Riemann-Hilbert approach to the modified nonlinear Schrödinger equation with non-vanishing asymptotic boundary conditions

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Abstract

The modified nonlinear Schrödinger (NLS) equation was proposed to describe the nonlinear propagation of the Alfven waves and the femtosecond optical pulses in a nonlinear single-mode optical fiber. In this paper, the inverse scattering transform for the modified NLS equation with non-vanishing asymptotic boundary at infinity is presented. An appropriate two-sheeted Riemann surface is introduced to map the original spectral parameter $k$ into a single-valued parameter $z$. The asymptotic behaviors, analyticity and the symmetries of the Jost solutions of Lax pair for the modified NLS equation, as well as the spectral matrix are analyzed in details. Then a matrix Riemann-Hilbert (RH) problem associated with the problem of nonzero asymptotic boundary conditions is established, from which $N$-soliton solutions is obtained via the corresponding reconstruction formulae. As an illustrate examples of $N$-soliton formula, two kinds of one-soliton solutions and three kinds of two-soliton solutions are explicitly presented according to different distribution of the spectrum. The dynamical feature of those solutions are characterized in the particular case with a quartet of discrete eigenvalues. It is shown that distribution of the spectrum and non-vanishing boundary also affect feature of soliton solutions. Finally, we analyze the differences between our results and those on zero boundary case.

Keywords: the modified NLS equation; Lax pair; inverse scattering transformation; Riemann-Hilbert problem; $N$-soliton solution.

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1 Introduction

The nonlinear NLS equation \[1, 2\]

\[iu_t + u_{xx} + 2|u|^2u = 0 \quad (1.1)\]

is one of the most celebrated soliton equations, which has applications in a wide variety of fields such as plasma physics, nonlinear optics and many other fields. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied \[3, 4\]. Among them, there are three celebrated derivative nonlinear Schrödinger equations, including the Kaup-Newell equation \[5\], the Chen-Lee-Liu equation \[6\] and the Gerdjikov-Ivanov equation \[7, 9\]. It is known that these three equations may be transformed into each other by implicit gauge transformations, and the method of gauge transformation can also be applied to some generalized cases \[10, 11\]. 1970s, Wadati et al proposed a kind of mixed NLS equation

\[u_t - iu_{xx} + a(|u|^2u)_x + ib|u|^2u = 0, \quad (1.2)\]

which was shown to be completely integrable by inverse scattering transformation \[12\]. For \(a = 0\), the equation (1.2) reduces to the classical NLS equation (1.1); For \(b = 0\), the equation (1.2) reduces to the Kaup-Newell equation; For \(a, b \neq 0\), the equation (1.2) is equivalent to the modified NLS equation

\[iu_t + u_{xx} + 2|u|^2u + i\frac{1}{\alpha}(|u|^2u)_x = 0, \quad \alpha > 0, \quad (1.3)\]

which was also called the perturbation NLS equation \[13\], it can be used to describe Alfven waves propagating along the magnetic field in cold plasmas and the deep-water gravity waves \[14, 15\]. The term \(i(|u|^2u)_x\) in the equation (1.3) is called the self-steepening term, which causes an optical pulse to become asymmetric and steepen upward at the trailing edge \[16, 17\]. The equation (1.3) also describes the short pulses propagate in a long optical fiber characterized by a nonlinear refractive index \[18, 19\]. Brizhik et al showed that the modified NLS equation (1.3), unlike the classical NLS equation (1.1), possesses static localized solutions when the effective nonlinearity parameter is larger than a certain critical value \[20\].

The modified NLS equation (1.3) has been discussed extensively, for example, various solutions such as analytical solutions, soliton solutions, rational and multi-rogue wave so-
olutions were found by analytical method, Hirota bilinear method and Darboux transformation respectively \[21-24\]. The Hamiltonian structure for the equation \(1.3\) was given \[25\]. \(N\)-soliton solutions for the modified NLS equation \(1.3\) with zero boundary condition \(u(x, t) \to 0, \ x \to \pm \infty\) also were obtained by inverse scattering transform (IST) and dressing method \[26-28\]. Deift-Zhou nonlinear steepest decedent method was used to obtain long-time asymptotic solution of initial problem of the equation \(1.3\) \[29\]. In recent years, coupled modified NLS equations and vector modified NLS equations also were presented and studied \[30-34\].

Solitons are found in various areas of physics such as gravitation and field theory, plasma physics, nonlinear optics and solid state physics, which can be described by nonlinear equations. The IST procedure, as one of the most powerful tool to investigate solitons of nonlinear models, was first discovered by Gardner, Green, Kruskal and Miura \[35\]. The IST for the focusing NLS equation with zero boundary conditions was first developed by Zakharov and Shabat \[36\], later for the defocusing case with nonzero boundary conditions \[37\]. The next important steps of the development of IST method is the Riemann-Hilbert (RH) method as the modern version of IST was established by Zakharov and Shabat \[38\], which involves the determination of a analytic function in given sectors of the complex plane, from the knowledge of the jumps of this function across the boundaries of the sectors. It has since become clear that the RH method is applicable to construction of exact solutions and asymptotic analysis of solutions for a wide class of integrable systems \[39-47\].

To our knowledge, with the exception of IST and dressing method to the modified NLS equation with zero boundary case \[26-28\], there are almost no known results on IST or RH method for the modified NLS equation with nonzero boundary conditions. However, in many laboratory and field situations, the wave motion is initiated by what corresponds to the imposition of boundary conditions. In addition, the modulational instability has received renewed interest in recent years, and has also been suggested as a possible mechanism for the generation of rogue waves \[48-49\]. It was shown that rogue wave solutions of integrable systems can be obtained via IST or RH method, which provide an effective and perfect tool to study rogue waves and the nonlinear stage of modulational instability \[42, 50, 51\].

In this article, we consider the modified NLS equation \(1.3\) with the following nonzero asymptotic boundary conditions

\[
u(x, t) \sim u_{\pm} e^{-4i\alpha^2 t + 2i\alpha x}, \quad x \to \pm \infty,
\]

(1.4)
where \(|u_\pm| = u_0 > 0\), and \(u_\pm\) are independent of \(x, t\). Our aim here is, by using Riemann-Hilbert (RH) method, to establish a formulae of \(N\)-soliton solutions for the above nonzero boundary problem and characterize their features in the particular case with a quartet of discrete eigenvalues.

The structure of this work is the following. In section 2, we introduce an appropriate two-sheeted Riemann surface is introduced to map the original spectral parameter \(k\) into a single-valued parameter \(z\). In Section 3 and Section 4, starting from the Lax pair of the modified NLS equation, we construct Jost solutions and spectral matrix. Then in Section 5, we analyze analytical and symmetric properties the Jost solutions and spectral matrix. In section 6, we analyze asymptotic behaviors of the Jost solution, scattering matrix and reflection coefficients. In Section 7, we discuss the discrete spectrum and the residue conditions to analyze poles for meromorphic matrices appearing in the RH problem. In Section 8, we establish reconstruction formula between solution of the modified NLS equation and the RH problem. We obtain the trace formula as well as theta condition that reflection coefficients and discrete spectrum satisfy. In the section 9, in reflectionless case, we discuss solvability of the RH problem, from which \(N\)-soliton solutions of the modified NLS equation are obtained.

As an illustrate examples of \(N\)-soliton formula, according to different distribution of the spectrum, two kinds of one-soliton solutions and three kinds of two-soliton solutions are explicitly presented, and their dynamical features are characterized with a quartet of discrete eigenvalues. The affects of distribution of the spectrum on soliton solutions are analyzed.

2 Riemann surface and uniformization variable

The modified NLS equation \([1.3]\) admits the Lax pair \([28]\)

\[
\phi_x = U\phi, \quad \phi_t = V\phi,
\]

(2.1)

where

\[
U = -\alpha i(k^2 - 1)\sigma_3 + iQ,
\]

\[
V = -2i\alpha^2(k^2 - 1)^2\sigma_3 + 2i\alpha(k^2 - 1)Q + ik^2Q^2\sigma_3 - kQ_3Q_x - \frac{i}{\alpha}kQ^3,
\]

and

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}.
\]
To change (1.4) into constant boundary, we make a transformation
\[ u \rightarrow u e^{-4i\alpha^2 t + 2i\alpha x}, \]
\[ \phi \rightarrow e^{-i\alpha^2 t + i\alpha x} \sigma_3 \phi, \]
then the modified NLS equation (1.3) becomes
\[ iu_t + u_{xx} + 4i\alpha u_x + i \frac{1}{\alpha} (|u|^2 u)_x = 0 \] (2.2)
with corresponding boundary
\[ \lim_{x \to \pm \infty} u(x, t) = u_{\pm}, \] (2.3)
where \( u_{\pm} \) are constant independent of \( x, t \), and \( |u_{\pm}| = u_0 \). And the Lax pair (2.1), as the compatibility of modified NLS equation (2.2), is changed to
\[ \phi_x = X\phi, \quad \phi_t = T\phi, \] (2.4)
where
\[ X = -i\alpha k^2 \sigma_3 + ikQ, \quad Q = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}, \]
\[ T = (-2i\alpha^2 k^4 + 4i\alpha^2 k^2)\sigma_3 + ik^2 |u|^2 \sigma_3 + 2i\alpha k(2k^2 - 2)Q - i \frac{1}{\alpha} kQ^3 + kQ_x. \]

Under the nonzero asymptotic boundary condition (2.3), limit spectral problem of the Lax pair (2.4) is
\[ \psi_x = X_\pm \psi, \quad \psi_t = T_\pm \psi, \] (2.5)
where
\[ X_\pm = -i\alpha k^2 \sigma_3 + ikQ_\pm, \quad T_\pm = \left(2\alpha k^2 - 4\alpha - \frac{u_0^2}{\alpha}\right) X_\pm, \] (2.6)
and
\[ Q_\pm = \begin{pmatrix} 0 & u_\pm \\ u_\pm^* & 0 \end{pmatrix}. \]

The eigenvalues of the matrix \( X_\pm \) are \( \pm ik\lambda \), where \( \lambda \) satisfies
\[ \lambda^2 = \alpha^2 k^2 + u_0^2, \] (2.7)
which are doubly branched, and its branch points are \( k = \pm iu_0/\alpha \). Gluing two copies of the complex plane \( S_1 \) and \( S_2 \) along the segment \([-\frac{1}{\alpha} u_0, \frac{1}{\alpha} u_0]\), we then obtain a Riemann surface. By setting
\[ k\alpha + iu_0 = r_1 e^{i\theta_1}, \quad k\alpha - iu_0 = r_2 e^{i\theta_2}, \quad -\pi/2 < \theta_j < 3/2\pi, \quad j = 1, 2, \]
we get two single-valued analytic functions on the Riemann surface
\[
\lambda(k) = \begin{cases} 
\frac{1}{\alpha}(r_1 r_2)^{1/2} e^{i(\theta_1 + \theta_2)/2}, & \text{on } S_1, \\
-\frac{1}{\alpha}(r_1 r_2)^{1/2} e^{i(\theta_1 + \theta_2)/2}, & \text{on } S_2.
\end{cases}
\] (2.8)

To avoid multi-valued case of eigenvalue \(\lambda\), we introduce a uniformization variable
\[
z = \alpha k + \lambda,
\] (2.9)
and obtain two single-valued functions
\[
k(z) = \frac{1}{2\alpha} (z - \frac{u_0^2}{z}), \quad \lambda(z) = \frac{1}{2} (z + \frac{u_0^2}{z}),
\] (2.10)
which allow us to discuss the IST on a standard \(z\)-plane instead of the more cumbersome two-sheeted Riemann surface. The second transformation of (2.10) is well-known Joukowsky transformation. However, the limit \(k \to \infty\) on two-sheeted Riemann surface will cause two limits \(z\)-plane: \(z \to 0\) and \(z \to \infty\). In fact, for \(k \in S_1\), we have
\[
\begin{align*}
z &= \alpha k + \sqrt{\alpha^2 k^2 + u_0^2} = \alpha k + \alpha k \left(1 + \frac{u_0^2}{\alpha^2 k^2}\right)^{1/2} \\
&= 2\alpha k + O(k^{-1}) \to \infty, \quad k \to \infty.
\end{align*}
\]
While for \(k \in S_2\), we find that
\[
z = \alpha k - \sqrt{\alpha^2 k^2 + u_0^2} = \frac{-u_0^2}{\alpha k + \sqrt{\alpha^2 k^2 + u_0^2}} \to 0, \quad k \to \infty.
\]
Therefore we should consider two kinds of asymptotic behaviors of Jost solution, scattering data and Riemann-Hilbert problem on the \(z\)-plane hereafter.

In addition to, we should explain how the two-sheeted Riemann surface is mapped into the \(z\)-plane. The transformation (2.8) possesses the following properties

- Map \(\text{Im } k > 0\) of \(S_1\) and \(\text{Im } k < 0\) of \(S_2\) together into the \(\text{Im } \lambda > 0\) of \(\lambda\)-plane;
- Map \(\text{Im } k < 0\) of \(S_1\) and \(\text{Im } k > 0\) of \(S_2\) together into the \(\text{Im } \lambda < 0\) of \(\lambda\)-plane;
- Map the segment \([-u_0, u_0]\) into a \([-\frac{1}{\alpha} u_0, \frac{1}{\alpha} u_0]\) on \(\lambda\)-plane.

Let’s consider map from the \(\lambda\)-plane to the \(z\)-plane again. By using the relation
\[
\lambda(z) = \frac{z^2 + u_0^2}{2z} = \frac{(|z|^2 - u_0^2)z + u_0^2(z + \bar{z})}{2|z|^2}
\]
\[
= \frac{1}{2|z|^2} \left[ (|z|^2 - u_0^2)z + 2u_0^2 \text{Re} z \right],
\]
Figure 1: Transformation relation from \( k \) two-sheeted Riemann surface, \( \lambda \)-plane and \( z \)-plane

we have

\[
\text{Im}\lambda(z) = \frac{1}{2|z|^2}(|z|^2 - u_0^4)\text{Im}z,
\]

which implies that the Joukowsky transformation admits the following properties

- Map \( \text{Im}\lambda > 0 \) into the domain \( \{ z \in C : (|z|^2 - u_0^4)\text{Im}z > 0 \} \) in \( z \)-plane;
- Map \( \text{Im}\lambda < 0 \) into the domain \( \{ z \in C : (|z|^2 - u_0^4)\text{Im}z < 0 \} \) in \( z \)-plane;
- Map \([-u_0, u_0]\) into circle \( C_0 = \{ |z| = u_0, z \in C \} \) in \( z \)-plane.

Transformation relations from \( k \) two-sheeted Riemann surface, \( \lambda \)-plane and \( z \)-plane are shown in Figure 1. It will be seen later that the circle \( C_0 \) is not the boundary of analytical domains for the Jost solutions and scattering data, but it will affect their symmetry. These are very different from classical focusing NLS equation with nonzero boundary conditions, where the boundary of analytical domains is \( \mathbb{R} \cup C_0 \) \[42\], but ours is \( \mathbb{R} \cup i\mathbb{R} \).

By using the relation

\[
\text{Im}(k(z)\lambda(z)) = \text{Im} \frac{z^4 - u_0^4}{4\alpha z^2} = \text{Im} \frac{(|z|^4 + q_0^4)z^2 - 2q_0^4((\text{Re}z)^2 - (\text{Im}z)^2)}{4\alpha |z|^4}
= \frac{1}{4\alpha |z|^4}(|z|^4 + u_0^4)\text{Im}z^2 = \frac{1}{2\alpha |z|^4}(|z|^4 + u_0^4)\text{Re}z\text{Im}z,
\]

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we define two domains $D^+, D^-$ and their boundary $\Sigma$ on $z$-plane by

$$D^- = \{ z : \text{Re}z \text{Im}z > 0 \}, \quad D^+ = \{ z : \text{Re}z \text{Im}z < 0 \},$$

$$\Sigma = \{ z : \text{Re}z \text{Im}z = 0 \} = \mathbb{R} \cup i\mathbb{R}\{0\},$$

which are shown in Figure 2.

Figure 2: The domains $D^-, D^+$ and boundary $\Sigma$.

3 Jost Solutions

From eigenvalues $\pm ik\lambda$, we can get the eigenvector matrix of $X_\pm$ and $T_\pm$ as

$$Y_\pm = \begin{pmatrix} 1 & 0 \\ \frac{u_\pm}{z} & 1 \end{pmatrix} = I + \frac{1}{z}\sigma_3 Q_\pm,$$  \hspace{1cm} (3.1)

by which $X_\pm$ and $T_\pm$ are diagonalized simultaneously

$$X_\pm = Y_\pm(-ik\lambda\sigma_3)Y_\pm^{-1}, \quad T_\pm = Y_\pm \left[-ik\lambda\sigma_3(2\alpha k^2 - 4\alpha - \frac{u_0^2}{\alpha})\right]Y_\pm^{-1}. \hspace{1cm} (3.2)$$

Direct computation shows that

$$\det(Y_\pm) = 1 + \frac{u_0^2}{z^2} \triangleq \gamma,$$  \hspace{1cm} (3.3)

and

$$Y_\pm^{-1} = \frac{1}{\gamma} \begin{pmatrix} \frac{1}{z} & \frac{u_\pm}{z} \\ \frac{u_\pm}{z} & 1 \end{pmatrix} = \frac{1}{\gamma}(I - \frac{1}{z}\sigma_3 Q_\pm). \hspace{1cm} (3.4)$$

Substituting (3.2) into (2.5), we immediately obtain

$$(Y_\pm^{-1}\psi)_x = -ik\lambda\sigma_3(Y_\pm^{-1}\psi), \quad (Y_\pm^{-1}\psi)_t = -ik\lambda \left(2\alpha k^2 - 4\alpha - \frac{u_0^2}{\alpha}\right)\sigma_3(Y_\pm^{-1}\psi), \hspace{1cm} (3.5)$$

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from which we can derive the solution of the asymptotic spectral problem \[ \psi_\pm = Y_\pm e^{i\theta(z)\sigma_3}, \] (3.6)

where

\[ \theta(z) = -k(z)\lambda(z)[x + (2\alpha k^2(z) - 4\alpha - \frac{u_0^2}{\alpha})t]. \]

It follows that the Jost solutions \( \phi_\pm(x,t,z) \) of the Lax pair (2.4) possess the following asymptotics

\[ \phi_\pm \sim Y_\pm e^{i\theta(z)\sigma_3}, \quad x \to \pm \infty. \] (3.7)

By making transformation

\[ \varphi_\pm = \phi_\pm e^{-i\theta(z)\sigma_3}, \] (3.8)

we then have

\[ \varphi_\pm \sim Y_\pm, \quad x \to \pm \infty. \]

Moreover, \( \varphi_\pm \) satisfy an equivalent Lax pair

\[ (Y^{-1}_-\varphi_\pm)_x = ik\lambda[Y^{-1}_-\varphi_\pm, \sigma_3] + Y^{-1}_-\Delta X_\pm\varphi_\pm, \] (3.9)

\[ (Y^{-1}_-\varphi_\pm)_t = ik(2\alpha k^2 - 4\alpha - \frac{1}{\alpha}u_0^2)[Y^{-1}_-\varphi_\pm, \sigma_3] + Y^{-1}_-\Delta T_\pm\varphi_\pm, \] (3.10)

where \( \Delta X_\pm = X - X_\pm = ik(Q - Q_\pm) \) and \( \Delta T_\pm = T - T_\pm \). Above two equations (3.9) and (3.10) can be written in full derivative form

\[ d(e^{-i\theta(z)\sigma_3}Y^{-1}_-\varphi_\pm) = e^{-i\theta(z)\sigma_3}[Y^{-1}_-(\Delta X_\pm dx + \Delta T_\pm dt)\varphi_\pm], \] (3.11)

which lead to two Volterra integral equations

\[ \varphi_- (x,t,z) = Y_- + \int_{-\infty}^{x} Y_- e^{-ik\lambda(x-y)\sigma_3}[Y^{-1}_-\Delta X_-\varphi_-(y,t,z)]dy, \] (3.12)

\[ \varphi_+ (x,t,z) = Y_+ - \int_{x}^{\infty} Y_+ e^{-ik\lambda(x-y)\sigma_3}[Y^{-1}_+\Delta X_+\varphi_+(y,t,z)]dy, \] (3.13)

where \( z \neq iu_0 \).

We define \( \varphi_\pm = (\varphi_{\pm,1}, \varphi_{\pm,2}) \) with \( \varphi_{\pm,1} \) and \( \varphi_{\pm,2} \) denoting the first and second column of \( \varphi_\pm \) respectively, then the first column of the equation (3.12) can be written as

\[ Y^{-1}_-\varphi_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} G(x - y, z)\Delta X_-\varphi_{-,1}dy, \] (3.14)
where
\[
G(x - y, z) = \frac{1}{\gamma} \begin{pmatrix}
\frac{1}{2} e^{2ik\lambda(x-y)} & -\frac{u_x}{z} e^{2ik\lambda(x-y)} \\
u_x e^{2ik\lambda(x-y)} & e^{2ik\lambda(x-y)}
\end{pmatrix}.
\] (3.15)

Note that \( e^{2ik\lambda(x-y)} = e^{2i(x-y) \text{Re}(k\lambda)} e^{2(x-y) \text{Im}(k\lambda)} \) and \( x - y > 0 \), we demonstrate that the first column of \( \varphi_- \) is analytical on \( D_- \), denoted by \( \varphi_--1 \). The same argument shows that the second column of \( \varphi_- \) is analytical on \( D_+ \), and denoted by \( \varphi_+2 \). And \( \varphi_+ = (\varphi_+1, \varphi_+2) \) which denote the first and second column are analytical on the \( D^+ \) and \( D^- \) respectively.

4 Scattering Matrix

Since \( trX = trT = 0 \) in (2.4), then by using Able formula, we have
\[
(det \phi_\pm)_x = (det \phi_\pm)_t = 0.
\] (4.1)

Again by using the relation
\[
det(\varphi_\pm) = det(\phi_\pm e^{-i\theta(z)\sigma_3}) = det(\phi_\pm),
\]
we get \( (det \varphi_\pm)_x = (det \varphi_\pm)_t = 0 \), which means that \( det(\varphi_\pm) \) is independent of \( x, t \). So we obtain that
\[
det \varphi_\pm = \lim_{x \to \pm \infty} det(\varphi_\pm) = det Y_\pm = \gamma \neq 0, \quad z \in D^+ \cup D^-,
\] (4.2)
which implies that \( \varphi_\pm \) are inverse matrices.

Since \( \phi_\pm \) are two fundamental matrix solutions of the Lax pair (2.4), there exists a linear relation between \( \phi_+ \) and \( \phi_- \), namely
\[
\phi_+(x, t, z) = \varphi_-(x, t, z)S(z),
\] (4.3)
where \( S(z) \) is called scattering matrix and (4.2) implies that \( det S(z) = 1 \).

Denoting the scattering matrix by \( S(z) = (s_{ij}(z))_{2 \times 2} \), then individual columns of the matrix equation (4.3) are
\[
\begin{align*}
\phi_{+,1} &= s_{11}(z)\phi_{-,1} + s_{21}(z)\phi_{-,2}, \\
\phi_{+,2} &= s_{12}(z)\phi_{-,1} + s_{22}(z)\phi_{-,2}, \\
\varphi_{+,1} &= s_{11}(z)\varphi_{-,1} + s_{21}(z)e^{-2i\theta}\varphi_{-,2}, \\
\varphi_{+,2} &= s_{12}(z)e^{2i\theta}\varphi_{-,1} + s_{22}(z)\varphi_{-,2},
\end{align*}
\] (4.4, 4.5)
in which \( s_{ij}(z), \ i,j = 1,2 \) are called scattering data, and the reflection coefficients are defined by
\[
\begin{align*}
\rho(z) &= \frac{s_{21}(z)}{s_{11}(z)}, & \tilde{\rho}(z) &= \frac{s_{12}(z)}{s_{22}(z)},
\end{align*}
\] (4.6)
Solving above four linear systems (4.4) and (4.5), we find that
\[ s_{11}(z) = \frac{\text{Wr}(\phi_{+1}, \phi_{-2})}{\gamma} = \frac{\text{Wr}(\varphi_{+1}, \varphi_{-2})}{\gamma}, \]  
(4.7)
\[ s_{12}(z) = \frac{\text{Wr}(\phi_{+2}, \phi_{-2})}{\gamma} = \frac{\text{Wr}(\varphi_{+2}, \varphi_{-2})}{e^{2i\theta}}, \]  
(4.8)
\[ s_{21}(z) = \frac{\text{Wr}(\phi_{-1}, \phi_{+1})}{\gamma} = \frac{\text{Wr}(\varphi_{-1}, \varphi_{+1})}{e^{-2i\theta}}, \]  
(4.9)
\[ s_{22}(z) = \frac{\text{Wr}(\phi_{-1}, \phi_{+2})}{\gamma} = \frac{\text{Wr}(\varphi_{-1}, \varphi_{+2})}{\gamma}, \]  
(4.10)
which together with analyticity of \( \varphi_{\pm} \) show that scattering data \( s_{11}(z) \) is analytic in \( D^+ \), \( s_{22}(z) \) is analytic in \( D^- \), and \( s_{12}(z), s_{21}(z) \) are continuous to \( \Sigma \).

5 Symmetry of \( \varphi_{\pm} \) and \( S(z) \)

**Proposition 1.** For \( z \in D^+ \), the Jost solution, scattering matrix and reflection coefficients admit the following two kinds of symmetries

- The first symmetry reduction
  \[ \varphi_{\pm}(x, t, z) = -\sigma_\ast \varphi_{\pm}^*(x, t, z^*)\sigma_\ast, \quad S(z) = -\sigma_\ast S^*(z^*)\sigma_\ast, \quad \rho(z) = -\tilde{\rho}^*(z^*), \]  
(5.1)
  \[ \varphi_{\pm}(x, t, z) = \sigma_1 \varphi_{\pm}^*(x, t, -z^*)\sigma_1, \quad S(z) = \sigma_1 S^*(-z^*)\sigma_1, \quad \rho(z) = \tilde{\rho}^*(-z^*), \]  
(5.2)
where \( \sigma_\ast = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

- The second symmetry reduction
  \[ \varphi_{\pm}(x, t, z) = \frac{1}{z} \varphi_{\pm} \left( x, t, \frac{-u_0^2}{z} \right) \sigma_3 Q_{\pm}, \]  
  \[ S(z) = (\sigma_3 Q_{\pm})^{-1} S \left( \frac{-u_0^2}{z} \right) \sigma_3 Q_{\pm}, \quad \rho(z) = \frac{u_+}{u_-} \tilde{\rho} \left( \frac{-u_0^2}{z} \right). \]  
(5.3)

**Proof.** We just need show that \( -\sigma_\ast \varphi_{\pm}^*(x, t, z^*)\sigma_\ast \) is also a solution of (3.9) and admits the same asymptotic behavior like the Jost solutions \( \varphi_{\pm}(x, t, z) \).

By using (5.1), we calculate \((-Y_{\pm}^{-1} \sigma \varphi_{\pm}^*(x, t, z^*)\sigma_\ast)_x\) and obtain
\begin{align*}
(-Y_{\pm}^{-1} \sigma \varphi_{\pm}^*(x, t, z^*)\sigma_\ast)_x &= -Y_{\pm}^{-1} \sigma Y_{\pm}[-ik(z^*)\lambda^*(z^*)[(Y_{\pm}^{-1})^*(x, t, z^*)\varphi_{\pm}(x, t, z^*)]\sigma_3 \\
&+ (Y_{\pm}^{-1})^*(x, t, z^*)\Delta X_{\pm}\sigma_3^* \varphi_{\pm}^*(x, t, z^*)].
\end{align*}
(5.4)
Noticing that
\[ Y_{\pm}^{-1}(x, t, z) = -\sigma_\ast (Y_{\pm}^{-1})^*(x, t, z^*)\sigma_\ast, \quad \sigma_\ast \sigma_3 \sigma_\ast = \sigma_3, \quad \sigma_\ast (\Delta X_{\pm}(z^*))^* \sigma_\ast = \Delta X_{\pm}(z), \]
then we have

\[
(-Y_{\pm}^{-1}(z)\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3)_x = -ik(z)\lambda(z)[Y_{\pm}^{-1}(z)(-\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3),\sigma_3]
+ Y_{\pm}^{-1}(z)\Delta X_{\pm}(-\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3),
\]

which implies that \(-\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3\) is a solution of (3.9) with asymptotic behaviors

\[-\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3 \sim Y_{\pm}, \quad x \to \pm \infty.\]

From the uniqueness of the solution for the equation (3.9), we have

\[
\varphi_{\pm}(x,t,z) = -\sigma_3\varphi_{\pm}(x,t,z^*)\sigma_3.
\]

Similarly, by using the symmetry relations

\[
Y_{\pm}^{-1}(x,t,z) = \sigma_1(Y_{\pm}^{-1})^*(x,t,-z^*)\sigma_1, \quad \sigma_3\sigma_1 = -\sigma_3,
\]

\[
\theta^*(-z^*) = \theta(z), \quad \sigma_1\sigma_3\sigma_1 = -\sigma_3,
\]

it is easy to derive the following symmetry

\[
\varphi_{\pm}(x,t,z) = \sigma_1\varphi_{\pm}(x,t,-z^*)\sigma_1, \quad S(z) = \sigma_1S^*(-z^*)\sigma_1, \quad \rho(z) = \tilde{\rho}^*(-z^*).
\]

Next, we prove the second kind of symmetries (5.3). From the relations

\[
k\left(-\frac{u_0^2}{z}\right) = k(z), \quad \lambda\left(-\frac{u_0^2}{z}\right) = \lambda(z), \quad \theta\left(-\frac{u_0^2}{z}\right) = -\theta(z),
\]

we know that if \(\phi(x,t,z)\) is a solution of the scattering problem (2.4), then \(\phi(x,t,-\frac{u_0^2}{z})C\) is also the solution of (2.4), where \(C\) is a determined \(2 \times 2\) matrix independent of \(x\) and \(t\). To obtain symmetric relation \(\phi(x,t,z) = \phi(x,t,-\frac{u_0^2}{z})C\), we require \(\phi(x,t,-\frac{u_0^2}{z})C\) has the same asymptotic condition (3.7) with \(\phi(x,t,z)\), that is,

\[
\phi_{\pm}\left(x,t,-\frac{u_0^2}{z}\right) C \sim Y_{\pm}\left(-\frac{u_0^2}{z}\right) e^{-i\theta(z)\sigma_3} C = Y_{\pm}(z)e^{i\theta\sigma_3}, \quad x \to \pm \infty,
\]

from which we find that \(C = \frac{1}{2}\sigma_3Q_{\pm}\). By uniqueness of the solution of the spectral problem (2.4), we get

\[
\phi(x,t,z) = \frac{1}{z}\phi\left(x,t,-\frac{u_0^2}{z}\right) \sigma_3Q_{\pm}.
\]

Again by using (3.8), we have

\[
\varphi_{\pm}(x,t,z) = \frac{1}{z}\varphi_{\pm}\left(x,t,-\frac{u_0^2}{z}\right) \sigma_3Q_{\pm}.
\]
What will come next are the symmetries of scattering matrix. For the individual columns, the above symmetries come to

\[
\varphi_{\pm,1}(x,t,z) = \sigma_3 \varphi_{\pm,2}(x,t,\pm z), \quad \varphi_{\pm,2}(x,t,z) = -\sigma_3 \varphi_{\pm,1}(x,t,\pm z),
\]

\( (5.9) \)

\[
\varphi_{\pm,1}(x,t,z) = \sigma_1 \varphi_{\pm,2}(x,t,-z^*), \quad \varphi_{\pm,2}(x,t,z) = \sigma_1 \varphi_{\pm,1}(x,t,-z^*),
\]

\( (5.10) \)

\[
\varphi_{\pm,1}(x,t,z) = \frac{u_+}{z} q_\pm \varphi_{\pm,2}, \quad \varphi_{\pm,2}(x,t,z) = \frac{u_+}{z} q_\pm \varphi_{\pm,1} \left( -\frac{u_0^2}{z} \right). \quad (5.11)
\]

Combining (5.9) and (5.10) we can get

\[
\varphi_{\pm,1}(x,t,z) = \sigma_3 \varphi_{\pm,1}(x,t,-z), \quad \varphi_{\pm,2}(x,t,z) = -\sigma_3 \varphi_{\pm,2}(x,t,-z).
\]

(5.12)

By using (5.9) and (4.3), we obtain a symmetry of scattering matrix

\[
S^*(z^*) = -\sigma_3 S(z) \sigma_3, \quad S(z) = \sigma_1 S^*(-z^*) \sigma_1,
\]

(5.13)

which gives

\[
s_{11}(z) = s_{22}(z^*), \quad s_{12}(z) = -s_{21}(z^*), \quad (5.14)
\]

\[
s_{11}(z) = s_{22}(-z^*), \quad s_{12}(z) = s_{21}(-z^*). \quad (5.15)
\]

Combining (5.14) and (5.15) we obtain that \( s_{11}(z) \) and \( s_{22}(z) \) are even function, and \( s_{12}(z) \) and \( s_{21}(z) \) are odd function.

By using (5.11) and (4.3), we obtain another symmetry of the scattering matrix

\[
S(z) = (\sigma_3 Q_-)^{-1} S \left( -\frac{u_0^2}{z} \right) \sigma_3 Q_+,
\]

(5.16)

which leads to

\[
s_{11}^*(z^*) = \frac{u_+}{u_-} s_{11} \left( -\frac{u_0^2}{z} \right), \quad s_{12}^*(z^*) = -\frac{u_+}{u_-} s_{12} \left( -\frac{u_0^2}{z} \right), \quad (5.17)
\]

\[
s_{21}^*(z^*) = \frac{u_+}{u_-} s_{21} \left( -\frac{u_0^2}{z} \right), \quad s_{22}^*(z^*) = \frac{u_+}{u_-} s_{22} \left( -\frac{u_0^2}{z} \right). \quad (5.18)
\]

Finally, all the above symmetries then give the symmetries for the reflection coefficients

\[
\rho(z) = \rho^*(-z^*) = -\rho^*(z^*) = \frac{u_-}{u_+} \hat{\rho} \left( -\frac{u_0^2}{z} \right) = -\frac{u_-}{u_+} \rho^* \left( -\frac{u_0^2}{z} \right).
\]

(5.19)

So we have done the proof. \( \square \)
6 Asymptotics of $\varphi_{\pm}$ and $S(z)$

To get the Riemann-Hilbert problem in the next section, it is necessary to discuss the asymptotic behaviors of the Jost solutions and scattering matrix as $z \to \infty$ and $z \to 0$.

**Proposition 2.** The Jost solutions possess the following asymptotic behaviors

$$
\varphi_{\pm}(x, t, z) = e^{i\nu_{\pm}(x, t)\sigma_3} + O(z^{-1}), \quad z \to \infty,
$$

$$
\varphi_{\pm}(x, t, z) = \frac{1}{z} e^{i\nu_{\pm}(x, t)\sigma_3} S_{\pm} + O(1), \quad z \to 0,
$$

where

$$
\nu_{\pm}(x, t) = \frac{1}{2\alpha} \int_{x}^{\infty} (u_0^2 - |u|^2)dy.
$$

**Proof.** We consider the following asymptotic expansions

$$
\varphi_{\pm}(x, t, z) = \varphi_{\pm}^{(0)}(x, t) + \frac{\varphi_{\pm}^{(1)}(x, t)}{z} + \frac{\varphi_{\pm}^{(2)}(x, t)}{z^2} + O(z^{-3}), \quad \text{as } z \to \infty.
$$

Substituting (6.4) into the Lax pair (3.9) leads to

$$
[\varphi_{\pm}^{(0)}(x, t), \sigma_3] = 0,
$$

$$
[\varphi_{\pm}^{(1)}(x, t) - \sigma_3 Q_{\pm} \varphi_{\pm}^{(0)}(x, t), \sigma_3] + 2(Q - Q_{\pm})\varphi_{\pm}^{(0)}(x, t) = 0,
$$

$$
\frac{i}{4\alpha} [\varphi_{\pm}^{(2)}(x, t) - \sigma_3 Q_{\pm} \varphi_{\pm}^{(1)}(x, t), \sigma_3] + \frac{i}{2\alpha} ((Q - Q_{\pm})\varphi_{\pm}^{(1)}(x, t) - \sigma_3 Q_{\pm} (Q - Q_{\pm}) \varphi_{\pm}^{(0)}(x, t))
$$

$$
- \frac{i}{2\alpha} (u_0^2 - |u|^2) \sigma_3 \varphi_{\pm}^{(0)}(x, t) = 0,
$$

$$
(\varphi_{\pm}^{(0)}(x, t))_x = \frac{i}{4\alpha} [\varphi_{\pm}^{(2)}(x, t) - \sigma_3 Q_{\pm} \varphi_{\pm}^{(1)}(x, t), \sigma_3] + \frac{i}{2\alpha} ((Q - Q_{\pm})\varphi_{\pm}^{(1)}(x, t) - \sigma_3 Q_{\pm} (Q - Q_{\pm}) \varphi_{\pm}^{(0)}(x, t))
$$

$$
= \frac{i}{2\alpha} (u_0^2 - |u|^2) \sigma_3 \varphi_{\pm}^{(0)}(x, t),
$$

from which we can know that $\varphi_{\pm}^{(0)}(x, t)$ is a diagonal matrix, and

$$
\varphi_{\pm}^{(0)}(x, t) = e^{i\nu_{\pm}(x, t)\sigma_3},
$$

where

$$
\nu_{\pm}(x, t) = \frac{1}{2\alpha} \int_{x}^{\infty} (u_0^2 - |u|^2)dy.
$$

Therefore, we get the asymptotic behavior of the modified Jost solution

$$
\varphi_{\pm}(x, t, z) = e^{i\nu_{\pm}(x, t)\sigma_3} + O(z^{-1}), \quad z \to \infty,
$$
Again from (6.6) and (6.9), we find that

\[ u = \lim_{z \to \infty} e^{i\nu}(x,t)\sigma_3(z\varphi_\pm)_{12} \]  

(6.10)

In a similar way, substituting the expansion

\[ \varphi_\pm(x,t,z) = \tilde{\varphi}_\pm(0) + \frac{\varphi_\pm(x,t)}{z} + O(z) \]

(6.11)

into the Lax equation (3.9), we obtain that

\[ \tilde{\varphi}_\pm = e^{i\nu}(x,t)\sigma_3 C, \]

where \( C \) is a constant matrix. Again from the expansion (6.11), we have

\[ \lim_{x \to \pm\infty} z\tilde{\varphi}_\pm = zY_\pm = z(I + \frac{1}{z}\sigma_3Q_\pm) = \lim_{x \to \pm\infty} (\tilde{\varphi}_\pm(1) + z\tilde{\varphi}_\pm(0) + \cdots). \]

(6.12)

Therefore \( C = \sigma_3Q_\pm \) and \( \tilde{\varphi}_\pm(1) = e^{i\nu}(x,t)\sigma_3\sigma_3Q_\pm \). Finally, we get the asymptotic behavior

\[ \varphi_\pm(x,t,z) = \frac{1}{z} e^{i\nu}(x,t)\sigma_3\sigma_3Q_\pm + O(1), \quad z \to 0. \]

\[ \square \]

**Proposition 3.** The scattering matrices admit asymptotic behaviors

\[ S(z) = e^{-i\nu_0}\sigma_3 + O(z^{-1}), \quad z \to \infty, \]

(6.13)

\[ S(z) = \text{diag} \left( \frac{u_-}{u_+}, \frac{u_+}{u_-} \right) e^{i\nu_0}\sigma_3 + O(z), \quad z \to 0, \]

(6.14)

where

\[ \nu_0 = \frac{1}{2\alpha} \int_{-\infty}^{+\infty} (u_0^2 - |u|^2)dy. \]

(6.15)

**Proof.** By using (4.3) and (6.1), for \( z \to \infty \), we have

\[ e^{i\theta(z)\sigma_3} S(z) = \varphi^{-1}(z)\varphi = (e^{-i\nu_-(x,t)\sigma_3} + O(z^{-1}))(e^{i\nu_+(x,t)\sigma_3} + O(z^{-1})) = e^{-i(\nu_- - \nu_+)}\sigma_3 + O(z^{-1}) = e^{-i\nu_0}\sigma_3 + O(z^{-1}), \]

where

\[ \nu_0 = \frac{1}{2\alpha} \int_{-\infty}^{+\infty} (u_0^2 - |u|^2)dy. \]

Similarly,

\[ s_{11} = \frac{Wr(\phi_{+1}, \phi_{-2})}{\gamma} = \det \left( \begin{array}{cc} O(1) & \frac{1}{z} u_- e^{i\nu_-} + O(1) \\ \frac{1}{z} u_+ e^{i\nu_+} + O(1) & O(1) \end{array} \right) \left( \frac{z^2}{u_0^4} - \frac{z^4}{u_0^4} + \cdots \right) \]

\[ = \frac{u_-}{u_+} e^{i\nu_0} + O(z), \quad z \to 0. \]

\[ \square \]
7 Discrete Spectrum and Residue Conditions

The discrete spectrum of the scattering problem is the set of all values \( z \in \mathbb{C} \setminus \Sigma \) which satisfy the eigenfunctions exist in \( L^2(\mathbb{R}) \). We suppose that \( s_{11}(z) \) has \( N_1 \) simple zeros \( z_1, \ldots, z_{N_1} \) on \( D^+ \cap \{ z \in \mathbb{C} : \text{Im} z > 0, |z| > u_0 \} \), and \( N_2 \) simple zeros \( w_1, \ldots, w_m \) on the circle \( \{ z = u_0 e^{i\varphi} : \pi/2 < \varphi < \pi \} \). The symmetries (5.1)-(5.3) imply that

\[
    s_{11}(\pm z_n) = 0 \iff s_{22}^*(\pm z_n^*) = 0 \iff s_{22} \left( \pm \frac{u_0^2}{z_n} \right) = 0 \iff s_{11} \left( \pm \frac{u_0^2}{z_n^*} \right) = 0, \quad n = 1, \ldots, N_1,
\]

and on the circle

\[
    s_{11}(\pm w_m) = 0 \iff s_{22}^*(\pm w_m^*) = 0, \quad m = 1, \ldots, N_2.
\]

It is convenient to define \( \zeta_n = z_n \) and \( \zeta_{n+N_1} = -\frac{u_0^2}{z_n} \) for \( n = 1, \ldots, N_1 \), \( \zeta_m = w_{m-2N_1} \) for \( m = 2N_1 + 1, \ldots, 2N_1 + N_2 \). Therefore the discrete spectrum is

\[
    Z = \{ \pm \zeta_n, \pm \zeta_{n+N_1} \}_{n=1}^{2N_1+N_2}, \tag{7.1}
\]

whose distribution on the \( z \)-plane are shown in Figure 3.

![Figure 3: Distribution of the discrete spectrum.](image)

All residues on the discrete spectrum are calculated as follows.

\[ 1. \text{ For } s_{11}(\zeta_n) = 0, \; n = 1, \cdots, N_1. \]

From determinant (4.17), we know that the eigenfunction \( \varphi_{+1}(\zeta_n) \) and \( \varphi_{-2}(\zeta_n) \) must be proportional, then there exist a constant \( b_n \neq 0 \), such that

\[
    \phi_{+1}(x, t, \zeta_n) = b_n \phi_{-2}(x, t, \zeta_n),
\]
equivalently,

$$\varphi_{+,1}(x, t, \zeta_n) = b_n e^{2i\theta(\zeta_n)} \varphi_{-,2}(x, t, \zeta_n),$$  \hspace{1cm} (7.2)

by which, the residue on \( z = \zeta_n \) is given by

$$\text{Res}_{z=\zeta_n} \left[ \frac{\varphi_{+,1}(x, t, z)}{s_{11}(z)} \right] = \lim_{z \to \zeta_n} \frac{\varphi_{+,1}(x, t, z)}{s_{11}(z)} = C_n e^{-2i\theta(\zeta_n)} \varphi_{-,2}(x, t, \zeta_n),$$  \hspace{1cm} (7.3)

where \( C_n = \frac{b_n}{s_{11}(\zeta_n)} \).

II. For \( s_{11}(-\zeta_n) = 0 \), \( n = 1, \cdots, N_1 \).

With the symmetries (5.12) and (4.7), we get

$$\varphi_{+,1}(x, t, -\zeta_n) = -b_n e^{-2i\theta(-\zeta_n)} \varphi_{-,2}(x, t, -\zeta_n),$$  \hspace{1cm} (7.4)

$$\varphi_{-,2}(x, t, -\zeta_n) = -\sigma_3 \varphi_{-,2}(x, t, \zeta_n).$$  \hspace{1cm} (7.5)

Also noticing that \( (s_{11}(-\zeta_n))^\prime = -s_{11}^\prime(\zeta_n) \) and \( \theta(-\zeta_n) = \theta(\zeta_n) \), direct computation shows that

$$\text{Res}_{z=-\zeta_n} \left[ \frac{\varphi_{+,1}(x, t, z)}{s_{11}(z)} \right] = \frac{-b_n e^{-2i\theta(-\zeta_n)} \varphi_{-,2}(x, t, -\zeta_n)}{-s_{11}^\prime(\zeta_n)} = -C_n e^{-2i\theta(\zeta_n)} \sigma_3 \varphi_{-,2}(x, t, \zeta_n).$$  \hspace{1cm} (7.6)

III. For \( s_{22}(\zeta_n^*) = 0 \), \( n = 1, \cdots, N_1 \).

There exist a constant \( \tilde{b}_n \neq 0 \), such that

$$\varphi_{+,2}(x, t, \zeta_n^*) = \tilde{b}_n e^{2i\theta(\zeta_n^*)} \varphi_{-,1}(x, t, \zeta_n^*),$$  \hspace{1cm} (7.7)

by which we can derive that

$$\text{Res}_{z=\zeta_n^*} \left[ \frac{\varphi_{+,2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_n e^{2i\theta(\zeta_n^*)} \varphi_{-,1}(x, t, \zeta_n^*),$$  \hspace{1cm} (7.8)

where \( \tilde{C}_n = \frac{\tilde{b}_n}{s_{22}(\zeta_n^*)} \).

Taking the conjugate on both sides of the equation \( (7.7) \) and multiplying by \( \sigma^* \) leads to

$$\sigma^* \varphi_{+,2}^*(x, t, \zeta_n^*) = \tilde{b}_n^* e^{-2i\theta(\zeta_n^*)} \sigma^* \varphi_{-,1}^*(x, t, \zeta_n^*),$$

Again by the symmetry (5.1), we get

$$\varphi_{+,1}(x, t, \zeta_n) = -\tilde{b}_n^* e^{2i\theta(\zeta_n)} \varphi_{-,2}(x, t, \zeta_n).$$  \hspace{1cm} (7.9)
which comparing with (7.2) gives $b_n = -\tilde{b}_n^*$. And from the (5.14) shows that $s_{11}^t(z) = s_{22}^*(z^*)'$, so we have $\tilde{C}_n^* = -C_n^*$.

IV. For $s_{22}(-\zeta_n^*) = 0$, $n = 1, \cdots, N_1$.

From (5.12) and (4.10), we obtain

$$\varphi_{-1}(x, t, -\zeta_n^*) = \sigma_3 \varphi_{-1}(x, t, \zeta_n^*),$$

$$\varphi_{+2}(x, t, -\zeta_n^*) = -\tilde{b}_n e^{2i\theta(-\zeta_n^*)} \varphi_{-1}(x, t, -\zeta_n^*),$$

by which the residue is given way

$$\text{Res}_{z=-\zeta_n^*} \left[ \frac{\varphi_{+2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_n e^{2i\theta(\zeta_n^*)} \sigma_3 \varphi_{-1}(x, t, \zeta_n^*). \tag{7.10}$$

V. For $s_{11} \left( \pm \frac{u_0}{\zeta_n^*} \right) = 0$, $s_{22} \left( \pm \frac{u_0}{\zeta_n^*} \right) = 0$, $n = 1, \cdots, N_1$.

From (5.17), we get the relation

$$s_{11} \left( -\frac{u_0^2}{\zeta_n^*} \right) = \frac{u_-}{u_+} s_{11}^* (\zeta_n^*),$$

which implies that

$$s_{11}^t \left( \pm \frac{u_0^2}{\zeta_n^*} \right) = \left( \frac{\zeta_n^*}{u_0} \right)^2 \frac{u_-}{u_+} (s_{11}^* (\zeta_n^*))'. \tag{7.11}$$

Similarly,

$$s_{22}^t \left( -\frac{u_0^2}{\zeta_n^*} \right) = \left( \frac{\zeta_n^*}{u_0} \right)^2 \frac{u_-}{u_+} (s_{22}^* (\zeta_n^*))',$$

$$s_{11}^t \left( \frac{u_0^2}{\zeta_n^*} \right) = - \left( \frac{\zeta_n^*}{u_0} \right)^2 \frac{u_-}{u_+} (s_{11}^* (\zeta_n^*))',$n

$$s_{22}^t \left( \frac{u_0^2}{\zeta_n^*} \right) = - \left( \frac{\zeta_n^*}{u_0} \right)^2 \frac{u_-}{u_+} (s_{22}^* (\zeta_n^*))'.$
Based on above results analyticity, symmetry and asymptotic of Jost solutions with the solution of the modified NLS equation with nonzero boundary condition.

Finally, combining the above relations, we get

\[
\text{Res}_{z=-\frac{\nu}{\kappa}n} \left[ \frac{\varphi_{+1}(x, t, z)}{s_{11}(z)} \right] = C_{N_1+n} e^{-2i\theta\left(-\frac{\nu}{\kappa}n\right)} \varphi_{-2}(x, t, -\frac{u_0^2}{\kappa_0 n}),
\]

(7.12)

\[
\text{Res}_{z=-\frac{\nu}{\kappa}n} \left[ \frac{\varphi_{+2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_{N_1+n} e^{2i\theta\left(-\frac{\nu}{\kappa}n\right)} \varphi_{-1}(x, t, -\frac{u_0^2}{\kappa_0 n}),
\]

(7.13)

\[
\text{Res}_{z=-\frac{\nu}{\kappa}n} \left[ \frac{\varphi_{+1}(x, t, z)}{s_{11}(z)} \right] = -C_{N_1+n} e^{-2i\theta\left(-\frac{\nu}{\kappa}n\right)} \varphi_{-2}(x, t, -\frac{u_0^2}{\kappa_0 n}),
\]

(7.14)

\[
\text{Res}_{z=-\frac{\nu}{\kappa}n} \left[ \frac{\varphi_{+2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_{N_1+n} e^{2i\theta\left(-\frac{\nu}{\kappa}n\right)} \varphi_{-1}(x, t, -\frac{u_0^2}{\kappa_0 n}),
\]

(7.15)

where

\[
C_{N_1+n} = \frac{u_+^*}{u_-} \left(\frac{u_0}{\kappa_0 n}\right)^2 \tilde{C}_n, \quad \tilde{C}_{N_1+n} = \frac{u_+^*}{u_-} \left(\frac{u_0}{\kappa_0 n}\right)^2 C_n,
\]

(7.16)

with the relation

\[
\tilde{C}_{N_1+n} = -C_{N_1+n}^*.
\]

VI. For \(s_{11}(\pm u_m^*) = 0, \ s_{22}(\pm u_m^*) = 0, \ n = 1, \cdots, N_2\).

Analogously, we consider the residue conditions at \(\pm u_m^*\) and \(\pm u_m^*\) and get

\[
\text{Res}_{z=u_m^*} \left[ \frac{\varphi_{+1}(x, t, z)}{s_{11}(z)} \right] = C_{2N_1+m} e^{-2i\theta(u_m^*)} \varphi_{-2}(x, t, u_m^*),
\]

(7.17)

\[
\text{Res}_{z=-u_m^*} \left[ \frac{\varphi_{+1}(x, t, z)}{s_{11}(z)} \right] = -C_{2N_1+m} e^{-2i\theta(u_m^*)} \varphi_{-2}(x, t, u_m^*),
\]

(7.18)

\[
\text{Res}_{z=u_m^*} \left[ \frac{\varphi_{+2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_{2N_1+m} e^{2i\theta(u_m^*)} \varphi_{-1}(x, t, u_m^*),
\]

(7.19)

\[
\text{Res}_{z=-u_m^*} \left[ \frac{\varphi_{+2}(x, t, z)}{s_{22}(z)} \right] = \tilde{C}_{2N_1+m} e^{2i\theta(u_m^*)} \varphi_{-1}(x, t, u_m^*),
\]

(7.20)

where \(C_{2N_1+m} = \frac{b_{2N_1+m}}{s_{11}(u_m^*)}, \ \tilde{C}_{2N_1+m} = -C_{2N_1+m}^*\) and \(b_{2N_1+m}\) are arbitrary constants.

8 Riemann-Hilbert Problem

Based on above results analyticity, symmetry and asymptotic of Jost solutions \(\varphi \pm\) and scattering data \(s_{ij}(z)\), we now can establish a generalized Riemann-Hilbert Problem associated with the solution of the modified NLS equation with nonzero boundary condition.
8.1 The Riemann-Hilbert Problem

**Proposition 4.** Define the sectionally meromorphic matrices

\[
M(x, t, z) = \begin{cases} 
M^+ = \left( \frac{\varphi_{+,1}}{s_{11}}, \frac{\varphi_{-,2}}{s_{22}} \right), & \text{as } z \in D^+, \\
M^- = \left( \frac{\varphi_{-,1}}{s_{11}}, \frac{\varphi_{+,2}}{s_{22}} \right), & \text{as } z \in D^-, 
\end{cases} \tag{8.1}
\]

then a multiplicative matrix Riemann-Hilbert problem is proposed:

- **Analyticity:** \(M(x, t, z)\) is analytic in \(\mathbb{C} \setminus \Sigma\) and has single poles.
- **Jump condition**

\[
M^-(x, t, z) = M^+(x, t, z)(I - G(x, t, z)), \quad z \in \Sigma, \tag{8.2}
\]

where

\[
G(x, t, z) = \begin{pmatrix} 0 & e^{-2i\theta}\tilde{\rho}(z) \\
e^{2i\theta}\rho(z) & \rho(z)\tilde{\rho}(z) \end{pmatrix}. \tag{8.3}
\]

- **Asymptotic behaviors**

\[
M(x, t, z) \sim e^{i\nu - \sigma_3} + O(z^{-1}), \quad z \to \infty, \tag{8.4}
\]

\[
M(x, t, z) \sim \frac{1}{z}e^{i\nu - \sigma_3}Q_+ + O(1), \quad z \to 0. \tag{8.5}
\]

**Proof.** The analyticity of \(M_{\pm}\) can be found out from the analyticity of the \(\varphi_{\pm}\) and \(S(z)\). From (4.4), we get

\[
\varphi_{-,2}(x, t, z) = -\tilde{\rho}(z)\varphi_{-,1}(x, t, z) + \frac{\varphi_{+,2}(x, t, z)}{s_{22}(z)}, \tag{8.6}
\]

\[
\frac{\varphi_{+,1}(x, t, z)}{s_{11}(z)} = (1 - \rho(z)\tilde{\rho}(z))\varphi_{-,1}(x, t, z) + \rho(z)\frac{\varphi_{+,2}(x, t, z)}{s_{22}(z)}, \tag{8.7}
\]

which leads to the jump condition (8.2).

With the asymptotic behaviors of the Jost solution (6.2) and scattering matrix (6.14), we can derive that

\[
M^+(x, t, z) \sim \frac{1}{z}e^{i\nu - \sigma_3}Q_+ + O(1), \quad z \to 0, \tag{8.8}
\]

\[
M^-(x, t, z) \sim \frac{1}{z}e^{i\nu - \sigma_3}Q_- + O(1), \quad z \to 0. \tag{8.9}
\]

Similarly, we can get another asymptotic behavior (8.4) immediately. \(\square\)
8.2 Reconstruction Formula

Based on the results in section 7, we can obtain the Residue Conditions on the meromorphic matrices $M^\pm$ as follow:

$$\text{Res } M^+ = \left( C_n e^{-2\theta(\zeta_n)} \varphi_{-2}(x, t, \zeta_n) \right), \quad n = 1, \cdots, 2N_1 + N_2, \quad (8.10)$$

$$\text{Res } M^+ = \left( -C_n e^{-2\theta(\zeta_n)} \sigma_3 \varphi_{-2}(x, t, \zeta_n) \right), \quad n = 1, \cdots, 2N_1 + N_2, \quad (8.11)$$

$$\text{Res } M^- = \left( 0 \right) \quad (8.12)$$

Subtracting out the asymptotic behaviors and the pole contributions, we obtain that

$$M^--e^{i\nu-\sigma_3} - \frac{1}{z} e^{i\nu-\sigma_3} \sigma_3 Q - \sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^-}{z - \zeta_n^*} - \sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^+}{z - \zeta_n} \quad (8.14)$$

$$\sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^+}{z + \zeta_n} - \sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^-}{z + \zeta_n^*} = M^+ G.$$

Apparently, the left-hand side is analytic in $D^-$ and is $O(z^{-1})$ as $z \to \infty$, while the sum of the first five terms of the right-hand side is analytic in $D^+$ and is $O(z^{-1})$ as $z \to \infty$.

Using Plemelj’s formula, we obtain

$$M(x, t, z) = e^{i\nu-\sigma_3} + \frac{1}{z} e^{i\nu-\sigma_3} \sigma_3 Q - \sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^+}{z - \zeta_n} + \sum_{n=1}^{2N_1+N_2} \frac{\text{Res}_{\zeta_n} M^-}{z - \zeta_n^*} + \frac{2N_1+N_2}{z + \zeta_n} \frac{\text{Res}_{\zeta_n} M^+}{z + \zeta_n} + \frac{2N_1+N_2}{z + \zeta_n^*} \frac{\text{Res}_{\zeta_n} M^-}{z + \zeta_n^*} + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x, t, \zeta)}{\zeta - z} G(x, t, z) d\zeta, \quad (8.15)$$

Where $z \in \mathbb{C} \setminus \Sigma$.

In the remaining parts of this section we would like to give the reconstruction formula from the solution of the Riemann-Hilbert problem. When $z \to \infty$, from the previous equa-
tion we can get
\[ M(x,t,z) = e^{i\nu_+ - \sigma_3} + \frac{1}{z} \left( e^{i\nu_- - \sigma_3} q_+ + \sum_{n=1}^{2N_1+N_2} \left( \text{Res}_{z=\zeta_n} M^+ + \text{Res}_{z=-\zeta_n} M^- + \text{Res}_{z=-\zeta_n^*} M^+ + \text{Res}_{z=\zeta_n^*} M^- \right) \right) \]
\[ - \frac{1}{2\pi i} \int_{\Sigma} M^+(x,t,\zeta) G(x,t,\zeta) d\zeta + O(z^{-2}). \]
(8.16)

When \( z = \zeta_n \), we can calculate the second column of \( M^+ \) in (8.15). Then we obtain
\[ \varphi_{-2}(x,t,\zeta_n) = \left( \frac{1}{\zeta_n} u_\nu e^{-i\nu_-} \right) + 2 \sum_{k=1}^{2N_1+N_2} \frac{\tilde{C}_k e^{2i\theta(\zeta_k^*)}}{\zeta_n^2 - \zeta_k^2} Z_1 \varphi_{-1}(x,t,\zeta_k^*) \]
\[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x,t,\zeta)}{\zeta - \zeta_n} G(x,t,\zeta) d\zeta, \]
(8.17)
where \( Z_1 = \text{diag}(\zeta_n, \zeta_n^*) \), for \( n = 1, \ldots, 2N_1 + N_2 \).

In the same way, when \( z = \zeta_n^* \) we can evaluate the first column of \( M^- \) and obtain
\[ \varphi_{-1}(x,t,\zeta_n^*) = \left( -\frac{e^{i\nu_-}}{\zeta_n^*} u_\nu e^{-i\nu_-} \right) + 2 \sum_{j=1}^{2N_1+N_2} \frac{\tilde{C}_j e^{-2i\theta(\zeta_j)}}{\zeta_n^{*2} - \zeta_j^2} Z_2 \varphi_{-2}(x,t,\zeta_j) \]
\[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^-(x,t,\zeta)}{\zeta - \zeta_n^*} G(x,t,\zeta) d\zeta, \]
(8.18)
where \( Z_2 = \text{diag}(\zeta_j, \zeta_n^*) \), for \( n = 1, \ldots, 2N_1 + N_2 \).

By using (8.10) and (8.16), we get the reconstruction formula for the potential
\[ u(x,t) = e^{2i\nu_-} u_\nu + e^{i\nu_+} \left\{ 2 \sum_{n=1}^{2N_1+N_2} \frac{\tilde{C}_n e^{2i\theta(\zeta_n^*)} \varphi_{-11}(\zeta_n^*)}{2\pi i} \int_{\Sigma} (M^+ G)_{12}(x,t,\zeta) d\zeta \right\}. \]
(8.19)

### 8.3 Trace formula and theta condition

Define two functions \( \beta^\pm(z) \) as follow
\[ \beta^+(z) = s_{11}(z) \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^{*2}}{z^2 - \zeta_n^2} e^{i\nu_0}, \]
(8.20)
\[ \beta^-(z) = s_{22}(z) \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^2}{z^2 - \zeta_n^{*2}} e^{-i\nu_0}, \]
which implies the relation \( \beta^+(z)\beta^-(z) = s_{11}(z)s_{22}(z) \) and \( \beta^\pm(z) \rightarrow 1, \ z \rightarrow \pm \infty \).

From the analyticity of the scattering matrix, we see that the above functions are analytic and have no zeros in \( D^+ \) and \( D^- \) respectively.
From $\det S(z) = 1$, we obtain that
\[
\frac{1}{s_{11} s_{22}} = \frac{s_{11} s_{22} - s_{12} s_{21}}{s_{11} s_{22}} = 1 - \rho(z) \bar{\rho}(z) = 1 + \rho(z) \rho^*(z^*),
\]
so that
\[
\beta^+(z) \beta^-(z) = \frac{1}{1 + \rho(z) \rho^*(z^*)}, \quad z \in \Sigma.
\]
Taking logarithms to the above relation and using the Plemelj’ formula, we have
\[
\log \beta_{\pm}(z) = \pm \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 + \rho(\zeta) \rho^*(\zeta^*)]}{\zeta - z} d\zeta, \quad z \in D^\pm. \tag{8.21}
\]
Substituting them back into (8.20) leads to
\[
s_{11}(z) = \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 + \rho(\zeta) \rho^*(\zeta^*)]}{\zeta - z} d\zeta \right] \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^2}{z^2 - \zeta_n^*} e^{-i\nu_0}, \quad z \in D^+, \tag{8.22}
\]
\[
s_{22}(z) = \exp \left[ \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 + \rho(\zeta) \rho^*(\zeta^*)]}{\zeta - z} d\zeta \right] \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^*}{z^2 - \zeta_n^2} e^{i\nu_0}, \quad z \in D^-. \tag{8.23}
\]
The above formulas are called trace formulas, which express the analytic scattering coefficient in terms of the discrete eigenvalues and the reflection coefficient.

Taking the limit $z \to 0$ in (8.22), and using the asymptotic behavior of the scattering matrix (6.14) we then obtain the following theta condition
\[
\arg \left( \frac{u_-}{u_+} \right) + 2\nu_0 = 8 \sum_{n=1}^{2N_1+N_2} \arg \zeta_n + \frac{1}{2\pi} \int_{\Sigma} \frac{\log[1 + \rho(\zeta) \rho^*(\zeta^*)]}{\zeta} d\zeta. \tag{8.24}
\]

9 Solving the Riemann-Hilbert problem

9.1 The formula for $N$-soliton solutions

We consider the reflectionless potential with the reflection coefficient $\rho(z) = 0$, then (8.19) becomes
\[
u(x, t) = e^{2i\nu_0} u_- + e^{i\nu_0} - 2 \sum_{n=1}^{2N_1+N_2} \tilde{C}_n e^{2i\theta(\zeta_n^*)} \varphi_{-11}(\zeta_n^*). \tag{9.1}
\]
Denote
\[
c_j(x, t, \zeta) = \frac{C_j}{z^2 - \zeta_j^2} e^{-2i\theta(x, t; \zeta_j)}, \quad j = 1, \ldots, 2N_1 + N_2. \tag{9.2}
\]
We can obtain that
\[
c_j^*(x, t, \zeta^*_k) = \frac{C_j^*}{\zeta_k^2 - \zeta_j^*} e^{2i\theta(x, t; \zeta_k^*)}, \quad j = 1, \ldots, 2N_1 + N_2. \tag{9.3}
\]
Then from (8.17) and (8.18), we can drive
\[ \varphi_{-12}(x, t, \zeta_j) = \frac{1}{\zeta_j} u_- e^{i\nu} + \sum_{k=1}^{2N_1+N_2} 2\zeta_j c_k^*(\zeta_j^*) \varphi_{-11}(x, t, \zeta_k^*), \quad j = 1, \cdots, 2N_1 + N_2. \] (9.4)

\[ \varphi_{-11}(x, t, \zeta_n^*) = e^{i\nu} + \sum_{j=1}^{2N_1+N_2} 2\zeta_j c_j(\zeta_n^*) \varphi_{-12}(x, t, \zeta_j), \quad n = 1, \cdots, 2N_1 + N_2. \] (9.5)

Substituting (9.4) into (9.5), we get
\[ \varphi_{-11}(x, t, \zeta_n^*) = e^{i\nu} + 2e^{i\nu} - \sum_{j=1}^{2N_1+N_2} u_- c_j(\zeta_n^*) + \sum_{j=1}^{2N_1+N_2} \sum_{k=1}^{2N_1+N_2} 4\zeta_j^2 c_j(\zeta_n^*) c_k^*(\zeta_j^*) \varphi_{-11}(x, t, \zeta_k^*), \quad n = 1, \cdots, 2N_1 + N_2. \] (9.6)

Next, we would like to write above system (9.6) as a matrix form, so let
\[ X = (X_1, \ldots, X_{2N_1+N_2})^t, \quad B = (B_1, \ldots, B_{2N_1+N_2})^t, \]
where
\[ X_n = \varphi_{-11}(x, t, \zeta_n^*), \quad B_n = 1 + 2u_- \sum_{j=1}^{2N_1+N_2} c_j(\zeta_n^*), \quad n = 1, \ldots, 2N_1 + N_2. \]

Again define the \((2N_1 + N_2) \times (2N_1 + N_2)\) matrix \(A = (A_{nk})\), where
\[ A_{nk} = - \sum_{j=1}^{2N_1+N_2} 4\zeta_j^2 c_j(\zeta_n^*) c_k^*(\zeta_j^*), \quad n, k = 1, \ldots, 2N_1 + N_2. \]

Then the linear system (9.6) becomes
\[ M X = e^{i\nu} B, \]
where \(M = I + A = (M_1, \ldots, M_{2N_1+N_2})\). The solution of the system is
\[ X_n = e^{i\nu} \frac{\det M_{n}^{ext}}{\det M}, \quad n = 1, \ldots, 2N_1 + N_2, \] (9.7)

where
\[ M_{n}^{ext} = (M_1, \ldots, M_{n-1}, B, M_{n+1}, \ldots, M_{2N_1+N_2}). \]

Therefore, substituting the above \(X_1, \ldots, X_{2N_1+N_2}\) back into the reconstruction formula (9.1), we then get formulae for \(N\)-soliton solutions
\[ u(x, t) = e^{2i\nu} - u_- + 2e^{2i\nu} \frac{\det M^{aug}}{\det M}, \] (9.8)
where the \((2N_1 + N_2 + 1) \times (2N_1 + N_2 + 1)\) matrix is given by

\[
M^{\text{aug}} = \begin{pmatrix} 0 & Y \\ B & M \end{pmatrix}, \quad Y = (Y_1, ..., Y_{2N_1+N_2}),
\]

and

\[
Y_n = \tilde{C}_n e^{2i\theta(x,t,\zeta^*_n)}, \quad n = 1, ..., 2N_1 + N_2.
\]

9.2 One-Soliton Solutions

As an application of \(N\)-soliton solution formula (9.8), we present two kinds of the one-soliton solutions of modified NLS equation according to different distribution of the spectrum (7.1). Without loss of generality, we take \(u_0 = 1\), due to the fact if \(u(x,t)\) is a solution of (1.3), then \(cu(x,t)\) is also the solution of (1.3).

Case I. \(N_1 = 1, N_2 = 0\):

In this case, one eigenvalue fall to outside of circle, and suppose that \(\zeta_1 = Ze^{i\gamma}\) with \(Z > 1, \gamma \in (\frac{\pi}{2}, \pi)\), then the other points in discrete spectrum are

\[
-\zeta_1 = -Ze^{i\gamma}, \quad \zeta_2 = -\frac{1}{Z} e^{i\gamma}, \quad -\zeta_2 = \frac{1}{Z} e^{i\gamma},
\]

\[
\zeta_1^* = Ze^{-i\gamma}, \quad -\zeta_1^* = -Ze^{-i\gamma}, \quad \zeta_2^* = -\frac{1}{Z} e^{-i\gamma}, \quad -\zeta_2^* = \frac{1}{Z} e^{-i\gamma}.
\]

By using the theta condition (8.24), we have

\[
\arg\left(\frac{u_-}{u_+}\right) + 2\nu_0 = 16\gamma,
\]

which allow us to set \(u_- = 1\) and \(u_+ = e^{i(2\nu_0 - 16\gamma)}\). And we can also know that \(C_1 = e^{\xi + i\varphi}\), with \(\xi, \varphi \in \mathbb{R}\) and \(C_2 = -\frac{1}{2\pi} e^{\xi + i(2\gamma - \varphi)}\).

Substituting above data into formulae (9.8), we get the one-soliton solution

\[
u(x,t) = e^{2i\nu_-} + 2e^{2i\nu_-} \frac{\det \left( \begin{array}{ccc} 0 & Y_1 & Y_2 \\ B_1 & 1 + A_{11} & A_{12} \\ B_2 & A_{21} & 1 + A_{22} \end{array} \right)}{\det \left( \begin{array}{ccc} 1 + A_{11} & A_{12} \\ A_{21} & 1 + A_{22} \end{array} \right)}, \quad (9.9)
\]
where

\[
\theta(x,t,\zeta_j) = -\frac{1}{4\alpha}(\zeta_j^2 - \frac{1}{\zeta_j^2}) \left[ x + \frac{1}{2\alpha}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 4\alpha - \frac{2}{\alpha}t \right], \quad j = 1, 2,
\]

\[
c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^2} e^{-2i\theta(x,t,\zeta_j)}, \quad j = 1, 2,
\]

\[
B_n = 1 + 2 \sum_{j=1}^{2} c_j(\zeta_n^*), \quad Y_n = -C_n^* e^{2i\theta(x,t,\zeta_n^*)}, \quad n = 1, 2,
\]

\[
A_{nk} = -2 \sum_{j=1}^{2} 4\zeta_j^2 c_j(\zeta_n^*) c_k(\zeta_j^*), \quad n, k = 1, 2.
\]

Specially taking parameters \(\alpha = 1, Z = 2, \gamma = \frac{3\pi}{4}, \xi = 0, \varphi = 0\), we draw a graphics of wave propagation in perspective view and along x- and t-orientation for the solution \(|u(x,t)|\) in Figure 4 and Figure 5. This kind of solution wave exhibits localized breather wave; While propagation of the wave along x and t orientation are locally oscillatory in the middle, but two tail parts are stationary and go to a nonzero constant, unlike soliton with zero boundary condition, where it go to zero.

![Figure 4: Breather solution: (a) Perspective view of the wave; (b) The contour of the wave.](image)

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Case II. \( N_1 = 0, \ N_2 = 1: \)

Taking a eigenvalue \( \zeta_1 = e^{i\beta} \) with \( \beta \in (\frac{\pi}{2}, \pi) \) on the circle \(|z| = 1\), then the discrete spectrum are \( \{e^{i\beta}, -e^{i\beta}, e^{-i\beta}, -e^{-i\beta}\} \). By using theta condition \((8.24)\), we obtain that

\[
\arg(u_-/u_+) = 8\beta - 2\nu_0, \quad (9.10)
\]

which allow us to let \( u_- = 1 \) and \( u_+ = e^{i(2\nu_0-8\beta)} \). Take \( C_1 = e^{i\tau+\kappa} \) with \( \tau, \kappa \in \mathbb{R} \), we find another kind of one-soliton solution

\[
u(x,t) = e^{2i\nu_-} + 2e^{2i\nu_-} \frac{\det \left( \begin{array}{cc} 0 & Y \\ B & 1 + A \end{array} \right)}{1 + A}, \quad (9.11)
\]

where

\[
\theta(x, t, \zeta_1) = -\frac{1}{2\alpha} \sinh(\beta) \left[ x + \left( \frac{1}{\alpha} \cosh(\beta) - 4\alpha - \frac{2}{\alpha} \right)t \right],
\]

\[
c_1(x, t, z) = \frac{C_1}{z^2 - \zeta_1^*} e^{-2i\theta(x, t, \zeta_1)}, \quad B = 1 + 2c_1(\zeta_1^*),
\]

\[
A = -4\zeta_1^2|c_1(\zeta_1^*)|^2, \quad Y = -C_1 e^{2i\theta(x, t, \zeta_1^*)}.
\]

Specially taking parameters \( \alpha = 1, \beta = \frac{3\pi}{4}, \tau = \frac{\pi}{2}, \kappa = 0 \), propagation of the wave in perspective view and along x- and t-orientation for the solution \(|\nu(x, t)|\) are given in Figure 6 and Figure 7. This kind of one-soliton wave exhibits bell-type localized wave like soliton wave zero boundary condition. While propagation of wave along x and t orientation are locally oscillatory in the middle, but two tail parts are stationary and go to zero.
Figure 6: One-solition solution: (a) Perspective view of the wave; (b) The contour of the wave.

Figure 7: One-soliton solution: (a) Propagation pattern of the wave along $x$-orientation with $t = -0.1$ (blue), $t = 0$ (orange), $t = 0.1$ (green); (b) Propagation pattern of the wave along $t$-orientation with $x = -1$ (blue), $x = 0$ (orange), $x = 1$ (green).

9.3 Two-Soliton Solutions

Here we consider three kinds of two-soliton solutions according to different distribution of the spectrum $Z = \{ \pm \zeta_n, \pm \zeta_n^* \}_{n=1}^{2N_1+N_2}$.

Case I. $N_1 = 1, N_2 = 1$:

We take two eigenvalues

$$\zeta_1 = Ze^{i\gamma}, \quad \zeta_3 = e^{i\beta}, \quad Z > 1, \quad \gamma, \beta \in (\frac{\pi}{2}, \pi).$$
then the other points in discrete spectrum are
\[-\zeta_1 = -Ze^{i\gamma}, \quad \zeta_2 = -\frac{1}{Z}e^{i\gamma}, \quad -\zeta_2 = \frac{1}{Z}e^{i\gamma}, \quad \zeta_1^* = Ze^{-i\gamma}, \quad -\zeta_1^* = -Ze^{-i\gamma}, \]
\[\zeta_2^* = -\frac{1}{Z}e^{-i\gamma}, \quad -\zeta_2^* = \frac{1}{Z}e^{-i\gamma}, \quad -\zeta_3 = -e^{i\beta}, \quad \zeta_3^* = e^{-i\beta}, \quad -\zeta_3^* = -e^{-i\beta}.\]

By using the theta condition (8.24), we have
\[
\arg (\frac{u_-}{u_+}) + 2\nu_0 = 16\gamma + 8\beta,
\]
from which we choose \(u_- = 1\) and \(u_+ = e^{i(2\nu_0 - 16\gamma - 8\beta)}\). Let \(C_1 = e^{\xi+\nu}, C_3 = e^{\tau+\kappa}\), with \(\xi, \varphi, \tau, \kappa \in \mathbb{R}\), then \(C_2 = -\frac{1}{Z}e^{\xi+i(2\gamma-\nu)}\).

Substituting above data into formulae (9.8), we get one kind of two-soliton solution
\[
u(x, t) = e^{2\nu_+ - 2e^{2\nu_-}}det\left(\begin{array}{cccc}
0 & Y_1 & Y_2 & Y_3 \\
B_1 & 1 + A_{11} & A_{12} & A_{13} \\
B_2 & A_{21} & 1 + A_{22} & A_{23} \\
B_3 & A_{31} & A_{32} & 1 + A_{33}
\end{array}\right), \quad (9.12)
\]
where
\[
\theta(x, t, \zeta_j) = -\frac{1}{4\alpha}(\zeta_j^2 - 1)\left[ x + \left( \frac{1}{2\alpha}(\zeta_j^2 + 1) - 4\alpha - \frac{2}{\alpha} \right) t \right], \quad j = 1, 2, 3,
\]
\[
c_j(x, t, z) = \frac{C_j}{z^2 - \zeta_j^2}e^{-2i\theta(x, t, \zeta_j)}, \quad j = 1, 2, 3,
\]
\[
B_n = 1 + 2\sum_{j=1}^3 c_j(\zeta_j^n), \quad Y_n = -C_n e^{2i\theta(x, t, \zeta_n^n)}, \quad n = 1, 2, 3.
\]
\[A_{nk} = -\sum_{j=1}^3 4\zeta_j^2 c_j(\zeta_j^n)c_j^*(\zeta_j^k), \quad n, k = 1, 2, 3.
\]

The propagation feature of this two-soliton wave are shown in Figure 8 and Figure 9. This kind of two-soliton wave exhibits two-peak localized wave, there is small oscillatory waves on one peak, and another peak is smooth. While propagation pattern of the two-soliton wave along x-orientation and t-orientation are locally oscillatory in the middle, and two tail parts are stationary and go to a nonzero constant.
Case II. $N_1 = 0, N_2 = 2$:

Choosing two eigenvalues $\zeta_j = e^{i\beta_j}$, with $\beta_j \in (\frac{\pi}{2}, \pi)$, $j = 1, 2$, then discrete spectrum are $\{e^{i\beta_j}, e^{-i\beta_j}, e^{i(-\beta_j)}, e^{-i(-\beta_j)}\}_{j=1,2}$. By using theta condition (8.24), we get

$$\arg(u_-/u_+) = 8\beta_1 + 8\beta_2 - 2\nu_0.$$ (9.13)

We set $u_- = 1$ then $u_+ = e^{i(2\nu_0 - 8\beta_1 - 8\beta_2)}$. Let $C_j = e^{i\tau_j + \kappa_j}$, with $\tau_j, \kappa_j \in \mathbb{R}$, $j = 1, 2$. Once again we get the soliton solution with parameters $N_1 = 0$ and $N_2 = 2$ by substituting above
The wave propagation properties of this two-soliton solution are shown in Figure 10 and Figure 11. This kind of two-soliton wave exhibits two-peak localized wave, there are no oscillatory waves and smooth on both peaks; While propagation of the wave along \( x \)-orientation and \( t \)-orientation are locally oscillatory in the middle, and two tail parts are stationary and go to zero.
Substituting above data into formulae (9.8), we get the two-soliton solution

$$u(x, t) = e^{2\nu_-} + 2e^{2\nu_-} \frac{\det \begin{pmatrix} 0 & Y_1 & Y_2 & Y_3 & Y_4 \\ B_1 & 1 + A_{11} & A_{12} & A_{13} & A_{14} \\ B_2 & A_{21} & 1 + A_{22} & A_{23} & A_{24} \\ B_3 & A_{31} & A_{32} & 1 + A_{33} & A_{34} \\ B_4 & A_{41} & A_{42} & A_{43} & 1 + A_{44} \end{pmatrix}}{\det \begin{pmatrix} 1 + A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & 1 + A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & 1 + A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & 1 + A_{44} \end{pmatrix} \}}.$$

(9.15)

Figure 11: Two-soliton solution: (a) Propagation pattern of the wave along x-orientation with $t = -0.2$ (blue), $t = 0$ (orange), $t = 0.2$ (green); (b) Propagation pattern of the wave along t-orientation with $x = -1$ (blue), $x = 0$ (orange), $x = 1$ (green).

**Case II.** $N_1 = 2$, $N_2 = 0$:

Similarly, let $\zeta_j = Z_j e^{i\gamma_j}$, with $Z_j > 1$ and $\gamma_j \in (\frac{\pi}{2}, \pi)$, $j = 1, 2$, $\zeta_1 \neq \zeta_2$. Then the other points in discrete spectrum are

$$-\zeta_1 = -Z_1 e^{i\gamma_1}, \zeta_1^* = Z_1 e^{-i\gamma_1}, -\zeta_1^* = -Z_1 e^{-i\gamma_1},$$
$$\zeta_3 = -\frac{1}{Z_1} e^{i\gamma_1}, -\zeta_3^* = \frac{1}{Z_1} e^{-i\gamma_1}, \zeta_3^* = \frac{1}{Z_1} e^{-i\gamma_1},$$
$$-\zeta_2 = -Z_2 e^{i\gamma_2}, \zeta_2^* = Z_2 e^{-i\gamma_2}, -\zeta_2^* = -Z_2 e^{-i\gamma_2},$$
$$\zeta_4 = -\frac{1}{Z_2} e^{i\gamma_2}, -\zeta_4^* = \frac{1}{Z_2} e^{-i\gamma_2}, \zeta_4^* = \frac{1}{Z_2} e^{-i\gamma_2}.$$

By using the theta condition (8.24), we have

$$\arg (u_-/u_+) + 2\nu_0 = 16\gamma_1 + 16\gamma_2.$$
where

\[
\theta(x,t,\zeta_j) = -\frac{1}{4\alpha} \left( \zeta_j^2 - \frac{1}{\zeta_j^2} \right) \left[ x + \left( \frac{1}{2\alpha}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 4\alpha - \frac{2}{\alpha} \right)t \right], \quad j = 1, 2, 3, 4,
\]

\[
c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^2} e^{-2i\theta(x,t,\zeta_j)}, \quad j = 1, 2, 3, 4,
\]

\[
B_n = 1 + 2 \sum_{j=1}^{4} c_j(\zeta_n^*), \quad n = 1, 2, 3, 4,
\]

\[
A_{nk} = -\sum_{j=1}^{4} 4\zeta_j^2 c_j(\zeta_n^*) e_k(\zeta_j^*), \quad n, k = 1, 2, 3, 4,
\]

\[
Y_n = -C_n^* e^{2i\theta(x,t,\zeta_n^*)}, \quad n = 1, 2, 3, 4.
\]

In this case, the wave propagation properties of two-soliton solution (9.15) are shown in Figure 11 and Figure 12. This kind of two-soliton wave exhibits two-peak localized wave, there are no oscillatory waves and smooth on both peaks; While propagation of the wave along x-orientation and t-orientation are locally stable in the middle, and two tail parts go to a nonzero constant.

Of course the N-soliton solution expressions (9.8) are not limited to one-soliton and two-soliton solutions above, and it allows us to obtain explicit solutions with an arbitrary number of soliton solutions for the modified NLS equation. Finally, we should remark that in the limit \( u_0 \to 0 \), both kinds of one-soliton and two-soliton solutions reduce to the same, soliton solutions of the modified NLS equation with zero boundary condition [28].

![Figure 12: Two-soliton solution: (a) Perspective view of the wave; (b) The contour of the wave.](image)
Figure 13: Two-soliton solution: (a) Propagation pattern of the wave along $x$-orientation with $t = -0.2$ (blue), $t = 0$ (orange), $t = 0.2$ (green); (b) Propagation pattern of the wave along $t$-orientation with $x = -1$ (blue), $x = 0$ (orange), $x = 1$ (green).

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