Normal conformal metrics on $\mathbb{R}^4$ with $Q$-curvature having power-like growth

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Abstract

Answering a question by M. Struwe [21] related to the blow-up behaviour in the Nirenberg problem, we show that the prescribed $Q$-curvature equation

$$\Delta^2 u = (1 - |x|^p)e^{4u} \text{ in } \mathbb{R}^4, \quad \Lambda := \int_{\mathbb{R}^4} (1 - |x|^p)e^{4u} dx < \infty$$

has normal solutions (namely solutions which can be written in integral form, and hence satisfy $\Delta u(x) = O(|x|^{-2})$ as $|x| \to \infty$) if and only if $p \in (0, 4)$ and

$$\left(1 + \frac{p}{4}\right) \frac{8\pi^2}{4} \leq \Lambda < 16\pi^2.$$

We also prove existence and non-existence results for the positive curvature case, namely for $\Delta^2 u = (1 + |x|^p)e^{4u}$ in $\mathbb{R}^4$, and discuss some open questions.

1 Introduction

Given the prescribed $Q$-curvature equation on $\mathbb{R}^4$

$$\Delta^2 u = Ke^{4u} \text{ in } \mathbb{R}^4,$$  \hfill (1)

where $K \in L^\infty_{\text{loc}}(\mathbb{R}^4)$ is a given function, we say that $u$ is a normal solution to (1) if $Ke^{4u} \in L^1(\mathbb{R}^4)$ and $u$ solves the integral equation

$$u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left(\frac{|y|}{|x-y|}\right) K(y)e^{4u(y)} dy + c,$$  \hfill (2)

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where \( c \in \mathbb{R} \). It is well known that (2) implies (1), while the converse is not true, see e.g. [3, 14].

If the right-hand side of (1) is slightly more integrable, more precisely if \( \log(|\cdot|)Ke^{4u} \in L^1(\mathbb{R}^4) \), then (2) is equivalent to

\[
    u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x-y|} \right) K(y)e^{4u(y)}dy + c'.
\]

(3)

We will often use this second version for convenience.

A solution \( u \) to (1) has the geometric property that the conformal metric \( e^{2u}|dx|^2 \) on \( \mathbb{R}^4 \) has \( Q \)-curvature equal to \( K \). For this reason equation (1) has received a lot of attention in the last decades, including lower and higher order analogs, both when \( K \) is constant and non-constant, see e.g. [4, 5, 15] and the references therein.

Part of the interest in solutions to (1) arises from the Nirenberg problem, i.e. the problem of finding whether a given function on a given smooth Riemannian manifold \((M, g)\), usually compact and without boundary, and of dimension 4 in our case (similar considerations hold in any dimension), can be the \( Q \)-curvature of a conformal metric \( e^{2u}g \). The variational methods or geometric flows used to study such problems, usually lead to lack of compactness issues, and upon suitable scaling at a blow-up point one often obtain as solution of (1). Moreover, because of a priori gradient and volume bounds, usually such solutions are normal, in the sense of (2), with \( Ke^{4u} \in L^1(\mathbb{R}^4) \), and if the prescribed curvature function is always positive and continuous, a blow-up argument will lead to a normal solution to (1) with \( K \) constant. Such solutions have been studied by [14] (in other dimensions by [3, 8, 10, 13, 15, 22] and others), and always take the form, when \( K = 6 \),

\[
    u(x) = \log \left( \frac{2\lambda}{1 + \lambda^2|x-x_0|^2} \right), \quad \lambda > 0, \quad x_0 \in \mathbb{R}^4.
\]

(4)

More recently, though, Borer, Galimberti and Struwe [2] studied a sign-changing prescribed Gaussian curvature problem in dimension 2, and, under the generic assumption that the prescribed curvature has only non-degenerate maxima, their blow-up analysis led possibly to either normal solutions to (1) with \( K > 0 \) constant, or to normal solutions to

\[
    -\Delta u(z) = (1 + A(x, x))e^{2u(x)}, \quad \text{in } \mathbb{R}^2,
\]

(5)

where \( A \) is a negative definite bilinear map. Later Struwe [20] showed that in fact (5) admits no normal solutions.

A similar analysis was done by Galimberti [6] and Ngô-Zhang [18] in dimension 4, which led to normal solution to (1) with \( K(x) = (1 + A(x, x)) \), again with \( A(x, x) \) a negative definite bilinear form, assuming that the prescribed curvature \( f \) has non-degenerate maxima. In this case, it was not possible to use the ideas of [20] to rule out the existence of such normal solutions. On the other hand in case the prescribed curvature function \( f \) has derivatives vanishing up to order 3, and 4-th order derivative negative definite, blow-up leads to normal solutions to (1) with \( K(x) = (1 + A(x, x, x, x)) \) and \( A \) a negative
definite symmetric 4-linear map, and in this case Struwe [21] has recently proven that such solutions do not exist.

Then the non-degenerate case remained open, namely whether solutions to (1) with $K(x) = (1 + A(x, x))$, $A$ bilinear and negative definite, do exist. In this paper we answer this question in the affirmative. In fact, focusing on the case $K(x) = (1 - |x|^p)$ for any $p \in (0, 4)$, we shall see that (1) has a normal solution with prescribed total curvature $\Lambda$, if and only if $\Lambda$ lies in a certain range.

More precisely, for $p > 0$ we set

$$\Lambda_{\text{sph}} := 16\pi^2, \quad \Lambda_{\ast,p} := (4 + p)2\pi^2 = \left(1 + \frac{p}{4}\right)\frac{\Lambda_{\text{sph}}}{2}. \quad (6)$$

The constant $\Lambda_{\text{sph}} = 6|S^4|$ is the total $Q$-curvature of the sphere $S^4$.

We start with a non-existence result.

\textbf{Theorem 1.1} Fix $p > 0$. For any $\Lambda \in (-\infty, \Lambda_{\ast,p}) \cup [\Lambda_{\text{sph}}, \infty)$ the problem

$$\Delta^2 u = (1 - |x|^p)e^{4u} \quad \text{in } \mathbb{R}^4, \quad \Lambda = \int_{\mathbb{R}^4} (1 - |x|^p)e^{4u} dx, \quad (7)$$

admits no normal solutions. In particular, for $p \geq 4$, Problem (7) admits no normal solutions.

Theorem 1.1 is based on a Pohozaev identity (Proposition 2.1) and some asymptotic estimates at infinity.

Based on a variational approach of A. Chang and W. Cheng [3], together with a blow-up argument, we shall then prove the following existence result.

\textbf{Theorem 1.2} Let $p \in (0, 4)$ be fixed. Then for every $\Lambda \in (\Lambda_{\ast,p}, \Lambda_{\text{sph}})$ there exists a (radially symmetric) normal solution to Problem (7). Such solutions (in fact, every normal solution to (7)) have the asymptotic behavior

$$u(x) = -\frac{\Lambda}{8\pi^2} \log |x| + C + O(|x|^{-\alpha}), \quad \text{as } |x| \to \infty, \quad (8)$$

for every $\alpha \in [0, 1]$ such that $\alpha < \frac{\Lambda - \Lambda_{\ast,p}}{2\pi^2}$, and

$$|\nabla^\ell u(x)| = O(|x|^{-\ell}), \quad \text{as } |x| \to \infty, \quad \ell = 1, 2, 3. \quad (9)$$

Theorems 1.1 and 1.2 leave open the case $\Lambda = \Lambda_{\ast,p}$, which is borderline from the point of view of integrability, in the sense that (8) is compatible with the integrability condition in (7) if $\Lambda > \Lambda_{\ast,p}$, but for $\Lambda = \Lambda_{\ast,p}$, (8) degenerates to

$$-\frac{\Lambda + o(1)}{8\pi^2} \log |x| + O(1) \leq u(x) \leq -\frac{\Lambda}{8\pi^2} \log |x| + O(1), \quad \text{as } |x| \to \infty,$$
(see Lemma 2.3 and Lemma 2.5), which is not incompatible the integrability of $(1-|x|)e^{4u}$.

We shall study the case $\Lambda = \Lambda_{s,p}$ from the point of view of compactness: while solutions of Theorem 1.2 must necessarily blow up as $\Lambda \uparrow \Lambda_{sph}$ (see Theorem 1.4), we find that such solutions remain compact as $\Lambda \downarrow \Lambda_{s,p}$.

**Theorem 1.3** Fix $p \in (0, 4)$. Given any sequence $(u_k)$ of radial normal solutions to (7) with $\Lambda = \Lambda_k \in [\Lambda_{s,p}, \Lambda_{sph})$ and $\Lambda_k \to \bar{\Lambda} \in [\Lambda_{s,p}, \Lambda_{sph})$, up to a subsequence $u_k \to \bar{u}$ locally uniformly, where $\bar{u}$ is a normal (and radial) solution to (7) with $\Lambda = \bar{\Lambda}$.

In particular, choosing $\Lambda_k \downarrow \Lambda_{s,p}$ and $u_k$ given by Theorem 1.2, we obtain that (7) has a normal solution $u$ also for $\Lambda = \Lambda_{s,p}$. Moreover such $u$ satisfies

$$u(x) \leq -\frac{\Lambda_{s,p}}{8\pi^2} \log |x| - \left(\frac{1}{2} + o(1)\right) \log \log |x|, \quad \text{as } |x| \to \infty,$$

and (9).

**Open problem** The solutions given by Theorems 1.2 and 1.3 are radially symmetric by construction. It is open whether all normal solutions are radially symmetric (compare to [14]), and whether they are unique, for every given $\Lambda \in [\Lambda_{s,p}, \Lambda_{sph})$.

**Open problem** Is the inequality in (10) actually an equality (see Lemma 2.9 for a sharper version of (10))?  

The proof of Theorem 1.3 relies on blow-up analysis and quantization, as studied in [19], [16], which implies that in case of loss of compactness the total $Q$-curvature converges to $\Lambda_{sph}$, which is a contradiction if $\Lambda_k \to \bar{\Lambda} \in [\Lambda_{s,p}, \Lambda_{sph})$. An important part of this argument is to rule out loss of curvature at infinity, see Lemma 4.5.

On the other hand, as $\Lambda \uparrow \Lambda_{sph}$, the non-existence result of Theorem 1.1 leaves open only two possibilities: loss of curvature at infinity, or loss of compactness. In the next theorem we show that the second case occurs.

**Theorem 1.4** Fix $p \in (0, 4)$ and let $(u_k)$ be a sequence of radial normal solutions of (4) with $\Lambda = \Lambda_k \uparrow \Lambda_{sph}$ (compare to Theorem 1.2) as $k \to \infty$. Then

$$(1-|x|^p)e^{4u_k} \to \Lambda_{sph} \delta_0 \quad \text{as } k \to \infty,$$

weakly in the sense of measures, and setting

$$\eta_k(x) := u_k(r_k x) - u_k(0) + \log 2, \quad r_k := 12e^{-u_k(0)},$$

we have

$$\eta_k(x) \to \log \left(\frac{2}{1 + |x|^2}\right) \quad \text{locally uniformly in } \mathbb{R}^4.$$  

Having addressed the case $K(x) = 1 - |x|^p$, we now analize the case $K(x) = 1 + |x|^p$. 


Similar to Theorems 1.1, 1.2 and 1.3 one can ask for which values of $\Lambda$ problem
\[ \Delta^2 u = (1 + |x|^p)e^{4u} \quad \text{in} \quad \mathbb{R}^4, \quad \Lambda := \int_{\mathbb{R}^4} (1 + |x|^p)e^{4u}dx < \infty \] (13)
admits a normal solution. The following result gives a complete answer for $p \in (0, 4]$ and a partial answer for $p > 4$. Let $\Lambda_{s,p}$ be as in (6).

**Theorem 1.5** For $p \in (0, 4]$, Problem (13) has a normal solution if and only if
\[ \Lambda_{sph} < \Lambda < 2\Lambda_{s,p}. \] (14)
For $p > 4$
\[ \Lambda_{s,p} < \Lambda < 2\Lambda_{s,p} \] (15)
is a necessary condition for the existence of normal solutions to (13), and there exists $\varepsilon_p > 0$ such that
\[ \Lambda_{s,p} + \varepsilon_p < \Lambda < 2\Lambda_{s,p} \] (16)
is a necessary condition for the existence of radial normal solutions to (13). Finally, for $p > 4$ and for every
\[ \frac{p\Lambda_{sph}}{4} < \Lambda < 2\Lambda_{s,p} \] (17)
there exists a radially symmetric normal solution to (13).

While the necessary condition (14)-(15) follow from the Pohozaev identity, the existence part and the more restrictive condition (16) are based on blow-up analysis. To study blow-up at the origin we use again the methods of [19] and [16], and to avoid vanishing of curvature at infinity, which can be seen as a form of blow-up at infinity or, equivalently, as blow-up of the Kelvin transform at the origin, we will use the blow-up analysis and classification result of [12] for normal solutions of (1) with $K(x) = |x|^p$.

**Open problems** In the case $p > 4$ it is not known whether the condition (17) is also necessary. This is also related to the problem of uniqueness/multiplicity of solutions to (13) for a given $\Lambda$ (open also in the case $p \in (0, 4]$), and to the problem of the existence of a minimal value of $\Lambda$ for which (13) admits a solution, in analogy with Theorem 1.3.

**Open problem** Every radial solution to (1) with $K(x) = 1 - |x|^p$, $p > 0$ must have finite total curvature, namely $Ke^{4u} \in L^1(\mathbb{R}^4)$, see Proposition 2.10. The same happens when $K = \text{const} > 0$, but in this case Albesino [1] has recently proven the existence of non-radial solutions $u$ with $Ke^{4u} \notin L^1(\mathbb{R}^4)$. It would be interesting to see whether there exist non-radial solutions to (1) with infinite total Q-curvature also in the case $K(x) = 1 - |x|^p$.

**Open problem** Inspired by [7, 9, 17], can one find (non-normal) solutions to (1) with $K = (1 - |x|^p)$ and arbitrarily large but finite total Q-curvature $\Lambda = \int_{\mathbb{R}^4} Ke^{4u}dx$? In the case of $K = (1 + |x|^p)$, using the methods from [12] it should be possible to prove the upper bound $\Lambda < 2\Lambda_{s,p}$ for (not necessarily normal) radial solutions.
2 Some preliminary results and proof of Theorem 1.1

We start with a Pohozaev-type identity that will be used several times.

**Proposition 2.1 (Pohozaev identity)** Let $K(x) = (1 \pm |x|^p)$ and let $u$ be a solution to the integral equation

$$u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) K(y)e^{4u(y)} \, dy + c$$

for some $c \in \mathbb{R}$, with $Ke^{4u} \in L^1(\mathbb{R}^4)$ and $\cdot^p e^{4u} \in L^1(\mathbb{R}^4)$. If

$$\lim_{R \to \infty} R^{4+p}\max_{|x| = R} e^{4u(x)} = 0,$$

then, denoting $\Lambda := \int_{\mathbb{R}^4} (1 \pm |x|^p)e^{4u(x)} \, dx$, we have

$$\Lambda = \Lambda_{sph} \left( \Lambda - \Lambda_{sph} \right) = \pm \frac{p}{4} \int_{\mathbb{R}^4} |x|^p e^{4u(x)} \, dx.$$ (19)

**Proof.** Following the proof of Proposition A.1 in [12], we need to show that as $R \to \infty$

$$R \int_{\partial B_R} |x|^p e^{4u} \, d\sigma \to 0 \quad \text{and} \quad \int_{|x| \leq R} \int_{|y| \geq R} \frac{|x+y|}{|x-y|} |x|^p e^{4u(x)} |y|^p e^{4u(y)} \, dy \, dx \to 0.$$

By (18) the boundary term goes to 0 as $R \to \infty$. For the double integral term we divide the domain of $B_R^c$ into $B_{2R}^c$ and $B_{2R} \setminus B_R$. Now using $\cdot^p e^{4u} \in L^1(\mathbb{R}^4)$ and (18) we estimate

$$\int_{|x| \leq R} \int_{|y| \leq 2R} \frac{|x+y|}{|x-y|} |x|^p e^{4u(x)} |y|^p e^{4u(y)} \, dy \, dx = o(1), \quad \text{as} \quad R \to \infty$$

and

$$\int_{|x| \leq R} \int_{|y| \geq 2R} \frac{|x+y|}{|x-y|} |x|^p e^{4u(x)} |y|^p e^{4u(y)} \, dy \, dx \leq C \int_{|x| \leq R} |x|^p e^{4u(x)} \, dx \int_{|y| \geq 2R} |y|^p e^{4u(y)} \, dy = o(1), \quad \text{as} \quad R \to \infty.$$

Another basic tool often used is the Kelvin transform.

**Proposition 2.2** Let $u$ be a normal solution to (1) with $K \in L^\infty(\mathbb{R}^4)$ and $Ke^{4u} \in L^1(\mathbb{R}^4)$. Then the function

$$\tilde{u}(x) = u \left( \frac{x}{|x|^2} \right) - \alpha \log |x|, \quad \text{for} \quad x \neq 0, \quad \alpha := \frac{1}{8\pi^2} \int_{\mathbb{R}^4} Ke^{2u} \, dx,$$ (20)
satisfies
\[ \tilde{u}(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x-y|} \right) K \left( \frac{y}{|y|^2} \right) e^{4\tilde{u}(y)} \frac{|y|}{|y|^{8-4\alpha}} dy + c, \]

namely \( \tilde{u} \) is a normal solution to
\[ \Delta^2 \tilde{u}(x) = K \left( \frac{x}{|x|^2} \right) e^{4\tilde{u}} \frac{|x|}{|x|^{8-4\alpha}}. \]

Proof. Starting from (2), with a change of variables, and using that
\[ |x||y| |\frac{x}{|x|^2} - \frac{y}{|y|^2}| = |x-y|, \]
we obtain
\[ \tilde{u}(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x|} \right) \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| K \left( \frac{y}{|y|^2} \right) e^{4\tilde{u}(y)} \frac{|y|}{|y|^{8-4\alpha}} dy + c. \]

(21)

We now start studying the asymptotic behavior of normal solutions to (1) under various assumptions.

Lemma 2.3 Let \( u \) solve the integral equation
\[ u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) K(y) e^{4u(y)} dy + c, \]

where \( K(y) \leq 0 \) for \( |y| \geq R_0 \), for a given \( R_0 \geq 0 \), and \( K e^{4u} \in L^1 \). Then we have
\[ u(x) \leq -\frac{\Lambda}{8\pi^2} \log |x| + O(1), \quad \text{as } |x| \to \infty, \]

(23)

where
\[ \Lambda = \int_{\mathbb{R}^4} K(y) e^{4u(y)} dy \in \mathbb{R}. \]

Proof. Choose \( x \) such that \( |x| \geq 2R_0 \). Without loss of generality we can assume \( R_0 \geq 2 \). Split \( \mathbb{R}^4 = A_1 \cup A_2 \cup A_3 \) where
\[ A_1 = B_{\frac{|x|}{2}}(x), \quad A_2 = B_{R_0}(0), \quad A_3 = \mathbb{R}^4 \setminus (A_1 \cup A_2). \]
Using that
\[
\log \left( \frac{|y|}{|x-y|} \right) \geq 0, \quad K(y) \leq 0 \quad \text{in } A_1,
\]
we get
\[
\int_{A_1} \log \left( \frac{|y|}{|x-y|} \right) K(y)e^{4u(y)} \, dy \leq 0.
\]
For \( y \in A_2 \) we have
\[
\log \left( \frac{|y|}{|x-y|} \right) = -\log |x| + O(1) \quad \text{as } |x| \to \infty,
\]
so that
\[
\int_{A_2} \log \left( \frac{|y|}{|x-y|} \right) K(y)e^{4u(y)} \, dy = -\log |x| \int_{A_2} K e^{4u} \, dy + O(1).
\]
For \(|y| \geq R_0 \geq 2\) and \(|x-y| > \frac{|x|}{2}\) we have \(|x-y| \leq |x| + |y| \leq |x||y|\) so that
\[
\int_{A_3} \log \left( \frac{|y|}{|x-y|} \right) K(y)e^{4u(y)} \, dy \leq -\log |x| \int_{A_3} K(y)e^{4u(y)} \, dy.
\]
Summing up
\[
\sum_{A_i} \log \left( \frac{|y|}{|x-y|} \right) K(y)e^{4u(y)} \, dy \leq -\log |x| \int_{A_2 \cup A_3} K e^{4u} \, dy + O(1),
\]
where again we used that \( K \leq 0 \) in \( A_1 \). \( \Box \)

**Corollary 2.4** Given \( p \in (0, 4) \) there is no normal solution to (7) for \( \Lambda \geq \Lambda_{\text{sph}} = 16\pi^2 \).

**Proof.** Assume that \( u \) solves (7) for some \( \Lambda \geq \Lambda_{\text{sph}} \). Then, by Lemma 2.3, \( u \) satisfies (23), which implies that assumption (18) in Proposition 2.1 is satisfied. Then (19) implies \( \Lambda < \Lambda_{\text{sph}} \), a contradiction. \( \Box \)

**Lemma 2.5** Given \( p > 0 \) let \( u \) be a normal solution to (7) for some \( \Lambda \in \mathbb{R} \). Then \( \Lambda \geq \Lambda_{*,p} \) and
\[
u(x) = -\frac{\Lambda + o(1)}{8\pi^2} \log |x| \quad \text{as } |x| \to \infty.
\]

**Proof.** We start by proving (24). We write \( u = u_1 + u_2 \), where
\[
u_2(x) = -\frac{1}{8\pi^2} \int_{B_1(x)} \log \frac{1}{|x-y|} |y|^p e^{4u(y)} \, dy.
\]
Then we have
\[ u_1(x) = -\frac{\Lambda}{8\pi^2} \log |x| \quad \text{as } |x| \to \infty. \]

We now claim that \( \frac{\Lambda}{2\pi^2} > p \). Then we have that \( |y|^p e^{4u_1(y)} \leq C \) on \( \mathbb{R}^4 \). This, and as \( u_2 \leq 0 \), we easily get that
\[ |u_2(x)| \leq C \int_{B_1(x)} \log \frac{1}{|x-y|} dy \leq C, \]

hence (24) is proven.

In order to prove the claim, given \( R \gg 1 \) and \( |x| \geq R + 1 \) we write
\[ -u_2(x) = \int_{B_R} h(R) \log \frac{1}{|x-y|} \chi_{|x-y| \leq 1} d\mu(y), \quad d\mu(y) = \frac{|y|^p e^{4u}}{|y|^p e^{4u} dy}, \]

where
\[ h(R) = \frac{1}{8\pi^2} \int_{B_R} |y|^p e^{4u} dy = o_R(1) \xrightarrow{R \to \infty} 0. \]

By Jensen’s inequality and Fubini’s theorem we obtain
\[ \int_{R+1 < |x| < 2R} e^{-4u_2} dx \leq \int_{B_R} \int_{R+1 < |x| < 2R} \left( 1 + \frac{1}{|x-y|^{4b(R)}} \right) dx d\mu(y) \leq CR^4. \]

Therefore, by Hölder inequality
\[ R^4 \approx \int_{R+1 < |x| < 2R} e^{2u_2} e^{-2u_2} dx \leq C R^2 \left( \int_{R+1 < |x| < 2R} e^{4u_2} dx \right)^{\frac{1}{2}}. \]

If \( \frac{\Lambda}{2\pi^2} \leq p \), then we have that \( |y|^p e^{4u_1(y)} \geq \frac{1}{|y|} \) for \( |y| \) large. Hence,
\[ o_R(1) = \int_{R+1 < |x| < 2R} |x|^p e^{4u_1} e^{4u_2} dx \gtrsim \frac{1}{R} \int_{R+1 < |x| < 2R} e^{4u_2} dx, \]
a contradiction.

Now that (24) is proven, we have that \( \Lambda < \Lambda_{*,p} \) contradicts \( (1 - |x|^p) e^{4u} \in L^1(\mathbb{R}^4) \), hence we must have \( \Lambda \geq \Lambda_{*,p} \). \( \square \)

**Lemma 2.6** Let \( w \in C^0(B_1 \setminus \{0\}) \) be given by
\[ w(x) = \int_{B_1} \log \left( \frac{1}{|x-y|} \right) f(y) dy, \]
for some nonnegative \( f \in L^1(B_1) \). If \( w(x_k) = O(1) \) for some \( x_k \to 0 \) then
\[ \int_{B_1} \log \left( \frac{1}{|y|} \right) f(y) dy < \infty. \]
Proof. Let \( x_k \to 0 \) be such that \( w(x_k) = O(1) \) as \( k \to \infty \). Then we have

\[
O(1) = \int_{B_1} \log \left( \frac{1}{|x - y|} \right) f(y) dy \geq O(1) + \int_{2|x_k| \leq |y| \leq 1} \log \left( \frac{1}{|x - y|} \right) f(y) dy
\]

\[
= O(1) + \int_{2|x_k| \leq |y| \leq 1} \log \left( \frac{1}{|y|} \right) f(y) dy.
\]

The lemma follows by taking \( k \to \infty \). \( \square \)

Lemma 2.7 Let \( u \) be a normal solution to (7) with \( \Lambda = \Lambda_{*,p} \), and let \( \tilde{u} \) as in (20) be its Kelvin transform, namely

\[
\tilde{u}(x) = u \left( \frac{x}{|x|^2} \right) - \left( 1 + \frac{p}{4} \right) \log |x|, \quad x \neq 0.
\]  

Then

\[
\lim_{x \to 0} \tilde{u}(x) = -\infty, \quad \lim_{x \to 0} \Delta \tilde{u}(x) = +\infty.
\]  

Proof. Observe that \( \sup_{B_1} \tilde{u} < \infty \) by Lemma 2.3. To prove (26) we assume by contradiction that \( \tilde{u}(x_k) = O(1) \) for a sequence \( x_k \to 0 \). By Proposition 2.2 we have

\[
\tilde{u}(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x - y|} \right) \left( 1 - \frac{1}{|y|^p} \right) \frac{e^{4\tilde{u}(y)}}{|y|^{4-p}} dy + c, \quad x \neq 0.
\]  

Since \( \tilde{u} \leq C \) in \( B_1 \), from (25) and the continuity of \( u \) it follows that

\[
\left( 1 + \frac{1}{|y|^p} \right) \frac{e^{4\tilde{u}(y)}}{|y|^{4-p}} \leq \frac{C}{|y|^8} \text{ in } B_1^c.
\]  

Then from (28) we obtain

\[
\tilde{u}(x) = -\frac{1}{8\pi^2} \int_{B_1} \log \left( \frac{1}{|x - y|} \right) \frac{e^{4\tilde{u}(y)}}{|y|^{4-p}} dy + O(1), \quad 0 < |x| < 1.
\]  

Then by Lemma 2.6 applied to (30), and with a change of variables, we get

\[
\int_{B_1} \log(|y|) |y|^p e^{4\tilde{u}(y)} dy = \int_{B_1} \log \left( \frac{1}{|y|} \right) \frac{e^{4\tilde{u}(y)}}{|y|^{4-p}} dy < \infty.
\]  

Then, as \( |x| \to \infty \) we obtain

\[
\int_{|y| \leq \sqrt{x}} (1 - |y|^p) e^{4u(y)} dy = \Lambda_{*,p} - \int_{|y| > \sqrt{x}} (1 - |y|^p) e^{4u(y)} dy
\]

\[
= \Lambda_{*,p} + O \left( \frac{1}{\log(|x|)} \right) \int_{|y| > \sqrt{x}} \log(|y|)(1 - |y|^p) e^{4u(y)} dy
\]

\[
= \Lambda_{*,p} + O \left( \frac{1}{\log(|x|)} \right).
\]
Moreover, $u$ can be given by (3) with $K = 1 - |x|^p$. Hence, for $|x| >> 1$

$$u(x) = O(1) + \frac{1}{8\pi^2} \left( \int_{|y| \leq \sqrt{|x|}} + \int_{|x| \leq |y| \leq 2|x|} + \int_{|y| \geq 2|x|} \right) \log \left( \frac{1}{|x - y|} \right) (1 - |y|^p) e^{4u(y)} dy$$

$$\geq O(1) - \frac{\Lambda_{s,p}}{8\pi^2} \left( 1 + O \left( \frac{1}{\log |x|} \right) \right) \log |x|$$

$$= -\frac{\Lambda_{s,p}}{8\pi^2} \log |x| + O(1),$$

a contradiction to $|\cdot|^p e^{4u} \in L^1(\mathbb{R}^4)$, which completes the proof of (26).

To prove (27) we differentiate into (28) and obtain

$$\Delta \tilde{u}(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x - y|^2} \left( 1 - \frac{1}{|y|^p} \right) e^{4\tilde{u}(y)} |y|^{4-p} dy,$$

and as before, we use (29) to get

$$\Delta \tilde{u}(x) = \frac{1}{2\pi^2} \int_{B_1} \frac{1}{|x - y|^2} \frac{e^{4\tilde{u}(y)}}{|y|^4} dy + O(1)$$

$$\geq C \int_{B_1} \log \left( \frac{1}{|x - y|} \right) \frac{e^{4\tilde{u}(y)}}{|y|^4} dy + O(1)$$

$$= -C 8\pi^2 \tilde{u}(x) + O(1) \to \infty \text{ as } x \to 0.$$

\(\square\)

**Proposition 2.8** There exists no normal solution to (7) for $p \geq 4$.

**Proof.** Assume by contradiction that there exists a normal solution $u$ to (7) for some $p \geq 4$. Then necessarily we have that $\Lambda \geq \Lambda_{s,p}$, thanks to (24). Now we distinguish the following two cases.

**Case 1** $\Lambda > \Lambda_{s,p}$.

Since $u \leq -\frac{\Lambda}{8\pi^2} \log |x| + C$ for $|x| \geq 1$ by Lemma 2.3, we see that $u$ satisfies (18). Hence, by (19),

$$\Lambda_{sph} \leq \Lambda_{s,p} < \Lambda < \Lambda_{sph},$$

a contradiction.

**Case 2** $\Lambda = \Lambda_{s,p}$. By Lemma 2.7 we see that (18) is satisfied, and we arrive at a contradiction as in Case 1. \(\square\)

**Proof of Theorem 1.1** Combine Corollary 2.4, Lemma 2.5 and Proposition 2.8.

For $\Lambda = \Lambda_{s,p}$ we obtain a sharper version of (26) if $u$ is radial.
Lemma 2.9 Given $p \in (0, 4)$, let $u$ be a radially symmetric normal solution to (7) with $\Lambda = \Lambda_{*,p}$. Then
\[
\limsup_{|x| \to \infty} \frac{u(x) + (1 + \frac{p}{4}) \log |x|}{\log \log |x|} = -\frac{1}{2}
\]

Proof. We set $\tilde{u}$ as in (25), so that it satisfies (28). By Lemma 2.7 we get
\[
\lim_{r \to 0} \tilde{u}(r) = -\infty, \quad \lim_{r \to 0} \Delta \tilde{u}(r) = +\infty.
\]
In particular, $\tilde{u}$ is monotone increasing in a small neighborhood of the origin. Using this and (30) we estimate for $|x| \to 0$
\[
-\tilde{u}(x) + O(1) \geq \frac{1}{8\pi^2} \int_{2|x| \leq |y| < 1} \log \left( \frac{1}{|x-y|} \right) \frac{e^{4\tilde{u}(y)}}{|y|^4} dy + O(1)
\]
\[
= \frac{1}{8\pi^2} \int_{2|x| \leq |y| < 1} \log \left( \frac{1}{|y|} \right) \frac{e^{4\tilde{u}(y)}}{|y|^4} dy + O(1)
\]
\[
\geq \frac{e^{4\tilde{u}(x)}}{8\pi^2} \int_{2|x| \leq |y| < 1} \log \left( \frac{1}{|y|} \right) \frac{dy}{|y|^4} + O(1)
\]
\[
= \frac{e^{4\tilde{u}(x)}}{8\pi^2} |S^3| \int_{2|x|}^{1} \log \frac{1}{t} dt + O(1).
\]
Computing the integral and considering that $|S^3| = 2\pi^2$ we get
\[
-\tilde{u}(x) + O(1) \geq \frac{e^{4\tilde{u}(x)}}{8} (\log(2|x|))^2, \quad \text{as } x \to 0.
\]
Taking the logarithm and rearranging we finally get
\[
\limsup_{x \to 0} \frac{\tilde{u}(x)}{\log \log \left( \frac{1}{|x|} \right)} \leq -\frac{1}{2}.
\]
Next we show that the above limsup is actually $-\frac{1}{2}$.

We assume by contradiction that the above lim sup is less than $-\frac{1}{2}$. Then there exists $\varepsilon > 0$ such that for $|x|$ small we have
\[
\tilde{u}(x) \leq - \left( \frac{1}{2} + \frac{\varepsilon}{4} \right) \log \log \frac{1}{|x|}.
\]
Hence, from (30) we obtain for $|x|$ small
\[
-\tilde{u}(x) \leq C \int_{B_1} \log \left( \frac{1}{|x-y|} \right) \frac{dy}{|y|^4 \log |y|^{2+\varepsilon}} + O(1)
\]
\[
= C(I_1 + I_2 + I_3) + O(1),
\]
where
\[ I_i = \int_{A_i} \log \frac{1}{|x-y|} dy \]
\[ A_1 = B_{\frac{x}{2}}, \quad A_2 = B_{2|x|} \setminus B_{\frac{x}{2}}, \quad A_3 = B_1 \setminus B_{2|x|}. \]

One easily gets
\[ I_1 \leq C \frac{1}{|\log |x||^{\varepsilon}}, \quad I_2 \leq \frac{C}{|\log |x||^{1+\varepsilon}}, \quad I_3 \leq C, \]
a contradiction to \( \tilde{u}(x) \to -\infty \) as \( |x| \to 0 \).

Proposition 2.10 Let \( u \) be a radial solution to
\[ \Delta^2 u = (1 - |x|^p)e^{4u} \quad \text{in} \ \mathbb{R}^4, \quad (32) \]
for some \( p > 0 \). Then
\[ \int_{\mathbb{R}^4} (1 + |x|^p)e^{4u} dx < \infty. \quad (33) \]

Proof. If (33) is false then there exists \( R > 0 \) such that
\[ \int_{B_R} (1 - |x|^p)e^{4u} dx < 0. \]
In particular, \( (\Delta u)'(r) < 0 \) for \( r \geq R \). We can consider the following two cases:

Case 1: \( \lim_{r \to \infty} \Delta u(r) \geq 0 \). Then as \( \Delta u \) is monotone decreasing on \( (R, \infty) \), we see that \( \Delta u > 0 \) on \( (R, \infty) \). Therefore, by (37) we get that \( u \geq -C \) in \( \mathbb{R}^4 \). This, (32) and (37) imply that \( \Delta u(r) \leq -r^{p+2} \) as \( r \to \infty \), a contradiction.

Case 2: \( \lim_{r \to \infty} \Delta u(r) < 0 \). In this case, by (37), we have that \( u(r) \lesssim -r^2 \) as \( r \to \infty \), a contradiction to the assumption \( (1 + |x|^p)e^{4u} \notin L^1(\mathbb{R}^4) \). \( \square \)

3 Proof of Theorem 1.2

3.1 Existence

The existence part in Theorem 1.2 will be based on the following result, which will be proven in Section 7 using methods from [3] and [14].

Proposition 3.1 For every \( 0 < \Lambda < \Lambda_{\text{sph}} \) and \( \lambda > 0 \), there exists a radial solution \( u_\lambda \) to
\[ \Delta^2 u_\lambda = (\lambda - |x|^p)e^{-|x|^2}e^{4u_\lambda} \quad \text{in} \ \mathbb{R}^4, \quad (34) \]
\[ \int_{\mathbb{R}^4} (\lambda - |x|^p)e^{-|x|^2}e^{4u_\lambda} dx = \Lambda, \quad (35) \]
which is normal, namely $u_\lambda$ solves the integral equation

$$u_\lambda(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x-y|} \right) (\lambda - |y|^p) e^{-|y|^2} e^{4u_\lambda(y)} dy + c_\lambda,$$

(36)

for some constant $c_\lambda \in \mathbb{R}$.

We will often use the identity

$$w(R) - w(r) = \int_r^R \frac{1}{\omega_3 t^3} \int_{B_t} \Delta w dx dt, \quad 0 \leq r < R, \ w \in C^2_{rad}(\mathbb{R}^4), \ \omega_3 = 2\pi^2, \ (37)$$

which follows at once from the divergence theorem and the fundamental theorem of calculus.

Let $u_\lambda$ be given as in Proposition 3.1.

**Lemma 3.2** For every $\lambda > 0$ we have $u_\lambda(x) \downarrow -\infty$ and $\Delta u_\lambda(x) \uparrow 0$ as $|x| \to \infty$.

**Proof.** The function

$$r \mapsto \int_{B_r} \Delta^2 u_\lambda(x) dx = \int_{B_r} (\lambda - |x|^p) e^{-|x|^2} e^{4u_\lambda(x)} dx$$

is increasing on $[0, \lambda^\frac{1}{p}]$, and decreasing to $\Lambda$ on $[\lambda^\frac{1}{p}, \infty)$. In particular it is positive for every $r > 0$. Then, by (37) (applied with $w = \Delta u_\lambda$) we infer that $\Delta u_\lambda(x)$ is an increasing function of $|x|$.

Differentiating under the integral sign from (36) we obtain

$$|\Delta u_\lambda(x)| \leq C \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} (1 + |y|^p) e^{-|y|^2} dy \xrightarrow{|x| \to \infty} 0,$$

where in the first inequality we have used that $u_\lambda \leq C$ on $\mathbb{R}^4$, thanks to Lemma 2.3.

This in turn implies that $\Delta u_\lambda < 0$, hence $u_\lambda$ is decreasing by (37). Finally $u_\lambda \to -\infty$ as $|x| \to \infty$ follows from Lemma 2.3.

□

**Lemma 3.3** We have $\lambda e^{4u_\lambda(0)} \to \infty$ as $\lambda \downarrow 0$.

**Proof.** Assume by contradiction that $\lambda e^{4u_\lambda(0)} \leq C$ as $\lambda \to 0$. Then

$$\Lambda = \int_{\mathbb{R}^4} (\lambda - |x|^p) e^{-|x|^2} e^{4u_\lambda} dx \leq \int_{B_{\lambda^\frac{1}{p}}} \lambda e^{-|x|^2} e^{4u_\lambda(0)} \xrightarrow{\lambda \to 0} 0,$$

which is absurd.

□

Now we set

$$\eta_\lambda(x) = u_\lambda(r_\lambda x) - u_\lambda(0), \quad \lambda r_\lambda^4 e^{4u_\lambda(0)} := 1. \ (38)$$

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Notice that $r_\lambda \to 0$ by Lemma 3.3. By definition and Lemma 3.2 we have that

$$\eta_\lambda \leq 0 = \eta_\lambda(0), \quad \Delta \eta_\lambda(x) \uparrow 0 \quad \text{as } |x| \to \infty,$$

and by a change of variables in (36) we see that $\eta_\lambda$ is a normal solution to

$$\Delta^2 \eta_\lambda = \left(1 - \frac{r^p_\lambda}{\lambda |x|^p}\right)e^{-r^2_\lambda |x|^2}e^{4\eta_\lambda}.$$ 

With a similar change of variables in (35) we also get

$$\int_{\mathbb{R}^4} \left(1 - \frac{r^p_\lambda}{\lambda |x|^p}\right)e^{-r^2_\lambda |x|^2}e^{4\eta_\lambda}dx = \Lambda.$$ \hspace{1cm} (39) 

Since $\eta_\lambda \leq 0$, we have

$$0 < \Lambda < \int_{B_{\frac{1}{\lambda^{1/p}}}} e^{-r^2_\lambda |x|^2}e^{4\eta_\lambda}dx \leq \text{meas} \left(B_{\frac{1}{\lambda^{1/p}}}ight),$$

which implies that

$$\limsup_{\lambda \to 0} \frac{r^p_\lambda}{\lambda} < \infty.$$ \hspace{1cm} (40) 

**Lemma 3.4** We have

$$\limsup_{\lambda \to 0} |\Delta \eta_\lambda(0)| < \infty.$$ \hspace{1cm} (41) 

**Proof.** Assume by contradiction that

$$\limsup_{\lambda \to 0} |\Delta \eta_\lambda(0)| = \infty.$$ \hspace{1cm} (42) 

Then, using that $\Delta \eta_\lambda(x) \uparrow 0$ as $|x| \to \infty$ for every $\lambda > 0$, there exists $R_\lambda > 0$ such that $\Delta \eta_\lambda(R_\lambda) = -1$. Then, as $\Delta \eta_\lambda \leq -1$ on $[0, R_\lambda]$, we have that

$$\eta_\lambda(r) \leq -\frac{1}{8}r^2 \quad \text{for } 0 \leq r \leq R_\lambda.$$ 

From this, and using (37) and (40) one obtains

$$\Delta \eta_\lambda(R_\lambda) - \Delta \eta_\lambda(0) = O \left(\int_0^{R_\lambda} \frac{1}{t^3} \int_{B_t} \Delta^2 \eta_\lambda dx dt\right)$$

$$= O \left(\int_0^{R_\lambda} \frac{1}{t^3} \int_{B_t} (1 + |x|^p)e^{4\eta_\lambda} dx dt\right) = O(1),$$

a contradiction to (42) and the definition of $R_\lambda$. \hfill \square
Using that $\eta_\lambda(0) = 0$ and Lemma 3.3, together with ODE theory we get that, up to a subsequence,

$$\eta_\lambda \to \eta \quad \text{in} \quad C^4_{\text{loc}}(\mathbb{R}^4), \quad \text{as} \quad \lambda \to 0, \quad (43)$$

where the limit function $\eta$ satisfies

$$\Delta^2 \eta = (1 - \mu |x|^p) e^{4\eta} \quad \text{in} \quad \mathbb{R}^4, \quad \mu := \lim_{\lambda \to 0} \frac{r^p}{\lambda} \in [0, \infty).$$

Notice that at this stage we do not know whether $\eta$ is a normal solution, $\mu > 0$, and

$$\int_{\mathbb{R}^4} (1 - |x|^p) e^{4\eta} \, dx = \Lambda. \quad \text{This is what we are going to prove next.}$$

**Lemma 3.5** If $\mu = 0$ then $e^{4\eta} \in L^1(\mathbb{R}^4)$.

**Proof.** It follows from Lemma 3.2 and (43) that $\Delta \eta$ is increasing, and $\lim_{r \to \infty} \Delta \eta(r) =: c_0 \in [-\infty, 0]$. If $c_0 < 0$ then $\eta(r) \lesssim -r^2$, and hence $e^{4\eta} \in L^1(\mathbb{R}^4)$. Therefore, if the lemma were false then necessarily we have $c_0 = 0$ and $e^{4\eta} \notin L^1(\mathbb{R}^4)$. Then using (37) one can show that for any $M > 0$ large we have $\Delta \eta(r) \leq -\frac{M}{r^2}$ for $r \gg 1$. This in turn implies that $\eta(r) \leq -2 \log r$ for $r \gg 1$, and hence $e^{4\eta} \in L^1(\mathbb{R}^4)$, a contradiction. $\square$

**Lemma 3.6** We have $\mu > 0$.

**Proof.** Assume by contradiction that $\mu = 0$. Then $\eta$ is a radial solution to

$$\Delta^2 \eta = e^{4\eta} \quad \text{in} \quad \mathbb{R}^4,$$

with $e^{4\eta} \in L^1(\mathbb{R}^4)$ by Lemma 3.5. By [14, Theorem 2.1], either $\eta$ is spherical, namely $\eta(x) = \log \left( \frac{2\lambda}{1+\lambda^2|x|^2} \right) + \log \frac{6}{4}$, for some $\lambda > 0$, or there exists $c_0 > 0$ such that

$$-c_0 := \lim_{|x| \to \infty} \Delta \eta(x) < 0. \quad (44)$$

We shall now show that each of these two cases leads to a contradiction.

**Case 1:** (44) holds.

By Lemma 3.2, for every $\lambda > 0$ we can find $0 < R_{1,\lambda} < R_{2,\lambda}$ such that

$$\eta_\lambda(R_{1,\lambda}) = -\frac{c_0}{2}, \quad \eta_\lambda(R_{2,\lambda}) = -\frac{c_0}{4}.$$ 

Moreover (43) implies that $R_{1,\lambda}, R_{2,\lambda} \to \infty$ as $\lambda \downarrow 0$.

Again by Lemma 3.2 and (43) we have $\Delta \eta_\lambda \leq -\frac{c_0}{4}$ in $[0, R_{2,\lambda}]$ and (37) implies that

$$\eta_\lambda(r) \leq -\frac{c_0}{32} r^2 \quad \text{for} \quad 0 \leq r \leq R_{2,\lambda}.$$
Applying (37) with \( w = \Delta \eta_\lambda \), and using (40), we finally get
\[
0 < \frac{c_0}{4} = \Delta \eta_\lambda (R_{2, \lambda}) - \Delta \eta_\lambda (R_{1, \lambda}) = \int_{R_{1, \lambda}}^{R_{2, \lambda}} \frac{1}{\omega_3 t^2} \int_{B_t} \left( 1 - \frac{r_\lambda^p}{\lambda} |x|^p \right) e^{-r_\lambda^2 |x|^2} e^{4\eta_\lambda} \, dx \, dt
\]
\[
= O \left( \int_{R_{1, \lambda}}^{\infty} \frac{1}{t^2} \int_{B_t} (1 + |x|^p) e^{-c_0 |x|^2} \, dx \, dt \right) \xrightarrow{\lambda \to 0} 0,
\]
which is a contradiction.

**Case 2:** \( \eta \) is spherical, and in particular
\[
\int_{\mathbb{R}^4} e^{4\eta} \, dx = \Lambda_{\text{sph}}.
\]
Since \( \Lambda < \Lambda_{\text{sph}} \), we can fix \( R_0 > 0 \) such that
\[
\int_{B_{R_0}} e^{4\eta} \, dx > \Lambda.
\]
Taking into account that \( r_\lambda \to 0 \) and by assumption \( \frac{r_\lambda^p}{\lambda} \to 0 \) as \( \lambda \downarrow 0 \), we can find \( \lambda_0 = \lambda_0(R_0) \) such that for
\[
\int_{B_{R_0}} \left( 1 - \frac{r_\lambda^p}{\lambda} |x|^p \right) e^{-r_\lambda^2 |x|^2} e^{4\eta_\lambda} \, dx \geq \Lambda \quad \text{for } 0 < \lambda < \lambda_0.
\]
(45)

Setting
\[
\Gamma_\lambda(t) := \int_{B_t} \left( 1 - \frac{r_\lambda^p}{\lambda} |x|^p \right) e^{-r_\lambda^2 |x|^2} e^{4\eta_\lambda} \, dx,
\]
we see that \( \Gamma_\lambda(0) = 0 \), \( \Gamma_\lambda \) is monotone increasing on \([0, \frac{\Lambda^{\frac{1}{p}}}{r_\lambda}]\), and then it decreases to \( \Lambda \) on the interval \([\frac{\Lambda^{\frac{1}{p}}}{r_\lambda}, \infty)\). Together with (45) it follows that
\[
\Gamma_\lambda(t) \geq \Lambda \quad \text{for } t \geq R_0, \; 0 < \lambda < \lambda_0.
\]
(46)

Applying (37) with \( R = \infty, w = \Delta \eta_\lambda \), and recalling that \( \lim_{|x| \to \infty} \Delta \eta_\lambda(x) = 0 \), we get for \( r \geq R_0 \)
\[
\Delta \eta_\lambda(r) = - \int_r^{\infty} \frac{\Gamma(t)}{\omega_3 t^2} \, dt \leq - \frac{\Lambda}{2\omega_3} \frac{1}{2r^2} = -(4 + \delta) \frac{1}{2r^2},
\]
where \( \delta > 0 \) is such that \( \Lambda_{\ast, p} + 2\delta \pi^2 = \Lambda \). Hence, for \( t > R_0 \),
\[
\int_{B_t} \Delta \eta_\lambda \, dx \leq \int_{B_t \setminus B_{R_0}} \Delta \eta_\lambda \, dx \leq - \frac{4 + \delta}{4} \omega_3 (t^2 - R_0^2).
\]

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Again by (37), as \( \eta_\lambda(R_0) = O(1) \) by (43), we have
\[
\eta_\lambda(r) \leq \eta_\lambda(R_0) + \int_{R_0}^{r} -\frac{(4 + p + \delta)(t^2 - R_0^2)}{4t^3} dt dx dt = C(R_0) - \frac{4 + p + \delta}{4} \log r, \quad r \geq R_0.
\]
This implies that
\[
\lim_{R \to \infty} \lim_{\lambda \to 0} \int_{B_R} (1 + |x|^p)e^{4\eta_\lambda} dx = 0.
\]
It follows from (48) that as \( \lambda \downarrow 0, \)
\[
\Lambda = \int_{\mathbb{R}^4} \left( 1 - \frac{r^p}{\lambda} |x|^p \right) e^{-r^2|x|^2} e^{4\eta_\lambda} dx \to \int_{\mathbb{R}^4} e^{4\eta} dx = \Lambda_{\text{sph}},
\]
a contradiction.

This completes the proof of the lemma. \( \square \)

**Remark 1** The above proof also works without using the fact that \( e^{4\eta} \in L^1(\mathbb{R}^4) \). Indeed, trivially one can find \( R_0 > 0 \) as in Case 2, and proceed in a similar way.

**Proof of the existence part (completed).** From Lemma 3.6, choosing \( R = \left( \frac{4}{\mu} \right)^{1/p} \) we obtain for \( \lambda \) sufficiently small
\[
1 - \frac{r^p}{\lambda} |x|^p \leq 1 - \frac{\mu}{2} |x|^p \leq -\frac{\mu}{4} |x|^p, \quad |x| \geq R,
\]
hence from (39) we obtain
\[
\int_{B_R} \frac{\mu}{4} |x|^p e^{-r^2|x|^2} e^{4\eta_\lambda} dx \leq -\int_{B_R} \left( 1 - \frac{r^p}{\lambda} |x|^p \right) e^{-r^2|x|^2} e^{4\eta_\lambda} dx
\]
\[
= \int_{B_R} \left( 1 - \frac{r^p}{\lambda} |x|^p \right) e^{-r^2|x|^2} e^{4\eta_\lambda} dx - \Lambda \leq C,
\]
where in the last inequality we also used that \( \eta_\lambda \leq 0 \). Since, the integrand in \( B_R \) is uniformly bounded, it follows at once that
\[
\int_{\mathbb{R}^4} (1 + |x|^p)e^{-r^2|x|^2} e^{4\eta_\lambda} dx \leq C.
\]
Moreover (49) also implies
\[
\int_{B_R} e^{-r^2|x|^2} e^{4\eta_\lambda} dx \leq \frac{C}{R^p} \to 0, \quad \text{as} \ R \to \infty,
\]
uniformly with respect to $\lambda$, which in turn yields

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^4} e^{-r_2^2|x|^2} e^{4\eta_{\lambda}} \, dx = \int_{\mathbb{R}^4} e^{4\eta} \, dx. \tag{51}$$

By Fatou’s lemma

$$\int_{\mathbb{R}^4} |x|^p e^{4\eta} \, dx \leq \lim_{\lambda \to 0} \int_{\mathbb{R}^4} |x|^p e^{-r_2^2|x|^2} e^{4\eta_{\lambda}} \, dx. \tag{52}$$

Then (51) and (52) give

$$\int_{\mathbb{R}^4} (1 - \mu |x|^p) e^{4\eta} \, dx \geq \Lambda. \tag{53}$$

We now proceed as in Case 2 of the proof of Lemma 3.6 to show that the above inequality is actually an equality and that $\eta$ is a normal solution. Since

$$\frac{\lambda^2}{r_\lambda} \to \frac{1}{\mu^2} > 0,$$

we have for $R_0 = 2\mu^{-\frac{1}{p}}$ and $\lambda_0 = \lambda_0(R_0)$ sufficiently small that (46) holds, hence, as before (47)-(48) follow. In particular (48) implies that

$$\int_{\mathbb{R}^4} (1 - \mu |x|^p) e^{4\eta} \, dx = \Lambda,$$

and by taking the limit using (43) and (47) we obtain

$$\eta(x) \leftarrow \eta_\lambda(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x - y|} \right) \left( 1 - \frac{r_\lambda^2}{\lambda} |y|^p \right) e^{-r_2^2|y|^2} e^{4\eta_{\lambda}(y)} \, dy + c_\lambda \to \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x - y|} \right) (1 - \mu |y|^p) e^{4\eta(y)} \, dy + c, \tag{54}$$

where the identity in the first line follows from (46) and (48). In particular we have shown that $\eta$ is a normal solution.

We now set

$$u(x) = \eta(\rho x) + \log \rho, \quad \rho := \mu^{-\frac{1}{p}},$$

and with a simple change of variable we get

$$u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x - y|} \right) (1 - |y|^p) e^{4u(y)} \, dy + c \tag{55}$$

so that $u$ is a normal solution to (17). \qed
3.2 Asymptotic behaviour

Proof of (8) Consider the Kelvin transform of $u$ given by (20). By Proposition 2.2 $\tilde{u}$ satisfies

$$\tilde{u}(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x-y|} \right) \left( 1 - \frac{1}{|y|^p} \right) e^{4\tilde{u}(y)} \frac{dy}{|y|^{8-\Lambda/2\pi^2}} + c.$$ \hspace{1cm} (56)

In particular

$$\Delta^2 \tilde{u}(x) = \left( 1 - \frac{1}{|x|^p} \right) e^{4\tilde{u}(x)} = O \left( \frac{1}{|x|^{8+\Lambda/2\pi^2}} \right), \quad \text{as } |x| \to 0.$$ 

Observing that for $\Lambda > \Lambda_{*,p}$ we have

$$8 + \frac{\Lambda}{2\pi^2} = 4 - \frac{\Lambda - \Lambda_{*,p}}{2\pi^2} < 4,$$

we get

$$\Delta^2 \tilde{u} \in L^q_{\text{loc}}(\mathbb{R}^4) \quad \text{for } 1 \leq q < \frac{1}{1 - \frac{\Lambda - \Lambda_{*,p}}{8\pi^2}}, \hspace{1cm} (57)$$

hence by elliptic estimates $\tilde{u} \in W^{4,q}_{\text{loc}}(\mathbb{R}^4)$ with $p$ as in (57), and by the Morrey-Sobolev embedding $\tilde{u} \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^4)$ for $\alpha \in [0,1]$ such that $\alpha < \frac{\Lambda - \Lambda_{*,p}}{2\pi^2}$. Then (8) follows. Alternatively to the elliptic estimates, the same $C^\alpha$ regularity can also be obtained directly from (56), using the Hölder inequality and the following estimate: For any $r > 0$

$$\int_{B_1} |\log |z-h| - \log |z||^r \, dz \leq C(r) \begin{cases} |h|^r & \text{for } r < 4 \\ |h|^r \log |h| & \text{for } r = 4 \\ |h|^4 & \text{for } r > 4, \end{cases}$$

for $|h| > 0$ small. \hfill \Box

Proof of (9) For $\ell = 1, 2, 3$ we differentiate in (55) to get

$$|\nabla^\ell u(x)| = O \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^{\ell}(1 + |y|^p)} e^{4u(y)} \, dy \right)$$

Since $\Lambda > \Lambda_{*,p}$, by (8) we have that

$$(1 + |x|^p) e^{4u(x)} \leq \frac{C}{1 + |x|^{4+\delta}},$$

for some $\delta > 0$. Therefore, for $|x|$ large

$$|\nabla^\ell u(x)| \leq C \left( \int_{B_{|x|^2}} + \int_{B_{|x|2} \setminus B_{|x|^2}} + \int_{B_{|x|2} \setminus B_{|x|^2}} \right) \frac{1}{|x-y|^{\ell}} \frac{dy}{1 + |y|^{4+\delta}}$$

$$\leq \frac{C}{|x|^\ell} + \frac{C}{|x|^{4+\delta}} \int_{B_{|x|2} \setminus B_{|x|^2}} \frac{dy}{|x-y|^{\ell}}$$

$$\leq \frac{C}{|x|^\ell}.$$
4 Proof of Theorem 1.3

Let \((u_k)\) be a sequence of radial normal solutions to (7) with \(\Lambda = \Lambda_k \in [\Lambda_{s,p}, \Lambda_{sph})\), i.e.

\[
    u_k(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p) e^{4u_k(y)} dy + c_k,
\]

and

\[
    \Lambda_k = \int_{\mathbb{R}^4} (1 - |x|^p) e^{4u_k(x)} dx \to \bar{\Lambda} \in [\Lambda_{s,p}, \Lambda_{sph}).
\]

We want to prove the following:

**Proposition 4.1** Up to a subsequence we have \(u_k \to \bar{u}\) uniformly locally in \(\mathbb{R}^4\) where \(\bar{u}\) is a normal solution to (7) with \(\Lambda = \bar{\Lambda}\)

In the following we shall use several times that \(u_k\) is radially decreasing. This follows with the same proof of Lemma 3.2.

**Lemma 4.2** We have \(u_k(0) \geq -C\) where \(C\) only depends on \(\inf_k \Lambda_k\).

**Proof.** We have

\[
    \Lambda_k = \int_{\mathbb{R}^4} (1 - |x|^p) e^{4u_k} dx \leq \int_{B_1} e^{4u_k(x)} dx \leq |B_1| e^{4u_k(0)},
\]

where in the last inequality we used that \(u_k\) is monotone decreasing. \(\square\)

Since \(\Lambda_k \in [\Lambda, \Lambda_{sph})\), we have the following Pohozaev identity (see Proposition 2.1, which can be applied thanks to Lemma 2.3 if \(\Lambda \in (\Lambda_{s,p}, \Lambda_{sph})\) and thanks to Lemma 2.9 if \(\Lambda = \Lambda_{s,p}\)):

\[
    \frac{\Lambda_k}{\Lambda_{sph}} (\Lambda_k - \Lambda_{sph}) = -\frac{p}{4} \int_{\mathbb{R}^4} |x|^p e^{4u_k} dx.
\]

Therefore, by (7) we get that

\[
    \int_{\mathbb{R}^4} e^{4u_k} dx = \Lambda_k + \frac{4\Lambda_k}{p\Lambda_{sph}} (\Lambda_{sph} - \Lambda_k).
\]

This yields

\[
    \lim_{k \to \infty} \int_{\mathbb{R}^4} e^{4u_k} dx = \bar{\Lambda} + \frac{4\bar{\Lambda}}{p\Lambda_{sph}} (\Lambda_{sph} - \bar{\Lambda}).
\]
Lemma 4.3 We have
\[ \limsup_{k \to \infty} u_k(0) < \infty. \]

Proof. It follows from (59), (60) and (61) that
\[ \limsup_{k \to \infty} \int_{\mathbb{R}^4} (1 + |x|^p)e^{4u_k} \, dx < \infty. \]
(63)

Then, differentiating in (58), integrating over \( B_1 \) and using Fubini’s theorem and (63), one obtains
\[ \int_{B_1} |\nabla u_k| \, dx \leq C \int_{\mathbb{R}^4} \left( \int_{B_1} \frac{1}{|x-y|} \, dx \right) (1 + |y|^p)e^{4u_k(y)} \, dy \leq C. \]
(64)

Hence, if (up to a subsequence) \( u_k(0) \to \infty \) as \( k \to \infty \), by [16, Theorem 2] (see also [11] and [19]) the blow-up at the origin is spherical, i.e.
\[ u_k(r_k x) - u_k(0) + \log(2) =: \eta_k(x) \to \log\frac{2}{1 + |x|^2}, \quad \text{locally uniformly}, \]
where \( r_k := 12e^{-u_k(0)} \to 0 \) as \( k \to \infty \), and, we have quantization of mass in the sense that
\[ \lim_{k \to \infty} \int_{B_1} (1 - |x|^p)e^{4u_k} \, dx = \Lambda_{\text{sph}}. \]
As \( u_k \) is monotone decreasing, we have that \( u_k \to -\infty \) locally uniformly in \( \mathbb{R}^4 \setminus \{0\} \). Consequently, using (63) we get
\[ \lim_{k \to \infty} \int_{\mathbb{R}^4} e^{4u_k} \, dx = \Lambda_{\text{sph}}. \]
(65)

On the other hand, comparing (65) with (62), and recalling that \( \Lambda_{s,p} \leq \bar{\Lambda} < \Lambda_{\text{sph}} \) and \( \Lambda_{s,p} = \frac{1}{8}(4 + p)\Lambda_{\text{sph}} \), we obtain
\[ 1 = \frac{4\bar{\Lambda}}{p\Lambda_{\text{sph}}} = \frac{4\Lambda_{s,p}}{p\Lambda_{\text{sph}}} = \frac{4 + p}{2p} > 1 \]
for \( p \in (0, 4) \), a contradiction. \( \square \)

Lemma 4.4 We have \( u_k \to \bar{u} \), where \( \bar{u} \) is a normal solution to (7) for some \( \Lambda = \bar{\Lambda} \geq \bar{\Lambda} \).

Proof. Since \( u_k \leq u_k(0) = O(1) \) by Lemma 4.2 and Lemma 4.3 we have
\[ \Delta^2 u_k = O_k(1) \quad \text{on } B_R. \]
Differentiating under the integral sign in (58), integrating over $B_R$ and using Fubini’s theorem, together with (63), we get
\[
\int_{B_R} |\Delta u_k| dx \leq CR^2, \quad \text{for every } R > 1.
\]
Hence, by elliptic estimate, up to a subsequence, $u_k \to \bar{u}$ in $C^0_{\text{loc}}(\mathbb{R}^4)$.

To prove that $\bar{u}$ is normal, first note that the constant $c_k = u_k(0)$ in (58). Moreover
\[
\text{For a fixed } x \in \mathbb{R}^4 \text{ we have as } R \to \infty \int_{B_c_R} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p)e^{4u_k(y)}dy = O \left( \frac{|x|}{R} \right) + u_k(0).
\]
thanks to (63). Therefore, using the convergence $u_k \to \bar{u}$ in $C^0_{\text{loc}}(\mathbb{R}^4)$, we conclude from (58) that
\[
\bar{u}(x) \xrightarrow{k \to \infty} u_k(x) = \frac{1}{8\pi^2} \int_{B_R} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p)e^{4u_k(y)}dy + O \left( \frac{|x|}{R} \right) + u_k(0)
\]
\[
R \to \infty \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p)e^{4\bar{u}(y)}dy + \bar{u}(0),
\]
where in the last line we used dominated convergence, which is possible since $\log \left( \frac{|y|}{|x-y|} \right) = O(1)$ as $|y| \to \infty$, and $(1 + |\cdot|^p)e^{4u} \in L^1(\mathbb{R}^4)$ by (63) and Fatou’s lemma.

Still by Fatou’s lemma, we also get
\[
\tilde{\Lambda} := \int_{\mathbb{R}^4} (1 - |x|^p)e^{4\tilde{u}}dx \geq \lim_{k \to \infty} \int_{\mathbb{R}^4} (1 - |x|^p)e^{4u_k}dx = \tilde{\Lambda}.
\]

\[\blacksquare\]

**Lemma 4.5** We have $\tilde{\Lambda} = \tilde{\Lambda}$.

**Proof.** We assume by contradiction that $\tilde{\Lambda} > \tilde{\Lambda}$. Since $u_k \to \bar{u}$ in $C^0_{\text{loc}}(\mathbb{R}^4)$, this is equivalent to
\[
\rho := \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R^c} |x|^p - 1)e^{4u_k}dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R^c} |x|^pe^{4u_k}dx = \tilde{\Lambda} - \tilde{\Lambda} > 0. \quad (66)
\]
We consider the Kelvin transform
\[
\tilde{u}_k(x) = u_k \left( \frac{x}{|x|^2} \right) - \frac{\Lambda_k}{8\pi^2} \log |x|, \quad x \neq 0.
\]
By Proposition 2.2 we have
\[ \tilde{u}_k(x) = \frac{1}{8\pi^2} \int_{B_k} \log \left( \frac{1}{|x-y|} \right) \left( 1 - \frac{1}{|y|^p} \right) e^{4\tilde{u}_k(y)} \frac{1}{|y|^{4-p-\delta_k}} dy, \quad \delta_k := \frac{\Lambda_k - \Lambda_{s,p}}{2\pi^2}. \]
In fact, with the same proof of (30) we obtain
\[ \tilde{u}_k(x) = -\frac{1}{8\pi^2} \int_{B_1} \log \left( \frac{1}{|x-y|} \right) e^{4\tilde{u}_k(y)} \frac{1}{|y|^{4-\delta_k}} dy + O(1) \quad \text{for} \quad x \in B_1. \tag{67} \]
If \( \delta_k \not\to 0 \) then from (67) we easily see that \( \tilde{u}_k = O(1) \) in \( B_1 \), a contradiction to our assumption that \( \rho > 0 \). Let us then assume that \( \delta_k \to 0 \), i.e. \( \Lambda_k \to \Lambda_{s,p} \), and let \( \varepsilon_k > 0 \) be such that
\[ \int_{B_{\varepsilon_k}} \frac{e^{4\tilde{u}_k(y)}}{|y|^{4-\delta_k}} dy = \frac{\rho}{2}. \]
Then clearly \( \varepsilon_k \to 0 \) as \( k \to \infty \). Using that \( \log \left( \frac{1}{|x-y|} \right) = \log \left( \frac{1}{|x|} \right) + O(1) \) for \( |y| \leq \varepsilon_k \), \( |x| \geq 2\varepsilon_k \), and that \( \log \left( \frac{1}{|x-y|} \right) \) is lower bounded for \( y \in B_1 \) and \( x \to 0 \), we get
\[ \tilde{u}_k(x) \leq -\frac{\rho}{16\pi^2} \log \left( \frac{1}{|x|} \right) + C \quad \text{for} \quad 2\varepsilon_k \leq |x| \leq 1, \tag{68} \]
which, in particular implies
\[ \lim_{r \to 0} \lim_{k \to \infty} \sup_{B_r} \tilde{u}_k = -\infty. \tag{69} \]
From (68) we immediately infer
\[ \lim_{r \to 0} \lim_{k \to \infty} \int_{B_r \setminus B_{2\varepsilon_k}} \frac{e^{4\tilde{u}_k(y)}}{|y|^{4-\delta_k}} dy = 0, \]
hence, also recalling (66),
\[ \lim_{k \to \infty} \int_{B_{2\varepsilon_k}} \frac{e^{4\tilde{u}_k(y)}}{|y|^{4-\delta_k}} dy = \rho. \]
This, and using (69) we get
\[ \frac{\rho}{2} = \lim_{k \to \infty} \int_{B_{2\varepsilon_k} \setminus B_{\varepsilon_k}} \frac{e^{4\tilde{u}_k(y)}}{|y|^{4-\delta_k}} dy = o(1) \int_{B_{2\varepsilon_k} \setminus B_{\varepsilon_k}} \frac{dy}{|y|^{4}} = o(1), \quad \text{as} \quad k \to \infty, \]contradiction. \( \square \)

**Proof of Theorem 1.3.** With Lemma 4.4 and Lemma 4.5 we have the desired convergence (up to a subsequence) of \( u_k \) to \( \bar{u} \), a normal solution of (7) with \( \Lambda = \bar{\Lambda} \). The asymptotic behaviour (10) follows from Lemma 2.9, while for (9), the same proof used for \( \Lambda \in (\Lambda_{s,p}, \Lambda_{sph}) \) also works for the case \( \Lambda = \Lambda_{s,p} \). \( \square \)
5 Proof of Theorem 1.4

Lemma 5.1 Let \((u_k)\) be a sequence solving (7) with \(\Lambda = \Lambda_k \uparrow \Lambda_{sph}\). Then we have \(u_k(0) \to \infty\) as \(k \to \infty\).

Proof. By Lemma 4.2 we have \(u_k(0) \geq -C\). Assume by contradiction that, up to a subsequence, \(u_k(0) \to \ell \in \mathbb{R}\). Then, by Lemma 4.4 (or, rather, following its proof) we have \(u_k \to \bar{u}\), normal solution to (7) for some \(\Lambda \geq \Lambda_{sph}\), contradicting Theorem 1.1. \(\square\)

Differentiating under the integral sign in (58) and integrating over \(B_1\) we obtain (64).

By Lemma 5.1, the sequence \((u_k)\) blows up at the origin. This, (64) and [16, Theorem 2] imply (12). This completes the proof of Theorem 1.4. \(\square\)

6 Proof of Theorem 1.5

We start by looking for normal solutions with prescribed value at the origin.

Theorem 6.1 For every \(p > 0\) and \(\rho \in \mathbb{R}\) there exists a unique radially symmetric normal solution to

\[
\Delta^2 u = (1 + |x|^p)e^{4u} \quad \text{in} \quad \mathbb{R}^4, \quad u(0) = \rho, \quad (1 + |x|^p)e^{4u} \in L^1(\mathbb{R}^4).
\]

(70)

Proof. For every \(\varepsilon > 0\) we claim that there exists a radial normal solution to

\[
\Delta^2 v_\varepsilon = (1 + |x|^p)e^{-\varepsilon|x|^2}e^{4v_\varepsilon} \quad \text{in} \quad \mathbb{R}^4, \quad v_\varepsilon(0) = \rho.
\]

(71)

To this end, we set

\[X = \{v \in C^0_{rad}(\mathbb{R}^4) : \|v\|_X < \infty\}, \quad \|v\|_X := \sup_{x \in \mathbb{R}^4} \frac{|v(x)|}{\log(2 + |x|)},\]

and define the operator \(T_\varepsilon : X \to X, T_\varepsilon v = \bar{v}\), where

\[\bar{v}(x) := \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left(\frac{|y|}{|x-y|}\right) (1 + |y|^p)e^{-\varepsilon|y|^2}e^{4v(y)} \, dy + \rho.\]

By the Arzerl`a-Ascoli theorem it follows that \(T_\varepsilon\) is compact.

Notice that \(\Delta \bar{v} < 0\), hence \(\bar{v}\) is monotone decreasing by (37), and

\[\bar{v} \leq \bar{v}(0) = \rho.\]

In particular, if \(v\) is a solution to \(v = tT_\varepsilon(v)\) with \(0 < t \leq 1\), then

\[|v(x)| \leq t\rho + \frac{te^{4\rho}}{8\pi^2} \int_{\mathbb{R}^4} \log \left(\frac{|y|}{|x-y|}\right) (1 + |y|^p)e^{-\varepsilon|y|^2} \, dy \leq C \log(2 + |x|),\]

where \(C\) depends only on \(\rho\). This completes the proof of Theorem 1.5. \(\square\)
where \( C > 0 \) is independent of \( v \) and \( t \). Then, by the Schauder fixed-point theorem, \( T_\varepsilon \) has a fixed point, which we call \( v_\varepsilon \), and which is a radial normal solution to (71) by definition, so the claim is proven.

Setting 
\[
\Lambda_\varepsilon = \int_{\mathbb{R}^4} (1 + |x|^p)e^{-\varepsilon|x|^2}e^{4v_\varepsilon(x)}dx,
\]
and using the Pohozaev identity (Proposition A.1 in [12] or a minor modification of Proposition 2.1, with \( K(x) = (1 + |x|^p)e^{-\varepsilon|x|^2} \)), we get 
\[
\frac{\Lambda_\varepsilon}{\Lambda_{\text{sph}}}(\Lambda_\varepsilon - \Lambda_{\text{sph}}) < \frac{p}{4} \int_{\mathbb{R}^4} |x|^p e^{-\varepsilon|x|^2}e^{4v_\varepsilon(x)}dx < \frac{p}{4} \Lambda_\varepsilon,
\]
which implies 
\[
\Lambda_\varepsilon < \left(1 + \frac{p}{4}\right) \Lambda_{\text{sph}}.
\]
Now one can follow the proof of Lemma 4.4 to conclude that \( v_\varepsilon \to v \), where \( v \) is a normal solution with \( v(0) = \rho \).

The uniqueness follows by the monotonicity property of solutions to ODEs with respect to the initial data. \( \square \)

Notice that the result of Theorem 6.1 does not hold in the case of Problem (7), see Lemma 4.2.

Lemma 6.2 Let \( u \) be a normal solution to (13) for some \( \Lambda > 0 \). Then we have
\[
u(x) \geq -\frac{\Lambda}{8\pi^2} \log |x| + \mathcal{O}(1) \quad \text{as } |x| \to \infty, \quad (72)
\]
hence \( \Lambda > \Lambda_{*,p} \).

Proof. The proof of (72) follows as in Lemma 2.3 by changing the sign of \( K \). Using that \( |\cdot|^p e^{4u} \in L^1(\mathbb{R}^4) \) together with (72) we then infer that \( \Lambda > (1 + \frac{p}{4})\Lambda_{\text{sph}} = \Lambda_{*,p} \). \( \square \)

Lemma 6.3 Let \( u \) be a normal solution to (13) for some \( \Lambda > 0 \). Then we have
\[
u(x) \leq -\frac{\Lambda}{8\pi^2} \log |x| + o(\log |x|) \quad \text{as } |x| \to \infty, \quad (73)
\]

Proof. From the same proof of [14, p. 213], we easily get that for every \( \varepsilon > 0 \) there exists \( R(\varepsilon) > 0 \) such that
\[
u(x) \leq \left(-\frac{\Lambda}{8\pi^2} + \varepsilon\right) \log |x| + \frac{1}{8\pi^2} \int_{B_1(x)} \log \left(\frac{1}{|x - y|}\right) K(y)e^{4u(y)}dy, \quad |x| \geq R(\varepsilon), \quad (74)
\]
where $K(y) = 1 + |y|^p$. As in [12, Lemma 3.5], from (74) and Jensen’s inequality we get that for every $\varepsilon' > 0$ and $q \geq 1$ there is a constant $C = C(\varepsilon', q)$ such that

$$
\int_{B_1(x)} e^{4u} dy \leq C |x|^{-\left(\frac{\Lambda}{2\pi^2} - \varepsilon'\right)q}.
$$

With Hölder’s inequality we then infer for $|x|$ large

$$
\int_{B_1(x)} \log \left(\frac{1}{|x - y|}\right) K(y) e^{4u(y)} dy \leq C |x|^p \int_{B_1(x)} \log \left(\frac{1}{|x - y|}\right) e^{4u(y)} dy
$$

$$
\leq C |x|^p \| e^{4u} \|_{L^q(B_1(x))}
$$

(75)

$$
\leq C |x|^{p - \frac{\Lambda}{2\pi^2} + \varepsilon'} \leq C
$$

since by Lemma 6.2 we have $\Lambda > \Lambda_{*,p} > \frac{p\Lambda_{sph}}{8} = 2\pi^2 p$ and we can choose $0 < \varepsilon' < \frac{\Lambda}{2\pi^2} - p$. Plugging (75) into (74), we obtain (73). □

**Lemma 6.4** Let $u$ be a normal solution to (13) for some $\Lambda \in \mathbb{R}$. Then $u$ satisfied the Pohozaev identity (19) and

$$
\max\{\Lambda_{sph}, \Lambda_{*,p}\} < \Lambda < 2\Lambda_{*,p}.
$$

(76)

**Proof.** From Lemma 6.2 we have $\Lambda > \Lambda_{*,p}$, which together with Lemma 6.3 implies that (18) is satisfied, hence the Pohozaev identity (19) holds. From it we obtain

$$
\frac{\Lambda}{\Lambda_{sph}} (\Lambda - \Lambda_{sph}) = \frac{p}{4} \int_{\mathbb{R}^4} |x|^p e^{4u} dx.
$$

(77)

Since

$$
0 < \int_{\mathbb{R}^4} |x|^p e^{4u} dx < \Lambda,
$$

(77) implies

$$
\Lambda_{sph} < \Lambda < \left(1 + \frac{p}{4}\right) \Lambda_{sph} = 2\Lambda_{*,p},
$$

and since we have already proven that $\Lambda > \Lambda_{*,p}$, (76) follows. □

**Lemma 6.5** Let $(u_k)$ be a sequence of radially symmetric normal solutions to (13) with $\Lambda = \Lambda_k$. If the sequence $(u_k(0))$ is bounded, then also the sequence $(\Lambda_k)$ is bounded and, up to a subsequence, $u_k \to \bar{u}$, $\Lambda_k \to \bar{\Lambda} \in (0, \infty)$, where $\bar{u}$ is a normal solution to (13) with $\Lambda = \bar{\Lambda}$.

**Proof.** By Lemma 6.4 the Pohozaev identity (19) holds, hence

$$
\frac{\Lambda_k}{\Lambda_{sph}} (\Lambda_k - \Lambda_{sph}) = \frac{p}{4} \int_{\mathbb{R}^4} |x|^p e^{4u_k} dx < \frac{p}{4} \Lambda_k,
$$

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which then implies
\[ \Lambda_k < \left( 1 + \frac{p}{4} \right) \Lambda_{sph} = 2\Lambda_{s,p}. \]

Following the proof of Lemma 4.4 (with inequality reversed in the last line because of the positivity of \( K \)) we obtain that, up to a subsequence \( u_k \to \bar{u} \), normal solution to (13) for some \( \Lambda = \tilde{\Lambda} \leq \bar{\Lambda} \).

To prove that \( \tilde{\Lambda} = \bar{\Lambda} \) it suffices to show that
\[ \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R} |x|^p \, e^{4u_k} \, dx = 0. \] (78)

Upon the Kelvin transform
\[ \tilde{u}_k(x) := u_k \left( \frac{x}{|x|^2} \right) - \frac{\Lambda_k}{8\pi^2} \log |x|, \]
(78) is equivalent to
\[ \lim_{r \to 0} \lim_{k \to \infty} \int_{B_r} |x|^{p_k} e^{4\tilde{u}_k} \, dx = 0, \quad p_k := \frac{\Lambda_k}{2\pi^2} - p - 8. \] (79)

By Proposition 2.2 \( \tilde{u}_k \) is a normal solution to
\[ \Delta^2 \tilde{u}_k = (1 + |x|^p) |x|^{p_k} e^{4\tilde{u}_k}, \quad \int_{\mathbb{R}^4} (1 + |x|^p) |x|^{p_k} e^{4\tilde{u}_k} \, dx = \Lambda_k. \]

Since \( \bar{u} \) is a normal solution, by Lemma 6.2 we have \( \tilde{\Lambda} > \Lambda_{s,p} \). Therefore \( \Lambda_{s,p} < \tilde{\Lambda} \leq \bar{\Lambda} \), which implies that for some \( \delta > 0 \) and \( k \) large, we have \( p_k \geq -4 + \delta \). Hence, (79) will follow if we show that
\[ \lim_{k \to \infty} \sup_{B_1} \tilde{u}_k < \infty. \] (80)

First we note that by differentiating the integral formula of \( \tilde{u}_k \) we obtain \( \Delta u_k < 0 \), hence by (37) we have that \( \tilde{u}_k \) is monotone decreasing. Therefore, if \( \tilde{u}_k(0) \to \infty \), then up to a subsequence, the rescaled function
\[ \eta_k(x) = \tilde{u}_k(r_k x) - \tilde{u}_k(0), \quad r_k := e^{-\frac{4}{4 + p_k} \tilde{u}_k(0)}, \]
which is a normal solution to
\[ \Delta^2 \eta_k = (1 + o(1)) |x|^{p_k} e^{4\eta_k}, \]
with \( o(1) \to 0 \) locally uniformly, converges to a limit function \( \eta \) (see the proof of Proposition 4.1 in [12] for details), where \( \eta \) is a normal solution to
\[ \Delta^2 \eta = |x|^{p_\infty} e^{4\eta}, \quad p_\infty := \lim_{k \to \infty} p_k > -4. \]
Then by [12, Theorem 1] we have
\[
\int_{\mathbb{R}^4} |x|^{p_\infty} e^{4\tilde{u}} \, dx = \left(1 + \frac{p_\infty}{4}\right) \Lambda_{\text{sph}}.
\]

Then, from Fatou’s lemma we have
\[
\lim_{r \to 0} \lim_{k \to \infty} \int_{B_r} |y|^{p_k} e^{4\tilde{u}_k} \, dy \geq \left(1 + \frac{p_\infty}{4}\right) \Lambda_{\text{sph}}. \tag{81}
\]

Moreover, as in the proof of Lemma 4.5 we estimate
\[
\tilde{u}_k(x) \geq \frac{1}{8\pi^2} \int_{B_1} \log \left(\frac{1}{|x - y|}\right) |y|^{p_k} e^{4\tilde{u}_k(y)} \, dy + O(1), \quad x \in B_1.
\]

Since \( \tilde{u}_k \to \tilde{u} \) outside the origin, where \( \tilde{u} = \tilde{\bar{u}} \) is the Kelvin transform of \( \bar{u} \), together with (81) we get that
\[
\tilde{u}(x) \geq \frac{1}{8\pi^2} \left(1 + \frac{p_\infty}{4}\right) \Lambda_{\text{sph}} \log \left(\frac{1}{|x|}\right) + O(1), \quad x \in B_1.
\]

In particular, as \( p_\infty > -4 \), we have
\[
|x|^{p_\infty} e^{4\tilde{u}(x)} \geq \delta |x|^{-4} \quad x \in B_1,
\]

for some \( \delta > 0 \). This shows that \( |x|^{p_\infty} e^{4\tilde{u}(x)} \notin L^1(B_1) \), however, by Fatou’s lemma
\[
\int_{B_1} |x|^{p_\infty} e^{4\tilde{u}(x)} \, dx \leq \liminf_{k \to \infty} \int_{B_1} |x|^{p_k} e^{4\tilde{u}_k(x)} \leq \liminf_{k \to \infty} \Lambda_k \leq 2\Lambda_{*,p}.
\]

This contradiction completes the proof of (80), hence of the lemma. \( \Box \)

Proof of Theorem 1.5 (completed). We have already proven the necessary conditions (14)–(15) in Lemma 6.4 so it remains to prove the existence part and the necessary condition (16) in the radial case with \( p > 4 \).

By Lemma 6.5 the map
\[
\mathbb{R} \ni \rho \mapsto \Lambda_{\rho} := \int_{\mathbb{R}^4} (1 + |x|^p) e^{4u_{\rho}} \, dx
\]
is continuous, where \( u_\rho \) is the solution to (70) given by Theorem 6.1.

We now have
\[
\lim_{\rho \to -\infty} \Lambda_{\rho} = \left(1 + \frac{p}{4}\right) \Lambda_{\text{sph}} = 2\Lambda_{*,p}, \tag{82}
\]

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which is a consequence of (61) and
\[ \int_{\mathbb{R}^4} |x|^p e^{4u_\rho} dx \leq C \Rightarrow \int_{\mathbb{R}^4} e^{4u_\rho} dx \to 0. \]
Taking \( \rho \to \infty \) we see that the blow-up around the origin is spherical (see e.g. [16]), and
\[ \lim_{\rho \to +\infty} \int_{\mathbb{R}^4} e^{4u_\rho} dx = \Lambda_{\text{sph}}. \]
Again by (61), and as \( \Lambda_\rho > \max\{\Lambda_{\text{sph}}, \Lambda_{s,p}\} \), we conclude that
\[ \lim_{\rho \to \infty} \Lambda_\rho = \max\left\{\Lambda_{\text{sph}}, \frac{p}{4} \Lambda_{\text{sph}}\right\}. \tag{83} \]
Then, by continuity, we have existence for every \( \max\{\Lambda_{\text{sph}}, \Lambda_{s,p}\} < \Lambda < 2 \Lambda_{s,p} \).

It remains to prove the stronger necessary condition (10) for \( p > 4 \) in the radial case.

Assume by contradiction that for a sequence \( (\Lambda_k) \) with \( \Lambda_k \downarrow \Lambda_{s,p} \) there are radial solutions \( u_k \) to (13) with \( \Lambda = \Lambda_k \). Since
\[ \Lambda_{s,p} < \frac{p}{4} \Lambda_{\text{sph}} < 2 \Lambda_{s,p}, \]
from (82)-(83) we obtain that the sequence \( (u_k(0)) \) is bounded. Then by Lemma 6.5 we have that (up to a subsequence) \( u_k \to \bar{u} \) locally uniformly, where \( \bar{u} \) is a normal solution to (13) with \( \Lambda = \Lambda_{s,p} \), and this contradicts Lemma 6.4. \( \square \)

7 Proof of Proposition 3.1

By [3, Theorem 2.1] (and its proof), setting \( K_\lambda = (\lambda - |x|^p) e^{-|x|^2} \) and given \( \mu = 1 - \frac{\Lambda}{\Lambda_{\text{sph}}} \in (0,1) \) one can find a solution \( u_\lambda \) to
\[ \Delta^2 u_\lambda = K_\lambda e^{4u_\lambda} \text{ in } \mathbb{R}^4, \]
such that
\[ \int_{\mathbb{R}^4} K_\lambda e^{4u_\lambda} dx = (1 - \mu) \Lambda_{\text{sph}} = \Lambda. \]
Moreover \( u_\lambda \) is of the form \( u_\lambda = w \circ \Pi^{-1} + (1 - \mu) \eta_0 \) where \( \eta_0(x) = \log \left( \frac{2}{1 + |x|^2} \right), \Pi : S^4 \to \mathbb{R}^4 \)
denotes the stereographic projection, and \( w \in H^2(S^4) \) minimizes a certain functional on \( S^4 \). This leads to the Euler-Lagrange equation
\[ P_{g_0}^4 w + 6(1 - \mu) = (K_\lambda \circ \Pi) e^{-4\mu(\eta_0 \circ \Pi)} e^{4w}, \]
where \( P_{g_0}^4 = \Delta_{g_0} (\Delta_{g_0} - 2) \) is the Paneitz operator on \( S^4 \) with respect to the round metric \( g_0 \). Since \( (K_\lambda \circ \Pi) e^{-4\mu(\eta_0 \circ \Pi)} \in L^\infty(S^4) \), and \( e^{4w} \in L^q(S^4) \) for every \( q \in [1, \infty) \) by the
Moser-Trudinger inequality, by elliptic estimates, we obtain $w \in C^{3,\alpha}(S^4)$ for $\alpha \in (0, 1)$. In particular $w$ is continuous at the South pole $S = (0, 0, 0, -1)$ (the singularity of the stereographic projection), hence

$$u_\lambda(x) = (1 - \mu)\eta_0(x) + w(S) + o(1) = \frac{\Lambda}{8\pi^2} \log |x| + C + o(1) \quad \text{as} \quad |x| \to \infty.$$ 

Now, setting

$$v_\lambda = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{1}{|x-y|} \right) K_\lambda(y)e^{4u_\lambda(y)}dy$$

we observe that $h_\lambda := u_\lambda - v_\lambda$ satisfies

$$\Delta^2 h_\lambda = 0, \quad h_\lambda(x) = O(\log |x|) \quad \text{as} \quad |x| \to \infty.$$ 

Hence by the Liouville theorem we get $h_\lambda = \text{const}$. In particular $u_\lambda$ is a normal solution, i.e. it satisfies (36). This completes the proof of Proposition 3.1. □

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