Brane cosmology, Weyl fluid, and density perturbations

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Abstract

We develop a technique to study relativistic perturbations in the generalised brane cosmological scenario, which is a generalisation of the multi-fluid cosmological perturbations to brane cosmology. The novelty of the technique lies in the inclusion of a radiative bulk which is responsible for bulk-brane energy exchange, and in turn, modifies the standard perturbative analysis to a great extent. The analysis involves a geometric fluid – called the Weyl fluid – whose nature and role have been studied extensively both for the empty bulk and the radiative bulk scenario. Subsequently, we find that this Weyl fluid can be a possible geometric candidate for dark matter in this generalised brane cosmological framework.

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I. INTRODUCTION

During the last few years braneworld gravity has emerged as a more general theory of gravity, mainly due to the possibility of explaining the gravitational phenomena observed in the four dimensional universe from a broader perspective \[1, 2\]. Subsequent developments of the theory in the cosmological sector \[3\] came as an inevitable outcome since the challenges any theory of cosmology, be it a theory based on General Relativity or any other phenomenologically motivated theory, faces in explaining predictions from the highly accurate observational data \[4, 5, 6\]. In spite of great complications involved, the cosmological aspects of this scenario did show some promising features. To mention a few, brane cosmology naturally gives rise to singularity-free bouncing and cyclic universes \[7\]. Also, in this theory, the universe does not need any special initial condition for the inflation to start so that the isotropy is built in the theory \[8\]. Even the possibility of inflation without any 4D inflaton field is in vogue \[9\]. Brane cosmology thus results in interesting physics which needs further investigations.

In this scenario, the bulk spacetime is either AdS$_5$ \[10\] or a generalised version of it. The generalised global structure depends upon whether the bulk has only a cosmological constant or there is any non-standard model fields minimally or non-minimally coupled to gravity or to brane matter. When the bulk is empty consisting only of a cosmological constant, the bulk metric in which an FRW brane can be consistently embedded, is given by a 5-dimensional Schwarzschild-Anti de Sitter (Sch-AdS$_5$) or a Reissner-Nördstrom Anti de Sitter (RNAdS$_5$) black hole \[2, 3, 7, 11, 12\] . A subsequent generalisation of this scenario can be obtained when the bulk is not necessarily empty but it consists of a radiative field, resulting in a Vaidya-Anti de Sitter (VAdS$_5$) black hole for the bulk metric \[3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22\]. A ‘black hole in the bulk’ scenario provides us with a novel way of visualising cosmological phenomena on the 4D universe. In this scenario, the brane is moving in the bulk, with its radial trajectory being identified with the scale factor of the 4D world, so that the expansion of the universe is a realisation of the radial trajectory of the brane in the bulk.

The most notable contribution from bulk geometry on the brane is, perhaps, an additional term in the Friedmann equations, which arise from the projection of the bulk Weyl tensor onto the brane. The precise role of this term, compatible to FRW background on the brane, is to supply a geometric perfect fluid whose nature is governed by the contents of the bulk (in
turn, bulk geometry) we choose. For an empty bulk, it is radiation-like and is called the dark radiation. There is extensive study in the literature either by setting it to zero for practical purpose or by attributing a very small value to it, constrained by Nucleosynthesis data (< 3% of total radiation energy density of the universe) [2, 23]. Examples include metric-based perturbations [24], density perturbations on large scales neglecting dark radiation [25], or including its effects [26], curvature perturbations [27] and the Sachs-Wolfe effect [28], vector perturbations [29], tensor perturbations [30], and CMB anisotropies [31]. In all the cases, the effect has been found to modify the standard analysis very little, as expected from its radiation-like behaviour.

On contrary, when the bulk is not necessarily empty, the nature of this entity is no longer radiation-like, rather it depends upon the contents in the bulk, which is reflected by the $V_{AdS_5}$ bulk geometry [3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 33, 34]. It is thus important to determine the nature as well as the role of this entity, called in general the Weyl fluid, in the cosmological dynamics and perturbations, and find if this scenario has some advantages over others. A recent work [21] has shown its significance as a possible dark matter candidate by Newtonian analysis of perturbations, followed by some confrontation with observations [35]. However, as in GR, the Newtonian analysis of gravitational instability is limited in the sense that it fails to account for the perturbations on scales larger than the Hubble radius. One needs relativistic analysis valid at super-Hubble scales as well. Further, in order to test braneworld scenario observationally, we need a complete description of the evolution of density perturbations in the most general brane cosmological scenario provided by this $V_{AdS_5}$ bulk. With these motivations, we develop here a technique for relativistic density perturbations valid for this generalised brane cosmology, which will act as a natural extension of the covariant perturbations of General Relativistic framework [36] to braneworld scenario. We further show in the subsequent discussions that the Weyl fluid can play a crucial role in late time cosmologies as a geometric candidate for dark matter albeit its actual material existence.
II. BRANE DYNAMICS WITH WEYL FLUID

As mentioned, we shall concentrate on the most general bulk scenario, for which the bulk geometry is given by a Vaidya-anti de Sitter metric

$$dS_5^2 = -f(r, v) \, dv^2 + 2 \, dr \, dv + r^2 d\Sigma_3^2$$

(2.1)

where $\Sigma_3$ is the 3-space having flat, spherical or hyperboloidal symmetry, $f(r, v) = k - \frac{\Delta v^2}{6} - \frac{m(v)}{r^2}$, and $m(v)$ is the resultant of the variable mass of the Vaidya black hole and radiation field. This type of bulk can exchange energy with the brane as a null flow along the radial direction $[3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]$. Consequently, the brane matter conservation equation is modified to

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = -2 \psi$$

(2.2)

where $\psi$ is the null flow characterising the $\text{VAdS}_5$ bulk by the radiation field of a null dust $T_{\text{bulk}}^{\mu \nu} = \psi q_A q_B$, which leads to the above equation by using

$$\nabla^\mu T_{\mu \nu} = -2 T_{\text{bulk}}^{\mu \nu} n^A g^B$$

(2.3)

(where $n^A$ are the normals to the surface), which gives

$$\nabla^\mu T_{\mu \nu} = -2 \psi u_\mu$$

(2.4)

($u_\mu$ are the unit velocity vectors), and readily leads to Eq (2.2). This modified conservation equation, with the help of the Bianchi identity on the brane $\nabla^\mu G_{\mu \nu} = 0$, leads to another constraint equation

$$\nabla^\mu \varepsilon_{\mu \nu} = \frac{6 \kappa^2}{\lambda} \nabla^\mu S_{\mu \nu} + \frac{2}{3} \left[ \kappa^2 \left( \dot{\psi} + 3 \frac{\dot{a}}{a} \psi \right) - 3 \kappa^2 \psi \right] u_\mu + \frac{2}{3} \kappa^2 \nabla_\mu \psi$$

(2.5)

where $\lambda$ is the brane-tension and $\varepsilon_{\mu \nu}$ and $S_{\mu \nu}$ are, respectively, the projected bulk Weyl tensor and the quadratic contribution from the brane energy-momentum tensor to the Einstein equation on the brane $[21]$. The above equation governs the evolution of the Weyl fluid $\rho^*$ (so named since it is a fluid-like contribution from the bulk Weyl tensor to the brane). For FRW geometry on the brane, this is given by

$$\dot{\rho}^* + 4 \frac{\dot{a}}{a} \rho^* = 2 \psi - \frac{2}{3} \left( \frac{\kappa_5}{\kappa} \right)^2 \left( \dot{\psi} + 3 \frac{\dot{a}}{a} \psi \right)$$

(2.6)
so that this quantity evolves as \[ \rho^* = \frac{C(\tau)}{a^4} \propto \frac{1}{a^{4-\alpha}} \] (2.7)

which gives a general, physically relevant behaviour for the Weyl fluid. Here, \( \tau \) is the proper time on the brane. Obviously, contrary to the Sch-AdS\(_5\) bulk, here \( C(\tau) \) is evolving, and consequently, the Weyl fluid no longer behaves like radiation. To a brane-based observer, the cosmological dynamics is now governed by an effective perfect fluid, the components of which are given by [15, 21]

\[
\rho^{\text{eff}} = \rho + \frac{\rho^2}{2\lambda} + \frac{C(\tau)}{a^4} \quad (2.8) \\
p^{\text{eff}} = p + \frac{\rho}{2\lambda}(\rho + 2p) + \frac{C(\tau)}{3a^4} \quad (2.9)
\]

The anisotropic components of the Weyl fluid, \( v_i, q^*_\mu \) and \( \pi^*_\mu\nu \) vanish, in order that the VAdS\(_5\) bulk be compatible to FRW geometry on the brane. The Friedmann equation and the covariant Raychaudhuri equation, expressed in terms of these effective quantities, are respectively [22]

\[
H^2 = \frac{\kappa_4^2}{3} \rho^{\text{eff}} + \frac{\Lambda}{3} - \frac{k}{a^2} \quad (2.10) \\
\dot{H} = -\frac{\kappa_4^2}{2} \left( \rho^{\text{eff}} + p^{\text{eff}} \right) + \frac{k}{a^2} - \frac{\kappa_5^2}{3} \psi \quad (2.11)
\]

In the brane-based Newtonian analysis of perturbations by considering small fluctuations of the effective density \( \rho^{\text{eff}}(\vec{x}, \tau) = \bar{\rho}^{\text{eff}}(\tau)(1 + \delta^{\text{eff}}(\vec{x}, \tau)) \) and the effective gravitational potential \( \Phi^{\text{eff}}(\vec{x}, \tau) = \Phi_0^{\text{eff}} + \phi^{\text{eff}} \) on the hydrodynamic equations for this effective perfect fluid, one obtains for a barotropic fluid a single second order equation in terms of Fourier mode

\[
\frac{d^2 \delta_k^{\text{eff}}}{d\tau^2} + \frac{\dot{a}}{a} d\delta_k^{\text{eff}} - \left[ 4\pi G \bar{\rho}^{\text{eff}} - \left( \frac{c_s^{2\text{eff}} k}{a} \right) \right] \delta_k^{\text{eff}} = 0 \quad (2.12)
\]

where \( c_s^{2\text{eff}} \) is the square of the effective sound speed [2, 21]. The above perturbation equation of the effective fluid can account for the required amount of gravitational instability if the Weyl density redshifts more slowly than baryonic matter density, so that it can dominate over matter at late times, which is realised when \( 1 < \alpha < 4 \) in Eq (2.7). Now, for late time behaviour, we can drop the quadratic terms in equations (2.8) and (2.9) so that the effective density is given by \( \rho^{\text{eff}} = \rho^{(b)} + \rho^* \) which is now constituted of the usual matter (baryonic) density \( \rho^{(b)} \) and an additional density contribution from the Weyl fluid. This Weyl density,
being geometric, is essentially non-baryonic. Consequently, we can decompose Eq (2.12) to get the individual evolution equations for the perturbation for each of the fluids

\[
\frac{d^2 \delta^{(b)}}{d\tau^2} + 2 \frac{\dot{a}}{a} \frac{d\delta^{(b)}}{d\tau} = 4\pi G \bar{\rho}^{(b)} \delta^{(b)} + 4\pi G \bar{\rho}^* \delta^* \tag{2.13}
\]
\[
\frac{d^2 \delta^*}{d\tau^2} + 2 \frac{\dot{a}}{a} \frac{d\delta^*}{d\tau} = 4\pi G \bar{\rho}^* \delta^* + 4\pi G \bar{\rho}^{(b)} \delta^{(b)} \tag{2.14}
\]

where \( \delta^{(b)} \) and \( \delta^* \) are the fluctuations of baryonic matter and Weyl fluid respectively. With \( \Omega^{(b)} \ll \Omega^* \), the relevant growing mode solutions are given by [21]

\[
\delta^*(z) = \delta^*(0)(1 + z)^{-1} \tag{2.15}
\]
\[
\delta^{(b)}(z) = \delta^*(z) \left(1 - \frac{1 + z}{1 + z_N}\right) \tag{2.16}
\]

with the input that the late time behaviour of the expansion of the universe in RS II is the same as the standard cosmological solution for the scale factor [13, 32] where the scale factor is related to the redshift function by \( a \propto (1 + z)^{-1} \).

The solutions reveal that at a redshift close to \( z_N \), the baryonic fluctuation \( \delta^{(b)} \) almost vanishes but the Weyl fluctuation \( \delta^* \) still remains finite. So, even if the baryonic fluctuation is very small at a redshift of \( z_N \approx 1000 \), as confirmed by CMB data [4], the fluctuations of the Weyl fluid still had a finite amplitude during that time, whereas at a redshift much less than \( z_N \) the baryonic matter fluctuations are of equal amplitude as the Weyl fluid fluctuations. This is precisely what is required to explain the formation of structures we see today. Thus, the Newtonian analysis of perturbations on the brane is capable of explaining structure formation (within its limit) by Weyl fluid, devoid of any material existence of dark matter. Hence, the Weyl fluid acts as a possible geometric candidate for dark matter.

III. RELATIVISTIC PERTURBATIONS WITH WEYL FLUID

The Newtonian analysis depicted so far turn out to be an useful tool to study perturbations on the brane after the universe enters the Hubble length. A more complete picture can be obtained if one studies relativistic analysis of perturbations, which include the evolution of the universe at the super-Hubble scale as well. In this section we shall develop a multi-fluid perturbative technique in order to discuss relativistic perturbation relevant for brane cosmology. This will be carried through in the subsequent sections for the purpose of analysis for different braneworld scenarios. Our basic motivation in the attempt to develop
a multi-fluid perturbative technique is governed by the realisation obtained from Newtonian analysis that, contrary to the Sch-AdS bulk scenario, the Weyl fluid may not be that much insignificant so as to neglect its effects at late time, if we have a general bulk geometry. Consequently, in a general brane cosmological scenario, along with baryonic matter, the universe consists of a considerable amount of Weyl fluid as well.

Before going into the details, let us jot down here the major points in addressing relativistic perturbations on the brane.

- Here the cosmological dynamics is governed by a two-fluid system. One of the components of the system is a material fluid $\rho^{(b)}$ – the baryonic matter content on the brane. The second component is a geometric fluid $\rho^*$ – the Weyl fluid. The total (effective) density for the system on the brane is given by $\rho^{\text{eff}} = \rho^{(b)} + \rho^*$.

- Though there are two components of the effective fluid, the Weyl fluid being a geometric entity, there is a single material fluid in the analysis. As a result, there will be no entropy perturbation as such.

- For the same reason, there is no peculiar velocity for the Weyl component, leading to $v^* = 0$.

- The anisotropic components of the Weyl fluid being absent so as to fit it into an FRW background, we will set $q^*_\mu = 0 = \pi^*_\mu$ right from the beginning. As a result, each component of the two-fluid system behaves individually like a perfect fluid, resulting in a perfect fluid behaviour for the effective fluid as a whole.

- These two fluids interact and exchange energy between them, which is governed via the bulk-brane energy exchange and the backreaction of the system on the brane.

- Since there is energy exchange between these two fluids, the conservation equation for each individual is now modified. These modified forms of the conservation equations have been explained in equations (2.2) and (2.6).

Because of the interaction between the two fluid components, each of the two modified conservation equations can be written in terms of the contribution from the interaction as

$$\dot{\rho}^{(i)} + \Theta(\rho^{(i)} + p^{(i)}) = I^{(i)}$$

(3.1)
where $\Theta = 3 \frac{a}{a}$ is the volume expansion rate, a superscript $(i)$ denotes the quantities for the \( i \)-th fluid and $I^{(i)}$ is the corresponding interaction term. It readily follows from equations (2.2) and (2.6) that the interaction terms, when written explicitly, are given by

$$I^{(b)} = -2\psi$$
$$I^* = 2\psi - \frac{2}{3} \left( \frac{\kappa_5}{\kappa} \right)^2 \left( \dot{\psi} + 3\frac{\dot{a}}{a} \psi \right)$$

For relativistic perturbations, we express the densities of each of the contributing fluid components in terms of dimensionless parameters as

$$\Omega_{\rho^{(b)}} = \frac{\kappa^2 \rho^{(b)}}{3H_0^2}, \quad \Omega_{\rho^*} = \frac{\kappa^2 \rho^*}{3H_0^2}$$

with the first one for baryonic matter and the second one for Weyl fluid. Considering the nature of the Weyl fluid as discussed in the previous section, we find that the density parameter for the Weyl fluid is given by

$$\Omega_{\rho^*} = \frac{2C_0}{a_0^{4-\alpha}H_0^2}$$

where $C_0$ is the onset value for the Weyl parameter $C(\tau)$.

For completion, we mention here that there can, in principle, appear two more dimensionless parameters, one each for the cosmological constant and the brane tension arising in the brane cosmological context. They are

$$\Omega_{\Lambda} = \frac{\Lambda}{3H_0^2}, \quad \Omega_{\lambda} = \frac{\kappa^2 \rho_0^2}{6\lambda H_0^2}$$

with the total density satisfying the critical value

$$\Omega_{\text{tot}} = \sum_i \Omega_i = \Omega_{\rho^{(b)}} + \Omega_{\rho^*} + \Omega_{\Lambda} + \Omega_{\lambda} = 1$$

Here, and throughout the rest of this article, we have considered a spatially flat universe with $k = 0$. In Eq (3.6), the first one is relevant if one considers cosmological constant in this brane universe while studying the expansion history of the universe whereas the one due to the brane tension is relevant in the high energy early universe (inflationary) phase but is negligible for low energy late time phenomena such as structure formation. Thus the baryonic density and the Weyl density are the only two relevant contributions in the scenario being discussed here. In what follows we shall restrict ourselves to the discussion of the Einstein-de Sitter brane universe for which $\Omega_{\Lambda} = 0$ leading to $\Omega_{\text{tot}} \approx \Omega_{\rho^{(b)}} + \Omega_{\rho^*} = 1$. 
We now express the comoving fractional gradients of the effective density and expansion relevant in the brane cosmology as

\[
\Delta_{\mu}^{(i)} = \frac{a}{\rho^{(i)}} D_{\mu} \rho^{(i)}
\]

\[
Z_{\mu}^{\text{eff}} = a D_{\mu} \Theta
\]

\[
\Delta_{\mu}^{\text{eff}} = \frac{a}{\rho^{\text{eff}}} D_{\mu} \rho^{\text{eff}}
\]

As already discussed, both baryonic matter and Weyl fluid behave individually as perfect fluid components, which means the effective flux arising from the peculiar velocities of each component vanish to zero order, confirming that the perturbations considered here are gauge-invariant at the first order.

With the above notations, the linearised evolution equation for the density perturbations in the braneworld is obtained by taking spatial gradient of the modified conservation equations. After linearisation, it turns out to be

\[
\dot{\Delta}_{\mu}^{(i)} = \left(3Hw^{(i)} - \frac{I^{(i)}}{\rho^{(i)}}\right)\Delta_{\mu}^{(i)} - (1+w^{(i)})Z_{\mu}^{\text{eff}} - \frac{c_{s}^{2\text{eff}} I^{(i)}}{\rho^{(i)}(1+w^{\text{eff}})} \Delta_{\mu}^{\text{eff}} - \frac{3aHI^{(i)}}{\rho^{(i)}} + \frac{a}{\rho^{(i)}} D_{\mu} I^{(i)}
\]

where \(w^{(i)} = p^{(i)}/\rho^{(i)}\) is the equation of state for \(i\)-th fluid and \(c_{s}^{2(i)} = \ddot{p}^{(i)}/\dot{\rho}^{(i)}\) is the sound speed squared for that species, with the corresponding quantities for the effective total fluid are, respectively,

\[
w^{\text{eff}} = \frac{1}{\rho^{\text{eff}}} \sum_{i} \rho^{(i)} w^{(i)}
\]

\[
c_{s}^{2\text{eff}} = \frac{1}{\rho^{\text{eff}}(1+w^{\text{eff}})} \sum_{i} c_{s}^{2(i)} \rho^{(i)}(1+w^{(i)})
\]

In the relativistic perturbations, contrary to the Newtonian analysis, we further have an evolution equation for the effective expansion gradient, which depends on the effective fluid. This is obtained by taking spatial gradient of the modified Raychaudhuri equation (2.11) and is given in the braneworld scenario by

\[
\dot{Z}_{\mu}^{\text{eff}} + 2HZ_{\mu}^{\text{eff}} = -\frac{\kappa_{5}^{2}}{2} \rho^{\text{eff}} \Delta_{\mu}^{\text{eff}} - \frac{c_{s}^{2\text{eff}}}{1+w^{\text{eff}}} D_{\mu} D_{\nu} \Delta^{\text{eff}}_{\nu} + \frac{\kappa_{5}^{2} \psi^{2}}{1+w^{\text{eff}}} c_{s}^{2\text{eff}} \Delta_{\mu}^{\text{eff}} - a\kappa_{5}^{2} D_{\mu} \psi
\]

It should be mentioned here that since the evolution of the expansion gradient is dependent on the curvature perturbations, the later should not remain strictly constant in this multi-fluid perturbation scenario. However, though there is a significant energy exchange between brane matter and Weyl fluid at early times, we shall see from the next section that
the energy exchange between the two fluids are almost in equilibrium at late times, so that
the local curvature perturbations can safely be considered to be constant for all practical
purpose. One should, however, consider the variation of this term while analysing inflationary
phase for an instance. We follow this argument right from here in order to avoid
mathematical complexity.

As in GR, we find that while discussing perturbations in brane cosmology, it is advan-
tageous to express the above equations in terms of covariant quantities. These density
perturbations are governed by the fluctuation of the following covariant projections
\[ \Delta^{(i)} = a D^\mu \Delta^{(i)}_{\mu} \]  
\[ \Delta^{\text{eff}} = a D^\mu \Delta^{\text{eff}}_{\mu} \]  
\[ Z^{\text{eff}} = a D^\mu Z^{\text{eff}}_{\mu} \]

Consequently, the covariant density perturbation equation and expansion gradient on the
brane, when expressed in terms of the above covariant quantities, are obtained straightaway
from equations (3.11) and (3.14). They are given by
\[ \dot{\Delta}^{(i)} = \left(3H w^{(i)} - \frac{I^{(i)}}{\rho^{(i)}}\right) \Delta^{(i)} - (1 + w^{(i)}) \Delta^{\text{eff}} - \frac{c_s^{2\text{eff}} I^{(i)}}{\rho^{(i)}(1 + w^{\text{eff}})} \Delta^{\text{eff}} \]
\[ - \frac{3a^2 H D^\mu I^{(i)}_{\mu}}{\rho^{(i)}} + \frac{a^2}{\rho^{(i)}} D^2 I^{(i)} \]
\[ \dot{Z}^{\text{eff}} + 2HZ^{\text{eff}} = -\frac{\kappa^2}{2} \rho^{\text{eff}} \Delta^{\text{eff}} - \frac{a c_s^{2\text{eff}}}{1 + w^{\text{eff}}} D^2 \Delta^{\text{eff}} + \frac{\kappa_s^2 \psi}{1 + w^{\text{eff}}} c_s^{2\text{eff}} \Delta^{\text{eff}} - a^2 \kappa_s^2 D^2 \psi \]

In deriving the above covariant perturbation equations, we have considered those kind of
perturbations for which, like the unperturbed Weyl fluid, the anisotropic stresses and fluxes
for the perturbed Weyl fluid are also vanishing.

These set of equations provide the key information about the perturbation in brane
cosmology. In the subsequent section, we shall try to analyse these relativistic perturbation
equations and obtain possible consequences.

IV. SOLUTIONS AND ANALYSIS

A. Empty bulk : Non-interacting fluids

Let us now discuss the special scenario when the bulk is empty for which the VAdS\textsubscript{5}
bulk reduces to Sch-AdS\textsubscript{5}. In this case, there is no question of energy exchange between the
brane and the bulk. Consequently, there is no interaction between brane matter and Weyl fluid as such, which reveals from Eq (3.1) the fact that the individual conservation equation for each of the components are preserved. Thus, the Weyl fluid evolves in this case as

$$\rho^* \propto a^{-4}$$  \hspace{1cm} (4.1)

with the Weyl parameter $\alpha$ now being zero, so that for empty bulk, the Weyl fluid behaves like radiation, for which this is called dark radiation.

Since in this case, there is no interaction between the two components of the effective fluid and also, there is no null flow from the bulk to the brane (or vice versa), we can drop the interaction terms and the terms involving $\psi$ in the analysis. As a result, the covariant perturbation equations (3.18) and (3.19) are vastly simplified. They are now given by

$$\dot{\Delta}^{(i)} = 3H w^{(i)} \Delta^{(i)} - (1 + w^{(i)}) Z_{\text{eff}}$$  \hspace{1cm} (4.2)

$$\dot{Z}_{\text{eff}} + 2HZ_{\text{eff}} = -\frac{\kappa^2}{2} \rho_{\text{eff}} \Delta_{\text{eff}} - \frac{\alpha c_s^2}{1 + w_{\text{eff}}} D^2 \Delta_{\text{eff}}$$  \hspace{1cm} (4.3)

Taking the time derivative of the above two equations and combining them, we obtain the evolution equations for density perturbations of the two fluids

$$\ddot{\Delta}^{(b)} + 2H \dot{\Delta}^{(b)} = \frac{\kappa^2}{2} \rho_{\text{eff}} \Delta_{\text{eff}}$$  \hspace{1cm} (4.4)

$$\ddot{\Delta}^* + 2H \dot{\Delta}^* = \frac{4}{3} \frac{\kappa^2}{2} \rho_{\text{eff}} \Delta_{\text{eff}} + \Delta^* \left(2H^2 - \frac{\kappa^2}{2}\right) + H \dot{\Delta}^*$$  \hspace{1cm} (4.5)

where the first equation is for baryonic matter while the second one for dark radiation.

Recall that the amount of dark radiation is constrained by the Nucleosynthesis data to be at most 3% of the total radiation density of the universe. So, it redshifts at a faster rate than ordinary matter on the brane so that the matter on the brane becomes dominant on the Weyl fluid at late time. Hence, it is expected that the dark radiation does not play any significant role in late time cosmologies. It is obvious from the fact that in this case, $\Omega^{(b)} \gg \Omega^*$, which when put back into the above equations, leads to $\Delta^{(b)} \gg \Delta^*$, so that the dark radiation fluctuation does not contribute substantially at late times. The Sch-AdS$_5$ bulk scenario thus fails to explain structure formation with only baryonic matter and dark radiation. One needs cold dark matter in the theory and the dark radiation can, at best, slightly modify the standard perturbative analysis.
B. General bulk : Interacting Weyl fluid

The general scenario, however, is different from the empty bulk case since now the Weyl fluid exchanges energy with brane matter through interactions and is the dominant contribution of the effective fluid in the perturbation equations. Here, the evolution of perturbations for the individual fluids are governed by equations (3.18) and (3.19), which now include the effects of the interaction terms as well as of the effect of null radiation through the term involving $\psi$. With these inclusions, the equations become a bit too complicated and it is almost impossible to have an analytical solution from these complicated equations. However, the equations turn out to be tractable if we incorporate certain simplifications following physical arguments, without losing any essential information as such. The simplifications we incorporate are as follows:

- The null flow from the brane to the bulk $\psi$ is a function of time only. This means that we are considering only the time-evolution for the null radiation, at least on the brane, which is relevant for its late time behaviour in perturbation analysis.

- The energy exchange between the two fluids is in equilibrium, i.e., the energy received by the Weyl fluid is the same as the energy released by brane matter, so that $\sum_i I_i^{(i)} = I_b^{(b)} + I^* = 0$. Hence, no extra energy is leaked to the bulk from the brane at late time (though at early time there may be some leakage of energy from the brane to the bulk). This basically describes the late time behaviour, consistent with the fact that the standard evolution history (scale factor) are regained in this scenario at the “matter-dominated” era [13, 32].

Now, we have shown that in this generalised braneworld scenario, the Weyl fluid, in general, evolves as (Ref Eq (2.7))

$$\rho^* = C_0 a^{-(4-\alpha)}$$

with the parameter $\alpha$ in the range $1 < \alpha < 4$ so that it is the dominant contribution in the two-fluid system. The energy exchange between the components of the system being in equilibrium, we find from Eq (2.7) that the Weyl fluid now behaves as

$$\rho^* \propto a^{-3/2}$$

with the parameter $\alpha = \frac{5}{2}$. This readily suggests that the Weyl fluid actually redshifts more slowly than ordinary matter and hence, can dominate over matter at late times, reflecting one
of the fundamental properties of dark matter. This also provides a more stringent bound for
the value of \( \alpha \) from theoretical ground alone (which was predicted from Newtonian analysis
to fall within 1 to 4).

We now take the time derivative of the covariant perturbation equations (3.18) and (3.19),
and rearrange terms so as to obtain a single second order differential equation for each of
the fluids. Thus, the equation describing evolution of scalar perturbations of matter on the
brane turn out to be

\[
\ddot{\Delta}^{(b)} + 2H \dot{\Delta}^{(b)} = \frac{\kappa^2}{2} \rho^{\text{eff}} \Delta^{\text{eff}} - \frac{c_s^{\text{eff}}}{1 + w^{\text{eff}}} \Delta^{\text{eff}} + \frac{4H \psi}{\rho} \left( \Delta^{(b)} + \frac{c_s^{\text{eff}} \Delta^{\text{eff}}}{1 + w^{\text{eff}}} \right) \left[ \frac{2\psi}{\rho} \left( \Delta^{(b)} + \frac{c_s^{\text{eff}} \Delta^{\text{eff}}}{1 + w^{\text{eff}}} \right) \right].
\]

whereas the scalar perturbation equation for the Weyl fluid on the brane is given by

\[
\ddot{\Delta}^{*} + \left( \frac{4 \kappa^2}{3} \rho^{\text{eff}} \Delta^{\text{eff}} \right) - \frac{c_s^{\text{eff}} \Delta^{\text{eff}}}{1 + w^{\text{eff}}} \left( \frac{7H \psi}{\rho} + \frac{2\psi}{\rho} \right) - \frac{c_s^{\text{eff}} \Delta^{\text{eff}}}{1 + w^{\text{eff}}} \frac{2\psi}{\rho} + \Delta^{*} \left( 2H^2 - \kappa^2 \rho + \frac{7H \psi}{\rho} - \frac{2\psi}{\rho} \right) + \dot{\Delta}^{*} \left( H - \frac{2\psi}{\rho} - \frac{\kappa^2 \psi}{3} \right)
\]

Recall from the discussions following Eq (4.7) that in this scenario the Weyl fluid is the
dominant component of the effective fluid. Consequently, the evolution equation for the Weyl fluid at late times is radically simplified by using \( \Delta^{(b)} \ll \Delta^{*} \) since the Weyl fluid is
now the dominant contribution. With the energy exchange between the two fluids being in
equilibrium, the expression for the null flow further simplifies the above equation so that it
can now be recast in the following form

\[
\ddot{\Delta}^{*} + \frac{A}{t} \dot{\Delta}^{*} - \left( \frac{B}{t} + \frac{C}{t^2} \right) \Delta^{*} = 0
\]

where the constants \( A, B, C \) are readily determined from the constraint equations. These
constants are given by

\[
A = \frac{2}{3} + \frac{5}{2} \left( \frac{\psi}{\rho} \right) \left( \frac{2a_0}{3H_0} \right)^{2/3}
\]

\[
B = \frac{2}{3} \kappa^2 \rho_0 \left( \frac{2a_0}{3H_0} \right)^{3/2} + \left( 1 + \frac{\kappa^2}{6} \rho_0 \right) \left( \frac{2a_0}{3H_0} \right)^{2/3}
\]

\[
C = \frac{A}{4} - \frac{19}{18}
\]

The above equation (4.10) for \( \Delta^{*} \) turns out to be somewhat tractable. One of its solutions
is given by

\[
\Delta^{*} \sim t^{\frac{1}{2} + \frac{A}{4}} \text{BesselI} \left[ \sqrt{1 - 2A + A^2 + 4C}, 2\sqrt{B} \sqrt{t} \right]
\]
The above solution, consisting of a Bessel function, is found to be a growing function. Therefore, the evolution equation for the Weyl fluid, indeed, shows a growing mode solution, which is required to explain the growth of perturbations at late times. Thus, the relativistic perturbation theory relevant in brane cosmology gives rise to a fluid which is very different from dark matter in origin and nature but has the potentiality to play the role of dark matter in cosmological context. It is worthwhile to note that, to a brane-based observer, the nature of the Weyl fluid is determined from bulk geometry arising from the radiation flow in the bulk. That is why the Weyl fluid can be treated as a geometric candidate for dark matter.

The following figure depicts a qualitative behaviour of the growth of Weyl fluid perturbations with time. The figure once again shows that the evolution of perturbations of Weyl fluid is very different from cold dark matter (CDM), which makes the theory distinct from standard analysis with CDM. We, however, note that since the dynamics here is completely different from the standard one involving CDM, one cannot comment conclusively on the merits/demerits of Weyl fluid over CDM right from here. One has to reformulate and estimate different cosmological parameters in this context and confront them with observations for a more conclusive remark. For example, the relation of the transfer function with the potential will now be replaced by a novel relation with the effective potential discussed in the brane cosmological context [21]. As a result, the variation of the growth function with the scale factor may not be the same as usually needed in the standard cosmological paradigm. It is to be seen if this analysis of perturbation with the Weyl fluid fits in this new, brane cosmological framework, which is not a trivial exercise, we suppose. The interested reader may further refer to [37] for an overall view on how different cosmological parameters are

FIG. 1: Growth of Weyl fluid perturbations with time

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developed in a specific theoretical framework.

However, even at this stage, our model does show some agreement with observational results. From the recent studies on confronting braneworld models with observations by obtaining the luminosity distance for FRW branes with the Weyl fluid, it is found that a certain amount of Weyl fluid with $2 \leq \alpha \leq 3$ is in nice agreement with Supernovae data. From the relativistic perturbations discussed in this article we have found a specific value for \( \alpha \), namely \( \alpha = \frac{5}{2} \), which falls within this region. Thus the braneworld model of perturbations fits well in this observational scenario. We hope an extensive study in this direction will lead to more interesting results to make a more conclusive remark.

V. SUMMARY AND OPEN ISSUES

In this article, we have developed a technique for relativistic perturbations valid for a general brane cosmological scenario. The essential distinction of our analysis from the studies on brane cosmological perturbations available in the literature is that, here the geometrical effect of the bulk on the brane – the so-called Weyl fluid – plays a very crucial role in determining the nature of the evolution of density perturbations. This is materialised from the realisation that in the general brane cosmological scenario obtained from Vaidya-anti de Sitter bulk, the Weyl fluid plays a significant role in controlling the dynamics on the brane, contrary to the earlier results based on dark radiation. Our results are, in a sense, a generalisation of the multi-fluid covariant perturbation formalism in brane cosmological framework. Further, we have solved the perturbation equations and found that the perturbation of the Weyl fluid grows at late time, and thus, this component of braneworld gravity plays a significant role in late time cosmology to act as a possible geometric candidate for dark matter. We have discussed some of the implications of fluctuations involving it and have mentioned some observable sides of this model as well.

An important issue is to fit this theoretical model with current observational data. Recently there has been some progress in this direction. An extensive study on confronting this braneworld model with observations in a more rigorous method can provide us with necessary information on the merits and demerits of the formalism. To this end, a thorough study of different parameters related to cosmological perturbation is to be performed. As mentioned in the previous section, the different cosmological parameters need to be refor-
mulated in this framework. The next step is to estimate them and confront them with observations. For example, it is to be seen if the power spectrum, redefined in this paradigm with the Weyl fluid acting as a dark matter candidate, fits with the highly accurate observational data.

Further, analysis of different types of metric-based perturbations, namely, scalar, vector and tensor as well as related issues like CMB anisotropy, Sachs-Wolfe effect etc has to be be studied in details in this brane cosmological framework with a significant Weyl fluid. An extensive study in this direction is essential, which we hope to address in near future. Also, to apply this formalism in the framework of branworld models of dark energy \[38\] remains as another interesting issue.

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[1] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62 (2000) 024012
[2] R. Maartens, Living Rev. Relativity 7 (2004) 7
[3] D. Langlois, Prog. Theor. Phys. Suppl. 148 (2003) 181
[4] D. N. Spergel et al., Astrophys. J. 148 (2003) 175
[5] L. Bergstrom, Rep. Prog. Phys. 63 (2000) 793; E. Hayashi and J. F. Navarro, in: 201st American Astronomical Society Meeting, Bull. Am. Astron. Soc. 34 (2002) 1270
[6] A. Doroshkevich et. al., Astron. Astrophys. 481 (2004) 7
[7] S. Mukherji and M. Peloso, Phys. Lett. B 547, 297 (2002); S. Mukherji and S. Pal, arXiv: 0806.2507 [gr-qc]
[8] N. Goheer and P. K. S. Dunsby, Phys. Rev. D 66, 043527 (2002); A. Coley, Y. He and W.C. Lim, Class. Quant. Grav. 21, 1311 (2004); P. K. S. Dunsby, N. Goheer, M. Bruni and A. Coley, Phys. Rev. D 69, 101303, (2004)
[9] Y. Himemoto and M. Sasaki, Phys. Rev. D63 (2001) 044015; N. Sago, Y. Himemoto and M. Sasaki, Phys. Rev. D65 (2001) 024014; S. Kobayashi, K. Koyama and J. Soda, Phys. Lett. B
501 (2001) 157

[10] J. Garriga and M. Sasaki, Phys. Rev. D62 (2000) 043523

[11] P. Kraus, JHEP 12, 011 (1999); D. Ida, JHEP 09, 014 (2000); S. Mukohyama, T. Shiromizu and K. Maeda, Phys. Rev. D 62, 024028 (2000)

[12] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000)

[13] E. Leeper, R. Maartens and C. Sopuerta, Class. Quant. Grav. 21 (2004) 1125

[14] D. Langlois, L. Sorbo and M. Rodriguez-Martinez, Phys. Rev. Lett. 89 (2002) 171301; D. Langlois and L. Sorbo, Phys. Rev. D68 (2003) 084006; D. Langlois, Astrophys. Space Sci. 283 (2003) 469

[15] A. Chamblin, A. Karch and A. Nayeri, Phys. Lett. B509 (2001) 163

[16] D. Langlois and L. Sorbo, Phys. Rev. D68 (2003) 084006

[17] L. A. Gergely, Phys. Rev. D68 (2003) 124011

[18] L. A. Gergely, E. Leeper and R. Maartens, Phys. Rev. D70 (2004) 104025; I. R. Vernon and D. Jennings, JCAP 07 (2005) 011

[19] D. Langlois, Prog. Theor. Phys. Suppl. 163 (2006) 258

[20] D. Jennings, I. R. Vernon, A. C. Davis and C. van de Bruck, JCAP 04 (2005) 013; L. A. Gergely and Z. Keresztes, JCAP 01 (2006) 022; A. R. Frey and A. Maharana, JHEP 08 (2006) 021; Z. Keresztes, I. Kepiro and L. A. Gergely, JCAP 05 (2006) 020

[21] S. Pal, Phys. Rev. D 74, 024005 (2006)

[22] S. Pal, Phys. Rev. D 74, 124019 (2006)

[23] R. Maartens, Prog. Theor. Phys. Suppl. 148, 213 (2003)

[24] H. A. Bridgman, K. A. Malik and D. Wands, Phys. Rev. D65 (2002) 043502

[25] R. Maartens, Phys. Rev. D62 (2000) 084023; C. Gordon and R. Maartens, Phys. Rev. D63 (2001) 044022

[26] B. Leong, P. K. S. Dunsby, A. D. Challinor and A. N. Lasenby, Phys. Rev. D65 (2002) 104012; B. Gumjudpai, R. Maartens and C. Gordon, Class. Quant. Grav. 20 (2003) 3592; N. Goheer, P.K.S. Dunsby, A. Coley and M. Bruni, Phys. Rev. D70, 123517 (2004)

[27] D. Langlois, R. Maartens, M. Sasaki and D. Wands, Phys. Rev. D63 (2001) 084009

[28] K. Koyama, Phys. Rev. D66 (2002) 084003; J. D. Barrow and R. Maartens, Phys. Lett. B532 (2002) 153

[29] R. Maartens, in Reference Frames and Gravitomagnetism, ed. J. F. Pascual-Sanchez et al.
[30] B. Leong, A. D. Challinor, R. Maartens and A. N. Lasenby, Phys. Rev. D66 (2002) 104014
[31] J. Soda and S. Kanno, Phys. Rev. D66 (2002) 083506; T. Shiromizu and K. Koyama, Phys. Rev. D67 (2003) 084022
[32] J. D. Barrow and R. Maartens, Phys. Lett. B532 (2002) 153
[33] K. Maeda, Lect. Notes Phys. 646 (2004) 323; D. Langlois and M. Rodriguez-Martinez, Phys. Rev. D64 (2001) 123507; G. Kofinas, G. Panotopoulos and T. N. Tomaras, JHEP 0601 (2006) 107; I. Brevik, J. Mattis Borven and S. Ng, Gen. Relativ. Gravit. 38 (2006) 907
[34] P. S. Apostolopoulos and N. Tetradis, Phys. Rev. D71 (2005) 043506; P. S. Apostolopoulos, N. Brouzakis, E. N. Saridakis and N. Tetradis, Phys. Rev. D72 (2005) 044013; P. S. Apostolopoulos and N. Tetradis, Phys. Lett. B633 (2006) 409
[35] Z. Keresztes, L. A. Gergely, B. Nagy and G. M. Szabo, PMC Physics A1 (2007) 4 [arXiv: astro-ph/0606698]; G. M. Szabo, L. A. Gergely and Z. Keresztes, PMC Physics A1 (2007) 8 [arXiv: astro-ph/0702610]; L. A. Gergely, Z. Keresztes and G. M. Szabo, AIP Conf. Proc. 957 (2007) 391 [arXiv: 0709.0933 [astro-ph]]
[36] J. M. Bardeen, Phys. Rev. D 22, 1882 (1980); H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78, 1 (1984); M. Bruni, P. K. S. Dunsby and G. F. R. Ellis, Astrophys. J. 395, 34 (1992); P. K. S. Dunsby, M. Bruni and G. F. R. Ellis, Astrophys. J. 395, 54 (1992); C. G. Tsagas, A. Challinor and R. Maartens, Phys. Reports (to appear) [arXiv: 0705.4397 [astro-ph]]
[37] S. Doledson, Modern Cosmology, Elsevier Ltd. (2003)
[38] V. Sahni and Y. Shtanov, J. Cosmol. Astropart. Phys. 11 (2003) 014; U. Alam and V. Sahni, Phys. Rev. D73 (2006) 084024