Topological Hall effect and Berry phase in magnetic nanostructures

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We discuss the anomalous Hall effect in a two-dimensional electron gas subject to a spatially varying magnetization. This topological Hall effect (THE) does not require any spin-orbit coupling, and arises solely from Berry phase acquired by an electron moving in a smoothly varying magnetization.

We propose an experiment with a structure containing 2D electrons or holes of diluted magnetic semiconductor subject to the stray field of a lattice of magnetic nanocylinders. The striking behavior predicted for such a system (of which all relevant parameters are well known) allows to observe unambiguously the THE and to distinguish it from other mechanisms.

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After half a century of theoretical efforts, the Hall effect of ferromagnets (usually called anomalous or extraordinary Hall effect) remains a puzzling and controversial topic. Until recently, it was considered that it originates from the combined effect of exchange and spin-orbit (SO) interactions. Two mechanisms of anomalous Hall effect (AHE) have been identified (the skew scattering [1, 2] and side-jump [3]) and studied thoroughly [4]. Recently, a new point of view has been proposed [5], in which the AHE is expressed in terms of a Berry curvature in momentum space. However, one should note that a generally accepted theory treating on an equal footing all the above mentioned contributions to the SO-induced AHE is still missing.

Recently, it has been suggested that (in addition to the above mentioned SO-induced mechanism) a new mechanism may give rise to a non-vanishing Hall effect in ferromagnets having a topologically non-trivial (chiral) spin-texture, such as manganites or pyrochlore-type compounds [6]. To distinguish this mechanism from the SO-based mechanism, we shall refer to it hereafter as the topological Hall effect (THE).

Several theoretical papers have been devoted to the THE. In order to explain the AHE observed in manganites, a model of 3D ferromagnet with thermally excited skyrmion strings (topological dipoles) has been proposed [7], showing that a THE can be induced by the Berry phase [8] related to the spatial variation of magnetization in the vicinity of the string. The case of disordered ferromagnets in the limit of small exchange splitting has been addressed in Ref. [9]. In both cases, in order to get a nonvanishing THE may be obtained, as discussed for a 2D kagomé lattice or a 3D pyrochlore lattice [10].

All the above mentioned discussions of the THE concern systems with spin-chirality at the microscopic scale (e.g., pyrochlore lattice) or due to skyrmion-strings. In both cases, quantitative experimental information on the chirality is not easily available. Furthermore, the SO-mechanism is usually also present, which makes complicated the quantitative interpretation of the observations.

In the present Letter, we propose to investigate the THE in nanostructures, namely in a 2D electron (or hole) gas, in which a well-controlled artificial chirality can be induced from the stray field of a lattice of magnetic nanocylinders. The great advantage of such a model system is that, in contrast to the above mentioned cases, all relevant parameters are well known and (to some extent) adjustable. We show that it is possible to obtain a significant net topological field, and that the latter has a characteristic variation with respect to a (uniform) external field, providing an unambiguous signature of the THE. Finally, we suggest a system appropriate for an experimental check of our theory.

We start from a model of 2DEG in a smoothly varying magnetization $\mathbf{M}(\mathbf{r})$. The Hamiltonian has the following form

$$
H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}^2} - g \mathbf{\sigma} \cdot \mathbf{M}(\mathbf{r}),
$$

where $g$ is the coupling constant and $\mathbf{\sigma}$ is the vector of Pauli matrices. We assume that the amplitude of magnetization is constant, $\mathbf{M}(\mathbf{r}) = M \mathbf{n}(\mathbf{r})$, and that the 3D unit vector $\mathbf{n}(\mathbf{r})$ is a slowly varying function of coordinates.

We use a gauge transformation $T(\mathbf{r})$, which makes the quantization axis oriented along vector $\mathbf{n}(\mathbf{r})$ at each point $\mathbf{r}$. It transforms the last term in (1) as $T^\dagger(\mathbf{r}) [(\mathbf{\sigma} \cdot \mathbf{n}(\mathbf{r})) T(\mathbf{r}) = \sigma_z$, corresponding to a local rotation of the quantization axis from z axis to the axis along $\mathbf{n}(\mathbf{r})$.

The transformed Hamiltonian describes the electrons moving in a (spinor) gauge potential $\mathbf{A}(\mathbf{r})$,

$$
H' = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial \mathbf{r}} - \frac{ie}{\hbar c} \mathbf{A}(\mathbf{r}) \right)^2 - gM \sigma_z,
$$

where $A_i(\mathbf{r}) = 2\pi i \phi_0 T^\dagger(\mathbf{r}) \partial_i T(\mathbf{r})$, $\phi_0 = hc/e$ is the
flux quantum, and \( i = x, y \). For convenience, we define the gauge potential \( \mathbf{A}(\mathbf{r}) \) to have the same dimension as the electromagnetic vector potential. The components of \( \mathbf{A}(\mathbf{r}) \) can be found easily using an explicit form of \( T(\mathbf{r}) \).

Hamiltonian (2) with the spinor \( \mathbf{A}(\mathbf{r}) \) contains terms inducing transitions between the spin-polarized states. We consider the case when spin-flip transitions can be neglected, i.e., where the spin adiabatically follows \( \mathbf{n}(\mathbf{r}) \). It corresponds to the condition

\[
\lambda \equiv \frac{\langle \varepsilon_F / \varepsilon_0 \rangle (k_F \xi)^{-1}}{1},
\]

where \( \xi \) is a characteristic length of the variation of \( \mathbf{n}(\mathbf{r}) \), \( \varepsilon_F \) is the Fermi energy, and \( \varepsilon_0 = 2gM \) is the spin splitting.

We also assume that the 2DEG is half-metallic with the Fermi level located in the spin-up subband. Then we can neglect the spin-down electrons, and we obtain the following effective Hamiltonian for spinless electrons:

\[
\hat{H} = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial \mathbf{r}} - i\frac{e}{\hbar} \mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}),
\]

where

\[
a_i(\mathbf{r}) = -\frac{\pi \phi_0}{2} \left( n_x \partial_x n_y - n_y \partial_x n_x \right) \frac{1 + n_z}{1 + n_z},
\]

the potential \( V(\mathbf{r}) = (\hbar^2/8m) (\partial_i n_\mu)^2 \) results from the second order in \( \mathbf{A}(\mathbf{r}) \) terms, and \( \mu = x, y, z \). For spin-down electrons, the sign of the gauge field \( \mathbf{a}(\mathbf{r}) \) in Eq. (4) is reversed.

The topological field is defined as \( B_\mathbf{t} = \partial_x a_y - \partial_y a_x \). It acts on the electrons within the spin polarized subband like the ordinary magnetic field, and in particular gives rise to a Lorentz-type force [12]. Using (4) we find

\[
B_\mathbf{t} = -\frac{\phi_0}{4\pi} \epsilon_{\mu\nu\lambda} n_\mu \left( \partial_x n_\nu \right) \left( \partial_y n_{\lambda} \right),
\]

where \( \epsilon_{\mu\nu\lambda} \) is the unit antisymmetric tensor. The integral over the area \( S_0 \) enclosed by an arbitrary contour \( L_0 \)

\[
\Omega(L) = \frac{1}{2} \int_{S_0} d^2 r \epsilon_{\mu\nu\lambda} n_\mu \left( \partial_x n_\nu \right) \left( \partial_y n_{\lambda} \right)
\]

is the Berry phase calculated as the spherical angle spanned by an area \( S \) inside the contour \( L \) in \( \mathbf{n} \)-space. This result from the mapping of the contour \( L_0 \) onto the contour \( L \) in the mapping space \( S_2 \). In the 2D case with a constant magnetization at infinity, we can compactify 2D plane to a sphere \( S_2 \), and the quantity \( \Omega \equiv \Omega/4\pi \) calculated with the integral (6) over \( S_2 \) is the topological invariant corresponding to the number of covering the mapping space.

We can consider different topologically nontrivial types of the distribution of magnetization field in 2D plane like, for example, separated magnetic domains embedded into homogeneous magnetization. Each domain creates a unit flux of topological field \(+\phi_0\) or \(−\phi_0\), where the sign is related to the magnetic polarization inside the domain with respect to the ambient. The important point is that this topological flux does not depend on a specific shape or a size of the domain. Besides, it does not depend on whether the domain is separated by the domain wall of Bloch or Néel type. The same result will be also for the skyrmion - a topological excitation similar to the circular magnetic domain with a Néel domain wall.

Next we consider a periodic magnetization created by a regular 2D lattice of the elements with a nontrivial topology. For example, it can be a square or triangular lattice of circular domains. Each of the domains creates the same magnetic flux. We can estimate the THE for this system using the formula for the Hall conductivity in the average topological field \( \mathbf{B}_t \):

\[
\sigma_{xy}(\mathbf{r}) = \frac{2\pi \hbar e^2 n_\uparrow \tau_\uparrow}{m^2 \phi_0} \mathbf{B}_t, \tag{7}
\]

where \( n_\uparrow \) and \( \tau_\uparrow \) are the concentration and momentum relaxation time of spin-up electrons, respectively. If the system is not half-metallic, there is also a corresponding additional contribution to \( \sigma_{xy} \), which differs from (7) by sign.

Using (7) and the standard formula for conductivity, \( \sigma_{xx} = n_\uparrow e^2 \tau_\uparrow / m \), in the case of unfilled spin-up subband we find

\[
\rho_{xy} \simeq \frac{\sigma_{xy}}{\sigma_{xx}} = \frac{\mathbf{B}_t}{\mathbf{B}_t}. \tag{8}
\]

The periodic magnetization can be realized using a structure with a lattice of magnetic nanocylinders [13] on top of 2D electron gas. To observe the THE we suggest using II-VI diluted magnetic semiconductor (DMS), as shown schematically in Fig. 1. The stray field from the magnetic nanocylinders, penetrates into semiconductor and polarizes the magnetic impurities, creating a large spin splitting of electrons or holes in the DMS [14]. This effect is much stronger than the direct diamagnetic or Zeeman actions of external magnetic field. The main
idea is to design a system, which provides a non-uniform magnetic field of zero average within the semiconductor but induces a topological field of nonzero average.

We have calculated the stray field and topological field for a triangular lattice of nanocylinders (of magnetization $M_s$ along the $z$-axis), using realistic values of parameters for the nanocylinder lattice. Namely, we take a lattice constant of 100 nm, a cylinder radius of 37 nm (i.e., a filling ratio of 50 percent), and the gap $D$ between the lattice and semiconductor equal to 20 nm. The distribution of $z$-component of the dipolar field (normal to the semiconductor surface) is shown in Fig. 2. The corresponding topological field calculated from Eq. (5) is presented in Fig. 3. For Fe cylinders, the dipolar field $B$ inside the semiconductor is about 2 kG in absolute value. The characteristic length for the variations of $n(r)$ is of the order of the gap $D$, i.e., $\xi \simeq 20$ nm.

One can see quite easily (for example by considering the Berry phase from the mapping of the plane onto the sphere $S_2$) that the net topological flux per unit cell, $\phi_t$, has to be an integer multiple of $\phi_0$, and that $\phi_t / \phi_0$ is given by the number (per unit cell) of lines $B_z = 0$ enclosing a region with $B_z < 0$ minus the number of lines $B_z = 0$ enclosing a region with $B_z > 0$. From Fig. 2, one thus sees that we have a triangular lattice of lines $B_z = 0$ enclosing regions with $B_z < 0$, yielding $\phi_t = -\phi_0$ for a vanishing external field. For the considered geometry this corresponds to an average topological field of about $-5$ kG, with local values ranging between $-15$ kG and $+5$ kG.

If a negative external field is applied (assuming that the magnetization of the nanocylinders remains unchanged, due to some large coercivity), the lines with $B_z = 0$ shrink without changing their topology until they collapse to a single point and eventually disappear at a critical field $B_1$, beyond which $\phi_t = 0$. If a positive field is applied, the lines with $B_z = 0$ expand, until they connect each other at a critical field $B_2$ and change their topology to a honeycomb lattice (the dual of the triangular lattice) of lines $B_z = 0$ enclosing regions with $B_z > 0$ (dashed lines in Fig. 2), giving $\phi_t = +2\phi_0$. Increasing further the external field leads to a collapse of the lines with $B_z = 0$ at a critical field $B_3$, beyond which $\phi_t = 0$. For Fe nanocylinders in the above geometry, the critical fields are, respectively, $B_1 \simeq -2$ kG, $B_2 \simeq +0.9$ kG, and $B_3 \simeq +1.3$ kG. This change of topology under application of an external magnetic field results in a striking field dependence of the Hall resistivity, as sketched in Fig. 4, and constitutes an unambiguous signature of the THE.

For the practical realization, we propose to use p-type DMS since the exchange constants for holes are much larger than for electrons [14]. For the estimation of the topological field acting on holes, we use the Luttinger

FIG. 2: Distribution of the $z$-component of dipolar field $B/4\pi M_s$ inside the semiconductor film for the triangular lattice of magnetic nanocylinders, for a zero external field. The black solid circles correspond to the lines with $B_z = 0$. Dashed lines correspond to the lines with $B_z = 0$ under an uniform external magnetic field $B_{ext}/4\pi M_s = +0.058$.

FIG. 4: Dependence of the Hall conductivity on external magnetic field (schematically). The slope corresponds to the contribution of the normal Hall effect; $\sigma^\phi_{xy}$ is the Hall conductivity corresponding to a total (magnetic + topological) flux per unit cell equal to $\phi_0$. 

FIG. 3: Topological field $B_t(r)$ (in units of $\phi_0$ per unit cell area) for the triangular lattice of magnetic nanocylinders. Black lines correspond to $B_t = 0$. 

FIG. 4: Dependence of the Hall conductivity on external magnetic field (schematically). The slope corresponds to the contribution of the normal Hall effect; $\sigma^\phi_{xy}$ is the Hall conductivity corresponding to a total (magnetic + topological) flux per unit cell equal to $\phi_0$. 

$\sigma_{xy}(B)/\sigma^\phi_{xy}$

$B_t$ $B_2$ $B_3$ $B$

$0$ $2$ $1$ $-1$ $B_1$ $B_2$ $B_3$ $B$
\[ H_h = \frac{\hbar^2}{2m_0} \left[ \left( \gamma_1 + \frac{5}{2} \gamma_2 \right) \nabla^2 - 2\gamma_2 (\mathbf{J} \cdot \nabla)^2 \right] + E_{ex} \mathbf{J} \cdot \mathbf{n}(r), \]

where \( J_\mu \) are the matrices of momentum \( J = 3/2, \gamma_1 \) and \( \gamma_2 \) are the Luttinger parameters, \( m_0 \) is the free electron mass, \( \mathbf{n}(r) \) is the unit vector along the direction of exchange field acting on 2D gas of holes, \( E_{ex} \) is the average exchange field created by magnetic impurities,

\[ E_{ex} = -JSxN_0\beta B_S(g_L\mu_B B/k_B T), \]

\( N_0\beta \) is the \( p-d \) exchange constant, \( S \) is the magnetic moment of impurity, \( xN_0 \) is the concentration of magnetic atoms, \( B_S(z) \) is the modified Brillouin function, and \( g_L \) is the Landé factor of magnetic atom.

We calculate the topological field for holes described by Hamiltonian (9), using the transformation \( T(r) = \exp \left( i \mathbf{J} \cdot \Omega(r)/2 \right) \), where \( \Omega(r) \) is the vector of rotation. We find the topological field \( B_T \) considering small deviations of the vector \( \mathbf{n}(r) \) in a vicinity of some point \( r \). Assuming that the magnetic splitting under exchange field is strong, we find that the gauge potential and the topological field for holes in the valence band, being proportional to \( J \), are given by the same expressions \( \Theta \) and \( \Theta_p \) as for electrons, multiplied by a factor 3.

After transforming (9) and restricting ourselves by the subband \( J_s = -3/2 \), we obtain for spin-polarized holes, the same Hamiltonian as (8), with \(-1/m^* \) replaced by \( 1/m^* = (\gamma_1 + \gamma_2)/m_0 \).

The spin splitting in II-VI semiconductor can be large enough to provide 100% polarization of holes under the dipolar field of 2 kG. Indeed, we can take the exchange coupling \( N_0\beta = -1.2 \) eV, which is the typical magnitude for different compounds (Cd$_{1-x}$Mn$_x$Se, Zn$_{1-x}$Mn$_x$Se, etc.), and the atomic density of magnetic atoms \( x = 0.05 \).

For \( B = 2 \) kG, \( g_L = 2 \), and \( T = 4.2 \) K the Brillouin function \( B_S(2g_L\mu_B B/k_B T) \approx 0.075 \). Then using (10) we obtain \( E_{ex} \approx 11 \) meV. The Fermi energy of 2D holes filling the spin-splitted subband is determined by the density of holes \( n_p \), \( E_F = 2\pi\hbar^2 n_p/m^* \), where \( m^* \) is the effective mass of holes. By taking \( m^* = 0.5 m_0 \), and \( n_p = 10^{11} \) cm$^{-2}$, we obtain \( E_F \approx 1 \) meV. Thus, the condition of 100% polarization of holes, \( E_F < E_{ex} \), is quite realistic for II-VI diluted magnetic semiconductors. The adiabaticity parameter for this choice of parameters is \( \lambda \approx 0.04 \ll 1 \). It should be noted that much stronger splitting can be reached for Zn$_{1-x}$Cr$_x$Te compounds, for which \( N_0\beta = 3.6 \) eV \( \Theta \).

For electrons, on the other hand, in order to fulfill the conditions of full polarization and adiabaticity, lower temperatures and/or very low electron densities would be needed. Taking the exchange constant of \( s-d \) interaction \( N_0\alpha = 0.22 \) eV \( \Theta \), the electron effective mass \( m_e = 0.22 m_0 \), \( T = 4.2 \) K, and the concentration of electrons \( n_e = 10^{11} \) cm$^{-2}$, we obtain \( E_F \approx 4.4 \) meV, \( E_{ex} \approx 2 \) meV, and \( \lambda \approx 1 \); the condition of adiabaticity is poorly fulfilled, resulting in a reduction of the effect. Also the gas is not fully polarized, reducing further the THE due to the partial compensation of contributions from spin up and down subbands. Although not fully developed, the THE should nevertheless be observable with electrons.

In conclusion, we have proposed that the THE can be observed in suitably chosen nanostructures, and that its striking behavior under an external magnetic field provides an unambiguous experimental signature of the THE.

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