A NEW DATA-DRIVEN ROBUST OPTIMIZATION APPROACH TO MULTI-ITEM NEWSBOY PROBLEMS

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Abstract. A newsboy problem is a typical stochastic inventory management problem and has extensive applications in the fields of operational research, management sciences and marketing sciences. One of the challenges underlying such problems is to handle the uncertainty of demands. In the existing results, it is often to assume that the demand distribution is given to facilitate solution of the problems. In this paper, a novel data-driven robust optimization model for solving multi-item newsboy problems is proposed by combining the absolute robust optimization with a data-driven uncertainty set, and the latter is leveraged to address the uncertainty of demands. For the single-item situation, a closed-form solution is obtained and influences of parameters on the optimal solutions are analyzed. Owing to complexity of the multi-item situation, a uniform smoothing function is leveraged to smooth the proposed model. Then, an algorithm, called a modified Frank-Wolfe feasible direction algorithm, is developed to solve a series of smooth subproblems. Numerical simulation demonstrates that the proposed model in this paper can reduce over-conservation of robust optimization methods and is more robust than other similar well-established methods in the literature. By numerical simulation and sensitivity analysis, it is concluded that: (1) The proposed method can provide more stable optimal order policy and profits than the existing ones; (2) For a product with a higher unit purchase price, the optimal order quantities are more sensitive to its change; (3) In view of profitability, the newsboy should not be too risk-averse.

1. Introduction.

1.1. Background. A newsboy problem is a typical stochastic inventory management problem, which needs to seek an optimal inventory or sales policy of perishable items with uncertain demands. Since this problem has extensive applications in the fields of operational research, management sciences and marketing sciences, it has attracted great attention from the researchers and practitioners in these fields [24,29].

One of the major challenges to study the newsboy problems lies in development of a nice method to hedge the uncertainty of demands such that an “optimal” inventory or sales policy is provided. In practice, it is really difficult to provide...
such an “optimal” policy since the decision must be made before one can observe real demands [16].

Previous research mainly captures the uncertain demand by assuming that the demand is a random variable conforming to a given distribution. Then, different variants of deterministic equivalent formulation (DEF) are derived to find an “optimal” policy with a certain meaning of optimality [11, 31]. Clearly, in practice, the demands of products may not be random variables, let alone their distributions are known. In this case, it is valuable to propose a new optimization approach to provide an “optimal” policy for the newsboy problems without the above mentioned assumption.

In this paper, we intend to remove the above inappropriate assumption by developing a new data-driven robust optimization model to solve the multi-item newsboy problems.

1.2. Related results and our research intention. The classical newsboy model was first proposed in 1951 by Arrow et al. [3]. Owing to its wide applications, a great number of extended models have been presented under assumption of given demand distribution in the past seven decades [8, 19, 34, 40]. Especially, for the recent research works on the newsboy problems, one can see [13, 17, 39] and the references therein.

Since we mainly concern the methods to treat the newsboy problems without assuming that the demand distribution is known, we now summarize the related results available in the literature in this connection.

Owing to uncertainty of demand, robust optimization is an efficient way to avoid the risk of decision making. Indeed, many robust optimization models have been built to study the newsboy problems. In [28], a max-min rule was first proposed to solve a random inventory problem, and only by the mean and standard deviation of demand, a closed-form robust solution of this problem was obtained. Vairaktaras [30] extended the model with three different robust methods, and presented a closed-form robust solution by using the bounds of demand intervals. Carrizosa et al. [7] presented a robust optimization model of newsboy problems with a p-order time series demand. Qiu et al. [25] proposed a distributionally robust multi-product inventory optimization model with budget constraints where the means and standard deviations of the uncertain demands and yields are known.

By forming different uncertainty sets from data of demands, another type of robust optimization models have been proposed to solve the newsboy problems. For example, a box uncertainty set [4], an ellipsoidal uncertainty set [5] and a polyhedral uncertainty set [5] have been proposed to capture the uncertainty of demands [18]. Based on interval, box, ellipsoidal, polyhedral uncertainty sets, Abdel-Aal and Selim [1] proposed different robust optimization models of selective newsboy problems, and developed a polynomial solution algorithm to solve the proposed mixed integer linear programming (MILP) models. Zhang et al. [41] presented a robust optimization model for substitutable multi-item newsboy problem under the assumption that demand and substitution are in box uncertainty sets, and proposed a branch and cut algorithm to solve MILP formulations of the proposed model.

With an improvement of data availability in the era of “Big Data”, data-driven approaches were also used to treat the uncertainty of demands in the newsboy problems. Sachs and Minner [27] presented a data-driven method to solve the newsboy problem by assuming the optimal inventory level is a linear function of temperature and price, and estimated the parameters directly from data. Levi et al. [20]
proposed a data-driven newsboy model based on the sample average approximation method such that the expected cost of newsboy is approximated by a function of the samplings of demand historical data. Huber et al. [16] presented a data-driven single-item newsboy model based on machine learning methods, where the relationship between optimal order quantities and features was described by the artificial neural network method. Punia et al. [23] extended the model proposed in [16] to the multi-item case and developed a machine learning-based quantile regression algorithm to obtain optimal solutions of newsboy problems. Support vector clustering was used in [26] to construct a data-driven uncertainty set of demand, then a robust optimization model of the inventory management problem was built. Cao and Shen [6] proposed a novel double parallel feedforward network, which was applied to quantile forecasting and determine the stock levels of newsboys. Xu et al. [35] presented a robust optimization model based on a data-driven distribution ambiguity set of demand. Chen et al. [9] proposed a mixture distribution-based uncertainty set to solve the chance constrained newsboy problems, then reformulated this problem in terms of convex problems based on the piecewise linear approximation technique.

Although there are many impressive data-driven models of newsboy problems, little of them combine the robust optimization approach and data-driven techniques to study the newsboy problems. Particularly, for the multi-item newsboy problems, no any related works handle the interaction between the demands by using data-driven techniques. It is noted that Ning and You [21] proposed a data-driven uncertainty set based on the principal component analysis and the kernel smoothing method. In this paper, we attempt to apply this method to solve the multi-item newsboy problems, and build a new robust optimization model of this problem based on a data-driven uncertainty set, which can take into account the interaction between different demands. Specifically, we will answer the following questions:

(1) How to construct a data-driven uncertainty set to describe uncertain demands of products in the multi-item newsboy problems? In particular, such a set can handle the interaction of demands between the different products.

(2) For multi-item newsboy problem, how to form a new data-driven robust optimization model such that it can have advantages of the existing robust optimization models?

(3) Owing to complexity of the multi-item situation, how to develop an efficient algorithm to solve the built model such that an optimal policy is found for the multi-item newsboy problems?

(4) What are the managerial implications from the proposed model in this paper? How to validate its advantages by comparative study?

To address these issues, we organize the rest of this paper as follows. In next section, a new data-driven robust optimization model of multi-item newsboy problems is constructed. Section 3 is devoted to analytical solution of single-item newsboy problems, and impacts of model parameters on optimal solutions are proved. In Section 4, we reformulate the proposed complicated model of multi-item newsboy problems such that an efficient algorithm is developed to find its numerical solution. To show advantages of our model, numerical simulation and comparative study among different methods are conducted in Section 5. In Section 6, sensitivity analysis is conducted and a number of managerial implications are revealed from the constructed model. Conclusions and future research directions are stated in Section 7.
2. Problem description and a new data-driven robust optimization model.
In this section, we first state what is a multi-item newsboy problem, then construct a new data-driven robust optimization model for this problem.

2.1. Problem description and its intrinsic difficulties. We consider a multi-item newsboy problem, where a supplier provides a variety of items to a retailer (called newsboy) who needs to determine order quantity of each item before starting a selling season in the case that the demands of items are all random variables with unknown distribution. Especially, we suppose that there are \( n \) items in total sold by the newsboy, which are sufficiently supplied by a supplier with fixed wholesale prices. For the newsboy, owing to randomness of demands, it is still often difficult to determine optimal order quantities of all the items to maximize the total retailing profit. Actually, the demand of any item may be greater or smaller than the order quantity in practice, thus the retailer does not know how to measure the total retailing profit: either being associated with shortage loss or holding cost. In a great number of articles available in the literature, it is first assumed that the distribution of all the random demands are given (see \([8,34,40]\)), the retailer determines optimal order quantities by minimizing the expectation of total retailing cost or maximizing the expectation of profit. Clearly, in a real-world situation, the distribution of demands is always difficult to capture for a decision-maker. In other words, the obtained order quantities by assuming the distribution of demands are known may be less valuable for the newsboy.

Note that the newsboy has often collected a lot of retailing data such as those on the market demands. Thus, without requirement of given demand distribution, it is an interesting issue to address how to make sufficient use of these data to provide the newsboy a more valuable policy to order items from the supplier. In this paper, we intend to investigate the above mentioned problem by developing a new optimization model.

2.2. New data-driven robust optimization model. Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) be the random demand vector of all the items. Let \( \mathbf{c} = (c_1, c_2, \ldots, c_n)^T \) represent the unit whole-sale price vector of items. Let \( \mathbf{v} = (v_1, v_2, \ldots, v_n)^T \) stand for the unit salvage value vector of items, \( \mathbf{p} = (p_1, p_2, \ldots, p_n)^T \) the unit shortage loss vector of items, and \( \mathbf{s} = (s_1, s_2, \ldots, s_n)^T \) the unit retail price vector of items.

Denote \( \mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)^T \) the decision variable vector of the newsboy, i.e., the order quantities of all the items. Then, the total cost is

\[
TC(\mathbf{Q}; \mathbf{x}) = (\mathbf{c} - \mathbf{v})^T \max \{0, \mathbf{Q} - \mathbf{x}\} + \mathbf{p}^T \max \{0, \mathbf{x} - \mathbf{Q}\} + \mathbf{c}^T \min \{\mathbf{Q}, \mathbf{x}\},
\]

(1)

where the first term is the holding cost, the second term is the shortage loss and the last term is the ordering cost. Consequently, the profit of the newsboy reads

\[
R(\mathbf{Q}; \mathbf{x}) = s^T \min \{\mathbf{Q}, \mathbf{x}\} - TC(\mathbf{Q}; \mathbf{x}),
\]

(2)

while the profit of Item \( i \) is

\[
R(Q_i; x_i) = s_i \min \{Q_i, x_i\} + v_i \max \{0, Q_i - x_i\} - p_i \max \{0, x_i - Q_i\} - c_i Q_i.
\]

By choosing an optimal vector \( \mathbf{Q} \), the goal of the newsboy is to maximize \( R(\mathbf{Q}; \mathbf{x}) \). Therefore, a multi-item newsboy problem is formulated as the following stochastic optimization model:

\[
\max_{\mathbf{Q}} R(\mathbf{Q}; \mathbf{x}) = s^T \min \{\mathbf{Q}, \mathbf{x}\} + \mathbf{v}^T \max \{0, \mathbf{Q} - \mathbf{x}\} - \mathbf{p}^T \max \{0, \mathbf{x} - \mathbf{Q}\} - \mathbf{c}^T \mathbf{Q}.
\]

(3)
Owing to randomness of demand, optimal order quantities cannot be obtained directly by solving Model (3). Many methods find the optimal order quantities by maximizing the expected profit $R(Q; x)$ [11, 42] under the assumption that the distributions of the demands are known. Without a given distribution of demands, robust optimization methods based on uncertainty sets are more applicable to deal with stochastic optimization problems than the expectation methods (see [28, 30]). However, these robust optimization models are often over-conservative in reality [16]. Since a method of data-driven uncertainty set recently proposed in [21] can handle relevance of the multi-item demands by principal component analysis, and give the bounds of the demands by kernel density estimation technique, we attempt to extend such a method to treat the newsboy problem (3) in this paper, which is different from the existing models available in the literature [28, 30, 38].

Specifically, let $\mu_0$ be the mean vector of the demand data $\mathbf{C}X = [x(1), x(2), \ldots, x(N)]^T$ collected by the newsboy, i.e.,

$$\mu_0 = \frac{1}{N} \sum_{j=1}^{N} x^{(j)}.$$ 

Denote

$$x_0^{(j)} = x^{(j)} - e\mu_0,$$

and $x_0 = [x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)}]^T$. Then, the sample covariance matrix is:

$$S = \frac{1}{N-1} x_0^T x_0.$$

For $S$, let $P = [p_1, \ldots, p_n] \in \mathbb{R}^{n \times n}$ represent a matrix consisting of $n$ eigenvectors corresponding to the principal components, i.e.,

$$S = PKP^T,$$

where $K = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ is a diagonal matrix defined by all the eigenvalues with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Denote $t_i^{(j)}$ the parameter of $j$-th uncertain demand data projected to the $i$-th principal component. Then, it is calculated by

$$t_i^{(j)} = p_i^T (x^{(j)} - \mu_0).$$

Denote $\xi_i$ the latent uncertainty of uncertain demand $x$ along the $i$-th principal component. Let $F_{KDE}^{(i)}(\xi_i)$ be the cumulative density function of the uncertain parameter $\xi_i$ with the following probability density function:

$$f_{KDE}^{(i)}(\xi_i) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\sqrt{2\pi h}} \exp \left(-\frac{(\xi_i - t_i^{(j)})^2}{2h^2}\right).$$

For a fixed $0 < \alpha < 1$, let $(1 - 2\alpha)$ be the confidence level of the uncertainty set. We define

$$\begin{align*}
F_{KDE}^{(i)}(\alpha)^{-1} &= \min \left\{ \xi_i \in \mathbb{R} | F_{KDE}^{(i)}(\xi_i) \geq \alpha \right\}, \\
F_{KDE}^{(i)}(1 - \alpha)^{-1} &= \min \left\{ \xi_i \in \mathbb{R} | F_{KDE}^{(i)}(\xi_i) \geq 1 - \alpha \right\}.
\end{align*}$$

Let $z^-$ represent the backward deviation vector, $z^+$ the forward deviation vector, $e$ a column vector of all ones and $B$ an uncertainty budget. Then, with the above
data analysis, we now approximate the unknown random demands by the following uncertainty set:

\[ X_{KDE + PCA} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in \mathbb{R}^n \\
\mathbf{x} = \mathbf{\mu}_0 + \mathbf{P}\mathbf{\xi}, \mathbf{\xi} = \mathbf{\xi} \circ \mathbf{z}^- + \mathbf{\xi} \circ \mathbf{z}^+, \\
0 \leq \mathbf{z}^-, \mathbf{z}^+ \leq \mathbf{e}, \mathbf{z}^- + \mathbf{z}^+ \leq \mathbf{e}, \mathbf{e}^T(\mathbf{z}^- + \mathbf{z}^+) \leq \mathbf{B}, \\
\xi = \left( F_{KDE}^{(1)}(\alpha), F_{KDE}^{(2)}(\alpha), ..., F_{KDE}^{(n)}(\alpha) \right)^T, \\
\mathcal{F} = \left( F_{KDE}^{(1)}(1 - \alpha), F_{KDE}^{(2)}(1 - \alpha), ..., F_{KDE}^{(n)}(1 - \alpha) \right)^T \end{cases} \]  

Consequently, we obtain a new data-driven robust optimization model of Problem (3) as follows

\[
\max_{\mathbf{Q}} \min_{x \in X_{KDE + PCA}} R(\mathbf{Q}; \mathbf{x}) = s^T \min \{ \mathbf{Q}, \mathbf{x} \} + \mathbf{v}^T \max \{ 0, \mathbf{Q} - \mathbf{x} \} \\
- \mathbf{p}^T \max \{ 0, \mathbf{x} - \mathbf{Q} \} - \mathbf{c}^T \mathbf{Q}.
\]

Remark 1. Compared with the models built in [30], our model (5) combines the absolute robust optimization with the data-driven uncertainty set (4) in which principal component analysis can handle the correlation between the demands of products while kernel density estimation method is used to estimate the bounds of the uncertainty set through data.

2.3. Properties of the data-driven robust optimization model.

Proposition 1. In Model (5), define

\[ f(\mathbf{x}) = \min_{x \in X_{KDE + PCA}} R(\mathbf{Q}; \mathbf{x}). \]

Then, (1) \( f \) is continuous in \( \mathbf{x} \); (2) \( f \) is not differentiable at \( x_i = Q_i \); (3) \( f \) is nonconvex.

Proof. From

\[
\lim_{x_i \to Q_i^+} R(Q_i; x_i) = \lim_{x_i \to Q_i^-} R(Q_i; x_i) = (s_i - c_i)Q_i,
\]

it follows that \( R(Q_i; x_i) \) is continuous in \( x_i \) at the point \( x_i = Q_i \). When \( x_i \neq Q_i \), it is easy to see that \( f \) is continuous. We have proved the first result. Since

\[
\lim_{x_i \to Q_i^+} \frac{\partial R(Q_i; x_i)}{\partial x_i} = -p_i,
\]

and

\[
\lim_{x_i \to Q_i^-} \frac{\partial R(Q_i; x_i)}{\partial x_i} = s_i - v_i,
\]

we know

\[
\lim_{x_i \to Q_i^+} \frac{\partial R(Q_i; x_i)}{\partial x_i} \neq \lim_{x_i \to Q_i^-} \frac{\partial R(Q_i; x_i)}{\partial x_i}.
\]

Actually, \( s_i - v_i > 0 \) and \( -p_i < 0 \). By definition, we conclude that the objective function \( f \) is nondifferentiable in \( x_i \), i.e., the second result holds.

For any \( \mathbf{x}, \mathbf{y} \in X \) and \( 0 < \alpha < 1 \), denote \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_n)^T \). In the case that \( (x_i - Q_i)(y_i - Q_i) \geq 0, i = 1, 2, \ldots, n \), it is easy to see that

\[
R(Q_i; \alpha x_i + (1 - \alpha)y_i) = \alpha R(Q_i; x_i) + (1 - \alpha)R(Q_i; y_i).
\]
In the case that \((x_i - Q_i)(y_i - Q_i) < 0\), without loss of generality, we suppose that \(x_i < Q_i\) and \(y_i > Q_i\), then
\[
R(Q_i; \alpha x_i + (1 - \alpha)y_i) - \alpha R(Q_i; x_i) - (1 - \alpha)R(Q_i; y_i) = \alpha(s_i - v_i + p_i)(Q_i - x_i), \tag{7}
\]
if \(\alpha x_i + (1 - \alpha)y_i > Q_i\) is satisfied. Consequently, owing to \(x_i < Q_i\), it follows that
\[
R(Q_i; \alpha x_i + (1 - \alpha)y_i) > \alpha R(Q_i; x_i) + (1 - \alpha)R(Q_i; y_i).
\]
If \(\alpha x_i + (1 - \alpha)y_i < Q_i\), then
\[
R(Q_i; \alpha x_i + (1 - \alpha)y_i) - \alpha R(Q_i; x_i) - (1 - \alpha)R(Q_i; y_i) = (1 - \alpha)(s_i - v_i + p_i)(y_i - Q_i).
\]
Since \(y_i > Q_i\), we also get
\[
R(Q_i; \alpha x_i + (1 - \alpha)y_i) > \alpha R(Q_i; x_i) + (1 - \alpha)R(Q_i; y_i).
\]
Summarily, by the definition of convexity, we have proved that \(f\) is a nonconvex function. The third result has been proved.

**Remark 2.** Model (5) is a nonsmooth and nonconvex optimization model. The nonsmooth property of (5) leads to two major difficulties to find optimal solutions. Firstly, there are nondifferentiable points of objective function, most existing gradient algorithms cannot solve this optimization problem directly. Secondly, since the gradient information is not available, it is difficult to find a descending direction. The nonconvexity of (5) leads to the existence of multiple local optimal solutions in feasible domain.

In previous research, for nonconvex optimization, Fuduli et al. [14] extended the classical cutting plane algorithm to solve an unconstrained nonconvex optimization model. Clarke generalized gradient was used to deal with nonsmooth optimization problems in [22, 33]. A smoothing approximation for nonsmooth and nonconvex optimization problems was proposed in [10] by using the function \((\cdot)^+\), where an approximation of \(\max\{x_1, x_2\}\) was written as (9). Similarly, Yang et al. [36] proposed a block successive convex approximation by the following function:
\[
\Phi_\varepsilon(x_1, x_2) = \varepsilon \ln \left( \exp \left( \frac{x_1}{\varepsilon} \right) + \exp \left( \frac{-x_2}{\varepsilon} \right) \right). \tag{9}
\]
In this paper, for the nonsmooth and nonconvex problem (5), we will build a uniform smooth approximation of the objective function.

3. **Closed-form optimal solution in single-item case.** In this section, we will construct a data-driven robust optimization model for the single-item newsboy problem, and derive its closed-form solution.

Since no correlation between items needs to be considered in the single-item case, it is unnecessary to use the principal component analysis (PCA) method to determine the uncertainty set. Instead, we construct a data-driven uncertainty set of the random demand by the kernel density estimation method based on the collected historical demand data.

Let \(\mathbf{X} = \{x^{(1)}, \ldots, x^{(N)}\}\) be the data (samples) of demands collected by the newsboy. Then, the mean of these samples is
\[
\mu_0 = \frac{1}{N} \sum_{j=1}^{N} x^{(j)},
\]
and the deviation of all the data from the mean $\mu_0$ is $t_j = x^{(j)} - \mu_0$. Consequently, by kernel density estimation method, we get an estimated probability density function of the random demand $x$, specified by

$$f_{KDE}(x) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\sqrt{2\pi h}} \exp \left( -\frac{(x - t_j)^2}{2h^2} \right),$$

(10)

where $h$ is the bandwidth parameter of the Gaussian kernel density function. Let $F_{KDE}(x)$ be the cumulative density function of $x$. Then, the $\alpha$-quantile of the random demand is calculated by

$$F_{KDE}^{-1}(\alpha) = \min \{ x \in \mathbb{R} | F_{KDE}(x) \geq \alpha \} , \forall 0 \leq \alpha \leq 0.5.$$  

Thus, for a given confidence level $1 - 2\alpha$, $\underline{x} = \mu_0 + F_{KDE}^{-1}(\alpha)$ and $\bar{x} = \mu_0 + F_{KDE}^{-1}(1 - \alpha)$ define the lower and upper bounds of the demand $x$, respectively. In other words, the demand $x$ can be approximated by the following deterministic set:

$$X_{KDE} = \left\{ x \right\} = \mu_0 + F_{KDE}^{-1}(\alpha) \cdot z_1 + F_{KDE}^{-1}(1 - \alpha) \cdot z_2, \quad 0 \leq z_1 \leq 1, 0 \leq z_2 \leq 1, z_1 + z_2 \leq 1.$$  

(11)

We call $X_{KDE}$ in (11) a data-driven uncertainty set, where $z_1$ and $z_2$ represent the backward and forward deviations, respectively. Intuitively, with an increasing value of $1 - 2\alpha$, $X_{KDE}$ can contain more elements in the dataset of demands such that $X_{KDE}$ has stronger conservative property.

Remark 3. Since $\underline{x}$ and $\bar{x}$ in the definition of $X_{KDE}$ are asymmetric in general, the proposed data-driven uncertainty set in this paper is different from the box uncertainty set in [5].

In virtue of $X_{KDE}$ in (11), we now construct a data-driven robust optimization model to formulate the single-item newsboy problem.

Let $s$ stand for the unit selling price of the item, let $c$ be the unit purchase price of the item, $v$ the unit salvage value of the item and $p$ the unit shortage loss of the item. Similar to (2), $Q$ is the order quantity of the item and $x$ is the random demand of the item, then the profit function of single-item newsboy problems reads

$$R(Q;x) = \left\{ \begin{array}{ll} \ (s - c)Q - p(x - Q), & \text{if } x \geq Q; \\ sx - cQ + v(Q - x), & \text{if } x < Q. \end{array} \right.$$  

(12)

The so-called data-driven robust optimization model is given by

$$\max_Q \min_{x \in X_{KDE}} R(Q;x).$$  

(13)

It can be seen that the uncertainty set $X_{KDE}$ plays a critical role in guaranteeing robustness of the optimal solution of (13).

We are in a position to study a closed-form solution of Model (13).

As done in [7], if both of the shortage loss and the salvage value are not taken into consideration, then

$$\underline{x} = \mu_0 + F_{KDE}^{-1}(\alpha)$$  

solves Model (13). Otherwise, the following inner optimization problem:

$$\min_{x \in X_{KDE}} R(Q;x),$$  

(14)

is a nonlinear optimization model with linear constraints and a piecewise linear objective function.
Note that $R(Q; x)$ attains its minimum value when $x$ is the lower bound or the upper bound of $X_{KDE}$ in the case $Q \in X_{KDE}$. If $Q > x$, then $R(Q; x)$ achieves its minimum value at $x = x$. If $Q < x$, then the minimizer of $R(Q; x)$ is $x = x$. Specifically, we have

$$
\min_{X_{KDE}} R(Q; x) = \begin{cases} 
(s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)), & \text{if } Q < x; \\
\min \left\{ (s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)), (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c) \right\}, & \text{if } Q \in X_{KDE}; \\
(s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c), & \text{if } Q > x.
\end{cases}
$$

(15)

Denote

$$Q_\epsilon = \mu_0 + \frac{s - v}{s + p - v} F_{KDE}^{-1}(\alpha) + \frac{p}{s + p - v} F_{KDE}^{-1}(1 - \alpha).$$

Since

$$0 \leq \frac{s - v}{s + p - v} \leq 1, 0 \leq \frac{p}{s + p - v} \leq 1, 0 \leq \frac{s - v}{s + p - v} + \frac{p}{s + p - v} \leq 1,$$

it holds that $Q_\epsilon \in X_{KDE}$, where we take $z_1 = \frac{s - v}{s + p - v}$ and $z_2 = \frac{p}{s + p - v}$ in $X_{KDE}$.

If $Q = Q_\epsilon \in X_{KDE}$, it is easy to prove that

$$(s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)) = (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c).$$

Clearly, if $Q \in X_{KDE}$ and $Q \leq Q_\epsilon$, then

$$\min \{(s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)), (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c)\} = (s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)).$$

If $Q \in X_{KDE}$ and $Q \geq Q_\epsilon$, then

$$\min \{(s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)), (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c)\} = (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c).$$

Thus, (15) is simplified as

$$\min_{X_{KDE}} R(Q; x) = \begin{cases} 
(s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)), & \text{if } Q \leq Q_\epsilon; \\
(s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c), & \text{if } Q > Q_\epsilon.
\end{cases}
$$

(16)

Consequently, the solution of Model (13), i.e., the optimal order quantity in the single-item case is given by

$$Q^* = Q_\epsilon = \mu_0 + \frac{s - v}{s + p - v} F_{KDE}^{-1}(\alpha) + \frac{p}{s + p - v} F_{KDE}^{-1}(1 - \alpha).$$

(17)

The corresponding maximal profit is

$$R^* = (s - c)\mu_0 + \frac{(s - v)(s + p - c)}{s + p - v} F_{KDE}^{-1}(\alpha) + \frac{p(v - c)}{s + p - v} F_{KDE}^{-1}(1 - \alpha).$$

(18)

From (17) and (18), we can prove the following results.

**Proposition 2.** For an arbitrarily chosen confidence level $1 - 2\alpha$, the following statements are true.

1. As the unit selling price increases, the optimal order quantity of the single-item newsboy problems decreases, but the maximal profit increases.

2. As the unit shortage loss or the unit salvage value increases, the optimal order quantity of the single-item newsboy problems increases, but the maximal profit decreases with an increasing shortage loss (or a decreasing salvage value).
(3) An increasing unit purchase price does not affect the optimal order quantity, but causes a decreasing profit.

Proof. We prove the first result as follows.
Let \( Q^* \) and \( R^* \) be defined by (17) and (18), respectively. Then,

\[
\frac{\partial Q^*}{\partial s} = \frac{p}{(s + p - v)^2} \left( F_{KDE}^{-1}(\alpha) - F_{KDE}^{-1}(1 - \alpha) \right).
\]

Clearly, \( F_{KDE}^{-1}(\alpha) < F_{KDE}^{-1}(1 - \alpha) \). Therefore, \( \frac{\partial Q^*}{\partial s} < 0 \). It says that the optimal order quantity of the single-item newsboy problems decreases with an increasing unit selling price.

On the other hand,

\[
\frac{\partial R^*}{\partial s} = \frac{(s + p - v)(s - v)}{(s + p - v)^2} \left( \mu_0 + F_{KDE}^{-1}(\alpha) \right) + \frac{p(s + p - c)}{(s + p - v)^2} \left( \mu_0 + F_{KDE}^{-1}(\alpha) \right)
+ \frac{p(c - v)}{(s + p - v)^2} \left( \mu_0 + F_{KDE}^{-1}(1 - \alpha) \right),
\]

where \( \mu_0 + F_{KDE}^{-1}(\alpha) \) and \( \mu_0 + F_{KDE}^{-1}(1 - \alpha) \) represent the lower bound and upper bounds of the random demand, respectively. Thus, both of them are greater than zero. Consequently, \( \frac{\partial R^*}{\partial s} > 0 \). That is to say, the maximal profit increases if the unit selling price increases.

We have completed the proof of the result (1). In a similar fashion, we can prove the second and third results. \( \square \)

Proposition 3. For the single-item newsboy problems, the following statements are true:

1. With an increasing confidence level \( 1 - 2\alpha \) of uncertainty set, whether the optimal order quantity changes or not depends on the values of \( \alpha \) and the values of the other model parameters.
2. The maximal profit always decreases as the confidence level increases.

Proof. We define two lines

\[
l_1 : R = (s - c + p)Q - p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)),
\]

and

\[
l_2 : R = (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) + Q(v - c).
\]

As shown in Figure 1, \( b_1(\alpha) = -p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)) \) and \( b_2(\alpha) = (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) \) are the intercepts of \( l_1 \) and \( l_2 \), respectively.

For \( \alpha_1 > \alpha \), the two lines \( l_1 \) and \( l_2 \) have the same slopes, but their intercepts change.

Denote \( \Delta \alpha = \alpha_1 - \alpha \). Then, the changes of the two intercepts are

\[
\begin{align*}
\Delta b_1 &= \left[ -p(\mu_0 + F_{KDE}^{-1}(1 - \alpha)) + p(\mu_0 + F_{KDE}^{-1}(1 - \alpha - \Delta \alpha)) \right], \\
\Delta b_2 &= \left[ (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha)) - (s - v)(\mu_0 + F_{KDE}^{-1}(\alpha + \Delta \alpha)) \right].
\end{align*}
\]

As indicated in Figure 1, the intersection points of \( l_1 \) and \( l_2 \) is exactly the optimal solution of Model (13), where the horizontal ordinate of the intersection point is the optimal order quantity and its vertical ordinate is the maximal profit. Thus, with regard to impacts of the confidence level on the optimal solution and the maximal profit, we conclude that:

1. If the confidence level and the other model parameters are chosen such that \( \Delta b_1 = \Delta b_2 \) holds, then the optimal order quantity remains unchanged (see Figure
Figure 1. Impacts of confidence level on order quantities and profits

1(a)). Otherwise, in the case that $\Delta b_1 < \Delta b_2$, it is concluded that the optimal order quantity increases when the confidence level decreases (see Figure 1(b)). In the case that $\Delta b_1 > \Delta b_2$, the optimal order quantity increases when the confidence level increases (see Figure 1(c)).

(2) As the confidence level of the uncertainty set increases, the maximal profit always decreases.

We have completed the proof of Proposition 3.

Remark 4. By Proposition 2, it is seen that there exist a number of differences between the classical newsboy model (see Model (32) in Section 5.1) and the proposed data-driven robust optimization model (13) in this paper. For example, when the unit selling price decreases, the optimal order quantity derived from Model (32) decreases, which contradicts the first result in Proposition 2. More importantly, different from the classical newsboy model, Model (13) can directly provide an optimal order quantity and its corresponding profit by mining the demand data, rather than assuming that the distribution function of the random demand is known in prior. Clearly, in the real-world situation, the distribution of demand is often unknown and is even time-changing [16].

Remark 5. Comparison between the proposed Model (13) and the absolute robust optimization model based on an interval demand (see Model (33) in Section 5.1) demonstrates that the optimal order quantity and the maximal profit from either
Model (13) or Model (33) have the same monotonicity with respect to the fixed parameters of items (for example, the unit shortage loss). However, in the multi-item situation, we need to use the principal component analysis method to handle the correlation between items such that Model (5) can provide a more realistic and valuable solution than that with assumption of interval demands (also see Figure 2(a) in Section 5.2).

**Remark 6.** It is noted that Model (13) is constructed and analyzed in the simplest single-item situation. In this case, the objective function of Model (13) is a univariate piecewise function which has been employed to directly derive its closed-form solution. For a multi-item newsboy problem, it is often difficult to conduct similar analysis owing to complicacy underlying the constructed model, such as the difficulties in determining the uncertainty set by mining the dataset of demands. We will further address the multi-item newsboy problems with a uniform smooth approximation function in next section.

4. Reformulation of model in multi-item case and development of algorithm. In this section, to treat the complicated newsboy model (5) in the multi-item case, we intend to reformulate it as a sequence of smooth optimization problems such that it can be solved by the subsequently developed algorithm.

4.1. **Reformulation of model in multi-item case.** By Proposition 1, the profit function (2) is nonconvex and nondifferentiable. Thus, it is valuable to find a way to transform Model (5) as a sequence of ordinary smooth optimization problems.

Since
\[
\max \{a, b\} = \frac{a + b}{2} + \frac{|a - b|}{2}, \quad \min \{a, b\} = \frac{a + b}{2} - \frac{|a - b|}{2},
\]
(19)

Model (5) can be rewritten as
\[
\max_Q \min_{x \in X_{KDE+PCA}} R_1(Q; x) = x^T Q + \frac{(s - v - c)Q + (s - v + p) - (s - v - c)Q - (s - v + p)}{2} ||x - Q||^2.
\]
(20)
The following result provides a method to approximate the absolute value function in (20).

**Proposition 4.** For the absolute value function \( \phi(t) = |t| \), define
\[
\Phi_\epsilon(t) = \epsilon \ln \left( \frac{1}{2} \exp \left( \frac{t}{\epsilon} \right) + \frac{1}{2} \exp \left( -\frac{t}{\epsilon} \right) \right), \quad t \in \mathbb{R}.
\]
(21)

Then, (1) the function \( \Phi_\epsilon \) is continuously differentiable, and
\[
\Phi_\epsilon(0) = 0, \quad \left| \frac{d\Phi_\epsilon(t)}{dt} \right| < 1.
\]
(2) As \( \epsilon \to 0^+ \), \( \Phi_\epsilon(t) \to \phi(t) \).
(3) For any \( \epsilon > 0 \) small enough, \( 0 \leq \phi(t) - \Phi_\epsilon(t) \leq \epsilon \ln 2 \).

**Proof.** The proof of Proposition 4 can be found in [37].

By Proposition 4, we replace \( |x_i - Q_i| \) in (20) with

\[
\Phi_\epsilon(x_i - Q_i) = \epsilon \ln \left( \frac{1}{2} \exp \left( \frac{x_i - Q_i}{\epsilon} \right) + \frac{1}{2} \exp \left( -\frac{x_i - Q_i}{\epsilon} \right) \right).
\]

Thus, Model (5) is approximated as a nonlinear smooth optimization problem:
\[
\max_Q \min_{x \in X_{KDE+PCA}} R_1(Q; x),
\]
where $R_1(Q; x)$ is specified by

$$
R_1(Q; x) = \sum_{i=1}^{n} \left( \frac{1}{2} (s_i + v_i - p_i - 2c_i)Q_i + \frac{1}{2} (s_i - v_i - p_i)x_i \right.
\left. - \frac{1}{2} (s_i - v_i + p_i)\varepsilon \ln \left( \frac{1}{2} \exp \left( \frac{x_i - Q_i}{\varepsilon} \right) + \frac{1}{2} \exp \left( \frac{Q_i - x_i}{\varepsilon} \right) \right) \right).
$$

(22)

If we replace $z^-$, $z^+$ and $P$ in the uncertainty set $X_{KDE+PCA}$ by

$$\begin{aligned}
\begin{cases}
z^- \triangleq a = (a_1, a_2, \ldots, a_n)^T,
\vspace{0.5em}
z^+ \triangleq b = (b_1, b_2, \ldots, b_n)^T,
\vspace{0.5em}
P = (k_{ij})_{n \times n}.
\end{cases}
\end{aligned}$$

Then, Model (5) is reformulated as

$$
\max_{Q} \min_{a, b} R_1(Q; x)
$$

s.t. 

$$
\begin{aligned}
\mu_0^{(1)} + k_{11}(\xi^{(1)}a_1 + \tilde{\xi}^{(1)}b_1) + k_{12}(\xi^{(2)}a_2 + \tilde{\xi}^{(2)}b_2) \\
\vdots \\
\mu_0^{(n)} + k_{n1}(\xi^{(1)}a_1 + \tilde{\xi}^{(1)}b_1) + k_{n2}(\xi^{(2)}a_2 + \tilde{\xi}^{(2)}b_2)
\end{aligned}
$$

$$x =
\begin{cases}
0 \leq a_i, b_i \leq 1, & i = 1, 2, \ldots, n, \\
a_i + b_i \leq 1, & i = 1, 2, \ldots, n, \\
\sum_{i=1}^{n} a_i + b_i \leq B,
\end{cases}
$$

Remark 7. It is easy to see that Model (23) is a max-min optimization problem with linear constraints only being associated with the decision variables $a_i$ and $b_i$, $i = 1, 2, \ldots, n$. In virtue of this property, we next develop an efficient algorithm to find its optimal solution of Model (23). Specifically, we consider Model (23) as a two-stage optimization model, and at the first stage, we only solve a nonconvex optimization model with linear constraints. At the second stage, we solve the following unconstrained convex optimization model:

$$
\max_{Q} R_1(Q; x).
$$

(24)

Proposition 5. For Item $i$, $i = 1, 2$, let $x_i^*$ be a solution of the first stage of Model (23). Then, the optimal order quantity $Q_i^*$ of Model (23) is

$$
Q_i^* = \varepsilon \ln \left( \frac{s_i + p_i - c_i}{c_i - v_i} \right) + x_i^*.
$$

Proof. In order to simplify the proof of this proposition, we first consider the second stage optimization Model (24). Let $x^*$ be the optimal solution of the first stage optimization. Then, Model (24) can be reformulated as

$$
\max_{Q} R_1(Q; x^*) = \min_{Q} (-R_1(Q; x^*)).
$$

(25)
It is noted that Model (25) is a convex optimization problem and the gradient of its objective function is calculated by

\[ \nabla Q(-R_1(Q;x^*)) = \begin{bmatrix} \frac{\partial R_1(Q;\mathbf{x}^*)}{\partial Q_1} \\ \vdots \\ \frac{\partial R_1(Q;\mathbf{x}^*)}{\partial Q_n} \end{bmatrix}, \]

where for any \( i \in \{1, 2, \ldots, n\} \),

\[
\frac{-\partial R_1(Q;\mathbf{x}^*)}{\partial Q_i} = -\frac{1}{2}(s_i + v_i + p_i - 2c_i) + \frac{1}{2}(s_i - v_i + p_i) - \exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right) \exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right).
\]

The first-order optimality condition of Model (25) yields

\[
(c_i - v_i) \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right) = (s_i + p_i - c_i) \exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right).
\]

From \( s_i > c_i > v_i, p_i > 0 \), we have

\[
Q_i^* = \varepsilon \ln\left(\frac{s_i + p_i - c_i}{c_i - v_i}\right) + x_i^*, \quad i = 1, 2, \ldots, n.
\]  

(26)

Denote

\[
G_{ij} = -\frac{\partial^2 R_1(Q;\mathbf{x}^*)}{\partial Q_i \partial Q_j}.
\]

Then,

\[
G_{ij} = \begin{cases} 
\frac{(s_i - v_i + p_i) \left(1 + \exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right)\right) - \exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right)}{2\varepsilon}, & \text{if } i = j; \\
0, & \text{if } i \neq j.
\end{cases}
\]

Since

\[
\begin{cases} 
s_i - v_i + p_i > 0, \quad i = 1, \ldots, n, \\
1 + \left(\frac{-\exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right)}{\exp\left(\frac{x_i^* - Q_i}{\varepsilon}\right) + \exp\left(-\frac{x_i^* - Q_i}{\varepsilon}\right)}\right)^2 > 1, \quad i = 1, \ldots, n,
\end{cases}
\]

we know that for all \( x_i \in R, G_{ii} > 0 \) are true. Consequently, the Hessian matrix of \(-R_1(Q;\mathbf{x}^*)\):

\[
\nabla^2 Q(-R_1(Q;\mathbf{x}^*)) = \begin{bmatrix} G_{11} & 0 & \cdots & 0 \\ 0 & G_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & G_{nn} \end{bmatrix}
\]  

(27)

is positive. Thus, by the necessary and sufficient second-order optimality conditions, we get the unique optimal solution of Model (24):

\[
Q_i^* = \frac{\varepsilon \ln\left(\frac{s_i + p_i - c_i}{c_i - v_i}\right)}{2} + x_i^*, \quad i = 1, 2.
\]  

(28)
The proof has been completed.

4.2. Development of an efficient algorithm. By the properties of Model (23), where the objective function is nonconvex and the constraints are linear, we state an efficient algorithm to solve this model by modifying the well-known Frank-Wolfe algorithm. Let $\tau_B$ be the proportion of demand data in the uncertainty set $X_{KDE+PCA}$ with different budgets $B$. Denote $g = (a_1, \ldots, a_n, b_1, \ldots, b_n)^T$ and

$$D = \{ g \in R^{2n} | \begin{array}{l} 0 \leq a_i, b_i \leq 1, \quad i = 1, 2, \ldots, n, \\ a_i + b_i \leq 1, \quad i = 1, 2, \ldots, n, \\ \sum_{i=1}^{n} a_i + b_i \leq B \end{array} \}.$$ 

the vector of decision variables and the feasible domain of Model (23), respectively. Clearly,

$$\nabla_g R_1(Q; x) = \begin{pmatrix} \frac{\partial R_1(Q; x)}{\partial x_1} & \ldots & \frac{\partial R_1(Q; x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial R_n(Q; x)}{\partial x_1} & \ldots & \frac{\partial R_n(Q; x)}{\partial x_n} \end{pmatrix},$$

where for any $i \in \{1, 2, \ldots, n\}$,

$$\frac{\partial R_1(Q; x)}{\partial x_i} = \frac{1}{2} (s_i - v_i - p_i) - \frac{1}{2} (s_i - v_i + p_i) = \frac{\exp\left(\frac{z_i - Q_i}{\varepsilon}\right) - \exp\left(\frac{Q_i - z_i}{\varepsilon}\right)}{\exp\left(\frac{z_i}{\varepsilon}\right) + \exp\left(\frac{Q_i}{\varepsilon}\right)}.$$

Then, we state an efficient algorithm to solve Model (23) as follows.

**Algorithm 1. (Modified Frank-Wolfe Feasible Direction Algorithm)**

**Step 0 (Initialization).** Take two positive numbers $\eta_1$ and $\eta_2$. Choose a prediction error $\delta > 0$, an approximation bias $\vartheta > 0$, and coverage difference $\gamma > 0$. Set an initial $\varepsilon > 0$, $B = B_0$. Given the initial value of $g$, $g^{(0)} = (0, \ldots, 0)^T$. Then,

$$Q^{(0)} = \left( \frac{\varepsilon \ln\left(\frac{2n + p_n - c_n}{c_n - v_n}\right)}{2}, \ldots, \frac{\varepsilon \ln\left(\frac{2n + p_n - c_n}{c_n - v_n}\right)}{2} \right).$$

Set $k := 0$.

**Step 1 (Select B).**

$$\tau_B = \frac{\text{Quantity of demand data in the set of } X_{KDE+PCA}}{\text{Quantity of demand data}}$$

if $\|\tau_B - \tau_{B-\eta_1}\| < \gamma$, then go to Step 2; Otherwise, set $B := B + \eta_1$. Return to Step 1.

**Step 2 (Optimization).** Substitute $Q^{(k)}$ into $\nabla_g R_1(g^{(k)})$, solve the following linear programming:

$$\min \nabla_g R_1 \left( g^{(k)} \right) g, g \in D.$$  \hspace{1cm} (29)

The obtained optimal solution is referred to as $y^{(k)}$. 

□
Step 3 (Termination). If \( |\nabla_g R_1(g^{(k)})^T (y^{(k)} - g^{(k)})| < \delta \) and \( \|R_1(g^{(k)}) - R_1(g^{(k-1)})\| < \vartheta \), then the algorithm stops and output the optimal solution \( Q^* = Q^{(k)} \); Otherwise, go to Step 4.

Step 4 (Line Search). Starting at the point \( g^{(k)} \), conduct line search along the direction of \( d_k = y^{(k)} - g^{(k)} \), i.e., compute \( t_k \) such that it satisfies
\[
R_1(g^{(k)} + t_k d_k) = \min_{0 \leq t \leq 1} R_1(g^{(k)} + t d_k).
\]

Step 5 (Update). Update
\[
\begin{cases}
  g^{(k+1)} \leftarrow g^{(k)} + t_k d_k, \\
  Q^{(k+1)} \leftarrow \begin{cases}
    \prod_{i=1}^n (x(g^{(k)} + t_k d_k)^{(l)}) + \varepsilon \ln \left( \frac{s_i + p_1 - c_1}{c_1 - v_1} \right), \\
    \ldots, x(g^{(k)} + t_k d_k)^{(n)} + \varepsilon \ln \left( \frac{s_n + p_n - c_n}{c_n - v_n} \right)
  \end{cases}, \\
  \varepsilon \leftarrow \eta \varepsilon, \\
  k \leftarrow k + 1.
\end{cases}
\]

Return to Step 2.

Theorem 4.1. Let \( \{Q^{(k)}\} \) be a sequence generated by Algorithm 1. Then, any accumulation point, \( Q^* \), of this sequence is a KKT point of Model (23). Thus, \( Q^* \) is a data-driven robust optimal solution of the random optimization problem (3).

Proof. The Lagrangian function of Model (23) is
\[
L(Q, \omega; x) = -R_1(Q; x) + \sum_{i=1}^n \omega_1(a_i - 1) + \sum_{i=1}^n \omega_2(b_i - 1) + \sum_{i=1}^n \omega_3(-a_i) + \sum_{i=1}^n \omega_4(-b_i) + \sum_{i=1}^n \omega_5(a_i + b_i - 1) + \omega_6 \left( \sum_{i=1}^n (a_i + b_i) - B \right),
\]
where \( \omega_k = (\omega_{k1}, \omega_{k2}, \ldots, \omega_{kn}) \), \( k = 1, 2, \ldots, 5 \), and \( \omega_6 \) are the Lagrangian multipliers. Consequently, the KKT conditions of Model (23) is
\[
\begin{cases}
  \nabla Q L(Q, \omega; x) = \nabla Q (-R_1(Q; x)) = 0, \\
  0 \leq a_i, b_i \leq 1, \quad i = 1, 2, \ldots, n, \\
  a_i + b_i \leq 1, \quad i = 1, 2, \ldots, n, \\
  \sum_{i=1}^n a_i + b_i \leq B, \\
  \omega_k \geq 0, \quad k = 1, 2, \ldots, 5, \\
  \omega_6 \geq 0, \\
  \sum_{i=1}^n \omega_{1i}(a_i - 1) = 0, \sum_{i=1}^n \omega_{2i}(b_i - 1) = 0, \\
  \sum_{i=1}^n \omega_{3i}(-a_i) = 0, \sum_{i=1}^n \omega_{4i}(-b_i) = 0, \\
  \sum_{i=1}^n \omega_{5i}(a_i + b_i - 1) = 0, \\
  \omega_6 \left( \sum_{i=1}^n (a_i + b_i) - B \right) = 0.
\end{cases}
\]
From (28), we get \( \nabla Q L(Q^{(k)}, \omega; x^{(k)}) = 0 \). Thus, \( Q^* \) is an accumulation point of \( Q^{(k)} \) and \( x^* \) is also the accumulation point of \( x^{(k)} \). In other words,
\[
\nabla Q L(Q^*, \omega; x^*) = \nabla Q (-R_1(Q^*; x^*)) = 0.
\]
Furthermore, since \( g^{(k)} \in D \), \( Q^{(k)} \) satisfies the rest conditions in (31). So, \( Q^* \) satisfies the rest conditions in (31).

We have proved that \( Q^* \) is the data-driven robust optimal solution of Model (3).

**Remark 8.** Different from the used heuristic algorithms in the literature to solve a complicated newsboy problem, Algorithm 1 is an efficient algorithm since it can generate better and better iterative points \( Q^{(k)} \) without any random search in the feasible domain \( D \). With such a property, Algorithm 1 is useful to conduct numerical simulations in Section 5 such that effectiveness and practicability of the proposed data-driven robust approach in this paper are further validated.

5. **Numerical simulation and comparison of models.** In this section, we first state five well-established models of newsboy problems in the literature, which are closely related with the proposed one in this paper. Then, by numerical simulation, we compare the solutions obtained by these methods to validate effectiveness and robustness of our data-driven robust model (23).

5.1. **Reference methods.** For simplification, we only state the optimal order quantities by the relevant methods in the case of a two-item situation, i.e. \( i = 1, 2 \).

To show advantages of the proposed data-driven robust model (23), we first present other five different reference optimization methods of newsboy problems. Four of them are from [30], which include the classical newsboy model under the assumption of demand conforming to the normal distribution, the absolute robust optimization model, the robust deviation model, and the relative robustness model based on interval demands where the upper and lower bounds is the 95% quartile and 5% quartile of changing demands which are assumed to follow normal distributions. Another method, called Sample Average Approximation (SAA), can be found in [16,20].

(1) By the classical newsboy model in [30], the optimal order quantity is calculated by
\[
Q^C_i = F_i^{-1} \left( \frac{s_i - c_i + p_i}{s_i - v_i + p_i} \right),
\]  
(32)

where
\[
F_i(x) = \frac{1}{\sqrt{2\pi} \sigma_i} \int_{-\infty}^{x} \exp \left( -\frac{(t - \mu_i)^2}{2\sigma_i^2} \right) dt
\]
is the cumulative distribution function of demand for Item \( i \), \( \mu = (\mu_1, \mu_2)^T \), \( \delta = (\delta_1, \delta_2)^T \) is the mean vector and deviation vector of demand data, respectively.

(2) For the model, called the Absolute Robustness Model Based on Interval Demand in [30], the optimal order quantity is calculated by
\[
Q^A_i = \frac{s_i - v_i}{s_i - v_i + p_i} x_i + \frac{p_i}{s_i - v_i + p_i} \pi_i.
\]  
(33)

(3) For the model, called the Robust Deviation Based on Interval Demand in [30], the optimal order quantity is calculated by
\[
Q^D_i = \frac{c_i - v_i}{s_i - v_i + p_i} x_i + \frac{s_i - c_i + p_i}{s_i - v_i + p_i} \pi_i.
\]  
(34)
(4) For the model, called the Relative Robustness Based on Interval Demand in [30], the optimal order quantity is calculated by
\[ Q_i^R = \frac{(s_i - v_i + p_i)E_i - x_i}{(s_i - c_i + p_i)E_i + (c_i - v_i)E_i}. \] (35)

(5) For the data-driven optimization model, called the SAA method, the optimal order quantities are obtained by solving the following optimization problem:
\[
\max \hat{R}_n(Q_1, Q_2) = \max \frac{1}{n} \sum_{k=1}^{2n} \sum_{i=1}^{2} s_i \min \{Q_i, x_i^k\} + v_i \max \{0, Q_i - x_i^k\} - p_i \max \{0, x_i^k - Q_i\} - c_i Q_i,
\] (36)
where \(\{x_1^1, x_1^2, \ldots, x_1^N\}\) and \(\{x_2^1, x_2^2, \ldots, x_2^N\}\) are two random independent samples with size \(N\), i.e., the demand data of Items A and B. Only in view of building the objective functions, it is easy to see that our model (23) is a data-driven method different from Model (36).

5.2. Numerical simulation and comparison of models. In order to evaluate robustness of Model (23), the optimal solution of Model (23) is compared with those of the other three robust methods described in Section 5.1 in the case that the demand data is time-varying.

Let \(\mu_1\) and \(\mu_2\) be the means of time-varying demand data for Items A and B, respectively. Let \(\sigma_1\) and \(\sigma_2\) stand for the corresponding standard deviations. Then, as done in [26], we generate such demand samples by simulating the mixed Gaussian distribution. Specifically, there are a total of respective 300 demand data of Items A and B (\(N = 300\)), randomly drawn from the two mixed Gaussian distribution with mean vectors \(X_1 = (235, 80)\) and \(X_2 = (245, 85)\), respectively. The covariance matrices of the mixed Gaussian distribution are given by
\[
\Sigma_1 = \Sigma_2 = \begin{pmatrix}
15 & 7 \\
7 & 20
\end{pmatrix}.
\]
The weight vector is chosen to be \(w = (0.4, 0.6)\), and the values of other model parameters in this case study are given in Table 1.

| Item | \(s\) (Yuan RMB) | \(v\) (Yuan RMB) | \(p\) (Yuan RMB) | \(c\) (Yuan RMB) |
|------|------------------|------------------|------------------|------------------|
| A    | 80               | 3                | 6                | 12               |
| B    | 97               | 10               | 16               | 32               |

With the above preparation, we can evaluate the impacts of time-varying demands on the maximum profits obtained by different robust optimization models. Specifically, by changing the means and deviation of demand data, we first generate nine demand data sets with different means and standard deviations (see the first two columns in Table 2). Then, we implement Algorithm 1 to solve Model (23) and also compute the maximum profits determined by the other three robust methods (33), (34) and (35). By taking \(\vartheta = 0.01\), \(\alpha = 0.05\), \(\varepsilon = 0.01\) and \(B_0 = 1.35\), we report the maximum profits of all the four methods in Table 2, where \(R_U\), \(R_A\), \(R_D\) and \(R_R\) represent the maximum profits corresponding to models (23), (33), (34) and (35), respectively. Initial values of demands are given in the second row of
Table 2, the changes of demands are described by increasing or decreasing means and deviations (see other rows of Table 2), where ‘+’ (‘-’) stands for a value greater (smaller) than the initial one, and ‘0’ stands for an unchanged value.

**Table 2. Profits of different robust methods with changing demands**

| \((\mu_1, \sigma_1)\)       | \((\mu_2, \sigma_2)\)       | \(RU\)     | \(RA\)     | \(RD\)     | \(RR\)     |
|-----------------------------|-----------------------------|------------|------------|------------|------------|
| (239.55, 5.80)             | (82.34, 3.87)               | 20824.79   | 20523.29   | 20205.75   | 20214.74   |
| (+10.82, 0)                | (0, 0)                      | +767.31    | +737.58    | +738.20    | +738.10    |
| (-8.85, 0)                 | (0, 0)                      | -625.43    | -601.57    | -601.61    | -601.57    |
| (0, 0)                     | (+10.20, 0)                 | +691.21    | +663.47    | +663.91    | +662.99    |
| (0, 0)                     | (-5.13, 0)                  | -350.61    | -331.46    | -330.69    | -330.23    |
| (0, +9.61)                 | (0, 0)                      | -702.95    | -1096.27   | -1329.40   | -1320.59   |
| (0, -1.42)                 | (0, 0)                      | +101.46    | +161.14    | +195.23    | +194.66    |
| (0, 0)                     | (0, +5.55)                  | -608.36    | -656.60    | -910.03    | -869.37    |
| (0, 0)                     | (0, -0.48)                  | +42.60     | +57.51     | +79.44     | +77.63     |

Numerical results in Table 2 demonstrate that:

1. An increasing demand mean brings about growth of the profits obtained by all the four robust models. The growth of profit by our data-driven robust model (23) is 691.21 Yuan (RMB), which is more than those by the other three robust methods (about 663 Yuan (RMB)) (see the last four columns in the fifth row of Table 2). That is to say, our model can reduce the over-conservation of robust methods.

2. When the standard deviations of demand data increase, the profits decrease. This fact is in line with the definition of robustness, as well as coinciding with common senses. The reduction of profits by our data-driven robust model (23) is 702.95 Yuan (RMB), which is lower than those by the other three robust methods (more than 1000 Yuan (RMB)) (see the last four columns in the seventh row of Table 2). Since the standard deviation represents the data volatility, these results indicate that Model (23) is more robust than the other ones.

We now further substantiate the above statements by studying the impacts of time-varying demands on optimal order quantities. Particularly, we intend to explore how the time-varying demands affect the optimal order quantities, in comparison with all the other five reference methods stated in Section 5.1.

By changing the means of the mixed Gaussian distribution \(X_1\) and \(X_2\) with given deviations, we generate distinct data sets of demands. Then, with these demand data, we compute optimal order quantities by the above-mentioned six models. Numerical results are presented in Figure 2.

From Figure 2, it is seen that the optimal order quantities obtained by different models increase as the demand mean increases. However, not all of them have the same stability. In particular, the optimal order quantities of Items A and B generated by Model (23) (the data-driven robust model in this paper) is greater than that by Model (33) (absolute robust method). The curve of order quantities corresponding to Model (23) is more stable than the those obtained by the other compared methods (see Figure 2(a)). That is to say, our data-driven method can reduce over-conservation of robust optimization methods, apart from ensuring robustness.

In summary, in terms of economic benefits and robustness, our data-driven model (23) outperforms all the other models.
6. Sensitivity analysis. In any model of newsboy problems, the model parameters play a fundamental role in choosing optimal order quantities. Particularly, as a robust optimization model, approximation levels and confidence levels may directly affect formation of uncertainty sets used to build the robust model. In this section, by sensitivity analysis, we intend to investigate how the model parameters affect the optimal solutions of Model (23).

6.1. Impacts of approximation levels. We first explore impacts of the approximation level on the optimal solutions, which is used to get the uniform smooth approximation function in our model (23).

We change the value of the approximation level $\varepsilon$ in Model (23) from 0.01 to 0.5 by a step length 0.01. Then, we implement Algorithm 1 to solve the models defined by (23), corresponding to different approximation functions. The relationship between the approximation level and the numerical solutions of models are presented in Figure 3.

From Figure 3, it is concluded that:

(1) An increment of uniform approximation levels causes increasing optimal order quantities of Items A and B. However, the increment rate is small. Specifically, when the value of uniform approximation levels changes from 0.01 to 0.5, the increment of the optimal order quantities for Items A and B are both lower than 1 (see Figures 3(a) and 3(b)). In other words, the optimal order policy of multi-item newsboy problems obtained by our model (23) is insensitive to the uniform approximation level.

(2) An increment of uniform approximation levels generates linear growth of the profit (see Figure 3(c)). Actually, let the approximation level $\varepsilon$ change from $\varepsilon_j$ to $\varepsilon_k$. Denote $\Delta R$ the generated difference of profits. Then, by our model (23) and Proposition 5, we know

$$
\Delta R = \frac{\varepsilon_k - \varepsilon_j}{2} \sum_{i=1}^{2} (s_i + v_i + p_i - 2c_i) \ln \left( \frac{s_i + p_i - c_i}{c_i - v_i} \right) - (s_i - v_i + p_i) \left( \sqrt{\frac{s_i + p_i - c_i}{c_i - v_i}} + \sqrt{\frac{c_i - v_i}{s_i + p_i - c_i}} \right),
$$

(37)
6.2. Impacts of confidence levels. In a data-driven robust optimization model, the confidence level of uncertainty sets represents the degree of risk aversion of a newsboy, being proportional to the degree of risk aversion. Generally, a lower degree of risk aversion often allows the newsboy to order more items and get greater profit. We now attempt to more accurately model the relation between the confidence level and the optimal order quantity, and that between the confidence level and the profit of the newsboy. Additionally, since standard deviation of demands reflects risk of decision making, we also take into account the impacts of the standard deviations of changing demand data when we conduct the following numerical analysis.

By changing the confidence level of the data-driven uncertainty set $1 - 2\alpha$ from 0 to 0.98 with a step length 0.02, we obtain the corresponding optimal solutions, shown in Figures 4 and 5.

From Figure 4, it is seen that:

(1) An increment of confidence levels makes the optimal order quantity of the newsboy decrease. That is to say, as the degree of risk aversion increases, the optimal order quantity of the newsboy decreases. Besides, the decreasing rate of optimal order quantities becomes greater when the confidence level is in interval $[0.9, 0.98]$, smaller in interval $[0, 0.1]$ (see Figures 4(a) and 4(b)). Accurately, the relation
between the confidence level and the optimal order quantity can be specified by the following nonlinear function \((\sigma_1 = 30.30 \text{ in Figure 4(a))):\)

\[
Q^*(\alpha) = 16.73 \exp(-38.12(1 - 2\alpha)) + 227 \exp(-0.15(1 - 2\alpha)), \quad \alpha \in (0, 0.5).
\]  

(2) The reduction of the optimal order quantity with a standard deviation of 30.30 is 19.84, which is seven times the corresponding result with standard deviation of 10.42 when the confidence level changes from 0 to 0.1 (see Figure 4(a)). In other words, as the volatility of changing demand data increases, the decreasing degree of the optimal order quantities increases, which means the optimal order quantity is inversely proportional to the risk of decision making.

From Figure 5, it is revealed that:

(1) An increment of confidence levels makes the profit of the newsboy decrease. That is to say, higher degree of risk aversion has negative impacts to the profit of newsboys, which coincides with general knowledge. In addition, the decreasing rate of profits becomes greater when the confidence level is in the interval \([0.9, 0.98]\) (see Figures 5(a) and 5(b)). In conclusion, a greater degree of risk aversion can cause stronger impacts on the profit. In view of profitability of newsboys, it suggests that the newsboy should not to be too risk-averse.
(2) As the volatility of demand data becomes stronger, the profit basically decreases (see the curves with three different colors in Figures 5(a) and 5(b)). Accurately, the relation between the confidence level and the maximum profit can be specified by the following nonlinear function ($\sigma_2 = 12.88$ in Figure 5(b)):

$$P^*(\alpha) = -1.41 \times 10^{-5} \exp(17.58(1 - 2\alpha)) + 2.19 \times 10^4 \exp(-0.06(1 - 2\alpha)), \ \alpha \in (0, 0.5).$$

(39)

Remark 9. Note that the observed result that profit decreases as the confidence level increases is in line with the theoretical results in Proposition 3. However, because the other parameters of the products are fixed in this numerical simulation, the result that the optimal order quantities of products decrease as the confidence level is different from that in Proposition 3.

6.3. Impacts of unit purchase prices. In the end of this section, we study what are the differences of optimal order quantities caused by changing unit purchase prices when different newsboy models are used.

We change the unit purchase prices of Items A and B in Model (23) by a step length of 3 Yuan (RMB) increment. In line with the actual situation, the salvage values are also adjusted corresponding to changes of the unit purchase prices. Then, we compute the optimal order quantities by solving the six different models. Numerical results are presented in Figure 6.

**Figure 6.** Impacts of unit purchase prices on optimal order quantities

From Figure 6, it is clear that:

(1) For all the six compared models, an increment of unit purchase prices makes the optimal order quantity decrease. It coincides with common knowledge. It seems true that for higher unit purchase prices, the decline rate of optimal order quantities is more greater than that for lower unit purchase prices (see the slopes of the curves shown in Figures 6(a) and 6(b)). In other words, for a product with higher unit purchase prices in the newsboy problems, its optimal order quantities are more sensitive to the change of unit purchase prices, compared with the scenarios when the unit purchase price of this product is lower.

(2) By our model (23) and the absolute robust model (33), the obtained optimal order quantities are smaller than those by the other four models as the unit purchase price changes. However, our model (23) can provide a more stable ordering policy as the unit purchase price increases.
Conclusions and future directions of research. In this paper, we have proposed a novel data-driven robust optimization approach to solution of multi-item newsboy problems by combining the absolute robust optimization with a data-driven uncertainty set. We have presented a method of describing uncertain demands, which can remove an assumption that the distribution of demands is given and take into account interaction of demands between different products by the PCA method.

For the single-item situation, different from the existing results, we have obtained a closed-form optimal solution of our model and proved new useful properties with respect to the optimal order quantity and the maximum profit.

Owing to complexity of the multi-item situation, we have analyzed the model properties, and in virtue of these nice properties, we have developed an efficient algorithm to solve the complicated model.

By numerical simulation and sensitivity analysis, we have shown the advantages of the proposed model in this paper. Specifically, compared to the other five well-established methods to treat the newsboy problems, our data-driven robust model has stronger robustness and can reduce the degree of over-conservation.

Particularly, from our model, the following managerial implications have been revealed:

1. As the demand mean increases, the optimal order quantities and the profits increase. But different models of newsboy problems have distinct stability of the order policy. The proposed model in this paper can provide a more stable order policy and maximum profits.

2. For a product with higher unit purchase prices in the newsboy problems, its optimal order quantities are more sensitive to the change of unit purchase prices, compared with the scenarios when the unit purchase price of this product is lower.

3. A greater degree of risk aversion can cause stronger impacts on the profit. In view of profitability of newsboys, the newsboy should not to be too risk-averse.

4. An increment of uniform approximation levels generates linear growth of the profit, but the optimal order policy of multi-item newsboy problems is insensitive to the uniform approximation level.

Compared with the similar results available in the literature, it is remarkable that the above findings benefit from efficiency of the developed algorithm in this paper. For complicated models of the newsboy problems, it is more important to develop efficient algorithms than building of these models.

In future research, it is interesting and important to apply the proposed method in this paper to analyze ordering strategies of retailers in logistic industry, especially find real-world applications before and during the COVID-19 pandemic.

In addition, on the one hand, the proposed model can be extended by taking into account the budget constraints [2], reservation discounts or supplier’s strategy of discounts [8,40] such that it is more realistic. On the other hand, it is also valuable to extend the proposed single-period model to a multi-period situation. Particularly, for the multi-period situation, it is critical to form an efficient uncertainty set to describe the multi-period uncertainty of demands by machine learning techniques. Besides, it is also interesting to leverage the distributionally robust optimization method [12,15] to handle newsboy problems.

Generally, in a classical newsboy problem, only the order quantity is the decision variable of retailers. The proposed data-driven robust optimization model in this paper is only applicable to such a multi-item newsboy problem. However, in
practice, it is often that the demands are associated with selling prices of products [19, 32, 42]. Therefore, it is significant to further investigate new data-driven robust optimization models of the newsboy problems with the selling prices being endogenous variables.

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