Let $f$ be the density function associated to a matrix-exponential distribution of parameters $(\alpha, T, s)$. By exponentially tilting $f$, we find a probabilistic interpretation which generalizes the one associated to phase-type distributions. More specifically, we show that for any sufficiently large $\lambda \geq 0$, the function $x \mapsto \int_0^\infty e^{-\lambda s} f(s) ds e^{-\lambda x} f(x)$ can be described in terms of a finite-state Markov jump process whose generator is tied to $T$. Finally, we show how to revert the exponential tilting in order to assign a probabilistic interpretation to $f$ itself.

Keywords: Phase-type distribution; matrix-analytic methods; exponential tilting; finite-state Markov jump process

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1. Introduction

A phase-type distribution corresponds to the law of $Y := \inf\{t \geq 0 : J_t = \star\}$ where $\{J_t\}_{t \geq 0}$ is a Markov jump process with state space $\{1, \ldots, p\} \cup \{\star\}$, with $\{1, \ldots, p\}$ assumed to be transient states and $\{\star\}$ absorbing. If $\{J_t\}_{t \geq 0}$ has a block-partitioned initial distribution $(\pi, 0)$ and intensity matrix given by

\[
\begin{bmatrix}
A & b \\
0 & 0
\end{bmatrix}
\]

with $b = -A \mathbf{1}$, (1)

where $\mathbf{0}$ represents a $p$-dimensional row vector of 0s and $\mathbf{1}$ a $p$-dimensional column vector of 1s, then we say that the phase-type distribution is of parameters $(\pi, A)$. Via simple probabilistic arguments, it can be shown that the density function of a phase-type distribution of parameters $(\pi, A)$ is of the form

\[
g(x) = \pi e^{Ax} b, \quad x \geq 0.
\]

Indeed, the vector $\pi e^{Ax}$ yields the probabilities of $\{J_t\}_{t \geq 0}$ being in some state $\{1, \ldots, p\}$ at time $x$, and $b$ corresponds to the intensity vector of an absorption happening immediately after. Phase-type distributions were first introduced in [14] with the aim of constructing a robust and tractable class of distributions on $\mathbb{R}_+$ to be used in econometric problems. A more
comprehensive study of phase-type distributions was carried on by Neuts [15, 16], whose work popularized their use in more general stochastic models.

On the other hand, a *matrix-exponential distribution* of dimension \( p \geq 1 \) is an absolutely continuous distribution on \((0, \infty)\) whose density function can be written as

\[
f(x) = \alpha e^{Tx}s, \quad x \geq 0,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_p) \) is a \( p \)-dimensional row vector, \( T = \{t_{ij} \}_{i,j \in \{1, \ldots, p\}} \) is a \((p \times p)\)-dimensional square matrix, and \( s = (s_1, \ldots, s_p)^T \) is a \( p \)-dimensional column vector, all with complex entries. If the dimension need not be specified, we refer to such a distribution simply as *matrix-exponential*. It follows from (2) and (3) that the class of phase-type distributions is a subset of those that are matrix-exponential, with the inclusion being strict (see [17] for details on the latter).

Matrix-exponential distributions were first studied in [8, 9] through the concept of complex-valued transition probabilities. More precisely, these papers showed that certain systems with complex-valued elements can be formally studied by analytical means without assigning a specific physical interpretation to their components. While their method provided mathematical rigour to systems ‘driven’ by complex-valued intensity matrices, it failed to provide a physical meaning to each individual component, as opposed to the case of Markov jump processes with genuine intensity matrices. Later on, it was proved in [5, 17] that matrix-exponential distributions have an interpretation in terms of a Markov process with continuous state space, as opposed to the finite-state-space one that phase-type distributions enjoy. Even after the discovery of these physical interpretations of matrix-exponential distributions, however, properties of this class of distributions are still not as well understood as they are for its phase-type counterpart. One of the main reasons for this is that processes with continuous state space are more difficult to handle, so that studying matrix-exponential distributions by physical means requires a more sophisticated framework. For example, this is the case in [2, 4, 3], where the theory of piecewise deterministic Markov processes is used to study models with matrix-exponential components. Thus, having a finite-state system interpretation for matrix-exponential distributions available may potentially lead to the discovery of new properties, as has traditionally been the case for phase-type distributions.

Functions of the form (3) also play an important role in control theory, more specifically, in the topic of single-input–single-output (SISO) linear systems. Such systems are described by a column-vector function \( x: \mathbb{R}_+ \to \mathbb{R}^p \) and \( y: \mathbb{R}_+ \to \mathbb{R} \) which satisfy the ordinary differential equations

\[
\frac{dx(t)}{dt} = T_0x(t) + b_0u(t),
\]

\[
y(t) = \alpha_0x(t);
\]

here \( u \) is called the input function, \( x \) the state function, and \( y \) the output function. SISO linear systems which produce a nonnegative output from a nonnegative input are said to be externally positive. It can be shown [12, Theorem 1] that if \( x(0) = 0 \), then the output function takes the form

\[
y(t) = \int_0^t h_0(t - z)u(s)dz,
\]

where \( h_0(z) = \alpha_0e^{T_0z}s_0 \). From this, one can deduce that the system is externally positive if and only if \( h_0 \) is a nonnegative function. If, additionally, \( h_0 \) is bounded, then \( h_0 \) is essentially a
Markov jump processes for ME distributions

A. The elements of $\alpha$, $T$ and $s$ are real.

B. The dominant eigenvalue of $T$, denoted by $\sigma_0$, is real and strictly negative.

Since it can be shown that for a given matrix-exponential density of the form (3) the parameters $(\alpha, T, s)$ can be chosen is such a way that A1 and A2 hold (see [1]), the interpretation that we develop essentially completes the picture laid out in [8, 9]. Our method, inspired by the recent work in [18], provides a transparent interpretation of $(\alpha, T, s)$ in terms of a finite-state Markov jump process. To do so, we employ the technique known as exponential tilting, which means that we focus on the density proportional to $e^{-\lambda}f(\cdot)$ for large enough $\lambda > 0$. After we perform this transformation, we construct a Markov jump process on a finite state space formed by two groups: the original states and the anti-states, the latter being a copy of the former. Heuristically, jumps within the set of original states or within the set of anti-states occur according to the off-diagonal nonnegative ’jump intensities’ of $T$, while jumps between the original and the anti-states occur according to the negative ’jump intensities’ of $T$. Eventual absorption or termination happens, and each realization ’carries’ a positive or negative sign depending only on its initial and final state. Our main contribution is to show that this mechanism yields the exponentially tilted matrix-exponential distribution, and, by reverting the exponential tilting, to provide some probabilistic insight into the original matrix-exponential distribution as well.

The structure of the paper is as follows. In Section 2 we provide a brief exposition on exponential tilting and how it affects the representation of a matrix-exponential distribution. In Section 3 we present our main results, Theorem 3.2 and Corollary 3.1, where we give a precise interpretation of an exponentially tilted matrix-exponential density in terms of a Markov jump process. Finally, in Section 4 we provide methods to recover formulae and probabilistic interpretations for matrix-exponential distributions for which the assumptions A1 and A2 hold, based on the results of their exponentially tilted versions.

2. Preliminaries

Exponential tilting, also known as the Escher transform, is a technique which transforms any probability density function $f$ with support on $[0, \infty)$ into a new probability density function $f_\lambda$ defined by

$$f_\lambda(x) = \frac{e^{-\lambda x}f(x)}{\int_0^{\infty} e^{-\lambda r}f(r)dr}, \quad x \geq 0,$$

where $\lambda \geq 0$ is the tilting rate. The use of exponential tilting goes back at least to [11], where it was used to build upon Cramér’s classical actuarial models [10]. Later on, the exponential tilting method played a prominent role in the theory of option pricing [13].

The exponentially tilted version of a matrix-exponential distribution has a simple form which happens to be matrix-exponential itself. To see this, notice that if $f$ is of the
form (3), then
\[
\int_0^\infty e^{-\lambda r} f(r) dr = \int_0^\infty e^{-\lambda r} (\alpha e^{Tr}) dr = \alpha (\lambda I - T)^{-1} s,
\]
where we used the fact that \( T - \lambda I \) has eigenvalues with strictly negative real parts and thus \( e^{(T - \lambda I)r} \) vanishes as \( r \to \infty \). Thus,
\[
f_\lambda(x) = \frac{e^{-\lambda x} (\alpha e^{Tx})}{\alpha (\lambda I - T)^{-1} s} = \left( \frac{\alpha}{\alpha (\lambda I - T)^{-1} s} \right) e^{(T - \lambda I)x} s, \quad x \geq 0,
\]
(4)

implying that \( f_\lambda \) corresponds to the density function of a matrix-exponential distribution of parameters \( \left( \frac{\alpha}{\alpha (\lambda I - T)^{-1} s}, T - \lambda I, s \right) \).

Recall that the parameters \((\alpha, T, s)\) need not have a probabilistic meaning in terms of a finite-state-space Markov chain, as opposed to the parameters associated to phase-type distributions. For instance, the parameters
\[
\alpha = (1, 0, 0), \quad T = \begin{bmatrix}
-1 & -1 & 2/3 \\
1 & -1 & -2/3 \\
0 & 0 & -1
\end{bmatrix}, \quad s = \begin{bmatrix} 4/3 \\ 2/3 \\ 1 \end{bmatrix}
\]
(5)
yield a valid matrix-exponential distribution whose density function is given by \( f(x) = \frac{2}{3} e^{-x} (1 + \cos(x)) \), and where the dominant eigenvalue of \( T \) is \(-1\) (see [6, Example 4.5.21] for details). In the following section we show how to assign a probabilistic meaning to the exponentially tilted version of (5), and more generally to those having the properties \( A1 \) and \( A2 \), in terms of a finite-state Markov jump process.

3. Main results

Let \((\alpha, T, s)\) be parameters associated to a \( p \)-dimensional matrix-exponential distribution which have the properties \( A1 \) and \( A2 \). For \( 1 \leq i, j \leq p \) denote by \( t_{ij} \) the \((i, j)\) entry of \( T \), and denote by \( s_i \) the \( i\)th entry of \( s \). For \( \ell \in \{+,-\} \), define the \((p \times p)\)-dimensional matrix \( T^\ell = \left\{ t_{ij}^{\ell} \right\}_{1 \leq i,j \leq p} \) and the \( p \)-dimensional column vector \( s^\ell = \left( s_1^\ell, \ldots, s_p^\ell \right)^T \) where
\[
t_{ij}^+ = \max\{0, \pm t_{ij}\} \quad \forall i \neq j,
\]
\[
t_{ii}^\pm = \pm \min\{0, \pm t_{ii}\} \quad \forall i, \quad \text{and}
\]
\[
s_i^\pm = \max\{0, \pm s_i\} \quad \forall i.
\]

It follows that \( T^+ \) has nonnegative off-diagonal elements and nonpositive diagonal elements, \( T^- \) is a nonnegative matrix, \( s^+ \) and \( s^- \) are nonnegative column vectors, \( T = T^+ - T^- \), and \( s = s^+ - s^- \). Now, let
\[
\lambda_0 = \min \left\{ r \geq 0 : s_i^+ + s_i^- + \sum_{j=1}^p (t_{ij}^+ + t_{ij}^-) \leq r \text{ for all } 1 \leq i \leq p \right\}.
\]
For some fixed \( \lambda \geq \lambda_0 \), consider a (possibly) terminating Markov jump process \( \varphi_t^\lambda \) of \( i \geq 0 \) driven by the block-partitioned subintensity matrix

\[
G = \begin{bmatrix}
T^+ - \lambda I & T^- & s^+ & s^- \\
T^- & T^+ - \lambda I & s^- & s^+ \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  

(6)

evolving on the state space \( \mathcal{E} = \mathcal{E}^o \cup \mathcal{E}^a \cup \{ \Delta^o \} \cup \{ \Delta^a \} \) where \( \mathcal{E}^o := \{ 1^o, 2^o, \ldots, p^o \} \) and \( \mathcal{E}^a := \{ 1^a, 2^a, \ldots, p^a \} \). The state space \( \mathcal{E} \) may be thought as the union of two sets: a collection of original states \( \mathcal{E}^o \cup \{ \Delta^o \} \) and a collection of anti-states \( \mathcal{E}^a \cup \{ \Delta^a \} \), where both \( \Delta^o \) and \( \Delta^a \) are absorbing. In the case \( \lambda > \lambda_0 \), the process \( \{ \varphi_t^\lambda \}_{i \geq 0} \) alternates between sojourn times in \( \mathcal{E}^o \) and \( \mathcal{E}^a \) up until one of the following happens: (a) get absorbed into \( \Delta^o \), (b) get absorbed into \( \Delta^a \), or (c) undergo termination due to the defect of (6). If \( \lambda = \lambda_0 \), the states \( \mathcal{E}^o \cup \mathcal{E}^a \) may or may not be transient, their status depending on the values of \( T \).

In Theorem 3.1 we establish a link between the absorption probabilities of \( \{ \varphi_t^\lambda \}_{i \geq 0} \) and the vector \( e^{(T-\lambda I)x}s \) appearing in the exponentially tilted matrix-exponential density (4). More specifically, we express each element in \( e^{(T-\lambda I)x}s \) as the sum of some positive density function and some negative density function, where the positive density is associated to an absorption of \( \{ \varphi_t^\lambda \}_{i \geq 0} \) to \( \Delta^o \), while the negative density function corresponds to an absorption of \( \{ \varphi_t^\lambda \}_{i \geq 0} \) to \( \Delta^a \). To shorten notation, from now on we denote by \( \mathbb{P}_j (\mathbb{E}_j) \), \( j \in \mathcal{E} \), the probability measure (expectation) associated to \( \{ \varphi_t^\lambda \}_{i \geq 0} \) conditional on the event \( \{ \varphi_0^\lambda = j \} \).

**Theorem 3.1.** Let \( \lambda \geq \lambda_0 \) be such that the states \( \mathcal{E}^o \cup \mathcal{E}^a \) are transient. Define

\[
\tau = \inf\{ x \geq 0 : \varphi_x^\lambda \notin \mathcal{E}^o \cup \mathcal{E}^a \}.
\]  

(7)

Then, for \( i \in \{ 1, \ldots, p \} \) and \( x \geq 0 \),

\[
(e_i^T e^{(T-\lambda I)x}s)dx = \mathbb{E}_p \left[ 1\{ \tau \in [x, x + dx] \} \beta(\varphi_x^\lambda) \right]
\]  

(8)

\[
= (e_i^T, 0) \exp\left( \begin{bmatrix} T^+ - \lambda I & T^- \\ T^- & T^+ - \lambda I \end{bmatrix} x \right) \begin{bmatrix} s^- \\ -s^+ \end{bmatrix} dx,
\]  

(9)

where \( e_i \) denotes the column vector with \( 1 \) as its \( i \)th entry and \( 0 \) elsewhere, and \( \beta(j) := 1\{ j = \Delta^o \} - 1\{ j = \Delta^a \} \). Moreover,

\[
(-e_i^T e^{(T-\lambda I)x}s)dx = \mathbb{E}_p \left[ 1\{ \tau \in [x, x + dx] \} \beta(\varphi_x^\lambda) \right]
\]  

(10)

\[
= (0, e_i^T) \exp\left( \begin{bmatrix} T^+ - \lambda I & T^- \\ T^- & T^+ - \lambda I \end{bmatrix} x \right) \begin{bmatrix} s^- \\ -s^+ \end{bmatrix} dx.
\]  

(11)

**Proof.** The block structure of (6) implies that

\[
\mathbb{P}_p(\tau \in [x, x + dx], \varphi_x^\lambda = \Delta^o) = (e_i^T, 0) \exp\left( \begin{bmatrix} T^+ - \lambda I & T^- \\ T^- & T^+ - \lambda I \end{bmatrix} x \right) \begin{bmatrix} s^+ \\ s^- \end{bmatrix} dx,
\]

\[
\mathbb{P}_p(\tau \in [x, x + dx], \varphi_x^\lambda = \Delta^a) = (e_i^T, 0) \exp\left( \begin{bmatrix} T^+ - \lambda I & T^- \\ T^- & T^+ - \lambda I \end{bmatrix} x \right) \begin{bmatrix} s^- \\ s^+ \end{bmatrix} dx;
\]
therefore, the right-hand side of (8) is equal to (9). Next, we prove that (8) holds.

Define the collection of \((p \times p)\)-dimensional matrices \(\{\Phi_{ao}(x)\}_{x \geq 0}, \{\Phi_{ao}(x)\}_{x \geq 0}, \{\Phi_{ao}(x)\}_{x \geq 0}\) by

\[
\begin{align*}
(\Phi_{ao}(x))_{ij} &= \mathbb{P}_P(\tau > x, \varphi_x^i = j^o), \\
(\Phi_{ao}(x))_{ij} &= \mathbb{P}_P(\tau > x, \varphi_x^i = j^o), \\
(\Phi_{aa}(x))_{ij} &= \mathbb{P}_P(\tau > x, \varphi_x^i = j^a),
\end{align*}
\]

for all \(i, j \in \{1, \ldots, p\}\). By the symmetry of the subintensity matrix \(G\) it is clear that for all \(x \geq 0, \Phi_{aa}(x) = \Phi_{aa}(x)\) and \(\Phi_{ao}(x) = \Phi_{ao}(x)\), even if their probabilistic interpretations differ.

For all \(x \geq 0\), let \(\Phi_o(x) := \Phi_{ao}(x) - \Phi_{ao}(x)\) and \(\Phi_a(x) := \Phi_{aa}(x) - \Phi_{aa}(x)\). Define

\[
y = \inf\{r \geq 0 : \varphi_x^1 \notin \mathcal{E}^o\}.
\]

Then, for \(i, j \in \{1, \ldots, p\}\),

\[
e_i^T \Phi_o(x) e_j = \mathbb{P}_P(\tau > x, \varphi_x^i = j^o) - \mathbb{P}_P(\tau > x, \varphi_x^i = j^a) \\
= \left\{\mathbb{P}_P(\gamma > x, \tau > x, \varphi_x^i = j^o) + \mathbb{P}_P(\gamma \leq x, \tau > x, \varphi_x^i = j^o)\right\} \\
- \left\{\mathbb{P}_P(\gamma > x, \tau > x, \varphi_x^i = j^a) + \mathbb{P}_P(\gamma \leq x, \tau > x, \varphi_x^i = j^a)\right\} \\
= \left\{\mathbb{P}_P(\gamma > x, \varphi_x^i = j^o) + \int_0^x \mathbb{P}_P(\gamma \in [r, r + dr], \tau > x, \varphi_x^i = j^o) dr\right\} \\
- \left\{\mathbb{P}_P(\gamma > x, \varphi_x^i = j^a) + \int_0^x \mathbb{P}_P(\gamma \in [r, r + dr], \tau > x, \varphi_x^i = j^a) dr\right\} \\
= \mathbb{P}_P(\gamma > x, \varphi_x^i = j^o) \\
+ \int_0^x \sum_{k=1}^p \mathbb{P}_P(\gamma \in [r, r + dr], \varphi_y^k = k^a) \mathbb{P}_k^a(\tau > x - r, \varphi_{x-r}^k = j^o) dr \\
- \int_0^x \sum_{k=1}^p \mathbb{P}_P(\gamma \in [r, r + dr], \varphi_y^k = k^a) \mathbb{P}_k^a(\tau > x - r, \varphi_{x-r}^k = j^a), \tag{12}
\]

where in the last equality we used that \(\{\gamma > x, \varphi_x^i = j^a\} = \emptyset\) and the Markov property of \(\{\varphi_x^i\}_{x \geq 0}\). Note that all the elements in (12) correspond to transition probabilities or intensities that can be expressed in matricial form as follows:

\[
\begin{align*}
\mathbb{P}_P(\gamma > x, \varphi_x^i = j^o) &= e_i^T e^{(T^+ - \lambda^I)x} e_j, \\
\mathbb{P}_P(\gamma \in [r, r + dr], \varphi_y^k = k^a) &= e_i^T e^{(T^+ - \lambda^I)r} T^- e_k dr, \\
\mathbb{P}_k^a(\tau > x - r, \varphi_{x-r}^k = j^o) &= e_k^T \Phi_{ao}(x-r) e_j, \\
\mathbb{P}_k^a(\tau > x - r, \varphi_{x-r}^k = j^a) &= e_k^T \Phi_{aa}(x-r) e_j.
\end{align*}
\]
By Theorem 3.10, substituting these expressions into (12) and using the identity $I = \sum_{k=1}^{p} e_k e_k^T$ gives

$$e_i^T \Phi_\alpha(x)e_j = e_i^T e^{(T^- - \lambda I)x} e_j + \int_0^x \sum_{k=1}^{p} (e_i^T e^{(T^- - \lambda I)y} e_k) (e_k^T \Phi_\alpha(x - r)e_j) \, dr$$

$$- \int_0^x \sum_{k=1}^{p} (e_i^T e^{(T^- - \lambda I)y} e_k) (e_k^T \Phi_\alpha(x - r)e_j) \, dr$$

$$= e_i^T \left( e^{(T^- - \lambda I)x} + \int_0^x e^{(T^- - \lambda I)y} T^- [\Phi_\alpha(x - r) - \Phi_\alpha(x - r)] \, dr \right) e_j$$

$$= e_i^T \left( e^{(T^- - \lambda I)x} + \int_0^x e^{(T^- - \lambda I)y} T^- [\Phi_\alpha(x - r) - \Phi_\alpha(x - r)] \, dr \right) e_j$$

$$= e_i^T \left( e^{(T^- - \lambda I)x} + \int_0^x e^{(T^- - \lambda I)y} (-T^-) \Phi_\alpha(x - r) \, dr \right) e_j,$$

so that $\{\Phi_\alpha(x)\}_{x \geq 0}$ is the bounded solution to the matrix-integral equation

$$\Phi_\alpha(x) = e^{(T^- - \lambda I)x} + \int_0^x e^{(T^- - \lambda I)y} (-T^-) \Phi_\alpha(x - r) \, dr.$$

By [3, Theorem 3.10],

$$\Phi_\alpha(x) = e^{(T^- - \lambda I)x + (-T^-)x} = e^{(T^- - \lambda I)x}.$$

The Markov property implies that

$$\mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^\alpha)$$

$$= \sum_{k=1}^{p} \mathbb{P}^\rho(\tau > x, \varphi_\tau^\lambda = k^\alpha) \mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^\alpha)$$

$$+ \sum_{k=1}^{p} \mathbb{P}^\rho(\tau > x, \varphi_\tau^\lambda = k^a) \mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^\alpha)$$

$$= \sum_{k=1}^{p} (e_i^T \Phi_\alpha(x)e_k)(e_k^T s^+)dx + \sum_{k=1}^{p} (e_i^T \Phi_\alpha(x)e_k)(e_k^T s^- dx)$$

$$= e_i^T (\Phi_\alpha(x)s^+ + \Phi_\alpha(x)s^-)dx.$$

Similarly,

$$\mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^\alpha) = e_i^T (\Phi_\alpha(x)s^+ + \Phi_\alpha(x)s^-)dx.$$

Thus,

$$\mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^\alpha) - \mathbb{P}^\rho(\tau \in [x, x + dx], \varphi_\tau^\lambda = \Delta^a)$$

$$= e_i^T ((\Phi_\alpha(x)s^+ + \Phi_\alpha(x)s^-) - [\Phi_\alpha(x)s^+ + \Phi_\alpha(x)s^-])dx$$

$$= e_i^T ((\Phi_\alpha(x) - \Phi_\alpha(x))s^+ - [\Phi_\alpha(x) - \Phi_\alpha(x)]s^-)dx$$

$$= e_i^T \Phi_\alpha(x)sdx = e_i^T e^{(T^- - \lambda I)x} sdx.$$
Theorem 3.2. follows.

Proof. so that (8) holds. Analogous arguments follow for (10) and (11), which completes the proof.

Heuristically, Equations (8) and (10) imply that initiating \( \{q_t^\lambda\}_{t \geq 0} \) in the anti-state \( i^a \) has the opposite effect, in terms of sign, to initiating in the original state \( i^o \). In the following we exploit this fact to provide a probabilistic interpretation not only for the elements of \( e^{(T-\lambda I)x} \), but also for the exponentially tilted matrix-exponential density \( \alpha e^{(T-\lambda I)x} \).

Define \( w^+ \) and \( w^- \) by

\[
w^\pm = \sum_{i=1}^{p} \max\{0, \pm \alpha_i\},
\]

and define \( \alpha^+ = (\alpha_1^+, \ldots, \alpha_p^+) \) and \( \alpha^- = (\alpha_1^-, \ldots, \alpha_p^-) \) by

\[
\alpha_i^\pm = \begin{cases} 
\frac{1}{w^\pm} \max\{0, \pm \alpha_i\} & \text{if } w^\pm > 0, \\
0 & \text{if } w^\pm = 0.
\end{cases}
\]

If \( w^\pm > 0 \), then \( \alpha^\pm \) is a probability vector, and in general,

\[
\alpha = w^+ \alpha^+ - w^- \alpha^-.
\]

In some sense, \( (w^+ + w^-)^{-1} \alpha \) can be thought as a mixture of the probability vectors \( \alpha^+ \) and \( \alpha^- \), with the latter contributing ‘negative mass’. Fortunately, this ‘negative mass’ in the context of \( \alpha e^{(T-\lambda I)x} \) can be given a precise probabilistic interpretation by means of anti-states as follows.

**Theorem 3.2.** Let \( f_\lambda(x) = (\alpha(\lambda I - T)^{-1}s)^{-1} \alpha e^{(T-\lambda I)x} \), \( x \geq 0 \), be the density of the exponentially tilted matrix-exponential distribution of parameters \( (\alpha, T, s) \). Define the vectors

\[
\hat{\alpha}^+ := \frac{w^+}{w^+ + w^-} \alpha^+ \quad \text{and} \quad \hat{\alpha}^- := \frac{w^-}{w^+ + w^-} \alpha^-,
\]

and suppose \( \phi_0^\lambda \sim (\hat{\alpha}^+, \hat{\alpha}^-) \). Then

\[
f_\lambda(x)dx = \frac{(w^+ + w^-)}{\alpha(\lambda I - T)^{-1}s} \mathbb{E}[\mathbb{1}\{\tau \in [x, x + dx]\} \beta(\phi_0^\lambda)]
\]

\[
= \frac{(w^+ + w^-)}{\alpha(\lambda I - T)^{-1}s} \left(\hat{\alpha}^+, \hat{\alpha}^-\right) \exp\left(\begin{bmatrix} T^+ - \lambda I & T^- \\ T^- & T^+ - \lambda I \end{bmatrix}x\right) \begin{bmatrix} s \\ -s \end{bmatrix} dx,
\]

where \( \tau \) and \( \beta(\cdot) \) are defined as in Theorem 3.1.

**Proof.** Equation (13) implies that

\[
f_\lambda(x) = \frac{1}{\alpha(\lambda I - T)^{-1}s} \left( \sum_{i=1}^{p} w^+ \alpha_i^+ \left(e_i^T e^{(T-\lambda I)x} \right) + \sum_{i=1}^{p} w^- \alpha_i^- \left(-e_i^T e^{(T-\lambda I)x} \right) \right)
\]

\[
= \frac{(w^+ + w^-)}{\alpha(\lambda I - T)^{-1}s} \left( \sum_{i=1}^{p} w^+ \alpha_i^+ \left(e_i^T e^{(T-\lambda I)x} \right) + \sum_{i=1}^{p} w^- \alpha_i^- \left(-e_i^T e^{(T-\lambda I)x} \right) \right).
\]

(16)
Equality (14) follows from (16), (8), and (10). Equality (15) follows from (16), (9), and (11).

**Example 3.1.** Let \((\alpha, T, s)\) be the matrix-exponential parameters corresponding to (5). As noted previously, these parameters by themselves lack a probabilistic interpretation, so we apply Theorem 3.1 to construct one. For such parameters we take the tilting parameter \(\lambda := \lambda_0 = 2\), leading to the block-partitioned matrices

\[
\begin{bmatrix}
  T^+ - \lambda I & T^- \\
  T^- & T^+ - \lambda I
\end{bmatrix} =
\begin{bmatrix}
  -3 & 0 & 2/3 & 0 & 1 & 0 \\
  1 & -3 & 0 & 0 & 0 & 2/3 \\
  0 & 0 & -3 & 0 & 0 & 0 \\
  0 & 1 & 0 & -3 & 0 & 2/3 \\
  0 & 0 & 2/3 & 1 & -3 & 0 \\
  0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix},
\]

\[
(\tilde{\alpha}^+, \tilde{\alpha}^-) = (1, 0, 0, 0, 0, 0), \quad \begin{bmatrix}
  s \\
  -s
\end{bmatrix} =
\begin{bmatrix}
  4/3 \\
  2/3 \\
  1 \\
  -4/3 \\
  -2/3 \\
  -1
\end{bmatrix},
\]

and \(w^+ = 1, w^- = 0\). We can then verify that

\[
\frac{(w^+ + w^-)}{\alpha(\lambda I - T)^{-1}s} \left( (\tilde{\alpha}^+, \tilde{\alpha}^-) \exp \left( \begin{bmatrix} T^+ - \lambda I & T^- \\
  T^- & T^+ - \lambda I \end{bmatrix} x \right) \begin{bmatrix} s \\
  -s \end{bmatrix} \right)
\]

\[
= \frac{1}{\alpha(\lambda I - T)^{-1}s} \left( \frac{2}{3} e^{-3x} (1 + \cos(x)) \right) = \frac{e^{-2x}}{\alpha(\lambda I - T)^{-1}s} \left( \frac{2}{3} e^{-x} (1 + \cos(x)) \right),
\]

the latter corresponding to the exponentially tilted matrix-exponential density function \(f(x) = \frac{2}{3} e^{-x} (1 + \cos(x))\).

A probabilistic interpretation of \(f_\lambda\) alternative to that of (14) is the following.

**Corollary 3.1.** Define \(d = (d_1, \ldots, d_p)^T := -(T^+ - \lambda I)1 - T^- 1\) to be the termination intensities vector from \(E^o\) or \(E^a\), and define \(q^\pm = (q^\pm_1, \ldots, q^\pm_p)^T\) by

\[
q^\pm_i = \begin{cases}
  \frac{s^\pm_i}{d_i} & \text{if } d_i > 0, \\
  0 & \text{if } d_i = 0.
\end{cases}
\]

Let \(\tilde{q} : E^o \cup E^a \mapsto \mathbb{R}\) be defined by

\[
\tilde{q}(i^o) = q^+_i - q^-_i \quad \text{and} \quad \tilde{q}(i^a) = q^-_i - q^+_i \quad \text{for } i \in \{1, \ldots, p\}.
\]
Then
\[ f_\lambda(x)dx = \left( \frac{w^+ + w^-}{\alpha(\lambda I - T)^{-1}s} \right) \mathbb{E} \left[ 1\{ \tau \in [x, x + dx] \} \tilde{q}(\varphi_\tau^{\lambda-}) \right], \]  

where \( \{ \varphi_\tau^{\lambda}\} \geq 0 \) and \( \tau \) are defined as in Theorem 3.2.

**Proof.** First, notice that the jump mechanism of \( \{ \varphi_\tau^{\lambda}\} \geq 0 \) described in (6) implies that for \( i \in \{1, \ldots, p\} \),
\[
\mathbb{P}(\varphi_\tau^{\lambda} = \Delta^o \mid \tau, \varphi_\tau^{\lambda-} = i^o) = q_i^+, \\
\mathbb{P}(\varphi_\tau^{\lambda} = \Delta^a \mid \tau, \varphi_\tau^{\lambda-} = i^o) = q_i^-, \\
\mathbb{P}(\varphi_\tau^{\lambda} = \Delta^a \mid \tau, \varphi_\tau^{\lambda-} = i^o) = q_i^-, \\
\mathbb{P}(\varphi_\tau^{\lambda} = \Delta^a \mid \tau, \varphi_\tau^{\lambda-} = i^o) = q_i^+, \\
\]
which in turn implies that
\[
\mathbb{E}[\beta(\varphi_\tau^{\lambda}) \mid \tau, \varphi_\tau^{\lambda-}] = \mathbb{P}(\varphi_\tau^{\lambda} = \Delta^o \mid \tau, \varphi_\tau^{\lambda-}) - \mathbb{P}(\varphi_\tau^{\lambda} = \Delta^a \mid \tau, \varphi_\tau^{\lambda-}) = \tilde{q}(\varphi_\tau^{\lambda-}).
\]

Consequently,
\[
\mathbb{E}\left[ 1\{ \tau \in [x, x + dx] \} \beta(\varphi_\tau^{\lambda}) \right] = \mathbb{E}\left[ \mathbb{E}\left[ 1\{ \tau \in [x, x + dx] \} \beta(\varphi_\tau^{\lambda}) \mid \tau, \varphi_\tau^{\lambda-} \right] \right] \\
= \mathbb{E}\left[ 1\{ \tau \in [x, x + dx] \} \mathbb{E}[\beta(\varphi_\tau^{\lambda}) \mid \tau, \varphi_\tau^{\lambda-}] \right] \\
= \mathbb{E}\left[ 1\{ \tau \in [x, x + dx] \} \tilde{q}(\varphi_\tau^{\lambda-}) \right],
\]
and the result follows from (14). □

Though closely related, the interpretation provided by Corollary 3.1 is more suitable than that of Theorem 3.2 for Monte Carlo applications. Indeed, a realization of \( \{ \varphi_\tau^{\lambda}\} \geq 0 \) may get absorbed in \( \Delta^o \), \( \Delta^a \) or terminated. If termination occurs, such a realization contributes nothing to the term in the right-hand side of (14). In contrast, by observing the process until its exit time from \( \mathcal{E}^o \cup \mathcal{E}^a \) and disregarding its landing point as in Corollary 3.1, we make sure that each realization contributes towards the mass in the right-hand side of (17).

4. Recovering the untilted distribution

Once the exponentially tilted density \( f_\lambda \) of a matrix-exponential distribution of parameters \( (\alpha, T, s) \) has a tractable known form, say as in (15), in principle it is straightforward to recover the original untilted density \( f \) by taking
\[
f(x) = (\alpha(\lambda I - T)s) e^{\lambda x} f_\lambda(x) \\
= (w^+ + w^-)(\alpha^+, \alpha^-) \exp\left( \begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x \right) \begin{bmatrix} s \\ -s \end{bmatrix}, \quad x \geq 0. \]  

While (18) is a legitimate matrix-exponential representation of \( f \), it has two drawbacks:
1. The matrix \[
\begin{bmatrix}
T^+ & T^- \\
T^- & T^+
\end{bmatrix}
\] may no longer be a subintensity matrix.

2. The dominant eigenvalue of \[
\begin{bmatrix}
T^+ & T^- \\
T^- & T^+
\end{bmatrix}
\] may be nonnegative.

The first item may affect the probabilistic interpretation of \( f \), while the second one may make integration of certain functions (with respect to the density \( f \)) more difficult to handle. For instance, in the context of Example 3.1, the matrix

\[
\begin{bmatrix}
-1 & 0 & 2/3 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 2/3 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 2/3 \\
0 & 0 & 2/3 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

is not a subintensity matrix since some row sums are strictly positive, and it has 0 as its dominant eigenvalue. Having 0 as an eigenvalue implies that some entries of \( \exp\left(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x \right) \) may potentially be of order \( e^{0\cdot x} = 1 \), meaning that the matrix integral

\[
\int_0^\infty h(x) \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \, dx \tag{19}
\]

may only be well-defined for functions \( h : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) that decrease to 0 fast enough. In comparison, \( \exp(Tx) \) with \( T \) as in (5) has entries of order \( e^{0\cdot x} = e^{-x} \) or less, so that

\[
\int_0^\infty h(x) \exp(Tx) \, dx \tag{20}
\]

is well-defined and finite for every function \( h : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) of order \( e^{-\rho x} \) for any \( \rho > -1 \). This apparent disagreement between the applicability of (19) and (20) vanishes when we multiply the elements of the vector \( \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \) cancelling each other when we multiply the matrix function by \( \begin{bmatrix} s \\ -s \end{bmatrix} \).

In the general case, this cancellation of higher-order terms is implied by Theorem 3.1 via the following arguments. If \( \sigma_0 \) is the dominant eigenvalue of \( T \) and has multiplicity \( m_0 \), then the order of \( e^T \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \) is at most \( x^{m_0} \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \). By (9) and (11), the elements of \( \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \) are also of order less than or equal to \( x^{m_0} \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \). Finally, if we multiply the previous by \( e^{\lambda x} \), then we get that \( \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \) is of order at most \( x^{m_0} \exp(\begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix} x) \), which coincides with the order of \( e^{Tx} \).

In terms of expectations, (14) and (17) provide alternative ways to recover properties of any matrix-exponential density \( f \) of parameters \((\alpha, T, s)\) in terms of the exponentially tilted
density $f_\lambda$. Indeed, for any function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ of order $e^{-\rho x}$ or less, $\rho > \sigma_0$, we have that
\[
\int_0^\infty h(x)f(x)dx = (\alpha(\lambda I - T)s) \int_0^\infty h(x)e^{\lambda x}f_\lambda(x)dx
\]
\[= (w^+ + w^-)\mathbb{E}\left[h(\tau)e^{\lambda \tau} \beta(\varphi^{\lambda}_\tau)\right]
\]
\[= (w^+ + w^-)\mathbb{E}\left[h(\tau)e^{\lambda \tau} \bar{q}(\varphi^{\lambda}_{\tau-})\right],
\]
where $(\varphi^{\lambda}_t)_{t \geq 0}$ and $\tau$ are as in Theorem 3.2. Existence and finiteness of the first moment of $h(\tau)e^{\lambda \tau} \beta(\varphi^{\lambda}_\tau)$ in (22) is guaranteed by noting that the order $e^{-\rho + \lambda}$ of $h(x)e^{\lambda x}$ is dominated by that of $f_\lambda$. Notice that, as opposed to the formula in (18), the representations (22) and (23) still have probabilistic interpretations in terms of the Markov jump process $(\varphi^{\lambda}_t)_{t \geq 0}$.

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