COUNTING THE NUMBER OF ELEMENTS IN THE MUTATION CLASSES
OF QUIVERS OF TYPE $\tilde{A}_n$

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Abstract. In this article we prove explicit formulae counting the elements in the mutation classes of quivers of type $\tilde{A}_n$. In particular, we obtain the number of non-isomorphic cluster-tilted algebras of type $\tilde{A}_n$. Furthermore, we give an alternative proof for the number of quivers of Dynkin type $D_n$ which was first determined by Buan and Torkildsen in [4].

1. Introduction

Quiver mutation is a central element in the recent theory of cluster algebras introduced by Fomin and Zelevinsky in [8]. It is an elementary operation on quivers which generates an equivalence relation. The mutation class of a quiver $Q$ is the class of all quivers which are mutation equivalent to $Q$.

The mutation class of quivers of type $A_n$ is the class containing all quivers mutation equivalent to a quiver whose underlying graph is the Dynkin diagram of type $A_n$, shown in Figure 1(a). The quivers in the mutation class were characterized by Buan and Vatne in [5]. The corresponding task for type $D_n$, shown in Figure 1(b), was accomplished by Vatne in [11]. Furthermore, an explicit formula for the number of quivers in the mutation class of type $A_n$ was given by Torkildsen in [10] and of type $D_n$ by Buan and Torkildsen in [4].

In this article, we consider quivers of type $\tilde{A}_{n-1}$. That is, all quivers mutation equivalent to a quiver whose underlying graph is the extended Dynkin diagram of type $\tilde{A}_{n-1}$, i.e., the $n$-cycle see Figure 1(c). If all arrows of this cycle go clockwise or all arrows go anti-clockwise, then we get the mutation class of $D_n$, see Fomin et al. in [7] and Type IV in [11]. If the cycle is non-oriented, we get the mutation classes of $\tilde{A}_{n-1}$, studied by one of the authors in [2]. The purpose of this paper is to give an explicit formula for the number of quivers in the mutation classes of quivers of type $\tilde{A}_{n-1}$.

Since a cluster-tilted algebra $C$ of type $\tilde{A}_{n-1}$ is finite dimensional over an algebraically closed field $K$, there exists a quiver $Q$ which is in the mutation classes of $\tilde{A}_{n-1}$ (see for instance Assem et al. in [1]) and an admissible ideal $I$ of the path algebra $KQ$ of $Q$ such that $C \cong KQ/I$. Therefore, we also obtain the number of non-isomorphic cluster-tilted algebras of type $\tilde{A}_{n-1}$. In fact, we prove a more refined counting theorem:
Theorem. The number of quivers mutation equivalent to a non-oriented $n$-cycle with $r$ arrows oriented in one direction and $s$ arrows oriented in the other direction is given by

\[
\tilde{a}(r, s) = \begin{cases} 
\frac{1}{2} \sum_{k|r, k|s} \phi(k) \left( \frac{2r/k}{r/k} \right) \left( \frac{2s/k}{s/k} \right) & \text{if } r \neq s, \\
\frac{1}{2} \left( \frac{1}{2} \left( \frac{2r}{r} \right) + \sum_{k|r} \phi(k) \left( \frac{2r/k}{r/k} \right)^2 \right) & \text{if } r = s.
\end{cases}
\]

where $\phi(k)$ is Euler’s totient function, i.e., the number of $1 \leq d \leq k$ coprime to $k$.

Additionally, we obtain the number of quivers in the mutation class of a quiver of Dynkin type $D_n$. This formula was first determined in [4]:

Corollary. The number of quivers of type $D_n$, for $n \geq 5$, is given by

\[
\tilde{a}(0, n) = \sum_{d|n} \phi(n/d) \left( \frac{2d}{d} \right).
\]

The number of quivers of type $D_4$ is 6.

The paper is organized as follows. In Section 2 we collect some basic notions about quiver mutation. Furthermore, we present the classification of quivers of type $\tilde{A}_{n-1}$ according to the parameters $r$ and $s$ mentioned above, as given in [2]. In Section 3 we restate the classification as a combinatorial grammar. Using ‘generating functionology’ we obtain $\tilde{a}(r, s)$ for $r \neq s$. For the case $r = s$ some additional combinatorial considerations, counting the number of quivers invariant under reflection, yield the result stated above. At the end of this section we give the proof for the number of quivers in the mutation class of type $D_n$, by exhibiting an appropriate bijection between these and a subclass of the objects counted in Section 3.2.

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2. Preliminaries

A quiver $Q$ is a (finite) directed graph where loops and multiple arrows are allowed. Formally, $Q$ is a quadruple $Q = (Q_0, Q_1, h, t)$ consisting of two finite sets $Q_0, Q_1$ whose elements are called vertices and arrows respectively, and two functions $h : Q_1 \to Q_0, \quad t : Q_1 \to Q_0,$
assigning a head $h(\alpha)$ and a tail $t(\alpha)$ to each arrow $\alpha \in Q_1$.

Moreover, if $t(\alpha) = i$ and $h(\alpha) = j$ for $i, j \in Q_0$, we say $\alpha$ is an arrow from $i$ to $j$ and write $i \xrightarrow{\alpha} j$. In this case, $i$ and $\alpha$ as well as $j$ and $\alpha$ are called incident to each other. As usual, two quivers are considered to be equal if they are isomorphic as directed graphs. The underlying graph of a quiver $Q$ is the graph obtained from $Q$ by replacing the arrows in $Q$ by undirected edges.

A quiver $Q' = (Q'_0, Q'_1, h', t')$ is a subquiver of a quiver $Q = (Q_0, Q_1, h, t)$ if $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$ and where $h'(\alpha) = h(\alpha) \in Q'_0$, $t'(\alpha) = t(\alpha) \in Q'_0$ for any arrow $\alpha \in Q'_1$. A subquiver is called a full subquiver if for any two vertices $i$ and $j$ in the subquiver, the subquiver also will contain all arrows between $i$ and $j$ present in $Q$.

An oriented cycle is a full subquiver of a quiver whose underlying graph is a cycle, and whose arrows are all oriented in the same direction. By contrast, a non-oriented cycle is a full subquiver of a quiver whose underlying graph is a cycle, but not all of its arrows are oriented in the same direction.

Throughout the paper, unless explicitly stated, we assume that

- quivers do not have loops or oriented 2-cycles, i.e., $h(\alpha) \neq t(\alpha)$ for any arrow $\alpha$ and there do not exist arrows $\alpha, \beta$ such that $h(\alpha) = t(\beta)$ and $h(\beta) = t(\alpha)$;
- quivers are connected.

2.1. Quiver mutation. In [5], Fomin and Zelevinsky introduced the quiver mutation of a quiver $Q$ without loops and oriented 2-cycles at a given vertex of $Q$:

**Definition 2.1.** Let $Q$ be a quiver. The mutation of $Q$ at a vertex $k$ is defined to be the quiver $Q^* := \mu_k(Q)$ given as follows.

1. Add a new vertex $k^*$.
2. Suppose that the number of arrows $i \to k$ in $Q$ equals $r$, the number of arrows $k \to j$ equals $s$ and the number of arrows $j \to i$ equals $t \in \mathbb{Z}$. Then we have $t - rs$ arrows $j \to i$ in $Q^*$.
   Here, a negative number of arrows means arrows in the opposite direction.
3. For any arrow $i \to k$ (resp. $k \to j$) in $Q$ add an arrow $k^* \to i$ (resp. $j \to k^*$) in $Q^*$.
4. Remove the vertex $k$ and all its incident arrows.

Note that mutation at sinks or sources only means changing the direction of all incoming and outgoing arrows. Mutation at a vertex $k$ is an involution on quivers, that is, $\mu_k(\mu_k(Q)) = Q$. It follows that mutation generates an equivalence relation and we call two quivers mutation equivalent if they can be obtained from each other by a finite sequence of mutations. The mutation class of a quiver $Q$ is the class of all quivers which are mutation equivalent to $Q$.

We have the following well-known lemma:

**Lemma 2.2.** If quivers $Q, Q'$ have the same underlying graph which is a tree, then $Q$ and $Q'$ are mutation equivalent.

This lemma implies that one can speak of quivers associated to a simply-laced Dynkin diagram, i.e., the Dynkin diagram of type $A_n, D_n$ or $E_n$: a quiver of type $A_n$ (resp. $D_n, E_n$) is defined to be a quiver in the mutation class of all quivers whose underlying graph is the Dynkin diagram of type $A_n$ (resp. $D_n, E_n$). One can easily check that the oriented $n$-cycle is also of type $D_n$, as has been done in [11, Type IV]. Two non-oriented $n$-cycles are mutation equivalent if and only if the number of arrows oriented clockwise coincide, or the number of arrows oriented clockwise in one cycle agrees with the number of arrows oriented anti-clockwise in the other cycle. This was shown in [2, 7] and is restated in Theorem [2.10]. Quivers in those mutation classes are called quivers of...
type \( \tilde{A}_{n-1} \) and are described in more detail in Section 2.2. In Figure 1, the Dynkin diagrams of types \( A_n \) and \( D_n \) and the extended Dynkin diagram of \( \tilde{A}_{n-1} \) are shown.

Example 2.3. The mutation class of type \( \tilde{A}_3 \) of the non-oriented cycles with two arrows in each direction is given by

\[
\begin{array}{c}
\text{Example 2.3. The mutation class of type } \tilde{A}_3 \text{ of the non-oriented cycles with two arrows in each direction is given by} \\
\end{array}
\]

The mutation class of type \( \tilde{A}_3 \) of the non-oriented cycle with 3 arrows in one direction and 1 arrow in the other is given by

\[
\begin{array}{c}
\text{The mutation class of type } \tilde{A}_3 \text{ of the non-oriented cycle with 3 arrows in one direction and 1 arrow in the other is given by} \\
\end{array}
\]

2.2. Mutation classes of \( \tilde{A}_{n-1} \)-quivers. Following [2], we now describe the mutation classes of quivers of type \( \tilde{A}_{n-1} \) in more detail:

Definition 2.4. Let \( Q_{n-1} \) be the class of quivers with \( n \) vertices which satisfy the following conditions:

1. There exists precisely one full subquiver which is a non-oriented cycle.
2. All other cycles (if present) are oriented cycles of length 3.
3. Every vertex has at most four incident arrows.
4. If a vertex has four incident arrows, then two of them belong to precisely one oriented 3-cycle, and the other two belong to precisely one other oriented 3-cycle, i.e., there is no incident arrow which belongs to two oriented 3-cycles.
5. If a vertex has exactly three incident arrows, then two of its incident arrows belong to one oriented 3-cycle, and the third arrow does not belong to any oriented 3-cycle.

Remark 2.5.

1. If the length of the non-oriented is two, it is a double arrow. Our convention is to count only one oriented 3-cycle in the following case:

2. According to Definition 2.4, a quiver \( Q \in Q_{n-1} \) contains a non-oriented cycle of length \( \geq 2 \). For each arrow \( x \xrightarrow{\alpha} y \) in this cycle, there may (or may not) be a vertex \( z_\alpha \) which is not on the non-oriented cycle, such that there is an oriented 3-cycle:
Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the non-oriented cycle.

Now, if we remove all vertices in the non-oriented cycle and their incident arrows, the result is a disconnected union of quivers, one for each \( z_\alpha \). These are of type \( A_{k_\alpha} \) for \( k_\alpha \geq 1 \) (see [5] for the mutation class of \( A_n \)), and the vertices \( z_\alpha \) have at most two other incident arrows. Furthermore, if such a vertex \( z_\alpha \) has 2 other incident arrows, then \( z_\alpha \) is a vertex in an oriented 3-cycle. We call these quivers rooted quivers of type \( A \) with root \( z_\alpha \). Note that this is a similar description as for Type IV in [11].

The rooted quiver of type \( A \) with root \( z_\alpha \) is called attached to the arrow \( \alpha \).

**Example 2.6.** The following quiver is of type \( \tilde{A}_{21} \):

![Diagram](image)
(1) base arrows and oriented anti-clockwise, or
(2) contained in a rooted quiver of type $A$ attached to a base arrow $\alpha$ which is oriented anti-clockwise.

Let $r_2$ be the number of oriented 3-cycles
(1) which share an arrow $\alpha$ with the non-oriented cycle and $\alpha$ (a base arrow) is oriented anti-clockwise, or
(2) which are contained in a rooted quiver of type $A$ attached to a base arrow $\alpha$ which is oriented anti-clockwise.

Similarly we define the parameters $s_1$ and $s_2$ with 'anti-clockwise' replaced by 'clockwise'.

Example 2.9. We indicate the arrows which count for the parameter $r_1$ by $\bullet\cdots\bullet\rightarrow$ and the arrows which count for $s_1$ by $\rightarrow$. Furthermore, the oriented 3-cycles counting for $r_2$ are indicated by $\triangleleft\triangleleft\triangleleft$ and the oriented 3-cycles counting for $s_2$ are indicated by $\triangle$.

Consider the following realization of a quiver in $Q_{16}$:

Here, we have $r_1 = 3$, $r_2 = 3$, $s_1 = 4$ and $s_2 = 2$.

In [2] an explicit description of the mutation classes of quivers of type $\tilde{A}_n$ is given as follows:
Theorem 2.10. [2, Theorem 3.14] Let $Q_1, Q_2 \in \mathcal{Q}_{n-1}$ with realizations having parameters $r_1, r_2, s_1$ and $s_2$, respectively $\tilde{r}_1, \tilde{r}_2, \tilde{s}_1$ and $\tilde{s}_2$. Then $Q_1$ is mutation equivalent to $Q_2$ if and only if $r := r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2$ and $s := s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$ (or $r_1 + 2r_2 = \tilde{s}_1 + 2\tilde{s}_2$ and $s_1 + 2s_2 = \tilde{r}_1 + 2\tilde{r}_2$).

3. A Combinatorial Grammar

In this section we describe the elements of the mutation classes of type $\tilde{A}_{n-1}$ by a combinatorial grammar. This can be viewed as an exercise in the theory of species (introduced by Joyal, see the book [3] by Bergeron, Labelle and Leroux) or the symbolic method (as detailed in the recent book [6] by Flajolet and Sedgewick). We first give a recursive description of rooted quivers of type $A$ as defined in 2.5. A quiver of type $\tilde{A}_{n-1}$ will then be roughly a cycle of rooted quivers of type $A$.

3.1. A recursive description of rooted quivers of type $A$. Let $A^*$ be the set of all rooted quivers of type $A$. We can then describe the elements of $A^*$ recursively. A rooted quiver of type $A$ is one of the following:

- the root;
- the root, incident to an arrow, and a rooted quiver of type $A$ incident to the other end of the arrow. The arrow may be directed either way.
- the root, incident to an oriented 3-cycle, and two rooted quivers of type $A$, each being incident to one of the other two vertices of the 3-cycle.

We obtain the following combinatorial grammar:

$$A^* = \bullet \cup A^* \cup A^* \cup A^* \cup A^*$$

We set the weight of an arrow equal to $z$ and of an oriented 3-cycle equal to $z^2$. Hence, the weight of a rooted quiver $Q$ of type $A$ is $z^\#\{\text{vertices in } Q\} - 1$. This choice of weight is in accordance with Theorem 2.10 where we count oriented 3-cycles in quivers (the number of which we denoted $r_2$, resp. $s_2$) twice.

Thus, let

$$A^*(z) = \sum_{Q \in A^*} z^{\#\{\text{vertices in } Q\} - 1}$$

be the generating function (in particular: the formal power series) associated to rooted quivers of type $A$. From the recursive description, we obtain

$$A^*(z) = 1 + 2zA^*(z) + z^2A^*(z)^2$$

$$= \frac{1 - 2z - \sqrt{1 - 4z}}{2z^2}$$

$$= \sum_{n \geq 1} \frac{1}{n+1} \binom{2n}{n} z^n,$$

which is the generating function of the Catalan numbers shifted by 1, see e.g. [3, Section 3.0 Eq. (3)].
To give a combinatorial description of the realizations of quivers in the mutation classes of type \( \tilde{\mathcal{A}}_{n-1} \) corresponding to Definition 2.8 we need auxiliary objects, which are one of the following:

1. a single (base) arrow, oriented from left to right, or
2. a rooted quiver of type \( A \) attached to an oriented 3-cycle, whose base arrow (see Definition 2.8) is oriented from left to right, or
3. a single (base) arrow, oriented from right to left, or
4. a rooted quiver of type \( A \) attached to an oriented 3-cycle, whose base arrow is oriented from right to left.

Remark 3.1. The ‘base arrows’ in (1)–(4) above will become precisely the arrows of the non-oriented cycle, which justifies the usage of the name.

Thus, we again obtain a combinatorial grammar:

The weight of \( Q \in B \) is \( p^{\# \{ \text{vertices in } Q \}} - 1 \) if it is of type (1) or (2), and \( q^{\# \{ \text{vertices in } Q \}} - 1 \) if it is of type (3) or (4). In particular, the weight of \( Q \) depends only on the orientation of the base arrow and on the total number of vertices of \( Q \). Passing to generating functions, we obtain

\[
B(p,q) = p + p^2 A^*(p) + q + q^2 A^*(q)
= \frac{1 - \sqrt{1 - 4p}}{2} + \frac{1 - \sqrt{1 - 4q}}{2}
= pC(p) + qC(q),
\]

where \( C(z) \) is the generating function of the Catalan numbers,

\[
C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n.
\]

3.2. The number of quivers of type \( \tilde{\mathcal{A}}_{n-1} \). By Remark 2.5, a realization of a quiver of type \( \tilde{\mathcal{A}}_{n-1} \) is simply a cyclic arrangement of elements in \( B \) with a total of \( n \) vertices. For example, the quiver in Example 2.9 consists of five elements of \( B \), three of which are just arrows, the two others are rooted quivers of type \( A \) attached to an oriented 3-cycle.

The following Lemma is the so called cycle construction, which is well known in combinatorics, see eg. [3] Eq. (18), Section 1.4] or [6] Theorem I.1, Section I.2.2].

Lemma 3.2. Let \( B(z) \) be the generating function for a family of unlabelled objects, where \( z \) marks size. Then the generating function for cycles of such objects is

\[
\sum_{k \geq 1} \frac{\phi(k)}{k} \log \left( \frac{1}{1 - B(z^k)} \right),
\]

where \( \phi(k) \) is Euler’s totient function, i.e., the number of \( 1 \leq d \leq k \) coprime to \( k \).
Thus, we obtain for the generating function for realizations of quivers of type $\tilde{A}_n$ with $p$ marking $r_1 + 2r_2$ and $q$ marking $s_1 + 2s_2$

$$\tilde{A}(p, q) = \sum_{k \geq 1} \frac{\phi(k)}{k} \log \left( \frac{1}{1 - B(p^k, q^k)} \right).$$

Let us first determine $\log \frac{1}{1 - B(p, q)}$. Using a computer, we generated the first few values of the Taylor coefficients of this formal power series. Then, using a guessing package (to be precise, guessBinRat in FriCAS, see [9] written by one of the authors), we found experimentally the statement of the following lemma:

**Lemma 3.3.**

$$[p^rq^s] \log \left( \frac{1}{1 - B(p, q)} \right) = \frac{1}{2} \binom{r}{2} \binom{s}{2},$$

where we use the convention that $\binom{-1}{1} = 1$, and $[p^rq^s]B(p, q)$ denotes the coefficient of $p^rq^s$ in the formal power series $B(p, q)$.

We give two proofs of this lemma. The first proof is computer assisted.

**Proof.** Using a guessing package (e.g., guessHolo in FriCAS), we find experimentally that $f(z) = \log \left( \frac{1}{1 - B(pz, qz)} \right)$ is a holonomic function in $z$, i.e., satisfies a linear differential equation where the coefficients are polynomials in $z$. In our particular case the package comes up with the equation

$$(16pqz^3 - 4(p + q)z^2 + z)f''(z) + (32pqz^2 - 6(p + q)z + 1)f'(z) + 8pqz - (p + 1) = 0.$$ 

Plugging in $\log \left( \frac{1}{1 - B(p, q)} \right)$ (preferably using a computer algebra system) and checking the initial values we see that this is indeed the case.

Since $f(z)$ is holonomic in $z$, its coefficients $f_n$ (with $n = r + s$) satisfy a polynomial recurrence, which turns out to be

$$(n + 2)^2 f_{n+2} - 2(p + q)(n + 1)(2n + 3)f_{n+1} + 16n(n + 1)pq f_n = 0.$$ 

It remains to check that $\sum_{r=0}^{n} p^r q^{n-r} \frac{1}{2n} \binom{2r}{r} \binom{2(n-r)}{n-r}$ satisfies the same recurrence, with the same initial conditions. Extracting the coefficient of $p^r q^{n+2-r}$ we are led to check that

$$(n + 2) \binom{2r}{r} \frac{2(n + 2 - r)}{n + 2 - r}$$

$$-2(2n + 3) \binom{2r}{r} \frac{2(n + 1 - r)}{n + 1 - r}$$

$$-2(2n + 3) \binom{2r}{r} \frac{2(n + 2 - r)}{n + 2 - r}$$

$$+ 16(n + 1) \binom{2r}{r} \frac{2(n + 1 - r)}{n + 1 - r} = 0,$$

which is a tedious, but straightforward exercise, best left to the computer. □

The second proof, using elementary transformations, was provided by Christian Krattenthaler, when he saw a preliminary version of this article.
Proof. Following Eq. (11), we obtain $B(p, q) = pC(p) + qC(q)$ with $[z^n](C(z))^m = \frac{m}{n + m} \binom{2n + m - 1}{n}$. Therefore,

$$
\log \left( \frac{1}{1 - B(p, q)} \right) = \sum_{k \geq 1} \frac{(B(p, q))^k}{k} = \sum_{k \geq 1} \frac{1}{k} \sum_{l=0}^{k} \binom{k}{l} p^l C(p)^l q^{k-l} C(q)^{k-l}
$$

and moreover,

$$
[p^r q^s] \log \left( \frac{1}{1 - B(p, q)} \right) = \sum_{l \geq 0} \frac{1}{l!} \sum_{r, s \geq 0} \binom{2r - l - 2}{r - l - 1} \binom{2s - k - 2}{s - k - 1} \frac{(2r - l - 1)^{s-1}}{k! (s-k-1)!} \sum_{k \geq 0} \frac{(k + l + 1)! (2s - k - 2)!}{k! (s-k-1)!}.
$$

Putting the pieces together we obtain:

**Theorem 3.4.** The number of realizations of quivers of type $\tilde{A}_{r+s-1}$ with parameters $r, s > 0$ is given by

$$
\frac{1}{2} \sum_{k \mid r, k \mid s} \frac{\phi(k)}{r + s} \binom{2r/k}{r/k} \binom{2s/k}{s/k}.
$$

Proof. Observe that for any $F(p, q) = \sum_{r,s} f_{r,s} p^r q^s$ we have

$$
[p^r q^s] F(p^k, q^k) = \begin{cases} f_{r/k, s/k} & \text{when } k \mid r \text{ and } k \mid s, \\ 0 & \text{otherwise.} \end{cases}
$$
We get
\[
[p^r q^s] \sum_{k \geq 1} \frac{\phi(k)}{k} \log \left( \frac{1}{1 - B(p^r, q^s)} \right) = \sum_{k \geq 1} \frac{\phi(k)}{k} [p^r q^s] \log \left( \frac{1}{1 - B(p^r, q^s)} \right) \\
= \sum_{k | r | k | s} \frac{\phi(k)}{k} \frac{k}{2(r + s)} \left( \frac{2r/k}{s/k} \right) \left( \frac{2s/k}{r/k} \right).
\]

\[\square\]

Using this theorem, we can immediately compute the number of quivers of type \( \tilde{A}_{r+s-1} \) for \( r \neq s \):

**Corollary 3.5.** For \( 0 < r < s \), the number of quivers of type \( \tilde{A}_{r+s-1} \) with parameters \( r, s \) equals
\[
\tilde{a}(r, s) = \frac{1}{2} \sum_{k | r | k | s} \phi(k) \left( \frac{2r/k}{s/k} \right) \left( \frac{2s/k}{r/k} \right).
\]

**Proof.** As \( r \neq s \), every quiver of type \( \tilde{A}_{r+s-1} \) with parameters \( r, s \) has exactly two realizations. Therefore, the sets of realizations of quivers of type \( \tilde{A}_{r+s-1} \) with parameters \( r, s \) and with parameters \( s, r \) are disjoint. As both sets have the same cardinality, the statement follows immediately from Theorem 3.4. \( \square \)

We have seen that a quiver of type \( \tilde{A}_{2r-1} \) is a non-oriented cycle of elements in \( B \) with a total number of \( 2r \) vertices. To count quivers of type \( \tilde{A}_{2r-1} \), we first have to consider symmetric quivers of type \( \tilde{A}_{2r-1} \), i.e., quivers where both possible realizations coincide. To do so, we have to count lists of elements in \( B \):

**Lemma 3.6.** The number of lists \( (B_1, \ldots, B_\ell) \) of elements in \( B \) with a total of \( r + 1 \) vertices is given by the central binomial coefficient \( \binom{2r}{r} \).

**Proof.** If we consider different orientations of base arrows, there are \( 2 \) cases. We have to consider \( B(z, z) = 1 - \sqrt{1 - 4z} \), see Eq. (1). We obtain that the number of lists of elements in \( B \) with \( r + 1 \) vertices is given by
\[
[z^{2r}] \frac{1}{1 - B(z, z)} = [z^{2r}] \frac{1}{\sqrt{1 - 4z}} = [z^{2r}] \sum_{n \geq 0} \binom{2n}{n} z^n = \binom{2r}{r}.
\]

\[\square\]

Given a list \( L = (B_1, \ldots, B_\ell) \) of elements in \( B \), we identify \( L \) with the quiver obtained from \( L \) by gluing together the right vertex in the base arrow of \( B_i \) and the left vertex in the base arrow of \( B_{i+1} \) for \( 1 \leq i < \ell \). For a list \( L = (B_1, \ldots, B_\ell) \) of elements in \( B \) define the reversed list \( \overline{L} := (\overline{B}_\ell, \ldots, \overline{B}_1) \), where \( \overline{B}_i \) is obtained from \( B_i \) by reversing the direction of the base arrow of \( B_i \) (and eventually of the associated oriented 3-cycle). See Figures 2(b) and 2(c) for an example. Obviously, we have \( \overline{(\overline{L})} = L \).

**Theorem 3.7.** The number of symmetric quivers of type \( \tilde{A}_{2r-1} \) is equal to \( \frac{1}{2} \binom{2r}{r} \).

**Proof.** Starting with a list \( L \) of elements in \( B \) with a total of \( r + 1 \) vertices, we obtain a symmetric quiver of type \( \tilde{A}_{2r-1} \) by taking \( L \) and \( \overline{L} \), and gluing together the end point of \( L \) with the start point of \( \overline{L} \) and vice versa. E.g., the symmetric quiver in Figure 2(a) is obtained from the lists \( L \) and \( \overline{L} \) shown in Figures 2(b) and 2(c).
To prove the statement it remains to show that exactly two different lists belong to a given symmetric quiver \( Q \). Observe first, that \( Q \) is of the form \( Q = (L, \overline{L}) \) where the end point of \( L \) is glued together with the start point of \( \overline{L} \) and vice versa. It may happen that \( L \) is itself symmetric i.e., \( L = \overline{L} \). However, it is always possible to find a non-symmetric \( X \) such that \( Y := \overline{X} \neq X \) and \( Q = (L, \overline{L}) = (L', \overline{L'}) \) where \( L = (X, Y, X, \ldots, Y) \) and \( L' = (Y, X, Y, \ldots, X) \). That is, any symmetric quiver is of the following form:

\[
\begin{array}{c}
X & & Y \\
& Y & \quad X \\
\end{array}
\]

This proves that there exist exactly two different lists that correspond to a symmetric quiver \( Q \), namely \( L \) and \( L' \). \( \square \)

We now know the number of realizations of quivers as well as the number of symmetric quivers of type \( \tilde{A}_{2r-1} \) with parameters \( r, r \). Therefore, we can compute the number of quivers of type \( \tilde{A}_{2r-1} \) with parameters \( r, r \):
Table 1. Number of quivers of type $\tilde{A}_{n-1}$ according to the parameter $r$ for $n$ in $\{2, 3, \ldots, 10\}$

| $n$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|
|     | 1  | 2  | 5  | 4  | 14 | 12 | 42 | 36 | 22 |
|     | 132| 108| 100|    | 429| 349| 315| 172|    |
|     | 1430|1144|1028|980|4862|3868|3432|3240|1651|

Corollary 3.8. The number of quivers of type $\tilde{A}_{2r-1}$ with parameters $r, r$ is given by

$$\frac{1}{2} \left( \frac{1}{r} \binom{2r}{r} + \sum_{k|r} \frac{\phi(k)}{k} \binom{2r/k}{r/k}^2 \right).$$

Proof. According to Theorem 3.4, the expression $\sum_{k|r} \frac{\phi(k)}{k} \binom{2r/k}{r/k}^2$ counts realizations of quivers with parameters $r, r$. Therefore, it counts non-symmetric quivers with parameters $r, r$ twice and symmetric quivers with parameters $r, r$ once. By Theorem 3.7, the number of symmetric quivers with parameters $r, r$ is given by $\frac{1}{2} \binom{2r}{r}$. In total, we get the desired expression. \(\square\)

3.3. The number of quivers of type $D_n$. With the help of Corollary 3.5 and a little extra work we obtain the number of quivers in the mutation class of Dynkin type $D_n$. This result was first determined by Buan and Torkildsen in [4].

Corollary 3.9. The number of quivers of type $D_n$, for $n \geq 5$, is given by

$$\tilde{a}(0, n) = \sum_{d|n} \frac{\phi(n/d)}{2n} \binom{2d}{d}.$$

The number of quivers of type $D_4$ is 6.

Proof. For $n = 4$, the quivers can be explicitly listed, see [4]. We remark that their number does not agree with the general formula. Now, let $\tilde{D}_n$, $n \geq 5$, be the family of cyclic arrangements of elements in $B$, with all base arrows oriented clockwise and a total of $n$ vertices. Thus, the elements in $\tilde{D}_n$ are quivers with a distinguished oriented cycle, which we call the main cycle. Note that the main cycle may be an oriented 2-cycle or even a loop.

We want to show that the quivers of type $D_n$ are in bijection with those in $\tilde{D}_n$. To do so, we use the classification given by Vatne [11], who distinguishes four types I–IV. Quivers in $D_n$ of type IV coincide with those objects in $\tilde{D}_n$ whose main cycle consists of at least three arrows. The other three types are as in Figure 3.

Suppose that the main cycle of $\tilde{Q} \in \tilde{D}_n$ is an oriented 2-cycle. By deleting these two arrows we obtain one of the following:
Figure 3. Quivers in $D_n$ of type I–III.

(1) a quiver in $D_n$ of type I, where precisely one of the two distinguished arrows incident to the root is oriented towards it, or

(2) a quiver in $D_n$ of type III, i.e., a quiver having a unique oriented 4-cycle.

It remains to describe the bijection in the case where the main cycle of $\tilde{Q} \in \tilde{D}_n$ is a loop. In a first step, we delete the vertex of this loop and all arrows incident to it, to obtain a rooted quiver $\bar{Q}$ of type $A$. For the second and final step, we distinguish two cases:

1) the root of $\bar{Q}$ is incident to a single arrow $\alpha$. In this case we obtain a quiver $Q$ in $D_n$ of type I by adding a second arrow, oriented in the same way as $\alpha$, to the other vertex $\alpha$ is incident to.

2) On the other hand, consider the case that the root of $\bar{Q}$ is incident to an oriented 3-cycle $\gamma$. Then, we glue a second 3-cycle, oriented in the same way as $\gamma$, along the arrow of $\gamma$ opposite to the root. In this way we we create a quiver in $D_n$ of type II.

This transformation is invertible:

- a quiver $Q$ in $D_n$ is of type I if and only if it has a uniquely determined root, and two distinguished arrows incident to it. If they are oriented in opposite directions, then the main cycle in the preimage of the transformation is an oriented 2-cycle. Otherwise, the preimage is a loop.
- $Q$ is of type II, if and only if it has two oriented 3-cycles sharing an arrow.
- Finally, $Q$ is of type III if and only if it has a unique oriented 4-cycle.

To conclude, we compute the number of elements in $\tilde{D}_n$. This is easy, since we can use the degenerate case of $r = 0$ and $s = n$ of Corollary 3.5:

$$\tilde{a}(0, n) = \frac{1}{2} \sum_{k|n} \phi(k) \left( \frac{2n}{k} \right) \left( \frac{n}{k} \right)$$

$$= \frac{1}{2} \sum_{d|n} \phi(n/d) \left( \frac{2d}{d} \right), \text{ for } d := \frac{n}{k}.$$

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