Classical analysis of Bianchi types I and II in Ashtekar variables

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We solve the complex Einstein equations for Bianchi I and II models formulated in the Ashtekar variables. We then solve the reality conditions to obtain a parametrization of the space of Lorentzian solutions in terms of real canonically conjugate variables. In the Ashtekar variables, the dynamics of the universe point particle is governed by only a curved supermetric – there is no potential term. In the usual metric formulation the particle bounces off a potential wall in flat superspace. We consider possible characterizations of this “bounce” in the potential-free Ashtekar variables.

I. INTRODUCTION

The Ashtekar variables for general relativity [1] are complex canonical coordinates on the real phase space, in terms of which Einstein’s equations become low-order polynomials. This has led to several simplifications and progress on various fronts [2,3]. Classically, the use of the new variables has led to a greater understanding of the space of selfdual solutions of Einstein’s equations [4], as well as of solutions corresponding to degenerate metrics [5] and some efforts towards solving the classical constraints [6]. In the canonical quantum theory, the use of new variables has led to the discovery of large classes of solutions to the quantum scalar constraint equations and quantum states approximating given 3-geometries. Much of this progress has resulted from the simplified form of the constraints, in particular, the scalar constraint is a low-order polynomial, homogeneous and quadratic in the canonical momenta, albeit the supermetric it defines is in general complex and non-flat. Thus, dynamical trajectories correspond to null geodesics of this supermetric and this simplification may allow one to solve the classical complex equations of motion, or at least to delve deeper into the structure of the space of solutions.

This progress, however, has come at certain expense. In order to recover real Lorentzian general relativity, the new variables have to satisfy certain reality conditions. The configuration variable is a complex SU(2) connection on the spatial 3-manifold Σ and the canonically conjugate momentum is a complex triad (of density weight +1). To recover real Lorentzian 4-geometries, one requires that the triads be real and that the real part of the complex SU(2) connection be the spin connection compatible with the triad.

Can these reality conditions be solved? If one reintroduces the geometrodynamical variables one obtains the usual formulation which is manifestly real, but in which the constraints and equations of motion are of a complicated form. The key question is: Can one first exploit the simple form of the constraints to solve the complex equations of motion and then attempt to solve the reality conditions? How does the space of Lorentzian solutions “sit” inside the space of complex solutions? Is there a simple characterization of the Lorentzian solutions in terms of real canonical variables? Clearly, such a real parametrization of the space of Lorentzian solutions can be obtained by starting from the geometrodynamical variables, however, our interest is to learn about the new variables themselves and the role of the reality conditions, since they appear to have greater potential for the full theory itself.

This issue – that of the “reality” structure of the space of solutions – is also likely to be important from the point of view of the quantum theory. A criterion for the selection of a physical inner product on the space of solutions to the quantum constraint equation is to require that the reality conditions on physical observables be represented by Hermiticity conditions on the corresponding operators [7]. Functions on the space of solutions are classical physical observables and thus may provide guides to constructing quantum Dirac observables.

In the metric variables, the dynamics is that of a “point particle” (corresponding to a 3-geometry) moving under the influence of a potential in a real flat background superspace. In the new variables the dynamics of the system is

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The connection. These variables are related to the usual canonical metric variables via

$$\tilde{A}^i_{ab} = a^{iab} \text{ and } A^i_a = \Gamma^i_a - iK^i_a,$$  \hspace{1cm} (1)

where $a^{iab}$ is the density weight +2 contravariant 3-metric on the spatial slices, $\Gamma^i_a$ is the spin connection of the triad and $K^i_a$ is the (triplet component of the) extrinsic curvature of the spatial manifold.

Now let us consider the spatially homogeneous Bianchi models. The reduction of the Ashtekar variables to these cosmologies has been carried out previously \[8,12\]. Here we will review the canonical formulation, following the conventions and notation of Ashtekar and Pullin \[8\]. The Bianchi models are spatially homogeneous models which admit a three dimensional isometry group which acts simply and transitively on the preferred homogeneous spatial slices. One introduces a basis (and its dual) of group invariant 1-forms $\omega^I$, $I = 1 - 3$ which satisfy $d\omega^I = -\frac{1}{2}C^I_{JK}\omega^J \wedge \omega^K$, where $C^I_{JK}$ are the structure constants of the Lie algebra associated with the Bianchi group. Components of homogeneous tensors in this basis are spatial constants; e.g. the 4-metric in this basis is

$$ds^2 = -N^2(t)dt^2 + \sum_{IJ} q_{IJ}(t)\omega^I \otimes \omega^J.$$
where the lapse function $N(t)$ and the 3-metric components $q_{IJ}$ are constant on the homogeneous spatial slices and are thus functions only of the time $t$.

Class A models—distinguished by the vanishing of the trace of $C$—readily admit a Hamiltonian formulation \[^3\]. The structure constants for the Class A models can be written as

$$C^I_{JK} = n^{(I)} \epsilon_{IJK} \text{ (no sum over } I),$$

where $\epsilon_{IJK}$ is the totally antisymmetric tensor and the set of numbers $\{n^{(I)}\}$—each of which can take the values $0, \pm 1$—are used to classify the different models. In this paper we are interested in the Bianchi types I and II, which are both in Class A.

For simplicity we will further confine our attention to the diagonal gauge, in which the off-diagonal components of the spatially homogeneous triad and connection vanish. The resulting phase space is 6 dimensional. In this diagonal gauge, the 6 canonically conjugate phase space variables are $\{A_I, -iE_I\}$, with $I = 1, 2, 3$ and the symplectic form is given by

$$\Omega = -i \sum_I dE_I \wedge dA_I. \quad (3)$$

For future reference note that the relationship between the 3-metric and the triads for the Bianchi cosmologies is

$$q_I = \left| \begin{array}{c} E_1 E_2 E_3 \\ E_I^2 \end{array} \right|. \quad (4)$$

In the diagonal gauge the diffeomorphism and Gauss constraints vanish identically, and we are left with the scalar constraint:

$$S \equiv G_N G^{IJ}(A,n) p_IP_J
- \sum_I \frac{1}{G_N} n^{(K)}(\epsilon_{KIJ})^2 A_K \left( -iE_I \right) \left( -iE_J \right), \quad (5)$$

where $G_N$ is Newton’s constant, $G^{IJ}$ is the (contravariant) supermetric on the configuration space of connections, and varies from one Bianchi model to another through its dependence on $n^{(I)}$. Note that we have made a choice for the real lapse function: $\mathcal{N} = 1$, thus $N = (E_1 E_2 E_3)^{1/2}$.

In general relativity, classical dynamics is generated by the (vanishing) scalar constraint. Let $\lambda$ be an affine parameter along the orbits generated by the scalar constraint. Then the evolution of any function on phase space is given by $f = df/d\lambda = \{f, \mathcal{S}\}$. Since the scalar constraint function in the new variables consists of only a kinetic term quadratic in the momenta, and is constrained to vanish, the dynamics it generates is that of a massless free particle in a curved spacetime with a (super)metric given by (3). Thus, in configuration space the dynamical trajectories are simply the null geodesics of the above supermetric. Even though the supermetric is complex, its geodesics and its null directions are both well defined.

As we can see from the canonical transformation (1), the coordinates $(E_I, A_I)$ are in general complex. In order to recover the real Lorentzian solutions, we have to impose “reality conditions” on the canonical variables. These conditions are:

$$E_I \in \mathbb{R} \quad \text{Re}(A_I) = -\frac{1}{2G_N} \frac{\partial}{\partial E_I} \left( \sum_J \frac{E_I E_J n^{(J)}}{E_J^2} \right), \quad (6)$$

where the $n^{(I)}$ defined in (2) distinguish the various Class A Bianchi models. Both the scalar constraint (3) and the symplectic form (3) are real when evaluated on triads and connections satisfying the above reality conditions. Since the triads are also required to be real, the parameter along the real evolution generated by the (real) scalar constraint should also be real. We see that the reality conditions in the form $E_I \in \mathbb{R}, \dot{E}_I \in \mathbb{R}$—where the dot · indicates derivative with respect to a real parameter along the orbits generated by a scalar constraint smeared with a real lapse—\(\mathcal{N} = 1\)—are completely equivalent to the reality conditions in the “canonical” form (1).

In the rest of this paper we will consider the Bianchi type I and II models, which are specified by $n^{(1)} = 0$ and $n^{(1)} = 1, n^{(2)} = n^{(3)} = 0$ respectively.
III. BIANCHI I

Now let us consider Bianchi type I, where the \( n^{(I)} = 0 \). The supermetric is then

\[
G^{IJ} = -\frac{1}{2} \begin{bmatrix}
0 & A_1 A_2 & A_1 A_3 \\
A_1 A_2 & 0 & A_2 A_3 \\
A_1 A_3 & A_2 A_3 & 0
\end{bmatrix}.
\]

(7)

In order to simplify the calculations we will change coordinates on the space of connections, \( \{A_I\} \rightarrow \{x^i = t, x, y\} \) as follows:

\[
t = A_2 A_3 \\
x = G N A_2 A_3 / A_1 \\
y = A_2 / A_3.
\]

(8)

In these coordinates the metric \( G \) is now diagonal:

\[
G^{ij} = \text{diag}(-t^2, x^2, y^2),
\]

(9)

and the connection components are

\[
A_1 = G N t / x \\
A_2 = \sqrt{t y} \\
A_3 = \sqrt{t / y}.
\]

(10)

Note that there are many ways to diagonalize the supermetric, this particular choice of coordinates has been made in order to follow as closely as possible the change of coordinates which simplifies the calculations for the Bianchi II model.

Since we are interested in the evolution of the triads, and will impose reality conditions on the triads, let us express them in terms of the new coordinates and momenta. Let \( p_i = \{p_t, p_x, p_y\} \) be the momenta canonically conjugate to the new coordinates \( t, x, y \) respectively, so that the triads are given by

\[
E_1 = -\frac{i}{A_1} x p_x \\
E_2 = \frac{t}{A_2} (tp_t + xp_x + y p_y) \\
E_3 = \frac{1}{A_3} (tp_t + xp_x - y p_y),
\]

and the symplectic form is

\[
\Omega = \sum dp_i \wedge dx^i.
\]

(12)

From (8) we see that under the transformation \( (A_2, A_3) \rightarrow (-A_2, -A_3) \) the new coordinates are left unchanged, and there is the related ambiguity in the square roots in (10). We will resolve this ambiguity by taking the principal value of the square roots, and thus work with a quarter of the original phase space. The other parts are recovered by choosing the other signs of the square roots. We are justified in considering these “quadrants” separately since they are dynamically disconnected, i.e. the Hamiltonian vector field of the constraint is tangential to the boundary between any two of them. Furthermore, the spacetime geometries that result from the different choices of sign are in fact the same.

Let us solve the (complex) equations of motion. For Bianchi type I the null geodesics are easy to find since the metric has three Killing vector fields: \( K_i = x^i \partial / \partial x^i \). Let \( \lambda \in \mathbb{C} \) be a parameter along the orbits generated by the scalar constraint, and let \( x^i(\lambda) \) be an orbit in configuration space. Then, associated with the Killing vectors we have three conserved quantities:

\[
c_i = x^i p_i = G_{jk} K_i^j \dot{x}^k
\]

i.e.

\[
\{c_t, c_x, c_y\} = \{tp_t, xp_x, yp_y\} = \{-t / t, \dot{x} / x, \dot{y} / y\},
\]

(13)

We require the trajectories to be null, in order to satisfy the scalar constraint. This yields a condition on the above constants:

\[
-c_t^2 + c_x^2 + c_y^2 = 0.
\]

(14)
We can directly solve the equations of motion (13) for the orbits to obtain

\[
\begin{align*}
t(\lambda) &= t_0 e^{-c_t \lambda} \\
x(\lambda) &= e^{c_x(\lambda-\lambda_0)} \\
y(\lambda) &= y_0 e^{c_y \lambda},
\end{align*}
\]

(15)

where the complex constants \((t_0, \lambda_0, y_0)\) correspond to the initial values of the coordinates \((t, x, y)\) respectively (We have chosen \(\lambda_0\) as a parameter instead of \(x_0\) following the most convenient choice for Bianchi II).

If we substitute (15) into (10) and (11), we obtain the general complex solutions to the scalar constraint: we still have

\[
\begin{align*}
E_1 &= \frac{\epsilon c_x}{A_1} = -i e^{c_x} \exp((c_t + c_x)\tau - \epsilon c_x \tau t_0 + i c_x (\phi - \phi_0)) \\
E_2 &= i \frac{c_t + \epsilon c_x + c_y}{A_2} = i \frac{c_t + e^{c_x} + c_y}{\sqrt{T_0 Y_0}} \exp((c_t - c_y)\tau/2) \\
E_3 &= i \frac{c_t + e^{c_x} - c_y}{A_3} = i (c_t + e^{c_x} - c_y) \sqrt{\frac{1}{T_0}} \exp((c_t + c_y)\tau/2).
\end{align*}
\]

(19)

Let us impose the rest of the reality conditions. Since we only have exponential functions of \(\tau\), we see that \(d \ln E_1/d\tau\) are real if and only if all \(c_t\) are real (and hence \(c_t^2 \geq c_y^2\)). Next, we see that \(E_2\) and \(E_3\) are real if and only if in addition \(T_0, Y_0\) are real and satisfy \(T_0 Y_0 < 0\). Then, \(E_1\) will be real only if \(\exp(i c_x (\phi - \phi_0)) = \pm i\), from which we conclude that \(\cos(c_x(\phi - \phi_0)) = 0\).

Collecting all reality conditions, we conclude that the triads are real if and only if

\[
\begin{align*}
c_t, c_x, c_y &\in \mathbb{R} \quad \text{and} \quad c_t^2 \geq c_y^2 \\
T_0, Y_0 &\in \mathbb{R} \quad \text{and} \quad T_0 Y_0 < 0 \\
\text{and} \quad &\cos(c_x(\phi - \phi_0)) = 0.
\end{align*}
\]

(20)

Note that \(\tau_0\) corresponds to a choice of the initial value of \(x\) and has been left as a free parameter, but \(\phi\) and \(\phi_0\) have disappeared as orbit parameters, fixed by the reality conditions. The real solution for the triads is then simply

\[
\begin{align*}
E_1 &= \pm \frac{c_t + c_x}{G N T_0} \exp((\epsilon c_x + c_t)\tau - \epsilon c_x \tau t_0) \\
E_2 &= \frac{c_t + \epsilon c_x + c_y}{\sqrt{-T_0 Y_0}} \exp((c_t - c_y)\tau/2) \\
E_3 &= (c_t + e^{c_x} - c_y) \sqrt{\frac{1}{T_0}} \exp((c_t + c_y)\tau/2).
\end{align*}
\]

(21)
Note that the sign ambiguities in the above solution for the triads lead to distinct triad solutions. However, the 4-dimensional spacetime geometries (quadratic in the triads) are unaffected by the choice of sign and thus the solutions are physically equivalent.

When the reality conditions are satisfied, the connections are pure imaginary:
\[
A_1 = \pm i G_N T_0 e^{c_x \tau_0} \exp(-(c_t + c_x) \tau)
\]
\[
A_2 = i \sqrt{-T_0} \exp(-c_y \tau/2)
\]
\[
A_3 = i \sqrt{-T_0} \exp(-(c_t + c_y) \tau/2).
\]
(22)

All the connections have a simple exponential behavior, vanishing (at different rates) as \(c_t \tau \rightarrow \infty\). Note that there are sign ambiguities in all three connection components arising due to either the phase \(c_x (\phi - \phi_0)\) or the square root. These sign ambiguities also lead to different solutions which however correspond to the same spacetime geometry.

Finally, the pull-back of the symplectic form to the space of real solutions still looks very much as in (17):
\[
\hat{\Omega} = dc_t \wedge d \ln T_0 + dc_y \wedge d \ln Y_0.
\]
(23)

Thus, the chosen coordinates on the reduced phase space are real as well as canonically conjugate.

From the above solutions and (4), one can obtain the components of the metric. The metric is the Kasner solution
\[
d s^2 = -d t^2 + t^{2 p_1} d x^2 + t^{2 p_2} d y^2 + t^{2 p_3} d z^2,
\]
(24)

where we have rescaled the time function
\[
e^{(c_x + 2 c_t) \tau} \rightarrow t
\]
(25)

and the set of Kasner parameters \(p = \{p_1, p_2, p_3\}\) are functions of the orbit parameters
\[
\{c_t, c_x, c_y\} \rightarrow p = \{c_x - c_t, c_x - c_y, c_y - c_t\}/(c_x + 2 c_t).
\]
(26)

The properties of the Kasner parameters \(\sum p_i^2 = \sum p_i = 1\) are derived from (14). Belinskii et al [14] introduced a single parameter \(u \in [0, 1]\) as a solution to the above constraints, where \((1 + u + u^2) p(u) = \{u, 1 + u, u(1 + u)\}\). In terms of the parameters we have used, \(u^2 = (|c_t| + |c_y|)/(|c_t| + |c_y|)\) for the ratio \(|c_y/c_t|\) in the range \([0, 1]\). The solutions with parameters \(|c_t/c_y| = 1\) correspond to flat Rindler spacetime (i.e., Minkowski spacetime in an accelerated frame). The “trivial” solution with all \(c_t = 0\) is, of course, Minkowski spacetime. In the non-trivial cases, the initial singularity is approached in the limit \(-c_t \tau \rightarrow \infty\).

IV. BIANCHI II

In this section we will solve the classical dynamics of the Bianchi type II model, which is specified by the structure constants \(n^{(1)} = 1 = C_{23}^1\). We will repeat the procedure step-by-step as for Bianchi type I.

The Bianchi type II supermetric is
\[
G^{IJ} = -\frac{1}{2} \begin{pmatrix}
0 & A_1 A_2 & A_1 A_3 \\
A_1 A_2 & 0 & A_2 A_3 + A_1/G_N \\
A_1 A_3 & A_2 A_3 + A_1/G_N & 0
\end{pmatrix}.
\]
(27)

As for Bianchi type I, we will change coordinates to a set adapted to the Killing symmetries of the supermetric, \(\{A_I\} \rightarrow \{x^i = t, x, y\}\) as follows:
\[
t = A_2 A_3 - A_1/G_N
\]
\[
x = G_N A_2 A_3/A_1
\]
\[
y = A_2/A_3.
\]
(28)

In these coordinates the metric \(G\) is now diagonal:
\[
G^{ij} = \text{diag}(-\frac{x}{x-1}\cdot t^2, x(x-1), \frac{x+1}{x}\cdot y^2).
\]
(29)

The connection components are expressed in terms of the Killing coordinates by
\[
A_1 = \frac{G_{Nt}}{(x-1)} \\
A_2 = \sqrt{\frac{G_{xy}}{(x-1)}} \\
A_3 = \sqrt{G_{y(x-1)}}.
\] (30)

(The ambiguity in the sign of \(A_2, A_3\) we resolve as for Bianchi type I; see the discussion after (11).) Let \(p_t = \{p_t, p_x, p_y\}\) be the momenta canonically conjugate to the new coordinates \(t, x, y\) respectively. Then the triads are given by

\[
E_1 = -\frac{i}{A_1} (\frac{tp_t}{x-1} + xp_x) \\
E_2 = \frac{i}{A_2} (\frac{x}{x-1} tp_t + xp_x + yp_y) \\
E_3 = \frac{i}{A_3} (\frac{x}{x-1} tp_t + xp_x - yp_y).
\] (31)

For Bianchi type II the supermetric has only two linearly independent Killing vector fields. However, since the space is 3-dimensional, the system is separable, and we will be left with three ordinary differential equations to solve. The two Killing vector fields are: \(K_t = t\partial/\partial t\) and \(K_y = y\partial/\partial y\). As before, let \(\lambda \in \mathbb{C}\) be a parameter along the orbits generated by the scalar constraint, and let \(x^i(\lambda)\) be an orbit in configuration space. Then, associated with the Killing vectors we have two conserved quantities:

\[
c_t = tp_t = tG_{tt}, \quad i = \frac{1-x}{x} i \\
c_y = yp_y = yG_{yy}, \quad \dot{y} = \frac{x+1}{y} y
\] (32)

Let us define \(c_x\) similar to the Bianchi I parameter:

\[
c_x = \text{pr.val.} \sqrt{c_t^2 - c_y^2}.
\] (33)

The constraint (i.e. that the trajectories be null) yields an equation for \(\dot{x}\):

\[
\dot{x}^2 = x^2 c_x^2 + c_y^2.
\] (34)

This equation can be readily solved for \(x(\lambda)\):

\[
x(\lambda) = \epsilon \frac{c_y}{c_x} \sinh (c_x (\lambda - \lambda_0)),
\] (35)

where \(\epsilon = \pm 1\) corresponds to the two possible choices of sign in the square root of (34).

Substituting the above solution \(x(\lambda)\) in the equations (31) for the conserved quantities we obtain ordinary linear differential equations for \(t(\lambda)\) and \(y(\lambda)\) which can be solved to yield the complete equations of motion:

\[
t(\lambda) = t_0 e^{-c_t \lambda} \frac{c_x}{c_y} \frac{x(t) + xc_t}{x - c_t} = t_0 e^{-c_t \lambda} \frac{c_x \cosh (c_x (\lambda - \lambda_0)) + c_t \sinh (c_x (\lambda - \lambda_0))}{c_y \cosh (c_x (\lambda - \lambda_0)) - c_t} \] (36)

\[
y(\lambda) = y_0 e^{c_x \lambda} \frac{c_y}{xc_x} \frac{c_x}{x} = y_0 e^{c_x \lambda} \frac{\cosh (c_x (\lambda - \lambda_0)) - \epsilon}{\sinh (c_x (\lambda - \lambda_0))},
\]

where the complex constants \(t_0, y_0\) correspond to the initial values of the coordinates \(t, y\) respectively.

A lengthy calculation shows that the symplectic structure evaluated on the solutions is simply

\[
\Omega = dc_t \wedge \ln t_0 + dc_y \wedge \ln y_0,
\] (37)

from which it is clear that \(\{c_t, c_y, \ln t_0, \ln y_0\}\) is a set of canonical coordinates on the complex reduced phase space. The calculation of the symplectic form on the reduced phase space is considerably simplified when we note that the pull-back of the symplectic structure to the constraint surface has two properties: \(i)\) it is degenerate – with the degenerate direction along the Hamiltonian vector field of the constraint – and \(ii)\) its Lie derivative along the above
vector field vanishes. Hence, there can be no terms in $\dot{\Omega}$ containing $d\lambda$, and the coefficients of the remaining terms will be independent of $\lambda$. Thus, we can simply evaluate the symplectic structure on some $\lambda = const$ cross-section of the constraint surface.

We now impose the reality conditions \([3]\) on the complex solutions and find the reduced phase space of the real degrees of freedom. As before we can identify $\tau = \text{Re}(\lambda)$ with the real time, and require that the triads and their derivatives with respect to $\tau$ should be real. It is more convenient to (partially) solve the reality conditions in terms of $E_2 \cdot E_3$ and $E_2/E_3$. First, we substitute for the momenta $p_i$ in terms of the velocities $\dot{x}^i$ in the expressions \([3]\). Various simplifications then lead to the following expressions:

\[
E_r(\tau) := E_2/E_3 = \frac{1}{Y_0} \frac{c_x}{c_x - c_y} e^{-c_y \tau} \\
E_p(\tau) := E_2 \cdot E_3 = -\frac{c_x c_y e^{c_y \tau}}{T_0},
\]

where we have defined the new orbit parameters $T_0, Y_0$ exactly as for Bianchi I \([8]\). Now $d \ln(E_r)/d\tau$ and $d \ln(E_p))/d\tau$ are real if and only if

\[
c_y, c_t \in \mathbb{R},
\]

and the reality of $E_p, E_r$ is then equivalent to

\[
c_x/Y_0, c_x/T_0 \in \mathbb{R},
\]

where $c_x$ is either real or imaginary. In order to guarantee that $E_2, E_3$ are themselves real, and not possibly pure imaginary, we also require that $E_r \cdot E_p \geq 0$, this condition on the orbit parameters is

\[
c_y(c_t + c_y)Y_0 T_0 \leq 0.
\]

Now let us consider the reality of $E_1$. With the above reality conditions on $E_2, E_3$ imposed, the solution for $E_1$ is

\[
E_1(\tau) = -\frac{i}{T_0} e^{c_y \tau} (c_t \sinh(c_x(\tau - \tau_0 + i(\phi - \phi_0))) - c_x \cosh(c_x(\tau - \tau_0 + i(\phi - \phi_0)))),
\]

where, as for Bianchi type I we have defined $\lambda_0 = \tau_0 + i\phi_0$. We will impose the reality conditions on $E_1, \dot{E}_1$ at $\tau = \tau_0$. Using the previously solved reality conditions on the orbit parameters, the remaining reality conditions are reduced to

\[
E_1(\tau_0) \in \mathbb{R} \iff \frac{c_t}{c_x} \sin(c_x(\phi - \phi_0)) + i \cos(c_x(\phi - \phi_0)) \in \mathbb{R}
\]

and

\[
\dot{E}_1(\tau_0) \in \mathbb{R} \iff c_x \sin(c_x(\phi - \phi_0)) + ic_t \cos(c_x(\phi - \phi_0)) \in \mathbb{R}.
\]

Now we know that $c_x$ is either real or pure imaginary; in every case the first terms in the above equations are real, and the second terms are pure imaginary and hence should vanish. Thus, we should have $\cos(c_x(\phi - \phi_0)) = 0$, which has no solutions for imaginary $c_x$, and we conclude that $c_x \in \mathbb{R}$ and thus $c_t^2 \geq c_y^2$. Putting this back into \([4]\), we see that these (necessary) conditions are sufficient to guarantee that $E_1$ is real for all $\tau$.

Collecting all the reality conditions, we conclude that the solutions for the triads in Bianchi type II will be real if and only if

\[
\begin{align*}
&c_t, c_y \in \mathbb{R} \quad \text{and} \quad c_t^2 \geq c_y^2 \\
&T_0, Y_0 \in \mathbb{R} \quad \text{and} \quad c_x c_T Y_0^2 T_0 \leq 0 \quad \text{and} \quad \cos(c_x(\phi - \phi_0)) = 0.
\end{align*}
\]

(In the inequality \([4]\) the sign of $(c_t + c_y)$ is determined by the sign of $c_t$. Note also that $\tau_0$, which corresponds to the initial value of $x$, is left undetermined by any of the reality conditions, and is a free parameter.)

The triads evaluated on the real trajectories are:

\[
\begin{align*}
E_1 &= \pm \frac{c_y}{G_0 Y_0} \exp(c_t \tau) \cosh(c_x(\tau - \tau_0) - \phi) \\
E_2 &= \sqrt{-\frac{2}{T_0 Y_0}} c_y (c_t + c_y) \exp(c_t - c_y) \tau / 2 \\
E_3 &= \sqrt{-\frac{2 Y_0}{T_y}} c_y (c_t - c_y) \exp(c_t + c_y) \tau / 2,
\end{align*}
\]

8
\[ \phi = \text{arctanh}(c_x/c_t). \]

If we evaluate the metric, we find the solution quoted by Belinskii et al [10] (as an approximate solution for a “Kasner epoch” transition in Bianchi IX) with the same identification of parameters we made in Bianchi I in (26), with \( \epsilon = +1 \). It is also, of course, the solution first quoted for Bianchi II models by Taub [15], with a similar identification of parameters \( k = c_x, c_1 = c_t - c_y \) and \( c_2 = c_t + c_y \).

The connections \( A_I \) evaluated on the real trajectories are:

\[
\begin{align*}
A_1 &= -G_N(c_x/c_y)T_0 \exp(-c_t \tau) (1 + i\epsilon f(\tau))^{-1} \\
A_2^2 &= T_0Y_0 \exp(-(c_t - c_y)\tau)(1 - i\epsilon \sin(c_x(\tau - \tau_0))) (1 - i\epsilon f(\tau))^{-1} \\
A_3^3 &= -(T_0/Y_0) \exp(-(c_t + c_y)\tau)(1 + i\epsilon \sin(c_x(\tau - \tau_0))) (1 - i\epsilon f(\tau))^{-1}.
\end{align*}
\]

where \( f(\tau) = \delta \sin(c_x(\tau - \tau_0)) - \phi \), with \( \delta = \text{sgn}(c_tc_y) \).

As \( \tau \to \pm \infty \), the connections behave respectively as:

\[
\begin{align*}
A_1 &\to -iG_N\sqrt{T_0Y_0} \exp(\pm c_t \tau_0)/(1 \mp c_t/c_x) \exp(-c_t \pm c_x \tau) \\
A_2 &\to i\sqrt{T_0Y_0} \left| [c_t \pm c_x/c_y] \exp(-(c_t - c_y)\tau/2) \\
A_3 &\to i\sqrt{T_0/Y_0} \left| [c_t \pm c_x/c_y] \exp(-(c_t + c_y)\tau/2).
\end{align*}
\]

We can see that these limits correspond to Bianchi I solutions, that is, Bianchi II solutions approximate (different) Bianchi I solutions as \( \tau \to \pm \infty \). These Bianchi I solutions differ in the value of the discrete parameter \( \epsilon \) and renormalizations of \( T_0, Y_0 \).

As in the case of Bianchi type I, the pull-back of the symplectic form to the space of real solutions is:

\[ \hat{\Omega} = dc_t \wedge d\ln T_0 + dc_y \wedge d\ln Y_0, \]

where \( c_t, c_y, T_0, Y_0 \) are Dirac observables for the theory.

V. CHARACTERIZATION OF THE “BOUNCE”

When solving Einstein’s equations in the Hamiltonian formulation for the “diagonal models”, the usual geometro-dynamical description [14] is in terms of variables \( \{\beta^A, p_A\} \), with the index \( A \) taking values 0, \pm 1 where \( \beta^0 \) is the log of the 3-volume and \( \beta^\pm \) are the anisotropies. The Hamiltonian constraint has a term quadratic in momenta \( (\eta^{AB} p_A p_B) \) and a potential term which for all Bianchi types (except Bianchi I) has exponential dependence on the 2-dimensional \( \beta^\pm \) plane. Therefore the dynamics is that of a particle (the “universe point”) moving on a flat background in the given potential. In the regions where the potential vanishes the particle moves essentially free along a (null) straight line as if it were a solution to Bianchi I equations (Kasner epoch), until it encounters a potential wall and bounces back along a (reflected) straight line. In Bianchi II there is only one “wall” in the potential, and therefore the world-particle bounces once between the initial singularity and the expanding evolution. In Bianchi IX models, there are 3 walls and in general the particle keeps bouncing within the walls, approaching the singularity as \( t \to 0 \).

This classical behaviour was first analyzed by Belinskii et al [10]. They use the Bianchi II solution as a transition between Kasner epochs in the general Bianchi IX solution. In each Kasner epoch, the solution is approximately the Kasner solution, where the metric component in one spatial direction decreases with time (it has a negative Kasner exponent) and the other two increase with time (they have positive exponents). In the transition between Kasner epochs, the negative power of time is transferred from one spatial direction to another, so one metric component reaches a minimum and another a maximum, while the third one increases monotonically during the transition. Bianchi II models have a single transition, and the “bounce” is then defined as the point along the trajectory when one of the metric components reaches its maximum (which is different from the point when the other metric component reaches its minimum).

Now, if we consider the Hamiltonian formulation in terms of the new variables, the Hamiltonian constraint contains just a term quadratic in momenta. Therefore, the dynamics can be interpreted as that of a free particle moving on a null geodesic in the complex 3-dimensional space \( \{A_I\} \), where the supermetric is given by \( G^{IJ} \) in (4). Since there is no potential, there is neither a wall nor any particular coordinate time to identify with the “bounce” time that characterizes Bianchi II in the geometro-dynamical formulation. If we translate the bounce time defined in the
functions of $\tau$ solutions. On real Lorentzian Bianchi I solutions, the connection components are pure imaginary, exponential $\tau$ zero in the asymptotic regions as $\tau \rightarrow \pm \infty$. Again, however, this symmetry does not survive when we consider other Bianchi models. Hence, we would like to find a way of describing a turn-around point that does not depend on the use of symmetries particular to the Bianchi II solutions. One possibility is to look for features such as maxima or singularities in (the absolute values of) various superspace curvature scalars like the Ricci scalar $R_{IJ}G^{IJ}$, or $R_{IJ}R^{IJ}$. Unfortunately, all the real metric scalars up to second order in the Ricci tensor have several extrema as functions of $\tau$ when evaluated on the real trajectories, and therefore do not uniquely identify the bounces in higher Bianchi models, like type IX.

There is however, another option. The Bianchi II solutions behave asymptotically ($\tau \rightarrow \pm \infty$) like specific Bianchi I solutions. On real Lorentzian Bianchi I solutions, the connection components are pure imaginary, exponential functions of $\tau$. Hence we can look for a characterization of the “transition region”, by studying the ratio of real to imaginary parts of the solutions for the connection components in Bianchi type II: we know that this ratio tends to zero in the asymptotic regions as $\tau \rightarrow \pm \infty$, and since the Bianchi II connections are in general complex, this ratio is non-zero at finite $\tau$. The functions $\Phi_I = Re(A_I)/Im(A_I)$ turn out to be very simple:

$$\Phi^{-1}_1 = c \delta \sinh(c_\tau (\tau - \tau_0) - \phi)$$

$$\Phi^{-1}_2 = \epsilon \cosh(c_\tau (\tau - \tau_0) - \phi/2) / \sinh(\phi/2) \quad \text{if} \quad \delta = +1$$

$$\Phi^{-1}_3 = -\epsilon \cosh(c_\tau (\tau - \tau_0) - \phi/2) / \sinh(\phi/2) \quad \text{if} \quad \delta = -1$$

where we recall that $\delta = \text{sgn}(c_\tau c_y) = -\text{sgn}(T_0 Y_0)$ and $\phi = \text{arctanh}(c_\tau / c_\delta)$. Each of these functions has a unique and very obvious “transition point” (where they either have a maximum or diverge) at either $c_\tau (\tau - \tau_0) = \phi$ (for $\Phi_1$) or $c_\tau (\tau - \tau_0) = \phi/2$ (for $\Phi_2, \Phi_3$). The zero of $\Phi^{-1}_1$ is exactly the “bounce time” as defined in [10]. Thus, the “Kasner epochs” in the Ashtekar variables are identified with pure imaginary, exponential connections and the “bounces” are either maxima or divergences of the (tangent of the) phase of the connection components, or in other words, the most extreme departure from Bianchi I connections. Both before and after the bounce, the connections have an asymptotic exponential behavior, but with different logarithmic velocities. This description is very similar to that by Belinskii et al in terms of the behaviour of metric components. As in the geometrodynamical description where the maximum of one metric component does not coincide with the minimum of the other (forbidding an invariant definition of the bounce time), here too the divergences and maxima in the phase of the connection components do not coincide.

VI. CONCLUSION

Let us briefly recapitulate the results we have obtained. For both Bianchi types I and II, we first solved the complex equations of motion by finding the null geodesics of the complex supermetric on the space of connections. The space of solutions is parametrized by 4 complex variables, which form 2 canonically conjugate pairs. Next, to find the real Lorentzian solutions, we required that the triads are real throughout the real evolution. This condition is satisfied when the parameters on the space of solutions are all real and satisfy in addition certain nonholonomic constraints. (Note that we have made a simplifying choice of variables in order to obtain a real canonical parametrization of the reduced phase space.)

All Bianchi II solutions approach a (different) Bianchi I solution asymptotically in the past or the future, and deviations from Bianchi I like behaviour and the transition from one asymptotic Bianchi solution to another occurs at some finite time. Now for Lorentzian Bianchi I, $Re(A_I(t)) = 0$, whereas the Lorentzian connections for Bianchi type II are in general complex. So it is particularly nice to characterize the bounce in Bianchi II as a deviation from Bianchi I like behaviour via maxima or divergences in $\Phi_I = Re(A_I(t))/Im(A_I(t))$. We have seen that for any given solution, one of the phases $\Phi_I$ has a maximum while the other two have a singularity. This does uniquely characterize the bounce and could be used for the counting of bounces in the numerical study of Bianchi IX in the new variables.
Recall that we failed to find a description independent of the coordinates on the (minisuper)space of connections. In retrospect it is not surprising that there is no characterization in terms of some supermetric curvature scalars, since the Lorentzian solutions have not been described as null geodesics in some real supergeometry, though the complex solutions are null geodesics in a complex superspace.

One can proceed to construct the reduced space quantum theory for these models. This construction is straightforward since the reduced phase space is coordinatized by two pairs of real canonically conjugate variables. However, from the point of view of learning something about the Dirac quantum theory in terms of the connections themselves, this is not a very useful approach. In particular, one does not see clearly the role of the complex reality conditions in finding an inner product on the space of solutions to the quantum scalar constraint equation \[7\], or whether this is even feasible. These issues will be explored in a later paper [17], in which we construct the complete Dirac quantum theory for the Bianchi II model. The classical Dirac observables \((c_t, c_y, T_0, Y_0)\) we have constructed here do have quantum analogs, and the Hermiticity conditions on them serve to select an inner product uniquely on the space of solutions to the quantum scalar constraint equation.

**VII. APPENDIX: SPECIAL CASE SOLUTIONS**

For special values of the parameters \(c_y, c_t\), the limits of the general solutions we found in section 4 are not themselves solutions to the limiting equations of motion. In order to find the correct solutions at these limiting values, we have to solve directly the reduced equations of motion obtained by taking the appropriate limits of the general ones. There are three special cases and we consider them one by one.

**Case 1: \(c_t = 0\)**

In this limit, the equations of motion \([32\,34]\) reduce to

\[
\begin{align*}
\dot{t} &= 0 \\
\dot{y}/y &= c_y(1 + 1/x) \\
\dot{x}^2 &= c_y^2(1 - x^2).
\end{align*}
\]

The solution to the above equations is

\[
\begin{align*}
t(\lambda) &= i\epsilon t_0 \\
y(\lambda) &= iy_0 e^{c_y \lambda} \frac{\sin(c_y(\lambda - \lambda_0))}{1 + \epsilon \cos(c_y(\lambda - \lambda_0))} \\
x(\lambda) &= \epsilon \sin(c_y(\lambda - \lambda_0)).
\end{align*}
\]

Note that the order and the form of the limiting equations are the same as in the general case, and thus we expect that the above solution to the limiting equations can be obtained by taking the limit \((c_t \to 0)\) of the general solution \([32\,35]\), as can be confirmed by direct calculation. Thus we can directly take the limits of the reality conditions \([44]\), and conclude that \(c_y = 0\). The triads \(E_I's\) in this case (with \(c_t = c_y = 0\)) vanish, and the connections \(A_I's\) are constants.

**Case 2: \(c_y = 0\)**

The equations of motion in this case are:

\[
\begin{align*}
\dot{t}/t &= c_t x/(1 - x) \\
\dot{y} &= 0 \\
\dot{x}^2 &= c_y^2 x^2.
\end{align*}
\]

and the solution is

\[
\begin{align*}
x(\lambda) &= e^{c_t(\lambda - \lambda_0)} \\
y(\lambda) &= y_0 \\
t(\lambda) &= t_0(x(\lambda) - 1)^\epsilon.
\end{align*}
\]
If we take $\epsilon = -1$, then $E_2 = E_3 = 0$ and $E_1 = i c_t/(G_N t_0)$, and the connections are $A_1 = t_0$, $A_2 = y_0$, $A_3 = \sqrt{t_0 y_0} \exp(-c_t (\lambda - \lambda_0)/2)$. The reality conditions only require that $c_t/t_0$ is pure imaginary.

If we take $\epsilon = +1$, then the triads are:

$$E_1 = \frac{i c_t}{G_N t_0} (1 - e^{2c_t (\lambda - \lambda_0)})$$
$$E_2 = E_3/y_0 = \frac{2 c_t}{\sqrt{t_0 y_0}} e^{c_t (\lambda - \lambda_0)/2}. \tag{54}$$

The reality conditions then require $c_t$, $i t_0$, $y_0 \in \mathbb{R}$, $\cos(c_t (\phi - \phi_0)) = 0$, and $i t_0 y_0 \sin(c_t (\phi - \phi_0)) > 0$. Let us define $\delta = \sin(c_t (\phi - \phi_0)) = \pm 1$, and $T_0 = i \delta t_0$, $Y_0 = y_0$ (so $T_0 Y_0 > 0$). The Lorentzian solutions for triads and connections are, then:

$$E_1 = -\frac{\delta c_t}{T_0} (1 - e^{2c_t (\tau - \tau_0)})$$
$$E_2 = E_3/Y_0 = \frac{2 c_t}{\sqrt{T_0 Y_0}} e^{c_t (\tau - \tau_0)/2}$$
$$A_1 = i \delta T_0 (1 - i \delta \exp(c_t (\tau - \tau_0)))^{-}\right)^{-2}$$
$$A_2 = Y_0 A_3 = i \sqrt{t_0 Y_0} \left[ \frac{1 - \delta \exp(c_t (\tau - \tau_0))}{1 - \delta \exp(c_t (\tau - \tau_0))} \right] \tag{55}$$

Notice that although the triads have the same form as (43) (ignoring constant multiplicative factors, and taking $c_y = 0, c_x = c_t$ in the exponentials), the connections do not have the same behavior as in (46). In other words, the limit $c_y \to 0$ can be taken smoothly in the metric variables, but not so in the connection space.

**Case 3: $|c_y| = |c_t|$**

We set $c_y = \delta c_t$, where $\delta = \pm 1$. The equations of motion are:

$$\dot{x}/t = c_t x/(1 - x)$$
$$\dot{y}/y = \delta c_t (1 + 1/x)$$
$$x^2 = c_t^2, \tag{56}$$

and yield the solution

$$x(\lambda) = e c_t (\lambda - \lambda_0)$$
$$y(\lambda) = y_0 (\lambda - \lambda_0) e^{\delta c_t \lambda}$$
$$t(\lambda) = t_0 (1 - e c_t (\lambda - \lambda_0))^{-\epsilon} e^{-c_t \lambda}. \tag{57}$$

We see that $\delta \to -\delta$ is equivalent to $y \to y_0^2/y$, and this in turn is equivalent to $\{A_2 \leftrightarrow y_0 A_3$, $E_2 \leftrightarrow y_0 E_3\}$, so we only need to consider in detail the case $\delta = 1$.

If $\epsilon = +1$, the triads are:

$$E_1 = \frac{-2 i c_t}{t_0} e^{c_t \lambda} (1 - c_t (\lambda - \lambda_0))$$
$$E_2 = \sqrt{\frac{c_t}{t_0}} \frac{1 + c_t (\lambda - \lambda_0)}{\lambda - \lambda_0}$$
$$E_3 = \sqrt{\frac{c_t y_0}{t_0}} e^{c_t \lambda} (1 + c_t (\lambda - \lambda_0)). \tag{58}$$

We can see by inspection that the only solution to the reality conditions is the trivial one, $c_t = 0$, (vanishing triads and constant connections).

If $\epsilon = -1$, then the triads are:

$$E_1 = 0$$
$$E_2 = i \sqrt{\frac{c_t}{t_0 y_0}} \frac{1 + c_t (\lambda - \lambda_0)}{1 - c_t (\lambda - \lambda_0)}$$
$$E_3 = i \sqrt{\frac{c_t y_0}{t_0}} e^{c_t \lambda} \frac{1 + c_t (\lambda - \lambda_0)}{1 - c_t (\lambda - \lambda_0)}. \tag{59}$$
Here, the reality conditions are satisfied when $\phi = \phi_0 = 0$ and $c_t, t_0$, and $y_0$ are real and satisfy $c_t t_0 y_0 < 0$.

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