On the level-dependence of Wess-Zumino-Witten three-point functions

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Abstract

Three-point functions of Wess-Zumino-Witten models are investigated. In particular, we study the level-dependence of three-point functions in the models based on algebras $su(3)$ and $su(4)$. We find a correspondence with Berenstein-Zelevinsky triangles. Using previous work connecting those triangles to the fusion multiplicities, and the Gepner-Witten depth rule, we explain how to construct the full three-point functions. We show how their level-dependence is similar to that of the related fusion multiplicity. For example, the concept of threshold level plays a prominent role, as it does for fusion.

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1. Introduction

A general study of Wess-Zumino-Witten (WZW) three-point functions was initiated in [1]. At sufficiently high level $k$, three-point functions for so-called generating-function primary fields were written as

$$\langle \phi_\lambda(z_1, x_1) \phi_\mu(z_2, x_2) \phi_\nu(z_3, x_3) \rangle = \Delta(z_1, z_2, z_3) F_{\lambda,\mu,\nu}^{(k)}(x_1, x_2, x_3).$$

(1.1)

A (generating-function) primary field $\phi_\lambda(z, x)$ is labelled by a dominant weight $\lambda$ in

$$P^k = \{ \sigma = \sum_{j=1}^r \sigma_j \Lambda^j | \sigma_j \in \mathbb{Z}_{\geq}, \forall j = 1, \ldots, r; \sum_{j=1}^r \sigma_j a^\vee j \leq k \} ,$$

(1.2)

so that its dependence on the level $k$ is implicit. Here $\Lambda^j$ is the $j$-th fundamental weight of a simple Lie algebra $X_r$, of rank $r$, and $a^\vee j$ is the corresponding co-mark. That is, the highest root of $X_r$ has co-root $\theta^\vee = \sum_{j=1}^r a^\vee j \alpha^\vee j$, where $\alpha^\vee j$ is co-root to the $j$-th simple root $\alpha_j$. Henceforth, we will normalise the long roots $\alpha^2_{\text{long}} = 2$, so that $\theta^\vee = \theta$.

In $\phi_\lambda(z, x)$, $z$ is the holomorphic world-sheet coordinate, and $x$ represents the flag variables - there is an independent variable $x^\alpha$ for every positive root $\alpha \in R_\geq$ of the simple Lie algebra $X_r$.

In (1.1),

$$\Delta(z_1, z_2, z_3) := (z_1 - z_2)^{-\Delta_1 - \Delta_2 + \Delta_3} (z_2 - z_3)^{\Delta_1 - \Delta_2 - \Delta_3} (z_3 - z_1)^{-\Delta_1 + \Delta_2 - \Delta_3},$$

(1.3)

where $\Delta_j$, $j = 1, 2, 3$, denote the conformal weights of the three primary fields. That is, the primary field $\phi_\lambda(z_1, x_1)$ obeys the following operator product expansions (OPEs) with the energy-momentum tensor $T(z)$

$$T(z) \phi_\lambda(z_1, x_1) \sim \frac{\Delta_1}{(z - z_1)^2} \phi_\lambda(z_1, x_1) + \frac{1}{z - z_1} \frac{\partial \phi_\lambda(z_1, x_1)}{\partial z_1},$$

(1.4)

and the currents $J_a(z)$

$$J_a(z) \phi_\lambda(z_1, x_1) \sim \frac{-1}{z - z_1} \omega(J_a)(x_1, \partial, \lambda) \phi_\lambda(z_1, x_1).$$

(1.5)

Similar OPEs are obeyed by $\phi_\mu(z_2, x_2)$ and $\phi_\nu(z_3, x_3)$. $J_a(x_1, \partial, \lambda)$ denotes a differential operator (in $x_1$) that realises the corresponding finite-dimensional Lie algebra generator $J_a$ - see [1,2] for more details. In (1.3), $\omega(J_a)$ is the generator of $X_r$ obtained from $J_a$ by the Chevalley involution $\omega$. Since the conformal weights depend on the level, the function $\Delta(z_1, z_2, z_3)$ has an implicit and well-known dependence on the level.
Here we investigate the extension of (1.1) from high level to all levels \( k \in \mathbb{Z}_\geq \). We will treat different levels together, investigating the so-called level-dependence of the three-point functions. The implicit level-dependence of the factor \( \Delta(z_1, z_2, z_3) \) is well-known, so the object of study will be \( F_{\lambda, \mu, \nu}^{(k)}(x_1, x_2, x_3) \).

Section 2 is a general discussion of three-point functions. It also contains a review of threshold levels in fusion and indicates how they enter consideration of three-point functions. Sections 3 and 4 treat the cases \( su(3) \) and \( su(4) \), respectively. (The case of \( su(2) \) is covered in Appendix A.) Section 5 is a concluding discussion.

2. Three-point functions

It is important here that the Ward identities can be written in terms of the differential operators \( J_a(x, \partial, \lambda) \). For example, we have

\[
0 = \left( \sum_{j=1}^{3} J_a(x_j, \partial, \Lambda(j)) \right) F_{\Lambda(1), \Lambda(2), \Lambda(3)}^{(k)}(x_1, x_2, x_3). \tag{2.1}
\]

Here we used \( \Lambda(1) := \lambda, \Lambda(2) := \mu, \Lambda(3) := \nu \), for convenience of notation.

Ward identities encoding the level-dependence, however, are not imposed by (2.1). Those were used by Gepner and Witten in the derivation of their depth rule [3]. Let \( E_\theta \) denote the generator of \( X_r \) that raises the weight by the highest root \( \theta \) of \( X_r \). In the form useful to us, a symmetrical version of the Gepner-Witten constraint is

\[
0 = \prod_{j=1}^{3} \left( E_\theta(x_j, \partial, \Lambda(j)) \right)^{\ell_j} F_{\Lambda(1), \Lambda(2), \Lambda(3)}^{(k)}(x_1, x_2, x_3), \tag{2.2}
\]

\[\forall (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_\geq^3, \text{ such that } \ell_1 + \ell_2 + \ell_3 > k.\]

The differential operator expression for \( E_\theta \) is particularly simple:

\[
E_\theta(x, \partial, \lambda) = \frac{\partial}{\partial x^{\theta}}. \tag{2.3}
\]

As a consequence, the differential operator realisation is ideally suited to the implementation of the Gepner-Witten depth rule: (2.2) takes a simple, useful form,

\[
0 = \prod_{j=1}^{3} \left( \frac{\partial}{\partial x^{\theta_j}} \right)^{\ell_j} F_{\Lambda(1), \Lambda(2), \Lambda(3)}^{(k)}(x_1, x_2, x_3), \tag{2.4}
\]

\[\forall (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_\geq^3, \text{ such that } \ell_1 + \ell_2 + \ell_3 > k.\]
This says that the level $k$ must be greater than or equal to the highest power of $x^\theta$ in $F_{\lambda,\mu,\nu}^{(k)}$, using $x_j^\theta = x^\theta$ for all $j = 1, 2, 3$.

Now $J_a|0\rangle = 0$, for all generators $J_a$ of the simple Lie algebra, where $|0\rangle$ represents the scalar state for the diagonal subalgebra of $X_r \oplus X_r \oplus X_r$. From this, we will see shortly that

$$F_{\lambda,\mu,\nu}^{(k)}(x_1, x_2, x_3) = \left( \bigotimes_{j=1}^{3} \langle \Lambda_{(j)} | G_+(x_j) P \rangle \right) |0\rangle \rangle \tag{2.5}$$

satisfies the Ward identities (2.1). Here we follow the notation of [1], so that

$$G_+(x) := \exp \left( \sum_{\alpha \in R_+} x^\alpha E_\alpha \right), \tag{2.6}$$

with $E_\alpha$ being the raising operator of $X_r$ associated to $\alpha \in R_+$. $P$ is the projector from the algebra $X_r \oplus X_r \oplus X_r$ to its diagonal subalgebra. Because

$$0 = \bigotimes_{j=1}^{3} \langle \Lambda_{(j)} | G_+(x_j) P \rangle (J_a|0\rangle \rangle), \tag{2.7}$$

we get

$$0 = \left( \bigotimes_{j=1}^{3} \langle \Lambda_{(j)} | G_+(x_j) \rangle \Delta(J_a) \right) |0\rangle \rangle \tag{2.8}$$

$$= \sum_{\ell=1}^{3} J_a(x_\ell, \partial, \Lambda_{(\ell)}) \bigotimes_{j=1}^{3} \langle \Lambda_{(j)} | G_+(x_j) |0\rangle \rangle, \tag{2.8}$$

using $PJ_a = \Delta(J_a)P$ with $\Delta(J_a) = (J_a \otimes I \otimes I) \oplus (I \otimes J_a \otimes I) \oplus (I \otimes I \otimes J_a)$. So (2.3) obeys (2.1), but the condition (2.2) must still be imposed.

Let $L(\lambda)$ denote the representation of $X_r$ of highest weight $\lambda$. We can decompose the polynomial (2.5) as

$$F_{\lambda,\mu,\nu}^{(k)}(x_1, x_2, x_3) = \left( \sum_{a} W_{\lambda,\mu,\nu}^{[a]}(x_1, x_2, x_3) \langle \langle 0 |_a \rangle \rangle \right) |0\rangle \rangle = \sum_{a} W_{\lambda,\mu,\nu}^{[a]}(x_1, x_2, x_3), \tag{2.9}$$

where $\langle \langle 0 |_a \rangle \rangle$ denotes one of the (normalised) singlet states in the triple tensor product $L(\lambda) \otimes L(\mu) \otimes L(\nu)$. The $k$-dependence will enter in the summation range (see (2.15) below).

Incidentally, we note that $W_{\lambda,\mu,\nu}^{[a]}(x_1, x_2, x_3)$ is a generating function for the Clebsch-Gordan coefficients of a coupling $L(\lambda) \otimes L(\mu) \otimes L(\nu) \supset L(0)$. We can write

$$|0\rangle \rangle_a = \sum_{|u\rangle \in L(\lambda)} \sum_{|v\rangle \in L(\mu)} \sum_{|w\rangle \in L(\nu)} C_{u,v,w}^{[a]} |u\rangle \otimes |v\rangle \otimes |w\rangle, \tag{2.10}$$
where \( C_{u,v,w}^{[a]} \) denotes the Clebsch-Gordan coefficient appropriate to the coupling indicated by \( a \). Define
\[
K^\lambda_u(x) := \langle \lambda | G_+(x) | u \rangle ,
\]
for \( |u\rangle \in L(\lambda) \). Then (2.5) and (2.9) yield
\[
W^{[a]}_{\lambda,\mu,\nu}(x_1, x_2, x_3) = \sum_{|u\rangle \in L(\lambda)} \sum_{|v\rangle \in L(\mu)} \sum_{|w\rangle \in L(\nu)} C_{u,v,w}^{[a]} K^\lambda_u(x_1) K^\mu_v(x_2) K^\nu_w(x_3) .
\]
\((2.12)\)

A key step in [1] was to write the polynomials \( F^{(k)}_{\lambda,\mu,\nu}(x_1, x_2, x_3) \) as sums of products of elementary polynomials:
\[
F^{(k)}_{\lambda,\mu,\nu}(x) = \sum_a \prod_{E \in \mathcal{E}} [R^E(x)]^{p_a(E)} .
\]
\((2.13)\)

Here \( \mathcal{E} \) denotes the set of elementary (three-point) couplings of the algebra \( X_r \), \( R^E(x) \) the corresponding polynomial, and \( (x) \) is short for \( (x_1, x_2, x_3) \). One necessary constraint on the products appearing in the decomposition (2.13) is determined by the weight \( \{\lambda, \mu, \nu\} \). If we define \( \text{wt}(E) \) to be the weight of an elementary coupling \( E \), then
\[
\sum_{E \in \mathcal{E}} p(E) \text{wt}(E) = \{\lambda, \mu, \nu\}
\]
\((2.14)\)
must hold. The sets \( \mathcal{E} \) must be found for each algebra (see [4], for example). There are also certain algebraic relations among the elementary couplings (and so among the elementary polynomials) that must be taken into account. These relations are sometimes called syzygies. They can be implemented by excluding certain products as redundant from the sum in (2.13).

Once the syzygies are implemented, each summand in (2.13) counts a coupling between the primary fields \( \phi_\lambda(z_1, x_1) \), \( \phi_\mu(z_2, x_2) \) and \( \phi_\nu(z_3, x_3) \). That is, each summand contributes 1 to the fusion multiplicity \( T^{(k)}_{\lambda,\mu,\nu} \) that corresponds to the three-point function (1.1). We therefore rewrite (2.9) as
\[
F^{(k)}_{\lambda,\mu,\nu}(x) = \sum_{a=1}^{T^{(k)}_{\lambda,\mu,\nu}} W^{[a]}_{\lambda,\mu,\nu}(x) ,
\]
\((2.15)\)
with
\[
W^{[a]}_{\lambda,\mu,\nu}(x) = \prod_{E \in \mathcal{E}} [R^E(x)]^{p_a(E)} .
\]
\((2.16)\)
Since each factor $R^E(x)$ is itself a valid polynomial, the $X_r$-symmetry is respected automatically. Incidentally, here one also sees the usefulness of the $x$-dependence of the generating function primary fields: it makes the different summands $W^{[a]}_{\lambda,\mu,\nu}(x)$ independent.

The $k$-dependence of a fixed $W^{[a]}_{\lambda,\mu,\nu}(x)$ is straightforward, since by (2.4) we must have

$$0 = \prod_{j=1}^{3} \left( \frac{\partial}{\partial x_j^\theta} \right) \ell_j W^{[a]}_{\Lambda_1,\Lambda_2,\Lambda_3}(x_1, x_2, x_3),$$

$$\forall (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_3^\geq, \text{ such that } \ell_1 + \ell_2 + \ell_3 > k. \quad (2.17)$$

2.1. Threshold levels

Let $t_a$ denote the maximum power of $x^\theta$ in $W^{[a]}_{\lambda,\mu,\nu}(x)$ with non-zero coefficient:

$$t_a = \min \left\{ t \in \mathbb{Z}_\geq \mid \prod_{j=1}^{3} \left( \frac{\partial}{\partial x_j^\theta} \right) \ell_j W^{[a]}_{\lambda,\mu,\nu}(x) = 0, \right. \quad (2.18)$$

$$\forall (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_3^\geq, \sum_{j=1}^{3} \ell_j = t + 1 \left. \right\}.$$

Then $W^{[a]}_{\lambda,\mu,\nu}(x)$ will not contribute to the sum (2.15) if $t_a$ is greater than the level $k$, but does contribute if $t_a \leq k$. That is, $t_a$ is a threshold level for $W^{[a]}_{\lambda,\mu,\nu}(x)$.

Since (2.18) only involves $x_j^\alpha$ for $\alpha = \theta$, it appears to violate the symmetry of the horizontal algebra $X_r$. It therefore must be interpreted with care - one could just drop the higher powers of $x^\theta$, for example, to obtain incorrect results. The condition should be used to restrict the $X_r$-invariant summands in (2.15), as we’ll discuss below.

The concept of threshold level has been useful in WZW fusion, when the fusion is viewed as a truncation of the tensor product of simple Lie algebras [5–8]. The decomposition of a tensor product $L(\lambda) \otimes L(\mu)$ of such representations can be written as

$$L(\lambda) \otimes L(\mu) = \bigoplus_{\nu \in P_\geq} T^{\nu}_{\lambda,\mu} L(\nu), \quad (2.19)$$

where

$$P_\geq = \{ \sigma = \sum_{j=1}^{r} \sigma_j A_j \mid \sigma_j \in \mathbb{Z}_\geq, \forall j = 1, \ldots, r \} \quad (2.20)$$

(compare to (1.2)). More important for three-point functions is

$$L(\lambda) \otimes L(\mu) \otimes L(\nu) \supset T^{\nu}_{\lambda,\mu} L(0), \quad (2.21)$$
where the triple product multiplicities $T_{\lambda,\mu,\nu}$ are related to the conventional tensor product multiplicities $T^{\nu}_{\lambda,\mu}$ by

$$T_{\lambda,\mu,\nu} = T^{\nu+}_{\lambda,\mu},$$

(2.22)

$\nu^+$ being the weight conjugate to $\nu$. Fusion products $\otimes_k$ can be written in a similar way:

$$L(\lambda) \otimes_k L(\mu) = \bigoplus_{\nu \in P_k} T^{(k)}_{\lambda,\mu,\nu} L(\nu)$$

(2.23)

and

$$L(\lambda) \otimes_k L(\mu) \otimes_k L(\nu) \supset T^{(k)}_{\lambda,\mu,\nu} L(0),$$

(2.24)

where

$$T^{(k)}_{\lambda,\mu,\nu} = T^{(k)}_{\lambda,\mu} \nu^+.$$

(2.25)

That fusion is a truncated tensor product,

$$T^{(k)}_{\lambda,\mu,\nu} \leq T^{(k+1)}_{\lambda,\mu,\nu}, \quad \lim_{k \to \infty} T^{(k)}_{\lambda,\mu,\nu} = T_{\lambda,\mu,\nu},$$

(2.26)

follows from the Gepner-Witten depth rule [3,6].

One can encode all this in a simple fashion using the threshold level. The triple product multiplicities can be written as a sum over terms of specific threshold level:

$$T_{\lambda,\mu,\nu} = \sum_{t=0}^{\infty} n^{[t]}_{\lambda,\mu,\nu},$$

(2.27)

where the threshold multiplicity $n^{[t]}_{\lambda,\mu,\nu}$ is the number of couplings of threshold level $t$. We can write the fusion triple product multiplicity in a similar way:

$$T^{(k)}_{\lambda,\mu,\nu} = \sum_{t=0}^{k} n^{[t]}_{\lambda,\mu,\nu}.$$

(2.28)

To compare this with a similar result for three-point functions, we define $W^{[a]}_{\lambda,\mu,\nu}(x; t) := W_{\lambda,\mu,\nu}(x)$, if $t_a = t$. The goal is to write

$$F^{(k)}_{\lambda,\mu,\nu}(x) = \sum_{t=0}^{k} \sum_{a=1}^{n^{[t]}_{\lambda,\mu,\nu}} W^{[a]}_{\lambda,\mu,\nu}(x; t),$$

(2.29)

where the values of the index $a$ have been redefined appropriately.

We emphasise here that a choice must be made to write (2.29). Due to the syzygies, different representations of $W^{[a]}$ will exist, in general with different threshold levels $t_a$. 

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By (2.18), one sees that the threshold level of a product of polynomials is just the sum of their individual threshold levels. We define the threshold level \( t(E) \) of an elementary coupling \( E \in \mathcal{E} \) by

\[
t(E) := \min \left\{ t \in \mathbb{Z} \geq 0 \mid \prod_{j=1}^{3} \left( \frac{\partial}{\partial x_j^E} \right)^{\ell_j} R^E(x_1, x_2, x_3) = 0 , \forall \sum_{j=1}^{3} \ell_j = t + 1 \right\} .
\]

Then when (2.10) gets updated to

\[
W_{\lambda, \mu, \nu}^{[a]}(x; t_a) = \prod_{E \in \mathcal{E}} [R^E(x)]^{p_a(E)},
\]

we must have

\[
t_a = \sum_{E \in \mathcal{E}} p_a(E) t(E) .
\]

For \( X_r = A_r \cong su(N) \) (so that \( N = r + 1 \)), the couplings counted in \( T_{\lambda, \mu, \nu} \) are in one-to-one correspondence with Berenstein-Zelevinsky (BZ) triangles \( \mathcal{T} \) of weight \( \{\lambda, \mu, \nu\} \). The BZ triangles were used to study \( su(3) \) fusion in [3][7]; \( su(4) \) fusion was studied in [8], which also alluded to \( su(N) \) generalisation. For \( su(3) \) and \( su(4) \), it was found that every BZ triangle could be assigned a threshold level. We will now “lift” those results to the corresponding polynomials, by associating a BZ triangle to each polynomial \( W_{\lambda, \mu, \nu}^{[a]}(x; t) \). Not only will this exercise promote the previous results to something containing more information, it will also provide the justification for some of them.

3. \( su(3) \) three-point functions, triangles and fusion

An \( su(3) \) BZ triangle of weight \( \{\lambda, \mu, \nu\} \) looks like:

\[
\begin{array}{ccc}
m_{13} & m_{12} & m_{23} \\
12 & & \\
n_{13} & l_{12} & n_{23} \\
& m_{12} & n_{23} \\
\end{array}
\]

(3.1)

Its entries \( l_{ij}, m_{ij}, n_{ij} \in \mathbb{Z}_{\geq} \) determine the Dynkin labels of \( \lambda, \mu, \nu \in \mathcal{P}_{\geq} \):

\[
\begin{align*}
m_{13} + n_{12} &= \lambda_1 , & n_{13} + l_{12} &= \mu_1 , & l_{13} + m_{12} &= \nu_1 , \\
m_{23} + n_{13} &= \lambda_2 , & n_{23} + l_{13} &= \mu_2 , & l_{23} + m_{13} &= \nu_2 .
\end{align*}
\]

(3.2)

We call these the outer constraints on the BZ entries.
For general $su(N)$, the BZ entries determine three elements of $\mathbb{Z}_{\geq} R>$:

$$\lambda + \mu - \nu^+ = \sum_{\alpha \in R>} n_{\alpha} \alpha =: n \cdot R>$$

$$\nu + \lambda - \mu^+ = \sum_{\alpha \in R>} m_{\alpha} \alpha =: m \cdot R>$$

$$\mu + \nu - \lambda^+ = \sum_{\alpha \in R>} l_{\alpha} \alpha =: l \cdot R> .$$

Using an orthonormal basis $\{ e_j \mid j = 1, \ldots, N; e_i \cdot e_j = \delta_{i,j} \}$ of $\mathbb{R}^N$, the positive roots can be written as

$$R> = \{ e_i - e_j \mid 1 \leq i < j \leq N \} .$$

This explains the indices $ij$ on the BZ entries.

If the $su(3)$ version of (3.3) is written, by applying the conjugation operation $^+$, one can derive the consistency conditions

$$n_{12} + m_{23} = n_{23} + m_{12} ,$$

$$l_{12} + m_{23} = l_{23} + m_{12} ,$$

$$l_{12} + n_{23} = l_{23} + n_{12} .$$

These are the hexagon constraints, so named because of the geometry of (3.4).

The triple tensor product multiplicity $T_{\lambda,\mu,\nu}$ equals the number of BZ triangles with non-negative integer entries that satisfy both the outer and hexagon constraints. Now a sum of BZ triangles is also a BZ triangle, but with a new weight equalling the sum of the weights of the added triangles. Compare this to (2.14); a product of “three-point polynomials” ($R^E$’s, and so by extension $W_{\lambda,\mu,\nu}^{[a]}$’s) is also a three-point polynomial, but with the weights added. Thus, adding triangles corresponds to multiplying polynomials. Clearly then, we should find the triangles associated to the elementary polynomials, the elementary triangles. Then the triangle associated with an arbitrary polynomial $W_{\lambda,\mu,\nu}^{[a]}(x)$ can be found simply from its factorisation into powers of elementary polynomials.

The elementary polynomials for $su(3)$ and $su(4)$ were worked out completely in [1]. Recall that $\alpha_b$ denotes the $b$-th simple root. For $su(3)$, $R_> = \{ \alpha_1, \theta = \alpha_1 + \alpha_2, \alpha_2 \}$. We write $x_j^{\alpha_b} =: x_j^b$ for the flag coordinates related to the simple roots. The variable $x^\theta$ only appears in the combinations

$$x_j^\pm := \frac{1}{2} x_j^1 x_j^2 \pm x_j^\theta .$$
This can be seen as follows. Let $F_\alpha$ denote the lowering operator of $X_r$ associated with $\alpha \in R_>$. In the differential operator realisation of the algebra $X_r$, one finds

$$F_\alpha(x, \partial, \lambda) = \sum_{\beta \in R_>} V^\beta_{-\alpha}(x) \frac{\partial}{\partial x^\beta} + \sum_{j=1}^r P^j_{-\alpha}(x) \lambda_j.$$  

(3.7)

Here $V^\beta_{-\alpha}(x)$ and $P^j_{-\alpha}(x)$ denote polynomials, explicitly written in [2], and only involving $x^\theta$ in the combinations (3.6). This property is carried over to the elementary polynomials, by a straightforward analysis.

Although for general $X_r$ it does not hold, for $su(3)$ and $su(4)$, an elementary coupling $E \in \mathcal{E}$ is uniquely specified by its weight. We will label the corresponding polynomials with those weights. $su(3)$ has 8 elementary polynomials. Six are related to $R^1, R^2, 0$ and $R^2, 1, 0$.

$R^{1,2,0}(x) := R^{A^1, A^2, 0}(x_1, x_2, x_3) = x_1^+ + x_2^- - x_1^1 x_2^2$ ;

(3.8)

they can be obtained from (3.8) by permuting the subscripts of $x_1, x_2, x_3$ and the three weights $A^1, A^2, 0$ in the identical way. These six elementary polynomials are actually elementary polynomials for two-point couplings - see [10]. The other two elementary polynomials are

$$R^{1,1,1}(x) := R^{A^1, A^1, A^1}(x_1, x_2, x_3) = x_2^1 x_3^1 - x_2^+ x_3^1 + x_1^1 (x_2^+ - x_3^+) + x_1^+ (x_3^1 - x_2^1) ,$$

(3.9)

and $R^{2,2,2}(x) := R^{A^2, A^2, A^2}(x_1, x_2, x_3)$, obtained from $R^{1,1,1}(x)$ by $x_j^+ \to x_j^-$ and $x_j^1 \to x_j^2$.

The elementary $su(3)$ BZ triangles were written in [4]. They are

$$
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{array}
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}$$

(3.10)

and those obtained by applying the triangle symmetries that generate the dihedral group $D_3 \cong S_3$. The $S_3$ orbits of the first two triangles of (3.10) are 3-dimensional, and that of the last is 2-dimensional, giving the correct number of 8 elementary triangles.

It is simple to assign these elementary triangles to elementary polynomials. For example, those of (3.10) correspond to $R^{1,2,0}$, $R^{2,1,0}$ and $R^{1,1,1}$, respectively. The threshold levels are also easily found. Because $x^\theta$ only appears in the combinations $x^\pm$ of (3.6), the threshold level is simply the maximum power of $x^\pm$. By (3.8) and (3.9), we see that all 8 elementary polynomials (and so elementary couplings $E$) have threshold level $t(E) = 1$. 

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If we write $\lambda = \lambda_1 A^1 + \lambda_2 A^2 = (\lambda_1, \lambda_2)$, then consider the example of the three-point function with weight $\{(1,1), (1,1), (1,1)\}$. The tensor product multiplicity is 2, with threshold levels 2 and 3 for the two couplings. We write

$$W_{(1,1),(1,1),(1,1)}^{[1]}(x; 2) = R_{(1,1),(1,1)}^{1,1,1}(x) R_{(1,1),(1,1)}^{2,2,2}(x)$$ (3.11)

and

$$W_{(1,1),(1,1),(1,1)}^{[2]}(x; 3) = R_{(1,1),(1,1)}^{2,1,0}(x) R_{(1,1),(1,1)}^{0,2,1}(x) R_{(1,1),(1,1)}^{1,0,2}(x) ,$$ (3.12)

for the polynomials of threshold levels 2 and 3, respectively.

But another product $R_{(1,1),(1,1),(1,1)}^{1,2,0} R_{(1,1),(1,1),(1,1)}^{0,1,2} R_{(1,1),(1,1),(1,1)}^{2,0,1}$ has weight $\{(1,1), (1,1), (1,1)\}$ - why is it not included? For $su(3)$, there is a single syzygy

$$R_{(1,1),(1,1),(1,1)}^{1,1,1} R_{(1,1),(1,1),(1,1)}^{2,2,2} = R_{(1,1),(1,1),(1,1)}^{1,2,0} R_{(1,1),(1,1),(1,1)}^{0,1,2} R_{(1,1),(1,1),(1,1)}^{2,0,1} + R_{(1,1),(1,1),(1,1)}^{2,1,0} R_{(1,1),(1,1),(1,1)}^{0,2,1} R_{(1,1),(1,1),(1,1)}^{1,0,2}$$ (3.13)

to be taken into account. Only two linear combinations of the three $R$-monomials in (3.13) can be included, since three are not independent. Here the triangle correspondence may guide us, since there are no relations on the BZ triangles themselves - they are simply counted to get the tensor product multiplicity. Associating a BZ triangle to each elementary polynomial implements the syzygies automatically: both $R_{(1,1),(1,1),(1,1)}^{1,1,1} R_{(1,1),(1,1),(1,1)}^{2,2,2}$ and $R_{(1,1),(1,1),(1,1)}^{1,2,0} R_{(1,1),(1,1),(1,1)}^{0,1,2} R_{(1,1),(1,1),(1,1)}^{2,0,1}$ correspond to the same BZ triangle. In order to produce the threshold levels $\{2,3\}$, it is the product $R_{(1,1),(1,1),(1,1)}^{1,2,0} R_{(1,1),(1,1),(1,1)}^{0,1,2} R_{(1,1),(1,1),(1,1)}^{2,0,1}$ that should be excluded.

That means that if

$$W_{\lambda, \mu, \nu}^{[\ell]} = (R_{(1,1),(1,1),(1,1)}^{1,2,0})^{p_{1,2,0}} (R_{(1,1),(1,1),(1,1)}^{0,1,2})^{p_{0,1,2}} (R_{(1,1),(1,1),(1,1)}^{2,0,1})^{p_{2,0,1}}$$

$$\times (R_{(1,1),(1,1),(1,1)}^{2,1,0})^{p_{2,1,0}} (R_{(1,1),(1,1),(1,1)}^{0,2,1})^{p_{0,2,1}} (R_{(1,1),(1,1),(1,1)}^{1,0,2})^{p_{1,0,2}} (R_{(1,1),(1,1),(1,1)}^{1,1,1})^{p_{1,1,1}} (R_{(1,1),(1,1),(1,1)}^{2,2,2})^{p_{2,2,2}} ,$$ (3.14)

then at least one of $p_{1,2,0}, p_{0,1,2}, p_{2,0,1}$ must vanish. Now, (3.14) corresponds to the triangle

$$p_{1,0,2}
\begin{align*}
p_{1,2,0} + p_{1,1,1} & \quad p_{0,1,2} + p_{2,2,2} \\
p_{2,0,1} + p_{2,2,2} & \quad p_{2,0,1} + p_{1,1,1} \\
p_{2,1,0} & \quad p_{0,1,2} + p_{1,1,1} \quad p_{1,2,0} + p_{2,2,2} \quad p_{0,2,1}
\end{align*}$$ (3.15)

Let's assume that $p_{1,2,0} = 0$. Because $t(E) = 1$ for all elementary couplings, we see that the threshold level of (3.14) is just the sum of all the powers $p_{i,j,k}$, or $t = n_{13} + m_{13} +$
\[ l_{23} + m_{12} + l_{13} = n_{13} + \nu_1 + \nu_2. \]

Notice that \( p_{1,2,0} = 0 \) ensures that \( n_{13} + \nu_1 + \nu_2 \geq \max\{ m_{13} + \mu_1 + \mu_2, l_{13} + \lambda_1 + \lambda_2 \} \). By considering the other possibilities, \( p_{0,1,2} = 0 \) and \( p_{2,0,1} = 0 \), we find the cyclically symmetric formula

\[
t = \max \{ l_{13} + \lambda_1 + \lambda_2, m_{13} + \mu_1 + \mu_2, n_{13} + \nu_1 + \nu_2 \},
\]

confirming the result of [3].

Of course, our results constitute much more than a verification of the results of [3] on fusion multiplicities. We have shown how to construct the full three-point functions (3.14). In addition, we have demonstrated that the fusion results “lift” in a straightforward manner to those for three-point functions.

Furthermore, in [3] certain obstacles were found in a direct application of the Gepner-Witten depth rule. Here we have shown how to implement the depth rule in a very simple way, using a differential operator realisation of \( X_r \). We will have more to say on this subject below.

The triangles helped us implement the syzygy (3.13), by choosing a product to forbid, \( R^{1,2,0} R^{0,1,2} R^{2,0,1} \). But a different choice could certainly have been made, with no effect on the results at large level. Because the syzygy is inhomogeneous in the threshold level, however, the choice becomes important at smaller levels. For example, if we had forbidden the product \( R^{1,1,1} R^{2,2,2} \), then the additivity (2.32) of threshold levels would be lost. Another possibility is to forbid \( R^{2,1,0} R^{0,2,1} R^{1,0,2} \). This is equivalent to choosing the orientation around the BZ triangle that is opposite to that used in (3.1). Of course, one could also forbid a linear combination of the \( R \)-monomials in (3.13), but such a choice is not convenient for us.

It is often preferable to work directly with the polynomials \( W_{\lambda,\mu,\nu}^{[a]}(x) \), rather than to first express them as products of the elementary polynomials \( R^E(x) \). But we can find the triangle of a polynomial \( W_{\lambda,\mu,\nu}^{[a]}(x) \) directly. For our purposes, rewrite (3.3) as

\[
\begin{align*}
\lambda + \left( \mu - \sum_{\alpha \in R_>} n_\alpha \alpha \right) + (-\nu^+) &= 0 \\
\nu + \left( \lambda - \sum_{\alpha \in R_>} m_\alpha \alpha \right) + (-\mu^+) &= 0 \\
\mu + \left( \nu - \sum_{\alpha \in R_>} l_\alpha \alpha \right) + (-\lambda^+) &= 0.
\end{align*}
\]

(3.17)
The first of these expressions suggests that we look at the contribution $|\lambda\rangle \otimes |\mu'\rangle \otimes |-\nu^+\rangle$.

Here $|\lambda\rangle$ is the highest weight state of $L(\lambda)$, $|−\nu^+\rangle$ is the lowest weight state of $L(\nu)$, and $|\mu'\rangle$ denotes a state of $L(\mu)$, of weight $\mu' := \mu - n \cdot R$.

Let $x_3^{[\beta]}$ denote any linear combination of products

$$\prod_{\alpha \in R_>} (x_3^\alpha)^{b_\alpha}$$

such that $\sum b_\alpha \alpha = \beta$. By construction, $x_1^{[0]} = 1$ isolates the highest weight state $|\lambda\rangle$, while the lowest weight state $|−\nu^+\rangle$ corresponds to $x_3^{[\nu^+\nu^+]}$ (see (3.11), for example). Now, a fixed $W_{\lambda,\mu,\nu}^{[\alpha]}$ should include a unique term (with non-vanishing coefficient) $x_1^{[0]} x_2^{[\mu'-\mu]} x_3^{[\nu^+\nu^+]}$, or $x_1^{[0]} x_2^{[n \cdot R>]\nu^+\nu^+]$. Let that term be denoted $W_{\lambda,\mu,\nu}^{[\alpha]}(x_1^{[0]} x_2^{[n \cdot R>]\nu^+\nu^+]$. We find that its factor $x_2^{[n \cdot R>]}$ suffices to determine the $\{n_\alpha\}$, i.e. one third of the BZ entries. The other two thirds are obtained in a similar way, using the other two equations of (3.17).

Simply put, we find

$$x_2^{[n \cdot R>]\nu^+\nu^+] = (x_2^1)^{n_{12}} (x_2^2)^{n_{13}} (x_2^3)^{n_{23}}.$$  

This result and its two siblings can be encoded in the form of a BZ triangle:

$$\begin{array}{ccc}
x_1^1 & x_2^1 & x_3^1 \\
x_2^1 & x_2^2 & x_3^2 \\
x_2^2 & x_2^3 & x_3^3 \end{array}$$

This diagram can be used to find the BZ triangle corresponding to a given polynomial $W_{\lambda,\mu,\nu}^{[\alpha]}(x)$. We should emphasise, however, that it does not allow us to construct the full polynomial from a prescribed BZ triangle, only a few terms in it.

A refined version of the Gepner-Witten depth rule [3] was conjectured in [11,12] (see also [12]). It is less symmetric than (2.2), but is a natural generalisation of a well-known formula in the theory of simple Lie algebras. It will prove to be more convenient in some computations. The realisation-independent form of the formula is

$$T_{\lambda,\mu,\nu}^{(k)} = \dim \{ v \in L(\mu;\nu^+ - \lambda) \mid F_{\alpha_i}^{\nu^+_i + 1} v = 0, \forall i \in \{1, 2, \ldots, r\}; E_{\theta}^{k-\nu^+\theta+1} v = 0 \}.$$  

Here $L(\mu;\nu^+ - \lambda)$ is the subspace of $L(\mu)$ of weight $\nu^+ - \lambda$, and $\nu^+_i$ is the $i$-th Dynkin label of $\nu^+$. The well-known classical formula for tensor product multiplicities (see [13],
for example) is obtained in the limit of large level $k \to \infty$. Then the constraint $E_{\theta}^k - \nu \cdot \theta + 1 = 0$ is always satisfied, and can be dropped.

If written similar to (2.17), it is

$$ 0 = \left( \frac{\partial}{\partial x_2^\theta} \right)^t W_{\lambda(1),\lambda(2),\lambda(3)}^{[a]}(x) x_1^{[0]} x_2^{[n-R>] x_3^{[\nu+\nu^+]}}, $$

$$ \forall \ell \in \mathbb{Z}_\geq, \text{ such that } \ell > k - \nu \cdot \theta. \quad (3.22) $$

Consequently, we find

$$ t_a = \min \left\{ t \in \mathbb{Z}_\geq \mid \left( \frac{\partial}{\partial x_2^\theta} \right)^{t+1} W_{\lambda,\mu,\nu}^{[a]}(x) x_1^{[0]} x_2^{[n-R>] x_3^{[\nu+\nu^+]}] = 0 \right\} + \nu \cdot \theta. \quad (3.23) $$

Clearly, the last two formulas treat the 3 weights in asymmetric manner. Consequently, 5 other formulas of this type can be written, based on the 5 other permutations of $\lambda, \mu, \nu$.

Using these formulas, we easily verify the threshold levels computed using the symmetric version of the depth rule. This provides evidence for the refinement of the depth rule (3.21), conjectured in [6,11]. This asymmetric form has the advantage that it effectively reduces consideration from the coupling of three states to a single state, one that necessarily couples to a simple highest-weight state and a simple lowest-weight state. Consequently, to use (3.22), we need only a small part of the polynomial $W_{\lambda,\mu,\nu}^{[a]}$; for (2.17), the full polynomial is required.

4. $su(4)$ three-point functions, triangles and fusion

The results for $su(4)$ are technically more complicated than those for $su(3)$. They also include an interesting new feature, discussed below.

The 18 elementary couplings may be specified by their weights, $\{\lambda, \mu, \nu\}$, or $\{\Lambda(1), \Lambda(2), \Lambda(3)\}$. Permuting 1,2,3 in $\{\Lambda(1), \Lambda(2), \Lambda(3)\}$ and $x_1, x_2, x_3$ produces other elementary couplings. The polynomial-triangle correspondence breaks the permutation $S_3$ symmetry to the cyclic $\mathbb{Z}_3$ symmetry (see (3.17), e.g.). So only cyclic permutations of 1,2,3 will produce other elementary couplings with related BZ triangles.

Of the 18 elementary couplings, 9 are of the two-point type [10]. They separate into three cyclic-permutation orbits, with representative polynomials

$$ R^{1,3,0} := R^{\Lambda^1,\Lambda^3,0}, \quad R^{3,1,0} := R^{\Lambda^3,\Lambda^1,0} \quad \text{and} \quad R^{2,2,0} := R^{\Lambda^2,\Lambda^2,0}. \quad (4.1) $$
The 9 elementary polynomials of three-point type form another three orbits, with representatives
\[ R^{1,1,2} := R^{\Lambda_1^1, \Lambda_1^2}, \quad R^{2,3,3} := R^{\Lambda_2^2, \Lambda_3^3}, \quad \text{and} \quad R^{13,2,2} := R^{\Lambda_1^1 + \Lambda_3^3, \Lambda_2^2}. \]  
(4.2)

By (2.18) or (3.23), we find that all these couplings have threshold level 1, except \( R^{13,2,2}, R^{2,13,2}, R^{2,2,13} \), which have \( t = 2 \).

For \( su(4) \) the BZ triangle is
\[
\begin{array}{ccccccc}
& m_{14} & & & & \left\{ n_{12} \right\} \\
n_{12} & l_{34} & m_{24} & m_{13} & n_{13} & l_{23} & n_{23} & l_{24} \\
m_{34} & n_{14} & m_{23} & m_{12} & m_{34} & n_{12} & l_{24} & l_{34} \\
& & l_{12} & n_{24} & l_{13} & n_{34} & l_{14} & \\
\end{array}
\]  
(4.3)

with corresponding Dynkin labels given by the outer constraints
\[
\begin{align*}
  m_{14} + n_{12} &= \lambda_1, & n_{14} + l_{12} &= \mu_1, & l_{14} + m_{12} &= \nu_1, \\
  m_{24} + n_{13} &= \lambda_2, & n_{24} + l_{13} &= \mu_2, & l_{24} + m_{13} &= \nu_2, \\
  m_{34} + n_{14} &= \lambda_3, & n_{34} + l_{14} &= \mu_3, & l_{34} + m_{14} &= \nu_3.
\end{align*}
\]  
(4.4)

The \( su(4) \) BZ triangle contains three hexagons, and there are inner constraints (hexagon identities) related to each of them in the obvious way.

The elementary polynomials of (4.1) and (4.2) can be related easily to elementary BZ triangles. The non-zero BZ entries corresponding to (4.1) are
\[
\{ n_{12} = n_{23} = n_{34} = 1 \}, \quad \{ n_{14} = 1 \} \quad \text{and} \quad \{ n_{13} = n_{24} = 1 \};
\]  
(4.5)

and those related to (4.2) are
\[
\begin{align*}
  \{ n_{12} = l_{12} = m_{13} = l_{23} = 1 \}, & \quad \{ m_{24} = l_{34} = n_{23} = n_{34} = 1 \} \\
\text{and} \quad \{ n_{12} = m_{34} = l_{23} = n_{24} = m_{13} = 1 \}. & \quad (4.6)
\end{align*}
\]

The \( su(4) \) polynomial \( \rightarrow \) triangle correspondence cannot be given as simply as for \( su(3) \), in (3.13), (3.20). Instead, we write
\[
\begin{align*}
  K_{1,0,0}(x_2) & \rightarrow \{ \} \\
  K_{-1,1,0}(x_2) & \rightarrow \{ n_{12} = 1 \} \\
  K_{0,-1,1}(x_2) & \rightarrow \{ n_{13} = 1 \} \\
  K_{0,0,-1}(x_2) & \rightarrow \{ n_{14} = 1 \}.
\end{align*}
\]  
(4.7)
\[ K_{0,1,0}(x_2) \rightarrow \{ \} \]
\[ K_{1,-1,1}(x_2) \rightarrow \{n_{23} = 1\} \]
\[ K_{-1,0,1}(x_2) \rightarrow \{n_{12} = n_{23} = 1\} \]
\[ K_{1,0,-1}(x_2) \rightarrow \{n_{24} = 1\} \]
\[ K_{-1,1,-1}(x_2) \rightarrow \{n_{12} = n_{24} = 1\} \]
\[ K_{0,-1,0}(x_2) \rightarrow \{n_{13} = n_{24} = 1\} \]
\[ K_{0,0,1}(x_2) \rightarrow \{\} \]
\[ K_{0,1,-1}(x_2) \rightarrow \{n_{34} = 1\} \]
\[ K_{1,-1,0}(x_2) \rightarrow \{n_{34} = n_{23} = 1\} \]
\[ K_{-1,0,0}(x_2) \rightarrow \{n_{34} = n_{12} = n_{23} = 1\} \]

\[-L_1(x_2) + 2L_2(x_2) - L_3(x_2) \rightarrow \{n_{13} = n_{34} = 1\}\]  \hspace{1cm} (4.10)

The \(K\)- and \(L\)-polynomials were defined in \[\text{Ref. 1}\]. Those in (4.7), (4.8), (4.9) are in one-to-one correspondence with the states in \(L(\Lambda^1)\), \(L(\Lambda^2)\), \(L(\Lambda^3)\), respectively. Eqn. (4.10) picks out one state in the three-dimensional subspace of weight 0 in the adjoint representation \(L(\Lambda^1 + \Lambda^3)\).

The rules (4.7),(4.8),(4.9) and (4.10), can be used to find the triangle that corresponds to a given polynomial directly, without first decomposing the polynomial into its elementary factors.

For \(su(4)\), there are 15 syzygies. They can be obtained from the following 5, by permuting the superscripts cyclically in all polynomials:

\[ S\{ R^{13,2,2}R^{2,13,2}, R^{2,2,0}R^{1,1,2}R^{3,3,2}, R^{0,2,2}R^{2,0,2}R^{3,1,0}R^{1,3,0} \} , \]  \hspace{1cm} (4.11)
\[ S\{ R^{13,2,2}R^{2,3,3}, R^{1,0,3}R^{2,2,0}R^{3,3,2}, R^{1,3,0}R^{2,0,2}R^{3,2,3} \} , \]  \hspace{1cm} (4.12)
\[ S\{ R^{2,2,13}R^{1,1,2}, R^{1,0,3}R^{0,2,2}R^{2,1,1}, R^{0,1,3}R^{2,0,2}R^{1,2,1} \} , \]  \hspace{1cm} (4.13)
\[ S\{ R^{13,2,2}R^{0,1,3}, R^{1,1,2}R^{3,2,3}, R^{1,0,3}R^{3,1,0}R^{0,2,2} \} , \]  \hspace{1cm} (4.14)
\[ S\{ R^{2,13,2}R^{1,0,3}, R^{1,1,2}R^{2,3,3}, R^{0,1,3}R^{1,3,0}R^{2,0,2} \} . \]  \hspace{1cm} (4.15)

Here \(S\{A, B, C\}\) indicates there exists one independent vanishing linear combination \(\alpha A + \beta B + \gamma C = 0\), with \(\alpha, \beta, \gamma \neq 0\).\(^1\)

\(^1\) The explicit linear combinations are given in Ref. [4]. We take this opportunity to correct a sign error there: the parameter \(u_5 = -2\).
We will implement each syzygy by choosing a forbidden product from among its "component" $R$-monomials. That is, we will not consider eliminating linear combinations of the component $R$-monomials as redundant, although that is certainly possible. Even with this limitation, however, the choice is not unique. As we did for $su(3)$, we will use the triangle correspondence, and specific fusion multiplicities to guide us. We will preserve the additivity of the threshold level, and by choosing a symmetric set of forbidden couplings, the cyclic $\mathbb{Z}_3$ symmetry of the triangles.

The fusions relevant to $(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,1)$, are

\begin{align}
L(1,1,1) \otimes L(1,1,1) & \supset L(0,2,0) , \\
L(1,1,1) \otimes L(0,1,1) & \supset L(0,0,0) , \\
L(1,1,0) \otimes L(1,1,0) & \supset L(0,0,0) , \\
L(1,0,1) \otimes L(1,0,1) & \supset L(0,0,0) , \\
L(1,0,0) \otimes L(1,0,1) & \supset L(0,0,0) ,
\end{align}

respectively. Here the notation is $L(\lambda_1,\lambda_2,\lambda_3) = L(\lambda)$, with $\lambda = \sum_{i=1}^{3} \lambda_i \Lambda_i$. Also, in $(4.16)$, e.g., $L(0,0,0)_{3,4}$ indicates that $T_{(1,1,1),(1,1,1),(0,2,0)} = 2$, and that the corresponding set of threshold levels is $\{3,4\}$. Compare these threshold levels with those of $L(1,1,1)$: $\{4,3,2\}$. Either $R^{13,2,2}R^{2,13,2}$ or $R^0 R^0 R^{2,0,2} R^{3,1,0} R^{1,3,0}$ must be forbidden (recall that we do not consider the linear combinations thereof as possible forbidden terms). Since $R^{13,2,2}R^{2,13,2}$ and $R^{2,2,0}R^{1,1,2}R^{3,3,2}$ correspond to the same BZ triangle, however, we forbid $R^{13,2,2}R^{2,13,2}$.

Analysing the syzygies $(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,1)$, yields the forbidden products

\begin{align}
R^{13,2,2}R^{2,13,2} , \\
R^{13,2,2}R^{2,3,3} \text{ or } R^{13,2,2}R^{2,3,2} , \\
R^{2,2,13}R^{1,1,2} \text{ or } R^{0,1,3}R^{2,0,2}R^{1,2,1} , \\
R^{13,2,2}R^{0,1,3} , \\
R^{0,1,3}R^{1,3,0}R^{2,0,2} ,
\end{align}

respectively. The consequences of the other 10 syzygies are obtained by cyclically permuting the superscripts in $(4.21)$-$(4.25)$. We will specify a choice of forbidden products later.
First we treat a new feature that arises in the $su(4)$ case. When $\lambda = \mu = \nu = \Lambda^1 + \Lambda^3$, the highest weight of the adjoint representation, the tensor product multiplicity is 2, and the threshold levels are 2 and 3. But the possible $R$-monomials of this weight are

$$R^{1,3,0}R^{0,1,3}R^{3,0,1} \quad \text{and} \quad R^{3,1,0}R^{0,3,1}R^{1,0,3},$$

both with threshold level 3. This discrepancy was noticed first in [8], where considerations were based entirely on BZ triangles, and was verified in [14]. Here we can explain why it happens: the terms in the $R$-monomials (4.26) of maximum degree in $x^\theta$ are cancelled in their sum. That is,

$$R^{13,13,13} := R^{1,3,0}R^{0,1,3}R^{3,0,1} + R^{3,1,0}R^{0,3,1}R^{1,0,3}$$

has threshold level 2, as found by (2.18).

One can persist in using the original set of elementary couplings obtained from (4.1),(4.2). When considering three-point functions of arbitrary weight $\{\lambda, \mu, \nu\}$, then linear combinations such as (4.27) may have to be found, in order to be consistent with the fusion multiplicities. In general, however, this is a tedious task.

Alternatively, as suggested by (4.27), we can continue to work in the same framework by introducing $R^{13,13,13}$ as a new elementary polynomial. This follows the procedure of [8] and [14]. The price to be paid is the introduction of extra syzygies involving the new elementary coupling, so that a single coupling is not counted more than once. In particular, we must implement

$$S\{R^{13,13,13}, R^{1,3,0}R^{0,1,3}R^{3,0,1}, R^{3,1,0}R^{0,3,1}R^{1,0,3}\}$$

as a new syzygy, by forbidding either $R^{1,3,0}R^{0,1,3}R^{3,0,1}$ or $R^{3,1,0}R^{0,3,1}R^{1,0,3}$ (for example). Since $R^{3,1,0}R^{0,3,1}R^{1,0,3}$ corresponds to the simpler triangle (with non-vanishing entries $l_{14} = m_{14} = n_{14} = 1$) we choose to forbid it, so that the new coupling then corresponds to this simple triangle.

This introduction of a new elementary coupling is perhaps not too surprising, if one looks back at the $su(3)$ case. Recall the single syzygy (3.13) for $su(3)$. As far as tensor products are concerned, it can be implemented by forbidding the product $R^{1,1,1}R^{2,2,2}$. As noted above, however, two other possibilities (forbidding $R^{1,2,0}R^{0,1,2}R^{2,0,1}$ or forbidding $R^{2,1,0}R^{0,2,1}R^{1,0,2}$) are more convenient for fusion, since they preserve in a simple way the additivity of the threshold level. If we insist on forbidding $R^{1,1,1}R^{2,2,2}$, however, the situation is completely analogous to the $su(4)$ case just considered. The
$R$-monomials for the coupling $\lambda = \mu = \nu = \Lambda^1 + \Lambda^2$ are then $R^{1,2,0}R^{0,1,2}R^{2,0,1}$ and $R^{2,1,0}R^{0,2,1}R^{1,0,2}$, each with threshold level 3; the correct threshold levels are 2,3, however. But a new elementary polynomial $R^{12,12,12} := R^{1,2,0}R^{0,1,2}R^{2,0,1} + R^{2,1,0}R^{0,2,1}R^{1,0,2}$ can be introduced, and it has threshold level 2. Of course, a new syzygy $S\{ R^{12,12,12}, R^{1,2,0}R^{0,1,2}R^{2,0,1}, R^{2,1,0}R^{0,2,1}R^{1,0,2} \}$ is also necessary.

The difference between the $su(3)$ and $su(4)$ cases originates in the syzygies. For $su(3)$, implementing the syzygy (3.13) makes the linear combination $R^{1,2,0}R^{0,1,2}R^{2,0,1} + R^{2,1,0}R^{0,2,1}R^{1,0,2}$ inaccessible. No such syzygy is present in the $su(4)$ case, however. The combination $R^{1,3,0}R^{0,1,3}R^{3,0,1} + R^{3,1,0}R^{0,3,1}R^{1,0,3}$ can be included, and so generates a new elementary coupling.

It is interesting to use the asymmetric depth rule, (3.21) and (3.22), to analyse these phenomena at the level of states. A single relation among $R$-monomials implies an $x^{[m-R>]}_i$ relation, an $x^{[n-R>]}_2$ relation, and an $x^{[l-R>]}_3$ relation. The $su(3)$ syzygy (3.13), for example, yields

\[ x^1 x^2 = x^- + x^+ \quad (4.29) \]

three times, once for each $x \rightarrow x_i$, $i = 1, 2, 3$. In terms of lowering operators, (4.29) is a consequence of the relation

\[ [F_{\alpha_1}, F_{\alpha_2}] = -F_\theta. \quad (4.30) \]

It is identical in the three cases because of the symmetry of (3.13), and encodes a relation between three states of weight 0 in the $su(3)$ representation $L(1,1)$. Since only two such states are independent, a relation like (4.29) is inevitable. A basis of the two-dimensional space can be chosen with depths \{1,1\}, or with \{1,0\}, giving rise to threshold levels \{3,3\} or \{3,2\}. Since the latter is correct for $L(1,1)^{\otimes 3} \supset L(0)$, we see that choosing the correct $R$-monomial $R^{1,1,1}R^{2,2,2} = R^{1,2,0}R^{0,1,2}R^{2,0,1} + R^{2,1,0}R^{0,2,1}R^{2,0,1}$ corresponds to choosing a basis of states of the required depths.

Many of the $su(4)$ syzygies are not symmetric and so lead through the asymmetric depth rule to three distinct $x_i$-relations. But the definition of the new elementary coupling (4.27), is symmetric. It yields

\[ x^1 x^{23} + x^{12} x^3 = -K_{-100}(x) + K_{00-1}(x), \quad (4.31) \]

in the notation of [1], where

\[
\begin{align*}
K_{00-1} &= x^\theta + \frac{1}{2} x^1 x^{23} + \frac{1}{2} x^{12} x^3 + \frac{1}{6} x^1 x^2 x^3, \\
K_{-100} &= x^\theta - \frac{1}{2} x^1 x^{23} - \frac{1}{2} x^{12} x^3 + \frac{1}{6} x^1 x^2 x^3.
\end{align*}
\]
This indicates that a linear combination of two depth-1 states of weight 0 in $L(1,0,1)$ is a state of depth 0.

In an arbitrary basis, finding such linear combinations are necessary when using the depth rule $[6,11]$. It is not surprising, therefore, that the new coupling $(4.27)$ must be introduced; it is perhaps surprising that $(4.27)$ is the only new coupling needed. This has not been proved, however. We defer to $[14]$, where some counting arguments supported this assumption.

We conclude our construction of $su(4)$ three-point functions by listing a choice of a set of forbidden couplings:

$$\{ R^{13,2,2} R^{2,13,2}, R^{2,13,2} R^{2,2,13}, R^{2,2,13} R^{13,2,2},$$
$$R^{13,2,2} R^{2,3,3}, R^{2,13,2} R^{3,2,3}, R^{2,2,13} R^{3,3,2},$$
$$R^{2,2,13} R^{1,1,2}, R^{13,2,2} R^{2,1,1}, R^{2,13,2} R^{1,2,1},$$
$$R^{13,2,2} R^{0,1,3}, R^{2,13,2} R^{3,0,1}, R^{2,2,13} R^{1,3,0},$$
$$R^{0,1,3} R^{1,3,0} R^{2,0,2}, R^{3,0,1} R^{0,1,3} R^{2,2,0}, R^{1,3,0} R^{3,0,1} R^{0,2,2},$$
$$R^{3,1,0} R^{0,3,1} R^{1,0,3} \} \tag{4.33}$$

The three-point function of weight $\{\lambda, \mu, \nu\}$ can be constructed from the sum over all $R$-monomials of that weight (as in $(2.15),(2.16)$) that do not contain the forbidden products $\tag{4.33}$.

To avoid confusion, we should state that our choice of forbidden products will not lead to the formula of $[8,14]$ assigning a threshold level to each $su(4)$ BZ triangle. Since it is only the set of threshold levels for a fixed weight $\{\lambda, \mu, \nu\}$ that is physically relevant, this is not a problem. We have not attempted to find a different formula, resulting from the choice $\tag{4.33}$, assigning a threshold level to a BZ triangle. Threshold levels can be very easily found from the relevant $R$-monomials (using $(2.32)$), and these $R$-monomials must be constructed in order to write the three-point function. When the interest is just fusion, and not the three-point functions, a formula like that of $[8,14]$ is very helpful (see $[15]$, e.g.). Here it represents an intermediate step we can eliminate.

5. Discussion

Building on the work $[1]$, we have shown how to construct WZW three-point functions for generating-function primary fields, for the algebras $su(3)$ and $su(4)$. More generally,
our approach is ideally suited to the use on three-point functions of the Gepner-Witten depth rule, and makes it clear that the level-dependence of WZW three-point functions mimics that of the corresponding fusion multiplicities, for all algebras.

In addition, we showed that a refined version \cite{6,11,12} of the original depth rule leads to a correspondence between three-point functions and BZ triangles. This correspondence was very helpful for \( su(3) \) and \( su(4) \). For example, it allowed us to use previous work on \( su(3) \) and \( su(4) \) fusion and BZ triangles \cite{4,5}. It also helped eliminate the redundancies encoded in the syzygies.

We will conclude with a brief discussion of possible uses and improvements of this work. We hope to address some of these issues in the future.

To be more precise, we found a correspondence between the polynomials \( R^E(x) \) and \( W^{[a]}_{\lambda,\mu,\nu}(x) \) and the BZ triangles, for the algebras \( su(3) \) and \( su(4) \). (For completeness, the \( su(2) \) case is treated in a short appendix.) That is, given such a polynomial, we showed how to find the associated triangle. The polynomials themselves are more important than the triangles, however. It would be interesting to try to find an algorithm to construct the appropriate polynomial from a BZ triangle.

As described above, when respecting the additivity of the threshold level, the triangles may favour certain ways of imposing the syzygies. However, it is sometimes preferable to impose them in other ways. Furthermore, BZ triangles, or their analogues, have not been constructed for all algebras. If one is interested in \( G_2 \) three-point functions, for example, one cannot rely on BZ constructions as a guide.

In those cases, one should consider the polynomials themselves, instead of first projecting to the corresponding triangles, and working with them. For example, one can write polytope volume formulas for the fusion multiplicities \cite{17} that are analogous to those written in \cite{18} for the tensor product multiplicities for \( su(N) \). These multiple sum formulas generalise the expressions for fusion multiplicities in \cite{4,13,15}, derived using BZ techniques. In particular, they are derived using virtual three-point functions, generalisations of the virtual BZ triangles and diagrams used in \cite{7,18,16,15}.

Finally, consider the generalisation to higher rank – for \( su(N) \) or other algebras. Our method of studying the level-dependence should work simply for any algebra, of any rank. However, we have relied on the method of elementary couplings to construct the polynomials \( W^{[a]}_{\lambda,\mu,\nu} \), before the level-dependent constraints are imposed. Since the numbers of elementary couplings and syzygies climb very rapidly, this part of the construction becomes
impractical for higher ranks. A method of finding the polynomials \( W^{[a]}_{\lambda,\mu,\nu} \) directly, without first factoring into elementary polynomials, might make larger ranks more tractable.

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Appendix A. \( su(2) \) three-point functions, triangles and fusion

Since there is only one positive root, \( \alpha_1 = \theta \), we use \( x_j^1 = x_j^\theta = x_j, \ j = 1, 2, 3 \). Similarly, we write

\[
E_{\alpha_1}(x, \partial, \lambda) =: E(x, \partial, \lambda) = \frac{\partial}{\partial x}
\]

\[
H_{\alpha_1}(x, \partial, \lambda) =: H(x, \partial, \lambda) = -2x \frac{\partial}{\partial x} + \lambda
\]

\[
F_{\alpha_1}(x, \partial, \lambda) =: F(x, \partial, \lambda) = -x^2 \frac{\partial}{\partial x} + \lambda x
\]

for the differential-operator realisation of \( su(2) \).

There are three elementary couplings,

\[
R^{1,1,0}(x) = x_1 - x_2, \quad R^{0,1,1}(x) = x_2 - x_3, \quad R^{1,0,1}(x) = x_3 - x_1,
\]

(A.2)

with no syzygies. For a weight \( \{\lambda, \mu, \nu\} = \{\lambda_1 \Lambda^1, \mu_1 \Lambda^1, \nu_1 \Lambda^1\} \), we therefore find

\[
W_{\lambda,\mu,\nu} = [R^{1,1,0}]^{(\lambda_1 + \mu_1 - \nu_1)/2} [R^{0,1,1}]^{(-\lambda_1 + \mu_1 + \nu_1)/2} [R^{1,0,1}]^{(\lambda_1 - \mu_1 + \nu_1)/2}.
\]

(A.3)

An \( su(2) \) BZ triangle is very simple:

\[
m \quad n \quad l.
\]

(A.4)

To be consistent with (3.1) and (4.3), we should write \( m = m_{12} \), etc., since \( \alpha_1 = e_1 - e_2 \), where \( \{e_1, e_2\} \) is an orthonormal basis of \( \mathbb{R}^2 \). We dropped the subscripts for simplicity, however. The outer constraints are

\[
\lambda_1 = m + n, \quad \mu_1 = n + l, \quad \nu_1 = l + m,
\]

(A.5)

and there are no inner constraints, since the triangle contains no hexagons.
The elementary polynomials of (A.2) are associated to the triangles
\[
\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}
\] (A.6)
respectively. Comparing this correspondence with (A.3), we see that
\[
\frac{\lambda_1 + \mu_1 - \nu_1}{2} = n, \quad \frac{-\lambda_1 + \mu_1 + \nu_1}{2} = l, \quad \frac{\lambda_1 - \mu_1 + \nu_1}{2} = m, \tag{A.7}
\]
as must be, by the outer constraints (A.5).

The symmetric form (2.17) of the depth rule gives threshold level \(t = 1\) for the three elementary couplings, and threshold \(t = l + m + n = (\lambda_1 + \mu_1 + \nu_1)/2\) for the general polynomial (A.3). This is consistent with the additivity of the threshold level under multiplication of \(R\)-monomials.

The asymmetric form (3.22) of the depth rule is in agreement. For the general polynomial (A.3), we find
\[
\begin{align*}
x_1^{[m \cdot R]} &= (-x_1)^m, & x_2^{[n \cdot R]} &= (-x_2)^n, & x_3^{[l \cdot R]} &= (-x_3)^l.
\end{align*}
\tag{A.8}
\]
The second of these corresponds to a state of depth \(d = n\), and so gives \(t = d + \nu \cdot \theta = n + (l + m)\). The first and third can be analysed using formulas obtained by permuting \(\lambda, \mu, \nu\) and \(x_1, x_2, x_3\) cyclically in (3.24). We get \(t = d + \mu \cdot \theta = m + (n + l)\) from the first, and \(t = d + \lambda \cdot \theta = l + (m + n)\) from the third monomial of (A.8).

From (A.8), we see that
\[
\begin{align*}
x_1 \\
x_2 \\
x_3
\end{align*}
\tag{A.9}
\]
is the \(su(2)\) analogue of the \(su(3)\) result (3.21).
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