Testing Closeness With Unequal Sized Samples

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Abstract

We consider the problem of closeness testing for two discrete distributions in the practically relevant setting of unequal sized samples drawn from each of them. Specifically, given a target error parameter $\varepsilon > 0$, $m_1$ independent draws from an unknown distribution $p$, and $m_2$ draws from an unknown distribution $q$, we describe a test for distinguishing the case that $p = q$ from the case that $||p - q||_1 \geq \varepsilon$. If $p$ and $q$ are supported on at most $n$ elements, then our test is successful with high probability provided $m_1 \geq n^{2/3}/\varepsilon^{4/3}$ and $m_2 = \Omega\left(\max\left\{\frac{n}{\sqrt{m_1 \varepsilon^2}}, \frac{\sqrt{n}}{\varepsilon} \right\}\right)$; we show that this tradeoff is optimal throughout this range, to constant factors. These results extend the recent work of Chan et al. [9] who established the sample complexity when the two samples have equal sizes, and tightens the results of Acharya et al. [3] by polynomials factors in both $n$ and $\varepsilon$. As a consequence, we obtain an algorithm for estimating the mixing time of a Markov chain on $n$ states up to a $\log n$ factor that uses $\tilde{O}(n^{3/5} \tau_{mix})$ queries to a “next node” oracle, improving upon the $\tilde{O}(n^{5/3} \tau_{mix})$ query algorithm of [8]. Finally, we note that the core of our testing algorithm is a relatively simple statistic that seems to perform well in practice, both on synthetic data and on natural language data.

1 Introduction

One of the most fundamental problems in statistical hypothesis testing is the question of distinguishing whether two unknown distributions are very similar, or significantly different. Classical tests, like the Chi-squared test or the Kolmogorov-Smirnov statistic, are optimal in the asymptotic regime, for fixed distributions as the sample sizes tend towards infinity. Nevertheless, in many modern settings—such as the analysis of customer data, web logs, natural language processing, and genomics, despite the quantity of available data—the support sizes and complexity of the underlying distributions are far larger than the datasets, as evidenced by the fact that many phenomena are observed only a single time in the datasets, and the empirical distributions of the samples are poor representations of the true underlying distributions. In such

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1To give some specific examples, two recent independent studies [19][25] each considered the genetic sequences of over 14,000 individuals, and found that rare variants are extremely abundant, with over 80% of mutations observed just once in the sample. A separate recent paper [16] found that the discrepancy in rare mutation abundance cited in different demographic modeling studies
settings, we must understand these statistical tasks not only in the asymptotic regime (in which the amount of available data goes to infinity), but in the “undersampled” regime in which the dataset is significantly smaller than the size or complexity of the distribution in question. Surprisingly, despite an intense history of study by the statistics, information theory, and computer science communities, aspects of basic hypothesis testing and estimation questions—especially in the undersampled regime—remain unresolved, and require both new algorithms, and new analysis techniques.

In this work, we examine the basic hypothesis testing question of deciding whether two unknown distributions over discrete supports are identical (or extremely similar), versus have total variation distance at least \( \varepsilon \), for some specified parameter \( \varepsilon > 0 \). We consider (and largely resolve) this question in the extremely practically relevant setting of unequal sample sizes. Informally, taking \( \varepsilon \) to be a small constant, we show that provided \( p \) and \( q \) are supported on at most \( n \) elements, for any \( \gamma \in [0, 1/3] \), the hypothesis test can be successfully performed (with high probability over the random samples) given samples of size \( m_1 = \Theta(n^{2/3+\gamma}) \) from \( p \), and \( m_2 = \Theta(n^{2/3-\gamma/2}) \) from \( q \). Furthermore, for every \( \gamma \) in this range, this tradeoff between \( m_1 \) and \( m_2 \) is necessary, up to constant factors. Thus our results smoothly interpolate between the known bounds of \( \Theta(n^{2/3}) \) on the sample size necessary in the setting where one is given two equal-sized samples \([5, 9]\), and the bound of \( \Theta(\sqrt{n}) \) on the sample size in the setting in which the sample is drawn from one distribution and the other distribution is known to the algorithm \([22, 29]\). Throughout most of the regime of parameters, when \( m_1 \ll m_2 \), our algorithm is a natural extension of the algorithm proposed in \([9]\), and is similar to the algorithm proposed in \([3]\) except with the addition of a normalizing term. In the extreme regime when \( m_1 \approx n \), our algorithm requires an additional statistic which appears to be new. Throughout the regime of parameters, our algorithm is relatively simple, and appears to be practically viable. In section \([4]\) we illustrate the efficacy of our approach on both synthetic data, and on the real-world problem of deducing whether two words are synonyms, based on a small sample of the bi-grams in which they occur.

We also note that, as pointed out in several related works \([3, 12, 5]\), this hypothesis testing question has several applications to other problems, such as estimating or testing the mixing time of Markov processes, and our results yield improved algorithms in these settings.

1.1 Related Work

The general question of how to estimate or test properties of distributions using fewer samples than would be necessary to actually learn the distribution, has been studied extensively since the late '90s. Most of the work has focussed on “symmetric” properties (properties whose value is invariant to relabeling domain elements) such as entropy, support size, and distance metrics between distributions (such as \( \ell_1 \) distance). This has included both algorithmic work (e.g. \([4, 6, 7, 10, 13, 20, 21, 26, 27, 29, 28]\)), and results on developing techniques and tools for establishing lower bounds (e.g. \([23, 30, 26]\)). See the recent survey by Rubinfeld for a more thorough summary of the developments in this area \([24]\).

The specific problem of “closeness testing” or “identity testing”, that is, deciding whether two distributions, \( p \) and \( q \), are similar, versus have significant distance, has two main variants: the one-unknown-distribution setting in which \( q \) is known and a sample is drawn from \( p \), and the two-unknown-distributions settings in which both \( p \) and \( q \) are unknown and samples are drawn from both. We briefly summarize the previous results for these two settings.
In the one-unknown-distribution setting (which can be thought of as the limiting setting in the case that we have an arbitrarily large sample drawn from distribution \( q \), and a relatively modest sized sample from \( p \)), initial work of Goldreich and Ron [12] considered the problem of testing whether \( p \) is the uniform distribution over \([n]\), versus has distance at least \( \varepsilon \). The tight bounds of \( \Theta(\sqrt{n}/\varepsilon^2) \) were later shown by Paninski [22], essentially leveraging the birthday paradox and the intuition that, among distributions supported on \( n \) elements, the uniform distribution minimizes the number of domain elements that will be observed more than once. Batu et al. [7] showed that, up to polylogarithmic factors of \( n \), and polynomial factors of \( \varepsilon \), this dependence was optimal for worst-case distributions over \([n]\). Recently, an “instance–optimal” algorithm and matching lower bound was shown: for any distribution \( q \), up to constant factors, \( m \leq \frac{1}{\varepsilon^2} \max\{ \epsilon^{-2} ||q - \Theta(\varepsilon)||_{2/3} \} \) samples from \( p \) are both necessary and sufficient to test \( p = q \) versus \( ||p - q|| \geq \varepsilon \), where \( ||q - \Theta(\varepsilon)||_{2/3} \) is the 2/3-rd norm of the vector of probabilities of distribution \( q \) after the maximum element has been removed, and the smallest elements up to \( \Theta(\varepsilon) \) total mass have been removed. (This immediately implies the tight bounds that if \( q \) is any distribution supported on \([n]\), \( O(\sqrt{n}/\varepsilon^2) \) samples are sufficient to test its identity.

The two-unknown-distribution setting was introduced to this community by Batu et al. [5] (refer to [8] for the journal version), and using collision statistics, they proposed an algorithm that requires \( m = O(\varepsilon^{-8/3}n^{2/3} \log n) \) samples from each distribution. Later, Valiant [30] proved a lower bound of \( m = \Omega(n^{2/3}) \), which was tight up to logarithmic factors in \( n \). Recently, Chan et al. [9] determined the optimal sample complexity for this problem: they showed that \( m = \Theta(\max\{ n^{2/3}/\varepsilon^{4/3}, \sqrt{n}/\varepsilon^2 \}) \) samples are necessary and sufficient for closeness testing, up to constant factors. In a slightly different vein, Acharya et al. [11, 2] recently considered the question of closeness testing with two unknown distributions from the standpoint of competitive analysis. They proposed an algorithm that performs the desired task using \( O(n^{3/2} \text{polylog } n) \) samples, and a lower bound of \( \Omega(n^{7/6}) \), where \( n \) represents the number of samples required to determine whether a set of samples were drawn from \( p \) versus \( q \), in the setting where \( p \) and \( q \) are explicitly known.

A natural generalization of this hypothesis testing problem, which interpolates between the two-unknown-distribution setting and the one-unknown-distribution setting, is to consider unequal sized samples from the two distributions. More formally, given \( m_1 \) samples from the distribution \( p \), the asymmetric closeness testing problem is to determine how many samples, \( m_2 \), are required from the distribution \( q \) such that the hypothesis \( p = q \) versus \( ||p - q|| > \varepsilon \) can be distinguished with large constant probability (say 2/3). Note that the results of Chan et al. [9] imply that it is sufficient to consider \( m_1 \geq \Theta(\max\{ n^{2/3}/\varepsilon^{4/3}, \sqrt{n}/\varepsilon^2 \}) \). This problem was studied recently by Acharya et al. [3]: they gave an algorithm that given \( m_1 \) samples from the distribution \( p \) uses \( m_2 = O(\max\{ \frac{n \log n}{\varepsilon \sqrt{m_1}}, \frac{\sqrt{n \log n}}{\varepsilon^2 m_1^2} \}) \) samples from \( q \), to distinguish the two distributions with high probability. They also proved a lower bound of \( m_2 = \Omega(\max\{ \frac{\sqrt{n}}{\varepsilon^2}, \frac{n^2}{\varepsilon^4 m_1^2} \}) \). There is a polynomial gap in these upper and lower bounds in the dependence on \( n, \sqrt{m_1} \) and \( \varepsilon \).

As a corollary to our main hypothesis testing result, we obtain an improved algorithm for testing the mixing time of a Markov chain. The idea of testing mixing properties of a Markov chain goes back to the work of Goldreich and Ron [12], which conjectured an algorithm for testing expansion of bounded-degree graphs. Their test is based on picking a random node and testing whether random walks from this node reach a distribution that is close to the uniform distribution on the nodes of the graph. They conjectured that their algorithm had \( O(\sqrt{n}) \) query complexity. Later, Czumaj and Sohler [11], Kale and Seshadhri [15], and Nachmias and Shapira [18] have independently concluded that the algorithm of Goldreich and Ron is provably a test for expansion property of graphs. Rapid mixing of a chain can also be tested using eigenvalue computations. Mixing is related to the separation between the two largest eigenvalues [14, 17],

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and eigenvalues of a dense $n \times n$ matrix can be approximated in $O(n^3)$ time and $O(n^2)$ space. However, for a sparse $n \times n$ symmetric matrix with $m$ nonzero entries, the same task can be achieved in $O((n+m) \log n)$ operations and $O(n+m)$ space. Later, Batu et al. [3] used their $\ell_1$ distance test on the $t$-step distributions, to test mixing properties of Markov chains. Given a finite Markov chain with state space $[n]$ and transition matrix $P = ((P(x,y)))$, they essentially show that one can estimate the mixing time $\tau_{mix}$ up to a factor of $\log n$ using $O(n^{5/3})$ queries to a next node oracle, which takes a state $x \in [n]$ and outputs the state $y \in [n]$ drawn from the probability $P(x, y)$. Such an oracle can often be simulated significantly more easily than actually computing the transition matrix $P(x, y)$.

We conclude this related work section with a comment on “robust” hypothesis testing and distance estimation. A natural hope would be to simply estimate $||p - q||$ to within some additive $\varepsilon$, which is a strictly more difficult task than distinguishing $p = q$ from $||p - q|| \geq \varepsilon$. The results of Valiant and Valiant [26, 27, 29, 28] show that this problem is significantly more difficult than hypothesis testing: the distance can be estimated to additive error $\varepsilon$ for distributions supported on at most $n$ elements using samples of size $O(n/ \log n)$ (in both the setting where either one, or both distributions are unknown). Moreover, $\Omega(n/ \log n)$ samples are information-theoretically necessary, even if $q$ is the uniform distribution over $[n]$, and one wants to distinguish the case that $||p - q|| \leq \frac{1}{\sqrt{n}}$ from the case that $||p - q|| \geq \frac{9}{10}$. Recall that the non-robust test of distinguishing $p = q$ versus $||p - q|| > 9/10$ requires a sample of size only $O(\sqrt{n/\varepsilon})$. The exact worst-case sample complexity of distinguishing whether $||p - q|| \leq \frac{1}{\sqrt{n}}$ versus $||p - q|| \geq \varepsilon$ is not well understood, though in the case of constant $\varepsilon$, up to logarithmic factors, the required sample size seems to scale linearly in the exponent between $n^{2/3}$ and $n$ as $c$ goes from $1/3$ to $0$.

### 1.2 Our results

Our main result resolves the closeness testing problem in the unequal sample setting, to constant factors, in terms of the worst-case distributions of support size $\leq n$:

**Theorem 1.** Given $m_1 \geq n^{2/3}/\varepsilon^{4/3}$ and $\varepsilon > n^{-1/12}$, and sample access to distributions $p$ and $q$ over $[n]$, there is an $O(m_1)$ time algorithm which takes $\Theta(m_1)$ samples from $p$ and $m_2 = O(\max\{\frac{n}{\sqrt{m_1 \varepsilon^2}}, \frac{\sqrt{n}}{\varepsilon^2}\})$ samples from $q$, and with probability at least $2/3$ distinguishes whether

$$||p - q||_1 \leq O\left(\frac{1}{m_2}\right) \quad \text{versus} \quad ||p - q||_1 \geq \varepsilon.$$  \hspace{1cm} (1)

Moreover, given $\Theta(m_1)$ samples from $p$, $\Omega\left(\max\{\frac{n}{\sqrt{m_1 \varepsilon^2}}, \frac{\sqrt{n}}{\varepsilon^2}\}\right)$ samples from $q$ are information-theoretically necessary to distinguish $p = q$ from $||p - q||_1 \geq \varepsilon$ with any constant probability bounded above by $1/2$.

The lower bound in the above theorem is proved using the machinery developed in Valiant [30], and “interpolates” between the $\Theta(\sqrt{n}/\varepsilon^2)$ lower bound in the one-unknown-distribution setting of testing uniformity [22] and the $\Theta(n^{2/3}/\varepsilon^{4/3})$ lowerbound in the setting of equal sample sizes from two unknown distributions [9]. The upper bound is proved in several steps. We begin by proposing two algorithms for the hypothesis testing problem $p = q$ versus $||p - q||_1 > \varepsilon$ depending on the value of $m_1$: the non-extreme regime, that is, $m_1 = O(n/\varepsilon^2)^{1-\gamma}$, and the extreme case where $m_1 = O(n)$. In the non-extreme regime, our algorithm is an extension of the algorithm proposed in [9], and is similar to the algorithm proposed in [3] except with the addition of a normalizing term. In the extreme regime when $m_1 \approx n$, we incorporate an additional statistic that has not appeared before in the literature.$^2$

$^2$We note that a further extension of this algorithm yields a stronger robustness parameter, distinguishing between $||p - q||_1 \leq O\left(\max\{\frac{1}{\varepsilon}, \frac{\sqrt{n}}{\varepsilon}\}\right)$ versus $||p - q||_1 \geq \varepsilon$. 

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As an application of Theorem 1 in the extreme regime when \( m_1 = O(n) \), we obtain an improved algorithm for estimating the mixing time of a Markov chain:

**Corollary 1.** Consider a finite Markov chain with state space \([n]\) and a next node oracle; there is an algorithm that estimates the mixing time, \( \tau_{\text{mix}} \), up to a multiplicative factor of \( \log n \), that uses \( \tilde{O}(n^{3/2}\tau_{\text{mix}}) \) time and queries to the next node oracle.

It remains an intriguing open question whether this query complexity is optimal; we are not aware of any lower bounds beyond the trivial \( \Omega(n\tau_{\text{mix}}) \).

### 1.3 Outline

We begin by stating our testing algorithms, and describe both the intuition behind the algorithms, as well as the high level proof approach. Throughout the theoretical portion of the paper, we will work in the “Poissonized” setting, where we assume that we have access to \( \text{Pois}(m_1) \) samples from distribution \( p \), and \( \text{Pois}(m_2) \) samples drawn distribution \( q \). This assumption that the sample size is a random variable renders the number of occurrences of different domain elements independent. Because \( \text{Pois}(\lambda) \) is tightly concentrated about its expectation, both the upper and lower bounds on the sample complexities proved in this “Poissonized” setting also hold (up to factors of \( 1 \pm o(1) \)) in the setting in which one obtains samples of a fixed size.

The complete proofs require rather involved calculations of the moments of the various statistics employed by our algorithms, and are deferred to Appendix A. The applications of our testing results to the problem of testing or estimating the mixing time of a Markov chain is discussed in Section 3. Finally, Section 4 contains some empirical results, suggesting that the statistic at the core of our algorithms performs very well in practice. This section contains both results on synthetic data, as well as an illustration of how to apply these ideas to the problem of estimating some notion of the semantic similarity of two words based on samples of the \( n \)-grams that contain the words in a corpus of text. The construction and proof of our lower bounds, showing the optimality of our testing algorithms is given in Appendix D.

### 2 Algorithms for \( \ell_1 \) Testing

In this section we describe algorithms for \( \ell_1 \) testing with unequal samples, which give the upper bound in Theorem 1. We propose two algorithms depending on the value of \( m_1 \): the non-extreme regime, that is, \( m_1 = O((n/\varepsilon^2)^{1-\gamma}) \), and the extreme case where \( m_1 \approx n \).

#### 2.1 Algorithms for \( \ell_1 \) Testing: Non-Extreme Case

We begin with the basic algorithm (Algorithm 1), which is optimal in the non-extreme regime, for constant \( \varepsilon \). All the subsequent algorithms are modifications of this basic algorithm.
Algorithm 1 Closeness Testing: Non-Extreme Case (The Basic Algorithm)

Suppose $\varepsilon = \Omega(1)$ and $m_1 = O(n^{1-\gamma})$ for some $\gamma > 0$. Let $S_1, S_2$ denote two independent sets of Pois($m_1$) samples from $p$ and let $T_1, T_2$ denote two independent sets of Pois($m_2$) samples drawn from $q$. We wish to test $p = q$ versus $||p - q||_1 > \varepsilon$.

1. Let $b = \frac{256 \log n}{\varepsilon^2 m_2}$, and define the set $B = \{i \in [n]: \frac{X_i^{S_1}}{m_1} > b\} \cup \{i \in [n]: \frac{Y_i^{T_1}}{m_2} > b\}$, where $X_i^{S_1}$ denotes the number of occurrences of $i$ in $S_1$, and $Y_i^{T_1}$ denotes the number of occurrences of $i$ in $T_1$.

2. Let $X_i$ denote the number of occurrences of element $i$ in $S_2$, and $Y_i$ denote the number of occurrences of element $i$ in $T_2$:

   1. Check if

   $$\sum_{i \in B} \left| \frac{X_i}{m_1} - \frac{Y_i}{m_2} \right| \leq \varepsilon / 6. \tag{2}$$

   2. Check if

   $$\sum_{i \in [n] \setminus B} \frac{(m_2 X_i - m_1 Y_i)^2}{X_i + Y_i} \leq C \gamma m_1^{3/2} m_2, \tag{3}$$

   for an appropriately chosen constant $C\gamma$ (depending on $\gamma$).

3. If (2), and (3) hold, then ACCEPT. Otherwise, REJECT.

The intuition behind the above algorithm is as follows: with high probability, all elements in the set $B$ satisfy either $p_i > b/2$, or $q_i > b/2$ (or both). Given that these elements are “heavy”, their contribution to the $\ell_1$ distance will be accurately captured by the $\ell_1$ distance of their empirical frequencies (where these empirical frequencies are based on the second set of samples, $S_2, T_2$). For the elements that are not in set $B$—the “light” elements—we use a modification of the statistic used by Chan et al. [9], where the terms are re-weighted according to the unequal sample sizes. This is similar to the algorithm proposed in [3], where instead of (3) the authors used the numerator of (3) to distinguish the light elements. However, just using the numerator only gives an estimate of the $\ell_2$ distance between $p$ and $q$. The normalization by $X_i + Y_i$ in (3) “linearizes” the statistic, which gives some estimate of the $\ell_1$ distance between the two distributions for the light elements. Similar results can possibly be obtained by using other linear functions of $X_i$ and $Y_i$ in the denominator, though we note that the “obvious” normalizing factor of $X_i + \frac{m_1}{m_2} Y_i$ does not seem to work theoretically, and seems to have extremely poor performance in practice. Additionally, the unweighted $X_i + Y_i$ normalization is easier to analyze.

Finally, we should emphasize that the crude step of using two independent batches of samples—the first to obtain the partition of the domain into “heavy” and “light” elements, and the second to actually compute the statistics, is for ease of analysis. As our empirical results of Section 4 suggest, for practical applications one might want to use only the $Z$-statistic of (3), and one certainly should not “waste” half the samples to perform the “heavy”/”light” partition.

To get the optimal dependence on $\varepsilon$, the above algorithm needs to be slightly modified. Algorithm 2 gives the optimal sample complexity in the non-extreme case, for any $\varepsilon \geq n^{-\frac{12}{11}}$. We state the algorithm here, as the algorithm in the extreme case where $m_1 \approx n$ and $m_2 \approx \sqrt{n}$ leverages some of its components. The analysis of the algorithm and the proof of the following proposition are given in Appendix B.
Algorithm 2 Asymmetric Closeness Testing: Non-Extreme Case

Suppose $m_1 = O((n/\varepsilon^2)^{1-\gamma}) \leq n$ for some $\gamma > 0$. Let $S_1, S_2$ denote two independent sets of Pois($m_1$) samples from $p$ and let $T_1, T_2$ denote two independent sets of Pois($m_2$) samples drawn from $q$. We wish to test $p = q$ versus $||p - q||_1 > \varepsilon$.

- Let $b = \frac{256 \log n}{\varepsilon^2 m_2}$, and $b' = \frac{256 \log n}{m_2}$, and let $X_{i}^{S_1}$ denote the number of occurrences of $i$ in $S_1$, and $Y_{i}^{T_1}$ denote the number of occurrences of $i$ in $T_1$.

- Define the “heavy” set $B = \{i \in [n] : \frac{X_{i}^{S_1}}{m_1} > b\} \cup \{i \in [n] : \frac{Y_{i}^{T_1}}{m_2} > b\}$.

- Define the “medium” set $M = \left\{\left. i \in [n] \right| b' \leq \max\left\{\frac{X_{i}^{S_1}}{m_1}, \frac{Y_{i}^{T_1}}{m_2}\right\} \leq b\right\}$.

- Define the “light” set $H = [m] \setminus (B \cup M)$.

- Let $X_i$ denote the number of occurrences of element $i$ in $S_2$, and $Y_i$ denote the number of occurrences of element $i$ in $T_2$.

1. Check if

   $$V_B := \sum_{i \in B} V_i := \sum_{i \in B} \left|\frac{X_i}{m_1} - \frac{Y_i}{m_2}\right| \leq \varepsilon / 6. \quad (4)$$

2. Check if

   $$W_M := \sum_{i \in M} W_i := \sum_{i \in M} (m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i) \leq \frac{\varepsilon^2 m_1^2 m_2 \log n}{2}. \quad (5)$$

3. Check if

   $$Z_H := \sum_{i \in H} Z_i := \sum_{i \in H} \frac{(m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i)}{X_i + Y_i} \leq C_\gamma m_1^{3/2} m_2. \quad (6)$$

   Where $C_\gamma$ is an appropriately chosen absolute constant, dependent on $\gamma$.

4. If (4), (5), and (6) hold, then ACCEPT. Otherwise, REJECT.

**Proposition 1.** Suppose $m_1 = O((n/\varepsilon^2)^{1-\gamma}) \leq n$ for some $\gamma > 0$, and $\varepsilon > n^{-1/12}$. Then algorithm (2) takes $\Theta(m_1)$ samples from $p$ and $O(\max\{\frac{n}{\sqrt{m_1 \varepsilon}}, \frac{\sqrt{m_1}}{\varepsilon^2}\})$ samples from $q$, and with probability at least 2/3 distinguishes whether $p = q$ versus $||p - q||_1 \geq \varepsilon$.

### 2.2 Algorithm for $\ell_1$ Testing: Extreme Case

For the extreme case, $m_1 \approx n$ and $m_2 \approx \sqrt{n}$, the re-weighted statistic $Z_H$ might have large variance, necessitating a modification to the algorithm in this extreme case. To see the cause of such variance, consider the case where the samples are drawn from the uniform distribution, Unif[$n$]. By the birthday paradox, we might see a constant number of indices $i$ for which $Y_i = 2$, but $X_i = 0$. Such domain elements themselves
contribute $O(n^4)$ to the variance of $Z_{HH}$, which is at the threshold of what can be tolerated. The statistic (7) introduced below, is tailored to deal with these cases, and captures the intuition that we are more tolerant of indices $i$ for which $Y_i = 2$ if the corresponding $X_i$ is larger.

These modifications allow us to solve the closeness testing problem in the extreme case. In fact, the following algorithm works whenever $\Omega(\left(\frac{n}{\varepsilon^2}\right)^{8/9+\gamma})$, overlapping with the non-extreme case for $\gamma \in (0, 1/9)$.

**Algorithm 3** Asymmetric Closeness Testing: Extreme Case

Suppose $m_1 = \Omega(\left(\frac{n}{\varepsilon^2}\right)^{8/9+\gamma})$ for some $\gamma > 0$. Let $S_1, S_2$ denote two independent sets of Pois($m_1$) samples from $p$ and let $T_1, T_2$ denote two independent sets of Pois($m_2$) samples drawn from $q$. We wish to test $p = q$ versus $\|p - q\|_1 > \varepsilon$.

- Define $b, b', B, M, H$ as in Algorithm 2.
- Let $X_i$ denote the number of occurrences of element $i$ in $S_2$, and $Y_i$ denote the number of occurrences of element $i$ in $T_2$:

1. REJECT if there exists $i \in [n]$ such that $Y_i \geq 3$ and $X_i \leq \frac{m_1 \varepsilon^2 / 3}{10 m_2 n^{17/9}}$.
2. Check if
   \[
   R_H := \sum_{i \in H} \frac{1_{\{Y_i = 2\}}}{X_i + 1} \leq C_1 m_2^2 m_1
   \]
   where $C_1$ is an appropriately chosen absolute constant.
3. If step (1) is not rejected and (2), (3), (4), and (5) are satisfied, then ACCEPT. Otherwise, REJECT.

Proposition 3 below summarizes the performance of the above algorithm. The proof is given in Appendix C.

**Proposition 2.** Suppose $m_1 = \Omega(\left(\frac{n}{\varepsilon^2}\right)^{8/9+\gamma})$ for some $\gamma > 0$ and $\varepsilon > n^{-1/12}$. Then algorithm (3) takes $\Theta(m_1)$ samples from $p$ and $O(\max\{n, \sqrt{n \varepsilon^{-4}}, \sqrt{n \varepsilon^{-2}}\})$ samples from $q$, and with probability at least $2/3$ distinguishes whether $p = q$ versus $\|p - q\|_1 \geq \varepsilon$.

It is worth noting that one can also define a natural analog of the $R_H$ statistic corresponding to the indices $i$ for which $Y_i = 3$, etc., and that the use of such statistics improves the robustness parameter of the test.

### 3 Estimating Mixing in Markov Chains

Consider a finite Markov chain with state space $[n]$, transition matrix $P = ((P(x, y)))$, with stationary distribution $\pi$. The $t$-step distribution starting at the point $x \in [n]$, $P_x^t(\cdot)$ is the probability distribution on $[n]$ obtained by running the chain for $t$ steps starting from $x$. More formally, for $A \subseteq [n]$, $P_x^t(A) = \Pr[X_t \in A|X_0 = x]$, where $(X_0, X_1, \ldots, X_t)$ are the steps of the chain. The $t$-step distribution $P_x^t$ can be computed as a vector matrix product $\vec{e}_x P^t$, where $\vec{e}_x \in \mathbb{R}^n$ is the standard basis vector which has 1 at position $x$ and zeros everywhere else.
Definition 1. The ε-mixing time of a Markov chain with transition matrix \( P = ((P(x, y))) \) is defined as

\[
t_{\text{mix}}(\varepsilon) := \inf \left\{ t \in [n] : \sup_{x \in [n]} \frac{1}{2} \sum_{y \in [n]} |P_x^t(y) - \pi(y)| \leq \varepsilon \right\}.
\]

Definition 2. The average \( t \)-step distribution of a Markov chain \( P \) with \( n \) states is the distribution \( \overline{P}^t = \frac{1}{n} \sum_{x \in [n]} P_x^t(A) \), that is, the distribution obtained by choosing \( x \) uniformly from \([n]\) and walking \( t \) steps from the state \( x \).

As observed by Batu et al. [8], \( \ell_1 \) closeness testing can be used to test whether a Markov chain is close to mixing after some specified number of steps, \( t_0 \). Here, we note that asymmetric closeness testing (as opposed to the case of equal sized samples as employed in [8]), yields an improvement in the performance of the testing algorithm for Markov chain mixing.

The algorithm to test mixing proposed by Batu et al. [8] involves testing the \( \ell_1 \) difference between distributions \( P_x^{t_0} \) and \( \overline{P}^{t_0} \), for every \( x \in [n] \). The algorithm uses their \( \ell_1 \) distance test which draws \( \tilde{O}(n^{2/3} \log n) \) samples from both the distributions \( P_x^{t_0} \) and \( \overline{P}^{t_0} \), and has a overall running time of \( \tilde{O}(n^{5/3} t_0) \). However, the distribution \( \overline{P}^{t_0} \) does not depend to the starting state \( x \) and using Algorithm 3 it suffices to take \( \tilde{O}(n) \) samples from \( \overline{P}^{t_0} \) once and \( \tilde{O}(\sqrt{n}) \) samples from \( P_x^{t_0} \), for every \( x \in [n] \). This results in a query and runtime complexity of \( \tilde{O}(n^{3/2} t_0) \).

Algorithm 4 Testing for Mixing Times in Markov Chains

Given \( t_0 \in \mathbb{R} \) and a finite Markov chain with state space \([n]\) and transition matrix \( P = ((P(x, y))) \), we wish to test

\[
H_0 : t_{\text{mix}}(O\left(\frac{\varepsilon^2}{\sqrt{n}}\right)) \leq t_0, \quad \text{versus} \quad H_1 : t_{\text{mix}}(\varepsilon) > t_0.
\] (8)

1. Draw \( O(\log n) \) samples \( S_1, \ldots, S_{O(\log n)} \), each of size \( \text{Pois}(C_1 n) \) from the average \( t_{0} \)-step distribution.

2. For each state \( x \in [n] \) we will distinguish whether \( ||P_x^{t_0} - \overline{P}^{t_0}||_1 \leq O\left(\frac{\varepsilon^2}{\sqrt{n}}\right) \), versus \( ||P_x^{t_0} - \overline{P}^{t_0}||_1 > \varepsilon \), with probability of error \( < 1/n \). We do this by running \( O(\log n) \) runs of Algorithm 3 with the \( i \)-th run using \( S_i \) and a fresh set of \( \text{Pois}(O(\varepsilon^{-2} \sqrt{n})) \) samples from \( P_x^{t_0} \).

3. If all \( n \) of the \( \ell_1 \) closeness testing problems are accepted, then we ACCEPT \( H_0 \).

The above testing algorithm can be leveraged to estimate the mixing time of a Markov chain, via the basic observation that if \( t_{\text{mix}}(1/4) \leq t_0 \), then for any \( \varepsilon, t_{\text{mix}}(\varepsilon) \leq \frac{\log \varepsilon}{\log 1/2} t_0 \), and thus \( t_{\text{mix}}(1/\sqrt{n}) \leq 2 \log n \cdot t_{\text{mix}}(1/4) \). Because \( t_{\text{mix}}(1/4) \) and \( t_{\text{mix}}(O(1/\sqrt{n})) \) differ by at most a factor of \( \log n \), by applying Algorithm 4 for a geometrically increasing sequence of \( t_0 \)'s, and repeating each test \( O(\log t_0 + \log n) \) times, one obtains Corollary 1.

4 Empirical Results

Both our formal algorithms and the corresponding theorems involve some unwieldy constant factors (that can likely be reduced significantly). Nevertheless, in this section we provide some evidence that the statistic at the core of our algorithms can be fruitfully used in practice, even for surprisingly small sample sizes.
4.1 Testing similarity of words

An extremely important primitive in natural language processing is the ability to estimate the semantic similarity of two words. Here, we show that the Z statistic, 

\[ Z = \sum_i \left( \frac{(m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i)}{m_1^{3/2} m_2 (X_i + Y_i)} \right), \]

which is the core of our testing algorithms, can accurately distinguish whether two words are very similar based on surprisingly small samples of the contexts in which they occur. Specifically, for each pair of words, \( a, b \) that we consider, we select \( m_1 \) random occurrences of \( a \) and \( m_2 \) random occurrences of word \( b \) from the Google books corpus, using the Google Books Ngram Dataset.\(^3\) We then compare the sample of words that follow \( a \) with the sample of words that follow \( b \). Henceforth, we refer to these as samples of the set of bi-grams involving each word, although for convenience, we only considered the bigrams whose first word was the word in question.

Figure 1 illustrates the Z statistic for various pairs of words that range from rather similar words like “smart” and “intelligent”, to essentially identical word pairs such as “grey” and “gray” (whose usage differs mainly as a result of historical variation in the preference for one spelling over the other); the sample size of bi-grams containing the first word is fixed at \( m_1 = 1,000 \), and the sample size corresponding to the second word varies from \( m_2 = 50 \) through \( m_2 = 1,000 \). To provide a frame of reference, we also compute the value of the statistic for independent samples corresponding to the same word (i.e. two different samples of words that follow “wolf”); these are depicted in red. For comparison, we also plot the total variation distance between the empirical distributions of the pair of samples, which does not clearly differentiate between pairs of identical words, versus different words, particularly for the smaller sample sizes.

One subtle point is that the issue with using the empirical distance between the distributions goes beyond simply not having a consistent reference point. For example, let \( X \) denote a large sample of size \( m_1 \) from distribution \( p \), \( X' \) denote a small sample of size \( m_2 \) from \( p \), and \( Y \) denote a small sample of size \( m_2 \) from a different distribution \( q \). It might be tempting to hope that the empirical distance between \( X \) and \( X' \) will be smaller than the empirical distance between \( X \) and \( Y \). As Figure 2 illustrates, this is not always the case, even for natural distributions: for this specific example, over much of the range of \( m_2 \), the empirical distance between \( X \) and \( X' \) is indistinguishable from that of \( X \) and \( Y \), and yet, as our statistic easy discerns, these distributions are very different.

This point is further emphasized in Figure 3, which depicts this phenomena in the synthetic setting where \( p = \text{Unif}[5, 000] \) is the uniform distribution over 5,000 elements, and \( q \) is the distribution whose elements have probabilities \((1 \pm \varepsilon)/5000\), for \( \varepsilon = 1/4 \). The right plot represents the empirical probability that the distance between two empirical distributions of the samples from \( p \) is larger than the distance between the empirical distributions of the samples from \( p \) and \( q \); the left plot represents the analogous probability involving the Z statistic. In both plots, \( m_1 \) ranges between \( n^{2/3} \) and \( n \), and \( m_2 \) ranges between \( n^{1/2} \) and \( n \), for \( n = 5,000 \).

\(^3\)The Google Books Ngram Dataset is freely available here: [http//storage.googleapis.com/books/ngrams/books/datasetsv2.html](http://storage.googleapis.com/books/ngrams/books/datasetsv2.html)
Figure 1: Two measures of the similarity between words, based on samples of the bi-grams containing each word. Each line represents a pair of words, and is obtained by taking a sample of $m_1 = 1,000$ bi-grams containing the first word, and $m_2 = 50, \ldots, 1,000$ bi-grams containing the second word, where $m_2$ is depicted along the $x$-axis in logarithmic scale. In both plots, the red lines represent pairs of identical words (e.g. “wolf/wolf”, “almost/almost”, \ldots). The blue lines represent pairs of similar words (e.g. “wolf/fox”, “almost/nearly”, \ldots), and the black line represents the pair “grey/gray” whose distribution of bi-grams differ because of historical variations in preference for each spelling. Solid lines indicate the average over 200 trials for each word pair and choice of $m_2$, with error bars of one standard deviation depicted. The left plot depicts our statistic, which clearly distinguishes identical words, and demonstrates some intuitive sense of semantic distance. The right plot depicts the total variation distance between the empirical distributions—which does not successfully distinguish the identical words, given the range of sample sizes considered. The plot would not be significantly different if other distance metrics between the empirical distributions, such as f-divergence, were used in place of total variation distance. Finally, note the extremely uniform magnitudes of the error bars in the left plot, as $m_2$ increases, which is a result of the $X_i + Y_i$ normalization term in the $Z$ statistic.
Figure 2: Illustration of how the empirical distance can be misleading: here, the empirical distance between the distributions of samples of bi-grams for “wolf/wolf” is indistinguishable from that for the pair “wolf/fox*” over much of the range of $m_2$; nevertheless, our statistic clearly discerns that these are significantly different distributions. Here, “fox*” denotes the distribution of bi-grams whose first word is “fox”, restricted to only the most common 100 bi-grams. As in Figure 1, $m_1 = 1,000$, and $m_2$ ranges from 50 to 1,000, with solid lines depicted the average of 200 trials, and error bars depicting one standard deviation.

Figure 3: A comparison of the $Z$ statistic versus the empirical distribution for distinguishing whether two samples of respective sizes $m_1, m_2$, were both drawn from distribution $p := \text{Unif}[5,000]$, versus one sample being drawn from $p$ and the other drawn from a distribution $q$ in which domain elements have probability $(1 \pm \varepsilon)/5000$, for $\varepsilon = 1/4$, and hence $||p - q|| = 1/4$. The color signifies the fraction of 120 repetitions for which the statistic correctly distinguishes these cases, as $m_1$ varies between $n^{2/3}$ and $n$, and $m_2$ varies between $n^{1/2}$ and $n$. 
References

[1] J. Acharya, H. Das, A. Jafarpour, A. Orlitsky, and S. Pan, Competitive closeness testing, Conference on Learning Theory (COLT), 2011.

[2] J. Acharya, H. Das, A. Jafarpour, A. Orlitsky, and S. Pan, Competitive classification and closeness testing. Proc. 25th Conference on Learning Theory (COLT), 2012.

[3] J. Acharya, A. Jafarpour, A. Orlitsky, and A. T. Suresh, Sublinear algorithms for outlier detection and generalized closeness testing, Proceedings of the International Symposium on Information Theory (ISIT), 3200–3204, 2014.

[4] Z. Bar-Yossef, R. Kumar, and D. Sivakumar. Sampling algorithms: lower bounds and applications Symposium on Theory of Computing (STOC), 2001.

[5] T. Batu, L. Fortnow, R. Rubinfeld, W. D. Smith, and P. White, Testing that distributions are close, IEEE Symposium on Foundations of Computer Science (FOCS), 259–269, 2000.

[6] T. Batu, S. Dasgupta, R. Kumar, and R. Rubinfeld, The complexity of approximating the entropy, SIAM Journal on Computing, 2005.

[7] T. Batu, E. Fischer, L. Fortnow, R. Kumar, R. Rubinfeld, and P. White, Testing random variables for independence and identity, IEEE Symposium on Foundations of Computer Science (FOCS), 2001.

[8] T. Batu, L. Fortnow, R. Rubinfeld, W. D. Smith, and P. White, Testing closeness of discrete distributions, J. ACM, Vol. 60 (1), 4, 2013.

[9] S.-on Chan, I. Diakonikolas, P. Valiant, G. Valiant, Optimal Algorithms for Testing Closeness of Discrete Distributions, Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA), 1193–1203, 2014.

[10] M. Charikar, S. Chaudhuri, R. Motwani, and V.R. Narasayya, Towards estimation error guarantees for distinct values, Symposium on Principles of Database Systems (PODS), 2000.

[11] A. Czumaj and C. Sohler, Testing expansion in bounded-degree graphs, IEEE Symposium on Foundations of Computer Science (FOCS), 570–578, 2007.

[12] O. Goldreich and D. Ron, On testing expansion in bounded-degree graphs, Technical Report TR00-020, Electronic Colloquium on Computational Complexity, 2000.

[13] S. Guha, A. McGregor, and S. Venkatasubramanian, Streaming and sublinear approximation of entropy and information distances, Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA), 2006.

[14] A. Sinclair and M. Jerrum, Approximate counting, uniform generation and rapidly mixing Markov chains, Information and Computation, Vol. 82(1), 93–133, 1989.

[15] S. Kale and C. Seshadhri, An expansion tester for bounded degree graphs, ICALP (1), Lecture Notes in Computer Science, Vol. 5125, 527–538, 2008.
[16] A. Keinan and A. G. Clark. Recent explosive human population growth has resulted in an excess of rare genetic variants. *Science*, 336(6082):740743, 2012.

[17] D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov Chains and Mixing Times*, Amer. Math. Soc., Providence, RI, 2009.

[18] A. Nachmias and A. Shapira, Testing the expansion of a graph, *Electronic Colloquium on Computational Complexity (ECCC)*, Vol. 14 (118), 2007.

[19] M. R. Nelson and D. Wegmann et al., An abundance of rare functional variants in 202 drug target genes sequenced in 14,002 people. *Science*, 337(6090):100104, 2012.

[20] L. Paninski, Estimation of entropy and mutual information, *Neural Computation*, Vol. 15 (6), 1191–1253, 2003.

[21] L. Paninski, Estimating entropy on m bins given fewer than m samples, *IEEE Transactions on Information Theory*, Vol. 50 (9), 2200–2203, 2004.

[22] L. Paninski, A coincidence-based test for uniformity given very sparsely-sampled discrete data, *IEEE Transactions on Information Theory*, Vol. 54, 4750–4755, 2008.

[23] S. Raskhodnikova, D. Ron, A. Shpilka, and A. Smith, Strong lower bounds for approximating distribution support size and the distinct elements problem, *SIAM Journal on Computing*, Vol. 39(3), 813–842, 2009.

[24] R. Rubinfeld, Taming big probability distributions, *XRDS*, Vol. 19(1), 24–28, 2012.

[25] J. A. Tennessen, A.W. Bigham, and T.D. O’Connor et al. Evolution and functional impact of rare coding variation from deep sequencing of human exomes. *Science*, 337(6090):6469, 2012

[26] G. Valiant and P. Valiant, Estimating the unseen: an n/ log n-sample estimator for entropy and support size, shown optimal via new CLTs, *Proceedings of the ACM Symposium on Theory of Computing (STOC)*, 2011.

[27] G. Valiant and P. Valiant, Estimating the unseen: improved estimators for entropy and other properties, *Neural Information Processing Systems (NIPS)*, 2013.

[28] G. Valiant and P. Valiant, The power of linear estimators, *IEEE Symposium on Foundations of Computer Science (FOCS)*, 2011.

[29] G. Valiant and P. Valiant, An Automatic Inequality Prover and Instance Optimal Identity Testing, *IEEE Symposium on Foundations of Computer Science (FOCS)*, 51–60, 2014.

[30] P. Valiant, Testing symmetric properties of distributions, *Symposium on Theory of Computing (STOC)*, 2008.

[31] P. Valiant, *Testing Symmetric Properties of Distributions*, PhD thesis, M.I.T., 2008.
A  Expectation and Variance Bounds

Before beginning the analysis of the algorithms we need bounds on the expectation and variance of the different statistics used in the algorithms. Throughout this section, fix any set \( A \subseteq [n] \), and let \( X_i \) denote the number of occurrences of the \( i \)-th domain element in set \( S_2 \)—a set of \( \text{Pois}(m_1) \) samples from distribution \( p \), and analogously let \( Y_i \) denote the number of occurrences of the \( i \)-th domain element in set \( T_2 \)—a set of \( \text{Pois}(m_2) \) samples from distribution \( q \). Throughout this section, we bound the moments of the following statistics:

- \( V_A = \sum_{i \in A} V_i = \sum_{i \in A} \left| \frac{X_i}{m_1} - \frac{Y_i}{m_2} \right| \).
- \( W_A = \sum_{i \in A} W_i = \sum_{i \in A} \left( (m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i) \right) \).
- \( Z_A = \sum_{i \in A} Z_i = \sum_{i \in A} \frac{(m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i)}{X_i + Y_i} \).

A.1 Expectation and Variance of \( V_A \)

Lemma 1. For any fixed set \( A \subseteq [n] \),

\[
\sum_{i \in A} |p_i - q_i| \leq \mathbb{E}[V_A] \leq \sum_{i \in A} |p_i - q_i| + \left( \frac{|A|}{m_1} + \frac{|A|}{m_2} \right)^{\frac{1}{2}} \leq \sum_{i \in A} |p_i - q_i| + \left( \frac{2|A|}{m_2} \right)^{\frac{1}{2}},
\]

and

\[
\text{Var}[V_A] \leq \frac{1}{m_1} + \frac{1}{m_2}.
\]

Proof. For the lower bound on the expectation, note that \( \mathbb{E} \left[ \left| \frac{X_i}{m_1} - \frac{Y_i}{m_2} \right| \right] \geq \mathbb{E} \left[ \frac{X_i}{m_1} - \frac{Y_i}{m_2} \right] = |p_i - q_i| \).

To prove the upper bound, observe that

\[
\mathbb{E}[V_i^2] = \frac{p_i}{m_1} + \frac{q_i}{m_2} + (p_i - q_i)^2.
\]

By the Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ \sum_{i \in A} V_i \right] \leq \sum_{i \in A} \mathbb{E}[V_i^2]^{\frac{1}{2}} \leq \sum_{i \in A} |p_i - q_i| + \sum_{i \in A} \left( \frac{p_i}{m_1} + \frac{q_i}{m_2} \right)^{\frac{1}{2}} \leq \sum_{i \in A} |p_i - q_i| + \left( \frac{|A|}{m_1} + \frac{|A|}{m_2} \right)^{\frac{1}{2}}.
\]

Finally, \( \text{Var}[V_A] = \sum_{i \in A} (\mathbb{E}[V_i^2] - \mathbb{E}[V_i])^2 \leq \sum_{i \in A} \frac{p_i}{m_1} + \sum_{i \in A} \frac{q_i}{m_2} \leq \frac{1}{m_1} + \frac{1}{m_2} \). \( \Box \)

A.2 Expectation and Variance of \( W_A \)

For \( A \subseteq [n] \), define \( W_A = \sum_{i \in A} W_i = \sum_{i \in A} (m_2 X_i - m_1 Y_i)^2 - (m_2^2 X_i + m_1^2 Y_i) \). Using the facts that \( X_i \sim \text{Pois}(m_1 p_i) \) and \( Y_i \sim \text{Pois}(m_2 q_i) \) and plugging in the expressions for the moments of Poissons, the following lemma follows immediately:
**Lemma 2.** For any $A \subseteq [n]$, $W_A/(m_1^2m_2^2)$ is an unbiased estimate of $\|p_A - q_A\|_2^2$. Namely,

$$\mathbb{E}[W_A] = m_1^2m_2^2 \sum_{i \in A} (p_i - q_i)^2, \quad (12)$$

Moreover,

$$\text{Var}[W_A] = 2m_1^2m_2^2 \sum_{i \in A} z_i^2 + 4m_1^3m_2^2 \sum_{i \in A} z_i(p_i - q_i)^2, \quad (13)$$

where $z_i = m_2p_i + m_1q_i$.

**A.3 Moments of $Z_A$**

Recall that

$$Z_i := \frac{(m_2X_i - m_1Y_i)^2 - (m_2^2X_i + m_1^2Y_i)}{X_i + Y_i},$$

and for $A \subseteq [n]$, $Z_A := \sum_{i \in A} Z_i$. We show that if $p = q$, then $\mathbb{E}[\sum_{i \in A} Z_i] = 0$, and otherwise, we give a lower bound on the expectation of the sum:

**Lemma 3.** If $p = q$, then $\mathbb{E}[\sum_{i \in A} Z_i] = 0$, and otherwise, $\mathbb{E}[\sum_{i \in A} Z_i] \geq \frac{m_1^2m_2(\sum_{i \in A} |p_i - q_i|)^2}{4n + m_1 + m_2}$.

**Proof.** Conditioned on the denominator,

$$X_i, X_i + Y_i = \sigma \sim \text{Bin} \left( \sigma, \frac{m_1p_i}{m_1p_i + m_2q_i} \right).$$

Set $\beta_i = \frac{m_1p_i}{m_1p_i + m_2q_i}$. Then using binomial moments we get,

$$\mathbb{E}[\left( (m_2X_i - m_1Y_i)^2 \right) | X_i + Y_i = \sigma] = \sigma^2(1 - \beta_i)(m_1 + m_2)^2 + \sigma^2(m_2\beta_i - m_1(1 - \beta_i))^2$$

$$= \left( m_1 + m_2 \right)^2 \left( \sigma\beta_i(1 - \beta_i) + \sigma^2 \left( \frac{m_1}{m_1 + m_2} - \beta_i \right)^2 \right). \quad (14)$$

Similarly,

$$\mathbb{E}[m_2^2X_i + m_1^2Y_i | X_i + Y_i = \sigma] = m_1^2\sigma + (m_2^2 - m_1^2)\mathbb{E}[X_i | X_i + Y_i = \sigma]$$

$$= m_1^2\sigma + (m_2^2 - m_1^2)\sigma\beta_i$$

Therefore, the conditional expectation of the numerator is

$$\mathbb{E} \left[ (m_2X_i - m_1Y_i)^2 - (m_2^2X_i + m_1^2Y_i) \right] | X_i + Y_i = \sigma] = (m_1 + m_2)^2(\sigma - 1) \left( \frac{m_1}{m_1 + m_2} - \beta_i \right)^2$$

$$= \sigma(\sigma - 1) \left( \frac{m_1m_2(q_i - p_i)}{m_1p_i + m_2q_i} \right)^2. \quad (15)$$

This implies

$$\mathbb{E} \left[ \sum_{i \in A} Z_i/m_1^2m_2^2 \right] = \sum_{i \in A} \frac{(q_i - p_i)^2}{z_i} \left( 1 - \frac{1 - e^{-z_i}}{z_i} \right),$$

where $z_i = m_1p_i + m_2q_i$. This implies that the expectation of the sum is zero if $p = q$. Let $g(z) = z/(1 - 1/e^{-z})$. Now, using the fact that $g(z) \leq 2 + z$ and the Cauchy-Schwarz inequality, the result follows. \qed
Lemma 4. For \( i \in [n] \) and \( p = q \),
\[
\text{Var}[Z_i] \leq 2m_1^2m_2^2 \text{Pr}[X_i + Y_i > 0], \text{ and hence } \text{Var}[Z_A] = O(m_1^3m_2^2).
\]

For \( p_i \geq q_i \), \( \text{Var}[Z_i] \leq O(m_1^3m_2^2p_i) \), and for \( p_i < q_i \)
\[
\text{Var}[Z_i] \leq O(m_1^3m_2^2) \min \left\{ \frac{q_i^2}{p_i}, m_1q_i^2 \right\}.
\]

(16)

Proof. The variance of \( Z_i \) can be computed by using the formula for conditional variance. Define,
\[
G_i(\sigma) := \text{Var}[(m_2 X_i - m_1 Y_i)^2 - (m_2 X_i + m_1 Y_i)|X_i + Y_i = \sigma].
\]
Let \( \beta_i = \frac{m_1 p_i}{m_1 p_i + m_2 q_i} \). Using formulas for binomial moments the conditional variance
\[
G_i(\sigma) = F_i(\sigma) + L_i(\sigma),
\]
where
\[
F_i(\sigma) = 2\beta_i^2(1-\beta_i)^2\sigma(-1)(m_1+m_2)^4, \quad L_i(\sigma) = 4\beta_i(1-\beta_i)\sigma(\sigma-1)(m_1+m_2)^4 \left( \frac{m_1}{m_1 + m_2} - \beta_i \right)^2.
\]

For \( p_i = q_i \), \( \beta_i = \frac{m_1}{m_1 + m_2} \) and \( L_i(\sigma) = 0 \). Also, from the proof of Lemma 4 it can be seen that \( \text{Var}[E[Z_i|X_i + Y_i = \sigma]] = 0 \), when \( p_i = q_i \). Therefore, for \( p_i = q_i \),
\[
\text{Var}[Z_i] = \text{E}[G_i(\sigma)/\sigma^2] = \text{E}[F_i(\sigma)/\sigma^2] \leq 2m_1^2m_2^2 \text{Pr}[X_i + Y_i > 0].
\]

Let \( z_i = m_1 p_i + m_2 q_i \). Then \( \text{Pr}[X_i + Y_i > 0] = 1 - e^{-z_i} \leq z_i \), and \( \text{Var}[Z_A] = \sum_{i \in A} \text{Var}[Z_i] = O(m_1^3m_2^2) \).

To prove the bound in the case \( p_i \neq q_i \), note that \( F_i(\sigma) = 0 \), for \( \sigma = 0, 1 \) and \( F_i(\sigma) \leq 2\beta_i^2(1-\beta_i)^2\sigma^2(m_1+m_2)^4 \), for \( \sigma \geq 2 \). Therefore,
\[
\text{E} \left( \frac{F_i(\sigma)}{\sigma^2} \right) \leq 2(m_1 + m_2)^4 \beta_i^2(1-\beta_i)^2 \text{Pr}[\sigma \geq 2] \leq 2(m_1m_2)^2(m_1 + m_2)^4 \left\{ \frac{p_i^2q_i^2(1 - e^{-z_i} - z_ie^{-z_i})}{z_i^4} \right\} \leq O(m_1^3m_2^2) \left\{ \frac{p_i^2q_i^2}{z_i^4} \min \{ z_i, z_i^2 \} \right\}.
\]

(17)

Now, for \( p_i \geq q_i \), \( z_i \geq \frac{m_1+m_2}{2}(p_i + q_i) \), and
\[
\text{E} \left( \frac{F_i(\sigma)}{\sigma^2} \right) \leq O(m_1^3m_2^2) \left\{ \frac{p_i^2q_i^2}{z_i^4} \min \{ z_i, z_i^2 \} \right\} \leq O(m_1^3m_2^2) \left\{ \frac{p_i^2q_i^2}{(p_i + q_i)^3} \right\} \leq O(m_1^3m_2^2p_i).
\]

The remaining terms in the variance can be bounded similarly, and for \( p_i < q_i \), it follows that \( \text{Var}[Z_i] \leq O(m_1^3m_2^2p_i) \).

For the case \( p_i < q_i \), use the bound \( z_i \geq m_1 p_i \) in (17) to get
\[
\text{E} \left[ \frac{F_i(\sigma)}{\sigma^2} \right] \leq O(m_1^3m_2^2) \min \left\{ \frac{q_i^2}{p_i}, m_1q_i^2 \right\}.
\]

(18)
Similarly, $L_i(\sigma) = 0$ for $\sigma = 0, 1$ and $L_i(\sigma) \leq 4\beta_i(1 - \beta_i)\sigma^3(m_1 + m_2)^4\left(\frac{m_1}{m_1 + m_2} - \beta_i\right)^2$. Therefore, for the case $p_i < q_i$, using the bound $z_i^3 \geq m_1^2m_2^2q_i^2$, for $z_i \leq 1$, and $z_i^2 \geq m_1m_2p_iq_i$, for $z_i \geq 1$ we get

$$\begin{align*}
\mathbb{E}\left(\frac{L_i(\sigma)}{\sigma^2}\right) &\leq 4(m_1 + m_2)^4\beta_i(1 - \beta_i)\left(\frac{m_1}{m_1 + m_2} - \beta_i\right)^2 \mathbb{E}[\sigma 1\{\sigma \geq 2\}] \\
&= 4m_1^3m_2^3(m_1 + m_2)^2p_iq_i(p_i - q_i)^2z_i(1 - e^{-z_i}) \\
&\leq O(m_1^3m_2^3)p_iq_i(p_i - q_i)^2 \min\{1, z_i\} \\
&= O(m_1^3m_2^3) \min\left\{\frac{q_i^2}{p_i}, m_1q_i^2\right\}. 
\end{align*}$$

(19)

Finally, from Lemma 3 when $p_i < q_i$

$$\begin{align*}
\text{Var}[\mathbb{E}[Z_i|X_i + Y_i = \sigma]] &\leq (m_1 + m_2)^2 \text{Var}[\sigma] \left(\frac{m_1}{m_1 + m_2} - \beta_i\right)^2 \\
&= m_1^4m_2^4(q_i - p_i)^4 \frac{s^2}{z_i^3} \\
&\leq O(m_1^3m_2^3) \min\left\{\frac{q_i^2}{p_i}, m_1q_i^2\right\}. 
\end{align*}$$

(20)

Combining (18), (19), and (20), the variance (16) follows.

For the analysis of the algorithms we also need bounds on the $s$-th moment of $Z_A$ corresponding to a set $A$ with the property that for all $i \in A$, $p_i \leq 2b'$ and $q_i \leq 2b'$, where $b' = \frac{256\log n}{m_2}$, as define in Algorithm 2.

**Lemma 5.** For any $s \in \mathbb{N}$, and set $A \subset [n]$ such that for all $i \in A$, $p_i \leq 2b'$ and $q_i \leq 2b'$,

$$\mathbb{E}[|Z_A - \mathbb{E}[Z_A]|^s] \leq \tilde{O}_s(m_1^2m_2),$$

where $\tilde{O}_s$ suppresses factor of $\log^{O(s)} n$.

**Proof.** Trivially, $|Z_i| \leq 3m_2^2X_i + 3m_1^2Y_i$. Since $\mathbb{E}[X_i^s]$ is a degree $s$ polynomial in $m_1p_i$, $\mathbb{E}[X_i^s] = O_s(\max\{m_1^sp_i^s, m_1p_i\})$. Similarly, for $\mathbb{E}[Y_i^s] = O_s(\max\{m_2^sq_i^s, m_2q_i\})$. Therefore, for $i \in A$,

$$\begin{align*}
\mathbb{E}[|Z_i|^s] &= O_s(m_2^{2s}\mathbb{E}[X_i^s] + m_1^{2s}\mathbb{E}[Y_i^s]) \\
&= \tilde{O}_s(m_1^2m_2^2 \max\{p_i, q_i\}). 
\end{align*}$$

(21)

Similarly, $\mathbb{E}[|Z_i|^s] = \tilde{O}_s(m_1^{2s}m_2 \max\{p_i, q_i\})$, and

$$\begin{align*}
\mathbb{E}[|Z_A - \mathbb{E}[Z_A]|^s] &\leq O_s\left(\sum_{i \in A} \mathbb{E}[|Z_i|^s] + \mathbb{E}[|Z_i|^s]\right) \\
&\leq \tilde{O}_s(m_1^{2s}m_2). 
\end{align*}$$

(22)

Combining (21) and (26) yields the lemma.

For the analysis of the algorithm in the extreme case, we will bounds on the $s$-th moment of $Z_A$ corresponding to a set $A$ with the property that for all $i \in A$, $\frac{\epsilon^{2/3}}{20m_2n^{1/3}} \leq p_i \leq 2b'$ and $q_i \leq 2b'$. In this case, a more careful analysis gives a better bound on moments of $Z_A$. 

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Lemma 6. For any $s \in \mathbb{N}$, and set $A \subset [n]$ such that for all $i \in A$, $\epsilon^{2/3} \leq p_i \leq 2b'$ and $q_i \leq 2b'$, 
\[
\mathbb{E}[|Z_A - \mathbb{E}[Z_A]|^s] \leq \tilde{O}_s \left( \frac{n^{s/3}m_1^sm_2^{s+1}}{q_i^{2s/3}} \right),
\]
where $\tilde{O}_s$ suppresses factor of $\log^{O(s)} n$.

Proof. From the definition $Z_i$, 
\[
|Z_i| \leq O \left( \frac{m_2^2X_i^2 + m_1^2Y_i^2}{X_i + Y_i} \right).
\]
Conditioned on $X_i + Y_i = \sigma$, $X_i \sim \text{Bin}(\sigma, m_1p_i/z_i)$ and $Y_i \sim \text{Bin}(\sigma, m_2q_i/z_i)$, where $z_i = m_1p_i + m_2q_i$. Then, $\mathbb{E}[X_i] = \sigma m_2q_i/z_i := x_i$, and for any $s \geq 1$,
\[
\mathbb{E}[X_i^s|X_i + Y_i = \sigma] = O(\max\{x_i^s, y_i^s\}).
\]
Similarly,
\[
\mathbb{E}[Y_i^s|X_i + Y_i = \sigma] = O(\max\{y_i^s, y_i^s\}) \text{ where } \mathbb{E}[Y_i] = \sigma m_2q_i/z_i := y_i.
\]
Therefore, for $\sigma > 0$,
\[
\mathbb{E}[|Z_i|^s|X_i + Y_i = \sigma] \leq O_s \left( \max \left\{ \frac{m_2^2m_2^2q_i^{2s}}{z_i^{2s}}, \frac{m_1^2m_2q_i^s}{\sigma - 1, z_i^{s+1}} \right\} \right) \leq O_s \left( \max \left\{ \frac{m_1^2m_2^2q_i^{2s}}{z_i^{2s}}, \frac{m_1^2m_2q_i^s}{z_i^{s+1}} \right\} \right). \tag{23}
\]
Note that $\mathbb{E}[\sigma] = z_i$ and $\mathbb{E}[\sigma^s] = O_s(z_i^s)$ because $z_i \geq 1$ by assumption. Using $q_i \leq 2b'$ we get
\[
O_s \left( \frac{m_1^2m_2^2q_i}{z_i^{s+1}} \right) \leq O_s \left( \frac{m_1^2m_2^2q_i^{2s}}{p_i^{s+1}} \right) \leq O_s \left( \frac{m_1^2m_2^2q_i^{2s}}{p_i^s} \right) = \tilde{O}_s \left( \frac{m_1^2m_2^2q_i}{p_i^s} \right). \tag{24}
\]
Moreover, because $m_1p_i \geq 1$,
\[
O_s \left( \frac{m_1^2m_2q_i}{z_i^{s+1}} \right) \leq O_s \left( \frac{m_1^2m_2q_i^{-1}}{p_i^{s+1}} \right) \leq O_s \left( \frac{m_1^2m_2q_i}{p_i^s} \right). \tag{25}
\]
Combining (24) and (25) with (23) and using $p_i \geq \epsilon^{2/3} / 20m_2n^{2/3}$ (since $i \in A$) gives
\[
\mathbb{E}[|Z_i|^s] \leq \tilde{O}_s \left( \frac{m_1^2m_2q_i}{p_i^s} \right) \leq \tilde{O}_s \left( \frac{n^{s/3}m_1^s m_2^{s+1} q_i}{\epsilon^{2s/3}} \right).
\]
Similarly, it can be shown that $\mathbb{E}[|Z_i|^s] = \tilde{O}_s \left( \frac{n^{s/3}m_1^s m_2^{s+1} q_i}{\epsilon^{2s/3}} \right)$, and
\[
\mathbb{E}[|Z_A - \mathbb{E}[Z_A]|^s] \leq O_s \left( \sum_{i \in A} \mathbb{E}[|Z_i|^s] + \mathbb{E}[|Z_i|^s] \right) \leq \tilde{O}_s \left( \frac{n^{s/3}m_1^s m_2^{s+1}}{\epsilon^{2s/3}} \right). \tag{26}
\]
completing the proof of the lemma. 

\[\square\]
B Proof of Proposition 1

We begin by establishing that, with high probability over the first set of samples, $S_1, T_1$, the sets $B, M, H$ successfully partition the elements in the “heavy”, “medium”, and “light” sets. This proof follows from a union bound over Poisson tail bounds. The proof of Proposition 1 will then proceed by arguing that, with high probability over the randomness of the second set of samples, $S_2, T_2$, the algorithm will be successful, provided that the sets $B, M, H$, were a reasonable partition.

**Definition 3.** Let $b, b'$ be as defined in Algorithm 2. The set $B$ is said to be faithful if for all $i \in B$, $p_i > b/2$ or $q_i > b/2$. Similarly, $M$ is said to be faithful if for all $i \in M$, $b'/2 \leq \max\{p_i, q_i\} \leq 2b$. Finally, $H$ is said to be faithful if for all $i \in H$, $p_i < 2b'$ and $q_i < 2b'$, for all $i \in H$.

**Lemma 7.** With probability at $1 - o(1/n)$ over the randomness in the samples $S_1, T_1$, the sets $B, M, H$ will be “faithful”.

**Proof.** We leverage the following Chernoff style bound for Poisson distributions: for any $\lambda \leq c$, and $\delta \in (0,1)$,
\[
\Pr\left[|\text{Pois}(\lambda) - \lambda| > \delta \lambda\right] \leq 2e^{-\delta^2 \lambda / 3}.
\]
Let $X_i^{S_1}$ denote the number of occurrences of $i$ in the Pois($m_1$) samples, $S_1$, drawn from $p$, and $Y_i^{T_1}$ denote the number of occurrences of $i$ in the Pois($m_2$) samples from $q$ that comprise $T_1$. For any domain element $i$ with probability $p_i \geq b'/2$,
\[
\Pr\left[|X_i^{S_1} - m_1 p_i| \geq \frac{1}{2} m_1 p_i\right] \leq 2e^{-\frac{1}{4} m_1 p_i} \leq 2e^{-20 \log n} = o(1/n^2).
\]
Similarly, for any domain element $i$ with probability $q_i \geq b'/2$,
\[
\Pr\left[|Y_i^{T_1} - m_2 q_i| \geq \frac{1}{2} m_2 q_i\right] \leq 2e^{-\frac{1}{4} m_2 q_i} \leq 2e^{-20 \log n} = o(1/n^2).
\]
So far, this ensures that common elements do not occur too infrequently. To ensure that none of the rare elements occur too frequently, note that the same bound implies that for any domain element $i$ with probability $p_i \leq b'/2$,
\[
\Pr\left[X_i^{S_1} \geq b' m_1\right] \leq \Pr\left[|X_i^{S_1} - m_1 p_i| \geq b' m_1/2\right] \leq 2e^{-b' m_1/6} \leq 2e^{-20 \log n} = o(1/n^2).
\]
Analogously for any domain element $i$ with probability $q_i \leq b'/2$,
\[
\Pr\left[Y_i^{T_1} \geq b' m_2\right] \leq \Pr\left[|Y_i^{T_1} - m_2 q_i| \geq b' m_2/2\right] \leq 2e^{-b' m_2/6} \leq 2e^{-20 \log n} = o(1/n^2).
\]
Note that if, for all domain elements $i$ with $p_i \geq b'/2$, $|X_i^{S_1} - m_1 p_i| < \frac{1}{2} m_1 p_i$, and for all elements $i$ with $p_i \leq b'/2$, $X_i^{S_1} \leq b' m_1$, and the analogous statements hold for $q_i$ and $Y_i^{T_1}$, then the sets $B, M, H$ will all be “faithful”. By our above bounds, and a union bound over the $n$ elements, with probability at least $1 - o(1/n)$ this occurs.

We now prove the correctness of Algorithm 2 by establishing that in the case that $p = q$, the algorithm will output ACCEPT with probability at least $2/3$, and in the case that $\|p - q\|_1 \geq \varepsilon$ the algorithm will output REJECT with probability at least $2/3$. The analysis of these two cases is split into Lemmas 8 and 12. Together with Lemma 7 this establishes Proposition 1.
We analyze the statistics of the algorithm in the case that $p = q$, with respect to the randomness in the samples $S_2, T_2$ under the assumption that the sets $B, M, H$ are faithful.

**Lemma 8.** Given that the sets $B, M,$ and $H$ are “faithful” and that $p = q$, then with high probability over the randomness in $S_2, T_2$, Algorithm 2 will output ACCEPT.

**Proof.**

**B.1.1 The statistic $V_B$:**

By Lemma 1, 

$$E[V_B] \leq \left( \frac{2|B|}{m_2} \right)^{1/2} + \sum_{i \in B} |p_i - q_i| = \left( \frac{2|B|}{m_2} \right)^{1/2}. $$

From our definition of “faithful”, every element of $i \in B$ must have $p_i + q_i \geq b/2 = \frac{128 \log n}{\epsilon^2 m_2}$, hence $|B| \leq \frac{2 \epsilon^2 m_2}{128 \log n} < \frac{\epsilon^2 m_2}{64 \log n}$, and

$$E[V_B] \leq \epsilon \frac{\sqrt{2}}{8 \sqrt{\log n}} < \epsilon/8, \text{ for } n > 2.$$

From Lemma 1, $\text{Var}[V_B] \leq \frac{1}{m_1} + \frac{1}{m_2} \leq \frac{\epsilon^2}{\sqrt{n}} = o(\epsilon^2)$. Hence by Chebyshev’s inequality, $\Pr[V_B > \epsilon/6] \leq o(1)$, and hence the first check of Algorithm 2 will pass.

**B.1.2 The statistic $W_M$:**

From Lemma 2, $E[W_M] = m_1^2 m_2^2 \sum_{i \in M} (p_i - q_i)^2 = 0$. Additionally,

$$\text{Var}[W_M] = 2m_1^2 m_2^2 \sum_{i \in M} (m_2 p_i + m_1 q_i)^2 \leq 2m_1^2 m_2^2 \cdot \max_i \{m_2 p_i + m_1 q_i\} \sum_{i} (m_2 p_i + m_1 q_i).$$

From the fact that $M$ is faithful, $\max_i \{m_2 p_i + m_1 q_i\} \leq O \left( \frac{m_1 \log n}{m_2 \epsilon^2} \right)$, and hence we conclude that $\text{Var}[W_M] = O \left( \frac{m_1^3 m_2 \log n}{\epsilon^2} \right)$.

By Chebyshiev’s inequality, and the assumption that $\epsilon > 1/n^{1/12}$,

$$\Pr \left[ W_M \geq \frac{\epsilon^2 m_1^3 m_2 \log n}{2} \right] = o(1),$$

and hence the second check of Algorithm 2 will pass.

**B.1.3 The statistic $Z_H$:**

By Lemma 3, $E[Z_H] = 0$, and by Lemma 4, $\text{Var}[Z_H] = O(m_1^3 m_2^3)$. Therefore, by Chebyshiev inequality $\Pr[Z_H \geq \gamma \cdot m_1^{3/2} m_2^3] \leq O \left( \frac{1}{\gamma^2} \right)$, which can be made arbitrarily small for a sufficiently large constant $C_{\gamma}$, and hence the third check of Algorithm 2 will pass.
B.2 \( ||p - q||_1 \geq \varepsilon \)

We now consider the execution of the algorithm when \( ||p - q||_1 \geq \varepsilon \).

**Lemma 9.** Given that the sets \( B, M, \) and \( H \) are “faithful” and that \( ||p - q||_1 \geq \varepsilon \), then with high probability over the randomness in \( S_2, T_2 \), Algorithm 3 will output REJECT.

**Proof.** The proof proceeds by considering the following three cases, at least one of which holds: 1) \( \sum_{i \in M} |p_i - q_i| \geq \varepsilon / 3 \), 2) \( \sum_{i \in B} |p_i - q_i| \geq \varepsilon / 3 \), and 3) \( \sum_{i \in H} |p_i - q_i| \geq \varepsilon / 3 \).

**B.2.1 \( \sum_{i \in B} |p_i - q_i| \geq \varepsilon / 3 \)**

By Lemma 1, \( \mathbb{E}[V_B] \geq \sum_{i \in B} |p_i - q_i| \geq \varepsilon / 3 \) and \( \text{Var}[V_B] \leq \frac{1}{m_1} + \frac{1}{m_2} \leq 2/\sqrt{n} \). Therefore by Chebyshev’s inequality, \( \Pr[V_B < \varepsilon / 6] = o(1) \), and hence the algorithm will output REJECT with high probability.

**B.2.2 \( \sum_{i \in M} |p_i - q_i| \geq \varepsilon / 3 \)**

From Lemma 2, \( \mathbb{E}[W_M] = m_1^2 m_2^2 \sum_{i \in M} (p_i - q_i)^2 \). From the definition of “faithful”, it follows that \( |M| \leq 2 \frac{m_2}{128 \log n} \), and hence by Cauchy-Schwarz,

\[
(m_1^2 m_2^2) \sum_{i \in M} (p_i - q_i)^2 \geq (m_1^2 m_2^2) \left( \frac{\sum_{i \in M} |p_i - q_i|}{|M|} \right)^2 \geq (m_1^2 m_2^2) \frac{128 \varepsilon^2 \log n}{18 m_2} \geq 7 \varepsilon^2 m_1^2 m_2 \log n.
\]

Furthermore, from Lemma 2,

\[
\text{Var}[W_M] \leq 2 m_1^2 m_2^2 \sum_{i \in M} z_i^2 + 4 m_1^3 m_2^2 \sum_{i \in M} z_i (p_i - q_i)^2,
\]

where \( z_i = m_1 q_i + m_2 p_i \). As in the proof of Lemma 8, the first term is \( O(\frac{m_1^2 m_2 \log n}{\varepsilon^2}) \). For the second term, noting that \( \sum_i z_i \leq m_1 + m_2 \), and \( (p_i - q_i)^2 \leq O(\frac{\log^2 n}{\varepsilon^4 m_2}) \), we get the bound of \( O(\frac{m_1^2 m_2 \log n}{\varepsilon^2}) \).

By Chebyshev’s inequality and the assumption that \( \varepsilon > 1/n^{1/12} \), with probability \( 1 - o(1) \), \( W_M > \varepsilon^2 m_1^2 m_2 \log n \), and the algorithm will output REJECT.

**B.2.3 \( \sum_{i \in H} |p_i - q_i| \geq \varepsilon / 3 \)**

From Lemma 3, \( \mathbb{E}[Z_H] \geq \Omega(\frac{m_1^2 m_2 \varepsilon^2}{n}) \). Using the assumption in the statement of Proposition 1 that \( m_2 = \Omega(\frac{n}{\varepsilon^2 \sqrt{m_1}}) \), we conclude that

\[
\mathbb{E}[Z_H] = \Omega(m_1^{3/2} m_2).
\]

Using the moment bounds from Lemma 5 and the definition of “faithful”, for any integer \( s > 0 \), \( \mathbb{E}[|Z_H - \mathbb{E}[Z_H]|^s] \leq \tilde{O}_s(m_1^2 m_2) \). By Markov’s inequality,

\[
\Pr[Z_H \leq C \gamma m_1^{3/2} m_2] \leq \Pr\left[|Z_H - \mathbb{E}[Z_H]| \geq \Omega(m_1^{3/2} m_2)\right] \\
= \Pr\left[|Z_H - \mathbb{E}[Z_H]|^s \geq \Omega(m_1^{3s/2} m_2^s)\right] \\
\leq \tilde{O}_s \left( \frac{m_1^{3s/2} m_2^s}{m_1^{3s/2} m_2^s} \right) = \tilde{O}_s \left( \frac{m_1^{3s/2}}{m_1^{3s/2} - 1} \right).
\]
As long as \( \frac{m_1^4}{m_2^5} \leq 1/n^c \) for some positive constant \( c \), there will be some integer \( s_c \), dependent on \( c \) for which this probability is \( o(1) \). Note that the stipulation in the proposition statement, that \( m_1 = O \left( (n/\varepsilon^2)^{1-\gamma} \right) \), for some constant \( \gamma > 0 \), ensures that \( \frac{m_1}{m_2^2} = O(1/n^{-2\gamma}) \), and hence the algorithm will output \( \text{REJECT} \) with probability \( 1 - o(1) \) in this case.

\[ \text{C Proof of Proposition 2} \]

In this section we prove Proposition 2, showing that Algorithm 3 performs as claimed in the extreme case where \( m_1 \approx n \). The algorithm is a slight modification of Algorithm 2, tailored to handle the imbalance between the sample sizes from \( p \) and \( q \). We prove that this algorithm works whenever \( m_1 = \Omega\left((n/\varepsilon^2)^{8/9+\gamma}\right) \) for some \( \gamma > 0 \), and overlaps with the regime of parameters for which the non-extreme algorithm, Algorithm 2, will succeed.

We begin the proof of the above proposition by considering the statistic \( R_H \).

**Observation 1.** Define \( R_A = \sum_{i \in A} \frac{1[Y_i=2]}{X_i+1} \), for \( A \subseteq [n] \). Then

\[
\mathbb{E}[R_A] = \sum_{i=1}^n \frac{m_2^2 q_i^2 (1 - e^{-m_1 p_i}) e^{-m_2 q_i}}{2m_1 p_i}. \tag{27}
\]

**Proof.** Since \( X_i \sim \text{Pois}(m_1 p_i) \), \( \mathbb{E}\left[\frac{1}{X_i+1}\right] = \frac{1 - e^{-m_1 p_i}}{m_1 p_i} \). Also, \( Y_i \sim \text{Pois}(m_2 q_i) \) implies \( \Pr[Y_i = 2] = \frac{(m_2 q_i)^2 e^{-m_2 q_i}}{2m_2 q_i} \). The expectation of \( R_A \) now follows from linearity of expectation and the independence of \( X_i \) and \( Y_i \).

As mentioned before, in the extreme case the statistic \( Z_A \) can incur a variance of \( O(n^4) \), which is at the threshold of what can be tolerated. The statistic \( R_A \) is tailored to deal with these cases. This is formalized in the following lemmas: whenever the variance of \( Z_A \) is at least the tolerance threshold \( \Omega(m_1^3 m_2^3) \), the expected values of \( R_A \) in the case \( p = q \) is well separated from the likely values of \( R_A \) in case \( ||p - q||_1 > \varepsilon \).

**Lemma 10.** If \( p = q \), \( \mathbb{E}[R_A] \leq \frac{m_2^2}{2m_1} \). If \( p \neq q \) and \( \max_{i \in A} q_i \leq \frac{10}{m_2} \) and \( \text{Var}[Z_A] = \Omega(m_1^3 m_2^3) \), then \( \mathbb{E}[R_A] \geq \Omega(m_2^2/m_1) \).

**Proof.** If \( p = q \), then

\[
\mathbb{E}[R_A] = \frac{m_2^2}{2m_1} \sum_{i \in A} q_i^2 \left(1 - e^{-m_1 p_i}\right) e^{-m_2 q_i} \leq \frac{m_2^2}{2m_1} \sum_{i \in A} \frac{q_i^2}{2p_i} \leq \frac{m_2^2}{2m_1}.
\]

Now, suppose \( p \neq q \). Let \( A_0 := \{ i \in A : m_1 p_i \geq 1/2 \} \).

Note that \( \text{Var}[Z_A] \geq \Omega(m_1^3 m_2^3) \) implies that either \( \sum_{i \in A_0} \frac{q_i^2}{p_i} \geq C \) or \( m_1 \sum_{i \in A \setminus A_0} \frac{q_i^2}{p_i} \geq C \) for some constant \( C \) (since by Lemma 4, \( \text{Var}[Z_A] \leq O(m_1^3 m_2^3) \sum_{i \in A} \min \left\{ \frac{q_i^2}{p_i}, m_1 q_i^2 \right\} \)). We consider the two cases separately:
1 Suppose \( \sum_{i \in A_0} \frac{q_i^2}{p_i} \geq C \). Since \( q_i \leq 10/m_2 \) for all \( i \in A \), it holds that for \( i \in A_0 \), \( e^{-m_2 q_i} \geq e^{-10} \). Moreover, \( i \in A_0 \) implies \( 1 - e^{-m_1 p_i} \geq 1 - e^{-1/2} \). Therefore,

\[
\sum_{i \in A_0} \frac{m_i^2 q_i^2 (1 - e^{-m_1 p_i}) e^{-m_2 q_i}}{2 m_1 p_i} \geq \frac{e^{-12 m_2^2}}{m_1} \sum_{i \in A_0} q_i^2 p_i \geq \frac{C \cdot e^{-12 m_2^2}}{m_1}.
\]

2 Suppose \( m_1 \sum_{i \in A \setminus A_0} q_i^2 \geq C \). Using the inequality \( 1 - e^{-x} \geq x - x^2/2 \),

\[
\sum_{i \in A \setminus A_0} \frac{m_i^2 q_i^2 (1 - e^{-m_1 p_i}) e^{-m_2 q_i}}{2 m_1 p_i} \geq \frac{e^{-10 m_2^2}}{2 m_1} \sum_{i \in A \setminus A_0} q_i^2 \left( m_1 p_i - \frac{m_i^2 q_i^2}{2} \right)
\]

\[
= \frac{e^{-10 m_2^2}}{2 m_1} \sum_{i \in A \setminus A_0} \left( m_1 q_i^2 - m_1^2 q_i^2 p_i / 2 \right)
\]

\[
\geq \frac{e^{-10 m_2^2}}{2} \sum_{i \in A \setminus A_0} \left( q_i^2 - q_i^2 / 4 \right)
\]

\[
= \frac{e^{-10 m_2^2}}{2} \sum_{i \in A \setminus A_0} 3q_i^2 / 4 \geq \frac{C \cdot 3e^{-10 m_2^2}}{8},
\]

where the second to last inequality uses that assumption that \( m_1 p_i < 1/2 \) for \( i \in A \setminus A_0 \).

Combining the above cases it follows that \( \mathbb{E}[R_A] \geq \Omega(m_2^2/m_1) \).

From the proof of the above lemma it is clear that we can choose some absolute constant \( K \) such that whenever if \( p \neq q \) and

\[
\max_{i \in A} |q_i| \leq 10/m_2, \quad \text{Var}[Z_A] \geq Km_1^2 m_2^2,
\]

(28)

then \( \mathbb{E}[R_A] \geq 11m_2^2/2m_1 \). Hereafter, fix this constant \( K \).

C.1 \( p = q \)

Suppose, \( m_1 = \Omega((n/\varepsilon^2)^{8/9+\gamma}) \) for some \( \gamma > 0 \). We analyze the statistics in Algorithm \( S \) in the case that \( p = q \), with respect to the randomness in the samples \( S_2, T_2 \) under the assumption that the sets \( B, M, H \) are faithful.

**Lemma 11.** Given that the sets \( B, M, \) and \( H \) are “faithful” and that \( p = q \), then with high probability over the randomness in \( S_2, T_2, \) Algorithm \( S \) will output ACCEPT.

**Proof.** From calculations identical to those in case \( [B.1.1] \) \( [B.1.2] \) it follows that

\[
\Pr[V_B \geq \varepsilon/6] \leq \frac{1}{100}, \quad \Pr[W_M \geq \varepsilon^2 m_1^2 m_2 \log n] \leq \frac{1}{100}, \quad \Pr[Z_H \geq C_2 m_1^3 m_2] \leq \frac{1}{100},
\]

when \( p = q \). Therefore, the unknown distributions will pass the checks in Algorithm \( S \) that correspond to the statistics \( V_B, W_M, \) and \( Z_H \).

It remains to verify the additional two checks in Algorithm \( S \).
C.1.1 Check (1) in Algorithm 3

To show that the first check in Algorithm 3 passes, we will show that when \( p = q \),
\[
\Pr \left[ \text{there exists } i \in [n] \text{ such that } Y_i \geq 3 \text{ and } X_i \leq \frac{m_1 \epsilon^{2/3}}{10m_2 n^{1/3}} \right] < 1/50.
\]

Denote \( \lambda = \frac{m_1 \epsilon^{2/3}}{10m_2 n^{1/3}} = \Omega \left( \frac{m_1 \epsilon^{3/2} \epsilon^{2/3}}{n^{4/3}} \right) = \Omega(\gamma) \) for some constant \( \gamma > 0 \), since by assumption, \( m_1 = \Omega((n/\epsilon^{2})^{8/9} + \gamma) \) for some \( \gamma > 0 \).

If \( p_i > \frac{2\lambda}{m_1} \). Then \( \Pr[X_i \leq \lambda] \leq \Pr[\text{Pois}(2\lambda) \leq \lambda] = o(1/n^2) \). On the other hand, if \( p_i = q_i \leq \frac{2\lambda}{m_1} \), then
\[
\Pr[Y_i \geq 3] \leq \Pr \left[ \text{Pois} \left( \frac{2\lambda m_2}{m_1} \right) \geq 3 \right] = \Pr \left[ \text{Pois} \left( \frac{2\epsilon^{2/3}}{10n^{1/3}} \right) \geq 3 \right] < \frac{1}{100n}.
\]

Hence by a union bound over all \( i \in [n] \), check (1) in Algorithm 3 passes.

C.1.2 The statistic \( R \)

Recall that \( H = [n] \setminus (B \cup M) \), where \( B \) and \( M \) are defined in \ref{eq:2}. Note that by Lemma \ref{lem:10} when \( p = q \),
\[
\mathbb{E}[R_H] \leq \frac{m_2^2}{2m_1}.
\]

Recall that \( m_2^2/m_1 \geq 1 \), and the second criteria for Algorithm 3 rejecting is \( R_H > Cm_2^2/m_1 \), for a large constant \( C \). Since \( R_H \) is a sum of independent random variables, each of which is in the range \((0,1)\), a standard Chernoff bound applies, yielding that the probability the algorithm rejects due to this \( R_H \) is at most 1/100.

\[\square\]

C.2 \( \|p - q\|_1 \geq \epsilon \)

Lemma 12. Given that the sets \( B, M, \) and \( H \) are “faithful” and that \( \|p - q\|_1 \geq \epsilon \), then with high probability over the randomness in \( S_2, T_2, \) Algorithm 2 will output REJECT.

Proof. The proof proceeds by considering the following three cases, at least one of which holds: 1) \( \sum_{i \in B} |p_i - q_i| \geq \epsilon/3 \), 2) \( \sum_{i \in M} |p_i - q_i| \geq \epsilon/3 \), and 3) \( \sum_{i \in H} |p_i - q_i| \geq \epsilon/3 \). Now, if either \( \sum_{i \in B} |p_i - q_i| \geq \epsilon/3 \) or \( \sum_{i \in M} |p_i - q_i| \geq \epsilon/3 \), then from calculations identical to those in Sections \ref{sec:b.2.1} \ref{sec:b.2.2} it follows that the algorithm will output REJECT.

Therefore, assume that \( \sum_{i \in H} |p_i - q_i| \geq \epsilon/3 \). We begin the proof with the following observation:

Observation 2. Suppose there exists \( j \in [n] \) such that \( q_j \geq \frac{10}{m_2} \) and \( p_j \leq \frac{\epsilon^{2/3}}{20m_2 n^{1/3}} \), then
\[
\Pr \left[ \exists i \in [n] \text{ s.t. } Y_i \geq 3 \text{ and } X_i \leq \frac{m_1 \epsilon^{2/3}}{10m_2 n^{1/3}} \right] \geq \frac{9}{10}.
\]

that is, Algorithm 3 fails the first check and REJECTS.

Proof. Given \( j \) with \( q_j \geq \frac{10}{m_2} \) and \( p_j \leq \frac{\epsilon^{2/3}}{20m_2 n^{1/3}} \), \( \Pr[Y_j \geq 3] > 0.99 \), and \( \Pr \left[ X_j < \frac{m_1 \epsilon^{2/3}}{10m_2 n^{1/3}} \right] > 1 - o(1) \).

\[\square\]

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Given this observation, we may continue under the assumption that for all \( i \in [n] \) such that \( q_i \geq \frac{10}{m_2} \),
\[
p_i \geq \frac{\varepsilon^{2/3}}{20m_2n^{1/3}}.
\]
Now, define
\[
S_0 := \{ i \in [n] : q_i \leq 10/m_2 \},
\]
and consider the following cases:

**Case 1** \( \sum_{i \in S_0} |p_i - q_i| \geq \varepsilon/6 \). To begin with suppose that \( \text{Var}[Z_{S_0}] \leq K m_1^3 m_2^2 \), with \( K \) as defined in (28). Then by Chebyshev’s inequality \( \Pr[Z_H \leq C_2 m_1^{3/2} m_2] \leq \frac{1}{20} \) (since \( \text{Var}[Z_{S_0}] \geq \Omega(m_1^{3/2} m_2) \)) by Lemma 3. Otherwise, \( \text{Var}[Z_{S_0}] \geq K m_1^3 m_2^2 \), in which case, by Lemma 10 \( \text{E}[R_{S_0}] \geq \frac{11m_2^2}{2m_1} \); since \( R_H \geq R_{S_0} \) is a sum of independent random variables, with values between 0 and 1, a Chernoff bound yields that with probability at least 0.99, \( R_H \) will exceed the threshold and the second check of Algorithm 3 will fail.

**Case 2** Finally, suppose that \( \sum_{i \in H \setminus S_0} |p_i - q_i| \geq \varepsilon/6 \). Since \( q_i > 10/m_2 \) for all \( i \in H \setminus S_0 \), it suffices to assume that \( p_i \geq \frac{\varepsilon^{2/3}}{20m_2n^{1/3}} \). From Lemma 6 letting \( T = H \setminus S_0 \), we have that \( \text{E}[Z_T] \geq O(\varepsilon^{2/3} m_1^2 m_2^3/36n) \), and
\[
\text{E}[[Z_T - \text{E}[Z_T]]^s] = O\left(\frac{n^{s/3} m_1^s m_2^{s+1}}{\varepsilon^{2s/3}}\right).
\]
By Markov’s inequality,
\[
\Pr[Z_T \leq C_7 m_1^{3/2} m_2/2] \leq \Pr[[Z_T - \text{E}[Z_T]] \geq \Omega(m_1^{3/2} m_2)]
\leq \tilde{O}_s\left(\frac{n^{s/3} m_1^s m_2^{s+1}}{\varepsilon^{2s/3} m_1^{3s/2} m_2^{s}}\right)
\leq \tilde{O}_s\left(\frac{n^{s/3} m_2}{\varepsilon^{2s/3} m_1^{s/2}}\right).
\]
If \( m_2 = \frac{n}{\sqrt{m_1} \varepsilon^2} \) then (30) becomes \( \tilde{O}_s\left(\frac{(n/\varepsilon^2)^{s/3} m_2^{s+1}}{m_1^{s/3} m_2^{s}}\right) \). Since \( m_2 \geq \Omega((n/\varepsilon^2)^{8/9}) \), by taking \( s = 5 \), we can make the probability in (30) \( o(1) \). Similarly, if \( m_1 = n \) and \( m_2 = \sqrt{n}/\varepsilon^2 \), then with \( s = 6 \), (30) becomes \( \tilde{O}_s\left(\frac{1}{\varepsilon^{2s/3} n^{s/2}}\right) = o(1) \) as \( \varepsilon \geq n^{-1/4} \). Together with the concentration of \( Z_{S_0} \) from Chebyshev’s inequality, we get that in this case, the \( Z \) statistic check will fail and the algorithm will output REJECT with probability at least 0.99 in this case.

\[\square\]

**D Lower Bound for \( \ell_1 \) Testing**

In this section, we present lower bounds for the closeness testing problem under the \( \ell_1 \) norm using the machinery developed in Valiant [30, 31]. To this end, define the \((k_1, k_2)\)-based moments \( m(r, s) \) of a distribution pair \((p, q)\) as \( k_1 k_2 \sum_{i=1}^n p_i^r q_i^s \). Valiant [31, Theorem 4.6.9] showed that if the distributions \( p_1^+, p_2^+ \) have probabilities at most \( 1/1000k_1 \), and \( p_1^-, p_2^- \) have probabilities at most \( 1/1000k_2 \), and
\[
\sum_{r+s > 1} \frac{|m^+(r, s) - m^-(r, s)|}{\sqrt{1 + \max\{m^+(r, s), m^-(r, s)\}}} < \frac{1}{1000},
\]

(31)
samples from Proposition 3. Let $m$.

Observe, clearly $m \geq 1$ or $-1$ depending on whether the index is even or odd (this is done so that $\sum_{i=1}^{n} q_i = 1$). Then clearly $\|p - q\|_1 = \delta \varepsilon = \varepsilon/4$.

Define $k_1 = cm_1$ and $k_2 = ce^{-2}n/\sqrt{m_1}$, where $c$ is a sufficiently small constant. Then $\|p\|_{\infty} = b \leq \frac{1}{1000n}$ and $\|p\|_{\infty} = b \leq \frac{1}{1000n^2}$, whenever $m_1 \geq n^{2/3}/\varepsilon^{4/3}$ and $b \geq a$.

Let $(p, p) = (p_1^+, p_2^+)$ and $(p, q) = (p_1^-, p_2^-)$ and computing the $(k_1, k_2)$-based moments gives:

$$m^+(r, s) = k_1^r k_2^s (1 - \delta) b^r s^{r+s-1} + k_1^r k_2^s \delta^r s^{r+s} a^{r+s-1},$$

and

$$m^-(r, s) = k_1^r k_2^s (1 - \delta) b^r s^{r+s-1} + k_1^r k_2^s \delta^r s^{r+s} a^{r+s-1} \left( \frac{(1 + \varepsilon)^s + (1 - \varepsilon)^s}{2} \right).$$

By Theorem 4.6.9 of Valiant [31], to show that $(k_1, k_2)$ samples are not enough, it suffices to have (31). Observe,

$$\frac{|m^+(r, s) - m^-(r, s)|}{\sqrt{1 + \max\{m^+(r, s), m^-(r, s)\}}} \leq \frac{k_1^r k_2^s \delta^r s^{r+s} a^{r+s-1} \left( 1 - \frac{1}{2} ((1 + \varepsilon)^s + (1 - \varepsilon)^s) \right)}{\sqrt{k_1^r k_2^s (1 - \delta) b^r s^{r+s-1}}}.$$ 

For any $s \geq 0$, define $h(\varepsilon, s) = 1 - \frac{(1+\varepsilon)^s + (1-\varepsilon)^s}{2}$. Observe that $h(\varepsilon, 1) = 0$, and $|h(\varepsilon, s)| \leq 1$, for $s \neq 1$. Note that $m_1 \geq n^{2/3}/\varepsilon^{4/3}$, implies that $\varepsilon \geq n^{-1/4}$. Therefore, for every fixed $r \geq 0$ and $s \neq 1$,

$$h(\varepsilon, s)k_1^r k_2^s b^{-(r+s-1)/2} a^{r+s-1} \leq c^{\frac{r+s}{2}} \left( \frac{m_1}{n} \right)^{\frac{r}{2}} \left( \frac{m_2}{\varepsilon n^2} \right)^{\frac{s}{2}} \leq c^{\frac{r+s}{2}} \left( \frac{m_1}{n} \right)^{\frac{r}{2} + \frac{s}{2} + \frac{1}{2}} < c^{\frac{r+s}{2}},$$

since $m_1 \leq n$ by assumption. This shows (31) if $c$ is chosen small enough. □

The optimality of the $\ell_1$ tester, establishing the lower bound in Theorem [1], follows from the above proposition together with the lower bound of $\sqrt{n}/\varepsilon^2$ for testing uniformity given in Paninski [22].