The Equivalence of Darmois-Israel- and Distributional-method for Thin Shells in General Relativity

R. Mansouri†‡, M. Khorrami‡

†Universität Potsdam, Mathematisches Institut, Kosmologie Gruppe, PF 60 15 53, 14415 Potsdam, Germany.
‡University of Tehran, Department of Physics, Tehran, Iran.

Abstract

A distributional method to solve the Einstein’s field equations for thin shells is formulated. The familiar field equations and jump conditions of Darmois-Israel formalism are derived. A careful analysis of the Bianchi identities shows that, for cases under consideration, they make sense as distributions and lead to jump conditions of Darmois-Israel formalism.

PACS numbers: 04.20,
1 Introduction

The study of hypersurfaces of discontinuity in general relativity has begun in early twenties[1-3]. But it has been revived through new questions raised in cosmology and black-hole physics. Domain walls separating two coexisting different phases in inflationary scenarios[4], bubble dynamics[5], wormholes[6], signature changes[7], and interior structures of black-holes[8] are just some of the recent applications of the thin shell formalism of general relativity.

The traditional and mostly used method of handling such problems is that of Darmois-Israel (DI), based on the Gauss-Kodazzi decomposition of space-time[9, 10]. It expresses the surface properties in terms of the jump of extrinsic curvature across the layer directly as functions of the layer’s intrinsic coordinates. Thus the four-dimensional coordinates may be chosen freely and independently, adapted to the symmetry requirements, on the two sides of the layer. This is the very practical advantage of that method which has found its final formulation in the outstanding paper of Israel[9]. The geometric conditions for the layer to be considered as a boundary of two different manifolds glued together at this boundary are first formulated by Darmois[3]. Those are minimum conditions which has been assumed by Israel. There are other conditions formulated by Lichnerowicz[11, 12], which means basically continuous coordinates across the layer, and seems to be necessary for using distributional tensor calculus. It is interesting to note that Sen[2], in this relatively unknown paper, uses the same conditions, without any further discussion, derives the reduced Einstein’s equations for the general case, and solve it for the spheric symmetric 2+1 dimensional mass distribution. The O’Brien- Synge conditions[13, 14] are in most cases equivalent to Lichnerowicz ones, so we are not going to consider them here[15]. Because of the restrictive choice of coordinates the Lichnerowicz conditions are not usually used. But there are cases where the distributional method, and therefore the Lichnerowicz conditions, are more suitable for calculation. The case of cylindrically symmetric thin layer, for example, has been solved by distributional method using these conditions[16].

There have been attempts to formulate the problem of thin layer in general relativity using distributional methods, familiar in other part of physics. In fact many physical systems, classical and relativistic, undergo very rapid transitions of their state of motion. Think of shock waves in hydrodynam-
ics. Although the state of the system need not be described by discontinuous functions of space and time, or by functions having discontinuity in their first or second derivatives, but a mathematical description of the system which is based on distribution valued states of the system give an accurate picture of some important aspects of the physical problem. Usually such a description is more amenable to treatment than the treatment which contains a smooth description of the physical state. This has not only been used in classical hydrodynamics, but also has been applied in the relativistic case. Lichnerowicz [11, 12] has given a discussion of hydrodynamic and gravitational shock wave problems by using curvature tensors for space-time which contain a Dirac $\delta$ function with support on a submanifold. Y. Choquet-Bruhat has used similar methods to treat high-frequency gravitational waves [17].

Rapid changes of physical quantities occur also in electromagnetism. One might have, for example, some charge distribution which is confined to a one- or two-dimensional region of space small compared with characteristic distances of the problem. This distribution of charge can be replaced by a concentrated source, and the problem can be formulated in the sense of distributions, and not of smooth functions. There is a natural mathematical framework in electromagnetism. Recall that linear operations, including differentiations, make sense when applied to distributions. Hence, Maxwell’s equations, by virtue of their linearity in both fields and sources, make sense as equations on these distributions. This means that the machinery of distribution theory is available in electromagnetism, and guarantee that distributional Maxwell fields with distributional charge-currents make physical sense, and at the same time gives a well defined and detailed sense in which a distributional charge density must approximate our real smooth charge density distribution.

In general relativity, where the field equations are non-linear, the use of distributional objects seem not to be trivial. Although Raju[18] claims to give an analytical formalism to deal with the occurrence of jump discontinuities in the metric across a hypersurface, using a non-linear theory of distributions[18], his method does not seem to be conclusive. The application he mentions is for a continuous metric where no non-linear distributional operations are needed. Anyhow, here the mathematical framework cannot be as simple as for electromagnetism. Efforts to implement distributional methods in general relativity goes back to works of Papapetrou and Treder[20], Nariai[21], aiming to understand the O’Brien-Synge junction conditions, and
demanding the Einstein’s tensor to be free of $\delta$ functions, uses continuous metrics and so brings in distributions in the formalism, which is consequently used by Kumar[22]. Papapetrou and Hamoui[23, 24] then try to formulate a general method with application to spherical symmetric thin layer. Their method has then been reformulated and corrected by Evans[25]. Lichnerowicz [12] gives a detailed mathematical analysis of distributions in curved space-time and comes to the conclusion that the classical properties of the covariant derivatives and all of the corresponding formulas are valid for tensor distributions. Barrabes [26] uses tensor distributions specially to include null hyper-surfaces in the shell dynamics.

Taub[27] is interested in relativistic hydrodynamics and shock waves but also discusses the previous accounts on the concentrated 2- and 3-dimensional sources. Israel[28] and Taub[27] give a formulation for a 2-dimensional concentrated mass distribution. These attempts have been criticized by Geroch and Traschen[28] who give an extensive and thorough analysis of concentrated mass distributions in general relativity. This work is a milestone in all the discussions about the validity of distributional Einstein’s field equations and its applications to concentrated sources. There the authors define some regularity conditions for metrics, for which the distributional tensor calculations are allowed. Hence, for example, the line source case should be handled with care, although some authors have criticized it[30]. The (2+1)-dimensional case, as a result of this work, should not arise any problem, as far as the continuity of the metric is assured. We have thereafter used distributional tensors to solve the Einstein’s equations directly, without any use of Gauss-Kodazzi decompositions. Although the coordinates have to be prepared to make them continuous at the hypersurface of discontinuity, but the method has been applied easily for several cases[16, 31, 32]. Naturally, there should be no difference in the results using either of the distributional- or Gauss-Kodazzi-method. But the complete equivalence has never been demonstrated explicitly, and the role of jump conditions by using the distributional method has never been clearly stated.

With the results of Geroch and Traschen in mind, we show here that all the dynamical- and constraint-equations derived by the DI-formalism results very naturally in the distributional method, without any needs to define a new covariant derivative.

In section 2 we review shortly the D-I formalism and give the necessary formulae. Section III begins first with the formulation of the distributional
method. Covariant derivative of distributions and some useful formulae are given in section 3.1, and the Einstein’s equations for a thin shell in section 3.2. Section 3.3 deals with the conservation laws and the Bianchi identities. In this section we will see the full equivalence of the two methods. We end with a conclusion in section 4.

Conventions and definitions:
We use the signature (− + ++), and follow the curvature conventions of Misner, Thorn, and Wheeler (MTW)[10]. However, our sign convention for extrinsic curvature is that of Israel[9], which is the opposite of MTW. The greek indices run from 0 to 3 and latin indices from 1 to 3. A semicolon indicates covariant derivatives with respect to either the four-metric of the whole space-time or to the three-metric of the layer. There will, however, be no confusion because the kind of indices and objects used makes the difference transparent. The symbol $\nabla^\pm$ denotes the covariant derivative with respect to either of the metrics of partial manifolds $M^\pm$ which are to be glued together.

The square brackets $[F]$ are used to indicate the jump of any quantity $F$ at the layer, and bars $\overline{F}$ the arithmetic mean of it. As we are going to work with distributional valued tensors, there may be terms in a tensor quantity $F$ proportional to some $\delta$-function. These terms are indicated by $\tilde{F}$.

2 Darmois-Isreal Formalism

Assume two space-times $M^+$ and $M^-$ with boundaries $\Sigma^+$ and $\Sigma^-$. $M^+$ and $M^-$ may have been cut from space-times $M_1$ and $M_2$, respectively, but this is irrelevant for our task of glueing these together. Coordinates on the two space-time manifolds are defined independently as $x^\mu_+$ and $x^\mu_-$, and the metrics denoted by $g^{\alpha\beta}_+(x^\mu_+)$ and $g^{\alpha\beta}_-(x^\mu_-)$. The induced metrics on the boundaries are called $g^{ij}_+(\xi^k_+)$ and $g^{ij}_-(\xi^k_-)$, where $\xi^k_\pm$ are intrinsic coordinates on $\Sigma^\pm$, respectively. Bringing these 3- and 4-dimensional quantities in connection is trivially done with the help of tetrads defining on $\Sigma[34]$.

Now, to paste the manifolds together we demand the boundaries to be isometric having the same coordinates

$$\xi^k_+ = \xi^k_- = \xi^k.$$
The identification
\[ \Sigma_+ = \Sigma_- =: \Sigma \]
gives us the single glued manifold \( M = M_+ \cup M_- \).
This is the minimum requirement for gluing two manifolds together. Formulated as
\[ [g_{ij}] = 0 \] (1)
gives together with the continuity of the second fundamental form on \( \Sigma \)
\[ [K_{ij}] = 0 \] (2)
the Darmois conditions. Both conditions should be satisfied if \( \Sigma \) is just a boundary surface. But in case of a thin shell we do not expect the second condition to be satisfied. In fact, the matter content of the shell should lead to a jump in the extrinsic curvature \( K_{ij} \).
The condition (1) leaves the coordinates in \( M^\pm \) free. If we assume the continuity of the coordinates \( x^\mu_\pm \) at \( \Sigma \) we then have to require
\[ [g_{\mu\nu}] = 0, \] (3)
which together with the corresponding equation for derivatives of the metric
\[ \left[ \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right] = 0 \] (4)
gives the Lichnerowicz conditions. In the following we just assume the condition (1) or (3) respectively.
On \( \Sigma \) we define a three-bein
\[ e_i = \frac{\partial}{\partial \xi^i} \]
having the components
\[ e^\mu_i = \frac{\partial x^\mu}{\partial \xi^i} \] (5)
The induced metric on \( \Sigma \) is given by the scalar product
\[ g_{ij} = e_i \cdot e_j = g_{\mu\nu} e^\mu_i e^\nu_j \] (6)
Note that, because of the assumed isometry, this metric is the same on both faces \( \Sigma_+ \) and \( \Sigma_- \). We note that the subscripts of the three-beins on \( \Sigma \) are
not the component indices, but as this distinction is trivial we prefer for the sake of simplicity not to use parentheses to distinguish them, as is usually done.

We choose the parametric equation for $\Sigma$ in the form

$$\Phi(x^\mu(\xi^i)) = 0,$$

(7)

having the unit normal four-vector $n^\mu$ given by

$$n_\mu = \alpha^{-1} \partial_\mu \Phi,$$

(8)

where

$$\alpha = \pm \sqrt{\left| g^{\nu\gamma} \frac{\partial f}{\partial x^\nu} \frac{\partial f}{\partial x^\gamma} \right|}.$$  

(9)

Therefore

$$n_\mu e^\mu_i = 0,$$  

(10)

and

$$n_\mu n^\mu = \epsilon,$$  

(11)

where $\epsilon = +1$ or $-1$ for $\Sigma$ to be time- or space-like, respectively. We suppose $n^\mu$ directed from $M^-$ to $M^+$, i.e. in the direction of increasing a space- or time-like coordinate corresponding to time- or space-like $\Sigma$. Therefore we have to take the positive (negative) sign in (9) for time- (space-)like $\Sigma$. This choice gives us the useful relation

$$\text{sign } \alpha = \frac{|\alpha|}{\alpha} = \epsilon.$$  

The choice of Lichnerowicz condition (3) makes it possible to have a unique normal vector for each case. As we want to concentrate on the formulation of the distributional method and avoid any undue complications, we leave aside the case of null hypersurfaces.

Now, in general the metrics in $M^+$ and $M^-$ need not to be continuous at $\Sigma$, but they could be. However, the normal extrinsic curvature (second fundamental form) is not continuous for a thin shell. It is defined by

$$K^\pm_{ij} = e^\mu_i e^\nu_j \nabla^\pm_\mu n_\nu = -n_\mu e^\nu_j \nabla^\pm_\nu e^\mu_i$$

$$= -n_\mu e^\nu_i \nabla^\pm_\nu e^\mu_j = K^\pm_{ji}$$

(12)
Now, we have all the prerequisites to write the Einstein’s equation for the hypersurface. These are 10 equations which will be written in components normal and tangent to the hypersurface. The first and second contracted Gauss-Kodazzi equations are \[ G_{\mu \nu} n^\mu n^\nu = \frac{1}{2}(K^2 - K_{ij}K^{ij} - \epsilon^3 R) \] (13) \[ G_{\mu \nu} e_i^\mu n^\nu = K^j_{i,j} - K_{i,i} \] (14) where \(3R\) and \(3G\) are the Ricci scalar and Einstein tensor of the three metric \(g_{ij}\), respectively. Now, to discover the effect the energy-momentum tensor \(S_{ij}\) of \(\Sigma\) on the space-time geometry, we perform a ”pill-box” integration of Einstein’s equations across \(\Sigma\):

\[ S_{\mu \nu} = \lim_{\Sigma \to 0} \int_{\Sigma} \left( T_{\mu \nu} - g_{\mu \nu} \frac{\Lambda}{k} \right) \, dn = \frac{1}{k} \lim_{\Sigma \to 0} \int_{\Sigma} G_{\mu \nu} \, dn, \] (15)

where \(n\) is the proper distance through \(\Sigma\) in the direction of the normal \(n_\mu\). \(S_{\mu \nu}\) is the associated 4-tensor of energy momentum of the shell. The equations (13, 14) have the physical meaning that no moment associated with the surface layer flows out of \(\Sigma\). Therefore \(S_{\mu \nu}\) vanishes off the hypersurface \(\Sigma\), which is expressed as

\[ S_{\mu \nu} n^\nu = 0 \] (16)

The energy momentum 4- and 3-tensors are related as

\[ S^{\mu \nu} = e_i^\mu e_j^\nu S_{ij} \] (17)

The covariant derivative of such a tensor relative to the corresponding connections is given by [9]

\[ \nabla_\nu S^{\mu \nu} = e_i^\mu S_{ij}^{\nu} - \epsilon S^{ij} K^{i,j} n^\mu, \] (18)

which leads to the following useful relation

\[ e_i^\mu \nabla_\nu S_{\mu \nu} = S_{i;j} \] (19)

Similarly we can associate to the 3-tensor \(K_{ij}\) defined on \(\Sigma\), the corresponding 4-dimensional tensor:

\[ K^{\mu \nu} = K^{ij} e_i^\mu e_j^\nu, \] (20)

satisfying

\[ K^{\mu \nu} n_\nu = 0. \] (21)

The remaining components of the Einstein’s equations lead to the following non-vanishing result
\[
\lim_{\Sigma \to 0} \int_{-\Sigma}^{\Sigma} G_{\mu\nu} e^\mu_i e^\nu_j \, dn = \\
\epsilon \left( [K_{ij}] - g_{ij}[K] \right) = \kappa S_{ij}
\]

This distributional equivalent of Einstein’s equations is called \textit{Lanczos} equation, which partly determines the dynamic of the thin shell. The other dynamical equations come from the defining equation of matter contents of the shell. Now, the two Gauss-Kodazzi equations act as constraints. The first one (13) is the so-called ”Hamiltonian”- and the second one (14) the ”ADM”- constraint. Note however that these equations are valid in \( M^+ \) and \( M^- \) on taking the limits as one approaches the layer \( \Sigma \). Therefore we are actually faced with 8 equations, the sum and difference of which give us the junction conditions. The Hamiltonian constraint along with the Einstein’s or Lanczos equations then give the \textit{evolution identity}:

\[
S^{ij} \overline{K}_{ij} = -[T_{\mu\nu} n^\mu n^\nu - \Lambda / \kappa]
\]  \hspace{1cm} (23)

and

\[
3R + (\overline{K}_{ij} K^{ij} - K^2) = 2\epsilon\kappa(T_{\mu\nu} n^\mu n^\nu - \Lambda / \kappa) + \frac{\epsilon\kappa^2}{4}(S^{ij} S_{ij} - S^2 / 2).
\]  \hspace{1cm} (24)

The ADM constraint gives the \textit{conservation identity}

\[
S^{ij}_{;i} = -\epsilon [T_{\mu\nu} n^\mu e^\nu_j]
\]  \hspace{1cm} (25)

and

\[
\overline{K}^{ij}_{;i} - K^i_{;i} = \kappa(T_{\mu\nu} n^\mu e^\nu_j)
\]  \hspace{1cm} (26)

Not all of these jump conditions are independent. Usually one takes the evolution identity (23) and the conservation identity (25) as the proper junction conditions, which in addition to the Lanczos equation should be satisfied[35].

3 \hspace{1cm} \textbf{Distributional Method}

Here we intend to give a formulation of the Einstein’s equations for the case where there exists a hypersurface of concentrated source immersed in
an otherwise arbitrary space-time, not necessarily vacuum. We assume the metric to be continuous at the hypersurface:

\[ [g_{\mu\nu}] = 0 \]  

(27)

Otherwise we would have to consider non-linear operations of distributions such as \( \delta\theta \) or \( \delta\delta \). The disadvantage of having a continuous metric across the shell pays off by the simplicity of the method to calculate specific solutions[16, 31, 32].

Write the metric in the following form

\[ g_{\mu\nu} = g_{\mu\nu}^+ \theta(\Phi(x)) + g_{\mu\nu}^- \theta(-\Phi(x)) \]  

(28)

where \( \theta \) is the step function and

\[ g_{\mu\nu}^+|_{\Phi(x)=0} = g_{\mu\nu}^-|_{\Phi(x)=0} \]  

(29)

This condition guarantees the smoothness of the metric on the hypersurface. Should this not be the case we try a coordinate transformation \( x = x(x') \) having a jump in the first derivative:

\[ \frac{\partial x^\mu}{\partial x'^\rho} = \alpha^+_{\rho} \theta(\Phi(x)) + \alpha^-_{\rho} \theta(-\Phi(x)) \]  

(30)

The condition for the new metric to be continuous comes out to be

\[ \alpha^+_{\rho} \alpha^+_{\sigma} g_{\mu\nu}^+|_{\Phi(x)=0} = \alpha^-_{\rho} \alpha^-_{\sigma} g_{\mu\nu}^-|_{\Phi(x)=0} \]  

(31)

We assume from now on that the metric is smooth everywhere, \( C^1 \) at the hypersurface and \( C^\infty \) on both sides of it.

Although the metric is continuous on \( \Sigma \), its derivatives, and so the corresponding connections, are discontinuous. Nevertheless the connection corresponding to the metric \( g_{\mu\nu} \) can be written in the following compact form:

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \]

\[ = \theta(\Phi(x)) \Gamma^+_{\mu\nu} + \theta(-\Phi(x)) \Gamma^-_{\mu\nu}, \]  

(32)

where \( \Gamma^\pm_{\mu\nu} \) are the ordinary connections on \( M^\pm \). The above connection has jump discontinuities on \( \Sigma \).
To write the field equations for the hypersurface we need the formulation of the energy-momentum tensor of the shell. Generally it can be written in the form

\[ \tilde{T}_{\mu\nu} = CS_{\mu\nu}\delta(\Phi(x)) \] (33)

where \( C \) is a constant to be calculated. We integrate the above equation in the direction of the normal to the hypersurface

\[ \int \tilde{T}_{\mu\nu} \, dn = CS_{\mu\nu} \int \delta(\Phi(x)) \, dn = CS_{\mu\nu}\left|\frac{d\Phi}{d\Phi}\right| \] (34)

Therefore, using the definition (15), we obtain

\[ C = \left|\frac{d\Phi}{dn}\right| \] (35)

and

\[ \tilde{T}_{\mu\nu} = S_{\mu\nu}\left|\frac{d\Phi}{dn}\right|\delta(\Phi(x)) \] (36)

Note that in the literature one usually takes \( C = 1 \), which is correct just for special cases. But in general the factor \( C \) is necessary (see also [33]). Now, the derivative of \( \Phi \) in the normal direction can be written in terms of unit normal vector \( n^\mu \):

\[ C = \left|\frac{d\Phi}{dn}\right| = |n^\mu \partial_\mu \Phi| = |\epsilon \alpha| = |\alpha|, \] (37)

where we have used (8-11). Therefore (33) will be written in the form

\[ \tilde{T}_{\mu\nu} = S_{\mu\nu}|n^\sigma \partial_\sigma \Phi| \delta(\Phi(x)) \]
\[ = CS_{\mu\nu}\delta(\Phi(x)) = |\alpha|S_{\mu\nu}\delta(\Phi(x)) \] (38)

### 3.1 Covariant Derivative of Distributional valued Tensors

There is no need to change of the ordinary concept of covariant derivative, as it has been carefully shown by Lichnerowicz [12]. In fact, all the known properties of covariant derivative and the corresponding formulae in a pseudo-Riemannian manifold are valid for tensor distributions. But for the sake of
convenience of calculation we refer to some useful formulae. Consider first an arbitrary vector $A^\mu$ defined as

$$A^\mu = \theta(\Phi)A^+ + \theta(-\Phi)A^-, \quad (39)$$

where $A^\pm$ has the support on $M^\pm$. It is therefore useful to define the operator

$$\nabla = \theta(\Phi)\nabla^+ + \theta(-\Phi)\nabla^- \quad (40)$$

We can know write the covariant derivative of a distributional valued vector $A^\mu$ in terms of the covariant derivatives of its defining parts in $M^\pm$. The following relation is easily obtained:

$$A_{\mu\nu}^\nu = \nabla_\nu A^\mu + [A^\mu] \partial_\nu \Phi \delta(\Phi) \quad (41)$$

This relation can be generalized easily for a distributional tensor of any rank. The covariant derivative of the tensor $T^{(\rho)} = \theta(\Phi)T^{+(\rho)} + \theta(-\Phi)T^{-(\rho)}$, where $(\rho)$ stands for any number of indices, is calculated to be

$$T^{(\rho)}_{\nu} = \nabla_\nu T^{(\rho)} + [T^{(\rho)}] \partial_\nu \Phi \delta(\Phi) \quad (43)$$

In the case a tensor has the support on $\Sigma$ its covariant derivative is in the usual form. Take the tensor $\tilde{T}^{\mu\nu}$ from (33). Its covariant derivative can be written

$$\tilde{T}^{\mu\nu}_{\rho} = (CS^{\mu\nu})_{\rho} \delta(\Phi) + CS^{\mu\nu} \delta(\Phi)_{\rho}$$

$$= (CS^{\mu\nu})_{\rho} \delta(\Phi) + CS^{\mu\nu} \partial_\rho \Phi \delta(\Phi) \quad (44)$$

We will need this relation later to discuss the conservation laws.

### 3.2 The Field Equations

The Einstein’s field equations are valid on both sides of the hypersurface as usual. So we concentrate our procedure on $\Sigma$, where we expect the curvature and Einstein tensor to be proportional to $\delta$. That means in calculating the
connection coefficients and the components of the Ricci tensor we can ignore terms not proportional to $\delta$. Hence, e.g., the terms in the Ricci tensor

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\nu,\rho} + \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\rho\sigma}$$  (45)

proportional to $\Gamma$'s can be ignored. The only relevant terms are

$$\tilde{R}_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu,\rho} - \tilde{\Gamma}^\rho_{\mu\nu,\rho}$$  (46)

Now,

$$\Gamma^\rho_{\mu\rho} = \frac{1}{2} g_{,\mu}$$  (47)

where $g$ is the determinant of the metric. The $\delta$ distribution can only occur in the second derivatives of the metric. Therefore

$$\tilde{\Gamma}^\rho_{\mu\nu,\rho} = \frac{1}{2 g} \tilde{g}_{,\mu\nu}$$  (48)

Similarly, for the second term in the Ricci tensor we have

$$\tilde{\Gamma}^\rho_{\mu\nu,\rho} = \frac{1}{2} g^{\rho\sigma} (\tilde{g}_{\rho\sigma,\nu,\rho} + \tilde{g}_{\rho\sigma,\mu,\rho} - \tilde{g}_{\rho\sigma,\mu\nu,\rho})$$  (49)

Having the metric in the form (28) we obtain

$$\tilde{g}_{\alpha\beta,\mu\nu} = [g_{\alpha\beta,\mu}] (\partial_\nu \Phi) \delta(\Phi(x))$$  (50)

and

$$\tilde{g}_{,\mu\nu} = [g_{,\mu}] (\partial_\nu \Phi) \delta(\Phi(x))$$  (51)

As the result we obtain for terms in the Ricci tensor proportional to $\delta$

$$\tilde{R}_{\mu\nu} = \frac{1}{2 g} g_{,\mu} (\partial_\nu \Phi - g^{\rho\sigma} ([g_{\sigma\mu,\nu}] + [g_{\sigma\nu,\mu}] - [g_{\rho\nu,\sigma}]) \partial_\rho f) \delta(f(x))$$

$$= \frac{1}{2 g} [g_{,\mu}] (\partial_\nu \Phi - [\Gamma^\rho_{\mu\nu}] \partial_\nu \Phi) \delta(\Phi(x))$$  (52)

This enable us to write the Einstein’s equations for the layer:

$$\tilde{G}_{\mu\nu} = \kappa \tilde{T}_{\mu\nu}$$  (53)
Defining
\[ Q_{\mu \nu} = (\alpha)^{-1} \left( \frac{1}{2g} [g_{\mu\nu}] \delta_\rho^\rho - [\Gamma^\rho_{\mu \nu}] \right) \partial_\rho \Phi \]
\[ = \left( \frac{1}{2g} [g_{\mu\nu}] \delta_\rho^\rho - [\Gamma^\rho_{\mu \nu}] \right) n_\rho \]
we obtain, using (38) and (52) for the energy momentum tensor the field equations in the 4-dimensional form
\[ Q_{\mu \nu} - \frac{1}{2} g_{\mu \nu} Q = \epsilon \kappa S_{\mu \nu}, \]
where \( Q = Q_{\mu \nu} g^{\mu \nu} \), and we have used the relation \( \epsilon = \frac{|\alpha|}{\alpha} \). Note that \( Q_{\mu \nu} \) is a tensor with support on \( \Sigma \). This equation, for the time-like case, has been first derived, without using the hitherto unknown distributional calculus, by Sen[2]. We would like, therefore, to coin it by Sen equation. The three dimensional form of the Sen equation is readily obtained by decomposing it to tangential- and normal-components to \( \Sigma \). Multiplying (54) with \( n^\mu \) we obtain
\[ S_{\mu \nu} n^\nu = 0, \]
which is the same relation as (16). This tell us immediately that the components corresponding to \( S_{\mu \nu} n^\mu n^\nu \) and \( S_{\mu \nu} n^\mu e_i^\nu \) identically vanishes. To obtain the proper 3-dimensional components we notice first that
\[ Q_{ij} = Q_{\mu \nu} e_i^\mu e_j^\nu = -[\Gamma^\rho_{\mu \nu}] n_\rho e_i^\mu e_j^\nu = [K_{ij}]. \]
Therefore, we obtain from the Sen equation
\[ Q_{ij} = \epsilon \kappa \left( S_{ij} - \frac{1}{2} g_{ij} S \right), \]
which is equivalent to the Lanczos equation (22).
We have therefore seen that the explicit method of writing the Einstein’s field equations for a regular metric which is continuous without having continuous derivatives leads to the equation (55) and is equivalent to the DI-formalism based on the Gauss-Codazzi formalism. In practice, one begins with known solutions of the Einstein’s equations in \( M^\pm \), and after making sure the continuity of the metric on \( \Sigma \), tries to solve the equations (55). In the following we will show that the jump conditions of DI-formalism follows from the Bianchi identities corresponding to the metric (28), and are therefore implicit in the equations we have used.
3.3 Conservation Laws

We have now all the prerequisites to evaluate the Bianchi identities and the conservation of energy momentum tensor of our pasted space-time. The energy momentum of the whole space-time, including the cosmological terms \( \Lambda^{\pm} \) is

\[
T^{\mu\nu} = \bar{T}^{\mu\nu} + (T^{+\mu\nu} - \Lambda^{+}/\kappa g^{+\mu\nu})\theta(\Phi) + (T^{-\mu\nu} - \Lambda^{-}/\kappa g^{-\mu\nu})\theta(-\Phi),
\]

where \( \bar{T}^{\mu\nu} \) is defined in (38). Having in mind that the covariant divergences of \( T^{\pm\mu\nu} \) and \( g^{\pm\mu\nu} \) with respect to the corresponding connections vanishes, we obtain

\[
T^{\mu\nu} \, ;_{\nu} = (\bar{T}^{\mu\nu}) \, ;_{\nu} + \left[ T^{\mu\nu} - \Lambda/\kappa \right] \partial_{\nu} \Phi \, \delta(\Phi),
\]

where we have used the relation (43). Now inserting for \( \bar{T}^{\mu\nu} \) from (38) and using (44) we obtain an equation having terms proportional to \( \delta(\Phi) \) and \( \delta'(\Phi) \). Each term vanishes independently. The term proportional to \( \delta'(\Phi) \) gives

\[
(S^{\mu\sigma} \partial_{\sigma} \Phi)(n^{\sigma} \partial_{\sigma} \Phi) = 0,
\]

which leads to the (56) and ensures the orthogonality of the energy-momentum tensor of \( \Sigma \) to the hypersurface normal \( n^{\mu} \). We use in the following this relation to simplify the remaining calculations. The term proportional to \( \delta \) is

\[
\left( CS^{\mu\nu} \right)_{,\nu} + CS^{\mu\rho} \Gamma^\nu_{\rho\nu} + CS^{\rho\nu} \Gamma^\mu_{\rho\nu} + \left[ T^{\mu\nu} - \Lambda/\kappa \right] g^{\mu\nu} \partial_{\nu} \Phi \right) \delta(\Phi) = 0.
\]

We have left \( \delta(\Phi) \) as proportionality factor to stress its influence specially on terms containing \( \Gamma \)'s and \( g^{\mu\nu} \)'s. Note that the third term containing \( \Gamma \)'s contains terms like \( \theta \cdot \delta \), i.e. product of distributions. This is in analogy to the elementary problem of evaluating the electrostatic force on a sheet of charge[5]. There the linearity of electrostatic equation resolve the ambiguity. But how about our case where the Einstein equations are non-linear? We have already shown that in the case of thin shells the only terms contributing to the Einstein tensor are the derivatives of the connection, or the second derivative of the metric, which appear linearly. It is then easily seen that is case of a layer the Einstein's equations leads to a Poisson-like equation corresponding to the Sen equation (55) for concentrated distribution, where
the second derivative can be replaced by (50) and (51). Therefore, in analogy to the electromagnetic case we can use the linearity of (55) to show that

$$\Gamma_{\rho\nu}^{\mu} \delta(\Phi) = \frac{1}{2}(\Gamma_{\rho\nu}^{\mu} + \Gamma_{\nu\rho}^{\mu}) \delta(\Phi) \quad (63)$$

Using this result and multiplying the equation (62) with \( n_{\mu} \), we obtain the component in the normal direction.

$$(CS^{\mu\nu})_{,\nu} n_{\mu} + CS^{\rho\nu} \Gamma_{\rho\nu}^{\mu} n_{\mu} = \epsilon[T^{\mu\nu} - \Lambda/\kappa g^{\mu\nu}] n_{\mu} \partial_{\nu} \Phi. \quad (64)$$

Using the relations (63) and the definition of the extrinsic curvature (12), we obtain the final result

$$S_{ij} K_{ij} = \epsilon[T^{\mu\nu} n_{\mu} n_{\nu} - \Lambda/\kappa] \quad (65)$$

This is the evolution identity (23) derived as one of the jump conditions in the Darmois-Israel method. Here it is just a consequence of the Bianchi identities. To obtain the remaining three equation we multiply (64) with \( e_i^\mu \).

Using (19) we obtain

$$S_{j;\ i} = -\epsilon[T_{\mu\nu} n_{\mu} e_i^{\nu}], \quad (66)$$

which is the conservation identity (25). It gives the conservation law for the energy momentum tensor of the layer. We therefore see that our explicit distributional method of solving the Einstein’s equations gives all the dynamical and constraint equations of Darmois-Israel method, and is therefore equivalent to it.

4 Conclusion

We have seen that, based on the Lichnerowicz condition (3), a distributional method can be formulated to solve the Einstein’s field equations for a thin shell, which is equivalent to the Darmois-Israel formalism, and gives all the necessary equations and jump conditions formulated there. In fact, it has been shown that the jump conditions are consequence of the Bianchi identities, and therefore implicit in the formalism, once the Lichnerowicz condition is satisfied. This makes the distributional formalism easy to apply, specially when explicit solutions are to be found, and it pays off the disadvantage of the continuity of the coordinates across the shell.
References

[1] C. Lanczos, Phys. Z. 23, 539 (1922); Ann. Phys. (Leipzig), 74, 518 (1924).

[2] N. Sen, Ann. Phys. (Leipzig), 73, 365 (1924).

[3] G. Darmois, Memorial de Sciences Mathematiques, Fascicule XXV, ”Les equations de la gravitation einsteinienne”, Chapitre V (1927).

[4] A. Vilenkin, Phys. Rep. 121, 263 (1984).

[5] S.K. Blau, E.I. Guendelman, and I.I. Tkachev, Phys. Rev. D35, 2919 (1987).

[6] M. Visser, Nucl. Phys. B328, 203 (1989).

[7] Ch. Hellaby and T. Dray, Phys. Rev. D49, 5096 (1994).

[8] C. Barrabes and V.P. Frolov, gr-qc/95 1136.

[9] W. Israel, Nouvo Cimento 44B, 1 (1966); Corrections in 44B, 463.

[10] C.W. Misner, K.S. Thorn and J.A. Wheeler, Gravitation, Freeman, San Francisco, (1973).

[11] A. Lichnerowicz, in Theories Relativistes de la Gravitation et de l’Electromagnetisme, Masson, Paris (1955).

[12] A. Lichnerowicz, in Relativity, Quanta, and Cosmology (Einstein 1879-1979), Vol II, Johnson, New York (1979).

[13] S. O’Brien and J.L. Synge, Commun. Dublin. Inst. Adv. Stud. A., no 9.. (1952).

[14] W.B. Bonnor and P.A. Vickers, Gen. Rel. Grav., 13, 29 (1981).

[15] R. Mansouri, The Art of Glueing Space-Time Manifolds: Methods and Applications, Lectures given at Kosmologie Gruppe, Mathematisches Institut, University of Potsdam, Germany, (1996).

[16] M. Khorrami and R. Mansouri, J. Math. Phys. 35, 951 (1994).
[17] Y. Choquet-Bruhat, Comm. Math. Phys. 12, 16 (1969).

[18] C.K. Raju, J. Phys. A 15, 1785 (1982).

[19] C.K. Raju, J. Phys. A 15, 381 (1982).

[20] A. Papapetrou and H. Treder, Math. Nachr. 23, 371 (1961).

[21] H. Nariai, Prog. Theor. Phys. 34, 173 (1965).

[22] M.M. Kumar, Prog. Theor. Phys. 44, 2 (1970).

[23] A. Papapetrou and A. Hamoui, Ann. Inst. H. Poincare 9, 179 (1968).

[24] A. Papapetrou and A. Hamoui, Gen. Rel. Grav. 10, 253 (1979).

[25] A.B. Evans, Gen. Rel. Grav. 8, 155 (1977).

[26] C. Barrabes, Class. Quantum Grav. 6, 581 (1989).

[27] A.H. Taub, J. Math. Phys. 21, 1423 (1980).

[28] W. Israel, Phys. Rev. D15, 935 (1977).

[29] R. Geroch and J. Traschen, Phys. Rev. D36, 1017 (1987).

[30] G. Hayward and J. Louko, Phys. Rev. D42, 4033 (1990).

[31] M. Khorrami and R. Mansouri, Phys. Rev. D44, 557 (1991).

[32] R. Mansouri, Gravitaional Field of Plane Domain Walls, Sharif University of Technology, report, unpublished, (1990).

[33] C. Barrabes and W. Israel, Phys. Rev. D43, 1129 (1991).

[34] K. Kuchar, Czeck. J. Phys. B18, 435 (1968).

[35] P. Musgrave and K. Lake, gr-qc/95 10 052.