Path regularity of coupled McKean-Vlasov FBSDEs

Christoph Reisinger*  Wolfgang Stockinger*  Yufei Zhang*

Abstract. This paper establishes Hölder time regularity of solutions to coupled McKean-Vlasov forward-backward stochastic differential equations (MV-FBSDEs). This is not only of fundamental mathematical interest, but also essential for their numerical approximation. We show that a solution triple to a MV-FBSDE with Lipschitz coefficients is $1/2$-Hölder continuous in time in the $L^p$-norm provided that it admits a Lipschitz decoupling field. Special examples include decoupled MV-FBSDEs, coupled MV-FBSDEs with a small time horizon and coupled stochastic Pontryagin systems arising from mean field control problems.

Key words. path regularity, Malliavin differentiability, mean field forward-backward stochastic differential equation.

AMS subject classifications. 60G17, 60H07, 49N60

1 Introduction

In this paper, we establish path regularity of solutions to fully-coupled McKean-Vlasov forward-backward stochastic differential equations (MV-FBSDEs). Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a $d$-dimensional Brownian motion $W = (W_t^{(1)}, \ldots, W_t^{(d)})_{t \in [0,T]}$ is defined, and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the natural filtration of $W$ augmented with an independent $\sigma$-algebra $\mathcal{F}_0$. We consider the following MV-FBSDEs: for $t \in [0,T]$,

\begin{align}
\mathrm{d}X_t &= b(t, X_t, Y_t, Z_t, \mathbb{P}(X_t, Y_t, Z_t)) \mathrm{d}t + \sigma(t, X_t, Y_t, Z_t, \mathbb{P}(X_t, Y_t, Z_t)) \mathrm{d}W_t, \quad X_0 = \xi_0, \\
\mathrm{d}Y_t &= -f(t, X_t, Y_t, Z_t, \mathbb{P}(X_t, Y_t, Z_t)) \mathrm{d}t + Z_t \mathrm{d}W_t, \quad Y_T = g(X_T, \mathbb{P}_{X_T}),
\end{align}

where $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, $(b, \sigma, f, g)$ are given Lipschitz continuous functions, $\mathbb{P}_U$ denotes the law of a given random variable $U$, and a solution triple $(X, Y, Z)$ is an $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$-valued square-integrable adapted process satisfying (1.1) $\mathbb{P}$-almost surely.

Such equations play an important role in large population optimization problems (see e.g. [6, 3, 4, 5, 9, 11]). In particular, the solutions of (1.1) give a stochastic representation of the solutions and their derivatives to certain nonlinear nonlocal partial differential equations (PDEs) defined on the Wasserstein space, the so-called nonlinear Feynman-Kac representation formula (see [6, 11]). Moreover, by applying the stochastic maximum principle, we can construct both the equilibria of mean field games and the solutions to mean field control problems based on solutions of (1.1).

Let $(X, Y, Z)$ be a given solution triple to (1.1). We aim to establish its path regularity of the

*Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK (christoph.reisinger@maths.ox.ac.uk, wolfgang.stockinger@maths.ox.ac.uk, yufei.zhang@maths.ox.ac.uk)
following form: for all $p \in \mathbb{N}, t, s \in [0, T]$ and for every partition $\pi = \{0 = t_0 < \cdots < t_N = T\}$,

$$
\mathbb{E} \left[ \sup_{0 \leq r \leq T} |Z_r|^p \right] \leq C_{(p, \xi_0)}, \quad \mathbb{E} \left[ \sup_{s \leq r \leq t} |X_r - X_s|^p \right] + \mathbb{E} \left[ \sup_{s \leq r \leq t} |Y_r - Y_s|^p \right] \leq C_{(p, \xi_0)} |t - s|^{p/2}, \quad (1.2)
$$

$$
\sum_{i=0}^{N-1} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} |Z_r - Z_{t_i}|^2 \, dr \right]^{p/2} + \left( \int_{t_i}^{t_{i+1}} |Z_r - Z_{t_{i+1}}|^2 \, dr \right)^{p/2} \leq C_{(p, \xi_0)} |\pi|^{p/2}, \quad (1.3)
$$

where $|\pi| = \max_{i=0,\ldots,N-1}(t_{i+1} - t_i)$ and $C_{(p, \xi_0)}$ is a constant depending only on $p$ and the initial condition. Such path regularity results are crucial for practical applications of MV-FBSDEs. For example, an estimate of $(X,Y,Z)$ in the norm $\mathbb{E} \left[ \sup_{0 \leq r \leq T} | \cdot |^p \right]$ with a sufficiently large $p \in \mathbb{N}$ is essential for establishing convergence rates of the particle approximation for (1.1) (so-called quantitative propagation of chaos results), where the marginal law $\mathbb{P}_{(X_t,Y_t,Z_t)}$ is approximated by the empirical distribution of interacting particles at each $t \in [0, T]$ (see e.g. [11, Proposition 5]). Moreover, by applying the stochastic maximum principle and studying the path regularity of the associated MV-FBSDEs (i.e., the stochastic Pontryagin systems), we can analyze the time regularity and discrete-time approximations of open-loop optimal controls of (extended) mean field control problems via purely probabilistic arguments, without analyzing the classical solutions to the associated infinite-dimensional PDEs on the measure spaces (see [15]). Finally, it is well-known that the Hölder regularity of the process $Z$ in (1.3) plays a crucial role in quantifying convergence rates of time-stepping schemes for (MV-)FBSDEs, in particular the approximation error of the process $Z$; see e.g. [16, 10] for classical BSDEs and [14, Theorem 4.2] for MV-FBSDEs.

Although such path regularity has been proved for classical FBSDEs (without mean field interaction) in various papers (e.g. [2, 10, 17]), to the best of our knowledge, there is no published work on the path regularity of solutions to MV-FBSDE (1.1) with possibly degenerate diffusion coefficient $\sigma$ and general Lipschitz continuous $f$, even for the decoupled cases. In this work, we shall close the gap by showing that a given solution triple to (1.1) enjoys the regularity estimates (1.2)-(1.3) provided that the process $Y$ admits a Lipschitz decoupling field (see Theorem 2.1). Such condition holds if (1.1) is uniquely solvable and stochastically stable with respect to the initial condition, which can be verified for several practically important cases, including decoupled MV-FBSDEs, coupled MV-FBSDEs with a small time horizon (see Corollary 2.3) and coupled MV-FBSDEs whose coefficients satisfy a generalized monotonicity condition (see Corollary 2.4).

The mean field interaction and the strong coupling in (1.1) pose a significant challenge for establishing the path regularity beyond those encountered in [2, 10, 17]. Recall that a crucial step in deriving these path regularity results for classical decoupled FBSDEs is to represent the Malliavin derivatives of the solutions by using the first variation processes (i.e., the derivatives of the solutions with respect to the initial condition) and their inverse. However, such a representation no longer holds for solutions to MV-FBSDEs, since equations for the first variation processes will involve the derivatives of marginal distributions of the solutions with respect to the initial condition, which do not appear in equations for the Malliavin derivatives of the solutions. Moreover, due to the strong coupling between the forward and backward equations, the first variation of the forward component of the solution will depend on the first variations of the backward components, which creates an essential difficulty in establishing the invertibility of the first variation processes.

We shall overcome the above difficulties by employing the decoupling field of the solution, which enables us to express the backward component $Y$ of the solution as a function of the forward component $X$ and then rewrite the coupled MV-FBSDE as a decoupled FBSDE, whose coefficients depend on the decoupling field and the flow $(\mathbb{P}_{(X_t,Y_t,Z_t)})_{t \in [0,T]}$. Note that these modified coefficients are in general merely Lipschitz continuous in space and square integrable in time, due to the lack of time regularity of the decoupling field and the flow $t \mapsto \mathbb{P}_{Z_t}$. We then establish a representation
formula of the process $Z$, based on “partial” first variation processes of the solutions and the weak derivatives of the decoupling field and these irregular coefficients, which subsequently leads us to the desired path regularity of (1.1); see the discussion at the end of Section 3 for details.

We state the path regularity results for coupled MV-FBSDE (1.1) in Section 2 and present its proof in Section 3. Appendix A is devoted to the proofs of some technical results.

Notation. We end this section by introducing some notation used throughout this paper. For any given $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we denote by $I_n$ the $n \times n$ identity matrix, by $0_n$ the zero element of $\mathbb{R}^n$ and by $\delta_x$ the Dirac measure supported at $x$. We shall denote by $\langle \cdot, \cdot \rangle$ the usual inner product in a given Euclidean space and by $|\cdot|$ the norm induced by $\langle \cdot, \cdot \rangle$, which in particular satisfy for all $n, m, d \in \mathbb{N}$ and $\theta_1 = (x_1, y_1, z_1), \theta_2 = (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ that $\langle z_1, z_2 \rangle = \text{trace}(z_1^* z_2)$ and $\langle \theta_1, \theta_2 \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \langle z_1, z_2 \rangle$, where $\cdot^*$ denotes the transposition of a matrix.

We then introduce several spaces: for each $p \geq 1, k \in \mathbb{N}$, $t \in [0, T]$ and Euclidean space $(E, |\cdot|)$, $L^p(\Omega; E)$ is the space of $E$-valued $\mathcal{F}$-measurable random variables $X$ satisfying $\|X\|_{L^p} = \mathbb{E}[^{|X|^p}]^{1/p} < \infty$, and $L^p(\mathcal{F}_t; E)$ is the subspace of $L^p(\Omega; E)$ containing all $\mathcal{F}_t$-measurable random variables; $\mathcal{S}^p(t, T; E)$ is the space of $\mathbb{F}$-progressively measurable processes $Y : \Omega \times [t, T] \rightarrow E$ satisfying $\|Y\|_{\mathcal{S}^p} = \mathbb{E}[^{\text{ess sup}_{s \leq t} |Y_s|^p}]^{1/p} < \infty$, and $\mathcal{S}^\infty$ is the subspace of $\mathcal{S}^p(t, T; E)$ containing all uniformly bounded processes $Y$ satisfying $\|Y\|_{\mathcal{S}^\infty} = \text{ess sup}_{s \leq t} |Y_s| < \infty$; $\mathcal{H}^p(t, T; E)$ is the space of $\mathbb{F}$-progressively measurable processes $Z : \Omega \times [t, T] \rightarrow E$ satisfying $\|Z\|_{\mathcal{H}^p} = \mathbb{E}[^{(\int_t^T |Z_s|^p \, ds)^{p/2}}]^{1/p} < \infty$; $\mathcal{D}^1(\mathcal{F}_t; E)$ is the space of Malliavin differentiable random variables. For notational simplicity, when $t = 0$, we often denote $\mathcal{S}^p = \mathcal{S}^p(0, T; E)$ and $\mathcal{H}^p = \mathcal{H}^p(0, T; E)$, if no confusion occurs.

Moreover, for every Euclidean space $(E, |\cdot|)$, we denote by $\mathcal{P}_2(E)$ the metric space of probability measures $\mu$ on $E$ satisfying $\|\mu\|_2 = (\int_E |x|^2 \, d\mu(x))^{1/2} < \infty$, endowed with the 2-Wasserstein metric defined by

$$\mathcal{W}_2(\mu_1, \mu_2) := \inf_{\kappa \in \Pi(\mu_1, \mu_2)} \left( \int_{E \times E} |x - y|^2 \, d\kappa(x, y) \right)^{1/2}, \quad \mu_1, \mu_2 \in \mathcal{P}_2(E),$$

where $\Pi(\mu_1, \mu_2)$ is the set of all couplings of $\mu_1$ and $\mu_2$, i.e., $\kappa \in \Pi(\mu_1, \mu_2)$ is a probability measure on $E \times E$ such that $\kappa(\cdot \times E) = \mu_1$ and $\kappa(E \times \cdot) = \mu_2$.

2 Path regularity of fully coupled MV-FBSDEs

In this section, we establish an $L^p$-path regularity result for (1.1), whose coefficients $(b, \sigma, f, g)$ satisfy the following standing assumptions:

**H.1.** Let $n, m, d \in \mathbb{N}, T \in [0, \infty)$ and let $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \rightarrow \mathbb{R}^{n \times d}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^m$ be measurable functions satisfying for some $L, K \in [0, \infty)$ that:

1. For all $t \in [0, T]$, the functions $b(t, \cdot), \sigma(t, \cdot), f(t, \cdot)$ and $g(\cdot)$ are uniformly Lipschitz continuous in all variables with a Lipschitz constant $L$.

2. $\|b(\cdot, 0, 0, \delta_{0_n+m+m+d})\|_{L^2(0, T)} + \|f(\cdot, 0, 0, 0, \delta_{0_n+m+m+d})\|_{L^2(0, T)} + |g(0, \delta_{0_n})| \leq K$, and it holds for all $\mu \in \mathcal{P}_2(\mathbb{R}^{m \times d})$ that $\|\sigma(\cdot, 0, 0, \delta_{0_n+m} \times \mu)\|_{L^\infty(0, T)} \leq K$.

Remark 2.1. Throughout this paper, we shall denote by $C \in [0, \infty)$ a generic constant, which is independent of the initial condition $\xi_0$, though it may depend on the constants appearing in the assumptions and may take a different value at each occurrence. Dependence of $C$ on additional parameters will be indicated explicitly by $C_{(\cdot)}$, e.g., $C_{(p)}$ for some $p \in \mathbb{N}$. 3
The following theorem presents a general path regularity result for a given solution to (1.1), provided that the solution admits a decoupling field and enjoys a natural moment estimate. In the subsequent analysis, for a given triple \((X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d})\) and a constant \(L_v \in [0, \infty)\), we say \(Y\) admits an \(L_v\)-Lipschitz decoupling field if there exists a measurable function \(v : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m\) such that \(E(vt \in [0, T], Y_t = v(t, X_t)) = 1\) and it holds for all \(t \in [0, T] \) and \(x, x' \in \mathbb{R}^n\) that \(|v(t, x) - v(t, x')| \leq L_v|x - x'|\). For the sake of readability, we postpone the proof of Theorem 2.1 to Section 3.

**Theorem 2.1.** Suppose (H.1) holds. Let \(\xi_0 \in L^2(F_0; \mathbb{R}^n)\), \(L_v, M \in [0, \infty)\) and \((X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d})\) be a solution to (1.1) satisfying the following two conditions: (1) \(Y\) admits a \(L_v\)-Lipschitz decoupling field; (2) \(\|X\|_{\mathcal{S}^2} + \|Y\|_{\mathcal{S}^2} + \|Z\|_{\mathcal{H}^2} \leq M(1 + \|\xi_0\|_{L^2})\). Then we have that:

1. There exists a constant \(C > 0\) such that for \(dP \otimes dt\) a.e., \(|Z_t| \leq C|\sigma(t, X_t, Y_t, P_{(X_t, Y_t, Z_t)})|\).

Consequently, for all \(p \geq 2\), there exists a constant \(C(p) > 0\) such that \(\|X\|_{S^p} + \|Y\|_{S^p} + \|Z\|_{S^p} \leq C(p)(1 + \|\xi_0\|_{L^p})t\) for all \(t \in [0, T]\).

2. For any \(p \geq 2\), there exists a constant \(C(p) > 0\) such that it holds for all \(0 \leq s \leq t \leq T\) that

\[
E\left[\sup_{s \leq r \leq t} |X_r - X_s|^p\right]^{1/p} + E\left[\sup_{s \leq r \leq t} |Y_r - Y_s|^p\right]^{1/p} \leq C(p)(1 + \|\xi_0\|_{L^p})t^{1/2}
\]

3. Assume further that for each \((t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m\), the function \(\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \ni \eta \mapsto \sigma(t, x, y, \eta) \in \mathbb{R}^{n \times d}\) depends only on the marginal \(\pi_{1,2}^{t,\eta} = \eta(\cdot \times \mathbb{R}^{m \times d})\) of the measure \(\eta\), and there exists a constant \(C_\sigma \in [0, \infty)\) such that for all \(s, t \in [0, T], (x, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})\),

\[
|\sigma(s, x, y, \eta) - \sigma(t, x, y, \eta)| \leq C_\sigma(1 + |x| + |y| + \|\pi_{1,2}^{t,\eta}\|_2)|s - t|^{1/2}.
\]

Then for any \(p \geq 2\) and \(\varepsilon > 0\), there exists a constant \(C_{(p, \varepsilon)} > 0\) such that it holds for every partition \(\pi = \{0 = t_0 < \cdots < t_N = T\}\) with stepsize \(\pi = \max_{i=0, \ldots, N-1}(t_{i+1} - t_i)\) that,

\[
\sum_{i=0}^{N-1} E\left[\left(\int_{t_i}^{t_{i+1}} |Z_r - Z_t|^2 \, dr\right)^{p/2} + \left(\int_{t_i}^{t_{i+1}} |Z_r - Z_{t_{i+1}}|^2 \, dr\right)^{p/2}\right]^{1/p} \leq C_{(p, \varepsilon)}(1 + \|\xi_0\|_{L^{p^+} + \varepsilon})^{1/2}.
\]

**Remark 2.2.** Note that the Hölder regularity of the processes \(X, Y\) in Item (2) has the optimal dependence on the integrability of the initial condition \(\xi_0\) and the time regularity of the coefficients \(b, \sigma\) and \(f\). The dependence on \(\|\xi_0\|_{L^{p^+} + \varepsilon}\) in (2.2) appears due to the application of Hölder’s inequality in the analysis, which is sharp for deterministic initial data since \(\|x_0\|_{L^p} = |x_0|\) for all \(p \geq 2\). Even though we only assume that \(\xi_0 \in L^2(F_0; \mathbb{R}^n)\), the moment bounds and regularity estimates in Theorem 2.1 obviously hold if \(\xi_0 \notin L^p(F_0; \mathbb{R}^n)\) for some \(p > 2\), since the right-hand side would be infinite.

The conditions in Theorem 2.1 are satisfied by most MV-FBSDEs appearing in practice. In particular, the following proposition shows that the solution to (1.1) admits a Lipschitz decoupling field if (1.1) is uniquely solvable and stochastically stable, whose proof follows from similar arguments as that of [4, Proposition 5.7].

**Proposition 2.2.** Assume for all \(t \in [0, T]\) and \(\xi \in L^2(F_t; \mathbb{R}^n)\) that there exists a unique triple of processes \((X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi}) \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}^m) \times \mathcal{H}^2(t, T; \mathbb{R}^{m \times d})\) satisfying (1.1) on \([t, T]\) with the initial condition \(X^{t,\xi}_t = \xi\). Assume further that there exists a constant \(L > 0\) such that
it holds for all $t \in [0, T]$ and $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^m)$ that $\|Y^t_{i}\xi - Y^t_{i}\xi'\|_{L^2} \leq \bar{L}\|\xi - \xi'\|_{L^2}$. Then for all $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, (1.1) admits a unique solution $(X, Y, Z) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m \times d})$ and the process $Y$ admits an $L$-Lipschitz decoupling field.

**Remark 2.3.** Note that due to the mean field interaction in (1.1), the decoupling field $v$ depends on the law of the initial condition $\xi_0$. Hence unlike for the classical FBSDEs, in general we do not have for all $(t, x) \in [0, T] \times \mathbb{R}^n$ that $Y^t_{i,x} = v(t, x)$. This creates a significant challenge in establishing the Hölder regularity of the mapping $t \mapsto v(t, x)$ for a given $x \in \mathbb{R}^d$. In fact, a common approach in the existing literature to analyze the time regularity of $v$ usually involves establishing the relation $v(t, \cdot) = \mathcal{U}(t, \cdot, X_t)$ for all $t \in [0, T]$, identifying the map $\mathcal{U}$ as a solution to an infinite-dimensional PDE on $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$, and then analyzing this PDE under strong regularity assumptions on the coefficients of (1.1), such as the boundedness and high-order differentiability conditions (see e.g. [6]).

We now present two concrete structural assumptions for the coefficients of (1.1) under which (1.1) is uniquely solvable, stochastically stable and the solution enjoys a natural moment estimate. The first one shows that Theorem 2.1 holds if the terminal time $T$ is sufficiently small compared to the coupling between (1.1a) and (1.1b) (see e.g. [7, 6, 9]), which includes decoupled MV-FBSDEs as special cases. Similar results can be extended to weakly coupled (MV-)FBSDEs as in [2], where the function $b$ is strongly decreasing in $x$ or the function $f$ is strongly decreasing in $y$.

**Corollary 2.3.** Suppose (H.1) holds and let the functions $\sigma$ and $g$ satisfy for some $L^2, L^2_{\sigma} \in [0, \infty)$, for all $(t, x, y, Y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times L^2(\Omega; \mathbb{R}^n)$, $X, X' \in L^2(\Omega; \mathbb{R}^n)$, $Z, Z' \in L^2(\Omega; \mathbb{R}^{m \times d})$ that $|\sigma(t, x, y, \mathbb{P}(X, Y, Z)) - \sigma(t, x, y, \mathbb{P}(X, Y, Z'))| \leq L_2^\sigma \|W_2(\mathbb{P}_Z, \mathbb{P}_{Z'})\|_{L^2}$ and $\|g(X, \mathbb{P}_X) - g(X', \mathbb{P}_X')\|_{L^2} \leq L^g \|X - X'\|_{L^2}$. If $c_0 := L^\sigma_2 L^g < 1$, then there exists a constant $C(\xi, c_0) > 0$ such that it holds for all $T \leq C(\xi, c_0)$ and $\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)$ that (1.1) admits a unique solution $(X, Y, Z) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{m \times d})$ satisfying the regularity estimates in Theorem 2.1, Items (1)–(3).

In particular, if (1.1) is decoupled in the sense that (1.1a) depends only on the process $X$ and the flow $(\mathbb{P}_t)_{t \in [0, T]}$, then the regularity estimates in Theorem 2.1, Items (1)–(3) hold for all $T \in [0, \infty)$.

**Proof.** One can establish the desired properties in Proposition 2.2 by extending the fixed-point arguments in [17, Theorem 8.2.1 and Corollary 8.2.2] for classical FBSDEs to the present setting with mean field interaction, whose detailed steps are omitted.

We now show that Theorem 2.1 holds for coupled MV-FBSDEs with an arbitrary terminal time, provided that the coefficients satisfy a monotonicity condition.

**Corollary 2.4.** Suppose (H.1) holds. Assume further the monotonicity condition holds: there exist $\alpha_1, \beta_1, \beta_2, L_\phi \in [0, \infty)$, $G \in \mathbb{R}^m \times \mathbb{R}^n$ and measurable functions $\phi_i : L^2(\Omega; \mathbb{R}^n)^{\otimes n} \to [0, \infty)$, $\phi_2 : [0, T] \times L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})^{\otimes n} \to [0, \infty)$ such that for all $t \in [0, T]$, $i \in \{1, 2\}$, $\Theta_i := (X_i, Y_i, Z_i) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$,

$$
\begin{align*}
\mathbb{E}[(b(t, X_1, Y_1, \mathbb{P}_{\Theta_1}) - b(t, X_2, Y_2, \mathbb{P}_{\Theta_2}), G^*(Y_1 - Y_2))] \\
+ \mathbb{E}[(\sigma(t, X_1, Y_1, \mathbb{P}_{\Theta_1}) - \sigma(t, X_2, Y_2, \mathbb{P}_{\Theta_2}), G^*(Z_1 - Z_2))] \\
+ \mathbb{E}[-f(t, \Theta_1, \mathbb{P}_{\Theta_1}) + f(t, \Theta_2, \mathbb{P}_{\Theta_2}, G(X_1 - X_2))] \\
\leq -\beta_1 \phi_1(X_1, X_2) - \beta_2 \phi_2(t, \Theta_1, \Theta_2), \\
\mathbb{E}[(g(X_1, \mathbb{P}_{X_1}) - g(X_2, \mathbb{P}_{X_2}), G(X_1 - X_2))] \geq \alpha_1 \phi_1(X_1, X_2).
\end{align*}
$$

Moreover, one of the following two conditions is satisfied: (1) $\beta_2 > 0$ and for all $t \in [0, T]$,

$$
\|(b, \sigma)(t, X_1, Y_1, \mathbb{P}_{\Theta_1}) - (b, \sigma)(t, X_2, Y_2, \mathbb{P}_{\Theta_2})\|_{L^2}^2 \leq L_\phi(\|X_1 - X_2\|_{L^2}^2 + \phi_2(t, \Theta_1, \Theta_2));
$$

(2.3)
or (2) \( \alpha_1, \beta_1 > 0 \) and for all \( t \in [0,T] \) that
\[
\| f(t, X_1, Y_2, Z_2, P_{(X_1,Y_2,Z_2)}) - f(t, X_2, Y_2, Z_2, P_{(X_2,Y_2,Z_2)}) \|_{L^2}^2 \\
+ \| g(X_1, P_{X_1}) - g(X_2, P_{X_2}) \|_{L^2}^2 \leq L_\phi \phi_1(X_1, X_2).
\] (2.4)

Then it holds for all \( \xi_0 \in L^2(F_0; \mathbb{R}^n) \) that (1.1) admits a unique solution \( (X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d}) \) satisfying the regularity estimates in Theorem 2.1, Items (1)-(3).

**Proof.** The desired unique solvability, stochastic stability and the moment estimates follow from a stability analysis of (1.1) under the monotonicity condition and the continuation method as in [13, 3], whose detailed steps can be found in Appendix A. \( \square \)

**Remark 2.4.** The monotonicity condition in Corollary 2.4 is a natural generalization of the well-known \( G \)-monotonicity condition in the existing literature (see (H2.2) in [13] for FBSDEs or Assumption (A.1) in [3] for MV-FBSDEs), which corresponds to the case where \( G \in \mathbb{R}^{m \times n} \) is a full-rank matrix, \( \phi_1(X_1, X_2) = \| G(X_1 - X_2) \|_{L^2}^2 \) and \( \phi_2(t, \Theta_1, \Theta_2) = \| G^*(Y_1 - Y_2) \|_{L^2}^2 + \| G^*(Z_1 - Z_2) \|_{L^2}^2 \).

More importantly, the generalized monotonicity condition can be applied to many FBSDEs arising from control problems whose coefficients enjoy specific structural conditions but fail to satisfy the \( G \)-monotonicity condition. For example, one can consider MV-FBSDEs with \( n = m, \ b(t, X, Y, P_{(X,Y)}) = b(t, X, \tilde{\alpha}(t, X, Y, P_{(X,Y)})) \) and \( \sigma(t, X, Y, P_{(X,Y)}) = \sigma(t, X, P_{X}) \) arising from applying the stochastic maximum principle to extended mean field control problems (see e.g. [4, 15]). In this case, the coefficients in general do not satisfy the \( G \)-monotonicity condition by virtue of the non-monotonicity of the function \( \tilde{\alpha} \). However, by choosing \( G = I_n \) and \( \phi_2 = \| \tilde{\alpha}(t, X_1, Y_1, P_{(X_1,Y_1)}) - \tilde{\alpha}(t, X_2, Y_2, P_{(X_2,Y_2)}) \|_{L^2}^2 \), the generalized monotonicity condition can still be satisfied under natural convexity conditions; see [15, Proposition 3.3] for details.

## 3 Proof of Theorem 2.1

In this section, we prove the \( L^p \)-path regularity results given in Section 2. Throughout this proof, let \((X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d})\) be a given solution to (1.1) with the decoupling field \( v : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m \). For notational simplicity, we define the following functions \( \tilde{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \tilde{\sigma} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}, \tilde{f} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \) and \( \tilde{g} : \mathbb{R}^n \to \mathbb{R}^m \) for all \((t, x, y, z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \)
\[
\tilde{b}(t, x) := b(t, x, v(t, x), P_{(X_t,Y_t,Z_t)}), \quad \tilde{\sigma}(t, x) := \sigma(t, x, v(t, x), P_{(X_t,Y_t,Z_t)}), \\
\tilde{f}(t, x, y, z) := f(t, x, y, z, P_{(X_t,Y_t,Z_t)}), \quad \tilde{g}(x) := g(x, P_{X_T}).
\]

Then it is clear that \((X, Y, Z)\) satisfies the following FBSDE:
\[
dX_t = \tilde{b}(t, X_t) \, dt + \tilde{\sigma}(t, X_t) \, dW_t, \quad X_0 = \xi_0, \quad \tag{3.1a}

dY_t = -\tilde{f}(t, X_t, Y_t, Z_t) \, dt + Z_t \, dW_t, \quad Y_T = \tilde{g}(X_T). \quad \tag{3.1b}
\]

The following lemma presents several regularity properties of the functions \((\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})\). Note that these functions are in general **discontinuous** in time, due to the lack of time regularity of the decoupling field \( v \) and the flow \( t \mapsto P_{Z_t} \).

**Lemma 3.1.** Assume the setting in the Theorem 2.1. Then there exists a constant \( C \) such that \((\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})\) are measurable functions which are \( C \)-Lipschitz continuous with respect to the spatial variables uniformly with respect to \( t \), and satisfy that \( \| \tilde{b}(\cdot, 0) \|_{L^2(0,T)} + \| \tilde{\sigma}(\cdot, 0) \|_{L^\infty(0,T)} + + \| \tilde{f}(\cdot, 0, 0, 0) \|_{L^2(0,T)} + \| \tilde{g}(0) \| \leq C(1 + \| \xi_0 \|_{L^2}). \)
Proof. Note that [1, Lemma 2.2] shows that the mapping \( [0, T] \ni t \mapsto \mathbb{P}(X_t, Y_t, Z_t) \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \) is measurable if for all continuous function \( \phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R} \) with quadratic growth, the map \( [0, T] \ni t \mapsto \mathbb{E}[\phi(X_t, Y_t, Z_t)] \) \in \mathbb{R} \) is measurable, which holds in the present setting due to the fact that \( (X, Y, Z) \in \mathcal{S}_2^2(\mathbb{R}^n) \times \mathcal{S}_2^2(\mathbb{R}^m) \times \mathcal{H}_2^2(\mathbb{R}^{m \times d}) \) and Fubini’s theorem. Then, we can deduce from the measurability of the functions \( (b, \sigma, f, g) \) that the functions \( (\hat{b}, \hat{\sigma}, \hat{f}, \hat{g}) \) are measurable. The \( L_v \)-Lipschitz continuity of the decoupling field \( v \) and the \( L \)-Lipschitz continuity of the functions \( (b, \sigma, f, g) \) in (H.1) imply that the functions \( (\hat{b}, \hat{\sigma}, \hat{f}, \hat{g}) \) are Lipschitz continuous with respect to their spatial variables, uniformly with respect to \( t \). Hence, it remains to show the integrability condition. Note that the Lipschitz continuity of the decoupling field \( v \) and Hölder’s inequality imply that
\[
\sup_{t \in [0, T]} |v(t, 0)| \leq \sup_{t \in [0, T]} (\mathbb{E}[|Y_t|] + \mathbb{E}[|v(t, 0) - v(t, X_t)|])
\leq \|Y\|_{\mathcal{S}_2^2} + L_v \|X\|_{\mathcal{S}_2^2} \leq C(1 + \|\xi_0\|_{L^2}). \tag{3.2}
\]
Thus we can obtain for a.e. \( t \in [0, T] \) that
\[
|\hat{b}(t, 0)| + |\hat{f}(t, 0, 0, 0)| = |b(t, 0, v(t, 0), \mathbb{P}(X_t, Y_t, Z_t))| + |f(t, 0, v(t, 0), 0, \mathbb{P}(X_t, Y_t, Z_t))|
\leq |b(t, 0, 0, \delta_{0_{n+m+md}})| + |f(t, 0, 0, 0, \delta_{0_{n+m+md}})| + \mathbb{E}[|v(t, 0)| + \mathbb{E}[|v(t, 0)| + C(1 + \|\xi_0\|_{L^2} + \|X_t, Y_t, Z_t\|_{L^2})]]
\leq C(1 + \|\xi_0\|_{L^2} + \|X_t, Y_t, Z_t\|_{L^2}),
\]
which together with the assumption that \( \|X\|_{\mathcal{S}_2^2} + \|Y\|_{\mathcal{S}_2^2} + \|Z\|_{\mathcal{H}_2^2} \leq M(1 + \|\xi_0\|_{L^2}) \) gives us that
\[
||\hat{b}(\cdot, 0)||_{L_2^2(0, T)} + ||\hat{f}(\cdot, 0, 0, 0)||_{L_2^2(0, T)} \leq C(1 + \|\xi_0\|_{L^2} + \|X, Y, Z\|_{\mathcal{H}_2^2}) \leq C(1 + \|\xi_0\|_{L^2}).
\]
Similarly, by setting \( 0_{n+m} \) to be the \( \mathbb{R}^n \times \mathbb{R}^m \)-valued zero random variable, we have for a.e. \( t \in [0, T] \) that \( \mathbb{P}(0_{n+m}, Z_t) = \delta_{0_{n+m}} \times \mathbb{P}_Z t \) and hence that
\[
|\hat{\sigma}(t, 0)| = |\sigma(t, 0, v(t, 0), \mathbb{P}(X_t, Y_t, Z_t))|
\leq |\sigma(t, 0, 0, \mathbb{P}(0_{n+m}, Z_t))| + \mathbb{E}[|v(t, 0)| + C(1 + \|\xi_0\|_{L^2} + \|X_t, Y_t\|_{L^2})]
\leq C(1 + \|\xi_0\|_{L^2} + \|X, Y\|_{\mathcal{S}_2^2}),
\]
which implies that \( ||\hat{\sigma}(\cdot, 0)||_{L_\infty(0, T)} \leq C(1 + \|\xi_0\|_{L^2} + \|X, Y\|_{\mathcal{S}_2^2}) \leq C(1 + \|\xi_0\|_{L^2}). \) This shows the desired integrability conditions and finishes the proof. \( \square \)

The Lipschitz continuity of the functions \( \hat{b}, \hat{\sigma} \) implies that the unique solution \( X \) to (3.1a) is Malliavin differentiable. This proof naturally extends [12, Theorem 2.2.1] to (3.1a) (whose initial condition is random and coefficients are merely integrable in time) and hence is omitted.

**Proposition 3.2.** Assume the setting in the Theorem 2.1. Then it holds for all \( t \in [0, T] \) that \( X_t \in \mathcal{D}^{1, 2}(\mathbb{R}^n) \), and the derivative \( DX = (DX^{(1)}, \ldots, DX^{(d)}) \), which is \( \mathbb{R}^{n \times d} \)-valued, satisfies for \( 0 \leq t < s \leq T \) that \( D_s X_t = 0 \) and for \( 0 \leq s \leq t \leq T \) that
\[
D_s X_t = \hat{\sigma}(s, X_s) + \int_s^t \hat{\partial}_b D_r X_r \, dr + \sum_{k=1}^d \int_s^t \hat{\partial}_f(\hat{\sigma}^{(k)}) D_r X_r \, dW_r^{(k)}, \tag{3.4}
\]
where \( \{\hat{\partial}_b, (\hat{\partial}_f)^{(k)}\}_{k=1}^d \subset \mathcal{S}_\infty(\mathbb{R}^{n \times n}) \) are uniformly bounded by some constant \( C \).

We then establish the Malliavin differentiability of the processes \( Y, Z \) in Theorem 2.1, which extends [8, Proposition 5.9] to BSDEs with non-differentiable coefficients.
Proposition 3.3. Assume the setting in the Theorem 2.1. Then it holds for a.e. \( t \in [0,T] \) that \((Y_t, Z_t) \in \mathbb{D}^{1,2}(\mathbb{R}^m) \times \mathbb{D}^{1,2}(\mathbb{R}^{m \times d})\), and the derivatives \( DY = (DY^{(1)}, \ldots, DY^{(d)}) \) and \( DZ = (DZ^{(1)}, \ldots, DZ^{(d)}) \), which are \( \mathbb{R}^{m \times d} \) and \( \mathbb{R}^{(m \times d) \times d} \)-valued, respectively, satisfy for \( 0 \leq t < s \leq T \) that \( D_s Y_t = D_s Z_t = 0 \) and for \( 0 \leq s \leq t \leq T \) that

\[
D_s Y_t^{(j)} = \partial \tilde{g}_T D_s X_t^{(j)} + \int_t^T \partial \tilde{f}_r \cdot D_s \Theta_r^{(j)} \, dr - \int_t^T D_s Z_r^{(j)} \, dW_r, \quad j = 1, \ldots, d
\]

with \( \partial \tilde{f}_r \cdot D_s \Theta_r^{(j)} := \partial \tilde{f}_r D_s X_r^{(j)} + \partial \tilde{g}_r D_s Y_r^{(j)} + \partial \tilde{g}_r D_s Z_r^{(j)} \), where \( DX \) is the Malliavin derivative of \( X \), and the random variable \( \partial \tilde{g}_T \in L^2(\mathcal{F}_T; \mathbb{R}^{m \times n}) \) and the processes \( \partial \tilde{f}_r \in \mathcal{S}_T^\infty(\mathbb{R}^{m \times m}), \partial \tilde{g}_r \in \mathcal{S}_T^\infty(\mathbb{R}^{m \times (m \times d)}) \) are uniformly bounded by some constant \( C \). Moreover, it holds for \( dW \otimes dt \) a.e. that \( D_t Y_t = Z_t \).

Proof. By using the Lipschitz continuity of \((\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})\) in their spatial variables, we can obtain by using the standard mollification argument a sequence of coefficients \((\tilde{b}^\varepsilon, \tilde{\sigma}^\varepsilon, \tilde{f}^\varepsilon, \tilde{g}^\varepsilon)_{\varepsilon > 0}\) that converge pointwise to \((\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})\) as \( \varepsilon \to 0 \), and are smooth and \( L\)-Lipschitz continuous with respect to the spatial variables. For each \( \varepsilon > 0 \), let \( \Theta^\varepsilon = (X^\varepsilon, Y^\varepsilon, Z^\varepsilon) \) be the solution to (3.1) with coefficients \((\tilde{b}^\varepsilon, \tilde{\sigma}^\varepsilon, \tilde{f}^\varepsilon, \tilde{g}^\varepsilon)\). Then we can obtain from standard stability results of (3.1) that \((X^\varepsilon, Y^\varepsilon, Z^\varepsilon) \rightarrow (X, Y, Z)\) as \( \varepsilon \to 0 \) in \( \mathcal{S}_T^2 \times \mathcal{S}_T^2 \times \mathcal{H}_T^2 \) (see e.g. [16, Lemma 2.4(ii)]).

For each \( \varepsilon > 0 \), we have \( X_t^\varepsilon \in \mathbb{D}^{1,2}(\mathbb{R}^n) \) for all \( t \in [0,T] \) and the derivative satisfies for \( 0 \leq s \leq t \leq T \) that

\[
D_s X_t^\varepsilon = \tilde{\sigma}^\varepsilon(s, X_s^\varepsilon) + \int_s^t \nabla_x \tilde{b}^\varepsilon(r, X_r^\varepsilon) D_r X_r^\varepsilon \, dr + \sum_{k=1}^d \int_s^t \nabla_x \tilde{\sigma}^\varepsilon(k)(r, X_r^\varepsilon) D_r X_r^\varepsilon \, dW_r^k.
\]

Moreover, by noticing that the function

\[
\mathbb{R}^m \times \mathbb{R}^{m \times d} \ni (y, z) \mapsto D_s \tilde{b}^\varepsilon(t, X_t^\varepsilon(\omega), y, z) = (\nabla_x \tilde{b}^\varepsilon)(t, X_t^\varepsilon(\omega), y, z) D_s X_t^\varepsilon(\omega) \in \mathbb{R}^{m \times d},
\]

is continuous for all \( t \in [0, T] \) and a.s. \( \omega \in \Omega \), and using the boundedness of the function \( \nabla_x \tilde{f}^\varepsilon \) and the fact that \( \mathbb{E} \left[ \int_0^T \int_0^T |D_s X_t|^2 \, dt \, ds \right] < \infty \), we can extend [8, Proposition 5.3] and establish that \( Y_t^\varepsilon \in \mathbb{D}^{1,2}(\mathbb{R}^m) \) for all \( t \in [0,T] \), \( Z_t^\varepsilon \in \mathbb{D}^{1,2}(\mathbb{R}^{m \times d}) \) for a.e. \( t \in [0,T] \), and the derivatives satisfy for \( 0 \leq s \leq t \leq T, j = 1, \ldots, d \) that

\[
D_t Y_t^\varepsilon(j) = (\nabla \tilde{g}^\varepsilon)(X_T^\varepsilon) D_s X_T^\varepsilon(j) + \int_t^T \nabla_{xyz} \tilde{b}^\varepsilon(r, \Theta_r^\varepsilon) \cdot D_s \Theta_r^{(j)} \, dr - \int_t^T D_s Z_r^{(j)} \, dW_r.
\]

Standard moment estimates of FBSDEs (see e.g. [17, Theorem 4.4.4]) show for all \( 0 \leq s \leq T \) that

\[
\mathbb{E} \left[ \sup_{s \leq r \leq T} |D_s Y_r^\varepsilon|^2 \right] + \mathbb{E} \left[ \int_s^T |D_s Z_r^k|^2 \, dr \right] \leq C \left( \mathbb{E}[|X_T^\varepsilon|^2] + \mathbb{E} \left[ \int_t^T |\nabla_x \tilde{f}^\varepsilon(r, \Theta_r^\varepsilon) D_r X_r^\varepsilon|^2 \, dr \right] \right) \quad (3.6)
\]

\[
\leq C \left( |\tilde{\sigma}(s, 0)|^2 + ||\tilde{\sigma}(\cdot, 0)||^2_{L^2(0,T)} + ||\tilde{b}(\cdot, 0)||^2_{L^2(0,T)} + ||\tilde{\sigma}(t, \cdot)||^2_{L^2(0,T)} + 1 \right),
\]

where for the last inequality we have used the estimates of \( \mathbb{E}[\sup_{s \leq r \leq T} |D_s X_r^\varepsilon|^2] \) and \( ||X^\varepsilon||_{\mathcal{S}_T^2} \), and the fact that \( |\tilde{b}(t, 0)| \leq C(|\tilde{b}(t, 0)| + 1) \) and \( |\tilde{\sigma}(t, 0)| \leq C(|\tilde{\sigma}(t, 0)| + 1) \) for all \( t \in [0,T] \), which follows from the Lipschitz continuity of \( \tilde{b}(t, \cdot) \) and \( \tilde{\sigma}(t, \cdot) \).
We now show the Malliavin differentiability of the processes \( Y \) and \( Z \). For each \( t \in [0, T] \), we have \( \lim_{\varepsilon \to 0} \| Y_\varepsilon - Y_t \|_{L^2} = 0 \) and \( \sup_{\varepsilon > 0} \mathbb{E} \left[ \int_0^T |D_s Y_\varepsilon|^2 \, ds \right] < \infty \) (see (3.6)), which together with [12, Lemma 1.2.3] imply that \( Y_t \in D^{1,2}(\mathbb{R}^m) \) for all \( t \in [0, T] \). To show the differentiability of the process \( Z \), we first introduce the random variable \( M = \int_0^T Z_r \, dW_r \). The fact that \( Z^\varepsilon \to Z \) in \( \mathcal{H}^2 \) implies that \( M_\varepsilon := \int_0^T Z_r^\varepsilon \, dW_r \to M \) in \( L^2(\Omega) \). Moreover, for all \( \varepsilon > 0 \), we can deduce from the fact that \( Z^\varepsilon_t \in D^{1,2}(\mathbb{R}^{m \times d}) \) for a.e. \( t \in [0, T] \), the convergence of \( (Z^\varepsilon)_{\varepsilon > 0} \) in \( \mathcal{H}^2 \) and the estimate (3.6) that \( M_\varepsilon \in D^{1,2}(\mathbb{R}^m) \) and

\[
\mathbb{E} \left[ \int_0^T |D_s M_\varepsilon|^2 \, ds \right] = \mathbb{E} \left[ \int_0^T |Z_s + \int_s^T D_s Z_r^\varepsilon \, dW_r|^2 \, ds \right] \leq C < \infty
\]

with a constant \( C \) uniformly with respect to \( \varepsilon \). This along with [12, Lemmas 1.2.3] shows that \( M \in D^{1,2}(\mathbb{R}^m) \), and hence \( Z_t \in D^{1,2}(\mathbb{R}^{m \times d}) \) for a.e. \( t \in [0, T] \) (see [12, Lemma 1.3.4]). Therefore, by using the Lipschitz continuity of \((\tilde{f}, \tilde{g})\) and the differentiability of \((X, Y, Z)\), one can easily deduce the linear FBSDE (3.5) by applying the operator \( D \) to (3.1b) and the chain rule (see [12, Proposition 1.2.4]). In particular, the random variable \( \partial \tilde{g} T \) and the process \( \partial \tilde{f} \) can be obtained as the weak limits of the sequences \(((\nabla \tilde{g}^\varepsilon)(X_T))_{\varepsilon > 0} \in L^2(\mathcal{F}_T; \mathbb{R}^m) \) and \((\nabla_{xy\varepsilon} \tilde{f}^\varepsilon(\cdot, \Theta))_{\varepsilon > 0} \in \mathcal{H}^2 \), respectively, which implies the desired measurability.

Finally, for each \( 0 \leq \theta < t \leq T \), we have \( Y_t - Y_\theta = -\int_\theta^t \tilde{f}(r, X_r, Y_r, Z_r) \, dr + \int_\theta^t Z_r \, dW_r \). Then for all \( 0 \leq \theta < s \leq t \leq T \), we have \( D_s Y_t = -\int_\theta^t D_s \tilde{f}(r, X_r, Y_r, Z_r) \, dr + Z_s + \int_s^t D_s Z_r \, dW_r \), from which we can conclude that \( D_t Y_t = Z_t \) by setting \( s = t \).

We then give a more concrete representation of the process \( Z \) based on the relation that \( D_t Y_t = Z_t \), which is essential for the regularity estimates of (1.1) with non-differentiable coefficients; see the discussion at the end of this section for details. Let us introduce the processes \((\partial X, \partial Y, \partial Z) \in S^2(\mathbb{R}^{n \times n}) \times S^2(\mathbb{R}^{m \times n}) \times \mathcal{H}^2(\mathbb{R}^{(m \times d) \times n}) \) satisfying for all \( t \in [0, T] \), \( j = 1, \ldots, n \) that

\[
\partial X_t = \mathbb{I}_n + \int_0^t \partial \tilde{b}_r \, dX_r \, dr + \sum_{k=1}^d \int_0^t \partial \tilde{\sigma}^{(k)}_r \, dX_r \, dW_r^{(k)},
\]

\[
\partial Y_t^{(j)} = \partial \tilde{g}_T \partial X_T^{(j)} + \int_t^T \partial \tilde{f}_r \cdot \partial \Theta_r^{(j)} \, dr - \int_t^T \partial Z_r^{(j)} \, dW_r,
\]

with \( \partial \tilde{b}, (\partial \tilde{\sigma}^{(k)})_{k=1}^d \) are the uniformly bounded processes in (3.4), and \( \partial \tilde{g}_T \) (resp. \( \partial \tilde{f} \)) is the bounded \( \mathcal{F}_T \)-measurable random variable (resp. uniformly bounded process) in (3.5).

The next proposition represents the process \( Z \) by using \( \partial Y \) and the inverse of \( \partial X \), which extends [16, Equation (2.13)] and [10, Equation (3.15)] to the present setting where the coefficients are non-differentiable in the spatial variables and merely measurable in the time variable.

We emphasize that unlike for the classical FBSDEs, the processes \((\partial X, \partial Y, \partial Z)\) do not agree with the first variation processes of solutions \((X, Y, Z)\) to the MV-FBSDE (1.1) (i.e., the derivatives of the solutions with respect to the initial condition \( \xi_0 \)), since the latter ones also involve the derivatives of marginal distributions of the solutions with respect to the initial condition \( \xi_0 \).

**Proposition 3.4.** Assume the setting in Theorem 2.1. Then we have that:

1. (3.7) admits a unique solution \((\partial X, \partial Y, \partial Z) \in S^2(\mathbb{R}^{n \times n}) \times S^2(\mathbb{R}^{m \times n}) \times \mathcal{H}^2(\mathbb{R}^{(m \times d) \times n})\) satisfying for all \( p \geq 2 \) that \( \| \partial X \|_{\mathcal{S}^p} + \| \partial Y \|_{\mathcal{S}^p} + \| \partial Z \|_{\mathcal{H}^p} \leq C_p < \infty \).
(2) For all \( t \in [0, T] \), \( \partial X_t \) is invertible. Moreover, for the inverse \( (\partial X_t)^{-1} \), it holds for some constant \( C_\rho > 0 \) that \( \| \partial X^{-1} \|_{\mathcal{S}_p} \leq C_\rho \) and \( \| \partial X_t^{-1} - \partial X_s^{-1} \|_{\mathcal{L}_p} \leq C_\rho |t - s|^{\frac{1}{2}} \) for all \( t, s \in [0, T] \).

(3) There exists a uniformly bounded process \( \partial v \in \mathcal{S}_\infty(\mathbb{R}^{m \times n}) \) such that it holds for \( d\mathbb{P} \otimes dt \) a.e. that \( Z_t = \partial Y_t \partial X_t^{-1} \tilde{\sigma}(t, X_t) = \partial v(t)\hat{\sigma}(t, X_t) \).

Proof. Due to the boundedness and adaptedness of the coefficients, it is clear that (3.7) is well-posed and admits the moment bounds in Item (1). We then show Item (2) by first introducing the process \( M \) as the solution to the following linear SDE:

\[
M_t = \mathbb{I}_n - \int_0^t M_s \left[ \partial b_s - \sum_{k=1}^d \partial \sigma_s^{(k)} \partial \tilde{\sigma}_s^{(k)} \right] ds - \sum_{k=1}^d \int_0^t M_s \partial \tilde{\sigma}_s^{(k)} dW_s^{(k)}.
\]

Then Itô's formula shows that \( M_t \partial X_t = \partial X_t M_t = \mathbb{I}_n \) for all \( t \in [0, T] \), which implies for all \( t \in [0, T] \) that \( \partial X_t \) is invertible with the inverse \( \partial X_t^{-1} = M_t \) (see [12, p. 126] for details). Standard estimates for linear SDEs then lead to the desired \textit{a priori} estimates of \( \partial X^{-1} \).

Finally, by comparing (3.7) with (3.4) and (3.5), we can deduce from the uniqueness of solutions to linear FBSDEs that \( D_s X_t = \partial X_t \partial X_s^{-1} \tilde{\sigma}(s, X_s) \) and \( D_s Y_t = \partial Y_t \partial X_s^{-1} \tilde{\sigma}(s, X_s) \) for all \( 0 \leq s \leq t \leq T \). Hence, since it holds for \( d\mathbb{P} \otimes dt \) a.e. that \( Z_t = D_t Y_t \), we can obtain the first identity in Item (3) by setting \( s = t \) in \( D_s Y_t = \partial Y_t \partial X_s^{-1} \tilde{\sigma}(s, X_s) \). On the other hand, by using the Lipschitz decoupling field \( v \) of the process \( Y \) and the fact that \( X_t \in D^{1,2}(\mathbb{R}^n) \) and \( Y_t \in D^{1,2}(\mathbb{R}^m) \) for all \( t \in [0, T] \), we can deduce from the chain rule that \( D_s Y_t = \partial v(t)D_s X_t \). In particular, the process \( \partial v \) can be obtained as a weak limit of \( \nabla_x [v * \rho^k](\cdot, X_\cdot) \) in \( H^2(\mathbb{R}^{m \times n}) \) with standard mollifiers \( \rho^k \) of order \( C \). Therefore, by setting \( s = t \) and using the identity that \( D_s X_t = \partial X_t \partial X_s^{-1} \tilde{\sigma}(s, X_s) \), we have that \( Z_t = D_t Y_t = \partial v(t)D_t X_t = \partial v(t)\hat{\sigma}(t, X_t) \), which completes the proof of the second identity in Item (3).

With Proposition 3.4 in hand, we are now ready to prove Theorem 2.1.

\textbf{Proof of Theorem 2.1.} We adapt the arguments for [10, Theorem 3.5] to (3.1) with irregular coefficients, and present the main steps for the reader’s convenience.

Lemma 3.1 and standard moment estimates of FBSDE (3.1) (see e.g. [17, Theorems 3.4.3 and 4.4.4]) give us that \( \| X \|_{\mathcal{S}_p} + \| Y \|_{\mathcal{S}_p} \leq C_\rho (1 + \| \xi_0 \|_{\mathcal{L}_p}) \). Moreover, Proposition 3.4, Item (3) shows for \( d\mathbb{P} \otimes dt \) a.e. that \( |Z_t| \leq C|\tilde{\sigma}(t, X_t)| = C|\sigma(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)})| \), which together with the assumption of \( \sigma \) in (H.1) shows that \( \| Z \|_{\mathcal{S}_p} \leq C_\rho (1 + \| \xi_0 \|_{\mathcal{L}_p}) \).

The Hölder regularity of the processes \( X \) and \( \hat{Y} \) follows directly from the fact that \( (X, Y, Z) \) solves (1.1) (or equivalently (3.1)), together with Hölder’s inequality, the estimate of \( \| (X, Y, Z) \|_{\mathcal{S}_p} \) and the Burkholder-Davis-Gundy inequality (see e.g. [10, Theorem 3.5 (ii)]).

Finally, we establish Item (3) by using the additional assumptions that \( \sigma \) depends only on the flow \( (\mathbb{P}_{(X_t, Y_t)})_{t \in [0, T]} \) and satisfies (2.1). Then, for any \( p \geq 2 \) and \( s, t \in [0, T] \), we can obtain from \( \mathcal{W}_2(\mathbb{P}_U, \mathbb{P}_V) \leq \| U - V \|_{\mathcal{L}_p} \) that

\[
\begin{align*}
&\| \tilde{\sigma}(s, X_s) - \hat{\sigma}(t, X_t) \|_{\mathcal{L}_p} \\
&\leq \| \sigma(s, X_s, Y_s, \mathbb{P}_{(X_s, Y_s)}) - \sigma(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)}) \|_{\mathcal{L}_p} \\
&\quad + \| \sigma(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)}) - \sigma(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)}) \|_{\mathcal{L}_p} \\
&\leq C \{ (1 + \| X_s \|_{\mathcal{L}_p} + \| Y_s \|_{\mathcal{L}_p}) |s - t|^{1/2} + \| X_s - X_t \|_{\mathcal{L}_p} + \| Y_s - Y_t \|_{\mathcal{L}_p} \},
\end{align*}
\]
which along with the estimates in Items (1)-(2) gives us that \( \|\tilde{\sigma}(s,X_s) - \tilde{\sigma}(t,X_t)\|_{L^p} \leq C(p)(1 + \|\xi_0\|_{L^p})|s-t|^{1/2} \) for all \( s, t \in [0,T] \).

Now let \( p \geq 2 \) and \( \varepsilon > 0 \) be arbitrary given constants. Let \( p_\varepsilon := \frac{p(1 + \varepsilon)}{p - \varepsilon} > 1 \) and \( q_\varepsilon := \frac{2(p + \varepsilon)}{p - \varepsilon} > 1 \), we have \( 1/p_\varepsilon + 1/q_\varepsilon + 1/q_\varepsilon = 1 \), which together with Hölder’s inequality shows that \( \|\xi\|_{L^p} \leq \|\xi\|_{L^{p_{\varepsilon}}} \|\psi\|_{L^{q_{\varepsilon}}} \). Let \( \pi = \{0 = t_0 < \cdots < t_N = T\} \) be a partition with stepsize \( |\pi| = \max_i (t_{i+1} - t_i) \geq N^{-1} \). For any given \( i \in \{0, \ldots, N-1\} \) and \( r \in (t_i, t_{i+1}) \), we can deduce from Proposition 3.4 Item (3) that \( Z_r - Z_{t_i} = I_{1,r} + I_{2,r} + I_{3,r}, \) where \( I_{1,r} := \partial Y_r(\partial X_r^{-1} - \partial X_{t_i}^{-1})\tilde{\sigma}(r,X_r) \), \( I_{2,r} := \partial Y_r \partial X_r^{-1} [\tilde{\sigma}(r,X_r) - \tilde{\sigma}(t_i,X_{t_i})] \) and \( I_{3,r} := (\partial Y_r - \partial Y_{t_i})\partial X_r^{-1}\tilde{\sigma}(t_i,X_{t_i}) \). By using Hölder’s inequality, Proposition 3.4 Items (1)-(2) and (3.8), we can deduce that

\[
\|I_{2,r}\|_{L^p} \leq \|\partial Y_r\|_{L^{p_{\varepsilon}}} \|\partial X_r^{-1}\|_{L^{p_{\varepsilon}}} \|\tilde{\sigma}(r,X_r) - \tilde{\sigma}(t_i,X_{t_i})\|_{L^{p+\varepsilon}} \leq C(p,\varepsilon)(1 + \|\xi_0\|_{L^{p+\varepsilon}})|\pi|^\frac{1}{2}. \tag{3.9}
\]

Then we can proceed along the lines of the proof of [10, Theorem 3.5 (iii)] to estimate \( \|I_{1,r}\|_{L^p} \) and \( \|I_{3,r}\|_{L^p} \), and then establish the desired estimate of \( \sum_{i=0}^{N-1} E[\int_{t_i}^{t_{i+1}} \|Z_r - Z_{t_i}\|^2 dr]^{p/2} \) \( \leq C(p,\varepsilon)(1 + \|\xi_0\|_{L^{p+\varepsilon}})|\pi|^\frac{1}{2} \). The term \( \sum_{i=0}^{N-1} E[\int_{t_i}^{t_{i+1}} \|Z_r - Z_{t_i}\|^2 dr]^{p/2} \) can be estimated by using similar arguments, which completes the proof of the estimates in Item (3).

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A Proof of Corollary 2.4

In this section, we prove Corollary 2.4 by adapting the method of continuation in [13, 3] to the present setting.

We first present a stability result for the following family of MV-FBSDEs: for \( t \in [0,T] \),
\[
\begin{align*}
    dX_t &= (\lambda b(t, X_t, Y_t, \mathbb{P}(X_t, Y_t, Z_t)) + \mathcal{I}_{b}^T) \, dt + (\lambda \sigma(t, X_t, Y_t, \mathbb{P}(X_t, Y_t, Z_t)) + \mathcal{I}_{\sigma}^T) \, dW_t, \\
    dY_t &= -\lambda f(t, X_t, Y_t, Z_t, \mathbb{P}(X_t, Y_t, Z_t)) + \mathcal{I}_{f}^T \, dt + Z_t \, dW_t, \\
    X_0 &= \xi, \quad Y_T = \lambda \theta(X_T, \mathbb{P}_{X_T}) + \mathcal{I}_{\theta}^T,
\end{align*}
\]
(A.1)

where \( \lambda \in [0,1] \), \( \xi \in L^2(F_0; \mathbb{R}^n) \), \((\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m) \) and \( \mathcal{I}_{\theta}^T \in L^2(F_T; \mathbb{R}^m) \) are given.

**Lemma A.1.** Suppose the functions \((b, \sigma, f, g)\) satisfy the assumptions in Corollary 2.4. Then there exists a constant \( C > 0 \) such that, for all \( \lambda_0 \in [0,1] \), for every \( \Theta := (X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d} \times \mathbb{R}^m) \) satisfying (A.1) with \( \lambda = \lambda_0 \), functions \((b, \sigma, f, g)\) and some \((\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m), \mathcal{I}_{\theta}^T \in L^2(F_T; \mathbb{R}^m), \xi \in L^2(F_0; \mathbb{R}^n), \) and for every \( \Theta := (X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d} \times \mathbb{R}^m) \) satisfying (A.1) with \( \lambda = \lambda_0 \), another 4-tuple of functions \((\bar{b}, \bar{\sigma}, \bar{f}, \bar{g})\) satisfying merely (H.1), and some \((\bar{\mathcal{I}}^b, \bar{\mathcal{I}}^\sigma, \bar{\mathcal{I}}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m), \bar{\mathcal{I}}_{\theta}^T \in L^2(F_T; \mathbb{R}^m), \xi \in L^2(F_0; \mathbb{R}^n), \) we have that
\[
\begin{align*}
    \|X - \bar{X}\|_{\mathcal{H}^2}^2 + \|Y - \bar{Y}\|_{\mathcal{H}^2}^2 + \|Z - \bar{Z}\|_{\mathcal{H}^2}^2 \\
    &\leq C \left\{ \|\xi - \bar{\xi}\|_{\mathcal{H}^2}^2 + \|\lambda_0(g(X_T, \mathbb{P}_{X_T}) - \bar{g}(\bar{X}_T, \mathbb{P}_{X_T})) + \mathcal{I}_{\theta}^T - \bar{\mathcal{I}}_{\theta}^T\|_{\mathcal{H}^2}^2 \\
    &\quad + \|\lambda_0(b(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta}) - \bar{b}(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta})) + \mathcal{I}^b - \bar{\mathcal{I}}^b\|_{\mathcal{H}^2}^2 \\
    &\quad + \|\lambda_0(\sigma(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta}) - \bar{\sigma}(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta})) + \mathcal{I}^\sigma - \bar{\mathcal{I}}^\sigma\|_{\mathcal{H}^2}^2 \\
    &\quad + \|\lambda_0(f(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta}) - \bar{f}(\cdot, \bar{X}_T, \bar{Y}_T, \mathbb{P}_{\Theta})) + \mathcal{I}^f - \bar{\mathcal{I}}^f\|_{\mathcal{H}^2}^2 \right\},
\end{align*}
\]
(A.2)

**Proof of Lemma A.1.** Throughout this proof, let \( G \in \mathbb{R}^{m \times n} \) be the matrix in Corollary 2.4, let \( \delta \xi = \xi - \bar{\xi}, \delta \mathcal{I}_{\theta}^T = \mathcal{I}_{\theta}^T - \bar{\mathcal{I}}_{\theta}^T, g(X_T) = g(X_T, \mathbb{P}_{X_T}), g(\bar{X}_T) = \bar{g}(\bar{X}_T, \mathbb{P}_{X_T}) \) and \( \bar{g}(\bar{X}_T) = \bar{g}(\bar{X}_T, \mathbb{P}_{X_T}) \), for each \( t \in [0,T] \) let \( \delta \mathcal{I}_{b}^T = \mathcal{I}_{b}^T - \bar{\mathcal{I}}_{b}^T, \delta \mathcal{I}_{\sigma}^T = \mathcal{I}_{\sigma}^T - \bar{\mathcal{I}}_{\sigma}^T, \delta \mathcal{I}_{f}^T = \mathcal{I}_{f}^T - \bar{\mathcal{I}}_{f}^T, b(\Theta_t) = b(t, X_t, Y_t, \mathbb{P}_{\Theta_t}), b(\bar{\Theta}_t) = b(t, \bar{X}_t, \bar{Y}_t, \mathbb{P}_{\Theta_t}) \) and \( b(\bar{\Theta}_t) = b(t, \bar{X}_t, \bar{Y}_t, \mathbb{P}_{\Theta_t}) \). Similarly, we introduce the notation \( \ell(\Theta_t), \ell(\bar{\Theta}_t), \bar{\ell}(\bar{\Theta}_t) \) for \( \ell = \sigma, f \) and \( t \in [0,T] \). We also denote by \( C \) a generic constant, which depends only on the dimensions, the constant \( L \) in (H.1) and the constants \( G, \alpha_1, \beta_1, \beta_2, L_\phi \) in Corollary 2.4, and may take a different value at each occurrence.

By applying Itô’s formula to \((Y_t - \bar{Y}_t, G(X_t - \bar{X}_t))\), we obtain that
\[
\begin{align*}
    \mathbb{E}[\lambda_0(g(X_T) - \bar{g}(\bar{X}_T)) + \delta \mathcal{I}_{\theta}^T, G(X_T - \bar{X}_T)] - \mathbb{E}[\{Y_0 - \bar{Y}_0, G\delta \xi\}] \\
    = \mathbb{E} \left[ \int_0^T \lambda_0(b(\Theta_t) - \bar{b}(\bar{\Theta}_t)) + \delta \mathcal{I}_{b}^T, G^*(Y_t - \bar{Y}_t) + \lambda_0(\sigma(\Theta_t) - \bar{\sigma}(\bar{\Theta}_t)) + \delta \mathcal{I}_{\sigma}^T, G^*(Z_t - \bar{Z}_t) \\
    + (-(\lambda_0(f(\Theta_t) - \bar{f}(\bar{\Theta}_t)) + \delta \mathcal{I}_{f}^T), G(X_t - \bar{X}_t)) \, dt \right].
\end{align*}
\]

Then, by adding and subtracting the terms \( g(\bar{X}_T), b(\bar{\Theta}_t), \sigma(\bar{\Theta}_t), f(\bar{\Theta}_t) \) and applying the mono-
tonicity condition, we can deduce that
\[
\begin{align*}
\lambda_0\alpha_1\phi_1(X_T, \bar{X}_T) & + E[(\lambda_0 g(X_T) - \bar{g}(X_T)) + \delta I_T^g, G(X_T - \bar{X}_T)] - E[(Y_0 - \bar{Y}_0, G\delta \xi)] \\
& \leq E \left[ \int_0^T \left( \lambda_0 (b(\bar{\Theta}_t) - \bar{b}(\Theta_t)) + \delta I_t^b, G^*(Y_t - \bar{Y}_t) \right) + \left( \lambda_0 (\sigma(\bar{\Theta}_t) - \bar{\sigma}(\Theta_t)) + \delta I_t^\sigma, G^*(Z_t - \bar{Z}_t) \right) \right] \\
& \quad + \left( - \lambda_0 (f(\bar{\Theta}_t) - \bar{f}(\Theta_t)) + \delta I_t^f, G(X_t - \bar{X}_t) \right) dt \\
& - \lambda_0 \left( \beta_1\phi_1(X_t, \bar{X}_t) + \beta_2\phi_2(t, \Theta_t, \bar{\Theta}_t) \right) dt,
\end{align*}
\]
which together with Young’s inequality yields for each \( \varepsilon > 0 \) that
\[
\begin{align*}
\lambda_0\alpha_1\phi_1(X_T, \bar{X}_T) & + \lambda_0 \int_0^T \left( \beta_1\phi_1(X_t, \bar{X}_t) + \beta_2\phi_2(t, \Theta_t, \bar{\Theta}_t) \right) dt \\
& \leq \varepsilon (\|X_T - \bar{X}_T\|_{S^2}^2 + \|Y_0 - \bar{Y}_0\|_{S^2}^2 + \|\Theta - \bar{\Theta}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS},
\end{align*}
\]
where RHS denotes the right-hand side of (A.2).

We now separate our discussion into two cases: (1) \( \beta_2 > 0 \) and the estimate (2.3) holds; (2) \( \alpha_1, \beta_1 > 0 \) and the estimate (2.4) holds. For the first case, we can obtain from (A.3) and \( \lambda_0, \alpha_1, \beta_1 \geq 0 \) that it holds for all \( \varepsilon > 0 \) that,
\[
\begin{align*}
\lambda_0 \int_0^T \phi_2(t, \Theta_t, \bar{\Theta}_t) dt & \leq \varepsilon (\|X_T - \bar{X}_T\|_{S^2}^2 + \|Y - \bar{Y}\|_{S^2}^2 + \|Z - \bar{Z}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS}.
\end{align*}
\]
Then, by using the Burkholder-Davis-Gundy inequality, (2.3), Gronwall’s inequality and the fact that \( \lambda_0 \in [0, 1] \), we can deduce that
\[
\begin{align*}
\|X - \bar{X}\|_{S^2}^2 & \leq C \left( \int_0^T \lambda_0\phi_2(t, \Theta_t, \bar{\Theta}_t) dt + \|\xi - \bar{\xi}\|_{L^2}^2 \\
& \quad + \|\lambda_0 (b(\bar{\Theta}) - \bar{b}(\Theta)) + \delta I^b\|_{H^2}^2 + \|\lambda_0 (\sigma(\bar{\Theta}) - \bar{\sigma}(\Theta)) + \delta I^\sigma\|_{H^2}^2 \right),
\end{align*}
\]
which together with (A.4) yields for all small enough \( \varepsilon > 0 \) that
\[
\|X - \bar{X}\|_{S^2}^2 \leq \varepsilon (\|Y - \bar{Y}\|_{S^2}^2 + \|Z - \bar{Z}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS}.
\]
Moreover, by standard estimates for MV-BSDEs (1.1b), we can obtain that
\[
\begin{align*}
\|Y - \bar{Y}\|_{S^2}^2 + \|Z - \bar{Z}\|_{H^2}^2 \\
& \leq C \left( \|X - \bar{X}\|_{S^2}^2 + \|\lambda_0 (g(X_T) - \bar{g}(X_T)) + \delta I^g\|_{L^2}^2 + \|\lambda_0 (f(\Theta) - \bar{f}(\Theta)) + \delta I^f\|_{H^2}^2 \right),
\end{align*}
\]
which completes the desired estimate (A.2) for the first case.

For the second case with \( \alpha_1, \beta_1 > 0 \), we can obtain from (A.3) that it holds for all \( \varepsilon > 0 \) that,
\[
\begin{align*}
\lambda_0\phi_1(X_T, \bar{X}_T) & + \lambda_0 \int_0^T \phi_1(X_t, \bar{X}_t) dt \\
& \leq \varepsilon (\|X - \bar{X}\|_{S^2}^2 + \|Y - \bar{Y}\|_{S^2}^2 + \|Z - \bar{Z}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS}.
\end{align*}
\]

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Standard stability estimates for MV-BSDEs with Lipschitz coefficients (see e.g. [17, Theorem 4.2.3]) shows that

\[
\|Y - \bar{Y}\|_{\mathcal{S}^2}^2 + \|Z - \bar{Z}\|_{\mathcal{H}^2}^2 \\
\leq C \left( \|\lambda_0(g(X_T) - \bar{g}(\bar{X}_T))\|_{\mathcal{L}^2}^2 + \|\lambda_0(g(Y_T) - \bar{g}(\bar{Y}_T))\|_{\mathcal{L}^2}^2 \\
+ \|\lambda_0(f(\cdot, X, Y, \bar{Z}, \mathbb{P}(X, \bar{Y}, \bar{Z})) - \bar{f}(\bar{\Theta}))\|_{\mathcal{H}^2}^2 + \|\lambda_0(f(\Theta) - \bar{f}(\bar{\Theta}))\|_{\mathcal{H}^2}^2 \right)
\]

from which, by using (2.4), the fact that \(\lambda_0 \in [0, 1]\) and (A.5), we can deduce for all sufficiently small \(\varepsilon > 0\) that

\[
\|Y - \bar{Y}\|_{\mathcal{S}^2}^2 + \|Z - \bar{Z}\|_{\mathcal{H}^2}^2 \\
\leq C \left( \|\lambda_0(g(X_T) - g(\bar{X}_T))\|_{\mathcal{L}^2}^2 + \|\lambda_0(g(\bar{X}_T))\|_{\mathcal{L}^2}^2 \\
+ \|\lambda_0(f(\cdot, X, \bar{Y}, \bar{Z}, \mathbb{P}(X, \bar{Y}, \bar{Z})) - f(\bar{\Theta}))\|_{\mathcal{H}^2}^2 + \|\lambda_0(f(\bar{\Theta}) - \bar{f}(\bar{\Theta}))\|_{\mathcal{H}^2}^2 \right)
\]

\[
\leq \varepsilon \|X - \bar{X}\|_{\mathcal{S}^2}^2 + C \varepsilon^{-1} \text{RHS}.
\]

Then, standard stability estimates for MV-SDEs with Lipschitz coefficients give that

\[
\|X - \bar{X}\|_{\mathcal{S}^2}^2 \\
\leq C \left( \|\lambda_0(b(\cdot, X, Y, \mathbb{P}(X, Y, Z)) - \bar{b}(\bar{\Theta}))\|_{\mathcal{L}^2}^2 + \|\lambda_0(\sigma(\cdot, X, Y, \mathbb{P}(X, Y, Z)) - \bar{\sigma}(\bar{\Theta}))\|_{\mathcal{L}^2}^2 \right)
\]

\[
\leq C \left( \|Y - \bar{Y}\|_{\mathcal{S}^2}^2 + \|Z - \bar{Z}\|_{\mathcal{H}^2}^2 + \|\lambda_0(b(\Theta) - \bar{b}(\bar{\Theta}))\|_{\mathcal{L}^2}^2 + \|\lambda_0(\sigma(\Theta) - \bar{\sigma}(\bar{\Theta}))\|_{\mathcal{L}^2}^2 \right) \\
\leq C \cdot \text{RHS},
\]

which completes the proof of the desired estimate (A.2) for the second case. \(\square\)

We are now ready to present the proof of Corollary 2.4.

Proof of Corollary 2.4. We shall establish the well-posedness, stability and a priori estimates for (1.1) with an initial time \(t = 0\) and initial state \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) by applying Lemma A.1. Similar arguments apply to a general initial time \(t \in [0, T]\) and initial state \(\xi \in L^2(\mathcal{F}_T; \mathbb{R}^n)\).

Let us start by proving the unique solvability of (1.1) with a given \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\). To simplify the notation, for every \(\lambda_0 \in [0, 1]\), we say \((\mathcal{P}_{\lambda_0})\) holds if for any \(\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n), (\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) \in \mathcal{H}_t^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m)\) and \(\mathcal{I}^f \in L^2(\mathcal{F}_T; \mathbb{R}^m), (A.1)\) with \(\lambda = \lambda_0\) admits a unique solution in \(B := S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^m) \times \mathcal{H}_t^2(\mathbb{R}^{n \times d})\). It is clear that \((\mathcal{P}_0)\) holds since (A.1) is decoupled. Now we show there exists a constant \(\delta > 0\), such that if \((\mathcal{P}_{\lambda_0})\) holds for some \(\lambda_0 \in [0, 1]\), then \((\mathcal{P}_{\lambda'_0})\) also holds for all \(\lambda'_0 \in (\lambda_0, \lambda_0 + \delta) \cap [0, 1]\). Note that this claim along with the method of continuation implies the desired unique solvability of (1.1) (i.e., (A.1) with \(\lambda = 1\), \((\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) = 0\), \(\xi = \xi_0\)).

To establish the desired claim, let \(\lambda_0 \in [0, 1]\) be a constant for which \((\mathcal{P}_{\lambda_0})\) holds, \(\eta \in [0, 1]\) and \((\tilde{\mathcal{I}}^b, \tilde{\mathcal{I}}^\sigma, \tilde{\mathcal{I}}^f) \in \mathcal{H}_t^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m), \tilde{\mathcal{I}}^f \in L^2(\mathcal{F}_T; \mathbb{R}^m), \xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) be arbitrarily given coefficients, we introduce the following mapping \(\Xi : B \to B\) such that for all \(\Theta = (X, Y, Z) \in B, \Xi(\Theta) \in B\) is the solution to (A.1) with \(\lambda = \lambda_0\), \(\tilde{\mathcal{I}}^b = \eta b(t, X_t, Y_t, \mathbb{P}_{\Theta_t}) + \tilde{I}_t^b, \tilde{\mathcal{I}}^\sigma = \eta \sigma(t, X_t, Y_t, \mathbb{P}_{\Theta_t}) + \tilde{I}_t^\sigma, \tilde{\mathcal{I}}^f = \eta f(t, \Theta_t, \mathbb{P}_{\Theta_t}) + \tilde{I}_t^f\), and \(\mathcal{I}^f = \eta g(X_T, \mathbb{P}_{X_T}) + I_T^f\), which is well-defined due to the fact that \(\lambda_0 \in (0, 1)\) satisfies the induction hypothesis. Observe that by setting \((b, \sigma, f, g) = (b, \sigma, f, g)\)
in Lemma A.1, we see that there exists a constant $C > 0$, independent of $\lambda_0$, such that it holds for all $\Theta, \Theta' \in \mathbb{B}$ that
\[
\|\Xi(\Theta) - \Xi(\Theta')\|^2 \leq C \left\{ \|\eta(g(X_T, \mathbb{P}_T) - g(X'_T, \mathbb{P}_T'))\|^2_{L^2} + \|\eta(b(\cdot, X, \mathbb{P}_\Theta) - b(\cdot, X', Y', \mathbb{P}_{\Theta'}))\|^2_{H^2} \\
+ \|\eta(\sigma(\cdot, X, Y, \mathbb{P}_\Theta) - \sigma(\cdot, X', Y', \mathbb{P}_{\Theta'}))\|^2_{H^2} + \|\eta(f(\cdot, \Theta, \mathbb{P}_\Theta) - f(\cdot, \Theta', \mathbb{P}_{\Theta'}))\|^2_{H^2} \right\}
\leq C\eta^2\|\Theta - \Theta'\|^2_{\mathbb{B}},
\]
which shows that $\Xi$ is a contraction when $\eta$ is sufficiently small (independent of $\lambda_0$), and subsequently leads to the desired claim due to Banach’s fixed point theorem.

For any given $\xi, \xi' \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, the desired stochastic stability of (1.1) follows directly from Lemma A.1 by setting $\lambda = 1$, $(\xi, \xi', f, g) = (\xi, \xi', f, g)$, $(I^b, I^c, I^f) = (I^b, I^c, I^f)$, and $\xi = \xi'$. Moreover, for any given $\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, by setting $\lambda = 1$, $(\xi, \xi', f, g) = (\xi, \xi', f, g)$ (which clearly satisfies (H.1)), $(I^b, I^c, I^f) = (I^b, I^c, I^f)$, and $\xi = 0$ and $(X, Y, Z) = 0$ in Lemma A.1, we can deduce the estimate that
\[
\|X\|^2_{L^2} + \|Y\|^2_{L^2} + \|Z\|^2_{H^2} \leq C \left\{ \|\xi\|^2_{L^2(0,T)} + \|\xi(0, \delta_{0,m})\|^2_{L^2(0,T)} + \|\xi(0, \delta_{0,m-1})\|^2_{L^2(0,T)} \right\} \leq C(1 + \|\xi\|^2_{L^2(0,T)}),
\]
which shows the desired moment bound of the processes $(X, Y, Z)$.

Finally, for a given initial condition $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, we can conclude the desired regularity estimates for solutions to (1.1) from Theorem 2.1 and Proposition 2.2, which completes the proof of Corollary 2.4.

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