominance to calculate the partial width for the decay of \(X\) into \(J/\psi\gamma\). The T-matrix element is

\[
\mathcal{T}[X \to J/\psi\gamma] = G_{v\gamma}F_\pi^2e\left(\frac{G_{X\psi\rho} + G_{X\psi\omega}\cos\theta_v/\sqrt{3}}{m_\omega^2 - im_\omega\Gamma_\omega}\right)\varepsilon_\mu\alpha\beta Q^\mu\varepsilon^\alpha\psi^*\varepsilon^\beta\gamma, \tag{39}
\]

where \(Q\) is the 4-momentum of the photon. The value of the coupling constant \(G_{v\gamma}\) is given in Eq. (A2a). The result for the decay rate is

\[
\Gamma[X \to J/\psi\gamma] = \frac{\alpha em^2 G_{v\gamma}^2 F_\pi^4(m_\pi^2 + m_\omega^2)(m_X^2 - m_\psi^2)^3}{24m_X^5m_\psi^2} \times \left|\frac{G_{X\psi\rho} + G_{X\psi\omega}\cos\theta_v/\sqrt{3}}{m_\rho^2 - im_\rho\Gamma_\rho} + \frac{G_{X\psi\omega}}{m_\omega^2 - im_\omega\Gamma_\omega}\right|^2. \tag{40}
\]

If the widths in the vector meson propagators are neglected and if we use \(m_\rho \approx m_\omega\), the decay rate in Eq. (40) reduces to

\[
\Gamma[X \to J/\psi\gamma] = |G_{X\psi\rho}|^2(5.51\text{ keV}). \tag{41}
\]

Our estimate in Eq. (36) implies that \(|G_{X\psi\omega}|\) is much larger than \(|G_{X\psi\rho}|\). However, the larger magnitude of \(G_{X\psi\omega}\) is compensated by the vector meson mixing factor \(\cos\theta_v/\sqrt{3} = 0.30\), so the \(G_{X\psi\rho}\) and \(G_{X\psi\omega}\) terms may be equally important. Using the partial widths in Eqs. (32) and (34), we can relate the branching fractions for \(J/\psi\gamma\) to those for \(J/\psi\pi^+\pi^-\) and \(J/\psi\pi^+\pi^-\pi^0\):

\[
\text{Br}[X \to J/\psi\gamma] = 0.025 \text{Br}[X \to J/\psi\pi^+\pi^-] + 0.026 \text{Br}[X \to J/\psi\pi^+\pi^-\pi^0] + 0.050 \cos\phi \left(\text{Br}[X \to J/\psi\pi^+\pi^-] \text{Br}[X \to J/\psi\pi^+\pi^-\pi^0]\right)^{1/2}, \tag{42}
\]

where \(\exp(i\phi)\) is the relative phase between \(G_{X\psi\omega}\) and \(G_{X\psi\rho}\). This prediction is compatible with the measurements of the branching ratios in Eqs. (4) and (5) if the angle \(\phi\) is small.

**F. Factorization of short-distance decay rates**

The decay modes of the \(X(3872)\) can be classified into long-distance decays and short-distance decays. The long-distance decay modes are \(D^0\bar{D}^0\pi^0\) and \(D^0\bar{D}^0\gamma\), which proceed through the decay of a constituent \(D^*\) or \(\bar{D}^*\). These decays are dominated by a component of the wavefunction of the \(X\) in which the separation of the \(D\) and \(D^*\) is of order \(|a|\). These long-distance decays involve interesting interference effects between the \(D^0\bar{D}^0\) and
$D^0\bar{D}^0$ components of the wavefunction [11]. The short-distance decays are dominated by a component of the wavefunction in which the separation of the $D$ and $D^*$ is of order $m_\pi$ or smaller. Examples are the observed decay modes $J/\psi\pi^+\pi^-$, $J/\psi\pi^+\pi^-\pi^0$, and $J/\psi\gamma$.

Short-distance decays of the $X$ into a hadronic final state $H$ involve well-separated momentum scales. The $DD^*$ wavefunction of the $X$ involves the momentum scale $1/|a|$ set by the large scattering length. The transition of the $DD^*$ to $H$ involves momentum scales $m_\pi$ and larger. The separation of scales $|a| \gg 1/m_\pi$ can be exploited by using a factorization formula for the decay rate [19]. In limit $|a| \gg 1/m_\pi$, the leading term in the T-matrix element for the decay $X \to H$ can be separated into a short-distance factor and a long-distance factor:

$$T[X \to H] = A_{\text{short}}[X \to H] \times A_X.$$  \hspace{1cm} (43)

The short-distance factor $A_{\text{short}}$ in Eq. (43) has a well-behaved limit as $|a| \to \infty$. The long-distance factor $A_X$ is the universal amplitude given in Eq. (12). If the complex scattering length is parameterized as in Eq. (13), this factor is

$$A_X = \left(\sqrt{2\pi}/\mu\right) (\gamma_{re} + i\gamma_{im})^{1/2}.$$ \hspace{1cm} (44)

When applied to decays of $X$ into $J/\psi$ and light hadrons, the factorization formula in Eq. (43) implies that the coupling constants $G_{X\psi\rho}$ and $G_{X\psi\omega}$ have a long-distance factor $A_X$.

The factorization formula for the T-matrix element in Eq. (43) implies a factorization formula for the decay rate:

$$\Gamma[X \to H] = \Gamma_{\text{short}}[X \to H] \times |A_X|^2.$$ \hspace{1cm} (45)

The short-distance factor $\Gamma_{\text{short}}$ in Eq. (45) has a well-behaved limit as $|a| \to \infty$. Using the expressions in Eqs. (15) and (14) for the binding energy and the width of the molecule, the long-distance factor in Eq. (45) can be expressed as

$$|A_X|^2 = \sqrt{8\pi^2/\mu^3} \left[E_X + \Gamma^2_X/(16E_X)\right]^{1/2}.$$ \hspace{1cm} (46)

Predictions for the rates for short-distance decays of the $X$ can be obtained from models for low-energy hadrons in which the parameters have been tuned to obtain a small binding energy $E_X$, such as Swanson’s model [14]. In such models, calculations using the most straightforward numerical methods tend to become increasingly unstable as the binding energy is tuned toward 0, because the small binding energy results from a delicate cancellation. The factorization formula in Eqs. (45) and (46) can be useful for extrapolating the predictions of a model to other values of the binding energy $E_X$. In many models, it is difficult to take into account effects of the width $\Gamma_X$ of the molecule. Given the prediction of a model in which the width has been neglected, the factorization formula in Eqs. (45) and (46) can be used to take into account the nonzero width $\Gamma_X$ consistently.

In order to use the factorization formula in Eqs. (45) and (46) to extrapolate a partial width calculated using a model to other values of the binding energy and the width, the calculation must be carried out for small enough binding energy that the model is in the universal scaling regime where observables scale as powers of the binding energy. For example, the probabilities for components of the wavefunction other than $D^0\bar{D}^*$ and $D^*\bar{D}^0$ should
scale as \( E_X^{1/2} \). In Swanson’s model with \( E_X = 0.7 \) MeV, this universal scaling behavior is satisfied by the probability \( Z_{\psi\omega} \) but not by \( Z_{\psi\rho} \), as is evident in Fig. 1. The delayed onset of the universal behavior for the probability \( Z_{\psi\rho} \) can perhaps be attributed to the weaker coupling of \( X \) to isospin-1 states. In the next section, we will use Swanson’s result for \( Z_{\psi\omega} \) to estimate the coupling constant \( G_{X\psi\omega} \).

V. PARTIAL WIDTH FOR \( X \to J/\psi \pi^+\pi^-\pi^0 \)

The partial widths of the \( X \) calculated in Section IV are expressed in terms of unknown coupling constants \( G_{X\psi\rho} \) and \( G_{X\psi\omega} \). In this section, we use a simple 2-channel scattering model to show that \( |G_{X\psi\omega}| \) can be deduced from the probability \( Z_{\psi\omega} \) for the \( J/\psi \omega \) component of the \( X \). We then use the probability \( Z_{\psi\omega} \) in Swanson’s model to give a quantitative prediction for the partial width for the \( X \) to decay into \( J/\psi \pi^+\pi^-\pi^0 \).

A. Two-channel scattering model

Cohen, Gelman, and van Kolck have constructed a renormalizable effective field theory that describes two scattering channels with S-wave contact interactions [47]. We will refer to this model as the two-channel scattering model. An essentially equivalent model has been used to describe the effects of \( \Delta\Delta \) states on the two-nucleon system [48]. The parameters of this model can be tuned to produce a large scattering length in the lower energy channel. It can be used as a simple model for the effects on the \( D^0\bar{D}^{*0}/D^{*0}\bar{D}^0 \) system of other hadronic channels with nearby thresholds, such as \( J/\psi \rho \) and \( J/\psi \omega \).

The two-channel model of Ref. [47] describes two scattering channels with S-wave contact interactions only. We label the particles in the first channel 1a and 1b and those in the second channel 2a and 2b. We denote the reduced masses in the two channels by \( \mu \) and \( \mu_2 \). Renormalized observables in the 2-body sector are expressed in terms of 4 parameters: three interaction parameters \( a_{11}, a_{22}, \) and \( a_{12} = a_{21} \) with dimensions of length and the energy gap \( \Delta \) between the two scattering channels, which is determined by the masses of the particles:

\[
\Delta = m_{2a} + m_{2b} - (m_{1a} + m_{1b}).
\]

The scattering parameters in Ref. [47] were defined in such a way that \( a_{11} \) and \( a_{22} \) reduce in the limit \( a_{12} \to \pm \infty \) to the scattering lengths for the two channels. The truncated connected transition amplitude \( \mathcal{A}(E) \) for this coupled-channel system is a \( 2 \times 2 \) matrix that depends on the energy \( E \) in the center-of-mass frame. If that energy is measured relative to the threshold \( m_{1a} + m_{1b} \) for the first scattering channel, the inverse of the matrix \( \mathcal{A}(E) \) is

\[
\mathcal{A}(E)^{-1} = \frac{1}{2\pi} \left( \mu \left[ -1/a_{11} + \sqrt{-2\mu E} \right] \sqrt{\mu_2/a_{12}} \mu_2 \left[ -1/a_{22} + \sqrt{2\mu_2(\Delta - E)} \right] \right).
\]

The square roots are defined for negative real arguments by the prescription \( E \to E + i\epsilon \) with \( \epsilon \to 0^+ \). The explicit expressions for the 11 and 12 entries of this matrix are

\[
\mathcal{A}_{11}(E) = \frac{2\pi}{\mu} \left( -1/a_{11} + \sqrt{-2\mu E} - \frac{1}{a_{12}} \left[ -1/a_{22} + \sqrt{2\mu_2(\Delta - E)} \right]^{-1} \right)^{-1},
\]

\[
\mathcal{A}_{12}(E) = \frac{2\pi}{\sqrt{\mu_2}} \left( \frac{1}{a_{12}} - a_{12} \left[ -\frac{1}{a_{11}} + \sqrt{-2\mu E} \right] \left[ -\frac{1}{a_{22}} + \sqrt{2\mu_2(\Delta - E)} \right] \right)^{-1}.
\]
The amplitudes defined by (48) are for transitions between states with the standard non-relativistic normalizations. The transitions between states with the standard relativistic normalizations are obtained by multiplying by a factor $\sqrt{2m_i}$ for every particle in the initial and final state. The T-matrix element $T_{11}(p)$ for the elastic scattering of particles in the first channel with relative momentum $p$ is obtained by evaluating $A_{11}(E)$ at the energy $E = p^2/(2\mu)$. The scattering length is determined by the T-matrix element at $p = 0$: $T_{11}(0) = -2\pi a/\mu$. The inverse scattering length $1/a$ is therefore

$$\frac{1}{a} = \frac{1}{a_{11}} + \frac{1}{a_{12}^2} \left[ \sqrt{2\mu_2\Delta} - 1/a_{22} \right]^{-1}. \quad (50)$$

If the matrix $A(E)$ given by Eq. (48) has a pole on the physical sheet at $E = -\kappa^2/(2\mu)$, there is a bound state below the scattering threshold for the first channel with binding energy $E_X = \kappa^2/(2\mu)$. The binding momentum $\kappa$ satisfies

$$\kappa = \frac{1}{a_{11}} + \frac{1}{a_{12}^2} \left[ -1/a_{22} + \sqrt{2\mu_2\Delta + (\mu_2/\mu)\kappa^2} \right]^{-1}. \quad (51)$$

The momentum-space wavefunction $\psi(p)$ for the bound state is a column vector whose two components are the amplitudes for the bound state to consist of particles with relative momentum $p$ in the first and second channel, respectively. The wavefunction can be deduced from the behavior of $A(E)$ near the bound-state pole:

$$A(E) \rightarrow -\frac{1}{E + \kappa^2/(2\mu)} \begin{pmatrix} A_{X1} \\ A_{X2} \end{pmatrix} \otimes \begin{pmatrix} A_{X1} & A_{X2} \end{pmatrix}. \quad (52)$$

The components $A_{X1}$ and $A_{X2}$ of the column vector are the amplitudes for transitions from the bound state to particles in the first and second channel, respectively. They satisfy

$$\mu[-1/a_{11} + \kappa] A_{X1} + \left[ \sqrt{\mu_2/a_{12}} \right] A_{X2} = 0. \quad (53)$$

Because the only interactions in the two-channel model are contact interactions, the dependence of the wavefunction on the relative momentum of the constituents comes only from propagators. The wavefunction can be expressed in the form

$$\psi(p) = N \begin{pmatrix} 2\mu A_{X1} [p^2 + \kappa^2]^{-1} \\ 2\mu_2 A_{X2} [p^2 + 2\mu_2\Delta + (\mu_2/\mu)\kappa^2]^{-1} \end{pmatrix}, \quad (54)$$

where $N$ is a normalization constant. The normalization condition

$$\int \frac{d^3p}{(2\pi)^3} (|\psi_1(p)|^2 + |\psi_2(p)|^2) = 1 \quad (55)$$

can be expressed as $Z_1 + Z_2 = 1$, where $Z_1$ and $Z_2$ are the probabilities for the bound state to consist of the particles in the first and second channels, respectively. The probability $Z_1$ for the first channel is given by

$$Z_1^{-1} = 1 + \frac{(\mu_2/\mu)a_{12}^2(-1/a_{11} + \kappa)^2\kappa}{\sqrt{2\mu_2\Delta + (\mu_2/\mu)\kappa^2}}. \quad (56)$$
B. Two-channel model with large scattering length

In the two-channel model of Ref. [47], a large scattering length \( a \) in the first channel can be obtained by fine-tuning the parameters \( a_{11}, a_{22}, a_{12}, \) and \( \Delta \). The natural momentum scale \( \Lambda \) associated with low-energy elastic scattering in the first channel is set by the magnitudes of \( a_{11}^{-1}, a_{22}^{-1}, a_{12}^{-1} \), and \( (2\mu_2\Delta)^{1/2} \). There are various ways to tune the parameters so that \( |a| \) is large compared to \( \Lambda^{-1} \). For example, \( a \) can be tuned to \( \pm \infty \) by tuning the scattering parameter \( a_{11} \) to the critical value \( -a_{12}^{\sqrt{2\mu_2\Delta}} - 1/a_{22} \).

As \( a \) is tuned to be much larger than the natural momentum scale, the amplitude \( A_{11}(E) \) for \( |E| \ll \Lambda^2/(2\mu) \) approaches the universal expression given in Eq. (9). The solution to Eq. (51) for the binding momentum \( \kappa \) approaches \( 1/a \), so if \( a > 0 \), there is a bound state with the universal binding energy in Eq. (10). The first component of the wavefunction in Eq. (54) approaches the universal expression in Eq. (11), while the probability of the second component approaches 0 as \( 1/a \). The amplitude \( A_{X1} \) for the transition from the bound state to particles in the first channel also approaches the universal amplitude \( A_X \) in Eq. (12).

There are also universal features associated with transitions from the bound state to particles in the second channel. If \( |a| \gg \Lambda^{-1} \), the leading term in the amplitude for the transition of the weakly-bound state \( X \) to particles in the second channel is

\[
A_{X2} = -\frac{\sqrt{\mu/\mu_2}}{a_{12}} \left[ \frac{\sqrt{2\mu_2\Delta}}{1/a_{22}} \right]^{-1} A_X,
\]

where \( A_X \) is the universal amplitude given in Eq. (12). This equation is a factorization formula that expresses the transition amplitude as the product of a short-distance factor and the universal long-distance factor \( A_X \). Using Eq. (56), the probability \( Z_2 = 1 - Z_1 \) for the bound state to consist of particles in the second channel reduces to

\[
Z_2 = \left( \frac{\mu_2/\mu}{a_{12}^2 \sqrt{2\mu_2\Delta}} \right) \left[ \frac{\sqrt{2\mu_2\Delta}}{1/a_{22}} \right]^{-2} \frac{1}{a}.
\]

Note that the probability \( Z_2 \) differs from \( |A_{X2}|^2 \) only by kinematic factors:

\[
|A_{X2}|^2 = \sqrt{8\pi^2 \Delta/\mu_3^2} Z_2.
\]

This relation also follows directly from the wavefunction in Eq. (54) if we use the fact that the normalization factor \( N \) approaches 1 as \( a \to \infty \). Thus the relation between the probability and the transition amplitude in Eq. (59) is not specific to the 2-channel model. It applies more generally to any 2-particle component of the bound state whose wavefunction can be approximated by \( (p^2 + 2\mu_2\Delta)^{-1} \), where \( \Delta \) is the energy gap. It requires only that \( \Delta \) is small enough that the interaction in that channel can be approximated by an S-wave contact interaction at momenta comparable to \( \sqrt{2\mu_2\Delta} \).

C. Partial width into \( J/\psi \pi^+ \pi^- \pi^0 \)

We can use results from Swanson’s model to estimate \( |G_{X\omega}| \), thereby determining the unknown constant in the expression in Eq. (34) for the partial width for \( X \to J/\psi \pi^+ \pi^- \pi^0 \). The relativistic amplitude for the transition from \( X \) to \( J/\psi \omega \) is given by the contraction

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of the amplitude $A_\mu$ in Eq. (29) with a polarization vector for the $\omega$. The corresponding nonrelativistic amplitude $A_{X\psi\omega}$ is the analog of the transition amplitude $A_{X2}$ in Eq. (57) for the 2-channel model. In the rest frame of the $X$, the relativistic amplitude differs from the nonrelativistic amplitude by a factor of $\sqrt{2m_i}$ for every external particle:

$$\epsilon_\omega^\mu A_\mu[X \rightarrow J/\psi \omega] = (8m_X m_\psi m_\omega)^{1/2} A_{X\psi\omega}.$$  (60)

Using the expression for the amplitude $A_\mu$ in Eq. (29) and the fact that the rest frame of the $X$ is almost identical to that of the $\omega$, the left side of Eq. (60) is

$$\epsilon_\omega^\mu A_\mu[X \rightarrow J/\psi \omega] = G_{X\psi\omega} m_\omega \epsilon_X \cdot (\epsilon_\psi \times \epsilon_\omega)^*.$$  (61)

The transition amplitude $A_{X\psi\omega}$ on the right side of Eq. (60) must have the same dependence on the polarization vectors of the $J/\psi$ and $\omega$. There are two independent pairs of spin states for $J/\psi$ and $\omega$ that couple to any given spin state of $X$. If the analog of the factorization formula in Eq. (59) is summed over the spin states of the $J/\psi$ and $\omega$, it gives

$$\sum_{\text{spins}} |A_{X\psi\omega}|^2 = \sqrt{8\pi^2 \Delta_\psi \omega / \mu_{\psi\omega}^3} Z_{\psi\omega},$$  (62)

where $\Delta_\psi \omega$ is the energy gap in Eq. (20c), $\mu_{\psi\omega}$ is the reduced mass of the $J/\psi$ and $\omega$, and $Z_{\psi\omega}$ is the probability for the $J/\psi \omega$ component of $X$. Squaring both sides of Eq. (60) and summing over the spin states of $J/\psi$ and $\omega$, we get

$$2m_\omega^2 |G_{X\psi\omega}|^2 = 16\pi m_X (m_\psi + m_\omega) \sqrt{2\Delta_\psi \omega / \mu_{\psi\omega}^3} Z_{\psi\omega}.$$  (63)

Inserting Swanson’s result $Z_{\psi\rho} = 9.6\%$ for $E_X = 0.7$ MeV and using the factorization formula in Eqs. (45) and (46), this reduces to

$$|G_{X\psi\rho}|^2 = 9.59 \left( \frac{E_X + \Gamma_X^2/(16E_X)}{0.7 \text{ MeV}} \right)^{1/2}. $$  (64)

Inserting the result into the expression in Eq. (34), we get a quantitative result for the partial width:

$$\Gamma[X \rightarrow J/\psi \pi^+ \pi^- \pi^0] = (222 \text{ keV}) \left( \frac{E_X + \Gamma_X^2/(16E_X)}{1 \text{ MeV}} \right)^{1/2}. $$  (65)

We can use the result in Eq. (65) to set a lower bound on the partial width into $J/\psi \pi^+ \pi^- \pi^0$. As a function of the binding energy $E_X$, the right side of Eq. (65) is minimized at $E_X = \Gamma_X/4$. The lower bound on the width is $\Gamma_X > 2\Gamma[D^{*0}] = 136 \pm 32$ keV. Thus the lower bound on the partial width into $J/\psi \pi^+ \pi^- \pi^0$ in Swanson’s model is about 58 keV.

As is evident in Fig. 1, Swanson did not calculate the probability $Z_{\psi\rho}$ for the $J/\psi \rho$ component of $X$ for a binding energy small enough to be in the scaling region where $Z_{\psi\rho}$ scales like $E_X^{1/2}$. If he had, we could use an equation analogous to Eq. (63) to determine $|G_{X\psi\rho}|$. If we assume that the smallest binding energy considered by Swanson is close to the scaling region, we can use his value $Z_{\psi\rho} = 0.86\%$ for $E_X = 0.7$ MeV to estimate $|G_{X\psi\rho}|$. 

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In the analog of Eq. (63), we should set $\Delta_\psi = \Delta_\omega$ rather than using the value $\Delta_\rho$ in Eq. (20b), because Swanson set $m_\rho = m_\omega$ in his calculation. The resulting estimate is

$$|G_{X\psi}\rangle^2 \approx 0.86 \left( \frac{E_X + \Gamma_X/16E_X}{0.7 \text{MeV}} \right)^{1/2}. \quad (66)$$

We can insert this estimate into Eq. (32) to get an estimate of the partial width for decay into $J/\psi \pi^+ \pi^-$. We can also insert this estimate of $|G_{X\psi}\rangle^2$ and the value of $|G_{X\omega}\rangle^2$ from Eq. (64) into Eqs. (38) and (41) to get ranges of estimates of the partial widths for the decays into $J/\psi \pi^0 \gamma$ and $J/\psi \gamma$. The ranges arise from the unknown relative phase between $G_{X\omega}$ and $G_{X\rho}$.

VI. SUMMARY

Evidence is accumulating that the $X(3872)$ is a loosely-bound S-wave molecule corresponding to a $C = +$ superposition of $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$. Because its binding energy is small compared to the natural energy scale associated with pion exchange, this molecule has universal properties that are completely determined by the large scattering length $a$ in the $C = +$ channel of $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$.

We have analyzed the decays of $X$ into $J/\psi$ plus light hadrons under the assumption that $X$ is a $DD^*$ molecule and that these decays proceed through transitions to $J/\psi \rho$ and $J/\psi \omega$. The differential decay rates were calculated in terms of unknown coupling constants $G_{X\psi\rho}$ and $G_{X\psi\omega}$ by using an effective lagrangian that reproduces the decays of the light vector mesons. The dependence on the unknown coupling constants enters only through multiplicative factors, so the angular distributions are completely determined.

Quantitative predictions of the partial widths for the decays of $X$ into $J/\psi$ plus light hadrons require numerical values for the coupling constants $G_{X\psi\rho}$ and $G_{X\psi\omega}$. We pointed out that the dependence of these coupling constants on the binding energy $E_X$ and the total width $\Gamma_X$ are determined by factorization formulas. We showed how $|G_{X\psi\omega}\rangle^2$ could be determined from the probability $Z_{\psi\omega}$ for the $J/\psi \omega$ component of $X$ in Swanson’s model. We used this result to give a quantitative prediction for the partial width for $X \to J/\psi \pi^+ \pi^- \pi^0$ as a function of $E_X$ and $\Gamma_X$.

APPENDIX A: DECAY AMPLITUDES FOR VECTOR MESONS

In this appendix, we present an updated determination of the coupling constants in the effective lagrangian for the light pseudoscalar and vector mesons that was used in Ref. [46] to calculate the semileptonic branching fractions for the $\tau$ lepton. The same effective lagrangian is used in Section IV to calculate the decay rates of the $X$ into $J/\psi$ and light hadrons.

The pion decay constant $F_\pi = 93$ MeV and the hadron masses have all been determined accurately [45]. The other parameters in the effective lagrangian can be determined from the partial widths for decays of $\rho^0$, $\rho^\pm$, and $\omega$ given in Table I. The most useful combinations of the parameters in the amplitudes for the decays of the vector mesons into pions in Eqs. (24)
TABLE I: Inputs that are used to determine the coupling constants in the vector meson decay amplitudes. The partial widths are taken from Ref. [45].

| Decay mode | Partial width |
|------------|--------------|
| $\rho^0 \rightarrow \pi^+\pi^-$ | $150.3 \pm 1.6$ MeV |
| $\rho^0 \rightarrow e^+e^-$ | $7.02 \pm 0.11$ keV |
| $\rho^- \rightarrow \pi^-\gamma$ | $67.6 \pm 7.5$ keV |
| $\omega \rightarrow e^+e^-$ | $0.60 \pm 0.02$ keV |
| $\omega \rightarrow \pi^0\gamma$ | $0.76 \pm 0.05$ MeV |
| $\omega \rightarrow \pi^0\mu^+\mu^-$ | $0.82 \pm 0.20$ keV |
| $\omega \rightarrow \pi^+\pi^-\pi^0$ | $7.56 \pm 0.093$ MeV |

and (25) are

\[ G_{v\pi\pi} = 11.99 \pm 0.06, \]  
\[ C_{v3\pi} + G_{v\pi\pi}C_{v\pi\pi}F_\pi^2/m_v^2 = (8.03 \pm 0.48)/(16\pi^2), \]  
\[ G_{v\pi\pi}C_{v\pi\pi}F_\pi^2/m_v^2 = (10.2 \pm 1.3)/(16\pi^2). \]  

The coupling constant $G_{v\gamma}$ associated with vector meson dominance and the most useful combinations of parameters in the amplitudes for the radiative decays of the vector mesons in Eqs. (27) are

\[ G_{v\gamma} = 14.01 \pm 0.11, \]  
\[ C_{v\pi\gamma} + G_{v\gamma}C_{v\pi\pi}F_\pi^2/m_v^2 = (7.99 \pm 0.45)/(16\pi^2), \]  
\[ G_{v\gamma}C_{v\pi\pi}F_\pi^2/m_v^2 = (11.9 \pm 1.5)/(16\pi^2). \]  

The vector meson mixing angle is given by

\[ \cos \theta_v = 0.51 \pm 0.01. \]  

Another function of $\theta_v$ that is often encountered is $\cos \theta_v + \sqrt{2} \sin \theta_v \approx 1.73 \pm 0.01$. The errors in the parameters in Eqs. (A1), (A2), and (A3) are determined using the uncertainties in the measurements of the vector meson decay widths only. The uncertainties in the hadron masses and the pion decay constant are negligible in comparison. Variations in the parameters associated with $U(3) \times U(3)$ symmetry breaking are neglected in this analysis.

The inputs that were used to determine the parameters in Eqs. (A1), (A2), and (A3) are listed in Table I. Following Ref. [46], we determine the parameters by the following steps:

1. The coupling constant $G_{v\pi\pi}$ in Eq. (A1a) is determined from the partial width for $\rho \rightarrow \pi^+\pi^-$:

\[ \Gamma[\rho \rightarrow \pi^+\pi^-] = \frac{G_{v\pi\pi}^2 m_\rho}{192\pi} \left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{3/2}. \]  

2. The coupling constant $G_{v\gamma}$ in Eq. (A2a) is determined from the partial width for $\rho \rightarrow e^+e^-$:

\[ \Gamma[\rho \rightarrow e^+e^-] = \frac{4\pi\alpha_{em}^2 G_{v\gamma}^2 F_\pi^4}{3m_\rho^3}. \]  

\[ 1 \text{ The cosine of the angle } \theta_v \text{ here is the sine of the vector meson mixing angle used in Ref. [46].} \]
3. The combination of parameters in Eq. (A2b) is determined from the partial width for 
\( \rho^- \rightarrow \pi^- \gamma \):

\[
\Gamma[\rho^- \rightarrow \pi^- \gamma] = \frac{2\alpha_{em} m_\rho^3}{27 F_\pi^2} \left( C_{\pi\pi\gamma} + \frac{G_{\rho\gamma} C_{\pi\pi\gamma} F_\pi^2}{m_\rho^2} \right)^2 (1 - \frac{m_\pi^2}{m_\rho^2})^3. \tag{A6}
\]

4. The combination of parameters \( G_{\rho\gamma} C_{\pi\pi\gamma} F_\pi^2 / m_\rho^2 \) in Eq. (A2c) is determined from the ratio of the partial widths for \( \omega \rightarrow \pi^0 \mu^+ \mu^- \) and \( \omega \rightarrow \pi^0 \gamma \). The possibility of a relative phase between \( C_{\pi\pi\gamma} \) and \( G_{\rho\gamma} C_{\pi\pi\gamma} F_\pi^2 / m_\rho^2 \) is ignored. The partial width for \( \omega \rightarrow \pi^0 \mu^+ \mu^- \) is

\[
\Gamma[\omega \rightarrow \pi^0 \mu^+ \mu^-] = \frac{1}{256\pi^3 m_\omega^5} \int ds_{12} \int ds_{23} \sum |A[\omega \rightarrow \pi^0 \mu^+ \mu^-]|^2. \tag{A7}
\]

The squared amplitude, averaged over initial spin states and summed over final spin states, is

\[
\sum |A[\omega \rightarrow \pi^0 \mu^+ \mu^-]|^2 = \frac{128\pi^2 \alpha_{em}^2}{9 F_\pi^2} (\cos \theta_v + \sqrt{2} \sin \theta_v)^2 \times \left[ (s_{23}^2 + 4m_\mu^2) \left( (m_\omega^2 - s_{23} - m_\pi^2)^2 - 4m_\pi^2 s_{23} + s_{23} (s_{12} - s_{31})^2 \right) \right] \times \frac{1}{s_{23}^2} \left| C_{\pi\pi\gamma} + \frac{G_{\rho\gamma} C_{\pi\pi\gamma} F_\pi^2}{m_\rho^2} (1 - f_\rho (s_{23})) \right|^2, \tag{A8}
\]

where \( s_{12}, s_{23}, \) and \( s_{31} \) are the squares of the invariant masses of the \( \pi^0 \mu^+, \mu^+ \mu^-, \) and \( \mu^- \pi^0 \), respectively. The partial width for \( \omega \rightarrow \pi^0 \gamma \) is

\[
\Gamma[\omega \rightarrow \pi^0 \gamma] = 3(\cos \theta_v + \sqrt{2} \sin \theta_v)^2 \frac{m_\omega^3 (1 - \frac{m_\pi^2}{m_\rho^2})^3}{m_\omega^3 (1 - \frac{m_\pi^2}{m_\rho^2})^3} \Gamma[\rho^- \rightarrow \pi^- \gamma], \tag{A9}
\]

where \( \Gamma[\rho^- \rightarrow \pi^- \gamma] \) is given in Eq. (A6). Note that the factor \( (\cos \theta_v + \sqrt{2} \sin \theta_v)^2 \) cancels in the ratio of Eqs. (A7) and (A9).

5. The combination of parameters \( G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2 / m_\nu^2 \) appearing in Eq. (A1c) is determined by multiplying the combination of parameters in Eq. (A2c) by the ratio \( G_{\nu\pi\pi} / G_{\nu\gamma} \) obtained from Eqs. (A1a) and (A2a).

6. The combination of parameters in Eq. (A1b) is determined from the ratio of the partial widths for \( \omega \) to decay into \( \pi^+ \pi^- \pi^0 \) and \( \pi^0 \gamma \) and from the value of the combination of parameters in Eq. (A1c). The possibility of a relative phase between \( C_{\nu\pi\pi} \) and \( G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2 / m_\nu^2 \) is ignored. The partial width for \( \omega \rightarrow \pi^0 \gamma \) is given in Eq. (A9). The partial width for \( \omega \rightarrow \pi^+ \pi^- \pi^0 \) is

\[
\Gamma[\omega \rightarrow \pi^+ \pi^- \pi^0] = \frac{1}{256\pi^3 m_\omega^5} \int ds_{12} \int ds_{23} \sum |A[\omega \rightarrow \pi^+ \pi^- \pi^0]|^2. \tag{A10}
\]

The squared amplitude, averaged over the spin states of \( \omega \), is

\[
\sum |A[\omega \rightarrow \pi^+ \pi^- \pi^0]|^2 = \frac{4(\cos \theta_v + \sqrt{2} \sin \theta_v)^2}{F_\pi^6} (s_{12} s_{23} s_{31} - m_\pi^2 (m_\omega^2 - m_\pi^2)^2) \times \left| C_{\nu\pi\pi} + \frac{G_{\nu\pi\pi} C_{\nu\pi\pi} F_\pi^2}{m_\nu^2} (1 - \frac{1}{3} [f_\rho (s_{12}) + f_\rho (s_{23}) + f_\rho (s_{31})]) \right|^2. \tag{A11}
\]

Note that the factor \( (\cos \theta_v + \sqrt{2} \sin \theta_v)^2 \) cancels in the ratio of Eqs. (A10) and (A9).
7. Finally, the cosine of the vector meson mixing angle in Eq. (A3) is determined from
the ratio of the partial widths for $\omega \rightarrow e^+e^-$ and $\rho^0 \rightarrow e^+e^-$:

$$\Gamma[\omega \rightarrow e^+e^-] = \frac{\cos^2 \theta_v m^3_{\rho}}{3m^3_{\omega}} \Gamma[\rho^0 \rightarrow e^+e^-].$$

(A12)

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[1] S. K. Choi et al. [Belle Collaboration], Phys. Rev. Lett. 91, 262001 (2003). [arXiv:hep-ex/0309032].
[2] D. Acosta et al. [CDF II Collaboration], Phys. Rev. Lett. 93, 072001 (2004). [arXiv:hep-ex/0312021].
[3] V. M. Abazov et al. [D0 Collaboration], Phys. Rev. Lett. 93, 162002 (2004) [arXiv:hep-ex/0405004].
[4] B. Aubert et al. [BABAR Collaboration], Phys. Rev. D 71, 071103 (2005) [arXiv:hep-ex/0406022].
[5] S. L. Olsen [Belle Collaboration], Int. J. Mod. Phys. A 20, 240 (2005) [arXiv:hep-ex/0407033].
[6] T. Barnes and S. Godfrey, Phys. Rev. D 69, 054008 (2004). [arXiv:hep-ph/0311162].
[7] E. J. Eichten, K. Lane and C. Quigg, Phys. Rev. D 69, 094019 (2004). [arXiv:hep-ph/0401210].
[8] C. Quigg, arXiv:hep-ph/0310397.
[9] C. Quigg, Nucl. Phys. Proc. Suppl. 142, 87 (2005) [arXiv:hep-ph/0407124].
[10] N.A. Tornqvist, arXiv:hep-ph/0308277; Phys. Lett. B 590, 209 (2004).
[11] M. B. Voloshin, Phys. Lett. B 579, 316 (2004). [arXiv:hep-ph/0309307].
[12] C. Y. Wong, Phys. Rev. C 69, 055202 (2004). [arXiv:hep-ph/0311088].
[13] E. Braaten and M. Kusunoki, Phys. Rev. D 69, 074005 (2004). [arXiv:hep-ph/0311147].
[14] E. S. Swanson, Phys. Lett. B 588, 189 (2004). [arXiv:hep-ph/0311299].
[15] E. Braaten, M. Kusunoki and S. Nussinov, Phys. Rev. Lett. 93, 162001 (2004). [arXiv:hep-ph/0401461].
[16] E. S. Swanson, Phys. Lett. B 598, 197 (2004) [arXiv:hep-ph/0406080].
[17] M. B. Voloshin, Phys. Lett. B 604, 69 (2004) [arXiv:hep-ph/0408321].
[18] E. Braaten and M. Kusunoki, Phys. Rev. D 71, 074005 (2005) [arXiv:hep-ph/0412268].
[19] E. Braaten and M. Kusunoki, arXiv:hep-ph/0506087.
[20] M. T. AlFiky, F. Gabbiani and A. A. Petrov, arXiv:hep-ph/0506141.
[21] D. V. Bugg, Phys. Lett. B 598, 8 (2004). [arXiv:hep-ph/0406293].
[22] D. V. Bugg, Phys. Rev. D 71, 016006 (2005) [arXiv:hep-ph/0410168].
[23] J. Vijande, F. Fernandez and A. Valcarce, Int. J. Mod. Phys. A 20, 702 (2005) [arXiv:hep-ph/0407136].
[24] F. E. Close and S. Godfrey, Phys. Lett. B 574, 210 (2003) [arXiv:hep-ph/0305285].
[25] B. A. Li, Phys. Lett. B 605, 306 (2005) [arXiv:hep-ph/0410264].
[26] K. K. Seth, Phys. Lett. B 612, 1 (2005) [arXiv:hep-ph/0411122].
[27] L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D 71, 014028 (2005) [arXiv:hep-ph/0412098].
[28] F. E. Close and P. R. Page, Phys. Lett. B 578, 119 (2004). [arXiv:hep-ph/0309253].
[29] S. Pakvasa and M. Suzuki, Phys. Lett. B 579, 67 (2004). [arXiv:hep-ph/0309294].
[30] J. L. Rosner, Phys. Rev. D 70, 094023 (2004) [arXiv:hep-ph/0408334].
[31] T. Kim and P. Ko, Phys. Rev. D 71, 034025 (2005) [arXiv:hep-ph/0405265].
[32] K. Abe, arXiv:hep-ex/0505038.
[33] K. Abe, arXiv:hep-ex/0505037.
[34] K. Abe et al. [Belle Collaboration], Phys. Rev. Lett. 93, 051803 (2004). [arXiv:hep-ex/0307061].
[35] K. Abe et al. [Belle Collaboration], arXiv:hep-ex/0408116.
[36] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 93, 041801 (2004). [arXiv:hep-ex/0402025].
[37] Z. Metreveli et al. [CLEO Collaboration], arXiv:hep-ex/0408057.
[38] C. Z. Yuan, X. H. Mo and P. Wang, Phys. Lett. B 579, 74 (2004). [arXiv:hep-ph/0310261].
[39] M. Bander, G. L. Shaw, P. Thomas and S. Meshkov, Phys. Rev. Lett. 36, 695 (1976).
[40] M. B. Voloshin and L. B. Okun, JETP Lett. 23, 333 (1976).
[41] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. Lett. 38, 317 (1977).
[42] S. Nussinov and D. P. Sidhu, Nuovo Cim. A 44, 230 (1978).
[43] N. A. Tornqvist, Z. Phys. C 61, 525 (1994). [arXiv:hep-ph/9310247].
[44] E. Braaten and H. W. Hammer, arXiv:cond-mat/0410417.
[45] S. Eidelman et al. [Particle Data Group Collaboration], Phys. Lett. B 592, 1 (2004).
[46] E. Braaten, R. J. Oakes and S. M. Tse, Int. J. Mod. Phys. A 5, 2737 (1990); Phys. Rev. D 36, 2188 (1987).
[47] T. D. Cohen, B. A. Gelman and U. van Kolck, Phys. Lett. B 588, 57 (2004). [arXiv:nucl-th/0402054].
[48] M. J. Savage, Phys. Rev. C 55, 2185 (1997) [arXiv:nucl-th/9611022].