Quantum information aspects of approximate position measurement

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Abstract

We perform a quantum information analysis for multi-mode Gaussian approximate position measurements, underlying noisy homodyning in quantum optics. The “Gaussian maximizer” property is established for the entropy reduction of these measurements which provides explicit formulas for computations including their entanglement-assisted capacity. The case of one mode is discussed in detail.

Keywords: approximate position measurement, entropy reduction, Gaussian maximizers, energy constraint, entanglement-assisted capacity, continuous variable system

1 Introduction

Among quantitative characteristics of quantum measurement, the entropy reduction and the entanglement-assisted classical information capacity (called also measurement strength or information gain, depending on different operational interpretations) play a significant role. We refer a reader to [16], [15], [6], [11], [12] and to [2] where one can find also a detailed survey and further references. In [10] we presented a study of these quantities for a class of multi-mode Gaussian approximate joint position-momentum measurements (or, in the context of quantum optics, “noisy heterodyning”). In particular, for this class of measurements we established the “Gaussian maximizer” property of the entropy reduction which allowed for an explicit computation of this quantity and of the related entanglement-assisted capacity. In the present paper we perform a similar analysis for another important class of
multi-mode Gaussian measurements, namely, of approximate position measurements underlying the “noisy homodyning” which is in a sense opposite to the “heterodyning” (see [3], [5], [14] for a detailed physical description of the relevant measurement processes). The “Gaussian maximizer” property is established in theorem 1 which also gives the explicit formula for computations. The entanglement-assisted capacity is considered in sec. 4 and the case of one mode is discussed in detail in sec. 5.

2 Entropy reduction of quantum observable

Let $H$ be the separable Hilbert space of a quantum system, and let $\mathcal{S}(H)$ be the convex set of quantum states (density operators on $H$). Let $(\mathcal{X}, \mathcal{F})$ be a standard measurable space of the outcomes of a measurement described by an completely positive (c.p.) instrument $\mathcal{M} = \{\mathcal{M}(A)[\cdot], A \in \mathcal{F}\}$, i.e. operation-valued measure where for each $A$ the map $\rho \to \mathcal{M}(A)[\rho], \rho \in \mathcal{S}(H)$, is c.p. trace-nonincreasing, and $\mathcal{M}(\mathcal{X})$ is trace-preserving (t.p.) (see, e.g. [1]). Let $M = \{M(A), A \in \mathcal{F}\}$ be the associated observable i.e. a probability operator-valued measure (POVM) on $(\mathcal{X}, \mathcal{F})$ such that $M(A) = M^*(A)[I]$, where $I$ is the identity operator on $H$. The probability distribution of the observable $M$ in the state $\rho \in \mathcal{S}(H)$ is given by the formula

$$ P_\rho(A) = \text{Tr}_\rho M(A), \quad A \in \mathcal{F}. $$

As shown in [13] for each $\rho \in \mathcal{S}(H)$ there is a family of posterior states $\{\hat{\rho}(x)\}$ such that

$$ \mathcal{M}(A)[\rho] = \int_A \hat{\rho}(x)P_\rho(dx). $$

for any state $\rho \in \mathcal{S}(H)$. The entropy reduction of the instrument is defined as

$$ ER(\rho, \mathcal{M}) = H(\rho) - \int_\mathcal{X} H(\hat{\rho}(x))P_\rho(dx), $$

where $H(\rho) = -\text{Tr}_\rho \log \rho$, provided $H(\rho) < \infty$ [13], [15].

We will deal with the special class of instrument, the observables of which have bounded operator-valued density, more precisely,

$$ M(A) = \int_A m(x) \mu(dx), \quad A \in \mathcal{F}, $$

(1)
where $\mu$ is a $\sigma$-finite measure on the $\sigma$-algebra $\mathcal{F}$ such that $m(x) = V(x)^*V(x)$, where $V(x)$ is a weakly measurable function with values in the algebra of bounded operators on $\mathcal{H}$ satisfying

$$\int_X V(x)^* V(x) \mu(dx) = I,$$

and the integral weakly converges \cite{11}. With any measurable factorization of $m(x)$ one can associate an efficient instrument (see \cite{13}, \cite{15})

$$\mathcal{M}(A)[\rho] = \int_A V(x) \rho V(x)^* \mu(dx).$$

Then the probability distribution $P_\rho(dx)$ has the density

$$p_\rho(x) = \text{Tr}\rho V(x)^* V(x)$$

with respect to the measure $\mu$, while the family of posterior states is

$$\dot{\rho}(x) = p_\rho(x)^{-1} V(x) \rho V(x)^*$$

The entropy reduction of the efficient instrument is then

$$ER(\rho, M) = H(\rho) - \int_\Omega p_\rho(x) H(\dot{\rho}(x)) \mu(dx),$$

where the posterior states are given by (3). In \cite{13} it was shown that the entropy reduction of an efficient instrument is nonnegative. In \cite{15} the entropy reduction was related to the quantum mutual information of the instrument and hence it is a concave, subadditive, lower semicontinuous function of $\rho$.

An essential observation made in \cite{6}, \cite{10} is that $H(\dot{\rho}(x))$ in the entropy reduction (4) depend only on $m(x)$ (i.e. on the observable $M$) and not on the way of its measurable factorization (i.e. the choice of an efficient instrument), which justifies the notation $ER(\rho, M)$.

### 3 Gaussian maximizers for the entropy reduction

Our framework will be a bosonic system with $s$ modes described by the canonical position-momentum operators $q = [q_1, \ldots, q_s]^t$, $p = [p_1, \ldots, p_s]^t$. 


(see e.g. [7], [14]). It will be convenient to take the Schrödinger (position) representation space $\mathcal{H} = L^2(\mathbb{R}^s)$ with the operators $q_j = \xi_j$, $p_j = i^{-1}\frac{\partial}{\partial \xi_j}$. We denote by

$$D(x, y) = e^{-ix^t y/2}e^{iy^t q}e^{-ix^t p}, \quad x, y \in \mathbb{R}^s$$

the unitary displacement operators.

We will be interested in approximate position measurement in $s$ modes. In quantum optics this underlies the multi-mode noisy homodyning, as measurement of any quadrature of the radiation field can be reduced to the measurement of position by a Bogoljubov transformation. In this case $\mathcal{X} = \mathbb{R}^s$ and the POVM is

$$M(d^s x) = \exp \left[ -\frac{1}{2}(q - x)^t \beta^{-1} (q - x) \right] \frac{d^s x}{\sqrt{(2\pi)^s \det \beta}}$$

where $\beta$ is a real positive definite covariance matrix of the measurement noise, and $D(x) = D(x, 0) = \exp (-ix^t p)$ is the position displacement operator. Notice that POVM $M$ has the form (1) with the bounded operator-valued density $m(x)$, hence we are in the situation where the formulas (3) and (4) are applicable.

Denote

$$V(x) = V^*(x) = \sqrt{m(x)} = [(2\pi)^s \det \beta]^{-1/4} \exp \left[ -\frac{1}{4}(q - x)^t \beta^{-1} (q - x) \right].$$

Let $\rho$ be a state and $\rho_{u, v} = D(u, v)\rho D(u, v)^*$ a displaced state. Then

$$V(x) \rho_{u, v} V(x) =$$

$$= D(u, v) (D(u, v)^* V(x) D(u, v)) \rho (D(u, v)^* V(x) \rho D(u, v)) D(u, v)^*$$

$$= D(u, v) V(x + u) \rho V(x + u) D(u, v)^*.$$

It follows that

$$p_{\rho_{u, v}}(x) = p_\rho(x + u), \quad \hat{\rho}_{u, v}(x) = D(u, v) \hat{\rho}(x + u) D(u, v)^*,$$

and

$$ER(\rho_{u, v}, M) = ER(\rho, M) \quad (6)$$
by the unitary invariance of the entropy. Therefore in what follows we can restrict to centered states \( \rho \) (\( \text{Tr} \rho q_j = \text{Tr} \rho p_j = 0 \)).

Consider the quantum state \( \rho_\alpha \in \mathcal{S}(\mathcal{H}) \) which is centered Gaussian density operator in \( \mathcal{H} \) with the covariance \( 2s \times 2s \)-matrix

\[
\alpha = \begin{bmatrix}
\alpha_{qq} & \alpha_{qp} \\
\alpha_{pq} & \alpha_{pp}
\end{bmatrix},
\]

(7)

where

\[
\alpha_{qq} = \alpha_{qq}^t = \text{Tr} \rho_\alpha q_j q_k |_{j,k=1,\ldots,s}, \quad \alpha_{pp} = \alpha_{pp}^t = \text{Tr} \rho_\alpha p_j p_k |_{j,k=1,\ldots,s},
\]

\[
\alpha_{qp} = \alpha_{pq}^t = \frac{1}{2} \text{Tr} \rho_\alpha (q_j p_k + p_k q_j) |_{j,k=1,\ldots,s}.
\]

It is explicitly given by the kernel in the Schrödinger representation

\[
\langle \xi | \rho_\alpha | \xi' \rangle = \frac{1}{\sqrt{(2\pi)^s \det \alpha_{qq}}} \exp \left[ -\frac{1}{2} \theta (\xi, \xi') \right],
\]

(8)

where

\[
\theta (\xi, \xi') = \frac{1}{4} (\xi' + \xi)^t \alpha_{qq}^{-1} (\xi' + \xi) + (\xi' - \xi)^t (\alpha_{pp} - \alpha_{pq} \alpha_{qq}^{-1} \alpha_{qp}) (\xi' - \xi)
\]

\[
+ i (\xi' + \xi)^t \alpha_{qq}^{-1} \alpha_{qp} (\xi' - \xi),
\]

(9)

see Appendix.

Let

\[
\rho_\alpha (x) = V(x) \rho_\alpha V(x),
\]

then the probability distribution of the observable (5) has the density

\[
p_\alpha (x) = \text{Tr} \rho_\alpha (x)
\]

(10)

and the posterior states are

\[
\hat{\rho}_\alpha (x) = \rho_\alpha (x) / p_\alpha (x).
\]

(11)

**Lemma 1.** The probability density (10) is Gaussian:

\[
p_\alpha (x) = \frac{1}{\sqrt{(2\pi)^s \det (\alpha_{qq} + \beta)}} \exp \left[ -\frac{1}{2} x^t (\alpha_{qq} + \beta)^{-1} x \right],
\]

(12)
and the posterior states \((11)\) are Gaussian
\[
\hat{\rho}_\alpha(x) = D(K_q x, K_p x) \rho_\alpha D(K_q x, K_p x)^*,
\]
where
\[
K_q = \alpha_{qq} (\alpha_{qq} + \beta)^{-1}, \quad K_p = \alpha_{pq} (\alpha_{qq} + \beta)^{-1},
\]
and \(\rho_\alpha\) is centered Gaussian state with the covariance matrix
\[
\hat{\alpha} = \begin{bmatrix}
\hat{\alpha}_{qq} & \hat{\alpha}_{qp} \\
\hat{\alpha}_{pq} & \hat{\alpha}_{pp}
\end{bmatrix}
\]
with
\[
\hat{\alpha}_{qq} = (\alpha_{qq}^{-1} + \beta^{-1})^{-1},
\]
\[
\hat{\alpha}_{pp} = (\alpha_{pp} - \alpha_{pq} (\alpha_{qq} + \beta)^{-1} \alpha_{qp}) + \frac{1}{4} \beta^{-1},
\]
\[
\hat{\alpha}_{qp} = (\alpha_{qq}^{-1} + \beta^{-1} - \alpha_{qq}^{-1} \alpha_{qp}).
\]

Proof. From the quantum characteristic function of \(\rho_\alpha\) we have
\[
\text{Tr} \rho_\alpha \exp (iu^t q) = \exp \left( -\frac{1}{2} u^t \alpha_{qq} u \right),
\]
hence the diagonal value of the kernel of \(\rho_\alpha\) in the position representation is the inverse Fourier transform
\[
\langle \xi | \rho_\alpha | \xi \rangle = \frac{1}{\sqrt{(2\pi)^s \det \alpha_{qq}}} \exp \left[ -\frac{1}{2} \xi^t \alpha_{qq}^{-1} \xi \right],
\]
and
\[
p_\alpha(x) = \text{Tr} \rho_\alpha(x) = \int \langle \xi | \rho_\alpha | \xi \rangle \exp \left[ -\frac{1}{2} (\xi - x)^t \beta^{-1} (\xi - x) \right] \frac{d^s \xi}{\sqrt{(2\pi)^s \det \beta}}
\]
is the convolution of the two Gaussian probability densities giving the right-hand side of \((12)\).

The posterior state \((11)\) is Gaussian since it has the Gaussian kernel in the Schrödinger representation
\[
\langle \xi | \hat{\rho}_\alpha(x) | \xi' \rangle = \frac{1}{\sqrt{(2\pi)^s \det \beta}} \exp \left[ -\frac{1}{4} (\xi - x)^t \beta^{-1} (\xi - x) - \frac{1}{4} (\xi' - x)^t \beta^{-1} (\xi' - x) \right] \langle \xi | \rho_\alpha | \xi' \rangle / p_\alpha(x),
\]
where $\langle \xi | \rho_\alpha | \xi' \rangle$ is given by (8) and $p_\alpha (x)$ by (12). Substituting (8) and (12), we obtain

$$
\langle \xi | \hat{\rho}_\alpha (x) | \xi' \rangle = \sqrt{\det (\alpha_{qq} + \beta)} \pi^{s \det \beta} \exp \left[ \hat{\mu} (x; \xi, \xi') - \frac{1}{2} \hat{\theta} (\xi, \xi') \right],
$$

where

$$
\hat{\mu} (x; \xi, \xi') = \frac{1}{2} x^t \beta^{-1} \xi + \frac{1}{2} x^t \beta^{-1} \xi' - \frac{1}{2} x^t \beta^{-1} x + \frac{1}{2} x^t (\alpha_{qq} + \beta)^{-1} x,
$$

$$
\hat{\theta} (\xi, \xi') = \frac{1}{2} \xi^t \beta^{-1} \xi + \frac{1}{2} (\xi')^t \beta^{-1} \xi' + \theta (\xi, \xi'),
$$

and $\theta (\xi, \xi')$ is given by (9). Let us first consider the terms independent of $x$. We have

$$
\hat{\theta} (\xi, \xi') = \frac{1}{4} (\xi' + \xi)^t \beta^{-1} (\xi' + \xi) + \frac{1}{4} (\xi' - \xi)^t \beta^{-1} (\xi' - \xi) + \theta (\xi, \xi')
$$

$$
= \frac{1}{4} (\xi' + \xi)^t \hat{\alpha}_{qq}^{-1} (\xi' + \xi) + (\xi' - \xi)^t (\hat{\alpha}_{pp} - \hat{\alpha}_{pq} \hat{\alpha}_{qq}^{-1} \hat{\alpha}_{qp}) (\xi' - \xi)
$$

$$
+ i (\xi' + \xi)^t \hat{\alpha}_{qq}^{-1} \hat{\alpha}_{qp} (\xi' - \xi),
$$

where

$$
\hat{\alpha}_{qq}^{-1} = \alpha_{qq}^{-1} + \beta^{-1},
$$

$$
\hat{\alpha}_{qq}^{-1} \hat{\alpha}_{qp} = \alpha_{qq}^{-1} \alpha_{qp},
$$

$$
\hat{\alpha}_{pp} - \hat{\alpha}_{pq} \hat{\alpha}_{qq}^{-1} \hat{\alpha}_{qp} = (\alpha_{pp} - \alpha_{pq} \alpha_{qq}^{-1} \alpha_{qp}) + \frac{1}{4} \beta^{-1},
$$

resulting in the elements of the posterior covariance matrix (14).

To find the posterior mean values $\hat{m}_q, \hat{m}_p$, we have

$$
\hat{\mu} (x; \xi, \xi') = \frac{1}{2} x^t \beta^{-1} (\xi + \xi') - \frac{1}{2} x^t \beta^{-1} (\alpha_{qq}^{-1} + \beta^{-1})^{-1} \beta^{-1} x
$$

$$
= \frac{1}{2} (K_q x)^t \hat{\alpha}_{qq}^{-1} (\xi + \xi') - \frac{1}{2} (K_q x)^t \hat{\alpha}_{qq}^{-1} (K_q x),
$$

where $K_q = \alpha_{qq} (\alpha_{qq} + \beta)^{-1}$. Comparing with the corresponding terms under the exponent in (39) (see Appendix), we should have for the posterior mean values:

$$
\hat{m}_q = K_q x, \quad \hat{m}_p = \hat{\alpha}_{pq} \hat{\alpha}_{qq}^{-1} \hat{m}_q.
$$
where the second relation follows from the fact that the second term in (39) is zero in our case. Thus \( \hat{m}_p = K_p x \), where \( K_p = \alpha_{pq} (\alpha_{qq} + \beta)^{-1} \). \( \square \)

Let \( \mathcal{G}(\alpha) \) be the set of all (not necessarily centered) states \( \rho \) with the covariance matrix \( \alpha \). We will study the following entropic characteristic of the Gaussian measurement \( M \)

\[
ER(M; \alpha) = \sup_{\rho \in \mathcal{G}(\alpha)} ER(\rho, M),
\]

which is strictly related to the entanglement-assisted capacity of \( M \) (see sec. 4). Due to (6), in (15) we can restrict to centered states \( \rho \) for which \( \alpha \) coincides with the matrix of second moments. We denote

\[
g(x) = (x + 1) \log(x + 1) - x \log x.
\]

**Theorem 1.** The supremum in (15) is attained on the Gaussian state \( \rho_\alpha \) and it is equal to

\[
ER(M; \alpha) = \frac{1}{2} \left[ \text{Sp} \left( \text{abs}(\Delta^{-1}) - \frac{I_2}{2} \right) \right] - \text{Sp} \left( \text{abs}(\Delta^{-1}) \hat{\alpha} - \frac{I_2}{2} \right),
\]

(16)

where \( \hat{\alpha} \) is given by (14), and \( \text{abs}(\Delta^{-1}) \) denotes the matrix with eigenvalues equal to modulus of eigenvalues of \( \Delta^{-1} \) and with the same eigenvectors.

**Proof.** Let \( \rho \) be a centered density operator from \( \mathcal{G}(\alpha) \), then denote \( \rho(x) = V(x) \rho V(x) \) and \( p(x) = \text{Tr} \rho(x) \). Also introduce the channel \( \mathcal{E}[\rho] = \{V(x) \rho V(x)\} \) with quantum input and hybrid classical-quantum (cq)-output. For the Gaussian state \( \rho_\alpha \) we have

\[
ER(\rho_\alpha, M) - ER(\rho, M) = H(\rho || \rho_\alpha) + \text{Tr}(\rho - \rho_\alpha) \log \rho_\alpha
\]

- \( H_{cq}(\mathcal{E}[\rho] || \mathcal{E}[\rho_\alpha]) + H_c(p || p_\alpha) \)

+ \( \int \text{Tr}(\rho(x) - \rho(x)) \log \rho_\alpha(x) dt^x \).

Here

\[
H_c(p || p_\alpha) = \int p(x) \log \left( \frac{p(x)}{p_\alpha(x)} \right) dt^x
\]

is the classical relative entropy between \( p, p_\alpha \) and

\[
H_{cq}(\mathcal{E}[\rho] || \mathcal{E}[\rho_\alpha]) = \int \text{Tr} \rho(x) \left( \log \rho(x) - \log \rho_\alpha(x) \right) dt^x,
\]

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is the relative entropy of the cq-states (see Eq. (3) in [1]).

Monotonicity of the relative entropy for cq-states ([1], theorem 1) then implies

\[ H_{cq}(E[\rho] || E[\rho_\alpha]) \leq H(\rho || \rho_\alpha), \]

hence we have for the first three terms in the right-hand side of (17)

\[ H(\rho || \rho_\alpha) - H_{cq}(E[\rho] || E[\rho_\alpha]) + H_c(p || p_\alpha) \geq 0. \] (18)

Without loss of generality we can assume that \( \rho_\alpha \) is non-degenerate so that \( \log \rho_\alpha \) exists and it is a polynomial in \( q, p \) of the second order. This follows from the exponential form of the density operator (theorem 12.23 in [7]). Since the first and second moments of the states \( \rho \) and \( \rho_\alpha \) coincide, we have

\[ \text{Tr}(\rho - \rho_\alpha) \log \rho_\alpha = 0. \] (19)

It remains to show that also

\[ \int \text{Tr}(\rho_\alpha(x) - \rho(x)) \log \rho_\alpha(x) d^nx = 0. \] (20)

Substituting the posterior state (13) into the right-hand side of (20), we obtain

\[ \int \text{Tr}(\rho_\alpha(x) - \rho(x)) \log \rho_\alpha(x) d^nx = \text{Tr} \Phi_M[\rho_\alpha - \rho] \log \rho_\alpha \]

\[ = \text{Tr} (\rho_\alpha - \rho) \Phi_M^* [\log \rho_\alpha], \] (21)

where we have introduced the channel

\[ \Phi_M[\sigma] = \int D(K_q x, K_p x)^* V(x) \sigma V(x) D(K_q x, K_p x) d^nx. \] (23)

The channel \( \Phi_M \) is Gaussian. First, it is a c.p.t.p. map. Complete positivity is apparent from the structure of the map (23). Trace preservation follows from

\[ \Phi_M^*[I] = \int V(x)^2 d^nx = \int M(d^nx) = I. \]

A routine calculation shows that if \( \sigma \) is a Gaussian state then \( \Phi_M[\sigma] \) is again Gaussian. Then by result of [4], \( \Phi_M \) is a Gaussian channel.

Since \( \rho_\alpha \) is Gaussian state, \( \log \rho_\alpha \) is a polynomial in \( q, p \) of the second order. The dual Gaussian channel \( \Phi_M^* \) takes it into another polynomial in
$q, p$ of the second order (see sec. 12.4.2 of [7]). Since the first and second moments of the states $\rho$ and $\rho_\alpha$ coincide, one obtains (20). Then (17), (18), (19) and (20) imply

$$ER(\rho_\alpha, M) - ER(\rho, M) \geq 0$$

for arbitrary $\rho \in S_\alpha$ proving that the supremum (15) is attained on the Gaussian state $\rho_\alpha$.

Taking into account the formula (13) for the posterior states, we obtain

$$ER(M; \alpha) = ER(\rho_\alpha, M) = H(\rho_\alpha) - \int_\Omega p_\alpha(x) H(\hat{\rho}_\alpha(x)) d^s x = H(\rho_\alpha) - \int_\Omega p_\alpha(x) H(D(K_q x, K_p x) \rho_\alpha D(K_q x, K_p x)^* d^s x = H(\rho_\alpha) - H(\rho_{\hat{\alpha}}).$$

Then (16) follows by applying the formula for the entropy of a Gaussian state

$$H(\rho_\alpha) = \frac{1}{2} Sp g \left( \text{abs} (\Delta^{-1} \alpha) - \frac{I_2}{2} \right),$$

given in [7], Eq. (12.110). □

4 The entanglement-assisted capacity

Consider a quadratic Hamiltonian of the form

$$H = \sum_{j,k=1}^n \left( q^j \epsilon_{qq} q + q^j \epsilon_{qp} p + p^j \epsilon_{pq} q + p^j \epsilon_{pp} p \right),$$

where

$$\epsilon = \begin{bmatrix} \epsilon_{qq} & \epsilon_{qp} \\ \epsilon_{pq} & \epsilon_{pp} \end{bmatrix}$$

is a real symmetric positive definite energy matrix. The importance of the quantity (15) is that it underlies the energy-constrained entanglement-assisted capacity, given by the formula (see e.g. [10])

$$C_{ea}(M, H, E) = \max_{\rho : Tr\rho H \leq E} ER(\rho, M),$$

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where $E$ is the energy level. For any state $\rho \in \mathcal{G}(\alpha)$ the mean energy
\[
\text{Tr}\rho H = \text{Sp} \epsilon \alpha_2 \geq \text{Sp} \epsilon \alpha,
\]
where $\alpha_2$ is the matrix of second moments of $\rho$, with the equality attained for centered states, therefore
\[
C_{ea}(M; H, E) = \max_{\alpha: \text{Sp} \epsilon \alpha \leq E} ER(M; \alpha).
\] (25)

It is easy to check that
\[
\sup_{\alpha} ER(M; \alpha) = +\infty,
\]
therefore theorem 1 of [10] applies showing that the maximum in (25) is attained on $\alpha$ satisfying $\text{Sp} \epsilon \alpha = E$.

In the case of the oscillator-type Hamiltonian
\[
H = \sum_{j,k=1}^{s} (q^j \epsilon_{qq} q^k + p^j \epsilon_{pp} p^k),
\]
the energy constraint is
\[
\text{Sp} \epsilon_{qq} \alpha_{qq} + \text{Sp} \epsilon_{pp} \alpha_{pp} \leq E.
\] (26)

Then the maximization in (25) can be taken over only block-diagonal covariance matrices $\alpha$ (satisfying (26) with equality). To see this consider the transformation $T : q_j \rightarrow q_j, p_j \rightarrow -p_j$ which changes the sign of the commutators between $q_j$ and $p_j$ and which is implemented by anti-unitary operator of complex conjugation in the Schrödinger representation. If $\rho$ is a centered state with the covariance matrix \footnote{We denote $\text{Sp}$ trace of matrices as distinct from the trace of operators in $\mathcal{H}$.}
\[
\alpha = \begin{bmatrix} \alpha_{qq} & \alpha_{qp} \\ \alpha_{pq} & \alpha_{pp} \end{bmatrix} \geq \pm i/2 \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix},
\]
then $T|\rho|$ has the covariance matrix
\[
\tilde{\alpha} = \begin{bmatrix} \alpha_{qq} & -\alpha_{qp} \\ -\alpha_{pq} & \alpha_{pp} \end{bmatrix} \geq \mp i/2 \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix},
\]
then $T|\rho|$ has the covariance matrix

\footnote{We denote by $I_s$ the unit $s \times s$–matrix.}
because it is the covariance matrix for $q_j, -p_j$. The mixture $\frac{1}{2} (\rho + T[\rho])$ has the covariance matrix
\[
\begin{bmatrix}
\alpha_{qq} & 0 \\
0 & \alpha_{pp}
\end{bmatrix} = \frac{1}{2} (\alpha + \bar{\alpha}) \geq \pm \frac{i}{2} \begin{bmatrix}
0 & I_s \\
I_s & 0
\end{bmatrix},
\]
while
\[
ER\left(\frac{1}{2} (\rho + T[\rho]), M\right) \geq \frac{1}{2} [ER(\rho, M) + ER(T[\rho], M)] = ER(\rho, M)
\]
by the concavity of the entropy reduction and its invariance under the transformation $T$.

5 The case of one mode

Consider the approximate measurement of position for one bosonic mode $q, p$. The corresponding POVM is
\[
M(dx) = \exp\left[ -\frac{(q - x)^2}{2\beta} \right] \frac{dx}{\sqrt{2\pi\beta}} = D(x)e^{-q^2/2\beta}D(x)^* \frac{dx}{\sqrt{2\pi\beta}},
\]
with the covariance $2 \times 2$–matrix (7), where $\alpha_{qq}, \alpha_{qp}, \alpha_{pp}$ are real numbers satisfying
\[
\alpha_{qq}\alpha_{pp} - \alpha_{qp}^2 \geq 1/4.
\]
The formula (24) for the entropy amounts to (see Example 12.25 in [7])
\[
H(\rho_\alpha) = g \left( \sqrt{\alpha_{qq}\alpha_{pp} - \alpha_{qp}^2} - \frac{1}{2} \right).
\]

According to eq. (13) posterior states have the covariance $2 \times 2$-matrix $\hat{\alpha}$ with the elements
\[
\hat{\alpha}_{qq} = \frac{\alpha_{qq}\beta}{\alpha_{qq} + \beta}, \quad \hat{\alpha}_{pp} = \left( \alpha_{pp} - \frac{\alpha_{qp}^2}{\alpha_{qq} + \beta} \right) + 1/(4\beta), \quad \hat{\alpha}_{qp} = \frac{\alpha_{qp}\beta}{\alpha_{qq} + \beta}.
\]
Then theorem 1 gives

\begin{align}
ER(M; \alpha) &= g \left( \sqrt{\alpha_{qq} \alpha_{pp} - \alpha_{qp}^2} - \frac{1}{2} \right) - g \left( \sqrt{\hat{\alpha}_{qq} \hat{\alpha}_{pp} - \hat{\alpha}_{qp}^2} - \frac{1}{2} \right) \\
&= g \left( \sqrt{\alpha_{qq} \alpha_{pp} - \alpha_{qp}^2} - \frac{1}{2} \right) - g \left( \sqrt{\frac{\alpha_{qq} (\beta \alpha_{pp} + 1/4) - \beta \alpha_{qp}^2}{\alpha_{qq} + \beta}} - \frac{1}{2} \right). \quad (30)
\end{align}

For the oscillator Hamiltonian \( H = \frac{1}{2} (q^2 + p^2) \), the energy constraint has the form \( \alpha_{qq} + \alpha_{pp} \leq 2E \) and the entanglement-assisted capacity (25) is equal to

\[ C_{ea}(M; H, E) = \]

(31)
\[ \max \left[ g \left( \sqrt{\alpha_{qq} \alpha_{pp}} - \frac{1}{2} \right) - g \left( \sqrt{\frac{\alpha_{qq} (\beta \alpha_{pp} + 1/4)}{\alpha_{qq} + \beta}} - \frac{1}{2} \right) \right], \]

where the maximum over positive \( \alpha_{qq}, \alpha_{pp} \) satisfying \( \alpha_{qq} + \alpha_{pp} = 2E, \alpha_{qq} \alpha_{pp} \geq 1/4, \) and \( \alpha_{qp} \) is taken to be 0 (see Fig. 1).

Let us now consider the limit \( \beta \to 0 \) corresponding to the exact measurement of position, where
\[ M_{ex}(dx) = |x\rangle \langle x|dx \]
is the spectral measure of the operator \( q \). Notice that strictly speaking it is not of the form (1), i.e. it does not have a bounded operator-valued density. In the limit \( \beta \to 0 \) we have
\[ ER(M; \alpha) \to g \left( \sqrt{\alpha_{qq} \alpha_{pp}} - \frac{1}{2} \right). \]

We obtain the entanglement-assisted capacity
\[ C_{ea}(M_{ex}, H, E) = \max g \left( \sqrt{\alpha_{qq} \alpha_{pp}} - \frac{1}{2} \right), \]
where the maximum over positive \( \alpha_{qq}, \alpha_{pp} \) satisfying \( \alpha_{qq} + \alpha_{pp} = 2E, \alpha_{qq} \alpha_{pp} \geq 1/4, \) By the concavity of \( g(x) \), this maximum is attained for \( \alpha_{qq} = \alpha_{pp} = E \) and is equal to
\[ C_{ea}(M_{ex}, H, E) = g(E - 1/2). \]

In [12] the POVM (32) was considered as associated with the instrument
\[ M_{ex}[\rho](dx) = |e\rangle \langle e| \rho \langle x|e\rangle dx, \]
where \( |e\rangle \) is an arbitrary unit vector. Notice that here \( V(x) = |e\rangle \langle x| \) is an unbounded and even non-closable operator. Then the posterior state \( |e\rangle \langle e| \) is pure and has zero entropy, resulting in \( ER(M_{ex}; \rho) = H(\rho) \). Thus
\[ ER(M_{ex}; \alpha) = H(\rho_{\alpha}) = g \left( \sqrt{\alpha_{qq} \alpha_{pp}} - \frac{1}{2} \right), \]
which agrees with (33) and leads again to the capacity (35).

Let us consider another realization of the exact measurement – the squeezed joint measurement of \( q, p \), described by POVM
\[ M_{sq}(dx \, dy) = |x, y\rangle \beta \langle x, y| \frac{dx \, dy}{2\pi} = D(x, y) \rho_{sq, \beta} D(x, y)^\dagger \frac{dx \, dy}{2\pi}, \]
where $\rho_{sq,\beta}$ is the squeezed vacuum with the covariance matrix $\begin{bmatrix} \beta & 0 \\ 0 & 1/(4\beta) \end{bmatrix}$.

Then
\[
\int_y M_{sq}(dx \, dy) = M(dx). \tag{37}
\]

To see this it is sufficient to show that
\[
\int D(x, y)\rho_{sq,\beta}D(x, y)^* \frac{dy}{2\pi} = \frac{1}{\sqrt{2\pi\beta}} \exp \left[ -\frac{(q-x)^2}{2\beta} \right]. \tag{38}
\]

Indeed, in the position representation
\[
\langle \xi | \int D(x, y)\rho_{sq,\beta}D(x, y)^* \frac{dy}{2\pi} | \xi' \rangle \\
= \delta(\xi - \xi')\langle \xi - x | \rho_{sq,\beta} | \xi - x \rangle \\
= \langle \xi | \xi' \rangle \langle \xi - x | \rho_{sq,\beta} | \xi - x \rangle.
\]

The classical characteristic function is
\[
\int \langle \xi | \rho_{sq,\beta} | \xi \rangle e^{i\xi\lambda} d\xi = \text{Tr} \rho_{sq,\beta} e^{iq\lambda} = \exp \left( -\frac{1}{2} \beta \lambda^2 \right),
\]

whence
\[
\langle \xi - x | \rho_{sq,\beta} | \xi - x \rangle = \frac{1}{\sqrt{2\pi\beta}} \exp \left[ -\frac{(\xi - x)^2}{2\beta} \right].
\]

which proves (38).

Similarly to (36) we have
\[
ER(M_{sq}; \alpha) = H(\rho_{\alpha}) = g \left( \sqrt{\alpha_{qq}\alpha_{pp}} - \frac{1}{2} \right).
\]

The difference between this expression and (30) reflects the loss of information due to the averaging (37) over the values of the momentum.

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6 Appendix

The kernel of a Gaussian density operator $\rho_{m,\alpha}$ with the mean $m_q, m_p$ and the covariance matrix $\alpha$ in the Schrödinger representation has the form

$$\langle \xi | \rho_{\alpha} | \xi' \rangle = \frac{1}{\sqrt{(2\pi)^s \det \alpha_{qq}}} \exp \left[ \mu(m; \xi, \xi') - \frac{1}{2} \theta(\xi, \xi') \right], \quad (39)$$

where

$$\mu(m; \xi, \xi') = \frac{1}{2} m^t q \alpha_{qq}^{-1} (\xi' + \xi) - i \left( m_p - \alpha_{pq} \alpha_{qq}^{-1} m_q \right)^t (\xi' - \xi) - \frac{1}{2} m^t q \alpha_{qq}^{-1} m_q, \quad (40)$$

$$\theta(\xi, \xi') = \frac{1}{4} \left( \xi' + \xi \right)^t \alpha^{-1}_{qq} (\xi' + \xi) + (\xi' - \xi)^t \left( \alpha_{pp} - \alpha_{pq} \alpha_{qq}^{-1} \alpha_{qp} \right) (\xi' - \xi) + i (\xi' + \xi)^t \alpha^{-1}_{qq} \alpha_{qp} (\xi' - \xi). \quad (41)$$

**Proof.** The quantum characteristic function of $\rho_{m,\alpha}$ is

$$\varphi(x, y) = \text{Tr} \rho_{m,\alpha} W(x, y) = \exp \left[ im(x, y) - \frac{1}{2} \alpha(x, y) \right],$$

where

$$m(x, y) = m_q x + m_p y,$$

$$\alpha(x, y) = \alpha_{qq} x + \alpha_{qp} y + \alpha_{qp} x + \alpha_{pp} y$$

and $W(x, y) = e^{ixq/2} e^{iyq} e^{ipq}$ is the Weyl operator. The kernel of the Weyl operator is

$$\langle \xi | W(x, y) | \xi' \rangle = \exp \left[ i \left( \frac{\xi + \xi'}{2} \right)^t x \right] \delta(y - (\xi' - \xi)) = \exp (iu^t x) \delta(y - v).$$

Here we introduced the variables $u = \frac{\xi + \xi'}{2}$, $v = \xi' - \xi$. Using the inversion formula quantum Fourier transform, we readily compute the kernel of the Gaussian state:

$$\langle \xi | \rho_{m,\alpha} | \xi' \rangle = \frac{d^s x d^s y}{(2\pi)^s} \varphi(x, y) \langle \xi | W(-x, -y) | \xi' \rangle =$$

$$= \int \frac{d^s x}{(2\pi)^s} \exp \left[ -iu^t x + im(x, -v) - \frac{1}{2} \alpha(x, -v) \right],$$

16
the expression under the exponent expands as
\[ -im_p^tv - \frac{1}{2}v'^{\alpha_{pp}}v - i(u'^t - m_q^t + iv'^{\alpha_{pq}}x - \frac{1}{2}x'^{\alpha_{qq}}x. \]

Taking the \(s\)-dimensional Gaussian integral yields
\[
\langle \xi | \rho_\alpha | \xi' \rangle = \frac{e^{-im_p^tv - \frac{1}{2}v'^{\alpha_{pp}}v}}{\sqrt{(2\pi)^s \det \alpha_{qq}}} \exp \left[ -\frac{1}{2}(u - m_q + i\alpha_{qp}v)^{\alpha_{qq}^{-1}}(u - m_q + i\alpha_{qp}v) \right] = \]
\[
= \frac{1}{\sqrt{(2\pi)^s \det \alpha_{qq}}} \exp \left[ \mu(m; \xi, \xi') - \frac{1}{2} \theta(\xi, \xi') \right],
\]
where \(\mu(m; \xi, \xi')\) and \(\theta(\xi, \xi')\) are given by \((40), (41)\). □

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