Violation of Leggett-Garg type inequalities in a driven two level atom interacting with a squeezed thermal reservoir

Javid Naikoo,1,* Subhashish Banerjee,1,+ Arun M. Jayannavar2,†
1Indian Institute of Technology Jodhpur, Jodhpur 342011, India
2Institute of Physics, Bhubaneswar, India

Abstract The violation of Leggett-Garg type inequalities (LGtIs) is studied on a two level atom, driven by an external field in the presence of a squeezed thermal reservoir. The violations are observed in the underdamped regime where the spontaneous transition rate is much smaller compared to the Rabi frequency. Increase in thermal effects is found to decrease the extent of violation as well as the time over which the violation lasts. With increase in the value squeezing parameter the extent of violation of LGtIs is seen to reduce. The violation of LGtIs is favored by increase in the driving frequency. Further, the interplay of the degree of violation and strength of the measurements is studied. It is found that the maximum violation occurs for ideal projective measurements.

I. INTRODUCTION

Quantum mechanics is so far the most elegant interpretation of nature whose predictions have been verified in various experiments. Central to quantum mechanics are the notions like coherence and entanglement arising from the superposition principle [1, 2]. Various approaches have been developed for quantification of quantumness leading to computable measures of nonclassicality [3, 4]. Another way of assessing the quantum coherent evolution is via inequalities based on the time correlation functions, known as Leggett-Garg inequalities (LGIs).

The LGIs have been developed to test the quantum coherence at macroscopic level [5, 6]. These inequalities are based on the assumptions of macrorealism and noninvasive measurability. The former assigns well defined macroscopically distinct states to an observable irrespective of the observation, while the later ensures that the post measurement dynamics is unaffected by the act of measurement. A quantum mechanical system does not obey these assumptions. The superposition principle violates macrorealism and the collapse postulate nullifies the possibility of a noninvasive measurement.

The verification of LGIs involve a single system being measured at different times unlike Bell inequality which involves multiple parties spatially separated from each other [7]. The simplest Leggett-Garg inequality is the one corresponding to three time measurements made at times \( t_0, t_1 \) and \( t_2 \) such that \( t_0 < t_1 < t_2 \). For a dichotomic operator \( \mathcal{M}(t) \), we define the two time correlation function \( C(t_i, t_j) = \langle \mathcal{M}(t_i)\mathcal{M}(t_j) \rangle = \text{Tr}[\rho\mathcal{M}(t_i)\mathcal{M}(t_j)] \). For the three time measurement case, we define the following combination of the two time correlation functions \( K_3 = C(t_0, t_1) + C(t_1, t_2) - C(t_0, t_2) \), such that the simplest LGI reads

\[
-3 \leq K_3 \leq 1. \tag{1}
\]

A violation of either lower or the upper bound is a signature of the “quantumness” of the system. The two time correlation function can be evaluated as follows,

\[
C(t_i, t_j) = \sum_{m,n=\pm} mn \text{Tr} \left[ \Pi^m \mathcal{E}_{t_i \leftarrow t_f} [\Pi^n \rho(t_i)\Pi^n] \right]. \tag{2}
\]

Here, \( \mathcal{E}_{t_i \leftarrow t_f} \) is the map governing the time evolution of the state, i.e., \( \rho(t_0) = \mathcal{E}_{t_0 \leftarrow t_f} [\rho(t_f)] \). The LGIs have been part of many theoretic [8–19] and experimental [20–27] studies.

In this work, we deviate from the original formulation of LGI and study instead a variant form of it, known as Leggett-Garg type inequalities (LGtIs) introduced in [28–30] and experimentally verified in [31, 32]. These inequalities were derived to avoid the requirement of noninvasive measurements at intermediate times. This feature makes them more suitable for the experimental verification as compared to LGIs. The assumption of NIM is replaced by a weaker condition known as stationarity. This asserts that the conditional probability \( p(\phi,t_j|\psi,t_i) \) that the system is in state \( \phi \) at time \( t_j \) given that it was in state \( \psi \) at time \( t_i \) is a function of the time difference \( t_j - t_i \). Invoking stationarity leads to the following form of LGtIs

\[
K_\pm = \pm 2C(t_0, t) - C(t_0, 2t) \leq 1. \tag{3}
\]

Here, \( t = t_2 - t_1 = t_1 - t_0 \), is the time between two successive measurements. From here on, we will call \( K_\pm \) as LG parameter. Though the assumption of stationarity helps to put the inequalities into easily testable forms, it reduces the class of macrorealist theories which are put to the test [28]. The stationarity condition holds provided the system can be prepared in a well-defined state and the system evolves under Markovian dynamics. These conditions are satisfied in the model considered in this work. Therefore, for a suitable experimental setup, inequalities (3) provide a tool to quantitatively probe the coherence effects in this system.

Here we study the violation of LGtIs in a driven two-level atom interacting with a squeezed thermal reservoir. The paper is organized as follows. In Sec. (II), we discuss
in detail the model considered. Section (III) is devoted to the description of LGIs in the context of the model considered. The results and their discussion are given in Sec. (IV). We conclude in Sec. (V).

II. MODEL: A DRIVEN TWO LEVEL SYSTEM

Here, we sketch the essential details of a driven two-level system in contact with a squeezed thermal bath [33–37]. The model consists of a two level system whose Hilbert space is spanned by two states, the ground state |g⟩ and the excited state |e⟩, Fig.(1). The description of such a system is analogous to that of a spin -1/2 system. The Pauli operators in terms of these basis vectors are σ₁ = |e⟩⟨g| + |g⟩⟨e|, σ₂ = −i|e⟩⟨g| + i|g⟩⟨e| and σ₃ = |e⟩⟨e| − |g⟩⟨g|, and satisfy the usual commutation [σᵢ, σⱼ] = 2iεᵢⱼkσₖ and the anticommutation {σᵢ, σⱼ} = 2δᵢⱼ. The raising and lowering operators can be defined as

σ⁺ = |e⟩⟨g| = 1/2(σ₁ + iσ₂),
σ⁻ = |g⟩⟨e| = 1/2(σ₁ − iσ₂). (4)

With this setting, we can define the system Hamiltonian Hₛ to be diagonal in basis {|e⟩, |g⟩}. With ω₀ denoting the transition frequency between the two levels (setting ħ = 1), we have

Hₛ = 1/2ω₀σ₃. (5)

A detailed account of two level systems and their application can be found in [38].

We now consider the case when a two level atomic transition |e⟩ ↔ |g⟩ is driven by an external source. The source is assumed to be a coherent single mode field on resonance. Under dipole approximation, the Hamiltonian (in the interaction picture) is given by

Hᵢ = −\vec{E}_L(t) . \vec{D}(t). Here, \vec{E}_L(t) = \vec{E}e^{-iω₀t} + \vec{E}^*e^{+iω₀t} is the electric field strength of the driving mode. Also, \vec{D}(t) = d_0e^{-iω₀t} + d_0^*e^{+iω₀t} is the atomic dipole operator in the interaction picture and d₀ = ⟨g|D|e⟩ is the transition matrix element of the dipole operator. The atom-field interaction can be written in the rotating wave approximation as follows,

Hᵢ = −Ω/2(σ⁺ + σ⁻). (6)

Here, Ω = 2\vec{E}d₀, is referred to as the Rabi frequency. Now coupling the system to a thermal reservoir leads to the quantum master equation

\[ \frac{d\rho(t)}{dt} = i\Omega \left[ \frac{1}{2} \sigma_+ + \sigma_- \right] \rho(t) \\
+ \gamma_0 n \left( \sigma_- \rho(t) \sigma_- - \frac{1}{2} \sigma_- \sigma_+ \rho(t) - \frac{1}{2} \rho(t) \sigma_- \sigma_+ \right) \\
+ \gamma_0(n+1) \left( \sigma_- \rho(t) \sigma_- - \frac{1}{2} \sigma_- \sigma_+ \rho(t) - \frac{1}{2} \rho(t) \sigma_- \sigma_+ \right) \\
- \gamma_0 M \sigma_- \rho(t) \sigma_- - \gamma_0 M^* \sigma_- \rho(t) \sigma_- . (7) \]

Here, γ = γ₀(2n + 1) is the total transition rate with γ₀ being the spontaneous emission rate. Further,

\[ n = n_{th}(\cosh^2(s) + \sinh^2(s)) + \sinh^2(s), \]

and \[ M = -\cosh(s) \sinh(s)e^{i\theta}(2n_{th} + 1) . (8) \]

where s and θ are the squeezing parameters and n_{th} = 1/(exp[βω₀] − 1) is the Planck distribution at transition frequency. In what follows, we will set θ = 0 for the purpose of calculations.

In order to solve Eq. (7), we write the density matrix as

\[ \rho(t) = \frac{1}{2}(I + \vec{\sigma}(t)) \]

with \[ \vec{\sigma}(t) = \langle \vec{\sigma}(t) \rangle = Tr[\vec{\sigma}\rho(t)] \], is known as the Bloch vector. With this notation, the master equation, Eq.
Having obtained the solution, one can calculate the survival probability of the atom being in the ground state

\[ p_g(t) = \frac{1}{2} \left( 1 + \langle \sigma_3 \rangle_s \right) = \frac{1}{2} \left[ 1 - \gamma_0 (\gamma - 2\gamma_0 M) \right]. \]

In the strong driving limit, \( \Omega \gg \gamma_s \), we have \( p_e^s = 1/2 \) and \( \langle \sigma_+ \rangle_s = -i\gamma_0/2\Omega \).

In order to solve the time dependent Bloch equation, Eq. (10), it is convenient to introduce the vector

\[ \langle \vec{\Sigma}(t) \rangle = \langle \vec{\sigma}(t) \rangle - \langle \vec{\sigma} \rangle_s. \]

This vector satisfies the homogeneous equation

\[ \frac{d}{dt} \langle \vec{\Sigma}(t) \rangle = \mathcal{G} \langle \vec{\Sigma}(t) \rangle. \] (14)

This equation can be easily solved by diagonalizing \( \mathcal{G} \), which has the eigenvalues

\[ \lambda_1 = \frac{-\gamma}{2} - \gamma_0 M, \]

\[ \lambda_{2,3} = \frac{\gamma_0 M}{2} - \frac{3\gamma}{4} \pm i\mu_s, \] (15)

where,

\[ \mu_s = \sqrt{\Omega^2 - \left( \frac{\gamma_s}{4} \right)^2} \text{ with } \gamma_s = \gamma + 2\gamma_0 M. \] (16)

Assuming the atom to be initially in the ground state \( \rho(0) = |g \rangle \langle g| \), we have

\[ \langle \sigma_3(0) \rangle = -1 \text{ or } \langle \Sigma_3(0) \rangle = -1 - \langle \sigma_3 \rangle_s, \] (17)

and

\[ \langle \sigma_{\pm}(0) \rangle = 0 \text{ or } \langle \Sigma_{\pm}(0) \rangle = -\langle \sigma_{\pm} \rangle_s. \] (18)

With these initial conditions, the solution of Eq. (14) is given by

\[ \langle \vec{\Sigma}(t) \rangle = \begin{pmatrix}
  e^{-i(\gamma + 2\gamma_0 M)t/4} \langle \Sigma_1(0) \rangle \\
  e^{-(3\gamma + 2\gamma_0 M)t/4} 
  \left[ \cos(\mu_s t) + \frac{\gamma + 2\gamma_0 M}{4\mu_s} \sin(\mu_s t) \right] \langle \Sigma_2(0) \rangle + \frac{\Omega}{\mu_s} \sin(\mu_s t) \langle \Sigma_3(0) \rangle \\
  e^{-(3\gamma + 2\gamma_0 M)t/4} 
  \left[ (1 - \frac{\gamma + 2\gamma_0 M}{2\mu_s}) \cos(\mu_s t) - \frac{\gamma}{4\mu_s} \sin(\mu_s t) \right] \langle \Sigma_3(0) \rangle + \frac{\Omega}{\mu_s} e^{-(3\gamma + 2\gamma_0 M)t/4} \sin(\mu_s t) \left[ \langle \Sigma_4(0) \rangle - \langle \Sigma_{-}(0) \rangle \right]
\end{pmatrix}. \] (19)

Having obtained the solution, one can calculate the survival probability of the atom being in the ground state

\[ p_g(t) = \frac{1}{2} \left[ (\langle \Sigma_4(t) \rangle + \langle \sigma_3 \rangle_s) \right]. \] (20)
Further, the degree of coherence is proportional to the off-diagonal element

$$\langle \sigma_+ (t) \rangle = \langle \sigma_1(t) \rangle + i \langle \sigma_2(t) \rangle \quad \text{and} \quad \langle \sigma_+ \rangle = \langle \sigma_1 \rangle + i \langle \sigma_2 \rangle. \quad (21)$$

The dynamics is underdamped or overdamped depending on whether $\mu s$, defined in Eq. (16), is real or imaginary. As a result, in underdamped regime, the probabilities as well as the coherence exhibit exponentially damped oscillations, while in the over damped case, they monotonically approach to their stationary values, Fig. (2). Throughout this paper, we work in units with $\hbar = k_B = 1$.

III. LEGGETT-GARG TYPE INEQUALITY FOR THE TWO LEVEL DRIVEN SYSTEM

Let $\mathcal{E}_{t_j-t_i}$ be the map corresponding to the evolution given by Eq. (7), such that the system in state $\rho(t_i)$ at time $t_i$ evolves to state $\rho(t_j)$ at some later time $t_j > t_i$

$$\rho(t_j) = \mathcal{E}_{t_j-t_i}[\rho(t_i)]. \quad (22)$$

Let at time $t_0$ the system be in the ground state $|g\rangle$. We define the dichotomic observable $\mathcal{M} = |g\rangle\langle g| - |e\rangle\langle e|$, such that $\mathcal{M} = \Pi^+ - \Pi^-$. Using Eq. (2), with the notation $t_1 - t_0 = t$, the two time correlation $C(t_0, t_1)$ is

$$C(t_0, t_1) = \text{Tr}[\Pi^+ \rho(t_0)] \text{Tr}[\Pi^+ \mathcal{E}_{t_1-t_0}\left[\Pi^+ \rho(t_0)\Pi^+ \text{Tr}[\Pi^+ \rho(t_0)]\right]]$$

$$- \text{Tr}[\Pi^+ \rho(t_0)] \text{Tr}[\Pi^- \mathcal{E}_{t_1-t_0}\left[\Pi^+ \rho(t_0)\Pi^+ \text{Tr}[\Pi^+ \rho(t_0)]\right]]$$

$$- \text{Tr}[\Pi^- \rho(t_0)] \text{Tr}[\Pi^+ \mathcal{E}_{t_1-t_0}\left[\Pi^- \rho(t_0)\Pi^+ \text{Tr}[\Pi^- \rho(t_0)]\right]]$$

$$+ \text{Tr}[\Pi^- \rho(t_0)] \text{Tr}[\Pi^- \mathcal{E}_{t_1-t_0}\left[\Pi^- \rho(t_0)\Pi^- \text{Tr}[\Pi^- \rho(t_0)]\right]]$$

$$= p_+(t) - p_-(t) = 2p_+(t) - 1. \quad (23)$$

Plugging in the expressions of probabilities, we have

$$K_{\pm} = \pm 2F(t) - F(2t) \mp 1. \quad (24)$$

Here,

$$F(t) = A [B + Ce^{-(3\gamma - 2\gamma_0)\mu t}/4\cos(\mu_st) + D \sin(\mu_st)] - 1, \quad (25)$$

with coefficients given by

$$A = [4\mu_s(\gamma^2 - 2\gamma_0 M + 2\Omega^2)]^{-1},$$

$$B = 4(\gamma + \gamma_0)(\gamma - 2\gamma_0 M)\mu_s + 8\mu_s\Omega^2, $$

$$C = -2(\gamma_0 M - 2\mu_s)[(\gamma - \gamma_0)(\gamma - 2\gamma_0 M) + 2\Omega^2],$$

$$D = -\gamma - \gamma_0(\gamma - 2\gamma_0 M) - 2(\gamma - 4\gamma_0)\Omega^2. \quad (26)$$

In the strong driving limit, $\Omega \gg \gamma_0$, the coefficients can be approximated as $A \approx \Omega^{-3}$, $B \approx C \approx \Omega^3$ and $D \approx \Omega^2$, such that in this limit, $F(t) \propto \cos(\Omega t)$ and therefore

$$K_{\pm} \approx \pm 2\cos(\Omega t) - \cos(2\Omega t). \quad (27)$$

Effect of weak measurement: The two time correlation function $C(t_0, t)$, Eq. (23), was obtained by assuming that the measurements are ideal or projective. However, it would be interesting to see how weak measurements...
affect the behavior of $C(t_0, t)$ and thereby of the LG parameters $K_{\pm}$. The weak measurements are characterized by invoking a parameter $\xi$ [39, 40], such that the ideal projectors $\Pi_{\pm}$ are replaced by the “weak projectors” $W_{\pm}$ defined as

$$W_{\pm} = \left(\frac{1 \pm \xi}{2}\right)\Pi^+ + \left(\frac{1 \mp \xi}{2}\right)\Pi^-.$$  

Here, $0 < \xi \leq 1$, such that when $\xi = 1$, $W_{\pm}$ reduce to the ideal projection operators $\Pi_{\pm}$. Invoking weak projectors leads to the following form of the two time correlation function becomes $C(t_0, t)|_{\text{weak}} = \xi^2 C(t_0, t)$, and consequently

$$K_{\pm}|_{\text{weak}} = \xi^2 K_{\pm}.$$  

Therefore, the maximum violation of LGtI occurs for an ideal projective measurement.

IV. RESULTS AND DISCUSSION

The LGtIs given by inequality (3) are studied in the context of a two level atom with the ground and excited states labelled as $|g\rangle$ and $|e\rangle$, respectively. An external field is driving the transition between the two levels. Further, the atom is allowed to interact with a squeezed thermal bath. The inequalities thus obtained are in terms of experimentally relevant parameters. The violation of LGtIs occur predominantly in the underdamped regime which is characterized by the real values of parameter $\mu_s$ defined in Eq. (16), such that

$$\Omega > \frac{\gamma_s}{4} = \frac{\gamma_0}{4} \left(\frac{2n + 1}{2} + \frac{2M}{4}ight) \quad \text{underdamped},$$

$$\Omega < \frac{\gamma_s}{4} = \frac{\gamma_0}{4} \left(\frac{2n + 1}{2} + \frac{2M}{4}\right) \quad \text{overdamped}. \quad (30)$$

Figure (3) depicts the behavior of LG parameters $K_{\pm}$ with respect to time $t$, for different values of the ratio $R = \gamma_0/\Omega$. The violations of LGtIs are observed mainly in the underdamped regime and fade quickly with the increase in $R$. In other words, strong driving favors the violation of LGtIs to their maximum quantum bound. The right most panel of the figure shows coherence parameter $C$ [41, 42] which is defined as

$$C = \sum_{i \neq j} |\rho_{ij}|.$$  

The extent of violation of LGtIs can be seen as a signature of the degree of coherence in the system.

In the strong driving limit, i.e., $\Omega \gg \gamma_0$, the LG parameters are given by Eq. (27) and are plotted in Fig. (4). The parameters $K_+$ and $K_-$ show complementary behavior in the sense that when one of these parameters does not show a violation, the other does, together covering the entire parameter range.

The interaction with the squeezed thermal reservoir leads to enhancement in the transition rate which is given by $\gamma = \gamma_0(2n + 1)$, where $\gamma_0$ is the spontaneous emission rate and $\gamma_0n$ is the squeezed thermal induced emission and absorption rate. The interactions with the reservoir are expected to decrease the quantumness in the system. This feature is depicted in Fig. (5), where $K_+$ shows enhanced violations for larger values of the parameter $\beta$ i.e., for smaller temperature.

The squeezing parameter as defined in Eq. (8), controls the degree of violation of LGtIs, since it affects the total photon distribution. Figure (6) exhibits the variation of the LG parameter $K_+$ for different values of squeezing parameter $s$. The increase in $s$ is found to decrease the extent of violation of LGtI.

The effect of weak measurement on the LG parameters is depicted in Fig. (7). The ideal projective measurements are characterized by $\xi = 1$, while $\xi = 0$ corresponds to no measurement. It is clear from the figure that the maximum violation occurs for ideal projective measurements.
ties in a driven two level atom interacting with a squeezed thermal bath. The effect of various experimentally relevant parameters on the violation of the inequality were examined carefully. The violations were seen to be prominent in the underdamped case. The increase in temperature was found to decrease the degree of violation as well as the time over which the violation is sustained. Squeezing the thermal state of the reservoir was also found to reduce the violation of LGtIs. Enhanced violations, reaching to the quantum bound, were witnessed in the strong driving limit. Further, we studied the effect of the weak measurements on the extent of violation of LGtI. The weak measurements are characterized by the parameter ξ such that ξ = 0 (ξ = 1) corresponds to no measurement (ideal projective measurement). The maximum violation was found to occur for the ideal projective measurements.

V. CONCLUSION

We studied the violation of Leggett-Garg type inequalities in a driven two level atom interacting with a squeezed

FIG. 7: (color online). Variation LG parameter $K_\varphi$ with respect to $t$ and $\xi$. With $\beta = 5$, $\omega_0 = 0.5$ and $s = 0$, we have $R \approx 0 (\mu \approx 1)$ depicted by blue plane-surface, $R = 0.05 (\mu \approx 0.9)$ represented by yellow lined-surface. Both these correspond to underdamped case. The maximum violation corresponds to $\xi = 1$, the ideal projective measurement.

Acknowledgment

AMJ thanks DST India for J C Bose National Fellowship.

[1] E. Schrödinger, Naturwissenschaften 23, 823 (1935).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[3] J. S. Bell, Physics Physique Fizika 1, 195 (1964).
[4] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Reviews of Modern Physics 81, 865 (2009).
[5] A. J. Leggett and A. Garg, Physical Review Letters 54, 857 (1985).
[6] C. Emary, N. Lambert, and F. Nori, Reports on Progress in Physics 77, 016001 (2013).
[7] I. Chakrabarty, S. Banerjee, and N. Siddharth, Quantum Info. Comput. 11, 541 (2011).
[8] M. Barbieri, Physical Review A 80, 034102 (2009).
[9] D. Avis, P. Hayden, and M. M. Wilde, Physical Review A 82, 030102 (2010).
[10] N. Lambert, C. Emary, Y.-N. Chen, and F. Nori, Physical Review Letters 105, 176801 (2010).
[11] N. Lambert, R. Johansson, and F. Nori, Physical Review B 84, 245421 (2011).
[12] A. Montina, Physical Review Letters 108, 160501 (2012).
[13] J. Kofler and C. Brukner, Physical Review A 87, 052115 (2013).
[14] C. Budroni, T. Moroder, M. Kleinmann, and O. Gühne, Physical Review Letters 111, 020403 (2013).
[15] S. Kumari and A. Pan, EPL (Europhysics Letters) 118, 50002 (2017).
[16] J. Naikoo, A. K. Alok, and S. Banerjee, Phys. Rev. D 97, 053008 (2018).
[17] J. Naikoo, A. K. Alok, S. Banerjee, and S. U. Sankar, Phys. Rev. D 99, 095001 (2019).
[18] J. Naikoo, S. Banerjee, and R. Srikanth, arXiv preprint arXiv:1806.00537 (2018).
[19] J. Naikoo and S. Banerjee, The European Physical Journal C 78, 602 (2018).
[20] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nature Physics 6, 442 (2010).
[21] J. Groen, D. Risté, L. Tornberg, J. Cramer, P. C. De Groot, T. Picot, G. Johansson, and L. DiCarlo, Physical Review Letters 111, 090506 (2013).
[22] M. Goggin, M. Almeida, M. Barbieri, B. Lanyon, J. O’Brien, A. White, and G. Pryde, Proceedings of the National Academy of Sciences 108, 1256 (2011).
[23] J. Dressel, C. Broadbent, J. Howell, and A. N. Jordan, Physical Review Letters 106, 040402 (2011).
[24] Y. Suzuki, M. Iinuma, and H. F. Hofmann, New Journal of Physics 14, 103022 (2012).
[25] V. Athalye, S. S. Roy, and T. Mahesh, Physical Review Letters 107, 130402 (2011).
[26] A. Souza, I. Oliveira, and R. Sarthour, New Journal of Physics 13, 053023 (2011).
[27] H. Katiyar, A. Shukla, K. R. K. Rao, and T. Mahesh, Physical Review A 87, 052102 (2013).
[28] S. F. Huelga, T. W. Marshall, and E. Santos, Phys. Rev. A 52, R2497 (1995).
[29] S. F. Huelga, T. W. Marshall, and E. Santos, Phys. Rev. A 54, 1798 (1996).
[30] G. Waldherr, P. Neumann, S. F. Huelga, F. Jelezko, and J. Wrachtrup, Phys. Rev. Lett. 107, 090401 (2011).
[31] J.-S. Xu, C.-F. Li, X.-B. Zou, and G.-C. Guo, Scientific reports 1, 101 (2011).
[32] Z.-Q. Zhou, S. F. Huelga, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. 115, 113002 (2015).
[33] R. Srikanth and S. Banerjee, Phys. Rev. A 77, 012318 (2008).
[34] S. Banerjee and R. Srikanth, The European Physical Journal D 46, 335 (2008).
[35] H.-P. Breuer, F. Petruccione, et al., The theory of open
quantum systems (Oxford University Press on Demand, 2002).

[36] S. Banerjee, *Open Quantum Systems: Dynamics of Non-classical Evolution*, Vol. 20 (Springer, 2018).

[37] S. Omkar, R. Srikanth, and S. Banerjee, *Quantum Information Processing* **12**, 3725 (2013).

[38] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics, Vol. III: The New Millennium Edition: Quantum Mechanics*, The Feynman Lectures on Physics (Basic Books, 2011).

[39] P. Busch, *Phys. Rev. D* **33**, 2253 (1986).

[40] D. Saha, S. Mal, P. K. Panigrahi, and D. Home, *Phys. Rev. A* **91**, 032117 (2015).

[41] A. K. Alok, S. Banerjee, and S. U. Sankar, *Nuclear Physics B* **909**, 65 (2016).

[42] S. Bhattacharya, S. Banerjee, and A. K. Pati, *Quantum Information Processing* **17**, 236 (2018).