STÄCKEL EQUIVALENCE OF NON-DEGENERATE SUPERINTEGRABLE SYSTEMS, 
AND INVARIANT QUADRICS

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ABSTRACT. A non-degenerate second-order maximally conformally superintegrable system in dimension 2 naturally gives rise to a quadric with position dependent coefficients. It is shown how the system’s Stäckel equivalence class can be obtained from this associated quadric.

1. INTRODUCTION

Second-order superintegrable systems in dimension 2 (2D) are classified [KKPM01, KKM05b, KKM05a, KS18, KPM02]. Their equivalence classes under Stäckel (i.e., conformal) transformations have been characterised [DY06, Kre07]. Particularly, in [Kre07] a method is developed that allows one to identify the Stäckel equivalence class of a non-degenerate superintegrable system from the properties of its associated quadratic algebra, see also [Pos11]. The present paper presents an alternative method to determine the conformal equivalence class, exploiting the existence of invariant quadrics associated to conformally superintegrable systems.

Let \( g \) be a (pseudo-)Riemannian metric on a 2-dimensional manifold \( M \) and consider the Hamiltonian \( H(x, p) = g^{ij}(x)p_ip_j + V(x) \). Here \( x \) and \( p \) stand, respectively, for position coordinates \( x^i \) and canonical momenta (fibre coordinates) \( p_i \) on the cotangent space \( T^*M \). Note that in what follows we shall consider two Hamiltonians \( H_1, H_2 \) to be equal if they are constant multiples as functions on \( T^*M \).

A second-order integral (of motion) for \( H \) is a function \( F(x, p) = K^{ij}(x)p_ip_j + W(x) \) such that \( H \) and \( F \) commute w.r.t. the canonical Poisson bracket on \( M \) (Einstein’s summation convention applies),

\[
\{F, H\} = \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i} \frac{\partial F}{\partial p_i} = 0. \tag{1.1}
\]

More generally, the function \( F \) is called a second-order conformal integral if

\[
\{F, H\} = \omega H \tag{1.2}
\]

holds, for some polynomial \( \omega = \omega^ip_i \) linear in momenta. Obviously, every integral is also a conformal integral for \( H \), with \( \omega = 0 \).

We remark that Condition (1.1) is equivalent to requiring that the coefficients \( K_{ij} \) (indices are lowered using \( g \)) form the components of a Killing tensor \( K \) and that \( K \) is compatible with \( V \) according to the Bertrand-Darboux condition [Ber57, Dar01]

\[
\nabla_{[i} K_{j]} a \nabla_a V = K^a_{[i} \nabla^2 a V \tag{1.3}
\]

where square brackets denote antisymmetrisation; \( \nabla \) is the Levi-Civita connection for \( g \) and \( \nabla^2 \) denotes the Hessian. If instead of (1.1) we require (1.2), then \( K_{ij} \) is a conformal Killing tensor, and we obtain an equation similar to (1.3), but with additional terms involving \( \omega \) [KKM11, KSV20].

Remark 1.1. Note that we work over the field \( \mathbb{C} \) of complex numbers unless otherwise indicated.

Definition 1.2. (i) A (2D maximally) second-order superintegrable system is the linear span \( \mathcal{F} \) of a functionally independent triple \( (H, F_1, F_2) \) where \( H \) is the Hamiltonian and where \( F_1, F_2 \) are second-order integrals for \( H \).
Definition 2.1. A second-order superintegrable system (in 2D) is non-degenerate if for any integral \( F \) Equation (1.3) admits a linear space of solutions, say \( \mathcal{U} \), of dimension \( n + 2 = 4 \) [KKPM01, KPM02].

(iv) We can consider the Hamiltonian with a fixed potential \( V \in \mathcal{U} \) or with the full \((n + 2)\)-parameter family. In the latter case we write \( \mathcal{V}^H \) for clarity. Note that \( \mathcal{V}^H \) denotes a specific parametrisation of \( \mathcal{U} \).

The following equivalence relation of second-order maximally superintegrable systems is well known.

**Definition 1.3.** Let \((H, F_1, F_2)\) be a second-order non-degenerate superintegrable system, and let \( U \in \mathcal{U} \) be one of its compatible potentials. Then \( \langle \tilde{H}, \tilde{F}_1, \tilde{F}_2 \rangle \) with

\[
\tilde{H} = U^{-1} H, \quad \tilde{F}_1 = F_i + (1 - W_i) U^{-1} H
\]

is called the Stäckel transform of \((H, F_1, F_2)\).

**Remark 1.4.** (i) The integrals \( \tilde{F}_i \) are indeed integrals for \( \tilde{H} \) and, provided they are functionally independent, the Stäckel transform is again a second-order superintegrable system.

(ii) The Stäckel transform is also known under the name coupling constant metamorphosis [BKM86, HGDR84], but these two concepts are not the same in other contexts [Pos10].

(iii) If in (1.4) we would allow for arbitrary functions \( U \), not necessarily compatible potentials of \( H \), the resulting functions \( \tilde{F}_i \) are not necessarily integrals any more, but in general are transformed into conformal integrals. In this way, a conformal equivalence relation for conformally superintegrable systems is obtained, see for instance [KSV20, BKM86, KKM05a].

2. Method

The aim of the current chapter is to construct a conformally invariant variety, defined by a quadric, for a given second-order conformally superintegrable system in 2D. This variety is used, in the next section, to characterise conformal equivalence classes of non-degenerate superintegrable systems in 2D.

**Definition 2.1.** A quadric in projective space \( \mathbb{P}^m \) is the subset defined by the zero set of a homogeneous quadratic polynomial equation in \( m + 1 \) variables.

Note that we do not require the polynomial equation to be irreducible.

We observe that for a conformally superintegrable Hamiltonian \( H = g^{ij} p_i p_j + V \), the product \( \mathcal{U}g \) is invariant under conformal (and particularly Stäckel) transformations (1.4),

\[
\mathcal{U}g \rightarrow \frac{U}{U} g = \mathcal{U}g,
\]

where \( U \) can be any function. For a non-degenerate Hamiltonian, it defines an \((n + 2)\)-dimensional linear space \( \mathcal{V} \) that is invariant under conformal transformations. Inside this linear space, however, the origin clearly never corresponds to a metric. Moreover, constant multiples give rise to equal Hamiltonians. It is therefore useful to reconsider the \((n + 2)\)-dimensional linear space \( \mathcal{V} \) as an \((n + 1)\)-dimensional projective space, which we denote by

\[
\mathcal{W} = (\mathcal{V} \setminus \{0\})/\sim,
\]

where \( h \sim k \) for \( h, k \in \mathcal{V} \) if \( h = ak \) for constant \( a \neq 0 \). We emphasize that elements \( q \in \mathcal{W} \) are (classes of) symmetric 2-tensors (and in fact metrics). The vanishing of their Riemannian curvature tensor \( \text{Riem}(q) \) is independent of the choice of representative for \( q \). We thus introduce a subset of \( \mathcal{W} \),

\[
\mathcal{Q} = \{q \in \mathcal{W} : \text{Riem}(q) = 0\},
\]

which, by construction, is invariant under conformal transformations.

**Lemma 2.2.** In dimension 2, the space \( \mathcal{Q} \subset \mathcal{W} \) is defined by a homogeneous quadratic polynomial equation with coefficients that depend on the position \( x \in \mathcal{M} \).

**Proof.** In 2D, the Riemannian curvature tensor is determined by its (unique) sectional curvature, or, alternatively, by its scalar curvature. Moreover, in suitable local coordinates, any 2D metric can be written as \( g = \phi^2 dx dy \), such that the requirement of vanishing Riemannian curvature becomes

\[
V^{U} V_{xy} \phi^2 + 2(V^{U})^2 \phi_x \phi_y - V_x V_y V^{U} \phi^2 - 2(V^{U})^2 \phi_x \phi_y = 0,
\]

where we recall that \( V^{U} \) is a parametrisation of \( \mathcal{U} \); the subscripts \( x, y \) denote usual derivatives. Therefore (2.1) is homogeneously quadratic in the \( n + 2 \) parameters of the potential \( V^{U} \), with coefficients depending on the position. \( \square \)
The space \( \mathcal{Q} \) gives rise to a tangible object: Write \( \mathcal{F} \) for the (invariant) intersection of \( \mathcal{Q} \) over all points in a neighborhood. Then \( \mathcal{F} \) is the space of all flat realisations of the Stäckel equivalence class arising from the initially given superintegrable system. Computationally, \( \mathcal{F} \) is easy to handle, and the known normal forms (2.1) can typically be written as a polynomial in \( x \) and \( y \).

As an explicit example, take the system [E7] from [KKPM01], whose underlying Hamiltonian is

\[
H = p_1 p_2 + a_3 xy + a_2 \frac{y}{\sqrt{y^2 - c^2}} + a_1 \frac{x}{\sqrt{y^2 - c^2}} (y + \sqrt{y^2 - c^2})^2 + a_0 =: p_1 p_2 + V[a_3, a_2, a_1, a_0].
\]

In what follows, the potential is to be considered modulo multiplication by an irrelevant constant, which we denote as \( V[a_3 : a_2 : a_1 : a_0] \). We find

\[\mathcal{Q} = \{V[a_3 : a_2 : a_1 : a_0] \, dx \, dy : 2a_1 a_2 + 6a_1 a_0 y^2 - 6a_2 a_3 y^4 - 2a_0 a_3 y^6 = 0\}\]

\[\mathcal{F} = \{V[a_3 : a_2 : a_1 : a_0] \, dx \, dy : a_1 a_2 = 0, a_1 a_0 = 0, a_2 a_3 = 0, a_0 a_3 = 0\}.
\]

One therefore obtains a space with two distinct components,

\[\mathcal{F} = \{V[a_3 : a_2 : a_1 : a_0] \, dx \, dy : a_2 = 0 = a_0\} \cup \{V[a_3 : a_2 : a_1 : a_0] \, dx \, dy : a_1 = 0 = a_3\}.
\]

3. Findings

We implement the method set out in the previous section for all cases of the classification [KKPM01], and conclude that \( \mathcal{F} \) carries enough information to identify the Stäckel class of a non-degenerate system. In a second step, we outline how \( \mathcal{F} \) contains, for a given equivalence class, all flat realisations of this class.

3.1. Characterisation of 2D Stäckel classes. Since the non-degenerate systems in 2D are classified [KKPM01], we can use the explicitly known normal forms to straightforwardly solve (2.1).

The results are summarised in Table 1. Note that the quadric varieties \( \mathcal{F} \) can be parametrised in various equivalent ways, but that the varieties themselves are characteristic to each class. For the reader’s convenience, two additional numbers are specified: (1) The Hilbert dimension \( h \) of \( \mathcal{F} \). (2) The number \( k \): Let \( H = g^{ij} p_i p_j \) be a non-degenerate superintegrable Hamiltonian admitting the integrals \( F_1 = K_1^j p_i p_j + W_1 \) and \( F_2 = K_2^j p_i p_j + W_2 \). Then the restriction \( R_0^2 = R^2 |_{h=0} \) of the square of \( R = \{F_1, F_2\} \) is a cubic polynomial in \( F_1, F_2 \) [Kre07]. The number \( k \) counts the distinct complex roots of \( R_0^2 \), where by convention we set \( k(0) = 0 \).

**Theorem 3.1.** A second-order non-degenerate conformally superintegrable system in dimension 2 is uniquely identified from its associated variety \( \mathcal{Q} \subset \mathcal{W} \).

**Proof.** Any second-order conformally superintegrable system in 2D is conformally equivalent to a superintegrable system on a constant curvature space [Cap14, KKM05a]. Therefore Table 1 covers all cases and we immediately infer the asserted statement. □

The theorem provides an alternative way to determine the Stäckel class of a non-degenerate superintegrable system or conformally superintegrable system. Other characterizations are given in [Kre07, DY06]. The advantage of the criterion outlined here is that in many situations it can be checked more quickly when using computer algebra.
3.2. Individual systems within the invariant variety. The invariant variety \(\mathcal{F}\) is nothing but the space of all flat superintegrable systems realised within each respective class (note that by construction we do not get flat conformally superintegrable realisations, unless they are actually superintegrable). Taking this viewpoint, we are now posing the question to describe the individual superintegrable systems within each variety \(\mathcal{F}\). To this end, we compute \(\mathcal{U}_g\) for a Hamiltonian \(H = g^{ij}p_ip_j + V^U\) from each class in Table 1. Due to the invariance of \(\mathcal{F}\) it does not matter which actual realisation we select for the computation. Furthermore, for convenience, we can chose coordinates \((x, y)\) such that \(g = dx^2 + dy^2\). With these coordinate, the isometry operations are \(x \rightarrow \lambda x + a_1\) and \(y \rightarrow \frac{y}{\lambda} + a_2\) for constants \(a_1, a_2\) and \(\lambda \neq 0\), which we use to identify the normal form given in [KKPM01]. We find:

Class (111,11). This is the simplest case, \(\mathcal{F} = \emptyset\): No flat Hamiltonians realising this Stäckel class exist.

Class (3,2). The variety \(\mathcal{F}\) consists of one projective point, corresponding to the system [E2] of [KKPM01].

Class (21,2), (21,0). Two disjoined projective lines are contained in \(\mathcal{F}\). One line is generically [E19], containing one point that is [E17]. The other line is [E7] generically and contains a point that is [E8].

Class (0,11). The most interesting variety contains the Harmonic Oscillator and is governed by the position-independent quadric \(a_1^2 + b_1^2 = uv\) for \(a, b, u, v \in \mathbb{C}\). In the quadric defining \(\mathcal{Q}\) the position dependent contributions factor out. Somewhat surprisingly, the system [E20] is realised generically, when \(\mathcal{F}\) is described by \(u = \frac{a^2 + b^2}{a} (u \neq 0)\). The system [E11] is realised if \(u = 0, a \neq 0\) and \(b \neq 0\). The quadric is \(a^2 + b^2 = (a + ib)(a - ib) = 0\). The projective point with \(u = a = b = 0\) realises [E3].

4. Higher dimension

The current paper focuses on dimension \(n = 2\). In dimensions \(n > 2\) it is still possible, by the same reasoning, to construct \(\mathcal{Q} \subset \mathcal{W}\), but it is generally not described by a single quadric. Also, one quickly finds that \(\mathcal{Q}\) cannot be used to identify the Stäckel class in higher dimension. This is already true in dimension 3, where we can use the explicit normal forms of second-order non-degenerate superintegrable systems [KKM06, KKM07, Cap14].

However, in most cases, \(\mathcal{F}\) is just one projective point. A similar ambiguity should be expected in any higher dimension, pointing to fundamental structural particularities of 2D second-order superintegrable systems [KS18, KSV19, KSV20].

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\footnote{We denote the 2D Harmonic Oscillator system by [E3], following [KKPM01], where however it is not written with all \(n + 2\) parameters. The full potential is given in [Kre07], for example, where the full system is distinguished by a prime, [E3].}
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