SOLITONS, SURFACES, CURVES, AND THE SPIN DESCRIPTION OF NONLINEAR EVOLUTION EQUATIONS

R. Myrzakulov\textsuperscript{a,b}

\textsuperscript{a}Physical-Thechnical Institute, NAS RK, 480082, Alma-Ata-82, Kazakhstan
\textsuperscript{b}Centre for Nonlinear Problems, PO Box 30, 480035, Alma-Ata-35, Kazakhstan

Abstract

The briefly review on the common spin description of the nonlinear evolution equations.
I. INTRODUCTION

In 1977 Lakshmanan[1] discovered that between the isotropic Landau-Lifshitz equation (LLE)
\[ \vec{S}_t = \vec{S} \wedge \vec{S}_{xx} \]  (1)
and the NLSE
\[ iq_t + q_{xx} + 2q^2 \bar{q} = 0 \]  (2)
takes place the equivalence, which following to the terminology of ref[2] we call the Lakshmanan equivalence (shortly, L-equivalence). On the other hand, as well known equations (1) and (2) are gauge equivalent to each other[3]. This equivalence we call gauge equivalence or briefly g-equivalence. In a series of papers [2] were presented the new class integrable and nonintegrale spin equations and the unit spin description of soliton equations. In this paper using the geometrical method we will present the Lakshmanan equivalent counterparts of the some spin equations.

Consider a n-dimensional space with the basic unit vectors: \( \vec{e}_1 = \vec{S}, \vec{e}_2, ..., \vec{e}_n \). Then the M-0 equation has the form[2]
\[ \vec{S}_t = \sum_{i=2}^{n} a_i \vec{e}_i \]  (3)
where \( a_i \) are real functions, \( \vec{S} = (S_1, S_2, ..., S_n), S^2 = E = \pm 1 \). This equation admits the many interesting class integrable and nonintegrable reductions[2]. Below we present the L-equivalent counterparts of the some reductions and only for the cases \( n = 2, 3 \).

The paper is organized as follows. In Sec.II and Sec.III we briefly review some necessary informations about the some surfaces/curves approaches to the nonlinear evolution equations. The L-equivalent counterparts of the some spin equations are presented in Sec.VI and Sec.V. Integrals of motion are discussed in Sec.VI. In Sec.VII the Lax representations are derived. A non-isospectral problems are discussed in Sec.VIII. The solitons (line and curved), domain walls and dromion-like solutions of the M-I equation as well as their breaking analogues are obtained in Sec.IX and breaking solitons and dromions of the Zakharov equation are obtained in Sec.X. In Sec.XI we present the L-equivalent counterparts of the some spin-phonon systems and we conclude in Sec.XII.

II. THE D-APPROACH

IIa. The n-dimensional case

Consider a space \( R_n \) with an orthonormal basis \( \vec{\xi}_i, i = 1, 2, ..., n \). In the D-approach the fundamental set of equations for example in 2+1 dimensions looks like[2]
\[ \vec{r}_t = \sum_{i=1}^{n} b_i \vec{\xi}_i \]  (4a)
\[ \tilde{\xi}_{1x} = k_1 \tilde{\xi}_2 \quad (4b) \]
\[ \tilde{\xi}_{jx} = Ek_j \tilde{\xi}_{j-1} + k_{j+1} \tilde{\xi}_{j+1} \quad (4c) \]
\[ \tilde{\xi}_{nx} = Ek_n \tilde{\xi}_{n-1} \quad (4d) \]
\[ \tilde{\xi}_{iy} = \sum_{m=1}^{n} a_{im} \tilde{\xi}_m \quad (4e) \]
\[ \tilde{\xi}_{it} = \sum_{m=1}^{n} b_{im} \tilde{\xi}_m \quad (4f) \]

where \( j = 2, 3, ..., n - 1; a_{ii} = b_{ii} = 0 \). Note that eqs (4b)-(4d) are the Serret-Frenet equations (SFE).

\textbf{IIb. The (2+1)-dimensions: a space curves}

In this case equations of the curves have the forms [2]

\[
\begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix}_x = C \begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix} 
\quad (5a)
\]

\[
\begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix}_y = D \begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix} 
\quad (5b)
\]

Here

\[
C = \begin{pmatrix}
0 & k & 0 \\
-Ek & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & m_3 & -m_2 \\
-Em_3 & 0 & m_1 \\
Em_2 & -m_1 & 0
\end{pmatrix}
\]

Hence now we have

\[ C_y - D_x + [C, D] = 0 \quad (6) \]

The time evolution of \( \tilde{e}_i \) we can write in the form [1]

\[
\begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix}_t = G \begin{pmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3 \\
\end{pmatrix} \quad (7)
\]

with

\[
G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-E\omega_3 & 0 & \omega_1 \\
E\omega_2 & -\omega_1 & 0
\end{pmatrix},
\]

So, we have

\[ C_t - G_x + [C, G] = 0 \quad (8a) \]
\[ D_t - G_y + [D, G] = 0 \quad (8b) \]
IIc. The (2+1)-dimensions: a plane curves

In this case we have[2]

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}_x = C \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}
\] (9a)

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}_y = D \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}
\] (9b)

Here

\[
C = \begin{pmatrix}
0 & k \\
-Ek & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & m_3 \\
-Em_3 & 0
\end{pmatrix}
\]

Now from the MPCE (6) we obtain

\[
m_3 = \partial^{-1}_x k_y
\] (10)

For the time evolution we get[1]

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}_t = G \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2
\end{pmatrix}
\] (11)

where

\[
G = \begin{pmatrix}
0 & \omega_3 \\
-E\omega_3 & 0
\end{pmatrix}
\]

Now already we ready using this formalism to construct the L-equivalent counterparts of the some spin equations.

III. THE C - APPROACH

IIIa. A hypersurface \( V_{n-1} \) in \( R_n \)

In this case we consider a time-dependent (n-1)-dimensional hypersurface \( V_{n-1}(t, \vec{r}) \) in \( R_n[2] \). Define

\[
\vec{\xi}_j = \vec{r}_{x^j}, \quad j = 1, 2, ..., n - 1
\] (12)

and \( \vec{\nu} \) is the unit normal vector to the hypersurface. The first and second fundamental forms are given by

\[
I = d\vec{r} d\vec{r} = \sum_{i,j=1}^{n-1} g_{ij} dx^i dx^j
\] (13a)
\[
II = -d\bar{r}d\bar{\nu} = \sum_{i,j=1}^{n-1} b_{ij} dx^i dx^j
\]  

(13b)

In the C-approach[2], the fundamental set of equations reads as

\[
\bar{r}_t = \sum_{i=1}^{n-1} b_i \bar{\xi}_i + b_0 \bar{\nu}
\]  

(14a)

\[
\nabla_i \bar{\xi}_j = b_{ij} \bar{\nu}
\]  

(14b)

\[
\nabla_i \bar{\nu} = -Eb^*_i \bar{\xi}_j
\]  

(14c)

here that eqs(14b,c) are the Gauss-Weingarten equations. Then the Peterson-
Mainardi-Codazzi equation (MPCE) takes the form

\[
R_{ij,kl} = E(b_{ik} b_{jl} - b_{jk} b_{il})
\]  

(15a)

\[
\nabla_i b_{jk} = \nabla_j b_{ik}
\]  

(15b)

where \( \nabla_i \) is the covariant derivative.

**IIIb. The 2+1 dimensional case**

In this case, starting from the results of the ref.[4](see, also, for example, [5-
7]) we consider the motion of surface in the 3-dimensional space which generated
by a position vector \( \vec{r}(x, y, t) = \vec{r}(x^1, x^2, t) \). According to the C-approach[2],
the main elements of which we present in this section, let \( x \) and \( y \) are local
coordinates on the surface. The first and second fundamental forms in the
usual notation are given by

\[
I = d\bar{r}d\bar{r} = E dx^2 + 2F dx dy + G dy^2
\]  

(16a)

\[
II = -d\bar{r}d\bar{n} = L dx^2 + 2M dx dy + N dy^2
\]  

(16b)

where

\[
E = \bar{r}_x \bar{r}_x = g_{11}, F = \bar{r}_x \bar{r}_y = g_{12}, G = \bar{r}_y^2 = g_{22},
\]

\[
L = \bar{n} \bar{r}_{xx} = b_{11}, M = \bar{n} \bar{r}_{xy} = b_{12}, N = \bar{n} \bar{r}_{yy} = b_{22}, \bar{n} = \frac{(\bar{r}_x \wedge \bar{r}_y)}{[\bar{r}_x \wedge \bar{r}_y]}
\]

In this case, the starting set of equations of the C-approach[2], becomes

\[
\bar{r}_t = a \bar{r}_x + b \bar{r}_y + c \bar{n}
\]  

(17a)

\[
\bar{r}_{xx} = \Gamma^1_{11} \bar{r}_x + \Gamma^2_{11} \bar{r}_y + L \bar{n}
\]  

(17b)

\[
\bar{r}_{xy} = \Gamma^1_{12} \bar{r}_x + \Gamma^2_{12} \bar{r}_y + M \bar{n}
\]  

(17c)

\[
\bar{r}_{yy} = \Gamma^1_{22} \bar{r}_x + \Gamma^2_{22} \bar{r}_y + N \bar{n}
\]  

(17d)

\[
\bar{n}_x = p_1 \bar{r}_x + p_2 \bar{r}_y
\]  

(17e)

\[
\bar{n}_y = q_1 \bar{r}_x + q_2 \bar{r}_y
\]  

(17f)
where $\Gamma_{ij}^k$ are the Christoffel symbols of the second kind defined by the metric $g_{ij}$ and $g^{ij} = (g_{ij})^{-1}$ as

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \quad (18)$$

The coefficients $p_i, q_i$ are given by

$$p_i = -b_{1j} g^{ji}, \quad q_i = -b_{2j} g^{ji} \quad (19)$$

The compatibility conditions $\vec{r}_{xyx} = \vec{r}_{xyy}$ and $\vec{r}_{yyx} = \vec{r}_{yyy}$ yield the following Mainardi-Peterson-Codazzi equations (MPCE)

$$R_{ijk}^l = b_{ij} b_{lk}^l - b_{ik} b_{lj}^l \quad (20a)$$

$$\frac{\partial b_{ij}}{\partial x^k} - \frac{\partial b_{ik}}{\partial x^j} = \Gamma_{is}^s b_{ks} - \Gamma_{js}^s b_{ks} \quad (20b)$$

where $b_{ij}^l = g^{il} b_{ij}$ and the curvature tensor has the form

$$R_{ijk}^l = \frac{\partial \Gamma_{ij}^k}{\partial x^l} - \frac{\partial \Gamma_{ik}^j}{\partial x^l} + \Gamma_{is}^s \Gamma_{ks}^l - \Gamma_{js}^s \Gamma_{ks}^l \quad (21)$$

Let $\vec{Z} = (r_x, r_y, n) \, t$. Then

$$\vec{Z}_x = A \vec{Z} \quad (22a)$$

$$\vec{Z}_y = B \vec{Z} \quad (22b)$$

where

$$A = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ p_1 & p_2 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \\ q_1 & q_2 & 0 \end{pmatrix}$$

Hence we get the new form of the MPCE (20)

$$A_y - B_x + [A, B] = 0 \quad (23)$$

Let us introduce the orthogonal trihedral[1]

$$\vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \frac{\vec{r}_y}{\sqrt{G}}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 = \vec{n} \quad (24a)$$

or

$$\vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \vec{n}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 \quad (24b)$$

Let $\vec{r}_x^2 = E = \pm 1$. Then from the previous equations after some algebra we get[2]

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \quad (25a)$$
\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_y = D \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix} \quad (25b)
\]

Here
\[
C = \begin{pmatrix}
0 & k & -\sigma \\
-Ek & 0 & \tau \\
E\sigma & -\tau & 0
\end{pmatrix}
\]
\[
D = \begin{pmatrix}
0 & m_3 & -m_2 \\
-Em_3 & 0 & m_1 \\
Em_2 & -m_1 & 0
\end{pmatrix}
\]

Now the MPCE (20) and/or (23) becomes
\[
C_y - D_x + [C, D] = 0 \quad (26)
\]

Hence as \( \sigma = 0 \) we obtain
\[
(m_1, m_2, m_3) = (\partial_x^{-1}(\tau y + km_2), m_2, \partial_x^{-1}(k y - \tau m_2)) \quad (27a)
\]
\[
m_2 = \partial_x^{-1}(\tau m_3 - km_1) \quad (27b)
\]

The time evolution of \( \vec{e}_i \) we can write in the form[2]
\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_t = G \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix} \quad (28)
\]

with
\[
G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-E\omega_3 & 0 & \omega_1 \\
E\omega_2 & -\omega_1 & 0
\end{pmatrix}
\]

So, we have
\[
C_t - G_x + [C, G] = 0 \quad (29a)
\]
\[
D_t - G_y + [D, G] = 0 \quad (29b)
\]

Below using these C-, and D-approches we will construct the L-equivalent counterparts of the some spin systems.

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**IV. THE LAKSHMANAN EQUIVALENT COUNTERPARTS OF THE SOME MYRZAKULOV EQUATIONS: the 2-dimensional case**

In this case the Myrzakulov-0 equation(3) becomes
\[
\vec{S}_t = a_2 \vec{e}_2 \quad (30)
\]
and $\vec{S} = (S_1, S_2)$, $\vec{S}^2 = E = \pm 1, \tau = c = 0$. So, we have[2]

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}_x = C \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$  \hspace{1cm} (31a)

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}_y = D \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$  \hspace{1cm} (31b)

Here

$$C = \begin{pmatrix} 0 & k \\ -Ek & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & m_3 \\ -Em_3 & 0 \end{pmatrix}.$$  

Now from the MPCE (20) we obtain

$$m_3 = \partial_x^{-1} k_y$$  \hspace{1cm} (32)

For the time evolution we get[1]

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}_t = G \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$  \hspace{1cm} (33)

where

$$G = \begin{pmatrix} 0 & \omega_3 \\ -E\omega_3 & 0 \end{pmatrix}.$$  

Now already we ready using this formalism to construct the L-equivalent counterparts of the some Myrzakulov equations.

**Examples**

1) The Myrzakulov-IV(M-IV) equation[2]

$$\vec{S}_t + \{\vec{S}_{xy} + V\vec{S} + E\vec{S}_x \land (\vec{S} \land \vec{S}_y)\}_x = 0$$  \hspace{1cm} (34a)

$$V_x = \frac{E}{2}(\vec{S}_x)_y$$  \hspace{1cm} (34b)

In this case

$$\vec{r}_t = W\vec{e}_1 + U\vec{e}_2$$  \hspace{1cm} (35a)

or

$$\vec{r}_t = -V\vec{e}_1 - k_y\vec{e}_2$$  \hspace{1cm} (35b)

So, we have

$$W = -V, \quad U = -k_y$$  \hspace{1cm} (36a)

$$W_x = EkU$$  \hspace{1cm} (36b)

$$k_t = U_{xx} + Ek_x\partial_x^{-1}(kU) + Ek^2U$$  \hspace{1cm} (37b)
Hence we get the following L-equivalent counterpart of the M-IV equation (34)

\[ k_t + k_{xxy} + (kV)_x = 0 \]  \hspace{1cm} (38a)

\[ V_x = Ekk_y \]  \hspace{1cm} (38b)

which is the 2+1 dimensional mKdV[10]. Note that in the our case \( E = \pm 1 \).

Similarly we can find the L-equivalent counterparts of the some other Myrzakulov equations. Now we present the final results. Details of these calculations are given in the our ealier papers (see, for example, [8-9]).

2) The M-XXI equation has the following L-equivalent counterpart

\[ k_t + k_{xxy} - 2(kV)_x + 2V_2k = 0 \]  \hspace{1cm} (39a)

\[ V_{1x} = Ekk_y \]  \hspace{1cm} (39b)

\[ V_{2x} = -Ekk_{xy} \]  \hspace{1cm} (39c)

which is the 2+1 dimensional KdV[10]. Note that in the our case \( E = \pm 1 \).

Now after the some minor modifications of the above presented formalism we obtain the following L-equivalents of the some spin systems.

3) The M-XXIII equation has the following L-equivalent

\[ k_{xx} - \sigma^2 k_{yy} + \frac{1}{2}R(x,y,t)e^{2k} + (2k_{tt} + 3k^2)e^{2k} = 0 \]  \hspace{1cm} (40)

4) The M-XXIV equation has the following L-equivalent

\[ k_{xx} + \sigma^2 k_{yy} + \frac{1}{4}R\sin k + \sin k(k_{tt} - \frac{1}{2}k_t^2) = 0 \]  \hspace{1cm} (41)

5) The M-XXV equation has the following L-equivalent

\[ \frac{k_{tx}}{\cos k} \sin k - \sigma^2 \frac{k_{ty}}{\sin k} \cos k + \frac{1}{2}(k_{x}^2)_t + \frac{1}{2}(k_{y}^2)_t + (k_{xx} + \sigma^2 k_{yy})k_t = \frac{1}{2}R(3) \sin k \cos k k_t. \]  \hspace{1cm} (42)

6) The M-XXVI equation has the following L-equivalent

\[ (k_{xx} + k_{yy})t + (k_{xx} + k_{yy})k_t = (\frac{-1}{2}R - 3)e^{2k}k_t \]  \hspace{1cm} (43)

7) The M-XXVII equation has the following L-equivalent

\[ k_{xx} + k_{yy} - A(x,y)e^{-k} + (1 + \frac{R(x,y)}{3})e^{2k} = 0 \]  \hspace{1cm} (44)

8) The M-XXVIII equation has the following L-equivalent

\[ k_{xx} + k_{yy} + k_{tt} - \frac{R}{8}k^5 = 0 \]  \hspace{1cm} (45)

Of course we can get these results using the other approaches (see, for example, [8-9]). Note that eqs (40)-(45) were considered in [11] from the other point of view.
V. THE LAKSHMANAN EQUIVALENT COUNTERPARTS OF THE ISHIMORI and SOME MYRZAKULOV EQUATIONS: the 3-dimensional case

In this case work equations (25)-(29) and the Myrzakulov-0 equation becomes

\[ \vec{S}_t = a_2 \vec{e}_2 + a_3 \vec{e}_3 \]  

and \( \vec{S} = (S_1, S_2, S_3), \vec{S}^2 = E = \pm 1 \). Using these equations we construct the L-equivalent counterparts of the some Myrzakulov and Ishimori equations[2]. Below we present only the final results.

Examples

1) The Myrzakulov-I(M-I) equation looks like[2]

\[ \vec{S}_t = (\vec{S} \land \vec{S}_y + u \vec{S})_x \]  

\[ u_x = -\vec{S}(\vec{S}_x \land \vec{S}_y) \]  

In this case the fundamental set of the equations of the C-approach[2] becomes

\[ \vec{r}_t = u \vec{r}_x - M(\frac{E}{G})^{1/2} \vec{r}_y + \frac{G_x}{2} (\frac{E}{G})^{1/2} m_2 \vec{n} \]  

\[ \vec{r}_{xx} = L \vec{n} \]  

\[ \vec{r}_{xy} = \frac{G_x}{2E} \vec{r}_x + M \vec{n} \]  

\[ \vec{r}_{yy} = -\frac{G_x}{2E} \vec{r}_x + \frac{G_y}{2G} \vec{r}_y + N \vec{n} \]  

\[ \vec{n}_x = -EL \vec{r}_x - \frac{M}{G} \vec{r}_y \]  

\[ \vec{n}_y = -EM \vec{r}_x - N \vec{r}_y \]

So the MPCE(23) take the form

\[ K = \frac{LN - M^2}{G} = \frac{G_x^2}{4G^2} - \frac{G_{xx}}{2G} \]  

\[ L_y - M_x = M \frac{G_x}{2G} \]  

\[ M_y - N_x = -\frac{EG_x}{2} L + \frac{G_y}{2G} M - \frac{G_x}{2G} N \]

Now let us pass to the new variable

\[ q = \frac{L}{2} \exp(-i\partial_x^{-1} MG^{-1/2}) \]  

Then this variable satisfies the following Zakharov equation

\[ iq_t = q_{xy} + 2d^2 V q, \]  

(51a)
\[ V_x = 2E(|q|^2)_y. \] (51b)

which is the L-equivalent (and the g-equivalent[12]) counterpart of the M-I equation(47). Similarly we can find the L-equivalents of the some other spin equations. Details are given for example in the refs[8-9].

2) The M-II equation[2]

\[ \vec{S}_t = (\vec{S} \wedge \vec{S}_y + u \vec{S})_x + 2cb^2 \vec{S}_y - 4cv \vec{S}_x \] (52a)

\[ u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y), \] (52b)

\[ v_x = \frac{1}{16b^2c^2} (\vec{S}_{1x}^2)_y \] (52c)

has the following L-equivalent

\[ iq_t = q_{xy} - 4ic(V q)_x, \] (53a)

\[ V_x = 2E(|q|^2)_y. \] (53b)

which is the Strachan equation[10].

3) The M-III equation[2]

\[ \vec{S}_t = (\vec{S} \wedge \vec{S}_y + u \vec{S})_x + 2h(cb + d) \vec{S}_y - 4cv \vec{S}_x \] (54a)

\[ u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y), \] (54b)

\[ v_x = \frac{1}{4(2bc + d)^2} (\vec{S}_{1x}^2)_y \] (54c)

in this case

\[ (m_1, m_2, m_3) = (\partial_x^{-1}(\tau_y + km_2), -\frac{u_x}{k}, \partial_x^{-1}(k_y - \tau m_2)) \] (55)

and the L-equivalent is the following set of equations [2]

\[ iq_t = q_{xy} - 4ic(V q)_x + 2d^2V q, \] (56a)

\[ V_x = 2E(|q|^2)_y. \] (56b)

Note that equations (54) and (56) admit the following integrable reductions: a) the M-I[2] and the Zakharov[14] equations, as c=0; b) the M-II[2] and Strachan[10] equations, as d=0, respectively [2]. Note that between the M-I,M-II,M-III equations and eqs(51),(53),(56) take place the g-equivalence, respectively [12].

4) The M-VIII equation looks like[2]

\[ iS_t = \frac{1}{2}[S_{xx}, S] + iu_x S_x \] (57a)

\[ u_{xy} = \frac{1}{4i} tr(S[S_y, S_x]) \] (57b)

where the subscripts denote partial derivatives and S denotes the spin matrix \((r^2 = \pm 1)\).
\[ S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^2 = I \tag{58} \]

Equations (57) are integrable, i.e. admits Lax representation and different type soliton solutions [2]. The Lakshmanan equivalent counterpart of the M-VIII equation (57) has the form [2]

\[ iq_t + q_{xx} + vq = 0, \tag{59a} \]
\[ ip_t - p_{xx} - vp = 0, \tag{59b} \]
\[ v_y = 2(pq)_x \tag{59c} \]

where \( p = E\bar{q} \). On the other hand, in [12] was shown that eqs.(57) and (59) are gauge equivalent to each other.

5) The Ishimori equation

\[ iS_t + \frac{1}{2}[S, M_{10}S] + A_{20}S_x + A_{10}S_y = 0 \tag{60a} \]

\[ M_{20}u = \frac{\alpha}{4i} tr(S[S_y, S_x]) \tag{60b} \]

where \( \alpha, b, a = \text{consts} \) and

\[ M_{j0} = M_j, \quad A_{j0} = A_j \quad \text{as} \quad a = b = -\frac{1}{2}. \]

The L-equivalent counterpart has the form [1]

\[ iq_t + M_{10}q + vq = 0 \tag{61a} \]
\[ ip_t - M_{10}p - vp = 0 \tag{61b} \]
\[ M_{20}v = M_{10}(pq) \tag{61c} \]

which is the Davey-Stewartson equation, where \( p = E\bar{q} \). As well known these equations are too gauge equivalent to each other[13].

6) The M-IX equation has the form[2]

\[ iS_t + \frac{1}{2}[S, M_1S] + A_2S_x + A_1S_y = 0 \tag{62a} \]

\[ M_2u = \frac{\alpha}{4i} tr(S[S_y, S_x]) \tag{62b} \]

where \( \alpha, b, a = \text{consts} \) and

\[ M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(b-a) \frac{\partial^2}{\partial x \partial y} + (a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}; \]
\[ M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + a(a + 1) \frac{\partial^2}{\partial x^2}, \]
\[ A_1 = i\alpha \{(2ab + a + b)u_x - (2b + 1)\alpha u_y \} \]
\[ A_2 = i\{\alpha(2ab + a + b)u_y - (2a^2b + a^2 + 2ab + b)u_x \}. \]

Eqs. (62) admit the two integrable reductions. As \( b=0 \), eqs. (62) after the same manipulations reduces to the M-VIII equation (57) and as \( a = b = -\frac{1}{2} \) to the Ishimori equation (60). In general we have the two integrable cases: the M-IXA equation as \( \alpha^2 = 1 \), the M-IXB equation as \( \alpha^2 = -1 \). We note that the M-IX equation is integrable and admits the following Lax representation [2]

\[ \alpha \Phi_y = \frac{1}{2}[S + (2a + 1)I] \Phi_x \]  
(63a)

\[ \Phi_t = \frac{i}{2} [S + (2b + 1)I] \Phi_{xx} + \frac{i}{2} W \Phi_x \]  
(63b)

where

\[ W_1 = W - W_2 = (2b + 1)E + (2b - a + \frac{1}{2})SS_x + (2b + 1)FS \]

\[ W_2 = W - W_1 = FI + \frac{1}{2}S_x + ES + \alpha SS_y \]

\[ E = -\frac{i}{2\alpha}u_x, \quad F = \frac{i}{2} (\frac{(2a + 1)u_x}{\alpha} - 2uy) \]

Hence we get the Lax representations of the M-VIII (26) as \( b = 0 \) and for the Ishimori equation (29) as \( a = b = -\frac{1}{2} \). The M-IX equation (62) admit the different type exact solutions (solitons, lumps, vortex-like, dromion-like and so on). As shown in [2] eqs. (62) have the following L-equivalent counterpart

\[ iq_t + M_1q + vq = 0 \]  
(64a)

\[ ip_t - M_1p - vp = 0 \]  
(64b)

\[ M_2v = M_1(pq) \]  
(64c)

where \( p = Eq \). As well known these equations are too integrable [14] and as in the previous case, equations (64) have the two integrable reductions: equations (59) as \( b = 0 \) and the Davey-Stewartson equation (61) as \( a = b = -\frac{1}{2} \). Note that the M-IX (62) and (64) equations are g-equivalent to each other [15].

7) The M-XXII equation has the form [2]

\[ -iS_t = \frac{1}{2}([S, S_y] + 2iuS)_x + \frac{i}{2} V_1S_x - 2ia^2S_y \]  
(65a)

\[ u_x = -\bar{S}(\bar{S}_x \wedge \bar{S}_y) \]  
(65b)

\[ V_{1x} = \frac{1}{4a^2}(\bar{S}_x^2)_y \]  
(65c)

The L-equivalent of these equations are given by [1]

\[ q_t = iq_{yx} - \frac{1}{2}[(V_1q)_x - qV_2 - qrq_y] \]  
(66a)

\[ r_t = -ir_{yx} - \frac{1}{2}[(V_1r)_x - qrr_y + rV_2] \]  
(66b)

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\[ V_{1x} = (qr)_y \]  
\[ V_{2x} = r_{yx}q - rq_{yx} \]

where \( r = E\bar{q} \). Both these set of equations are integrable and the corresponding Lax representations were presented in [2].

8) The M-XXIX equation looks like [2]

\[ \vec{S}_t + \{ \vec{S}_{xy} + V\vec{S} + E\vec{S} \wedge (\vec{S} \wedge \vec{S}_y) \}_x = 0 \]  
\[ V_x = \frac{E}{2} (\vec{S}_x^2)_y \]

and the corresponding L-equivalent counterpart of the M-XXX equation (67) has the form

\[ q_t + q_{xxy} - (qV_1)_x - qV_2 = 0, \]  
\[ V_{1x} = 2E(q\bar{q})_y \]  
\[ V_{2x} = 2E(q\bar{q}_{xy} - \bar{q}_{xy}q) \]

which is the 2+1 dimensional complex mKdV. Note that in the our case \( E = \pm 1 \).

9) The M-XXX equation has the following L-equivalent [2]

\[ q_t = iq_{yx} - \frac{1}{2} [(V_1q)_x - qV_2 - Eqq_y] \]  
\[ V_{1x} = Eq_y \]  
\[ V_{2x} = -Eq_{yx} \]

\( E = \pm 1 \). Both these set of equations are integrable and the corresponding Lax representations exist [2].

VI. ON THE INTEGRALS OF MOTION

Of course the integrable nonlinear evolution equations allows an infinite number of integrals of motion. The above presented geometry allows construct the important integrals of motion. So in 2+1 dimensions we have the following remarkable

**Theorema-1:**

The 2+1 dimensional nonlinear evolution equations admit the following integrals of motion

\[ K_1 = \int \kappa m_2 dx dy, \quad K_2 = \int \tau m_2 dx dy \]

In particular for the 2+1 dimensional spin systems this theorema we can reformulate in the following way

**Theorema-2:**
The 2+1 dimensional spin systems admit the following integrals of motion

\[ K_1 = \int \vec{S} \cdot (\vec{S}_x \wedge \vec{S}_y) dxdy \] (71a)

\[ K_2 = \int \frac{[\vec{S} \cdot (\vec{S}_x \wedge \vec{S}_y)])[\vec{S} \cdot (\vec{S}_x \wedge \vec{S}_{xx})]}{|\vec{S}_x|^2} dxdy \] (71b)

Note that in the last case \( G = \frac{1}{4\pi}K_1 \) is the well known topological charge.

If \( \sigma \neq 0 \) then we have the following

**Theorema-3:**

The 2+1 dimensional nonlinear evolution equations admit the following integrals of motion

\[ K_3 = \int (\kappa m_2 + \sigma m_3) dxdy \] (72a)

\[ K_4 = \int (\tau m_2 + \sigma m_1) dxdy \] (72b)

\[ K_5 = \int (\tau m_3 - km_1) dxdy \] (72c)

The proves of these theoremes are given in[2].

**VII. ON THE LAX REPRESENTATIONS IN 2+1 DIMENSIONS**

It is of interest that the above considered formalism allows construct the "Lax representations" the above presented nonlinear equations [16].

**VIIa. The 2-dimensional case**

Let us introduce now the complex function

\[ \phi_l = \frac{e_2l + ie_3l}{1 - e_{1l}}, \quad e_1^2 + e_2^2 + e_3^2 = 1, \quad l = 1, 2, 3 \] (73)

The spatial and temporal evolution of this function can be written as a set of the following equations:

\[ \phi_{lx} = \frac{1}{2}k\phi_l^2 + \frac{1}{2}k, \] (74a)

\[ \phi_{ly} = \frac{1}{2}m_3\phi_l^2 + \frac{1}{2}m_3, \] (74b)

\[ \phi_{lt} = \frac{1}{2}\omega_3\phi_l^2 + \frac{1}{2}\omega_3. \] (74c)

Further introducing the transformation

\[ \phi_l = \frac{v_2}{v_1}, \] (75)
eq.(74) can be written as

\[
\begin{pmatrix}
  v_{1x} \\
  v_{2x}
\end{pmatrix} = \begin{pmatrix}
  0 & -\frac{k}{2} \\
  \frac{k}{2} & 0
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix},
\]
(76a)

\[
\begin{pmatrix}
  v_{1y} \\
  v_{2y}
\end{pmatrix} = \begin{pmatrix}
  0 & -\frac{1}{2}m_3 \\
  \frac{1}{2}m_3 & 0
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix},
\]
(76b)

\[
\begin{pmatrix}
  v_{1t} \\
  v_{2t}
\end{pmatrix} = \begin{pmatrix}
  0 & -\frac{1}{2}\omega_3 \\
  \frac{1}{2}\omega_3 & 0
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix}.
\]
(76c)

These systems we can consider as the "Lax representation".

VIIb. The 3-dimensional case

Introducing now the complex variable corresponding to an orthogonal rotation

\[
\phi_l = \frac{e^{2lt} + ie^{3lt}}{1 - e^{4lt}}, \quad e_l^2 + e_2^2 + e_3^2 = 1, \quad l = 1, 2, 3
\]
(77)

the spatial and temporal evolution of the trihedral can be rewritten as a set of the following three Riccati equations:

\[
\phi_{lx} = -i\tau \phi_l + \frac{1}{2} \left[ k - i\sigma \right] \phi_l^2 + \frac{1}{2} \left[ k + \sigma \right],
\]
(78a)

\[
\phi_{ly} = -im_1 \phi_l + \frac{1}{2} \left[ m_3 + im_2 \right] \phi_l^2 + \frac{1}{2} \left[ m_3 - im_2 \right],
\]
(78b)

\[
\phi_{lt} = -i\omega_1 \phi_l + \frac{1}{2} \left[ \omega_3 + i\omega_2 \right] \phi_l^2 + \frac{1}{2} \left[ \omega_3 - i\omega_2 \right].
\]
(78c)

Further introducing the transformation

\[
\phi_l = \frac{v_2}{v_1},
\]
(79)

eq.(78) can be written as a system of three coupled two component first order equations,

\[
\begin{pmatrix}
  v_{1x} \\
  v_{2x}
\end{pmatrix} = \begin{pmatrix}
  i\tau & -\frac{k}{2} \\
  \frac{k}{2} & -i\tau
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix},
\]
(80a)

\[
\begin{pmatrix}
  v_{1y} \\
  v_{2y}
\end{pmatrix} = \begin{pmatrix}
  \frac{-im_1}{2} & \frac{-1}{2} \left( m_3 + im_2 \right) \\
  \frac{1}{2} \left( m_3 - im_2 \right) & \frac{-im_1}{2}
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix},
\]
(80b)

\[
\begin{pmatrix}
  v_{1t} \\
  v_{2t}
\end{pmatrix} = \begin{pmatrix}
  \frac{-i\omega_1}{2} & \frac{-1}{2} \left( \omega_3 + i\omega_2 \right) \\
  \frac{1}{2} \left( \omega_3 - i\omega_2 \right) & \frac{-i\omega_1}{2}
\end{pmatrix} \begin{pmatrix}
  v_{1} \\
  v_{2}
\end{pmatrix}.
\]
(80c)

These equations are the new form of the Lax representations of the above considered 2+1 dimensional equations.
VIII. A NON-ISOSPECTRAL PROBLEMS

Unlike the 1+1 dimensions, where $\lambda_t = 0$, as consequence the equation $\lambda_t \neq \text{const}$ the 2+1 dimensional equations must be solve with help the non-isospectral version of the inverse scattering transform (IST) method. In fact that for example for the M-III equations (54) the spectral parameter $\lambda$ satisfies the following equation [2]

$$\lambda_t = (2c\lambda^2 + 2d\lambda)\lambda_y$$

Of course that we can solve this eq. using the following Lax representation

$$\Phi_x = [ic(\lambda^2 - b^2) + id(\lambda - b)]\sigma_3 \Phi \quad (82a)$$
$$\Phi_t = (2c\lambda^2 + 2d\lambda)\Phi_y \quad (82b)$$

For example we have the following two special solutions of eq (81) as $c = d - 1/2 = 0$ [2]

$$\lambda = \lambda_1 = \text{const} \quad (83a)$$
$$\lambda = \lambda_2 = \frac{y + k + i\eta}{b - t} \quad (83b)$$

where $b$, $k$ and $\eta$ are real constants. The corresponding solutions of the soliton eqs. is we called the overlapping or breaking solutions [17].

Note that unlike the 1+1 dimensions, where $\lambda_t = 0$, in 2+1 dimensions we have the following integral of motion for the spectral parameter

$$K = \int \lambda dy \quad (84)$$

So $K_t = 0$.

IX. ON THE SOLUTIONS OF THE M-I EQUATION

In this section we present the some exact solution of the M-I equation (47). Details are given in [18]. The Hirota bilinear form of eq (47) are given by

$$(iD_t - D_x D_y)(f^* \circ g) = 0, \quad (85a)$$
$$(iD_t - D_x D_y)(f^* \circ f - g^* \circ g) = 0, \quad (85b)$$
$$D_x (f^* \circ f + g^* \circ g) = 0, \quad (85c)$$

while the potential $u$ takes the form

$$u(x, y, t) = -\frac{iD_y (f^* \circ f + g^* \circ g)}{f^* \circ f + g^* \circ g}, \quad (86)$$

where $g$ and $f$ are complex valued functions. Here $D_x$ is the Hirota bilinear operator, defined by
\[ D^k D^m D^n_t (f \circ g) = (\partial_x - \partial_x')^k (\partial_y - \partial_y')^m (\partial_t - \partial_t')^n f(x, y, t)g(x, y, t)|_{x = x', y = y', t = t'} \]  
\[ (87) \]

Using the above definition of the \( D \)-operator, we get from (31d) that

\[ u_x = -2i [D_y (f \circ g) D_x (f^* \circ g^*) - c.c]. \]  
\[ (88a) \]

In terms of \( g \) and \( f \), the spin field takes the form

\[ S^+ = \frac{2f^* g}{|f|^2 + |g|^2}, \quad S_3 = \frac{|f|^2 - |g|^2}{|f|^2 + |g|^2}. \]  
\[ (88b) \]

Eq.(85) represents the starting point to obtain interesting classes of solutions for the spin system (47). The construction of the solutions is standard. One expands the functions \( g \) and \( f \) as a series

\[ g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \cdots, \]  
\[ (89a) \]

\[ f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \epsilon^6 f_6 + \cdots \]  
\[ (89b) \]

Substituting these expansions into (85 a,b,c) and equating the coefficients of \( \epsilon \), one obtains the following system of equations from (85a):

\[ \epsilon^1 : i g_{1t} + g_{1xy} = 0, \]  
\[ (90a) \]

\[ \epsilon^3 : [i \partial_t + \partial_x \partial_y] g_3 = [i D_t - D_x D_y] (f^*_2 \cdot g_1), \]  
\[ (90b) \]

\[ \vdots \]

\[ \vdots \]

\[ \epsilon^{2n+1} : [i \partial_t + \partial_x \partial_y] g_{2n+1} = \sum_{k+m=n} [i D_t - D_x D_y] (f^*_{2k} \cdot g_{2m+1}), \]  
\[ (90c) \]

and

\[ \epsilon^2 : [i \partial_t + \partial_x \partial_y] (f^*_2 - f_2) = [i D_t - D_x D_y] (g^*_1 \cdot g_1), \]  
\[ (91a) \]

\[ \epsilon^4 : [i \partial_t + \partial_x \partial_y] (f^*_4 - f_4) = [i D_t - D_x D_y] (g^*_1 \cdot g_3 + g^*_3 \cdot g_1 - f^*_2 \cdot f_2), \]  
\[ (91b) \]

\[ \vdots \]

\[ \vdots \]
\[
\epsilon^{2n} : [i\partial_t + \partial_x \partial_y] (f_{2n}^* - f_{2n}) = (iD_t - D_x D_y) \left( \sum_{n_1+n_2=n-1}^{n+n_2} g_{2n_1+1}^* g_{2n_2+1} \right) \\
- (iD_t - D_x D_y) \left( \sum_{m_1+m_2=n} f_{2n_1}^* f_{2n_2} \right).
\] (91c)

Further from (31c), we have the following:

\[
\epsilon^2 : \partial_x (f_2^* - f_2) = D_x (g_1^* g_1), 
\] (92a)

\[
\epsilon^4 : \partial_x (f_4^* - f_4) = D_x (g_1^* g_3 + g_3^* g_1 + f_2^* f_2), 
\] (92b)

.. 

.. 

.. 

\[
\epsilon^{2n} : \partial_x (f_{2n}^* - f_{2n}) = D_x \left[ \sum_{n_1+n_2=n-1}^{n+n_2} (g_{2n_1+1}^* g_{2n_2+1}) + \sum_{n_1+n_2=n} f_{2n_1}^* f_{2n_2} \right]. 
\] (92c)

Solving recursively the above equations, we obtain many interesting classes of solutions to eq.(47).

Using the results of the previous section, we are in a position to construct many exact solutions such as solitons, domain walls, breaking solitons and induced dromions of eq.(4). To obtain such solutions, we can use eqs.(85) as the starting point.

**A. The 1-line soliton solution.**

In order to construct exact 1-line soliton solutions (1-SS) of eq.(47), we take the ansatz

\[
g_1 = \exp \chi_1, \quad \chi_1 = a_1 x + b_1 y + c_1 t + e_1.
\] (93)

where \(a_1, b_1, c_1\) and \(e_1\) are complex constants. By substituting the above value of \(g_1\) in eq.(90a), we get

\[
c_1 = ia_1 b_1.
\]

Hence \(g_1\) reads as

\[
g_1 = \exp \chi_1, \quad \chi_1 = a_1 x + b_1 y - ia_1 b_1 t + e_1.
\] (94)

By picking up the appropriate equations from (90) to (92), we obtain the expression for \(f_2\) as
\[ f_2 = \exp (\chi_1 + \chi_1^* + \psi), \]  
\[ \exp \psi = \frac{-a_1^2}{(a_1 + a_1^*)^2}. \]

By substituting the above values of \( g_1 \) and \( f_2 \) in eqs. (88b) and (88c), we obtain the expressions for the spin components and for the potential.

B. The 2-soliton and N-soliton solutions.

In this subsection we present the 2-SS of eq.(47). To generate a 2-line soliton solution (2-SS), we take

\[ g_1 = \exp \chi_1 + \exp \chi_2. \]  
(96)

Substituting (96) in (88)-(90), after some calculation we obtain

\[ f_2 = N_{11} \exp (\chi_1 + \chi_1^*) + N_{12} \exp (\chi_1 + \chi_2^*) + N_{21} \exp (\chi_1^* + \chi_2) + N_{22} \exp (\chi_2 + \chi_2^*), \]  
(97a)

\[ g_3 = L_{12} N_{11} N_{12} \exp (\chi_1 + \chi_1^* + \chi_2) + L_{12} N_{22} N_{21} \exp (\chi_1 + \chi_2 + \chi_2^*), \]  
(97b)

\[ f_4 = L_{12} L_{12}^* N_{11} N_{12} N_{21} N_{22} \exp (\chi_1 + \chi_1^* + \chi_2 + \chi_2^*), \]  
(97c)

where

\[ N_{rs} = \frac{-k_r^2}{(k_r + k_s^*)^2}, \quad L_{rs} = \frac{-(k_r - k_s)^2}{k_s^2}. \]  
(98)

Inserting (97) into (88b) we get

\[ S^+(x,y,t) = 2 \frac{(1 + f_2^* + f_4^*)(g_1 + g_3)}{|1 + f_2 + f_4|^2 + |g_1 + g_3|^2}, \]  
(99a)

\[ S_3(x, y, t) = \frac{|1 + f_2 + f_4|^2 - |g_1 + g_3|^2}{|1 + f_2 + f_4|^2 + |g_1 + g_3|^2}, \]  
(99b)

and similarly the expression for the potential can also be obtained.

By taking

\[ g_N = \sum_{j=1}^{N} \exp \chi_j \]

and extending the above procedure, one can obtain the N-SS also.

C. The 1-curved soliton solution.
In the one soliton solution, we can write the general form of $g$ as
\[ g_1 = \exp \chi_1, \quad \chi_1 = a_1 x + b_1(y, t) + e_1, \quad (100) \]
where $b_1(y, t)$ is an arbitrary function of $y$ and $t$, satisfying the relation
\[ b_1(y, t) = b_1(y + ia_1 t). \quad (101) \]
Here also $a_1$ and $e_1$ are complex constants as stated earlier. Now for the above general choice of the arbitrary function $b_1(y, t)$, the 1-SS of eq.(47) takes the form
\[ S_3(x,y,t) = 1 - \frac{2a_1^2}{a_1^2 + \alpha^2} \sech^2 \chi_1 R, \quad (102a) \]
\[ S^+(x,y,t) = \frac{2a_1 R}{a_1^2 + \alpha^2} [ia_1 R - a_1 R \tanh \chi_1 R] \sech \chi_1 R, \quad (102b) \]
while for the potential
\[ u(x,y,t) = \frac{2a_1 R}{a_1^2 + \alpha^2} (a_1 R b_1' - a_1 R \eta b_1') \sech \chi_1 R, \quad (102c) \]
in which the prime has been used to denote the differentiation with respect to the real part of the argument. For fixed $(y, t)$, it follows from (102) that $\vec{S} \to (0, 0, \pm 1)$ as $x \to \pm \infty$ and the wavefront itself is defined by the equation $\chi_1 R = \eta x + m_1 R(\rho) + c_1 R = 0$. For other choices of $m_1$, we can obtain more interesting solutions.

**D. The domain wall type solution**

The soliton solutions of (98) and (102) correspond to the boundary condition
\[ \vec{S}(x,y,t) = (0,0,1), \quad \text{as} \quad x^2 + y^2 \to \pm \infty. \quad (103) \]
Another class of physically interesting solutions are the domain wall type solutions, which have the asymptotic form
\[ \vec{S}(x,y,t) = (0,0,\pm1), \quad \text{as} \quad x^2 + y^2 \to \pm \infty \quad (104) \]
In order to obtain domain wall solutions, we make the choice
\[ \omega(x,y,t) = g(x,y,t), \quad f(x,y,t) = 1 \quad (105) \]
Then, eq.(85) reduce to
\[ ig_t + g_{xy} = 0, \quad (106a) \]
\[ g^* g_y + g^* g_x = 0, \quad (106b) \]
\[ g^* g - g^* g_x = 0, \quad (106c) \]
which is consistent with eq.(13). Identically, we can use another way
\[ \omega(x, y, t) = \frac{1}{f(x, y, t)}, \quad g(x, y, t) = 1 \] (107)

Here also it follows from (85) that

\[ if_t^* - f_{xy}^* = 0 \] (108a)

\[ f_x^* f_y + f_x f_y^* = 0 \] (108b)

\[ f_x^* f - f^* f_x = 0. \] (108c)

Comparing eqs. (106) and (108), we see that eq.(47) is invariant under the inversions, that is if \( \omega(x, y, t) \) is the solution of eq.(47) then

\[ \omega'(x, y, t) = \pm \frac{1}{\omega(x, y, t)}, \] (109)

are also the solutions of eq.(47).

Now, we find the simplest non-trivial solutions for example of eq.(106). Let us take the ansatz

\[ g = \exp(ax + iby - abt) \] (110)

where \( a, b \) are real constants. The components of the spin vector \( \vec{S} \) are given by

\[ S^+(x, y, t) = \frac{\exp iby}{\cosh[a(x - bt - x_0)]}, \] (111a)

\[ S_3(x, y, t) = -\tanh[a(x - bt - x_0)]. \] (111b)

We can also have a more general solution of the form

\[ g = \exp[ax + m(y, t)], \] (112)

where \( a \) is a real constant and \( m(y, t) \) is an arbitrary function of \( y \) and \( t \). From eq.(106a), it follows that

\[ m = m(y + iat). \] (113)

Expressions for the spin components are then given by

\[ S^+(x, y, t) = \frac{\exp[iIm(m(y, t))]\cosh[ax + Re(m(y, t))]}{\cosh[ax + Re(m(y, t))]}, \] (114a)

\[ S_3(x, y, t) = -\tanh[ax + Re(m(y, t))], \] (114b)

and the potential is

\[ u(x, y, t) = -2Im(m(y, t))\{1 + \exp[-2(ax + m_R)]\}^{-1}, \] (114c)
E. The breaking soliton solution.

We have already noted in Sec.II that for the present system (47), we have a non-isospectral problem, as the spectral parameter $\lambda$ satisfies eq.(81). The above presented solutions all correspond to the constant solution of eq.(83a), namely $\lambda = \lambda_1 = \text{constant}$. One may consider other interesting solutions of eq.(81). For example, one can have a special solution

$$\lambda = \lambda_1 = \eta(y,t) + i\xi(y,t) = \frac{y + k + i\eta}{b - t},$$

(115)

where $b$, $k$ and $\eta$ are real constants. Corresponding to this case, we may call the solutions of eqs. (47) and (51) as breaking solitons[19]. Using the Hirota method, one can also construct the breaking 1-SS of eq.(47) associated with (115). For this purpose, we take $g_1$ in the form

$$g = g_1 = \exp \chi, \quad \chi = ax + m + c = \chi_R + i\chi_I,$$

(116)

where $a = a(y,t)$, $m = m(y,t)$ and $c = c(t)$ are functions to be determined. Substituting (116) into the first of eq.(88), we get

$$ia_t + aa_y = 0, \quad im_t + am_y = 0, \quad iA_t + Aa_y = 0,$$

(117)

where $A = \exp(c)$. Particular solutions of eqs.(117) have the forms

$$a = -i\lambda = \frac{\eta - i(y + k)}{b - t}, \quad m = m\left(\frac{y + k + i\eta}{b - t}\right), \quad A = \frac{A_0}{b - t},$$

(118)

where $\eta$, $k$, $b$ and $A_0$ are some constants. From eqs. (88)-(90), we obtain

$$f_2 = B \exp 2\chi_R, \quad B = \frac{|A_0|^2 (y + k + i\eta)^2}{4\eta^2(b - t)^2}.$$  

(119)

Now, we can write the breaking 1-SS of eq.(47) (using equations (85b), (116)-(119)),

$$S^+(x,y,t) = \frac{2\eta\exp i(\chi_I + \phi)(y + k - i\eta) \left[(y + k)\cosh z - i\eta \sinh z\right]}{[(y + k)^2 + \eta^2]^\frac{1}{2} \cosh^2 z},$$

(120a)

$$S_3(x,y,t) = 1 - \frac{2\eta^2}{[(y + k)^2 + \eta^2]} \text{sech}^2 z,$$

(120b)

where $z = \frac{\eta}{b - t}x - \frac{1}{2}\ln[(y + \alpha)^2 - \eta^2] + \psi, \quad \psi = \ln \left|\frac{A_0(y + k + i\eta)}{2\eta(b - t)}\right|, \quad \chi_I = \frac{\psi - i(y + k + i\eta)}{(b - t)x + m_I}$. We see that the solution (120) corresponds to an algebraically decaying solution for large $x$, $y$.

D. Localized coherent structures (dromions).
Particularly, we present the dromion type localized solutions of eq.(4), the so-called induced localized structures/or induced dromions[14] for the potential \( u(x, y, t) \). This is possible by utilising the freedom in the choice of the arbitrary functions \( b_{1R} \) and \( b_{1I} \) in eqs.(101). For example, if we make the ansatz

\[
b_{1I}(\rho_R) = \kappa b_{1R}(\rho_R) = \tanh(\rho_R),
\]

where \( \rho_R = y - a_{1I}t \). Similarly, the expressions for the spin can be obtained from eqs.(98). The solution (122) for \( u(x, y, t) \) decays exponentially in all the directions, even though the spin \( \vec{S} \) itself is not fully localized. Analogously we can construct another type of “induced dromion” solution with the choice

\[
b_{1I} = \kappa b_{1R} = \int \frac{d\rho_R}{(\rho_R + \rho_0)^2 + 1} + b_0,
\]

where \( \rho_0 \) and \( b_0 \) are constants, so that

\[
u(x, y, t) = \frac{2\eta(\xi - \eta y)}{(\rho_R + \rho_0)^2 + 1} \text{sech}^2 \left[ \eta x + \int \frac{d\rho_R}{(\rho_R + \rho_0)^2 + 1} - \eta x_0 \right].
\]

X. SOLUTIONS of (2+1) DIMENSIONAL NLSE

In this section, we wish to consider briefly the corresponding solutions of the equivalent generalized NLSE eq.(51). Already this equation has received some attention in the literature. The following types of solutions are available: In a similar way, we can construct the N-breaking soliton solutions of eq.(51). As an example, let us obtain the 1-breaking soliton solution of eq.(51). The Hirota bilinear form of eq.(51) can be obtained by using the transformation

\[
\psi = \frac{h}{\phi},
\]

as [19]

\[
[iD_t + D_x D_y](h \circ \phi) = 0,
\]

\[
D_x^2(\phi \circ \phi) = 2hh^*.
\]

We look for the 1-breaking soliton solution in the following form:

\[
h = \exp \chi,
\]

\[
\phi = 1 + \phi_2,
\]

where \( \chi = b(y, t)x + n(y, t) + c(t) \). Substituting (127) into (126), we get
\text{ib}_t + \text{bb}_y = 0, \quad (128a)
\text{in}_t + \text{bn}_y = 0, \quad (128b)
\text{iB}_t + \text{Bb}_y = 0, \quad (128c)

\text{and}

\phi_2 = \frac{|B|^2}{(b + b^*)^2} \exp \chi + \chi^* = \exp 2(b_R x + n_R + \chi_0), \quad (129)

where \exp 2\chi_0 = \frac{|B|^2}{\delta b_R}, B = \exp c(t) \text{ and } b_R = b_R(t) = Re(b). \text{ Now, the formulae (125) provide us the 1-breaking soliton solution of eq.}(24)

\psi(x, y, t) = \frac{b_R(t) \exp i \left[ b_I(y, t)x + n_I(y, t) + c_0 \right]}{\cosh \left[ b_R x + n_R(y, t) + \chi_0 \right]}, \quad (130)

where \(b(y, t) = b_R + ib_I, \ n(y, t) = n_R + in_I \) and \(B(t)\) are the solutions of eqs.(128). \text{ As for the case of eqs.}(117), \text{ if we take the following particular solutions of the system of eqs.}(128);

\lambda = \frac{y + k + i\eta}{b - t}, \ n = \frac{y + k + i\eta}{b - t}, \ B = \frac{B_0}{(b - t)}, \quad (131)

\text{then the 1-breaking soliton solution of eq.}(51) \text{ takes the form}

\psi(x, y, t) = -\frac{n}{b - t} \exp i \left[ \frac{y + k}{b - t} x + n_I(y, t) + c_0 \right] \text{sech} Z, \quad (132)

where \(Z = \frac{n}{b - t} x + n_R(y, t) + \chi_0\) and \(c_0, \chi_0\) are constants.

\text{Similarly, we obtain the breaking N-SS of (51). In this case we can take the ansatz}

\text{g}_1 = \sum_{j=1}^{N} \exp \chi_j \quad (133)

with \(\chi_j = b_j(y, t)x + n_j(y, t) + c_j(t)\). \text{ Inserting (133) into (132) leads to}

\text{i}b_{jt} + b_jb_{xy} = 0, \quad (134a)
\text{i}n_{jt} + bjn_j = 0, \quad (134b)
\text{iB}_{jt} + B_Bb_{yy} = 0, \quad (134c)

\text{Proceeding as before, one can obtain breaking N-soliton solution(see, also,[20]).}
XI. ON THE SPIN-PHONON SYSTEMS

Let us find the L-equivalent of the $M^{33}_{00}$ equation\[2\]

$$\vec{S}_t = (\vec{S} \wedge \vec{S}_x + u\vec{S} \wedge \vec{S}_x)_x$$  \hspace{0.5cm} (135a)

$$u_t + u_x + \lambda (\vec{S}_x^2)_x = 0$$  \hspace{0.5cm} (135b)

Let $q = \tau - ik$; $\tilde{e}_2 \equiv \vec{S}, \sigma = \frac{1}{2}u - \frac{1}{4}u^2$. Then the L-equivalent of the equations (135) has the form

$$iqt + q_{xx} + (uq)_{xx} - \frac{i}{\lambda}[u - \frac{u^3}{2} + \frac{u^2}{2}]q_x - \frac{1}{\lambda}\frac{3i}{2}uux + (\frac{1}{4\lambda} - 4)u^2 + \frac{i}{2}u_x + \frac{u}{\lambda} + \frac{u^3}{4\lambda} - \lambda|q|^2 q = 0,$$

$$u_t + u_x + \lambda|q|^2 = 0.$$  \hspace{0.5cm} (136a)

Similarly we can construct the L-equivalents of the other spin-phonon systems\[2\].

XII. CONCLUSION

In this paper using the C-approach\[2\] we have presented the L-equivalent soliton equations of the Ishimori and some Myrzakul equations. Finally we note that using the C-approach we can see to the older problems from the new point of view. For example, the isotropic Landau-Lifshitz equation

$$\vec{S}_t = \vec{S} \wedge \vec{S}_x$$  \hspace{0.5cm} (137)

and the NLSE

$$iq_t + q_{xx} + 2E\bar{q}q = 0$$  \hspace{0.5cm} (138)

are L-equivalence to each other\[2\]. In our case $E = \pm 1$. At the same time the 1+1 dimensional M-IV and M-XXI equations have the following L-equivalents:

the 1+1 dimensional mKdV

$$q_t + q_{xxx} + 6E\bar{q}q_x = 0$$  \hspace{0.5cm} (139)

the 1+1 dimensional KdV

$$q_t + q_{xxx} + 6Eqq_x = 0$$  \hspace{0.5cm} (140)

respectively.

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