GERSTENHABER BRACKET ON DOUBLE HOCHSCHILD COMPLEX AND DEFORMATION THEORY

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ABSTRACT. We construct a differential and a Lie bracket on the space \( \{ \text{Hom}(A \otimes_k, A \otimes_l) \}_{k, l \geq 0} \) for any associative algebra \( A \). The restriction of this bracket to the space \( \{ \text{Hom}(A \otimes_k, A) \}_{k \geq 0} \) is exactly the Gerstenhaber bracket. We discuss some formality conjecture related with this construction. We also discuss some applications to deformation theory.

0. There exists a well-known way (due to Jim Stasheff) to define the Hochschild differential and the Gerstenhaber bracket via coderivations on the cofree coalgebra cogenerated by the vector space \( A^1 \) (\( A \) is an associative algebra). Analogously, one can define a dual differential and bracket on the graded space \( \{ \text{Hom}(A, A \otimes_k) \}_{k \geq 0} \) using derivations of tensor algebra, generated by the space \( A^1 \) for any coalgebra \( A \). In both cases the bracket does not depend on the (co)algebra structure, and only the differential does.

In these notes we generalize this construction involving all “differential operators” on the tensor algebra \( T^\bullet(A^1) \), not only of the first order. In such a way, we define a bidifferential and a bracket on the bigraded space \( \{ \text{Hom}(A \otimes_k, A \otimes_l) \}_{k, l \geq 0} \). It turns out that for any associative algebra \( A \) with unit the total cohomology of the bicomplex are equal examples to zero. So, the interesting examples appear for the algebras without unit, for example, for the algebras of polynomials without unit.

1. Let \( V \) be a vector space, and let \( \Psi_1: V^\otimes k_1 \to V^\otimes l_1 \), \( \Psi_2: V^\otimes k_2 \to V^\otimes l_2 \) be any two maps. We are going to define the bracket \( [\Psi_1, \Psi_2] \).

Let \( \Psi: V^\otimes k \to V^\otimes l \) be any map. It defines a map \( i(\Psi): V^\otimes N \to V^\otimes (N-k+l) \), \( N \gg 0 \), as follows:

\[
(1) \quad i(\Psi)(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = \Psi(v_1 \otimes \ldots \otimes v_k) \otimes v_{k+1} \otimes \ldots \otimes v_N +
+ v_1 \otimes \Psi(v_2 \otimes \ldots \otimes v_{k+1}) \otimes \ldots \otimes v_N + \ldots +
+ v_1 \otimes v_2 \otimes \ldots \otimes \Psi(v_{N-k+1} \otimes \ldots \otimes v_N).
\]

It is clear that the composition \( i(\Psi_1) \circ i(\Psi_2) \) has not the form \( i(\Psi_3) \) for some \( \Psi_3 \). Nevertheless, one has the following statement.

**Lemma.** The bracket \( i(\Psi_1) \circ i(\Psi_2) - i(\Psi_2) \circ i(\Psi_1) \) has a form \( i \left( \sum_{s \geq 0} \Psi_{3,s} \right) \) where \( \Psi_{3,s} \) is a map \( V^\otimes (k_1+k_2-s) \to V^\otimes (l_1+l_2-s) \).

**Proof.** It is clear.

\( \square \)
We set:

\[ [\Psi_1, \Psi_2] = \sum_s \Psi_{3,s}. \]

**Remark.** The case \( s = 0 \) may appear only when \( k_1 = k_2 = 0 \) or \( l_1 = l_2 = 0 \). In the general case, \( s \geq 1 \).

We have constructed a “Lie algebra of differential operators” on the tensor algebra \( T^*(V) \). Let us note that this Lie algebra is not corresponded to an associative algebra.

2. Let \( V = A[1] \) where \( A \) is an associative algebra, \( m_A : A^{\otimes 2} \to A \) is the product. Let \( \Psi : A^{\otimes k} \to A^{\otimes l} \) be any map.

**Lemma.** The bracket \([m_A, \Psi] = \Psi_1 + \Psi_2\), where \( \Psi_1 : A^{\otimes (k+1)} \to A^{\otimes l} \) and \( \Psi_2 : A^{\otimes k} \to A^{\otimes (l-1)} \) are defined as follows:

\[
\begin{align*}
(3) & \quad \Psi_1(a_1 \otimes \ldots \otimes a_{k+1}) = (a_1 \otimes 1 \otimes 1 \otimes \ldots) \cdot \Psi(a_2 \otimes \ldots \otimes a_{k+1}) - \\
& \quad - \Psi(a_1a_2 \otimes a_3 \otimes \ldots \otimes a_{k+1}) + \Psi(a_1 \otimes a_2a_3 \otimes \ldots) \pm \\
& \quad \pm \Psi(a_1 \otimes \ldots \otimes a_k) \cdot (1 \otimes 1 \otimes \ldots \otimes a_{k+1}); \\
(4) & \quad \Psi_2(a_1 \otimes \ldots \otimes a_k) = \pm (m_{12} - m_{23} + \ldots \pm m_{l-1,l}) \circ \Psi(a_1 \otimes \ldots \otimes a_k), \\
\end{align*}
\]

where

\[
\begin{align*}
(5) & \quad m_{i,i+1}(b_1 \otimes \ldots \otimes b_l) = b_1 \otimes \ldots \otimes b_{i-1} \otimes b_i \cdot b_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_l.
\end{align*}
\]

**Proof.** It is a direct calculation. \( \square \)

**Corollary.**

\[ [m_A, m_A] = 0 \]

Let us denote \( \Psi_1(\Psi) = d_1(\Psi), \Psi_2(\Psi) = d_2(\Psi) \) (in the notations of Lemma above); it follows from Corollary that \( d_1 + d_2 \) is a bidifferential, i.e. \( d_1^2 = 0, \quad d_2^2 = 0, \quad d_1 d_2 = \pm d_2 d_1 \). We have the following bicomplex:

\[
\begin{align*}
0 & \longrightarrow A^{\otimes 2} \longrightarrow A^* \\
& \downarrow d_1 \downarrow 0 \downarrow 0 \downarrow 0 \downarrow 0 \\
0 & \longrightarrow A \longrightarrow \text{Hom}(A,A) \longrightarrow \text{Hom}(A^{\otimes 2}, A) \longrightarrow \text{Hom}(A^{\otimes 3}, A) \longrightarrow \text{Hom}(A^{\otimes 4}, A) \longrightarrow \text{Hom}(A^{\otimes 5}, A) \longrightarrow \text{Hom}(A^{\otimes 6}, A) \longrightarrow \text{Hom}(A^{\otimes 7}, A) \longrightarrow \text{Hom}(A^{\otimes 8}, A) \longrightarrow \text{Hom}(A^{\otimes 9}, A) \longrightarrow \text{Hom}(A^{\otimes 10}, A) \longrightarrow \text{Hom}(A^{\otimes 11}, A) \longrightarrow \text{Hom}(A^{\otimes 12}, A) \longrightarrow \cdots \\
& \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \downarrow d_2 \\
0 & \longrightarrow \text{Hom}(A, A^{\otimes 2}) \longrightarrow \text{Hom}(A, A^{\otimes 3}) \longrightarrow \text{Hom}(A, A^{\otimes 4}) \longrightarrow \text{Hom}(A, A^{\otimes 5}) \longrightarrow \text{Hom}(A, A^{\otimes 6}) \longrightarrow \text{Hom}(A, A^{\otimes 7}) \longrightarrow \text{Hom}(A, A^{\otimes 8}) \longrightarrow \text{Hom}(A, A^{\otimes 9}) \longrightarrow \text{Hom}(A, A^{\otimes 10}) \longrightarrow \text{Hom}(A, A^{\otimes 11}) \longrightarrow \text{Hom}(A, A^{\otimes 12}) \longrightarrow \cdots
\end{align*}
\]

Let us note that the second row is exactly the Hochschild complex, and the first column is the bar-complex.

Let us introduce the grading on this bicomplex as follows:

\[
\text{Hom}^i(A) = \{ A^{\otimes k} \to A^{\otimes l}, \quad k - l = i \}.
\]

It is clear that \( \text{Hom}^* \) is a dg Lie algebra with the bracket constructed in Section 1.
3.

**Theorem.** For any associative algebra $A$ with unit the cohomology of the complex $\text{Hom}^\bullet(A)$ (see formula (6)) is equal to $\mathbb{C}[0]$.

**Proof.** It is clear that the complex $\text{Hom}^\bullet(A)$ is equal to the Hochschild complex $\text{Hoch}^\bullet(A, B^\bullet)$ where

$$B^\bullet = 0 \leftarrow \mathbb{C} \leftarrow^0 A \leftarrow^2 A \otimes A \leftarrow^3 A \otimes^3 \leftarrow \cdots$$

considered as a complex of $A$-bimodules, and $\mathbb{C}$ is considered as $A$-bimodule with zero action. On the other hand, $B^\bullet$ is the bar-complex, it is quasi-isomorphic to $\mathbb{C}[0]$ for any algebra $A$ with unit as complex of bimodules, where $\mathbb{C}[0]$ is equipped with zero action.

Then, $\text{Hoch}^\bullet(A, B^\bullet) \cong \text{Hoch}^\bullet(A, \mathbb{C})$. The last complex is dual to the bar-complex.

\[\square\]

4.

**Theorem.** Let $A = S^\bullet(V)_0$ be the algebra of polynomials on a vector space $V$ without unit. Then

$$H^\bullet(\text{Hom}^\bullet(A)) \simeq \bigwedge^\bullet(V) \otimes \bigwedge^\bullet(V^*).$$

**Proof.** The proof is analogous to the proof of theorem in Section 3, but now the bar-complex $B^\bullet$ quasi-isomorphic to $\bigwedge^\bullet(V)$ as a complex of $A$-bimodules, when $\bigwedge^\bullet(V)$ is equipped with zero structure of $S^\bullet(V)_0$-bimodule.

\[\square\]

**Theorem.** The induced Lie algebra structure on $\bigwedge^\bullet(V) \otimes \bigwedge^\bullet(V^*)$ is the structure arising from the Clifford algebra $\text{Clif}(V \oplus V^*)$ (with the bracket $[a, b] = a \cdot b \pm b \cdot a$).

Then $\bigwedge^\bullet(V) \otimes \bigwedge^\bullet(V^*)$ is a graded Lie algebra $\text{Clif}^\bullet(V \oplus V^*)$, where

$$\text{Clif}^i(V \oplus V^*) = \{\text{Hom}(V^\otimes k, V^\otimes l), k - l = i\}.$$

5. **Formality conjecture for Clifford algebra.**

**Conjecture.** The differential graded Lie algebra $\text{Hom}^\bullet(S(V)_0)$ is quasi-isomorphic to the Clifford Lie algebra $\text{Clif}^\bullet(V \oplus V^*)$.

It seems that the Hochschild cohomology $HH^\bullet(S(V)_0)$ are equal to polyvector fields vanishing at zero. Then the Hochschild complex $\text{Hoch}^\bullet(S(V)_0)$ is not formal as Lie algebra (the graph

![Figure 1.](image)

defines a map $S(V)_0^2 \to S(V)$, where $\alpha$ is a linear polyvector field, see [K]).

As a consequence, there may exist many deformation quantizations on the algebra $S(V)_0$, corresponding to a Lie algebra structure on $V$ or, more generally, to a Poisson
bivector field vanishing at 0, which define the same (gauge equivalent) star-products on the algebra $S(V)$. But any star-product on $S(V)_0$, i.e. a map $\Psi: S(V)_0^\otimes 2 \to S(V)_0$ (satisfying the Maurer–Cartan equation), defines a solution of the Maurer–Cartan equation in the dg Lie algebra $\text{Hom}^\bullet(S(V)_0)$ (because $d_2 = 0$ at this place). Then, if the Conjecture is true, it defines a solution of the Maurer–Cartan equation in $\text{Clif}(V \oplus V^*)$ modulo the gauge equivalence, i.e. a map $Q: \bigwedge^\bullet V \to \bigwedge^{\bullet-1} V$

of degree $-1$ such that $Q^2 = 0$, modulo the action $Q \mapsto Q' = AQA^{-1}$, where $A$ is a map $\bigwedge^\bullet V \to \bigwedge^\bullet V$ preserving the grading. It is exactly a structure of complex on the graded space $\bigwedge^\bullet V$

$$0 \to \mathbb{C} \xrightarrow{Q_0} V^* \xrightarrow{Q_1} \bigwedge^2 V^* \xrightarrow{Q_2} \bigwedge^3 V^* \to \ldots \xrightarrow{Q_{n-1}} \bigwedge^n V^* \to 0,$$

i.e. $Q_iQ_{i+1} = 0$, modulo changes $Q_i \to A_iQ_iA_i^{-1}$ for linear automorphisms $(A_0, \ldots, A_n)$, $A_i \in \text{Aut} \bigwedge^i V$. Such a data is described by dimensions of the cohomology spaces, i.e. by numbers $b_0, \ldots, b_n$ such that

$$0 \leq b_i \leq \binom{i}{n}, \quad \text{and} \quad \sum_{i=0}^n (-1)^i b_i = \sum_{i=0}^n (-1)^i \binom{i}{n} = 0. \tag{7}$$

In the case when $Q = \sum c_{ij}^k v_k \otimes v_i^* \otimes v_j^* \in \bigwedge^2 V^* \otimes V$ is correponded to a Lie algebra structure on $V$, then $b_i = \dim H^i_{\text{Lie}}(V; \mathbb{C})$.

At the moment, I don’t know any examples of star-products which would be interesting from this viewpoint.

Thus, a star-product on $S^\bullet(V)_0$ is described (may be not completely) by the corresponding bivector field and by a sequence $(b_i)$ satisfying (7).

6. In any case, the dg Lie algebra structure on $\text{Hom}^\bullet(A)$ (for any associative algebra $A$) produces a quite strange definition of a “homotopy algebra structure” on $A$ (via the Maurer–Cartan equation). The theorem of Section 3 shows that for algebras with unit this structure is nondeformable; in particular, any star-product is equal to usual commutative product in this sense. But the algebras without unit may give us some interesting examples.

References

[1] M. Kontsevich, Deformation quantization of Poisson manifolds, I, preprint math.dg/9709040

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