BRUNELLA–KHANEDANI–SUWA VARIATIONAL RESIDUES FOR INVARIANT CURRENTS

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To Israel Vainsencher on the occasion of his 70th birthday

ABSTRACT. In this work we prove a Brunella–Khanedani–Suwa variational type residue theorem for currents invariant by holomorphic foliations. As a consequence, we give conditions for the leaves of a singular holomorphic foliation to accumulate in the intersection of the singular set of the foliation with the support of an invariant current.

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1. INTRODUCTION

In [18] B. Khanedani and T. Suwa introduced an index for singular holomorphic foliations on complex compact surfaces called the Variational index. In [19] D. Lehmann and T. Suwa generalized the variational index for higher dimensional holomorphic foliations. In particular, they showed that if $V$ is a complex subvariety invariant by a holomorphic foliation $\mathcal{F}$ of dimension $k \geq 1$ on a $n$-dimensional complex compact manifold $X$, then

$$c_1^{n-k}(\det(N^*\mathcal{F})) \cdot [V] = (-1)^{n-k} \sum_{\lambda} \operatorname{Res}_{c_1^{n-k}}(\mathcal{F}; S_{\lambda}),$$

where $S_{\lambda}$ is a connected component of $S(\mathcal{F}, V) := (\text{Sing}(\mathcal{F}) \cap V) \cup \text{Sing}(V)$ (here Sing($\mathcal{F}$) and Sing($V$) denote the singular sets of $\mathcal{F}$ and $V$, respectively), $[V]$ is the integration current of $V$ and $N^*\mathcal{F}$ is the conormal sheaf of $\mathcal{F}$. In the case $X$ is a complex surface and $S(\mathcal{F}, V)$ is an isolated set, then for each $p \in S(\mathcal{F}, V)$,

$$- \operatorname{Res}_{c_1}(\mathcal{F}; p) = \operatorname{Var}(\mathcal{F}, V, p),$$

where $\operatorname{Var}(\mathcal{F}, V, p)$ denotes the Variational index of $\mathcal{F}$ along $V$ at $p$, as defined by Khanedani-Suwa in [18].

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M. Brunella in [5] studied the Khanedani-Suwa variational index and its relations with the GSV and the Camacho-Sad indices. See also [22, II, Proposition 1.2.1].

M. McQuillan, in his proof of the Green-Griffiths conjecture (for a projective surface \(X\) with \(c_2^e(X) > c_2(X)\), [22], showed that if \(X\) is a complex surface of general type and \(\mathcal{F}\) is a holomorphic foliation on \(X\), then \(\mathcal{F}\) has no entire leaf which is Zariski dense. See [12, 23, 16, 13] for more details about the Green-Griffiths conjecture and generalizations. M. Brunella in [1] provided an alternative proof of McQuillan’s result by showing that if \([T_f]\) is the Ahlfors current associated to a Zariski dense entire curve \(f : \mathbb{C} \to X\) which is tangent to \(\mathcal{F}\), then

\[
c_1(N^*\mathcal{F}) \cdot [T_f] = \frac{1}{2\pi i} \sum_{p \in \text{Sing}(\mathcal{F}) \cap \text{Supp}(T_f)} [T_f](\chi_{U_p} d(\phi_p \beta_p)) \leq 0,
\]

where \(\chi_{U_p}\) denotes the characteristic function of a neighborhood \(U_p\) of \(p \in \text{Sing}(\mathcal{F}) \cap \text{Supp}(T_f)\), see section 3 for more details.

To continue we consider \(\mathcal{F}\) a singular holomorphic foliation of dimension \(k \geq 1\) on a complex compact manifold \(X\) of dimension at least two. We recall that a positive closed current \(T\) in \(X\) is invariant by \(\mathcal{F}\) if \(T|\mathcal{F} \equiv 0\), that is, \(T(\eta) = 0\) for every test form \(\eta\) vanishing along the leaves of \(\mathcal{F}\), so that \(T(\eta)\) depends only on the restriction of \(\eta\) to the leaves.

In [4] M. Brunella proved a more general variational index type Theorem for positive closed currents of bidimension \((1, 1)\) invariant by one-dimensional holomorphic foliations, with isolated singularities, on complex compact manifolds. More precisely, he showed that if \(T\) is an invariant positive closed current of bidimension \((1, 1)\), then

\[
c_1(\det(N^*\mathcal{F})) \cdot [T] = \sum_{p \in \text{Sing}(\mathcal{F}) \cap \text{Supp}(T)} \frac{1}{2\pi i} [T](\chi_{U_p} d(\phi_p \beta_p)).
\]

Compare this formula with the so called asymptotic Chern Class of a foliation on a complex surface introduced in [11]. Moreover, M. Brunella showed, in the same work, that a generic one-dimensional holomorphic foliation on complex projective spaces has no invariant measure. In [17, Corollary 1.2] L. Kaufmann showed that there is no diffuse foliated cycle directed by embedded Lipschitz laminations of dimension \(k \geq n/2\) on \(\mathbb{P}^n\).

We denote the class of a closed current \(T\) of bidimension \((p, p)\) in the cohomology group \(H^{n-p,n-p}(X)\) by \([T]\). In order to provide a generalization of the above results, we define the residue of \(\mathcal{F}\) relative to \(T\) along a connected component of the singular set of \(\mathcal{F}\), (see for instance Def. 3.1 in Sect. 3.). In this work we prove the following result.

**Theorem 1.1.** Let \(\mathcal{F}\) be a holomorphic foliation, of dimension \(k \geq 1\), on a complex compact manifold \(X\) with \(\dim(\text{Sing}(\mathcal{F})) \leq k - 1\). Write \(\text{Sing}(\mathcal{F}) = \bigcup_{\lambda} Z_{\lambda}\), a decomposition into connected components and let \(U_{\lambda}\) be a regular neighborhood of \(Z_{\lambda}\). For \(p \geq k\), if \(T\) is a positive closed current of bidimension \((p, p)\) invariant by \(\mathcal{F}\), then

\[
c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T] = \sum_{Z_{\lambda} \subset \text{Supp}(T) \cap \text{Sing}(\mathcal{F})} \text{Res}(\mathcal{F}, T, Z_{\lambda}).
\]
A compact non-empty subset $M \subset X$ is said to be a **minimal set** for $\mathcal{F}$ if the following properties are satisfied

(i) $M$ is invariant by $\mathcal{F}$;
(ii) $M \cap \text{Sing}(\mathcal{F}) = \emptyset$;
(iii) $M$ is minimal with respect to these properties.

The problem of existence of minimal sets for codimension one holomorphic foliations on $\mathbb{P}^n$ was considered by Camacho–Lins Neto–Sad in [11]. To our knowledge, this problem remains open for $n = 2$. If $\mathcal{F}$ is a codimension one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$, Lins Neto [20] proved that $\mathcal{F}$ has no minimal sets.

M. Brunella posed in [2] the following question:

**Conjecture.** Let $X$ be a compact connected complex manifold of dimension $n \geq 3$, and let $\mathcal{F}$ be a codimension one holomorphic foliation on $X$ such that $N\mathcal{F}$ is ample. Then every leaf of $\mathcal{F}$ accumulates to $\text{Sing}(\mathcal{F})$.

In [3], Brunella–Perrone proved the above Conjecture for codimension-one holomorphic foliations on a projective manifold with cyclic Picard group. In [8] the natural conjecture has been stated:

**Conjecture (Generalized Brunella’s conjecture).** Let $X$ be a compact connected complex manifold of dimension $n \geq 3$, and let $\mathcal{F}$ be a holomorphic foliation of codimension $r < n$ on $X$ such that $\det(N\mathcal{F})$ is ample. Then every leaf of $\mathcal{F}$ accumulates to $\text{Sing}(\mathcal{F})$, provided $n \geq 2r + 1$.

The main result in [8] suggests that the property of accumulation of the leaves of a foliation $\mathcal{F}$ to the its singular set (or nonexistence of minimal sets of $\mathcal{F}$) depends on the existence of strongly $q$-convex spaces which contains the singularities of $\mathcal{F}$. In [11] it was proved that there is no invariant measure with support on a nontrivial minimal set of a foliation on $\mathbb{P}^2$. We observe that in $\mathbb{P}^n$ we have that $\det(N\mathcal{F})$ is ample for every foliation $\mathcal{F}$. The following Corollary 1.2 generalize the result in [11] Theorem 2.

**Corollary 1.2.** Let $\mathcal{F}$ be a holomorphic foliation, of dimension $k \geq 1$, on a projective manifold $X$ such that $\dim(\text{Sing}(\mathcal{F})) \leq k - 1$ and $\det(N\mathcal{F})$ ample. Suppose that $h^{n-p,n-p}(X) = 1$, for some $p \geq k$. If $T$ is a positive closed current of bidimension $(p,p)$ invariant by $\mathcal{F}$, then $\text{Supp}(T) \cap \text{Sing}(\mathcal{F}) \neq \emptyset$. In particular, there is no invariant positive closed current of bidimension $(p,p)$ with support on a nontrivial minimal set of $\mathcal{F}$.

Compare Corollary 1.2 with [17] Corollary 5.5. Since $h^{n-p,n-p}(\mathbb{P}^n) = 1$, this result holds for foliations on $\mathbb{P}^n$, in particular if $V \subset \mathbb{P}^n$ is an $\mathcal{F}$-invariant complex subvariety, then $V \cap \text{Sing}(\mathcal{F}) \neq \emptyset$. This is the Esteves–Kleiman’s result [15 12, Proposition 3.4, pp. 12].

We can also apply Theorem 1.1 to the Ahlfors currents associated to $f : \mathbb{C}^k \to X$ a holomorphic map of generic maximal rank which is a leaf of the foliation $\mathcal{F}$. Fix a Kähler form $\omega$ on $X$. On $\mathbb{C}^k$ we take the homogeneous metric form

$$\omega_0 := dd^c \ln |z|^2,$$

and denote by

$$\sigma = d^c \ln |z|^2 \wedge \omega_0^{k-1}$$
the Poincaré form. Consider \( \eta \in A^{1,1}(X) \) and for any \( r > 0 \) define
\[
T_{f,r}(\eta) = \int_0^r \frac{dt}{t} \int_{B_t} f^* \eta \wedge \omega_0^{k-1},
\]
where \( B_t \subset \mathbb{C}^k \) is the ball of radius \( t \). Then we consider the positive currents \( \Phi_r \in A^{1,1}(X)' \) defined by
\[
\Phi_r(\eta) := \frac{T_{f,r}(\eta)}{T_{f,r}(\omega)}.
\]
This gives a family of positive currents of bounded mass from which we can extract a subsequence \( \Phi_{r_n} \) which converges to a current \( [T_f] \in A^{1,1}(X)' \) called the Ahlfors current of \( f \), see [16, Claim 2.1].

This construction has been generalized in [6] by Burns–Sibony and [14] by De Thélin. In order to associate to \( f : \mathbb{C}^k \to X \) a positive closed current of any bidimension \((s, s)\), \( 1 \leq s \leq k \) (also called Ahlfors currents) it is necessary to impose some technical conditions.

We obtain another consequence of Theorem 1.1 as follows:

**Corollary 1.3.** Let \( \mathcal{F} \) be a holomorphic foliation, of dimension \( k \geq 1 \), on a projective manifold \( X \) such that \( \dim(\text{Sing}(\mathcal{F})) \leq k - 1 \) and \( \det(N_{\mathcal{F}}) \) ample. Let \( f : \mathbb{C}^k \to X \) be a holomorphic map of generic maximal rank which is a leaf of the foliation. Suppose that \( h^{n-p,p}(X) = 1 \), for some \( p \geq k \), and that there exists an Ahlfors current of bidimension \( (p, p) \) associated to \( f \). Then \( \overline{f(\mathbb{C}^k) \cap \text{Sing}(\mathcal{F})} \neq \emptyset \).

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### 2. Singular holomorphic foliations

Let \( X \) be a connected compact complex manifold of dimension \( n \). To define a (singular) holomorphic foliation \( \mathcal{F} \) on \( X \) we adopt the following point of view: such a \( \mathcal{F} \) is determined by a coherent subsheaf \( N^* \mathcal{F} \) of the cotangent sheaf \( T^* X = \Omega^1_X \) of \( X \) which satisfies

1) integrability: \( dN^* \mathcal{F} \subset N^* \mathcal{F} \wedge \Omega^1_X \) and
2) \( \Omega^1_X/N^* \mathcal{F} \) is torsion free.

The generic rank of \( N^* \mathcal{F} \) is the codimension of \( \mathcal{F} \), the dual \( (N^* \mathcal{F})^* = N\mathcal{F} \) is the normal sheaf of the foliation and the singular locus of \( \mathcal{F} \) is
\[
\text{Sing}(\mathcal{F}) = \{ p \in X : (\Omega^1_X/N^* \mathcal{F})_p \text{ is not a free } \mathcal{O}_p \text{ module} \}.
\]
Condition 2 above implies \( \text{codim}(\text{Sing}(\mathcal{F})) \geq 2 \).

Remark that, on \( X \setminus \text{Sing}(\mathcal{F}) \), we have an exact sequence of holomorphic vector bundles
\[
0 \to N^* \mathcal{F} \to \Omega^1_X \to T^* \mathcal{F} \to 0
\]
and, dualizing
\[
0 \to T\mathcal{F} \to TX \to N\mathcal{F} \to 0,
\]
where $T\mathcal{F}$ is called the tangent bundle of $\mathcal{F}$, of dimension $k = (n - \text{codim}(\mathcal{F}))$. Also, since the singular set has codimension greater than 1 we have the adjunction formula

$$KX = K\mathcal{F} \otimes \det(N^*\mathcal{F}),$$

where $K\mathcal{F} = \det(T\mathcal{F})^*$ denotes the canonical bundle of $\mathcal{F}$.

If $\mathcal{F}$ has codimension $(n - k)$ then, by taking the $(n - k)$-th wedge product of the inclusion

$$N^*\mathcal{F} \rightarrow \Omega^1_X,$$

we get a $(n - k)$-form $\omega$ with coefficients in the line bundle $(\wedge^{n-k} N\mathcal{F})^* = \det(N\mathcal{F})$.

### 2.1. Holomorphic foliations on complex projective spaces.

Let $\omega \in H^0(\mathbb{P}^n, \Omega^{n-k}_{\mathbb{P}^n}(m))$ be the twisted $(n-k)$-form induced by a holomorphic foliation $\mathcal{F}$ of dimension $k$ on $\mathbb{P}^n$.

Take a generic non-invariant linearly embedded subspace $i : L \cong \mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$. We have an induced non-trivial section $i^*\omega \in H^0(L, \Omega^{n-k}_{\mathbb{P}^n}(m)) \simeq H^0(\mathbb{P}^{n-k}, \mathcal{O}_{\mathbb{P}^{n-k}}(k - n - 1 + m))$, since $\Omega^{n-k}_{\mathbb{P}^n-k} = \mathcal{O}_{\mathbb{P}^{n-k}}(k - n - 1)$. The degree of $\mathcal{F}$ is defined by

$$\text{deg}(\mathcal{F}) := \text{deg}(Z(i^*\omega)) = k - n - 1 + m.$$  

In particular, $\omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}((\text{deg}(\mathcal{F})+n-k+1)))$. That is, $\det(N\mathcal{F}) = \mathcal{O}_{\mathbb{P}^n}(\text{deg}(\mathcal{F})+n-k+1)$ is ample.

A holomorphic foliation, of degree $d$, can be induced by a polynomial $(n - k)$-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$, see for instance [9].

### 3. Variational residue and proof of Theorem 1.1

Hence, a holomorphic foliation of dimension $k$ is given by a family $(\{V_\mu\}, \{\omega_\mu\})_{\mu \in \Lambda}$, where $\mathcal{V} = \{V_\mu\}_{\mu \in \Lambda}$ is an open cover of $X$ by Stein open sets, $\omega_\mu$ is an integrable holomorphic $(n-k)$-form defined in $V_\mu$ and locally decomposable in $V_\mu \setminus \text{Sing}(\mathcal{F})$. This means that, for each $p \in V_\mu$, there is an open neighborhood $V_p \subset V_\mu$ of $p$ such that

$$\omega_\mu|_{V_p} = \omega_1^\mu \wedge \cdots \wedge \omega_{{n-k}}^\mu,$$

where $\omega_j^\mu$ is a holomorphic 1-form and $d\omega_j^\mu \wedge \omega_\mu = 0$ for $1 \leq j \leq n - k$.

The integrability condition tells us that, in $V_\mu \setminus \text{Sing}(\mathcal{F})$, there is a $C^\infty$ 1-form $\alpha_\mu$ satisfying:

(i) $d\omega_\mu = \alpha_\mu \wedge \omega_\mu$, for all $\mu \in \Lambda$. $\alpha_\mu$ is not unique, but its restriction to the leaves of $\mathcal{F}$ is, provided $\omega_\mu$ is fixed.

(ii) $\alpha_\mu$ is of type $(1,0)$ since $\omega_\mu$ is holomorphic and $\alpha_\mu|_{\mathcal{F}}$ is holomorphic. This last fact follows from: if we assume that around a regular point the foliation $\mathcal{F}$ is generated by $\partial/\partial z_i$, $i = 1, \ldots, k$, then $t_{\partial/\partial z_i}(d\omega_\mu) = (t_{\partial/\partial z_i} \alpha_\mu) \omega_\mu$. In particular, if $k = 1$ then $\alpha_\mu|_{\mathcal{F}}$ is closed and $d\alpha_\mu|_{\mathcal{F}} = 0$.

(iii) In the overlapping $V_{\mu\nu}$ we have $\omega_\mu = f_{\mu\nu} \omega_\nu$, with $f_{\mu\nu} \in \mathcal{O}^*(V_{\mu\nu})$ and the cocycle $\{f_{\mu\nu}\}_{\mu,\nu \in \Lambda}$ determines the line bundle $\det(N\mathcal{F})$. Hence

$$d(\alpha_\mu - \alpha_\nu - \frac{df_{\mu\nu}}{f_{\mu\nu}}) \wedge \omega_\mu = 0.$$  

(3.1)
This shows that $\alpha_\mu - \alpha_\nu - \frac{df_{\mu\nu}}{f_{\mu\nu}}$ is a $C^\infty$ local section of the conormal bundle $N^*\mathcal{F}$ of the regular foliation $\mathcal{F}|_{X\setminus\text{Sing}(\mathcal{F})}$. Since the sheaf of smooth sections of $N^*\mathcal{F}$ is acyclic, we have that there exist $C^\infty$ 1-forms $\gamma_\mu$ in $V_\mu$ satisfying: $\gamma_\mu$ is a local section of $N^*\mathcal{F}$ and

$$\alpha_\mu - \alpha_\nu - \frac{df_{\mu\nu}}{f_{\mu\nu}} = \gamma_\mu - \gamma_\nu,$$

so that

$$\alpha_\mu - \gamma_\mu = \alpha_\nu - \gamma_\nu + \frac{df_{\mu\nu}}{f_{\mu\nu}}.$$

Call $\beta_\mu = \alpha_\mu - \gamma_\mu$, hence

$$(3.2) \quad \beta_\mu = \beta_\nu + \frac{df_{\mu\nu}}{f_{\mu\nu}}, \quad d\beta_\mu = d\beta_\nu \text{ in } V_\mu, \quad d\omega_\mu = \beta_\mu \wedge \omega_\mu \text{ and } d\beta_\mu \wedge \omega_\mu = 0.$$

By the second equality in (3.2), the 2-forms $\{d\beta_\mu\}$ piece together and we have a global $C^\infty$ 2-form on $X \setminus \text{Sing}(\mathcal{F})$ which we denote by $d\beta$.

We shall briefly digress on the geometric meaning of this smooth 2-form $d\beta$ (see [7] 6.2.4): the first equality in (3.2) tells us that the 1-forms $\{\beta_\mu\}$ behave as connection matrices of $\det(N\mathcal{F})$, in $V_\mu$, for some connection. In this case it’s natural to consider the basic connections (in the sense of Bott, see [10]).

Fix a $C^\infty$ decomposition

$$TX|_{X\setminus\text{Sing}(\mathcal{F})} = N\mathcal{F} \oplus T\mathcal{F},$$

where $N\mathcal{F}$ and $T\mathcal{F}$ are the normal and tangent bundles, respectively, of the regular foliation $\mathcal{F}|_{X\setminus\text{Sing}(\mathcal{F})}$.

Let $V_\mu$ be the domain of a local trivialization of $N\mathcal{F}$ and $\{v_1^\mu, \ldots, v_{n-k}^\mu\}$ be a local frame for $N\mathcal{F}|_{V_\mu}$ such that $\omega_\mu(v_1^\mu, \ldots, v_{n-k}^\mu) \equiv 1$. For a suitable basic connection $\nabla$ and $\zeta$ any section of $T\mathcal{F}|_{V_\mu}$, we have that

$$\beta_\mu(\zeta) = -\text{tr}(\theta^\mu)(\zeta)$$

if, and only if, $d\omega_\mu = \beta_\mu \wedge \omega_\mu$, where $\theta^\mu$ is the connection matrix in $V_\mu$ of $\nabla$ relative to the frame $\{v_1^\mu, \ldots, v_{n-k}^\mu\}$. In particular, the 1-forms $\{\beta_\mu\}$ piece together to give a well defined global form $\beta$ on $X \setminus \text{Sing}(\mathcal{F})$. It follows that $d\beta = -\text{tr}(K\nabla) = -c_1(K\nabla)$ where $K\nabla = \{K^\mu\}_{\mu \in \Lambda}$ is the curvature form of $\nabla$ and the class $d\beta = -c_1(N\mathcal{F}) = c_1(\det N^*\mathcal{F})$.

**Definition 3.1.** Let $\mathcal{F}$ be a singular foliation of dimension $k \geq 1$, as above, and consider

$$\text{Sing}(\mathcal{F}) = \bigcup_{\lambda} Z_\lambda$$

a decomposition of its singular locus into connected components. For $p \geq k$, suppose $T$ is a positive closed current of bidimension $(p, p)$ which is invariant by $\mathcal{F}$. The residue of $\mathcal{F}$ relative to $T$ along $Z_\lambda$ is

$$\text{Res}(\mathcal{F}, T, Z_\lambda) = \frac{T(d(\varphi_{\lambda}^{-1} \beta)_{n-k+1} \wedge \chi_{Z_\lambda} v_{Z_\lambda})}{\text{vol}(Z_\lambda)} \cdot [Z_\lambda],$$

where $\chi_{Z_\lambda}$ denotes the characteristic function, $v_{Z_\lambda}$ is a volume element of $Z_\lambda$ and $\varphi_{\lambda} : X \to \mathbb{R}$ is a $C^\infty$ function satisfying $\varphi_{\lambda}^{-1}(0) = Z_\lambda$, $0 < \varphi_{\lambda} \leq 1$ in $X \setminus Z_\lambda$ and $\varphi_{\lambda} = 1$ in $X \setminus U_\lambda$. 


Now, we are able to prove Theorem 1.1.

**Theorem.** Let $\mathcal{F}$ be a holomorphic foliation of dimension $k$ on a complex compact manifold $X$ with $\dim(\text{Sing}(\mathcal{F})) \leq k - 1$. Write $\text{Sing}(\mathcal{F}) = \bigcup_{\lambda \in L} Z_\lambda$, a decomposition into connected components and let $U_\lambda$ be a regular neighborhood of $Z_\lambda$. For $p \geq k$, if $T$ is a positive closed current of bidimension $(p,p)$ invariant by $\mathcal{F}$ then,

$$c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T] = \sum_{Z_\lambda \subset \text{Supp}(T) \cap \text{Sing}(\mathcal{F})} \text{Res}(\mathcal{F}, T, Z_\lambda).$$

**Proof.** In order to show geometrically that $c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T]$ localizes at $\text{Supp}(T) \cap \text{Sing}(\mathcal{F})$ we will use the concept of regular neighborhood.

An open set $U_\lambda$, $Z_\lambda \subset U_\lambda \subset X$, is a regular neighborhood of $Z_\lambda$ provided $\overline{U}_\lambda$ is a (real) $C^0$ manifold of dimension $n$ with boundary $\partial U_\lambda$. Regular neighborhoods can be obtained in the following way: take a Whitney stratification $S$ of $Z_\lambda$ and let $W_\lambda$ be any open set containing $Z_\lambda$. By the proof of Proposition 7.1 of [21], we can construct a family of tubular neighborhoods $\{T_{S,\rho_S}\}$, with $[T_{S,\rho_S}] \subset W_\lambda$ ($\rho_S$ is the tubular function), of the strata $S$ of $S$, satisfying the commutation relations which give control data for $S$: if $S$ and $S'$ are strata with $S < S'$ then

$$\pi_S \pi_{S'}(p) = \pi_S(p)$$
$$\rho_S \pi_{S'}(p) = \rho_S(p).$$

This allows for the construction of $U_\lambda$ as a subset of $W_\lambda$ and, by shrinking $W_\lambda$, we may assume $U_\lambda \cap U_{\check{\lambda}} = \emptyset$ for $\lambda \neq \check{\lambda}$. We call $\{U_\lambda\}_{\lambda \in L}$ a system of regular neighborhoods of $Z$.

Let $\{U_\lambda\}_{\lambda \in L}$ be a system of regular neighborhoods of $\text{Sing}(\mathcal{F})$. Since

$$d(\varphi_\lambda \beta) = d\varphi_\lambda \wedge \beta + \varphi_\lambda d\beta = \varphi_\lambda d\beta$$

outside $U_\lambda$, $d\beta|_\mathcal{F} = 0$ in $X \setminus \text{Sing}(\mathcal{F})$ and $T$ is $\mathcal{F}$-invariant, we have

$$(T \wedge \chi_{z_\lambda} u_{z_\lambda}) \left( d(\varphi_\lambda \beta)^{p-k+1} \right) = 0$$

in $X \setminus U_\lambda$. By reducing the tubular function of $U_\lambda$ we conclude that

$$\text{Supp} \left( T \wedge \chi_{z_\lambda} u_{z_\lambda} \right) \left( d(\varphi_\lambda \beta)^{p-k+1} \right) \subseteq Z_\lambda$$

which gives

$$T \left( d(\varphi_\lambda \beta)^{p-k+1} \wedge \chi_{z_\lambda} u_{z_\lambda} \right) = \mu_{z_\lambda} [Z_\lambda] (u_{z_\lambda})$$

and

$$\mu_{z_\lambda} = \text{Res}(\mathcal{F}, T, Z_\lambda).$$

Since $d(\varphi_\lambda \beta)$ represents $c_1(\det N^*\mathcal{F})$, we get

$$c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T] = \sum_{Z_\lambda \subset \text{Supp}(T) \cap \text{Sing}(\mathcal{F})} \text{Res}(\mathcal{F}, T, Z_\lambda).$$

□
3.1. **Proof of Corollaries 1.2 and 1.3** It is enough to prove the Corollary 1.2. The result is a straightforward consequence of Theorem 1.1. In fact, suppose by contradiction that $T$ is a closed positive current of bidimension $(p, p)$ invariant by $\mathcal{F}$ and that $\text{Supp}(T) \cap \text{Sing}(\mathcal{F}) = \emptyset$. Then, it follows from Theorem 1.1 that

$$c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T] = 0.$$ 

Since $h^{n-p,n-p}(X) = 1$ and $\det(N^*\mathcal{F})$ is ample, then $[T] = b \cdot c_1^{n-p}(\det(N^*\mathcal{F})) \in H^{n-p,n-p}(X)$, for some $b > 0$. Therefore, we have

$$c_1^{p-k+1}(\det(N^*\mathcal{F})) \cdot [T] = (-1)^{p-k+1}b \cdot c_1^{n-k+1}(\det(N^*\mathcal{F})) \neq 0.$$ 

This is a contradiction.

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