Classical and Quantum Dynamics of a Periodically Driven Particle in a Triangular Well

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Abstract

We investigate the correspondence between classical and quantum mechanics for periodically time dependent Hamiltonian systems, using the example of a periodically forced particle in a one-dimensional triangular well potential. In particular, we consider quantum mechanical Floquet states associated with resonances in the classical phase space. When the classical motion exhibits subharmonic resonances, the corresponding Floquet states maintain the driving field’s periodicity through dynamical tunneling. This principle applies both to Floquet states associated with classical invariant vortex tubes surrounding stable, elliptic periodic orbits and to Floquet states that are associated with unstable, hyperbolic periodic orbits. The triangular well model also poses a yet unsolved mathematical problem, related to perturbation theory for systems with a dense pure point spectrum. The present approximate analytical and numerical results indicate that quantum tunneling between different resonance zones is of crucial importance for the question whether the driven triangular well has a dense point or an absolutely continuous quasienergy spectrum, or whether there is a transition from the one to the other.

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I. INTRODUCTION

Ever since Bohr’s formulation of the correspondence principle [1], the relation between classical and quantum mechanics has been a subject of great conceptual interest. The simplest type of time-independent, bounded classical Hamiltonian system is integrable, i.e., if there are as many \((n)\) independent constants of motion as degrees of freedom. The phase space of such a system is completely stratified into invariant \(n\)-tori [2] and the quantum mechanical energy eigenfunctions and eigenvalues can be determined approximately with the help of the Einstein-Brillouin-Keller (EBK) quantization rules [3–6]. If, on the other hand, the classical system is chaotic, then semiclassical quantization relies on classical periodic orbits [7–11]. “Typical” classical Hamiltonian systems, however, are neither completely regular nor completely chaotic, but exhibit an intricate coexistence of regular and apparently stochastic motion [12]. An important example of this type is the three-body Coulomb system. The application of both torus quantization and a periodic orbit cycle expansion to this system has resulted in a semiclassical understanding of the spectrum of the Helium atom [13].

Much insight into the classical-quantum correspondence has been gained by the study of simple model systems. A popular model, the periodically kicked rotor, has led to the discovery of quantum mechanical suppression of classical diffusion [14], a phenomenon that could subsequently be explained by a mechanism closely related to Anderson localization in disordered one-dimensional crystal lattices [15,16]. For a general quantum system defined by a periodically time dependent Hamiltonian operator \(H(t) = H(t + T)\) the role of the stationary states is taken over by the Floquet states [17,18]. If the operator

\[
\mathcal{H} := H(t) - i\hbar \partial_t , \tag{1}
\]

acting on an extended Hilbert space of \(T\)-periodic functions [19], has \(T\)-periodic eigenfunctions, i.e., if the eigenvalue equation

\[
[H(t) - i\hbar \partial_t] u_\alpha(t) = \varepsilon_\alpha u_\alpha(t) \tag{2}
\]
can be solved with periodic boundary conditions in time, $u_{\alpha}(t) = u_{\alpha}(t+T)$, then the Floquet states

$$\psi_{\alpha}(t) := u_{\alpha}(t) \exp(-i\varepsilon_{\alpha}t/\hbar)$$  \hspace{1cm} (3)

are solutions to the time-dependent Schrödinger equation. The eigenvalue equation (2) now plays a role analogous to that of the stationary Schrödinger equation for time-independent systems, and the objective of a semiclassical theory is the approximate computation of the Floquet eigenfunctions $u_{\alpha}(t)$ and the quasienergies $\varepsilon_{\alpha}$, starting again from invariant objects in the classical phase space.

However, periodically time dependent quantum systems pose some mathematical difficulties that are rarely met in the time-independent case. Consider a Hamiltonian operator of the form $H(t) = H_0 + \beta H_1(t)$, where only $H_1(t) = H_1(t+T)$ carries the time dependence, and $\beta$ is a dimensionless coupling constant. Assume further that $H_0$ possesses a discrete spectrum of energy eigenvalues $E_n$ ($n = 1, 2, \ldots, \infty$), with corresponding eigenfunctions $\phi_n$. Then for $\beta = 0$ the Floquet states can be written as

$$\psi_{(n,m)}(t) = (\phi_n e^{im\omega t}) \exp[-i(E_n + m\hbar\omega)t/\hbar]$$  \hspace{1cm} (4)

with $\omega = 2\pi/T$. If $m$ is an integer number, the Floquet function $u_{(n,m)}(t) = \phi_n e^{im\omega t}$ is $T$-periodic, as required; the corresponding quasienergies are $\varepsilon_{(n,m)} = E_n + m\hbar\omega$. The index $\alpha$ in (2) thus becomes a double index, $\alpha = (n,m)$, and the quasienergy spectrum is given by the energy eigenvalues modulo $\hbar\omega$. Therefore it can be divided into Brillouin zones of width $\hbar\omega$.

Apart from special cases, this spectrum will be dense on the real axis. But what happens when $\beta > 0$? Rigorous perturbation theory for such systems with a dense point spectrum is anything but trivial [20,21]. The conditions for the perturbed system to retain a pure point spectrum constitute a crucial unsolved problem. Partial answers to this problem of “quantum stability” [21] have been obtained only recently: it has been shown by Howland [22] that even for arbitrarily large $\beta$ the quasienergy spectrum remains a dense pure point spectrum, provided the gaps $E_{n+1} - E_n$ between the energy eigenvalues of $H_0$ grow sufficiently
rapidly with \( n \), and the perturbation \( H_1(t) \) is bounded. This result applies, for instance, to a periodically forced particle in a one-dimensional box. The condition on the boundedness of the perturbation can be relaxed in the case of certain other periodically forced anharmonic oscillators \([23]\), but the condition on the growth of the energy gaps between the eigenvalues of \( H_0 \) remained essential also in subsequent studies \([24, 25]\).

A model that violates both propositions of Howland’s theorem is given by a periodically driven particle in a triangular well potential: the unperturbed system has a pure point spectrum, with the gaps between successive energy eigenvalues \( E_n \) decreasing asymptotically as \( n^{-1/3} \), and its dipole operator is unbounded. This model emerges, for example, in the description of far-infrared irradiated semiconductor heterojunctions \([26]\), or of microwave-driven charge carriers in superfluid helium \([27]\).

This model is the subject of the present work. Although fairly simple, it provides interesting insights into the classical-quantum correspondence for periodically time-dependent systems with a mixed classical phase space. As we will demonstrate, many features of the quantum system have a very close link to its classical counterpart, both in the classically regular (elliptic) and chaotic (hyperbolic) case. Nevertheless, quantum mechanical tunneling is of crucial importance to understand prominent features of both the quasienergy spectrum and the Floquet eigenfunctions.

The tunnel effect might also play a central role in the question whether there is a pure point spectrum or an absolutely continuous one. From a rigorous point of view this difficult question is still open, but it has been argued that the quasienergy spectrum of the periodically driven triangular well exhibits a transition from a point spectrum to a continuous spectrum at a nonzero, finite value of the driving amplitude \([28, 29]\). Although there are well studied examples of perturbations through which a discrete spectrum becomes continuous \([30]\), a spectral transition in a system like the driven triangular well would be quite unusual, and therefore interesting. If there is such a transition, there must be a mechanism producing it; however, what could that be?

Thus, the aim of this work is twofold. First, we wish to provide instructive examples
of classical-quantum correspondence and non-correspondence. Due to its simplicity, the triangular well model might even be useful for courses in elementary quantum mechanics, despite the open mathematical problems. Second, we wish to stimulate further rigorous research on the nature of the quasienergy spectrum of periodically time dependent quantum systems. Previous work [20–25] has addressed this problem in a quite formal way. However, a physically oriented starting point, such as a semiclassical analysis, might also prove fruitful, and provide complementary information.

We proceed as follows. Section 2 introduces the classical model and states the results of a standard resonance analysis. Unlike the periodically forced particle in a box considered by Reichl and Lin [31,32], the triangular well model has the property that even for arbitrarily small driving amplitude there exist arbitrarily large primary resonances in the classical phase space. Therefore, even for small amplitudes an infinite number of quantum mechanical Floquet states will be influenced by these resonances. Section 3 contains a semiclassically-motivated description of near-resonant Floquet states, followed in Section 4 by a more refined analysis, which is an extension of previous work by Berman and Zaslavsky [33]. Section 5 compares the analytically calculated quasienergy spectrum with numerical data. The central Section 6 explores the influence of classical phase space structures on the Floquet states. We do not focus on questions pertaining to the resonance overlap [31,32,34,35], but rather try to stretch the single-resonance picture as far as possible. An interesting result that will be obtained in this way is the existence of a closed link between states associated with regular classical dynamics and others linked with apparently chaotic parts of the classical phase space: Floquet states associated with unstable, hyperbolic periodic orbits appear as certain excitations of states most closely associated with invariant manifolds surrounding stable elliptic periodic orbits. The final discussion then ties together some loose ends, and returns to the question whether or not there could be a spectral transition.
II. THE CLASSICAL TRIANGULAR WELL

A classical particle of mass $m$ that is confined to a one-dimensional triangular well and driven by a monochromatic force of amplitude $F$ and frequency $\omega$ is described by the Hamiltonian

$$\tilde{H}(P,X,t) = \frac{P^2}{2m} + V_a(X) + FX \cos(\omega t)$$  \hspace{1cm} (5)

with $V_a(X) = aX$ for $X \geq 0$ and $V_a(X) = \infty$ for $X < 0$. Introducing dimensionless variables,

$$x = \frac{m\omega^2 X}{a}$$
$$p = \frac{\omega P}{a}$$
$$\tau = \omega t$$  \hspace{1cm} (6)

this Hamiltonian can be written in the form $\tilde{H}(P,X,t) = a^2/(m\omega^2)H(p,x,\tau)$, where the scaled Hamiltonian

$$H(p,x,\tau) = \frac{p^2}{2} + V_1(x) + \beta x \cos(\tau) \equiv H_0(p,x) + \beta x \cos(\tau)$$  \hspace{1cm} (7)

depends only on the scaled amplitude $\beta = F/a$. A detailed analysis of the “kicked” version of this system, where the sinusoidal time dependence is replaced by a sequence of $\delta$ kicks with alternating signs, has been presented by Shimshoni and Smilansky [27].

To express $H(p,x,\tau)$ in terms of the action-angle variables of the undriven system defined by $H_0(p,x)$, we first exploit the relation between the action $I$ and the energy $E$:

$$I = \frac{1}{\pi} \int_0^E \sqrt{2(E-x)} \, dx = \frac{1}{3\pi} (2E)^{3/2}.$$  \hspace{1cm} (8)

Hence,

$$H_0(I) = (3\pi I)^{2/3}/2.$$  \hspace{1cm} (9)

The oscillation frequency $\Omega(I)$ of the undriven particle in the triangular well is then found from Hamilton’s equation

6
\[ \Omega(I) = \frac{\partial H_0}{\partial I} = \left( \frac{\pi^2}{3I} \right)^{1/3}, \]  

and the angle variable is \( \theta = \Omega(I) \tau \), restricted to the interval from 0 to \( 2\pi \).

Imposing the initial condition \( x(\tau = 0) = 0 \), the unperturbed trajectories can be written as

\[ x(\theta) = \frac{3I^{2/3}}{(3\pi)^{1/3}} \left( \theta - \frac{\theta^2}{2\pi} \right) = \frac{I^{2/3}}{(3\pi)^{1/3}} \left( \pi - \sum_{n=1}^{\infty} \frac{6}{\pi n^2} \cos(n\theta) \right). \]  

Thus, the full Hamiltonian becomes

\[ H(I, \theta, \tau) = \frac{(3\pi I)^{2/3}}{2} + \frac{\beta}{3}(3\pi I)^{2/3} \cos(\tau) - \frac{\beta}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} [\cos(\tau + n\theta) + \cos(\tau - n\theta)]. \]

Primary resonances occur when the particle’s action is such that the cycle time of unperturbed motion coincides with an integer multiple, \( N \), of the period of the driving force \([12, 36]\). Then the argument \( \tau - n\theta \) in the above expansion becomes stationary for \( n = N \). The resonant action \( I_N \) can thus be found from the equation \( \Omega(I_N) = 1/N \). By (10),

\[ I_N = \frac{\pi^2 N^3}{3}. \]

Approximating \( H_0(I) \) quadratically around \( I_N \) and keeping only the resonant term of the perturbation (taken at \( I = I_N \)), one obtains the familiar pendulum Hamiltonian \([12, 36]\)

\[ H(I, \theta, \tau) \approx H_0(I_N) + \Omega(I_N)(I - I_N) + \frac{1}{2M} (I - I_N)^2 - \beta \cos(\tau - N\theta), \]

where

\[ M^{-1} = \frac{d\Omega(I)}{dI} \bigg|_{I_N} = -\frac{1}{\pi^2 N^4}. \]

This Hamiltonian describes the regular, resonant motion close to \( I_N \). Within the pendulum approximation, the half-width of the resonances is
\[ \Delta I_N = 2\pi N^2 \sqrt{\beta} . \] (16)

According to the Chirikov criterion \[ \text{[12,36]} \], the \( N \)-th resonance overlaps with the \( (N+1) \)-th when \( I_{N+1} - I_N = \Delta I_{N+1} + \Delta I_N \). The corresponding critical values of the driving strength are

\[ \beta_N^{(c)} = \frac{\pi^2}{16} \left( \frac{N^2 + N + 1/3}{N^2 + N + 1/2} \right)^2 . \] (17)

For example, \( \beta_1^{(c)} = 0.537 \) and \( \beta_2^{(c)} = 0.586 \).

Summing up, the model of the classical driven particle in a triangular well has the following three essential features: (i) there are an infinite number of primary resonances, with the \( N \)-th resonant action \( I_N \) being proportional to \( N^3 \); (ii) the half-widths \( \Delta I_N \) grow quadratically with the order \( N \) of resonance; (iii) within the pendulum approximation, all resonance overlaps occur in the interval \( 0.537 < \beta < 0.617 \) of the control parameter \( \beta \).

Of course, the pendulum approximation can only give a coarse description of the actual dynamics, and is restricted to moderate values of \( \beta \). The larger \( \beta \), the more important are the stochastic layers surrounding the resonances \[ \text{[12]} \]. When \( \beta \) is increased beyond the perturbative regime, the widths of the zones of resonant, mainly regular motion in phase space do no longer grow proportional to \( \sqrt{\beta} \), as suggested by \[ \text{[16]} \], but they actually shrink. The \( \sqrt{\beta} \) growth is overcompensated by the growth of the stochastic layers. This fact is illustrated by the following two figures. Fig. 1 shows a Poincaré surface of section for the driven particle in a triangular well with \( \beta = 0.5 \), taken for \( \tau = 3\pi/2 \mod 2\pi \). Apart from small secondary resonances, the most prominent features are the island of regular motion for \( N = 1 \) in the lower left corner, and the two islands for \( N = 2 \). The stochastic layers, which for smaller \( \beta \) merely cover limited areas bordering the individual resonance zones, have already merged into a connected sea of stochastic motion. The locations of the resonant islands agree reasonably well with the predictions of \[ \text{[13]} \], but for \( N = 2 \) the half-width is clearly smaller than \( \Delta I_2 \), as given by \[ \text{[16]} \]. As shown in Fig. 2, for \( \beta = 1.0 \) the islands are squeezed to a small fraction of their previous size. But still, there exists regular motion “inside” the resonant islands, although \( \beta \) is significantly larger than \( \beta_\infty^{(c)} \).
The scaled classical Hamiltonian \( \mathcal{H} \) depends only on a single dimensionless parameter. A second one appears in quantum mechanics: Planck’s constant \( \hbar \) is scaled to 

\[
\hbar_{\text{eff}} := m\omega^3 \hbar/a^2
\]

as a consequence of the transformations (14). The model of the periodically driven triangular well, therefore, is of paradigmatic simplicity for a study of the classical-quantum correspondence: one parameter, \( \beta \), controls the degree of nonintegrability in the classical system, the other, \( \hbar_{\text{eff}} \), controls the scale at which its quantum mechanical counterpart can resolve classical phase space structures.

Since \( I - I_N \) and \( \theta \) constitute a canonically conjugate pair, the approximate pendulum Hamiltonian (14) can be quantized by the replacement

\[
I - I_N \rightarrow \frac{\hbar_{\text{eff}}}{i} \frac{\partial}{\partial \theta}
\]

This is essentially a semiclassical argument, since it ignores the fact that the canonical transformation \((p, x) \rightarrow (I, \theta)\) does not commute with the quantization operation. The resulting Schrödinger equation

\[
i\hbar_{\text{eff}} \frac{\partial}{\partial \tau} \psi(\theta, \tau) = \left[ H_0(I_N) + \Omega(I_N) \frac{\hbar_{\text{eff}}}{i} \frac{\partial}{\partial \theta} - \frac{\hbar_{\text{eff}}^2}{2M} \frac{\partial^2}{\partial \theta^2} - \beta \cos(N\theta - \tau) \right] \psi(\theta, \tau)
\]

can be solved with the help of the ansatz

\[
\psi(\theta, \tau) \equiv \chi(z) \exp(-iW\tau/\hbar_{\text{eff}}),
\]

where the new variable \( z \) is given by

\[
z = (N\theta - \tau)/2.
\]

Inserting (21) into (20) and using the resonance condition \( \Omega(I_N) = 1/N \), one obtains the Mathieu equation [37]
\[
\left( \frac{d^2}{dz^2} + a - 2q \cos(2z) \right) \chi(z) = 0 \tag{23}
\]

with parameters

\[
a = \frac{8|M|}{\hbar_{\text{eff}}^2 N^2} (H_0(I_N) - W) \tag{24}
\]

\[
q = \frac{4|M|}{\hbar_{\text{eff}}^2 N^2} \beta. \tag{25}
\]

The general solution to this equation has the Floquet form \( \chi(z) = P_\nu(z) e^{i\nu z} \), with a characteristic exponent \( \nu \) and a \( \pi \)-periodic function \( P_\nu(z) = P_\nu(z + \pi) \). Thus, the corresponding wave functions \( \psi(\theta, \tau) \) have the form

\[
\psi(\theta, \tau) = P_\nu \left( \frac{N\theta - \tau}{2} \right) \exp \left( i\nu \frac{N\theta - \tau}{2} - i\frac{W\tau}{\hbar_{\text{eff}}} \right). \tag{26}
\]

Since \( \theta \) is an angle variable, \( \psi \) must be \( 2\pi \)-periodic in \( \theta \). This requirement now restricts the possible values of \( \nu \) to the discrete set

\[
\nu = \nu(j) = \frac{2j}{N}, \quad j = 0, 1, 2, \ldots, N - 1. \tag{27}
\]

According to (25), the parameter \( q \) entering the Mathieu equation is fixed by the choice of the amplitude \( \beta \). The allowed values \( \nu(j) \) can occur as characteristic exponents of solutions to the Mathieu equation only for certain values (labelled by the integer \( k \)) \( a_k(\nu(j), q) \) of the other parameter \( a \). These values \( a_k(\nu(j), q) \) finally determine, by (24), the allowed \( W_{j,k} \).

The wave functions (26) are now fully specified. Since they already are of the required form \( \psi_{j,k}(\theta, \tau) = u_{j,k}(\theta, \tau) \exp(-i\varepsilon_{j,k}\tau/\hbar_{\text{eff}}) \), with \( u_{j,k}(\theta, \tau) = u_{j,k}(\theta, \tau + 2\pi) \), the quasienergies for the approximate equation (20) have been found:

\[
\varepsilon_{j,k} = \left[ \hbar_{\text{eff}} \frac{\nu(j)}{2} + W_{j,k} \right] \mod \hbar_{\text{eff}}
\]

\[
= \left[ H_0(I_N) - \frac{\hbar_{\text{eff}}^2 N^2}{8|M|} a_k(\nu(j), q) + \hbar_{\text{eff}} \frac{j}{N} \right] \mod \hbar_{\text{eff}}. \tag{28}
\]

The cases \( N = 1 \) and \( N = 2 \) are of particular interest. For \( N = 1 \) there is \( j = 0 \) only, and the \( a_k(0, q) \) coincide with the well known characteristic values \( \varepsilon_{1,k} \) that give rise
to $\pi$-periodic Mathieu functions. Those values associated with even Mathieu functions are customarily denoted \[37\] as $a_0(q), a_2(q), a_4(q), \ldots$, and those associated with odd functions as $b_2(q), b_4(q), b_6(q), \ldots$. With the definition

$$
\alpha_{0,k}(q) := \begin{cases} 
a_k(q), & k = 0, 2, 4, \ldots 
b_{k+1}(q), & k = 1, 3, 5, \ldots
\end{cases}
$$

(29)

the approximate quasienergies for an $N = 1$ resonance can be written as

$$
\varepsilon_k = \left\lceil H_0(I_N) - \frac{\hbar_{\text{eff}}^2}{8|\lambda|} \alpha_{0,k}(q) \mod \hbar_{\text{eff}} \right\rceil.
$$

(30)

For $N = 2$ there are two groups of states, labelled by $j = 0$ and $j = 1$. Those with $j = 1$ have to be constructed from the $2\pi$-periodic Mathieu functions. The required characteristic values $\alpha_{1,k}(q) \equiv a_k(1, q)$ are

$$
\alpha_{1,k}(q) := \begin{cases} 
b_{k+1}(q), & k = 0, 2, 4, \ldots 
a_k(q), & k = 1, 3, 5, \ldots
\end{cases}
$$

(31)

Thus, the quasienergies are given by

$$
\varepsilon_{j,k} = \left\lceil H_0(I_N) - \frac{\hbar_{\text{eff}}^2}{2|\lambda|} \alpha_{j,k}(q) + \hbar_{\text{eff}} \frac{j}{2} \right\rceil \mod \hbar_{\text{eff}}
$$

(32)

for $j = 0, 1$ and $k = 0, 1, 2, \ldots$

This spectrum has a simple interpretation \[38\]. The Mathieu equation \[23\] can be regarded as a stationary Schrödinger equation for a fictitious particle moving in a cosine lattice, and the solution for the $N$-th resonance requires periodic boundary conditions after $N$ cosine wells. Thus, $N = 1$ leads to a single-well potential, and $N = 2$ to a double well. The quasi-energies of the original problem correspond to the energies of the fictitious particle. The eigenstates in the single well are simply labelled by the quantum number $k$. The spectrum of the double well, however, consists of a sequence of doublets, each one associated with a symmetric and an antisymmetric eigenfunction. The absolute value of the difference $\varepsilon_{1,k} - \varepsilon_{0,k} - \hbar_{\text{eff}}/2 =: (\Delta \varepsilon)_k$ of eigenvalues within each doublet corresponds to the tunnel splitting. The well known asymptotic behavior of the characteristic values \[37\],
\[ b_{k+1}(q) - a_k(q) \sim 2^{4k+5} \left(\frac{2}{\pi}\right)^{k/2+3/4} \frac{q^{k/2+3/4}}{k!} \exp(-4\sqrt{q}) , \]  

shows that the tunnel splitting becomes exponentially small with \( \sqrt{q} \), i.e., with the square root of the driving amplitude \( \beta \) (see (25)). That is easy to understand: \( q \) determines the depth of the individual cosine wells; the larger \( q \), the weaker the tunneling between them. In particular, the ground state splitting, i.e., the (quasi-) energy difference between the two states most tightly bound by the double well potential, is

\[
(\Delta \varepsilon)_0 = \frac{\hbar^2_{\text{eff}}}{2|\mathcal{M}|} (b_1(q) - a_0(q)) 
\sim \frac{\hbar^2_{\text{eff}}}{\pi^2} \frac{2}{\sqrt{q}} q^{3/4} \exp(-4\sqrt{q}) \]  

with \( q = 16\pi^2 \beta / \hbar^2_{\text{eff}} \). It is quite remarkable that the approximate construction of near-resonant Floquet states for \( N > 1 \) accounts for the tunnel effect, even within the present semiclassical approach.

**IV. QUANTUM MECHANICAL APPROACH**

The considerations in the previous section, albeit instructive, have several flaws. Most of all, it is not easy to control the accuracy of the semiclassical replacement (19), and an expression like \( H_0(I_N) \) will, in general, not correspond directly to a quantum mechanical energy eigenvalue. It is, therefore, desirable to analyze the role of classical resonances in quantum systems in a strictly quantum mechanical way, without direct recourse to classical mechanics. In this section, we sketch such a quantum mechanical approach. It is basically a generalization of the previous work by Berman and Zaslavsky [33]. Although it also confirms the main conclusions of the semiclassical computation, its primary merit lies in the fact that it clearly shows the physical and mathematical difficulties that will be encountered in the attempt to improve the approximation.

We consider the Hamiltonian (7) and denote the unperturbed energy eigenstates and eigenvalues of the triangular well by \( |n\rangle \) and \( E_n \), respectively. We assume that the \( r\)-th
eigenstate is “resonant”, i.e., that the level spacing close to the $r$-th eigenstate is approximately equal to $\hbar_{\text{eff}}/N$ for some integer $N$:

$$E'_r \equiv \left. \frac{dE_n}{dn} \right|_{n=r} \approx \frac{\hbar_{\text{eff}}}{N}.$$  \hspace{1cm} (35)

This is the quantum mechanical equivalent of the classical resonance condition $\Omega(I_N) = 1/N$. Only for special choices of $\hbar_{\text{eff}}$ will it be possible to obtain an exact identity $E'_r = \hbar_{\text{eff}}/N$; in the general case we choose $r$ such that the absolute value of the difference

$$\delta := E'_r - \frac{\hbar_{\text{eff}}}{N}$$  \hspace{1cm} (36)

is minimized. At most, $\delta$ will be of the order of $E''_r$. We also stipulate that $r$ be so large that for levels close to $E_r$ the spacing between $E_n$ and $E_{n+N}$ varies only slowly on the scale of $\hbar_{\text{eff}}$, which implies the inequality

$$E''_r \ll E'_r.$$  \hspace{1cm} (37)

We are interested in a solution $|\psi(\tau)\rangle$ to the time-dependent Schrödinger equation that consists mainly of a superposition of near-resonant eigenstates. Thus, a reasonable ansatz is provided by

$$|\psi(\tau)\rangle = \sum_n c_n(\tau)|n\rangle \exp \left\{ -i \left( E_r + (n - r)\frac{\hbar_{\text{eff}}}{N} \right) \frac{\tau}{\hbar_{\text{eff}}} \right\},$$  \hspace{1cm} (38)

which immediately leads to

$$i\hbar_{\text{eff}} \dot{c}_n(\tau) = \left( E_n - E_r - \hbar_{\text{eff}} \frac{n - r}{N} \right) c_n(\tau) + \beta \cos(\tau) \sum_m \langle n|x|m\rangle c_m(\tau) \exp[i(n-m)\tau/N].$$  \hspace{1cm} (39)

Keeping in mind the inequality (37), we expand the eigenvalues quadratically around $E_r$,

$$E_n \approx E_r + (n - r)E'_r + \frac{1}{2}(n - r)^2E''_r.$$  \hspace{1cm} (40)

According to our propositions, the time-dependence of the coefficients $c_n(\tau)$ should be weak, compared to that of the exponentials $\exp(i\ell\tau/N)$ with a nonzero integer $\ell$. Correspondingly, we keep only the stationary terms $m = n \pm N$ on the r.h.s. of (39). Thus each
coefficient \( c_n(\tau) \) is coupled only to \( c_{n+\pm N}(\tau) \), \( c_{n\pm 2N}(\tau) \), and so forth. One obtains \( N \) disjoint sets of coefficients, which we denote by \( c_n^{(j)} \). Within each set, \( j = 0, 1, \ldots, N-1 \) is constant, and \( n \) is restricted to the values \( n = r + j + mN \), where \( m \) is an integer. In addition, we assume that the dipole matrix elements \( \beta \langle n|\alpha|n\pm N \rangle \) can be approximated by a constant \( V \), independent of \( n \). Using now the new index \( m \) to label the coefficients, one arrives at the \( N \) decoupled sets of equations:

\[
i \hbar_{\text{eff}} c_m^{(j)}(\tau) = \frac{1}{2} N^2 E''_r (m + j/N)^2 c_m^{(j)}(\tau) + N\delta(m + j/N) c_m^{(j)}(\tau) + \frac{V}{2} \left( c_{m+1}^{(j)}(\tau) + c_{m-1}^{(j)}(\tau) \right). \tag{41}
\]

To solve these equations, we set

\[
c_m^{(j)}(\tau) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi f_j(\varphi) e^{-i m\varphi} e^{-i W_j \tau/\hbar_{\text{eff}}} = \frac{1}{2N\pi} \int_0^{2N\pi} d\varphi g_j(\varphi) e^{-i (m+j/N)\varphi} e^{-i W_j \tau/\hbar_{\text{eff}}} , \tag{42}
\]

with \( W_j \) still to be determined. Assuming that \( f_j(\varphi) \) is \( 2\pi \)-periodic, the auxiliary function \( g_j(\varphi) \equiv f_j(\varphi) \exp(i j\varphi/N) \) is \( 2N\pi \)-periodic. The resulting equation for \( g_j(\varphi) \), i.e.,

\[
\left( -\frac{1}{2} N^2 E''_r \frac{d^2}{d\varphi^2} + \frac{N\delta}{i} \frac{d}{d\varphi} + V \cos(\varphi) - W_j \right) g_j(\varphi) = 0 , \tag{43}
\]

can then be transformed into a standard Mathieu equation; the ansatz

\[
g_j(\varphi) = \exp[-2i z\delta/(N E''_r)] \chi_j(z) \tag{44}
\]

with \( \varphi = 2z \) leads to

\[
\left( \frac{d^2}{dz^2} + a - 2q \cos(2z) \right) \chi_j(z) = 0 , \tag{45}
\]

with the parameters

\[
a = \frac{8W}{N^2 E''_r} + \left( \frac{2\delta}{NE''_r} \right)^2 \tag{46}
\]
\[
q = \frac{4V}{N^2 E''_r} . \tag{47}
\]
Starting from a general Floquet solution to the Mathieu equation (45),
\[ \chi_j(z) = P_{\nu(j)}(z) \exp(1 \nu(j)z) \],
the function \( g_j(\varphi) \) can now be expressed as
\[ g_j(\varphi) = P_{\nu(j)}(\varphi/2) \exp \left( \frac{i \nu(j) \varphi}{2} - i \frac{\varphi \delta}{NE''_r} \right). \]  
(48)

Since, on the other hand, \( g_j(\varphi) = f_j(\varphi) \exp(i j \varphi/N) \), one finds \( f_j(\varphi) = P_{\nu(j)}(\varphi/2) \), together with
\[ \nu(j) = \frac{2j}{N} + \frac{2\delta}{NE''_r} ; \quad j = 1, \ldots, N - 1. \]  
(49)

As in the previous section, these characteristic exponents admit only a discrete set of allowed values for the parameter \( a \); these values are again denoted by \( a_k(\nu(j), q) \). By (46), they lead to a discrete set of possible values \( W_{j,k} \) for the quantities \( W_j \). Now the expansion coefficients are completely specified:
\[ c^{(j)}_m(\tau) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi P_{\nu(j),k}(\varphi/2) \exp(-im\varphi - iW_{j,k}\tau/h_{eff}) \]
\[ = \chi^{(m)}_{j,k} \exp(-iW_{j,k}\tau/h_{eff}). \]  
(50)

Thus, the resulting approximate Floquet solutions \( |\psi_{j,k}(\tau)\rangle \) to the Schrödinger equation can finally be written as
\[ |\psi_{j,k}(\tau)\rangle = \sum_m \chi^{(m)}_{j,k} |r + j + mN\rangle e^{-im\tau} \exp \left\{ -i \left( E_r + W_{j,k} + h_{eff} \frac{j}{N} \right) \frac{\tau}{h_{eff}} \right\} \]
\[ \equiv |u_{j,k}(\tau)\rangle \exp(-i\varepsilon_{j,k}\tau/h_{eff}). \]  
(51)

Using (46), the quasienergies \( \varepsilon_{j,k} = E_r + W_{j,k} + h_{eff}j/N \) (mod \( h_{eff} \)) are
\[ \varepsilon_{j,k} = \left[ E_r + \frac{1}{3} \frac{N^2 E''_r a_k(\nu(j), q) - 4\delta^2}{2E''_r} + h_{eff} \frac{j}{N} \right] \text{ mod } h_{eff}. \]  
(52)

The structure of the near-resonant quasienergy spectrum obtained in this way is equal to the structure of (28), but the parameters differ. These differences are small in the regime where the approximations are valid. First, the exact energy eigenvalues for the triangular well potential are given by
\[ E_n = \left( \frac{h_{eff}^2}{2} \right)^{1/3} z_n, \]  
(53)
where the numbers $z_n$ ($n = 1, 2, 3, \ldots$) denote the negative zeros of the Airy function $Ai(z)$ [37]:

$$z_n = f \left( \frac{3\pi}{2} (n - \frac{1}{4}) \right)$$

(54)

with

$$f(z) \sim z^{2/3} \left( 1 + \frac{5}{48} z^{-2} - \frac{5}{36} z^{-4} + \ldots \right).$$

(55)

If this expansion is approximated by its leading term, and if the Maslov correction resulting from the right turning point of the motion in the triangular well potential is neglected, one has

$$z_n \approx \left( \frac{3\pi}{2} (n - \frac{1}{4}) \right)^{2/3} \approx (3\pi n/2)^{2/3},$$

(56)

and hence

$$E_n \approx \frac{1}{2} (3\pi h_{\text{eff}} n)^{2/3}.$$  

(57)

Since the quantum number $r$ of the resonant state is large by assumption, these “semiclassical” approximations are appropriate. Thus, within these approximations the eigenvalues $E_n$ agree with those obtained from the classical Hamiltonian $H_0(I)$ by the Bohr-Sommerfeld replacement $I \to h_{\text{eff}} n$, see (8). With $I_N \approx h_{\text{eff}} r$, the parameter $M^{-1}$ in (28) approximates $E''_r/h_{\text{eff}}^2$ in (52), see (13).

The dipole matrix elements of the triangular well can be computed analytically [39]:

$$\langle n|x|m \rangle = \begin{cases} (2h_{\text{eff}})^{2/3} z_n/3 & \text{, } n = m \\ -(2h_{\text{eff}})^{2/3}/(z_n - z_m)^2 & \text{, } n \neq m \end{cases}.$$  

(58)

Using the arithmetic mean

$$V = \frac{\beta}{2} (\langle r|x|r + N \rangle + \langle r|x|r - N \rangle)$$

(59)

for the coupling constant $V$ appearing in (41), and expanding $z_{r \pm N}$ to first order in $N/r$, one finds
Then the semiclassical and the quantum expression for the Mathieu parameter $q$, (25) and (47), coincide.

The characteristic exponents (27) are exactly equal to those in (49) only if $\delta$ vanishes, i.e., if $E'_r$ is exactly equal to $\hbar_{\text{eff}}/N$. In the general case, there can therefore be a slight difference between the values of the parameters $a_k(\nu(j), q)$ in both approximation schemes. However, since these parameters lie between the characteristic values $a_k(q)$ and $b_{k+1}(q)$, and since $b_{k+1}(q) - a_k(q)$ vanishes for large $q$ according to (33), this difference is visible only for small $q$. The term $-\delta^2/(2E''_r)$ appearing in (52) makes sure that the approximate quasienergies merge into the quadratically approximated eigenvalues (40) for vanishing $q$. Since $\delta$ is, at most, of the order of $E''_r$, one sees from (37) and (36) that this term is small compared to $\hbar_{\text{eff}}/N$. Thus, there is a one-to-one correspondence between the semiclassically calculated spectrum (28) and (52).

The approximations made in the quantum calculation have their obvious analogues in the previous section. There is one detail that deserves particular attention: it has tacitly been assumed that the Fourier coefficients $\chi_{j,k}^{(m)}$ of the approximate Floquet states (51) decay so rapidly that the non-existing “eigenstates below the ground state $|1\rangle$” carry zero weight.

The approximations leading to the Mathieu equation introduce an additional symmetry (the translational invariance of the cosine lattice) which renders the problem integrable. The larger $q$, the broader the distribution of the Fourier coefficients. When the distribution becomes so broad that the wave function can feel deviations from the lattice symmetry, the Mathieu approximation becomes insufficient.

**V. THE QUASIENERGY SPECTRUM: ANALYTICAL VERSUS NUMERICAL RESULTS**

To compute quasienergies and Floquet states for the periodically forced triangular well (7) numerically, we expand the wave functions in a truncated basis of unperturbed
eigenfunctions $\langle x|n \rangle = \varphi_n(x)$, with $n = 1, \ldots, n_{\text{max}}$. Within this basis set, we compute the monodromy operator (i.e., the “one-cycle evolution operator”) $U(2\pi, 0)$. Its eigenvalues are the Floquet multipliers $\exp(-i2\pi\varepsilon_n/h_{\text{eff}})$; the eigenvectors yield the Floquet functions $u_n(x, 0)$ at time $\tau = 0$. The numerical stability of the results obtained in this way has been checked by varying the truncation border $n_{\text{max}}$. However, even though the results appear stable, it is by no means certain whether such a numerical procedure can, for sufficiently large basis size, actually “converge” in the strict mathematical sense. This question is related to a deeper physical problem which we shall discuss in the final section; for the moment, we take the numerical results at face value.

The following two figures show parts of the quasienergy spectrum versus the driving amplitude $\beta$. Because of the mod $h_{\text{eff}}$ structure of the spectrum, each of the (infinitely many) Floquet states has a representative quasienergy within the first quasienergy Brillouin zone $-h_{\text{eff}}/2 \leq \varepsilon < +h_{\text{eff}}/2$. That brings about the necessity to select from all computed quasienergies only a small fraction for graphical display. The computed spectra contain a plethora of avoided crossings, and it is not always possible to select the quasienergies in such a way that the displayed eigenvalues appear to vary smoothly with $\beta$. Therefore, abruptly ending or beginning lines visible in the figures are not caused by some sort of numerical instability; such discontinuities are merely unavoidable in the selection procedure.

First, we choose $h_{\text{eff}} = 1.33325$. With this choice, one has $E'_{20} = h_{\text{eff}}/2$, so that the energy eigenstate $r = 20$ is exactly resonant for $N = 2$. Fig. 3 shows only quasienergies originating from states close to the resonant one. From (32) one expects two almost identical groups of eigenvalues, separated by $h_{\text{eff}}/2$. This expectation is fully confirmed by the numerical result; quasienergies belonging to one of the two groups are labelled by the Mathieu quantum numbers $k$ in the right margin of Fig. 3. In view of the corresponding classical mechanics, the fact that for $\beta = 0.5$ at least $2 \times 8$ states are accessible to the Mathieu approximation is quite remarkable: the upper right corner of the phase space portrait displayed in Fig. 1 shows the area $2\pi h_{\text{eff}}$, corresponding to the area “occupied” by a single Floquet state. Judging by the total areas of the islands of regular motion, one might expect
the classical $N = 2$ resonance to influence roughly $2 \times 4$ Floquet states, merely half the actual number. However, it must be borne in mind that the actual regime of influence of a classical resonance is not restricted to the regular islands. In their vicinity there are still lots of fragments of invariant manifolds, such as Cantori.

Fig. 4 displays quasienergies for $\hbar_{\text{eff}} = 0.66219$, so that $r = 40$ is exactly resonant for $N = 2$, and the number of states accessible to the Mathieu approximation is doubled. To check the accuracy of the previous analytical considerations, Fig. 5 shows the evaluation of (52) for the same parameters as used in Fig. 4, and $k = 0, \ldots, 9$. The semiclassical expression $M^{-1}$ here differs by merely $0.047\%$ from $E_{40}''/\hbar_{\text{eff}}^2$; the Mathieu parameter $q$ is approximately given by $16\pi^2\beta/\hbar_{\text{eff}}^2$. The characteristic values $a_k(\nu(j), q)$ have been computed by an algorithm suggested in [40], which has been found to be both efficient and accurate. The agreement between Figs. 4 and 5 is quite good, although one of the approximations used in the derivation of (52) is certainly critical: the assumption that the matrix elements $\langle n|x|n \pm 2 \rangle$ close to $n = r = 40$ can simply be approximated by a constant is investigated in Fig. 6. Although these matrix elements vary substantially in the range from $n = 30$ to 50, the comparison of Figs. 4 and 5 indicates that the somewhat crude approximation is still viable. This observation is rendered plausible by a perturbative argument: a better approximation to the “nearest neighbor couplings” in (11) would be given by a linear fit, $\beta \langle n|x|n \pm N \rangle \approx V + (n - r)V'$, leading to a perturbation

$$H_{\text{pert}} \approx \frac{V'}{2} \sum_n (n - r) \langle n|n + N| + |n\rangle\langle n - N| \rangle.$$  \hspace{1cm} (61)

The matrix elements $\langle u_{j,k}(\tau)|H_{\text{pert}}|u_{j,k}(\tau)\rangle$, taken with the approximate Floquet states (51), then contain only contributions proportional to either $e^{-i\tau}$ or to $e^{+i\tau}$. Therefore, to first order in $H_{\text{pert}}$ the approximate quasienergies (52) remain unchanged:

$$\int_0^{2\pi} d\tau \langle u_{j,k}(\tau)|H_{\text{pert}}|u_{j,k}(\tau)\rangle = 0.$$  \hspace{1cm} (62)

However, the significance of the Rayleigh-Schrödinger perturbation series for operators acting in the extended Hilbert space of $2\pi$-periodic functions [19] is questionable. This is again
a problem with perturbing of a dense point spectrum; the mod $\hbar_{eff}$ structure gives rise to an infinite number of small denominators in the expression for the second order perturbative contribution of $H_{pert}$.

VI. CLASSICAL-QUANTUM CORRESPONDENCE

Figs. 7 and 8 show contour plots of the probability density $|u_{j,k}(x,\tau)|^2$ of a (numerically computed) member of the ground state doublet ($k = 0$) for the $N = 2$ resonance and $\beta = 0.5$, with $\hbar_{eff} = 1.33325$ and 0.66219, respectively. In these and the following figures, the spatial coordinate $x$ ranges from $x_{min} = 0$ to $x_{max} = 25$, and the displayed interval of time is $8\pi$, corresponding to 4 cycles of the driving force.

The structure of these Floquet states, as well as that of the quasienergy spectrum of near-resonant states, can intuitively be understood if the quantum dynamics is connected in a direct way to the corresponding classical mechanics. Whereas quantum mechanical energy eigenstates of systems with a (partially) integrable classical counterpart are associated with invariant manifolds in the usual phase space spanned by momentum and position coordinates [11], Floquet states of a (partially) integrable system with a Hamiltonian that is $2\pi$-periodic in $\tau$ are associated with invariant, $2\pi$-periodic vortex tubes [2] in the odd-dimensional, extended phase space $\{(p,x,\tau)\}$ [42]. Consequently, there are semiclassical quantization rules for Floquet states that resemble closely the well-known Einstein-Brillouin-Keller rules [3–6] for energy eigenstates. In the present case of a system with merely one spatial degree of freedom and an effective Planck constant $\hbar_{eff}$, these rules can be formulated as follows: First, the correct “quantized” vortex tubes are selected by

$$\oint_{\gamma_1} p\,dx = 2\pi \hbar_{eff} \left( k + \frac{1}{2} \right),$$

where the quantization path $\gamma_1$ winds once around a $2\pi$-periodic tube in a plane $\tau = \text{const.}$, and $k = 0, 1, 2, \ldots$ is an integer. The quasienergies then follow from the second rule

$$\varepsilon = -\frac{1}{2\pi} \oint_{\gamma_2} (p\,dx - H\,d\tau) + m\hbar_{eff},$$

(64)
where the path $\gamma_2$ is $2\pi$-periodic in $\tau$ and lies on such a tube, with the integration extending over one period. The integer $m$ accounts for the $\hbar_{\text{eff}}$ structure of the quasienergy spectrum [12].

It is now crucial to realize that the direct computation of Floquet states by these rules is possible only if there actually are vortex tubes that inherit the $2\pi$-periodicity of the underlying Hamiltonian. The rules can, therefore, be applied to vortex tubes that are merely perturbative (and, hence, $2\pi$-periodic) deformations of energy manifolds of the undriven system, or to the vortex tubes of an $N = 1$ resonance. Such a resonance emerges if the time of one cycle of unperturbed motion is equal to the driving period, $2\pi$. Hence the central elliptic periodic orbit of such a resonance is also $2\pi$-periodic, and the rules (63), (64) refer to the vortex tubes surrounding it. The fact that these tubes are not simply deformations of energy manifolds of the undriven system is reflected by the emergence of a new quantum number, $k$. According to (63), the resonant ground state, $k = 0$, is associated with the innermost quantized tube surrounding the elliptic periodic orbit; the first excited state, $k = 1$, with the next largest quantized tube, and so forth. Naively, one might expect the hierarchy of resonant states to extend as far as there are still preserved tubes.

Obviously, the quantum number $k$ that appears in the semiclassical rule (63) can be identified with the Mathieu quantum number $k$ in (30); for example, the innermost quantized vortex tube corresponds to the state that is most tightly bound in the cosine potential. In the example shown in Fig. 1, the closed curves seen inside the $N = 1$ island in the lower left corner are sections of $2\pi$-periodic vortex tubes with the plane $\tau = 3\pi/2$, and (63) applies, provided that $\hbar_{\text{eff}}$ is small enough so that the required tubes actually “fit inside” the resonance.

An $N = 2$ resonance, however, requires a different reasoning [38]. In this case, the central elliptic periodic orbits, as well as the surrounding vortex tubes, are $4\pi$-periodic. Of course, one can still use (63) to select “quantized” vortex tubes and compute the associated semiclassical wave functions. In principle, such a procedure can be carried through in some detail [13], but the wave functions constructed in this way are not Floquet states. Generally,
the association of a wave function with an invariant classical object living in the extended phase space is witnessed by the concentration of the probability density along the projection of that object to the “configuration space” \( \{(x, \tau)\} \). Hence, wave functions associated with a \( 4\pi \)-periodic vortex tube would also be \( 4\pi \)-periodic in time, whereas a Floquet state necessarily has to be \( 2\pi \)-periodic.

This difficulty can only be resolved if a Floquet state is not associated with a single, \( 4\pi \)-periodic tube, but rather with both parts that result from its projection to the fundamental piece \( \{(p, x, \tau), 0 \leq \tau \leq 2\pi\} \) of extended phase space \(^{38}\). In this way, \( 2\pi \)-periodicity can be restored. The existence of two equivalent families of classical \( 4\pi \)-periodic tubes, displaced from each other in time by a single period, imply that the associated quantum mechanical Floquet states must appear in pairs. That corresponds precisely to the Mathieu approximation for \( N = 2 \); the eigenfunctions of the double cosine well are not localized in the individual wells (as the corresponding Floquet states of the original problem are not associated with a single invariant vortex tube), but are equally extended over both wells (as the Floquet states are equally associated with both tubes), so that doublets of states emerge, separated in (quasi-) energy by the tunnel splitting.

The additional quasienergy separation of \( \hbar_{\text{eff}}/2 \) between the two groups of states with \( j = 0 \) and \( j = 1 \) in \(^{32}\) has a simple explanation in terms of an analogue borrowed from solid state physics. If the lattice constant of a one-dimensional lattice is doubled from \( d \) to \( 2d \), the width of the crystal momentum Brillouin zone shrinks from \( 2\pi/d \) to \( \pi/d \). A similar “dimerization” occurs here: in the vicinity of an \( N = 2 \) resonance the classical invariant manifolds are \( 2 \cdot 2\pi \)-periodic, so that the width of the corresponding quasienergy Brillouin zone should be \( \hbar_{\text{eff}}/2 \), instead of \( \hbar_{\text{eff}} \). Thus, the separation by \( \hbar_{\text{eff}}/2 \) can be regarded as the manifestation of a purely classical effect. In contrast, the additional tunnel splitting is of genuinely quantum mechanical origin.

The “ground state” Floquet states displayed in Figs. 7 and 8 likewise both reflect the dynamics of the corresponding classical system and exhibit a purely quantum mechanical effect. A particle following the stable elliptic periodic orbit of the \( N = 2 \) resonance bounces
against the wall of the triangular well, is reflected, runs against the linear slope, is deceler-
ated, reverses its direction and hits the wall again precisely \(4\pi\) after the previous bounce.
There is an equivalent periodic orbit displaced in time from the first one by \(2\pi\). The two
quantized vortex tubes with \(k = 0\) surround these orbits in the extended phase space, and,
as seen in Figs. 7 and 8, the quantum mechanical probability density of an associated Flo-
quyet state is concentrated along the projection of both tubes to the \((x, \tau)\) plane. Now the
projection of a tube that describes classical particles moving away from the wall intersects,
at certain moments, the projection of a tube describing motion towards the wall. In the
intersection regions there is a strong interference of both quantum mechanical “possibili-
ties”. The two equivalent classical tubes are isolated in the extended phase space, but the
associated wave functions communicate.

If one forms even or odd linear combinations of the eigenfunctions of a symmetric double
well, one obtains functions that are localized in the individual wells. In complete analogy, it is
possible to form superpositions of the two members of the resonant ground state doublet that
are localized along the projection of only one of its two tubes. This is demonstrated in Fig. 9,
again for \(\hbar_{\text{eff}} = 0.66219\) and \(\beta = 0.5\). The wave function displayed there appears to follow
only one of the two equivalent tubes and, thus, manifestly exhibits subharmonic motion.
However, it remains coupled to the other tube by quantum tunneling: after the tunneling
time \(\tau_{\text{tunnel}} := \hbar_{\text{eff}} \pi / (\Delta \varepsilon)_0\) the density would be concentrated along the projection of the
other tube, and then start to tunnel back. The approximation (34) yields \((\Delta \varepsilon)_0 \approx 8.5 \cdot 10^{-24}\),
and thus a tunneling time \(\tau_{\text{tunnel}} \approx 4 \cdot 10^{22} \cdot 2\pi\).

The same principle holds for the excited resonant states, i.e., for those states with higher
Mathieu quantum number \(k\). For example, Fig. 10 shows the density of a pure Floquet
state with \(k = 6\), for the same parameters as before. According to (34), its associated tubes
are much wider than those for \(k = 0\). Hence, the projections are larger than those of the
ground state tubes, which is clearly reflected by the wave functions. Again, it is possible to
form a superposition that is localized along only one of the tubes (see Fig. 11), but now the
tunneling time is 11 orders of magnitude shorter than that for \(k = 0\).
The vortex tubes for \( k = 6, \bar{h}_{\text{eff}} = 0.66219 \) and \( \beta = 0.5 \) still roughly “fit inside” the \( N = 2 \) islands of regular motion, see Fig. 1. But the quasienergy spectrum displayed in Fig. 4 indicates that the number of states accessible to the Mathieu approximation is larger than the area of the regular islands, divided by \( 2\pi \bar{h}_{\text{eff}} \); the hierarchy of near-resonant states extends farther than one might expect. Further evidence for this statement is provided by the following two figures. Fig. 12 shows the probability density of a Floquet state with \( k = 10 \), whereas Fig. 13 shows the density of a superposition with its partner state. The tubes for \( k = 10 \) do not fit into the islands of regular motion; they would lie in the surrounding stochastic sea. Nevertheless, the wave functions are structured similarly to those in Figs. 10, 11, as if there still were “quantizable” vortex tubes. Hence, although fully preserved vortex tubes do no longer exist outside the islands, it must be possible to use their remnants, such as Cantori, in a semiclassical quantization procedure. In this way, the quantum mechanical regularity associated with regular classical resonant motion is continued into a regime where a major part of the corresponding classical dynamics is chaotic. Expressed differently, the Mathieu quantum number \( k \) is not necessarily related to invariant vortex tubes; the number of Floquet states that can be labelled by \( k \) is larger than the number of vortex tubes that fit into the regular islands. Even though the transition from (mainly) regular to (mainly) stochastic classical motion appears to be quite sharp on the borders of the islands of stability, there is no correspondingly sharp transition in quantum mechanics.

It had already been pointed out that the Mathieu equation (23) can be regarded as the Schrödinger equation for a fictitious particle in a cosine lattice. When the energy of such a particle is close to the top of the barrier, i.e., when \( a_k(\nu(j), q) \approx 2q \), its probability density is strongly enhanced in the vicinity of the potential maxima, reflecting the fact that a corresponding classical particle would spend most of its time there. Just as the minima of the cosine potential correspond, within the pendulum approximation, to the stable, elliptic periodic orbits, the maxima correspond to the unstable, hyperbolic ones. According to this qualitative consideration, Floquet states with a Mathieu quantum number \( k \) such that \( a_k(\nu(j), q) \approx 2q \) should be separatrix states, with a probability density that is strongly
enhanced along the unstable, hyperbolic periodic orbits.

Quantitatively, the simple Mathieu approximation becomes less accurate for large quantum numbers \( k \). For the present parameters, a characteristic value with \( k = 17 \) is closest to \( 2q \), whereas it is actually a state with \( k = 16 \) whose density exhibits the clearest traces of the unstable classical orbit (see Fig. 14). Nevertheless, the approximation still yields the correct order of magnitude for the tunnel splitting: analytically, one obtains \(|(\Delta \varepsilon)_{16}/\hbar_{\text{eff}}| \approx 1.4 \cdot 10^{-2} \), compared to the numerically found value of \( 4.6 \cdot 10^{-2} \). Naturally, close to the top of the cosine wells the splitting is quite large (in the present example, the absolute value of \((\Delta \varepsilon)_{16} \) is 21 orders of magnitude larger than the ground state splitting!), and the probability density of both members of a doublet shows significant differences. In spite of that, Fig. 15 demonstrates that it is still possible to form a superposition of a separatrix state with its partner state such that, at least for a short time, the resulting density is strongly enhanced along only one of the two unstable \( 4\pi \)-periodic orbits. But complete destructive interference along the other one is no longer possible. The large tunnel splitting results in a quite short tunneling time for this doublet, \( \tau_{\text{tunnel}} \approx 10.9 \cdot 2\pi \). Thus, after merely 11 cycles of the driving force the density will be concentrated along the other hyperbolic orbit.

Quantum mechanical energy eigenstates with a density that peaks along unstable, hyperbolic periodic orbits, first investigated by McDonald and Kaufmann \cite{44} and Heller \cite{45}, have been dubbed “scars”. In the context of periodically driven systems, such scars have attracted some interest since it was speculated that the experimentally observed anomalous stability of certain hydrogenic Rydberg states against microwave ionization \cite{46} could be related to a scarred Floquet state \cite{47}. In fact, the scar investigated by Jensen et al. \cite{47} turns out to be simply the separatrix state of an \( N = 1 \) resonance. The present simple model shows, first of all, why an unstable classical orbit can “attract” quantum mechanical probability density, and it also shows under which conditions this phenomenon can occur in Floquet states. To get further insight into a possible connection between scarring and enhanced stability against ionization, it might be interesting to investigate also an \( N = 2 \) res-
onance in highly excited, microwave-driven hydrogen atoms: are there, again, anomalously stable states?

**VII. IMPLICATIONS**

The preceding investigation of Floquet states and quasienergies linked to a primary $N = 2$ resonance has demonstrated that the incompatibility of the $2\pi$-periodic boundary conditions imposed on the quantum mechanical Floquet states, and the $2 \cdot 2\pi$ periodicity of classical invariant manifolds, necessarily leads to the tunnel effect. Pictorially speaking, there is quantum tunneling from one of the $N = 2$ islands seen in Fig. 1 through the stochastic sea to the other. Thus, one is faced with a special type of “dynamical tunneling”, as studied by Davis and Heller [48] in the context of energy eigenstates. The generalization to the case of an arbitrary $N$ is obvious: the Mathieu approximation leads to a cosine lattice with periodic boundary conditions after $N$ wells. The wave functions associated with classical, $N \cdot 2\pi$-periodic vortex tubes correspond to the single-well “atomic” states of that lattice, whereas the actual near-resonant Floquet states correspond to delocalized Bloch waves.

Besides dynamical tunneling, another feature known from the investigation of time-independent systems carries over to the periodically time-dependent case. Jaffé, Shirts and Reinhardt [49,50] have argued that remnants of destroyed invariant manifolds, so-called “vague tori” existing in the apparently chaotic part of phase space, may introduce enough regularity to allow EBK-like quantization even in the absence of complete tori. Analogously, “vague vortex tubes” should be responsible for the surprisingly regular appearance of the Floquet state displayed in Fig. 12. The notion of a vague torus, although somewhat hard to formulate in precise mathematical terms, provides a bridge between regular and fully chaotic classical dynamics, and gives some insight how a quantum system behaves when its classical counterpart falls into this transition regime. The numerical example of the previous section also shows that there can be a closed link, expressed by the existence of a common quantum number $k$, between Floquet states associated with invariant manifolds surrounding stable,
elliptic periodic orbits and states associated with unstable, hyperbolic periodic orbits. In the general case, one might wish to apply torus quantization to the classically regular, elliptic motion and a periodic orbit cycle expansion to the chaotic, homoclinic motion \cite{13}. But to which extent, if at all, is it possible to “interpolate” between both quantization schemes? One-dimensional, periodically time-dependent systems like the one studied here, with their merely three-dimensional phase space \{(p, x, \tau)\}, might provide the simplest access to this important problem.

Returning now to the specific properties of the triangular well model, it has been suggested that there exists a transition from a pure point quasienergy spectrum to a continuous one at a nonzero value of the amplitude \(\beta\) \cite{28,29}. A point spectrum would be connected with square-integrable Floquet states that have a “localized” distribution of coefficients when expanded in the basis of unperturbed triangular well eigenstates; a continuous spectrum would be connected with “extended” states. The classical dynamics indeed shows a qualitative change when \(\beta\) is enlarged: for small amplitudes, all trajectories are either tied to invariant vortex tubes or move in the stochastic layers surrounding the individual resonance zones, but remain bounded. Within a comparatively small interval of \(\beta\), all resonances overlap (see (17)), so that the individual stochastic layers merge into a connected stochastic sea. We then have a dichotomy: the trajectories are either confined to the remnants of the regular resonant islands or they move in the stochastic sea (cf. Fig. 2), where they can gain an unlimited amount of energy. It would be tempting to speculate that the latter type of behavior is linked to “extended” Floquet states in the quantum system.

The numerical evidence remains inconclusive. Fig. 16 shows the near-resonant quasienergy spectrum \((N = 2)\) for \(\hbar_{eff} = 0.66219\), for amplitudes \(\beta\) between 0 and 1.0. With increasing \(\beta\), more and more resonant vortex tubes are destroyed (cf. Figs. 1 and 2), and the number of states that can be described by the Mathieu approximation is diminished. Nevertheless, there are clear signs of the resonance-induced regularity even for \(\beta = 1.0\). Fig. 17 shows the probability density of a member of the resonant ground state doublet for this amplitude, whereas Fig. 18 shows the density of a “one-tube” superposition with its
partner state. Although these wave functions appear considerably more “ragged” than they
did for $\beta = 0.5$ (cf. Figs. 8 and 9), there does not seem to be a sharp transition, but merely
a gradual change of behavior. However, the quantum mechanics of the periodically driven
triangular well is more subtle than a numerical investigation might suggest.

The previous analytical deliberations were concerned with single resonances. But for
a rigorous study of the nature of the quasienergy spectrum the single-resonance approxi-
mation is insufficient. To get a glimpse of the problem, consider the approximate Mathieu
spectrum (52) for the $N$-th resonance, together with that for the $(N+1)$-th, projected into
one Brillouin zone. At certain values of $\beta$, approximate eigenvalues belonging to the $N$-th
resonance will cross others of the $(N+1)$-th. However, the full Hamiltonian of the driven
triangular well has no symmetry which could allow these crossings: the terms neglected in
the derivation of the two near-resonant spectra will lead to a small, but nonvanishing cou-
pling of the Floquet states belonging to the two different resonances. Hence, the eigenvalues
will repel each other [51], so that the crossing of the approximate eigenvalues become anti-
crossings, most of them much too tiny to be detectable by numerical means. Accordingly,
the Floquet states become linear combinations of the approximate single-resonance states.
In other words, there is quantum tunneling not only between invariant manifolds belonging
to the same $N$, but also between those belonging to different $N$. Now there are not only two
primary resonances, but infinitely many. According to (16), the width $\Delta I_N$ of the classical
resonance zones grows quadratically with $N$, so that even for arbitrarily small $\beta$ there will
always be an infinite number of primary resonances that support an arbitrarily large number
of Floquet states, no matter how large (or small) the effective Planck constant $\hbar_{eff}$. If one
imagines all the infinitely many near-resonant quasienergy spectra superimposed in a single
Brillouin zone for, say, $0 \leq \beta \leq 1$, with all crossings replaced by anticrossings, one gets an
extremely complicated net, with arbitrarily narrow anticrossings appearing everywhere. It
seems unavoidable to conclude that the true quasienergies of the driven triangular well can
not be differentiable with respect to the amplitude $\beta$.

A question that could shed some light on the nature of the quasienergy spectrum can
therefore be phrased as follows: if one starts from the approximate near-resonant Floquet states, and then takes into account quantum tunneling between different resonances, are the resulting Floquet states “localized” over a finite number of resonances, or “extended”? It appears quite possible that, because of tunneling, there is no spectral transition except for \( \beta = 0 \), and all Floquet states might be extended for every \( \beta > 0 \). Or could localized and extended states coexist, such that the number of extended states gradually increases when \( \beta \) is enlarged?

For a physicist, the approximate analytical and numerical results provide a description of the dynamics over time scales that are short compared to inter-resonance tunneling times, which may be sufficient for most practical purposes. From the point of view of a rigorous mathematician, the present study of the periodically driven triangular well certainly poses additional questions. It suggests, however, a more direct starting point for the investigation of periodically time dependent quantum systems than the one adopted in [20–25]. The model of the kicked rotor [14], which was motivated by a single-resonance approximation, has meanwhile been understood in some detail [15,16,20,21]. The next generation of problems seems to arise from the interaction of different resonances; a detailed analysis of the role of quantum tunneling is required. The triangular well model [27–29] appears to be ideally suited for that purpose.
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FIG. 1. Poincaré surface of section for the driven particle in a triangular well potential (7), taken at \( \tau = 3\pi/2 \mod 2\pi \). The island in the lower left corner originates from the \( N = 1 \) resonance, the two large islands from the resonance with \( N = 2 \). The driving amplitude is \( \beta = 0.5 \), lower than \( \beta_1^{(c)} = 0.537 \) which is the value required by the Chirikov criterion for the overlap of these two resonances. The two boxes in the upper right corner enclose areas \( 2\pi\hbar_{\text{eff}} \), with \( \hbar_{\text{eff}} = 1.33325 \) and \( 0.66219 \), respectively.

FIG. 2. Poincaré section for \( \beta = 1.0 \). The driving amplitude now is significantly higher than \( \beta_\infty^{(c)} = 0.617 \), above which, according to the Chirikov criterion, all resonances overlap. Nevertheless, there are still remnants of the regular islands for \( N = 1 \) and \( N = 2 \). The two boxes in the upper right corner have the same meaning as in Fig. 1.

FIG. 3. Part of the numerically computed quasienergy spectrum for \( \hbar_{\text{eff}} = 1.33325 \), restricted to one Brillouin zone. Energies of unperturbed triangular well eigenstates close to the “resonant” state \( r = 20 \) differ by approximately \( \hbar_{\text{eff}}/2 \). The behavior of the quasienergies of near-resonant states is well described by the Mathieu approximation (32); integers in the right margin are the quantum numbers \( k \) for one of the two resonant ladders.

FIG. 4. Part of the numerically computed quasienergy spectrum for \( \hbar_{\text{eff}} = 0.66219 \). For this value of \( \hbar_{\text{eff}} \), energies of unperturbed triangular well eigenstates close to the “resonant” state \( r = 40 \) differ by approximately \( \hbar_{\text{eff}}/2 \).

FIG. 5. Approximate quasienergy spectrum according to the Mathieu approximation (52), for the same parameters as used in Fig. 4, and \( k = 0, \ldots, 9 \).

FIG. 6. Negative dipole matrix elements \( \langle n|x|n+2 \rangle \) in the vicinity of \( n = 40 \). Within the Mathieu approximation leading to Fig. 5, it is assumed that all these elements are equal to unity.
FIG. 7. Contour plot of the probability density of a member of the ground state doublet \((k = 0)\), for \(\beta = 0.5\) and \(\hbar_{\text{eff}} = 1.33325\). The density is concentrated along the two classically equivalent “innermost quantized vortex tubes” surrounding the two central elliptic periodic orbits of the \(N = 2\) resonance, projected to the \((x, \tau)\) plane. Strong interference occurs where both projections intersect. In this and the following figures, the displayed interval of space ranges from \(x_{\text{min}} = 0\) to \(x_{\text{max}} = 25\), the interval of time corresponds to \(4 \cdot 2\pi\), i.e., to 4 cycles of the driving force.

FIG. 8. Probability density of a member of the ground state doublet \((k = 0)\), for \(\beta = 0.5\) and \(\hbar_{\text{eff}} = 0.66219\). The wave function is now tied closer to the associated classical object than it is in the case of the larger \(\hbar_{\text{eff}}\), see Fig. 7.

FIG. 9. Probability density of a solution to the time-dependent Schrödinger equation that consists of a superposition of both members of the ground state doublet. The parameters are again \(\beta = 0.5, \hbar_{\text{eff}} = 0.66219\). The interval of time corresponds to 4 cycles of the driving force. The wave function appears to follow only one of the two quantized vortex tubes, and thus exhibits subharmonic motion. However, it is coupled to the other tube by the tunnel effect. The tunneling time is \(\tau_{\text{tunnel}} \approx 4 \cdot 10^{22} \cdot 2\pi\).

FIG. 10. Probability density of a member of the doublet with \(k = 6\). The associated vortex tubes still roughly fit inside the regular elliptic islands, cf. Fig. 1 \((\beta = 0.5, \hbar_{\text{eff}} = 0.66219)\).

FIG. 11. Solution to the time-dependent Schrödinger equation that consists of a “one-tube” superposition of the \(k = 6\) doublet \((\beta = 0.5, \hbar_{\text{eff}} = 0.66219)\).

FIG. 12. Probability density of a member of the doublet with \(k = 10\). Although the associated vortex tubes would not fit inside the regular elliptic islands seen in Fig. 1, this eigenfunction is structured analogously to that displayed in Fig. 10 \((\beta = 0.5, \hbar_{\text{eff}} = 0.66219)\).

FIG. 13. Solution to the time-dependent Schrödinger equation that consists of a superposition of both members of the \(k = 10\) doublet \((\beta = 0.5, \hbar_{\text{eff}} = 0.66219)\).
FIG. 14. Probability density of a member of the $k = 16$ doublet. This is a separatrix state: its density is concentrated along the projections of the two hyperbolic periodic orbits originating from the $N = 2$ resonance ($\beta = 0.5, \hbar_{\text{eff}} = 0.66219$). The arrows on the bottom indicate the moments where the stable, elliptic periodic orbits hit the wall, cf. Fig. 7.

FIG. 15. Solution to the time-dependent Schrödinger equation that consists of both members of the $k = 16$ doublet. This wave function shows signs of “scarring”, i.e., its density appears strongly enhanced along the projection of a single hyperbolic periodic orbit of the $N = 2$ resonance. However, the tunneling time for this doublet is merely $10.9 \cdot 2\pi$, so that the density will be enhanced along the other hyperbolic periodic orbit after merely 11 cycles ($\beta = 0.5, \hbar_{\text{eff}} = 0.66219$).

FIG. 16. Part of the quasienergy spectrum for $\hbar_{\text{eff}} = 0.66219$, and amplitudes $\beta$ ranging from 0 to 1.0. Even for the highest amplitude, there are clear signs of the resonance-induced regularity.

FIG. 17. Probability density of a Floquet state with $k = 0$ for $\hbar_{\text{eff}} = 0.66219$ and $\beta = 1.0$, cf. Fig. 2.

FIG. 18. Superposition of both members of the $N = 2$ ground state doublet for $\hbar_{\text{eff}} = 0.66219$ and $\beta = 1.0$. 