Black hole remnants due to Planck-length deformed QFT

Alain R. P. Dirkes, Michael Maziashvili, and Zurab K. Silagadze

1Frankfurt Institute for Advanced Studies (FIAS), Johann Wolfgang Goethe Universit"at, Ruth-Moufang-Strasse 1, Frankfurt am Main, D-60438, Germany
2Particle Physics & Cosmology Group, Ilia State University, 3/5 Cholokashvili Ave., Tbilisi 0162, Georgia
3Budker Institute of Nuclear Physics SB RAS & Novosibirsk State University, 630 090, Novosibirsk, Russia

It was shown in a number of papers that the gravitational potential calculated by using the propagator that follows from the minimum-length deformed QFT implies the existence of black-hole remnants of the order of the Planck mass. Here we examine the behaviour of the potential that follows from the Planck-length deformed QFT, which in general does not entail the concept of the non-zero minimum length, and analyse whether the existence of the black hole remnants is intimately related to the concept of the minimum length or not. The answer to this question turns out to be negative.

PACS numbers: 04.60.-m, 04.70.Dy, 04.50.Gh

A. Introduction

The aim of this paper is to demonstrate that the existence of black hole remnants in the framework of this discussion that uses Planck-length deformed QFT does not necessarily require the presence of the minimum length. Before proceeding to our analysis, we remark briefly on the relationship between the behaviour of the potential and the black hole remnants. In what follows we adopt natural system of units $c = \hbar = 1$. The starting point is the modified Schwarzschild (-Tangherlini) spacetime

$$ds^2 = \left[1 - r_g^{n+1}(r)\right]dt^2 - \left[1 - r_g^{n+1}(r)\right]^{-1}dr^2 - r^2d\Omega_{n+2}^2,$$  \hspace{2cm} (1)

where $d\Omega_{n+2}^2$ is a line element of a $2+n$ dimensional unit sphere,

$$r_g(M) = (G_NM)^{\frac{1}{n+1}} \left[\frac{16\pi}{(n+2) \text{Vol}(S^{n+2})}\right]^{\frac{1}{n+1}},$$ \hspace{2cm} (2)

and the gravitational potential $V(r)$ is calculated by using the modified propagator that follows from the Planck-length deformed QFT. Essentially, the existence of the zero-temperature black hole remnants in the framework of this discussion is based on the following facts. The potential appears to be a monotonically decreasing function, finite at the origin with vanishing derivative at this point, that is: $V'(r) < 0$ for $r > 0$; $V(0) < \infty$, $V(0) = 0$. To see how these conditions provide the black hole remnant, let us assume for simplicity $n = 0$. In view of the Eq. (1) the gravitational radius turns out to be the solution of the equation: $1/2G_NV = V(r)$ and as the potential reaches its maximum at the origin, this equation does not have a solution for $M < 1/2G_NV(0)$. Thus, one infers that for $M_{\text{remnant}} = 1/2G_NV(0)$ the black hole horizon disappears and, on the other hand, its temperature, for it is proportional to the surface gravity: $V'(0)$, vanishes. It is worth noticing that typically, the mass of such remnants are of the order of the quantum gravity scale.

Even in presence of the minimum length it is not self-evident why the potential estimated through the modified propagator should behave this way but, in this case, it completes well the intuitive picture that follows simply from the Poisson’s equation: $\Delta \Phi = 4\pi G_N \rho$. The presence of the non-zero minimum position uncertainty in QM engenders the smearing of the delta function $\rho = M\delta(r)$ thus replacing the point-like source with a regular distribution. So, in presence of the minimum length, the result can be recognized as an over-all picture that follows from the implementation of the minimum length into quantum theory.

In what follows we will use the Hilbert space representation for a relatively broad class of Planck-length deformed QM constructed in [11].

$$\hat{\mathbf{x}}_i = \hat{x}_i, \quad \hat{\mathbf{p}}_j = \hat{p}_j \left(1 - \frac{2\beta(\alpha - 1)}{\alpha}r_P^\alpha p^\alpha\right)^{-rac{1}{2\beta}},$$ \hspace{2cm} (3)

where $\mathbf{x}, \mathbf{p}$ stands for the standard position and momentum operators, $\beta$ is a numerical factor of order unity and $l_P$ denotes the Planck length, and show that the above mentioned behaviour of the potential takes place even in absence of the minimum position uncertainty when in particular $\alpha > 1/2$. In the case $\alpha = 2$, the Eq. (3) reduces to the well-known result found in [12]. Let us notice that in view of Eq. (3), there is a cutoff $p^\alpha < \alpha/2\beta(\alpha - 1)l_P^\alpha$ when $\alpha > 1$. This cutoff arises merely from the fact that when small $p$ runs over this region - large $P$ covers the whole region from 0 to $\infty$ (for more details see [11]). Indeed this cutoff is responsible to the existence of the non-zero minimum position uncertainty. In the case $\alpha < 1$ there is no lower bound on position uncertainty and no cutoff on $p$, respectively.

In light of the Eq. (3), the dispersion relation for a free
massless particle and the correspondingly modified field theory read

\[ \varepsilon = P^2, \quad A[\Phi] = - \int d^4x \frac{1}{2} [\Phi \partial^2 \Phi + \Phi P^2 \Phi]. \]  

(4)

In what follows, we will use the propagator following from Planck-length deformed field theory (4) to estimate the modified potential, which will then be used in Eq. (1).

B. The behaviour of the potential: 3D case

1. No minimum length: 0 < \alpha < 1

In this case the potential calculated by using the modified propagator, Eqs. (3, 4), takes the form

\[ V(r) = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi e^{ikr}}{k^2 (1 + \beta k^2)^{2/(1-\alpha)}}, \]  

(5)

where

\[ \beta \equiv \frac{2\beta(1-\alpha)}{\alpha}. \]

It is easy to observe that

\[ V(0) = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2 (1 + \beta k^2)^{2/(1-\alpha)}} = \frac{2}{\pi} \int_0^\infty \frac{dk}{(1 + \beta k^2)^{2/(1-\alpha)}}, \]  

(6)

is finite when \alpha > 1/3, otherwise it is divergent. Let us see how \( V(r) \) behaves when \( r \to 0 \). The Eq. (5) can be written as (\( q \equiv kr \))

\[ V(r) = \frac{2}{\pi r} \int_0^\infty \frac{dq \sin(q)}{q [1 + \beta (q/r)^\alpha]^{2/(1-\alpha)}}. \]  

(7)

Let us split the integral in Eq. (7) (\( \epsilon \ll 1 \))

\[ \int_0^\infty \frac{dq \sin(q)}{q [1 + \beta (q/r)^\alpha]^{2/(1-\alpha)}} = \epsilon \int_0^\epsilon \frac{dq \sin(q)}{q [1 + \beta (q/r)^\alpha]^{2/(1-\alpha)}} + \epsilon \int_\epsilon^\infty \frac{dq \sin(q)}{q [1 + \beta (q/r)^\alpha]^{2/(1-\alpha)}}. \]

When \( r \to 0 \), the integral over the region (\( \epsilon, \infty \)) goes to zero - so, one can omit this part. Thus, the short distance behaviour of this integral is essentially determined by the expression

\[ \int_0^\epsilon \frac{dq}{[1 + \beta (q/r)^\alpha]^{2/(1-\alpha)}} \to r \int_0^{\epsilon/r} \frac{dk}{[1 + \beta (k)^\alpha]^{2/(1-\alpha)}}. \]  

(8)

Hence, the short distance behaviour of the potential is

\[ V(r \to 0) \to \frac{2}{\pi} \int_0^{1/r} \frac{dk}{[1 + \beta k^2]^{2/(1-\alpha)}}. \]  

(9)

Now let us show that \( V'(r) < 0 \) for \( r > 0 \). For the potential derivative one finds

\[ V'(r) = \nabla V(r) \cdot \frac{r}{r} = \frac{i}{\pi} \int_0^\infty \frac{dk}{(1 + \beta k^2)^{2/(1-\alpha)}} \int_0^\pi d\theta \sin(\theta) \cos(\theta) e^{ikr \cos(\theta)} = -\frac{1}{\pi} \int_{-1}^1 d\tau \int_0^\infty \frac{dk}{(1 + \beta k^2)^{2/(1-\alpha)}} \int_0^\infty \frac{d\tau}{(1 + \beta k^2)^{2/(1-\alpha)}}. \]  

(10)

Since in the standard case, \( \beta = 0 \), the double integral entering the Eq. (10) is positive, then by taking into account that the integrand is now divided by the monotonically increasing function, one concludes that its positiveness is again guaranteed. The point is that the amplitude of the \( \sin \) function is monotonically suppressed now and as the integral starts from zero\(^1\) the negative contribution coming from this function in the integral is now more suppressed as compared to its positive contribution. Let us

\(^1\) One can write the integral with respect to the \( \tau \) over the region \([0^-, 1]\) as the product \( \tau \sin(kr\tau) \) is even function of \( \tau \).
notice that this argument may sound somewhat formal and heuristic, as one may wonder about the well definiteness of the integral with respect to the $k$ in Eq. (10). Indeed this integral may result in a singular function of $\tau$ but one should take into account that it is an integrable singularity. It is easy to see this by considering the worse case: $\beta = 0$,

$$V'(r) = -\frac{1}{\pi} \int_{-1}^{1} d\tau \int_{0}^{\infty} dk \frac{r}{k} \sin(kr\tau) = -\frac{1}{\pi r^2} \times$$

$$\int_{-1}^{1} d\tau \int_{0}^{\infty} dq \frac{r}{q} \sin(q\tau) = \frac{1}{2\pi r^2} \int_{-1}^{1} d\tau \frac{d}{d\tau} \int_{-\infty}^{\infty} dq e^{iq\tau} = \frac{1}{r^2} \int_{-1}^{1} d\tau \frac{d\delta(\tau)}{d\tau} = -\frac{1}{r^2}. \quad (11)$$

So, it is obvious that the integral over $k$ in Eq. (10) is no more singular than the derivative of the delta function. Let us notice that when $\alpha > 1/2$ then the integral (10) is evidently convergent and one does not need to operate with the generalized functions (distributions). Having seen this, one can make the above argument mathematically rigorous for $0 < \alpha \leq 1/2$ by introducing a cutoff $\Lambda$ on $k$ (merely by taking the integral over the region $(0, \Lambda)$ or including the exponential function $\exp(-|k|/\Lambda)$ in the integrand) and defining the integral as a limit $\Lambda \to \infty$. For a detailed account of generalized functions (distributions) we refer the reader to the review article [13].

From Eq. (10) one easily concludes that for $\alpha > 1/2$

$$V'(0) = \int_{0}^{\infty} \frac{dk}{(1 + Bk^2)^{2/(\alpha - 1)}} \int_{0}^{\pi} d\theta \sin(\theta) \cos(\theta) = 0. \quad (12)$$

This result can be obtained from Eq. (9) as well

$$V'(r \to 0) \propto -r^{\frac{4\alpha - 2}{\alpha - 1}}. \quad (13)$$

2. Minimum length: $\alpha > 1$

In presence of the minimum length all of the steps mentioned in the previous section can be proved straightforwardly for in this case there is a cutoff on $k$. The gravitational potential looks like

$$V(r) = \int_{k^2 < B^{-1}} \frac{d^3k}{(2\pi)^3} \frac{4\pi (1 - Bk^2)^{2/(\alpha - 1)} e^{ikr}}{k^2}, \quad (13)$$

where now

$$\beta \equiv \frac{2\beta(\alpha - 1)}{\alpha}. \quad (14)$$

It is easy to see that $V(0) < \infty$. One also finds without much trouble that $V'(0) = 0$. Namely,

$$V'(0) \propto \int_{0}^{k < B^{-1/\alpha}} dk \frac{r}{k} (1 - Bk^2)^{2/(\alpha - 1)} \times$$

$$\int_{0}^{\pi} d\theta \sin(\theta) \cos(\theta) = 0. \quad (15)$$

The argument put forward in the previous subsection for proving that $V'(r) < 0$ when $r > 0$ works now readily for there is the cutoff: $k^2 < \beta^{-1}$. The behaviour of the potential when $r \to 0$ looks as follows

$$V(r) = \frac{2}{\pi} \int_{0}^{k < B^{-1/\alpha}} dk \frac{r}{k} (1 - Bk^2)^{2/(\alpha - 1)} \frac{\sin(kr)}{kr} =$$

$$\frac{2}{\pi} \int_{0}^{k < B^{-1/\alpha}} dk \frac{r}{k} (1 - Bk^2)^{2/(\alpha - 1)} -$$

$$\frac{r^2}{3\pi} \int_{k < B^{-1/\alpha}} dk k^2 (1 - Bk^2)^{2/(\alpha - 1)} + O(r^4), \quad (14)$$

where we have just used the Taylor series expansion of the sin function near the origin.

C. Higher dimensional case

1. No minimum length: $0 < \alpha < 1$

The Fourier integrals in higher-dimensional case are more divergent but the basic arguments are essentially the same. Now the potential takes the form

$$V(r) = \frac{(1 + n)\text{Vol}(S^{n+2})}{(2\pi)^{3+n}} \int \frac{d^3k}{k^2(1 + Bk^2)^{2/(\alpha - 1)}}, \quad (15)$$

where $n$ denotes the number of extra dimensions. From this expression one finds that $V(0)$ is finite as long as $\alpha > (1 + n)/(3 + n)$. To work out the behaviour of the potential for $r \to 0$, let us write the Eq. (15) in the form
\[ V(r) = \frac{(1 + n) \text{Vol}(S^{n+2}) \text{Vol}(S^{n+1})}{(2\pi)^{3+n}} \int_0^\infty \frac{dk \, k^n}{(1 + Bk^n)^{2/(1-\alpha)}} \int_0^\pi d\theta_{n+1} \sin^{n+1}(\theta_{n+1})e^{ikr\cos(\theta_{n+1})} = \]

\[ \frac{(1 + n) \text{Vol}(S^{n+2}) \text{Vol}(S^{n+1})}{(2\pi)^{3+n} r^{n+1}} \int_{-1}^1 d\tau \left( 1 - \tau^2 \right)^{\frac{n}{2}} \int_0^\infty dq q^n \cos(qr) \frac{1}{[1 + B(q/r)^\alpha]^{2/(1-\alpha)}} \]  

(16)

First of all, let us notice that the integral with respect to \( q \) in Eq. (26) is understood in the sense of the generalized functions. It may be divergent for some values of \( \alpha \) but nevertheless the second integral with respect to \( \tau \) gives the finite result (see the discussion after Eq. (19)). Keeping this in mind, one can again split the integral

\[ \int_{-1}^1 d\tau \left( 1 - \tau^2 \right)^{\frac{n}{2}} \int_0^\infty dq q^n \cos(qr) \frac{1}{[1 + B(q/r)^\alpha]^{2/(1-\alpha)}} = \int_{-1}^1 d\tau \left( 1 - \tau^2 \right)^{\frac{n}{2}} \int_0^\infty dq q^n \cos(qr) \]

(\( \epsilon \ll 1 \)) and omit the second term as it goes to zero when \( \tau \to 0 \). Thus, one infers that the short distance behaviour \( (r \ll \epsilon B^{1/\alpha}) \) of the potential is essentially determined by the integral

\[ V(r \ll \epsilon B^{1/\alpha}) \simeq \frac{(1 + n) \text{Vol}(S^{n+2}) \text{Vol}(S^{n+1})}{(2\pi)^{3+n} 2^{2+n} \Gamma(\frac{n+3}{2})} \int_0^{\epsilon/r} \frac{dk \, k^n}{[1 + Bk^n]^{2/(1-\alpha)}} , \]  

(17)

As in the previous case, it can be seen that the integral

\[ \int_0^{\epsilon/r} \frac{dk \, k^{n+1}}{[1 + Bk^n]^{2/(1-\alpha)}} = \Theta \int_0^\infty \frac{dk \, k^{n+1} e^{iqr}}{[1 + Bk^n]^{2/(1-\alpha)}} , \]

that enters the Eq. (19) may be divergent for certain values of \( \alpha \) (for a given \( n \)) but still integrable with respect to \( \tau \). Again, this integral should be understood by introducing the factor \( e^{-\epsilon q} \). So, one can use the following relation (14) \((P \) stands for the principal value)

\[ \int_0^\infty dq e^{iq(\tau + i\epsilon)} = \frac{\epsilon}{\tau^2 + \epsilon^2} + \frac{i\tau}{\tau^2 + \epsilon^2} = \pi \delta(\tau) + P \frac{i}{\tau} , \]

which is tantamount to using
\[
\int_0^\infty dq \cos(q\tau) = \pi \delta(\tau), \quad \int_0^\infty dq \sin(q\tau) = \mathcal{P} \frac{1}{\tau}.
\]

Since in the standard case, \(\beta = 0\), the double integral entering the Eq. \((19)\) is positive, then by taking into account that the integrand is now divided by the monotonically increasing function, one concludes that its positiveness is again guaranteed. The argument is the same as in the previous case: one can write the integral with respect to \(\tau\) over the region \([0^-, 1]\), that is, the integration starts from zero. The amplitude of the \(\sin\) function is monotonically suppressed now, so the negative contribution coming from this function in the integral is now more suppressed as compared to its positive contribution. Considering the worse case, \(\beta = 0\), one can calculate the \((19)\) straightforwardly with the use of equation

\[
\int_0^\infty dq q^{n+1} e^{i\tau} = (-i)^{n+1} \frac{d}{d\tau} + \int_0^\infty e^{i\tau} =
\]

\((-i)^{n+1} \pi \delta^{(n+1)}(\tau) - (-1)^n(-i)^n (n+1)! \mathcal{P} \frac{1}{\tau^{n+2}} \), \((20)\)

where the principal value \(\mathcal{P}\) is understood in the sense of Hadamard \([15]\)

\[
\int_a^b dx \frac{f(x)}{(x-u)^{n+1}} = \frac{f(a)}{n(a-u)^n} - \frac{f(b)}{n(b-u)^n} + \frac{1}{n} \mathcal{P} \int_a^b dx \frac{f^{(1)}(x)}{(x-u)^n}. \quad (21)
\]

Instead of using Eqs. \((20)\) \((21)\) straightforwardly, one could calculate this type of integrals by using the \(\epsilon\) prescription but this approach is somewhat more cumbersome; see the Appendix. Using the Eqs. \((20)\) \((21)\), in the case \(n = 1\) one finds

\[
\int_{-1}^1 d\tau \sqrt{1 - \tau^2} \tau \int_0^\infty dq q^2 \sin(\tau q) = - \int_{-1}^1 d\tau \sqrt{1 - \tau^2} \tau \times
\]

\[
\mathcal{P} \int_0^\infty dq \sin(\tau q) = -2 \mathcal{P} \int_{-1}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} = 2 \mathcal{P} \int_{-1}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} = 2 \arcsin(\tau)|_{-1}^1 = 2\pi.
\]

Thus, the derivative of the potential \((19)\) takes the form

\[
V'(r) = -\frac{4\pi \text{Vol}(S^3) \text{Vol}(S^2)}{\left(2\pi\right)^5 r^3}.
\]

For \(n = 2\) from Eqs. \((19)\) \((20)\) one finds

\[
\int_0^\infty dq q^3 \sin(q\tau) = \frac{d^3}{d\tau^3} \int_0^\infty dq \cos(q\tau) = \pi \frac{d^3}{d\tau^3} \tau,
\]

and correspondingly

\[
V'(r) = -\frac{3\pi \text{Vol}(S^4) \text{Vol}(S^3)}{\left(2\pi\right)^5 r^4} \times
\]

\[
\int_{-1}^1 d\tau \left(1 - \tau^2\right) \frac{d^3}{d\tau^3} \frac{1}{\tau^3} = -\frac{18\pi \text{Vol}(S^4) \text{Vol}(S^3)}{\left(2\pi\right)^5 r^4}. \quad (23)
\]

When \(n = 3\) one can again use the Eqs. \((20)\) \((21)\)

\[
\int_{-1}^1 d\tau \left(1 - \tau^2\right) \frac{d^3}{d\tau^3} \int_0^\infty dq q^4 \sin(\tau q) = \int_{-1}^1 d\tau \left(1 - \tau^2\right) \frac{d^3}{d\tau^3} \tau \times
\]

\[
\int_0^\infty dq \sin(\tau q) = 4! \mathcal{P} \int_{-1}^1 d\tau \left(1 - \tau^2\right)^{3/2} \tau = -4! \mathcal{P} \int_{-1}^1 d\tau \left(1 - \tau^2\right)^{3/2} \tau = 4\pi,
\]

(last integral is calculated after Eq. \((21)\) \)) that results in

\[
V'(r) = -\frac{4! \cdot 4\pi \text{Vol}(S^5) \text{Vol}(S^4)}{\left(2\pi\right)^5 r^5}. \quad (24)
\]

The transition to arbitrary \(n\) is then achieved by making use of Eqs. \((20)\) \((21)\) step by step.

\[2. \quad \text{Minimum length: } \alpha > 1\]

Because of the presence of cut-off \(k < \beta^{-1/\alpha}\) in the case of minimum length, one can readily show all of the features required for the potential for the existence of the zero-temperature BH remnants. The potential looks like
\[ V(r) = \frac{(1 + n)\text{Vol}(S^{n+2})\text{Vol}(S^{n+1})}{(2\pi)^{3+n}} r^{-1/\alpha} \int_0^\infty dk k^n (1 - B k^n)^{2/(\alpha - 1)} \int_{-1}^1 d\tau (1 - \tau^2)^{-\frac{\alpha}{2}} \cos(k r \tau) . \] (25)

It is evident from this expression that \( V(0) \) is finite, \( V'(0) = 0 \) and \( V'(r) < 0 \). The last statement can be proved much in the same way as it was done in the case \( n = 1 \). Its asymptotic behaviour for \( r \to 0 \) can easily be found by expanding the \( \cos(k r \tau) \) into the Taylor series. As in the previous case, see Eq.\([14]\), one finds

\[ V(r) = A - Br^2 + O(r^4) , \] (26)

where \( A \) and \( B \) are positive quantities.

**D. Concluding remarks**

We have seen that the Planck-length deformed propagator for \( \alpha > 1/3 \) results in the potential, which after being using in the Schwarzschild metric shows up the existence of the black hole remnants. If \( 1/3 < \alpha < 1/2 \) the temperature of the black hole goes to infinity when the black hole mass approaches \( M_{\text{remnant}} \). For \( \alpha > 1/2 \) the black hole remnants are characterized with the zero temperature. The key observation made throughout this paper is that the black hole remnants exist even when there is no minimum length: \( \alpha < 1 \).

Interestingly enough, in 3D case the hot black hole remnants start to take place for the Planck-length deformed quantum theory that follows from the Károlyházy uncertainty relation (\( \alpha = 1/3 \)) while the zero-temperature black hole remnants show up already for deformed QFT that corresponds to the random fluctuations of the background space (\( \alpha = 1/2 \)). Let us notice that when \( 1/3 < \alpha < 1/2 \), the temperature of the black hole remnants goes to infinity.

It is worth noticing that the Planck-length modified Schwarzschild space-time is free of the curvature singularity at the origin when \( \alpha > 3/5 \) as in this case the metric as well as the first and second derivatives of it do not diverge when \( r \to 0 \) (see Eqs.\([9]\)\([12]\)\([14]\)\([17]\)\([18]\)\([25]\)).

Finally let us comment on the validity of an approximation assumed tacitly throughout the above discussion. We have presumed the gravitational field put on the equal footing with other interactions, that is, QFT picture for gravity is taken as a starting point. That means that the graviton field is defined as the difference between the full metric and its Minkowski background value. Let us distinguish between two cases: \( \alpha < 1/2 \) and \( \alpha > 1/2 \). In the former case the gravitational force does not go to zero when \( r \to 0 \) and one can not justify the ignorance of the radiative corrections near the Planck scale. In the latter case, one might think of gravity as an asymptotically free interaction and correspondingly the radiative corrections close to the Planck scale could be ignored.

To be more strict, we do not know what the full implementation of minimum-length deformed quantum mechanics in GR might look like. If one truncates the modified field theory given by Eqs.\([3]\)\([4]\) to some power of \( B \), then it will result in a so called Lifshitz like theory and therefore one might expect the corresponding gravity theory to look something like the Horava-Lifshitz gravity \([16]\). But so far we can say nothing definitively about the full implementation of minimum-length deformed quantum mechanics in GR. Another way of modifying the GR with respect to the minimum-length concept might be a non-local theory of GR \([17]\)\([19]\); the question of black hole remnants in this sort of theory was addressed in \([20]\).

Finally, let us mention the papers known to us addressing the question of modified potential due to deformed propagator \([21]\)\([22]\) and some of the papers devoted to the black hole remnants due to Planck-length deformed field theory \([20]\)\([26]\)\([33]\).

**Appendix**

Using the \( \epsilon \) prescription, in the case \( B = 0 \) the double integral entering the Eq.\([19]\) can be written in the form

\[ I_n = \int_{-1}^1 d\tau (1 - \tau^2)^{n/2} \int_0^\infty dq q^{n+1} e^{i q (\tau + i \epsilon)} = \]

\[ (-1)^{n+1} \int_{-1}^1 d\tau (1 - \tau^2)^{n/2} \int_0^\infty dq e^{i q (\tau + i \epsilon)} = \]

\[ i(-1)^{n+1} \int_{-1}^1 d\tau (1 - \tau^2)^{n/2} \int_0^\infty dq \frac{\tau^2 + \epsilon^2 - \epsilon^2}{\tau^2 + \epsilon^2} = \]

\[ i(-1)^{n+1} \int_{-1}^1 d\tau (1 - \tau^2)^{n/2} \frac{\epsilon^2}{\tau^2 + \epsilon^2} \int_{-\pi/2}^{\pi/2} d\theta \frac{\cos^{n+1} \theta}{\sin^{n+1} \theta + \epsilon^2} = \]

Making the substitution \( \tau = \sin \theta \), one finds

\[ I_n = i(-1)^n \int_{-\pi/2}^{\pi/2} d\theta \frac{\cos^{n+1} \theta}{\sin^{n+1} \theta + \epsilon^2} \equiv \]

\[ i(-1)^n \int_{-\pi/2}^{\pi/2} \left[ \epsilon^2 \mathcal{I}_{n+1}(\epsilon) \right] . \] (27)
So, we have

\[ I_n(\epsilon) = \int_{-\pi/2}^{\pi/2} \frac{\cos^n \theta}{\sin^2 \theta + \epsilon^2} d\theta = \frac{\pi}{2} \frac{\cos^{n-2} \theta(1 - \sin^2 \theta - \epsilon^2 + \epsilon^2)}{\sin^2 \theta + \epsilon^2} d\theta, \]  

and hence

\[ I_n(\epsilon) = (1 + \epsilon^2)I_{n-2}(\epsilon) - K_{n-2}, \]  

where

\[ K_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta. \]

Using (29), we can prove by induction that

\[ I_{2m} = (1 + \epsilon^2)^mI_0 - \sum_{j=0}^{m-1} (1 + \epsilon^2)^j K_{2(m-1)-2j}, \]

and

\[ I_{2m+1} = (1 + \epsilon^2)^mI_1 - \sum_{j=0}^{m-1} (1 + \epsilon^2)^j K_{2(m-1)-2j}. \]

Hence, when \( n = 2m - 1 \) is odd, we get from Eqs. (27, 30)

\[ I_{2m-1} = -i \frac{d^{2m}}{d\epsilon^{2m}} \left[ \epsilon^2 (1 + \epsilon^2)^m I_0(\epsilon) - \epsilon^2 (1 + \epsilon^2)^{m-1} K_0 \right], \]

all other terms in Eq. (30) are giving zero contribution. Going further, one easily finds the values of \( I_0 \) and \( K_0 \)

\[ I_0(\epsilon) = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2 \theta + \epsilon^2} = \frac{\pi}{\epsilon \sqrt{1 + \epsilon^2}}, \quad K_0 = \pi, \]

and, therefore, the Eq. (32) reduces to

\[ I_{2m-1} = (2m)!\pi - i\pi \frac{d^{2m}}{d\epsilon^{2m}} \left[ \epsilon(1 + \epsilon^2)^{m-1/2} \right], \]

which is the same as

\[ I_{2m-1} = (2m)!\pi - \frac{i\pi}{2m + 1} \frac{d^{2m+1}}{d\epsilon^{2m+1}} (1 + \epsilon^2)^{m+1/2}. \]  

In the \( \epsilon \to 0 \) limit, the second term in Eq. (33) goes to zero as the Taylor expansion of \( (1 + \epsilon^2)^{m+1/2} \) around \( \epsilon = 0 \) contains only even powers of \( \epsilon \). Hence,

\[ \lim_{\epsilon \to 0} I_{2m-1} = (2m)!\pi. \]  

Similarly, when \( n \) is even \( n = 2m \), we get

\[ I_{2m} = \frac{d^{2m+1}}{d\epsilon^{2m+1}} \left[ \epsilon^2 (1 + \epsilon^2)^m I_1(\epsilon) \right], \]

as no term containing \( K \)-factors survives after taking the \( (2m+1) \)-th derivative. Using the relation

\[ I_1(\epsilon) = \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{\sin^2 \theta + \epsilon^2} d\theta = \frac{2}{\epsilon} \arctan \left( \frac{1}{\epsilon} \right), \]

this equation reduces to

\[ I_{2m} = 2i \frac{d^{2m+1}}{d\epsilon^{2m+1}} \left[ \epsilon(1 + \epsilon^2)^m \arctan \left( \frac{1}{\epsilon} \right) \right]. \]  

Now we can take the limit \( \epsilon \to 0 \) by taking into account that \( \arctan(1/\epsilon) \) and its derivatives are not singular in this limit. Exploiting the Leibniz rule to the Eq. (35) one finds

\[ \lim_{\epsilon \to 0} I_{2m} = 2i \lim_{\epsilon \to 0} \arctan \left( \frac{1}{\epsilon} \right) \frac{d^{2m+1}}{d\epsilon^{2m+1}} \left[ \epsilon(1 + \epsilon^2)^m \right] = i\pi (2m + 1)!. \]

All other terms vanish because for any \( j \geq 1 \) either

\[ \lim_{\epsilon \to 0} \frac{d^j}{d\epsilon^j} \epsilon(1 + \epsilon^2)^m = 0, \]

since binomial expansion of \( \epsilon(1 + \epsilon^2)^m \) contains only odd powers of \( \epsilon \), or

\[ \lim_{\epsilon \to 0} \frac{d^j}{d\epsilon^j} \arctan \left( \frac{1}{\epsilon} \right) = -\lim_{\epsilon \to 0} \frac{d^{j-1}}{d\epsilon^{j-1}} \frac{1}{1 + \epsilon^2} = 0. \]

Equations (34) and (36) can be unified in the final result

\[ \lim_{\epsilon \to 0} I_n = i\pi (n + 1)!. \]

Acknowledgments

Authors are greatly indebted to Hendrik van Hees for useful discussions and to Igor Khavkine for his suggestion regarding the straightforward calculation of integrals for odd values of \( n \) by using the Eq. (20). M. M. is indebted to Marcus Bleicher and Piero Nicolini for their hospitality at the Frankfurt Institute for Advanced Studies. This research was supported in part by the Shota Rustaveli National Science Foundation under contract number 31/89, the DAAD research fellowship for university teachers and researchers, by the Ministry of Education and Science of the Russian Federation, Russian Federation President Grant for support of scientific schools NSh-5320.2012.2 and RFBR grant 13-02-00418-a.
[1] M. Maziashvili, Fortschr. Phys. 61, 685 (2013) [arXiv:1110.0649 [gr-qc]].

[2] M. Maziashvili, JCAP 1303, 042 (2013) [arXiv:1208.5570 [hep-th]].

[3] M. Maziashvili and P. Nicolini, will appear soon on arXiv.

[4] F. R. Tangherlini, Nuovo Cim. 27, 636 (1963).

[5] R. C. Myers and M. J. Perry, Annals Phys. 172, 304 (1986).

[6] P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 632, 547 (2006) [gr-qc/0510112].

[7] S. Ansoldi, [arXiv:0802.0330 [gr-qc]].

[8] E. Spallucci and S. Ansoldi, Phys. Lett. B 701, 471 (2011) [arXiv:1101.2760 [hep-th]].

[9] L. Modesto, J. W. Moffat and P. Nicolini, Phys. Lett. B 695, 397 (2011) [arXiv:1010.0608 [gr-qc]].

[10] M. Sprenger, P. Nicolini and M. Bleicher, Eur. J. Phys. 33, 853 (2012) [arXiv:1202.1500 [physics.ed-ph]].

[11] M. Maziashvili, Phys. Rev. D 86, 104066 (2012) [arXiv:1206.4388 [gr-qc]].

[12] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997) [hep-th/9612084].

[13] W. Güttinger, Fortschr. Phys. 14, 483 (1966).

[14] N. N. Bogolyubov and D. V. Shirkov, "Introduction to the Theory of Quantized Field" (John Wiley & Sons Inc; 3rd edition, 1980), Page - 603.

[15] C. Fox, Canad. J. Math. 9, 110 (1957).

[16] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].

[17] N. V. Krasnikov, Teor. Math. Phys. 73, 1184 (1987) [Teor. Mat. Fiz., 73, 235 (1987)].

[18] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, Phys. Rev. Lett. 108, 031101 (2012) [arXiv:1110.5249 [gr-qc]].

[19] L. Modesto, [arXiv:1202.3151 [hep-th]].

[20] P. Nicolini, [arXiv:1202.2102 [hep-th]].

[21] V. M. Tkachuk, J. Phys. Stud. 11, 41 (2007).

[22] R. C. Helling and J. You, JHEP 0806, 067 (2008) [arXiv:0707.1885 [hep-th]].

[23] G. Amelino-Camelia, N. Lorent, G. Mandanici and F. Mercati, Int. J. Mod. Phys. D 19, 2385 (2010) [arXiv:1007.0851 [gr-qc]].

[24] S. K. Moayed, M. R. Setare and B. Khosropour, Adv. High Energy Phys. 2013, 657870 (2013) [arXiv:1303.0100 [hep-th]].

[25] S. K. Moayed, M. R. Setare and B. Khosropour, [arXiv:1306.1070 [hep-th]].

[26] R. J. Adler, P. Chen and D. I. Santiago, Gen. Rel. Grav. 33, 2101 (2001) [gr-qc/0106080].

[27] B. Koch, M. Bleicher and S. Hossenfelder, JHEP 0510, 053 (2005) [hep-ph/0507138].

[28] K. Nozari and S. H. Mehdipour, Mod. Phys. Lett. A 20, 2937 (2005) [arXiv:0809.3144 [gr-qc]].

[29] P. Nicolini, Int. J. Mod. Phys. A 24, 1229 (2009) [arXiv:0807.1939 [hep-th]].

[30] B. Koch, M. Bleicher and H. Stöcker, Phys. Lett. B 672, 71 (2009) [arXiv:0807.3349 [hep-ph]].

[31] R. Banerjee and S. Gosh, Phys. Lett. B 688, 224 (2010) [arXiv:1002.2302 [gr-qc]].

[32] M. Bleicher and P. Nicolini, J. Phys. Conf. Ser. 237, 012008 (2010) [arXiv:1001.2211 [hep-ph]].

[33] P. Nicolini and E. Winstanley, JHEP 1111, 075 (2011) [arXiv:1108.4419 [hep-ph]].

[34] M. Bleicher, P. Nicolini, M. Sprenger and E. Winstanley, Int. J. Mod. Phys. E 20S2, 7 (2011) [arXiv:1111.0657 [hep-th]].

[35] P. Nicolini, J. Mureika, E. Spallucci, E. Winstanley and M. Bleicher, [arXiv:1302.2640 [hep-th]].