An asymptotic reduction of a Painlevé VI equation to a Painlevé III

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Abstract
When the independent variable is close to a critical point, it is shown that PVI can be asymptotically reduced to PIII. In this way, it is possible to compute the leading term of the critical behaviors of PVI transcendents starting from the behaviors of PIII transcendents.

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1. Introduction
As is well known, the first five Painlevé equations, PI, PII, PII, PIV, PV, can be obtained from the sixth PVI, by a step-by-step degeneration process [5].

Here, we present a different reduction of PVI to PIII. When the independent variable is close to a critical point, we show that it is possible to reduce PVI, with \( \alpha = (2\mu - 1)^2/2 \in \mathbb{C}, \beta = \gamma = 0, \delta = 1/2, \) namely

\[
y_{ss} = \frac{1}{2} \left[ \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - s} \right] (y_s)^2 - \left[ \frac{1}{s} + \frac{1}{s - 1} + \frac{1}{y - s} \right] y_s \\
+ \frac{y(y - 1)(y - s)}{2s^2(s - 1)^2} \left[ (2\mu - 1)^2 + \frac{s(s - 1)}{(y - s)^2} \right],
\]

(1)

to the following PIII, with \( \alpha = \beta = 0, \gamma = -\delta = 1, \) namely

\[
y_{\theta\theta} = \frac{1}{y} (y_s)^2 - \frac{1}{\theta} \tilde{y}_\theta + \tilde{y}^3 - \frac{1}{\tilde{y}}.
\]

(2)
The notation \( y_s \) stands for \( dy/ds \). We consider here, for definiteness, the case of the critical point \( s = 0 \). The reduction will be done accordingly for \( s \rightarrow 0 \) (and \( s \sim \theta^2 \)). The convergence is intended for bounded \( \arg(s) \). This is an asymptotic reduction. We show that, remarkably, it
allows us to reproduce the correct leading term of the critical behavior of PVI transcendents, which are classified in [4]. Equation (1) is important in the theory of semi-simple Frobenius manifolds of dimension 3 (see [1, 2]).

1.1. A $3 \times 3$ isomonodromy representation

Usually, PVI is regarded as the isomonodromy deformation equation for a $2 \times 2$ Fuchsian system with four singularities [8]. Here, we regard (1) as the isomonodromy deformation condition of the $3 \times 3$ linear system (L1) below, having a Fuchsian singularity at $z = 0$ and an irregular singularity of rank 1 at $z = \infty$. This system for the general PVI is described in [7]. For the particular PVI we consider here, related to Frobenius manifolds, (L1) is introduced and studied in [1, 2], where the following $3 \times 3$ Lax pair is given:

$$(L1) : \frac{dY}{dz} = \left[ U + \frac{V}{z} \right] Y, \quad (L2) : \frac{\partial Y}{\partial u_i} = [zE_i + V_i] Y, \quad i = 1, 2, 3.$$ 

The $3 \times 3$ matrix coefficients are

$$U = \text{diag}(u_1, u_2, u_3), \quad V = V(u_1, u_2, u_3), \quad V^T = -V, \quad \text{V has diagonal form} = \text{diag}(\mu, 0, -\mu).$$

$$(E_k)_{kk} = 1, \quad (E_k)_{ij} = 0, i, j \neq k. \quad (V_k)_{ij} = \delta_{ki} - \delta_{kj} V_{ij}.$$ 

Let $(\mu_1, \mu_2, \mu_3) := (\mu, 0, -\mu)$. A fundamental solution of (L1) at $z = 0$ has the representation

$$Y(z) = \sum_{p=0}^{\infty} \phi_p(u) z^p \frac{1}{\Lambda_1},$$

where \(\Lambda_1\) if \(\mu_i - \mu_j \neq n > 0, n \in \mathbb{Z}\). The necessary and sufficient condition for the dependence of the system (L1) on $(u_1, u_2, u_3)$ to be isomonodromic is [8]

$$\frac{\partial V}{\partial u_i} = [V_i, V], \quad \frac{\partial \phi_0}{\partial u_i} = V_i \phi_0.$$ 

The equations \(\sum_i \frac{\partial V}{\partial u_i} = 0\) and \(\sum_i u_i \frac{\partial V}{\partial u_i} = 0\) imply that \(V = V(s)\), where \(s = \frac{u_1-u_2}{u_2-u_1}\). Thus, if we let

$$V(s) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix},$$

the equation for $V$ becomes

$$\begin{cases} \frac{d\Omega_1}{ds} = \frac{1}{s} \Omega_2 \Omega_3 \\ \frac{d\Omega_2}{ds} = \frac{1}{1-s} \Omega_1 \Omega_3 \\ \frac{d\Omega_3}{ds} = \frac{1}{s(s-1)} \Omega_1 \Omega_2 \end{cases} \quad (4)$$

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = -\mu^2.$$ 

This system is equivalent to (1), and the following holds (see [3] for details):

$$y(s) = \frac{-s A(s)}{1 - s(1 + A(s))}, \quad A(s) := \left[ \Omega_1 \Omega_2 + \mu \Omega_3 \right]^2 = \left[ \frac{\Omega_1 \Omega_2 + \mu \Omega_3}{\Omega_1^2 + \Omega_2^2} \right]^2.$$ 

This last equation allows us to compute the critical behavior of $y(s)$ if we know that of the $\Omega_j(s)$'s.
2. Asymptotic reduction of PVI to PIII

As $s \to 0$, we do the following approximation to the system (4):

$$\frac{d\Omega_1^{(a)}}{ds} = \frac{1}{s} \Omega_2^{(a)} \Omega_3^{(a)}, \quad \frac{d\Omega_2^{(a)}}{ds} = \Omega_1^{(a)} \Omega_3^{(a)}, \quad \frac{d\Omega_3^{(a)}}{ds} = -\frac{1}{s} \Omega_1^{(a)} \Omega_2^{(a)},$$

with $\Omega_j(s) \sim \Omega_j^{(a)}(s)$ for $s \to 0$. The superscript $(a)$ stands for ‘asymptotic’. The reduced system has a first integral:

$$\left(\Omega_1^{(a)}\right)^2 + \left(\Omega_3^{(a)}\right)^2 = R^2 \in \mathbb{C}.$$

This implies that we can introduce the new dependent variable $\phi(s)$ as follows:

$$\Omega_1^{(a)} = R \sin \phi, \quad \Omega_2^{(a)} = R \cos \phi, \quad R \neq 0.$$

The system becomes

$$\frac{d\phi}{ds} = -\frac{1}{s} \frac{\Omega_2^{(a)}}{\Omega_1^{(a)}}, \quad \frac{d\Omega_2^{(a)}}{ds} = R \sin(2\phi).$$

Thus,

$$\frac{d^2\phi}{ds^2} + \frac{1}{s} \frac{d\phi}{ds} + \frac{R^2}{2} \sin \phi = 0.$$

With another change of variables,

$$u := 2\phi, \quad s = \frac{x^2}{4R^2}, \quad \text{(namely } 2\phi(s) = u(2R\sqrt{s}))$$

we obtain the following particular form of the Painlevé III equation:

$$u_{xx} + \frac{1}{x} u_x + \sin(u) = 0. \quad (6)$$

A last change of variables is necessary:

$$x = 2i\theta, \quad \hat{y} = \exp\left(\frac{iu}{2}\right).$$

This gives the PIII equation in standard form (2):

$$\hat{y}_{\theta\theta} = \frac{1}{\hat{y}}(\hat{y}_\theta)^2 - \frac{1}{\hat{y}} \hat{y}_\theta + \hat{y}^{3/2} - \frac{1}{\hat{y}}.$$

To summarize, the change of variables is

$$\Omega_1^{(a)} = \frac{R}{2}(\hat{y}^{-1} + \hat{y}), \quad \Omega_2^{(a)} = i\frac{R}{2}(\hat{y}^{-1} - \hat{y}), \quad \Omega_3^{(a)} = i\frac{d\hat{y}}{ds}.$$

$$\hat{y}(s) := \hat{y}(\theta(s)) = \hat{y}\left(\frac{R\sqrt{s}}{i}\right).$$

3. From asymptotic behaviors of PIII to behaviors of PVI

Let $s \to 0$, $|\arg s| < \pi$. In [4] we classified the critical behaviors of the PVI transcendents into a few classes, in the case when there is a one to one correspondence between branches of PVI transcendents and points in the space of the associated monodromy data. The critical behaviors are decided by the value of a complex ‘exponent’ $\sigma$ such that $0 \leq \Re \sigma < 1$. Let $\nu$ be a real number. Let also $a \neq 0$ and $C$ be two complex numbers, which, together with $\sigma$, play the role of constants of integration. According to [4], equation (1) has solutions with branches admitting the following critical behaviors for $s \to 0$. 


(1) Small-power-type behaviors [6]—4 real parameters:

\[ y(s) = ax^{1-\sigma}(1 + O(s^{\sigma} + s^{1-\sigma})), \quad 0 < \Re \sigma < 1. \]

(2) Sine-type oscillatory behaviors—3 real parameters:

\[ y(s) = s \sin^2(\nu \ln s + C) + O(s), \quad \sigma = 2i\nu. \]

(3) Inverse sine-type oscillatory behaviors—3 real parameters:

\[ y(s) = \frac{1}{1 - \frac{4s^2(2\mu - 1)^2}{4s^2 + 1}} + O(s^{\sigma} + s^{1-\sigma}), \quad \sigma = 1 + 2i\nu. \]

(4) Log-type behaviors and Taylor expansions—2 real parameters:

\[ y(s) = \frac{-4}{(2\mu - 1)^2} \left[ 1 + O \left( \frac{1}{\ln x} \right) \right], \quad \sigma = 1. \]

\[ y(x) = ax + O(s^2), \quad \sigma = 0. \]

The last is a convergent Taylor series, degeneration of a log behavior which occurs when \( \beta = \frac{1}{2} - 1 = 0 \) in the general PVI. The higher order terms in (1), (2), (3) can be written as convergent expansions in \( s \) and \( s^\sigma \), as is explained in [4].

We show that we can obtain the leading term of the above behaviors, when we substitute into the \( /\Omega_1(a) \)'s the asymptotic expansions of the solutions of (2), according to the formulas

\[ y(s) = \frac{-sA(s)}{1 - s(1 + A(s))}, \]

\[ A(s) \approx \left[ \frac{\Omega_1^{(a)} \Omega_2^{(a)} + \mu \Omega_3^{(a)}}{\mu^2 + (\Omega_2^{(a)})^2} \right]^2 = \left[ \frac{\Omega_1^{(a)} \Omega_2^{(a)} + \mu \Omega_3^{(a)}}{(\Omega_1^{(a)})^2 + (\Omega_3^{(a)})^2} \right]^2, \quad s \to 0, \]

and

\[ \Omega_1^{(a)} = \frac{R}{2} (\hat{y} - 1), \quad \Omega_2^{(a)} = \frac{iR}{2} (\hat{y} - 1 - \hat{y}), \quad \Omega_3^{(a)} = i\frac{d\hat{y}}{ds}. \]

At this point, we need the asymptotic behaviors of the solutions of the PIII equation (2). We can find them in [9].

(i) The first expansion we consider is (already in variable \( s \))

\[ \hat{y}(s) = Bs^2 \left( 1 + \frac{1}{4} \left( \frac{R}{B} \right)^2 s^{3-\sigma} - \frac{1}{4} \left( \frac{R\sigma^2}{1 - \sigma^2} \right)^2 s^{1+\sigma} - \sum_{j=3}^{+\infty} \sum_{k=2}^{j+1} c_{jk}s^{j-\sigma(j+2-k)} \right), \]

\[ B, \sigma \in \mathbb{C}, \quad -1 < \Re \sigma < 1, \quad c_{jk} \in \mathbb{C}. \]

\( B \) and \( \sigma \) are integration constants. The \( c_{jk} \)'s are certain rational functions of \( B \) and \( \sigma \). For symmetry reasons, we can restrict to the case

\[ 0 \leq \Re \sigma < 1. \]
It follows that
\[
\Omega_1^{(s)}(s) = bs^{-\frac{1}{2}}\left(1 - \frac{b^2}{(1 - \sigma)^2} s^{1-\sigma} + \frac{a^2}{(1 + \sigma)^2} s^{1+\sigma} + \cdots\right) + as^{-\frac{1}{2}}\left(1 + \frac{b^2}{(1 - \sigma)^2} s^{1-\sigma} - \frac{a^2}{(1 + \sigma)^2} s^{1+\sigma} + \cdots\right),
\]
\[
\Omega_2^{(s)}(s) = i \left[ bs^{-\frac{1}{2}}\left(1 - \frac{b^2}{(1 - \sigma)^2} s^{1-\sigma} + \frac{a^2}{(1 + \sigma)^2} s^{1+\sigma} + \cdots\right) - as^{-\frac{1}{2}}\left(1 + \frac{b^2}{(1 - \sigma)^2} s^{1-\sigma} - \frac{a^2}{(1 + \sigma)^2} s^{1+\sigma} + \cdots\right) \right],
\]
\[
\Omega_3^{(s)}(s) = \frac{\sigma}{2} + i\frac{b^2}{1 - \sigma} s^{1-\sigma} - i\frac{a^2}{1 + \sigma} s^{1+\sigma} + \cdots,
\]
where
\[
b := \frac{R}{2B}, \quad a := \frac{RB}{2}, \quad \mu^2 - \frac{\sigma^2}{4} = -R^2 + o(1).
\]
The dots are higher order corrections, and the ordering depends on the specific value of \( \sigma \). If we substitute into (7) and we keep the dominant term, we obtain
\[
y(s) \sim \frac{4b^2}{(2\mu - \sigma)^2} s^{1-\sigma}, \quad \text{for} \quad 0 < 2\sigma < 1 \quad \text{and} \quad \sigma = 0,
\]
\[
y(s) \sim s[\sin^2(\nu \ln s + C(R, B)) + O(s)], \quad \text{for} \quad \sigma = 2i\nu, \nu \in \mathbb{R} \setminus \{0\}.
\]
Thus, we have obtained the small-power-type behaviors (1), the Taylor series (4) and the sine-type oscillatory behavior (2).

(ii) Next, we consider the case corresponding to \( \sigma = 1 \). Let \( \omega = \ln \frac{\theta}{4} + \gamma \), where \( \gamma \) is Euler’s constant. In [9] we find the solution
\[
y(\theta) = -\theta\omega - \frac{\theta^5}{128} (8\omega^3 - 8\omega^2 + 4\omega - 1) + O(\theta^9\omega^5)
\]
\[
= -\theta \left(\ln \frac{\theta}{4} + \gamma\right) + O(\theta^5\ln^3 \theta), \quad \theta > 0, \quad \theta \to 0.
\]
In the variable \( s \),
\[
y(s) = i \frac{R}{2} s^{\frac{i}{2}} (\ln s + C)[1 + O(s^2 \ln^2 s)], \quad C := 2 \ln \frac{R}{4i} + 2\gamma.
\]
Now we compute
\[
\Omega_1^{(s)} = \frac{1}{s} \frac{d}{ds} \left[ i + \frac{i}{s^{\frac{i}{2}}(\ln s + C)} + O(s^2 \ln^2 s) \right] (1 + O(s^2 \ln^2 s))
\]
\[
\Omega_2^{(s)} = \frac{R}{2} (y^{-1} + \hat{y}) = \left[ -\frac{i}{s^{\frac{i}{2}}(\ln s + C)} + i \frac{R^2}{4} s^{\frac{i}{2}} (\ln s + C) \right] (1 + O(s^2 \ln^2 s))
\]
\[
= -\frac{i}{s^{\frac{i}{2}}(\ln s + C)} + O(s^2 \ln s)
\]
\[
\Omega_3^{(s)} = i \frac{R}{2} (y^{-1} - \hat{y}) = \left[ \frac{1}{s^{\frac{i}{2}}(\ln s + C)} + \frac{R^2}{4} s^{\frac{i}{2}} (\ln s + C) \right] (1 + O(s^2 \ln^2 s))
\]
\[
= \frac{1}{s^{\frac{i}{2}}(\ln s + C)} + O(s^2 \ln s).
\]
Observe that \( \mu^2 - \frac{1}{4} = -R^2 + o(1) \). Now, if we substitute into (7) and we keep the dominant term, we obtain
\[
y(s) \sim -\frac{4}{(2\mu - 1)^2} \frac{1}{(\ln s + C)^2}.
\]
Thus, we have obtained a log-type behavior (4).

(iii) The last case to be studied is
\[
\sigma = 1 + 2iv, \quad v \in \mathbb{R}, \quad v \neq 0.
\]

In [9] we find the solution
\[
y(\theta) = -\frac{1}{2v} \theta \sin \left(2v \ln \left(\frac{\theta}{4}\right) + 2\varphi(v)\right) + O(\theta^3), \quad \theta \to 0, \quad \varphi(v) = \arg \Gamma(iv).
\]

In variable \( s \),
\[
y(s) = i \frac{R}{2v} s^2 \sin(v \ln s + D) + O(s^2), \quad s \to 0, \quad D = 2v \ln \left(-i\frac{R}{4}\right) + 2\varphi(v).
\]
Therefore,
\[
\Omega_1^{(a)} = i s \frac{d \hat{y}}{ds} = \frac{i}{\sin(v \ln s + D) + O(s)} \left[\frac{1}{2} \sin(v \ln s + D) + v \cos(v \ln s + D) + O(s)\right],
\]
\[
\Omega_1^{(a)} = \frac{R}{2} (\hat{y}^{-1} + \hat{y}) = -\frac{1}{s^2} \sin(v \ln s + D) + O(s) + i \frac{R^2}{4v} s^2 (\sin(v \ln s + D) + O(s)),
\]
\[
\Omega_1^{(a)} = \frac{R}{2} (\hat{y}^{-1} - \hat{y}) = \frac{1}{s^2} \sin(v \ln s + D) + O(s) + \frac{R^2}{4v} s^2 (\sin(v \ln s + D) + O(s)).
\]

**Remark.** There is a sequence of poles (accumulating at \( s = 0 \)) corresponding to the roots of \( \sin(v \ln s + D) + O(s) = 0 \). We stress that it is not possible to write \( \frac{1}{\sin(v \ln s + D) + O(s)} \) as \( \frac{1}{\sin(v \ln s + D)} (1 + O(s)) \), because when we collect \( \sin(v \ln s + D) \) in the denominator we divide \( O(s) \) by \( \sin(v \ln s + D) \) itself, so we introduce poles (the roots of \( \sin(v \ln s + D) = 0 \)) in the \( O(s) \) terms!

The computation of \( y(s) \) is slightly more complicated than before. We have
\[
\Omega_1^{(a)} = i \Omega_2^{(a)} + O(s^2),
\]
\[
A(s) \sim \left[\frac{\Omega_1^{(a)} \Omega_2^{(a)} + i \mu \Omega_2^{(a)} + O(\sqrt{s})}{(\Omega_2^{(a)})^2 + \mu^2} \right] \left[\frac{\Omega_1^{(a)}}{\Omega_2^{(a)} - i \mu} + O(\sqrt{s})\right]^{-2} = 1 + O(\sqrt{s})
\]
\[
= \frac{1}{s^2} \left(\frac{2s-1}{2v} \sin(v \ln s + D) - \cos(v \ln s + D) + O(s)\right)^2.
\]

Note that in the computation we have used only the leading term of \( \Omega_1^{(a)} \). The higher order terms \( O(\sqrt{s}) \) have been divided by \( (\Omega_2^{(a)})^2 + (\Omega_2^{(a)})^2 = R^2 \), while the leading part \( \Omega_1^{(a)} \Omega_2^{(a)} + i \mu \Omega_2^{(a)} \) has been divided by \( (\Omega_2^{(a)})^2 + \mu^2 \), with the assumption \( (\Omega_2^{(a)})^2 + (\Omega_2^{(a)})^2 + (\Omega_2^{(a)})^2 \equiv -\mu^2 \). If we substitute the above result into (7) and we keep the dominant term, we obtain
\[ y(s) \sim \frac{1}{1 - \left( \frac{2\mu - 1}{2\nu} \sin(\nu \ln s + D) - \cos(\nu \ln s + D) \right)^2 + O(\sqrt{s})} = \frac{1}{1 - \frac{4\nu^2 + (2\nu - 1)^2}{4\nu^2} \sin^2(\nu \ln s + C) + O(\sqrt{s})}, \]
\[ C := D - \frac{i}{2} \ln \frac{2\mu - 1 - 2\nu}{2\mu - 1 + 2\nu}. \]

Thus, we have found an inverse sine-type oscillatory behavior (3).

4. On the error in the approximation of PVI with PIII

We may estimate the error in the approximation of \( \Omega_1 \) with \( \Omega_1^{(a)} \). Let us estimate the error in \( \Omega_3 \). We put
\[ \Omega_3 := \Omega_3^{(a)} + \delta \Omega_3 \]
and we substitute into the left-hand side of the third equation of (4). On the right-hand side we substitute \( \Omega_1^{(a)} \) and \( \Omega_2^{(a)} \):
\[ \frac{d\Omega_1^{(a)}}{ds} + \frac{d\delta \Omega_3}{ds} = -\frac{1}{s} (1 + s + \cdots) \Omega_1^{(a)} \Omega_2^{(a)}. \]

Recalling that \( \frac{d\Omega_2^{(a)}}{ds} = -\frac{1}{\gamma} \Omega_1^{(a)} \Omega_2^{(a)} \) we obtain
\[ \frac{d\delta \Omega_3}{ds} = -\Omega_1^{(a)} \Omega_2^{(a)} + \cdots \sim -i \frac{\sigma}{2} (bs^{-\frac{\sigma}{2}} + as^{\frac{\sigma}{2}}) + \cdots \]
which implies
\[ \delta \Omega_3 \cong O(s^{1-\frac{\sigma}{2}} + s^{1+\frac{\sigma}{2}}). \]

We write \( \Omega_1 := \Omega_1^{(a)} + \delta \Omega_1 \) and substitute on the left-hand side of the first equation of (4), while on the right-hand side we substitute \( \Omega_2^{(a)} \), \( \Omega_4^{(a)} \). The same procedure yields
\[ \delta \Omega_1 \cong O(s^{1-\frac{\sigma}{2}} + s^{1+\frac{\sigma}{2}}). \]
In fact, the terms of order \( s^{1-\frac{\sigma}{2}} \) are missing in the approximated solutions \( \Omega_1^{(a)} \) and \( \Omega_3^{(a)} \), but they appear in the true formal expansion of \( \Omega_1 \) and \( \Omega_2 \), which is computed in subsection 4.1. As for \( \Omega_2 \), we proceed as above making use of the second equation of (4), which becomes
\[ \frac{d\delta \Omega_2}{ds} = s \Omega_1^{(a)} \Omega_3^{(a)} = ib^2 s^{1-\sigma} - ia^2 s^{1+\sigma} + \cdots. \]
Thus,
\[ \delta \Omega_2 = i \frac{b^2}{2 - \sigma} s^{2-\sigma} - i \frac{a^2}{2 + \sigma} s^{2+\sigma} + \text{higher orders}. \]

4.1. Expansion with respect to a small parameter

The error in the asymptotic reduction is more precisely evaluated if we write the true formal expansion of the \( \Omega_j \)'s. In order to do this, a small parameter expansion can be used, as it is done in [3]. Let \( s := \epsilon z \) where \( \epsilon \) is the small parameter. The system (4) becomes
\[ \frac{d\Omega_1}{dz} = \frac{1}{z} \Omega_2 \Omega_3, \quad \frac{d\Omega_2}{dz} = \frac{\epsilon}{1 - \epsilon z} \Omega_1 \Omega_3, \quad \frac{d\Omega_3}{dz} = \frac{1}{\epsilon (z - 1)} \Omega_1 \Omega_2. \]

The coefficients of the new system are holomorphic for \( \epsilon \in E := [\epsilon \in \mathbb{C} ||\epsilon|| \leq \epsilon_0] \) and for \( 0 < |z| < \frac{1}{|\epsilon_0|} \), in particular for \( z \in D := [z \in \mathbb{C} | R_1 \leq |z| \leq R_2] \), where \( R_1 \) and \( R_2 \) are...
Then, we solve the linear system for $\epsilon$ independent of $J$. The small parameter expansion is a formal way to compute the expansions of the $\Omega_j$’s for $\epsilon \to 0$. To our knowledge, the procedure does not give a rigorous justification of the uniform convergence of the $s$-expansions of the $\Omega_j$’s. For $\epsilon \in E$ and $z \in D$ we can expand the fractions as follows:

$$
\frac{d\Omega_1}{dz} = \frac{1}{z} \Omega_2 \Omega_3, \quad \frac{d\Omega_2}{dz} = \epsilon \sum_{n=0}^\infty z^n e^n \Omega_1 \Omega_3, \quad \frac{d\Omega_3}{dz} = -\frac{1}{z} \sum_{n=0}^\infty z^n e^n \Omega_1 \Omega_2
$$

and we look for a solution expanded in powers of $\epsilon$:

$$
\Omega_j(z, \epsilon) = \sum_{n=0}^\infty \Omega_j^{(n)}(z) \epsilon^n, \quad j = 1, 2, 3.
$$

We find the $\Omega_j^{(n)}$’s substituting (10) into (9). At order $\epsilon^0$ we find

\[
\begin{align*}
\Omega_2^{(0)} &= 0 \\
\Omega_1^{(0)'} &= \frac{1}{z} \Omega_2^{(0)} \Omega_3^{(0)} \\
\Omega_3^{(0)'} &= -\frac{1}{z} \Omega_2^{(0)} \Omega_1^{(0)}.
\end{align*}
\]

The prime denotes the derivative w.r.t. $z$. Thus,

$$
\Omega_2^{(0)} = i\sigma z \frac{d}{dz}.
$$

Then, we solve the linear system for $\Omega_1^{(0)}$ and $\Omega_3^{(0)}$ and find

$$
\begin{align*}
\Omega_1^{(0)} &= b \zeta^{-\frac{1}{2}} + \tilde{a} \zeta^{\frac{1}{2}} = (b \zeta^{-\frac{1}{2}}) z^{-\frac{1}{2}} + (\tilde{a} e^{-\frac{1}{2}}) z^{-\frac{1}{2}} \\
\Omega_3^{(0)} &= ib \zeta^{-\frac{1}{2}} - i\tilde{a} \zeta^{\frac{1}{2}} = i(b \zeta^{-\frac{1}{2}}) z^{-\frac{1}{2}} - i(\tilde{a} e^{-\frac{1}{2}}) z^{-\frac{1}{2}},
\end{align*}
$$

where $\tilde{a}$ and $b$ are integration constants. We will require that $b := b \zeta^{-\frac{1}{2}}$, $a := \tilde{a} e^{-\frac{1}{2}}$ are finite, when $\epsilon \to 0$. The higher orders are

\[
\begin{align*}
\Omega_2^{(n)}(z) &= \int \cdots \int \sum_{k=0}^{n-1} \sum_{l=0}^{n-1-k} \Omega_2^{(n)}(\zeta) \Omega_3^{(n-1-k-l)}(\zeta) \\
\Omega_1^{(n)'} &= \frac{1}{z} \Omega_2^{(n)} \Omega_3^{(n)} + A_1^{(n)}(z) \\
\Omega_3^{(n)'} &= -\frac{1}{z} \Omega_2^{(n)} \Omega_1^{(n)} + A_3^{(n)}(z),
\end{align*}
\]

where

\[
\begin{align*}
A_1^{(n)}(z) &= \frac{1}{z} \sum_{k=1}^{n} \Omega_2^{(k)}(z) \Omega_3^{(n-k)}(z) \\
A_3(z) &= -\frac{1}{z} \left[ \sum_{k=1}^{n} \Omega_2^{(k)}(z) \Omega_1^{(n-k)}(z) + \sum_{k=1}^{n} \sum_{l=0}^{n-k} \Omega_1^{(k)}(z) \Omega_2^{(n-k-l)}(z) \right].
\end{align*}
\]

The system for $\Omega_1^{(n)}$, $\Omega_3^{(n)}$ is closed and non-homogeneous. By variation of parameters we find the particular solution

$$
\begin{align*}
\Omega_1^{(n)}(z) &= \frac{z^{\sigma/2}}{\sigma} \int \cdots \int \zeta^{1-\frac{1}{2}} R_1^{(n)}(\zeta) - \frac{z^{\sigma/2}}{\sigma} \int \cdots \int \zeta^{1+\frac{1}{2}} R_1^{(n)}(\zeta) \\
\Omega_3^{(n)}(z) &= \frac{z^{\sigma/2}}{\sigma} \left( \Omega_1^{(n)}(z) - A_1^{(n)}(z) \right)
\end{align*}
$$

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where

$$R_1^{(1)}(z) = \frac{1}{z} A_1^{(0)}(z) + \frac{i\sigma}{2z} A_3^{(0)}(z) + A_1^{(1)}(z).$$

Thus,

$$\begin{align*}
\Omega_1(s) &= s^{-\frac{i\sigma}{2}} \sum_{k,q=0}^{\infty} b_{kq}^{(i)} s^k \epsilon^{1-\sigma} q + s^{\frac{i\sigma}{2}} \sum_{k,q=0}^{\infty} d_{kq}^{(i)} s^{k+1+\sigma} q, \quad j = 1, 3 \\
\Omega_2 &= \sum_{k,q=0}^{\infty} b_{kq}^{(2)} s^k \epsilon^{1-\sigma} q + \sum_{k,q=0}^{\infty} d_{kq}^{(2)} s^{k+1+\sigma} q.
\end{align*}$$

(11)

The coefficients $a_{kj}^{(1)}$ and $b_{kj}^{(1)}$ contain $\epsilon$. In fact, they are functions of $a := \tilde{a} \epsilon^{-\frac{i\sigma}{2}}$, $b := \tilde{b} \epsilon^{\frac{i\sigma}{2}}$. The re-normalization after restoring $s$ is possible if $\sigma \neq 2n + 1$, $n \in \mathbb{Z}$ (no ln $z$ terms in $\Omega_2$), and if the additive constant in the integration of $\Omega_2^{(0)}$ is zero. If this is not the case, some coefficients of the expansions for the $\Omega_j$’s diverge.

We can fix the range of $\sigma$ according to the condition that the first term $\Omega_2^{(0)}$ in $\Omega_2$ be the leading one. The approximation at order 0 for $\Omega_2$ is

$$\Omega_2 \approx \frac{i\sigma}{2} \equiv \text{constant}.$$ 

The approximation at order 1 contains powers $\epsilon^{1-\sigma}, \epsilon^{1+\sigma}$. If we assume that the approximation at order 0 in $\epsilon$ is actually the limit of $\Omega_2$ as $s = \epsilon z \to 0$, then we need

$$-1 < \Re \sigma < 1.$$ 

The ordering of expansion (11) is somehow conventional: namely, we could transfer some terms multiplied by $s^{\frac{i\sigma}{2}}$ in the series multiplied by $s^{-\frac{i\sigma}{2}}$, and conversely. I report the first terms

$$\begin{align*}
\Omega_1(s) &= bs^{-\frac{i\sigma}{2}} \left( 1 - \frac{b^2}{(1-\sigma)^2} s^{1-\sigma} + \frac{a^2}{4(1-\sigma)} s + \frac{a^2}{(1+\sigma)^2} s^{1+\sigma} + \ldots \right) \\
&\quad + as^{\frac{i\sigma}{2}} \left( 1 + \frac{b^2}{(1-\sigma)^2} s^{1-\sigma} + \frac{a^2}{4(1+\sigma)} s - \frac{a^2}{(1-\sigma)^2} s^{1+\sigma} + \ldots \right) \\
\Omega_2(s) &= ibs^{-\frac{i\sigma}{2}} \left( 1 - \frac{b^2}{(1-\sigma)^2} s^{1-\sigma} + \frac{a^2}{4(1+\sigma)} s + \frac{a^2}{(1+\sigma)^2} s^{1+\sigma} + \ldots \right) \\
&\quad - ias^{\frac{i\sigma}{2}} \left( 1 + \frac{b^2}{(1-\sigma)^2} s^{1-\sigma} + \frac{a^2}{4(1-\sigma)} s - \frac{a^2}{(1-\sigma)^2} s^{1+\sigma} + \ldots \right) \\
\Omega_3(s) &= i\sigma s^{-\frac{i\sigma}{2}} + i - \frac{b^2}{1-\sigma} s^{1-\sigma} - i \frac{a^2}{1+\sigma} s^{1+\sigma} + \ldots.
\end{align*}$$

Note that the dots do not mean higher order terms. There may be terms bigger than those written above (which are computed through the expansion in the small parameter up to order $\epsilon$) depending on the particular value of $\Re \sigma$ in $(-1, 1)$. Finally, we note that we can always assume

$$0 \leq \Re \sigma < 1,$$

because that would not affect the expansion of the solutions but for the change of two signs.

It is worth observing that if we pretend that the solutions (11) are still valid for $\Re \sigma = 1$ and if we extract the terms where $s$ has an exponent with negative or vanishing real part, we have

$$\Omega_1 = bs^{-\frac{i\sigma}{2}} \sum_{q=0}^{\infty} (-1)^q \left[ \frac{b^2}{(1-\sigma)^2} \right]^q s^{(1-\sigma)q} \big|_{|\sigma| = 1+2i} + \ldots.$$
where

\[ \Omega_1 = \frac{i\sigma}{2} + \frac{b^2}{1 - \sigma} s^{-2i\nu} \sum_{q=0}^{\infty} (-1)^q \left( \frac{b^2}{(1 - \sigma)^2} \right)^q s^{(1 - \sigma)q} |_{\sigma = 1 + 2i\nu} + \cdots. \]

\[ \Omega_2 = \frac{i\sigma}{2} + i \frac{b^2}{1 - \sigma} s^{-2i\nu} \sum_{q=0}^{\infty} (-1)^q \left( \frac{b^2}{(1 - \sigma)^2} \right)^q s^{(1 - \sigma)q} |_{\sigma = 1 + 2i\nu} + \cdots. \]

For \(|\Re\sigma| < 1\) and \(s\) small, we sum the series

\[ \sum_{q=0}^{\infty} (-1)^q \left( \frac{b}{1 - \sigma} \right)^2 s^{1 - \sigma} = \frac{1}{1 + \left( \frac{b}{1 - \sigma} \right)^2 s^{1 - \sigma}}. \]

Then, we analytically extend the result at \(\sigma = 1 + 2i\nu\). In this way

\[ \Omega_1 = -\frac{iv}{s^2 \sin(v \ln s + C)} + \cdots, \]

\[ \Omega_2 = \frac{iv}{s^2 \sin(v \ln s + C)} + \cdots, \]

\[ \Omega_3 = \frac{1}{2} + iv \cot(v \ln s + C) + \cdots, \]

where \(C = -i \ln(2\nu/b)\). The result is similar to that obtained for the \(\Omega_f^{(a)}\)'s, but the \(O(s)\) terms in the denominator do not appear.

**Note.** Consider the system (4) and expand the fractions as \(s \to 0\). We find

\[
\frac{d\Omega_1}{ds} = \frac{1}{s} \Omega_2 \Omega_3, \quad \frac{d\Omega_2}{ds} = \sum_{n=0}^{\infty} s^n \Omega_1 \Omega_3, \quad \frac{d\Omega_3}{ds} = -\frac{1}{s} \sum_{n=0}^{\infty} s^n \Omega_1 \Omega_2. \tag{12}
\]

We can look for a formal solution

\[ \Omega_j(s) = s^{-\frac{j}{2}} \sum_{k,q=0}^{\infty} b_{kq}^{(j)} s^{k(1 - \sigma)q} + s^{\frac{j}{2}} \sum_{k,q=0}^{\infty} a_{kq}^{(j)} s^{k(1 + \sigma)q}, \quad j = 1, 3 \]

\[ \Omega_2 = \sum_{k,q=0}^{\infty} b_{kq}^{(2)} s^{k(1 - \sigma)q} + \sum_{k,q=0}^{\infty} a_{kq}^{(2)} s^{k(1 + \sigma)q}. \]

Plugging the series into the equation we find solvable relations between the coefficients and we can determine them. For example, the first relations give

\[ \Omega_2 = \frac{i\sigma}{2} + \left( \frac{ib_{00}^{(1)}}{1 - \sigma} s^{-\frac{1}{2}} \right) + \left( \frac{i a_{00}^{(1)}}{1 + \sigma} s^{-\frac{1}{2}} \right) + \cdots \]

\[ \Omega_1 = \left( b_{00}^{(1)} s^{-\frac{1}{2}} \right) + \left( a_{00}^{(1)} s^{-\frac{1}{2}} \right) + \cdots \]

\[ \Omega_3 = \left( ib_{00}^{(1)} s^{-\frac{1}{2}} \right) + \left( -ia_{00}^{(1)} s^{-\frac{1}{2}} \right) + \cdots. \]

All the coefficients determined by successive relations are functions of \(\sigma, b_{00}^{(1)}, a_{00}^{(1)}\). These are the three parameters on which the solution of (4) must depend. We can identify \(b_{00}^{(1)}\) with \(b\) and \(a_{00}^{(1)}\) with \(a\).

The case \(\sigma = 1\) in the small parameter formalism is more complicated. If we perform the small parameter expansions as before, we find the same \(\Omega_j^{(0)}\) than before. But due to the exponent \(z^{-1/2}\) the integration for \(\Omega_2^{(1)}\) gives
\[ \Omega_2^{(1)} = -\frac{i}{2} a^2 z^2 + i b^2 \ln(z). \]

In this way, we find for \( \Omega_1 \) and \( \Omega_3 \) an expansion in power of \( \epsilon \) with coefficients which are polynomials in \( \ln(z) \); also the powers \( z^{-1/2}, z^{1/2}, \ldots, z^n/2 \) (\( n > 0 \)) appear in the coefficients. \( \Omega_2 \) is an expansion in power of \( \epsilon \) with coefficients which are polynomials in \( \ln(z) \) and \( z \). It is not obvious how to recombine \( z \) and \( \epsilon \) when logarithms appear. We can put \( \epsilon = 1 \). Anyway, we see that the first correction to the constant \( \frac{1}{2} \) in \( \Omega_2 \) is \( \ln(s) \), which is not a correction to the constant when \( s \to 0 \), because it diverges.

We can try an expansion which already contains logarithms of \( \epsilon \):

\[ \Omega_j(z, \epsilon) = \sum_{k=-1}^{+\infty} \sum_{n=0}^{+\infty} \Omega_{j,n}^{(k)}(z) \frac{\epsilon^{k+1}}{(\ln \epsilon)^n}, \quad j = 1, 3 \]

Then we substitute in (9) and we equate powers of \( \epsilon \) and \( \ln \epsilon \). The requirement that we could re-compose the powers of \( \ln \epsilon \) and \( \ln z \) appearing in the expansion in the form \( \ln(z\epsilon) \) imposes very strong relations on the integration constants. The result which we obtain, when we solve the equations for the coefficients \( \Omega_{j,n}^{(k)} \) equating powers up to \( \frac{1}{\ln \epsilon} \) and \( \epsilon^{-1/2} \), is

\[ \Omega_1 = \frac{i}{s^2} \left( \frac{1}{\ln(s) + C} \right) + O\left( \frac{1}{(\ln \epsilon)^n} \right) + O(\epsilon^{1/2}), \]

\[ \Omega_3 = \frac{-1}{s^2} \left( \frac{1}{\ln(s) + C} \right) + O\left( \frac{1}{(\ln \epsilon)^n} \right) + O(\epsilon^{1/2}), \]

\[ \Omega_2 = \frac{i}{2} + \frac{i}{\ln(s) + C} + O\left( \frac{1}{(\ln \epsilon)^n} \right) + O(\epsilon^{1/2}). \]

These are the log-type behaviors.

5. Conclusions

The asymptotic reduction of PVI to PIII produces the correct leading term of the critical behavior of branches of PVI transcendents, classified in [4], starting from the asymptotic behaviors of PIII transcendents, computed in [9]. We have evaluated the error of the asymptotic reduction, showing that it does not affect the leading term of the PVI transcendents obtained from asymptotic behaviors of PIII transcendents.

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