Topological Edge and Interface states at Bulk disorder-to-order Quantum Critical Points

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We study the interplay between two nontrivial boundary effects: (1) the two dimensional (2d) edge states of three dimensional (3d) strongly interacting bosonic symmetry protected topological states, and (2) the boundary fluctuations of 3d bulk disorder-to-order phase transitions. We then generalize our study to 2d gapless states localized at an interface embedded in a 3d bulk, when the bulk undergoes a quantum phase transition. Our study is based on generic long wavelength descriptions of these systems and controlled analytic calculations. Our results are summarized as follows: (i.) The edge state of a prototype bosonic symmetry protected states can be driven to a new fixed point by coupling to the boundary fluctuations of a bulk quantum phase transition; (ii.) the states localized at a 2d interface of a 3d SU(N) quantum antiferromagnet may be driven to a new fixed point by coupling to the bulk quantum critical modes. Properties of the new fixed points identified are also studied.

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I. INTRODUCTION

The most prominent feature of topological insulators (TI)¹² and more generally symmetry protected topological (SPT) states,⁸,⁹ is the contrast between the boundary and the bulk of the system. In particular the 2d edge of 3d SPT states hosts the most diverse zoo of exotic phenomena that keep attracting attentions and efforts from theoretical physics. It has been shown that many exotic phenomena such as anomalous topological order,⁴⁻¹¹ deconfined quantum critical points,¹² self-dual field theories¹⁸⁻²¹ can all occur on the 2d edge of 3d SPT stats. Sometimes the symmetry of the system is secretly realized as a self-dual transformation of the field theories at the boundary²²,²³. All these suggest that the 2d boundary of a 3d system is an ideal platform of studying physics beyond the standard frameworks of condensed matter theory.

On the other hand, even the boundary of an ordinary Landau-Ginzburg type of quantum phase transition can have nontrivial behaviors. It was studied and understood in the past that the boundary of a bulk conformal field theory (CFT) follows a very different critical behavior from the bulk due to the strong boundary condition imposed on the CFT. The boundary fluctuations (or the boundary CFT) of the Landau-Ginzburg phase transitions were studied through the standard $\epsilon$-expansion, and it was shown that the critical exponents are very different from the bulk. Hence if experiments are performed at the boundary of the system, one should refer to the predictions of the boundary instead of the bulk CFT. These two different boundary effects were studied separately in the past. In this work we will study the interplay of these two distinct boundary effects. Our goal is to seek for new physics, ideally new fixed points under renormalization group (RG) flow due to the coupling of the two boundary effects.

For our purpose we give the system under study a virtual two-layer structure Fig. 1. Layer-1 is a SPT or TI with nontrivial edge states, and it is not tuned to a bulk phase transition; layer-2 is a topological trivial system which undergoes an ordinary Landau-Ginzburg disorder-to-order phase transition. Then as a starting point we assume a weak coupling between the boundary of the two layers, and study the RG flow of the coupling. Besides the edge state localized at the boundary of a SPT state, we will also consider symmetry protected gapless states localized at a 2d interface embedded in a 3d bulk. We will demonstrate that in several cases, including the edge state of a prototype bosonic SPT state, the 2d boundary or interface will flow to a new fixed point due to the bulk quantum phase transition.

Previous works have explored related ideas with different approaches. Exactly soluble 1d and 2d Hamiltonians have been constructed for gapless systems with protected
edge states\cite{32} fate of edge states was also studied for 1d and 2d SPT states\cite{33,34}. But the 2d edge of 3d bosonic SPT systems coupled with boundary modes which originate from bulk quantum critical points, i.e., the situation that potentially hosts the richest and most exotic phenomena, have not been studied to our knowledge. We note that the interaction between bulk quantum critical modes and the boundary of free or weakly interacting fermion topological insulator (or topological superconductor) was studied in Ref. \cite{32} but the coupling in that case was strongly irrelevant hence will not lead to new physics in the infrared (we will review the interplay between the bulk quantum critical modes and the edge states of free fermion topological insulator in the next section). We will focus on bosonic SPT state with intrinsic strong interaction in this work. We use the generic long wavelength field theory description of both the bulk bosonic SPT states and the edge states. Due to the lack of exact results of strongly interacting \((2+1)d\) field theories, we seek for a controlled calculation procedure that allows us to identify new fixed points under RG flow. Indeed, in several scenarios we will explore in this work, new fixed points are identified based on controlled calculations.

II. EDGE STATES OF 3d SPT AT BULK QCP

A. Edge states of noninteracting 3d TIs

We first consider the edge state of 3d topological insulator (TI) and symmetry protected topological states. The edge state of free fermion TI is described by the action

\[ S = \int d^2xd\tau \sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha \gamma_{\mu} \partial_{\mu} \psi_\alpha, \]

with \(\gamma^1 = \sigma^2, \gamma^2 = -\sigma^1, \gamma^3 = \sigma^3, \bar{\psi} = \psi^\dagger \gamma^0\). Based on the “ten-fold way classification”\cite{35,36}, for the AIII class, at the noninteracting level the TI is always nontrivial and topologically different from each other for arbitrary integer\(-N_f\); while for the AII class the TI is nontrivial only for odd integer \(N_f\), and they are all topologically equivalent to the simplest case with \(N_f = 1\). In both cases the fermion mass term \(\sum_{\alpha} \bar{\psi}_\alpha \psi_\alpha\) is forbidden by the time-reversal symmetry. Hence let us consider the disorder-to-order phase transition in the 3d bulk associated with a spontaneous time-reversal symmetry breaking, which is described by an ordinary \((3+1)d\) Landau-Ginzburg quantum Ising theory:

\[ S_b = \int d^3xd\tau (\partial \phi)^2 + u \phi^4. \]

Because \(u\) is a marginally irrelevant coupling at the \((3+1)d\) noninteracting Gaussian fixed point, the scaling dimension of \(\phi\) in the bulk is precisely \([\phi] = 1\).

Here we stress that the disorder-to-order transition is driven by the physics in the bulk. Without the bulk, the boundary alone does not support an ordered phase. To study the fate of the edge state when the bulk is tuned to the quantum critical point, we view the bulk as a “two layer” system (Fig. [1]): layer-1 is a 3d TI which is not tuned to the quantum phase transition; while layer-2 is at the disorder-to-order bulk quantum phase transition between a time-reversal invariant trivial insulator and a spontaneous time-reversal symmetry breaking phase. Now both layers have nontrivial physics at the edge. The quantum critical fluctuation (from layer-2) at the 2d boundary must satisfy the boundary scaling law. When we impose the most natural boundary condition \(\phi(z \geq 0) = 0\), the leading field at the boundary which carries the same quantum number as \(\phi\) is \(\Phi \sim \partial_z \phi\). Since \(\phi\) has scaling dimension 1, \(\Phi\) should have scaling dimension \([\Phi] = 2\), i.e.

\[ (\Phi(x, z = 0)\Phi(0, z = 0)) \sim 1/|x|^4, \]

where \(x = (\tau, x, y)\). Eq. 3 is a much weaker correlation than \(\phi\) in the bulk (more detailed derivation of boundary correlation functions can be found in Ref. \cite{23} or \cite{24}).

Now we turn on coupling between the two 2d boundaries of the two layers. The edge state of the TI in layer-1 is affected by the boundary fluctuations of layer-2 through the “proximity effect”. The coupling between the two layers at the 2d boundary is described by the following term in the action:

\[ S_c = \int d^2xd\tau \sum_{\alpha} g \bar{\psi}_\alpha \psi_\alpha. \]

Since \(\Phi \sim \partial_z \phi\) has scaling dimension 2, \(g\) will have scaling dimension \([g] = -1\), i.e., it is strongly irrelevant. This conclusion is consistent with previous study Ref. \cite{32} A negative “mass term” \(\Phi^2\) will be generated through the standard fermion loop diagram, but since \(\Phi\) has scaling dimension 2, this mass term will be irrelevant. Hence the edge state of a 3d TI is stable even at the bulk quantum critical point where the time-reversal symmetry is spontaneously broken, and the properties of the edge states (such as electron Green’s function) should be identical to the edge state of TI in the infrared. To make the coupling \(g\) relevant, the quantum critical modes also need to localize on the boundary, which is one of the situations studied in Ref. \cite{32}.

B. Edge states of bosonic SPT states

The situation of bosonic SPT phases can be much more interesting. The bosonic SPT state can only exist in strongly interacting systems. We use the prototype 3d bosonic SPT phase with \((U(1) \times U(1)) \times Z_2^f\) symmetry as an example, since this phase can be viewed as the parent state of many 3d bosonic SPT phases by breaking the symmetry down to its subgroups, without fully trivializing the SPT phase. The topological feature of this phase
can be conveniently captured by the following nonlinear sigma model in the \((3+1)\) bulk:

\[
S = \int d^3x \frac{1}{g} \langle \partial n \rangle^2 + \frac{i2\pi}{\Omega_4} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_t n^e.
\]

where \(n\) is a five component vector field with unit length, and \(\Omega_4\) is the volume of the four dimension sphere with unit radius. \((n_1, n_2)\), and \((n_3, n_4)\) transform as a vector under the two \(U(1)\) symmetries respectively, and the \(Z_2^\dagger\) changes the sign of all components of the vector \(n\). The nonlinear sigma model Eq. 5 is invariant under all the transformations.

The 2d edge state of this SPT phase can be described by the following \((2+1)\) d action:

\[
S = \int d^2x \sum_{\alpha=1,2} \frac{1}{e^2} (da)^2 + \frac{1}{e^2} (da)^2,
\]

where \(a_\mu\) is a noncompact \(U(1)\) gauge field. The theory Eq. 6 is referred to as the “easy-plane noncompact \(\mathbb{CP}^1\)” (EP-NCCP\(^1\)) model. We are most interested in the point \(r = 0\). The term \(\sum_\alpha r|z_\alpha|^2\) would be forbidden if there is an extra \(Z_2\) self-dual symmetry that exchanges the two \(U(1)\) symmetries, while without the self-duality symmetry \(r\) needs to be tuned to zero, and the point \(r = 0\) becomes the transition point between two ordered phases that spontaneously breaks the two \(U(1)\) symmetries respectively. At \(r = 0\), starting with the UV fixed point with noninteracting \(z_\alpha\) and \(a_\mu\), both \(u\) and \(e\) are expected (though not proven) to flow to a fixed point with \(u = u_\ast, e = e_\ast\).

The putative conformal field theory at \(r = 0\) and its fate under coupling to the boundary fluctuations (boundary modes) of the bulk quantum critical points is the goal of our study in this section. As was discussed in previous literatures, it is expected that there is an emergent \(O(4)\) symmetry in Eq. 6 at \(r = 0\), when we fully explore all the duality features of Eq. 6. In the EP-NCCP\(^1\) action, the following operators form a vector under \(O(4)\):

\[
(n_1, n_2, n_3, n_4) \sim (z^1 \sigma^1, z^1 \sigma^2, \text{Re}[\mathcal{M}_a], \text{Im}[\mathcal{M}_a]),
\]

where \(\mathcal{M}_a\) is the monopole operator (the operator that annihilates a quantized flux of \(a_\mu\)). In the equation above, \((n_1, n_2)\) and \((n_3, n_4)\) form vectors under the two \(U(1)\) symmetries respectively. The emergent \(O(4)\) includes the self-dual \(Z_2\) symmetry of the EP-NCCP\(^1\), i.e. the operation that exchanges the two \(U(1)\) symmetries.

Now we consider the 3d bulk quantum phase transition between the SPT phase and the ordered phases that break part of the defining symmetries of the SPT phase. We first consider two order parameters: \(\phi_0, \phi_2\). \(\phi_0\) is the order parameter that corresponds to the self-dual \(Z_2\) symmetry; and \(\phi_2\) is a singlet under the emergent \(SO(4)\) but odd under the improper rotation of the emergent \(O(4)\), and also odd under \(Z_2^\dagger\). Again we view our system as a two layer structure: layer-1 is a SPT phase with solid edge states described by Eq. 6; layer-2 is a topological-trivial system that undergoes the transition of condensation of either \(\phi_0\) or \(\phi_2\). Both order parameters have an ordinary mean field like transition in the bulk of layer-2. Again at the boundary, both order parameters will have very different scalings from the bulk. We assume that system under study fills the entire semi-infinite space at \(z < 0\), then at the boundary plane \(z = 0\), the most natural boundary condition is that \(\phi_0(z \geq 0) = \phi_2(z \geq 0) = 0\), hence all order parameters near but inside the bulk should be replaced by the following representations: \(\Phi_0 \sim \partial_z \phi_0, \Phi_3 \sim \partial_z \phi_3\). Both order parameters have scaling dimensions 2 at the \((2+1)\)d boundary of layer-2.

Now we couple \(\Phi_0\) and \(\Phi_3\) to the edge states of layer-1. The coupling will take the following form:

\[
\mathcal{L}_{c\phi} = \sum_\alpha \Phi_0 |\phi_\alpha|^2, \quad \mathcal{L}_{c\Phi} = g_3 \Phi_3 \Phi_3^* \alpha^2 z^2.
\]

The RG flow of coupling constants \(g_0, g_3\) can be systematically evaluated in certain large-\(N\) generalization of the action in Eq. 6.

\[
S = \int d^3x \sum_{\alpha=1,2} \sum_{j=1}^{N/2} (\partial - ia)z_j \alpha,^2 + u \sum_j |z_j \alpha|^2.
\]

The large-\(N\) generalization facilitate calculations of the RG flow, but the down side is that the duality structure and emergent symmetries no longer exist for \(N > 2\). In the large-\(N\) limit of Eq. 9 the scaling dimension of the operators under study is

\[
N \to +\infty: \ [z^\dagger \sigma^3 z] = [[|z|^2] = 2.
\]

In the equation above, each operator has a sum of index \(j\), which was not written explicitly. Apparently coupling constants \(g_{0,3}\) are both irrelevant with large-\(N\) due to the weakened boundary correlation of \(\Phi_0\) and \(\Phi_3\).

We are seeking for more interesting scenarios when the boundary is driven to a new fixed point due to the bulk quantum criticality. For this purpose we consider another order parameter \(\phi\) which transforms as a vector under one of the two \(U(1)\) symmetries. Here we no longer assume the \(Z_2\) self-dual symmetry on the lattice scale. Again at the boundary should be replaced by \(\hat{\Phi} \sim \partial_z \hat{\phi}\). At the 2d boundary, the coupling between \(\hat{\Phi}\) and the edge state of layer-2 reads

\[
\mathcal{L}_{cv} = g_v (\Phi_1 z^\dagger \sigma^1 z + \Phi_2 z^\dagger \sigma^2 z).
\]

In the large-\(N\) limit of Eq. 9 the scaling dimension of the operators under study is

\[
N \to +\infty: \ [z^\dagger \sigma^1 z] = [z^\dagger \sigma^2 z] = 1.
\]

Hence \(g_v\) is marginal in the large-\(N\) limit, and there is a chance that \(g_v\) could drive the system to a new fixed point with \(1/N\) corrections.
We introduce the following action in order to compute the RG flow of $g_v$ with finite but large $N$:

$$S = \int d^2x d\tau \sum_{\alpha=1}^{N/2} \sum_{j=1}^{\frac{N}{2}} |(\partial - i\alpha)z_j,\alpha|^2 + i\lambda_+ |z_j,\alpha|^2$$

$$+ i\lambda_- z_j^+ \sigma^3 z_j + g_{\psi,\Phi} \cdot \Phi \cdot \pm \frac{1}{|\Phi|} \Phi. \quad (13)$$

The $\lambda_{\pm}$ are two Hubbard-Stratonovich (HS) fields introduced for the standard $1/N$ calculations. The scaling of $|z|^2$ and $z^+ \sigma^3 z$ in Eq. (9) are replaced by the HS fields $\lambda_+$, $\lambda_-$ in the new action Eq. (13) respectively. A coefficient "i" is introduced in the definition of $g_v$ by redefining $\Phi \to i\Phi$ for convenience of calculation.

The schematic beta function of $g_v$ reads

$$\frac{dg_v}{d\ln l} = (1 - \Delta_v)g_v - Bg_v^3 + O(v^5). \quad (14)$$

$\Delta_v$ is the scaling dimension of $z_j^+ \sigma^3 z_j$ in the large-$N$ generalization of the EP-NCCP model Eq. (9) with $\vec{\sigma} = (\sigma^1, \sigma^2)$. The standard $1/N$ calculation leads to

$$\Delta_v = 1 - \frac{56}{3\pi^2 N} + O\left(\frac{1}{N^2}\right). \quad (15)$$

The 1/$N$ correction of $\Delta_v$ comes from diagram Fig. 2(a–d), where the wavy line is the gauge boson propagator, and the dashed line represents propagators of both $\lambda_{\pm}$. The first term of Eq. (15) implies that $g_v$ is indeed weakly relevant with finite but large $-N$.

The constant $B$ in the beta function arises from the operator product expansion of the coupling term Eq. (11) which is equivalent to the diagrams Fig. 2(e, f). This computation leads to $B = 1/(3\pi^2)$. The two diagrams in Fig. 3 which are also at $g_v^2$ order cancel each other for arbitrary gauge choices. Similar two-loop diagrams at the same order of $1/N$ do not enter the RG equation due to lack of logarithmic contribution, as was explained in Ref. 43. $\Phi$ does not receive a wave function renormalization due to the singular form of its action. Hence with finite but large $-N$, $g_v$ indeed flows to a new fixed point:

$$g_v^2 = \frac{56}{N} + O\left(\frac{1}{N^2}\right). \quad (16)$$

We stress that this result is drawn from a controlled calculation and it is valid to the leading order of $1/N$.

![FIG. 2: (a, b) the 1/N contribution to $z^1 \sigma^1 z$ and $\bar{\psi} \tau^1 \psi$ from the gauge field fluctuation, the solid lines represent either the propagator of $z_\alpha$ or $\psi_\alpha$, the wavy line represents the propagator of the photon; (c, d) the 1/N contribution to $z^1 \bar{\sigma} z$ from $\lambda_\pm$ in Eq. (13) (e, f) the contribution to $B$ in Eq. (14)](Image)

![FIG. 3: The two diagrams at $g_v^3$ order which cancel each other for arbitrary gauge choices.](Image)

![FIG. 4: The $g_v^2$ diagrams that contributes to the scaling dimension of $[\lambda_+]$. Here the solid line represents the propagator of $z_j,\alpha$, the dotted line represents the vector operator $\Phi$, and the dashed line represents $\lambda_+$.](Image)

As we explained before, the point $r = 0$ is a direct transition between two ordered phases that spontaneously break the two U(1) symmetries. This transition will be driven to a new fixed point by coupling to the boundary fluctuations of bulk critical points as we demonstrated above. At this new fixed point, the critical exponent $\nu$ follows from the relation

$$\nu^{-1} = 3 - [\lambda_+]. \quad (17)$$

To evaluate the scaling dimension $[\lambda_+]$ we have to incorporate the contributions of $g_v^2$ from the diagrams shown.
in Fig.[1] and combined with 1/N calculations performed previously[13,14]. Then in the end we obtain
\[ \nu_*^{-1} = 1 + \frac{160}{3\pi^2 N} + \frac{4g_e^2}{3\pi^2} + O\left(\frac{1}{N^2}\right) \]
\[ = 1 + \frac{128}{\pi^2 N} + O\left(\frac{1}{N^2}\right). \] (18)

Again, there are other loop diagrams which appear to be at the same order of 1/N but do not make any logarithmic contributions.[13]

III. INTERFACE STATES EMBEDDED IN 3d BULK

A. Interface states of noninteracting electron systems

![Image](91x280 to 262x510)

**FIG. 5:** We consider a SU(N) antiferromagnet with self-conjugate representation on each site. The system forms a background VBS pattern, with opposite dimerizations between semi-infinite spaces when the entire 3d bulk undergoes a phase transition simultaneously due to the mirror (reflection) symmetry that connects the two sides of the domain wall.

In previous examples we studied topological edge states at the boundary of a 3d system. In this case there is no physical reason to impose the strong boundary condition at the interface embedded in the 3d space, hence the quantum critical modes at the interface follow the ordinary bulk scalings, instead of the weakened correlation of boundary CFT.

Again we will consider free fermion systems first. Let us first recall that the AIII class TI has a Z classification which is characterized by a topological index \( n_T \). \( n_T \) will appear as the coefficient of the electromagnetic response of the TI: \( \mathcal{L} \sim i\pi n_T \mathbf{E} \cdot \mathbf{B} \). \( n_T \) must change sign under spatial reflection transformation \( \mathcal{M}_z : z \rightarrow -z \). To construct the desired system, we assume the semi-infinite space \( z < 0 \) is occupied with the AIII class TI with Hamiltonian \( \hat{H} \), whose topological index is \( n_T \); and its “reflection conjugate” \( \mathcal{M}_z^{-1} \hat{H} \mathcal{M}_z \) fills the semi-infinite space \( z > 0 \). Then there are \( N_f = 2n_T \) flavors of massless Dirac fermions localized at the 2d plane \( z = 0 \), which are still protected by time-reversal symmetry. Now we assume the entire bulk undergoes a quantum phase transition with a spontaneous time-reversal symmetry breaking, whose order parameter couples to the domain wall Dirac fermions as

\[ S = \int d^2xd\tau \sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha \gamma_\mu \partial_\mu \psi_\alpha + g \phi \bar{\psi}_\alpha \psi_\alpha + \frac{1}{2} \phi (-\partial^2)^{1/2} \phi. \] (19)

The last term in the action is still defined in the (2+1)d interface, and it reproduces the correlation of \( \phi \) in the bulk: \( \langle \phi(0)\phi(r) \rangle \sim 1/r^2 \). We stress that, since now the order parameter resides in the entire bulk, \( \phi \) no longer obeys the boundary scaling as we discussed in previous examples. A negative boson mass term \( -r^2 \phi^2 \) can be generated through the standard fermion mass loop diagram, hence we need to tune an extra term at the interface to make sure the mass term of \( \phi \) vanishes.

In this case the coupling constant \( g \) is a marginal perturbation based on simple power-counting. But \( g \) will flow under renormalization group (RG) with loop corrections in Fig.[2](e, f):

\[ \beta(g) = \frac{dg}{d\ln l} = -\frac{2}{3\pi^2} g^3 + O(g^5). \] (20)

Hence even in this case, the coupling between the domain wall states and the bulk quantum critical modes is perturbatively marginally irrelevant.

So far we have assumed that the velocity of the interface state is identical with the bulk. Now let us tune the velocity of the domain wall Dirac fermions slightly differently, which can be captured by the following term in the Lagrangian:

\[ \sum_\alpha \delta \bar{\psi}_\alpha (\gamma^1 \partial_x + \gamma^2 \partial_y - 2\gamma^3 \partial_z) \psi_\alpha. \] (21)
the leading order beta function of $\delta$:

$$\beta(\delta) = \frac{d\delta}{d \ln l} = -\frac{1}{5\pi^2 g^2} \delta. \quad (22)$$

Together with $\beta(g)$, the velocity anisotropy is also perturbatively irrelevant.

**FIG. 6**: The Feynman diagrams that renormalizes the extra velocity $\delta$ in Eq. 21. The box represents the vertex $\delta$, and all three diagrams contribute to the fermion self-energy and renormalize $\delta$.

**FIG. 7**: The extra diagrams that contribute to the scaling dimension of $\sum_\alpha \bar{\psi}_\alpha \psi_\alpha$ at the leading order of $1/N_f$ in QED$_4$. Again the wavy lines are photon propagators.

### B. Interface states of quantum antiferromagnet

We now consider a SU($N$) quantum antiferromagnet on a tetragonal lattice with a self-conjugate representation on each site (we assume $N$ is an even integer). With large--$N$, an antiferromagentic Heisenberg SU($N$) model has a dimerized ground state, where the two SU($N$) spins on two nearest neighbor sites form a spin singlet (valence bond). We consider the following background configuration of valence bond solid (VBS): the spins form VBS along the $\hat{z}$ direction which spontaneously break the translation symmetry, while there is a domain wall between two opposite dimerizations at the 2d XY plane. Notice that due to the reflection symmetry $z \rightarrow -z$ of the background VBS configuration, the two sides of the domain wall will reach the quantum critical point simultaneously. The bulk transition is still described by Eq. 2.

When we couple the Ising order parameter $\phi$ to the domain wall QED$_3$, the total $(2+1)d$ action reads

$$S = \int d^2xd\tau \sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha \gamma_\mu(\partial_\mu - i a_\mu) \psi_\alpha + g\phi \bar{\psi}_\alpha \psi_\alpha + \frac{1}{2} \phi(-\partial^2)^{1/2} \phi. \quad (24)$$

If the gauge field fluctuation is ignored, or equivalently in the large--$N_f$ limit, the scaling dimension of $\bar{\psi} \psi$ is $[\bar{\psi} \psi] = 2$, and hence the scaling dimension of $g$ is $[g] = 0$, i.e. $g$ is a marginal perturbation. The $1/N_f$ correction to the RG flow arises from the Feynman diagrams (Fig. 2(a, b) and Fig. 7) which involves one or two photon propagators:

$$G^a_{\mu \nu}(\vec{p}) = \frac{16}{N_f p_f} \left( \delta^{a \nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (25)$$

Again in this case the fermions will generate a mass term for the order parameter at the interface, which we need to tune to zero. At the leading order of $1/N_f$ the corrected beta function for $g$ reads

$$\beta(g) = \frac{dg}{d \ln l} = -\frac{128}{3\pi^2 N_f} g - \frac{2}{3\pi^2} g^3 + O(g^3). \quad (26)$$

But this beta function does not lead to a new unitary fixed point other than the decoupled fixed point $g = 0$. Hence in this case the domain wall state is decoupled from the bulk quantum critical modes in the infrared limit.

A more interesting scenario is when the bulk undergoes a transition which spontaneously breaks the translation and $C_4$ rotation symmetry by developing an extra VBS order within the XY plane. The inplane VBS order parameters are $V_{\alpha} \sim \psi \tau^1 \psi$, and $V_{\alpha} \sim \psi \tau^2 \psi$, where $\tau^{1,2}$ are the Pauli matrices operating in the Dirac valley space.
The coupling between the VBS order parameter and the domain wall QED$_3$ reads
\[ S_c = \int d^2 x d\tau \, g (\phi^* \bar{\psi} \tau^- \psi + \phi \bar{\psi} \tau^+ \psi) + \phi^*(-\partial^2)^{1/2}\phi. \] (27)

Here $\tau^\pm = (\tau^1 \pm i\tau^2)/2$. The scaling dimension of the VBS order parameter at the QED$_3$ fixed point has been computed previously \cite{48, 50} $[\bar{\psi} \tau^3 \psi] = 2-64/(3\pi^2 N_f)$, and the beta function of $g$ to the leading order of $1/N_f$ reads
\[ \beta(g) = \frac{64}{3\pi^2 N_f} g - \frac{1}{6\pi^2} g^3 + O(g^3). \] (28)

In the large--$N_f$ limit, the coupling $g$ is marginally irrelevant; but with finite and large--$N_f$, $g$ is weakly relevant at the noninteracting fixed point, and it will flow to an interacting fixed point
\[ g^2 = \frac{128}{N_f} + O(1/N_f^2). \] (29)

This new fixed point will break the emergent SU($N_f$) flavor symmetry down to SU($N$) $\times$ U(1) symmetry, where U(1) corresponds to the rotation of the Dirac valley space. The following gauge invariant operators receive different corrections to their scaling dimensions from coupling to the bulk quantum critical modes:
\[ [\bar{\psi} \psi] = 2 + \frac{128}{3\pi^2 N_f} + \frac{2}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} T^a \psi] = 2 - \frac{64}{3\pi^2 N_f} + \frac{2}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} \tau^3 \psi] = 2 - \frac{64}{3\pi^2 N_f} - \frac{1}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} \tau^{1,2} \psi] = 2 - \frac{64}{3\pi^2 N_f} + \frac{1}{6\pi^2} g^2. \] (30)

The operators $[\bar{\psi} \tau^{1,2} \psi]$ have exactly the same scaling dimension 2, the Feynman diagram contributions from Fig. 2 cancel each other for operator $[\bar{\psi} \tau^{1,2} \psi]$ as they should. Notice that the last three operators in Eq. (30) should have the same scaling dimension in the original QED$_3$ fixed point due to the large SU($N_f$) flavor symmetry, but at this new fixed point they will acquire different corrections.

Another interesting scenario is that the bulk is at a critical point whose order parameter couples to the Ising like operator $\bar{\psi} \tau^3 \psi$, which breaks the inplane parity but preserves the time-reversal:
\[ S_c = \int d^2 x d\tau \, g \bar{\psi} \tau^3 \psi + \frac{1}{2} \phi(-\partial^2)^{1/2}\phi. \] (31)

The microscopic representation of the operator $\bar{\psi} \tau^3 \psi$ can be found in Ref. 48. The beta function of the coupling $g$ reads
\[ \beta(g) = \frac{64}{3\pi^2 N_f} g - \frac{2}{3\pi^2} g^3 + O(g^3), \] (32)
and once again there is new stable fixed point $g^2 = 32/N_f + O(1/N_f^2)$. And at this fixed point,
\[ [\bar{\psi} \psi] = 2 + \frac{128}{3\pi^2 N_f} + \frac{2}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} T^a \psi] = 2 - \frac{64}{3\pi^2 N_f} + \frac{2}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} \tau^3 \psi] = 2 - \frac{64}{3\pi^2 N_f} - \frac{1}{3\pi^2} g^2 + O(1/N_f^2); \]
\[ [\bar{\psi} \tau^{1,2} \psi] = 2 - \frac{64}{3\pi^2 N_f} + \frac{1}{6\pi^2} g^2. \] (33)

The domain wall state considered here is formally equivalent to the boundary state of a 3d bosonic SPT state with $\text{pSU}(N) \times \text{U}(1)$ symmetry, which can also be embedded to the 3d SPT with $\text{pSU}(N_f)$ symmetry discussed in Ref. 51. This SPT state can be constructed as follows: we first break the U(1) symmetry in the 3d bulk by driving the bulk $z < 0$ into a superfluid phase, and then decorate the vortex loop of the superfluid phase with a 1d Haldane phase with $\text{pSU}(N)$ symmetry. Eventually we proliferate the decorated vortex loops to restore all the symmetries in the bulk. A 1d $\text{pSU}(N)$ Haldane phase can be constructed as a spin-chain with a $\text{pSU}(N)$ spin on each site, and there is a dangling self-conjugate representation of $\text{SU}(N)$ on each end of the chain. And this dangling spin will also exist in the U(1) vortex at the boundary of the $\text{pSU}(N) \times \text{U}(1)$ SPT state. Notice that the self-conjugate representation of $\text{SU}(N)$ is a projective representation of $\text{pSU}(N)$.

IV. DISCUSSION

In this work we systematically studied the interplay of two different nontrivial boundary effects: the 2d edge states of 3d symmetry protected topological states, and the boundary fluctuations of 3d bulk quantum phase transitions. New fixed points were identified through generic field theory descriptions of these systems and controlled calculations. We then generalized our study to the 2d states localized at the interface embedded in the 3d bulk.

The last case studied in Eq. 32, 33 is special when $N_f = 2$, and when the gauge field is noncompact. This is the theory that has been shown to be dual to the EP-NCCP$^1$ mode\cite{49} studied in Eq. 6: the operator $\sum_\alpha r|z_\alpha|^2$ is dual to $r\bar{\psi} \tau^3 \psi$, and both theories are self-dual. By coupling the operator $\bar{\psi} \tau^3 \psi$ to the bulk critical modes (rather than the boundary fluctuations of the bulk critical points), we have shown that this (2 + 1)d theory is driven to a new fixed point, and the self-duality structure still holds. The self-duality transformation of Eq. 3 now is combined with the Ising symmetry of the order parameter $\phi$. However, the O(4) emergent symmetry no longer exists at this new fixed point, due to the nonzero fixed point of $g$ in Eq. 31.
The methodology used in this work can have many potential extensions. We can apply the same field theory and RG calculation to the 1d boundary of 2d SPT states (for instance the AKLT state), which was studied through exactly soluble lattice Hamiltonians, and also numerical methods. Also, 1d defect in a 3d topological state can also have gapless modes, it would be interesting to investigate the fate of a 1d defect embedded in a 3d bulk at the bulk quantum phase transition. Defects of free or weakly interacting fermionic topological insulator and topological superconductor coupled with bulk critical modes was studied in Ref. [36] but we expect the defect of an intrinsic strongly interacting topological state can lead to much richer physics. Last but not least, the “higher order topological insulator” has nontrivial critical points and corner topological modes is also worth exploration.

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