Estimating drift parameters in a non-ergodic Gaussian Vasicek-type model

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Abstract

We study the problem of parameter estimation for a non-ergodic Gaussian Vasicek-type model defined as $dX_t = (\mu + \theta X_t) dt + dG_t$, $t \geq 0$ with unknown parameters $\theta > 0$ and $\mu \in \mathbb{R}$, where $G$ is a Gaussian process. We provide least square-type estimators $\tilde{\theta}_T$ and $\tilde{\mu}_T$ respectively for the drift parameters $\theta$ and $\mu$ based on continuous-time observations $\{X_t, t \in [0, T]\}$ as $T \to \infty$. Our aim is to derive some sufficient conditions on the driving Gaussian process $G$ in order to ensure that $\tilde{\theta}_T$ and $\tilde{\mu}_T$ are strongly consistent, the limit distribution of $\tilde{\theta}_T$ is a Cauchy-type distribution and $\tilde{\mu}_T$ is asymptotically normal. We apply our result to fractional Vasicek, subfractional Vasicek and bifractional Vasicek processes. In addition, this work extends the result of \cite{13} studied in the case where $\mu = 0$.

Key words: Gaussian Vasicek-type model; Parameter estimation; Strong consistency; Asymptotic behavior in distribution; Fractional Gaussian processes; Young integral.

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1 Introduction

Let $G$ be a centered Gaussian process satisfying the following hypothesis

$(A_1)$ There exist constants $c > 0$ and $\gamma \in (0, 1)$ such that, for every $s, t \geq 0$,

$$E \left[ (G_t - G_s)^2 \right] \leq c |t - s|^{2\gamma}. $$

Note that, if $(A_1)$ holds, then by the Kolmogorov-Centsov theorem, we can conclude that for all $\varepsilon \in (0, \gamma)$, the process $G$ admits a modification with $(\gamma - \varepsilon)$-Hölder continuous paths, still denoted $G$ in the sequel.

In the present paper, our goal is to investigate least squares-type estimators for the drift parameters of the Gaussian Vasicek-type process $X = \{X_t, t \geq 0\}$ that is defined as the unique (pathwise) solution to

$$\begin{cases}
  dX_t = \theta (\mu + X_t) dt + dG_t, & t \geq 0, \\
  X_0 = 0,
\end{cases}$$

(1.1)
where $\theta > 0$ and $\mu \in \mathbb{R}$ are considered as unknown parameters.

In recent years, the study of various problems related to the model (1.1) has attracted interest. In finance modeling $\mu$ can be interpreted as the long-run equilibrium value of $X$ whereas $\theta$ represents the speed of reversion. For a motivation in mathematical finance and further references, we refer the reader to [7, 8, 9, 10]. When $G$ is a standard Brownian motion, the model (1.1) with $\mu = 0$ was originally proposed by Ornstein and Uhlenbeck and then it was generalized by Vasicek, see [26].

An example of interesting problem related to (1.5) is the statistical estimation of $\mu$ and $\theta$ when one observes the whole trajectory of $X$. In order to estimate the unknown parameters $\mu$ and $\theta$ when the whole trajectory of $X$ defined in (1.1) is observed, we will consider the following least squares-type estimators (LSEs)

\[
\tilde{\theta}_T = \frac{\frac{1}{2}TX^2_T - X_T T \int_0^T X_s ds}{T \int_0^T X^2_s ds - \left( \int_0^T X_s ds \right)^2} \quad (1.2)
\]

and

\[
\tilde{\mu}_T = \frac{\int_0^T X^2_s ds - \frac{1}{2}X_T T \int_0^T X_s ds}{\frac{1}{2}TX_T - \int_0^T X_s ds} \quad (1.3)
\]

as statistics to estimate $\theta$ and $\mu$ respectively.

Our motivation for considering these estimators $\tilde{\theta}_T$ and $\tilde{\mu}_T$ comes from the fact that $\tilde{\theta}_T$ and $\tilde{\mu}_T$ are extensions of the classical least squares estimators $\hat{\theta}_T$ and $\hat{\mu}_T$ respectively, defined, in the case when $\gamma \in \left( \frac{1}{2}, 1 \right)$, by

\[
\hat{\theta}_T = \frac{T \int_0^T X_s dX_s - X_T T \int_0^T X_s ds}{T \int_0^T X^2_s ds - \left( \int_0^T X_s ds \right)^2}
\]

and

\[
\hat{\mu}_T = \frac{X_T T \int_0^T X^2_s ds - \int_0^T X_s dX_s \int_0^T X_s ds}{T \int_0^T X_s dX_s - X_T T \int_0^T X_s ds},
\]

which are obtained by minimizing (formally) the function

\[
(\theta, \mu) \mapsto \int_0^T \left| \dot{X}_s - \theta (\mu + X_s) \right|^2 ds.
\]

More precisely, $\hat{\theta}_T$ and $\hat{\mu}_T$ are the solutions of the system

\[
\begin{cases}
\hat{\mu}_T \hat{\theta}_T \int_0^T X_s ds + \hat{\theta}_T \int_0^T X^2_s ds = \int_0^T X_s dX_s, \\
\hat{\mu}_T \hat{\theta}_T T + \hat{\theta}_T \int_0^T X_s ds = X_T,
\end{cases} \quad (1.4)
\]

where the stochastic integral $\int_0^T X_s dX_s$ is interpreted as a pathwise (Young) integral (see Appendix).
The estimators $\tilde{\theta}_T$ and $\tilde{\mu}_T$ are extensions of $\hat{\theta}_T$ and $\hat{\mu}_T$ respectively, because for any $\gamma \in (\frac{1}{2}, 1)$, we have $\tilde{\theta}_T = \theta_T$ and $\tilde{\mu}_T = \mu_T$. Indeed, using $\gamma \in (\frac{1}{2}, 1)$, (2.8) and Lemma 2.1, we can conclude that $X$ has Hölder continuous paths of order in $(\frac{1}{2}, 1)$. Therefore, the integral $\int_0^T X_s dX_s$ is well defined in the Young sense. Moreover, thanks to (4.44), we can write $\int_0^T X_s dX_s = X_T^2$, which implies the desired result.

Now, notice that the estimators (1.2) and (1.3) exist for all $\gamma \in (0, 1)$, and not only for $\gamma \in (\frac{1}{2}, 1)$. This then allows us to consider (1.2) and (1.3) as estimators to estimate respectively the drift coefficients $\theta$ and $\mu$ of the equation (1.1) for all $\gamma \in (0, 1)$.

We apply our approach to some Vasicek Gaussian processes as follows:

**Fractional Vasicek process:**
Suppose that the process $G$ given in (1.1) is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. When $H \in (\frac{1}{2}, 1)$, the parameter estimation for $\theta$ and $\mu$ has been studied in [27] by using the LSEs $\hat{\theta}_t$ and $\hat{\mu}_t$ which coincide respectively with $\tilde{\theta}_t$ and $\tilde{\mu}_t$ for $H \in (0, \frac{1}{2})$. Here we present a study valid for all $H \in (0, 1)$ (see Section 3.1).

**Subfractional Vasicek process:**
Assume that the process $G$ given in (1.1) is a subfractional Brownian motion with parameter $H \in (0, 1)$. For $H > \frac{1}{2}$, using the LSEs $\hat{\theta}_t$ and $\hat{\mu}_t$ which also coincide respectively with $\tilde{\theta}_t$ and $\tilde{\mu}_t$, the statistical estimation for $\theta$ and $\mu$ has been discussed in [28]. Here, we extend the result of [28] to the case $H \in (0, 1)$ (see Section 3.2).

**Bifractional Vasicek process:**
To the best of our knowledge there is no study of the problem of estimating the drift of (1.1) in the case when $G$ is a bifractional Brownian motion with parameters $(H, K) \in (0, 1)^2$. Section 3.3 is devoted to this question.

Recently, the paper [2] considered the least square-type estimators (1.2) and (1.3) as estimators for the drift parameters $\theta$ and $\mu$ for the so-called mean-reverting Ornstein-Uhlenbeck process of the second kind $\{X_t, t \geq 0\}$ defined as $dX_t = \theta(\mu + X_t)dt + dB^H_t, \ t \geq 0$, where $Y^{(1)}_{t,G} := \int_0^t e^{-s}dG_a$ with $a_t = He^\frac{t}{\gamma}$, and $\{G_t, t \geq 0\}$ is a Gaussian process.

Mention also that similar drift statistical problems for other Vasicek models were recently studied. Let us describe what is known about these parameter estimation problems. Let $B^H := \{B^H_t, t \geq 0\}$ denote a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. Consider the following fractional Vasicek-type model driven by $B^H$:

$$ dX_t = \theta (\mu + X_t) dt + dB^H_t, \ X_0 = 0, $$

where $\theta, \mu \in \mathbb{R}$ are unknown parameters. The process $X := \{X_t, t \geq 0\}$ given by (1.5) is called ergodic if $\theta < 0$ and non-ergodic if $\theta > 0$.

Recently, several researchers have been interested in studying statistical estimation problems for (1.5). Let us mention some works in this direction: in the case when $\theta < 0$, the
statistical estimation for the parameters \( \mu, \theta \) based on continuous-time observations of \( \{X_t, \ t \in [0, T]\} \) as \( T \to \infty \), has been studied by several papers, for instance \cite{11, 5, 24, 27} and the references therein. When \( \mu = 0 \) in (1.5), the estimation of \( \theta \) has been investigated by using least squares method as follows: the case of ergodic fractional Ornstein-Uhlenbeck processes, corresponding to \( \theta < 0 \), has been considered in \cite{18, 14, 19}, and the case non-ergodic fractional Ornstein-Uhlenbeck processes has been studied in \cite{13, 15}.

On the other hand, using Malliavin-calculus advances (see \cite{23}), the work \cite{17} provided new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and its result has been used to study drift parameter estimation problems for some stochastic differential equations driven by fractional Brownian motion with fixed-time-step observations, in particular for the fractional Ornstein-Uhlenbeck given in (1.5), where \( \mu = 0 \) and \( \theta < 0 \). Similarly, in \cite{12} the authors studied an estimator problem for the parameter \( \theta \) in (1.5), where the fractional Brownian motion is replaced with a general Gaussian process. More recently, using similar arguments as in \cite{16, 6}, the author of \cite{30} considered the problem of drift parameter estimation for (1.1) when \( G \) is a self-similar Gaussian process with index \( L \in (\frac{1}{2}, 1) \). In this work we provide a general technique that also allows to extend the results of \cite{30} for a general self-similarity index \( L \in (0, 1) \).

The rest of the paper is structured as follows. In Section 2 we first analyze some pathwise properties of the Vasicek process (1.1). Then we derive some sufficient conditions on the driving Gaussian process \( G \) ensuring the asymptotic consistency and the asymptotic distribution of \( \tilde{\theta}_T \) and \( \tilde{\mu}_T \). Section 3 is devoted to apply our approach to fractional Vasicek, subfractional Vasicek and bifractional Vasicek processes.

### 2 Asymptotic behavior of the least squares-type estimator

In this section we first study pathwise properties of the non-ergodic Vasicek-type model (1.1). These properties will be needed in order to analyze the asymptotic behavior of the LSEs \( \tilde{\theta}_T \) and \( \tilde{\mu}_T \). Because (1.1) is linear, it is immediate to solve it explicitly; one then gets the following formula

\[
X_t = \mu (e^{\theta t} - 1) + e^{\theta t} \int_0^t e^{-\theta s} dG_s, \quad t \geq 0,
\]  

(2.6)

the integral with respect to \( G \) being a Young integral.

Let us introduce the following processes, for every \( t \geq 0 \),

\[
\zeta_t := \int_0^t e^{-\theta s} dG_s; \quad \Sigma_t := \int_0^t X_s ds.
\]  

(2.7)

Thus, using (2.6), we can write

\[
X_t = \mu (e^{\theta t} - 1) + e^{\theta t} \zeta_t.
\]  

(2.8)
Furthermore, by (1.1),

\[ X_t = \mu \theta t + \theta \Sigma_t + G_t. \]

(2.9)

Moreover, applying the formula (4.44), we have

\[ \zeta_t = e^{-\theta t} G_t + \theta \int_0^t e^{-\theta s} G_s ds. \]

(2.10)

Here we will discuss some pathwise properties of \( \zeta \) and \( X \).

**Lemma 2.1** (13). Assume that (\( A_1 \)) holds with \( \gamma \in (0, 1) \). Let \( \zeta \) be given by (2.7). Then for all \( \varepsilon \in (0, \gamma) \) the process \( \zeta \) admits a modification with \((\gamma - \varepsilon)\)-Hölder continuous paths, still denoted \( \zeta \) in the sequel.

Moreover,

\[ \zeta_T \to \zeta_\infty := \theta \int_0^\infty e^{-\theta s} G_s ds \]

(2.11)

almost surely and in \( L^2(\Omega) \) as \( T \to \infty \).

**Lemma 2.2.** Assume that (\( A_1 \)) holds with \( \gamma \in (0, 1) \). Then, almost surely, as \( T \to \infty \),

\[ e^{-\theta T} X_T \to \mu + \zeta_\infty, \]

(2.12)

\[ e^{-2\theta T} \int_0^T X_s^2 ds \to \frac{1}{2\theta} (\mu + \zeta_\infty)^2, \]

(2.13)

\[ \frac{e^{-\theta T}}{T^\alpha} \int_0^T X_s ds \to 0 \quad \text{for any } \alpha > 0, \]

(2.14)

with \( \zeta_\infty \) defined in Lemma 2.1.

**Proof.** Notice first that the convergence (2.12) is a direct consequence of (2.8) and (2.11).

On the other hand, from [13] we have

\[ \lim_{T \to \infty} \int_0^T e^{2\theta s} \zeta_s^2 ds = \infty \] almost surely.

Combined with (2.8) we deduce

\[ \lim_{T \to \infty} \int_0^T X_s^2 ds = \infty \] almost surely.

Hence, using l’Hôpital’s rule, (2.8) and (2.11), we can conclude

\[ \lim_{T \to \infty} \frac{\int_0^T X_s^2 ds}{e^{2\theta T}} = \lim_{T \to \infty} \frac{X_T^2}{2\theta e^{2\theta T}} = \lim_{T \to \infty} \frac{(\mu (1 - e^{-\theta T}) + \zeta_T)^2}{2\theta} = \frac{1}{2\theta} (\mu + \zeta_\infty)^2 \] almost surely,
which proves (2.13).

Similarly, using (see [13])

$$\lim_{T \to \infty} \int_0^T e^{\theta s} |\zeta_s| ds = \infty$$

almost surely, we can also conclude (2.14).

We will now analyze the asymptotic behavior of the least squares-type estimators \( \tilde{\theta}_T \) and \( \tilde{\mu}_T \) when \( T \to \infty \).

### 2.1 Strong consistency

The following theorem proves the strong consistency of the estimators \( \tilde{\theta}_T \) and \( \tilde{\mu}_T \).

**Theorem 2.1.** Assume that \((A_1)\) holds and let \( \tilde{\theta}_T \) and \( \tilde{\mu}_T \) be given by (1.2) and (1.3) for every \( T \geq 0 \). Then

$$\tilde{\theta}_T \rightarrow \theta, \quad (2.15)$$

and

$$\tilde{\mu}_T \rightarrow \mu \quad (2.16)$$

almost surely, as \( T \to \infty \).

**Proof.** Using (1.2) we get

$$\tilde{\theta}_T = \frac{1}{2} \left( e^{-\theta T} X_T^2 \right) \left( e^{-\theta T} X_T - e^{-\theta T} \frac{\int_0^T X_s ds}{\sqrt{T}} \right)$$

$$\rightarrow \theta, \quad \text{almost surely, as } T \to \infty,$$

where the last convergence comes from the convergences (2.12), (2.13) and (2.14). Thus the convergence (2.15) is obtained.

Let us now prove (2.16). It follows from (1.3) that \( \tilde{\theta}_T \) can be written as follows

$$\tilde{\mu}_T = \frac{e^{-\theta T}}{T} \left[ \frac{\int_0^T X_s^2 ds - X_T}{2} \int_0^T X_s ds \right] \times \frac{1}{\frac{1}{2} e^{-\theta T} X_T - e^{-\theta T} \frac{\int_0^T X_s ds}{\sqrt{T}}}.$$

According to the convergences (2.12) and (2.14) we have, almost surely, as \( T \to \infty \),

$$\frac{1}{\frac{1}{2} e^{-\theta T} X_T - e^{-\theta T} \frac{\int_0^T X_s ds}{\sqrt{T}}} \rightarrow \frac{2}{\mu + \zeta_\infty}.$$
Therefore, it remains to prove
\[
e^{-\theta T} \left[ \int_0^T X_s^2 ds - \frac{X_T}{2} \int_0^T X_s ds \right] \rightarrow \frac{\mu}{2} (\mu + \zeta_\infty) \tag{2.17}
\]
almost surely, as \( T \to \infty \).

Using the formula (4.44) and the equation (1.1), we have
\[
\int_0^T X_s^2 ds - \frac{X_T}{2} \int_0^T X_s ds = \int_0^T X_s d\Sigma_s - \frac{1}{2} (\mu \theta + \theta \Sigma_T + G_T) \Sigma_T
\]
\[
= \int_0^T (\mu s + \theta \Sigma_s + G_s) d\Sigma_s - \frac{\mu \theta}{2} T \Sigma_T - \frac{\theta}{2} \Sigma_T^2 - \frac{1}{2} G_T \Sigma_T
\]
\[
= \mu \theta \int_0^T s X_s ds + \frac{\theta}{2} \Sigma_T^2 + \int_0^T G_s d\Sigma_s - \frac{\mu \theta}{2} T \Sigma_T - \frac{\theta}{2} \Sigma_T^2 - \frac{1}{2} G_T \Sigma_T
\]
\[
= \left( \mu \theta \int_0^T s X_s ds - \frac{\mu \theta}{2} T \Sigma_T \right) + \left( \int_0^T G_s d\Sigma_s - \frac{1}{2} G_T \Sigma_T \right)
\]
\[
=: I_T + J_T.
\]
Moreover, by L’Hôpital’s rule and (2.12) we have
\[
e^{-\theta T} I_T = e^{-\theta T} \left( \mu \theta \int_0^T s X_s ds - \frac{\mu \theta}{2} T \Sigma_T \right)
\]
\[
\rightarrow \frac{\mu}{2} (\mu + \zeta_\infty)
\]
almost surely, as \( T \to \infty \).

On the other hand, for any \( \gamma < \delta < 1 \), we have almost surely, as \( T \to \infty \),
\[
\frac{G_T}{T^\delta} \rightarrow 0, \tag{2.18}
\]
by \((A_1)\), \( G \) is Gaussian and Borel-Cantelli lemma.

Hence, taking \( \gamma < \delta < 1 \),
\[
e^{-\theta T} |J_T| = e^{-\theta T} \left| \int_0^T G_s d\Sigma_s - \frac{1}{2} G_T \Sigma_T \right|
\]
\[
= e^{-\theta T} \left| \int_0^T G_s X_s ds - \frac{1}{2} G_T \Sigma_T \right|
\]
\[
\leq C e^{-\theta T} \int_0^T |X_s| ds + \frac{1}{2} e^{-\theta T} |G_T \Sigma_T|
\]
\[
\rightarrow 0
\]
almost surely, as \( T \to \infty \), where we used (2.18) and (2.13).

Consequently, the convergence (2.17) is proved. Thus the proof of the theorem is done. \( \square \)
2.2 Asymptotic distribution

In this section the following assumptions are required:

\((A_2)\) There exist \(\lambda_G > 0\) and \(\eta \in (0, 1)\) such that, as \(T \to \infty\)
\[
\frac{E(G_T^2)}{T^{2\eta}} \to \lambda_G^2.
\]

\((A_3)\) The limiting variance of \(e^{-\theta T} \int_0^T e^{\theta s} dG_s\) exists as \(T \to \infty\) i.e., there exists a constant \(\sigma_G > 0\) such that
\[
\lim_{T \to \infty} E \left( \left( e^{-\theta T} \int_0^T e^{\theta s} dG_s \right)^2 \right) \to \sigma_G^2.
\]

\((A_4)\) For all fixed \(s \geq 0\)
\[
\lim_{T \to \infty} E \left( G_s e^{-\theta T} \int_0^T e^{\theta r} dG_r \right) = 0.
\]

In order to investigate the asymptotic behavior in distribution of the estimators \(\tilde{\theta}_T\) and \(\tilde{\mu}_T\), as \(T \to \infty\), we need the following lemmas.

**Lemma 2.3.** Assume that \((A_1)\) holds and let \(X\) be the process given by (1.1). Then we have for every \(T \geq 0\),
\[
\frac{1}{2} X_T^2 - \frac{X_T^T}{T} \int_0^T X_t dt = \theta \left( \int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \right) + (\mu + \theta Z_T) \int_0^T e^{\theta t} dG_t + R_T,
\]
where \(Z_T := \int_0^T e^{-\theta s} G_s ds\), and the process \(R_T\) is defined by
\[
R_T := \frac{1}{2} (\mu T)^2 + \frac{1}{2} (G_T)^2 - \mu G_T - \frac{\theta^2}{2} \int_0^T X_t dt - \theta \int_0^T (G_t)^2 dt + \theta^2 \int_0^T e^{-\theta t} G_t \int_0^t e^{\theta s} G_s ds dt.
\]
Moreover, as \(T \to \infty\),
\[
Z_T \to \frac{\zeta_\infty}{\theta},
\]
\[
e^{-\theta T} R_T \to 0
\]
almost surely.
Proof. In order to prove (2.19) we first need to introduce the following processes, for every \( t \geq 0 \),

\[
A_t := \mu \left( e^{\theta t} - 1 \right).
\]

Thus, by (2.6) and (4.44) we have

\[
X_t = \mu \left( e^{\theta t} - 1 \right) + e^{\theta t} \int_0^t e^{-\theta s} dG_s = A_t + G_t + \theta e^{\theta t} Z_t. \tag{2.22}
\]

On the other hand, using (2.9), we have

\[
\frac{1}{2} X_T^2 = \frac{1}{2} (\mu T)^2 + G_T^2 + \mu T G_T + \mu \theta^2 T \Sigma_T + \frac{\theta^2}{2} \Sigma_T + \theta \Sigma_T G_T, \tag{2.23}
\]

where, according to (4.44) and (2.9),

\[
\frac{\theta^2}{2} \Sigma_T = \theta^2 \int_0^T \Sigma_t dt
\]

\[
= \theta^2 \int_0^T \Sigma_t X_t dt
\]

\[
= \theta \int_0^T X_t^2 dt - \mu \theta^2 \int_0^T t d\Sigma_t - \theta \int_0^T G_t X_t dt
\]

\[
= \theta \int_0^T X_t^2 dt - \mu \theta^2 T \Sigma_T + \mu \theta^2 \int_0^T \Sigma_t dt - \theta \int_0^T G_t X_t dt. \tag{2.24}
\]

Thus, by (2.22) and (4.44),

\[
-\theta \int_0^T G_t X_t dt = -\theta \int_0^T G_t A_t dt - \theta \int_0^T (G_t)^2 dt - \theta^2 \int_0^T G_t e^{\theta t} Z_t dt
\]

\[
= -\theta \int_0^T G_t A_t dt - \theta \int_0^T G_t^2 dt - \theta^2 \int_0^T G_t e^{\theta t} Z_t dt
\]

\[
= -\theta \int_0^T G_t A_t dt - \theta \int_0^T G_t^2 dt - \theta^2 \left( Z_T G_T - \int_0^T G_t dZ_t \right)
\]

\[
= -\theta \int_0^T G_t A_t dt - \theta \int_0^T G_t^2 dt - \theta^2 Z_T G_T
\]

\[
+ \theta^2 \int_0^T e^{-\theta t} G_t \int_0^t e^{\theta s} G_s dtdt. \tag{2.25}
\]

Also, by (2.9) and (2.22),

\[
\theta \Sigma_T G_T = G_T \left( X_T - \mu T - G_T \right)
\]

\[
= G_T \left( -\mu T + A_T + \theta e^{\theta T} Z_T \right). \tag{2.26}
\]
Combining (2.23), (2.24), (2.25) and (2.23), we can conclude that
\[
\frac{1}{2} X_T^2 = \frac{1}{2} (\mu T)^2 + \frac{1}{2} G_T^2 + \mu \theta T \Sigma_T + \theta \int_0^T X_t^2 dt - \mu \theta T \Sigma_T + \mu \theta^2 \int_0^T \Sigma_t dt \\
-\theta \int_0^T G_t A_t dt - \theta \int_0^T (G_t)^2 dt - \theta^2 Z_T G_T + \theta^2 \int_0^T e^{-\theta t} G_t \int_0^t e^{\theta s} G_s ds dt + G_T (A_T + \theta e^{\theta T} Z_T).
\] (2.27)

Using (2.9), we have
\[
-\theta^2 Z_T G_T + \theta e^{\theta T} Z_T G_T = -\theta Z_T (\theta G_T - e^{\theta T} G_T) \\
= -\theta Z_T \int_0^T e^{\theta t} dG_t,
\] (2.28)

and
\[
G_T A_T - \theta \int_0^T G_t A_t dt = -\mu G_T + \mu \int_0^T G_t dt + \mu e^{\theta T} G_T - \mu \int_0^T G_t e^{\theta t} dt \\
= -\mu G_T + \mu \int_0^T G_t dt + \mu \int_0^T e^{\theta t} dG_t.
\] (2.29)

Now, combining (2.27), (2.28) and (2.29), we obtain
\[
\frac{1}{2} X_T^2 = \frac{1}{2} (\mu T)^2 + \frac{1}{2} G_T^2 + \theta \int_0^T X_t^2 dt + \mu \theta^2 \int_0^T \Sigma_t dt \\
-\theta \int_0^T G_t^2 dt + \theta^2 \int_0^T e^{-\theta t} G_t \int_0^t e^{\theta s} G_s ds dt \\
-\theta \int_0^T e^{\theta t} dG_t - \mu G_T + \mu \int_0^T G_t dt + \mu \int_0^T e^{\theta t} dG_t.
\] (2.30)

On the other hand, using (2.9),
\[
\frac{-X_T}{T} \int_0^T X_t dt = -\mu \theta \Sigma_T - \frac{\theta}{T} \Sigma_T^2 - \frac{G_T}{T} \Sigma_T.
\] (2.31)

Combining (2.30), (2.31) and the fact that
\[
\mu \theta \int_0^T \Sigma_t dt - \mu \theta \Sigma_T + \mu \theta \int_0^T G_t dt = -\frac{(\mu T)^2}{2},
\]
we get therefore (2.19).

For (2.20), by (4.44), we have
\[
Z_T = \frac{1}{\theta} (\zeta_T - e^{\theta T} G_T).
\]
Thus, using (2.18) and (2.11) we obtain (2.20).

Finally, the convergence (2.21) is a direct consequence of (2.18) and (2.14).
Lemma 2.4 ([13]). Assume that \((A_1), (A_3)\) and \((A_4)\) hold. Let \(F\) be any \(\sigma\{G_t, t \geq 0\}\)-measurable random variable such that
\[
P(F < \infty) = 1.
\]
Then, as \(T \to \infty\)
\[
\left( F, e^{-\theta T} \int_0^T e^{\theta t} dG_t \right) \xrightarrow{\text{Law}} (F, \sigma_G N), \tag{2.32}
\]
where \(N \sim N(0,1)\) is independent of \(G\).

Recall that if \(X \sim N(m_1, \sigma_1)\) and \(Y \sim N(m_2, \sigma_2)\) are two independent random variables, then \(X/Y\) follows a Cauchy-type distribution. For a motivation and further references, we refer the reader to [25], as well as [21].

Theorem 2.2. Assume that \((A_1), (A_2), (A_3)\) and \((A_4)\) hold. Let \(N \sim N(0,1)\) independent of \(G\). Then, as \(T \to \infty\)
\[
e^{\theta T} (\bar{\theta}_T - \theta) \xrightarrow{\text{Law}} \frac{2\theta \sigma_G N}{\mu + \zeta_{\infty}}, \tag{2.33}
\]
and
\[
T^{1-\eta} (\bar{\mu}_T - \mu) \xrightarrow{\text{Law}} N \left( 0, \frac{\lambda^2_G}{\theta^2} \right). \tag{2.34}
\]

Proof. First we prove (2.33). From (1.2) and (2.19) we can write
\[
e^{\theta T} (\bar{\theta}_T - \theta) = (\mu + \theta Z_T) e^{-\theta T} \int_0^T e^{\theta t} dG_t + e^{-\theta T} R_T e^{-2\theta T} \left( \int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \right) \times \frac{(\mu + \zeta_{\infty}) (\mu + \theta Z_T)}{e^{-2\theta T} \left( \int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \right)}
\]
\[
+ e^{-\theta T} R_T \left( \mu + \zeta_{\infty} \right) \times \frac{e^{-\theta T} \int_0^T e^{\theta t} dG_t}{\mu + \zeta_{\infty}} \times \frac{(\mu + \zeta_{\infty}) (\mu + \theta Z_T)}{e^{-2\theta T} \left( \int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \right)}
\]
\[
:= a_T \times b_T + c_T.
\]
Lemma 2.4 yields, as \(T \to \infty\),
\[
a_T \xrightarrow{\text{Law}} \frac{\sigma_H N}{\mu + \zeta_{\infty}}, \tag{2.33}
\]
where \(N \sim N(0,1)\) is independent of \(G\), whereas (2.13), (2.14) and (2.20) imply that \(b_T \to 2\theta\) almost surely as \(T \to \infty\). On the other hand, by (2.13), (2.14) and (2.21), we
obtain that $c_T \rightarrow 0$ almost surely as $T \rightarrow \infty$.

By putting all these facts together, we get that, as $T \rightarrow \infty$,

$$e^{\theta T} (\bar{\theta}_T - \theta) \xrightarrow{\text{law}} \frac{2\theta \sigma_H N}{\mu + \zeta_\infty}.$$  

Moreover, $(\mu + \zeta_\infty) \sim \mathcal{N}(\mu, E(\zeta_\infty^2))$ and independent of $N$ which completes the proof of (2.33).

Let us now do the proof of (2.34). Using (1.2) and (1.3), a straightforward calculation shows that $\bar{\theta}_T$ and $\bar{\mu}_T$ verify

$$\bar{\theta}_T \bar{\mu}_T = \bar{\theta}_T \bar{\mu}_T X_T T = X_T - \bar{\theta}_T \int_0^T X_t dt.$$  

Combining this with (1.1), we have

$$T^{1-\eta} (\bar{\mu}_T - \mu) = \frac{1}{\bar{\theta}_T} \left[ -e^{\theta T} (\bar{\theta}_T - \theta) e^{-\theta T} \int_0^T X_t dt - \mu T^{1-\eta} e^{\theta T} (\bar{\theta}_T - \theta) + G_T \frac{T}{\eta} \right].$$  

Now, using (2.15), (2.14), (A2) and Slutsky’s theorem, we deduce (2.34).

3 Applications to Gaussian Vasicek processes

This section is devoted to some examples of Gaussian Vasicek processes. We will discuss the following three cases of the driving Gaussian process $G$ of (1.1): fractional Brownian motion, subfractional Brownian motion and bifractional Brownian motion. We will need the following technical lemma.

**Lemma 3.1.** For every $H \in (0, 1)$, we have

$$a_H := \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} t^{2H} ds dt = \frac{\Gamma(2H + 1)}{\theta^{2H+2}},$$  

$$b_H := \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} |t - s|^{2H} ds dt = \frac{\Gamma(2H + 1)}{\theta^{2H+2}},$$  

$$d_H := \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} |t + s|^{2H} ds dt = \frac{\Gamma(2H + 2)}{\theta^{2H+2}}.$$  

**Proof.** We prove (3.35),

$$a_H = \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} t^{2H} ds dt = \frac{1}{\bar{\theta}} \int_0^\infty e^{-\theta t} t^{2H} dt = \frac{\Gamma(2H + 1)}{\theta^{2H+2}}.$$  

For (3.36), we have

\[
\begin{align*}
b_H &= \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} |t - s|^{2H} ds dt \\
&= 2 \int_0^\infty \int_0^t e^{-\theta t} e^{-\theta s} (t - s)^{2H} ds dt \\
&= 2 \int_0^\infty e^{-2\theta t} \int_0^t e^{\theta u} u^{2H} du dt \\
&= 2 \int_0^\infty e^{\theta u} u^{2H} \int_u^\infty e^{-2\theta t} dt du \\
&= \frac{1}{\theta} \int_0^\infty e^{-\theta u} u^{2H} du \\
&= \frac{\Gamma(2H + 1)}{\theta^{2H+2}}.
\end{align*}
\]

Finally, for (3.37), we have

\[
\begin{align*}
d_H &= \int_0^\infty \int_0^\infty e^{-\theta t} e^{-\theta s} (t + s)^{2H} ds dt \\
&= \int_0^\infty \int_t^\infty e^{\theta u} u^{2H} du dt \\
&= \int_0^\infty e^{-\theta u} u^{2H+1} du \\
&= \frac{\Gamma(2H + 2)}{\theta^{2H+2}}.
\end{align*}
\]

\[
\square
\]

### 3.1 Fractional Vasicek process

The fractional Brownian motion (fBm) \( B^H := \{ B^H_t, t \geq 0 \} \) with Hurst parameter \( H \in (0, 1) \), is defined as a centered Gaussian process starting from zero with covariance

\[
E \left( B^H_t B^H_s \right) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
\]

Note that, when \( H = \frac{1}{2} \), \( B^{\frac{1}{2}} \) is a standard Brownian motion.

We have

\[
E \left[ (B^H_t - B^H_s)^2 \right] = |s - t|^{2H}; \quad s, t \geq 0.
\]

Let us first start with the following simulated path of the fractional Vasicek process, i.e., when \( G = B^H \) in (1.1),

\[
X_0 = 0; \quad dX_t = \theta (\mu + X_t) dt + dB^H_t.
\]
First, we generate the fractional Brownian motion using the wavelet method (see [1]).

After that we simulate the process (3.39) using the Euler-Maruyama method for different values of $H$, $\theta$ and $\mu$ (see Figure 1).

We simulate a sample path on the interval $[0, 1]$ using a regular partition of 10,000 intervals.

![Sample paths for different values of $H$, $\theta$, and $\mu$.](image)

Figure 1: The sample path of Fractional Vasicek process.

Let us now discuss the asymptotic behavior of $\tilde{\theta}_T$ and $\tilde{\mu}_T$. Thanks to (3.38), the process $B^H$ satisfies the assumptions ($A_1$) and ($A_2$) for $G = B^H$ and $\gamma = \eta = H$. Moreover, by [13, Proposition 3.1.], the assumptions ($A_3$) and ($A_4$) hold, with

$$\sigma_{BH}^2 = \frac{H\Gamma(2H)}{\theta^{2H}}; \quad \lambda_{BH} = 1.$$  

(3.40)
Furthermore, by Lemma 3.1 we have for every \( H \in (0, 1) \), the variance of \( \zeta_{B^H,\infty} := \theta \int_0^\infty e^{-\theta s} B^H_s ds \), given in (2.11) when \( G = B^H \), is equal to
\[
E(\zeta_{B^H,\infty}^2) = \theta^2 \int_0^\infty \int_0^\infty e^{-\theta s} e^{-\theta t} E(B^H_s B^H_t) ds dt
\]
\[
= \theta^2 \left( a_H - \frac{1}{2} b_H \right)
\]
\[
= \frac{H \Gamma(2H)}{\theta^{2H}}
\]
which is equal in this case to \( \sigma_{B^H}^2 \) given in (3.40).

Thus, we obtain the following result.

**Proposition 3.1.** Assume that \( H \in (0, 1) \) and the process \( G \), given in (1.1), is a fBm \( B^H \).
Let \( N \sim N(0, 1) \) independent of \( B^H \). Then, almost surely, as \( T \to \infty \),
\[
\left( \tilde{\theta}_T, \tilde{\mu}_T \right) \to (\theta, \mu).
\]

In addition, as \( T \to \infty \),
\[
e^{\theta T} (\tilde{\theta}_T - \theta) \xrightarrow{\text{Law}} 2\theta \sigma_{B^H} N \frac{N}{\mu + \zeta_{B^H,\infty}},
\]
and
\[
T^{1-H} (\tilde{\mu}_T - \mu) \xrightarrow{\text{Law}} N \left( 0, \frac{1}{\theta^2} \right),
\]
where \( \sigma_{B^H} \) is defined in (3.40), and \( \zeta_{B^H,\infty} \sim N(0, \sigma_{B^H}^2) \).

### 3.2 Subfractional Vasicek process

The subfractional Brownian motion (subfBm) \( S^H := \{ S^H_t, t \geq 0 \} \) with parameter \( H \in (0, 1) \) is a centered Gaussian process with covariance function
\[
E \left( S^H_t S^H_s \right) = t^{2H} + s^{2H} - \frac{1}{2} \left( (t + s)^{2H} + |t - s|^{2H} \right); \quad s, t \geq 0.
\]
In particular, for every \( T \geq 0 \),
\[
\frac{E[(S^H_T)^2]}{T^{2H}} = 2 - 2^{2H-1}.
\]
Note that, when \( H = \frac{1}{2} \), \( S^{\frac{1}{2}} \) is a standard Brownian motion. Moreover, it is known that
\[
E \left[ (S^H_t - S^H_s)^2 \right] \leq (2 - 2^{2H-1}) |s - t|^{2H}; \quad s, t \geq 0.
\]
So, the process $S^H$ satisfies the assumptions $(A_1)$ and $(A_2)$ for $G = S^H$ and $\gamma = \eta = H$. On the other hand, from [33, Proposition 3.2], the assumptions $(A_3)$ and $(A_4)$ hold, with
\[
\sigma_{S^H}^2 = \frac{H\Gamma(2H)}{\theta^{2H}}; \quad \lambda_{S^H}^2 = 2 - 2^{2H-1}. \tag{3.41}
\]
Furthermore, by Lemma 3.1, we have for every $H \in (0, 1)$, the variance of $\zeta_{S^H,\infty} := \theta \int_0^\infty e^{-\theta s} S^H_s ds$, given in (2.11) when $G = S^H$, is equal to
\[
E(\zeta_{S^H,\infty}^2) = \theta^2 \int_0^\infty \int_0^\infty e^{-\theta s} e^{-\theta t} E(S^H_s S^H_t)ds \tag{3.42}
\]
We therefore obtain the following result.

**Proposition 3.2.** Assume that $H \in (0, 1)$ and the process $G$, given in (1.1), is a subfBm $S^H$. Let $N \sim \mathcal{N}(0, 1)$ independent of $S^H$. Then, almost surely, as $T \to \infty$,
\[
(\tilde{\theta}_T, \tilde{\mu}_T) \longrightarrow (\theta, \mu).
\]
In addition, as $T \to \infty$,
\[
e^{\theta T}(\tilde{\theta}_T - \theta) \xrightarrow{Law} 2\theta \sigma_{S^H} \frac{N}{\mu + \zeta_{S^H,\infty}},
\]
and
\[
T^{1-H}(\tilde{\mu}_T - \mu) \xrightarrow{Law} \mathcal{N}\left(0, \frac{2 - 2^{2H-1}}{\theta^2}\right),
\]
where $\sigma_{S^H}$ is defined in (3.41), and $\zeta_{S^H,\infty} \sim \mathcal{N}(0, E(\zeta_{S^H,\infty}^2))$ with $E(\zeta_{S^H,\infty}^2)$ is given in (3.42).

### 3.3 Bifractional Vasicek process

Let $B^{H,K}_t := \{B^{H,K}_t, t \geq 0\}$ be a bifractional Brownian motion (bifBm) with parameters $H \in (0, 1)$ and $K \in (0, 1]$. This means that $B^{H,K}$ is a centered Gaussian process with the covariance function
\[
E(B^{H,K}_s B^{H,K}_t) = \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK}\right).
\]
In particular, for every \( T \geq 0 \),
\[
\frac{E[(B_{T}^{H,K})^2]}{T^{2HK}} = 1.
\]
Note that the case \( K = 1 \) corresponds to the fBm with Hurst parameter \( H \).
In addition, the process \( B_{H,K}^{H} \) verifies,
\[
E \left( \left| B_{t}^{H,K} - B_{s}^{H,K} \right|^2 \right) \leq 2^{1-K} |t-s|^{2HK}.
\]
Hence, the process \( B^{H} \) satisfies the assumptions \((\mathcal{A}_1)\) and \((\mathcal{A}_2)\) for \( G = B^{H,K} \) and \( \gamma = \eta = HK \).
Furthermore, by [13, Proposition 3.3.], the assumptions \((\mathcal{A}_3)\) and \((\mathcal{A}_4)\) are satisfied, with
\[
\sigma_{B^{H,K}}^2 = \frac{HK \Gamma(2HK)}{\theta^{2HK}}; \quad \lambda_{B^{H,K}} = 1.
\]
(3.43)

Then, we obtain the following result.

**Proposition 3.3.** Assume that \((H, K) \in (0,1) \times (0,1)\) and the process \( G \), given in (1.1), is a bifBm \( B^{H,K} \). Let \( N \sim \mathcal{N}(0,1) \) independent of \( B^{H,K} \). Then, almost surely, as \( T \to \infty \),
\[
\left( \tilde{\theta}_T, \tilde{\mu}_T \right) \longrightarrow (\theta, \mu).
\]
In addition, as \( T \to \infty \),
\[
e^{\theta T} (\tilde{\theta}_T - \theta) \xrightarrow{Law} 2\theta \sigma_{B^{H,K}} \frac{N}{\mu + \zeta_{B^{H,K},\infty}},
\]
and
\[
T^{1-HK} (\tilde{\mu}_T - \mu) \xrightarrow{Law} \mathcal{N} \left( 0, \frac{1}{\theta^2} \right),
\]
where \( \sigma_{B^{H,K}} \) is defined in (3.43), and \( \zeta_{B^{H,K},\infty} = \theta \int_{0}^{\infty} e^{-\theta s} B_{s}^{H,K} ds \sim \mathcal{N}(0, E(\zeta_{B^{H,K},\infty}^2)) \),
with \( E(\zeta_{B^{H,K},\infty}^2) < \infty \) by Lemma 3.1.

4 Appendix: Young integral

In this section, we briefly recall some basic elements of Young integral (see [29]), which are helpful for some of the arguments we use. For any \( \alpha \in [0, 1] \), we denote by \( \mathcal{H}^\alpha([0, T]) \) the set of \( \alpha \)-Hölder continuous functions, that is, the set of functions \( f : [0, T] \to \mathbb{R} \) such that
\[
|f|_\alpha := \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.
\]
We also set $|f|_\infty = \sup_{t \in [0,T]} |f(t)|$, and we equip $H^\alpha([0,T])$ with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty.$$ 

Let $f \in H^\alpha([0,T])$, and consider the operator $T_f : C^1([0,T]) \to C^0([0,T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0,T].$$

It can be shown (see, e.g., [22, Section 3.1]) that, for any $\beta \in (1 - \alpha, 1)$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on $\alpha$, $\beta$ and $T$ such that, for any $g \in H^\beta([0,T])$,

$$\left\| \int_0^t f(u)g'(u)du \right\|_\beta \leq C_{\alpha,\beta,T} \|f\|_\alpha \|g\|_\beta.$$

We deduce that, for any $\alpha \in (0,1)$, any $f \in H^\alpha([0,T])$ and any $\beta \in (1 - \alpha, 1)$, the linear operator $T_f : C^1([0,T]) \subset H^\beta([0,T]) \to H^\beta([0,T])$, defined as $T_f(g) = \int_0^t f(u)g'(u)du$, is continuous with respect to the norm $\| \cdot \|_\beta$. By density, it extends (in an unique way) to an operator defined on $H^\beta$. As consequence, if $f \in H^\alpha([0,T])$, if $g \in H^\beta([0,T])$ and if $\alpha + \beta > 1$, then the (so-called) Young integral $\int_0^t f(u)dg(u)$ is (well) defined as being $T_f(g)$.

The Young integral obeys the following formula. Let $f \in H^\alpha([0,T])$ with $\alpha \in (0,1)$ and $g \in H^\beta([0,T])$ for all $\beta \in (0,1)$. Then $\int_0^t g_udf_u$ and $\int_0^t f_udg_u$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$,

$$f_t g_t = f_0g_0 + \int_0^t g_udf_u + \int_0^t f_udg_u. \quad (4.44)$$

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