Commuting quantities and exceptional $W$-algebras

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Abstract

Sets of commuting charges constructed from the current of a $\mathfrak{u}(1)$ Kac-Moody algebra are found. There exists a set $S_n$ of such charges for each positive integer $n > 1$; the corresponding value of the central charge in the Feigin-Fuchs realization of the stress tensor is $c = 13 - 6n - 6/n$. The charges in each series can be written in terms of the generators of an exceptional $W$-algebra.
1 Introduction

Classical integrable systems have been studied for many years. Two of the most familiar examples are the KP hierarchy, with its reductions to the KdV and higher hierarchies (and also the modified KdV hierarchies), and affine Toda field theory, of which the simplest is sinh-Gordon theory. The integrability of these systems stems from the presence of an infinite number of conserved quantities that commute with each other. The existence of these conserved quantities can be shown in a number of different ways, for example by using pseudodifferential operators, or by Hamiltonian reduction from affine Kac-Moody algebras. There is in fact a link between the KdV, mKdV and sinh-Gordon equations, coming from the fact that, after a change of variables, the conserved quantities for the KdV equation coincide with those of the mKdV and sinh-Gordon equations. There are analogous links for the other hierarchies of integrable systems referred to above.

The KdV, mKdV and sinh-Gordon equations can all be written in Hamiltonian form, and in fact for the KdV equation there is more than one way to do this. There is an intriguing connection between the so-called second Poisson bracket structure of the KdV equation and the Virasoro algebra, satisfied by the $T_{zz}$-component of the energy-momentum tensor in conformal field theories. In fact, the components of the field variable $u$ of the KdV equation have Poisson brackets given precisely by the Virasoro algebra.

More recently the quantum analogues of the above integrable systems have been investigated. A significant contribution was made by Sasaki and Yamanaka, who investigated the quantum Sine-Gordon equation and its relation with the quantum KdV (qKdV) and quantum mKdV equations. They found that at low levels the sine-Gordon equation did indeed possess conserved quantities, and they suggested that conserved quantities would exist at all levels. Furthermore, they realized that these conserved quantities could be expressed in terms of the energy-momentum tensor associated with the related qKdV equation.
In another development Zamolodchikov [3] showed that certain perturbations of conformal field theories lead to conserved quantities in addition to those associated with the two-dimensional Poincaré group. These additional quantities were certain polynomials in the energy-momentum tensor, and were shown to exist at low levels by an elegant counting argument. Furthermore, by explicit calculation it was shown that the first two non-trivial charges commuted. It was suggested that there were in fact an infinite number of conserved quantities, and that they commuted amongst themselves. Affine Toda field theories were studied as deformations of conformal field theories by Eguchi and Yang [4] and by Hollowood and Mansfield [5], generalizing the work of ref. [2], and Kupershmidt and Mathieu [6] considered the qKdV equation in the context of deformed conformal field theories.

Unfortunately the arguments that can be used to determine the form and properties of the conserved quantities in the classical theory do not seem to have straightforward analogues in the quantum domain. One can, for example, write the qKdV equation in terms of a Lax pair involving the operator $T$, but the absence of a natural associative normal-ordered product for differential polynomials in $T$ means that this Lax pair cannot be used to prove the existence of infinitely many commuting charges. To show the commutativity of even the low level conserved quantities requires a lengthy calculation, and although the first few levels have been checked by computer the existence of these quantities appears somewhat miraculous. Recently Feigin and Frenkel [7] have shown the existence of an infinite set of commuting quantities in a free field realization, using homology arguments, but this proof does not give explicit forms for the conserved charges.

Some partial results are known, however. For the central charge $c$ having the value $-2$, the general form of the conserved quantities was conjectured by Sasaki and Yamanaka [2] and recently proven by DiFrancesco, Mathieu and Senechal [8] using a free fermion realisation of the $c = -2$ Virasoro algebra. Also, for the $(p, p') = (2, 2k + 1)$ series of minimal models, the density of the charge with spin $2k - 1$ is equal to the singular vector that occurs in the
vacuum representation of the Virasoro algebra at level $2k$ for $c = 1 - 3(2k - 1)^2/(2k + 1)$ [9, 10, 11, 12].

Clearly, one of the outstanding problems in this development is to gain a better understanding of the conserved quantities and their properties. One way in which this might be achieved would be if the conserved quantities were contained in some larger algebra within which they could be viewed as some sort of Cartan subalgebra. There are two situations in which this picture does indeed arise. The first is the case of $c = -2$. In this case the enveloping algebra of the Virasoro algebra contains a truncation of $\mathcal{W}_\infty$ [13, 14] as a linear subalgebra, and it is easy to see that this subalgebra contains a ‘wedge subalgebra’ [14] which includes infinitely many commuting charges. In fact these charges are simply the integrals of the basic even-spin quasiprimary fields $V^i$ of $\mathcal{W}_\infty$. The second situation in which the picture we are thinking of arises is the classical case, viewed from the perspective of $\tau$-functions. Here the commuting quantities of the KdV equation can be embedded in the affine Lie algebra $\widehat{SU}(2)$ [15], and in fact they are simply the elements $J^3_n$, $n > 0$ of this Kac-Moody Lie algebra.

In this paper we attempt to gain some insight into these issues by looking for commuting charges that can be constructed from the generators of a $U(1)$ Kac-Moody algebra. We began our investigation by using a computer to look for commuting charges, and we then interpreted our results analytically. Since the Virasoro algebra has a free field realization we expected to obtain the charges corresponding to the qKdV equation, but we found in addition that it is possible to construct extra commuting quantities. In particular we found that there exists an infinite number of series of commuting charges, with each series itself containing an infinite number of charges. In terms of the free field realization of the Virasoro algebra each series exists only for a particular value of $c$. The $n$'th series contains even spin currents for every even value of the spin, but in addition contains odd spin currents at spins $m(2n - 2) + 1$ for $m$ a positive integer. The lowest such non-trivial odd spin is $2n - 1$, and the corresponding current can be taken to be a primary
field. Furthermore this field and the identity operator generate a $W$-algebra, denoted by $W(2n - 1)$, which plays an important role in determining the structure of the charges. We were able to prove analytically that these charges did commute, and furthermore we found that the $W$-algebra and an extension of it provide a natural algebraic framework in which this result can be understood. Eguchi and Yang [10], in their work on the restricted sine-Gordon equation [16, 17], noted that for the values of $c$ we considered there should exist additional even-spin charges. However, the forms of the charges they suggest differ from ours, and they did not consider any connection with an underlying $W$-algebra.

2 Computer results

We consider the $U(1)$ Kac-Moody algebra $\hat{U}(1)$,

$$[J_n, J_m] = n\delta_{n+m,0},$$  \hspace{1cm} (2.1)

with generators $J_n = \oint dz \, z^n J(z)$ and corresponding operator product expansion

$$J(z)J(w) = \frac{1}{(z-w)^2} + \ldots.$$ \hspace{1cm} (2.2)

We wish to address the question of what sets of mutually commuting quantities can be constructed as integrals of polynomials in $J$ and its derivatives. To put this another way, we want to find sets of mutually commuting operators in the enveloping algebra of $\hat{U}(1)$. We know that it is possible to write a stress energy tensor in terms of $J$,

$$T = \frac{1}{2} : J^2 : + \alpha J',$$  \hspace{1cm} (2.3)

and so we certainly expect to find the commuting charges that correspond to the KdV equation. We shall therefore look for sets of mutually commuting charges that include operators not contained in this series of charges.

In this section we report on our investigations of the above question using a computer to carry out the operator product expansions. To do this we used Mathematica [18] and
the operator product package OPEdefs. As a consequence of doing the algebra by computer, however, these results are necessarily restricted to currents of relatively low spin; we have looked at currents of spin 13 and less. In subsequent sections we give analytic derivations of many of these results, thereby extending them to arbitrary spins.

Given currents \( A \) and \( B \) with charges
\[
Q_A = \oint dz A(z), \quad Q_B = \oint dz B(z),
\]
the commutator of \( Q_A \) with \( Q_B \) is given by
\[
[Q_A, Q_B] = \oint dw \oint_w dz A(z)B(w).
\]
The only contribution to this commutator comes from the single pole term in the OPE of \( A \) and \( B \), so \( Q_A \) and \( Q_B \) will commute precisely when this single pole term is a derivative.

It is straightforward to see that
\[
Q_1 \equiv \oint dz J(z)
\]
commutes with any charge constructed from \( J \) and its derivatives. It is also true that
\[
Q_2 \equiv \oint dz :J^2(z):
\]
commutes with all other charges, since if \( P \) is an arbitrary differential polynomial in \( J \), the single pole term in :\( J^2(z) : P(w) \) comes from the single contraction term, namely
\[
2 \sum_{n=0}^{\infty} \frac{(n+1)!}{(z-w)^{n+2}} :J(z) \frac{\partial P(w)}{\partial J^{(n)}(w)} :,
\]
and the coefficient of the single pole in this expression is just
\[
2 \sum_{n=0}^{\infty} :J^{(n+1)} : \frac{\partial P}{\partial J^{(n)}} = 2P'
\]
This is also apparent from the Feigin-Fuchs representation—the integral of 1/2 :\( J^2 : \) is just \( L_{-1} \), the generator of translations.
In order to go beyond these somewhat trivial commuting charges, we looked for charges commuting
with the integral $Q_4$ of the spin-4 current

$$p_4 = :J^4: + g :J^2:$$ (2.10)

for some value of $g$, using a computer to calculate the OPE’s. We found that there were a
number of different sets of such charges. All of the currents in a given set gave rise to
charges that commuted with a given $Q_4$, with the same value of the coupling $g$ for each
set. Each set contained a unique current at every even spin, modulo total derivatives,
corresponding to those of the qKdV equation. There were also currents occurring at certain
odd spins. In fact each set was uniquely fixed by the spin of the lowest non-trivial odd
spin current. If the spin of this current is $h \equiv 2n - 1$, we call the corresponding set of
currents $S_n$. The odd spin currents in $S_n$ then occur at spins $1, h, 2h - 1, 3h - 2, \ldots$, 
increasing in steps of $h - 1$; the charges constructed from these currents commute with
$Q_4$ provided that $g$ is given in terms of $h$ by

$$g = \frac{1}{h + 1}(h^2 - 4h - 1)$$ (2.11)

We found such sets of odd spin currents for every odd integer $h$ larger than 1. Thus the
first series $S_2$ has $h = 3$ and has a unique current at every odd spin as well as at every
even spin. The second series $S_3$ has unique odd spin currents at spins $5, 9, 13, \ldots$ as well
as currents at every even spin.

Given a set of charges commuting with a given spin-3 charge $Q_4$, it is natural to ask
whether these charges commute with each other. We have verified that this is indeed the
case for the currents we found using the computer, and we shall give a general analytic
proof in section 4.

Although it would clearly be possible to give expressions for the currents series by series, it
is remarkable that there is a relatively simple formula that gives all the currents in all the
series, at least up to the dimension to which we have calculated. In order to write down
such a formula we include a parameter $x$ that is given in terms of the coupling constant
by \( x = -3(g + 1) \). The reason for this choice of parametrization is that for \( x = 0 \) the currents take particularly simple forms; this will be explained fully in a later section. Since the currents are fixed only up to total derivatives it is useful to make a choice of basis for the space of fields modulo derivatives. Making such a choice, the expression for \( p_r \) is

\[
p_r = :J^r: + :J^{r-4}J^2: g_1(r, x) + :J^{r-6}J^2: g_2(r, x) + :J^{r-8}J^2: g_3(r, x) + :J^{r-10}J^2: g_4(r, x) + \ldots
\]

where the coefficients \( g_l(r, x) \) are polynomials in the expansion parameter \( x \). The coefficients \( g_1(r, x), \ldots, g_5(r, x) \) are given by

\[
g_1(r, x) = -\frac{1}{3} \left( \frac{r}{4} \right) \frac{1}{(r-3)} [3(r-3) + x]
\]

\[
g_2(r, x) = \frac{1}{6} \left( \frac{r}{6} \right) \frac{1}{(r-3)(r-5)} [6(r-3)(r-5) + 5x(r-4) + x^2]
\]

\[
g_3(r, x) = -\frac{1}{9} \left( \frac{r}{8} \right) \frac{1}{(r-3)(r-5)(r-7)} \times
\]

\[
[9(r-3)(r-5)(r-7) + \frac{7}{10} x (19r^2 - 192r + 450) + \frac{1}{30} x^2 (193r - 962) + x^3]
\]

\[
g_4(r, x) = -\frac{7}{27} \left( \frac{r}{8} \right) \frac{1}{(r-3)(r-5)(r-7)} \times
\]

\[
[27(r-3)(r-5)(r-7) + 6x(4r^2 - 42r + 105) + x^2 (5r - 31)]
\]

\[
g_5(r, x) = \frac{10}{9} \left( \frac{r}{9} \right) \frac{1}{(r-3)(r-5)(r-7)} \times
\]

\[
[9(r-3)(r-5)(r-7) + \frac{63}{100} x (21r^2 - 213r + 500) + \frac{1}{100} x^2 (641r - 3209) + x^3].
\]

These coefficients describe the fields \( p_r \) up to \( r = 9 \) exactly and give the first terms for all the higher dimensional fields. For completeness we list explicitly the additional terms in
the fields of spin 10 to 13 that are not described by the coefficients $g_1, \ldots, g_5$:

\[
\begin{align*}
p_{10} & = \ldots + \left( 90 + \frac{152}{3} x + \frac{2381}{315} x^2 + \frac{17}{63} x^3 \right) J^2 J''^2 \\
& \quad + \left( 1 + \frac{565}{756} x + \frac{36311}{158760} x^2 + \frac{178}{6615} x^3 + \frac{1}{1134} x^4 \right) J^{(4)}^2 \\
p_{11} & = \ldots + 11 \left( 90 + \frac{6263}{192} x + \frac{689}{192} x^2 + \frac{5}{48} x^3 \right) J J''^2 J''^2 \\
& \quad - 11 \left( 5 + \frac{26329}{10752} x + \frac{12407}{24192} x^2 + \frac{13135}{290304} x^3 + \frac{25}{20736} x^4 \right) J'' J^{(3)} J^2 \\
& \quad + 11 \left( 1 + \frac{16519}{32256} x + \frac{643}{6048} x^2 + \frac{2663}{290304} x^3 + \frac{5}{20736} x^4 \right) J J^{(4)}^2 \\
p_{12} & = \ldots - 11 \left( 27 + \frac{251}{25} x + \frac{14939}{14175} x^2 + \frac{79}{2835} x^3 \right) J^6 \\
& \quad + 22 \left( 270 + \frac{378}{5} x + \frac{6319}{945} x^2 + \frac{31}{189} x^3 \right) J^2 J''^2 J''^2 \\
& \quad + 11 \left( 20 + \frac{1228}{135} x + \frac{6929}{3969} x^2 + \frac{8173}{59535} x^3 + \frac{11}{3402} x^4 \right) J''^4 \\
& \quad - 11 \left( 15 + \frac{478}{45} x + \frac{100301}{39690} x^2 + \frac{4309}{19845} x^3 + \frac{1}{89} x^4 \right) J''^2 J^{(3)} J^2 \\
& \quad - 11 \left( 60 + \frac{1054}{45} x + \frac{450913}{119070} x^2 + \frac{3151}{11907} x^3 + \frac{10}{1701} x^4 \right) J J''^2 J^{(3)} J^2 \\
& \quad + 11 \left( 6 + \frac{109}{45} x + \frac{92891}{238140} x^2 + \frac{319}{11907} x^3 + \frac{1}{1701} x^4 \right) J^2 J''^2 J^{(4)} J^2 \\
& \quad - \left( 1 + \frac{7601}{9450} x + \frac{6457273}{21432600} x^2 + \frac{294211}{5358150} x^3 + \frac{4511}{1071630} x^4 + \frac{1}{1020600} x^5 \right) J^{(5)}^2 \\
p_{13} & = \ldots - 143 \left( 27 + \frac{3623}{480} x + \frac{4579}{7200} x^2 + \frac{31}{2160} x^3 \right) J J^6 \\
& \quad + 143 \left( 180 + \frac{4003}{96} x + \frac{1117}{360} x^2 + \frac{19}{288} x^3 \right) J^2 J''^2 J''^2 \\
& \quad + 143 \left( 30 + \frac{114091}{8064} x + \frac{76861}{30240} x^2 + \frac{343}{1920} x^3 + \frac{97}{25920} x^4 \right) J^2 J''^3 \\
& \quad + 143 \left( 20 + \frac{57319}{8064} x + \frac{252527}{241920} x^2 + \frac{283}{4320} x^3 + \frac{1}{768} x^4 \right) J J''^4 \\
& \quad - 143 \left( 15 + \frac{28435}{4032} x + \frac{29731}{24192} x^2 + \frac{1453}{17280} x^3 + \frac{1}{576} x^4 \right) J J''^2 J^{(3)} J^2 \\
& \quad - 143 \left( 30 + \frac{158489}{16128} x + \frac{9883}{7560} x^2 + \frac{175}{2304} x^3 + \frac{5}{3456} x^4 \right) J^2 J'' J^{(3)} J^2 \\
& \quad + 143 \left( 2 + \frac{32587}{48384} x + \frac{3245}{36288} x^2 + \frac{59}{11520} x^3 + \frac{1}{10368} x^4 \right) J^3 J^{(4)} + \\
\end{align*}
\]
We shall see that the existence of the infinitely many sets of currents $S_n$ is related to
the occurrence of certain $W$-algebras. The first step in this process is the verification that
the even spin currents can be expressed in terms of the Feigin-Fuchs energy-momentum
tensor given in eqn (2.3), which has background charge $\alpha$ and central charge $c = 1 - 12\alpha^2$.
Evidently $Q_2 = \oint : J^2 : = 2L_{-1}$. The charge $Q_4$ can also be written in terms of $T(z)$,
provided we relate the coupling $g$ and the central charge $c$ by

$$g = -\frac{1}{3}(c + 5);$$

for this value of $g$ we have

$$Q_4 \equiv \oint : J^4 : + g : J'^2 :$$

$$= 4 \oint : T^2 : .$$

Here $T^2$ is normal-ordered according to the standard prescription in conformal field theory,
which takes the normal product of two operators $A(z)$ and $B(w)$ to be the term of order
$(z - w)^0$ in their OPE [20],

$$(AB)(Z) \equiv \oint dw \frac{A(w)B(z)}{w - z}. \quad (2.16)$$

To reconcile the two expressions for $Q_4$ in (2.15), we must relate the normal ordering
in terms of $J$, which is the usual Wick ordering, to that in terms of $T$. Using the formula [20]

$$(AB)(C) = A(BC) + [(AB), C] - A[B, C] - [A, C]B \quad (2.17)$$

with $A = B = J$ and $C = : J^2 :$, and also making use of

$${J} : J^2 : = - : J^2 : J = -J''$$

it is straightforward to recover eqn (2.15).
In this way it is possible to express the even spin currents in terms of $T$ alone, provided we use the freedom to add and subtract total derivatives, but the odd spin currents cannot be expressed only in terms of $T$.

The value $c_n$ of the central charge corresponding to the series $S_n$ can be found from equations (2.11) and (2.14) to be

$$c_n = 13 - 6n - \frac{6}{n}.$$  \hfill (2.19)

It is known for $n = 2, 3$ and 4 that a $W$-algebra $W(2n-1)$ exists \cite{21, 22}, where $W(2n-1)$ is an algebra generated by the identity operator together with a single primary field of spin $2n-1$. For $n = 2$ this algebra is the $W(3)$-algebra of Zamolodchikov \cite{23}, which exists for any value of $c$, while for $n = 3$ the algebra $W(5)$ exists for only five values of $c$, namely $6/7$, $-350/11$, $-7$, $134 \pm 60/\sqrt{5}$, and for $n = 4$ $W(7)$ exists for the single value $-25/2$ of $c$. There is an argument to suggest that an algebra $W(2n-1)$ exists for any $n$ at the value of $c$ given in eqn (2.19). This can be seen by examining the primary field algebra for the conformal field theory with $c$ given by (2.19). In this model the field $\phi(3,1)$ has dimension $2n-1$ and is expected to have an OPE with itself of the form \cite{23}

$$[\phi(3,1)][\phi(3,1)] = [1] + [\phi(3,1)] + [\phi(5,1)].$$  \hfill (2.20)

The dimension of $[\phi(5,1)]$ is $6n - 2$, however, which is too high to appear in the OPE of a spin $2n-1$ field with itself, and so we expect the identity operator together with $[\phi(3,1)]$ to form a closed operator algebra. This observation has also been made previously by Kausch \cite{26}. We note in addition that for the values of $c$ we are considering the field $[\phi(3,1)]$ has odd spin, and as a consequence cannot appear on the right-hand side of the OPE (2.20). Thus the OPE of $[\phi(3,1)]$ with itself gives only the identity operator in this case.

It is natural to suppose that the series $S_n$ is associated in some way with the algebra $W(2n-1)$. This is confirmed by the realization that the first non-trivial field of odd spin in $S_n$ can be chosen to be a primary field and that its OPE with itself is just that for
$\mathcal{W}(2n - 1)$. This connection between commuting charges and $\mathcal{W}$-algebras will be further discussed in section 4.

Of course, all the above deliberations were based on computer results involving currents of spin 13 and less. We now summarize the above and in the next sections prove by analytic means some of the statements.

We have amassed sufficient evidence to conclude that:

(A) There exists an infinite number of sets of mutually commuting operators constructed from the Kac-Moody generators $J(z)$. The series $S_n$ has a unique current at every even spin and unique odd spin currents at spins $1 + m(h - 1)$ for $h = 2n - 1$ and $m = 0, 1, 2, \ldots$.

(B) The even spin currents can be built in terms of the energy-momentum tensor $T = \frac{1}{2} J^2 + \frac{3(h-1)}{\sqrt{h+1}} J'$ where the central charge is $c = \frac{1}{h+1}(-3h^2 + 7h - 2)$.

(C) The series $S_n$ can be associated with the algebra $\mathcal{W}(2n - 1)$, the current $p_{2n-1}$ being a primary field which is the generator of $\mathcal{W}(2n - 1)$. All odd spin currents in $S_n$ are descendants of $p_{2n-1}$.

One might ask how the above sets of commuting quantities compare with those that are known from classical integrable systems. For the $n$’th KdV hierarchy there are conserved currents with spins $2, 3, \ldots, n + 1$ mod $n + 1$, so that the conserved charges have spins $1, 2, \ldots, n$ mod $n + 1$. These are the exponents of the Lie algebra $sl(n)$ repeated modulo the Coxeter number, and in fact for any Lie algebra $\mathcal{G}$ there exists an integrable system for which the charges have spins equal to the exponents of $\mathcal{G}$ [27, 28]. These do not correspond to the spins of the currents for any of the series found above.
3 Commuting currents with a $J^3$

In this section we study in more detail the first series of currents, $S_2$, containing a current $J^3$. Some features of this series will generalize to the other series, although there are aspects of this series that will have no analogues for the others.

Let us first show that this series, which is defined by the existence of commuting currents of spins 3 and 4, corresponds to the central charge having the value $c = -2$. If we take an arbitrary current $p_r$ of spin $r$, defined modulo derivatives,

$$p_r = J^r + g(r)J^{r-4}(J')^2 + \ldots,$$

it is straightforward to calculate its OPE with $J^3$. We find that, up to derivatives, the coefficient of the single pole is given by

$$r(r-1)(r-2)(r-3)\frac{J^{r-4}(J')^3}{4} + 6g(r)J^{r-4}(J')^3 + \ldots,$$

so we must have

$$g(r) = -\left(\frac{r}{4}\right)$$

in accordance with the computer results of the previous section. For $r = 4$ we find the current is $J^4 - (J')^2$, and comparing with (2.15) we find that $c = -2$. Hence we conclude that a spin-3 and a spin-4 current can commute only if $c = -2$.

From the computer results of the previous section it is apparent that for $c = -2$ or $x = 0$ there is a considerable simplification in the form of the currents. In fact they are consistent with the formula

$$p_r = :e^{-\phi} \partial^r e^\phi: ,$$

up to derivatives. The generating function for this series is

$$:e^{-\phi(z)+\phi(z+a)}: = \sum_{r=0}^{\infty} \frac{\alpha^r}{r!} p_r(z).$$

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1 We will loosely say that two currents commute if their corresponding charges commute.
We can use this to show that integrals of the currents $p_r$ commute—to this end we consider

\[ I \equiv :e^{-\phi(z)+\phi(z+\alpha)}:e^{-\phi(w)+\phi(w+\beta)}: \]

\[ = \frac{(z - w)(z + \alpha - w - \beta)}{(z - w - \beta)(z + \alpha - w)} :\exp(-\phi(z) + \phi(z + \alpha) - \phi(w) + \phi(w + \beta)):+. \quad (3.6) \]

Integrating this gives

\[ \oint dz \oint dz I = \frac{\alpha \beta}{\alpha + \beta} \oint dz \{ :\exp(-\phi(z - \beta) + \phi(z + \alpha)):+ - :\exp(-\phi(z) + \phi(z + \alpha + \beta)):\}, \quad (3.7) \]

since $\alpha$ and $\beta$ are small and so contained within the $w$ contour around $z$. This can be seen to vanish by a shift of the $z$ integration contour, and as a consequence we can conclude that

\[ [Q_r, Q_s] \equiv \oint dz \oint dz p_r(z)p_s(w) = 0, \quad \forall r, s. \quad (3.8) \]

These results for the series $S_2$ can understood in terms of the algebra $W_\infty$ of reference \[13, 14\]. This is a linear algebra containing a quasiprimary field $V^i(z)$ with spin $i + 2$ for $i = 0, 1, 2, \ldots$. Examining the commutation relations of $W_\infty$, as given for example in eqn(3.2) of \[24\], it can be seen that the modes $V^i_{-i-1}$ form an infinite set of commuting operators, which we might think of as a Cartan subalgebra. Since $V^0(z) = T(z)$, we recognize $V^0_{-1}$ as $L_{-1}$, and similarly $V^1_{-2}$ is the mode $W_{-2}$ of the spin-3 primary field $W(z)$.

For the case of $c = -2$, however, the $W_\infty$ algebra has a realization in terms of a complex fermion, and this can be bosonized to give a realization in terms of a single real scalar field. In terms of this scalar field $\phi$, the field $V^i$ is proportional to $:e^{-\phi\partial^{i+2}}e^{\phi}:$, up to derivatives. We recognize this as the current given in eqn (3.4). We also note that for $c = -2$ the fields $V^i$ for $i \geq 2$ can be written as composites of the stress tensor $T$ and the spin-3 primary field $W$, so that the enveloping algebra of the Zamolodchikov $W(3)$-algebra contains $W_\infty$ as a linear subalgebra for $c = -2$. This thus gives a nice explanation of the
presence of the infinitely many commuting quantities in the enveloping algebra of $W(3)$ in terms of a linear algebra.

Let us summarize the main results of this section. Starting from a $U(1)$ Kac-Moody algebra, we demanded a set of commuting currents that included currents of spins 3 and 4. We found that we could supplement these two currents by an infinite number of other currents, one at each spin, that formed a mutually commuting set. These could be expressed in terms of a stress tensor $T$ and a spin-3 primary field $W$ alone, where $T$ has a central charge $c = -2$. Furthermore these currents can be identified with a Cartan subalgebra of $W_\infty$, which in turn can be written in terms of $T$ and $W$. We therefore see that in terms of the algebra $W_\infty$ it is possible to understand very easily the existence of the commuting charges. Although we do not know of a generalization of $W_\infty$ that enables us to gain a similar understanding of all the series of commuting charges, we shall see in the next section that, for each series, there is a relation between the odd-spin currents that allows to us to give a simple proof of the commutativity of the corresponding charges.

4 Analytic results

In this section we wish to give analytic derivations of some of the results that we have obtained earlier. More specifically, we shall give explicit formulae for all of the odd spin fields in each of the series $S_n$, and we shall prove that these fields commute with each other and with all of the even spin fields in the series$^2$. We shall also spell out in detail the connection of the series $S_n$ to a $W$-algebra.

Let us consider the forms of the fields of odd spin in $S_n$. The series $S_n$ exists for $c$ having the value $13 - 6n - 6/n$, with $n$ a positive integer, and the first non-trivial field of odd spin

$^2$Here, as before, we refer to fields commuting when we really mean that their integrals commute.
has dimension $2n - 1$ and can be chosen to be primary. From the work of Zamolodchikov \cite{Zamolodchikov} we expect there to be only two primary fields that commute with the even spin fields built out of the stress tensor, namely the two fields with null descendants at level 3. If we write an arbitrary value of the central charge as $c = 13 - 6t - 6/t$, the two corresponding fields will have dimensions $h_{(3,1)} = 2t - 1$ and $h_{(1,3)} = 2/t - 1$. These two values of the dimension $h$ are related to $c$ by

$$c = 13 - 3(h + 1) - \frac{12}{h + 1} = \frac{1}{h + 1}(-3h^2 + 7h - 2).$$

Taking $t$ to be a positive integer $n$ we indeed expect to find a field of dimension $2n - 1$ that commutes with the commuting charges constructed from $T$. In order to write down an explicit expression for this field in terms of the currents $J$ alone, we first consider the two fields of this dimension that can be written as exponentials of $\phi$, namely

$$V_{-\alpha_+} \equiv e^{-\alpha_+\phi} \quad \text{(4.2)}$$

and

$$V_{2\alpha_+ + \alpha_-} \equiv e^{(2\alpha_+ + \alpha_-)\phi}, \quad \text{(4.3)}$$

where $\alpha_+ = \sqrt{2n}$ and $\alpha_- = -\sqrt{2/n}$. Each of these fields has a null descendant at level 3, obtained by acting with the operator

$$\mathcal{O}_{(3,1)} \equiv L_{-3} - \frac{2}{2n + 1}L_{-1}L_{-2} + \frac{1}{2n(2n + 1)}L_{-1}^3. \quad \text{(4.4)}$$

It is only for $V_{-\alpha_+}$, however, that this null descendant is actually zero, and so it is only in this case that the commutator of the corresponding charges will actually vanish, rather than simply giving zero in correlation functions. We shall therefore concentrate our attention on this field in what follows.

When $c = 13 - 6n - 6/n$, we can use the screening charge $Q_+ \equiv \oint \exp \alpha_+\phi$ to construct another primary field of weight $2n - 1$ that is annihilated by $\mathcal{O}_{(3,1)}$. The commutator of a Virasoro generator $L_n$ with $\exp \alpha_+\phi$ is a total derivative, so provided this field is
single-valued the commutator of $Q_+$ with a primary field will be another primary field of the same weight. In general this new primary field could be zero, but that will not be the case here. For the values of $c$ in which we are interested, $\exp\alpha_+\phi$ is indeed local with respect to $\exp -\alpha_+\phi$, and we can therefore construct another primary field of weight $2n - 1$ by

$$p_{2n-1} \equiv [Q_+, \exp -\alpha_+\phi(z)] = \oint_z dw \exp(\alpha_+\phi(w)) \exp(-\alpha_+\phi(z)).$$  (4.5)

This field has vanishing background charge and so is expressible entirely in terms of the current $J$ and its derivatives, and in fact it is given explicitly by the term of order $\partial^0$ in the differential operator

$$(\partial + \alpha_+ J)^{2n-1},$$  (4.6)

in which the derivatives are taken to act on everything that occurs to the right of them. We claim that this is the first non-trivial field of odd spin occurring in $S_n$. For the first few series this can be checked explicitly against the computer results given in Section 2. To show that it is true in general, we need to check that the charge constructed from this field commutes with all of the even spin charges, which correspond to those that occur in the quantum KdV equation. This is a consequence of the fact that the screening charge $\exp\alpha_+\phi$ commutes with any polynomial in $T$ and its derivatives, and in particular with the even spin charges. Since these even spin charges are even under $\phi \to -\phi$, it follows that $\oint \exp -\alpha_+\phi$ also commutes with these charges, as explained in ref [2].

The above considerations immediately suggest a way to construct infinitely many odd spin currents whose integrals commute with the even spin charges. We take $Q_+$, as defined earlier, to be $\oint \exp\alpha_+\phi$, and we define also $Q_- = \oint \exp -\alpha_+\phi$. These two charges have conformal weights 0 and $2n - 2$ respectively, and each commutes with all of the even spin charges. We then define an operator $\Delta$ that acts on an arbitrary field $\Phi$ by

$$\Delta \Phi(z) = [Q_+, [Q_-, \Phi(z)]] = \oint_z dy \oint_{y,z} dx : e^{\alpha_+\phi(x)} : e^{-\alpha_+\phi(y)} : \Phi(z).$$  (4.7)
If the integral of $\Phi$ commutes with the even spin charges, it is clear that the integral of $\Delta \Phi$ will also commute with these charges. Our strategy is then to apply $\Delta$ repeatedly to the primary field constructed above. In fact it is interesting to note that the primary field $p_{2n-1}$ can itself be written as

$$p_{2n-1} = \Delta J,$$  \hspace{1cm} (4.8)

since $[Q_-, J(z)] = \exp -\alpha_+ \phi(z)$. If we write $\Phi^{(m)}$ for the $m$th odd spin field in $S_n$, and $Q^{(m)}$ for the corresponding charge, we have

$$\Phi^{(m)} = \Delta \Phi^{(m-1)}, \quad \Phi^{(0)} = J,$$  \hspace{1cm} (4.9)

with similar formulae holding for $Q^{(m)}$. Thus $\Phi^{(m)}$ is given by the multiple commutator

$$\Phi^{(m)} = \Delta^m J = [Q_+, [Q_-, \ldots [Q_+, [Q_-, J], m \text{ times}].$$  \hspace{1cm} (4.10)

which is given explicitly by

$$\oint dx_m \oint dy_m \ldots \oint dx_1 \oint dy_1 \left\{ \prod_{i=1}^m e^{\alpha_+ \phi(z_x) + \alpha_+ \phi(z_y)} \right\} J(z) \hspace{1cm} (4.11)$$

where the integration contours satisfy $|x_m| > |y_m| \ldots |x_1| > |y_1| > 0$. Furthermore it is readily seen to be expressible as a sum of terms, with each term being a product of derivatives of exponentials of $\phi$, with the total $U(1)$ charge for each term in the sum being zero. Thus $\Phi^{(m)}$ can be written in terms of $J$ alone, with no exponentials of $\phi$. While it is conceivable that this multiple commutator could vanish, and indeed this will be the case for some orderings of the $Q_+$ and $Q_-$, we believe this will not happen in general for the ordering we have chosen. Let us demonstrate this for the first series, $S_2$. To do this we take the generating functional

$$G[\beta] = \oint dz e^{-\phi(z) + \phi(z + \beta)} = \sum_n \frac{\beta^n}{n!} Q^{(n)}$$  \hspace{1cm} (4.12)

where the integration contour surrounds both the origin and $\beta$, and then form

$$\Delta G[\beta] \equiv \oint dy \oint dx e^{-2\phi(x)} e^{2\phi(y)} G[\beta] \hspace{1cm} (4.13)$$
which equals
\[ \oint_0 \oint \oint \int \int dxdzdydz \frac{(x-z)^2(y-z-\beta)^2}{(x-y)^4(x-z-\beta)^2(y-z)^2} \exp(-2\phi(x) + 2\phi(y) - \phi(z) + \phi(z + \beta)). \] (4.14)

Carrying out the \( y \) integration we obtain
\[ -2\beta \frac{\partial}{\partial \beta} \oint \oint \int \int dx dz \frac{\beta}{(x-z)^2(x-z-\beta)^2} \exp(-2\phi(x) + \phi(z) + \phi(z + \beta)). \] (4.15)

We can now do the \( x \) integration, giving
\[ 4 \left( \frac{2}{\beta^2} - \frac{2}{\beta} \frac{\partial}{\partial \beta} + \frac{\partial^2}{\partial \beta^2} \right) \oint dz \{ \exp(-\phi(z) + \phi(z + \beta)) - \exp(\phi(z) - \phi(z + \beta)) \}. \] (4.16)

This then leads to
\[ \Delta Q^{(n)} = \begin{cases} \frac{8n(n+1)}{n!} Q^{(n+2)}, & \text{otherwise.} \\ 0, & \text{n even;} \end{cases} \] (4.17)

Hence we can indeed construct an infinite sequence of non-zero charges as multiple commutators in this way for \( c = -2 \), and as far as we are able to calculate in the higher series we obtain non-zero charges in these cases also.

Having seen how to write explicit formulae for an infinite set of charges commuting with the even spin charges, our next task is to understand why these charges commute amongst themselves. We start from the recursive definition for the \( m \)'th charge \( Q^{(m)} \),
\[
Q^{(1)} = [Q_+, Q_-] \\
Q^{(m+1)} = [Q_+, [Q_-, Q^{(m)}]].
\]

We shall prove that \([Q^{(n)}, Q^{(m)}] = 0\) for general \( m \) and \( n \), but we begin by proving this at lowest order. A crucial identity both in this simple case and in general is that
\[ [Q_+, [Q_+, [Q_+, Q_-]]] = 0, \] (4.18)
and similarly
\[ [Q_-, [Q_-, [Q_-, Q_+]]] = 0. \] (4.19)
These relations can be obtained very simply just from looking at the contour integrals that need to be done in order to evaluate these expressions. An essentially equivalent derivation for the first of these two relations follows from the observation that \([Q_+,[Q_+,[Q_+,\exp(-\alpha_+\phi)]]]\) has the same conformal weight as \(\exp(-\alpha_+\phi)\), namely \(2n-1\), but that it must be a descendant of \(\exp 2\alpha_+\phi\), which has weight \(2n+2\). Evidently this is possible only if \([Q_+,[Q_+,[Q_+,\exp(-\alpha_+\phi)]]]\) vanishes. We now explore some of the consequences of the relations (4.18) and (4.19), which we choose to rewrite in the form

\[
[Q_+, [Q_+, Q^{(1)}]] = 0, \quad [Q_-, [Q_-, Q^{(1)}]] = 0. \quad (4.20)
\]

It will be convenient to introduce the notation \(Q_+X\) for the commutator \([Q_+, X]\), where \(X\) is an arbitrary field, and similarly for \(Q_-\). The above relations then become

\[
Q_+^2 Q^{(1)} = 0, \quad Q_-^2 Q^{(1)} = 0. \quad (4.21)
\]

Let us start from

\[
Q^{(2)} = [Q_+, [Q_-, Q^{(1)}]] = [Q_-, [Q_+, Q^{(1)}]], \quad (4.22)
\]

which we write as

\[
Q^{(2)} = Q_+ Q_- Q^{(1)} = Q_- Q_+ Q^{(1)}; \quad (4.23)
\]

the equality of these two expressions follows from the Jacobi identity. Taking the commutator of these relations with \(Q_+\) and \(Q_-\) implies, using the Jacobi identity again, that

\[
Q_+ Q^{(2)} = [Q^{(1)}, Q_+ Q^{(1)}], \quad Q_- Q^{(2)} = -[Q^{(1)}, Q_- Q^{(1)}], \quad (4.24)
\]

and so

\[
Q_+ Q_- Q^{(2)} = -[Q_+ Q^{(1)}, Q_- Q^{(1)}] - [Q^{(1)}, Q^{(2)}] \]

\[
Q_- Q_+ Q^{(2)} = [Q_- Q^{(1)}, Q_+ Q^{(1)}] + [Q^{(1)}, Q^{(2)}].
\]

This implies that

\[
Q_+ Q_- Q^{(2)} - Q_- Q_+ Q^{(2)} = -2[Q^{(1)}, Q^{(2)}], \quad (4.25)
\]
whereas the Jacobi identity implies
\[ Q_+Q_-Q^{(2)} - Q_-Q_+Q^{(2)} = [Q^{(1)}, Q^{(2)}]. \]  
(4.26)
From this we conclude that
\[ [Q^{(1)}, Q^{(2)}] = 0, \]  
(4.27)
and hence
\[ Q^{(3)} = Q_+Q_-Q^{(2)} = [Q_-Q^{(1)}, Q_+Q^{(1)}]. \]  
(4.28)
We have also that
\[ Q^2_+Q^{(2)} = Q^2_-Q^{(2)} = 0. \]  
(4.29)
This begins to suggest the following pattern:
\[ [Q^{(i)}, Q^{(j)}] = 0, \text{ for } i + j = n, \]
\[ Q^{(n)} \equiv Q_+Q_-Q^{(n-1)} \]
\[ = [Q_-Q^{(k)}, Q_+Q^{(l)}], \text{ for } k + l = n - 1, \]
\[ Q^2_+Q^{(n-1)} = Q^2_-Q^{(n-1)} = 0. \]  
(4.30)
We now prove this by induction. We have seen that \((4.30)\) holds for \(n = 3\), so now assume that it is true for \(n \leq N\), for some \(N\). Then
\[ Q^2_+Q^{(N)} = Q^2_+[Q_-Q^{(i)}, Q_+Q^{(j)}], \text{ for any } i + j = N - 1 \]
\[ = Q_+[Q^{(i+1)}, Q_+Q^{(j)}] \]
\[ = [Q_+Q^{(i+1)}, Q_+Q^{(j)}] \]
\[ = \frac{1}{2}Q^2_+[Q^{(i+1)}, Q^{(j)}] = 0, \]  
(4.31)
and similarly we can prove that \(Q^2_-Q^{(N)} = 0\). Let us now look at \(Q_+Q_-Q^{(N)}\) and \(Q_-Q_+Q^{(N)}\). We have
\[ Q_+Q_-Q^{(N)} = Q_+Q_-[Q_-Q^{(i)}, Q_+Q^{(j)}], \text{ for any } i + j = N - 1 \]
\[ = Q_+Q_-[Q_-Q^{(i)}, Q^{(j+1)}] \]
\[ = [Q^{(i+1)}, Q^{(j+1)}] + [Q_-Q^{(i)}, Q_+Q^{(j+1)}] \]  
(4.32)
and

\[ Q_- Q_+ Q^{(N)} = Q_- Q_+ [Q_- Q^{(j)}, Q_+ Q^{(i)}] \]
\[ = Q_- [Q^{(j+1)}, Q_+ Q^{(i)}] \]
\[ = [Q^{(j+1)}, Q^{(i+1)}] + [Q_- Q^{(j+1)}, Q_+ Q^{(i)}]. \] (4.33)

Hence

\[ [Q^{(1)}, Q^{(N)}] = Q_+ Q_- Q^{(N)} - Q_- Q_+ Q^{(N)} \]
\[ = 2[Q^{(i+1)}, Q^{(j+1)}] + [Q_- Q^{(i)}, Q_+ Q^{(j+1)}] + [Q_+ Q^{(i)}, Q_- Q^{(j+1)}] \]
\[ = 2[Q^{(i+1)}, Q^{(j+1)}] + Q_+ Q_- [Q^{(i)}, Q^{(j+1)}] - [Q^{(i+1)}, Q^{(j+1)}] - [Q^{(i)}, Q^{(j+2)}] \]
\[ = [Q^{(i+1)}, Q^{(j+1)}] - [Q^{(i)}, Q^{(j+2)}]. \] (4.34)

This final expression can be solved for \([Q^{(k)}, Q^{(N+1-k)}]\) in terms of \([Q^{(1)}, Q^{(N)}]\), giving

\[ [Q^{(k)}, Q^{(N+1-k)}] = k [Q^{(1)}, Q^{(N)}]. \] (4.35)

In particular this implies \([Q^{(N)}, Q^{(1)}] = N [Q^{(1)}, Q^{(N)}]\), and so \([Q^{(1)}, Q^{(N)}]\) must vanish. This in turn implies that \([Q^{(k)}, Q^{(N+1-k)}]\) is zero. It then follows that \(Q^{(N+1)} = [Q_- Q^{(i)}, Q_+ Q^{(j)}]\) for any \(i\) and \(j\) such that \(i + j = N\). This completes the induction process. We conclude that \([Q^{(i)}, Q^{(j)}] = 0\) for all \(i\) and \(j\).

It is interesting to ask to what perturbation of a conformal field theory with \(c = 13 - 6n - 6/n\) the commuting charges found in this paper correspond. This amounts to finding an operator that commutes with these charges. The charges constructed from the even spin currents are differential polynomials in \(T\) and so will commute with the screening charges \(\oint \exp \alpha_\pm \phi\). They also commute with \(\oint \exp -\alpha_\pm \phi\), on account of being even polynomials in \(\phi\). The charges for the odd spin currents, however, are not constructed from \(T\) alone, and so will not commute with the above charges in general. Nevertheless, since they are constructed from \(\oint \exp \alpha_+ \phi\) and \(\oint -\exp \alpha_+ \phi\), and since \(\alpha_+ \alpha_- = -2\) implies that both of these charges commute with \(\oint \exp \pm \alpha_- \phi\), all the charges we have found commute with
$\exp \alpha \phi$ and $\exp -\alpha \phi$. While the former is the screening charge, the latter is the field $\phi_{(1,3)}$, which has weight $2/n - 1$. Thus the commuting charges are conserved in the presence of the $\phi_{(1,3)}$ perturbation. These remarks are in agreement with those of Eguchi and Yang [10], who considered the quantum sine-Gordon theory with Hamiltonian $\oint e^{\alpha - \phi} - e^{-\alpha - \phi}$. They observed that for particular values of the sine-Gordon coupling constant extra odd-spin conserved currents existed for spins $2n - 1 \mod (2n - 2)$. The detailed form of their charges is different from ours, however, and it is not possible to write them in terms of $J$ alone. Their first charge, for example, is given in our notation by $Q_+ - Q_-$. 

We now turn to the connection between the series of commuting charges and certain $\mathcal{W}$-algebras. We explained in the previous section that the first series of charges was related to Zamolodchikov’s $\mathcal{W}(3)$-algebra with the central charge taking the value $c = -2$, and that for this value of $c$ the enveloping algebra of the $\mathcal{W}(3)$-algebra contains a linear subalgebra which is just $\mathcal{W}_\infty$. In order to make a connection between $\mathcal{W}$-algebras and the higher series, there was another aspect of the algebras that played an important role in our proof of the commutativity of the charges constructed from the odd-spin currents. This was the fact that we had not just a single primary field of conformal weight $2n - 1$ for $c = 13 - 6n - 6/n$, but in fact we made use of three distinct primary fields having this dimension. In our free-field representation, these were $\exp (-\alpha_+ \phi)$, $Q_+ \exp (-\alpha_+ \phi)$ and $Q_+^2 \exp (-\alpha_+ \phi)$. It is known that for the values of $c$ that we are considering there exist $\mathcal{W}$-algebras generated by the stress tensor and three primary fields each having spin $2n - 1$ [20]. Let us denote these algebras by $\mathcal{W}((2n - 1)^3)$. If we denote the primary fields by $W^i(z)$, for $i = 1, 2$ and 3, they have operator product expansions of the form

$$[W^i][W^j] = \delta^i_j [I] + \epsilon^{ijk}[W^k],$$

(4.36)

so that there is an $\text{SU}(2)$-like structure present. The operator $Q_+$ can be considered as an $\text{SU}(2)$ raising operator. We have seen that acting repeatedly with $Q_-$ on the multiplet of primary fields gives other spin-1 $\text{SU}(2)$ multiplets of higher conformal weight. The fields in these higher multiplets are no longer primary, but their integrals give rise to the infinite
sets of commuting charges we have found.

5 Discussion

One of the most interesting aspects of the results we have found is the presence of higher symmetry algebras underlying the series of commuting charges, namely $\mathcal{W}((2n - 1)^3)$ for the series $S_n$. This symmetry played an important role in our proof of the commutativity of the odd spin currents, and in a sense it controlled the structure of the commuting set.

For the first series of charges, we found that the $\mathcal{W}(3)$ algebra could be extended to $\mathcal{W}_\infty$. The commuting quantities could then be thought of as some sort of Cartan subalgebra of $\mathcal{W}_\infty$. One may hope that for the higher series $S_n$, the algebra $\mathcal{W}(2n - 1)$ will have a similar extension to an infinite-dimensional linear algebra and the same phenomenon occur.

One might also expect other groups to play a role in understanding the commuting currents. A large class of solutions of the KP hierarchy and its reductions have been formulated using the Grassmanian approach of Sato and of Segal and Wilson—see for example [15, 29]. The KdV hierarchy is associated to the group affine $\mathfrak{U}(2)$, and the Grassmanian is the coset space of this group divided by $\mathfrak{U}(2)$. To each point of the Grassmanian is associated the Baker function, and the logarithm of this function contains as coefficients in its power series all of the conserved quantities. Furthermore, it is possible to view the commuting charges as the elements $J_n^3$, $n > 0$, of an $\mathfrak{SU}(2)$ Kac-Moody algebra. It is natural to ask how one might generalize the construction of conserved quantities via the Baker function to the quantum case, and to suppose that a quantum analogue of affine $\mathfrak{U}(2)$ will play a role.

The general form of the even spin currents has so far not been found explicitly except
for $c = -2$. However, the form of the currents in terms of $J$ given in eqn (2.12) strongly suggests that a closed form expression can be found. Such an expression would be likely to involve the factors $\exp \alpha \pm \phi$ and $\exp -\alpha \pm \phi$. Indeed, one way to obtain quantities commuting with $\exp \pm \alpha \phi$ is to exploit the identity $(\exp \pm \alpha \phi)^n = 0$ [30], where $\alpha^2 = 2/n$, as a consequence of which $(\exp \pm \alpha \phi)^{n-1}X$ will commute with $\exp \pm \alpha \phi$ for any $X$. For the first series $n = 2$, and it can be checked that in this case the currents can be written as $\oint dw : e^{\alpha - \phi(w)} :: e^{\alpha + \phi(z)} \partial^m e^{\alpha - \phi(z)} :$. Unfortunately we do not know how to generalise this formula to the higher series.

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