A construction of representations for quantum groups: an example of $\mathcal{U}_q(\mathfrak{so}(5))$

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Abstract
A short description is given of a construction of representations for quantum groups. The method uses infinitesimal dressing transformation on quantum homogeneous spaces and is illustrated on an example of $\mathcal{U}_q(\mathfrak{so}(5))$.

1 Introduction
The purpose of this paper is to illustrate a construction of representations on an explicit example, namely the deformed enveloping algebra $\mathcal{U}_q(\mathfrak{so}(5))$. We are going to describe the construction as well, however, its detailed presentation will appear elsewhere. The basic ingredient is the infinitesimal dressing transformation on a quantum homogeneous space, in analogy with the celebrated method of orbits due to Kirillov and Kostant.

The construction generalizes and simplifies some results derived in the papers [1, 2, 3, 4, 5] and also [6, 7]. Let us mention just a few additional papers dealing also with constructions of representations of quantum groups and/or with quantum homogeneous spaces [8, 9, 10, 11], but taking a different point of view or applying other methods.

Concerning the deformation parameter, we assume that $q > 0$, $q \neq 1$. All fractional powers of $q$ are supposed to be positive.
2 Construction

We assume that we are given a bialgebra \( U \) with the counit denoted by \( \varepsilon \) and the comultiplication denoted by \( \Delta \), and a unital algebra \( C \). Moreover, \( C \) is supposed to be a left \( U \)-module with the action denoted by \( \xi \), and fulfilling two conditions:

\[
\begin{align*}
\xi_x \cdot 1 &= \varepsilon(x) 1, \quad \forall x \in U, \\
\xi_x \cdot (fg) &= (\xi_{x(1)} \cdot f)(\xi_{x(2)} \cdot g), \quad \forall x \in U, \ \forall f, g \in C.
\end{align*}
\]

If convenient we shall write \( \xi(x) \cdot f \) instead of \( \xi_x \cdot f \). The second condition (2) is nothing but Leibniz rule. Here and everywhere in what follows we use Sweedler’s notation: \( \Delta x = x(1) \otimes x(2) \).

**Proposition 1** Suppose that a linear mapping \( \varphi : U \to C \) satisfies \( \varphi(1) = 1 \) and

\[
\varphi(xy) = (\xi_{x(1)} \cdot \varphi(y))\varphi(x(2)), \quad \forall x, y \in U.
\]

Then the prescription

\[
\begin{align*}
x \cdot f := (\xi_{x(1)} \cdot f) \varphi(x(2)), \quad \forall x \in U, \ \forall f \in C,
\end{align*}
\]

defines a left \( U \)-module structure on \( C \) and it holds

\[
\begin{align*}
x \cdot (fg) &= (\xi_{x(1)} \cdot f)(x(2) \cdot g), \quad \forall x \in U, \ \forall f, g \in C.
\end{align*}
\]

Particularly,

\[
\varphi(x) = x \cdot 1, \quad \forall x \in U.
\]

Conversely, suppose that \( U \otimes C \to C : x \otimes f \mapsto x \cdot f \) is a left \( U \)-module structure on \( C \) such that the rule (5) is satisfied. Then the linear mapping \( \varphi : U \to C \) defined by the equality (6) fulfills (3), and consequently the prescription (4) holds true.

Let us suppose, as usual, that \( U \) is generated as an algebra by a set of generators \( \mathcal{M} \subset U \). Let \( F \) be the free algebra generated by \( \mathcal{M} \). Thus \( U \) is identified with a quotient \( F/\langle R \rangle \) where \( \langle R \rangle \) is the ideal generated by a set of defining relations \( R \subset F \). Let \( \pi \) be the factor morphism, \( \pi : F \to U \). We set \( \tilde{\varepsilon} := \varepsilon \circ \pi \) and

\[
\tilde{\xi}_x \cdot f := \xi_{\pi(x)} \cdot f, \quad \forall x \in F, \ \forall f \in C.
\]

In addition we impose the following condition on the set of generators \( \mathcal{M} \subset U \):

\[
\Delta(\mathcal{M}) \subset \text{span}_C(\mathcal{M}_1 \otimes \mathcal{M}_1) \quad \text{where} \quad \mathcal{M}_1 := \mathcal{M} \cup \{1\}.
\]
Then it is natural to define a comultiplication \( \tilde{\Delta} \) on \( \mathcal{F} \) by the equality
\[
\tilde{\Delta}(x_1 \ldots x_n) := \Delta(x_1) \ldots \Delta(x_n), \quad x_i \in \mathcal{M}.
\]
As \( \mathcal{U} \) is a bialgebra \( \langle \mathcal{R} \rangle \) must be, at the same time, a coideal.

It is not difficult to check that \( \mathcal{F} \) becomes this way a bialgebra and that the triple \( (\mathcal{F}, \tilde{\xi}, \mathcal{C}) \) fulfills the original conditions (1) and (2), just replacing \( \mathcal{U} \) with \( \mathcal{F} \) and \( \xi \) with \( \tilde{\xi} \). One finds that to any mapping \( \varphi : \mathcal{M} \to \mathcal{C} \) there exists a unique linear extension \( \tilde{\varphi} : \mathcal{F} \to \mathcal{C} \) such that \( \tilde{\varphi}(1) = 1 \) and the property
\[
\tilde{\varphi}(xy) = (\tilde{\xi}_{x(1)} \cdot \tilde{\varphi}(y))\tilde{\varphi}(x(2)),
\]
is satisfied for all \( x, y \in \mathcal{F} \).

The final step in the construction is to decide when the mapping \( \tilde{\varphi} \) can be factorized from \( \mathcal{F} \) to \( \mathcal{U} = \mathcal{F}/\langle \mathcal{R} \rangle \).

**Proposition 2** Suppose that there is given a mapping \( \varphi : \mathcal{M} \to \mathcal{C} \) and let \( \tilde{\varphi} \) be its extension to \( \mathcal{F} \) as described above. If
\[
(\pi \otimes \tilde{\varphi}) \circ \tilde{\Delta}(\mathcal{R}) = 0
\]
then \( \tilde{\varphi}(\langle \mathcal{R} \rangle) = 0 \) and so there exists a unique linear mapping \( \varphi' : \mathcal{U} \to \mathcal{C} \) such that \( \tilde{\varphi} = \varphi' \circ \pi \). Moreover, \( \varphi' = 1 \) and \( \varphi' \) satisfies the condition (3).

The same conclusions hold true provided \( \mathcal{R} \) fulfills a stronger condition than that of being a coideal, namely
\[
\tilde{\Delta}(\mathcal{R}) \subset \langle \mathcal{R} \rangle \otimes \mathcal{F} + \mathcal{F} \otimes \mathcal{F} \mathcal{R},
\]
and \( \tilde{\varphi} \) satisfies a weaker condition
\[
\tilde{\varphi}(\mathcal{R}) = 0.
\]

Particularly this construction goes through for the standard deformed enveloping algebras \( \mathcal{U} = \mathcal{U}_q(\mathfrak{g}) \) in the FRT description \[12\] where \( \mathfrak{g} \) is any simple complex Lie algebra from the four principal series \( A_\ell, B_\ell, C_\ell \) and \( D_\ell \). So the generators are arranged in respectively upper and lower triangular matrices \( L^+ \) and \( L^- \), and the set \( \mathcal{R} \) is given by the usual RLL relations.

On the other hand the unital algebra \( \mathcal{C} \) is generated by quantum anti-holomorphic coordinate functions \( z^*_{jk}, \ j < k \), on the generic dressing orbit of dimension \( (\dim_{\mathbb{C}} \mathfrak{g} - \text{rank} \mathfrak{g})/2 \). The elements are arranged in an upper triangular matrix \( Z \) with units on the diagonal, and the defining relations are given in terms of its Hermitian adjoint \( Z^* \), namely
\[
R_{12}Z^*_2QQ_1Q^{-1} = Z^*_1QQZ^*_2Q^{-1}R_{12}
\]
where \( Q \) is the diagonal part of the R-matrix \( R \).
The infinitesimal dressing transformation $\xi$ is prescribed on the generators,
\[ \xi(L_1^+ \cdot Z_r^s) = R_{21}^{-1} Z_2 Q, \quad \xi(L_1^- \cdot Z_2^s) = Z_1^q Z_2^q (Z_1^q)^{-1}. \] (14)
It can be extended to an arbitrary element from $C$ with the aid of Leibniz rule (2). The mapping $\varphi$ is defined on the generators as well,
\[ \varphi(L^+) = D^{-1}, \quad \varphi(L^-) = Z^* D^2 (Z^*)^{-1} D^{-1} \] (15)
where $D$ is an arbitrary complex diagonal matrix obeying the conditions
\[ \det(D) = 1 \quad \text{and} \quad K_{12} D_1 D_2 = K_{12}. \] (16)
Here $K$ is a matrix related to the R-matrix via the equality
\[ R_{12} - R_{21}^{-1} = (q - q^{-1})(P - K_{12}), \] (17)
P stands for the flip operator.

3 Example: $U_q(\mathfrak{so}(5))$

We shall use the Drinfeld–Jimbo description of $U_q(\mathfrak{so}(5))$ [13, 14], with the six generators $q^{H_1}$, $q^{H_2}$, $X_1^+$, $X_2^+$, $X_1^-$, $X_2^-$, the relations
\[
\begin{align*}
[q^{H_1}, q^{H_2}] &= 0, \\
q^{H_1} X_1^\pm &= q^{\mp 1} q^{H_1} X_1^\pm, \quad q^{H_1} X_2^\pm = q^{\mp 1} q^{H_1} X_2^\pm, \\
q^{H_2} X_1^\pm &= q^{\mp 1} q^{H_2} X_1^\pm, \quad q^{H_2} X_2^\pm = q^{\mp 2} q^{H_2} X_2^\pm, \\
[X_1^+, X_1^-] &= \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}, \quad [X_2^+, X_2^-] = \frac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \\
[X_1^+, X_2^-] &= 0, \quad [X_1^-, X_2^+] = 0, \\
(X_2^\pm)^2 X_1^\pm - (q^{-1} + q) X_2^\pm X_1^\pm X_2^\pm + X_1^\pm (X_2^\pm)^2 &= 0, \\
(X_1^\pm)^3 X_2^\pm - (q^{-1} + 1 + q) (X_1^\pm)^2 X_2^\pm X_1^\pm + (q^{-1} + 1 + q) X_1^\pm X_2^\pm (X_1^\pm)^2 - X_2^\pm (X_1^\pm)^3 &= 0.
\end{align*}
\] (18)
and the comultiplication
\[ \Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}, \quad \Delta(X_i^\pm) = X_i^\pm \otimes q^{-\frac{1}{2} H_i} + q^\frac{1}{2} H_i \otimes X_i^\pm, \quad i = 1, 2. \] (19)
One can pass from the FRT description to the Drinfeld–Jimbo generators using the equalities
\[
\begin{align*}
L_{11}^+ &= L_{55}^- = q^{H_1 + H_2}, & L_{22}^+ &= L_{44}^- = q^{H_1}, \\
L_{12}^+ &= (q - q^{-1}) q^{-1/2} X_2^- q^{\frac{1}{2} H_1 + \frac{1}{2} H_2}, & L_{23}^- &= (q - q^{-1}) q^{-1/2} X_1^+ q^{\frac{1}{2} H_2}, \\
L_{34}^- &= (q - q^{-1}) q^{1/2} X_1^+ q^{\frac{1}{2} H_1}, & L_{45}^- &= (q - q^{-1}) q^{1/2} X_2^+ q^{H_1 + \frac{1}{2} H_2}.
\end{align*}
\] (20)
Only 4 among the 10 generators \( z_{jk}^* \), \( 1 \leq j < k \leq 5 \), are independent. We denote the independent generators by \( w_1, \ldots, w_4 \) and make the following choice:
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \sigma \\
\end{align*}
\]

The remaining entries can be expressed in terms of \( w_1, \ldots, w_4 \) as well,
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \\
\end{align*}
\]

The algebra \( \mathcal{C} \) is then determined by the relations
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \\
\end{align*}
\]

Consequently, the ordered monomials \( w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} \), \( n_1, n_2, n_3, n_4 \in \mathbb{Z}_+ \), form an algebraic basis of \( \mathcal{C} \).

The infinitesimal dressing transformation is prescribed on the generators as follows:
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \\
\end{align*}
\]

Let us turn to the mapping \( \varphi \). The constraints (16) imply that
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \\
\end{align*}
\]

where \( \sigma_1, \sigma_2 \in \mathbb{C} \) are parameters. A straightforward calculation gives
\[
\begin{align*}
\mathcal{C} & \quad \text{choice:} \quad w \\
\end{align*}
\]
where \( [m]_p := (p^m - p^{-m})/(p - p^{-1}) \).

The final step is to calculate the modified action according to the prescription (4). Here is the result:

\[
q^{H_1} \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = q^{-n_1+n_3+n_4-\frac{1}{2}\sigma_1} w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
q^{H_2} \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = q^{2n_1+n_2-n_4-\sigma_2} w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
X^-_1 \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
- q^{\frac{1}{2}(-n_1+n_2-n_3-n_4)+\frac{1}{4}\sigma_1} [n_2]_{q^{1/2}} w_1^{n_1+1} w_2^{n_2-1} w_3^{n_3} w_4^{n_4} \\
+ q^{\frac{1}{2}(-n_1+n_2-n_3-n_4)+\frac{1}{4}\sigma_1} [n_3]_{q^1} w_1^{n_1} w_2^{n_2+1} w_3^{n_3-1} w_4^{n_4} \\
- q^{\frac{1}{2}(-n_1+n_3)+\frac{1}{4}\sigma_1} [n_4]_{q^{1/2}} w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4-1}, \\
X^-_2 \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
- q^{\frac{1}{2}(1-n_2+n_4+\sigma_2)} [n_1]_{q^1} w_1^{n_1-1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
X^+_1 \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
- q^{-1+\frac{1}{2}(n_1-n_3-n_4)+\frac{1}{4}\sigma_1} [n_1]_{q} w_1^{n_1+1} w_2^{n_2+1} w_3^{n_3} w_4^{n_4} \\
+ q^{-1+\frac{1}{2}(n_1-n_2-n_3-n_4)+\frac{1}{4}\sigma_1} [n_2]_{q^{1/2}} w_1^{n_1} w_2^{n_2-1} w_3^{n_3+1} w_4^{n_4} \\
+ q^{\frac{1}{2}(1-n_1+n_3)-\frac{1}{4}\sigma_1} \\
\frac{1}{1+q} [n_4 - \sigma_1]_{q^{1/2}} w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4+1}, \\
X^+_2 \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
q^{-\frac{1}{2}(1-n_2+n_4+\sigma_2)} [n_1 + 1-n_2 - n_4 - \sigma_2]_{q} w_1^{n_1+1} w_2^{n_2} w_3^{n_3} w_4^{n_4} \\
- q^{-1+n_1+\frac{1}{2}n_2+n_3-\frac{1}{2}(n_4+\sigma_2)} \\
[1+q] [n_3]_{q^1} w_1^{n_1} w_2^{n_2+2} w_3^{n_3-1} w_4^{n_4} \\
+ q^{1-n_1+\frac{1}{2}n_2+n_3-n_4-\frac{1}{2}\sigma_2} [n_4]_{q^{1/2}} w_1^{n_1} w_2^{n_2+1} w_3^{n_3} w_4^{n_4-1} \\
- (q - 1) q^{-\frac{1}{2}+n_2+n_4} w_1^{n_1} w_2^{n_2} w_3^{n_3-1} w_4^{n_4-2}.
\]

Note that \( 1 \in \mathcal{C} \) is a lowest weight vector (\( X^-_1 \cdot 1 = X^-_2 \cdot 1 = 0 \)), with the lowest weight determined by \( q^{H_1} \cdot 1 = q^{-\frac{1}{2}\sigma_1} \), \( q^{H_2} \cdot 1 = q^{-\sigma_2} \). Consequently, the cyclic submodule \( \mathcal{U} \cdot 1 \) is finite-dimensional and irreducible provided \( \sigma_1, \sigma_2 \in \mathbb{Z}_+ \), and this way one can obtain, in principle, all finite-dimensional irreducible representations of \( \mathcal{U}_q(\mathfrak{so}(5)) \). For example, if \( \sigma_1 = 1, \sigma_2 = 0 \), then \( \mathcal{U} \cdot 1 \) is a 4-dimensional vector space spanned by the vectors: \( 1, w_4, w_2 - q w_1 w_4, (1 + q) w_3 + q^{3/2} w_2 w_4 \).

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