1 Introduction

The following theorem was proven by Agol in [1]. The aim of these notes is to make the proof accessible to a wider audience; we retain the underlying ideas and constructions from [1], but substantially change or add to many parts of the argument to give a more transparent and detailed account.

**Theorem 1.1. (Agol’s Theorem)**
Let $G$ be a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex $X$. Then $G$ has a finite index subgroup $G'$ that acts freely on $X$ such that the quotient $X/G'$ is special.

This theorem bridges the gap between two areas of activity within geometric group theory. On the one hand there has been a lot of work about cubulating groups: finding for a given group a proper cocompact action on a CAT(0) cube complex. On the other hand there is the theory of special cube complexes.

The topic of cubulating groups, a programme initiated by Gromov [6] and driven largely by Dani Wise, begins with Sageev’s construction of a CAT(0) cube complex from a poset, which provides a cubulation of a group $G$ if it acts on a poset in a certain way (see for instance [14]). Many classes of groups have been cubulated using this method, such as a class of Coxeter groups in [8] and small cancellation groups in [12]. A general criterion for cubulating hyperbolic groups was given in [3]. Consequences of cubulating a group include that the group is bi-automatic [10], and that it satisfies the Tits Alternative, namely every subgroup is either virtually finitely generated abelian or has a non-abelian free subgroup [9].

Special cube complexes were introduced by Haglund and Wise in [7], and in that paper it was proven that fundamental groups of compact special cube complexes are subgroups of right-angled Artin groups, and hence subgroups of $SL_n(\mathbb{Z})$. That paper also showed that for $G$ a hyperbolic fundamental group of a compact nonpositively curved cube complex, the complex is virtually special if and only if every quasi-convex subgroup of $G$ is separable. A stronger Tits Alternative was proven for fundamental groups of compact special cube complexes in [11, 14.10], namely every subgroup is virtually abelian or large (has a finite index subgroup surjecting to $\mathbb{Z} \ast \mathbb{Z}$).

One famous problem that was answered affirmatively by Theorem 1.1 is Thurston’s conjecture that every closed hyperbolic 3-manifold $M$ has a finite-sheeted cover that fibres over the circle. Kahn and Markovich’s construction of quasi-fuchsian subgroups of $\pi_1(M)$, together with work of Bergeron and Wise [3], shows that $\pi_1(M)$ can be cubulated. Thus Theorem 1.1 implies $\pi_1(M)$ is virtually the fundamental group of a compact special cube complex. The required finite-sheeted cover is then constructed using the largeness of $\pi_1(M)$ (see [2] for more details on this).

The strategy of the proof of Theorem 1.1 is to cut $X$ along all of its walls and start with orbit representatives of these pieces; then glue these back together one wall at a time until we get a cube complex with universal cover $X$ and deck transformations a subgroup of $G$. At each stage we will
have a collection of finite virtually special cube complexes whose universal covers are convex subspaces of $X$. To ensure the complexes remain virtually special at each stage we employ theorems from [11] and [4]. The main condition we need for these theorems is that the walls we want to glue along are acylindrical subspaces; this requires keeping preimages of these gluing walls far apart in the universal cover. This is where the key idea of Agol’s proof comes in: we endow each vertex in each complex with local data that colours nearby walls in the universal cover in such a way that:

- finitely many colours are used,
- the colours determine which walls are preimages of gluing walls,
- walls that are close have different colours,
- adjacent vertices have compatible colouring data, in that they agree about the colours of nearby walls,
- if there are two vertices with compatible colouring data next to different gluing walls, then all vertices next to these walls have compatible colouring data (allowing us to ‘zip’ the walls together).

That last point is important as it ensures that the colouring data retains its properties from one stage of the construction to the next. Devising colourings that satisfy all this is difficult, in fact it requires a more complicated definition of local colouring than simply colouring in walls that are near to the vertex in the universal cover - and the presence of infinite walls in the universal cover further messes things up. So what we actually do is pass to a quotient complex $X = X/K$ with finite walls, and we colour walls in $X'$ rather than $X$ using local colourings that form a sort of ‘hierarchy’. The existence of a suitable $X'$ is a deep fact - it relies on the appendix to [1] which in turn relies on Wise’s Malnormal Special Quotient Theorem. Note that we do not revisit the proofs contained in the appendix to [1].

The notion of a hierarchy is central in the work of Haglund and Wise, and has subsequently played an important role in many applications. The hierarchy of coloured walls we use depends on Wise’s hierarchy of hyperbolic groups from [11].

Here is summary of what happens in each section:

2. Background on walls: We provide some basic theory about walls in cube complexes, with a focus on CAT(0) cube complexes.

3. Special cube complexes: We recall some results about special cube complexes, which underpin many later arguments.

4. Making walls finite: This section constructs the quotient complex $X'$ with finite walls.

5. Invariant colouring measures: The main result here is the existence of a measure on the space of colourings of a graph.

6. Colouring walls: This is where we define the local colouring data.

7. Starting the gluing construction: Here we set up the main inductive construction, which at each stage gives a collection of finite virtually special cube complexes. We use Section 5 to carefully choose colouring data for the first stage of the construction, to ensure that later on we can always pair up gluing walls with compatible colouring data.

8. Controlling boundary walls: Here we prove the zipping property of gluing walls mentioned above, and prove that gluing walls are acylindrical subspaces.

9. Gluing up walls: In this final section we perform the inductive step by gluing the walls together; the subtlety here is that two gluing walls with compatible colouring data might be different shapes (so the zipping gets stuck), to solve this we take finite covers of our complexes.
The sections with the most significant modifications to Agol’s proof are 4, 7, 8 and 9.

Acknowledgements: Thanks to Ric Wade and my supervisor Martin Bridson for their careful proofreading and helpful comments.

2 Background on walls

In this section we cover some basic definitions and results about cube complexes and their walls; most of this material can be found in [7].

Definition 2.1. (Cube complex)
A cube complex is a metric polyhedral complex in which all polyhedra are unit Euclidean cubes. For $X$ a cube complex, $V(X)$ will denote the vertex set and $E(X)$ will denote the edge set. All edges in these notes will be unoriented. We will denote the metric by $d$ and also use $d$ for the distance between subsets of a cube complex, $d(A, B) = \inf \{ d(x,y) \mid x \in A, y \in B \}$. Any finite dimensional cube complex is a complete geodesic space [5, I.7.33].

Definition 2.2. (Walls)
An $n$-cube $C = [-1,1]^n \subset \mathbb{R}^n$ has $n$ midcubes, each obtained by setting one coordinate to zero. The face of a midcube of $C$ is naturally the midcube of a face of $C$, and so the collection of midcubes of a cube complex $X$ can be given the structure of a cube complex. We call this the wall complex; it has a natural immersion $q : W \to X$ induced by inclusions of the midcubes - note this immersion is neither combinatorial nor an embedding, but it can be made combinatorial by passing to the barycentric subdivisions of $W$ and $X$. A component of the wall complex is called a wall (it is also called a hyperplane by other authors), but when talking about a wall we will usually be referring to its image under $q$. A picture of a wall in a cube complex is given below.

For an edge $e$ in a cube complex $X$, write $W(e)$ for the unique wall that it intersects, and say that $e$ is dual to $W(e)$. If a group $G$ acts on $X$ then it also acts on the set of walls, and clearly $gW(e) = W(ge)$ for $g \in G, e \in E(X)$. For a wall $W$, let $G_W$ denote its stabiliser. Given an edge $e$ dual to $W$, $g \in G_W$ if and only if $W(ge) = W$ - thus $G_W$ is also the stabiliser of the set of edges dual to $W$.

Remark 2.3. If the action of $G$ on $X$ is proper and cocompact then so is the action of $G_W$ on $W$. Properness is immediate, and cocompactness is a consequence of the following easy argument. Let
\{e_1, ..., e_k\} be edges dual to \(W\) which are representatives of those \(G\)-orbits of edges in \(X\) that include edges dual to \(W\). For an edge \(e\) dual to \(W\), there exists \(g \in G\) with \(ge = e_j\) some \(1 \leq j \leq k\), but then \(g \in G_W\). Therefore the union of midcubes that intersect \(\{e_1, ..., e_k\}\) is a compact subset of \(W\) with \(G_W\)-translates that cover \(W\).

**Definition 2.4.** (NPC and \(CAT(0)\) cube complexes)

A cube complex is nonpositively curved (we will use the shorthand NPC) if the link of each vertex is a flag complex, and an NPC cube complex is \(CAT(0)\) if it is simply connected. For the general definition of a \(CAT(0)\) space, and why it is equivalent to ours for cube complexes, see [5, II.1 and II.5.20].

For any cube complex \(X\), the map \(q: W \to X\) is a local isometry. If \(X\) is \(CAT(0)\) we have that, for any wall \(W\), \(q: W \to X\) is an embedding with convex image (because in a \(CAT(0)\) space, local geodesics are geodesics and geodesics are unique), and so by identifying \(W\) with its image we can view \(W\) as a closed convex subspace of \(X\). In particular, each cube of \(X\) will have at most one midcube belonging to \(W\).

**Definition 2.5.** For \(W\) a wall in any cube complex, let \(N(W)\) be the union of cubes that intersect \(W\). Define an equivalence relation on the vertices of \(N(W)\) in which \(x \sim y\) if \(x, y\) are joined by an edge path in \(N(W)\) that never crosses \(W\).

**Proposition 2.6.** If \(W\) is an embedded wall in \(X\), meaning that \(q: W \to X\) is an embedding, then there will either be one or two \(\sim\)-equivalence classes. If there are two then we say that \(W\) is two-sided and we denote the two classes of vertices by \(W^+\) and \(W^-\).

**Proof.**

Let \(e\) be an edge with endpoints \(x, y\) that is dual to \(W\). Given a vertex \(z \in N(W)\), we wish to show that it is equivalent to either \(x\) or \(y\). Take an edge \(e'\) incident at \(z\) that is dual to \(W\), and an edge path in the cube structure of \(W\) joining \(e'\) with \(e\). As shown, this edge path sits inside a sequence of squares containing \(x, y, z\), and along the boundary of these squares there is an edge path in \(X\) from \(z\) to one of \(x, y\) and this path doesn’t cross \(W\) because each square of \(X\) contains at most one midcube of \(W\).

**Proposition 2.7.** In a \(CAT(0)\) cube complex \(X\), each wall is two-sided and separates \(X\) into two connected components.

**Proof.**

Let \(W\) be a wall in \(X\). Consider an edge loop \(\gamma\) in \(X\). \(X\) is \(CAT(0)\), so in particular it is simply connected, thus \(\gamma\) can be homotoped down to a constant loop by a sequence of moves that add/remove backtracks or push a subpath of \(\gamma\) across a square in \(X\) as shown (see [5, I.8A.4]). The parity of the number of times \(\gamma\) crosses \(W\) is preserved by these moves, so \(\gamma\) must originally have crossed \(W\) an even number of times.
Consider an equivalence relation similar to that of Definition 2.5, but defined on all vertices of $X$: $x \sim y$ if $x$ can be joined to $y$ by an edge path that never crosses $W$. Write $\{x\}$ for the equivalence class of $x$. Let $e$ be an edge with endpoints $x, y$ that is dual to $W$. Note that $\{x\} \neq \{y\}$ because loops containing $e$ cross $W$ an even number of times. Clearly any vertex in $X$ is equivalent to one in $N(W)$, so by Proposition 2.6 $\{x\}$ and $\{y\}$ are the only classes.

Let $X(x) := \{z \in X - W \mid d(z, \{x\}) < d(z, \{y\})\}$ - this is the union of all cubes containing only vertices in $\{x\}$ plus, for every cube with a midcube in $W$, the open half cube that has a vertex in $\{x\}$. Define $X(y)$ similarly. By construction $X(x)$ and $X(y)$ are open, disjoint and their union is $X - W$. □

The closures of $X(x)$ and $X(y)$ above are called half-spaces. Each half-space is a convex subcomplex of $X$ (where the dot denotes barycentric subdivision), because if a geodesic joining two points in the same half-space crosses into the other half-space this would create a geodesic between points in $W$ that leaves $W$, contradicting the convexity of $W \subset X$.

**Proposition 2.8.** Let $e_1, e_2$ be edges incident at a vertex $x$ in a CAT(0) cube complex $X$. If $W(e_1) = W(e_2)$ then $e_1 = e_2$.

**Proof.** Let $W = W(e_1) = W(e_2)$. $W$ is a closed convex subspace of $X$, so there is a well-defined closest point projection $p : X \to W$. If $y_1$ is the midpoint of $e_1$, then $d(x, y_1) = 1/2$, and no other point of $W$ can be closer to $x$ (the open ball of radius $1/2$ about $x$ is contained in the cubes incident at $x$ and doesn’t touch any wall), so $p(x) = y_1$. But similarly if $y_2$ is the midpoint of $e_2$ then $p(x) = y_2$; hence $y_1 = y_2$ and $e_1 = e_2$. □

**Proposition 2.9.** Let $x, y$ be vertices in a CAT(0) cube complex $X$. An edge path between $x$ and $y$ will be of minimal length if and only if it only crosses walls that separate $x$ and $y$ and it crosses each of these once.

**Proof.** An edge path between $x$ and $y$ must cross every wall that separates $x$ and $y$, so any edge path from $x$ to $y$ that only crosses these walls, and crosses each of them once, is necessarily of minimal length.

For the converse implication, let $\gamma$ be a shortest edge path from $x$ to $y$ and suppose for contradiction that it crosses some wall twice. Say $e_1, \ldots, e_n$ are the edges of a subpath of $\gamma$ with $W(e_1), \ldots, W(e_{n-1})$ distinct and $W = W(e_1) = W(e_n)$.

Suppose first that $e_2, \ldots, e_{n-1}$ all lie in $N(W)$. Then there must be a square containing $e_2$ that intersects $W$; and if $e'_1$ is the edge of this square that crosses $W$ and meets $e_2$ at the same vertex as $e_1$, then Proposition 2.8 implies that $e_1 = e'_1$. But then we can replace $e_1, e_2$ with edges $f_1, f_2$ going the other way round the square. The result is that we have pushed the first crossing of $W$ further along the subpath, whilst preserving the length of $\gamma$. Repeating this we can replace the subpath with $f_1, \ldots, f_{n-1}, e_n$ where $W = W(f_{n-1})$. But then, by Proposition 2.8 $f_{n-1}, e_n$ must be a backtrack, contradicting the minimality of $\gamma$.

Suppose now that $e_i$ is the first edge in the subpath that leaves $N(W)$. Note that $N(W)$ is the $\frac{1}{2}$-neighbourhood of $W$, so it is convex. The first half of $e_i$ is a geodesic $\eta$ between $N(W)$ and $W(e_i)$; if $N(W)$ and $W(e_i)$ intersect then we can form a geodesic triangle between a point of the intersection and $\eta$, but the angles at either end of $\eta$ are at least $\pi/2$, contradicting $X$ being CAT(0). Thus $N(W) \cap W(e_i) = \emptyset$ and $W(e_i)$ is disjoint from $W$. But then our subpath must recross $W(e_i)$ before it can get back to $W$, which is a final contradiction. □

**Proposition 2.10.** (Helly’s Theorem for walls of CAT(0) cube complexes)[2, lemma 13.13] If $W_1, \ldots, W_n$ are pairwise intersecting walls in a CAT(0) cube complex $X$, then there is a cube $C$ with $C \cap W_1 \cap \ldots \cap W_n \neq \emptyset$. 

5
3 Special cube complexes

**Definition 3.1.** A cube complex is *simple* if the link of each vertex is a simplicial complex. In particular, a pair of edges meeting at a vertex $x$ cannot form two different corners of squares at $x$, as this would be a double edge in the link of $x$.

**Definition 3.2.** (Special cube complex)
A simple cube complex $X$ is *special* if:

1. $W(e_1) \neq W(e_2)$ for any two distinct edges $e_1, e_2$ incident at a vertex $x$.

2. Given distinct edges $e_1, e_2$ incident at a vertex $x$ with $W(e_1) \cap W(e_2) \neq \emptyset$, we have that $e_1, e_2$ form the corner of a square in $X$.

3. $X^{(1)}$ is a bipartite graph.

Note this is the definition of $C$-special from [3.2] rather than the definition of special, I am using this definition simply because it is easier to state. The results we use hold for either definition of special, in fact if a finite cube complex satisfies one definition of special then it has a finite cover that satisfies the other definition.

**Remark 3.3.** We can also think of the definition of special cube complex as ruling out certain behaviours of hyperplanes. Property (1) of the definition rules out self-intersections and self-osculations of hyperplanes as illustrated below (hyperplanes in red, edges of the cube complex in blue). Property (2) of the definition forbids inter-osculations of hyperplanes.

![Diagram showing self-intersection, self-osculation, and inter-osculation]
Being special grants a cube complex some of the nice geometry of CAT(0) cube complexes, but without the strong restriction of being simply connected. We now state an assortment of results about special cube complexes that we will need later. Let’s start with three easy propositions.

**Proposition 3.4.** Any covering of a special cube complex is special.

*Proof.* Suppose \( \phi : Y \to X \) is a covering of cube complexes with \( X \) special. Being simple is a local condition about links of vertices, so the simplicity of \( Y \) follows from the simplicity of \( X \). We need to check that \( Y \) satisfies properties (1)-(3) of Definition 3.2. \( \phi(W(e)) = W(f(e)) \), so \( W(e_1) = W(f(e_1)) = f(W(e_1)) = f(W(e_2)) = W(f(e_2)) \), therefore \( Y \) satisfies (1). Similarly \( W(e_1) \cap W(e_2) \neq \emptyset \) implies \( W(f(e_1)) \cap W(f(e_2)) \neq \emptyset \), so (2) holds. Finally (3) is true by taking the two vertex classes in \( Y \) to be preimages of the vertex classes in \( X \). \( \square \)

**Proposition 3.5.** Any locally convex subcomplex of a special cube complex is special.

*Proof.* Properties (1) and (3) obviously pass to any subcomplex. Property (2) passes to any locally convex subcomplex \( Y \) of a special cube complex \( X \), because if edges \( e_1, e_2 \) are in \( Y \) and they form the corner of a square in \( X \), then by considering little geodesics that cut the corner of this square, we see that the square must also be in \( Y \). \( \square \)

**Proposition 3.6.** If \( X_1, \ldots, X_n \) are special cube complexes and \( x_i \in X_i \) are choices of base vertex, then the cube complex obtained from \( \sqcup_i X_i \) by identifying \( x_1, \ldots, x_n \) is also special.

*Proof.* This follows immediately from the definition of special cube complex because each wall will be contained in a single \( X_i \) factor. \( \square \)

A cube complex is **virtually special** if it has a finite degree cover that is special. We will now state some powerful theorems showing that hyperbolic groups enjoy some strong properties if they are fundamental groups of virtually special cube complexes. The first of these theorems, due to Wise, characterises the fundamental groups of compact NPC virtually special cube complexes using the following hierarchy of hyperbolic groups.

**Definition 3.7.** Let \( QVH \) denote the smallest class of hyperbolic groups that is closed under the following operations:

1. \( 1 \in QVH \).
2. If \( G = A \ast_B C \) and \( A, C \in QVH \) and \( B \) is finitely generated and quasi-convex in \( G \), then \( G \in QVH \).
3. If \( G = A \ast B \) and \( A \in QVH \) and \( B \) is finitely generated and quasi-convex in \( G \), then \( G \in QVH \).
4. Let \( H < G \) with \( |G : H| < \infty \). If \( H \in QVH \) then \( G \in QVH \).

**Theorem 3.8.** (Wise, 2011)\[^{[11]}\], Theorem 13.3]

A torsion-free hyperbolic group is in \( QVH \) if and only if it is the fundamental group of a compact NPC virtually special cube complex.

**Definition 3.9.** (Separable subgroup)

Let \( G \) be a group. A subgroup \( H \) of \( G \) is **separable** (in \( G \)) if for every \( g \in G - H \) there is a homomorphism \( \phi : G \to F \) such that \( F \) is finite and \( \phi(g) \notin \phi(H) \).

**Theorem 3.10.** (Haglund-Wise, 2008)\[^{[2]}\], Theorems 1.3 and 1.4]

Let \( X \) be a compact NPC cube complex with \( \pi_1(X) \) hyperbolic. Then \( X \) is virtually special if and only if every quasi-convex subgroup of \( \pi_1(X) \) is separable.
Corollary 3.11. Let $X$ and $Y$ be compact NPC cube complexes with $\pi_1(X) \cong \pi_1(Y)$ hyperbolic. Then $X$ is virtually special if and only if $Y$ is virtually special.

These theorems will allow us to glue together two compact NPC virtually special cube complexes along locally convex subcomplexes, to produce a larger virtually special cube complex $X$ - one just needs that all three cube complexes have hyperbolic fundamental group. Indeed in this case the locally convex subcomplex $Z$ being glued along will be locally convex in $X$, and so lifting to universal covers we have that $\tilde{Z}$ is convex in $\tilde{X}$. By the ˇSarc-Milnor Lemma [5, I.8.19] we have a quasi-isometry $\pi_1(X) \to \tilde{X}$ that restricts to a quasi-isometry $\pi_1(Z) \to \tilde{Z}$, so it follows that $\pi_1(Z)$ is quasi-convex in $\pi_1(X)$ (using stability of quasi-geodesics [5, III.H.1.7]). We may then apply Theorem 3.8 to deduce that $\pi_1(Z)$ is quasi-convex in $\pi_1(X)$ (using stability of quasi-geodesics [5, III.H.1.7]). We may then apply Theorem 3.8 to deduce that $\pi_1(Z)$ is quasi-convex in $\pi_1(X)$ (using stability of quasi-geodesics [5, III.H.1.7]). We may then apply Theorem 3.8 to deduce that $\pi_1(Z)$ is quasi-convex in $\pi_1(X)$ (using stability of quasi-geodesics [5, III.H.1.7]).

The third big theorem we state in this section, due to Agol, Groves and Manning, appeared in the appendix of [1] - which is the main paper these notes are based on. We will use this theorem in the next section to take a quotient of the CAT(0) cube complex $X$ from Theorem 1.1 that makes the walls finite.

Theorem 3.12. (Agol-Groves-Manning, 2013)[2, Theorem A.1]

Let $G$ be a hyperbolic group and $H < G$ a quasi-convex subgroup which is virtually the fundamental group of a compact special cube complex. Then for any $g \in G - H$, there is a hyperbolic group $G'$ and a homomorphism $\phi : G \to G'$ such that $\phi(g) \notin \phi(H)$ and $\phi(H)$ is finite.

4 Making walls finite

From now on let $G$ be a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex $X$, as in Theorem 1.1. The object of this section is to construct a quotient map $X \to \mathcal{X}$ such that walls in $\mathcal{X}$ are finite, and so that distinct walls in $X$ which are close together map to distinct walls in $\mathcal{X}$. The quotient complex $\mathcal{X}$ will be important for defining the local colouring data used in later sections. This section is based on [1, §4], but with considerably more detail added.

We may assume $X$ is unbounded since the theorem is trivial otherwise. For $x, y \in X$ we will use $[x, y]$ to denote the unique geodesic segment between them.

Remark 4.1. By passing to the barycentric subdivision of $X$, we can assume that for every wall $W$ in $X$, $G_W$ does not exchange the sides of $W$, so $gW^\pm = W^\pm$ for all $g \in G_W$. By appropriate choice of labelling we can also assume that $gW^\pm = (gW)^\pm$ for all $g \in G$.

Proposition 4.2. $X$ is finite dimensional, locally finite and $\delta$-hyperbolic (for some $\delta$).

Proof. $X$ is finite dimensional because $G$ acts cocompactly on it. Now suppose $X$ is not locally finite and that $x \in X$ is a vertex contained in infinitely many cubes. By cocompactness there is a cube $C$ and $A \subset G$ such that $A \cdot C$ is an infinite family of cubes each containing $x$; but then $C$ must have a vertex $x'$ such that $gx' = x$ for infinitely many $g \in A$, contradicting properness at $x$. Lastly, by Milnor-Schwarz, for any $x \in X$ the map $G \to X$, $g \mapsto gx$ is a quasi-isometry; hyperbolicity is a quasi-isometry invariant for geodesic spaces, and so $X$ is $\delta$-hyperbolic for some $\delta$. \qed
The remainder of this paper will be a proof of Theorem 1.1 by induction on \( \dim X \) (the case \( \dim X = 0 \) is trivial), so from now on assume that the theorem holds for lower dimensional cases - we will need this in the next two lemmas (in fact only for those lemmas).

These lemmas will use a few standard facts about hyperbolic and CAT(0) spaces, which we now recall.

1. For a geodesic \( n \)-gon in a \( \delta \)-hyperbolic space \( Y \), each side is within the \((n - 2)\delta \) neighbourhood of the union of the other sides (proof: subdivide the \( n \)-gon into triangles).

2. If \( C \) is a closed convex subspace of a CAT(0) space \( Y \), then there is a well-defined closest point projection map \( p: X \to C \), and this map is distance non-increasing (see [5, II.2.4]). In addition, \( p \) commutes with any isometry of \( Y \) that preserves \( C \).

3. If \( C \) is a closed convex subspace of a CAT(0) \( \delta \)-hyperbolic space \( Y \), \( p: Y \to C \) the closest point projection map, and \( A \subset Y \) another convex subspace with \( p(A) \) unbounded, then \( d(A, C) < 2\delta \).

Moreover, \( N_{2\delta}(A) \cap C \) will be unbounded.

**Proof.** Let \( x, y \in A \) and suppose \( z \in [p(x), p(y)] \subset C \) with \( d(z, p(x)), d(z, p(y)) > 4\delta \).

\[ X \quad \begin{array}{c} \leq 2\delta' \\vdash \end{array} \quad Y \]

\[ p(x) \quad Z \quad p(y) \]

We must have \( d(z, [x, p(x)]) \geq 2\delta \) as otherwise \( z \) would be closer to \( x \) than \( p(x) \) is. Similarly we must have \( d(z, [y, p(y)]) \geq 2\delta \). By applying fact (1) to the geodesic quadrilateral shown, we deduce that \( d(z, [x, y]) < 2\delta \). Thus \( N_{2\delta}(A) \cap [p(x), p(y)] \) contains all of \([p(x), p(y)]\) except possibly the end segments of length \( 4\delta \). As we can have \( d(p(x), p(y)) \) arbitrarily large, we see that \( N_{2\delta}(A) \cap C \) must be unbounded. \( \square \)

**Lemma 4.3.** Either we can choose \( W_1, ..., W_m \) orbit representatives for the walls of \( X \) such that \( d(W_i, W_j) > 3\delta \) for all \( 1 \leq i < j \leq m \), or Theorem 1.1 holds.

**Proof.** \( G \) is hyperbolic, so contains an infinite order element \( b \) [5, \textsection 2.22]. All isometries of \( X \) are semi-simple [5, II.6.10(2)], so \( b \) acts hyperbolically on \( X \), and acts by translations on an axis \( \gamma \) in \( X \) (a geodesic line in \( X \) [5, II.6.8(1)]). Let \( p : X \to \gamma \) be the closest point projection map to \( \gamma \).

First suppose there is a wall \( W \), such that \( p(gW) \) is unbounded for all \( g \in G \). By fact (3), we know that \( \gamma \cap N_{2\delta}(gW) \) is unbounded for all \( g \in G \). Because we are in a CAT(0) space, and \( \gamma \) and \( gW \) are convex, we see that \( \gamma \cap N_{2\delta}(gW) \) is convex, so contains an infinite subinterval of \( \gamma \). We deduce that there are only finitely many distinct translates \( gW \), else infinitely many of them would be within \( 2\delta \) of some point on \( \gamma \), contradicting local finiteness of \( X \). This means that the stabiliser \( G_W \) is finite index in \( G \). But \( G_W \) acts properly cocompactly on the CAT(0) cube complex \( W \), so by the lower dimensional case of theorem 1.1, there is a finite index subgroup \( G' < G_W \) acting freely on \( W \) such that \( W/G' \) is special. Then \( G' \) also acts freely on \( X \), for if \( g \in G' \) fixed \( x \in X \) then \( g \) would also fix \( p(x) \). Then \( X/G' \) is virtually special by Corollary 3.1.1 and Theorem 1.1 holds by replacing \( G' \) with a further finite index subgroup.

Conversely, suppose that for every wall \( W \) there exists \( g \in G \) with \( p(gW) \) bounded. Let \( W_1, ..., W_m \) be orbit representatives for the walls of \( X \) such that \( p(W_i) \) is bounded for each \( i \). Each \( p(W_i) \) is contained in a finite subinterval of \( \gamma \), and \( b \) acts as a translation along \( \gamma \), so we may choose \( n_1, ..., n_m \in \mathbb{Z} \) such that \( d(p(b^nW_i), p(b^nW_j)) = d(b^n p(W_i), b^n p(W_j)) > 3\delta \) for all \( 1 \leq i < j \leq m \). But \( p \) is distance non-increasing by fact (2), thus \( d(b^nW_i, b^nW_j) > 3\delta \) for all \( 1 \leq i < j \leq m \), as required. \( \square \)
Let Claim: Theorem 3.10 tells us in particular that \( H \) sets \( W := H \) there exists all bounded. By Remark 2.3 and induction on the lower dimensional cases of Theorem 1.1, for each trivial.

Proof: that \( H \) the finite index subgroup we can assume that \( d(W_i, W_j) > 3\delta \) for all \( 1 \leq i, j \leq m \). We are now ready for the main lemma of this section, which will produce a quotient of 

\[ \text{Lemma 4.4.} \quad \text{For any } R > 1 \text{ large enough so that } G \cdot B = X \text{ for any } R \text{-ball } B \text{ in } X, \text{ there exists a surjective homomorphism } \phi : G \rightarrow G \text{ with kernel } K \text{ and } H_i \triangleleft G_{W_i} \text{ finite index such that} \]

\[(1) \ \phi(H_i) \text{ are all finite},\]
\[(2) \text{ if } g \in G - H_i \text{ with } d(gW_i, W_i) \leq 2R \text{ then } \phi(g) \notin \phi(H_i),\]
\[(3) K \text{ is torsion-free} \ (\text{so acts freely on } X).\]

The proof will use the following variant of the ping-pong lemma.

**Lemma 4.5.** (Ping-pong Lemma)

Let \( H \) be a group that acts on a set \( Y \). If \( Y_1, \ldots, Y_n \subset Y \) and \( H_1, \ldots, H_n < H \) and \( y_0 \in Y - \cup_i Y_i \) are such that \( hY_j \subset Y_i \) and \( hy_0 \in Y_i \) whenever \( 1 \neq h \in H_i \) and \( j \neq i \), then \( H \) splits as a free product \( H = H_1 \ast \cdots \ast H_n \).

**Proof.** A product \( h = h_1 \cdots h_k \) with \( 1 \neq h_i \in H_m \), and \( m_i \neq m_{i+1} \) clearly maps \( y_0 \) into \( Y_{m_1} \), and so is the identity.

**Proof of Lemma 4.4.**

As in fact (2) from earlier, let \( p_i : X \rightarrow W_i \) be the closest point projection map to the wall \( W_i \). As the \( W_i \) are at least \( 3\delta \) apart from each other, fact (3) tells us that the images \( p_i(W_i) \) for \( i \neq j \) are all bounded. By Remark \ref{remark:important} and induction on the lower dimensional cases of Theorem \ref{theorem:main}, for each \( i \) there exists \( H_i < G_{W_i} \) finite index acting freely on \( W_i \) with \( W_i/H_i \) special. Define bounded subspaces

\[ A_i := N_{14\delta + 2R + 1}(\cup_{j \neq i} p_i(W_j)). \]

Theorem \ref{theorem:technical} tells us in particular that \( H_i \) is residually finite, so by replacing \( H_i \) with a further finite index subgroup we can assume that \( d(hA_i, A_i) > 1 \) for all \( 1 \neq h \in H_i \). We can also assume that \( H_i \triangleleft G_{W_i} \) by intersecting it with its finitely many conjugates, and \( W_i/H_i \) will still be special by Proposition \ref{proposition:reduction}. Note that some \( W_i \) might be finite and have \( A_i = W_i \) - in these cases \( H_i \) will be trivial.

Let \( X_i := \text{ball}(W_i - A_i) \). Pick \( x_0 \in p_1(W_2) \subset A_1 \) and note that \( x_0 \) is not in any of the \( X_i \). The next part of the proof does ping-pong with \( x_0, H_i \) and \( X_i \) to prove that we get a free splitting \( H := \langle H_1, \ldots, H_m \rangle \cong H_1 \ast \cdots \ast H_m \). By ignoring the \( i \) for which \( H_i \) is trivial we can assume that the sets \( W_i - A_i \) and \( X_i \) are non-empty.

Claim: \( p_i(X_j) \subset A_i \) for \( j \neq i \).

**Proof:** Let \( x \in X_j \) and suppose for contradiction that \( p_i(x) \notin A_i \).

We have the geodesic pentagon shown, where \( y \) is any point in \( p_j(W_i) \) and \( z \) is a point on \([p_i(x), p_j(y)] \cap A_i \). We have defined \( A_i \) to include a \( 14\delta \) buffer zone around \( \cup_{j \neq i} p_j(W_j) \), and \( p_i(x) \notin A_i \), therefore we can choose \( z \) to satisfy \( d(z, p_i(x)), d(z, p_j(W_j)) > 7\delta \). By fact (1) from earlier, \( z \) is within \( 3\delta \) of one of the other sides of the pentagon, we now check each of these four sides in turn:
1. If \( z' \in [x, p_j(x)] \) then \( p_j(z) = p_j(x) \notin A_j \), and so \( d(p_j(z), p_j(z')) \geq d(p_j(W), p_j(x)) \), which is greater than \( 3\delta \) because of the buffer zone in \( A_j \). But \( p \) is distance non-increasing, so \( d(z, z') > 3\delta \).

2. \( d(z, W_j) \geq d(W_i, W_j) > 3\delta \) by choice of the walls \( W_k \).

3. If \( z' \in [y, p_i(y)] \) then
   \[
   d(y, z) \leq d(y, z') + d(z', z) \\
   = d(y, p_i(y)) - d(p_i(y), z') + d(z', z) \\
   \leq d(y, p_i(y)) - d(z, p_i(y)) + 2d(z, z') \\
   \leq d(y, p_i(y)) - 7\delta + 2d(z, z') \\
   \leq d(y, z) - 7\delta + 2d(z, z'),
   \]
   (the last inequality by definition of \( p_i \)). This implies \( d(z, z') > 3\delta \).

4. The same argument as 3. shows that \( z \) cannot be within \( 3\delta \) of one of the other sides of the pentagon, contradicting fact (1). The claim follows.

**Claim:** For \( j \neq i \) and \( 1 \neq h \in H_i \) we have \( hX_j \subset X_i \) and \( hx_0 \in X_i \).

**Proof:** Let \( x \in X_j \). By the previous claim we have \( p_i(hx) = hp_i(x) \in hA_i \), hence \( p_i(hx) \notin A_i \) and so \( hx \in X_i \). Additionally, \( p_i(hx_0) = hp_i(x_0) \in hA_i \) and so \( hx_0 \in X_i \).

This last claim allows us to do ping-pong, as in Lemma 4.5 to obtain the desired splitting \( H \cong H_1 \ast \ast H_m \).

**Claim:** \( H < G \) is quasi-convex.

**Proof:** \( G \to X, g \to gx_0 \) is a quasi-isometry, so it suffices to show that \( H \cdot x_0 \) is quasi-convex in \( X \).

Let \( h = h_1 h_2 \ldots h_k \) with \( 1 \neq h_i \in H_{n_i} \) and \( n_i \neq n_{i+1} \). Put \( g_i = h_1 \ldots h_i \) and \( g_0 = 1 \). Our strategy will be to show that all points on the geodesic \( [x_0, hx_0] \) are close to one of the walls \( g_iW_{n_i} \) for \( 1 \leq i \leq k \).

First we consider projections to such a wall. Let \( q_i : X \to g_iW_{n_i} \) be the closest point projection map to \( g_iW_{n_i} \). For \( x \in X \), \( q_i(x) \) is the closest point on \( g_iW_{n_i} \) to \( x \), left multiplying by \( g_i^{-1} \) then tells us that \( g_i^{-1}q_i(x) \) is the closest point on \( W_{n_i} \) to \( g_i^{-1}x \), so \( g_i^{-1}q_i(x) = p_{n_i}(g_i^{-1}x) \). Therefore

\[
q_i = g_i p_{n_i} g_i^{-1}. \tag{4.1}
\]

We can then compute for \( 1 \leq i < k \),

\[
q_i(hx_0) = g_i p_{n_i} (g_i^{-1} hx_0) \\
= g_i p_{n_i} (h_{i+1} \ldots h_k x_0) \\
\in g_i p_{n_i} (X_{n_{i+1}}) \quad \text{by the second claim,} \\
\subset g_i A_{n_i} \quad \text{by the first claim.} \tag{4.2}
\]

Similarly,

\[
q_k(hx_0) = h p_{n_k}(x_0) \in h A_{n_k}. \tag{4.3}
\]

Next observe that \( g_i W_{n_i} = g_{i-1} W_{n_i} \), and so analogously to (4.1) we have \( q_i = g_i p_{n_i} g_i^{-1} \) (\( 1 \leq i \leq k \)).
We then compute for $1 < i \leq k$,
\[
q_i(x_0) = g_{i-1}p_n, (g_{i-1}^{-1}x_0) \\
= g_{i-1}p_n, (h_{i-1}^{-1}...h_1^{-1}x_0) \\
\in g_{i-1}p_n, (X_{n_{i-1}}) \\
\subset g_{i-1}A_n,
\]
by the second claim, 
\[
\text{by the first claim. (4.4)}
\]
And similarly
\[
q_1(x_0) = p_n, (x_0) \in A_n. 
\]
(4.5)

We now consider the concatenation of geodesics joining the following points pairwise in order.
\[
x_0, q_1(x_0), q_1(hx_0), q_2(x_0), q_2(hx_0), ..., q_k(x_0), q_k(hx_0), hx_0
\]
Call this path $\gamma$, and refer to the above points as the vertices of $\gamma$. Recalling that $x_0 \in A_1$, we can bound every other gap between consecutive vertices as follows.
\[
D := \text{diam}(\cup A_j) \geq \begin{cases} 
\text{by (4.3)} \\
\frac{d(q_i(hx_0), q_{i+1}(x_0))}{1} & \text{for } 1 \leq i < k, \text{ by (4.2) and (4.4)} \\
\frac{d(q_k(hx_0), hx_0)}{1} & \text{by (4.3)}
\end{cases}
\]

The other gaps between consecutive vertices are spanned by segments $\gamma_i := [q_i(x_0), q_i(hx_0)] \subset g_iW_{n_i}$. Since $H_{n_i}$ acts cocompactly on $W_{n_i}$ and $g_i \in H$, we deduce that each $\gamma_i$ is contained within $N_M(H \cdot x_0)$ for some constant $M$ that is independent of $h$. Hence $\gamma \subset N_{M+D}(H \cdot x_0)$.

To complete the proof of the claim it remains to show that $\sigma := [x_0, hx_0] \subset N_{L}(\gamma)$ for some constant $L$ that is independent of $h$.

Consider $z \in \gamma_i$ at least $5\delta$ away from the endpoints of $\gamma_i$. By fact (1), $z$ is within $2\delta$ of one of the other sides of the geodesic quadrilateral shown, so it must be within $2\delta$ of $\sigma$ otherwise it contradicts the definition of closest point projection. Therefore $\gamma_i \subset N_{7\delta}(\sigma)$ and $\gamma \subset N_{D+7\delta}(\sigma)$.

Finally note that projection from $\gamma$ to $\sigma$ is continuous; and, as the paths share endpoints $x_0$ and $hx_0$, the image is the whole of $\sigma$. So in fact $\sigma \subset N_{D+7\delta}(\gamma)$.

By Proposition 3.6, $H \cong H_1 \ast ... \ast H_m$ is the fundamental group of a special cube complex. By taking direct products of the homomorphisms in [3.12] we deduce that for any finite $A \subset G - H$ there is a quotient homomorphism $\phi : G \rightarrow G$ such that $\phi(A) \cap \phi(H) = \emptyset$ and $\phi(H)$ is finite. We now show that conclusions (1)-(3) of the lemma can be satisfied by a certain choice of $A$.

(1) $\phi(H_i) < \phi(H)$ so must be finite.
(2) For each \( i \) the collection of double cosets

\[
    \mathcal{A}_i := \{H_i g H_i \mid g \in G, \, d(g W_i, W_i) \leq 2R\} - \{H_i\}
\]

is finite. To see this, fix \( y \in W_i \) and consider \( g \in G \) with \( d(g W_i, W_i) \leq 2R \) - say \( x, x' \in W_i \) satisfy \( d(g x', x) \leq 2R \). Suppose that \( Q > 0 \) with \( W_i \subset H_i \cdot B_Q(y) \). Now pick \( h, h' \in H_i \) so that \( d(h x, y), d(h'y, x') < Q \); then \( d(h g h' y, y) \leq d(h g h' y, h g x') + d(h g x', h x) + d(h x, y) < 2Q + 2R \). The finiteness of \( \mathcal{A}_i \) then follows because \( X \) is locally finite and the action of \( G \) is proper.

We are then done provided \( A \) contains representatives for all of the double cosets in the \( \mathcal{A}_i \).

(3) If \( g \in G \) is a torsion element then by [\( R \) II.2.8] it has a fixed point \( x \in X \). By assumption of the lemma, there exists \( k \in G \) with \( k x \in B_R(x_0) \). Then \( d(k g k^{-1} x_0, x_0) < 2R \). Therefore there is a finite set \( T \) of representatives for conjugacy classes of torsion elements in \( G \). Each \( H_i \) is torsion-free because it acts freely on \( W_i \), so \( H \) is also torsion-free and \( H \cap T = \emptyset \). Adding \( T \) to \( A \) will ensure that \( K \) is torsion-free.

The point of Lemma 4.4 is that it allows us to define the following quotient complex.

**Definition 4.6. (Quotient Complex \( X' \))**

As a result of Lemma 4.4 we can define the NPC cube complex \( X' := X/K \). The value of \( R \) we use will be some constant large enough to satisfy the cocompactness condition of the lemma, and we also require \( R \geq \delta + 2 \sqrt{\dim X} \) (this inequality will be demystified in section 8). The metric on \( X' \) will be denoted \( d' \), the same as for \( X \).

As was the aim of this section, this quotient complex satisfies the following properties.

**Lemma 4.7. (Properties of \( X' \))**

(1) There are natural cocompact actions of \( G \) and \( G \) on \( X' \).

(2) All walls of \( X' \) are finite.

(3) For any wall \( W \) in \( X \), the \( R \)-neighbourhood \( N_R(W) \) quotiented by \( K \cap G_W \) embeds in \( X' \). In particular this implies that all walls of \( X' \) are embedded, and that distinct walls in \( X \) which are less than \( R \) apart map to distinct walls in \( X' \).

**Proof.** (1) holds because \( K \) is normal in \( G \) and \( G \) acts cocompactly on \( X \). By Lemma 4.3(1) we know that, for any \( g \in G, \, K \cap g G_W, g^{-1} \) has finite index in \( g G_W, g^{-1} = G_{gW} \), and so acts cocompactly on \( g W_i \) - property (2) follows. Lemma 4.3(2) tells us that the \( R \)-neighbourhood \( N_R(W_i) \) quotiented by \( K \cap H_i = K \cap G_W \) embeds in \( X' \) - property (3) follows by considering translates of the \( W_i \) and conjugates of the \( H_i \).

To finish the section we introduce some notation.

**Notation:** The quotient map \( m : X \to X' \) will send a vertex \( x \) (resp. an edge \( e \) and wall \( W \)) to a vertex \( \bar{x} \) (resp. an edge \( \bar{e} \) and wall \( \bar{W} \)). And \( \bar{W}(\bar{e}) \) will denote the wall dual to \( \bar{e} \). By Remark 4.1 we know that walls in \( X' \) are two-sided and that no element of \( G \) can exchange the sides of any wall. We can define \( \bar{W}^e := (\bar{W} \bar{e}) \).
5 Invariant colouring measures

This section is preparatory. More specifically, before we can start colouring walls in the next section, we need to establish some theory about colouring graphs. This section is essentially the same as [1, §5], but with a little more detail regarding weak* convergence.

Definition 5.1. (Colourings)
An n-colouring of a graph Γ is a map $c : V(Γ) \to [n] := \{1, ..., n\}$ such that $c(v_1) \neq c(v_2)$ whenever $\{v_1, v_2\} \in E(Γ)$. Let $C_n(Γ)$ denote the set of n-colourings. If vertex degrees are bounded by $k$ then it is clear that $C_{k+1}(Γ) \neq \emptyset$.

Suppose a group $H$ acts on $Γ$. Then we have an action of $H$ on $C_n(Γ)$ by $h : c \mapsto c \circ h^{-1}$.

Definition 5.2. (Colourings as a measurable space)
Consider $C_n(Γ)$ as a closed subspace of $[n]^{V(Γ)}$ with the product topology. We will consider $[n]^{V(Γ)}$ as a measurable space with $σ$-algebra generated by the sets $A_{v,j} := \{f \in [n]^{V(Γ)} \mid f(v) = j\}$ for $v \in V(Γ)$ and $j \in [n]$ - note that if $V(Γ)$ is countable then this is also the $σ$-algebra generated by the open subsets of $[n]^{V(Γ)}$. The action of $H$ on $C_n(Γ)$ extends naturally to $[n]^{V(Γ)}$ by $h : f \mapsto f \circ h^{-1}$. These are measurable functions so the family of measurable subsets of $[n]^{V(Γ)}$ is $H$-invariant.

Theorem 5.3. Suppose a group $H$ acts cocompactly on a countable graph $Γ$ with vertex degrees bounded by $k$. Then there exists an $H$-invariant probability measure $μ$ on $C_{k+1}(Γ)$.

Proof. Let $M_H(n)$ denote the set of $H$-invariant probability measures on $[n]^{V(Γ)}$. We have that $C_{k+1}(Γ) \subset [k+1]^{V(Γ)}$ and our task is to find $μ \in M_H(k+1)$ with $μ(C_{k+1}(Γ)) = 1$.

For an edge $e = \{v_1, v_2\} \in E(Γ)$ let $B_e(n) := \{f \in [n]^{V(Γ)} \mid f(v_1) = f(v_2)\}$; it is clear that $[n]^{V(Γ)}$ splits as a disjoint union $[n]^{V(Γ)} = C_n(Γ) \cup \bigcup_{e \in E(Γ)} B_e(n)$, so our task reduces to finding $μ \in M_H(k+1)$ with $μ(B_e(k+1)) = 0$ for all $e \in E(Γ)$. Let $\{e_1, ..., e_m\} \subset E(Γ)$ be a complete set of orbit representatives for the action of $H$ on $E(Γ)$; since $ν(B_{e_i}(k+1)) = ν(hB_{e_i}(k+1)) = ν(B_{e_i}(k+1))$ for all $ν \in M_H(k+1)$, it suffices to find $μ \in M_H(k+1)$ with $μ(B_{e_i}(k+1)) = 0$ for $i = 1, ..., m$.

For $ν \in M_H(n)$ define $weight(ν) := \sum_{e \in E(Γ)} ν(B_e(n))$. We will use a limiting argument to construct $μ \in M_H(k+1)$ with zero weight. Define a $H$-invariant probability measure $μ_n$ on $[n]^{V(Γ)}$ as the product of uniform measures on $[n]$. Then $μ_n$ is the unique measure with $μ_n(A_{v,j}) = 1/n$ for every $A_{v,j}$ - this is $H$-invariant because $A_{v,j}$ is $H$-invariant. $μ_n(B_e(n)) = μ_n(\bigcup_j (A_{v,j} \cap A_{w,j})) = 1/n$ for $e = \{v, w\} \in E(Γ)$, so $weight(μ_n) = m/n$.

For $n > k+1$ define a map $p_n : [n]^{V(Γ)} \to [n-1]^{V(Γ)}$ by

$$p_n(c)(v) := \begin{cases} c(v), & c(v) < n \\ \min([n-1] - \{c(u) \mid \{u, v\} \in E(Γ)\}), & c(v) = n \end{cases}$$

for $c \in [n]^{V(Γ)}$ and $v \in V(Γ)$. In other words $p_n$ changes the colour of each vertex coloured $n$ to the smallest colour not used by its neighbours, and leaves other vertices with the same colour. This is well-defined because vertex degrees are at most $k$. It is clear that $p_n$ is $H$-equivariant and continuous, so it induces a well-defined push-forward $p_{n*} : M_H(n) \to M_H(n-1)$ given by $p_{n*}(ν)(A) = ν(p_n^{-1}(A))$ for $ν \in M_H(n)$ and $A \subset [n-1]^{V(Γ)}$ measurable. Furthermore, for $\{v_1, v_2\} \in E(Γ)$ and $c \in [n]^{V(Γ)}$, if $p_n(c)(v_1) = p_n(c)(v_2)$ then $c(v_1) = c(v_2)$. Therefore $p_n^{-1}(B_c(n-1)) \subset B_c(n)$ for any edge $e$; consequently $weight(p_{n*}(ν)) \leq weight(ν)$ for any $ν \in M_H(n)$.

Now define $P_{n*} = p_{k+2*} \circ p_{k+3*} \circ \cdots \circ p_n : M_H(n) \to M_H(k+1)$. We will then have that $weight(P_{n*}(μ_n)) \leq weight(μ_n) = m/n \to 0$ as $n \to \infty$.

By [13] the set of all probability measures on $[k+1]^{V(Γ)}$ is compact metrizable in the weak* topology (measures $(ν_n)$ converge to $ν$ in the weak* topology if and only if $\int f \, dν_n \to \int f \, dν$ for every
continuous map \( \alpha : [k+1]^V_\Gamma \to \mathbb{R} \). \( M_H(k+1) \) is a closed subspace with respect to this topology; to see this suppose a sequence \( (\nu_n) \) in \( M_H(k+1) \) converges to a measure \( \nu \). Let \( \Sigma \) be the algebra generated by the \( A_{v,j} \) (the smallest family containing the \( A_{v,j} \) that is closed under finite union and complementation), all sets in \( \Sigma \) are clopen in \([k+1]^V_\Gamma\) so by considering characteristic functions we have \( \nu_n(A) \to \nu(A) \) for every \( A \in \Sigma \). For \( h \in H \) define \( \nu_h \) by \( \nu_h(A) = \nu(hA) \) for measurable sets \( A \); for \( A \in \Sigma \) we have \( \nu_h(A) = \lim_{n \to \infty} \nu_n(hA) = \lim_{n \to \infty} \nu_n(A) = \nu(A) \), so by Caratheodory’s Extension theorem we get \( \nu = \nu_h \) thus \( \nu \in M_H(k+1) \).

We conclude that \( P_n(\mu_n) \) has a convergent subsequence converging to some \( \mu \in M_H(k+1) \). The weight is continuous with respect to the weak* topology because \( B_{e_i}(k+1) \in \Sigma \), thus weight(\( \mu \)) = 0 as required.

Remark 5.4. When we apply this theorem later in the paper it would be enough for \( \mu \) to only be defined on the algebra \( \{ C_{k+1}(\Gamma) \cap A \mid A \in \Sigma \} \), where \( \Sigma \) is the algebra generated by the \( A_{v,j} \), rather than having \( \mu \) defined on all measurable subsets of \( C_{k+1}(\Gamma) \). If we had modified the theorem to only require this, then the last part of the proof that argues about convergence would be easier because we wouldn’t need weak* convergence or Caratheodory’s Extension theorem (the reason is basically that \( \Sigma \) is countable). However keeping the theorem as it is makes for a cleaner statement.

6 Colouring walls

In this section we define the local colouring data that will play a key role in the following sections. This local colouring data will be defined as equivalence classes on the space of all colourings of walls in \( \mathcal{X} \); first we define this space of colourings by building a graph out of the walls of \( \mathcal{X} \). This section follows Definitions 6.6-6.8 of [1], but with a simplification implied by Agol’s ICM notes [2, 7.4] and with some differences in notation.

Definition 6.1. (The graph \( \Gamma \))

Let \( \Gamma = \Gamma(\mathcal{X}) \) be a graph whose vertices are the walls of \( \mathcal{X} \), and with walls \( \overline{W}_1, \overline{W}_2 \) joined by an edge in \( \Gamma \) if \( d(\overline{W}_1, \overline{W}_2) \leq R \). We have a natural action of \( G \) on \( \Gamma \). As \( \mathcal{X} \) is locally finite, cocompact and with finite walls (see Lemma 4.7) it follows that the degree of vertices in \( \Gamma \) is bounded by some \( k \in \mathbb{N} \).

As in Definition 5.1 we have an action of \( G \) on \( C_{k+1}(\Gamma) \) by \( c \mapsto gc := c \circ g^{-1} \) - this also induces an action of \( G \) on \( C_{k+1}(\Gamma) \) by \( c \mapsto gc := c \circ \phi(g^{-1}) \).

Definition 6.2. (Equivalent colourings)

For \( W \) a wall in \( X \), we will define equivalence classes \([\cdot]_W \) in \( C_{k+1}(\Gamma) \) that depend only on the colour of vertices ‘near’ to \( \overline{W} \) in \( \Gamma \). What we mean by vertices ‘near’ to \( \overline{W} \) will depend on the colouring \( c \in C_{k+1}(\Gamma) \) in question. Specifically, define the equivalence class \([c]_W \) by

\[
[c]_W := \{ c' \in C_{k+1}(\Gamma) \mid c' = c \text{ on the ball of radius } c(\overline{W}) \text{ in } \Gamma \text{ centred at } \overline{W} \}.
\]
For \( e \in E(X) \) that crosses a wall \( W \), we will use \([-\cdot]_W \) as an alternative notation for the equivalence class \([-\cdot]_W \). We will also use the notation \( c(e) := c(W) \). Note that \( c' \in [c]_x \) implies \( c'(e) = c(e) \). For \( x \in V(X) \) we will require the finer equivalence classes \([-\cdot]_x \) in \( C_{k+1}(\Gamma) \) defined by

\[
[c]_x := \cap \{[c]_e \mid e \in E(X) \text{ incident at } x \}.
\]

We want to combine these families of equivalence relations into two \( G \)-invariant equivalence relations, one for edges and one for vertices. Do this as follows. Define equivalence classes on \( E(X) \times C_{k+1}(\Gamma) \) by \([e, c] := \{e\} \times [c]_e \). Similarly, define equivalence classes on \( V(X) \times C_{k+1}(\Gamma) \) by \([x, c] := \{x\} \times [c]_x \).

The action of \( G \) on \( E(X), V(X) \) and \( C_{k+1}(\Gamma) \) induces actions on the product spaces and on the two sets of equivalence classes by \( g[e, c] = [gc, gc] \) and \( g[x, c] = [gx, gc] \). It is straightforward to check that these are well-defined (first check that \( g[c]_W = [gc]_{gW} \)).

**Remark 6.3.** For each \( e \in E(X) \), the classes \([-\cdot]_e \) only depend on the colour of vertices in some \((k+1)\)-ball of \( \Gamma \), and so there are only finitely many of these equivalence classes. Similarly, for each \( v \in V(X) \), there are only finitely many classes \([-\cdot]_v \). As there are finitely many \( G \)-orbits in \( E(X) \) and \( V(X) \), there must only be finitely many \( G \)-orbits of equivalence classes on \( E(X) \times C_{k+1}(\Gamma) \) and \( V(X) \times C_{k+1}(\Gamma) \).

**Remark 6.4.** If edges \( e, e' \in E(X) \) are both incident at a vertex \( x \), then \( d(W(e), W(e')) \leq 1 < R \), so \( c(e) = c(W(e)) \neq c(W(e')) = c(e') \) for any \( c \in C_{k+1}(\Gamma) \).

## 7 Starting the gluing construction

In this section we introduce the main construction used to prove Theorem [11]. We will implement the first step of the construction and show how the theorem follows from the final step. The process of going between steps will be left to the last two sections. The construction is similar in spirit to [1, §8], but we work with subspaces of \( X \) rather than orbi-complexes; in other words we work on the space where \( G \) acts rather than in the quotient. Lemma [74] is based on the ideas of [1, §7].

To prove Theorem [11] we will inductively construct \( V_{k+1}, V_k, \ldots, V_0 \) (with the \( k \) from Definition 6.1). Each \( V_j \) will be a non-empty collection of triples \((Z, H, (c_x))\) where \( Z \subset X \) is a non-empty intersection of half-spaces (so is closed and convex), and for each \( x \in Z \) a vertex we have \( c_x \in C_{k+1}(\Gamma) \) a colouring. \( H \) will be a subgroup of \( G \) that acts on \( Z \) freely and cocompactly, and \( c_hx = hcx \) for \( h \in H \). We will permit \( V_j \) to contain duplicates of some triples. Where there is no danger of ambiguity, we will write \( Z \in V_j \) as shorthand for \((Z, H, (c_x)) \in V_j \).

\( V_j \) will satisfy four properties; before stating these formally we give some loose motivation for them. We will often work with the finite complex \( Z/H \) which has universal cover \( Z \) (as \( Z \) is \( \text{CAT}(0) \)). Technically \( Z \) and \( Z/H \) are not quite cube complexes, rather they are something like ‘cube complexes with boundary walls’ - we’ll just have to live with this, but we will get a genuine cube complex once we’ve finished all the gluing. Think of the colourings \( c_x \) as giving information about some of the walls nearby \( x \), such as which walls mark the boundary of \( Z \). The rough idea of the construction is to glue the complexes \( Z/H \) along walls coloured \( j \) to form \( V_{j-1} \). Neighbouring vertices in \( Z \) must agree about colours of nearby walls, and so vertices next to boundary walls that will later be glued up must have a potential matching in which colourings are compatible.
Here are the properties that \( V_j \) must satisfy (which use notation from Definitions 6.2 and 6.7):

1. If \( e \in E(X) \) joins vertices \( x, y \in Z \in V_j \), then \([e, c_x] = [e, c_y]\). So adjacent vertices are equipped with similar colourings.

2. If \( e \in E(X) \) joins vertices \( x, y \) with \( x \in Z \in V_j \), then
   \[ y \in Z \iff c_x(e) > j. \]
   So walls in the interior of \( Z \) are coloured \( > j \) by their neighbouring vertices, whereas walls on the boundary of \( Z \) are coloured \( \leq j \).

3. (Gluing Equations)
   Say \( e \in E(X) \) has endpoints \( x_+ \in W(e)^+ \) and \( x_- \in W(e)^- \) (notation from definition 2.5), and let \( c \in C_{k+1}(\Gamma) \). Define sets
   \[ V_j^\pm(e, c) := \{(H \cdot x, Z) \mid x \in Z \in V_j, \exists y \in G : gx = x_\pm, [e, gc_x] = [e, c]\}, \]
   where any duplicates of triples \((Z, H, (c_x)) \in V_j \) are counted separately. In other words, \( V_j^\pm(e, c) \) is the set of vertices in the complexes \( Z/H \) that, modulo the action of \( G \), correspond to \( x_\pm \) and have colouring in the class \([c]_e\).
   The Gluing Equations are given by
   \[ |V_j^+(e, c)| = |V_j^-(e, c)|, \]
   where \( e \) ranges over \( E(X) \) and \( c \) ranges over \( C_{k+1}(\Gamma) \). Roughly speaking, these equations will ensure that walls on the boundary of complexes \( Z/H \) that look like \( W(e) \) on the \( W(e)^\pm \) side can be matched up with those that look like \( W(e) \) on the \( W(e)^- \) side, with compatible colourings matched together - but there is more work to be done later to arrange this precisely.

4. \( H \in QVH \) for any triple \((Z, H, (c_x)) \in V_j \). This will allow us to make use of Theorems 3.8 and 3.10

It will be convenient to also define colourings \( c_e \in C_{k+1}(\Gamma) \) for edges \( e \in E(X) \) that intersect \( Z \), such that if \( e \) is incident at a vertex \( x \in Z \) then \([e, c_x] = [e, c_x]\). This is possible by property (1). We will only ever care about the class \([e, c_x]\), so the colourings \( (c_e) \) are not extra data that needs to be added to the triple \((Z, H, (c_x))\).

To start the inductive construction of the \( V_i \), we first define \( V_{k+1} \).

**Lemma 7.1.** There exists \( V_{k+1} \) satisfying all of the above conditions.

**Proof.** Let \( \{x_1, \ldots, x_t\} \) be a complete set of \( G \)-orbit representatives in \( V(X) \). For each \( j \) let

\[ C_{k+1}(\Gamma) = \bigcup_{1 \leq l \leq n_j} [c_{jl}]_{x_j} \]  \hspace{1cm} (7.1)

be a partition, it is finite by Remark 6.3. For each \((j, l)\) let \( Z_{jl} \) be the intersection of all half-spaces containing \( x_j \). Note that \( Z_{jl} \) will be compact, in fact it will be the union of cubes in \( \hat{X} \) (the barycentric subdivision of \( X \)) that contain \( x_j \), so \( Z_{jl} \cap V(X) = \{x_j\} \). We will then define \( V_{k+1} \) to be the collection of triples \((Z_{jl}, \{1\}, c_{jl})\).

We must check that this definition of \( V_{k+1} \) satisfies properties (1)-(4) above. Each \( Z_{jl} \) only contains one vertex of \( X \), so properties (1) and (2) hold vacuously, and (4) is also immediate since the trivial group is in \( QVH \). However (3) might not hold. To rectify this we will make \( V_{k+1} \) contain \( \alpha_{jl} \) copies of \( Z_{jl} \) for appropriate integers \( \alpha_{jl} \), which we will spend the rest of the proof constructing.

Take \( e \in E(X) \) with endpoints \( x_+ \in W(e)^+ \) and \( x_- \in W(e)^- \), and take \( c \in C_{k+1}(\Gamma) \). Say \( x_+ \) is in the orbit of \( x_1 \). How can we count \( V_{k+1}^+(e, c) \)? Well the contributions will come from precisely the pairs
Fix and putting equivalent to \( \mu \). We do this by taking the measure not all zero, but as a start we will exhibit positive real numbers using the measure from Theorem 5.3. \( \alpha \)

\[ \text{Claim: } M^+(e, c) = |\text{Stab}_G([e, c])||V^+_{k+1}(e, c)| \]

\[ \text{Proof: } \text{Fix } l \text{ that contributes something to } M^+(e, c). \text{ Then there must be some } g_0 \in G \text{ with } g_0 x_i = x_+ \text{ and } [gc_i]_e = [c]_e, \text{ and so } c_i (g_0^{-1} e) = gc_i (e) = c(e). \text{ By Remark 6.3, } e_l := g_0^{-1} e \text{ is the unique edge incident at } x_+, \text{ that is coloured } c(e) \text{ by } c_i. \]

With the same \( l \) fixed, we now claim that \( (g, l) \) contributes to \( M^+(e, c) \) if and only if \( g [e_l, c_i] = [e, c] \). Indeed on the one hand, any \( (g, l) \) contributing to \( M^+(e, c) \) has \( c_i (g^{-1} e) = c(e) \) like we had for \( g_0 \) above, so by uniqueness of \( e_l \) we get \( g[e_l, c_i] = [e, c] \). On the other hand, \( g[e_l, c_i] = [e, c] \) implies \( gc_i (e) = c(e) \), so \( gc_i (e) = x_+ \) or \( x_- \); but it can’t be \( x_- \), hence \( gc_i (e) = x_+ \) and \( (g, l) \) contributes to \( M^+(e, c) \). Since \( G \) acts on the equivalence classes of \( E(X) \times C_{k+1}(\Gamma) \), we deduce that there are \( |\text{Stab}_G([e, c])| \) pairs \( (g, l) \) contributing to \( M^+(e, c) \).

Now \( l \) contributes to \( M^+(e, c) \) if and only if \( \{x_i\} \) \( Z_d \) contributes to \( |V^+_{k+1}(e, c)| \); and in this case the contributions will be \( \alpha_d |\text{Stab}_G([e, c])| \) to \( M^+(e, c) \) and \( \alpha_d |V^+_{k+1}(e, c)| \).

Why on earth is it useful to over-count \( V^+_{k+1}(e, c) \)? Well the factor of over-counting we obtained only depends on \( e \) and \( c \), so if we over-count \( V^+_{k+1}(e, c) \) in the same way to produce a total \( M^-(e, c) \), then the factor of over-counting is the same. Hence the Gluing Equation \( |V^+_{j}(e, c)| = |V^-_{j}(e, c)| \) is equivalent to \( M^+(e, c) = M^-(e, c) \). The trick now is to solve these transformed Gluing Equations by using the measure from Theorem 5.3.

The \( M^+(e, c) \) are just integer sums of the \( \alpha_j \). We want the \( \alpha_j \) to be non-negative integers (that are not all zero), but as a start we will exhibit positive real numbers \( \alpha_j \) that solve the Gluing Equations. We do this by taking the measure \( \mu \) from Theorem 5.3, applied to the graph \( \Gamma \) with the action of \( G \), and putting

\[ \alpha_j = \frac{\mu([c_j]_x)}{|\text{Stab}_G(x_j)|}. \]

(7.2)

Next, observe that \([c]_x\) can be partitioned into \([-]_{x_+}\) equivalence classes, which can be written

\[ [c]_x = \bigcup_b [c_b]_{x_+}. \]

(7.3)

for some \( c_b \in C_{k+1}(\Gamma) \) (and this partition is finite by Remark 6.3). A pair \((g, l)\) contributes to \( M^+(e, c) \) if and only if \( g[x_i, c_i] = [x_+, c_b] \) for some \( b \), so we can count \( M^+(e, c) \) by adding up the contributions from each \( c_b \).

There will be \( |\text{Stab}_G(x_i)| \) choices of \( g \) with \( gx_i = x_+ \), and for each pair \((g, b)\) there will be a unique \( l \) with \( g[x_i, c_i] = [x_+, c_b] \). As \( \mu \) is \( G \)-invariant, we see from (7.2) that each \( \alpha_d \) only depends on the \( G \)-orbit of \([x_i, c_i]\), and so the contribution to \( M^+(e, c) \) from a given \( c_b \) will equal \( \alpha_d |\text{Stab}_G(x_i)| \) for any \( l \) with \([x_+, c_b] \in G \cdot [x_i, c_i]\). For any such \( l \), the \( G \)-invariance of \( \mu \) implies that \( \mu([c_b]_{x_+}) = \mu([c_i]_x) \), hence

\[ M^+(e, c) = \sum_b \frac{\mu([c_b]_{x_+})}{|\text{Stab}_G(x_i)|} |\text{Stab}_G(x_i)| \]

\[ = \sum_b \mu([c_b]_{x_+}) \]

\[ = \mu([c]_x). \]

Again this only depends on \( e \) and \( c \), so our clever choices of \( \alpha_j \) will also give \( M^-(e, c) = \mu([c]_x) \). This will hold for all \( e \) and \( c \), thus solving the Gluing Equations.

All that remains is to convert this into a non-negative integer solution of the Gluing Equations. Note that, as functions of the \( \alpha_j \), \( M^\pm(e, c) \) only depend on the \( G \)-orbit of \([e, c]\); there are finitely many
such orbits by Remark 6.3, so there are actually just finitely many Gluing Equations (as equations in the $\alpha_{jl}$). Note that the values of $\alpha_{jl}$ from (7.2) are not all zero, because, for fixed $j$, $C_{k+1}(\Gamma)$ can be expressed as a finite partition of $[-\infty, \infty]$ equivalence classes as in (7.1), and $\mu(C_{k+1}(\Gamma)) = 1$. We can then promote our non-negative real number solution of the Gluing Equations to a non-negative integer solution using the following claim. Moreover, since our real number solution isn’t identically zero, we can arrange that the integer solution isn’t identically zero either. This is an example of linear programming, a technique which has been widely used in topology, an early instance being Haken’s work on normal surface theory [15].

Claim: Let $A$ be an integer matrix defining a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. If $\exists v \in \ker A - \{0\}$ with non-negative entries, then $\exists w \in \ker A - \{0\}$ with non-negative integer entries.

Proof: Let $v \in \ker A - \{0\}$ have non-negative entries. In fact we may assume that all entries of $v$ are strictly positive (else delete columns in $A$ corresponding to the zero entries of $v$ and solve the claim for this matrix, and reintroduce the zero entries to $w$ afterwards). It suffices to find $w$ with non-negative rational entries since we can multiply out denominators to make the entries integers. Now $A\mathbb{R}^n$ is the closure of $A\mathbb{Q}^n$ so both have the same dimension as vector spaces over $\mathbb{R}$ and $\mathbb{Q}$ respectively; thus $\ker (A)$ and $\ker (A) \cap \mathbb{Q}^n$ also have the same dimensions, and so the former must be the closure of the latter. Therefore we can choose $w \in \ker (A) \cap \mathbb{Q}^n$ to be a rational approximation of $v$, close enough so that it has positive entries. ■

The inductive construction of $\mathcal{V}_k, \ldots, \mathcal{V}_0$ will be left to the final two sections. To close this section we show that Theorem 1.1 follows from the existence of $\mathcal{V}_0$.

Proof of theorem 1.1 given $\mathcal{V}_0$. Take some triple $(Z, H, (c_x)) \in \mathcal{V}_0$. Property (2) and the connectedness of $X$ imply that $Z = X$. $H$ acts cocompactly on $X$ so must be finite index in $G$. $H$ acts freely on $X$ by definition of $\mathcal{V}_0$. Property (4) in conjunction with Theorem 3.8 and Corollary 3.11 tells us that $X/H$ is virtually special. We can then take $G' < H$ finite index such that $X/G'$ is special. ■

8 Controlling boundary walls

To go from $\mathcal{V}_j$ to $\mathcal{V}_{j-1}$ we will glue together the various complexes $Z/H$ along the quotients of certain ‘boundary walls’. In this section we will establish what boundary walls are and which ones we are gluing along, and we will prove some technical lemmas (to be used later) that control the behaviour of these walls. Lemma 8.4 comes from [1, p1062], and Lemma 8.8 comes from [1, p1063], but both are recast to fit with our definitions.

For this section fix $(Z, H, (c_x)) \in \mathcal{V}_j$.

Definition 8.1. (Boundary walls)
For $e$ an edge crossing out of $Z$ we call $W(e)$ a boundary wall of $Z$. Equivalently, boundary walls are walls $W(e)$ for $e$ an edge intersecting $Z$ and $c_x(e) \leq j$. $Z$ is an intersection of half-spaces, so if $W$ is a
boundary wall then \( Z \) is contained in one half-space of \( W \). Let \( \partial Z \subset Z \) be the union of all boundary walls intersected with \( Z \).

**Remark 8.2.** For vertices \( x, y \in V(X) \) with \( x \in Z \), let \( \gamma \) be a shortest edge path from \( x \) to \( y \), then the following are equivalent,

1. \( y \notin Z \),
2. \( \gamma \) crosses a boundary wall,
3. \( y \) and \( z \) are separated by a boundary wall.

Indeed if \( y \notin Z \) then the first time \( \gamma \) leaves \( Z \) it must cross a boundary wall, so (1) implies (2). (2) implies (3) follows from Proposition 2.9. (3) implies (1) because \( Z \) is contained in one half-space of the boundary wall. In particular this shows that any two vertices in \( Z \) are connected by an edge path that stays in \( Z \).

**Lemma 8.3.** Let \( W_1, \ldots, W_n \) be pairwise intersecting walls, with \( Z \cap W_i \neq \emptyset \) for each \( i \), then \( Z \) contains a vertex \( x \) incident at edges \( e_1, \ldots, e_n \) that are dual to \( W_1, \ldots, W_n \) respectively (and so \( e_1, \ldots, e_n \) form the corner of an \( n \)-cube in \( X \)).

**Proof.** It suffices to prove the lemma for \( W_1, \ldots, W_n \) a maximal family of pairwise intersecting walls with \( Z \cap W_i \neq \emptyset \) for each \( i \). Let \( y \in Z \) be a vertex. By Proposition 2.10 there is a vertex \( x \) incident at edges \( e_1, \ldots, e_n \) that are dual to \( W_1, \ldots, W_n \) respectively such that no \( W_i \) separates \( x \) and \( y \). Consider a shortest edge path from \( y \) to \( x \); if it crosses no boundary walls then \( x \in Z \) and we are done. Suppose it does cross a boundary wall, \( W \) say. By Proposition 2.9 \( W \) divides \( X \) into two half-spaces, one containing \( x \) and the other containing \( y \) and \( Z \). For each \( i \), part of \( W_i \) is in the half-space containing \( x \), but \( Z \cap W_i \neq \emptyset \), so we must also have \( W \cap W_i \neq \emptyset \), contradicting the maximality of \( W_1, \ldots, W_n \). \( \square \)

If there are two edges crossing different boundary walls (possibly in different triples of \( V_j \)), and these edges give the same colouring equivalence class, then we want to be able to ‘zip’ together these boundary walls in a colour-compatible way. The following lemma will help us to achieve this, although we won’t actually do the zipping until Section 9.

**Lemma 8.4.** (Zipping Lemma)

Let \( W \) be a boundary wall of \( Z \). Then the edges \( e \) crossing out of \( Z \) with \( W = W(e) \) all induce the same class \( [c_e]_W \) and hence give the same colour \( c_e(\overline{W}) = c_e(e) \) (this can be thought of as the colour of \( W \), and we’ll refer to it as such).

**Proof.**

Let \( S \) be a square in \( X \) with an edge \( e \) joining vertices \( x_1, x_2 \in Z \), let \( e_1, e_2 \) be the other edges incident at \( x_1, x_2 \) respectively and suppose they cross out of \( Z \) with \( W(e_1) = W(e_2) = W \). By property (2) of \( V_j \), \( c_{x_1}(e) > j \geq c_{x_1}(e_1) \). \( W(e) \) and \( \overline{W} \) intersect, so are adjacent vertices in \( \Gamma \). Now \( [c_{x_1}]_W(e) = [c_e]_W(e) \), so \( c_{x_1} \) agrees with \( c_e \) on the ball of radius \( c_{x_1}(e) \) about \( W(e) \) in \( \Gamma \). But this ball contains the ball of radius \( c_{x_1}(e_1) \) about \( \overline{W} \), hence \( [c_{x_1}]_W = [c_e]_W \). Similarly \( [c_{x_2}]_W = [c_e]_W \). So \( [c_{x_1}]_W = [c_{x_2}]_W = [c_{e_1}]_W = [c_{e_2}]_W \).

\( Z \) is an intersection of half-spaces, so \( W \cap Z \) is an intersection of half-spaces in the induced cube structure on \( W \); so by Remark 8.2 (applied to \( W \cap Z \subset W \) instead of \( Z \subset X \)) any two vertices in \( W \) that lie in \( Z \) are joined by an edge path in \( W \) that stays in \( Z \). Vertices in \( W \) that lie in \( Z \).
correspond to edges dual to $W$ that cross out of $Z$, and an edge in $W$ that lies in $Z$ corresponds to a square, as above, joining edges $e_1, e_2$ dual to $W$ that cross out of $Z$. Thus the lemma follows from the fact that $[e_{e_1}]_W = [e_{e_2}]_W$.

**Definition 8.5.** ($j$-boundary walls and portals)

If a boundary wall of $Z \in V_j$ has colour $j$ (in the sense of the Zipping Lemma), call it a $j$-boundary wall. For $W$ a $j$-boundary wall, let $P(W) := Z \cap W$ be the portal of $W$ leading to $Z$. If an edge $e$ dual to $W$ crosses out of $Z$, say that $e$ is dual to $P(W)$. These are shown in the picture to the left, with $j$-boundary walls in red and other boundary walls in black. Note that portals need not be bounded. Let $\partial_j Z \subseteq \partial Z$ denote the union of all portals leading to $Z$.

We said at the beginning of this section that we will be gluing together the various $Z/H$ along quotients of certain boundary walls. We can now be a little more precise and say that we will glue together the $Z/H$ along the $H$-quotients of their portals. To facilitate this we now establish some lemmas that control the behaviour of $j$-boundary walls and portals.

**Lemma 8.6.** A vertex in $Z$ cannot be incident at distinct edges dual to $j$-boundary walls. Moreover, any two $j$-boundary walls are disjoint.

**Proof.** Suppose there is a vertex $x \in Z$ incident at distinct edges dual to $j$-boundary walls $W_1$ and $W_2$. By Proposition 2.8, $W_1 \neq W_2$. Furthermore, we know from Lemma 4.7(3) that $W_1$ and $W_2$ map to distinct walls $\overline{W_1}$ and $\overline{W_2}$ in $\mathcal{X}$. Since $W_1$ and $W_2$ are $j$-boundary walls, we have that $c_x(\overline{W_1}) = c_x(\overline{W_2}) = j$. But $d(\overline{W_1}, \overline{W_2}) \leq d(W_1, W_2) \leq 1 < R$, contradicting $c_x$ being a colouring in $C_{k+1}(\Gamma)$. For the second part of the lemma, if we have two intersecting $j$-boundary walls then we can apply Lemma 8.3 to reduce to the first part of the lemma.

**Definition 8.7.** (Acylindricity)

Let $U$ be a locally convex subcomplex of an NPC cube complex $Y$. We say that $U$ is acylindrical in $Y$ if any map of pairs

$$\lambda : ([0,1] \times S^1, \{0,1\} \times S^1) \to (Y, U)$$

which is injective on $\pi_1$ is relatively homotopic (meaning the restriction to $\{0,1\} \times S^1$ is fixed throughout the homotopy) to a map with image in $U$. 

21
Lemma 8.8. $\partial_j Z/H$ is acylindrical in $Z/H$.

Proof.

Let $\lambda : (\{0, 1\} \times \{0, 1\}, \{0, 1\} \times \{0, 1\}) \to (Z/H, \partial_j Z/H)$, with $\lambda(-, 0) = \lambda(-, 1)$, be an essential cylinder (meaning $\lambda(0, -)$ is an essential loop in $Z/H$). Let $\lambda$ be a lift of $\lambda$ to $Z$, so now there will be $1 \neq h \in H$ with $\lambda(-, 1) = h\lambda(-, 0)$. We know that $\lambda(0, -)$ and $\lambda(1, -)$ are in components of $\partial_j Z$, let’s say these correspond to portals in $j$-boundary walls $W_0$ and $W_1$. We have two cases according to whether $W_0$ and $W_1$ are distinct. The rough strategy of the proof is the following. If they are distinct then we’ll show that they must be within distance $R$ of each other, and arrive at a contradiction by proving that some vertex colours both of them $j$. Conversely, if they are the same wall, then we will homotope $\hat{\lambda}$ into the wall by a projection.

First suppose that $W_0$ and $W_1$ are distinct. Now the portals $P_0 := Z \cap W_0$ and $P_1 := Z \cap W_1$ are both $h$-invariant CAT(0) subspaces of $X$, and $h$ has no fixed points in $Z$, so $h$ must restrict to hyperbolic isometries of $P_0$ and $P_1$ with translation axes $a_1, a_2$ respectively (see II.6.10(3])). Any two translation axes of $h$ are asymptotic, hence we can apply [I.2.13] to see that $a_1$ and $a_2$ bound a flat strip, and by $\delta$-hyperbolicity this strip can have width at most $\delta$. Any point $p$ on $P_0$ is contained in a cube $C$ of $X$; and one of the edges of $C$ closest to $p$ will be dual to $P_0$ and have endpoint $x$ in $Z$, with $d(p, x) \leq \frac{1}{4}\sqrt{\dim X}$. The same is true for $P_1$, and so there is a path $\beta$ in $Z$ between vertices $x_0, x_1 \in Z$ of length at most $\delta + \sqrt{\dim X}$ with $x_0, x_1$ being adjacent at edges dual to $P_0, P_1$ respectively. By considering the sequence of cubes that $\beta$ travels through, there is an edge path $\gamma$ in $Z$ from $x_0$ to $x_1$ with $\gamma \subset N_{\sqrt{\dim X}}(\beta)$. Let $\gamma$ have edges $e_1, \ldots, e_n$ and vertices $x_0 = y_0, y_1, \ldots, y_n = x_1$. Since $R \geq \delta + 2\sqrt{\dim X}$ (and thus the mystery of $R$ is revealed!), we have that $d(W(e_i), W_0) \leq d(W(e_i), P_0) \leq R$ for $1 \leq i \leq n$, and so $W(e_1)$ and $W_0$ are adjacent vertices in $\Gamma$.

For $1 \leq i \leq n$, we know from property (1) of $V_j$ that $[c_{y_{i-1}}]_{e_i} = [c_{y_i}]_{e_i}$, so $c_{y_i}(W_0) = c_{y_{i-1}}(W_0)$. We deduce that $c_{x_i}(W_0) = c_{x_0}(W_0)$. And $c_{x_0}(W_0) = j = c_{x_1}(W_1)$ since $W_0$ and $W_1$ are $j$-boundary walls. But $d(W_1, W_0) \leq d(W_1, W_0) \leq \delta \leq R$, and by II.6.10(3) we know that $W_0 \neq W_1$, hence $W_0$ and $W_1$ are adjacent vertices in $\Gamma$ which are given the same colour by $c_{x_1}$, a contradiction.

Conversely suppose that $W_0 = W_1 = W$. Then there is a relative homotopy of $\hat{\lambda}$ into $P := Z \cap W \subset \partial_j Z$ using the closest point projection $Z \to P$, noting that both $Z$ and $P$ are convex. □

Definition 8.9. (Splitting along colourings)

For $W$ a wall in $X$ and $c \in C_{k+1}(\Gamma)$, let $B(W, c) := W \cap \bigcup c^{-1}([1, j])$ be the intersection of $W$ with other walls in $X$ that are coloured $\leq j$ by $c$. Define $W$ split along $c$ by $W - c := W - B(W, c)$ (this will of course depend on $j$, but $j$ is fixed for the rest of the notes so we don’t include it in the notation).

Working in the barycentric subdivision of $W$, $W - c$ will be a cube complex with some missing faces corresponding to where we have removed $B(W, c)$. In general $W - c$ will be disconnected, so for a vertex $\bar{x}$ in $W$, let $(W - c)(\bar{x})$ denote the component of $W - c$ containing $\bar{x}$.

![Diagram showing the definition of splitting along colourings.](image)
Lemma 8.10. (Portal covers)
Let $W$ be a $j$-boundary wall with portal $P = Z \cap W$ and let $e$ be an edge dual to $P$. Let $x_0$ denote the midpoint of $e$ - so $x_0$ is a vertex of $W$. Then the quotient map $m : X \to \hat{X}$ restricts to a universal covering map
\[ m|_P : \hat{P} \to (\hat{W} - c_e)(\bar{x}_0), \]
where $\hat{P}$ is the interior of $P$ with respect to the metric topology of $W$ (equivalently $\hat{P}$ is $P$ minus all boundary walls $W' \neq W$ that intersect $W$). Moreover, $(\hat{W} - c_e)(\bar{x}_0) = (\hat{W} - c)(\bar{x})$ for any other $x \in P$ a vertex of $W$ and any $c \in \pi_1(P) - W$ - in particular, by the Zipping Lemma, $(\hat{W} - c_e)(\bar{x}_0)$ is independent of the choice of $e$ dual to $P$. Furthermore, the group of deck transformations of $m|_P$ is $K_P := \{ g \in K \mid gx_0 \in P \}$ (where $K$ is from Lemma 7.4).

Proof. Consider a wall $W_1$ with $W_1 \cap W \cap Z \neq \emptyset$. By Lemma 8.3, there is a vertex $x \in Z$ with edges $e_1, e_2$ dual to $W_1, W$ respectively. Then
\[ c_{e_1}(\hat{W}_1) = c_{e_2}(\hat{W}_1) = c_{e_3}(\hat{W}_1) = c_e(\hat{W}_1), \]
with the third equality due to the Zipping Lemma. By property (2) of $\mathcal{V}_j$, $W_1$ is a boundary wall if and only if $c_{e_1}(\hat{W}_1) \leq j$. So $W_1$ implies that $W_1$ is a boundary wall if and only if $c_{e}(\hat{W}_1) \leq j$.

The restriction of the quotient map $m : X \to \hat{X} = X/K$ certainly defines a map $m|_P : \hat{P} \to \hat{W} \subset \hat{X}$ with $m(x_0) = \bar{x}_0$. Now a path $\gamma$ in $P$ based at $x_0$ can go anywhere in $W$ except cross over a boundary wall $W_1$, which by the above arguments is equivalent to $\gamma$ not crossing a wall $W_1$ with $c_{e}(\hat{W}_1) \leq j$, which in turn is equivalent to $m \circ \gamma$ not crossing $F(W, c_e)$ (see Definition 8.9). This establishes that $m|_P$ is a covering. Note that $\hat{P}$ is equal to $W$ intersected with open half-spaces corresponding to other boundary walls, so it is convex in $X$ and hence simply connected, making $m|_P$ a universal covering.

If $c \in \pi_1(W)$ then the definition of $[-]_W$ tells us that $c$ agrees with $c_e$ on which walls intersecting $W$ have colour $\leq j$, so $W - c = \hat{W} - c_e$. If we also replace $x_0$ by a different $x \in P$ then clearly $\bar{x} = m_P(x) \in \hat{W} - c$, and so $(\hat{W} - c_e)(x) = (\hat{W} - c)(x) = (\hat{W} - c)(\bar{x}_0)$.

Finally, we know that $m : X \to \hat{X}$ has $K$ as the group of deck transformations, and that $\hat{P}$ is a component of $m^{-1}(\hat{W} - c_e)(\bar{x}_0)$, and that $K$ acts on these components. So $K_P$ is the stabiliser in $K$ of $\hat{P}$ (and of $P$), it preserves the covering $m|_P$ and it acts transitively on $m|_P^{-1}(\bar{x}_0) = P \cap m^{-1}(\bar{x}_0)$ - thus $K_P$ is exactly the group of deck transformations of $m|_P$. \qed

9 Gluing up walls

We are now ready to start constructing $\mathcal{V}_{j-1}$ from $\mathcal{V}_j$. Our strategy will be to glue together different complexes $Z/H$ along the $H$-quotients of portals with ‘compatible colourings’. To glue together two quotient portals, we need them to be isomorphic as complexes and for this isomorphism to be ‘colour compatible’; initially it may not be possible to glue up all the portals, but the idea is to make it possible by passing to finite covers of the $Z/H$. The arguments in this section are based on [1, Theorem 3.1], for example we quote the same theorems of Haglund-Wise (Theorem 3.10) and Bestvina-Feighn (Theorem 3.8) - but our arguments contain considerably more detail and will appear quite different as they are recast to work for our set-up.
Definition 9.1. (Compatible portals)
We say that two portals $P$ and $P'$ leading to $(Z, H, (c_e)), (Z', H', (c'_e)) \in V_j$ respectively are compatible if there are edges $e$ and $f$ dual to $P$ and $P'$ respectively such that $[e, c_e] \in G \cdot [f, c'_f]$.

Let $P$ and $P'$ be compatible portals as above, and say they lie in walls $W$ and $W'$. Take $g \in G$ and edges $e$ and $f$ dual to $P$ and $P'$ such that $[e, c_e] = g[f, c'_f]$, and let $x_0$ and $y_0$ be the midpoints of $e$ and $f$. So $e = gf$, $W = gW'$ and $x_0 = yg_0$. As $K$ is normal in $G$, we have the following commuting diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow^m & & \downarrow^m \\
X & \xrightarrow{g} & X
\end{array}
\]

Then $g$ acts on the wall $\overline{W'}$ to produce
\[
g(\overline{W'} - c'_f)(\overline{y_0}) = (\overline{W} - gc'_f)(\overline{x_0}) = (\overline{W} - c_e)(\overline{x_0})
\]
by Lemma 8.10 and the fact that $[c_e]_W = [gc'_f]_W$.

As the maps are coverings for $P$ and $P'$, we deduce that $g$ restricts to a cube isomorphism $\bar{P'} \rightarrow \bar{P}$ and also $\bar{P}' \rightarrow \bar{P}$. In fact $P' \rightarrow P$ is equivariant with respect to the group isomorphism $K_{P'} \rightarrow K_P; k \mapsto gkg^{-1}$. This can all be put into the following commuting diagram.

\[
\begin{array}{ccc}
K_{P'} & \xrightarrow{g(-)g^{-1}} & K_P \\
\downarrow & & \downarrow \\
\bar{P}' & \xrightarrow{g} & \bar{P} \\
\downarrow^m & & \downarrow^m \\
(\overline{W'} - c'_f)(\overline{y_0}) & \xrightarrow{g} & (\overline{W} - c_e)(\overline{x_0})
\end{array}
\]

We also have the following lemma and corollary, which are basically consequences of the Zipping Lemma. These will allow us to group together compatible portals into compatibility classes.

Lemma 9.2. (Teleports)
Portals $P$ and $P'$ leading to $(Z, H, (c_e)), (Z', H', (c'_e)) \in V_j$ are compatible if and only if there exists $g \in G$ such that
\[
\{ [e, c_e] | e \text{ is dual to } P \} = g \{ [f, c'_f] | f \text{ is dual to } P' \}. \tag{9.3}
\]
In this case we say that $P$ is a $g$-teleport of $P'$ (note that $P$ could be a $g$-teleport of $P'$ for several different $g$, and that $P'$ could have several different $g$-teleports corresponding to portals that lead to different $Z \in V_j$).

Proof. Suppose $P$ and $P'$ are compatible. Then there exist edges $e$ and $f$ dual to $P$ and $P'$, and $g \in G$, such that $[e, c_e] = g \cdot [f, c'_f]$. If $f_1$ is another edge dual to $P'$, then $e_1 := gf_1$ is dual to $P$ because, as we showed above, $g : P' \rightarrow P$ is an isomorphism. Suppose that $P$ and $P'$ lie in walls $W$ and $W'$. We then have
\[
g[f_1, c'_f] = \{gf_1\} \times \{gc'_f\}_W
\]
by the Zipping Lemma
\[
= \{e_1\} \times \{gc'_f\}_W
\]
since $[e, c_e] = g \cdot [f, c'_f]$;
\[
= \{e_1\} \times \{c_e\}_W
\]
by the Zipping Lemma
\[
= \{e_1\} \times \{c_{e_1}\}_W
\]
This gives the $\supset$ inclusion in (9.3), and the $\subset$ inclusion follows similarly by considering $g^{-1} : P \rightarrow P'$.

Conversely, it is immediate that (9.3) implies compatibility of $P$ and $P'$.
Corollary 9.3. Compatibility of portals is an equivalence relation. We will refer to the equivalence classes as compatibility classes.

Proof. If $P$ is a $g$-teleport of $P'$, then $P'$ is a $g^{-1}$-teleport of $P$. If in addition $P'$ is a $h$-teleport of $P''$, then $P$ will be a $gh$-teleport of $P''$ (teleports form a groupoid on the set of portals, and the components of this groupoid are the compatibility classes).

The isomorphism $g : P' \to P$ for compatible portals $P, P'$ as in Lemma 9.2 is good news for gluing together $Z$ and $Z'$ along $P$ and $P'$, however we actually want to glue $Z/H$ and $Z'/H'$ along their respective quotients of $P$ and $P'$. The problem is that these quotients of $P$ and $P'$ may not be isomorphic; we will overcome this by taking finite covers of the complexes $Z/H$, or equivalently by replacing the groups $H$ with finite index subgroups.

What exactly do we mean by the quotient of $P$? We want this to be the image of $P$ under the quotient map $Z \to Z/H$, but it will be more convenient to write this directly as a quotient by a group acting on $P$. The appropriate group will be $H_P := \{g \in H | gP = P\}$, the stabiliser of $P$ in $H$, but we must take a moment to see why $P/H_P \to Z/H$ is an embedding (and hence why $H_P$ acts cocompactly on $P$). Indeed if $x \in P$ and $g \in H$ with $gx \in P$, then $gW$ is a wall with $gx \in Z \cap W \cap gW$. Moreover we can assume $x$ is a vertex of $W$, the midpoint of an edge $e \in X$, and then we get $c_{ge}(W) = j = c_{e}(W) = c_{ge}(gW)$. But $c_{pe}$ is a colouring, so $W = gW$ and $W = gW$, thus $gP = g(Z \cap W) = gZ \cap gW = Z \cap W = P$, implying $g \in H_P$.

As a first step towards forming an isomorphism $g : P'/H_{P'} \to P/H_P$, we will modify $V_j$ to ensure that $H_{P'}, H_P < K$ (Lemma 9.5), from which we see that $P'/H_{P'}$ and $P/H_P$ are both finite covers of $P/K_P$ via the following diagram, where horizontal maps are cube isomorphisms induced by $g$ and all others are covering maps induced by quotients.

First we need a short lemma about subgroup separability that we will use several times in this section.

Lemma 9.4. Let $H_1 < H_2 < H_3$ be groups with $H_1$ separable in $H_3$ and $H_1$ finite index in $H_2$. Then there exists $N < H_3$ a finite index normal subgroup such that $N \cap H_2 < H_1$.

Proof. Let $H_1, g_1H_1, g_2H_1, \ldots, g_lH_1$ be the cosets of $H_1$ in $H_2$. By separability of $H_1$ in $H_3$, there exist $N_i < H_3$ finite index normal subgroups with $g_i \notin H_1N_i$ for $i = 1, \ldots, l$. Since $g_iH_1 \cap H_1N_i = \emptyset$, we deduce that $N = \cap_i N_i$ satisfies $N \cap H_2 < H_1$.

Lemma 9.5. By modifying $V_j$, we can ensure that $H_P < K$ for every $Z \in V_j$ and every portal $P$.

Proof. Let $G_P$ be the stabiliser in $G$ of a portal $P$ leading to $Z \in V_j$. It is clear from Lemma 8.10 that $K_P$ acts cocompactly on $P$, so by the properness of the $G$-action $K_P < G_P$ has finite index. Then $K \cap H_P = K_P \cap H_P$ is a finite index subgroup of $H_P$.

$P$ is convex in $Z$, so $H_P$ and $K \cap H_P$ are quasi-convex in $H$. Now $H \in QVH$, so by Theorems 3.8 and 3.10 $K \cap H_P$ is separable in $H$; thus we may use Lemma 9.4 to take $H_0 < H$ finite index and normal such that $H_0 \cap H_P < K$. Since $H$ acts cocompactly on $Z$ there are only finitely many $H$-orbits of portals, hence by passing to a further finite index subgroup we can assume that $H_0 \cap H_P < K$ for all
from a finite set of $H$-orbit representatives of portals. In fact this would mean that $H_0 \cap H_P < K$ for all portals $P$ for the following reason: if $h \in H$ and $P$ is a portal with $H_0 \cap H_P < K$ then

$$H_{hP} = hH_P h^{-1}$$

and

$$H_0 \cap H_{hP} = h(H_0 \cap H_P)h^{-1} < K$$

because $H_0 \triangleleft H$ and $K \triangleleft G$.

Replacing $(Z, H, (c_x))$ by $(Z, H_0, (c_x))$ would preserve all the properties of $V_j$ except the Gluing Equations. $(Z, H_0, (c_x))$ would contribute $|H : H_0|$ times more to each set $V_j^\pm(\bar{e}, c)$ than $(Z, H, (c_x))$. But making $|H : H_0|$ copies of $(Z, H, (c_x))$ in $V_j$ would have the same effect. Therefore, for each triple $(Z, H, (c_x)) \in V_j$, we can take $H_0$ as above and replace $(Z, H, (c_x))$ by some number of copies of $(Z, H_0, (c_x))$, and we can do this in such a way that the Gluing Equations are preserved and so that $V_j$ now satisfies the conclusion of the lemma.

Since we now have that compatible portals are finite covers of a common complex, it follows that compatible portals can be made isomorphic to each other by passing to a common finite cover; the hope is to do this by taking finite covers of different $Z/H$ for $(Z, H, (c_x)) \in V_j$; the difficulty is doing this simultaneously for all $Z/H$ - to facilitate this we will add in ‘scaffolding’ in the form of a graph of spaces.

**Definition 9.6.** (A graph of spaces: $\mathcal{Y}$)

Define the (finite) graph of spaces $\mathcal{Y}$ as follows. The vertex spaces come in two types, (a) and (b).

- Type (a) vertex spaces are $Z/H$ for $(Z, H, (c_x)) \in V_j$.
- Type (b) vertex spaces are $P_i/KP_i$, where $\{P_i\}$ is some choice of representatives for the compatibility classes of portals.
- Edge spaces will be portal quotients $P/H_P$ for $P$ a portal leading to $(Z, H, (c_x)) \in V_j$, but for each triple in $V_j$ we will choose just one $P$ from each $H$-orbit of portals (so the edge spaces correspond to components of $\partial_j Z/H$).

Maps from edge spaces into type (a) vertex spaces will be inclusions

$$P/H_P \hookrightarrow Z/H$$

and into type (b) vertex spaces will be covers

$$P/H_P \rightarrow P/KP \overset{\partial_t}{\rightarrow} P_i/KP_i$$

where $g \in G$ is chosen so that $P_i$ is a $g$-teleport of $P$ (for the case $g : P_i \rightarrow P_i$ we will take $g = 1$). For this choice of $g$, we say that $P_i$ is a $g$-teleport of $P$ within $\mathcal{Y}$. On the level of $\pi_1$, these maps are

$$H \leftarrow H_P \hookrightarrow KP \cong KP_i,$$

which are both injective. Thus we have a graph of spaces with $\pi_1$-injective edge maps.

Suppose that, within $\mathcal{Y}$, $P_i$ is a $g$-teleport of $P$ and a $g'$-teleport of $P'$. Then similarly to (9.2) we get the following equivariant maps.

$$H_{P'} \leftrightarrow K_{P'} \overset{g'(-)(g')^{-1}}{\sim} K_{P_i} \leftarrow \overset{g(-)g^{-1}}{\sim} K_P \leftrightarrow H_P$$

$$\bigcap_{P'} \bigcap_{P'} \bigcap_{P_i} \bigcap_{g} \bigcap_{P} = P$$

(9.8)
Moreover, the induced maps \( P'/H' \to P_i/K_P \leftarrow P/H \) are exactly the maps from edge spaces to type (b) vertex spaces given in (9.6). The top row of (9.8) gives inclusions from edge groups to the type (b) vertex group \( K_{P_i} \) in the graph of groups corresponding to \( \mathcal{Y} \).

Putting \( t = g^{-1}g' \), we will refer to \( P \) as a \( t \)-teleport of \( P' \) within \( \mathcal{Y} \) (note that \( P \) is certainly a \( t \)-teleport of \( P' \) by the composition property of teleports). (9.8) then contracts to the following diagram, in which \( K_{P'} \) and \( K_P \) both represent the same type (b) vertex group, and are identified by the isomorphism shown.

\[
K_{P'} \xrightarrow{(\sim)^{-1}} K_P
\]

Let \( P_i \) lie in the wall \( W_i \). If \( P_i \) is a \( g \)-teleport (not necessarily within \( \mathcal{Y} \)) of a portal \( P \) that leads to \( Z \in \mathcal{V}_j \), then \( gZ \) must lie on one of the two sides of \( W_i \), and which side it lies on is independent of the choice of \( g \) by Remark 4.1. Thus we get two cases: say that \( P \) is a \( P_i^+ \)-portal if \( gZ \cap W_i^+ \neq \emptyset \) and a \( P_i^- \)-portal if \( gZ \cap W_i^- \neq \emptyset \). If \( P \) is a \( P_i^+ \)-portal leading to \( Z \) and \( P' \) is a \( P_i^- \)-portal leading to \( Z' \), and \( P \) is a \( t \)-teleport of \( P' \), then we get that \( Z\) and \( tZ' \) lie on opposite sides of \( P \). In this case we get the following picture of what’s going on upstairs in \( X \) and downstairs in the graph of spaces \( \mathcal{Y} \) (the blue parts commute if \( P \) is a \( t \)-teleport of \( P' \) within \( \mathcal{Y} \)).

\[
\begin{array}{ccc}
X & \xrightarrow{t} & \mathcal{Y} \\
\downarrow & & \downarrow \\
Z/P & \xrightarrow{t} & Z'/P' \\
\end{array}
\]

This picture is the prototype for how we will glue the different spaces \( Z \) together to create the spaces of \( \mathcal{V}_{j-1} \). We effectively ‘zip’ together \( Z \) and \( tZ' \) along \( P \) in a colour-compatible way. The Zipping Lemma (via Lemma 9.2) is what makes this work, justifying its name. More precisely, the fact that \( P \) is a \( t \)-teleport of \( P' \) means that the colourings \( (c_x) \) can be combined with \( t \)-translates of the colourings.
(c′) to give a family of colourings for $Z \cup tZ'$ that satisfy property (1) of $V_j$ (Lemma 9.2 implies that the colourings match up along the edges dual to $P$).

Now let $Y$ be a connected component of $Y$. Before taking a cover of $Y$ we need two lemmas about its fundamental group. This will rely on the following theorem of Bestvina-Feighn that gives a sufficient condition for a graph of spaces to have hyperbolic fundamental group. The original wording of the theorem in [4] refers to ‘negatively curved spaces’ rather than spaces with hyperbolic fundamental group, but these notions are equivalent - at least for finite cell complexes. More specifically, when [4] refers to a finite complex $Z$ being negatively curved it means that the universal cover $\tilde{Z}$ satisfies a linear isoperimetric inequality, this is equivalent to $\pi_1(Z)$ having linear Dehn function which is in turn equivalent to $\pi_1(Z)$ being hyperbolic by [3, III.Γ.2.7].

**Definition 9.7.** (Essential annulus)
In a graph of spaces $U$, an annulus is defined as a $\pi_1$-injective map $\lambda : [0, 1] \times S^1 \to U$ (viewing $U$ as a total space, with vertex spaces connected by pieces homeomorphic to $[0, 1] \times$ edge space), such that each cross section $\lambda(\{t\} \times S^1)$ is contained in a single edge or vertex space and $\lambda(\{0, 1\} \times S^1)$ is contained in two edge spaces (possibly the same). We say that $\lambda$ is an essential annulus if it cannot be homotoped to an annulus that visits fewer vertex spaces. The length of an essential annulus is the number of vertex spaces it visits.

**Theorem 9.8.** (Bestvina-Feighn Combination Theorem)[4, first corollary in §7]
A connected graph of spaces $U$ has hyperbolic fundamental group if the following conditions are satisfied:

- The vertex spaces are finite cell complexes with hyperbolic fundamental groups.
- Each map from an edge space to an adjacent vertex space is an embedding and lifts to a quasi-isometric embedding between their universal covers.
- Essential annuli have bounded length.

**Lemma 9.9.** $\pi_1(Y)$ is hyperbolic.

**Proof.** We just need to check that the conditions of Theorem 9.8 are satisfied.

- The vertex spaces take the form $Z/H$ and $P_i/K_{P_i}$, which are both quotients of a hyperbolic cube complex by a free cocompact group action.
- The maps of edge spaces given in (9.5) and (9.6) induce the following inclusions of universal covers
  $$Z \leftarrow P \rightarrow P',$$
  and these are isometric embeddings of hyperbolic spaces.
- In our case essential annuli cannot pass through a type (a) vertex space $Z/H$, because the union of the embedded edge spaces is $\partial_j Z/H$, which is an acylindrical subspace by Lemma 8.8. Hence essential annuli can have length at most one.

**Lemma 9.10.** $\pi_1(Y) \in QVH$.

**Proof.** We have already shown that $\pi_1(Y)$ is hyperbolic, and by construction it is the fundamental group of a graph of groups with vertex groups $H$ and $K_{P_i}$, so we just require these vertex groups to be in $QVH$ and the edge groups to be quasi-convex in $\pi_1(Y)$.

- Property (4) of $V_j$ tells us that $H \in QVH$.  

28
• $K_P$ acts freely cocompactly on the CAT(0) cube complex $\hat{P} \subset \hat{X}$ (we barycentrically subdivide to make $P$ a cube complex), so by the lower dimensional case of Theorem 1.1 we know that $P_1/K_P$ is virtually special. By Theorem 3.8 and Definition 3.7 we deduce that $K_P \in \mathcal{QVH}$.

• The edge groups $H_P$ are quasi-convex in $\pi_1(Y)$, because $Y$ can be built as an NPC cube complex itself, with edge spaces attached by gluing in product cube complexes $\hat{P}/H_P \times [0,1]$, and then $H_P$ will be the fundamental group of a hyperplane of $Y$ (which lifts to a convex subcomplex of the universal cover $\hat{Y}$).

\[ \]

The idea now is to take a finite cover of the graph of spaces $Y$ so that edge spaces of compatible portals become isomorphic.

For each type (a) vertex space choose a preferred embedding $H \to \pi_1(Y)$ (different homotopy classes of paths joining the basepoint of $Y$ with the basepoint of the vertex space $Z/H$ will give different embeddings, but it will be notationally easier for us to fix one). These preferred embeddings of type (a) vertex groups induce preferred embeddings of edge groups $H_P \leftarrow \pi_1(Y)$. By theorems 3.5 and 3.10 the edge groups are separable in $\pi_1(Y)$. For each edge group $H_P$, the adjacent type (b) vertex group $K_P$ is identified with $K_P$ as in (9.7), so we get a preferred embedding $K_P \leftarrow \pi_1(Y)$ such that $H_P \prec K_P \prec \pi_1(Y)$ and $H_P$ has finite index in $K_P$. We may then use Lemma 9.4 to obtain $N \triangleleft \pi_1(Y)$ of finite index, such that the intersection with each type (b) vertex group $K_P$ satisfies $N \cap K_P \prec H_P$.

Thus we have

\[ \hat{K}_P := N \cap K_P = N \cap H_P \]  

(9.10)

for all portals $P$. Also introduce the notation $\hat{H} := N \cap H$. Note that $\hat{K}_P$ will be the stabiliser in $\hat{H}$ of $P$. Let $\hat{Y} \to Y$ be a finite cover corresponding to $N$.

In $\hat{Y}$ the edge and vertex spaces will be components of preimages (which, following Wise \cite{11}, we will refer to as elevations) of edge and vertex spaces of $Y$; and restricting $\hat{Y} \to Y$ to an edge or vertex space of $\hat{Y}$ will give a cover of an edge or vertex space of $Y$. Since the cover is normal, all the elevations of a given edge or vertex space of $Y$ will be isomorphic. Say that elevations of type (a)/(b) vertex spaces in $Y$ are respectively type (a)/(b) vertex spaces in $\hat{Y}$.

If $P/H_P \to Z/H$ is an inclusion of an edge space in $Y$, then $P/\hat{K}_P \to Z/\hat{H}$ will be an inclusion of an edge space in $\hat{Y}$ (we might say that the second inclusion is an elevation of the first inclusion). However, there will be extra inclusions of edge spaces in $\hat{Y}$ corresponding to other elevations, which will be of the form

\[ hP/\hat{K}_P \hookrightarrow Z/\hat{H}, \]  

(9.11)

where $h \in H$ and $\hat{K}_P := h\hat{K}_P h^{-1} = N \cap hK_P h^{-1}$ is the stabiliser in $\hat{H}$ of the portal $hP$ (in fact we will have one such inclusion for each left coset of $\hat{H}H_P$ in $\hat{H}$, but this is not important). These additional edge spaces are related in that $hP$ is a $h$-teleport of $P$ for any $h \in H$ (since $hc_x = c_{hx}$ for any vertex $x \in Z$). Furthermore, we have that $P$ is a $P_1^+$-portal if and only if $hP$ is.

If $P$ is a $t$-teleport of $P'$ within $Y$, then (9.9) tells us that $t(-)t^{-1} : K_{P'} \to K_P$ is an identification between two ways of representing the same type (b) vertex group in $Y$, so it follows that $t(-)t^{-1} : \hat{K}_{P'} \to \hat{K}_P$ is an identification between two ways of representing the same type (b) vertex group in $\hat{Y}$.

By composing this $t$ with $h$ from above, we can conclude the following:

• Any edge space in $\hat{Y}$ can be written as $P/\hat{K}_P$ for some portal $P$ (possibly a $H$-translate of a portal coming from an edge space in $Y$).

• If $P/\hat{K}_P$ and $P'/\hat{K}_{P'}$ are edge spaces in $\hat{Y}$ with $P$ and $P'$ compatible portals, then there exists $g \in G$ such that $P$ is a $g$-teleport of $P'$ and

\[ \hat{K}_P = g\hat{K}_{P'} g^{-1}. \]  

(9.12)

Therefore $g : P'/\hat{K}_{P'} \to P/\hat{K}_P$ is an isomorphism between the edge spaces.
We will now modify $\tilde{Y}$ so that edge spaces get glued in pairs - removing the ‘scaffolding’ if you like. But first we must prove that a suitable pairing exists.

**Definition 9.11.** (Size of edge spaces)
Define the size of an edge space in $Y$ as

$$\text{size}(P/H_P) := |\{H_P \cdot e \mid e \text{ is dual to } P\}|,$$

and similarly for an edge space in $\tilde{Y}$ by

$$\text{size}(P/\tilde{K}_P) := |\{\tilde{K}_P \cdot e \mid e \text{ is dual to } P\}|.$$

**Definition 9.12.** ($P_i^+$ and $P_i^-$ -edge spaces)
Say that an edge space $P/H_P$ in $Y$ or $P/\tilde{K}_P$ in $\tilde{Y}$ is a $P_i^+$-edge space (resp. $P_i^-$-edge space) if $P$ is a $P_i^+$-portal (resp. $P_i^-$-portal).

It follows from the above discussion about elevations of edge spaces that any elevation of a $P_i^+$ ($P_i^-$)-edge space in $Y$ is a $P_i^+$ ($P_i^-$)-edge space in $\tilde{Y}$.

**Lemma 9.13.** For each type (b) vertex space $P_i/K_{P_i}$ in $Y$, $\tilde{Y}$ contains the same number of $P_i^+$-edge spaces as $P_i^-$-edge spaces.

**Proof.** Fix a type (b) vertex space $P_i/K_{P_i}$, with $P_i$ contained in a wall $W$, and let $[c]_W$ be the colouring equivalence class associated to $P_i$ (well-defined by the Zipping Lemma). The idea is to show that, in $Y$, the total size of $P_i^+$-edge spaces equals the total size of $P_i^-$-edge spaces. Each elevation of an edge space $P/H_P$ is isomorphic to $P/\tilde{K}_P$, so its size is $|H_P : \tilde{K}_P|$ times bigger than that of $P/H_P$; $\tilde{Y}$ is a degree $|\pi_1(Y) : N|$ covering of $Y$, so there are $|\pi_1(Y) : N|/|H_P : \tilde{K}_P|$ elevations of $P/H_P$. We deduce that the total size of $P_i^+$-edge spaces in $Y$ is $|\pi_1(Y) : N|$ times bigger than the total size in $Y$, and the same goes for $P_i^-$-edges. But all $P_i^+$ and $P_i^-$-edge spaces in $\tilde{Y}$ are isomorphic by (9.12), so they all have the same size. The lemma then follows by a simple counting argument.

It remains to show that, in $\tilde{Y}$, the total size of $P_i^+$-edge spaces equals the total size of $P_i^-$-edge spaces. This is quite technical as it involves linking together many of our definitions. The key is to show that the following map is a bijection:

$$\Phi^+ : \left\{ (H_P \cdot e, Z) \middle| P/H_P \text{ a } P_i^+\text{-edge space with } e \text{ dual to } P \text{ and } (Z, H, (c_x)) \in V_j \right\} \rightarrow \bigcup_{e \text{ dual to } P_i} V_j^+(e, c),$$

where $x$ is the endpoint of $e$ in $Z$.

- Firstly let’s see why $\Phi^+$ is well-defined. Given $(H_P \cdot e, Z)$ in the domain of $\Phi^+$, if $P_i$ is a $g$-teleport of $P$, then $ge$ is dual to $P_i$ and $[ge, gc_x] = [ge, c]$. Since $P/H_P$ is a $P_i^+$-edge space, we also have $gx \in W^+$. And $[e, c_x] = [e, c]$, so $[ge, gc_x] = [ge, c]$, from which we see that $(H \cdot x, Z) \in V_j^+(ge, c)$.
- $\Phi^+$ is injective since we choose $H$-orbit representatives of portals to form our edge spaces, and because we cannot have distinct edges $e, e'$ both dual to portals that are incident at the same vertex $x \in Z$ (Lemma 8.6).
- $\Phi^+$ is surjective because if $(H \cdot x, Z) \in V_j^+(e', c)$ for some $e'$ dual to $P_i$, then by definition there is $g \in G$ and an edge $e$ incident at $x$ with $ge = e'$, $gx \in W^+$ and $[e', gc_x] = [e', c_x] = [e', c]$. We have $c_x(e) = c(e') = j$, so $e$ is dual to some portal $P$, and by possibly $H$-translating $e$ and $x$ we can assume that $P/H_P$ is one of our chosen edge spaces; therefore $(H_P \cdot e, Z) \in (\Phi^+)^{-1}(H \cdot x, Z)$.

Note that $V_j^+(e, c)$ only depends on the $G$-orbit of the equivalence class $[e, c]$. In fact edges $e_1, e_2$ dual to $P_i$ with $[e_1, c] \notin G \cdot [e_2, c]$ must have $V_j^+(e_1, c) \cap V_j^+(e_2, c) = \emptyset$, as we’ll now see. Indeed if
\((H \cdot x, Z) \in \mathcal{V}_j^+(e_1, c) \cap \mathcal{V}_j^+(e_2, c),\) then there exist \(g_1, g_2 \in G\) with \(g_1x, g_2x \in W^+\) and \(g_1x, g_2x\) incident at \(e_1, e_2\) respectively, and

\[
[e_1, g_1c_x] = [e_1, c] \\
[e_2, g_2c_x] = [e_2, c].
\]

We then get \(c_x(g_2^{-1}e_1) = c_x(g_2^{-1}e_2) = j,\) so Lemma 8.8 implies \(g_1^{-1}e_1 = g_2^{-1}e_2,\) so \(g_2g_1^{-1}[e_1, c] = [e_2, c],\) contrary to our assumption.

The upshot is that the image of \(\Phi^+\) can be written as a disjoint union if we index over a finite set of edges dual to \(P_i\) that give distinct \(G\)-orbits \(G \cdot [e, c].\) Of course we can define \(\Phi^-\) in a similar manner to \(\Phi^+,\) and then the Gluing Equations tell us that \(\Phi^+\) and \(\Phi^-\) have images of equal size. Hence the domains of \(\Phi^+\) and \(\Phi^-\) also have equal size, which exactly means that, in \(Y,\) the total size of \(P_i^+\)-edge spaces equals the total size of \(P_i^-\)-edge spaces.

**Definition 9.14.** (Another graph of spaces: \(T\))

For each \(P_i/K_P,\) a type (b) vertex space in \(Y,\) take some arbitrary matching between \(P_i^+\)-edge spaces and \(P_i^-\)-edge spaces in \(\hat{Y}.\) Now transform \(\hat{Y}\) into a new graph of spaces, \(T,\) as follows:

- The vertex spaces of \(T\) will be the type (a) vertex spaces of \(\hat{Y}\) - so will be of the form \(Z/\hat{H}.\)
- For two edge spaces \(P/\hat{K}_P\) and \(P'/\hat{K}_{P'}\) in \(\hat{Y}\) that form a \((P_i^+, P_i^-)\)-edge space pair in the matching, we identify them using an isometry \(g : P' \to P\) as in (9.12) to create a single edge space in \(T.\) The maps into vertex spaces will be the natural inclusions \(P/\hat{K}_P \to Z/\hat{H}\) and \(P'/\hat{K}_{P'} \to Z'/\hat{H}'\) (where \(P\) leads to \(Z\) and \(P'\) leads to \(Z').\)
- Edge maps into a given vertex space are embeddings with disjoint images, so we can build \(T\) as a total space by gluing together vertex spaces along images of edge maps instead of inserting (edge space) \(\times [0,1]\) between appropriate vertex spaces - the two constructions are homotopy equivalent.

The next step is to embed the universal cover of each connected component of \(T\) into \(X\) - these will form the triples in \(V_{j-1}.\)

Consider a connected component \(T\) of \(T.\) Take two vertex spaces \(Z/\hat{H}\) and \(Z'/\hat{H}'\) in \(T\) which are connected by an (oriented) edge \(f\) corresponding to the edge space \(P/\hat{K}_P \cong P'/\hat{K}_{P'}\) as in (9.12) with \(g_f = g) \in G,\) we get \(P\) is a \(g_f\)-teleport of \(P'.\) Furthermore, by definition of \(P_i^\pm\)-edge spaces we know that \(Z\) and \(g_fZ'\) lie on opposite sides of \(W = g_fW'\) (where \(P, P'\) lie in walls \(W, W'\)), and that

\[
Z \cap g_fZ' = P = g_fP'.
\]

Thus \(Z \cup g_fZ'\) is an embedding of a small section of the universal cover of \(T\) into \(X.\)

We want to extend this to an embedding \(\hat{T} \subset X\) of the entire universal cover of \(T\) in \(X.\) First we need to define some basic paths in \(T.\)

**Definition 9.15.** (\(\gamma_h, \alpha_f, \beta_f\) paths)

Equip each \((Z, \hat{H})\) with a basepoint \(x,\) and for \(h \in \hat{H}\) let \(\gamma_h\) be a loop in \(Z/\hat{H}\) which lifts to a path from \(x\) to \(hx.\) For \(Z, Z'\) as in (9.13), with basepoints \(x, x',\) let \(\alpha_f\) be a path in \(Z \cup g_fZ'\) from \(x\) to \(g_fx'\) (there is only one up to homotopy because \(Z, g_fZ'\) and \(Z \cup g_fZ'\) are all simply connected). Say that \(\alpha_f\) descends to a path \(\beta_f\) in \(T\) via \(Z \to Z/\hat{H}\) and \(g_fZ' \to Z'/\hat{H}'.\) Letting \(-f\) denote \(f\) with reversed orientation, we can assume that \(g_{-f} = g_f^{-1}\) and that \(\beta_{-f}\) is the reverse path of \(\beta_f.\)

**Definition 9.16.** (Embedding \(\tilde{T} \subset X\))

If \(f_1, ..., f_n\) are edges in \(T\) that form a path through vertex spaces \(Z_0/\hat{H}^{(0)}, Z_1/\hat{H}^{(1)}, ..., Z_n/\hat{H}^{(n)},\) with \(Z_0/\hat{H}^{(0)}\) a fixed base vertex space, and \(h_i \in \hat{H}^{(i)}\) for \(i = 0, 1, ..., n,\) then the concatenation

\[
\gamma = \gamma_0 \cdot \beta_1 \cdot \gamma_1 \cdots \beta_n \cdot \gamma_n
\]

(9.14)
Lemma 9.17. \( \mu : \tilde{T} \rightarrow T \) as constructed above is a well-defined universal covering.

Proof. For \( Z, Z' \) as in (9.13), and \( \gamma \) a path as above from \( Z_0/\hat{\hat{H}}(0) \) to \( Z/\hat{\hat{H}} \), the translate \( g(\gamma)Z \) will be glued to \( g(\gamma)hg_fh'Z' \) along the portal translate \( g(\gamma)hP \) for every \( h \in \hat{H}, h' \in \hat{H}' \). This is because

\[
g(\gamma)Z \cap g(\gamma)hg_fh'Z' = g(\gamma)h(h^{-1}Z \cap g_fh'Z') = g(\gamma)h(Z \cap g_fZ') = g(\gamma)hP
data by (9.13).
\]

Varying \( h' \) doesn’t change what’s being glued together since \( h'Z' = Z' \), and the restriction of \( \mu \) to \( g(\gamma)Z \) and \( g(\gamma)hg_fZ' \) agree on \( g(\gamma)hP \), because the gluing defining \( T \) ensures that the bottom rectangle of the following diagram commutes (it is clear why the rest commutes).

Next note that elements \( h \) in distinct left cosets of \( \hat{K}_P < \hat{H} \) will give distinct portals \( g(\gamma)hP \) - in fact these portals will lie in disjoint \( j \)-boundary walls by Lemma 8.6, so the different \( g(\gamma)hg_fZ' \) will be in disjoint halfspaces of \( X \). Conversely if \( h \in \hat{K}_P \) then \( hg_f = g_fh' \) for some \( h' \in \hat{K}_P \), so there is really just one \( G \)-translate of \( Z' \) being glued for each coset of \( \hat{K}_P \). Extending \( \gamma \) using other edges in \( T \) that leave \( Z/\hat{H} \) will result in gluing yet more \( G \)-translates of various \( Z \in V_j \), along portals in disjoint \( j \)-boundary walls by Lemma 8.6. Thus \( \tilde{T} \) will be made up of \( G \)-translates of different \( Z \) glued in a tree structure as shown in the picture (\( j \)-boundary wall translates in red), hence \( \mu \) is well-defined and \( \tilde{T} \) is simply connected. The above discussion also shows that every portal translate in \( g(\gamma)Z \) will have another \( Z \in V_j \) glued along it, so \( \mu \) is a covering of \( T \).

Finally we promote \( \tilde{T} \) to a triple in \( V_{j-1} \) by adding the data of a group action and colourings. We define these now, and afterwards we will verify the numbered properties of \( V_{j-1} \) given in Section 7.

Definition 9.18. (Group action and colourings for \( \tilde{T} \))

For a loop \( \gamma \) of the form (9.14), and \( g(\beta)Z \) in \( \tilde{T} \), we have that \( g(\gamma)g(\beta) = g(\gamma \cdot \beta) \), hence \( g(\gamma)g(\beta)Z = g(\gamma \cdot \beta)Z \subset \tilde{T} \). This holds for all translates \( g(\beta)Z \) in \( \tilde{T} \), thus \( g(\gamma)\tilde{T} \subset \tilde{T} \) and \( \mu \circ g(\gamma) = \mu \). So

\[
H(T) := \{ g(\gamma) \mid \gamma \text{ is a loop of form } (9.14) \}< G
\]
is a subgroup of the group of deck transformations of $\mu : \tilde{T} \to T$. What's more, if $x_0$ is a basepoint of $Z_0$, then by construction of $\mu$ and $T$ we have $H(T) \cdot x_0 = \mu^{-1}(x_0)$, hence $H(T)$ is the full group of deck transformations, and acts freely cocompactly on $\tilde{T}$.

To complete the triple we need to provide colourings $(c^T_x)$ for every vertex in $\tilde{T}$: if $x \in Z$ is a vertex and $(Z, H, (c_x)) \in V_j$ then endow $g(\gamma)x$ with the colouring $c^T_{g(\gamma)x} := g(\gamma)c_x$.

**Definition 9.19.** (Construction of $V_{j-1}$)

Of course everything we have done works for every connected component $T$ of $\mathcal{T}$ and every connected component $Y$ of $\mathcal{Y}$. By possibly duplicating some $\mathcal{Y}$'s, we can assume that the collection of $\mathcal{Y}$'s forms a cover $\tilde{Y} \to Y$ of degree $d$ say (we will need this below for the Gluing Equations). For each connected component of $\tilde{Y}$ we form the graph of spaces $\mathcal{T}$, and the universal cover of each connected component of $\mathcal{T}$ forms a triple $(\tilde{T}, H(T), (c^T_x)) \in V_{j-1}$.

We have already seen that $H(T)$ acts freely cocompactly on $\tilde{T}$, and it is clear from the definition that the colourings $(c^T_x)$ are invariant under the action of $H(T)$. There is just one other thing we need to check before going on to the numbered properties of $V_{j-1}$, and this is the following lemma.

**Lemma 9.20.** $\tilde{T}$ is an intersection of half-spaces in $X$.

**Proof.** For each $g(\gamma)Z$ in $\tilde{T}$ and boundary wall $W$ of $Z$ of colour $< j$ consider the wall $g(\gamma)W$. We claim that $\tilde{T}$ is an intersection of half-spaces corresponding to these walls. Any edge leaving $\tilde{T}$ must cross one of these walls, so it suffices to show that each of these walls has a half-space containing $\tilde{T}$. Indeed suppose $Z, Z', P$ and $W$ are as in Definition 9.13, and consider $g(\gamma)Z$ glued to $g(\gamma)hg_1Z'$ along $g(\gamma)hP$ (with $h \in H$ and $g(\gamma)Z$ as in Definition 9.13). Suppose $W_0$ is a boundary wall of $Z$ of colour $< j$ and let $M$ be the part of $\tilde{T}$ on the opposite side of $g(\gamma)hW$ to $g(\gamma)Z$. We will show that $M$ and $g(\gamma)Z$ are on the same side of $g(\gamma)W_0$. If $g(\gamma)W_0 \cap g(\gamma)hW = \emptyset$, then this is immediate. Now suppose $g(\gamma)W_0$ and $g(\gamma)hW$ intersect, then we may apply Lemma 9.9 to find a vertex $x \in Z$ incident at edges $e_0, e$ which are dual to $W_0$ and $hW$ respectively, and form the corner of a square in $X$. Let $(hg_1)y$ be the vertex at the other end of $e$ (so $y \in Z'$), then $(hg_1)y$ is incident at an edge dual to $W_0$. Since $hP$ is a $hg_1$-teleport of $P'$, $[e, e_0] = [e, hg_1c_e']$

where $e' := (hg_1)^{-1}e$ and $c, c'$ are the colourings for $Z, Z'$. So $j > c_0(\overline{W_0}) = hg_1c'_e(\overline{W_0}) = hg_1c'_e(\overline{W_0})$.

Thus $W_0' := (hg_1)^{-1}W_0$ is a boundary wall of $Z_2$ of colour $< j$. This implies that $g(\gamma)hP_1Z'$ and $g(\gamma)(hg_1)y$ and $g(\gamma)Z$ are on the same side of $g(\gamma)W_0 = g(\gamma)hg_1W_0'$. Iterating this argument along the various branches of $M$ shows that $M$ and $g(\gamma)Z$ are on the same side of $g(\gamma)W_0$, as required.

\[\square\]
Lastly we check the numbered properties of $\mathcal{V}_{j-1}$:

1. If $e \in E(X)$ joins vertices $x, y \in \hat{T}$, we need $[e, c^T_x] = [e, c^T_y]$ to hold. This is clearly true if $e$ is contained within a translate $g(\gamma)Z$ because $c^T_x$ is a translate of the colourings for $Z$. From the proof of Lemma 9.17, we know that the different $g(\gamma)Z$ translates are separated by walls in a tree structure, and that two translates are adjacent only if they are of the form $g(\gamma)Z$ and $g(\gamma)hg_jZ'$ (with $Z, Z', f$ as in (9.13) and $h \in H$). If $e$ crosses from $g(\gamma)Z$ to $g(\gamma)hg_jZ'$, then it crosses the portal translate $g(\gamma)hP = g(\gamma)hg_pP'$, and can be written $e = g(\gamma)he_1 = g(\gamma)hg_f e_2$ for $e_1$ dual to $P$ and $e_2$ dual to $P'$. If $c, c'$ are the colourings for $Z, Z'$, then putting $x = g(\gamma)hx_1$ and $y = g(\gamma)hg_f y_1$ we have

$$[e, c^T_x] = [e, g(\gamma)hc_{x_1}],$$
$$= g(\gamma)h[e_1, c_{x_1}],$$
$$= g(\gamma)h[e_1, c_{e_1}],$$
$$= g(\gamma)hg_f[e_2, c_{e_2}],$$
$$= g(\gamma)hg_f[e_2, c_{y_2}],$$
$$= [e, g(\gamma)hg_f c'_{y_2}],$$
$$= [e, c'_{y}],$$

since $P$ is a $g_f$-teleport of $P'$

2. Given $e \in E(X)$ joining vertices $x, y$ with $x \in \hat{T}$, $c^T_x(e) > j$ if and only if $e$ is contained within some translate $g(\gamma)Z$; $c^T_x(e) = j$ if and only if $e$ crosses a portal translate into a different $g(\gamma)Z$; so $y \in \hat{T}$ if and only if $c^T_x(e) > j - 1$ as required.

3. Each pair $(H \cdot x, Z)$, for $(Z, H, (c_x)) \in \mathcal{V}_j$ and $x \in Z$ a vertex, corresponds to a vertex in a type (a) vertex space in $\mathcal{Y}$. This vertex will have $d$ lifts in $\hat{\mathcal{Y}}$, each of which gives one vertex in one of the $T$’s, which in turn makes a pair $(H(T) \cdot \hat{x}, \hat{T})$ for $(\hat{T}, H(T), (c^T_x)) \in \mathcal{V}_{j-1}$. Furthermore, by construction of $\hat{T}$, we know that $\hat{x} = g(\gamma)x$ for some $g(\gamma) \in G$, and $c^T_x = g(\gamma)c_x$. Therefore, for a given edge $e$ and colouring $c$, $(H(T) \cdot \hat{x}, \hat{T}) \in \mathcal{V}_{j-1}^\pm(e, c)$ if and only if $(H \cdot x, Z) \in \mathcal{V}_j^\pm(e, c)$. The Gluing Equations hold in $\mathcal{V}_j$, so we deduce that they also hold in $\mathcal{V}_{j-1}$ (with each side of each equation multiplied by $d$).

4. $H(T) \cong \pi_1(T)$ is hyperbolic and in $QVH$ by the same arguments used in Lemmas 9.9 and 9.10 (this time essential annuli have length zero). Alternatively, the fact it is hyperbolic follows from $\hat{T} \subset X$ being convex, which is a result of Lemma 9.20.

References

[1] Ian Agol (with appendix by Daniel Groves and Jason Manning), The Virtual Haken Conjecture, Documenta Mathematica 18 (2013), 1045-1087.
[2] Ian Agol, Virtual properties of 3-manifolds, Proceedings of the International Congress of Mathematicians 2014 Vol. 1, 141-170.
[3] Nicolas Bergeron and Daniel T. Wise, A boundary criterion for cubulation, American Journal of Mathematics Vol. 134, No. 3 (2012), 843-859.
[4] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, J. Differential Geom. 35 (1992), no. 1, 85-101.
[5] Martin R. Bridson and André Haefliger, *Metric Spaces of Non-positive Curvature*, Springer (1999).

[6] Mikhael Gromov, *Hyperbolic groups*, Essays in Group Theory (S. M. Gersten, ed.), Mathematical Sciences Research Institute Publications, vol. 8, Springer-Verlag (1987), 75-264.

[7] Frédéric Haglund and Daniel T. Wise, *Special Cube Complexes*, GAFA Vol. 17 (2008), 1551-1620.

[8] G. A. Niblo and L. D. Reeves, *Coxeter groups act on CAT(0) cube complexes*, J. Group Theory 6 (2003), 399-413.

[9] Michah Sageev and Daniel T. Wise, *The Tits alternative for CAT(0) cubical complexes*, Bull. London Math. Soc. 37 (2005) 706–710.

[10] Jacek Swiatkowski, *Regular path systems and (bi)automatic groups*, Geometriae Dedicata Vol. 118 (2006), 23-48.

[11] Daniel T. Wise, *The structure of groups with a quasiconvex hierarchy*, preprint, 2011.

[12] Daniel T. Wise, *Cubulating small cancellation groups*, GAFA Vol. 14 (2004), 150-214.

[13] https://personalpages.manchester.ac.uk/staff/charles.walkden/ergodic-theory/lecture10.pdf

[14] Michah Sageev, *CAT(0) cube complexes and groups*, Geometric group theory, 7–54, IAS/Park City Math. Ser. 21, Amer. Math. Soc., Providence, RI, 2014.

[15] Wolfgang Haken, *Theorie der Normalflächen*, Acta Math. 105 (1961), 245–375.