CLASSIFYING COMPLEMENTS FOR ASSOCIATIVE ALGEBRAS

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Abstract. For a given extension $A \subset E$ of associative algebras we describe and classify up to an isomorphism all $A$-complements of $E$, i.e. all subalgebras $X$ of $E$ such that $E = A + X$ and $A \cap X = \{0\}$. Let $X$ be a given complement and $(A, X, \triangleright, \triangleleft, \rhd, \lhd)$ the canonical matched pair associated with the factorization $E = A + X$. We introduce a new type of deformation of the algebra $X$ by means of the given matched pair and prove that all $A$-complements of $E$ are isomorphic to such a deformation of $X$. Several explicit examples involving the matrix algebra are provided.

Introduction

The concept of a matched pair first appeared in the group theory setting ([14]). Since then, the corresponding concepts were introduced for several other categories such as Lie algebras ([9]), Hopf algebras ([10]), groupoids ([5]), Leibniz algebras ([1]), locally compact quantum groups ([15]), etc. With any matched pair of groups (resp. Lie algebras, Hopf algebras, etc.) we can associate a new group (resp. Lie algebra, Hopf algebra, etc.) called the bicrossed product. The bicrossed product construction is responsible for the so-called factorization problem, which asks for the description and classification of all objects $E$ (groups, Lie algebras, Hopf algebras, etc.) which can be written as a 'product' of two subobjects $A$ and $X$ having 'minimal intersection' in $E$ - we refer to [3] for more details, a historical background and additional references. In the setting of associative algebras, the bicrossed product was recently introduced in [4] as a special case of the more general unified product. However, in this paper we use a slightly more general construction than the one from [4], leaving aside the unitary condition on the algebras.

The classifying complements problem (CCP) was introduced in [2] in a very general, categorical setting, as a sort of converse of the factorization problem. A similar problem, called invariance under twisting, was studied in [11] for Brzezinski’s crossed products. In this paper we deal with the (CCP) in the context of associative algebras:

Classifying complements problem (CCP): Let $A \subset E$ be a given subalgebra of $E$. If an $A$-complement of $E$ exists, describe explicitly, classify all $A$-complements of $E$ and compute the cardinal of the (possibly empty) isomorphism classes of all $A$-complements of $E$ (which will be called the factorization index $[E : A]^f$ of $A$ in $E$).

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Another related problem which will not be discussed in this paper is that concerning the existence of complements whose natural approach is the computational one. In the sequel the existence of a complement will, however, be a priori assumed and we will be interested in describing all complements of an algebra extension \( A \subset E \) in terms of one given complement.

The paper is organized as follows. In Section 1 we recall the bicrossed product for associative algebras introduced in [4]. However, the construction used in this paper is slightly more general as we drop the unitary assumption on the algebras. Section 2 contains the main results of the paper which provide the complete answer to the (CCP) for associative algebras. Let \( A \subset E \) be a given extension of algebras. If \( X \) is a given \( A \)-complement of \( E \) then Theorem 2.3 provides the description of all complements of \( A \) in \( E \): any \( A \)-complement of \( E \) is isomorphic to an \( r \)-deformation of \( X \), as defined by (6).

In other words, exactly as in the case of Hopf algebras, Lie algebras or Leibniz algebras, given \( X \) an \( A \)-complement of \( E \) all the other \( A \)-complements of \( E \) are deformations of the algebra \( X \) by certain maps \( r : X \rightarrow A \) associated with the canonical matched pair which arises from the factorization \( E = A + X \). The theoretical answer to the (CCP) is given in Theorem 2.6 where we explicitly construct a cohomological type object \( \mathcal{H}A^2(X, A \mid (\triangleright, \triangleleft, \leftarrow, \rightarrow)) \) which parameterizes all \( A \)-complements of \( E \). We introduce the factorization index \( [E : A]^I \) of a given extension \( A \subset E \) as the cardinal of the (possibly empty) isomorphism classes of all \( A \)-complements. Moreover, we prove that the factorization index is computed by the formula: \( [E : A]^I = |\mathcal{H}A^2(X, A \mid (\triangleright, \triangleleft, \leftarrow, \rightarrow))| \).

Several explicit examples are provided. More precisely, we indicate associative algebra extensions whose factorization index is 1, 2 or 3. We end the paper with an extension of index at least 4.

1. Preliminaries

Unless otherwise stated, all vector spaces, linear or bilinear maps are over an arbitrary field \( K \) of characteristic zero. A map \( f : V \rightarrow W \) between two vector spaces is called the trivial map if \( f(v) = 0 \), for all \( v \in V \). By an algebra \( A \) we mean an associative, not necessarily unital algebra over \( K \). The concept of left/right \( A \)-module or \( A \)-bimodule is defined as in the case of unital algebras except of course for the unitary condition. \( A \mathcal{M}_A \) stands for the category of all \( A \)-bimodules, i.e. triples \((V, \triangleright, \triangleleft)\) consisting of a vector space \( V \) and two bilinear maps \( \triangleright : A \times V \rightarrow V \), \( \triangleleft : V \times A \rightarrow V \) such that \((V, \triangleright)\) is a left \( A \)-module, \((V, \triangleleft)\) is a right \( A \)-module and \( a \triangleright (x \triangleleft b) = (a \triangleright x) \triangleleft b \), for all \( a, b \in A \) and \( x \in V \).

Let \( A \subseteq E \) be a subalgebra. Another subalgebra \( X \) of \( E \) is called an \( A \)-complement of \( E \) if \( E = A + X \) and \( A \cap X = \{0\} \). For an arbitrary integer \( n \geq 2 \) let \( \mathcal{M}_n(K) \) be the algebra of \( n \times n \) matrices over the field \( K \). We denote by \( e_{ij} \in \mathcal{M}_n(K) \) the matrix having 1 in the \((i, j)^{th}\) position and zeros elsewhere.

Bicrossed products revisited. We recall the construction of the bicrossed product for associative algebras as defined in [4] but rephrased into the present setting. More precisely, working with associative not necessarily unital algebras will result in dropping the normalization assumption on the matched pair.
**Definition 1.1.** A matched pair of algebras is a system \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\) consisting of two algebras \(A, X\) and four bilinear maps

\[
\triangleleft : X \times A \to A, \quad \triangleright : X \times A \to A, \quad \leftarrow : A \times X \to A, \quad \rightarrow : A \times X \to X
\]

such that \((X, \rightarrow, \triangleleft)\) \(\in A \mathcal{M}_{A}\) is an \(A\)-bimodule, \((A, \triangleright, \leftarrow)\) \(\in X \mathcal{M}_{X}\) is an \(X\)-bimodule and the following compatibilities hold for any \(a, b \in A, x, y \in X\):

\[
\begin{align*}
\text{(MP1)} & \quad a \rightarrow (xy) = (a \rightarrow x) y + (a \leftarrow x) \rightarrow y; \\
\text{(MP2)} & \quad (ab) \leftarrow x = a (b \leftarrow x) + a \leftarrow (b \rightarrow x); \\
\text{(MP3)} & \quad x \triangleright (ab) = (x \triangleright a) b + (x \triangleleft a) \triangleright b; \\
\text{(MP4)} & \quad (xy) \triangleleft a = x \triangleleft (y \triangleright a) + x (y \triangleleft a); \\
\text{(MP5)} & \quad a (x \triangleright b) + a \leftarrow (x \triangleleft b) = (a \leftarrow x) b + (a \rightarrow x) \triangleright b; \\
\text{(MP6)} & \quad x \triangleleft (a \leftarrow y) + x (a \rightarrow y) = (x \triangleright a) \rightarrow y + (x \triangleleft a) y;
\end{align*}
\]

Let \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\) be a matched pair of algebras. Then, \(A \bowtie X = A \times X\), as a vector space, with the bilinear map defined by

\[
(a, x) \bullet (b, y) := (ab + a \leftarrow y + x \triangleright b, a \rightarrow y + x \triangleleft b + xy)
\]

for all \(a, b \in A\) and \(x, y \in X\) is an associative algebra called the bicrossed product associated with the matched pair \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\). As we will see in the following examples, matched pairs of algebras appear quite naturally from minimal sets of data.

**Examples 1.2.** 1) Let \(A\) be an algebra and \((X, \rightarrow, \triangleleft)\) \(\in A \mathcal{M}_{A}\) an \(A\)-bimodule. We see \(X\) as an algebra with the trivial multiplication, i.e. \(xy = 0\) for all \(x, y \in X\). It is straightforward to see that \((A, X, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) is a matched pair of \(\mathcal{A}\) and \(\mathcal{X}\) are the trivial actions. The multiplication on the corresponding bicrossed product \(A \bowtie X\) is given as follows:

\[
(a, x) \bullet (b, y) := (ab, a \rightarrow y + x \triangleleft b)
\]

The above bicrossed product is precisely the trivial extension of \(A\) by the \(A\)-bimodule \(X\).

2) The previous example can be slightly generalized by considering \(A\) and \(X\) to be both algebras such that \((X, \rightarrow, \triangleleft)\) \(\in A \mathcal{M}_{A}\) is an \(A\)-bimodule for which the following compatibilities hold

\[
a \rightarrow (xy) = (a \rightarrow x) y, \quad (xy) \triangleleft a = x (y \triangleleft a), \quad x (a \rightarrow y) = (x \triangleleft a) y
\]

for all \(a \in A, x, y \in X\). In \cite[Definition, pg. 212]{12} a bimodule \(X\) satisfying (3) is called a multiplicative \(A\)-bimodule. Then, the bicrossed product associated with the matched pair \((A, X, \triangleleft, \triangleright, \leftarrow, \rightarrow)\), where \(\triangleright_{0}, \leftarrow_{0}\) are the trivial actions will be called, following \cite[pg. 20]{4}, a semidirect product of \(A\) and \(X\). The multiplication on the corresponding bicrossed product \(A \bowtie X\) is given as follows:

\[
(a, x) \bullet (b, y) := (ab, a \rightarrow y + x \triangleleft b + xy)
\]

This construction originates in \cite[Lemma a]{12} where is presented in a different form.

The bicrossed product of two algebras is the construction responsible for the so-called factorization problem, which in the case of associative algebras comes down to:
Let $A$ and $X$ be two algebras. Describe and classify all algebras $E$ that factorize through $A$ and $X$, i.e. $E$ contains $A$ and $X$ as subalgebras such that $E = A + X$ and $A \cap X = \{0\}$. Recall from [4, Corollary 3.7] that an algebra $E$ factorizes through two subalgebras $A$ and $X$ if and only if there exists a matched pair of algebras $(A, X, \triangleright, \triangleleft, \blacktriangleright, \blacktriangleleft)$ such that $E \cong A \bowtie X$. More precisely, if $E$ factorizes through $A$ and $X$ we can construct a matched pair of algebras as follows:

$$x \triangleright a + x \triangleleft a := xa, \quad a \leftarrow x + a \rightarrow x := ax$$

(4)

for all $a \in A$, $x \in X$. Throughout, the above matched pair will be called the canonical matched pair associated with the factorization of $E$ through $A$ and $X$. Besides the factorizable algebras mentioned above, several other classes of associative algebras were studied recently: for instance, in [8], all complex finite-dimensional algebras of level one are described.

Examples 1.3. 1) Let $n \in \mathbb{N}$, $n \geq 2$. It can be easily seen that $\mathcal{M}_n(K)$ factorizes through the subalgebra of strictly lower triangular matrices $A = \{(a_{ij})_{i,j=1,n} \mid a_{ij} = 0 \text{ for } i \leq j\}$ and the subalgebra of upper triangular matrices $X = \{(x_{ij})_{i,j=1,n} \mid x_{ij} = 0 \text{ for } i > j\}$. We denote by $B_A := \{e_{ij} \mid i, j \in \mathbb{1,n}, i \geq j\}$ and $B_X := \{e_{ij} \mid i, j \in \mathbb{1,n}, i < j\}$ the $K$-basis of $A$, respectively $X$. Then, the canonical matched pair associated with this factorization is given as follows:

$$e_{ij} \leftarrow e_{lk} = \begin{cases} e_{ik}, & \text{if } i > k \geq j = l \\ 0, & \text{otherwise} \end{cases}, \quad e_{ij} \rightarrow e_{lk} = \begin{cases} e_{ik}, & \text{if } l = j < i \leq k \\ 0, & \text{otherwise} \end{cases}$$

$$e_{rs} \triangleright e_{pt} = \begin{cases} e_{rt}, & \text{if } t < r \leq s = p \\ 0, & \text{otherwise} \end{cases}, \quad e_{rs} \triangleleft e_{pt} = \begin{cases} e_{rt}, & \text{if } r \leq t < s = p \\ 0, & \text{otherwise} \end{cases}$$

2) Consider $n \in \mathbb{N}$, $n \geq 2$. Then $\mathcal{M}_n(K)$ factorizes also through the subalgebras $A = \{(a_{ij})_{i,j=1,n} \mid a_{nu} = 0 \text{ for all } u = \mathbb{1,n}\}$ and $X = \{(x_{ij})_{i,j=1,n} \mid x_{kl} = 0 \text{ for all } k = \mathbb{1,n-1} \text{ and } l = \mathbb{1,n}\}$. We denote by $B_A := \{e_{ij} \mid i = \mathbb{1,n-1}, j = \mathbb{1,n}\}$ and $B_X := \{e_{nj} \mid j = \mathbb{1,n}\}$ the $K$-basis of $A$, respectively $X$. The canonical matched pair associated with this factorization is given as follows:

$$e_{nu} \triangleleft e_{vt} = \begin{cases} e_{nt}, & \text{if } u = v \\ 0, & \text{otherwise} \end{cases}, \quad e_{vt} \leftarrow e_{nu} = \begin{cases} e_{vt}, & \text{if } t = n \\ 0, & \text{otherwise} \end{cases}$$

while the other two actions are both trivial.

3) Let $R$, $S$ be $K$-algebras and $M \in r\mathcal{M}_S$. Then the algebra $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ factorizes through the subalgebras $A := \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$ and $X = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. More precisely, the associated matched pair is given as follows for all $r \in R$, $s \in S$, $m \in M$:

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \triangleleft \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & ms \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \rightarrow \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & rm \\ 0 & 0 \end{pmatrix}$$

while the other two actions are both trivial.
2. Classifying complements. Applications

In this section we prove the main result of this paper which answers the (CCP) for associative algebras. First we need to introduce the following concept:

**Definition 2.1.** Let \((A, X, \triangleright, \triangleleft, \rightharpoonup, \leftharpoondown)\) be a matched pair of algebras. A linear map \(r : X \to A\) is called a deformation map of the matched pair \((A, X, \triangleright, \triangleleft, \rightharpoonup, \leftharpoondown)\) if the following compatibility holds for all \(x, y \in X\):

\[
r(x) r(y) - r(x y) = r(r(x) \rightharpoonup y + x \triangleleft r(y)) - r(x) \leftharpoondown y - x \triangleright r(y)
\]

We denote by \(\mathcal{DM}(A, X | (\triangleright, \triangleleft, \rightharpoonup, \leftharpoondown))\) the set of all deformation maps of the matched pair \((A, X, \triangleright, \triangleleft, \rightharpoonup, \leftharpoondown)\). The trivial map \(r : X \to A, r(x) = 0, \text{for all } x \in X\) is of course a deformation map. The right hand side of (5) measures how far \(r : X \to A\) is from being an algebra map.

The next example shows that computing all deformation maps associated with a given matched pair is a highly non-trivial problem.

**Examples 2.2.** Consider \(M_n(K)\) with the factorization given in Example 1.3. Then \(\mathcal{DM}(A, X | (\triangleright, \triangleleft, \rightharpoonup, \leftharpoondown))\) is in bijection with the families of scalars \(\{ \alpha_{cd}^{ab} \}_{a, b, c, d \in \mathbb{N}}\) subject to the compatibility condition:

\[
\sum_{q < t < k} \alpha_{ij}^{kt} \alpha_{rs}^{tq} = \delta_{jr} \alpha_{is}^{kq} + \sum_{r < u \leq s} \alpha_{ij}^{ur} \alpha_{is}^{kq} + \sum_{i \leq v < j} \alpha_{rs}^{jv} \alpha_{iv}^{kq} - \alpha_{ij}^{kr} \delta_{sq} - \alpha_{ij}^{jr} \delta_{ki}
\]

for all \(k > q, i \leq j\) and \(r \leq s\), where \(\delta_{jr}\) is the Kronecker symbol. The bijection is such that any deformation map \(r : X \to A\) is implemented from a family of scalars by the following formula:

\[
r(e_{ij}) = \sum_{k > t} \alpha_{ij}^{kt} e_{kt}, \text{ for all } i \leq j
\]

The following theorem is the key result in solving the (CCP) for associative algebras:

**Theorem 2.3.** Let \(A\) be a subalgebra of \(E\) and \(X\) a given \(A\)-complement of \(E\) with the associated canonical matched pair \((A, X, \triangleright, \triangleleft, \rightharpoonup, \leftharpoondown)\).

1. Let \(r : X \to A\) be a deformation map of the above matched pair. Then \(X_r := X\), as a vector space, with the new multiplication defined for any \(x, y \in X\) by:

\[
x \cdot_r y := xy + r(x) \rightharpoonup y + x \triangleleft r(y)
\]

is an associative algebra called the \(r\)-deformation of \(X\). Furthermore, \(X_r\) is an \(A\)-complement of \(E\).

2. \(X\) is an \(A\)-complement of \(E\) if and only if there exists an isomorphism of algebras \(X \cong X_r\), for some deformation map \(r : X \to A\) of the matched pair \((A, X, \triangleright, \triangleleft, \rightharpoonup, \leftharpoondown)\).

**Proof.** 1) The fact that the multiplication \(\cdot_r\) defined by (6) is associative follows by a long but straightforward computation which relies on the axioms (MP1)-(MP6). However, we present here a different and more natural proof which will shed some light on the
way we arrived at the multiplication given by (6). For a deformation map \( r : X \to A \), consider \( f_r : X \to E = A \bowtie X \) to be the \( K \)-linear map defined for any \( x \in X \) by:

\[
f_r(x) = (r(x), x)
\]

We will prove that \( \tilde{X} := \text{Im}(f_r) \) is an \( A \)-complement of \( E \). We start by showing that \( \tilde{X} \)

is a subalgebra of \( A \bowtie X \). Indeed, for all \( x, y \in X \) we have:

\[
(r(x), x)(r(y), y) = (r(x)r(y) + r(x) - y + x \triangleright r(y), r(x) - y + x \triangleleft r(y) + xy)
\]

Therefore \( \tilde{X} \) is a subalgebra of \( A \bowtie X \). We are left to prove that \( A \cap \tilde{X} = \{0\} \). To this end, consider \( (a, x) \in A \cap \tilde{X} \). Since in particular we have \((a, x) \in \tilde{X}\), it follows that \( a = r(x) \). As we also have \((r(x), x) \in A \) we obtain \( x = 0 \) and thus \( A \cap \tilde{X} = \{0\} \). Moreover, if \((b, y) \in E = A \bowtie X \) we can write \((b, y) = (b - r(y), 0) + (r(y), y) \in A + \tilde{X} \). Hence, \( \tilde{X} \) is an \( A \)-complement of \( E \). The proof will be finished once we prove that \( X_r \) and \( \tilde{X} \) are isomorphic as algebras. Denote by \( \tilde{f}_r \) the linear isomorphism from \( X \) to \( \tilde{X} \) induced by \( f_r \). We will prove that \( \tilde{f}_r \) is an algebra morphism if we consider \( X \) endowed with the multiplication given by (6). For all \( x, y \in X \) we have:

\[
\tilde{f}_r(x \cdot y) = \tilde{f}_r(y + r(x) - y + x \triangleleft r(y)) = (r(x)y + r(x) - y + x \triangleleft r(y), xy + r(x) - y + x \triangleleft r(y))
\]

Therefore, \( X_r \) is an algebra and the proof is now finished.

2) Let \( \overline{X} \) be an arbitrary \( A \)-complement of \( E \). Since \( E = A \oplus X = A \oplus \overline{X} \) we can find four \( K \)-linear maps:

\[
u : X \to A, \quad v : X \to \overline{X}, \quad t : \overline{X} \to A, \quad w : \overline{X} \to X
\]

such that for all \( x \in X \) and \( y \in \overline{X} \) we have:

\[
x = u(x) \oplus v(x), \quad y = t(y) \oplus w(y)
\]

By an easy computation it follows that \( v : X \to \overline{X} \) is a linear isomorphism of vector spaces. We denote by \( \tilde{v} : X \to A \bowtie X \) the composition \( \tilde{v} := i \circ v \) where \( i : \overline{X} \to E = A \bowtie X \) is the canonical inclusion. Therefore, we have \( \tilde{v}(x) = (-u(x), x) \), for all \( x \in X \). In what follows we will prove that \( r := -u \) is a deformation map and \( \overline{X} \cong X_r \). Indeed, \( \overline{X} = \text{Im}(v) = \text{Im}(\tilde{v}) \) is a subalgebra of \( E = A \bowtie X \) and we have:

\[
(r(x), x)(r(y), y) = (r(x)r(y) + r(x) - y + x \triangleright r(y), r(x) - y + x \triangleleft r(y) + xy)
\]

\[
= (r(z), z)
\]
for some \( z \in X \). Thus, we obtain:
\[
    r(z) = r(x) r(y) + r(x) \leftarrow y + x \triangleright r(y), \quad z = r(x) \rightarrow y + x \triangleleft r(y) + xy
\]
(8)

By applying \( r \) to the second part of (8) it follows that \( r \) is a deformation map of the matched pair \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\). Furthermore, (6) and (8) show that \( v : X_r \rightarrow X \) is also an algebra map. The proof is now finished.

**Remark 2.4.** We should point out that in the context of associative algebras there exists another type of deformation in the literature, not related to the one we introduce in Theorem 2.3 (see for instance [7]).

We will see in Example 2.7, different deformation maps can give rise to isomorphic deformations. Therefore, in order to classify all complements we introduce the following:

**Definition 2.5.** Let \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\) be a matched pair of algebras. Two deformation maps \( r, R : X \rightarrow A \) are called equivalent and we denote this by \( r \sim R \) if there exists \( \sigma : X \rightarrow X \) a \( K \)-linear automorphism of \( X \) such that for any \( x, y \in X \) we have:
\[
    \sigma(xy) - \sigma(x)\sigma(y) = \sigma(x) \triangleleft R(\sigma(y)) + R(\sigma(x)) \rightarrow \sigma(y) - \sigma(x \triangleleft r(y)) - \sigma(r(x) \rightarrow y)
\]

The classification of complements now follows:

**Theorem 2.6.** Let \( A \) be a subalgebra of \( E \), \( X \) an \( A \)-complement of \( E \) and \((A, X, \triangleright, \triangleleft, \leftarrow, \rightarrow)\) the associated canonical matched pair. Then \( \sim \) is an equivalence relation on the set \( \mathcal{DM}(X, A | (\triangleright, \triangleleft, \leftarrow, \rightarrow)) \) and the map
\[
    \mathcal{HA}^2(X, A | (\triangleright, \triangleleft, \leftarrow, \rightarrow)) := \mathcal{DM}(X, A | (\triangleright, \triangleleft, \leftarrow, \rightarrow))/ \sim \longrightarrow \mathcal{F}(A, E), \quad \mathcal{F} \mapsto X_r
\]
is a bijection between \( \mathcal{HA}^2(X, A | (\triangleright, \triangleleft, \leftarrow, \rightarrow)) \) and the isomorphism classes of all \( A \)-complements of \( E \). In particular, the factorization index of \( A \) in \( E \) is computed by the formula:
\[
    [E : A]^f = |\mathcal{HA}^2(X, A | (\triangleright, \triangleleft, \leftarrow, \rightarrow))|
\]

**Proof.** Two deformation maps \( r \) and \( R \) are equivalent in the sense of Definition 2.5 if and only if the corresponding algebras \( X_r \) and \( X_R \) are isomorphic. The conclusion follows by Theorem 2.3.

We end the paper with a few examples illustrating our theory:

**Examples 2.7.** 1) Let \( A \) be a two-sided ideal of \( E \). Then \([E : A]^f \leq 1\). Indeed, if an \( A \)-complement exists then it should be isomorphic to the factor algebra \( E/A \). Therefore, the factorization index is at most 1.

2) Let \( A = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \) be a subalgebra of \( E = M_2(K) \). In this case \([E : A]^f = 2\). Indeed, using Example 1.3 1), it follows that \( X = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \) is an \( A \)-complement of \( E \). The non-zero values of the canonical matched pair are given by:
\[
    e_{21} \leftarrow e_{11} = e_{21}, \quad e_{21} \rightarrow e_{12} = e_{22}
\]
\[
    e_{12} \triangleleft e_{21} = e_{11}, \quad e_{22} \triangleright e_{21} = e_{21}
\]
By a straightforward computation it can be seen that the associated deformation maps are as follows:

\[ r_a(e_{11}) = a e_{21}, \quad r_a(e_{12}) = a^2 e_{21}, \quad r_a(e_{22}) = -a e_{21}, \quad a \in K \]

The multiplication on the \( r_a \)-deformation of \( X \) is described below:

| \( e_{ij} \) | \( e_{11} \) | \( e_{12} \) | \( e_{22} \) |
|-----------|-------------|-------------|-------------|
| \( e_{11} \) | \( e_{11} \) | \( e_{12} + a e_{22} \) | \( 0 \) |
| \( e_{12} \) | \( a e_{11} \) | \( a^2 e_{22} \) | \( e_{12} \) |
| \( e_{22} \) | \( 0 \) | \( -a e_{22} \) | \( e_{22} \) |

If \( a = 0 \) then \( r_0 \) is the trivial map and \( X_{r_0} = X \). On the other hand, for any \( a \in K^* \) we have an isomorphism of algebras \( X_{r_a} \) and \( X_{r_1} \) given as follows:

\[ \varphi : X_{r_a} \to X_{r_1}, \quad \varphi(e_{11}) = e_{11}, \quad \varphi(a^{-1} e_{12}) = e_{12}, \quad \varphi(e_{22}) = e_{22} \]

To end with, it can be easily seen that \( X_{r_1} \) is not isomorphic to \( X \). Indeed, it is enough to observe that \( e_{11} + e_{22} \) is a unit for \( X \) while \( X_{r_1} \) is not unital. Therefore the factorization index is equal to 2.

3) Let \( M \in K \), and consider \( A = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \) a subalgebra of \( E = \begin{pmatrix} K & M \\ 0 & K \end{pmatrix} \). According to Example 1.3 3) for \( R = S := K \) we obtain that \( X = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \) is an \( A \)-complement of \( E \). Then any deformation map \( r : \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \) is uniquely implemented by two \( K \)-linear maps \( \alpha, \beta : M \to K \) such that \( r \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(m) & 0 \\ 0 & \beta(m) \end{pmatrix} \). It can be easily seen that \( r \) satisfies (5) if and only if \( \alpha(m) \beta(n) = 0 \) for all \( m, n \in M \). Therefore, we either have \( \alpha(m) = 0 \) for all \( m \in M \) or \( \beta(m) = 0 \) for all \( m \in M \). If \( \alpha(m) = 0 \) for all \( m \in M \) we obtain a deformation map \( r_\beta \) defined by \( r_\beta \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta(m) \end{pmatrix} \) for all \( m \in M \), where \( \beta : M \to K \) is an arbitrary \( K \)-linear map. The multiplication induced by this deformation map is given as follows:

\[ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \cdot_{r_\beta} \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m \beta(n) \\ 0 & 0 \end{pmatrix} \]

On the other hand, if \( \beta(m) = 0 \) for all \( m \in M \) we obtain a deformation map \( r_\alpha \) defined by \( r_\alpha \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(m) & 0 \\ 0 & 0 \end{pmatrix} \) for all \( m \in M \), where \( \alpha : M \to K \) is an arbitrary \( K \)-linear map. The multiplication induced by this deformation map is given as follows:

\[ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \cdot_{r_\alpha} \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha(m)n \\ 0 & 0 \end{pmatrix} \]

By a straightforward computation it can be proved that if the \( K \)-linear maps \( \alpha, \alpha' : M \to K \) are different from the trivial map then \( r_\alpha \) is equivalent in the sense of Definition 2.5 to \( r_\alpha' \). In the same manner, if the \( K \)-linear maps \( \beta, \beta' : M \to K \) are different from the trivial map then \( r_\beta \) is equivalent in the sense of Definition 2.5 to \( r_{\beta'} \). Finally, \( r_\alpha \) is never
equivalent to \( r_\beta \) in the sense of Definition 2.5, except for the case when both \( \alpha \) and \( \beta \) are equal to the trivial map. Therefore the factorization index \( |E : A|^1 = 3 \).

4) Let \( A = \begin{pmatrix} K & K & K \\ K & K & K \\ 0 & 0 & 0 \end{pmatrix} \) be a subalgebra of \( E = M_3(K) \). Then, by Example 1.3 2), it follows that \( X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & K & K \end{pmatrix} \) is an \( A \)-complement of \( E \). The non-zero values of the canonical matched pair are given as follows:

\[
\begin{align*}
\epsilon_{31} &\prec \epsilon_{11} = \epsilon_{31}, & \epsilon_{31} &\prec \epsilon_{12} = \epsilon_{32}, & \epsilon_{31} &\prec \epsilon_{13} = \epsilon_{33} \\
\epsilon_{32} &\prec \epsilon_{21} = \epsilon_{31}, & \epsilon_{32} &\prec \epsilon_{22} = \epsilon_{32}, & \epsilon_{32} &\prec \epsilon_{23} = \epsilon_{33} \\
\epsilon_{33} &\prec \epsilon_{31} = \epsilon_{31}, & \epsilon_{33} &\prec \epsilon_{32} = \epsilon_{32}, & \epsilon_{33} &\prec \epsilon_{33} = \epsilon_{33} \\
\end{align*}
\]

In this case the computational complexity increases dramatically, making it very difficult to compute all associated deformation maps. However, we are still able to check by a straightforward computation that the following maps are deformations of the above canonical matched pair:

\[
\begin{align*}
r_1 : X &\rightarrow A, & r_1(\epsilon_{33}) &= \epsilon_{22} \\
r_2 : X &\rightarrow A, & r_2(\epsilon_{33}) &= \epsilon_{11} + \epsilon_{22} \\
r_3 : X &\rightarrow A, & r_3(\epsilon_{31}) &= \epsilon_{12}, & r_3(\epsilon_{33}) &= \epsilon_{11} + \epsilon_{22} \\
\end{align*}
\]

We denote by \( X_i \) the \( r_i \)-deformation of \( X \), for all \( i = \overline{1,3} \). The multiplication tables of the \( X_i \)'s, \( i = \overline{1,3} \), are depicted below:

\[
\begin{align*}
X_1 : &\quad \epsilon_{33}\epsilon_{31} = \epsilon_{31}, & \epsilon_{33}\epsilon_{32} = \epsilon_{32}\epsilon_{33} = \epsilon_{32}, & \epsilon_{33}\epsilon_{33} = \epsilon_{33} \\
X_2 : &\quad \epsilon_{31}\epsilon_{33} = \epsilon_{33}\epsilon_{31} = \epsilon_{31}, & \epsilon_{32}\epsilon_{33} = \epsilon_{33}\epsilon_{32} = \epsilon_{32}, & \epsilon_{33}\epsilon_{33} = \epsilon_{33} \\
X_3 : &\quad \epsilon_{31}\epsilon_{31} = \epsilon_{32}, & \epsilon_{31}\epsilon_{33} = \epsilon_{33}\epsilon_{31} = \epsilon_{31}, & \epsilon_{32}\epsilon_{33} = \epsilon_{33}\epsilon_{32} = \epsilon_{32}, & \epsilon_{33}\epsilon_{33} = \epsilon_{33} \\
\end{align*}
\]

It can be easily seen that the \( X_i \)'s, \( i = \overline{1,3} \), are isomorphic to the algebras \( A_{3\overline{0}} \), \( A_{3\overline{4}} \) and respectively \( A_{3\overline{12}} \) listed in [13] (for a complete list of 3-dimensional associative algebras over \( \mathbb{C} \) we refer to [6]). In each case the isomorphism sends \( \epsilon_{31} \) to \( e_1 \), \( \epsilon_{32} \) to \( e_2 \) and \( \epsilon_{33} \) to \( e_3 \), where according to the notations of [13], \( \{e_1, e_2, e_3\} \) is a \( K \)-basis for the 3-dimensional algebras mentioned above. In particular, we obtain that the \( X_i \)'s are two by two non-isomorphic. Moreover, none of the three algebras listed above is isomorphic to \( X \). Indeed, to start with, we should notice that \( X \) is not commutative and therefore it cannot be isomorphic to the commutative algebras \( X_2 \) or \( X_3 \). We prove now that \( X \) is not isomorphic to \( X_1 \). Assume that \( \varphi : X_1 \rightarrow X \) is an isomorphism of algebras given by:

\[
\varphi(\epsilon_{31}) = \sum_{i=1}^3 a_i \epsilon_{3i}, \quad \varphi(\epsilon_{32}) = \sum_{i=1}^3 b_i \epsilon_{3i}, \quad \varphi(\epsilon_{33}) = \sum_{i=1}^3 c_i \epsilon_{3i}
\]

where \( a_i, b_i, c_i \in K \) for all \( i = \overline{1,3} \). Since \( \varphi(\epsilon_{31}) = \varphi(\epsilon_{33}\epsilon_{31}) = \varphi(\epsilon_{33})\varphi(\epsilon_{31}) = c_3\varphi(\epsilon_{31}) \) and \( \varphi(\epsilon_{31}) \neq 0 \) (as \( \varphi \) is an isomorphism) we obtain that \( c_3 = 1 \). Moreover, from \( 0 = \varphi(\epsilon_{32}\epsilon_{31}) = \varphi(\epsilon_{32})\varphi(\epsilon_{31}) = b_3\varphi(\epsilon_{31}) \) it follows that \( b_3 = 0 \). Finally, as \( \varphi(\epsilon_{32}) = \).
\[ \varphi(e_{32}e_{33}) = \varphi(e_{32})\varphi(e_{33}) = b_3\varphi(e_{33}) = 0 \] we have reached a contradiction. Therefore \( X_1 \) is not isomorphic to \( X \) and the factorization index \( [E : A]^f \geq 4 \).

**Remark 2.8.** After a careful analysis of Example 2.7 we can easily conclude that the deformations of a given algebra \( X \) do not necessarily preserve the properties of \( X \). For instance, in Example 2.7, 2) we obtained a non-unital algebra as a deformation of a unital one, while in Example 2.7, 4) we construct commutative deformations of a non-commutative algebra. This is not the case for the classical deformations studied in [7] as it is well known that finite dimensional unital algebras only deform to unital algebras.

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