Tight Lower Bound for Average Number of Terms in Optimal Double-base Number System

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Abstract
We show in this note that the average number of terms in the optimal double-base number system is in $\Omega(n/\log n)$. The lower bound matches the upper bound shown earlier by Dimitrov, Imbert, and Mishra (Math. of Comp. 2008).

Keywords: Double-base Number System, Asymptotic Analysis

1. Introduction and Notations

Given a non-negative integer $m$. A tuple $[k, X = \langle x_i \rangle_{i=1}^k, Y = \langle y_i \rangle_{i=1}^k]$ where $k, x_i, y_i$ are non-negative integers is a representation of $m$ in double-base number system if $\sum_{i=1}^k 2^{x_i} 3^{y_i} = m$. For some $m$, there might be more than one representations of $m$ in the system. For example, both $[2, \langle 0, 2 \rangle, \langle 1, 1 \rangle]$ and $[4, \langle 0, 1, 2, 3 \rangle, \langle 0, 0, 0, 0 \rangle]$ are representations of 15.

For a representation $[k, X = \langle x_i \rangle_{i=1}^k, Y = \langle y_i \rangle_{i=1}^k]$, we call $k$ as the number of terms of the representation. A representation with $k$ terms is a minimum representation of $m$ if there is no representation of $m$ with $k'$ terms such as $k' < k$. The number of terms of a minimum representation is denoted by $k_m^*$ in this paper.

To speed up the calculation of scalar multiplication in elliptic curve cryptography, many researchers devise algorithms to calculate a representation with small number of terms for a given $m$. Those include the algorithm by Dimitrov, Imbert, and Mishra [1]. Let $k_m'$ be the number of terms of representations obtained from the algorithm. The authors have shown that
\(k_m \in O(\lg m/\lg \lg m)\) for all \(m\). The result is surprising as, in all variations of binary representations, there are infinite number of \(m\) of which all representations have \(\Omega(\log m)\) terms. It leads to a smaller asymptotic complexity for calculating the scalar multiplication.

One may ask if the asymptotic complexity can be even smaller with the double-base number system. However, in [3], we have shown that a smaller asymptotic complexity cannot be obtained. There are infinite number of \(m\) of which \(k_m^* \in \Omega(\lg m/\lg \lg m)\).

Until now, we have discussed the worst-case computation time of the scalar multiplication. However, in literature, it is more common to analyze the average-case computation time than to analyze the worst-case computation time [4, 5, 6]. The average-case computation time usually depends on

\[
A'(n) := \sum_{m=0}^{2^n-1} \frac{k_m'}{2^n}
\]

when \(n\) is a positive integer and \(k_m'^*\) is the number of terms in the representation obtained from an algorithm. When the algorithm is that proposed in [1], we have

\[
A'(n) = \sum_{m=0}^{2^n-1} \frac{k_m'}{2^n} = \sum_{m=0}^{2^n-1} O\left(\frac{\lg m}{\lg \lg m}\right)/2^n = \sum_{m=0}^{2^n-1} O(n/\lg n)/2^n = O(n/\lg n).
\]

Let \(A^*(n) := \sum_{m=0}^{2^n-1} k_m^*/2^n\). We know that \(A^*(n) \leq A'(n) = O(n/\lg n)\).

Although it is known that the worst-case asymptotic complexity using double base number system cannot be further improved, we may be able to further improve the average-case asymptotic complexity. We have tried to improve our algorithm and our analysis to have \(A'(n) \in o(n/\lg n)\), but have not been successful.

2. Our Result

We show that the asymptotic of \(A'(n)\) cannot be further improved, i.e. \(A^*(n) \in \Omega(n/\lg n)\). This implies that the upper bound of \(A'(n)\) is asymptotically tight, and the algorithm in [1] is asymptotically optimal also on average case.

We will use the concept of prefix code and its properties to show the tightness. The prefix code can be defined as follows:

**Definition 1 (Prefix code).** Let \(N\) be a positive integer and, for \(i \in \{0,\ldots,N-1\}\), let \(c_i \in \{0,1\}^*\). We say that \(c_0,\ldots,c_{N-1}\) is a prefix code of \(0,\ldots,N-1\) if \(c_i\) is not a prefix of \(c_j\) for \(i \neq j\).
We can define an optimal prefix code as follows:

**Definition 2** (Optimal prefix code). Let \( c_0, \ldots, c_{N-1} \) be a prefix code of \( 0, \ldots, N-1 \), let \( p_0, \ldots, p_{N-1} \) be a probability distribution on \( 0, \ldots, N-1 \), and let \(|c|\) be the length of \( c \in \{0, 1\}^* \).

We say that \( c_0, \ldots, c_{N-1} \) is an optimal prefix code under the probability distribution \( p_0, \ldots, p_{N-1} \) if there is no prefix code \( c'_0, \ldots, c'_{N-1} \) of \( 0, \ldots, N-1 \) such that

\[
\sum_{i=0}^{N-1} |c'_i| p_i < \sum_{i=0}^{N-1} |c_i| p_i.
\]

We will use an algorithm in [7] to show our result. The algorithm takes a probability distribution of \( 0, \ldots, N-1 \) and gives a prefix code \( c_0, \ldots, c_{N-1} \) as an output. The code obtained from the algorithm is called Huffman code. The author of the paper has shown the following lemma.

**Lemma 1** (Optimality of Huffman code [7]). The output code obtained from the algorithm in [7] is an optimal prefix code under the input probability distribution.

By Lemma 1, we can obtain the following lemma.

**Lemma 2.** Suppose that \( N \) can be written in the form of \( 2^n \) for some \( n \in \mathbb{Z}_{>0} \). Then, there is no prefix code \( c_0, \ldots, c_{N-1} \) such that

\[
\sum_{i=0}^{N-1} |c_i| N < n.
\]

**Proof.** We omit the description of the algorithm of [7] in this paper. However, when \( N = 2^n \) for some \( n, p_i = 1/N \) for all \( i \in \{0, \ldots, N-1\} \), and \( c_0, \ldots, c_{N-1} \) is a prefix code obtained from the algorithm, we know that \(|c_i| = n\) for all \( i \). By the optimality of \( c_1, \ldots, c_{N-1} \), there is no prefix code \( c'_1, \ldots, c'_{N-1} \) such that

\[
\sum_{i=0}^{N-1} |c'_i| p_i < \sum_{i=0}^{N-1} |c_i| p_i \quad \text{or} \quad \sum_{i=0}^{N-1} |c'_i| / N < \sum_{i=0}^{N-1} |c_i| / N = n. \quad \square
\]

Let a tuple \([k_m, X^* = \langle x_i^* \rangle_{i=1}^{k_m}, Y^* = \langle y_i^* \rangle_{i=1}^{k_m}\) be a minimum representation of \( m \) in double-base number system. Recall from Section 1 that \( \sum_{i=1}^{k_m} 2^{x_i^*} 3^{y_i^*} = m \). We know that, for all \( i, x_i^*, y_i^* \leq \lg m \), otherwise the term \( 2^{x_i^*} 3^{y_i^*} \) and the summation \( \sum_{i=1}^{k} 2^{x_i^*} 3^{y_i^*} \) would be larger than \( m \). Therefore, we can represent \( x_i^* \) and \( y_i^* \) using a binary representation length \( \lg n \) when
Let \( b[x] \in \{0, 1\}^{\lg n} \) be a \((\lg n)\)-bit binary representation of a non-negative integer \( x \in \{0, \ldots, n-1\} \). Note that the length of \( b[x] \) is fixed to \( \lg n \) and the bit string \( b[x] \) begins with 0 when \( x < 2^{n-1} \). We define the code \( c''_0, \ldots, c''_{2^n-1} \) of 0, \ldots, 2\(^n\) - 1 in the way that, for all \( m \),

\[
c''_m = b[k_m^*]b[x_1^*]b[y_1^*] \ldots b[x_{k_m^*}^*]b[y_{k_m^*}^*].
\]

We can show the following lemma for the code.

**Lemma 3.** The code \( c''_0, \ldots, c''_{2^n-1} \) of 0, \ldots, 2\(^n\) - 1 defined above is a prefix code.

**Proof.** Assume a contradictory statement that there are \( i \neq j \) such that \( c''_i \) is a prefix of \( c''_j \). We must have \( b[k_i^*] = b[k_j^*] \) and \( k_i^* = k_j^* \). We can then conclude that the code \( c''_i \) and \( c''_j \) have the same length. As \( c''_i \) is a prefix of \( c''_j \), we have \( c''_i = c''_j \). Thus, for \( 1 \leq p \leq k_m^* \), we have same \( x_p^* \) and \( y_p^* \) in the representations of \( i \) and \( j \). The integer \( i \) and \( j \) have the same representation in double-base number system. That is not possible when \( i \neq j \).

We are now ready to show our main result.

**Theorem 4.** The average number of terms in the double-base number system, denoted by \( A^*(n) \), is in \( \Omega(n/\lg n) \).

**Proof.** By the construction of \( c''_m \), we have \(|c''_m| = (2k_m^* + 1) \cdot \lg n. \) Then,

\[
\sum_{i=0}^{2^n-1} |c''_i|/2^n = \sum_{i=0}^{2^n-1} (2k_m^* + 1) \cdot \lg n/2^n = (2A^*(n) + 1) \cdot \lg n.
\]

If \( A^*(n) \in o(n/\lg n) \), then

\[
\sum_{i=0}^{2^n-1} |c''_i|/2^n = (2 \cdot o(n/\lg n) + 1) \cdot \lg n = o(n).
\]

There is \( n \) such that \( \sum_{i=0}^{2^n-1} |c''_i|/2^n < n \). This contradicts our result in Lemma 2.
3. Concluding Remarks

In this paper, we have shown a tight lower bound for the double-base number system in the previous section. Indeed, we can use the same argument to show the tight lower bound for the multi-base number system. In other words, for any constant \( q \) and for any \( b_1, \ldots, b_q \) such that \( b_i \) and \( b_j \) are co-prime for \( i \neq j \), the average number of terms in the summation \( \sum_{i=1}^{k} b_1^{\beta_1} \cdots b_q^{\beta_q} = m \) would not be smaller than \( \Omega(n/\lg n) \) for \( m \in \{0, \ldots, 2^n - 1\} \).

The result also holds for the case that digit set is not \( \{0, 1\} \) as far as the exponent is in \( O(\log n) \). In other words, let the digit set be \( D_S \), the average number of terms in the summation \( \sum_{i=1}^{k} d_i b_1^{\beta_1} \cdots b_q^{\beta_q} = m \) for \( d_i \in D_S \) is not asymptotically smaller than that without \( d_i \) unless we allow \( \beta_i \) to be in \( \Omega(\log n) \). This implies that our algorithm for the multi-base number system in [3] is tight also on the average case.

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