NONABELIAN GLOBAL CHIRAL SYMMETRY REALISATION IN THE TWO-DIMENSIONAL $N$ FLAVOUR MASSLESS SCHWINGER MODEL

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The nonabelian global chiral symmetries of the two-dimensional $N$ flavour massless Schwinger model are realised through bosonisation and a vertex operator construction.

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1. Introduction

Recently,\textsuperscript{1} the quantum two-dimensional $N$ flavour massless Schwinger model has been revisited without any gauge fixing but using the method of Dirac quantisation. The physical spectrum of this ideal “theoretical laboratory” for nonperturbative quantum field theory is known, and consists of one massive pseudoscalar field which is essentially the electric field with squared mass $(N/\pi)e^2$, $e$ being the U(1) gauge coupling constant, and $(N-1)$ massless scalar fields, none of which are interacting. At the quantum level, this model has SU($N$)$_-$×SU($N$)$_+$×U(1)$_V$ chiral symmetries, where the factor SU($N$)$_-$×SU($N$)$_+$ mixes separately each the chiral Dirac fermionic field components of given chirality, while the factor U(1)$_V$ is a common phase symmetry associated to the total fermionic number. The associated SU($N$)$_\pm$ and U(1)$_V$ Noether currents are bosonised. Using implicitly techniques from two-dimensional conformal field theory and string theory developed in the 1990’s,\textsuperscript{2} one may construct vertex operators in direct relationship with these global chiral symmetries. From the modes of both these bosonised Noether currents and these vertex operators, we realise two commuting affine Kac–Moody algebras, of which the zero modes of the vertex operators are shown to correspond to the generators of the nonabelian global chiral symmetries.

This paper is organised as follows. In Sec. 2, we introduce the two-dimensional $N$ flavour massless Schwinger model. In Sec. 3, we identify the chiral symmetries of the model and specify our notations by also defining the Hilbert space in which we work. In Sec. 4, we construct the relevant vertex operators. Section 5 is devoted to the Kac–Moody algebra. In Sec. 6, we realise the nonabelian global symmetries. Concluding remarks appear in Sec. 7.
2. The Two-Dimensional $N$ Flavour Massless Schwinger Model

2.1. The classical formulation

Let us consider the two-dimensional $N$ flavour massless Schwinger model with a dynamics described by the following Lagrangian density,

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \sum_{j=1}^{N} \bar{\psi}^j \gamma^\mu \partial_\mu \psi^j - \frac{i}{2} \sum_{j=1}^{N} \partial_\mu \bar{\psi}^j \gamma^\mu \psi^j - e \sum_{j=1}^{N} \bar{\psi}^j \gamma^\mu A^\mu \psi^j,
$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As usual, the spacetime coordinate indices take the values $\mu = (0, 1)$, while the Minkowski spacetime metric signature is $\text{diag} \eta_{\mu\nu} = (+, -)$. We also assume a system of units such that $c = 1 = \hbar$. This dynamics is singular, and, reducing the second-class constraints through the introduction of Dirac brackets, the fundamental first-class Hamiltonian reads as

$$
\mathcal{H} = \frac{1}{2} \pi_1^2 - \frac{i}{2} \sum_{j=1}^{N} \bar{\psi}^j \gamma_5 (\partial_1 - i e A^1) \psi^j + \frac{i}{2} \sum_{j=1}^{N} (\partial_1 + i e A^1) \psi^j \gamma_5 \psi^j - \partial_1 [A^0 \pi_1],
$$

(2)

where the first-class constraint, related to the U(1) local gauge invariance of the system, is simply

$$
\sigma = \partial_1 \pi_1 + e \sum_{j=1}^{N} \bar{\psi}^j \psi^j.
$$

(3)

2.2. Quantum formulation

Through canonical quantisation and within the Schrödinger picture, bosonisation of the Dirac fermionic operators is achieved as

$$
\hat{\psi}^j_\pm (z) = e^{i\pi \lambda \sum_{k=1}^{N} (\alpha^j_+ k \hat{p}_k^+ + \beta^j_+ k \hat{p}_k^-)} e^{\pm i\lambda \hat{\phi}^j_\pm (z)},
$$

(4)

where the first factor on the r.h.s of this expression represents the Klein factor necessary in order to have fermionic operators of different flavours or chiralities that anticommute with one another, while the quantities $\hat{\phi}^j_\pm (z)$ are real chiral bosons. Applying the point splitting regularisation procedure
and some field redefinitions, the fundamental quantum Hamiltonian is given as

$$\hat{H} = \frac{1}{2} \hat{\pi}_\varphi^2 + \frac{1}{2} \mu^2 \hat{\varphi}^2 + \frac{1}{2} (\hat{\sigma}/\mu)^2 + \frac{1}{2} \frac{(\partial_1 \hat{\varphi})^2}{\hat{\varphi}} + \frac{1}{4\pi} \sum_{j=1}^{N-1} (\hat{\partial}_1 \hat{\Phi}_j^0)^2 + \frac{1}{4\pi} \sum_{j=1}^{N-1} (\hat{\partial}_1 \hat{\Phi}_j^1)^2,$$

(5)

where $\hat{\varphi}$ is essentially (up to a normalisation factor) the electric field with squared mass $\mu^2 = (N/\pi)e^2$. Here, $\hat{\Phi}_j^\pm$ are massless chiral bosons defined by

$$\hat{\Phi}_j^\pm = \frac{1}{\sqrt{U+1}} \left( \sum_{k=1}^j \phi_k^\pm - j \phi_{j+1}^\pm \right), \quad j \in \{1, \ldots, (N-1)\}. \quad (6)$$

3. Chiral Symmetries

3.1. SU(N)$_\pm$ currents

The SU(N)$_\pm$ currents associated to the chiral symmetries are defined by

$$\hat{J}_{\pm}^{\mu} = \lambda L \sqrt{2} \sum_{i,j=1}^N \hat{\psi}_i^\dagger \hat{\psi}_j \left( \mp \lambda L \sqrt{2} \lambda^a \right)_{ij} \hat{\phi}_\pm^a, \quad \mu = 0, 1, \quad (7)$$

where the matrices $\lambda^a$, $a \in \{1, \ldots, (N^2-1)\}$, are $(N^2-1)$ independent hermitian traceless matrices spanning the SU(N) algebra. These matrices are a generalisation of the Pauli matrices in the SU(2) case or the Gell-Mann matrices in the SU(3) case. In particular, the matrices associated to the Cartan subalgebra U(1)$^{N-1}$ of the Lie algebra $su(N)$ may be chosen to be given by

$$\left( \lambda^i \right)_{ij} = \frac{1}{\sqrt{2i(i+1)}} \left( \sum_{k=1}^i \delta_{jk} \delta_{lk} - i \delta_{j,i+1} \delta_{l,i+1} \right), \quad i \in \{1, \ldots, (N-1)\}, \quad (8)$$

leading to the following associated currents,

$$\hat{J}_{0}^{\pm} = (\mp \lambda L \sqrt{2}) \sum_{i,j=1}^N \hat{\psi}_i^\dagger \hat{\psi}_j \left( \lambda^i \right)_{ij} \hat{\phi}_\pm^i, \quad (9)$$

$$\hat{J}_{1}^{\pm} = (\mp \lambda L \sqrt{2}) \sum_{i,j=1}^N \hat{\psi}_i^\dagger \gamma_5 \left( \lambda^i \right)_{ij} \hat{\phi}_\pm^i. \quad (10)$$
Through the bosonisation procedure of the fermionic operators, one finds,

\[ \hat{J}^{i0}_{\pm} = \left( \pm \frac{L}{2\pi} \right) \partial_1 \hat{\Phi}_i^\pm, \quad \hat{J}^{i1}_{\pm} = \left( -\frac{L}{2\pi} \right) \partial_1 \hat{\Phi}_i^\pm. \]  

(11)

Let us consider the U(1)\(^{N-1}\) currents given by

\[ \hat{J}^{i\pm}_i(x) = \left( \pm \frac{L}{2\pi} \right) \partial_1 \hat{\Phi}_i^\pm(x), \quad i \in \{1, \ldots, (N-1)\}. \]  

(12)

In terms of modes, these currents may be written as

\[ \hat{J}^{i\pm}_i(z) = \hat{P}^{i\pm}_i + \sum_{n \geq 1} \left( \hat{J}^{i\pm}_{i,n} z^n + \hat{\bar{J}}^{i\pm}_{i,n} z^{-n} \right), \]  

(13)

where the modes are function of the modes of the \((N-1)\) real bosonic fields \(\Phi_i^\pm\), and satisfy the following algebra

\[ \left[ \hat{J}^{i\pm}_{i,n}, \hat{J}^{j\mp}_{j,m} \right] = n \delta_{ij} \delta_{mn}. \]  

(14)

### 3.2. The quantum Hilbert space

From the algebra (14), we conclude that the operators \(\hat{J}^{i\pm}_{i,n}\) and \(\hat{\bar{J}}^{i\pm}_{i,n}\), with \(i, j = \{1, \ldots, (N-1)\}\) and \(n = 1, \ldots, \infty\), form a set of independent harmonic oscillators. Therefore, the state space considered here is a Fock space built up from the simultaneously normalized vacua of all these oscillators, \(|0\rangle\),

\[ \hat{J}^{i\pm}_{i,n} |0\rangle = 0, \quad n > 0, \quad \hat{P}^{i\pm}_i |0\rangle = 0. \]  

(15)

Let us consider \(\Sigma\), the set of roots of the Lie algebra \(su(N)\), and \(\Lambda_\Sigma\) the root lattice of \(su(N)\). States \(|\lambda\rangle\) can be added to the above states by acting with the plane wave operator \(e^{i\lambda \cdot \hat{Q}^\pm}\). We denote

\[ |\lambda\rangle = e^{i\lambda \cdot \hat{Q}^\pm} |0\rangle, \]  

(16)

where \(\lambda \in \Lambda_\Sigma\). Later we shall assume that all \(\alpha \in \Sigma\) have length \(\sqrt{2}\).

### 4. Vertex Operators

Given any complex number \(z\) and any root \(\alpha\), let us introduce the vertex operator

\[ \hat{U}^{\alpha}_\pm(z) = z^{\alpha^2} \epsilon^{i \alpha \cdot \hat{\Phi}_\pm(z)} : \]  

(17)

where

\[ \alpha \cdot \hat{\Phi}_\pm(z) = \sum_{j=1}^{N-1} \alpha^j \hat{\Phi}_\pm^j(z). \]  

(18)
In order to make (17) single valued in $z$, the following condition is required,

$$\left( \frac{\alpha^2}{2} + \alpha \cdot \hat{P}_{\pm} \right) \in \mathbb{Z}. \quad (19)$$

Therefore (17) is analytic and has a Laurent expansion

$$\hat{U}^\alpha_{\pm}(z) = \sum_{m \in \mathbb{Z}} \hat{U}^\alpha_{\pm,m} z^{-m}, \quad (20)$$

where the modes can be written as

$$\hat{U}^\alpha_{\pm,m} = \oint_0 \frac{dz}{2i\pi z} z^m \hat{U}^\alpha_{\pm}(z) \quad (21)$$

and satisfy the hermiticity condition

$$\hat{U}^\alpha_{\pm,m} = \hat{U}^{\alpha}_{\mp,-m}. \quad (22)$$

5. The Kac–Moody Algebra

5.1. Almost commutation relations

Using the modes of currents and those of vertex operators, we can build almost commutation relations by

$$[\hat{J}^i_{\pm,n},\hat{J}^j_{\pm,m}] = m \delta_{ij} \delta_{m,n}, \quad [\hat{J}^i_{\pm,n}, \hat{U}^\alpha_{\pm,m}] = \alpha^i \hat{U}^\alpha_{\pm,m+n}, \quad (23)$$

$$\hat{U}^\alpha_{\pm,m} \hat{U}^\beta_{\pm,n} - (-1)^{\alpha \cdot \beta} \hat{U}^\beta_{\pm,n} \hat{U}^\alpha_{\pm,m} = \begin{cases} 0 & \text{if } \alpha \cdot \beta \geq 0, \\ \hat{U}^{\alpha+\beta}_{\pm,m+n} & \text{if } \alpha \cdot \beta = -1, \\ \alpha \cdot \hat{J}^i_{\pm,m+n} + m \delta_{m+n,0} & \text{if } \alpha \cdot \beta = -2, \end{cases} \quad (24)$$

where $\alpha \cdot \hat{J}^i_{\pm,m+n} = \sum_{j=1}^{N-1} \alpha^j \hat{J}^j_{\pm,m+n}$.

In order to have commutation relations, we have to correct the sign $(-1)^{\alpha \cdot \beta}$ which appears in (24).

5.2. Sign compensation

Let us set $V = \left( \mathcal{F}, \left\{ e^{i\lambda \hat{Q}}_{\pm} \right\}_{\lambda \in \Lambda_{\Sigma}} \right)$, where $\mathcal{F}$ denotes the Fock space. We define a sign compensation operator$^3$ by

$$C_{\pm,\alpha} = \sum_{\beta \in \Lambda_{\Sigma}} \epsilon(\alpha,\beta) |\beta\rangle \langle \beta|, \quad (25)$$
which only acts on the wave plane factor and satisfies the following conditions

\[
\epsilon(\alpha, \beta) \in \{-1, 1\}, \quad \epsilon(\alpha, \beta) = (-1)^{\alpha \beta + \alpha^2 \beta^2} \epsilon(\beta, \alpha), \quad (26)
\]

\[
\epsilon(\alpha, \beta) \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma) \epsilon(\beta, \gamma). \quad (27)
\]

Let us set

\[
\hat{E}_\pm^\alpha_{\pm, n} = \hat{U}_\pm^\alpha_{\pm, n} C_\pm^\alpha, \quad \hat{E}_\pm^\beta_{\pm, m} = \hat{U}_\pm^\beta_{\pm, m} C_\pm^\beta.
\]

(28)

We then have the following commutation relations, known as the affine Kac–Moody algebra (in fact, we obtain two such algebras, one for each of the chiral sectors of the model, which commute with one another),

\[
\left[ \hat{J}_\pm^i_{\pm, m}, \hat{J}_\pm^{i+1}_{\pm, n} \right] = m \delta_{ij} \delta_{m,n}, \quad \left[ \hat{J}_\pm^i_{\pm, m}, \hat{E}_\pm^\alpha_{\pm, n} \right] = \alpha^i \hat{E}_\pm^\alpha_{\pm, m+n}, \quad (29)
\]

\[
\left[ \hat{E}_\pm^\alpha_{\pm, m}, \hat{E}_\pm^\beta_{\pm, n} \right] = \begin{cases} 
\epsilon(\alpha, \beta) \hat{E}_\pm^{\alpha+\beta}_{\pm, m+n} & \text{if } \alpha \cdot \beta = -1, \\
\alpha \cdot \hat{J}_\pm_{\pm, m+n} + m \delta_{m,n} & \text{if } \alpha \cdot \beta = -2, \\
0 & \text{if } \alpha \cdot \beta \geq 0.
\end{cases} \quad (30)
\]

6. Nonabelian Global Chiral Symmetries

We are now ready to realise the nonabelian global chiral symmetries of the model. In fact, having defined the affine Kac–Moody algebra, it is well known from the literature\(^2\) that the following algebra is isomorphic to the Lie algebra \(su(N)\),

\[
\hat{J}_\pm^i_{\pm, 0}, \quad \hat{E}_\pm^\alpha_{\pm, 0}, \quad 1 \leq i \leq (N - 1), \quad \alpha \in \Lambda.
\]

(31)

The Cartan subalgebra is generated by \(\hat{J}_\pm^i_{\pm, 0}, i \in \{1, ..., (N - 1)\}\), while the nonabelian global symmetries are realised by \(\hat{E}_\pm^\alpha_{\pm, 0}, \alpha \in \Lambda\).

7. Concluding remarks

We have realised the nonabelian global chiral symmetries of the two-dimensional \(N\) flavour massless Schwinger model through bosonisation of the massless Dirac spinors. Our results generalise the recent work of Michael Creutz.\(^4\) An important aspect relative to the nature of the action of these nonabelian global chiral symmetries on the one-particle quantum states, and beyond, of the model, would merit further investigations.
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