Ideal Whitehead Graphs in $Out(F_r)$
III: Achieved Graphs in Rank 3

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Abstract

By proving precisely which singularity index lists arise from the pair of invariant foliations for a pseudo-Anosov, Masur and Smillie, in [MS93], determined a Teichmüller flow invariant stratification of the space of quadratic differentials. In this final paper of a three-paper series, we give a first step to an $Out(F_r)$ analog of the [MS93] theorem. Since the ideal Whitehead graphs of [HM11] give a strictly finer invariant in the analogous $Out(F_r)$ setting of a fully irreducible outer automorphism, we determined which of the twenty-one connected, simplicial, five-vertex graphs are ideal Whitehead graphs of fully irreducible outer automorphisms in $Out(F_3)$. It can be noted that our methods are valid in any rank.

1 Introduction

Our main theorem (Theorem 4.1) is motivated by the [MS93] theorem of Masur and Smillie listing precisely which invariant singular measured foliation singularity index lists arise from pseudo-Anosov mapping classes. The Masur-Smillie theorem was significant in its determining a stratification of the space of quadratic differentials invariant under the Teichmüller flow. For several results on the stratification of the space of quadratic differentials and on the Teichmüller flow, one can see, for example, [KZ03], [Lan04], [Lan05], [AB06], [Ath06], [EM08], and [Zor10]. This paper is the first step to proving an analog to the [MS93] theorem for outer automorphism groups of free groups.

For a free group $F_r$ of rank $r$, we denote the outer automorphism group by $Out(F_r)$. In this paper we analyze outer automorphisms by topological representatives: Let $R_r$ be the $r$-petaled rose. Given a graph $\Gamma$ with no valence-one vertices, we can assign to $\Gamma$ a marking via a homotopy equivalence $R_r \to \Gamma$. One calls such a graph, together with its marking, a marked graph. Each outer automorphism $\phi \in Out(F_r)$ can be represented by a homotopy equivalence $g: \Gamma \to \Gamma$ of a marked graph, where $\phi = g_*$ is the induced map of fundamental groups. Analogous to pseudo-Anosov mapping classes are fully irreducible (iwip) outer automorphisms, i.e. those such that no representative of a power leaves invariant a subgraph with a nontrivial component. Thus, the analog theorem would involve fully irreducible outer automorphisms.

A beauty in studying the groups $Out(F_r)$ is how they are actually richly more complicated than mapping class groups. A particularly good example of this arises when trying to generalize the Masur-Smillie pseudo-Anosov index theorem. Unlike in the surface case where one has the Poincare-Hopf index equality $i(\psi) = \chi(S)$, for a pseudo-Anosov $\psi$ on a surface $S$, Gaboriau, Jaeger, Levitt, and Lustig proved in [GJLL98] that there is instead an index sum inequality $i(\phi) \geq 1 - r$ for the fully irreducible $\phi \in Out(F_r)$. The index lists of geometric (induced by homeomorphisms of compact surfaces with boundary) fully irreducibles are understood by the Masur-Smillie theorem, but complexity of the nongeometric case prompted the following question [HM11]:

1
Question 1.1. Which index types, satisfying \( i(\phi) > 1 - r \), are achieved by nongeometric, fully irreducible \( \phi \in \text{Out}(F_r) \)?

Unlike in the surface case, the ideal Whitehead graph \( \mathcal{IW}(\phi) \) for a fully irreducible \( \phi \in \text{Out}(F_r) \) (see [HM11], [Pfa12a], or Subsection 2.2) gives a strictly finer outer automorphism invariant than just the corresponding index list. Indeed, for an ageometric \( \phi \in \mathcal{FI}(r) \), the index of a component in \( \mathcal{IW}(\phi) \) is simply \( 1 - \frac{k}{2} \), where \( k \) is the number of vertices in the component. One can think of an ideal Whitehead graph as describing the structure of singular leaves, in analog to the boundary curves of principle regions in Nielsen theory [?].

Since an ideal Whitehead graph is a strictly finer invariant than a singularity index list, the deeper, more appropriate question was thus:

Question 1.2. Which isomorphism types of graphs occur as \( \mathcal{IW}(\phi) \) for fully irreducible \( \phi \)?

We focus on the index sum \( \frac{3}{2} - r \), the closest possible to that of \( 1 - r \), achieved by geometries, without being achieved by any geometric outer automorphism. As in [Pfa12c], we denote the set of connected \((2r-1)\)-vertex simplicial graphs by \( \mathcal{PI}(r;\frac{3}{2} - r) \). Our partial answer (Theorem 4.1) to Question 1.2 completely answers the following subquestion posed in person by Mosher and Feighn:

Question 1.3. Which of the twenty-one graphs in \( \mathcal{PI}(3;\frac{3}{2} - r) \) are the ideal Whitehead graph \( \mathcal{IW}(\phi) \) for a fully irreducible \( \phi \in \text{Out}(F_3) \)?

The complete answer to Question 1.3 is our main theorem, Theorem 4.1.

Theorem. Exactly eighteen of the twenty-one connected, simplicial five-vertex graphs are the ideal Whitehead graph \( \mathcal{IW}(\phi) \) for a fully irreducible outer automorphism \( \phi \in \text{Out}(F_3) \).

The twenty-one graphs in \( \mathcal{PI}(3;\frac{3}{2} - r) \) ([CP84]) are:

\[
\begin{array}{cccccc}
I & II & III & IV & V & VI \\
VII & VIII & IX & X & XI & XII \\
XIII & XIV & XV & XVI & XVII & XVIII \\
XIX & XX & XXI & & & \\
\end{array}
\]

The graphs in \( \mathcal{PI}(3;\frac{3}{2} - r) \) that are not ideal Whitehead graphs for fully irreducible \( \phi \in \text{Out}(F_3) \) are:

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Outline:
Recall [BF94] that, for a train track \( g : \Gamma \to \Gamma \), a periodic Nielsen path \( (p\text{Np}) \) is a nontrivial path \( \rho \) in \( \Gamma \) such that, for some \( k \), \( g^k(\rho) \simeq \rho \) rel endpoints. Also [GJLL98], an outer automorphism is ageometric whose stable representative, in the sense of [HM10], has no pNp’s (closed or otherwise). We use \( \mathcal{AFI}_r \) to denote the subset of \( \mathcal{FI}_r \) comprised of its ageometric elements.

Feighn and Handel defined rotationless train tracks and outer automorphisms in [FH11]. Recall [HM11]: Let a \( \phi \in \mathcal{AFI}_r \) be such that \( \mathcal{IW}(\phi) \in \mathcal{PI}_{(r,(3/2-r))} \), then \( \phi \) is rotationless if and only if the vertices of \( \mathcal{IW}(\phi) \in \mathcal{PI}_{(r,(3/2-r))} \) are fixed by the action of \( \phi \).

Finally, recall [Pfa12c] that if \( \phi \in \mathcal{AFI}_r \) with \( \mathcal{IW}(\phi) \in \mathcal{PL}_{(r,(3/2-r))} \) have pNp-free representatives of a rotationless power whose Stallings fold decomposition [Sta83] consists entirely of proper full folds of roses. We call such representatives ideally decomposed. The first ingredient in our Theorem 4.1 proof is [Pfa12c] Proposition 3.3, implying ideal decomposition existence:

**Proposition.** Let \( \phi \in \mathcal{FI}_r \) be ageometric with \( \mathcal{IW}(\phi) \in \mathcal{PI}_{(r,(3/2-r))} \). There exists a pNp-free train track on the rose representing a rotationless power \( \psi = \phi^R \) and decomposing as \( \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_n = \Gamma \), where:

I) the index set \( \{1, \ldots, n\} \) is viewed as the set \( \mathbb{Z}/n\mathbb{Z} \) with its natural cyclic ordering;

II) each \( \Gamma_k \) is a rose with an indexing \( \{e(k,1), e(k,2), \ldots, e(k,2r-1), e(k,2r)\} \) of the edge set such that:

(a) one can index the edge set of \( \Gamma \) with \( \mathcal{E}(\Gamma) = \{e_1, e_2, \ldots, e_{2r-1}, e_{2r}\} \) where, for each \( t \) with \( 1 \leq t \leq 2r \),

\[
g(e_t) = e_1 \cdots e_i \quad \text{where} \quad (g_n \circ \cdots \circ g_1)(e_{0,t}) = e_{n,i_1} \cdots e_{n,i_s};
\]

(b) for some \( i_k, j_k \) with \( e_k,i_k \neq e_{k,j_k}^{\pm 1} \)

\[
g_k(e_{k-1,t}) = \begin{cases} e_{k,t}e_{k,j_k} & \text{for } t = i_k \\ e_{k,t} & \text{for all } e_{k-1,t} \neq e_{k-1,j_k}^{\pm 1} \end{cases};
\]

(c) for each \( e_t \in \mathcal{E}(\Gamma) \) such that \( t \neq j_n \), we have

\[
Dh(d_t) = d_t, \quad \text{where} \quad d_t = D_0(e_t).
\]

The next proof ingredient is the “lamination train track (ltt) structures” of [Pfa12c]. Using smooth paths in ltt structures (see [Pfa12b]), we “construct” subgraphs of the ideal Whitehead graphs using the construction compositions of [Pfa12b]. To determine which construction compositions to compose, we use the “ideal decomposition (ID) diagrams” of [Pfa12c]. Recall that, if there is \( \phi \in \mathcal{AFI}_r \) with \( \mathcal{IW}(\phi) \cong \mathcal{G} \), where \( \mathcal{G} \in \mathcal{PL}_{(r,(3/2-r))} \), then there is a loop in the ID diagram for \( \mathcal{G} \) corresponding to an ideally decomposed representative of some \( \phi^R \) (Proposition 2).

In Section 3 we describe the three main categories of strategies we used to produce the representatives for the main theorem.

Finally, in order to show that our maps represent \( \phi \in \mathcal{AFI}_r \), we use the “Full Irreducibility Criterion (FIC)” proved in [Pfa12b] (Lemma 4.1):

**Lemma.** (The Full Irreducibility Criterion) Let \( g : \Gamma \to \Gamma \) be a pNp-free, irreducible train track representative of \( \phi \in \text{Out}(F_r) \). Suppose that the transition matrix for \( g \) is Perron-Frobenius and that all the local Whitehead graphs are connected. Then \( \phi \) is fully irreducible.

To apply the criterion we use the [Pfa12b] method for identifying ideally decomposed train track representative pNps.

We proved Graphs II, V, and VII are unachievable in [Pfa12c].

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2 Preliminary definitions and notation

We use this section to establish notation and to remind the reader of background used throughout this document. One familiar with \textbf{[Pfa12c]} and \textbf{[Pfa12b]} may simply skip to Section 3.

We continue with the introduction’s notation. Additionally, unless otherwise stated, we assume throughout this document that outer automorphism representatives are train track representatives in the sense of \textbf{[BH92]}. Further, unless otherwise specified, \( g : \Gamma \to \Gamma \) will represent \( \phi \in \Out(F_r) \).

2.1. Directions and turns

In general, we use definitions of \textbf{[BH92]} and \textbf{[BP90]} for discussing train tracks. We remind the reader here of additional definitions and notation given in \textbf{[Pfa12c]}. \( E^+(\Gamma) := \{ E_1, \ldots, E_n \} = \{ e_1, e_1, \ldots, e_{2n-1}, e_{2n} \} \) will be the edge set of \( \Gamma \) with a prescribed orientation. \( E(\Gamma) := \{ E_1, \overline{E}_1, \ldots, E_n, \overline{E}_n \} \), where \( \overline{E}_i \) denotes \( E_i \) oppositely oriented. If an edge indexing \( \{ E_1, \ldots, E_n \} \) (thus indexing \( \{ e_1, e_1, \ldots, e_{2n-1}, e_{2n} \} \)) is prescribed, we call \( \Gamma \) edge-indexed. \( V(\Gamma) \) will denote the vertex set of \( \Gamma \) and \( D(\Gamma) := \bigcup_{v \in V(\Gamma)} D(v) \), where \( D(v) \) is the set of directions at \( v \). For each \( e \in E(\Gamma) \), \( D_0(e) \) will denote the initial direction of \( e \) and \( D_0(\gamma) := D_0(e_1) \) for each path \( \gamma = e_1 \ldots e_k \) in \( \Gamma \). \( Dg \) will denote the direction map induced by \( g \). We call \( d \in D(\Gamma) \) periodic if \( Dg^k(d) = d \) for some \( k > 0 \) and fixed if \( k = 1 \).

\( T(v) \) will denote the set of turns at a \( v \in V(\Gamma) \) and \( D^t g \) the induced turn map. Sometimes we abusively write \( \{ e_i, e_j \} \) for \( \{ D_0(e_i), D_0(e_j) \} \). For a path \( \gamma = e_1 e_2 \ldots e_k \) in \( \Gamma \), we say \( \gamma \) traverses \( \{ e_i, e_{i+1} \} \) for each \( 1 \leq i < k \). Recall, a turn is called illegal for \( g \) if \( Dg^k(d_i) = Dg^k(d_j) \) for some \( k \).

2.2. Ideal Whitehead graphs and lamination train track (ltt) structures.

Ideal Whitehead graphs were defined in \textbf{[HM11]} and lamination train track structures in \textbf{[Pfa12a]} (and \textbf{[Pfa12c]}). We recount relevant definitions here. Further expositions can be found in \textbf{[HM11]}, \textbf{[Pfa12a]}, and \textbf{[Pfa12c]}. Let \( \Gamma \) be a marked graph, \( v \in \Gamma \) a singularity (vertex with at least three periodic directions), and \( g : \Gamma \to \Gamma \) a tt representing \( \phi \in \Out(F_r) \). The local Whitehead graph \( LW(g; v) \) for \( g \) at \( v \) has:

1. a vertex for each direction \( d \in D(v) \) and
2. edges connecting vertices for \( d_1, d_2 \in D(v) \) when \( g^k(e) \), with \( e \in E(\Gamma) \), traverses \( \{ d_1, d_2 \} \).

The local Stable Whitehead graph \( SW(g; v) \) is the subgraph obtained by restricting precisely to vertices with periodic direction labels. For a rose \( \Gamma \) with vertex \( v \), we denote the single local stable Whitehead graph \( SW(g; v) \) by \( SW(g) \) and the single local Whitehead graph \( LW(g; v) \) by \( LW(g) \).

For a pNP-free \( g \), the ideal Whitehead graph \( LW(\phi) \) of \( \phi \) is isomorphic to \( \bigcup_{\text{vertices } v \in \Gamma} SW(g; v) \). In particular, when \( \Gamma \) is a rose, \( LW(\phi) \cong SW(g) \).

Let \( g : \Gamma \to \Gamma \) be a pNP-free tt on a marked rose with vertex \( v \). Recall from \textbf{[Pfa12c]} the definition of the lamination train track (ltt) structure \( G(g) \) for \( g \). The colored local Whitehead graph \( CW(g) \) at \( v \) is \( LW(g) \), but with the subgraph \( SW(g) \) colored purple and \( LW(g) - SW(g) \) colored red (nonperiodic direction vertices are red). Let \( \Gamma_N = \Gamma - N(\Gamma) \) where \( N(\Gamma) \) is a contractible neighborhood of \( v \). For each \( E_i \in E^+ \), add vertices \( D_0(E_i) \) and \( \overline{D}_0(E_i) \) at the corresponding boundary points of the partial edge \( E_i - (N(\Gamma) \cap E_i) \). The lamination train track (ltt) structure \( G(g) \) for \( g \) is formed from \( \Gamma_N \cup CW(g) \) by identifying vertex \( d_i \) in \( \Gamma_N \) with vertex \( d_i \) in \( CW(g) \). Vertices for nonperiodic directions are red, edges of
Whitehead graphs record at turns taken by immersions of 1-manifolds into graphs. In our case, the 1-structure for $G$

1. Structure for $G$

2. Vertices are indexed by edge pairs. Index pair-labeled ltt structures are the vertex labels. Additionally, $G$ identification, we say $G$

3. Edges are of 3 types (ltt2: ltt1: ltt3: No pair of vertices is connected by two distinct colored edges. We consider ltt structures if the black edge and red vertices are nonperiodic ltt structures, refer the reader to the Standard Notation and Terminology 2. 2 of [Pfa12c]. In particular, in abstract and nonabstract ltt structures, $G$

4. A single black edge connects each pair of (edge-pair)-labeled vertices. There are no other black edges. In particular, each vertex is contained in a unique black edge.

5. A colored edge is red if and only if at least one of its endpoint vertices is red.

Purple Edges: A colored edge is purple if and only if both endpoint vertices are purple.

We denote the purple subgraph of $G$ (from $\mathcal{GW}(g)$) by $\mathcal{PI}(G)$ and, if $G \cong \mathcal{PI}(G)$, say $G$ is an ltt structure for $G$. An $(r; (\frac{3}{2} - r))$ ltt structure is an ltt structure $G$ for a $G \in \mathcal{PI}_{(r; (\frac{3}{2} - r))}$ such that:

**ltt(*)4:** $G$ has precisely 2r-1 purple vertices, a unique red vertex, and a unique red edge. We consider ltt structures equivalent that differ by an ornamentation-preserving homeomorphism and refer the reader to the Standard Notation and Terminology 2.2 of [Pfa12c]. In particular, in abstract and nonabstract ltt structures, $[d_i, d_j]$ is the edge connecting a vertex pair $[d_i, d_j]$, $[e_i]$ denotes the black edge $[d_i, d_j]$ for $e_i \in \mathcal{E}(\Gamma)$, and $C(G)$ denotes the colored subgraph (from $\mathcal{GW}(g)$). Purple vertices are periodic if birecurrent as a train track structure (i.e has a locally smoothly embedded line traversing each edge infinitely many times as $\mathbb{R} \rightarrow \infty$ and as $\mathbb{R} \rightarrow -\infty$).

For an $(r; (\frac{3}{2} - r))$ ltt structure $G$ for $G$, additionally:

1. $d^u$ labels the unique red vertex and is called the unachieved direction.

2. $e^R = [t^R]$ denotes the unique red edge, $\overline{d^u}$ labels its purple vertex, thus $t^R = \{d^u, \overline{d^u}\}$ ($e^R = [d^u, \overline{d^u}]$).

3. $\overline{d^u}$ is contained in a unique black edge, which we call the twice-achieved edge.

4. $d^u$ will label the other twice-achieved edge vertex and be called the twice-achieved direction.

5. If $G$ has a subscript, the subscript carries over to all relevant notation. For example, in $G_k$, $d_k^u$ will label the red vertex and $e_k^R$ the red edge.

We call a 2r-element set of the form $\{x_1, \overline{x_1}, \ldots, x_r, \overline{x_r}\}$, elements paired into edge pairs $\{x_i, \overline{x_i}\}$, a rank-r edge pair labeling set (we write $\overline{x_i} = x_i$). We call a graph with vertices labeled by an edge pair labeling set a pair-labeled graph, and an indexed pair-labeled graph if an indexing is prescribed.

An ltt structure, index pair-labeled as a graph, is an indexed pair-labeled ltt structure if the black edge vertices are indexed by edge pairs. Index pair-labeled ltt structures are equivalent that are equivalent as ltt structures preserving the indexing of the vertex labeling set.

By rank-r index pair-labeling an $(r; (\frac{3}{2} - r))$ ltt structure $G$ and edge-indexing the edges of an r-petaled rose $\Gamma$, one creates an identification of the vertices in $G$ with $D(v)$, where $v$ is the vertex of $\Gamma$. With this identification, we say $G$ is based at $\Gamma$. In such a case we may use the notation $\{d_1, d_2, \ldots, d_{2r-1}, d_{2r}\}$ for the vertex labels. Additionally, $[d_i]$ denotes $[D_0(e_i), D_0(\overline{e_i})] = [d_i, \overline{d_i}]$ for each edge $e_i \in \mathcal{E}(\Gamma)$.

To resolve a reoccurring point of confusion, we remind the reader that the Whitehead graphs used here (defined in [HM11]) differ from Whitehead graphs mentioned elsewhere in the literature. In general, Whitehead graphs record at turns taken by immersions of 1-manifolds into graphs. In our case, the 1-manifold is a set of lines, the attracting lamination. In much of the literature the 1-manifolds are circuits representing conjugacy classes of free group elements. For example, for the Whitehead graphs of [CV86], edge images are viewed as cyclic words. This is not true of ours.
The invariance of the ideal Whitehead graph is explained in [Pfa12a], as is its connection to the expanding lamination for a fully irreducible outer automorphism.

2.3. Ideal decompositions.

Recall that tt’s satisfying (I)-(II) of [Pfa12c] Proposition 3.3 (see the introduction) are called ideally decomposable (ID) with an ideal decomposition (ID). When we additionally require $\phi \in \mathcal{AFL}_r$ and $\mathcal{IW}(\phi) \in \mathcal{P}_{\mathcal{T}}(r; (\frac{3}{2} - r))$, we will say $g$ has type $(r; (\frac{3}{2} - r))$. (By saying $g$ has type $(r; (\frac{3}{2} - r))$, it will be implicit that, not only is $\phi \in \mathcal{AFL}_r$, but $\phi$ is ideally decomposed, or at least ideally decomposable.)

Again we denote $e_{k-1,jk}$ by $e_{k-1}^r$, denote $e_{k,jk}$ by $e_{k}^r$, denote $e_{k,ik}$ by $e_{k}^a$, and denote $e_{k-1,ik}$ by $e_{k-1}^{pa}$. $\mathcal{D}_k$ will denote the set of directions at the vertex of $\Gamma_k$ and $\mathcal{E}_k := \mathcal{E}(\Gamma_k)$. Further recall [Pfa12c] that, for a $(r; (\frac{3}{2} - r))$ tt $g: \Gamma \to \Gamma$, $G(g)$ is an $(r; (\frac{3}{2} - r))$ ltt structure with base $\Gamma$. We denote the ltt structure $G(f_k)$ by $G_k$ where $f_k := g_k \circ \cdots \circ g_1 \circ g_0 \circ \cdots \circ g_{k+1}: \Gamma_k \to \Gamma_k$,

$$g_{k,i} := \begin{cases} g_k \circ \cdots \circ g_i : \Gamma_{i-1} \to \Gamma_k & \text{if } k > i \\ g_k \circ \cdots \circ g_i \circ g_{i+1} \circ \cdots \circ g_k & \text{if } k < i. \end{cases}$$

$\mathcal{C}(G_k)$ will denote the subgraph of $G_k$, from $\mathcal{LW}(f_k)$, containing all colored (red and purple) edges of $G_k$. Sometimes $\mathcal{PI}(G_k)$ will be used to denote the purple subgraph of $G_k$ from $\mathcal{SW}(f_k)$.

In [Pfa12c] we proved $D_0(e_k^r) = d_k^r$, $D_0(e_k^a) = d_k^a$, $D_0(e_{k-1}^{pa}) = d_{k-1}^{pa}$, and $D_0(e_{k-1}^{pa}) = d_{k-1}^{pa}$. As described in [Pfa12c], for any $k,l$, there exists a direction map $D_{g_{k,l}}$, induced turn map $D_{g_{k,l}}^t$, and induced ltt structure map $D_{f_{k,l}}: \Gamma_{l-1} \to \Gamma_k$. The restriction of $D_{f_{k,l}}$ to $\mathcal{C}(\Gamma_{l-1})$ is denoted $D_{g_{k,l}}^C$.

2.4. Extensions and switches.

As in [Pfa12c], a triple $(g_k, G_{k-1}, G_k)$ is an ordered set of three objects where $g_k: \Gamma_k \to \Gamma_k$ is a proper full fold of roses and, for $i = k - 1, k$, $G_i$ is an ltt structure with base $\Gamma_i$. Recall from [Pfa12c] that each triple $(g_k, G_{k-1}, G_k)$ in an ideal decomposition of a representative of $(r; (\frac{3}{2} - r))$ type is either a “switch” or an “extension.”

A generating triple (GT) is a triple $(g_k, G_{k-1}, G_k)$ where

(gtI) $g_k: \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of edge-indexed roses defined by

\begin{itemize}
    \item a. $g_k(e_{k-1,jk}) = e_{k,ik} e_{k,jk}$ where $d_k^a = D_0(e_{k,ik})$, $d_k^r = D_0(e_{k,jk})$, and $e_{k,ik} \neq (e_{k,jk})^{\pm 1}$ and
    \item b. $g_k(e_{k-1,i}) = e_k,i$ for all $e_{k-1,j} \neq (e_{k,jk})^{\pm 1}$;
\end{itemize}

(gtII) $G_i$ is an indexed pair-labeled $(r; (\frac{3}{2} - r))$ ltt structure with base $\Gamma_i$ for $i = k - 1, k$; and

(gtIII) The induced map of based ltt structures $D^T(g_k): G_{k-1} \to G_k$ exists and, in particular, restricts to an isomorphism from $\mathcal{PI}(G_{k-1})$ to $\mathcal{PI}(G_k)$.

The triple will be called admissible if $G_k$ and $G_{k-1}$ are both birecurrent (and thus are actually indexed (edge-pair)-labeled $(r; (\frac{3}{2} - r))$ admissible ltt structures) and if either $d_{k-1}^a = d_{k-1,jk}$ or $d_{k-1}^a = d_{k-1,ik}$. In this case $g_k$ will also be considered admissible.

We call $G_{k-1}$ the source ltt structure and $G_k$ the destination ltt structure. We sometimes write $g_k: e_{k-1,jk}^{pa} \to e_{k,jk}^{a} e_{k}^{a}$ for $g_k$, write $d_{k-1}^{pa}$ for $d_{k-1,jk}$, and write $e_{k-1,ik}$ for $e_{k-1}^{pa}$.

For $g_{k-1}$ and $g_{k-1}$, we denote pair-labeled $(r; (\frac{3}{2} - r))$ ltt structures for $G$, then $(g_k, G_{k-1}, G_k)$ will be a GT for $G$.

For a purple edge $[d_{k,1}^r, d_{k,l}]$ in $G_k$, the extension determined by $[d_{k,1}^r, d_{k,l}]$, is the GT $(g_k, G_{k-1}, G_k)$ for $G$ satisfying:

(extI): The restriction of $D^T(g_k)$ to $\mathcal{PI}(G_{k-1})$ is defined by sending, for each $j$, the vertex labeled $d_{k-1,j}$ to the vertex labeled $d_{k,j}$ and extending linearly over edges.

(extII): $d_{k-1}^a = d_{k-1}^{pa}$, i.e. $d_{k-1}^a = d_{k-1,jk}$ labels the single red vertex in $G_{k-1}$.

(extIII): $d_{k-1}^{pa} = d_{k-1,1}$. 

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The switch determined by a purple edge \([d_k^u, d_{(k,t)}]\) in \(G_k\) is the GT \((g_k, G_{k-1}, G_k)\) for \(\mathcal{G}\) where:

\[[\text{swI}]: \quad D^T(g_k) \text{ restricts to an isomorphism from } \mathcal{P}I(G_{k-1}) \text{ to } \mathcal{P}I(G_k) \text{ defined by} \]

\[
\mathcal{P}I(G_{k-1}) \xrightarrow{d^\alpha_{k-1} = d^u_{k-1}} \mathcal{P}I(G_k)
\]

\((d_{k-1,t} \mapsto d_{k,t} \text{ for } d_{k-1,t} \neq d^\alpha_{k-1})\) and extended linearly over edges.

\[[\text{swII}]: \quad d^\alpha_{k-1} = d^u_{k-1}.\]

\[[\text{swIII}]: \quad d^u_{k-1} = d_{k-1,t}.\]

Recall from [Pfa12c] that admissible switches and extensions satisfy the “admissible map properties” \(\mathcal{A}\mathcal{M} \text{ I-VII of } \text{Pfa12c}\) and that the converse holds by [Pfa12c] Proposition 7.8. These facts motivated our defining “ideal decomposition diagrams” (see the final subsection of this section) in [Pfa12c].

2.5. Construction paths.

As in [Pfa12b], to ensure the entire ideal Whitehead graphs are realized, we use “building block” compositions of extensions, “construction compositions.”

**Definition 2.1.** A *preamblemissible composition* \((g_{i-k}, \ldots, g_i, G_{i-k-1}, \ldots, G_i)\) for a \(\mathcal{G} \in \mathcal{PI}(r; (\frac{3}{2} - r))\) is a sequence of proper full folds of (edge-pair)-indexed roses, \(\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i\), with associated sequence of \((r; (\frac{3}{2} - r))\) ltt structures for \(\mathcal{G}\), \(G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_i\), where, for each \(i - k - 1 \leq j < i\), \((g_{j+1}, G_j, G_{j+1})\) is an extension or switch for \(\mathcal{G}\).

The Definition 2.1 notation is standard. A composition is *admissible* if each \(G_j\) is. We call \(g_{i-i-k}\) the associated automorphism, \(G_{i-k-1}\) the source ltt structure, and \(G_k\) the destination ltt structure.

If each \((g_j, G_{j-1}, G_j)\) with \(i - k < j \leq i\) is an admissible extension and \((g_{i-k}, G_{i-k-1}, G_{i-k})\) is an admissible switch, then we call \((g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)\) an *admissible construction composition* for \(\mathcal{G}\). We call \(g_{i-i-k}\) a construction automorphism. Leaving out the switch, gives a *purified construction automorphism* \(g_p = g_i \circ \cdots \circ g_{i-k+1}\) and purified construction composition \((g_{i-k+1}, \ldots, g_i; G_{i-k}, \ldots, G_i)\).

Recall from [Pfa12b]:

**Lemma.** Let \((g_1, \ldots, g_n, G_0, \ldots, G_n)\) be an TD for a \(\mathcal{G} \in \mathcal{PI}(r; (\frac{3}{2} - r))\) and \((g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)\) a construction composition. Then \([d^a_1, d^u_1, d^l_1, d_i, d_{i-1}, \ldots, d^a_{i-k+1}, d_{i-k+1}, d^u_{i-k}, \ldots, d^a_{i-k}, d^u_{i-k}] = [d^a_1, d^u_1, d^l_1, d^u_{i-k-1}, \ldots, d^a_{i-k}, d^u_{i-k}]\) is a smooth path in the ltt structure \(G_i\).
2.6. Switch paths and sequences.

Switch sequences and switch paths were introduced in [Pfa12b] as useful tools in representative construction.

**Definition 2.3.** An admissible switch sequence for a \((r;\left(\frac{t}{2}-r\right))\) graph \(G\) is an admissible composition \((g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)\) for \(G\) such that

1. each \((g_j; G_{j-1}, G_j)\) with \(i-k \leq j \leq i\) is a switch and
2. \(d_{n+1}^a = d_n^a \neq d_i^l = d_{i+1}^a\) and \(d_l^a \neq d_n^a = d_{n+1}^a\) for all \(i \geq n > l \geq i-k\).

We call the associated automorphism \(g_{i,i-k} = g_i \circ \cdots \circ g_{i-k}\) a switch sequence automorphism.

**Definition 2.4.** Let \((g_j, \ldots, g_k; G_{j-1}, \ldots, G_k)\) be an admissible switch sequence. Its switch path is a path in the destination ltt structure \(G_k\) traversing the red edge \([d_i^l, d_k^a]\) from its red vertex \(d_k^a\) to \(d_i^l\), the black edge \([d_k^b, d_k^a]\) from \(d_k^b\) to \(d_k^a\), what is the red edge \([d_k^a, d_{k-1}^a]\) in \(G_{k-1}\) (purple edge in \(G_k\)) from \(d_k^a\) to \(d_{k-1}^a\), the black edge \([d_{k-1}^a, d_{k-1}^a]\) from \(d_{k-1}^a\) to \(d_{k-1}^a\), continues as such through the red edges for the \(G_i\) with \(j \leq i \leq k\) (inserting black edges between), and ends by traversing the black edge \([d_{j-1}^a, d_j^a]\) from \(d_{j-1}^a\) to \(d_j^a\), what is the red edge \([d_j^a, d_j^a]\) in \(G_j\) (purple edge in \(G_k\)), and then the black edge \([d_j^a, d_j^a]\) from \(d_j^a\) to \(d_j^a\). In other words, a switch path alternates between the red edges (oriented from \(d_j^a\) to \(d_j^a\)) for the \(G_j\) (for descending \(j\)) and the black edges between.
The following lemma was proved in [Pfa12b] and shows that switch paths are indeed smooth paths in destination LTT structures. It is important to note that this only holds when (SS1) and (SS2) hold.

**Lemma.** Let \((g_1, \ldots, g_n, G_0, \ldots, G_n)\) be an \(TD\) for a \(G \in \mathcal{PI}(r; (\frac{3}{2} - r))\) and \((g_{i-k}, \ldots, g_i; G_{i-k-1}, \ldots, G_i)\) a switch sequence. Then the associated switch path forms a smooth path in the LTT structure \(G_k\).

**Example 2.5.** In the LTT structure \(G_i\), we number the colored edges of a switch path:

The switch sequence constructed from the switch path is:

The red edge \(e_k^r\) in \(G_k\) is (0), the red edge \(e_{k-1}^r\) in \(G_{k-1}\) is (1), and the red edge \(e_{k-2}^r\) in \(G_{k-2}\) is (2).

**2.7. Ideal decomposition (ID) diagrams.**

As in [Pfa12c] and [Pfa12b], we use in this paper the fact that type \((r; (\frac{3}{2} - r))\) representatives can be realized as loops in “ideal decomposition diagrams.”

Recall from [Pfa12c] that, for a \(G \in \mathcal{PI}(r; (\frac{3}{2} - r))\), the preliminary ideal decomposition diagram for \(G\) is the directed graph where:

1. the nodes correspond to equivalence classes of admissible indexed pair-labeled \((r; (\frac{3}{2} - r))\) LTT structures for \(G\) and
2. for each equivalence class of an admissible GT \((g_i, G_{i-1}, G_i)\) for \(G\), there exists a directed edge \(E(g_i, G_{i-1}, G_i)\) from the node \([G_{i-1}]\) to the node \([G_i]\).

We called the disjoint union of the maximal strongly connected subgraphs of the preliminary ideal decomposition diagram for \(G\) the ideal decomposition diagram for \(G\) (or \(ID(G)\)). [Pfa12a] gives a procedure for constructing \(TD\) diagrams (there called “AM Diagrams”).

We use the following to prove that, for a \(G \in \mathcal{PI}(r; (\frac{3}{2} - r))\), representatives form loops in \(ID(G)\), satisfying certain properties, are indeed tt representatives of \(\phi \in \mathcal{AFI}_r\) with \(\mathcal{IW}(\phi) \cong G\).

**Lemma.** ([Pfa12b], Lemma 4.2) Suppose \(G \in \mathcal{PI}(r; (\frac{3}{2} - r))\) and \(L(g_1, \ldots, g_k; G_0, G_1, \ldots, G_k) = E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)\) is a loop in \(ID(G)\) satisfying:

1. Each purple edge of \(G(g)\) corresponds to a turn taken by some \(g^k(E_j)\) where \(E_j \in \mathcal{E}(\Gamma)\);
2. for each \(1 \leq i, j \leq q\), there exists some \(k \geq 1\) such that \(g^k(E_j)\) contains either \(E_i\) or \(\bar{E}_i\); and
3. \(g\) has no periodic Nielsen paths.

Then \(g: \Gamma \rightarrow \Gamma\) is a train track representative of an ageometric \(\phi \in \mathcal{FI}_r\) such that \(IW(\phi) = G\).

**3 Representative construction strategies**

We describe here three categories of strategies for constructing \((r; (\frac{3}{2} - r))\) tt representatives. Different strategies work better in different circumstances. For example, if most LTT structures \(G\) with \(\mathcal{PI}(G) = G\) are birecurrent, then category II and III strategies are better suited (\(ID(G)\) may be large and impractical to construct). On the other hand, if only a few LTT structures \(G\) with \(\mathcal{PI}(G) = G\) are birecurrent, then
constructing \( TD(G) \) is simpler than using “guess and check” strategies, so category I strategies are often more practical.

Before actually describing the strategies, we establish in Subsection 3.1 some additional terminology, prove a useful fact (Lemma 3.1), and give (in Example 3.5) a method used repeatedly for checking whether the entire ideal Whitehead graph is “achieved.”

### 3.1 Preliminary definitions and tools

The following lemma gives a precondition for a \((r; (\frac{3}{2} - r))\) ltt structure to be birecurrent (thus admissible). A *valence-1 edge* will mean an edge with a valence-1 vertex. A \( G \in \mathcal{PI}(r; (\frac{3}{2} - r)) \) will be called *edge-pair (index)-labeled* if its vertices are labeled by a \( 2r - 1 \) element subset of the rank \( r \) (indexed) edge pair labeling set.

**Lemma 3.1.** If a \((r; (\frac{3}{2} - r))\) ltt structure \( G \) is birecurrent, then \( \mathcal{C}(G) \) can have at most one valence-1 edge-pair-(labeled) edge \([x_i, \overline{x_i}]\).

**Proof.** Suppose, for contradiction’s sake, \( G \) is a birecurrent with some birecurrent line \( l \). Without generality loss assume \( x_i \) had valence-1 in \( \mathcal{C}(G) \). Since \( l \) must be birecurrent with both orientations, we can focus on the situation where \( l \) traverses \([x_i, \overline{x_i}]\) oriented from \( x_i \) to \( \overline{x_i} \). Since \( l \) is smooth, \( l \) must traverse a black edge after \([x_i, \overline{x_i}]\). But the only black edge at \( \overline{x_i} \) is \([\overline{x_i}, x_i]\). To be smooth, after traversing the black edge \([\overline{x_i}, x_i]\), it must traverse a colored edge containing \( x_i \). Since \( x_i \) had valence-1 in \( \mathcal{C}(G) \), this would imply \( l \) would traverse \([x_i, \overline{x_i}]\) again. Inductively, one sees that \( l \) will get caught in this loop formed by the colored edge \([x_i, \overline{x_i}]\) and black edge \([\overline{x_i}, x_i]\), never again traversing a colored edge other than \([x_i, \overline{x_i}]\) as it heads toward this end, violating birecurrency.

**Definition 3.2.** Motivated by the above lemma, the edge-pair labeling will be considered *preadmissible* if \( \mathcal{G} \) contains no more than one (edge-pair)-labeled edge.

**Remark 3.3.** If the Lemma 3.1 condition is violated, regardless of the red edge, the \( \mathcal{C}(G) \) for any \((r; (\frac{3}{2} - r))\) ltt structure \( G \) with \( \mathcal{PI}(G) \cong \mathcal{G} \), has at least one (edge-pair)-labeled edge, violating birecurrency.

The different strategies we describe here frequently require we track our progress in ensuring all edges of \( \mathcal{G} \) are actually in the ideal Whitehead graph for a given representative \( g_G \). In this section we give methods and terminology we use for this purpose. We start by establishing the notion of a “preimage subgraph.”

**Definition 3.4.** For an admissible map \((g_{k,m}; G_{m-1}, \ldots, G_k)\), the *preimage subgraph* under \((g_{k,m}; G_{m-1}, \ldots, G_k)\) for a subgraph \( H \subset \mathcal{PI}(G_i) \) will be denoted \( H^{-g_{k,m}} \). It is obtained from \( H \) by replacing each edge of \( H \) with its preimage under the isomorphism from \( \mathcal{PI}(G_{m-1}) \) to \( \mathcal{PI}(G_k) \).

**Example 3.5.** Consider a subgraph \( H \) of an ltt structure \( G_i \):

On the left we show the preimage subgraph \( H^{-g_i} \) under the direction map \( Dg_i : \overline{a} \mapsto \overline{b} \) for \( g_i : a \mapsto \overline{b} \):
Remark 3.6. Recall that, for an extension, \((g_i, G_{i-1}, G_i)\), there exists an isomorphism from \(\mathcal{PI}(G_{i-1})\) to \(\mathcal{PI}(G_i)\) fixing the second index of the labels of each vertex of \(\mathcal{PI}(G_{i-1})\) (it sends the vertex labeled \(d_{i-1,j}\) in \(\mathcal{PI}(G_{i-1})\) to the vertex labeled \(d_{i,j}\) in \(\mathcal{PI}(G_i)\) for all \(d_{i-1,j}\)). The isomorphism extends naturally from vertices to edges. Thus, for each edge \([d_{(i,j)}, d_{(i,j)}']\) in \(H\), there is an edge \([d_{(i-1,j)}, d_{(i-1,j)'}]\) in \(H^{-g_i}\). And, for a purified extension \((g(k,m); G_m-1, \ldots, G_k)\), \(\mathcal{H}^{-g_{(k,m)}}\) is obtained from \(H\) by changing the first indices of all vertex labels from \(k\) to \(m-1\). If instead \((g_i, G_{i-1}, G_i)\) is a switch, the isomorphism from \(\mathcal{PI}(G_{i-1})\) to \(\mathcal{PI}(G_i)\) sends the vertex labeled \(d_{i-1,j}\) to \(d_{i,j}\) and fixes the second index of the labels of all other vertices of \(\mathcal{PI}(G_{i-1})\) (it sends the vertex labeled \(d_{i,j}\) in \(\mathcal{PI}(G_{i-1})\) to the vertex labeled \(d_{i,j}\) in \(\mathcal{PI}(G_i)\) for all \(d_{i,j}\) \(\neq d_{i,j}^{pu}\)). Again the isomorphism extends naturally from vertices to edges and so, for each edge \([d_{(i,j)}, d_{i,j}']\) in \(H\) there is an edge \([d_{(i-1,j)}, d_{i,j}']\) in \(H^{-g_i}\) and for each edge \([d_{(i,j)}, d_{(i,j)'}]\) in \(H\), where \(d_i^{a} \neq d_{i,j}\) and \(d_i^{a} \neq d_{i,j}'\), there is an edge \([d_{(i-1,j)}, d_{(i-1,j)'}]\) in \(H^{-g_i}\). Consequently, if \((g(k,m); G_m-1, \ldots, G_k)\) is a construction composition, for each edge \([d_{(m,j)}, d_{m}]\) in \(H\), there is an edge \([d_{(m-1,j)}, d_{m}]\) in \(\mathcal{H}^{-g_{(k,m)}}\) and for each edge \([d_{(j,k)}, d_{(k,j)'}]\) in \(H\), where \(d_k^{a} \neq d_{k,j}\) and \(d_k^{a} \neq d_{(k,j)'}\), there is an edge \([d_{(m-1,j)}, d_{(m-1,j)'}]\) in \(\mathcal{H}^{-g_{(k,m)}}\).

We now define further notation used for tracking progress in ensuring all edges of \(G\) are actually \(\mathcal{TW}(g)\). What we define is a graph \(G_k^{a}\) encoding what edges have been “constructed” thus far.

Definition 3.7. Let \(g_{\gamma} = s_n \circ h_n^{p} \circ s_{n-1} \circ h_{n-1}^{p} \circ \cdots \circ s_1 \circ h_1^{p}\) where each \(h_k^{p}\) is a purified construction composition with destination ltt structure \(G_h^{k}\) and each \(s_k\) is a switch. We define \(G_i^{a}\) as the subgraph of \(G_i\) consisting of precisely the purple edges in the construction path for \(h = s_1 \circ h_1^{p}\). Let \(P(\gamma_{h_k})\) denote the set of purple edges in the construction path \(\gamma_{h_k}\) for \(h_k = s_k \circ h_k^{p}\). Then \(G_i^{a} = P(\gamma_{h_1})\) and we inductively define \(G_k^{a}\) as the subgraph \(P(\gamma_{h_k}) \cup (G_{k-1}^{a})^{-s_{k-1}}\) of \(G_{t_k}\).

In addition to tracing preimage subgraphs, one can check that the entire graph is built by taking images of the red edges created by \(g_i\), as in the following example:

Example 3.8. We show here an example of how to check that all of \(G\) is “built” (we iteratively take the image under each \(Dg_{h_k}\) of the edges “created” thus far):

\[
\begin{align*}
\text{a} & \rightarrow \text{ba} \\
\text{b} & \rightarrow \text{bc} \\
\end{align*}
\]

We include subgraphs \(H_i\) of the ltt structures \(G_i\) to track how edges are “built.” \(g_1\) is defined by \(c \rightarrow \text{bc}\). Thus, the red edge \(e_1^{R}\) in \(G_1\) will be \([c, b]\), where the red periodic vertex label is \(c\). \(g_2\) is defined by \(b \rightarrow \text{bc}\). Thus, the red edge \(e_2^{R}\) in \(G_2\) will be \([b, c]\), where the red direction vertex label is \(c\). This ltt structure will also contain the image \([c, b]\) of the red edge \([c, b]\) under \(Dg_2 : b \rightarrow c\). \(g_3\) is defined again by \(b \rightarrow \text{bc}\). Thus, the red edge \(e_3^{R}\) in \(G_3\) will be again \([b, c]\), where the red periodic vertex label is \(c\). This ltt structure will also contain the image \([c, c]\) of the red edge \([b, c]\) and the image \([c, b]\) of the purple edge \([c, b]\) under \(Dg_3 : b \rightarrow c\). \(g_4\) is defined by \(a \rightarrow \text{ba}\). Thus, the red edge \(e_4^{R}\) in \(G_4\) will be \([a, b]\), where the red periodic vertex label is \(a\). This ltt structure will also contain the image \([b, c]\) of the red edge \([b, c]\) and the images \([c, b]\) and \([c, c]\) of the purple edges \([c, b]\) and \([c, c]\) under \(Dg_4 : a \rightarrow b\). The remaining \(H_i\) are constructed similarly.
3.2 Category I strategies: Finding “test” loops when the entire ID diagram is known

Proposition 3.3 implies each achievable $\mathcal{G} \in \mathcal{PI}_{(r, (\frac{3}{2} - r))}$ has an ideally decomposed tt representative and Proposition 8.3 implies the representative is realized by a loop in $ID(\mathcal{G})$. This section provides guidance on finding these loops. However, before even attempting to find a loop, it is advisable to check the irreducibility potential of $ID(\mathcal{G})$ (see the Irreducibility Potential Test of [Pfa12c]).

**Irreducibility Potential Test:** Check whether, in each connected component of $ID(\mathcal{G})$, for each edge vertex pair $\{d_i, d_i\}$, there is a node $N$ in the component such that either $d_i$ or $d_i'$ labels the red vertex in the structure $N$. If it holds for no component, $\mathcal{G}$ is unachieved.

And then, once one finds a loop, they still must test the representative constructed from the loop to ensure that it is pNp-free (see the procedure of [Pfa12d] for identifying pNps), that the transition matrix is PF, and that $\mathcal{I}W(g) \cong \mathcal{G}$. These issues are addressed in Section 3.5 and, because these tests are not included before Section 3.5, we call the loops we find in this section “Test Loops”.

Suppose $\mathcal{G} \in \mathcal{PI}_{(r, (\frac{3}{2} - r))}$. We describe here two strategies (Strategy Ia and Ib) for finding the desired representative loop $L(g_1, \ldots, g_k; G_0, G_1, \ldots, G_{k-1}, G_k)$ when $ID(\mathcal{G})$ is known.

(Ia) In this strategy we use potential composition paths to build subgraphs of $\mathcal{G}$ (following progress using preimage subgraphs). We show in this subsection only how to find the paths. One can reference Strategy III for how to piece them together.

To find the paths, we identify the subdiagram of $ID(\mathcal{G})$ (“Extension Subdiagram”) where paths for construction compositions would have to live. We then find a subgraph (the “Potential Composition Subgraph”) of the lt structures in a component of the subdiagram their construction paths would live in that would actually contain the construction paths.

(1) Recall that each directed edge in $ID(\mathcal{G})$ corresponds to either a switch or an extension. The *extension subdiagram* $(ID(\mathcal{G}))_e$ of $ID(\mathcal{G})$ consists precisely of the directed edges (including their nodes) for extensions.
Example 3.9. Extension Subdiagram, $(\mathcal{D}(\mathcal{G}))_e$ where $\mathcal{G}$ is Graph III:
The following is a component of $\mathcal{D}(\mathcal{G})$, where $\mathcal{G}$ is Graph III.

The components of $(\mathcal{D}(\mathcal{G}))_e$ living in the component of $\mathcal{D}(\mathcal{G})$ given above (notice how the ltt structures in each component of $\mathcal{D}(\mathcal{G})$ have the same purple subgraph):

(2) Find the potential composition PI subgraph for each component of $(\mathcal{D}(\mathcal{G}))_e$:

All ltt structures (extension source and destination ltt structures) labeling nodes in a connected component of $(\mathcal{D}(\mathcal{G}))_e$ share a purple (edge-pair)-indexed subgraph, the potential composition PI subgraph for the component.
Example 3.10. Potential composition PI subgraphs for the components of \((\mathcal{ID}(G))_e\) in Example 3.9 (the left graph is for the left component and the right graph is for the right component):

(3) Find the potential composition subgraph for an \((\mathcal{ID}(G))_e\) connected component for \(C\):

We add black edges connecting edge-pair vertices in the potential composition PI subgraph, then recursively remove valence-1 edges (leaving the larger valence vertex each time a valence-1 edge is removed).

Example 3.11. The composition subgraph for the graph on the left in Example 3.10 is obtained by first adding the black edges \([a, \bar{a}], [b, \bar{b}], \) and \([c, \bar{c}]\) to obtain the graph on the left in Example 3.11 below and then removing both \([a, \bar{a}], (\bar{a}, c]\) to obtain the graph on the right not containing any valence-1 vertices. The potential composition subgraph for the graph on the left in Example 3.10 is:

Start with:

and remove valence-1 edges:

(4) Find a potential composition path in the potential composition subgraph for \(C\):

Find a directed smooth path \([d^0_i, d^1_{i,j_1}, d^2_{i,j_1}, \ldots, d^1_{i,j_n}, d^0_{i,j_n}]\) in a potential composition subgraph (where the potential composition subgraph is viewed as a subgraph of some ltt structure \(G_i\) in \((\mathcal{ID}(G))_e\)).

Example 3.12. A potential composition path in the potential composition subgraph of Example 3.11. The numbered colored edges, combined with the black edges between give a Graph III potential composition path. (Note: This path is not used to compute the representative below.)

(5) We check that the construction composition for the potential composition path of (4) (see \([\text{Pfa12b}], \text{Lemma 3.4}\) is actually realized in \(\mathcal{ID}(G)\). For example, it may be that the destination ltt structure for one of the extensions in the decomposition of the construction composition was not birecurrent (so the extension was not admissible) or even just that the directed edge in \(\text{pre}\mathcal{ID}(G)\) labeled by one of the extensions was not in a maximal strongly connected component of \(\text{pre}\mathcal{ID}(G)\).

If the construction composition is realized by a path in \(\mathcal{ID}(G)\), the path may give the final segment in the loop realizing a representative. Including this path in \(\mathcal{ID}(G)\) as the final segment of a loop in \(\mathcal{ID}(G)\) will guarantee that the purple edges of its construction path are in the ideal Whitehead graph (see \([\text{Pfa12b}], \text{Lemma 3.6}\)).

(6) One way to continue with this strategy is:

(a) We choose a node \(V_i\) in \(C\) such that \(d^1_i\) is the attaching red vertex in the ltt structure labeling \(V_i\).

(b) The final segment of our loop in \(\mathcal{ID}(G)\) will be the path in \(\mathcal{ID}(G)\) realizing the construction composition for the potential composition path of (4).

(c) We determine what edges of the ideal Whitehead graph are still missing (not contained in
the construction path).
(d) We trace those edges via their preimages to another component of \( \mathcal{ID}(G) \) where the PI subgraph contains (at least some) of the preimage edges (see Strategy III).
(e) We find a path in the potential composition subgraph containing those preimage edges.
(f) We continue as such until the entire ideal Whitehead graph is built.
(g) We conclude the loop with a path returning to \( V_i \).

(Ib) “Guess and Add” with PreTest Loops:
In this strategy we find a “pretest” loop in \( \mathcal{ID}(G) \) such that, for each vertex edge pair \( \{d_i, \overline{d}_i\} \), either \( d_i \) or \( \overline{d}_i \) is the red vertex for some ltt structure labeling a node in the loop. We add small loops until the entire graph is built (see Example 3.5 for how to check this).

Example 3.13. We find \( L(g_1, \ldots, g_k; G_0, \ldots, G_k) \), where \( G \) is Graph I: We find a loop in a component of \( \mathcal{ID}(G) \), where \( G \) is Graph I (ltt structure black edges are left out for simplicity’s sake). We start with a pretest loop in \( \mathcal{ID}(G) \) (the blue directed edges together give the loop). By the procedure illustrated in Example 3.8 we see that we do not get all of \( G \) (we do not get the edge \( \overline{\bar{b}, \bar{c}} \)), so we add a second loop (depicted in green) to the pretest loop.

Combining the two loops gave the representative yielding Graph I (the line). The colors and numbers correspond to those in the loops they came from.
3.3 Category II (Strategy II): Piecing together construction compositions

Again suppose \( G \in \Pi_{r,(\frac{3}{2} - r)} \) and let \( G \) be a \( (r; (\frac{3}{2} - r)) \) admissible ltt structure with \( \Pi(G) = G \) and the standard notation. Strategy II is similar to Strategy Ia but here we do not use \( ID(G) \). Instead we alternate between finding construction compositions using construction paths and using “guess and check” to find admissible switches and extensions leading to the next admissible ltt structure we find a construction path in.

**STEP 1: First building subgraph**

The first step we use in “building” a representative \( g_G \) with \( \mathcal{W}(g_G) = G \) is to determine the construction subgraph \( G_C \) of \( G \). We call this the “first building subgraph” for our test loop.

**STEP 2: A potential construction path in \( G_C \) and purified construction composition \( h_1^p \)**

The next step is to find a potential construction path \( \gamma = [d_1, d_2, \ldots, x_{k+1}, x_k] \) in \( G_C \) (see Example 2.2). From \( \gamma \) we obtain a purified construction composition, \( h_1^p \) (a good choice is one of minimal length among all potential construction paths traversing the maximum number of edges of \( G'_{ep} \)). If none can be found, we construct \( ID(G) \) and determine whether \( g_G \) exists at all.

**STEP 3: Determine the purple edges of \( G \) missed by the construction path in Step 2.**

One just looks at the purple subgraph of \( G \) and looks at what edges are not hit by the construction path. These are the remaining edges that still need to be “built” in the ideal Whitehead graph.

**STEP 4: SWITCH \( s_1 \)**

We determine a switch \( (g_{i-k}, G_{i-k-1}, G_{i-k}) \) to precede \( h_1^p \) in the decomposition of \( g_G \). To determine choices that may give the switch, we look at the source ltt structure \( G_{j_1} = G_{i-k} \) for the first generator in the purified construction composition. There is one switch for each purple edge \( [d_{j_1}, d] \) of \( G_{j_1} \) such that \( d \neq d_{j_1} \) in \( G_{j_1} \). We disregard nonadmissible switches (in particular, those with nonbirecurrent source ltt structure). We choose a remaining switches and call it \( s_1 \). The source ltt structure \( G_{i-k-1} \) is denoted \( G_{j_1} \).
Example 3.16. The two options for the switch proceeding the pure construction composition of Example 3.14 can be identified by giving their source ltt structures (the generator is determined to be \( a \mapsto ac \) by the red edge \([\overline{a}, c]\) in \( G_{i-k} \)). The two source ltt structures are (the black edges in the structures are left out, since they are easily ascertainable):

![Diagram of two ltt structures]

Both of the ltt structures are admissible \((r, (\frac{3}{2} - r))\) ltt structures, so are options.

STEP 5: RECURSIVE CONSTRUCTION COMPOSITION BUILDING

Recursive Process of Construction Composition Building:
Steps I-IV below are repeated recursively with the following assumptions until \( G_{a N} = \text{PI}(G_{iN}) \) for some \( N \). The assumptions are that 
\[ s_{n-1} \circ h_{n-1}^p \circ s_{n-2} \circ h_{n-2}^p \circ \cdots \circ s_1 \circ h_1^p \]
where each \( h_k^p \) is a purified construction composition with source ltt structure \( G_{jk} \) and destination ltt structure \( G_{ik} \) and each \( s_k \) is a switch with source ltt structure \( G_{j'k} = G_{i'k-1} \) and destination ltt structure \( G_{jk} \). (Notice that \( G_{i'k+1} = G_{j'k} \), for each \( 1 \leq k \leq n - 1 \).)

I. We determine the first building subgraph \((G_{j_{n-1}})_C\) for \( G_{j_{n-1}} \).

II. We find a potential construction path in \((G'_{j_{n-1}})_C\) (an “optimal strategy,” similar to that in Step 2, may involve choosing the path to be of minimal length among all potential construction paths transversing the maximum number of colored edges of \((G'_{j_{n-1}})_C - G_{aN}\)). Call the corresponding purified construction composition \( h_n^p \) and the construction path \( \gamma_h \). If no valid construction composition can be found via this method, one can try using different construction compositions in the previous steps. If this does not work, one can find \( \text{ID}(G) \) and determine whether \( g \) exists at all.

\[ \Gamma_{jn} = \Gamma_{(i_n-k_n)} \xrightarrow{g_{(n-k_n)+1}} \cdots \xrightarrow{g_{(i_n-1)}} \Gamma_{(i_n-1)} \xrightarrow{g_{i_n}} \Gamma_{i_n} \]
will denote the decomposition of \( h_n^p \) and \( G_{jn} = G_{(i_n-k_n)} \xrightarrow{D^T(g_{(n-k_n)+1})} \cdots \xrightarrow{D^T(g_{(i_n-1)})} G_{(i_n-1)} \xrightarrow{D^T(g_{i_n})} G_{i_n} \)
will denote the corresponding sequence of ltt structures.

III. Determining \( s_n \):
There is one switch for each purple edge \([d_{jn}, d]\) of \( G_{jn} = G_{i'k-n} \). We choose an admissible switches and call it \( s_n \). The source ltt structure for \( s_n \) is denoted \( G_{j'k} \).

IV. We repeat (I)-(III) recursively until \( G_{aN} = \text{PI}(G_{jN}) \) for some \( N \).

Example 3.17. We continue with the example for Graph XIII
This gives (all graphs here are birecurrent):

![Graphs](image)

The preimage of \( a \) under the direction map for \( c \) is \( a, c \).

The choices for the source ltt structure for the switch starting the composition are (in short-hand)

![Graphs](image)

We decide to continue with the Left-hand graph.

The ltt structure is:

![Ltt Structure](image)

Construct path maximizing blue edges crossed:

![Path Construction](image)

This gives (all graphs here are birecurrent):

![Final Graphs](image)

**STEP 6: CONCLUDING SWITCH SEQUENCE**

Once we have \( G_N^a = \mathcal{P}I(G_{jN}) \), we find the shortest possible admissible switch sequence

\[
G = G_{iN-kN} \xrightarrow{g_{iN-kN+1}} \cdots \xrightarrow{g_{iN-1}} G_{iN-1} \xrightarrow{g_{iN}} G_{jN} = G_N
\]

with \( G \) as the source ltt structure and \( G_{jN} \) as the destination ltt structure. A switch path in \( G_{jN} \) may be used for this purpose, though it will be necessary to check that the corresponding switch sequence is indeed an admissible switch sequence (in particular that each \( G_j \) with \( iN-kN \leq j \leq iN = jN \) is an admissible \( (r; (\frac{3}{2} - r)) \) ltt structure for \( G \)).

If it is not possible to get a pure sequence of switches, then one can try any admissible composition with \( G \) as its source ltt structure (and \( G_{jN} \) its destination ltt structure) or, if necessary, find a path in \( \mathcal{TD}(G) \) from \( G \) to \( G_{jN} \) (see [Pfa12a] for how to construct \( \mathcal{TD}(G) \)). It may be possible to find the path in \( \mathcal{TD}(G) \) without actually building the entire diagram by instead just looking at the portion of the permitted extension/switch web constructed starting with \( G_{jN} \) (see also [Pfa12a]).

**Example 3.18.** Concluding sequence of generators for Graph XIII example:

![Final Graphs](image)

We have the final map and get the entire representative for Graph XIII:
We showed that this map does not have any pNps in \([Pfa12b]\).

### 3.4 Category III (Strategy III): Inserting construction compositions from construction loops into switch sequences

(A) We find a switch sequence \((g_{(i,i-k)}, G_{i-k-1}, G_i)\) with \(G_{i-k-1} = G_i\) such that, for each vertex pair \(\{d_i, \bar{d_i}\}\), either \(d_i\) or \(\bar{d}_i\) is the red vertex in some ltt structure in the sequence. (Such a composition would be represented by a loop in \(\mathcal{ID}(G)\) and can be found as a loop in \(\mathcal{ID}(G)\), if not by switch paths or trial and error. It would also work to use a loop in \(\mathcal{ID}(G)\) that does not represent a switch sequence, but the condition on vertex pairs still holds.)

(B) As in Strategy II, we find a construction path in \((G_i)_{ep}'\) transversing as many edges of \((G_i)_{ep}'\) as possible, except that we now have the added condition that the corresponding purified construction composition must start and end with the same ltt structure.

(C) We proceed as in Strategy II with the added condition of (B) and with the condition that the switches between the purified construction compositions are determined by the switch sequence \((g_{(i,i-k)}, G_{i-k-1}, \ldots, G_i)\).

**Example 3.19.** We analyze Graph XX:

We start with the switch sequence of Example 2.5.

Our first construction composition (with construction automorphism \(a \mapsto ab\bar{c}\bar{b}b\bar{c}b\)) is given by the construction path in the following ltt structure:

What is still needed after that composition is:
We take the preimage of edges left under the direction map for the final switch and get:

![Diagram 1]

Since we could not obtain all these edges from a single construction composition, we take another preimage (the preimage under the direction map of a second switch in the switch sequence):

![Diagram 2]

We use the construction composition for the following construction path to obtain these edges:

![Diagram 3]

When composed we get:

![Diagram 4]

The automorphism obtained is:

\[ h = \begin{cases} 
    a \mapsto ab\bar{c}\bar{b}bc, \\
    b \mapsto bc, \\
    c \mapsto cab\bar{c}\bar{b}bcab\bar{c}\bar{b}bcbbcbcab\bar{c}\bar{b}bcbbcbc.
\end{cases} \]

Since the periodic directions for this map are not fixed, we compose \( h \) with itself to get \( g = h^2 \).

### 3.5 Final Checks

As mentioned before, the loops we find are only test loops and still have properties they must satisfy. The map is not acceptable if any of the following holds:

1. For some vertex edge pair \( \{d_i, \bar{d}_i\} \), neither \( d_i \) nor \( \bar{d}_i \) is the red vertex in any ltt structure in the decomposition.
   
   One can check (1) visually. If (1) fails in Strategy I, we “attach” small loops to the initial loop in \( \mathcal{ID}(G) \), where the red vertices of the added small loops include labels from each of the pairs \( \{d_i, \bar{d}_i\} \) not yet included. If (1) fails in Strategy II or III, one can try finding an alternative concluding switch sequence (or tacking on a concluding sequence) resolving the problem.

2. There are not \( 2r - 1 \) fixed directions.
   
   Notice first that there would still be \( 2r - 1 \) periodic directions, since we are dealing with admissible compositions. One can check (2) by composing generator direction maps. If (2) fails, one can take a power of the map fixing all periodic directions.
(3) The map constructed is pNp-free.

One can check (3) via the procedure in [Pfa12b]. (See [Pfa12b] for the procedure applied to show that the map we gave for Graph XIII in Example 3.18 was pNp-free).

(4) All of $G$ is “built.”

One can check (4) by looking at the union of the $[Dg_{k+1,n}(t^R)]$ (See Example 3.8). If (4) fails in Strategy I, again one can “attach” small loops to the initial loop in $ID(G)$ until the entire graph is built. This can be done strategically by using the potential composition PI subgraphs (determining potential composition paths to ensure inclusion of necessary remaining edges, keeping in mind that direction maps map purple edges of the construction path into the destination ltt structure). If this fails in Strategy II or Strategy III, one can add extra construction compositions or try using an alternative route to the current final sequence of admissible maps.

4 Achievable Graphs in Rank 3

This section includes our main theorem. The theorem gives a refinement of the achievability of the index list $(-\frac{3}{2})$ by fully irreducible $\phi \in \text{Out}(F_3)$.

**Theorem 4.1.** Precisely eighteen of the twenty-one connected, simplicial five-vertex graphs are the ideal Whitehead graph $IW(\phi)$ for a fully irreducible outer automorphism $\phi \in \text{Out}(F_3)$.

**Proof.** Graphs II, V, and VII were proved unachievable in [Pfa12c]. We give representatives for the remaining graphs, leaving it to the reader to prove they are pNp-free (using Proposition 5.2 [Pfa12b]), have PF transition matrices, and have the appropriate ideal Whitehead graphs. Then, by [Pfa12b] Lemma 4.2, they are representatives of $\phi \in \mathcal{AFL}_r$ with the desired ideal Whitehead graphs.

For each achieved graph we give a representative $g$ achieving it and then an ideal decomposition for $g$. When showing the ideal decomposition, in most cases, we leave out the black edges in the ltt structures. For Graphs X, XII, XV, and XIX we give a condensed description of the ideal decomposition where a pure construction composition starting and ending at an ltt structure $G_i$ is shown below as a path in $(G_i)_C$. For graphs XI and XVI, the pure construction compositions do not start and end with the same ltt structure, so are depicted as paths in $(G_i)_C$ below, but between, their initial and terminal ltt structures.

**GRAPH I (The Line):**

$$g = \begin{cases} 
  a \mapsto abcabcacacba \\
  b \mapsto abcabcacab \\
  c \mapsto cacbacbca 
\end{cases}$$

**GRAPH II:**
\[
g = \begin{cases} 
a \mapsto abca \\
b \mapsto b\overline{a}c\overline{a}c \\
c \mapsto c\overline{a}c\overline{a}c\overline{a}c \\
\end{cases}
\]
GRAPH IX:

\[ g = \begin{cases} 
  a &\mapsto abcba \bar{b} \bar{c} \bar{b} \bar{c} 
  b &\mapsto bcab \bar{c} \bar{b} \bar{c}abab 
  c &\mapsto cb \bar{b} \bar{a} 
\end{cases} \]

GRAPH X:

\[ g = \begin{cases} 
  a &\mapsto ab \bar{a} \bar{b} \bar{c} \bar{b} \bar{a} \bar{b} \bar{a} \bar{c} \bar{b}ababababababababababab 
  b &\mapsto bababac\bar{a} \bar{b} \bar{c} \bar{a} \bar{b}ababababababababababab 
  c &\mapsto babac\bar{a} \bar{b} \bar{c} \bar{a} \bar{b}ababababababababababababababababab 
\end{cases} \]

GRAPH XI:
Our ideal decomposition for this representative and further explanation were given in Example 3.17.
GRAPH XV:

\[ g = \begin{cases} 
  a \mapsto ab \bar{c} \bar{b} \bar{e} \bar{b} c \bar{b} c \bar{b} e \bar{c} \bar{b} c \\
  b \mapsto \bar{b} \bar{c} \bar{b} c \bar{b} c \bar{b} e \bar{c} \bar{b} e \\
  c \mapsto \bar{c} \bar{b} c \bar{b} c \bar{b} e \bar{c} \bar{b} e 
\end{cases} \]

GRAPH XVI:

\[ g = \begin{cases} 
  a \mapsto abce \bar{c} \bar{b}e \\
  b \mapsto \bar{b} \bar{c} \bar{b} e \bar{c} \bar{b} \bar{e} \bar{b} e \\
  c \mapsto abce \bar{b} \bar{e} \bar{b} \bar{e} \bar{b} e 
\end{cases} \]

GRAPH XVII:

\[ g = \begin{cases} 
  a \mapsto abce \bar{b} \bar{c} \bar{b} e \bar{c} \bar{b} \bar{e} \bar{b} e \\
  b \mapsto \bar{b} \bar{c} \bar{b} e \bar{c} \bar{b} \bar{e} \bar{b} e \\
  c \mapsto \bar{c} \bar{b} c \bar{b} e \bar{c} \bar{b} \bar{e} \bar{b} e 
\end{cases} \]
GRAPH XVIII:

\[ g = \begin{cases} 
  a \mapsto abcabb \\
  b \mapsto bababab \\
  c \mapsto cbabcbbabc 
\end{cases} \]

GRAPH XIX:

\[ g = \begin{cases} 
  a \mapsto accbabc \\
  b \mapsto bcbacacb \\
  c \mapsto cabcabcabc 
\end{cases} \]

GRAPH XX:

The representative \( g = h^2 \) having ideal Whitehead graph GRAPH XX, where

\[ h = \begin{cases} 
  a \mapsto abbcabc \\
  b \mapsto bc \\
  c \mapsto cabcabcabcabc 
\end{cases} \]
was constructed in the examples above.

GRAPH XXI (Complete Graph): This was given in [Pfa12b].

\[ g = \begin{cases} 
  a \mapsto ababaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaacbabaabc
\end{cases}
\]

Since we have either given representatives yielding or shown that they cannot exist for all twenty-one \( \mathcal{PI}_{(3;(-3/2))} \) graphs, we have completed the proof. \qed
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