A tropical morphism related to the hyperplane arrangement of the complete bipartite graph

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Abstract

We undertake a combinatorial study of the piecewise linear map $g : \mathbb{R}^{2(m+n)} \to \mathbb{R}^{m \times n}$ which assigns to the four vectors $a, A$ in $\mathbb{R}^m$ and $b, B$ in $\mathbb{R}^n$ the $m \times n$ matrix given by $g_{ij} = \min(a_i + b_j, A_i + B_j)$. This map arises naturally in Pachter and Sturmfels’s work on the tropical geometry of statistical models. The image of $g$ has been a subject of recent interest; it is the positive part of the tropical algebraic variety which parameterizes $n$-tuples of points on a tropical line in $m$-space.

The domains of linearity of $g$ are the regions of the real hyperplane arrangement $\mathcal{A}_{m,n}$, corresponding to the complete bipartite graph $K_{m,n}$.

We explain how the images of (some of) the regions provide two polyhedral subdivisions of the image of $g$, one of which is a refinement of the other. The finer subdivision is particularly nice enumeratively: it has $2^m \binom{n}{2} r_{m-2,n-2}$ maximum-dimensional cells, where $r_{m-2,n-2}$ is the number of regions of the arrangement $\mathcal{A}_{m-2,n-2}$.

1 Introduction

0. The goal of this paper is to undertake a combinatorial study of the piecewise linear map $g : \mathbb{R}^{2(m+n)} \to \mathbb{R}^{m \times n}$ given by

$$g(a, A, b, B)_{ij} = \min(a_i + b_j, A_i + B_j) \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n,$$

where $a$ and $A$ denote vectors in $\mathbb{R}^m$ and $b$ and $B$ denote vectors in $\mathbb{R}^n$.

The paper is organized as follows. In Section 2 we describe and enumerate the domains of linearity of the map $g$; they are precisely the faces of the arrangement $\mathcal{A}_{m,n}$, corresponding to the complete bipartite graph $K_{m,n}$. In Section 3 we describe the images $g(R)$, as $R$ ranges over the regions of the arrangement. In Section 4 we show that these images fit together in

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an unusual way to give the full image $\text{Im}(g)$; as a consequence, we obtain two different polyhedral subdivisions of $\text{Im}(g)$, one of which is a refinement of the other. The finer subdivision is particularly nice combinatorially; the number of its facets is $2 \binom{m}{2} \binom{n}{2} r_{m-2,n-2}$, where $r_{m-2,n-2}$ is the number of regions of the arrangement $A_{m-2,n-2}$.

This project was suggested in a recent paper of Pachter and Sturmfels [17]. It arose in their study of the tropical geometry of statistical models, and provides a new perspective on Develin, Santos and Sturmfels’s study of the image of $g$ [9, 10].

To explain their motivation, we start with a brief overview of some recent developments in tropical geometry which led to its consideration.

1. The tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is the set of real numbers augmented by infinity, together with the operations of tropical addition and multiplication, which are defined by $x \oplus y = \min(x, y)$ and $x \odot y = x + y$.

Tropical algebraic geometry is, roughly speaking, the geometry of the tropical semiring. Let $K = \mathbb{C}\{\{t\}\}$ be the (algebraically closed) ring of Puiseux series; its elements are formal power series of the form $f = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $a_1 < a_2 < \cdots$ are rational numbers with a common denominator. There is a natural valuation $\text{deg} : K^{\ast} \to \mathbb{Q}$, which sends the non-zero Puiseux series $f$ to its degree $a_1$.

For an ideal $I$ in $K[x_1, \ldots, x_n]$, let $V(I)$ be the corresponding algebraic variety, intersected with the torus $(K^\ast)^n$. It consists of the $n$-tuples $u(t) = (u_1(t), \ldots, u_n(t))$ of non-zero Puiseux series such that $f(u(t)) = 0$ for all $f \in I$. Define $\deg u(t) = (\deg u_1(t), \ldots, \deg u_n(t))$.

**Theorem.** [12, 19, 22] For an ideal $I$ in $K[x_1, \ldots, x_n]$, the following subsets of $\mathbb{R}^n$ coincide.

1. The topological closure of $\deg V(I)$.
2. The set of $w \in \mathbb{R}^n$ such that the initial ideal $\text{in}_w(I)$ contains no monomials.

This set is called the tropical variety of $I$, and denoted $\text{trop} V(I)$.

Let $V^+(I)$ be the set of $n$-tuples $u(t)$ in $V(I)$ such that the coefficient of the leading term of each $u_i(t)$ is a positive real number.

**Theorem.** [20] For an ideal $I$ in $K[x_1, \ldots, x_n]$, the following subsets of $\mathbb{R}^n$ coincide.

1. The topological closure of $\deg V^+(I)$. 
2. The set of $w \in \mathbb{R}^n$ such that $\text{in}_w(I) \cap \mathbb{R}^+[x_1, \ldots, x_n] = \emptyset$.

This set is called the positive part of the tropical variety of $I$, and denoted $\text{trop}^+ V(I)$.

For all $I$, $\text{trop} V(I)$ is a polyhedral complex, and $\text{trop}^+ V(I)$ is a subcomplex of it.

2. Let $f = (f_1, \ldots, f_n) : \mathbb{R}^d \to \mathbb{R}^n$ be a polynomial map. Say that $f$ is positive if each polynomial $f_i$ has only positive coefficients. Say that $f$ is surjectively positive if, additionally, every point in the image of $f$ whose coordinates are positive has a preimage whose coordinates are positive; i.e., $f(\mathbb{R}^d_{>0}) = \text{Im} f \cap \mathbb{R}^n_{>0}$.

The tropicalization of $f$ is the piecewise linear map $g : \mathbb{R}^d \to \mathbb{R}^n$ obtained from $f$ by replacing every $\times$ with a $\odot$ and every $+$ with a $\oplus$. Such a map is called a tropical morphism. [17]

The relationship between the positive part of a tropical variety and morphisms is outlined in the following theorem.

Theorem. [17, 20] Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a polynomial map, and let $g$ be the tropicalization of $f$. The image $\text{Im}(f)$ is an algebraic variety in $\mathbb{R}^n$; let $I$ be its corresponding ideal.

1. $\text{Im}(g) \subset \text{trop} V(I)$.

2. If $f$ is positive, $\text{Im}(g) \subset \text{trop}^+ V(I)$.

3. If $f$ is surjectively positive, $\text{Im}(g) = \text{trop}^+ V(I)$.

3. Pachter and Sturmfels [17] used this setup to study the tropical geometry of statistical models. The naive Bayes model with two features [13] is parameterized by the polynomial map $f : \mathbb{R}^{2(m+n)} \to \mathbb{R}^{m \times n}$ which maps an $m \times 2$ matrix and a $2 \times n$ matrix to their product. Its tropicalization, $g$, is the object of study of this paper.

The image of $f$ consists of the real matrices of rank at most 2. This is an algebraic variety, whose corresponding prime ideal $I$ is generated by the $3 \times 3$ subdeterminants of an $m \times n$ matrix.

The tropical variety $\text{trop} V(I)$ is the set of $m \times n$ real matrices having tropical rank at most 2, and the positive part $\text{trop}^+ V(I)$ is the set $B_{m,n}$ of $m \times n$ real matrices having Barvinok rank at most 2. For the relevant definitions and further information, see [9, 10, 17].

This model interests us because the map $f$ is surjectively positive:
**Theorem.** [5] Every positive $m \times n$ matrix of rank 2 can be written as the product of a positive $m \times 2$ matrix and a positive $2 \times n$ matrix.

It follows that $\text{Im}(g) = \text{trop}^+ V(I) = B_{m,n}$.

The study of the map $g$ is therefore closely related to Pachter and Sturmfels’s study of the tropical geometry of statistical models, and to Develin, Santos and Sturmfels’s study of the space $B_{m,n}$.

## 2 The morphism and its domains of linearity.

The piecewise linear map $g : \mathbb{R}^{2(m+n)} \to \mathbb{R}^{m \times n}$ which we wish to study is the tropicalization of matrix multiplication; it is given by

$$g \left( \begin{bmatrix} a & A \\ b & B \end{bmatrix} \right) = \begin{bmatrix} a & A \\ b & B \end{bmatrix} \odot \begin{bmatrix} b \\ B \end{bmatrix}.$$

Here $a$ and $A$ denote vectors in $\mathbb{R}^m$ and $b$ and $B$ denote vectors in $\mathbb{R}^n$. Let $x_i = a_i - A_i$ and $y_j = B_j - b_j$. The entry $i, j$ of $g$ is given by

$$g_{ij} = \min(a_i + b_j, A_i + B_j) = \begin{cases} a_i + b_j, & \text{if } x_i \leq y_j \\ A_i + B_j, & \text{if } x_i > y_j. \end{cases}$$

The piecewise linear map $g$ is linear on the faces of the hyperplane arrangement in $\mathbb{R}^{2(m+n)}$:

$$A_{m,n} : \quad x_i = y_j \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

This is essentially the **graphical arrangement** of the complete bipartite graph $K_{m,n}$; it contains one hyperplane for each edge of $K_{m,n}$.

Therefore, to understand the fibers of the map $g$, we first study the faces of $A_{m,n}$.

Let us briefly outline the close connection between the arrangement $A_{m,n}$ and the graph $K_{m,n}$. The matroid $M(A_{m,n})$ associated to the arrangement is isomorphic to the matroid $M(K_{m,n})$ associated to the graph. In other words, a subset of $k$ hyperplanes of $A_{m,n}$ intersects in codimension $k$ if and only if the corresponding subset of $k$ edges of $K_{m,n}$ does not contain a cycle. Let us call this matroid simply $M_{m,n}$; it encodes much of the combinatorial information of the arrangement of the graph. For more details, see [15 Ch. 2] or [16 Ch. 5].

The regions of the arrangement $A_{m,n}$ are in one-to-one correspondence with the acyclic orientations of the bipartite graph $K_{m,n}$, as follows: Denote the vertices of $K_{m,n}$ by $u_1, \ldots, u_m, v_1, \ldots, v_n$, and consider an acyclic
orientation $o$ of $K_{m,n}$. The corresponding region $R(o)$ consists of the points $(x, y) \in \mathbb{R}^{m+n}$ such that $x_i < y_j$ if the edge $u_iv_j$ is directed $u_i \to v_j$ in $o$, and $x_i > y_j$ otherwise.

The matroid $M_{m,n}$ appeared independently in the work of Martin and Reiner [14]. They used the finite field method of [1] to compute a generating function for $\overline{\chi}_{m,n}(q, t)$, the coboundary polynomial of $M_{m,n}$. This polynomial is a simple transformation of the Tutte polynomial, and it captures much of the interesting enumerative information of the matroid.

Proposition. [14] If $\overline{\chi}_{m,n}(q, t)$ is the coboundary polynomial of $M_{m,n}$,

$$1 + q \sum_{(m,n) \in \mathbb{N}^2 - \{(0,0)\}} \overline{\chi}_{m,n}(q, t) \frac{x^m y^n}{m! n!} = \left( \sum_{(m,n) \in \mathbb{N}^2} t^{mn} \frac{x^m y^n}{m! n!} \right)^q .$$

Corollary. [21, Ex. 5.6] Let $r_{m,n}$ be the number of acyclic orientations of $K_{m,n}$ (or equivalently, the number of regions of $A_{m,n}$). Then

$$\sum_{m,n \geq 1} r_{m,n} x^m y^n \frac{m! n!}{m! n!} = \frac{1}{e^{x+y} - e^x + e^y} .$$

Proof. This follows from the formula $r(m, n) = (-1)^{m+n-1} \overline{\chi}_{m,n}(-1, 0)$ [23] for the number of regions of a real arrangement. [QED]

We now proceed to describe and count the domains of linearity of $g$; that is, the faces of the arrangement $A_{m,n}$.

Proposition 1. Let $f_{k,m,n}$ be the number of $k$-dimensional faces of the arrangement $A_{m,n}$. Then

$$\sum_{k,m,n \geq 0} f_{k,m,n} t^k \frac{x^m y^n}{m! n!} = \frac{1}{e^{-tx} + e^{-ty} - t(e^x - 1)(e^y - 1) - 1} .$$

Proof. To describe a face $F$ of the arrangement we must specify, for each hyperplane $x_i = y_j$ ($1 \leq i \leq m$, $1 \leq j \leq n$), whether $F$ is in the halfspace $x_i > y_j$, in the halfspace $x_i < y_j$, or on the hyperplane $x_i = y_j$. Consider the inequalities or equalities $x_i \bigcirc y_j$ that define a face $F$. Each one of them puts a restriction on the relative order of the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$. For a point $(x, y)$ in $F$, the relative order of $x_i$ and $y_j$ is determined for all $i$ and $j$. The relative order of $x_{i_1}$ and $x_{i_2}$ is not always determined: an equality $x_{i_1} = x_{i_2}$ can only be deduced from two defining equalities of the form $x_{i_1} = y_j$ and $x_{i_2} = y_j$. Similarly, an inequality $x_{i_1} < x_{i_2}$ can only be
deduced as a consequence of two defining inequalities or equalities of the form \( x_{i_1} \, \triangleleft \, y_j \) and \( x_{i_2} \, \triangleleft \, y_j \).

The faces of \( A_{m,n} \) are described putting these restrictions together. For example, one face of \( A_{7,8} \) consists of the points \((x, y) \in \mathbb{R}^{15}\) such that

\[
x_1, x_3 < y_2, y_5, y_7 < x_5 < x_2 = y_6 < y_1 < x_6 < x_4 = x_7 = y_4 < y_8.
\]

Each face of \( A_{m,n} \) can be described in a similar way, as a sequence of blocks of variables. The variables \( x_{i_1} \) and \( x_{i_2} \) are in the same block if the comparisons \( x_{i_1} \, \triangleleft \, y_j \) and \( x_{i_2} \, \triangleleft \, y_j \) yield the same result for each \( j \). A similar statement holds for \( y_{j_1} \) and \( y_{j_2} \). The variables \( x_i \) and \( y_j \) are in the same block if \( x_i = y_j \) in \( F \).

Call a block positive if it only contains \( x_i \)'s, negative if it only contains \( y_j \)'s, and mixed otherwise. For a point to belong to the face, the variables within each mixed block are equal. Within an unmixed block, the relative order of the variables is not determined.

It follows that the faces of \( A_{m,n} \) are in one-to-one correspondence with the ordered partitions of the set \( \{x_1, \ldots, x_m, y_1, \ldots, y_n\} \) containing no two consecutive unmixed blocks of the same sign. The dimension of a face is easily determined from the partition: it is equal to the sum of the sizes of the unmixed blocks plus the number of mixed blocks. We can now use the methods of [21, Chapter 5] to compute the desired generating function.

The generating function for non-empty positive blocks is \( X(t, x, y) = \sum_{n \geq 1} t^n x^n y^n = e^{tx} - 1 \). The generating function for non-empty negative blocks is \( Y(t, x, y) = e^{ty} - 1 \). Therefore, the generating function for partitions of \( \{x_1, \ldots, x_m, y_1, \ldots, y_n\} \) into unmixed blocks of alternating sign is

\[
Z(t, x, y) = (1 + X)(1 + YX + YXYX + YXYXYX + \cdots)(1 + Y) = \frac{1}{e^{-tx} + e^{-ty} - 1}.
\]

On the other hand, the generating function for mixed blocks is given by

\[
M(t, x, y) = \sum_{m,n \geq 1} t^m x^m y^n = t(e^x - 1)(e^y - 1). \]

The partitions we wish to count are alternating sequences of partitions of the type counted by \( Z \) (which may be empty) and mixed blocks. Therefore

\[
\sum_{k,m,n \geq 0} f_{k,m,n} t^k x^m y^n = Z + ZMZ + ZMZMZ + \cdots,
\]

which is equal to the given expression.
Observe that, under the above correspondence, the regions of $A_{m,n}$ correspond to the ordered partitions which contain no mixed blocks. These are counted by $Z(t, x, y)$; setting $t = 1$ recovers the generating function for $r_{m,n}$.

It will be convenient to label each region $R$ of the arrangement with the permutation $\pi(R)$ of the set $[m, \pi] = \{1, 2, \ldots, m, \pi, \pi, \ldots, \pi\}$ obtained by reading the blocks from left to right. The variable $x_i$ corresponds to the letter $i$ (which we call a positive letter), and the variable $y_j$ corresponds to the letter $j$ (which we call a negative letter). Within each block, the letters are arranged in increasing order.

For example, the region

$$y_2, y_4 < x_3 < y_1 < x_1, x_2 < y_3, y_5$$

of $A_{3,5}$ is simply denoted by the permutation $\underline{2431235}$.

This labelling is a bijection between the regions of $A_{m,n}$ and the permutations of $[m, \pi]$ such that any two consecutive letters of the same sign are in increasing order.

3 The image of $g$ in each region

Consider a region $R$ of $A_{m,n}$. As we mentioned earlier, the restriction of the map $g$ to the region $R$ is a linear function. We now describe the image $g(R)$.

Color each entry of the $m \times n$ matrix $g$ either black or white: the entry $g_{ij}$ is black if $a_i - A_i > B_j - b_j$ in $R$, and white if $a_i - A_i < B_j - b_j$ in $R$. Permute the rows and columns of $g$ according to the order in which their labels appear in $\pi$. A path $P$ separates the white and black entries: it starts at the northwest corner of the matrix, and takes a step south for each positive letter in $\pi$ and a step east for each negative letter in $\pi$, in the order prescribed by $\pi$. Figure 1 shows the resulting matrix and path for

| 2 | 4 | 1 | 3 | 5 |
|---|---|---|---|---|
| 3 | $A_3 + B_2$ | $A_3 + B_4$ | $a_3 + b_1$ | $a_3 + b_2$ | $a_3 + b_5$ |
| 1 | $A_1 + B_2$ | $A_1 + B_4$ | $A_1 + B_1$ | $a_1 + b_3$ | $a_1 + b_2$ |
| 2 | $A_2 + B_2$ | $A_2 + B_4$ | $A_2 + B_1$ | $a_2 + b_3$ | $a_2 + b_5$ |

Figure 1: The diagram of the region $\underline{2431235}$. 

7
The entry $g_{ij}$ is $A_i + B_j$ if it is below $P$ or $a_i + b_j$ if it is above $P$. We call this picture the diagram of $R$ or of $g(R)$.

For $g \in \mathbb{R}^{m \times n}$, write
\[ \Delta_{i_1i_2j_1j_2}(g) = g_{i_1j_1} + g_{i_2j_2} - g_{i_1j_2} - g_{i_2j_1}. \]

For simplicity, we will omit $g$ from the notation and simply write $\Delta_{i_1i_2j_1j_2}$ for this expression. We will call an equality or inequality of the form $\Delta_{i_1i_2j_1j_2} \circ 0$ a rectangle relation. Write $i < j$ when $i$ appears before $j$ in $\pi(R)$.

**Proposition 2.** (Version 1.) The image $g(R)$ is an open polytope, described by the rectangle equalities and inequalities that it satisfies. They are the following:

\[
\begin{align*}
\Delta_{i_1i_2j_1j_2} &= 0 \text{ if } i_1, i_2 < j_1, j_2, \\
\Delta_{i_1i_2j_1j_2} &= 0 \text{ if } j_1, j_2 < i_1, i_2, \\
\Delta_{i_1i_2j_1j_2} &> 0 \text{ if } i_1 < j_1 < i_2 < j_2, \\
\Delta_{i_1i_2j_1j_2} &> 0 \text{ if } j_1 < i_1 < j_2 < i_2, \\
\Delta_{i_1i_2j_1j_2} &> 0 \text{ if } i_1 < j_1 < i < j_2 < i_2 \text{ for some } i, \\
\Delta_{i_1i_2j_1j_2} &> 0 \text{ if } j_1 < i_1 < i < j_2 < j_2 \text{ for some } j,
\end{align*}
\]

and no others.

In the diagram of $R$, let $\square_{i_1i_2j_1j_2}$ be the sub-rectangle of the diagram containing rows $i_1$ through $i_2$ and columns $j_1$ through $j_2$. (We implicitly assume that $i_1 < i_2$ and $j_1 < j_2$ in $\pi(R)$.) Call this sub-rectangle monochromatic if all its entries have the same color. Call it sliced if it has black and white entries, separated by a single vertical or horizontal line. Call it jagged otherwise. These definitions are illustrated in Figure 2.

![Figure 2: The three types of rectangle: monochromatic, sliced, and jagged.](image-url)
Proposition 2. (Version 2.) The image \( g(R) \) consists of those \( g \in \mathbb{R}^{m \times n} \) such that \( \Delta_{i_1j_1i_2j_2} \) is:

- equal to zero if \( \square_{i_1j_1i_2j_2} \) is monochromatic,
- positive if \( \square_{i_1j_1i_2j_2} \) is jagged.

(If \( \square_{i_1j_1i_2j_2} \) is sliced, \( \Delta_{i_1j_1i_2j_2} \) takes positive and negative values in \( R \).)

Proof. It is straightforward to verify that the two versions of Proposition 2 are equivalent. Consider a point \( (a, A, b, B) \in \mathbb{R}^{2(m+n)} \) in \( R \). The entry \( g_{ij} \) of \( g(a, A, b, B) \) is given by \( a_i + b_j \) if it is white (i.e. if \( i < j \) in \( \pi(R) \)), or \( A_i + B_j \) if it is black (i.e. if \( i > j \) in \( \pi(R) \)). These entries are easily seen to satisfy the given equalities and inequalities.

Conversely, consider a matrix \( g \in \mathbb{R}^{m \times n} \) which satisfies the given equalities and inequalities. Permute its columns and rows according to the order in which their labels appear in \( \pi \), and draw the path \( P \).

Choose appropriate values of the \( A_i \)s and \( B_j \)s so that \( g_{ij} = A_i + B_j \) for all the southernmost and westernmost black entries (the ones surrounded by a box in Figure 1). This can be done because the system of equations is independent, and has more unknowns than equations. Make the \( A_i \)s and \( B_j \)s which do not appear in this system of equations very large positive numbers. Similarly, choose appropriate values of the \( a_i \)s and \( b_j \)s so that \( g_{ij} = a_i + b_j \) for all the northernmost and easternmost white entries, and make the other \( a_i \)s and \( b_j \)s very large positive numbers.

Since \( g \) satisfies the rectangle equalities of \( g(R) \), \( g_{ij} = A_i + B_j \) for all black entries and \( g_{ij} = a_i + b_j \) for all the white entries.

Take a black entry \( g_{ij} \). If the northernmost entry on its column or the easternmost entry on its row is black, then \( a_i + b_j \) is a very large positive number, larger than \( A_i + B_j \). Otherwise, the rectangle determined by it and the northeasternmost corner of the matrix has three white entries and one black entry, and the corresponding rectangle inequality is equivalent to \( A_i + B_j < a_i + b_j \). Similarly, \( a_i + b_j < A_i + B_j \) for white entries \( g_{ij} \). Therefore \( (a, A, b, B) \in R \) and \( g = g(a, A, b, B) \).

Corollary. Let \( R \) be a region of \( A_{m,n} \). Then \( \text{span} \ g(R) \) is the subspace of \( \mathbb{R}^{m \times n} \) described by the equalities

\[ g_{i_1j_1} + g_{i_2j_2} = g_{i_1j_2} + g_{i_2j_1} \]

for those \( i_1, i_2, j_1, j_2 \) such that \( g_{i_1j_1}, g_{i_2j_2}, g_{i_1j_2} \) and \( g_{i_2j_1} \) have the same color.
For a region $R$ let $\text{first}(R)$ be the length of the first block of letters of $\pi(R)$ of the same sign (or equivalently, the number of sources of the corresponding orientation $o(R)$), and let $\text{last}(R)$ be the length of the last block of letters of $\pi(R)$ of the same sign (or equivalently, the number of sinks of $o(R)$).

Let $R_1$ be the region where $x_i < y_j$ for all $i$ and $j$, and let $R_2$ be the region where $x_i > y_j$ for all $i$ and $j$.

**Corollary.** Let $R$ be a region of $\mathcal{A}_{m,n}$. If $R \neq R_1, R_2$, then

$$\dim g(R) = 2m + 2n - 2 - \text{first}(R) - \text{last}(R).$$

Otherwise, $\dim g(R_1) = \dim g(R_2) = m + n - 1$.

**Proof.** The southernmost and westernmost black entries and the northernmost and easternmost white entries linearly generate the remaining ones, and there are no linear relations among them. The number of them is as claimed.

In particular, the regions where $g$ has maximum rank are those corresponding to acyclic orientations of $K_{m,n}$ with a unique source and a unique sink. This maximum rank is equal to $2m + 2n - 4$.

## 4 Two subdivisions of the image of $g$.

The closures of the regions of the hyperplane arrangement $\mathcal{A}_{m,n}$ give a polyhedral covering of $\mathbb{R}^{2(m+n)}$. Their images under the map $g$ give a covering of the full image, $\text{Im}(g)$. We now wish to understand how the images $g(R)$ of the closures of the regions $R$, which we call the cells, fit together.

First notice that if $R$ is a region of $\mathcal{A}_{m,n}$ and $-R$ is its negative (so the permutation $\pi(-R)$ is equal to the permutation $\pi(R)$ reversed), then it follows from Proposition 2 that $g(R) = g(-R)$. Therefore, we can restrict our attention only to the positive regions, where $x_1 < y_1$ (or equivalently, $1 < 1$ in $\pi(R)$).

Also, the following proposition shows that it suffices to study the maximum-dimensional cells.

**Proposition.** The image $\text{Im}(g) = B_{m,n}$ is pure.

Interestingly, though, our collection of cells is not pure-dimensional. For example, the image of the region $1\overline{1}2\ldots\overline{1}23\ldots m$ is a subspace which is not contained in any maximum-dimensional cell.
Recall that the cell $g(R)$ is maximum-dimensional if and only if the first and the last block of $\pi(R)$ are singletons. Call such a maximum-dimensional cell large if the second and the second-to-last blocks of $\pi(R)$ are not singletons, small if the second and the second-to-last blocks of $\pi(R)$ are both singletons, and medium if one of them is a singleton and the other one is not.

**Theorem 1.** The large cells form a polyhedral subdivision of $B_{m,n}$. The small cells form a finer subdivision of $B_{m,n}$.

**Proof.** The proof is divided into five steps. We start by showing that each small or medium cell is contained in a large cell; therefore, the large cells cover $\text{Im}(g)$. Secondly, we show that each large cell is subdivided into small cells; therefore, the small cells cover $\text{Im}(g)$ also. The third step is to show that large cells are pairwise interior-disjoint, and so are small cells. Then we prove that the covering of $\text{Im}(g)$ with small cells is a subdivision. Finally, we prove that the covering with large cells is also a subdivision.

1. Take any small or medium cell $g(R_1)$; we want to find a large cell containing it. Say $\pi(R_1) = \overline{j_1} i_1 j_2 \ldots \overline{j_r} i_2 \ldots$. Let $R$ be the region with the label $\pi(R) = i_1 \overline{j_1} \overline{j_2} \ldots \overline{j_r} i_2 \ldots$: we have adjusted the beginning of the permutation so that the second block is not a singleton. This construction is illustrated in Figure 3. We will show that $\overline{g(R_1)} \subset g(R)$. If $\overline{g(R_1)}$ is medium, then $g(R)$ is large and we are done. If it is small, then $g(R)$ is medium; we can then adjust the end of the permutation $\pi(R)$ in the same way, to obtain a large cell containing $\overline{g(R)}$.

![Figure 3: A medium cell and its associated large cell.](image)

We need to show that every rectangle relation satisfied by $g(R)$ is also satisfied by $g(R_1)$. This analysis is most easily carried out in terms of Version 2
of Proposition 2. Notice that $R_1$ and $R_1$ have exactly the same monochromatic sub-rectangles. Also, the only sub-rectangles which are sliced in one diagram and not in the other are $\square_{i_{i_1}j_{j_1}}$ for $2 \leq s \leq r$ and $i \neq i_1$.

It follows that $g(R_1)$ has the same rectangle relations that $g(R)$ has, and the additional relations $\Delta_{i_{i_1}j_{j_1}} > 0$ for $2 \leq s \leq r$ and $i \neq i_1$. The rectangle equalities imply that $\Delta_{i_{i_1}j_{j_1}} = \Delta_{i_{i_2}j_{j_1}}$ for $2 \leq s \leq r$ and $i \neq i_1$. In conclusion, $g(R_1)$ is the subset of $g(R)$ satisfying the extra relations:

$$g_{i_1j_1} - g_{i_2j_1} > g_{i_1j_s} - g_{i_2j_s}$$

for $2 \leq s \leq r$.

We conclude that any maximum-dimensional cell is contained in a large cell, so the large cells cover $\text{Im}(g)$ by themselves, as desired.

2. Now, let us describe which medium and small cells are contained in a given large cell. Let $R$ be as defined above, and let $R_s$ be the region with $\pi(R_s) = \overline{R_{i_1j_1} \ldots \overline{R_s} \ldots \overline{R_t} \overline{j_2} \ldots}$ for $1 \leq s \leq r$. Imitating the argument above, we conclude that $g(R_s)$ is the subset of $g(R)$ such that $g_{i_1j_s} - g_{i_2j_s}$ is the unique largest element in $\{g_{i_1j_t} - g_{i_2j_t} : 1 \leq t \leq r\}$. It follows that $\{g(R_s) : 1 \leq s \leq r\}$ is a subdivision of $g(R)$.

In general, the above argument shows that the large cell $\overline{g(R)}$ is subdivided into $st$ small cells, where $s$ and $t$ are the lengths of the second and second-to-last blocks of $\pi(R)$. (There are also $s + t$ medium cells in $\overline{g(R)}$. The first $s$ medium cells form a subdivision of $\overline{g(R)}$ and are subdivided into $t$ small cells each. The other $t$ medium cells also form a subdivision of $\overline{g(R)}$ and are subdivided into $s$ small cells each.) In particular, the small cells also cover $\text{Im}(g)$.

3. To show that the two coverings of $\text{Im}(g)$ are polyhedral subdivisions, let us start by showing that the interiors of any two large cells are disjoint, and the interiors of any two small cells are also disjoint.

Our strategy is to show that the linear spans of any two large cells (which are $(2m + 2n - 4)$-dimensional subspaces) are different; therefore the intersection of the two cells cannot be $(2m + 2n - 4)$-dimensional, and their interiors must be disjoint. We will use the first Corollary to Proposition 2.

Suppose, then, that we know which rectangle equalities an unknown large cell $\overline{g(R)}$ satisfies. We can recover $R$ as follows. Define an equivalence relation $\sim$ on $[m] \times [n]$ by declaring that $(i, j) \sim (i', j')$ if $g_{ij}$ and $g_{i'j'}$ are part of the same rectangle equality, and then taking the transitive closure.

This equivalence relation will have two non-trivial equivalence classes, and several singletons. (The only exceptions are the regions with $L$-shaped
diagrams, like $\tilde{112}\ldots \tilde{j}\ldots \tilde{n}12\ldots \hat{i}\ldots m\hat{j}$. Here there is only one non-trivial equivalence class, and we can immediately recover the diagram, and hence the region, from it.)

In one equivalence class, find an entry $(i, j)$ which appears in a rectangle relation with every other entry in the equivalence class. Then $g_{ij}$ must be, essentially, the southwest or northeast corner of the diagram of $R$ which we are trying to recover. More precisely, $i$ and $j$ are either in the last positive and first negative block of $\pi(R)$, or in the last positive and first negative block, respectively. Because $g(R) = g(-R)$, we can assume it is the former.

Let $\pi(R) = (i_0)\tilde{j}(i_1\ldots i_r)\sigma(j_s\ldots j_1)i\tilde{j_0})$ be the (unknown) label of $R$. Here $\sigma$ denotes the segment of $\pi(R)$ which starts at the second negative letter and ends at the second-to-last positive letter. Symbols in parenthesis denote letters which may or may not be in $\pi(R)$.

From Version 1 of Proposition 2, $\Delta_{ii'jj'} = 0$ holds in $g(R)$ if and only if $j, j' < i, i'$. For $i', j'$ in $\sigma$, this holds if and only if $j < i'$. This allows us to recover which letters are in $\sigma$, and in which order.

Any positive letters which do not appear in $\sigma$ must appear to the left of it. Their position in $\pi(R)$ is determined by the fact that $g(R)$ is large: If there is only one such letter, it must be $i_0$, and $i_1, \ldots, i_r$ do not exist. If there are several such letters, they must be $i_1, \ldots, i_r$, and $i_0$ does not exist. Similarly, we recover the positions of the negative letters which do not appear in $\sigma$. We have recovered the label of $R$, as claimed.

It follows that the interiors of the large cells are disjoint. Since each small cell is in a unique large cell, and each large cell is subdivided into small cells with disjoint interiors, if follows that the interiors of the small cells are disjoint also.

4. Let us now show that the covering of $\text{Im}(g)$ by the small cells is actually a polyhedral subdivision. Consider two small cells $A_1 = g(R_1)$ and $A_2 = g(R_2)$. We wish to show that their intersection is a face of $A_1$.

A first description of $A_1 \cap A_2$ is given by the rectangle relations of $A_1$ and $A_2$. We need to find a second description, which only uses defining equalities and inequalities of $A_1$ (corresponding to monochromatic and jagged rectangles in $R_1$), and equalities which define facets of $A_1$ (corresponding to jagged rectangles in $R_1$).

Consider a rectangle relation $\Delta_{ii'jj'} = 0$ which holds in $A_1 \cap A_2$. If $\square_{ii'jj'}$ is monochromatic in $R_1$, the given relation is satisfied in $A_1$. If it is jagged, then the relation defines a facet of $A_1$. Therefore, we can assume that it is sliced. Assume that it is sliced horizontally, so $g_{ij}$ and $g_{ij'}$ are white, and $g_{i'j}$ and $g_{i'j'}$ are black. The sign of $\Delta_{ii'jj'}$ is undetermined in $A_1$, and therefore
the equality $\Delta_{i'i'jj'} = 0$ must hold in $A_2$. The rectangle $\Box_{i'i'jj'}$ must then be monochromatic in $R_2$; assume it is black. Also assume that $i \leq i'$ and $j \leq j'$ in $\pi(R_2)$; our arguments extend immediately to the other cases. Figure 4 shows the position of $\Box_{i'i'jj'}$ in the diagrams of $R_1$ and $R_2$.

Since $A_1$ is small, $\overline{j}$ and $\overline{j'}$ are not in the first negative block of $\pi(R_1)$. Let $\overline{j}_1$ be the first negative letter in $\pi(R_1)$. Then $\Box_{i'i'jj_1}$ is jagged in $R_1$, and $\Delta_{i'i'jj_1} \geq 0$ in $A_1$. If $\overline{j}_1 \leq \overline{j}$ in $\pi(R_2)$ then $\Box_{i'i'jj_1}$ is monochromatic in $R_2$ and $\Delta_{i'i'jj_1} = 0$ in $A_2$; if $\overline{j}_1 > \overline{j}$ in $\pi(R_2)$ then $\Box_{i'i'jj_1}$ is either monochromatic or jagged in $R_2$ and $\Delta_{i'i'jj_1} \geq 0$ in $A_2$. In any case, $\Delta_{i'i'jj_1} = 0$ in $A_1 \cap A_2$, and this is a facet equality of $A_1$. Similarly, $\Delta_{i'i'jj'} = 0$ in $A_1 \cap A_2$, and this is a facet equality of $A_1$. Therefore the equality $\Delta_{i'i'jj'} = 0$ is a consequence of two facet equalities of $A_1$.

Exactly the same argument shows that any rectangle inequality satisfied by $A_1 \cap A_2$ is a consequence of the relations of $A_1$ and its facet equalities. It follows that the small cells are actually a polyhedral subdivision of $\text{Im}(g)$.

5. We now use a similar argument to show that the covering of $\text{Im}(g)$ into large cells is also a polyhedral subdivision.

Consider two large cells $A_1 = \overline{g(R_1)}$ and $A_2 = \overline{g(R_2)}$, and a rectangle relation $\Delta_{i'i'jj'} = 0$ which holds in $A_1 \cap A_2$. As before, assume that $\Box_{i'i'jj'}$ is sliced horizontally in $R_1$ and monochromatic black in $R_2$, and that $i \leq i'$ and $\overline{j} \leq \overline{j'}$ in $\pi(R_2)$.

The argument that we used for small cells carries over to this situation, unless $\overline{j}$ and $\overline{j'}$ are in the first block of negative letters of $\pi(R_1)$. Since $R_1$ is large, $i$ must then be the unique first positive letter of $\pi(R_1)$. Figure 5 shows the position of $\Box_{i'i'jj'}$ in the diagrams of $R_1$ and $R_2$ in this case.

Since $A_1$ is large, $\overline{j}$ and $\overline{j'}$ are not in the last negative block of $\pi(R_1)$. Let
be the last negative letter in $\pi(R_1)$. The rectangles $\square_{ii'jj_n}$ and $\square_{ii'j'j_n}$ are jagged in $R_1$. They are either both monochromatic or both jagged in $R_2$. If they are monochromatic, then $\Delta_{i'i'jj_n} = 0$ and $\Delta_{i'i'j'j_n} = 0$ in $R_2$, and hence in $R_1 \cap R_2$. These two equalities, which define facets of $A_1$, will imply the desired equality $\Delta_{i'i'jj_n} = 0$. Therefore, assume that the two rectangles are jagged in $R_2$, with $g_{ijn}$ white.

Now, since $A_2$ is large, $i$ and $i'$ are not in the first positive block of $\pi(R_2)$. Let $i_1$ be the first positive letter in $\pi(R_2)$. The rectangle $\square_{ii_1jj_n}$ is jagged in $R_1$, so $\Delta_{ii_1jj_n} \geq 0$ in $A_1$. The rectangle $\square_{i_1ijj_n}$ is jagged in $R_2$, so $\Delta_{i_1ijj_n} \geq 0$ in $A_2$. Therefore $\Delta_{ii_1jj_n} = 0$ in $A_1 \cap A_2$, and this is a facet equality of $A_1$. Similarly, $\Delta_{i_1i'jj_n} = 0$ in $A_1 \cap A_2$, and this is a facet equality of $A_1$.

It follows that $\Delta_{ii_1jj_n} = 0$ in $A_1 \cap A_2$. We also have that $\square_{ii_1jj_n}$ is monochromatic in $R_1$ and $\Delta_{ii_1jj_n} = 0$ holds in $A_1$. The last two equalities imply that $\Delta_{i'i'jj_n} = 0$ in $A_1 \cap A_2$, and this equality is implied by two of the facet equalities and one of the defining equalities of $A_1$. This completes the proof.

Putting together the considerations in the proofs of Proposition 2 and Theorem 1, it is now easy to describe the fiber of a generic point $g$ in the image. Such a point lies in a unique small cell, in two medium cells, and in a unique large cell. Therefore, it has preimages in exactly four regions of $\mathcal{A}_{m,n}$. The diagrams of these four regions are almost equal; they only differ in the color of the northwesternmost and southeasternmost entry; assume for simplicity that they are $g_{11}$ and $g_{mn}$, respectively.

Notice that we can add a constant to the $a_i$s and subtract it from the $b_j$s,
or add a constant to the $A_i$s and subtract it from the $B_j$s, without affecting $g(a, A, b, B)$. Therefore, we can focus our attention on the preimages of $g$ with, say, $A_m$ and $b_n$ fixed.

First consider the preimages in the region where $g_{11}$ and $g_{mn}$ are black. With $A_m$ and $b_n$ fixed, forcing $g(a, A, b, B)$ to have the correct border entries determines almost all of $(a, A, b, B)$. Only $a_m$ and $b_1$ are not determined; $a_m$ can take any value larger than $g_{mn} - b_n$ (the value it would have if $g_{mn}$ was white), and $b_1$ can take any value larger than $g_{11} - a_1 = g_{11} - g_{1n} + b_n$ (the value it would have if $g_{11}$ was white). The preimages form a two-dimensional quadrant parallel to the $a_m b_1$ plane of $\mathbb{R}^{2(m+n)}$.

The remaining three cells give similar preimages. Putting them all together, we are left with four two-dimensional quadrants parallel to the $a_m b_1, b_1 B_n, B_n A_1$ and $A_1 a_m$ planes of $\mathbb{R}^{2(m+n)}$. Their common apex is the point with $a_m = g_{mn} - b_n, b_1 = g_{11} - a_1, B_n = g_{mn} - A_m, A_1 = g_{11} - B_1$.

We have two extra degrees of freedom, given by the choices of $A_m$ and $B_n$. The preimage of a generic point described is a four-dimensional polyhedral complex; this is consistent with the dimension drop from the $(2m + 2n)$-dimensional range to the $(2m + 2n - 4)$-dimensional image.

We now have all the tools to enumerate the cells in the two subdivisions of $B_{m,n}$ that we have constructed.

**Proposition 3.** Let $\text{small}_{m,n}$ be the number of small cells in $B_{m,n}$. For $m, n \geq 2$,

$$\text{small}_{m,n} = 2 \left( \begin{array}{c} m \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) r_{m-2,n-2};$$

where $r_{m-2,n-2}$ is the number of regions of the real arrangement $A_{m-2,n-2}$.

**Proof.** If we remove the top and bottom rows and leftmost and rightmost columns from the diagram of a small cell of $B_{m,n}$, we obtain the diagram of a region of an arrangement combinatorially isomorphic to $A_{m-2,n-2}$.

To recover the diagram of the cell from the diagram of the region, we need to choose the labels of the top and bottom rows (for which there are $m(m-1)$ options to choose from), and the leftmost and rightmost columns (for which there are $n(n-1)$ options) which we deleted. We also need to extend the path that separates the black and white cells; there is only one way of doing this that gives the diagram of a small cell. Finally, remember that each cell has two diagrams that represent it, which differ by a $180^\circ$ rotation and a color switch. The desired result follows. \qed
Proposition 4. Let \( \text{large}_{m,n} \) be the number of large cells in \( B_{m,n} \). Then
\[
\sum_{m,n \geq 0} \text{large}_{m,n} \frac{x^m y^n}{m! n!} = \frac{1}{2} (x X + y Y) + \frac{2 X Y + X^2 (e^x - 1) + Y^2 (e^y - 1)}{2 (e^x + e^y - e^{x+y})},
\]
where \( X = x(e^y - y - 1) \) and \( Y = y(e^x - x - 1) \).

Proof. Imitate the proof of Proposition \( \square \)

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