Reach-Avoid Differential Games Based on Invariant Generation

Bai Xue¹ and Qiuye Wang¹ and Naijun Zhan¹ and Martin Fränzle²

Abstract

Reach-avoid differential games play an important role in collision avoidance, motion planning and control of aircrafts, and related applications. The central problem is the computation of the set of initial states from which the ego player can enforce a stay within the safe set over a specified time horizon. Previous methods addressing this problem mostly focus on finite time horizons. We study this problem in the context of the infinite time horizon, where the ego player aims to perpetually maintain the system within the safe set while the mutual other player attempts to enforce a visit to an unsafe state. The problem is studied within the Hamilton-Jacobi reachability framework with unique Lipschitz continuous viscosity solutions. The continuity and uniqueness property of the viscosity solution facilitates use of contemporary numerical methods to solve this problem with an appropriate number of state variables. Moreover, the game is proven to have value without Isaacs’ assumption. An example adopted from a Moore-Greitzer jet-engine model is employed to illustrate our approach.

Keywords

Differential Games; Hamilton-Jacobi Equation; Invariant Sets

I. INTRODUCTION

Differential games, i.e. dynamic games featuring an evolution governed by differential equations, have many important applications in engineering domains, e.g., in the analysis of collision avoidance [21], energy management [9] and safe reinforcement learning [24]. They model a form of strategic interactions among rational players, where each player makes decisions in light of its own preference while expecting adversarial actions from the mutual other player. As the resulting winning strategies are robust against any possible action of the adversary, differential games have in recent years received growing interest as a model facilitating synthesis of reliable control strategies for safety-critical systems.

Differential games were initiated by Rufus Isaacs in the early 1950s when he studied military pursuit-evasion problems while working in the Rand Corporation. The pursuit-evasion game he studied is a two-player zero-sum game, where the players have completely opposite interests. A challenging class of differential games is known as reach-avoid games, which are to determine the set of states from which the ego player is able to drive the system to reach a desired target set of states while staying away from an avoid set, regardless of the opposing actions of the mutual other player - this set goes by many names in the literature, e.g., discriminating kernels [2], backward reachable sets [20] and stable bridges [26]. The present work studies this problem within the Hamilton-Jacobi reachability framework. Hamilton-Jacobi reachability analysis addresses reachability problem by exploiting the link to optimal control through viscosity solutions of Hamilton-Jacobi equations [4]. It extends the use of Hamilton-Jacobi equations, which are widely used in optimal control theory [5], to perform reachability analysis over both finite time horizons [17], [20], [18], [1], [12] and the infinite time horizon [8], [15], [16]. While computationally intensive, Hamilton-Jacobi reachability approaches are still appealing nowadays due to the availability of modern numerical tools such as [19], [6], [10], which allow solving associated game problems conveniently with appropriate number of state variables. Within the Hamilton-Jacobi framework, continuity of viscosity solutions is a desirable property from a theoretical point of view since discontinuities may invalidate uniqueness of the solution [4], [11]. Continuity is also desirable from a numeric computation point of view, since rigorous convergence results for numerical approximations to the derived Hamilton-Jacobi equation usually require continuity of the solution. Unfortunately, reach-avoid analysis under state constraints may induce discontinuities in the viscosity solutions, see for instance [4], [11], unless the dynamics satisfies special assumptions at the boundary of state constraints, e.g., inward pointing qualification assumption [25] and outward pointing condition [13]. These conditions are, however, restrictive and viscosity solution can therefore be discontinuous in general. Recently, without requiring such assumptions, [7] infers a modified Hamilton-Jacobi equation and considers reachability problems over finite time horizons for state-constrained problems with control inputs. The modified Hamilton-Jacobi equation exhibits a unique continuous viscosity solution. Based on such Hamilton-Jacobi formulation in [7], [18] studies the finite-time reach-avoid games for state-constrained systems. [12] further investigates differential games over finite time horizons where the target set, the state constraint set, and dynamics are allowed to be time-varying. Recently, [15] considers the generation of the region of attraction over the infinite time horizon. The region of attraction here is the set of initial states that are controllable in that they can be driven, using an admissible control while

1. State Key Lab. of Computer Science, Institute of Software, CAS, China. (xuebai,wangqiuye,znj)@ios.ac.cn).
2. Carl von Ossietzky Universität Oldenburg, Germany. (martin.franzle@uni-oldenburg.de)
respecting a set of state constraints, to asymptotically approach an equilibrium state. To the best of our knowledge, there is no previous work on the use of Hamilton-Jacobi equations having continuous viscosity solutions to address the infinite time reach-avoid differential game for state-constrained systems.

In this paper we therefore extend the Hamilton-Jacobi formulation from [2] to address infinite time reach-avoid games for state-constrained systems. In the reach-avoid game, we consider computation of the lower robust controlled invariant set and the upper robust controlled invariant set. The lower controlled robust invariant set is a set of initial states such that there exists a nonanticipative strategy for the ego player which perpetually maintains the system within the safe set, irrespective of actions of the mutual other player. The upper robust controlled invariant set is a set of initial states such that for each nonanticipative strategy of the mutual other player there exists a corresponding action for the ego player which maintains the system within the safe set. We characterize the lower robust controlled invariant set as the zero level set of a unique Lipschitz continuous viscosity solution to a Hamilton-Jacobi equation with sup-inf Hamiltonian and the upper controlled robust invariant set as the zero level set of a unique Lipschitz continuous viscosity solution to a Hamilton-Jacobi equation with inf-sup Hamiltonian, respectively. The continuity of viscosity solutions facilitates use of existing numerical methods to solve the Hamilton-Jacobi equations. More importantly, the reach-avoid game is proved to have value without Isaacs’ assumption. An example adopted from modern Moore-Greitzer jet engine model [23] is employed to demonstrate our approach.

This paper is structured as follows: Section II gives a detailed introduction of the differential game of interest in this paper, including the notion of lower and upper controlled robust invariants. Section III formulates the computation of both lower and upper controlled robust invariant within the framework of Hamilton-Jacobi type partial differential equation. After demonstrating our approach on one example in Section IV, we conclude this paper in Section V.

II. DIFFERENTIAL GAME FORMULATION

In this section we introduce the definitions and notations which are employed in the rest of this paper. The following basic notations will be used in what follows: \( \mathbb{R}^n \) for the set of n-dimensional real vectors, \( \|x\| \) denotes the 2-norm, i.e., \( \|x\| := \sqrt{\sum_{i=1}^{n} x_i^2} \), where \( x = (x_1, \ldots, x_n) \). \( C^\infty(\mathbb{R}^n) \) denotes the set of smooth functions over \( \mathbb{R}^n \). Vectors are denoted by boldface letters.

We consider a reach-avoid differential game with dynamics given by

\[
\begin{align*}
\dot{x}(s) &= f(x(s), u(s), d(s)) \\
x(0) &= x_0 \in X.
\end{align*}
\]

(1)

Here we assume that \( f(x, u, d) \in \mathbb{R}^n \times U \times D \mapsto \mathbb{R}^n \) is continuous over \( x, u \) and \( d \), and locally Lipschitz in \( x \) uniformly in \( u \) and \( d \). The sets \( X, U \), and \( D \) are compact subsets of finite dimensional spaces \( \mathbb{R}^n, \mathbb{R}^m \), and \( \mathbb{R}^l \) respectively, and the controls \( u(\cdot) : [0, \infty) \mapsto U \) and \( d(\cdot) : [0, \infty) \mapsto D \) are measurable functions. We define

\[
\begin{align*}
U &= \{u(\cdot) : [0, \infty) \mapsto U, \text{measurable}\} \\
D &= \{d(\cdot) : [0, \infty) \mapsto D, \text{measurable}\}
\end{align*}
\]

as the respective sets of control functions.

As pointwise limits of measurable functions are measurable, \( U \) is a closed subset, and consequently compact [22]. Analogously, \( D \) is also compact. Throughout this paper we will investigate the situation in which the ego player wants to control the system to stay within the set \( X \) while the other player attempts to prevent this. For this reason, we will usually interpret \( u(\cdot) \) as a control action while we consider \( d(\cdot) \) as an adversarial perturbation. The trajectory of system (1) under the control of \( u(\cdot) \in U \) and \( d(\cdot) \in D \) is denoted by \( \phi_{x_0}^{u(\cdot),d(\cdot)} : \mathbb{R} \mapsto \mathbb{R}^n \) with \( \phi_{x_0}^{u(\cdot),d(\cdot)}(0) = x_0 \). The game is investigated in the framework of non-anticipative strategy, whose concept is formally presented in Definition 1.

Definition 1: We say that a map \( \alpha(\cdot) : D \mapsto U \) is a non-anticipative strategy (for the ego player) if it satisfies the following condition:

For \( d_1(\cdot), d_2(\cdot) \in D \) with \( d_1 = d_2 \) almost everywhere on \( [0, s] \) for any \( s \geq 0 \), \( \alpha(d_1) \) and \( \alpha(d_2) \) coincide almost everywhere on \( [0, s] \). The set of non-anticipative strategies \( \alpha \) for the ego player are denoted by \( \Gamma \).

Non-anticipative strategies for the other player \( \beta(\cdot) : U \mapsto D \) are defined similarly. Its corresponding set is denoted by \( \Delta \).

According to Remark 5.9 in [22], \( \Gamma \) and \( \Delta \) are compact in the product topology of point-wise convergence. Based on the non-anticipative strategies in Definition 1 we define two types of robust controlled invariant sets, i.e., lower robust controlled invariant set and upper robust controlled invariant set. Their generation is the focus of this paper.

Definition 2: Let \( X = \{x \in \mathbb{R}^n \mid h(x) \leq 0\} \), where \( h(x) \) is a bounded and locally Lipschitz continuous function in \( \mathbb{R}^n \).

1) The lower robust controlled invariant set \( R^- \) of system (1) is the set of states \( x \)’s such that there exists a non-anticipative strategy \( \alpha_x(\cdot) \in \Gamma \) such that for any perturbation \( d(\cdot) \in D \) the corresponding trajectory \( \phi_{x_0}^{u(\cdot),d(\cdot)}(t) \) stay inside \( X \) for \( t \geq 0 \), i.e.,

\[
R^- = \{x \in \mathbb{R}^n \mid \exists \alpha_x(\cdot) \in \Gamma, \forall d(\cdot) \in D, \phi_{x_0}^{u(\cdot),d(\cdot)}(t) \in X, t \in [0, \infty)\}.
\]
2) The upper robust controlled invariant set $R^+$ of system \([\textit{1}]\) is the set of states $x$'s such that for any non-anticipative strategy $\beta(\cdot) \in \Delta$ there exists a corresponding control $u_{\beta,x}(\cdot) \in \mathcal{U}$ such that the trajectory $\phi^{u_{\beta,x}, R(u_{\beta,x})}_x(t)$ stays inside $X$ for $t \geq 0$, i.e.,

$$R^+ = \{ x \in \mathbb{R}^n | \forall \beta(\cdot) \in \Delta, \exists u_{\beta,x}(\cdot) \in \mathcal{U}, \phi^{u_{\beta,x}, R(u_{\beta,x})}_x(t) \in X \text{ for } t \in [0, \infty) \}.$$  

Note that the assumption on the boundedness of $h(x)$ over $x \in \mathbb{R}^n$ is not strict since if $h(x)$ is unbounded, then $h(x) := \frac{1}{1 + |x|^2}$ is bounded and $X$ is still equal to $\{ x \in \mathbb{R}^n | h(x) \leq 0 \}$ holds.

We have an immediate conclusion from Definition \([\textit{2}]\).

**Corollary 1**: The lower robust controlled invariant set $R^-$ is a subset of the upper robust controlled invariant set $R^+$, i.e., $R^- \subset R^+$. 

### III. Characterization of $R^\pm$ using HJI

In this section we characterize the lower and upper robust controlled invariants $R^-$ and $R^+$ using Hamilton-Jacobi equations with sup-inf and inf-sup Hamiltonians respectively.

In order to obtain Hamilton-Jacobi equations for characterizing these two robust controlled invariants $R^-$ and $R^+$, for any solution $\phi^{u,d}_x(\cdot)$ of \([\textit{1}]\) with initial value $x$ we associate a payoff which depends on $u(\cdot) \in \mathcal{U}$ and $d(\cdot) \in \mathcal{D}$ and is denoted by

$$J(x, u, d) := \sup_{t \in [0, \infty)} e^{-\gamma t} h(\phi^{u,d}_x(t)),$$

where $\gamma$ is a scalar constant valued in $(0, \infty)$.

Note that we only assume that $f(x, u, d)$ in system \([\textit{1}]\) is locally Lipschitz continuous over $x$ uniformly in $u \in U$ and $d \in D$, this can not guarantee the global existence of the Caratheodory solution $\phi^{u,d}_x(t)$ for $t \in [0, \infty)$ for every $x \in \mathbb{R}^n$. Thanks to Kirszbraun’s extension theorem for Lipschitz maps \([\textit{27}]\), we can construct a global Lipschitz function $F(x, u, d)$ such that $F(x, u, d) = f(x, u, d)$ over $x \in B$ and its global Lipschitz constant $L_F$ is equal to $L_f$, where $L_f$ is the Lipschitz constant of the function $f(x, u, d)$ over $B$ and $\mathcal{X} \subset B$. For instance, $F(x, d) := \inf_{y \in B}(f(y, u, d) + AL_f \| x - y \|$ satisfies such requirement, where $A$ is an $n$-dimensional vector with each component equaling to one. Since $F(x, u, d) = f(x, u, d)$ over $x \in \mathcal{X}$, the dynamics of the system \([\textit{1}]\) and the system $\dot{x} = F(x, u, d)$ are the same within the set $\mathcal{X}$, consequently the sets $R^-$ and $R^+$ under the system $\dot{x} = F(x, u, d)$ remain the same. Also, the original system \([\textit{1}]\) is sufficient for computing $R^-$ and $R^+$ on the set $B$ since $F(x, u, d) = f(x, u, d)$ over $x \in B$. Therefore, for ease exposition we still use the original system \([\textit{1}]\) for theoretical analysis in the remainder of this paper with assumed global existence of solutions for each $x \in \mathbb{R}^n$. In the sequel we continue exploring properties of the function $J(x, u, d)$ in \([\textit{1}]\).

**Lemma 1**: $J(x, u, d)$ in \([\textit{2}]\) is continuous over $(u(\cdot), d(\cdot)) \in \mathcal{U} \times \mathcal{D}$.

**Proof**: Assume that $\lim_{n \to \infty} u_n(t) = u(t)$ and $\lim_{n \to \infty} d_n(t) = d(t)$ point-wise, where $u_n(\cdot) \in \mathcal{U}$ and $d_n(\cdot) \in \mathcal{D}$ for $n \geq 1$, we will prove that for every $\epsilon > 0$, there exists $N > 0$ such that

$$|J(x, u, d) - J(x, u_n, d_n)| < \epsilon, \forall n > N.$$

Since $h(x)$ is bounded over $\mathbb{R}^n$, there exists $M \in [0, \infty)$ such that $|h(x)| \leq M$ over $\mathbb{R}^n$. Consequently, we have that for given $\epsilon > 0$, there exists $T > 0$ such that

$$|e^{-\gamma t} h(\phi^{u,d}_x(t)) - e^{-\gamma t} h(\phi^{u_n,d_n}_x(t))| \leq 2Me^{-\gamma t} < \frac{\epsilon}{2}, \forall u(\cdot) \in \mathcal{U}, \forall d(\cdot) \in \mathcal{D}, \forall t \geq T$$

holds. Therefore,

$$|J(x, u, d) - J(x, u_n, d_n)| \leq \sup_{t \in [0, \infty)} |e^{-\gamma t} h(\phi^{u,d}_x(t)) - e^{-\gamma t} h(\phi^{u_n,d_n}_x(t))|$$

$$\leq \sup_{t \in [0, T]} |e^{-\gamma t} h(\phi^{u,d}_x(t)) - e^{-\gamma t} h(\phi^{u_n,d_n}_x(t))| + \frac{\epsilon}{2}, \forall t \geq T$$

According to Lemma 5.8 in \([\textit{22}]\) stating that if $\lim_{n \to \infty} u_n(t) = u(t)$ and $\lim_{n \to \infty} d_n(t) = d(t)$ point-wise, $\lim_{n \to \infty} \phi^{u_n,d_n}_x(t) = \phi^{u,d}_x(t)$ uniformly on $[0, T]$, we finally have that for given $\epsilon > 0$, there exists $N > 0$ such that

$$|J(x, u, d) - J(x, u_n, d_n)| < \epsilon, \forall n \geq N.$$  

For the payoff $J(x, u, d)$, we respectively define the lower value function $V^-$ and upper value function $V^+$ as follows:

$$V^-(x) := \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in \mathcal{D}} J(x, \alpha(d), d)$$

(4)
and
\[ V^+(x) := \sup_{\beta(\cdot) \in \Delta} \inf_{u(\cdot) \in U} J(x, u, \beta(u)). \]  
(5)

We show that the zero level sets of the upper value function \( V^+ \) and the lower value function \( V^- \) are respectively equal to the upper robust controlled invariant set \( R^+ \) and the lower robust controlled invariant set \( R^- \), i.e. \( R^+ = \{ x \in \mathbb{R}^n \mid V^+(x) = 0 \} \) and \( R^- = \{ x \in \mathbb{R}^n \mid V^-(x) = 0 \} \). Before justifying this statement, we need an intermediate proposition stating that both the lower value function \( V^- \) and the upper value function \( V^+ \) are positive and bounded over \( \mathbb{R}^n \).

**Proposition 1:** \( V^- (x) \) is non-negative and bounded over \( x \in \mathbb{R}^n \). Analogously, \( V^+ (x) \) is non-negative and bounded over \( x \in \mathbb{R}^n \) as well.

**Proof:** We just prove the statement pertinent to \( V^- (x) \). The same proof procedure applies to \( V^+ \) as well.

Since \( h(x) \) is bounded over \( \mathbb{R}^n \), we have that
\[
\lim_{t \to \infty} e^{-\gamma t} h(\phi_x^x(t)) = 0, \forall \alpha(\cdot) \in \Gamma, \forall d(\cdot) \in D, \forall x \in \mathbb{R}^n.
\]

Thus,
\[
\sup_{d(\cdot) \in D} \sup_{t \in [0, \infty)} e^{-\gamma t} h(\phi_x^x(t)) \geq 0, \forall \alpha(\cdot) \in \Gamma.
\]

Consequently, \( V^-(x) \geq 0 \) for \( x \in \mathbb{R}^n \).

The boundedness of \( V^- \) is guaranteed by the fact that
\[
J(x, \alpha(d), d) \leq M, \forall \alpha(\cdot) \in \Gamma, \forall d(\cdot) \in D, \forall x \in \mathbb{R}^n,
\]
where \( M \) is a positive value such that \( |h(x)| \leq M \) over \( x \in \mathbb{R}^n \). Thus, \( V^-(x) \leq M \) over \( x \in \mathbb{R}^n \).

**Lemma 2:** \( R^- = \{ x \mid V^-(x) = 0 \} \) and \( R^+ = \{ x \mid V^+(x) = 0 \} \).

**Proof:** 1. For the statement \( R^- = \{ x \mid V^-(x) = 0 \} \), we first prove \( R^- \subseteq \{ x \mid V^-(x) = 0 \} \).

Consider \( x \in \mathbb{R}^- \). According to the concept of \( \mathcal{R}^- \) in Definition \( \mathbb{R}^- \), we have that there exists a non-anticipative strategy \( \alpha_x(\cdot) \) such that for all \( d(\cdot) \in D \), \( \phi_x^x(d)(t) \in \mathcal{X} \) for \( t \geq 0 \). This implies
\[
\sup_{d(\cdot) \in D} \sup_{t \in [0, \infty)} h(\phi_x^x(d)(t)) \leq 0
\]
and consequently
\[
\sup_{d(\cdot) \in D} \sup_{t \in [0, \infty)} e^{-\gamma t} h(\phi_x^x(d)(t)) \leq 0.
\]

Thus,
\[
V^-(x) = \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} J(x, \alpha(d), d) \leq \sup_{d(\cdot) \in D} \sup_{d(\cdot) \in D} J(x, \alpha_x(d), d) \leq 0.
\]  
(6)

In addition, according to Proposition \( \mathbb{R}^- \), which states that \( V^-(x) \geq 0 \) over \( \mathbb{R}^n \), we conclude that \( R^- \subseteq \{ x \mid V^-(x) = 0 \} \).

Next, we show that \( \{ x \mid V^-(x) = 0 \} \subseteq \mathcal{R}^- \). Let \( x_0 \in \{ x \mid V^-(x) = 0 \} \) and \( V(x_0, \alpha) = \sup_{d \in D} J(x_0, \alpha, d) \). Thus, \( V^-(x_0) = \inf_{\alpha \in \Gamma} V(x_0, \alpha) \). Since
\[
|V(x_0, \alpha_1) - V(x_0, \alpha_2)| = \sup_{d \in D} |J(x_0, \alpha_1(d), d) - J(x_0, \alpha_2(d), d)|
\]
\[
\leq \sup_{d \in D} |J(x_0, \alpha_1(d), d) - J(x_0, \alpha_2(d), d)|, \forall \alpha_1(\cdot), \alpha_2(\cdot) \in \Gamma.
\]  
(7)

Also, since \( J(x_0, u, d) \) is continuous over \((u, d) \in U \times D\) according to Lemma \( \mathbb{R}^- \), we have that \( V(x_0, \alpha) \) is continuous over \( \alpha \in \Gamma \). Such continuity can be proved by showing \( \lim_{\alpha_1 \to \alpha_2} \sup_{d \in D} |J(x_0, \alpha_1, d) - J(x_0, \alpha_2, d)| = 0 \). Assume that there exists a positive value \( \epsilon \) such that \( \sup_{d \in D} |J(x_0, \alpha_1(d), d) - J(x_0, \alpha_2(d), d)| \geq \epsilon \) for \( \alpha_1 \to \alpha_2 \), then there exists \( d_1 \in D \) such that \( J(x_0, \alpha_1(d_1), d_1) - J(x_0, \alpha_2(d_1), d_1) \geq \frac{\epsilon}{2} \) for \( \alpha_1 \to \alpha_2 \), contradicting Lemma \( \mathbb{R}^- \). In addition, \( \Gamma \) is compact, there definitely exists an \( \alpha(\cdot) \in \Gamma \) such that \( V^-(x_0) = V(x_0, \alpha) = \sup_{d \in D} J(x_0, \alpha, d) \). Therefore, \( \sup_{d \in D} J(x_0, \alpha, d) = 0 \), implying that \( J(x_0, \alpha, d) \leq 0 \) for all \( d \in D \). Consequently, \( h(\phi_x^x(d)) \leq 0 \) for all \( d \in D \) and all \( t \geq 0 \). Therefore, \( x_0 \in \mathcal{R}^- \) and thus \( \{ x \mid V^-(x) = 0 \} \subseteq \mathcal{R}^- \).

Hence, \( \{ x \mid V^+(x) = 0 \} = \mathcal{R}^+ \).

2. For the second statement that \( \mathcal{R}^+ = \{ x \mid V^+(x) = 0 \} \), we first consider \( \mathcal{R}^+ \subseteq \{ x \mid V^+(x) \leq 0 \} \). Let \( x \in \mathcal{R}^+ \) and \( V^+(x) = \delta > 0 \). We will derive a contradiction. Due to \( V^+(x) = \delta > 0 \), there exists \( \beta_1(\cdot) \in \Delta \) such that \( \inf_{u(\cdot) \in U} J(x, u, \beta_1) >\)
\( \frac{\delta}{2} \), implying that \( J(x, u, \beta_1) > \frac{\delta}{4} \) for all \( u(\cdot) \in \mathcal{U} \). Therefore, for every \( u(\cdot) \in \mathcal{U} \), there exists \( T_u \in [0, \infty) \) such that \( e^{-\gamma T_u} h(\phi_{u(\cdot)}^{\frac{\delta}{2}}(T_u)) > \frac{\delta}{4} \) and therefore, \( \phi_{u(\cdot)}^{\beta_1}(T_u) \notin \mathcal{X} \), contradicting \( x \in \mathcal{R}^+ \). \( \mathcal{R}^+ \subset \{ x \mid V^+(x) \leq 0 \} \) holds.

Secondly, consider \( \{ x \in \mathbb{R}^n \mid V^+(x) \leq 0 \} \subset \mathcal{R}^+ \). Let \( \mathcal{A} = \{ \mathcal{B} : \mathcal{U} \to \mathcal{D} \text{ Borel-measurable} \} \). It is obvious that \( \mathcal{A} \subset \Delta \). We have

\[
0 \geq \sup_{\beta(\cdot) \in \Delta} \inf_{u(\cdot) \in \mathcal{U}} J(x, u, \beta(u)) \geq \sup_{\mathcal{A}(\cdot) \in \mathcal{A}} \inf_{u(\cdot) \in \mathcal{U}} J(x, u, \mathcal{A}(u)) = \min \max_{u(\cdot) \in \mathcal{U}} J(x, u, d).
\]

(8)

The equality in (8) is gained according to Lemma 1 and Lemma 5.6 in [23].

Hence there exists \( u_1(\cdot) \in \mathcal{U} \) such that \( J(x, u_1, d) \leq 0 \), \( \forall d(\cdot) \in \mathcal{D} \) as min, max extrema will happen for some concrete \( u_1(\cdot) \in \mathcal{U} \), \( d(\cdot) \in \mathcal{D} \). Since \( u_1 \) applies to all possible values \( \beta(\cdot) \in \Delta \), we have there exists \( u_1(\cdot) \in \mathcal{U} \) such that \( h(\phi_{u_1}^{\beta_1}(u_1)) \leq 0 \) for all \( \forall \beta(\cdot) \in \Delta \) and all \( t \in [0, \infty) \). Consequently, \( x \in \mathcal{R}^+ \). Therefore, we have that \( x \in \mathcal{R}^+ \) iff \( V^+(x) \leq 0 \). In addition, according to Proposition 1 which states that \( V^+(x) \geq 0 \) for \( x \in \mathbb{R}^n \), we conclude that \( \mathcal{R}^+ = \{ x \in \mathbb{R}^n \mid V^+(x) = 0 \} \).

According to Lemma 2 if \( V^-(x) \) and \( V^+(x) \) are computed, we can obtain \( \mathcal{R}^- \) and \( \mathcal{R}^+ \). In order to compute them, we study more about them and consequently exploit more properties related to them below.

**Lemma 3:** Both the lower value function \( V^- \) and the upper value function \( V^+ \) are locally Lipschitz continuous over \( \mathbb{R}^n \).

**Proof:** We just prove the statement related to \( V^- \). The one for \( V^+ \) can be justified following the same procedure.

Let \( \epsilon > 0 \) and choose \( \alpha_1(\cdot) \in \Gamma \) such that

\[
V^-(x_1) \geq \sup_{d(\cdot) \in \mathcal{D}} J(x_1, \alpha_1(d), d) - \epsilon.
\]

For \( V^-(x_2) \), we have that

\[
V^-(x_2) \leq \sup_{d(\cdot) \in \mathcal{D}} J(x_2, \alpha_1(d), d),
\]

Moreover, we can choose \( d_1(\cdot) \in \mathcal{D} \) such that

\[
V^-(x_2) \leq J(x_2, \alpha_1(d_1), d_1) + \epsilon.
\]

Therefore,

\[
\begin{align*}
V^-(x_2) - V^-(x_1) & \leq J(x_2, \alpha_1(d_1), d_1) - J(x_1, \alpha_1(d_1), d_1) + 2\epsilon \\
& \leq \sup_{t \in [0, \infty)} e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) - \sup_{t \in [0, \infty)} e^{-\gamma t} h(\phi_{x_1}^{\alpha_1(d_1)}(d_1)(t)) + 2\epsilon \\
& \leq \sup_{t \in [0, \infty)} (e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) - e^{-\gamma t} h(\phi_{x_1}^{\alpha_1(d_1)}(d_1)(t))) + 2\epsilon.
\end{align*}
\]

(9)

Since \( h(x) \) is bounded over \( x \in \mathbb{R}^n \), we have that

\[
\lim_{t \to \infty} e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) = 0.
\]

As a consequence, we obtain that there exists \( T > 0 \) such that

\[
e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) - e^{-\gamma t} h(\phi_{x_1}^{\alpha_1(d_1)}(d_1)(t)) \leq \epsilon, \forall t \geq T.
\]

Therefore, we infer that

\[
\begin{align*}
V^-(x_2) - V^-(x_1) & \leq \max \{ \sup_{t \in [0, T]} (e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) - e^{-\gamma t} h(\phi_{x_1}^{\alpha_1(d_1)}(d_1)(t))), \\
& \sup_{t \in [T, \infty)} \{ e^{-\gamma t} h(\phi_{x_2}^{\alpha_1(d_1)}(d_1)(t)) - e^{-\gamma t} h(\phi_{x_1}^{\alpha_1(d_1)}(d_1)(t)) \} \} + 2\epsilon \\
& \leq e^{-\gamma T} L_h e^{L_f T} \| x_1 - x_2 \| + 3\epsilon,
\end{align*}
\]

(10)

where \( L_h \) and \( L_f \) are the Lipschitz constants of \( h \) and \( f \) over \( \Omega(B_1) = \{ x \mid x = \phi_{u_1}^{\alpha_1(d_1)}(d_1)(t), t \in [0, T], x_0 \in B_1 \} \) with \( B_1 \) being a compact set covering \( x_1 \) and \( x_2 \) respectively. The same argument with the role of \( x_1 \), \( x_2 \) reversed establishes that

\[
V^-(x_2) - V^-(x_1) \geq -e^{-\gamma T} L_h e^{L_f T} \| x_1 - x_2 \| - 3\epsilon.
\]

Since \( \epsilon \) is arbitrary, there is a constant \( L \) such that \( |V^-(x_1) - V^-(x_2)| \leq L \| x_1 - x_2 \| \), where \( L \) is larger than or equal to \( e^{-\gamma T} L_h e^{L_f T} \).
The Lipschitz continuity of $V^+$ can be assured by following the arguments as $V^-$. Besides the Lipschitz continuity of $V^-$ and $V^+$, both $V^-$ and $V^+$ satisfy the dynamic programming principle. 

**Lemma 4:** For $x \in \mathbb{R}^n$ and $t \geq 0$, we have

$$
V^-(x) = \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} \max \left\{ e^{-\gamma t} V^- (\phi_x^{\alpha(d),d}(t)), \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha(d),d}(\tau)) \right\}
$$

(11)

and

$$
V^+(x) = \sup_{\beta(\cdot) \in \Delta} \inf_{u(\cdot) \in U} \max \left\{ e^{-\gamma t} V^+ (\phi_x^{u(\cdot),\beta(u)}(t)), \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{u(\cdot),\beta(u)}(\tau)) \right\}.
$$

(12)

**Proof:** Let

$$W(x,t) := \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} \max \left\{ e^{-\gamma t} V^- (\phi_x^{\alpha(d),d}(t)), \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha(d),d}(\tau)) \right\}.
$$

We will show that for every $\epsilon > 0$, $V^-(x) \leq W(x,t) + 2\epsilon$ and $V^-(x) \geq W(x,t) - 3\epsilon$. Then since $\epsilon > 0$ is arbitrary, $V^-(x) = W(x,t)$.

1. $V^-(x) \leq W(x,t) + 2\epsilon$. Fix $\epsilon > 0$ and choose $\alpha_1(\cdot) \in \Gamma$ such that

$$W(x,t) \geq \sup_{d_1(\cdot) \in D} \max \left\{ e^{-\gamma t} V^- (\phi_x^{\alpha_1(d_1),d_1}(t)), \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha_1(d_1),d_1}(\tau)) \right\} - \epsilon.
$$

Similarly, choose $\alpha_2(\cdot) \in \Gamma$ such that

$$V^-(y) \geq \sup_{d_2(\cdot) \in D} \sup_{\tau \in [t,\infty)} e^{-\gamma (\tau-t)} h(\phi_y^{\alpha_2(d_2),d_2}(\tau-t)) - \epsilon,
$$

where $y = \phi_x^{\alpha_1(d_1),d_1}(t)$.

Let

$$d(\tau) = \begin{cases} 
\alpha_1(\tau) & \text{if } \tau \in [0,t) \\
\alpha_2(\tau-t) & \text{if } \tau \in [t,\infty)
\end{cases}
$$

and

$$\alpha(d(\tau)) = \begin{cases} 
\alpha_1(d(\tau)) & \text{if } \tau \in [0,t) \\
\alpha_2(d(\tau-t)) & \text{if } \tau \in [t,\infty).
\end{cases}
$$

(13)

It is easy to see that $\alpha(\cdot) : \mathcal{D} \mapsto \mathcal{U}$ is non-anticipative. By uniqueness, $\phi_x^{\alpha(d),d}(\tau) = \phi_x^{\alpha_1(d_1),d_1}(\tau)$ if $\tau \in [0,t)$, and $\phi_x^{\alpha(d),d}(\tau) = \phi_y^{\alpha_2(d_2),d_2}(\tau-t)$ if $\tau \in [t,\infty)$.

Hence,

$$W(x,t) \geq \sup_{d_1(\cdot) \in D} \sup_{d_2(\cdot) \in D} \max \left\{ \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_y^{\alpha_2(d_2),d_2}(\tau-t)), \sup_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha_1(d_1),d_1}(\tau)) \right\} - 2\epsilon
$$

(14)

$$\geq \sup_{d(\cdot) \in D} \max_{\tau \in [0,\infty)} e^{-\gamma \tau} h(\phi_x^{\alpha(d),d}(\tau)) - 2\epsilon
$$

$$\geq V^-(x) - 2\epsilon.
$$

Therefore, $V^-(x) \leq W(x,t) + 2\epsilon$.

2. $V^-(x) \geq W(x,t) - 3\epsilon$. Fix $\epsilon > 0$ and choose $\alpha(\cdot) \in \Gamma$ such that

$$V^-(x) \geq \sup_{d(\cdot) \in D} \sup_{\tau \in [0,\infty)} e^{-\gamma \tau} h(\phi_x^{\alpha(d),d}(\tau)) - \epsilon.
$$

(15)

By the definition of $W(x,t)$, we have

$$W(x,t) \leq \sup_{d(\cdot) \in D} \max \left\{ \max_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha(d),d}(\tau)), e^{-\gamma t} V^-(\phi_x^{\alpha(d),d}(t)) \right\}.
$$

Hence there exists $d_1(\cdot) \in D$ such that

$$W(x,t) \leq \max \left\{ \max_{\tau \in [0,t]} e^{-\gamma \tau} h(\phi_x^{\alpha_1(d_1),d_1}(\tau)), e^{-\gamma t} V(y) \right\} + \epsilon.
$$

(16)
where \( y = \phi^{(d_1)}_x(t) \). Moreover, we have
\[
V^{-1}(y) \leq \sup_{d(.) \in \mathcal{D}} \max_{\tau \in [t, \infty)} e^{-\gamma(\tau-t)} h(\phi^{(d)}_y(\tau-t)), \forall \tau \in [t, \infty).
\]  
(17)
so there exists \( d_2(.) \in \mathcal{D} \) such that
\[
V^{-1}(y) \leq \max_{\tau \in [t, \infty)} e^{-\gamma(\tau-t)} h(\phi^{(d_2)}_y(\tau-t)) + \epsilon.
\]  
(18)
We define
\[
d(\tau) = \begin{cases} 
  d_1(\tau) & \text{if } \tau \in [0, t) \\
  d_2(\tau-t) & \text{if } \tau \in [t, \infty)
\end{cases}.
\]
(19)
Therefore, combining (16) and (18), we have
\[
W(x, t) \leq \sup_{\tau \in [0, \infty)} e^{-\gamma \tau} h(\phi^{(d)}_x(\tau)) + 2\epsilon,
\]
which together with (15) implies \( V^{-}(x) \geq W(x, t) - 3\epsilon \).

The above procedure can be applied to prove that \( V^+ \) satisfies the dynamic programming principle (12).

Based on the established dynamic programming principle in Lemma 4, we construct Hamilton-Jacobi partial differential equations associated with \( V^- \) and \( V^+ \) respectively,
\[
\min \{ \gamma V(x) - H^-(x, \frac{\partial V(x)}{\partial x}), V(x) - h(x) \} = 0
\]
(20)
and
\[
\min \{ \gamma V(x) - H^+(x, \frac{\partial V(x)}{\partial x}), V(x) - h(x) \} = 0,
\]
(21)
where
\[
H^-(x, p) = \sup_{d(.) \in \mathcal{D}} \inf_{u(.) \in U} p \cdot f(x, u, d)
\]
(22)
and
\[
H^+(x, p) = \inf_{u(.) \in \mathcal{U}} \sup_{d(.) \in \mathcal{D}} p \cdot f(x, u, d)
\]
(23)
are the sup-inf and inf-sup Hamiltonians respectively. These two equations are the core focus of this paper. We in the sequel will show that \( V^- \) and \( V^+ \) are respectively the unique Lipschitz continuous and bounded viscosity solution to (20) and (21).

Before this, we first recall the concept of viscosity solutions to (20) (or (21)).

**Definition 3:** A locally bounded continuous function \( V(x) \) on \( \mathbb{R}^n \) is a viscosity solution of (20) (21), if 1) for any test function \( v \in C^\infty(\mathbb{R}^n) \) such that \( V - v \) attains a local minimum at \( x_0 \in \mathbb{R}^n \),
\[
\min \{ \gamma V(x_0) - H^-(x_0, \frac{\partial v(x_0)}{\partial x}), V(x_0) - h(x_0) \} \geq 0
\]
(24)
holds (i.e., \( V \) is a viscosity supersolution); 2) for any test function \( v \in C^\infty(\mathbb{R}^n) \) such that \( V - v \) attains a local maximum at \( x_0 \in \mathbb{R}^n \),
\[
\min \{ \gamma V(x_0) - H^+(x_0, \frac{\partial v(x_0)}{\partial x}), V(x_0) - h(x_0) \} \leq 0
\]
(25)
holds (i.e., \( V \) is a viscosity subsolution).

In order to prove that \( V^- (x) \) and \( V^+ (x) \) are respectively a viscosity solution to (20) and (21), we need an intermediate lemma below.

**Lemma 5:** Let \( v \in C^\infty(\mathbb{R}^n) \).
1) If \( \gamma v(x_0) - H^-(x_0, \frac{\partial v(x_0)}{\partial x}) \mid_{x_0} \leq -\theta < 0 \), then, for sufficiently small \( \delta > 0 \), there exists an input \( d(.) \in \mathcal{D} \) such that for all \( \alpha(.) \in \Gamma \) and all \( s \in [0, \delta] \),
\[
\gamma v(\phi^{(d)}_{x_0}(s)) - \frac{\partial v(x_0)}{\partial x} \mid_{x_0 = \phi^{(d)}_{x_0}(s)} \cdot f(\phi^{(d)}_{x_0}(s), d(s), \alpha(s)) \leq -\theta \frac{\delta}{2}.
\]
2) If \( \gamma v(x_0) - H^-(x_0, \frac{\partial v(x)}{\partial x} | x = x_0) \geq \theta > 0 \), then, for sufficiently small \( \delta > 0 \), there exists a strategy \( \alpha(\cdot) \in \Gamma \) such that for all \( d(\cdot) \in D \) and all \( s \in [0, \delta] \),

\[
\gamma(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s)) = \frac{\partial v(x)}{\partial x} | x = \phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s) \cdot f(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s), \alpha(d(s)), d(s)) \geq \frac{\theta}{2}. \]

3) If \( \gamma v(x_0) - H^+(x_0, \frac{\partial v(x)}{\partial x} | x = x_0) \geq \theta > 0 \), then, for sufficiently small \( \delta > 0 \), there exists an input \( u(\cdot) \in \mathcal{U} \) such that for all \( \beta(\cdot) \in \Delta \) and all \( s \in [0, \delta] \),

\[
\gamma(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s)) = \frac{\partial v(x)}{\partial x} | x = \phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s) \cdot f(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s), \alpha(d(s)), d(s)) \geq \frac{\theta}{2}. \]

4) If \( \gamma v(x_0) - H^+(x_0, \frac{\partial v(x)}{\partial x} | x = x_0) \leq -\theta < 0 \), then, for sufficiently small \( \delta > 0 \), there exists a strategy \( \beta(\cdot) \in \Delta \) such that for all \( u(\cdot) \in \mathcal{U} \) and all \( s \in [0, \delta] \),

\[
\gamma(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s)) = \frac{\partial v(x)}{\partial x} | x = \phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s) \cdot f(\phi_{x_0}^{\alpha(\cdot), d(\cdot)}(s), \alpha(d(s)), d(s)) \leq -\frac{\theta}{2}. \]

Proof: The proofs of statements 1 and 2 are given. The statements 3 and 4 can be justified similarly.

1. Since \( \gamma v(x_0) - H^-(x_0, \frac{\partial v(x)}{\partial x} | x = x_0) \leq -\theta < 0 \), there exists \( d_0 \in D \) such that

\[
\gamma v(x_0) - \frac{\partial v(x)}{\partial x} | x = x_0 \cdot f(x_0, u_0, d_0) \leq -\frac{3}{4} \theta < 0, \forall u_0 \in \mathcal{U}. \]

Also, since \( v \in C^\infty \), \( f(x, u, d) \) is continuous over \((x, u, d)\), there exists \( \delta_{u_0} \) for \( u_0 \in \mathcal{U} \) such that for \( x \) satisfying \( \|x - x_0\| \leq \delta_{u_0} \),

\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, u_0, d_0) \leq -\frac{\theta}{2} < 0. \]

Since \( U \) is compact, there exist finitely many distinct points \( u_{0,1}, \ldots, u_{0,l} \in \mathcal{U} \) with positive values \( \delta_1, \ldots, \delta_l \) such that

\[
U \subset \bigcup_{i=1}^l \{ u \mid \|u - u_{0,i}\| \leq \delta_i \} \]

and

\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, u_0, d_0) \leq -\frac{1}{2} \theta < 0 \]

for \( x \) satisfying \( \|x - x_0\| \leq \delta_1 \) and \( u_0 \) satisfying \( \|u_0 - u_{0,i}\| \leq \delta_i \), where \( i = 1, \ldots, l \). Therefore,

\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, u_0, d_0) \leq -\frac{1}{2} \theta < 0, \forall u_0 \in \mathcal{U} \]

for \( x \) satisfying \( \|x - x_0\| \leq \delta' = \min_{i=1}^l \delta_i \).

Let \( \Omega \) be a compact set which covers all states traversed by trajectories starting from \( x_0 \) within a finite time interval \([0, \delta']\), and \( M \) be the upper bound of \( f(x, u, d) \) over \( \Omega \times U \times D \). We have

\[
\|\phi_{x_0}^{u, d}(t) - x_0\| = \|\int_{\tau=0}^t f(x(\tau), u(\tau), d(\tau))\|d\tau \leq M t, \forall u \in \mathcal{U}, \forall d \in \mathcal{D}. \]

Therefore, there exists \( \delta > 0 \) such that

\[
\|\phi_{x_0}^{u, d}(t) - x_0\| \leq \delta', \forall t \in [0, \delta], \forall u \in \mathcal{U}, \forall d \in \mathcal{D}. \quad (26) \]

We choose a measurable function \( d' : [0, \infty) \to D \) with \( d'(s) = d_0 \) for \( s \in [0, \infty) \). Obviously, \( d' \in D \). Therefore, we have

\[
\gamma(\phi_{x_0}^{u, d'}(s)) - \frac{\partial v(x)}{\partial x} | x = \phi_{x_0}^{u, d'}(s) \cdot f(\phi_{x_0}^{u, d'}(s), u(s), d'(s)) \leq -\frac{\theta}{2}, \forall u \in \mathcal{U}, \forall s \in [0, \delta], \]

implying that for all \( \alpha(\cdot) \in \Gamma \) and all \( s \in [0, \delta], \)

\[
\gamma(\phi_{x_0}^{u, d'}(s)) - \frac{\partial v(x)}{\partial x} | x = \phi_{x_0}^{u, d'}(s) \cdot f(\phi_{x_0}^{u, d'}(s), \alpha(d'(s)), d'(s)) \leq -\frac{\theta}{2}. \]

2. Since \( \gamma v(x_0) - H^+(x_0, \frac{\partial v(x)}{\partial x} | x = x_0) \geq \theta > 0 \), there exists a corresponding \( u_{d_0} \in \mathcal{U} \) for every \( d_0 \in \mathcal{D} \) such that

\[
\gamma v(x_0) - \frac{\partial v(x)}{\partial x} | x = x_0 \cdot f(x_0, u_{d_0}, d_0) \geq \frac{3}{4} \theta > 0. \]
Since \( v \in C^\infty \) and \( f(x, u, d) \) is continuous over \( x, u \) and \( d \), there exists \( \delta' > 0 \) such that for \( d \in D \) satisfying \( \|d - d_0\| \leq \delta' \) and \( x \) satisfying \( \|x - x_0\| \leq \delta' \),
\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, u_{d_0}, d) \geq \frac{1}{2} \theta > 0.
\]

Since \( D \) is compact, there exist finitely many distinct points \( d_1, \ldots, d_l \in D \) with positive values \( \delta_1, \ldots, \delta_l \) such that \( D \subset \bigcup_{i=1}^l \{d | \|d - d_i\| \leq \delta_i \} \).

Moreover, there exists \( u_{d_i} \in U \) such that for \( d \) satisfying \( \|d - d_i\| \leq \delta_i \) and \( x \) satisfying \( \|x - x_0\| \leq \delta_i \),
\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, u_{d_i}, d) \geq \frac{1}{2} \theta > 0
\]
holds, where \( i = 1, \ldots, l \).

Setting \( \nu : D \mapsto U \) such that \( \nu(d) = u_{d_i} \) if \( d \in \{d | \|d - d_i\| \leq \delta_i \} \), we have that for \( x \) satisfying \( \|x - x_0\| \leq \delta' = \min_{i=1,\ldots,l} \delta_i \),
\[
\gamma v(x) - \frac{\partial v(x)}{\partial x} \cdot f(x, \nu(d), d) \geq \frac{1}{2} \theta > 0, \forall d \in D.
\]

Furthermore, like (26), we obtain that there exists \( \delta > 0 \) such that
\[
\phi_{\alpha_0}(d, s) = \{x | \|x - x_0\| \leq \delta', \forall s \in [0, \delta], \forall d \in D \}.
\]

Let \( \alpha(\cdot) : D \mapsto \mathcal{U} \) be \( \alpha(d(s)) = \nu(d(s)) \) for \( s \geq 0 \). It is obvious that \( \alpha(\cdot) \in \Gamma \). Consequently, there exist \( \delta > 0 \) and a strategy \( \alpha(\cdot) \in \Gamma \) such that for all \( d(\cdot) \in D \) and all \( s \in [0, \delta] \),
\[
\gamma v(\phi_{\alpha_0}(d, s)) - \frac{\partial v(x)}{\partial x} \cdot f(\phi_{\alpha_0}(d, s), \alpha(d(s)), d(s)) \geq \frac{\theta}{2}.
\]

We in the following reduce \( V^-(x) \) and \( V^+(x) \) to the viscosity solution to (20) and (21) respectively.

**Theorem 1:** \( V^- \) and \( V^+ \) are respectively the viscosity solution to Hamilton-Jacobi equations (20) and (21).

**Proof:** Likewise, we just prove the statement pertinent to \( V^- \). We will prove that \( V^- \) is both viscosity sub and super-solution to (20) according to Definition 8.

Firstly, we prove that \( V^- \) is a sub-solution to (20). Let \( v \in C^\infty(\mathbb{R}^n) \) such that \( V^- - v \) attains a local maximum at \( x_0 \). Without loss of generality, assume that this maximum is zero, i.e., \( V^-(x_0) = v(x_0) \). According to the continuity of \( V^- \) and \( v \), there exists a positive value \( \delta \) such that
\[
V^-(x) - v(x) \leq 0
\]
for \( x \) satisfying \( \|x - x_0\| \leq \delta \). Suppose (25) is false. Then there definitely exists a positive value \( \epsilon_1 \) such that
\[
h(x_0) \leq v(x_0) - \epsilon_1
\] (27)
and
\[
\gamma v(x_0) - H^-(x_0, \frac{\partial v(x)}{\partial x} |_{x=x_0}) \geq \epsilon_1
\] (28)
hold. Therefore, for the former inequality, i.e., \( h(x_0) \leq v(x_0) - \epsilon_1 \), there exists a sufficiently small \( \delta_1 > 0 \) with \( \delta_1 \leq \delta \) such that for \( x \) satisfying \( \|x - x_0\| \leq \delta_1 \) and \( t \) satisfying \( 0 \leq t \leq \delta_1 \),
\[
e^{-\gamma t} h(x) \leq v(x_0) - \frac{\epsilon_1}{2}.
\]

According to Lemma 5, (28) implies that for sufficiently small \( \delta > 0 \), there exists a strategy \( \alpha_1(\cdot) \in \Gamma \) such that for all \( d \in D \) and all \( s \in [0, \delta] \),
\[
\gamma v(\phi_{\alpha_1}(d, s)) - \frac{\partial v(x)}{\partial x} \bigg|_{x=\phi_{\alpha_1}(d, s)} \cdot f(\phi_{\alpha_1}(d, s), \alpha_1(d(s)), d(s)) \geq \frac{\epsilon_1}{2}.
\] (29)

\( \delta \) can be chosen such that \( \|\phi_{\alpha_1}(d, s) - x_0\| \leq \delta_1, \forall s \in [0, \delta], \forall d(\cdot) \in D \).

Since \( v \in C^\infty(\mathbb{R}^n) \), by applying Grönwall’s inequality (14) to (29) with the time interval \([0, \delta]\), we have
\[
v(\phi_{\alpha_1}(d, \delta)) \leq e^{\gamma \delta} v(x_0) + \frac{\epsilon_1}{2\gamma} (1 - e^{\delta \gamma}).
\] (30)
Therefore,
\[ e^{-\delta \gamma} v(\phi^{\alpha(d),d}(\delta)) \leq v(x_0) - \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta \gamma}). \]  

(31)

Furthermore, since \( V^{-}(x) \leq v(x) \) for \( x \) satisfying \( \|x - x_0\| \leq \delta_1 \) with \( V^{-}(x_0) = v(x_0) \) as well as \( V^{-} \geq 0 \), we have
\[ e^{-\delta \gamma} V^{-}(\phi^{\alpha(d),d}(\delta)) \leq V^{-}(x_0) - \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta \gamma}). \]

Therefore, according to (11), we finally have
\[
V^{-}(x_0) = \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} \max \{e^{-\gamma \delta} V^{-}(\phi^{\alpha(d),d}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi^{\alpha(d),d}(\tau)) \} \\
\leq \sup_{d(\cdot) \in D} \max \{e^{-\gamma \delta} V^{-}(\phi^{\alpha_1(d_1),d_1}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi^{\alpha_1(d_1),d_1}(\tau)) \} \\
\leq \max \{e^{-\gamma \delta} V^{-}(\phi^{\alpha_1(d_1),d_1}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi^{\alpha_1(d_1),d_1}(\tau)) \} + \epsilon_3 \\
\leq V^{-}(x_0) - \min \{\frac{\epsilon_1}{2}, \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta \gamma})\} + \epsilon_3 < V^{-}(x_0),
\]

which is a contradiction. In (13), \( d_1(\cdot) \in D \) satisfies
\[
\sup_{d(\cdot) \in D} \max \{e^{-\gamma \delta} V^{-}(\phi^{\alpha_1(d_1),d_1}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi^{\alpha_1(d_1),d_1}(\tau)) \} \\
\leq \max \{e^{-\gamma \delta} V^{-}(\phi^{\alpha_1(d_1),d_1}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi^{\alpha_1(d_1),d_1}(\tau)) \} + \epsilon_3
\]

with \( 0 < \epsilon_3 < \min \{\frac{\epsilon_1}{\delta}, \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta \gamma})\} \). Consequently, \( V^{-} \) is a subsolution to (20).

In what follows we prove that \( V^{-} \) is a super-solution to (20). Let \( v \in C^\infty(\mathbb{R}^n) \) such that \( V^{-} - v \) attains a local minimum at \( x_0 \). Without loss of generality, assume that this minimum is zero, i.e., \( V^{-}(x_0) = v(x_0) \). Therefore, there exists a positive value \( \delta_1 \) such that \( V^{-}(x) - v(x) \geq 0 \) for \( x \) satisfying \( \|x - x_0\| \leq \delta_1 \). Assume that (24) is false. Since \( V^{-}(x) \geq h(x) \) for \( x \in \mathbb{R}^n \) according to (11), \( v(x_0) \geq h(x_0) \) holds. Therefore,
\[
\gamma v(x_0) - H^{-}(x_0, \frac{\partial v(x)}{\partial x} |_{x=x_0}) < 0
\]

(34)

holds, i.e., there exists a positive value \( \theta > 0 \) such that
\[
\gamma v(x_0) - H^{-}(x_0, \frac{\partial v(x)}{\partial x} |_{x=x_0}) < -\theta.
\]

(35)

According to Lemma 5 we have that for sufficiently small \( \delta > 0 \), there exists \( d_1(\cdot) \in D \) such that for all strategies \( \alpha(\cdot) \in \Gamma \) and all \( s \in [0,\delta] \),
\[
\gamma v(\phi^{\alpha(d_1),d_1}(s)) - \frac{\partial v(x)}{\partial x} |_{x=\phi^{\alpha(d_1),d_1}(s)} \cdot f(\phi^{\alpha(d_1),d_1}(s), \alpha(d_1(s)), d_1(s)) \leq -\frac{\theta}{2}.
\]

(36)

\( \delta \) can be chosen such that
\[
\|\phi^{\alpha(d_1),d_1}(s) - x_0\| \leq \delta_1, \forall s \in [0,\delta], \forall \alpha(\cdot) \in \Gamma.
\]

By applying Grönwall’s inequality [14] to (36) with the time interval \([0, \delta]\), we obtain
\[
v(\phi^{\alpha(d_1),d_1}(\delta)) \geq e^{\delta \gamma} v(x_0) - \frac{\theta}{2\gamma}(1 - e^{\delta \gamma}).
\]

(37)

Therefore,
\[
e^{-\delta \gamma} v(\phi^{\alpha(d_1),d_1}(\delta)) \geq v(x_0) + \frac{\theta}{\gamma}(1 - e^{-\delta \gamma}).
\]

(38)

Furthermore, since \( V^{-} \geq v \) for \( x \in \{x \mid \|x - x_0\| \leq \delta_1\} \) with \( V^{-}(x_0) = v(x_0) \) as well as \( V^{-}(x) \geq 0 \) over \( x \in \mathbb{R}^n \), we have
\[
e^{-\delta \gamma} V^{-}(\phi^{\alpha(d),d}(\delta)) \geq V^{-}(x_0) + \frac{\theta}{\gamma}(1 - e^{-\delta \gamma}).
\]
Therefore, according to (11), we finally have

\[ V^-(x_0) = \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} \max \left\{ e^{-\gamma \delta} V^-(\phi_{x_0}^{(d)}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi_{x_0}^{(d)}(\tau)) \right\} \]

\[ \geq \sup_{d(\cdot) \in D} \max \left\{ e^{-\gamma \delta} V^-(\phi_{x_0}^{(d)}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi_{x_0}^{(d)}(\tau)) \right\} - \epsilon_1 \]

(39)

\[ \geq V^-(x_0) + \frac{\theta}{2\gamma} (1 - e^{-\delta \gamma}) - \epsilon_1 > V^-(x_0), \]

which is a contradiction. In (39), \( \alpha_1(\cdot) \in \Gamma \) satisfies

\[ \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} \max \left\{ e^{-\gamma \delta} V^-(\phi_{x_0}^{(d)}(\delta)), \sup_{\tau \in [0,\delta]} e^{-\gamma \tau} h(\phi_{x_0}^{(d)}(\tau)) \right\} \]

(40)

with \( 0 < \epsilon_1 < \frac{\theta}{2\gamma} (1 - e^{-\delta \gamma}) \). Thus, \( V^- \) is a supersolution to (20).

Therefore, we conclude that \( V^- \) is a viscosity solution to (20).

Furthermore, we show the uniqueness of the Lipschitz continuous and bounded viscosity solutions to (20) and (21).

**Theorem 2:** \( V^- \) and \( V^+ \) are respectively the unique bounded and Lipschitz continuous viscosity solution to (20) and (21).

**Proof:** We just show the uniqueness of the Lipschitz continuous and bounded viscosity solution to (20). We first prove a comparison principle: If \( V_1 \) and \( V_2 \) are bounded Lipschitz continuous functions over \( x \in \mathbb{R}^n \), and they are respectively a viscosity sub and supersolution to (20), then \( V_1 \leq V_2 \) in \( \mathbb{R}^n \). Obviously, if such comparison principle holds, the uniqueness of bounded Lipschitz continuous solutions to (20) is guaranteed.

Let

\[ \Phi(x, y) = V_1(x) - V_2(y) - \frac{\| x - y \|^2}{2\epsilon} - \delta(|x|^m + |y|^m), \]

where \( \langle x \rangle = (1 + \| x \|^2)^{\frac{1}{2}} \), and \( \epsilon, \delta, m \) are positive parameters. Assume that there are \( \beta > 0 \) and \( z \) such that \( V_1(z) - V_2(z) = \beta \).

We choose \( \delta > 0 \) such that \( 2\delta(z) \leq \frac{\beta}{2} \) such that for \( 0 < m \leq 1 \),

\[ \frac{\beta}{2} \leq \beta - 2\delta(z)^m = \Phi(z, z) \leq \sup \Phi(x, y). \]

(41)

Since \( \Phi \) is continuous and \( \lim_{\| x \| + \| y \| \to \infty} \Phi(x, y) = -\infty \), there exist \( \overline{x}, \overline{y} \) such that

\[ \Phi(\overline{x}, \overline{y}) = \sup \Phi(x, y). \]

(42)

From the inequality \( \Phi(\overline{x}, \overline{x}) + \Phi(\overline{y}, \overline{y}) \leq 2\Phi(\overline{x}, \overline{y}) \), we easily get

\[ \frac{\| \overline{x} - \overline{y} \|^2}{\epsilon} \leq V_1(\overline{x}) - V_1(\overline{y}) + V_2(\overline{x}) - V_2(\overline{y}). \]

(43)

Then the boundedness of \( V_1 \) and \( V_2 \) implies that

\[ \| \overline{x} - \overline{y} \| \leq c\sqrt{\epsilon} \]

(44)

for a suitable constant \( c \). By plugging (44) into (43) and using the Lipschitz continuity of \( V_1 \) and \( V_2 \) we get

\[ \frac{\| \overline{x} - \overline{y} \|}{\epsilon} \leq w\sqrt{\epsilon} \]

(45)

for some constant \( w \).

Next, define the continuously differentiable functions

\[ \phi(x) := V_2(\overline{y}) + \frac{\| x - \overline{y} \|^2}{2\epsilon} + \delta(|x|^m + |\overline{y}|^m), \]

\[ \psi(y) := V_1(\overline{x}) - \frac{\| \overline{x} - \overline{y} \|^2}{2\epsilon} - \delta(|\overline{x}|^m + |\overline{y}|^m), \]

(46)
and observe that $V_1 - \phi$ attains its maximum at $\boldsymbol{x}$ and $V_2 - \psi$ attains its minimum at $\boldsymbol{y}$. It is easy to compute

$$
\frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}} = \frac{\boldsymbol{x} - \boldsymbol{y}}{\epsilon} + \lambda \boldsymbol{x}, \lambda = \delta m(\boldsymbol{x})^{m-2}, \\
\frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}} = \frac{\boldsymbol{x} - \boldsymbol{y}}{\epsilon} - \tau \boldsymbol{y}, \tau = \delta m(\boldsymbol{y})^{m-2}.
$$

(47)

Thus, we obtain that

$$
\min \{ \gamma V_1(\boldsymbol{x}) - H^{-}(\boldsymbol{x}, \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}}), V_1(\boldsymbol{x}) - h(\boldsymbol{x}) \} \leq \min \{ \gamma V_2(\boldsymbol{y}) - H^{-}(\boldsymbol{y}, \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}}), V_2(\boldsymbol{y}) - h(\boldsymbol{y}) \}. 
$$

(48)

Thus, we obtain that

$$
\min \{ \gamma V_1(\boldsymbol{x}) - H^{-}(\boldsymbol{x}, \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}}) - (\gamma V_2(\boldsymbol{y}) - H^{-}(\boldsymbol{y}, \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}}), V_1(\boldsymbol{x}) - h(\boldsymbol{x}) - (V_2(\boldsymbol{y}) - h(\boldsymbol{y})) \} \leq 0.
$$

(49)

Obviously, either

$$
\gamma V_1(\boldsymbol{x}) - H^{-}(\boldsymbol{x}, \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}}) - (\gamma V_2(\boldsymbol{y}) - H^{-}(\boldsymbol{y}, \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}})) \leq 0
$$

(50)

or

$$
V_1(\boldsymbol{x}) - h(\boldsymbol{x}) - (V_2(\boldsymbol{y}) - h(\boldsymbol{y})) \leq 0
$$

(51)

holds. We will obtain a contradiction separately.

If (50) holds,

$$
V_1(\boldsymbol{x}) - V_2(\boldsymbol{y}) \leq \frac{1}{\gamma} (H^{-}(\boldsymbol{x}, \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}}) - H^{-}(\boldsymbol{y}, \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}}))
$$

(52)

$$
\leq \frac{1}{\gamma} (L_f w \sqrt{\epsilon} + \delta m (\frac{\tau_\gamma}{\lambda} + \tau_\gamma) m + (\frac{\tau_\gamma}{\lambda} + \tau_\gamma)^m + \epsilon)
$$

where $K = L_f + \sup_{u \in U, d \in D} \{ \| f(0, u, d) \| \}$ and the last inequality can be obtained as follows:

$$
H^{-}(\boldsymbol{x}, \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}}) - H^{-}(\boldsymbol{y}, \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}}) = \sup_{d \in D} \inf_{u \in U} \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}} \cdot f(\boldsymbol{x}, u, d) - \sup_{d \in D} \inf_{u \in U} \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}} \cdot f(\boldsymbol{y}, u, d)
$$

$$
\leq \sup_{d \in D} \left( \inf_{u \in U} \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}} \cdot f(\boldsymbol{x}, u, d) - \inf_{u \in U} \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}} \cdot f(\boldsymbol{y}, u, d) \right)
$$

(53)

$$
\leq \inf_{u \in U} \frac{\partial \phi(x)}{\partial x} \bigg|_{x=\boldsymbol{x}} \cdot f(\boldsymbol{x}, u, d_1) - \inf_{u \in U} \frac{\partial \psi(y)}{\partial y} \bigg|_{y=\boldsymbol{y}} \cdot f(\boldsymbol{y}, u, d_1) + \frac{\epsilon}{2}
$$

$$
\leq \frac{\epsilon}{\epsilon} \cdot f(\boldsymbol{x}, u_2, d_1) - \frac{\epsilon}{\epsilon} \cdot f(\boldsymbol{y}, u_2, d_1) + \epsilon
$$

$$
= (\frac{\epsilon}{\epsilon} + \lambda \boldsymbol{x}) \cdot f(\boldsymbol{x}, u_2, d_1) - (\frac{\epsilon}{\epsilon} + \tau \boldsymbol{y}) \cdot f(\boldsymbol{y}, u_2, d_1) + \epsilon
$$

$$
\leq \frac{\epsilon}{\epsilon} \cdot L_f + \lambda \boldsymbol{x} \cdot f(\boldsymbol{x}, u_2, d_1) + \tau \boldsymbol{y} \cdot f(\boldsymbol{y}, u_2, d_1) + \epsilon
$$

$$
\leq \frac{\epsilon}{\epsilon} \cdot L_f + \lambda \boldsymbol{x} + \lambda \| f(0, u_2, d_1) \| + \tau \| f(\boldsymbol{y}, u_2, d_1) \| + \tau \| f(\boldsymbol{y}, u_2, d_1) \| + \epsilon
$$

$$
\leq \frac{\epsilon}{\epsilon} \cdot L_f + \lambda \| f(0, u_2, d_1) \| + \tau \| f(\boldsymbol{y}, u_2, d_1) \| + \epsilon
$$

$$
\leq \frac{\epsilon}{\epsilon} \cdot L_f + \lambda \| f(0, u_2, d_1) \| + \tau \| f(\boldsymbol{y}, u_2, d_1) \| + \epsilon
$$

$$
\leq L_f w \sqrt{\epsilon} + \delta m (\frac{\tau_\gamma}{\lambda} + \tau_\gamma) m + (\frac{\tau_\gamma}{\lambda} + \tau_\gamma)^m + \epsilon
$$
where $d_1$ satisfies
\begin{align}
\sup_{d \in D} \left( \inf_{u \in U} \frac{\partial \varphi(x)}{\partial x} |_{x=\varphi(f(\overline{x}, u, d))} - \inf_{u \in U} \frac{\partial \psi(y)}{\partial y} |_{y=\overline{y} \cdot f(\overline{y}, u, d)} \right) \\
\leq \inf_{u \in U} \frac{\partial \varphi(x)}{\partial x} |_{x=\varphi(f(\overline{x}, u_1, d_1))} - \inf_{u \in U} \frac{\partial \psi(y)}{\partial y} |_{y=\overline{y} \cdot f(\overline{y}, u_1, d_1) + \epsilon/2}
\end{align}

and $u_2$ satisfies
\begin{align}
\inf_{u \in U} \frac{\partial \psi(y)}{\partial y} |_{y=\overline{y} \cdot f(\overline{y}, u, d_1)} \geq \frac{\partial \psi(y)}{\partial y} |_{y=\overline{y} \cdot f(\overline{y}, u_2, d_1)} - \frac{\epsilon}{2}.
\end{align}

Therefore, choosing $0 < m \leq \frac{\beta}{2}$, we obtain
\begin{align}
\Phi(\overline{x}, \overline{y}) \leq V_1(\overline{x}) - V_2(\overline{y}) - \delta((\overline{x})^m + (\overline{y})^m) \leq \frac{1}{\gamma}(L_f w \sqrt{\epsilon} + \epsilon).
\end{align}

$\Phi(\overline{x}, \overline{y})$ can be smaller than $\frac{\beta}{2}$ for $\epsilon$ small enough, contradicting (41) and (42).

If (51) holds,
\begin{align}
\Phi(\overline{x}, \overline{y}) \leq V_1(\overline{x}) - V_2(\overline{y}) \leq h(\overline{x}) - h(\overline{y}) \leq L_h c \sqrt{\epsilon},
\end{align}

where $L_h$ is the Lipschitz constant over a local compact region covering $\overline{x}$ and $\overline{y}$. $\Phi(\overline{x}, \overline{y})$ can be smaller than $\frac{\beta}{2}$ for $\epsilon$ small enough, contradicting (41).

Above all, $V_1 \leq V_2$ over $x \in \mathbb{R}^n$. It is evident that if $U(x)$ is a bounded Lipschitz continuous viscosity solution to (20), then $U(x) = V^-(x)$ over $x \in \mathbb{R}^n$, due to the fact that $U(x)$ and $V^-(x)$ are both sub and superviscosity solutions. Therefore, the uniqueness of the bounded Lipschitz continuous solutions to (20) is guaranteed.

We continue exploiting more on $V^-$ and $V^+$ based on (20) and (21). According to Lemma 2 stating that $V^-$ and $V^+$ are Lipschitz continuous, we have the following corollary.

**Corollary 2**: The Hamilton-Jacobi equations (20) and (21) for $V^-$, $V^+$ hold classically a.e. in $\mathbb{R}^n$, i.e. except on a set of measure 0.

**Proof**: By Lemma 3, $V^-$ and $V^+$ are Lipschitz and, hence, by Rademacher’s theorem, they are differentiable a.e., which implies that the equations (20) and (21) that $V^-$ and $V^+$ satisfy in Theorem 1 hold classically in these points.

Moreover, we have that $V^-=V^+$.

**Theorem 3**: $V^- = V^+$ holds, i.e., the game has values. Furthermore, $\mathcal{R}^- = \mathcal{R}^+$.

Before justifying the statement in Theorem 3 we need the following Lemma.

**Lemma 6**: Let $u$ be a bounded Lipschitz viscosity subsolution of (20) and $v$ be a bounded Lipschitz viscosity supersolution of (21), then $u \leq v$ on $\mathbb{R}^n$.

**Proof**: $v$ is a supersolution of $\min \{ \gamma V(x) - H^+(x, \frac{\partial V(x)}{\partial x}), V(x) - h(x) \} = 0$ if
\begin{align}
\min \{ \gamma v(x) - H^+(x, \frac{\partial p(x)}{\partial x}), v(x) - h(x) \} \geq 0
\end{align}

for $p(x) \in C^\infty(\mathbb{R}^n)$ and $v-p$ attains a local minimum at $x$. Thus, $v$ is also a supersolution of $\min \{ \gamma V(x) - H^-(x, \frac{\partial V(x)}{\partial x}), v(x) - h(x) \} = 0$, i.e.,
\begin{align}
\min \{ \gamma v(x) - H^-(x, \frac{\partial p(x)}{\partial x}), v(x) - h(x) \} \geq 0
\end{align}

for $p(x) \in C^\infty(\mathbb{R}^n)$ and $v-p$ attains a local minimum at $x$, which follows from (55) and $H^- \leq H^+$. Following the comparison statement in the proof of Theorem 2 we have $u \leq v$.

**Proof of Theorem 3**

**Proof**: Lemma 6 implies that $V^- \leq V^+$. In the following, we prove that $V^- \geq V^+$. According to (43) and (45), we have the following deduction:
\begin{align}
V^-(x) = \inf_{\alpha(\cdot) \in \Gamma} \sup_{d(\cdot) \in D} J(x, \alpha(d), d) \\
\geq \inf_{u(\cdot) \in \mathcal{U}} \sup_{d(\cdot) \in D} J(x, u, d) \\
\geq \sup_{d(\cdot) \in D} \inf_{u(\cdot) \in \mathcal{U}} J(x, u, d) \\
\geq \sup_{\beta(\cdot) \in \Delta} \inf_{u(\cdot) \in \mathcal{U}} J(x, u, \beta(u)) = V^+(x).
\end{align}

The first and third inequalities are obtained due to the fact that $\alpha(d(\cdot)) \in \mathcal{U}$ for $\alpha(\cdot) \in \Gamma$ and $d(\cdot) \in D$, $\beta(u(\cdot)) \in D$ for $\beta(\cdot) \in \Delta$ and $u(\cdot) \in \mathcal{U}$, respectively. Therefore, we have that $V^- = V^+$. Thus, $\{x \mid V^-(x) = 0\} = \{x \mid V^+(x) = 0\}$ is an immediate result. According to Lemma 2 $\mathcal{R}^- = \mathcal{R}^+$. 
IV. Examples

In this section we illustrate our approach on one example. All computations were performed on an i7-7500U 2.70GHz CPU with 4GB RAM running Ubuntu 17. For numerical implementation, we employ the ROC-HJ solver [6] for solving Hamilton-Jacobi equations (20) and (21).

Example 1: Moore-Greitzer jet engine model. We test our approach on the following polynomial system coming from [23], corresponding to a Moore-Greitzer model of a jet engine:

\[
\begin{align*}
\dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d, \\
\dot{y} &= (0.8076 + u)x - 0.9424y,
\end{align*}
\]

where \(X = \{ x \mid h(x) \leq 0 \}\) with \( h(x) = \frac{x^2 + y^2 - 0.25}{1 + (x^2 + y^2 - 0.25)^2} \), \( d \in [-0.02, 0.02] \) and \( u \in [-0.01, 0.01] \).

From [23], we know that \( u(x) = 0.8076x - 0.9424y \) is a controller that guarantees the existence of a robust invariant set of the following system

\[
\begin{align*}
\dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d, \\
\dot{y} &= u
\end{align*}
\]

where \( d \in [-0.02, 0.02] \). In our example we change the coefficient 0.8076 of the variable \( x \) in \( u(x) \) to 0.8076 + \( u \) with \( u \in [-0.01, 0.01] \).

The level sets of value functions \( V^- (x) \) and \( V^+ (x) \) to (20) and (21) are illustrated in Fig. 1 and 3 respectively. The visualized results in Fig. 1 and 3 justify the statement in Proposition 1 that both \( V^- (x) \) and \( V^+ (x) \) are non-negative over \( x \in \mathbb{R}^n \). On the other side, the corresponding zero level sets \( \{ x \mid V^- (x) = 0 \} \) and \( \{ x \mid V^+ (x) = 0 \} \) are respectively showcased in Fig. 2 and 4. According to Lemma 2, \( R^- = \{ x \mid V^- (x) = 0 \} \) and \( R^+ = \{ x \mid V^+ (x) = 0 \} \), the comparison between the two level sets \( R^- \) and \( R^+ \) is demonstrated in Fig. 5. From the visualized results in Fig. 5, it is difficult to distinguish \( R^- \) and \( R^+ \) since they are the same according to Theorem 3.

V. Conclusion

In this paper we considered infinite time reach-avoid differential game, in which the ego player aims to maintain the system inside a safe set perpetually while the mutual other player attempts to prevent the ego player from succeeding. This game was studied within the Hamilton-Jacobi reachability framework, in which the lower robust controlled invariant set is the zero level set of the unique bounded Lipschitz continuous viscosity solution to a Hamilton-Jacobi equation with sup-inf Hamiltonian while the

1https://uma.ensta-paristech.fr/soft/ROC-HJ/
upper robust controlled invariant set is characterized as the zero level set of the unique bounded Lipschitz continuous viscosity solution to a Hamilton-Jacobi equation with inf-sup Hamiltonian. In this formulation, the game was proved to have values without Isaacs’ assumption. One example adopted from Moore-Greitzer model of a jet engine was employed to illustrate our approach.

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Fig. 4: An illustration of the upper robust controlled invariant set $R^+$ for Example 1. Blue region – $R^+$.

Fig. 5: A comparison of the lower and upper robust controlled invariant sets $R^-$ and $R^+$ for Example 1. Red region – $R^+$; Blue region – $R^-$.

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