MODULI OF EINSTEIN-HERMITIAN HARMONIC MAPPINGS OF THE PROJECTIVE LINE INTO QUADRICS

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Abstract. The present article studies the class of Einstein-Hermitian harmonic maps of constant Kähler angle from the projective line into quadrics. We provide a description of their moduli spaces up to image, and gauge-equivalence using the language of vector bundles and representation theory. It is shown that the dimension of the moduli spaces is independent of the Einstein-Hermitian constant, and rigidity of the associated real standard, and totally real maps is examined. Finally, certain classical results concerning embeddings of two-dimensional spheres into spheres are rephrased and derived in our formalism.

1. Introduction

Let $S, Q \to \text{Gr}_n(\mathbb{R}^{n+2})$ denote respectively the tautological, and universal quotient bundles over the complex hyperquadric. Given a Riemannian manifold $(M, g)$ and a mapping $f : M \to \text{Gr}_n(\mathbb{R}^{n+2})$, write $V$ for $f^{-1}Q$. Then, the mean curvature operator of $f$ is the bundle endomorphism $A \in \Gamma(\text{End}V)$ defined [10] as:

$$A = \text{Tr}(H \circ K)$$

where $H, K$ are respectively the pull-backs of the second fundamental forms of $S, Q \to \text{Gr}_n(\mathbb{R}^{n+2})$, and the trace of the $V$-valued two-tensor $H \circ K$ is taken with respect to $g$. The mapping $f$ is said to be Einstein-Hermitian (EH, for short) if its mean curvature operator satisfies the strong Einstein condition,

$$A = \varphi \, \text{id}_V$$

for some constant $\varphi$, which we term the EH-constant. In the holomorphic setting $A$ coincides up to a sign with the mean curvature in the sense of Kobayashi [7], where the strong Einstein condition was introduced to define the notion of Einstein–Hermitian vector bundle. Moreover, [11] characterises the minima of the functional

$$\int_M |A|^2 \, dv_M$$

which generalises some instances of the Yang–Mills functional ([10] §4, [7], p.111) Although we do not pursue the functional approach of EH maps in this article, many of its properties are examined in §3.

The present work deals with the classification (up to suitable notions of equivalence) of full, EH harmonic maps $\mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ of degree $k$ and EH-constant $l$, with constant Kähler angle. If the classifying criterium is...
gauge–equivalence of maps (resp. image–equivalence, aka congruence), the resulting moduli space is denoted by \( \mathcal{M}_{k,l} \) (resp. \( \mathbb{M}_{k,l} \)). These moduli spaces are fully analysed in §4. To avoid long repetitions we will write ‘(\( k, \text{EH}(l) \)) mapping’ instead of ‘EH mapping of degree \( k \) and EH-constant \( l \)’.

It has been conjectured from remarks in [9] that the moduli space \( \mathcal{M}_{k,l} \), and the moduli space \( \mathcal{M}_k \) of gauge–equivalence classes of holomorphic isometric embeddings \( \mathbb{C}P^1 \to \text{Gr}_g(\mathbb{R}^{n+2}) \) of degree \( k \), satisfying the gauge condition are diffeomorphic. The present work (§4) decides the question in the affirmative as a consequence of the

**Main Theorem 1** (Theorem 4.4) If \( f : \mathbb{C}P^1 \to \text{Gr}_g(\mathbb{R}^{n+2}) \) is a full, \((k,\text{EH}(l))\) harmonic map of constant Kähler angle, then

1. \( n \leq 2(|k| + 2l) \).

Assuming \( n \) to be maximal, let \( \mathcal{M}_{k,l} \) be the moduli space up to gauge equivalence of maps, and denote its closure by the inner product by \( \mathcal{M}_{k,l} \). Then,

2. \( \mathcal{M}_{k,l} \) can be regarded as an open bounded convex body in

\[
\bigoplus_{r=|l|+1}^{2r\leq |k|+2l} S^{2(|k|+2l-2r)} \mathbb{C}^2.
\]

3. The boundary points of \( \mathcal{M}_{k,l} \) describe those maps whose images are included in some totally geodesic submanifold

\[
\text{Gr}_p(\mathbb{R}^{p+2}) \subset \text{Gr}_g(\mathbb{R}^{2(|k|+2l+1)}), \quad p < 2(|k| + 2l)
\]

4. The totally geodesic submanifold \( \text{Gr}_p(\mathbb{R}^{p+2}) \) can be regarded as the common zero set of some sections of \( Q \to \text{Gr}_g(\mathbb{R}^{2(|k|+2l+1)}), \) which belongs to \( (\mathbb{R}^{p+2})^\perp \subset \Gamma(Q) \).

The above characterisation of \( \mathcal{M}_{k,l} \) coincides with the description of \( \mathcal{M}_k \) given in [9], Theorem 7.4. The key tool in the proof of Main Theorem 1 is the so called *contraction operator* \( \mathcal{C} \), introduced in the same section.

Other connections between different moduli spaces are also studied in this article. As further properties of the contraction operator are explored in §5, we use the *modified contraction operator* \( \tilde{\mathcal{C}} \) to make clear the relation between moduli spaces \( \mathcal{M}_{k,l} \) for different EH-constants. We prove

**Main Theorem 2** (Theorem 5.1) There is a one-to-one correspondence between \( \mathcal{M}_{k,l} \) and \( \mathcal{M}_{k,l-1} \) which associates the gauge–equivalence class of full \((k,\text{EH}(l))\) harmonic maps determined by \( T = (\text{id} + D)^{\frac{1}{2}} \) to the gauge–equivalence class of full \((k,\text{EH}(l-1))\) harmonic maps determined by

\[
\text{id} + \frac{D|_{\text{op}}}{|\tilde{\mathcal{C}}^2(D)|_{\text{op}}} \tilde{\mathcal{C}}^2(D)^{\frac{1}{2}}
\]

where \( \tilde{\mathcal{C}} \) denotes the modified contraction operator [8].

The following two sections describe rigidity results for certain classes of mappings naturally associated to EH harmonic maps. §6 deals with the
real standard map and concludes (Corollary 5.2) that it is strongly rigid, i.e., admits no deformation at the gauge-equivalence moduli space level. Totally real (in the sense of Chen & Ogiue [3]) EH maps are discussed in §7. Strong rigidity of totally real, full, EH minimal immersions is proved (Theorem 7.1). The dependence of this last result with the strong Einstein condition for the mean curvature operator is clarified.

Finally, §8 diverts from the abstract setting and turns to applications. We recover a well-known result by Ruh & Vilms [11] on harmonicity of the Gauss map (Theorem 8.2) from our analysis of the moduli spaces in §4. From it, we are able to give new proofs of classical theorems by Calabi [2], and do Carmo & Wallach [5] regarding isometric minimal immersions of two-dimensional spheres into spheres (Theorem 8.1).

2. Mean curvature operator and do Carmo–Wallach theory

Here we introduce some preparatory material needed in the remaining sections. To avoid repetition of what could be easily found elsewhere, the presentation does not intend to be exhaustive but builds on the preliminaries expounded in §2. Hence, we make free use of concepts defined there, remarkably the notions of fullness, induced map by \((V \rightarrow M, W)\), standard map, gauge equivalence of maps, and image equivalence of maps.

Let \(W\) be a real (oriented) or complex \(N\)-dimensional vector space together with a fixed scalar product (an inner product or a Hermitian inner product). Then, we have the following homomorphisms of vector bundles over the Grassmannian \(\text{Gr}_p(W)\) of (oriented) \(p\)-planes in \(W\):

\[
\begin{array}{cccc}
0 & \rightarrow & S & \xrightarrow{i_S} W & \xrightarrow{\pi_Q} Q & \rightarrow & 0
\end{array}
\]

where \(S \rightarrow \text{Gr}_p(W)\) is the tautological bundle, \(Q \rightarrow \text{Gr}_p(W)\) the universal quotient bundle, and \(W \rightarrow \text{Gr}_p(W)\) the trivial vector bundle with fibre \(W\). The natural injection \(i_S\), and projection \(\pi_Q\) form a exact sequence. Moreover, \(\pi_Q\) allows one to regard \(W\) a subspace of the space of sections \(\Gamma(Q)\). The orthogonal projection defined via the scalar product on \(W\) induces the bundle homomorphisms \(\pi_S\) and \(i_Q\), together with fibre metrics \(g_S\), \(g_Q\).

Regarding sections \(t \in \Gamma(Q)\) as \(W\)-valued functions \(i_Q(t)\) the differential \(di_Q(t)\) splits as

\[
di_Q(t) = K(t) + \nabla^Q(t) = \pi_Sdi_Q(t) + \pi_Qdi_Q(t)
\]

where \(\nabla^Q = \pi_Qdi_Q\) is the so-called canonical connection, while the \((Q^* \otimes S)\) valued 1-form \(K = \pi_Sdi_Q\) is the second fundamental form in the sense of Kobayashi [7]. In a similar way, a connection \(\nabla^S = \pi_Sdi_S\) and a second fundamental form \(H = \pi_Qdi_S\) are defined. Note that from the identification of the (holomorphic) tangent bundle \(T \rightarrow \text{Gr}_p(W)\) with \(S^* \otimes Q \rightarrow \text{Gr}_p(W)\),

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the Levi-Civita connection $\nabla$ is induced from $\nabla^S$ and $\nabla^Q$. The next Lemma then follows (cf [10], §2):

**Lemma 2.1.** If $H, K$ are the second fundamental forms defined above,

1. $\nabla H = \nabla K = 0$
2. $g_Q(H_s, t) = -g_S(s, K_t)$.

Further properties of submanifolds with parallel second fundamental forms have been studied using the present formalism in [8].

If $f : M \to \text{Gr}_p(W)$ is smooth, we pull back $g_Q$ and $\nabla_Q$ to obtain a fibre metric $g_V$ and a connection $\nabla_V$ on the pull-back of the universal quotient bundle, denoted by $V \to M$. The second fundamental forms $H, K$ are also pulled back and denoted by the same symbols. If we restrict bundle-valued linear forms $H$ and $K$ on the pull-back bundle $f^{-1}T^* \to M$ to linear forms on $M$, then they are just the second fundamental forms of subbundles $f^{-1}S \to W$ and $V \to W$, where now $W = M \times W$ with certain abuse of the notation. The bundle epimorphism $\pi_V : W \to V$ defines a not necessarily injective linear map $W \to \Gamma(V)$. Even if this is the case, we shall still refer to $W \subset \Gamma(V)$ as a space of sections.

Next, assume that $(M, g)$ is an $m$-dimensional Riemannian manifold and let $\{e_i\}_{i=1, \ldots, m}$ be an orthonormal frame of $M$. The Riemannian structure on $M$, and the pull-back connection on $V \to M$ define the Laplace operator $\Delta_V : \Gamma(V) \to \Gamma(V)$

$$\Delta_V = -\sum_{i=1}^n \nabla_{e_i}^V (\nabla_V)(e_i)$$

We will also introduce a bundle homomorphism $A \in \Gamma(\text{End} V)$ defined as the trace of the composition of the second fundamental forms:

$$A := \sum_{i=1}^m H_{e_i} K_{e_i}.$$  

We call $A \in \Gamma(\text{End} V)$ the **mean curvature operator** of $f : M \to \text{Gr}_p(W)$. The following properties of $A$ are easily proved (cf [10], §3):

**Lemma 2.2.** Let $A$ be the mean curvature operator of $f : M \to \text{Gr}_p(W)$ as defined above. Then,

1. $A$ is a non-positive symmetric (or Hermitian) operator.
2. The energy density $e(f)$ is equal to $-\text{Tr} A$.

We use the mean curvature operator $A$ to introduce the concept of *Einstein-Hermitian mapping*:

**Definition 1.** Let $f$ be a map from a Riemannian manifold into a Grassmannian. Then, $f$ is *Einstein-Hermitian* (for short, EH map) if the mean curvature operator $A$ is proportional to the identity, i.e if

$$A = -\mu \text{id}_V$$

for some non-negative constant $\mu$.

To end this section we state the harmonic version of the generalisation of the theorem of do Carmo–Wallach for reader’s benefit.
Theorem 2.3. [10] Let \( M = G/K_0 \) be a compact reductive Riemannian homogeneous space with decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \). Fix a homogeneous complex line bundle \( L = G \times_{K_0} V_0 \to M \) with invariant metric \( h \) and canonical connection \( \nabla \). Regard \( L \to M \) as a real vector bundle with complex structure \( J \). If \( f : M \to \text{Gr}_n(\mathbb{R}^{n+2}) \) is a full harmonic map satisfying the following two conditions:

- \( (G) \) The pull-back \( V \to M \) of the universal quotient bundle, with the pull-back metric, connection and complex structure is gauge equivalent to \( L \to M \) with \( h \), \( \nabla \) and \( J \).
- \( (EH) \) The mean curvature operator of \( f \) equals \( -\mu \text{id}_V \) for some positive real number \( \mu \), such that \( e(f) = 2\mu \).

Then there exist an eigenspace \( W \subset \Gamma(V) \) of the Laplacian with eigenvalue \( \mu \), equipped with \( L^2 \)-scalar product \( (\cdot, \cdot)_W \), and a semi-positive Hermitian endomorphism \( T \in \text{End}(W) \), such that

1. The vector bundle \( L \to M \) is globally generated by a subspace \( \mathbb{R}^{n+2} \) of \( W \). Denote the inclusion by \( \iota : \mathbb{R}^{n+2} \to W \).
2. As a subspace, \( \mathbb{R}^{n+2} = (\ker T)^\perp \) and the restriction \( T|\mathbb{R}^{n+2} \) is a positive Hermitian transformation.
3. Regard \( W \) as \( \mathfrak{g} \)-representation \((\varrho, W)\). The endomorphism \( T \) satisfies
   \[
   \begin{align*}
   (T^2 - \text{id}_W, \text{GH}(V_0, V_0))_H &= 0 \\
   (T^2, \text{GH}(\varrho(m)V_0, V_0))_H &= 0
   \end{align*}
   \]
   where \( V_0 \) is regarded as a subspace of \( W \) by Frobenius reciprocity and the presence of the scalar product \( (\cdot, \cdot)_W \).
4. The endomorphism \( T \) determines an embedding
   \[
   \text{Gr}_n(\mathbb{R}^{n+2}) \to \text{Gr}_{n'}(W), \quad n' = n + \dim \ker T
   \]
   and a fixed bundle isomorphism \( \phi : L \to V \).

Then, \( f : M \to \text{Gr}_n(\mathbb{R}^{n+2}) \) can be expressed as

\[
(2) \quad f ([g]) = (\iota^*T\iota)^{-1} \left( f_0 ([g]) \cap (\ker T)^\perp \right),
\]

where \( \iota^* \) denotes the adjoint operator of \( \iota \) under the scalar product on \( \mathbb{R}^{n+2} \), induced from \( (\cdot, \cdot)_W \) on \( W \) and \( f_0 \) is the standard map by \( W \). The pairs \( (f_1, \phi_1) \) and \( (f_2, \phi_2) \) are gauge equivalent if and only if

\[
\iota_1^*T_1\iota_1 = \iota_2^*T_2\iota_2,
\]

where \( (T_i, \iota_i) \) correspond to \( f_i \) \( (i = 1, 2) \) under the expression \( (2) \), respectively.

The converse of the theorem is also true.

Remark 1. Let \( \tau : \text{Gr}_n(\mathbb{R}^{n+2}) \to \text{Gr}_n(\mathbb{R}^{n+2}) \) be the map obtained by switching the orientation of \( n \)-dimensional subspaces of \( \mathbb{R}^{n+2} \). Then \( \tau : \text{Gr}_n(\mathbb{R}_{n+2}^+) \to \text{Gr}_n(\mathbb{R}_{n+2}^+) \) is an isometry. In the sequel, we do not distinguish a map \( f : M \to \text{Gr}_n(\mathbb{R}^{n+2}) \) from a map \( \tau \circ f : M \to \text{Gr}_n(\mathbb{R}^{n+2}) \).
3. EH harmonic maps

The universal quotient bundle $Q \to \text{Gr}_n(\mathbb{R}^{n+2})$ has a holomorphic vector bundle structure induced by the canonical connection. To obtain a characterisation of EH harmonic maps into $\text{Gr}_n(\mathbb{R}^{n+2})$ we use the natural embedding $i$ of $\text{Gr}_n(\mathbb{R}^{n+2})$ into complex projective space $\mathbb{CP}^n \cong \text{Gr}_{n+1}(\mathbb{C}^{n+2})$. Then, the pull-back of the hyperplane bundle $\mathcal{O}(1) \to \text{Gr}_{n+1}(\mathbb{C}^{n+2})$ is just $Q \to \text{Gr}_n(\mathbb{R}^{n+2})$, and the pull-back connection is the canonical connection by equivariance of $i$. Moreover the embedding $i$ induces a real structure on $\mathbb{C}^{n+2}$, which is to be regarded as a space of holomorphic sections of $Q \to \text{Gr}_{n+2}(\mathbb{R}^{n+2})$. The real structure distinguishes a real subspace $\mathbb{R}^{n+2} \subset \mathbb{C}^{n+2}$. We denote by $J$ the complex structure of $Q \to \text{Gr}_{n+2}(\mathbb{R}^{n+2})$. Thus, if $t \in \mathbb{R}^{n+2}$, then $Jt \in \mathbb{C}^{n+2}$.

**Proposition 3.1.** Let $f : M \to \text{Gr}_n(\mathbb{R}^{n+2})$ be a harmonic map. Then $f$ is EH harmonic if and only if the composition $i \circ f : M \to \text{Gr}_{n+1}(\mathbb{C}^{n+2})$ is a harmonic map of constant energy density.

**Proof.** Suppose that $f : M \to \text{Gr}_n(\mathbb{R}^{n+2})$ is an EH harmonic map with $A = -\mu \text{id}_V$. From the generalisation of the Theorem of Takahashi [10, §3], we see that

$$\Delta^V t = \mu t,$$

for any $t \in \mathbb{R}^{n+2}$. Thus, using the complex structure $J$ of $Q \to \text{Gr}_n(\mathbb{R}^{n+2})$, we have that the same is true for any $t \in \mathbb{C}^{n+2}$. When $Q \to \text{Gr}_n(\mathbb{R}^{n+2})$ is regarded as a complex vector bundle with the canonical connection, the pull-back bundle $V \to M$ is also regarded as the pull-back of $\mathcal{O}(1) \to \text{Gr}_{n+1}(\mathbb{C}^{n+2})$ with the canonical connection by the composition of $i$ and $f$. Using again the generalisation of the Theorem of Takahashi, the induced map $i \circ f$ by the pair $(V \to M, \mathbb{C}^{n+2})$ is harmonic. Lemma 2.2 yields that $i \circ f$ has constant energy density.

Conversely, suppose that $i \circ f : M \to \text{Gr}_{n+1}(\mathbb{C}^{n+2})$ is a harmonic map with constant energy density. Since $\mathcal{O}(1) \to \text{Gr}_{n+1}(\mathbb{C}^{n+2})$ is of complex rank 1 and $i \circ f$ has constant energy density, $i \circ f$ is automatically an EH map with $A = -\mu \text{id}_V$. It then follows from the generalisation of the Theorem of Takahashi that relation (3) holds for all $t \in \mathbb{C}^{n+2}$, and also for any $t \in \mathbb{R}^{n+2} \subset \mathbb{C}^{n+2}$. Hence, if we regard $(i \circ f)^{-1}\mathcal{O}(1) \to M$ as a real vector bundle, then we recover $f : M \to \text{Gr}_n(\mathbb{R}^{n+2})$, which is an EH harmonic map by the generalisation of the Theorem of Takahashi. \hfill $\square$

In what follows we particularise the previous theory to the case in which $M$ is the complex projective line $\mathbb{CP}^1$.

Let $\mathcal{O}(k) \to \mathbb{CP}^1$ be the holomorphic line bundle of degree $k$ over $\mathbb{CP}^1$ equipped with the standard metric and canonical connection. Using the theory of spherical harmonics [12], we have a decomposition of $\Gamma(\mathcal{O}(k))$ in the $L^2$-sense:

$$\Gamma(\mathcal{O}(k)) = \sum_{l=0}^{\infty} S^{[k]+2l} \mathbb{C}^2.$$

Moreover, $S^{[k]+2l} \mathbb{C}^2$ is an eigenspace of the Laplacian induced by the canonical connection, and its eigenvalue is $2\pi \{2l([k]+l+1)+|k|\}$. 

Let \( f : \mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2}) \) be a smooth map. Then, the pull-back of the universal quotient bundle is isomorphic, as a complex line bundle, to \( \mathcal{O}(k) \to \mathbb{C}P^1 \) for some \( k \in \mathbb{Z} \), termed the degree of \( f \). In addition, we say that \( f \) satisfies the gauge condition (with regard to \( \mathcal{O}(k) \to \mathbb{C}P^1 \) together with its canonical connection) if there exists a bundle isomorphism preserving metrics and connections between \( V \to \mathbb{C}P^1 \) and \( \mathcal{O}(k) \to \mathbb{C}P^1 \). This is just condition (G) in Theorem 2.3.

Since the bundles are of complex rank one, a map of degree \( k \) satisfies the gauge condition if and only if

\[
(5) \quad f^* \omega_Q = k \omega_1
\]

where \( \omega_1 \) is the fundamental two-form of \( \mathbb{C}P^1 \), and \( \omega_Q \) is the fundamental two-form of \( \text{Gr}_n(\mathbb{R}^{n+2}) \).

**Remark 2.** If \( f : \mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2}) \) is harmonic, then it is conformal (cf Eells & Lemaire [6]). Having constant energy density, \( f \) is an isometric immersion up to homothety. Then, it follows from harmonicity that \( f \) is minimal.

The gauge condition for maps (5) is intimately related to the more familiar concept of Kähler angle (eg Chern & Wolfson [4], Bolton, Jensen, Rigoli & Woodward [1]). In the following paragraphs we would like to clarify this relationship.

In the general situation, let \( N \) be a Kähler manifold with metric \( g_n \) and Kähler two-form \( \omega_n \), and let \( \phi : \mathbb{C}P^1 \to N \) be a harmonic map with constant energy density \( e(\phi) \). Regarded as an isometric immersion (up to homothety), it satisfies \( \phi^* g_n = mg_1 \) for some positive number \( m > 0 \), where \( g_1 \) denotes the metric on \( \mathbb{C}P^1 \). Declaring \( \{e_1, e_2\} = \{e, J_1 e\} \) to be an oriented orthonormal (local) frame of \( \mathbb{C}P^1 \) with respect to \( g_1 \), the homothety condition can be written as

\[
g_n (d\phi(e_i), d\phi(e_j)) = m \delta_{ij}, \quad m > 0.
\]

In the context of submanifold theory, the Kähler angle \( \theta_\phi \) for the map \( \phi \), is defined by the relation

\[
\cos \theta_\phi = \frac{g_n (J_n d\phi(e), d\phi(J_1 e))}{|J_n d\phi(e)|_n |d\phi(J_1 e)|_n}.
\]

Now, further assume that \( \phi \) is subject to a condition similar to (5). That is, to \( \phi^* \omega_n = k \omega_1 \) for some constant degree \( k \). Then, constancy of the Kähler angle follows

\[
\cos \theta_\phi = \frac{k}{m}
\]

and \( \theta_\phi \) depends only on the degree \( k \) and the EH constant \( l \).

In the particular case of our interest, if \( f : \mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2}) \) is an EH harmonic map, then by Remark 2 we can regard it as an isometric immersion up to homothety. The previous paragraph shows that for an EH harmonic mapping, satisfying the gauge condition (5) is equivalent to the more familiar requirement of constancy of the Kähler angle. Hence, statements in terms of this latter condition are thus favoured in the sequel.
Lemma 3.2. Let $f : \mathbb{CP}^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ be an EH harmonic map of degree $k$, and of constant Kähler angle. Then there exists a non-positive integer $l$ such that $e(f) = 4\pi \{2\{k| + l + 1\} + |k|\}$.

Proof. Since $f$ is a harmonic map of degree $k$, it follows from the generalisation of the Theorem of Takahashi that $\Delta f + A f = 0$ for any section $t$ of $\mathbb{R}^{n+2} \subset \Gamma(V)$. By the EH condition, we have a non-negative $\mu$ such that $A = -\mu \text{id}_V$. Combined with the gauge condition implied by the constancy of the Kähler angle assumption, $\mathbb{R}^{n+2} \subset \Gamma(V)$ can be considered as the eigenspace with eigenvalue $\mu$ of the Laplacian induced by the canonical connection, acting on $\Gamma(O(k))$. Thus, $\mu = 2\pi \{2\{k| + l + 1\} + |k|\}$, and Lemma 2.2 yields the result. □

Definition 2. Let $f : \mathbb{CP}^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ be an EH harmonic map of degree $k$, and of constant Kähler angle. If the energy density of $f$ is $4\pi \{2\{k| + l + 1\} + |k|\}$, then $l$ is said to be the Einstein-Hermitian constant of $f$ (EH constant, for short).

Remark 3. In the above situation, $A = -2\pi \{2\{k| + l + 1\} + |k|\} \text{id}_{O(k)}$. Hence, $2\pi \{2\{k| + l + 1\} + |k|\} \mu$ should more accurately be called the EH constant of $f$, in accordance with what stated in §1. However, in what follows we adopt the above convention by simplicity. Moreover, we will often be making reference to ‘Einstein-Hermitian harmonic mappings of degree $k$ and Einstein-Hermitian constant $l$’. We will shorten this to ‘$(k, \text{EH}(l))$ harmonic map’, instead.

4. Moduli space by gauge, and image equivalence

In this section we describe the moduli space of gauge equivalence classes of $(k, \text{EH}(l))$ harmonic maps $\mathbb{CP}^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ of constant Kähler angle. It is a direct application of the generalised version of the theory of do Carmo–Wallach [10], §5. The particular version of the theorem needed has been introduced in §2 as Theorem 2.3.

The proof of our Main Theorem 1 is similar to the one of the Main Theorem in [9] except by the use made here of the contraction operator. Hence, we use the same notation and conventions used there without further explanation. The representation theory needed can be consulted in [9] §4.

Let $W$ denote the space of the $l$-th eigensections of $O(k) \to \mathbb{CP}^1$, in the ordering defined by (4), regarded as the SU(2)–representation $S^{|k|+2}\mathbb{C}^2$. It decomposes under the action of the subgroup U(1) as

$$W = C_{-|k|+2} \oplus C_{-|k|+2+2} \oplus \cdots \oplus C_{|k|+2-2} \oplus C_{|k|+2},$$

where $C_\lambda$ denotes the irreducible U(1)-module of weight $\lambda$.

The homogeneous description of $O(k) \to \mathbb{CP}^1$ is $SU(2) \times_U SU(1) V_0 \to \mathbb{CP}^1$ with $V_0 = C_{-k} \subset W$. Following the generalisation of do Carmo–Wallach theory [10] §5, we shall regard the universal quotient bundle as a real vector bundle of rank 2, and determine the subspaces GS($V_0, V_0$) and GS($mV_0, V_0$) of $S(W)$. In the sequel $V_0$ and $W$ shall stand either for the complex or underlying real vector spaces whenever the meaning is clear. Since GH($V_0, V_0$) ⊆ GS($V_0, V_0$), we have that $H(W) \subset GS(V_0, V_0)$ and we must determine the
Next, we use the symplectic form $\omega$

$\text{GS}(V_0, V_0) \cap \sigma H_+(W) \oplus J\sigma H_+(W)$,

and

$\text{GS}(mV_0, V_0) \cap \sigma H_+(W) \oplus J\sigma H_+(W)$.

**Lemma 4.1.** $mV_0 = C_{-k-2} \oplus C_{-k+2}$.

**Proof.** Decompose $S^2C^2$ into $U(1)$-irreducible representation $S^2C^2|U(1) = C_2 \oplus C_0 \oplus C_{-2}$ and using the real structure we have $(S^2C^2)^R \cong su(2)$, $(C_0)^R \cong u(1)$ therefore $(C_2 \oplus C_{-2})^R \cong m$. Then,

$m \otimes V_0 = (C_2 \oplus C_{-2}) \otimes C_{-k} = C_{-k+2} \oplus C_{-k-2}$

The action of $m$ on $V_0$ is then obtained by projecting $m \otimes V_0$ back to $S^{k+2l}C^2$; therefore

$mV_0 = (m \otimes V_0) \cap S^{k+2l}C^2|U(1) = C_{-k-2} \oplus C_{-k+2}$.

$\square$

**Proposition 4.2.** Let $W = S^{k+2l}C^2$ be an irreducible representation of $SU(2)$ with $k, l > 0$. Then $\text{GS}(mV_0, V_0) \cap (\sigma H_+(W) \oplus J\sigma H_+(W))$ is the direct sum of representations $S^mC^2$ of $SU(2)$ with $m < 4k$ appeared in

$\sigma H_+(S^{k+2l}C^2) \oplus J\sigma H_+(S^{k+2l}C^2) = \bigoplus_{r=0}^{2r \leq k+2l} S^{2(k+2l) - 4r}C^2$.

Before proving Proposition 4.2, we introduce a necessary technical tool, the contraction operator $C$ and some of its properties. Let $\{e_1, e_2\}$ be the standard basis of $C^2$, that is, $\{e_1, e_2\}$ is a unitary basis and satisfies $\omega(e_1, e_2) = 1$ where $\omega$ is the invariant symplectic form on $C^2$. The totally symmetrised product of $n-p$ copies of $e_1$ and $p$ copies of $e_2$ will be denoted by juxtaposition, $e_1^{n-p} e_2^p$. Defining $u_{n-2p} = \binom{n}{p}^{-\frac{1}{2}} e_1^{n-p} e_2^p$, $\{u_n, u_{n-2}, \ldots, u_{n+2}, u_{-n}\}$ is a unitary, weight basis of $S^nC^2$. It is an easy matter to check that $S^nC^2$ sits in $S^{n-1}C^2 \otimes C^2$ (resp. in $C^2 \otimes S^{n-1}C^2$) as follows:

$e_1^{n-p} e_2^p = e_1^{n-p-1} e_2^p \otimes e_1 + e_1^{n-p} e_2^{p-1} \otimes e_2 \in S^{n-1}C^2 \otimes C^2$,

$= e_1 \otimes e_1^{n-p-1} e_2^p + e_2 \otimes e_1^{n-p} e_2^{p-1} \in C^2 \otimes S^{n-1}C^2$.

Next, we use the symplectic form $\omega$ to define an equivariant contraction operator $C : S^nC^2 \otimes S^nC^2 \to S^{n-1}C^2 \otimes S^{n-1}C^2$ given by:

$S^nC^2 \otimes S^nC^2 \underset{\omega}{\hookrightarrow} S^{n-1}C^2 \otimes C^2 \otimes C^2 \otimes S^{n-1}C^2 \to S^{n-1}C^2 \otimes S^{n-1}C^2$.

Explicitly,

$C \left( e_1^{n-p} e_2^p \otimes e_1^{n-q} e_2^q \right) = e_1^{n-p-1} e_2^p \otimes e_1^{n-q} e_2^{q-1} - e_1^{n-p} e_2^{p-1} \otimes e_1^{n-q-1} e_2^q$.

It follows from the equivariance of $C$, and Schur’s lemma that

$\ker C = S^{2n}C^2$. 

(6)
Moreover, explicit computation allows to establish the useful formula
\begin{equation}
C^{2r} \left( e_{1}^{r} e_{2}^{p} r^{q} e_{2}^{q} + e_{1}^{r} e_{2}^{p} r^{q} e_{2}^{q} \right)
= \sum_{s} (-1)^{s} \binom{2r}{s} \left( e_{1}^{r-p-s} e_{2}^{p-s-2r} \otimes e_{1}^{r-p-q-s-2r} e_{2}^{q-s}
+ e_{1}^{r-p-q-s} e_{2}^{q-s} \otimes e_{1}^{r-p-1} e_{2}^{p-s-2r} \right),
\end{equation}
where \( s \leq n - p, q, \) and \( s \geq 2r - p, 2r + q - n. \) If \( s \) is not in this range, the corresponding term is regarded as 0.

**Proof of Proposition 4.2.**

The real structure map of \( \sigma H_{\pm}(W) \oplus J \sigma H_{\pm}(W), \) interchanges \( \pm \) signs on weights. Therefore, we can assume \( k > 0 \) without loss of generality. Fix a positive integer \( k \) and let \( W = S^{k+2l} \mathbb{C}^{2}. \) Notice that \( \sigma H_{\pm}(W) \oplus J \sigma H_{\pm}(W) \) is a complex \( SU(2) \) representation and so the following decomposition follows from Clebsch–Gordan formulae:
\[
\sigma H_{+}(S^{k+2l} \mathbb{C}^{2}) \oplus J \sigma H_{+}(S^{k+2l} \mathbb{C}^{2}) = \bigoplus_{r=0}^{2r \leq k+2l} S^{2(k+2l)-4r} \mathbb{C}^{2},
\]
Consequently, it can be identified with the space \( S^{2}(S^{k+2l} \mathbb{C}^{2}) \) of symmetric powers of \( S^{k+2l} \mathbb{C}^{2}. \) Let us make the identification explicit. Define \( X \in \sigma H_{+}(S^{k+2l} \mathbb{C}^{2}) \oplus J \sigma H_{+}(S^{k+2l} \mathbb{C}^{2}) \) as
\[
X = S(u_{-k+2}, u_{-k}) - S(J u_{-k+2}, J u_{-k})
\]
Since \( \sigma(u_{-n-2p}) = (-1)^{p} u_{-n+2p}, \) we have
\[
2\sigma X = (\cdot, u_{-k+2})(-1)^{k+l} u_{k} + (\cdot, u_{-k})(-1)^{k+l-1} u_{k-2}
+ (\cdot, J u_{-k+2})(-1)^{k+l} J u_{k} + (\cdot, J u_{-k})(-1)^{k+l-1} J u_{k-2}
= (-1)^{k+l}(h(\cdot, u_{-k+2})u_{k} - h(\cdot, u_{-k})u_{k-2})
= (-1)^{k+1}(h(\cdot, \sigma(u_{-k-2}))u_{k} + h(\cdot, \sigma(u_{k}))u_{k-2}).
\]
Therefore, when we regard \( \sigma H_{+}(W) \oplus J \sigma H_{+}(W) \) as a subspace of \( S^{k+2l} \mathbb{C}^{2} \otimes S^{k+2l} \mathbb{C}^{2}, \) \( X \) corresponds to \( u_{k-2} \otimes u_{k} \oplus u_{k} \otimes u_{k-2}, \) that is
\[
X = e_{1}^{k+2l-\ell} e_{1}^{\ell+1} \otimes e_{1}^{k+2l-\ell} e_{2}^{\ell+1} + e_{1}^{k+2l-\ell} e_{1}^{k+2l-\ell} \otimes e_{1}^{k+2l-\ell} e_{2}^{\ell+1}
= e_{1}^{k+\ell-1} e_{2}^{\ell+1} \otimes e_{1}^{k+\ell} e_{2}^{\ell} + e_{1}^{k+\ell} e_{2}^{\ell} \otimes e_{1}^{k+\ell-1} e_{2}^{\ell+1},
\]
up to a constant multiple. Applying equation (7) to this last expression
\[
C^{2r} \left( e_{1}^{k+\ell-1} e_{2}^{\ell+1} \otimes e_{1}^{k+\ell} e_{2}^{\ell} + e_{1}^{k+\ell} e_{2}^{\ell} \otimes e_{1}^{k+\ell-1} e_{2}^{\ell+1} \right) = 0 \quad \text{iff } r \geq l + 1
\]
A similar argument is possible for \( Y = S(u_{-k+2}, J u_{-k}) + S(J u_{-k+2}, u_{-k}). \) It then follows from (6) that
\[
\text{GS}(mV_{0}, V_{0}) \perp \bigoplus_{r \geq l} S^{2(k+2l)-4r} \mathbb{C}^{2}.
\]
Corollary 4.3. The orthogonal complement to $\text{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbb{R} \text{Id}$ in $S(W)$ is
\[
\bigoplus_{r \geq |k| + 2l + 1} S^{2(|k| + 4l - 4r)} \mathbb{C}^2
\]

Theorem 4.4. If $f : \mathbb{CP}^1 \to \text{Gr}_n(\mathbb{R}^{|k| + 2l})$ is a full, $(k, \text{EH}(l))$ harmonic map of constant Kähler angle, then

(1) $n \leq 2(|k| + 2l)$.

Assuming $n$ to be maximal, let $\mathcal{M}_{k,l}$ be the moduli space up to gauge equivalence of maps, and denote its closure by the inner product by $\overline{\mathcal{M}_{k,l}}$. Then,

(2) $\mathcal{M}_{k,l}$ can be regarded as an open bounded convex body in
\[
\bigoplus_{r = |k| + 2l + 1} S^{2(|k| + 2l - 4r)} \mathbb{C}^2.
\]

(3) The boundary points of $\overline{\mathcal{M}_{k,l}}$ describe those maps whose images are included in some totally geodesic submanifold
\[
\text{Gr}_p(\mathbb{R}^{p+2}) \subset \text{Gr}_{2(|k| + 2l)}(\mathbb{R}^{2(|k| + 2l + 1)}), \quad p < 2(|k| + 2l).
\]

(4) The totally geodesic submanifold $\text{Gr}_p(\mathbb{R}^{p+2})$ can be regarded as the common zero set of some sections of $Q \to \text{Gr}_{2(|k| + 2l)}(\mathbb{R}^{2(|k| + 2l + 1)})$, which belongs to $(\mathbb{R}^{p+2})^\perp \subset \Gamma(Q)$.

Proof. The restriction $n \leq 2(|k| + 2l)$ follows from (1) in Theorem 2.3 and the dimension of the corresponding eigenspace. □

Remark 4. The previous theorem establishes a diffeomorphism between the moduli space $\mathcal{M}_{k,l}$ of full, $(k, \text{EH}(l))$ harmonic mappings of constant Kähler angle and the moduli space $\mathcal{M}_k$ of full, holomorphic isometric embeddings of degree $k$. Due to this, proofs of propositions about $\mathcal{M}_{k,l}$ are, with minor changes, identical to proofs about $\mathcal{M}_k$. We now recall some important properties of the moduli space $\mathcal{M}_{k,l}$, which in virtue of the previous Theorem are derivative from those in [9] §8.

The complex structure on $Q \to \text{Gr}_{2(|k| + 2l)}(\mathbb{R}^{2(|k| + 2l + 1)})$ induces a similar one on $\mathcal{M}_{k,l}$, so it is a complex submanifold of $\bigoplus_{r = |k| + 2l + 1} S^{2(|k| + 4l - 4r)} \mathbb{C}^2$. As the centraliser of the holonomy group acts on $\mathcal{M}_{k,l}$ with weight $-k$, we get

Theorem 4.5. Let $\mathcal{M}_{k,l}$ be the moduli space (up to image equivalence) of full, $(k, \text{EH}(l))$ harmonic maps $\mathbb{CP}^1 \to \text{Gr}_{2(|k| + 2l)}(\mathbb{R}^{2(|k| + 2l + 1)})$ of constant Kähler angle. Then $\mathcal{M}_{k,l} = \mathcal{M}_{k,l}/S^1$.

Again, as in the holomorphic isometric embedding case, $\mathcal{M}_{k,l}$ is a Kähler manifold together with an $S^1$-action preserving the Kähler structure, and is therefore equipped with the moment map $\mu = |T|^2$.

Corollary 4.6. There exists a one-parameter family $\{f_t\}, \ t \in [0, 1]$, of $\text{SU}(2)$-equivariant image–inequivalent isometric minimal immersions of even degree of $\mathbb{CP}^1$ into complex quadrics where $f_0$ corresponds to the standard map.
Remark 5. In this setting $f_1$ would coincide with the real standard map to be introduced in the proof of Proposition 6.1.

5. A one-to-one correspondence between moduli spaces

From Theorem 4.4, the dimension of the moduli space $M_{k,l}$ of full $(k,\text{EH}(l))$ harmonic maps of constant Kähler angle is independent of the EH constant $l$. In the present section, we will obtain a one-to-one correspondence between the moduli spaces $M_{k,l}$ and $M_{k,l-1}$ for each $k, l$.

For this purpose, we modify the contraction operator $C$ (cf, §3) as follows:

\begin{equation}
\tilde{C} = \bigoplus_{r=1}^{n} c_{2n-2r} C_{S^{2n-2r}C^2}
\end{equation}

preserves the Hermitian inner product. The adjoint operator $\tilde{C}^*$ of $\tilde{C}$ coincides with the ‘inverse’ $\tilde{C}^{-1}$ since $\tilde{C}^* \tilde{C}$ is a positive Hermitian operator preserving the Hermitian product.

Next, we introduce the operator norm:

$$|D|_{op} = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } D \}.$$  

If $f$ is a full $(k,\text{EH}(l))$ harmonic map then by Theorem 2.3

$$f([g]) = T^{-1} \ker \ ev,$$

where the positive symmetric automorphism $T = (\text{id} + D)^{\frac{1}{2}} \in \text{Aut} \ R^{2(|k|+2l+1)}$ satisfies

\begin{equation}
(D, \text{GS}(mV_0, V_0)) = 0.
\end{equation}

Note that $|D|_{op} < 1$ due to the positivity of $T$.

Then $\tilde{C}^2(D)$ can be regarded as a symmetric endomorphism on $R^{2(|k|+2l-1)+1}$. Hence,

$$\text{id} + \frac{|D|_{op}}{|\tilde{C}^2(D)|_{op}} \tilde{C}^2(D)$$

is also a positive symmetric automorphism on $R^{2(|k|+2l-1)+1}$. Moreover, since $\tilde{C}$ is equivariant, $\tilde{C}^2(D)$ also satisfies (9). Consequently, Theorem 2.3 implies that

$$\left( \text{id} + \frac{|D|_{op}}{|\tilde{C}^2(D)|_{op}} \tilde{C}^2(D) \right)^{\frac{1}{2}} \ker ev,$$

is also a full $(k,\text{EH}(l-1))$ harmonic map. The inverse construction is straightforward: we may correspond

$$\text{id} + \frac{|D|_{op}}{|\tilde{C}^* \tilde{C}^2(D)|_{op}} \tilde{C}^* \tilde{C}^2(D)$$

to $\text{id} + D$ to obtain a one-to-one correspondence from $M_{k,l}$ to $M_{k,l+1}$. Thus we have established the following
Theorem 5.1. There is a one-to-one correspondence between $\mathcal{M}_{k,l}$ and $\mathcal{M}_{k,l-1}$ which associates the gauge-equivalence class of full $(k, EH(l))$ harmonic maps determined by $T = (id + D)^{\frac{1}{2}}$ to the gauge-equivalence class of full $(k, EH(l - 1))$ harmonic maps determined by

$$\left(id + \frac{|D|_{op}}{|C^2(D)|_{op}}\tilde{C}^2(D)\right)^{\frac{1}{2}}$$

where $\tilde{C}$ denotes the modified contraction operator $[3]$. 

6. Rigidity of the real standard map

An SU(2)-irreducible representation is a class-one representation of the pair $(SU(2), U(1))$, if it contains non-zero $U(1)$-invariant elements.

Proposition 6.1.

1. Let $S^{[k]+2l}C^2$ be the $l$-th eigenspace of the vector bundle $O(k) \rightarrow \mathbb{C}P^1$ and $V_0$ the $U(1)$-representation regarded as its standard fibre. Then, $GH(V_0, V_0) = H(S^{[k]+2l}C^2)$.

2. Let $S^{2[k]+2l}C^2$ be the indicated representation space of SU(2), of which an invariant real subspace is $S^{[k]+\sigma}R^3 \cong R^{2([k]+l)+1}$ and $V_0$ the $U(1)$-representation regarded as the standard fibre for $O(2k) \rightarrow \mathbb{C}P^1$. Then, $GS(V_0, V_0) = S(S^{[k]+\sigma}R^3)$.

Proof. (1) The proof is by reductio ad absurdum and follows, with minor modifications, the same lines as Theorem 5.4 in [9]. We sketch the argument. The SU(2)-module $W = S^{[k]+2l}C^2$ decomposes under $U(1)$ as $S^{[k]+2l}C^2 = C_{-|k|-2l} \oplus C_{-|k|-2l+2} \oplus \cdots \oplus C_{|k|+2l}$.

Then, $V_0 = C_{-k}$ by Frobenius reciprocity and the invariance of the Hermitian inner product, and the decomposition is normal (cf [5], or [9] §6). Consider a class-one representation of $(SU(2), U(1))$ in the orthogonal complement to $GH(V_0, V_0)$ in $H(W)$, and let $C$ be a non-zero $U(1)$-invariant element in it. Polarisation of the orthogonality condition $(C, gH(v_1, v_2))_{H(W)} = 0$ leads to $(Cgv_1, g_{v_2}) = 0$ for all $g \in SU(2), v_1, v_2 \in V_0$. A positive Hermitian operator $T$ is then defined by $T^2 = id + C$, for $C$ small enough. Being $U(1)$-equivariant, Schur’s lemma implies that $T = id$ and so $C = 0$ against the hypothesis. Therefore, every class-one subrepresentation of $(SU(2), U(1))$ in $H(W)$ is in $GH(V_0, V_0)$.

(2) Regarding $\mathbb{C}P^1$ as $SU(2)/U(1)$, the space of sections $\Gamma(O(2k))$ becomes an $SU(2)$-module and the $l$-th eigenspace is identified with $S^{2[k]+2l}C^2$. This decomposes under $U(1)$ as

$$S^{2[k]+2l}C^2 = \bigoplus_{r=0}^{2[k]+2l} C_{2[k]+2l-2r}$$

By the same arguments above, the typical fibre of $O(2k) \rightarrow \mathbb{C}P^1$ is identified with $C_{-2[k]}$ in [10]. Although $S^{2[k]+2l}C^2$ has a real invariant subspace $S^{[k]+\sigma}R^3$ of dimension $2[k] + 2l + 1$, the irreducible components in the right-hand side of [10] are not invariant under the real structure $\sigma$, but
\[ \sigma(C_{2|k|+2l-2r}) = C_{-2|k|-2l+2r}. \] Therefore \( \sigma \) leaves \( (C_{2|k|+2l-2r} \oplus C_{-2|k|-2l+2r}) \) stable for each \( r = 0, \ldots, |k| + l + 1 \), which splits in two isomorphic real irreducible \( U(1) \)-modules, denoted \( (C_{2|k|+2l-2r} \oplus C_{-2|k|-2l+2r})^R \). If \( r = |k| + l \), then \( (C)^R = R \), the trivial real representation. Thus

\[ S_0^{[k]+l} R^3 = \bigoplus_{r=0, r \neq |k|+l}^{2|k|+2l} (C_{2|k|+2l-2r} \oplus C_{-2|k|-2l+2r})^R \oplus R. \]

The space \( S^{[k]+l} R^3 \) globally generates \( \mathcal{O}(2k) \to \mathbb{CP}^1 \), and so determines a real standard map \( f_0 : \mathbb{CP}^1 \to \text{Gr}_{2k-1}(R^{2k+1}) \) which turns out to be an \( (2k, \text{EH}(l)) \) isometric minimal immersion by Lemma 2.3 in [9]. It is then possible to define the adjoint of the evaluation map,

\[ ev^*_{\omega} : \mathcal{O}(2k) \to S^{[k]+l} R^3, \]

such that its image at the reference point of \( \mathbb{CP}^1 \), is \( (C_{2|k|} \oplus C_{-2|k|})^R \).

Now, \( S^{[k]+l} R^3 \) gives the normal decomposition of \( S_0^{[k]+l} R^3 \) where \( V_0 = (C_{-2k} \oplus C_{2k})^R \). The space of symmetric endomorphisms of \( S_0^{[k]+l} R^3 \) can be identified using representation theory as in [9] §4 to give

\[ S(S_0^{[k]+l} R^3) \subset \text{End}(S_0^{[k]+l} R^3) = S_0^{[k]+l} R^3 \otimes_R (S_0^{[k]+l} R^3)^* \cong \otimes^2 S_0^{[k]+l} R^3, \]

where

\[ S(S_0^{[k]+l} R^3) = \bigoplus_{r=0}^{[k]+l} S_0^{4|k|+4l-2r} R^3 \subset \otimes^2 S_0^{[k]+l} R^3 = \bigoplus_{r=0}^{2|k|+2l} S_0^{4|k|+4l-2r} R^3. \]

The real standard map induced by \( \mathcal{O}(2k) \to \mathbb{CP}^1 \) and \( S_0^{[k]+l} R^3 \) has just been depicted in the above proof. Since its deformation space is, up to gauge equivalence, \( GS(V_0, V_0) \perp \subset S(S_0^{[k]+l} R^3) \) we obtain the following

**Corollary 6.2.** Let \( S_0^{[k]+l} R^3 \cong R^{2([k]+l)+1} \) be the real invariant subspace of the \( SU(2) \)-module \( S^{2([k]+l)+2} \mathbb{C}^2 \). If \( f : \mathbb{CP}^1 \to \text{Gr}_{2([k]+l)-1}(R^{2([k]+l)+1}) \) is a \( (2k, \text{EH}(l)) \) isometric minimal immersion, then it is the standard map induced by \( S_0^{[k]+l} R^3 \) up to gauge equivalence.

### 7. Rigidity of totally real EH harmonic maps

We start our discussion by recalling the relation between certain special class of harmonic maps, and totally real immersions: Let \( f : \mathbb{CP}^1 \to \text{Gr}_n(R^{n+2}) \) be a harmonic map of degree 0 with constant energy density satisfying the gauge condition [5]. Since the canonical connection on the trivial complex line bundle is flat, we see from the gauge condition that

\[ f^* \omega_n = 0. \]

Since \( f \) is conformal, we conclude that it is a totally real isometric immersion up to a constant multiple. Conversely, if \( f : \mathbb{CP}^1 \to \text{Gr}_n(R^{n+2}) \) is an isometric minimal totally real immersion, then \( f \) satisfies the gauge condition above. Thus \( f \) is a harmonic map of degree 0, with constant energy density, satisfying [5].

We will show the rigidity of totally real minimal immersion with the EH condition in this section.
Theorem 7.1. Let $f : \mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ be a totally real, full, $(0,\text{EH}(l))$ minimal immersion. Then $n = 4l$ and $f$ is image-equivalent to the standard map.

Proof. Since $f$ is EH harmonic, it follows from the generalisation of the Theorem of Takahashi [10] that $\mathbb{R}^{n+2}$ is an eigenspace of the Laplacian acting on $\mathbb{R}^2$-valued functions. Thus, $\mathbb{R}^{n+2}$ is a subspace of $S^l\mathbb{C}^2$, where $l$ is an even number. To define the standard map $f_0$, we consider the weight decomposition of $S^l\mathbb{C}^2$:

$$S^l\mathbb{C}^2 = C_l \oplus C_{l-2} \oplus \cdots \oplus C_0 \oplus \cdots \oplus C_{-l}.$$ 

Then $f_0([g]) = gC^l$. The normal decomposition of the standard map is given by

$$\text{Im}B_p = C_{2p} \oplus C_{-2p}.$$ 

Then the same argument as in the proof of Theorem 5.4 in [9] gives the result. □

Actually, the result would still hold at the level of gauge-equivalence of maps.

The EH condition is indispensable for the previous rigidity argument, as dropping it leads to the following counterexample by Wang and Jiao [13].

We use an invariant real subspace $\mathbb{R}^{2m+1} \subset S^{2m}\mathbb{C}^2$ to define a totally real isometric minimal immersion $f_1 : \mathbb{C}P^1 \to \text{Gr}_{2m}(\mathbb{R}^{2m+2})$. The weight decomposition of $\mathbb{R}^{2m+1}$ is

$$\mathbb{R}^{2m+1} = C_{2m} \oplus C_{2m-2} \oplus \cdots \oplus C_2 \oplus R_0.$$ 

Consider the orthogonal direct sum of $\mathbb{R}^{2m+1}$ and a trivial representation $R$. The orthogonal complement of $R_0 \oplus R$ in $\mathbb{R}^{2m+1} \oplus R$ is denoted by $(R_0 \oplus R)^\perp$. Then we define $f_1$ as

$$f_1([g]) := g(R_0 \oplus R)^\perp = gR^\perp_0,$$ 

where $R^\perp_0$ denotes the orthogonal complement of $R_0$ in $\mathbb{R}^{2m+1}$. The orientation of $R_0 \oplus R$ is determined by this ordering.

Now, $f_1$ is of degree 0 and the induced connection on the pull-back of the universal quotient bundle is a product connection. By definition of $f_1$, the mean curvature operator $A$ has 0 as eigenvalue. Moreover, $A$ is parallel because its eigenvalues are constant, and the eigenspace decomposition is invariant under the connection. The generalisation of Theorem of Takahashi yields that $f_1$ is harmonic. Then, by the above argument, $f_1$ is a totally real, isometric minimal immersion with $\det A = 0$ and $\nabla A = 0$.

Notice that $f_0$ and $f_1$ are not image-equivalent because their mean curvature operators are different. Indeed, the image-equivalence class of $f_1$ is characterised by the following

Theorem 7.2. Let $f : \mathbb{C}P^1 \to \text{Gr}_n(\mathbb{R}^{n+2})$ be a totally real, full, minimal immersion with $\det A = 0$ and $\nabla A = 0$. Then, $n$ is an even integer and $f$ is image equivalent to $f_1$.

Proof. The pull-back of the universal quotient bundle has the eigenspace decomposition of $A$, which is invariant under the pull-back connection, because $A$ is parallel. Since $\det A = 0$, its only eigenvalues are 0 and $-2e(f)$. The
orthogonal decomposition of $V \to \mathbb{C}P^1$ is denoted by $V_1 \oplus V_2$ corresponding to eigenvalues $0$ and $-2e(f)$, respectively. The real vector space $\mathbb{R}^{n+2}$ determines a family of sections of $V_1 \to \mathbb{C}P^1$ which, by the generalisation of the Theorem of Takahashi, are constant. Hence the image of $V_1 \to \mathbb{C}P^1$ under the adjoint of the evaluation homomorphism gives a subspace $U_1$ of $\mathbb{R}^{n+2}$. It follows from the fullness of $f$ that $U_1$ is of dimension one and so, $V_1 \to \mathbb{C}P^1$ is identified with $U_1 \to \mathbb{C}P^1$.

The orthogonal complement of $U_1$ is denoted by $U_2$. Using again the generalisation of the Theorem of Takahashi, there exists an integer $m$ such that $e(f) = 2\pi \{2m(m+1)\}$. Moreover, it follows from the fullness of $f$ that $U_2$ is a subspace of $\mathbb{R}^{2m+1}$ which is an invariant real subspace of $S^{2m} \mathbb{C}^2$.

Therefore, by the discussion at the beginning of the section, the problem is equivalent to that of classifying an $(0, \text{EH}(m))$ harmonic map $\mathbb{C}P^1 \to \mathbb{R}P^{2m}$, fulfilling the gauge condition.

The standard map induced by a rank-one, trivial vector bundle and $\mathbb{R}^{2m+1}$ has the normal decomposition:

$$\text{Im} B_p = C_{2p}, \quad p \geq 1$$

and then again the result follows as in the proof of Theorem 5.4 [9]. □

8. Applications

In this final section we use our classification theorem (in particular, Corollary 4.6) to give a new proof of the following classical result, originally stated in its different incarnations by Calabi [2], and do Carmo & Wallach [5].

**Theorem 8.1.** Let $i : S^2 \to S^n \subset \mathbb{R}^{n+1}$ be a full isometric minimal immersion, where $S^n$ is the unit sphere. Then $i$ is $SU(2)$-equivariant.

The proof of the theorem will follow from a reinterpretation (in the sense of [10]) of the well-known theory developed by Ruh & Vilms [11]. To this end, let $M$ be a Riemannian manifold and $I : M \to \mathbb{R}^N$ an isometric immersion. Using the inner product on $\mathbb{R}^N$, we consider a bundle homomorphism $ev : \mathbb{R}^N \to T M$. In fact, $ev$ is the adjoint homomorphism of $dI : TM \to \mathbb{R}^N$. Then, we have an exact sequence of vector bundles:

$$0 \to NM \to \mathbb{R}^N \to TM \to 0,$$

where $NM \to M$ is the normal bundle of $M$. Denote the second fundamental form of the tangent bundle by $K \in \Omega^1(TM^* \otimes NM)$. Since $I$ is an isometric immersion, $K$ is also regarded as the second fundamental form of submanifold geometry. Hence, it satisfies

$$K_XY = K_YX, \quad X, Y \in TM.$$

The bundle homomorphism $ev$, together with the orientation of $M$ induces a Gauss map $f : M \to \text{Gr}_p(\mathbb{R}^N)$:

$$f(x) = \ker ev_x, \quad x \in M,$$

where $p = N - \text{dim } M$. In this context (cf §2) the pull-back of the universal quotient bundle $Q \to \text{Gr}_p(\mathbb{R}^N)$ coincides with the tangent bundle of $M$, and
$K$ can also be regarded as the pull-back of the second fundamental form of $Q \to \text{Gr}_p(\mathbb{R}^N)$ in the exact sequence

$$0 \to S \to \mathbb{R}^N \to Q \to 0.$$ 

Let $n$ denote the mean curvature of $I : M \to \mathbb{R}^N$. Explicitly, $n = \sum K_{e_i}e_i$, where $\{e_1, e_2, \cdots \}$ is an orthonormal frame of $M$. The Gauss-Codazzi equations, \cite{7}, p.23, then yield

$$\nabla_X n = \sum (\nabla_X K)(e_i, e_i) = (\nabla e_i K)(e_i, X) = K^*_{\tau(f)} X,$$

where $\tau(f)$ is the tension field of the Gauss map $f$. Hence, we recover Ruh & Vilms result \cite{11}

**Theorem 8.2.** Let $M$ be a Riemannian manifold and $I : M \to \mathbb{R}^N$ an isometric immersion. The Gauss map is denoted by $f : M \to \text{Gr}_p(\mathbb{R}^N)$. Then the mean curvature of $I$ is parallel if and only if $f$ is a harmonic map.

Moreover, since $I$ is an isometric immersion, the Gauss map $f$ satisfies the gauge condition; in other words, the pull-back connection is the Levi-Civita connection on $V \cong TM \to M$.

Finally, we use the Gauss-Codazzi equations to compute the mean curvature operator $A$ of $f : M \to \text{Gr}_p(\mathbb{R}^N)$:

$$AX = -\sum K^*_{e_i} K_{e_i} X = K^*_X n - \text{Ric}^M(X),$$

where $\text{Ric}^M$ is the Ricci operator of $M$.

Next, let $i : M \to S^n \subset \mathbb{R}^{n+1}$ be an isometric minimal immersion, where $S^n$ is the unit sphere. By composition, we get an isometric immersion $I : M \to \mathbb{R}^{n+1}$ with parallel mean curvature $n$. In this case, $n(x) = -mI(x)$, where $m = \dim M$, and so, $K^*_X n = -m X$. Summarising,

**Lemma 8.3.** Let $i : M \to S^n \subset \mathbb{R}^{n+1}$ be an isometric minimal immersion. Using the standard embedding $S^n \subset \mathbb{R}^{n+1}$, we get an isometric immersion $I : M \to \mathbb{R}^{n+1}$ with parallel mean curvature. The Gauss map of $I$ is an EH harmonic mapping if and only if $M$ is an Einstein manifold.

Now, we can proceed with the

**Proof of Theorem 8.1.**

By composition, we consider an isometric immersion $I : S^2 \to \mathbb{R}^{n+1}$ with parallel mean curvature. Since $S^2$ is an Einstein manifold, the Gauss map of $I$ denoted by $f$ is an EH harmonic map with the gauge condition. Since the pull-back of the universal quotient bundle is identified with the tangent bundle, $f$ is of degree 2. We use Theorem 4.5 and Corollary 4.6 to conclude that $f$ is an SU(2)-equivariant map. Since $f$ can be considered as the differential of $I$, $I$ and $i$ themselves are SU(2)-equivariant maps. \hfill $\square$

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