Evolutionary game dynamics with three strategies in finite populations

Jing Wang\textsuperscript{a,b}, Feng Fu\textsuperscript{a,b}, Long Wang\textsuperscript{a,b,*}, Guangming Xie\textsuperscript{a,b}

\textsuperscript{a}Intelligent Control Laboratory, Center for Systems and Control, Department of Mechanics and Space Technologies, College of Engineering, Peking University, Beijing 100871, China

\textsuperscript{b}Department of Industrial Engineering and Management, College of Engineering, Peking University, Beijing 100871, China

Abstract

We propose a model for evolutionary game dynamics with three strategies $A$, $B$ and $C$ in the framework of Moran process in finite populations. The model can be described as a stochastic process which can be numerically computed from a system of linear equations. Furthermore, to capture the feature of the evolutionary process, we define two essential variables, the global and the local fixation probability. If the global fixation probability of strategy $A$ exceeds the neutral fixation probability, the selection favors $A$ replacing $B$ or $C$ no matter what the initial ratio of $B$ to $C$ is. Similarly, if the local fixation probability of $A$ exceeds the neutral one, the selection favors $A$ replacing $B$ or $C$ only in some appropriate initial ratios of $B$ to $C$. Besides, using our model, the famous game with AllC, AllD and TFT is analyzed. Meanwhile, we find that a single individual TFT could invade the entire population under proper conditions.

Key words: evolutionary game theory, three strategies, finite populations, global fixation probability, local fixation probability

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1 Introduction

Since the theory of games was first explicitly applied in evolutionary biology by Lewontin [1], it has undergone extensive development. Evolutionary game
theory as a mathematical framework to depict the population dynamics under natural selection has attracted much attention for a long time. It is a way of thinking about particular phenotypes depending on their frequencies in the population [2]. Much more important results in terms of the corresponding replicator dynamics [3,4], in which the size of the well-mixed population is infinite, promote the development of the evolutionary game theory. Replicator dynamics, which is due to Taylor and Jonker [5], is a system of deterministic differential equations which could describe the evolutionary dynamics of multi-species. Besides replicator dynamics, the Lotka-Volterra equations which were devised by Lotka and Volterra, have received much attention. They are the most common models for the dynamics of the population numbers [3,6], whereas replicator dynamics is the most common model for the evolution of the frequencies of strategies in a population. These two deterministic models both fail to account for stochastic effects. Thus the theory of stochastic processes plays an extraordinarily important role in depicting the evolutionary dynamics.

In nature, however, populations are finite in size. Finite population effects can be neglected in infinite populations, but affect the evolution in finite size. In fact, with high probability, the state of the process for large size $N$ remains close to the results of corresponding deterministic replicator dynamics for some large time $T$ [7]. Recently, an explicit mean-field description in the form of Fokker-Planck equation was derived for frequency-dependent selection in finite populations [8,9,10]. It is an approach which could connect the situation of finite populations with that of infinite populations. The explicit rules governing the interaction of a finite number of individuals with each other are embodied in a master equation. And the finite size effects are captured in the drift and diffusion terms of a Fokker-Planck equation where the diffusion term vanishes with $1/N$ for increasing population sizes. This framework was extended to an evolutionary game with an arbitrary number of strategies [11]. The stochastic evolutionary processes would be characterized by the stochastic replicator-mutator equation $\dot{x}_i = a_i(\vec{x}) + \sum_{j=1}^{d-1} c_{ij}(\vec{x}) \xi_j(t)$, here $x_i$ means the density of the $i$th individuals using one of the arbitrary $d$ strategies. Note that the difference between above equation and the replicator dynamics in infinite populations is the uncorrelated Gaussian noises term $\sum_{j=1}^{d-1} c_{ij}(\vec{x}) \xi_j(t)$. Thus the finite size effects can be viewed as combinations of some uncorrelated Gaussian noises.

Nowak introduced the frequency-dependent Moran process into evolutionary game in finite populations [12,13,14]. The frequency-dependent Moran process is a stochastic birth-death process. It follows two steps: selection, a player is selected to reproduce with a probability proportional to its fitness, and the offspring will use the same strategy as its parent; replacement, a randomly selected individual is replaced by the offspring. Hence, the population size, $N$, is strictly constant [15]. Suppose a population consists of individuals who use either strategy $A$ or $B$. Individual using strategy $A$ receives payoff $a$ or $b$ playing with $A$ or $B$ individual; individual using strategy $B$ obtains payoff
c or d playing with A or B individual. Viewing individuals using strategy A as mutants, we get the probability \( \bar{x}(i) \) that \( i \) mutants could invade and take over the whole populations. The fitness of individuals using strategy A and B is respectively given by:

\[
\begin{align*}
  f_i &= 1 - w + w[a(i - 1) + b(N - i)]/(N - 1) \\
  g_i &= 1 - w + w[ci + d(N - i - 1)]/(N - 1)
\end{align*}
\]

Here \( i = 1, 2, \ldots, N - 1 \), \( w \in [0, 1] \) describes the contribution of the game to the fitness. For neutral selection, it needs \( w = 0 \); for weak selection, it must satisfy the condition \( w \ll 1 \). Accordingly, the fixation probability \( \bar{x}(i) \) is given by [16]:

\[
\bar{x}(0) = 0, \bar{x}(N) = 1 \\
\bar{x}(i) = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{g_k}{f_k}}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^{j} \frac{g_k}{f_k}}, \quad i = 1, 2, \ldots, N - 1
\]

In the limit of weak selection, the 1/3 law can be obtained. If A and B are strict Nash equilibria and the unstable equilibrium occurs at a frequency of A which is less than 1/3, then selection favors replacement of B by A. Moreover, stochastic evolution of finite populations need not choose the strict Nash equilibrium and can therefore favor cooperation over defection [17].

Obviously, the characteristic timescales also play a crucial role in the evolutionary dynamics. Sometimes, although the mutant could invade the entire population, it takes such a long time that the population typically consists of coexisting strategies [18]. It can be shown that a single mutant following strategy A fixates in the same average time as a single B individual does in a given game, although the fixation probability for the two strategies are different [19]. Furthermore, if the population size is appropriate for the fixation of the cooperative strategy, then this fixation will be fast [20]. Besides the standard Moran process, Wright-Fisher model and pairwise comparison using Fermi function are brought into the analysis of the evolutionary game in finite populations [21,22]. Moreover, spatial structure effects can not be ignored in the real world. Much results reveal that a proper spatial structure could enhance the fixation probability [23,24].

Most results state the situation with two strategies. But in reality, there may be many strategies in a game. Furthermore, in coevolution of three strategies, how and why a single A individual could invade a finite population of B and C individuals, what kinds of strategists would be washed out by the natural selection, and how cooperation could emerge in finite populations are unclear.
Motivated by these, here we study the evolutionary game of finite populations with three strategies. This paper is organized as follows. Using the stochastic processes theory, we formulate the evolutionary game dynamics in finite populations as a system of linear equations in Section 2. The variable of these equations is fixation probability $x(i, j)$, which represents the probability that $i$ individuals using strategy $A$ could dominate a population in which $j$ of them follow strategy $B$ and $N - i - j$ follow strategy $C$. Two probabilities, the global and the local fixation probability, act crucial roles in the evolutionary game dynamics with three strategies. If the global fixation probability of a single $A$ individual exceeds the neutral fixation probability $1/N$, the selection favors $A$ replacing $B$ or $C$ no matter what the initial ratio of $B$ to $C$ is. Similarly, if the local fixation probability exceeds the neutral one $1/N$, the selection favors $A$ replacing $B$ or $C$ only in some appropriate initial ratios of $B$ to $C$. In Section 3, some numeric computations of evolutionary game with AllC, AllD and TFT are adopted to investigate the emergence of cooperation in some specified situations. For weak selection and sufficiently large size $N$, we find a condition in terms of the number of rounds $n$ and the ratio $r$ of cost to benefit, under which the selection favors only one TFT replacing AllC or AllD individuals. Furthermore, the condition under which a single TFT could invade the entire population is also obtained. Finally, the results are summarized and discussed in Section 4.

2 Model

Let us consider a well-mixed population of constant and finite $N$ individuals. Suppose the strategy set in our model is $A$, $B$ and $C$. The payoff matrix of the three strategies is

$$
\begin{array}{ccc}
A & B & C \\
A & a & b & p \\
B & c & d & q \\
C & m & n & l \\
\end{array}
$$

The fitness of individuals using $A$, $B$ and $C$ is respectively as follows:

$$
\begin{align}
    f_{i,j} &= 1 - w + w[a(i - 1) + bj + p(N - i - j)]/(N - 1) \\
    g_{i,j} &= 1 - w + w[ci + dj(j - 1) + q(N - i - j)]/(N - 1) \\
    h_{i,j} &= 1 - w + w[mi + nj + l(N - i - j - 1)]/(N - 1)
\end{align}
$$

(3)
Here \( i \) denotes the number of individuals using strategy \( A \), \( j \) denotes the number of those using strategy \( B \), and there are \( N = i + j \) players using strategy \( C \). The balance between selection and drift can be described by a frequency-dependent Moran process. At each time step, the number of \( A \) individuals increases by one corresponding to two situations. One is eliminating a \( B \) individual whereas the number of \( C \) players keeps unchanged. The other is eliminating a \( C \) individual whereas the number of \( B \) players keeps unchanged. The transition probabilities can be formulated as:

\[
\begin{align*}
    p_{i,i+1}^{j,j} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{N - i - j}{N} \\
    p_{i,i+1}^{j,j-1} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{i}{N} \\
    p_{i,i-1}^{j,j} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{jg_{i,j}}{i} \\
    p_{i,i-1}^{j,j+1} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{N - i - j}{N} \\
    p_{i,i}^{j,j+1} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{i}{N} \\
    p_{i,i}^{j,j-1} &= \frac{if_{i,j}}{if_{i,j} + jg_{i,j} + (N - i - j)h_{i,j}} \frac{jg_{i,j}}{i} \\
    p_{i,i}^{j,j} &= 1 - p_{i,i}^{j,j+1} - p_{i,i}^{j,j-1} - p_{i,i}^{j,j+1} - p_{i,i+1}^{j,j} - p_{i,i+1}^{j,j+1}
\end{align*}
\]

Here \( p_{i,i+1}^{j,j} \) is the transition probability from the state of \( i \) \( A \), \( j \) \( B \) and \( N - i - j \) \( C \) individuals to that of \( i + 1 \) \( A \), \( j \) \( B \) and \( N - i - j - 1 \) \( C \) individuals. Let \( x(i, j) \) denotes the fixation probability that \( i \) \( A \) individuals could invade the population of \( j \) \( B \) and \( N - i - j \) \( C \) individuals. We have the recursive relation:

\[
\begin{align*}
x(i, j) &= x(i + 1, j)p_{i,i+1}^{j,j} + x(i + 1, j - 1)p_{i,i+1}^{j,j-1} \\
&\quad + x(i - 1, j)p_{i,i-1}^{j,j} + x(i - 1, j + 1)p_{i,i-1}^{j,j+1} \\
&\quad + x(i, j + 1)p_{i,i}^{j,j+1} + x(i, j - 1)p_{i,i}^{j,j-1} + x(i, j)p_{i,i}^{j,j}
\end{align*}
\]

Researchers reported that in a well-mixed environment, two of the initial three kinds of strategists would go extinct after some finite time, while coexistence of the populations was never observed [25]. In finite populations, no matter how many kinds of individuals initially, only one type of strategists can survive in the evolutionary game eventually. Hence, the fixation probabilities of the game with two strategies can be viewed as special boundary conditions of our model. There are three types of boundary conditions:

(1) obviously, \( x(0, j) = 0, j = 0, 1, \ldots, N \);

(2) \( x(i, 0) = \bar{x}(i), i = 0, 1, \ldots, N \), here \( \bar{x}(i) \) is the fixation probability that \( i \) \( A \) individuals could take over the population of \( N - i \) \( C \) and no \( B \) players;

(3) similarly, \( x(i, N - i) = \bar{x}(i), i = 0, 1, \ldots, N \), here \( \bar{x}(i) \) means the fixation
probability that \( i \) \( A \) individuals could invade the population of \( N - i \) \( B \) and no \( C \) players. Note that \( \bar{x}(i) \) can be formulated as Eq. 2. Similarly, \( \tilde{x}(i) \) is written as

\[
\tilde{x}(i) = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} h_k}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^{j} h_k}
\]

here \( i = 1, 2, \ldots, N - 1 \), \( f'_i = 1 - w + w[a(i - 1) + p(N - i)]/(N - 1) \), \( h_i = 1 - w + w[m + l(N - i - 1)]/(N - 1) \), and the corresponding boundary conditions are \( \tilde{x}(0) = 0, \tilde{x}(N) = 1 \).

The relationship among the solutions of the system of equations can be depicted by Fig. 1. The point \((i, j)\) marked by full black dot denotes the boundary condition of the equations, while this marked by empty dot denotes the unknown of the equations. Thus in what follows, the unknown element \( x(i, j) \) of our interest is discussed.

![Fig. 1](image)

Fig. 1. The point \((i, j)\) marked by full black dot means the boundary condition of the equations, while this marked by empty dot means the unknown of the equations which could be formulated by the boundary conditions.

Eq. 5 can be transformed to Eq. 6 which is a system of linear equations in \((N - 1)(N - 2)/2\) variables.

\[
\begin{align*}
[i(N - i)f_{i,j} + j(N - j)g_{i,j} + (i + j)(N - i - j)h_{i,j}]x(i, j) &= i(N - i - j)h_{i,j}x(i - 1, j) + ijg_{i,j}x(i - 1, j + 1) \\
&+ i(N - i - j)f_{i,j}x(i + 1, j) + ijf_{i,j}x(i + 1, j - 1) \\
&+ j(N - i - j)g_{i,j}x(i, j + 1) + j(N - i - j)h_{i,j}x(i, j - 1)
\end{align*}
\]

(6)

Where \( i = 1, 2, \ldots, N - 2, j = 1, 2, \ldots, N - i - 1 \). Accordingly, Eq. 5 can also be simplified to \( A\bar{x} = \bar{b} \), where \( \bar{x} \) is a vector \((x(1, 1), x(1, 2), \ldots, x(1, N - 2), x(2, 1), \ldots, x(N - 2, 1))\).
Let us compare the evolutionary game dynamics in infinite populations under some suitable conditions. Whatever the ratio of $B$ to $x$, it favors $A$ to $x$. When $x > 1/N$, natural selection always favors $A$ replacing $B$ or $C$. We define $x_{\text{min}}(i)$ as global fixation probability and $x_{\text{max}}(i)$ as local fixation probability of $i$ A individuals. If $x_{\text{min}}(1) > 1/N$, natural selection favors $A$ replacing $B$ or $C$ no matter what the ratio of $B$ to $C$ is, that is global. Similarly, if $x_{\text{max}}(1) > 1/N$, the selection favors $A$ replacing $B$ or $C$ in some proper ratios of $B$ to $C$, not any ratios, that is local. Accordingly, there may be three situations:

1. if $x_{\text{max}}(1) < 1/N$, natural selection never favors $A$ replacing $B$ and $C$;

2. if $x_{\text{min}}(1) > 1/N$, natural selection always favors $A$ replacing $B$ or $C$ whatever the ratio of $B$ to $C$ is. It is likely that $A$ could invade the population;

3. if $x_{\text{min}}(1) < 1/N$ and $x_{\text{max}}(1) > 1/N$, natural selection favors $A$ replacing $B$ or $C$ in some proper ratios of $B$ to $C$. Thus the fixation of $A$ is possible under some suitable conditions.

Let us compare the evolutionary game dynamics in infinite populations with that in finite populations. The finite size effects bring stochastic factor to the evolution. A individuals which are eliminated in infinite populations may be
favored by natural selection replacing $B$ or $C$ individuals. In Fig. 2, the left column shows evolutionary game with a small frequency of $A$ individuals initially in infinite populations, and the right column shows that with a single $A$ individual at first in finite populations. The first row shows situations with payoff matrix $a = 2, b = 4, c = 3, d = 5, q = 1, m = 3, n = 1, l = 5$, the size $N = 50, w = 0.1$. $A$ individuals will disappear in infinite populations no matter how many $B$ and $C$ individuals are initially. While in finite populations, the selection won't favor $A$ replacing $B$ or $C$. The second row shows the situations with payoff matrix $a = 5, b = 4, p = 4, c = 4, d = 3, q = 4, m = 4, n = 4, l = 3$, the size $N = 50, w = 0.1$. $A$ individuals will always invade the population of $B$ and $C$ individuals in infinite populations. And in finite populations, natural selection will all the time favor $A$ replacing $B$ or $C$ no matter what the ratio of $B$ to $C$ is. The third row shows situations with payoff matrix $a = 5, b = 5, p = 1.6, c = 5, d = 5, q = 0, m = 3, n = 7, l = 2, N = 50, w = 0.1$. $A$ individuals will disappear in infinite populations no matter how many $B$ and $C$ individuals are initially. Furthermore, from plenty of computations, we find that $A$ individuals will monotonously decrease to zero in infinite situation. Whereas sometimes $A$ will tend to replace $B$ or $C$ in finite populations.

For the three payoff matrices in Fig. 2, it is clear that the fixation probability of $A$ individual is not always monotonic function of the number of $B$ individuals (see Fig. 3). Thus, the global and local fixation probability of $A$ may nontrivially occur at intermediate ratio of $B$ to $C$, not always end points of the ratio range of $B$ to $C$.

3 AllC-AllD-TFT

Let us consider a very interesting and famous repeated game with three strategies AllC (cooperate all the time), AllD (defect all the time) and TFT (tit-for-tat). TFT is an adaptive cooperative strategy which is one of the most successful strategies proved by experiments. Individuals using TFT cooperate in the first round generally, and then do whatever the opponents did in the previous round. The number of rounds $n$, by definition, can be $1, 2, \cdots, \infty$. If the rounds are infinite, this game is out of our consideration. In one-round repeated Prisoners' Dilemma game, a cooperator can obtain a benefit of $b$ or 0 if it meets a cooperator or defector, meanwhile, it must cost $c$ whomever its opponent is; a defector can obtain a benefit of $b$ or 0 if it meets a cooperator or defector, but costs nothing in the whole process. We bring the ratio of cost to benefit, by definition, $r$ ($r = c/b \in [0, 1]$), into our game, the payoff matrix
Fig. 2. The comparisons between different situations corresponding to infinite and finite populations respectively. In the left column figures, the $x$–axis denotes the percentage of B individuals initially, meanwhile, the $y$–axis denotes the percentage of A individuals after 1000 steps. In the right column figures, the $x$–axis shows the size of the population, and the $y$–axis shows the fixation probability of a single A. Let NFP, LFP and GFP represent the neutral fixation probability $1/N$, the local and the global fixation probability respectively.

between cooperator and defector can be simplified as follows:

$$
\begin{align*}
C & \quad D \\
C & \quad 1 \quad 0 \\
D & \quad 1 + r \quad r
\end{align*}
$$
Fig. 3. The fixation probability of only one A individual as a function of the number of B individuals. Where \( N = 50 \), NFP is the neutral fixation probability, PM\( i \) means the situation with payoff matrix \( i \) in Fig. 2, \( i = 1, 2, 3 \). The fixation probability of only one A individual is not always monotonic function of the number of B individuals. The global and local fixation probability may be not at the end points of the curve.

Thus the payoff matrix of TFT, AllC and AllD with \( n \) rounds is

\[
\begin{array}{ccc}
TFT & AllC & AllD \\
TFT & n & n & (n-1)r \\
AllC & n & n & 0 \\
AllD & 1 + nr & n(1 + r) & nr \\
\end{array}
\]

The pairwise comparison of the three strategies leads to the following conclusions.

(1) AllC is dominated by AllD, which means it is best to play AllD against both AllC and AllD;

(2) TFT is equal to AllC when TFT plays with AllC;

(3) If the average number of rounds exceeds a minimum value, \( n > 1/(1 - r) \), then TFT and AllD are bistable.

Suppose that a single individual using strategy TFT is brought in the population in which some individuals adopt strategy AllC and the others use strategy AllD originally. Provided that the number of rounds is finite and greater than \( 1/(1 - r) \), the strategies TFT and AllD are both strict Nash equilibrium and evolutionary stable strategies (ESS) [2]. If \( n = 1 \), TFT becomes strategy C which is out of our discussion. Thus we need finite \( n \gg 2 \). For a fixed number of individuals and a value of the rounds, we find that there is a barrier of \( r \) which can determine whether or not the selection favors TFT replacing AllC.
or AllD. The barrier also has two types: one is \( r_l \), which represents the barrier of local situation; the other is \( r_g \), which represents that of global situation. If \( r < r_l \), the natural selection favors TFT replacing AllC or AllD locally, whereas if \( r > r_l \), TFT tends to be washed out by selection. The results about \( r_g \) are similar. Under the condition of weak selection, for sufficiently large population size \( N \) and large number of rounds \( n \), the barrier ratio \( r \) as the function of \( n \) is approximately followed by \( (n - 1)/(n + \theta) \), where \( \theta \) is a parameter dependent with \( N \). In Fig. 4, we can fit \( r_l \) and \( r_g \) as \( r_l = (n - 1)/(n + 2) \) and \( r_g = (n - 1)/(n + \theta_g) \), here \( \theta_g \) is also a parameter dependent with \( N \) and \( \theta_g > 2 \). Therefore, \( r_l > r_g \). In other words, the ratio \( r \) which could lead the selection to favor TFT replacing AllC or AllD globally can also induce local replacement, but not vice versa.

Deterministic replicator dynamics with three strategies in infinite populations

![Graph](image)

Fig. 4. The global and local barrier ratios \( r_g \) and \( r_l \) as a function of the number of rounds \( n \). Here \( N = 50, w = 0.1 \). \( r'^{g}_g \) represents the fitting curve of \( r_g \), \( r'^{l}_l \) represents the fitting curve of \( r_l \). We can fit \( r_l \) and \( r_g \) as \( r_l = \frac{n - 1}{n + 2} \) and \( r_g = \frac{n - 1}{n + \theta_g} \). In other words, TFT admits two interior equilibria at frequency of TFT given by \( x^* = \frac{r}{(n - 1)(1 - r)} \) and \( x^{**} = \frac{nr}{(n - 1)} \). For local situation, substitute \( r < (n - 1)/(n + 2) \) into \( x^* \) and \( x^{**} \), we get \( x^* < \frac{1}{3} \) and \( x^{**} < \frac{n}{(n + 2)} \); for global situation, substitute \( r < (n - 1)/(n + \theta_g) \) into \( x^* \) and \( x^{**} \), we obtain \( x^* < \frac{1}{(1 + \theta_g)} \) and \( x^{**} < \frac{n}{(n + \theta_g)} \). If the frequency of TFT at the equilibrium is \( x^* < \frac{1}{3} \) or \( x^{**} < \frac{n}{(n + 2)} \) in infinite populations, it will be favored replacing AllC or AllD locally in finite situation; if the frequency of TFT at the equilibrium of infinite situation is \( x^* < \frac{1}{(1 + \theta_g)} \) or \( x^{**} < \frac{n}{(n + \theta_g)} \), it will tend to replace AllC or AllD globally by TFT in finite populations.

The 1/3 law proposed by Nowak in [12] is still valid in our case, that is, the selection favors TFT replacing AllC or AllD in finite populations, if its frequency at the equilibrium is \( x^* < \frac{1}{3} \) in infinite populations. However, when
there are three strategies TFT, AllC and AllD, in which TFT’s frequency at one equilibrium is \( x^* \), the corresponding frequency of AllC is zero. Thus in this situation, our results to some extent validate the conjecture in which AllC is eliminated by natural selection so quickly that the effect of AllC can be neglected in finite populations. And then the evolutionary game dynamics with the two left strategies TFT and AllD, is equivalent to the situation of the situation with these two strategies initially. Nevertheless not any size of AllC individuals could be wiped out quickly, their effects can not be ignored in the dynamics. Hence \( x^* < 1/3 \) can only determine the replacement locally (in some certain circumstances). As for global fixation situation (the fixation is certain for any ratio of AllC to AllD), we have \( x^* < 1/(1 + \theta_g) < 1/3 \) for \( \theta_g > 2 \). The conditions that natural selection favors global replacement of AllC or AllD by TFT are more intensified than those of local situation. 

Let us discuss the other equilibrium. The \( n/(n + \theta) \) is a monotony increasing function of \( n \). This \( n/(n + \theta) \) approaches one for increasing \( n \). That is to say, when the number of rounds increases, the condition \( x^{**} < n/(n + \theta) \) can be satisfied with higher probability, and TFT may have more opportunities to replace AllC or AllD locally and globally. In the standard evolutionary model of the finitely repeated Prisoner’s Dilemma, TFT can not invade AllD. But interestingly, we find that for intermediate \( n \), if \( x^{**} < n/(n + \theta) \), nature selection favors TFT replacing AllC or AllD as the fixation probability of a single TFT (\( \rho_{TFT} \)) is larger than that of a single AllC (\( \rho_{AllC} \)) or AllD (\( \rho_{AllD} \)). Actually, in this case, \( \rho_{TFT} > 1/N > \max(\rho_{AllD}, \rho_{AllC}) \). Therefore, a single TFT is likely to invade the entire population consisting of AllC and AllD finally. In this case, cooperation tends to emerge in the evolution process. And yet, for large limit \( n \), the situation is out of our consideration due to its extraordinary intricacy. However, as \( n \) increasing to infinite, the probability that the selection favors TFT taking over the whole population also approaches one. Accordingly, the fixation of cooperation is enhanced in finite populations. It is because that when TFT meets AllD, its loss in the first round can be diluted by many rounds games. In this case, the total fitness of TFT is almost the same as that of AllD and they are a pair of nip and tuck opponents. But TFT receives more payoff than AllD when they both play with TFT. To sum up, TFT is superior to AllD for limit large rounds because of its adaption. As a result of this property of TFT, natural selection mostly prefers to choose TFT to reproduce offspring, and then TFT is most likely to dominate the population at last. Therefore, cooperation has more opportunities to win in finite populations contrasting against infinite situation.
4 Conclusion

We have proposed a model of evolutionary game dynamics with three strategies in finite populations. It can be characterized by a frequency-dependent Moran process which could be stated by a system of linear equations. By the comparative study of evolution in finite and infinite populations, we show that a single $A$ individual which can not invade infinite populations may have an opportunity to replace $B$ or $C$ in finite situation. In other words, a single $A$ individual could be eliminated by selection with smaller probability in finite populations than situation in infinite populations. In addition, a famous game with AllC, AllD, and TFT is adopted to illuminate our results by numeric computations. Furthermore, under the condition of weak selection, for sufficiently large population size $N$ and appropriate number of rounds $n$, a single TFT could invade the population composed of AllC and AllD with high probability almost one. In this situation, the emergence of cooperation is attributed to the finite population size effects. Our results may help understand the coevolution of multi-species and diversity of natural world.

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