Complexity of computing the anti-Ramsey numbers

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Abstract

The anti-Ramsey numbers are a fundamental notion in graph theory, introduced in 1978, by Erdős, Simonovits and Sós. For given graphs G and H the anti-Ramsey number ar(G, H) is defined to be the maximum number k such that there exists an assignment of k colors to the edges of G in which every copy of H in G has at least two edges with the same color. Precolored version of the problem is defined in a similar way except that the input graph is given with some fixed colors on some of the edges.

Usually, combinatorists study extremal values of anti-Ramsey numbers for various classes of graphs. In this paper we study the complexity of computing the anti-Ramsey number ar(G, Pk), where Pk is a path of length k. First We show that computing the ar(G, P3) is hard to approximate to a factor of n\(^{-1/2-\epsilon}\) even in 3-partite graphs, unless NP = ZPP. We observe the hardness of the problem when k is not fixed and we study the exact complexity of precolored version and show that there is no subexponential algorithm for the problem unless ETH fails already for k = 3. On the positive side we provide polynomial time algorithm for trees and we show approximability of the problem on special classes of graphs.
1 Introduction

For given graphs $G$ and $H$ the anti-Ramsey number $ar(G, H)$ is defined to be the maximum number $k$ such that there exists an assignment of $k$ colors to the edges of $G$ in which every copy of $H$ in $G$ has at least two edges with the same color. Classically, the graph $G$ is a large complete graph or complete bipartite graph and the graph $H$ is from a particular graph class.

The study of anti-Ramsey numbers was initiated by Erdős, Simonovits and Sós in 1975 [10]. Since then, there is a large number of papers on the topic. There are papers that study the case when $G = K_n$ and $H$ is a: cycle, e.g., [10] 28 5 24, e.g., tree [23] 22, e.g., clique [14] 10 6, e.g., matching [31] 8 17 26 and others, e.g., [10] 21 4.

The anti-Ramsey numbers are connected with the rainbow number $rb(G, H)$ which is defined as the minimum number $k$ such that in any coloring of the edges of $G$ with $k$ colors, there exists a rainbow copy of $H$. Thus, $ar(G, H) = rb(G, H) + 1$. We call a coloring without rainbow copy of $H$, a $H$-free coloring.

Various combinatorial work studied the case when $H$ is a path or a cycle, for instance the work of Simonovits and Sos [33] shows that there exists a constant $c$ such that for sufficiently long path $ar(K_n, P_t) \in O(t \cdot n)$, the combinatorial analysis of the problem is extremely difficult when instead of $K_n$ we use an arbitrary graph as the host graph. For a more detailed exposition of the combinatorial results on anti-Ramsey numbers, we refer the reader to the following surveys: [32] [15].

Besides the extremal results, the anti-Ramsey numbers have been studied from the computational point of view in several papers. The anti-Ramsey numbers when $G$ is an arbitrary graph was studied for the case when $H$ is a star. The problem was introduced by Feng et al. [12] [11] [13], motivated by applications in wireless mesh networks and was termed the maximum edge $q$-coloring. The maximum edge $q$-coloring models interference in a new type of wireless mesh network where each computer has $q$ interface cards. Thus, the nodes of the graph correspond to computers, the edges with the communication links and the colors with the frequencies on which two computers communicate (see more details in [12] [11] [13]).

They provide a 2-approximation algorithm for $q = 2$ and a $(1 + \frac{4q-2}{3q^2-5q+2})$-approximation for $q > 2$. They show that the problem is solvable in polynomial time for trees and complete graphs in the case $q = 2$, later, Adamszeck and Popa [1] show that the problem is APX-hard and present a $5/3$-approximation algorithm for graphs which have a perfect matching. For more results related to the maximum edge $q$-coloring, the reader can refer to [2].

We study the complexity of the problem on paths. In [7], Bujtas et al. study a similar problem to $P_3$-free coloring. They named it the 3-consecutive edge coloring of a graph. In this problem we are required to color the edges
of the graph with the largest number of colors such that for any three consecutive edges \( e_1, e_2, e_3 \) (i.e., a path of length 3 or a triangle) it holds that \( e_1 \) and \( e_2 \) have the same color or \( e_2 \) and \( e_3 \) have the same color. Notice that, although the definitions of two problems seem similar, they are actually different even for a simple case of a triangle. Their problem has a very tight relation to the stable cut problem so they obtained hardness result on deciding whether the 3-consecutive coloring number of the graph is 1 or 2. They also provide an algorithm for trees.

Our results. Our general goal is to develop the understanding of the anti-Ramsey numbers by focusing on the computational complexity of the problem. It is intriguing that the problem has been studied extensively in the combinatorics community, while in the algorithms community the problem has been considered only recently. Due to its practical applications in networking [29, 30, 12, 11, 13] and its success in combinatorics community, the problem is interesting to study from the computational point of view.

In this paper, we study the problem for the case when \( H \) is a path and \( G \) is either an arbitrary graph or a restricted class of graphs such as trees or bipartite graphs. We obtain both algorithmic and hardness of approximation results.

1. We prove the inapproximability of \( ar(G, P_3) \) by factor \( n^{-1/2-\epsilon} \) via a reduction from the maximum independent set problem even on 3-partite graphs. Our inapproximability holds under the condition \( NP \neq ZPP \) — similarly it works under \( P \neq NP \) with a slightly worse factor. The key of the reduction is to analyze how do the edges of the graph influence each other. Thus, we also provide a better understanding on the global behavior of the problem.

2. We show a slight variant of the \( P_3 \)-free coloring problem, namely, Precolored \( P_3 \)-free coloring, does not admit an exact algorithm with running time \( 2^{o(|E(G)|)} \) assuming ETH. We obtain this with a fine grained reduction from 3-SAT. To provide the desirable gadgets and wire them together we introduce some new ideas.

3. On the positive side, we provide a general algorithmic idea which we call it color connected coloring which for instance yields in linear time algorithm for trees. The known combinatorial results for cycles of length three on outerplanar graphs [16] and algorithm for trees for 3 consecutive coloring of [17] are closest works to ours. Our algorithm is completely independent of the latter, however, at the end of Section 5 we will see that the two problems are essentially the same when the host graph is a tree. Color connectedness property combined with some other attributes could lead to the design of exact algorithms with running time \( 2^{O(n)} \), matching the lower bound of general graphs.
The paper is organized as follows. In Section 2 we introduce preliminaries and prove several useful lemmas. Then, we prove the hardness of inapproximability results for $P_3$-free coloring in Section 3 and in Section 4 we show the exact complexity result for Precolored $P_3$-free coloring. In Section 5 we show exact polynomial time algorithms for trees and approximation algorithms for some other classes of graphs. Finally, in Section 6 we present directions for future work.

## 2 Preliminaries and useful lemmas

We use $\mathbb{N}$ to denote the set of natural numbers and we write $[n]$ to denote the set $\{1, \ldots, n\}$. We refer the reader to [9] for basic notions related to graph theory. All the graphs considered in this paper are simple and undirected. Let $G$ be a graph, we write $V(G)$ for its vertices and $E(G)$ for its edges. For any vertex $v \in V(G)$ we define $N(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ to be the open neighborhood of $v$, and $N[v] = N(v) \cup \{v\}$ as its closed neighborhood. Similarly for any subset of vertices $A \subseteq V(G)$ we define $N(A) = \bigcup_{v \in A} N(v)$, and $N[A] = \bigcup_{v \in A} N[v]$. For $k \in \mathbb{N}^+$ we denote by $P_k$ a path with $k + 1$ vertices. The length of $P_k$ is $k$. Also let $p$ be a $P_k$, depending on the context we may write $p = (e_1, \ldots, e_k)$ where $e_i \in E(p)$ or $p = (v_1, \ldots, v_{k+1})$ where $v_i \in V(p)$ to describe a path. For two vertices $u, v \in V(G)$ we denote by $\text{dist}(u, v)$ the length of a shortest path connecting the two. Also, we denote by $P_{uv}$ such a shortest path. We define the distance of two subgraphs $S_1, S_2$ of $G$, denoted by $\text{dist}(S_1, S_2)$, as the minimum $\text{dist}(u, v)$ where $u \in V(S_1)$ and $v \in V(S_2)$.

**Definition 1** (Coloring). Given an undirected graph $G = (V, E)$, the coloring of the edges of $G$ is a function $c : E \to \mathbb{N}$. Similarly for any subset $A \subseteq E$ we define $c(A) = \bigcup_{e \in A} c(e)$.

We call a coloring of the edges of a graph $G$ a rainbow coloring if for every pair of edges $e \neq e' \in E$ we have $c(e) \neq c(e')$. Let $G, H$ be two graphs, an edge coloring $c$ of $G$ is $H$-free coloring if there is no rainbow subgraph of $G$ isomorphic to $H$. We denote the number of distinct colors used in $c$ by $c_{G,H}$. Let $C$ be the set of all $H$-free colorings of $G$. The anti-$H$ number of $G$ is $\text{ar}(G, H) = \max_{c \in C} c_{G,H}$.

As we already discussed, in this paper we will be focusing on paths and in particular path of length three, thus in the following we present upper bounds on the number of colors when the graph $H$ is a path.

**Lemma 2.** In any $P_3$-free coloring of $G$ there are at most $|V(G)|$ distinct colors.

**Proof.** Let $c$ be a $P_3$-free coloring of $G$ with maximum number of distinct colors and let $G' \subseteq G$ be an edge minimal subgraph of $G$ which is colored
by \(ar(G, P_3)\) distinct colors w.r.t. \(c\) and let \(G_1, G_2, \ldots, G_k \subseteq G'\) be the components of \(G'\). Each \(G_i, i \in [k]\), is rainbow colored otherwise it contradicts to the edge minimality condition of our choice of \(G'\).

We prove that the number of edges in each \(G_i\) is at most \(|V(G_i)|\) and thus, the lemma follows. In particular, we prove that for all \(t \in [k]\), it holds that \(G_t\) is either a star or a triangle.

Fix \(t\) and let \(v\) be a vertex of maximum degree in \(G_t\). Let \(v_1, \ldots, v_{|N(v)|}\) be neighbors of \(v\). If \(|N(v)| = 1\) then \(G_t\) is a star.

If \(|N(v)| = 2\), then \(G_t\) is a star, otherwise there exists an edge \(e = \{v_1, u\}\) or \(e = \{v_2, u\}\). Assume w.l.o.g. that \(e = \{v_1, u\}\). If \(u = v_2\), it’s a triangle.

Otherwise we have path of length 3: \((v_2, v, v_1, u)\).

If \(|N(v)| \geq 3\), then \(G_t\) is star. Otherwise, there are two possibilities:

a) there is an edge \(e = \{v_i, u\}\) \((i, j \in [|N(v)|], i \neq j)\) and we have \(P_3(v_j, v_i, v)\) \((z \in [|N(v)|], z \neq j, i)\); or b) there is an edge \(e' = \{v_i, u\}\) \((i \in [|N(v)|])\) and we have \(P_3(u, v_i, v)\) \((z \in [|N(v)|], z \neq i)\).

If \(G_t\) is a star then \(|E(G_t)| + 1 = |V(G_t)|\). If \(G_t\) is a triangle then \(|E(G_t)| = |V(G_t)| = 3\).

Thus, the lemma follows.

\[\square\]

The following lemma is the generalization of the previous lemma with a slightly weaker upperbound.

**Lemma 3.** \(ar(G, P_k) \in O(k \log k|V(G)|)\).

*Proof.* Similar to the previous lemma let \(c\) be a \(P_k\)-free coloring of \(G\) with the maximum number of colors, we take the maximum size set of edges of distinct colors w.r.t. \(c\). The resulting graph has no \(P_k\) as a subgraph and hence it does not have any \(P_k\) as a minor so by Mader’s theorem [27] [9] it does have at most \(c_k|V(G)|\) edges where \(c_k \in O(k \log k)\).

Note that the proof of above lemma does not work for arbitrary \(H\). For instance, if a graph does not have a cycle of length \(k\) \((C_k)\), then it might have a minor of \(C_k\) and hence we cannot apply Mader’s theorem.

Next lemma shows that if \(k\) is part of the input, then the problem of computing \(ar(G, P_k)\) is at least as hard as finding a Hamiltonian path in the graph.

**Lemma 4.** Let \(k \in \mathbb{N}\) and \(G\) be a graph then computing \(ar(G, P_k)\) is NP-hard.

*Proof.* For a graph \(G\) with \(n + 1\) vertices and \(m\) edges let suppose \(k = n\). Then \(ar(G, P_n) = m\) if and only if \(G\) does not have a Hamiltonian Path. Thus, if we can compute the \(ar(G, P_n)\) in polynomial time, then we can solve the Hamiltonian Path in polynomial time.

\[\square\]
Fine-grained complexity

In the classical complexity setting the goal is to group the problems in broad classes such as polynomial, NP-hard, FPT and so on. In the fine-grained complexity our goal is to do a more precise classification of algorithms according to their running times. One of the main tools to prove hardness results in this setting is the Exponential Time Hypothesis (ETH) introduced by Impagliazzo and Paturi [19]. The ETH states that there exists $\delta > 0$ such that there is no algorithm that solves 3-SAT in $O(2^{\delta n})$.

3 Inapproximability of $P_3$ Anti-Ramesy Coloring

In this section we show that for every $\varepsilon > 0$ there is no polynomial time $\frac{1}{\sqrt{|V(G)|}}$-approximation for $P_3$-free coloring unless $NP = ZPP$ [18], or similarly there is no polynomial $\frac{1}{\sqrt{|V(G)|}}$-approximation to estimate $ar(G, P_3)$ unless $P = NP$. We prove our hardness result via a gap preserving reduction from the maximum independent set problem.

Given an instance of the maximum independent set, i.e. an undirected graph $G$, we construct a graph $G'$ as follows:

1. For each $v \in V(G)$ we introduce two new vertices $s_v, t_v \in V(G')$ and $6|V(G)|$ internally disjoint paths of length 2, $\mathcal{P}^v = \{P^v_1, \ldots, P^v_6|V(G)|\}$, connecting $s_v$ to $t_v$.

2. For each edge $\{v, u\} \in E(G)$, add 2 new edges in $E(G')$: $\{s_v, t_u\}, \{t_v, s_u\}$. We call this set of edges $E^v_t$.

We abuse a notation and say an edge coloring is valid if it is a $P_3$-free coloring. We start by providing some lemmas and observations on the structure of valid colorings of $G'$.

**Lemma 5.** In any $P_3$-free coloring of $G'$ the set of edges $E^v_t$ will receive at most $2|V(G)|$ distinct colors.

**Proof.** The subgraph of $G'$ induced on endpoints of edges in $E^v_t$ has exactly $2|V(G)|$ edges hence lemma follows from Lemma 2 $\square$

**Observation 6.** If $G$ is a cycle of length 4 then $ar(G, P_3) = 2$.

**Lemma 7.** There is not any valid coloring of $G'$ with more than $6|V(G)|$ colors in one $\mathcal{P}^v$ for $v \in V(G)$.

**Proof.** For the sake of contradiction suppose there is a valid coloring of $G'$ so that $\mathcal{P}^v$ is colored with more than $6|V(G)|$ colors. At least one $P^v_i$ has two edges with distinct colors $c_1$ and $c_2$. By Observation 6 all other edges in $\mathcal{P}^v$ should be colored either with $c_1$ or $c_2$ contradicting that $\mathcal{P}^v$ has $6|V(G)|$ colors. $\square$
Lemma 8. Let \( \{v,u\} \in E(G) \). In any \( P_3 \)-free coloring of \( G' \) if there are at least 3 distinct colors in \( P^u \) then \( P^u \) is colored with at most 2 colors.

Proof. Firstly, we claim that if \( P^u \) is colored by at least 3 distinct colors then \( s_v \) and \( t_v \) are incident to three edges with distinct colors. Assume the contrary, then w.l.o.g. \( s_v \) is incident to two edges \( e_1 = \{s_v, w_1\}, e_2 = \{s_v, w_2\} \) of distinct colors and then there is an edge \( e_3 = \{t_v, w_3\} \) incident to \( t_v \) with \( c(e_1) \neq c(e_2) \neq c(e_3) \). Then, it holds that \( w_3 \neq w_2 \) as otherwise we get a rainbow colored path \((t_v, w_2, s_v, w_1)\), similarly \( w_3 \neq w_1 \). Then consider an edge \( e' = \{w_3, s_v\} \). We show that \( c(e') = c(e_3) \) and thus we obtain a contradiction.

Assume that we have \( c(e') \in \{c(e_1), c(e_2)\} \). Then if \( c(e') = c(e_1) \) (or \( c(e') = c(e_2) \)) the path \((t_v, w_3, s_v, w_2)\) (or \((t_v, w_3, s_v, w_1)\)) is a rainbow colored path, hence \( c(e') = c(e_3) \) thus there are 3 edges of distinct colors incident to \( s_v \) and follows there are at least 3 edges of distinct colors incident to \( t_v \).

Now suppose \( P^u \) has at least 3 distinct colors then similarly both of its ends \((s_u, t_u)\) are incident to 3 edges of distinct colors but those edges with edge \( e = \{s_u, t_u\} \) (or \{(s_u, t_u)\}) and 3 edges of distinct colors incident to \( s_u, t_v \) will result in a rainbow path of length 3.

Observation 9. For each \( v \in V(G) \), there is a valid coloring so that \( P^v \) colored with \( 6|V(G)| \) distinct colors.

Proof. \( P^v = \{P^v_1, \ldots, P^v_{6|V(G)|}\} \). Assign color \( i \) to the edges of \( P^v_i \) for \( i \in [6|V(G)|] \). This is a valid coloring as any \( P_3 \) contains one of the \( P^v_i \)'s.

Let \( I \) be a maximum independent set of \( G \).

Lemma 10. \( ar(G', P_3) > 6|V(G)| \cdot |I| \)

Proof. For \( v \in I \) color \( P^v \) with \( 6|V(G)| \) different colors and all other edges of \( G' \) with the same color \( c_0 \). By Observation 8 there is no rainbow colored \( P_3 \) in \( P^v \) for all \( v \in I \) and all other \( P_3 \)'s have at least 2 edges with color \( c_0 \) or they contain exactly one of the \( P^v_i \)'s.

Theorem 11. Unless \( NP = ZPP \), for any fixed \( \varepsilon > 0 \), there is no polynomial time \( \frac{1}{\sqrt{|V(G)|}} \) -approximation for \( P_3 \)-free coloring even in 3-partite graphs.

Proof. First of all note that the graph \( G' \) constructed above is a 3-partite graph: put every \( s_v \) for \( v \in V(G) \) in part 1, every \( t_v \) in part 2 and every other vertex in part 3. We provide a gap preserving reduction from independent set problem. More precisely we know there is no polynomial time \( \frac{1}{\sqrt{|V(G)|}} \) approximation for MIS for any fixed \( \varepsilon > 0 \) [18] unless \( NP = ZPP \). We show that if there is a \( \frac{1}{\sqrt{|V(G)|}} \) -approximation for \( P_3 \)-free coloring (for any
constant \( \epsilon' \) then there is a \( \frac{1}{|V(G)|^{1-\epsilon'}} \)-approximation for MIS in polynomial time.

The graph \( G' \) has \( 6|V(G)|^2 + 2|V(G)| \) vertices. Hence \( \frac{1}{\sqrt{|V(G)|}} \in O(1) \), so we prove there is no \( \frac{O(1)}{|V(G)|^{1-\epsilon'}} \) approximation of \( ar(G', P_3) \) unless there is such an approximation for MIS in \( G \).

By Lemma 10 we know that we have at least
\[
\left\lceil \frac{1}{|V(G)|^{1-\epsilon}} \cdot |I| \cdot 6|V(G)| \right\rceil
\]
colors.

Let \( X = \{v \in G \mid \mathcal{P}^v \text{ has more than 3 colors} \} \). By Lemma 8 we know that \( X \) is an independent set.

In the rest of proof we show \( |X| \geq \left\lceil \frac{1}{|V(G)|^{1-\epsilon}} \cdot |I| \right\rceil \). Before completing the proof note that given the coloring, it is easy to obtain the set \( X \) in polynomial time. Hence, proving this claim implies that we find a large independent set \( X \), contradicting the known inapproximability of the independent set problem in general graphs. So the size of set \( X \) is less than \( \sqrt{|V(G')|^{1-\epsilon'}} \) for any \( \epsilon' > 0 \), hence the number of colors used in a coloring cannot be in \( O\left( \frac{1}{\sqrt{|V(G)|^{1-\epsilon'}}} ar(G', P_3) \right) \) for any constant \( \epsilon' \).

To aim contradiction assume \( |X| < \left\lceil \frac{1}{|V(G)|^{1-\epsilon}} \cdot |I| \right\rceil \). We calculate the maximum number of colors and prove that it is less than \( \left\lceil \frac{1}{|V(G)|^{1-\epsilon}} \cdot |I| \cdot 6|V(G)| \right\rceil \) (a contradiction). By Observation 7 we have at most \( 6|V(G)| \cdot |X| \) colors for \( \mathcal{P}^v \)'s in \( X \), \( 2(|V(G)| - |X|) \) colors for other \( \mathcal{P}^v \)'s and \( 2|V(G)| \) for the remaining edges by Lemma 5. So we have:

\[
6|V(G)| \cdot |X| + 2(|V(G)| - |X|) + 2|V(G)| < \\
6|V(G)| \cdot (|X| + 1) \leq \\
\left\lceil \frac{1}{|V(G)|^{1-\epsilon}} \cdot |I| \cdot 6|V(G)| \right\rceil.
\]

\( \square \)

4 Precoloring \( ar(G, P_k) \) has no subexponential algorithm already for \( k = 3 \)

In this section we study the complexity of exact algorithms computing the anti-Ramsey number \( ar(G, P_k) \) where \( P_k \) is a path with \( k \) edges. For any connected graph \( ar(G, P_2) \) is always 1 as we cannot color two consecutive edges of \( G \) with different colors. We now consider a variant of problem for exact time complexity of the problem.
Problem 12 (Precolored \( ar(G, H) \)). The input consists of a graph \( G = (V, E) \) where \( E = E_1 \cup E_2 \). The edges in \( E_1 \) have assigned a color while the edges in \( E_2 \) are uncolored. The goal is to color the edges in \( E_2 \) with as many colors as possible such that there is no rainbow copy of \( H \) in \( G \).

In the following we provide a fine grained reduction from 3SAT to show hardness of the problem. That is we provide an instance of Precolored \( ar(G, P_3) \) problem, i.e., a graph \( G \) where some of the edges are precolored, that asymptotically has a same size as the instance of the 3SAT problem, hence if there is a \( 2^{o(|E|)} \) algorithm to compute precolored \( ar(G, P_3) \) then there is a subexponential algorithm to solve 3SAT problem and this is impossible unless ETH fails.

Lemma 13. The Precolored \( ar(G, P_3) \) is NP-hard.

Proof. We show the hardness using a reduction from the 3-SAT problem. Given a Boolean formula \( \phi \) with \( n \) variables and \( m \) clauses, we create a graph \( G = (V, E) \) as follows. To simplify the understanding we abuse a notation and color some edges with colors \( T \) or \( F \) — one may assume \( T, F \) are two distinct integers.

- For each variable \( X_i \in \phi \) we create two vertices in \( V \), namely \( x_i \) and \( \bar{x}_i \) as corresponding literals of \( X_i \). Moreover, we add the edge \( (x_i, \bar{x}_i) \in E \) and we do not precolor it. In the next step we construct clause gadgets and connect them to the literal gadgets.
- For a clause \( C_i = (z_1 \lor z_2 \lor z_3) \) we distinguish two cases:
  - Either all literals are negated variables, or all are non-negated. In this case, we add vertices \( c_0, c_1, c_2, c_3 \in V \) and we add edges \( \{c_0, c_1\}, \{c_0, c_2\}, \{c_0, c_3\} \in E \) which are not precolored. Then we add edges \( \{c_1, z_1\}, \{c_2, z_2\}, \{c_3, z_3\} \in E \), which are precolored.

Figure 1: The left part shows that how the gadget is constructed for the case when precisely one of the literals in a clause are negations of a variable. A color of a node shows the final graph is a bipartite graph. In the right side a valid color assignment is shown in the presence of a satisfiable clause.
with $T$, if $z_i$’s are non-negated variables or with $F$ if $z_i$’s are negated variables.

- Two of the literals are either non-negated or negated. Assume without loss of generality that $z_1$ and $z_2$ are both variables or both negations of variables. Then we add vertices $c_0', c_1', c_2, c_3, \in V$ and we add edges $\{c_0, c_1\}, \{c_0, c_2\}, \{c_0', c_3\}, \{c_0, c_0'\} \in E$ which are not precolored. Then we add edges $\{c_1, z_1\}, \{c_2, z_2\}, \{c_3, z_3\} \in E$. The edge $\{c_i, z_i\}$ is precolored with $T$, if $z_i$ is a positive instance of a variable and it is precolored with $F$ if $z_i$ is the negation of a variable.

See Figure 1 for sketch of construction of the gadgets.

W.l.o.g., we assume that for every $i \in [n]$ both variable $x_i$ and its negation appear in some clauses as literals. Otherwise, if a variable appears only negated or non-negated, we can simply satisfy all the clauses that contain that variable. The above assumption enforces any valid coloring to color $\{x_i, \bar{x}_i\}$ by either $T$ or $F$.

We claim that the formula $\phi$ is satisfiable if and only if $ar(G, P_3) = m + 2$, that is there is a coloring of the edges of $G$ with $m + 2$ colors ($T,F$ and another new $m$ colors, one for each clause).

For the direct implication, if the formula $\phi$ is satisfiable we color the edges of $G$ as follows. For each variable $x_i$, if $x_i$ is assigned to True, then we color the edge $\{x_i, \bar{x}_i\}$ with $T$, otherwise we color this edge with $F$.

Let $C_i = (z_1 \lor z_2 \lor z_3)$. Assume without loss of generality that $C_i$ is satisfied by the literal $z_1$. Then we color the edge $\{c_0, c_1\}$ with a new color (or the edge $\{c_0', c_1\}$). Then, if $z_1$ corresponds to a negation of a variable, we color all the other edges in the clause gadget with $F$. Otherwise, we color all the other edges with $T$.

Now we show that the coloring is valid. We have two cases:

1. All the literals in a clause are either negated variables or non-negated variables. Assume w.l.o.g., that $C_i$ is satisfied by $z_1$. Then since $\{c_0, c_2\}$ and $\{c_2, z_2\}$ have the same color, then any $P_2$ containing $(c_0, c_2, z_2)$ is not rainbow. Thus, the only path that we have to check is $(c_0, c_1, z_1, \bar{z}_1)$. This path is not rainbow since the edge $\{c_1, z_1\}$, has the same color as the edge $\{z_1, \bar{z}_1\}$.

2. Two of the literals in a clause are false and the other is true, or vice-versa. The same argument as in the previous case holds. We distinguish two cases. First, assume that the clause is satisfied by the literal that is different from the other two, i.e. $z_3$. Observe that the color of the edge $\{c_0', c_0\}$ is the same as the color of the edges $\{c_0, c_1\}$ and $\{c_0, c_2\}$. Thus, all the paths containing $(c_1, c_0, c_0')$ and $(c_1, c_0, c_2)$ are not rainbow. The only path in the clause gadget left to check is $(z_3, c_3, c_0', c_0)$, which is
not rainbow since from our assignment \(\{z_3, c_3\}\) has the same color as \(\{c_0, c'_0\}\).

In the second case, the clause is satisfied by \(z_1\) or \(z_2\). Assume, w.l.o.g., that \(z_1\) satisfies the clause and that is not the negation of a variable.

Then, similarly as in the previous case, observe that the color of the edge \(\{c'_0, c_0\}\) is the same as the color of the edges \(\{c_0, c_2\}\) and \(\{c_0, c_3\}\).

Thus, all the paths containing \(\{c_3, c_0, c'_0\}\) and \(\{c_2, c_0, c'_0\}\) are not rainbow. The only two paths in the clause gadget left to check is \(\{z_1, c_1, c_0, c'_0\}\) and \(\{z_1, c_1, c_0, c_2\}\), which are not rainbow since from our assignment \(\{z_1, c_1\}\) has the same color as \(\{c_0, c'_0\}\) and \(\{c_0, c_2\}\).

For the reverse implication, assume that we are given a coloring of \(G\) with \(m + 2\) colors. We show how to recover a satisfying assignment for \(\phi\).

First of all, notice that in a clause gadget we can add at most one new color. Assume, for the sake of contradiction that there are two edges from a clause gadget that have two distinct new colors. If these two edges are incident, then they form a rainbow path with another incident edge since the edges \(\{c_1, z_i\}\) have either color \(T\) or \(F\). If these two edges are not incident (e.g., \(\{c_0, z_1\}, \{c'_0, z_3\}\)) then either the edge incident between them has a different color as well, and thus we have a rainbow path, or the edges between them has one of the two new colors and then, we have the contradiction form the previous case.

Thus, we achieve the satisfying assignment as follows. If the edge \(\{x_i, \bar{x}_i\}\) is set to \(T\) then we set \(x_i\) to True, otherwise we set \(x_i\) to False.

Finally, we show that this is a satisfying assignment for \(\phi\). We have two cases:

1. If \(\{c_0, c_1\}\) or \(\{c'_0, c_1\}\) is the edge colored with the new color. Then the color of \(\{c_1, z_i\}\) is equal to the color of \(\{z_i, z_i\}\), since otherwise we would have a rainbow path. However, this implies that \(z_i\) satisfies \(C_i\).

2. The edge \(\{c_0, c'_0\}\) is colored with the new color. Assume by contradiction that the clause is not satisfied. Then \(\{c_0, c_1\}, \{c_0, c_2\}, \{c'_0, c_3\}\) are colored with the same color as \(\{c_1, z_1\}, \{c_2, z_2\}, \{c_3, z_3\}\), respectively (otherwise we have a rainbow path). However, since \(\{c_1, z_1\}\) has a different color than \(\{c_3, z_3\}\) (this is how the clause was constructed), then \(\{c_1, c_0, c'_0, c_3\}\) is a rainbow path, leading to a contradiction.

\(\square\)

Given the above lemma and sparsification lemma we conclude the following theorem.

**Theorem 14.** There is no \(2^{o(|E(G)|)}\) algorithm for Precolored ar\((G, P_3)\) unless ETH fails.
Proof. We may assume the 3SAT instance used in the construction of the graph $G$ in the proof of Lemma 13 is sparse, that is the number of clauses $m$ is in order of number of variables $n$, i.e. $m \in O(n)$. Thus by sparsification lemma [20] there is no $2^{o(n)}$ algorithm to solve Precolored $ar(G, P_3)$ (unless ETH fails).

On the other hand in the construction of the graph $G$ for each variable we have one edge and for each clause we have at most 7 edges so in total the number of edges in the graph is bounded above by $7m + n$ hence $|E(G)| \in O(n)$ therefore there is no $2^{o(|E(G)|)}$ algorithm for Precolored $ar(G, P_3)$ unless ETH fails.

5 Color Connected Coloring and its Applications

In this section, we introduce an idea of color connected coloring and using that we provide a polynomial time algorithm to compute $ar(T, P_3)$, where $T$ is a tree. Roughly speaking in color connected coloring we try to color the graph with the maximum number of colors so that the set of edges of every color class induces a connected subgraph.

Our proof implicitly shows that the algorithm works for 3-consecutive coloring as well and more generally it shows that anti-Ramsey coloring and 3-consecutive coloring are essentially the same when the host graph is a tree. Algorithm of [7] is based on the observation that they have to find a stable separator that maximizes the number of remaining components. However, we need more ideas to arrive at that point.

Definition 15 (rainbow and monochrome vertices). Let $G$ be a graph with coloring $c$. Vertex $v$ is monochrome if all the edges incident to $v$ have the same color, i.e. $|c(\bigcup_{u \in N(v)} \{v, u\})| = 1$. Vertex $v$ is rainbow if edges incident to $v$ have distinct colors, i.e. $|c(\bigcup_{u \in N(v)} \{v, u\})| = |N(v)|$.

Definition 16 (Monochrome($v$), Rainbow($v$)). Let $T$ be a rooted tree and $v \in V(T)$. We define Monochrome($v$) as the maximum number of colors in the subtree rooted at $v$ if $v$ is monochrome. Similarly let Rainbow($v$) be the maximum number of colors in the subtree rooted at $v$ if $v$ is rainbow.

Theorem 17. Let $T$ be a rooted tree. Algorithm [7] computes $ar(T, P_3)$ in $O(|V(T)|)$.

Before proving Theorem 17 we introduce the main idea we use in this section and we need some definitions and lemmas to take advantage of it.

Let $c$ be a $P_3$-free coloring of a graph $G$ and let $c_1$ be one of such colors used in $c$. Then we call the induced graph $G[\{v \mid \exists u \in V(G), e = \{u, v\} \in E(G), c(e) = c_1\}]$ as an induced $c_1$-graph and we write it $G[c_1]$. If $G[c_1]$ is a connected component then we say $c_1$ is a connected color otherwise it is a disjoint color.
colors so that for every color $c$ coloring $c$ connected color. A graph $G$ coloring $c$ connected coloring if every color used in $G$ is a connected color. A graph $G$ is color connected coloring if there is a color connected coloring $c$ of $G$ with $ar(G, P_3)$ many distinct colors.

Let $c$ be a $P_3$-free coloring of $T$ with $ar(G, P_3)$ colors and minimum number of disjoint colors. If there is no disjoint color used in $c$ we are done, otherwise towards the contradiction let suppose for a color $c_1$ of $c$, $T[c_1] = \{T_1, \ldots, T_k\}$ for some $k > 1$. W.l.o.g. suppose $T_1$ is the first component of $T[c_1]$ one visits by preorder traversal of $T$ starting at its root. Then as $T$ is rooted we know that for $i > 1$ the root $r_i$ of every subtree $T_i$ has a parent and hence there is a parental edge $e_i = \{parentr_i, r_i\} \in E(T)$ and in addition to that we know $c(e_i) \neq c_1$. We recolor every component $T_i$ with a color $c(e_i)$ for $i > 1$. This clearly creates a new coloring $c'$ with the exact same set of colors used in $c$ however it has one less disjoint color contradicting to our assumption on $c$ so to complete the contradiction it is sufficient to show $c'$ is a $P_3$-free coloring.

Suppose $c'$ is not a $P_3$-free coloring and let $p = (e_1, e_2, e_3)$ be a rainbow $P_3$. Since $c$ was a $P_3$-free coloring and we only recolored $c_1$ colored edges, there are three cases:

- $c(e_1) = c(e_2) = c_1$. $e_1$ and $e_2$ are in the same component, therefore, after our update $c'(e_1) = c'(e_2)$, a contradiction.

- $c(e_2) = c(e_3) = c_1$. It is similar to the previous case.

- $c(e_1) = c(e_3) = c_1 \neq c(e_2)$. So $e_1$ and $e_3$ are in different components w.r.t. color classes, and $e_2$ is incident to both of them, hence $e_2$ should be the parental edge for one of them, therefore, we have either $c'(e_2) = c'(e_1)$ or $c'(e_2) = c'(e_3)$, a contradiction.

**Definition 18** (Color Connected Coloring). Given a graph $G$, a $P_3$-free coloring $c$ of $G$ is color connected coloring if every color used in $c$ is a connected color. A graph $G$ is color connected coloring if there is a color connected coloring $c$ of $G$ with $ar(G, P_3)$ many distinct colors.

**Lemma 19.** There is a $P_3$-free coloring of $T$ with $m \in [ar(T, P_3)]$ distinct colors so that for every color $c_i$ of $c$ the graph $T[c_i]$ is a connected graph.

**Proof.** Let $c$ be a $P_3$-free coloring of $T$ with $ar(G, P_3)$ colors and minimum number of disjoint colors. If there is no disjoint color used in $c$ we are done, otherwise towards the contradiction let suppose for a color $c_1$ of $c$, $T[c_1] = \{T_1, \ldots, T_k\}$ for some $k > 1$. W.l.o.g. suppose $T_1$ is the first component of $T[c_1]$ one visits by preorder traversal of $T$ starting at its root. Then as $T$ is rooted we know that for $i > 1$ the root $r_i$ of every subtree $T_i$ has a parent and hence there is a parental edge $e_i = \{parentr_i, r_i\} \in E(T)$ and in addition to that we know $c(e_i) \neq c_1$. We recolor every component $T_i$ with a color $c(e_i)$ for $i > 1$. This clearly creates a new coloring $c'$ with the exact same set of colors used in $c$ however it has one less disjoint color contradicting to our assumption on $c$ so to complete the contradiction it is sufficient to show $c'$ is a $P_3$-free coloring.

Suppose $c'$ is not a $P_3$-free coloring and let $p = (e_1, e_2, e_3)$ be a rainbow $P_3$. Since $c$ was a $P_3$-free coloring and we only recolored $c_1$ colored edges, there are three cases:

- $c(e_1) = c(e_2) = c_1$. $e_1$ and $e_2$ are in the same component, therefore, after our update $c'(e_1) = c'(e_2)$, a contradiction.

- $c(e_2) = c(e_3) = c_1$. It is similar to the previous case.

- $c(e_1) = c(e_3) = c_1 \neq c(e_2)$. So $e_1$ and $e_3$ are in different components w.r.t. color classes, and $e_2$ is incident to both of them, hence $e_2$ should be the parental edge for one of them, therefore, we have either $c'(e_2) = c'(e_1)$ or $c'(e_2) = c'(e_3)$, a contradiction.

**Algorithm 1:** Computing Anti-Ramsey Number in Trees (CARNIT)

1. **CARNIT**($T$, $v$, $P_3$)
2. **input:** Tree $T$ rooted at a vertex $v$
3. **output:** $ar(T, P_3)$
4. $\forall u \in N(v)$
5. **let** $T'$ be the subtree rooted at $u$
6. $\text{CARNIT}(T', u, P_3)$
7. $\text{Monochrome}(v) \leftarrow 1 + \Sigma_{u \in N(v)} \max\{\text{Monochrome}(u), \text{Rainbow}(u)\}$
8. $\text{Rainbow}(v) \leftarrow \Sigma_{u \in N(v)} \text{Monochrome}(u)$
9. **return** $\max\{\text{Rainbow}(v), \text{Monochrome}(v)\}$
Hence there is no rainbow $P_3$ with the new coloring scheme and the lemma follows.

**Lemma 20.** Let $T$ be a tree and $\{u, v\} \in E(T)$. Let $c$ be a color connected $P_3$-free coloring of $T$. Then at least one of $u, v$ is monochrome.

**Proof.** Proof is by contradiction. Assume that there is an edge $\{u, v\} \in E(T)$ where none of its endpoints are monochrome. There is a $x \in N(u)$ and a $y \in N(v)$ such that $c(\{u, x\}) \neq c(\{u, v\})$, $c(\{u, v\}) \neq c(\{v, y\})$. Since $c$ is a color connected coloring $c(\{u, x\}) \neq c(\{v, y\})$ which immediately yields a rainbow $P_3$, a contradiction.

**Lemma 21.** Let $T$ be a tree. There is a color connected $P_3$-free coloring $c$ of $T$ with $ar(T, P_3)$ colors such that for every vertex $v \in V(T)$, $v$ is either rainbow or monochrome.

**Proof.** By Lemma 19 there exists a color connected coloring with $ar(T, P_3)$ distinct colors. We show that any $v \in V(T)$ is rainbow or monochrome. For the sake of contradiction assume $u$ is a vertex with $x, y, z \in N(u)$ such that $c(\{u, x\}) = c(\{u, y\}) \neq c(\{u, z\})$ and for simplicity assume that $u$ is the root of $T$. We show that we can use an extra color contradicting the optimality of $c$. By Lemma 20 all the children of $u$, e.g. $x, y, z$, are monochrome. Let New be a new color. Recolor $\{u, x\}$ and every edge $e$ in subtree of $x$ with New.

It is enough to show that $c$ is still a $P_3$-free coloring. Let $p = (v_1, v_2, v_3, v_4)$ be a rainbow $P_3$. Since we recolored the subtree of $x$, $p$ should contain edges from this subtree. There are two cases:

- $p$ is completely in subtree of $x$. In this case switching back the New colored edges to color $c(\{u, x\})$ again leaves us with a rainbow $P_3$, a contradiction.

- Otherwise there are two subcases:
  
  1. $p = (v_1, v_2, u, x)$. Then $v_2 \in N(u) \setminus \{x\}$ and $v_2$ is monochrome, hence $c(\{v_1, v_2\}) = c(\{v_2, u\})$.
  
  2. $p = (v_1, u, x, v_4)$. Similarly $v_4 \in N(x) \setminus \{u\}$ and $x$ is monochrome, thus $c(\{u, x\}) = c(\{x, v_4\}) = \text{New}$.

Therefore the new coloring is $P_3$-free. Hence, $c$ was not an optimum coloring.

Now we are ready to prove Theorem 17.

**Proof.** According to Lemma 21 there is an optimum $P_3$-free coloring of $T$ so that each vertex is either monochrome or rainbow. Algorithm 1 uses a dynamic programming approach to solve the problem for those two roles, in
the subtree rooted at vertex \( v \). Afterward the appropriate choice for vertex \( v \) can be decided when the algorithm is solving subtree of \( \text{parent}(v) \).

If \( v \) is rainbow then all of its children \( u_i \) should be monochrome with color \( c(\{v, u_i\}) \) thus:

\[
Rainbow(v) = \Sigma_{u \in \text{Children}(v)} Monochrome(u)
\]

If \( v \) is monochrome then its children can be either monochrome or rainbow, hence they pick the one that maximizes the number of colors.

\[
Monochrome(v) = 1 + \Sigma_{u \in \text{Children}(v)} \max\{Monochrome(u), \ Rainbow(u)\}
\]

And finally the answer for tree \( T \) rooted at \( r \) is \( \max\{Rainbow(r), Monochrome(r)\} \).

To show that the algorithm produces a \( P_3 \)-free coloring it is enough to show that in each step the edges of \( v \) are not part of a rainbow \( P_3 \) in the subtree which is rooted at \( v \). Let \( p = (v, x, y, z) \) be an arbitrary \( P_3 \) starting from \( v \) and going down in the tree (getting far from the root).

- \( v \) is monochrome. Then \( \text{Children}(v) = X_{\text{monochrome}} \cup X_{\text{rainbow}} \) where \( X_{\text{monochrome}} \) is the set of monochrome children of \( v \) and \( X_{\text{rainbow}} \) is the set of rainbow children. Then we have two cases:
  1. \( x \in X_{\text{monochrome}} \). Then \( c(\{v, x\}) = c(\{x, y\}) \).
  2. \( x \in X_{\text{rainbow}} \). Then \( y \) is monochrome, hence \( c(\{x, y\}) = c(\{y, z\}) \).

- \( v \) is rainbow. Then for all \( x \in \text{Children}(v) \), \( x \) is monochrome, hence \( c(\{v, x\}) = c(\{x, y\}) \).

Therefore \( c \) is also a \( P_3 \)-free coloring.

5.1 Bounded Degree and Bipartite Graphs

We end the section by providing a simple constant factor approximation algorithm on bounded degree graphs and a more refined version for bounded degree bipartite graphs. Note that the independent set problem remains hard in bounded degree graphs [3] so the reduction of previous sections still proves the hardness of the problem on bounded degree graphs. Recall that the problem is hard on 3-partite graphs, however, we do not know how hard it is on bipartite graphs. In this section we concentrate on \( P_3 \)-free coloring, hence, we simply write valid coloring for such a coloring.

Let \( \Delta \) be the maximum degree of a graph. For bounded degree graphs a greedy choice will provide a constant factor approximation to the optimum solution i.e. we have the following lemma.

**Lemma 22.** There is a \( \frac{1}{\log \Delta} \)-approximation for \( P_3 \)-free coloring.
Proof. We color the graph iteratively. In round $i \geq 1$, choose an uncolored edge $e = \{u, v\}$. Let $\mathcal{P}^e = \{P^e_1, \ldots, P^e_k\}$ be the set of paths of length at most 2 intersecting either $u$ or $v$. We color $e$ with $c_i$ and color every other edges in paths of $\mathcal{P}^e$ with $c_0$. In each iteration we colored at most $O(\Delta^2)$ edges and we used at least one new color, note that we may color an edge of color $c_0$ multiple times, but once an edge gets a color other than $c_0$ it never changes to any other color. We may suppose there are at least $n - 1$ edges in the graph. So we colored the graph with at least $\frac{1}{O(\Delta^2)} \cdot ar(G, P_3) \cdot \Delta + 1$ distinct colors.

Let $G = (A, B, E)$ be a bipartite graph with parts $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$ and $E$ denoting the edges of the graph. Also for the rest of this section assume that $|A| \geq |B|$. The proof of the following is similar to Proposition 18 of [7].

Lemma 23. There is a $\frac{1}{2}$-approximation for $P_3$-free coloring in bipartite graphs.

Proof. First observe that there is a $P_3$-free coloring of $G$ with $|A|$ distinct colors. Just color all edges incident to $a_i$ with color $c_i$ for $i \in [|A|]$. Then by considering $|A| \geq |B|$ the lemma is a direct consequence of Lemma 2.

Observe that the above observation helps to provide a better approximation on bounded degree bipartite graphs, for instance in grid graphs (subgraph of infinite grid) it is easy to obtain a $\frac{5}{8}$-approximation in linear time.

Corollary 24. Let $G$ be a bipartite graph of maximum degree $\Delta$, then the algorithm of Lemma 23 is a $\frac{\Delta + 1}{\Delta}$ approximation for $P_3$-free coloring.

Proof. Let $c$ be the optimal coloring on $G$ and $S$ be an edge maximal subgraph of disjoint union of edges of $G$ such that each of them colored with a distinct color. $G$ is bipartite then $S$ is a set of disjoint stars and the largest star of $S$ has at most $\Delta$ leaves but it has $\Delta + 1$ vertices. Hence total number of possible colors is bounded above by $\frac{\Delta}{\Delta + 1} \cdot |V(G)|$. The lemma follows immediately from this observation and Lemma 23.

6 Conclusions and open problems

In this paper, we studied the complexity of computing the anti-Ramsey number $ar(G, P_3)$. We showed that computing the $ar(G, P_3)$ is hard to approximate to a factor of $n^{-1/2-\epsilon}$. This result holds even when $G$ is a 3-partite graph. Then, we proved that there is no subexponential algorithm for the problem unless ETH fails.
On the positive side, we introduce the idea of color connected coloring and as a positive example we show that it results in a polynomial time algorithm on trees.

In our inapproximability construction, if we split each node of the original graph into paths of length $2k$, instead of paths of length 2, then by a careful analysis it is possible to show that for every constant $k$ the $P_{2k+1}$-free coloring is NP-hard, however, the construction does not preserve the inapproximability for $k > 1$. An interesting observation is that we cannot use a similar idea for paths of length 4. Hence, another research direction is to understand whether computing $ar(G,H)$ is easy when $H$ is a path of even length? On the other hand, given the hardness results for $P_3$, Lemma 4 and the fact that longest path is hard to approximate w.r.t. reasonable complexity assumptions, we anticipate that the problem is hard to approximate within any constant factor when $H$ is an odd path.

From the algorithmic point of view, it is not clear what is the complexity of the problem for bounded treewidth graphs. Moreover, in the bipartite graphs, even though there is a straightforward $\frac{1}{2}$-approximation algorithm, the complexity of the problem is still open. On the positive side, we just know the trivial exact algorithm that runs in $n^{|E(G)|}$ which is very far from a tight bound w.r.t. the possible lower bound. Hence, similar to other coloring problems it is interesting to understand whether one can get rid of the logarithmic factor in the exponent? Or even more, what happens in the dense graphs, is it possible to make it single exponential?

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