Topological and Algebraic Structures of the Space of Atanassov’s Intuitionistic Fuzzy Values

Xinxing Wu, Tao Wang, Peide Liu, Gül Deniz Çaylı, Xu Zhang

Abstract—We demonstrate that the space of intuitionistic fuzzy values (IFVs) with the linear order based on a score function and an accuracy function has the same algebraic structure as the one induced by the linear order based on a similarity function and an accuracy function. By introducing a new operator for IFVs via the linear order based on a score function and an accuracy function, we present that such an operator is a strong negation on IFVs. Moreover, we propose that the space of IFVs is a complete lattice and a Kleene algebra with the new operator. We also observe that the topological space of IFVs with the order topology induced by the above two linear orders is not separable and metrizable but compact and connected. From exactly new perspectives, our results partially answer three open problems posed by Atanassov [Intuitionistic Fuzzy Sets: Theory and Applications, Springer, 1999] and [On Intuitionistic Fuzzy Sets Theory, Soft Computing, 2012]. Furthermore, we construct an isomorphism between the spaces of IFVs and q-rung orthopair fuzzy values (q-ROFVs) under the corresponding linear orders. Meanwhile, we introduce the concept of the admissible similarity measures with particular orders for IFVs, extending the previous definition of the similarity measure for IFSs, and construct an admissible similarity measure with the linear order based on a score function and an accuracy function, which is effectively applied to a pattern recognition problem about the classification of building materials.

Index Terms—Intuitionistic fuzzy set; Q-rung orthopair fuzzy set; Complete lattice; Kleene algebra; Isomorphism.

I. INTRODUCTION

Atanassov [1], [2] extended Zadeh’s fuzzy set theory by introducing the concept of intuitionistic fuzzy sets (IFSs), characterized by a membership function and a non-membership function meeting the condition that the sum of the membership degree and the non-membership degree for every point is less than or equal to one. Atanassov and Gargov [3] further extended IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs) (see also [4]), whose membership degree and non-membership degree are the closed intervals contained in the unit interval rather than the real numbers in the unit interval. Gau and Buehrer [5] defined the concept of vague sets, which was proven to be equivalent to IFSs by Bustince and Burillo [6]. Every pair of membership and non-membership degrees for IFSs was called an intuitionistic fuzzy value (IFV) by Xu [7]. However, in the theory of IFSs, the condition that the sum of the membership degree and the non-membership degree is less than or equal to one induces some decision evaluation information to be not expressed effectively. Hence, the range of their application is limited. To overcome this shortcoming, Yager [8], [9], [10] proposed the concepts of Pythagorean fuzzy sets (PFSs) and q-rung orthopair fuzzy sets (q-ROFSs). These sets satisfy the condition that the square sum or the qth power sum of the membership degree and the non-membership degree is less than or equal to one.

Atanassov [1] and De et al. [11] introduced some basic operational laws for IFSs, including “intersection”, “union”, “supplement”, “sum”, “product”, “power”, “necessity” and “possibility” operators. Since all operations on IFSs act on IFVs point by point, it is essential to study the operational laws for IFVs, which inspired Xu and Yager [12] to define some of them. To obtain effective decision-making under an intuitionistic fuzzy environment, we need to rank any two IFVs. For this reason, Xu and Yager [12] introduced a linear order ‘≤xy’ for ranking any two IFVs by using a score function and an accuracy function. Then, by using a similarity function and an accuracy function, Zhang and Xu [13] introduced another linear order ‘≤xz’ to compare any two IFVs. For more results on linear orders for ranking IFVs or IVIFVs, we refer to [14], [15], [16], [17], [18]. Based on linear orders, the above operational laws for IFSs and IVIFSs were successfully used for intuitionistic fuzzy information aggregation [19], [20], [21], [22], [23], [24] and decision-making [25], [26], [27], [28], [29], [30]. In practical applications, the linear orders ≤xy and ≤xz are fundamental for decision-making under the intuitionistic fuzzy environment.

With the development of the theory of IFSs, more and more unsolved or unformulated problems were presented. In particular, Atanassov [2], [31] posed the following three open ones:

Problem 1 ([31, Open problem 15], [2, Open problem 3]): What other operations, relations, operators, norms, and metrics (essential from the standpoint of the applications of IFSs) can be defined over IFSs and their extensions, and what properties will they have?
Problem 2 ([31, Open problem 18]): To study IF algebraic objects.

Problem 3 ([31, Open problem 22]): To introduce elements of IFS in topology and geometry.

As previously stated, all operations on IFSs act on IFVs point by point, so it is essential to study the operational laws on IFVs. Accordingly, we focus on Problems 1–3 for IFVs. Although the linear order \( \leq_{XY} \) was introduced by Xu and Yager [12] many years ago, the topological and algebraic structures of the space of IFVs under this linear order have never been included in the research field of vision. Therefore, this paper is devoted to systematically study the topological and algebraic structures of the space of IFVs under some famous linear orders in [7], [12], [13], [19].

The rest of this study is organized as follows. In Section II, some basic concepts of orders, lattices, and IFSs are introduced. In Section III, we obtain that the spaces of IFVs under linear orders \( \leq_{XY} \) and \( \leq_{2X} \) are isomorphic; namely, they have the same algebraic structures. In Section IV, we introduce a new operator for IFVs by using the linear order \( \leq_{XY} \), which turns out to be a strong negation on IFVs under this linear order. Furthermore, we prove that the space of IFVs is a complete lattice and a Kleene algebra under the linear order \( \leq_{XY} \) with new strong negation. In Section V, we prove that the space of IFVs under the order topology with the linear order \( \leq_{XY} \) is not separable and metrizable but compact and connected. These results partially answer Problems 1–3 from exactly new perspectives. Moreover, in Section VI, we construct an isomorphism between the space of IFVs and the space of q-ROFNs under some linear orders in [32], [33], which allows us to transform q-ROFNs to IFVs equivalently. In Section VII, as applications, we introduce the concept of an admissible similarity measure with particular orders for IFSs, extending the previous definition of the similarity measure for IFSs in [34]. We also construct an admissible similarity measure with the linear order \( \leq_{XY} \) effectively applied to a pattern recognition problem about the classification of building materials. Finally, Section VIII gives some concluding remarks.

II. PRELIMINARIES

A. ORDER

Definition 2.1 ([35, Definition 1.1.3]): A partial order is a binary relation \( \preceq \) on a set \( L \) with the following properties:

1. (Reflexivity) \( a \preceq a \).
2. (Antisymmetry) If \( a \preceq b \) and \( b \preceq a \), then \( a = b \).
3. (Transitivity) If \( a \preceq b \) and \( b \preceq c \), then \( a \preceq c \).

A partially ordered set, or poset for short, is a nonempty set \( L \) equipped with a partial order \( \preceq \). A bounded poset is a structure \( (L, \preceq, 0, 1) \) such that \( (L, \preceq) \) is a poset, and \( 0, 1 \in L \) are its bottom and top elements, respectively.

A function \( f : X \to Y \) between two posets is called order preserving or monotone if and only if \( x \preceq y \) implies \( f(x) \preceq f(y) \). A bijection \( f : X \to Y \) is called an isomorphism if \( f \) and \( f^{-1} \) are monotone. Two posets \( (X, \preceq) \) and \( (Y, \preceq) \) are called (order-) isomorphic, denoted by \( X \simeq Y \), if and only if there is an isomorphism between them.

Let \( (L, \preceq) \) be a poset and \( X \subseteq L \). An element \( u \in L \) is said to be an upper bound of \( X \) if \( x \preceq u \) for all \( x \in X \). An upper bound \( u \) of \( X \) is said to be its smallest upper bound or supremum, written as \( \bigvee X \) or sup \( X \), if \( u \preceq y \) for each upper bound \( y \) of \( X \). Dually, we can define the greatest lower bound or infimum of \( X \), written as \( \bigwedge X \) or inf \( X \). In the case of pairs of elements, it is customary to write

\[
\bigvee y = \sup \{x, y\} \quad \text{and} \quad \bigwedge y = \inf \{x, y\}.
\]

For \( x, y \in X \), the notation \( x \prec y \) means that \( x \preceq y \) and \( x \neq y \). Suppose that \( (X, \preceq) \) is a poset. Given elements \( a, b \in X \) such that \( a \preceq b \), there exist some subsets of \( X \), called the intervals, as below:

\[
(a, b) = \{x \in X \mid a \prec x \prec b\},
\]

\[
(a, b] = \{x \in X \mid a \prec x \leq b\},
\]

\[
[a, b) = \{x \in X \mid a \leq x \prec b\},
\]

\[
[a, b] = \{x \in X \mid a \leq x \leq b\},
\]

\[
(\leftarrow, a) = \{x \in X \mid x \preceq a\},
\]

\[
(\leftarrow, a) = \{x \in X \mid x < a\},
\]

\[
[b, \to) = \{x \in X \mid b \preceq x\},
\]

\[
(b, \to) = \{x \in X \mid b < x\}.
\]

Definition 2.2 ([36], [37]): A linear order is a binary relation \( \prec \) on a set \( X \) with the following properties:

\((\text{LO}1)\) (Comparability) If \( x \neq y \), then either \( x \prec y \) or \( y \prec x \).

\((\text{LO}2)\) (Nonreflexivity) For no \( x \in X \), the relation \( x \prec x \) holds.

\((\text{LO}3)\) (Transitivity) If \( x \prec y \) and \( y \prec z \), then \( x \prec z \).

A linearly ordered set is a nonempty set \( X \) by a linear order \( \prec \).

Definition 2.3 ([37]): Let \( X \) be a linearly ordered set with more than one element. Assume that \( \mathcal{B} \) is the collection of all sets in the following types:

1. (1) All open intervals \( (a, b) \) in \( X \).
2. (2) All intervals of the form \( (\leftarrow, b) \).
3. (3) All intervals of the form \( (a, \to) \).

The collection \( \mathcal{B} \) is a basis for a topology on \( X \), called the linear order topology.

Any pair \( (D, E) \) of subsets of \( X \) that is a linearly ordered set by \( \prec \) is called a cut of \( X \) if the following statements hold:

1. \( D \cup E = X \);
2. \( D \neq \emptyset \) and \( E \neq \emptyset \);
3. \( x \prec y \) for all \( x \in D \) and \( y \in E \).

The sets \( D \) and \( E \) are called the lower section and the upper section of the cut, respectively. These sections are disjoint. For every cut of a linearly ordered set, one of the following four conditions is exactly satisfied:

(i) The lower and the upper sections have the largest and the smallest elements, respectively.

(ii) The lower section has the largest element while the upper section has no smallest one.

(iii) The upper section has the smallest element while the lower section has no largest one.

(iv) The lower and the upper sections have no largest and smallest elements, respectively.

When the condition (i) (resp. (iv)) holds, we say that the cut is a jump (resp. gap). If no cut of a linearly ordered set \( X \) is a jump (resp. gap), then it is called a densely ordered set (resp. continuously ordered set).
B. Lattice

Definition 2.4 ([38]): A lattice is a poset such that every pair of two elements have the greatest lower bound and the smallest upper bound. A lattice is bounded if it has the bottom and the top elements.

Definition 2.5 ([38]): A lattice is an algebra \((L, \lor, \land)\) with two binary operations \(\land\) and \(\lor\), which are called meet and join, respectively, if the following conditions hold:

\begin{itemize}
  \item[(L1)] (Idempotent law) \(x \land x = x\) and \(x \lor x = x\).
  \item[(L2)] (Commutative law) \(x \land y = y \land x\) and \(x \lor y = y \lor x\).
  \item[(L3)] (Associative law) \(x \land (y \land z) = (x \land y) \land z\) and \(x \lor (y \lor z) = (x \lor y) \lor z\).
  \item[(L4)] (Absorption law) \(x \land (x \lor y) = x\) and \(x \lor (x \land y) = x\).
\end{itemize}

For the equivalence of Definitions 2.4 and 2.5, we refer to [38, Theorem 1]. In fact, for a lattice, the meet \(\land\) and join \(\lor\) operations in Definition 2.5 can be defined as the infimum and supremum of two elements, respectively.

Definition 2.6: A lattice \((L, \leq)\) is complete if every subset \(A\) of \(L\) has a supremum and an infimum under the partial order \(\leq\).

Lemma 2.1 ([38]): For a bounded lattice \(L\), the following statements are equivalent:

\begin{itemize}
  \item[(i)] \(L\) is a complete lattice;
  \item[(ii)] Every nonempty subset of \(L\) has an infimum.
  \item[(iii)] Every nonempty subset of \(L\) has a supremum.
\end{itemize}

Definition 2.7 ([35, Definition 1.1.4]): A distributive lattice is a lattice \((L, \lor, \land)\) that satisfies the following distributive laws: for any \(x, y, z \in L\),

\begin{align*}
  (1) & \quad x \land (y \lor z) = (x \land y) \lor (x \land z); \\
  (2) & \quad x \lor (y \land z) = (x \lor y) \land (x \lor z).
\end{align*}

Definition 2.8 ([35, Definition 1.1.6]): Let \(L\) be a bounded lattice. A negation on \(L\) is a decreasing mapping \(N : L \to L\) such that \(N(0) = 1\) and \(N(1) = 0\). If additionally, \(N(N(x)) = x\) holds for all \(x \in L\), then \(L\) is called a strong negation.

Definition 2.9 ([35, Definition 1.1.7]): A De Morgan algebra is a bounded distributive lattice with a strong negation \(\tilde{\cdot}\).

Definition 2.10 ([35, Definition 1.1.8]): A Kleene algebra is a De Morgan algebra \((L, \lor, \land, \tilde{\cdot})\) that satisfies Kleene’s inequality: for any \(x, y \in L\),

\[x \land (\neg x) \leq y \lor (\neg y).\]

C. Intuitionistic fuzzy set (IFS)

Definition 2.11 ([2, Definition 1.1]): Let \(X\) be the universe of discourse. An intuitionistic fuzzy set (IFS) \(I\) in \(X\) is defined as an object in the following form

\[I = \{\langle x, \mu_i(x), \nu_i(x)\rangle \mid x \in X\},\]  

where the functions

\[
\mu_i : X \to [0, 1],
\]

and

\[
\nu_i : X \to [0, 1]
\]

define the degree of membership and the degree of non-membership of the element \(x \in X\) to the set \(I\), respectively, and for every \(x \in X\),

\[
\mu_i(x) + \nu_i(x) \leq 1. \tag{II.3}
\]

Let IFS\((X)\) denote the set of all IFSs in the universe of discourse \(X\). For \(I \in\) IFS\((X)\), the indeterminacy degree \(\pi_i(x)\) of an element \(x\) belonging to \(I\) is defined by \(\pi_i(x) = 1 - \mu_i(x) - \nu_i(x)\). In [7], [19], the pair \(\langle \mu_i(x), \nu_i(x)\rangle\) is called an intuitionistic fuzzy value (IFV) or an intuitionistic fuzzy number (IFN). For convenience, we use \(\alpha = \langle \mu_\alpha, \nu_\alpha \rangle\) to represent any IFV \(\alpha\), which satisfies \(\mu_\alpha \in [0, 1]\), \(\nu_\alpha \in [0, 1]\), and \(0 \leq \mu_\alpha + \nu_\alpha \leq 1\). Additionally, \(\bar{s}(\alpha) = \mu_\alpha - \nu_\alpha\) and \(h(\alpha) = \mu_\alpha + \nu_\alpha\) are called the score degree and the accuracy degree of \(\alpha\), respectively. Let \(\bar{I}\) denote the set of all IFVs, i.e., \(\bar{I} = \{\langle \mu, \nu \rangle \in [0, 1]^2 \mid \mu + \nu \leq 1\}\).

Motivated by the basic operations on IFSs, Xu et al. [12], [19] introduced the following basic operational laws for IFVs.

Definition 2.12 ([19, Definition 1.2.2]): Let \(\alpha = \langle \mu_\alpha, \nu_\alpha \rangle\), \(\beta = \langle \mu_\beta, \nu_\beta \rangle \in \bar{I}\). Define

\begin{itemize}
  \item[(i)] \(\overline{\alpha} = \langle \nu_\alpha, \mu_\alpha \rangle\).
  \item[(ii)] \(\alpha \cap \beta = (\min{\{\mu_\alpha, \mu_\beta\}}, \max{\{\nu_\alpha, \nu_\beta\}})\).
  \item[(iii)] \(\alpha \cup \beta = (\max{\{\mu_\alpha, \mu_\beta\}}, \min{\{\nu_\alpha, \nu_\beta\}})\).
  \item[(iv)] \(\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta, \nu_\alpha \nu_\beta)\).
  \item[(v)] \(\alpha \otimes \beta = (\mu_\alpha \mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta)\).
  \item[(vi)] \(\lambda^\alpha = (1 - (1 - \mu_\alpha)^\lambda, (\nu_\alpha)^\lambda), \lambda > 0\).
  \item[(vii)] \(\mu^{\alpha} = (\langle \mu_\alpha \lambda^\lambda, 1 - (1 - \nu_\alpha)^\lambda \rangle, \lambda > 0\).
\end{itemize}

The order \(\subseteq\), defined by \(\alpha \subseteq \beta\) if and only if \(\alpha \cap \beta = \alpha\), is a partial order on \(\bar{I}\). To compare any two IFVs, Xu and Yager [12] introduced the following linear order \(\preceq_{xy}\) (see also [7, Definition 3.1] and [19, Definition 1.1.3]):

Definition 2.13 ([12, Definition 1.1]): Let \(\alpha_1\) and \(\alpha_2\) be two IFVs.

\begin{itemize}
  \item If \(s(\alpha_1) < s(\alpha_2)\), then \(\alpha_1\) is smaller than \(\alpha_2\), denoted by \(\alpha_1 \prec_{xy} \alpha_2\).
  \item If \(s(\alpha_1) = s(\alpha_2)\), then
    \begin{itemize}
      \item if \(h(\alpha_1) = h(\alpha_2)\), then \(\alpha_1 = \alpha_2\);
      \item if \(h(\alpha_1) < h(\alpha_2)\), then \(\alpha_1 \prec_{xy} \alpha_2\).
    \end{itemize}
\end{itemize}

If \(\alpha_1 \prec_{xy} \alpha_2\) or \(\alpha_1 = \alpha_2\), we will denote it by \(\alpha_1 \leq_{xy} \alpha_2\).

Alongside Xu and Yager’s order \(\preceq_{xy}\) in Definition 2.13, Szmidt and Kacprzyk [39] proposed another comparison function \(\rho(\alpha) = \frac{1}{2}(1 + \pi(\alpha))(1 - \mu(\alpha))\) for IFVs, which is a partial order. However, it sometimes cannot distinguish between any two IFVs. Although Xu’s method [7] constructs a linear order for ranking any two IFVs, this procedure has the following disadvantages: (1) It may result in that the less we know; the better the IFV, which contradicts our intuition. (2) It is sensitive for a slight change of the parameters. (3) It is not preserved under multiplication by a scalar, namely, \(\alpha \leq_{xy} \beta\) might not imply \(\lambda \alpha \leq_{xy} \lambda \beta\), where \(\lambda\) is a scalar (see [40, Example 1]). To overcome some shortcomings of the above two ranking methods, Zhang and Xu [13] improved Szmidt and Kacprzyk’s one [39], according to Hwang and Yoon’s idea [41] of the technique for order preference by similarity to an ideal solution. They also gave the similarity function \(L(\alpha)\), for any IFV \(\alpha = \langle \mu_\alpha, \nu_\alpha \rangle\), as follows:

\[
L(\alpha) = \frac{1 - \nu_\alpha}{(1 - \mu_\alpha) + (1 - \nu_\alpha)} = \frac{1 - \nu_\alpha}{1 + \pi_\alpha}. \tag{II.4}
\]
In particular, if \( \nu_\alpha < 1 \), then
\[
L(\alpha) = \frac{1}{1 - \mu_\alpha + \mu_\alpha 
\frac{2 - 4L(\alpha)}{L(\alpha)}}.
\] (II.5)

Furthermore, they [13] introduced the following order ‘\( \leq_{x_2} \)’ for IFVs by applying the similarity function \( L(\_). \)

**Definition 2.14 ([13]):** Let \( \alpha_1 \) and \( \alpha_2 \) be two IFVs.

- If \( L(\alpha_1) < L(\alpha_2) \), then \( \alpha_1 \) is smaller than \( \alpha_2 \), denoted by \( \alpha_1 \leq_{x_2} \alpha_2 \);
- If \( L(\alpha_1) = L(\alpha_2) \), then
  - if \( h(\alpha_1) = h(\alpha_2) \), then \( \alpha_1 = \alpha_2 \);
  - if \( h(\alpha_1) < h(\alpha_2) \), then \( \alpha_1 \leq_{x_2} \alpha_2 \).

If \( \alpha_1 \leq_{x_2} \alpha_2 \) or \( \alpha_1 = \alpha_2 \), we will denote it by \( \alpha_1 \leq_{x_2} \alpha_2 \).

**Remark 1:** The bottom and the top elements of \( \bar{I} \) are \( (0, 1) \) and \( (1, 0) \), respectively, under both the order \( \leq_{x_1} \) and the order \( \leq_{x_2} \).

### III. AN ISOMORPHISM BETWEEN LATTICES \( (\bar{I}, \leq_{x_2}) \) AND \( (\bar{I}, \leq_{x_1}) \)

In this section, we construct an isomorphism between \( (\bar{I}, \leq_{x_2}) \) and \( (\bar{I}, \leq_{x_2}) \): namely, we show that they are isomorphic to each other. Therefore, we only consider the constructions of \( (\bar{I}, \leq_{x_1}) \) in the following sections.

Let us define a mapping \( \hat{\Gamma} : (\bar{I}, \leq_{x_2}) \to (\bar{I}, \leq_{x_2}) \) as follows:

\[
\hat{\Gamma}(\alpha) = \begin{cases} 
(0, 1), & L(\alpha) = 0, \\
(1, 0), & L(\alpha) = 1, \\
\alpha, & L(\alpha) = \frac{1}{2}, \\
\left( \frac{\mu_\alpha}{2(1-L(\alpha))}, \frac{\mu_\alpha+2-4L(\alpha)}{2(1-L(\alpha))} \right), & 0 < L(\alpha) < \frac{1}{2}, \\
\left( \frac{1}{2} \left( \frac{1}{2} - \frac{L(\alpha)\mu_\alpha - (2L(\alpha)-1)}{2L(\alpha)(1-L(\alpha))} \right), \frac{1}{2} < L(\alpha) < 1. 
\end{cases}
\] (III.1)

The following is the geometric description of \( \hat{\Gamma}(\alpha) \).

**Theorem 3.1:** The mapping \( \hat{\Gamma} \) defined by the formula (III.1) is an isomorphism between \( (\bar{I}, \leq_{x_2}) \) and \( (\bar{I}, \leq_{x_2}) \).

**Proof:** By the formula (III.1), it is easy to see that \( \hat{\Gamma} \) is a bijection. For \( \alpha = (\mu_\alpha, \nu_\alpha) \) and \( \beta = (\mu_\beta, \nu_\beta) \in \bar{I} \) such that \( \alpha <_{x_2} \beta \), we prove that \( \hat{\Gamma}(\alpha) <_{x_1} \hat{\Gamma}(\beta) \). For this purpose, we consider the following cases:

1. If \( L(\alpha) = 0 \) or \( L(\beta) = 1 \), i.e., \( \alpha = (0, 1) \) or \( \beta = (1, 0) \), then \( \hat{\Gamma}(\alpha) = (0, 1) \) or \( \hat{\Gamma}(\beta) = (1, 0) \). Thus, \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

2. If \( L(\alpha) = L(\beta) = \frac{1}{2} \), then we have \( \hat{\Gamma}(\alpha) = \alpha \) and \( \hat{\Gamma}(\beta) = \beta \) by the formula (III.1). Since \( \alpha <_{x_2} \beta \), we obtain that \( h(\alpha) < h(\beta) \), which implies that \( \alpha <_{x_1} \beta \). Thus, \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

3. If \( 0 < L(\alpha) = L(\beta) < \frac{1}{2} \), then, by the formula (III.1), we have that
\[
\hat{\Gamma}(\alpha) = \left( \frac{\mu_\alpha}{2(1-L(\alpha))}, \frac{\mu_\alpha + 2 - 4L(\alpha)}{2(1-L(\alpha))} \right),
\]
and
\[
\hat{\Gamma}(\beta) = \left( \frac{\mu_\beta}{2(1-L(\beta))}, \frac{\mu_\beta + 2 - 4L(\beta)}{2(1-L(\beta))} \right).
\]

Since \( \alpha <_{x_2} \beta \), we have that \( h(\alpha) < h(\beta) \). This, together with \( \frac{1}{2} \leq L(\alpha) = L(\beta) < \frac{1}{2} \), implies that \( \nu_\alpha < \nu_\beta \). Thus, \( \mu_\alpha < \mu_\beta \) by formula (II.5). In this case, \( h(\hat{\Gamma}(\alpha)) = \frac{\mu_\alpha}{2(1-L(\alpha))} < \frac{\mu_\beta}{2(1-L(\beta))} = h(\hat{\Gamma}(\beta)) \). By using that \( s(\hat{\Gamma}(\alpha)) = s(\hat{\Gamma}(\beta)) \), there holds \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

4. If \( \frac{1}{2} < L(\alpha) = L(\beta) < 1 \), then, similarly to the proof of (3), it is verified that \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

5. If \( 0 < L(\alpha) = L(\beta) < \frac{1}{4} \), then, by the formula (III.1), we have that \( s(\hat{\Gamma}(\alpha)) = \frac{2L(\alpha)-1}{L(\alpha)} < \frac{2L(\beta)-1}{L(\beta)} = s(\hat{\Gamma}(\beta)) \). Thus, \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

6. If \( \frac{1}{2} \leq L(\alpha) < L(\beta) < 1 \), then, by the formula (III.1), we have that \( s(\hat{\Gamma}(\alpha)) = \frac{2L(\alpha)-1}{L(\alpha)} < \frac{2L(\beta)-1}{L(\beta)} = s(\hat{\Gamma}(\beta)) \). Thus, \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

7. If \( 0 < L(\alpha) < \frac{1}{2} < L(\beta) < 1 \), then, by the formula (III.1), we have that \( s(\hat{\Gamma}(\alpha)) = \frac{2L(\alpha)-1}{L(\alpha)} < \frac{2L(\beta)-1}{L(\beta)} = s(\hat{\Gamma}(\beta)) \). Thus, \( \hat{\Gamma}(\alpha) \leq_{x_1} \hat{\Gamma}(\beta) \).

By the direct calculation, it is not difficult to check that, for \( \alpha = (\mu_\alpha, \nu_\alpha) \in \bar{I} \),
\[
\hat{\Gamma}^{-1}(\alpha) = \begin{cases} 
(0, 1), & s(\alpha) = -1, \\
(1, 0), & s(\alpha) = 1, \\
\alpha, & s(\alpha) = 0, \\
\left( 2\mu_\alpha - s(\alpha), \frac{2\mu_\alpha - 2s(\alpha)}{2-s(\alpha)} \right), & 0 < s(\alpha) < 1, \\
\left( \frac{2\mu_\alpha}{2-s(\alpha)}, \frac{2\mu_\alpha(1+s(\alpha)) - 2s(\alpha)}{2-s(\alpha)} \right), & -1 < s(\alpha) < 0. 
\end{cases}
\] (III.2)

Similarly to the above proof, one can show that, for any \( \alpha, \beta \in \bar{I} \) with \( \alpha <_{x_1} \beta \), \( \hat{\Gamma}^{-1}(\alpha) <_{x_2} \hat{\Gamma}^{-1}(\beta) \). Hence, \( \bar{I} \) is an isomorphism.

### IV. ALGEBRAIC STRUCTURES OF \( (\bar{I}, \leq_{x_2}) \)

In Theorem 3.1, we observe that \( (\bar{I}, \leq_{x_2}) \simeq (\bar{I}, \leq_{x_2}) \): namely, \( (\bar{I}, \leq_{x_2}) \) and \( (\bar{I}, \leq_{x_2}) \) have the same algebraic structures. Accordingly, it is enough to study the algebraic structures for \( (\bar{I}, \leq_{x_2}) \). In the following, we first describe that \( (\bar{I}, \leq_{x_2}) \) is a complete lattice. Then, by introducing the operator ‘\( \sim \)’ for IFVs, we demonstrate in Corollary 4.1 that it is a strong
negation on \( \tilde{\Omega}, \preceq_{xy} \). We also present in Theorem 4.3 that \( \tilde{\Omega} \) is a Kleene algebra.

**Proposition 4.1:** (i) The order \( \preceq_{xy} \) defined in Definition 2.13 is a linear order on \( \tilde{\Omega} \).

(ii) Let \( \alpha \leq_{xy} \beta \) and \( s(\alpha) = s(\beta) \) for \( \alpha = \langle \mu_\alpha, \nu_\alpha \rangle \) and \( \beta = \langle \mu_\beta, \nu_\beta \rangle \in \tilde{\Omega} \). Then, there holds \( (\alpha, \beta) = (\gamma \in \tilde{\Omega} | \alpha \leq_{xy} \gamma) \cdot (\nu, \delta) \in \tilde{\Omega} \mid \mu - \nu = s(\alpha) \) and \( \mu_\alpha < \mu < \mu_\beta \).

**Theorem 4.1:** \( \tilde{\Omega}, \preceq_{xy} \) is a complete lattice.

**Proof:** Given a nonempty subset \( \Omega \subset \tilde{\Omega} \), we claim that the greatest lower bound of \( \Omega \) exists.

Let \( \mathcal{S}(\Omega) = \{\mu_\alpha - \nu_\alpha \mid \langle \mu_\alpha, \nu_\alpha \rangle \in \Omega \} \) and \( \xi = \inf \mathcal{S}(\Omega) \).

We consider the following two cases:

1. Assume that \( \xi \in \mathcal{S}(\Omega) \). Then, there exists \( \alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle \in \Omega \) such that \( \mu_{\alpha_1} - \nu_{\alpha_1} = \xi \). This means that the set \( \Omega^\xi = \{\mu_\alpha \mid \langle \mu_\alpha, \mu_\alpha - \xi \rangle \in \Omega \} \) is a nonempty set. Let \( \hat{\mu} = \inf \Omega^\xi \) and \( \hat{\alpha} = \langle \hat{\mu}, \hat{\mu} - \xi \rangle \). It is obvious that \( s(\hat{\alpha}) = \xi \).

Notice that

\[
\hat{\mu} - \xi = \inf \Omega^\xi - \xi = \inf \{\mu_\alpha - \xi \mid \langle \mu_\alpha, \mu_\alpha - \xi \rangle \in \Omega \}
\]

is a lower bound of \( \hat{\alpha} \). Hence, we get that \( \hat{\alpha} \in \tilde{\Omega} \), and, we show that \( \hat{\alpha} \) is the greatest lower bound of \( \Omega \).

1.2) Given a lower bound \( \alpha' = \langle \mu_{\alpha'}, \nu_{\alpha'} \rangle \in \tilde{\Omega} \), it follows from Definition 2.13 that \( s(\alpha') \leq \xi \).

1.2.1) If \( s(\alpha') < \xi \), then we have that \( \alpha' <_{xy} \hat{\alpha} \) since \( s(\hat{\alpha}) = \xi \).

1.2.2) If \( s(\alpha') = \xi \), it follows from Definition 2.13 that, for any \( \alpha = \langle \mu_\alpha, \nu_\alpha \rangle \in \Omega \) with \( \mu_\alpha - \nu_\alpha = \xi \), \( h(\alpha') \leq h(\alpha) = \mu_\alpha + \nu_\alpha \). This fact means that

\[
h(\alpha') \leq \inf \{\mu_\alpha + \nu_\alpha \mid \langle \mu_\alpha, \nu_\alpha \rangle \in \Omega \text{ and } \mu_\alpha - \nu_\alpha = \xi\}
\]

is a lower bound of \( \Omega \).

2.1) It follows from \( \xi \notin \mathcal{S}(\Omega) \) that, for any \( \alpha \in \Omega \), \( s(\alpha) > \xi = s(\hat{\alpha}) \). This observation means that \( s(\hat{\alpha}) = \xi \). Now, we show that \( \hat{\alpha} \) is the greatest lower bound of \( \Omega \).

2.2) Given a lower bound \( \alpha'' = \langle \mu_{\alpha''}, \nu_{\alpha''} \rangle \in \tilde{\Omega} \), it follows from Definition 2.13 that \( s(\alpha'') \leq \xi \).

2.2.1) If \( s(\alpha'') < \xi \), then we have that \( \alpha'' <_{xy} \hat{\alpha} \) since \( s(\hat{\alpha}) = \xi \).

2.2.2) If \( s(\alpha'') = \xi \), then we have that \( \alpha'' \leq_{xy} \hat{\alpha} \) since \( h(\hat{\alpha}) = \frac{1 + \xi}{2} = 1 \geq \mu_{\alpha''} + \nu_{\alpha''} = h(\alpha'') \) and \( s(\hat{\alpha}) = \xi \).

Summing up 2.1 and 2.2, we get that \( \hat{\alpha} \) is the greatest lower bound of \( \Omega \).

Hence, \( \tilde{\Omega}, \preceq_{xy} \) is a complete lattice by applying Lemma 2.1.

Given any nonempty subset \( \Omega \subset \tilde{\Omega} \), let \( \mathcal{S}(\Omega) = \{\mu_\alpha - \nu_\alpha \mid \langle \mu_\alpha, \nu_\alpha \rangle \in \Omega \}, \xi(\Omega) = \inf \mathcal{S}(\Omega), \) and \( \eta(\Omega) = \sup \mathcal{S}(\Omega) \).

**Remark 2:** According to the proof of Theorem 4.1, it holds that

\[
\inf \Omega = \left\{\langle \hat{\mu}, \hat{\mu} - \xi(\Omega) \rangle, \xi(\Omega) \in \mathcal{S}(\Omega), \frac{1 + \xi(\Omega)}{2} \right\}, \xi(\Omega) \notin \mathcal{S}(\Omega), \text{ (IV.1)}
\]

where \( \hat{\mu} = \inf \{\mu_\alpha \mid \langle \mu_\alpha, \mu_\alpha - \xi(\Omega) \rangle \in \Omega \} \).

Similarly to the proof of Theorem 4.1, it is not difficult to check that

\[
\sup \Omega = \left\{\langle \hat{\mu}, \hat{\mu} - \eta(\Omega) \rangle, \eta(\Omega) \in \mathcal{S}(\Omega), 0, -\eta(\Omega) \rangle, \eta(\Omega) \notin \mathcal{S}(\Omega) \) and \( \eta(\Omega) \leq 0, \eta(\Omega), 0 \rangle, \eta(\Omega) \notin \mathcal{S}(\Omega) \) and \( \eta(\Omega) > 0, \text{ (IV.2)} \)
\]

The operator \( \neg \) in Definition 2.12 is not a negation on \( \tilde{\Omega} \) under the order \( \preceq_{xy} \). In the following, by introducing a new operator \( \neg^{xy} \) for IFVs, we show that it is a strong negation on \( \tilde{\Omega}, \preceq_{xy} \) (see Corollary 4.1). Moreover, we propose that \( \tilde{\Omega}, \preceq_{xy}, \lor, \neg^{xy} \) is a Kleene algebra (see Theorem 4.3).

For \( \alpha = \langle \mu_\alpha, \nu_\alpha \rangle \in \tilde{\Omega} \), we define

\[
\neg \alpha = \left\{\langle 1 - \frac{1}{2} \mu_\alpha - \frac{1}{2} \nu_\alpha, \frac{1}{2} + \frac{1}{2} \mu_\alpha - \frac{1}{2} \nu_\alpha \rangle, \mu_\alpha > \nu_\alpha, \frac{1}{2} - \mu_\alpha - \frac{1}{2} \nu_\alpha, \frac{1}{2} + \mu_\alpha + \frac{1}{2} \nu_\alpha \rangle, \mu_\alpha = \nu_\alpha, \frac{1}{2} + \nu_\alpha - \frac{1}{2} \mu_\alpha, \frac{1}{2} - \nu_\alpha - \frac{1}{2} \mu_\alpha \rangle, \mu_\alpha < \nu_\alpha \right\}, \text{ (IV.3)}
\]

The following is the geometric description of \( \neg \alpha \).

![Fig. 2. Geometrical interpretation of \( \neg \alpha \)](image-url)
We consider the following cases:

- If $\mu_\alpha = \nu_\alpha$, then it is clear that $-\alpha = -(1/2 - \mu_\alpha, 1/2 - \nu_\alpha) = (\mu_\alpha, \nu_\alpha) = \alpha$.

- If $\mu_\alpha > \nu_\alpha$, then, by the formula (IV.3), we have that $-\alpha = (1/2 + \mu_\alpha - 3\nu_\alpha, 1 + \mu_\alpha - 3\nu_\alpha)$. This, together with $1/2 + \mu_\alpha - 3\nu_\alpha = (1/2 - \mu_\alpha, 1/2 - \nu_\alpha)$ and the formula (IV.3), implies that

\[
-\langle \alpha \rangle = \left\{ \begin{array}{ll}
\frac{1}{2} \left[ 1 + \frac{1 + \mu_\alpha - 3\nu_\alpha}{2} - \frac{3}{2} \left( 1 - \mu_\alpha - \nu_\alpha \right) \right], \\
\frac{1}{2} \left[ 1 - \frac{1 - \mu_\alpha - \nu_\alpha}{2} - \frac{3}{2} \left( 1 + \mu_\alpha - 3\nu_\alpha \right) \right]
\end{array} \right.
\]  

\[= \langle \mu_\alpha, \nu_\alpha \rangle = \alpha.
\]

- If $\mu_\alpha < \nu_\alpha$, then, similarly to the above proof, $-\langle \alpha \rangle = \alpha$.

(4) The equality that $-(\alpha \wedge \beta) = (\alpha \vee \beta)$ holds trivially when $\alpha = \beta$. Without loss of generality, assume that $\alpha \leq_X \beta$.

We consider the following cases:

- Let $s(\alpha) = s(\beta)$ and $h(\alpha) < h(\beta)$. This means that $\mu_\beta > \mu_\alpha$ and $\nu_\beta > \nu_\alpha$.

  - If $s(\alpha) = s(\beta) = 0$, then $\langle \alpha \wedge \beta \rangle = \alpha = \langle 1/2 - \mu_\alpha, 1/2 - \nu_\alpha \rangle$ and $\langle \beta \rangle = \langle 1/2 - \mu_\beta, 1/2 - \nu_\beta \rangle$. Observing that $s(\alpha) = s(\beta)$ and $h(\alpha) = 1 - h(\alpha) > 1 - h(\beta) = h(\beta)$, i.e., $\alpha \leq_X \beta$. Thus, $-(\alpha \wedge \beta) = -(\alpha \vee \beta)$. Then, by the formula (IV.4), we have that

\[
-(\alpha \wedge \beta) = \alpha = -(\alpha \vee \beta),
\]

and

\[
-(\beta) = \left\{ \begin{array}{ll}
1/2 + \mu_\beta - \nu_\beta, \\
1 + \mu_\beta - 3\nu_\beta
\end{array} \right.
\]

This observation means that $s(\alpha) = s(\beta) = 0$ and $h(\alpha) = 1 - 2\nu_\alpha > 1 - 2\nu_\beta = h(\beta)$. Hence, $-(\alpha \wedge \beta) = -(\alpha \vee \beta)$. Thus, the strong negation operator $\neg$ defined by the formula (IV.3) is a strong negation on $(\mathbb{I}, \leq_X)$.

**Corollary 4.1:** The operator $\neg$ defined by the formula (IV.3) is a strong negation on $(\mathbb{I}, \leq_X)$.

**Proof:** It follows directly from Proposition 4.1 and Theorem 4.2.

**Example 1:** Take $\alpha = (0, 0)$ and $\beta = (1/2, 1/2)$. It is easy to see that $\alpha \cap \beta = (0, 1/2)$, $\alpha \cup \beta = (1/2, 0)$, and $\neg \alpha = \beta$. Therefore, $\neg(\alpha \cap \beta) = (3/2, 1/2)$ and $\neg(\alpha \cup \beta) = \beta \cup \alpha = (1/2, 0)$. This means that $-(\alpha \cap \beta) = (\alpha \vee \beta)$.

**Theorem 4.3:** $(\mathbb{I}, \wedge, \vee, -)$ is a Kleene algebra, where $\wedge$ and $\vee$ are infimum and supremum operations under the order $\leq_X$, respectively.

**Proof:** By Proposition 4.1, we obtain that $(\mathbb{I}, \wedge, \vee)$ is a distributive lattice since $\leq_X$ is a linear order. This, together with Corollary 4.1, implies that $(\mathbb{I}, \wedge, \vee, -)$ is a De Morgan algebra. So, we only need to show that $(\mathbb{I}, \wedge, \vee, -)$ satisfies Kleene’s inequality.

For any $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$, $\beta = \langle \mu_\beta, \nu_\beta \rangle \in \mathbb{I}$, we consider the following cases:

1. If $\alpha \leq_X \beta$, then $\alpha \wedge (\neg \alpha) \leq_X \beta \leq_X \beta \vee (\neg \beta)$.
2. If $\alpha \geq_X \beta$, then we have the following:

- If $s(\alpha) > s(\beta)$, then it follows from Theorem 4.2 (2) that $s(\alpha \wedge (\neg \alpha)) = \min\{s(\alpha), s(\alpha)\} \leq 0$ and $s(\beta \vee (\neg \beta)) = \max\{s(\beta), s(\beta)\} \geq 0$. Thus,

\[
\begin{align*}
\langle \alpha \wedge (\neg \alpha) \rangle & = \langle \min\{s(\alpha), s(\alpha)\} \rangle \leq_X \beta \leq_X \beta \vee (\neg \beta), \\
& \text{which implies that } \alpha \wedge (\neg \alpha) \leq_X \beta \vee (\neg \beta); \\
\langle \beta \vee (\neg \beta) \rangle & = \langle \max\{s(\beta), s(\beta)\} \rangle \geq_X \beta \vee (\neg \beta), \\
& \text{which implies that } \beta \vee (\neg \beta) \geq_X \beta \vee (\neg \beta);
\end{align*}
\]

Therefore, $\alpha \wedge (\neg \alpha) \leq_X \beta \vee (\neg \beta)$.

2. If $\alpha = s(\beta)$ and $h(\alpha) > h(\beta)$, it follows from Theorem 4.2 (2) that $\langle s(\alpha \wedge (\neg \alpha)) \rangle = \min\{s(\alpha \wedge (\neg \alpha)) \rangle \leq_X \beta \leq_X \beta \vee (\neg \beta)$.

Thus, $\alpha \wedge (\neg \alpha) \leq_X \beta \vee (\neg \beta)$.

Therefore, $\alpha \wedge (\neg \alpha) \leq_X \beta \veq (\neg \beta)$.

Summing up the above, we have that $\alpha \wedge (\neg \alpha) \leq_X \beta \veq (\neg \beta)$.

By Theorems 4.1, 3.1, and 4.3, we have that the following results.

**Theorem 4.4:** $(\mathbb{I}, \leq_X)$ is a complete lattice.

**Theorem 4.5:** $(\mathbb{I}, \wedge, \vee, \Gamma^{-1})$ is a Kleene algebra, where $\wedge$ and $\vee$ are infimum and supremum operations under the order $\leq_X$, respectively, and $\Gamma$ is defined by the formula (III.2).

**Remark 3:** Theorems 4.2 and 4.5 and Corollary 4.1 show that the strong negation operators $\neg$ and $\Gamma^{-1} \circ \neg \circ \Gamma$ are new...
operators on both IFVs and IFSs. This fact partially answers Problem 1.

Remark 4: Through the linear orders \( \leq_{xy} \) and \( \leq_{yx} \), and the strong negation operators \( \neg \) and \( \neg^{-1} \circ \neg \), we obtain in Theorems 4.1, 4.3, 4.4, and 4.5 some IF algebraic properties from exactly different perspectives, which partially answers Problem 2.

Remark 5: Because both \( (\tilde{I}, \leq_{xy}) \) and \( (\tilde{I}, \leq_{yx}) \) are complete lattice, we can establish the decomposition theorem and Zadeh’s extension principle for IFSs as follows: for an IFS \( I \in IFS(X) \).

(Decomposition Theorem) For every \( x \in X \),

\[
I(x) = \bigvee \{ \alpha \in \tilde{I} \mid x \in I_\alpha \},
\]

where \( I(x) = (\mu_x(x), \nu_x(x)) \), \( I_\alpha = \{ z \in X \mid I(z) \geq \alpha \} \), \( \vee \) is the supremum under the linear order \( \leq_{xy} \).

(Zadeh’s Extension Principle) Let \( X \) and \( Y \) be two nonempty sets and \( f : X \to Y \) be a mapping from \( X \) to \( Y \). Define a mapping \( \tilde{g} : \tilde{I}^X \to \tilde{I}^Y \) by

\[
\tilde{g} : \tilde{I}^X \to \tilde{I}^Y
\]

\[
I \mapsto \tilde{g}(I)(y) = \left\{ \begin{array}{ll}
(0, 1), & f^{-1}(\{y\}) = \emptyset, \\
\vee_{x \notin f^{-1}(\{y\})} I(x), & f^{-1}(\{y\}) \neq \emptyset,
\end{array} \right.
\]

which is called the Zadeh’s extension mapping of \( f \) in the sense of IFSs.

V. TOPOLOGICAL STRUCTURES OF \((\tilde{I}, \leq_{xy})\) UNDER THE ORDER TOPOLOGY

This section is devoted to investigating the topological structures of \((\tilde{I}, \leq_{xy})\) under the order topology. In particular, we present that the space \((\tilde{I}, \leq_{xy})\) under the order topology induced by the linear order \( \leq_{xy} \) is not separable and metrizable but compact and connected. We also obtain that \((\tilde{I}, \leq_{xy})\) has the same topological structures as \((\tilde{I}, \leq_{yx})\) since they are homeomorphic.

Definition 5.1 ([36]): A topological space with a countable dense subset is called separable.

Definition 5.2 ([36]): A topological space \( X \) is called a compact space if \( X \) is a Hausdorff space and every open cover of \( X \) has a finite subcover; namely, there exists a finite set \( \{ s_1, s_2, \ldots, s_n \} \subset \mathcal{G} \) such that \( U_{s_1} \cup U_{s_2} \cup \cdots \cup U_{s_n} = X \) for every open cover \( \{ U_s \}_{s \in \mathcal{G}} \) of the space \( X \).

Definition 5.3 ([36], [37]): Let \( X \) be a topological space. A separation of \( X \) is a pair \((U, V)\) of disjoint nonempty open subsets of \( X \) whose union is \( X \). The space \( X \) is said to be connected if there does not exist a separation of \( X \).

Definition 5.4 ([36]): Let \( X \) be a topological space. It is called metrizable if there is a metric \( \rho \) on \( X \) such that the topology induced by this metric coincides with the original topology of \( X \).

Lemma 5.1 ([36, Problem 3.12.3 (a)], [42]): A space \( X \) with the topology induced by a linear order \( \prec \) is compact if and only if every subset \( A \) of \( X \) has the least upper bound.

Lemma 5.2 ([36, Problem 6.3.2 (a)]): A space \( X \) with the topology induced by a linear order \( \prec \) is connected if and only if it is a continuously ordered set by \( \prec \).

Proposition 5.1: The space \( \tilde{I} \) with the order topology induced by the linear order \( \leq_{xy} \) defined in Definition 2.13 is compact.

Proof: It follows directly from Theorem 4.1 and Lemma 5.1.

Proposition 5.2: The space \( \tilde{I} \) with the order topology induced by the linear order \( \leq_{xy} \) defined in Definition 2.13 is connected.

Proof: Let \((D, E)\) be a cut of \( \tilde{I} \).

\( (1) \) We show that \((D, E)\) is not a gap.

Suppose, on the contrary, that \((D, E)\) is a gap. Then, we get that \( D \) and \( E \) have the largest and the smallest elements, respectively, denoted by \( \xi \) and \( \eta \), respectively. This implies that \( X = D \cup E \subset (\langle \xi, \eta \rangle] \cup [\eta, \to) \). Thus, \( \langle \xi, \eta \rangle = X \setminus (\langle \xi, \xi \rangle \cup [\eta, \to) \) = \emptyset \). This is impossible since \( \frac{1}{2}(\langle 1_s(\xi) + s(\eta) + h(\xi) + h(\eta) \rangle, \frac{1}{2}(h(\xi) + h(\eta) - s(\xi) - s(\eta)) \rangle \in \langle \xi, \eta \rangle \). Therefore, \((D, E)\) is not a gap.

\( (2) \) We show that \((D, E)\) is not a gap.

Suppose, on the contrary, that \((D, E)\) is a gap. Then, we get that \( D \) and \( E \) have no largest and smallest elements, respectively. It follows from Theorem 4.1 that \( \sup D \) exists. It is verified that

\( (i) \) \( \sup D \notin D \). This, together with \( D \cup E = \tilde{I} \), implies that \( \sup D \notin E \).

\( (ii) \) For any \( \beta \in E \), since \( \beta \) is an upper bound of \( D \), we get that \( \sup D \leq_D \beta \).

Therefore, \( \sup D \) is the smallest element of \( E \). This is a contradiction. Hence, \((D, E)\) is not a gap.

Summing up (1) and (2), we observe that \( \tilde{I} \) is a continuously ordered set. Hence, \( \tilde{I} \) is connected by applying Lemma 5.2.

Proposition 5.3: The space \( \tilde{I} \) with the order topology induced by the linear order \( \leq_{xy} \), defined in Definition 2.13, is not separable.

Proof: Consider any dense subset \( D \) of \( \tilde{I} \). Then, by Proposition 4.1, we have that \( I_\gamma = \langle \gamma, 0 \rangle \) \( \langle \frac{1}{2} + \gamma, \frac{1}{2} \rangle \) = \( \{ \langle \mu, \mu - \gamma \rangle \in \tilde{I} \mid | \gamma \prec_{xy} \mu \prec_{xy} \frac{1}{2} + \gamma \} \) is a nonempty open subset of \( \tilde{I} \) for every \( \gamma \in (0, 1) \). Thus, \( D \cap I_\gamma \neq \emptyset \).

For every \( \gamma \in (0, 1) \), choose a point \( \alpha_\gamma \in D \cap I_\gamma \). Since \( I_\gamma \cap I_{\gamma'} \neq \emptyset \) for any \( \gamma, \gamma' \in (0, 1) \) with \( \gamma \neq \gamma' \), we have that \( 0 < 2^{\alpha_\gamma} = |(0, 1)| = |(\alpha_\gamma, | \gamma \in (0, 1)|) |\) \( |D| \). Therefore, \( \tilde{I} \) is not separable.

Corollary 5.1: The space \( \tilde{I} \) with the order topology induced by the linear order \( \leq_{xy} \), defined in Definition 2.13, is not homeomorphic to [0, 1] with the natural topology.

Proof: It follows directly from Proposition 5.3 and the fact that [0, 1] is separable under the natural topology.

Corollary 5.2: The space \( \tilde{I} \) with the order topology induced by the linear order \( \leq_{xy} \), defined in Definition 2.13, is not metrizable.

Proof: Suppose, on the contrary, that \( \tilde{I} \) is metrizable. Then, there is a metric \( \rho \) on \( \tilde{I} \) such that the topology induced by this metric and the order topology coincide. By Proposition 5.1, we get that \((X, \rho)\) is a compact metric space. Thus, it is separable, which contradicts Proposition 5.3. Therefore, \( \tilde{I} \) is not metrizable.

Corollary 5.3: Two spaces \([0, 1],[\preceq]\) and \((\tilde{I}, \leq_{xy})\) are not isomorphic.
Proof: Suppose, on the contrary, that \((0,1,\leq)\) and 
\((\bar{\mathbb{I}}, \leq_{\mathbb{I}_X})\) are isomorphic. In this case, there exists a bijection \(f : [0,1] \rightarrow \bar{\mathbb{I}}\) such that both \(f\) and \(f^{-1}\) are order preserving. Take \(\mathcal{A} = f([0,1] \cap \mathbb{Q})\). For any \(\alpha, \beta \in \bar{\mathbb{I}}\) with \(\alpha <_{\mathbb{I}_X} \beta\), we have that \(f^{-1}(\alpha) < f^{-1}(\beta)\). It is obvious that \((f^{-1}(\alpha), f^{-1}(\beta)) \cap \mathbb{Q} \neq \emptyset\). This implies that 
\[
\emptyset \neq f((f^{-1}(\alpha), f^{-1}(\beta)) \cap \mathbb{Q}) = (\alpha, \beta) \cap \mathcal{A}.
\]
Since \(\mathcal{A}\) is countable, we get that \(\bar{\mathbb{I}}\) with the order topology is separable. This fact contradicts Proposition 5.3. Therefore, \((0,1,\leq)\) and 
\((\bar{\mathbb{I}}, \leq_{\mathbb{I}_X})\) are not isomorphic.

Remark 6: (1) By the formula (III.1), it is verified that the space \(I\) with the order topology induced by the linear order \(<_{\mathbb{I}_X}\) is homeomorphic to \(\bar{\mathbb{I}}\) with the order topology induced by the linear order \(<_{\mathbb{I}_X}\). Thus, they have the same topological structures.

(2) Propositions 5.1–5.3 and Corollaries 5.1 and 5.2 partially answer Problem 3.

VI. A NEW LOOK AT Q-RUNG ORTHOPAIR FUZZY SETS

The concept of Pythagorean fuzzy set (PFS) was introduced by Yager [8] as follows:

Definition 6.1 ([8]): Let \(X\) be the universe of discourse. A Pythagorean fuzzy set (PFS) \(P\) in \(X\) is defined as an object in the following form

\[
P = \{ (x, \mu_p(x), \nu_p(x)) \mid x \in X \},
\]

where the functions

\[
\mu_p : X \rightarrow [0,1],
\]

and

\[
\nu_p : X \rightarrow [0,1],
\]

transform the degree of membership and the degree of non-membership of the element \(x \in X\) to the set \(P\), respectively, and for every \(x \in X\),

\[
(\mu_p(x))^2 + (\nu_p(x))^2 \leq 1.
\]

The indeterminacy degree \(\pi_p(x)\) of element \(x\) belonging to the PFS \(P\) is defined by \(\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}\). For convenience, Chang and Xu [43] termed \((\mu_p(x), \nu_p(x))\) a Pythagorean fuzzy number (PFN). Let \(\mathbb{P}\) denote the set of all PFNs, i.e., \(\mathbb{P} = \{ (\mu, \nu) \mid [\mu, \nu] \subset [0,1] \}\).

Recently, Yager et al. [9], [10] proposed the concept of q-rung orthopair fuzzy sets (q-ROFSs).

Definition 6.2 ([9]): Let \(X\) be the universe of discourse. A q-rung orthopair fuzzy set (q-ROFS) \(Q\) in \(X\) is defined as an object in the following form

\[
Q = \{ (x, \mu_Q(x), \nu_Q(x)) \mid x \in X \},
\]

where the functions

\[
\mu_Q : X \rightarrow [0,1],
\]

and

\[
\nu_Q : X \rightarrow [0,1],
\]

transform the degree of membership and the degree of non-membership of the element \(x \in X\) to the set \(Q\), respectively, and for every \(x \in X\),

\[
(\mu_Q(x))^q + (\nu_Q(x))^q \leq 1, \quad (q \geq 1).
\]

The indeterminacy degree \(\pi_Q(x)\) of element \(x\) belonging to the q-ROFS \(Q\) is defined by \(\pi_Q(x) = \sqrt[2q]{1 - (\mu_Q(x))^q - (\nu_Q(x))^q}\). For convenience, Yager [9] termed \((\mu_Q(x), \nu_Q(x))\) a q-rung orthopair fuzzy number (q-ROFN). Let \(\mathbb{Q}\) denote the set of all q-ROFNs, i.e., \(\mathbb{Q} = \{ (\mu, \nu) \in [0,1]^2 \mid \mu^q + \nu^q \leq 1 \}\).

It is clear that a q-ROFS reduces an IFS (resp. PFS) when \(q = 1\) (resp. \(q = 2\)). When \(q = 3\), Senapati and Yager [44] called q-rung orthopair fuzzy sets as Fermatean fuzzy sets (FFSs).

Additionally, for a q-ROFN \(\alpha = (\mu_\alpha, \nu_\alpha) \in \mathbb{Q}\), \(s_\alpha(x) = (\mu_\alpha(x))^q - (\nu_\alpha(x))^q\), and \(h_\alpha(x) = (\mu_\alpha)^q + (\nu_\alpha)^q\) are called the score value and the accuracy value of \(\alpha\), respectively, by Liu and Wang [32].

Based on score values and accuracy values, a comparison method was given by Liu and Wang [32].

Definition 6.3 ([32]): Let \(\alpha_1\) and \(\alpha_2\) be two q-ROFNs.

- If \(s_\alpha(\alpha_1) < s_\alpha(\alpha_2)\), then \(\alpha_1\) is smaller than \(\alpha_2\), denoted by \(\alpha_1 <_\alpha \alpha_2\).
- If \(h_\alpha(\alpha_1) = h_\alpha(\alpha_2)\), then \(\alpha_1 = \alpha_2\); if \(h_\alpha(\alpha_1) < h_\alpha(\alpha_2)\), then \(\alpha_1 <_\alpha \alpha_2\).

If \(\alpha_1 <_\alpha \alpha_2\) or \(\alpha_1 = \alpha_2\), we will denote it by \(\alpha_1 \leq_\alpha \alpha_2\).

Observe that the score function and the accuracy function, introduced in Definition 6.3, are compressed after squaring each difference value. Hence, to rank any two q-ROFNs, Xing et al. [33] proposed the modified score function and accuracy function for q-ROFNs.

Definition 6.4 ([33], Definitions 5 and 6): Let \(\alpha = (\mu, \nu)\) be a q-ROFN and \(q \geq 1\). The modified score function \(s_\alpha (\alpha)\) and the modified accuracy function \(h_\alpha (\alpha)\) are defined as follows:

\[
s_\alpha (\alpha) = \begin{cases} 
\sqrt[2q]{1 - (\mu^q - \nu^q)}, & \mu \geq \nu, \\
\sqrt[2q]{1 - (\mu^q + \nu^q)}, & \mu < \nu,
\end{cases}
\]

and

\[
h_\alpha (\alpha) = \sqrt[2q]{\mu^q + \nu^q}.
\]

Definition 6.5 ([33], Definitions 7): Let \(\alpha_1\) and \(\alpha_2\) be two q-ROFNs.

- If \(s_\alpha(\alpha_1) < s_\alpha(\alpha_2)\), then \(\alpha_1\) is smaller than \(\alpha_2\), denoted by \(\alpha_1 <_\alpha \alpha_2\).
- If \(s_\alpha(\alpha_1) = s_\alpha(\alpha_2)\), then
  - if \(h_\alpha(\alpha_1) = h_\alpha(\alpha_2)\), then \(\alpha_1 = \alpha_2\);
  - if \(h_\alpha(\alpha_1) < h_\alpha(\alpha_2)\), then \(\alpha_1 <_\alpha \alpha_2\).

If \(\alpha_1 <_\alpha \alpha_2\) or \(\alpha_1 = \alpha_2\), we will denote it by \(\alpha_1 \leq_\alpha \alpha_2\).

Proposition 6.1: The orders \(\leq_{\alpha}\) and \(\leq_{\alpha}\) are equivalent, i.e., for \(\alpha_1, \alpha_2 \in \mathbb{Q}\), \(\alpha_1 \leq_{\alpha} \alpha_2\) if and only if \(\alpha_1 \leq_{\alpha} \alpha_2\).

Proof: It follows directly from the facts that \(s_\alpha(\alpha_1) < s_\alpha(\alpha_2) \iff s_\alpha(\alpha_1) < s_\alpha(\alpha_2), s_\alpha(\alpha_1) = s_\alpha(\alpha_2) \iff s_\alpha(\alpha_1) = s_\alpha(\alpha_2), h_\alpha(\alpha_1) < h_\alpha(\alpha_2) \iff h_\alpha(\alpha_1) < h_\alpha(\alpha_2), h_\alpha(\alpha_1) = h_\alpha(\alpha_2) \iff h_\alpha(\alpha_1) = h_\alpha(\alpha_2).\)

Fix \(q \geq 1\). Define a mapping \(\Gamma : \mathbb{Q} \rightarrow \bar{\mathbb{I}}\) by

\[
\Gamma : \mathbb{Q} \rightarrow \bar{\mathbb{I}},
\]

\[
(\mu, \nu) \mapsto (\mu^q, \nu^q).
\]

It is easy to check that \(\Gamma\) is a bijection. Every partial order \(\leq_{\alpha}\) on \(\mathbb{I}\) can induce a partial order \(\leq_{\alpha}\) defined by \(\alpha \leq_{\alpha} \beta\)
by the following way: $\Gamma((\mu_1,\nu_1),\cdots, (\mu_n,\nu_n)) = \cap f((\mu_1^q,\nu_1^q),\cdots, (\mu_n^q,\nu_n^q))$. \hfill (VI.8)

Denote $\Gamma^{(n)} = \Gamma \times \Gamma \times \cdots \times \Gamma$ (product mapping). Then, it is easy to see that $\mathcal{J}(f) = \Gamma^{-1} \circ f \circ \Gamma^{(n)}$. By using the definition of $\mathcal{J}(\cdot)$, the following theorem trivially follows from Proposition 6.1.

**Theorem 6.3:** The following statements are equivalent:

1. The mapping $f : (\tilde{I}^n, \leq_{\tilde{I}}) \to (\tilde{I}, \leq_{\tilde{I}})$ is idempotent (resp., increasing, commutative, or bounded);
2. The mapping $\mathcal{J}(f) : (\tilde{Q}^n, \leq_{\tilde{Q}}) \to (\tilde{Q}, \leq_{\tilde{Q}})$ is idempotent (resp., increasing, commutative, or bounded);
3. The mapping $\mathcal{J}(f) : (\tilde{Q}^n, \leq_{\tilde{Q}}) \to (\tilde{Q}, \leq_{\tilde{Q}})$ is idempotent (resp., increasing, commutative, or bounded).

By applying the mapping $\mathcal{J}(\cdot)$, we obtain from Theorem 6.3 many known aggregation operations for q-ROFNs.

1. Let $q = 2$. If $f$ is an intuitionistic fuzzy weighted averaging (IFWA) operator in [7, Definition 3.3], then $\mathcal{J}(f)$ reduces to PFWA operator in [45, Definition 2.10]. If $f$ is an intuitionistic fuzzy ordered weighted averaging (IFOWA) operator in [7, Definition 3.4], then $\mathcal{J}(f)$ reduces to PFOWA operator in [45, Definition 2.11].

2. Let $q \geq 1$. If we take $f$ as $\oplus$, $\odot$, $\lambda_\alpha$, or $\ominus_\alpha$ in Definition 2.12, then $\mathcal{J}(f)$ reduces to the formulas (4)–(7) in [32], respectively. By [19, Theorem 1.2.3], we observe that [32, Theorem 1] is true. Moreover, if $f$ is an IFWA operator in [7, Definition 3.3], then $\mathcal{J}(f)$ reduces to q-ROFWA operator in [32, Definition 5]. If $f$ is an IFWG operator in [12, Definition 2], then $\mathcal{J}(f)$ reduces to q-ROFWG operator in [32, Definition 6]. By Theorem 6.3, [12, Theorems 3–6], and [7, Theorems 3.4 and 3.5], we trivially obtain [32, Theorems 2–9].

3. Let $q \geq 1$. If $f$ is an intuitionistic fuzzy Maclaurin symmetric mean (IFMSM) operator in [46, Definition 4], then $\mathcal{J}(f)$ reduces to q-rung orthopair fuzzy MSM (q-ROFMSM) operator in [27, Definition 11]. By [27, Properties 1–4] and Theorem 6.3, we observe that q-ROFMSM is idempotent, increasing, commutative, and bounded. By [46, Theorem 1], we have that, for $(\mu_1,\nu_1),\cdots, (\mu_n,\nu_n) \in \tilde{Q}^n$,

$$\mathcal{J}(\text{IFMSM})((\mu_1,\nu_1),\cdots, (\mu_n,\nu_n)) = \left\langle \left(1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} (\mu_{i_j}^q)^q\right)^{\frac{1}{k^n}} \right)^\frac{1}{q}, \left(1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} (1 - \nu_{i_j}^q)^q\right)^{\frac{1}{k^n}} \right)^\frac{1}{q} \right)$$

which is exactly [27, Theorem 6].

4. Let $q \geq 1$. If $f$ is the Archimedean t-conorm and t-norm based intuitionistic fuzzy weighted averaging (ATS-IFWA)
If for all $I$ and $\tau$ where $\tau$ is an additive generator of a strict t-norm and $\zeta(x) = \tau(1-x)$. By Theorem 6.3 and [23, Properties 1–3], we observe that $\mathcal{S}$ is idempotent, increasing, and bounded.

(5) Let $q \geq 1$. If $f$ is an Atanassov’s intuitionistic extended Bonferroni mean (AIF-EBM) operator in [21, Definition 8], then, by [21, Theorem 1] and the formula (VI.8), we have that, for $(\mu_1, \nu_1, \cdots, \mu_n, \nu_n) \in \mathbb{Q}^n$,

$$\mathcal{S}(\mu_1, \nu_1, \cdots, \mu_n, \nu_n) = \left(\sqrt{\lambda_1 \zeta(1-\lambda_1 \zeta(\mu_1))} + \sqrt{\lambda_2 \zeta(1-\lambda_2 \zeta(\mu_2))} + \cdots + \sqrt{\lambda_n \zeta(1-\lambda_n \zeta(\mu_n))}\right),$$

where $\zeta$ is an additive generator of a strict t-norm, $\lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_n > 0$. By Theorem 6.3 and [21, Property 3], we observe that $\mathcal{S}$ is idempotent, increasing, and bounded.

Similarly, it is verified that all operators in [48], [49], [50], [51], [52] can be obtained by the formula (VI.8) and certain operators on IFVs.

VII. AN ADMISSIBLE SIMILARITY MEASURE WITH THE ORDER $\leq_{XY}$ AND ITS APPLICATIONS

Li and Cheng [34] introduced the concept of the similarity measure for IFSs, which was then improved by Mitchell [53] as follows. More results on the similarity measure, we refer to [54].

Definition 7.1 ([53]): Let $X$ be a universe of discourse and $S : IFS(X) \times IFS(X) \to [0, 1]$ be a mapping. $S(\_\_)$ is called an admissible similarity measure with the order $\leq$ on IFS($X$) if it satisfies the following conditions: for any $I_1, I_2, I_3 \in IFS(X)$,

1. $0 \leq S(I_1, I_2) \leq 1$.
2. $S(I_1, I_2) = 1$ if and only if $I_1 = I_2$.
3. $S(I_1, I_2) = S(I_2, I_1)$.
4. $S(I_1, I_2) \leq S(I_1, I_3) \leq S(I_1, I_2)$ and $S(I_1, I_3) \leq S(I_2, I_3)$.

Definition 7.2: Let $X$ be a universe of discourse and $I_1, I_2 \in IFS(X)$. If $(\mu_1(x), \nu_1(x)) \leq_{XY} (\mu_2(x), \nu_2(x))$ holds for all $x \in X$, then we say that $I_1$ is smaller than or equal to $I_2$ under the linear order $\leq_{XY}$, denoted by $I_1 \leq_{XY} I_2$.

Based on Definition 7.2, we introduce the improved similarity measure for IFSs as below:

Definition 7.3: Let $X$ be a universe of discourse and $S : IFS(X) \times IFS(X) \to [0, 1]$ be a mapping. $S(\_\_)$ is called an admissible similarity measure with the order $\leq_{XY}$ on IFS($X$) if it satisfies the conditions (1)–(3) in Definition 7.1, and the following one (4'):

(4') For any $I_1, I_2, I_3 \in IFS(X)$, if $I_1 \leq_{XY} I_2 \leq_{XY} I_3$, then $S(I_1, I_2) \leq S(I_1, I_3)$ and $S(I_1, I_2) \leq S(I_2, I_3)$.

If $I_1 \subset I_2$, then we have that $I_1 \leq_{XY} I_2$. This fact implies that the function $S(\_\_)$, being the similarity measure under the order $\leq_{XY}$ in Definition 7.3, is also the similarity measure in Definition 7.1. Namely, Definition 7.3 is more substantial than Definition 7.1. Definition 7.3 includes the linear order $\leq_{XY}$ on $\_\_$. In consideration; however, Definition 7.1 only considers the partial order $\subseteq$ induced by $\cap$. Therefore, the similarity measure in Definition 7.3 is more effective than that of Definition 7.1.

In the following, we construct the similarity measure $S(\_\_)$ satisfying the conditions in Definition 7.3.

Let $\alpha, \beta, \gamma \in \mathbb{I}$, define $\rho(\alpha, \beta) = \frac{1}{3}(1 + |s(\alpha) - s(\beta)|)$, $s(\alpha) \neq s(\beta)$, $\frac{1}{3}(|h(\alpha) - h(\beta)|)$, $s(\alpha) = s(\beta)$, (VII.1)

where $S(\alpha)$ and $h(\alpha)$ are the score degree and the accuracy degree of $\alpha$, respectively.

Now, we can investigate some desirable properties of the function $\rho$.

Property 1: (1) $\rho(\alpha, \beta) \in [0, 1]$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.

(2) $\rho(\alpha, \beta) = 1$ if and only if $\alpha = \beta$ and $\beta = \alpha$.

Property 2: $\rho(\alpha, \beta) = \rho(\beta, \alpha)$.

Property 3: For any $\alpha, \beta, \gamma \in \mathbb{I}$, $\rho(\alpha, \beta) + \rho(\beta, \gamma) \geq \rho(\alpha, \gamma)$.

Proof of Property 3. Consider the following cases:

1. If $s(\alpha) = s(\beta) = s(\gamma)$, then $\rho(\alpha, \beta) + \rho(\beta, \gamma) = \frac{1}{3}(|h(\alpha) - h(\beta)|) + \frac{1}{3}(|h(\beta) - h(\gamma)|) \geq \frac{1}{3}(|h(\alpha) - h(\gamma)|) = \rho(\alpha, \gamma)$.

2. If $s(\alpha) = s(\beta) \neq s(\gamma)$, then $\rho(\alpha, \beta) + \rho(\beta, \gamma) = \frac{1}{3}(|h(\alpha) - h(\beta)|) + \frac{1}{3}(|1 + |s(\beta) - s(\gamma)|) \geq \frac{1}{3}(1 + |s(\beta) - s(\gamma)|) = \rho(\alpha, \gamma)$. 

Fig. 3. Values obtained from (VII.1) for any IFV and $(0, 0, 0)$.
Fig. 4. Values obtained from (VII.1) for any IFV and $\varrho$.

(3) If $s(\alpha) \neq s(\beta) = s(\gamma)$, then, similarly to the proof of (2), it is verified that $g(\alpha, \beta) + g(\beta, \gamma) \geq g(\alpha, \gamma)$.

(4) If $s(\alpha) = s(\gamma) \neq s(\beta)$, then $g(\alpha, \beta) + g(\beta, \gamma) = \frac{1}{2}(1 + |s(\alpha) - s(\beta)|) + \frac{1}{2}(h(\beta) - h(\alpha)) \geq \frac{1}{3} \geq \frac{1}{2}((h(\alpha) - h(\gamma))) = g(\alpha, \gamma)$.

(5) If $s(\alpha) \neq s(\beta)$, $s(\alpha) \neq s(\gamma)$, and $s(\beta) \neq s(\gamma)$, then $g(\alpha, \beta) + g(\beta, \gamma) = \frac{1}{2}(1 + |s(\alpha) - s(\beta)|) + \frac{1}{2}(1 + |s(\beta) - s(\gamma)|) \geq \frac{1}{2}(2 + |s(\alpha) - s(\gamma)|) > g(\alpha, \gamma)$.

**Property 4:** For any $\alpha, \beta, \gamma \in \mathbb{I}$, if $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, then $g(\alpha, \beta) \leq g(\alpha, \gamma)$ and $g(\beta, \gamma) \leq g(\alpha, \gamma)$.

**Proof of Property 4.** Consider the following cases:

1. If $s(\alpha) = s(\beta) = s(\gamma)$, then, by $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, we have that $h(\alpha) \leq h(\beta) \leq h(\gamma)$. In this case, $g(\alpha, \beta) = \frac{1}{2}((h(\beta) - h(\alpha))) \leq \frac{1}{2}((h(\gamma) - h(\alpha))) = g(\alpha, \gamma)$ and $g(\beta, \gamma) = \frac{1}{2}((h(\beta) - h(\alpha))) \leq \frac{1}{2}((h(\gamma) - h(\alpha))) = g(\alpha, \gamma)$.

2. If $s(\alpha) = s(\beta) \neq s(\gamma)$, then, by applying $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, we have that $h(\alpha) \leq h(\beta)$ and $s(\alpha) = s(\beta) < s(\gamma)$. In this case, $g(\alpha, \beta) = \frac{1}{2}((h(\beta) - h(\alpha))) \leq \frac{1}{2} \leq \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$ and $g(\beta, \gamma) = \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$.

3. If $s(\alpha) \neq s(\beta) = s(\gamma)$, then, by applying $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, we have that $h(\beta) \leq h(\gamma)$ and $s(\beta) = s(\gamma) = s(\alpha)$. In this case, $g(\alpha, \beta) = \frac{1}{2}(1 + |s(\alpha) - s(\beta)|) = \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$ and $g(\beta, \gamma) = \frac{1}{2}((h(\beta) - h(\gamma))) \leq \frac{1}{2} \leq \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$.

4. If $s(\alpha) = s(\gamma) \neq s(\beta)$, then, by applying $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, we have that $s(\alpha) < s(\beta)$ and $s(\beta) < s(\gamma)$. This is impossible.

5. If $s(\alpha) \neq s(\beta)$, $s(\alpha) \neq s(\gamma)$, and $s(\beta) \neq s(\gamma)$, then, by applying $\alpha \leq_{XY} \beta \leq_{XY} \gamma$, we have that $s(\alpha) < s(\beta) < s(\gamma)$. In this case, $g(\alpha, \beta) = \frac{1}{2}(1 + |s(\alpha) - s(\beta)|) < \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$ and $g(\beta, \gamma) = \frac{1}{2}(1 + |s(\beta) - s(\gamma)|) < \frac{1}{2}(1 + |s(\alpha) - s(\gamma)|) = g(\alpha, \gamma)$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $x_j$ ($j = 1, 2, \ldots, n$) such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

For $I_1, I_2 \in \text{IFS}(X)$, define

$$d(I_1, I_2) = \sum_{j=1}^n \omega_j g(I_1(x_j), I_2(x_j)), \quad (VII.2)$$

and

$$S(I_1, I_2) = 1 - \sum_{j=1}^n \omega_j g(I_1(x_j), I_2(x_j)), \quad (VII.3)$$

where $I_1(x_j) = (\mu_{I_1}(x_j), \nu_{I_1}(x_j))$ and $I_2(x_j) = (\mu_{I_2}(x_j), \nu_{I_2}(x_j))$.

**Theorem 7.1:** (1) The function $d(\cdot \cdot)$ defined by the formula (VII.2) is a metric on IFSs ($\mathcal{X}$).

(2) The function $S(\cdot \cdot)$ defined by the formula (VII.3) is a similarity measure on IFSs ($\mathcal{X}$) under the order $\leq_{XY}$.

**Proof:** It follows directly from Properties 1–4.

**Remark 8:** (1) From Theorem 7.1, we observe that $d(\cdot \cdot)$ is a new metric on IFSs, which partially also answers Problem 1.

(2) The topology $\mathcal{T}_1$ on the space $\mathbb{I}$ induced by the metric $\varrho$, given by the formula (VII.1), is weaker than the order topology $\mathcal{T}_2$ by $\leq_{XY}$. Notice that ($\mathbb{I}, \mathcal{T}_2$) is not metrizable from Corollary 5.2, which means that $\mathcal{T}_1$ induced by the metric $\varrho$, given by the formula (VII.1), is strictly weaker than $\mathcal{T}_2$.

**Example 2** ([19, Example 3.3.1], [55, Example 4.1]): (A pattern recognition problem about the classification of building materials) Consider four building materials: sealant, floor varnish, wall paint, and polyvinyl chloride flooring represented by the IFSs $I_j$ ($j = 1, 2, 3, 4$) in the feature space $X = \{x_1, x_2, \ldots, x_{12}\}$. Now, given another kind of unknown building material $I$ with data as listed in Table I, we can use the similarity measure in Theorem 7.1 to identify which type of $I_j$ the unknown material $I$ belongs.

If we consider that the weight vector $\omega$ of $x_j$ ($j = 1, 2, \ldots, 12$) is:

$$\omega = (0.06, 0.10, 0.08, 0.05, 0.10, 0.11, 0.09, 0.06, 0.12, 0.10, 0.07, 0.06)^T,$$

then, by the formula (VII.3), we have that $S(I_1, I) = 0.3873$, $S(I_2, I) = 0.3828$, $S(I_3, I) = 0.5437$, and $S(I_4, I) = 0.6491$. In this case,

$$S(I_4, I) > S(I_3, I) > S(I_1, I) > S(I_2, I),$$

which means that the unknown building material $I$ should approach $I_4$. This result coincides with the one in [19].

(2) If the weights of $x_j$ ($j = 1, 2, \ldots, 12$) are equal, i.e., $\omega_1 = \omega_2 = \cdots = \omega_{12} = \frac{1}{12}$, then, by the formula (VII.3), we have that $S(I_1, I) = 0.3793$, $S(I_2, I) = 0.3773$, $S(I_3, I) = 0.5354$, and $S(I_4, I) = 0.6500$. In this case,

$$S(I_4, I) > S(I_3, I) > S(I_1, I) > S(I_2, I),$$

which means that the unknown building material $I$ should approach $I_4$. This result coincides with the one in [55].

**VIII. Conclusion**

This paper was devoted to answering three open problems proposed by Atanassov [2], [31] and systematically studying the topological and algebraic structures of the spaces $(\mathbb{I}, \leq_{XY})$ and $(\mathbb{I}, \leq_{XX})$. We first obtained that the two spaces $(\mathbb{I}, \leq_{XY})$ and $(\mathbb{I}, \leq_{XX})$ are isomorphic. We next introduced a new operator “$\rightarrow$” for IFVs using the linear order $\leq_{XY}$. We also demonstrated that this operator is a strong negation on $(\mathbb{I}, \leq_{XY})$. Moreover, we presented the facts as follows: (1) $(\mathbb{I}, \leq_{XY})$ and $(\mathbb{I}, \leq_{XX})$ are
complete lattices and Kleene algebras. (2) $\langle I, \leq_{IA} \rangle$ and $\langle I, \leq_{\leq_{IA}} \rangle$ are not separable and metrizable but compact and connected topological spaces. These results partially answer Problems 1–3 from exactly new perspectives. Furthermore, we construct an isomorphism between $\langle I, \leq_{IA} \rangle$ and $\langle Q, \leq_{QIA} \rangle$. Finally, we introduce the concept of the admissible similarity measures with particular orders for IFSs, extending the previous definition of the similarity measure for IFSs in [34]. We also construct an admissible similarity measure with the linear order $\leq_{IA}$ effectively applied to a pattern recognition problem about the classification of building materials.

References

[1] K. T. Atanassov, “Intuitionistic fuzzy set,” Fuzzy Sets Syst., vol. 20, pp. 87–96, 1986.
[2] ———, Intuitionistic Fuzzy Sets: Theory and Applications, ser. Studies in Fuzziness and Soft Computing. Springer, Berlin, Heidelberg, 1999, vol. 35.
[3] K. T. Atanassov and G. Gargov, “Interval valued intuitionistic fuzzy sets,” Fuzzy Sets Syst., vol. 31, pp. 343–349, 1989.
[4] K. T. Atanassov, Interval-Valued Intuitionistic Fuzzy Sets, ser. Studies in Fuzziness and Soft Computing. Springer, Berlin, Heidelberg, 2020, vol. 388.
[5] W. L. Gau and D. J. Buehrer, “Vague sets,” IEEE Trans. Syst., Man, Cybern., vol. 23, no. 2, pp. 610–614, 1993.
[6] H. Bustince and P. Burillo, “Vague sets are intuitionistic fuzzy sets,” Fuzzy Sets Syst., vol. 79, pp. 403–405, 1996.
[7] Z. Xu, “Intuitionistic fuzzy aggregation operators,” IEEE Trans. Fuzzy Syst., vol. 15, no. 6, pp. 1179–1187, 2007.
[8] R. R. Yager, “Pythagorean membership grades in multicriteria decision making,” IEEE Trans. Fuzzy Syst., vol. 22, no. 4, pp. 958–965, 2014.
[9] ———, Generalized orthopair fuzzy sets,” IEEE Trans. Fuzzy Syst., vol. 25, no. 5, pp. 1222–1230, 2017.
[10] R. R. Yager and N. Alajlan, “Approximate reasoning with generalized orthopair fuzzy sets,” Inf. Fusion, vol. 38, pp. 65–73, 2017.
[11] S. K. De, R. Biswas, and A. R. Roy, “Some operations on intuitionistic fuzzy sets,” Fuzzy Sets Syst., vol. 114, pp. 477–484, 2000.
[12] Z. Xu and R. R. Yager, “Some geometric aggregation operators based on intuitionistic fuzzy sets,” Int. J. Gen. Syst., vol. 35, pp. 417–433, 2006.
[13] X. Zhang and Z. Xu, “A new method for ranking intuitionistic fuzzy values and its application in multiple-attribute decision making,” Fuzzy Optim. Decis. Making, vol. 11, pp. 135–146, 2012.
[14] K. Guo, “Amount of information and attitudinal-based method for ranking Atanassov’s intuitionistic fuzzy values,” IEEE Trans. Fuzzy Syst., vol. 22, no. 1, pp. 127–137, 2020.
[15] L. De Miguel, H. Bustince, B. Pekala, U. Bentkowska, I. Da Silva, B. Bedregal, R. Mesiar, and G. Ochoa, “Interval-valued Atanassov intuitionistic OWA aggregations using admissible linear orders and their application to decision making,” IEEE Trans. Fuzzy Syst., vol. 24, no. 6, pp. 1586–1597, 2016.
[16] L. De Miguel, H. Bustince, J. Fernandez, E. Indurain, A. Kolesárová, and R. Mesiar, “Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making,” Inf. Fusion, vol. 27, pp. 189–197, 2016.
[17] Z. Xing, W. Xiong, and H. Liu, “A Euclidean approach for ranking intuitionistic fuzzy values,” IEEE Trans. Fuzzy Syst., vol. 26, no. 1, pp. 353–365, 2017.
[18] M. I. Ali, F. Feng, T. Mahmood, I. Mahmood, and H. Faizan, “A graphical method for ranking Atanassov’s intuitionistic fuzzy values using the uncertainty index and entropy,” Int. J. Intell. Syst., vol. 34, no. 10, pp. 2092–2712, 2019.
[19] Z. Xu and X. Cai, Intuitionistic Fuzzy Information Aggregation: Theory and Applications, ser. Mathematics Monograph Series. Science Press, 2012, vol. 20.
[20] W. Wang and X. Liu, “Intuitionistic fuzzy information aggregation using Einstein operations,” IEEE Trans. Fuzzy Syst., vol. 20, no. 5, pp. 923–938, 2012.
[21] S. Das, D. Guha, and R. Mesiar, “Extended Bonferroni mean under intuitionistic fuzzy environment based on a strict t-conorm,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 47, no. 8, pp. 2083–2099, 2017.
[22] Z. Xu and M. Xia, “Induced generalized intuitionistic fuzzy operators,” Knowledge-Based Systems, vol. 24, pp. 197–209, 2011.
[23] M. Xia, Z. Xu, and B. Zhu, “Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-norm and t-conorm,” Knowledge-Based Systems, vol. 31, pp. 78–88, 2012.
[24] S.-P. Wan and Z.-H. Yi, “Power average of trapezoidal intuitionistic fuzzy numbers using strict t-norms and t-conorms,” IEEE Trans. Fuzzy Syst., vol. 24, pp. 1035–1047, 2016.
[25] P. Gupta, C.-T. Lin, M. K. Mehlawat, and N. Grover, “A new method for intuitionistic fuzzy multiattribute decision making,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 46, no. 9, pp. 1167–1179, 2015.
[26] P. Liu, “Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making,” IEEE Trans. Fuzzy Syst., vol. 22, no. 1, pp. 83–97, 2014.
[27] P. Liu, S.-M. Chen, and P. Wang, “Multiple-attribute group decision-making based on q-rung orthopair fuzzy power Maclaurin symmetric mean operators,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 50, no. 10, pp. 3741–3756, 2020.
[28] H.-W. Liu and G.-J. Wang, “Multi-criteria decision-making methods based on intuitionistic fuzzy sets,” Eur. J. Oper. Res., vol. 179, no. 1, pp. 220–233, 2007.
[29] Z. Xu and N. Zhao, “Information fusion for intuitionistic fuzzy decision making: an overview,” Inf. Fusion, vol. 28, pp. 10–23, 2016.
[30] D. Li, “Topsis-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets,” IEEE Trans. Fuzzy Syst., vol. 18, no. 2, pp. 299–311, 2010.
[31] K. T. Atanassov, On Intuitionistic Fuzzy Sets Theory, ser. Studies in Fuzziness and Soft Computing. Springer, Berlin, Heidelberg, 2012, vol. 283.
[32] P. Liu and P. Wang, “Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making,” Int. J. Intell. Syst., vol. 33, no. 2, pp. 259–280, 2018.
[33] Y. Xing, R. Zhang, Z. Zhou, and J. Wang, “Some q-rung orthopair fuzzy point weighted aggregation operators for multi-attribute decision making,” Soft Computing, vol. 23, pp. 11 627–11 649, 2019.
[34] D. Li and C. Cheng, “New similarity measures of intuitionistic fuzzy sets and application to pattern recognition,” Pattern Recognit. Lett., vol. 23, pp. 221–225, 2002.
[35] J. Harding, C. Walker, and E. Walker, The Truth Value Algebra of Type-2 Fuzzy Sets: Order Convolutions of Functions on the Unit Interval. CRC Press, 2016.
[36] R. Engelking, General Topology. Polish Scientific Publishers, 1977.
[37] I. R. Munkres, Topology, Prentice-Hall, 1975.
[38] G. Birkhoff, Lattice Theory, revised ed. American Mathematical Society Colloquium Publications, Rhode Island, 1948, vol. XXV.
[39] E. Szmidt and J. Kacprzyk, “Amount of information and its reliability in the ranking of Atanassov’s intuitionistic fuzzy alternatives,” in Recent Advances in Decision Making, E. Rakus-Andrsson, R. R. Yager, N. Ichalkaranje, and L. C. Jain, Eds. Springer Berlin Heidelberg, 2009, pp. 7–19.
[40] G. Beliakov, H. Bustince, D. P. Goswami, U. K. Mukherjee, and N. R. Pal, “On averaging operators for Atanassov’s intuitionistic fuzzy sets,” Inf. Sci., vol. 181, no. 6, pp. 1116–1124, 2011.
[41] C.-L. Hwang and K. Yoon, Multiple Attribute Decision Making: Methods and Applications A State-of-the-Art Survey, ser. Lecture Notes in Economics and Mathematical Systems. Springer-Verlag Berlin Heidelberg, 1981, vol. 186.
[42] A. Haar and D. König, “Ueber einfach geordnete mengen,” J. Reine Angew. Math., vol. 139, pp. 16–28, 1910.
[43] X. Zhang and Z. Xu, “Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets,” Int. J. Intell. Syst., vol. 29, no. 12, pp. 1061–1078, 2014.
[44] T. Senapati and R. R. Yager, “Fermatean fuzzy sets,” J. Ambient Intell. Humanized Comput., vol. 11, pp. 663–674, 2020.
[45] X. Zhang, “A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making,” Int. J. Intell. Syst., vol. 31, no. 6, pp. 593–611, 2016.
[46] J. Qin and X. Liu, “An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators,” J. Intell. Fuzzy Syst., vol. 27, no. 5, pp. 2177–2190, 2014.
[47] Z. Liu, P. Liu, and X. Liang, “Multiple attribute decision-making method for dealing with heterogeneous relationship among attributes and unknown attribute weight information under q-rung orthopair fuzzy environment,” Int. J. Intell. Syst., vol. 33, no. 9, pp. 1900–1928, 2018.
[48] P. Liu and P. Wang, “Multiple-attribute decision-making based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers,” IEEE Trans. Fuzzy Syst., vol. 27, no. 5, pp. 834–848, 2018.
[49] P. Liu and Y. Wang, “Multiple attribute decision making based on q-rung orthopair fuzzy generalized Maclaurin symmetric mean operators,” Inf. Sci., vol. 518, pp. 181–210, 2020.
[50] A. P. Darko and D. Liang, “Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method,” Eng. Appl. Artif. Intell., vol. 87, p. 103259 (17 pages), 2020.
[51] C. Jana, G. Muliuddin, and M. Pal, “Some Dombi aggregation of q-rung orthopair fuzzy numbers in multiple-attribute decision making,” Int. J. Intell. Syst., vol. 34, no. 12, pp. 3220–3240, 2019.
[52] Y. Zhong, H. Gao, X. Guo, Y. Qin, M. Huang, and X. Luo, “Dombi power partitioned Heronian mean operators of q-rung orthopair fuzzy numbers for multiple attribute group decision making,” PloS ONE, vol. 14, no. 10, p. e0222007, 2019.
[53] H. B. Mitchell, “On the Dengfeng-Chuntian similarity measure and its application to pattern recognition,” Pattern Recognit. Lett., vol. 24, pp. 3101–3104, 2003.
[54] E. Szmidt, Distances and Similarities in Intuitionistic Fuzzy Sets, ser. Studies in Fuzziness and Soft Computing. Springer, Berlin, Heidelberg, 2014, vol. 307.
[55] W. Wang and X. Xin, “Distance measure between intuitionistic fuzzy sets,” Pattern Recognit. Lett., vol. 26, no. 13, pp. 2063–2069, 2005.