GROUP-FREENESS AND CERTAIN AMALGAMATED FREENESS

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Abstract. In this paper, we will consider certain amalgamated free product structure in crossed product algebras. Let $M$ be a von Neumann algebra acting on a Hilbert space $H$ and $G$, a group and let $\alpha : G \rightarrow \text{Aut} M$ be an action of $G$ on $M$, where $\text{Aut} M$ is the group of all automorphisms on $M$. Then the crossed product $M = M \times_\alpha G$ of $M$ and $G$ with respect to $\alpha$ is a von Neumann algebra acting on $H \otimes l^2(G)$, generated by $M$ and $\{u_g\}_{g \in G}$, where $u_g$ is the unitary representation of $g$ on $l^2(G)$. We show that $M \times_\alpha (G_1 \ast G_2) = (M \times_\alpha G_1) \ast_M (M \times_\alpha G_2)$. We compute moments and cumulants of operators in $M$. By doing that, we can verify that there is a close relation between Group Freeness and Amalgamated Freeness under the crossed product. As an application, we can show that if $F_N$ is the free group with $N$-generators, then the crossed product algebra $L_M(F_n) \equiv M \times_\alpha G$ satisfies that $L_M(F_n) = L_M(F_{k_1}) \ast_M L_M(F_{k_2})$, whenever $n = k_1 + k_2$, for $n, k_1, k_2 \in \mathbb{N}$.

In this paper, we will consider a relation between a free product of groups and a certain free product of von Neumann algebras with amalgamation over a fixed von Neumann subalgebra. In particular, we observe such relation when we have crossed product algebras. Crossed product algebras have been studied by various mathematicians. Let $M$ be a von Neumann algebra acting on a Hilbert space $H$ and $G$, a group, and let $M = M \times_\alpha G$ be the crossed product of $M$ and $G$ via an action $\alpha : G \rightarrow \text{Aut} M$ of $G$ on $M$, where $\text{Aut} M$ is the automorphism group of $M$. This new von Neumann algebra $M$ acts on the Hilbert space $H \otimes l^2(G)$, where $l^2(G)$ is the group Hilbert space. Each element $x$ in $M$ has its Fourier expansion

$$x = \sum_{g \in G} m_g u_g, \text{ for } m_g \in M$$

where $u_g$ is the (left regular) unitary representation of $g \in G$ on $l^2(G)$.

On $M$, we have the following basic computations;

(0.1) If $u_h$ is the unitary representation of $h \in G$, as an element in $M$, then
\( u_g u_g = u_{g, g} \) and \( u_g^* = u_{g^{-1}} \), for all \( g, g_1, g_2 \in G \)

(0.2) If \( m_1, m_2 \in M \) and \( g_1, g_2 \in G \), then

\[
(m_1 u_{g_1}) (m_2 u_{g_2}) = m_1 u_{g_1} m_2 (u_{g_1^{-1}} u_{g_1}) u_{g_2} = (m_1 (\alpha_{g_1} (m_2))) u_{g_1 g_2}
\]

(0.3) If \( m u_g \in M \), then

\[
(m u_g)^* = u_g^* m^* = u_{g^{-1}} m^* (u_g u_{g^{-1}}) = (\alpha_{g^{-1}} (m^*)) u_{g^{-1}} = (\alpha_{g^{-1}} (m^*)) u_g^*.
\]

(0.4) If \( m \in M \) and \( g \in G \), then

\[
u_g m = u_g m u_{g^{-1}} u_g = \alpha_g (m) u_g.
\]

and

\[
u_m g = u_g u_{g^{-1}} m u_g = u_g \cdot \alpha_{g^{-1}} (m)
\]

The element \( u_g m \) is of course contained in \( M \), since it can be regarded as \( u_g m u_{e_G} \), where \( e_G \) is the group identity of \( G \), for \( m \in M \) and \( g \in G \).

Free Probability has been researched from mid 1980's. There are two approaches to study it; the Voiculescu's original analytic approach and the Speicher's combinatorial approach. We will use the Speicher's approach. Let \( M \) be a von Neumann algebra and \( N \), a \( W^* \)-subalgebra and assume that there is a conditional expectation \( E : M \to N \) satisfying that (i) \( E \) is a continuous \( \mathbb{C} \)-linear map, (ii) \( E (n) = n \), for all \( n \in N \), (iii) \( E (n_1 m n_2) = n_1 E (m) n_2 \), for all \( m \in M \) and \( n_1, n_2 \in N \), and (iv) \( E (m^*) = E (m)^* \), for all \( m \in M \). If \( N = \mathbb{C} \), then \( E \) is a continuous linear functional on \( M \), satisfying that \( E (m^*) = E (m) \), for all \( m \in M \). The algebraic pair \( (M, E) \) is called an \( N \)-valued \( W^* \)-probability space. All operators \( m \) in \( (M, E) \) are said to be \( N \)-valued random variables. Let \( x_1, \ldots, x_s \in (M, E) \) be \( N \)-valued random variables, for \( s \in \mathbb{N} \). Then \( x_1, \ldots, x_s \) contain the following free distributional data.

\( \circ \) \( (i_1, \ldots, i_n) \)-th joint \( * \)-moment : \( E (x_{i_1}^{u_{i_1}} \cdots x_{i_n}^{u_{i_n}}) \)

\( \circ \) \( (j_1, \ldots, j_m) \)-th joint \( * \)-cumulant : \( k_m (x_{j_1}^{u_{j_1}} \cdots x_{j_m}^{u_{j_m}}) \) such that

\[
k_m (x_{j_1}^{u_{j_1}} \cdots x_{j_m}^{u_{j_m}}) \overset{def}{=} \sum_{\pi \in NC(m)} E_{\pi} (x_{j_1}^{u_{j_1}} \cdots x_{j_m}^{u_{j_m}}) \mu (\pi, 1_m),
\]

for \( (i_1, \ldots, i_n) \in \{ 1, \ldots, s \}^n \), \( (j_1, \ldots, j_m) \in \{ 1, \ldots, s \}^m \), for \( n, m \in \mathbb{N} \), and \( u_{i_k}, u_{j_l} \in \{ 1, * \} \), and where \( NC(m) \) is the lattice of all noncrossing partitions over \( \{ 1, \ldots, s \} \).
\[ \ldots, m \} \text{ with its minimal element } 0_m = \{(1), \ldots, (m)\} \text{ and its maximal element } 1_n = \{(1), \ldots, (m)\} \text{ and } \mu \text{ is the Möbius functional in the incidence algebra and } E_\pi(\ldots) \text{ is the partition-depending moment of } x_{j_1}, \ldots, x_{j_m} \text{ (See [19]).} \]

For instance, \( \pi = \{(1, 4), (2, 3)\} \) is in \( NC(4) \). We say that the elements \((1, 4)\) and \((2, 3)\) of \( \pi \) are blocks of \( \pi \), and write \((1, 4) \in \pi \) and \((2, 3) \in \pi \). In this case, the partition-depending moment \( E_\pi(x_{j_1}, \ldots, x_{j_k}) \) is determined by

\[ E_\pi(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}) = E(x_{j_1}E(x_{j_2}x_{j_3})x_{j_4}). \]

The ordering on \( NC(m) \) is defined by

\[ \pi \leq \theta \iff \text{for any block } B \in \pi, \text{ there is } V \in \theta \text{ such that } B \subseteq V, \]

for \( \pi, \theta \in NC(m) \), where \( \subseteq \) means the usual set-inclusion.

Suppose \( M_1 \) and \( M_2 \) are \( W^* \)-subalgebras of \( M \) containing their common subalgebra \( N \). The \( W^* \)-subalgebras \( M_1 \) and \( M_2 \) are said to be free over \( N \) in \( (M, E) \), if all mixed cumulants of \( M_1 \) and \( M_2 \) vanish. The subsets \( X_1 \) and \( X_2 \) of \( M \) are said to be free over \( N \) in \( (M, E) \), if the \( W^* \)-subalgebras \( vN(X_1, N) \) and \( vN(X_2, N) \) are free over \( N \) in \( (M, E) \), where \( vN(S_1, S_2) \) is the von Neumann algebra generated by arbitrary sets \( S_1 \) and \( S_2 \). In particular, we say that the \( N \)-valued random variables \( x \) and \( y \) are free over \( N \) in \( (M, E) \) if and only if \( \{x\} \) and \( \{y\} \) are free over \( N \) in \( (M, E) \). Notice that the \( N \)-freeness is totally depending on the conditional expectation \( E \). If \( M_1 \) and \( M_2 \) are free over \( N \) in \( (M, E) \), then the \( N \)-free product von Neumann algebra \( M_1 \star_N M_2 \) is a \( W^* \)-subalgebra of \( M \), where

\[ M_1 \star_N M_2 = N \oplus \left( \bigoplus_{n=1}^{\infty} \left( \bigoplus_{i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n} (M_{i_1}^p \otimes \ldots \otimes M_{i_n}^p) \right) \right), \]

where

\[ M_{i_j}^p = M_{i_j} \ominus N, \text{ for all } j = 1, \ldots, n. \]

Here, all algebraic operations \( \oplus, \otimes \) and \( \ominus \) are defined under \( W^* \)-topology.

Also, if \( (M_1, E_1) \) and \( (M_2, E_2) \) are \( N \)-valued \( W^* \)-probability space with their conditional expectation \( E_j : M_j \to N \), for \( j = 1, 2 \). Then we can construct the free product conditional expectation \( E = E_1 \ast E_2 : M_1 \ast_N M_2 \to N \) making its cumulant \( k_n^{(E)}(\ldots) \) vanish for mixed \( n \)-tuples of \( M_1 \) and \( M_2 \) (See [19]).

The main result of this paper is that if \( G_1 \ast G_2 \) is a free product of groups \( G_1 \) and \( G_2 \), then

\[ (0.5) \quad M \times_\alpha (G_1 \ast G_2) = (M \times_\alpha G_1) \ast_M (M \times_\alpha G_2), \]
where $M$ is a von Neumann algebra and $\alpha : G_1 \ast G_2 \to AutM$ is an action. This shows that the group-freeness implies a certain freeness on von Neumann algebras with amalgamation. Also, this shows that, under the crossed product structure, the amalgamated freeness determines the group freeness.

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1. **Crossed Product Probability Spaces**

In this chapter, we will observe some computations of $N$-valued moments and cumulants of operators in the crossed product algebra $\mathcal{M} = M \times_\alpha G$, with respect to the canonical conditional expectation from $\mathcal{M}$ onto $M$. Throughout this chapter, let $M$ be a von Neumann algebra and $G$, a group and let $\alpha : G \to AutM$ be an action of $G$ on $M$, where $AutM$ is the automorphism group of $M$.

Denote the group identity of $G$ by $e_G$. Consider the trivial subgroup $G_0 = \langle e_G \rangle$ of $G$ and the crossed product algebra $\mathcal{M}_0 = M \times_\alpha G_0$. Then this algebra $\mathcal{M}_0$ is a $W^*$-subalgebra of $\mathcal{M}$ and it satisfies that

\[(1.1) \quad \mathcal{M}_0 = M,\]

where the equality “=” means “$\ast$-isomorphic”. Indeed, there exists a linear map sending $m \in M$ to $m u_{e_G}$ in $\mathcal{M}_0$. This is the $\ast$-isomorphism from $M$ onto $\mathcal{M}_0$, since

\[(1.2) \quad m_1 m_2 \mapsto \begin{cases} (m_1 m_2) u_{e_G} = m_1 \alpha_{e_G}(m_2) u_{e_G} \\ = m_1 u_{e_G} m_2 u_{e_G} u_{e_G} \\ = (m_1 u_{e_G})(m_2 u_{e_G}), \end{cases} \]

for all $m_1, m_2 \in M$. The first equality of the above formula holds, because $\alpha_{e_G}$ is the identity automorphism on $M$ satisfying that $\alpha_{e_G}(m) = m$, for all $m \in M$. Also, the third equality holds, because $u_{e_G} u_{e_G} = u_{e_G} = u_{e_G}$ on $G_0$ (and also on $G$).

**Proposition 1.1.** Let $G_0 = \langle e_G \rangle$ be the trivial subgroup of $G$ and let $\mathcal{M}_0 = M \times_\alpha G_0$ be the crossed product algebra, where $\alpha$ is the given action of $G$ on $M$. Then the von Neumann algebra $\mathcal{M}_0$ and $M$ are $\ast$-isomorphic. i.e., $\mathcal{M}_0 = M$. $\square$

From now, we will identify $M$ and $\mathcal{M}_0$, as $\ast$-isomorphic von Neumann algebras.

**Definition 1.1.** Let $\mathcal{M} = M \times_\alpha G$ be the given crossed product algebra. Define a canonical conditional expectation $E_M : \mathcal{M} \to M$ by
\( (1.3) \quad E_M \left( \sum_{g \in G} m_g u_g \right) = m_{e_G}, \text{ for all } \sum_{g \in G} m_g u_g \in \mathbb{M}. \)

By (0.4), we have \( u_{e_G} m = \alpha_{e_G}(m) u_{e_G} = m u_{e_G} \). So, indeed, the \( \mathbb{C} \)-linear map \( E \) is a conditional expectation; By the very definition, \( E \) is continuous and

(i) \( E_M(m) = E_M(mu_{e_G}) = E_M(u_{e_G} m) = m, \) for all \( m \in M \),

(ii) \( E_M(m_1(mu_g)m_2) = m_1 E_M(m_2u_g)m_2 \)

\[
= \begin{cases} 
  m_1 m_2 = m_1 E_M(m_2u_g)m_2 & \text{if } g = e_G \\
  0_M = m_1 E_M(m_2u_g)m_2 & \text{otherwise},
\end{cases}
\]

for all \( m_1, m_2 \in M \) and \( mu_g \in \mathbb{M} \). Therefore, we can conclude that \( E_M(m_1x_m_2) = m_1 E_M(x)m_2, \) for \( m_1, m_2 \in M \) and \( x \in \mathbb{M} \).

(iii) For \( \sum_{g \in G} m_g u_g \in \mathbb{M}, \)

\[
E_M \left( \left( \sum_{g \in G} m_g u_g \right)^* \right) = E_M \left( \sum_{g \in G} u_g^* m_g^* \right)
= E_M \left( \sum_{g \in G} \alpha_g(m_g^*) u_{g^{-1}} \right)
= \alpha_{e_G}(m_{e_G}^*) = m_{e_G}^* = E_M \left( \sum_{g \in G} m_g u_g \right)^* ,
\]

Therefore, by (i), (ii) and (iii), the map \( E_M \) is a conditional expectation. Thus the pair \( (M, E_M) \) is a \( M \)-valued \( W^* \)-probability space.

**Definition 1.2.** The \( M \)-valued \( W^* \)-probability space \( (M, E_M) \) is called the \( M \)-valued crossed product probability space.

It is trivial that \( \mathbb{C} \cdot 1_M \) is a \( W^* \)-subalgebra of \( M \). Consider the crossed product \( \mathbb{M}_G = \mathbb{C} \times_\alpha G \), as a \( W^* \)-subalgebra of \( \mathbb{M} \). Recall the group von Neumann algebra \( L(G) \) defined by

\[
L(G) = \overline{\mathbb{C}[G]}^\vee .
\]
Since every element $y$ in $\mathbb{M}_G$ has its Fourier expansion $y = \sum_{g \in G} t_g u_g$ and since every element in $L(G)$ has its Fourier expansion $\sum_{g \in G} r_g u_g$, there exists a $*$-isomorphism, which is the generator-preserving linear map, between $\mathbb{M}_G$ and $L(G)$.

**Proposition 1.2.** Let $\mathbb{M}_G \equiv \mathbb{C} \cdot 1_M \times_\alpha G$ be the crossed product algebra. Then $\mathbb{M}_G = L(G)$. □

2. Moments and Cumulants on $(\mathbb{M}, E_M)$

In the previous section, we defined an amalgamated $W^*$-probability space for the given crossed product algebra $\mathbb{M} = M \times_\alpha G$. Throughout this chapter, we will let $M$ be a von Neumann algebra and $G$, a group and let $\alpha : G \to Aut M$ be an action of $G$ on $M$. We will compute the amalgamated moments and cumulants of operators in $M$. These computations will play a key role to get our main results (0.5), in Chapter 3. Let $(\mathbb{M}, E_M)$ be the $M$-valued crossed product probability space.

**Notation** From now, we denote $\alpha_g(m)$ by $m^g$, for convenience. □

Consider group von Neumann algebras $L(G)$, which are $*$-isomorphic to $\mathbb{M}_G = \mathbb{C} \times_\alpha G$, with its canonical trace $tr$ on it. On $L(G)$, we can always define its canonical trace $tr$ as follows,

\[ tr \left( \sum_{g \in G} r_g u_g \right) = r e_G, \text{ for all } \sum_{g \in G} r_g u_g \in L(G), \]

where $r_g \in \mathbb{C}$, for $g \in G$. So, the pair $(L(G), tr)$ is a $\mathbb{C}$-valued $W^*$-probability space. We can see that the unitary representations $\{ u_g \}_{g \in G}$ in $(\mathbb{M}, E)$ and $\{ u_g \}_{g \in G}$ in $(L(G), tr)$ are identically distributed.

By using the above new notation, we have

\[ (m_{g_1} u_{g_1})(m_{g_2} u_{g_2}) \cdots (m_{g_n} u_{g_n}) \]

\[ = (m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 g_2 \cdots g_{n-1}}) u_{g_1 \cdots g_n}, \]

for all $m_{g_j}, u_{g_j} \in \mathbb{M}, j = 1, ..., n$, where $n \in \mathbb{N}$. The following lemma shows us that a certain collection of $M$-valued random variables in $(\mathbb{M}, E_M)$ and the generators of group von Neumann algebra $(L(G), tr)$ are identically distributed (over $\mathbb{C}$).
Lemma 2.1. Let $u_{g_1}, \ldots, u_{g_n} \in \mathbb{M}$ (i.e., $u_{g_k} = 1_M \cdot u_{g_k}$ in $\mathbb{M}$, for $k = 1, \ldots, n$). Then

\begin{equation}
E_M (u_{g_1} \ldots u_{g_n}) = tr (u_{g_1} \ldots u_{g_n}) \cdot 1_M,
\end{equation}

where $tr$ is the canonical trace on the group von Neumann algebra $L(G)$.

Proof. By definition of $E_M$,

\begin{align*}
E_M (u_{g_1} \ldots u_{g_n}) &= E_M (\left(1_M \cdot 1^g_1 \cdot 1^g_2 \cdot \ldots \cdot 1^g_{n-1}\right) u_{g_1} \ldots u_{g_n}) \\
&= E_M (u_{g_1} \ldots u_{g_n}) \quad \text{since } 1^g_M = u_g 1_M u_g^{-1} = u_g u_g^{-1} = u_{e G} = 1_M, \quad \text{for all } g \in G \\
&= \begin{cases}
1_M & \text{if } g_1 \ldots g_n = e_G \\
0_M & \text{otherwise},
\end{cases}
\end{align*}

for all $n \in \mathbb{N}$. By definition of $tr$ on $L(G)$, we have that

\begin{equation*}
tr (u_{g_1} \ldots u_{g_n}) = tr (u_{g_1} \ldots u_{g_n}) = \begin{cases}
1 & \text{if } g_1 \ldots g_n = e_G \\
0 & \text{otherwise},
\end{cases}
\end{equation*}

for all $n \in \mathbb{N}$.

We want to compute the $M$-valued cumulant $k_n^{E_M} (m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n})$, for all $m_{g_k} u_{g_k} \in \mathbb{M}$ and $n \in \mathbb{N}$. If this $M$-valued cumulant has a “good” relation with the cumulant $k_n^{tr} (u_{g_1}, \ldots, u_{g_n})$, then we might find the relation between a group free product in $G$ and $M$-valued free product in $\mathbb{M}$. The following three lemmas are the preparation for computing the $M$-valued cumulant $k_n^{E_M} (m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n})$.

Lemma 2.2. Let $(\mathbb{M}, E_M)$ be the $M$-valued crossed product probability space and let $m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n}$ be $M$-valued random variables in $(\mathbb{M}, E_M)$, for $n \in \mathbb{N}$. Then

\begin{equation}
E_M (m_{g_1} u_{g_1} \ldots m_{g_n} u_{g_n}) = \begin{cases}
m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \ldots m_{g_n}^{g_1 \ldots g_{n-1}} & \text{if } g_1 \ldots g_n = e_G \\
0_M & \text{otherwise},
\end{cases}
\end{equation}

in $\mathbb{M}$.

Proof. By the straightforward computation, we can get that

\begin{equation*}
E_M (m_{g_1} u_{g_1} \ldots m_{g_n} u_{g_n})
\end{equation*}
\[
E_M \left( m_1 g_1^1 m_2 g_2^2 ... m_n g_n^{n-1} \cdot u_{g_1} u_{g_2} ... u_{g_n} \right)
\]

by \((0.2)\)

\[
= E_M \left( (m_1 g_1^1 m_2 g_2^2 ... m_n g_n^{n-1}) u_{g_1} ... u_{g_n} \right)
\]

\[
= (m_1 g_1^1 m_2 g_2^2 ... m_n g_n^{n-1}) E_M (u_{g_1} ... u_{g_n})
\]

since \(E_M : \mathbb{M} \rightarrow M = M \times_t < e_G > \) is a conditional expectation.

\[
= \begin{cases} 
  m_1 g_1^1 m_2 g_2^2 ... m_n g_n^{n-1} & \text{if } g_1 ... g_n = e_G \\
  0_M & \text{otherwise,}
\end{cases}
\]

by the previous lemma. \(\blacksquare\)

Based on the previous lemma, we will compute the partition-depending moments of \(M\)-valued random variables. But first, we need the following observation.

**Lemma 2.3.** Let \(mu_g \in (\mathbb{M}, E_M)\) be a \(M\)-valued random variable. Then \(E_M (u_g m) = m^g E_M (u_g)\).

**Proof.** Compute

\[
E_M (u_g m) = E_M (u_g m u_g -1 u_g) = E_M (m^g u_g) = m^g E_M (u_g).
\]

Since \(E_M\) is a conditional expectation, \(E_M (u_g m) = E_M (u_g) m\), too. So, by the previous lemma, we have that

\[(2.6) \quad E_M (u_g) m = E (u_g m) = m^g E (u_g) . \]

In the following lemma, we will extend this observation \((2.6)\) to the general case. Notice that since \(E_M\) is a \(M\)-valued conditional expectation, we have to consider the insertion property (See [19]). i.e., in general,

\[
E_{M,\pi} (x_1, ..., x_n) \neq \prod_{V \in \pi} E_{M,V} (x_1, ..., x_n),
\]

for \(x_1, ..., x_n \in \mathbb{M}\), where \(E_{M,V}(\ldots)\) is the block-depending moments. But, if \(x_k = u_{g_k} = 1_M \cdot u_{g_k}\) in \(\mathbb{M}\), then we can have that

\[
E_{M,\pi} (u_{g_1}, ..., u_{g_n}) = \prod_{B \in \pi} E_{M,V} (u_{g_1}, ..., u_{g_n}).
\]
are done, by (2.3) and (2.4). Assume that \( \pi \)

**Proof.** If we have a partition \( \pi \) be the \( V \).

Remark that it is possible there are several innerest blocks in a certain noncrossing partition. Also, we say that \( V \) is innerest if there is no other block inner in \( V \). For instance, if we have a partition

\[
\pi = \{(1,6), (2,5), (3,4)\} \text{ in } NC(6).
\]

Then the block (2, 5) is inner in the block (1, 6) and the block (3, 4) is inner in the block (2, 5). Clearly, the block (3, 4) is inner in both (2, 5) and (1, 6), and there is no other block inner in (3, 4). So, the block (3, 4) is an innerest block in \( \pi \).

Remark that it is possible there are several innerest blocks in a certain noncrossing partition. Also, notice that if \( V \) is an innerest block, then there exists \( j \) such that \( V = (j, j + 1, ..., j + |V| - 1) \), where \( |V| \) means the cardinality of entries of \( V \).

**Lemma 2.4.** Let \( n \in \mathbb{N} \) and \( \pi \in NC(n) \), and let \( m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n} \in (M, E) \) be the \( M \)-valued random variables. Then

\[
E_{M,\pi}(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n})
\]

(2.7) 

\[= (m_{g_1} m_{g_2} \ldots m_{g_{n-1}}) \text{tr}_\pi(u_{g_1}, \ldots, u_{g_n}),\]

where \( \text{tr} \) is the canonical trace on the group von Neumann algebra \( L(G) \).

**Proof.** If \( \pi = 1_n \), then \( E_{M,1_n}(\ldots) = E_M(\ldots) \) and \( \text{tr}_{1_n}(\ldots) = \text{tr}(\ldots) \), and hence we are done, by (2.3) and (2.4). Assume that \( \pi \neq 1_n \) in \( NC(n) \). Assume that \( V = (j, j + 1, ..., j + k) \) is an innerest block of \( \pi \). Then

\[
T_V \overset{def}{=} E_{M, V}(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n})
\]

\[= E_M(m_{g_1} u_{g_1} m_{g_{j+1}} u_{g_{j+1}} \ldots m_{g_{j+k}} u_{g_{j+k}})
\]

\[= (m_{g_1} ^{g_j} m_{g_{j+1}}^{g_{j+1}} \ldots m_{g_{j+k}}^{g_{j+k-1}}) \cdot \text{tr}(u_{g_j \ldots g_{j+k}}).
\]
Suppose \( V \) is inner in a block \( B \) of \( \pi \) and \( B \) is inner in all other blocks \( B' \) where \( V \) is inner in \( B' \). Let \( B = (i_1, \ldots, i_k) \) and assume that there is \( k_0 \in \{2, \ldots, k-1\} \) such that \( i_{k_0} < t < i_{k_0+1}, \) for all \( t = j, j+1, \ldots, j+k \). Then the \( B \)-depending moment goes to

\[
E_M \left( m_{g_{i_1} u_{g_{i_1}} \cdots m_{g_{i_k}} u_{g_{i_k}}} (TV) m_{g_{k_0+1} u_{g_{k_0+1}} \cdots m_{g_{i_k}} u_{g_{i_k}}} \right)
\]

\[
= E_M \left( m_{g_{i_1}} m_{g_{i_1} g_{j+1}} \cdots m_{g_{k_0}} \cdot m_{g_{j+1}} \cdots m_{u_{g_{i_k}}} \right) E_M \left( u_{g_{i_1} g_{j+1} \cdots g_{j+k} \cdots g_{i_k}} \right)
\]

By doing the above process for all block-depending moments in the \( \pi \)-depending moments, we can get that

\[
E_{M, \pi} \left( m_{g_1 u_{g_1}, \ldots, m_{g_n} u_{g_n}} \right)
\]

\[
= \left( m_{g_1} m_{g_2} m_{g_3} \cdots m_{g_{n-1}} \right) E_\pi \left( u_{g_1}, \ldots, u_{g_n} \right).
\]

By (2.3), we know \( E_\pi \left( u_{g_1}, \ldots, u_{g_n} \right) = tr_\pi (u_{g_1}, \ldots, u_{g_n}) \cdot 1_M \), where \( tr \) is the canonical trace on the group von Neumann algebra \( L(G) \).

By the previous lemmas and proposition, we have the following theorem.

**Theorem 2.5.** Let \( m_{g_1 u_{g_1}, \ldots, m_{g_n} u_{g_n}} \in (\mathbb{M}, E_M) \) be the \( M \)-valued random variables, for \( n \in \mathbb{N} \). Then

\[
k_n^{E_M} \left( m_{g_1 u_{g_1}, \ldots, m_{g_n} u_{g_n}} \right)
\]

\[
(2.8) \quad = \left( m_{g_1} m_{g_2} m_{g_3} \cdots m_{g_{n-1}} \right) k_n^{tr} \left( u_{g_1}, \ldots, u_{g_n} \right).
\]

**Proof.** Observe that

\[
k_n^M \left( m_{g_1 u_{g_1}, \ldots, m_{g_n} u_{g_n}} \right)
\]
\[
E_M, \pi \left( m_{g_1}, u_{g_1}, \ldots, m_{g_n}, u_{g_n} \right) \mu(\pi, 1_n)
\]
\[
= \sum_{\pi \in NC(n)} E_M, \pi \left( m_{g_1}, u_{g_1}, \ldots, m_{g_n}, u_{g_n} \right) \mu(\pi, 1_n)
\]
\[
= \sum_{\pi \in NC(n)} \left( \left( m_{g_1} m_{g_2} \ldots m_{g_n} \right) \right) \mu(\pi, 1_n)
\]
by (2.7)
\[
\left( m_{g_1} m_{g_2} m_{g_3} \ldots m_{g_n} \right) \sum_{\pi \in NC(n)} \mu(\pi, 1_n)
\]
\[
= \left( m_{g_1} m_{g_2} m_{g_3} \ldots m_{g_n} \right) k_n(\mu, 1_n).
\]

The above theorem shows us that there is close relation between the \( M \)-valued cumulant on \((\mathbb{M}, E_M)\) and \( \mathbb{C} \)-valued cumulant on \((L(G), tr)\).

**Example 2.1.** In this example, instead of using (2.7) directly, we will compute the \( \pi \)-depending moment of \( m_{g_1}, u_{g_1}, \ldots, m_{g_n}, u_{g_n} \) in \( \mathbb{M} \), only by using the simple computations (0.1) - (0.4). By doing this, we can understand why (2.7) holds concretely. Let \( \pi = \{ (1, 4), (2, 3), (5) \} \) in NC(5). Then

\[
E_M, \pi \left( m_{g_1}, u_{g_1}, \ldots, m_{g_n}, u_{g_n} \right)
\]
\[
= E_M \left( m_{g_1} u_{g_1} \right) E_M \left( m_{g_2} u_{g_2} m_{g_3} u_{g_3} \right) m_{g_4} u_{g_4} \right) E_M \left( m_{g_5} u_{g_5} \right)
\]
\[
= m_{g_1} E_M \left( u_{g_1} \right) E_M \left( m_{g_2} m_{g_3} u_{g_2} m_{g_3} u_{g_3} \right) m_{g_4} u_{g_4} \right) \left( m_{g_5} E_M \left( u_{g_5} \right) \right)
\]
\[
= m_{g_1} E_M \left( u_{g_1} \right) E_M \left( m_{g_2} m_{g_3} m_{g_4} u_{g_2} m_{g_3} u_{g_3} \right) m_{g_4} u_{g_4} \right) \left( m_{g_5} E_M \left( u_{g_5} \right) \right)
\]
\[
= m_{g_1} E_M \left( m_{g_2} m_{g_3} m_{g_4} u_{g_1} \right) E_M \left( u_{g_2} m_{g_3} u_{g_3} \right) m_{g_4} u_{g_4} \right) \left( m_{g_5} E_M \left( u_{g_5} \right) \right)
\]
\[
= m_{g_1} E_M \left( m_{g_2} m_{g_3} m_{g_4} u_{g_1} \right) m_{g_5} \left( E_M \left( u_{g_2} m_{g_3} u_{g_3} \right) \right) m_{g_4} u_{g_4} \right) \left( m_{g_5} E_M \left( u_{g_5} \right) \right)
\]
\[
= m_{g_1} m_{g_2} m_{g_3} m_{g_4} \left( u_{g_1} \right) E_M \left( u_{g_2} \right) \left( m_{g_3} \left( E_M \left( u_{g_3} \right) \right) \right) m_{g_4} \left( E_M \left( u_{g_4} \right) \right) \left( E_M \left( u_{g_5} \right) \right)
\]
\[
= m_{g_1} m_{g_2} m_{g_3} m_{g_4} \left( u_{g_1} \right) \left( E_M \left( u_{g_2} \right) \right) \left( E_M \left( u_{g_3} \right) \right) \left( E_M \left( u_{g_4} \right) \right) \left( E_M \left( u_{g_5} \right) \right)
\]
\[ E_M(u_{g_1}m_{g_2}m_{g_3}m_{g_4}E_M(u_{g_2}u_{g_3})u_{g_4})(E_M(u_{g_5})) \]

\[ = m_{g_1}m_{g_2}m_{g_3}m_{g_4}E_M(u_{g_1}m_{g_2}m_{g_3}u_{g_4})(E_M(u_{g_5})) \]

\[ = (m_{g_1}m_{g_2}m_{g_3}m_{g_4}m_{g_5})(E_M(u_{g_1}E_M(u_{g_2}u_{g_3})u_{g_4})(E_M(u_{g_5}))) \]

\[ = (m_{g_1}m_{g_2}m_{g_3}m_{g_4}m_{g_5}) (tr(u_{g_1}tr(u_{g_2}u_{g_3})u_{g_4}) (tr(u_{g_5}))) \]

\[ = (m_{g_1}m_{g_2}m_{g_3}m_{g_4}m_{g_5}) (tr_\pi(u_{g_1}, u_{g_2}, u_{g_3}, u_{g_4}, u_{g_5})). \]

**Example 2.2.** We can compute the following $M$-valued cumulant, by applying (2.8).

\[ k_3^{EM}(m_{g_1}u_{g_1}, m_{g_2}u_{g_2}, m_{g_3}u_{g_3}) = (m_{g_1}m_{g_2}m_{g_3}) \cdot k_3^{tr}(u_{g_1}, u_{g_2}, u_{g_3}) \]

\[ = (m_{g_1}m_{g_2}m_{g_3}) (tr(u_{g_1}), tr(u_{g_2}), tr(u_{g_3})) - tr(u_{g_1})tr(u_{g_2})tr(u_{g_3}) \]

\[ - tr(u_{g_1}tr(u_{g_2}, u_{g_3})). \]

3. **The Main Result (0.5)**

In this chapter, we will prove our main result (0.5). Like before, throughout this chapter, let $M$ be a von Neumann algebra and $G$, a group and let $\alpha : M \rightarrow AutM$ be an action of $G$ on $M$. Assume that a group $G$ is a group free product $G_1 \ast G_2$ of groups $G_1$ and $G_2$. (Also, we can assume that there is a subgroup $G_1 \ast G_2$ in the group $G$, and $M \times_\alpha (G_1 \ast G_2)$ is a $W^*$-subalgebra of $M$.) Recall that, by Voiculescu, it is well-known that

\[ L(G_1 \ast G_2) = L(G_1) \ast L(G_2), \]

where “$\ast$” in the left-hand side is the group free product and “$\ast$” in the right-hand side is the von Neumann algebra free product, where $L(K)$ is a group von Neumann algebra of an arbitrary group $K$. This says that the $C^*$-freeness on $(L(G))$, $tr$ is depending on the group freeness on $G = G_1 \ast G_2$. In other words, if the groups $G_1$ and $G_2$ are free in $G = G_1 \ast G_2$, then the group von Neumann algebras $L(G_1)$ and $L(G_2)$ are free in $(L(G), tr)$. Also, if two group von Neumann algebras $L(G_1)$ and $L(G_2)$ are given and if we construct the $C^*$-free product $L(G_1) \ast L(G_2)$ of them, with respect to the canonical trace $tr_G = tr_{G_1} \ast tr_{G_2}$, where $tr_{G_k}$ is the canonical trace on $L(G_k)$, for $k = 1, 2$, then this $C^*$-free product is $\ast$-isomorphic to
a group von Neumann algebra $L(G)$, where $G$ is the group free product $G_1 \ast G_2$ of $G_1$ and $G_2$.

**Theorem 3.1.** Let $M = M \times_\alpha G$ be a crossed product algebra, where $G = G_1 \ast G_2$ is the group free product of $G_1$ and $G_2$. Then

\[(3.1) \quad M = (M \times_\alpha G_1) \ast_M (M \times_\alpha G_2),\]

where “$\ast_M$” is the $M$-valued free product of von Neumann algebras.

**Proof.** Let $G = G_1 \ast G_2$ be the group free product of $G_1$ and $G_2$. By Chapter 1, the crossed product algebra $M$ has its $W^*$-subalgebra $M = M \times_\alpha \langle \epsilon_G \rangle$, where $\langle \epsilon_G \rangle$ is the trivial subgroup of $G$ generated by the group identity $\epsilon_G \in G$. Define the canonical conditional expectation $E_M : M \to M$ by

\[E_M \left( \sum_{g \in G} m_g u_g \right) = m_{\epsilon_G}, \quad \text{for all } \sum_{g \in G} m_g u_g \in M.\]

By (2.8), if $m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n} \in (M, E_M)$ are $M$-valued random variables, then

\[k_n^E \left( m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n} \right) = \left( m_{g_1} m_{g_1}^{g_2} m_{g_3}^{g_2} \ldots m_{g_n}^{g_2 \ldots g_{n-1}} \right) k_n^t \left( u_{g_1}, \ldots, u_{g_n} \right),\]

for all $n \in \mathbb{N}$, where $t^r$ is the canonical trace on $L(G)$. As we mentioned in the previous paragraph, the $\mathbb{C}$-freeness on $L(G)$ is completely determined by the group freeness of $G_1$ and $G_2$ on $G$ and vice versa. By the previous cumulant relation, the $M$-freeness on $M$ is totally determined by the $\mathbb{C}$-freeness on $L(G)$. Therefore, the $M$-freeness on $M$ is determined by the group freeness on $G$. Thus, we can conclude that

\[M \times_\alpha (G_1 \ast G_2) = (M \times_\alpha G_1) \ast_M (M \times_\alpha G_2).\]

If $F_N$ is the free group with $N$-generators, then $L(F_N) = \ast_{k=1}^N L(\mathbb{Z})_k$, where $L(\mathbb{Z})_k = L(\mathbb{Z})$, for all $k = 1, \ldots, N$. Also, $L(F_N) = L(F_{k_1}) \ast L(F_{k_2})$, for all $k_1, k_2 \in \mathbb{C}$ such that $k_1 + k_2 = N$.

**Corollary 3.2.** Let $F_N$ be the free group with $N$-generators, for $N \in \mathbb{N}$. Then

\[(3.2) \quad M \times_\alpha F_N = (M \times_\alpha \mathbb{Z}) \ast_M \ldots \ast_M (M \times_\alpha \mathbb{Z}) \\quad \text{(N-times)}\]

and

\[(3.3) \quad M \times_\alpha F_N = (M \times_\alpha F_{k_1}) \ast_M (M \times_\alpha F_{k_2}),\]
whenever \( k_1 + k_2 = N \), for \( k_1, k_2 \in \mathbb{N} \). □

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