\((n,m)\)-SG RINGS

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ABSTRACT. This paper is a continuation of the paper Int. Electron. J. Algebra 6 (2009), 219–227. Namely, we introduce and study a doubly filtered set of classes of rings of finite Gorenstein global dimension, which are called \((n,m)\)-SG for integers \(n \geq 1\) and \(m \geq 0\). Examples of \((n,m)\)-SG rings, for \(n = 1\) and 2 and every \(m \geq 0\), are given.

1. INTRODUCTION

Throughout the paper all rings are associative with identity, and all modules are, if not specified otherwise, left modules.

Let \(R\) be a ring and let \(M\) be an \(R\)-module. For an \(R\)-module \(M\), we use \(\text{pd}_R(M)\), \(\text{id}_R(M)\), and \(\text{fd}_R(M)\) to denote, respectively, the classical projective, injective and flat dimension of \(M\). We use \(l.\text{gldim}(R)\) and \(r.\text{gldim}(R)\) to denote, respectively, the classical left and right global dimension of \(R\), and \(\text{wgldim}(R)\) to denote the weak global dimension of \(R\).

The Gorenstein homological dimensions theory originated in the works of Auslander and Bridger [1] and [2], where they introduced the G-dimension of any finitely generated module \(M\) and over any Noetherian ring \(R\). The G-dimension is analogous to the classical projective dimension and shares some of its principal properties (see [12] for more details). However, to complete the analogy an extension of the G-dimension to non-necessarily finitely generated modules is needed. This is done in [16, 15], where the Gorenstein projective dimension was defined over arbitrary rings (as an extension of the G-dimension to modules that are not necessarily finitely generated), and the Gorenstein injective dimension was defined as a dual notion of the Gorenstein projective dimension:

**Definition 1.1.** Let \(R\) be a ring.

- An \(R\)-module \(M\) is called Gorenstein projective (G-projective for short) if there exists an exact sequence of projective \(R\)-modules,
  \[
P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots,
  \]
  such that \(M \cong \text{Im}(P_0 \rightarrow P_{-1})\) and such that \(\text{Hom}_R(-, Q)\) leaves the sequence \(P\) exact whenever \(Q\) is a projective \(R\)-module.
  The exact sequence \(P\) is called a complete projective resolution of \(M\).

  For a positive integer \(n\), we say that \(M\) has Gorenstein projective dimension at most \(n\), and we write \(\text{Gpd}_R(M) \leq n\) (or simply \(\text{Gpd}(M) \leq n\)), if there is an exact sequence of \(R\)-modules,
  \[
  0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0,
  \]
  where each \(G_i\) is G-projective.
Dually, the Gorenstein injective module (G-injective for short) is defined, and so the Gorenstein injective dimension, $\text{Gid}_R(M) \leq n$, of an $R$-module $M$ is defined.

Also to complete the analogy with the classical homological dimensions theory, the Gorenstein flat dimension was introduced in [17] as follows:

**Definition 1.2.** Let $R$ be a ring. An $R$-module $M$ is called Gorenstein flat (G-flat for short) if there exists an exact sequence of flat $R$-modules,

\[ F = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots, \]

such that $M \cong \text{Im}(F_0 \to F^0)$ and such that $I \otimes_R -$ leaves the sequence $F$ exact whenever $I$ is an injective right $R$-module.

The exact sequence $F$ is called a complete flat resolution of $M$.

For a positive integer $n$, we say that $M$ has Gorenstein flat dimension at most $n$, and we write $\text{Gfd}_R(M) \leq n$, if there is an exact sequence of $R$-modules,

\[ 0 \to G_n \to \cdots \to G_0 \to M \to 0, \]

where each $G_i$ is G-flat.

The Gorenstein homological dimensions have been extensively studied by many others, who proved that these dimensions share many nice properties of the classical homological dimensions (see for instance [12, 13, 14]). Recently, in [3], a particular case of modules of finite Gorenstein projective dimension is introduced as follows:

**Definition 1.3.** Let $R$ be a ring and let $n \geq 1$ and $m \geq 0$ be integers. An $R$-module $M$ is called $(n, m)$-strongly Gorenstein projective ($(n, m)$-SG-projective for short) if there exists an exact sequence of $R$-modules,

\[ 0 \to M \to Q_n \to \cdots \to Q_1 \to M \to 0, \]

where $\text{pd}_R(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\text{Ext}_R^i(M, Q) = 0$ for every $i > m$ and every projective $R$-module $Q$.

The $(1,0)$-SG-projective modules are already investigated in [4] (see [4 Proposition 2.9]). They are called strongly Gorenstein projective modules (SG-projective modules for short) (see also [20] and [21]). In [6], $(n,0)$-SG-projective modules are first studied, and they are called $n$-strongly Gorenstein projective modules ($n$-SG-projective modules for short). In general, $(n,m)$-SG-projective modules are a particular case of modules with Gorenstein projective dimension at most $m$ [3 Theorem 2.4]. The $(1,m)$-SG-projective modules are served to characterize modules of Gorenstein projective dimension at most $m$ in a similar way to the way SG-projective modules characterize G-projective modules (see [3 Corollary 2.8] and [4 Theorem 2.7]). Namely, we have that a module $M$ has Gorenstein projective dimension at most a positive integer $m$ if and only if $M$ is a direct summand of a $(1,m)$-SG-projective module. As mentioned at the end of the paper [3], dually the $(n,m)$-SG-injective modules are defined.

In this paper, we continue the investigation of $(n,m)$-SG-projective and $(n,m)$-SG-injective modules. Namely, we are interested in studying rings over which all modules are $(n,m)$-SG-projective (resp., $(n,m)$-SG-injective). First, we show, for a ring $R$, that the assertions “all $R$-modules are $(n,m)$-SG-projective” and “all $R$-modules are $(n,m)$-SG-injective” are equivalent (Proposition 2.1). A ring that satisfies one of these equivalent
assertions is called left \((n,m)\)-SG (Definition 2.2). In the main result of this paper (Theorem 2.5), left \((n,m)\)-SG rings are characterized in terms of left Gorenstein global dimension: the left Gorenstein global dimension of a ring \(R\), \(\text{gldim}(R)\), is defined in [7] as the common value of the equal quantities [2 Theorem 1.1]:

\[
\sup\{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}.
\]

Namely, after giving a characterization of \((n,m)\)-SG-projective (resp., \((n,m)\)-SG-injective) modules (Lemmas 2.3 and 2.4), we show, in Theorem 2.5, that the \((n,m)\)-SG rings are particular cases of rings with left Gorenstein global dimension at most \(m\). Then, for Noetherian rings, \((n,m)\)-SG rings are particular cases of \(m\)-Gorenstein rings (see Theorem 2.6 and Corollary 2.7). So the \((n,0)\)-SG rings are particular examples of the well-known quasi-Frobenius rings (see Corollary 2.8). In particular, \((1,0)\)-SG commutative rings are investigated in [9], and they are called SG-semisimple. It is proved that a local \((1,0)\)-SG commutative ring is just a ring with only one non-trivial ideal [9 Theorem 3.7].

After investigating some relationships between \((n,m)\)-SG rings (Proposition 2.10), the remain of the paper is devoted to establish examples of \((n,m)\)-SG rings. For that, we study the notion of \((n,m)\)-SG rings in direct product of rings, such that we prove (Proposition 2.13):

A direct product of rings \(\prod_{i=1}^n R_i\) is left \((n,m)\)-SG if and only if each \(R_i\) is left \((n,m)\)-SG.

Then, a family of left \((1,i)\)-SG rings which are not left \((1,i-1)\)-SG, for every \(i \geq 1\), are given (Example 2.14). In Proposition 2.15 we show that if \(R\) is a commutative ring with \(\text{gldim}(R) = m\) for an integer \(m \geq 1\), such that \(R\) contains a non-zero divisor element \(x\), then the quotient ring \(R/xR\) is \((2,m-1)\)-SG. As a consequence, we give examples of left \((2,0)\)-SG rings which are not left \((1,0)\)-SG; and, for \(i \geq 1\), we construct examples of left \((2,i)\)-SG rings which are neither left \((1,i)\)-SG nor \((2,i-1)\)-SG.

2. Main results

We start with the following result:

**Proposition 2.1.** Let \(n \geq 1\) and \(m \geq 0\) be integers. For a ring \(R\), the following assertions are equivalent:

1. Every \(R\)-module is \((n,m)\)-SG-projective;
2. Every \(R\)-module is \((n,m)\)-SG-injective.

**Proof.** We prove only the implication (1) \(\Rightarrow\) (2). The implication (2) \(\Rightarrow\) (1) has a dual proof.

First, using [18 Proposition 2.3], note that every \(R\)-module \(P\) with finite projective dimension has injective dimension at most \(m\). Indeed, \(\text{Ext}^i(N,P) = 0\) for any \(i > m\) and every \(R\)-module \(N\) (since, by hypothesis, \(N\) is \((n,m)\)-SG-projective). Also, note that every injective \(R\)-module \(I\) has projective dimension at most \(m\). In fact, as \((n,m)\)-SG-projective, \(I\) embeds in an \(R\)-module with projective dimension at most \(m\), and since \(I\) is injective it is a direct summand of a such \(R\)-module.

Now, consider an \(R\)-module \(M\). Since \(M\) is \((n,m)\)-SG-projective, there exists an exact sequence of \(R\)-modules,

\[
0 \rightarrow M \rightarrow Q_n \rightarrow \cdots \rightarrow Q_1 \rightarrow M \rightarrow 0,
\]
where \( \text{pd}_R(Q_i) \leq m \). By the reason above \( \text{id}_R(Q_i) \leq m \); and also \( \text{Ext}_R^i(J, M) = 0 \) for any \( i > m \) and every injective \( R \)-module \( J \) (since, by the reason above, \( \text{pd}_R(J) \leq m \)). Therefore, \( M \) is \( (n, m) \)-SG-injective.

**Definition 2.2.** A ring \( R \) is called left (resp., right) \( (n, m) \)-SG for some integers \( n \geq 1 \) and \( m \geq 0 \), if \( R \) satisfies one of the equivalent conditions of Proposition 2.1 for left (resp., right) \( R \)-modules. If \( R \) is both left and right \( (n, m) \)-SG, we simply say that it is \( (n, m) \)-SG.

Later, we give examples of \( (n, m) \)-SG rings. Now, we set the main result of this paper which gives a characterization of \( (n, m) \)-SG rings in terms of left Gorenstein global dimension (for a background on left Gorenstein global dimension, see \([5, 7, 8]\)). For that, we need the following key lemma which gives a characterization of \( (n, m) \)-SG-projective modules (see also its \((n, m) \)-SG-injective version, Lemma 2.4). This result is a generalization of \([3, \text{Theorem 2.7}]\) and so it gives an affirmative answer to the question concerning the converse of \([3, \text{Theorem 2.4}]\) (see the note before \([3, \text{Lemma 2.5}]\)).

Recall, for a projective resolution of a module \( M \),

\[
\cdots \to P_1 \to P_0 \to M \to 0,
\]

that the module \( K_i = \text{Im}(P_i \to P_{i-1}) \) for \( i \geq 1 \), is called an \( i \)th syzygy of \( M \).

**Lemma 2.3.** Let \( R \) be a ring and let \( n \geq 1 \) and \( m \geq 0 \) be integers. For an \( R \)-module \( M \) the following assertions are equivalent:

1. \( M \) is \( (n, m) \)-SG-projective;
2. \( \text{Gpd}_R(M) \leq m \) and an \( m \)th syzygy of \( M \) is \( (n, 0) \)-SG-projective.
3. There exists a short exact sequence of \( R \)-modules, \( 0 \to P \to G \to M \to 0 \), where \( G \) is \( (n, 0) \)-SG-projective and \( \text{pd}_R(P) \leq m - 1 \);
4. There exists a short exact sequence of \( R \)-modules, \( 0 \to M \to Q \to H \to 0 \), where \( H \) is \( (n, 0) \)-SG-projective and \( \text{pd}_R(Q) \leq m \).

**Proof.** (1) \( \Rightarrow \) (2). Follows from \([3, \text{Theorem 2.4}]\).

(2) \( \Rightarrow \) (3). A similar proof to the one of \([3, \text{Theorem 2.4(3)}]\) shows that any \( k \)th syzygy of \( M \) is \( (n, 0) \)-SG-projective, where \( k = \text{Gpd}_R(M) \leq m \). Then, we have an exact sequence of \( R \)-modules,

\[
0 \to G_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0,
\]

where \( P_i \) are projective and \( G_k \) is \( (n, 0) \)-SG-projective. Consider a right half of a complete projective resolution of \( G_k \),

\[
0 \to G_k \to Q_{k-1} \to \cdots \to Q_0 \to N \to 0,
\]

where \( Q_i \) are projective and, by \([3, \text{Lemma 1.2(2)}]\), \( N \) is \( (n, 0) \)-SG-projective. Then, from \([8, \text{Proposition 1.8}]\), we get the following commutative diagram:

\[
\begin{array}{c}
0 \to G_k \to Q_{k-1} \to \cdots \to Q_0 \to N \to 0 \\
\| \quad \downarrow \quad \quad \downarrow \\
0 \to G_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0
\end{array}
\]

This diagram gives a chain map between complexes,

\[
\begin{array}{c}
0 \to Q_{k-1} \to \cdots \to Q_0 \to N \to 0 \\
\downarrow \quad \quad \quad \downarrow \\
0 \to P_{k-1} \to \cdots \to P_0 \to M \to 0
\end{array}
\]
which induces an isomorphism in homology. Then, its mapping cone is exact (see [22, Section 1.5]). That is, the following exact sequence:

\[0 \to Q_{k-1} \to P_{k-1} \oplus Q_{k-2} \to \cdots \to P_1 \oplus Q_0 \to P_0 \oplus N \to M \to 0\]

Therefore, the sequence, \(0 \to P \to G \to M \to 0\), where \(G = P_0 \oplus N\) and \(P = \text{Ker}(P_0 \oplus N \to M)\), is the desired sequence.

(3) \(\Rightarrow\) (4). Since \(G\) is \((n,0)\)-SG-projective, there exists a short exact sequence of \(R\)-modules, \(0 \to G \to F \to H \to 0\), where \(F\) is projective and \(H\) is \((n,0)\)-SG-projective. Then, with the sequence \(0 \to P \to G \to M \to 0\) we get the following pushout diagram:

\[
\begin{array}{ccccccccccc}
0 & & 0 & & & & & & & & & & & & \\
& & P & & & & & & & & & & & & \\
& & & & & & & & & & & & & & \\
& & G & & F & & H & & 0 & & & & & & \\
& & & & & & & & & & & & & & \\
& & 0 & & M & & Q & & H & & 0 & & & & & & \\
& & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & \\
& & 0 & & 0 & & 0 & & 0 & & & & & & \\
\end{array}
\]

From the middle exact sequence, \(\text{pd}_R(Q) = \text{pd}_R(P) + 1 \leq m\). Therefore, the bottom sequence is the desired short exact sequence.

(4) \(\Rightarrow\) (1). First, using the short exact sequence \(0 \to M \to Q \to H \to 0\), one can show that \(\text{Ext}^i_R(M, L) = 0\) for every \(i > m\) and every projective \(R\)-module \(L\). Then, it remains, by definition, to prove the existence of an exact sequence of the form:

\[0 \to M \to L_n \to \cdots \to L_1 \to M \to 0,\]

where every \(L_i\) has projective dimension at most \(m\).

Since \(H\) is \((n,0)\)-SG-projective, there exists an exact sequence of modules:

\[0 \to H \to Q_n \to \cdots \to Q_1 \to H \to 0\]

Decomposing this sequence into short exact sequences

\[0 \to H_{i+1} \to Q_i \to H_i \to 0,\]

where \(H_{n+1} = H = H_1\). And consider the following family of short exact sequences

\[(\alpha_i) \quad 0 \to G_i \to F_i \to H_i \to 0,\]

where, for \(i = 2, \ldots, n\), \(P_i\) is projective and \(G_i\) is \(G\)-projective, and, for \(i = 1\) and \(n\), the short exact sequence \((\alpha_i)\) is the sequence \(0 \to M \to Q \to H \to 0\). Then, since \(\text{Ext}^1_R(H_i, L) = 0\) for every \(R\)-module \(L\) with finite projective dimension [18, Proposition 2.3], we get, from the Horseshoe Lemma (a dual version of [18, Lemma 1.7]), the following family of
commutative diagrams ($\beta_i$):

\[
\begin{array}{ccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & H_{i+1} & \to & Q_i \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & F_{i+1} & \to & Q_i \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & G_{i+1} & \to & L_i \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & 0 & \to & 0
\end{array}
\]

Then, we obtain a family of short exact sequences ($\theta_i$):

\[
\begin{align*}
(\theta_1) : & \quad 0 \to G_2 \to L_1 \to M \to 0, \\
(\theta_i) : & \quad 0 \to G_{i+1} \to L_i \to G_i \to 0, \text{ for } i = 2, \ldots, n - 1 \text{ and} \\
(\theta_n) : & \quad 0 \to M \to L_n \to G_n \to 0
\end{align*}
\]

such that, from the middle sequences of the commutative diagrams ($\beta_i$), $L_i$ is projective for $i = 2, \ldots, n - 1$, $\text{pd}_R(L_1) \leq m$, and $\text{pd}_R(L_n) \leq m$. Therefore, we get the desired sequence by assembling the short exact sequences ($\theta_i$).

Also, one can prove the following dual version of Lemma 2.3. Recall, for an injective resolution of a module $M$,

\[
0 \to M \to I_0 \to I_1 \to \cdots,
\]

that the module $K_i = \text{Im}(P_{i-1} \to P_i)$ for $i \geq 1$, is called an $i^{th}$ cosyzygy of $M$.

**Lemma 2.4.** Let $R$ be a ring and let $n \geq 1$ and $m \geq 0$ be integers. For an $R$-module $M$ the following assertions are equivalent:

1. $M$ is $(n,m)$-SG-injective;
2. $\text{Gldim}(M) \leq m$ and an $m^{th}$ cosyzygy of $M$ is $(n,0)$-SG-injective;
3. There exists a short exact sequence of $R$-modules, $0 \to M \to G \to I \to 0$, where $G$ is $(n,0)$-SG-injective and $\text{id}_R(I) \leq m - 1$;
4. There exists a short exact sequence of $R$-modules, $0 \to H \to J \to M \to 0$, where $H$ is $(n,0)$-SG-injective and $\text{id}_R(J) \leq m$.

Now, we can prove our main result:

**Theorem 2.5.** Let $n \geq 1$ and $m \geq 0$ be integers. For a ring $R$, the following assertions are equivalent:

1. $R$ is $(n,m)$-SG;
2. $\text{Gldim}(R) \leq m$ and every $G$-projective $R$-module is $(n,0)$-SG-projective;
3. $\text{Gldim}(R) \leq m$ and every $G$-injective $R$-module is $(n,0)$-SG-injective.

**Proof.** We prove only the equivalence $(1) \iff (2)$. The equivalence $(1) \iff (3)$ has a dual proof.

$(1) \Rightarrow (2)$. We have $\text{Gldim}(R) \leq m$ since every $(n,m)$-SG-projective $R$-module has Gorenstein projective dimension at most $m$ (by Lemma 2.3(1) $\Rightarrow (2)$). From [3] Lemma 2.6(1), we get that every $G$-projective $R$-module is $(n,0)$-SG-projective.

$(2) \Rightarrow (1)$. Let $M$ be an $R$-module. Since $\text{Gldim}(R) \leq m$, every $m^{th}$ syzygy of $M$ is $G$-projective, which is, by hypothesis, $(n,0)$-SG-projective. Therefore, by Lemma 2.3(2) $\Rightarrow (1)$, $M$ is $(n,m)$-SG-projective. \qed
As a consequence, the Noetherian \((n,m)\)-SG rings are particular cases of \(m\)-Gorenstein rings: a ring \(R\) is said to be \(m\)-Gorenstein, for a positive integer \(m\), if it is left and right Noetherian with self-injective dimension at most \(m\) on both the left and the right sides [14 Definitions 9.1.1 and 9.1.9]. The \(m\)-Gorenstein rings are characterized in terms of Gorenstein homological dimensions (see [14 Theorem 12.3.1]) and in terms of classical homological dimensions (see [14 Theorem 9.1.11]). In the following result, we rewrite this list of properties that characterize the \(m\)-Gorenstein rings, and we enlarge it using the notions of \((n,m)\)-SG-projective and \((n,m)\)-SG-injective modules.

**Theorem 2.6** ([13]. Theorems 9.1.11 and 12.3.1). If \(R\) is a left and right Noetherian ring, then, for a positive integer \(m\), the following are equivalent:

1. \(R\) is \(m\)-Gorenstein;
2. \(l.Ggldim(R) \leq m\);
3. \(r.Ggldim(R) \leq m\);
4. \(id_r(M) \leq m\) for every projective left (resp., right) \(R\)-module \(M\);
5. \(pd_r(M) \leq m\) for every injective left (resp., right) \(R\)-module \(M\);
6. \(Gid_l(M) \leq m\) for every \(G\)-projective left (resp., right) \(R\)-module \(M\);
7. \(Gpd_r(M) \leq m\) for every \(G\)-injective left (resp., right) \(R\)-module \(M\);
8. For every integer \(n \geq 1\), every \((n,0)\)-SG-projective left (resp., right) \(R\)-module is \((n,m)\)-SG-injective;
9. For every integer \(n \geq 1\), every \((n,0)\)-SG-injective left (resp., right) \(R\)-module is \((n,m)\)-SG-projective.

**Proof.** The equivalences \((1) \iff (2) \iff (3)\) follow from [14 Theorem 9.1.11]. Then, trivially, these equivalent assertions imply the assertions \((6)\) and \((7)\).

The equivalences \((1) \iff (4) \iff (5)\) are the same as [14 Theorems 12.3.1 \((1) \iff (2) \iff (3)\)]. Then, easily we show that these equivalent assertions imply the assertions \((8)\) and \((9)\).

For the implications \((6) \Rightarrow (4)\) and \((7) \Rightarrow (5)\), use [19 Theorems 2.1 and 2.2].

To prove the implication \((8) \Rightarrow (5)\), consider a projective \(R\)-module \(M\). Then, it is \((n,0)\)-SG-projective for every \(n \geq 1\), and so, by hypothesis, \(M\) is \((n,m)\)-SG-injective. This means that \(M\) is a quotient of an \(R\)-module \(I\) with injective dimension at most \(m\). Then, as a projective \(R\)-module, \(M\) is a direct summand of \(I\). Therefore, \(id_r(M) \leq m\).

Similarly we prove the implication \((9) \Rightarrow (6)\).

\[\square\]

Note that [7 Proposition 2.6] shows that we do not need, in Theorem 2.6, to assume first that the ring is Noetherian when \(m = 0\). In this case the ring \(R\) is quasi-Frobenius (i.e., \(0\)-Gorenstein).

As a consequence, we have for Noetherian \((n,m)\)-SG rings the following result:

**Corollary 2.7.** Let \(n \geq 1\) and \(m \geq 0\) be integers. For a left and right Noetherian ring \(R\), the following assertions are equivalent:

1. \(R\) is left \((n,m)\)-SG rings;
2. \(R\) is right \((n,m)\)-SG rings;
3. \(R\) is \(m\)-Gorenstein and every \(G\)-projective (left or right) \(R\)-module is \((n,0)\)-SG-projective
4. \(R\) is \(m\)-Gorenstein and every \(G\)-injective (left or right) \(R\)-module is \((n,0)\)-SG-injective.

**Proof.** Apply Theorems 2.5 and 2.6. \[\square\]
In [9, Section 3], (1,0)-SG commutative rings are studied and they are called SG-semisimple. It is proved that a local (1,0)-SG commutative ring is just a ring with only one non-trivial ideal [9, Theorem 3.7]. For (n,0)-SG rings we have:

**Corollary 2.8.** Let \( n \geq 1 \) be an integer. For a ring \( R \), the following assertions are equivalent:

1. \( R \) is left \( (n,0) \)-SG rings;
2. \( R \) is right \( (n,0) \)-SG rings;
3. \( R \) is quasi-Frobenius and every \( G \)-projective (left or right) \( R \)-module is \( (n,0) \)-SG-projective
4. \( R \) is quasi-Frobenius and every \( G \)-injective (left or right) \( R \)-module is \( (n,0) \)-SG-injective.

**Proof.** Apply [7, Proposition 2.6] and Corollary 2.7. □

Also as a consequence of the main result, we get the following result which study the relation between rings of finite left global dimension and left \((n,m)\)-SG rings.

**Corollary 2.9.** Let \( R \) be a ring and let \( m \geq 0 \) be an integer. Then, \( l\text{gldim}(R) \leq m \) if and only if \( R \) is left \((n,m)\)-SG for every integer \( n \geq 1 \) and \( \text{wgldim}(R) < \infty \).

**Proof.** Follows easily from Theorem 2.5 and [7, Corollary 1.2]. □

The following establish some relations between left \((n,m)\)-SG rings.

**Proposition 2.10.** For two integers \( n \geq 1 \) and \( m \geq 0 \), we have the following assertions:

1. Every left \((n,m)\)-SG ring is \((n,m')\)-SG for every \( m' \geq m \).
2. Every left \((n,m)\)-SG ring is left \((nk,m)\)-SG for every \( k \geq 1 \).

In particular, every left \((1,m)\)-SG ring is left \((n,m)\)-SG for every \( n \geq 1 \).

**Proof.** A simple consequence of [3, Proposition 2.2]. □

Naturally, one would like to have examples of \((n,m)\)-SG rings which are neither \((n-1,m)\)-SG nor \((n,m-1)\)-SG for every integers \( n \geq 1 \) and \( m \geq 0 \). In what follows, we give examples for \( n = 1 \) and 2. For that, we need some change of ring results.

The next result studies the notion of left \((n,m)\)-SG rings in direct products of rings. For the convenience of the reader, we recall some properties concerning the structure of modules and homomorphisms over direct products of rings (for more details please see [11, Section 2.6]).

Let \( R = \prod_{i=1}^{n} R_i \) be a direct product of rings. If \( M_i \) is a left (resp., right) \( R_i \)-module for \( i = 1, \ldots, n \), then \( M = M_1 \oplus \cdots \oplus M_n \) is a left (resp., right) \( R \)-module. Conversely, if \( M \) is a left (resp., right) \( R \)-module, then it is of the form \( M = M_1 \oplus \cdots \oplus M_n \), where \( M_i \) is a left (resp., right) \( R_i \)-module for \( i = 1, \ldots, n \) [11, Subsection 2.6.6]. Also, the homomorphisms of \( R \)-modules are determined by their actions on the \( R_i \)-module components. This is summarized in the following result:

**Lemma 2.11 ([11], Theorem 2.6.8).** Let \( R = \prod_{i=1}^{n} R_i \) be a direct product of rings and let \( M = M_1 \oplus \cdots \oplus M_n \) and \( N = N_1 \oplus \cdots \oplus N_n \) be decompositions of left (resp., right) \( R \)-modules into left (resp., right) \( R_i \)-modules. Then, the following hold:
(1) There is a natural isomorphism of abelian groups:
\[ \text{Hom}_R(M,N) \cong \text{Hom}_{R_1}(M_1,N_1) \oplus \cdots \oplus \text{Hom}_{R_n}(M_n,N_n) \]
where the homomorphism \( \alpha_1 \oplus \cdots \oplus \alpha_n \) is defined by:
\[ (\alpha_1 \oplus \cdots \oplus \alpha_n)(m_1, \ldots, m_n) = (\alpha_1 m_1, \ldots, \alpha_n m_n). \]

(2) The homomorphism \( \alpha \) is injective (resp., surjective) if and only if each \( \alpha_i \) is injective (resp., surjective).

Using this result with [11, Corollary 2.6.9], we get the following known result:

Lemma 2.12. Let \( R = \prod_{i=1}^n R_i \) be a direct product of rings and let \( M = M_1 \oplus \cdots \oplus M_n \) be an \( R \)-module. Then,
\[ \text{pd}_R(M) = \sup\{\text{pd}_{R_i}(M_i) \mid 1 \leq i \leq n\}. \]
Consequently,
\[ \text{lgldim}(R) = \sup\{\text{lgldim}(R_i) \mid 1 \leq i \leq n\}. \]

Then, using these results, we get the following result:

Proposition 2.13. Let \( R = \prod_{i=1}^n R_i \) be a direct product of rings and let \( M = M_1 \oplus \cdots \oplus M_n \)
be an \( R \)-module. Then, for two integers \( n \geq 1 \) and \( m \geq 0 \), \( M \) is an \((n,m)\)-SG-projective \( R \)-module if and only if each \( M_i \) is an \((n,m)\)-SG-projective \( R_i \)-module.
Consequently, \( R \) is left \((n,m)\)-SG if and only if each \( R_i \) is left \((n,m)\)-SG.

Now, we can give the first example.

Example 2.14. Consider the the quotient ring \( R = \mathbb{Z}/4\mathbb{Z} \), where \( \mathbb{Z} \) denotes the ring of integers, and consider a family of rings \( S_i \) for \( i \geq 1 \), such that \( l\text{gldim}(S_i) = i \). Then, for every \( i \geq 1 \), the direct product of rings \( R \times S_i \) is left \((1,i)\)-SG, but it is not left \((1,i-1)\)-SG.

Proof. From [9, Corollary 3.9], \( R \) is a commutative \((1,0)\)-SG ring. Then, using Proposition 2.13 and Lemma 2.12, we get, for every \( i \geq 1 \), that the ring \( R \times S_i \) is left \((1,i)\)-SG, but it is not left \((1,i-1)\)-SG. \( \square \)

We end with examples of \((2,m)\)-SG rings.

Proposition 2.15. Let \( R \) be a commutative ring with \( \text{gldim}(R) = m \) for some integer \( m \geq 1 \), such that \( R \) contains a non-zero divisor element \( x \). Then, the quotient ring \( R/xR \) is \((2,m-1)\)-SG.

Proof. Let \( M \) be an \( R/xR \)-module. Then, using the canonical surjection of rings \( R \to R/xR \), \( M \) is an \( R \)-module. Thus, there exists an exact sequence of \( R \)-modules,
\[ 0 \to Q \to P \to M \to 0, \]
where \( P \) is projective and \( \text{pd}_R(Q) \leq m-1 \). Tensoring this sequence by \( R/xR \), we get:
\[ \text{Tor}_1^R(R/xR,P) = 0 \to \text{Tor}_1^R(R/xR,M) \to Q/xQ \to P/xP \to M \to 0. \]
From [10] Examples (1), p. 102, \( \text{Tor}_1^R(R/xR,M) = M \) since \( xM = 0 \), and so we get an exact of \( R/xR \)-modules of the form:
\[ 0 \to M \to Q/xQ \to P/xP \to M \to 0. \]
The $R/xR$-module $P/xP$ is projective, and since $Q$ is a submodule of the projective $R$-module $P$, $x$ is also a non-zero divisor element on $Q$, then $pd_{R/xR}(Q/xQ) \leq pd_R(Q) \leq m - 1$ (from [22 Theorem 4.3.5]). Then, to see that $M$ is $(2,m - 1)$-SG-projective, it remains to show that $Ext^{i}_{R/xR}(M,F)$ for every projective $R/xR$-module $F$ and for every $i \geq m$. For that, it suffices to consider $F$ to be a free $R/xR$-module. In that case $F$ is of the form $L/xL$, where $L$ is a free $R$-module. From Rees’s theorem and since $\text{gldim}(R) = m$, we have for every $i \geq m$:

$$\text{Ext}^{i}_{R/xR}(M,F) \cong \text{Ext}^{i}_{R/xR}(M,L/xL) \cong \text{Ext}^{i+1}_{R}(M,L) = 0.$$ 

This completes the proof. \hfill \Box

**Example 2.16.** Consider the the quotient ring $R = \mathbb{Z}/8\mathbb{Z}$, where $\mathbb{Z}$ denotes the ring of integers, and consider a family of rings $S_i$, for $i \geq 0$, such that $l \text{ gldim}(S_i) = i$. Then:

- the direct product of rings $R \times S_0$ is left $(2,0)$-SG which is not left $(1,0)$-SG, and
- for $i \geq 1$, the direct product of rings $R \times S_i$ is left $(2,i)$-SG, but it is neither left $(1,i)$-SG nor $(2,i-1)$-SG.

**Proof.** From Proposition [2.15], $R$ is $(2,0)$-SG, and, by Lemma [9 Corollary 3.10], $R$ is not $(1,0)$-SG. Then, from Proposition [2.13], $R \times S_0$ is left $(2,0)$-SG, but it is not left $(1,0)$-SG. Using Lemma [2.12] the same argument as above shows, for $i \geq 1$, that $R \times S_i$ is left $(2,i)$-SG, but it is neither left $(1,i)$-SG nor $(2,i-1)$-SG. \hfill \Box

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