LOCAL MATRIX GENERALIZATIONS OF W-ALGEBRAS

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Abstract. In this paper, we propose local matrix generalizations of the classical W-algebras based on the second Hamiltonian structure of the \( \mathbb{Z}_m \)-valued KP hierarchy, where \( \mathbb{Z}_m \) is a maximal commutative subalgebra of \( gl(m, \mathbb{C}) \). More precisely, firstly we give a brief discussion about the Sato theory of the \( \mathbb{Z}_m \)-KP hierarchy. Secondly we construct its bihamiltonian structure and propose two kinds of local matrix generalizations of the \( W_n^{(m)} \)-algebra and the \( W_{\infty}^{(n)} \)-algebra, namely the \( W_{\text{KP}}^{(m,n)} \)-algebra and the \( W_{\infty}^{(m,n)} \)-algebra. Furthermore by constructing a Miura map we describe free-field realizations of the above W-type algebras. Afterwards, we study the dispersionless analogue of the \( \mathbb{Z}_m \)-KP hierarchy including the bi-hamiltonian structure, the Miura map, local matrix w-type algebras and their free-field realizations. Finally we discuss the relation between Frobenius manifolds and the dispersionless \( \mathbb{Z}_m \)-GD\(_n\) hierarchy. When \( m > 1 \), the corresponding Frobenius manifolds are nonsemisimple.

1. Introduction

Since the pioneering discovery by Zamolodchikov in [8], W-algebras have been an active field of theoretical and mathematical physics, see e.g. [20, 21, 32, 33, 43, 51, 47, 48, 49, 50] and references therein. It is well known that classical realizations of W-algebras appear naturally as Poisson brackets of integrable hierarchies of Lax type. For example, the Virasoro algebra \( W_2 \) is realized as the Magri bracket for the KdV hierarchy [2, 9], and the Zamolodchikov-Fateev-Lykyanov \( W_n \)-algebra as the second Adler-Gelfand-Dickey (AGD briefly) bracket for the \( n^{th} \)-order Gelfand-Dickey (GD\(_n\)) hierarchy [10, 11, 12, 13]. The extension of the \( W_n \)-algebra to the nonlinear \( W_{\infty} \)-algebra containing all higher spin \( s \) currents with \( s \geq 2 \) has been intensively studied. For instance, a class of such algebras defined by the \( W_{\text{KP}}^{(n)} \)-algebra and its reduction the \( W_{\infty}^{(n)} \)-algebra are realized as the second Hamiltonian structure in the \( n^{th} \)-order Hamiltonian pair of the Kadomtsev-Petviashvili (KP) hierarchy. Free-field relations of W-algebras have also been obtained by constructing the related Miura maps, please see e.g. [18, 22, 26, 27] and references therein for details.

In [25], A.Bilal proposed a non-local matrix generalization of the well-known \( W_n \)-algebra, called the \( V_{m,n} \)-algebra, by constructing the second AGD bracket associated

\[
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\]

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with a matrix differential operator of order \( n \)

\[
\mathcal{L} = -I_m \partial^n + U_1 \partial^{n-1} + U_2 \partial^{n-2} + \cdots + U_n
\]

\( = -(I_m \partial - P_1) \cdots (I_m \partial - P_n) \), \( \partial = \frac{\partial}{\partial x} \), \( P_j, U_j \in gl(m, \mathbb{C}) \).

The key point of his construction was to use the usual matrix trace map which ensures the identity

\[
\text{trace} \int \text{res} AB \, dx = \text{trace} \int \text{res} BA \, dx,
\]

where \( A \) and \( B \) are two arbitrary matrix (pseudo)-differential operators and \( \text{res} A = \) the coefficient of \( \partial^{-1} \). Upon reducing to \( U_1 = 0 \), the non-commutativity of matrices implies the presence of non-local terms in the \( V_{m,n} \)-algebra. A Miura transformation relates these Poisson brackets of the \( U_j \) to much simpler ones of a set of \( P_i \in gl(m, \mathbb{C}) \), i.e., the Kupershmidt-Wilson type theorem. Contrary to the scalar case, generally \( P_i \) are not free fields. It is difficult to give such a free-field realization because of the non-local terms except some special cases \([23, 24]\). Related results have also been generalized to the matrix-KP hierarchy in \([29, 35, 37]\). Besides these, we want to comment that in the scalar case when one takes the dispersionless limit of the GD\(_n\) hierarchy with \( U_1 = 0 \), the corresponding \( W_n \)-algebra reduces to the \( w_n \)-algebra. But because of the non-locality, the \( V_{m,n} \)-algebra for \( m > 1 \) will diverge under the dispersionless limit. In other words, there is no matrix analogue of the \( w_n \)-algebra according to this construction.

The aim of this paper is to give a new local matrix generalization of the classical \( W \)-algebra, called the \( W_{m,n}^{\text{KP}} \)-algebra, by constructing the second AGD bracket associated with a \( Z_m \)-valued pseudo-differential operator (ΨDO)

\[
L = I_m \partial + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots, \quad U_j \in Z_m,
\]

where \( Z_m = \mathbb{C}[\Lambda]/(\Lambda^m) \) is a maximal commutative subalgebra of \( gl(m, \mathbb{C}) \) and \( \Lambda = (\delta_{i,j+1}) \in gl(m, \mathbb{C}) \). Under some reductions, we could get other \( W \) algebras including the \( W_{\infty}^{(m,n)} \)-algebra, the \( W_{\text{GD}}^{(m,n)} \)-algebra and the \( W_{(m,n)} \)-algebra. This paper is organized as follows.

In section 2, firstly we introduce the \( Z_m \)-KP hierarchy given by

\[
\frac{\partial L}{\partial t_r} = [L^r_+, L],
\]

where \( t_r, r = 1, 2, \cdots \), are some variables and \( L^r_+ \) is the pure differential part of the operator \( L^r \). Let us remark that the \( Z_1 \)-KP hierarchy is exactly the KP hierarchy (cf.\([5, 6, 38]\)). The \( Z_m \)-KP hierarchy has been proposed in \([35]\) as a commutative KP hierarchy, which is also equivalent to the coupled KP hierarchy introduced by P.Casati and G.Ortenzi in \([41]\). The reason is as follows. In \([41]\), P.Casati and G.Ortenzi used vertex operator representations of polynomial Lie algebras and a boson-fermion type of correspondence to
obtain coupled Hirota bilinear equations of this hierarchy. J. Van de Leur in [42] obtained elementary Bäcklund-Darboux transformations and the Lax representation with the \( \mathbb{Z}_m \)-valued Lax operator \( L \) in (1.1).

Afterwards, we use the canonical AGD-method ([1, 3, 38]) to construct Hamiltonian structures of the \( \mathbb{Z}_m \)-KP hierarchy. The crucial point is to introduce a new trace-type map \( \text{tr}_m : gl(m, \mathbb{C}) \rightarrow \mathbb{C} \) defined by

\[
\text{tr}_m(Z) = \text{the trace of } \begin{pmatrix} 1/m & 1/m-1 & \cdots & 1 \\ 0 & 1/m & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/m \end{pmatrix} Z, \tag{1.2}
\]

instead of the usual matrix trace because of the speciality of \( \mathbb{Z}_m \). Furthermore, we propose local matrix generalizations of the classical W-algebras based on the second Hamiltonian structure of the \( \mathbb{Z}_m \)-KP hierarchy. Finally we give some examples to illustrate our constructions.

In section 3, we firstly discuss the transformation of the second Hamiltonian structure by the factorization \( L^n = \mathcal{L}_r \mathcal{L}_{r-1} \cdots \mathcal{L}_1 \), where \( \mathcal{L}_j = I_m \partial^{n_j} + P_{j,n_j-1} \partial^{n_j-1} + \cdots \) are \( \mathbb{Z}_m \)-valued \( \Psi \)DOs. Afterwards, we construct free-field realizations of the \( W^{(m,n)}_{\text{KP}} \)-algebra and the \( W^{(m,n)}_{\infty} \)-algebra. By considering the reduction \( L^n = L^n_+ \), we obtain free-field realizations of the \( W^{(m,n)}_{\text{GD}} \)-algebra and the \( W_{(m,n)} \)-algebra. Finally we give an example of the free-field realization of the \( W_{2,2} \)-algebra.

In section 4, we study the dispersionless analogue of the \( \mathbb{Z}_m \)-KP. We describe similar results about the bi-hamiltonian structure, the Miura map, local matrix \( w \)-type algebras and their free-field realizations without proofs.

In section 5, we propose a conjectural result about the relation between Frobenius manifolds and the dispersionless \( \mathbb{Z}_m \)-GD\(_n \) (briefly \( \mathbb{Z}_m \)-dGD\(_n \)) hierarchy and check the validity of the conjecture for the dispersionless \( \mathbb{Z}_m \)-KdV (briefly \( \mathbb{Z}_m \)-dKdV) hierarchy and the dispersionless \( \mathbb{Z}_2 \)-Boussinesq (briefly \( \mathbb{Z}_2 \)-dBoussinesq) hierarchy. The last section is devoted to the conclusion.

2. Hamiltonian structures of the \( \mathbb{Z}_m \)-KP hierarchy

In this section, we shall introduce the \( \mathbb{Z}_m \)-KP hierarchy and construct its Hamiltonian structures. We will use the following notations: for a \( \mathbb{Z}_m \)-valued operator \( P = \sum_i P_i \partial^i \), \( P_+ \) is the pure differential part of the operator \( P \) and

\[
P_- = P - P_+, \quad \text{res}(P) = P_{-1}, \quad P^* = \sum_i (-1)^i \partial^i P_i, \quad \partial = \frac{\partial}{\partial x}.
\]
2.1. The $\mathbb{Z}_m$-KP hierarchy. Let

$$L = I_m \partial + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots$$

(2.1)

be a $\mathbb{Z}_m$-valued $\Psi$DO with coefficients $U_1, U_2, \cdots$ being smooth $\mathbb{Z}_m$-valued functions of an infinite many variables $t = (t_1, t_2, \cdots)$ and $t_1 = x$.

**Definition 2.1.** The $\mathbb{Z}_m$-KP hierarchy is the set of equations

$$\frac{\partial L}{\partial t_r} = [B_r, L], \quad B_r = L^r_+, \quad r = 1, 2, \cdots$$

(2.2)

or equivalently,

$$\frac{\partial B_l}{\partial t_r} - \frac{\partial B_r}{\partial t_l} + [B_l, B_r] = 0.$$  

(2.3)

**Example 2.2.** [ $r = 2$ and $l = 3$ ]. By using $B_2 = \partial^2 + 2U_1$ and $B_3 = \partial^3 + 3U_1 \partial + 3U_2 + 3U_1 x$, the system (2.3) becomes

$$U_{1,t_2} = U_{1,xx} + 2U_{2,x}, \quad 2U_{1,t_3} = 2U_{1,xxx} + 3U_{2,xx} + 3U_{2,t_2} + 6U_1 U_{1,x}.$$  

(2.4)

If we eliminate $U_2$ in (2.4) and rename $t_2 = y$, $t_3 = t$ and $U = U_1$, we obtain

$$(4U_t - 12UU_x - U_{xxx})_x - 3U_{yy} = 0.$$  

(2.5)

When $U = U(x, y, t)$ is a smooth scalar function, the system (2.5) is the classical KP equation. When we choose a $\mathbb{Z}_2$-valued smooth function $U = \begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix}$, the system (2.5) reads the coupled KP equation

$$\begin{cases}
(4u_{0t} - 12u_0u_{0x} - u_{0xxx})_x - 3u_{0yy} = 0, \\
(4u_{1t} - 12u_0u_{1x} - 12u_{0x}u_1 - u_{1xxx})_x - 3u_{1yy} = 0.
\end{cases}$$

Especially if $u_{0y} = u_{1y} = 0$, the coupled KP equation reduces to the coupled KdV equation

$$\begin{cases}
4u_{0t} - 12u_0u_{0x} - u_{0xxx} = 0, \\
4u_{1t} - 12u_0u_{1x} - 12u_{0x}u_1 - u_{1xxx} = 0.
\end{cases}$$

(2.6)

Generally, by imposing the constraint $(L^n)_- = 0$, the $\mathbb{Z}_m$-KP hierarchy (2.2) reduces to the $\mathbb{Z}_m$-GD$_n$ hierarchy.

Let us remark that at the level of equations, there are subtle differences between the scalar KP hierarchy and the $\mathbb{Z}_m$-KP hierarchy by using $\mathbb{Z}_m$-valued function instead of the scalar function. Therefore, we easily generalize the Sato theory and the symmetry theory of the KP hierarchy to the $\mathbb{Z}_m$-KP hierarchy ([5, 38]). For instance,
Theorem 2.3. (1). There exists a $\mathbb{Z}_m$-valued function $\tau = \tau(t_1, t_2, \cdots)$ describing the whole system such that

$$\text{res } L^i = \frac{\partial}{\partial t_i} (\tau \tau^{-1})$$

(2.7)

The $\tau$-function is determined up to multiplication by $C_0 \exp(\sum_{i=1}^{\infty} C_i t_i)$, where all $C_j \in \mathbb{Z}_m$ are constant matrices. Moreover

(2). Let $\tau_0 = \tau_0(t; a_1, a_2, \cdots)$ be a $\tau$-function of the $\mathbb{Z}_1$-KP hierarchy depending on constants $a_1, a_2, \cdots$. Then $\tau = I_m \tau_0(t; A_1, A_2, \cdots)$ is a $\mathbb{Z}_m$-valued $\tau$-function of the $\mathbb{Z}_m$-KP hierarchy, where all $A_i = a_i I_m + \sum_{k=1}^{m-1} b_{k,i} \Lambda^k \in \mathbb{Z}_m$ are constant matrices.

Proof. The proof of the first part is an easy and direct generalization of the scalar case (e.g., Chapters 1,6,7 in [38]). The second part follows from (1) and the definition of the $\mathbb{Z}_m$-KP hierarchy.

Let us come back to the coupled KdV equation (2.6) again. Assume that

$$\tau = \begin{pmatrix} \tau_0 & 0 \\ \tau_1 & \tau_0 \end{pmatrix}$$

is its $\mathbb{Z}_2$-valued $\tau$-function, then we have

$$\begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix} = U = \frac{\partial}{\partial x} (\tau \tau^{-1}) = \begin{pmatrix} (\log \tau_0)_{xx} & 0 \\ (\tau_1)_x & (\log \tau_0)_{xx} \end{pmatrix}.$$ 

So

$$u_0 = (\log \tau_0)_{xx}, \quad u_1 = \frac{\tau_1}{\tau_0}_x.$$ 

(2.8)

We remark that the variable transformation (2.8) has been used to derive the coupled KdV equation from the Hirota equation in [41]. Furthermore, taking a $\tau$-function $\tau_0 = 1 + \exp(2ax + 2a^3t)$ of the KdV equation, it follows from (2) in the theorem 2.3 that for $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, the function

$$\tau = I_2 + \exp(2Ax + 2A^3t) = \begin{pmatrix} 1 + \exp(2ax + 2a^3t) & 0 \\ (2bx + 2b^3t) \exp(2ax + 2a^3t) & 1 + \exp(2ax + 2a^3t) \end{pmatrix}$$

is a $\tau$-function of the coupled KdV equation (2.6). Consequently, we obtain a solution of (2.6) given by

$$u_0 = \frac{a^2}{\cosh^2(ax + a^3t)}, \quad u_1 = \frac{b + 2abx + 2ab^3t + b \exp(2ax + 2a^3t)}{2\cosh^2(ax + a^3t)}.$$
2.2. Hamiltonian structures of the \( \mathbb{Z}_m \)-KP hierarchy. In this subsection, we will use the AGD-scheme to construct Hamiltonian structures of the \( \mathbb{Z}_m \)-KP hierarchy. As said in the introduction, the key point is to use the new trace-type map \( \text{tr}_m: gl(m, \mathbb{C}) \rightarrow \mathbb{C} \) in [1.2].

Let
\[
L = I_m \partial + U_0 + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots ,
\]
be a \( \mathbb{Z}_m \)-valued \( \Psi \)DO with an additional term \( U_0 \). Denoting
\[
L := L_n = I_m \partial^n + V_{n-1} \partial^{n-1} + V_{n-2} \partial^{n-2} + \cdots ,
\]
then we have
\[
V_i = \sum_{q=0}^{m-1} v_{[i]q} \Lambda^q = nU_{n-i-1} + Q_i(U_0, \cdots, U_{n-i-2}),
\]
where \( Q_i, i = n-1, n-2, \cdots, \) are \( \mathbb{Z}_m \)-valued differential polynomials of its arguments. In the following our Hamiltonian structures will be established in terms of the “dynamical coordinates” \( \{v_{[i]q}\} \).

We denote by \( \mathcal{A} \) the differential algebra of polynomials in formal symbols \( \{v_{[i]q}\} \), where
\[
v_{[i]q}^{(j)} = \frac{\partial^j v_{[i]q}}{\partial x^j} \quad \text{for} \quad q = 0, \cdots, m-1 \quad \text{and} \quad j = 0, 1, \cdots .
\]
For any \( f(v) \in \mathcal{A} \), it is easy to see that \( f(v) \) could be written as \( \text{tr}_m f(V) \), where \( f(V) \) is now regarded as a \( \mathbb{Z}_m \)-valued differential polynomial of \( \{V_i\} \). Besides the ordinary differentiation \( \partial \), we may construct many other differentiations in \( \mathcal{A} \). Among them, the differentiation commuting with \( \partial \) will play an exceptional role. Suppose \( a = (a_{n-1}, a_{n-2}, \cdots) \) with the elements
\[
a_i = \sum_{q=0}^{m-1} a_{[i]q} \Lambda^q \in \mathbb{Z}_m, \quad i = n-1, n-2, \cdots ,
\]
we define a vector field for \( a \)
\[
\partial_a = \sum_{i=-\infty}^{n-1} \sum_{j=0}^{\infty} \sum_{q=0}^{m-1} a_{[i]q}^{(j)} \partial \frac{\partial}{\partial V_{[j]q}}.
\]
Obviously, \( \partial_a \) and \( \partial \) commute, i.e.,
\[
\partial \partial_a f = \partial_a \partial f, \quad \text{for} \quad f \in \mathcal{A}.
\]
The set of all vector fields \( \partial_a \) will be denoted by \( \mathcal{V} \). For our purpose, we would like to write the vector field \( \partial_a \) in (2.12) as the form
\[
\partial_a = \sum_{i=-\infty}^{n-1} \sum_{j=0}^{\infty} \text{tr}_m a_i^{(j)} \partial \frac{\partial}{\partial V_{[j]}}.
\]
where
\[
\frac{\partial}{\partial V^{(j)}} = I_m \frac{\partial}{\partial v_{m-1}^{(j)}} + \sum_{q=1}^{m-1} \Lambda^q \left( \frac{\partial}{\partial v_{m-q}^{(j)}} - \frac{\partial}{\partial v_{m-q}^{(j-1)}} \right). \tag{2.15}
\]

We denote the space of functionals by
\[
\tilde{A} = \left\{ \tilde{f} = \int f(v) dx = \int \text{tr}_m f(V) dx \mid f \in A \right\}.
\]
The variational derivative \(\frac{\delta f}{\delta V}\) is defined by
\[
\tilde{f}(v + \delta v) - \tilde{f}(v) = \int \text{tr}_m \left( \frac{\delta f}{\delta V} \delta V + o(\delta V) \right) dx
\]
\[
= \int \sum_{q=0}^{m-1} \left( \frac{\delta f}{\delta v_q} \delta v_q + o(\delta v) \right) dx,
\]
where \(V = \sum_{q=0}^{m-1} v_q \Lambda^q, \delta V \in \mathcal{Z}_m\) and \(\delta v = (\delta v_0, \cdots, \delta v_{m-1})\). A direct computation gives
\[
\frac{\delta f}{\delta V} = \sum_{j=0}^{\infty} (-\partial)^j \frac{\partial f}{\partial V^{(j)}} \in \mathcal{Z}_m, \quad \frac{\delta f}{\delta v_q} = \sum_{j=0}^{\infty} (-\partial)^j \frac{\partial f}{\partial v_q^{(j)}}. \tag{2.16}
\]
Due to the formula (2.13), the action of \(V\) on \(A\) can be transferred to \(\tilde{A}\):
\[
\partial_a \tilde{f} = \partial_a \int f dx = \int \partial_a f dx
\]
\[
= \text{tr}_m \int \sum_{i=-\infty}^{n-1} \sum_{j=0}^{\infty} a_i^{(j)} \frac{\partial f}{\partial V_i^{(j)}} dx
\]
\[
= \text{tr}_m \int \sum_{i=-\infty}^{n-1} a_i \frac{\delta f}{\delta V_i} dx. \tag{2.17}
\]
If we identify the vector \(a = (a_{n-1}, a_{n-2}, \cdots)\) with the \(\mathcal{Z}_m\)-valued \(\Psi DO\) \(a = \sum_{i=-\infty}^{n-1} a_i \partial^i\), then the formula (2.17) can be written as
\[
\partial_a \tilde{f} = \text{tr}_m \int \text{res} a \frac{\delta f}{\delta \mathcal{L}} dx, \tag{2.18}
\]
where the operator \(\frac{\delta f}{\delta \mathcal{L}}\) is defined by
\[
\frac{\delta f}{\delta \mathcal{L}} = \sum_{i=-\infty}^{n-1} \partial^{-i-1} \frac{\delta f}{\delta V_i}. \tag{2.19}
\]
Furthermore, it is easy to verify that
\[
(1) \quad \partial_a L^n = a; \quad (2) \quad [\partial_a, \partial_b] = \partial_a \partial_b - \partial_b \partial_a = \partial_{[a,b]} = \partial_{\partial a - \partial_a b}.
\]
therefore \((\mathcal{V}, [, ]\)) is a Lie algebra. Let \(\Omega^1\) be the dual space of \(\mathcal{V}\) consisting of formal \(\mathbb{Z}_m\)-valued integral operators
\[
X = \sum_{i=-\infty}^{n-1} \partial^{-i-1}X_i, \quad X_i \in \mathbb{Z}_m
\]
with the pairing
\[
\langle \partial_a, X \rangle = \langle a, X \rangle = \text{tr}_m \int \text{res} \, aX \, dx. \quad (2.20)
\]
So by using the formula (2.18), we see that
\[
\langle \partial_a, \delta f \delta L \rangle = \partial_a \tilde{f} = \langle \partial_a, d \tilde{f} \rangle, \quad d \tilde{f} = \frac{\delta f}{\delta L} \in \Omega^1. \quad (2.21)
\]

**Lemma 2.4.** Suppose \(A = \sum_{i=-\infty}^{k} A_i \partial^i\) and \(B = \sum_{j=-\infty}^{l} B_j \partial^j\) are two \(\mathbb{Z}_m\)-valued \(\Psi\) DOs, then there exists a \(\mathbb{Z}_m\)-valued function \(h\) such that
\[
\text{res} \left[ A, B \right] = \frac{\partial h}{\partial x}. \quad (2.22)
\]

**Proof.** By linearity, it is sufficient to prove (2.23) for any two \(\mathbb{Z}_m\)-valued monomials \(A = A_i \partial^i, B = B_j \partial^j\). If \(i, j \geq 0\) or \(i + j < 1\), then \(\text{res} \left[ A, B \right] = 0\) and so \(h = 0\). Thus we only need consider the case \(i \geq 0, j < 0\) and \(i + j \geq 1\). A direct computation gives
\[
\text{res} \left[ A, B \right] = C_{i+j+1}^i \left( A_i B_j^{i+j+1} + (-1)^{i+j} B_j A_i^{i+j+1} \right)
\]
\[
= \frac{\partial}{\partial x} \left( C_{i+j+1}^i \sum_{s=0}^{i+j} (-1)^s A_i^{(s)} B_j^{i+j-s} \right) := \frac{\partial}{\partial x} h.
\]
Obviously \(h\) is \(\mathbb{Z}_m\)-valued. \(\square\)

Taking the trace-type map \(\text{tr}_m\) on both sides of (2.22), we obtain
\[
\text{tr}_m \, \text{res} \left[ A, B \right] = \text{tr}_m \, \frac{\partial h}{\partial x} \quad (2.23)
\]
and
\[
\text{tr}_m \, \int \text{res} \, AB \, dx = \text{tr}_m \, \int \text{res} \, BA \, dx. \quad (2.24)
\]
When \(m = 1\), this formula is very crucial in the construction of Hamiltonian structures of the KP hierarchy. Similar to the KP hierarchy, we have

**Theorem 2.5.** Let us define the Adler map \(H : \Omega^1 \rightarrow \mathcal{V}\) based on \(\mathcal{L}\) by
\[
H^n(X) = (\hat{\mathcal{L}}X)_+ \hat{\mathcal{L}} - \hat{\mathcal{L}}(X \hat{\mathcal{L}})_+,
\]
(2.25)
where \(\hat{\mathcal{L}} = \mathcal{L} - z^n\) and \(z \in \mathbb{C}\). Then the Adler map \(H\) is a Hamiltonian mapping.
Proof. Observe that $\mathcal{Z}_m$ is a commutative subalgebra of $gl(m, \mathbb{C})$, and using the formulas (2.23) and (2.24), it is easy to know that the proof is a straightforward generalization of the scalar case (e.g., Chapters 2, 3, 5 in [38]).

Denoting $H^{n(0)}(X) = (\mathcal{L}X)_+\mathcal{L} - \mathcal{L}(X\mathcal{L})_+$ and $H^{n(\infty)}(X) = [\mathcal{L}_-, X_+]_+ - [\mathcal{L}_+, X_-]_+$, obviously we have

$$H^n(X) = H^{n(0)}(X) + z^n H^{n(\infty)}(X).$$

This means that both $H^{n(0)}$ and $H^{n(\infty)}$ are Hamiltonian mappings, hence the first and the second Poisson brackets of the $\mathcal{Z}_m$-KP hierarchy associated with the $\mathcal{Z}_m$-valued $\Psi$DO $L^n$ are given by

$$\{\tilde{f}, \tilde{g}\}^{n(\infty)} = \text{tr}_m \int \text{res} H^{n(\infty)}(X) \frac{\delta f}{\delta \mathcal{L}} \frac{\delta g}{\delta \mathcal{L}} dx + \mathcal{L}_r dx,$$

and

$$\{\tilde{f}, \tilde{g}\}^{n(0)} = \text{tr}_m \int \text{res} H^{n(0)}(X) \frac{\delta f}{\delta \mathcal{L}} \frac{\delta g}{\delta \mathcal{L}} dx + \mathcal{L}_r dx,$$

where $\tilde{f}, \tilde{g} \in \tilde{A}$ are two functionals. Since $n$ is arbitrary, therefore we have infinite series of pairs of the Hamiltonian structures.

**Proposition 2.6.** The Hamiltonians of the $\mathcal{Z}_m$-KP hierarchy corresponding to two Poisson brackets (2.28) and (2.29) in the $n^{th}$ pair are

$$\tilde{h}_r = -\frac{n}{r + n} \text{tr}_m \int \text{res} L^{n+r} dx$$

and

$$\tilde{g}_r = \frac{n}{r} \text{tr}_m \int \text{res} L^r dx.$$

Proof. Observe that by using (2.24), we could get

$$\delta \text{tr}_m \int \text{res} L^r dx = \frac{r}{n} \text{tr}_m \int \text{res} L^{r-n} \delta \mathcal{L} dx,$$

which means

$$\frac{\delta}{\delta \mathcal{L}} \text{tr}_m \int \text{res} L^r dx = \frac{r}{n} L^{r-n}$$.
So we have
\[ \partial_t L = H^{(\infty)} \left( \frac{\delta \tilde{h}_V}{\delta \mathcal{L}} \right) = -[L^n, L^*_+] + [L^n_+, L^*_+] = [L^*_+, L] \]
and
\[ \partial_t L = H^{(0)} \left( \frac{\delta \tilde{h}_V}{\delta \mathcal{L}} \right) = (L^n L^{r-n})_+ L^n - L^n (L^n L^{r-n})_+ = [L^*_+, L]. \]
We thus complete the proof. \(\square\)

Similarly, if we restrict to \(V_{n-1} = 0\), it is easy to check that the first Hamiltonian structure automatically reduces to this submanifold, but the second one is reducible if and only if
\[ \text{res} \left[ \mathcal{L}, \frac{\delta f}{\delta \mathcal{L}} \right] = 0. \] (2.30)

We denote the corresponding reduced brackets by \(\{ , \}^{(\infty)}\) and \(\{ , \}^{(0)}_D\). Assume that \(V_n = I_m\) and \(X_i = \frac{\delta f}{\delta V_i} \in \mathcal{Z}_m\), we then get
\[ \text{res} \left[ \mathcal{L}, \frac{\delta f}{\delta \mathcal{L}} \right] = \text{res} \left[ \sum_{j=\infty}^n V_j \partial_j \sum_{i=\infty}^{n-1} \partial^{-i-1}X_i \right] = - \sum_{i=\infty}^{n-1} \sum_{j=i+1}^n \left( -i - 1 \right) \left( \frac{j - i}{j - i} \right) (X_i V_j)^{(j-i)} \]
\[ = nX_{n-1} - \sum_{i=\infty}^{n-2} \left( -i - 1 \right) X_i^{(n-i-1)} - \sum_{i=\infty}^{n-1} \sum_{j=i+1}^{n-1} \left( -i - 1 \right) \left( X_i V_j \right)^{(j-i)}. \]
This means that the condition (2.30) is equivalent to
\[ X_{n-1} = \frac{1}{n} \sum_{i=\infty}^{n-2} \left( -i - 1 \right) X_i^{(n-i-1)} + \sum_{j=i+1}^{n-1} \left( -i - 1 \right) \left( X_i V_j \right)^{(j-i-1)}. \] (2.31)

Observe that there is no nonlocal term in \(X_{n-1}\) because of \(V_i, X_j \in \mathcal{Z}_m\), which is different from the results in [25, 29].

**Definition 2.7.** In terms of the basis \(\{ v_{[i]} \}\), the second Poisson bracket \(\{ , \}^{(0)}\) for \(L^n\) in (2.10) and the reduced bracket \(\{ , \}^{(0)}_D\) for \(L^n\) with the constraint \(V_{n-1} = 0\) will provide two kinds of W-type algebras, we call them the \(W_{KP}^{(n)}\)-algebra and the \(W_{\infty}^{(n)}\)-algebra respectively. Under the reduction \(L^n = 0\), the corresponding algebras are called the \(W_{GD}^{(m,n)}\)-algebra and the \(W_{(m,n)}\)-algebra respectively. All of them are local matrix generalizations.

To end up this section, we give some examples to illustrate our constructions.

**Example 2.8.** Taking \(\varphi(x) = \text{tr}_m V_{n-2}(x)\), it follows from (2.31) and (2.29) that the reduced Poisson bracket is given by
\[ \{ \varphi(x), \varphi(y) \}^{(0)}_D = - \left( \frac{n^3 - n}{12} \partial^3 + \varphi \partial + \partial \varphi \right) \delta(x - y). \]
This means that both the \( W^{(m,n)} \)-algebra and the \( W_{(m,n)} \)-algebra contain the Virasoro algebra as its subalgebra. Besides these, by analogy with tenuous and similar computations in [25], one can show that the \( W_{(m,n)} \)-algebra has the conformal property. In other words, the combinations of the \( V_k \) can be formed that are \( \mathbb{Z}_m \)-valued primary fields of spin \( k \).

Example 2.9. [The \( \mathbb{Z}_m \)-KdV hierarchy] In this case, \( L^2 = 0 \) and we denote
\[
L^2 = I_m \partial^2 + V, \quad X = \partial^{-2} X_1 + \partial^{-1} X_0, \quad Y = \partial^{-2} Y_1 + \partial^{-1} Y_0.
\]
The condition (2.31) becomes \( X_1 = \frac{1}{2} X_0' \), then we have
\[
H^{2(\infty)} = [X, L^2]_+ = -2X_0'
\]
and
\[
H^{2(0)}(X) = (L^2 X)_+ L^2 - L^2(XL^2)_+ = (X_0'' - 2X_1') \partial + 2V X_0' + X_0 V' - X_1'' + X_0'' = 2V X_0' + X_0 V' + \frac{1}{2} X_0''.
\]
Thus two Poisson brackets of the \( \mathbb{Z}_m \)-KdV hierarchy are given by
\[
\{ \tilde{f}, \tilde{g} \}^{2(\infty)} = 2 \text{tr}_m \int \frac{\delta f}{\delta V} \frac{\delta g}{\delta V} \partial \delta V dx
\]
and
\[
\{ \tilde{f}, \tilde{g} \}^{2(0)}_D = -\frac{1}{2} \text{tr}_m \int \frac{\delta f}{\delta V} \left( \frac{\partial^3}{\partial x^3} + 2V \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial x} V \right) \frac{\delta g}{\delta V} dx.
\]
More precisely, for \( V = \sum_{q=0}^{m-1} v_q \Lambda_q \in \mathbb{Z}_m \), we have
\[
\{ v_q(x), v_r(y) \}^{2(\infty)} = 2(\delta_{q,m-1-r} - \delta_{q,m-r}) \delta'(x - y)
\]
and
\[
\{ v_q(x), v_r(y) \}^{2(0)}_D = \sum_{s=0}^{m-1} \delta_{q,m-1-r+s}(J_{s-1} - J_s) \delta(x - y),
\]
where \( J_{-1} = 0 \), \( J_0 = \frac{1}{2} \partial^3 + v_0 \partial + \partial v_0 \) and \( J_k = v_k \partial + \partial v_k \), \( k = 1, \ldots, m-1 \).

Example 2.10. [The \( \mathbb{Z}_m \)-Boussinesq hierarchy] In this case, we have \( \mathcal{L} = I_m \partial^3 + V_1 \partial + V_0 \). Let us take
\[
\tilde{f} = \int f dx \in \tilde{A}, \quad \tilde{g} = \int g dx \in \tilde{A},
\]
and denote
\[
X_j = \frac{\delta f}{\delta V_j}, \quad Y_j = \frac{\delta g}{\delta V_j}, \quad j = 0, 1.
\]
Using the condition (2.31), we have
\[
\begin{align*}
\frac{\delta f}{\delta L} &= \partial^{-3}X_2 + \partial^{-2}X_1 + \partial^{-1}X_0, \\
\frac{\delta g}{\delta L} &= \partial^{-3}Y_2 + \partial^{-2}Y_1 + \partial^{-1}Y_0.
\end{align*}
\]
where \(X_2 = X'_1 - \frac{1}{3}X''_0 - \frac{1}{3}X_0V_1\) and \(Y_2 = Y'_1 - \frac{1}{3}Y''_0 - \frac{1}{3}Y_0V_1\).

A direct calculation gives two Poisson brackets of the \(Z_m\)-Boussinesq hierarchy
\[
\{ \tilde{f}, \tilde{g} \}^{3(\infty)} = 3 \text{tr}_m \int (X_1Y'_0 + X_0Y'_1)dx
\]
and
\[
\begin{align*}
\{ \tilde{f}, \tilde{g} \}^{3(0)} &= \text{tr}_m \int \left( \frac{2}{3} X_0Y''_0 - \frac{1}{3} X_0Y'_0 \right) V_1^2 dx \\
&+ \text{tr}_m \int \left( \frac{2}{3} X_1Y'_0 - \frac{2}{3} X_0Y'_0 + X'_1Y_0 - X_0Y''_0 \right) V_1 dx \\
&+ \text{tr}_m \int (X_0Y''_0 - X''_0Y_0 + 2X'_1Y_0 - X_1Y'_0 + X'_1Y_1 - X_1Y'_1) V_0 dx.
\end{align*}
\]
Especially, by analogy with the classical \(W\)-algebra in \([13, 14]\), we set
\[
W_2 = V_1, \quad W_3 = V_0 - \frac{1}{2}V'_1,
\]
then for any two \(Z_m\)-valued test functions \(F\) and \(G\), we have
\[
\begin{align*}
\{ \text{tr}_m \int FW_2 dx, \text{tr}_m \int GW_2 dx \}^{3(0)}_{D} &= \text{tr}_m \int \left( 2F^{(3)} + 2W_2F' + W_3F \right) G dx,
\end{align*}
\]
and
\[
\begin{align*}
\{ \text{tr}_m \int FW_3 dx, \text{tr}_m \int GW_3 dx \}^{3(0)}_{D} &= \text{tr}_m \int \left( 3W_3F' + W_3F \right) G dx,
\end{align*}
\]
and
\[
\begin{align*}
\{ \text{tr}_m \int FW_3 dx, \text{tr}_m \int GW_3 dx \}^{3(0)}_{D} &= \frac{1}{6} \text{tr}_m \int \left( 2FG - 2F'G \right) W_2^2 \\
&+ FG^{(5)} dx + \frac{1}{12} \text{tr}_m \int \left( 2FG^{(3)} - 2F^{(3)}G + 3F''G' - 3F'G'' \right) W_2 dx.
\end{align*}
\]
We thus confirm that \(W_k\) for \(k = 2, 3\) are spin-\(k\) conformally primary \(Z_m\)-valued fields. But notice that the equation \(\text{tr}_m FW_2^2 = (\text{tr}_m FW_2)^2\) has no \(Z_m\)-valued non-zero solution, which means the classical \(W_3\)-algebra is not a subalgebra of the \(W_{(m,3)}\)-algebra for \(m > 1\).
3. Free-field realizations of W-algebras

In this section, we want to construct free-field realizations of W-algebras obtained in the section 2.

3.1. Modifying the second Hamiltonian structure. In order to construct free-field realizations of W-algebras, firstly we discuss the transformation of the second Hamiltonian structure \( \{ , \}^{n(0)} \) by the following factorization

\[
\mathcal{L} = \mathcal{L}_r \mathcal{L}_{r-1} \cdots \mathcal{L}_1,
\]

(3.1)

where \( \mathcal{L} \) is defined in (2.10) and \( \mathcal{L}_j = I_m \partial^{n_j} + P_{j,n_j-1} \partial^{n_j-1} + \cdots \) are \( \mathbb{Z}_m \)-valued \( \Psi \)DOs and \( \sum_{j=1}^r n_r = n \). Comparing the same powers of \( \partial \) in both sides of (3.1), all \( V_s \) could be expressed as the differential polynomials in \( P_{j,n_j-k} \). Generally, this expression is called the general Miura map, which could be used to construct the modified \( \mathbb{Z}_m \)-KP hierarchy.

**Theorem 3.1.** The factorization (3.1) leads to

\[
\{ \tilde{f}, \tilde{g} \}^{n(0)} = \sum_{j=1}^r \{ \tilde{f}, \tilde{g} \}^{n_j(0)}
\]

(3.2)

and the constraint condition \( V_{n-1} = 0 \) is equivalent to

\[
\text{res} [\mathcal{L}, \frac{\delta f}{\delta \mathcal{L}}] = \sum_{j=1}^r \text{res} [\mathcal{L}_j, \frac{\delta f}{\delta \mathcal{L}_j}] = 0.
\]

(3.3)

**Proof.** Using the property of the trace-type map in (2.21), we firstly express \( \frac{\delta f}{\delta \mathcal{L}_j} \) in terms of \( \frac{\delta f}{\delta \mathcal{L}} \) by

\[
\delta \tilde{f} = \text{tr}_m \int \text{res} \frac{\delta f}{\delta \mathcal{L}} \mathcal{L} dx = \sum_{j=1}^r \text{tr}_m \int \text{res} \frac{\delta f}{\delta \mathcal{L}_j} \delta \mathcal{L}_j dx
\]

\[
= \sum_{j=1}^r \text{tr}_m \int \text{res} \frac{\delta f}{\delta \mathcal{L}} \mathcal{L}_r \cdots \mathcal{L}_{j+1} \delta \mathcal{L}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_1 dx
\]

\[
= \sum_{j=1}^r \text{tr}_m \int \text{res} \mathcal{L}_{j-1} \cdots \mathcal{L}_1 \frac{\delta f}{\delta \mathcal{L}} \mathcal{L}_r \cdots \mathcal{L}_{j+1} \delta \mathcal{L}_j dx
\]

which implies

\[
\frac{\delta f}{\delta \mathcal{L}_j} = \mathcal{L}_{j-1} \cdots \mathcal{L}_1 \frac{\delta f}{\delta \mathcal{L}} \mathcal{L}_r \cdots \mathcal{L}_{j+1} \mod R(-\infty, -n_j - 1).
\]

(3.4)
Here $R(-\infty, -k)$ contains all of the $\mathbb{Z}_m$-valued operators of the form $\sum_{j=-\infty}^{-k} A_j \partial^j$. It follows from (3.4) that

$$\sum_{j=1}^{r} \text{res} [\mathcal{L}_j, \frac{\delta f}{\delta \mathcal{L}_j}] = \text{res} (\mathcal{L}, \frac{\delta f}{\delta \mathcal{L}_1} - \frac{\delta f}{\delta \mathcal{L}_{-1}}) = \text{res} [\mathcal{L}, \frac{\delta f}{\delta \mathcal{L}}]$$

and

$$\mathcal{L}_{-1} \frac{\delta f}{\delta \mathcal{L}_{-1}} = \mathcal{L}_{j+1} = \mathcal{L}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_1 \frac{\delta f}{\delta \mathcal{L}} \mathcal{L}_{r} \cdots \mathcal{L}_{j+1} \mod R(-\infty, -1),$$

which yields the formula (3.2). \qed

3.2. The free-field realization for the $W^{(m,n)}_{\text{KP}}$-algebra. According to the theorem 3.1, we could simplify the construction of the free-field realization for $W^{(m,n)}_{\text{KP}}$-algebra to the construction of the free-field realization for each copy of the $W^{(m,1)}_{\text{KP}}$-algebra. For simplicity, for a given $s \in \mathbb{N}$, we define

$$\sigma_i := \begin{cases} 
1 & \text{if } 1 \leq i \leq s; \\
-1 & \text{if } s + 1 \leq i, j \leq n;
\end{cases}$$

$$\delta_{ij} := \begin{cases} 
1 & \text{if } i = j; \\
-1 & \text{if } i \neq j;
\end{cases}$$

$$\sigma_{ji} = \sigma_{ij} := \begin{cases} 
1 & \text{if } 1 \leq i, j \leq s; \\
0 & \text{if } 1 \leq i \leq s; s + 1 \leq j \leq n; \\
-1 & \text{if } s + 1 \leq i, j \leq n.
\end{cases}$$

Firstly, we factorize $\mathcal{L}$ in (2.10) as

$$\mathcal{L} = \mathcal{L}_1 \cdots \mathcal{L}_s \mathcal{L}_{s+1}^{-1} \cdots \mathcal{L}_n^{-1},$$

where

$$\mathcal{L}_j = I_m \partial + P_j, \quad P_j = \sum_{q=0}^{m-1} P_{[j]q} A^q \in \mathbb{Z}_m, \quad j = 1, \cdots, s_1$$

It follows from the theorem 3.1 that

$$\{\tilde{f}, \tilde{g}\}^{n(0)} = \sum_{j=1}^{n} \sigma_j \{\tilde{f}, \tilde{g}\}^{1(0)}_{\mathcal{L}_j} = \sum_{j=1}^{n} \sigma_j \text{tr}_m \int \frac{\delta f}{\delta P_j} \frac{\partial}{\partial x} \frac{\delta g}{\delta P_j} \, dx.$$ 

(3.6)

More precisely, for any two $\mathbb{Z}_m$-valued test functions $F$ and $G$, we have

$$\left\{ \text{tr}_m \int FPdx, \text{tr}_m \int GPdx \right\}^{n(0)} = \text{tr}_m \int \sigma_{ij} \delta_{ij} FG' \, dx,$$

(3.7)

Secondly we introduce $mn$ free fields $\{\varphi_{[j]q}(x)\}$ with the currents $j_{[j]q} = \varphi'_{[j]q}$ together with the Poisson bracket

$$\{\varphi_{[j]q}(x), \varphi'_{[j]r}(y)\}^{n(0)} = \sigma_{ij} \delta_{ij} \delta_{qr} \delta'(x - y),$$

(3.8)
where $i,j = 1, \cdots, n$ and $q,r = 0, \cdots, m-1$. Furthermore, we denote $K = (K_{ij}) \in gl(m, \mathbb{R})$ with the element $K_{ij} = \delta_{i,m+1-j} - \delta_{i,m+2-j}$. It is well known that every real symmetric matrix can be diagonalized. In other words, there exists an orthogonal matrix $Q$ such that $K = Q \text{diag}(\lambda_1, \cdots, \lambda_m)Q^t$, where $\lambda_j$ are eigenvalues of $K$ and $Q^t$ is the transpose of $Q$. Suppose that

$$A = (a_{ij}) = Q \text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_m}) \in gl(m, \mathbb{C})$$

and

$$\psi_{[j]q} = \sum_{s=0}^{m-1} a_{r+1,s+1} \varphi_{[j]s},$$

then $AA^t = K$ and

$$\{\psi_{[j]q}(x), \psi_{[j]r}(y)\}_{n(0)} = (\delta_{q,m-1-r} - \delta_{q,m-r})\sigma_{ij} \delta'(x-y),$$

By comparing (3.10) and (3.7), we get

**Proposition 3.2.** Let us assume that $P_{[j]q} = \psi'_{[j]q}$, i.e., $\mathcal{L}_j = I_m \partial + P_j = I_m \partial + \sum_{q=0}^{m-1} \psi'_{[j]q} \Lambda^q$, then the identification (3.5) gives rise to a free-field realization of the $W_{\mathbb{KP}}^{(m,n)}$-algebra.

3.3. The free-field realization for the $W_{\infty}^{(m,n)}$-algebra. If we restrict it to the submanifold $V_{n-1} = \sum_{j=1}^{n} P_j = 0$, we should take into account the following condition

$$\sum_{j=1}^{n} \sigma_j \frac{\delta f}{\delta P_j} x = 0.$$

More precisely, for any two $\mathcal{Z}_m$-valued test functions $F$ and $G$, we have

$$\left\{\text{tr}_m \int FP_1 dx, \text{tr}_m \int GP_2 dx\right\}_{D}^{n(0)} = \text{tr}_m \int (\sigma_{ij} \delta_{ij} - \frac{1}{n}) FG' dx.$$

Similarly, we introduce $m(n-1)$ free fields $\{\varphi_{[j]q}(x)\}$ with the currents $j_{[j]q} = \varphi'_{[j]q}$ together with Poisson bracket

$$\{\varphi_{[j]q}(x), \varphi'_{[j]r}(y)\}_{D}^{n(0)} = \sigma_{ij} \delta_{ij} \delta_{qr} \delta'(x-y),$$

where $i,j = 1, \cdots, n-1$ and $q,r = 0, \cdots, m-1$. Similarly taking

$$\psi_{[j]r} = \sum_{s=0}^{m-1} a_{r+1,s+1} \varphi_{[j]s},$$

as in (3.9), then we have

$$\{\psi_{[j]q}(x), \psi_{[k]r}(y)\}_{D}^{n(0)} = \sigma_{ij} (\delta_{q,m-1-r} - \delta_{q,m-r}) \delta_{ij} \delta'(x-y),$$
For simplicity, we write
\[ \psi_q = (\psi_{q[1]}, \cdots, \psi_{q[n-1]}), \quad q = 0, \cdots, m - 1. \]

Considering an \((n - 1)\)-dimensional Euclidean space and choosing an overcomplete set of \(n\) vectors \(\mathbf{h}_i = (h_1^i, \cdots, h_{n-1}^i),\ i = 1, \cdots, n\) such that
\[ \sum_{j=1}^n \sigma_j \mathbf{h}_j = 0, \quad \mathbf{h}_i \cdot \mathbf{h}_j = \sigma_{ij} \delta_{ij} - \frac{1}{n}. \quad (3.14) \]

**Theorem 3.3.** Let us assume that \(P_j[q] = \mathbf{h}_j \cdot \psi'_q\), i.e.,
\[ \mathcal{L}_j = I_m \partial + P_j = I_m \partial + \sum_{q=0}^{m-1} (\mathbf{h}_j \cdot \psi'_q) \Lambda_q \quad (3.15) \]
then the identification \((3.5)\) gives rise to a free-field realization for the \(W_{\infty}^{(m,n)}\)-algebra.

**Proof.** The constrained condition \(V_{n-1} = \sum_{j=1}^n P_j = 0\) follows from the first identity in \((3.14)\). Furthermore, using \((3.13)\) and \((3.14)\), we get
\[
\{ P_{[j]q}(x), P_{[k]r}(y) \}_{D}\n = \{ \mathbf{h}_j \cdot \psi'_q, \mathbf{h}_k \cdot \psi'_r \}_{D}\n
= \sum_{s,t=1}^{n-1} \{ h_j^s \psi'_{[s]q}(x), h_k^t \psi'_{[t]r}(y) \}_{D}\n
= \sum_{s,t=1}^{n-1} h_j^s h_k^t (\delta_{q,m-1-r} - \delta_{q,m-r}) \delta_{st} \delta'(x-y) \n
= (\delta_{q,m-1-r} - \delta_{q,m-r})(\sigma_{jk} \delta_{jk} - \frac{1}{n}) \delta_{qr} \delta'(x-y),
\]
which coincides with the reduced Poisson bracket \((3.11)\) because of the formula \(P_{[j]q} = \text{tr}_m (\Lambda^{m-q-1} P_j - \Lambda^{m-q} P_j)\). We thereby obtain the free-field realization of the \(W_{\infty}^{(m,n)}\)-algebra. \(\Box\)

**Remark 3.4.** Under the reduction \(L_n^m = 0\), the above results also present free-field realizations for the \(W_{GD}^{(m,n)}\)-algebra and the \(W_{(m,n)}\)-algebra.

To end up this section, we give an example about the free-field realization of the \(W_{2,2}\)-algebra.

**Example 3.5.** [The \(W_{2,2}\)-algebra]. In the example \(2.9\), we have obtained the second Hamiltonian structure for the \(\mathbb{Z}_2\)-KdV hierarchy. According to the above discussions, we factorize
\[
L^2 = I_2 \partial^2 + V = (I_2 \partial^2 - P)(I_2 \partial^2 + P), \quad P = \begin{pmatrix} p_0 & 0 \\ p_1 & p_0 \end{pmatrix},
\]
then we have
\[ v_0 = p_{0,x} - p_0^2, \quad v_1 = p_{1,x} - 2p_0p_1. \]
and
\[ \{ p_0(x), p_0(y) \}^{2(0)}_D = 0, \quad \{ p_0(x), p_1(y) \}^{2(0)}_D = \delta'(x - y), \quad \{ p_1(x), p_1(y) \}^{2(0)}_D = -\delta'(x - y). \]

For instance, taking \( \varphi'_0 = \sqrt{-3}(p_0 - p_1) \) and \( \varphi'_1(x) = \frac{2p_0 + p_1}{\sqrt{3}} \), we get
\[ \{ \varphi'_i(x), \varphi'_j(y) \}^{2(0)}_D = \delta_{ij} \delta'(x - y). \]

4. The dispersionless analogue of the \( \mathbb{Z}_m \)-KP hierarchy

In this section, we are interested in the dispersionless limit of the \( \mathbb{Z}_m \)-KP hierarchy, i.e., the dispersionless \( \mathbb{Z}_m \)-KP hierarchy (\( \mathbb{Z}_m \)-dKP in brief). As discussed in [40], using the Moyal bracket \( \{ \, , \} \), we could unify to discuss Hamiltonian structures of the KP hierarchy (\( \kappa = \frac{1}{2} \)) and the dKP hierarchy (\( \kappa = 0 \)). When \( \kappa \) approaches to 0, actually this process is the dispersionless limit. This means there are much similarities, so here we only list the results about the \( \mathbb{Z}_m \)-dKP hierarchy without proofs.

We will use the following notations in this part. For a \( \mathbb{Z}_m \)-valued Laurent series of the form \( A = \sum_i A_i p^i \), we denote by \( A_+ \) the polynomial part of the Laurent series \( A \) and \( A_- = A - A_+ \), \( \text{res} \ (A) = a_{-1} \). Let
\[ L = I_m p + U_1 p^{-1} + U_2 p^{-2} + \cdots, \tag{4.1} \]
be a \( \mathbb{Z}_m \)-valued Laurent series.

**Definition 4.1.** The \( \mathbb{Z}_m \)-dKP hierarchy is the set of equations of motion

\[ \partial_r L = \{ L^r_+, L \}, \quad \partial_r = \frac{\partial}{\partial t_r}, \tag{4.2} \]
where \( \{ \, , \} \) is defined by
\[ \{ A, B \} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial p}. \]

Let us assume that \( L^n, n \in \mathbb{N} \), is of the form
\[ \mathcal{L} := L^n = I_m \partial^n + V_{n-1} p^{n-1} + \cdots. \tag{4.3} \]
If imposing the constraint \( L^n_- = 0 \), the \( \mathbb{Z}_m \)-dKP hierarchy (4.2) reduces to the \( \mathbb{Z}_m \)-dGD\(_n\) hierarchy.
Taking a dispersionless limit of Hamiltonian structures for the $\mathbb{Z}_m$-KP hierarchy, we get the first and the second Poisson brackets of the $\mathbb{Z}_m$-dKP hierarchy associated with $\mathcal{L}$ in (4.3) as follows

$$\{\tilde{f}, \tilde{g}\}^{n(\infty)} = \text{tr}_m \int \text{res} \left( \left\{ \mathcal{L}_-, \left(\frac{\delta f}{\delta \mathcal{L}}\right)_+ \right\} - \left\{ \mathcal{L}_+, \left(\frac{\delta f}{\delta \mathcal{L}}\right)_- \right\} \right) \frac{\delta g}{\delta \mathcal{L}} \, dx \quad (4.4)$$

and

$$\{\tilde{f}, \tilde{g}\}^{n(0)} = \text{tr}_m \int \text{res} \left( (\mathcal{L} \frac{\delta f}{\delta \mathcal{L}})_+ \mathcal{L} - \mathcal{L} (\frac{\delta f}{\delta \mathcal{L}})_+ \right) \frac{\delta g}{\delta \mathcal{L}} \, dx, \quad (4.5)$$

where $\tilde{f}, \tilde{g} \in \tilde{\mathcal{A}}$ are two functionals. The variational derivative $\frac{\delta f}{\delta \mathcal{L}}$ is given by

$$\frac{\delta f}{\delta \mathcal{L}} = \sum_{i=-\infty}^{n-1} \frac{\delta f}{\delta V_i^p} - (i - 1), \quad (4.6)$$

where $\frac{\delta \mathcal{L}}{\delta V_i}$ is defined in (2.16). Similarly, when we restrict these to the submanifold $V_{n-1} = 0$, the first Hamiltonian structure automatically reduces to this sub manifold, but the second one is reducible if and only if

$$\text{res} \left\{ \mathcal{L}, \frac{\delta f}{\delta \mathcal{L}} \right\} = 0. \quad (4.7)$$

**Definition 4.2.** In terms of the basis $\{v_{ij}\}$, the second Poisson bracket $\{ , \}^{n(0)}$ for $L^n$ in (4.3) and the reduced bracket $\{ , \}^{D(n(0))}_D$ for $L^n$ with the constraint $V_{n-1} = 0$ will provide two kinds of $w$-type algebras, we call them the $w_{dKP}^{(m,n)}$-algebra and the $w_\infty^{(m,n)}$-algebra respectively.

**Theorem 4.3.** Let $\mathcal{L}$ in (1.3) be factorized by

$$\mathcal{L} = \mathcal{L}_r \mathcal{L}_{r-1} \cdots \mathcal{L}_1, \quad (4.8)$$

where $\mathcal{L}_j = I_m p^{n_j} + P_{j,n_j-1} p^{n_j-1} + \cdots$, are $\mathbb{Z}_m$-valued and $\sum_{j=1}^r n_r = n$. Then we have

$$\{\tilde{f}, \tilde{g}\}^{n(0)} = \sum_{j=1}^r \{\tilde{f}, \tilde{g}\}^{n_j(0)} \quad (4.9)$$

and the constraint condition $V_{n-1} = 0$ is equivalent to

$$\text{res} \left\{ \mathcal{L}, \frac{\delta f}{\delta \mathcal{L}} \right\} = \sum_{j=1}^r \text{res} \left\{ \mathcal{L}_j, \frac{\delta f}{\delta \mathcal{L}_j} \right\} = 0. \quad (4.10)$$

By using the theorem 4.3, we could construct free-field realizations of $w$-type algebras. For instance, under the reduction $L^n_- = 0$ we factorize

$$L^n = I_m \partial^n + \sum_{j=0}^{n-2} V_j p^j = \prod_{j=1}^r (I_m p + P_j)^{n_j}, \quad n = \sum_{j=1}^r n_j, \quad n_j \in \mathbb{N},$$
the associated w-algebra denoted by \( w_{[m,n_1,\ldots,n_j]} \) is a \( \mathbb{Z}_m \)-version of the \( w_{[n_1,\ldots,n_j]} \)-algebra in [28]. By analogy with the section 3, we could derive its free-field realization. The only difference is to choose vectors \( \vec{h}_j \) as follows

\[
\sum_{j=1}^n n_j \vec{h}_j = 0, \quad \vec{h}_i \cdot \vec{h}_j = \frac{1}{n_j} \delta_{ij} - \frac{1}{n}
\]

instead of (3.14). Especially, when all \( n_j = 1 \), the \( w_{[m,1,\ldots,1]} \)-algebra is the dispersionless limit of the \( W_{(m,n)} \)-algebra.

5. Frobenius manifolds and the dispersionless \( \mathbb{Z}_m \)-GD\(_n\) hierarchy

The concept of Frobenius manifold \( \mathcal{M} \) was introduced by B. Dubrovin [30] as a geometric formalism of the Witten-Dijkgraaf-E. Verlinde-H. Verlinde (WDVV) equation in 2D topological field theory [16, 17] given by

\[
\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \eta^{\mu \nu} \frac{\partial^3 F}{\partial v^\mu \partial v^\delta \partial v^\gamma} = \frac{\partial^3 F}{\partial v^\delta \partial t^\beta \partial v^\gamma} \eta^{\mu \nu} \frac{\partial^3 F}{\partial v^\mu \partial t^\alpha \partial v^\gamma},
\]

with a quasihomogeneity condition

\[
\mathcal{L}_E F = (3 - d) F + \text{quadratic terms}
\]

and a nondegenerate matrix \( (\eta^\alpha^\beta) \) with the component

\[
\frac{\partial^3 F}{\partial t^\alpha \partial v^\beta \partial v^1} = \eta^\alpha^\beta,
\]

where \( F = F(v) \) is a smooth function named as potential of the Frobenius manifold and \( v = (v^1, \ldots, v^m) \) is a flat coordinate on \( \mathcal{M} \). A Frobenius manifold \( \mathcal{M} \) is said to be semisimple if the Frobenius algebras \( T_v \mathcal{M} \) are semisimple for generic points \( v \in \mathcal{M} \).

Associated to every Frobenius manifold, there is a so-called principal hierarchy of Hamiltonian equations of hydrodynamic type, in which the unknown functions depend on one scalar spatial variable and some time variables. Conversely, from the bi-hamiltonian structure of the principal hierarchy, one can reconstruct the Frobenius manifold. Please see [30, 36] and references therein for details. The simplest example is \( \mathcal{M} = \mathbb{C} \), i.e. \( F(v^1) = \frac{1}{6}(v^1)^3 \), the corresponding principal hierarchy is the \( \mathbb{Z}_1 \)-dKdV hierarchy.

In this section, we want to propose a conjectural result between the \( \mathbb{Z}_m \)-dGD\(_n\) hierarchy based on the Lax operator

\[
\mathcal{L}_m = I_m p^n + V_{n-2} p^{n-2} + \cdots + V_0
\]

and the associated Frobenius manifold \( \mathcal{M}_m \), where \( V_j \) are \( \mathbb{Z}_m \)-valued functions. To make the expression clear, when \( m = 1 \), we write \( V_j = v_j \) for \( j = 0, \ldots, n - 2 \). It is well known
that when $m = 1$, the associated Frobenius manifold $\mathcal{M}_1 := (\mathcal{F}(v), e, E)$ is given by

$$\mathcal{F}(v) = \frac{n^2}{2(n+1)} \sum_{i=1}^{n-1} \frac{n+1-i}{i(n^2-i^2)} \text{res}_{p=\infty} L_n^{\frac{n-i}{n}} \text{res}_{p=\infty} L_n^{\frac{n-i}{n}}$$

and

$$e = \frac{\partial}{\partial v_0}, \quad E = \sum_{k=0}^{n-2} \frac{n-k}{n} v_k \frac{\partial}{\partial v_k}.$$ 

We remark that this Frobenius manifold is semisimple and the potential $\mathcal{F}(v)$ is a polynomial function of its arguments.

**Conjecture 5.1.** Suppose that $\mathcal{F}(v) = \mathcal{F}(v_0, v_1, \cdots, v_{n-2})$ is the potential function of the Frobenius manifold $\mathcal{M}_1$. Then when $m > 1$, the associated Frobenius manifold $\mathcal{M}_m := (\mathcal{F}(V), e, E)$ is nonsemisimple and given by

$$F(V) = \text{tr}_m \mathcal{F}(V_0, V_1, \cdots, V_{n-2}), \quad e = \text{tr}_m \frac{\partial}{\partial V_0}, \quad E = \sum_{k=0}^{n-2} \frac{n-k}{n} \text{tr}_m \left( V_k \frac{\partial}{\partial V_k} \right).$$

This conjecture is in some manner surprising, since it implies a method generating infinite nonsemisimple Frobenius manifolds from a given semisimple Frobenius manifold $\mathcal{M}_1$. A complete proof of this conjecture will be given in the subsequent paper [54]. In the following we verify the validity of this conjecture for the case $(n = 2, m)$ and the case $(n = 3, m = 2)$.

**Example 5.2. [The $\mathcal{Z}_m$-dKdV hierarchy].** In this case, the Lax operator of the $\mathcal{Z}_m$-dKdV hierarchy is given by

$$\mathcal{L} = I_m p^2 + V_0, \quad V_0 = \sum_{q=1}^{m} v^q \Lambda^{q-1} \in \mathcal{Z}_m.$$ 

Taking the dispersionless limit in the example 2.9, we obtain the bi-hamiltonian structure of the $\mathcal{Z}_m$-dKdV hierarchy denoted by

$$\{v^q(x), v^r(y)\}_1 = 2(\delta_{q,m+1-r} - \delta_{q,m+2-r}) \delta'(x - y) \quad (5.4)$$

and

$$\{v^q(x), v^r(y)\}_2 = \sum_{s=1}^{m} \delta_{q,m-r+s}(J_{s-1} - J_s) \delta(x - y), \quad (5.5)$$

where $J_0 = 0$ and $J_k = v^k \partial + \partial v^k$, $k = 1, \cdots, m$. From (5.4) and (5.5), using the result in [17] we could get a flat pencil of metrics given by $(\eta^{qr}(v))$ and $(g^{qr}(v))$ with

$$\eta^{qr}(v) = \delta_{q,m+1-r} - \delta_{q,m+2-r}, \quad g^{qr}(v) = \sum_{s=1}^{m} \delta_{q,m-r+s}(v^s - v^{s-1}),$$
where \( v^{-1} = 0 \). It is easy to check that from this pencil, we have the unit vector filed \( e \), the Euler vector field \( E \) and the potential \( F \) as follows

\[
F(v) = \frac{1}{6} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{\alpha} \sum_{\gamma=1}^{\beta} v^{\alpha+1-\beta} v^{\beta+1-\gamma} v^{\gamma} = \frac{1}{6} \text{tr}_m V_0^3
\]

and

\[
e = \frac{\partial}{\partial v^1} = \text{tr}_m \frac{\partial}{\partial V_0}, \quad E = \sum_{\alpha=1}^{m} v^\alpha \frac{\partial}{\partial v^\alpha} = \text{tr}_m (V_0 \frac{\partial}{\partial V_0}).
\]

**Example 5.3. [The \( \mathcal{Z}_m \)-dBoussinesq hierarchy].** In this case, the Lax operator is given by

\[
\mathcal{L} = I_m p^3 + V_1 p + V_0, \quad V_k \in \mathcal{Z}_m.
\]

Taking the dispersionless limit in the example 2.10, we obtain the bi-hamiltonian structure of the \( \mathcal{Z}_m \)-dBoussinesq hierarchy denoted by

\[
\left\{ \tilde{f}, \tilde{g} \right\}_1 = 3 \text{tr}_m \int (X_1Y_0' + X_0Y_1') dx, \quad \text{here} \quad X_k = \frac{\delta f}{\delta V_k}, Y_k = \frac{\delta f}{\delta V_k} \quad (5.6)
\]

and

\[
\left\{ \tilde{f}, \tilde{g} \right\}_2 = \frac{1}{3} \text{tr}_m \int (X_0Y'_0 - X_0'Y_0) V_1^2 dx + \text{tr}_m \int (X'_1Y_1 - X_1Y'_1) V_1 dx
\]
\[
+ \text{tr}_m \int (2X_1Y_0 - X_1Y_0' + X_0'Y_1 - 2X_0Y'_1) V_0 dx. \quad (5.7)
\]

When \( m = 1 \), it is easy to get a semisimple Frobenius manifold as follows

\[
F(V) = \frac{1}{2} V_0^2 V_1 - \frac{1}{72} V_1^4, \quad e = \frac{\partial}{\partial V_0}, \quad E = V_0 \frac{\partial}{\partial V_0} + \frac{2}{3} V_1 \frac{\partial}{\partial V_1}.
\]

When \( m = 2 \), we write

\[
V_0 = \begin{pmatrix} v_1 & 0 \\ v_2 & v_1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} v_3 & 0 \\ v_4 & v_3 \end{pmatrix} \in \mathcal{Z}_2.
\]

By using (5.6) and (5.7), we could get explicit formulas of \( \{ v_i(x), v_j(y) \}_k, k = 1, 2 \). According to the result in [7], we write

\[
\{ v_i(x), v_j(y) \}_1 = 3 \eta^{ij}(v) \delta'(x - y) + h_1(v; v_x) \delta(x - y)
\]

and

\[
\{ v_i(x), v_j(y) \}_2 = 3 g^{ij}(v) \delta'(x - y) + h_2(v; v_x) \delta(x - y)
\]
for known functions $h_k(v;v_x)$, where $\eta^{ij}(v)$ and $g^{ij}(v)$ form a flat pencil of metrics given by

$$(\eta^{ij}(v)) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}$$

and

$$(g^{ij}(v)) = \begin{pmatrix}
0 & -\frac{2}{9}v_3^2 & \frac{2}{9}v_3^2 & 0 & v_1 \\
-\frac{2}{9}v_3^2 & \frac{2}{9}v_3^2 - \frac{4}{9}v_3v_4 & v_1 & v_2 - v_1 \\
0 & v_1 & 0 & 2/3v_3 \\
v_1 & v_2 - v_1 & 2/3v_3 & 2/3(v_4 - v_3)
\end{pmatrix}.$$  

A direct calculation gives $(F(v), e, E)$ as follows

$$e = \frac{\partial}{\partial v_1} = \text{tr}_m \frac{\partial}{\partial V_0}$$

and

$$E = v_1 \frac{\partial}{\partial v_1} + v_2 \frac{\partial}{\partial v_2} + \frac{2}{3}v_3 \frac{\partial}{\partial v_3} + \frac{2}{3}v_4 \frac{\partial}{\partial v_4} = \text{tr}_m \left( V_0 \frac{\partial}{\partial V_0} \right) + \frac{2}{3} \text{tr}_m \left( V_1 \frac{\partial}{\partial V_1} \right)$$

and

$$F(v) = \frac{1}{2}v_1^2v_4 + \frac{1}{2}v_1^2v_3 + v_1v_2v_3 - \frac{1}{18}v_3^3v_4 - \frac{1}{72}v_3^4 = \frac{1}{2} \text{tr}_m (V_0^2V_1) - \frac{1}{72} \text{tr}_m V_1^4.$$  

6. Conclusions

We have described Hamiltonian structures of the (dispersionless) $\mathbb{Z}_m$-KP hierarchy and proposed local matrix versions of classical ($\mathbb{W}$-algebras) $W$-algebras and also constructed their free-field realizations and discussed the relation between (nonsemisimple) Frobenius manifolds and the $\mathbb{Z}_m$-dGD$_n$ hierarchy.

Actually, as discussed in [35], the $\mathbb{Z}_m$-KP hierarchy is exactly a commutative version of the KP-hierarchy. So using the trace-type map $\text{tr}_m : \mathfrak{gl}(m, \mathbb{C}) \longrightarrow \mathbb{C}$ defined in (1.2), it is not difficult to generalize our results to the following cases:

(1). the $\mathbb{Z}_m$-(2)BKP hierarchy and the $\mathbb{Z}_m$-CKP hierarchy; and

(2). the $\mathbb{Z}_m$-2Toda hierarchy and the extended $\mathbb{Z}_m$-Toda hierarchy; and

(3). the modified $\mathbb{Z}_m$-KP hierarchy and the $\mathbb{Z}_m$-Harry-Dym hierarchy etc; and their reductions, their constraints and their dispersionless analogues.

A more challenging problem is to study the relation between the $\mathbb{Z}_m$-dKP (or 2dBKP or 2dToda) hierarchy and infinite-dimensional Frobenius manifolds for $m > 1$. When $m = 1$, it has been studied in [14, 45, 46, 52, 53]. We will address these problems in subsequent publications.
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