A New Approach to Quantum Gravity from a Model of an Elastic Solid

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Abstract

We show that the dynamics of an elastic solid embedded in a Minkowski space consist of a set of coupled equations describing a spin-1/2 field, \( \Psi \), obeying Dirac’s equation, a vector potential, \( A_\mu \), obeying Maxwell’s equations and a metric, \( g_{\mu\nu} \), which satisfies the Einstein field equations. The combined set of Dirac’s, Maxwell’s and the Einstein field equations all emerge from a simple elastic model in which the field variables \( \Psi \), \( A_\mu \) and \( g_{\mu\nu} \) are each identified as derived quantities from the field displacements of ordinary elasticity theory. By quantizing the elastic field displacements, a quantization of all of the derived fields are obtained even though they do not explicitly appear in the Lagrangian. We demonstrate the approach in a three dimensional setting where explicit solutions of the Dirac field in terms of fractional derivatives are obtained. A higher dimensional version of the theory would provide an alternate approach to theories of quantum gravity.
I. INTRODUCTION

In constructing a quantum field theory, the usual prescription is to start with a known equation of motion, such as Dirac’s equation, and "invent" a suitable Lagrangian that reproduces the equation of motion when Lagrange’s equations are applied. While this prescription has been successful it is not unique. In other words it is possible for two different Lagrangian’s to lead to the same equation of motion. For example, section 12 – 2 of Reference[1] provides a good example of two different Lagrangian’s that lead to the same equation of motion for the density variations in an acoustic field.

In a theory of quantum gravity, this traditional approach would involve using a Lagrangian with an appropriate set of terms such that when Lagrange’s equations are applied, the Einstein field equations are reproduced. The Lagrangian obtained in this manner explicitly contains the gravitational metric, as well as any other field variables that are coupled to it. For instance an attempt at merging gravity with QED would produce a Lagrangian that explicitly includes the Dirac Spinor field Ψ, the electromagnetic vector potential $A_\mu$ and the gravitational metric $g_{\mu\nu}$.

In this paper we demonstrate an alternate approach to quantum gravity based on a model of an elastic solid. In the this model, the only field variables that appear in the Lagrangian are the field displacements, $u_i$, that occur in elasticity theory. Using the methods of fractional calculus, we will show that the equations of motion of the system describe excitations that can be identified as massless, non-interacting, spin-1/2 particles obeying Dirac’s equation. We then assume that one of our coordinates is periodic and use a dimensional reduction technique to reduce the dimensionality from three-dimensions to two-dimensions.

When terms beyond the linear approximation are included, this dimensional reduction produces a new set of equations in which the spin field, $Ψ$, is shown to interact via a vector potential $A_\mu$ and a metric with field variables $g_{\mu\nu}$. The compatibility equations of St. Venant are shown to reproduce Maxwell’s equation for $A_\mu$ and the Einstein field equations for $g_{\mu\nu}$. We quantize the field displacements using standard approaches and thereby produce a quantization of $Ψ$, $A_\mu$ and $g_{\mu\nu}$ even though none of these quantities appears explicitly in the Lagrangian. We demonstrate the basic methods in a three-dimensional setting where exact expressions for the Dirac field can be obtained.

When quantized, this theory provides a low dimensional version of a quantum description
electrodynamics coupled to gravity. If this procedure could be extended to higher dimensions it would provide an alternate approach to theories of quantum gravity.

II. ELASTICITY THEORY

The theory of elasticity is usually concerned with the infinitesimal deformations of an elastic body\textsuperscript{2,3,4,5,6}. We assume that the material points of a body are continuous and can be assigned a unique label $\vec{a}$. For a three-dimensional solid each point of the body may be labeled with three coordinate numbers $a^i$ with $i = 1, 2, 3$.

If this three dimensional elastic body is placed in a large ambient three dimensional space then the material coordinates $a^i$ can be described by their positions in the 3-D fixed space coordinates $x^i$ with $i = 1, 2, 3$. We imagine that the solid is free to distort within the fixed ambient space described with coordinates $x^i$. In this description the material points $a^i(x^1, x^2, x^3)$ are functions of $\vec{x}$. A deformation of the elastic body results in infinitesimal displacements of these material points. If before deformation, a material point $a^0$ is located at fixed space coordinates $x^{01}, x^{02}, x^{03}$ then after deformation it will be located at some other coordinate $x^1, x^2, x^3$. The deformation of the medium is characterized at each point by the displacement vector

$$u^i = x^i - x^{0i}$$

which measures the displacement of each point in the body after deformation. We will assume that our elastic solid is periodic in the coordinate $a^3$ and at various points in this paper we will Fourier transform the $a^3$ coordinate.

It is one of the aims of this paper to take this model of an elastic medium and derive from it equations of motion that have the same form as Dirac’s equation. In doing so we have to distinguish between the intrinsic coordinates of the medium which we will call ”internal” coordinates and the fixed space coordinates which facilitates our derivation of the equations of motion. In the undeformed state we may take the external coordinates to coincide with the material coordinates $a^i = x^{0i}$. The approach that we will use in this paper is to derive equations of motion using the fixed space coordinates and then translate this to the internal coordinates of our space.
A. Strain Tensor

Let us assume that we have an elastic solid embedded in a three-dimensional Minkowski space with metric

\[
\eta_{ij} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

We first consider the effect of a deformation on the measurement of distance. After the elastic body is deformed, the distances between its points changes as measured with the fixed space coordinates. If two points which are very close together are separated by a radius vector \( dx^0 \) before deformation, these same two points are separated by a vector \( dx^i = dx^0 + du^i \) afterwards. The squared distance between the points before deformation is then \( ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \). Since these coincide with the material points in the undeformed state, this can be written \( ds^2 = \sum_{i,j} da^i \eta_{ij} da^j \).

The squared distance after deformation can be written

\[
\sum_{ij} (dx^i) \eta_{ij} (dx^j) = \sum_{ij} (da^i + du^i) \eta_{ij} (da^j + du^j)
\]

\[
= \sum_{ij} \left( da^i + \sum_k \frac{\partial u^i}{\partial a^k} da^k \right) \eta_{ij} \left( da^j + \sum_l \frac{\partial u^j}{\partial a^l} da^l \right)
\]

\[
= \sum_{ij} \eta_{ij} da^i da^j + \sum_{ijk} \eta_{ij} \frac{\partial u^i}{\partial a^k} da^k da^j + \sum_{ijl} \eta_{ij} \frac{\partial u^j}{\partial a^l} da^i da^l + \sum_{ijkl} \eta_{ij} \frac{\partial u^i}{\partial a^k} \frac{\partial u^j}{\partial a^l} da^k da^l
\]

\[
= \sum_{ij} \left( \eta_{ij} + \sum_k \left( \eta_{ik} \frac{\partial u^k}{\partial a^j} + \eta_{jk} \frac{\partial u^k}{\partial a^i} \right) + \sum_{kl} \eta_{kl} \frac{\partial u^k}{\partial a^i} \frac{\partial u^l}{\partial a^j} \right) da^i da^j
\]

\[
= \sum_{ik} (\eta_{ij} + 2\epsilon_{ij}) da^i da^j
\]

where \( \epsilon_{ij} \) is

\[
\epsilon_{ij} = \frac{1}{2} \sum_k \left( \eta_{ik} \frac{\partial u^k}{\partial a^j} + \eta_{jk} \frac{\partial u^k}{\partial a^i} + \sum_l \eta_{kl} \frac{\partial u^k}{\partial a^i} \frac{\partial u^l}{\partial a^j} \right)
\]

and the presence of the matrix \( \eta_{ij} \) simply reflects the fact that we are assuming our solid is embedded in a Minkowski space with a pseudo-Euclidean metric.
The quantity $\epsilon_{ik}$ is known as the strain tensor. It is fundamental in the theory of elasticity. In the above derivation, the material or internal coordinates were treated as functions of the fixed space coordinates. As is well known in elasticity theory, we could just as well treat the fixed space coordinates as functions of the material coordinates. In this case, the strain tensor has the form

$$\epsilon_{ij} = \frac{1}{2} \sum_k \left( \eta_{ik} \frac{\partial u^k}{\partial x^j} + \eta_{jk} \frac{\partial u^k}{\partial x^i} - \sum_l \eta_{kl} \frac{\partial u^k}{\partial x^i} \frac{\partial u^l}{\partial x^j} \right)$$

(3)

These two different approaches to the strain tensor are known in elasticity theory as the Lagrangian and Eulerian perspectives. In this work we will derive the equations of motion using the fixed space coordinates which simplifies the derivation and we will translate the result, when necessary, to the internal coordinates.

In most treatments of elasticity it is assumed that the displacements $u^i$ as well as their derivatives are infinitesimal so the last term in Equation (2) is dropped. In this work, we will treat the strain components as small but finite. We will then examine the structure of the equations of motion when the higher order terms are treated as small perturbations on the infinitesimal strain results.

### B. Metric Tensor

The quantity

$$g_{ij} = \eta_{ij} + \sum_k \left( \eta_{ik} \frac{\partial u^k}{\partial x^j} + \eta_{jk} \frac{\partial u^k}{\partial x^i} + \sum_l \eta_{kl} \frac{\partial u^k}{\partial x^i} \frac{\partial u^l}{\partial x^j} \right)$$

(4)

is the metric for our system and determines the distance between any two points. One interesting aspect of the elasticity theory approach is that it provides a natural metric on the system in terms of the strain components expressed entirely in terms of the internal coordinates of the elastic body. This means that at any point in space the distance measurement can be made without reference to the fixed space coordinates. In other words if you were an ant living in this elastic medium, Equation (4) would be the metric that you would use.

Even though the metric in Equation (4) does not have the Euclidean form, the space in which we are working is still intrinsically flat. The metric that we derived is due simply to a coordinate transformation and so cannot describe the curved space of general relativity.
That this metric is simply the result of a coordinate transformation from the Minkowski metric can be seen by writing the metric in the form

\[
g_{\mu\nu} = \begin{pmatrix}
\frac{\partial x^1}{\partial a^1} & \frac{\partial x^2}{\partial a^1} & \frac{\partial x^3}{\partial a^1} \\
\frac{\partial x^1}{\partial a^2} & \frac{\partial x^2}{\partial a^2} & \frac{\partial x^3}{\partial a^2} \\
\frac{\partial x^1}{\partial a^3} & \frac{\partial x^2}{\partial a^3} & \frac{\partial x^3}{\partial a^3}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x^1}{\partial a^1} & \frac{\partial x^1}{\partial a^2} & \frac{\partial x^1}{\partial a^3} \\
\frac{\partial x^2}{\partial a^1} & \frac{\partial x^2}{\partial a^2} & \frac{\partial x^2}{\partial a^3} \\
\frac{\partial x^3}{\partial a^1} & \frac{\partial x^3}{\partial a^2} & \frac{\partial x^3}{\partial a^3}
\end{pmatrix}
\]

\[
= J^T \eta J
\]

where

\[
\frac{\partial x^\mu}{\partial a^\nu} = \delta_{\mu\nu} + \frac{\partial u^\mu}{\partial a^\nu}.
\]

and \(J\) is the Jacobian of the transformation. Later in section VI, however we will use a dimensional reduction technique borrowed from Kaluza-Klein theories to reduce the three-dimensional flat space to a two-dimensional curved space. We will show that the metric for the Fourier modes of this two dimensional system is not a simple coordinate transformation.

The inverse metric, which is written with upper indices as \(g^{ik}\), can be obtained by explicitly inverting Equation (4) or we can write \((g^{ik}) = (J^{-1})^{-1}(J^{-1})^T\) where

\[
J^{-1} = \begin{pmatrix}
\frac{\partial a^1}{\partial x^1} & \frac{\partial a^1}{\partial x^2} & \frac{\partial a^1}{\partial x^3} \\
\frac{\partial a^2}{\partial x^1} & \frac{\partial a^2}{\partial x^2} & \frac{\partial a^2}{\partial x^3} \\
\frac{\partial a^3}{\partial x^1} & \frac{\partial a^3}{\partial x^2} & \frac{\partial a^3}{\partial x^3}
\end{pmatrix}
\]

This yields for the inverse metric

\[
g^{\mu\nu} = \eta_{\mu\nu} + \sum_\alpha \left(-\eta_{\nu\alpha} \frac{\partial u^\mu}{\partial x^\alpha} - \eta_{\mu\alpha} \frac{\partial u^\nu}{\partial x^\alpha} + \sum_{\alpha\beta} \eta_{\alpha\beta} \frac{\partial u^\mu}{\partial x^\alpha} \frac{\partial u^\nu}{\partial x^\beta}\right)
\]

Equation (6) shows that we can write the inverse matrix directly in terms of derivatives of \(u_i\) with respect to the fixed space coordinates. This form of the inverse metric will be useful in later sections.

1. Internal vs. External Coordinates and Summation Convention

The change in form of the metric between that given in Equation (11) and Equation (14) is due simply to a change in coordinates between the fixed space coordinates and the material
coordinates. In this regard the transformation is similar to changing from Cartesian to spherical coordinates. This change is useful because it allows us to derive equations in the fixed space coordinates where the calculations are simplified, and then when necessary we can switch to the internal coordinates using \( u^i = x^i - a^i \).

We would like to be able to use the notation that a raised index on a variable indicates a contraction with the metric tensor and that a raised index and a lower index with the same label implies a summation (i.e., the Einstein summation convention). We have to be careful, however, to point out which set of coordinates, and hence which metric we are using, so we will be explicit in each section as to which coordinate system the raised indices refer to. For instance we can write Equation (2) more compactly as

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial a^j} + \frac{\partial u_j}{\partial a^i} + \frac{\partial u_i}{\partial a^i} \frac{\partial u^l}{\partial a^j} \right).
\]

(7)

where \( u^i u^l = \sum_k \eta_{kl} u^i u^k \) so that upper/lower indices indicate contraction with the fixed space metric, Equation (1).

C. St Venant’s Equations of Compatibility

In Section (II D) we will derive the equations of motion of the elastic solid using the Lagrangian formalism. There are, however, additional constraints that an elastic solid must also satisfy. These constraints are called the St Venant equations of compatibility in classical elasticity theory\(^3\). The usual description of these compatibility equations is that they are integrability conditions or, a restriction on the strain components \( \epsilon_{ij} \) such that they can be considered partial derivatives of a function \( u \) as displayed in Equation (2). In other words if \( \epsilon_{ij} \) is a function that is composed of the partial derivatives of \( u \) then it has to satisfy certain conditions and these are the compatibility equations.

However, from a geometric standpoint these equations are simply a re-statement of the fact that space is flat\(^6\). In other words the compatibility equations are equivalent to\(^6\)

\[
R_{\alpha\beta\mu\nu} = \frac{1}{2} \left( \frac{\partial^2 g_{\alpha\nu}}{\partial a^\beta a^\mu} - \frac{\partial^2 g_{\alpha\mu}}{\partial a^\beta a^\nu} + \frac{\partial^2 g_{\beta\mu}}{\partial a^\alpha a^\nu} - \frac{\partial^2 g_{\beta\nu}}{\partial a^\alpha a^\mu} \right) + \Gamma^\rho_{\beta\mu} \Gamma_{\rho,\alpha\nu} - \Gamma^\rho_{\alpha\mu} \Gamma_{\rho,\beta\nu} = 0,
\]

(8)

where \( R_{\alpha\beta\mu\nu} \) is the Riemann Curvature tensor and \( \Gamma^\rho_{\alpha\beta} \) are the Christoffel symbols given by\(^8\)

\[
\Gamma^\rho_{\alpha\beta} = g^{\lambda\rho} \left( \frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \right).
\]
In the above equations an upper/lower index implies a contraction with the metric in the internal coordinates.

One of the interesting aspects of an elastic solid is that this setting gives you for "free" an explicit expression for the metric, Equation (4), and a statement about the curvature of space, Equation (8). We get these equations even though, as we will see shortly, the metric is not a dynamical variable appearing in the Lagrangian.

D. Equation of Motion

In the following we will use the notation

\[ u_{\mu\nu} = \frac{\partial u_\mu}{\partial x^\nu} \]

and therefore the strain tensor is

\[ \epsilon_{\mu\nu} = \frac{1}{2} \left( u_{\mu\nu} + u_{\nu\mu} + u_{\mu} u_{\nu} \right) . \]  

(9)

and all contractions are with the fixed space metric, Equation (1).

We work in the fixed space coordinates and take the negative of the strain energy as the lagrangian density of our system. This approach leads to the usual equations of equilibrium in elasticity theory. The strain energy is quadratic in the strain tensor \( \epsilon^{\mu\nu} \) and therefore the Lagrangian can be written

\[ L = -\sum_{\mu\nu\alpha\rho} C^{\mu\nu\alpha\rho} \epsilon_{\mu\nu} \epsilon_{\alpha\rho} \]

The quantities \( C^{\mu\nu\alpha\rho} \) are known as the elastic stiffness constants of the material. For an isotropic space most of the coefficients are zero and in fact there are only two independent elastic constants in a three-dimensional isotropic space. The lagrangian density then reduces to

\[ L = (\lambda + 2\mu) \left[ \epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 \right] + 2\lambda \left[ \epsilon_{22} \epsilon_{33} - \epsilon_{11} \epsilon_{22} - \epsilon_{11} \epsilon_{33} \right] + 4\mu \left[ \epsilon_{23}^2 - \epsilon_{12}^2 - \epsilon_{13}^2 \right] \]  

(10)

where \( \lambda \) and \( \mu \) are known as Lamé constants.

We first derive the equations of motion of the system in the approximation where the strain components \( u_{ij} \) are infinitesimal. In the infinitesimal strain approximation, the quadratic terms in Equation (10) are dropped giving

\[ \epsilon_{\mu\nu} = \frac{1}{2} \left( u_{\mu\nu} + u_{\nu\mu} \right) . \]
The usual Lagrange equations,
\[ \sum_{\nu} \frac{d}{dx^\nu} \left( \frac{\partial L}{\partial (\partial u^\nu_{\rho})} \right) - \frac{\partial L}{\partial u^\rho} = 0, \]
apply with each component of the displacement vector, \( u^\rho \), treated as an independent field variable.

Using the above form of the Lagrangian one can write
\[ \frac{\partial L}{\partial u^\rho_{\nu}} = 2\lambda(\sigma \eta^\rho_{\nu}) + 4\mu \epsilon^\rho_{\nu} \]
where the divergence of the displacement field is \( \sigma \equiv (-u_{11} + u_{22} + u_{33}) \). In classical elasticity theory \( \sigma \) is known as the dilatation and physically represents the fractional change in density of a medium due to a deformation.

We now have three field equations (one for each value of \( \rho \)),
\[ \frac{\partial}{\partial x^\nu} \frac{\partial L}{\partial u^\rho_{\nu}} = (2\lambda + 2\mu) \frac{\partial}{\partial x^\rho} \sigma + 2\mu \nabla^2 (u^\rho) = 0 \] (11)
where \( \nabla^2 = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \). Applying the operator \( \partial/\partial x^\rho \) to Equation (11) yields the wave equation
\[ \left( -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \sigma = 0 \] (12)

Equation (12) shows that the classical dilatation in the medium obeys the wave equation. In Section (III) we will demonstrate a new method for reducing the wave equation (12) to Dirac’s equation and compare this method to the traditional Dirac reduction. But first we turn our attention to quantizing the field displacements \( u_i \) in this elastic model.

**E. Quantization**

The Lagrangian density of the system is given by Equation (10). The coordinate \( x_1 \) plays the role of time in this three-dimensional space so the canonical momenta associated with the field variable \( u^\rho \) are given by
\[ P_\rho = \frac{\partial L}{\partial (\partial u^\rho_1)} \]
which gives
\[ P_1 = -2\lambda(\epsilon_{11} - \epsilon_{22} - \epsilon_{33}) - 4\mu \epsilon_{11} \] (13)
\[ P_2 = 4\mu \epsilon_{12} \]
\[ P_3 = 4\mu \epsilon_{13}. \]

The Hamiltonian density is defined as
\[ \mathcal{H} = \sum \rho P_{\rho}^2 \mu - L \]
\[ = -P_1 u_{11} + P_2 u_{21} + P_3 u_{31} - L \]

and the total Hamiltonian is the integral over all space of the Hamiltonian density
\[ H = \int d^3x \mathcal{H} \]

Inverting Equation (13) allows us to replace the variables \( u_{\rho 1} \) with \( P_{\rho} \) in the result. This gives

\[ \mathcal{H} = \frac{3P_1^2}{4(\lambda + 2\mu)} + \frac{P_2^2 + P_3^2}{4\mu} - \frac{\lambda (u_{22} + u_{33}) P_1}{\lambda + 2\mu} - P_2 u_{12} - P_3 u_{13} \]
\[ + \mu (u_{23} + u_{32})^2 - \frac{\lambda^2 (u_{22} + u_{33})^2}{\lambda + 2\mu} + 2\lambda u_{22} u_{33} \]
\[ + (\lambda + 2\mu) \left( u_{22}^2 + u_{33}^2 \right) \]

(14)

We now Fourier transform the field variables making the assumption that one of our coordinates, \( a_3 \) is compact with the topology of a circle. Therefore, when we Fourier transform the field variables, the \( q_3 \) component is associated with a discrete spectrum while the other two coordinates are continuous. Writing out the coordinate dependencies explicitly we have,

\[ P_{\rho}(x_1, x_2, x_3) = \sum_{q_3} \int dq_{1} \int dq_{2} P_{\rho, q} e^{iq\cdot x} \]
\[ u_{\rho}(x_1, x_2, x_3) = \sum_{q_3} \int dq_{1} \int dq_{2} u_{\rho, q} e^{iq\cdot x}. \]

The Fourier transform results in terms in the Hamiltonian that mix field variables associated with \( q \) and \(-q\). For instance, the contribution to the total Hamiltonian from the \( P_1^2 \) term in Equation (14) becomes

\[ \int d^3x P_1^2 = \int d^3x \left( \sum_{q_3} \int dq_{1} \int dq_{2} P_{1,q} P_{1,-q} e^{(q+q')\cdot x} \right) \]
\[ = \sum_{q_3} \int dq_{1} \int dq_{2} P_{1,q} P_{1,-q} \]

10
The total Hamiltonian then becomes

\[ H = \sum_q H_q \]

where \( H_q \) is written symmetrically in \( q \) and \(-q\) as

\[
H_q = \frac{q_2^2 u_{2,-q} u_{2,q} \lambda^2}{\lambda + 2\mu} - \frac{q_2 q_3 u_{2,-q} u_{3,q} \lambda^2}{\lambda + 2\mu} - \frac{q_3^2 u_{3,-q} u_{3,q} \lambda^2}{\lambda + 2\mu}
\]

\[
+ \frac{iq_2 P_{1,q} u_{2,-q} \lambda}{2(\lambda + 2\mu)} - \frac{iq_2 P_{1,-q} u_{2,q} \lambda}{2(\lambda + 2\mu)} + q_2^2 u_{2,-q} u_{2,q} \lambda + \frac{iq_3 P_{1,q} u_{3,-q} \lambda}{2(\lambda + 2\mu)}
\]

\[
+ q_2 q_3 u_{2,q} u_{3,-q} \lambda - \frac{iq_3 P_{1,-q} u_{3,q} \lambda}{2(\lambda + 2\mu)} + q_2 q_3 u_{2,-q} u_{3,q} \lambda + q_3^2 u_{3,-q} u_{3,q} \lambda
\]

\[
+ \frac{3P_{1,-q} P_{1,q}}{4(\lambda + 2\mu)} + \frac{P_{2,-q} P_{2,q}}{4\mu} + \frac{P_{3,-q} P_{3,q}}{4\mu} + \frac{1}{2} iq_2 P_{2,q} u_{1,-q}
\]

\[
+ \frac{1}{2} iq_3 P_{3,q} u_{1,-q} - \frac{1}{2} iq_2 P_{2,-q} u_{1,q} - \frac{1}{2} iq_3 P_{3,-q} u_{1,q} + 2\mu q_2^2 u_{2,-q} u_{2,q}
\]

\[
+ \mu q_3^2 u_{2,-q} u_{2,q} + \mu q_2 q_3 u_{2,q} u_{3,-q} + \mu q_2 q_3 u_{2,-q} u_{3,q} + \mu q_2^2 u_{3,-q} u_{3,q} + 2\mu q_3^2 u_{3,-q} u_{3,q}
\]

(15)

Since terms in the Hamiltonian with different values of \( q \) are not mixed, the function \( H_q \) in Equation (15) can be solved independently for each \( q \). \( H_q \) is a bilinear function in the variables \( u_{i,\pm q} \) and \( P_{i,\pm q} \) and can be diagonalized exactly, using the methods of Biougliobov\(^9\,10\,11\).

1. Exact Diagonalization

The methods used in diagonalizing the Hamiltonian in Equation (15) are summarized in the references\(^9\,10\,11\). The idea is to rewrite the Hamiltonian in terms of a set of creation and annihilation operators, \( b_{i,q} \) and \( b_{i,q}^\dagger \) such that the Hamiltonian has the form

\[ H_q = \sum_i \omega_{i,q} b_{i,q}^\dagger b_{i,q} \]

and the operators satisfy the commutation relations

\[ [b_{i,q}, b_{j,q'}^\dagger] = i\delta_{i,j} \delta_{q,q'}. \]

The details of this procedure are included in Appendix (A). Of particular interest are the energy eigenvalues of the modes. There are three distinct positive energies for the states \( b_i \). They are

\[ E_{1,q} = \frac{1}{4}\sqrt{q_2^2 + q_3^2} \]

(18)
\[
E_{2,q} = \frac{1}{4} \left[ 1 - \frac{4\mu(\lambda + \mu)}{\sqrt{\mu(\lambda + \mu)(\lambda + 2\mu)^2}} \sqrt{q_2^2 + q_3^2} \right]
\]

(19)

\[
E_{3,q} = \frac{1}{4} \left[ 1 + \frac{4\mu(\lambda + \mu)}{\sqrt{\mu(\lambda + \mu)(\lambda + 2\mu)^2}} \sqrt{q_2^2 + q_3^2} \right]
\]

(20)

With the operators \(b_i\) calculated, the field variables \(u_i\) can be written as a linear combination of creation and annihilation operators as

\[
u_i = \sum_{q=1}^{3} \sum_{i=1}^{6} \left( c_{ij} b_i e^{i\vec{q} \cdot \vec{x}} + c'_{ij} b_i^\dagger e^{-i\vec{q} \cdot \vec{x}} \right)
\]

(21)

where the \(c_{ij}\) are coefficients given in the appendix and \(q_1\) is the energy of a given mode. These eigenstates are the linear approximation obtained from keeping the lowest terms in the strain components \(u_{ij}\) in Equation (10). The higher order terms that were left out of the Lagrangian can be incorporated by treating them as perturbations. In other words we can use standard perturbation theory to find new strain components that are nonlinear in the creation and annihilation operators. These field components can then serve as the basis for a theory of finite strain as we do in the next section.

Strictly speaking the field displacements as expressed in Equation (21) are not energy eigenstates since the \(b_i\) have different energies. We will mainly be concerned however, with a low energy approximation of the spectrum of this elastic solid. For positive values of \(\lambda\) and \(\mu\) and \(\lambda > (6 + 4\sqrt{3})\mu\), the energies \(E_{3,q}\) can be arbitrarily small compared to \(E_{1,q}\) and \(E_{2,q}\). For instance with \(\lambda = 13\mu\) and \(\mu = .1\), the energies \(E_{1,q}\) and \(E_{2,q}\) are more than 20 times greater than \(E_{3,q}\). This suggests that in a low energy theory only the excitation corresponding to energies \(E_{3,q}\) will be present for suitably defined Lame constants. We will not investigate the mechanical properties of such a solid but merely point out that in such a theory at low energies, both the field displacements \(u_i\) and the dilatation, \(\nabla \cdot \vec{u}\), are energy eigenstates. We also note that each of the energies is proportional to \(\sqrt{q_2^2 + q_3^2}\) with \(q_3\) taking on discrete values. So one would expect in the lowest energy approximation that only the modes with \(q_3 = 0\) will be present and at slightly higher energies the mode with \(q_3 = 1\) will be present. This low energy approximation will be exploited in later sections.

One of the things that we have gained from this formalism is the ability to calculate any quantity, that depends on the field displacements, quantum mechanically. For instance we can now calculate the metric given in Equation (6) using the form of the field decomposition.
given in Equation (21) even though the metric itself is not a dynamical variable appearing in the Lagrangian.

In the finite strain theory treated in Section (V) we will need to take Fourier transforms of field variables in the internal coordinates rather than the fixed space coordinates. Since we will be keeping the nonlinear terms in all of our equations, then for consistency we assume that the field variables in Equation (21) have been properly treated to the same order in perturbation theory. We will not explicitly calculate the field variables in perturbation theory rather we will focus on the form of the field equations when terms beyond the linear approximation are kept.

We will now give a new derivation of Dirac’s equation as the equation of motion of the elastic solid.

III. DERIVATION OF DIRAC’S EQUATION OF MOTION

A. Cartan’s Spinors

The concept of Spinors was introduced by Eli Cartan in 1913. In Cartan’s original formulation spinors were motivated by studying isotropic vectors which are vectors of zero length. In three dimensional Minkowski space the equation of an isotropic vector is

\[-x_1^2 + x_2^2 + x_3^2 = 0\]  \hspace{1cm} (22)

for generally complex quantities \(x_i\). A closed form solution to this equation is realized as

\[x_1 = \xi_0^2 + \xi_1^2, \quad x_2 = \xi_0^2 - \xi_1^2, \quad \text{and} \quad x_3 = -2\xi_0\xi_1\]  \hspace{1cm} (23)

where the two quantities \(\xi_i\) are then

\[\xi_0 = \pm \sqrt{\frac{x_1 + x_2}{2}} \quad \text{and} \quad \xi_1 = \pm \sqrt{\frac{x_1 - x_2}{2}}.\]

The two component object \(\xi = (\xi_0, \xi_1)\) has the rotational properties of a spinor and any equation of the form (22) has a spinor solution.

In the following we use the notation \(\partial_\mu \equiv \partial / \partial x^\mu\) and the wave equation is written

\[\left(-\partial_1^2 + \partial_2^2 + \partial_3^2\right) \phi = 0.\]
This equation can be viewed as an isotropic vector in the following way. The components of the vector are the partial derivative operators $\partial/\partial x^\mu$ acting on the quantity $\phi$. As long as the partial derivatives are restricted to acting on the scalar field $\phi$ it has a spinor solution given by

$$\hat{\xi}^2_0 = \frac{1}{2} \left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \right)$$

and

$$\hat{\xi}^2_1 = \frac{1}{2} \left( \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \right)$$

where the "hat" notation indicates that the quantities $\hat{\xi}$ are operators. Let us now introduce the variables

$$z_0 = x_1 + x_2 \text{ and } z_1 = x_1 - x_2$$

Equations (24) and (25) are now

$$\hat{\xi}^2_0 = \frac{\partial}{\partial z^0}$$

and

$$\hat{\xi}^2_1 = \frac{\partial}{\partial z^1}.$$ 

These are equations of fractional derivatives of order $1/2$ (also called semiderivatives) denoted $\hat{\xi}_0 = D^{1/2}_z$ and $\hat{\xi}_1 = D^{1/2}_z$. Fractional derivatives have the property that

$$D^{1/2}_z D^{1/2}_z = \frac{\partial}{\partial z}$$

and various methods exist for writing closed form solutions for these operators. The exact form for these fractional derivatives however, is not important here. The important thing to note is that a solution to the wave equation can be written in terms of spinors which are fractional derivatives.

One of the interesting properties of suitably defined fractional derivatives (for instance the Weyl fractional derivative) that will be exploited in later sections is their action on the exponential function. While the derivative of an exponential is given by

$$\frac{\partial}{\partial x} e^{\alpha x} = \alpha e^{\alpha x}$$

the semiderivative of the exponential function is given by

$$D^{1/2}_z e^{\alpha x} = \sqrt{\alpha} e^{\alpha x}$$

This will prove useful later when we Fourier transform the equations of motion.
B. Matrix Form

It can be readily verified that our spinors satisfy the following equations

\[
\begin{align*}
\hat{\xi}_0 \frac{\partial}{\partial x^3} + \hat{\xi}_1 \left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \right) \phi &= 0 \\
\hat{\xi}_0 \left( \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1} \right) - \hat{\xi}_1 \frac{\partial}{\partial x^3} \phi &= 0
\end{align*}
\]

and in matrix form

\[
\begin{pmatrix}
\frac{\partial}{\partial x^3} \\
\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1} \\
\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1} \\
\frac{\partial}{\partial x^3}
\end{pmatrix}
\begin{pmatrix}
\hat{\xi}_0 \\
\hat{\xi}_1 \\
\end{pmatrix} = 0
\]

(27)

The matrix

\[
X = \begin{pmatrix}
\frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1} & -\frac{\partial}{\partial x^3}
\end{pmatrix}
\]

is equal to the dot product of the vector \( \partial_\mu \equiv \partial/\partial x^\mu \) with the pauli spin matrices

\[
X = \frac{\partial}{\partial x^1} \gamma^1 + \frac{\partial}{\partial x^2} \gamma^2 + \frac{\partial}{\partial x^3} \gamma^3
\]

where

\[
\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are proportional to the Pauli matrices and satisfy the anticommutation relations

\[
\{ \gamma^\mu, \gamma^\nu \} = 2I \eta^{\mu\nu}.
\]

(28)

where \( I \) is the identity matrix.

Equation (27) can be written

\[
\sum_{\mu=1}^3 \partial_\mu \gamma^\mu \hat{\xi} \phi = 0.
\]

(29)

This equation has the form of Dirac’s equation in three-dimensions for a noninteracting, massless, spin-1/2 field, \( \hat{\xi} \phi \).

C. Relation to the Dirac Decomposition

The fact that the wave equation and Dirac’s equation are related is not new. However the decomposition used here is not the same as that used by Dirac. The usual method of
connecting the second order wave equation to the first order Dirac equation is to operate on Equation (29) from the left with $\sum_{\nu=1}^{3} \gamma^{\nu} \partial_{\nu}$ giving

$$0 = \sum_{\mu,\nu=1}^{3} \gamma^{\nu} \gamma^{\mu} \partial_{\nu} \partial_{\mu} \Psi(x)$$

$$= \sum_{\mu,\nu=1}^{3} \frac{1}{2} (\gamma^{\nu} \gamma^{\mu} + \gamma^{\mu} \gamma^{\nu}) \partial_{\nu} \partial_{\mu} \Psi(x)$$

$$= \left(-\partial_{1}^{2} + \partial_{2}^{2} + \partial_{3}^{2}\right) \Psi(x)$$

(30)

where $\Psi = (\alpha_1, \alpha_2)$ is a two component spinor and Equation (28) has been used in the last step.

This shows that Dirac’s equation does in fact imply the wave equation. The important thing to note about Equation (30) however, is that the three-dimensional Dirac’s equation implies not one wave equation but two in the sense that each component of the spinor $\Psi$ satisfies this equation. Explicitly stated, Equation (30) reads

$$\begin{pmatrix}
-\partial_{1}^{2} + \partial_{2}^{2} + \partial_{3}^{2} & 0 \\
0 & -\partial_{1}^{2} + \partial_{2}^{2} + \partial_{3}^{2}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 0$$

for the independent scalars $\alpha_1, \alpha_2$.

Conversely, if one starts with the wave equation and tries to recover Dirac’s equation, it is necessary to start with two independent scalars each independently satisfying the wave equation. In other words, using the usual methods, it is not possible to take a single scalar field that satisfies the wave equation and recover Dirac’s equation for a two component spinor.

What has been demonstrated in the preceding sections is that starting with only one scalar quantity satisfying the wave equation, Dirac’s equation for a two component spinor may be derived. Furthermore any medium (such as an elastic solid) that has a single scalar that satisfies the wave equation must have a spinor that satisfies Dirac’s equation and such a derivation necessitates the use of fractional derivatives.

IV. DIMENSIONAL REDUCTION IN INFINITESIMAL STRAIN

In this section we take a closer look at the equation of motion, Equation (29), and the spinor (represented as a fractional derivative) when the field displacements are Fourier trans-
formed. In the infinitesimal theory of elasticity, all terms in $u_{ij}$ beyond the linear term are dropped. In the infinitesimal theory therefore, no distinction is made between transforming the coordinates $a_i$ and $x_i$. Later when we assume small but finite strain components, we will need to distinguish these coordinates.

The Dirac field in Equation (29) is given explicitly by

$$\hat{\xi}\phi(x_1, x_2, x_3) = \begin{pmatrix} D_{z_0}^{1/2} \\ D_{z_1}^{1/2} \end{pmatrix} \phi(x_1, x_2, x_3)$$

with $z_0 = x_1 + x_2$ and $z_1 = x_1 - x_2$. Because the semiderivatives $D_{z_0, z_1}^{1/2}$ are independent of $x_3$ we can bring $e^{iz_3}$ through the operator $\hat{\xi}$ when we Fourier transform $\phi$.

Transforming the dilatation first in the periodic coordinate $q_3$, Equation (29) becomes

$$0 = \sum_{\mu=1}^3 \gamma^\mu \partial_\mu \hat{\xi}\phi$$

$$= \left( \sum_{\mu=1}^2 \gamma^\mu \partial_\mu + \gamma^3 \partial_3 \right) \hat{\xi}\phi$$

$$= \sum_q e^{iq_3 x_3} \left( \sum_{\mu=1}^2 \gamma^\mu \partial_\mu + \gamma^3 q_3 \right) \hat{\xi}\phi_{q_3}$$

$$= \sum_q e^{iq_3 x_3} \left( \sum_{\mu=1}^2 \gamma^3 \gamma^\mu \partial_\mu + q_3 \partial_3 \right) \hat{\xi}\phi_{q_3}$$

(32)

where we used $\phi = \sum_{q_3} \phi_{q_3} e^{iq_3 x_3}$, and $\gamma^3 \gamma^3 = 1$.

Equation (32) is equal to zero only if the coefficients of $e^{iq_3 x_3}$ are zero for each value of $q_3$ giving

$$\left( \sum_{\mu=1}^2 \gamma^\mu \partial_\mu - q_3 \right) \hat{\xi}\phi_{q_3} = 0$$

(33)

where $\gamma^\mu = \gamma^3 \gamma^\mu$ and satisfies the conditions for a two dimensional metric $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(34)

Equation (33) shows that the fourier modes of the elastic solid obey a two dimensional version of Dirac’s equation for spin-1/2 particles with a mass $q_3$. The two continuous variables left in the problem are $x_1$ and $x_2$ with $x_1$ playing the role of time.
Let us examine the form of the spinor further by Fourier transforming the two continuous
coordinates $x_1$ and $x_2$. Using Equation (26) we further transform the spinor as

$$\hat{\xi} \phi_{q_3}(x_1, x_2) = \begin{pmatrix} D_{z_0}^{1/2} \\ D_{z_1}^{1/2} \end{pmatrix} \phi_{q_3}(x_1, x_2)$$

$$= \int dq_1 dq_2 \begin{pmatrix} D_{z_0}^{1/2} e^{i(q_1 x_1 + q_2 x_2)} \\ D_{z_1}^{1/2} e^{i(q_1 x_1 + q_2 x_2)} \end{pmatrix} \phi_{q_1, q_2, q_3}$$

$$= \int dq_1 dq_2 \begin{pmatrix} D_{z_0}^{1/2} e^{\frac{i}{2} z_0 (q_1 + q_2)} + \frac{1}{12} z_1 (q_1 - q_2) \\ D_{z_1}^{1/2} e^{\frac{i}{2} z_0 (q_1 + q_2)} + \frac{1}{12} z_1 (q_1 - q_2) \end{pmatrix} \phi_{q_1, q_2, q_3}$$

$$= \int dq_1 dq_2 e^{i(q_1 x_1 + q_2 x_2)} \begin{pmatrix} \sqrt{q_1 + q_2} \\ \sqrt{q_1 - q_2} \end{pmatrix} \sqrt{i} \phi_{q_1, q_2, q_3} \quad (35)$$

When the fields $b_i$ are viewed as a time dependent quantity (i.e., the Heisenberg picture
in quantum mechanics), the wavevector $q_1$ is equal to the energy of the $q^{th}$ mode. This
allows us to write the column vector in Equation (35) as

$$u(q) = \begin{pmatrix} \sqrt{E + q_2} \\ \sqrt{E - q_2} \end{pmatrix}$$

Compare this formula to the expression for the four dimensional Dirac spinor in a definite
state of helicity.

$$u(p) = \begin{pmatrix} \sqrt{E + P} \\ \sqrt{E - P} \end{pmatrix}$$

The dimensional reduction in the infinitesimal theory of elasticity has produced a two
dimensional version of Dirac’s equation for a particle with a bare mass $q_3$ and a spinor that
has a consistent form to the known 4 dimensional version.

Incidentally, with the identification $q_3$ as a mass term, the energy eigenstates given in
Equation (18) are seen to have the relativistic form $E_q \sim \sqrt{q_2^2 + m^2}$.

The only problem with the interpretation as relativistic particles is the question of quantum
statistics. Relativistic spin-1/2 particles obey anticommutation relations while the boson
operators $b_i$ that we have defined obey commutation relations. This situation could be "rescued" by defining new operators

$$c_k = \theta_k b_k$$
where the $\theta_k$ are complex Grassman numbers. Grassman numbers satisfy

$$\theta_k \theta_{k'} = -\theta_{k'} \theta_k, \quad (\theta_k)^2 = 0$$

this would imply

$$\{c_k, c_{k'}\} = \{\theta_k b_k, \theta_{k'} b_{k'}\}$$

$$= \theta_k \theta_{k'} b_k b_{k'} + \theta_{k'} \theta_k b_{k'} b_k$$

$$= \theta_k \theta_{k'} b_k b_{k'} - \theta_{k'} \theta_k b_{k'} b_k$$

$$= \theta_k \theta_{k'} [b_k, b_{k'}]$$

$$= 0$$

Similarly we have,

$$\{c_k^\dagger, c_{k'}^\dagger\} = 0, \quad \text{and} \quad \{c, c_{k'}^\dagger\} = i \theta_k \theta_{k'}^* \delta_{k,k'}$$

With this definition, the fields $c_k$ satisfy appropriate quantum statistics and still obeys Dirac’s equation. We do not wish to dwell on this admittedly ad-hoc procedure for getting fermion statistics into this theory. Rather we wish to focus on the form of the equations of motion that are derived and demonstrate that the all ingredients for a theory of quantum gravity are present in this model.

In this section, we have demonstrated that the equation of motion of this model of an elastic solid, in the infinitesimal strain approximation has as its equation of motion a two dimensional version of Dirac’s equation for spin-$1/2$ particles. However, these spin-$1/2$ particles described in Equation (33) do not interact with each other or the outside world. In the next section we will treat the dimensional reduction problem again in the context of the finite theory of strain. When terms beyond the linear approximation are kept, we show that the equations obtained describe particles that interact gravitationally and electromagnetically.

V. FINITE STRAIN

A. Internal Coordinates

In this section we will need to Fourier transform our field variables in $a_3$ and therefore need to translate the equations of motion from the fixed space coordinates to the internal
coordinates. For clarity and to adopt a more consistent convention, in the remainder of this text we change notation and write the internal coordinates not as $a^i$ but as $x^i$ and the fixed space coordinates will continue to be unprimed and denoted $x^i$. Now using $u^i = x^i - x'^i$ we can write

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j}$$

$$= \sum_j \left( \frac{\partial x^j}{\partial x^i} - \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x'^j}$$

$$= \sum_j \left( \delta_{ij} - \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x'^j}$$

$$= \frac{\partial}{\partial x'^i} - \sum_j \frac{\partial u^j}{\partial x^i} \frac{\partial}{\partial x'^j}$$  \hspace{1cm} (36)$$

Equation (36) relates derivatives in the fixed space coordinates $x^i$ to derivatives in the material coordinates $x'^i$. We can now re-write the three-dimensional Dirac’s equation as

$$\sum_{\mu=1}^{3} \gamma^\mu \partial_{\mu} \Psi = \sum_{\mu=1}^{3} \gamma^\mu \left( \partial'_\mu - \sum_{\nu} \frac{\partial u^\nu}{\partial x'^\nu} \partial_{\nu} \right) \Psi$$

$$= \sum_{\mu=1}^{3} \gamma'^\mu \partial_{\mu} \Psi = 0$$  \hspace{1cm} (37)$$

where $\Psi \equiv \hat{\xi} \phi$, $\partial'_\mu = \partial/\partial x'_\mu$ and $\gamma'^\mu$ is given by

$$\gamma'^\mu = \gamma^\mu - \sum_{\alpha=1}^{3} u^\mu_{\alpha} \gamma^\alpha.$$  \hspace{1cm} (38)$$

The $\gamma^\mu$ are simply the gamma matrices expressed in the primed coordinate system. The anticommutator of these matrices is

$$\{ \gamma'^\mu, \gamma'^\nu \} = \{ \gamma^\mu - \sum_{\alpha} u^\mu_{\alpha} \gamma^\alpha, \gamma^\nu - \sum_{\beta} u^\nu_{\beta} \gamma^\beta \}$$

$$= \{ \gamma^\mu, \gamma^\nu \} - \sum_{\beta} u^\nu_{\beta} \{ \gamma^\mu, \gamma^\beta \} - \sum_{\alpha} u^\mu_{\alpha} \{ \gamma^\alpha, \gamma^\nu \} + \sum_{\alpha} u^\mu_{\alpha} u^\nu_{\beta} \{ \gamma^\alpha, \gamma^\beta \}$$

$$= 2I \left( \eta^{\mu\nu} - \sum_{\beta} u^\nu_{\beta} \eta^{\mu\beta} - \sum_{\alpha} u^\mu_{\alpha} \eta^{\alpha\nu} + \sum_{\alpha} \sum_{\beta} u^\mu_{\alpha} u^\nu_{\beta} \eta^{\alpha\beta} \right)$$

Comparison with Equation (6) shows that

$$\{ \gamma'^\mu, \gamma'^\nu \} = 2I g^{\mu\nu}$$  \hspace{1cm} (39)$$
These gamma matrices have the form of the usual Dirac’s matrices in a curved space\cite{17}. To further develop the form of Equation (37) we have to transform the spinor properties of $\xi$. Much like a normal vector, the components of a spinor are altered under a change of coordinates and as currently written $\xi$ is a spinor with respect to the $x_i$ coordinates not the $x'_i$ coordinates. To transform its spinor properties we assume (similar to Brill and Wheler\cite{17}) a real similarity transformation and write $\Psi = S\Psi'$ where $S$ is a transformation that takes the spinor in $x_{\mu}$ to a spinor in $x'_{\mu}$.

We then have

$$\partial'_\mu \Psi = (\partial'_\mu S)\Psi' + S\partial'_\mu \Psi'.$$

Equation (37) then becomes

$$0 = \gamma'^\mu [S\partial'_\mu \Psi' + (\partial'_\mu S)\Psi']$$
$$= \gamma'^\mu S[\partial'_\mu \Psi' + S^{-1}(\partial'_\mu S)\Psi']$$
$$= S^{-1}\gamma'^\mu S[\partial'_\mu \Psi' + S^{-1}(\partial'_\mu S)\Psi']$$

Using $(\partial'_\mu S^{-1})S = -S^{-1}(\partial'_\mu S)$. This can finally be written

$$\tilde{\gamma}^\mu [\partial'_\mu - \Gamma_\mu] \Psi' = 0 \quad (40)$$

where $\Gamma_\mu = (\partial'_\mu S^{-1})S$ and $\tilde{\gamma}^\mu = S^{-1}\gamma'^\mu S$. The new gamma matrices $\tilde{\gamma}^\mu$ still satisfy the appropriate anticommutation condition

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2Ig^{\mu\nu} \quad (41)$$

What we have done in the above manipulations is re-write Equation (29) completely in terms of the internal coordinates of the elastic solid. Equation (40) has the same physical content as Equation (29), it is just expressed in different coordinates.

Notice however that Equation (40) now superficially has the form of the Einstein-Dirac equation in three-dimensions for a massless, noninteracting, spin-1/2 particle\cite{17}. The quantity $\partial'_\mu - \Gamma_\mu$ is the covariant derivative for an object with spin. In order to make this identification, the field $\Gamma_\mu$ must satisfy the additional equation\cite{17,18}

$$\frac{\partial\tilde{\gamma}^\mu}{\partial x^\nu} + \tilde{\gamma}^\beta\Gamma^\mu_{\beta\nu} - \Gamma_\nu\tilde{\gamma}^\mu + \tilde{\gamma}^\mu\Gamma_\nu = 0$$

where $\Gamma^\mu_{\beta\nu}$ is the usual Christoffel symbol. That this equation holds is shown in Appendix (B).

Just as was done in the infinitesimal strain approach we will use a dimensional reduction to transform Equation (40). In this case however, the gamma matrices couple the metric into the problem and this metric will also need to be reduced from three to two dimensions.

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VI. DIMENSIONAL REDUCTION IN FINITE STRAIN

The dimensional reduction method that we use is borrowed from Kaluza Klein theory. It consists of two parts. First we need to remove the dependence of the field variables on the coordinate $x'_3$. This is accomplished by Fourier transforming the field variables as before. The second part consists of reducing the $3 \times 3$ matrix, $g_{\mu\nu}$, in three dimensions to a $2 \times 2$ matrix suitable for a two dimensional space. It is this reduction of the metric that will introduce the electromagnetic vector potential into our equations.

Let us denote the $3 \times 3$, three dimensional metric as $\tilde{g}_{\mu\nu}$ and the $2 \times 2$, two dimensional metric as $g_{\mu\nu}$. We now write the Kaluza Klein ansatz\textsuperscript{19,20}

$$\tilde{g}_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} - \Phi^2 A_\alpha & -\Phi^2 A_\alpha \\ -\Phi^2 A_\beta & -\Phi^2 \end{pmatrix} \quad g^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & (-\Phi^{-2} + A^\mu A_\mu) \end{pmatrix}$$ (42)

where the vector $A_\mu$ is the electromagnetic vector potential, $\Phi$ is a scalar and the upper/lower indices indicate a contraction with $g_{\mu\nu}$.

With this ansatz for the metric, it can be shown that the flat space condition $R_{\alpha\beta} = 0$ in three dimensions, reduces to the Einstein Field equations in curved two-dimensional space and Maxwell’s equations for the field $A_\mu$. The details of this derivation are given by Liu and Wesson\textsuperscript{20} where the reduction is performed in a five dimensional setting. This is a standard dimensional reduction used in Kaluza Klein theories and has been shown to produce a consistent treatment of four-dimensional gravity and electromagnetism from five-dimensional flat space. Therefore, we will not repeat this derivation here only noting that there is nothing in the derivation that is particular to five dimensions. The exact same treatment works in a dimensional reduction from three to two dimensions so that we will freely quote the results of that work which gives the following equations for the metric and the electromagnetic vector potential

$$F^\lambda_{\alpha;\lambda} = -3\Phi^{-1}\Phi^\lambda F_{\lambda\alpha}$$

$$R_{\alpha\beta} = -\frac{1}{2}\Phi^2 F_{\lambda\beta} F^\lambda_\beta + \Phi^{-1}\Phi_{\alpha;\beta}$$

where $F_{\alpha\beta} \equiv A_{\beta;\alpha} - A_{\alpha;\beta}$ and we have used the notation where a comma indicates an ordinary derivative and a semicolon indicates covariant differentiation.
The second of these equations, may be written in the more traditional form

\[ G^\alpha\beta \equiv R^\alpha\beta - \frac{1}{2}g^\alpha\beta R = 8\pi \left( T^\alpha\beta_{\text{em}} + T^\alpha\beta_{s} \right) \]  \hspace{1cm} (43)

where the quantities \( T^\alpha\beta_{\text{em}} \) and \( T^\alpha\beta_{s} \) are effective energy momentum tensors given by

\[ T^\alpha\beta_{\text{em}} = -\frac{1}{2}\Phi^2 \left( F^\alpha_\lambda F^\beta\lambda - \frac{1}{4}g^\alpha\beta F^\mu\nu F_\mu\nu \right) \]
\[ T^\alpha\beta_{s} = \Phi^{-1} \left( \Phi^\alpha\beta - g^\alpha\beta \Phi^\mu_\mu \right) \]

So what we see is that by using the dimensional reduction technique of Kaluza Klein theory we automatically obtained the Einstein field equations and Maxwell’s equations. What we now show is that the same dimensional reduction technique not only produces these equations but also changes the non-interacting Einstein-Dirac equation in three dimensions, into an interacting theory for a massive spin-1/2 particle in two dimensions.

A. Dimensional Reduction of the Dirac Equation

We begin by rewriting Equation (40) as

\[ \sum_{\mu=1}^{2} \tilde{\gamma}^\mu [\partial'_\mu - \Gamma_\mu] \Psi' + \tilde{\gamma}^3 [\partial'_3 - \Gamma_3] \Psi' = 0 \]  \hspace{1cm} (44)

We now rewrite the matrix \( \tilde{\gamma}^3 \). From Equations (42) and (11) we can write

\[ \tilde{\gamma}^3 \tilde{\gamma}^3 = g^{33} = A^\mu A_\mu - \Phi^{-2} \]

This allows us to write

\[ \tilde{\gamma}^3 = \sum_{\alpha=1}^{2} \tilde{\gamma}^\alpha A_\alpha + \gamma_\perp \Phi^{-1} \]  \hspace{1cm} (45)

where the matrix \( \gamma_\perp \) satisfies

\[ \{\tilde{\gamma}^1, \gamma_\perp\} = 0, \{\tilde{\gamma}^2, \gamma_\perp\} = 0, \gamma_\perp^2 = 1 \]

An explicit expression for \( \gamma_\perp \) is given in Appendix (C). It can be seen that this form for \( \tilde{\gamma}^3 \) correctly gives \( \tilde{\gamma}^3 \tilde{\gamma}^3 = g^{33} \). We can now write Equation (44) as

\[ 0 = \sum_{\mu=1}^{2} \tilde{\gamma}^\mu [\partial'_\mu - \Gamma_\mu] \Psi' + (\sum_{\alpha=1}^{2} \tilde{\gamma}^\alpha A_\alpha + \gamma_\perp \Phi^{-1}) [\partial'_3 - \Gamma_3] \Psi' \]
\[ = \sum_{\mu=1}^{2} \tilde{\gamma}^\mu [\partial'_\mu + A_\mu (\partial'_3 - \Gamma_3) - \Gamma_\mu] \Psi' + \gamma_\perp \Phi^{-1} [\partial'_3 - \Gamma_3] \Psi' \]
\[ = \sum_{\mu=1}^{2} \gamma'^\mu [\partial'_\mu + A_\mu (\partial'_3 - \Gamma_3) - \Gamma_\mu] \Psi' + \Phi^{-1} [\partial'_3 - \Gamma_3] \Psi' \]  \hspace{1cm} (46)

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where $\gamma''^\mu = \gamma_0 \gamma^\mu$. This equation has the same form as the classical Einstein-Dirac-Maxwell equation\textsuperscript{17,18,21,22,23}. The metric enters in the equation through the $\gamma''^\mu$ matrices which satisfy
\[ \{\gamma''^\mu, \gamma''^\nu\} = 2I g''^{\mu\nu} \] and the electromagnetic vector potential enters in the same way as the minimal coupling prescription $\partial_\mu \rightarrow \partial_\mu + ie/c A_\mu$. Additionally, a mass term has been created in Equation (46) with $m = \Phi^{-1}[\partial^3_3 - \Gamma_3]$.

The field variables in Equation (46) still are dependent on $x'_3$. To complete the derivation we need to remove this dependence by Fourier Transforming our field variables. We first write the spinor field
\[ \Psi = \sum_q \Psi_q e^{iqx'_3} \]
this gives
\[ \sum_q e^{iqx'_3} \left\{ \sum_{\mu=1}^2 \gamma''^\mu [\partial'_\mu + A_\mu(iq - \Gamma_3) - \Gamma_\mu] \Psi' + \Phi^{-1}[iq - \Gamma_3] \Psi' \right\} = 0 \]
Now we write
\[ A_\mu = \sum_k A_{\mu,k} e^{ikx'_3}, \quad \Gamma_\mu = \sum_k \Gamma_{\mu,k} e^{ikx'_3}, \quad \gamma''^\mu = \sum_{k'} \gamma''_{k'} e^{ik'x'_3}, \quad \Phi^{-1} = \sum_{k'} \Phi^{-1}_{k'} e^{ik'x'_3} \]
Which gives
\[ \sum_{q,k,k',k''} e^{iqx'_3(q+k+k'+k'')} e^{ikx'_3} \left\{ \sum_{\mu=1}^2 \gamma''^\mu [\partial'_\mu \delta_{k,0} \delta_{k'',0} + A_{\mu,k}(iq \delta_{k'',0} - \Gamma_{3,k''}) - \Gamma_{\mu,k} \delta_{k'',0}] \Psi'_q \\
+ \delta_{k'',0} \Phi^{-1}_{k'} [iq \delta_{k,0} - \Gamma_{3,k}] \Psi'_q \right\} = 0 \]
This equation is true only if the coefficients of the exponential are independently true. Let $m = q + k + k' + k''$ then
\[ \sum_{k,k',k''} \left\{ \sum_{\mu=1}^2 \gamma''^\mu \partial'_\mu \delta_{k,0} \delta_{k'',0} + A_{\mu,k}(i(m - k - k' - k'') \delta_{k'',0} - \Gamma_{3,k''}) - \Gamma_{\mu,k} \delta_{k'',0} \right\} \Psi'_{m-k-k'-k''} \\
+ \delta_{k'',0} \Phi^{-1}_{k'} [i(m - k - k' - k'') \delta_{k,0} - \Gamma_{3,k}] \Psi'_{m-k-k'-k''} \right\} = 0. \quad (47) \]
This is a series of equations, one for each distinct value of $m$. The dynamics contained in Equation (47) describes an infinite series of spin particles $\Psi'_{m-k-k'-k''}$ interacting via an infinite series of vector potentials $A_{\mu,k}$. Up until now we have kept all terms in this series of equations. We will now examine Equation (47) in the low energy approximation.
B. Spectrum of Lowest modes

Each of the quantities in Equation (47) is a function of the field displacements \( u_i \). We showed in previous sections that the energy of the Fourier modes of field displacements increases with \( q_3 \). We therefore expect that in a system where the energy is arbitrarily low, not all of the modes in Equation (47) will be excited. At the lowest energies only the mode \( \vec{u}_{q=0} \) will be excited, as the energy of the system increases modes \( \vec{u}_{q=\pm1} \) becomes excited and so on. As this energy is increased Equation (47) rapidly becomes more complex but at sufficiently low energies its form is quite simple.

We illustrate this by considering a theory in which only the lowest modes are present. Strictly speaking, our Hamiltonian formalism in Section (II E) produced energy eigenstates for modes \( \vec{u}_q \) in a basis \( e^{iqx} \) while we want to make a statement about the energy of the modes, \( \vec{u}_q' \) in the basis \( e^{iq'x'} \), with \( x'=x^i - u^i \). In Appendix (D) we show that the two are related and that if the only mode present in the system is \( q=0 \) in the first basis then, the only mode present is \( q'=0 \) in the second.

Let us now consider the case where there is insufficient energy to excite the mode \( \vec{u}_{q=\pm1} \) and only \( \vec{u}_{q=0} \) is excited. This would represent the lowest energy possible in our system. In this case the wavevectors in Equation (47) must equal to \( m = k = k' = k'' = 0 \) and Equation (47) reduces to

\[
\sum_{\mu=1}^{2} \chi_{\mu}^0 [\tilde{g}_{\mu}^0 - A_{\mu,0} \Gamma_{3,0} - \Gamma_{\mu,0}] \Psi'_0 - \Phi^{-1}_{0} \Gamma_{3,0} \Psi'_0 = 0. \tag{48}
\]

From Equation (6) we also have

\[
\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \sum_{\alpha=1}^{2} \left( -\eta^{\nu\alpha} \frac{\partial u_0^{\mu}}{\partial x^\alpha} - \eta^{\mu\alpha} \frac{\partial u_0^{\nu}}{\partial x^\alpha} + \sum_{\beta=1}^{2} \eta^{\alpha\beta} \frac{\partial u_0^{\mu}}{\partial x^\alpha} \frac{\partial u_0^{\nu}}{\partial x^\beta} \right), \quad (\mu, \nu = 1, 2, 3) \tag{49}
\]

which via Equation (42), provides an explicit expression for the two dimensional metric and the electromagnetic vector potential

\[
g^{\mu\nu} = \eta^{\mu\nu} + \sum_{\alpha=1}^{2} \left( -\eta^{\nu\alpha} \frac{\partial u_0^{\mu}}{\partial x^\alpha} - \eta^{\mu\alpha} \frac{\partial u_0^{\nu}}{\partial x^\alpha} + \sum_{\beta=1}^{2} \eta^{\alpha\beta} \frac{\partial u_0^{\mu}}{\partial x^\alpha} \frac{\partial u_0^{\nu}}{\partial x^\beta} \right), \quad (\mu, \nu = 1, 2) \tag{49}
\]

and

\[
A^\mu = -\sum_{\alpha=1}^{2} \left( -\eta^{\nu\alpha} \frac{\partial u_0^{3}}{\partial x^\alpha} + \sum_{\beta=1}^{2} \eta^{\alpha\beta} \frac{\partial u_0^{\mu}}{\partial x^\alpha} \frac{\partial u_0^{3}}{\partial x^\beta} \right), \quad (\mu = 1, 2) \tag{50}
\]
and where

\[ \eta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (51)

Note that the 2D metric in Equation (49) is no longer due to a simple coordinate change. There is no coordinate transformation (or any other transformation) that involves only the coordinates \( x'_1 \) and \( x'_2 \) that will remove the Fourier transforms in this equation and globally create the form given in Equation (1).

Let us summarize what we have done in this section. We began with an equation for Dirac’s equation in three-dimensions, Equation (40), which described a spin-1/2 particle with zero mass and no interactions with the outside world. Using dimensional reduction we obtained a two-dimensional equation for a massive spin-1/2 particle in curved space, which interacts via the gravitational metric \( g_{\mu\nu} \) and the electromagnetic potential \( A_\mu \).

We also began with a trivial flat space metric in three-dimensions, Equation (4) and using dimensional reduction, derived a two-dimensional curved metric, Equation (49), and an electromagnetic vector potential, Equation (50).

Equations (48), (49) and (50) represent the main result of this paper. They provide a quantum mechanical treatment of a combined system of the Dirac Equation, electromagnetism and gravity, albeit in a low dimensional setting. We note however that this whole procedure can be carried out in higher dimensions. The only part of the derivation that is lacking in higher dimensions is an explicit solution of the Dirac spinor in terms of fractional derivatives like those given in Equations (III A). If a higher dimensional version of this theory is constructed it would provide an alternate approach to current theories of quantum gravity.

VII. CONCLUSIONS

We have taken a model of an elastic medium and derived an equation of motion that has the same form as Dirac’s equation in the presence of electromagnetism and gravity. We derived this equation by using the formalism of Cartan to reduce the quadratic form of the wave equation to the linear form of Dirac’s equation. We showed that the dimensional reduction technique from Kaluza Klein theory produces not only the Einstein Field and
Maxwell’s equations but also induces both mass and interaction terms into Dirac’s Equation. The formalism demonstrates that a quantum mechanical treatment of the Einstein-Dirac-Maxwell equations can be derived from the equations of motion of the Fourier modes of an elastic solid and provides a new approach to theories of quantum gravity.

APPENDIX A: QUANTIZATION COEFFICIENTS

In diagonalizing the Hamiltonian given in Equation (15) we first write the field operators in terms of an intermediate set of ladder operators

\[ P_{n,q} = \frac{i}{\sqrt{2}} (a_{n,q}^\dagger - a_{n,-q}) \]
\[ u_{n,q} = \frac{1}{\sqrt{2}} (a_{n,q} + a_{n,-q}) \]

Now define the vectors

\[ Q_q = (a_{1,q}, a_{2,q}, a_{3,q}, a_{1,-q}, a_{2,-q}, a_{3,-q}, a_{1,q}^\dagger, a_{2,q}^\dagger, a_{3,q}^\dagger, a_{1,-q}^\dagger, a_{2,-q}^\dagger, a_{3,-q}^\dagger) \]
\[ Q_q^\dagger = (a_{1,q}^\dagger, a_{2,q}^\dagger, a_{3,q}^\dagger, a_{1,-q}^\dagger, a_{2,-q}^\dagger, a_{3,-q}^\dagger, a_{1,q}, a_{2,q}, a_{3,q}, a_{1,-q}, a_{2,-q}, a_{3,-q}) \]

This allows us to write \( H_q = Q_q^\dagger AQ_q \) where the matrix \( A \) is given by

\[ A = \begin{pmatrix} T & S \\ S & T \end{pmatrix} \]

and the nonzero elements of the matrices \( T \) and \( S \) are

\[ T_{1,1} = \frac{3}{16(\lambda + 2\mu)} \]
\[ T_{1,5} = \frac{\mu q_2}{4(\lambda + 2\mu)} \]
\[ T_{1,6} = \frac{\mu q_3}{4(\lambda + 2\mu)} \]
\[ T_{2,2} = \frac{8(2q_2^2 + q_3^2)\mu^3 + 2\mu + \lambda (4(4q_2^2 + q_3^2)\mu^2 + 1)}{16\mu(\lambda + 2\mu)} \]
\[ T_{2,3} = \frac{\mu(3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ T_{2,4} = -\frac{\mu q_2}{4(\lambda + 2\mu)} \]
\[ T_{3,2} = \frac{\mu(3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ T_{3,3} = \frac{8 (q_2^2 + 2q_3^2) \mu^3 + 2 \mu + \lambda (4 (q_2^2 + 4q_3^2) \mu^2 + 1)}{16\mu(\lambda + 2\mu)} \]
\[ T_{3,4} = -\frac{\mu q_3}{4(\lambda + 2\mu)} \]
\[ T_{4,2} = -\frac{\mu q_2}{4(\lambda + 2\mu)} \]
\[ T_{4,3} = -\frac{\mu q_3}{4(\lambda + 2\mu)} \]
\[ T_{4,4} = 3 \]
\[ T_{5,1} = \frac{\mu q_2}{4(\lambda + 2\mu)} \]
\[ T_{5,5} = \frac{8 (2q_2^2 + q_3^2) \mu^3 + 2 \mu + \lambda (4 (q_2^2 + 4q_3^2) \mu^2 + 1)}{16\mu(\lambda + 2\mu)} \]
\[ T_{5,6} = \frac{\mu (3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ T_{6,1} = \frac{\mu q_3}{4(\lambda + 2\mu)} \]
\[ T_{6,5} = \frac{\mu (3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ T_{6,6} = \frac{8 (q_2^2 + 2q_3^2) \mu^3 + 2 \mu + \lambda (4 (q_2^2 + 4q_3^2) \mu^2 + 1)}{16\mu(\lambda + 2\mu)} \]
\[ S_{1,8} = -\frac{(2\lambda + 2\mu)q_2}{8(\lambda + 2\mu)} \]
\[ S_{1,9} = -\frac{(2\lambda + 2\mu)q_3}{8(\lambda + 2\mu)} \]
\[ S_{1,10} = 3 \]
\[ S_{2,7} = -\frac{(2\lambda + 2\mu)q_2}{8(\lambda + 2\mu)} \]
\[ S_{2,11} = \frac{2\mu (4\mu^2 (2q_2^2 + q_3^2) - 1) + \lambda (4\mu^2 (4q_2^2 + q_3^2) - 1)}{16\mu(\lambda + 2\mu)} \]
\[ S_{2,12} = \frac{\mu (3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ S_{3,7} = -\frac{(2\lambda + 2\mu)q_3}{8(\lambda + 2\mu)} \]
\[ S_{3,11} = \frac{\mu (3\lambda + 2\mu)q_2q_3}{4(\lambda + 2\mu)} \]
\[ S_{3,12} = \frac{2\mu (4\mu^2 (q_2^2 + 2q_3^2) - 1) + \lambda (4\mu^2 (q_2^2 + 4q_3^2) - 1)}{16\mu(\lambda + 2\mu)} \]
\[ S_{4,7} = -\frac{3}{16(\lambda + 2\mu)} \]
The methods outlined by Tikochinsky and Tsallis\textsuperscript{10,11} can now be applied to the matrix $A$. The final results allow us to define six creation operators $b_i^\dagger$ and six annihilation operators $b_i$ that satisfy

$$[b_{i,q}, b_{j,q'}^\dagger] = i\delta_{i,j}\delta_{q,q'} \quad [b_{i,q}, b_{j,q'}] = 0 \quad [b_{i,q}^\dagger, b_{j,q'}^\dagger] = 0$$

These operators are Eigenstates of the Hamiltonian with eigenvalues

$$E_{1,q} = \frac{1}{4}\sqrt{q_2^2 + q_3^2}$$

$$E_{2,q} = \frac{1}{4}\sqrt{q_2^2 + q_3^2}$$

$$E_{3,q} = \frac{1}{4}\sqrt{1 - \frac{2\sqrt{2}\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}{(\lambda + 2\mu)^2}\sqrt{q_2^2 + q_3^2}}$$

$$E_{4,q} = \frac{1}{4}\sqrt{1 - \frac{2\sqrt{2}\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}{(\lambda + 2\mu)^2}\sqrt{q_2^2 + q_3^2}}$$

$$E_{5,q} = \frac{1}{4}\sqrt{1 + \frac{2\sqrt{2}\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}{(\lambda + 2\mu)^2}\sqrt{q_2^2 + q_3^2}}$$

$$E_{6,q} = \frac{1}{4}\sqrt{1 + \frac{2\sqrt{2}\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}{(\lambda + 2\mu)^2}\sqrt{q_2^2 + q_3^2}}$$
and the Hamiltonian has the diagonal form

\[ H_q = \sum_i E_{i,q} b_{i,q}^\dagger b_{i,q} \]

The field displacement operators, \( u_{ij} \) can be written in terms of these ladder operators. Denote the vector of field and ladder operators as

\[ X_q = (P_{1,q}, P_{2,q}, P_{3,q}, u_{1,q}, u_{2,q}, u_{3,q}, P_{1,-q}, P_{2,-q}, P_{3,-q}, u_{1,-q}, u_{2,-q}, u_{3,-q}) \]

\[ B_q = (b_{1,q}, b_{2,q}, b_{3,q}, b_{4,q}, b_{5,q}, b_{6,q}, b_{1,q}^\dagger, b_{2,q}^\dagger, b_{3,q}^\dagger, b_{4,q}^\dagger, b_{5,q}^\dagger, b_{6,q}^\dagger) \]

This allows us to write

\[ X_{i,q} = \sum_j c_{i,j} B_{i,q} \]

where the coefficients \( c_{i,j} \) are listed below. In writing the \( c_{i,j} \) coefficients we have defined the following quantities

\[ a = \sqrt{2} \sqrt{\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)} \]

\[ \lambda_2 = \sqrt{1 - \frac{2\sqrt{2} \sqrt{\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}}{(\lambda + 2\mu)^2}} \]

\[ \lambda_3 = \sqrt{\frac{2\sqrt{2} \sqrt{\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)}}{(\lambda + 2\mu)^2} + 1} \]

\[ d_1 = 8\mu(2\lambda + 2\mu) \left( 16E_\text{q}^2\mu^3 + 16E_\text{q}^2\lambda\mu^2 - 2\mu - \lambda \right) \]

\[ + a \left( 32E_\text{q}^2\mu^3 + 32E_\text{q}^2\lambda\mu^2 - 10\mu - 2\lambda \right) \]

\[ d_2 = 8\mu(2\lambda + 2\mu) \left( 16E_\text{q}^2\mu^3 + 16E_\text{q}^2\lambda\mu^2 - 2\mu - \lambda \right) \]

\[ + a \left( -32E_\text{q}^2\mu^3 + 10\mu + \lambda (2 - 32E_\text{q}^2\mu^2) \right) \]

\[ d_3 = a \left( 32E_\text{q}^2\mu^3 + 32E_\text{q}^2\lambda\mu^2 - 10\mu - 2\lambda \right) \]

\[ - 8\mu(2\lambda + 2\mu) \left( 16E_\text{q}^2\mu^3 + 16E_\text{q}^2\lambda\mu^2 - 2\mu - \lambda \right) \]

\[ d_4 = 12a\mu + (\lambda + 2\mu) \left( 64E_\text{q}^2\mu^4 + 96E_\text{q}^2\lambda\mu^3 + 4 \left( 8E_\text{q}^2\lambda^2 + 3 \right) \mu^2 + 18\lambda\mu - 2\lambda^2 \right) \]

\[ d_5 = (\lambda + 2\mu) \left( 64E_\text{q}^2\mu^4 + 96E_\text{q}^2\lambda\mu^3 + 4 \left( 8E_\text{q}^2\lambda^2 + 3 \right) \mu^2 + 18\lambda\mu - 2\lambda^2 \right) - 12a\mu \]

\[ n_1 = 2\sqrt{2} \sqrt{\frac{\mu (q_2^3 + q_3^3)}{q_2^3 (2\mu\omega_q - 1)^2}} \]

\[ n_2 = 2\sqrt{2} \sqrt{\frac{\mu (q_2^3 + q_3^3)}{q_2^3 (2\mu\omega_q - 1)^2}} \]
\[ n_3 = 8\sqrt{2}(8\mu^2(2\lambda + 2\mu)\omega_q^2(2(2\mu(\lambda + 2\mu)\lambda^2_2\omega_q^2 + 2\mu(3\lambda + 4\mu)\omega_q^2
+ \lambda_2(16\omega_q^2\mu^3 + 2\mu + \lambda(16\mu^2\omega_q^2 + 1))\omega_q)a^2 - (\lambda + 2\mu)(4\mu(\lambda + 2\mu)(2\lambda + 2\mu)\lambda^3_2\omega_q^2
+ 4\mu(2\lambda + 2\mu)(\lambda + 10\mu)\omega_q^2 + (\lambda + 2\mu)\lambda_2(2\mu(16\mu^2\omega_q^2 + 5) + \lambda(32\mu^2\omega_q^2 + 2))\omega_q)a
+ 4\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)(2\mu(\lambda + 2\mu)\lambda^3_2\omega_q^2 + 2\mu(3\lambda + 4\mu)\omega_q^2
+ \lambda_2(16\omega_q^2\mu^3 + 2\mu + \lambda(16\mu^2\omega_q^2 + 1))\omega_q)) \right)^{1/2}
/ \left( (\lambda + 2\mu)q_3^2(a(32\omega_q^2\mu^3 - 10\mu
+ \lambda(32\mu^2\omega_q^2 - 2)) - 8\mu(2\lambda + 2\mu)(16\omega_q^2\mu^3 - 2\mu + \lambda(16\mu^2\omega_q^2 - 1))) \right)^{1/2}
\]

\[ n_4 = 8\left( -\mu(\lambda + 2\mu)^2(8(\lambda + 2\mu)(2\lambda + 2\mu)(4\lambda\omega_q^2 - 3\lambda_2\omega_q)\mu^2
+ a(\lambda_2\omega_q((32\mu^2\omega_q^2 + 2)\lambda^2 + 6\mu(16\mu^2\omega_q^2 - 3)\lambda + 4\mu^2(16\mu^2\omega_q^2 - 3))
- 8(\lambda - 6\mu)(2\lambda + 2\mu)\omega_q^2))^\frac{1}{2}
/ \left( (12a\mu + (\lambda + 2\mu)(32\mu^2\omega_q^2 - 2)\lambda^2
+ 6\mu(16\mu^2\omega_q^2 + 3)\lambda + 4\mu^2(16\mu^2\omega_q^2 + 3)) \right)^{1/2}
\]

\[ n_5 = 8\sqrt{2}(\mu^2(2\lambda + 2\mu)\omega_q^2(2(2\mu(\lambda + 2\mu)\lambda^2_2\omega_q^2 + 2\mu(3\lambda + 4\mu)\omega_q^2
+ \lambda_3(16\omega_q^2\mu^3 + 2\mu + \lambda(16\mu^2\omega_q^2 + 1))\omega_q)a^2 - (\lambda + 2\mu)(4\mu(\lambda + 2\mu)(2\lambda + 2\mu)\lambda^3_2\omega_q^2
+ 4\mu(2\lambda + 2\mu)(\lambda + 10\mu)\omega_q^2 + (\lambda + 2\mu)\lambda_3(2\mu(16\mu^2\omega_q^2 + 5) + \lambda(32\mu^2\omega_q^2 + 2))\omega_q)a
+ 4\mu(\lambda + 2\mu)^2(2\lambda + 2\mu)(2\mu(\lambda + 2\mu)\lambda^3_2\omega_q^2 + 2\mu(3\lambda + 4\mu)\omega_q^2 + \lambda_3(16\omega_q^2\mu^3 + 2\mu
+ \lambda(16\mu^2\omega_q^2 + 1))\omega_q)) \right)^{1/2}
/ \left( (\lambda + 2\mu)q_3^2(8\mu(2\lambda + 2\mu)(16\omega_q^2\mu^3 - 2\mu
+ \lambda(16\mu^2\omega_q^2 - 1)) + a(32\omega_q^2\mu^3 - 10\mu + \lambda(32\mu^2\omega_q^2 - 2)) \right)^{1/2}
\]

\[ n_6 = 8\left( \mu(\lambda + 2\mu)^2(8(\lambda + 2\mu)(2\lambda + 2\mu)(3\lambda_3\omega_q - 4\lambda\omega_q^2)\mu^2
+ a(\lambda_3\omega((32\mu^2\omega_q^2 + 2)\lambda^2 + 6\mu(16\mu^2\omega_q^2 - 3)\lambda + 4\mu^2(16\mu^2\omega_q^2 - 3))
- 8(\lambda - 6\mu)(2\lambda + 2\mu)\omega_q^2))^\frac{1}{2}
/ \left( ((\lambda + 2\mu)(32\mu^2\omega_q^2 - 2)\lambda^2 + 6\mu(16\mu^2\omega_q^2 + 3)\lambda
+ 4\mu^2(16\mu^2\omega_q^2 + 3)) - 12a\mu \right)^{1/2}
\]

Finally, the nonzero components of the matrix \( c_{ij} \) are

\[ c_{1,3} = \frac{iv\sqrt{2}\omega_q(a(2\lambda - 6\mu)(\lambda + 2\mu)\lambda_2 - 4\mu(2\lambda + 2\mu)(4\mu(\lambda + 2\mu) - a(\lambda - 6\mu))\omega_q)}{(\lambda + 2\mu)d_2n_3q_1}
\]

\[ c_{1,5} = \frac{-iv\sqrt{2}\omega_q(a(2\lambda - 6\mu)(\lambda + 2\mu)\lambda_3 + 4\mu(2\lambda + 2\mu)(a(\lambda - 6\mu) + 4\mu(\lambda + 2\mu))\omega_q)}{(\lambda + 2\mu)d_1n_3q_1}
\]

\[ c_{1,10} = \frac{i\sqrt{2}(12a\mu + (\lambda + 2\mu)(-2\lambda^2 + 18\mu\lambda + 12\mu^2 + 4\mu(2\mu + (\lambda + 2\mu)(2\lambda + 2\mu))\lambda_2^2\omega_q))}{d_4n_4}
\]

\[ c_{1,12} = \frac{i\sqrt{2}((\lambda + 2\mu)(-2\lambda^2 + 18\mu\lambda + 12\mu^2 + 4\mu(2\mu + (\lambda + 2\mu)(2\lambda + 2\mu) - 2a)\lambda_3\omega_q)) - 12a\mu}{d_5n_6}
\]
\[ C_{2,2} = -\frac{i\sqrt{2}q_1}{n_2q_2 - 2m_2q_2\omega} \]

\[ C_{2,4} = -i\sqrt{2}(\lambda + 2\mu)q_2((\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega) - 8\mu\omega^2) \]

\[ C_{2,6} = -\frac{i\sqrt{2}(\lambda + 2\mu)q_2(8\mu\omega^2 + (\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega))}{d_5n_6\omega^2} \]

\[ C_{2,7} = -\frac{i\sqrt{2}q_2}{n_1q_2 - 2m_1q_2\omega} \]

\[ C_{2,9} = i\sqrt{2}(\lambda + 2\mu)q_2((\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega) - 8\mu(\lambda + 2\mu)(2\lambda + 2\mu)(2\mu\lambda_2\omega + 1)) \]

\[ C_{2,11} = -i\sqrt{2}(\lambda + 2\mu)q_2(8\mu(\lambda + 2\mu)(2\mu\lambda_2\omega + 1) + 2\mu(2\lambda + 2\mu)\lambda_2\omega)) \]

\[ C_{3,2} = \frac{i\sqrt{2}}{n_1 - 2m_1\omega} \]

\[ C_{3,4} = i\sqrt{2}(\lambda + 2\mu)q_2((\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega) - 8\mu\omega^2) \]

\[ C_{3,6} = -\frac{i\sqrt{2}(\lambda + 2\mu)q_2(8\mu\omega^2 + (\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega))}{d_5n_6\omega^2} \]

\[ C_{3,7} = \frac{i\sqrt{2}}{n_1 - 2m_1\omega} \]

\[ C_{3,9} = i\sqrt{2}(\lambda + 2\mu)q_2((\lambda + 2\mu)(-2\lambda\lambda_2 + 6\mu\lambda_2 + 4\mu(2\lambda + 2\mu)\omega) - 8\mu(\lambda + 2\mu)(2\lambda + 2\mu)(2\mu\lambda_2\omega + 1)) \]

\[ C_{3,11} = -i\sqrt{2}(\lambda + 2\mu)q_2(8\mu(\lambda + 2\mu)(2\mu\lambda_2\omega + 1) + 2\mu(2\lambda + 2\mu)\lambda_2\omega)) \]

\[ C_{4,4} = 4\sqrt{2}\mu(\lambda + 2\mu)\omega^2((\lambda + 2\mu)(-\mu\omega - \lambda_2) - 2\mu\lambda_2) \]

\[ C_{4,6} = 4\sqrt{2}\mu(\lambda + 2\mu)\omega^2(2\mu\lambda_3 + (\lambda + 2\mu)(2\lambda + 2\mu)(2\mu\lambda_2\omega + 1)) \]

\[ C_{4,9} = -4\sqrt{2}\mu(\lambda + 2\mu)\omega^2(4\mu\lambda_2\omega a + a + 4\mu(\lambda + 2\mu)) \]

\[ C_{4,11} = 4\sqrt{2}\mu(\lambda + 2\mu)\omega^2(4\mu\lambda_2\omega a + a + 4\mu(\lambda + 2\mu)) \]

\[ C_{5,1} = \frac{2\sqrt{2}\mu_2q_2\omega}{n_2q_2 - 2m_2q_2\omega} \]

\[ C_{5,3} = 4\sqrt{2}\mu(2\lambda + 2\mu)q_2(\omega((\lambda + 2\mu)\lambda_2 + 4\mu(2\lambda + 2\mu)\omega)) \]

\[ C_{5,5} = 4\sqrt{2}\mu(2\lambda + 2\mu)q_2(\omega((\lambda + 2\mu)\lambda_2 + 4\mu(2\lambda + 2\mu)\omega)) \]

\[ C_{5,8} = -\frac{2\sqrt{2}\muq_2\omega}{n_2q_2 - 2m_2q_2\omega} \]

\[ C_{5,10} = -4\sqrt{2}\muq_2((2\lambda + (\lambda + 2\mu)(2\lambda + 2\mu)(-\lambda + 6\mu + 4\mu(\lambda + 2\mu)\lambda_2\omega)) \]

\[ C_{5,12} = -4\sqrt{2}\muq_2((2\lambda + (\lambda + 2\mu)(-\lambda + 6\mu + 4\mu(\lambda + 2\mu)\lambda_2\omega)) - 2\lambda) \]

\[ C_{6,1} = \frac{2\sqrt{2}\muq_2\omega}{n_1 - 2m_1\omega} \]

\[ C_{6,3} = 4\sqrt{2}\mu(2\lambda + 2\mu)\omega(\omega((\lambda + 2\mu)\lambda_2 + 4\mu(2\lambda + 2\mu)\omega)) \]

\[ C_{6,5} = 4\sqrt{2}\mu(2\lambda + 2\mu)\omega((\lambda + 2\mu)\lambda_2 + 4\mu(2\lambda + 2\mu)\omega)) \]

\[ C_{6,8} = \frac{2\sqrt{2}\muq_2\omega}{n_2 - 2m_2q_2\omega} \]

\[ C_{6,10} = -4\sqrt{2}\muq_2((2\lambda + (\lambda + 2\mu)(2\lambda + 2\mu)(-\lambda + 6\mu + 4\mu(\lambda + 2\mu)\lambda_2\omega)) \]

\[ C_{6,12} = -4\sqrt{2}\muq_2((2\lambda + (\lambda + 2\mu)(-\lambda + 6\mu + 4\mu(\lambda + 2\mu)\lambda_2\omega)) - 2\lambda) \]
\[ C_{7,4} = \frac{i \sqrt{2} \left( 12\mu \mp (\lambda \mp 2\mu) \left( -2\lambda^2 + 18\mu \lambda + 12\mu^2 + 4\mu (2\alpha + (\lambda + 2\mu)(2\lambda + 2\mu)) \lambda_2 \omega_1 \right) \right)}{d_{4n4}} \]
\[ C_{7,6} = i \frac{\sqrt{2} \left( \lambda + 2\mu \right) \left( -2\lambda^2 + 18\mu \lambda + 12\mu^2 + 4\mu (\lambda + 2\mu)(2\lambda + 2\mu) - 2\alpha \lambda_3 \omega_3 \right) - 12\mu}{d_{5n6}} \]
\[ C_{7,9} = -i \frac{\sqrt{2} \omega_3 \left( a(2\lambda - 6\mu)(\lambda + 2\mu) \lambda_3 - 4\mu (2\lambda + 2\mu) \left( 4\mu (\lambda + 2\mu) - \alpha (\lambda - 6\mu) \omega_3 \right) \right)}{d_{5n3q3}} \]
\[ C_{7,11} = -i \frac{\sqrt{2} \omega_1 \left( a(2\lambda - 6\mu)(\lambda + 2\mu) \lambda_1 + 4\mu (2\lambda + 2\mu) (\alpha (\lambda - 6\mu) + 4\mu (\lambda + 2\mu)) \omega_3 \right)}{(\lambda + 2\mu) d_{1n5q1}} \]
\[ C_{8,1} = \frac{i \sqrt{2} q_1}{n_1 q_2 - 2m_1 q_2 \omega_q} \]
\[ C_{8,3} = \frac{i \sqrt{2} \left( \lambda + 2\mu \right) q_2 \left( a(2\lambda + 10\mu + 4\mu (2\lambda + 2\mu) \omega_q \omega_q) - 8\mu (\lambda + 2\mu)(2\lambda + 2\mu) (2\mu \lambda_2 \omega_q + 1) \right)}{d_{2n3q3}} \]
\[ C_{8,5} = \frac{i \sqrt{2} \omega_1 q_2 \left( 8\mu (\lambda + 2\mu)(2\mu \lambda_2 \omega_q + 1) + a(2\lambda + 10\mu + 4\mu (2\lambda + 2\mu) \lambda_1 \omega_q) \right)}{d_{1n5q3}} \]
\[ C_{8,8} = \frac{i \sqrt{2} q_1}{n_2 q_2 - 2m_2 q_2 \omega_q} \]
\[ C_{8,10} = \frac{i \sqrt{2} \left( \lambda + 2\mu \right) q_2 \left( a(2\lambda + 10\mu + 4\mu (2\lambda + 2\mu) \lambda_2 \omega_q) - 8\mu (\lambda + 2\mu)(2\lambda + 2\mu) (2\mu \lambda_2 \omega_q + 1) \right)}{d_{3n4q4}} \]
\[ C_{9,9} = \frac{i \sqrt{2} q_1}{n_1 - 2m_1 \omega_q} \]
\[ C_{9,10} = \frac{i \sqrt{2} q_2 \left( a(2\lambda + 10\mu + 4\mu (2\lambda + 2\mu) \lambda_1 \omega_q) - 8\mu (2\lambda + 2\mu) (2\mu \lambda_2 \omega_q + 1) \right)}{d_{2n3q4}} \]
\[ C_{9,12} = \frac{i \sqrt{2} q_1 \left( a(2\lambda + 10\mu + 4\mu (2\lambda + 2\mu) \lambda_1 \omega_q) - 8\mu (2\lambda + 2\mu) (2\mu \lambda_2 \omega_q + 1) \right)}{d_{3n4q6}} \]
\[ C_{10,3} = \frac{4 \sqrt{2} \left( \lambda + 2\mu \right) \omega_2^2 (4 \mu \lambda_2 \omega_2 \alpha + a - 4 \mu (\lambda + 2\mu))}{d_{3n4q3}} \]
\[ C_{10,5} = -4 \frac{\sqrt{2} \left( \lambda + 2\mu \right) \omega_2^2 (4 \mu \lambda_3 \omega_2 \alpha + a + 4 \mu (\lambda + 2\mu))}{d_{1n3q3}} \]
\[ C_{10,10} = -4 \frac{\sqrt{2} \left( \lambda + 2\mu \right) \omega_2 (2 \mu \lambda_3 + (\lambda + 2\mu) (4 \mu \omega_2 - \lambda_2) - 2 \nu \lambda_2)}{d_{1n4}} \]
\[ C_{10,12} = -4 \frac{\sqrt{2} \left( \lambda + 2\mu \right) \omega_2 (2 \mu \lambda_3 + (\lambda + 2\mu) (4 \mu \omega_2 - \lambda_3))}{d_{3n6}} \]
\[ C_{11,2} = \frac{2 \sqrt{2} \mu q_3 \omega_q}{n_2 q_2 - 2m_2 q_2 \omega_q} \]
\[ C_{11,4} = \frac{4 \sqrt{2} q_2 (2\alpha \lambda + (\lambda + 2\mu) (-\alpha + 4 \mu (\lambda + 2\mu) \lambda_3 \omega_q))}{d_{4n4}} \]
\[ C_{11,6} = \frac{4 \sqrt{2} q_2 ((\lambda + 2\mu) (2\lambda + 2\mu) (-\lambda + 6 \mu + 4 \mu (\lambda + 2\mu) \lambda_2 \omega_q))}{d_{5n6}} \]
\[ C_{11,7} = \frac{2 \sqrt{2} \mu q_3 \omega_q}{n_1 q_2 - 2m_1 q_2 \omega_q} \]
\[ C_{11,9} = -4 \frac{\sqrt{2} q_2 (2\lambda + 2\mu) \omega_2 \omega_3 \left( a(\lambda_3 + 4 \mu \omega_2) - 4 \mu ((\lambda + 2\mu) \lambda_3 + 4 \mu (2\lambda + 2\mu) \omega_q) \right)}{d_{3n3q3}} \]
\[ C_{11,11} = -4 \frac{\sqrt{2} q_1 (2\lambda + 2\mu) q_2 \omega_2 \left( a(\lambda_3 + 4 \mu \omega_2) + 4 \mu ((\lambda + 2\mu) \lambda_3 + 4 \mu (2\lambda + 2\mu) \omega_q) \right)}{d_{1n5q3}} \]
\[ C_{12,2} = \frac{2 \sqrt{2} \mu q_3 \omega_q}{n_2 - 2m_2 \omega_q} \]
\[ C_{12,4} = \frac{4 \sqrt{2} q_2 (2 \mu + (\lambda + 2\mu) (2\lambda + 2\mu) (-\lambda + 6 \mu + 4 \mu (\lambda + 2\mu) \lambda_2 \omega_q))}{d_{4n4}} \]
\[ c_{12,6} = \frac{4\sqrt{2} \mu n_1 ((\lambda + 2\mu)(2\lambda + 2\mu)(-\lambda + 6\mu + 4\mu(\lambda + 2\mu)\lambda_3 \omega_q) - 2a\lambda)}{d_5 n_6} \]
\[ c_{12,7} = \frac{2\sqrt{2} \mu_\omega_q}{n_1 - 2\mu n_1 \omega_q} \]
\[ c_{12,9} = -\frac{4\sqrt{2} \mu (2\lambda + 2\mu) \omega_q (a(\lambda + 4\mu \omega_q) - 4\mu(\lambda + 2\mu)\lambda_3 \omega_q) - 2a\lambda)}{d_5 n_3} \]
\[ c_{12,11} = -\frac{4\sqrt{2} \mu (2\lambda + 2\mu) \omega_q (a(\lambda + 4\mu \omega_q) + 4\mu(\lambda + 2\mu)\lambda_3 + 4\mu(2\lambda + 2\mu)\omega_q))}{d_5 n_5} \]

**APPENDIX B: AUXILIARY EQUATION**

We wish to show that
\[ \frac{\partial \tilde{\gamma}^\mu}{\partial x^\nu} + \tilde{\gamma}^\beta \Gamma^\mu_{\beta \nu} - \Gamma^\nu_{\nu} \tilde{\gamma}^\mu + \tilde{\gamma}^\mu \Gamma^\nu_{\nu} = 0 \]

We first consider the equation
\[ \frac{\partial \gamma^\mu}{\partial x^\nu} = 0 \]
true in the unprimed coordinate system. But since the unprimed coordinate system is Euclidean space, the Christoffel symbols are identically zero. This allows us to write
\[ \frac{\partial \gamma^\mu}{\partial x^\nu} + \gamma^\beta \Gamma^\mu_{\beta \nu} = 0 \]

Since this is a tensor equation true in all frames, in the primed coordinate system we can immediately write
\[ \partial'_\nu \gamma^\mu + \gamma'_{\beta} \Gamma'^\mu_{\beta \nu} = 0 \]

Using \( \gamma^\mu = S \tilde{\gamma}^\mu S^{-1} \), we have
\[ \partial'_\nu (S \tilde{\gamma}^\mu S^{-1}) + (S \tilde{\gamma}^\beta S^{-1}) \Gamma'^\mu_{\beta \nu} = 0 \]

or
\[ (\partial'_\nu S) \tilde{\gamma}^\mu S^{-1} + S(\partial'_\nu \tilde{\gamma}^\mu) S^{-1} + S \tilde{\gamma}^\mu (\partial'_\nu S^{-1}) + (S \tilde{\gamma}^\beta S^{-1}) \Gamma'^\mu_{\beta \nu} = 0. \]

Multiplying by \( S^{-1} \) on the left and \( S \) on the right yields
\[ S^{-1}(\partial'_\nu S) \tilde{\gamma}^\mu + (\partial'_\nu \tilde{\gamma}^\mu) + \tilde{\gamma}^\mu (\partial'_\nu S^{-1}) + \tilde{\gamma}^\beta \Gamma'^\mu_{\beta \nu} = 0 \]

Finally, using \( \Gamma^\nu = (\partial'_\nu S^{-1})S \) and again noting that \( \partial'_\nu S^{-1}S = -S^{-1} \partial'_\nu S \) we have,
\[ \tilde{\gamma}^\mu \Gamma^\nu - \Gamma^\nu \tilde{\gamma}^\mu + \left( \partial'_\nu \tilde{\gamma}^\mu + \tilde{\gamma}^\beta \Gamma'^\mu_{\beta \nu} \right) = 0 \]  (B1)
APPENDIX C: DECOMPOSITION OF \( \tilde{\gamma}'^3 \)

In decomposing the matrix \( \tilde{\gamma}'^3 \) we seek a matrix \( \gamma_\perp \) that satisfies

\[
\{ \tilde{\gamma}'^1, \gamma_\perp \} = 0, \quad \{ \tilde{\gamma}'^2, \gamma_\perp \} = 0, \quad \gamma_\perp^2 = 1 \tag{C1}
\]

where from Equations (41) and (38) we can write

\[
\tilde{\gamma}'^1 = S^{-1}(1 - u_1^1)\gamma^1 + u_2^1\gamma^2 + u_3^1\gamma^3 S \\
\tilde{\gamma}'^2 = S^{-1}(u_1^2\gamma^1 + (1 - u_2^2)\gamma^2 + u_3^2\gamma^3 S
\]

We can treat the matrices \( \gamma^\mu \) as vectors and use the cross product formula to compute an orthogonal vector. In other words we can write

\[
\gamma_\perp = \tilde{\gamma}'^1 \times \tilde{\gamma}'^2
\]

using the rules

\[
\gamma^1 \times \gamma^2 = \gamma^3 \quad \gamma^2 \times \gamma^3 = \gamma^1 \quad \gamma^1 \times \gamma^3 = -\gamma^2
\]

This gives

\[
v_\perp = S^{-1}\left([u_2^1u_3^2 - u_3^1(1 - u_2^2)]\gamma^1 + [u_3^1u_1^2 - (1 - u_1^1)u_2^2]\gamma^2 + [(1 - u_1^1)(1 - u_2^2) - u_2^1u_1^2]\gamma^3 S
\]

with

\[
\gamma_\perp \equiv \frac{v_\perp}{|v_\perp|}
\]

and \( |v_\perp| = \sqrt{v_\perp^2} \). It can be verified directly that \( \gamma_\perp \) satisfies Equation (C1).

APPENDIX D: FOURIER COMPONENTS

We wish to relate the Fourier components of the field displacements \( u_i \), when expanded in the basis \( e^{iq\cdot x'} \) to those expanded in the basis \( e^{iqx} \). The two coordinate systems are related by \( u_\mu = x_\mu - x'_\mu \) where the field displacements \( u_i \) are assumed to be small. We write

\[
u_\mu = \sum q u_{\mu q} e^{iqx} = \sum q u_{\mu q} e^{iq(x'+u)}
\]
correction to second order in $u_{\mu,q}$.

Setting $q + k = q'$ we have

$$u_{\mu} = \sum_{q'k} u_{\mu,q'-k}(\delta_{k,0} + iq' - k) e^{iq'-x'}$$

so that the components of the field displacements in the primed frame are

$$u_{m'} = \sum_{k} u_{m'-k}(\delta_{k,0} + i(m' - k))$$

We can now relate the energy eigenstates which are expressed in the unprimed frame to the Fourier components in the primed frame. In a low energy theory in which only $u_0$ is present, to second order in $u_i$, the only Fourier component in the primed frame that is nonzero is $u_{m'=0}$. If there is enough energy to excite the fields $u_0$ and $u_{\pm 1}$. Then the only nonzero modes in the primed frame are $u_{m'=0}$ and $u_{m'=\pm 1}$ and so on.

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