A COMPLEX CASE OF VOJTA’S GENERAL ABC CONJECTURE AND
CASES OF CAMPANA’S ORBIFOLD CONJECTURE

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Abstract. We showed a truncated second main theorem of level one with explicit exceptional
sets for analytic maps into $\mathbb{P}^2$ intersecting the coordinate lines with sufficiently high multiplicities.
The proof is based on a GCD theorem for an analytic map $f: \mathbb{C} \to \mathbb{P}^n$ and two homogeneous
polynomials in $n+1$ variables with coefficients which are meromorphic functions of the same
growth as the analytic map $f$. As applications, we studied some cases of Campana’s orbifold
conjecture for $\mathbb{P}^2$ and finite ramified covers of $\mathbb{P}^2$ with three components admitting sufficiently
large multiplicities. Moreover, we explain how to adapt our methods to show the strong Green-
Griffiths-Lang conjecture for a finite ramified covers of $\mathbb{G}_m^2$.

1. Introduction

The Green-Griffiths-Lang conjecture in the non-compact case (see [22, Proposition 15.3]) reads
as follows: If $X$ is a complex smooth projective variety, $D$ is a normal crossing divisor on $X$, and
$X \setminus D$ is a variety of log general type, then a holomorphic map $f: \mathbb{C} \to X \setminus D$ cannot have Zariski-
dense image. Instead of considering a holomorphic map $f: \mathbb{C} \to X$ with image not intersecting
the support of $D$, Campana took into account the the multiplicities of $f$ intersecting the support
of $D$ in [3]. To formulate Campana’s conjecture, let $X$ be a complex smooth projective variety.
Recall that an orbifold divisor $\Delta$ is a linear combination $\sum_{Y \subset X} c_{\Delta}(Y) \cdot Y$, where $Y$ ranges over all
irreducible divisors of $X$, and the orbifold coefficients are rational numbers $c_{\Delta}(Y) \in [0,1] \cap \mathbb{Q}$ such
that all but finitely many are zero. Equivalently,

$$\Delta = \sum_{\{Y \subset X\}} (1 - m_{\Delta}^{-1}(Y)) \cdot Y,$$

where only finitely $m_{\Delta}(Y) \in [1, \infty] \cap \mathbb{Q}$ are larger than 1. An orbifold pair is a pair $(X, \Delta)$, where
$\Delta$ is an orbifold divisor. The pair interpolates between the compact case where $\Delta = \emptyset$ and the
pair $(X, \emptyset) = X$ has no orbifold structure, and the open, or logarithmic case where $c_{\Delta}(D) = 1$ for

2020 Mathematics Subject Classification. Primary 30D35, Secondary 11J97, 30A99.

The first author is supported by National Natural Science Foundation of China NO. 12201643. The second named
is supported in part by Taiwan’s MoST grant 110-2115-M-001-009-MY3.
all $c_\Delta(D) \neq 0$, and we identify $(X, \Delta)$ with $X \setminus \text{Supp}(\Delta)$. Let $D_1, \ldots, D_q$ be irreducible divisors of $X$. Let $m_1, \ldots, m_q \in (1, \infty] \cap \mathbb{Q}$ and $\Delta = (1 - m_1^{-1})D_1 + \cdots + (1 - m_q^{-1})D_q$. Here, consider an orbifold entire curve $f : \mathbb{C} \to (X, \Delta)$, i.e. an entire curve $f : \mathbb{C} \to X$ such that $f(\mathbb{C}) \not\subset \text{Supp}(\Delta)$ and $\text{mult}_t(f^*D_i) \geq m_i$ for all $i$ and all $t \in \mathbb{C}$ with $f(t) \in D_i$. Finally, we say that an orbifold pair $(X, \Delta)$ is of general type if $K_X + \Delta$ is big, where $K_X$ is a canonical divisor on $X$.

Recall the following natural generalization to the orbifold category of the (strong) Green-Griffiths-Lang conjecture.

**Conjecture** (Campana). If $(X, \Delta)$ is an orbifold pair of general type, then there exists a proper closed subvariety $Z \subset X$ containing the images of all nonconstant orbifold entire curves $f : \mathbb{C} \to (X, \Delta)$.

This conjecture has been proved by Brotbek and Deng in [2] for $(X, \Delta)$ with $\Delta$ consisting of only one (general) component and sufficiently large multiplicity, and by Campana, Darondeau and Rousseau in [4] for the case $X = \mathbb{P}^2$ and $\Delta$ consisting of 11 lines with orbifold multiplicity 2. Indeed, they both proved orbifold hyperbolicity, i.e. any orbifold entire curve $f : \mathbb{C} \to (X, \Delta)$ in their situation is constant. It has been shown recently by Rousseau, Turchet and the second author in [17] that the orbifold entire curve $f$ is algebraically degenerate for the case of smooth projective surfaces with $\Delta = \sum_{i=1}^q(1 - \frac{1}{m_i})D_i$, where $m_i$ is sufficiently large for each $i$ and $q \geq 4$ by the intersection criteria imposed on the $D_i$.

The first major purpose of this article is to study Campana’s conjecture for $\mathbb{P}^2$ and its ramified covers with at least three components admitting sufficiently large multiplicities. We now state the results in this direction.

**Theorem 1.1.** Let $\Delta_0$ be an orbifold divisor of $\mathbb{P}^2(\mathbb{C})$ and $H_1, H_2, H_3$ be three distinct lines in $\mathbb{P}^2(\mathbb{C})$. Assume that the support of $\Delta_0$ and $H_1, H_2, H_3$ are in general position. Let $m_i \in (1, \infty] \cap \mathbb{Q}$, $1 \leq i \leq n$, and $\Delta = \Delta_0 + (1 - \frac{1}{m_1})H_1 + (1 - \frac{1}{m_2})H_2 + (1 - \frac{1}{m_3})H_3$. Assume that $\deg \Delta > 3$. Then there exists a proper Zariski closed subset $W$ of $\mathbb{P}^2$ and an effectively computable positive integer $\ell$ such that the image of any nonconstant orbifold entire curve $f : \mathbb{C} \to (\mathbb{P}^2, \Delta)$ with $\min\{m_1, m_2, m_3\} \geq \ell$ must be contained in $W$.

**Remark.**

1. The condition that $\deg \Delta > 3$ is equivalent to that $(\mathbb{P}^2, \Delta)$ is of general type.
2. The proper Zariski closed subset $W$ of $\mathbb{P}^2$ can be constructed explicitly.

**Theorem 1.2.** Let $F_i$, $1 \leq i \leq 3$, be homogeneous irreducible polynomials of positive degrees in $\mathbb{C}[x_0, x_1, x_2]$. Assume that the plane curves $D_i := [F_i = 0] \subset \mathbb{P}^2(\mathbb{C})$, $1 \leq i \leq 3$, intersect
transversally. Let \( m_i \in (1, \infty] \cap \mathbb{Q} \), \( 1 \leq i \leq 3 \) and \( \Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2 + (1 - \frac{1}{m_3})D_3 \).

Suppose that \( \text{deg} \Delta > 3 \). Then there exist two effectively computable positive integers \( \ell \) and \( N \) such that if \( \min\{m_1, m_2, m_3\} \geq \ell \) then the image of any orbifold entire curve \( f : \mathbb{C} \to (\mathbb{P}^2, \Delta) \) is contained in a plane curve of degree bounded by \( N \).

**Theorem 1.3.** Let \( X \) be a complex smooth projective surface of dimension 2 with a finite morphism \( \pi : X \to \mathbb{P}^2 \). Let \( H_i = [x_{i-1} = 0] \), \( 1 \leq i \leq 3 \), be the coordinate hyperplane divisors of \( \mathbb{P}^2 \), and \( D_i \) be the support of \( \pi^*H_i \) (i.e. the sum of the components of \( \pi^*H_i \) counted with multiplicity 1). Let \( m_i \in (1, \infty] \cap \mathbb{Q} \), \( 1 \leq i \leq 3 \) and \( \Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2 + (1 - \frac{1}{m_3})D_3 \). Let \( Z \subset X \) be the ramification divisor of \( \pi \) omitting components from the support of \( \Delta \). Assume that \( \pi(Z) \) does not intersect the set of points \( \{(1,0,0),(0,1,0),(0,0,1)\} \) in \( \mathbb{P}^2 \). If the orbifold pair \((X, \Delta)\) is of log-general type, then there exist two positive integers \( \ell \) and \( N \) such that if \( \min\{m_0, m_1, m_2\} \geq \ell \), then the image any \( f : \mathbb{C} \to (X, \Delta) \) is contained in an algebraic curve in \( X \) with degree bounded by \( N \).

**Remark.** Let \( D = D_1 + D_2 + D_3 \). Recall that \( X \setminus D \) is said to be of log-general type if the divisor \( D + K_X \) is big. It’s clear that \( X \setminus D \) is of log-general type if \( (X, \Delta) \) is of general type. On the other hand, the condition that \( X \setminus D \) is of log-general type implies \( (X, \Delta) \) is of general type if each \( m_i, 1 \leq i \leq 3 \), is sufficiently large. (See [13] [Corollary 2.2.24].)

As noted in the beginning, the theorems above recover the corresponding results for Green-Griffiths-Lang conjecture when \( m_i = \infty \) for \( 1 \leq i \leq 3 \). Indeed, we can modify our proof of Theorem 1.2 (resp. Theorem 1.3 ) to obtain the strong Green-Griffiths-Lang conjecture, i.e. there exists a proper Zariski closed subset \( W \) of \( X \) such that all non-constant entire curves \( f : \mathbb{C} \to X \setminus D \) are contained in \( W \). When \( X = \mathbb{P}^n \), the condition for \( X \setminus D \) to be of log-general type is equivalent to the inequality \( \text{deg} D \geq n + 2 \). When \( D \) has \( n + 1 \) components, the Green-Griffiths-Lang conjecture is verified by Green for \( n = 2 \) with \( \text{deg} D = 4 \) in [8] under the assumption that \( f \) is of finite order, and is solved for general \( n \) by Noguchi, Winkelmann and Yamanoi in [15]. Moreover, the strong Green-Griffiths-Lang conjecture is also achieved in [15] for smooth surfaces of log-general type with a proper finite morphism \( \pi : X \to A \), where \( A \) is a semi-abelian surface. In [9] and [11], the case of \( X = \mathbb{P}^n \) with \( D \) consisting of \( n + 1 \) irreducible hypersurfaces parameterized by small functions, i.e. moving targets of slow growth, are studied for \( \text{deg} D = n + 2 \) and \( \text{deg} D \geq n + 2 \) respectively.

The proofs of our main results related to Campana’s orbifold conjecture are based on the following theorem, which is of its own interest.

**Theorem 1.4.** Let \( G \) be a non-constant homogeneous polynomial in \( \mathbb{C}[x_0, x_1, x_2] \) with no monomial factors and no repeated factors. Let \( H_i = [x_i = 0] \), \( 0 \leq i \leq 2 \), be the coordinate hyperplane divisors of \( \mathbb{P}^2 \), \( m_i \in (1, \infty] \cap \mathbb{Q} \), \( 0 \leq i \leq 2 \), and \( \Delta = (1 - \frac{1}{m_0})H_0 + (1 - \frac{1}{m_1})H_1 + (1 - \frac{1}{m_2})H_2 \). Assume that
the plane curve \([G = 0] \) and \(H_i, 0 \leq i \leq 2, \) are in general position. Then for any \(\epsilon > 0, \) there exists a proper Zariski closed subset \(W \) and effectively computable positive integers \(\ell\) and \(n\) such that for any non-constant orbifold entire curve \(g : \mathbb{C} \rightarrow (\mathbb{P}^2, \Delta)\) with \(\min\{m_0, m_1, m_2\} \geq \ell\) and the image of \(g\) not contained in \(W,\) the following two inequalities hold.

(i) \(N_{G(g)}(0, r) - N_{G(g)}^{(1)}(0, r) \leq \text{exc} \epsilon T_g(r), \) and

(ii) \(N_{G(g)}^{(1)}(0, r) \geq \text{exc} (\deg G - \epsilon) \cdot T_g(r). \)

Furthermore, the exceptional set \(W\) is a finite union of closed subsets of \(\mathbb{P}^2\) of the following type:

\[ x_n^0 x_1^{n_1} x_2^{-n_0 - n_2} = \beta, \]

where \(\beta \in \mathbb{C}\) and \((n_0, n_1)\) is a pair of integers with \(\max\{|n_0|, |n_1|\} \leq n.\)

Remark. It is clear from our proof that the exceptional set \(W\) can be constructed explicitly.

Indeed, the assertion (ii) in Theorem 1.4 is a complex case of Vojta’s general abc conjecture as follows. (See [22, Conjecture 15.2] and [22, Conjecture 23.4].)

Conjecture. Let \(X\) be a smooth complex projective variety, \(D\) be a normal crossing divisor on \(X,\) \(K_X\) be a canonical divisor on \(X,\) and \(A\) be an ample divisor on \(X.\) Then

(a) If \(f : \mathbb{C} \rightarrow X\) is an algebraically nondegenerate analytic map, then

\[ N_f^{(1)}(D, r) \geq \text{exc} T_{K_X + D, f}(r) - o(T_{A, f}(r)). \]

(b) For any \(\epsilon > 0,\) there exists a proper Zariski-closed subset \(Z\) of \(X,\) depending only on \(X, D, A,\) and \(\epsilon\) such that for any analytic map \(f : \mathbb{C} \rightarrow X\) whose image is not contained in \(Z,\) the following

\[ N_f^{(1)}(D, r) \geq \text{exc} T_{K_X + D, f}(r) - \epsilon T_{A, f}(r) \]

holds.

Here, for each positive integer \(n, N_f^{(n)}(D, r)\) is the \(n\)-truncated counting function with respect to \(D\) given by

\[ N_f^{(n)}(D, r) = \sum_{0 < |z| < r} \min\{\text{ord}_z f^* D, n\} \log \frac{r}{|z|} + \min\{\text{ord}_0 f^* D, n\} \log r, \]

\(T_{D, f}(r)\) is the (Nevanlinna) height function relative to the divisor \(D\) (referring to [22, Section 12]), and the notion \(\leq \text{exc}\) means that the estimate holds for all \(r\) outside a set of finite Lebesgue measure.
If we reformulate Theorem 1.4 by taking \( D = [G = 0] + H_0 + H_1 + H_2 \), where \( H_i := [x_i = 0] \), \( 0 \leq i \leq 2 \); then \( K_T + D \) is linearly equivalent to \( [G = 0] \). Since \( g_0, g_1, g_2 \) are entire functions with no common zeros and sufficiently large zero multiplicity \( \ell \), we can see that \( N_{g_i}^{(1)}(H_i, r) = N_g^{(1)}(0, r) \leq \frac{1}{\ell} T_g(r) \). Therefore, the assertion (ii) of Theorem 1.4 implies Eq.(1.2). Moreover, the exceptional set \( W \) in Theorem 1.4 can be constructed explicitly. This also allows us to derive the strong Green-Griffiths-Lang conjecture for Theorem 1.2 and Theorem 1.3 with \( m_i = \infty \), \( 1 \leq i \leq 3 \).

There are many results in this direction with a high truncated level, but very few with level one. Additionally, the ability to construct an explicit exceptional set is quite limited. The following are some known results. First, the conjecture holds for \( \dim X = 1 \). When \( X \) is a semiabelian variety, Noguchi, Winkleman and Yamanoi in [16] showed that the inequality (1.1) holds with \( N_f^{(1)}(D, r) \) replaced by \( N_f^{(k_0)}(D, r) \) for some positive integer \( k_0 \), and (1.2) holds if the map is algebraically nondegenerate. In [2], Brotbek and Deng also proved (1.1) for general hypersurfaces in a smooth projective variety. The above conjecture is much harder for the case of moving targets, i.e. the divisor \( D \) is defined over a field of “small functions” with respect to the map \( f \). The only existing results in the moving case with level one are due to Yamanoi in [26] for \( \dim X = 1 \), and in the joint work [11] of the two authors and Sun, where the inequality (1.2) is derived for complex tori with slowly growth moving targets under the assumption that the map is multiplicatively independent over the small fields.

The proof of Theorem 1.4 is based on the machinery developed in [9] and [11] for complex tori, i.e. \( g = (g_0, \ldots, g_n) \), where the \( g_i \)'s are entire functions without zeros. It is motivated by the work of Corvaja and Zannier in [5]. Consider the following example to explain the proof for Theorem 1.4 (i). Let \( G = x_0^2 + x_1^2 + x_2^3 \) and \( g = (g_0, g_1, g_2) \), where \( g_i \), \( 0 \leq i \leq 2 \), are entire functions without common zeros. Let \( D_\mathbf{g}(G) := 2 \frac{g_0}{g_1} x_0^2 + 2 \frac{g_0}{g_2} x_1^2 + 2 \frac{g_0}{g_2} x_2^3 \). Then \( D_\mathbf{g}(G)(\mathbf{g}) = G(\mathbf{g})' \) and hence \( N_{G(\mathbf{g})}(0, r) - N_{G(\mathbf{g})}^{(1)}(0, r) \leq N_{\gcd(G(\mathbf{g}), D_\mathbf{g}(G)(\mathbf{g}), r)}(\mathbf{g}), \) (See Section 4.1 for definition.) Therefore, the assertion (i) can be achieved by showing that \( N_{\gcd(G(\mathbf{g}), D_\mathbf{g}(G)(\mathbf{g}), r)}(\mathbf{g}), \) \( r \) \( \leq \epsilon T_\mathbf{g}(r) \). When \( g_i \)'s are entire functions without zeros, we have \( T_\mathbf{g}(r) \leq o(T_\mathbf{g}(r)) \) and hence the coefficients of \( D_\mathbf{g}(G) \) are small functions w.r.t. \( g \). In this case, we can apply the GCD theorem established by Levin and the second author in [14]. When the zero multiplicity of each \( g_i \) is at least \( \ell \), then \( T_\mathbf{g}(r) \leq \frac{2}{\ell} T_\mathbf{g}(r) \) and hence the coefficients of \( D_\mathbf{g}(G) \) are in the same growth w.r.t. \( g \). Therefore, we will need to extend the GCD theorem in [14] to the case where the coefficients of the polynomial are in the same growth as the entire curves. In this step, we can conclude (i) under the assumption that \( T_{\left( \frac{m_1}{m_2} \right)}^{m_1}(\frac{m_1}{m_2})^{m_2} \leq \frac{c}{\ell} T_g(r) \) for some computable constants \( m_1, m_2 \) and \( c \) if the zero multiplicity of each \( g_i \) is sufficiently large. The next step is to do an algebraic reduction, which also enables us to find the exceptional sets for Theorem 1.4 explicitly.
Some background materials will be given in the next session. In Section 3, we develop some lemmas to deal with orbifold curves with sufficiently large multiplicities. In Session 4, we formulate a version of Nevanlinna’s second main theorem with moving targets of the same growth and establish the corresponding GCD theorem. The proof of Theorem 1.4 will be given in Section 5, and the proofs of the other theorems will be given in Section 6. Finally, we explain in Section 6.4 how to adapt our proofs of Theorem 1.2 and Theorem 1.3 to show the strong Green-Griffiths-Lang conjecture, i.e. finding exceptional sets under the assumption that the multiplicities $m_i = \infty$, $1 \leq i \leq 3$ in both theorems.

2. Preliminaries

We will give relevant materials and derive some basic results in this session.

2.1. Nevanlinna Theory. We will set up some notation and definitions in Nevanlinna theory and recall some basic results. We refer to [22], [19], and [9] for details.

Let $f$ be a meromorphic function and $z \in \mathbb{C}$ be a complex number. Denote $v_z(f) := \text{ord}_z(f)$,

$v_z^+(f) := \max\{0, v_z(f)\}$, and $v_z^-(f) := -\min\{0, v_z(f)\}$.

$$N_f(\infty, r) = \sum_{0 < |z| \leq r} v_z^-(f) \log \frac{|r_z|}{z} + v_0^-(f) \log r,$$

and

$$N_f^{(Q)}(\infty, r) = \sum_{0 < |z| \leq r} \min\{Q, v_z^-(f)\} \log \frac{|r_z|}{z} + \min\{Q, v_0^-(f)\} \log r.$$  

Then define the counting function $N_f(r, a)$ and the truncated counting function $N_f^{(Q)}(r, a)$ for $a \in \mathbb{C}$ as

$N_f(a, r) := N_1/(f-a)(r, \infty)$ and $N_f^{(Q)}(a, r) := N_1^{(Q)}(f-a)(\infty, r)$.  

The proximity function $m_f(\infty, r)$ is defined by

$$m_f(\infty, r) := \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

where $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$. For any $a \in \mathbb{C}$, the proximity function $m_f(a, r)$ is defined by

$$m_f(a, r) := m_1/(f-a)(\infty, r).$$

The characteristic function is defined by

$$T_f(r) := m_f(\infty, r) + N_f(\infty, r).$$
Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and $(f_0, \ldots, f_n)$ be a reduced representation of $f$, i.e. $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \int_0^{2\pi} \log \max\{|f_0(re^{i\theta})|, \ldots, |f_n(re^{i\theta})|\} \frac{d\theta}{2\pi}.$$ 

This definition is independent, up to an additive constant, of the choice of the reduced representation of $f$.

We will make use of the following elementary inequality.

**Proposition 2.1.** Let $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be holomorphic curve, where $f_0, \ldots, f_n$ are entire functions without common zeros. Then

$$T_{f_i/f_0}(r) + O(1) \leq T_f(r) \leq \sum_{j=0}^n T_{f_j/f_0}(r) + O(1).$$

Recall the following truncated second main theorem due to Ru and the second author.

**Theorem 2.2 ([20, Theorem 2.1]).** Let $f = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with $f_0, \ldots, f_n$ entire and no common zeros. Assume that $f_{n+1}$ is a holomorphic function satisfying the equation $f_0 + \cdots + f_n + f_{n+1} = 0$. If $\sum_{i \in I} f_i \neq 0$ for any proper subset $I \subset \{0, \ldots, n+1\}$, then

$$T_f(r) \leq \text{exc} \sum_{i=0}^{n+1} N_{f_i}^{(n)}(0, r) + O(\log T_f(r)).$$

We will use the following second main theorem for hypersurfaces with truncation and bounded degeneration degree, which can be obtained from [1] easily.

**Theorem 2.3 ([1]).** Let $f$ be a nonconstant holomorphic map of $\mathbb{C}$ into $\mathbb{P}^n$. Let $D_i$, $1 \leq i \leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_i$, in general position. Let $0 < \epsilon < 1$. Then there exist two positive integers $M(\geq O(\epsilon^{-2}))$ and $N(= O(\epsilon^{-1}))$ depend only on $\epsilon$, $n$ and $d_i$, $1 \leq i \leq q$, such that for any holomorphic map of $f : \mathbb{C} \to \mathbb{P}^n$ either the following inequality holds:

$$(q - n - 1 - \epsilon)T_f(r) \leq \text{exc} \sum_{i=1}^q \frac{1}{d_i} N_i^{(M)}(D_i, r),$$

or the image of $f$ is contained in a hypersurface in $\mathbb{P}^n(\mathbb{C})$ with degree bounded by $N$.

3. **Orbifold curves with sufficiently large multiplicities**

We show some basic propositions and develop a version of Borel lemma for orbifold curves with sufficiently large multiplicities.
Proposition 3.1. Let $f$ be a non-constant entire function on $\mathbb{C}$. Suppose that the zero multiplicity of $f$ at each $z \in \mathbb{C}$ is either zero or bigger than $\ell \geq 1$. Then
\[
T_{f'/f}(r) \leq \frac{1}{\ell} T_f(r) + O(\log T_f(r)).
\]

Proof. The assertion follows from the lemma of logarithmic and the following estimate:
\[
N_{f'/f}(\infty, r) = N_f^{(1)}(0, r) \leq \frac{1}{\ell} N_f(0, r) \leq \frac{1}{\ell} T_f(r) + O(1).
\]

Proposition 3.2. Let $f_0, \ldots, f_n$ be non-constant entire functions with no common zeros and the zero multiplicity of each $f_i$ be either zero or bigger than a positive integer $\ell$. Let $u_i = f_i/f_0$ for $1 \leq i \leq n$. Then for any $\alpha \in \mathbb{C}(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0})$, there is a positive constant $c$ independent of $\ell$ such that
\[
T_\alpha(r) \leq c \frac{1}{\ell} T_f(r),
\]
where $\mathbf{f} := (f_0, \ldots, f_n)$.

Proof. Let $\mathbf{u} := \left[ 1 : \frac{u_1}{u_0} : \cdots : \frac{u_n}{u_0} \right]$. Since $\alpha \in \mathbb{C}(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0})$, we may find two coprime homogeneous polynomials $P, Q \in \mathbb{C}[x_0, \ldots, x_n]$ such that $\alpha = P(\mathbf{u})/Q(\mathbf{u})$. Then
\[
T_\alpha(r) \leq T_{P(\mathbf{u})}(r) + T_{Q(\mathbf{u})}(r) \leq (\deg P + \deg Q) T_\mathbf{u}(r).
\]
By Proposition 2.1, we have
\[
T_\mathbf{u}(r) \leq \sum_{j=1}^{n} T_{\frac{u_j}{u_0}}(r) + O(1).
\]
By the lemma of logarithmic, we have
\[
m_{u_0}(\infty, r) \leq \log T_{u_0}(r) \leq O(\log T_f(r)).
\]
For the properties of counting functions, we have
\[
N_{\frac{u_j}{u_0}}(\infty, r) \leq N_{u_j}(0, r) + N_{u_j}(\infty, r) \leq N_{f_j}(0, r) + N_{f_0}^{(1)}(0, r)
\leq \frac{1}{\ell} N_{f_j}(0, r) + \frac{1}{\ell} N_{f_0}(0, r) \leq \frac{2}{\ell} T_f(r).
\]
Therefore, $T_\mathbf{u}(r) \leq 2n T_f(r) + O(\log T_f(r))$, and
\[
T_\alpha(r) \leq 2n (\deg P + \deg Q) \frac{T_f(r)}{\ell} + O(\log T_f(r)).
\]

We also need the following version of the Borel Lemma for orbifold curves.
Lemma 3.3. Let \( f_0, \ldots, f_n \) be non-constant entire functions with no common zeros, and \((a_0, \ldots, a_n) \neq (0, \ldots, 0)\) be an \((n+1)\) tuple of meromorphic functions. Assume that the zero multiplicity of each \( f_i \) is either zero or bigger than a positive integer \( \ell \). Suppose that \( a_0f_0 + a_1f_1 + \ldots + a_nf_n = 0 \). Then for each \( i \) with \( a_i \neq 0 \), there exists \( j \neq i \) such that

\[
T_{f_i/f_j}(r) \leq \text{exc} 3n \cdot T_a(r) + \frac{n^2 - 1}{\ell} T_f(r) + O(\log T_f(r)),
\]

where \( a = [a_0 : \ldots : a_n] \), and \( f := (f_0, \ldots, f_n) \).

Proof. By multiplying an appropriate meromorphic function to each \( a_i \), we may assume that the non-trivial \( a_i, 0 \leq i \leq n \), are entire functions without common zeros. Let \( H_i \) be the coordinate hyperplane defined by \( x_i = 0, 0 \leq i \leq n \). Then

\[
N_a(0, r) = N_a(H_i, r) \leq T_a(r) + O(1).
\]

For a given \( i \) with non-trivial \( a_i \), there exists a vanishing subsum of \( a_0f_0 + \ldots + a_nf_n = 0 \) consisting of the term \( a_if_i \) and without any vanishing proper subsum. By reindexing, we may assume that \( i = 1 \) and this vanishing subsum is

\[
a_0f_0 + \ldots + a_{m-1}f_{m-1} = 0.
\]

If \( m = 1 \), then \( T_{f_1/f_0}(r) = T_{a_0/a_1}(r) \leq T_a(r) \). Therefore we assume that \( m \geq 2 \). Let \( g \) be an entire function such that \( \tilde{f}_0 := f_0/g, \ldots, \tilde{f}_{m-1} := f_{m-1}/g \) are entire functions with no common zeros. Let

\[
\tilde{f} := (\tilde{f}_0, \ldots, \tilde{f}_{m-1}).
\]

Let \( h \) be an entire function such that \( a_0\tilde{f}_0/h, \ldots, a_{m-1}\tilde{f}_{m-1}/h \) are entire functions with no common zeros. Let

\[
F := (a_0\tilde{f}_0/h, \ldots, a_{m-1}\tilde{f}_{m-1}/h).
\]

We can deduce from the definition of characteristic functions that

\[
T_F(r) \leq T_{\tilde{f}}(r) + T_a(r) \leq T_{\tilde{f}}(r) + T_a(r).
\]

On the other hand, we may write \( \tilde{f}_i = \frac{h}{a_i} \cdot \frac{a_i\tilde{f}_i}{h} \). Then

\[
\max_{0 \leq i \leq m-1} \{ \log |\tilde{f}_i| \} \leq \max_{0 \leq i \leq m-1} \{ \log |a_i\tilde{f}_i/h| \} + \log |h| + \max_{0 \leq i \leq m-1} \{ \log 1/|a_i| \}.
\]

Hence,

\[
T_{\tilde{f}}(r) \leq T_F(r) + (m - 1)T_a(r) + N_h(0, r).
\]
Since \( \tilde{f}_0, \ldots, \tilde{f}_{m-1} \) have no common zeros, the zeros of \( h \) must be zeros of some \( a_i, 0 \leq i \leq m-1 \). Therefore,
\[
N_h(0, r) \leq \sum_{i=0}^{m-1} N_{a_i}(0, r) \leq mT_a(r) + O(1)
\]
by (3.3). Consequently, we derive from (3.6) that
\[
T_{\tilde{f}}(r) \leq T_F(r) + (2m - 1)T_a(r) + O(1).
\]

Applying Theorem 2.2 to the map \( F \) with the equation
\[
a_0\tilde{f}_0 + \cdots + a_m\tilde{f}_m = 0,
\]
we have
\[
T_F(r) \leq \text{exc} \sum_{i=0}^{m-1} N_{a_i}(0, r) + O(\log T_F(r))
\]
\[
\leq \text{exc} \sum_{i=0}^{m-1} N_{f_i}(0, r) + \sum_{i=0}^{m-1} N_{f_i}^{(m-1)}(0, r) + O(\log T_F(r)) \quad \text{(as } \tilde{f}_i = f_i/g)\)
\]
\[
\leq \text{exc} \sum_{i=0}^{m-1} N_{f_i}(0, r) + \frac{m^2 - 1}{\ell} T_F(r) + O(\log T_F(r)).
\]
Together with Proposition 2.1, (3.7) and the fact \( m \leq n \), this yields
\[
T_{f_i/f_j}(r) \leq T_{\tilde{f}}(r) + O(1) \leq \text{exc} 3n \cdot T_a(r) + \frac{n^2 - 1}{\ell} T_F(r) + O(\log T_F(r))
\]
for any \( 0 \leq j \leq m \). 

Let \( Q \in \mathcal{M}[x_1, \ldots, x_n] \), where \( \mathcal{M} \) is the field of meromorphic functions. We may express \( Q = \sum_{i \in I_Q} a_i \cdot x^i \), where \( i = (i_1, \ldots, i_n) \), \( x^i = x_1^{i_1} \cdots x_n^{i_n} \), and \( a_i \neq 0 \) if \( i \in I_Q \). We let
\[
\|Q\|_z := \max_{i \in I_A} |a_i(z)|,
\]
and define the characteristic function of \( Q \) as
\[
T_Q(r) := T_{[\cdots a_i : \cdots]}(r),
\]
where \( a_i \) is taken for every \( i \in I_Q \).
Corollary 3.4. Let $f_0, \ldots, f_n$ be non-constant entire functions with no common zeros and the zero multiplicity of each $f_i$ be either zero or larger than a positive integer $\ell$. Let $Q$ be a non-constant homogeneous polynomial in $\mathbb{C}(\frac{u_1}{w_1}, \ldots, \frac{u_n}{w_n})[x_0, \ldots, x_n]$, where $u_i = f_i/f_0$ for $1 \leq i \leq n$. Then

\[(3.12) \quad T_Q(r) \leq \text{exc} \frac{c_1}{\ell} \cdot T_F(r) \quad \text{for some positive constant } c_1 \text{ independent of } \ell,\]

where $f = (f_0, \ldots, f_n)$. Furthermore, if $Q(f_0, \ldots, f_n) = 0$, then there exists a non-trivial $n$-tuple of integers $(j_1, \ldots, j_n)$ with $|j_1| + \cdots + |j_n| \leq 2 \deg Q$ and a positive real $c_2$ independent of $\ell$ such that

\[T_{u_1^{j_1} \cdots u_n^{j_n}}(r) \leq \text{exc} \frac{c_2}{\ell} \cdot T_F(r).\]

Proof. Let $m = \deg Q$ and $Q = \sum_{i \in I_0} a_i \cdot x^i$, where $i = (i_0, \ldots, i_n)$, with $|i| = i_0 + \cdots + i_n = m$, $x^i = x_0^{i_0} \cdots x_n^{i_n}$, and $a_i \neq 0$, if $i \in I_Q$. Let $i_0 \in I_Q$. By Proposition 2.1, we have

\[T_Q(r) \leq \sum_{i \in I_Q} T_{u_1^{i_1} \cdots u_n^{i_n}}(r).\]

Then the first assertion follows from Corollary 3.2.

If $Q(f_0, \ldots, f_n) = 0$, then $\sum_{i \in I_Q} a_i \cdot f_0^{i_0} \cdots f_n^{i_n} = 0$. Let $F = \{f_0^{i_0} \cdots f_n^{i_n}, \ldots\}$, where the index set $i$ runs over all $i_0 + \cdots + i_n = m$. Then the zero multiplicities of $f_0^{i_0} \cdots f_n^{i_n}$ are at least $\ell$ if they are not zero. We note that the entries of $F$ have no common zeros and $T_F(r) \leq mT_F(r) + O(1)$. Then by Lemma 3.3 and (3.12), we find $i \in I_Q$ distinct from $i_0 = (i_0', \ldots, i_n')$ such that

\[(3.13) \quad T_{f_0^{i_0'-i_0} \cdots f_n^{i_n'-i_n}}(r) \leq \text{exc} \frac{c_2}{\ell} \cdot T_F(r),\]

for some positive constant $c_2$ independent of $\ell$. Finally, since $|i| = |i_0|$ and $u_i = f_i/f_0$, we have $f_0^{i_0'-i_0} \cdots f_n^{i_n'-i_n} = u_1^{i_1'-i_1} \cdots u_n^{i_n'-i_n}$ to conclude the second assertion by (3.13). \hfill $\square$

4. GCD THEOREM

4.1. GCD with moving targets of the same growth. Let $f$ and $g$ be meromorphic functions. We let

\[n(f, g, r) := \sum_{|z| \leq r} \min\{v_{z}^{+}(f), v_{z}^{+}(g)\}\]

and

\[N_{\gcd}(f, g, r) := \int_{0}^{r} \frac{n(f, g, t) - n(f, g, 0)}{t} dt + n(f, g, 0) \log r.\]

We will need the following GCD theorem.
Theorem 4.1. Let \( n \geq 2 \) be a positive integer. Let \( g_0, \ldots, g_n \) be non-constant entire functions with no common zeros and the zero multiplicity of each \( g_i \) is either zero or larger than a positive integer \( \ell \). Let \( u_i = g_i/g_0 \) for \( 1 \leq i \leq n \). Let \( F \) and \( G \) be coprime homogeneous polynomials in \( \mathbb{C}(u_1, \ldots, u_n)[x_0, \ldots, x_n] \) such that not both of them are identically zero at \((1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\). Then for any sufficiently small \( \epsilon > 0 \), there exist integers \( N \geq O(\epsilon^{-3}) \) and \( m = O(\epsilon^{-1}) \), both depending only on \( \epsilon \), such that if \( \ell > N \), then we have either

\[
N_{\gcd}(F(g_0, \ldots, g_n), G(g_0, \ldots, g_n), r) \leq \text{exc} \epsilon T_{g}(r),
\]

or

\[
T(z_0)^{m_1} \cdots (z_0)^{m_n} \leq \text{exc} \epsilon^3 T_{g}(r)
\]

for some non-trivial tuple of integers \( (m_1, \ldots, m_n) \) with \( m_1 + \cdots + m_n \leq 2m \), where \( g = (g_0, \ldots, g_n) \).

Let \( K_g := \{ a : a \) is a meromorphic function with \( T_a(r) \leq o(T_g(r)) \} \). In [14], Levin and the second author showed that (4.1) holds for \( F \) and \( G \) with coefficients in \( \mathbb{C} \) or \( K_g \) under the assumption that \( g_i, 0 \leq i \leq n \), are entire functions with no zeros and they are multiplicatively independent over \( \mathbb{C} \) or \( K_g \) respectively. In view of Proposition 3.2, the coefficients of \( F \) and \( G \) is in \( K_g := \{ a : a \) is a meromorphic function with \( T_a(r) \leq O(T_g(r)) \} \), which is a field of the same growth w.r.t. \( g \).

Therefore, we need to develop a Nevanlinna’s second main theorem with moving targets of the same growth in order to adapt the proof in [14].

4.2. Nevanlinna Theory with Moving Targets of the Same Growth. In view of the second main theorem for function fields with moving targets developed in [24] and [25], it comes naturally to develop a Nevanlinna’s second main theorem with moving targets of the same growth. We will follow the ideas in [24]. Let \( f = (f_0, \ldots, f_n) \) be a holomorphic map from \( \mathbb{C} \) to \( \mathbb{P}^n \) where \( f_0, f_1, \ldots, f_n \) are entire functions without common zeros. Let \( a_0, \ldots, a_n \) be meromorphic functions in \( K_f \), and \( L := a_0X_0 + \cdots + a_nX_n \). Then \( L \) defines a hyperplane \( H \) in \( \mathbb{P}^n \), over the field \( K_f \). We note that \( H(z) \) is the hyperplane determined by the linear form \( L(z) = a_0(z)X_0 + \cdots + a_n(z)X_n \) for \( z \in \mathbb{C} \) that is not a common zero of \( a_0, \ldots, a_n \), or a pole of any \( a_k, 0 \leq k \leq n \). The definition of the Weil function, proximity function and counting function can be easily extended to moving hyperplanes. For example,

\[
\lambda_H(z)(P) = -\log \frac{|(ha_0)(z)x_0 + \cdots + (ha_n)(z)x_n|}{\max\{|x_0|, \ldots, |x_n|\} \max\{|(ha_0)(z)|, \ldots, |(ha_n)(z)|\}},
\]
where $h$ is a meromorphic function such that $ha_0, \ldots, ha_n$ are entire functions without common zeros, $P = (x_0, \ldots, x_n) \in \mathbb{P}^n(\mathbb{C})$ and $z \in \mathbb{C}$. It’s clear that

$$\lambda_{H(z)}(P) = -\log \frac{|a_0(z)x_0 + \cdots + a_n(z)x_n|}{\max\{|x_0|, \ldots, |x_n|\} \max\{|a_0(z)|, \ldots, |a_n(z)|\}}, \quad (4.3)$$

for $z \in \mathbb{C}$ which is not a common zero of $a_0, \ldots, a_n$, or a pole of any $a_k$, $0 \leq k \leq n$. The first main theorem for a moving hyperplane $H$ can be stated as

$$T_\ell(r) = N_\ell(H, r) + m_\ell(H, r) + T_n(r) + O(1), \quad (4.4)$$

where $a := [a_0 : \cdots : a_n]$.

We will reformulate the second main theorem with moving targets stated in [19, Theorem A6.2.1] to the situation where the coefficients of the underlying linear forms are in $\mathbb{K}_x$. Let $a_{j_0}, \ldots, a_{j_n} \in \mathbb{K}_x$ and let $L_j := a_{j_0}x_0 + \cdots + a_{j_n}x_n$. Without loss of generality, we will normalize the linear forms $L_j$, $1 \leq j \leq q$, such that for each $1 \leq j \leq q$, there exists $0 \leq j' \leq n$ such that $a_{j'} = 1$. Let $t$ be a positive integer and let $V(t)$ be the complex vector space spanned by the elements

$$\left\{ \prod a_{j_k}^{n_j} : n_j \geq 0, \sum n_j = t \right\},$$

where the products and sums run over $1 \leq j \leq q$ and $0 \leq k \leq n$. Let $1 = b_1, \ldots, b_u$ be a basis of $V(t)$ and $b_1, \ldots, b_w$ a basis of $V(t+1)$. It’s clear that $u \leq w$. Moreover, we have

$$\liminf_{t \to \infty} \frac{\dim V(t+1)}{\dim V(t)} = 1. \quad (4.5)$$

**Definition 4.2.** Let $E$ be a $\mathbb{C}$-vector space spanned by finitely many meromorphic functions. We say that an analytic map $f = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n$ is linearly nondegenerate over $E$ if whenever we have a linear combination $\sum_{i=1}^m a_i f_i = 0$ with $a_i \in E$, then $a_i = 0$ for each $i$; otherwise we say that $f$ is linearly degenerate over $E$.

The following formulation of the second main theorem with moving targets in $\mathbb{K}_x$ follows from the proof of [19, Theorem A6.2.1] by adding the Wronskian term and computing the error term explicitly when applying the second main theorem.

**Theorem 4.3.** Let $f = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve where $f_0, f_1, \ldots, f_n$ are entire functions without common zeros. Let $H_j$, $1 \leq j \leq q$, be arbitrary (moving) hyperplanes given by $L_j := a_{j_0}x_0 + \cdots + a_{j_n}x_n$ where $a_{j_0}, \ldots, a_{j_n} \in \mathbb{K}_x$. Denote by $W$ the Wronskian of $\{hb_m f_k | 1 \leq m \leq w, 0 \leq k \leq n\}$, where $h$ is a meromorphic function such that $hb_1, \ldots, hb_w$ are...
entire functions without common zeros. If $f$ is linearly non-degenerate over $V(t + 1)$, then for any $\varepsilon > 0$, we have the following inequality:

$$
\int_0^{2\pi} \max_j \sum_{k \in J} \lambda_{H_k(re^{i\theta})}(f(re^{i\theta})) \frac{d\theta}{2\pi} + \frac{1}{u} N_W(0, r)
\leq \text{exc} \left(\frac{w}{u}(n + 1) + \varepsilon\right) T(r) + \frac{w}{u}(t + 2) \max_{1 \leq j \leq q} T_{a_j}(r),
$$

where $w = \dim V(t + 1)$, $u = \dim V(t)$, $a_j = [a_{j0} : \cdots : a_{jn}]$ and the maximum is taken over all subsets $J$ of $\{1, \ldots, q\}$ such that $H_j(re^{i\theta})$, $j \in J$, are in general position.

4.3. The key theorem and the proof of Theorem 4.1. The following is the key theorem to prove Theorem 4.1.

**Theorem 4.4.** Let $n \geq 2$ be a positive integer. Let $g_0, \ldots, g_n$ be entire functions without common zeros and let $g = (g_0, \ldots, g_n)$. Let $F, G$ be coprime homogeneous polynomials in $n + 1$ variables of the same degree $d > 0$ over $\mathbb{K}_g$. Assume that one of the coefficients in each expansion of $F$ and $G$ is 1. Let $I$ be the set of exponents $i$ such that $x^i$ appears with a nonzero coefficient in either $F$ or $G$. For every positive integer $t$, we denote by $V_{F,G}(t)$ the (finite-dimensional) $\mathbb{C}$-vector space spanned by $\prod_0 \alpha^{n_\alpha}$, where $\alpha$ runs through all non-zero coefficients of $F$ and $G$, $n_\alpha \geq 0$ and $\sum n_\alpha = t$; we also put $d_i := \dim V_{F,G}(t)$. For every integer $m \geq d$, we let $M = M_{m,n,d} = 2\binom{m+n-d}{n} - \binom{m+n-2d}{n}$. Then there exists a positive real $c$ and a positive integer $b$ such that if the set $\{g_0^{i_0} \cdots g_n^{i_n} : i_0 + \cdots + i_n = m\}$ is linearly non-degenerate over $V_{F,G}(Mb + 1)$, then we have that the following estimate.

$$
MN_{\text{gcd}}(F(g), G(g), r) \leq \text{exc} \left(\frac{M'}{d_{M(b-1)}} M - M\right) m n T_g(r)
+ c_{m,n,d} \sum_{i=0}^n N_{g_i}^{(L)}(0, r) + \frac{m}{n+1} - c_{m,n,d} M' m \sum_{i=0}^n N_{g_i}(0, r)
+ \binom{m+n-2d}{n} N_{\text{gcd}}(\{g_i^1\}_{i \in I}, r) + c(T_F(r) + T_G(r)) + O(\log T_g(r)),
$$

where $c_{m,n,d} = 2\binom{m+n-d}{n+1} - \binom{m+n-2d}{n+1}$, $M'$ is an integer of order $O(m^{n-2})$, $L = \frac{1}{2} M(M - 1)c_{m,n,d}^{-1}$ and $c = \frac{d_{M(b-1)}}{d_{M(b-1)}(1 + M(b + 1))}.$

**Proof.** Most of the proof is identical to the one of [14, Theorem 5.7], except that it is necessary to estimate the characteristic functions of the linear forms when apply Theorem 4.3. We will point out the important differences and omit the proof which is similar to the one of [14, Theorem 5.7]. We also refer to [10, Theorem 26] for the structure of the proof and the explicit computations of all constants.
Since one of the coefficients in the expansion of \( F \) and \( G \) is 1, from the definition (3.10) we have

\[
\log \| F \|_z \geq 0 \quad \text{and} \quad \log \| G \|_z \geq 0
\]

for each \( z \in \mathbb{C} \) which is neither a zero nor a pole of the coefficients of \( F \) and \( G \). Let \((F,G)\) be the ideal generated by \( F \) and \( G \) in \( K_g[x] \). If \((F,G) = 1\), then it is elementary to show that

\[
N_{grd}(F(g), G(g), r) \leq c(T_F(r) + T_G(r)),
\]

where \( c \) is a positive constant independent of \( g \). Therefore, we assume that ideal \((F,G)\) is proper in \( K_g[x] \). Let \( (F,G)_m := K_g[x]_m \cap (F,G) \), where \( K_g[x]_m := \{ P \in K_g[x] : P \text{ is a homogeneous polynomial of degree } m \} \).

We choose \( \{\phi_1, \ldots, \phi_M\} \) to be a basis of the \( K_g \)-vector space \((F,G)_m\) consisting of elements of the form \( Fx^1, Gx^1 \). For each \( z \in \mathbb{C} \), we can construct a basis \( B_z \) of \( V_m := K_g[x]_m/(F,G)_m \) with monomial representatives \( x^1, \ldots, x^{1^m} \) as in the proof of [14, Theorem 5.7]. Let \( I_z := \{i_1, \ldots, i_{1^m}\} \).

For each \( i \in I_z \), there is a linear form

\[
L_e := \sum_{j \in I_z} b_{e,j}y_j \in K_g[y_1, \ldots, y_M]
\]

such that

\[
L_e (\Phi(x)) = \sum_{j \in I_z} c_{e,j}x_j \in (F,G)_m
\]

for some choice of \( c_{e,i,j} \in K_g \). Then for each such \( i \), there is a linear form

\[
\phi_\ell = \sum_{j \notin I_z, |j| = m} a_{\ell,j}x^j + \sum_{j \in I_z} a_{\ell,j}x^j,
\]

where \( a_{\ell,j} \in K^*_g \) will be chosen later. By the choice of \( \phi_\ell \), we may write

\[
\Phi := (\phi_1, \ldots, \phi_M). \quad (4.10)
\]

Combining (4.8) and (4.10), we have

\[
L_e (\Phi(x)) = \sum_{\ell=1}^M b_{e,\ell} \left( \sum_{j \notin I_z, |j| = m} a_{\ell,j}x^j + \sum_{j \in I_z} a_{\ell,j}x^j \right).
\]
Next we define the $M \times M$ matrices

$$A_z := (\alpha_{z,\ell,j})_{1 \leq \ell \leq M \atop j \notin I_z, |j| = m} \quad \text{and} \quad B_z := (b_{z,i,\ell})_{1 \leq \ell \leq M \atop i \notin I_z, |i| = m}.$$

By comparing (4.9) with (4.11), we see that $B_z = c_z A_z^{-1}$. From now on, we let $c_z := \det A_z$, which is in $V_{F,G}(M)$. Then

$$b_{z,i,\ell} \in V_{F,G}(M-1)$$

for each $|i| = m$, $i \notin I_z$ and $1 \leq \ell \leq M$ by Cramer’s rule. This comparison also gives

$$c_z = \sum_{\ell=1}^{M} b_{z,i,\ell} \alpha_{z,\ell,i} \in V_{F,G}(M) \quad \text{for each } |i| = m, i \notin I_z, \text{ and}$$

$$c_z c_{z,j} = \sum_{\ell=1}^{M} b_{z,i,\ell} \alpha_{z,\ell,j} \in V_{F,G}(M) \quad \text{for each } |i| = m, i \notin I_z \text{ and for each } j \in I_z.$$

Then from the proof of [14, Theorem 5.7], (4.6), (4.8), (4.9) and (4.13), we obtain

$$|L_{z,i}(\Phi(g(z)))| \leq \log |g(z)| + \max_{j \in I_z} \{\log |c_z(z)|, \log |c_z(z)c_{z,j}(z)|\} + O(1)$$

$$\leq \log |g(z)| + \max_{\ell} \log |b_{z,i,\ell}(z)| + \log \|F\|_z + \log \|G\|_z + O(1),$$

which gives the following key inequality for computing the corresponding Weil functions

$$|L_{z,i}(\Phi(g(z)))| - \|L_{z,i}\|_z \leq \log |g(z)| + \log \|F\|_z + \log \|G\|_z + O(1).$$

To apply Theorem 4.3, we first note that the coefficients of $L_{z,i}$ are in $V := V_{F,G}(M)$ by (4.13). Since the set $\{g_0^{i_0} \cdots g_n^{i_n} : i_0 + \cdots + i_n = m\}$ is linearly non-degenerate over $V_{F,G}(Mb+1)$, we must have that $\Phi(g) : \mathbb{C} \to \mathbb{P}^{M-1}$ is linearly nondegenerate over $V_{F,G}(Mb) = V(b)$ (as in Theorem 4.3). We also note that when computing Weil function (4.3), we may assume that the coefficients of $L_{z,i}$ are entire functions without common zeros. Then by the construction, for each $z \in \mathbb{C}$, the hyperplanes defined by evaluating the linear forms $L_{z,i}$ at $z$ with $|i| = m$ and $i \notin I_z$ are in general position.

The other part of the arguments is similar to the proof of [14, Theorem 5.7], so we omit the details. \qed
Proof of Theorem 4.1. Let \( \alpha \) and \( \beta \) be one of the nonzero coefficients of \( F \) and \( G \) respectively. Since 
\[
v_z^+(F(g)) \leq v_z^+(\frac{1}{\alpha} F(g)) + v_z^+(\alpha) \text{ and } v_z^+(G(g)) \leq v_z^+(\frac{1}{\beta} G(g)) + v_z^+(\beta)
\]
for each \( z \in \mathbb{C} \), we have
\[
N_{S, \gcd}(F(g), G(g)) \leq N_{S, \gcd}(\frac{1}{\alpha} F(g), \frac{1}{\beta} G(g)) + N_\alpha(0, r) + N_\beta(0, r)
\]
for some constant \( c_1 \). Therefore, we will assume that one of the coefficients in each expansion of 
\( F \) and \( G \) is 1. Recall that \( M := M_m := 2(m+n-d) - (m+n-2d) \) and \( M' := M'_m := (m+n) - M = O(m^{n-2}) \). Elementary computations give that
\[
\binom{m+n}{n} = \frac{m^n}{n!} + \frac{(n+1)m^{n-1}}{2(n-1)!} + O(m^{n-2}),
\]
\[
c_{m,n,d} = \frac{m^{n+1}}{(n+1)!} + \frac{m^n}{2(n-1)!} + O(m^{n-1}),
\]
\[
M = \frac{m^n}{n!} + O(m^{n-1}).
\]
Then
\[
\frac{m}{n+1} \left( \frac{m+n}{n} \right) - c_{m,n,d} = O(m^{n-1}).
\]
Let \( \epsilon > 0 \) be given. Due to the above estimates, we may choose \( m = O(\epsilon^{-1}) \) so that \( m \geq 2d \),
\[
\frac{M'mn}{M} \leq \frac{\epsilon}{4} \quad \text{and} \quad \frac{1}{M} \left( \frac{m}{n+1} \left( \frac{m+n}{n} \right) - c_{m,n,d} - M'm \right) \leq \frac{\epsilon}{4(n+1)}.
\]
By (4.5) we may then choose a sufficiently large integer \( b \in \mathbb{N} \) such that
\[
\frac{w}{u} - 1 \leq \frac{\epsilon}{4mn},
\]
where \( w := \dim_k V_{F,G}(Mb) \) and \( u := \dim_k V_{F,G}(Mb - M) \). Suppose that the set \( \{g_0, \ldots, g_n : i_0 + \cdots + i_n = m\} \) is linearly non-degenerate over \( V_{F,G}(Mb + 1) \). Then by Theorem 4.4, we have
\[
N_{\gcd}(F(g), G(g), r) \leq \text{exc} \left( \frac{M'}{M}mn + \left( \frac{w}{u} - 1 \right)mn \right) T_g(r)
\]
\[
+ \frac{1}{M} \left( \frac{m}{n+1} \left( \frac{m+n}{n} \right) - c_{m,n,d} - M'm \right) \sum_{i=0}^{n} N_{g_i}(0, r) + \frac{c_{m,n,d}}{M} \sum_{i=0}^{n} N_{g_i}^{(L)}(0, r)
\]
\[
+ \frac{1}{M} \left( \frac{m+n-2d}{n} \right) N_{\gcd}((g^i)_{i \in I}, r) + \frac{c}{M}(T_F(r) + T_G(r)).
\]
We note that the assumption that not both $F$ and $G$ vanish at any of the points of $\{[1 : 0 : \cdots : 0], [0 : \cdots : 0 : 1]\}$ implies that $(d, 0, \ldots, 0), \ldots, (0, \ldots, 0, d) \in I$. As $g_0, \ldots, g_n$ have no common zeros, we have $N_{\gcd}\{g^i\}_{i \in I, r} = 0$. By (4.15), (4.16) and that

\begin{align*}
(4.18) \quad N_{g_i}(0, r) &\leq T_g(r) \quad \text{for } 0 \leq i \leq n,
\end{align*}

we see that the sum of first two terms of the right hand side of (4.17) is bounded by $\frac{3\epsilon}{4}T_g(r)$. By Corollary 3.4, $T_F(r) + T_G(r) \leq \text{exc } \frac{c_2}{\ell} T_g(r)$ for some positive constant $c_2$. In conclusion, we derive from (4.17) that

\begin{align*}
N_{\gcd}(F(g), G(g), r) &\leq \text{exc } \frac{3\epsilon}{4}T_g(r) + m \sum_{i=0}^n N_{g_i}^{(L)}(0, r) + \frac{c_3}{M\ell}T_g(r) \\
&\leq \frac{3\epsilon}{4}T_g(r) + \frac{mL}{\ell} \sum_{i=0}^n N_{g_i}(0, r) + \frac{c_3}{M\ell}T_g(r) \\
&\leq \left( \frac{3\epsilon}{4} + \frac{m(n+1)L}{\ell} + \frac{c_3}{M\ell} \right)T_g(r) \quad \text{(by (4.18))}
\end{align*}

where $c_3 = c \cdot c_2$. Let $N \geq O(\epsilon^{-3})$ be an integer greater than $4((n+1)mL + c_3/M)\epsilon^{-1}$. Then for $\ell \geq N$, we have

\begin{align*}
N_{\gcd}(F(g), G(g), r) &\leq \text{exc } \epsilon T_g(r).
\end{align*}

Finally, if the set $\{g_0^{i_0} \cdots g_n^{i_n} : i_0 + \cdots + i_n = m\}$ is linearly degenerate over $V_{F,G}(Mb + 1)$, then there exists a homogeneous polynomial $A \in V_{F,G}(Mb + 1)[x_0, \ldots, x_n]$ of deg $A = m$ such that $A(g_0, \ldots, g_n) = 0$. Hence we may apply Corollary 3.4 to derive that there exists a non-trivial $n$-tuple of integers $(j_1, \ldots, j_n)$ with $|j_1| + \cdots + |j_n| \leq 2m$ and a positive real $c_4$ depending only on $A$ such that

\begin{align*}
T_{(g_1^{j_1} \cdots g_n^{j_n})n}(r) &\leq \text{exc } \frac{c_4}{\ell} T_g(r).
\end{align*}

Since we may enlarge $N$ whenever necessary, we can conclude the proof by taking $\ell \geq N \geq c_4 \cdot \epsilon^{-3}$. \quad \square

5. Proofs of Theorem 1.4

5.1. Polynomials over a ring with logarithmic differentials. Let $g = (g_0, \ldots, g_n)$, where $g_0, \ldots, g_n$ are non-constant entire functions without common zeros. Recall that

\begin{align*}
K_g := \{a : a \text{ is a meromorphic function with } T_a(r) \leq O(T_g(r))\},
\end{align*}

which is a field of the same growth w.r.t. $g$. We note $a' \in K_g$ if $a \in K_g$. Let $u_i = g_i/g_0$, for $0 \leq i \leq n$, and $u = (1, u_1, \ldots, u_n)$. Since $T_{u_i}(r) \leq T_g(r)$, we see that $u_i, u'_i$ and $u'_i/u_i$ are all in $K_g$. 

Let \( x := (x_0, \ldots, x_n) \). For \( i = (i_0, \ldots, i_n) \in \mathbb{Z}^n \), we let \( x^i := x_0^{i_0} \cdots x_n^{i_n} \) and \( u^i := u_0^{i_0} \cdots u_n^{i_n} \).

For a non-constant polynomial \( F(x) = \sum_i a_i x^i \in \mathbb{K}_g[x] := \mathbb{K}_g[x_0, \ldots, x_n] \), we define

\[
D_u(F)(x) := \sum_i \frac{(a_i u^i)^j}{u^i} x^i = \sum_i (a_i^j + a_i \cdot \sum_{j=1}^n i_j \frac{a_j'}{a_j}) x^i \in \mathbb{K}_g[x].
\]

A direct computation shows that

\[
F(u)^j = D_u(F)(u),
\]

and that the following product rule

\[
D_u(FG) = D_u(F)G + FD_u(G)
\]

holds for \( F, G \in \mathbb{K}_g[x] \). The following lemma is a modification of [9, Lemma 3.1].

**Lemma 5.1.** Let \( F \) be a nonconstant homogeneous polynomial in \( \mathbb{K}_g[x] \) with no monomial factors and no repeated factors. Then \( F \) and \( D_u(F) \) are coprime in \( \mathbb{K}_g[x] \) unless there exists a non-trivial tuple of integers \( (m_1, \ldots, m_n) \) with \( \sum_{i=1}^n |m_i| \leq 2 \deg F \) such that \( T_{u_1^{m_1} \cdots u_n^{m_n}} \leq T_F(r) \).

**Proof.** Let \( F = F_1 \cdots F_k \), where \( F_i, 1 \leq i \leq k \), are irreducible (homogeneous) polynomials in \( \mathbb{K}_g[x] \). By (5.3), we have

\[
D_u(F) = D_u(F_1)F_2 \cdots F_k + \cdots + F_1 \cdots F_{k-1}D_u(F_k).
\]

If \( F \) and \( D_u(F) \) are not coprime in \( \mathbb{K}_g[x] \), then the irreducible common factors are among the irreducible polynomials \( F_1, \ldots, F_k \in \mathbb{K}_g[x] \). Assume that \( F_j, 1 \leq j \leq k \), is a common factor. Then \( D_u(F_j) \) is divisible by \( F_j \). Write \( F_j = \sum_i b_i x^i \in \mathbb{K}_g[x] \), which contains at least two distinct terms, since \( F \) has no monomial factors. Then \( D_u(F_j)/F_j \) is a non-zero constant in \( \mathbb{K}_g \) since \( D_u(F_j) \) is not zero and \( \deg D_u(F_j) = \deg F_j \). Comparing the coefficients of \( F_j \) and \( D_u(F_j) \), for nonzero \( b_i \) and \( b_j \) in \( \mathbb{K}_g \) (\( i \neq j \)), we have from (5.1) that

\[
\frac{(b_i u^i)^j}{b_i u^i} = \frac{(b_j u^j)^j}{b_j u^j},
\]

where \( i = (i_0, i_1, \ldots, i_n) \), \( j = (j_0, j_1, \ldots, j_n) \) and \( \sum_{h=0}^n i_h = \sum_{h=0}^n j_h = \deg F_j \). It implies that \( \frac{b_i u^i}{b_j u^j} \in \mathbb{C} \). Therefore,

\[
T_{u_1^{i_1-j_1} \cdots u_n^{i_n-j_n}}(r) \leq T_{F_j}(r) \leq T_F(r).
\]

\( \square \)
5.2. A General Theorem. For convenience of discussion, we denote by $E$ the collection of entire functions. For a positive integer $\ell$, we let

$$\mathcal{E}_\ell := \{g \in E \setminus \{0\} : \text{the zero multiplicity of } g \text{ at each } z \in \mathbb{C} \text{ is at least } \ell \text{ if } g(z) = 0\}. $$

It’s clear that $f \cdot g \in \mathcal{E}_\ell$ if $f, g \in \mathcal{E}_\ell$.

We first show the following.

**Theorem 5.2.** Let $G$ be a non-constant homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ with no monomial factors and no repeated factors. Assume that the hypersurface defined by $G$ in $\mathbb{P}^n$ and the coordinate hyperplanes are in general position. Let $g_0, g_1, \ldots, g_n$ be entire functions in $E_\ell$ with no common zeros. Let $g = (g_0, g_1, \ldots, g_n) : \mathbb{C} \to \mathbb{P}^n$. Then for any sufficiently small $\epsilon > 0$, there exist positive integers $\ell_1 \geq O(\epsilon^{-3})$ and $\ell_2 \geq O(\epsilon^{-1})$ independent of $g$ such that if $\ell \geq \ell_1$, then either there is a non-trivial $n$-tuple $(i_1, \ldots, i_n)$ of integers with $\sum_{j=1}^n |i_j| \leq \ell_2$ such that

$$T_{(x_0^{i_1}, \ldots, x_0^{i_n})} \leq \epsilon^2 T_g(r),$$

or the following holds.

(i) $N_{G(G)}(0, r) - N_{G(G)}^{(1)}(0, r) \leq \epsilon \cdot T_g(r)$, and

(ii) $N_{G(G)}(0, r) \geq \epsilon \cdot (\deg G - \epsilon) \cdot T_g(r)$.

**Proof.** We first note that if $G(g) = 0$, then we have (5.6) following Corollary 3.4 if $\ell \geq \ell_1 \geq \epsilon^{-3}$ for some positive constant $c$. Therefore, it suffices to consider (i) when $G(g)$ is not identically zero. Let $u_i = g_i / g_0$, $0 \leq i \leq n$ and $u = (1, u_1, \ldots, u_n)$. We may assume that each $u_i$, $1 \leq i \leq n$, is not constant, otherwise $T_{u_i}(r) = O(1)$. Let $d = \deg G$. Then $G(g) = g_0^d G(u)$, $D_u(G) = D_u(G)(u)$ and

$$G(g') = d g_0' g_0^{d-1} G(u) + g_0^d G(u)' = d g_0' g_0 G(u) + D_u(G)(g)$$

by (5.2).

Let $z_0 \in \mathbb{C}$. If $v_{z_0}(G(g)) \geq 2$, it follows from (5.7) that $v_{z_0}(D_u(G)(g)) \geq v_{z_0}(G(g)) - 1$ since $v_{z_0}(G(g')) = v_{z_0}(G(g)) - 1$ and $g_0'/g_0$ has only at worst simple poles. Hence,

$$\min\{v_{z_0}^+(G(g)), v_{z_0}^+(D_u(G)(g))\} \geq v_{z_0}^+(G(g)) - \min\{1, v_{z_0}^+(G(g))\}$$

for any $z_0 \in \mathbb{C}$. Consequently,

$$N_{gcd}(G(G), D_u(G)(g), r) \geq N_{G(G)}(0, r) - N_{G(G)}^{(1)}(0, r).$$

On the other hand, by Lemma 5.1, $G$ and $D_u(G)$ are coprime in $\mathbb{K}_g[x] := \mathbb{K}_g[x_0, \ldots, x_n]$ or (5.6) holds with $\sum_{j=1}^n |i_j| \leq 4d$. Since the coefficients of $G$ are in $\mathbb{C}$, we have that $D_u(G)(g) \in
$\mathbb{C}(\frac{u_1}{u}, \ldots, \frac{u_n}{u_n})[x]$ and that $G$ and $D_u(G)$ are coprime in $\mathbb{C}(\frac{u_1}{u}, \ldots, \frac{u_n}{u_n})[x]$. Furthermore, since $[G = 0]$ and the coordinate hyperplanes are in general position, we see that $G(P) \neq 0$ for $P \in \{(1,0,\ldots,0), \ldots, (0,0,\ldots,1)\}$. Therefore, we can apply Theorem 4.1 to obtain that for any sufficiently small $\epsilon > 0$, there exist integers $\ell_1 \geq O(\epsilon^{-3})$ and $\ell_2 = O(\epsilon^{-1})$ such that either (5.6) holds, or

$$N_{gcd}(G(g), D_u(G(g)), r) \leq exc \epsilon T_g(r).$$

Then (5.8) implies that

$$N_G(g, 0, r) - N_{G}^{(1)}(0, r) \leq exc \epsilon T_g(r).$$

This proves the first assertion.

Since $[G = 0]$ and the coordinate hyperplanes are in general position, it follows from Theorem 2.3 with polynomials $G, x_0, \ldots, x_n$ that there exist positive integers $M \geq O(\epsilon^{-2})$ and $D = O(\epsilon^{-1})$ depending only on $\epsilon$ and $\deg G$ such that either the image of $g$ is contained in a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree bounded by $D$, which again imply (5.6) by Corollary 3.4 (enlarge $\ell_1$ if necessary), or

$$(1 - \frac{\epsilon}{2d})T_g(r) \leq exc \sum_{i=0}^{n} N_{g_i}^{(M)}(0, r) + \frac{1}{d}N_G(g, 0, r).$$

Hence

$$N_G(g, 0, r) \geq exc d(1 - \frac{\epsilon}{2d})T_g(r) - \frac{dM}{\ell} \sum_{i=0}^{n} N_{g_i}(0, r)$$

$$\geq (d - \frac{\epsilon}{2})T_g(r) - \frac{dM(n + 1)}{\ell}T_g(r).$$

Thus, we obtain the second assertion by taking $\ell_1 \geq 2dM(n + 1)\epsilon^{-1}$. \hfill \Box

5.3. **Proof of Theorem 1.4.** The proof of Theorem 1.4 is the combination of Theorem 5.2 for $n = 2$ and the following proposition.

**Proposition 5.3.** Let $G \in \mathbb{C}[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial with no monomial factors and no repeated factors. Assume that the plane curve $[G = 0]$ and $H_i = [x_i = 0]$, $0 \leq i \leq 2$, are in general position. Let $\epsilon$ be a sufficiently small positive real and $n_1, n_2$ be integers not both zeros such that $|n_i| \leq O(\epsilon^{-1})$. Then there exists a proper Zariski closed subset $W \subset \mathbb{P}^2(\mathbb{C})$ and an effectively computable positive integer $\ell \geq O(\epsilon^{-3})$ such that for any nonconstant orbifold entire curve $g = (g_0, g_1, g_2) : \mathbb{C} \to (\mathbb{P}^2, \Delta)$, where $\Delta = (1 - \frac{1}{m_1})H_0 + (1 - \frac{1}{m_2})H_1 + (1 - \frac{1}{m_3})H_2$ with $g_0 \neq 0, m_1, m_2, m_3 \geq \ell$, and

$$T^{(x_1)^{n_1}(x_2)^{n_2}}(r) \leq \epsilon^2 T_g(r),$$

(5.11)
we have the following two inequalities

\[ N_{G(\mathfrak{g})}(0, r) = N_{G(\mathfrak{g})}^{(1)}(0, r) \leq \text{exc } \epsilon T_\mathfrak{g}(r), \quad \text{and} \]

\[ N_{G(\mathfrak{g})}^{(1)}(0, r) \geq \text{exc } (\deg G - \epsilon) \cdot T_\mathfrak{g}(r), \]

if the image of \( \mathfrak{g} \) is not contained in \( W \). Furthermore, the exceptional set \( W \) is a finite union of closed subsets of the following types: \([x_1^{n_1}x_2^{n_2} = \beta x_0^{n_1+n_2}] \), if \( n_1 \geq 0 \), and \([x_2^{n_2}x_0^{-n_1-n_2} = \beta x_1^{-n_1}] \), if \( n_1 < 0 \).

**Remark.** For each pair \((n_1, n_2)\), the proper Zariski closed subset \( W \) can be constructed explicitly. Moreover, we have only finitely many choices of \( n_1 \) and \( n_2 \) for a given \( \epsilon \) when apply Theorem 5.2.

**Proof of Theorem 1.4.** Let \( \epsilon > 0 \). We may assume that \( \epsilon \) is sufficiently small. We first note that we only need to consider the nonconstant holomorphic map \( \mathfrak{g} = (g_0, g_1, g_2) : \mathbb{C} \to \mathbb{P}^2 \) with \( g_i, 0 \leq i \leq 2 \), not identically zero by including the coordinate line \([x_i = 0] \), \( 0 \leq i \leq 2 \), to \( W \). By Theorem 5.2, we find positive integers \( \ell_1 \geq O(\epsilon^{-3}) \) and \( \ell_2 = O(\epsilon^{-1}) \) independent of \( \mathfrak{g} \) such that for any nonconstant holomorphic map \( \mathfrak{g} = (g_0, g_1, g_2) : \mathbb{C} \to \mathbb{P}^2 \), where \( g_0, g_1, g_2 \in \mathcal{E}_{\ell_1} \) with no common zeros, we have either both (i) and (ii) are valid or (5.11) in Proposition 5.3 holds for some integers \( n_1 \) and \( n_2 \) whose absolute values are bounded by \( \ell_2 \). We then apply Proposition 5.3 for the latter situation for each possible pair of \((n_1, n_2)\) to conclude the proof. \( \square \)

5.4. **Proof of Proposition 5.3.** To prove Proposition 5.3, we need the following version of Hilbert Nullstellensatz reformulated from [23, Chapter XI] or [12, Chapter IX, Lemma 3.7].

**Proposition 5.4.** Let \( A \) be a ring. Let \( \{Q_i\}_{i=1}^{n+1} \) be a set of homogeneous polynomials in \( A[x_0, \ldots, x_n] \) such that their zero locus are in general position. Then there exists a positive integer \( s, R \in A \setminus \{0\} \) and \( P_{ji} \in A[x_0, \ldots, x_n], 1 \leq i, j \leq n+1 \), such that

\[ x_j^s \cdot R = \sum_{i=1}^{n+1} P_{ji}Q_i \]

for each \( 0 \leq j \leq n \).

**Proof of Proposition 5.3.** Suppose that (5.11) holds. Let \( u_1 = g_1/g_0 \) and \( u_2 = g_2/g_0 \). We may assume that \( n_1 \) and \( n_2 \) are coprime and let \( u_1^{n_1}u_2^{n_2} = \lambda \). Then there exist integers \( a \) and \( b \) such that \( n_1a + n_2b = 1 \) and

\[ T_\lambda(r) \leq \epsilon^3 T_\mathfrak{g}(r). \]

Since \( T_\lambda(r) = T_{\lambda-1}(r) \), we may exchange the sign of \( n_1 \) and \( n_2 \) simultaneously. Moreover, we can also rearrange the indices of \( u_1 \) and \( u_2 \). Therefore, we may assume that \( n_2 \geq n_1 \geq 0 \) if \( n_1n_2 \geq 0 \) and \( 0 < n_2 \leq -n_1 \) if \( n_1n_2 < 0 \). It’s clear that \( n_2 > 0 \) in this setting as \((n_1, n_2) \neq (0, 0)\). For the
Let $\beta = u_1^a u_2^{-a}$. We may write

\begin{equation}
(5.13) \quad u_1 = \lambda^a \beta^n \quad \text{and} \quad u_2 = \lambda^{b-\beta^n}.
\end{equation}

Then $T_\beta(r) = T_{[1:a_1:b_2]}(r) = T_{[1:a_2:b_n;\lambda^n\beta^{-n}]}(r)$. As we have set $n_2 > 0$, it suffices to consider for $n_1 \geq 0$ and $n_1 < 0$. From the definition of the characteristic function and that $|a| + |b| < |n_1| + |n_2|$, we have

\begin{equation}
(5.14) \quad T_\beta(r) = T_{[1:a_1:b_2;\lambda^n\beta^{-n}]}(r) \leq \max\{|n_1|, |n_2|\} T_\beta(r) + (|n_1| + |n_2|) T_\lambda(r)
\end{equation}

if $n_1 < 0$; and

\begin{equation}
(5.15) \quad T_\beta(r) = T_{[\lambda^n\beta^{-n};\lambda^n\beta^{-n}]}(r) \leq (n_1 + n_2) T_\beta(r) + (n_1 + n_2) T_\lambda(r)
\end{equation}

if $n_1 \geq 0$. Let

\begin{equation}
(5.16) \quad A = X^{n_1} Y^{n_2} \quad \text{and} \quad T = X^b Y^{-a}
\end{equation}

be two variables. Then

\begin{equation}
(5.17) \quad X = \Lambda^a T^{n_2} \quad \text{and} \quad Y = \Lambda^b T^{-n_1}.
\end{equation}

Let $G_1(X, Y) = G(1, X, Y)$. Let $B_\Lambda(T) \in \mathbb{C}[\Lambda, \Lambda^{-1}][T]$ be the polynomial such that $B_\Lambda(0) \neq 0$ and

\begin{equation}
(5.18) \quad G_1(X, Y) = G_1(\Lambda^a T^{n_2}, \Lambda^b T^{-n_1}) = T^{M_1} B_\Lambda(T)
\end{equation}

for some integer $M_1$. Let $B(\Lambda, T) \in \mathbb{C}[\Lambda, T]$ be the polynomial such that $B(0, T) \neq 0$ and

\begin{equation}
(5.19) \quad B(\Lambda, T) = \Lambda^{M_2} B_\Lambda(T)
\end{equation}

for some integer $M_2$. Then

\begin{equation}
(5.20) \quad G_1(X, Y) = G_1(\Lambda^a T^{n_2}, \Lambda^b T^{-n_1}) = T^{M_1} B_\Lambda(T) = T^{M_1} \Lambda^{M_2} B(\Lambda, T),
\end{equation}

and $B(\Lambda, 0) \in \mathbb{C}[\Lambda]$ is not identically zero. Therefore, there are at most finite $\gamma_1, \ldots, \gamma_s \in \mathbb{C}$ such that $B(\gamma_i, 0) = 0$ for $1 \leq i \leq s$.

We note that $B(\Lambda, T)$ cannot be constant as $G$ has no monomial factors. We now claim that $B(\Lambda, T) \in \mathbb{C}[\Lambda, T]$ is square free. We first rewrite (5.20) as

\begin{equation}
G_1(X, Y) = X^{bM_1+n_1M_2} Y^{-aM_1+n_2M_2} B(X^{n_1} Y^{n_2}, X^b Y^{-a}).
\end{equation}
If \( B_0(\Lambda, T) \) is an irreducible factor of \( B(\Lambda, T) \) in \( \mathbb{C}[\Lambda, T] \), then \( B_0(X^{n_1}Y^{n_2}, X^bY^{-a}) = X^{\ell_1}Y^{\ell_2}H(X, Y) \), where \( H(X, Y) \in \mathbb{C}[X, Y] \), \( H(X, 0) \neq 0 \), \( H(0, Y) \neq 0 \). If \( H(X, Y) \) is a constant \( \alpha \), then \( B_0(\Lambda, T) = B_0(X^{n_1}Y^{n_2}, X^bY^{-a}) = \alpha X^{\ell_1}Y^{\ell_2} \). We may express \( B_0(\Lambda, T) = \sum a_{(i_1, i_2)}\Lambda^{i_1}T^{i_2} \). Then the above equation implies that \( n_1i_1 + bi_2 = \ell_1 \) and \( n_2i_1 - ai_2 = \ell_2 \) and hence \( (i_1, i_2) = (a\ell_1 + b\ell_2, n_2\ell_1 - n_1\ell_2) \), if \( a_{(i_1, i_2)} \neq 0 \). This implies that \( B_0(\Lambda, T) \) is a monomial, which is not possible. Therefore, \( \deg H \geq 1 \).

Then \( H \) is not a monomial and it is a non-constant factor of \( G \). Since \( G \) is square-free, we conclude that \( B_0^2(\Lambda, T) \) is not a factor of \( B(\Lambda, T) \). In conclusion, \( B(\Lambda, T) \in \mathbb{C}[\Lambda, T] \) is square free.

Since \( B(\Lambda, T) \) is square free, the resultant \( R(B_\Lambda, B_\Lambda') \) of \( B_\Lambda \) and \( B_\Lambda'(T) \) is a Laurent polynomial in \( \mathbb{C}[\Lambda, \Lambda^{-1}] \), not identically zero. Let

\[
(5.21) \quad \alpha_i, \ 1 \leq i \leq t, \text{ be the zeros of the resultant } R(B_\Lambda, B_\Lambda').
\]

It is clear that \( \alpha_i \in \mathbb{C} \). Let \( B(T) := B_\Lambda(T) \in \mathbb{C}[\Lambda, \Lambda^{-1}]|T] \), the specialization of \( B_\Lambda(T) \) at \( \Lambda = \lambda \). Then \( B(T) \) has no multiple factors in \( \mathbb{C}[\lambda, \lambda^{-1}]|T] \) if \( \lambda \neq \alpha_i \) for any \( 1 \leq i \leq t \).

From now on, we assume that \( \lambda \neq \alpha_i \) for any \( 1 \leq i \leq t \) and \( \lambda \neq \gamma_j \) for any \( 1 \leq j \leq s \). Let \( \tilde{B} \in \mathbb{C}(\lambda)|Z, U \) be the homogenization of \( B \), i.e. \( \tilde{B}(1, T) = B(T) \). Let \( D_{(1, \beta)}(\tilde{B}) \in \mathbb{C}(\lambda, \lambda', \frac{\partial}{\partial \lambda'})|Z, U \) be as defined in (5.1) with \( u = (1, \beta) \). By Lemma 5.1, \( \tilde{B} \) and \( D_{(1, \beta)}(\tilde{B}) \) are coprime homogeneous polynomials in \( \mathbb{C}(\lambda, \lambda', \frac{\partial}{\partial \lambda'})|Z, U \) unless there exists a non-zero integer \( k \) such that \( T_{\beta^k}(r) \leq T_B(r) \), which implies

\[
(5.22) \quad T_{\beta}(r) \leq T_B(r) \leq (|a| + |b|) \deg G \cdot T_\lambda(r) \leq (|n_1| + |n_2|) \deg G \cdot T_\lambda(r).
\]

Together with (5.14) and (5.15), we have

\[
(5.23) \quad T_\mathbf{g}(r) \leq (|n_1| + |n_2|)(T_{\beta}(r) + T_\lambda(r)) \leq (|n_1| + |n_2|)(|n_1| + |n_2|) \deg G + 1)T_\lambda(r) \leq 2(|n_1| + |n_2|)^2 \deg G \cdot \ell T_\mathbf{g}(r)
\]

by (5.12). This is not possible since \( |n_1| + |n_2| \leq O(\epsilon^{-1}) \). Therefore, we will assume that \( \tilde{B} \) and \( D_{(1, \beta)}(\tilde{B}) \) are coprime homogeneous polynomials in \( \mathbb{C}(\lambda, \lambda', \frac{\partial}{\partial \lambda'})|Z, U \).

Let \( \beta = \beta_1/\beta_0 \), where \( \beta_0 \) and \( \beta_1 \) are entire functions without common zeros. Moreover, we have \( a < b \) and \( b > 0 \) in our setting. Then by (5.2) we have

\[
(5.24) \quad \tilde{B}(\beta_0, \beta_1)' = (\beta_0^{\deg B} B(\beta))' = \deg B \cdot \frac{\partial}{\beta_0} \tilde{B}(\beta_0, \beta_1) + D_{(1, \beta)}(\tilde{B})(\beta_0, \beta_1);
\]

and by (5.20) we have

\[
(5.25) \quad G(\mathbf{g}) := G(g_0, g_1, g_2) = g_0^d G_1(u_1, u_2) = g_0^d \beta^{M_1} B(\beta) = g_0^d \beta^{M_1} \beta_0^{-\deg B} \tilde{B}(\beta_0, \beta_1).
\]
Next, we prove the following.

Claim. There exists a proper Zariski closed set $W_1$ of $\mathbb{P}^2(\mathbb{C})$, independent of $g$ such that

\[(5.26) \quad N_{G(g)}(0, r) - N_{G_{(1)}}(0, r) \leq \text{exc} N_{\gcd(B)}(\beta_0, \beta_1), D(1, \beta)(\tilde{B})(\beta_0, \beta_1), r), \]

if the image of $g$ is not contained in $W_1$.

The condition that $[G = 0]$ is in general position with the coordinate hyperplanes of $\mathbb{P}^2$ implies that $G(1, X, Y) := G(1, X, Y)$ can be expanded as

\[(5.27) \quad G_1(X, Y) = a_0 + a_1 X^d + a_2 Y^d + \cdots \in \mathbb{C}[X, Y],\]

where $a_i \neq 0$, $0 \leq i \leq 2$. Then

\[(5.28) \quad G_1(\lambda^a T^{n_2}, \lambda^b T^{-n_1}) = a_0 + a_1 \lambda^{ad} T^{n_2} + a_2 \lambda^{bd} T^{-n_1} + \cdots.\]

We note that if $G(g(z_0)) = 0$ and $g_0(z_0) = 0$, then $g_i(z_0) \neq 0$ for $z_0 \in \mathbb{C}$, and $i = 1, 2$. This can be seen easily. For example, if $G(g(z_0)) = g_0(z_0) = g_1(z_0) = 0$, then $g_2(z_0) \neq 0$ and $G(0, 0, 1) = 0$, which contradicts the assumption that $[G = 0]$ and the coordinate hyperplanes of $\mathbb{P}^2$ are in general position.

We first consider when $n_1 < 0$ and $n_2 \neq -n_1$. Then $B(T) = G_1(\lambda^a T^{n_2}, \lambda^b T^{-n_1})$, which is a polynomial in $\mathbb{C}[\lambda, \lambda^{-1}][T]$ of degree $d \cdot \max\{-n_1, n_2\}$. By (5.25), we have

\[(5.29) \quad G(g) = g_0^d B(\beta) = g_0^d \beta^{-\text{deg}B} \tilde{B}(\beta_0, \beta_1).\]

To show (5.26), it suffices to consider when $v_{z_0}(G(g)) \geq 2$. In this case, we have

$$v_{z_0}(D(1, \beta)(\tilde{B})(\beta_0, \beta_1)) \geq v_{z_0}(\tilde{B}(\beta_0, \beta_1)) - 1 \geq v_{z_0}(G(g)) - 1$$

by (5.24); and

$$v_{z_0}(G(g)) \leq v_{z_0}(\tilde{B}(\beta_0, \beta_1)) + v_{z_0}(g_0^d \beta^{-\text{deg}B})$$

by (5.29). We will need to show that $v_{z_0}(g_0^d \beta^{-\text{deg}B}) = 0$ if $v_{z_0}(G(g)) \geq 2$. This is clear if $v_{z_0}(g_0) = 0$. Therefore, we assume in addition that $v_{z_0}(g_0) > 0$. Then $v_{z_0}(g_1) = v_{z_0}(g_2) = 0$ in this case as noted before. Consequently, $v_{z_0}(\beta_0) = (b - a)v_{z_0}(g_0) \geq v_{z_0}(g_0)$, since $\beta = u_1^b u_2^{-a} = g_1^b g_2^{-a} g_0^{-b}$ and $a < b$. As $\text{deg}B = d \cdot \max\{-n_1, n_2\} \geq d$, we conclude that $v_{z_0}^+(g_0^d \beta^{-\text{deg}B}) = 0$. This shows our claim (5.26) for this case.

Next, we consider when $n_1 < 0$ and $n_2 = -n_1$. Since we assume that $n_1$ and $n_2$ are coprime, we have $n_1 = -1$ and $n_2 = 1$ and $u_1^{-1} u_2 = \lambda$. Therefore, we can simply consider the pair $(u_1, \lambda u_1) = (u_1, u_2)$, i.e. taking $a = 0$, $b = 1$ and $\beta = u_1$. Consider the following expansion of $G$

\[(5.30) \quad G_1(X, Y) = \tilde{G}_d(X, Y) + \cdots + \tilde{G}_1(X, Y) + a_0 \in \mathbb{C}[X, Y],\]
where \( G_i(X,Y) \) is a homogeneous polynomial of degree \( i \) in \( \mathbb{C}[X,Y] \) and \( a_0 \neq 0 \). Expand

\[
\tilde{G}_d(X,Y) = (X - \delta_1 Y) \cdots (X - \delta_d Y).
\]

Then \( G_d(u_1, \lambda u_1) \neq 0 \) and \( B(T) = G_1(T, \lambda T) \) is a polynomial of degree \( d \) with \( B(0) = a_0 \neq 0 \) if \( \lambda \neq \delta_i, 1 \leq i \leq d \). Then

\[
G(g) = g_0^d G_1(u_1, u_2) = g_0^d B(\beta) = g_0^d \beta_0^{-d} \tilde{B}(\beta_0, \beta_1).
\]

Therefore the claim of (5.26) holds similarly in this case.

Finally, it remains to consider \( n_1 \geq 0 \). Then it follows from (5.28) that \( G_1(\lambda^a T^{n_2}, \lambda^b T^{n_1}) = T^{-n_1 d} \cdot B(T) \), where \( B(T) \) is a polynomial of degree \( (n_1 + n_2)d \) with \( B(0) \neq 0 \). Therefore,

\[
G(g) = g_0^d \beta^{-n_1 d} B(\beta) = g_0^d \beta^{-n_1 d} \beta_0^{n_1 + n_2 d} \tilde{B}(\beta_0, \beta_1) = g_0^d \beta_0^{-n_2 d} \beta_1^{-n_1 d} \tilde{B}(\beta_0, \beta_1).
\]

Similar to the previous arguments, it suffices to show that \( v_{z_0}^+(g_0^d \beta_0^{-n_2 d} \beta_1^{-n_1 d}) = 0 \) if \( v_{z_0}(G(g)) \geq 2 \) and \( v_{z_0}(g_0) > 0 \). This can be done as \( v_{z_0}(g_1) = v_{z_0}(g_2) = 0 \) and \( v_{z_0}(\beta_0) = (b-a) v_{z_0}(g_0) \geq v_{z_0}(g_0) \).

By (5.26), to complete the proof of (i), it remains to show the following:

\[
N_{\gcd}(\tilde{B}(\beta_0, \beta_1), D_{(1,\beta)}(\tilde{B})(\beta_0, \beta_1), r) \leq \epsilon_T g(r).
\]

Let \( A := \mathbb{C}[\lambda, \lambda^{-1}, \lambda', \frac{\partial}{\partial \beta}] \). Since \( \tilde{B} \) and \( D_{(1,\beta)}(\tilde{B}) \) are coprime homogeneous polynomials in \( A[Z, U] \), we may apply Proposition 5.4 to find an integer \( s, R \in A \setminus \{0\} \) and \( F_1, F_2, P_1, P_2 \in A[Z, U] \) such that

\[
Z^s \cdot R = F_1 \tilde{B} + F_2 D_{(1,\beta)}(\tilde{B}) \quad \text{and} \quad U^s \cdot R = P_1 \tilde{B} + P_2 D_{(1,\beta)}(\tilde{B}).
\]

As the coefficients of \( F_1, F_2, P_1, P_2 \) are in \( A = \mathbb{C}[\lambda, \lambda^{-1}, \lambda', \frac{\partial}{\partial \beta}] \), by multiplying some entire functions \( \eta_1 \) and \( \eta_2 \) on the both sides of the equations of (5.35), we may assume that the coefficients of \( F_i \) and \( P_i \) are entire functions. Then by evaluating (5.35) at \( (\beta_0, \beta_1) \), we have

\[
\beta_0^s \cdot R \eta_1 = F_1(\beta_0, \beta_1) \tilde{B}(\beta_0, \beta_1) + F_2(\beta_0, \beta_1) D_{(1,\beta)}(\tilde{B})(\beta_0, \beta_1),
\]

\[
\beta_0^s \cdot R \eta_2 = P_1(\beta_0, \beta_1) \tilde{B}(\beta_0, \beta_1) + P_2(\beta_0, \beta_1) D_{(1,\beta)}(\tilde{B})(\beta_0, \beta_1).
\]

Since \( \beta_0 \) and \( \beta_1 \) have no common zeros, we see that

\[
\min\{v_2^+(\tilde{B}(\beta_0, \beta_1)), v_2^+(D_{(1,\beta)}(\tilde{B})(\beta_0, \beta_1))\} \leq v_2^+(R) + v_2^+(\eta_1) + v_2^+(\eta_2)
\]

for each \( z \in \mathbb{C} \). Therefore,

\[
N_{\gcd}(\tilde{B}(\beta_0, \beta_1), D_{(1,\beta)}(\tilde{B})(\beta_0, \beta_1), r) \leq N_R(0, r) + N_{\eta_1}(0, r) + N_{\eta_2}(0, r).
\]
Since $\beta = u_1^a u_2^{-a} = g_2^a g_1^{-a} g_2^{a-b}$, we have $\frac{d\beta'}{\phi} = b \frac{d_1}{g_1} - a \frac{d_2}{g_2} + (a - b) \frac{d_3}{g_0}$. Hence

$$T_{\frac{\beta'}{\phi}}(r) \leq \sum_{i=0}^{2} T_{\frac{\beta_i}{\phi}}(r) + O(1) \leq \text{exc } \frac{3}{\ell} T_{\frac{c}{\phi}}(r) + O(1) \leq \epsilon^3 T_{\frac{c}{\phi}}(r)$$

by Proposition 3.1 and taking $\ell > 3\epsilon^{-3}$. We also note that $T_{\chi}(r) \leq \epsilon^3 T_{\phi}(r)$ and $T_{\lambda}(r) \leq \epsilon^3 T_{\phi}(r)$. Therefore, for any $\alpha \in A$, we have $T_{\alpha}(r) \leq c_\alpha \epsilon^3 T_{\phi}(r)$, where $c_\alpha$ is a positive constant independent of $\epsilon$ if $\ell > 3\epsilon^{-3}$. Since $\eta_1$ and $\eta_2$ are chosen such that the coefficients of $\eta_1 F_1, \eta_1 F_2, \eta_2 P_1, \eta_2 P_2$ are entire functions, we may assume that $N_{\eta_i}(0, r) \leq N_{Q_i}(\infty, r), i = 1, 2$, for some $Q_i \in A = \mathbb{C}[\lambda, \lambda^{-1}, \lambda', \frac{d_3}{\phi}]$.

Then for $i = 1, 2$, we have

$$(5.39) \quad N_{\eta_i}(0, r) \leq N_{Q_i}(\infty, r) \leq T_{Q_i}(r) \leq c_1 \epsilon^3 T_{\phi}(r)$$

for some constant $c_1$ independent of $\epsilon$. Then we derive from (5.38) that

$$N_{\text{gcd} \{ \hat{B}(\beta_0, \beta_1), D(1, \beta) \hat{B}(\beta_0, \beta_1), r \}} \leq \text{exc } c_2 \epsilon^3 T_{\phi}(r)$$

for some constant $c_2$. Since $c_2$ is independent of $\epsilon$ and $\epsilon$ is sufficiently small, we can now conclude (i).

To prove (ii), we first express $B(T) = \sum_{i \in I_B} b_i(\lambda) T^i \in \mathbb{C}[\lambda, \lambda^{-1}][T]$, where $b_i \neq 0$ if $i \in I_B$. Then

$$(5.40) \quad \hat{B}(\beta_0, \beta_1) - \sum_{i \in I_B} b_i(\lambda) \beta_0^{d_0^{a-b} - 1} \beta_1^i = 0,$$

where $d_B := \deg B$. If some proper sub-sum of (5.40) vanishes, then we have $\sum_{i \in I_C} b_i(\lambda) \beta^i = 0$, for some index subset $I_C$ of $I_B$. Let $k$ be the largest integer in $I_C$. Then $\beta^k + \sum_{i \in I_C \setminus \{k\}} b_i(\lambda) b_k^{-1}(\lambda) \beta^i = 0$. Hence, by [19, Theorem A3.1.6] and Proposition 2.1,

$$T_{\beta}(r) \leq \sum_{i \in I_C \setminus \{k\}} T_{\frac{\beta_i}{\phi}}(r) + O(1) \leq k T_{\beta}(r) + O(1) \leq d_B T_{\beta}(r) + O(1),$$

which leads to a contradiction by the same arguments for (5.23). Therefore, we can assume that no proper sub-sum of (5.40) vanishes. Therefore, we may apply Theorem 2.2 to (5.40) by noting that $d_B$ and 0 are in $I_B$ to obtained the following

$$N_{\hat{B}(\beta_0, \beta_1)}(0, r) + \sum_{i \in I_B} N_{\beta_0^{d_0^{a-b} - 1} \beta_1^i}(0, r) \geq \text{exc } d_B T_{\beta}(r) - \sum_{i \in I_B} N_{b_i(\lambda)}(0, r) \geq d_B T_{\beta}(r) - c_3 d_B T_{\lambda}(r),$$

(5.41)
for some positive constant $c_3$, independent of $\epsilon$. On the other hand, since the zeros of $\beta_0$ and $\beta_1$ come from the zeros of $g_i$, $0 \leq i \leq 2$, for any nonnegative integers $A$ and $B$ and a positive integer $n$, we have

\begin{equation}
N^{(n)}_{\beta_0, \beta_1}(0, r) \leq \sum_{i=0}^{2} N^{(n)}_{g_i}(0, r) \leq \frac{n}{\ell} \sum_{i=0}^{2} N_{g_i}(0, r) \leq \frac{3n}{\ell} T_g(r).
\end{equation}

Then by (5.41) and (5.42), we have

\begin{equation}
N^{(db)}_{B(\beta_0, \beta_1)}(0, r) \geq \text{exc } d_B T_\beta(r) - c_3 d_B^2 T_\lambda(r) - \frac{3d_B^2}{\ell} T_g(r).
\end{equation}

We now treat the case that $n_1 < 0$. We note that in this case $d_B = \max\{-n_1, n_2\} \cdot d = -n_1 d$ if we assume that $\lambda \neq \delta_i$ as in (5.31). In other words, the image of $\mathbf{g}$ is not contained in $[x_1 - \delta, x_2 = 0]$, $1 \leq i \leq d = \deg G$. By (5.29), we have

\begin{equation}
N^{(db)}_{G(\mathbf{g})}(0, r) \geq N^{(db)}_{B(\beta_0, \beta_1)}(0, r) - N^{(db)}_{g_1}(0, r) \geq N^{(db)}_{g_1}(0, r) - \frac{3d_B^2}{\ell} T_g(r)
\end{equation}

by (5.42). Since $d_B = \max\{-n_1, n_2\} \cdot d = -n_1 d$ in this case, from (5.14) we have

\[d_B T_\beta(r) \geq d T_g(r) - 2d|n_1| T_\lambda(r).\]

Thus, we can derive from (5.43) and (5.44) that

\begin{equation}
N_{G(\mathbf{g})}(0, r) \geq \text{exc } \left(d - \frac{4d^2 n_1^2}{\ell}ight) \cdot T_g(r) - (c_3 + 2) d|n_1| T_\lambda(r).
\end{equation}

Since $\epsilon$ is sufficiently small, $|n_1| + |n_2| \leq O(\epsilon^{-1})$ and $T_\lambda(r) \leq \epsilon^3 T_g(r)$, by choosing $\ell > O(\epsilon^{-3})$, we arrive at

\begin{equation}
N_{G(\mathbf{g})}(0, r) \geq \text{exc } (1 - \epsilon) d \cdot T_g(r).
\end{equation}

Together with (i), we have $N^{(1)}_{G(\mathbf{g})}(0, r) \geq \text{exc } (1 - 2\epsilon) d \cdot T_g(r)$.

For the case $n_1 \geq 0$, we have $d_B = (n_1 + n_2) d$, and from (5.33) we have

\begin{equation}
N^{(db)}_{G(\mathbf{g})}(0, r) \geq N^{(db)}_{B(\beta_0, \beta_1)}(0, r) - N^{(db)}_{g_1}(0, r) - \frac{6d_B^2}{\ell} T_g(r)
\end{equation}

by (5.42). The rest of the argument to derive (5.46) which we omit is similar to the previous one.

Finally, we note that the exceptional set $W$ consists of two types as follows. The first type is $[x_1^n x_2^{n_2} = \beta x_0^{n_1 + n_2}]$, if $n_1 \geq 0$; $[x_2^n x_0^{n_1 - n_2} = \beta x_1^{n_1}]$, if $n_1 < 0$, where $\beta \in \{\alpha_1, \ldots, \alpha_t, \gamma_1, \ldots, \gamma_s\}$ is a zero of the resultant defined in (5.21) or a (possible) zero of $B(\Lambda, 0)$. The second type is of the form $[x_1 - \delta_j x_2 = 0]$, where $\delta_j \in \mathbb{C}$ are defined in (5.31) with $n_1 = -1$ and $n_2 = 1$. \square
6. **Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3**

Theorem 1.1 is a direct consequence of Theorem 1.4. The proof of Theorem 1.3 is inspired by the arguments of Corvaja and Zannier in [6] for function fields.

6.1. **Proof of Theorem 1.1.**

*Proof of Theorem 1.1.* After a linear change of variables, we may assume that the $H_1, H_2, H_3$ are the coordinate hyperplanes of $\mathbb{P}^2$. We note that $\deg \Delta > 3$ implies $\Delta_0$ is not trivial. Therefore, we may express $\Delta_0 = (1 - \frac{1}{m_1})D_1 + \ldots + (1 - \frac{1}{m_q})D_q$, where $D_1, \ldots, D_q$ be distinct irreducible curves in $\mathbb{P}^2(\mathbb{C})$ and $n_i \in (1, \infty] \cap \mathbb{Q}$. For a non-constant orbifold entire curve $f : \mathbb{C} \to (\mathbb{P}^2, \Delta)$, we have $f(\mathbb{C}) \not\subseteq |\Delta|$ and $\mult_i(f^*D_i) \geq 2$ for all $1 \leq i \leq q$ and all $t \in \mathbb{C}$ with $f(t) \in D_i$ and $\mult_i(f^*H_j) \geq m_j$ for all $1 \leq j \leq 3$ and all $t \in \mathbb{C}$ with $f(t) \in H_j$. Let $f = (f_0, f_1, f_2)$ be a reduced form. Then the zero multiplicity of $f_i$, $0 \leq i \leq 2$, is at least $m_i$ if it is not zero. Let $D_i = [G_i = 0]$, $1 \leq i \leq q$, where $G_i \in \mathbb{C}[x_0, x_1, x_2]$ is irreducible. Let $G = G_1 \cdots G_q$. Then the zero multiplicity of $G(f) := G(f_0, f_1, f_2)$ at any $z_0 \in \mathbb{C}$ is either zero or at least 2. Hence,

\[(6.1)\quad N^{(1)}_{G(f)}(0, r) \leq \frac{1}{2} N_{G(t)}(0, r) \leq \frac{1}{2} \deg f \cdot T_f(r) + O(1) .\]

To apply Theorem 1.4, we let $0 < \epsilon < \frac{1}{3}$. Then there exists a proper Zariski closed subset $W$ and a positive integer $\ell$ independent of $f$ such that if $m_i \geq \ell$ and the image of $f$ is not contained in $W$, we have

\[N^{(1)}_{G(f)}(0, r) \geq \epsilon (\deg G - \epsilon) \cdot T_f(r) .\]

Together with (6.1), it yields

\[\frac{1}{2} \deg f \cdot T_f(r) \leq \epsilon \cdot T_f(r) + O(1) ,\]

which is not possible since $\epsilon < \frac{1}{3}$. This shows that the image of $f$ is contained in $W$. \qed

6.2. **Proof of Theorem 1.2.**

*Proof of Theorem 1.2.* Denote by $d_i$ the degree of the irreducible homogeneous polynomial $F_i \in \mathbb{C}[x_0, x_1, x_2]$, $D_i := [F_i = 0]$ for $1 \leq i \leq 3$, and $\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2 + (1 - \frac{1}{m_3})D_3$. The condition that $\deg \Delta > 3$ implies that $\sum_{i=1}^3 \deg F_i \geq 4$. Since the hypersurfaces $D_1, D_2, D_3$ intersect transversally, they do not have a common zero and the association $P \mapsto [F_1^a(P) : F_2^a(P) : F_3^a(P)]$, where $a_i := \gcd(d_1, d_2, d_3)/d_i$, defines a finite morphism $\pi : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$.

It is well-known that the ramification divisor of $\pi$ is the zero locus of the determinant $J \in \mathbb{C}[x_0, x_1, x_2]$ of the Jacobian matrix

\[\left( \frac{\partial F_i^a}{\partial x_j} \right)_{1 \leq i \leq 3, 0 \leq j \leq 2} .\]
of \( \pi \). Our plan is to show that there exists an irreducible factor \( \tilde{G} \) of \( J \) in \( \mathbb{C}[x_0, x_1, x_2] \) such that the corresponding hypersurfaces of \( \tilde{G}, D_1, D_2, D_3 \) are in general position. Furthermore, we will show that \( \tilde{G}(f) \) has very few zeros and hence conclude that the image of \( f \) is contained in a hypersurface of bounded degree in \( \mathbb{P}^2(\mathbb{C}) \) by applying Theorem 2.3 for the hypersurface defined by \( \tilde{G}, F_1, F_2, F_3 \).

Observing that \( J \) has a factor \( G \in \mathbb{C}[x_0, x_1, x_2] \) which denotes the determinant of

\[
M := \left( \frac{\partial F_i}{\partial x_j} \right)_{1 \leq i \leq 3, 0 \leq j \leq 2}.
\]

We note that \( G \) is not a constant since each \( F_i \) is homogeneous and irreducible and \( \sum_{i=1}^{3} \deg F_i \geq 4 \).

We claim that \([G = 0], D_1, D_2, D_3 \) are in general position. To prove this, it suffices to show that \( G \) does not vanish at any intersection point of any 2 divisors among \( D_1, D_2, D_3 \). By rearranging the indices, it suffices to consider that \( P \in D_1 \cap D_2 \) and show that \( G(P) \neq 0 \). Since \( D_1, D_2, D_3 \) have no common zeros, we see that \( F_3(P) \neq 0 \). Using the Euler formula

\[
\sum_{j=1}^{3} \frac{\partial F_i}{\partial x_j} x_j = d_i \cdot F_i,
\]

we obtain

\[
 x_0 G = \det \begin{pmatrix} d_1 F_1 & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ d_2 F_2 & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ d_3 F_3 & \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \end{pmatrix},
\]

and hence

\[
 x_0(P) G(P) = d_3 F_3(P) \det \left( \frac{\partial F_i}{\partial x_j}(P) \right)_{1 \leq i,j \leq 2}.
\]

Since \( D_1, D_2, D_3 \) intersect transversally, we see that \( \det \left( \frac{\partial F_i}{\partial x_j}(P) \right)_{1 \leq i,j \leq 2} \neq 0 \). Then \( G(P) \neq 0 \) as \( F_3(P) \neq 0 \). This proves our claim. Hence, there is a nonconstant irreducible factor \( \tilde{G} \) of \( G \) (and hence of \( J \)) in \( \mathbb{C}[x_0, x_1, x_2] \) such that \( Z := [\tilde{G} = 0], D_1, D_2, D_3 \) are in general position.

Since \( \pi \) is a finite morphism and \( \tilde{G} \) is irreducible, \( \pi(Z) \) is the zero locus of an irreducible homogeneous polynomial \( A \in \mathbb{C}[y_0, y_1, y_2] \) and the vanishing order of \( \pi^* A \) along \( Z \) is at least 2. Then this construction gives \( \pi^* \circ A = \tilde{G}^2 H \) for some \( H \in \mathbb{C}[x_0, x_1, x_2] \). Next, we verify that \( \pi(Z) = [A = 0], [y_0 = 0], [y_1 = 0], [y_2 = 0] \) are in general position. It suffices to show that none of the points \([0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0] \) is in \( \pi(Z) \). If any of the points, say \([0 : 0 : 1] \in \pi(Z) \), then there exists \( P \in Z \) such that \( F_1(P) = F_2(P) = 0 \), which is impossible since \( Z, D_1, D_2, D_3 \) are in general position.
Now let \( f = (f_0, f_1, f_2) : \mathbb{C} \to \mathbb{P}^2 \) be a holomorphic map, where \( f_0, f_1, f_2 \) are entire functions without common zeros, such that

\[
u := \pi(f) = (F_1(f)^{a_1}, F_2(f)^{a_2}, F_3(f)^{a_3})
\]
is a 3-tuple of entire functions with zero multiplicity at least \( a_i \cdot m_i \) for the \( i \)-th position, \( 1 \leq i \leq 3 \).

From the equality \( A(u) = (\pi \circ A)(f) = \tilde{G}^2(f)H(f) \), it follows that for each \( z \in \mathbb{C} \) with \( v_z(\tilde{G}(f)) > 0 \), we have

\[
v_z(A(u)) \geq 2v_z(\tilde{G}(f)) \geq v_z(\tilde{G}(f)) + 1
\]
as \( f_0, f_1, f_2 \) are entire functions. Therefore,

\[
N_{\tilde{G}(f)}(0, r) \leq N_A(u)(0, r) - N_A^{(1)}(0, r).
\]

Then we may apply Theorem 1.4 with the nonconstant polynomial \( A \in \mathbb{C}[y_0, y_1, y_2] \) as it is irreducible and the zero locus is in general position with the coordinate lines. Then for a given \( \epsilon > 0 \), there exists a proper Zariski closed subset \( W \subset \mathbb{P}^2 \) and a (sufficiently large) positive integer \( \ell_1 \) such that if \( m_i \geq \ell_1 \) for \( 1 \leq i \leq 3 \) and the image of the holomorphic map \( u \) as in (6.2) is not contained in \( W \), then

\[
N_A(u)(0, r) - N_A^{(1)}(0, r) \leq \epsilon T_{\mu}(r) = \epsilon T_f(r).
\]

Therefore,

\[
N_{\tilde{G}(f)}(0, r) \leq \epsilon T_f(r)
\]
if the image of \( f \) is not contained in \( \pi^{-1}(W) \).

Finally, since \( |\tilde{G} = 0|, D_1, D_2 \) and \( D_3 \) are in general position, Theorem 2.3 implies that for any \( 0 < \epsilon < \frac{1}{4} \) there exist two positive integers \( M \) and \( N \) (independent of \( f \)) such that

\[
(1 - \epsilon) T_f(r) \leq \frac{1}{\deg \tilde{G}} N_{\tilde{G}(f)}(0, r) + \sum_{j=1}^{3} \frac{1}{\deg F_j} N_{F_j}(0, r),\]
or the image of \( f \) is contained in a plane curves with degree bounded by \( N \). Let \( m_i \geq \ell_2 := 3M\epsilon^{-1} \).

Then

\[
\frac{1}{\deg F_j} N_{F_j}(0, r) \leq \frac{1}{\deg F_j} \frac{M}{\ell_2} N_{F_j}(0, r) \leq \frac{\epsilon}{3} \cdot T_f(r).
\]

Together with (6.4), we derive from (6.5) that

\[
(1 - \epsilon) T_f(r) \leq \epsilon 2\epsilon \cdot T_f(r),
\]
which is not possible since $\epsilon < \frac{1}{4}$. Therefore, we conclude that if $m_i \geq \ell := \max\{\ell_1, \ell_2\}$ for $1 \leq i \leq 3$, then the image of $f$ is contained in some plane curve of degree bounded by $N$, where $N$ is independent of $f$.

6.3. Proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let $\pi : X \to \mathbb{P}^2$ be a finite morphism. Let $H_i = [x_i = 0]$, $0 \leq i \leq 2$, $D_i$ be the support of $\pi^* H_i$ and $\Delta = (1 - \frac{1}{m_0})D_0 + (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2$. Let $f : \mathbb{C} \to (X, \Delta)$ be a non-constant orbifold entire curve, i.e. $f(\mathbb{C}) \not\subset \Delta$ and

\[
(6.8) \quad m_i \leq \text{mult}_i(f^*E_i) = \text{mult}_i((\pi \circ f)^*H_i)
\]

for $0 \leq i \leq 2$ and all $t \in \mathbb{C}$ with $f(t) \in E_i$, for any component $E_i$ of $D_i$. Let $(f_0, f_1, f_2)$ be a reduced representation of $\pi \circ f : \mathbb{C} \to \mathbb{P}^2$, i.e. $\pi \circ f = (f_0, f_1, f_2)$ and $f_0, f_1, \text{ and } f_2$ are entire functions with no common zeros. Then (6.8) implies that for $0 \leq i \leq 2$

\[
(6.9) \quad \text{mult}_i(f_i) \geq m_i \quad \text{for all } t \in \mathbb{C} \text{ with } f_i(t) = 0.
\]

We now recall some arguments from [6, Lemma 1]. By [7, (1.11)], the canonical divisor class $K_X$ on $X$ can be written as $K_X \sim \pi^*(K_{\mathbb{P}^2}) + \text{Ram}$, where $\text{Ram}$ is the ramification divisor of $\pi$. Let $\text{Ram} = Z + R_D$, where $R_D$ is the contribution coming from the support contained in $D$, i.e. $\pi^* H_1 + \pi^* H_2 + \pi^* H_3 = D + R_D$. Since $K_{\mathbb{P}^2} \sim -(H_1 + H_2 + H_3)$, we obtain

\[
(6.10) \quad Z \sim D_1 + D_2 + D_3 + K_X.
\]

Since $(X, \Delta)$ is of general type, $K_X + \Delta$ is big and hence $Z$ is big as well.

As $\pi(Z)$ is a curve in $\mathbb{P}^2$, it is the zero locus of a homogeneous polynomial $F \in [x_0, x_1, x_2]$. We note that $\pi(Z) = [F = 0]$ and the coordinate hyperplanes $[x_i = 0]$, $0 \leq i \leq 2$ are in general position by the assumption that $\pi(Z)$ does not intersect the set of points $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ in $\mathbb{P}^2$.

Let $Z_0$ be an irreducible component of $Z$ and $F_0$ be the irreducible factor of $F$ in $\mathbb{C}[x_0, x_1, x_2]$ such that its zero locus $R_0 := [F_0 = 0] = \pi(Z_0)$. Then $\pi^* R_0$ has multiplicity at least 2 along $Z_0$. Moreover, the vanishing order of $f$ along $\pi^* R_0$ equals the vanishing order of $\pi \circ f = (f_0, f_1, f_2)$ along $R_0$. Let $g_U$ be a local defining function of $Z_0$ in an open set $U$ of a point $x \in Z_0$. Then

\[
(6.11) \quad \text{ord}_t(F_0(f_0, f_1, f_2)) \geq 2\text{ord}_t(g_U \circ f)
\]

for $t \in \mathbb{C}$ such that $f(t) = x$. Therefore, the zero multiplicity of $F_0(f_0, f_1, f_2)$ at $t$ is at least twice of $\text{mult}_i(f^*Z_0)$. Therefore,

\[
(6.12) \quad N_f(Z_0, r) \leq N_{F_0(f_0, f_1, f_2)}(0, r) - N_{F_0(f_0, f_1, f_2)}^{(1)}(0, r).
\]
We are now in position to apply Theorem 1.4 for $F_0$. Then for any $\epsilon > 0$, there exists a proper Zariski closed subset $W$ of $\mathbb{P}^2$ and a positive integer $\ell_1$ independent of $f$ such that if $m_i \geq \ell_1$ for $i = 0, 1, 2$ and the image of $\pi \circ f$ is not contained in $W$, then $N_{F_0(f_0,f_1,f_2)}(0,r) - N_{f_0(f_0,f_1,f_2)}^{(1)}(0,r) \leq \epsilon T_{\pi \circ f}(r)$. Therefore,

\begin{equation}
(6.13) \quad N_f(Z_0,r) \leq \epsilon T_{\pi \circ f}(r)
\end{equation}

if $m_i \geq \ell_1$ for $i = 0, 1, 2$ and the image of $\pi \circ f$ is not contained in $W$.

On the other hand, since $[F_0 = 0], H_1, H_2,$ and $H_3$ are in general position, Theorem 2.3 implies that for any $0 < \epsilon < \frac{1}{3}$ there exist two positive integers $M$ and $N$ (independent of $f$) such that either the image of $\pi \circ f$ is contained in a curve in $\mathbb{P}^2$ with degree bounded by $N$, or

\begin{equation}
(6.14) \quad (1 - \epsilon) T_{\pi \circ f}(r) \leq \frac{1}{\deg F_0} N_{F_0(f_0,f_1,f_2)}(0,r) + \sum_{j=0}^{2} N_{f_j}^{(M)}(0,r)
\end{equation}

\begin{equation}
\leq \frac{1}{\deg F_0} N_{F_0(f_0,f_1,f_2)}(0,r) + \sum_{j=0}^{2} \frac{M}{m_j} N_{f_j}(0,r)
\end{equation}

if $\min\{m_0, m_1, m_2\} > 3M\epsilon^{-1}$.

By repeating the above arguments for each component of $Z$ and replacing $\epsilon, \ell_1, W, M$ and $N$ if necessary, then (6.14) remains valid by replacing $Z_0$ with $Z$ if the image of $f$ is not contained in $\pi^{-1}(W)$. Hence we have

\begin{equation}
(6.15) \quad N_{F_0(f_0,f_1,f_2)}(0,r) \geq \epsilon (1 - 2\epsilon) \deg F \cdot T_{\pi \circ f}(r)
\end{equation}

if $\min\{m_0, m_1, m_2\} \geq 3M\epsilon^{-1}$ and the image of $\pi \circ f$ is not contained in a curve in $\mathbb{P}^2$ with degree bounded by $N$. We can derive from (6.15) that

\begin{equation}
(6.16) \quad m_{F_0(f_0,f_1,f_2)}(0,r) \leq \epsilon 2\deg F \cdot T_{\pi \circ f}(r) + O(1).
\end{equation}

Then the functorial property, $\hat{Z} \leq \hat{\pi}^*([F = 0])$ (as divisors) implies that

\begin{equation}
(6.17) \quad m_f(Z,r) \leq m_{\pi \circ f}([F = 0],r) + O(1) \leq 2\epsilon \deg F \cdot T_{\pi \circ f}(r).
\end{equation}

Together with (6.13) for $Z$, we have

\begin{equation}
(6.18) \quad T_{\hat{Z}_f}(f,r) \leq \epsilon (2\deg F + 1) \epsilon \cdot T_{\pi \circ f}(r)
\end{equation}

if the image of $\pi \circ f$ is not contained in a curve in $\mathbb{P}^2$ with degree bounded by $\max\{N, \deg W\}$. 
Let $A$ be an ample divisor on $X$. Then by [22, Proposition 10.7], there exists a constant $c$ such that

$$T_{\pi \circ f}(r) = \frac{1}{\deg F} \cdot T_{\pi^*(F=0)}(f, r) + O(1) \leq cT_A(f, r) + O(1).$$

(6.19)

On the other hand, since $Z$ is big, there exists a constant $b > 0$ and a proper Zariski-closed set $W_0$ of $X$, depending only on $A$ and $Z$, such that

$$T_A(f, r) \leq bT_Z(f, r) + O(1),$$

(6.20)

if the image of $f$ is not contained in $W_0$. Combining this with (6.18) and (6.19), it yields

$$T_A(f, r) \leq \text{exc}_{bc}(2 \deg F + 1)\epsilon T_A(f, r) + O(1),$$

which is not possible as $b$ and $c$ are independent of $\epsilon$ and $\epsilon$ can be taken sufficiently small. In conclusion, the image of $\pi \circ f$ is contained in a curve in $\mathbb{P}^2$ with degree bounded by $N_1 := \max\{N, \deg W, \deg \pi(W_0)\}$. Since $\pi : X \to \mathbb{P}^2$ is a finite morphism, it implies that the image of $f$ is contained in a curve of degree bounded by $N_1 \cdot \deg \pi$, which is independent of $f$. \qed

6.4. Remark on Strong Green-Griffiths-Lang conjecture. We will discuss the exceptional sets for Theorem 1.2 and Theorem 1.3 under the assumption that the multiplicities $m_i = \infty$, $1 \leq i \leq 3$, i.e., the open case of the Green-Griffiths-Lang conjecture.

The proofs of Theorem 1.2 and Theorem 1.3 are based on Theorem 1.4 and Theorem 2.3. The exceptional set $W$ in Theorem 1.4 can be constructed explicitly. So, the main point is to find an alternative for Theorem 2.3. When $m_i = \infty$ for $1 \leq i \leq 3$, we consider units instead of entire functions with sufficiently large multiplicities. Therefore, we can replace Theorem 2.3 with the original theorem of Ru, where he basically considered counting functions without truncation. We also note that the proof of Ru’s theorem is an application of Cartan’s second main theorem, which is under the assumption that the entire curves are linearly nondegenerate. In [21], Vojta has weaken the linearly nondegenerate condition to that there exists $\mathcal{H}$, a finite union of proper linear subspaces, such that the non-constant entire curves are not contained in $\mathcal{H}$. Combining Vojta’s refinement of the second main theorem, we can reformulate the result of Ru as follows.

**Theorem 6.1 ([18]).** Let $f$ be a nonconstant holomorphic map of $\mathbb{C}$ into $\mathbb{P}^n$. Let $\{D_j\}$, $1 \leq i \leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_i$, in general position. Then for any $\epsilon > 0$, there exist a hypersurface $Z$ in $\mathbb{P}^n(\mathbb{C})$ such that the following inequality holds:

$$(q - n - 1 - \epsilon)T_f(r) \leq_{\text{exc}} \sum_{j=1}^{q} \frac{1}{d_j}N_{Q_j}(r)(0, r),$$

if the image of $f$ is contained in $Z$. 

Then it is clear that the use of Theorem 6.1 allows us to find exceptional set $W$ such that any non-constant entire curve $f$ in Theorem 1.2 (resp. Theorem 1.3) is contained in $W$ when $m_i = \infty$ for $1 \leq i \leq 3$.

References

[1] T. T. H. An and H. T. Phuong, An explicit estimate on multiplicity truncation in the second main theorem for holomorphic curves encountering hypersurfaces in general position in projective space. Houston J. Math. 35 (2009), no. 3, 775–786.
[2] D. Brotbek and Y. Deng, Kobayashi hyperbolicity of the components of general hypersurfaces of high degree. Geometric and Functional Analysis 29 (2019), no. 4, 690–750.
[3] F. Campana, Fibres multiples sur les surfaces: aspects geometriques, hyperboliques et arithmetiques. Manuscr. Math. 117 (2005), no. 4, 429–461.
[4] F. Campana, L. Darondeau, and E. Rousseau Orbifold hyperbolicity. Compos. Math. 156 (2020), no. 8, 1664–1698.
[5] P. Corvaja and U. Zannier, Some cases of Vojta’s conjecture on integral points over function fields, J. Algebraic Geom. 17 (2008), no. 2, 295–333.
[6] P. Corvaja and U. Zannier, Algebraic hyperbolicity of ramified covers of $\mathbb{G}_m^2$ (and integral points on affine subsets of $\mathbb{P}^2$), J. Differential Geom. 93 (2013), no. 3 355–377.
[7] O. Debarre, Higher-dimensional algebraic geometry. Universitext, Springer-Verlag, New York, 2001.
[8] M. Green, On the functional equation $f^2 = e^{2\phi_1} + e^{2\phi_2} + e^{2\phi_3}$ and a new Picard theorem, Trans. Amer. Math. Soc. 195 (1974), 223–230.
[9] J. Guo, C.-L. Sun and J. T.-Y. Wang, On the $d$-th roots of exponential polynomials and related problems arising from Green-Griffiths-Lang conjecture, J. of Geometric Analysis, 31 (2021), no. 5, 5201–5218.
[10] J. Guo, C.-L. Sun and J. T.-Y. Wang, On Pisot’s $d$-th root conjecture for function fields and related GCD estimates, J. of Number Theory, (2022), 401–432.
[11] J. Guo, C.-L. Sun and J. T.-Y. Wang, A truncated second main theorem for algebraic tori with moving targets and applications, accepted by J. Lond. Math. Soc. (2).
[12] S. Lang, Algebra, Springer-Verlag, New York, Revised Third Edition, 1993.
[13] R. Lazarsfeld, Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics vol. 48. Springer-Verlag, Berlin, 2004.
[14] A. Levin and J. T.-Y. Wang, Greatest common divisors of analytic functions and nevanlinna theory on algebraic tori, J. Reine Angew. Math., 767 (2020), 77–107.
[15] J. Noguchi, J. Winkelmann, and K. Yamanoi, Degeneracy of holomorphic curves into algebraic varieties, J. Math. Pures Appl. (9) 88 (2007), no. 3, 293–306.
[16] J. Noguchi, J. Winkelmann, and K. Yamanoi, The second main theorem for holomorphic curves into semi-abelian varieties II, *Forum Math.* **20** (2007), 469–503.

[17] E. Rousseau and A. Turchet and J. T.-Y. Wang, Nonspecial varieties and generalized Lang-Vojta conjectures, *Forum of Mathematics, Sigma* **9** (2021), e11.

[18] M. Ru, A defect relation for holomorphic curves intersecting hypersurfaces, *Amer. Journal of Math.* **126** (2004), 215–226.

[19] M. Ru, *Nevanlinna theory and its relation to Diophantine approximation*, World Scientific Publishing Co., Pte. Ltd., Hackensack, NJ, 2021.

[20] M. Ru and J. T.-Y. Wang, Truncated second main theorem with moving targets, *Trans. Amer. Math. Soc.* **356** (2004), no. 2, 557–571.

[21] P. Vojta, On Catan’s theorem and Cartan’s conjecture, *American Journal of Mathematics*, **119** (1997), no. 1, 1–17.

[22] P. Vojta, Diophantine approximation and Nevanlinna theory, in *Arithmetic geometry, Lecture Notes in Math.* **2009**, Springer, Berlin, 2011, 111–224.

[23] B. L. van der Waerden, *Moderne Algebra*, vol. 2, 5th ed., Springer-Verlag, Berlin, 1967; English transl., Ungar, New York, 1970.

[24] J. T.-Y. Wang, Cartan’s conjecture with moving targets of same growth and effective Wirsing’s theorem over function fields, *Math. Z.* **234** (2000), 739–754.

[25] J. T.-Y. Wang, An effective Schmidt’s subspace theorem over function fields, *Math. Z.* **246** (2004), no. 4, 811–844.

[26] K. Yamanoi, The second main theorem for small functions and related problems. *Acta Math.* **192** (2004), no. 2, 225–294

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