NONPERTURBATIVE ASPECTS IN $\mathcal{N}$-FOLD SUPERSYMMETRY

TOSHIAKI TANAKA, 1 AND MASATOSHI SATO 2

1 Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.
2 The Institute for Solid State Physics, The University of Tokyo, Kashiwanoha 5-1-5, Kashiwa-shi, Chiba 277-8581, Japan.

Through a nonperturbative analysis on a sextic triple-well potential, we reveal novel aspects of the dynamical property of the system in connection with $\mathcal{N}$-fold supersymmetry and quasi-solvability.

1 Introduction

Analytical understanding of nonperturbative aspects in quantum theories is a quite difficult problem. We will present novel features of the following quantum mechanical system:

$$
H = \frac{p^2}{2} + \frac{1}{2}q^2(1 - g^2q^2)^2 + \frac{\epsilon}{2}(1 - 3g^2q^2).
$$

This is a triple-well potential having three local minima at $q = 0$ and $q \simeq \pm 1/g$ for $\epsilon g^2 \ll 1$, see Fig. 1. It is a simple one-dimensional quantum mechanical system but the dynamics and the nonperturbative analysis are highly nontrivial.

![Figure 1: The form of the triple-well potential](image)
We can find that the traditional semi-classical approach fails. In the case of \( \epsilon \neq 0 \), there is a so-called bounce solution as the classical solution which has a negative mode in the fluctuations. The negative mode contributes non-zero imaginary part of the spectra in the approximation, showing instability of the system. Since the spectra of the model must be real, the instability in the approximation must be \textit{fake}. In the case of \( \epsilon = 0 \), the classical solution is now an instanton having no negative mode and does not cause fake instability. However, another difficulty takes place when we sum up the multi-instanton contribution with the aid of the dilute-gas approximation. The application of this approximation leads to the splitting of the ground-state spectra due to the quantum tunneling, in spite of the fact that the ground-state of the potential (1) does not degenerate in the tree-level. As we will see, both the difficulties are circumvented with the aid of the valley method briefly explained in the next section. Furthermore, the system (1) possesses significant analytic properties at specific values of the parameter, namely, \( \mathcal{N} \)-fold supersymmetry and quasi-solvability. These issues are summarized in Section 7 and 8.

2 Valley method

The main problem in quantum theories concerns with the evaluation of the Euclidean partition function: 
\[
Z = \mathcal{J} \int Dq e^{-S[q]}. 
\]
Since the evaluation cannot be done exactly in general, one must find out a proper method which enables one to get a good estimation of the quantity. The semi-classical approximation is known to be one of the most established methods. Especially, the uses of instantons have been succeeded in analyzing non-perturbative aspects of various quantum systems which have degenerate vacua [1]. However, validity of the approximation comes into question when the fluctuations around the classical configuration contain a negative mode. The appearance of a negative mode indicates that the classical action does not give the minimum but rather a saddle point in the functional space. In this case, one may expect that the path-integral is dominated by the configurations along the negative mode, which may intuitively constitute a \textit{valley} in the functional space. The valley method is a natural realization of this consideration.

At first, we give a geometrical definition of the valley in the functional space \( q(\tau) \) [2]:
\[
\frac{\delta}{\delta q(\tau)} \left[ \frac{1}{2} \int d\tau' \left( \frac{\delta S[q]}{\delta q(\tau')} \right)^2 - \lambda S[q] \right] = 0.
\] (2)
The above definition (2) can be interpreted as follows; for each fixed “height” \( S[q] \), the valley is defined at the point where the norm of the gradient vector becomes extremal. Introducing an auxiliary field \( F(q) \), we can make the valley equation (2) a more perspicuous form:

\[
\frac{\delta S[q]}{\delta q(\tau)} = F(\tau), \quad \int d\tau' D(\tau, \tau') F(\tau') = \lambda F(\tau). \tag{3}
\]

where the operator \( D \) is defined by \( D(\tau, \tau') = \delta^2 S[q]/\delta q(\tau)\delta q(\tau') \). It is now evident that any solution of the equation of motion is also a solution of the valley equation (3) with \( F(\tau) \equiv 0 \).

Next, we separate the integration along the valley line from the whole functional integration. We parametrize the valley line by a parameter \( R \) and denote the valley configuration by \( q_R(\tau) \). Introducing Faddeev-Popov determinant \( \Delta[\varphi_R] \), expanding the action \( S[q] \) around the valley configuration, and integrating up to the second order term in the fluctuation \( \varphi_R(\tau) = q(\tau) - q_R(\tau) \), we finally obtain the one-loop order result:

\[
Z = \mathcal{J} \int dR \int Dq \delta \left( \int d\tau \varphi_R(\tau) G_R(\tau) \right) \Delta[\varphi_R] e^{-S[q]}
\approx \mathcal{J} \int \frac{dR}{\sqrt{2\pi \det' D_R}} \Delta[\varphi_R] e^{-S[q_R]}, \tag{4}
\]

where \( G_R(\tau) \) is the normalized gradient vector on the valley configuration. In the above, \( \det' \) denotes the determinant in the functional subspace which is perpendicular to the gradient vector \( G_R(\tau) \). The valley equation (3) ensure that the subspace does not contain the eigenvector of the eigenvalue \( \lambda \). Therefore, we can safely perform the Gaussian integrations even when we encounter a non-positive mode. The extension to the multi-dimensional valley, which will be needed when there are multiple non-positive eigenvalues, is straightforward.

### 3 Valley-instantons

Let us investigate the solutions of the valley equation (3) for the system (1). In the case of \( \epsilon = 0 \), the three local minima of the potential (1) have the same potential value. Thus, there are (anti-)instanton solutions of the equation of motion which describe the quantum tunneling between the neighboring vacua:

\[
q^{(I)}_0(\tau - \tau_0) = \pm \frac{1}{g} \frac{1}{(1 + e^{\pm 2(\tau - \tau_0)})^{1/2}}, \quad q^{(F)}_0(\tau - \tau_0) = \pm \frac{1}{g} \frac{1}{(1 + e^{\pm 2(\tau - \tau_0)})^{1/2}}. \tag{5}
\]
When $\epsilon \neq 0$, the classical solutions drastically change into the so-called bounce solutions which cause fake instability. On the other hand, the solutions of the valley equation (3) contain a continuously deformed (anti-)instanton which connects the two non-degenerate local minima and is called \textit{(anti-)valley-instanton} \cite{3}.

The solutions of the valley equation (3) also contain a family of the configurations, which tends to the trivial vacuum configuration in the one limit and tends to well-separated valley-instanton and anti-valley-instanton configuration in the other limit. The latter configuration is called $\bar{I}I$-valley. The bounce solution is also realized as an intermediate configuration of this family, which is consistent with the fact that the solution of the equation of motion is also a solution of the valley equations. For details, see the numerical result in Ref. \cite{3}. It turns out that the quantum fluctuation around the bounce solution along this family of the valley configurations actually corresponds the negative mode. Therefore, we can separate the integration along the negative mode with the aid of the valley method and thus do not suffer from the problem of fake instability.

There are two distinct $\bar{I}I$-(\textit{I}-)valley configurations since the curvature at the central potential bottom (at $q = 0$) is different, even at the leading order of $g^2$, from the one at the side potential bottoms (at $q \simeq \pm 1/g$); the $\bar{I}I$-(\textit{I}-)valley which satisfy $q(\pm T/2) = 0$ ($T \gg 1$) are different from the ones which satisfy $q(\pm T/2) \simeq 1/g$ or $-1/g$ ($T \gg 1$). The Euclidean action of the former with large separation $R$ can be calculated by the perturbative expansion in $\lambda \sim O(e^{-2R})$ as follows:

$$S^{(\textit{I})}(R) = S^{(\textit{II})}(R) = 2S_0^{(I)} - \epsilon R + \frac{\epsilon}{2}(T - R) - \frac{1}{g^2}e^{-2R} + O(e^{-4R}), \quad (6)$$

while the one of the latter with large separation $\tilde{R}$ can be calculated in the same way as,

$$S^{(\textit{I})}(\tilde{R}) = S^{(\textit{II})}(\tilde{R}) = 2S_0^{(I)} + \frac{\epsilon}{2}\tilde{R} - \epsilon(\tilde{T} - \tilde{R}) - \frac{2}{g^2}e^{-\tilde{R}} + O(e^{-2\tilde{R}}), \quad (7)$$

where $S_0^{(I)}$ denotes the Euclidean action of one (anti-)instanton Eq. (5) and amounts to $S_0^{(I)} = 1/4g^2$. In Eqs. (6) and (7), the fourth term can be interpreted as the interaction term between the valley-instanton and the anti-valley-instanton. Therefore, the minus sign indicates that the interaction is attractive.

The other type of the solutions emerges in this case, which is asymptotically composed of two successive valley-instantons or two successive anti-valley-instantons.
instantons. We call them $\Pi$-valley and $\bar{\Pi}$-valley, respectively. These configurations do not appear in the case of double-well potentials since they connect every other vacuum. The Euclidean action of them with large separation $R$ can be also calculated in the same way as,

$$S^{(\Pi)}(\bar{R}) = S^{(\bar{\Pi})}(\bar{R}) = 2S_0^{(\Pi)} + \frac{\epsilon}{2} \bar{R} - \epsilon(T - \bar{R}) + \frac{2}{g^2}e^{-\bar{R}} + O(e^{-2\bar{R}}). \quad (8)$$

Note that the sign of the fourth term is plus and thus the interaction between the (anti-)valley-instantons in this case is repulsive. It turns out that these interaction terms in Eqs. (3)-(8) play an important role in circumventing the breakdown of the dilute-gas approximation and in producing physically acceptable results.

4 Multi-valley-instantons calculus

Utilizing the knowledge of the (anti-)valley-instantons and the interactions between them obtained previously, we can evaluate the partition function $Z = \text{tr} e^{-HT}$ by summing over those configurations made of several (anti-)valley-instantons. The sum of the contributions from the $2n$ valley-instantons configuration can be written as,

$$Z_{\text{NP}} = \sum_{n=1}^{\infty} \alpha^{2n} \sum_{n_{II}=0}^{[n/2]} \left( \frac{n}{2n_{II}} \right) J_{n,n_{II}}, \quad (9)$$

where $\alpha^2$ denotes the contribution of the Jacobian and the $R$-independent part of the determinant for one valley-instanton-pair and is calculated as $\alpha^2 = \sqrt{2e^{-1/2g^2}/\pi g^2}$. The function $J_{n,n_{II}}$ is given by,

$$J_{n,n_{II}} = \frac{T}{n} \int_0^{\infty} \left( \prod_{i=1}^{n} dR_i d\bar{R}_i \right) \delta \left( \sum_{i=1}^{n} (R_i + \bar{R}_i) - T \right) \exp \left[ -(1 - \epsilon) \sum_{i=1}^{n} R_i \right.$$

$$- \frac{1}{2} \sum_{i=1}^{n} \bar{R}_i + \frac{1}{g^2} \sum_{i=1}^{n} e^{-2R_i} - \frac{2}{g^2} \sum_{i=1}^{2n_{II}} e^{-\bar{R}_i} + \frac{2}{g^2} \sum_{i=2n_{II}+1}^{n} e^{-\bar{R}_i} \left], \quad (10)$$

where $R_i$ is the distance between the $(2i - 1)$-th and $2i$-th (anti-)valley-instanton and $\bar{R}_i$ the one between the $2i$-th and the $(2i + 1)$-th (anti-)valley-instanton (mod $n$), see Fig. 2.
Figure 2: The collective coordinates $R_i$ and $\tilde{R}_i$ for a $2n$ valley-instantons configuration.

5 Nonperturbative shifts of the spectra

From Eqs. (9) and (10), the nonperturbative contributions to the spectra are determined by the following equation:

\[
\alpha^2 \frac{(-E^{\frac{1}{2}} + \frac{1}{2}) \pm 1}{2} \left( \frac{2}{g^2} \right) \left( -E + \frac{1}{2} + \frac{\epsilon}{2} \right) \times \Gamma \left( \frac{E - 1}{2} + \frac{\epsilon}{2} \right) \times \left( \frac{1}{g^2} \right) \left( -E + \frac{1}{2} + \frac{\epsilon}{2} \right) = 1. \tag{11}
\]

We will solve the above equation by the series expansion in $\alpha$: $E_{n_0/n_\pm} = \sum_{r=0}^{\infty} E_{n_0/n_\pm}^{(r)} \alpha^r$, where $E_{n_0}$ stands for the spectra corresponding to, in the limit $g \to 0$, the eigenfunctions of the center potential well and $E_{n_\pm}$ for the ones corresponding to the parity eigenstates obtained by the linear combinations of the eigenfunctions of the each side potential well. The results are complicated because the calculation needs separate treatments depending on the value of the parameter $\epsilon$. For details, see Eqs. (5.22)–(5.27) in Ref. [6]. From the results, we find that the nonperturbative contribution to the spectra of the particular levels vanishes at the specific values of the parameter $\epsilon$ as follows:

1. $\epsilon = (4\mathcal{N} + 1)/3 \ (\mathcal{N} = 1, 2, 3, \ldots)$: $E_{n_+}^{(1)} = E_{n_+}^{(2)} = \cdots = 0 \ \text{for} \ n_+ < \mathcal{N}$. 
2. \( \epsilon = (4N - 1)/3 \) (\( N = 1, 2, 3, \ldots \)): \( E_{n-}^{(1)} = E_{n-}^{(2)} = \cdots = 0 \) for \( n_- < N \).

### 6 Large-order behavior of the perturbation series

If we evaluate the spectra \( E \) by means of the perturbation expansion in the coupling constant \( g^2 \) as 
\[
E_{n_0/n_\pm} = E_{n_0/n_\pm}^{(0)} + \sum_{r=1}^{\infty} a_{n_0/n_\pm}^{(r)} g^{2r},
\]
the large order behavior of the coefficients \( a^{(r)} (r \gg 1) \) can be estimated via the following dispersion relation [3]:
\[
a^{(r)} = -\frac{1}{\pi} \int_0^{\infty} dg^2 \frac{\text{Im} E_{NP}(g^2)}{g^{2r+2}}.
\]

From the results obtained by Eq. (11), the leading contributions for \( r \gg 1 \) reads,
\[
a_{n_0}^{(r)} \sim A_{n_0} (\epsilon) 2^r \Gamma \left( r + \frac{3}{2} n_0 + \frac{3}{4} + \frac{3}{4} \epsilon \right), \quad (13)
\]
\[
a_{n_\pm}^{(r)} \sim A_{n_\pm} (\epsilon) 2^r \Gamma \left( r + 3n_\pm + \frac{3}{2} - \frac{3}{2} \epsilon \right), \quad (14)
\]

where \( A_n (\epsilon) \)'s are some constants depending on \( n \) and \( \epsilon \). Equations (13) and (14) show that the perturbative coefficients diverge factorially unless the prefactor \( A_n (\epsilon) \)'s vanish. It turns out that the disappearance of the leading divergence takes place only when \( \epsilon = \pm (2n + 1)/3 \) (\( n = 1, 2, 3, \ldots \)), see Eqs. (5.31) in Ref. [3]. More precisely, we obtain the following results:

1. \( \epsilon = (4N - 1)/3 \) (\( N = 1, 2, 3, \ldots \)): \( A_{n_\pm} (\epsilon) = 0 \) for \( n_\pm < N \).

2. \( \epsilon = -(4N + 1)/3 \) (\( N = 1, 2, 3, \ldots \)): \( A_{2m_0+1} (\epsilon) = 0 \) for \( m_0 < N \).

3. \( \epsilon = -(4N - 1)/3 \) (\( N = 1, 2, 3, \ldots \)): \( A_{2m_0} (\epsilon) = 0 \) for \( m_0 < N \).

### 7 \( \mathcal{N} \)-fold supersymmetry

An \( \mathcal{N} \)-fold supersymmetric quantum mechanical system of one-degree of freedom is, roughly speaking, a system of a pair of Hamiltonians \( H_N^\pm \) which satisfies intertwining relations with respect to an \( \mathcal{N} \)-th order linear differential operator \( P_N \) as follows:
\[
P_N H_N^- - H_N^+ P_N = 0, \quad P_N^\dagger H_N^+ - H_N^- P_N^\dagger = 0. \quad (15)
\]
For the general treatment and discussion on $\mathcal{N}$-fold supersymmetry, see Ref. [4].

It turns out that the system (1) becomes $\mathcal{N}$-fold supersymmetric when $\epsilon = \pm (4\mathcal{N} \pm 1)/3$. More precisely, when $\epsilon = (4\mathcal{N} \pm 1)/3$, the relations (13) are satisfied for $H^\pm_\mathcal{N} = H$ and following $H^-_\mathcal{N}$ and $P_\mathcal{N}$:

$$H^-_\mathcal{N} = \frac{p^2}{2} + \frac{1}{2}q^2 (1 - g^2q^2)^2 - \frac{2N \mp 1}{6} (1 - 3g^2q^2) + \frac{(2N - 1 \pm 1)(2N + 1 \pm 1)}{8q^2},$$  \hfill (16) 

$$P_\mathcal{N} = (i)^{N/2} \prod_{k=-(N-1)/2}^{(N-1)/2} \left( \frac{d}{dq} + q \left( 1 - g^2q^2 \right) + \frac{N \pm 1 - 2k}{2q} \right). \hfill (17)$$

When $\epsilon = -(4\mathcal{N} + 1)/3$, the relations (15) are satisfied for $H^-_\mathcal{N} = H$ and following $H^+_\mathcal{N}$ and $P_\mathcal{N}$:

$$H^+_\mathcal{N} = \frac{p^2}{2} + \frac{1}{2}q^2 (1 - g^2q^2)^2 - \frac{2N \mp 1}{6} (1 - 3g^2q^2) + \frac{(2N - 1 \pm 1)(2N + 1 \pm 1)}{8q^2},$$  \hfill (18) 

$$P_\mathcal{N} = (i)^{N/2} \prod_{k=-(N-1)/2}^{(N-1)/2} \left( \frac{d}{dq} + q \left( 1 - g^2q^2 \right) - \frac{N \pm 1 + 2k}{2q} \right). \hfill (19)$$

8 Quasi-solvability

If we define $\mathcal{N}$-dimensional vector spaces $\mathcal{V}^\pm_\mathcal{N}$ by $\mathcal{V}^-_\mathcal{N} = \ker P_\mathcal{N}$ and $\mathcal{V}^+_\mathcal{N} = \ker P^\dagger_\mathcal{N}$, we see from Eq. (15) that $\mathcal{V}^\pm_\mathcal{N}$ are invariant under the action of the $\mathcal{N}$-fold supersymmetric Hamiltonians: $H^\pm_\mathcal{N} \mathcal{V}^\pm_\mathcal{N} \subset \mathcal{V}^\pm_\mathcal{N}$. This means that both $H^\pm_\mathcal{N}$ are quasi-solvable [3]. Therefore, the system (1) is quasi-solvable when $\epsilon = \pm (4\mathcal{N} \pm 1)/3$.

Actually, when $\epsilon = (4\mathcal{N} \pm 1)/3$, the solvable sector of $H = H^+_\mathcal{N}$ reads,

$$\mathcal{V}^+_\mathcal{N} = \text{span}\{\phi^+_n : n = 1, \ldots, \mathcal{N}\}, \quad \phi^+_n(q) = q^{2n-\frac{3}{2}+\frac{1}{2}} \exp \left( -\frac{g^2}{4}q^4 + \frac{1}{2}q^2 \right). \hfill (20)$$

We note that, since $\mathcal{V}^+_\mathcal{N} \subset L^2(\mathbb{R})$, $\mathcal{N}$ elements of the above $\mathcal{V}^+_\mathcal{N}$ are the exact eigenfunctions of the Hamiltonian (1). They are analytic at $g^2 = 0$ on the $g^2$-plane and thus, (i) the perturbative expansions of them in $g^2$ have nonzero convergent radii, and (ii) there is no nonperturbative contribution on them.
The results (i) and (ii) are also the cases for the corresponding eigenvalues, which explain the peculiar results for the case of \( \epsilon = (4N \pm 1)/3 \) in Sections 5 and 3, namely, the disappearance of the leading divergence and the vanishment of the nonperturbative spectral shifts.

Similarly, when \( \epsilon = -(4N \pm 1)/3 \), the solvable sector of \( H = H_{1N} \) reads,

\[
V_{1N} = \text{span} \{ \phi_n^- : n = 1, \ldots, N \}, \quad \phi_n^-(q) = q^{2n-\frac{3}{2}+\frac{1}{2}} \exp \left( \frac{g^2}{4} q^4 - \frac{1}{2} q^2 \right). \tag{21}
\]

In contrast to in the case of \( \epsilon = (4N \pm 1)/3 \), \( V_{1N} \not\subset L^2(\mathbb{R}) \) and thus any element of the above \( V_{1N} \) cannot be the exact eigenfunction of the Hamiltonian (1). However, if we expand them in power of \( g^2 \) as, \( \phi_n^-(q) = \sum_{r=0}^{\infty} \phi_n^{(r)}(q) g^{2r} \), they are normalizable at any finite order in \( g^2 \) and thus the elements of the \( V_{1N} \) give the perturbatively well-defined and correct eigenfunctions. Therefore, (iii) the perturbative expansions of them in \( g^2 \) have nonzero convergent radii too, but (iv) there are nonperturbative contributions on them. The results (iii) and (iv) are also the cases for the corresponding eigenvalues, which also explain the peculiar results for the case of \( \epsilon = -(4N \pm 1)/3 \) in Section 3, namely, the disappearance of the leading divergence and the nonvanishment of the nonperturbative spectral shifts.

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