

Riesz transform via heat kernel and harmonic functions on non-compact manifolds

Renjin Jiang

Abstract. Let $M$ be a complete non-compact manifold satisfying the volume doubling condition, with doubling index $N$ and reverse doubling index $n$, $n \leq N$, both for large balls. Assume a Gaussian upper bound for the heat kernel, and an $L^2$-Poincaré inequality outside a compact set.

If $2 < n$, then we show that for $p \in (2, n)$, $(R_p)$: $L^p$-boundedness of the Riesz transform, $(G_p)$: $L^p$-boundedness of the gradient of the heat semigroup, and $(RH_p)$: reverse $L^p$-Hölder inequality for the gradient of harmonic functions, are equivalent to each other. Our characterization implies that for $p \in (2, n)$, $(R_p)$ has an open ended property and is stable under gluing operations. This substantially extends the well known equivalence of $(R_p)$ and $(G_p)$ from [4] to more general settings, and is optimal in the sense that $(R_p)$ does not hold for any $p \geq n > 2$ on manifolds having at least two Euclidean ends of dimension $n$.

For $p \in (\max\{N, 2\}, \infty)$, the fact that $(R_p)$, $(G_p)$ and $(RH_p)$ are equivalent essentially follows from [22], moreover, if $M$ is non-parabolic, then any of these conditions implies that $M$ has only one end.

For the proof, we develop a new criteria for boundedness of the Riesz transform, which was nontrivially adapted from [4], and make an essential application of results from [22]. Our result allows extensions to non-smooth settings.

Contents

1 Introduction .................................................. 2
  1.1 Background and motivations .................................. 2
  1.2 Necessary and sufficient conditions for small $p$ ............. 5
  1.3 Necessary and sufficient conditions for large $p$ ............. 7
  1.4 Applications and comments ................................... 8
  1.5 Structure of the paper ...................................... 10

2 Poincaré inequality .......................................... 10

2010 Mathematics Subject Classification. Primary 58J35; 42B20; Secondary 58J05; 35B65; 35K05.

Key words and phrases: Riesz transform, harmonic functions, heat kernels, Ricci curvature, Poincaré inequality
1 Introduction

1.1 Background and motivations

Let $M$ be a complete, connected and non-compact Riemannian manifold. Denote by $d$ the geodesic distance, by $\mu$ the Riemannian measure, and by $\mathcal{L}$ be the non-negative Laplace-Beltrami operator on $M$. Let $\{e^{-t\mathcal{L}}\}_{t>0}$ be the heat semigroup. The negative square root of $\mathcal{L}$ is given by

$$\mathcal{L}^{-1/2} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-s\mathcal{L}} \frac{ds}{\sqrt{s}}.$$ 

Denote by $\nabla$ the Riemannian gradient.

The study of the Riesz transform $\nabla \mathcal{L}^{-1/2}$ is one of the central topics of analysis on manifolds. Strichartz in 1983 [48] and then Bakry in 1987 [5], provided sufficient conditions on non-compact manifolds such that the Riesz transform is bounded for all $1 < p < \infty$. Since then, many sufficient, or even in some cases necessary and sufficient, conditions for the boundedness of the Riesz transform have been provided; see for instance [1, 3, 4, 11, 13, 14, 16, 20, 21, 40, 43]. Let us review some related results. Since the boundedness of Riesz transform on compact manifolds is not an issue (cf. [48]), we will only consider non-compact cases.

For each $p \in (1, \infty)$, we say that $(R_p)$ holds, if the Riesz transform $|\nabla \mathcal{L}^{-1/2}|$ is bounded on $L^p(M)$. Notice that $(R_2)$ holds automatically which can be seen by integration by parts. In the metric measure space $(M, d, \mu)$, denote by $B(x, r)$ the open ball with centre $x \in M$ and radius $r > 0$ and by $V(x, r)$ its volume $\mu(B(x, r))$. One says that $M$ satisfies the volume doubling property (in short is doubling) if there exists a constant $C_D > 1$ such that

$$(D) \quad V(x, 2r) \leq C_D V(x, r),$$

for all $r > 0$ and $x \in M$. The heat semigroup has a smooth positive and symmetric kernel $p_t(x, y)$, meaning that

$$e^{-t\mathcal{L}}f(x) = \int_M p_t(x, y)f(y) \, d\mu(y)$$
Riesz transform on manifolds with ends

for suitable functions $f$. One says that the heat kernel satisfies a Gaussian upper bound if there exist $C, c > 0$ such that for all $t > 0$ and $x, y \in M$,

$$(UE) \quad p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}.$$  

Coulhon and Duong [20] showed that the doubling condition together with a Gaussian upper bound of heat kernel is sufficient for $(R_p)$ for all $p \in (1, 2)$. Recently, Chen et al. [15] showed, a bit surprisingly, that a sub-Gaussian upper bound of the heat kernel could replace the Gaussian upper bound in the above result; see [39] for further developments.

The case $p > 2$ is more difficult. Notice that if $(R_p)$ holds, then it follows from the analytic property of the heat semigroup that

$$(G_p) \quad \|\nabla e^{-t\Delta}\|_{p \to p} \leq \|\nabla L^{1/2} e^{-t\Delta} L^{1/2} f\|_{p \to p} \leq C\sqrt{t}$$

for all $t > 0$; see [47] or [4]. Above and in what follows, we use the notation $\|\cdot\|_p$ to denote the $L^p$ norm over $M$, and the notation $\|\cdot\|_{p \to p}$ for the operator norm from $L^p$ to $L^p$, for any $p \in [1, \infty]$. A natural and longstanding question is as following.

**Question 1.1.** Let $p \in (2, \infty)$. Does $(G_p)$ imply $(R_p)$?

Auscher, Coulhon, Duong and Hofmann in 2004 [4] established a remarkable result, which shows that, under $(D)$ and a scale-invariant $L^2$-Poincaré inequality, $(G_{p_0})$ implies $(R_p)$ for all $p \in (2, p_0)$, where $p_0 \in (2, \infty]$. The scale invariant $L^2$-Poincaré inequality means that there exists $C > 0$ such that for every ball $B$ and each $f \in C^1(\overline{B})$, it holds

$$(P_2) \quad \int_B |f - f_B|^2 \, d\mu \leq C_C^2 \int_B |\nabla f|^2 \, d\mu,$$

where $f_B$ denotes the average of the integral of $f$ on $B$. Notice that $(D)$ together with $(P_2)$ is equivalent to the heat kernel has two-sided Gaussian bounds; see [28, 45]. By recent results from [7, 22], one finally sees that $(G_p) \iff (R_p)$ for each $p \in (2, \infty)$, under $(D)$ and $(P_2)$.

However, Question 1.1 in generality is still open; see [4, Subsection 1.4] and also [7]. The requirement of $(P_2)$ is not necessary by looking at a manifold obtained by gluing two Euclidean ends through a compact manifold smoothly; see [13, 20, 29]. Here and below, an *end* means, an unbounded component of a complete noncompact manifold $M$ outside a compact subset $M_0$.

In [13], Carron, Coulhon and Hassell showed that the Riesz transform is $L^p$-bounded for $2 < p < n$, $n \geq 3$, if $M$ is an $n$-dimensional manifold with a finite number of Euclidean ends; the result has been further generalized to manifolds with conic ends by Guillarmou and Hassell [32], and by Carron [12] to manifolds with quadratic Ricci curvature decay, i.e., for a fixed $x_M \in M$ and $C_M \geq 0$, it holds

$$(QD) \quad \text{Ric}_M(x) \geq -\frac{C_M}{[d(x, x_M) + 1]^2}.$$
Moreover, in [13], it has been showed that if \( M \) has at least two ends, then the Riesz transform is not \( L^p \)-bounded for any \( p \geq n \). Indeed, by using \( L^p \)-cohomology, the following non-trivial result was proved in [13].

**Theorem 1.2** ([13]). Suppose that \( M \) has Ricci curvature bounded from below, and for some \( N > 2 \) \( V(x, r) \leq r^N \), for all \( x \in M \) and \( r \geq 1 \). If there exists \( C > 0 \) such that for any \( f \in C^\infty_c(M) \) it holds

\[
(S_{\frac{n}{n-2}}), \quad \|f\|_{\frac{n}{n-2}} \leq C\|\nabla f\|_2,
\]

and \( M \) has at least two ends, then the Riesz transform is not bounded on \( L^p(M) \) for any \( p \geq N \).

Notice that the Sobolev inequality \((S_{\frac{n}{n-2}})\) together with \( V(x, r) \leq r^N \) implies \((UE)\) (cf. [28, 29]), conversely \((UE)\) only implies a local Sobolev inequality (cf. [9, 28, 29]). In particular, under \((UE), (S_{\frac{n}{n-2}})\) may not hold; see [49].

The above result has been further refined by Carron [12, Theorem C]. Notice that, in particular, in the above theorem and Carron’s theorem, the ends are not necessarily Euclidean or conic. In view of this, in [13], several questions, regarding relaxing the requirement that ends are Euclidean, had been proposed; see following Question 1.13, Question 6.2 and Question 6.3.

In this paper, we provide a solution to Question 1.1 by relaxing the requirement of \((P_2)\), but only for \( p \) in the intervals \((2, n)\) and \((2 \vee N, \infty)\); see Theorem 1.5 and Theorem 1.11 below. As an application, we obtain stability under gluing operation and open ended property for the Riesz transform on manifolds with general ends. Notice that the case \( p \in (1, 2) \) was well understood by [20, 15], as we recalled above. We will only consider the case \( p > 2 \) in this work.

Throughout the paper, we assume that \( M \) is a non-compact, connected and complete manifold of the same (topological) dimension, that satisfies the doubling condition \((D)\). We shall simply recognize \( M \) as the union of a compact set \( M_0 \) and one or more but finitely many ends \( \{E_i\} \). We fix a point \( x_M \in M_0 \) and assume without loss of generality that \( \text{diam}(M_0) = 1 \).

The doubling condition \((D)\) together with connectivity implies that there exist \( 0 < \eta \leq \Upsilon < \infty \) such that for any \( x \in M \) and all \( 0 < r < R < \infty \) it holds

\[
\left(\frac{R}{r}\right)\eta \leq \frac{V(x, R)}{V(x, r)} \leq \left(\frac{R}{r}\right)\Upsilon;
\]

see for instance [35]. This further implies that there exist \( 0 < N < \infty \) such that

\[
(D_N) \quad \frac{V(x, R)}{V(x, r)} \leq \left(\frac{R}{r}\right)^N, \quad \forall x \in M \text{ and } \forall 1 < r < R < \infty,
\]

and there exists \( 0 < n \leq N \) such that

\[
(RD_n) \quad \left(\frac{R}{r}\right)^n \leq \frac{V(x_M, R)}{V(x_M, r)}, \quad \forall 1 < r < R < \infty.
\]

In what follows, we call \( n \) the lower dimension, and \( N \) the upper dimension, of \( M \). Moreover, we simply use \((D_N)\) to indicate that \( \mu \) is a doubling measure with \( N \) being the upper dimension.
Remark 1.3. (i) It holds obviously \( \nu \leq n \leq N \leq \Upsilon \). The examples of cocompact covering Riemannian manifolds with polynomial growth deck transformation group and Lie groups of polynomial growth show it may happen that \( \nu < n \) and \( N < \Upsilon \); see \([1, 26, 34, 49]\) for instance.

(ii) Notice that we only need \((RD_n)\) for a fixed point \( x_M \in M_0 \) and \( R > r > 1 \). Take weighted lines \((\mathbb{R}, (1 + |x|)^\alpha \, dx), \alpha > 0\), for example. A small calculation shows that \((D_{\alpha+1})\) and \((RD_{\alpha+1})\) hold, but \((1.1)\) holds with \( \nu = 1 \) and \( \Upsilon = \alpha + 1 \); see \([34]\) and also \([12]\). Moreover, by using the doubling property and the fact \( M_0 \) is compact, one sees that \((RD_n)\) holds if and only it holds for each \( o \in M_0 \) and all \( 1 < r < R < \infty \) that \((R/r)^n \lesssim V(o, R)/V(o, r)\).

(iii) In many cases, such as manifolds with conic ends, or with ends like cocompact covering Riemannian manifolds with polynomial growth deck transformation group or Lie groups of polynomial growth, one has \( n = N \).

By Theorem 1.2 and \([12, \text{Theorem C}]\), we already see that the (homogenous) dimension plays a key role in the Riesz transform. It is then naturally to split the case \( p > 2 \) into two categories: \( p \) less than the dimension and \( p \) bigger than the dimension. We will provide necessary and sufficient conditions for boundedness of the Riesz transform in both cases.

We first consider \( p > 2 \) that is smaller than the lower dimension \( n \), which means that \( n > 2 \) and the ends are non-parabolic; see Subsection 1.3 for the definition and \([12, 41]\) for more materials. Recall that if a manifold has two Euclidean or conic ends of dimension two, then the Riesz transform is not \( L^p \)-bounded for any \( p > 2 \) by \([12, 20]\).

For \( p > 2 \) that is bigger than the upper dimension \( N \), we will consider manifolds with general ends (including small ones). Notice that in this case boundedness of the Riesz transform will imply that the manifold can have only one end, if \( M \) is non-parabolic; see Theorem 1.11 below.

1.2 Necessary and sufficient conditions for small \( p \)

In this part, we provide a necessary and sufficient condition for \( L^p \)-boundedness of the Riesz transform for small \( p \), i.e., \( p \) less than the lower dimension. Our approach depends heavily on recent developments on the relation of regularities of harmonic functions and heat kernels from \([22, 36, 37]\), and is a nontrivial adaption of the criteria for the boundedness of the Riesz transform established in \([4]\) (see also \([2]\)) to our settings.

Definition 1.4 (Poincaré inequality). We say that a Poincaré inequality holds on ends \(((PE)\), for short) of \( M \), if there exists \( C > 0 \) such that for any ball \( B \) with \( 2B \cap M_0 = \emptyset \), and each \( f \in C^1(\bar{B}) \),

\[
(P_E^2) \quad \frac{1}{|B|} \int_B |f - f_B|^2 \, d\mu \leq C_r^2 \int_B |\nabla f|^2 \, d\mu.
\]

Our first main result provides a solution to Question 1.1. Notice that our assumptions below are much weaker than \((D)\) plus \((P_2)\), or equivalently \((LY)\) estimates.

Theorem 1.5. Assume that \((DN)\) and \((RD_n)\) hold on \( M \) with \( 2 < n \leq N < \infty \). Suppose that \((UE)\) and \((P_E^2)\) hold. Let \( p \in (2, n) \). Then the following statements are equivalent.

(i) \((R_p)\) holds.
(ii) \( (G_p) \) holds.

(iii) \( (RH_p) \) holds, where \( (RH_p) \) means that there exists \( C > 0 \) such that for any ball \( B \) with radius \( r_B \) and any harmonic function \( u \) on \( 3B \), it holds

\[
(RH_p) \quad \left( \frac{\int_B |\nabla u|^p \, d\mu}{r_B} \right)^{1/p} \leq \frac{C}{r_B} \int_{2B} |u| \, d\mu.
\]

**Remark 1.6.** (i) Under assumptions of the theorem, \( (R_p) \) holds for all \( p \in (1, 2] \) from Coulhon-Duong [20], and does not hold for any \( p \geq N \), if the manifold has at least two ends, by Carron [12, Theorem C].

(ii) The condition \( (RH_p) \) is different from the true reverse Hölder inequalities used in [3, 46]. Our formulation is natural since in case of manifolds with two Euclidean/conic ends, the true reverse Hölder inequalities fail for any \( p > 2 \), but \( (RH_p) \) holds for \( p \in (2, n) \); see [22, Section 7].

(iii) The equivalence \( (G_p) \iff (RH_p) \) was proved in [22] under a local Poincaré inequality \( (P_{2, \text{loc}}) \) instead of \( (P^E_2) \). Notice that \( (P^E_2) \) implies \( (P_{2, \text{loc}}) \); see Lemma 2.2 below.

In view of Theorem 1.2 and [12, Theorem C], the above result is rather optimal if the manifold has at least two ends. It is worth to note that our method are completely different from those from [11, 12, 13, 32], in particular, our assumptions \( (D) \), \( (UE) \) and \( (P^E_2) \) all are stable under quasi-isometries. As a consequence, our results work with the Laplace-Beltrami operator replaced by any *uniformly elliptic operator* of divergence form, and more generally, work on Dirichlet metric measure spaces; see Section 5.

The condition \( (P^E_2) \) is satisfied on an end, if the Ricci curvature has quadratic decay \( (QD) \) (see Buser [10] or Theorem 2.5), or if in the isometric sense it can be extended to a co-compact covering manifold with polynomial growth deck transformation group, Lie group of polynomial growth as well as conic manifold; see [1, 17, 23, 26, 38, 49, 50] for instance.

The condition \( (UE) \) turns out to be the most restrictive condition, some well known conditions requiring non-negative Ricci curvature, which certainly is not enough for our purpose. Fortunately, recent results by Grigor’yan and Saloff-Coste [29, 30] shed some light on this point. In particular, by [30] one sees that if each end, in the isometric sense, can be extended to a complete manifold satisfies \( (UE) \), then the manifold satisfies \( (UE) \); see the final section.

We next provide some further necessary and sufficient conditions for the boundedness of the Riesz transform.

**Definition 1.7.** Let \( p \in (2, \infty] \). We say that the reverse \( L^p \)-Hölder inequality for gradients of harmonic functions holds on ends of \( M \) (for short, \( (RH^E_p) \)), if there exists \( C > 0 \) such that for each ball \( B \) with \( 3B \cap M_0 = \emptyset \), and each \( u \) satisfying \( \Delta u = 0 \) in \( 3B \), it holds

\[
(RH^E_p) \quad \left( \frac{\int_B |\nabla u|^p \, d\mu}{r_B} \right)^{1/p} \leq \frac{C}{r_B} \int_{2B} |u| \, d\mu.
\]

The observation below is that, \( (RH_p) \) is stable under gluing operation, if \( p < n \); see Lemma 3.9 and Lemma 3.10 below.
Theorem 1.8. Assume that \((D_N)\) and \((R_{D_n})\) hold on \(M\) with \(2 < n \leq N < \infty\). Suppose that \((UE)\) and \((P^E_{2})\) hold. Let \(p \in (2, n)\). Then \((R_p)\) holds on \(M\), if and only if, \((RH^E_p)\) holds.

The advantage is that \((RH^E_p)\) is a condition much easier to verify than \((G_p)\). An immediate consequence of the above result is that compact metric perturbation does not effect \((R_p)\), if \(p < n\).

We also note that, the above result implies the stability of \((R_p)\) \((p < n)\) under gluing operations, see Theorem 1.14 and Corollary 1.15 below.

An open-ended property of the Riesz transform follows from the above result.

Corollary 1.9. Assume that \((D_N)\) and \((R_{D_n})\) hold on \(M\) with \(2 < n \leq N < \infty\). Suppose that \((UE)\) and \((P^E_{2})\) hold. Let \(p \in (2, n)\). If \((R_p)\) holds, then there exists \(\epsilon > 0\) such that \(p + \epsilon < n\) and \((R_{p+\epsilon})\) holds.

Further, \((RH^E_{\infty})\) and \((P^E_{2})\) hold if the Ricci curvature has quadratic decay.

Corollary 1.10. Assume that \((D_N)\) and \((R_{D_n})\) hold on \(M\) with \(2 < n \leq N < \infty\). If \((UE)\) holds, and there exists \(C_M > 0\) such that for each \(x \in M\),

\[
(QD) \quad \text{Ric}_M(x) \geq -\frac{C_M}{[d(x, x_M) + 1]^2},
\]

then \((R_p)\) holds for \(p \in (1, n)\).

Notice that, in [12, Theorem A], under \((QD)\) together with some additional geometric assumptions, Carron established a result similarly to Corollary 1.10. The approach in [12] used Li-Yau’s Harnack inequality (cf. [42]), which depends on the Ricci curvature, and the theory of pseudo-differential operators (cf. [13]). See also Devyver [24, Theorem 5] for a related result.

1.3 Necessary and sufficient conditions for large \(p\)

Theorem 1.5 seems to be rather optimal if the manifold has at least two ends, however, it is less satisfied if the manifold has only one end, where in general the Riesz transform may be bounded on \(L^p(M)\) for some \(p > N\); see [12, 52, 58] for instance. We next provide a necessary and sufficient condition for \(p > N\) under the same requirements as Theorem 1.5 except that we do not need the reverse doubling condition, which however holds automatically. The result in this part essentially follows from [22]. We shall denote \(\max\{A, B\}\) by \(A \vee B\).

Let us recall some notation regarding parabolic and hyperbolic manifolds; see [12] for instance. Let \(p \in (1, \infty)\). For a bounded open set \(O \subset M\), define its \(p\)-capacity by

\[
\text{Cap}_p(O) := \inf \left\{ \int_M |\nabla \psi|^p \, d\mu : \psi \in C_c^\infty(M), \psi \geq 1 \text{ on } O \right\}.
\]

We say that \(M\) is \(p\)-hyperbolic if the \(p\)-capacity of some (equivalently, any) bounded open subsets is positive. A non-\(p\)-hyperbolic manifold is called \(p\)-parabolic. A 2-hyperbolic manifold is called non-parabolic.
Theorem 1.11. Assume that \((D_N)\) holds on \(M\) with \(0 < N < \infty\), and that \((UE)\) and \((P_E^2)\) hold. Let \(p \in (N \lor 2, \infty)\). Then the following statements are equivalent.

(i) \((R_p)\) holds;

(ii) \((RH_p)\) holds;

(iii) \((G_p)\) holds.

Moreover, if \(M\) is non-parabolic, then any of the three conditions implies that \(M\) can have only one end.

Note that we did not assume \((P_2)\) above, however \((P_2)\) follows as a consequence of the proof; see Remark 4.1.

For the proof we will show that the validity \((P_E^2)\) guarantees scale-invariant Poincaré inequalities \((P_p)\) for any \(p > N \lor 2\) (see Theorem 2.3). The validity of these Poincaré inequalities allows us to use [22, Theorem 1.9], and then [12, Theorem C] to conclude the theorem.

We have the following unboundedness of the Riesz transform as an application of the above result.

Corollary 1.12. Assume that \((D_N)\) holds on \(M\) with \(0 < N < \infty\), and that \((UE)\) and \((P_E^2)\) hold. If there exists a non-constant harmonic function \(u\) on \(M\) with the growth

\[ u(x) = O(d(x, o)^p) \text{ as } d(x, o) \to \infty \]

for some \(\alpha \in [0, 1)\) and a fixed \(o \in M\), then \((R_p)\) does not hold for any \(p > N \lor 2\) satisfying \(p(1 - \alpha) \geq N\).

Using the Poincaré inequality \((P_p)\) for any \(p > N \lor 2\) established in Theorem 2.3 together with Theorem 1.11 allows us to conclude the claim via arguing by contradiction. We refer the reader to [12, 41] for more on existence and non-existence of non-constant harmonic functions of sublinear growth.

1.4 Applications and comments

As applications of our main results, in this part, we address the questions of stability of boundedness of the Riesz transform under gluing operations and make some final comments on our result.

The following question was asked in [13].

Question 1.13 (Part of Open Problem 8.2 [13]). Under which conditions is boundedness of the Riesz transform on \(L^p\) stable under the gluing operation on manifolds?

We refer the reader to [30, Section 3] and also [29] for a detailed description of the gluing operation. Here we only need to know that the gluing operation is smooth, and only changes structure and metric in a compact set. As shown by Theorem 1.11, the \(L^p\)-boundedness of the Riesz transform is not stable under the gluing operations if \(p\) is not less than the dimension \(N\) and bigger than two. Previously, Carron [11] and Devyver [25]...
had addressed this question under the requirement of lower Ricci curvature bound and Sobolev inequalities; see also [12] for a description of Devyver’s result.

Our Theorem 1.8 provides a solution to the above question in a different manner than [11, 25].

**Theorem 1.14.** Let \( k \geq 2 \). Suppose that for each \( 1 \leq i \leq k \), \( M_i \) is a complete non-compact manifold where \( (D) \), \( (UE) \) and \( (P_2^E) \) hold. Assume that the gluing manifold \( M := M_1 \# \cdots \# M_k \) satisfies \( (D_N) \) and \( (RD_n) \) for some \( 2 < n \leq N < \infty \). Then if for some \( p \in (2, n) \), \( (R_p) \) holds on each \( M_i \), \( (R_p) \) holds on \( M \).

It is worth to note that, our assumptions, \( (UE) \) and \( (P_2^E) \), are stable under gluing operations. Indeed, under gluing operations, it is straight to see that \( (P_2^E) \) is stable, on the other hand, the stability of \( (UE) \) follows by [29, 30] (see Theorem 6.1).

Since \( (P_2) \) implies \( (UE) \) and \( (P_2^E) \), we obtain the following corollary.

**Corollary 1.15.** Let \( k \geq 2 \). Suppose that for each \( 1 \leq i \leq k \), \( M_i \) is a complete non-compact manifold where \( (D) \) and \( (P_2) \) hold. Assume that the gluing manifold \( M := M_1 \# \cdots \# M_k \) satisfies \( (D_N) \) and \( (RD_n) \) for some \( 2 < n \leq N < \infty \).

(i) There exists \( \epsilon > 0 \) such that \( 2 + \epsilon < n \) and \( (R_{2+\epsilon}) \) holds.

(ii) If for some \( p \in (2, n) \), \( (R_p) \) holds on each \( M_i \), then \( (R_p) \) holds on \( M \).

In [13], some open questions regarding manifolds with conic ends or ends isometric to simply connected nilpotent Lie groups at infinity were also proposed. These two questions were solved by Guillarmou and Hassell [32] and Carron [11], respectively; see also Carron [12]. Our results also provide a new proof to the two questions; see Section 6.

Finally, let us make some comments. Notice that our main results, Theorem 1.5, Theorem 1.8 and Theorem 1.11 together with [12, Theorem C] and [20], give a more or less satisfied solution for the Riesz transform on manifolds with ends, for the two cases: (i) \( 1 < p < \max\{2, n\} \), (ii) \( p > N + 2 \). Recall that the case \( p = 2 \) is trivially true.

Note that for manifolds with ends like Euclidean ends, conic ends, or ends at infinity isometric to Lie groups of polynomial growth or cocompact covering Riemannian manifolds with polynomial growth deck transformation group, it holds that \( n = N \). It turns out that on these settings, \( (R_p) \) is stable under gluing operation for \( p < n = N \) by Theorem 1.14 and not stable for \( p \geq N \) by [12, Theorem C]. It is then somehow not restricted to assume \( (P_2) \) for \( p \geq N \) or may necessary to have \( (P_2) \), under which \( (R_p) \) for \( p > 2 \) is well understood by [4], see also [22] and Theorem 1.11.

However, for manifolds where one only has \( n < N \), the case \( p \in (2, \infty) \cap [n, N] \) is still unclear, and certainly deserves further study.

**Question 1.16.** Let \( M \) be a complete non-compact manifold, which satisfies \( (D_N) \) and \( (RD_n) \) for some \( 0 < n \leq N < \infty \) and \( N > 2 \). Suppose that \( (UE) \) and \( (P_2^E) \) hold. Then is \( (R_p) \) equivalent to \( (G_p) \) or \( (RH_p) \) for \( p \in (2, \infty) \cap [n, N] \)?

For each \( p \in (2, \infty) \), it was known from [22] that \( (G_p) \iff (RH_p) \), and it holds automatically that \( (R_p) \implies (G_p) \) (cf. [4]). So the only question left is, does \( (G_p) \) or \( (RH_p) \) imply \( (R_p) \) for \( p \in (2, \infty) \cap [n, N] \)?
1.5 Structure of the paper

The paper is organized as follows. In Section 2, we provide various versions of Poincaré inequalities for later use. In Section 3, we study the Riesz transform for $p$ less than the lower dimension, while in Section 4, we study the case $p$ bigger than the upper dimension. In Section 5, we provide some extensions of the main results to non-smooth settings. In the final section, we shall discuss the validity of $(UE)$, and provide examples that our results can be applied to, in particular, we give the proof of Theorem 1.14 and Corollary 1.15.

Throughout the work, we denote by $C, c$ positive constants which are independent of the main parameters, but which may vary from line to line. For a ball $B$, unless otherwise specified, we denote its radius and center by $r_B$ and $x_B$, respectively.

2 Poincaré inequality

In this section, we shall provide various versions of Poincaré inequalities for later use.

Definition 2.1 (Hardy-Littlewood maximal function). For any locally integrable function $f$ on $M$, its Hardy-Littlewood maximal function is defined as

$$Mf(x) := \sup_{B : x \in B} \int_B |f| \, d\mu,$$

where $B$ is any ball that contains $x$. For $p > 1$, we define the $p$-Hardy-Littlewood maximal function as

$$M_pf(x) := \sup_{B : x \in B} \left( \int_B |f|^p \, d\mu \right)^{1/p}.$$

We say that $M$ supports a local $L^2$-Poincaré inequality (for short, $(P^2, \text{loc})$), if for all $r_0 > 0$ there exists $C_p(r_0) > 0$ such that, for every ball $B$ with $r_B < r_0$ and each $f \in C^1(\bar{B})$, it holds

$$(P^2, \text{loc}) \quad \int_B |f - f_B|^2 \, d\mu \leq C_p(r_0) r_B^2 \int_B |\nabla f|^2 \, d\mu.
$$

Lemma 2.2. Assume that $(P^E_2)$ holds on $M$, then $(P^2, \text{loc})$ holds on $M$.

Proof. For any $r_0 > 0$, the Ricci curvature on the set $\{x \in M : \text{dist} (x, M_0) < 3r_0\}$ is bounded below by a constant $K(r_0)$ depending on $r_0$. Therefore, by Buser [10] (see also [33]), there exists $C_p(r_0)$ such that for every ball $B = B(x, r)$ with $r < r_0$ and $x$ with $\text{dist} (x, M_0) < 2r_0$ and each $f \in C^1(\bar{B})$, it holds

$$(P^2, \text{loc}) \quad \int_B |f - f_B|^2 \, d\mu \leq C_p(r_0) r_B^2 \int_B |\nabla f|^2 \, d\mu.
$$

On the other hand, by $(P^E_2)$, one sees that there exists $C$ such that for any ball $B(x, r)$ with center $x \not\in \{x \in M : \text{dist} (x, M_0) < 2r_0\}$ and $r < r_0$, it holds for each $f \in C^1(\bar{B})$ that

$$\int_B |f - f_B|^2 \, d\mu \leq Cr^2 \int_B |\nabla f|^2 \, d\mu,$$
For a real number $\gamma > 0$ we denote by $[\log_2 \gamma]$ the biggest integer not bigger than $\log_2 \gamma$.

**Theorem 2.3.** Assume that $(D_N)$ holds on $M$ with $0 < N < \infty$. If $(P^E_2)$ holds on $M$, then for any $p > N \vee 2$ there is a Poincaré inequality $(P_p)$, i.e., there exists $C > 0$ such that for any ball $B$ and any $f \in C^1(B)$ it holds

$$(P_p) \quad \int_B |f - f_B| \, d\mu \leq CR_B \left( \int_B |\nabla f|^p \, d\mu \right)^{1/p}.$$ 

**Proof.** Since $(M, d)$ is a geodesic space, by Hajłasz-Koskela [33] Section 9, it suffices to prove the following weaker version, i.e., for $f \in C^1(8B)$,

$$(\bar{P}_p) \quad \int_B |f - f_B| \, d\mu \leq CR_B \left( \int_{8B} |\nabla f|^p \, d\mu \right)^{1/p}.$$ 

By Lemma 2.2 a local Poincaré inequality $(P_{2,loc})$ holds. If $r_B \leq 100$, then the required estimate $(\bar{P}_p)$ follows from $(P_{2,loc})$.

Assume now $r_B > 100$. If $2B \cap M_0 = \emptyset$, then $(P_p)$ and hence $(\bar{P}_p)$ follows from $(P^E_2)$.

Suppose $2B \cap M_0 \neq \emptyset$. Let $f \in C^1(8B)$ and write

$$\int_B |f - f_B| \, d\mu \leq C \int_B \int_B |f(x) - f(y)| \, d\mu(x) \, d\mu(y).$$

**Claim:** For each $q \in (N \vee 2, \infty)$, there is a constant $C > 0$ such that for all $x, y \in B$ it holds

$$|f(x) - f(y)| \leq CR_B \left[ M_q (|\nabla f|_{8B}) (x) + M_q (|\nabla f|_{8B}) (y) \right]$$

If the claim holds, then by taking $q \in (N \vee 2, p)$, we conclude that

$$\int_B |f - f_B| \, d\mu \leq CR_B \int_B \int_B |M_q (|\nabla f|_{8B}) (x) + M_q (|\nabla f|_{8B}) (y)| \, d\mu(x) \, d\mu(y)$$

$$\leq CR_B \left( \int_B M_q (|\nabla f|_{8B}) (x)^p \, d\mu(x) \right)^{1/p}$$

$$\leq CR_B \left( \int_{8B} |\nabla f|^p \, d\mu \right)^{1/p},$$

which completes the proof of $(\bar{P}_p)$ and therefore the thesis.

Let us prove the claim. Take $x_{M_0} \in M_0 \cap 2B$. Recall that we assume $\text{diam}(M_0) = 1$. For all $x, y \in B$, there is one of them, assuming that is $x$, such that

$$|f(x) - f(y)| \leq 2|f(x) - f(x_{M_0})|.$$
If \( d(x,x_{M_0}) \leq 100 \), then by using \((P_{2,\text{loc}})\), the standard telescopic approach (cf. \[3\]) and the Hölder inequality, one has

\[
|f(x) - f(x_{M_0})| \leq C \frac{d(x,x_{M_0})}{V(x,2d(x,x_{M_0}))^{1/q}} \left( \int_{B(x,2d(x,x_{M_0}))} |\nabla f|^q \, dm \right)^{1/q} \leq C r_B M_q (|\nabla f|_{8B})(x).
\]

Suppose \( d(x,x_{M_0}) > 100 \) and let \( k_0 = \lfloor \log_2(d(x,x_{M_0}) - 8) \rfloor \). Take a geodesic connecting \( x \) to \( x_{M_0} \). Take \( x_0 \in \gamma \) such that \( d(x_0,x_{M_0}) = 8 \). For \( 0 \leq i \leq k_0 - 3 \), take \( x_{4i+1}, x_{4i+2}, x_{4i+3}, x_{4i+4} \in \gamma \) such that \( d(x_{4i+k}, x_{4i+k-1}) = 2^i \) for \( k = 1, 2, 3, 4 \). Notice that the length of \( \gamma \) between the points \( x_{M_0}, x_{4k_0-8} \) is \( 4 + 2^{k_0} \), which is less than \( d(x_0,x) \).

If \( d(x,x_{4k_0-8}) \leq 2^{k_0-2} \) then we stop the progress, otherwise we select \( x_{4k_0-7} \) such that \( d(x_{4k_0-7}, x_{4k_0-8}) = 2^{k_0-2} \). Since \( d(x,x_0) = d(x,x_{M_0}) \in [2^{k_0} + 2, 2^{k_0+1} + 2] \), adding up to four such points we can find the distance from \( x \) to the last selected points is not greater than \( 2^{k_0-2} \). The last selected point may have the lower index \( 4k_0 - k \), \( k \in \{4, \cdots, 8\} \), but nothing else. We denote such point by \( x_{4k_0-\ell} \), where \( \ell \) belongs to \( \{4, \cdots, 8\} \).

For each \( 1 \leq j \leq 4k_0 - k_1 \), set \( B_j := B(x_j, 2^j) \) where \( i \) is such that \( 4i + 1 \leq j \leq 4i + 4 \). Set also \( B_{x_{M_0}} = B(x_{M_0}, 16), B_0 = B(x_0, 1) \) and \( B_x = B(x, 2^{k_0-2}) \).

By the above construction, we see that for each \( 1 \leq j \leq 4k_0 - k_1 \), there is a ball \( B'_j \) such that \( B_{j-1} \cup B_j \subset B'_j, r_{B_{j+1}} \leq r_{B'_j} \leq 2r_{B'_j}, \) and \( 2B'_j \cap M_0 = \emptyset \). This can be seen as follows. For \( 0 \leq i \leq k_0 - 2 \), if \( j = 4i + 2, 4i + 4, 4i + 4 \), then by letting \( B'_j = B(x_j, 2^{i+1}) \) we see that \( B_{j-1} \cup B_j \subset B'_j \).

Moreover, as

\[
\text{dist} (x_j, M_0) \geq 8 - 1 + 2 \sum_{k=1}^{i-1} 2^k + 2^{i+1} \geq 8 - 1 + 2^{i+1} - 4 + 2^{i+1} > 2^{i+2},
\]

we have \( 2B'_j \cap M_0 = \emptyset \). For \( j = 4i + 1, 1 \leq i \leq k_0 - 2 \), then by letting \( B'_j := B(x_j, 3 \times 2^{i-1}) \) we see that \( B_{j-1} \cup B_j \subset B'_j \) and \( 2B'_j \cap M_0 = \emptyset \) since

\[
\text{dist} (x_j, M_0) \geq 8 - 1 + 2 \sum_{k=1}^{i-1} 2^k + 2^i \geq 8 - 1 + 2^{i+1} - 4 + 2^i > 3 \times 2^i.
\]

For \( j = 4k_0 - k_1 \), since \( d(x_{4k_0-k_1}, x) \leq 2^{k_0-2} \), it holds \( B_j \cup B_x \subset 2B_x \), and \( 2B_x \cap M_0 = \emptyset \).

Using the above construction of chain of balls we find

\[
|f(x) - f(x_{M_0})| \leq |f(x_{M_0}) - f_{B_{x_{M_0}}} - f_{B_0}| + \left| f_{B_{x_{M_0}}} - f_{B_0} \right| + |f_{B_0} - f_B| + \\
+ \left\{ \sum_{j=2}^{4k_0-k_1} \left| f_{B_j} - f_{B_{j-1}} \right| + \left| f_{B_{4k_0-k_1}} - f_{B_0} \right| + |f(x) - f_{B_0}| \right\}.
\]

Notice that \( B_0 \subset B_{x_{M_0}} \) and \( B_0 \cup B_1 \subset 2B_0 \). Applying \((P_{2,\text{loc}}), (D_N)\) and a telescopic approach, we find

\[
|f(x_{M_0}) - f_{B_{x_{M_0}}} - f_{B_0}| + \left| f_{B_{x_{M_0}}} - f_{B_0} \right| + |f_{B_0} - f_B|.
\]
which confirms the Claim and completes the proof.

Applying \((P^E)\) and \((D_N)\), using the fact \(q > N\), we conclude that

\[
\sum_{j=2}^{4k_0-k_1} |f_{B_{j-1}} - f_{B_{j-1}}| \leq 2 \sum_{j=1}^{4k_0-k_1} |f_{B_j} - f_{B'_j}| \leq C \sum_{j=1}^{4k_0-k_1} \frac{r_{B'_j}}{\mu(B'_j)^{1/q}} \left( \int_{B'_{j}} |\nabla f|^q \, d\mu \right)^{1/q}
\]

\[
\leq C \sum_{j=1}^{4k_0-k_1} 2^{j/4} \frac{d(x,x_{M_0})^{N/q}}{V(x_{M_0}, d(x, x_{M_0}))^{1/q}} \left( \int_{B(x_{M_0}, 4d(x, x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q}
\]

\[
\leq C \sum_{j=1}^{4k_0-k_1} 2^{j(1-N)/4} \frac{d(x,x_{M_0})^{N/q}}{V(x_{M_0}, d(x, x_{M_0}))^{1/q}} \left( \int_{B(x_{M_0}, 4d(x, x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q}
\]

\[
\leq C 2^{k_0(1-N)/4} \frac{d(x,x_{M_0})^{N/q}}{V(x_{M_0}, d(x, x_{M_0}))^{1/q}} \left( \int_{B(x_{M_0}, 4d(x, x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q}
\]

\[
\leq \frac{Cd(x,x_{M_0})}{V(x_{M_0}, d(x, x_{M_0}))^{1/q}} \left( \int_{B(x_{M_0}, 4d(x, x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q}.
\]

Finally, applying \((P^E)\), \((D_N)\) and a telescopic approach once more, we find

\[
|f_{B_{4k_0-k_1}} - f_{B_{1}}| + |f(x) - f_{B_1}| \leq C r_{B_1} \left( \int_{2B_1} |\nabla f|^2 \, d\mu \right)^{1/2} \leq \frac{Cd(x,x_{M_0})}{V(x_{M_0}, d(x, x_{M_0}))^{1/q}} \left( \int_{B(x_{M_0}, 4d(x, x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q}.
\]

Combining the above three estimates and noticing that \(B(x_{M_0}, 4d(x, x_{M_0})) \subset 8B\), we conclude that

\[
|f(x) - f(x_{M_0})| \leq Cd(x,x_{M_0}) \left( \int_{B(x_{M_0}, 4d(x,x_{M_0}))} |\nabla f|^q \, d\mu \right)^{1/q} \leq Cr_B M_q(|\nabla f|\chi_{8B})(x).
\]

This together with the choice of \(x\) gives

\[
|f(x) - f(y)| \leq 2|f(x) - f(x_{M_0})| \leq Cr_B M_q(|\nabla f|\chi_{8B})(x) \leq Cr_B \left[ M_q(|\nabla f|\chi_{8B})(x) + M_q(|\nabla f|\chi_{8B})(y) \right],
\]

which confirms the Claim and completes the proof. \(\Box\)
Proposition 2.4. Assume that $(D_N)$ holds on $M$ with $0 < N < \infty$ and that $(P_{2,\text{loc}})$ holds. Then there exist $N_\mu > 0$ and $C > 0$ such that for any ball $B = B(x_0, r)$, $r > 1$, and any $f \in C^1(\overline{B})$, it holds

$$(P_G) \quad \int_B |f - f_B|^2 \, d\mu \leq C r^{2N_\mu + 1 + N} \int_{2B} |\nabla f|^2 \, d\mu.$$ 

Proof. Based on the validity of $(P_{2,\text{loc}})$, we only need to show $(P_G)$ for balls $B = B(x_0, r)$ when $r$ is sufficiently large. Let us assume $r > 100$. Set $B_0 = B(x_0, 1)$. By $(D_N)$, we can find a sequence of balls, $\{B_i\}_{i \leq C(r)}$, where $C(r)$ is an integer not bigger than $Cr^{N_\mu}$, $N_\mu > 0$, such that each ball $B_i$ is of radius one, $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$ for any $i \neq j$, $0 \leq i, j \leq C(r)$, and $B \subset \bigcup_{i=0}^{C(r)} B_i \subset 2B$; see [35] for instance.

Write

$$\left( \int_B |f - f_B|^2 \, d\mu \right)^{1/2} \leq 2 \left( \int_B |f - f_{B_0}|^2 \, d\mu \right)^{1/2} \leq \frac{2}{\mu(B)^{1/2}} \sum_{i=0}^{C(r)} \left( \int_{B_i} |f - f_{B_i}|^2 \, d\mu \right)^{1/2}.$$ 

If $i = 0$, then $(P_{2,\text{loc}})$ implies

$$\int_{B_0} |f - f_{B_0}|^2 \, d\mu \leq C \int_{B_0} |\nabla f|^2 \, d\mu \leq C \int_{B_0} |\nabla f|^2 \, d\mu.$$ 

For other $i$'s, let $x_i$ be the center of the ball $B_i$, and there exists a geodesic that links $x_i$ to $x_0$ with length equaling $d(x_0, x_i)$. Along the geodesic, we may find a sequence of balls $\{B_{i,j}\}_{1 \leq j \leq C(i)}$ with $C(i)$ be an integer not bigger than $2d(x_0, d_i)$, such that each ball $B_{i,j}$ is of radius one, and $B_{i,1} \cap B_0 \neq \emptyset$, $B_{i,C(i)} \cap B_i \neq \emptyset$ and $B_{i,j} \cap B_{i,j-1} \neq \emptyset$ if $2 \leq j \leq C(i)$. A chain argument implies

$$\left( \int_{B_i} |f - f_{B_i}|^2 \, d\mu \right)^{1/2} \leq \left( \int_{B_i} |f - f_{B_{i,1}}|^2 \, d\mu \right)^{1/2} + |f_{B_i} - f_{B_{i,C(i)}}| + |f_{B_{i,0}} - f_{B_{i,1}}| + \sum_{j=2}^{C(i)} |f_{B_{i,j}} - f_{B_{i,j-1}}|.$$ 

Notice that, as $B_0 \cap B_{i,1} \neq \emptyset$, $B_{i,1} \subset 3B_0$, and therefore,

$$|f_{B_0} - f_{B_{i,1}}| \leq |f_{3B_0} - f_{B_0}| + |f_{3B_0} - f_{B_{i,1}}| \leq C \int_{3B_0} |f - f_{3B_0}| \, d\mu \leq C \left( \int_{3B_0} |\nabla f|^2 \, d\mu \right)^{1/2}.$$ 

Similarly, we conclude that, for each $1 \leq i \leq C(r)$, it holds

$$\left( \int_{B_i} |f - f_{B_i}|^2 \, d\mu \right)^{1/2} \leq C \left( \int_{3B_0} |\nabla f|^2 \, d\mu \right)^{1/2} + C \sum_{j=1}^{C(i)-1} \left( \int_{3B_{i,j}} |\nabla f|^2 \, d\mu \right)^{1/2} \leq C r^{N/2} \left[ \left( \int_{3B_0} |\nabla f|^2 \, d\mu \right)^{1/2} + \left( \int_{3B_{i,C(i)}} |\nabla f|^2 \, d\mu \right)^{1/2} + \sum_{j=1}^{C(i)-1} \left( \int_{3B_{i,j}} |\nabla f|^2 \, d\mu \right)^{1/2} \right].$$
Riesz transform on manifolds with ends

\[ \leq C \frac{r^{N/2}}{\mu(2B)^{1/2}} \left( \int_{2B} |\nabla f|^2 \, d\mu \right)^{1/2} \leq C \frac{r^{N/2+1/2}}{\mu(2B)^{1/2}} \left( \int_{2B} |\nabla f|^2 \, d\mu \right)^{1/2} \]

Summarizing these estimates, we conclude that

\[ \left( \int_B |f - f_B|^2 \, d\mu \right)^{1/2} \leq \frac{2}{\mu(B)^{1/2}} \sum_{i=0}^{C(r)} \left( \int_{B_i} |f - f_{B_i}|^2 \, d\mu \right)^{1/2} \]

\[ \leq \frac{C}{\mu(B)^{1/2}} \left( \int_{3B_0} |\nabla f|^2 \, d\mu \right)^{1/2} + \frac{C(r)^{1+N/2} \mu(B)^{1/2}}{\mu(2B)^{1/2}} \left( \int_{2B} |\nabla f|^2 \, d\mu \right)^{1/2} \]

\[ \leq \frac{CC(r)^{1+N/2}}{\mu(B)^{1/2}} \left( \int_{2B} |\nabla f|^2 \, d\mu \right)^{1/2} \]

\[ \leq Cr^{N/2+(1+N)/2} \left( \int_{2B} |\nabla f|^2 \, d\mu \right)^{1/2}, \]

which gives the desired estimate.  □

**Theorem 2.5.** If there exists \( C_M > 0 \) such that for each \( x \in M \), it holds

\[ \text{Ric}_M(x) \geq - \frac{C_M}{[d(x, x_M) + 1]^2}, \]

then \((P^E_2)\) holds on \( M \).

**Proof.** By Buser’s inequality (cf. [10, 33]), there exists a constant \( C > 0 \) depending only on the dimension such that, for any \( f \in C^1(\bar{B}) \),

\[ \int_B |f - f_B| \, d\mu \leq C e^{\sqrt{K} r_B} \int_B |\nabla f| \, d\mu; \]

where \( K \geq 0 \) and the Ricci curvature on \( B \) is not less than \(-K\).

For any \( B \subset M \) with \( 2B \cap M_0 = \emptyset \), we then have

\[ \text{Ric}_M(x) \geq - \frac{C_M}{[r_B + 1]^2}, \forall \ x \in B. \]

This together with Buser’s inequality implies

\[ \int_B |f - f_B| \, d\mu \leq C e^{\sqrt{\text{Ric}_M} r_B} \int_B |\nabla f| \, d\mu \leq Cr_B \int_B |\nabla f| \, d\mu, \]

which together with [33] Theorem 5.1 further implies

\[ \int_B |f - f_B|^2 \, d\mu \leq Cr_B^2 \int_B |\nabla f|^2 \, d\mu, \]

as desired.  □
3 Riesz transform for $p$ below the lower dimension

3.1 Riesz transform via heat kernel regularity

In this section, we study the behavior of the Riesz transform on $L^p(M)$, where $p \in (2, n)$. In what follows, let $A_r := I - (I - e^{-r^2\mathcal{L}})^m$, where $m \in \mathbb{N}$ is chosen such that $m > N/4$; see [4, p. 932]. Let $T := \nabla \mathcal{L}^{-1/2}$. The sharp maximal function $\mathcal{M}_{T,A}^\# f$ for every locally integrable function $f$ is given as

$$
\mathcal{M}_{T,A}^\# f(x) := \sup_{B : x \in B} \left( \int_B |T(1 - A_r) f|^2 \, d\mu \right)^{1/2}.
$$

Lemma 3.1. Assume that $(D_N)$ holds on $M$ with $0 < N < \infty$. There exists $C > 0$ such that for any $f \in L^2(M)$ and any ball $B$ and $x \in B$, it holds

$$
\left( \int_B |T(I - A_r) f|^2 \, d\mu \right)^{1/2} \leq \mathcal{M}_{T,A}^\# f(x) \leq CM_2(|f|(x)).
$$

Proof. See [4, Lemma 3.1].

Lemma 3.2. Assume that $(D_N)$ holds on $M$ with $0 < N < \infty$, and that $(U, E)$ and $(G_{p_0})$ for some $p_0 \in (2, \infty)$ hold. Then for every $p \in (2, p_0)$, there exist $C, \tau > 0$ such that for every ball $B$ with radius $r_B$ and every $f \in L^2(M)$ supported in $U_i = 2^{i+1}B \setminus 2B$, $i \geq 2$, or $U_1 = 4B$, one has

$$
\left( \int_B |\nabla A_{r_B} f|^p \, d\mu \right)^{1/p} \leq \frac{C e^{-\tau r_B}}{r_B} \left( \frac{1}{\mu(2B)} \int_{U_i} |f|^2 \, d\mu \right)^{1/2}.
$$

Proof. The lemma was proved in [4, Lemma 3.2]. Notice that, although in the statement of [4, Lemma 3.2], $(P_2)$ was assumed, its proof indeed only needs $(D_N)$, $(U, E)$ and $(G_{p_0})$. 

Recall that $x_M \in M_0$ is fixed, and we assume that $\text{diam}(M_0) = 1$. The reverse doubling condition only requires that for all $1 < r < R < \infty$ it holds

$$
(RD_n) \quad \left( \frac{R^n}{r^n} \right) \leq \frac{\text{V}(x_M, R)}{\text{V}(x_M, r)}.
$$

Lemma 3.3. Assume that $(D_N)$ and $(RD_n)$ hold on $M$ with $1 < n \leq N < \infty$, and that $(U, E)$ holds. Let $C_0 > 0$ be fixed. Then for any $p \in (1, n)$, there exists $C > 0$, depending only on $C_0, n, N$, such that for any ball $B$, with $r_B > 1$ and $C_0 B \cap M_0 \neq \emptyset$, and any $f \in L^p(M)$, it holds

$$
\left( \int_B |\mathcal{L}^{-1/2} f|^p \, d\mu \right)^{1/p} \leq \frac{C r_B}{\mu(B)^{1/p}} \|f\|_p.
$$

Proof. For each $x \in B$, write

$$
|\mathcal{L}^{-1/2} f(x)| \leq \frac{\sqrt{\pi}}{2} \int_0^{(2C_0 r_B)^2} \left| e^{-s\mathcal{L}} f(x) \right| \frac{ds}{\sqrt{s}} + \frac{\sqrt{\pi}}{2} \int_{(2C_0 r_B)^2}^{\infty} \left| e^{-s\mathcal{L}} f(x) \right| \frac{ds}{\sqrt{s}} =: I_1 + I_2.
$$
For the term $I_1$, one has via the Minkowski inequality that
\[
\left( \int_B |I_1|^p \, d\mu \right)^{1/p} \leq C \int_0^{(2C_0r_B)^2} \|e^{-sL}f\|_p \frac{ds}{\sqrt{s}} \leq C \int_0^{(2C_0r_B)^2} \|f\|_p \frac{ds}{\sqrt{s}} \leq Cr_B\|f\|_p.
\]

For the term $I_2$, notice that by $(D_N)$, the assumptions $C_0B \cap M_0 \neq \emptyset$ and diam $(M_0) = 1$, it holds
\[
V(x_B, \sqrt{t}) \sim V(x_M, \sqrt{t}) \sim V(x, \sqrt{t})
\]
for any $x \in B$ and $t \geq (2C_0r_B)^2$. From this together with $(UE)$, $(RD_n)$ and the Hölder inequality, we deduce that for each $x \in B$
\[
|I_2| \leq C \int_0^{(2C_0r_B)^2} \int_M \frac{C}{V(x, \sqrt{s})} \exp \left\{ -\frac{d^2(x, y)}{cs} \right\} |f(y)| \, d\mu(y) \frac{ds}{\sqrt{s}}
\]
\[
\leq C \int_0^{(2C_0r_B)^2} \|f\|_p \left( \int_M \frac{1}{V(x, \sqrt{s})^{p/(p-1)}} \exp \left\{ -\frac{pd^2(x, y)}{c(p-1)s} \right\} \, d\mu(y) \right)^{(p-1)/p} \frac{ds}{\sqrt{s}}
\]
\[
\leq C\|f\|_p \int_0^{(2C_0r_B)^2} \frac{C r_B^{n/p}}{V(x_M, 2C_0r_B)^{1/p} \sqrt{s}} \frac{ds}{\sqrt{s}} \leq \frac{C r_B}{\mu(B)^{1/p}} \|f\|_p,
\]
where in the last inequality we used the fact that $p < n$. Combining the estimates of $I_1$ and $I_2$, we conclude that
\[
\left( \int_B |L^{-1/2} f|^p \, d\mu \right)^{1/p} \leq \left( \int_B |I_1 + I_2|^p \, d\mu \right)^{1/p} \leq \frac{C r_B}{\mu(B)^{1/p}} \|f\|_p,
\]
as desired. \(\square\)

Using the the previous mapping property of the Riesz potentials together with the Lemma\ref{lemma} we deduce the following estimates.

**Proposition 3.4.** Assume that $(D_N)$ and $(RD_n)$ hold on $M$ with $2 < n \leq N < \infty$, and that $(UE)$ and $(P^E_2)$ hold. Suppose that $(G_{p_0})$ for some $p_0 \in (2, n)$ holds. Let $p \in (2, p_0)$ and $q \in (2, n)$. Let $10 < C_0, \alpha, \beta < \infty$. Then for each $B = B(x_B, r_B) \subset M$ and each $f \in C_c^\infty(M)$, the followings hold.

(i) If $r_B < \alpha$, then there exists $C_1 = C_1(n, N, p, p_0, \alpha)$ such that it holds
\[
\left( \int_B |TA_{r_B}f|^p \, d\mu \right)^{1/p} \leq C_1 \inf_{y \in B} M_2(|Tf|)(y).
\]

(ii) If $r_B \geq \beta$ and $d(x_B, x_M) < C_0r_B$, then there exists $C_2 = C_2(n, N, p, q, p_0, C_0, \beta)$ such that it holds
\[
\left( \int_B |TA_{r_B}f|^p \, d\mu \right)^{1/p} \leq \frac{C_2}{\mu(B)^{1/q}} \|f\|_q.
\]
(iii) If \( r_B \geq \beta \) and \( d(x_B, x_M) \geq C_0 r_B \), then there exist \( C_3, C_4 \), depending on \( n, N, p, q, p_0, C_0, \beta \), such that it holds

\[
\left( \frac{1}{ \int_B |TA_{r_B} f|^p d\mu} \right)^{1/p} \leq \frac{C_3 \|f\|_q}{V(x_M, d(x_B, x_M) + 1)^{1/q}} + C_4 \inf_{y \in B} M_2((Tf)(y)).
\]

Proof. Let \( g := L^{-1/2} f \) for \( f \in C^\infty_c(M) \). Let \( U_i = 2^{i+1} B \setminus 2^i B, i \geq 2, \) and \( U_1 = 4B \).

(i) The case \( r_B < \alpha \) follows from a proof similar to [41, p. 935], we provide a proof for completeness. By Lemma 3.2 one has

\[
\left( \frac{1}{ \int_B |TA_{r_B} f|^p d\mu} \right)^{1/p} \leq \sum_{i \geq 1} \left( \int_B |\nabla A_{r_B}[(g - g_{4B})\chi_{U_i}]|^p d\mu \right)^{1/p}
\]

By Lemma 2.2 and then using the Poincaré inequality \((P_G)\) from Proposition 2.4 one finds

\[
\left( \frac{1}{ \int_B |TA_{r_B} f|^p d\mu} \right)^{1/p} \leq \sum_{i \geq 1} \left( \int_{U_i} |g - g_{4B}|^2 d\mu \right)^{1/2},
\]

(ii) Suppose now \( d(x_B, x_M) < C_0 r_B \) and \( r_B \geq \beta \). By Lemma 3.2 Lemma 3.3 together with the Hölder inequality, one has

\[
\left( \frac{1}{ \int_B |TA_{r_B} f|^p d\mu} \right)^{1/p} \leq \sum_{i \geq 1} \left( \int_{U_i} |\nabla A_{r_B}[(g - g_{4B})\chi_{U_i}]|^p d\mu \right)^{1/p} \leq \sum_{i \geq 1} \frac{C e^{-r_B^4 i}}{r_B} \left( \int_{U_i} |g - g_{4B}|^2 d\mu \right)^{1/2},
\]

(iii) If \( d(x_B, x_M) \geq C_0 r_B \) and \( r_B \geq \beta \), then the ball \( B \) belongs to one end. Let \( k \in \mathbb{N} \) such that \( 2^{k+1} r_B \leq d(x_B, x_M) < 2^{k+2} r_B \) (recall that \( C_0 > 10 \)).
By Lemma 3.2 again one has
\[
\left( \int_B |TA_{r_B} f|^p \, d\mu \right)^{1/p} \leq \sum_{i=1}^{k} \left( \int_B |\nabla A_{r_B}(g-g_{4B})\chi_{U_i}|^p \, d\mu \right)^{1/p} \\
\leq \sum_{i=1}^{k} C e^{-r_B^{-q}} \left( \frac{1}{\mu(2^i B)} \int_{U_i} |g-g_{4B}|^2 \, d\mu \right)^{1/2} + \sum_{i>k} \cdot \\
=: I_1 + I_2.
\]

Since \( d(x_B, x_M) < 2^{k+2} r_B \), for each \( i > k \), \( 2^{i+1} B \cap M_0 \neq \emptyset \). By Lemma 3.3, \( q < n, (D_N) \) and \((RD_n)\), we obtain
\[
I_2 \leq \sum_{i=k}^{\infty} C e^{-r_B^{-q}} \left[ \frac{1}{\mu(2^i B)} \int_{U_i} |g-g_{4B}|^2 \, d\mu \right]^{1/2} \\
\leq \sum_{i=k}^{\infty} C e^{-r_B^{-q}} \left[ \frac{1}{\mu(2^i B)} \int_{2^{i+1} B} |g|^q \, d\mu \right]^{1/2} + \frac{2^i}{\mu(2^i B)} \int_{U_i} |g|^q \, d\mu \\
\leq \sum_{i=k}^{\infty} C \|f\|_q e^{-r_B^{-q}} \left[ \frac{2^{k+qN/2}}{\mu(2^i B)^{1/2}} + \frac{2^i}{\mu(2^i B)^{1/2}} \right] \\
\leq \sum_{i=k}^{\infty} C \|f\|_q e^{-r_B^{-q}} \left[ \frac{2^{k+qN/2}}{\mu(2^i B)^{1/2}} + \frac{2^{k+i(1-n/2)}}{\mu(2^i B)^{1/2}} \right] \\
\leq \frac{C e^{-r_B^{-q}} 2^{k+qN/2}}{\mu(2^i B)^{1/2}} \int_{U_i} |g|^q \, d\mu \\
\leq \frac{C_3}{\mu(2^i B)^{1/2}} \int_{U_i} |g|^q \, d\mu.
\]

Using \((P^E_2)\), we can estimate the term \( I_1 \) as
\[
I_1 \leq \sum_{i=1}^{k} \frac{C e^{-r_B^{-q}}}{\mu(2^i B)} \left( \int_{U_i} |g-g_{4B}|^2 \, d\mu \right)^{1/2} \\
\leq \sum_{i=1}^{k} \frac{C e^{-r_B^{-q}}}{\mu(2^i B)} \left( \int_{2^{i+1} B} |g-g_{2^{i+1} B}|^2 \, d\mu \right)^{1/2} + \sum_{i=2}^k \frac{C e^{-r_B^{-q}}}{\mu(2^i B)} \left( \int_{2^i B} |\nabla g|^2 \, d\mu \right)^{1/2} \\
\leq \sum_{i=1}^{k} \frac{C e^{-r_B^{-q}}}{\mu(2^i B)} \sum_{j=2}^{i+1} (2^j r_B) \left( \int_{2^j B} |\nabla g|^2 \, d\mu \right)^{1/2}.
\]
\[ \leq C_4 \inf_{y \in B} M_2(|Tf|(y)). \]

Using the estimates of \( I_1 \) and \( I_2 \), one can finally conclude that
\[ \left( \int_B |TA_{\gamma}f|^p \, d\mu \right)^{1/p} = \frac{C_3}{V(x_M, d(x_B, x_M) + 1)^{1/q}} \|f\|_q + C_4 \inf_{x \in B} M_2(|Tf|(x)), \]
as desired. \( \square \)

Using Proposition 3.4 and adapting the argument from [4], we are able to provide a modified good-\( \lambda \) inequality. The key ingredient is that for large balls, Proposition 3.4 allows us to deduce an small error term in the good-\( \lambda \) inequality; see Proposition 3.6 below.

**Proposition 3.5.** Assume that \((D_N)\) and \((R D_n)\) hold on \( M \) with \( 2 < n \leq N < \infty \), and that \((U E)\) and \((P^k_E)\) hold. Assume that \((G_{p_0})\) holds for some \( p_0 \in (2, n) \). Let \( \alpha > 1 \) and \( 2 < q < p_0 \). There exists \( K_0, C > 0 \) only depending on \( n, N, \alpha, q, p_0 \), such that for each \( f \in C^\infty_c(M) \), every \( \lambda > 0 \), \( K > K_0 \) and \( \gamma > 0 \), and every ball \( B_0 = B(x_B, r_B) \), \( r_B < \alpha \), if there exists \( x_0 \in B_0 \) such that \( M_2(|Tf|(x_0)) \leq \lambda \), then it holds
\[ (3.6) \quad \mu \left( \left\{ x \in B_0 : M_2([Tf](x)) > K\lambda, M^\#_{T,A}f(x) \leq \gamma \lambda \right\} \right) \leq C(\gamma^2 + K^{-q})\mu(B_0). \]

**Proof.** By using (i) of Proposition 3.4 the conclusion follows from [4] Lemma 2.2. One can also deduce the conclusion via going through the proof of following Proposition 3.6 by using Proposition 3.4 (ii) and (iii). \( \square \)

Recall again that we fix \( x_M \in M_0 \) and assume \( \text{diam}(M_0) = 1 \).

**Proposition 3.6.** Assume that \((D_N)\) and \((R D_n)\) hold on \( M \) with \( 2 < n \leq N < \infty \), and that \((U E)\) and \((P^k_E)\) hold. Suppose that \((G_{p_0})\) holds for some \( p_0 \in (2, n) \). Let \( \beta > 1 \) and \( 2 < p < q < p_0 \). There exists \( K_0 > 1 \) and \( C, \lambda > 0 \) only depending on \( n, N, p, q, p_0, \beta \), such that for every \( f \in C^\infty_c(M) \), every \( \lambda > 0 \), \( K > K_0 \) and \( \gamma > 0 \), and every ball \( B_0 = B(x_B, r_B) \), \( r_B > \beta \), if there exists \( x_0 \in B_0 \) such that \( M_2(|Tf|(x_0)) \leq \lambda \), then it holds
\[ (3.7) \quad \mu \left( \left\{ x \in B_0 : M_2(|Tf|(x)) > K\lambda, M^\#_{T,A}f(x) \leq \gamma \lambda, \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} \leq C_E \right\} \right) \leq C(\gamma^2 + K^{-q})\mu(B_0). \]

**Proof.** Let \( J, K > 1 \) and \( \gamma, C_E > 0 \) to be fixed later. For \( \lambda > 0 \) let
\[ E := \left\{ x \in B_0 : M_2(|Tf|(x)) > K\lambda, M^\#_{T,A}f(x) \leq \gamma \lambda, \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} \leq C_E \right\} \]
and
\[ \Omega := \left\{ x \in B_0 : M_2(|TA_{\gamma}f I_{\chi_{B_0}}(x)) > J\lambda, \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} \leq C_E \right\}. \]
**Claim 1.** There exists $C > 0$ such that

\begin{equation}
\mu(\Omega) \leq C J^{-q} \mu(B_0). \tag{3.8}
\end{equation}

Let us prove the claim. First assume $d(x_B, x_M) < 10r_B$. Notice that, if $\|f\|_p \geq \mu(B_0)^{1/p} \lambda$, then it follows from the doubling property $(D_N)$ that for each $x \in B$

$$V(x_M, d(x, x_M) + 1) \leq V(x_M, d(x, x_M) + r_B) \leq V(x_B, 3r_B + 2d(x_B, x_M)) \leq C \left( \frac{2d(x_B, x_M) + 3r_B}{r_B} \right)^N V(x_B, r_B) \leq C \mu(B_0),$$

and hence, there exists $c_1 > 0$ such that for each $x \in B_0$ it holds

$$\lambda \leq \|f\|_p \mu(B_0)^{-1/p} \leq c_1 \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}}.$$

By choosing $C_E < 1/c_1$ we see that

\begin{equation}
\mu(\Omega) = 0 \tag{3.9}
\end{equation}

Suppose now $\|f\|_p < \mu(B_0)^{1/p} \lambda$. By using (ii) of Proposition 3.4 and $(D_N)$ one has

$$\left( \int_{3B_0} |TA_{3r_B}f|^q \, d\mu \right)^{1/q} \leq C_2 \frac{\|f\|_p}{\mu(3B_0)^{1/p}} < C \lambda,$$

which together with the $(q, q)$ boundedness of $M_2$ implies that

\begin{equation}
\mu(\Omega) \leq \frac{1}{(J\lambda)^q} \int_{B_0} M_2(|TA_{3r_B}f|_{3B_0})^q \, d\mu \leq C \int_{3B_0} |TA_{3r_B}f|^q \, d\mu \leq C \frac{\|f\|_p}{J \lambda} \mu(B_0). \tag{3.10}
\end{equation}

If $d(x_B, x_M) \geq 10r_B$, then by using (iii) of Proposition 3.4 one has via the fact that $M_2(|Tf|)(x_0) \leq \lambda$ that

\begin{equation}
\left( \int_{3B_0} |TA_{3r_B}f|^q \, d\mu \right)^{1/q} \leq C_3 \frac{\|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}} + C_4 \inf_{y \in B_0} M_2(|Tf|)(y) \tag{3.11}
\end{equation}

$$\leq C_4 \lambda + \frac{C_3 \|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}}.$$

If $\|f\|_p \geq V(x_M, d(x_B, x_M) + 1)^{1/p} \lambda$, then by the fact that $d(x_B, x_M) \geq 10r_B$, we see that for each $x \in B_0$ it holds

$$\lambda \leq \frac{\|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}} \leq c_2 \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}}.$$

By choosing $C_E < 1/c_2$ we find

\begin{equation}
\mu(\Omega) = 0. \tag{3.12}
\end{equation}
Suppose now \( ||f||_p < V(x_M, d(x_B, x_M) + 1)^{1/p} \). Then by the \((q, q)\) boundedness of \( M_2 \) and (3.11) we obtain

\[
\mu(\Omega) \leq \frac{1}{(J \lambda)^q} \int_{3B_0} |T A 3_{3B_0}|^q d\mu \leq \frac{C}{(J \lambda)^q} \int_{3B_0} |T A 3_{3B_0}|^q d\mu \\
\leq \frac{C}{(J \lambda)^q} \left( \lambda + \frac{||f||_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}} \right)^q \mu(3B_0) \\
\leq \frac{C}{J^q \mu(B_0)}.
\]

(3.13)

By choosing \( 0 < C_E < \min\{1/c_1, 1/c_2\} \), the above estimates (3.9)–(3.13) confirm Claim 1.

Let us estimate the set \( E \setminus \Omega \). Notice that there exists \( c_0 > 0 \) depends only on the measure such that if \( c_0 K^2 > 1 \) then

\[ M_2(|T f|_{3B_0})(x) > K \lambda \]

if \( x \in E \). Indeed, since \( M_2(|T f|)(x) > K \lambda \) for \( x \in E \) and \( M_2(|T f|)(x_0) \leq \lambda \), there exists a ball \( B = B(z, r) \) such that \( x \in B, x_0 \notin B \) and

\[ \int_B |T f|^2 d\mu > K^2 \lambda^2 \mu(B), \]

and hence

\[ \int_{B(x, 2r)} |T f|^2 d\mu \geq \int_B |T f|^2 d\mu > K^2 \lambda^2 \mu(B) \geq c_0 K^2 \lambda^2 V(x, 2r) > \lambda^2 V(x, 2r). \]

This implies that \( r < r_B, B \subset 3B_0 \), and hence \( M_2(|T f|_{3B_0})(x) > K \lambda \).

Therefore, there exists \( K_0 > 0 \) large enough, such that for any \( K > K_0 \) and \( x \in E \) it holds

\[ M_2(|T f|_{3B_0})(x) > K \lambda. \]

By letting \( K = J + 1 > K_0 \) we find

\[
\mu(E \setminus \Omega) \\
\leq \mu \left( \left\{ x \in B_0 : M_2(|T A 3_{3B_0}|)(x) \leq J \lambda, M_2(|T f|_{3B_0})(x) > K \lambda, M_{n,A}^f(x) \leq \gamma \lambda \right\} \right) \\
\leq \mu \left( \left\{ x \in B_0 : M_2(|T(I - A 3_{3B_0})|)(x) > (K - J) \lambda, M_{n,A}^f(x) \leq \gamma \lambda \right\} \right) \\
\leq \frac{C}{(K - J)^2 \lambda^2} \int_{3B_0} |T(I - A 3_{3B_0}|^2 d\mu \leq \frac{C \mu(B_0)}{\lambda^2} \inf_{x \in B_0} \left( M_{n,A}^f(x) \right)^2 \\
\leq C \gamma^2 \mu(B_0).
\]

This together with the estimate (3.8) with \( J = K - 1 \) for \( \Omega \) gives the desired result.

We next show that \((G_{p_0})\) implies \((R_p)\) for \( p < p_0 \).

**Theorem 3.7.** Assume that \((D_N)\) and \((R_{D_N})\) hold on \( M \) with \( 2 < n \leq N < \infty \), and that \((UE)\) and \((P_{\lambda}^E)\) hold. Suppose that \((G_{p_0})\) for some \( p_0 \in (2, n) \) hold. Then \((R_p)\) holds for all \( p \in (2, p_0) \).
Proof. Let $p \in (2, p_0)$ and fix $q \in (p, p_0)$. Let $f \in C_c^\infty(M)$. Then we have $\| \nabla L^{-1/2} f \| \in L^2(M)$. For each $\lambda > 0$, let

$$E_\lambda := \bigl\{ x \in M : M_2|\nabla L^{-1/2} f|(x) > \lambda \bigr\}. $$

Then $\mu(E_\lambda) < \infty$ for each $\lambda > 0$.

By [19, Chapter III, Theorem 1.3], we can find a sequence of balls $\{B_i\}_i$ of boundedly overlapping, such that $E_\lambda = \cup_i B_i$. Moreover, there exists $C_W > 1$ such that there exists $\tilde{x}_i \in C WB_i$, $M_2(|\nabla L^{-1/2}|)(\tilde{x}_i) \leq \lambda$ for each $i$.

Fix a sufficient large $K_0 > 0$ such that Propositions 3.5 and 3.6 hold for any $K > K_0$. Let $\gamma > 0$ to be fixed later. Set

$$F_{\gamma\lambda} := \bigl\{ x \in M : M_2^{\#}f > \gamma \lambda \bigr\}$$

and

$$G_\lambda := \bigl\{ x \in M : \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} > C_E \lambda \bigr\},$$

where $C_E$ is the constant from Proposition 3.6. By noticing that $E_{\lambda} \cap \gamma \lambda F_{\lambda} \leq \sum_{B_i : r_B < 100} \mu((B_i \cap E_{\lambda}) \setminus F_{\gamma\lambda}) + \sum_{B_i : r_B \geq 100} \mu((B_i \cap E_{\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda))$

Using Proposition 3.5 one has that for each $B_i$ with $r_B < 100$

$$\mu((B_i \cap E_{\lambda}) \setminus F_{\gamma\lambda}) \leq \mu((C WB_i \cap E_{\lambda}) \setminus F_{\gamma\lambda}) \leq C \left( \gamma^2 + K^{-q} \right) \mu(C WB_i) \leq C \left( \gamma^2 + K^{-q} \right) \mu(B_i),$$

and hence by the boundedly overlapping property of $\{B_i\}$, we obtain

$$I_3 \leq \sum_{B_i : r_B < 100} C \left( \gamma^2 + K^{-q} \right) \mu(B_i) \leq C \left( \gamma^2 + K^{-q} \right) \mu(E_{\lambda}).$$

Meanwhile, Proposition 3.6 gives for each $B_i$ with $r_B \geq 100$ that

$$\mu((B_i \cap E_{\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda)) \leq \mu((C WB_i \cap E_{\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda)) \leq C \left( \gamma^2 + K^{-q} \right) \mu(B_i),$$

and hence by applying the boundedly overlapping property of $\{B_i\}$ once more, we obtain

$$I_4 \leq \sum_{B_i : r_B \geq 100} C \left( \gamma^2 + K^{-q} \right) \mu(B_i) \leq C \left( \gamma^2 + K^{-q} \right) \mu(E_{\lambda}).$$

By the estimates of $I_3$ and $I_4$, we conclude that

$$\mu(E_{\lambda}) \leq C \left( \gamma^2 + K^{-q} \right) \mu(E_{\lambda}) + \mu(F_{\gamma\lambda}) + \mu(G_\lambda).$$

(3.14)

It follows from Lemma 3.1 that

$$\mu(F_{\gamma\lambda}) \leq \frac{1}{(\gamma \lambda)^p} \int_M (M_2^{\#}f)^p \, d\mu \leq \frac{C}{(\gamma \lambda)^p} \int_M (M_2 f)^p \, d\mu \leq \frac{C \|f\|_p^p}{(\gamma \lambda)^p}. $$
Let us estimate $\mu(G_\lambda)$. If $V(x_M,1)(C_{E,A})^p \geq \|f\|_p^p$, then $G_\lambda = 0$. If $V(x_M,1)(C_{E,A})^p < \|f\|_p^p$, then we have

$$G_\lambda = \left\{ x \in M : V(x_M,d(x_M,x) + 1) < \left( \frac{\|f\|_p^p}{C_{E,A}} \right)^{1/p} \right\},$$

and hence

$$\mu(G_\lambda) \leq \left( \frac{\|f\|_p^p}{C_{E,A}} \right)^{1/p}.$$  

Inserting the estimates of $\mu(F_{\gamma,\lambda})$ and $\mu(G_\lambda)$ into the estimate (3.14), we see that

$$\mu(E_{K,\lambda}) \leq C \left( \gamma^2 + K^{-q} \right) \mu(E_\lambda) + C \frac{\|f\|_p^p}{(\gamma,\lambda)^p} + \left( \frac{\|f\|_p^p}{C_{E,A}} \right)^{1/p}.$$  

This implies that for each $\lambda > 0$

$$(K\lambda)^p \mu(E_{K,\lambda}) \leq C \left( \gamma^2 + K^{-q} \right) (K\lambda)^p \mu(E_\lambda) + CK^p \gamma^{-p} \|f\|_p^p + CK^p \|f\|_p^p.$$  

By taking $K$ large enough first and then $\gamma$ small enough, we see that

$$\|M_2|\nabla L^{-1/2}f\|_{L^p_{\text{loc}}} \leq \frac{1}{2} \|M_2|\nabla L^{-1/2}f\|_{L^p_{\text{loc}}} + C(K,\gamma,p)\|f\|_p^p,$$

which implies the Riesz transform is bounded from $L^p(M)$ to $L^{p,\infty}(M)$ for $p \in (2, p_0)$.

Since the Riesz transform is naturally $L^2$-bounded, we conclude that the Riesz transform is $L^p$-bounded for any $p \in (2, p_0)$ via the Marcinkiewicz interpolation theorem. \hfill \Box

### 3.2 Harmonic functions and Riesz transform

We need the following lemmas to conclude Theorem 1.5 and Theorem 1.8. Recall that $M$ is a complete, non-compact manifold with one or more but finitely many ends.

Let $p \in (2, \infty]$. We say that the local reverse $L^p$-Hölder inequality for gradients of harmonic functions holds on $M$, if for all $r_0 > 0$ there exists $C_H(r_0) > 0$ such that, for all ball $B$ with $r_B < r_0$, and each $u$ satisfying $Lu = 0$ in $3B$, it holds

$$(\text{RH}_{p,\text{loc}}) \quad \left( \int_B |\nabla u|_p^p \, d\mu \right)^{1/p} \leq \frac{C_H(r_0)}{r_B} \int_{2B} |u| \, d\mu.$$  

Notice that, if the constant $C_H(r_0)$ can be taken independent of $r_0$, then $(\text{RH}_{p,\text{loc}})$ becomes $(\text{RH}_p)$.

We shall need the Harnack inequality for harmonic functions (see [22 Proposition 2.1] for instance).

**Lemma 3.8.** Assume that $(D_N)$ holds on $M$ with $0 < N < \infty$, and that $(UE)$ holds. Then for any $\beta \in (0, 1)$ there exists $C > 0$ depending on $\beta$ such that if $Lu = 0$ in $B(x_0, r)$, then

$$\|u\|_{L^\infty(B(x_0,Br))} \leq C \int_{B(x_0,\beta r)} |u| \, d\mu.$$
Proof. The case $\beta = 1/2$ is a well-known fact as a consequence of Sobolev inequality; see [22], Proposition 2.1 for instance. The general case for $\beta \in (0, 1)$ follows from a simple covering argument. \hfill $\Box$

Lemma 3.9. Assume that $(D_N)$ and $(RD_n)$ hold on $M$ with $2 < n \leq N < \infty$, and that $(UE)$ and $(P^E_2)$ hold. Let $p \in (2, n)$. Then $(RH_p)$ holds if and only if $(RH_{p,\text{loc}})$ and $(RH^E_p)$ hold.

Proof. It is obvious that $(RH_p)$ implies $(RH_{p,\text{loc}})$ and $(RH^E_p)$. Let us prove the converse side.

If $3B \cap M_0 = \emptyset$, then $(RH_p)$ holds by $(RH^E_p)$, and if $r_B \leq 100$, then $(RH_p)$ holds by $(RH_{p,\text{loc}})$.

Assume now $3B \cap M_0 \neq \emptyset$ and $r_B > 100$. For any $x \in B$, if $d(x, x_M) \leq 100$, then by using $(RH_{p,\text{loc}})$ to the ball $B_x := B(x, 10)$, we see that

$$
\left( \int_{B_x} |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C}{r_x} \int_{2B_x} |u| \, d\mu \leq \frac{C}{r_x} \|u\|_{L^\infty(2B_x)} \leq \frac{C}{r_x} \int_{2B} |u| \, d\mu,
$$

where the last inequality follows from the fact $2B_x \subset B \subset 2B$ and Lemma 3.8.

If $d(x, x_M) > 100$, then we set $r_x := \min\{d(x, x_M)/10, r_B/10\}$. Notice that since $\text{diam} M_0 = 1$, $3B(x, r_x) \cap M_0 = \emptyset$. By applying $(RH^E_p)$ to $B_x := B(x, r_x)$, we conclude via Lemma 3.8 once more that

$$
\left( \int_{B_x} |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C}{r_x} \int_{2B_x} |u| \, d\mu \leq \frac{C}{r_x} \|u\|_{L^\infty(2B_x)} \leq \frac{C}{r_x} \int_{2B} |u| \, d\mu.
$$

Noticing that $d(x, x_M) < 4r_B$, and combining the above two estimates, we conclude that for each $x \in B$, it holds

$$
(3.15) \quad \int_{B_x} |\nabla u|^p \, d\mu \leq \frac{C}{[1 + d(x, x_M)]^p} \left( \int_{2B} |u| \, d\mu \right)^p.
$$

Let $k_0 = \lfloor \log_2(7r_B) \rfloor$. Since $p < n$, by using $(RD_n)$ and $(D_N)$, we have

$$
\int_{B} \frac{1}{[1 + d(x, x_M)]^p} \, d\mu(x) \leq \int_{B(x_M, 7r_B)} \frac{1}{[1 + d(x, x_M)]^p} \, d\mu(x)
\leq \sum_{k=0}^{k_0} \frac{1}{2^{k+1}B(x_M, 7r_B)} \left[ \int_{B(x_M, 7r_B)^{2k+1}} \frac{1}{[1 + d(x, x_M)]^p} \, d\mu(x) + \int_{B(x_M, 7r_B)} \frac{1}{[1 + d(x, x_M)]^p} \, d\mu(x) \right]
\leq \sum_{k=0}^{k_0} \frac{C}{[1 + 2^k r_B]^p} V(x_M, 2^k 7r_B) + V(x_M, 1)
\leq \sum_{k=0}^{k_0} \frac{C}{[1 + 2^k r_B]^p} 2^{k_0} V(x_M, 7r_B) + C 2^{-k_0} V(x_M, 7r_B)
$$
\[
\leq \sum_{k=0}^{-k_0} C 2^{k(n-p)} r_B^{-p} V(x_M, 7r_B) + C r_B^{-n} V(x_M, 7r_B)
\]

(3.16) \[ \leq C \mu(B) r_B^{-p}. \]

Recall that \( r_x = \max\{10, \min\{d(x, x_M)/10, r_B/10\}\}. \) Set \( r_y = \max\{10, \min\{d(y, x_M)/10, r_B/10\}\}. \) For each \( x \in B \), and every \( y \in B_x = B(x, r_x) \), one has

\[
d(x, x_M) - r_x \leq d(y, x_M) \leq d(x, x_M) + r_x.
\]

As a consequence of \( d(x, x_M) > 100 \) and \( r_B > 100 \), it holds \( 9r_x/10 \leq r_y \leq 11r_x/10. \) This together with the doubling condition leads to

\[
\int_B \int_{B_x} |\nabla u(y)|^p \, d\mu(y) \, d\mu(x) \geq C \int_B \int_B |\nabla u(y)|^p \frac{X_{B(x,r_x)}(y)}{V(x, r_x)} \, d\mu(y) \, d\mu(x)
\]

\[
\geq C \int_B \int_B |\nabla u(y)|^p \frac{X_{B(x,r_x)}(y)}{V(y, r_y)} \, d\mu(y) \, d\mu(x)
\]

\[
\geq C \int_B \int_B |\nabla u(y)|^p \frac{X_{B(y,10r_x/11)}(x)}{V(y, r_y)} \, d\mu(x) \, d\mu(y)
\]

\[
\geq C \int_B |\nabla u(y)|^p \, d\mu(y).
\]

This together with (3.15) and (3.16) gives that

\[
\int_B |\nabla u(y)|^p \, d\mu(y) \leq \int_B \int_{B_x} |\nabla u(y)|^p \, d\mu(y) \, d\mu(x)
\]

\[
\leq \int_B \frac{C}{1 + d(x, x_M)^p} \left( \int_{2B} |u| \, d\mu \right)^p \, d\mu(x)
\]

\[
\leq C \mu(B) r_B^{-p} \left( \int_{2B} |u| \, d\mu \right)^p,
\]

which is nothing but \((RH_p)\).

\[ \square \]

**Lemma 3.10.** Assume that \((D_N)\) holds on \( M \) with \( 0 < N < \infty \), and that \((P_F^E)\) and \((UE)\) hold. Let \( p \in (2, \infty]. \) If \((RH_p^E)\) holds on \( M \), then \((RH_{p,\text{loc}})\) holds on \( M \).

**Proof.** The proof is similar to that of Lemma 2.2 by using Yau’s gradient estimates for harmonic functions.

Recall that Yau’s gradient estimate states that if \( u \) is a harmonic function on \( 2B \), then it holds

\[
\sup_{x \in B} \frac{\|\nabla u(x)\|}{u(x)} \leq C(N) \left( \frac{1}{r_B} + \sqrt{K} \right),
\]

where \( K \geq 0 \), if every point in \( 2B \) has Ricci curvature not less than \( -K \); see [17, 50].
For any \( r_0 > 0 \), the Ricci curvature on the set \( \{ x \in M : \text{dist}(x, M_0) < 6r_0 \} \) is bounded below by a constant \(-K(r_0)\) depending on \( r_0 \), \( K(r_0) \geq 0 \). Suppose that \( u \) is a harmonic function on \( 3B \), where \( B = B(x, r) \) with \( r < r_0 \).

If \( \text{dist}(x, M_0) \leq 3r_0 \), then for an arbitrary \( \epsilon > 0 \), applying the pointwise Yau’s gradient estimate to \( u + \|u\|_{L^\infty(\tilde{2}B)} + \epsilon \), one has for each \( y \in B \)

\[
|\nabla u(y)| \leq C \left( u + \|u\|_{L^\infty(\tilde{2}B)} + \epsilon \right) \left( \frac{1}{r} + \sqrt{K(r_0)} \right) \leq \frac{C(r_0)}{r} \left( \|u\|_{L^\infty(\tilde{2}B)} + \epsilon \right).
\]

By Lemma 3.8 and letting \( \epsilon \to 0 \), we see that

\[
|\nabla u(x)| \leq \frac{C(r_0)}{r} \int_{2B} |u| \, d\mu,
\]

and hence,

\[
\left( \int_B |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C(r_0)}{r} \int_{2B} |u| \, d\mu.
\]

If \( \text{dist}(x, M_0) > 3r_0 \), then \( B(x, 3r) \cap M_0 = \emptyset \) for any \( r < r_0 \). By using \((RH^E_p)\), one sees that

\[
\left( \int_B |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C}{r} \int_{2B} |u| \, d\mu,
\]

as desired.

The above lemma leads to the following open-ended character of condition \((RH_p)\) for \( p < n \).

**Lemma 3.11.** Assume that \((D_N)\) and \((RD_n)\) hold on \( M \) with \( 2 < n \leq N < \infty \), and that \((UE)\) and \((P^{E}_2)\) hold. Let \( p \in (2, n) \). Then if \((RH_p)\) holds, there exists \( \epsilon > 0 \) such that \( p + \epsilon < n \) and \((RH_{p+\epsilon})\) holds.

**Proof.** Using \((P^{E}_2)\) and a standard Caccioppoli inequality (see [22] Lemma 2.4) for instance, one can conclude that for each ball \( B = B(x_B, r_B) \) with \( 3B \cap M_0 = \emptyset \) and \( v \) satisfying \( \mathcal{L}v = 0 \) in \( 3B \), it holds

\[
\left( \int_B |\nabla v|^p \, d\mu \right)^{1/p} \leq C \left( \int_{2B} |\nabla v|^2 \, d\mu \right)^{1/2}.
\]

Moreover, this estimate holds for any sub-ball \( \tilde{B} \) with \( 2\tilde{B} \subset 2B \), since \( 3\tilde{B} \cap M_0 = \emptyset \). Applying Gehring’s lemma (cf. [31]) and self-improvements of the reverse Hölder inequality (cf. [6] Appendix)), we see that it holds for some \( \epsilon > 0 \),

\[
\left( \int_B |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} \leq C \left( \int_{2B} |\nabla v|^2 \, d\mu \right)^{1/2} \leq \frac{C}{r_B} \int_{\tilde{2}B} |v| \, d\mu.
\]

This together with a simple covering argument implies that \((RH^{E}_{p+\epsilon})\) holds. The desired conclusion follows by combining this together with Lemma 3.9 and Lemma 3.10. \( \square \)
We can now finish the proof of Theorem 1.5.

**Proof of Theorem 1.5.** Since in our setting, \((D)\) and \((UE)\) hold, and \((P_{2,\text{loc}})\) follows from \((P^E_2)\) by Lemma 2.2. By [22, Theorem 1.5], we see that \((RH_p) \Leftrightarrow (G_p)\) holds for any \(p \in (2, \infty)\). The implication \((R_p) \Rightarrow (G_p)\) holds automatically by the analytic property of the heat semigroup; see [4] for instance.

Finally, if \((RH_p)\) holds, then by Lemma 3.11 there exists \(\varepsilon > 0\) such that \(p + \varepsilon < n\) and \((RH_{p+\varepsilon})\) holds. This implies \((G_{p+\varepsilon})\), which by Theorem 3.7 gives \((R_q)\) for all \(q \in (2, p + \varepsilon)\), in particular, \((R_p)\). The proof is complete. \(\Box\)

Theorem 1.8 follows from Lemma 3.9 and Theorem 1.5.

**Proof of Theorem 1.8.** By Theorem 1.5, \((R_p)\) is equivalent to \((RH_p)\). By Lemma 3.9 and Lemma 3.10 one sees that \((RH_p)\) is equivalent to \((RH^E_p)\). \(\Box\)

**Proof of Corollary 1.9.** This corollary follows from Theorem 1.8 and Lemma 3.11. \(\Box\)

**Lemma 3.12.** Assume that \((D_N)\) holds on \(M\) with \(0 < N < \infty\), and that \((QD)\) and \((UE)\) hold. Then \((RH^E_\infty)\) holds.

**Proof.** Suppose that \(u\) is a harmonic function on \(2B\), with \(3B \cap M_0 = \emptyset\). For an arbitrary \(\varepsilon > 0\), applying the pointwise Yau’s gradient estimate (see the proof of Lemma 3.10) to \(u + \|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon\), one has for each \(x \in B\)

\[
|\nabla u(x)| \leq C \left( u + \|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon \right) \left( \frac{1}{r_B} + \frac{C_M}{r_B + 1} \right) \leq C \left( \|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon \right) \leq C \int_{2B} |u| \, d\mu.
\]

Above in the last step we used Lemma 3.8. Applying this to the last estimate and letting \(\varepsilon \to 0\), we see that \((RH^E_\infty)\) holds. \(\Box\)

**Proof of Corollary 1.10.** By [20] it is known that under \((D)\) and \((UE)\), \((R_p)\) holds for all \(p \in (1, 2]\). Notice that under \((QD)\), \((P^E_2)\) holds by Theorem 2.5, and \((RH^E_\infty)\) holds by Lemma 3.12. Since \((RH^E_\infty)\) implies \((RH^E_p)\) for all \(p \in (2, \infty)\), we see that \((R_p)\) holds for all \(p \in (2, n)\) by Theorem 1.8. \(\Box\)

### 4 Riesz transform for \(p\) above the upper dimension

In this section, we provide the proofs for Theorem 1.11 and Corollary 1.12. The ideas employed come from recent developments of the elliptic theory for heat kernels from [6, 22].

**Proof of Theorem 1.11.** By Theorem 2.3 we see that \((P_p)\) holds for any \(p > N \vee 2\). Since our measure is doubling, and \((UE)\) holds, we can apply [22, Theorem 1.9] to show that the conditions \((R_p)\), \((RH_p)\), and \((G_p)\) are equivalent.

Notice that, by [20], \((D)\) and \((UE)\) implies that \((R_q)\) holds for all \(q \in (1, 2]\). Since \(p > N\), \(M\) is \(p\)-parabolic. By [12, Theorem C] and the assumption that \(M\) is non-parabolic, \((R_p)\) together with \((R^p_{p-1})\) implies that \(M\) can have only one end. \(\Box\)
Remark 4.1. Notice that \((P_p)\) together with \((G_p)\) implies \((P_2)\) by \([5]\). One may also use \((P_2)\) to show that there exists only one end if \(n > 2\).

Proof of Corollary \([1, 2]\) Suppose that there exists a non-constant harmonic function \(u\) on \(M\) with the growth

\[ u(x) = O(d(x, o)^p) \text{ as } d(x, o) \to \infty \]

for some \(\alpha \in [0, 1)\) and a fixed \(o \in M\).

Assume first \(\alpha = 0\). If \((R_p)\) holds for some \(p > N \vee 2\), then by Theorem \([1, 11]\) we have \((RH_p)\), which is

\[ \left( \int_B |\nabla v|^p d\mu \right)^{1/p} \leq \frac{C\mu(B)^{1/p}}{r_B} \int_{2B} |v| d\mu \leq \frac{CV(x_B, 1)^{1/p} r_B^{N/p}}{r_B} \int_{2B} |v| d\mu. \]

Applying this estimate to \(u\) and letting the radius of \(B\) tend to infinity, we see that \(\|\nabla u\|_p = 0\), which cannot be true. Therefore the Riesz transform is not bounded on \(L^p(M)\) for any \(p > N \vee 2\).

Assume now \(\alpha \in (0, 1)\). Suppose first that \(\frac{N}{1-\alpha} \leq 2\). Notice that it implies \(N < 2\). Assume \((R_p)\) holds for some \(p > 2\). Then the estimate (4.1) holds for \(u\), which further implies that

\[ \left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq \frac{C\mu(B)^{1/p}}{r_B} \int_{2B} |u| d\mu \leq CV(x_B, 1)^{1/p} r_B^{N/p} \to 0, \]

as \(r_B \to \infty\), since \(p > 2 \geq \frac{N}{1-\alpha}\). This implies \(\|\nabla u\|_p = 0\) which contradicts with \(u\) being non-constant. Therefore, \((R_p)\) does not hold for any \(p > 2\).

For the cases \(\frac{N}{1-\alpha} > 2\), we only need to show that the Riesz transform is not bounded on \(L^p(M)\) for \(p = \frac{N}{1-\alpha}\). Suppose this is not the case. By the validity of \((P_p)\) from Theorem \([2, 4]\) and \((R_p)\), we apply \([22]\) Corollary 1.10] to find that there exists \(\epsilon > 0\) such that for each \(v\) satisfying \(\mathcal{L}v = 0\) in \(3B\), it holds

\[ \left( \int_B |\nabla v|^{p+\epsilon} d\mu \right)^{1/(p+\epsilon)} \leq \frac{C}{r_B} \int_{2B} |v| d\mu. \]

This gives

\[ \left( \int_B |\nabla v|^{p+\epsilon} d\mu \right)^{1/(p+\epsilon)} \leq \frac{C\mu(B)^{1/(p+\epsilon)}}{r_B} \int_{2B} |v| d\mu. \]

Applying this estimate to \(u\) and using \((D_N)\), we conclude that

\[ \left( \int_B |\nabla u|^{p+\epsilon} d\mu \right)^{1/(p+\epsilon)} \leq CV(x_B, 1)^{1/(p+\epsilon)} r_B^{N/(p+\epsilon)-\alpha} \to 0, \]

as \(r_B \to \infty\). This contradicts with \(u\) being non-constant. Therefore, the Riesz transform cannot be bounded on \(L^p(M)\) for \(p = \frac{N}{1-\alpha}\). The proof is completed. \(\square\)

Carron \([12]\) Theorem D & Proposition E] had provided some sufficient conditions for both boundedness and unboundedness of the Riesz transform for \(p > N\), under the requirement of quadratic Ricci curvature decay \((QD)\). As an application of our criteria above, we can relax the requirement of Ricci curvature bound from \([12]\) to \((P_p^E)\) and \((RH_p^E)\) (see Lemma \([2, 5]\) and Lemma \([3, 12]\), and show that his condition \((HE_\alpha)\) is also necessary, if \(n = N\).
Theorem 4.2. Assume that \((D_N)\) and \((RD_n)\) hold on \(M\) with \(1 < n = N < \infty\). Suppose that \((UE)\), \((P^E_2)\) and \((RH^E_M)\) hold. Let \(p \in (N \vee 2, \infty)\). Then the following statements are equivalent.

(i) \((R_p)\) holds;
(ii) \((RH_p)\) holds;
(iii) \((HE_\alpha)\) holds for some \(\alpha \in (1 - \frac{N}{p}, 1]\), i.e., there exists \(C > 0\) such that for any ball \(B \subset M\) and any harmonic function \(u\) on \(3B\), it holds for any \(x, y \in B\) that

\[
(HE_\alpha) \quad |u(x) - u(y)| \leq C \left( \frac{d(x, y)}{r_B} \right)^\alpha \int_{2B} |u| d\mu.
\]

Proof. The equivalence \((R_p)\) and \((RH_p)\) is a special case of Theorem 1.11. Let us show that \((RH_p) \Leftrightarrow (HE_\alpha)\).

Step 1. \((HE_\alpha) \Rightarrow (RH_p)\). The case \(\alpha = 1\) is easy, since \((HE_\alpha)\) implies that for any \(x \in B\),

\[
|\nabla u(x)| \leq \limsup_{y, d(x, y) \to 0} \frac{|u(x) - u(y)|}{d(x, y)} \leq C \frac{1}{r_B} \int_{2B} |u| d\mu,
\]

which is \((RH_\infty)\).

Suppose now \(\alpha \in (1 - N/p, 1)\). Recall that \(p > N \vee 2\). If \(3B \cap M_0 = \emptyset\), then \((RH_\infty)\) holds on \(B\) by \((RH^E_M)\). If \(r_B \leq 100\), then by using \((RH_{\infty, \text{loc}})\) from Lemma 3.10, one has that for \(u\) satisfying \(L u = 0\) on \(3B\), it holds

\[
\|\nabla u\|_{L^\infty(B)} \leq C \frac{1}{r_B} \int_{2B} |u| d\mu.
\]

Let us consider the remaining case: \(3B \cap M_0 \neq \emptyset\) and \(r_B > 100\). For any \(x \in B\), if \(d(x, x_M) \leq 10\), then by using \((RH_{\infty, \text{loc}})\) to the ball \(B_x := B(x, d(x, x_M) + 1)\) and \(u - u_{2B}\), we see that

\[
\|\nabla u\|_{L^\infty(B_x)} \leq C \frac{1}{d(x, x_M) + 1} \int_{2B_x} \int_{2B_x} |u(y) - u(z)| d\mu(y) d\mu(z).
\]

Applying \((HE_\alpha)\) we find

\[
\|\nabla u\|_{L^\infty(B_x)} \leq C \left[ 1 + d(x, x_M) \right]^{\alpha} \left[ 1 + d(x, x_M) \right]^{\alpha} \frac{1}{r_B^\alpha} \int_{2B} |u| d\mu.
\]

If \(d(x, x_M) > 10\), then by applying \((RH^E_M)\) to \(B_x := B(x, r_x)\), where \(r_x = \max\{d(x, x_M)/10, r_B/10\}\), and using \((HE_\alpha)\), we conclude

\[
\|\nabla u\|_{L^\infty(B_x)} \leq C \frac{1}{r_B} \int_{2B_x} \int_{2B_x} |u(y) - u(z)| d\mu(y) d\mu(z) \leq C \frac{r_x^{1-\alpha}}{r_B^\alpha} \int_{2B} |u| d\mu.
\]

Noticing that \(d(x, x_M) < 4r_B\), and combining the above two estimates, we conclude that for each \(x \in B\), it holds

\[
|\nabla u(x)| \leq C \frac{1}{\left[ 1 + d(x, x_M) \right]^{1-\alpha} r_B^{\alpha}} \int_{2B} |u| d\mu.
\]
Notice that $p(1 - \alpha) < N$. One has via (5.16) that
\[
\left( \int_B |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C}{r_B^p} \int_{2B} |u| \, d\mu \left( \int_B \frac{1}{[1 + d(x, x_M)]^{(1-\alpha)p} r_B^p} \, d\mu(x) \right)^{1/p} \\
\leq \frac{C}{r_B^p} \int_{2B} |u| \, d\mu \left( \int_{B(x_M, 4r_B)} \frac{1}{[1 + d(x, x_M)]^{(1-\alpha)p}} \, d\mu(x) \right)^{1/p} \\
\leq C \mu(B)^{1/p} r_B^{-1} \int_{2B} |u| \, d\mu.
\]

That is nothing but $(RH_p)$.

**Step 2.** $(RH_p) \Rightarrow (HE_\alpha)$.

As $(RH_p^c)$ and hence $(RH_{\infty,\text{loc}})$ hold, for a ball $B$ satisfying $3B \cap M_0 = \emptyset$ or $r_B \leq 100$, one sees that for each $v$ satisfying $\mathcal{L}v = 0$ in $3B$, it holds that
\[
|v(x) - v(y)| \leq C \frac{d(x, y)}{r_B^\alpha} \int_{2B} |v| \, d\mu.
\]

Thus we only need to verify $(HE_\alpha)$ for balls $B$ with large radius and $3B \cap M_0 \neq \emptyset$. By Theorem 2.3 $(P_p)$ holds since $p > N \vee 2$. By using $(RH_p)$ and $(P_p)$, we apply [22 Corollary 1.10] to see that $(RH_{p+\epsilon})$ holds for some $\epsilon > 0$. Therefore, for each $v$ satisfying $\mathcal{L}v = 0$ in $3B$, it holds
\[
\left( \int_B |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} \leq \frac{C}{r_B^p} \int_{2B} |v| \, d\mu.
\]

Let $x, y \in B$. If $d(x, y) \geq r_B/100$, then by Lemma 3.8 one has that
\[
|v(x) - v(y)| \leq C \left( \frac{d(x, y)}{r_B^\alpha} \right)^\alpha (|v(x)| + |v(y)|) \leq C \left( \frac{d(x, y)}{r_B^\alpha} \right)^\alpha \int_{2B} |v| \, d\mu.
\]

Suppose that $d(x, y) < r_B/100$. Using $(P_{p+\epsilon}), (D_N), (RH_{p+\epsilon}),$ Lemma 3.8 and a standard telescopic argument (cf. [33,55]) gives that
\[
|v(x) - v(y)| \\
\leq C d(x, y) \sum_{j=0}^\infty 2^{-j} \left( \int_{B(x, 2^{-j}d(x, y))} |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} + \sum_{j=1}^\infty 2^{-j} \left( \int_{B(x, 2^{-j}d(x, y))} |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} \\
\leq C d(x, y) \left( \sum_{j=0}^\infty 2^{-j} \left( \frac{r_B^N}{2^{-j}d(x, y)^N V(x, r_B/8)} \right)^{1/(p+\epsilon)} \left( \int_{B(x, r_B/8)} |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} \right) \\
\leq C r_B \left( \frac{d(x, y)}{r_B} \right)^{1-\frac{N}{p+\epsilon}} \left( \int_{B(x, r_B/8)} |\nabla v|^{p+\epsilon} \, d\mu \right)^{1/(p+\epsilon)} \leq C \frac{d(x, y)}{r_B} \left( \frac{d(x, y)}{r_B} \right)^{-\frac{N}{p+\epsilon}} \int_{B(x, r_B/4)} |v| \, d\mu \\
\leq C \left( \frac{d(x, y)}{r_B} \right)^{-\frac{N}{p+\epsilon}} \|v\|_{L^{\infty}(\Omega_B)} \leq C \left( \frac{d(x, y)}{r_B} \right)^{-\frac{N}{p+\epsilon}} \int_{2B} |v| \, d\mu.
\]

This implies $(HE_\alpha)$ for $\alpha = 1 - \frac{N}{p+\epsilon}$, and completes the proof. \qed
5 Extensions to Dirichlet metric measure spaces

In this section, we discuss extensions of main results to the setting of Dirichlet metric measure spaces. Since in a non-smooth setting, local Poincaré inequality (see Lemma 2.2) and local smoothness of harmonic functions (see Lemma 3.10) do not follow automatically from the assumptions on ends, we need to assume them as additional assumptions. However, other assumptions are the same as in the smooth settings. As the proofs are basically identical to the smooth settings (see [4, 27]), we will sketch the proofs in the section.

Let \( X \) be a locally compact, separable, metrisable, and connected space equipped with a Borel measure \( \mu \) that is finite on compact sets and strictly positive on non-empty open sets. Consider a strongly local and regular Dirichlet form \( \mathcal{E} \) on \( L^2(X, \mu) \) with dense domain \( \mathcal{D} \subset L^2(X, \mu) \) (see [27] for precise definitions). According to Beurling and Deny [8], such a form can be written as

\[
\mathcal{E}(f, g) = \int_X d\Gamma(f, g)
\]

for all \( f, g \in \mathcal{D} \), where \( \Gamma \) is a measure-valued non-negative and symmetric bilinear form defined by the formula

\[
\int_X \varphi d\Gamma(f, g) := \frac{1}{2} [\mathcal{E}(f, \varphi g) + \mathcal{E}(g, \varphi f) - \mathcal{E}(fg, \varphi)]
\]

for all \( f, g \in \mathcal{D} \cap L^\infty(X, \mu) \) and \( \varphi \in \mathcal{D} \cap C_0(X) \). Here and in what follows, \( C(X) \) denotes the space of continuous functions on \( X \) and \( C_0(X) \) the space of functions in \( C(X) \) with compact support. We shall assume in addition that \( \mathcal{E} \) admits a “carré du champ”, meaning that \( \Gamma(f, g) \) is absolutely continuous with respect to \( \mu \), for all \( f, g \in \mathcal{D} \). In what follows, for simplicity of notation, we will denote by \( \langle Df, Dg \rangle \) the energy density \( \frac{d\Gamma(f, g)}{d\mu} \), and by \( |Df| \) the square root of \( \frac{d\Gamma(f, f)}{d\mu} \).

Since \( \mathcal{E} \) is strongly local, \( \Gamma \) is local and satisfies the Leibniz rule and the chain rule; see [27]. Therefore we can define \( \mathcal{E}(f, g) \) and \( \Gamma(f, g) \) locally. Denote by \( \mathcal{D}_{loc} \) the collection of all \( f \in L^2_{loc}(X) \) for which, for each relatively compact set \( K \subset X \), there exists a function \( h \in \mathcal{D} \) such that \( f = h \) almost everywhere on \( K \). The intrinsic (pseudo-)distance on \( X \) associated to \( \mathcal{E} \) is then defined by

\[
d(x, y) := \sup \{ f(x) - f(y) : f \in \mathcal{D}_{loc} \cap C(X), |Df| \leq 1 \text{ a.e.} \}.
\]

We always assume that \( d \) is indeed a distance (meaning that for \( x \neq y \), \( 0 < d(x, y) < +\infty \) and that the topology induced by \( d \) is equivalent to the original topology on \( X \). Moreover, we assume that \( (X, d) \) is a complete metric space.

Corresponding to such a Dirichlet form \( \mathcal{E} \), there exists an operator, denoted by \( \mathcal{L} \), acting on a dense domain \( \mathcal{D}(\mathcal{L}) \) in \( L^2(X, \mu) \), \( \mathcal{D}(\mathcal{L}) \subset \mathcal{D} \), such that for all \( f \in \mathcal{D}(\mathcal{L}) \) and each \( g \in \mathcal{D} \),

\[
\int_X f(x) \mathcal{L} g(x) \, d\mu(x) = \mathcal{E}(f, g).
\]

The opposite of \( \mathcal{L} \) is the infinitesimal generator of the heat semigroup \( H_t = e^{-t\mathcal{L}} \), \( t > 0 \).

We assume that \( X \) is the union of a compact set \( X_0 \) and some ends \( \{E_k\}_{k \in \mathbb{N}} \), \( k \in \mathbb{N} \). We simply adapt all the notions from previous sections with the Laplace-Beltrami operator \( \mathcal{L} \) replaced by \( \mathcal{L} \),
the Riemannian gradient $\nabla$ replaced by $D$, and $M_0$ replaced by $X_0$; see [6, 22] for more studies in such settings.

The following result generalizes Theorem 1.5 and Theorem 1.8 to the metric setting.

**Theorem 5.1.** Assume that the non-compact Dirichlet metric measure space $(X, d, \mu, \mathcal{E})$ satisfies $(D_N)$ and $(RD_n)$ with $2 < n \leq N < \infty$. Suppose that $(UE)$, $(P_{2, \text{loc}})$ and $(P^E_2)$ hold. Let $p \in (2, n)$. Then the following statements are equivalent.

(i) $(R_p)$ holds.

(ii) $(RH_p)$ holds;

(iii) $(RH^E_p)$ and $(RH_{p, \text{loc}})$ hold;

(iv) $(G_p)$ holds.

**Remark 5.2.** Comparing to Theorem 1.5 and Theorem 1.8, $(P_{2, \text{loc}})$ is an additional assumption. Notice that, in the smooth setting, $(P_{2, \text{loc}})$ follows from $(P^E_2)$ as in Lemma 2.2 however, in the non-smooth setting, this is not true in general. Also in the term (iii), we need $(RH_{p, \text{loc}})$ additionally, since in metric setting, harmonic functions are not necessarily smooth (see [22]), and $(RH_{p, \text{loc}})$ does not follow from $(RH^E_p)$, comparing to Lemma 3.10. For instance, one can glue two Euclidean ends via a smooth part removing a suitable fractal, where the local Poincaré inequality and local smoothness of harmonic functions may not hold.

**Proof of Theorem 5.1.** By [22, Theorem 1.6], we have the equivalence of $(RH_p)$ and $(G_p)$. Moreover, the same proof of Lemma 3.9 works in the metric setting, which implies that $(RH_p)$ is equivalent to $(RH^E_p)$ together with $(RH_{p, \text{loc}})$, for $p \in (2, n)$.

It remains to show that $(R_p)$ is equivalent to $(G_p)$. It holds automatically that $(R_p)$ implies $(G_p)$, see [4, 22] for instance. On the other hand, the same proof of Theorem 3.7 gives that $(G_p)$ implies $(R_q)$ for any $q \in (2, p)$. By the same proof of Lemma 3.11 one sees that there exists $\varepsilon > 0$ such that $(G_{p+\varepsilon})$ holds, which then implies $(R_p)$, and completes the proof. □

We have the following metric version of Theorem 1.11.

**Theorem 5.3.** Assume that the non-compact Dirichlet metric measure space $(X, d, \mu, \mathcal{E})$ satisfies $(D_N)$ with $0 < N < \infty$. Suppose that $(UE)$, $(P_{2, \text{loc}})$ and $(P^E_2)$ hold. Let $p \in (N \lor 2, \infty)$. Then the following statements are equivalent.

(i) $(R_p)$ holds;

(ii) $(RH_p)$ holds;

(iii) $(G_p)$ holds.

**Proof.** Notice that, by the same proof of Theorem 2.8 $(P_{2, \text{loc}})$ and $(P^E_2)$ imply that Poincaré inequality $(P_q)$ holds for any $p \in (N \lor 2, \infty)$. [22, Theorem 1.9] then gives the desired conclusion. □
6 Applications

A key tool in the paper is the Gaussian upper bound of heat kernel, i.e., there exist $C,c > 0$ such that for all $t > 0$ and $x,y \in M$,

$$(UE) \quad p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left\{-\frac{d^2(x,y)}{ct}\right\}.$$  

By [28, 44, 45], it is well known that $(D)$ together with $(UE)$ is equivalent to a Faber-Krahn inequality, and also equivalent to a local Sobolev inequality; see also [9]. Recent result by Grigor’yan and Saloff-Coste [29, 30] gives a very useful solution to the stability of $(UE)$ under gluing operations. The following result follows from [30, Theorem 3.5], see also [29, Corollary 4.6].

**Theorem 6.1.** Let $M$ be a manifold with finitely many ends $\{E_i\}_{1 \leq i \leq k}$, $k \in \mathbb{N}$. Suppose that $M$ satisfies $(D)$. If for each $i$, there exists a manifold $M_i$ satisfying $(D)$, and a compact subset $K_i \subset M_i$, such that $E_i$ is isometric to $M_i \setminus K_i$, and each $M_i$ satisfies $(D)$ and $(UE)$, then $(UE)$ holds on $M$.

Above by each $M_i$ satisfies $(UE)$, we mean the heat kernel on $M_i$ satisfies $(UE)$, with nothing to do with the gluing manifold $M$.

From Theorem 6.1 we see that, if $M$ is obtained by gluing some Riemannian manifolds with non-negative Ricci curvature, simply connected nilpotent Lie groups with polynomial growth as well as conic manifolds, together through a compact manifold smoothly, then $M$ satisfies $(UE)$, since $(UE)$ holds on the aforementioned manifolds; see [1, 4, 33, 22] for instance.

As a consequence, our Theorem 1.8 and Theorem 1.11 work, if $M$ is obtained by gluing Riemannian manifolds with non-negative Ricci curvature, simply connected nilpotent Lie groups as well as conic manifolds, together through a compact manifold smoothly.

Another class of gluing manifolds that our result can be applied to is the manifold obtained by gluing several cocompact covering Riemannian manifolds with polynomial growth deck transformation group together. Here, a manifold $\tilde{M}$ has a cocompact covering, if there is a finitely generated discrete group $G$ with polynomial volume growth of some order $D > 2$, that acts properly and freely on $\tilde{M}$ by isometries, such that the orbit space $M_G = \tilde{M}/G$ is a compact manifold. See [26, 22] for instance.

Note also, our result also work on the these settings with the Laplace-Beltrami operator replaced by any uniformly elliptic operators of divergence form, by Theorem 5.1 and Theorem 5.3.

Let us finish the proof of Theorem 1.14 and Corollary 1.15.

**Proof of Theorem 1.14.** Notice that $(P_E^2)$ holds automatically, as each $M_i$ satisfies $(P_E^2)$. Moreover, $(UE)$ follows from Theorem 6.1 since $M_i$ supports $(UE)$.

For each $p \in (2, n)$, since $(R_p)$ implies $(RH_p)$ on each $M_i$ by Theorem 1.5 we see that $(RH_p^E)$ holds on $M$ and the conclusion follows from Theorem 1.8. 

**Proof of Corollary 1.15.** Since $(D)$ plus $(P_2)$ imply $(UE)$ and $(P_E^2)$, we see that Theorem 1.14 applies.
By [3, Theorem 0.4], for each $M_i$, there exists $\epsilon_i > 0$ such that the Riesz transform is bounded on $L^{2+\epsilon_i}(M_i)$. This implies that there exists $\epsilon > 0$, possibly smaller than $\epsilon_i$, $1 \leq i \leq k$, such that $2 + \epsilon < n$ and $(R_{2+\epsilon})$ holds on each $M_i$. By Theorem 1.14 we see that $(R_{2+\epsilon})$ holds.

(ii) Since $(R_p)$ holds on each $M_i$, we apply Theorem 1.14 to conclude that $(R_p)$ holds on $M$. □

Consider an $n$-dimensional conic manifold $C(X)$ with compact basis $X$, $C(X) := \mathbb{R}^+ \times X$, where the metric is given by $dr^2 + r^2dX$. Let $\lambda_1$ be the smallest nonzero eigenvalue of the Laplacian on the basis $X$. By Li [38], the Riesz transform is bounded on $L^p(C(X))$ for all $p \in (1, p_0)$ and not bounded for $p \geq p_0$, where

$$p_0 := n \left( \frac{n}{2} - \sqrt{\left( \frac{n-2}{2} \right)^2 + \lambda_1} \right)^{-1}$$

if $\lambda_1 < n-1$ and $p_0 = \infty$ otherwise; see also [4]. The following question was also asked in [13].

Question 6.2 (Open Problem 8.1 [13]). Is a result similar to H.-Q. Li’s valid for smooth manifolds with one conic or asymptotically conic end? What happens for several conic ends?

Guillarmou and Hassell [32] had solved the above question, which was recovered by recent work of Carron [12]. Our result also gives a new proof to the above question.

Let us explain how the proof works.

Notice that the measure satisfies $V(x, r) \sim r^n$ for each $x \in M$ and each $r > 0$, where $n \geq 2$. Suppose that the manifold has at least two conic ends. If $n \geq 3$, then Corollary 1.10 applies to show that the Riesz transform is bounded on $L^p(M)$, for any $p \in (1, n)$, while [13] (see Theorem 1.2) already implies the Riesz transform cannot bounded for any $p \geq n$ if $n \geq 3$. If $n = 2$, $(R_p)$ holds for any $p \in (1, 2]$ by [20], and not bounded for any $p > 2$ by applying Corollary 1.12 and using the fact that there exists a non-constant harmonic function of logarithmic growth (cf. [12, Section 7]).

If the manifold has only one conic end, our Theorem 4.2 and Corollary 1.12 apply since the Ricci curvature satisfies

$$\text{Ric}_M(x) \geq -\frac{C_M}{[d(x, x_M) + 1]^2}$$

for some $C_M > 0$. The existence of harmonic functions of sub-linear growth, and the elliptic Hölder regularity of harmonic functions can be found in [18] and also [12, Section 7].

Let $(M, g_0)$ be a simply connected nilpotent Lie group of dimension $n > 2$ (endowed with a left-invariant metric), and $\nu$ be the homogenous dimension of $\tilde{M}$, i.e. for some $o \in \tilde{M}$

$$\nu := \lim_{R \to \infty} \frac{\log V(o, R)}{\log R}.$$  

Notice that $\nu \geq n > 2$. Let $(M, g)$ be a manifold obtained by gluing $k > 1$ copies of $(\tilde{M}, g_0)$. Carron-Coulhon-Hassell [13] showed that $(R_p)$ does not hold if $p \geq \nu$, and they asked

Question 6.3 (Open Problem 8.3 [13]). Show that the Riesz transform on $(M, g)$ is bounded on $L^p$ for $p \in (1, \nu)$.
Carron \cite{11} had solved the question. Our Corollary \cite{14,15} also provides a proof, by noticing that \((D_\nu)\) and \((RD_\nu)\) hold on \(M\), and on a Lie group of polynomial growth \((P_2)\) holds and \((R_p)\) holds for all \(p \in (1, \infty)\); see \cite{11,22,49}.

\textbf{Acknowledgments}

The author is indebted to Thierry Coulhon for his deep insights of this question, and for many helpful and inspiring discussions. He wishes to thank Hongquan Li for helpful discussions, and to thank Gilles Carron for helpful communications, in particular, for sending the author the paper \cite{12}, which inspires Corollary \cite{11,12} and Theorem \cite{4,2} of this paper. The author was partially supported by NNSF of China (11671039 & 11771043).

\textbf{References}

\begin{enumerate}
  \item G. Alexopoulos, An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth, \textit{Canad. J. Math.} 44 (1992), 691-727.
  \item P. Auscher, On necessary and sufficient conditions for \(L^p\)-estimates of Riesz transforms associated to elliptic operators on \(\mathbb{R}^n\) and related estimates, \textit{Mem. Amer. Math. Soc.} 186 (2007), no. 871, xviii+75 pp.
  \item P. Auscher, T. Coulhon, Riesz transform on manifolds and Poincaré inequalities, \textit{Ann. Sc. Norm. Super. Pisa Cl. Sci.}, (5) 4 (2005), 531-555.
  \item P. Auscher, T. Coulhon, X.T. Duong, S. Hofmann, Riesz transform on manifolds and heat kernel regularity, \textit{Ann. Sci. École Norm. Sup.}, (4) 37 (2004), 911-957.
  \item D. Bakry, Etude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée. \textit{Séminaire de Probabilités} Vol. XXI, Lecture Notes in Mathematics 1247, Springer-Verlag, Berlin, 1987, 137-172.
  \item F. Bernicot, T. Coulhon, D. Frey, Gaussian heat kernel bounds through elliptic Moser iteration, \textit{J. Math. Pures Appl.} (9) 106 (2016), 995-1037.
  \item F. Bernicot, D. Frey, Riesz transforms through reverse Hölder and Poincaré inequalities, \textit{Math. Z.} 284 (2016), 791-826.
  \item A. Beurling, J. Deny, Dirichlet spaces, \textit{Proc. Nat. Acad. Sci. USA}, 45 (1959) 208-215.
  \item S. Boutayeb, T. Coulhon, A. Sikora, A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces, \textit{Adv. Math.} 270 (2015), 302-374.
  \item P. Buser, A note on the isoperimetric constant, \textit{Ann. Sci. École Norm. Sup.} 15 (1982), 213-230.
  \item G. Carron, Riesz transforms on connected sums, Festival Yves Colin de Verdière, \textit{Ann. Inst. Fourier (Grenoble)} 57 (2007), 2329-2343.
  \item G. Carron, Riesz transform on manifolds with quadratic curvature decay, \textit{Rev. Mat. Iberoam}, 33 (2017), 749-788.
  \item G. Carron, T. Coulhon, A. Hassell, Riesz transform and \(L^p\)-cohomology for manifolds with Euclidean ends, \textit{Duke Math. J.} 133 (2006), 59-93.
\end{enumerate}
[14] J.C. Chen, Riesz transform on manifold, (Chinese) *Acta Math. Sinica (Chin. Ser.),* 43 (2000), 821-828.

[15] L. Chen, T. Coulhon, J. Feneuil, E. Russ, Riesz transform for $1 \leq p \leq 2$ without Gaussian heat kernel bound, *J. Geom. Anal.* 27 (2017), 1489-1514.

[16] P. Chen, J. Magniez, E.M. Ouhabaz, The Hodge-de Rham Laplacian and $L^p$-boundedness of Riesz transforms on non-compact manifolds, *Nonlinear Anal.* 125 (2015), 78-98.

[17] S.Y. Cheng, S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.,* 28 (3) (1975) 333-354.

[18] T. Christiansen, M. Zworski, Harmonic functions of polynomial growth on certain complete manifolds, *Geom. Funct. Anal.* 6 (1996), 619-627.

[19] R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977) 569-645.

[20] T. Coulhon, X.T. Duong, Riesz transforms for $1 \leq p \leq 2$, *Trans. Amer. Math. Soc.* 351 (1999), 1151-1169.

[21] T. Coulhon, X.T. Duong, Riesz transform and related inequalities on noncompact Riemannian manifolds, *Comm. Pure Appl. Math.* 56 (2003), 1728-1751.

[22] T. Coulhon, R. Jiang, P. Koskela, A. Sikora, Gradient estimates for heat kernels and harmonic functions, *arXiv:1703.02152*.

[23] T. Coulhon, H.Q. Li, Estimations inférieures du noyau de la chaleur sur les variétés coniques et transformée de Riesz. (French) [Estimates of lower bounds for the heat kernel on conical manifolds and Riesz transform] *Arch. Math.* (Basel) 83 (2004), 229-242.

[24] B. Devyver, A Gaussian estimate for the heat kernel on differential forms and application to the Riesz transform, *Math. Ann.* 358 (2014), 25-68.

[25] B. Devyver, A perturbation result for the Riesz transform, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 14 (2015), 937-964.

[26] N. Dungey, Heat kernel estimates and Riesz transforms on some Riemannian covering manifolds, *Math. Z.* 247 (2004), 765-794.

[27] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, 1994.

[28] A. Grigor’yan, The heat equation on noncompact Riemannian manifolds, (Russian) *Mat. Sb.* 182 (1991), 55-87; translation in Math. USSR-Sb. 72 (1992), 47-77.

[29] A. Grigor’yan, L. Saloff-Coste, Heat kernel on manifolds with ends, *Ann. Inst. Fourier (Grenoble)* 59 (2009), 1917-1997.

[30] A. Grigor’yan, L. Saloff-Coste, Surgery of the Faber-Krahn inequality and applications to heat kernel bounds, *Nonlinear Anal.* 131 (2016), 243-272.

[31] F.W. Gehring, The $L^p$-integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* 130 (1973), 265-277.

[32] C. Guillarmou, A. Hassell, Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. I, *Math. Ann.* 341 (2008), 859-896.

[33] P. Hajłasz, P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.*, 145 (2000).

[34] A. Hassell, A. Sikora, Riesz transforms in one dimension, *Indiana Univ. Math. J.* 58 (2009), 823-852.
[35] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*. New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015. xii+434 pp.

[36] R. Jiang, The Li-Yau inequality and heat kernels on metric measure spaces, *J. Math. Pures Appl.*, (9) 104 (2015), 29-57.

[37] R. Jiang, P. Koskela, D. Yang, Isoperimetric inequality via Lipschitz regularity of Cheeger-harmonic functions, *J. Math. Pures Appl.*, (9) 101 (2014), 583-598.

[38] H.Q. Li, La transformation de Riesz sur les variétés coniques, *J. Funct. Anal.*, 168 (1999), 145-238.

[39] H.Q. Li, J.X. Zhu, A Note on “Riesz Transform for $1 \leq p \leq 2$ Without Gaussian Heat Kernel Bound”, to appear in *J. Geom. Anal*.

[40] J.Y. Li, Gradient estimate for the heat kernel of a complete Riemannian manifold and its applications, *J. Funct. Anal.*, 97 (1991), 293-310.

[41] P. Li, L.F. Tam, Harmonic functions and the structure of complete manifolds, *J. Differential Geom.*, 35 (1992), 359-383.

[42] P. Li, S.T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, 156 (1986), 153-201.

[43] X.D. Li, Riesz transforms on forms and $L^p$-Hodge decomposition on complete Riemannian manifolds, *Rev. Mat. Iberoam.* 26 (2010), 481-528.

[44] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, *I.M.R.N.* (2) (1992) 27-38.

[45] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.*, 36 (1992), 417-450.

[46] Z.W. Shen, Bounds of Riesz transforms on $L^p$ spaces for second order elliptic operators, *Ann. Inst. Fourier (Grenoble)*, 55 (2005), 173-197.

[47] E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Princeton UP, 1970.

[48] R.S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, *J. Funct. Anal.* 52 (1983), 48-79.

[49] N.T. Varopoulos, Analysis on Lie groups, *J. Funct. Anal.* 76 (1988), 346-410.

[50] S.T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.*, 28 (1975), 201-228.

Renjin Jiang
Center for Applied Mathematics
Tianjin University
Tianjin 300072
China
rejiang@tju.edu.cn