Nolinear waves, differential resultant, computer algebra and completely integrable dynamical systems

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Abstract
The hierarchy of integrable equations are considered. The dynamical approach to theory of nonlinear waves is proposed. The special solutions (nonlinear waves) of considered equations are derived. We use powerful methods of computer algebra such as differential resultant and others.

1 Introduction
The paper is organized as follows: section 1 describes the basic facts of hierarchy of nonlinear equations, Baker-Akhiezer (BA) function theory for finite–gap integration method. Particular solutions in terms of generalized Novikov, Lamé, Hermite type polynomials are presented.

2 Preliminaries
We review in this section some basic facts about Baker–Akhieser function which will be used in the sequel. For systematic treatments and proofs, we refer the reader to [2, 3].

\[
\begin{align*}
[\partial_x - U, \partial_t - V^{(N)}] &= 0, \quad \text{(2.1)} \\
U &= U(x, t, \lambda), \quad V^{(N)} = V^{(N)}(x, t, \lambda), \quad \text{(2.2)}
\end{align*}
\]

where \( V^{(N)} = V_0 \lambda^N + V_1 \lambda^{N-1} + \ldots + V_N \). The nonlinear equations in the form (2.2) are obtain as compatibility condition of the following linear differential
equations
\[ \partial_x \Psi = U(x, t, \lambda) \Psi, \quad \partial_t \Psi = V^{(N)}_t(x, t, \lambda) \Psi. \] (2.3)

where \( \Psi \) is a vector-column \( (\Psi_1 \ldots \Psi_n)^T \). The system (2.3) play an important role in the theory of finite-gap integration method. The function \( \Psi \), which is general solution of the system (2.3), is meromorphic function on the Riemann surface and have essential singularities of prescribed form near fixed points on Riemann surface. This surface is defined by
\[ \det(L(x, t, \lambda) - \mu I) = 0. \] (2.4)

We call finite-gap solutions of (2.2) these for which there exists the meromorphic in \( \lambda \) matrix function \( L(x, t, \lambda) \) such that
\[ L_t = [V, L], \quad i = 1, \ldots, N \] (2.5)
\[ L_x = [U, L], \quad L = L_0 \lambda^N + L_1 \lambda^{N-1} + \ldots L_N. \] (2.6)

3 Resultant of two differential operators

Let us consider the following two differential operators
\[ A = \sum_{k=0}^{n} a_k D^k, \quad a_n \neq 0, \quad B = \sum_{l=0}^{m} b_l D^l, \quad b_m \neq 0, \quad a_k, b_l \in C_k^{(l)} \] (3.7)

where \( D = d/dx \), \( C_k^{(l)} \) is the set of \( k(l) \)-th differentiable functions on the interval \( I \) of real variable \( x \). Right resultant of operators \( A \) and \( B \) \( \text{RRes} (A, B) \) is called the determinant of resultant matrix \( R \) of degree \( (m+n) \). Analogically left resultant of operators \( A \) and \( B \) \( \text{LRes} (A, B) \) is called the resultant matrix \( R^* \) of degree \( (m+n) \), i.e. \( \text{RRes} (A, B) = \det(R), \text{LRes} (A, B) = \det(R^*) \). By definition \( \text{LRes} (A, B) = \text{RRes} (A^*, B^*) \), where \( A^* \) and \( B^* \) are the conjugated operators. The differential resultant answers to the question when the operators \( A \) and \( B \) have right(left) divisor i.e. when the overdetermined system \( Ay = 0, By = 0 \) (or \( A^*y = 0, B^*y = 0 \)) has solution. By construction to define right resultant \( \text{RRes} (A, B) \) we act with operators \( D^{m-1}, \ldots, D, D^0 = 1 \), by left to \( A \) and with operators \( D^{n-1}, \ldots, D, D^0 = 1 \) to \( B \). In this case the system \( Ay = 0, By = 0 \) has the following form
\[ \sum_{k=0}^{n+s} a_{k,s} y^{(k)} = 0, \quad s = 0 \div (m - 1), \quad \sum_{l=0}^{m+p} b_{l,p} y^{(l)} = 0, \quad p = 0 \div (n - 1), \] (3.8)

where the coefficients \( a_{k,s} \) and \( b_{l,p} \) are computed by:
\[ a_{k,s} = \sum_{i=0}^{s} \binom{s}{i} a_{k-i}^{(s-i)}, \quad b_{l,p} = \sum_{j=0}^{p} \binom{p}{j} b_{l-j}^{(p-j)}. \] (3.9)
The system (3.8) is constructed by homogeneous linear algebraic equations in terms of the following variables \(y^{(n+m-1)}, \ldots, y', y\). The matrix of coefficients of this system is the right resultant of the matrix \(R\), \(\det(R) = \text{RRes}(A, B)\).

\[
\text{RRes}(A, B) = \begin{vmatrix} a_{n+m-1,m-1} & a_{n+m-2,m-1} & \cdots & \cdots & a_{0,m-1} \\
0 & a_{n+m-1,m-2} & \cdots & \cdots & a_{0,m-2} \\
0 & \cdots & a_{n,0} & \cdots & a_{0,0} \\
b_{n+m-1,n-1} & b_{n+m-2,n-1} & \cdots & \cdots & b_{0,n-1} \\
0 & \cdots & b_{n,0} & \cdots & b_{0,0} \\
\end{vmatrix} \quad (3.10)
\]

If the system \(Ay = 0, By = 0\) is consistent then \(\det(R) = \text{RRes}(A, B) = 0\), in the opposite case the system is not consistent.

For applications of differential resultant see for example [?]. The most important of them are the following: the criterion of existing the greater right and left divisor of the system of operators; the criterion of conciseness of the system of linear ordinary differential equations with one unknown function; the criterion of commutation of two linear differential operators; the criterion of existing polynomial and exponential solutions of linear differential equations with variable coefficients; the criterion of factorization of operators of degree \(n\) into products of operators of degrees \(n-1\) and 1.

**Example.** Let us introduce the following equations,

\[
L_1 \equiv Ay = x^2 y'' + xy' - (x^2 + 1/4)y = 0, \quad (3.11)
\]

\[
L_2 \equiv By = 2xy'' + (3 - 4x)y' + (2x - 3)y = 0. \quad (3.12)
\]

Using (3.10) we obtain

\[
\text{RRes}(A, B) = \begin{vmatrix} x^2 & 3x & -x^2 + 3/4 & -2x \\
0 & x^2 & x & -x^2 - 1/4 \\
2x & 5x - 4 & 2x - 7 & 2 \\
0 & 2x & 3 - 4x & 2x - 3 \\
\end{vmatrix}
\]

**Problem 1.** Let us define two LODE:

\[
Ay = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad By = \sum_{l=0}^{m} b_l D^l y = 0, \quad b_m \neq 0, a_k, b_l \in C^{(k+l)}_I \quad (3.13)
\]

Find differential resultant of operators \(A\) and \(B\).

**Algorithm 1.** We solve problem 1 by the procedure DIFRESULT\((a, n, b, m, x)\):

**Input**
\(a\) is the array of coefficients of differential operator \(A\);  
\(n\) is the degree of differential operator \(A\);  
\(b\) is the array of coefficients of differential operator \(B\);  
\(m\) is the degree of differential operator \(B\);  
\(x\) is independent variable.  

Output:  
Using (3.9) with given coefficients of differential operator \(A\) and \(B\) we compute the elements of the resultant matrix, after that we find the determinant. The output is \(R_{\text{Res}}(A, B) = \det(R)\), where \(R\) is the resultant matrix.  

4 The generalized Riccati equation  
4.2.1. Prime right divisor of operator \(L\).  
Let us define the differential operator \(L\) of degree \(n\):  
\[
L = \sum_{k=0}^{n} a_k D^k, \quad a_n \neq 0, a_k \in \mathbb{C}, 
\]
A necessary and sufficient condition for the following factorization:  
\[
L = L_2(D - \alpha), \quad \text{ord} L_2 = n - 1
\]
is given by \(R_{\text{Res}}(L, D - \alpha) = 0\).  

\[
R_{\text{Res}}(L, D - \alpha) = \begin{vmatrix}
  a_n & a_{n-1} & \cdots & a_1 & a_0 \\
  1 & -\alpha & \cdots & (n-3) & (n-2) \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & 1 & -\alpha \\
  0 & 0 & \cdots & 0 & 1
\end{vmatrix} = 0 (4.14)
\]
Then we find the following generalized Riccati equation of first kind \(N(\alpha)\) of degree \((n - 1)\):  
\[
N(\alpha) = L(\alpha) + M(\alpha),
\]
where  
\[
L(\alpha) = \sum_{k=0}^{n} a_k \alpha^k,
\]
and \(L(\alpha)\) is called he pseudo-characteristic Riccati equation of first kind  
\[
M(\alpha) = \sum_{k=0}^{n} a_k M_{k-1}, \quad M_{k-1} = (D + \alpha)^{(k-1)} - \alpha^{(k-1)}
\]
and $M(\alpha)$ is called the reduced Riccati equation of first kind. When $n = 2$ the generalized Ricatti equation of first kind have the form

$$N(\alpha) = a_{2n'} + a_{2} \alpha^2 + a_{1} \alpha + a_{0}$$

4.2.2. Prime left divisor of operator $L$. Let us define the following differential operator $L$ of degree $n$:

$$L = \sum_{k=0}^{n} a_{k} D^{k}, \quad a_{n} \neq 0, \quad a_{k} \in C_{I}^{k}$$

To find factorization of following type,

$$L = L_{2}(D - \alpha), \quad \text{ord} L_{2} = n - 1$$

the necessary and sufficient condition is $\text{RRes}(L, D - \alpha) = 0$, or to find factorization of the following type,

$$L = (D - \alpha)L_{1}, \quad \text{ord} L_{1} = n - 1$$

the necessary and sufficient condition is $\text{LRes}(L, D - \alpha) = \text{RRes}(L^{*}, D + \alpha)$, i.e.

$$| a_{n}^{*} a_{n-1}^{*} \ldots a_{2}^{*} a_{1}^{*} a_{0}^{*} | 1 \alpha \ldots (n-1 \alpha^{n-3}) (n-2 \alpha^{n-2}) (n-1 \alpha^{n-1}) \frac{\alpha^{r-1}}{r-1} \alpha^{r-1} | = 0.$$ 

Thus we obtain the generalized Riccati equation $N^{*}(\alpha)$ of second kind and degree $(n - 1)$:

$$N^{*}(\alpha) = L^{*}(\alpha) + M^{*}(\alpha),$$

where

$$L^{*}(\alpha) \equiv \tau L = \sum_{r=0}^{n} \sum_{k=r}^{n} (-1)^{r} \binom{k}{r} a_{k}^{(r)} \alpha^{k} = L_{1}^{*}(D + \alpha). \quad (4.15)$$

Here $\tau$ is the conjugation operator, $L^{*}(\alpha)$ is pseudo-characteristic Riccati equation of second kind and

$$M^{*}(\alpha) = \sum_{r=1}^{n} \sum_{k=r}^{n} (-1)^{r} \binom{k}{r} a_{k}^{(r)} M_{r-1}^{*}, \quad M_{r-1}^{*} = (D + \alpha)^{(r-1)} - \alpha^{(r-1)} \quad (4.16)$$

$M^{*}(\alpha)$ is the reduced Riccati equation of second kind.

When $n = 2$ the generalized Riccati equation of second kind have the form,
\[ N(\alpha) = a_2\alpha' + a_2\alpha^2 + (-a_1 + 2a_2')\alpha + a_0 - a_1' + a_2'' = 0 \]

As a consequence of problem 1 we have:

**Problem 2.** Let us define the following equations,

\[ L_1 y = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C^k_f \quad (4.17) \]

\[ L_1^* y \equiv \tau L_1 = \sum_{r=0}^{n} \sum_{k=r}^{n} (-1)^k \binom{k}{r} a_k^{(k-r)} y^k = 0 \quad (4.18) \]

Find the generalized equation of first kind,

\[ N_1(\alpha) = R\text{Res}\left(L_1, D - \alpha\right) \]

and the generalized Riccati equation of second kind,

\[ N_2(\alpha) = R\text{Res}\left(L_1^*, D + \alpha\right) \]

**Algorithm 2.** We solve the problem 2 by the procedure 

\[ \text{RICCATI}(a, n, pp, x) \]

**Input**

- \( a \) is the array of coefficients of differential operator \( L_1 \);
- \( n \) is the degree of differential operator \( L_1 \);
- \( pp \) is the kind of seeking Riccati equation (\( pp = 1 \) and \( pp = 2 \));
- \( x \) is the independent variable.

**Output:** The generalized Riccati equation of first kind (\( pp = 1 \)) and of second kind (\( pp = 2 \)).

**R1:** If \( pp = 1 \) then go to R2 else go to R3;

**R2:** \( \ll L_2 = (D - \alpha)y; \)

- \( b \) is the array of coefficients of operator \( L_2 \);
- \( m := 1; \quad N_1(\alpha) := \text{DIFRESULT}(a, n, b, m, x); \quad \text{return } N_1(\alpha) \gg; \)

**R3:** \( \ll L_1^* = \tau L_1, \quad \tau \text{ is operator of conjugation,} \)

\[ L_1^* = \sum_{r=0}^{n} \sum_{k=r}^{n} (-1)^k \binom{k}{r} a_k^{(k-r)} y^k, \]

- \( a^* \) is the array of coefficients of differential operator \( L_1^* \);
- \( L_2^* = (D + \alpha)y; \)
- \( b^* \) is the array of the coefficients of differential operator \( L_2^* \);
- \( m := 1; \quad N_2(\alpha) := \text{DIFRESULT}(a^*, n, b^*, m, x); \quad \text{return } N_2(\alpha) \gg; \)
5 ODE resolved by algebraic means

4.3.1. Linear differential equations with exponential solutions of type \( y = \exp(-\alpha x) \), \( \alpha = \text{const.} \)

The necessary and sufficient condition for existence of such solutions is the pseudo-characteristic Riccati equation of first kind \( L(\alpha) \) and of second kind \( L^*(\alpha) \) to have solution \( \alpha = \text{const.} \), i.e. \( \text{RRes}(L, D + \alpha) = 0 \) or \( \text{RRes}(L^*, D - \alpha) = 0 \). Then the linear differential equation \( Ly = \sum_{k=0}^{n} a_k D^k y \) have the form \( Ly = L_2(D + \alpha)y \), where \( L_2 \) is differential operator, \( \text{ord}(L_2) = n - 1 \), and the differential equation have the exponential solution of following type \( y = \exp(-\alpha x) \). In the case of linear equation of second degree

\[
Ly = a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad a_k \neq 0, \quad a_k \in C^k_I, \quad k = 0 \div 2,
\]

the problem is to find the coefficients of factorization \( \alpha_1, \alpha_2 \) in \( Ly = (D - \alpha_1)(D - \alpha_2)y = 0 \), where \( \alpha_1 = \text{const} \) and \( \alpha_2 = \text{const.} \). i.e. we consider the classes of equations for which the associated Riccati equation is of first type or of second type in terms of \( \alpha \) have constant solutions. **Problem 3.** Given LEDE of degree \( n \),

\[
Ly = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, a_k \in C^k_I.
\]

Find exponential solution \( y = \exp(-\alpha x) \), where \( \alpha = \text{const.} \).

**Algorithm 3.** In our program the problem 3 is solved by the procedure \( LIVDIF(a, d, n, x) \).

**Input**

- \( a \) is the array of coefficients of operator \( L \);
- \( d \) is nonhomogeneous part of equation \( Ly \);
- \( n \) is the degree of differential operator \( L \);
- \( x \) is the independent variable.

**Output** If we find the particular solution of the following form \( y = \exp(-\alpha x) \), \( \alpha = \text{const} \) and differential equation is of degree two the coefficients of factorization \( \alpha_1, \alpha_2 \) and the fundamental system of solutions \( y_1(x), y_2(x) \) are obtained. If the degree of LODE is \( n > 2 \), then the degree of LODE is reduced by 1. If the exponential solution does not exists the message is received.

**D1:** Algorithm 2 for finding the generalized Riccati equation of first type \( N_1(\alpha) \), \( N_1(\alpha) := \text{RICCATI}(a, n, 1, x) \);
D2: \( L_1(\alpha) = \sum_{k=0}^{n} a_k \alpha^k \); \( L_1(\alpha) \) - pseudo-characteristic equation of first kind;

D3: \( M_1(\alpha) = N_1(\alpha) - L_1(\alpha) \); \( M_1(\alpha) \) - reduced equation of first kind;

D4: Subalgorithm for finding constant solution of algebraic equation \( L_1(\alpha) \) in terms of \( \alpha \).
   **Output:** constant solution \( \alpha_0 \), or obtain message that the solution of this type does not exists,

D5: If \( \alpha_0 \) exists and is solution of \( M_1(\alpha) = 0 \) then go to D11 else go to D6;

D6: Algorithm 2 for finding the generalized Riccati equation of second type \( N_2(\alpha) \).
   \( N_2(\alpha) := \text{RICCATI}(a, n, 2, x) \);

D7:
\[
L_2(\alpha) = \sum_{r=0}^{n} \sum_{k=r}^{n} (-1)^k \binom{k}{r} a_k^{(k-r)} \alpha^k
\]
   \((L_2(\alpha)-\text{pseudo-characteristic equation of second type})\)

D8:
\[
M_2(\alpha) = N_2(\alpha) - L_2(\alpha)
\]  \((M_2(\alpha)-\text{reduced Riccati equation of second type})\)

D9: Subalgorithm of finding constant solution of algebraic equation \( L_2(\alpha) \) in terms of \( \alpha \) (Output- const. the solution \( \alpha_0 \), or the message that such a solution does not exists)

D10: If \( \alpha_0 \) exists and is solution of \( M_2(\alpha) = 0 \) then go to D11 else go to D13;

D11: If \( n = 2 \) then go to D12 else go to D13;

D12: \( \alpha_2 = \alpha_0; \alpha_1 = -(\alpha_1 + \alpha_2) \);
\[
y_1 = \exp(\int \alpha_1 dx), \quad y_2 = y_1 \exp(\int (\alpha_2 - \alpha_1) dx) dx;
\]  \((5.21)\)
   Message ”The coefficients of factorization \( \alpha_1 \) and \( \alpha_2 \) and fundamental solutions \( y_1 \) and \( y_2 \) are obtained” \( \text{\textit{...}} \);

D13: \( y_1 = \exp(\int \alpha_0 dx) \), Message ”The degree of differential equation is reduced by 1”; New Lode := Sub(\( y = y_1 \int z dx, Ly \)); In equation
\[
Ly = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C^k_I
\]  \((5.22)\)
we make the change of variables \( y = y_1 \int z dx \) where \( z \) is new variable)
   Return New Lode \( \text{\textit{...}} \);
D14: Message “The solution of exponential type does not exists”; Subalgorithm for finding of constant solution of algebraic equation $L(\alpha)$ in terms of $\alpha$. In our program this subalgorithm is realized by procedure HAREQ($L(\alpha), \alpha, x$);
Output: $L(\alpha)$ is algebraic equation in term of $\alpha$. x is independent variable.
Output: We have const. is the solution $\alpha_0$ of equation $L(\alpha)$, or message, such a solution does not exists
Example:

$$Ly = y'' + (8 + \sin(x)^2)y' + 8\sin(x)^2y = 0; \quad (') = d/dx \quad (5.23)$$

In the array $a(n)$, $n = 0 \div 2$, we write the coefficients of the equation

$$a(2) := 1; \quad a(1) := 8 + \sin(x)^2; \quad a(0) := 8\sin(x)^2; \quad n := 2; \quad d := 0; \quad (5.24)$$

Using the procedure LIVDIF($a, d, n, x$); we seek particular solution of the following type $y = \exp(-\alpha x)$, where $\alpha = \text{const.}$, and if the differential equation is of degree $n = 2$, then we find the coefficients of factorization $\alpha_1, \alpha_2$ and the fundamental system of solutions $y_1(x), y_2(x)$, also the general solution. Then we have the solution of differential equation of type $y = \exp(-8x)$, and the coefficients of factorization are

$$\alpha_1 = 8; \quad \alpha_2 = \sin(x)^2, \quad (5.25)$$

and due to the fact that the equation is of second degree we may found second solution

$$y_2 = y_1 \int \exp(\int (\alpha_2 - \alpha_1)dx)dx, \quad (5.26)$$

and general solution

$$y = C_1 y_1 + C_2 y_2. \quad (5.27)$$

4.3.2. LODE with polynomial solutions To exists in the set of solutions of the equation

$$Ly = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C^k_i \quad (5.28)$$

polynomial of degree $p \neq m$ the necessary and sufficient condition is $\text{RRes}(L, D^{m+1}) = 0$, where

$$L = \sum_{k=0}^{n} a_k D^k, \quad D = d/dx, \quad D^{m+1} = d^{m+1}/dx, \quad (5.29)$$

i.e. using the differential resultant of operators $L$ and $D^{m+1}$ we find the degree of existing polynomial solution of LODE and after that by method of indefinite coefficients we find the coefficients of this polynomial and if the
degree of LODE is \( n = 2 \) we may obtain the general solution and when \( n > 2 \) we reduce by 1 the degree of LODE.

**Problem 4.** Given LODE of degree \( n \):

\[
Ly = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C_i^k
\]  

(5.30)

find the polynomial solution.

**Algorithm 4.** To solve the problem 4 we apply the procedure POLDIF(a,n,x,hs);

**Input**

- \( a \) is the array of coefficients of LODE \( Ly = 0 \),
- \( n \) is the degree of differential operator \( L \),
- \( x \) is independent variable,
- \( hs \) is the maximal degree of the polynomial solution,

**Output:**

If there exists the polynomial solution of degree \( p \leq hs \), then in the case of \( n = 2 \) we find the general solution, and when \( n > 2 \) the degree is reduced by 1. If there is no solution we have the message that polynomial solution does not exists.

**P1:** \( i := 1; \)

**P2:** Algorithm 1 for finding the differential resultant \( \text{RRRes}(L, D^i); \)

**P3:** If \( \text{RRRes}(L, D^i) = 0 \), then go to P6 else go to P4;

**P4:** If \( i > hs \) then go to P12 else go to P5;

**P5:** \( \ll i := i + 1; \) go to \( \text{P2} \gg; \)

**P6:** \( P := i - 1; \)

**P7:** Message “\( P \) is the degree of polynomial solution”;

**P8:** Subalgorithm for finding the coefficients of the polynomial solution (Output-polynomial solution)

**P9:** If \( n = 2 \) then go to P10 else go to P11;
**P10:** Find the fundamental system of LODE.

\[ y_2 = y_1 \int \exp \left( - \int a_1 \, dx \right) y_1^{-2} \, dx \]

Return \( y_2, y_1 \gg \);

**P11:** The degree of LODE is reduced by 1.

\[ y_1 = \text{polynomial solution}; \]

New Lode := Sub\((y = y_1 \int z \, dx, Ly)\), where in equation

\[ Ly = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C^k \]  \hspace{1cm} (5.31)

we make the change of variables \( y = y_1 \int z \, dx \) where \( z \) is the new variable

Return New Lode \( \ll \);

**P12:** Message “The found polynomial solution of the LODE”;

Subalgorithm for finding the coefficients of the polynomial solution of the LODE.

In our program this subalgorithm is realized by the procedure \text{DEPOLDIFF}(a, n, x, p);

**Input:**

- \( a \) is the array of coefficients of the differential equation \( y = 0 \)
- \( n \) is the degree of differential operator \( L \),
- \( x \) is independent variable,
- \( p \) is the degree of polynomial solution of \( Ly = 0 \).

**Output:** The coefficients of the polynomial solution by the method of indefinite coefficients.

**Example.**

\[ Ly = x^2(\ln(x) - 1)y'' - xy' + y = 0, \quad (') = d/dx \]

The array \( a(n), n = 0 \div 2 \), contain the coefficients of the given LODE.

\[
\begin{align*}
a(2) &:= x^2(\ln(x) - 1); \quad a(1) := -x; \quad a(0) := 1; \quad n := 2; \quad hs := 3; \\
\end{align*}
\]

By using the procedure \text{POLDIFF}(a, n, x, hs) we find the fundamental system of solution \( y_1 := x; y_2 := -\ln(x) \). Then the general solution is given by:

\[ y := C1x + C2 \ln(x); \]
4.3.3. The equations in terms of exact differentials. Let us define

\[ L_0 = \sum_{k=0}^{n} a_k D^k y = 0, \quad a_n \neq 0, \quad a_k \in C_I^k, \quad D = d/dx \quad (5.32) \]

This equation is expressed in terms of exact differentials of there exists the following form

\[ L_0 = D \sum_{k=0}^{n-1} b_k D^k y = 0, \quad D = d/dx, \quad (5.33) \]

where

\[ a_n = b_{n-1}(x), \quad a_0 = b'_0(x), \quad a_{k+1} = (b_k + b'_k(x)), \quad (') = d/dx, \quad k = 0 \div (n-2). \quad (5.34) \]

The necessary and sufficient condition the equation \( L_0 = 0 \) to be in exact differential is,

\[ \sum_{k=0}^{n} (-1)^k a_k(x) = 0, \quad (5.35) \]

If the equation

\[ L_0 = \sum_{k=0}^{n} a_k D^k y = 0, \quad (5.36) \]

is expressed in terms of exact differentials in the case \( n = 2 \) we may find the general solution of this equation and when \( n > 2 \) we may find fist integral.

**Problem 5.** Given LODE of degree \( n \):

\[ L_0 = \sum_{k=0}^{n} a_k D^k y = f(x), \quad a_n \neq 0, \quad a_k \in C_I^k \quad (5.37) \]

Check is this solution of LODE in exact differentials and if the equation is of this type find the general solution (when \( n = 2 \)) and first integral (when \( n > 2 \)).

**Algorithm 5.** To solve the problem 5 we apply the procedure

\[ \text{EQDIF}(a,n,dl,x); \]

**Input:**

- \( a \) is the array of coefficients of LODE
- \( n \) is the degree of differential operator \( L \),
- \( dl \) is the nonhomogeneous part of given LODE
- \( x \) is independent variable,

**Output:**

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Check is given equation in exact differentials and if it is true find the general solution in the case $n = 2$ and first integral if $n > 2$, and if it not the case, message that the equation is not in exact differentials

**E1:**

$$D := \sum_{k=0}^{n} (-1)^k a_k(x) = 0, \quad (5.38)$$

**E2:** If $D = 0$ then go to E4 else go to E3;

**E3:** Message: "This equation is not in exact differentials".

**E4:** \[ b(n-1) := a(n); \text{ For } k := (n-2) \text{ Step } (-1) \text{ Until } 0 \text{ Do } \]

$$b(k) := a(k+1) - b(k+1)', \quad (') = d/dx \quad (5.39)$$

**E5:** If $n = 2$ Then Go To E6 Else Go To E7;

**E6:**

$$y = \exp(- \int a(1)dx)(c_2 + \int (\int dldx + C_1) \exp(\int a(1)dx)dx); \quad (5.40)$$

( $C_1, C_2$ - are constant of integration, $y$ - general solution of LODE)

**E7:**

$$L_2y = D \sum_{k=0}^{n-1} b_k D^k y, \quad D = d/dx, \quad (5.41)$$

**Appendix 3:** Differential resultant and algebra of commuting differential operators.

**Example 1**

$$H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} q_1 q_2^2 + q_1^3. \quad (5.42)$$

We associate to this Hamiltonian (5.42) the following linear differential operators

$$L = D^2 + q_1, \quad (5.43)$$

$$M = 16D^5 + 40q_1D^3 + 60p_1D^2 - (120q_1^2 + 25q_2^3)D$$

$$-60q_1p_1 - 15q_2p_2, \quad (5.44)$$

This example show the abilities of the differential resultant to find the first integrals and algebraic curve associated to given linear differential operators. Using REDUCE 3.4 syntactis we have
depend qq1,x; depend qq2,x; depend pp1,x; depend pp2,x;
a(0):=-60*qq1*pp1-15*qq2*pp2-zz;
a(1):=-120*qq1**2-25*qq2**2; a(2):=60*pp1;
a(3):=40*qq1; a(4):=0; a(5):=16;
b(0):=qq1-ll; b(1):=0; b(2):=1;
b(3):=0;
out nik52kdv;
res52:=difresult(a,5,b,2,x)$
off mcd; res52:=num(res52)/den(res52);
write "denumerator:=" ,den(res52);
p1:=deg(res52,zz); write "p1:=" ,p1;
p2:=deg(res52,ll); write "p2:=" ,p2;
for i:=0:p1 do <<
d1(i):=coeffn(res52,zz,i);
LET df(qq1,x,2)=-3*qq1**2-qq2**2/2;
LET df(qq2,x,2)=-qq1*qq2;
LET df(pp1,x)=-3*qq1**2-qq2**2/2;
LET df(pp2,x)=-qq1*qq2;
LET df(qq2,x,3)=df(-qq1*qq2,x);
LET df(qq2,x)=pp2,df(qq1,x)=pp1;
LET df(qq2,x,5)=df(-qq1*qq2,x,3);
LET df(pp1,x,2)=df(-3*qq1**2-qq2**2/2,x);
LET df(pp2,x,2)=df(-qq1*qq2,x);
LET df(pp2,x,4)=df(-qq1*qq2,x,3);
Determinant of resultant matrix

i-degree of zz:=0

j - degree of ll:=0
d2(0,0):=0

***********************************
i-degree of zz:=0

j - degree of ll:=1

d2(0,1):=4*qq1*qq2 - 8*qq1*pp2 + qq2 + 8*qq2*pp1*pp2

***********************************
\begin{verbatim}
 i-degree of zz:=0

 j - degree of ll:=2

 \[ d2(0,2):= -32*qq1 - 16*qq1*qq2 - 16*pp1 - 16*pp2 \]

 ***********************************

 i-degree of zz:=0

 j - degree of ll:=3

 d2(0,3):=0

 ***********************************

 i-degree of zz:=0

 j - degree of ll:=4

 d2(0,4):=0

 ***********************************

 i-degree of zz:=0

 j - degree of ll:=5

 d2(0,5):=-256

 ***********************************

 p3:=0

 i-degree of zz:=1

 j - degree of ll:=0

 d2(1,0):=0

 ***********************************

 p3:=0

\end{verbatim}
The associated algebraic curve has the form (the results are obtained by differential resultant)

\[ z^2 = -256\lambda^5 - 32E\lambda^2 + 8K\lambda \]  

(5.45)

where

\[ E = \frac{1}{2}(p_1^2 + p_2^2) + q_1^3 + \frac{1}{2}q_1q_2^2, \]  

(5.46)

\[ K = q_2p_1p_2 - q_1p_2^2 + q_2^4 + 1/2q_1^2q_2^2. \]  

(5.47)

**Example 2** (Fordy example [?])

```plaintext
a(0):=-zz; a(1):=-5*qq2**2; a(2):=15*pp1;
a(3):=15*qq1; a(4):=0; a(5):=9;
b(0):=-ll; b(1):=qq1; b(2):=0;
b(3):=1;
out nik53sk;
res53:=difresult(a,5,b,3,x)$
```

```plaintext
off mcd; res53:=num(res53)/den(res53); write "denumerator:=";den(res53);
p1:=deg(res53,zz); write "p1:=";p1;
p2:=deg(res53,ll); write "p2:=";p2;
for i:=0:p1 do
  d1(i):=coeffn(res53,zz,i);
  LET df(qq1,x,2)=-qq1**2/2-qq2**2/2;
  LET df(qq2,x,2)=-qq1*qq2;
  LET df(pp1,x)=-qq1**2/2-qq2**2/2;
  LET df(pp2,x)=-qq1*qq2;
  LET df(qq2,x,3)=df(-qq1*qq2,x);
  LET df(qq2,x)=pp2,df(qq1,x)=pp1;
  LET df(qq2,x,5)=df(-qq1*qq2,x,3);
  LET df(pp1,x,2)=df(-qq1**2/2-qq2**2/2,x);
  LET df(pp2,x,2)=df(-qq1*qq2,x);
  LET df(pp2,x,4)=df(-qq1*qq2,x,3);

  d1(i):=d1(i);
```
%write "d1(" ,i ,"):=",d1(i); %write "***********************************

p3:=deg(d1(i),ll); write "p3:=",p3;
for j:=0:p3 do <<
<< d2(i,j):=coeffn(d1(i),ll,j);
write "i-degree of zz:=",i;
write "j - degree of ll:=",j;
write "d2(" ,i ,"," ,j ,"):=",d2(i,j);
write "***********************************" >> >>;

Determinant of resultant matrix
denumerator:=1

i-degree of zz:=0
j - degree of ll:=0
d2(0,0):=0

***********************************

i-degree of zz:=0
j - degree of ll:=1
d2(0,1):=
6 3 2 2 4 2
- 1/4*qq2 - 3*qq2 *pp1*pp2 - 9*pp1 *pp2

***********************************

i-degree of zz:=0
j - degree of ll:=2
d2(0,2):=0

***********************************
i-degree of $zz:=0$

j - degree of $ll:=3$

\[ d_{2}(0,3):=27*qq_{1} + 81*qq_{1}*qq_{2} + 81*pp_{1} + 81*pp_{2} \]

*****************************************************************************

i-degree of $zz:=0$

j - degree of $ll:=4$

\[ d_{2}(0,4):=0 \]

*****************************************************************************

i-degree of $zz:=0$

j - degree of $ll:=5$

\[ d_{2}(0,5):=-729 \]

*****************************************************************************

\[ p_{3}:=0 \]

i-degree of $zz:=1$

j - degree of $ll:=0$

\[ d_{2}(1,0):=0 \]

*****************************************************************************

\[ p_{3}:=0 \]

i-degree of $zz:=2$

j - degree of $ll:=0$

\[ d_{2}(2,0):=0 \]

*****************************************************************************
p3:=0

i - degree of zz:=3

j - degree of ll:=0

d2(3,0):=1

*******************************************************************************
The associated algebraic curve has the form (the results are obtained by differential resultant)

\[ z^3 = 729\lambda^5 - 162E\lambda^3 + K^2\lambda, \]  
(5.48)

where

\[ E = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{6}q_1^3 + \frac{1}{2}q_1q_2^2, \]  
(5.49)

\[ K = 3p_1p_2 + \frac{3}{2}q_2q_1^2 + \frac{1}{2}q_2^2, \]  
(5.50)

print('A program for calculation differential resultants');
print('written by N.A. Kostov, Z.T. Kostova 8 August 1994');

#find the elements of series
procoff:=proc(s,k,m,b,x)
    local i,s1,s2,cr;
    s1:=max(0,k-m); s2:=min(s,k);
    if k < 0 or k > (m+s) then cr:=0 else
        cr:=0; for i from s1 to s2 do
            if (s-i)=0 then cr:=cr+binomial(s,i)*b(k-i) else
                cr:=cr+binomial(s,i)*diff(b(k-i),x$s-i);
            fi;
        od;
    fi;
    RETURN(eval(cr))
end:

difresult:=proc(a,n,b,m,x)
    local i1,i2,i3,j2,i3,j3,k,s,l,p;
    **************************************************************************
    ** Programm for finding    **
    ** differential resultant **
    **************************************************************************
end:
r:=array(1..n+m,1..n+m):
for s from 0 to (m-1) do
    for k from 0 to (n+s) do
        ak(k,s):=procoff(s,k,n,a,x);
    od;
    od;
for p from 0 to (n-1) do
    for l from 0 to (m+p) do
        bk(l,p):=procoff(p,l,m,b,x);
    od;
    od;
for i1 from 1 to m do
    for j1 from 1 to (n+m) do
        if i1 > 1 and j1 < i1 then r[i1,j1]:=0 else
            r[i1,j1]:=ak(n+m-j1,m-i1);
        fi;
    od;
    od;
for i2 from 1 to n do
    for j2 from 1 to (n+m) do
        if i2>1 and j2<i2 then r[m+i2,j2]:=0 else
            r[m+i2,j2]:=bk(n+m-j2,n-i2);
        fi;
    od;
    od;
print ('"Element of resultant matrix"');
# for i3 from 1 to (n+m) do
#     for j3 from 1 to (n+m) do
#         print('r:=',r[i3,j3],i3,j3);
#     od;
# od;
with(linalg,det);
rres1:=det(r):
RETURN(rres1)
end:

#alias( dq1=dq1(x),q1=q1(x),q2=q2(x),dq2=dq2(x)):
#b(0):=dq1-z;
#b(1):=2*q1; b(2):=0; b(3):=1;
#a(0):=-5*q2*dq2-60*q1*dq1-ll;
#a(1):=-120*q1**2-(35/4)*q2**2; a(2):=45*dq1;
#a(3):=30*q1; a(4):=0; a(5):=9;
#res3:=difresult(a,b,3,x);
#alias(UU=uu(x), WW=ww(x)):
#alias(UUx=diff(uu(x),x), UUxx=diff(uu(x),x,x), UUxxx=diff(uu(x),x,x,x)):
#alias(WWx=diff(ww(x),x), WWxx=diff(ww(x),x,x), WWxxx=diff(ww(x),x,x,x)):
#alias(WWxxxx=diff(ww(x),x,x,x,x), WWxxxxx=diff(ww(x),x,x,x,x,x)):
#b(0):=-Z+diff(ww(x),x); b(1):=2*WW; b(2):=0; b(3):=1;
#a(0):=20*WW*diff(ww(x),x)+10*diff(ww(x),x$3)-L;
#a(1):=20*WW**2+35*diff(ww(x),x$2); a(2):=45*diff(ww(x),x);
#a(3):=30*WW; a(4):=0; a(5):=9;
#difresult(a,5,b,3,x);

#alias(DQ1=dq1(x), Q1=q1(x), Q2=q2(x), DQ2=dq2(x)):
#alias(DQ1x=diff(dq1(x),x), DQ1xx=diff(dq1(x),x,x), DQ1xxx=diff(dq1(x),x,x,x)):
#alias(DQ2x=diff(dq2(x),x), DQ2xx=diff(dq2(x),x,x), DQ2xxx=diff(dq2(x),x,x,x)):
#alias(DQ1xxxx=diff(dq1(x),x,x,x,x), DQ1xxxxx=diff(dq1(x),x,x,x,x,x)):
#alias(Q1x=diff(q1(x),x), Q1xx=diff(q1(x),x,x), Q1xxx=diff(q1(x),x,x,x)):
#alias(Q2x=diff(q2(x),x), Q2xx=diff(q2(x),x,x), Q2xxx=diff(q2(x),x,x,x)):
#alias(Q1xxxx=diff(q1(x),x,x,x,x), Q1xxxxx=diff(q1(x),x,x,x,x,x)):
#alias(Q2xxxx=diff(q2(x),x,x,x,x), Q2xxxxx=diff(q2(x),x,x,x,x,x)):
#alias(Q2Nxxx=diff(Q2xx(x),x));
#alias(Q2xx=Q2xx(x));
DQ1:=Q1x:
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\[
DQ2 := Q2x:
\]

\[
b(0) := DQ1 - Z;
b(1) := 2Q1; \quad b(2) := 0; \quad b(3) := 1;
a(0) := -5Q2 \cdot DQ2 - 60Q1 \cdot DQ1 - L;
a(1) := -120Q1^2 - \frac{35}{4}Q2^2; \quad a(2) := 45DQ1;
a(3) := 30Q1; \quad a(4) := 0; \quad a(5) := 9;
ppp := \text{difresult}(a, 5, b, 3, x):
\]

\[
rrr := \text{expand}(\text{subs}(DQ1 = Q1x, Q1xx = -4Q1^2 - Q2^2/4, Q2Nxxx = Q2xxx, Q2xx = -Q1 \cdot Q2/2, Q2xxx = -Q1 \cdot Q2/2 - Q1 \cdot Q2x/2, Q1xxxxx = -24Q1x \cdot Q1xx - 8Q1 \cdot Q1xxx - Q2x \cdot Q2xx \cdot 3/2 - Q2 \cdot Q2xxx/2, DQ2 = Q2x, Q1xxx = -8Q1x \cdot Q1 - Q2 \cdot Q2x/2, L, Z, Z, tres)):
\]

\[
\text{Pres} := \text{coeffs}(rrr, [L, Z], 'tres'):
\]

\[
\text{printdifres} := \text{proc}(\text{sss})
\quad \text{local} \ rrr;
\quad \text{for} \ i \ \text{from} \ 1 \ \text{to} \ \text{nops}([\text{tres}]) \ \text{do}
\quad \ rrr := \text{expand}(\text{Pres}[i]):
\quad \text{print}('\text{number}=');
\quad \text{print}(i);
\quad \text{print}('\text{spectral parameters}');
\quad \text{print}(\text{tres}[i]);
\quad \text{print}('\text{element}');
\quad \text{print}(rrr);
\quad \text{od};
\quad \text{end};
\]

\[
\text{print}('\\text{*******************************************************************')};
\]

\[
\text{print}('\text{spectral parameters}', \text{tres}[2]*162);
\text{EE} := \text{expand}((\text{subs}(Q2xx = -Q1 \cdot Q2/2, Q1xx = -4Q1^2 - Q2^2/4, \text{Pres}[2])));
\text{EE} := \text{expand}(\text{EE})/(81*2):
\text{print}('\text{energy is equal to res2}');
\text{print}(\text{EE});
\]

\[
\text{print}('\\text{*******************************************************************')};
\]

\[
\text{print}('\text{spectral parameters}', 3 \text{tres}[4]/4);
\text{K} := \text{expand}((\text{subs}(Q2xx = -Q1 \cdot Q2/2, Q1xx = -4Q1^2 - Q2^2/4, \text{Pres}[4]))):
\]

\[
22
\]
K:=-4*expand(K)/3:
print('second integral is res4');
print(K);

print('*******************************************************************');
print('spectral parameters',tres[7]);
res7:=expand(subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
        Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
        Pres[7]));
res7:=expand(res7):
print('element res7=');
print(res7);

print('*******************************************************************');
print('spectral parameters',tres[3]);
res3:=expand(subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
        Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
        Pres[3]));
res3:=expand(res3):
print('element res3 =');
print(res3);

print('*******************************************************************');
print('spectral parameters',tres[1]);
res1:=subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
        Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
        Pres[1]):
res1:=expand(res1):
print('element res1=');
print(res1);

print('*******************************************************************');
print('spectral parameters',tres[5]);
res5:=subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
        Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
        Pres[5]):
res5:=expand(res5):
print('element res5=');
print(res5);
print('*******************************************************************');
print('*******************************************************************');
print('spectral parameters',tres[8]);
res8:=subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
            Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
            Pres[8]);
res8:=expand(res8):
print('element res8=');
print(res8);
print('*******************************************************************');
print('*******************************************************************');
print('spectral parameters',tres[6]);
res6:=subs(Q2xx=-Q1*Q2/2,Q1xx=-4*Q1^2-Q2^2/4,
            Q1xxx=-8*Q1*Q1x-Q2*Q2x/2,Q2xxx=-Q1x*Q2/2-Q1*Q2x/2,
            Pres[6]);
res6:=expand(res6):
print('element res6=');
print(res6);
print('*******************************************************************');
print('*******************************************************************');
alias(q[1]=q[1](x),q[2]=q[2](x)); alias(wp=wp(x));
alias(u=u(x));
Eq[1]:=diff(q[1],x$2)+q[2]^2/2+3*q[1]^2+a[0]*q[1]-a[1]/4:
Eq[2]:=q[2]^3*diff(q[2],x$2)+q[2]^4*q[1]+q[2]^4*a[0]/4+a[4]/4:
wp1:=(4*wp^3-g2*wp-g3)^(1/2):
wp2:=eval(diff(wp1,x)):
wp3:=eval(expand(subs(diff(wp,x)=wp1,wp2))):
F:=6; B:=0; a[0]=0; G:=-144;
for i from 1 to 2 do
    SEq[i]:=eval(expand(subs(q[1]=F*wp+B,
                      q[2]=(G*wp^2+E)^(1/2),Eq[i]))):
    SSEq[i]:=expand(subs(diff(wp,x)^2=(4*wp^3-g2*wp-g3),
                       diff(wp,x$2)=wp3,expand(SEq[i]))):
    #print('SSEq(',i,') := ',SSEq[i]);
end do;
n[i]:=degree(SSEq[i],wp):
for j from 0 to n[i] do
Cowp[i,j]:=coeff(SSEq[i],wp,j):
#Cowp1[i,j]:=expand(subs(F=0,G=0,Cowp[i,j]));
od:
#FF1[i]:=seq(Cowp[i,i1]=0,i1=0..n[i]);
FF2[i]:=seq(Cowp[i,i3],i3=0..n[i]);
od:
for i1 from 1 to 2 do
for j1 from 0 to n[i1] do
#print('Cowp(',i1,',',j1,'):=', Cowp[i1,j1]);
print('Cowp(',i1,',',j1,'):=',subs(a[0]=0,E=36*g2, Cowp[i1,j1]));
od;
od;
#for i2 from 1 to 2 do
#for j2 from 0 to n[i2] do
# print('Cowp1(',i2,',',j2,'):=',Cowp1[i2,j2]);
#od;
#od:
#Fs:={FF1[1],FF1[2]};
#sols:=solve(Fs,{F,A,B,G,D,E,C1^2,C2^2,gama,beta});
with(grobner);
Fg:=[FF2[1],FF2[2]];
#Fg1:=subs(eta^(-1)=eta1,C1^2=CC1,C2^2=CC2,Fg);
#gtsolve(Fg);
#gbasis(Fg1,[A,b,F,G,D,E,CC1,CC2,r,s,sigma,gama,beta,eta,eta1,g2,g3],plex);
#gbasis(Fg1,[A,D,C1^2,C2^2,gama,beta],tdeg);
#gtsolve(Fg1,[A,D,C1^2,C2^2,gama,beta]);
#A1:=expand(subs(A=(-6*s-sigma*D)/eta,B=A^2/F-F*g2/4,G=-eta*F/sigma,
#E=D^2/G-G*eta2/4,Cowp[1,3]));
#D1:=expand(subs(A=(-6*s-sigma*D)/eta,B=A^2/F-F*g2/4,G=-eta*F/sigma,
#E=D^2/G-G*eta2/4,Cowp[2,3]));
#C12:=solve(Cowp1[1,0],C1^2);
#Gama1:=solve(Cowp1[1,1],gama);
#Gama2:=solve(Cowp1[1,2],gama);
#AA:=solve(Cowp1[1,3],A);
#C22:=solve(Cowp1[2,0],C2^2);
#Be1:=solve(Cowp1[2,1],beta);
#Be2:=solve(Cowp1[2,2],beta);
#DD:=solve(Cowp1[2,3],D);

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