Conservation Laws and Formation of Singularities in Relativistic Theories of Extended Objects

Jens Hoppe
Institut für Theoretische Physik
ETH Hönggerberg
CH-8093 Zürich

Abstract

The dynamics of an M-dimensional extended object whose M+1 dimensional world volume in M+2 dimensional space-time has vanishing mean curvature is formulated in term of geometrical variables (the first and second fundamental form of the time-dependent surface $\sum_M$), and simple relations involving the rate of change of the total area of $\sum_M$, the enclosed volume as well as the spatial mean – and intrinsic scalar curvature, integrated over $\sum_M$, are derived. It is shown that the non-linear equations of motion for $\sum_M(t)$ can be viewed as consistency conditions of an associated linear system that gives rise to the existence of non-local conserved quantities (involving the Christoffel-symbols of the flat M+1 dimensional euclidean submanifold swept out in $\mathbb{R}^{M+1}$). For M=1 one can show that all motions are necessarily singular (the curvature of a closed string in the plane can not be everywhere regular at all times) and for M=2, an explicit solution in terms of elliptic functions is exhibited, which is neither rotationally nor axially symmetric.

As a by-product, 3-fold-periodic spacelike maximal hypersurfaces in $\mathbb{R}^{1,3}$ are found.
I. Introduction

Consider the motion of an M-dimensional extended object \( \sum_M(t) \) in \( \mathbb{R}^{M+1} \). Any such motion gives rise to a \( (M+1) \)-dimensional manifold \( \mathbb{M} \) in \( (M+2) \)-dimensional space-time \( \mathbb{R}^{1,M+1} \), whose boundaries (if \( \sum_M \) is compact) are \( \sum_M \) (initial time \( t_i \)) and \( \sum_M \) (final time \( t_f \)). Relativistically invariant dynamics for \( \sum_M \) can be formulated by subjecting \( \mathbb{M} \) to a variational principle, like the extremization of the volume-functional (generalizing [1]). The volume of \( \mathbb{M} \) may be given by introducing coordinates \( (\phi^\alpha)_{\alpha=0,\ldots,M} \) on \( \sum_M \subset \mathbb{R}^{1,M+1} \) by the \( M+2 \) coordinate-functions \( x^\mu(\phi^0,\ldots,\phi^M) \), calculating the metric \( G_{\alpha\beta} \) induced by the flat Minkowski-metric \( (\eta_{\mu\nu})_{\mu\nu} = \text{diag}(1,-1,\ldots,-1) \) and integrating,

\[
S = \text{Vol}(\mathbb{M}) = \int d\phi^{M+1} \sqrt{G} = (-)^M \det(G_{\alpha\beta}), \quad G_{\alpha\beta} = \frac{\partial x^\mu}{\partial \phi^\alpha} \frac{\partial x^\nu}{\partial \phi^\beta} \eta_{\mu\nu}.
\]

Taking (1) as a starting point (with signature \( (\mathbb{M}) = (1,-1,\ldots,-1) \)) one may ask: what does extremality of \( S \) (considered as a functional of the \( x^\mu \)) imply for \( \sum_M(t,\varphi) \), the shape of the extended object? Choosing \( \phi^0 = x^0 = t \), and the time dependence of the spatial parameters \( \varphi = (\varphi^r)_{r=1,\ldots,M} \) such that the motion of \( \sum_M \) (described by \( \vec{x}(t,\varphi) = (x^1,\ldots,x^{M+1}) \)) is always normal, i.e.

\[
(G_{\alpha\beta}) = \begin{pmatrix}
1 - \dot{\vec{x}}^2 & 0 \cdots 0 \\
0 & \ddots & \vdots \\
0 & \ddots & -\partial_r \vec{x} \cdot \partial_s \vec{x} \\
0 & \ddots & 1
\end{pmatrix}
\]

the extremality condition(s)

\[
\frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = 0 \quad \mu = 0,\ldots,M+1
\]

read:

\[
\frac{\partial}{\partial t} \left( \sqrt{g} \frac{1}{\sqrt{1 - \dot{\vec{x}}^2}} \right) = 0
\]

\[
\rho \cdot \dot{\vec{x}} = \partial_r \frac{1}{\rho} gg^{rs} \partial_s \vec{x}
\]

\[
\rho = \rho(\varphi^1,\ldots,\varphi^M) := \sqrt{\frac{g}{1 - \dot{\vec{x}}^2}}
\]

where \( \cdot = \frac{\partial}{\partial t} \), and \( g \) and \( g^{rs} \) are the determinant and inverse, respectively, of the (positive definite) metric \( g_{rs} := \partial_r \vec{x} \partial_s \vec{x} \) on \( \sum_M(t) \). The conservation law (4), “large area(densitie)s have to slow down, while small area(densitie)s speed up” (anticipating singularities as well as periodicity), encodes almost the complete dynamical information. To see this, one first notes that on a fixed compact surface \( \sum_M(t = t_i) \) parameters \( (\varphi^r)_{r=1,\ldots,M} \) may be chosen such that the conserved (energy-)density is actually independent of \( \varphi \), i.e.

\[
\dot{\vec{x}}^2 + g/\lambda^{2M} = 1
\]
\[ \lambda = \text{const.} \]

as noted already in [2], (4) then ensures that (6) will hold for all \( t \). Furthermore, as (5) and the orthogonality conditions (cp. (2))

\[ \dot{x} \partial_r x = 0 \quad , \quad r = 1, \ldots, M \]

are invariant under

\[ x(t, \lambda) \rightarrow \lambda x\left(\frac{t}{\lambda}, \varphi\right) \]

(corresponding to \( x^\mu \rightarrow \lambda x^\mu \) in (3)), one could put \( \lambda = 1 \) in (6), with the understanding, that each motion with \( \lambda \neq 1 \) can be obtained from a \( \lambda = 1 \) motion via (8). In any case, one can show that, since (6) and (7), i.e.

\[ \dot{\vec{x}} = \pm \sqrt{1 - g/\lambda^2 \vec{n}}, \]

\( \vec{n} = \) surface normal, holds (cp[3]), the equations of motion (5) are automatically satisfied – apart from points where \( \dot{\vec{x}} = 0 \). As will be seen in the next section(s) it is convenient to write (9) in the form

\[ \dot{\vec{x}} = -\sin \theta \vec{n} \]

\[ \theta = \theta(t, \varphi^1, \cdots, \varphi^M) \in (-\pi/2, +\pi/2) \]

One should note that choosing the conserved energy density \( \rho \) to be constant on \( \Sigma \) (i.e. independent of \( \varphi \)) is a matter of convenience, not necessity; eq. (5) is a consequence of (4) and (7), resp. (10), for any \( \rho \), and in the considerations that will follow one could equally think of \( \sin^2 \theta \) as being given by \( 1 - g/\rho^2(\varphi) \), rather than \( 1 - g/\lambda^2 M \). Leaving the density \( \rho \) unspecified one would keep full Diff \( \Sigma \) invariance of the equations.

At first sight (9), with the normal velocity being a (specific) function of the area-density \( \sqrt{g} \) (see [4] for a Hamiltonian formulation, and [5] for a general dependence on \( \sqrt{g} \)) may look rather simple – perhaps simpler than the, by now fairly well understood, mean-curvature flow (defined by letting the normal velocity be equal to the mean curvature of \( \Sigma_t \) – thus involving second derivatives of \( \vec{x} \), rather than first ones); however, certain crucial techniques available for the mean curvature flow (see e.g. [6]) do not apply to (9).

II. Formulation of the Dynamics of \( \Sigma_M \) in Terms of Geometrical Variables

The simple first-order form of the dynamics, (10) (resp. (9)), allow one to easily derive the basic equations,

\[ \dot{g}_{rs} = -2\sin \theta \, h_{rs} \]

\[ \dot{h}_{rs} = (\nabla_r \nabla_s - h_{ra} g^{ab} h_{sb}) \sin \theta \]

for the components of the metric tensor, and the second fundamental form

\[ h_{rs} := -\partial^2_{rs} \vec{x} \cdot \vec{n} \]

(\( \nabla_r \)) are the covariant derivatives (with respect to \( \varphi^r \)) on \( \Sigma_t \), i.e.

\[ \nabla_a \nabla_b f = \partial^2_{ab} f - \gamma^c_{ab} \partial_c f \]

\[ \gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \]

3
for any function $f$: $\sum_t \to \mathbb{R}$. Note that the gauge-fixing (cp.(6)) has left one with a residual SDiff $\sum_t$-invariance, i.e. invariance of the equations under reparametrisations

$$\varphi^r \to \varphi^r(\varphi^1, \cdots \varphi^M), \quad J = \det \frac{\partial \varphi^r}{\partial \varphi^s} = 1$$

and that $\theta$ (remember that $\cos^2 \theta = g/\lambda^2 M$) is an ‘observable’. Also note that (5), with $\rho = \lambda^M = \text{const}$, implies

$$\ddot{\vec{x}} \cdot \vec{n} = -\cos^2 \theta \cdot H, \quad H = g^{rs}h_{rs}$$

(as well as $\ddot{\vec{x}} \partial_r \vec{x} = -\frac{1}{2} \partial_r (g/\lambda^2 M) = \sin \theta \cos \theta \partial_r \theta$ – which is zero at the turning points, $\dot{x}(\tilde{t}, \tilde{\varphi}) = 0$); taking the time-derivative of $\sin \theta := \vec{n} \dot{x}$ one obtains

$$\dot{\theta} = \cos \theta \cdot H$$

(for $\theta \neq 0$, this could have been obtained directly from (11), $-2 \sin \theta \cos \theta \lambda^2 M = \dot{g} = gg^{rs}g_{rs} = -2g \sin \theta H$). Calculating $\ddot{\vec{x}} n_\mu, n_\mu$ being normal to $M$ in $\mathbb{R}^{1,M+1}$,

$$n_\mu = \left( \begin{array}{c} -\tan \theta \\ \frac{\vec{n}}{\cos \theta} \end{array} \right)$$

one can check that $\frac{\dot{\theta}}{\cos \theta}$ is indeed the curvature of any $\varphi = \text{const}$ curve (worldline) in $M$ (as it should, according to (17), to add up to zero, with the spatial principal curvatures). In any case, (11) and (12) imply

$$\ddot{g}_{rs} = \cot \theta \dot{g}_{rs} \dot{\theta} + \frac{1}{2} \dot{g}_{ra}g^{ab}g_{bs} - 2 \sin \theta \nabla_r \nabla_s \sin \theta$$

(19)

(where $\dot{\theta}$, (cp) (17), could be replaced by $-\frac{1}{2} \cot \theta g^{ab} \dot{g}_{ab}$). Modulo the gauge-fixing, (19) is equivalent to the original minimal hypersurface equations. Note that only for $M = 1$, where $g_{rs} = \lambda^2 \cos^2 \theta(t, \varphi)$ yields $\dot{\theta} = \theta''$, one has decreased the number of equations. For any $M$, letting $\dot{T_g} := g^{ac} \dot{g}_{cb}$, they imply the matrix equation

$$\dot{T_g} = -\frac{1}{2} T_g^2 - \frac{1}{2} (\cot \theta)^2 (T \dot{T_g}) T_g - 2 \sin \theta \nabla \cdot \nabla \sin \theta.$$ (20)

In order to make all the $\theta$-dependence explicit, one could insert

$$g_{rs} = \lambda^2 (\cos \theta) \frac{\ddot{x}}{\dot{x}} g_{rs} \quad \tilde{g} = \det \tilde{g}_{rs} = 1$$

(21)

into (11)/(12), resp. (19) or (20), which then becomes an equation for the traceless matrix $\tilde{T} = T_g + \frac{2}{M} H \sin \theta 1$: on the other hand it is easy to see directly from (11)/(12) that

$$\ddot{T}^a_b := \ddot{g}^{ac} \dot{g}_{cb} = 2 \sin \theta (h^a_b - H^a_b) \frac{1}{2} \dot{T} T^2 = \frac{2}{M} \sin^2 \theta \nabla T \cdot (\kappa_r - \kappa_s)^2$$

(22)

and that the Weingarten map $T : T^a_b = g^{ac} h_{cb} = h^a_b$, whose eigenvalues are the principal curvatures $\kappa_r$, satisfies

$$\dot{T} = (T^2 + \nabla \cdot \nabla) \sin \theta.$$ (23)
Taking the trace of (23), and integrating over $\sum t$, one finds that
\[ \int \dot{H} \sqrt{g} d^M \varphi = \int \left( \sum_{r=1}^{M} \kappa_r^2 \right) \sin \theta \sqrt{g} d^M \varphi. \]  
(24)

As (11) and (12) were derived from (9) (resp. (10)) which describe the (time-)deformation of embedded hyper-surfaces, solutions $g_{rs}(t) : \sum_t \rightarrow \mathbb{R}, h_{rs}(t) : \sum_t \rightarrow \mathbb{R}$ will automatically (if they do so at $t = t_i$) satisfy the Gauss-equations
\[ R_{abcd} = h_{ac} h_{bd} - h_{ad} h_{bc}, \]  
(25)
in particular
\[ R := R_{abcd} g^{ac} g^{bd} = H^2 - TrT^2, \]  
(26)
and the Codazzi equations
\[ \nabla_a h_{bc} = \nabla_b h_{ac}. \]  
(27)
Due to (26), and $\sqrt{g} = -\sin \theta H \sqrt{g}$ (24) may also be stated as
\[ \frac{d}{dt} \int_{\Sigma_t} H = - \int_{\Sigma_t} R \sin \theta. \]  
(28)
(27), on the other hand, is useful when considering the evolution of $Q_m := Tr T^n$ for $n > 1$, e.g.
\[ \frac{1}{2} \dot{Q}_2 = Q_3 + T_a^a \nabla^b \nabla_a \sin \theta. \]  
(29)
Integrating over $\sum$ (using $\nabla^b T_a^a = \nabla^a H$, and $\triangle \sin \theta = \dot{H} - Q_2 \sin \theta$) yields
\[ \int \dot{H} \sqrt{g} d^M \varphi = 2 \int \left( HQ_2 - Q_3 \right) \sin \theta \sqrt{g} d^M \varphi, \]  
(30)
respectively
\[ \frac{d}{dt} \int_{\Sigma_t} R = \int_{\Sigma_t} (3HQ_2 - 2Q_3 - H^3) \sin \theta \]  
(31)
\[ = - \int_{a \neq b \neq c} \left( \sum \kappa_a \kappa_b \kappa_c \right) \sin \theta \]  
(32)
(recovering, for $M = 2$, a weak form of the Gauss-Bonnet theorem as a consequence of the dynamical equations). Also note that the rate of change of the volume enclosed by $\sum M$, respectively its total area, are given by
\[ \dot{V} = - \int \sin \theta \cos d^M \varphi, \dot{A} = - \int \sin \theta \cos Hd^M \varphi \]  
(33)

III. Zero Curvature Condition and Non-Local Conserved Quantities.

The fact that the dynamical equations (5) are automatically satisfied as a consequence of gauge-fixing conditions, (7), and a conservation law, (4), – which too can be stated as a condition on the metric of $\mathbb{M}$ – may also be used in the following way: Consider hypersurfaces
∑_{t_i} \sum_{t_f} \Delta t and motions in between such that for \( t_i \leq t \leq t_f \) all points of the surface have non-vanishing velocity. The projection of \( \mathcal{M} \) onto \( \mathbb{R}^{M+1} \) will then be a euclidean domain \( \mathcal{M}_E \subset \mathbb{R}^{M+1} \) (with \( \sum_{t_i} \) and \( \sum_{t_f} \) as boundary), parametrized by \( t \) and \( (\varphi^r)_{r=1,M} \), and with the euclidean metric
\[
(G^E_{ij})_{ij=1,\ldots,M+1} = \begin{pmatrix} g_{rs} = (\rho \cdot \cos \theta)^{\frac{2}{M}} \bar{g}_{rs} & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & 0 \end{pmatrix}.
\] (34)

Again one may choose \( \rho(\varphi) = \lambda^M = \text{const} \), for simplicity. As (34) contains the entire information about \( \mathcal{M}_E \), the minimal hypersurface equations should be equivalent to the flatness of \( \mathcal{M}_E \), i.e. the vanishing of the curvature-tensor
\[
R^E_{ijkl} = \frac{1}{2} (\partial^2 G^E_{jk} + \partial^2 G^E_{il} - \partial^2 G^E_{ik} - \partial^2 G^E_{jl}) + (G^E)^{mn} (\Gamma_{m,il} - \Gamma_{m,ik}) \Gamma_{n,jl} \quad .
\] (35)

Due to the special form of the metric, cp. (34), one has (with \( M + 1 =: N; a, b, c = 1 \ldots M \))
\[
\Gamma_{N,Na} = \sin \theta \cos \theta \partial_a \theta = -\Gamma_{a,NN} \quad (36)
\]
\[
\Gamma_{N,ab} = -\frac{1}{2} g_{ab} = -\Gamma_{a,Nb} \quad (\text{cp. (14)})
\]
\[
\Gamma_{N,NN} = \sin \theta \cos \theta \; \dot{\theta}, \quad \Gamma_{a,bc} = \gamma_{a,bc}
\]

Using (36) one indeed finds the following: \( R^E_{Nr,Ns} = 0 \) is equivalent to (19);
\[
R^E_{Nabc} = \frac{\sin \theta}{2} (\nabla_a (\frac{\dot{g}_{ab}}{\sin \theta}) - \nabla_b (\frac{\dot{g}_{ac}}{\sin \theta}))
\] (37)
so that, by defining \( h_{ab} \) according to (11), the vanishing of (37) is equivalent to the Codazzi-equations (27). Finally,
\[
R^E_{abcd} = R_{abcd} - \frac{1}{4 \sin^2 \theta} (\dot{g}_{ac} \dot{g}_{bd} - \dot{g}_{ad} \dot{g}_{bc})
\] (38)
so that the vanishing of (38) is equivalent to the Gauss-equations, (25). One major advantage of this formulation is that the minimal hyper-surface equations (due to the definition of the curvature tensor, \( (\nabla_i \nabla_l - \nabla_l \nabla_i) x^j \equiv -R^E_{ikj} x^k \)) are therefore the compatibility conditions \((\partial_i + \Gamma_i, \partial_l + \Gamma_l) x^j = 0 \) of the linear system of equations
\[
(\partial_i + \Gamma_i) \psi = 0 \quad i = 1 \ldots M + 1 \quad ,
\] (39)
with \( \times N \) Matrices \( (\Gamma_i)^j_k := \Gamma^j_{ik} \). Explicitely, one finds
\[
\Gamma_c = \begin{pmatrix} \gamma_{cb} & \frac{1}{2} \Gamma_{g^c} \\ -\frac{\dot{g}_{ac}}{\sin^2 \theta} \cot \theta \partial_c \theta \end{pmatrix}
\] (40)
\[ \Gamma_N = \left( \begin{array}{cc} \frac{1}{2} T_g & -\sin \theta \cos \theta \partial^\theta \\ \cot \theta \partial_b \theta & \hat{\theta} \cot \theta \end{array} \right) \] (41)

where \( T_g \) and \( \gamma^a_{\alpha} \) are as before (with \( g_{rs} = (\rho \cdot \cos \theta)^2 \bar{g}_{rs} \)), i.e. depending on \( \theta \) (and \( \rho \)) in the following way:

\[ T_g = \bar{T} - \frac{2}{M} \hat{\theta} \tan \theta \] (42)

\[ \gamma^a_{bc} = \bar{\gamma}^a_{bc} + \frac{1}{M} (\delta^a_c \partial_b \ln (\rho \cdot \cos \theta) + \delta^a_b \partial_c \ln (\rho \cdot \cos \theta) - \bar{g}_{bc} \bar{g}^{ad} \partial_d (\rho \cdot \cos \theta)) \] , (43)

\( \bar{\gamma}^a_{bc} \) being the Christoffel-symbols corresponding to the reduced metric \( \bar{g}_{rs}(\bar{g} = 1) \). For M=2, e.g., \( \bar{g}_{ab} \) could be conveniently parametrized as

\[ \bar{g}_{ab} = \left( \begin{array}{cc} \cosh \chi + \cos \phi \sinh \chi & \sin \phi \sinh \chi \\ \sin \phi \sinh \chi & \cosh \chi - \cos \phi \sinh \chi \end{array} \right) \] (44)

Considering

\[ \phi^{(r)}(\varphi^1 \cdots \varphi^M, t) = \psi(\varphi^1, \cdots, \varphi^r + \omega^r, \cdots \varphi^M, t) \psi^{-1}(\varphi^1, \cdots, \varphi^r, \cdots, \varphi^M, t) \] r = 1 \cdots M

(\( \psi \) the matrix of fundamental solutions of (39) and, for definiteness, taking \( \sum_M \) to be an M-torus, with \( \varphi^r \epsilon [0, \omega^r] \)), satisfying

\[ \partial_i \phi^{(r)} = [\phi^{(r)}, \Gamma_i] \] (46)

non-local conserved charges

\[ Q_{rm} = Tr (\phi^{(r)})^m \] (47)

can be deduced from (39) – expressable in terms of the Christoffel-symbols \( \Gamma^i_{jk} \) of \( \mathbb{M}_E \) via solving (46)\( i=r \) as a pathordered exponential,

\[ \phi^{(r)}(\varphi^1 \cdots \varphi^M, t) = \varphi e^{- \int_{\varphi^r=\omega^r}^{\varphi^r+\omega^r} \Gamma_r(\varphi^1 \cdots \varphi^r \cdots \varphi^M, t) d\varphi^r} \] (48)

It is extremely tempting to speculate that the hidden Lorentz-invariance together with the (S)Diff \( \sum \) invariance should allow one to introduce a spectral parameter into (39). This would imply an infinity of conserved quantities by expanding (47) in terms of this parameter (note that the scale-parameter \( \lambda \), cp (8), on which the \( \Gamma_i \) at first sight seem to depend non-trivially, eventually just leads to a conjugation of \( \phi^{(r)} \) by a \( \lambda \)-dependent matrix).

**IV. Singularity Structure and Conserved Quantities for M=1 (Strings)**

Due to the fact that for \( M = 1 \) eq. (5) (with \( \rho = \lambda = \text{const} \)) is trivial, having

\[ \bar{x}(t, \varphi) = \lambda (\bar{a} (\varphi + \frac{t}{\lambda}) + \bar{b} (\varphi - \frac{t}{\lambda})) \] (49)
as its general solution (with the components of \( \vec{a} \) and \( \vec{b} \) being \( 2\pi \) periodic functions, for closed strings) the possible motions of the string can be obtained explicitly, by inserting (49) into (6) and (7) (yielding \( \vec{a}^2 = \vec{b}^2 = \frac{1}{4} \)), so that

\[
\vec{x}(t, \varphi) = \lambda \cos (f - g) \begin{pmatrix} -\sin (f + g) \\ \cos (f + g) \end{pmatrix}
\]

\[
\dot{x}(t, \varphi) = -\sin (f - g) \begin{pmatrix} \cos (f + g) \\ \sin (f + g) \end{pmatrix}
\]

where \( f = f(\varphi + \frac{\lambda}{2}) \) and \( g = g(\varphi - \frac{\lambda}{2}) \); from now on, \( \lambda \) will be put equal to 1, for simplicity. Apart from the requirement that (44) should describe a closed curve, \( f \) and \( g \) are arbitrary. The closedness-condition is important, as it forbids, e.g., to choose \( f \) and \( g \) to be small on the entire interval \([0, 2\pi]\); moreover, as the range of \( f + g \) has to be at least \( 2\pi \), it is easy to see that even if \(|f - g| < \pi\) initially, there will always exist a finite time \( t_s \) at which \(|f - g| = \pi\) (and \( f' + g' \neq 0\)) for some point on the string (i.e. some \( \varphi_s \)). At \((t_s, \varphi_s)\) the worldsheet can not be regular – the curvature \( k \) of the string diverges as

\[
k(t, \varphi) = \frac{f' + g'}{\cos(f - g)}
\]

– hence one finds that any (!) closed string motion in \( \mathbb{R}^2 \) (that was supposed to extremize the area functional in Minkowski-space) must be (become) singular. Infinitely extended regular “minimal” hypersurfaces in \( \mathbb{R}^{1,2} \) of the topological type \( S^1 \times \mathbb{R} \) can not exist. This fact is known (see e.g. [14]) but not really well known. Considering the fact that in the case of membranes moving in \( \mathbb{R}^3 \) (i.e. 3+1 dimensional space-time) it has often been argued (and taken against such theories) that regular motions will not exist due to an impossibility of balancing the surface tension by rotation the lack of thought concerning singularities in string theories (which are rather commonly believed to be stabilized by rotation) is somewhat astonishing – in particular as these singularities appear to be one of their interesting (rather than disturbing) features. Due to (50) it is clear that for smooth \( f \) and \( g \) such singularities not only appear, but also go away smoothly (i.e., in the context of the orthonormal gauge, can be uniquely extended beyond the singularity). There also exist choices for \( f \) and \( g \), for which the number of singularities is constant in time, e.g.

\[
\vec{x} = \frac{1}{2m} \begin{pmatrix} \cos(m(\varphi + t)) \\ \sin(m(\varphi + t)) \end{pmatrix} + \frac{1}{2n} \begin{pmatrix} \cos(n(\varphi - t)) \\ \sin(n(\varphi - t)) \end{pmatrix}
\]

(\( m, n \) being two different integers with no common divisor \( \neq \pm 1 \)). (52) corresponds to choosing \( f = \frac{m}{n}(\varphi + t) \), \( g = \frac{n}{m}(\varphi - t) \) in (50), and describes a closed curve of time-independent shape, rotating with constant angular velocity \( \omega = \frac{2\pi}{m-n} \) around the origin, having \(|m-n|\) cusps - the minima of

\[
|\vec{x}| = r(\hat{\varphi} := \varphi + \frac{m+n}{m-n}t) = \frac{(m-n)^2}{4m^2n^2} + \frac{1}{mn} \cos^2\left(\frac{m-n}{2} \hat{\varphi}\right)
\]

Note that \( \hat{\varphi} \) does not coincide with the geometrical angle \( \arctan \frac{a_2}{a_1} \), and that the curves (52) have the length \( L = \int_0^{2\pi} |\vec{x}| d\varphi = 4 \), independent of \( m \) and \( n \) (and \( t \), of course).
In order to see why regular shapes can not be balanced by rotation, one can insert the Ansatz
\[ \vec{x}(t, \varphi) = e^{i\frac{t}{2}f(t)} \vec{m}(\varphi) \] (54)
into the \( \mu = 0 \) part of (3), giving up the orthogonality-condition (7); with
\[ \sqrt{GG^{\alpha\beta}} = \frac{-1}{\sqrt{\vec{m}^2(1 - \dot{f}^2\vec{m}^2) + \dot{f}^2((\vec{m} \times \vec{m})')}^2} \left( \begin{array}{cc} -\vec{m}^2 & \dot{f}((\vec{m} \times \vec{m})')' \\ \dot{f}((\vec{m} \times \vec{m})') & 1 - \dot{f}^2\vec{m}^2 \end{array} \right) \] , (55)
\( \vec{m} \times \vec{m}' := m_1m'_2 - m_2m'_1 \), one gets
\[ \partial_{\varphi} \left( \frac{|\vec{m}'|}{\sqrt{1 - \dot{f}^2r^2\cos^2 \theta}} \right) = \partial_t \left( \frac{\dot{f}r \sin \theta}{\sqrt{1 - \dot{f}^2r^2\cos^2 \theta}} \right) \] (56)
where \( r = |\vec{x}| = |\vec{m}| \) and \( \theta = \varphi (\vec{m}, \vec{m}') \) are functions of \( \varphi \) (which for a regular curve could be chosen to be the arclength, setting \( |\vec{m}'| = 1 \)). Dividing by \( \dot{f} \) one gets
\[ \ddot{f}r^2 \cos^2 \theta |\vec{m}'| + \dot{f}^2r^3(\sin \theta)' = (r \sin \theta)' \] (57)
Excluding the case \( r \cdot \sin \theta = \text{const} \) (which, too, can not correspond to a regular curve, s.b.) one finds \( f(t) = \omega \cdot t \) (as expected, due to the assumption of time-independent shape) and by integrating (57) (or directly from (56))
\[ \omega^2r^2 \cdot (1 + c \sin^2 \theta) = 1 \] (58)
This yields the “desired” conclusion, as for \( c < 0 \) \((c > 0) \sin^2 \theta \) would have to be minimal (maximal) when \( r \) assumes it minimum (maximum). A rather special class of solutions consists of string-motions with \( \ddot{x}(t = 0, \varphi) \equiv 0 \); from (50) it is clear that \( f \equiv g =: \frac{h}{2} \) in this case. As an example, consider the following “harmonic perturbations of the radially symmetric string solution”:
\[ h(\varphi) = \varphi + \epsilon \sin(m\varphi) \] (59)
\[ \ddot{x}(t, \varphi) = \cos(t + \epsilon \sin mt \cos m\varphi) \cdot \left( \begin{array}{c} -\sin(\varphi + \epsilon \sin m\varphi \cos mt) \\ \cos(\varphi + \epsilon \sin m\varphi \cos mt) \end{array} \right) \] (60)
\[ k(t, \varphi) = \frac{1 + m\epsilon \cos m\varphi \cos mt}{\cos(t + \epsilon \sin mt \cos m\varphi)} \] (61)
(so far, no approximation was made). Suppose now, that \(|m\epsilon| << 1 \); the two cases I) \( m \) odd, II) \( m \) even show drastically different behaviour. Case I: (60) becomes singular shortly before \( t = \pi/2 \), at \( m \) (equally distributed) discrete points \( \varphi_i \); for a small time-interval these \( m \) singularities move “along the string” (in particular, disappear instantaneously at the \( \varphi_i \)); shortly after \( t = \pi/2 \) all singularities disappear, and at \( t = \pi \) the string is back to its original shape (with \( \varphi \rightarrow \varphi + \pi \)) and at rest. Case II: as \( \sin mt \rightarrow 0 \) for even \( m \), \( t \rightarrow \pi/2 \), the string stays regular until \( t = \pi/2 \) (when it has shrunk to a point), grows again and reaches its initial conditions at \( t = \pi \).
It would be interesting to use the infinitely many known conservation laws for the string, including e.g. non-local charges [7]
\[ Q_{\mu_1 \cdots \mu_n}^\pm = \int_{\varphi}^{\varphi + 2\pi} d\varphi_1 \int_{\varphi}^{\varphi + 1} d\varphi_2 \cdots \int_{\varphi}^{\varphi + n-1} d\varphi_n \ u_{\mu_1 \mu_2 \cdots}^\pm + \text{all cyclic permutations} \] (62)
\[ Q_{F,G} = \int\frac{dx}{2\dot{\rho} + \rho^2} \cdot \left\{ \frac{1}{F'(p' + \sqrt{2\dot{\rho} + \rho^2})} + \frac{1}{G'(p' - \sqrt{2\dot{\rho} + \rho^2})} \right\} \]  

(63)

(where \( p = x^0 - x^2 \), expressed as a function of \( x = x^1 \) and \( \tau = \frac{x^0 + x^2}{2} \); \( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x} \) and \( F \) and \( G \) arbitrary functions) for an understanding of the string motion, e.g. (should the curve be star-shaped around the center of mass) in the radial (non-parametric) representation,

\[ H = \int_0^{2\pi} \sqrt{1 + \frac{\rho^2}{r^2}} \sqrt{r^2 + r'^2} d\varphi , \]  

(64)

with equations of motion

\[
\begin{align*}
\dot{r} &= \frac{p}{r} \sqrt{\frac{r^2 + r'^2}{p^2 + r^2}} , \\
\dot{\rho} &= -\sqrt{\frac{r^2 + r'^2}{p^2 + r^2}} + \frac{1}{r} (r') \sqrt{\frac{p^2 + r'^2}{r^2 + r'^2}} \]
\end{align*}

(65)

respectively

\[ \ddot{r}(r^2 + r'^2) - r''(1 - \dot{r}^2) - 2r'r'' + r(1 - \dot{r}^2) + \frac{2r'^2}{r} = 0 \]  

(66)

(note that \( \sqrt{r^2 + r'^2} d\varphi \) is the infinitesimal arc-length of the curve, and that in (66) the combination of terms not involving time-derivatives of \( r \) is proportional to the curvature). Simple properties may be directly deduced from the time-independence of \( H \), e.g.: As \( p/r = \frac{\dot{r}}{\sqrt{1 - \dot{r}^2 + \frac{\dot{r}^2}{r^2}}} \to 0 \) when \( \frac{\dot{r}^2}{r^2} \to 0 \) (implying \( \int_0^{2\pi} d\varphi |r'| \to \text{const} \), i.e. the string becoming infinitely rough, if \( r \to 0 \), but \( \sqrt{1 + \frac{\dot{r}^2}{r^2}} \) finite otherwise (implying \( p = \frac{\dot{r}}{\sqrt{(1 + \frac{\dot{r}^2}{r^2}) - i^2}} \to 0 \) for \( r \to 0, \dot{r}^2 - 1 \text{ not } \to 0 \) one finds that in the latter case one actually must have \( \frac{\dot{r}}{r} \to 0 \) and \( \dot{r}^2 \to 1 \), i.e. the singularity being light-cone like.

V. Some Explicit Hypersurface Solutions

In addition to the methods described in [9], solutions (of (3)) of the following form may be found:

\[ I) \sum_{\mu=0}^{M+1} f_{\mu}(x^\mu) = 0 \]  

(67)

\[ II) (\vec{x} - \vec{a}(x^0))^2 - r^2(x^0) = \text{const} . \]

In both cases it is easiest to use that the level sets \( u = \text{const} \) (cp. [3]) of functions \( u(x^0 \cdots x^{M+1}) \) satisfying

\[ (\eta^{\mu\nu} \eta^{\rho\lambda} - \eta^{\mu\rho} \eta^{\nu\lambda}) \frac{\partial u}{\partial x^\mu} \frac{\partial u}{\partial x^\nu} \frac{\partial^2 u}{\partial x^\rho \partial x^\lambda} = 0 \]  

are extremal hypersurfaces in \( \mathbb{R}^{1,M+1} \).
Ansatz I: Considering \( F_\mu := (f_\mu')^2 \) as functions of \( v_\mu := f_\mu(x_\mu) \) (implying \( f_\mu'' = \frac{1}{2} F_\mu'(v_\mu) \)), where ' always means derivative with respect to the relevant variable

\[
\frac{f_0'^2}{2} \sum_{i=1}^{M} f_i'' + f_0'' \sum_{i=1}^{M} f_i^2 = \sum_{i\neq j} f_0'^2 f_j'' \quad (69)
\]

becomes a first order (functional-) differential equation,

\[
F_0 \sum_{i=1}^{M} F_i' + F_0' \sum_{i=1}^{M} F_i = \sum_{i\neq j} F_i F_j' \quad (70)
\]

which is to be solved on the constraint surface \( \sum_{\mu=0}^{M+1} v_\mu = 0 \) (so \( F_0'(v_0) = -F_0''(\sum_{\mu=0}^{M} v_\mu) \) if \( F_0 \) is an even function of \( v_0 \)). Depending on the dimension, \( M \), (70) may admit “soliton” solutions of different type. While for \( M = 1 \) there are solutions of the form \( (a_\mu \neq 0) \)

\[
F_\mu = a_\mu + b_\mu e^{\kappa\nu_\mu} + c_\mu e^{-\kappa\nu_\mu} \quad (71)
\]

the constant term has to be zero for \( M = 2 \), where particular solutions are

\[
F_\mu = -\frac{\epsilon^2}{c_\mu} e^{\kappa\nu_\mu} + c_\mu e^{-\kappa\nu_\mu} > 0 \quad (72)
\]

Defining \( h \) as \( \frac{\epsilon^2}{4} e^{-\kappa\nu_\mu} \) if \( c_\mu > 0 \), respectively \( \frac{\epsilon^2}{4} e^{\kappa\nu_\mu} \) for \( c_\mu < 0 \), yielding \( h^2 = 4h^3 - \frac{\epsilon^2}{4} h \), i.e. (irrespective of the choice of \( c_\mu \)) the elliptic Weierstrass-function \( h(w) = \wp(\kappa w + w_0); g_2 = \frac{\epsilon^2}{4}, g_3 = 0 \) that up to scale transformations \( x^\mu \to \lambda x^\mu \), translations \( x^\mu \to x^\mu + d^\mu \), Lorentz transformations \( x^\mu \to \Lambda \delta x^\nu \), and permutations of the spatial coordinates one gets two inequivalent solutions, namely

\[
\wp(x) \wp(y) \wp(z) = \wp(t) \quad (73)
\]

and

\[
\wp(x) \wp(y) \wp(t) = \wp(z) \quad (74)
\]

with the invariants \( g_2 \) of the \( \wp \)-functions equal to 1 (hence all \( \wp \)'s having period \( 2\omega = \sqrt{2K(\frac{1}{2})} = \frac{1}{2\sqrt{2\pi}}(\Gamma(\frac{1}{4}))^2 \) and taking the value 1 as their minimum). Before discussing the solutions (73) and (74) as time-dependent surfaces \( \sum_{\mu} \) in \( \mathbb{R}^3 \), it seems worthwhile to note how they evolve from various other points of view. E.g., replacing the Ansatz I) by the (equivalent) Ansatz

\[
\tilde{\Gamma} u(x^0, \cdots, x^{M+1}) = \mathbb{T}_{\mu=0}^{N} g_\mu (x^\mu) (\equiv 1) ,
\]

to be inserted into (68), i.e. \( \sum_{\mu\neq\rho} u^\mu u_\mu - 2 \sum_{\mu<\nu} u^\mu u^\nu u_{\mu\nu} = 0 \), one is led to a first order (functional) differential equation by taking the unknown functions \( g_\mu(x^\mu) =: z_\mu \) as independent variables, and the square of their derivatives, \( g_\mu'^2 = G_\mu(z_\mu) \) as unknown functions of the new variables. While the form of the equation to be solved on the constraint surface \( z_0 z_1 \cdots z_N = 1 \) is slightly more involved than (70), one may now look for polynomial solutions. (73), e.g., respectively

\[
f_0(x^0) = -\ln \wp(x^0) \quad (75)
\]

\[
f_i(x^i) = + \ln \wp(x^i) =: X_i \quad i = 1, 2, 4
\]
corresponds to

\[ G_1(z) = -G_0(z) = 4z(z^2 - 1) =: G(z) \]  

satisfying

\[
\frac{1}{2} (z_0 G(z_0) z_1^2 G'(z_1) z_2^3 z_3^3 + 11 \text{ more terms}) - 2 (z_0 G(z_0) z_1 G(z_1) z_2^3 z_3^3 + 5 \text{ more}) \equiv 0
\]  

on \( z_0 z_1 z_2 z_3 \equiv 1 \). In this “derivation”, the fact that (up to a sign), the Weierstrass \( \wp \)-function satisfies the same differential equation as its inverse, \( \frac{1}{\wp} \), plays a crucial role (otherwise, the resulting equation would be much more complicated, than \( 77 \)).

It is easy to deduce from \( 73 \), \( 74 \) that in the first case, all points of the surface \( \sum t \) always move with a velocity \( \geq 1 \) (=1 if and only if at least 2 of the spatial coordinates are equal to \( \omega \) mod \( 2\omega \)), whereas with a velocity \( \leq 1 \) (=1 if and only if \( x = \omega = y \) mod \( 2\omega \)) in the latter case. Hence \( 73 \) defines a space-like maximal hypersurface of \( \mathbb{R}^{1,3} \); it provides e.g. a nice example in the context of theorems on isolated singularities (of area-maximizing hypersurfaces)\[11] and generalized Bernstein-theorems \[12\]. \( 73 \) may be described as follows (restricting to \( \sum \) in the context of theorems on isolated singularities (of area-maximizing hypersurfaces)\[11]\) and generalized Bernstein-theorems \[12\]. \( 73 \) may be described as follows (restricting to one unit cube \( C = \{ \vec{x} \in \mathbb{R}^3 | x, y, z \in [0, 2\omega] \} \): At \( t = \omega \), \( 73 \) implies \( x = y = z = \omega \) (a point); at \( t = 0 = 2\omega \), \( \sum t \) consists of all faces of \( C \). When \( t \) varies from \( 0 \) to \( \omega \), the – initially square-surface \( \sum t \) becomes rounder (always convex), finally vanishing as a round point, as can easily be seen by expanding \( \wp \) around \( \omega \), yielding in first order the light cone

\[
(x - \omega)^2 + (y - \omega)^2 + (z - \omega)^2 = (t - \omega)^2
\]  

as \( t \to \omega \). As \( t \) varies from \( \omega \) to \( 2\omega \), the reversed picture holds (the surface becoming more square, due to the fact that the further \( x \) and \( y \), e.g., are away from \( \omega \), the faster \( z(x, y) \) moves up, respectively down).

The mean-\( \text{respectively} \) Gauss-curvature of \( \sum t \) are

\[
H = \frac{1}{(X^2 + Y^2 + Z^2)^{3/2}} \cdot \left( X''(Y'^2 + Z'^2) + Y''(X'^2 + Z'^2) + Z''(X'^2 + Y'^2) \right)
\]  

\[
K = \frac{R}{2} = \frac{1}{(X^2 + Y^2 + Z^2)^2} \cdot \left( X''Y''Z^2 + X''Z''Y^2 + Y''Z''X^2 \right)
\]  

where \( X = \ln \wp(x), \ldots \), (cp. \( 75 \)), and \( X + Y + Z = \ln \wp(t) = \text{const} \) (on each \( \sum t \)). From \( \wp \) having a second order pole at \( 0 \) (with residue 1) one can easily find the exact form of the curvatures at those points of \( \sum t \) that approach the corners, resp. edges, of \( C \).

Solution \( 74 \), on the other hand (again restricting to one unit cube \( C \)), is such that the upper and lower edges of \( C \) always belong to \( \sum t \), acting like a fixed frame. At \( t = 0 \sum t \) is flat (covering the upper and lower face of \( C \)); as \( t \) grows, so do the upper part of \( \sum t \) (moving downwards) as well as the lower part of \( \sum t \) (moving upwards); at \( t = \omega \), the two parts touch at \( x = y = z = \omega \), where the curvatures diverge. For \( t > \omega \), the process reverses (\( \sum_{\omega^{-1}} = \sum_{\omega^{-1}} \)). In order to find solutions of the form \( 67 \), one inserts \( u(\vec{x}, t) = \frac{1}{2}(\vec{x} - \vec{a}(t))^2 - \frac{1}{2} r^2(t) \) into \( 68 \). With \( \vec{\wp} u = (\vec{x} - \vec{a}), (\vec{\wp} u)^2 = r^2(t) + 2u, \vec{\wp}^2 u = N, \vec{\wp} \vec{u} = -\vec{a} \) one can write the resulting equation in the form

\[
\dot{F} + (N - 2) F^2 = \frac{N - 1}{r^2(t)}
\]  

as

12
where
\[ F := \frac{\dot{u}}{r^2(t)} \quad . \] (82)

When \( N = 2 \), (81) implies
\[ F(t, \vec{x}) = f(\vec{x}) + \int_0^t \frac{d\tilde{t}}{r^2(\tilde{t})} \] (83)
while a comparison of \( \dot{u} = \dot{a} (\vec{a} - \vec{x}) - \dot{r} r \) with (82) then yields
\[ f(\vec{x}) = \vec{\lambda} \cdot \vec{x}, \quad \vec{a}(t) = \vec{a}_0 - \vec{\lambda} \int_0^t r^2(\tilde{t}) \, d\tilde{t} \] (84)
\[ \vec{\lambda} \int_0^t r^2 - \vec{\lambda} \cdot \vec{a}_0 - (\ln r) - \int_0^t \frac{1}{r^2} = 0 \] (85)

Differentiating (85), putting \( \ln r^2 = h(t) \), multiplying by \( \dot{h} \), integrating and letting \( h = -\ln H \), one finds that up to an additive constant \( \mu, H \) equals the Weierstrass \( \wp \)-function (with \( g_2 = 12\mu^2 - 4\vec{\lambda}^2 \) and \( g_3 = -4\mu(\mu^2 + \vec{\lambda}^2) \)).
So
\[ r(t) = \frac{1}{\sqrt{\wp(t) + \mu}} \] (86)

This solution was also found in [13].

Acknowledgement

I would like to thank A. Bobenko, M. Bordemann, A. Chamseddine, J. Fröhlich, Y. Giga, K. Happle, G. Huisken, M. Struwe, S. Theisen and K. Voss for helpful discussions, E. Heeb for a drawing of one of the curves (52), N. Bollow for visualizing solution (73) on a computer, the Deutsche Forschungsgemeinschaft for financial support, and the Institute for Theoretical Physics of the Swiss Federal Institute of Technology, Zürich, as well as the mathematics department of Tübingen- and Hokkaido University, for hospitality.
References

[1 ] J. Nambu; Copenhagen Summer Symposium 1970, unpublished.
   T. Goto; Progr. Theor. Physics 46 (1971) 1560.

[2 ] J. Hoppe; MIT Ph.D. Thesis (1982) [Elem. Part. Res. J. (Kyoto) 80 (1989) 145].

[3 ] M. Bordemann, J. Hoppe; Phys. Lett B 325 (1994) 359.

[4 ] J. Hoppe; “Canonical 3+1-Description of Relativistic Membranes”, Yukawa Institute preprint, K-1097-94.

[5 ] J. Hoppe; Phys. Lett. B 335 (1994) 41.

[6 ] G. Huisken; J. of Diff. Geometry 20 (1984) 237.

[7 ] K. Pohlmeyer; Phys. Lett. B 119 (1982) 100.

[8 ] J.M. Verosky; J. Math. Phys. 27 (1986) 3061.

[9 ] J. Hoppe; Phys. Lett. B 329 (1994) 10.

[10 ] I.S. Gradshteyn, I.M. Ryzhik; “Tables of Integrals, Series and Products”, Academic Press 1965.

[11 ] K. Ecker; Manuscripta Math. 56 (1986) 375.

[12 ] S.Y. Cheng, S.T. Yau; Annals of Mathematics 104 (1976) 407.

[13 ] K. Happle, T. Kornhass; “Klassische Zyklische Strings im 3-dimensionalen Minkowski-Raum”, Freiburg preprint, Oct. 1990, unpublished.

[14 ] G.P. Pron’ko, A.V. Razumov, L.D. Solov’ev; Sov. J. Part. Nucl. 14 (3) (1983) 229.