Mathematical Properties of a Class of Four-dimensional Neutral Signature Metrics

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Abstract

We will investigate the mathematical properties of four-dimensional neutral signature Ricci flat Walker and Kundt spaces for which all of the polynomial scalar curvature invariants vanish. To do this in an explicit manner we choose two subcases of the Ricci-flat VSI Walker metrics to examine. To compare these metrics we study the existence of a null, geodesic, expansion-free, shear-free and vorticity-free vector in the two examples, and classify these spaces using their holonomy algebras. We examine the geometric implications of the two distinct holonomy algebras by identifying the recurrent or covariantly constant null vectors whose existence is implied by holonomy structure in each example. These tools are best suited for showing inequivalence, they cannot give a complete description of these spaces. We conclude the paper with a simple example of the equivalence algorithm for these pseudo-Riemannian manifolds that provides all necessary information to determine equivalence.

Keywords: pseudo-Riemannian manifolds; neutral metrics; holonomy; vanishing scalar curvature invariants; Walker; Kundt; equivalence problem

1 Introduction

In this paper we study the mathematical properties of exact four-dimensional (4D) neutral signature solutions for which all of the polynomial scalar curvature invariants vanish (VSI spaces) [1], which generalize the 4D degenerate Kundt metrics [2] and the VSI metrics [3] in the Lorentzian case. The pseudo-Riemannian spaces (of arbitrary signature) were algebraically classified in terms of the boost weight decomposition and the so-called S$_i$- and N-properties are defined in [7]. It was shown that if the curvature tensors of the space possess the N-property then it is a VSI space.
In four dimensions, the neutral metrics are the least studied of the three distinct metric signatures, as such these manifolds were particularly emphasized and it was proven that VSI spacetimes were either Kundt (possess a geodesic, expansion-free, shear-free, and twist-free null-congruence) or Walker (admit a 2-dimensional invariant null plane [5]) form [7]. In [15] we presented examples of 4D neutral signature VSI metrics which are genuinely Walker spaces (i.e., not Kundt).

To illustrate this dichotomy in the neutral Ricci-flat VSI Walker metrics, we study two distinct subcases: one which is not Kundt in general [15, 1], that is, strictly Walker, and a second that is always Kundt. This comparison is achieved using two approaches, first by determining the existence of a null geodesic, expansion-free, shear-free, and vorticity-free vector for Walker metrics, and second using the Lie algebra classification provided in [6] and [11] to distinguish the two metrics. We show that this classification is well-suited for determining the existence of covariant constant null vectors, recurrent null vectors and more general invariant null distributions, however, it is not fine enough to determine the equivalence of metrics; at best it can be used to show inequivalence of metrics.

As an example it is clear that the two metrics are inequivalent as they have distinct two-dimensional Lie algebras, yet both metrics contain a subcase for which these Lie algebras become one-dimensional. This is notable as all one-dimensional Lie algebras are equivalent to the Lie algebra for a metric admitting two covariantly constant null vectors, implying that this metric is doubly Kundt. A natural question is to ask when, if at all, are the subcases of the two metrics equivalent. This question can only be resolved by implementing the equivalence algorithm for neutral metrics, which is a non-trivial task. We end this paper with an example of the equivalence algorithm applied to a simple subcase of the Kundt-Walker metric, this subcase parallels the plane-wave spacetimes in the Lorentzian case.

1.1 Holonomy Algebras for the Four Dimensional Neutral Walker Metrics

From theorem 8.5 in [10], the holonomy group must preserve the inner product of the neutral metric, and hence must be a subgroup of the generalized orthogonal group O(2,2), the matrix Lie group preserving the quadratic form:

\[ (s, x, t, y) \rightarrow -s^2 - t^2 + x^2 + y^2. \]

Hence, the holonomy algebras will be a subalgebras of \( \mathfrak{so}(2,2) \), with members of \( \phi \) represented as 2-forms. Alternatively we may represent the elements of \( \mathfrak{so}(2,2) \) as \( (1,1) \) tensors by raising one index of the 2-form representation of each element. In [6] a classification of all possible Lie subalgebras of \( \mathfrak{so}(2,2) \) is given by exploiting the isomorphism between \( \mathfrak{so}(2,2) \) and \( \mathfrak{su}(1,1) \times \mathfrak{su}(1,1) \), producing 32 possible classes of Lie subalgebras.

Using this classification, the authors were able to examine 31 of the 32 cases (the exception being their \( A_{13} \)) and determine whether the subalgebra in each case is achieved for a particular neutral metric in four dimensions as a holonomy algebra. Furthermore the geometric structure was determined for each subalgebra and summarized in Table II of [6], the proof of these results follow...
We call members of $\Lambda^m_{m}$ and $(0,2)$ due to the isomorphisms between them arising from the metric elements of this vector space may be represented as tensors of type $(2,0)$, $(1,1)$, and $(0,2)$ due to the isomorphisms between them arising from the metric $g_m$. The classification of the holonomy algebras is given on page 43 of [11].

There is an alternative approach based on geometric and algebraic considerations for the neutral metric manifolds. This formalism was used in Wang and Hall [11] to classify the holonomy subalgebras in order to study the problem of projectively related manifolds sharing similar holonomy groups. (an equivalent classification of the holonomy algebras is given on page 43 of [11]).

Given an arbitrary orthonormal basis for the tangent space $T_m M$ to $M$ at $m$ satisfying $g_m(x, x) = g_m(y, y) = -g_m(s, s) = -g_m(t, t) = 1$, it is easily shown that the vector space of 2-forms at $m$, $\Lambda^2_m M$ is six-dimensional. The elements of this vector space may be represented as tensors of type $(2,0)$, $(1,1)$, and $(0,2)$ due to the isomorphisms between them arising from the metric $g_m$. We call members of $\Lambda^2_m M$, bivectors and express these in component form as: $F \in \Lambda^2_m M$, $F \leftrightarrow F^{ab} = -F^{ba}$. The bivector representation of $\mathfrak{o}(2,2)$ is the Lie algebra $\{ \alpha \in M_2(\mathbb{R}) : \alpha \epsilon + (\alpha \epsilon)T \}$ with $\epsilon = \text{diag}(1, 1, -1, -1)$ and $T$ denoting matrix transpose. There is a natural metric $P$ on $\Lambda^2_m M$ for which the inner product $P(F, G)$ of $F, G \in \Lambda^2_m M$ is $P^{ab}G_{ab} = P_{abcd}F^{ab}G^{cd}$, with $P_{abcd} = \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

Due to the anti-symmetry of the indices, any $F \in \Lambda^2_m M$ will have even rank when expressed as a matrix. Furthermore, as the dimension of the manifold is four, the rank of any non-zero member of $\Lambda^2_m M$ is two or four. If the rank of $F$ is two, we say the bivector $F$ is simple, while if the rank is four, $F$ is called non-simple. If $F$ is simple one may write $F^{ab} = \rho^a \rho^b - q^a q^b = 2p^a \wedge q^b$, where $p, q \in T_m M$. By algebraically classifying the simple and non-simple elements, the authors of [11] are able to identify the possible subalgebras in terms of two simpler subalgebras, $S^+_m = \{ F \in \Lambda^2_m M : F^* = F \}$ and $S^-_m = \{ F \in \Lambda^2_m M : F^* = -F \}$. Noting that $\Lambda^2_m M = S^+_m \oplus S^-_m$, the authors enumerate all possible subalgebras of $\mathfrak{o}(2,2)$ in bivector form by examining the subalgebras of $S^+_m$ and $S^-_m$, producing a list of potential subalgebras of $\mathfrak{o}(2,2)$ given on page 43 of [11], with basis vectors taken from the following list of bivectors:

$$F_1 = \frac{1}{2}(l \wedge n - L \wedge N), \quad F_2 = \frac{1}{2}(l \wedge N), \quad F_3 = \frac{1}{2}(n \wedge L);$$

$$G_1 = \frac{1}{2}(l \wedge n + L \wedge N), \quad G_2 = \frac{1}{2}(l \wedge L), \quad G_3 = \frac{1}{2}(n \wedge N).$$

With all possible subalgebras of $\mathfrak{o}(2,2)$ identified, we may consider the holonomy group of $(M, g)$, $\Phi$, and holonomoy algebra $\phi$. The bivector representation of $\phi$ as a Lie subalgebra of $\mathfrak{o}(2,2)$ is important, due to the Ambrose-Singer theorem, which states that the range of the curvature tensor treated as a mapping $f : \Lambda^2_m M \rightarrow \Lambda^2_m M$, is a subspace of $\phi$. Furthermore if $0 \neq k \in T_m M$ such that $k$ is an eigenvector of each member of $\phi$ then $m$ admits a coordinate neighbourhood $U$ and a nowhere zero vector field $K$ on $U$ which agrees with $K$ at $m$ and is such that $K$ is recurrent on $U$ - see Theorem 8.6 in [10].
1.2 Equivalence of Neutral Signature Walker Metrics

We can also study the Cartan equivalence of neutral signature spaces \([8, 13]\) and determine the Cartan invariants of the special class of Ricci flat VSI spaces. To begin the equivalence algorithm for this particular class of neutral metrics, we must determine the effect of the frame transformations on the Riemann tensor. This may be done by computing individually the effect of each null rotation about \(\ell_1, n_1, \ell_2\) and \(n_2\) along with the effect of boosts in the \((\ell_1, n_1)\) and \((\ell_2, n_2)\) planes on the spin-coefficients as these quantities are of vital importance when computing the covariant derivatives of the curvature tensor. The goal of the equivalence algorithm is the computation of a finite list of invariants arising from the curvature tensor and its covariant derivatives which has been normalized by fixing all frame transformations affecting the form of the curvature tensor and its covariant derivatives.

For the Ricci-flat VSI Walker metrics, we may always fix the boost parameters so that two components of the curvature tensor to be constant. Thus the null rotations are left as potential members of the zeroth order isotropy group, that is those transformations that leave the Riemann tensor unchanged; as an example, in the context of a Kundt-Walker metric the zeroth order isotropy group is two-dimensional, consisting of null rotations about the null geodesic, expansion-free, shear-free, and vorticity-free vector. After recording the number of functionally independent invariants that appear at zeroth order, we proceed to compute the first covariant derivative of the curvature tensor.

From the components of this rank five tensor we may solve for a subset of the spin-coefficients \([9]\) as first order Cartan invariants. Identifying the spin-coefficients that appear at first order is incredibly helpful as we may use their simpler transformation rules to determine the remaining isotropy at first order, instead of considering the isotropy group of a rank five tensor. Using this approach, we fix all frame transformations not included in the first order isotropy group, and identify all new functionally independent and dependent invariants that appear at first order. The algorithm continues each iteration by computing higher order covariant derivatives of the tensor and identifying the isotropy group and functionally independent invariants at each order. The algorithm stops when it reaches the \(q\)-th iteration for which the dimension of the isotropy group and number of functionally independent invariants does not change from iteration \(q-1\) to \(q\).

In general it is not known how many iterations are required to compute the entire list of invariants for the equivalence problem for the neutral metrics. This is an important question, as each iteration \(n\) of the algorithm requires computing the \(n\)-th order covariant derivative of the curvature tensor, and this can be computationally infeasible for high enough \(n\). The theoretical upper-bound introduced by Cartan can at least provide some solace as it limits the number of iterations required to classify an arbitrary neutral metric: the upper-bound for the number of iterations, \(q\), needed to classify a neutral metric spacetime is determined by the largest isotropy subgroup of the Riemann curvature tensor \(\tilde{s}_0\) which must be less than six for spacetimes which are not locally homogeneous:

\[
q \leq n + \tilde{s}_0 + 1 = 4 + 5 + 1 = 10
\]

In the case of the Ricci-flat VSI neutral metrics, we may fix the two real-valued boost parameters to set two components of the curvature tensor to be constant,
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this reduces the upper-bound from ten to eight. In the Ricci-flat VSI Kundt-Walker metric this may be reduced further as the isotropy group consists of the two-dimensional null rotations about a particular null vector

\[ q \leq n + s_0 + 1 = 4 + 2 + 1 = 7. \]

This is the standard upper-bound for the equivalence algorithm for the Lorentzian metrics, however this has only been achieved for a simple subcase of the neutral metrics. An effective lowering of the upper-bound for all neutral metrics would require a classification akin to the Petrov classification for Lorentzian metrics.

2 Preliminaries

A metric is said to possess a 2-dimensional invariant plane if there exist vectors \( l \) and \( m \) such that the bivector \( l \wedge m \) is recurrent; that is,

\[ \nabla_a(l \wedge m) = k_a(l \wedge m) \quad (1) \]

for some covector \( k_a \). If these vectors are also null and orthogonal, the invariant plane is called null. 4D neutral signature spaces possessing an invariant null plane are known as Walker metrics \([5]\). In \([9]\), it was shown that a metric has an invariant null plane if and only if there exists a frame in which the spin coefficients \( \kappa = \rho = \sigma = \tau = 0 \).

A Kundt metric is a metric which possesses a non-zero null vector \( \ell \) which is geodesic, expansion-free, twist-free, and shear-free, which implies a particular form for the covariant derivative of \( \ell \) \([7]\). The condition imply that there exists a null frame in which the spin-coefficients require satisfy:

\[ \tilde{\kappa} = \kappa = \tilde{\rho} = \rho = \tilde{\sigma} = \sigma = 0 \]

this implies the covariant derivative of \( \ell_a \) is then

\[ \ell_{a;b} = -(\epsilon' - \epsilon')\ell_a\ell_b + (\tilde{\alpha}' + \beta')\ell_a m_b - \tilde{\tau} m_a \ell_b \]

This is automatically a geodesic vector in \( \{ n_a, \ell_a, \bar{m}_a, m_a \} \). We project this tensor into \( \{ n_a, \bar{m}_a, m_a \} \), and we find that the projected covariant derivative of \( \ell_a \) is expansion-free, shear-free and vorticity-free. Thus if a spacetime is Kundt, there exists a null frame \( X \) which is geodesic on the cotangent space, and when its covariant derivative is projected onto the the cotangent space orthogonal to \( X_a \) it is expansion-free, shear-free, and vorticity-free. [Note that the null vector \( \ell \) does not have to be recurrent (or covariantly constant). However, if \( \ell \) is covariantly constant the space is Kundt. But if the null vector \( \ell \) is recurrent, then the space does not necessarily have to be Kundt.]

We are especially interested in the case where the VSI metric is Walker, and hence admits an invariant null plane, but is not Kundt. For Walker metrics in 4D neutral spaces with an invariant 2-dimensional null plane it is always possible to find a null field that is geodesic, expansion-free, vorticity-free (see below). But, in general, this null vector is not necessarily shear-free.
2.1 Spin Coefficients

To determine if there is a preferred null frame, we need to use Law’s formalism for the spin-coefficients \[9\]. As an example, from \[9\] the following frame transformation

\[
\begin{align*}
\ell^a &\rightarrow \tilde{\lambda}\lambda^a, \quad n^a \rightarrow (\tilde{\lambda}\lambda)^{-1}n^a + \lambda^{-1}\tilde{\mu}n^a + \mu\lambda^{-1}n^a + \mu\tilde{\lambda}\ell^a \\
m^a &\rightarrow \lambda^{-1}\lambda m^a + \lambda\tilde{\mu}\ell^a, \quad \tilde{m}^a \rightarrow \lambda^{-1}\tilde{\lambda}m^a + \mu\lambda^{-1}\tilde{\lambda}a
\end{align*}
\]

we produce the following transformation rules for \(\tilde{\tau}, \tau, \sigma', \sigma, \kappa', \gamma, \alpha, \alpha'\) and \(a'\):

\[
\begin{align*}
\kappa &\rightarrow \lambda^3\tilde{\lambda}\kappa, \tilde{\kappa} \rightarrow \lambda^3\lambda\tilde{\kappa}, \\
\rho &\rightarrow \lambda\lambda\rho + \lambda^2\tilde{\mu}\kappa, \tilde{\rho} \rightarrow \lambda^{-1}\tilde{\lambda}\tilde{\rho} + \tilde{\lambda}^2\lambda\tilde{\kappa} \\
\sigma &\rightarrow \lambda^3\tilde{\lambda}\sigma + \lambda^3\tilde{\mu}\kappa, \tilde{\sigma} \rightarrow \tilde{\lambda}^3\lambda^{-1}\sigma + \tilde{\lambda}^3\mu\kappa \\
\tau &\rightarrow \lambda\tilde{\lambda}^{-1}\tau + \lambda\tilde{\lambda}\tilde{\rho} + \lambda^2\tilde{\lambda}^{-1}\mu\sigma + \lambda^2\mu\kappa, \tilde{\tau} \rightarrow \tilde{\lambda}^{-1}\tilde{\tau} + \lambda\tilde{\lambda}\tilde{\rho} + \tilde{\lambda}^2\lambda^{-1}\tilde{\sigma} + \tilde{\lambda}^2\mu\kappa.
\end{align*}
\]

The frame used in any particular computation (e.g., in the examples below) in which there is a different normalization of the null frame (and for Walker or Kundt) were given in \[7\]. In such a situation, to show, for example, a metric is Walker or Kundt it must be shown that there exists a frame under which the spin coefficients transform to their standard form. Suffice it to say that this is the situation in the examples considered below (and the conditions are consistent with \[7\]).

3 4D Neutral signature Walker metrics

We investigate a class of VSI-Walker Ricci-flat metrics \[15\] in the Walker form:

\[
ds^2 = 2du(dv + Adu + CdU) + 2dU(dV + BdU),
\]

where

\[
\begin{align*}
A &= vA_1(u, U) + BA_2(u, U) + A_0(u, U), \\
B &= VB_{10}(u, U) + v^2B_{02}(u, U) + vB_{01}(u, U) + B_{00}(u, U) \\
C &= vC_{11}(u, U) + VC_2(u, U) + C_0(u, U).
\end{align*}
\]

For the space to be Ricci flat, we must also have that \(A_2B_{02} = 0\). This metric does not in general possess the \(N\)-property, but rather the weaker requirement of the \(N^G\)-property \[7\].

The null tetrad frame \(\{\ell, n, m, -\tilde{m}\} = \{l_1, n_1, l_2, n_2\}\) is defined by:

\[
\begin{align*}
l_1 &= du, \quad n_1 = dv + A(u, v, U, V)du + \frac{C(u, v, U, V)}{2}dU \\
l_2 &= dU, \quad n_2 = dv + \frac{C(u, v, U, V)}{2}du + B(u, v, U, V)dU
\end{align*}
\]

The invariant null plane is given by the null orthogonal vectors \(l_1\) and \(l_2\):

\[
\nabla (l_1 \wedge l_2) = (l_1 \wedge l_2) \otimes \left( \frac{C_2}{2} + A_1 \right) du + \left( B_{10} + \frac{C_{11}}{2} \right) dU
\]

confirming that this is a Walker metric.
There are two subcases for $A_2B_{02} = 0$. The Walker-Kundt case $B_{02} \neq 0; A_2 = 0$ was investigated in [15]. Ricci flat solutions in the case $A_2 = C_2 = 0$, $B_{02} \neq 0$ were obtained, and the corresponding infinitesimal holonomy algebra was found to be $A_{26}$ [3], and so our metric has a null two-dimensional distribution containing a recurrent vector field. When $B_{10,u} = 0$, the infinitesimal holonomy algebra reduces to $A_{17}$, and so the metric has a null two-dimensional distribution containing one parallel vector field, which was displayed explicitly. The Cartan equivalence problem was considered in [8, 13] and the Cartan invariants of the special class of Ricci flat VSI spaces was determined - here a bifurcation again occurs when $B_{10,u}$ vanishes.

### 3.1 Kundt Condition for Walker VSI Metrics

Let us show explicitly that, in general, there exists no null vector $X_a$ for the VSI metric given in Theorem 2.1 of [15] which is null, geodesic and when restricted to the subspace of the tangent space $\{X_b \}$ is expansion-free ($X_b = 0$), shear-free ($X_{(i',j')} = 0$) and vorticity-free ($X_{[i',j']} = 0$).

Imposing the vorticity condition, we find that $X_a$ is the gradient of some function $X(u, v, U, V)$, then by expanding and equating powers of $v$ and $V$ we find that the conditions $X$ be null and geodesic both imply that $X_u = X_V = 0$. Thus $X_a = X_u \ell_a + X_v m_a$ and its covariant derivative is of the form:

$$X_{a,b} = (A_2 X_{U} + A_1 X_{u} + X_{uu})m_a m_b + (C_2 X_{u} + C_{11} X_{u} + X_{uu})[\ell_a m_b + m_a \ell_b] + (X_{u} B_{10} + 3 X_{u} m^2 B_{03} + 2 X_{u} v B_{02} + X_{u} B_{03} + X_{u} B_{02}) \ell_a \ell_b$$

To project this tensor to the subspace $\{e_i\}$ we assume that $X_U \neq 0$ and consider the following projection operator:

$$h^{b}_{a} = g^{b}_{a} + \frac{m_a X_{b}}{X_{U}} + \frac{X_{b} m_a}{X_{U}}.$$

As $h^{b}_{a} X_b = 0$ this will serve to project onto the subspace $\{e_i\}$. Applying this operator to the covariant derivative of $X_a$, $X_{c,d}h^{c}_{a}h^{d}_{b}$, we have one non-zero coefficient of $m_a m_b$, namely the component $X_{c,d}h^{c}_{a}h^{d}_{b} \tilde{m}^c \tilde{m}^b$ which is of the form:

$$X^{-2}_{U}[-2 X_{u} X_{U} V B_{11} - X_{u} X_{u} B_{10} - 3 X_{u}^2 V^2 B_{03} - 2 X_{u}^2 V B_{02}]$$

$$+ X^{-2}_{U}[-X_{u}^2 B_{01} - X_{u} X_{u} U + 2 X_{u}^2 C_2 + 2 X_{u} X_{u} C_{11}]$$

$$+ X^{-2}_{U}[2 X_{u} X_{u} U - A_1 X_{u}^3 V - A_2 X_{u}^3 V - 3 X_{u} X_{u} X_{U}^2]$$

Equating the $v$ and $V$ linear terms we must have $X_{u} = 0$ or $B_{11} = B_{03} = B_{02} = 0$. For the moment we assume that $X_{u} = 0$, producing the simpler expression for this coefficient of $m_a m_b$

$$X_{c,d}h^{c}_{a}h^{d}_{b} = -A_2 X_{U}^3 m_a m_b.$$

Thus, in order for these metrics to be Kundt, there must be a coordinate system where $A_2$ must vanish and $X_a$ is proportional to $\ell_a$. 

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Alternatively, assuming $X_u \neq 0$ we may produce a similar projection operator, and then by contracting this new projection operator twice with the covariant derivative of $X_{a,b}$ we find:

\[
X^{-2}_u[X_u(-2X_uX_uVB_{11} - X_uX_uB_{10} - 3X_u^2V^2B_{03} - 2X_u^2VB_{02})
+ X^{-2}_u[-X_u^2B_{01} - X_uX_uUU + 2X_u^2C_2 + 2X_uX_uC_{11}]
+ X^{-2}_u[2X_uX_uU - A_1X_u^2] - A_2X_u^2 - X_uuX_u^3]
\]

Again, equating the $v$ and $V$ linear terms we must have that either $X_U = 0$ or $B_{11} = B_{03} = B_{02} = 0$. It is not clear if there is a solution to the partial differential equation for $X(u,U)$ when these three functions vanish, thus we will assume that $X_U = 0$. Simplifying the equations, we find:

\[
\frac{3X^3_uu^2B_{03} + 2X^3_uvB_{02} + X^3_uB_{01}}{X^2_u}
\]

This metric will be Kundt if and only if $B_{01} = B_{02} = B_{03} = 0$ with $X_a$ proportional to $m_a$. With these observations we conclude with the following proposition.

**Proposition 3.1.** A four dimensional neutral signature Walker metric (2) is Kundt with a null geodesic, expansion-free, shear-free and vorticity-free vector $X_a$ which is proportional to:

- $\ell_a$ if and only if the metric component in $A$ in $\ell_1$ in (3) satisfies $A_v = 0$.
- $m_a$ if and only if the metric component in $B$ in $\ell_2$ in (4) satisfies $B_v = B_{uv} = 0$.

### 3.2 A Walker metric that is not Kundt

We examine the case in which $B_{02} = 0$ and $A_2$ may or may not be zero. From section 3.1, when $A_2 = 0$ or $B_{01} = B_{02} = 0$ we have a Kundt metric. In general these functions are non-zero here.

To satisfy the condition of Ricci flatness, the metric functions must be solutions to the following equations:

\[
\frac{\partial C_2}{\partial u} + AC_2 = 2\frac{\partial A}{\partial u} - 2BA_2, \tag{6}
\]

\[
2\frac{\partial B_{01}}{\partial u} + AB_{01} = \frac{\partial C_{11}}{\partial u} + BC_{11} \tag{7}
\]

\[
\frac{\partial A}{\partial u} + \frac{\partial B}{\partial u} - 2A_2B_{01} + \frac{1}{2}C_2C_{11} = 0 \tag{8}
\]

where $A \equiv A_1 - \frac{1}{2}C_2$ and $B \equiv B_{10} - \frac{1}{2}C_{11}$. As a simple example, we set $A = B = 0$ and obtain a simple solution for (6) and (7):

\[
A_1 = \frac{1}{2}C_2, \quad A_2 = aU + \frac{\beta}{u}, \quad B_{10} = \frac{1}{2}C_{11}, \quad B_{01} = \frac{1}{2}(\frac{\beta}{u} + du),
\]

\[
C_2 = 2au + \frac{\beta}{u}, \quad C_{11} = \frac{a}{u} + dU
\]

where $a, \alpha, \beta, c_1, c_2, d$ are constants. Then the Ricci tensor has one nonzero component:

\[
R_{13} = R_{31} = \frac{1}{2}(\beta c_1 - 2\alpha c_2) + (2ac_1 + \beta d - 2c_2a - 2ad)uU \tag{9}
\]
which for Ricci flatness gives us algebraic constraints, $\beta c_1 = 2\alpha c_2$ and $2\alpha c_1 + \beta d - 2c_2 a - 2\alpha d = 0$. If $c_1 \neq 0$, we must have $\beta = 2c_2 \alpha / c_1$ and $(c_1 - c_2) (a - \alpha d / c_1) = 0$. We chose to work with $a = \alpha d / c_1$, and assume $c_1$ is nonzero and $c_1 \neq c_2$ in order to make later calculations more manageable.

When $c_1$ vanishes, the Ricci-flat conditions produce five possible subcases where some of the constants must vanish or satisfy an identity:

- $\alpha = 0$, and $\beta = 2\alpha d / c_1$.
- $\alpha = 0$, $d = 0$, and $a = 0$.
- $\alpha = 0$, $d = 0$, and $c_2 = 0$.
- $c_2 = 0$, and $d = 0$.
- $c_2 = 0$, and $\beta = 2\alpha$.

while if $c_1$ is non-zero and $c_2 = c_1$ we find one more case:

- $\beta = 2\alpha$.

The exact form of the two recurrent vectors for these subcases are not discussed in this paper, however, the analysis is similar to the case studied in this paper.

### 3.2.1 Holonomy

We calculate the holonomy algebra in the null tetrad basis by contracting the Riemann tensor with bivectors constructed from the null tetrad vectors \[12\]. Again, only six such contractions are required. The matrices with indices lowered are presented, since these have a simpler form:

$$R_{abcd} l_1 c_1 d = \xi (l_1 \wedge l_2)_{ab}$$
$$R_{abcd} l_1 c_2 d = R_{abcd} l_1 c_2 d = R_{abcd} n_1 c_2 d = 0$$
$$R_{abcd} n_1 c_2 d = (l_1 \wedge n_1 + l_2 \wedge n_2 - \zeta (l_1 \wedge l_2))_{ab}$$
$$R_{abcd} l_2 c_2 d = \xi (l_1 \wedge l_2)_{ab}$$

where $\xi = c_2 u - 2\alpha c_2 x$ and $\zeta$ is a complicated expression.

Taking linear combinations of these, we find that our holonomy algebra is spanned by $\{l_1 \wedge l_2, l_1 \wedge n_1 + l_2 \wedge n_2\}$, corresponding to Hall’s subalgebra 2(d) \[11\]. For the subalgebra 2(d), so that $\phi_m =< F_1, G_2 >$, with $|F_1| = -1$ and $|G_2| = 0$. If a neutral metric, $g$, admits this holonomy subalgebra, the Ambrose-Singer theorem implies that the curvature tensor associated with $g$ may be expressed in the following way:

$$R_{abcd} = \alpha [F_1]_{ab} \odot [F_1]_{cd} + \beta [F_1]_{ab} \odot [G_2]_{cd} + \delta [G_2]_{ab} \odot [G_2]_{cd}$$

where $\alpha$, $\beta$ and $\delta$ are real-valued quantities which act as parameters in general.

From theorem 8.6 in \[10\], this metric admits two recurrent vectors $l$ and $L$ as these are shared eigenvectors of $F_1$ and $G_2$ (with differing eigen-values). Due to the symmetrization of the two-form representations of the Lie algebra
members in the Riemann tensor, this implies the vectors $\ell$ and $L$ may be seen as eigen-vectors with "eigen-two-forms" proportional to the Lie algebra members.

As matrices in the null tetrad basis with the first index up, we have

\[
\begin{bmatrix}
0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \begin{bmatrix}
+1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & +1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.
\]

Using the Jordan normal form, one may construct a transformation matrix $N$ showing that these two matrices are equivalent to the subcase $A_{10}$ in [6]. We have found a holonomy algebra isomorphic to $A_{10}$ which, according to Ghanam and Thompson, corresponds to a metric containing two recurrent vectors. A vector $l$ is called recurrent if $\nabla l = l \otimes \omega$ for some one-form $\omega$ [10].

It is worthwhile to consider when the holonomy algebra becomes one-dimensional, that is, when $\xi = \frac{c_2 u^2 - 2 a c_1 x^2}{2 c_1 v^2 + x^2} = 0$. This occurs when $c_1$ vanishes and either $a = 0$ or $c_2 = 0$. There are three possible cases where this can happen

1. Case 1: $c_1 = a = \beta = a = 0$
2. Case 2: $c_1 = a = \beta = c_2 = 0$
3. Case 3: $c_1 = c_2 = d = \beta = 0$

According to theorem 4.6 in [6], as each of these subcases admit a one-dimensional holonomy algebra $A_{9}$, each of these subcases admit two covariantly constant null vectors. This condition implies that these Walker metrics are Kundt with two null geodesic, expansion-free, shear-free and vorticity-free vectors. For this particular example, this implies there is a coordinate system where $A_2$ and $B_01$ both vanish. Looking at the five examples, it is not clear that one has found the appropriate coordinate system as only one of $A_2$ or $B_01$ vanishes in cases 1, 2, 4 and 5, while in case 3 neither function vanishes. This question cannot be answered by comparing holonomy algebras alone, one must examine these subcases in the context of the equivalence algorithm.

### 3.3 A Walker metric that is Kundt

To provide a simple illustrative example of the equivalence algorithm, we consider a case that is automatically Kundt and Ricci-flat, with $B_{02} \neq 0$ and $A_2 = C_2 = 0$; ensuring that this is indeed a Walker-Kundt metric, with only one null, geodesic, expansion-free, shear-free and vorticity free vector. Imposing the Ricci-flat conditions we have the following expressions for the metric functions:

\[
A_0 = \frac{1}{8} \left( -2 B_{10} C_{11} - 4 A_1 B_{01} + 4 B_{01},u - 2 C_{11},v + C_1^2 \right),
\]

\[
A_1 = \frac{1}{2} [\log(B_{02})],u,
\]

\[
C_{11} = 2 B_{10} + [\log(B_{02})],v + G(U)
\]
where $A_0$ has not been fully expanded in order to display it compactly. With these metric functions, the components of the Riemann tensor are now:

\[
R_{1224} = R_{2344} = B_{10,u}
\]

\[
R_{2323} = 2B_{02}
\]

\[
R_{2424} = -C_{0,u}B_{+} + B_{00,uu} - 3v^2A_1B_{02,u} - 2v^2A_1A_2B_{02} - B_{10}vC_{11,u} + B_{10}vA_{1,u}
\]

\[
+ A_1V B_{10,u} - A_1B_{10}C_0 + 2v^2A_1^2B_{02} + B_{10}C_{11}A_0 + (1/2)vC_{11}C_{11,u}
\]

\[
- 4A_0vB_{02,u} - 2A_0,uvB_{02} - vA_1B_{01,u} - vA_1A_2B_{01} + 4vA_1B_{02}A_0 + A_0,vU
\]

\[
+ v^2B_{02,uu} + vB_{10,uu} - 2A_0B_{01,u} - A_0,uvB_{01} - B_{10}C_{0,u} - B_{10,u}C_0
\]

\[
- A_1C_{0,u} - A_1V C_0 + vA_{1,u},vU + C_{11}A_0,U + C_{11},vA_0 + 2A_0,vB_{02}
\]

\[
+ (1/2)C_{11,u}C_0 - vC_{11},u,u + vB_{01,uu} + A_1B_{00,u} + B_{10}A_{0,u} - B_{10,u}vC_{11}
\]

For the remainder of the paper, the component $R_{2424}$ will be denoted as $\Psi$. It should be remarked that when the Ricci-flat conditions are imposed $\Psi$ is independent of $v$ and $V$. In the current form it is not entirely clear that this is the case.

### 3.3.1 Holonomy

With these Riemann components computed we can use the Ambrose-Singer theorem to identify the generators of the holonomy Lie algebra. Despite the differing order of variables, i.e., $\{V, v, U, u\}$ instead of $\{u, v, U, V\}$, we may compare the matrices arising from the curvature tensor with those in [6]. Note that the covariant derivatives of the Riemann tensor introduce no new generators.

When $B_{10,u} \neq 0$ the Lie algebra is three dimensional and it is a trivial exercise using the Jordan form to show that this is equivalent to $A_{26}$ with $\alpha = 0$ in [6], as the generators are:

\[
\left\{ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}
\]

If $B_{10,u} = 0$ there is a bifurcation and the Lie algebra is two dimensional, and hence is equivalent to $A_{17}$ in [6] with generators:

\[
\left\{ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}
\]

Finally, if $B_{10,u} = \Psi = 0$ the Lie algebra is equivalent to the one-dimensional Lie algebra $A_0$.

According to Table II in [6], when $B_{02} \neq 0$, $\Psi \neq 0$, and $B_{10,u} \neq 0$, we find that the infinitesimal holonomy algebra corresponds to $A_{26}$, and so our metric has a null two-dimensional distribution containing a recurrent vector field. When $B_{02} \neq 0$, $\Psi \neq 0$, and $B_{10,u} = 0$, we find that the infinitesimal holonomy algebra corresponds to $A_{17}$, implying that our metric has a null two-dimensional distribution containing one parallel vector field. We look for the recurrent vector explicitly. We expect that $\partial_V$ will be recurrent because it is an eigenvector of each of the members of the holonomy algebra. Calculating the covariant derivative of $\partial_V$, we find that $\nabla \partial_V = B_{10}\partial_V dU$, as expected. It is also clear that $\partial_V$ becomes parallel when $B_{10} = 0$. If $B_{10,u}$ and $\Psi$ both vanish, the null two-dimensional distribution contains two parallel vector fields.
4 Holonomy and Recurrent Vectors

Theorem 8.6 [10] suggests that in order to find the recurrent vectors, we calculate the eigenvectors of the elements of the holonomy algebra. In the tetrad basis, we find that each of the tetrad vectors is an eigenvector of \( l_1 \wedge n_1 + l_2 \wedge n_2 \), but only \( l_1 = \partial_c \) and \( l_2 = \partial_V \) are eigenvectors of \( l_1 \wedge l_2 \). The theorem tells us that for each \( m \in M \) there exist two recurrent vector fields on \( M \), one having the value \( \partial_c \) at \( m \) and one having the value \( \partial_V \) at \( m \). Thus, we look for two recurrent vectors: one of the form \( \ell_1 = \partial_c + f(u, v, U, V)\partial_V \) and another of the form \( \ell_2 = h(u, v, U, V)\partial_V + \partial_V \).

First we take the covariant derivative of \( \partial_c + f(u, v, U, V)\partial_V \) to find conditions on \( f \) that give recurrence for \( \nabla(\ell_1) \) as:

\[
\frac{\alpha f(u, v, U, V)\partial_V}{2c_1uU} + \frac{2f(u, v, U, V)\partial_V}{2c_1uU} + \frac{f(u, v, U, V)}{2c_1uU} \partial_V \odot du + \frac{f(u, v, U, V)}{2c_1uU} \partial_V \odot dU + \frac{f(u, v, U, V)}{2c_1uU} \partial_V \odot du.
\]

Thus we must have \( f(u, v, U, V) = f(u, U) \) and \( f(u, U) \) must satisfy a system of two PDEs:

\[
\alpha f(u, v, U, V) = \frac{1}{f_c(u, v, U, V) + f_c(v, U, V) - u f_u} - u f_u = 0, \tag{15}
\]

\[
f^2 \alpha \left( \frac{dU}{c_1 U} + 1 \right) - \frac{c_1}{\sqrt{2c_1}} - \frac{u f_u}{2} - U f_U = 0. \tag{16}
\]

Assuming \( \alpha \neq 0 \), these equations have a solution:

\[
f_{k_0}(u, U) = -\sqrt{\frac{2c_1}{\sqrt{2c_1}}} \tan \left( \frac{\alpha u}{\sqrt{2c_1}} (c_1 \ln |u| + c_2 \ln |U| + duU + k_0) \right)
\]

where \( k_0 \) is an arbitrary constant. This solution gives a one-parameter family of recurrent vectors; if we choose any \( k_1 \) and \( k_2 \) such that \( k_1 \neq k_2 \), then \( f_{k_1}(u, U) \neq f_{k_2}(u, U) \), and so the recurrent vectors \( \partial_c + f_{k_1}(u, U)\partial_V \) and \( \partial_c + f_{k_2}(u, U)\partial_V \) are linearly independent.

Repeating the above for \( \ell_2 \), we find \( h_{k_0}(u, U) = \frac{2c_1}{\sqrt{2c_1}} f_{k_0}(u, U) \). However, choosing \( k_1' = k_0 + i\pi \sqrt{\frac{2c_1}{\sqrt{2c_1}}} \in \mathbb{C} \) and noting that \( \tan(x + i\pi/2) = 1/\tan(x) \), we find that each \( h_{k_0}(u, U) \) corresponds to \( h_{k_0}(u, U) = \frac{1}{f_{k_0}(u, U)} \). Thus, these solutions are not quite as different from those for \( \ell_1 \) as they initially appear. If \( m \in M \) corresponds to \( (u_0, v_0, U_0, V_0) \), then choosing \( k_0 = -\ln |u_0| + c_2 \ln |U_0| + duU_0 \) and \( k_0 = k_0 \) gives \( f_{k_0}(u_0, v_0, U_0, V_0) = h_{k_0}(u_0, v_0, U_0, V_0) = 0 \) and so our recurrent vectors are such that \( [\partial_c + f_{k_0}\partial_V]_{m} = \partial_c \) and \( [h_{k_0}\partial_c + \partial_V]_{m} = \partial_V \), as Theorem 8.6 [10] predicts.

When \( \alpha = 0 \), (15) and (16) have the solution

\[
\tilde{f}_{k_0}(u, U) = -\frac{duU + c_1 \ln |u| + c_2 \ln |U| + k_0}{2c_1} = \lim_{\alpha \to 0} f_{k_0} \tag{17}
\]
and the corresponding PDEs for $h$ have a solution
\[ h_{k_0}(u, U) = \frac{1}{f_{k_0}(u, U)} = \lim_{\alpha \to 0} h_{k_0}. \]  

The recurrence of $\ell_1$ with $\alpha = 0$ is shown explicitly:
\[ \nabla \left( \partial_v + f_{k_0}(u, U)\partial_{\nu} \right) = \left( \partial_v + f_{k_0}(u, U)\partial_{\nu} \right) \otimes \left( \frac{c_1 + du}{2u} \right) du \]  
and so because $\frac{c_1 + du}{2u}$ does not depend on the value of the constant $k_0$, linear combinations of $\partial_v + f_{k_0}\partial_{\nu}$ (for different values of $k_0$) are recurrent: if for $\kappa, \lambda, k_1, k_2 \in \mathbb{R}$ we define
\[ v_1 \equiv \left( \partial_v + f_{k_1} \partial_{\nu} \right), \quad v_2 \equiv \left( \partial_v + f_{k_2} \partial_{\nu} \right) \]
then
\[ \nabla(\kappa v_1) = \kappa v_1 \otimes \left( \frac{c_1 + du}{2u} \right) du, \quad \nabla(\lambda v_2) = \lambda v_2 \otimes \left( \frac{c_1 + du}{2u} \right) du. \]

Thus we see the following recurrent relation
\[ \nabla(\kappa v_1 + \lambda v_2) = (\kappa v_1 + \lambda v_2) \otimes \left( \frac{c_1 + du}{2u} \right) du. \]

Setting $\kappa = -1, \lambda = 1, k_1 = -1$, and $k_2 = 1$, we find that
\[ \kappa v_1 + \lambda v_2 = - \left( \partial_v + f_0 \partial_{\nu} \right) + \left( \partial_v + f_1 \partial_{\nu} \right) = \left( -f_0 + f_1 \right) \partial_{\nu} = \partial_{\nu}. \]

### 4.1 Recurrent Vectors in a Kundt Subcase

As a simple example, we show that in the Kundt subcase where $\alpha = 0$, the tetrad vector $l_2 = \partial_{\nu}$ is a Kundt vector. We know that $l_2$ is null since it is a null tetrad member. $Vl_2l_2 = 0$ implies that $l_2$ is geodesic. It remains to check that the projection of $\nabla l_2 = \frac{c_1 + du}{2u} \partial_{\nu} du$ onto the tangent space orthogonal to $l_2$ is expansion-free, twist-free, and shear-free. The projection tensor is given by
\[ p_a^b = g_a^b - l_2a n_2^b - n_2a l_2^b \]
and by applying it to $\nabla l_2$ we find
\[ l_{2a;b}p_a^c p_d^b = 0. \]

Thus the projected covariant derivative is expansion-free, shear-free, and vorticity-free; $l_2$ is a Kundt vector when $A_2(u, U) = 0$.

The above recurrent and Kundt vectors were obtained under the assumption (from $\mathbf{(22)}$) that $a = ad/c_1$ (including $c_1 = c_2$). Now the only case not examined is $a \neq ad/c_1$ and $c_1 = c_2$ ($c_1 \neq 0$). Resuming from $\mathbf{(22)}$ and assuming $c_1 = c_2$ instead of $a = ad/c_1$ results in the same holonomy algebras. The recurrent vectors are found similarly, with $f(u, U)$ and $h(u, U)$ satisfying PDEs
\[ f^2 \left( u + \frac{\alpha}{a} \right) - \frac{c_1\alpha}{u} - \frac{du}{2} - f_u = 0, \quad f^2 \left( au + \frac{\alpha}{a} \right) - \frac{c_1\alpha}{u} - \frac{du}{2} - f_U = 0, \]
\[ h^{2 \frac{1}{2}} \left( dU + \frac{d}{d} \right) - au - \frac{\alpha}{u} - H_u = 0, \quad h^{2 \frac{1}{2}} \left( du + \frac{d}{d} \right) - au - \frac{\alpha}{u} - H_U = 0. \]

Evidently, when \( A_2 = 0 \) \( \Leftrightarrow a = \alpha = 0 \), these PDEs reduce to those found previously and so the recurrent vectors are the same as (17) and (18), so we find that we have a Kundt vector once again.

## 5 Equivalence

We now describe in detail the equivalence algorithm outline in [15] for those Ricci-flat Walker metrics with \( B_{02} \neq 0 \) and \( A_2 = C_2 = 0 \) with the following conditions on the remaining metric functions:

\[ B_{10} = f(U), B_{00} = 0, B_{02} = e^{W(u)} e^Z(U). \]

The metric functions \( A_0, A_1 \) and \( C_{11} \) now become:

\[
\begin{align*}
A_0 &= 1 - 2B_{10}C_{11} - 4A_1B_{01} - 2C_{11,u} + C_{11}^2 \quad / B_{02} \\
A_1 &= \frac{1}{2} W_u \\
C_{11} &= 2B_{10} + Z_u + G(U).
\end{align*}
\]

The non-zero components of the Riemann tensor are:

\[
\begin{align*}
R_{2323} &= 2B_{02} \\
R_{2424} &= -C_{0,u} + B_{00,uu} - 3v^2A_1B_{02,u} - 2v^2A_{1,u}B_{02} - B_{10}vC_{11,u} \\
&\quad + B_{10}vA_{1,U} - A_1B_{10}C_0 + 2v^2A_1^2B_{02} + B_{10}C_{11}A_0 \\
&\quad + (1/2)A_0vB_{02,u} - 2A_{0,u}vB_{02} - vA_1B_{01,u} - vA_{1,u}B_{01} \\
&\quad + 4vA_1B_{02}A_0 + A_0,uu + v^2B_{02,uu} + vB_{10,uu} - 2A_0B_{01,u} - A_{0,u}B_{01} \\
&\quad - B_{10}C_{0,u} - A_1C_{0,u} - A_1U_C_0 + vA_{1,U,U} + C_{11}A_{0,U} + C_{11,U}A_0 \\
&\quad + 2A_{0}^2B_{02} + (1/2)C_{11,u}C_0 - vC_{11,U,u} + vB_{01,uu} + A_1B_{00,u} + B_{10}A_{0,u},
\end{align*}
\]

where again we will denote \( R_{2424} \) as \( \Psi \).

Since we may fix the components of the Riemann curvature tensor to constants by performing boosts in both the \((\ell_1, n_1)\) and \((\ell_2, n_2)\) null planes, with \( z_1 = A, z_2 = B \) (See Appendix for transformation rules) as boost parameters:

\[
\begin{align*}
z_1^2 &= \frac{z_2^2}{2B_{02}}, \quad z_2^4 = \frac{2B_{02}}{\Psi}
\end{align*}
\]

no new functionally independent invariants appear at zeroth order [8, 13]. Thus we must compute the first covariant derivative of the curvature tensor, which requires knowledge of the spin-coefficients.
The non-vanishing spin coefficients arising from the metric coframe are:

\[
\gamma = f(U) + \frac{1}{4}(G(U) + Z_u),
\]

\[
\sigma' = \frac{1}{2} C_{11} A_0 - \frac{1}{2} v C_{11, u} - \frac{1}{2} C_{0,u} + v A_{1,u} + A_{0,u} - \frac{1}{2} A_1 C_0,
\]

\[
\kappa' = -\frac{1}{2} B_{10} v C_{11} - \frac{1}{2} B_{00} C_0 - 2 v^2 A_1 B_{02} - v A_1 B_{01} - 2 A_0 v B_{02} - A_0 B_{01} + V B_{10,u},
\]

\[
+ v^2 B_{02,u} + v B_{01,u} + v B_{01,u} + B_{00,u} - \frac{1}{2} v C_{11,u} - \frac{1}{2} C_{0,u} + \frac{1}{4} v C_{11} + \frac{1}{4} C_{11} C_0,
\]

\[
\beta' = \frac{1}{4} W_{u},
\]

\[
\tilde{\beta} = -\frac{1}{4} W_{u},
\]

\[
\tilde{\gamma} = -\frac{1}{2} (Z_u + G(U)),
\]

\[
\tilde{\rho'} = -f(U) - \frac{1}{2} f_{,u} - \frac{1}{2} G(u),
\]

\[
\tilde{\kappa}' = 2 v e^{W(u)} f(U) + B_{01}
\]

We notice that, as we have chosen that \(\ell_a n^a = 1\) and \(m_a \tilde{m}^a = -1\), we have the following relationships between spin-coefficients

\[
\epsilon = -\gamma', \alpha = \beta', \beta = \alpha', \gamma = -\epsilon', \tilde{\epsilon} = -\tilde{\gamma}', \tilde{\alpha} = \tilde{\beta}', \tilde{\beta} = \tilde{\alpha}', \tilde{\gamma} = -\tilde{\epsilon}'. \tag{24}
\]

Of the thirty-two spin-coefficients we may concentrate on twenty-four of them instead.

Performing boosts in both the \((\ell_1, n_1)\) and \((\ell_2, n_2)\) null planes and denoting our boosted spin coefficients with a subscript B (where \(\kappa_B\) would be the spin coefficient \(\kappa\) after boosts in the \((\ell_1, n_1)\) and \((\ell_2, n_2)\) null planes, we find the non-vanishing transformed spin coefficients to be

\[
\alpha_B' = -\frac{1}{2} \left( \frac{z_1 W}{z_2} + \frac{z_2 W}{z_1} \right), \quad \alpha_B = \frac{1}{2} z_2 \alpha, \quad \tilde{\alpha}_B = \frac{1}{2} z_2 \tilde{\alpha}, \quad \tilde{\alpha}_B = \frac{1}{2} \left( \frac{z_2 W}{z_1} + \frac{z_1 W}{z_2} \right) \tag{25}
\]

\[
\gamma_B = z_1 \gamma, \quad \tilde{\gamma}_B = \frac{1}{2} \left( \frac{z_1 W}{z_2} + \frac{z_2 W}{z_1} \right), \quad \tilde{\gamma}_B = \frac{1}{2} \left( \frac{z_1 W}{z_2} - \frac{z_2 W}{z_1} \right) \tag{26}
\]

\[
\sigma_B' = \frac{1}{2} z_1 z_2 \sigma', \quad \kappa_B = \frac{1}{2} z_2 z_1 \kappa', \quad \tilde{\rho}_B = \frac{1}{2} z_1 \tilde{\rho}, \quad \tilde{\kappa}_B = \frac{1}{2} \tilde{\kappa} \tag{27}
\]

From the components of this rank-five tensor, we may solve for the following boosted spin-coefficients [10] as first-order Cartan invariants:

\[
\{ \rho, \tau, \kappa, \sigma, \tilde{\rho}, \tilde{\sigma}, \tilde{\kappa}, \tilde{\alpha}, \tilde{\alpha}', \gamma, \gamma', \tilde{\gamma}, \tilde{\gamma}' \}
\]

Of which, the following are non-zero:

\[
\{ \alpha, \alpha', \tilde{\alpha}, \tilde{\alpha}', \gamma, \gamma', \tilde{\gamma}, \tilde{\gamma}' \}
\]

The remaining isotropy at first order consists of null rotations about \(\ell_1\), as null rotations about \(n_1, \ell_2\), and \(n_2\) change the number of non-zero components of the Riemann tensor, and thus do not belong to the first-order isotropy group. Computing null rotations about \(\ell_1\) and denoting boosted and rotated spin coefficients with a subscript \(R\), we obtain the following list of non-vanishing transformed
spin coefficients (with \( z_3, z_4 \) rotation parameters):

\[
\begin{align*}
\alpha_R &= \alpha_B + z_3 \gamma_B' \\
\alpha_R' &= \alpha_B' + z_4 \gamma_B' \\
\tilde{\alpha}_R &= \tilde{\alpha}_B - z_3 \gamma_B' \\
\tilde{\alpha}_R' &= \tilde{\alpha}_B' + z_3 \gamma_B' \\
\gamma_R &= z_4 \gamma_B - z_3 \alpha_B' + z_4 \beta_B' + \gamma_B \\
\gamma_R' &= \gamma_B \\
\tilde{\gamma}_R &= z_4 \gamma_B - z_3 \beta_B' + z_4 \tilde{\alpha}_B' + \tilde{\gamma}_B \\
\tilde{\gamma}_R' &= \tilde{\gamma}_B
\end{align*}
\]

\( \tau_R = 0, \quad \tilde{\tau}_R = 0, \quad \rho_R = 0, \quad \tilde{\rho}_R = 0, \quad \sigma_R = 0, \quad \tilde{\sigma}_R = 0, \quad \kappa_R = 0, \quad \tilde{\kappa}_R = 0 \)

As \( \gamma_B' \) and \( \tilde{\gamma}_B' \) are unaffected by the null rotation, they are invariant under such a transformation; that is, \( \gamma_B' = \gamma_B' \) and \( \tilde{\gamma}_B' = \tilde{\gamma}_B' \). Furthermore, their vanishing or non-vanishing affects the transformation rules for the remaining first order invariants, and hence indicates possible subcases. In the present work, we examine a simple subcase where

\[
\alpha_B = \alpha_B' = \tilde{\alpha}_B = \tilde{\alpha}_B' = \gamma_B' = \tilde{\gamma}_B' = 0
\]

in order to present a complete application of the equivalence algorithm.

It is worthwhile to outline the general case where \( \gamma_B' \) and \( \tilde{\gamma}_B' \) do not vanish: If these invariants are non-zero, one may always fix the null rotation parameter \( z_3 \) to fix either \( \beta_R \) or \( \tilde{\beta}_R \) to zero, and the null rotation parameter \( z_4 \) to fix either \( \beta_R \) or \( \tilde{\beta}_R \) to zero. Thus we may always fix all isotropy at first order, and the equivalence algorithm may be focused to consider the number of functionally independent invariants at the remaining iterations. Potentially, if \( \gamma_B' \) and \( \tilde{\gamma}_B' \) are constant, the equivalence algorithm will require at most seven iterations, \( q = 7 \), in order to fully classify the metrics. If these two invariants are non-constant, in general the algorithm will end on the third iteration, \( q = 3 \). Although it is possible that in special subcases \( 4 < q \leq 7 \), conditions for such metrics would be severe.

Returning to the simpler case, we set \( \alpha_B = \alpha_B' = \tilde{\alpha}_B = \tilde{\alpha}_B' = \gamma_B' = \tilde{\gamma}_B' = 0 \) to produce the following conditions on the metric functions:

\[
\begin{align*}
\alpha_1 &= \alpha_2 = 0, \\
\beta_1 &= \beta_2 = 0
\end{align*}
\]

Thus, the components of the Riemann tensor \( R^{\alpha\beta}_{\gamma\delta} \) are constant, implying that \( Z_U = W_{,u} = 0 \), and so the metric function \( A_1 \) vanishes. The constancy of the curvature component \( \Psi \) requires that that \( C_0 \) satisfies a complicated partial differential equation.

Simplifying the above expressions for our boosted and rotated spin coefficients, we obtain two non-zero first order invariants:

\[ \{ \gamma_B, \tilde{\gamma}_B' \} \]

However, since \( \beta_B' = \alpha_B' = \tilde{\alpha}_B' = \tilde{\beta}_B' = 0 \), our new first order invariants are invariant under null rotations, and so we cannot fix all of our isotropy after first order, and the dimension of the isotropy group after first order \( \text{dim}(I) = 2 \). Therefore, we must proceed to second order.

After taking the second order covariant derivative of the Riemann tensor, we may simplify the components of this rank six tensor to produce the following set of second order curvature invariants

\[ \{ \gamma_B, \tilde{\gamma}_B, D\gamma_B, \delta\gamma_B, D\tilde{\gamma}_B, \delta\tilde{\gamma}_B, \Delta\gamma_B, \Delta\tilde{\gamma}_B, D'\gamma_B, D'\tilde{\gamma}_B \} \]
Since we have that $\gamma_B$ and $\tilde{\gamma}_B$ are functions of $U$ alone, all of their derivatives taken with respect to $u$, $v$, and $V$ vanish. Therefore, our new frame derivatives simplify to $D\gamma_B = \delta \gamma_B = D\tilde{\gamma}_B = \delta \tilde{\gamma}_B = \Delta \gamma_B = \Delta \tilde{\gamma}_B = 0$, and $D'\gamma_B = \partial_U \gamma_B, D'\tilde{\gamma}_B = \partial_U \tilde{\gamma}_B$. Now, since $\gamma_B$ and $\tilde{\gamma}_B$ are invariants, they are expressions not involving $z_3$ or $z_4$; that is,

$$D'\gamma_B = D'\gamma_B = \partial_U [\gamma_B],$$

$$D'\tilde{\gamma}_B = D'\tilde{\gamma}_B = \partial_U [\tilde{\gamma}_B].$$

Therefore, we cannot manipulate these equations to find conditions on $z_3$ and $z_4$. Thus, we still cannot fix any isotropy, and the dimension of the isotropy group after second order remains $\dim(I) = 2$. After the second iteration the algorithm terminates, as $t_1 = t_2 = 1$ and $\dim H_1 = \dim H_2 = 2$. The resulting list of invariants up to second order allows one to completely classify these spaces, and no further iterations of the algorithm will yield new information.

6 Discussion

We have investigated the mathematical properties of a class of four-dimensional neutral signature metrics, with vanishing scalar curvature invariants (VSI). In this study a distinct collection of metrics were found that satisfy the VSI-property and are distinct from the Kundt class. To discuss the difference in the neutral Ricci-flat Walker metrics with vanishing scalar curvature invariants, we have studied two distinct subcases: one which is generally Walker but not Kundt, and a second that is always Kundt.

By giving conditions for the existence of a null geodesic, expansion-free, shear-free, and vorticity-free vector for Walker metrics we were able to compare the two examples. Then, using the Lie algebra classification provided in [6] and [11], we explicitly identified the geometrically special vectors that arise from the holonomy algebra in each example. This classification is well-suited for determining the existence of invariant null distributions, recurrent vectors and covariantly constant null vectors; however it is not fine enough to determine the equivalence of metrics. As an example it is clear that the two metrics are inequivalent as they have distinct two-dimensional Lie algebras, yet both metrics contain a subcase for which these Lie algebras become one-dimensional. This is notable as all one-dimensional Lie algebras are equivalent to the Lie algebra for a metric admitting two covariantly constant null vectors, implying that this metric is doubly Kundt.

A natural question is to ask when, if at all, are the subcases of the two metrics equivalent. This question can only be resolved by implementing the equivalence algorithm for neutral metrics, which is a non-trivial task. We have provided a simple example of the equivalence algorithm applied to a subcase of the Kundt-Walker metric, which parallels the plane-wave spacetimes in the Lorentzian case. We have shown that neutral signature "plane waves" require the same number of covariant derivatives as their Lorentzian counterparts. It is unknown whether this holds for neutral-signature metrics in general, due to the difference in the group of frame transformations, it is possible that the neutral signature metric require a higher number of covariant derivatives to complete the equivalence algorithm. In the context of the Ricci-flat Walker metrics this is
a particularly relevant question as one cannot simply compare scalar curvature invariants to determine inequivalence. \[8\]

### 7 Appendix: Transformation Rules for Spin Coefficients

Consider the boosts in the two null planes, given by the following transformation,

\[
\{n_a, \ell_a, \bar{m}_a, m_a\} \rightarrow \{An_a, A^{-1}\ell_a, B\bar{m}_a, B^{-1}m_a\}
\] (28)

the spin-coefficients transform as:

\[
\begin{align*}
\kappa_B &= \frac{\kappa_B}{\lambda_B}, \quad \rho_B = \frac{\rho_B}{\lambda_B}, \quad \sigma_B = \frac{\sigma_B}{\lambda_B}, \quad \tau_B = \frac{\tau_B}{\lambda_B}, \\
\tau_B &= Br', \quad \sigma_B = AB\sigma', \quad \rho_B = Ap', \quad \kappa_B = A^2B\kappa', \\
\gamma_B' &= \frac{1}{2} \left[\frac{D(A) + A\gamma' - A\gamma'}{\lambda_A} + \frac{D(B) + B\gamma' - B\gamma'}{\lambda_B}\right], \\
\beta_B' &= -\frac{1}{2} \left[\frac{D(A) - A\beta' - A\beta'}{\lambda_A} + \frac{D(B) - B\beta' - B\beta'}{\lambda_B}\right], \\
\alpha_B' &= -\frac{1}{2} \left[\frac{D(A) - A\alpha' - A\alpha'}{\lambda_A} - \frac{D(B) + B\alpha' + B\alpha'}{\lambda_B}\right], \\
\epsilon_B' &= \frac{1}{2} \left[D'(A) + A\epsilon' + A\epsilon' + A(D'(B) + B\epsilon' - B\epsilon')\right], \\
\end{align*}
\]

To produce a rotation about the null vector $\ell^a$ we make the transformation:

\[
\begin{align*}
\{n_a, \ell_a, \bar{m}_a, m_a\} &\rightarrow \{n_a + \mu\bar{m}_a - \mu m_a - \mu\ell_a, \ell_a, \bar{m}_a - \mu\ell_a, m_a + \mu\ell_a\} \\
\end{align*}
\] (29)
while the spin-coefficients transform as

\[
\begin{align*}
\kappa_R &= \kappa, \quad \rho_R = \rho - \mu \kappa, \\
\sigma_R &= \sigma + \mu \kappa, \quad \tau_R = \tau + \mu \rho - \mu \sigma - \mu \mu \kappa, \\
\tau'_R &= \tau' + D(\mu) - 2 \mu \gamma' + \mu^2 \kappa, \\
\sigma'_R &= \sigma' - \Delta(\mu) + \mu D(\mu) - 2 \mu \beta' - 2 \mu^2 \gamma' - \mu^2 \rho + \mu^3 \kappa + \mu \tau', \\
\rho'_R &= \rho' - \delta(\mu) - D(\mu) \mu - 2 \mu \alpha' + 2 \mu \mu \gamma' - \mu^2 \sigma - \mu^2 \mu \kappa - \mu \mu \tau, \\
\kappa'_R &= \kappa' + \Delta(\mu) \mu - \mu \delta(\mu) + D'(\mu) - \mu \mu D(\mu) + 2 \mu \beta' - 2 \mu^2 \alpha' - 2 \mu \gamma' + 2 \mu^2 \mu \gamma' + \mu^2 \mu \rho - \mu^2 \mu \sigma - \mu^2 \mu \kappa - \mu \sigma' + \mu \rho' + \mu \mu \tau', \\
\gamma'_R &= \gamma' - \mu \kappa, \\
\beta'_R &= \beta' + \mu \rho - \mu^2 \kappa + \mu \gamma', \\
\alpha'_R &= \alpha' + \mu \sigma + \mu \mu \kappa - \mu \mu \tau, \\
\epsilon'_R &= \epsilon' - \mu \gamma' + \mu \sigma' + \mu \mu \kappa - \mu \mu \rho - \mu \mu \tau.
\end{align*}
\]

Noting that the square of this transformation is identity, we may summarize the effect on the spin-coefficients as:

\[
\begin{align*}
\tilde{\tau}' &= \tilde{\tau}' - D(\tilde{\mu}) + 2 \mu \tilde{\gamma}' + \mu^2 \tilde{\kappa}, \\
\tilde{\rho}' &= \tilde{\rho}' + \Delta(\tilde{\mu}) - \mu D(\tilde{\mu}) + 2 \mu \tilde{\alpha}' + 2 \mu \mu \tilde{\gamma}' - \mu^2 \tilde{\sigma} + \mu \mu^2 \tilde{\kappa} + \mu \tilde{\tau}, \\
\tilde{\sigma}' &= \tilde{\sigma}' + \delta(\tilde{\mu}) + \mu D(\tilde{\mu}) + 2 \mu \tilde{\beta}' - 2 \mu^2 \tilde{\gamma}' - \mu^2 \tilde{\rho} - \mu^3 \tilde{\kappa} - \mu \mu \tilde{\tau}, \\
\tilde{\kappa}' &= \tilde{\kappa}' - \Delta(\tilde{\mu}) \tilde{\mu} + \mu \delta(\tilde{\mu}) - D'(\tilde{\mu}) - \mu \mu D(\tilde{\mu}) - 2 \mu \tilde{\alpha}' + 2 \mu \mu \tilde{\beta}' + 2 \mu \tilde{\gamma}' - 2 \mu \mu^2 \tilde{\gamma}' + \mu \mu^2 \tilde{\kappa} + \mu \mu \tilde{\tau}, \\
\tilde{\gamma}' &= \tilde{\gamma}' + \tilde{\mu} \tilde{\kappa}, \\
\tilde{\beta}' &= \tilde{\beta}' - \tilde{\mu} \tilde{\rho} - \mu^2 \tilde{\kappa} - \mu \tilde{\gamma}', \\
\tilde{\alpha}' &= \tilde{\alpha}' - \tilde{\mu} \tilde{\sigma} + \mu \mu \tilde{\kappa} + \mu \tilde{\gamma}', \\
\tilde{\epsilon}' &= \tilde{\epsilon}' - \mu \tilde{\mu} \tilde{\gamma}' + \mu \tilde{\beta}' - \tilde{\mu} \tilde{\alpha}' + \mu \tilde{\mu} \tilde{\rho} - \mu^2 \tilde{\kappa} - \mu \tilde{\mu} \tilde{\tau}.
\end{align*}
\]

To generate a rotation about \( n_a \), we may apply the prime operation \( \mathfrak{g} \) to the above spin-coefficients. Notice that \( \mu' = -\mu \) and \( \tilde{\mu}' = -\tilde{\mu} \), this is reflected in the resulting frame transformation on \( M \):

\[
\{ \ell_a, n_a, -a_m, -\tilde{m}_a \} \rightarrow \{ \ell_a - \tilde{\mu} m_a + \mu \tilde{m}_a - \mu \mu \tilde{m}_a, n_a, -a_m - \mu m_a, -\tilde{m}_a + \mu \tilde{m}_a \}
\]

to determine the effect of a null rotation about \( n \) on the spin-coefficients merely prime the above quantities.

There are twenty-four discrete transformations that will be important, although it is best seen on the level of vectors on \( M \) as the interchange of the order of the four null vectors. As an example, consider the following transformation which is relevant for the subcase of Ricci-flat VSI Walker metrics we have been studying:

\[
\ell^{a} = m^{a}, \quad m^{a} = \ell^{a}, \quad n^{a} = -\tilde{m}^{a}, \quad \tilde{m}^{a} = -n^{a}.
\]

Noting that the square of this transformation is identity, we may summarize the effect on the spin coefficients as:
\( \epsilon^x = \alpha', \alpha^x = \epsilon', \beta^x = -\gamma', \gamma^x = -\beta', \)
\( \kappa^x = -\sigma, \rho^x = \tau, \tau^x = \rho', \sigma^x = -\kappa', \)
\( \tilde{\epsilon}^x = -\tilde{\beta}', \tilde{\beta}^x = -\tilde{\epsilon}', \tilde{\alpha}^x = \tilde{\gamma}', \tilde{\gamma}^x = \tilde{\alpha}', \)
\( \tilde{\kappa}^x = \tilde{\sigma}', \tilde{\sigma}^x = \tilde{\kappa}', \tilde{\rho}^x = -\tilde{\tau}', \tilde{\tau}^x = -\tilde{\rho}'. \)

Although the priming operation leaves the formula unchanged for this example, this may not be the case with other re-orderings of the coframe.

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**References**

[1] A. Alcolado, A. MacDougall, A. Coley and S. Hervik, J. Geom. Phys. 62 594 [arXiv:1104.3799].

[2] A. Coley, S. Hervik and N. Pelavas, 2009, Class. Quant. Grav. 26, 025013 [arXiv:0901.0791] & 2006, ibid., 23, 3053 [arXiv:gr-qc/0509113]

[3] V. Pravda, A. Pravdova, A. Coley and R. Milson, 2002 Class. Quant. Grav. 19, 6213 [gr-qc/0209024]; A. Coley, A. Fuster, S. Hervik and N. Pelavas, 2006, Class. Quant. Grav. 23, 7431; A. Coley, 2008, Class. Quant. Grav. 25, 033001 [arXiv:0710.1598].

[4] S. Hervik and A. Coley, 2011, Class. Quant. Grav. 28, 015008 [arXiv:1008.2838] & [arXiv:1008.3021]; see also S Hervik and A. Coley, 2010, Class. Quant. Grav. 27, 095014 [arXiv:1002.0505].

[5] A. G. Walker, 1949, Quart. J. Math. (Oxford), 20, 135 & 1950, ibid., 1, 69.

[6] R. Ghanam and G. Thompson, 2001, J. Math. Phys. 42, 2266.

[7] S. Hervik, 2012, Class. Quant. Grav., 29 095011.

[8] D. McNutt, equivalence problem for neutral signature metrics.

[9] P. Law, Journal of Geometry and Physics, 59 1087 (2009).

[10] G. S. Hall, 'Symmetries and Curvature Structure in General Relativity' World Scientific Publishing Co. Pte. Ltd (2004).

[11] Z. Wang, G.S. Hall, Journal of Geometry and Physics 66 37-49 (2013).

[12] N. Musoke, 2013, Honours Thesis, Dalhousie University.

[13] D. Brooks, 2013, Honours Thesis, Dalhousie University.
[14] J. Davidov, G. Grantcharov, O. Mushkarov, 2008, *Geometry of neutral metrics in dimension four* [arXiv:0804.2132v1]

[15] A. Coley, S. Hervik, D. McNutt, N. Musoke and D. Brooks, 2013, preprint