Kinematic consistency relations of large-scale structures

Patrick Valageas
Institut de Physique Théorique,
CEA, IPht, F-91191 Gif-sur-Yvette, Cédex, France
CNRS, URA 2306, F-91191 Gif-sur-Yvette, Cédex, France
(Dated: November 19, 2013)

We describe how the kinematic consistency relations satisfied by density correlations of the large-scale structures of the Universe can be derived within the usual Newtonian framework. These relations express a kinematic effect and show how the $(l + n)$-density correlation factors in terms of the $n$-point correlation and $l$ linear power spectrum factors, in the limit where the $l$ soft wave numbers become linear and much smaller than the $n$ other wave numbers. We show how these relations extend to multi-fluid cases. These consistency relations are not equivalent to the Galilean invariance nor to the equivalence principle, as both can be violated and the relations remain valid.

We describe how these relations are due to a weak form of scale separation and that a detection of their violation would indicate non-Gaussian initial conditions or a modification of gravity that does not converge to General Relativity on large scales.

PACS numbers: 98.80.-k

I. INTRODUCTION

The large-scale structure of the Universe is the main probe of the recent evolution of the Universe and of the properties of still mysterious components such as dark matter and dark energy. Unfortunately, even without considering the very complex processes of galaxy and star formation and focusing on the large-scale properties where gravity is the dominant driver, theoretical progress is difficult. Large scales can be described by standard perturbative approaches [1, 2], which can be improved to some degree by using resummation schemes [3–11]. However, these methods cannot reach the truly non-linear regime where shell-crossing effects become important [12–14]. Small scales are studied through numerical simulations or phenomenological models [15] that rely on informations gained through these simulations. However, these scales are very difficult to model with a high accuracy, even with simulations, because of the complexities of galaxy formation processes and feedback effects such as AGN and supernovae outflows [16–19]. Therefore, exact results that do not depend on the small-scale nonlinear physics are very important.

Such results have been recently obtained [20] following the knowledge and methods developed in studies of the inflation era. Using the diffeomorphism invariance of General Relativity, one can express the bispectrum of the curvature fluctuations $\zeta$ in the squeezed limit (where one wave number becomes very small) in terms of the product of the linear soft-mode power spectrum with the possibly nonlinear hard-mode power spectrum [21, 22]. Recently, [20] used this approach to study these “consistency relations” and obtained their expression in the non-relativistic limit that is relevant for current surveys of large-scale structures. The great advantage of these relations is that they remain valid independently of the small-scale physics, which can be highly nonlinear and involve astrophysical processes such as star formation and supernovae outflows.

These relations have also been obtained within a Newtonian framework [27–50]. Indeed, it should not be necessary to go through the relativistic approach to obtain the final non-relativistic limit. As nicely described in [20], within the context of General Relativity and for standard scenarios, these consistency relations follow from the equivalence principle. This ensures that small-scale structures are transported without distortions by large-scale fluctuations, which at leading order correspond to a constant gravitational force over the extent of the small-scale region. Thus, these kinematic consistency relations express a kinematic effect and describe how small scales are transported with time by large scale gravitational forces.

This paper has two main goals. First, we wish to obtain a simple and more explicit derivation (without using the single-stream approximation) in the non-relativistic framework that is most often used for studies of large-scale structures. This also provides a generalization to an arbitrary number of soft wave numbers and fluid components. Second, this simple and explicit derivation allows us to distinguish which ingredients are required for their validity. In particular, it provides a clear framework to check that these consistency relations are not equivalent with the Galilean invariance nor with the equivalence principle. Nevertheless, most realistic models, which satisfy both properties on large scales (including most modified-gravity models) also satisfy the weak scale-separation property that guarantees the consistency relations.

This paper is organized as follows. We first derive the consistency relations in Sec. [1] within a very general framework based on Gaussian initial conditions, using an assumption of scale separation (which states that large-scale fluctuations have an almost uniform impact on small-scale structures). We also generalize these relations to an arbitrary number of soft wave numbers and
to cases with several fluid components. Next, we discuss in Sec. III the conditions of validity of these consistency relations. In particular, we show how they arise from a weak form of scale separation, which is satisfied in the standard cosmological scenario but also in most modified-gravity models. We also explain that this property is not equivalent to the Galilean invariance nor to the equivalence principle. We conclude in Sec. IV.

II. CONSISTENCY RELATIONS FOR DENSITY FIELD CORRELATIONS

A. Correlation and response functions

Let us consider a system fully determined by a field \( \varphi(x) \), which may be for instance the initial condition of a dynamical system [in our case \( \varphi \) will be the Fourier-space linear density contrast \( \delta_{L0}(k) \) today]. We also consider quantities \( \{ \rho_1, \ldots, \rho_n \} \) that are functionals of the field \( \varphi \) [in our case \( \rho_i \) will be the Fourier-space nonlinear density contrast \( \delta(k_i, t_1) \) at wave number \( k_i \) and time \( t_1 \)]. Then, general relations between correlation functions and response functions can be obtained from integrations by parts \[31\], \[32\]. Thus, considering the Gaussian case where the statistical properties of the field \( \varphi(x) \) are defined by its two-point correlation \( C_0(x_1, x_2) = \langle \varphi(x_1)\varphi(x_2) \rangle \), the mixed correlations can be written as the Gaussian average

\[
C^{\ell,n}(x_1, \ldots, x_\ell) \cdots C^{\ell,n}(x_1, \ldots, x_\ell) \rho_1 \cdots \rho_n
\]

\[
= \int D\varphi \ e^{-(1/2)\varphi C_0^{-1} \varphi} \varphi(x_1) \cdots \varphi(x_\ell) \rho_1 \cdots \rho_n
\]

(1)

If the inverse correlation matrix satisfies \( C_0^{-1}(x_i, x_j) = 0 \) for \( i \neq j \), we also have the functional derivatives

\[
\frac{D^n [e^{-(1/2)\varphi C_0^{-1} \varphi} \varphi(x_1) \cdots \varphi(x_\ell)]}{D\varphi(x_1) \cdots D\varphi(x_\ell)} = (-1)^\ell C_0^{-1}(x_1, x_1') \cdots \varphi(x_1')
\]

\[
\times \cdots \times C_0^{-1}(x_\ell, x_\ell') \cdots \varphi(x_\ell') e^{-(1/2)\varphi C_0^{-1} \varphi}.
\]

(2)

Therefore, we can write Eq. (1) as

\[
C^{\ell,n}(x_1, \ldots, x_\ell) = (-1)^\ell C_0(x_1, x_1') \cdots C_0(x_\ell, x_\ell') \int D\varphi \rho_1 \cdots \rho_n
\]

\[
\times \frac{D^n [e^{-(1/2)\varphi C_0^{-1} \varphi} \varphi(x_1) \cdots \varphi(x_\ell)]}{D\varphi(x_1') \cdots D\varphi(x_\ell')}
\]

\[
= C_0(x_1, x_1') \cdots C_0(x_\ell, x_\ell') \int D\varphi e^{-(1/2)\varphi C_0^{-1} \varphi}
\]

\[
\times \frac{D^n [\rho_1 \cdots \rho_n]}{D\varphi(x_1') \cdots D\varphi(x_\ell')}.
\]

(3)

where we made \( \ell \) integrations by parts. This gives the relation

\[
C^{\ell,n}(x_1, \ldots, x_\ell) = C_0(x_1, x_1') \cdots C_0(x_\ell, x_\ell') \cdot R^{\ell,n}(x_1', \ldots, x_\ell')
\]

(4)

between the correlation \( C^{\ell,n} \) and the response function \( R^{\ell,n} \) defined by

\[
R^{\ell,n}(x_1, \ldots, x_\ell) = \left( \frac{D^n \rho_1 \cdots \rho_n}{D\varphi(x_1) \cdots D\varphi(x_\ell)} \right).
\]

(5)

In the cosmological context, working in Fourier space, we take \( \varphi \) as the linear matter density contrast today, \( \delta_{L0}(k) \), and \( \rho_i \) as the nonlinear density contrast \( \delta(k_i, t_1) \) at wave number \( k_i \) and time \( t_1 \), where \( \delta = (\rho - \bar{\rho})/\bar{\rho} \). [The system is fully defined by \( \delta_{L0} \) because we assume that the linear decaying mode has had time to become negligible, so that \( \varphi \) also specifies the initial condition \( \delta_{L1} = D_+(t_1) \delta_{L0} \) at the initial time \( t_1 \rightarrow 0 \), where \( D_+ \) is the linear growth rate.] Then, the linear correlation function is

\[
C_{L0}(k_1, k_2) = \langle \delta_{L0}(k_1) \delta_{L0}(k_2) \rangle = P_{L0}(k_1) \delta_D(k_1 + k_2),
\]

(6)

where \( P_{L0} \) is the linear matter power spectrum, with the inverse

\[
C_{L0}^{-1}(k_1, k_2) = P_{L0}(k_1)^{-1} \delta_D(k_1 + k_2).
\]

(7)

Thus, if the wave numbers \( \{ k_i' \} \) satisfy \( k_i' + k_j' \neq 0 \) for all pairs \( \{ i, j \} \), Eq. (4) writes as

\[
C^{\ell,n}(k_1', \ldots, k_j', k_1, \ldots, k_n, t_n) = P_{L0}^{k_1'} \cdots P_{L0}^{k_j'} \times R^{\ell,n}(-k_1', \ldots, -k_j'; k_1, \ldots, k_n, t_n),
\]

(8)

where

\[
C^{\ell,n}(k_1', \ldots, k_j'; k_1, \ldots, k_n, t_n) = \langle \delta_{L0}(k_1') \cdots \delta_{L0}(k_j') \delta(k_1, t_1) \cdots \delta(k_n, t_n) \rangle
\]

(9)

and

\[
R^{\ell,n}(k_1', \ldots, k_j'; k_1, \ldots, k_n, t_n) = \left( \frac{D^n \delta(k_1, t_1) \cdots \delta(k_n, t_n)}{D\delta_{L0}(k_1') \cdots D\delta_{L0}(k_j')} \right).
\]

(10)

In this paper, we denote all wave numbers associated with the initial field \( \delta_{L0} \) or soft wave numbers with a prime.

B. Consistency relations

Turning to a Lagrangian point of view, matter particles follow trajectories \( x(q, t) \) labeled by their initial (Lagrangian) coordinate \( q \). The conservation of matter means that \((1 + \delta)dx = dq\), and the Fourier-space density contrast also writes as

\[
\tilde{\delta}(k, t) = \int \frac{dx}{(2\pi)^3} e^{-ik \cdot x} \delta(x, t) = \int \frac{dq}{(2\pi)^3} e^{-ik \cdot (q + \Psi)}
\]

(11)

where we discarded a Dirac factor that does not contribute for \( k \neq 0 \). Therefore, the density-contrast response functions write as

\[
R^{\ell,n}(q_1, \ldots, q_n) = \left( \int \frac{dq_1 \cdots dq_n}{(2\pi)^3} \right) \tilde{\delta}_L(q_1') \cdots \tilde{\delta}_L(q_n')
\]

\[
\times e^{-ik_1 \cdot (q_1 + \Psi_1) \cdots -ik_n \cdot (q_n + \Psi_n)},
\]

(12)
where we introduced the displacement field $\Psi(q,t) = x(q,t) - q$.

Let us first consider the case $\ell = 1$, where Eq. (12) reads as

$$R^{1,n}_{k' \to 0} = -i \int \frac{dq_1 \ldots dq_n}{(2\pi)^{3n}} \sum_{i=1}^{n} k_i \cdot \nabla \Psi_i \frac{\delta \Psi}{\delta \phi_0(k')} \times e^{-ik_i \cdot (q_1 + \Psi_1) - \ldots - ik_n \cdot (q_n + \Psi_n)}. \quad (13)$$

We now assume that, if we look at a fixed region of size $L$ and volume $V = 1^3$, a perturbation to the initial conditions $\delta_L$ at a larger-scale linear wave number $k' \ll 1/L$ gives rise to an almost uniform displacement of the small-size region, at leading order over $(k'L)$. Thus, we write

$$k' \rightarrow 0, \quad k'L \ll 1 : \frac{\nabla \Psi(q)}{\delta \phi_0(k')} \approx \int_V \frac{dq'}{V} \frac{\nabla \Psi(q')}{\delta \phi_0(k')}.$$

(14)

where we integrate over a volume $V$ centered on $q$. Next, in the limit $k' \rightarrow 0$ we can take for instance $L \sim 1/\sqrt{k'}$ so that the size $L$ also goes to infinity (while keeping much smaller than $1/k'$). Then, we also assume that on large scales we recover the linear theory,

$$k \rightarrow 0 : \frac{\nabla \Psi(k)}{\delta \phi_0(k)} \rightarrow \Psi_L(k), \quad (15)$$

so that Eq. (14) implies

$$k' \rightarrow 0 : \frac{\nabla \Psi(q)}{\delta \phi_0(k')} \rightarrow \frac{\nabla \Psi_L(q)}{\delta \phi_0(k')}.$$

(16)

On the other hand, the conservation of matter can also be expressed through the continuity equation,

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1 + \delta)v] = 0,$$

(17)

where $\tau = \int dt/a$ is the conformal time and $v$ the peculiar velocity ($v = dx/d\tau = d\Psi/d\tau$). At linear order this gives $\partial_\tau \delta_L + \nabla \cdot v_L = 0$, whence

$$\Psi_L(k,\tau) = \frac{k}{k^2} \delta_L(k,\tau) = \frac{k}{k^2} D_+(k,\tau) \delta_L(k). \quad (18)$$

The linear growth rate of the density contrast $D_+(\tau)$ (which we normalize to unity today) does not depend on scale in the standard $\Lambda$CDM cosmology, but this is no longer true in some modified-gravity scenarios. Therefore, we include a possible $k$-dependence for completeness. Substituting into Eq. (16) we obtain

$$k' \rightarrow 0 : \frac{\nabla \Psi(q)}{\delta \phi_0(k')} \rightarrow \frac{k'}{k^2} D_+(\tau), \quad (19)$$

where we note with the overbar the low-$k$ limit of the linear growth rate, $\bar{D}_+(\tau) = D_+(0,\tau)$. We postpone to Sec. [III] a more explicit derivation of Eq. (19) than the intuitive argument [14], as well as the discussion of its validity, because we first wish to show how consistency relations for arbitrary numbers of soft wave numbers and fluid components follow from this property.

Then, using the expression (19) in Eq. (13), we obtain

$$R^{1,n}_{k' \to 0} = \langle \delta(k_1,t_1) \delta(k_n,t_n) \rangle \sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i) \times e^{-ik_i \cdot (q_1 + \Psi_1) - \ldots - ik_n \cdot (q_n + \Psi_n)}. \quad (20)$$

Thus, the prefactor generated by the functional derivative in Eq. (13) has a deterministic large-scale limit, which does not depend on the initial conditions, and the statistical average gives [compare with Eq. (11)]

$$R^{1,n}_{k' \to 0} = \langle \delta(k_1,t_1) \delta(k_n,t_n) \rangle \sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i). \quad (21)$$

Substituting into Eq. (20) we obtain

$$\langle \delta_L(k') \delta(k_1,t_1) \delta(k_n,t_n) \rangle_{k' \to 0} = -\sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i) \times P_L(k') \langle \delta(k_1,t_1) \delta(k_n,t_n) \rangle', \quad (22)$$

Here and in the following, the prime in $\langle \ldots \rangle'$ denotes that we removed the Dirac factor $\delta_D(\sum k_i)$ from the correlation functions.

The result (22) can be extended at once to $\ell \geq 2$. Indeed, each derivative $D/D\delta_L(k')$ in Eq. (12) generates a constant prefactor, given by Eq. (19), which is not affected by the next derivatives. This yields

$$R^{\ell,n}_{k' \to 0} = \langle \delta(k_1,t_1) \ldots \delta(k_n,t_n) \rangle \prod_{j=1}^{\ell} \left( \sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i) \right). \quad (23)$$

Substituting into Eq. (20) we obtain

$$\langle \delta_L(k'_1) \ldots \delta_L(k'_\ell) \delta(k_1,t_1) \ldots \delta(k_n,t_n) \rangle_{k'_\to 0} = -\prod_{j=1}^{\ell} \left( -P_L(k'_j) \sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i) \right) \times \langle \delta(k_1,t_1) \ldots \delta(k_n,t_n) \rangle', \quad (24)$$

where the soft wave numbers must satisfy the condition $k'_i + k'_j \neq 0$ for all pairs $\{i,j\}$. Since on large scales we have $\delta_L(k,t) \approx D_+(k,t) \delta_L(k)$, Eq. (24) also writes as

$$\langle \delta(k'_1,t'_1) \ldots \delta(k'_\ell,t'_\ell) \delta(k_1,t_1) \ldots \delta(k_n,t_n) \rangle_{k'_\to 0} = P_L(k'_1,t'_1) \ldots P_L(k'_\ell,t'_\ell) \langle \delta(k_1,t_1) \ldots \delta(k_n,t_n) \rangle' \times \prod_{j=1}^{\ell} \left( -\sum_{i=1}^{n} \frac{k_i \cdot k'_i}{k^2} D_+(t_i) \right), \quad (25)$$

with the condition $k'_i + k'_j \neq 0$ for $i \neq j$. Thus, Eq. (25) shows how the density correlation functions $\langle \delta_1, \ldots \delta_{\ell+n} \rangle$ factorize when $\ell$ wave numbers are within the linear
regime and become very small as compared with the fixed \( n \) other wave numbers. This generalizes to \( \ell \geq 2 \) the consistency relations obtained in previous works \cite{22}.

We can check that the formula \( 25 \) is self-consistent, that is, when we first let \( \ell \) wave numbers go to zero, and next decrease the \( \ell + 1 \) soft wave numbers, we recover the expression \( 25 \) where we directly take \( \ell + 1 \) soft wave numbers. Indeed, the results obtained from the two procedures differ by terms of the form \( (k'_i + 1 \cdot k'_j)/k'^2 \) that are negligible with respect to the terms of the form \( (k \cdot k'_j)/k'^2 \). However, the general expression \( 25 \) is not a mere consequence of the iterated use of the equation at \( \ell = 1 \). Indeed, the iterative procedure only applies when there is a strong hierarchy between the soft wave numbers, \( k'_i \ll k'_j \ll \ldots \ll k'_f \), whereas Eq. \( 26 \) is also valid when the soft wave numbers are of the same order.

The remarkable property of these relations is that they do not require the hard wave numbers \( k \) in Eq. \( 25 \) to be in the linear or perturbative regime. In particular, they still apply when these high wave numbers \( k \) are in the highly nonlinear regime governed by shell-crossing effects and affected by baryon processes such as star formation and cooling. The only requirement is the “scale-separation” property \cite{13-19}, which states that long wavelength fluctuations have a uniform impact on small-scale structures, which are merely transported by the large-scale velocity flow without deformation, at leading order in the ratio of scales. We discuss in more details the derivation and the meaning of this property in Sec. 3 below.

In the lowest order case, \( \ell = 1 \) and \( n = 2 \), this gives

\[
\lim_{k' \to 0} B(k', t'; k_1; k_2; t_1) = -P_L(k', t') P(k_1; t_1, t_2) \times \left( \frac{k_1 \cdot k'}{k'^2} \bar{D}_+(t_1) + \frac{k_2 \cdot k'}{k'^2} \bar{D}_+(t_2) \right).
\]

\[ (26) \]

where we introduced the bispectrum defined by

\[
B(k_1; t_1; k_2; t_2; k_3; t_3) = \langle \tilde{\delta}(k_1, t_1) \tilde{\delta}(k_2, t_2) \tilde{\delta}(k_3, t_3) \rangle'.
\]

\[ (27) \]

C. Multi-component case

The results obtained in the previous section also apply to cases where there are several fluids, when their large-scale linear growth rates are identical. Thus, let us consider \( N \) fluids, which may interact with each other and with gravity (which may be “modified” for instance through additional scalar fields that mediate a fifth force). Then, each fluid \( (\alpha) \) satisfies its own continuity equation,

\[
\alpha = 1, \ldots, N: \quad \frac{\partial \delta^{(\alpha)}}{\partial \tau} + \nabla \cdot [(1 + \delta^{(\alpha)}) \mathbf{v}^{(\alpha)}] = 0.
\]

\[ (28) \]

We again assume that decaying or subdominant linear modes have had time to become negligible with respect to the fastest growing mode, so that we can define the initial conditions by a single field \( \tilde{\delta}_{LO}(k) \) and in the linear regime we have

\[
\tilde{\delta}^{(\alpha)}(k, \tau) = D_+(k, \tau) \tilde{\delta}_{LO}(k).
\]

\[ (29) \]

(The \( k \) dependence arises if we consider modified-gravity scenarios.) The normalization of \( \tilde{\delta}_{LO} \) is arbitrary and it is not necessarily equal to one of the density contrasts or to the total density contrast. As in Eq. \( 15 \), each linear displacement field obeys

\[
\tilde{\Psi}^{(\alpha)}_L(k, \tau) = \bar{i} \frac{k}{k^2} \tilde{\delta}^{(\alpha)}_L(k, \tau) = \bar{i} \frac{k}{k^2} D_+(k, \tau) \tilde{\delta}_{LO}(k).
\]

\[ (30) \]

Then, we can follow the derivation presented in Sec. 3.1. The only critical point is the assumption \( 15 \), which states that a large-scale perturbation of \( \tilde{\delta}_{LO} \) leads to a uniform displacement. It is clear that this requires the large-scale growing modes \( \bar{D}_+(\tau) \) to be identical for all fluids,

\[
k \to 0: \quad D_+(k, \tau) \to \bar{D}_+(\tau),
\]

\[ (31) \]

so that a distant large-scale perturbation does not give rise to a local relative velocity between the different fluids. [An alternative is for the different fluids to be independent (i.e., they are determined by the same initial conditions but do not interact), so that we only need each fluid to respond by its own uniform displacement. In the cosmological context, because all fluids interact through gravity, we only have the possibility \( 31 \).] The large-scale common limit \( 31 \) is satisfied in most cosmological scenarios, for instance when we consider dark matter and baryons in a Λ-CDM universe \cite{10-13}. Indeed, on large scales the dominant force is gravity, which acts in the same fashion on all particle species thanks to the equivalence principle, and we recover the same linear growing mode that is driven by the gravitational instability. Effects due to different initial velocities correspond to decaying modes, which we neglect throughout this paper. Therefore, in practice the condition \( 31 \) is not a serious limitation, it would break down if we consider instance modified-gravity models that do not converge to General Relativity on large scales, see Sec. 3.1.3 below.

Then, Eq. \( 24 \) becomes

\[
\langle \tilde{\delta}(\alpha; k_1, t_1) .. \tilde{\delta}(\alpha; k_0, t_0) \rangle_{k_j, t_j} = \prod_{j=1}^{\ell} \left( -P_{L_0}(k'_j) \sum_{i=1}^{n} \frac{v_j \cdot k'_j}{k'^2} \bar{D}_+(t_i) \right)
\times \langle \tilde{\delta}(\alpha; k_1, t_1) .. \tilde{\delta}(\alpha; k_n, t_n) \rangle'.
\]

\[ (32) \]

As in Eq. \( 25 \), this may also be written as

\[
\lim_{k''_j \to 0} \prod_{j=1}^{\ell} \tilde{\delta}(\alpha; k_j', t_j') \prod_{i=1}^{n} \tilde{\delta}(\alpha; k_i, t_i)' = \prod_{j=1}^{\ell} P_{L_j}^{(\alpha)}(k_j', t_j')
\times \prod_{i=1}^{n} \bar{\delta}(\alpha; k_i, t_i)'.
\]

\[ (33) \]
Thus, our approach provides a straightforward generalization to the multi-fluid case.

The constraint (34) agrees with Ref. 30, who also find that the usual consistency relations no longer hold when there is a large-scale velocity bias and the linear growth rates of the various fluids are different. This is also clear from the fact that these consistency relations express a kinematic effect, that is, how small-scale structures are moved about by large-scale modes. Then, new terms arise when different fluids respond in different fashions to large-scale modes [30].

III. CONDITIONS OF VALIDITY

A. Perturbative check

The derivation presented in Sec. II is very general, since it only relies on Gaussian initial conditions, the conservation of matter, the linear regime on large scales, and the scale-separation property (19).

In particular, it also applies to most modified-gravity scenarios and multi-fluid systems. Then, it is interesting to follow in details how this property appears in an explicit perturbative treatment of the equations of motion, independently of the form of the interaction vertices, as long as they respect the conditions above. For our purpose, we only check the “squeezed” bispectrum relation (26) at lowest order of perturbation theory. Following the notations used in Refs. 3, 34 for the Λ-CDM cosmology and Refs. 35, 36 for modified-gravity scenarios, we write the equations of motion as

\[
\mathcal{O}(x, x') \cdot \tilde{\psi}(x') = \sum_{n=2}^{\infty} K_n^s(x; x_1, \ldots, x_n) \cdot \tilde{\psi}(x_1) \ldots \tilde{\psi}(x_n),
\]

where we introduced the coordinate \( x = (k, \eta, i) \), where \( \eta = \ln(\alpha(t)) \) is the time coordinate, and \( i \) is the discrete index of the 2N-component vector \( \tilde{\psi} \). Here, we consider \( N \) fluids, which are described by their continuity equations (28) and their Euler equations, and focusing on the growing-mode curl-free velocity component, \( \tilde{\psi} \) can be written as

\[
\tilde{\psi}(k, \eta) = \left( \tilde{\delta}^{(1)} - \tilde{\theta}^{(1)} / \hat{\eta}, \ldots, \tilde{\delta}^{(N)} - \tilde{\theta}^{(N)} / \hat{\eta} \right),
\]

where \( \tilde{\theta}^{(a)} = \nabla \cdot \mathbf{v}^{(a)} \). These (matter) fluids are subject to the usual Newtonian gravitational potential \( \Phi_N \) as well as to possible fifth-force potentials \( \Phi^{(a)} \). This includes the case of \( f(R) \) theories and scalar field models, where using the quasi-static approximation we can write the additional scalar fields as functionals of the \( N \) (matter) density fields 35, 36. Then, if the coupling constants are different or the matter fields interact in a different manner with the various scalar fields, the new potentials \( \Phi^{(a)} \) can be different for the \( N \) matter fields. The linear operator \( \mathcal{O} \) contains the first-order time derivatives \( \partial / \partial \eta \) and other linear terms. The vertices \( K_n^s \) are equal-time vertices (within this quasi-static approximation) of the form

\[
K_n^s(x; x_1, \ldots, x_n) = \delta_D(\eta_1 - \eta) \ldots \delta_D(\eta_n - \eta) \times \delta_D(k_1 + \ldots + k_n - k) \gamma_n^s(\eta_1, \ldots, \eta_n)(k_1, \ldots, k_n; \eta).
\]

In the standard Λ-CDM case, the gravitational potential is a linear functional of the density field, thanks to the Poisson equation, and the nonlinearities only come from the terms \( \nabla \cdot [(1 + \delta(x))] \) and \( (\mathbf{v} \cdot \nabla)\delta \) of the continuity and Euler equations. Then, the equations of motion are quadratic and the only nonzero vertices are

\[
\begin{align*}
\gamma_{2a-1;2a-1}^s(k_1, k_2) &= \frac{(k_1 + k_2) \cdot k_1}{2k_1^2}, \\
\gamma_{2a-1;2a-2a-1}^s(k_1, k_2) &= \frac{(k_1 + k_2) \cdot k_1}{2k_1^2}, \\
\gamma_{2a;2a}^s(k_1, k_2) &= \frac{|k_1 + k_2|^2 (k_1 \cdot k_2)}{2k_1^2 k_2^2}.
\end{align*}
\]

In the case of modified-gravity scenarios, or nonlinear fluid interactions, the potentials \( \Phi^{(a)} \) can be nonlinear functionals of the density field that contain terms of all orders and give rise to vertices \( \gamma_{2a-1;2a-1}^s(k_1, k_2) \). They correspond to source terms, which only depend on the density fields, in the Euler equations. Solving the equation of motion (34) in a perturbative manner, we write the expansion

\[
\tilde{\psi} = \sum_{n=1}^{\infty} \tilde{\psi}^{(n)}, \quad \text{with} \quad \tilde{\psi}^{(n)} \propto \delta_{L, 0}^n
\]

and the first two terms read as

\[
\begin{align*}
\tilde{\psi}^{(1)} &= \tilde{\psi}_L, \\
\tilde{\psi}^{(2)} &= R_L \cdot K_2^s \tilde{\psi}_L \tilde{\psi}_L.
\end{align*}
\]

where \( \tilde{\psi}_L \) is the linear growing mode and \( R_L \) the linear response function (i.e., the retarded Green function),

\[
\mathcal{O} \cdot \tilde{\psi}_L = 0, \quad \mathcal{O} \cdot R_L = \delta_D.
\]

The linear growing mode also satisfies

\[
\eta > \eta' : \quad \tilde{\psi}_L(k, \eta) = \sum_j R_L, j(k; \eta, \eta') \tilde{\psi}_L, j(k; \eta'),
\]

where there is no integration over time. As in Eq. (29), we also write the linear growing mode as

\[
\tilde{\psi}_i(k, \eta) = D_i(k, \eta) \delta_{L, 0}(k), \quad \mathcal{O} \cdot D = 0
\]

where \( D_i(k, \eta) \) is the linear growth rate of the \( i \)-element of the vector \( \tilde{\psi} \) and \( D = (D_1, \ldots, D_{2N}) \). The linear growth
rate and the response function may depend on wave number, depending on the form of the potentials $\Phi^0(\alpha)$.

At lowest order, the density bispectrum $B$ reads as
\[
B(k'; k_1, k_2, \eta_2) = \langle \tilde{\delta}(k') \tilde{\delta}(k_1) \tilde{\delta}(k_2, \eta_2) \rangle = \langle \tilde{\delta}_L(\tilde{\delta}_L(\tilde{\delta}(k_1) \tilde{\delta}(k_2, \eta_2))) \rangle + \text{sym.}
\]
\[
= \langle \tilde{\delta}_L(\tilde{\delta}_L(k_1)) \tilde{\delta}(k_2, \eta_2) \rangle + \text{sym.}
\]
\[
= \langle \tilde{\delta}_L(\tilde{\delta}_L(\tilde{\delta}_L(\tilde{\psi}_L \tilde{\psi}_L))) \rangle + \text{sym.}
\]
where “sym.” stands for the symmetric term by $1 \leftrightarrow 2$, and we use simplified notations. Taking the Gaussian average gives
\[
B = 2P_L(k')P_L(k_2)\langle \tilde{\delta}_L(\tilde{\delta}_L(k_1)) \tilde{\delta}(k_2, \eta_2) \rangle D_2(k_2, \eta_2)
\times R_{L,1,1}(k_1, \eta_1) \tilde{\gamma}^s_{\alpha, \alpha'}(-k', -k_2; \eta_1) + \text{sym.}
\]
which is the common large-scale density growing mode. Then, the bispectrum in the large-scale limit $k' \to 0$, we are dominated by the vertices $\gamma_{2\alpha-1,2\alpha-1}^s$ and $\gamma_{2\alpha,2\alpha}^s$ of Eqs. (35) and (36), with $\gamma_{2\alpha-1,2\alpha-1}^s \simeq \gamma_{2\alpha,2\alpha}^s \simeq (k_2 \cdot k')/(2k^2)$. [We discuss the non-standard vertices below Eq. (51).] This yields
\[
B_0 = \frac{k_2 \cdot k'}{k'^{2}} P_L(k')P_L(k_2)D_2(k_2, \eta_2) \sum_{\alpha} D_{2\alpha}(\eta_1) D_{2\alpha}(\eta_2)
\times \left[ R_{L,1,1}(k_1, \eta_1) \tilde{\gamma}^s_{\alpha, \alpha'}(-k', -k_2; \eta_1) \right] + \text{sym.}
\]
Using the property (31), we can factor the term $D_{2\alpha}(\eta_1) \to \tilde{D}_2(\eta_1)$ out of the sum. Here $\tilde{D}_2$ is the common large-scale velocity growing mode and $\tilde{D}_1 = \tilde{D}_+ \tilde{D}_-$ is the common large-scale density growing mode. Then, the sum can be resummed at once from Eq. (44), using $k_2 \to k_1$ in the limit $k' \to 0$. This gives
\[
B_0 = \frac{k_2 \cdot k'}{k'^{2}} P_L(k')P_L(k_2)D_{2\alpha}(k_2, \eta_2) \tilde{D}_2(\eta_1) D_{2\alpha}(k_1, \eta_1) + \text{sym.}
\]
Next, we can integrate over the time $\eta_1$ [because of causality, in the equations above there was an implicit Heaviside term $\Theta(\eta_1 < \eta_1)$], which arises from Eq. (44), using the continuity equation which implies that $D_2(\eta) = dD_1(\eta)/d\eta$. This yields
\[
B_0 = \frac{k_2 \cdot k'}{k'^{2}} P_L(k')P_L(k_2)D_{\alpha, \alpha}(k_2, \eta_2) D_1(\eta_1) + \text{sym.},
\]
and using $k_2 \to -k_1$,
\[
B_0 = -P_L(k')P_L(k_1, \eta_1, \eta_2) \left[ \frac{k_1 \cdot k'}{k'^2} \tilde{D}_+ + \text{sym.} \right] \tilde{\gamma}^s_{\alpha, \alpha'}(k_1, \eta_1, \eta_2) \tilde{\gamma}^s_{\alpha, \alpha'}(-k', -k_2; \eta_1) + \text{sym.}
\]
This agrees with Eq. (42), and with Eq. (43) when we change variable from $\delta_L(k')$ to $\tilde{\delta}_L(k', \eta')$. This explicit derivation provides a general check of Eq. (20) at lowest order of perturbation theory. It explicitly shows that this result only relies on two ingredients:

(a) the linear growth rates of the different fluids coincide in the large scale limit, as in (31).

(b) the new vertices $\gamma^s$ associated with nonlinear interactions, that may arise for instance from modified-gravity scenarios (or models of baryonic physics) must be subdominant with respect to the standard vertices in the limit $k' \to 0$ in Eq. (17).

The point (a) was already noticed in Sec. II C and follows from the requirement (19). The point (b) is satisfied in usual $f(R)$ theories and scalar-field models, including nonlinear screening mechanism as for dilaton and symmetron models, as can be seen from the expressions of the vertices $\gamma^s_{2,1,1}$ given in (33). This remains valid at the general level, for higher-order vertices and up to the highly nonlinear regime, and for $n$-point correlation functions, as discussed in Sec. III C below.

B. Validity requirements

1. Separation of scales and kinematic response

The derivation presented in Sec. II B only relies on the property (19). If we assume this large-scale limit is correct, then the consistency relations (21) directly follow from the general property [8] and the expression (11), which merely expresses the conservation of matter. Then, since we have already assumed in Eq. (19) that we recover linear theory on large scales, Eq. (21) implies Eq. (25).

Using Eq. (11), the critical property (19) can also be written in terms of the nonlinear density contrast as
\[
k' \to 0 : \frac{\mathcal{D}\tilde{\delta}(k, t)}{\mathcal{D}\tilde{\delta}_L(k')} = \tilde{D}_+(t) \frac{k \cdot k'}{k'^2} \tilde{\delta}(k, t).
\]

Then, we do not need to introduce the displacement field and by substituting Eq. (22) into Eq. (11) we directly obtain Eqs. (23) and (24). This is more general and consistency relations such as Eq. (21) hold for any system, beyond the cosmological context, where the derivative (22) takes the form of a simple multiplicative factor in the low-k limit. An obvious example is the case where the field $\tilde{\delta}(k)$, which is no longer interpreted as a density field, is a functional of the form
\[
\tilde{\delta}(k) = \exp \left[ \sum_{n=1}^{\infty} \prod_{i=1}^{n} d\mathbf{k}_i \delta_D(\mathbf{k}_1 + \ldots + \mathbf{k}_n - \mathbf{k}) \right]
\times \mathcal{E}^s_\alpha(\mathbf{k}_1, \ldots, \mathbf{k}_n) \tilde{\delta}_L(\mathbf{k}_1) \ldots \tilde{\delta}_L(\mathbf{k}_n)
\]
where the symmetric kernels $\mathcal{E}^s_\alpha$ satisfy $\mathcal{E}^s_\alpha(0, \mathbf{k}_2, \ldots, \mathbf{k}_n) = 0$ for $n \geq 2$.

In the cosmological case, the property (22) means that if we perturb the initial condition $\tilde{\delta}_L$ by a small perturbation $\Delta \tilde{\delta}_L$ that only modifies large-scale linear modes (i.e., $\Delta \tilde{\delta}_L(k') = 0$ for $k' > k_c$ where the cutoff $k_c$ is far in
the linear regime and much below the other wave numbers of interest), the nonlinear density contrast transforms, at linear order over $\Delta \delta_{L0}$, as
\[
\delta_{L0} \rightarrow \delta_{L0} + \Delta \delta_{L0}
\]
and
\[
\tilde{\delta}(\mathbf{k}) \rightarrow \tilde{\delta}(\mathbf{k}) + \int \mathrm{d} k' \Delta \tilde{\delta}_{L0}(k') \tilde{D}_+(t) \frac{k' \cdot k'}{k'^2} \tilde{\delta}(k) = \delta(\mathbf{k}) e^{k \Delta x},
\] (54)
with
\[
\Delta x = \tilde{D}_+(t) \int \mathrm{d} k' \Delta \tilde{\delta}_{L0}(k') \left(\frac{k'}{k'^2}\right) x.
\] (55)

The last line in Eq. (54) simply means that $\exp(x) = 1 + x$ at linear order. Then, in configuration space this yields
\[
\delta(x, t) \rightarrow \delta(x + \Delta x, t).
\] (56)

This corresponds to a uniform translation, as was clear from Eq. (19), where the displacement field $\Psi(\mathbf{q})$ is modified by a uniform ($\mathbf{q}$-independent) amount.

Thus, the critical assumption that gives rise to the consistency relations (25) is that, at leading order, a very large-scale perturbation of the initial conditions only leads to an almost uniform translation of small structures. This is an hypothesis of “scale separation”: large scales do not strongly modify small-scale structures and only move them around. In fact, as noticed above, the hypothesis can be made more general as the leading order effect does not need to be a uniform shift, it could also be any uniform multiplicative factor. If this assumption is satisfied, then the details of the small scale structures are not important and the latter can be deep in the nonlinear regime, which is why the consistency relations (21) remain valid when the smaller-scale wavenumbers $k_i$ are in the nonlinear regime.

2. Derivation of the kinematic effect

In the standard cosmological case, the reason why the property (52), or equivalently (19), is valid can be seen as follows (see also [26, 27]). By definition of the functional derivative, an infinitesimal change of the initial condition $\Delta \delta_{L0}$ leads to a change of the nonlinear displacement field given by
\[
\Delta \Psi(\mathbf{q}) = \int \mathrm{d} k' \frac{\partial \Psi(\mathbf{q})}{\partial \delta_{L0}(k')} \Delta \tilde{\delta}_{L0}(k')
\] (57)

Therefore, to obtain the low-$k'$ limit of the functional derivative we can look at a perturbation $\Delta \delta_{L0}(k')$ that is restricted to $k' < k_c$ with $k'_c \rightarrow 0$. For instance, we can choose a Gaussian perturbation of size $R \rightarrow \infty$ centered on a point $\mathbf{q}_c$ at a large distance from point $\mathbf{q}$ ($(\mathbf{q}_c - \mathbf{q}) \gg R$). This limit also means that the distance $|\mathbf{q}_c - \mathbf{q}|$ is much greater than the scale associated with the transition to the linear regime, so that this localized perturbation always remains far away. Because we are perturbing the linear growing mode, by definition of the field $\delta_{L0}$, the perturbation $\Delta \delta_{L0}$ does not correspond to just adding a mass $\Delta M$ around $\mathbf{q}_c$. It also means that we are perturbing the initial velocity field $\mathbf{v}_{L0}$ by the precise amount that corresponds to the relation between velocity and density in the growing mode. In other words, we look at the impact of the change of linear growing mode
\[
\delta_L(\mathbf{q}, \tau) \rightarrow \delta_L = \tilde{D}_+ \Delta \delta_{L0},
\] (58)
\[
\mathbf{v}_L(\mathbf{q}, \tau) \rightarrow \mathbf{v}_L = \mathbf{v}_L - \frac{d \tilde{D}_+}{d \tau} \nabla^{-1} \cdot \Delta \delta_{L0}
\] (59)
(because $R \rightarrow \infty$ it is the large-scale limit $D_+(k' = 0, \tau)$ that appears). At the linear level, this means that the small-scale region around $\mathbf{q}$ is falling towards the distant large-scale mass $\Delta M$ centered on $\mathbf{q}_c$ as in the growing mode regime. In particular, if the fields are everywhere linear, we have at once the relation (19), which becomes exact, as well as the property (11). Thus, what we must show is that even when the small-scale region around $\mathbf{q}$ is nonlinear, the impact of the distant mass $\Delta M$ is still to attract the small region with the same acceleration as in the linear regime, and with negligible tidal effects. This is most easily seen from the equation of motion of the trajectories $\mathbf{x}(\mathbf{q}, \tau)$ of the particles,
\[
\frac{\partial^2 \mathbf{x}}{\partial \tau^2} + \mathcal{H} \frac{\partial \mathbf{x}}{\partial \tau} = -\nabla \Phi = \mathbf{F},
\] (60)
where $\mathcal{H} = d \ln a/d \tau$ is the conformal expansion rate and $\Phi$ and $\mathbf{F}$ are the Newtonian gravitational potential and force. When we add the perturbation $\Delta M$, the trajectories are modified as $\mathbf{x} \rightarrow \hat{\mathbf{x}}$ and the Newtonian force as $\mathbf{F} \rightarrow \hat{\mathbf{F}}$, and they follow Eq. (60) with a hat on each field. In a fashion similar to the method used for inflation consistency relations [26], we can look for a simple solution of this perturbed equation of motion built from the unperturbed one $\mathbf{x}(\mathbf{q}, \tau)$ by a simple transformation. In our case, we simply need to consider new trajectories $\mathbf{x}'$ defined by
\[
\mathbf{x}'(\mathbf{q}, \tau) \equiv \mathbf{x}(\mathbf{q}, \tau) + \tilde{D}_+ (\tau) \Delta \Psi_{L0}(\mathbf{q}).
\] (61)
where $\Delta \Psi_{L0} = -\nabla^{-1} \cdot \Delta \delta_{L0}$ is the perturbation to the linear displacement. Then, since the unperturbed trajectories obey Eq. (60), these auxiliary trajectories satisfy
\[
\frac{\partial^2 \mathbf{x}'}{\partial \tau^2} + \mathcal{H} \frac{\partial \mathbf{x}'}{\partial \tau} = \mathbf{F}'(\mathbf{x}', \tau) + \Delta \mathbf{F}_L(\mathbf{q}, \tau).
\] (62)
In the second line, we used the relation $\mathbf{F}'(\mathbf{x}') = \mathbf{F}(\mathbf{x})$ because the uniform translation (11) only gives rise to the same translation of the Newtonian force, since $\mathbf{F} \propto \nabla^{-1} \delta$. The last term follows from Eq. (60), which implies at linear order that the displacement field and the force
we consider. Moreover, since we consider an infinitesimal perturbation, whence $\Delta F$ is to induce the uniform translation (61), which is set by the Poisson equation with the linear density $\Phi_L$. Thus, at constant force field, to which the system reacts by uniform velocity $\delta \Phi_L$, the size of the distant perturbation $\Delta M$ is the condition to have a unique coordinate transformation (51), where $\delta \Phi_L$ and $\Phi_L$ is Newton’s potential, the requirement that the distant large-scale structure leads to the same displacement for all particles means that the inertial and gravitational masses are equal. Thus, in the standard framework, the consistency relations follow from the equivalence principle, in agreement with the analysis in [24]. In particular, for the multi-fluid case discussed in Sec. II C, we explicitly recover the condition (31) of identical large-scale linear growth rates. Indeed, this is the condition to have a unique coordinate transformation (61).

On the other hand, we check that only the large scale limits $D^{(a)}(\tau)$ need to coincide and that the growth rates $D^{(a)}(k, \tau)$ can be different for $k \neq 0$. This also means that the equivalence principle can be violated on small scales, associated with the hard wave numbers $k_f$ in Eq. (25), and the consistency relations still be valid. It is sufficient that the equivalence principle applies in the large-scale limit, that is, for $k' \to 0$ or $R \to \infty$, where $R$ is the size of the distant perturbation $\Delta M$. An example would be modified-gravity scenarios associated with a new scalar field that mediates a fifth-force. At the linear level, this gives rise to modified Newton’s constants $G_N \to [1 + \epsilon^{(a)}(k, \tau)]G_N$ in the equations of motion of the matter particles. If different fluids have different couplings to the scalar field, the factors $\epsilon^{(a)}$ can be different.

3. Conditions of validity and equivalence principle

The derivation above might seem a bit superfluous, as the result may look obvious. However, it helps to explicitly show which ingredients are required to obtain the consistency relations. In particular, it is clear that the argument does not depend on the structure of the small nonlinear object at $q$. It can be in the highly nonlinear regime where complex baryon astrophysical processes (e.g., star formation) are taking place. Thus, the consistency relations (21)-(25) hold even when the hard wave numbers $k_i$ are in the highly nonlinear regime and we take into account shell crossing and astrophysical processes (star formation, outflows,..).

Moreover, we did not use the Poisson equation and we did not need to specify the potential $\Phi$. Therefore, the consistency relations remain valid when we include (speculative) other long-range forces than the standard Newtonian (more precisely, General Relativity) gravity. The only requirement is that a weak form of the equivalence principle remains valid on large scales.

In the standard case, going back to Newton’s equation, $m_q \ddot{x} = -mg \nabla \Phi_N$, where $m_q$ and $mg$ are the inertial and gravitational masses and $\Phi_N$ is Newton’s potential, the requirement that the distant large-scale structure leads to the same displacement for all particles means that the inertial and gravitational masses are equal. Thus, in the standard framework, the consistency relations follow from the equivalence principle, in agreement with the analysis in [24]. In particular, for the multi-fluid case discussed in Sec. II C, we explicitly recover the condition (31) of identical large-scale linear growth rates. Indeed, this is the condition to have a unique coordinate transformation (61).

On the other hand, we check that only the large scale limits $D^{(a)}(\tau)$ need to coincide and that the growth rates $D^{(a)}(k, \tau)$ can be different for $k \neq 0$. This also means that the equivalence principle can be violated on small scales, associated with the hard wave numbers $k_i$ in Eq. (25), and the consistency relations still be valid. It is sufficient that the equivalence principle applies in the large-scale limit, that is, for $k' \to 0$ or $R \to \infty$, where $R$ is the size of the distant perturbation $\Delta M$. An example would be modified-gravity scenarios associated with a new scalar field that mediates a fifth-force. At the linear level, this gives rise to modified Newton’s constants $G_N \to [1 + \epsilon^{(a)}(k, \tau)]G_N$ in the equations of motion of the matter particles. If different fluids have different couplings to the scalar field, the factors $\epsilon^{(a)}$ can be different.

Of course, at higher orders over $k'$, the small-scale density field is sensitive to the large-scale overdensity, as in the peak-background split argument used to analyze halo bias [39], and its structure is modified. If the kinematic effect is not present, as in non-cosmological systems such as [59] where there is no velocity field and $\delta$ is some functional of $\tilde{\delta}_{L0}$, the effect of a large-scale perturbation is usually not uniform and we no longer have consistency relations of the form (21), where the correlation between one soft mode $k'$ and $n$ hard modes $k_i$ factors as a product of one soft-mode power spectrum and a hard mode $n$-correlation. A simple example is the functional $\delta(x) = \delta_{L0}(x) + \delta_{L0}(x)^2$. 


However, if they coincide at low k [typical models have $\epsilon(k) \propto k^2$, which vanishes at $k \to 0$, as discussed in Sec. III C below and in Eq. (70)], the consistency relations remain valid although the different fluids behave in a different fashion on small scales. We discuss in more details these scenario in Sec. III C below.

It is interesting to note that the consistency relations remain valid at an even more general level, where the equivalence principle can be violated on all scales. Thus, let us consider the following toy model, made of different particle species ($\alpha$) that obey the equations of motion

$$\frac{\partial^2 x^{(\alpha)}}{\partial \tau^2} + \left( H + \beta^{(\alpha)}(\tau) \right) \frac{\partial x^{(\alpha)}}{\partial \tau} = -\epsilon^{(\alpha)}(\tau) \nabla x \Phi_N, \quad (65)$$

where $\Phi_N = 4\pi G_N a^2 \nabla^2 - \sum_\alpha \delta \rho^{(\alpha)}$ is Newton’s potential. As compared with the standard case (60), we have added a friction term $\beta^{(\alpha)}$ and an effective Newton’s constant $\epsilon^{(\alpha)} G_N$ that depend on the particle species (and on time). (We could imagine that there is some friction with respect to a non-interacting component that exactly follows the Hubble flow and gravity is modified, but this example is not meant to be realistic.) This model clearly violates the equivalence principle on all scales when the coefficients $\epsilon^{(\alpha)}$ are different.

However, following the procedure described in Sec. III B 2 we can still build auxiliary trajectories as in Eq. (61), with a common displacement $\bar{D}_+(\tau) \Delta \Psi_{L0}$ so that all particles move by the same amount and the potential $\Phi_N$ is only displaced without deformation. Then, the right hand side of Eq. (62) contains a term $\left[ d^2 \bar{D}_+/d\tau^2 + (H + \beta^{(\alpha)}) d\bar{D}_+/d\tau \right] \Delta \Psi_{L0}$ that is again identical to $\Delta F^{(\alpha)}(\mathbf{q}, \tau)$ if all linear growing modes $\bar{D}_+^{(\alpha)}$ are equal to $\bar{D}_+$. Using the Poisson and continuity equations and the equation of motion (65) in its linear form, the different linear growing modes are identical if $\bar{D}_+$ is simultaneously the solution of

$$\frac{d^2 \bar{D}_+}{d\tau^2} + \left( H + \beta^{(\alpha)} \right) \frac{d\bar{D}_+}{d\tau} = \epsilon^{(\alpha)} \frac{3}{2} \nabla^2 \sum_\alpha \Omega^{(\alpha)} \bar{D}_+. \quad (66)$$

Choosing for instance for $\bar{D}_+$ the usual solution associated with the coefficients $\beta^{(\alpha)} = 0$, $\epsilon^{(\alpha)} = 1$, we can see that for any set of functions $\epsilon^{(\alpha)}(\tau)$ we can find functions $\beta^{(\alpha)}(\tau)$ so that Eq. (66) is satisfied. For such a choice, we obtain a toy-model that violates the equivalence principle on all scales, but where the consistency relations (24) - (26) remain valid.

The reason for this is that the demonstration of Sec. III B 2 involves a dynamic argument and not a static consideration. More precisely, as emphasized in Sec. III B 2 to derive the consistency relations we need the response of a small-scale objects to a large-scale perturbation of the linear growing mode (i.e., the initial conditions). This is not the same thing as adding a large mass $\Delta M$ far away, because we must modify the density and velocity fields in a coupled manner (as dictated by the constraint that we remain within the space of initial conditions spanned by linear growing modes). In the toy model (65), this means that when we add the distant mass $\Delta M$, which gives rise to different “gravitational” forces $\epsilon^{(\alpha)} \nabla \Phi_N$ on the different particle species, we are also adding different velocities to these particles. Then, for a suitable choice of the functions $\beta^{(\alpha)}$ and $\epsilon^{(\alpha)}$ the trajectories coincide at the linear level. This is sufficient to obtain the consistency relations.

We do not mean here that the toy model (65) is relevant for cosmology. However, it clarifies that the ingredients that underlie the consistency relations and it shows that they do not require the equivalence principle in a strict sense. They only require a very weak form of separation of scales: a large-scale perturbation to the initial growing mode must have the same “multiplicative” effect on all particles in a small-scale structure, independently of the particle species and features of the nonlinear structure. In our case, the Fourier-space “multiplicative” effect (62) is a translation (70) in configuration space, but in more general systems there could also be a uniform amplification. Moreover, as emphasized above, the restriction to the linear growing mode is an important point because it means that a generic large-scale perturbation does not need to satisfy this separation of scales. We only require this to be the case for the subset of linear growing mode perturbations.

C. Modified-gravity scenarios

Let us briefly consider the case of modified-gravity models, such as $f(R)$ theories or scalar field models. To simplify the analysis we focus on a single matter fluid (we have already seen the general conditions for multi-fluid cases above), which feels the usual Newtonian gravitational potential $\Phi_N$ and an additional fifth-force potential $\Phi_A$. For scalar-tensor theories, which involve a new field $\phi$ that couples to matter particles through a conformal rescaling of the Jordan-frame metric $36, 39 - 43$, $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$, this potential reads as

$$\Phi_A = c^2 \ln A(\phi), \quad (67)$$

while the scalar field obeys the Klein-Gordon equation

$$\frac{c^2}{a^2} \nabla^2 \phi = \frac{dV}{d\phi} + \rho \frac{dA}{d\phi}, \quad (68)$$

where $V(\phi)$ is the scalar-field potential. Here we used the quasi-static approximation (as well as the non-relativistic limit). In the weak field limit, we can linearize Eq. (68) around the background, $\phi = \bar{\phi} + \delta \phi$, and we obtain

weak field: \[ \tilde{\Phi}_A \propto \delta \bar{\phi} \propto \frac{\delta \bar{\phi}}{k^2 + a^2 m^2}, \quad (69) \]

where $c^2 m^2 = d^2 V/d\phi^2$ and we consider models where $A \simeq 1 + \beta \phi/M_{Pl}$ with $\beta \phi/M_{Pl} \ll 1$. Thus, the total
potentials \( \Phi = \Phi_N + \Phi_A \) is amplified with respect to the Newtonian potential by a factor \( 1 + \epsilon \) with

\[
\epsilon(k,t) \propto \frac{k^2}{k^2 + a^2 m^2}.
\]  

(70)

In very dense objects, a screening mechanism takes place \[40\], due to the nonlinearities of the Klein-Gordon equation \[68\]. As we take \( \rho \to \infty \), at fixed scale \( R \), the left hand side becomes negligible with respect to each term in the right hand side and the field \( \varphi \) in the objects settles down to the solution of \( dV/d\varphi + \rho A/d\varphi = 0 \) [e.g., for \( V = V_0 e^{-\varphi/M_p} \) we have \( \varphi \sim \ln(\beta \rho/V_0) \)]:

\[
\text{strong field: } \varphi \simeq \varphi_c \text{ with } dV/d\varphi_c(\varphi_c) + \frac{dA}{d\varphi}(\varphi_c) = 0.
\]  

(71)

Then, gradients of the scalar field \( \varphi \) and of the potential \( \Phi_A \) are negligible and the fifth force vanishes, so that we recover the usual Newtonian gravity.

As noticed in \[44\], the screening mechanism also means that a very dense object, which is screened, and a moderate density object, which is in the weak-field regime \[69\], do not feel the same fifth-force from a given distant object. Indeed, let us consider a small isolated dense object at a position \( x' \). Its density profile gives rise to a specific \( \varphi \)-profile set by Eq. \[69\] and at equilibrium the gravitational potential and the fifth force are balanced by internal pressure (gas) or velocity dispersion (dark matter). Next, let us add a second object of mass \( M \) at a distant position \( x \). The Newtonian potential due to this distant mass is proportional to the mass \( M \) and does not depend on the structure of the small object at \( x \), because the inertial and gravitational masses are identical, and the small object falls towards the mass \( M \) at the same speed independently of its structure, in agreement with the equivalence principle.

However, this property is not satisfied by the fifth force. Indeed, the fifth force due to the distant mass \( M \) acts on the small object at \( x \) through the local gradients of the potential \( \Phi_A \) at \( x \), whence, through the local gradients of the scalar field \( \varphi \). In the weak-field regime \[69\], the fifth force is proportional to the gravitational force, with a factor \( \epsilon \) that depends on the distance to the mass \( M \) \((k \sim 1/|x' - x|)\), and does not depend on the small object structure. This is due to the linear approximation: solutions to the Klein-Gordon equation and to the potential simply add up. In contrast, in the strong field regime \[74\], the field \( \varphi \) is pinned down to the solution \( \varphi_c \), with a very high curvature of the effective potential \( V + \rho A \), and adding a distant mass only gives rise to a small deviation of the local value of \( \varphi \). Then, the fifth force due to the distant object is negligible. Therefore, moderate-density and high-density objects do not respond in the same way to the distant mass \( M \), which corresponds to a violation of the equivalence principle \[44\].

This behavior may seem to rule out the ingredient \[19\], that is, the response of the local object to a large-scale perturbation should not be universal as in Eq. \[19\] (the response of the local displacement field is expected to depend on the nonlinear structure of the small object) and the consistency relation \[24\] should no longer be valid. This also seems to contradict the explicit check of the bispectrum relation presented in Sec. \[III A\] at lowest order of perturbation theory, which we have seen to remain valid for these modified gravity models (through the explicit expression of the vertices \( \gamma_{2,1} \) generated by these models).

The solution to this apparent paradox can be seen in Eq. \[70\]. To derive the consistency relations we need the response of a small possibly nonlinear object to a large-scale perturbation of the initial condition, of size \( R \to \infty \). Then, Eq. \[70\] shows that for \( k \sim 1/R \to 0 \), in the weak-field regime for the small object, the fifth force vanishes as \( k^2 \) as compared with the Newtonian gravity. This is because Newtonian gravity is a long range force, with \( \Phi_N \sim \delta/k^2 \), whereas the fifth force is a relatively “short-range” force mediated by the scalar field \( \varphi \), with a characteristic length \( \sim 1/m \) (realistic models take \( 1/m \lesssim 1\text{Mpc}/h \) because of observational constraints from the Solar System). This remains true when the small object is in the strong field regime (where the screening mechanism makes it insensitive to external fluctuations anyway) as the short-range nature of the fifth force does not depend on the structure of the objects. Therefore, the fifth-force is subdominant with respect to Newtonian gravity at leading order in \( 1/k \) and it does not contribute to the response \[19\] of the small object to a large-scale distant mass.

Going back to the explicit perturbative check presented in Sec. \[III A\], this feature explicitly appears as we go from Eq. \[47\] to Eq. \[48\], where we assume that the new nonlinear vertices generated by the fifth-force potential are subdominant with respect to the usual vertices \( \gamma_{1,1}^2 \) and \( \gamma_{2,2} \). As seen from the explicit expressions given by Eqs. \[78\)-\[79\] in Ref. \[36\], this is true because the vertices are rational functions with denominators of the form \( 1/(k^2 + a^2 m^2) \) that remain finite as \( k \to 0 \). This is the same denominator as in Eq. \[70\] and again it is due to the small-range character of the fifth force. The same result holds for the \( f(R) \) theories, for the same short-range reason, as can be checked in the explicit expression of the low-order vertices given by Eqs. \[75\]-\[76\] in Ref. \[36\]. Therefore, the consistency relation \[24\] remains valid in these modified-gravity scenarios.

If the mass \( m \) vanishes and the new vertices \( \gamma^+ \) show divergences of the form \( 1/k \) or stronger, then the fifth force is no longer subdominant and the consistency relations no longer apply. However, such a behavior does not correspond to the modified-gravity models described above and it would give a significant departure from General Relativity on very large scales. Moreover, because Solar Systems constraints leads to \( 1/m \lesssim 1\text{Mpc}/h \), we can see that on large scales where the consistency relations are relevant (for the soft mode limit \( k' \to 0 \) to apply), we are indeed in the regime \( k'^2 \ll a^2 m^2 \) and the fifth force is subdominant.
Therefore, in contrast with what could have been expected, usual modified-gravity scenarios do not lead to a violation of the consistency relations [24], because the latter do not rely on the equivalence principle in a strict sense. They only require that long-range forces satisfy the equivalence principle in the large-scale limit, or that the equations of motion possess some precise (unrealistic) balance as in the toy model [53], to ensure that the departure from the equivalence principle up to the largest scales cancels out for fluctuations that take the form of the linear growing mode.

D. Galilean invariance

1. Inequivalence with Galilean invariance

Because the effect of a long-wavelength perturbation is to move the small-scale structures as in Eq.(56), the net effect on equal-time density correlations vanishes, as can be checked in the consistency relation [24], using \[ \sum_i k_i = \sum_j k'_j \to 0. \] This would not have been the case if the impact of a long-wavelength perturbation had been a uniform amplification, as in the toy model [53].

The same cancellation for equal-time statistics appears in perturbation theory computations of the density correlations [45, 46]. This cancels the infrared divergent contributions from different diagrams that appear if the initial power spectrum has significant power on large scales (i.e., the variance of the initial velocity is infinite). In this context, this property is somewhat loosely referred to as “Galilean invariance”, by which it is meant that small scales are only transported without deformation by long-wavelength modes. This terminology refers to the usual case (in the laboratory or in a static Universe) where the Euler equation reads as \( \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \Phi \), which is invariant under a uniform velocity change \( \mathbf{v} \to \mathbf{v} + \mathbf{v}_0 \).

Thus, in physical coordinates \((\mathbf{r}, \mathbf{u}, t)\), the equations of motion (in the single-stream approximation) write as

\[
\begin{align*}
\partial_t \rho + \nabla_r (\rho \mathbf{u}) &= 0, \\
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla \Phi_{N}^{\text{tot}}, \\
\nabla^2 \Phi_{N}^{\text{tot}} &= 4\pi G_N \rho.
\end{align*}
\]

This system is invariant through the “extended Galilean transformation” (EGT), where we look at the system in a new frame, denoted with a prime, that is translated from the first one by a uniform time-dependent vector \( \mathbf{n}(t) \),

\[
\mathbf{r}' = \mathbf{r} - \mathbf{n}(t), \quad \mathbf{u}' = \mathbf{u} - \dot{\mathbf{n}}(t), \quad \rho'(r') = \rho(r).
\]

Then, we can check that the new fields \( \{ \rho', \mathbf{u}', \Phi_{N}^{\text{tot}'} \} \) satisfy the same equations of motion \((72)-(74)\), provided the gravitational field transforms as

\[
\Phi_{N}^{\text{tot}'}(\mathbf{r}', t) = \Phi_{N}^{\text{tot}}(\mathbf{r}, t) + \dot{\mathbf{n}} \cdot \mathbf{r}'.
\]

The usual Galilean transformation (GT) is the case \( \mathbf{n}(t) = \mathbf{x}_0 + \mathbf{v}_0 t \), with a constant velocity \( \mathbf{v}_0 \), so that \( \dot{\mathbf{n}} = 0 \) and \( \Phi_{N}^{\text{tot}} \) is transported without new contribution, as the density field. These symmetries are due to the Galilean invariance of the Lagrangian derivative \( \partial_t + \mathbf{u} \cdot \nabla_r \). Changing variable to comoving coordinates, with \( \mathbf{x} = \mathbf{r}/a(t), \mathbf{v} = \mathbf{u} - H \mathbf{r}, \rho = \tilde{\rho}(1 + \delta), \tau = \int dt/a, \) where \( H = \dot{a}/a \) is the Hubble expansion rate and \( \tilde{\rho} = -3H \rho \), we obtain

\[
\begin{align*}
\partial_r \delta + \nabla \cdot [(1 + \delta) \mathbf{v}] &= 0, \\
\partial_r \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + H \mathbf{v} &= -\nabla \Phi_N,
\end{align*}
\]

where \( H = a'/a \) is the conformal expansion rate (we denote time derivatives with respect to \( \tau \) with a prime), and

\[
\Phi_N = \Phi_{N}^{\text{tot}} + \tilde{\rho} \frac{x^2}{2} = \frac{3\Omega_m}{2} H^2 \nabla^{-2} \delta.
\]

[It is only for the Einstein-de Sitter Universe \( \Omega_m = 1 \) that we can go from the first to second (exact) equality in Eq.(79). This is a limitation of the Newtonian approach that starts from Eqs.(72)-(74) and the last expression requires the use of General Relativity for a rigorous derivation.] Then, the invariance through the EGT of the original system \((72)-(74)\) is expressed in the new system \((77)-(79)\) as

\[
\begin{align*}
\mathbf{x}' &= \mathbf{x} - \mathbf{n}(\tau), \quad \mathbf{v}' = \mathbf{v} - \mathbf{n}'(\tau), \quad \delta' = \delta,
\end{align*}
\]

\[
\Phi_N' = \Phi_N + (\mathbf{n}''' + H \mathbf{n}') \cdot \mathbf{x}',
\]

where we made the redefinition \( \mathbf{n} \to \mathbf{n}_0 \). Because of the term \( H \mathbf{v} \) in the comoving Euler equation \((78)\), the standard GT in physical coordinates leads to an EGT in the comoving system, so that we do not need to single out the case \( \mathbf{n}'' = 0 \). In any case, this invariance shows that uniform time-dependent displacements of the system generate new solutions (with an appropriate change of the gravitational potential by a linear term over \( x' \)).

As pointed out by Ref.[26], the transformation \((80)-(83)\) with the specific case \( \mathbf{n}(\tau) = \mathbf{n}_0 \tau \) is not the reason for the consistency relations \([24]-[25]\). Indeed, as these authors note, the transformation \((80)-(83)\) with \( \mathbf{n}'' = 0 \) does not have the form of a perturbation to the growing mode. The perturbation that is relevant implies both a change of the velocity field and of the gravitational potential, with a time-dependent uniform displacement that is proportional to the linear growing mode \( D_\tau(t) \), see Eqs.(58)-(63). In other words, the consistency relations rely on the invariance of the small-scale structure (at leading order over \( k' \)) as it falls towards a distant large-scale mass \( \Delta M \), with its displacement and velocity coupled as in the linear growing mode, rather than a pure constant velocity boost. In particular, the standard Galilean Invariance (GI) in this case corresponds to a displacement such that \( \mathbf{n}'' + H \mathbf{n}' = 0 \) (which recovers the standard constant-velocity transformation when \( H = 0 \)), where we move the system without modifying the density field at any place, whereas the consistency relations
involve a coupled modification of the density and velocity fields.

On the other hand, the fact that the standard GI comes along with the extended Galilean Invariance (EGI), as \( \mathbf{n}(\tau) \) can be an arbitrary function of time in the transformation \([80]-[81]\), blurs somewhat the matter because it usually means that when we have GI with \( \mathbf{n}' = 0 \) we also have EGI with \( \mathbf{n}(\tau) \propto \dot{D}_+ (\tau) \), and the consistency relations are also usually valid. However, this is rather misleading and the EGI is actually decoupled from the validity of the consistency relations.

A simple counter-example, where GI is violated but the consistency relations are still valid, is provided by the toy model \([85]\). Through the transformation \([80]-[81]\), we find that the equation of motion of the fluid (\( a \)) keeps the same form if the gravitational potential transforms as

\[
\Phi'_{\text{N}} = \Phi_{\text{N}} + \frac{1}{\epsilon (a)} \left[ \mathbf{n}'' + (\mathcal{H} + \beta (a)) \mathbf{n}' \right] \cdot \mathbf{x}'.
\]

This is only possible when the right hand side does not depend on (\( a \)), that is, when \( \mathbf{n}(\tau) \propto \dot{D}_+ (\tau) \) where \( \dot{D}_+ \) satisfies the conditions \([80]\). Thus, in this toy model, the standard GI is not satisfied and the EGI is satisfied by a single time-dependent function \( \mathbf{n}(\tau) \) (up to a proportionality factor), which is sufficient to yield the consistency relations.

Reversely, it is possible to have Extended Galilean Invariance without the consistency relations. For instance, let us introduce an additional fifth-force long-range potential \( \Xi \) in the modified Euler equation

\[
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathcal{H} \mathbf{v} = -\nabla \Phi_{\text{N}} - \nabla \Xi,
\]

where

\[
\Xi = \alpha \mathcal{Y}^2 \quad \text{with} \quad \mathcal{Y} = \nabla^{-2} \delta,
\]

with \( \alpha \) a parameter. The potential \( \Xi \) is proportional to the square of the gravitational potential, \( \Xi \propto \Phi_{\text{N}}^2 \), but it is more convenient to keep the distinction between both potentials. Then, the EGI is still satisfied, with the transformations \([80]-[81]\) supplemented by \( \Xi' = \Xi \) (\( \Xi \) and \( \mathcal{Y} \) are functionals of the density field and are transported in the same fashion, while the gravitational potential receives an additional contribution). The potential \( \Xi \) adds a new quadratic term in the Euler equation, and in terms of the equation of motion \([31]\) and the vertices \([30]\), this gives rise to the new vertex

\[
\gamma_{2,1,1}^s (k_1, k_2) = -\alpha \frac{k^2}{a^2 \mathcal{H}^2 k_1^2 k_2^2}.
\]

Then, going through the check of the bispectrum consistency relation at lowest order of perturbation theory, described in Sec. III A, we find that the new vertex \( \gamma_{2,1,1}^d (k_2, -k_2) \) can no longer be neglected as \( k' \rightarrow 0 \), because it diverges as \( 1/k'^2 \) whereas the standard vertices \( \gamma_{2,1,1}^s \) and \( \gamma_{2,2,2}^s \) only diverge as \( 1/k' \). Therefore, the consistency relations \([24]-[25]\) no longer apply. We can easily see where the demonstration presented in Sec. III B breaks down. The force associated with the potential \( \Xi \) reads as

\[
F = -\nabla \Xi = -2\alpha \mathcal{Y} \nabla \mathcal{Y} \propto \Phi_{\text{N}} \nabla \Phi_{\text{N}}.
\]

Then, the perturbation to the fifth force due to a distant large-scale perturbation \( \Delta M \) reads at linear order as

\[
\Delta F \propto (\Delta \Phi_{\text{N}}) \nabla \Phi_{\text{N}} + \Phi_{\text{N}} \nabla (\Delta \Phi_{\text{N}}) \sim (\Delta \Phi_{\text{N}}) \nabla \Phi_{\text{N}},
\]

where we used \( \nabla \Phi_{\text{N}} \sim \Phi_{\text{N}} / r \) and \( \nabla (\Delta \Phi_{\text{N}}) \sim \Delta \Phi_{\text{N}} / R \), where \( r \) and \( R \) are the size of the small object and of the distant large-scale perturbation, with \( r \ll R \). Thus, the distant large-scale mass \( \Delta M \) no longer generates an almost constant force \( \Delta F \) over the extent of the small object, because the slowly varying factor \( (\Delta \Phi_{\text{N}}) \) is modulated by the fast varying factor \( \nabla \Phi_{\text{N}} \). Therefore, we can no longer make the approximation \( \Delta F_L (q, \tau) \approx \Delta F (q', \tau) \) in Eq. (83) to prove that the auxiliary trajectories \([81]\) are solutions of the perturbed equations of motion (at lowest order over \( k' \)).

The toy model \([83]\) is an explicit example of the violation of the equivalence principle, in the sense that the force \(-\nabla \Xi\) due to a distant mass depends on the local structure of the small object, even when \( r \ll R \), and of the consistency relation, while the Extended Galilean Invariance is still preserved. The difference with the modified-gravity models discussed in Sec. III C is that the new force \(-\nabla \Xi\) is truly long range [i.e., \( m = 0 \) in terms of the parameter introduced in Eq. (70) and vertices show divergences of the form \( 1/k^2 \) instead of \( 1/(k^2 + \alpha^2 m^2) \)] and violations of the equivalence principle do not vanish in the large-scale limit.

This shows that at a conceptual level one must distinguish between Galilean Invariance and the validity of the consistency relations, as noticed in previous works \([26, 29, 30]\). We also note that zero-mode symmetry arguments [i.e., that rely on \( k = 0 \) or uniform transformations] as well as linearized procedures must be used with caution, as they can miss nonlinear behaviors as in the model \([89]\).

2. Moving frames

Even though we have seen that it is not possible to derive the consistency relations from Galilean invariance alone, there is a close relationship between Galilean invariance and the precise form of the consistency relations, because they both are related to the Lagrangian derivative \( \partial_t + \mathbf{v} \cdot \nabla \), which gives rise to the factor \( k' / k^2 \) in Eqs. (19) and (52). On the other hand, one can wonder whether the Extended Galilean Invariance \([80]-[81]\) alone can be used to obtain some constraints on the density correlations.

As recalled in Sec. III A in analytical studies of the dynamics of large-scale structures, we usually work with
the density contrast $\delta$ and the velocity divergence $\theta = \nabla \cdot \mathbf{v}$, because gravity does not source vorticity (in the single-stream regime) and the growing mode is a curl-free flow that remains curl free at all perturbative orders (until we take into account nonperturbative shell-crossing effects).

However, it is clear that we cannot handle the Galilean Transformation [30]-[31] in this framework, because the transformation of the velocity is lost when we only look at the divergence $\theta$. Indeed, when we work with the fields $\varsigma(\delta, \theta)$, we have already actually taken full advantage of the EGI and we have already chosen to work in the frame where the mean initial peculiar velocity is zero (and it remains so afterwards by conservation of momentum). Therefore, it is not possible to perform a second time a Galilean Transformation on the fields $(\delta, \theta)$ and we must go back to the density and velocity fields. Because the velocity is curl-free, it is convenient to introduce the velocity potential $\chi$, with $\mathbf{v} = \nabla \cdot \chi$, and the equations of motion read as

$$\partial_t \delta + \nabla \chi \cdot \nabla \delta + (1 + \delta) \nabla^2 \chi = 0,$$  \hspace{1cm} (88)

$$\partial_t \chi + \frac{1}{2} (\nabla \chi)^2 + \nabla \chi = -\Phi_N,$$  \hspace{1cm} (89)

where the gravitational potential is still given by the last expression in [20] (here we focus on the standard cosmological scenario with only the dark matter fluid).

In the usual $(\delta, \theta)$ framework [30]-[31], we write the generating functional $Z^{(0)}[\delta]$ of the density correlation functions as

$$Z^{(0)}[\delta] = \langle e^{i \delta} \rangle = \int \mathcal{D}\delta L \int \mathcal{D}\delta_0 \mathcal{D}\theta_0 \mathcal{D}\Phi_N \ e^{-\frac{1}{2} \delta L \cdot C^{-1}_{L_0} \delta L_0} \times e^{\frac{1}{2} \delta_0 \cdot C^{-1}_{L_0} \delta_0} \times e^{\frac{1}{2} \theta_0 \cdot \mathcal{L} \theta_0} \mathcal{P}(\mathbf{n}),$$  \hspace{1cm} (90)

where $\delta^{(0)}[\delta L_0]$ is the functional that affects the nonlinear density contrast $\delta(x, \tau)$ to each initial condition $\delta L_0$. Here and in the following, the superscript $(0)$ denotes that we consider the density and velocity fields that are uniquely defined by the initial density contrast $\delta L_0(\mathbf{k})$, at the initial time $\tau_I$, by setting the mean velocity to zero. As explained above, this is the implicit choice when we work with the density contrast $\delta$ and velocity divergence $\theta$ alone. It is convenient to introduce the nonlinear density and velocity divergence $(\delta^{(0)}, \theta^{(0)})$ through the a Dirac factor that imposes both the equation of motion (34) and the initial conditions. This gives [30]-[31]

$$Z^{(0)} = \int \mathcal{D}\delta L_0 \mathcal{D}\theta^{(0)} \mathcal{D}\theta_0 \ e^{\frac{-1}{2} \delta L_0 \cdot C^{-1}_{L_0} \delta L_0} \times \mathcal{L} \mathcal{P}(\mathbf{n}).$$  \hspace{1cm} (91)

where we did not write the source term $e^{\psi^{(0)}}$ and $\psi^{(0)} = (\delta^{(0)} - \theta^{(0)}/a)$. (The gravitational potential has been written in terms of the density in the Euler equation by using the Poisson equation.). Here we used the fact that the Jacobian $\text{Det}\left[\mathcal{D} (\mathcal{O} \cdot \psi^{(0)} - K_s \cdot \psi^{(0)}) / \mathcal{D} \psi^{(0)}\right]$ is unity, thanks to causality and symmetry. The term $M_I \cdot \delta L_0$ sets the initial conditions at time $\tau_I$, with a factor $\mathcal{L} \mathcal{P}(\mathbf{n})$, so that the integration is over all fields $\delta^{(0)}$ and $\theta^{(0)}$ that are zero at earlier times.

Next, we can change variable from $(\delta^{(0)}, \theta^{(0)})$ to $(\delta, \chi, \Phi_N)$. In terms of the fields $(\delta, \chi, \Phi_N)$, the EGT [30]-[31] means that we have

$$\delta(x, \tau) = \delta^{(0)}(x + \mathbf{n}, \tau),$$  \hspace{1cm} (92)

$$\chi(x, \tau) = \chi^{(0)}(x + \mathbf{n}, \tau) - \mathbf{n} \cdot \mathbf{x},$$  \hspace{1cm} (93)

$$\Phi_N = \Phi_N^{(0)}(x + \mathbf{n}, \tau) - \frac{1}{2} \mathbf{n}^2 + (\mathbf{n}^2 + \mathcal{H} \mathbf{n}) \cdot \mathbf{x},$$  \hspace{1cm} (94)

where $\chi^{(0)}$ and $\Phi_N^{(0)}$ are uniquely defined by $(\delta^{(0)}, \theta^{(0)})$ as the solutions of $\nabla^2 \chi^{(0)} = \theta^{(0)}$ and $\nabla^2 \Phi_N^{(0)} = \frac{3}{2} H^2 \delta^{(0)}$, that vanish at infinity (we can think of localized density and velocity fluctuations in a finite region of space). The function $\mathbf{n}(\tau)$ is an arbitrary function of time that is an additional degree of freedom that is not specified by the initial condition $\delta L_0$, which only sets the density contrast and velocity divergence at the initial time. Then, the generating functional $Z$ associated with the field $\psi = (\delta, \chi, \Phi_N)$ writes as

$$Z = \int \mathcal{D}\delta L_0 \mathcal{D}\delta \mathcal{D}\chi \mathcal{D}\Phi_N \ e^{-\frac{1}{2} \delta L_0 \cdot C^{-1}_{L_0} \delta L_0} \mathcal{P}(\mathbf{n}) \times \delta_D[\mathcal{O} \cdot \psi^{(0)} - K_s \cdot \psi^{(0)} \psi^{(0)} - M_I \cdot \delta L_0] \times \delta_D[\psi - \mathcal{L} \mathcal{P}(\mathbf{n})],$$  \hspace{1cm} (95)

where $\mathcal{L} \mathcal{P}(\psi^{(0)})$ is the affine operator, which depends on $\mathbf{n}(\tau)$, that affects $\psi^{(0)} = (\delta^{(0)} - \theta^{(0)}/a)$ the triplet $\psi = (\delta, \chi, \Phi_N)$, according to Eqs. (92)-(94). Here, $\mathcal{P}(\mathbf{n})$ is the arbitrary probability distribution of the function $\mathbf{n}(\tau)$. The action $S[\psi]$ that we can derive from the generating functional (91) is invariant through the EGT, which corresponds to a change of the function $\mathbf{n}(\tau)$, except for the part that comes from $\mathcal{P}(\mathbf{n})$. Then, for a given choice of this distribution $\mathcal{P}(\mathbf{n})$, which breaks the EGI, it is possible to derive identities between zero-mode (i.e., with at least one wave number $\mathbf{k} = 0$) correlations. However, it is clear that this is only a complex way to write that the functional $Z$ comes from the gauge-fixed functional $Z^{(0)}$, with some choice of $\mathcal{P}(\mathbf{n})$, but that it does not tell us anything more about the properties of the fields $(\delta^{(0)}, \theta^{(0)})$.

For the specific purpose of the consistency relations, the simplest way to convince oneself of this point is to recall that it is possible to have Galilean invariance and a violation of the consistency relations [24]-[24], as we explicitly showed with the toy model [33]. However, on a more general level, the EGI alone does not provide much information on density correlations as it is restricted to zero-mode vertices. Therefore, we do not pursue here to derive the Slavnov-Taylor identities satisfied by the generating functional $Z$ of Eq. (95). In the context of turbulence, similar conclusions based on physical arguments and on the computation of these Slavnov-Taylor
identities are presented in [47,48], where it is found that Extended Galilean invariance does not constrain fluctuations with respect to the mean flow and only constrains zero modes of the vertices.

IV. CONCLUSION

We have described in this paper how general exact relations between correlation and response functions for systems defined by Gaussian initial conditions (or parameters) give rise to kinematic consistency relations, where the \((\ell + n)\) correlation between \(\ell\) soft modes and \(n\) hard modes can be factorized in terms of the correlation of the hard modes alone (with prefactors that involve the Gaussian power spectrum of the soft modes), provided the system verifies a weak form of scale separation. We have also shown how these relations extend to multi-fluid systems.

These kinematic consistency relations (usually with a single soft mode or in the single-fluid case) have already been derived in previous works, using the formalism developed for studies of the inflation era or a Newtonian approach. Here, we have presented a simple non-relativistic derivation, which follows from the usual equations of motion that describe the formation of large-scale structures, and applies to arbitrary numbers of soft wave numbers and fluid components. This explicit derivation allows us to identify the key ingredients that underlie these consistency relations and to clarify some confusing statements in some previous works.

We describe how the scale-separation property that gives rise to these consistency relations is not equivalent to the Galilean invariance of the equations of motion nor to the equivalence principle. Explicit examples show how both of these properties can be violated while the consistency relations remain valid. On the other hand, the Galilean invariance can be satisfied and the consistency relations violated. These relations rely on a weak property of scale separation: large scale fluctuations (in the linear growing mode) must have a uniform impact on small-scale objects, which takes the form of a multiplicative factor. In our case, this Fourier-space multiplicative factor corresponds to a configuration-space displacement, and long wavelength modes merely transport small-scale structures without disturbing them. In general, there could also be a uniform amplification.

Two points make this property even more general than it may seem at first sight. First, the large scale limit means that most modified-gravity models respect these consistency relations because they converge to General Relativity on large scales. This means that small nonlinear structures, whether they are screened by nonlinear local mechanisms as in chameleon models or not, do not feel the effect of a fifth force produced by a very large-scale density fluctuation, because it vanishes in any case. Therefore, these kinematic consistency relations remain valid in most modified-gravity models, even though they violate the equivalence principle on small scales. Second, because the consistency relations involve the effect from a large scale fluctuation of the linear growing mode, which is only a subset of all possible large-scale fluctuations as it implies a specific relation between density and velocity fields, it is possible to have systems where large scale fluctuations generically do not have a uniform impact on small scales except when these large scale fluctuations follow the linear growing mode pattern. In particular, it is possible to build models, such as Eq. (66), where Galilean invariance does not hold and the equivalence principle is violated at all scales, but where the consistency relations still apply. Of course, these are rather academic toy models, but they show that at a conceptual level consistency relations can be distinguished from the equivalence principle and the Galilean invariance.

These examples and our derivation show that these kinematic consistency relations hold in a wide variety of scenarios. On a theoretical level, a very important advantage of these relations is that they do not depend on the details of small-scale physics and they remain valid whatever small-scale nonperturbative processes take place, such as shell crossing of dark matter trajectories or complex astrophysical processes like star formation and outflows due to supernovae. Thus, a detection of a violation of these relations would signal either non-Gaussian initial conditions, significant decaying mode contributions, or a modification of gravity that does not converge to General Relativity on large scales (more generally, long range forces that do not satisfy the above scale-separation property in the large scale limit).

For different-time statistics, the kinematic consistency relations have a specific form that comes from the transport of small-scale structures by the long wavelength modes between these different times. However, in practice it will be very difficult to observe different-time correlations and it is unlikely that this method is the most efficient tool to constrain non-standard scenarios. For equal-time statistics, the consistency relations vanish, which means that the correlations between the \(\ell\) soft modes and the \(n\) hard modes do not show the \([P_{L0}(k')/k']^\ell\) tail (of course, the exact correlations do not vanish, they show lower order contributions over \(k'\)). Then, from an observational point of view, non-standard scenarios (which do not converge to General Relativity on large scales) or significant primordial non-Gaussianities could better be detected through a possible nonzero \([P_{L0}(k')/k']^\ell\) tail (or even stronger) of equal-time statistics. However, this may not be the most efficient probe of such scenarios (which may show even stronger departures from the \(\Lambda\)-CDM cosmology on small nonlinear scales), especially since observations already show that large scales should not deviate too much from the “concordance” \(\Lambda\)-CDM model.
Acknowledgments

We thank F. Bernardeau, Ph. Brax, and F. Vernizzi for discussions. This work is supported in part by the French Agence Nationale de la Recherche under Grant ANR-12-BS05-0002.