Monte Carlo simulation of lattice $\mathbb{CP}^{N-1}$ models at large $N$.

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In order to check the validity and the range of applicability of the $1/N$ expansion, we performed numerical simulations of the two-dimensional lattice $\mathbb{CP}^{N-1}$ models at large $N$, in particular we considered the $\mathbb{CP}^{20}$ and the $\mathbb{CP}^{40}$ models.

Quantitative agreement with the large-$N$ predictions is found for the correlation length defined by the second moment of the correlation function, the topological susceptibility and the string tension. On the other hand, quantities involving the mass gap are still far from the large-$N$ results showing a very slow approach to the asymptotic regime.

To overcome the problems coming from the severe form of critical slowing down observed at large $N$ in the measurement of the topological susceptibility by using standard local algorithms, we performed our simulations implementing the Simulated Tempering method.

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I. INTRODUCTION

The most attractive feature of two dimensional $C^P^{N-1}$ models is their similarity with the Yang-Mills theories in four space-time dimensions. Most properties of $C^P^{N-1}$ models have been obtained in the context of the $1/N$ expansion around the large-$N$ saddle point solution [1,2,3].

An alternative and more general non-perturbative approach is the simulation of the theory on the lattice. Recently there has been considerable interest in simulations of lattice $C^P^{N-1}$ models [4,5,6,7,8,9,10,11,12]. Simulations up to $N = 10$ have shown a qualitative agreement with some of the features derived from the continuum $1/N$ expansion, especially those concerning the sector of theory closely connected to the dynamically generated gauge fields, such as topology and confinement. On the other hand, while the $1/N$ expansion predicts a complex mass spectrum, evidence of other bound states, beyond the fundamental one in the adjoint positive parity channel, is not found up to $N = 10$ [5]. This is not a surprise, in that the agreement with the $1/N$ expansion can only be reached at very large $N$, because of the very large coefficient in the effective expansion parameter $6\pi/N$ that can be extracted from a nonrelativistic Schrödinger equation analysis of the linear confining potential [2,3]. It is then important to obtain numerical results at large $N$ to check the validity and the range of applicability of the $1/N$ expansion.

In this paper we present the results of some simulations of the lattice $C^P^{N-1}$ models at large $N$, in particular $N = 21$ and $N = 41$, and we compare them to the large-$N$ predictions coming from the $1/N$ expansion.

At large $N$ a particularly severe form of critical slowing down has been observed in measuring the topological susceptibility $\chi_t$ by using local updatings. For $N = 10$ the autocorrelation time of $\chi_t$ seems to grow exponentially with respect to the correlation length [4]. Moreover, this phenomenon becomes stronger with increasing $N$, making the simulations effectively non ergodic, already at small $\xi$. In order to overcome this difficulty and to perform simulations sampling correctly the topological sectors, we used the Simulated
Tempering method proposed by Marinari and Parisi [13]. In this method the temperature becomes a dynamical variable, and it is changed while keeping the system at equilibrium.

This paper is organized as follows.

In Sec. II the lattice actions adopted for numerical simulations and the lattice definitions of physical observables are presented.

In Sec. III we describe the Monte Carlo algorithm.

In Sec. IV we present the numerical results comparing them with the large-\(N\) predictions.

II. LATTICE FORMULATION

We regularize the theory on the lattice by considering the following action:

\[
S_g = -N\beta \sum_{n,\mu} \left( \bar{z}_{n+\mu} z_n \lambda_{n,\mu} + \bar{z}_n z_{n+\mu} \bar{\lambda}_{n,\mu} - 2 \right),
\]

where \(z_n\) is an \(N\)-component complex scalar field, constrained by the condition \(\bar{z}_n z_n = 1\), and \(\lambda_{n,\mu}\) is a U(1) gauge field satisfying \(\bar{\lambda}_{n,\mu} \lambda_{n,\mu} = 1\). We also considered its tree Symanzik improved counterpart \(S_{g}^{\text{Sym}}\) [14,4] to test universality. Tests of rotation invariance and stability of dimensionless ratios of physical quantities showed that \(S_g\) and \(S_{g}^{\text{Sym}}\) lead to scaling for rather small correlation lengths [4].

An important class of observables can be constructed by considering the local gauge-invariant composite operator

\[
P_{ij}(x) = \bar{z}_i(x) z_j(x)
\]

and its group-invariant correlation function

\[
G_P(x) = \langle \text{Tr} P(x) P(0) \rangle_{\text{conn}}.
\]

The standard correlation length \(\xi_w\) is extracted from the long-distance behavior of the zero space momentum correlation function (“wall-wall” correlation). \(\xi_w\) should reproduce in the continuum limit the inverse mass gap, that is the inverse mass of the lowest positive parity state belonging to the adjoint representation.
An alternative definition of the correlation length $\xi_G$ comes from considering the second moment of the correlation function $G_P$. In the small momentum regime we expect the behavior

$$\tilde{G}_P(k) \approx \frac{Z_P}{\xi_G^{-2} + k^2}, \quad (4)$$

where $\tilde{G}_P(k)$ is the Fourier transform of $G_P(x)$. The zero component of $\tilde{G}_P(k)$ is by definition the magnetic susceptibility $\chi_m$. On the lattice from the two lowest components of $\tilde{G}_P(k)$ we can derive the following definition of $\xi_G$:

$$\xi_G^2 = \frac{1}{4 \sin^2 \pi / L} \left[ \tilde{G}_P(0,0) - \tilde{G}_P(0,1) \right]. \quad (5)$$

In the scaling region the ratio $\xi_G / \xi_w$ must be a constant, scale-independent number.

For $N = 2$ $\xi_G / \xi_w \simeq 1$ within 1% \[4\], while the large-$N$ expansion predicts \[18\]

$$\frac{\xi_G}{\xi_w} = \sqrt{\frac{2}{3}} + O \left( \frac{1}{N^{2/3}} \right). \quad (6)$$

The mass gap, and therefore $\xi_w$, is a non-analytic function of $1/N$ around $N = \infty$ depending on $N^{-2/3}$. Instead $\xi_G$ can be expand in power of $1/N$ \[3\]:

$$(\xi_G \Lambda_{SM})^{-1} = \sqrt{6} \left[ 1 + \frac{6.1325}{N} + O \left( \frac{1}{N^2} \right) \right] \quad (7)$$

where $\Lambda_{SM}$ is the $\Lambda$ parameter of the sharp-momentum cut-off regularization scheme \[15\]. Standard perturbative calculations give $\Lambda_{SM}/\Lambda_g = \sqrt{32} \exp(\pi/2N)$, where $\Lambda_g$ is the $\Lambda$ parameter of the lattice action $S_g$.

The quantity $Z_P = \chi_m \xi_G^{-2}$ is related to the renormalization of the composite operator $P_{ij}$. Its dependence on $\beta$ can therefore be determined by renormalization group considerations. One finds that

$$Z_P = c \beta^{-2} \left[ 1 + O \left( \frac{1}{N \beta} \right) \right], \quad (8)$$

where $c$ is a constant independent of the regularization scheme and therefore of the lattice action. In the large-$N$ limit it turns out to be \[3\].
Another important class of observables is that connected to the dynamically generated
gauge field, such as the topological susceptibility and the string tension.

The geometrical definition of the topological charge is [16]

\[ q_n = \frac{1}{2\pi} \text{Im}\{\ln[\text{Tr} \, P_{n+\mu} P_{n+\mu} P_n] + \ln[\text{Tr} \, P_n P_{n+\mu} P_{n+\mu}]\} , \quad \mu \neq \nu . \] (10)

The topological susceptibility should then be extracted by measuring the following expectation value

\[ \chi_t = \frac{1}{V} \left\langle \left( \sum_n q_n \right)^2 \right\rangle . \] (11)

For large \( N \) this definition is expected to reproduce the physical topological susceptibility [17,5].

The large-\( N \) predictions concerning the topological susceptibility are [18]

\[ \chi_t \xi_G^2 = \frac{1}{2\pi N} \left( 1 - \frac{0.3801}{N} \right) + O \left( \frac{1}{N^3} \right) , \] (12)

and [19]

\[ \chi_t \xi_w^2 = \frac{3}{4\pi N} + O \left( \frac{1}{N^{5/3}} \right) . \] (13)

Eq. (12) and Eq. (13) are not in contradiction with each other due to Eq. (6), but the first
one should be testable at lower values of \( N \) according to the powers of \( N \) in the neglected
terms.

The large-\( N \) expansion predicts an exponential area law behavior for sufficiently large
Wilson loops [3]:

\[ W(C) = \prod_{n,\mu \in C} \lambda_{n,\mu} \sim e^{-\sigma A(C) - \rho P(C)} \quad \text{for} \quad A(C) \gg \xi^2 , \] (14)

where \( \sigma \) is the Abelian string tension and \( \rho \) is a perimeter term. This implies also that
the dynamical matter fields do not screen the linear potential at any distance. Monte Carlo
simulations at $N = 4$ and $N = 10$ confirmed the absence of screening effects [5]. The large-$N$
prediction for $\sigma$ is

$$\sigma \xi^2 = \frac{\pi}{N} + O \left( \frac{1}{N^2} \right).$$

(15)

The string tension can be easily extracted by measuring the Creutz ratios defined by

$$\chi(l, m) = \ln \frac{W(l, m-1) W(l-1, m)}{W(l, m) W(l-1, m-1)}.$$  

(16)

In a 2-d finite lattice with periodic boundary conditions, the large abelian Wilson loops of a
confining theory are subject to large finite size effects. For sufficiently large $R$ the behavior
of the Creutz ratios $\eta(R) \equiv \chi(R, R)$, i.e. of those with equal arguments, should be [5]:

$$\eta(R) \simeq \sigma \left[ 1 - \left( \frac{2R - 1}{L} \right)^2 \right],$$

(17)

where $L$ is the lattice size. To compare data from different lattices it is convenient to define
a rescaled Creutz ratio

$$\eta_r(R) = \eta(R) \left[ 1 - \left( \frac{2R - 1}{L} \right)^2 \right]^{-1} \simeq \sigma.$$  

(18)

III. THE MONTE CARLO ALGORITHM

In most of our simulations we used the simulated tempering method proposed in Ref. [13].

The basic idea of this method consists in enlarging the configuration space of the system
by including the temperature, and changing it while remaining at statistical equilibrium.
Considering a finite set of temperatures $\beta_i$, $i = 1, ... N_\beta$, the probability distribution is
chosen to be

$$P(\beta_i, x) = e^{-K(\beta_i, x)},$$

(19)

where

$$K(\beta_i, x) = \beta_i H(x) - g_i,$$

(20)
where $x$ indicates the lattice variables, $H(x)$ is the hamiltonian of the statistical system, and $g_i$ is independent of $x$. The probability distribution induced by $K(\beta_i, x)$, restricted to the subspace $i$, is the Gibbs distribution for $\beta = \beta_i$.

By making the choice

$$g_i = \beta_i F_i,$$

where $F_i$ is the free energy at $\beta_i$, the probability of having a given value of $i$ becomes independent of $i$, i.e. $P_i = 1/N\beta$.

In practice the simulated tempering method is implemented by performing the following cycle [13]: (i) updating the lattice variables at the temperature $\beta_i$ by using a standard algorithm; (ii) updating the temperature according to the probability

$$P(\beta_i) = e^{-\beta_i E + g_i},$$

where $E$ is the energy of the configuration obtained in (i). The expectation values at a given $\beta$ can be obtained performing the measurements when $\beta_i = \beta$.

In the presence of free energy barriers separating different regions of the configuration space, the visits of the system to lower values of $\beta$ will make easier to jump, in that at lower $\beta$ free energy barriers are lower.

It is important to choose the values of $\beta_i$ so that the probability transition from one value of $\beta$ to another is not negligible. This can be achieved by requiring a non-negligible overlap in the values of the energy of the configurations coming from simulations at contiguous values of $\beta_i$, and using the following approximation for $g_i$:

$$g_{i+1} \approx g_i + (\beta_{i+1} - \beta_i) \left( \frac{E_{i+1} + E_i}{2} \right),$$

which is a good approximation when $\Delta \beta$ is small and it is simple to estimate.

In the case of the lattice CP$^{N-1}$ models, going to lower $\beta$ should make it easier to jump from one topological sector to another, and when the temperature will decrease again the system will be visiting a different topological sector with the correct equilibrium probability, providing us with a well representative ensemble of configurations at the given value of $\beta$. 

To update the lattice variables, we chose the over-heat bath algorithm [20] because it is very efficient in decorrelating the energy and, at the same time, contains a procedure of overrelaxation. Furthermore it requires less computational effort than a standard heat bath. The implementation of the over-heat bath method in the lattice CP$^{N-1}$ models is described in Ref. [4].

The difficulty in applying the simulated tempering method to the CP$^{N-1}$ models is that with increasing $N$ the fluctuations of the energy tend to be frozen (at $N = \infty$ only one configuration contributes to the path integral) making necessary to keep the difference between contiguous $\beta_i$ very small. Therefore in order to work with a wide range of temperatures we must introduce a large number of $\beta_i$.

We performed also some standard simulations by employing algorithms consisting in mixtures of over-heat bath and microcanonical algorithm [4].

IV. SIMULATIONS

In Table I we present a summary of the runs done by using the simulating tempering method. In each run we performed the measurements at two values of $\beta$, which can be read in Table II. In Table II we also give a summary of the runs done by using standard algorithms. Some preliminary results of the lattice CP$^{20}$ model were already presented in Ref. [4]. There the exponential growth of the autocorrelation time of $\chi_t$ allowed to obtain meaningful measurements of $\chi_t$ only at small correlation length, $\xi_G \simeq 2.5$, while simulations at larger $\xi$ did not sample correctly the topological sectors. By using the simulated tempering method we performed simulations up to $\xi_G \simeq 4.2$ obtaining reliable measurements of $\chi_t$. For the CP$^{40}$ model we performed a simulated tempering run with correlation lengths up to $\xi_G \simeq 2.5$. All simulations were performed setting periodic boundary conditions.

Since the measurements required much more computational time than the updating procedure and we were essentially interested in decorrelating the topological charge, when using the simulated tempering method we checked the value of $\beta$ every 4–5 sweeps and
performed the measures when $\beta_i = \beta$. Errors were estimated by a blocking procedure. Measurements at different values of $\beta$ but from the same simulated tempering simulation are not completely decorrelated, especially those regarding the topological susceptibility. In the standard runs the integrated autocorrelation times of the magnetic susceptibility were small, instead those relative to the topological susceptibility were very large. For $N = 21$ and at $\beta = 0.65$, we found $\tau_{\chi_m}^{\text{int}} = 3.2(1)$ and $\tau_{\chi_t}^{\text{int}} \simeq 100$; by using $S_g^{\text{Sym}}$ and at $\beta = 0.60$ $\tau_{\chi_t}^{\text{int}} \simeq 600$.

For large enough $N$, the finite size effects should be dominated by the size of the ground state and not by its mass. The $1/N$ expansion predicts a radius of the ground state proportional to $\xi N^{1/3}$. A comparison of the finite size scaling functions of $\chi_m$ and $\xi_G$ at $N = 10$ and $N = 21$ has shown that $z = L/\xi_G \simeq 4.5 N^{1/3}$ should be a safe value in order to have finite size effects smaller than 1% (at least for $\chi_m$ and $\xi_G$) [4]. We checked this further for $N = 41$ by comparing the results obtained at $\beta = 0.57$ on lattices with $L = 33$ ($z \simeq 16.5$) and with $L = 42$ ($z \simeq 21$), and finding agreement within errors of about 0.5%.

In Table III we list the correlation length $\xi_G$, the ratio $\xi_G/\xi_w$, the dimensionless quantity $\chi_t \xi_G^2$ and the combination $\beta^2 Z_P \equiv \beta^2 \chi_m \xi_G$. All these quantities were analyzed using the jackknife method.

$\xi_w$ was obtained by fitting the wall-wall correlations starting from a minimum distance $x_{\text{min}}$. We set $x_{\text{min}} \simeq 2\xi_w$ for the CP$^{20}$ model and $x_{\text{min}} \simeq 3\xi_w$ for the CP$^{40}$ model; fits using larger $x_{\text{min}}$ gave consistent results.

At all values of $\beta$ we performed a test of rotation invariance by comparing $\xi_w$ with the correlation length $\xi_d$ extracted from the long-distance behavior of the diagonal wall-wall correlations of $P_{i,j}$ [4]. We found $\xi_d/\xi_w \simeq 1$ within errors of about 0.5% in all cases.

In Fig. [4] the ratio $\xi_G/\xi_w$ is plotted versus $\xi_G$. We note that at $N = 41$ $\xi_G/\xi_w$ is still far from the large-$N$ prediction [3], indicating a very slow approach to the large-$N$ asymptotic regime.

Data for $\beta^2 Z_P$ show scaling and, for the CP$^{20}$ model, the two actions give close values. The small discrepancies can be imputed to the non-universal terms of order $(N/\beta)^{-1}$ in Eq. [3].
The comparison with Eq. (9), which gives \( c = 0.6717 \) for \( N = 21 \) and \( c = 0.5769 \) for \( N = 41 \), is satisfactory.

At large \( N \) the dynamically generated gauge field contains essentially two distinct types of modes at large distance: the gaussian fluctuations around the large \( N \) saddle point solution, which are responsible for confinement, and those determining the topological properties. We expect to find in the \( \text{CP}^{N-1} \) models a phenomenon similar to that observed in the 2-d \( U(1) \) gauge model, that is a large decoupling between the gaussian modes and the topological ones [21]. This picture is supported by the agreement found in the results concerning observables not related to the topological properties, obtained by the simulated tempering method and by standard simulations which did not sample correctly the topological sectors, whose results were reported in Ref. [4].

In Fig. 2 we plot the dimensionless quantity \( \chi t \xi_G^2 \). For both the \( \text{CP}^{20} \) and \( \text{CP}^{40} \) models data are consistent with the large-\( N \) prediction (12). In order to quote a value for \( \chi t \), we fitted to a constant the data selected by taking only those relative at the biggest correlation length of each simulated tempering run (to avoid introducing correlated data in the fit) and discarding the data at \( \xi_G < 3 \) obtained by using \( S_g \) (this is justified by the slow approach to scaling expected when using \( S_g \) [5]). Then for the \( \text{CP}^{20} \) model we found

\[
\chi t \xi_G^2 = 0.0076(3),
\]

to be compared with the value \( \chi t \xi_G^2 = 0.00744 \) coming from Eq. (12). However, the result (24) still disagrees with the Lüscher large-\( N \) prediction (13), which would require \( \xi_G^2 / \xi_w^2 \simeq 2/3 \), while we found \( \xi_G^2 / \xi_w^2 \simeq 0.91 \).

To extract the string tension we calculated the Creutz ratios \( \eta(R) \equiv \chi(R, R) \). The Wilson loops were measured using improved estimators obtained by replacing each \( \lambda_{n,\mu} \) with its average \( \lambda_{n,\mu}^{\text{imp}} \) in the field of its neighbors [4]. In Fig. 3 we plot the rescaled Creutz ratios \( \eta_r(R) \) defined in Eq. (18). Data show a good agreement with the large-\( N \) prediction (15). The data for the \( \text{CP}^{20} \) model shown in Fig. 3 were taken at \( \beta = 0.65 \) and \( \beta = 0.67 \). At larger \( \beta \) the signals were too noisy for distances larger than \( d \simeq 2\xi_G \), for \( d < 2\xi_G \) the
results were consistent with those shown in Fig. 3.

We checked asymptotic scaling, according to the two loop formula

$$f(\beta) = (2\pi \beta)^{2/N} \exp(-2\pi \beta),$$

by analyzing the quantity $M_G/\Lambda_g = [\xi_G f(\beta)]^{-1}$. We also analyzed the data by using the $\beta_E$ scheme, in which a new coupling $\beta_E$ is extracted from the energy and inserted in the two loop formula (25) [22,23]. In Fig. 4 we plot the values of $M_G/\Lambda_g$ obtained with the two actions $S_g$ and $S_g^{\text{Sym}}$ and by using the standard and the $\beta_E$ schemes. To report all data in terms of $\Lambda_g$, we used the ratios of the $\Lambda$ parameters given in Ref. [5].

At smaller $N$ the $\beta_E$ scheme showed a notable improvement in testing asymptotic scaling, giving also quite different values with respect to the standard scheme. For example, for the CP$^1$ (or $O(3)$ $\sigma$) model and by using $S_g$, the $\beta_E$ scheme gave a determination of $M_G/\Lambda_g$ in agreement with its analytical prediction (within errors of about 3%), while the standard scheme was out by about 30% at $\xi \simeq 30$ [5]. At $N = 21$ the discrepancy among the different determinations is still present, although it is reduced. Instead at $N = 41$ it has almost disappeared and the result is in good agreement with the large-$N$ prediction (7).

In conclusion, the results of the Monte Carlo simulations at $N = 21$ and $N = 41$ show a quantitative agreement with the large-$N$ predictions for those quantities which are analytical functions of $1/N$ around $N = \infty$ and which can be expanded in powers of $1/N$, such as $\xi_G$, $\chi_t$ and $\sigma$. On the other hand, the approach to the large-$N$ asymptotic regime of the quantities involving the mass gap appears very slow and the CP$^{40}$ should be still outside the region where the complete mass spectrum predicted by the $1/N$ expansion [24] could be observed.

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FIGURES

FIG. 1. The ratio $\xi_G/\xi_w$ versus $\xi_G$. The dashed line shows the large-$N$ prediction (1). The dotted lines are the results of a fit.

FIG. 2. Topological susceptibility versus $\xi_G$. The dashed lines show the large-$N$ prediction (12).

FIG. 3. The quantity $\eta_p(R)\xi_G^2$ as a function of the physical distance $R/\xi_G$. The dashed lines show the value of the string tension predicted by the large-$N$ expansion: $\sigma\xi_G^2 = \pi/N$.

FIG. 4. Asymptotic scaling test for $\xi_G$. The dashed lines show the large-$N$ prediction (7).
TABLE I. Summary of the simulation runs by using the simulated tempering algorithm. The runs are labeled by the letters a,b,c,d. We report: the action $S$ used in the simulation; the minimum and maximum value of $\beta$, “range”; the difference between two contiguous values of $\beta$, $\Delta \beta$; the number of $\beta_i$, $N_\beta$; the acceptance in the updating of $\beta$, $A_\beta$; the total number of iterations, “stat”.

|   | $N$ | $S$    | $L$ | range   | $\Delta \beta$ | $N_\beta$ | $A_\beta$ | stat |
|---|-----|--------|-----|---------|----------------|-----------|-----------|------|
| a | 21  | $S_g$  | 48  | 0.61–0.70 | 0.003          | 31        | 68%       | 400k |
| b | 21  | $S_g$  | 60  | 0.60–0.72 | 0.003          | 41        | 62%       | 700k |
| c | 21  | $S_g^{\text{Sym}}$ | 48 | 0.51–0.63 | 0.003          | 41        | 64%       | 700k |
| d | 41  | $S_g$  | 42  | 0.50–0.60 | 0.0025         | 41        | 64%       | 500k |
TABLE II. Summary of the measurements. An asterisk indicates runs with the Symanzik improved action. We use the notation “m,γ” for a stochastic mixture of microcanonical and over-heat bath updating with relative weight γ (see Ref. [4]) and “S.T.” for the simulated tempering algorithm. The letters a,b,c,d near the values of β indicate the simulated tempering run where the measures were performed. t is the percentage of time spent by the system at a given value of β.

| N | β   | L | Algor. | stat   | E      | χm   |
|---|-----|---|--------|--------|--------|------|
| 21 | 0.65 | 42 | m,1    | 100k   | 0.7995(1) | 12.14(3) |
| 21 | 0.67 a | 48 | S.T.   | 400k t=3.3% | 0.7741(1) | 14.70(3) |
| 21 | 0.70 a | 48 | S.T.   | 400k t=3.3% | 0.7391(1) | 19.54(5) |
| 21 | 0.69 b | 60 | S.T.   | 700k t=2.0% | 0.7504(1) | 17.73(3) |
| 21 | 0.72 b | 60 | S.T.   | 700k t=1.7% | 0.7174(1) | 23.65(6) |
| 21 | 0.60 * | 42 | m,1    | 150k   | 0.8582(1) | 15.08(3) |
| 21 | 0.60 * c | 48 | S.T.   | 700k t=2.4% | 0.8583(1) | 15.04(3) |
| 21 | 0.63 * c | 48 | S.T.   | 700k t=2.3% | 0.8161(1) | 19.89(4) |
| 41 | 0.57 d | 42 | S.T.   | 500k t=2.4% | 0.8890(1) | 7.756(8) |
| 41 | 0.60 d | 42 | S.T.   | 500k t=1.8% | 0.8454(1) | 10.105(15) |
### TABLE III. Results for the CP\(^{20}\) and the CP\(^ {40}\) models.

| \(N\) | \(\beta\) | \(L\) | \(\xi_G\) | \(\xi_G/\xi_w\) | \(\chi^2\xi_G^2\) | \(\beta^2 Z_P\) |
|---|---|---|---|---|---|---|
| 21 | 0.65 | 42 | 2.69(2) | 0.949(4) | 0.0088(5) | 0.708(8) |
| 21 | 0.67 a | 48 | 3.10(2) | 0.952(4) | 0.0074(6) | 0.687(7) |
| 21 | 0.70 a | 48 | 3.72(2) | 0.955(4) | 0.0081(8) | 0.694(6) |
| 21 | 0.69 b | 60 | 3.49(2) | 0.954(3) | 0.0083(7) | 0.693(8) |
| 21 | 0.72 b | 60 | 4.23(2) | 0.952(4) | 0.0078(7) | 0.685(7) |
| 21 | 0.60* | 42 | 2.887(13) | 0.957(5) | 0.0070(8) | 0.651(6) |
| 21 | 0.60* c | 48 | 2.872(15) | 0.961(3) | 0.0075(4) | 0.656(6) |
| 21 | 0.63* c | 48 | 3.506(16) | 0.955(3) | 0.0075(6) | 0.643(11) |
| 41 | 0.57 d | 42 | 2.011(10) | 0.926(4) | 0.0044(4) | 0.623(7) |
| 41 | 0.60 d | 42 | 2.431(11) | 0.930(8) | 0.0036(4) | 0.616(7) |