Three matching intersection property for matching covered graphs

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In connection with Fulkerson’s conjecture on cycle covers, Fan and Raspaud proposed a weaker conjecture: For every bridgeless cubic graph $G$, there are three perfect matchings $M_1$, $M_2$, and $M_3$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$. We call the property specified in this conjecture the three matching intersection property (and 3PM property for short). We study this property on matching covered graphs. The main results are a necessary and sufficient condition and its applications to characterization of special graphs, such as the Halin graphs and 4-regular graphs.

Keywords: matching-covered graph, Fan-Raspaud’s conjecture, 3PM-admissible graph

1 Introduction

Fulkerson’s conjecture asserts that every bridgeless cubic graph has six perfect matchings such that each edge appears in exactly two of them (cf. \cite{2, 4, 6}). If we take three of these six perfect matchings, then each edge appears in at most two of them. This motivates the following weaker conjecture proposed by Fan and Raspaud \cite{5}: In every bridgeless cubic graph there exist three perfect matchings $M_1$, $M_2$, and $M_3$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$. For brevity, this conjecture is referred to as the three matching intersection conjecture or 3PM conjecture.

A graph is said to be matching covered if it is connected and each edge is contained in a perfect matching. Note that every bridgeless cubic graph is matching covered (or 1-extendable in \cite{7}). So we generally discuss the matching covered graphs below. In a viewpoint of generalization to the 3PM conjecture, we propose the following.

Definition 1.1. A matching covered graph $G$ is called a 3PM-admissible graph (or $G$ admits the 3PM property) if there exist three perfect matchings $M_1$, $M_2$, and $M_3$ of $G$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Our goal is to characterize 3PM-admissible graphs. Within the realm of cubic graphs, this amounts to the 3PM conjecture. Many 3PM-admissible cubic graphs have been found to support this conjecture, such as the 3-edge-colourable cubic graphs (including bipartite graphs, hamiltonian graphs), the cubic graphs with independent perfect matching polytope $P(G)$ or with low dimension perfect matching polytope (see \cite{8, 9}). Here, a cubic graph $G$ is 3-edge-colorable if there are three perfect matchings of $G$ which form a partition of $E(G)$. Some basic cubic graphs are shown in Figure 1, which are 3PM-admissible.

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Furthermore, apart from those cubic graphs, there are more 3PM-admissible matching covered graphs. For example, a wheel $W_n$ is a cycle $C_n$ with every vertex joining to a single vertex, the hub. When $n$ is odd, $W_n$ is called an odd wheel, which is matching covered (see Figure 2(a)). The wheels form a basic family of 3-connected graphs in the sense that every 3-connected graph can be constructed from a wheel via some kind of operations (see Tutte’s theorem in [2]). When performing an ‘expansion’ at the hub of a wheel, we can obtain another matching covered graph, called the double wheel. An example is shown in Figure 2(b). Moreover, the tetrahedron $K_4$, the cube $Q_3$, the dodecahedron, the octahedron and the icosahedron, which are well-known platonic graphs, are matching covered and the last two are not cubic [2]. Here the octahedron is shown in Figure 2(c), and the icosahedron is shown in Figure 3. To see that these graphs are 3PM-admissible, we define the perfect matchings $M_i$ for $1 \leq i \leq 3$ in Figures 2 and 3, where $M_i$ is represented by the edges with label $i$ at the edges.

We can see from the above examples that in addition to the cubic graphs, there would be many 3PM-admissible matching covered graphs. In this paper, we consider the characterization of 3PM-admissibility for matching covered graphs. Especially, we are concerned with several special classes of matching covered graphs, such as the platonic graphs, wheels, Halin graphs, outerplanar graphs, 4-regular graphs on small size.
The organization of the paper is as follows. In Section 2, we present a necessary and sufficient condition and its consequences. Section 3 is dedicated to the 4-regular graphs. We give a short summary in Section 4. We shall follow the graph-theoretic terminology and notation of [2].

2 Basic theorems

Throughout the paper, we consider $G$ as a matching covered graph. So $G$ has a perfect matching and has even number of vertices.

Matching covered graphs have a basic property (see [7]): If $G'$ results from $G$ by subdividing an edge with two vertices, then $G'$ is matching covered if and only if $G$ is matching covered. For this, the subdivision from $G$ to $G'$ is called a bisubdivision. A graph results from $G$ by performing several times of this kind of operations is also called a bisubdivision of $G$. On the other hand, the inverse operation, namely, replacing a path of $G'$ whose length is three and whose internal vertices have degree two in $G'$ by an edge, is called a bicontraction. The resulting graph obtained from $G'$ by performing several times of this kind of operations is also called a bicontraction of $G'$.

A spanning subgraph $G'$ of $G$ is called a 2-factor if every vertex of $G'$ has degree two. We have the following basic criterion.

**Theorem 2.1** A graph $G$ is 3PM-admissible if and only if (1) $G$ has a 2-factor $G'$ with even components, or (2) $G$ has a spanning subgraph $G'$ which is a bisubdivision of a 3-edge-colorable cubic graph.

**Proof:** If $G$ is 3PM-admissible, then there exist three perfect matchings $M_1$, $M_2$, and $M_3$ such that $M_1 \cap M_2 \cap M_3 = \emptyset$. Consider the spanning subgraph $G' = G[M_1 \cup M_2 \cup M_3]$. Note that the maximum degree of $G'$ is at most three. If every vertex of $G'$ has degree two, then $G'$ is a 2-factor and so each of its components is a cycle. Since each $M_i$ ($1 \leq i \leq 3$) is a perfect matching, these cycle components must be $(M_i, M_j)$-alternating cycles, where $1 \leq i, j \leq 3$ and $i \neq j$. Thus they have even number of edges. Hence (1) holds. Otherwise, $G'$ has vertices of degree three. If every vertex of $G'$ has degree three, then $G'$ is a cubic graph with edge set $M_1 \cup M_2 \cup M_3$ and so is 3-edge-colorable. If this is not the case, then $G'$ has vertices of degree two. Suppose that a vertex $u$ has degree two and it is incident with two edges $xu$ and $uv$. Without loss of generality, assume that $xu \in M_1$ and $uv \in M_2 \cap M_3$. Then $v$ must be incident with an edge $vy \in M_1$. Thus $xuwy$ is a path of $G'$ whose length is three and whose
internal vertices have degree two in $G'$. Replacing this path by an edge $xy$, we get a bicontraction $H'$ of $G'$. Moreover, $(M_1 \setminus \{ux, vy\}) \cup \{xy\}, M_2 \setminus \{uv\}$ and $M_3 \setminus \{uv\}$ are three perfect matchings of $H'$ with empty intersection. If there are more vertices of degree two, then we can repeatedly perform this kind of bicontractions. As a result, we finally obtain a cubic graph $H$, and $G'$ is a bisubdivision of $H$. Furthermore, $H$ is 3-edge colorable. Hence (2) holds.

Conversely, if (1) holds, then $G$ has a 2-factor $G'$ with even components. Here, each component of $G'$ is an even cycle. So we can define perfect matchings $M_1$ and $M_2$ of $G'$ by making each even cycle in $G'$ to be an $(M_1, M_2)$-alternating cycle. Further, let $M_3 := M_2$. In this way, we obtain three perfect matchings $M_1, M_2$, and $M_3$ with $M_1 \cap M_2 \cap M_3 = \emptyset$.

On the other hand, if (2) holds, then $G$ has a spanning subgraph $G'$ which is a bisubdivision of a 3-edge-colorable cubic graph, say $H$. So $H$ has three perfect matchings which cover $E(H)$ and whose intersection is empty. We can extend these three perfect matchings to $G'$ as follows. Suppose that $H'$ is a graph whose edge set is covered by three perfect matchings $M_1, M_2$, and $M_3$ with $M_1 \cap M_2 \cap M_3 = \emptyset$. Initially, $H' := H$. Suppose that we have made a bisubdivision of $H'$ on $xy$ by subdividing it with two vertices $u$ and $v$. The resulting graph is also denoted by $H'$. Since $M_1 \cap M_2 \cap M_3 = \emptyset$, suppose, without loss of generality, that $xy \in M_1$ and $xy \notin M_3$. If $xy \in M_1 \setminus M_2$, then we delete $xy$ from $M_1$, add $xu, vy$ into $M_1$, and add $uv$ into $M_2 \cap M_3$. If $xy \in M_1 \cap M_2$, then we delete $xy$ from $M_1 \cap M_2$, add $xu, vy$ into $M_1 \cap M_2$, and add $uv$ into $M_3$. Then $M_1 \cup M_2 \cup M_3 = E(H')$ and $M_1 \cap M_2 \cap M_3 = \emptyset$. By this procedure, we construct three perfect matchings $M_1, M_2$, and $M_3$ in $G'$ (and thus in $G$) such that $M_1 \cup M_2 \cup M_3 = E(G')$ and $M_1 \cap M_2 \cap M_3 = \emptyset$. This completes the proof. \hfill $\Box$

In condition (2) of this theorem, the cubic graph $H$ is called the cubic skeleton of $G$. As we know, a graph is a minor of $G$ if it can be obtained from $G$ by a sequence of deleting vertices or edges, and contracting edges. So the cubic skeleton $H$ is in fact a minor of $G$, a cubic minor.

**Corollary 2.2** If $G$ is an odd wheel, a double wheel with even number of vertices, or the octahedron, then $G$ is 3PM-admissible.

**Proof:** First, an odd wheel $W_n$ has $K_4$ as its cubic skeleton, that is, it has a spanning subgraph $G'$ which is a bisubdivision of $K_4$. Second, a double wheel $G$ has the 3-prism $B_6 = K_3 \times K_2$ as its cubic skeleton. Moreover, the octahedron contains a 3-prism $B_6$ as spanning subgraphs (see Figure 2(c)). And it is known that $K_4$ and $B_6$ in Figure 1 are 3-edge-colorable. The result follows from Theorem 2.1. \hfill $\Box$

Theorem 2.1 also implies the following.

**Corollary 2.3** A hamiltonian graph is 3PM-admissible.

The well-known Tutte’s theorem says that every 4-connected planar graph is hamiltonian (see [1]). So we have the following.

**Corollary 2.4** Every 4-connected planar graph is 3PM-admissible.

A graph $G$ is called a Halin graph if it can be drawn in the plane as a tree $T$, with all non-end-vertices having minimum degree 3, together with a cycle $C$ passing through the end-vertices of $T$. Since Halin graphs are hamiltonian (see Exercise 10.2.4 of [2]), we have the following.

**Corollary 2.5** Every Halin graph is 3PM-admissible.
As we know, the dodecahedral is hamiltonian. Moreover, the icosahedron is hamiltonian. In fact, the edges with labels 1 and 2 in Figure 3 constitute a Hamilton cycle. An outerplanar graph (it has a planar embedding in which all vertices lie on the boundary of its outer face) is also hamiltonian. So they are 3PM-admissible.

Let us see one more example taken from [8] whose perfect matching polytope is independent, as shown in Figure 4. It is hamiltonian (the edges with labels 1 and 2 in Figure 4 constitute a Hamilton cycle). Also, it contains a 3-prism $B_6$ as its cubic skeleton.

![Figure 4. A 3-connected graph with independent polytope](image)

3 4-regular graphs

Corollary 3.4.3 in [7] says that if a graph is $(k - 1)$-edge-connected, $k$-regular, and has even number of vertices, then it is matching covered. A 3-connected 4-regular graph is 3-edge-connected and so is matching covered. Recall Jackson’s theorem: Every 2-connected $k$-regular graph on at most $3k$ vertices is hamiltonian (see [3]). From this, we have an observation as follows.

**Proposition 3.1** Every 3-connected 4-regular graph $G$ on at most 12 even number of vertices is 3PM-admissible.

For example, the octahedron in Figure 2(c) is 4-regular and has 6 vertices. So it is 3PM-admissible. The following is a stronger result.

**Theorem 3.2** Every 3-connected 4-regular simple graph $G$ on at most 18 even number of vertices is 3PM-admissible.

**Proof:** Let $M_1$ be a perfect matching of $G$ and let $G' = G - M_1$. Then $G'$ is a cubic subgraph of $G$. If $G'$ has a perfect matching $M_2$, then $G$ has two disjoint perfect matchings $M_1$ and $M_2$. Thus $G$ is 3PM-admissible. In the following, assume that $G'$ has no perfect matchings.

We shall apply Gallai-Edmonds structure theorem (see [7]) to $G'$. Denote by $D$ the set of all vertices not covered by at least one maximum matching of $G'$, by $A$ the set of neighbours of $D$ in $V(G') \setminus D$, and by $C$ the set of all other vertices of $G'$. Then

(a) each component of $G'[D]$ is factor critical;
(b) $G'[C]$ has a perfect matching;
(c) any maximum matching in $G'$ contains a perfect matching in $G'[C]$ and near-perfect matchings of components of $G'[D]$, and matches all vertices of $A$ to distinct components of $G'[D]$.

Here, a graph $H$ is factor critical if $H - v$ has a perfect matching for each $v \in V(H)$, and a matching of $H$ is near perfect if it covers all but one vertex in $H$. 
Let $D'$ be the vertex set of a component of $G'[D]$, and let $t$ be the number of edges in $G'$ connecting $A$ and $D'$. Then $G'[D']$ is factor critical, and so $|D'|$ is odd. Recall that $G'$ is a cubic graph. We have $3|D'| = 2|E(G'[D'])| + t$. This implies that $t$ is odd. Since $G$ is simple, if $t = 1$, then $|D'| \geq 5$. Let $\omega_1$ denote the number of components of $G'[D]$ each of which is connected by only one edge to $A$. Let $\omega$ denote the number of components of $G'[D]$. Since $G'$ has no perfect matchings, by Gallai-Edmonds structure theorem, we have $\omega > |A|$. Since the number of vertices of $G'$ is even, $\omega$ and $|A|$ have the same parity, and so $\omega \geq |A| + 2$.

When $|A| = 1$, we have $\omega \geq 3$. Since $G'$ is cubic, we have $\omega = \omega_1 = 3$. Let $u$ be the vertex in $A$, $G_1, G_2, G_3$ the three components in $G'[D]$, and $x \in V(G_1), y \in V(G_2), z \in V(G_3)$ three neighbours of $u$. Then $|V(G_i)| \geq 5, i = 1, 2, 3$. Moreover, by the definition of $C$, in this case $G'$ is the disjoint union of $G'[C]$ and $G'[A \cup D]$, both of which are cubic.

If $C = \emptyset$, then $G' - u = G'[D]$ (see an example in Figure 5). Since $G$ is 3-connected, $G - u$ is connected. Thus there exist at least two edges of $M_1, M_2, M_3$ which connect the components $G_1, G_2, G_3$. Suppose, without loss of generality, that $e$ connects $G_1$ and $G_2$ and $f$ connects $G_1$ and $G_3$. Let $G^* = G' + e + f$. Since $G_1, G_2, G_3$ are factor-critical, there exists a perfect matching $M_2$ of $G^*$ containing $\{e, uz\}$, and a perfect matching $M_3$ of $G^*$ containing $\{f, uy\}$. Then $M_1, M_2, M_3$ are three perfect matchings of $G$ such that $M_1 \cap M_2 = \{e\}, M_1 \cap M_3 = \{f\}$, and $M_2 \cap M_3$ may be nonempty. However, $M_1 \cap M_2 \cap M_3 = \emptyset$. Therefore, $G$ is 3PM-admissible.

If $G$ is not 3PM-admissible, then either $|A| = 1$ and $C \neq \emptyset$ or $|A| \geq 2$. For the former case, noting that $G'[C]$ is cubic and $G$ is simple, there are at least four vertices in $C$. Thus $|V(G)| = |V(G')| = |C| + |A| + \sum_{i=1}^{3} |V(G_i)| \geq 20$. For the latter case, when $|A| = 2$, we have $\omega \geq 4$. Combining the fact that the number of edges in $G'$ connecting $A$ and a component of $G'[D]$ is odd and $G'$ is a cubic graph, we have $\omega_1 \geq 3$. If $\omega_1 = 3$, then $\omega = 4$ and there is a component $D''$ of $G'[D]$ such that there are three edges in $G'$ connecting $A$ and $D''$. Since $G'$ is simple and $|D''|$ is odd, we have $|D''| \geq 3$. So $|V(G)| \geq |A| + |D''| + 5\omega_1 \geq 20$. If $\omega_1 \geq 4$, then $|V(G)| \geq |A| + 5\omega_1 \geq 22$. When $|A| \geq 3$, we have $\omega \geq 5$. By counting the number of edges which connect $A$ and $D$ in two ways, we have $\omega_1 + 3(\omega - \omega_1) \leq 3|A|$. Thus $\omega_1 \geq \frac{3}{2}(\omega - |A|) \geq 3$, and so $|V(G)| \geq |A| + (\omega - \omega_1) + 5\omega_1 = |A| + \omega + 4\omega_1 \geq 20$. Therefore, a graph with at most 18 vertices admits the 3PM property.

4 Concluding remarks

To look for 3PM-admissible graphs, traversing from cubic graphs to matching covered graphs, we can see some connections and some new features. Many problems remain to be investigated.
The concept of 3PM-admissible graphs is a generalization (relaxation) of that of the hamiltonian graphs. At the beginning we introduce five polyhedral graphs, the platonic graphs. They are all hamiltonian. In general, a graph is polyhedral if and only if it is planar and 3-connected (see [1]). Tutte presented a counterexample to show that a polyhedral graph is not necessarily hamiltonian. However, this counterexample is cubic and is 3PM-admissible. So it is not a counterexample for the statement that every polyhedral graph is 3PM-admissible. We can ask if this statement holds true.

Jackson’s theorem asserts that every 2-connected 4-regular graph on at most 12 vertices is hamiltonian. Further, Jackson conjectured that every 3-connected 4-regular graph on at most 16 vertices is hamiltonian (see [3]). Now, we obtain an easier assertion that every 3-connected 4-regular graph on at most 18 vertices is 3PM-admissible. Can we further improve this upper bound?

For a cubic graph $G$, we have proved that if the perfect matching polytope is independent, then $G$ is 3PM-admissible. In Figure 4, we show a 3-connected graph with independent polytope to be 3PM-admissible. Can we prove this for every 3-connected graph?

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