On indefinite Kirchhoff-type equations under the combined effect of linear and superlinear terms

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Abstract

We investigate a class of Kirchhoff type equations involving a combination of linear and superlinear terms as follows:

\[-\left(a \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + 1 \right) \Delta u + \mu V(x)u = \lambda f(x)u + g(x)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,\]

where $N \geq 3$, $2 < p < 2^\ast := \frac{2N}{N-2}$, $V \in C(\mathbb{R}^N)$ is a potential well with the bottom $\Omega := \text{int}\{x \in \mathbb{R}^N \mid V(x) = 0\}$. When $N = 3$ and $4 < p < 6$, for each $a > 0$ and $\mu$ sufficiently large, we obtain that at least one positive solution exists for $0 < \lambda < \lambda_1(f_\Omega)$ while at least two positive solutions exist for $\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_a$ without any assumption on the integral $\int_\Omega g(x)\phi_1^p \, dx$, where $\lambda_1(f_\Omega) > 0$ is the principal eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ with weight function $f_\Omega := f|_\Omega$, and $\phi_1 > 0$ is the corresponding principal eigenfunction.

Keywords: Kirchhoff type problem, steep potential well, eigenvalue problem, mountain pass theory.

1 Introduction

In this paper, we investigate the following Kirchhoff type equation:

\[
\begin{align*}
-\left(a \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + b \right) \Delta u + \mu V(x)u = h(x, u) \quad \text{in} \quad \mathbb{R}^N,
\end{align*}
\]

where $N \geq 3$, the parameters $a, b, \mu > 0$, and the potential $V$ satisfies the following conditions:
The Kirchhoff type equation is an extension of the classical D’Alembert wave equation, namely,

\[ u'' - M \left( \int_D |\nabla u|^2 \, dx \right) \Delta u = h(x, u), \quad (1.2) \]

which was proposed by Kirchhoff \cite{23} to describe the transversal oscillations of a stretched string, taking into account of the effect of changes in string length during the vibrations. Here \( D \) is a bounded domain, \( u \) denotes the displacement and \( h \) is the external force. In particular, if \( M(t) = at + b \), then \( b \) denotes the initial tension while \( a \) is related to the intrinsic properties of the string (such as Young’s modulus). It is notable that Eq. \((1.2)\) is often referred to as being non-local because of the presence of the integral over the domain \( D \). About the solvability of Eq. \((1.2)\), we refer to the reader to the papers \cite{5,13,26,27}.

In recent years, the stationary analogue of Eq. \((1.2)\) with specific formulations of \( M \) and \( h \) has been widely studied in bounded domain \cite{10,12,22,30,33} and in unbounded domain \cite{2,9,14,16,17,19–21,25,28,29,31,32,35,36} via variational methods. Now let us briefly comment some known results related to our work.

Sun and Wu \cite{29}, the first and third authors of the current paper, studied the existence of ground state solution for Eq. \((1.1)\) with \( N = 3 \), where \( V \) satisfies conditions \((V1) - (V2)\) and the nonlinearity \( h \) is required to be asymptotically linear, asymptotically 3-linear and asymptotically 4-linear at infinity on \( u \), respectively. The proof is based on mountain pass theorem and the Nehari manifold method. It is worth noting that the potential \( \mu V \), first introduced by Bartsch and Wang \cite{6}, is usually called the steep potential well whose depth is controlled by the parameter \( \mu \). Later, the corresponding results were further extended and improved by Jia and Luo \cite{21} and Zhang and Du \cite{35}.

Recently, Sun et al. \cite{28} considered the case of \( N \geq 4 \). They found that when \( h \) is superlinear and subcritical on \( u \), the geometric structure of the functional \( J \) related to Eq. \((1.1)\) is known to have a global minimum and a mountain pass, due to the forth power of the non-local term. As a result, two positive solutions of Eq. \((1.1)\) can be found. After that, Sun and Wu \cite{31} showed that when \( N = 3 \) and \( h(x, u) = g(x)|u|^{p-2}u \) with \( 2 < p < 4 \), the functional \( J \) related to Eq. \((1.1)\) also has a global minimum and a mountain pass. However, an additional assumption on \( V \) and \( g \) needs to be required as follows:

\begin{enumerate}
\item[(H3)] There exist two numbers \( c_*, R_* > 0 \) such that
\[ |x|^{p-2} g(x) \leq c_* [V(x)]^{4-p} \text{ for all } |x| > R*. \]
\end{enumerate}

We notice that there seems to be rarely concerned on Kirchhoff type equation involving a combination of linear and superlinear terms in the existing literature. We are only aware of the
work \cite{36}. Actually, the combined effect of linear and superlinear terms was first studied by Alama and Tarantello \cite{1} in the following indefinite semilinear elliptic equations in bounded domain:

\[
\begin{align*}
-\Delta u &= \lambda u + a(x)h(u) \quad \text{in } D, \\
u &= 0 \quad \text{on } \partial D,
\end{align*}
\]

where \(a \in C(D)\) changes sign in \(D\) and \(h\) is a nonlinear function with superquadratic growth both at zero and at infinity. They concluded that for \(\lambda\) in a small right neighborhood of \(\lambda_1\), the first eigenvalue of \(-\Delta\) in \(H^1_0(D)\), the condition \(\int_D a(x)\phi_1^p\,dx < 0\) is necessary and sufficient for existence of a positive solution, where \(\phi_1 > 0\) is the corresponding principal eigenfunction. Furthermore, the existence of two positive solutions for \(\lambda \in (\lambda_1, \gamma)\), for some \(\gamma > \lambda_1\), is also established in \cite{1}. For more similar results, we refer the reader to \cite{3,7,11,18}.

Very recently, Zhang et al. \cite{36} extended the analysis to Kirchhoff type equation with a combination of linear and superlinear terms, namely, Eq. (1.1) with \(h(x, u) = \lambda f(x)u + g(x)|u|^{p-2}u\), where \(2 < p < 2^*\) and \(f, g\) are both sign-changing in \(\mathbb{R}^N\). They illustrated the difference in the solution behavior which arises from the consideration of the nonlocal and eigenvalue problem effects. By using the Nehari manifold method and giving an approximation estimate of eigenvalue problem, they explored the existence and multiplicity of positive solutions when \(\lambda\) lies in the left and right neighborhood of \(\lambda_1(f_{\Omega})\), respectively, where \(\lambda_1(f_{\Omega})\) is the positive principal eigenvalue of the problem

\[
\begin{align*}
-\Delta u &= \lambda f_{\Omega}(x)u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

with \(\phi_1\) the corresponding positive principal eigenfunction, here \(\Omega\) is given in condition (V2) and the function \(f_{\Omega}\) is a restriction of \(f\) on \(\Omega\). Some of the results obtained in \cite{36} are summarized as follows (also see Figure 1):

\[\text{(a) } N = 3, 4 < p < 6 \text{ and } \int_{\Omega} g(x)\phi_1^p\,dx < 0 \]

\[\text{(b) } N \geq 3 \text{ and } 2 < p < \min\{4, 2^*\}\]

Figure 1: Bifurcation diagrams for result (i) on (a) and for results (ii) – (iii) on (b).

(i) \(N = 3\) and \(4 < p < 6\) : if \(\int_{\Omega} g(x)\phi_1^p\,dx < 0\), then for each \(a > 0\) there exists a number \(\delta_0 > 0\) such that for each \(\lambda_1(f_{\Omega}) < \lambda < \lambda_1(f_{\Omega}) + \delta_0\), Eq. (1.1) admits at least two positive solutions for \(\mu\) sufficiently large;
(ii) $N \geq 3$ and $2 < p < \min \{4, 2^\ast \}$: there exists a number $a_0 > 0$ such that for each $0 < a < a_0$ and $0 < \lambda < \Lambda \lambda_1(f_{\Omega})$, Eq. (1.1) admits at least two positive solutions for $\mu$ sufficiently large, where

$$\Lambda := 1 - 2 \left( \frac{4 - p}{4} \right)^{2/p} < 1 \text{ for } 2 < p < \min \{4, 2^\ast \};$$

(iii) $N \geq 3$ and $2 < p < \min \{4, 2^\ast \}$: there exists a number $a_0 > 0$ such that for each $0 < a < a_0$ and $\lambda \geq \Lambda \lambda_1(f_{\Omega})$, Eq. (1.1) admits at least a positive solution for $\mu$ sufficiently large.

From the results mentioned above, it is very natural for us to raise a series of interesting questions, such as the following

(I) The condition $\int_{\Omega} g(x) \phi_1^p \, dx < 0$ appears necessary in finding two positive solutions when $N = 3$ and $4 < p < 6$ in [36], as well as in the study of local elliptic equations [1, 3, 7, 11]. Can one obtain the same result as described in [36] without this condition?

(II) When $N \geq 3$ and $2 < p < \min \{4, 2^\ast \}$ in [36], the existence of two positive solutions is established only in the range of $0 < \lambda < \Lambda \lambda_1(f_{\Omega})$, while not including the range of $\Lambda \lambda_1(f_{\Omega}) \leq \lambda \leq \lambda_1(f_{\Omega})$. In view of this, we wonder if two positive solutions can be found when $\Lambda \lambda_1(f_{\Omega}) \leq \lambda \leq \lambda_1(f_{\Omega})$, like that in the case of $0 < \lambda < \Lambda \lambda_1(f_{\Omega})$.

(III) It is notable that Zhang et al. [36] only found one positive solution when $\lambda \geq \Lambda \lambda_1(f_{\Omega})$ for $N \geq 3$ and $2 < p < \min \{4, 2^\ast \}$. In other words, they can not conclude that $\lambda = \lambda_1(f_{\Omega})$ is a bifurcation point of Eq. (1.1) with positive solutions bifurcating to the right of $\lambda_1(f_{\Omega})$. Based on (II), we would like to further probe into whether there exists a bifurcation phenomenon at the point $\lambda = \lambda_1(f_{\Omega})$.

In the present paper, we are very interested in seeking definite answers to Questions (I)–(III) and establishing the multiplicity of positive solutions for Eq. (1.1) with $h(x,u) = \lambda f(x)u + g(x)|u|^{p-2}u$ by using the mountain pass theory and the direct sum decomposition of the function. Here we wish to point out that the Nehari manifold method used in [36] is not a good choice in our study. Indeed, for $N = 3$ and $4 < p < 6$, the condition $\int_{\Omega} g(x) \phi_1^p \, dx < 0$ is used to ensure that the Nehari manifold is a natural constraint and can be decomposed into two nonempty submanifolds. Moreover, for $N \geq 3$ and $2 < p < \min \{4, 2^\ast \}$, the filtration of the Nehari manifold is adopted to derive the boundedness of (PS)-sequence, which is available only for $0 < \lambda < \Lambda \lambda_1(f_{\Omega})$. For simplicity, we always assume that $b = 1$ in Eq. (1.1). The problem we consider is thus

$$\begin{cases}
- \left( a \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + 1 \right) \Delta u + \mu V(x)u = \lambda f(x)u + g(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases} \quad (K^\mu_{a,\lambda})$$

where $N \geq 3, 2 < p < 2^\ast$, the parameters $a, \mu, \lambda > 0$, the potential $V$ satisfies conditions (V1)–(V2), and the weight functions $f, g$ satisfy the following conditions:

(D1) $f \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\Omega)$ and $f^+ := \max\{f, 0\} \not\equiv 0$ in $\Omega$;
(D2) $g \in L^\infty(\mathbb{R}^N)$ and $g^+ \neq 0$ in $\Omega$.

**Remark 1.1** If $f$ is bounded in $\Omega$ and $|\{x \in \Omega \mid f(x) > 0\}| > 0$, then there exists a sequence of eigenvalues $\{\lambda_n(f_\Omega)\}$ of Eq. (1.3) with $0 < \lambda_1(f_\Omega) < \lambda_2(f_\Omega) \leq \cdots$ and each eigenvalue being of finite multiplicity. Denoting the positive principal eigenfunction by $\phi_0$, we have

$$\lambda_1(f_\Omega) = \int_\Omega |\nabla \phi_0|^2 dx = \inf \left\{ \int_\Omega |\nabla u|^2 dx \mid u \in H^1_0(\Omega), \int_\Omega f_\Omega(x)u^2 dx = 1 \right\}$$

and

$$\lambda_2(f_\Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \mid u \in H^1_0(\Omega), \int_\Omega f_\Omega(x)u^2 dx = 1, \int_\Omega \nabla u \nabla \phi_0 dx = 0 \right\}.$$

We now summarize our main results as follows.

**Theorem 1.1** Suppose that $N = 3, 4 < p < 6$ and conditions (V1) - (V2), (D1) - (D2) hold. Then for each $a > 0$, the following statements are true.

(i) For each $0 < \lambda \leq \lambda_1(f_\Omega)$, Eq. $K_{a,\lambda}^\mu$ has at least one positive solution for $\mu$ sufficiently large.

(ii) There exists a number $\delta_a > 0$ such that for every $\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_a$, Eq. $K_{a,\lambda}^\mu$ has at least two positive solutions for $\mu$ sufficiently large.

**Remark 1.2** (i) Theorem 1.1 (ii) give an answer to Question (I).

(ii) In fact, in the proof of Theorem 1.1 (ii) one can see that the length of right neighborhood of $\lambda_1(f_\Omega)$, i.e. $\delta_a$ can be given explicitly by

$$\delta_a = \min \left\{ a^{\frac{p-2}{2}} C_1, C_2 \right\},$$

where $C_1, C_2 > 0$. It means that $\delta_a$ depends on the parameter $a$ when $a$ is small while not depending on it when $a$ is large.

Let us consider the following problem:

$$\Gamma_p := \sup \left\{ \frac{\int_\Omega g(x)|u|^pdx}{\left( \int_\Omega |\nabla u|^2 dx \right)^{p/2}} \mid u \in H^1_0(\Omega) \backslash \{0\}, \int_\Omega f_\Omega(x)u^2 dx \geq 0 \right\}.$$

Under conditions (D1) - (D2), we can choose a function $\varphi \in H^1_0(\Omega)$ such that $\int_\Omega f_\Omega(x)\varphi^2 dx > 0$ and $\int_\Omega g(x)|\varphi|^p dx > 0$ (see [8, Proposition 6.2] for more details). Then, it is easy to deduce that $0 < \Gamma_p < \infty$ by conditions (V2), (D2) and Sobolev inequality. Now we set

$$a_0(p) = 2(p-2) \frac{4-p}{4} \left( \frac{\Gamma_p}{p} \right)^{\frac{4}{p^2}}\frac{2}{2}$$

for $2 < p < \min\{4, 2^*\}$.

**Theorem 1.2** Suppose that $N \geq 3, 2 < p < \min\{4, 2^*\}$ and conditions (V1) - (V2), (D1) - (D2) hold. In addition, for $N = 3$, we also assume that condition (H3) is satisfied. Then for each $0 < a < a_0(p)$, the following statements are true.
(i) For each $0 < \lambda < \lambda_1(f_\Omega)$, Eq. $(K_{a,\lambda}^\mu)$ has at least two positive solutions for $\mu$ sufficiently large.
(ii) If $\int_\Omega g(x)\phi_1^p dx < 0$, then we have

(iii) for $\lambda = \lambda_1(f_\Omega)$, Eq. $(K_{a,\lambda}^\mu)$ has at least two positive solutions for $\mu$ sufficiently large;

(iii) there exists $\delta_a > 0$ such that for every $\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_a$, Eq. $(K_{a,\lambda}^\mu)$ has at least three positive solutions for $\mu$ sufficiently large.

**Remark 1.3** (i) Theorem 1.2 answers Questions (II) and (III).

(ii) Similar to Remark 1.2 (ii), $\delta_a$ can also be given explicitly by

$$\delta_a = \min \left\{ a^{-\frac{p-2}{4-p}}C_3, C_4 \right\},$$

where $C_3, C_4 > 0$. It shows that $\delta_a$ does not depend on $a$ for a small while depending on it for a large.

To prove Theorems 1.1 and 1.2, we need to study the mountain pass geometry of the functional related to Eq. $(K_{a,\lambda}^\mu)$ when $\lambda$ lies in a right neighborhood of $\lambda_{1,\mu}(f)$ by decomposing each $u \in X$ (defined later) into the sum of a function in $\text{span}\{\phi_1,\mu\}$ and a function in $\{\text{span}\{\phi_1,\mu\}\}^\perp$, where $\lambda_{1,\mu}(f)$ is the positive principal eigenvalue of the problem

$$-\Delta u + \mu V(x)u = \lambda f(x)u \quad \text{in } X,$$

and $\phi_{1,\mu}$ is the corresponding positive principal eigenfunction. Then using the approximation estimate gives

$$\lambda_{1,\mu}(f) \rightarrow \lambda_1^1(f_\Omega) \quad \text{as } \mu \rightarrow \infty,$$

which can help us obtain the mountain pass geometry in a right neighborhood of $\lambda_1(f_\Omega)$. In the case of $N = 3$ and $4 < p < 6$, the non-local term in Eq. $(K_{a,\lambda}^\mu)$ can ensure the mountain pass geometry for $\lambda$ in a right neighborhood of $\lambda_1(f_\Omega)$ without any assumption on the integral $\int_\Omega g(x)\phi_1^p dx$. However, when $N \geq 3$ and $2 < p < \min\{4, 2^*\}$, we need to add the condition $\int_\Omega g(x)\phi_1^p dx < 0$ to ensure the mountain pass geometry for $\lambda$ in a right neighborhood of $\lambda_1(f_\Omega)$.

Next, we consider the case of $\int_\Omega g(x)\phi_1^p dx > 0$. Let

$$\lambda_a^+ = \lambda_1(f_\Omega) - (4 - p) \left( \frac{\int_\Omega g(x)\phi_1^p dx}{p} \right) \frac{2}{\lambda_1^1(f_\Omega)} \frac{2(p-2)}{a^{2-p}}.$$

Note that $0 \leq \lambda_a^+ < \lambda_1(f_\Omega)$ for each $a \geq a_0(p)$. Then we have the following result.

**Theorem 1.3** Suppose that $N \geq 3, 2 < p < \min\{4, 2^*\}$ and conditions (V1) - (V2), (D1) - (D2) hold. In addition, for $N = 3$, we also assume that condition (H3) is satisfied. If $\int_\Omega g(x)\phi_1^p dx > 0$, then for each $a \geq a_0(p)$ and $\lambda_a^+ < \lambda < \lambda_1(f_\Omega)$, Eq. $(K_{a,\lambda}^\mu)$ has at least two positive solutions for $\mu$ sufficiently large.
The results of Theorems 1.2 and 1.3 are illustrated in Figure 2. For \( a < a_0(p) \) assumed in (a) and (b), the turning from the middle to the upper solution branch occurs in the region \( \lambda < 0 \) demonstrating the result of Theorem 1.2 (i) permitting two positive solutions whenever \( 0 < \lambda < \lambda_1(f_{\Omega}) \). In (a), the upper solution branch extending passes the bifurcation point \( \lambda_1(f_{\Omega}) + \delta_a \), giving two positive solutions whenever \( \lambda = \lambda_1(f_{\Omega}) \) described by Theorem 1.2 (ii - 1) and three positive solutions whenever \( \lambda_1(f_{\Omega}) < \lambda < \lambda_1(f_{\Omega}) + \delta_a \) by Theorem 1.2 (ii - 2). For \( a \geq a_0(p) \) assumed in (b), the turning from the middle to the upper solution branch occurs at \( \lambda_a^+ \), thus describing the case of Theorem 1.3 for which two positive solutions are found for \( \lambda_a^+ < \lambda < \lambda_1(f_{\Omega}) \). Moreover, note that \( \lambda_a^+ \) approaches \( \lambda_1(f_{\Omega}) \) from the left with increasing values of \( a \). Consequently, the turning point can be seen to edge closer to the bifurcation point \( \lambda_1(f_{\Omega}) \) as a result of increasing \( a \).

The structure of this paper is as follows. After briefly introducing some technical lemmas in Section 2, we discuss the mountain pass geometry of the energy functional in Section 3. We demonstrate proofs of Theorem 1.1 in Section 4 and of Theorems 1.2 and 1.3 in Section 5, respectively.

## 2 Preliminaries

We denote the following notations which will be used in the paper. Denote by \( \| \cdot \|_r \) the \( L^r(\mathbb{R}^N) \)-norm for \( 1 \leq r \leq \infty \). A strong convergence is indicated using "\( \rightarrow \)" whereas the weak convergence "\( \rightharpoonup \)". We use \( o(1) \) to denote a quantity that goes to zero as \( n \to \infty \). If we take a subsequence of a sequence \( \{u_n\} \) we shall again denote it by \( \{u_n\} \). Let \( S \) be the best Sobolev constant for the embedding of \( D^{1,2}(\mathbb{R}^N) \) in \( L^{2^*}(\mathbb{R}^N) \), where \( D^{1,2}(\mathbb{R}^N) \) is the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm \( \|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx \).

Let

\[
X = \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}
\]

be equipped with the following inner product and norm

\[
\langle u, v \rangle_\mu = \int_{\mathbb{R}^N} (\nabla u \nabla v + \mu V(x)uv) dx \quad \text{and} \quad \|u\|_\mu = \langle u, u \rangle_\mu^{1/2}
\]
for $\mu > 0$. By condition (V1), we have

$$
\int_{\mathbb{R}^N} u^2 \, dx = \int_{\{V \geq c_0\}} u^2 \, dx + \int_{\{V < c_0\}} u^2 \, dx \leq \frac{1}{\mu c_0} \int_{\mathbb{R}^N} \mu V(x) u^2 \, dx + \frac{|\{V < c_0\}|^{\frac{2^*-2}{2}}}{S^2} \|u\|^2_{D^{1,2}},
$$

which implies that the embedding $X \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Furthermore, for all $2 \leq r \leq 2^*$, it holds

$$
\int_{\mathbb{R}^N} |u|^r \, dx \leq \left( \int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{2^*-r}{2^*-2}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{r-2}{2^*-2}} \left( \frac{\mu}{\mu c_0} \int_{\mathbb{R}^N} \mu V(x) u^2 \, dx + \frac{|\{V < c_0\}|^{\frac{2^*-2}{2}}}{S^2} \|u\|^2_{D^{1,2}} \right)^{\frac{2^*-r}{2^*-2}} \left( \frac{\|u\|^2_{D^{1,2}}}{S^2} \right)^{\frac{r-2}{2^*-2}} \quad (2.1)
$$

for all $\mu \geq \mu_0 := S^2 \left( c_0 |\{V < c_0\}|^{\frac{2^*-2}{2^*}} \right)^{-1}$.

Define the energy functional $J_{a,\lambda}^\mu : X \to \mathbb{R}$ by

$$
J_{a,\lambda}^\mu (u) = \frac{a}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \|u\|_{\mu}^2 - \frac{\lambda}{2} \int_{\mathbb{R}^N} f(x) u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u|^p \, dx.
$$

$J_{a,\lambda}^\mu$ is a $C^1$ functional with the derivative given by

$$
\langle (J_{a,\lambda}^\mu)'(u), \varphi \rangle = a \|u\|^4_{D^{1,2}} + \int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + \mu V(x) u \varphi) \, dx
$$

$$
- \lambda \int_{\mathbb{R}^N} f(x) u \varphi \, dx - \int_{\mathbb{R}^N} g(x) |u|^{p-2} u \varphi \, dx
$$

for all $\varphi \in X$, where $(J_{a,\lambda}^\mu)'$ denotes the Fréchet derivative of $J_{a,\lambda}^\mu$. One can see that the critical points of $J_{a,\lambda}^\mu$ are corresponding to the solutions of Eq. $(K_{a,\lambda}^\mu)$.

In what follows we consider the following eigenvalue problem:

$$
- \Delta u + \mu V(x) u = \lambda f(x) u \quad \text{in} \quad X. \quad (2.3)
$$

In order to find the positive principal eigenvalue of Eq. $(2.3)$, we need to solve the following minimization problem:

$$
\min \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + \mu V(x) u^2) \, dx \mid u \in X \text{ and } \int_{\mathbb{R}^N} f(x) u^2 \, dx = 1 \right\}.
$$

Denote

$$
\lambda_{1,\mu}(f) = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + \mu V(x) u^2) \, dx \mid u \in X \text{ and } \int_{\mathbb{R}^N} f(x) u^2 \, dx = 1 \right\}. \quad (2.4)
$$
Using condition \((D1)\) and Hölder inequality gives
\[
\int_{\mathbb{R}^N} \left( |\nabla u|^2 + \mu V(x)u^2 \right) dx \geq \frac{\|u\|_{D^{1,2}}^2}{\|f\|_{N/2S^{-2}}^2}\|u\|_{D^{1,2}}^2 > 0,
\]
which implies that \(\lambda_{1,\mu}(f) \geq S^2\|f\|_{N/2}^{-1} > 0\). Moreover, by condition \((V2)\) one has
\[
\inf_{u \in X \setminus \{0\}} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \mu V(x)u^2 \right) dx \leq \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \int_{\Omega} |\nabla u|^2 dx,
\]
which indicates that \(\lambda_{1,\mu}(f) \leq \lambda_1(f_\Omega)\) for all \(\mu > 0\). Then the following result is proved.

**Lemma 2.1** (\cite[Lemma 3.2]{36}) For each \(\mu > 0\) there exists a positive function \(\phi_{1,\mu} \in X\) with \(\int_{\mathbb{R}^N} f(x)\phi_{1,\mu}^2 dx = 1\) such that
\[
\lambda_{1,\mu}(f) = \int_{\mathbb{R}^N} \left( |\nabla \phi_{1,\mu}|^2 + \mu V(x)\phi_{1,\mu}^2 \right) dx < \lambda_1(f_\Omega).
\]
Furthermore, it holds
\[
\lambda_{1,\mu}(f) \to \lambda_1^-(f_\Omega) \quad \text{and} \quad \phi_{1,\mu} \to \phi_1 \quad \text{in} \quad X \quad \text{as} \quad \mu \to \infty. \quad (2.5)
\]

Note that we can find the other positive eigenvalues of Eq. \((2.3)\) by solving the following problem:
\[
\min \left\{ \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \mu V(x)u^2 \right) dx \mid u \in X, \int_{\mathbb{R}^N} f(x)u^2 dx = 1 \quad \text{and} \quad \langle u, \phi_{1,\mu} \rangle = 0 \right\}. \quad (2.6)
\]
Denote
\[
\lambda_{2,\mu}(f) = \inf \left\{ \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \mu V(x)u^2 \right) dx \mid u \in X, \int_{\mathbb{R}^N} f(x)u^2 dx = 1 \quad \text{and} \quad \langle u, \phi_{1,\mu} \rangle = 0 \right\}.
\]
In order to solve problem \((2.6)\), we need some known lemmas as follows.

**Lemma 2.2** (\cite[Lemma 2.13]{34}) If \(N \geq 3\) and \(f \in L^{N/2}(\mathbb{R}^N)\), the functional \(u \mapsto \int_{\mathbb{R}^N} f(x)u^2 dx\) is weakly continuous on \(H^1(\mathbb{R}^N)\).

**Lemma 2.3** (\cite[Lemma 3.1]{36}) Let \(\mu_n \to \infty\) as \(n \to \infty\) and \(\{v_n\} \subset X\) with \(\|v_n\|_{\mu_n} \leq C_0\) for some \(C_0 > 0\). Then there exist a subsequence \(\{v_n\}\) and \(v_0 \in H^1_0(\Omega)\) such that \(v_n \to v_0\) in \(X\) and \(v_n \to v_0\) in \(L^r(\mathbb{R}^N)\) for all \(2 \leq r < 2^\ast\).

Then we have the following result.

**Lemma 2.4** For each \(\mu > 0\) there exists a function \(\phi_{2,\mu} \in X\) with \(\int_{\mathbb{R}^N} f(x)\phi_{2,\mu}^2 dx = 1\) and \(\langle \phi_{2,\mu}, \phi_{1,\mu} \rangle = 0\) such that
\[
\lambda_{2,\mu}(f) = \int_{\mathbb{R}^N} \left( |\nabla \phi_{2,\mu}|^2 + \mu V(x)\phi_{2,\mu}^2 \right) dx.
\]
Furthermore, it holds
\[
\frac{\lambda_1(f_\Omega) + \lambda_2(f_\Omega)}{2} < \lambda_{2,\mu}(f) \quad \text{for} \quad \mu \quad \text{sufficiently large}. \quad (2.7)
\]
Proof. Let \( \{u_n\} \subset X \) be a minimizing sequence of problem \((2.6)\). Clearly, it is bounded. Then there exist a subsequence \( \{u_n\} \) and \( \phi_{2,\mu} \in X \) such that \( u_n \rightharpoonup \phi_{2,\mu} \) in \( X \), which implies that \( \langle \phi_{2,\mu}, \phi_{1,\mu} \rangle_\mu = 0 \). By Lemma \( 2.2 \) and the fact of \( X \hookrightarrow H^1(\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} f(x)\phi_{2,\mu}^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} f(x)u_n^2 \, dx = 1.
\]
We now prove that \( u_n \to \phi_{2,\mu} \) in \( X \). If it is false, then
\[
\int_{\mathbb{R}^N} (|\nabla \phi_{2,\mu}|^2 + \mu V(x)\phi_{2,\mu}^2) \, dx < \liminf_{n \to \infty} \|u_n\|^2_{\mu} = \lambda_{2,\mu}(f),
\]
which is impossible due to the definition of \( \lambda_{2,\mu}(f) \). So \( u_n \to \phi_{2,\mu} \) in \( X \) and \( \lambda_{2,\mu}(f) = \|\phi_{2,\mu}\|_{\mu}^2 \).

Next, we show that \( \frac{1}{2}(\lambda_1(f_\Omega) + \lambda_2(f_\Omega)) < \lambda_{2,\mu}(f) \) for all \( \mu \) sufficiently large. Suppose on the contrary. Then there exists a sequence \( \{\lambda_{2,\mu_n}(f)\} \) such that
\[
\lambda_{2,\mu_n}(f) \leq \frac{1}{2}(\lambda_1(f_\Omega) + \lambda_2(f_\Omega)) \quad \text{as} \quad n \to \infty.
\]
Let \( v_n = \phi_{2,\mu_n} \) be the minimizer of \( \lambda_{2,\mu_n}(f) \). Then it holds \( \int_{\mathbb{R}^N} f(x)v_n^2 \, dx = 1 \), \( \langle v_n, \phi_{1,\mu_n} \rangle_{\mu_n} = 0 \) and
\[
\|v_n\|^2_{\mu_n} = \lambda_{2,\mu_n}(f) \leq \frac{1}{2}(\lambda_1(f_\Omega) + \lambda_2(f_\Omega)). \tag{2.8}
\]
By \( \overline{(2.8)} \) and Lemma \( 2.3 \), there exist a subsequence \( \{v_n\} \) and \( v_0 \in H^1_0(\Omega) \) such that \( v_n \rightharpoonup v_0 \) in \( X \) and \( v_n \to v_0 \) in \( L^r(\mathbb{R}^N) \) for all \( 2 \leq r < 2^* \). Then we have
\[
\int_{\Omega} f_\Omega v_0^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} f(x)v_n^2 \, dx = 1 \tag{2.9}
\]
and
\[
\int_{\Omega} |\nabla v_0|^2 \, dx = \int_{\mathbb{R}^N} (|\nabla v_0|^2 + V(x)v_0^2) \, dx \leq \liminf_{n \to \infty} \|v_n\|_{\mu_n}^2 \leq \frac{1}{2}(\lambda_1(f_\Omega) + \lambda_2(f_\Omega)). \tag{2.10}
\]
According to \( \overline{(2.5)} \), we deduce that
\[
\|\phi_{1,\mu_n} - \phi_1\|_{\mu_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.11}
\]
It follows from \( \overline{(2.8)}, \overline{(2.11)} \) and \( v_n \rightharpoonup v_0 \) in \( X \) that
\[
\int_{\Omega} \nabla v_0 \nabla \phi_1 \, dx = \lim_{n \to \infty} \langle v_n, \phi_{1,\mu_n} \rangle_{\mu_n} = 0. \tag{2.12}
\]
From \( \overline{(2.9)} \) and \( \overline{(2.12)} \), we conclude that \( \int_{\Omega} |\nabla v_0|^2 \, dx \geq \lambda_2(f_\Omega) \), which is a contradiction with \( \overline{(2.10)} \). Consequently, this completes the proof.

\[\square\]
3 Mountain pass geometry

Let us start this section by recalling the well-known the mountain pass theorem \[4\] as follows.

**Theorem 3.1** Let \( E \) be a Banach space, \( J \in C^1(E,\mathbb{R}), v \in E \) and \( \rho > 0 \) be such that \( \|v\| > \rho \) and
\[
\mu := \inf_{\|u\|=\rho} J(u) > J(0) \geq J(v).
\]

If \( J \) satisfies the Palais-Smale condition at level \( \alpha := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \) with
\[
\Gamma := \{ \gamma \in C([0,1],E) \mid \gamma(0) = 0, \gamma(1) = v \},
\]
then \( \alpha \) is a critical value of \( J \) and \( \alpha \geq b \).

We say that the functional \( J_{a,\lambda}^\mu \) satisfies Palais-Smale condition at level \( \alpha \in \mathbb{R} ((PS)_\alpha \text{-condition for short}) if any sequence \( \{u_n\} \subset X \) with \( J_{a,\lambda}^\mu(u_n) \to \alpha \) and \( (J_{a,\lambda}^\mu)'(u_n) \to 0 \) has a convergent subsequence. Such sequence is called a Palais-Smale sequence at \( \alpha ((PS)_\alpha \text{-sequence for short}) \).

In what follows, we prove that the functional \( J_{a,\lambda}^\mu \) satisfies the mountain pass geometry.

**Lemma 3.2** Suppose that \( N = 3, 4 < p < 6 \) and conditions \((V1) - (V2), (D1) - (D2)\) hold. Then for each \( a > 0 \), there exists a number \( \delta_a > 0 \) such that for each \( 0 < \lambda < \lambda_1(f_\Omega) + \delta_a \), there exist \( \rho_{a,\lambda} > 0 \) and \( e_0 \in H^1_0(\Omega) \) such that
\[
\|e_0\|_\mu > \rho_{a,\lambda} \text{ and } \inf_{\|u\|_\mu = \rho_{a,\lambda}} J_{a,\lambda}^\mu(u) > J_{a,\lambda}^\mu(e_0)
\]
for \( \mu \) sufficiently large.

**Proof.** First of all, we show that there exists a number \( \delta_a > 0 \) such that for each \( 0 < \lambda < \lambda_1(f_\Omega) + \delta_a \), there exists a number \( \rho_{a,\lambda} > 0 \) such that \( \inf_{\|u\|_\mu = \rho_{a,\lambda}} J_{a,\lambda}^\mu(u) > 0 \) for \( \mu \) sufficiently large. Now we need to separate the proof in two cases as follows.

Case (i): \( 0 < \lambda < \lambda_1(f_\Omega) \). It follows from (2.5) that
\[
\lambda_{1,\mu}(f) \geq \frac{\lambda_1(f_\Omega) + \lambda}{2}
\]
for \( \mu \) sufficiently large,

which indicates that
\[
\frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{1,\mu}(f)}\right) \geq \frac{1}{2} \left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda}\right)
\]
for \( \mu \) sufficiently large. \(^{(3.1)}\)

By (2.2), (2.4) and (3.1), one has
\[
J_{a,\lambda}^\mu(u) \geq \frac{a}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda}\right) \|u\|_\mu^2 - \frac{\|g\|_\infty \|\{V < c_0\}\|^{(6-p)/6}}{pS^p} \|u\|_\mu^p
\]
for \( \mu \) sufficiently large. Let
\[
\rho_\lambda = \left[\frac{1}{4} \left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda}\right) \frac{pS^p}{\|g\|_\infty \|\{V < c_0\}\|^{(6-p)/6}}\right]^{1/(p-2)} > 0.
\]
Then for all \( u \in X \) with \( \|u\|_\mu = \rho_\lambda \), we have

\[
J_{a,\lambda}^\mu(u) \geq \frac{1}{4} \left( \frac{\lambda_1(f_{\Omega}) - \lambda}{\lambda_1(f_{\Omega}) + \lambda} \right) \rho_\lambda^2 > 0,
\]

which indicates that \( \inf_{\|u\|_\mu = \rho_\lambda} J_{a,\lambda}^\mu(u) > 0 \).

Case (ii) : \( \lambda \geq \lambda_1(f_{\Omega}) \). For each \( u \in X \), by the orthogonal decomposition theorem, there exist \( t \in \mathbb{R} \) and \( w \in X \) with \( \langle w, \phi_{1,\mu} \rangle = 0 \) such that \( u = t\phi_{1,\mu} + w \). Clearly, it holds

\[
\|u\|_\mu^2 = \lambda_{1,\mu}(f)t^2 + \|w\|_\mu^2. \tag{3.2}
\]

Moreover, we obtain

\[
\lambda_{2,\mu}(f) \int_{\mathbb{R}^3} f(x)w^2dx \leq \|w\|_\mu^2 \tag{3.3}
\]

and

\[
\lambda_{1,\mu}(f) \int_{\mathbb{R}^3} f(x)\phi_{1,\mu}wdx = \int_{\mathbb{R}^3} (\nabla \phi_{1,\mu} \nabla w + \mu V(x)\phi_{1,\mu}w)dx = 0. \tag{3.4}
\]

For the functional \( J_{a,\lambda}^\mu \), it follows from \([2.2] \), \([3.2] \) – \([3.4] \) that

\[
J_{a,\lambda}^\mu(u) = \frac{a}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} (\lambda_{1,\mu}(f)t^2 + \|w\|_\mu^2)
- \frac{\lambda}{2} \int_{\mathbb{R}^3} (t^2 f(x)\phi_{1,\mu}^2 + 2tf(x)\phi_{1,\mu}w + f(x)w^2)dx - \frac{1}{p} \int_{\mathbb{R}^3} g(x)|u|^pdx
\geq \frac{a}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{1,\mu}(f)} \right) \lambda_{1,\mu}(f)t^2 + \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{2,\mu}(f)} \right) \|w\|_\mu^2
- \frac{\|g\|_{\infty}\{V < c_0\}\langle(6-p)/6\rangle}{ps_p} \|u\|^p_\mu,
\]

where

\[
\theta_{1,\mu} := \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{1,\mu}(f)} \right) \quad \text{and} \quad \theta_{2,\mu} := \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{2,\mu}(f)} \right). \tag{3.6}
\]

Since \( \lambda_{1,\mu}(f) < \lambda_1(f_{\Omega}) \), by \([2.7] \) one has

\[
\theta_{2,\mu} - \theta_{1,\mu} \geq \frac{1}{2} \left( 1 - \frac{\lambda_{1,\mu}(f)}{\lambda_{2,\mu}(f)} \right) \geq \frac{\lambda_2(f_{\Omega}) - \lambda_1(f_{\Omega})}{2(\lambda_2(f_{\Omega}) + \lambda_1(f_{\Omega}))} =: \lambda_0 \tag{3.7}
\]

for \( \mu \) sufficiently large. Moreover, since \( \phi_{1,\mu} \rightarrow \phi_1 \) in \( X \) as \( \mu \rightarrow \infty \), we conclude that \( \phi_{1,\mu} \rightarrow \phi_1 \) in \( D^{1,2}(\mathbb{R}^3) \) as \( \mu \rightarrow \infty \), which implies that

\[
\|\phi_{1,\mu}\|_{D^{1,2}}^4 \geq \frac{1}{2} \lambda_1^2(f_{\Omega}) \quad \text{for} \; \mu \text{ sufficiently large.} \tag{3.8}
\]

For the non-local term, we deduce that

\[
4t^2 \|\phi_{1,\mu}\|_{D^{1,2}}^2 \int_{\mathbb{R}^3} \nabla \phi_{1,\mu} \nabla wdx \leq \frac{3}{4} t^4 \|\phi_{1,\mu}\|_{D^{1,2}}^4 + 4t^2 \left( \int_{\mathbb{R}^3} \nabla \phi_{1,\mu} \nabla wdx \right)^2 + 16 \|w\|_{D^{1,2}}^4 \tag{3.9}
\]
and
\[ 4t^2 w^2 D_{1.2} \int_{\mathbb{R}^3} \nabla \phi_{1,\mu} \nabla w \, dx \leq 2t^2 \| \phi_{1,\mu} \|_{D_{1.2}}^2 \| w \|_{D_{1.2}}^2 + 2 \| w \|_{D_{1.2}}^4. \] (3.10)

Then from (3.8) – (3.10) it follows that
\[
\| u \|_{D_{1.2}}^4 = \| t \phi_{1,\mu} + w \|_{D_{1.2}}^4 \geq \frac{\| \phi_{1,\mu} \|_{D_{1.2}}^4}{4 \lambda^2_{1,\mu}(f)} (\| u \|_{\mu}^2 - \| w \|_{\mu}^2)^2 - 17 \| w \|_{D_{1.2}}^4 \geq \frac{1}{16} \| u \|_{\mu}^4 - \frac{137}{8} \| w \|_{\mu}^4. \] (3.11)

Combining (3.5), (3.7) with (3.11), for \( \mu \) sufficiently large one has
\[
J_{a,\lambda}^\mu(u) \geq \frac{a}{64} \| u \|_{\mu}^4 - \frac{137}{32} a \| w \|_{\mu}^4 - |\theta_{1,\mu}| \| u \|_{\mu}^2 + \Lambda_0 \| u \|_{\mu}^2 - \| g \|_{\infty} \{ V < c_0 \} |(6-p)/6| \| u \|_{\mu}^p \\
= -|\theta_{1,\mu}| \| u \|_{\mu}^2 + \| u \|_{\mu}^4 \left( \frac{a}{64} - \frac{\| g \|_{\infty} \{ V < c_0 \} |(6-p)/6| \| u \|_{\mu}^{p-4} }{pS^p} \right) + \| u \|_{\mu}^2 \left( \Lambda_0 - \frac{137}{32} a \| w \|_{\mu}^2 \right). 
\]

This implies that there exists a number
\[
\rho_a = \min \left\{ \left( \frac{apS^p}{128 \| g \|_{\infty} \{ V < c_0 \} |(6-p)/6|} \right)^{1/(p-4)} : \frac{32 \Lambda_0}{137a} \right\} \] (3.12)
such that for all \( u \in X \) with \( \| u \|_{\mu} = \rho_a \),
\[
J_{a,\lambda}^\mu(u) \geq -|\theta_{1,\mu}| \rho_a^2 + \frac{a}{128} \rho_a^4. 
\]

Thus, we deduce that
\[
J_{a,\lambda}^\mu(u) \geq \frac{a}{256} \rho_a^4 > 0 
\]
for each \( \lambda_{1,\mu}(f) \leq \lambda < \lambda_{1,\mu}(f) + \delta_{a,\mu} \), where
\[
\delta_{a,\mu} := \frac{\lambda_{1,\mu}(f)}{128} a \rho_a^2. \] (3.13)

So, according to Case (i) – (ii), for each \( a > 0 \) and \( 0 < \lambda < \lambda_{1,\mu}(f) + \delta_{a,\mu} \), we have
\[
\inf_{\| u \|_{\mu} = \rho_{a,\lambda}} J_{a,\lambda}^\mu(u) > 0 \text{ for } \mu \text{ sufficiently large,}
\]
where
\[
\rho_{a,\lambda} := \begin{cases} 
\rho_\lambda & \text{for } 0 < \lambda < \lambda_1(f_\Omega), \\
\rho_a & \text{for } \lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_{a,\mu}.
\end{cases}
\]

Set
\[
\delta_a = \min \left\{ a^{p-2} C_1, C_2 \right\} > 0,
\]
where
\[
C_1 := \frac{\lambda_1(f_\Omega)}{4} \left( \frac{1}{128} \right) a^{p-2} \left( \frac{pS^p}{\| g \|_{\infty} \{ V < c_0 \} |(6-p)/6|} \right)^{2/(p-4)} 
\]
and
\[
\rho_{a,\lambda} := \begin{cases} 
\rho_\lambda & \text{for } 0 < \lambda < \lambda_1(f_\Omega), \\
\rho_a & \text{for } \lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_{a,\mu}.
\end{cases}
\]
and
\[ C_2 := \frac{\lambda_1(f_\Omega)(\lambda_2(f_\Omega) - \lambda_1(f_\Omega))}{4384(\lambda_2(f_\Omega) + \lambda_1(f_\Omega))}. \]

Then by (3.12) – (3.13), we obtain that for \( \mu \) sufficiently large,
\[ \lambda_1(f_\Omega) + \delta_a \leq \lambda_{1,\mu}(f) + 2\delta_a \leq \lambda_{1,\mu}(f) + \delta_{a,\mu}. \]

Hence, for each \( a > 0 \) and \( 0 < \lambda < \lambda_1(f_\Omega) + \delta_a \) it holds
\[ \inf_{\|u\|_\mu = \rho_{a,\lambda}} J^\mu_{a,\lambda}(u) > 0 \text{ for } \mu \text{ sufficiently large.} \]

Next, we show that there exists \( e_0 \in H^1_0(\Omega) \) such that \( \|e_0\|_\mu > \rho_{a,\lambda} \) and \( J^\mu_{a,\lambda}(e_0) < 0 \). Owing to condition (D2), we can take \( \varphi \in H^1_0(\Omega) \) such that \( \int_{\mathbb{R}^3} g(x)|\varphi|^p dx > 0 \). Then for any \( t > 0 \), we have
\[ J^\mu_{a,\lambda}(t\varphi) = \frac{1}{2} \left( \|\varphi\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f(x)|\varphi|^2 dx \right) t^2 + \frac{a}{4} \|\varphi\|_{D^1,2}^p t^4 - \frac{\int_{\mathbb{R}^3} g(x)|\varphi|^p dx}{p} t^p. \]

This implies that there exists \( t_0 > 0 \) such that \( \|t_0\varphi\|_\mu > \rho_{a,\lambda} \) and \( J^\mu_{a,\lambda}(t_0\varphi) < 0 \). Consequently, we complete the proof. \( \blacksquare \)

**Lemma 3.3** Suppose that \( N \geq 3, 2 < p < \min\{4, 2^*\} \) and conditions (V1) – (V2), (D1) – (D2) hold. Then for each \( 0 < a < a_0(p) \), we have the following results.

(i) For each \( 0 < \lambda < \lambda_1(f_\Omega) \), there exist a number \( \overline{\rho}_{a,\lambda} > 0 \) and \( e_0 \in H^1_0(\Omega) \) such that
\[ \|e_0\|_\mu > \overline{\rho}_{a,\lambda} \text{ and } \inf_{\|u\|_\mu = \overline{\rho}_{a,\lambda}} J^\mu_{a,\lambda}(u) > 0 > J^\mu_{a,\lambda}(e_0) \tag{3.14} \]

for \( \mu \) sufficiently large.

(ii) If \( \int_{\Omega} g(x)\phi_a^2 dx < 0 \), then there exists a number \( \overline{\delta}_a > 0 \) such that for each \( \lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \overline{\delta}_a \), there exist \( \overline{\rho}_{a,\lambda} > 0 \) and \( e_0 \in H^1_0(\Omega) \) such that (3.14) holds for \( \mu \) sufficiently large.

**Proof.** (i) It follows from (2.2), (2.4) and (3.1) that
\[ J^\mu_{a,\lambda}(u) \geq \frac{a}{4} \|u\|_{D^1,2}^4 + \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_1(f_\Omega)} \right) \|g\|_\infty \|V < c_0\|^{(2^*-p)/2^*} \frac{pS^p}{\mu} \|u\|_\mu^p \]
\[ \geq \frac{a}{4} \|u\|_{D^1,2}^4 + \frac{1}{2} \left( \frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda} \right) \|g\|_\infty \|V < c_0\|^{(2^*-p)/2^*} \frac{pS^p}{\mu} \|u\|_\mu^p. \tag{3.15} \]

Let
\[ \overline{\rho}_{a,\lambda} = \min\{\overline{\rho}_\lambda, \overline{\rho}_a\} > 0, \tag{3.16} \]
where
\[ \overline{\rho}_\lambda := \left[ \frac{1}{4} \left( \frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda} \right) \frac{pS^p}{\|g\|_\infty \|V < c_0\|^{(2^*-p)/2^*}} \right]^{1/(p-2)} \tag{3.17} \]
and
\[ \overline{\rho}_a := \left( \frac{(p-2)\Gamma_p}{ap} \right)^{1/(4-p)} \tag{3.18} \]
Then by (3.15), for all \( u \in X \) with \( \| u \|_\mu = \overline{p}_{a, \lambda} \), one has

\[
J_{a, \lambda}^\mu(u) \geq \frac{1}{4} \left( \frac{\lambda_1(f_{\Omega}) - \lambda}{\lambda_1(f_{\Omega}) + \lambda} \right) \overline{p}_{a, \lambda}^2 > 0.
\]

Since \( a < a_0(p) \), there exists \( \varphi_a \in H^1_0(\Omega) \) with \( \int_\Omega f_{\Omega}(x) \varphi_a^2 dx \geq 0 \) and \( \int_\Omega g(x)|\varphi_a|^p dx > 0 \) such that

\[
a < 2(p - 2)(4 - p)^{\frac{1-p}{2}} \left( \frac{\int_\Omega g(x)|\varphi_a|^p dx}{p (\int_\Omega |\nabla \varphi_a|^2 dx)^{p/2}} \right)^{2/(p-2)} \leq a_0(p).
\]

Let

\[
t_a = \left( \frac{(2p - 4)\Gamma_p}{ap} \right)^{1/(4-p)} \left( \int_\Omega |\varphi_a|^2 dx \right)^{-1/2}.
\]

Then by (3.16) and (3.19), we have \( \| t_a \varphi_a \|_\mu > \overline{p}_{a, \lambda} \) and

\[
J_{a, \lambda}^\mu(t_a \varphi_a)
= \frac{t_a^2}{2} \left( \int_\Omega |\nabla \varphi_a|^2 dx - \lambda \int_\Omega f_{\Omega} \varphi_a^2 dx \right) + \frac{a}{4} \left( \int_\Omega |\nabla \varphi_a|^2 dx \right)^2 t_a^4 - \frac{\int_\Omega g(x)|\varphi_a|^p dx}{p} \left( \frac{2(p - 2) \int_\Omega g(x)|\varphi_a|^p dx}{ap (\int_\Omega |\nabla \varphi_a|^2 dx)^{2}} \right)^{\frac{p-2}{p}}
= \frac{t_a^2}{2} \left[ \int_\Omega |\nabla \varphi_a|^2 dx - \lambda \int_\Omega f_{\Omega}(x) \varphi_a^2 dx - \frac{(4 - p) \int_\Omega g(x)|\varphi_a|^p dx}{p} \left( \frac{2(p - 2) \int_\Omega g(x)|\varphi_a|^p dx}{ap (\int_\Omega |\nabla \varphi_a|^2 dx)^{2}} \right)^{\frac{p-2}{p}} \right] - \lambda \int_\Omega f_{\Omega}(x) \varphi_a^2 dx < 0.
\]

(ii) For each \( u \in X \), by the orthogonal decomposition theorem, there exist \( t \in \mathbb{R} \) and \( w \in X \) such that \( u = t \phi_{1, \mu} + w \). Using the same process in (3.5) and (3.7) gives

\[
J_{a, \lambda}^\mu(u) \geq \frac{a}{4} \| u \|_{D,2}^4 - |\theta_{1, \mu}| \| u \|_\mu^2 + \Lambda_0 \| w \|_\mu^2
- \frac{1}{p} \int_{\mathbb{R}^N} g(x) |t \phi_{1, \mu}|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x) (|t \phi_{1, \mu} + w|^p - |t \phi_{1, \mu}|^p) dx
\]

(3.20)

for \( \mu \) sufficiently large, where \( \theta_{1, \mu} \) and \( \Lambda_0 \) are as in (3.6) and (3.7), respectively. By (2.2) and (2.5), we conclude that \( \int_{\mathbb{R}^N} g(x) \phi_{1, \mu}^p dx \rightarrow \int_\Omega g(x) \phi^p dx \) as \( \mu \rightarrow \infty \), which implies that

\[
\int_{\mathbb{R}^N} g(x) \phi_{1, \mu}^p dx \leq \frac{1}{2} \int_\Omega g(x) \phi^p dx < 0 \quad \text{for \( \mu \) sufficiently large.}
\]

(3.21)

By the mean value theorem, there exists \( 0 < \theta < 1 \) such that

\[
\frac{1}{p} \int_{\mathbb{R}^N} g(x)(|t \phi_{1, \mu} + w|^p - |t \phi_{1, \mu}|^p) dx = \int_{\mathbb{R}^N} g(x) |t \phi_{1, \mu} + \theta w|^{p-2} (t \phi_{1, \mu} + \theta w) w dx.
\]

(3.22)
Using Young’s inequality and (2.2) leads to
\[
\left| \int_{\mathbb{R}^N} g(x) |t \phi_{1, \mu} + \theta w|^p (t \phi_{1, \mu} + \theta w) \, dx \right| \\
\leq 2^{p-2} \| g \|_{\infty} \int_{\mathbb{R}^N} (|t \phi_{1, \mu}|^{p-1} + |\theta w|^{p-1}) |w| \, dx \\
\leq 2^{p-2} \| g \|_{\infty} \int_{\mathbb{R}^N} \left( \frac{p-1}{p} B^{p/(p-1)} |t \phi_{1, \mu}|^p + \frac{1}{pB^p} |w|^p + |w|^p \right) \, dx \\
\leq \frac{2^{p-2} \| g \|_{\infty}}{pB^p} \left\{ V < c_0 \right\}^{\frac{2-p}{2}} \left( B^{p/(p-1)} (p-1) |t|^p \| \phi_{1, \mu} \|_\mu^p + \frac{1 + pB^p}{B^p} \| w \|_\mu^p \right) \\
\leq \frac{\left| \int_{\mathbb{R}^N} g(x) \phi_{1, \mu}^p \, dx \right| |t|^p + 2^{p-2} \| g \|_{\infty} \left( 1 + pB^p \right) \left\{ V < c_0 \right\}^{\frac{2-p}{2}} \| w \|_\mu^p \text{ for } \mu \text{ sufficiently large} \right(3.23) \\
\text{where} \\
B := \left( \frac{\left| \int_{\mathbb{R}^N} g(x) \phi_{1, \mu}^p \, dx \right|}{2 \| g \|_{\infty} (p-1) \left\{ V < c_0 \right\} \frac{2-p}{2^2} \chi_{1/2}^2(f_\Omega)} \right)^{(p-1)/p}.
\]

Thus, it follows from \((3.20)\) to \((3.23)\) that
\[
J_{a, \lambda}^\mu(u) \geq \frac{a}{4} \| u \|_{D^{1,2}}^2 - |\theta_{1, \mu}| |u|_\mu^2 + \Lambda_0 \| w \|_\mu^2 + \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{2p} |t|^p \\
- \frac{1}{p} \int_{\mathbb{R}^N} g(x) (|t \phi_{1, \mu} + w|^p - |t \phi_{1, \mu}|^p) \, dx \\
\geq -|\theta_{1, \mu}| |u|_\mu^2 + \Lambda_0 \| w \|_\mu^2 + \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{2p} \left( \| u \|_\mu^2 - \| w \|_\mu^2 \right) \\
- \frac{2^{p-2} \| g \|_{\infty} \left\{ V < c_0 \right\}^{\frac{2-p}{2}}}{pB^p} \| w \|_\mu^p \\
\geq -|\theta_{1, \mu}| |u|_\mu^2 + \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{2(p^2)/2p} \| u \|_\mu^p \\
+ \| w \|_\mu^2 \left[ \Lambda_0 - \left( \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{4p} \chi_{1/2}^2(f_\Omega) \right) \right. \\
+ \frac{2^{p-2} \| g \|_{\infty} \left\{ V < c_0 \right\}^{\frac{2-p}{2}}}{pB^p} \| w \|_\mu^p \left. \right] \| w \|_\mu^{-2}. \right(3.24) \\
\]

Let
\[
\overline{a, \lambda} = \min\{\rho_0, \overline{a}\} \quad \text{and} \quad \overline{a, \mu} = \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \chi_{1, \mu}(f) \right|}{2(p^2)/2p} \overline{a, \mu}^p \\
\text{where} \\
\rho_0 := \Lambda_0^{1/(p-2)} \left( \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{4p \chi_{1/2}^2(f_\Omega)} + \frac{2^{p-2} \| g \|_{\infty} \left\{ V < c_0 \right\}^{(2-p)/2}}{pB^p} \right)^{-1/(p-2)} \\
\text{and } \overline{a} \text{ is as } (3.18). \] Then by \((3.24)\), for all \( u \in X \text{ with } \| u \|_\mu = \overline{a, \lambda} \text{ one has} \\
\[
J_{a, \lambda}^\mu(u) \geq -|\theta_{1, \mu}| \overline{a, \mu}^p + \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{2(p^2)/2p} \overline{a, \mu}^p \geq \frac{\left| \int_{\Omega} g(x) \phi_{1, \mu}^p \, dx \right|}{2(p^2)/2p} \overline{a, \lambda}^p. \right(3.25) \\
\]
for each \( \lambda_{1,\mu}(f) \leq \lambda < \lambda_{1,\mu}(f) + \delta_{a,\mu} \). Set
\[
\delta_a = \frac{\left| \int_\Omega g(x)\phi_\mu^p dx \right|}{2^{(p+6)/2}p\lambda_\mu^{(p-2)/2}(f_\Omega)} \rho_{a,\lambda}^{p-2}.
\]

Since \( \lambda_{1,\mu}(f) \to \lambda_1(f_\Omega) \) as \( \mu \to \infty \), we conclude that
\[
\lambda_1(f_\Omega) + \delta_a \leq \lambda_{1,\mu}(f) + 2\delta_a \leq \lambda_{1,\mu}(f) + \delta_{a,\mu}.
\]

Hence, it follows from (3.25) and (3.26) that for each \( \lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_a \),
\[
\inf_{\|u\|_\mu = \rho_{a,\lambda}} J_{a,\lambda}^\mu(u) > 0 \text{ for } \mu \text{ sufficiently large.}
\]

Next, repeating the same argument as in (i), we can conclude that there exists \( e_0 \in H^1_0(\Omega) \) such that \( \|e_0\|_\mu > \rho_{a,\lambda} \) and \( J_{a,\lambda}^\mu(e_0) < 0 \). Consequently, we complete the proof.

**Lemma 3.4** Suppose that \( N \geq 3, 2 < p < \min\{4, 2^*\} \) and conditions \((V1) - (V2), (D1) - (D2)\) hold. If \( \int_\Omega g(x)\phi_\mu^p dx > 0 \), then for each \( a \geq a_0(p) \) and \( \lambda_a^+ < \lambda < \lambda_1(f_\Omega) \), there exist \( \hat{\rho}_{a,\lambda} > 0 \) and \( e_0 \in H^1_0(\Omega) \) such that
\[
\|e_0\|_\mu > \hat{\rho}_{a,\lambda} \text{ and } \inf_{\|u\|_\mu = \hat{\rho}_{a,\lambda}} J_{a,\lambda}^\mu(u) > 0 > J_{a,\lambda}(e_0) \text{ for } \mu \text{ sufficiently large},
\]
where \( \lambda_a^+ \) is as in (1.4).

**Proof.** Let
\[
\hat{\rho}_{a,\lambda} := \min\{\rho_{\lambda}, \hat{\rho}_a\},
\]
where \( \rho_{\lambda} \) is as in (3.17) and
\[
\hat{\rho}_a := \left(\frac{(p-2)\int_\Omega g(x)\phi_\mu^p dx}{ap\lambda_\mu^{p/2}(f_\Omega)}\right)^{1/(4-p)}.
\]

Then, similar to the argument in Lemma 3.3(i), for all \( u \in X \) with \( \|u\|_\mu = \hat{\rho}_{a,\lambda} \) one has
\[
J_{a,\lambda}^\mu(u) \geq \frac{a}{4}\|u\|_{D^{1,2}}^4 + \frac{1}{2}\left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda}\right)\|u\|_\mu^2 - \frac{\|g\|_\infty\|\{V < c_0\}\|^{(2-p)/2}}{p^{p/2}}\|u\|_\mu^p
\]
\[
\geq \frac{1}{4}\left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega) + \lambda}\right)\hat{\rho}_{a,\lambda}^2 > 0 \text{ for } \mu \text{ sufficiently large.}
\]

We set
\[
t_a = \left(\frac{2(p-2)\int_\Omega g(x)\phi_\mu^p dx}{ap\lambda_\mu^{p/2}(f_\Omega)}\right)^{1/(4-p)}.
\]

Then it holds \( \|t_a\phi_1\|_\mu > \hat{\rho}_{a,\lambda} \) and
\[
J_{a,\lambda}(t_a\phi_1) = \frac{\lambda_1(f_\Omega) - \lambda}{2}t_a^2 + \frac{a\lambda_a^2(f_\Omega)}{4}t_a^4 - \frac{\int_\Omega g(x)\phi_\mu^p dx}{p}t_a^p
\]
\[
= \frac{t_a^2}{2}(\lambda_a^+ - \lambda) < 0 \text{ for } \lambda > \lambda_a^+.
\]
Consequently, we complete the proof. ■

Finally, we state the compactness condition for the functional $J_{a,\lambda}^\mu$, which has been proved in [36].

**Lemma 3.5** ([36, Proposition 2.5]) Suppose that $N \geq 3, 2 < p < 2^*$ and conditions (V1) – (V2), (D1) – (D2) hold. In addition, for $N = 3$, we also assume that condition (H3) holds. Let $\alpha \in \mathbb{R}$ and $\{u_n\}$ be a $(PS)_{\alpha}$-sequence for $J_{a,\lambda}^\mu$. If there exists $d_0 > 0$ such that $\|u_n\|_\mu < d_0$, then $\{u_n\}$ strongly converges in $X_\mu$ up to subsequence for $\mu$ sufficiently large.

### 4 The Proof of Theorem 1.1

We now prove Theorem 1.1. By Lemma 3.2, for each $a > 0$ and $\mu$ is sufficiently large, there exists $\delta_a > 0$ such that the functional $J_{a,\lambda}^\mu$ has the mountain pass geometry whenever $0 < \lambda < \lambda_1(f_1) + \delta_a$. Let

$$\alpha_\mu = \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} J_{a,\lambda}^\mu(\gamma(s)) \quad \text{with} \quad \Gamma = \{ \gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e_0 \},$$

where $e_0$ is as in Lemma 3.2. It is clear that

$$0 < \alpha_\mu \leq \max_{0 \leq s \leq 1} J_{a,\lambda}^\mu(se_0) =: D_0$$

and that $D_0$ is independent of $\mu$ due to $e_0 \in H_0^1(\Omega)$. Let $\{u_n\}$ be a $(PS)_{\alpha_\mu}$-sequence, that is $J_{a,\lambda}^\mu(u_n) = \alpha_\mu + o(1)$ and $(J_{a,\lambda}^\mu)'(u_n) = o(1)$. In fact, since $J_{a,\lambda}^\mu(u_n) = J_{a,\lambda}^\mu(|u_n|)$ for all $n$, we may assume that $u_n \geq 0$. Then we have

$$\alpha_\mu + 1 \geq p J_{a,\lambda}^\mu(u_n) - \langle (J_{a,\lambda}^\mu)'(u_n), u_n \rangle = \frac{a(p - 4)}{4} \|u_n\|_{D,1,2}^4 + \frac{p - 2}{2} \|u_n\|_\mu^2 - \frac{\lambda(p - 2)}{2} \int_{\mathbb{R}^3} f(x)u_n^2 dx. \quad (4.1)$$

Using condition (D1) and Young’s inequality gives

$$\frac{\lambda(p - 2)}{2} \int_{\mathbb{R}^3} f(x)u_n^2 dx \leq \frac{\lambda(p - 2)}{2} S^{-2} \|f\|_{3/2} \|u_n\|_{D,1,2}^2 \leq \frac{a(p - 4)}{4} \|u_n\|_{D,1,2}^4 + \frac{\lambda^2(p - 2)^2 \|f\|_{3/2}^2}{4(p - 4) a S^4}. \quad (4.2)$$

Combining (4.1) with (4.2) leads to

$$D_0(p + 1) \geq \alpha_\mu + 1 \geq \frac{p - 2}{2} \|u_n\|_\mu^2 - \frac{\lambda^2(p - 2)^2 \|f\|_{3/2}^2}{4(p - 4) a S^4},$$

which indicates that there exists $d_0 > 0$ such that $\|u_n\|_\mu < d_0$ for $\mu$ sufficiently large. Thus, by Lemma 3.5, the functional $J_{a,\lambda}^\mu$ satisfies the $(PS)_{\alpha_\mu}$-condition. Hence, there exists $0 \leq u_0^{(1)} \in X$ such that $J_{a,\lambda}^\mu(u_0^{(1)}) = \alpha_\mu$ and $(J_{a,\lambda}^\mu)'(u_0^{(1)}) = 0$ for $\mu$ sufficiently large, this implies that $u_0^{(1)}$ is...
a nontrivial nonnegative solution of Eq. \((K_{a,\lambda}^\mu)\). The strong Maximum Principle implies that \(u_0^{(1)} > 0\) in \(\mathbb{R}^3\). Therefore, the proof of part (i) is completed.

To prove part (ii), we consider the infimum of \(J_{a,\lambda}^\mu\) on the closed ball \(B_{\rho_{a,\lambda}} := \{u \in X \mid \|u\|_\mu \leq \rho_{a,\lambda}\}\) with \(\rho_{a,\lambda}\) being as in Lemma 3.2. Note that \(\rho_{a,\lambda}\) is independent of \(\mu\) for \(\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_a\). Set

\[
\beta_\mu = \inf_{\|u\|_\mu \leq \rho_{a,\lambda}} J_{a,\lambda}^\mu(u).
\]

Let

\[
J_{a,\lambda}^\mu(t\phi_1) = -\frac{\lambda - \lambda_1(f_\Omega)}{2} t^2 + \frac{a\lambda_1^2(f_\Omega)}{4} t^4 - \frac{\int_{\mathbb{R}^3} g(x)\phi_1^p dx}{p} t^p \quad \text{for} \ t > 0.
\]

Then for each \(\lambda > \lambda_1(f_\Omega)\), there exists \(t_0 > 0\) such that \(\|t_0\phi_1\|_\mu \leq \rho_{a,\lambda}\) and \(J_{a,\lambda}^\mu(t_0\phi_1) < 0\). Moreover, we have

\[
J_{a,\lambda}^\mu(u) \geq -\frac{\lambda\|f\|_{3/2}^2}{2S^2 - \rho_{a,\lambda}^2} \|u\|_\mu^2 - \frac{\|g\|_\infty\|\{V < c_0\}\|_{(6-p)/6}^p}{pS^p} \|u\|_\mu^p
\]

which implies that \(-\infty < \beta_\mu < 0\). By the Ekeland variational principle \([15]\) and \(J_{a,\lambda}^\mu(u) = J_{a,\lambda}^\mu(|u|)\), there exists a \((PS)_{\beta_\mu}\)-sequence \(\{u_n\} \subset B_{\rho_{a,\lambda}}\) with \(u_n \geq 0\) in \(\mathbb{R}^3\). Then by Lemma 3.5, there exists \(0 \leq u_0^{(2)} \in X\) such that \(J_{a,\lambda}^\mu(u_0^{(2)}) = \beta_\mu < 0\) and \((J_{a,\lambda}^\mu)'(u_0^{(2)}) = 0\) for \(\mu\) sufficiently large, this implies that \(u_0^{(2)}\) is a nontrivial nonnegative solution of Eq. \((K_{a,\lambda}^\mu)\). The strong Maximum Principle implies that \(u_0^{(2)} > 0\) in \(\mathbb{R}^3\). Consequently, this completes the proof of Theorem 1.1.

5 Proofs of Theorems 1.2 and 1.3

We start this section by showing that the functional \(J_{a,\lambda}^\mu\) is coercive and bounded below on \(X\) when \(N \geq 3\) and \(2 < p < \min\{4, 2^*\}\).

Lemma 5.1 Suppose that \(N \geq 3, 2 < p < \min\{4, 2^*\}\) and conditions \((V1) - (V2), (D1) - (D2)\) hold. In addition, we assume that condition \((H3)\) holds for \(N = 3\). Then for each \(a > 0\) and \(\lambda > 0\),

\[
J_{a,\lambda}^\mu(u) \geq \frac{1}{4} \|u\|_\mu^2 - C_{N,a,\lambda}
\]

for \(\mu\) sufficiently large, where the number \(C_{N,a,\lambda} > 0\) is independent of \(\mu\).

Proof. By condition \((D1)\), the Hölder and Young’s inequalities, we have

\[
\frac{\lambda}{2} \int_{\mathbb{R}^N} f(x)u^2 dx \leq \|f\|_{N/2} S^{-2} \|u\|_{D^{1,2}}^2 \leq \frac{a}{12} \|u\|_{D^{1,2}}^4 + \frac{3\lambda^2\|f\|_{N/2}^2}{4aS^4}.
\]
Then it holds
\[ J_{a,\lambda}^\mu(u) = \frac{a}{4} \| u \|_{D^{1,2}}^4 + \frac{1}{2} \| u \|_\mu^2 - \frac{\lambda}{2} \int_{\mathbb{R}^N} f(x) u^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u|^p dx \]
\[ \geq \frac{a}{4} \| u \|_{D^{1,2}}^4 + \frac{1}{2} \| u \|_\mu^2 - \left( \frac{a}{12} \| u \|_{D^{1,2}}^3 + \frac{3 \lambda^2 \| f \|_{L^{N/2}}^2}{4aS^4} \right) - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u|^p dx \]
\[ = \frac{1}{2} \| u \|_\mu^2 + \frac{a}{6} \| u \|_{D^{1,2}}^4 - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u|^p dx - \frac{3 \lambda^2 \| f \|_{L^{N/2}}^2}{4aS^4}. \]

Next, about the estimate of the nonlinear term \( \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u|^p dx \), we consider two cases as follows.

Case (i) : \( N = 3 \). Using the Sobolev, Hölder and Caffarelli-Kohn-Nirenberg inequalities and condition (H3) gives
\[ \frac{1}{p} \int_{\mathbb{R}^3} g(x) |u|^p dx \]
\[ \leq \frac{1}{p} \left( \int_{\{ x \in \mathbb{R}^3 \} \cap \{|x| > R \}} g(x) \frac{4}{6-p} u^2 dx \right)^{\frac{6-p}{4}} \left( \int_{\mathbb{R}^3} u^6 dx \right)^{\frac{4}{6-p}} \]
\[ \leq \frac{1}{p} \left( \int_{\{ x \in \mathbb{R}^3 \} \cap \{|x| > R \}} g(x) \frac{4}{6-p} u^2 dx + \int_{\{ x \in \mathbb{R}^3 \} \cap \{|x| \leq R \}} g(x) \frac{4}{6-p} u^2 dx \right)^{\frac{6-p}{4}} \left( \frac{\| u \|_{D^{1,2}}^4}{S^6} \right)^{\frac{p-2}{4}} \]
\[ \leq \frac{1}{p} \left( \frac{c_* \| g \|_{\infty}^2}{S^6} \int_{\mathbb{R}^3} V(x) u^2 dx \right)^{\frac{2(4-p)}{p}} \left( \int_{\mathbb{R}^3} u^2 dx \right)^{\frac{p-2}{4}} \left( \frac{\| g \|_{\infty}^2 |B_{R_*}(0)|^{\frac{2}{3}}}{S^2} \right)^{\frac{6-p}{4}} \left( \frac{\| u \|_{D^{1,2}}^4}{pS^2} \right)^{\frac{p-2}{4}} \]
\[ \leq \frac{2^{\frac{6-p}{4}} (c_* \| g \|_{\infty}^2 \overline{C}_0 \| u \|_{D^{1,2}}^4 \| u \|_{H^{2,p-2}}^4)}{pS^{\frac{1}{4}(p-2)}} \left( \int_{\mathbb{R}^3} V(x) u^2 dx \right)^{\frac{4-p}{4}} \| u \|_{D^{1,2}}^2 \]
\[ \leq \frac{2^{\frac{6-p}{4}} (c_* \| g \|_{\infty}^2 \overline{C}_0 \| u \|_{D^{1,2}}^4 \| u \|_{H^{2,p-2}}^4)}{pS^{\frac{1}{4}(p-2)}} \left( \int_{\mathbb{R}^3} V(x) u^2 dx \right)^{\frac{4-p}{4}} \| u \|_{H^{2,p-2}}^2 \]
\[ \leq \frac{a}{12} \| u \|_{D^{1,2}}^4 + (4 - p) \left( \frac{c_* \| g \|_{\infty}}{p^2} \right)^{\frac{4}{12}} \left( \frac{6\sqrt{2}(p-2)\overline{C}_0}{aS^4} \right)^{\frac{4-p}{4}} \right) \int_{\mathbb{R}^3} V(x) u^2 dx \]
(5.2)

and
\[ \frac{2^{\frac{6-p}{4}} \| g \|_{\infty} |B_{R_*}(0)|^{\frac{6-p}{6}}} {pS^p} \| u \|_{D^{1,2}}^p \]
\[ \leq \frac{a}{12} \| u \|_{D^{1,2}}^4 + \frac{4 - p}{p} \frac{2^{\frac{6-p}{4}} |B_{R_*}(0)|^{\frac{12-2p}{12}}}{pS^p} \left( \frac{\| g \|_{\infty}}{S^p} \right)^{\frac{4}{12}} \left( \frac{3}{a} \right)^{\frac{p}{12}}. \]
(5.3)
It follows from (5.1) – (5.3) that
\[ J_{a,\lambda}^\mu(u) \geq \frac{1}{2}\|u\|_\mu^2 - (4 - p) \left( \frac{c_0\|g\|_\infty}{p^2} \right)^\frac{1}{4-p} \left( \frac{6\sqrt{2}(p - 2)C_0}{aS^3} \right)^\frac{p-2}{4-p} \int_{\mathbb{R}^3} V(x)u^2\,dx - C_{3,a,\lambda} \]
\[ \geq \frac{1}{4}\|u\|_\mu^2 - C_{3,a,\lambda} \text{ for all } \mu \geq \mu_0, \]
where \[ \mu_0 := 4(4 - p) \left( \frac{c_0\|g\|_\infty}{p^2} \right)^\frac{1}{4-p} \left( \frac{6\sqrt{2}(p - 2)C_0}{aS^3} \right)^\frac{p-2}{4-p} \]
and \[ C_{3,a,\lambda} := \frac{4 - p}{p} 2^{\frac{p-2}{2-p}} |B_{R_*}(0)|^{12-2p \over 2} \left( \frac{\|g\|_\infty}{S^p} \right)^{\frac{4}{4-p}} \left( \frac{3}{a} \right)^\frac{p}{4-p} + \frac{3\lambda^2\|f\|_3^2}{4aS^4}. \]
Case (ii) \( N \geq 4 \). Using (2.1) gives
\[ \frac{1}{p} \int_{\mathbb{R}^N} g(x)|u|^p\,dx \]
\[ \leq \frac{\|g\|_\infty}{p} \left[ \frac{|\{V < c_0\}|}{S^2} \frac{2^{7-p}}{p^2} \|u\|_{D,1,2}^2 + \frac{1}{c_0} \int_{\mathbb{R}^N} V(x)u^2\,dx \right]^{\frac{2^{7-p}}{p-2}} \left( \frac{\|u\|_{D,1,2}^2}{S^{2^*}} \right)^{\frac{p-2}{p}} \]
\[ \leq \frac{2^{7-p} \|g\|_\infty}{pS^p} |\{V < c_0\}|^{\frac{2^{7-p}}{p^2}} \|u\|^p_{D,1,2} + \frac{\|g\|_\infty}{p} \left( \frac{2}{c_0} \int_{\mathbb{R}^N} V(x)u^2\,dx \right)^{\frac{2^{7-p}}{p-2}} \left( \frac{\|u\|_{D,1,2}^2}{S^{2^*}} \right)^{\frac{p-2}{p}}. \quad (5.4) \]
Moreover, by Young’s inequality one has
\[ \frac{2^{7-p} \|g\|_\infty}{pS^p} |\{V < c_0\}|^{\frac{2^{7-p}}{p^2}} \|u\|^p_{D,1,2} \]
\[ \leq \frac{a}{12} \|u\|^4_{D,1,2} + \frac{4 - p}{4p} \left( \frac{2^{7-p} \|g\|_\infty}{pS^p} |\{V < c_0\}|^{\frac{2^{7-p}}{p^2}} \right)^\frac{4}{4-p} \left( \frac{3}{a} \right)^\frac{p}{4-p}, \quad (5.5) \]
and
\[ \frac{\|g\|_\infty}{p} \left( \frac{2}{c_0} \int_{\mathbb{R}^N} V(x)u^2\,dx \right)^{\frac{2^{7-p}}{p-2}} \left( \frac{\|u\|_{D,1,2}^2}{S^{2^*}} \right)^{\frac{p-2}{p}} \]
\[ \leq \frac{a}{12} \|u\|^4_{D,1,2} + (4 - p) \left( \frac{\|g\|_\infty}{p} \right)^\frac{2^{7-p}}{p} \left( \frac{6(p - 2)}{aS^4} \right)^\frac{p}{p-1} \frac{1}{c_0} \int_{\mathbb{R}^N} V(x)u^2\,dx \quad (5.6) \]
for \( N = 4; \)
\[ \frac{\|g\|_\infty}{p} \left( \frac{2}{c_0} \int_{\mathbb{R}^N} V(x)u^2\,dx \right)^{\frac{2^{7-p}}{p-2}} \left( \frac{\|u\|_{D,1,2}^2}{S^{2^*}} \right)^{\frac{p-2}{p}} \]
\[ \leq \frac{2(2^* - p)\|g\|_\infty}{(2^* - 2)p} \int_{\mathbb{R}^N} V(x)u^2\,dx + \frac{a}{12} \|u\|^4_{D,1,2} + \frac{4 - 2^*}{4} \left( \frac{\|g\|_\infty}{(2^* - 2)pS^{2^*}} \right)^\frac{4}{4-p} \left( \frac{32}{a} \right)^\frac{2^*}{4-p}. \quad (5.7) \]
for $N \geq 5$. We now set

$$
\mu_1 = \begin{cases} 
\frac{4(4-p)c_p^{-1}}{8(2^r-p)||g||_\infty} \frac{\sqrt[4]{p}}{p} \left( \frac{6(p-2)}{aS^3} \right)^{\frac{2-p}{4}} & \text{for } N = 4, \\
\text{for } N \geq 5,
\end{cases}
$$

and

$$
C_{N,a,\lambda} = \begin{cases} 
\frac{(4-p)||V|<\infty||}{\left( \frac{||g||_\infty}{S^p} \right)^{\frac{3}{4}} \left( \frac{3}{a} \right)^{\frac{2-p}{4}} + \frac{3\lambda^2||f||^2}{4aS^4} & \text{for } N = 4, \\
\frac{4-p}{4p} \left( \frac{2^{2^r-2}||V|<\infty||}{S^p} \right)^{\frac{3}{4}} \left( \frac{3}{a} \right)^{\frac{2-p}{4}} + \frac{3\lambda^2||f||^2}{4aS^4} & \text{for } N \geq 5.
\end{cases}
$$

Thus, it follows from $(5.4) - (5.7)$ that

$$
J^\mu_{a,\lambda}(u) \geq \frac{1}{4} ||u||^2 - C_{N,a,\lambda} \text{ for all } \mu \geq \mu_1.
$$

Consequently, this completes the proof. ■

We are now ready to prove Theorem 1.2: $(i)$ By Lemma 3.3 $(i)$, for each $0 < a < a_0(p)$ and $0 < \lambda < \lambda_1(f_0)$, the functional $J^\mu_{a,\lambda}$ has the mountain pass geometry for $\mu$ sufficiently large. Let

$$
\alpha_\mu := \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} J^\mu_{a,\lambda}(\gamma(s)) \text{ with } \Gamma = \{ \gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e_0 \},
$$

where $e_0$ is as in Lemma 3.3. It is evident that

$$
0 < \alpha_\mu \leq \max_{0 \leq s \leq 1} J^\mu_{a,\lambda}(se_0) =: D_0 
$$

(5.8)

and that $D_0$ is independent of $\mu$ since $e_0 \in H^1_0(\Omega)$. Let $\{u_n\}$ be a $(PS)_{\alpha_\mu}$-sequence, that is $J^\mu_{a,\lambda}(u_n) = \alpha_\mu + o(1)$ and $(J^\mu_{a,\lambda})'(u_n) = o(1)$. In fact, since $J^\mu_{a,\lambda}(u_n) = J^\mu_{a,\lambda}(|u_n|)$ for all $n$, we may assume that $u_n \geq 0$. Moreover, by Lemma 5.1 and (5.8), we deduce that there exists $d_0 > 0$ such that the $(PS)_{\alpha_\mu}$-sequence $\{u_n\}$ satisfies $||u_n||_\mu < d_0$ for $\mu$ sufficiently large, which implies that the functional $J^\mu_{a,\lambda}$ satisfies the $(PS)_{\alpha_\mu}$-condition via Lemma 3.5. Therefore, there exists $0 \leq u^{(3)}_0 \in X$ such that $J^\mu_{a,\lambda}(u^{(3)}_0) = \alpha_\mu > 0$ and $(J^\mu_{a,\lambda})'(u^{(3)}_0) = 0$ for $\mu$ sufficiently large, this implies that $u^{(3)}_0$ is a nontrivial nonnegative solution of Eq. $(K^\mu_{a,\lambda})$. The strong Maximum Principle implies that $u^{(3)}_0 > 0$ in $\mathbb{R}^N$.

Next, we consider the infimum of $J^\mu_{a,\lambda}$ on the set $\{u \in X \mid \|u\|_\mu \geq \overline{p}_{a,\lambda}\}$ with $\overline{p}_{a,\lambda}$ as given in Lemma 3.3 $(i)$. Set

$$
\beta_\mu := \inf_{\|u\|_\mu \geq \overline{p}_{a,\lambda}} J^\mu_{a,\lambda}(u).
$$

By virtue of $\|e_0\|_\mu > \overline{p}_{a,\lambda}$, $J^\mu_{a,\lambda}(e_0) < 0$ and Lemma 5.1 we conclude that $-C_{N,a,\lambda} < \beta_\mu < 0$. It follows from $J^\mu_{a,\lambda}(u) = J^\mu_{a,\lambda}(|u|)$, the Ekeland variational principle [15] and Lemma 5.1 that there exists a bounded $(PS)_{\beta_\mu}$-sequence $\{u_n\} \subset X$ with $u_n \geq 0$ in $\mathbb{R}^N$. Hence, by Lemma 3.5 there exists $0 \leq u^{(4)}_0 \in X$ with $\|u^{(4)}_0\|_\mu \geq \overline{p}_{a,\lambda}$ such that $J^\mu_{a,\lambda}(u^{(4)}_0) = \beta_\mu < 0$ and $(J^\mu_{a,\lambda})'(u^{(4)}_0) = 0$ for...
μ sufficiently large, this implies that \( u_0^{(4)} \) is a nontrivial nonnegative solution of Eq. \( (K_{a,λ}^μ) \). The strong Maximum Principle implies that \( u_0^{(4)} > 0 \) in \( \mathbb{R}^N \).

(ii) Using Lemma 3.3 (ii) and repeating the same argument as in the proof of part (i), we conclude that there exist two positive solutions \( u_0^{(3)} \) and \( u_0^{(4)} \) with \( J_{a,λ}^μ(u_0^{(3)}) > 0 > J_{a,λ}^μ(u_0^{(4)}) \) and \( \|u_0^{(4)}\| ≥ \bar{p}_{a,λ} \) whenever \( \lambda_1(f_Ω) ≤ λ < \lambda_1(f_Ω) + \delta_a \), where \( \delta_a \) is as in Lemma 3.3 (ii). To complete the proof of part (ii), for \( \lambda_1(f_Ω) < λ < \lambda_1(f_Ω) + \delta_a \), we consider the infimum of \( J_{a,λ}^μ \) on the closed ball

\[ B_{\bar{p}_{a,λ}} := \{u \in X \mid \|u\|_μ ≤ \bar{p}_{a,λ}\} \]

with \( \bar{p}_{a,λ} \) as in Lemma 3.3 (ii). Note that \( \bar{p}_{a,λ} \) is independent of \( μ \). Set

\[ η_μ = \inf_{\|u\|_μ ≤ \bar{p}_{a,λ}} J_{a,λ}^μ(u). \]

For any \( t > 0 \), we deduce that

\[ J_{a,λ}^μ(tφ_1) = -\frac{λ - λ_1(f_Ω)}{2} t^2 + \frac{∫_Ω g(x)φ_1^p dx}{p} t^p + \frac{aλ^2_1(f_Ω)}{4} t^4. \]

Clearly, there exists \( t_0 > 0 \) such that \( \|t_0φ_1\|_μ < \bar{p}_{a,λ} \) and \( J_{a,λ}^μ(t_0φ_1) < 0 \), which implies that \( -∞ < η_μ < 0 \). By \( J_{a,λ}^μ(u) = J_{a,λ}^μ(|u|) \) and the Ekeland variational principle \([15]\), there exists a \((PS)_ημ\)-sequence \( \{u_n\} \subset B_{\bar{p}_{a,λ}} \) with \( u_n ≥ 0 \) in \( \mathbb{R}^N \). Then it follows from Lemma 3.5 that there exists \( 0 ≤ u_0^{(5)} ∈ X \) with \( \|u_0^{(5)}\|_μ < \bar{p}_{a,λ} \) such that \( J_{a,λ}^μ(u_0^{(5)}) = η_μ < 0 \) and \( (J_{a,λ}^μ)'(u_0^{(5)}) = 0 \) for \( μ \) sufficiently large, this implies that \( u_0^{(5)} \) is a nontrivial nonnegative solution of Eq. \( (K_{a,λ}^μ) \). The strong Maximum Principle implies that \( u_0^{(5)} > 0 \) in \( \mathbb{R}^N \). Consequently, we complete the proof of Theorem 1.2.

At the end of this section, we give the proof of Theorem 1.3: By virtue of Lemma 3.4, we can easily reach the conclusion by using the similar argument of Theorem 1.2 (i). We omit it here.

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