SMALL RESOLUTIONS OF MODULI SPACES OF SCALED CURVES

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ABSTRACT. We construct small resolutions of the moduli space $\mathcal{Q}_n$ of stable scaled $n$-marked lines of Ziltener and Ma’u–Woodward and of the moduli space $\mathcal{P}_n$ of stable $n$-marked $\mathbb{G}_a$-rational trees introduced in earlier work. The resolution of $\mathcal{P}_n$ is the augmented wonderful variety corresponding to the graphic matroid of the complete graph. The resolution of $\mathcal{Q}_n$ is a further blowup, also a wonderful model of an arrangement in $\mathbb{P}^{n-1}$.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

The results of this paper involve two related moduli spaces:

1. the moduli space $\mathcal{Q}_n$ of stable nodal scaled marked lines (also called stable scaled affine curves (or lines) in the literature) constructed by Ziltener [22, 23] and Ma’u and Woodward [17], denoted by $\mathcal{Q}_n(\mathbb{C})$ in [17, §10];
2. the moduli space $\mathcal{P}_n$ of stable $n$-marked $\mathbb{G}_a$-rational trees denoted by $\mathcal{P}_n,\mathbb{C}$ in [21, §1.3].

$\mathcal{Q}_n$ (respectively $\mathcal{P}_n$) is a compactification of the space of configurations of $n$ distinct (respectively not necessarily distinct) labeled points on an affine line modulo translation, that is, $(z_1, \ldots, z_n) \sim (z'_1, \ldots, z'_n)$ if $z'_1 - z_1 = \cdots = z'_n - z_n$. Both are moduli spaces of nodal genus 0 curves with markings and some additional structure that captures the rough idea of ‘scale’ (i.e. what is preserved by translation). For $\mathcal{P}_n$, please see [21, §1.3] for an elementary introduction and more details, and [21, Theorem 1.5] for its functor of points (with the trivial changes over $\mathbb{C}$), which serves as the definition. For $\mathcal{Q}_n$, please see [17, §10], but also [8, Definition 2.4 and Theorem 2.5] and [20, Example 4.2.(d)], where $\mathcal{Q}_n$ is called $\mathcal{M}_{n,1}(\mathbb{A})$. The equivalence of the definitions is stated in [20, Example 4.2.(d)].

A mysterious feature of $\mathcal{Q}_n$ and $\mathcal{P}_n$ is that they are mildly singular in general (in codimension 3) despite being fine moduli spaces. In this paper, we will construct small resolutions of singularities of $\mathcal{P}_n$ and $\mathcal{Q}_n$. Here, a small resolution of singularities means a proper birational morphism from a smooth variety, whose exceptional locus has codimension at least 2 in the source. (For $n \geq 7$, the small resolutions...
we will construct are in fact not small resolutions in the sense used in intersection homology \([9, \S 6.2]\) by Remark 5.12.)

In fact, the small resolution of \(\overline{\mathcal{F}}_n\) is the augmented wonderful variety corresponding to the graphic matroid of the complete graph \(K_n\). Augmented wonderful varieties play an important role \([10, 3, 4, 6, 13]\) in the major recent developments in the interactions between algebraic geometry and matroid theory. They are also resolutions of ‘matroid’ Schubert varieties; however, these resolutions aren’t small, so don’t include the one we will study.

For any subset \(S\) of \(\{1, \ldots, n\}\), \(|S| \geq 2\), we have a diagonal of \(\mathbb{P}^{n-1}\),
\[
\{[X_1 : \cdots : X_n] \in \mathbb{P}^{n-1} : X_i = X_j \text{ if } i,j \in S\}.
\]

Polydiagonals of \(\mathbb{P}^{n-1}\) are arbitrary intersections of diagonals, including \(\mathbb{P}^{n-1}\) itself. Let \(H\) be the hyperplane \(X_1 + \cdots + X_n = 0\). Let \(\mathcal{D}\) be the set of diagonals, and
\[
(1) \quad \mathcal{P}_0 = \{\Pi \cap H : \Pi \neq \{1 : \cdots : 1\}\} \text{ is a polydiagonal},
\]
which can be thought of as the projective version of the intersection semilattice of the Coxeter arrangement of type \(A_{n-1}\). Note that both \(\mathcal{P}_0\) and \(\mathcal{P}_0 \cup \mathcal{D}\) are building sets in \(\mathbb{P}^{n-1}\).

**Definition 1.1.** Let \(W_n\) and \(T_n\) be the De Concini–Procesi wonderful models \([5, \S 4]\) of \(\mathcal{P}_0\) respectively \(\mathcal{P}_0 \cup \mathcal{D}\).

Then \(W_n\) is the augmented wonderful variety corresponding to the graphic matroid of a complete graph \([3, \text{Remark 2.13}]\); please see Remark 5.5 for details. Note that \(T_n\) can be obtained from \(W_n\) by a sequence of blowups with smooth centers.

**Theorem 1.2.** There exists a small resolution \(W_n \to \mathcal{P}_n\) if \(n \geq 4\).

**Theorem 1.3.** There exists a small resolution \(T_n \to \mathcal{Q}_n\) if \(n \geq 4\).

It is a famous fact that \(\overline{\mathcal{M}}_{0,n}\) is the minimal (projective) wonderful model of the arrangement of type \(A_{n-2}\). The results above can be thought of as a version for scaled curves of this well-known fact, with the interesting twist that the wonderful models are instead the small resolutions of the moduli spaces – of course they couldn’t have been precisely the moduli spaces, since these are singular.

Theorem 1.2 also sheds some new light on this augmented wonderful variety. It shows that \(W_n\) admits a natural family of nodal curve over it with a logarithmic vector field, and is close to being a moduli space. In fact, this will be the point of view we will take to prove Theorem 1.2. It would be interesting to know the extent to which this applies to other augmented wonderful varieties.

It is very natural to wonder how these results can be understood from the point of view of Mori theory. This aspect will be discussed in a separate paper.

**Polydiagonal degenerations and outline of proof.** The idea to prove Theorems 1.2 and 1.3 is to obtain the resolutions from the representability of the moduli functors. We will work very little with \(\mathcal{P}_n\) and \(\mathcal{Q}_n\), and the paper can be read with only vague understanding of these moduli spaces.

The key object is a type of degeneration of some compactifications of configuration spaces \(\text{F}(X, n)\), which we will call polydiagonal degenerations \((\S 4)\) inspired by Ulyanov’s polydiagonal compactification \(X(n)\) \([19]\). Although a common generalization is likely possible, we will only need the case of the trivial compactification \(X^n\) (for Theorem 1.2) and that of the Fulton–MacPherson compactification \(X[n]\)
Polydiagonal degenerations are constructed using Li’s theory of wonderful compactifications [15] and still parametrize degenerations of $X$ à la Fulton–MacPherson cf. Theorem 4.6, so they’re not drastically new. A new feature is the fact that these degenerations of $X$ are, in some suitable sense, scaled. The scaling is defined in terms of a line bundle which is roughly the dim $X$-th root of the dualizing sheaf (§2).

When $X = C$ is a curve, $W_n$ is a codimension 2 subvariety of the polydiagonal degeneration of $C^n$. If we restrict the natural universal family to $W_n$, we can erase the components isomorphic to $C$, and we obtain an unstable $n$-marked $G_a$-rational tree. We stabilize it and obtain a map $W_n \to \mathcal{P}_n$ by [21, Theorem 1.5], which turns out to be the desired small resolution. The idea for $Q_n$ is similar, replacing $C^n$ with $C[n]$. Moreover, although we will not spell it out below, this approach makes it clear that similar ideas also apply to the spaces $M_{n,1}(C)$ from [20, Example 4.2.(e)] or [8, Definition 2.1].

Acknowledgements. I guessed that the small resolution of $\mathcal{P}_n$ (which was starting to take shape at the time) is an augmented wonderful variety when I saw [18]. I would like to thank Joel Kamnitzer and Nicholas Proudfoot for asking, respectively answering, the question on mathoverflow. I would also like to thank Joel Kamnitzer for interesting discussions related to this topic.

2. Naive Fulton–MacPherson degeneration spaces

Let $X$ be a smooth projective variety of dimension $d > 0$. The notion of Fulton–MacPherson degeneration space (or simply FM space) $(W \to S, W \to X)$ of $X$ over a base $S$ has been introduced in [11, Definition 2.1.1]. Such spaces will come up in §4, though we may even work with a smaller, more elementary, class.

Let $S$ be a smooth quasi-projective variety.

Definition 2.1. A naive FM space of $X$ over $S$ is an object of the smallest collection of pairs of morphisms $(\pi : W \to S, \xi : W \to X)$ with the following properties:

1. The pair of projections $(S \times X \to S, S \times X \to X)$ is a naive FM space of $X$ over $S$.
2. If $(W \to S, W \to X)$ is a naive FM space of $X$ over $S$, $\sigma : S \to W$ is a smooth section of $W \to S$, $D \subset S$ is a smooth divisor, and $W' = \text{Bl}_{\sigma(D)}W$, then $(W' \to S, W' \to X)$, where the maps are the compositions of $W' \to W$ with $W \to S$ and $W \to X$, is a naive FM space of $X$ over $S$.

Thus, any naive FM space of $X$ over $S$ can be obtained starting from $S \times X$, by repeatedly blowing up as in item 2 above.

Remark 2.2. Although it is true that naive FM spaces are FM spaces in the sense of [11], we won’t even prove it, as the proof seems to be quite technical. All we will need is that, with notation as in Definition 2.1, the fibers of $W \to S$ over $\mathbb{C}$-points of $S$ are FM spaces, and that $W \to S$ is flat. The former is elementary, while the latter follows from the ‘miracle flatness’ theorem [16, Theorem 23.1].

If $(W \to S, W \to X)$ is a naive FM space, then

$$\text{Pic}(W) = \text{Pic}(S \times X) \oplus F_{W,X/S},$$

where $F_{W,X/S}$ is a free abelian group freely generated by the irreducible components of the exceptional locus of $W \to S \times X$. 
Lemma 2.3. For any naive FM space \((\pi : W \to S, \xi : W \to X)\), there exists a unique line bundle \(\sqrt[\omega_{W,X/S}] \in F_{W,X/S}\) such that

\[
\omega_{W/S} = \xi^*\omega_X \otimes \sqrt[\omega_{W,X/S}]^d.
\]

Moreover, in the situation of item 2 in Definition 2.1, if \(\beta : W' \to W\) is the blowup, and \(E\) is the exceptional divisor, then

\[
\sqrt[\omega_{W',X/S}] = \mathcal{O}_{W'}(E) \otimes \beta^*\sqrt[\omega_{W,X/S}].
\]

Proof. Uniqueness is clear, since \(F_{W,X/S}\) is torsion-free. To prove existence, we turn \(\sqrt[\omega_{S \times S,X/S}] = \mathcal{O}_{S \times S}\) and (3) into the inductive definition of the root relative dualizing sheaf, and check (2). Since \(\mathcal{O}_{W'}(E) \in F_{W',X/S}\) and \(\beta^*(F_{W,X/S}) \subseteq F_{W',X/S}\), we have \(\sqrt[\omega_{W',X/S}] \in F_{W',X/S}\). By the well-known \(\omega_W = \omega_{W/S} \otimes \pi^*\omega_S\) [14, Ch. 6, Theorem 4.9.a]) \((\pi\) is lci because \(W,S\) are smooth varieties) and \(\omega_{S \times S} = \omega_S \otimes \omega_X\),

(2) can be rearranged equivalently as

\[
\omega_W = (\pi, \xi)^*\omega_{S \times X} \otimes \sqrt[\omega_{W,X/S}]^d,
\]

which follows inductively from (3) and \(K_{W'} = \beta^*K_W + dE\). □

Definition 2.4. The line bundle \(\sqrt[\omega_{W,X/S}]\) in Lemma 2.3 is the root relative dualizing sheaf of \((W \to S, W \to X)\). If \(d = 1\), we write \(\omega_{W,X/S}\) instead of \(\sqrt[\omega_{W,X/S}]\).

No doubt the root relative dualizing sheaf can be defined for any FM space, not just naive FM spaces, but we won’t need it.

Lemma 2.5. Let \((\pi : W \to S, \xi : W \to X)\) be a naive FM space, \(D \subseteq S\) a smooth divisor, \(\sigma : S \to W\) a smooth section of \(\pi\), and \(W' = \text{Bl}_{\sigma(D)}W\) with exceptional divisor \(E\). Then, for any

\[
\psi \in \text{Hom}_{\mathcal{O}_W} \left(\sqrt[\omega_{W,X/S}], \mathcal{I}_{\sigma(D)} \otimes \mathcal{O}_W\right),
\]

there exists a unique

\[
\psi' \in \text{Hom}_{\mathcal{O}_W} \left(\sqrt[\omega_{W',X/S}], \mathcal{O}_{W'}\right)
\]

such that the restrictions of \(\psi\) and \(\psi'\) to \(W \setminus \sigma(D)\) and \(W' \setminus E\) respectively correspond to each other under the isomorphism \(W \setminus \sigma(D) \cong W' \setminus E\). Moreover, the restriction of \(\psi'\) to \(E\) is equal to 0 if and only if \(\text{Im}(\psi) \subseteq \mathcal{I}_{\sigma(D)}^2 \otimes \mathcal{O}_{W'}\).

Proof. This follows from (3) and the following general fact: if \(W\) is a smooth variety, \(\beta : W' = \text{Bl}_YW \to W\) is the blowup of a smooth closed subvariety \(Y \subset W\) with exceptional divisor \(E\), and \(\mathcal{L} \in \text{Pic}(W)\), then there is an isomorphism

\[
\mathcal{L} \otimes \mathcal{I}_{Y,W} \simeq \beta_*\left(\beta^*\mathcal{L} \otimes \mathcal{I}_{E,W'}\right)
\]

which restricts to the trivial one on \(W \setminus Y\) and has the following property: if \(s\) is a section of \(\mathcal{L} \otimes \mathcal{I}_{Y,W}\) and \(s'\) the corresponding section of \(\beta^*\mathcal{L} \otimes \mathcal{I}_{E,W'}\) under (4), then \(s'|_E = 0\) if and only if \(s\) belongs to \(\mathcal{L} \otimes \mathcal{I}_{Y,W}^2\).

The projection formula boils (4) down to \(\mathcal{I}_{Y,W} \simeq \beta_*\mathcal{I}_{E,W'}\). Working locally on \(W\), we may assume \(\mathcal{L} = \mathcal{O}_W\). Recall that \(\beta^* : \mathcal{O}_W \to \beta_*\mathcal{O}_W\) is an isomorphism, and that \(\beta^*\mathcal{I}_{Y,W} = \beta_*(\mathcal{I}_{E,W'}^k)\) as subsheaves of \(\beta_*\mathcal{O}_W\). The case \(k = 1\) proves the desired isomorphism, while the case \(k = 2\) proves the claim regarding the restriction of sections to \(E\). □
3. Combinatorial language

This section is a summary of the notation of combinatorial nature that will be necessary in sections §4 and §5. There is very little new or of substance.

3.1. Partitions and trees. This is extremely similar to [19, §2], although some small differences are significant in practice. Let $L_{[n]}$ be the lattice of partitions of $[n] = \{1, \ldots, n\}$, with the (inverse, according to some conventions) refinement partial ordering $\rho_1 \leq \rho_2$ if each block of $\rho_2$ is contained in a block of $\rho_1$. Then $\bot = 12 \cdots n = \min L_{[n]}$ and $\top = 1[2] \cdots n = \max L_{[n]}$. For $\rho_1, \rho_2 \in L_{[n]}$, $\rho_1 \lor \rho_2$ and $\rho_1 \land \rho_2$ are the join and the meet respectively, for instance,

$$12|34|56 \land 123|456 = 123456, \quad 12|34|56 \lor 123|456 = 12|3|4|56.$$  

We have $|L_{[n]}| = B_n$, the $n$-th Bell number. For $\rho \in L_{[n]}$, $\sim_\rho$ denotes the corresponding equivalence relation on $[n]$, and $B(\rho)$ is the set of blocks of $\rho$. Recall that $L_{[n]}$ is also the set of flats of the graphic matroid of the complete graph $K_n$.

As in [19], the interplay between chains in $L_{[n]}$ and leveled trees is essential.

**Definition 3.1.** Let $H = \{\rho_1 < \rho_2 < \cdots < \rho_k\} \neq \emptyset$ be a chain in $L_{[n]}$.

The **leveled tree representation of $H$ with phantom vertices included** $T(H)$ is a rooted tree whose root $\star$ is the only level-0 vertex, whose set of level-$i$ vertices is $B(\rho_i)$, and in which edges correspond to inclusions of blocks on consecutive levels (and all level-1 vertices are adjacent to the root). Finally, to each level-$k$ vertex $B \in B(\rho_k)$ we attach legs indexed by the elements of $B$.

The **leveled tree representation of $H$ with phantom vertices excluded** $T'(H)$ is obtained from $T(H)$, by replacing each maximal chain with internal vertices of degree 2 (and vertex endpoints – leg ends are forbidden, see Figure 1, and not containing $\star$ as an internal vertex) with an edge and erasing the internal vertices.

![Figure 1.](image)

**Figure 1.** $H = \{123|45678, 12|3|45|67|8, 12|3|45|6|7|8\}$. We are not contracting all the way to the leg (visible for legs 3, 6, 7, 8).

Please compare with [19, Figure 7].

We will see later that $T(H)$ relates to $W_n$ and $T'(H)$ relates to $\overline{P}_n$, please see Remark 5.8. The following trivial lemma will be needed in §5.

**Lemma 3.2.** Let $H \subseteq L_{[n]}$ be a chain such that $\bot \in H$ and $|H| \geq 2$. If $T'(H) - \star$ is a star (that is, $K_{1,m}$, ignoring the legs) with the center adjacent to $\star$ in $T'(H)$, then $|H| = 2$ and $T'(H) = T(H)$.

**Proof.** The number of non-root, non-leaf vertices of degree larger than 2 is the same in $T(H)$ and $T'(H)$, so it’s at most 1 in $T(H)$ too. Since every level of $T(H)$ except level 0 and level $|H|$ contains at least one such vertex, $|H| - 1 \leq 1$. □
Remark 3.3. For later use, note that the partitions of \([n+1]\) which have \(\{n+1\}\) as a block are in bijection with \(L_{[n]}\), so we get \(\iota : L_{[n]} \rightarrow L_{[n+1]} \setminus L_{[n]} \cong U_{[n]}\), where \(U_{[n]} := \{ (\rho, B) : \rho \in L_{[n]} \text{ and } B \in B(\rho) \}\). By abuse of notation, we will sometimes omit \(\iota\).

3.2. Fulton–MacPherson nests. A family of sets is nested if any two of them are either disjoint or comparable (one is contained in the other). Let \(Z_n\) be set of subsets of \([n]\) with at least two elements, and

\[
Z_{n+1}^* = Z_{n+1} \setminus \{\{1, n+1\}, \ldots, \{n, n+1\}\}.
\]

Recall that \(Z_n\) and \(Z_{n+1}^*\) index the blowups in the construction of the Fulton–MacPherson space \(X[n]\) and its universal family \(X[n]^+\) respectively [7]. We will sometimes regard \(Z_{n}\) as a subset of \(Z_{n+1}\); note then that \(Z_{n+1}^+ \sim Z_n\) by \(T \mapsto T \setminus \{n+1\}\). Recall also that to any nested \(Y \subseteq Z_n\) we may associate a rooted tree \(R(Y)\) which describes the topology of the degenerations of \(X\) in the corresponding stratum of \(X[n]\). If the finite ‘ambient set’ is some \(S\) other than \([n]\), we write \(R_S(Y)\) if there is danger of confusion, or \(R(Y)\) again if not.

We can combine the objects above and those in §3.1 as follows. Let \(H_n\) be the set of nonempty chains in \(L_{[n]}\), and

\[
H_n^+ = \left\{ H \cup Y \right\} \left|\begin{array}{l} \text{H \in H}_n, \ Y \subseteq Z_n, \ Y \cup B(\max H) \text{ is nested, and} \\ \text{any } S \in B(\max H) \text{ is minimal in } Y \cup B(\max H) \end{array}\right\}.
\]

Here, \(Y\) may very well be empty, and this will happen frequently in practice. It would have probably been much more natural to write \((H, Y)\) instead of \(H \cup Y\), but ultimately this strange notation is more convenient. We associate trees \(T(N)\) and \(T'(N)\) to each \(N \in H_n^+\) as follows.

**Definition 3.4.** Let \(N = H \cup Y \in H_n^+\). For each \(B \in B(\max H)\), let

\[
Y|_B = \{ S \in Y : S \subseteq B \},
\]

and, in the rooted tree \(T(H)\) (respectively \(T'(H)\)) in Definition 3.1, we replace the leaf corresponding to \(B\) and the legs attached to this leaf with the rooted trees \(R_B(Y|_B)\), in such a way that the old leaf is replaced with the root of \(R_B(Y|_B)\). The resulting rooted tree with legs is \(T(N)\) (respectively \(T'(N)\)).

![Diagram](image.png)

**Figure 2.** \(N = \{1234|5678, 13|24|5678, \{1, 3\}, \{6, 7, 8\}, \{6, 7\}\}\).
An example for $T(N)$ is shown in Figure 2. Later, we will see that the three strips on the right correspond (after stabilizing if necessary, to obtain $T'(N)$) to infinite, finite, and 0 scaling (e.g. [8, §2]).

4. Polydiagonal degenerations

Throughout §4, $X$ is a smooth projective variety of dimension $d > 0$. For each $\rho \in L_{[n]}$, we have a polydiagonal of $X^n$

$$\Delta_\rho = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i \sim_\rho j\}.$$ 

By abuse of notation, if $S \subseteq [n]$, we have a diagonal

$$\Delta_S = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i, j \in S\}.$$ 

As usual, $F(X, n) = X^n \setminus \bigcup_{|S|>2} \Delta_S$ is the configuration space.

4.1. The polydiagonal degenerations of $X^n$ and $X[n]$. We will define two polydiagonal degenerations, one of $X^n$, and one of the Fulton–MacPherson compactification $X[n]$. Both $X^n$ and $X[n]$ are examples of Kuperberg–Thurston compactifications [12]. A Kuperberg–Thurston compactification can be defined for any finite simple graph $\Gamma$, please see [15, §4.3] for an algebraic construction. We will prefer the notation $X[\Gamma]$ over the notation $X^T$ used in [15] for the trivial reason of being able to write $X[\Gamma]^+$ for the universal family. However, with apologies for doing something so horribly inaesthetic:

From now on, assume that $\Gamma$ is either complete or has no edges.

The graph $\Gamma$ encodes which pairs of points are allowed to collide and which aren’t. Although the assumption is likely not truly needed in what follows, without it, the combinatorics gets harder, and the generality gained is not the most relevant to the story we want to tell ($\Gamma$ is unrelated to the complete graph whose graphic matroid is in some ways part of this story, cf. §1). Identify the set of vertices with $[n]$ in both cases. Thus, $X[\Gamma]$ is either $X^n$ or $X[n]$.

We start by introducing notation. To avoid confusion with diagonals in $X^n$, $\Delta_\Gamma \subseteq X^{n+1}$ is the diagonal corresponding to a subset $T$ of $[n+1]$, and $\Delta_{m} \subseteq X^{n+1}$ is the polydiagonal corresponding to $\varpi \in L_{[n+1]}$. Let

$$\nabla_\alpha = \begin{cases} 
\{0\} \times \Delta_\alpha & \text{if } \alpha \in L_{[n]} \\
A^1 \times \Delta_\alpha & \text{if } \alpha \in Z_n
\end{cases}$$

inside $A^1 \times X^n$ respectively $A^1 \times X^{n+1}$. Let also

$$I_\Gamma = \begin{cases} 
L_{[n]} & \text{if } \Gamma = K_n^c \\
L_{[n]} \cup Z_n & \text{if } \Gamma = K_n
\end{cases}$$

$$I_+ = \begin{cases} 
L_{[n+1]} & \text{if } \Gamma = K_n^c \\
L_{[n+1]} \cup Z_{n+1} & \text{if } \Gamma = K_n
\end{cases}$$

Finally, let $F(X, K_n^c) = X^n$ respectively $F(X, K_n) = F(X, n)$. We will sometimes make use implicitly of the fairly natural inclusions $L_{[n]} \hookrightarrow L_{[n+1]}$ and $Z_n \hookrightarrow Z_{n+1}$ described in §3.

Definition 4.1. The polydiagonal degeneration $X[[\Gamma]]$ of $X[\Gamma]$ is the wonderful compactification [15] of the building set $\mathcal{G}_\Gamma = \{\nabla_\alpha : \alpha \in I_\Gamma\}$ in $A^1 \times X^n$. Let $D_\alpha \subset X[[\Gamma]]$ be the divisor corresponding to $\nabla_\alpha$.

Lemma 4.2. If $\Gamma = K_n^c$, the $\mathcal{G}_\Gamma$-nests are precisely the subsets of the form $\{\nabla_\alpha : \alpha \in H\}$, where $H \in H_n$. If $\Gamma = K_n$, the $\mathcal{G}_\Gamma$-nests are precisely the subsets of the form $\{\nabla_\alpha : \alpha \in N\}$, where $N \in H_n^+$, or $\{\nabla_\alpha : \alpha \in Y\}$, where $Y \subseteq Z_n$ is nested.
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The last type of nest is not fundamentally different from the previous one, but it will not come up much in practice because we’ll be focusing on the fiber over 0 of \( X[[\Gamma]] \). The lemma follows easily from [15, Definition 2.3], and it is very similar to what happens in the usual Fulton–MacPherson setup.

Most of §4 is devoted to studying the universal family over \( X[[\Gamma]] \). In fact, we will construct two universal families \( X[[K_n]]^\wedge \) and \( X[[K_n]]^+ \) over \( X[[K_n]] \). The ‘correct’ one is the latter, but it is convenient to have both.

**Definition 4.3.** Consider the following building sets in \( A^1 \times X^{n+1} \):

\[
\begin{align*}
\mathcal{U}_{K_n}^- &= \{ \nabla_\beta : \beta \in L_{[n+1]} \} = G_{K_{n+1}}^- \\
\mathcal{U}_{K_n}^+ &= \{ \nabla'_\beta : \beta \in L_{[n+1]} \} \\
\mathcal{U}_{K_n}^+ &= \{ \nabla'_\beta : \beta \in L_{[n+1]} \cup Z_{n+1} = I_{K_n}^+ \},
\end{align*}
\]

Let \( X[[\Gamma]]^+ \) and \( X[[\Gamma]]^\wedge \) be the wonderful compactifications of \( \mathcal{U}_{K_n}^+ \) and \( \mathcal{U}_{K_n}^\wedge \) respectively.

**Proposition 4.4.** There exists a (unique) commutative diagram

\[
\begin{array}{ccc}
A^1 \times X[[\Gamma]]^\wedge & \xrightarrow{\xi_n} & X[[\Gamma]]^+ \\
\downarrow & & \downarrow \\
A^1 \times X[\Gamma] \times X & \leftarrow & X[[\Gamma]] \times X
\end{array}
\]

over \( A^1 \times X^{n+1} \), where all varieties are \( A^1 \times X^{n+1} \)-schemes naturally.

Note that most of this diagram is uninteresting when \( \Gamma \) is edgeless. For instance, \( \xi_n \) and the downward arrow are isomorphisms in that case.

**Proof.** The diagram is a fortiori commutative and unique since all varieties are canonically birational to \( A^1 \times X^{n+1} \) (and obviously separated). Hence, it suffices to construct any morphisms over \( A^1 \times X^{n+1} \).

**Lemma 4.5.** Consider enumerations as follows:

- \( l \) is an increasing enumeration of \( L_{[n]} \); first the partition with 1 block, then the partitions with 2 blocks, etc.;
- \( l^+ \) is the corresponding enumeration of \( L_{[n+1]} \setminus L_{[n]} \) by Remark 3.3: first all pairs whose first entry is the first partition in \( L_{[n]} \), then all pairs whose first entry is the second partition in \( L_{[n]} \), etc.;
- \( L \) is an increasing enumeration of \( L_{[n+1]} \);
- \( z \) is the Fulton–MacPherson enumeration [7, page 196] of \( Z_n \) if \( \Gamma \) is complete, respectively empty if \( \Gamma \) is edgeless;
- \( z^* \) is the Fulton–MacPherson enumeration of \( Z_{n+1} \setminus Z_n \) (that is, the FM enumeration of \( Z_{n+1} \) with the segments corresponding to \( Z_n \) and \( Z_{n+1} \setminus Z_n \) removed) if \( \Gamma \) is complete, respectively empty if \( \Gamma \) is edgeless.

The enumerations of the elements in a building set (either \( G_{\Gamma} \) or \( U_{K_n}^+ \)) corresponding to any of the following enumerations of indices: \( zl \) of \( I_{\Gamma} \); \( lzl^*z^* \) of \( I_{\Gamma}^+ \); \( zz^*L \) of \( I_{\Gamma}^+ \) satisfy condition (*) in [15, Theorem 1.3].

**Proof.** It is well-known that the FM enumerations \( z \) and \( zz^* \) (in the context of \( X^n \), though taking a direct product with \( A^1 \) doesn’t change much) satisfy condition (*) in [15, Theorem 1.3]. We will focus on the two enumerations of \( I_{\Gamma}^+ \) and the case...
$\Gamma = K_n$, since the other cases are similar but quite trivial. What needs to be checked is that, given any ‘initial segment’ $\varsigma$ in either enumeration of $I^+_{\Gamma}$ (for $z\sharp$ or $z\sharp^*L$), and any subset of indices $J$ in $\varsigma$, whose corresponding subvarieties intersect at $F_J \subset A^1 \times X^{n+1}$, the minimal subvarieties in $\varsigma$ which contain $F_J$ intersect transversally. If $J$ contains only sets $(z\sharp^*)$, no subvariety indexed by a partition can contain $F_J$, so we can invoke ($\ast$) for the FM enumeration. If $J$ contains at least one partition (from $II^*$ or $L$), $F_J$ must have already appeared on (the list of subvarieties indexed by) $\varsigma$, so the requirement is satisfied trivially in this case. □

In light of Lemma 4.5, [15, Theorem 1.3] gives morphisms:

1. $X[[\Gamma]] / \rightarrow A^1 \times X[[\Gamma]] / \rightarrow A^1 \times X^n$;
2. $X[[\Gamma]] / \rightarrow X[[\Gamma]] / \rightarrow X[[\Gamma]] \times X / \rightarrow A^1 \times X^{n+1}$;
3. $X[[\Gamma]] / \rightarrow A^1 \times X[[\Gamma]] / \rightarrow A^1 \times X[[\Gamma]] \times X / \rightarrow A^1 \times X^{n+1}$

corresponding respectively to the three enumerations in Lemma 4.5. For instance, for the first one, we start with $A^1 \times X^n$ and blow up in order until we get $X[[\Gamma]]$ thanks to [15, Theorem 1.3], but at some point in the sequence of blowups we will see $A^1 \times X[[\Gamma]]$. The other two sequences of morphisms are analogous. □

**Theorem 4.6.**

1. Both $X[[\Gamma]] /$ and $X[[\Gamma]] /$ (with the maps to $X[[\Gamma]]$ and $X$ obtained by composing the ‘diagonal’ maps in the diagram with the projections to the factors) are naive FM spaces of $X$ over $X[[\Gamma]]$.
2. Let $A^* = A^1 \backslash \{0\}$. There exist smooth sections

$$x_1^+, \ldots, x_n^+ : X[[\Gamma]] / \rightarrow X[[\Gamma]] /$$

of $X[[\Gamma]] /$ such that $x_i^+$ restricts on $A^* \times X(X, \Gamma) \subset X[[\Gamma]]$ and $A^* \times F(X, \Gamma) \times X \subset X[[\Gamma]] /$ to the graph of the projection to the $i$-th $X$ factor of $A^* \times F(X, \Gamma) \subset A^* \times X^n$. Moreover,

$$x_i^+(z) \neq x_j^+(z) \quad \text{for all } z \in X[[\Gamma]] / \text{ and } i, j \text{ adjacent in } \Gamma.$$

Furthermore, $x_i^+ = \xi_\alpha \circ x_i^+$ are smooth sections of $X[[\Gamma]] /$.

3. Let $z \in X[[\Gamma]] / (0, \mathbb{C})$ and $N = \{ \alpha \in I_\Gamma : z \in D_\alpha \}$. By Lemma 4.2, $N$ is an element of $H_n$ if $\Gamma$ is edgeless, respectively $H_n^+$ if $\Gamma$ is complete, and let $H$ be the image of $N$ in $H_n$ in either case. Then the dual tree of $X[[\Gamma]] /$ is $T(H)$ (Definition 3.1); the dual tree of $X[[\Gamma]] /$ is again $T(H)$ if $\Gamma = K_n$ since $X[[\Gamma]] / = X[[\Gamma]] /$, respectively $T(N)$ if $\Gamma = K_n$ (Definition 3.4).

4. By item 1 and Lemma 2.3, we may speak of a root relative dualizing sheaf on $X[[\Gamma]] /$. There exists a homomorphism

$$\psi : \sqrt{\omega} = \sqrt{\omega}_{X[[\Gamma]]}, X / X[[\Gamma]] / \rightarrow O_{X[[\Gamma]]}$$

which maps to $t \in \mathbb{C}[t, t^{-1}]$ under the restriction map

$$H^0(X[[\Gamma]] /, \sqrt{\omega}^\vee) \rightarrow H^0(X[[\Gamma]] / \times X[[\Gamma]] /, \sqrt{\omega}^\vee) \cong \mathbb{C}[t, t^{-1}].$$

5. The homomorphisms $(x_1^+) \psi, \ldots, (x_n^+) \psi$ are isomorphisms.

4.2. **Proof of Theorem 4.6.** The proof is a long induction, which follows the sequence of blowups indexed by $z\sharp^*L^*$ (please see Lemma 4.5) from $A^1 \times X^{n+1}$ to $X[[\Gamma]] /$. We split it into 3 parts as follows:

1. The blowups indexed by $I_\Gamma (lz)$;
2. The blowups indexed by $L_{[n+1]} \setminus L_{[n]} \simeq U_{[n]}$, cf. Remark 3.3 (I*);
3. Only in the case $\Gamma = K_n$, the blowups indexed by $Z_{n+1}^* \setminus Z_n (z^*)$.
The ‘scaling’ appears in part II. The other two parts are purely variations on well-known ideas.

**Proof of Theorem 4.6 – part I.** We will need the following simple technical fact.

**Lemma 4.7.** Let $\pi: W \to V$ be a smooth morphism between two smooth varieties, and $\sigma: V \to W$ a section of $\pi$. Let $R \subset V$ be a smooth closed subvariety, $V' = Bl_R(V)$, and $W' = W \times_V V' = Bl_{\pi^{-1}(R)}W$. Let $\pi': W' \to V'$ be the projection and $\sigma': V' \to W'$ the section of $\pi'$ induced by $\sigma$.

If $A \subset V$ and $B \subset W$ are smooth closed subvarieties such that $\sigma(A) = B$, and $A' \subset V'$ and $B' \subset W'$ are the dominant transforms of $A$ and $B$, then $\sigma'(A') = B'$.

To clarify, $Bl_{\pi^{-1}(R)}W = W \times_V V'$ follows from $\pi^*I_{W'}^m = I_{\pi^{-1}(R), W}^m$ for all $m \geq 0$ and the functoriality of the relative Proj construction.

**Proof.** After some reductions, we may assume without loss of generality that $W = V \times \mathbb{A}^m$, $\pi$ is the projection to the first factor, and $\sigma(v) = (v, 0, \ldots, 0)$. Everything is clear in this situation.

Let $x_1^0, \ldots, x_n^0: X[[\Gamma]] \to X[[\Gamma]] \times X$ be the sections induced by $X[[\Gamma]] \to X^n$, and $\Sigma_i^0 = x_i^0(X[[\Gamma]])$. Let $G_\beta^0 \subset X[[\Gamma]] \times X$ be the dominant transform of $\nabla_\beta$ in the sequence of blowups from $\mathbb{A}^1 \times X^{n+1}$ to $X[[\Gamma]] \times X$, for each $\beta \in I^+_1$.

**Lemma 4.8.** Let $\beta \in I^+_1$.

(1) Assume that either $\beta \in \iota(L_{[n]})$ or $\beta \in Z_n$. Then $G_\beta^0 = D_\beta \times X$.

(2) If $\beta \in L_{[n+1]} \setminus L_{[n]}$, and $\beta \mapsto (\rho, B) \in U_{[n]}$ (Remark 3.3), then $G_\beta^0 = x_i^0(D_\rho)$ for all $i \in B$.

(3) If $\beta = T \in Z_{n+1} \cap Z_n$, then $G_\beta^0 = x_i^0(D_{T \setminus \{n+1\}})$ for all $i \in T$.

**Proof.** The first claim is trivial. The others follow inductively from Lemma 4.7, going through the blowups one by one.

We write $G_{\rho, B}^0$ instead of $G_\beta^0$ if $\varpi \in L_{[n+1]} \setminus L_{[n]}$ corresponds to $(\rho, B) \in U_{[n]}$.

**Proof of Theorem 4.6 – part II.** The morphism $X[[\Gamma]]^1 \to X[[\Gamma]] \times X$ was obtained explicitly as a sequence of blowups with smooth centers indexed by $L_{[n+1]} \setminus L_{[n]}$, cf. Proposition 4.4 and Lemma 4.5.

Given a rooted tree $T$ with legs attached only to leaves (such as those in Definition 3.4), if a ‘star’ subset of vertices $S$ is specified, i.e. the shortest chain from the root to $v$ is contained in $S$, for all $v \in S$, then there is an obvious way to trim $T$ relative to $S$: the set of vertices is $S$, the edges of $T$ with both endpoints in $S$ remain edges, and the edges of $T$ with only one endpoint in $S$ become legs. In particular, if $H \subset L_{[n]}$ is a chain, and $T(H)$ is its representation described in Definition 3.1, then any initial segment in the enumeration $\Gamma$ (i.e. all entries up to a given one) specifies a trimming of $T(H)$, since the subset of vertices of $T(H)$ whose label is in this initial segment is a star subset.

**Claim 4.9.** Let $W$ be a blowup of $X[[\Gamma]] \times X$ in the sequence of blowups above. Let $\Sigma_i$ be the dominant transform of $\Sigma_i^0$ on $W$.

(1) $(W \to X[[\Gamma]], W \to X)$ is a naive FM space of $X$ over $X[[\Gamma]]$.

(2) The composition $\Sigma_i \mapsto W \to X[[\Gamma]]$ is an isomorphism.

(3) The morphism $W \to X[[\Gamma]]$ is smooth at all points of $\Sigma_i$. 


According to item 2, \( \Sigma_i \) is the image of a section \( x_i : X[[\Gamma]] \to W \). Let \( G_{\rho,B} \subset W \) be the dominant transform of \( G^0_{\rho,B} \), for \( (\rho,B) \in U[n] \).

4. Let \( x_i(D_\rho) \) if \( i \in B \) and the dominant transform of the polydiagonal corresponding to \( (\rho,B) \) hasn’t been blown up yet.

5. Let \( z \in X[[\Gamma]]_0(\mathbb{C}) \) and \( H = \{ \rho \in L_{[n]} : z \in D_\rho \} \), which is a chain by Lemma 4.2. Then the dual tree of \( W_z \) is the trimming of \( T(H) \) induced by the initial segment of \( I^* \) corresponding to \( W \) (Definition 3.1 and the paragraph above). In particular, if a vertex of the trimmed tree has a leg towards a block which contains the number \( i \in [n] \), then \( x_i(z) \) lives on the component of \( W_z \) which corresponds to the vertex.

The maps \( W \to X[[\Gamma]] \) and \( W \to X \) are obtained as the compositions of the blowdown \( W \to X[[\Gamma]] \times X \) with the projections to the factors.

**Proof.** We proceed inductively. The base case \( W = X[[\Gamma]] \times X \) is trivial, with the exception of item 4, which follows from Lemma 4.8. We make a very simple assumption that’s very hard to phrase: assume that the initial segment of \( I^* \) corresponding to \( W \) is a union of fibers of \( L_{[n+1]} \setminus L_{[n]} \to L_{[n]} \). Thus, at the next steps we have to blow up all \( G_{\hat{\rho},B}, B \in \mathcal{B}(\hat{\rho}) \) for fixed \( \hat{\rho} \in L_{[n]} \). Lemma 4.10 below implies among other things that there is no loss of generality in doing so.

**Lemma 4.10.** If \( B_1, B_2 \in \mathcal{B}(\hat{\rho}) \) and \( B_1 \neq B_2 \), then \( G_{\hat{\rho},B_1} \cap G_{\hat{\rho},B_2} = \emptyset \).

**Proof.** For \( i = 1,2 \), let \( \varpi_i \mapsto (\hat{\rho},B_i) \) by Remark 3.3. The enumeration \( lzl^*z^* \) looks like this:

\[
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \varpi_1 \wedge \varpi_2 \cdots & \cdots & \varpi_1 \ldots \varpi_2 \cdots & \cdots \\
1 & 1 & 1 & l' & 1 & 1 & l' \\
\end{array}
\]

where the bar ‘|’ marks the current state at \( W \). Let’s shuffle as follows:

\[
\begin{array}{cccccccc}
\varpi_1 \wedge \varpi_2 \cdots \varpi_1 \wedge \varpi_2 \cdots \cdots & \cdots & \varpi_1 \ldots \varpi_2 \cdots & \cdots \\
\text{missing} & 1 & 1 & 1' & 1' & 1' & 1' \\
\end{array}
\]

Then, nothing has changed after the current state, and, more importantly, the (enumeration of \( \{ \varpi_\beta : \beta \in I^*_\hat{\rho} \} \) corresponding to) enumeration (5) satisfies condition (\( * \)) in [15, Theorem 1.3]. This can be checked by an argument similar to Lemma 4.5. Indeed, the main thrust of the argument there still holds true: if the chosen intersection is not contained in the 0-fiber, then the property to check is clear from the fact that the FM enumeration satisfies Li’s condition; if the intersection is contained in the 0-fiber, then it must have already appeared on our list, which can be checked by a (slightly confusing but entirely straightforward) case analysis. Then the lemma follows from [15, Theorem 1.3] and the fact that, since \( \varpi_1, \varpi_2 \) are not comparable in \( L_{[n+1]} \), blowing up \( \{0\} \times \Delta_{\varpi_1} \wedge \varpi_2 \) in \( A^1 \times X_{[n+1]} \) separates the proper transforms of \( \{0\} \times \Delta_{\varpi_1} \) and \( \{0\} \times \Delta_{\varpi_2} \).

Assume that \( G_{\hat{\rho},B} \) is the locus to be blown up next. By items 3 and 4, \( G_{\hat{\rho},B} \) is a smooth section over a smooth divisor in \( X[[\Gamma]] \). Moreover, we claim that

\[
x_i(X[[\Gamma]]) \cap G_{\hat{\rho},B} = \begin{cases} 
  x_i(D_\rho), & \text{if } i \in \hat{B} \\
  \emptyset, & \text{if } i \notin \hat{B}.
\end{cases}
\]
Indeed, the case \( i \in \hat{B} \) follows from item 4, whereas the case \( i \notin \hat{B} \) follows from item 4 and Lemma 4.10: if \( B' \) is the block of \( \hat{\rho} \) which contains \( i \), then
\[
x_i(X[[\Gamma]]) \cap G_{\hat{\rho},\hat{B}} = x_i(X[[\Gamma]]) \cap (D_{\hat{\rho}} \times X[[\Gamma]]) W \cap G_{\hat{\rho},\hat{B}}
\]
\[
= x_i(D_{\hat{\rho}}) \cap G_{\hat{\rho},\hat{B}} = G_{\hat{\rho},\hat{B}} \cap G_{\hat{\rho},\hat{B}} = \emptyset.
\]
In either case, \( x_i(X[[\Gamma]]) \cap G_{\hat{\rho},\hat{B}} \) is a possibly empty smooth effective Cartier divisor on \( x_i(X[[\Gamma]]) \), so \( x_i(X[[\Gamma]]) \) remains the image of a section after blowing up, call it \( x_i' : X[[\Gamma]] \to W' \), where \( W' \to W \) is the blowup. Note also that \( W' \to X[[\Gamma]] \) is smooth at these sections. Thus, we’ve already checked items 1, 2, and 3. Item 4 is clear since the equality of the generic points remains true if \( (\rho, B) \neq (\hat{\rho}, \hat{B}) \). Item 5 follows by construction and (6).

\[
\square
\]

Let \( x_i' : X[[\Gamma]] \to X[[\Gamma]] \) be the section \( x_i \) in Claim 4.9 at the final step \( W = X[[\Gamma]] \). Of course, it will turn out that \( x_i' = x_i \), but we don’t really know this yet, so we use the notation \( x_i' \) for now.

The same induction as in Claim 4.9 allows us to define the homomorphism
\[
(7) \quad \psi : \sqrt{\omega_{X[[\Gamma]]}} : X[[\Gamma]] \to \mathcal{O}_{X[[\Gamma]]}.
\]
Specifically, we start with \( \psi_0 \) on \( X[[\Gamma]] \times X \) corresponding to the pullback of the regular function \( t \) on \( A^1 \) along \( X[[\Gamma]] \times X \to A^1 \). The property we need to check inductively is:

\[
(6) \quad \text{If } \psi_0 : \sqrt{\omega_{W,X[[\Gamma]]}} : W \to \mathcal{O}_W \text{ is the map on } W, \text{ and the blowup indexed by } (\rho, B) \text{ hasn’t been performed yet, then } \text{Im}(\psi) \subseteq \mathcal{I}_{G_{\rho,B},W}, \text{ where } G_{\rho,B} \text{ is the dominant transform of } G^0_{\rho,B} \text{ on } W.
\]

Property 6 ensures that Lemma 2.5 can be applied to produce \( \psi_{W'} \). It is clear that 6 continues to hold because the desired vanishing obviously continues to hold at least at the generic point of \( G_{\rho,B} \). This completes the construction of (7). For simplicity, we write \( \sqrt{\omega} \) instead of \( \sqrt{\omega_{X[[\Gamma]]}} : X[[\Gamma]] \to \mathcal{O}_{X[[\Gamma]]} \) for the rest of this section.

Claim 4.11. The map \( (x_i')^* \psi : (x_i')^* \sqrt{\omega} : X[[\Gamma]] \to \mathcal{O}_{X[[\Gamma]]} \) is an isomorphism.

Proof. Let \( W \) be a blowup of \( X[[\Gamma]] \times X \) in the sequence of blowups in Claim 4.9 and \( x_i,W : X[[\Gamma]] \to W \) the sections on \( W \). We claim that:

\[
(7) \quad \text{If the blowup indexed by } (\rho, B) \text{ has been performed already, then the restriction of } \psi_W \text{ to } x_i,W(D_{\rho}) \text{ is not identically } 0 \text{ for any } i \in B.
\]

We proceed inductively again. The base case is vacuous. Let \( W' \) be next variety in the sequence, obtained by performing the blowup \( \beta : W' \to W \) corresponding to \((B, \rho), \) with exceptional divisor \( E \subset W' \). It suffices to check the claim in 7 for the pair \((B, \rho)\) only. It is clear that \( \text{Im}(\psi_W) \) is not contained in \( \mathcal{I}^2_{x_i,W(D_{\rho}),W} \); if it was the case, we could remove the exceptional loci of all blowups so far and find that the same holds at least in a suitable open in \( X[[\Gamma]] \times X \) (note that \( x_i,W(D_{\rho}) \) is not contained in the union of the removed exceptional divisors), but this is certainly not the case given the definition of \( \psi_0 \). It follows from Lemma 2.5 that

\[
(8) \quad \psi_W|_E \neq 0.
\]

Clearly, \( E \to D_{\rho} \) is a projective bundle, and, for any \( z \in D_{\rho}(\mathbb{C}) \),

\[
(9) \quad \sqrt{\omega_{W',X[[\Gamma]]}}|_E \simeq \mathcal{O}_{E_z}(1),
\]

by Lemma 2.3. Moreover, \( \psi_W \) vanishes along the hyperplane \( H \subset E_z \), where \( E_z \) intersects the proper transform of \( W_0 \) because \( \psi_W \) vanishes in a neighbourhood in
Proof of Theorem 4.6 – part III. As explained above, only some claims in the case \( \Gamma = K_\bullet \) follow by an inductive argument similar to Claim 4.9, which will not be repeated. To lift all the way to \( X \), establishing if \( 7 \) is false, then \( 8 \) is false, contradiction.

Now consider the final case \( W = X[[\Gamma]]^\text{\#} \). By property 7 above and the obvious fact that \((x^+_i)^*\psi\) is an isomorphism away from the central fiber, \((x^+_i)^*\psi\) is an isomorphism in codimension 2 on \( X[[\Gamma]] \), hence an isomorphism. \( \square \)

Claims 4.9 and 4.11 complete the proof of Theorem 4.6 in the case \( \Gamma = K_\bullet \), and much of it in the case \( \Gamma = K_n \), specifically, everything about \( X[[\Gamma]]^\text{\#} \) pending establishing \( x^+_i = x^+_i \).

5. The Small Resolutions

Let \( C \) be any smooth complex projective curve. It is completely irrelevant but convenient to assume that \( C \) is not rational. We will use the results of §4 for \( X = C \). We will keep the notation from §4.1, but not from §4.2.

5.1. Stabilizing the curves. By Theorem 4.6 and Remark 2.2, \( C[[\Gamma]]^\text{\#} \rightarrow C[[\Gamma]] \) and \( C[[\Gamma]]^\text{\#} \rightarrow C[[\Gamma]] \) are prestable curves. In this section, we stabilize them (with their decorations) by contracting suitable bridges. For the sake of clarity, we isolate an example of the well-known contraction procedure in the lemma below.

**Lemma 5.1.** Let \( S \) be a variety over \( \mathbb{C} \) and \( \pi : C \rightarrow S \) a prestable curve over \( S \) of compact type. Let \( \mathcal{L} \) be a line bundle on \( C \) such that for any \( s \in S(\mathbb{C}) \), \( \deg \mathcal{L}_s > 0 \), and for any connected \( R \subset C_s \) which is the union of several irreducible components of \( C_s \), we have \( \deg \mathcal{L}|_R \geq 0 \), with strict inequality if \( R \) is not a rational chain. Then

\[
C' = \text{Proj}_S \bigoplus_{k \geq 0} \pi_*(\mathcal{L}^\otimes k)
\]

is a prestable curve over \( S \), and there exists a rational contraction (please see [21, Definition 2.1]) \( f : C \rightarrow C' \) which, for every \( s \in S(\mathbb{C}) \), contracts all irreducible components of \( C_s \) on which \( \mathcal{L}_s \) has degree 0 and nothing else.
Sketch of proof. This is completely analogous to the proof of [2, Proposition 3.10], so we will only sketch the main steps. The case \( S = \text{Spec} \ C \) is elementary, though we also need the following fact. If \( S = \text{Spec} \ C \) and \( L' = f_* L \), then \( L' \) is invertible, ample, and

\[
L'^\otimes k = f_* L^\otimes k \quad f^* L'^\otimes k = L^\otimes k \quad R^1 f_* L^\otimes k = 0 \quad H^1(L'^\otimes k) = H^1(L^\otimes k)
\]

for any integer \( k \geq 0 \), cf. [2, Lemma 3.11]. From this, we can deduce the analogue of [2, Proposition 3.10]. The analogues of Claims 1–5 in [2] hold, with precisely the same arguments. □

A rational bridge is an irreducible component of a nodal curve over \( \text{Spec} \ C \) which is smooth, rational, and contains precisely two nodes of the curve. Applying Lemma 5.1 in our setup, we have the following.

**Proposition 5.2.** For each symbol \( \star \in \{ \cdot, + \} \), there exists a prestable curve \( Y^\star \to C[[\Gamma]] \) and a rational contraction \( f^\star : C[[\Gamma]]^\star \to Y^\star \), such that for each closed point \( z \in C[[\Gamma]] \), \( f^\star_z \) contracts all rational bridges in \( C[[\Gamma]]^\star_z \) without markings and nothing else.

Note that even in the case \( \star = + \), such bridges are actually (non-leaf) vertices of \( T(H) \), where \( H \) is the piece in \( L_{[n]} \) of the element in \( H^+_n \) describing the dual tree of the respective fiber. This follows from Theorem 4.6. Informally, what is really happening is that the Fulton–MacPherson-like ends are already stable. A word on notation: from now on, we will often write \( \pi_{A/B} \) for a map \( A \to B \) that is ‘reasonably’ thought of as a projection; context will never leave any doubt about what the map is.

**Proof.** Let \( x_i = x^\star_i(C[[\Gamma]]) \), \( J \) an ample line bundle on \( C \), and

\[
L = \omega_{C[[\Gamma]]^\star/C[[\Gamma]]}(2x_1 + \cdots + 2x_n) \otimes \pi_{C[[\Gamma]]^\star/C[[\Gamma]]}^\star J.
\]

By Theorem 4.6, the restriction of \( L \) to \( C[[\Gamma]]^\star_z \) has degree 0 on all rational bridges with no markings, and strictly positive degree on all other components, for any \( z \in C[[\Gamma]](C) \), so we may apply Lemma 5.1 to conclude the existence of \( f^\star. \)

By [2, Lemma 2.2], there exists a morphism \( \mu : Y^+ \to Y^\star \) such that \( \mu f^+ = f^\star \xi_n \). In fact, it is clear that \( \mu \) must be a rational contraction, e.g. [21, Lemma 2.2]. We have the following commutative diagram.

\[
\begin{array}{ccc}
C[[\Gamma]]^+ & \xrightarrow{f^+} & Y^+ \\
\downarrow \xi_n & & \downarrow \mu \\
C[[\Gamma]]^\star & \xrightarrow{f^\star} & Y^\star \\
\end{array}
\]

We also need to show that the scaling descends along the rational contraction in Proposition 5.2. By Lemma 2.3, we have

\[
\text{Hom}(\omega_{C[[\Gamma]]^\star/C[[\Gamma]]}, \pi^\star_{C[[\Gamma]]^\star/C[[\Gamma]]} \omega_C) = \text{Hom}(\omega_{C[[\Gamma]]^\star/C[[\Gamma]]}, \mathcal{O}_{C[[\Gamma]]^\star}),
\]

so we may think of \( \psi \) as an element of either side.

**Lemma 5.3.** With notation as in Proposition 5.2, there exists

\[
\phi : \omega_{Y^\star/C[[\Gamma]]} \to \pi^\star_{Y^\star/C[[\Gamma]]} \omega_C,
\]
such that \( \phi|_U \) and \( \psi|_{(f^\lambda)^{-1}(U)} \) correspond to each other under the isomorphism \( U \simeq (f^\lambda)^{-1}(U) \) and (10), where \( U \subseteq Y^\lambda \) is the maximal open such that \( U \simeq (f^\lambda)^{-1}(U) \).

**Proof.** By [21, Lemma 2.6] and Lemma 2.3, we have

\[
(f^\lambda)^!(\omega_{Y^\lambda/C[[\Gamma]]} \otimes \pi_{Y^\lambda/C}^* \omega_C^\lambda) = \omega_{C[[\Gamma]]^{\lambda},C/C[[\Gamma]]} \otimes \pi_{C[[\Gamma]]^{\lambda},C/C[[\Gamma]]}^* \omega_C^\lambda.
\]

By [21, Proposition 2.11], we obtain a homomorphism

\[
f_\lambda^* \omega_C^\lambda/(C/C[[\Gamma]]) \to \omega_{Y^\lambda/C[[\Gamma]]} \otimes \pi_{Y^\lambda/C}^* \omega_C^\lambda.
\]

We define \( \phi \) as the image of \( \psi \) under the homomorphism above. \( \square \)

Lemma 5.3 descends the scaling from \( C[[\Gamma]]^{\lambda} \), which will suffice for Theorem 1.2. However, for the purposes of Theorem 1.3, we will need to rearrange this in terms of \( C[[\Gamma]]^+ \), which is done in the lemma below.

**Lemma 5.4.** There exists a section \( \sigma \) of the \( \mathbb{P}^1 \)-bundle

\[
P(\omega_{Y^\lambda/C[[\Gamma]]} \oplus \pi_{Y^\lambda/C}^* \omega_C^\lambda) \to C[[\Gamma]]
\]

which corresponds to the graph of \( \phi \) (see Lemma 5.3) on the maximal open \( U \subseteq Y^\lambda \) such that \( \mu^{-1}(U) \simeq U \), at least up to the natural \( \mathcal{O}_{C[[\Gamma]]}^\lambda \) automorphisms on the \( \mathbb{P}^1 \)-bundle with the two disjoint distinguished sections.

(If \( \mathcal{L}_1, \mathcal{L}_2 \) are line bundles on a variety \( X \), the graph of an \( \mathcal{O}_X \)-linear map \( \mathcal{L}_1 \to \mathcal{L}_2 \) gives a section of \( \xi : \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2) \to X \). If two sections of \( \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2) \) are identical up to the \( \xi^{-1} \mathcal{O}_X^\lambda \) automorphisms, then they have the same ‘zeroes and poles’, i.e. intersections with the two distinguished sections.)

**Proof.** We will construct a surjective homomorphism

\[
\omega_{Y^\lambda/C[[\Gamma]]} \oplus \pi_{Y^\lambda/C}^* \omega_C^\lambda \xrightarrow{\mu^*(\phi \otimes \text{id}) \otimes \text{id}} \mu^* \omega_{Y^\lambda/C[[\Gamma]]}.
\]

The first component is \(-d\mu\), where \( d\mu \) is roughly the (logarithmic) differential of \( \mu \). More precisely, by standard facts,

\[
\mu^! \omega_{Y^\lambda/C[[\Gamma]]} = \mu_! \pi_{Y^\lambda/C}^! \mathcal{O}_{C[[\Gamma]]} = \pi_{Y^\lambda/C}^! \mathcal{O}_{C[[\Gamma]]} = \omega_{Y^\lambda/C[[\Gamma]]},
\]

and then [21, Proposition 2.8] gives a homomorphism

\[
\mu^* \omega_{Y^\lambda/C[[\Gamma]]} \to \mu^! \omega_{Y^\lambda/C[[\Gamma]]} = \omega_{Y^\lambda/C[[\Gamma]]},
\]

and we define \( d\mu \) to be the dual of this map. Then \( d\mu \) is an isomorphism at all points where \( \mu \) is an isomorphism. On the other hand, \( \mu^* \phi \) induces a map

\[
\mu^* \omega_C^\lambda \otimes \pi_{Y^\lambda/C}^* \omega_C^\lambda \to \pi_{Y^\lambda/C}^* \omega_C^\lambda,
\]

which we take to be the second component of (11). It follows from Lemma 5.3 that this map is an isomorphism at the preimages by \( \mu \) of all points where \( f^\lambda \) is an isomorphism. Thus at least one of the components of (11) is nonzero at each point, so it is indeed surjective. Then \( \sigma \) is obtained from the well-known universal property, and the required property is clear by construction. \( \square \)
5.2. The wonderful models as subvarieties of $C[\Gamma]$. We start by rearranging Definition 1.1 in a more convenient form. Let $V_S, V_\rho \subseteq \mathbb{C}^n$ such that $\Lambda_S = P V_S$ and $\Lambda_\rho = P V_\rho$ be the diagonals and polydiagonals of $\mathbb{C}^n$ ($P$ refers to 1-dimensional subspaces). If we embed $\mathbb{C} \to \mathbb{C}^n$ diagonally and consider the isomorphism $\mathbb{C}^n \cong \mathbb{C} \oplus \mathbb{C}^\perp \cong \mathbb{C} \oplus \mathbb{C}^n / \mathbb{C}$ relative to the standard inner product $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$ and the corresponding $\mathbb{P}^{n-1} \cong \mathbb{F}(\mathbb{C} \oplus \mathbb{C}^n / \mathbb{C})$, then $\mathbb{F}(\mathbb{C} \oplus V_S / \mathbb{C})$ and $\mathbb{F}(0 \oplus V_\rho / \mathbb{C})$ map to $\Lambda_S$ and $\Lambda_\rho \cap H$ respectively. Thus, we may think of $W_n$ as the wonderful model of $\{ \mathbb{P}(0 \oplus V_\rho / \mathbb{C}) : \rho \in L_{[n]} \{ \perp \} \}$, and of $T_n$ as the wonderful model of the union of the previous arrangement with $\{ \mathbb{P}(\mathbb{C} \oplus V_S / \mathbb{C}) : S \subseteq [n], |S| \geq 2 \}$.

Remark 5.5. With the above in mind, it is clear that if we take

$$V = \{(z_2 - z_1, z_3 - z_1, \ldots, z_n - z_{n-1}) : z_1, \ldots, z_n \in \mathbb{C} \} \subset \mathbb{C}^{\binom{n}{2}}$$

in [3, Remark 2.13], then the matroid is the graphic matroid of $K_n$ and the augmented wonderful variety of loc. cit. is indeed $W_n$.

Let $W_\Gamma = \begin{cases} W_n & \text{if } \Gamma = K_n^c, \\ T_n & \text{if } \Gamma = K_n \end{cases}$, and let $\{D^w_\alpha : \alpha \in I^w\}$ be the boundary divisors of $W_\Gamma$. The codimension 0 and 1 strata are

$$W^w_\Gamma := W_\Gamma \setminus \bigcup_{\alpha \in I^w} D^w_\alpha \quad \text{and} \quad D^{w,\alpha}_{\alpha'} := D^w_\alpha \setminus \bigcup_{\alpha' \neq \alpha} D^{w,\alpha}_{\alpha'}.$$

Note that $W^w_\Gamma \cong F(\mathbb{A}^1, \Gamma) / \mathbb{G}_a$, where $F(\mathbb{A}^1, \Gamma) \subseteq \mathbb{A}^n$ if $\Gamma$ is edgeless, respectively $F(\mathbb{A}^1, n)$ if $\Gamma$ is complete, and $\mathbb{G}_a$ acts by $z \cdot (z_1, \ldots, z_n) = (z_1 + \ldots, z_n + z)$.

Let’s fix a (closed) point $p \in C$ and an isomorphism $T_p C \cong \mathbb{C}$.

**Proposition 5.6.** The variety $W_\Gamma$ is isomorphic to the fiber of $C[\Gamma] \to \mathbb{A}^1 \times \mathbb{C}^n$ over $(0, p, \ldots, p)$. Moreover, $D^w_\alpha = W_\Gamma \cap D_\alpha$ for any $\alpha \in I^w$, and the intersection is transverse.

**Proof.** Recall that $C[\Gamma]$ is a wonderful compactification. Blowing up $\mathbb{A}^1 \times \mathbb{C}^n$ at $\{0\} \times \Delta_\perp$ places us in an instance of the situation covered by [15, Proposition 2.8]. The fiber of the blowup over $(0, p, \ldots, p)$ is $\mathbb{F}(\mathbb{C} \oplus T_p \mathbb{C}^{[n]} / T_p C) = \mathbb{F}(\mathbb{C} \oplus \mathbb{C}^n / \mathbb{C})$. It is clear that the intersection of the proper transform of $\{0\} \times \Delta_\perp$ (assume $p \neq \perp$ in this case) or $\mathbb{A}^1_\perp \times \Delta_\perp$ with $\mathbb{F}(\mathbb{C} \oplus \mathbb{C}^n / \mathbb{C})$ is non-empty, connected, and transverse, so the proposition follows from Proposition A.2 (and the fact that the only impact removing a divisor from a building set has on a wonderful compactification is on what is considered a boundary divisor).

Let $* \in \{\perp, +\}$. We claim that $C[\Gamma][D_\perp] = D_\perp \times_C C[\Gamma][\Gamma]^*$ has an irreducible component isomorphic to $D_\perp \times C$. Recall that $C[\Gamma]^*$ was constructed as a blowup of $C[\Gamma] \times C$. It is elementary that at the first blowup in this sequence, the restriction of the family of curves to $D_\perp$ has a component isomorphic to $D_\perp \times C$. However, all subsequent blowups are actually disjoint from this component isomorphic to $D_\perp \times C$, since we’ve always blowing up loci inside the union of the marked sections, and these sections have already been moved away from $C$ above $D_\perp$. In particular, $C[\Gamma][W_\Gamma]$ has an irreducible component isomorphic to $W_\Gamma \times C$. This component intersects the union of the other components at $W_\Gamma \times \{p\}$. \[\square\]
In fact, not only does $C[[\Gamma]]^\sigma_{W^\Gamma}$ have an irreducible component isomorphic to $W_\Gamma \times C$, but the same is true of $Y^*_{W^\Gamma}$: the composition

$$W_\Gamma \times C \hookrightarrow C[[\Gamma]]^\sigma_{W^\Gamma} \xrightarrow{f^*} \text{restricted to } C[[\Gamma]]^\sigma_{W^\Gamma} \to Y^*_{W^\Gamma},$$

where $f^*$ is the rational contraction in Proposition 5.2, is the immersion of an irreducible component. Indeed, it is clear by simple dimension considerations that the image is an irreducible component, and since the map is fiberwise an immersion of a component by Proposition 5.2, it must be so globally.

Let $Z^*$ be the union of all the irreducible components of $Y^*_{W^\Gamma}$ other than the one isomorphic to $W_\Gamma \times C$ (in fact, there’s just one), $z^*_i = (x^*_i \circ f^*)|_{W^\Gamma} : W_\Gamma \to Z^*$ (the fact that the image of $z^*_i$ is contained in $Z^*$ follows from Theorem 4.6), and $z^*_\infty : W_\Gamma \to Z^*$ the section whose image is $z^*_\infty(W_\Gamma) = (W_\Gamma \times C) \cap Z^*$. The rational contraction $\mu : Y^+ \to Y^\lambda$ restricts to a morphism $\overline{\mu} : Z^+ \to Z^\lambda$, which is also a rational contraction by [21, Lemma 2.2]. Note that $\overline{\mu}$ restricts to an isomorphism above $W^\Gamma = F(A^1, \Gamma)/\mathbb{G}_a$. Let $\phi|_{Z^+}$ be the restriction of $\phi$ to $Z^\lambda$. Let $\sigma|_{Z^+}$ be the restriction of $\sigma$ to $Z^+$.

Here is the critical observation: the objects constructed above define a stable $n$-marked $\mathbb{G}_a$-rational tree [21, Theorem 1.5] over $W_\Gamma$ if $(\ast, \Gamma) = (\lambda, K_n^c)$, respectively a stable affine scaled curve [8, Definition 2.4] over $W_\Gamma$ if $(\ast, \Gamma) = (+, K_n)$. The technical details are sketched in Claim 5.7 below, although there is not much to say. Two quick warnings regarding notation and conventions. First, the marking indexed $\infty$ in our setup obviously corresponds to the marking indexed 0 in [8]. Second and more confusingly, although the vector field $\phi|_{Z^+}$ and the ‘scaling’ $\sigma|_{Z^+}$ contain roughly the same information, the notions of zeroes/poles are reversed: zeroes of the vector field correspond to infinite scalings.

**Claim 5.7.** Assume that $(\ast, \Gamma) = (\lambda, K_n^c)$ or $(+, K_n)$.

1. $Z^* \to W_\Gamma$ is a prestable curve of genus 0.
2. The sections $z^*_1, \ldots, z^*_n, z^*_\infty$ are smooth, $z^*_s(s) \neq z^*_\infty(s)$ for all $s \in W_\Gamma$, and, if $(\ast, \Gamma) = (+, K_n)$, then $z^*_s(s) \neq z^*_j(s)$, for all $s \in W_\Gamma$ and $i \neq j$.
3. The restriction $\phi|_{Z^+}$ is a global section of $\omega_{K_n^c/W_\Gamma}(-2z^*_\infty)$.
4. The restriction $\phi|_{Z^+}$ doesn’t vanish at $z^*_i(s)$ for any $s \in W_\Gamma$.
5. The scaling $\sigma|_{Z^+}$ is infinite at $z^+_i(s)$ and finite at $z^+_j(s)$, for any $s \in W_\Gamma$.
6. For $\ast = +$ and $\Gamma = K_n$, condition (a) in [8, Definition 2.4] is satisfied in each fiber.
7. The natural stability conditions hold. The stability condition for $(\ast, \Gamma) = (\lambda, K_n^c)$ is stated in [21, Theorem 1.5], and for $(\ast, \Gamma) = (+, K_n)$ in [8, Definition 2.4].

The stability conditions quoted above are equivalent to the non-existence of nontrivial automorphisms in the natural sense.

**Proof.** For item 1, $Z^* \to W_\Gamma$ is clearly proper, it is flat away from $z^*_\infty$ since $Y^*$ is flat and even smooth (hence flat in particular) at points $z^*_s(s)$, and its geometric fibers have nodal singularities since the same holds for $Y^*$. Items 2, 4, and 7 are clear by construction, keeping in mind Theorem 4.6, Proposition 5.2, and Lemma 5.3. Item 3 follows from the classical description of dualizing sheaves of nodal curves in terms of meromorphic differentials and residues (e.g. [1, Chapter X, §2]). A priori, $\phi|_{Z^+}$ is a global section of $\omega_{Y^+}/C[[\Gamma]]|_{Z^+} \simeq \omega_{Z^+}/W_\Gamma(-z^*_\infty)$, but it must vanish on $z^*_\infty$.
5.3. The small resolutions of $\mathcal{P}_n$ and $\mathcal{Q}_n$. Claim 5.7 and the moduli space definitions of $\mathcal{P}_n$ and $\mathcal{Q}_n$ provide the morphisms in Theorems 1.2 and 1.3, and all that remains to do is to check that they are indeed small resolutions. Let $P_n \subset \mathcal{P}_n$ and $Q_n \subset \mathcal{Q}_n$ be the open subsets where the universal curves are irreducible.

The case of $\mathcal{P}_n$. In this paragraph, we assume $\Gamma = K_n^\text{\tiny p}$, the edgeless $n$-vertex graph. Let $Z = Z^\gamma = Z^\dagger$, $z_i = z_i^\gamma = z_i^\dagger$ and $z_\infty = z_\infty^\gamma = z_\infty^\dagger$. Recall that $W_\Gamma = W_n$. We also write $W_n^\circ = W_\Gamma^\circ \cong \mathbb{A}^n/G_n$. By Claim 5.7, $(Z, z_1, \ldots, z_n, z_\infty, \phi|Z)$ is a stable $n$-marked $G_n$-rational tree, so, by [21, Theorem 1.5], we obtain a morphism $\gamma : W_n \to \mathcal{P}_n$.

Remark 5.8. The relation between the curve over $s \in W_n(\mathbb{C})$ and the curve which corresponds to $\gamma(s) \in \mathcal{P}_n$ is clear. If $H = \{ \rho \in L[n] : s \in D_\rho \}$, then the dual tree of $C[[K_n]]_s$ is $T(H)$ by Theorem 4.6, or, if we wish to remove the component isomorphic to $C$, the dual tree is $T(H) - \star$ (this is still connected, since $\bot \in L[n]$).

The dual tree of $Z$ (the curve corresponding to $\gamma(s) \in \mathcal{P}_n$) is $T'(H) - \star$ (Definition 3.1) by Proposition 5.2 and the construction above.

Proposition 5.9. The restriction of $\gamma$ to $W_n^\circ$ is an isomorphism, and $\gamma^{-1}(P_n) = W_n^\circ$. In particular, $\gamma$ is birational.

Proof. The fact that $\gamma^{-1}(P_n) = W_n^\circ$ follows by Theorem 4.6, Remark 5.8, and a trivial combinatorial argument. It remains to check that $\gamma$ restricts on $W_n^\circ$ to an isomorphism $W_n^\circ \simeq P_n$. Since $\gamma^{-1}(P_n) = W_n^\circ$ and $P_n$ is smooth (even normal suffices, which follows from [21, Proposition 5.6]), it suffices to check that $\gamma$ is injective on $W_n^\circ$.

This can be done by an explicit calculation, and requires only two blowups: the blowup of $\{0\} \times \Delta_\bot$ in $\mathbb{A}^1 \times C^n$ is needed to see $W_n^\circ$, and one more blowup is needed to see the family of curves $C[[K_n]]_s$ over $W_n^\circ$. Working étale locally, we may pretend that $C = \mathbb{A}^1, p = 0$. First, we blow up $\{0\} \times \Delta_\bot$ in $\mathbb{A}^1 \times \mathbb{A}^n$ and remove the proper transform of $\{0\} \times \mathbb{A}^n$ to obtain the base $B$ (thus, $B$ is the deformation to the normal cone of $\Delta_\bot$). If $t, u_1, \ldots, u_n$ are the coordinates on the original $\mathbb{A}^1 \times \mathbb{A}^n$, then $t, u_1, \delta_2, \ldots, \delta_n$, where $\delta_i = (u_i - u_1)/t$, are coordinates on $B$. The pullbacks $x_i : B \to B \times \mathbb{A}^1$ of the ‘coordinate sections’ are $x_1 = u_1$ and $x_i = u_i + \delta_i$ for $i \geq 2$.

Moreover, $W_n^\circ = \{ t = 0, u_1 = 0 \}$, with coordinates $\delta_2, \ldots, \delta_n$. Second, in $B \times \mathbb{A}^1$, we blow up the subvariety $J = x_1(B_0) = \cdots = x_n(B_0) = \{ t = 0, r = u_1 \}$, where $B_0 = \{ t = 0 \} \subset B$ and $r$ is the coordinate on the second factor of $B \times \mathbb{A}^1$. The $\mathbb{P}^1$ fiber over $(0, 0, \delta_2, \ldots, \delta_n)$ of the exceptional divisor of the blowup of $J$ has the marked points (i.e. intersections with the proper transforms of $x_i(B)$, $i = 1, \ldots, n$, and $B_0$) at $0, \delta_2, \ldots, \delta_n$ and $\infty$. Moreover, a straightforward calculation shows that its ‘scaling’ is a field independent of $\delta_2, \ldots, \delta_n$, so $\gamma$ is indeed injective on $W_n^\circ$. □

It remains to check that $\gamma$ is small, but not an isomorphism if $n \geq 4$.

Proposition 5.10. The morphism $\gamma$ is small.

Proof. Let $R \subset \mathcal{P}_n$ be the closed set where the fibers of $\gamma$ are positive dimensional and $E = \gamma^{-1}(R)$. The restriction $W_n \setminus E \to \mathcal{P}_n \setminus R$ of $\gamma$ is a finite (since proper and quasi-finite) birational morphism with normal target (since $\mathcal{P}_n$ is normal by [21,
Proposition 5.6], hence an isomorphism. Thus, if we manage to show that $E$ has codimension at least 2 in $W_n$, then the proposition follows.

By Proposition 5.9, $W_n^* \cap E = \emptyset$, so $E \subseteq \partial W_n = W_n \setminus W_n^*$.

By standard facts about stratifications of moduli stacks of prestable curves, for each $\rho \in L[n] \setminus \{\perp\}$ there exists a locally closed subset $P_n[\rho] \subset P_n$ which parametrizes curves whose dual tree is a star with $|B(\rho)|$ leaves (that is, $K_1,|B(\rho)|$), such that the $\infty$ marking is on the central component, and the tail component corresponding to the block $B \in B(\rho)$ contains the marked points indexed by $B$. Heuristically, it is clear that $\dim P_n[\rho] = n - 2$. In fact, the inequality

$$\dim P_n[\rho] \geq n - 2$$

suffices and is easier to justify very rigorously by constructing explicitly a family of $n$-marked $\mathbb{G}_a$-rational trees over an $(n - 2)$-dimensional base, such that the map to $P_n$ it induces by [21, Theorem 1.5] is quasi-finite and has its image contained in $P_n[\rho]$. For instance, it is easy to construct such a family as divisors in $\mathbb{P}^1 \times \mathbb{P}^1$ consisting of one horizontal line and $|B(\rho)|$ vertical lines in the rulings.

From now on, the fact that $D^{w,\circ}_\rho$ is nonempty and irreducible of dimension $n - 2$ for any $\rho \neq \perp$ will be utilized frequently but tacitly. We have

$$\dim \partial W_n \setminus \bigcup_{\rho \neq \perp} D^{w,\circ}_\rho = \dim \bigcup_{\rho_1 \neq \rho_2} (D^{w}_\rho \cap D^w_{\rho_2}) = n - 3,$$

provided that $n \geq 3$. Theorem 4.6, Remark 5.8, and Lemma 3.2 imply by a purely combinatorial argument that

$$\dim \partial W_n \setminus \bigcup_{\rho \neq \perp} D^{w,\circ}_\rho = \dim \bigcup_{\rho_1 \neq \rho_2} (D^{w}_\rho \cap D^w_{\rho_2}) = n - 3,$$

By (12) and (14), $P_n[\rho]$ is irreducible of dimension $n - 2$. Assume by way of contradiction that $E$ was of codimension 1 in $W_n$. By $E \subseteq \partial W_n$ and (13), there exists $\rho \neq \perp$ such that $D^{w,\circ}_\rho \subseteq E$. Then $\dim P_n[\rho] < \dim D^{w,\circ}_\rho = n - 2$ by (14) (keeping in mind that $P_n[\rho]$ is irreducible), which contradicts (12).

Proposition 5.11. The map $\gamma$ is not an isomorphism if $n \geq 4$.

Proof. Let $k = \lfloor \log_2 n \rfloor$, and for $j = 0, 1, \ldots, k$, let $\rho_j \in L[n]$ be the partition of $[n]$ into the fibers of the map $[n] \to \mathbb{Z}, a \mapsto \lfloor 2^{-j}(a - 1) \rfloor$. Let $H = \{\rho_0, \rho_1, \ldots, \rho_k\}$. Note that $\rho_k = \perp$, $T(H) - *$ is binary with one leg attached to each leaf, and $T'(H) = T(H)$. Thus $T'(H) - *$ is also binary with one leg per leaf, so there exists a unique $b \in P_n(\mathbb{C})$ such that the curve over $b$ has dual tree $T'(H) - *$. By the results of [15] yet again, $Y := D^{w}_{\rho_0} \cap \cdots \cap D^{w}_{\rho_{k-1}} \neq \emptyset$, and

$$\dim Y = \dim W_n - k = n - 1 - \lfloor \log_2 n \rfloor \geq 1$$

for $n \geq 4$. However, $\gamma(Y) = \{b\}$ by Remark 5.8.

Remark 5.12. The argument in the proof of Proposition 5.11 also shows that $\gamma$ is not small in the sense used in the context of intersection homology [9, §6.2] for $n \geq 7$. Specifically, if $r = n - 1 - \lfloor \log_2 n \rfloor$, then $\dim \gamma^{-1}(b) \geq r$, so $\operatorname{codim}\{p \in P_n : \dim \gamma^{-1}(p) \geq r\} \leq n - 1 \leq 2r$ for $n \geq 7$. The same argument will prove the analogous statement about $\mathcal{Q}_n$ once the work below is done.

Propositions 5.9, 5.10, and 5.11 complete the proof of Theorem 1.2. Moreover, a small birational morphism has smooth source and target only if it is an isomorphism, so we are now sure that $P_n$ is singular for all $n \geq 4$, and it makes sense to call $\gamma$ a resolution.
The case of $\overline{Q}_n$. The arguments for $\overline{Q}_n$ are in essence analogous, so we will focus on the (few) additional ingredients and differences.

In this paragraph, we assume $\Gamma = K_n$. Let $Z = Z^+ = Z^{+}_i$ and $z_\infty = z^{+}_i$. We have $W^\Gamma_n = T_n$ and $T_n^\circ = W^\Gamma_n \simeq F(\mathbb{A}^1, n) / \mathbb{G}_a$. By Claim 5.7 and [8, Definition 2.4 and Theorem 2.5] or [20, Example 4.2.(d)], $(Z, z_1, \ldots, z_n, z_\infty, \sigma|_Z)$ corresponds to a morphism $\eta: T_n \to \overline{Q}_n$.

Remark 5.13. Similarly to Remark 5.8, we can describe the relation between the curve over $s \in T_n(\mathbb{C})$ and the curve which corresponds to $\eta(s) \in \overline{Q}_n$. If $N = \{ \alpha \in I:\ s \in D_\alpha \}$, then the dual tree of $C([K_n])^+_{\alpha}$ is $T(N)$ by Theorem 4.6. After removing the component isomorphic to $C$, the dual tree becomes $T(N) - \star$. The dual tree of $Z$s is $T'(N) - \star$ (Definition 3.4) by Proposition 5.2.

Remark 5.14. Above, we used the fact that $P_n$ is normal, and normality should ideally be clarified before discussing resolutions of singularities. The author hasn’t been able to find an explicit statement in the literature that $\overline{Q}_n$ is normal, although it is true and essentially analogous to $P_n$, and follows from locatable statements. In [20, Example 4.2.(d)], it is stated and sketched that $\overline{Q}_{n+1}$ is the universal curve over $\overline{Q}_n$. Then we may argue inductively that $\overline{Q}_n$ is lci (prestable curves are lci over their base and compositions of lci maps are lci under very mild hypotheses) and regular in codimension 1, hence normal since Serre’s $S_2$ condition is implied by lci. As an alternative, we will actually not need the fact that all of $\overline{Q}_n$ is normal – it will only be needed (cf. the proofs of Propositions 5.15 and 5.16 below) for the open subset of $\overline{Q}_n$ which consists of the codimension 0 and 1 strata. However, this open subset is in fact even smooth, which can be checked by a direct calculation.

Proposition 5.15. The restriction of $\eta$ to $T^\circ_n$ is an isomorphism, and $\eta^{-1}(Q_n) = T^\circ_n$. In particular, $\eta$ is birational.

Proof. Similarly to the proof Proposition 5.9, the only real verification needed is that $\eta$ is injective on $T^\circ_n$. This is not just similar to the injectivity of $\gamma$ on $W^\alpha_n$, but actually follows from it, since $T^\circ_n$ can be naturally identified with $\{ 0 \neq \delta_i \neq \delta_j, \forall i \neq j \} \subset W^\alpha_n$ with the notation in the proof of Proposition 5.9, in a manner which also identifies the families of curves over them, their marked points, and scalings.

Proposition 5.16. The morphism $\eta$ is small.

Proof. This is similar to the proof of Proposition 5.10, with the ultimately irrelevant difference that now we have two types of boundary divisors. As in that proof, let $R \subset T_n$ be the set of points whose $\eta$-fiber has positive dimension, and $E = \eta^{-1}(R)$. Then $E \subseteq \partial T_n = T_n \setminus T^\circ_n$ by Proposition 5.15.

For each $\rho \in L_{[n]} \setminus \{ \perp \}$, we have a locally closed subset $Q_n[\rho] \subset \overline{Q}_n$ which parametrizes curves which fit the same description as the curves parametrized by $P_n[\rho]$, but this time, for each subset $S \subseteq [n], |S| \geq 2$, we also have a locally closed subset $Q_n[S] \subset \overline{Q}_n$ which parametrizes curves with two components, one of which contains all marked points indexed by $S$ and no others including $\infty$. We have

$$\dim Q_n[\rho] \geq n - 2 \quad \text{and} \quad \dim Q_n[S] \geq n - 2.$$  

The former is completely analogous to (12), while the latter can be proved by similar techniques.

If $E$ has codimension 1 in $T_n$, then arguments similar to those in the proof of Proposition 5.10 show that $E$ contains either some $Q_n[\rho]$, or some $Q_n[S]$, and
also rule out the first alternative. To rule out the second possibility, note that 
\( \eta^{-1}(Q_n[S]) = D^\infty_S \) by Remark 5.13 and Theorem 4.6, and then argue again as in 
the proof of Proposition 5.10. In conclusion, \( E \) has codimension at least 2 in \( T_n \).

We conclude the proof using Remark 5.14. \( \square \)

**Proposition 5.17.** The map \( \eta \) is not an isomorphism if \( n \geq 4 \).

**Proof.** The marked points on the ‘binary curve’ (corresponding to \( b \in \mathbb{P}_n \)) used in 
the proof of Proposition 5.11 are pairwise distinct, so the curve is also in \( Q_n \) and 
the same argument applies. \( \square \)

Theorem 1.3 follows from Propositions 5.15, 5.16 and 5.17.

**Example 5.18.** As explained at the end of [17], the threefold \( \overline{Q}_4 \) has 3 ordinary double points, namely, the points where the curves look like this \( \text{cf. [17, Figure 19]} \), and in fact the same holds for \( \mathbb{P}_4 \). The exceptional set of \( W_4 \to \mathbb{P}_4 \)

![Figure 3](image.jpg)

**Figure 3.** For \( n = 4 \): the arrangement \( \mathbb{P}_0 \) cf. (1) if \( H \) is the screen or paper (left); the lines and point in \( D \) (right).

(respectively \( T_4 \to \overline{Q}_4 \)) is the intersection of the preimage by \( W_4 \to \mathbb{P}^3 \) (respectively \( T_4 \to \mathbb{P}^3 \)) of the 3 points of the form \( \pm 1 : \pm 1 : \pm 1 : \pm 1 \) in \( H = \{ X_1 + X_2 + X_3 + X_4 = 0 \} \) (the 3 dots in Figure 3 which are intersections of just 2 lines) with the proper transform of \( H \).

**Appendix A. Functoriality of wonderful compactifications**

It is quite clear that there is a set of natural conditions, under which the formation of wonderful compactifications commutes with base change by a closed immersion. In this appendix, we state and sketch a proof of this uninteresting technicality. We use the notations and conventions from [15].

**Lemma A.1.** Let \( F, S, Y' \) be smooth subvarieties of a smooth variety \( Y \) such that
\( S \) and \( F \) intersect cleanly and \( S \cap F \) is connected if non-empty, and the intersection of \( Y' \) with each of \( F, S, S \cap F \) is nonempty, connected, and transverse, provided that the chosen scheme among \( F, S, S \cap F \) is nonempty (which may fail for \( S \cap F \)).

Consider the blowup of \( Y \) at \( F \) with exceptional divisor \( E \). Then the following hold.

1. \( \tilde{Y}' \) is isomorphic to the blowup of \( Y' \) at \( Y' \cap F \), and its intersection with \( \tilde{S} \)
is nonempty, connected, and transverse.

2. If \( S \not\subseteq F \) and \( S \cap F \neq \emptyset \), the intersection of \( \tilde{Y}' \) with \( \tilde{S} \cap E \) is nonempty, 
connected, and transverse.
Proof. The first claim in (1) is clear. The case $S \cap F = \emptyset$ is trivial and the case $S \subseteq F$ is elementary, so we assume that $S \not\subseteq F$ and $S \cap F \neq \emptyset$. By well-known facts, we have

$$\tilde{Y} \cap \tilde{S} \cap E \simeq \text{Proj}_{Y \cap \tilde{S} \cap F} \text{Sym} \left( \frac{T_S|_{Y \cap \tilde{S} \cap F} + T_F|_{Y \cap \tilde{S} \cap F}}{T_F|_{Y \cap \tilde{S} \cap F}} \right),$$

and, in particular, by (16) and the transversality assumptions again,

$$\dim \tilde{Y} \cap \tilde{S} \cap E = \dim Y' \cap S \cap F + \dim S - \dim(S \cap F) - 1$$

(17)

It is easy to deduce (2) and the transversality part in (1) from (16) and (17). Indeed, for transversalities, we may combine cleanliness with counting dimensions. Note also that given the transversality of the intersection of $Y'$ with $S \cap F$, $S \not\subseteq F$ implies the stronger statement $Y' \cap S \not\subseteq F$, so $\tilde{Y} \cap \tilde{S} \neq \emptyset$ is clear. Connectivity of $\tilde{Y} \cap \tilde{S}$ follows from the above, (17), and the fact that $\tilde{Y} \cap \tilde{S}$ is smooth with components of the same dimension by the transversality proved earlier.

\textbf{Proposition A.2.} Let $Y$ be a smooth variety, $\mathcal{G}$ a building set in $Y$ with induced arrangement $\mathcal{S}$, and $Y_{\mathcal{G}}$ the wonderful compactification of $\mathcal{G}$. Let $Y' \subset Y$ be a smooth closed subvariety whose intersection with any $S \in \mathcal{S}$ is transverse, and

$$\mathcal{S}' = \{ Y' \cap S : S \in \mathcal{S} \} \quad \text{and} \quad \mathcal{G}' = \{ Y' \cap G : G \in \mathcal{G} \}.$$

Assume also that all elements of $\mathcal{S}'$ are non-empty and connected.

Then $\mathcal{S}'$ is an arrangement in $Y'$, $\mathcal{G}'$ is a building set for $\mathcal{S}'$, and

$$Y_{\mathcal{G}'} = Y' \times_Y Y_{\mathcal{G}}.$$

Moreover, if $\mathcal{D}$ and $\mathcal{D}'$ are the collections of distinguished divisors on $Y_{\mathcal{G}}$ and $Y_{\mathcal{G}'}$, respectively (i.e. the divisors in [15, Theorem 1.2]), then $\mathcal{D}' = \{ Y_{\mathcal{G}'} \cap D : D \in \mathcal{D} \}$, where $Y_{\mathcal{G}'} \hookrightarrow Y_{\mathcal{G}}$ by (18), and $Y_{\mathcal{G}'} \cap D$.

Proof. Let $F' = Y' \cap F$, $S' = Y' \cap S$ for $S \in \mathcal{S}$, etc. Clearly, $\mathcal{S}'$ is an arrangement since $S_1' \cap S_2' = (S_1 \cap S_2)'$. Let $S \in \mathcal{S}$, and $G_1, \ldots, G_m$ the $\mathcal{G}$-factors of $S$. It is clear that $S' = G_1' \cap \cdots \cap G_m'$ transversally. We also have to check that $S' \subseteq G' = Y' \cap G$ for given $G \in \mathcal{G}$ implies $G \supseteq G_i$ for some $i$. Indeed, the assumption above plus the transversality of $Y'$ with $G$ and $S$ implies $S \subseteq G$, so $G \supseteq G_i$ for some $i$ by the definition of $\mathcal{G}$-factors. Hence, $\mathcal{G}'$ is a building set of $\mathcal{S}'$.

In the situation of [15, Proposition 2.8], we have $\text{Bl}_F Y' = Y' \times_Y \text{Bl}_F Y$ because $F$ and $Y'$ intersect transversally. Note that $F' = \emptyset$ is allowed. Finally, [15, Proposition 2.8(i)] and Lemma A.1 guarantee that the non-emptiness, connectivity, and transversality hypotheses on the intersection of the subvariety with anything in the arrangement continue to hold after blowing up $F$, so our induction is sound.

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