AdS$_5$–Schwarzschild deformed black branes and hydrodynamic transport coefficients

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Deformed AdS$_5$–Schwarzschild black branes are here derived, employing the membrane paradigm and the ADM procedure. AdS/CFT near-horizon methods are then implemented to compute the shear viscosity-to-entropy ratio of the deformed AdS$_5$–Schwarzschild black branes. It provides constraints for the deformed black brane free parameter, generating new black brane solutions.

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I. INTRODUCTION

AdS/CFT is a paradigm relating gravity in anti-de Sitter (AdS) space to a large-N conformal field theory (CFT), located on the AdS codimension one boundary. Perturbatively, considering an $1/N$ expansion, quantum fields in the bulk correspond to CFT operators [1–3]. The dynamics of Einstein’s equations, describing weakly coupled gravity, in an AdS space rules the corresponding dynamics of the energy-momentum tensor of strongly coupled QFTs on the AdS boundary. The AdS$_5$ boundary is a 4D Minkowski spacetime, and the $D_3$-brane near horizon geometry is the AdS$_5$ space, whereas the far away brane geometry remains flat [4]. In the $N \to \infty$ ’t Hooft regime, keeping a fixed coupling, the gauge theory on the boundary is an effective classical theory.

The AdS boundary is usually identified to a 4D brane. Braneworld models describe a brane that has tension, $\sigma$, constrained to both the bulk and the brane cosmological constants [5, 6]. General relativity (GR) describes gravity in an infinitely rigid brane, with an infinite tension. However, recent works derived a strong bound for the finite brane tension, lying in the bound $\sigma \gtrsim 3.2 \times 10^{-6}$ GeV$^4$ [7, 8]. This condition in fact produces a physically correct low energy limit, allowing the construction of an AdS/CFT membrane paradigm analogue of any classical GR solution [5, 6, 9–13]. One can also describe the AdS bulk gravity by a black hole, which behaves as a fluid at its own horizon, in the membrane paradigm. Einstein’s equations near the horizon of the black hole reduce to the Navier-Stokes equations for the fluid [1–3]. A fluid at the black hole horizon mimics a fluid at the AdS boundary [9, 14–16], introducing an useful dictionary, linking brane models and the membrane paradigm of AdS/CFT. Here we aim to derive new black brane solutions and use the shear viscosity-to-entropy density ratio, $\eta/s$, to impose viscosity bounds to the free parameters into these new solutions.

Previously, we have explored the technique employed here to derive a family of solutions that consists of a deformation in the AdS$_4$–Reissner–Nordström background, and its potential applications to AdS/CMT [17]. By embedding the brane into a higher dimensional bulk, we were able to mimic the Hamiltonian and momentum constraints from the ADM formalism for static configurations of the metric field [18, 19]. These equations turn out to be a weaker condition on the metric functions, allowing for a family of deformations of solutions from classical GR. In the present work we apply a similar procedure to the AdS$_5$–Schwarzschild black brane [20, 21].

The paper is organized as follows: in Sect. II the relevant results of linear response theory and fluid dynamics are briefly presented within the hydrodynamics formalism, followed by a presentation of the AdS/CFT duality and its methods in Sect. II B. Sect. III is then devoted to derive the AdS$_5$–Schwarzschild deformed gravitational background and to discuss the calculation of the $\eta/s$ ratio, whose explicit computation is carried out for the family AdS$_5$–Schwarzschild deformed black branes in Sect. IV. The saturation of $\eta/s$ therefore is shown to constrain the free parameter AdS$_5$–Schwarzschild deformed black brane, driving the family of deformed branes to two unique solutions: the standard AdS$_5$–Schwarzschild black brane and a new black brane solution. The concluding remarks are then presented.

II. HYDRODYNAMICS AND LINEAR RESPONSE THEORY

The so called hydrodynamic limit is characterized by the long-wavelength, low-energy regime [22], and is often applicable to describe conserved quantities. As an effective description of field theory, hydrodynamics naturally does not contain the details of a microscopic theory. These are encoded into the transport coefficients, among which the shear viscosity, $\eta$, plays a prominent role.

The macroscopic variables encoded in the energy-momentum stress tensor, $T^{\mu\nu}$, along with its conservation law, $\partial_\mu T^{\mu\nu} = 0$, describe a simple fluid. In general,
one introduces a constitutive equation by determining
the form of $T^{\mu\nu}$ in a derivative expansion, given in terms
of the normalized fluid velocity field $u^\mu(x^\nu)$, its pressure
field $p(x^\nu)$ and its rest-frame energy density $\rho(x^\nu)$.

To first order in the derivative expansion, the stress
tensor is expressed as [3, 22]

$$T^{\mu\nu} = p(\eta^{\mu\nu} + u^\mu u^\nu) + \rho u^\mu u^\nu + \tau^{\mu\nu},$$ (1)

where $\tau^{\mu\nu}$, the term which is first-order in derivatives,
carries dissipative effects. In the local rest frame, its
spatial components are

$$\tau_{ij} = -\zeta \delta_{ij} \partial^i u^j + \eta \left( \frac{2}{3} \delta_{ij} \partial^i u^j - \partial_i u_{ij} \right).$$ (2)

The constitutive equation for a viscous fluid, as de-
defined above, yields both the continuity and Navier–Stokes
equations, thus closing the system of equations of motion.
The shear and bulk viscosities, respectively $\eta$ and $\zeta$, were
introduced to account for dissipative effects.

For a theory described by an action functional $S$, the
coupling of an operator $O$ to an external source $\phi^{(0)}$ reads [23]

$$S \rightarrow S + \int d^4x \phi^{(0)}(t, x) O(t, x).$$ (3)

One is often interested in determining the response in $O$, which is given by

$$\delta \langle O(t, x) \rangle := \langle O(t, x) \rangle_S - \langle O(t, x) \rangle,$$ (4)

where $\langle O(t, x) \rangle_S$ denotes the ensemble average (one-
point function) of the operator $O$ in the presence of $\phi^{(0)}$.
The study of such response up to first order in $\phi^{(0)}$ is
known as linear response theory. In momentum space
the one-point function (4) reads [23]

$$\delta \langle \mathcal{O}(\omega, \mathbf{q}) \rangle = -G^{\mathcal{O}, \mathcal{O}}_{R}(\omega, \mathbf{q}) \phi^{(0)}(\omega, \mathbf{q}),$$ (5)

where $G^{\mathcal{O}, \mathcal{O}}_{R}(\omega, \mathbf{q})$ is the retarded Green’s function
associated to $\mathcal{O}$ in Fourier space [24]. As seen in Eq. (5),
the one-point function, $\delta \langle \mathcal{O} \rangle$, is reduced to the determination of $G^{\mathcal{O}, \mathcal{O}}_{R}(\omega, \mathbf{q})$. The Kubo formula relates the retarded
Green’s function to a transport coefficient. Computing the shear viscosity, $\eta$, the transport coefficient associated
to the viscous fluid that is dual to the deformed AdS5-
Schwarzschild black hole will restrict the possible values of the deformed black hole free parameter.

### A. The shear viscosity and its Kubo formula

By coupling a fictitious gravitational field to the stress
tensor of the fluid one is able to derive the shear viscosity
in agreement with the idea from GR that fluctuations
in the stress tensor induce spacetime fluctuations and vice-versa [2]. This approach, although seemingly a pure
analogy, has a natural interpretation in the context of AdS/CFT, as we will further detail in Sec. II B.

The response of $T^{\mu\nu}$ under gravitational fluctuation is
determined by the introduction of an off-diagonal pertur-
bation term, $h^{(0)}_{xy}(t)$, leading to the perturbed metric $g^{(0)}_{\mu\nu}$. In the $\{t, x, y, z\}$ coordinate system $g^{(0)}_{\mu\nu}$ is:

$$g^{(0)}_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + 2h^{(0)}_{xy}(t)dx^y.$$ (6)

The perturbation is off-diagonal so that its response in
$T^{\mu\nu}$ accounts for the shear effects encoded in $\eta$. The perturbation (6) requires the extension of the
constitutive equation (1) to curved spacetime, where the
dissipative term reads $[2, 3]

$$\tau^{\mu\nu} = -\zeta \Pi^{\mu\nu} \Pi^{\lambda\sigma} \nu_L \delta_{k^k} - \nu_L \Pi^{\mu\nu} \Pi^{\lambda\sigma} \nu_L = \epsilon^{\mu\nu} \Pi^{\lambda\sigma} \nu_L \delta_{k^k} - \nu_L \Pi^{\mu\nu} \Pi^{\lambda\sigma} \nu_L \delta_{k^k}.$$ (7)

beging $\nabla_\mu$ the covariant derivative with respect to $g^{(0)}_{\mu\nu}$. In order to write the constitutive equation in a covari-

tant way, the projection tensor $\Pi^{\mu\nu} = g^{\mu\nu(0)} + u^\mu u^\nu$ is

introduced.

Taking the appropriate covariant derivatives of the ve-

locity, and assuming that $u_i = u_i(t)$, leads to the response
in Fourier space [23]

$$\delta \langle \tau^{xy}(\omega, \mathbf{q}) \rangle = i \omega \eta h^{(0)}_{xy}.$$ (8)

The general result from linear response theory expressed in Eq. (5) is, in this case,

$$\delta \langle \tau^{xy} \rangle = -G^{\tau^{xy}, \tau^{xy}}_{R}(\omega, \mathbf{q}).$$ (9)

Comparing this with the result in Eq. (8), and solving for $\eta$, yields the Kubo formula

$$\eta = -\lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} \left( G^{\tau^{xy}, \tau^{xy}}_{R}(\omega, \mathbf{q}) \right).$$ (10)

It is given in the $\omega \rightarrow 0$ limit, since $\eta$ does not depend on $\omega$ or $\mathbf{q}$. Now, with Eq. (10), it is clear that $\eta$ is fully
determined once the retarded Green’s function $G^{\tau^{xy}, \tau^{xy}}_{R}$ is

found. Computation of the retarded Green’s function is

straightforwardly achieved in this context applying the
AdS/CFT duality.

### B. AdS/CFT and the GKP–Witten relation

The AdS/CFT duality [4], in simple terms, relates a

gravity theory defined on an asymptotically AdS spacetime,
in $D$ dimensions, to gauge theory in $D - 1$ space-
time dimensions. A further development of such duality
is the so called GKPW relation [25, 26], which relates the
partition functions of both gravity and gauge theo-

ries. Explicitly, the GKPW formula reads

$$\exp \left( i \mathcal{S}[\phi^{(0)}] \right) = \left\langle \exp \left( i \int \phi^{(0)} \mathcal{O} \right) \right\rangle,$$ (11)
where $\varphi$ is a field in the gravitational bulk theory; $\bar{S}$ is the on-shell action; $\langle \cdot \rangle$ denotes the ensemble average; and $\varphi^{(0)} = \varphi|_{u=0}$, in coordinates such that the AdS boundary is at $u = 0$, which is where the gauge theory is realized.

The gauge theory is said to live on the boundary of the bulk. In fact, one obtains the on-shell action by evaluating the integral for a field $\varphi$ which is the solution of the equations of motion subject to conditions imposed at the AdS boundary, $\varphi|_{u=0} = \varphi^{(0)}$. In this case, $\bar{S}$ reduces to a surface term on the AdS boundary, which, for $D = 5$, reduces the 5D action (left-hand side of Eq. (11)) to a 4D quantity (right-hand side of Eq. (11), identified with the partition function of the boundary gauge theory, when an external source $\varphi^{(0)}$ is added). In fact, from the 4D gauge theoretical point of view, $\varphi^{(0)}$ is an external source, whereas from the 5D gravitational point of view, $\varphi$ is seen as a field propagating across the bulk. Therefore, in the context of AdS/CFT, one can say that a bulk field behaves as an external source of an operator in the boundary theory.

In this framework, the GKPW relation yields the following expression for the one-point function, related to the response of a system when an external source is added [26, 27],

$$ \langle \mathcal{O} \rangle_S = \frac{\delta \bar{S}[\varphi^{(0)}]}{\delta \varphi^{(0)}}. $$

The one-point function in the absence of the external source is obtained by simply evaluating the expression above for $\varphi^{(0)} = 0$,

$$ \langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_S|_{\varphi^{(0)} = 0}. \quad (13) $$

One considers the bulk theory to be GR in 5D with negative cosmological constant, $\Lambda_5$. Therefore the action is simply the Einstein–Hilbert one, added to matter fields

$$ S = \frac{1}{16\pi} \int d^5x \sqrt{-g} (R - 2\Lambda_5) + S_{\text{mat}}, \quad (14) $$

where $S_{\text{mat}}$ is specified by the boundary theory of interest. Regarding massless scalar field yields

$$ S_{\text{mat}} = -\int d^5x \sqrt{-g} \left( g_{MN} \nabla^M \varphi \nabla^N \varphi \right). \quad (15) $$

The solution to the 5D bulk action will be asymptotically AdS spacetime. A particular case of interest is the AdS$_5$–Schwarzschild spacetime,

$$ ds^2 = -\frac{r^2_0}{u^2} f(u) dt^2 + \frac{1}{u^2 f(u)} du^2 + \frac{r^2_0}{u^2} \delta_{ij} dx^i dx^j, \quad (16) $$

where $f(u) = 1 - u^4$, with $u = r_0/r$ defining the radial coordinated such that $u = 1$ locates the horizon, whereas $u = 0$ is the spacetime boundary. Also, one notices that the AdS radius is set to unity, $L = 1$. For $u \to 0$, Eq. (16) reads

$$ ds^2 = \frac{r^2_0}{u^2} \left( -dt^2 + \frac{1}{r^2_0} du^2 + \delta_{ij} dx^i dx^j \right). \quad (17) $$

Since the Einstein–Hilbert term in Eq. (14) clearly does not depend on the scalar field, the one-point function, Eq. (12), depends only on the matter contribution when it comes to computing the on-shell action, $\bar{S}$. Then, in what follows one effectively considers $S = S_{\text{mat}}$. Assuming that the scalar field is static and homogeneous along the spatial directions of the boundary, i.e. $\varphi = \varphi(u)$, and considering the asymptotic behaviour of the metric, Eq. (17), the action for the massless scalar field at the boundary becomes

$$ S \sim \int d^4x \left( \frac{r^4_0}{2u^4} \varphi \varphi' \right) |_{u=0} + \int d^5x r^4_0 \left( \frac{1}{2u^3} \varphi' \right)' \varphi, \quad (18) $$

assuming that the scalar field vanishes at the horizon. Notice that the second term is just the EOM for the scalar field, whose asymptotic solution reads

$$ \varphi \sim \varphi^{(0)} \left( 1 + \varphi^{(1)} u^4 \right). \quad (19) $$

The second term in Eq. (18) vanishes, and the on-shell action reduces to the surface term on the AdS boundary $(u = 0)$. Finally, substituting the asymptotic form of the scalar field, Eq. (19), one obtains the on-shell action,

$$ \bar{S}[\varphi^{(0)}] \sim \int d^4x \left( 2r^4_0 \left( \varphi^{(0)} \right)^2 \varphi^{(1)} \right). \quad (20) $$

The one-point function, Eq. (12), reads

$$ \langle \mathcal{O} \rangle_S = 4r^4_0 \varphi^{(1)} \varphi^{(0)} \quad (21) $$

Notice that the absence of the external source implies that Eq. (13) vanishes, so that

$$ \delta \langle \mathcal{O} \rangle = 4r^4_0 \varphi^{(1)} \varphi^{(0)}. \quad (22) $$

Relating this result to Eq. (5) one determines the retarded Green’s function,

$$ G^O_{\mathcal{O}}(k = 0) = -4r^4_0 \varphi^{(1)} \varphi^{(0)}, \quad (23) $$

where the Green’s function does depend neither on $\omega$ nor on $\mathbf{q}$, since $\varphi^{(1)}$ also does not. As it was mentioned in the beginning of this subsection, $\eta$ is related to a gravitational perturbation. In fact, all the results obtained for the scalar field here apply in this case because, formally, the gravitational perturbation reduces to the EOM governing a massless scalar field [28]. Thus, one can use the results obtained in this section to compute $\delta \langle \tau^{xy} \rangle$ from Eq. (22).

III. THE ADS$_5$–SCHWARZSCHILD DEFORMED BLACK BRANE

The general solution to 5D vacuum Einstein gravity with a negative cosmological constant depends on the horizon metric $H_{ij}$ and an integration constant, $k$. Provided that the constraint $R_{ij} = 3kH_{ij}$ holds, the solution
for $k = 0$, leading to a planar horizon i.e. $H_{ij} = \delta_{ij}$, is the AdS$_5$–Schwarzschild black brane [29]

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 (dx^2 + dy^2 + dz^2), \quad (24)$$

with $f(r) = 1 - \frac{r_s}{r}$. The black brane entropy density is defined from the area law as $s = a/(4G_5)$, where the horizon area density, $a = A/V$, is written in terms of the horizon area,

$$A = \iiint \sqrt{g_{xx} g_{yy} g_{zz}} |_{r = r_0} dz \, dy \, dx.$$

Therefore, $a = A/V$ can be evaluated directly by substituting Eq. (24) into (25), and the entropy density of the AdS$_5$–Schwarzschild black brane reads

$$s = \frac{r_s^3}{4G_5}. \quad (26)$$

The theory dual to AdS$_5$–Schwarzschild black brane is a conformal fluid [30]. Hence its stress-energy tensor is traceless, fixing the bulk viscosity [1, 3], $\zeta = 0$, leaving the shear viscosity $\eta$ as the only non-trivial transport coefficient. To evaluate it, one first considers a gravitational perturbation and then compute the response to the stress-energy tensor, by solving the perturbation equation within the hydrodynamic limit [23, 28].

We will fully present the arguments and a similar calculation in the next section, when considering the deformed AdS$_5$–Schwarzschild black brane as the gravitational background. The calculation will not be exactly the same, but analogous. The saturation of the $\eta/s$ ratio in the AdS$_5$–Schwarzschild black brane gravitational background reads [31]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (27)$$

One does not need discuss specific bulk features, as the existence of solutions to the higher-dimensional Einstein’s equations describing gravity is undertaken by the Campbell–Magaard embedding theorems [32]. Therefore, considering a brane with finite tension embedded in an AdS bulk, the Gauss–Codazzi equations yield the electric part of the Weyl tensor.

In an AdS bulk with cosmological constant $\Lambda$, a solution must satisfy the effective Einstein’s equations

$$R_{AB} = \Lambda g_{AB} + \mathcal{E}_{AB}. \quad (28)$$

Projecting Eq. (28) onto the brane, which is timelike and has codimension 1, in Gaussian coordinates $(x^\mu, z)$, where $z = 0$ corresponds to the brane itself, one obtains constraints

$$R_{\mu z} = 0, \quad R = \Lambda, \quad (29)$$

for $\Lambda$ denoting the brane cosmological constant. Eqs. (29) mimics constraints in the ADM procedure [33]. The Hamiltonian constraint is equivalent to $R_{\mu \nu} = \mathcal{E}_{\mu \nu}$.

One supposes a general metric, performing the coordinate change $u = r_0/r$ and sets the AdS radius to unity,

$$ds^2 = -\frac{r_0^2}{u^2} N(u) dt^2 + \frac{1}{u^2 A(u)} du^2 + \frac{r_0^2}{u^2} \delta_{ij} dx^i dx^j, \quad (30)$$

where $r_0$ is the horizon radius. By demanding that the ADM constraint leads to the AdS$_5$–Schwarzschild metric when $\beta \to 1$, the Hamiltonian constraint reads,

$$\frac{2N''}{N} - \frac{N'^2}{N^2} + \frac{2A''}{A} + \frac{A'^2}{A^2} - \frac{N'A'}{NA} + \frac{4A}{r} \left( \frac{N'}{N} - \frac{A'}{A} \right) - \frac{4A}{r^2} = f(r, r_0, \beta), \quad (31)$$

where the function $f(r, r_0, \beta)$ is given by Eq. (A1) in the Appendix A. The constraint (31) is satisfied by

$$N(u) = 1 - u^4 + (\beta - 1) u^6, \quad (32)$$

$$A(u) = (1 - u^4) \left( \frac{2 - 3u^4}{2 - (4\beta - 1) u^4} \right). \quad (33)$$

The constant $\beta$ parameter is referred to as a deformation parameter. In the next section we will investigate how the shear-viscosity-to-entropy density ratio can drive specific values for $\beta$.

IV. $\eta/s$ FOR THE AdS$_5$–SCHWARZSCHILD DEFORMED BLACK BRANE

We now consider the AdS$_5$–Schwarzschild deformed black brane (30, 32, 33), and derive the $\eta/s$ ratio in this gravitational background. An important remark is the use of some results which are only valid as long as the gravitational action takes the form of the Einstein–Hilbert action. In fact, the AdS$_5$–Schwarzschild deformed metric arises as a deformation of the AdS$_5$–Schwarzschild one [11]. Hence the same action-dependent results may be applied to the AdS$_5$–Schwarzschild deformed black brane.

The metric determinant, $g$, is such that $\sqrt{-g} = \frac{r_0^4}{r^4} \sqrt{\frac{\Lambda}{g}}$, where, from now on, $N$ and $A$ refer respectively to $N(u)$ and $A(u)$, unless otherwise specified.

Let one considers a bulk perturbation $h_{xy}$, so that:

$$ds^2 = ds^2_{AdS_5-SD} + 2h_{xy} dx dy, \quad (34)$$

where $ds^2_{AdS_5-SD}$ denotes the AdS$_5$–Schwarzschild deformed black brane metric, Eq. (30). Now, one considers Eq. (8), for $h_{xy}^{(0)}$ being the perturbation added to the boundary theory, which is asymptotically related to $h_{xy}$, the bulk perturbation, by

$$g^{xy} h_{xy} \sim h_{xy}^{(0)} \left( 1 + h_{xy}^{(1)} u^4 \right), \quad (35)$$

according to Eq. (19). Notice that one can directly use the results for a massless scalar field, as $g^{\mu \nu} h_{\mu \nu}$ obeys
the EOM for a massless scalar field [23, 28]. Besides, the AdS$_5$–Schwarzschild deformed black brane has the same asymptotic behavior of the AdS$_5$–Schwarzschild black brane (namely, Eq. (17)). In the context of what was discussed in Sec. II B, one can identify $g^{x\tau}h_{xy}$ as the bulk field, $\varphi$, which plays the role of an external source of a boundary operator, in this case $\tau^y$. Therefore, one can directly obtain the response $\delta (\tau^y)$, from Eq. (22),

$$\delta (\tau^y) = \frac{r_0^4}{16\pi G_5} 4h_{xy}^{(1)} h_{xy}^{(0)}, \quad (36)$$

where it is now convenient to reintroduce the $1/16\pi G_5$ factor. Comparing Eqs. (8) and (36) yields

$$i\omega \eta = \frac{r_0^4}{4\pi G_5} h_{xy}^{(1)}. \quad (37)$$

Since both metrics, the AdS$_5$-Schwarzschild and its deformation, are obtained from the Einstein-Hilbert action, the entropy density is the same c.f. Eq. (26). Plugging this result into Eq. (37) yields

$$\frac{\eta}{s} = \frac{r_0}{\pi} \frac{h_{xy}^{(1)}}{i\omega}. \quad (38)$$

Now, $h_{xy}^{(1)}$ is found by solving the EOM for the perturbation $g^{x\tau}h_{xy} \equiv \varphi$, which is that of a massless scalar field [23, 28]

$$\nabla_M \left( \sqrt{-g}g^{MN} \nabla_N \varphi \right) = 0 . \quad (39)$$

Considering a stationary perturbation of the form $\varphi = \Phi(u)e^{-i\omega t}$, the perturbation equation reduces to a second-order ODE for $\Phi(u)$,

$$\Phi'' + \left( \frac{(N\lambda)' - 3}{2NA} \right) \Phi' + \frac{1}{NA} \frac{r_0^4}{\omega^2} \Phi = 0 . \quad (40)$$

To derive the solution of Eq. (40), two boundary conditions are imposed: the incoming wave boundary condition in the near-horizon region, corresponding to $u \to 1$, and a Dirichlet boundary condition at the AdS boundary, $\Phi(u \to 0) = \Phi_0$, where $h_{xy}^{(0)} = \Phi_0 e^{-i\omega t}$.

To incorporate the near-horizon incoming wave boundary condition, Eq. (40) is first solved in the limit $u \to 1$. After a straightforward computation one finds the following solution

$$\Phi \propto \exp \left( \pm \frac{i\omega}{r_0} \sqrt{\frac{4\beta - 3}{\beta - 1}} \sqrt{1 - u} \right) . \quad (41)$$

As discussed in Ref. [24], this solution has a natural interpretation using tortoise coordinates, which allows one to identify this solution to a plane wave. The positive exponent represents the wave outgoing from the horizon, whereas the negative one describes the wave incoming to the horizon, which, according to the near-horizon boundary condition, allows us to fix, in the $u \to 1$ regime,

$$\Phi \approx \exp \left( - \frac{i\omega}{r_0} \sqrt{\frac{4\beta - 3}{\beta - 1}} \sqrt{1 - u} \right) . \quad (42)$$

Now Eq. (40) will be solved for all $u \in [0, 1]$, with a power series in $\omega$, up to $O(\omega)$, as we are interested in a solution in the hydrodynamic limit, $\omega \to 0$:

$$\Phi(u) = \Phi_0(u) + \omega \Phi_1(u) . \quad (43)$$

Therefore, in the hydrodynamic limit, the second term in Eq. (40), which is proportional to $\omega^2$, is not considered. By direct integration of the equation the solution reads

$$\Phi_i = C_i + K_i \int \frac{u^3}{\sqrt{NA}} du , \quad (44)$$

for $C_i$ and $K_i$, the integration constants and $i = 0, 1$. Thus, according to Eq. (43), we have

$$\Phi = (C_0 + \omega C_1) + (K_0 + \omega K_1) \int \frac{u^3}{\sqrt{NA}} du . \quad (45)$$

In the $u \to 0$ and $u \to 1$ limits, in order to impose the boundary conditions, one can expand the integral in (45) around these extremal values. It yields, up to leading order in the respective expansions,

$$\int \frac{u^3}{\sqrt{NA}} du = \begin{cases} \frac{u^4}{4}, & \text{for } u \to 0, \\ -\sqrt{\frac{3 - 4\beta}{1 - \beta}} \sqrt{1 - u}, & \text{for } u \to 1. \end{cases} \quad (46)$$

The boundary condition at the AdS boundary then fixes the first pair of integration constants, as

$$\lim_{u \to 0} \Phi = (C_0 + \omega C_1) + (K_0 + \omega K_1) \lim_{u \to 0} \frac{u^4}{4} = \Phi(0) , \quad (47)$$

implying that $(C_0 + \omega C_1) = \Phi(0)$. Near the horizon, $u \to 1$, one has

$$\Phi \approx \Phi(0) - (K_0 + \omega K_1) \sqrt{\frac{3 - 4\beta}{\beta - 1}} \sqrt{1 - u} . \quad (48)$$

Expanding the near-horizon solution, Eq. (42), up to $O(\omega)$, yields

$$\Phi \propto 1 - i\frac{\omega}{r_0} \sqrt{\frac{4\beta - 3}{\beta - 1}} \sqrt{1 - u} . \quad (49)$$

It is straightforward to see that Eq. (48) fixes the proportionality according to

$$\Phi \approx \Phi(0) - i\frac{\omega}{r_0} \sqrt{\frac{4\beta - 3}{\beta - 1}} \sqrt{1 - u} . \quad (50)$$

Comparing Eqs. (48) and (50) allows us to fix the second pair of integration constants:

$$(K_0 + \omega K_1) = i\frac{\omega}{r_0} \left( \frac{\beta - 1}{4\beta - 3} \right) \frac{|4\beta - 3|}{|\beta - 1|} . \quad (51)$$

Then the full solution reads

$$\Phi = \Phi(0) \left( 1 + i\frac{\omega}{r_0} \left( \frac{\beta - 1}{4\beta - 3} \right) \frac{|4\beta - 3|}{|\beta - 1|} \int \frac{u^3}{\sqrt{NA}} du \right) . \quad (52)$$
Accordingly, the full time-dependent perturbation
\[ \varphi = g^{xx} h_{xy} = \phi(u)e^{-i\omega t}, \quad (53) \]
is asymptotically given by:
\[ g^{xx} h_{xy} \sim e^{-i\omega t} \phi(0) \left( 1 + i \frac{\omega}{r_0} \left( \frac{\beta - 1}{4\beta - 3} \right) \frac{|4\beta - 3|}{|\beta - 1|} u^4 \right). \quad (54) \]
Comparing Eq. (54) to Eq. (35), and identifying
\[ h_{xy}^{(0)} = \phi(0)e^{-i\omega t}, \quad (55) \]
yields
\[ h_{xy}^{(1)} = \frac{i \omega}{4r_0} \left( \frac{\beta - 1}{4\beta - 3} \right) \frac{|4\beta - 3|}{|\beta - 1|}. \quad (56) \]
Thus, substituting the above result into Eq. (38) implies that
\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( \frac{\beta - 1}{4\beta - 3} \right) \frac{|4\beta - 3|}{|\beta - 1|}. \quad (57) \]

For computations in gravitational backgrounds which are solutions of the EOM from the Einstein–Hilbert action with \( \Lambda < 0 \), the ratio bound \( \eta/s \geq 1/4\pi \) holds [31]. Both the AdS\(_5\)-Schwarzschild and the AdS\(_5\)-Schwarzschild deformed black branes satisfy this bound. In this case, the deformation parameter can attain the values
\[ \beta \leq \frac{3}{4} \text{ or } \beta \geq 1. \quad (58) \]
The saturation \( \eta/s = 1/4\pi \), corresponding to \( N_c \to \infty \), then implies \( \beta = 1 \). This result has been expected, as this case recovers the AdS\(_5\)-Schwarzschild black brane (16). On the other hand, the saturation of the shear viscosity-to-entropy density ratio also yields \( \beta = 3/4 \), generating a new black brane solution. In this case, the AdS\(_5\)-Schwarzschild deformed black brane is given by Eq. (30), with
\[ N(u) = 1 - u^4 - \frac{u^6}{4}, \quad (59) \]
\[ A(u) = 1 - \frac{3}{2} u^4. \quad (60) \]
This new solution can be an interesting result worthy further investigation, mainly in the AdS/QCD correspondence.

V. CONCLUDING REMARKS AND PERSPECTIVES

The ADM procedure was used to derive a family of AdS\(_5\)-Schwarzschild deformed gravitational backgrounds, involving a free parameter, \( \beta \), in the black brane metric (30, 32, 33). Computing the \( \eta/s \) ratio for this family provided two possible values to \( \beta \). The first one, \( \beta = 1 \), was physically expected, corresponding to the AdS\(_5\)-Schwarzschild black brane. The another one, \( \beta = 3/4 \), generates a new AdS\(_5\)-Schwarzschild-like deformed black brane (30, 59, 60). Besides the importance of the result itself, in particular for the membrane paradigm of AdS/CFT, it has a good potential for relevant applications, mainly in AdS/QCD.

As large-\( N_c \) gauge theories considered by AdS/CFT are good approximations to QCD, one could expect that the result of Eq. (27) may be applied to the quark-gluon plasma (QGP), which is a natural phenomenon in QCD, when at high enough temperature the quarks and gluons are deconfined from protons and neutrons to form the QGP [34]. In fact, experiments in the Relativistic Heavy Ion Collider (RHIC) have shown that the QGP behaves like a viscous fluid with very small viscosity, which implies that the QGP is strongly-coupled, thus discarding the possibility of using perturbative QCD to the study of the plasma. Therefore, the new AdS\(_5\)-Schwarzschild deformed black brane (30, 59, 60) can be widely used to probe additional properties in the AdS/QCD approach. As in the holographic soft-wall AdS/QCD the AdS\(_5\)-Schwarzschild black brane provides a reasonable description of mesons at finite temperature [20], we can test if using the AdS\(_5\)-Schwarzschild deformed black brane derives a more reliable meson mass spectra for the mesonic states and their resonances, better matching experimental results. Besides, the new AdS\(_5\)-Schwarzschild deformed black brane can be also explored in the context of the Hawking–Page transition and information entropy [35, 36].

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Appendix A

\[ f(r, r_0, \beta) = -\frac{1}{r^{10}} \left\{ -\left(10(\beta - 1) + r^6 - 3r^2 r_0^4 \right) \left(\beta + r^6 - r^2 r_0^4 - 1 \right) + \frac{4r^8 \left(-2\beta + r^6 + r^2 r_0^4 + 2\beta \right) \left(\beta + r^6 - r^2 r_0^4 - 1 \right)}{(2r^4 - 5r^4 r_0^4 + 3r_0^8) \left(2r^4 + (1 - 4\beta) r_0^8 \right)} + \frac{4r^8 \left(8(2 - 3\beta)r_0 r_0^4 + 2(2\beta - 23) r^4 r_0^4 + 3(4\beta - 1) r_0^4 r_0^4 \right)^2}{(2r^6 - 5r^4 r_0^4 + 3r_0^8) \left(2r^4 + (1 - 4\beta) r_0^8 \right)} \right\} \]  

(A1)

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