Maximum principle for optimal control of stochastic evolution equations with recursive utilities

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Abstract

We consider the optimal control problem of stochastic evolution equations in a Hilbert space under a recursive utility, which is described as the solution of a backward stochastic differential equation (BSDE). A very general maximum principle is given for the optimal control, allowing the control domain not to be convex and the generator of the BSDE to vary with the second unknown variable \( z \). The associated second-order adjoint process is characterized as a unique solution of a conditionally expected operator-valued backward stochastic integral equation.

Keywords. Stochastic evolution equations, nonconvex control domain, recursive optimal control, maximum principle, operator-valued backward stochastic integral equations.

AMS 2020 Subject Classifications. 93E20, 60H15, 60G07, 49K27, 60H20.

1 Introduction

In this paper, we consider the optimal control problem of stochastic evolution equations (SEEs)

\[
\begin{aligned}
dx(t) &= [A(t)x(t) + a(t, x(t), u(t))]dt + [B(t)x(t) + b(t, x(t), u(t))]dw(t), \quad t \in [0, T], \\
x(0) &= x_0 \in H: \text{a Hilbert space}
\end{aligned}
\]

with a recursive utility which solves the backward stochastic differential equation (BSDE)

\[
y(t) = h(x(T)) + \int_t^T k(s, x(s), y(s), z(s), u(s))ds - \int_t^T z(s)dw(s).
\]

Here, \( w(\cdot) \) is a Brownian motion, \((A(t), B(t))\) are random linear unbounded operators for \( t \in [0, T] \), \((a, b, h, k)\) are nonlinear functions and \( u(\cdot) \) is a control process, and taking values in a given metric space. The objective is to minimize the initial value \( y(0) \) as a functional of the control:

\[
J(u(\cdot)) := y(0).
\]

The notion of a recursive utility in continuous time was introduced by Duffie and Epstein [7] and generalized to the form of (1.2) in Peng [26] and El Karoui, Peng and Quenez [10]. When \( k \) is invariant with \((y, z)\), by taking expectation on both sides of (1.2), we get

\[
J(u(\cdot)) = \mathbb{E}[h(x(T))] + \int_0^T k(t, x(t), u(t))dt,
\]

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and the stochastic optimal control problem is reduced to the conventional one, which has been addressed in

\[ 6, 12, 20. \]

Pontryagin’s maximum principle for optimally controlled ordinary differential equations is a milestone in the modern optimal control theory. By now, the maximum principle for optimally controlled finite-dimensional systems is quite complete. The maximum principle for a general stochastic optimal control problem was finally given by Peng \[ 25, \] by introducing a second-order adjoint process which solves a matrix-valued BSDE. In the extension to incorporate the recursive utility, an essential difficulty is how to derive the second-order variational equation of the recursive BSDE \[ 14, 10. \] It was listed as an open problem by Peng \[ 27. \] Until recently, Hu \[ 16. \] completely solved this problem by developing a clever Taylor’s expansion, so to reduce the order of the variation of the recursive BSDEs.

To formulate the counterpart of the infinite-dimensional stochastic optimal control system, a crucial issue is the characterization of the second-order adjoint process \( P \), which takes values in the space \( \mathcal{L}(H) \) of all bounded linear operators from \( H \) to \( H \). Since the operator space \( \mathcal{L}(H) \) is not a (separable) Hilbert space, the dynamics of the second adjoint process could not be described by a conventional BSDE as in the finite-dimensional case. In the existing maximum principles for the conventional stochastic optimal control problem, the second-order process \( P \) is given in various ways. Lü and Zhang \[ 20, 21. \] utilize the notion of transposition solutions in the context of real-valued equations, assuming the coefficients, such as the terminal condition and the generator of the equation, to be strongly measurable (hence separably valued; see [17, Theorem 2.1]) and the space \( L^2(\mathcal{F}_T) \) to be separable. Derived from the limit of the quadratic terms in the variational calculation of the maximum principle, Du and Meng \[ 5. \] and Fuhrman, Hu and Tessitore \[ 12. \] define \( P \) through a stochastic bilinear form. In both approaches, no dynamics of the second adjoint process are given. On the other hand, Guatteri and Tessitore \[ 13, 14. \] characterize \( P \) using the mild solution of an operator-valued BSDE. They impose either the Hilbert-Schmidt assumption on the coefficients (which can be relaxed only for a suitable limit of solutions with such data, referred to as a generalized solution) or a rather restrictive regularity condition on the unbounded operators. Similarly, Stannat and Wessels \[ 30. \] employ a function-valued backward SPDE when the coefficients (of the system and the cost functional) depend on the state variables in a Nemytskii manner. However, their diffusion coefficient contains no unbounded operator (this also happens in \[ 12, 13, 14, 20, 21. \]) and is further required to have a very high regularity when the space dimension is greater than one (see [30, Remark 4.3]).

The aim of this paper is to study the maximum principle for the optimal control problem \( 1.3 \) of infinite-dimensional stochastic system with recursive utilities. To characterize the dynamics of the second-order adjoint process \( P \), we propose a notion of conditionally expected operator-valued backward stochastic integral equations (BSEIs in short) to serve as the second-order adjoint equations. The formulation of our BSIEs is very naturally inspired by the variation of constants method for operator-valued SPDEs (see Remark \[ 2.21 (i) \]). Under mild conditions, the existence and uniqueness of solutions to the operator-valued BSIEs is obtained in virtue of a concept of aggregated-defined operator-valued backward stochastic integral equations (BSIEs in short) to serve as the second-order adjoint equations. The formulation of

\[ \langle 4.10 \rangle \]

we propose a notion of conditionally expected operator-valued backward stochastic integral equations (BSEIs in short) to serve as the second-order adjoint equations. The formulation of our BSIEs is very naturally inspired by the variation of constants method for operator-valued SPDEs (see Remark \[ 2.21 (i) \]). Under mild conditions, the existence and uniqueness of solutions to the operator-valued BSIEs is obtained in virtue of a concept of aggregated-defined operator-valued conditional expectation and a contraction mechanism, without imposing additional separability assumption on the coefficients.

On the other hand, the Itô’s formulas (or the duality formulas) for \( \langle P(t)x(t), x(t) \rangle \) in the above mentioned works of characterizing \( P \) require that the homogeneous terms in both equations of \( P \) and \( x \) are dual (in a proper sense) so that they can cancel out in the final duality formula, which are not satisfied for our recursive utility context. In this paper, to obtain the maximum condition, we shall derive a more general Itô’s formula in which some homogeneous terms in both the equations of \( P \) and \( x \) remain to appear (see Theorem \[ 2.23 \] and Remark \[ 2.21 (iii) \]), by using the explicit formula of linear BSDEs and an approximation argument. Furthermore, unlike the finite-dimensional or non-recursive case, the variational equations of utility BSDE \[ 4.10 \] involve additional terms \( \langle p(\cdot), B(\cdot)x^{1,\rho}(\cdot) \rangle \) and \( \langle p(\cdot), B(\cdot)x^{2,\rho}(\cdot) \rangle \), which incorporate the unbounded operator \( B \) and thus cannot be handled using the usual estimates for \( p \) and \( x^{1,\rho}, x^{2,\rho} \) in \( H \). Here, \( p \) is the first-order adjoint process, \( x^{1,\rho} \) and \( x^{2,\rho} \) are the solutions of the first- and second-order variational equations for the state equation \[ 1.1 \] respectively. To overcome this difficulty, we deduce and utilize an \( L^\beta \)-estimate of \( p \) in the space \( V \) (see \[ 3.0 \]), the proof of Proposition \[ 5.5 \] and Remark \[ 5.6 \].

The rest of this paper is organized as follows. In Section 2, we introduce a conditionally expected
operator-valued BSIE and further give its Itô’s formula. We formulate our infinite-dimensional optimal control problem under a recursive utility and derive the maximum principle in Section 3. The appendix includes the proofs of some important technical results used in the paper.

2 Conditionally expected operator-valued BSIEs

In this section, we give an existence and uniqueness result for a conditionally expected operator-valued backward stochastic integral equation (BSIE). It will be used to characterize the dynamics of the second-order adjoint process in the maximum principle for optimally controlled stochastic evolution equations (SEEs).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. Fix a terminal time \( T > 0 \), let \( \mathcal{F} := \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) be a filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \) satisfying the usual conditions. We denote by \( \| \cdot \|_X \) the norm on a Banach space \( X \). By \( \mathcal{L}(X;Y) \), we denote the space of all bounded linear operators from \( X \) to another Banach space \( Y \), equipped with the operator norm. We write \( \mathcal{L}(X) \) for \( \mathcal{L}(X;X) \).

Let \( H \) be a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). We adopt the standard identification viewpoint of \( \mathcal{L}(H;\mathbb{R}) = H \). By \( M^* \), we denote the adjoint of an operator \( M \). We denote by \( I_d \) the identity operator on \( H \).

Given a sub-\( \sigma \)-algebra \( \mathcal{G} \) of \( \mathcal{F} \). For \( \alpha \geq 1 \), we denote by \( L^\alpha(\mathcal{G}, H) \) the space of \( H \)-valued \( \mathcal{G} \)-measurable mapping \( y \) with norm \( \|y\|_{L^\alpha(\mathcal{G}, H)} = \left\{ \mathbb{E}[\|y\|_{2, \mathcal{G}}^\alpha] \right\}^{\frac{1}{\alpha}} \), and by \( L^2(0, T; H) \) (resp. \( L^2(0, T; H) \)) the space of \( H \)-valued progressively measurable processes \( y(\cdot) \) with norm \( \|y\|_{L^2(0, T; H)} = \left\{ \mathbb{E}\left[ \int_0^T \|y(t)\|_{H}^2 dt \right] \right\}^{\frac{1}{2}} \) (resp. \( \|y\|_{L^2(0, T; H)} = \left\{ \mathbb{E}[\int_0^T \|y(t)\|_{H}^2 dt] \right\}^{\frac{1}{2}} \)). We write \( L^\alpha(\mathcal{G}), L^2(0, T) \) and \( L^2(0, T) \) for \( L^\alpha(\mathcal{G}, \mathbb{R}), L^2(0, T; \mathbb{R}) \) and \( L^2(0, T; \mathbb{R}) \), respectively.

We say a mapping \( Z : \Omega \rightarrow \mathcal{L}(H) \) is weakly \( \mathcal{G} \)-measurable if for each \( u, v \in H \times H \), \( \langle Zu, v \rangle : \Omega \rightarrow \mathbb{R} \) is \( \mathcal{G} \)-measurable. A process \( Y : \Omega \times [0, T] \rightarrow \mathcal{L}(H) \) is said to be weakly progressively measurable (weakly adapted, resp.) if for each \( u, v \in H \times H \), the process \( \langle Yu, v \rangle : \Omega \times [0, T] \rightarrow \mathbb{R} \) is progressively measurable (adapted, resp.).

By \( L^\alpha_{\mathcal{G}}(\mathcal{G}, \mathcal{L}(H)) \), we denote the space of \( \mathcal{L}(H) \)-valued weakly \( \mathcal{G} \)-measurable mapping \( F \) with norm \( \|F\|_{L^\alpha_{\mathcal{G}}(\mathcal{G}, \mathcal{L}(H))} = \left\{ \mathbb{E}[\|F\|_{\mathcal{L}(H)}^\alpha] \right\}^{\frac{1}{\alpha}} \). Since there is a countable dense subset \( V \) of \( H \) such that

\[
\|F(\omega)\|_{\mathcal{L}(H)} = \sup_{(u,v)\in V\times V, \|u\|,\|v\|\leq 1} |\langle F(\omega)u, v \rangle|, \quad \omega \in \Omega,
\]

the real-valued function \( \omega \mapsto \|F(\omega)\|_{\mathcal{L}(H)} \) is \( \mathcal{G} \)-measurable and the norm \( \|F\|_{L^\alpha_{\mathcal{G}}(\mathcal{G}, \mathcal{L}(H))} \) is well-defined. Similarly, we denote by \( L^\alpha_{\mathcal{F},w}(0, T; \mathcal{L}(H)) \) (resp. \( L^\alpha_{\mathcal{F},w}(0, T; \mathcal{L}(H)) \)) the space of \( \mathcal{L}(H) \)-valued weakly progressively measurable processes \( F(\cdot) \) with norm \( \|F\|_{L^\alpha_{\mathcal{F},w}(0, T; \mathcal{L}(H))} = \left\{ \mathbb{E}[\int_0^T \|F(t)\|_{\mathcal{L}(H)} dt] \right\}^{\frac{1}{\alpha}} \) (resp. \( \|F\|_{L^\alpha_{\mathcal{F},w}(0, T; \mathcal{L}(H))} = \left\{ \mathbb{E}[\int_0^T \|F(t)\|_{\mathcal{L}(H)}^2 dt] \right\}^{\frac{1}{2}} \)). From standard arguments, we can see that \( L^\alpha_{\mathcal{G}}(\mathcal{G}, \mathcal{L}(H)), L^2_{\mathcal{F},w}(0, T; \mathcal{L}(H)) \) and \( L^2_{\mathcal{F},w}(0, T; \mathcal{L}(H)) \) are all Banach spaces. In the following, we shall not distinguish two random variables if they coincide \( P \)-a.s. and two processes if one is a modification of the other, unless other stated.

Remark 2.1 In general, there are mainly three kinds of measurability notions for Banach space-valued random variables: strongly measurable (can be approximated by a sequence of simple measurable functions), measurable (the preimage of each Borel set is measurable) and weakly measurable (the composition with any element in the dual space or in a proper subspace (called a norming subspace; see [21, p. 2]) of the dual space is a real-valued measurable function). These three notions are equivalent in a separable Banach space (see [21, Theorem 1.5 and Prop. 1.8]) and it is not necessary to indicate the notion of measurability in the above for \( H \)-valued random mappings. Moreover, the notion of “measurable” does not work well in the non-separable case since even the sum of two measurable functions may not be measurable (see [23]). The
operator space $\mathcal{L}(H)$ is not separable in general (even when $H$ is; see [15] Solution 99), so these notions are quite different for it. We adopt the above weak measurability notion for $\mathcal{L}(H)$-valued mappings in which the test functions are from $H \times H$. Note that $H \times H$ can be regarded as a subset of the dual of $\mathcal{L}(H)$ by taking $f_{u,v}(z) = \langle z(u), v \rangle$, for $z \in \mathcal{L}(H)$ and $(u, v) \in H \times H$ and $\text{span}(H \times H)$ is a norming subspace of $\mathcal{L}(H)$. Thus this weak measurability notion is still one kind of standard forms.

Denote by $L_w$ the weak $\sigma$-algebra generated by all the sets in the form of
\[ \{ z \in \mathcal{L}(H) : \langle zu, v \rangle \in A \}, \quad u, v \in H, \; A \in \mathcal{B}(\mathbb{R}). \]
Then it is straightforward to verify that $Z : \Omega \to \mathcal{L}(H)$ is weakly $\mathcal{G}$-measurable if and only if it is measurable from $(\Omega, \mathcal{G})$ to $(\mathcal{L}(H), L_w)$ (see also [8]). Similarly, $Y : \Omega \times [0, T] \to \mathcal{L}(H)$ is weakly progressively measurable if and only if it is measurable from $(\Omega \times [0, T], \mathcal{P})$ to $(\mathcal{L}(H), L_w)$, where $\mathcal{P}$ is the progressive $\sigma$-algebra on $\Omega \times [0, T]$.

2.1 Conditional expectation for operator-valued random variables

The operator-valued BSIE is based on a notion of conditional expectations for random variables taking values in the operator space $\mathcal{L}(H)$. As is well known, the classical theory on the conditional expectations for Banach or Hilbert space-valued random variables requires the separability of the value spaces (see, e.g., [3, 29]). But in general the operator space $\mathcal{L}(H)$ is not separable and thus the above-mentioned result does not apply. In this subsection we shall construct a new kind of conditional expectations for operator-valued random variables by exploring the separability of $H$, rather than that of $\mathcal{L}(H)$ (which is the case when the classical Banach or Hilbert space-valued conditional expectation theory applies to this situation).

Recall that for any Banach space $X$, we have the identity (see [2] for more details)
\[ \mathcal{L}_{2}(H \times H; X) = \mathcal{L}(H; \mathcal{L}(H; X)) \]
by identifying $\tilde{\varphi} \in \mathcal{L}_{2}(H \times H; X)$ with $\varphi \in \mathcal{L}(H; \mathcal{L}(H; X))$ through
\[ \tilde{\varphi}(u, v) := \varphi(u)v, \quad \forall (u, v) \in H \times H, \]
where $\mathcal{L}_{2}(H \times H; X)$ is the space of all bounded bilinear operators from $H \times H$ to $X$, equipped with the operator norm. Thus,
\[ \mathcal{L}(H; \mathcal{L}(H; L^1(\mathcal{G}))) = \mathcal{L}_{2}(H \times H; L^1(\mathcal{G})). \]
From $\mathcal{L}(H; \mathbb{R}) = H$, we also have the isometry
\[ \mathcal{L}_{2}(H \times H) := \mathcal{L}_{2}(H \times H; \mathbb{R}) = \mathcal{L}(H; \mathcal{L}(H; \mathbb{R})) = \mathcal{L}(H; H) = \mathcal{L}(H). \]
This new space $\mathcal{L}_{2}(H \times H)$ is easier to work with than the previous $\mathcal{L}(H)$ and is more essential for us to construct the conditional expectations. So we shall state the construction (of the conditional expectations) for $\mathcal{L}_{2}(H \times H)$-valued random variables, and the original $\mathcal{L}(H)$ form can be obtained directly via the above isometry after this procedure completes.

2.1.1 Existence of the conditional expectation

We adopt the same weak measurability meaning for $\mathcal{L}_{2}(H \times H)$-valued random variables according to the isometry $\mathcal{L}_{2}(H \times H) = \mathcal{L}(H)$. That is, a mapping $Z : \Omega \to \mathcal{L}_{2}(H \times H)$ is called $\mathcal{G}$-weakly measurable if for each $(u, v) \in H \times H$, $Z(u, v) : \Omega \to \mathbb{R}$ is $\mathcal{G}$-measurable. The definition of weakly progressive measurability and weakly adaptedness for $\mathcal{L}_{2}(H \times H)$-valued processes is similar. In the same manner, we define $L^o_w(G, \mathcal{L}_{2}(H \times H))$ as the space of $\mathcal{L}_{2}(H \times H)$-valued weakly $\mathcal{G}$-measurable mapping $F$ with norm $\| F \|^L_\mathcal{L}_{2}(G, \mathcal{L}_{2}(H \times H)) = \{ \mathbb{E}[\| F \|^o] \}^{\frac{1}{2}}$ and have $L^o_w(G, \mathcal{L}_{2}(H \times H)) = L^w_w(G, \mathcal{L}(H))$. 

4
It is very natural to define the conditional expectation for $\mathcal{L}_2(H \times H)$-valued random variables, i.e., for an $\mathcal{L}_2(H \times H)$-valued $Y$, to find an $\mathcal{L}_2(H \times H)$-valued $\mathbb{E}[Y|\mathcal{G}]$ as its conditional expectation. But in the infinite-dimensional case, the quantity to be conditionally expected in the formulation of the BSEs later lies in a larger space $\mathcal{L}_2(H \times H; L^1(\mathcal{F}))$ (see (2.13)). So in the following we shall define conditional expectations for this larger class (the conditional expectation is still $\mathcal{L}_2(H \times H)$-valued), and the $\mathcal{L}_2(H \times H)$-valued situation can be regarded as a special case.

We first verify that

\[
L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H)) \subset \mathcal{L}_2(H \times H; L^1(\mathcal{G})).
\]

Indeed, for any $Y \in L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H))$ and $(u, v) \in H \times H$, from the definition of weak measurability, we see that $Y(u, v) \in \mathcal{G}$. Moreover,

\[
\mathbb{E}[|Y(u, v)|] \leq \mathbb{E}[\|Y\|_{\mathcal{L}_2(H \times H)}\|u\|_H\|v\|_H] = \mathbb{E}[\|Y\|_{\mathcal{L}_2(H \times H)}\|u\|_H\|v\|_H] < \infty,
\]

and thus the mapping $(u, v) \mapsto Y(u, v)$ is bounded bilinear from $H \times H$ to $L^1(\mathcal{G})$. So $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$.

The main difference between the elements in $L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H))$ and $\mathcal{L}_2(H \times H; L^1(\mathcal{G}))$ is that the definition and bilinearity for the one in the first space is pointwise or say, independent the effect arguments $(u, v) \in H \times H$, while the definition and bilinearity for the one in the second space is only in a rough way and may depend on its effect arguments $(u, v) \in H \times H$. To be more detailed, given any $Y \in L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H))$, for each (or at least $P$-a.s. $\omega$, we have $Y(\omega) \in \mathcal{L}_2(H \times H)$, which is also

\[
Y(\alpha_1 u_1 + u_2, v_1)(\omega) = \alpha_1 Y(u_1, v_1)(\omega) + Y(u_2, v_1)(\omega),
\]

for every $(u_1, v_1), (u_2, v_2) \in H \times H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ (The negligible set is universal for all $(u_1, v_1), (u_2, v_2) \in H \times H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$). Whereas for $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$, since we do not distinguish the $P$-a.s. equal elements in $L^1(\mathcal{G})$, we can only have that, for any $(u_1, v_1), (u_2, v_2) \in H \times H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

\[
Y(\alpha_1 u_1 + u_2, v_1) = \alpha_1 Y(u_1, v_1) + Y(u_2, v_1) \quad \text{and} \quad Y(u_1, \alpha_2 v_1 + v_2) = \alpha_2 Y(u_1, v_1) + Y(u_1, v_2), \quad P\text{-a.s.}
\]

(The negligible set depends on $(u_1, v_1), (u_2, v_2) \in H \times H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$).

For $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}))$, we call an $\mathcal{L}_2(H \times H)$-valued weakly $\mathcal{G}$-measurable mapping $Z$ the conditional expectation of $Y$ with respect to $\mathcal{G}$, denoted by $\mathbb{E}[Y|\mathcal{G}]$, if for each $(u, v) \in H \times H$,

\[
Z(u, v) = \mathbb{E}[Y(u, v)|\mathcal{G}], \quad P\text{-a.s.}
\]

meaning that $Z$ coincides with the classical conditional expectation at all the test points $(u, v)$.

In general, for $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}))$, we always have that the mapping defined by $H \times H \ni (u, v) \mapsto \mathbb{E}[Y(u, v)|\mathcal{G}]$ (we can still denote it $\mathbb{E}[Y|\mathcal{G}]$ by a slight abuse of the notations) belongs to $\mathcal{L}_2(H \times H; L^1(\mathcal{G}))$. Indeed,

\[
\mathbb{E}[\mathbb{E}[Y(u, v)|\mathcal{G}]] \leq \mathbb{E}[\|Y(u, v)\|_{\mathcal{L}_2(H \times H)}] \leq C\|u\|_H\|v\|_H,
\]

where the last inequality is due to $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}))$. But whether some of its versions can be operator $\mathcal{L}_2(H \times H)$-valued so that it is the conditional expectation we are searching for, is not known. To find such a version can be regarded as an aggregation problem of constructing a better version among all the equivalent admissible rough classes, which will be discussed in the next subsection.

We generally have the following existence and uniqueness theorem on the conditional expectation of an operator-valued random variable.

**Theorem 2.2** Let $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}))$. Then the conditional expectation $\mathbb{E}[Y|\mathcal{G}]$ exists and is integrable (i.e., $\mathbb{E}[Y|\mathcal{G}] \in L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H))$) if and only if the mapping $(u, v) \mapsto \mathbb{E}[Y(u, v)|\mathcal{G}] \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$ satisfies the domination condition

\[
\mathbb{E}[Y(u, v)|\mathcal{G}] \leq g\|u\|_H\|v\|_H, \quad P\text{-a.s., } \forall (u, v) \in H \times H,
\]

for some $0 \leq g \in L^1(\mathcal{G})$. Moreover, such an $\mathbb{E}[Y|\mathcal{G}]$ is unique (up to $P$-a.s. equality) and satisfies

\[
\|\mathbb{E}[Y|\mathcal{G}]\|_{\mathcal{L}_2(H \times H)} \leq g, \quad P\text{-a.s.}
\]
Moreover, such an integrable $H$ and in particular, when $k=1$, it constructs the conditional expectation for $H$-valued random variables in a slightly new way. Indeed, at this case, from $H = \mathcal{L}(H; \mathbb{R})$, the relationship \((\ref{2.3})\) becomes $L^1(\mathcal{G}, H) = L^1(\mathcal{L}(H; \mathbb{R})) \subset \mathcal{L}(H; L^1(\mathcal{G}))$ (we delete the subscript $w$ for the first and second spaces since the measurability and weak measurability are the same now due to the separability of $H$); the conditional expectation for $Y \in \mathcal{L}(H; L^1(\mathcal{F}))$ is a $H$-valued \(\mathcal{G}\)-measurable mapping $Z$ satisfying $\langle Z, u \rangle = Z(u) = \mathbb{E}[Y(u)|\mathcal{G}]$ $P$-a.s., for all $u \in H$; the above theorem reads: for $Y \in \mathcal{L}(H; L^1(\mathcal{F}))$, the conditional expectation $\mathbb{E}[Y|\mathcal{G}] \in L^1(\mathcal{G}, H)$ exists iff

$$\|\mathbb{E}[Y(u)|\mathcal{G}]\| \leq g\|u\|_H, \quad P\text{-a.s., } \forall u \in H,$$

for some $0 \leq g \in L^1(\mathcal{G})$, $\mathbb{E}[Y|\mathcal{G}]$ is unique and satisfies $\|\mathbb{E}[Y|\mathcal{G}]\|_H \leq g$, $P$-a.s. This generalizes the classical result for the conditional expectation of $H$-valued random variables since $Y$ does not need to be true $H$-valued.

Remark 2.4 From the proofs latter, the condition $Y \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}))$ in the definition of the conditional expectation and in Theorem \ref{2.3} can be weaken to $(u, v) \mapsto \mathbb{E}[Y(u,v)|\mathcal{G}] \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$. Note that, if the conditional expectation $\mathbb{E}[Y|\mathcal{G}] \in L^1_w(\mathcal{G}, \mathcal{L}_2(H \times H))$ exists, this new condition also holds (see \((\ref{2.3})\)), so it (plus the domination condition) is the weakest condition to guarantee the existence of integrable $\mathcal{L}_2(H \times H)$-valued conditional expectations. This generalization also holds for the $k$-linear operator case, and in particular, when $k=1$, it provides a necessary and sufficient characterization for the existence of integrable $H$-valued conditional expectations.

2.1.2 An aggregation theorem and proof of Theorem \ref{2.3}

For a mapping $G \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$, by a version of $G$, we mean another $G' : H \times H \mapsto L^1(\mathcal{G})$ satisfying $G(u,v) = G'(u,v)$ in $L^1(\mathcal{G})$ (which is also, $P$-a.s.), for each $(u,v) \in H \times H$. It is easy to check that $G' \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$.

The construction of the conditional expectation is based on the following aggregation theorem for operator-valued random variables in the space of bilinear mappings.

Theorem 2.5 The mapping $G \in \mathcal{L}_2(H \times H; L^1(\mathcal{G}))$ admits a version $\tilde{G} \in L^1(\mathcal{G}, \mathcal{L}_2(H \times H))$ if and only if the following domination condition holds: there exists some $0 \leq g \in L^1(\mathcal{G})$ such that

$$|G(u,v)| \leq g\|u\|_H\|v\|_H, \quad P\text{-a.s., } \forall (u,v) \in H \times H. \quad (2.5)$$

Moreover, such an $\mathcal{L}_2(H \times H)$-valued version is unique (up to $P$-a.s. equality) and satisfies

$$\|\tilde{G}\|_{\mathcal{L}_2(H \times H)} \leq g, \quad P\text{-a.s.} \quad (2.6)$$

Remark 2.6 The proof is based on an idea of extension from a countable dense subset of indexes, which is motivated from \cite{9}, see also \cite{6, 12, 31}.

Proof. Let $\{e_i\}_{i=1}^\infty$ be a countable basis of $H$.

Step 1: an auxiliary deterministic result. For any given real values $\{a_{ij}\}_{i,j=1}^\infty$, define

$$F(e_i,e_j) := a_{ij}, \quad \text{for } i,j \geq 1.$$
Then $F$ can be extended uniquely to be an element in $\mathcal{L}_2(H \times H)$, which we still denote by $F$, if and only if there exists some constant $C > 0$ such that

$$|\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j | \leq C \| \sum_{i=1}^{n} \alpha_i e_i \|_H \| \sum_{j=1}^{m} \beta_j e_j \|_H, \quad \text{for all } \alpha_i, \beta_j \in \mathbb{Q}, \text{ and integers } n, m \geq 1.$$ 

Moreover, this extension satisfies $\| F \|_\mathcal{L}_2(H \times H) \leq C$.

Indeed, we take a dense linear subspace with field $\mathbb{Q}$ of $H$

$$V := \{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{Q}, \; n \geq 1 \}.$$ 

We define on $V \times V$

$$F(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j) := \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j F(e_i, e_j).$$

It is easy to check that $F$ is a well-defined bilinear mapping with field $\mathbb{Q}$ on $V \times V$ and $\| F(u, v) \| \leq C \| u \|_H \| v \|_H$, for all $(u, v) \in V \times V$. Then by the continuous extension theorem (see, e.g., [13 Lemma 2.4]), $F$ can be extended to be an element in $\mathcal{L}_2(H \times H)$ satisfying $\| F(u, v) \| \leq C \| u \|_H \| v \|_H$, for all $(u, v) \in H \times H$, which is also $\| F \|_\mathcal{L}_2(H \times H) \leq C$.

Now we show that such an extension from basis $\{ e_i \}_{i=1}^{\infty}$ is unique. Let $F^1$, $F^2$ be two such extensions. Then $F^1(e_i, e_j) = F^2(e_i, e_j)$ for each $i, j$, which implies $F^1(u, v) = F^2(u, v)$ for all $(u, v) \in V \times V$ by the bilinearity. Thus from the continuity of the extension, we have $F^1(u, v) = F^2(u, v)$ for all $(u, v) \in H \times H$. That is, $F^1 = F^2$.

The converse of the assertion is trivial.

**Step 2: proof of the theorem.** We fix any versions of $G(e_i, e_j)$ for $i, j \geq 1$. For each given $\omega$, we define the effect of $\hat{G}(\cdot, \cdot)(\omega)$ on the basis:

$$\hat{G}(e_i, e_j)(\omega) := a^\omega_{ij} := G(e_i, e_j)(\omega), \quad i, j \geq 1. \quad (2.7)$$

Since the elements in $V \times V$ is countable, we have from (2.5) that, for $P$-a.s. $\omega$,

$$|\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j a^\omega_{ij} | = |G(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j)(\omega) | \leq g(\omega) \| \sum_{i=1}^{n} \alpha_i e_i \|_H \| \sum_{j=1}^{m} \beta_j e_j \|_H,$$

for all $(u, v) = (\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j) \in V \times V$. \quad (2.8)

We denote by $\Omega_0$ the $\mathcal{G}$-measurable set of full measure in which the inequality (2.8) holds. For each fixed $\omega \in \Omega_0$, we can apply Step 1 to extend $\hat{G}$ to be an element in $\mathcal{L}_2(H \times H)$, which satisfies $\| \hat{G}(\omega) \|_{\mathcal{L}_2(H \times H)} \leq g(\omega)$. On the exception set $\Omega_0 \setminus \Omega_0^P$, let $\hat{G}$ take the zero element in $\mathcal{L}_2(H \times H)$. Thus we obtain a $G \in L^1_\omega(\mathcal{G}, \mathcal{L}_2(H \times H))$ such that (2.6) holds.

Now we prove $\hat{G}$ is a version of $G$. From the construction of $\hat{G}$, we have $\hat{G}(e_i, e_j) = G(e_i, e_j)$ $P$-a.s., for each $i, j$. Assume $(u, v) = (\sum_{i=1}^{\infty} \alpha_i e_i, \sum_{j=1}^{\infty} \beta_j e_j)$, for $\alpha_i, \beta_j \in \mathbb{R}$, $i, j \geq 1$. Then

$$\hat{G}(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \hat{G}(e_i, e_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j G(e_i, e_j) = G(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j), \quad P$-a.s.$$

Letting $n, m \to \infty$ (on a subsequence if necessary), from the continuity of $\hat{G}$ and $G$ ($\hat{G}$ is continuous from $H \times H$ to $\mathbb{R}$ pointwise, and $G$ is continuous from $H \times H$ to $L^1(\mathcal{G})$ by (2.5)), we obtain

$$\hat{G}(u, v) = G(u, v), \quad P$-a.s.$
To see the uniqueness, consider two \( \Sigma_2(\mathcal{H} \times H) \)-valued versions \( \bar{G}^1 \) and \( \bar{G}^2 \) of \( G \). For each \( (u,v) \), we have \( \bar{G}^1(u,v) = G(u,v) = \bar{G}^2(u,v) \), \( P \)-a.s. Thus, \( \bar{G}^1(e_i,e_j) = \bar{G}^2(e_i,e_j) \) for all \( i,j \), \( P \)-a.s. From the uniqueness result in Step 1, we obtain that \( \bar{G}^1 = \bar{G}^2 \), \( P \)-a.s.

Taking \( g(\omega) = \| \bar{G}(\omega) \|_{\Sigma_2(\mathcal{H} \times H)} \) for \( \omega \in \Omega \), we have the converse of the theorem. \( \square \)

**Proof of Theorem 2.2** We define \( G(u,v) := \mathbb{E}[Y(u,v)|\mathcal{G}] \), for \( (u,v) \in H \times H \). In view of Theorem 2.5, there is a \( Z \in L_w^1(\mathcal{G}, \Sigma_2(\mathcal{H} \times H)) \) such that for each \( (u,v) \in H \times H \),

\[
Z(u,v) = G(u,v) = \mathbb{E}[Y(u,v)|\mathcal{G}], \quad P \)-a.s.,
\]

and satisfies (2.4) according to (2.6). It is the expectation of \( Y \) conditioned on \( \mathcal{G} \) and is unique by the uniqueness result of \( \Sigma_2(\mathcal{H} \times H) \)-valued versions in Theorem 2.5. On the contrary, assume there exists such a conditional expectation \( \mathbb{E}[Y|\mathcal{G}] \in L_w^1(\mathcal{G}, \Sigma_2(\mathcal{H} \times H)) \). Then for any \( (u,v) \in H \times H \), from the definition of the conditional expectation that

\[
\mathbb{E}[Y|\mathcal{G}](u,v) = \mathbb{E}[Y(u,v)|\mathcal{G}], \quad P \)-a.s.,
\]

we have

\[
|\mathbb{E}[Y|\mathcal{G}](u,v)| = |\mathbb{E}[Y(u,v)|\mathcal{G}]| \leq \|\mathbb{E}[Y|\mathcal{G}]\|_{\Sigma_2(\mathcal{H} \times H)} \|u\|_H \|v\|_H, \quad P \)-a.s.
\]

By taking \( g = \|\mathbb{E}[Y|\mathcal{G}]\|_{\Sigma_2(\mathcal{H} \times H)} \), we obtain the domination condition. \( \square \)

**Remark 2.7** In Theorem 2.2, it is not necessary that \( Y \) itself can be aggregated, for \( \mathbb{E}[Y|\mathcal{G}] \) (referred to the mapping defined by \( (u,v) \mapsto \mathbb{E}[Y(u,v)|\mathcal{G}] \in \Sigma_2(\mathcal{H} \times H; L^1(\mathcal{G})) \)) to have an aggregated version. This may not be true in the subsequent applications; see Remark 2.3. Thus, we take \( Y \) to be in the larger space \( \Sigma_2(\mathcal{H} \times H; L^1(\mathcal{F})) \) than \( L_w^1(\mathcal{F}, \Sigma_2(\mathcal{H} \times H)) \).

From \( \Sigma_2(\mathcal{H} \times H) = \Sigma(\mathcal{H}) \), we can also write (2.2) as, for weakly \( \mathcal{G} \)-measurable \( Z \) taking values in \( \Sigma(\mathcal{H}) = \Sigma_2(\mathcal{H} \times H) \) and \( Y \in \Sigma_2(\mathcal{H} \times H; L^1(\mathcal{F})) \),

\[
(Z(u,v) = Z(u,v) = \mathbb{E}[Y(u,v)|\mathcal{G}], \quad P \)-a.s., \( \forall (u,v) \in H \times H \).
\]

### 2.2 Formulation of the BSIE

By a stochastic evolution operator on \( H \), we mean a family of mappings

\[
\{L(t,s) \in \mathcal{L}(L^2(\mathcal{F}_t,H); L^2(\mathcal{F}_s,H)) : (t,s) \in \Delta\}
\]

with \( \Delta = \{(t,s) : 0 \leq t \leq s \leq T\} \). We adopt a definition of the following formal adjoint \( L^* \) for \( L \): For any fixed \( (t,s) \in \Delta \) and \( u \in L^1(\mathcal{F}_s,H) \), define \( L^*(t,s)u \) by

\[
(L^*(t,s)u)(v) := \langle u, L(t,s)v \rangle \quad P \)-a.s., \quad \forall v \in L^2(\mathcal{F}_t,H).
\]

Motivated by the constants of variation method for operator-valued SPDEs (see (i) of Remark 2.2), we shall consider a conditionally expected \( \Sigma(\mathcal{H}) \)-valued BSIE (i.e., \( \Sigma(\mathcal{H}) \)-valued BSIE in the conditional expectation form):

\[
P(t) = \mathbb{E}[L^*(t,T)\xi L(t,T)] + \int_t^T L^*(t,s)f(s,P(s))L(t,s)ds|\mathcal{F}_t|, \quad t \in [0,T],
\]

where the coefficients \( \xi, f \) and \( L \) are given and subject to the following assumptions:

**\((H1)\)** There exists some constant \( \Lambda \geq 0 \) such that for each \( (t,s) \in \Delta \) and \( u \in L^1(\mathcal{F}_s,H) \), it holds that

\[
\|L(t,s)u\|_H^4|\mathcal{F}_s| \leq \Lambda \|u\|_H^4, \quad P \)-a.s.,
\]

and \( (\omega, t,s) \mapsto (L(t,s)u)(\omega) \) admits a jointly measurable version.
\[(H2) \ \xi \in L^2_p(F_T, L(H))\); the function \(f(w, t, p) : \Omega \times [0, T] \times L(H) \to L(H)\) is \(P \otimes L_w/L_w\)-measurable and satisfies the Lipschitz condition in \(p\) with constant \(\lambda \geq 0\); \(f(\cdot, \cdot, 0) \in L^2_{F_w}(0, T; L(H))\).

**Remark 2.8** Fix any \(u \in L^1(F_s, H)\). For each \(v \in L^2(F_t, H), L^*(t, s)u\) maps \(v\) to a real-valued \(F_s\)-measurable random variable \((u, L(t, s)v)\). But the quantity \((u, L(t, s)v)\) is not necessarily integrable. It is integrable if, according to the Hölder inequality, one of the following is imposed: (i) \(u \in L^2(F_s, H)\); (ii) \(u \in L^2(F_s, H), v \in L^1(F_t, H)\) and \((H1)\) holds.

We first show that the operator-valued conditional expectation on the right hand side of the equation is meaningful. To apply the result in Theorem 2.2 we begin with assigning a rigorous meaning to the term \(L^*(t, T)\xi L(t, T) + \int_{t}^{T} L^*(t, s)f(s, P(s))L(t, s)ds\) inside the conditional expectation and demonstrate that it belongs to \(L^2(H \times H; L^1(F_T))\).

**Remark 2.9** From the settings for \(L\), we know that \(L(t, s)\) is not \(L(H)\)-valued for pointwise \(\omega\) (see also subsection 2.4 for the explanations on this setting), and so \(L^*(t, T)\xi L(t, T) + \int_{t}^{T} L^*(t, s)f(s, P(s))L(t, s)ds\) is not. That is, we cannot expect that this term belongs to \(L^2(H \times H; L^1(F_T))\), but rather, as we shall see later, is an element in \(L^2(H \times H; L^1(F_T))\).

**Remark 2.10** (i) For any sub-\(\sigma\)-algebra \(G\) of \(F\) and a mapping \(\eta : \Omega \to L(H)\), the following four statements are equivalent:

(a) \(\eta\) is weakly \(G\)-measurable;

(b) For any \(u \in H, \eta u : \Omega \to H\) is (strongly) \(G\)-measurable (note that since \(H\) is separable, the notions of measurable, weakly measurable and strongly measurable are the same);

(c) For any (strongly) \(G\)-measurable \(u, v : \Omega \to H\), the real-valued function \((\eta u, v)\) is \(G\)-measurable;

(d) For any (strongly) \(G\)-measurable \(u : \Omega \to H\), the function \(\eta u : \Omega \to H\) is (strongly) \(G\)-measurable.

Indeed, it can be proved as follows:

(a)\(\Rightarrow\)(b): The real-valued function \((\eta u, v)\) is \(F_s\)-measurable for each \(v \in H\). This means that \(\eta u : \Omega \to H\) is weakly \(F_s\)-measurable. Noting that \(H\) is separable, this is equivalent to stating that \(\eta u : \Omega \to H\) is (strongly) \(F_s\)-measurable.

(b)\(\Rightarrow\)(a): Since \(\eta u : \Omega \to H\) is (strongly) \(G\)-measurable, then it is weakly measurable, i.e., for any \(u \in H\), the real-valued function \((\eta u, v)\) is \(G\)-measurable.

Surely, (c)\(\Rightarrow\)(a) and (d)\(\Rightarrow\)(b).

Now we only prove (b)\(\Rightarrow\)(d), and the proof of (a)\(\Rightarrow\)(c) is similar. First for any simple

\[
  u = \sum_{i=1}^{N} u_i I_{A_i}, \quad \text{with } u_i \in H, \ A_i \in F_s,
\]

we have that

\[
  \eta u = \sum_{i=1}^{N} (\eta u_i) I_{A_i}
\]

is (strongly) \(G\)-measurable. Finally, for any \(H\)-valued (strongly) \(G\)-measurable \(u\), we can take a simple sequence

\[
  u_k \to u \text{ pointwise, as } k \to \infty.
\]

Then

\[
  \eta u = \eta(\lim_{k \to \infty} u_k) = \lim_{k \to \infty} \eta u_k
\]
is (strongly) $G$-measurable. The proof is complete.

(ii) From (i), we know that, the weakly measurability notion used in this paper is coincide with the notion of strongly measurability used in [3]. But we prefer to call it weak measurability since it is weak than the usual (norm-) measurability. According to (i), we know that for any $\eta \in L^2_{w}(\mathcal{F}_t, \mathcal{L}(H))$ and $u \in H$, the random mapping $\eta L(t,s)u : \Omega \to H$ is (strongly) $\mathcal{F}_t$-measurable.

It is easy to see that similar results hold for weakly adapted and progressively measurable processes. Moreover, by a similar proof, the above equivalence relationship also holds for different separable Hilbert spaces $H_1, H_2$ and mappings taking values in $\mathcal{L}(H_1, H_2)$.

Under the assumption $(H_1)$, given any $\eta \in L^2_{w}(\mathcal{F}_s, \mathcal{L}(H))$ and $(u, v) \in H \times H$, from the Hölder inequality and the condition $(H_1)$, it is straightforward to check that

$$E[|\eta L(t,s)u|^\frac{\lambda}{4}] \leq E[|\eta|^2_{\mathcal{H}}]^\frac{\lambda}{2} (E[|L(t,s)u|^2_{\mathcal{H}}])^\frac{\lambda}{2} \leq \Lambda^\frac{\lambda}{2} (E[|\eta|^2_{\mathcal{H}}])^\frac{\lambda}{2} |u|^\frac{\lambda}{4} < \infty.$$  

Thus the random function $\eta L(t,s)u \in L^2(\mathcal{F}_s, H)$.

Moreover,

$$E[|L^*(t,s)\eta L(t,s)u(v)|] = E[|\langle \eta L(t,s)u, L(t,s)v \rangle|] \leq (E[|L(t,s)u|^2_{\mathcal{H}}])^\frac{\lambda}{2} (E[|\eta|^2_{\mathcal{H}}])^\frac{\lambda}{2} \leq \Lambda^\frac{\lambda}{2} (E[|\eta|^2_{\mathcal{H}}])^\frac{\lambda}{2} |u| |v|.$$  

Thus, we have $L^*(t,s)\eta L(t,s)u \in \mathcal{L}(H; \mathcal{L}(H; L^1(\mathcal{F}_s))) = \mathcal{L}_2(H \times H; L^1(\mathcal{F}_s))$ and we can also write that $(L^*(t,s)\eta L(t,s)u)(v) = L^*(t,s)\eta L(t,s)(u,v)$. In particular, $L^*(t,T)\xi L(t,T) \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}_T))$.

Now we consider the integral term. In general, for a $g \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}_T))$, following the idea of Pettis integration (see, e.g., [28]), we define its integral with respect to time $\int_t^T g(s)ds$ in a weak sense by

$$(\int_t^T g(s)ds)(u,v) := \int_t^T g(s)(u,v)ds \text{  P-a.s.,  } \forall (u,v) \in H \times H.$$  

Then $\int_t^T g(s)ds \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}_T))$ by the observation that

$$E[|\int_t^T g(s)ds(u,v)|] \leq E[\int_t^T |g(s)(u,v)|ds] \leq C |u| H |v|.$$  

Note that for any $h \in L^2_{\mathcal{F}_w}(t,T; \mathcal{L}(H))$ and $(u, v) \in H \times H$,

$$E[\int_t^T |L^*(t,s)h(s) L(t,s)(u,v)|ds] = E[\int_t^T |\langle h(s) L(t,s)u, L(t,s)v \rangle|ds] \leq (\int_t^T E[|L(t,s)u|^2_{\mathcal{H}}]ds)^{\frac{\lambda}{2}} (\int_t^T E[|h(s)|^2_{\mathcal{H}}]ds)^{\frac{\lambda}{2}} (\int_t^T E[|L(t,s)v|^2_{\mathcal{H}}]ds)^{\frac{\lambda}{2}} \leq \Lambda^{\frac{\lambda}{2}} T^{\frac{\lambda}{2}} (E[\int_t^T |h(s)|^2_{\mathcal{H}}]ds)^{\frac{\lambda}{2}} |u| H |v|.$$  

Thus $t \mapsto L^*(t,s)h(s)L(t,s)$ is in $\mathcal{L}_2(H \times H; L^1(\mathcal{F}_T))$ and the integral $\int_t^T L^*(t,s)h(s)L(t,s)ds \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}_T))$ is defined.

The BSIE is considered as an equation in the space $\mathcal{L}(H)$ as follows.

**Definition 2.11** A process $P \in L^2_{\mathcal{F}_w}(0,T; \mathcal{L}(H))$ is called a solution of (2.11) if for each $0 \leq t \leq T$,

$$P(t) = E[L^*(t,T)\xi L(t,T)] + \int_t^T L^*(t,s) f(s, P(s)) L(t,s)ds |\mathcal{F}_t], \text{  P-a.s.} \quad (2.12)$$
Given any \( P \in L^2_{F_p, w}(0, T; \mathcal{L}(H)) \). Since it is \( \mathcal{P}/L_w \)-measurable, we deduce by (H2) and the measurability of composition that \( f(\cdot , P(\cdot )) \) is \( \mathcal{P}/L_w \)-measurable, i.e., weakly progressively measurable. From this and the Lipschitz continuity of \( f \), we obtain that \( f(\cdot , P(\cdot )) \in L^2_{F_p, w}(0, T; \mathcal{L}(H)) \). Thus,

\[
L^*(t, T)\xi_L(t, T) + \int_t^T L^*(t, s)f(s, P(s))L(t, s)ds \in \Sigma_2(H \times H; L^1(F_T)).
\]  

(2.13)

Then the conditional expectation on right hand side of BSIE (2.12) is a well-defined operator-valued random variable as long as we check the domination condition (2.4), which shall be done in the next subsection. In what follows, \( C > 0 \) will denote a constant which may vary from line to line.

### 2.3 Existence and uniqueness of solutions

We have the following well-posedness result on BSIEs.

**Theorem 2.12** Let Assumptions (H1) and (H2) be satisfied. Then there exists a unique (up to modification) solution \( P \) to BSIE (2.11). Moreover, for each \( t \in [0, T] \),

\[
\|P(t)\|^2_{\Sigma(H)} \leq C\mathbb{E}[\|\xi\|^2_{\Sigma(H)}] + \int_t^T \|f(s, 0)\|^2_{\Sigma(H)}ds|\mathcal{F}_t], \quad \text{P.-a.s.,}
\]

for some constant \( C \) depending on \( \Lambda \) and \( \lambda \).

To prove this theorem, we need the following lemmas. First we see that the conditional expectation on the right-hand side of (2.12) is well-defined.

**Lemma 2.13** Suppose (H1) and (H2) hold. For any \( p \in L^2_{F_p, w}(0, T; \mathcal{L}(H)) \) and \( 0 \leq t \leq T \), we define

\[
Y^p_{t,T} := L^*(t, T)\xi_L(t, T) + \int_t^T L^*(t, s)f(s, P(s))L(t, s)ds.
\]

Then \( \mathbb{E}[Y^p_{t,T} | \mathcal{F}_t] \in L^2_w(F_t, \mathcal{L}(H)) \), and there exists some constant \( C > 0 \) depending on \( \Lambda \) and \( \lambda \) such that

\[
\|\mathbb{E}[Y^p_{t,T} | \mathcal{F}_t]\|_{\Sigma(H)} \leq C(\mathbb{E}[\|\xi\|^2_{\Sigma(H)}] + \int_t^T \|f(s, 0)\|^2_{\Sigma(H)}ds + \int_t^T \|f(s, 0)\|^2_{\Sigma(H)}ds|\mathcal{F}_t])^{\frac{1}{2}}, \quad \text{P.-a.s.}
\]

(2.15)

Moreover, \( \{\mathbb{E}[Y^p_{t,T} | \mathcal{F}_t]\}_{t \in [0, T]} \in L^2_{F_p, w}(0, T; \mathcal{L}(H)) \).

**Proof.** First we have \( Y^p_{t,T} \in \Sigma_2(H \times H; L^1(F_T)) \) from the discussions in the last subsection. For any \( (u, v) \in H \times H \), we directly calculate

\[
\mathbb{E}[Y^p_{t,T}(u, v)|\mathcal{F}_t] = |\mathbb{E}[(\xi_L(t, T)u, L(t, T)v) + \int_t^T (f(s, p(s))L(t, s)u, L(t, s)v)ds|\mathcal{F}_t]|
\]

\[
\leq (\mathbb{E}[\|L(t, T)u\|^2_{H}|\mathcal{F}_t])^{\frac{1}{2}}(\mathbb{E}[\|\xi\|^2_{\Sigma(H)}|\mathcal{F}_t])^{\frac{1}{2}}(\mathbb{E}[\|L(t, T)v\|^2_{H}|\mathcal{F}_t])^{\frac{1}{2}}
\]

\[
+ (\int_t^T \mathbb{E}[\|L(t, s)u\|^2_{H}|\mathcal{F}_t]ds)^{\frac{1}{2}}(\mathbb{E}[\|f(s, p(s))\|^2_{\Sigma(H)}ds|\mathcal{F}_t])^{\frac{1}{2}}(\int_t^T \mathbb{E}[\|L(t, s)v\|^2_{H}|\mathcal{F}_t]ds)^{\frac{1}{2}}
\]

\[
\leq C\|u\|_H\|v\|_H \{\mathbb{E}[\|\xi\|^2_{\Sigma(H)}|\mathcal{F}_t]^{\frac{1}{2}} + (\mathbb{E}[\int_t^T \|f(s, p(s))\|^2_{\Sigma(H)}ds|\mathcal{F}_t])^{\frac{1}{2}}\}
\]

\[
\leq C\|u\|_H\|v\|_H (\mathbb{E}[\|\xi\|^2_{\Sigma(H)}] + \int_t^T \|p(s)\|^2_{\Sigma(H)}ds + \int_t^T \|f(s, 0)\|^2_{\Sigma(H)}ds|\mathcal{F}_t])^{\frac{1}{2}}, \quad \text{P.-a.s.}
\]

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Then by Theorem 2.2, $E[Y_{t,T}^p|\mathcal{F}_t]$ is a well-defined $\mathcal{L}(H)$-valued random variables (see 2.10), and (2.15) follows from (2.4). Thus, $E[Y_{t,T}^p|\mathcal{F}_t] \in \mathcal{L}_{w}(\mathcal{F}_t, \mathcal{L}(H))$.

It remains to show that $\{E[Y_{t,T}^p|\mathcal{F}_t]\}_{t \in [0,T]}$ has a weakly progressively measurable version. This is obtained from the following Lemma 2.14 and the fact that, for each $(u,v) \in H \times H$, $\{E[Y_{t,T}^p(u,v)|\mathcal{F}_t]\}_{t \in [0,T]}$ has a progressively measurable version by considering its optional projection (see [1, Corollary 7.6.8]). □

**Lemma 2.14** Let $Y$ be an $\mathcal{L}(H)$-valued weakly adapted process satisfying $Y_t \in \mathcal{L}_w(\mathcal{F}_t, \mathcal{L}(H))$ for $0 \leq t \leq T$. Then there exists a constant $C$ such that $\mathbb{E}[\|Y_t\|_{\mathcal{L}(H)}^{2}]$ is a well-defined $\mathcal{L}(H)$-valued random variables (see 2.10), and (2.15) follows from (2.4). Thus, $E[Y_{t,T}^p|\mathcal{F}_t] \in \mathcal{L}_{w}(\mathcal{F}_t, \mathcal{L}(H))$.

**Proof.** We look for the desired process by a variant of Step 2 in the proof of Theorem 2.2 in the space $\mathcal{L}_2(H \times H)$ of bilinear mapping, and derive the result in the original form can be obtained via the isometry $\mathcal{L}_2(H \times H) = \mathcal{L}(H)$. For any $(u,v) \in H \times H$, we denote by $\{y_t(u,v)\}_{0 \leq t \leq T}$ and $\{h_t\}_{0 \leq t \leq T}$ the progressively measurable modifications of $\{Y_t(u,v)\}_{0 \leq t \leq T}$ and $\{\mathbb{E}[Y_t(u,v)|\mathcal{F}_t]\}_{0 \leq t \leq T}$, respectively. Then for any $t \in [0,T]$, $|y_t(u,v)| \leq \|Y_t\|_{\mathcal{L}(H)}\|u\|_H\|v\|_H = h_t\|u\|_H\|v\|_H, \quad P\text{-a.s.}$

Thus, $y_t \in \mathcal{L}_2(H \times H; L^1(\mathcal{F}_t))$. Adopt the notations in the proof of Theorem 2.5 and fix any versions of process $y(e_i, e_j)$ for $i, j \geq 1$. For every $t$, define $\tilde{Y}_t(e_i, e_j)(\omega) := a_{i,j}^w := y_t(e_i, e_j)(\omega), \quad i, j \geq 1, \text{ for each } \omega.$

For any fixed $t$, according to (2.11), we have $P\text{-a.s.}$ that $\left|\sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j}^w u_{i,j} \right| = \|y_t(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j)(\omega)\| \leq \|h_t(\omega)\| \|\sum_{i=1}^{n} \alpha_i e_i\|_H \|\sum_{j=1}^{m} \beta_j e_j\|_H,$

for all $(u,v) = (\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{m} \beta_j e_j) \in V \times V$.

and we denote the set (on $\Omega$) in which the above relationship holds by $\Omega_t$. Similar to Step 2 in the proof of Theorem 2.5, $\tilde{Y}_t(\omega)$ has an extension in $\mathcal{L}_2(H \times H)$ on $\Omega_t$ and we set $\bar{Y}_t = 0$ on $\Omega_t^c$. Then $\bar{Y}_t \leq h_t$ $P\text{-a.s.}$ Denote by $A$ the progressively measurable set of all points $(t, \omega) \in \Omega \times [0,T]$ such that (2.11) holds. Note $\Omega_t$ is the section of $A$ for each $t$. Then the $\mathcal{L}_2(H \times H)$-valued process $\bar{Y}$ is automatically weakly progressively measurable and $\bar{Y}_t(u,v) = y_t(u,v)$ $P\text{-a.s.}$, for any $(u,v) \in H \times H$ and $t \in [0,T]$, by a similar analysis as in the proof of Theorem 2.5. Since for each $t, Y_t$ and $\bar{Y}_t$ are both aggregated versions of $y_t$ in the sense of Theorem 2.5, we deduce from the uniqueness result in that theorem that $Y_t = \bar{Y}_t$ $P\text{-a.s.}$ That is, $Y_t$ is a modification of $\bar{Y}_t$.

The inversed assertion is trivial, by noting that for each $(u,v) \in H \times H$, $\{|\bar{Y}_t(u,v)\|_{\mathcal{L}(H)}\}_{0 \leq t \leq T}$ is a progressively measurable modification of $\{Y_t(u,v)\}_{0 \leq t \leq T}$. □

The following is the a priori estimate for the difference between two solutions.

**Theorem 2.15** Let $L$ satisfy (H1) and $(\xi, f)$ and $(\tilde{\xi}, \tilde{f})$ satisfy (H2). Assume that $P, \tilde{P} \in L^2_{w}(0,T; \mathcal{L}(H))$ are solutions to BSIEs

$$P(t) = \mathbb{E}[L^*(t,T)\xi L(t,T) + \int_t^T L^*(s,t)f(s, P(s))L(t,s)ds|\mathcal{F}_t], \quad t \in [0,T]$$

and

$$\tilde{P}(t) = \mathbb{E}[L^*(t,T)\tilde{\xi} L(t,T) + \int_t^T L^*(s,t)\tilde{f}(s, \tilde{P}(s))L(t,s)ds|\mathcal{F}_t], \quad t \in [0,T].$$

Then there exists a constant $C > 0$ which depends on $\Lambda$ and $\lambda$ such that, for each $t \in [0,T]$,

$$\|P(t) - \tilde{P}(t)\|_{\mathcal{L}(H)}^2 \leq C\mathbb{E}[\|\xi - \tilde{\xi}\|_{\mathcal{L}(H)}^2 + \int_0^T \|f(s, P(s)) - \tilde{f}(s, \tilde{P}(s))\|_{\mathcal{L}(H)}^2 ds|\mathcal{F}_t], \quad P\text{-a.s.}$$

(2.18)
Proof. For any $t \in [0, T]$, we have $P$-a.s. that
\[
P(t) - \hat{P}(t) = \mathbb{E}[\ell^*(t, T)(\bar{\xi} - \bar{\xi})L(t, T) + \int_t^T \ell^*(t, s)(f(s, \bar{P}(s)) - \bar{f}(s, \bar{P}(s)))L(t, s)ds|\mathcal{F}_t].
\]
Applying Lemma 2.13, we obtain
\[
\|P(t) - \hat{P}(t)\|_{L^2(H)}^2 \leq C\mathbb{E}\|\xi - \bar{\xi}\|_{L^2(H)}^2 + \int_t^T \mathbb{E}\|f(s, \bar{P}(s)) - \bar{f}(s, \bar{P}(s))\|_{L^2(H)}^2 ds|\mathcal{F}_t] + \mathbb{E}[\int_t^T \|P(s) - \hat{P}(s)\|_{L^2(H)}^2 ds|\mathcal{F}_t], \quad P\text{-a.s.}
\]
Fix any $r \leq T$ and any $A \in \mathcal{F}_r$. For $t \in [r, T]$, multiplying by $I_A$ and taking expectation on both sides, we obtain that
\[
\mathbb{E}[\|P(t) - \hat{P}(t)\|_{L^2(H)}^2 I_A] \leq C\mathbb{E}[(\|\xi - \bar{\xi}\|_{L^2(H)}^2 + \int_r^T \|f(s, \bar{P}(s)) - \bar{f}(s, \bar{P}(s))\|_{L^2(H)}^2 ds)I_A] + \int_r^T \mathbb{E}[\|P(s) - \hat{P}(s)\|_{L^2(H)}^2] ds.
\]
Then an application of Gronwall’s inequality yields
\[
\mathbb{E}[\|P(t) - \hat{P}(t)\|_{L^2(H)}^2 I_A] \leq C\mathbb{E}[\|\xi - \bar{\xi}\|_{L^2(H)}^2] + \int_r^T \mathbb{E}[\|f(s, \bar{P}(s)) - \bar{f}(s, \bar{P}(s))\|_{L^2(H)}^2 ds]I_A, \quad t \in [r, T].
\]
From the arbitrariness of $A$, this implies for $t \in [r, T]$ that
\[
\mathbb{E}[\|P(t) - \hat{P}(t)\|_{L^2(H)}^2 |\mathcal{F}_r] \leq C\mathbb{E}[\|\xi - \bar{\xi}\|_{L^2(H)}^2 |\mathcal{F}_r] + \mathbb{E}[\int_r^T \|f(s, \bar{P}(s)) - \bar{f}(s, \bar{P}(s))\|_{L^2(H)}^2 ds |\mathcal{F}_r], \quad P\text{-a.s.}
\]
Letting $t = r$, we obtain (2.18). □

Now we prove Theorem 2.12.

Proof. We define the solution mapping $I : L^2_{F,w}(0, T; \mathcal{L}(H)) \to L^2_{F,w}(0, T; \mathcal{L}(H))$ by $I(p) := P$ for $p \in L^2_{F,w}(0, T; \mathcal{L}(H))$ with
\[
P(t) := \mathbb{E}[\ell^*(t, T)\xi L(t, T) + \int_t^T \ell^*(t, s)f(s, p(s))L(t, s)ds|\mathcal{F}_t], \quad t \in [0, T].
\]
In view of Lemma 2.13, we have $I(p) \in L^2_{F,w}(0, T; \mathcal{L}(H))$. Thus, the mapping $I$ is well-defined.

Now we show that the mapping $I$ is a contraction on the interval $[T - \delta, T]$ when $\delta > 0$ is sufficiently small. Set $P := I(p)$ and $\hat{P} := I(\hat{p})$ for $p, \hat{p} \in L^2_{F,w}(0, T; \mathcal{L}(H))$. From Theorem 2.13, we have
\[
\|P(t) - \hat{P}(t)\|_{L^2(H)}^2 \leq C\mathbb{E}[\int_t^T \|f(s, p(s)) - f(s, \hat{p}(s))\|_{L^2(H)}^2 ds |\mathcal{F}_t] \leq C\mathbb{E}[\int_t^T \|p(s) - \hat{p}(s)\|_{L^2(H)}^2 ds |\mathcal{F}_t], \quad t \in [0, T].
\]
For any $0 < \delta < T$, taking expectation on both sides and integrating over time on $[T - \delta, T]$, we get
\[
\mathbb{E}[\int_T^{T - \delta} \|P(t) - \hat{P}(t)\|_{L^2(H)}^2] \leq C\delta \mathbb{E}[\int_T^{T - \delta} \|p(s) - \hat{p}(s)\|_{L^2(H)}^2 ds].
\]
So for sufficiently small $\delta > 0$, we obtain a unique $P \in L^2_{w,w}(T - \delta, T; \mathfrak{L}(H))$ such that $P = I(P)$ in $L^2_{w,w}(T - \delta, T; \mathfrak{L}(H))$, which is also

$$P(t) = \mathbb{E}[L^*(t, T)\xi L(t, T) + \int_t^T L^*(t, s)f(s, P(s))L(t, s)ds|\mathcal{F}_t], \quad P\text{-a.s., a.e. on } [T - \delta, T]. \tag{2.19}$$

We take

$$\tilde{P}(t) := \mathbb{E}[L^*(t, T)\xi L(t, T) + \int_t^T L^*(t, s)f(s, \tilde{P}(s))L(t, s)ds|\mathcal{F}_t], \quad t \in [T - \delta, T].$$

Then $\tilde{P}$ satisfies

$$\tilde{P}(t) = \mathbb{E}[L^*(t, T)\xi L(t, T) + \int_t^T L^*(t, s)f(s, \tilde{P}(s))L(t, s)ds|\mathcal{F}_t], \quad P\text{-a.s., } \forall t \in [T - \delta, T],$$

and thus is a solution of BSIE (2.11) in the meaning of Definition 2.11 on $[T - \delta, T]$. The uniqueness of $\tilde{P}$ in this sense follows from that in the meaning of (2.19). Indeed, on $[T - \delta, T]$, if $P'$ is another solution of BSIE (2.11) in the sense of Definition 2.11, then they are both solutions of (2.19) in the meaning of (2.19). Thus $P = P'$ $P$-a.s., a.e. on $[T - \delta, T]$. From the identity (2.12) on $[T - \delta, T]$, we then obtain that $P(t) = \tilde{P}(t)$ $P$-a.s., for all $t \in [T - \delta, T]$.

Denoting $\tilde{P}$ by $P$, we can apply a backward iteration procedure to obtain a $P \in L^2_{w,w}(0, T; \mathfrak{L}(H))$ such that

$$P(t) = \mathbb{E}[L^*(t, T)\xi L(t, T) + \int_t^T L^*(t, s)f(s, P(s))L(t, s)ds|\mathcal{F}_t], \quad P\text{-a.s., } \forall t \in [0, T], \tag{2.20}$$

since the constant $\delta$ can be chosen to be independent of the terminal time in each step. The uniqueness of $P$ follows from the one on each interval. □

We end this section with the following continuity of $P$, and the proof is given in the appendix. We first note that, if $P$ is a solution of (2.11), then for each $(u, v) \in H \times H$,

$$\langle P(t)u, v \rangle = \langle \mathbb{E}[L^*(t, T)\xi L(t, T) + \int_t^T L^*(t, s)f(s, P(s))L(t, s)ds|\mathcal{F}_t]u, v \rangle$$

$$\quad = \mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle + \int_t^T \langle f(s, P(s))L(t, s)u, L(t, s)v \rangle ds|\mathcal{F}_t], \quad P\text{-a.s.} \tag{2.21}$$

From this and an approximation of simple random variables, we can obtain that (2.21) holds for $(u, v) \in L^4(\mathcal{F}_t, H) \times L^4(\mathcal{F}_t, H)$.

**Proposition 2.16** For some $\alpha \geq 1$, suppose (H1), (H2) and

(H3) $(\xi, f(\cdot, \cdot, 0)) \in L^2_{w,w}(\mathcal{F}_T, \mathfrak{L}(H)) \times L^2_{w,w}(0, T; \mathfrak{L}(H))$ and there exists some constant $\Lambda_\alpha \geq 0$ such that for each $0 \leq t \leq r \leq s \leq T$ and $u \in L^{4\alpha}(\mathcal{F}_t, H)$, it holds that $L(t, s) = L(t, r)L(r, s)$,

$$\mathbb{E}[\|L(t, s)u\|_{H}^{4\alpha}|\mathcal{F}_t] \leq \Lambda_\alpha\|u\|_{H}^{4\alpha} \quad P\text{-a.s.} \quad \text{and } [t, T] \ni s \mapsto L(t, s)u \text{ is strongly continuous in } L^{4\alpha}(\mathcal{F}_t, H).$$

Let $P$ be the solution of (2.11). Then, for each $t \in [0, T)$ and $u, v \in L^{4\alpha}(\mathcal{F}_t, H)$, we have

$$\lim_{\delta \downarrow 0} \mathbb{E}[\{(P(t + \delta)u, v) - (P(t)u, v)\}^{\alpha}] = 0.$$
Remark 2.17 According to similar proofs, the discussions and results in this section hold for a more general setting that the bilinear framework is replaced by the $k$-linear framework, for $k = 1, 2, 3 \ldots$, for possibly different separable Hilbert spaces (and even more generally, Banach spaces with Schauder basis) $H_j$ and stochastic evolution operators $L_j(t, s)$ on $H_j$, for $1 \leq j \leq k$. We only give a short description of BSIEs for convenience as follows.

We make use of the similar weakly measurable meaning for $L_k(H_1 \times H_2 \times \cdots \times H_k)$-valued random variables and stochastic processes as in the bilinear case (with a direct modification from the case of $k = 2$ to the general $k$). Given terminal $\xi \in L^2_\mathbb{F} (\mathcal{F}_T, \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k))$, generator $f(w, t, p) : \Omega \times [0, T] \times \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k) \to \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k)$ satisfying the Lipschitz condition and $f(\cdot, \cdot, 0) \in L^2_{\mathbb{F}, w} (0, T; \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k))$, and stochastic evolution operator $L_j(t, s)$ satisfying $\mathbb{E} [\|L_j(t, s)u\|_{H_j}^{2k}] \leq \Lambda \|u\|_{H_j}^{2k}$ for some $\Lambda \geq 0$, for $j = 1, 2, \cdots, k$; other measurability assumptions are imposed similarly as in (H1) and (H2) (with some possible direct modifications).

For an $\eta \in L^2_{\mathbb{F}_T} (\mathcal{F}_s, \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k))$, we define the mapping

$$\eta(L_1(t, s), L_2(t, s), \cdots, L_k(t, s)) \in \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k; L^1(\mathcal{F}_t))$$

by

$$(u_1, u_2, \cdots, u_k) \mapsto \eta(L_1(t, s)u_1, L_2(t, s)u_2, \cdots, L_k(t, s)u_k), \quad \forall (u_1, u_2, \cdots, u_k) \in H_1 \times H_2 \times \cdots \times H_k.$$

For a mapping $g \in \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k; L^1(\mathcal{F}_t))$, we define $\int^T_t g(s)ds \in \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k; L^1(\mathcal{F}_T))$ by

$$\int^T_t g(s)ds(u_1, u_2, \cdots, u_k) := \int^T_t g(s)(u_1, u_2, \cdots, u_k)ds \quad \text{P-a.s.}, \quad \forall (u_1, u_2, \cdots, u_k) \in H_1 \times H_2 \times \cdots \times H_k.$$

Then for any $h \in L^2_{\mathbb{F}_T}(t, T; \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k))$,

$$h(s)(L_1(t, s), L_2(t, s), \cdots, L_k(t, s)) \in \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k; L^2(\mathcal{F}_T)).$$

Therefore,

$$\mathbb{E}[\xi(L_1(t, T), \cdots, L_k(t, T))] + \int^T_t f(s, P(s))(L_1(t, T), \cdots, L_k(t, T))ds$$

\in $\mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k; L^1(\mathcal{F}_T))$.

We consider the $\mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k)$-valued conditionally expected BSIE

$$P(t) = \mathbb{E}[\xi(L_1(t, T), \cdots, L_k(t, T)) + \int^T_t f(s, P(s))(L_1(t, T), \cdots, L_k(t, T))ds|\mathcal{F}_t],$$

$t \in [0, T]$.

By a solution of it, we mean a process $P \in L^2_{\mathbb{F}_T}(0, T; \mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k))$ satisfying: for $t \in [0, T]$, it hold P-a.s. that in $\mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k)$

$$P(t) = \mathbb{E}[\xi(L_1(t, T), \cdots, L_k(t, T)) + \int^T_t f(s, P(s))(L_1(t, T), \cdots, L_k(t, T))ds|\mathcal{F}_t].$$

This equation can be solved by firstly defining and constructing the $k$-linear operator $\mathcal{L}_k (H_1 \times H_2 \times \cdots \times H_k)$-valued conditional expectations, and then making use a contraction argument, similarly as in the bilinear situation (Whereas in the multilinear situation, it seems awkward to introduce the formal adjoint operators for $L$, which will make the notations complicated).

This is a multilinear operator-valued backward stochastic evolution equations.
Remark 2.18 If we strength the growth assumption for $L$ in (H1) to: for each $t$, $L(t, s)u$ is continuous in $s$ and
\[ E[\sup_{t \leq s \leq T} \|L(t, s)u\|^4_H | F_t] \leq \Lambda \|u\|^4_H, \quad P\text{-a.s.} \]
or more generally, for each $t$,
\[ E[\text{ess sup}_{0 \leq t \leq T} \|L(t, s)u\|^4_H | F_t] \leq \Lambda \|u\|^4_H, \quad P\text{-a.s.} \]
By a standard modifications of the proofs, we can weaken the assumption $f(\cdot, \cdot, 0) \in L^2_{\mathbb{P}, w}(0, T; \mathfrak{L}(H))$ of $f$ to $f(\cdot, \cdot, 0) \in L^1_{\mathbb{P}, w}(0, T; \mathfrak{L}(H))$ in (H2) for the well-posedness result and estimates of the solutions, as well as other results (For Proposition 2.15 we also need a similar modification for the condition (H3)) obtained for the BSIEs in this paper. Here, $L^1_{\mathbb{P}, w}(0, T; \mathfrak{L}(H))$ is the space of $\mathfrak{L}(H)$-valued weakly progressively measurable processes $F(\cdot)$ with norm $\|F\|_{L^1_{\mathbb{P}, w}(0, T; \mathfrak{L}(H))} = \{E[\int_0^T \|L(t, s)u\|^2_H dt]\}^{1/2}$. We illustrate this change for the first condition in the proof of Lemma 2.15
\[
\begin{align*}
|E[\int_t^T (f(s, p(s))L(t, s)u, L(t, s)v)ds] | & \\
& \leq |E[\sup_{t \leq s \leq T} \|L(t, s)u\|_H \sup_{t \leq s \leq T} \|L(t, s)v\|_H \int_t^T \|f(s, p(s))\|_{\mathfrak{L}(H)} ds] | \\
& \leq (E[\sup_{t \leq s \leq T} \|L(t, s)u\|_H^4 | F_t])^{1/2} (E[(\int_t^T \|f(s, p(s))\|_{\mathfrak{L}(H)} ds)^2 | F_t])^{1/2} (E[\sup_{t \leq s \leq T} \|L(t, s)v\|_H^4 | F_t])^{1/2} \\
& \leq C \|u\|_H \|v\|_H \{E[\int_t^T \|f(s, 0)\|_{\mathfrak{L}(H)} ds]^2 + \int_t^T \|p(s)\|_{\mathfrak{L}(H)}^2 ds] \}, \quad P\text{-a.s.}
\end{align*}
\]
For reader’s convenience, we present the improved result for the well-posedness of the BSIEs: Under one of the above new conditions, there exists a unique solution $P$ to BSIE 2.11. Moreover, for each $t \in [0, T]$,
\[
\|P(t)\|^2_{\mathfrak{L}(H)} \leq C E[\|\xi\|^2_{\mathfrak{L}(H)}] + (\int_t^T \|f(s, 0)\|^2_{\mathfrak{L}(H)} ds)^2 | F_t|, \quad P\text{-a.s.,}
\]
for some constant $C$ depending on $\Lambda$ and $\lambda$.

2.4 Itô’s formula

A typical example and main prototype of the stochastic evolution operator $L$ is the formal solution of forward operator-valued SDEs, which can be rigorously defined as the solution map of forward vector-valued SEE. In this section, we shall derive an Itô’s formula for the product of BSIEs with two forward SEE when $L$ takes this concrete form. It is needed in the derivation of the maximum principle.

2.4.1 Evolution operators associated to forward SEEs

Let $V$ be a separable Hilbert space densely embedded in $H$. Denote $V^* := \mathfrak{L}(V; \mathbb{R})$, then $V \subset H \subset V^*$ form a Gelfand triple. We denote by $\langle \cdot, \cdot \rangle_*$ the duality between $V^*$ and $V$.

Let $w := \{w(t)\}_{t \geq 0}$ be a one-dimensional standard Brownian motion with respect to $\mathbb{F}$. Consider the following linear homogeneous SEE on $[t, T]$: 
\[
\begin{aligned}
\begin{cases}
\frac{du^{t, u_0}(s)}{ds} &= A(s)u^{t, u_0}(s)ds + B(s)u^{t, u_0}(s)dw(s), & s \in [t, T], \\
u^{t, u_0}(t) &= u_0,
\end{cases}
\end{aligned}
\]
where \( u_0 \in L^2(\mathcal{F}_t, H) \) and \((A, B) : [0, T] \times \Omega \to \mathcal{L}(V; V^* \times H)\).

**Remark 2.19** We only write the one-dimensional Brownian motion case for simplicity of presentation. With direct modifications, the results throughout this paper still hold for the more general case that \( w \) is a Hilbert space \( K \)-valued cylindrical \( Q \)-Brownian motion (including multi-dimensional Brownian motion, finite-trace \( Q \)-Brownian motion, cylindrical Brownian motion as special cases) and the integrands \( f \) takes valued in the Hilbert-Schmidt space \( L_2(Q^2(K); H) \); see \[19\] and \[22\] for more discussions on this direction.

We make the following assumption.

\((H4)\) For each \( u \in V, A(t, \omega)u \) and \( B(t, \omega)u \) are progressively measurable and satisfying: There exist some constants \( \delta > 0 \) and \( K \geq 0 \) such that the following two assertions hold: for each \( t, \omega \) and \( u \in V, \)

(i) coercivity condition:

\[
2\langle A(t, \omega)u, u \rangle + \|B(t, \omega)u\|_{H}^2 \leq -\delta\|u\|_{V}^2 + K\|u\|_{V}^2 \quad \text{and} \quad \|A(t, \omega)u\|_{V^*} \leq K\|u\|_{V};
\]

(ii) quasi-skew-symmetry condition:

\[
\|B(t, \omega)u, u\| \leq K\|u\|_{H}^2.
\]

From \[17\], Equation \(2.23\) has a unique solution \( u^{t, u_0}(\cdot) \in L^2(t, T; V) \cap S^2(t, T; H) \), where \( S^2(t, T; H) \) is the space of adapted \( H \)-valued processes \( y \) with continuous paths such that \( \mathbb{E}[\sup_{t \leq s \leq T} \|y(s)\|_{H}^2] < \infty \).

Through this solution, we define a stochastic evolution operator \( L_{A, B} \) as follows:

\[
L_{A, B}(t, s)(u_0) := u^{t, u_0}(s) \in L^2(\mathcal{F}_s, H), \quad \text{for } t \leq s \leq T \text{ and } u_0 \in L^2(\mathcal{F}_t, H). \tag{2.24}
\]

From the basic estimates for SEEs, it satisfies the assumptions \((H1)\) and \((H3)\). In fact, in general, if \( y \) is the solution to the SEE

\[
\begin{align*}
\left\{\begin{array}{ll}
\mathrm{d}y(s) = [A(s)y(s) + a(s)]\mathrm{d}s + [B(s)y(s) + b(s)]\mathrm{d}w(s), & s \in [t, T], \\
y(t) = y_0,
\end{array}\right.
\end{align*}
\]

for \( a, b \in L^1_{\mathcal{F}}([t, T]; H) \times L^2_{\mathcal{F}}([t, T]; H) \) and \( y_0 \in L^2(\mathcal{F}_t, H) \), with \( \alpha \geq 1 \) and \( L^1_{\mathcal{F}}([0, T]; H) \) being the space of \( H \)-valued progressively measurable processes \( \varphi(\cdot) \) with norm \( \|\varphi\|_{L^1_{\mathcal{F}}([0, T]; H)} = \{\mathbb{E}[\int_0^T \|\varphi(t)\|_{H}^2] \}^{\frac{1}{2}} \)

then there exists a constant \( C > 0 \) depending on \( \delta, K \) and \( \alpha \) (see \([6\] Lemma 3.1)) such that

\[
\mathbb{E}[\sup_{s \in [t, T]} \|y(s)\|_{H}^{2\alpha}] \leq C\mathbb{E}[\|y_0\|_{H}^{2\alpha}] + (\int_t^T \|a(s)\|_{H}^2 + (\int_t^T \|b(s)\|_{H}^2)^{\alpha}] \tag{2.25}
\]

This implies

\[
\mathbb{E}[\sup_{s \in [t, T]} \|y(s)\|_{H}^{2\alpha} | \mathcal{F}_t] \leq C\{(\|y_0\|_{H}^{2\alpha}] + \mathbb{E}[\int_t^T \|a(s)\|_{H}^{2\alpha} + \int_t^T \|b(s)\|_{H}^2 \}^{\alpha}]
\]

by noting that, with \( y \) denoted by \( y^{t, y_0; a, b} \), for any \( D \in \mathcal{F}_t, \)

\[
\mathbb{E}[I_D \cdot \sup_{s \in [t, T]} \|y^{t, y_0; a, b}(s)\|_{H}^{2\alpha}] = \mathbb{E}[\sup_{s \in [t, T]} \|y^{t, I_D y_0, I_D a, I_D b}(s)\|_{H}^{2\alpha}]
\]

\[
\leq C\mathbb{E}[\|I_D \cdot y_0\|_{H}^{2\alpha} + \int_t^T \|I_D \cdot a(s)\|_{H}^2 + \int_t^T \|I_D \cdot b(s)\|_{H}^2 \}^{\alpha}]
\]

Furthermore, the continuity in \((H3)\) for \( L_{A, B} \) follows from the continuity property of solutions for SEEs.

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Remark 2.20 The operator $L_{A,B}$ can be regarded as the formal solution of the following $\mathcal{L}(H)$-valued SDEs

$$\begin{cases}
  dL_{A,B}(t, s) = A(s)L_{A,B}(t, s)dt + B(s)L_{A,B}(t, s)dw(s), & s \in [t, T], \\
  L_{A,B}(t, t) = I_A.
\end{cases} \tag{2.26}$$

When $H$ is finite dimensional (i.e., $H = \mathbb{R}^n$ for some integer $n \geq 1$, then $\mathcal{L}(H) = \mathcal{L}(\mathbb{R}^n) = \mathbb{R}^{n \times n}$), it is indeed the classical (matrix-valued) solution of (2.26). In the infinite-dimensional situation, such an equation is far from being well understood (it is not known that it admits an $\mathcal{L}(H)$-valued solution).

Now, in virtue of Theorem 2.12, the $\mathcal{L}(H)$-valued BSIE

$$P(t) = \mathbb{E}[L_{A,B}(t, T)\xi] + \int_t^T L_{A,B}^*(t, s)P(s)L_{A,B}(t, s)ds|\mathcal{F}_t], \quad t \in [0, T], \tag{2.27}$$

has a unique solution $P \in L^2_{\mathbb{F}, \mathbb{Q}}(0, T; \mathcal{L}(H))$.

In the following, we shall always assume that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the augmented natural filtration of Brownian motion $\{w(t)\}_{t \geq 0}$.

Remark 2.21 Let $H$ be finite dimensional.

(i) BSIE (2.27) is equivalent to the following matrix-valued BSDE

$$P(t) = \xi + \int_t^T \left[ A^*(s)P(s)P(s)A(s) + B^*(s)P(s)B(s) + B^*(s)Q(s) + Q(s)B(s) \right] ds - \int_t^T Q(s)dw(s). \tag{2.28}$$

In fact, recall that in the matrix case, $L_{A,B}(t, s)$ is the solution of matrix-valued SDE (2.26) and $L_{A,B}^*(t, s)$ is its transpose which satisfies

$$\begin{cases}
  dL_{A,B}^*(t, s) = L_{A,B}^*(t, s)A^*(s)ds + L_{A,B}^*(t, s)B^*(s)dw(s), & s \in [t, T], \\
  L_{A,B}^*(t, t) = I_A.
\end{cases}$$

Then using Itô’s formula to $L_{A,B}^*(t, s)P(s)L_{A,B}(t, s)$ on $[t, T]$, we get

$$P(t) = L_{A,B}^*(t, T)\xi L_{A,B}(t, T) + \int_t^T L_{A,B}^*(t, s)f(s, P(s))L_{A,B}(t, s)ds - \int_t^T L_{A,B}^*(t, s)(P(s)B(s) + Q(s) + B^*(s)P(s))L_{A,B}(t, s)dw(s).$$

Taking conditional expectation on both sides, we obtain

$$P(t) = \mathbb{E}[L_{A,B}^*(t, T)\xi L_{A,B}(t, T) + \int_t^T L_{A,B}^*(t, s)f(s, P(s))L_{A,B}(t, s)ds|\mathcal{F}_t]. \tag{2.29}$$

Naturally, BSDE is preferred in the characterization of the adjoint process. Unfortunately, in an infinite-dimensional space without separability, the stochastic integral and unbounded operators in BSDE (2.28) find difficult to be well defined. This is why we appeal to a conditionally expected BSIE to characterize the adjoint process.

(ii) We can also give the integral equation of the following matrix-valued BSDEs in a more general form, which will be used in the recursive optimal control problem latter. Consider

$$P(t) = \xi + \int_t^T \left[ A^*(s)P(s) + P(s)A(s) + B^*(s)P(s)B(s) + B^*(s)Q(s) + Q(s)B(s) + \beta(s)Q(s) \right] ds - \int_t^T Q(s)dw(s),$$
where \( \beta \in L^\infty_F(0,T) \). We can write it into

\[
P(t) = \xi + \int_t^T [(A(s) - \frac{\beta(s)}{2}B(s) - \frac{\beta^2(s)}{8}I_d)^*P(s) + P(s)(A(s) - \frac{\beta(s)}{2}B(s) - \frac{\beta^2(s)}{8}I_d) + (B(s) + \frac{\beta(s)}{2}I_d)^*P(s)(B(s) + \frac{\beta(s)}{2}I_d) + (B(s) + \frac{\beta(s)}{2}I_d)^*Q(s) + Q(s)(B(s) + \frac{\beta(s)}{2}I_d)]ds - \int_t^T Q(s)dw(s).
\]

Then from (i), we have

\[
P(t) = E[\tilde{L}^*(t,T)\xi \tilde{L}(t,T) + \int_t^T \tilde{L}^*(t,s)f(s,P(s))\tilde{L}^*(t,s)ds|\mathcal{F}_t]
\]

with

\[
\tilde{L}(t,s) := L_{\tilde{A},\tilde{B}}(t,s), \quad \text{for } \tilde{A}(s) := A(s) - \frac{\beta(s)}{2}B(s) - \frac{\beta^2(s)}{8}I_d \text{ and } \tilde{B}(s) := B(s) + \frac{\beta(s)}{2}I_d.
\]

**Remark 2.22** The above (2.22) is the solution of SEE's under the variational solution framework. Also, as examples, in the same way, the solution of vector-valued SEE's under other framework (or conditions, settings) may also generates such kind of stochastic evolution operator \( \tilde{L} \) that satisfies (H1), and then the corresponding conditionally expected BSIE is well-posed.

We give a detailed mathematical description on mild solution (semigroup solution) case. Consider the SEE's

\[
\begin{align*}
du(t,u_0(s)) &= Au(t,u_0(s))ds + \tilde{A}(s)u(t,u_0(s))ds + \tilde{B}(s)u(t,u_0(s))dw(s), \quad s \in [t,T], \\
u(t,u_0(t)) &= u_0,
\end{align*}
\]

where \( u_0 \in L^2(\mathcal{F}_t,H) \), the operator \( A : D(A) \subset H \to H \) is the infinitesimal generator of a \( C_0 \)-semigroup \( \{e^{tA} \in \mathcal{L}(H); t \geq 0\} \), and \( \tilde{A}, \tilde{B} : [0,T] \times \Omega \to \mathcal{L}(H) \) are bounded and satisfying: for each \( u \in H, \tilde{A}u, \tilde{B}u \) are progressively measurable. This SEE have a unique solution \( u^{t,u_0}(-) \in \mathcal{S}^2_F(t,T;H) \) (see [3]). Using the same approach as in [22,23], it also defines a stochastic evolution operator \( \tilde{L} \) satisfying the assumption (H1). Then the corresponding conditionally expected BSIE is also well-posed. In this kind of concrete mild solution situation, in [13] the authors also describe a variation of constant formula characterization for their generalized solutions. Compared with that, our BSIE is an operator-valued equation (i.e., the equation itself is operator-valued) and has a fully nonlinear generator for \( P \).

### 2.4.2 Itô's formula in a weak formulation

Now we derive an Itô’s formula by an approximation argument for the product of the operator-valued BSIE

\[
P(t) = E[\tilde{L}^*(t,T)\xi \tilde{L}(t,T) + \int_t^T \tilde{L}^*(t,s)f(s,P(s))\tilde{L}(t,s)ds|\mathcal{F}_t], \quad t \in [0,T], \quad (2.30)
\]

and two forward SEE's in the form of

\[
\begin{align*}
dx(t) &= A(t)x(t)dt + [B(t)x(t) + \zeta(t)I_{E_n}(t)]dw(t), \quad t \in [0,T], \\
x(0) &= 0,
\end{align*}
\]

where, for some \( \beta \in L^\infty_F(0,T) \),

\[
\tilde{L}(t,s) := L_{\tilde{A},\tilde{B}}(t,s) \quad \text{with } \tilde{A} = A(s) + \frac{\beta(s)}{2}B(s) - \frac{\beta^2(s)}{8}I_d \text{ and } \tilde{B} = B(s) + \frac{\beta(s)}{2}I_d,
\]

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and $ζ$ is an $H$-valued process, $E_ρ = [t_0, t_0 + ρ]$ for some $t_0 \in [0, T)$ and $ρ \in [0, T - t_0]$.

Then we have the following Itô’s formula. The proof is lengthy and technical, and is thus put in the appendix.

**Theorem 2.23** Let Assumptions (H2) and (H4) be satisfied and for some $α > 1$,

$$(ξ, f(\cdot, \cdot, 0), ζ) \in L_w^2(\mathcal{F}_T, \mathcal{L}(H)) \times L_w^2, δw(0, T, \mathcal{L}(H)) \times L_w^4(0, T; H). \quad (2.32)$$

Then

$$\langle P(t)x(t), x(t) \rangle + σ(t) = \langle \xi x(T), x(T) \rangle + \int_t^T \{[f(s, P(s))x(s), x(s)] + β(s)Z(s)

- \langle P(s)ζ(s), ζ(s)⟩ I_{E, s}(s)\} ds - \int_t^T Z(s)dw(s), \quad t \in [0, T], \quad (2.33)$$

for a unique couple of processes $(σ, Z) \in L_w^2(0, T) \times L_w^2, δw(0, T)$ satisfying

$$\sup_{t \in [0, T]} \mathbb{E}(|σ(t)|^α) = o(ρ^α), \quad (2.34)$$

$$\mathbb{E}([\int_0^T |Z(t)|^2 dt]^{α}) = O(ρ^α). \quad (2.35)$$

**Remark 2.24** When solving the stochastic optimal control problem for SEEIs in the conventional case (see Remark 3.3), only the form of Theorem 2.23 when $β ≡ 0$ and $f$ is independent of $p$ is needed, and it corresponds to [6] Equality (5.11), [12] Equality (5.17), and [20] Equality (9.61) (plus estimates (9.62), (9.63) and (9.82)).

**Remark 2.25** To understand the above Itô’s formula, let us look at how this is derived in the finite dimensional case. The differential form (taking $A_1 = A + βB$, $B_1 = B$ in Remark 2.21 (ii)) of BSIE (2.30) is

$$P(t) = \xi + \int_t^T \{A^*(s)P(s) + P(s)A(s) + β(s)(B^*(s)P(s) + P(s)B(s)) + f(s, P(s))

+ B^*(s)P(s)B(s) + B^*(s)Q(s) + Q(s)B(s) + β(s)Q(s)\} ds - \int_t^T Q(s)dw(s). \quad$$

We apply Itô’s formula to $\langle P(t)x(t), x(t) \rangle$ and obtain

$$\langle P(t)x(t), x(t) \rangle = \langle ξ x(T), x(T) \rangle + \int_t^T \{β(s)((B(s)^*P(s) + P(s)B(s) + Q(s))x(s), x(s))

+ \langle f(s, P(s))x(s), x(s) \rangle - \langle P(s)ζ(s), ζ(s)⟩ I_{E, s}(s)

- \langle Q(s)x(s), ζ(s) \rangle + Q(s)ζ(s), x(s) \rangle + ⟨P(s)B(s)x(s), ζ(s)⟩

+ ⟨B^*(s)P(s)ζ(s), x(s)]I_{E, s}(s)\} ds - \int_t^T [([B^*(s)P(s) + P(s)B(s) + Q(s))x(s), x(s))

+ ⟨P(s)ζ(s), x(s)]I_{E, s}(s) + ⟨P(s)x(s), ζ(s)⟩ I_{E, s}(s)\} dw(s).$$

Since the depiction of $Q$ is unavailable, we try to merge the martingale terms and the small terms together and determine them via the solution of BSDEs, as follows. We take

$$Z_1(s) = \langle (B^*(s)P(s) + P(s)B(s) + Q(s))x(s), x(s) \rangle + \langle [P(s)ζ(s), x(s)] + ⟨P(s)x(s), ζ(s)⟩ I_{E, s}(s)$$

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and
\[ k(s) = \langle Q(s)x(s), \zeta(s) \rangle + \langle Q(s)x(s), \zeta(s) \rangle + \langle P(s)B(s)x(s), \zeta(s) \rangle + \langle B^*(s)P(s)\zeta(s), x(s) \rangle + \langle P(s)\zeta(s), x(s) \rangle + \langle P(s)x(s), \zeta(s) \rangle ]_{E_p}(s). \]

Then
\[ (P(t)x(t), x(t)) = (\xi x(T), x(T)) + \int_t^T [(f(s, P(s))x(s), x(s)) + \beta(s)Z_1(s) \]
\[ - (P(s)\zeta(s), \zeta(s)) ]_{E_p}(s) - k(s)]ds - \int_t^T Z_1(s)d\omega(s). \]

Let \((\sigma, b)\) be the solution of BSDE
\[ -\sigma(t) = \int_t^T [\beta b(s) - k(s)]ds - \int_t^T b(s)d\omega(s), \quad t \in [0, T]. \]
and set
\[ Z(t) := Z_1(t) - b(t). \]

Subtracting (2.37) from (2.36), we have
\[ (P(t)x(t), x(t)) + \sigma(t) = (\xi x(T), x(T)) + \int_t^T [(f(s, P(s))x(s), x(s)) + \beta(s)Z(s) \]
\[ - (P(s)\zeta(s), \zeta(s)) ]_{E_p}(s)ds - \int_t^T Z(s)d\omega(s), \]
and the corresponding estimates can be obtained from the standard BSDE theory.

So Theorem 2.23 can be regarded as a weak formulation of the classical Itô’s formula in the infinite dimensional framework. It is also worth noting that the above analysis does not apply to our infinite dimensional situation, since we do not have a differential form for operator-valued BSDE now.

3 Stochastic maximum principle for optimally controlled SEE

3.1 Formulation of the problem

Consider the following controlled SEE:
\[
\begin{aligned}
  dz(t) &= [A(t)x(t) + a(t, x(t), u(t))]dt + [B(t)x(t) + b(t, x(t), u(t))]d\omega(t), \\
  x(0) &= x_0,
\end{aligned}
\]
where \(x_0 \in H\),
\[(A, B) : [0, T] \times \Omega \rightarrow \mathcal{L}(V; V^* \times H)\]
are linear unbounded operators satisfying the coercivity and quasi-skew-symmetry condition \((H4)\) and
\[(a, b) : [0, T] \times \Omega \times H \times U \rightarrow H \times H\]
are nonlinear functions. Define the cost functional \(J(\cdot)\) as
\[ J(u(\cdot)) := y(0), \]
where $y$ is the recursive utility subject to a BSDE:

$$y(t) = h(x(T)) + \int_t^T k(s, x(s), y(s), z(s), u(s))ds - \int_t^T z(s)dw(s).$$

(3.2)

Here,

$$k : [0, T] \times \Omega \times H \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R} \quad \text{and} \quad h : H \times \Omega \to \mathbb{R}.$$

The control domain $U$ is a separable metric space with distance $d(\cdot, \cdot)$. By fixing an element $0$ in $U$, we define the length $|u|_U := d(u, 0)$. We define the admissible control set

$$\mathcal{U}[0, T] := \{ u : [0, T] \times \Omega \to U \text{ is progressively measurable and } \mathbb{E}\left[ \int_0^T |u(t)|^2_U dt \right] < \infty, \text{ for each } \alpha \geq 1 \}.$$

Our optimal control problem is to find an admissible control $\hat{u}(\cdot)$ such that the cost functional $J(u(\cdot))$ is minimized at $\hat{u}(\cdot)$ over the control set $\mathcal{U}[0, T]$

$$J(\hat{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot)).$$

We make the following assumption for $a, b, h$ and $k$.

(H5) For each $(x, y, z, u), (x, u), (x, y, z, u)$ are progressively measurable and $h(\cdot, \cdot)$ is $\mathcal{F}_t$-measurable. For each $(t, \omega, u), (x, y, z, u)$, $a, b, h$ are twice continuously differentiable with respect to $(x, y, z)$; for each $(t, \omega, u), (x, y, z, u)$, $DKk$ are continuous in $(x, y, z, u)$, where $DKk$ are the gradient and Hessian matrix of $k$ with respect to $(x, y, z)$, respectively; $a, b, h$ are bounded; $a$ is bounded by $C(1 + \|x\|_H + |u|_U)$ and $k$ is bounded by $C(1 + \|x\|_H + |y| + |z| + |u|_U)$.

### 3.2 Adjoint equations and the maximum principle

We introduce the following simplified notations: for $\psi = a, b, a_x, b_x, a_{xx}, b_{xx}$ and $v \in U$, define

$$\tilde{\psi}(t) := \psi(t, \bar{x}(t), \bar{u}(t)), \quad \delta \psi(t; v) := \psi(t, \bar{x}(t), v) - \tilde{\psi}(t)$$

and

$$\bar{A} := A + a_x, \quad \bar{B} := B + b_x.$$ 

Consider the following first-order $H$-valued adjoint backward stochastic evolution equation (BSEE for short, and the well-posedness result is referred to [3]):

$$\begin{cases}
-dp(t) = \{[A^* + k_y(t) + k_z(t)B^*]p(t) + [B^* + k_y(t)q(t) + k_z(t)]dt - q(t)dw(t), \\
p(T) = h_x(\bar{x}(T)),
\end{cases}$$

and the following second-order $\mathcal{L}(H)$-valued adjoint BSIE

$$P(t) = \mathbb{E}[\bar{L}^*(t, T)h_{xx}(\bar{x}(T))\bar{L}(t, T) + \int_t^T \bar{L}^*(t, s)(k_y(s)P(s) + G(s))\bar{L}(t, s)ds|\mathcal{F}_t], \quad 0 \leq t \leq T,$$

where

$$\phi(t) := \phi(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), \quad \text{for } \phi = k_x, k_y, k_z, D^2 k,$$

$$L(t, s) = \mathbb{E}[L_{A, B}(t, s), \quad \text{for } A(s) := \mathbb{E}(s) + \frac{k_z(s)}{2}B(s) - \frac{(k_z(s))^2}{8}I_d \quad \text{and} \quad B(s) := B(s) + \frac{k_z(s)}{2}I_d,$$

$$G(t) := D^2 k(t)[(I_d, p(t), B^*(t)p(t) + q(t))] + (I_d, p(t), B^*(t)p(t) + q(t))] + (p(t), a_{xx}(t)) + k_z(t)p(t, b_{xx}(t)) + (q(t), b_{xx}(t)).$$
Remark 3.1 Letting the coefficients for the first- and second-order adjoint equations wait to be determined and plugging the Itô’s formulas into the derivation of maximum principle, we can use a similar analysis as in \([16]\) to derive heuristically the proper generators for the first- and second-order adjoint equations \((3.3)\) and \((3.4)\). We may also give their formulations based on the adjoint equations in \([16]\) and the discussion in (ii) of Remark 2.2.

Our maximum principle is stated as follows.

**Theorem 3.2** Let Assumptions \((H4)-(H5)\) be satisfied. Assume that \(\bar{x}(\cdot)\) and \((\bar{y}(\cdot), \bar{z}(\cdot))\) are the solutions of \(\text{SEE} (3.7)\) and \(\text{BSDE} (3.3)\) corresponding to the optimal control \(\bar{u}(\cdot)\). Denote by processes \((p, q) \in L^2_{\mathcal{F}}(0, T; V \times H)\) and \(P \in L^2_{\mathcal{F}}(0, T; \mathcal{L}(H))\) the solutions of \(\text{BSEE} (3.5)\) and \(\text{BSIE} (3.3)\), respectively. Then

\[
\inf_{v \in U} \{\mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), v, p(t), q(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), p(t), q(t))
+ \frac{1}{2} \langle P(t)(b(t, \bar{x}(t), v) - b(t, \bar{x}(t), \bar{u}(t))), b(t, \bar{x}(t), v) - b(t, \bar{x}(t), \bar{u}(t)) \rangle(\cdot) \} = 0, \quad P\text{-a.s. a.e.,}
\]

where the Hamiltonian

\[
\mathcal{H}(t, x, y, z, v, p, q) := \langle p, a(t, x, v) \rangle + \langle q, b(t, x, v) \rangle + k(t, x, y, z, v, p, b(t, x, v) - b(t, \bar{x}(t), \bar{u}(t))), v), \quad (t, \omega, x, y, z, v, p, q) \in [0, T] \times \Omega \times \mathbb{H} \times \mathbb{R} \times \mathbb{R} \times U \times H \times H.
\]

Remark 3.3 When \(k\) is independent of \(y\) and \(z\), Theorem 3.2 degenerates to the conventional maximum principle without utilities, which was obtained in \([6, 12, 20]\).

### 3.3 Proof of Theorem 3.2

**Step 1: Spike variation and dual analysis for SEEs.** Given any admissible control \(u(\cdot) \in U[0, T]\) and \(t_0 \in [0, T]\), we consider the spike variation perturbation

\[
u^\rho(t) := \begin{cases}
u(t), & t \in E_\rho, \\
\bar{u}(t), & t \in [0, T] \setminus E_\rho,
\end{cases}
\]

with \(E_\rho = [t_0, t_0 + \rho)\) for \(\rho \in [0, T - t_0]\). We denote

\[
\delta \psi(t) := \delta \psi(t; u(t)), \quad \text{for} \quad \psi = a, b, a_x, b_x, a_{xx}, b_{xx}.
\]

Let \((x^\rho(\cdot), y^\rho(\cdot), z^\rho(\cdot))\) solve the system corresponding to the control \(u^\rho(\cdot)\). Consider the following linearized variational systems:

\[
x^{1, \rho}(t) = \int_0^t \bar{A}(s)x^{1, \rho}(s)ds + \int_0^t \bar{B}(s)x^{1, \rho}(s)ds + \delta b(s)I_{E_\rho}(s)dw(s)
\]

and

\[
x^{2, \rho}(t) = \int_0^t \bar{A}(s)x^{2, \rho}(s) + \frac{1}{2}\bar{a}_{xx}(s)(x^{1, \rho}(s), x^{1, \rho}(s)) + \delta a(s)I_{E_\rho}(s)ds
+ \int_0^t \bar{B}(s)x^{2, \rho}(s) + \frac{1}{2}\bar{b}_{xx}(s)(x^{1, \rho}(s), x^{1, \rho}(s)) + \delta b(s)x^{1, \rho}(s)I_{E_\rho}(s)dw(s).
\]
Proposition 3.4 Assume that (H4) and (H5) hold. Then for $\alpha \geq 1$,

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|x^\rho(t) - \bar{x}(t)\|_{H}^{2\alpha} \right] = O(\rho^\alpha),
$$

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t)\|_{H}^{2\alpha} \right] = O(\rho^\alpha),
$$

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{2,\rho}(t)\|_{H}^{2\alpha} \right] = O(\rho^{2\alpha}),
$$

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|x^\rho(t) - \bar{x}(t) - x^{1,\rho}(t) - x^{2,\rho}(t)\|_{H}^{2\alpha} \right] = o(\rho^{2\alpha}).
$$

Proof. The proof is quite standard. As an illustration, we give the proof of the second estimate. By (2225) and the Lebesgue differentiation theorem, we have (for a.e. $t_0$) that

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t)\|_{H}^{2\alpha} \right] \leq C \mathbb{E}\left[ \int_{0}^{T} I_{E_\rho}(t) \|\delta b(t)\|_{V}^{2\alpha} dt \right] \leq C \mathbb{E}\left[ \int_{0}^{T} I_{E_\rho}(t) (1 + |u(t)|^2 + |\bar{u}(t)|^2) dt \right] \leq C \rho^{\alpha-1} \mathbb{E}\left[ \int_{E_\rho} (1 + |u(t)|^2 + |\bar{u}(t)|^2) dt \right] = O(\rho^\alpha).
$$

According the assumptions on the coefficients, the adjoint processes $(p,q)$ and $P$ satisfy (see Appendix for the proofs): for any $\beta \geq 2$,

$$
\sup_{t \in [0,T]} \mathbb{E}[||p(t)||_{H}^{\beta}] + \mathbb{E}[\int_{0}^{T} ||p(t)||_{V}^{2\beta} dt] + \mathbb{E}[\int_{0}^{T} ||q(t)||_{H}^{2\beta} dt] < \infty \quad \text{and} \quad \sup_{t \in [0,T]} \mathbb{E}[||P(t)||_{2(\beta)}] < \infty. \tag{3.6}
$$

We have the following Itô’s formula for the first-order adjoint equation (see (17)):

$$
\langle p(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle = \langle h_x(\bar{x}(T)), x^{1,\rho}(T) + x^{2,\rho}(T) \rangle + \int_{t}^{T} J_1(s) ds - \int_{t}^{T} J_2(s) dw(s), \tag{3.7}
$$

where

$$
J_1(t) := \langle k_x(t) + k_y(t)p(t) + k_z(t)q(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + \langle k_z(t)p(t), \bar{B}(t)(x^{1,\rho}(t) + x^{2,\rho}(t)) \rangle - \langle [p(t), \delta a(t)], \bar{B}(t)(x^{1,\rho}(t) + x^{2,\rho}(t)) \rangle + \langle q(t), \delta b(t) + \delta b_x(t)x^{1,\rho}(t) \rangle I_{E_\rho}(t) - \frac{1}{2} \langle [p(t), \delta b_x(t)x^{1,\rho}(t)], I_{E_\rho}(t) \rangle - \langle [p(t), \delta b_x(t)x^{2,\rho}(t)], I_{E_\rho}(t) \rangle - \frac{1}{2} \langle [p(t), \delta b_y(t)x^{1,\rho}(t), x^{2,\rho}(t)], I_{E_\rho}(t) \rangle - \langle [p(t), \delta b_y(t)x^{2,\rho}(t)], I_{E_\rho}(t) \rangle,
$$

$$
J_2(t) := \langle p(t), \bar{B}(t)(x^{1,\rho}(t) + x^{2,\rho}(t)) \rangle + \langle q(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + \langle p(t), \delta b(t) + \delta b_x(t)x^{1,\rho}(t) \rangle I_{E_\rho}(t) + \frac{1}{2} \langle [p(t), \delta b_x(t)x^{1,\rho}(t), x^{2,\rho}(t)], I_{E_\rho}(t) \rangle.
$$

By Theorem (2223) we also have the following Itô’s formula for the second-order adjoint equation:

$$
\langle P(t)x^{1,\rho}(t), x^{1,\rho}(t) \rangle + \sigma(t) = \langle h_{xx}(\bar{x}(T))x^{1,\rho}(T), x^{1,\rho}(T) \rangle + \int_{t}^{T} k_y(s) \langle P(s)x^{1,\rho}(s), x^{1,\rho}(s) \rangle ds + \frac{1}{2} \langle [P(s), \delta b(s), \delta b(s)], I_{E_\rho}(s) \rangle ds - \int_{t}^{T} \mathbb{Z}(s) dw(s), \tag{3.8}
$$

$$
+ k_z(s) \mathbb{Z}(s) + \langle G(s)x^{1,\rho}(s), x^{1,\rho}(s) \rangle - \langle P(s) \delta b(s), \delta b(s) \rangle I_{E_\rho}(s) ds - \int_{t}^{T} \mathbb{Z}(s) dw(s),
$$

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for some processes \((\sigma, \mathcal{Z}) \in L^{2p}(0, T) \times L^{2,2\alpha}(0, T)\) satisfying
\[
\sup_{t \in [0,T]} E[|\sigma(t)|^{2\alpha}] = o(\rho^{2\alpha}) \quad \text{and} \quad E\left[\left( \int_0^T |\mathcal{Z}(t)|^{2}dt \right)^{\alpha} \right] = O(\rho^{2\alpha}), \quad \text{for any } \alpha \geq 1. \quad (3.9)
\]
Thus,
\[
\langle p(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + \frac{1}{2} \langle P(t)x^{1,\rho}(t), x^{1,\rho}(t) \rangle + \frac{1}{2} \sigma(t) = \langle h_x(\bar{x}(T)), x^{1,\rho}(T) + x^{2,\rho}(T) \rangle
\]
\[
+ \frac{1}{2} \langle h_{xx}(\bar{x}(T))x^{1,\rho}(T), x^{1,\rho}(T) \rangle + \int_t^T I_1(s)ds - \int_t^T [I_2(s) + \langle p(s), \delta b(s) \rangle] E_v(s)dw(s),
\]
where
\[
I_1(t) := \langle k_x(t) + k_y(t)p(t) + k_z(t)q(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + k_z(t)\bar{B}_t(x^{1,\rho}(t) + x^{2,\rho}(t)), \quad \bar{B}_t := \frac{1}{2}(k_y(t)P(t)
\]
\[
+ D^2 k(t)[I_d, p(t), \bar{B}_t p(t) + q(t)], [I_d, p(t), \bar{B}_t p(t) + q(t)] + k_z(t)\bar{B}_t x(t), \quad \bar{x}(t), \bar{\mathcal{Z}}(t) \rangle - \langle [p(t), \delta a(t)] + \langle q(t), \delta b(t) + \delta b_x(t)x^{1,\rho}(t) \rangle + \frac{1}{2} \langle P(t)\delta b(t), \delta b(t) \rangle ] E_v(t)
\]
and
\[
I_2(t) := \langle p(t), \bar{B}_t(x^{1,\rho}(t) + x^{2,\rho}(t)) \rangle + \langle q(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + \langle p(t), \delta b_x(t)x^{1,\rho}(t) \rangle E_v(t)
\]
\[
+ \frac{1}{2} \langle P(t)\delta b(t), \delta b(t) \rangle ] E_v(t)
\]
\[
\frac{1}{2} \mathcal{Z}(t). \quad (3.10)
\]
Step 2: Variation calculation. To obtain the maximum principle, we consider the variation
\[
g^\rho(t) - \frac{1}{2} \sigma(t) = h(x^\rho(T)) - h(\bar{x}(T)) - \langle h_x(\bar{x}(T)), x^{1,\rho}(T) + x^{2,\rho}(T) \rangle - \frac{1}{2} \langle h_{xx}(\bar{x}(T))x^{1,\rho}(T), x^{1,\rho}(T) \rangle
\]
\[
+ \int_t^T \{ k(s,x^\rho(s), y^\rho(s), z^\rho(s), u^\rho(s)) - k(s,\bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) - I_1(s) \} ds - \int_t^T \tilde{z}^\rho(s)dw(s),
\]
where
\[
g^\rho(t) := y^\rho(t) - \bar{y}(t) - \langle p(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle - \frac{1}{2} \langle P(t)x^{1,\rho}(t), x^{1,\rho}(t) \rangle,
\]
\[
\tilde{z}^\rho(t) := z^\rho(t) - \tilde{z}(t) - I_2(t) - \langle p(t), \delta b(t) \rangle E_v(t).
\]
Motivated from the Taylor’s expansion of the above equation, we introduce the following BSDE:
\[
g(t) = \int_t^T \{ k_y(s)\tilde{y}(s) + k_z(s)\tilde{z}(s) + \langle [p(s), \delta a(s)] + \langle q(s), \delta b(s) \rangle + k(s,\bar{x}(s), \bar{y}(s), \bar{z}(s)) \rangle \}
\]
\[
+ \langle p(s), \delta b(s), u(s) \rangle - k(s,\bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) + \frac{1}{2} \langle P(s)\delta b(s), \delta b(s) \rangle ] E_v(s) \} ds - \int_t^T \tilde{z}(s)dw(s).
\]
Proposition 3.5 Assume that (H4) and (H5) hold. Then for \(\alpha \geq 1\),
\[
\sup_{t \in [0,T]} E[|\tilde{y}(t)|^{2\alpha}] + E\left[\left( \int_0^T |\tilde{z}(t)|^{2}dt \right)^{\alpha} \right] = o(\rho^{\alpha}), \quad (3.12)
\]
\[
\sup_{t \in [0,T]} E[|\tilde{y}^\rho(t)|^{2\alpha}] + E\left[\left( \int_0^T |\tilde{z}^\rho(t)|^{2}dt \right)^{\alpha} \right] = o(\rho^{\alpha}), \quad (3.13)
\]
\[
\sup_{t \in [0,T]} E[|\tilde{y}^\rho(t) - \tilde{y}(t)|^2] + E\left[\left( \int_0^T |\tilde{z}^\rho(t) - \tilde{z}(t)|^{2}dt \right)^{\alpha} \right] = o(\rho^2). \quad (3.14)
\]
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Proof. We first prove (3.12). Denote

\[ I_3(t) := \langle p(t), x^{1,\rho}(t) + x^{2,\rho}(t) \rangle + \frac{1}{2} \langle P(t)x^{1,\rho}(t), x^{1,\rho}(t) \rangle, \]
\[ I_4(t) := \langle p(t), \delta a(t) \rangle + \langle q(t), \delta b(t) \rangle + \frac{1}{2} \langle P(t)\delta b(t), \delta b(t) \rangle, \]
\[ I_5(t) := k(t, \bar{z}(t), \bar{y}(t), \bar{z}(t) + \langle p(t), \delta b(t) \rangle, u(t)) - k(t, \bar{z}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)). \]

We calculate directly that

\[
\mathbb{E}[\int_0^T |q(t), \delta b(t)||I_{E_p}(t)dt|^{2\alpha}] \leq \left( \mathbb{E}\left[ \int_0^T |q(t)||I_{E_p}(t)dt|^2 \right] \right)^{\frac{\alpha}{2}} \left( \mathbb{E}\left[ \int_0^T \||\delta b(t)||I_{E_p}(t)dt|^{2\alpha} \right] \right)^{\frac{1}{2}}
\leq \rho \sup_{t\in[0,T]} \mathbb{E}[|\hat{y}(t)|^{2\alpha}] + \mathbb{E}\left[ \int_0^T |\hat{z}(t)|^2 dt \right]^\alpha
\leq C \rho^{\alpha} \mathbb{E}[\int_0^T |q(t)|^2 I_{E_p}(t)dt] + C \mathbb{E}[\int_0^T |q(t), \delta b(t)||I_{E_p}(t)dt|^{2\alpha}]
= O(\rho^{2\alpha}) + o(\rho^{\alpha}) = o(\rho^\alpha).
\]

Then from the a priori estimates for BSDEs and the Lebesgue differentiation theorem, we have

\[
\sup_{t\in[0,T]} \mathbb{E}[|\hat{y}(t)|^{2\alpha}] + \mathbb{E}[\int_0^T |\hat{z}(t)|^2 dt]^\alpha
\leq C \rho^{\alpha} \mathbb{E}[\int_0^T |p(t), \delta a(t)| + \frac{1}{2} \langle P(t)\delta b(t), \delta b(t) \rangle + I_5(t)|^{2\alpha} I_{E_p}(t)dt] + C \mathbb{E}[\int_0^T |q(t), \delta b(t)||I_{E_p}(t)dt|^{2\alpha}]
= O(\rho^{2\alpha}) + o(\rho^{\alpha}) = o(\rho^\alpha).
\]

We first consider (3.13). By the Taylor’s expansion,

\[
\hat{y}^\rho(t) - \frac{1}{2}\sigma(t) = J_4 + \int_1^T \{ \bar{k}_y(s)(\hat{y}^\rho(s) - \frac{1}{2}\sigma(s)) + \bar{k}_z(s)\bar{z}^\rho(s) + J_3(s) + \frac{1}{2} J_5(s) + \frac{1}{2} \bar{k}_y(s)\sigma(s) + \frac{1}{2} J_6(s) \}
+ [J_4(s) + \langle q(s), \delta b_x(s)x^{1,\rho}(s) \rangle + k_z(s)\langle p(s), \delta b_x(s)x^{1,\rho}(s) \rangle] I_{E_p}(s) ds - \int_0^T \bar{z}^\rho(s) dw(s),
\]
where

\[
\bar{k}_y(t) := \int_0^1 k_y(t, \bar{x}(t) + x^{1,\rho}(t) + x^{2,\rho}(t), \bar{y}(t) + I_3(t) + \mu \bar{y}^\rho(t), \bar{z}(t) + I_2(t) + \mu \bar{z}^\rho(t), \bar{u}(t)) \mu dt,
\]
\[
\bar{k}_z(t) := \int_0^1 k_z(t, \bar{x}(t) + x^{1,\rho}(t) + x^{2,\rho}(t), \bar{y}(t) + I_3(t) + \mu \bar{y}^\rho(t), \bar{z}(t) + I_2(t) + \mu \bar{z}^\rho(t), \bar{u}(t)) \mu dt,
\]
\[
J_3(t) := k(t, x^{\rho}(t), y^{\rho}(t), z^{\rho}(t), w^{\rho}(t))
- k(t, \bar{x}(t) + x^{1,\rho}(t) + x^{2,\rho}(t), \bar{y}(t) + I_3(t) + \bar{y}^\rho(t), \bar{z}(t) + I_2(t) + \bar{z}^\rho(t), \bar{u}(t)),
\]
\[
J_4 := h(x^{\rho}(T)) - h(\bar{x}(T)) - \langle h_x(\bar{x}(T)) + x^{1,\rho}(T) + x^{2,\rho}(T) \rangle - \frac{1}{2} \langle h_{xx}(\bar{x}(T))x^{1,\rho}(T), x^{1,\rho}(T) \rangle,
\]
\[
J_5(t) := \overline{D^2 k}(t)[x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)], [x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)], [x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)]
- \langle \overline{D^2 k}(t)[I_3(t), p(t), B^*(p)] p(t) + q(t), [I_3(t), p(t), B^*(p)] p(t) + q(t) \rangle x^{1,\rho}(t), x^{1,\rho}(t),
\]
with

\[
\overline{D^2 k}(t) := 2 \int_0^1 \int_0^1 \mu \overline{D^2 k}(t, \bar{x}(t) + x^{1,\rho}(t) + x^{2,\rho}(t), \bar{y}(t) + \mu \bar{y}^\rho(t), \bar{z}(t) + \mu \bar{z}^\rho(t), \bar{u}(t)) \mu dw.
\]

We can write

\[
J_5(t) = J_6(t) + J_7(t),
\]
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where

\[ J_0(t) := (\bar{D}^2k(t)([I_d, p(t), \bar{B}^*(t)p(t) + q(t)], [I_d, p(t), \bar{B}^*(t)p(t) + q(t)])x^{1,\rho}(t), x^{1,\rho}(t)) \]
\[ - (\bar{D}^2k(t)([I_d, p(t), \bar{B}^*(t)p(t) + q(t)], [I_d, p(t), \bar{B}^*(t)p(t) + q(t)])x^{1,\rho}(t), x^{1,\rho}(t)), \]
\[ J_1(t) := \bar{D}^2k(t)([x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)], [x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)]) \]
\[ - (\bar{D}^2k(t)([I_d, p(t), \bar{B}^*(t)p(t) + q(t)], [I_d, p(t), \bar{B}^*(t)p(t) + q(t)])x^{1,\rho}(t), x^{1,\rho}(t)). \]

Indeed, for any \((t, \omega) \in [0, T] \times \Omega\), recall that the coercivity condition ((1) in the assumption (H4)) implies
\[ \|B(t, \omega)v\|_{\mathcal{L}(V)} \leq C(K)\|v\|_V, \quad \text{for } v \in V. \]

That is,
\[ \|B(t, \omega)\|_{\mathcal{L}(V, H)} \leq C(K). \]

Moreover, according to [6] Remark 2.4 (2), we have \(B(t, \omega) + \bar{B}^*(t, \omega) \in \mathcal{L}(H)\). From (2) in the assumption (H4), we also have \(|\langle v, (B(t, \omega) + \bar{B}^*(t, \omega))v \rangle| = 2\|v, B(t, \omega)v\| \leq 2K\|v\|_H^2, \text{ for } v \in V. \]

Then by [33] Theorem VII.3.3, we have
\[ \|B(t, \omega) + \bar{B}^*(t, \omega)\|_{\mathcal{L}(V, H)} \leq C(K). \]

Thus from \(B^*(t, \omega) = (B(t, \omega) + \bar{B}^*(t, \omega)) - B(t, \omega)\), we deduce that \(\|B^*(t, \omega)\|_{\mathcal{L}(V, H)} \leq C(K)\), which implies (5.17). Now denoting \(\|\bar{D}^2k(t) - D^2k(t)\| := \|\bar{D}^2k(t) - D^2k(t)\|_{\mathcal{L}(H \times R \times R \times H \times R \times R)}\), from (5.17) we have
\[ \mathbb{E}[\int_0^T |J_0(t)|dt]^{2\alpha} \leq C\mathbb{E}[\int_0^T \|\bar{D}^2k(t) - D^2k(t)\|((1 + \|p(t)\|_H^2)||x^{1,\rho}(t)||_H^2 \]
\[ + \|p(t)\|_H^2||x^{1,\rho}(t)||_H^2 + \|q(t)\|_H^2||x^{1,\rho}(t)||_H^2)dt)^{2\alpha} \]
\[ \leq C\mathbb{E}[\int_0^T \|\bar{D}^2k(t) - D^2k(t)\|^4(1 + \|p(t)\|_H^2)dt)^{\frac{\alpha}{2}}(\mathbb{E}[\int_0^T ||x^{1,\rho}(t)||_H^2 dt)^{\frac{\alpha}{2}} \]
\[ + C\mathbb{E}[\int_0^T \|\bar{D}^2k(t) - D^2k(t)\|^4(1 + \|p(t)\|_H^2)dt)^{\frac{\alpha}{2}}(\mathbb{E}[\sup_{t \in [0, T]} ||x^{1,\rho}(t)||_H^2 dt)^{\frac{\alpha}{2}} \]
\[ + C\mathbb{E}[\int_0^T \|\bar{D}^2k(t) - D^2k(t)\|^4(1 + \|q(t)\|_H^2)dt)^{\frac{\alpha}{2}}(\mathbb{E}[\sup_{t \in [0, T]} ||x^{1,\rho}(t)||_H^2 dt)^{\frac{\alpha}{2}} \]
\[ = o(\rho^{2\alpha}). \]

Furthermore, we can decompose
\[ J_7(t) = J_{7a}(t) + J_{7b}(t), \]
where
\[ J_{7a}(t) := \bar{D}^2k(t)([x^{1,\rho}(t) + x^{2,\rho}(t), I_3(t), I_2(t)], [x^{2,\rho}(t), q(t), x^{2,\rho}(t)] + \frac{1}{2}(P(t)x^{1,\rho}(t), x^{1,\rho}(t)), I_2(t) \]
\[ - (p(t), \bar{B}(t)x^{1,\rho}(t)) - \langle q(t), x^{1,\rho}(t) \rangle). \]
and

\[ J_{76}(t) := (\hat{D}^2 k(t)[x^2(\rho(t), (p(t), x^2(\rho(t)))] + \frac{1}{2} (P(t)x^1(\rho(t), x^1(\rho(t)), I_2(t) - \langle p(t), \hat{B}(t)x^1(\rho(t)) \rangle - \langle q(t), x^1(\rho(t)) \rangle, [x^1(\rho(t), (p(t), x^1(\rho(t)), (p(t), \hat{B}(t)x^1(\rho(t)) + \langle q(t), x^1(\rho(t)) \rangle). \]

From a similar analysis as for \( J_6 \), we have

\[
\mathbb{E}[(\int_0^T |J_{76}(t)| dt)^{2\alpha}] \\
\leq C\mathbb{E}[(\int_0^T (\|x^2(\rho(t))\|_H + |\langle p(t), x^2(\rho(t) \rangle + \frac{1}{2} (P(t)x^1(\rho(t), x^1(\rho(t))) | + I_2(t) \\
- \langle p(t), \hat{B}(t)x^1(\rho(t)) - \langle q(t), x^1(\rho(t)) \rangle (\|x^1(\rho(t))\|_H + |\langle p(t), x^1(\rho(t) \rangle + |\langle p(t), \hat{B}(t)x^1(\rho(t)) \\
+ \langle q(t), x^1(\rho(t)) \rangle dt)^{2\alpha} \\
\leq C(\mathbb{E}[(\int_0^T \|x^2(\rho(t))\|_H^2 + \|p(t)\|_H^2 \|x^2(\rho(t))\|_H^2 + \|p(t)\|_H^2 \|x^2(\rho(t))\|_H^2 + \|q(t)\|_H^2 \|x^1(\rho(t))\|_H^2 \\
+ \|p(t)\|_H^2 \|x^1(\rho(t))\|_H^2 I_{E_\rho(t)}(t) + (\|p(t)\|_H^2 + \|P(t)\|_H^2 \|x^1(\rho(t))\|_H^2 + \|Z(t)\| dt)^{2\alpha} )) = O(\rho^{2\alpha}). \]

Thus, \( \mathbb{E}[(\int_0^T |J_{76}(t)| dt)^{2\alpha}] = O(\rho^{2\alpha}). \)

From Proposition 3.4, it is direct to check that \( \mathbb{E}[|I_4|^2\alpha] = o(\rho^{2\alpha}) \) and \( \mathbb{E}[(\int_0^T |J_5(t)| dt)^{2\alpha}] = O(\rho^{2\alpha}). \) Recall that in (3.9) we have obtained that

\[
\sup_{t \in [0, T]} \mathbb{E}[|\sigma(t)|^{2\alpha}] = o(\rho^{2\alpha}). \tag{3.18}
\]

Then by (3.15), (3.18) and the a priori estimates for classical BSDEs,

\[
\sup_{t \in [0, T]} \mathbb{E}[|\hat{\sigma}(t)^2\alpha] + \mathbb{E}[\int_0^T |\hat{\sigma}(t)|^{2\alpha} dt] = o(\rho^{2\alpha}).
\]

Making use of (3.18) again, we obtain (3.13).

Now we prove the last estimate. Denote

\[
\tilde{x}(t) = x(\rho(t)) - \bar{x}(t) - x^1(\rho(t)) - x^2(\rho(t)), \\
\tilde{y}(t) = \hat{y}(t) - \hat{y}(t), \\
\tilde{z}(t) = \hat{z}(t) - \hat{z}(t).
\]
Then from (3.11) and (3.16),
\[
\tilde{y}^\rho(t) - \frac{1}{2} \sigma(t) = J_4 + \int_t^T \left\{ k_y(s)(\tilde{y}^\rho(t) - \frac{1}{2} \sigma(s)) + k_z(s)\tilde{z}^\rho(s) + \frac{1}{2} \tilde{k}_y(s)\sigma(s) \\
+ (\tilde{k}_y(s) - k_y(s))(\tilde{y}^\rho(s) - \frac{1}{2} \sigma(s)) + (\tilde{k}_z(s) - k_z(s))\tilde{z}^\rho(s) + \frac{1}{2} J_5(s) \\
+ [\{g(s), \delta b_x(s)x^{1,\rho}(s)\} + k_z(s)p(s), \delta b_x(s)x^{1,\rho}(s)]I_{E_\rho}(s) \\
+ J_3(s) - I_5(s)I_{E_\rho}(s)\right\} ds - \int_t^T \tilde{z}^\rho(s)dw(s).
\]

Note that
\[
|J_3(t) - I_5(t)I_{E_\rho}(t)|
\]
\[
\leq C[\|\tilde{y}^\rho(t)\|_H + \|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + |\tilde{y}^\rho(t)| + |\tilde{z}^\rho(t)| + |I_2(t)| + |I_3(t)||I_{E_\rho}(t)|
\]
\[
\leq C[\|\tilde{y}^\rho(t)\|_H + |\tilde{y}^\rho(t)| + |\tilde{z}^\rho(t)| + (1 + \|p(t)\|_H + \|q(t)\|_H)\|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + \|p(t)\|_H \|x^{1,\rho}(t)\|_H
\]
\[
+ \|p(t)\|_V \|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + |\tilde{Z}(t)| + (\|p(t)\|_H + \|P(t)\|_{L(\mathcal{H})})\|x^{1,\rho}(t)\|_H^2)I_{E_\rho}(t).
\]

We have
\[
\mathbb{E}[\int_0^T |J_3(t) - I_5(t)I_{E_\rho}(t)|dt]^2]
\]
\[
\leq C\mathbb{E}\left[ \int_0^T \|\tilde{y}^\rho(t)\|^2_H dt + C\rho \mathbb{E}\left[ \int_0^T (|\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 + |\tilde{Z}(t)|^2)dt \right] \right. \\
+ \left. (\mathbb{E}\left[ \int_0^T (1 + \|p(t)\|_H)I_{E_\rho}(t)|dt\right])^{\frac{1}{2}} (\mathbb{E}\left[ \int_0^T \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_H dt\right])^{\frac{1}{2}} \right]
\]
\[
+ \left( \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H I_{E_\rho}(t)|dt\right] \right)^{\frac{1}{2}} \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_H \right] \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E}\left[ \int_0^T \|q(t)\|^2_H I_{E_\rho}(t)|dt\right] \right)^{\frac{1}{2}} \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_H \right] \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H I_{E_\rho}(t)|dt\right] \right)^{\frac{1}{2}} \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_H \right] \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E}\left[ \int_0^T \|p(t)\|^4_H I_{E_\rho}(t)|dt\right] \right)^{\frac{1}{4}} \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^4_H \right] \right)^{\frac{1}{4}}
\]
\[
= o(\rho^2).
\]

Analogously, from
\[
|\tilde{k}_y(t) - k_y(t)| + |\tilde{k}_z(t) - k_z(t)|
\]
\[
\leq C[\|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + |I_2(t)| + |I_3(t)| + |\tilde{y}^\rho(t)| + |\tilde{z}^\rho(t)|]
\]
\[
\leq C[|\tilde{y}^\rho(t)| + |\tilde{z}^\rho(t)| + (1 + \|p(t)\|_H + \|q(t)\|_H)\|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + \|p(t)\|_H \|x^{1,\rho}(t)\|_H
\]
\[
+ \|p(t)\|_V \|x^{1,\rho}(t) + x^{2,\rho}(t)\|_H + |\tilde{Z}(t)| + (\|p(t)\|_H + \|P(t)\|_{L(\mathcal{H})})\|x^{1,\rho}(t)\|_H^2).
\]
we also obtain

\[
\mathbb{E}\left[ \int_0^T (\tilde{k}_y(t) - k_y(t))\tilde{y}^\rho(t) + (\tilde{k}_z(t) - k_z(t))\tilde{z}^\rho(t)dt \right] \\
\leq C\mathbb{E}\left[ \int_0^T (|\tilde{k}_y(t) - k_y(t)|^2 + |\tilde{k}_z(t) - k_z(t)|^2) |\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 dt \right]^{\frac{1}{2}} + \mathbb{E}\left[ \int_0^T |\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 dt \right]^{\frac{1}{2}} \\
\leq C\mathbb{E}\left[ \int_0^T (|\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 + (1 + \|p(t)\|^2_{L^2} + \|q(t)\|^2_{L^2})|x^{1,\rho}(t) + x^{2,\rho}(t)|^2_{L^2} + \|p(t)\|^2_{L^2} \|x^{1,\rho}(t)\|^2_{L^2} \\
+ \|p(t)\|^2_{L^2} |x^{1,\rho}(t) + x^{2,\rho}(t)|^2_{L^2} + |Z(t)|^2 + (\|p(t)\|^2_{L^2} + \|P(t)\|^2_{L^2})|x^{1,\rho}(t)|^2_{L^2} dt \right]^{\frac{1}{2}} \\
\cdot \mathbb{E}\left[ \int_0^T (|\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 dt) \right]^{\frac{1}{2}} \\
\leq C\left\{ \mathbb{E}\left[ \int_0^T (|\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 + |Z(t)|^2) dt \right]^{\frac{1}{2}} \\
+ \mathbb{E}\left[ \int_0^T (1 + \|p(t)\|^2_{H^1}) dt \right]^{\frac{1}{2}} \mathbb{E}\left[ \int_0^T \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_{L^2} dt \right]^{\frac{1}{2}} \\
+ \mathbb{E}\left[ \int_0^T (\|q(t)\|^2_{L^2} dt)^{\frac{1}{4}} \mathbb{E}\left[ \sup_{t\in[0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_{L^2} \right]^{\frac{1}{2}} + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_{L^2} dt \right]^{\frac{1}{2}} \mathbb{E}\left[ \int_0^T \|x^{1,\rho}(t)\|^2_{L^2} dt \right]^{\frac{1}{2}} \\
+ \mathbb{E}\left[ \int_0^T \|p(t)\|^2_{H^1} dt \right]^{\frac{1}{2}} \mathbb{E}\left[ \sup_{t\in[0,T]} \|x^{1,\rho}(t) + x^{2,\rho}(t)\|^2_{L^2} \right]^{\frac{1}{2}} \\
+ \mathbb{E}\left[ \int_0^T (\|p(t)\|^2_{L^2} + \|P(t)\|^2_{L^2}) dt \right]^{\frac{1}{2}} \mathbb{E}\left[ \int_0^T \|x^{1,\rho}(t)\|^2_{L^2} dt \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} \mathbb{E}\left[ \int_0^T (|\tilde{y}^\rho(t)|^2 + |\tilde{z}^\rho(t)|^2 dt) \right]^{\frac{1}{2}} \\
= o(\rho^2). \]

Therefore,

\[
\sup_{t\in[0,T]} \mathbb{E}\|\tilde{y}^\rho(t) - \frac{1}{2}\sigma(t)\|^2 + \mathbb{E}\int_0^T |\tilde{z}^\rho(t)|^2 dt = o(\rho^2).
\]

This, together with (3.13), implies (3.14). \(\square\)

**Remark 3.6** From the proofs we can know that if \(B \equiv 0\) or \(k\) does not contain \(z\), it is not necessary to estimate \(p\) in the space \(V\) in (3.10).

**Step 3: Duality for BSDEs and the completion of the proof.** Consider the following adjoint equation for BSDE (3.11):

\[
\lambda(t) = 1 + \int_0^t k_y(s) \lambda(s) ds + \int_0^t k_z(s) \lambda(s) dw(s). \tag{3.19}
\]

Applying Itô’s formula to \(\lambda(t)\tilde{y}(t)\), we get

\[
\tilde{y}(0) = \mathbb{E}\int_0^T \lambda(t)(p(t),\delta a(t)) + (q(t),\delta b(t)) + k(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t) + p(t),\delta b(t)), u(t)) \\
- k(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)) + \frac{1}{2}(P(t)\delta b(t),\delta b(t))I_{E^u}(t) dt.
\]
Note that from the optimization assumption and (3.14),
\[
0 \leq J(\hat{u}(\cdot)) - J(u(\cdot)) = y^\rho(0) - \hat{y}(0)
\]
\[
= \hat{y}(0) + o(\rho) + \frac{1}{2}(P(0)x^{1,\rho}(0),x^{1,\rho}(0)) + \frac{1}{2}(P(t)x^{1,\rho}(0),x^{1,\rho}(0))
\]
\[
= \hat{y}(0) + o(\rho)
\]
\[
= \mathbb{E} \int_0^T \lambda(t)[\langle p(t), \delta a(t) \rangle + \langle q(t), \delta b(t) \rangle] + k(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t) + \langle p(t), \delta b(t) \rangle, u(t)) - k(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t), \tilde{u}(t))
\]
\[
+ \frac{1}{2}(P(t)\delta b(t), \delta b(t))I_{E^c}(t)dt + o(\rho).
\]
Note that \(\lambda(t) > 0\) for \(t \in [0, T]\), then we obtain the pointwise maximum principle as
\[
\langle p(t), \delta a(t; v) \rangle + \langle q(t), \delta b(t; v) \rangle + k(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t) + \langle p(t), \delta b(t; v) \rangle, v) - k(t, \tilde{x}(t), \tilde{y}(t), \tilde{z}(t), \tilde{u}(t))
\]
\[
+ \frac{1}{2}(P(t)\delta b(t; v), \delta b(t; v)) \geq 0, \quad \forall v \in U, \text{ } P \text{-a.s. } \text{a.e.,}
\]
which can also be written as (3.5). The proof is now complete.

### 3.4 Application on controlled SPDEs

We present an example of controlled SPDEs that fits our framework. Let \(G\) be a bounded domain in \(\mathbb{R}^n\). Consider super-parabolic stochastic PDE (cf. [29])
\[
\begin{cases}
    dx(t, \zeta) = [\sum_{i,j=1}^n \partial_{\zeta_i}(\alpha_{ij}(t, \zeta)\partial_{\zeta_j}x(t, \zeta)) + a(t, \zeta, u(t), x(t, \zeta))]dt + [\sum_{i=1}^n \beta_i(t, \zeta)\partial_{\zeta_i}x(t, \zeta) + b(t, \zeta, u(t), x(t, \zeta))]dw(t), \quad (t, \zeta) \in [0, T] \times G, \\
x(0, \zeta) = x_0(\zeta), \quad \zeta \in G, \\
x(x, \zeta) = 0, \quad (t, \zeta) \in [0, T] \times \partial G,
\end{cases}
\]
Here \(\alpha_{ij}, \beta_i, a, b\) and \(x_0\) are given coefficients and initial value, respectively. The control \(u(t)\) is a progressive process taking values in some metric space \(U\). We consider the problem of minimizing the cost functional
\[
J(u(\cdot)) = y(0),
\]
where \(y\) is the recursive utility subjected to a BSDE:
\[
y(t) = \int_G h(\zeta, x(T, \zeta))d\zeta + \int_t^T \int_G k(s, \zeta, y(s), z(s), u(s), x(s, \zeta))d\zeta ds - \int_t^T z(s)dw(s).
\]
We impose standard measurability conditions on the coefficients. We take
\[
H = L^2(G), \quad V = H_0^2(G), \quad A = \sum_{i,j=1}^n \partial_{\zeta_i}(\alpha_{ij}(t, \zeta)\partial_{\zeta_j}), \quad B = \sum_{i=1}^n \beta_i(t, \zeta)\partial_{\zeta_i}.
\]
To guarantee the condition (H4), we assume there exist some constants \(0 < \kappa \leq K\) such that
\[
\kappa I_{n \times n} + (\beta_i\beta_j)_{n \times n} \leq 2(\alpha_{ij})_{n \times n} \leq K I_{n \times n},
\]
the function \(\beta_i\) is continuously differentiable with respect to \(\zeta\), and \(\alpha_{ij}, \beta_i, \partial_{\zeta_i} \beta_i\) are bounded by \(K\). Indeed, the proof for the coercivity condition is standard and can be found in [29] and the quasi-skew-symmetry condition can be deduced by the observation that
\[
\int_G (\beta_i(t, \zeta)\partial_{\zeta_i}x(t, \zeta))x(t, \zeta)d\zeta = -\int_G x(t, \zeta)\partial_{\zeta_i}\beta_i(t, \zeta)x(t, \zeta)d\zeta
\]
\[
= -\int_G x(t, \zeta)\beta_i(t, \zeta)\partial_{\zeta_i}x(t, \zeta)d\zeta - \int_G \partial_{\zeta_i}\beta_i(t, \zeta)x(t, \zeta)^2d\zeta.
\]
Next, provided the corresponding differentiation and growth conditions on the coefficients $a, b, h$ and $k$, the assumption (H5) can be verified (cf. [21]). Therefore, we obtain the maximum principle for the above stochastic optimal control problem.

4 Appendix

4.1 Proof of Proposition 2.16

We have the decomposition:

\[
|\langle P(t+\delta)u, v \rangle - \langle P(t)u, v \rangle| \leq |\mathbb{E}[\langle \xi L(t+\delta, T)u, L(t+\delta, T)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle \xi L(t, T)u, L(t+\delta, T)v \rangle |\mathcal{F}_{t+\delta}]|
\]

\[
+ |\mathbb{E}[\langle \xi L(t, T)u, L(t+\delta, T)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle |\mathcal{F}_{t+\delta}]|
\]

\[
+ |\mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle |\mathcal{F}_{t}]|
\]

\[
+ \mathbb{E} \int_{t+\delta}^{T} |\langle f(s)u, L(t+\delta, s)v \rangle - \langle f(s)u, L(t+\delta, s)v \rangle| ds |\mathcal{F}_{t+\delta}|
\]

\[
+ \mathbb{E} \int_{t+\delta}^{T} |\langle f(s)u, L(t+\delta, s)v \rangle - \langle f(s)u, L(t+\delta, s)v \rangle| ds |\mathcal{F}_{t+\delta}|
\]

\[
+ \mathbb{E} \int_{t+\delta}^{T} |\langle f(s)u, L(t+\delta, s)v \rangle - \langle f(s)u, L(t, s)v \rangle| ds |\mathcal{F}_{t+\delta}|
\]

\[
+ |\mathbb{E}[\langle f(s)u, L(t, s)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle f(s)u, L(t, s)v \rangle |\mathcal{F}_{t+\delta}]|
\]

\[
+ |\mathbb{E}[\langle f(s)u, L(t, s)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle f(s)u, L(t, s)v \rangle |\mathcal{F}_{t}]|
\]

We only show the convergence of the first, third and fourth terms, and the others can be estimated in the same manner. As $\delta \downarrow 0$, we have by the assumption (H3)

\[
\mathbb{E}[|\mathbb{E}[\langle \xi L(t+\delta, T)u, L(t+\delta, T)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle \xi L(t, T)u, L(t+\delta, T)v \rangle |\mathcal{F}_{t+\delta}]|^a]^{\frac{1}{a}}
\]

\[
\leq (\mathbb{E}[\|\xi\|^2_{L^2(H)}])^{\frac{1}{a}}(\mathbb{E}[\|L(t+\delta, T)u - L(t+\delta)u\|^2_{H^a}])^{\frac{1}{a}}(\mathbb{E}[\|L(t+\delta, T)v\|^2_{H^a}])^{\frac{1}{a}}
\]

\[
\leq C_1(\mathbb{E}[\|u - L(t, t+\delta)u\|^2_{H^a}])^{\frac{1}{a}} \rightarrow 0,
\]

where $C_1$ is a constant independent of $\delta$, and by the martingale convergence theorem

\[
\mathbb{E}[\mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle |\mathcal{F}_{t+\delta}] - \mathbb{E}[\langle \xi L(t, T)u, L(t, T)v \rangle |\mathcal{F}_{t}]|^a] \rightarrow 0.
\]

Making use of the assumption (H3) again, we also obtain that, as $\delta \downarrow 0$,

\[
\mathbb{E}[\mathbb{E}[\int_{t+\delta}^{T} |\langle f(s)u, L(t+\delta, s)v \rangle | ds |\mathcal{F}_{t+\delta}] - \mathbb{E}[\int_{t+\delta}^{T} |\langle f(s)u, L(t+\delta, s)v \rangle | ds |\mathcal{F}_{t+\delta}]|^a]^{\frac{1}{a}}
\]

\[
\leq (\mathbb{E}[\int_{t+\delta}^{T} |f(s)u|^2_{L^2(H)} ds])^{\frac{1}{a}}(\mathbb{E}[\int_{t+\delta}^{T} |L(t+\delta, s)u - L(t+\delta)u\|^2_{H^a} ds])^{\frac{1}{a}}(\mathbb{E}[\int_{t+\delta}^{T} |L(t+\delta, T)v\|^2_{H^a} ds])^{\frac{1}{a}}
\]

\[
\leq C_1(\mathbb{E}[\int_{t+\delta}^{T} |u - L(t, t+\delta)u|^2_{H^a}])^{\frac{1}{a}} \rightarrow 0.
\]

4.2 Proof of Theorem 2.23

One crucial ingredient in the proof is the following estimate.
Theorem 4.1 Let the assumptions of Theorem 2.23 hold. Define, for \( t \in [0, T] \),
\[
\begin{align*}
\sigma(t) := & E \left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi(x(T), x(T)) \rangle + \int_0^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))x(s), x(s) \rangle ds \right] \\
& - \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle P(s)\zeta(s), \zeta(s) \rangle I_{E_F}(s) ds|F_t] - \langle P(t)x(t), x(t) \rangle 
\end{align*}
\]  
(4.1)
with
\[
\lambda(t) := e^{\int_0^t \beta^2(s)ds + \beta(s)dw(s)}.
\]  
(4.2)
Then the process \( \sigma \) satisfies (2.34).

Let us admit for a moment the following result on moving the nonhomogeneous term from the diffusion to the initial point.

Proposition 4.2 Suppose (H4) holds. Given any \( \alpha \geq 1 \) and \( \zeta_0 \in L^{2\alpha}(F_{t_0}, V) \), let \( y \) solve SEE
\[
\begin{align*}
\frac{dy(t)}{dt} & = A(t)y(t)dt + [B(t)y(t) + \zeta_0 I_{E_F}(t)]dw(t), \\
y(0) & = 0,
\end{align*}
\]
and define
\[
z(t) := \begin{cases} 
0, & t < t_0, \\
\eta(t), & t_0 \leq t < t_0 + \rho, \\
z(t) : z(t) solves \, z(t) = \eta(t_0 + \rho) + \int_{t_0+\rho}^t A(s)z(s)ds + \int_{t_0+\rho}^t B(s)z(s)dw(s), & t \geq t_0 + \rho,
\end{cases}
\]
where
\[
\eta(t) := \frac{1}{\sqrt{\rho}} \zeta_0 \int_{t_0}^t I_{E_F}(s) dw(s), \quad t \geq t_0.
\]
Then there exists some constant \( C > 0 \) depending on \( \alpha, \delta \) and \( K \) such that
\[
E[ \sup_{t \in [0, T]} \| y(t) - \sqrt{\rho} z(t) \|^2_{L^2} ] \leq C E[\| \zeta_0 \|^2_{L^2}]^{2\alpha}.
\]

Proof of Theorem 4.1 The proof is divided into the following three steps. Moreover, we only need to give the estimate of \( E[|\sigma(t)|^\alpha] \) for any given \( t \), since this bound can be chosen to be independent of \( t \) according to the latter proof.

Step 1: an auxiliary approximation result. By the following Lemma 4.3 we have
\[
\tilde{L}(\hat{t}, s) = \frac{\lambda_1(s)}{\lambda_1(\hat{t})} L(\hat{t}, s), \quad \text{for any } \hat{t} \leq s \leq T,
\]
with
\[
L(\hat{t}, s) := L_{A,B}(\hat{t}, s) \quad \text{and} \quad \lambda_1(s) := e^{\int_0^s \frac{1}{2} \beta^2(r)dr + \frac{1}{2} \beta(r)dw(r)}.
\]
Noting that \( \lambda = \lambda_1 \cdot \lambda_1 \), then
\[
P(\hat{t}) = E_0 \left[ \frac{\lambda(T)}{\lambda(\hat{t})} \right] \tilde{L}(\hat{t}, T) \xi L(\hat{t}, T) + \int_{\hat{t}}^T \frac{\lambda(s)}{\lambda(\hat{t})} L^*(\hat{t}, s)f(s, P(s))L^*(\hat{t}, s)ds|F_{\hat{t}}].
\]
Given any $\zeta_0 \in L^{4\alpha}(\mathcal{F}_{t_0}, V)$, we define $z(t)$ as in Proposition 4.2. For $t \geq t_0 + \rho$, it holds that $L(\hat{t}, s)z(\hat{t}) = z(s)$ for $s \geq t$, and thus

$$
\langle P(\hat{t})z(\hat{t}), z(\hat{t}) \rangle = \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi L(\hat{t}, T)z(\hat{t}), L(\hat{t}, T)z(\hat{t}) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))L(\hat{t}, s)z(\hat{t}), L(\hat{t}, s)z(\hat{t}) \rangle ds | \mathcal{F}_{\hat{t}} \right] \\
= \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi z(T), z(T) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_{\hat{t}} \right].
$$

(4.3)

Fix any $t \in [0, T]$. Based on (4.3), we separate our discussions into two cases: (1) $t > t_0$; (2) $t \leq t_0$. For the first case, when $\rho$ is small, it holds that $t \geq t_0 + \rho$ and then

$$
\langle P(t)z(t), z(t) \rangle = \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi z(T), z(T) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_t \right].
$$

(4.4)

For the second case, we have

$$
\langle P(t_0 + \rho)z(t_0 + \rho), z(t_0 + \rho) \rangle = \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t_0 + \rho)} \langle \xi z(T), z(T) \rangle + \int_{t_0 + \rho}^T \frac{\lambda(s)}{\lambda(t_0 + \rho)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_{t_0 + \rho} \right] \\
= \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t_0 + \rho)} \langle \xi z(T), z(T) \rangle + \int_{t_0}^T \frac{\lambda(s)}{\lambda(t_0 + \rho)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_{t_0 + \rho} \right] \\
- \int_{t_0}^{t_0 + \rho} \frac{\lambda(s)}{\lambda(t_0 + \rho)} \langle f(s, P(s))z(s), z(s) \rangle ds.
$$

Taking $\mathcal{F}_t$-conditional expectation on both sides, we then get

$$
\mathbb{E}\left[ \frac{\lambda(t_0 + \rho)}{\lambda(t)} \langle P(t_0 + \rho)z(t_0 + \rho), z(t_0 + \rho) \rangle | \mathcal{F}_t \right] = \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi z(T), z(T) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_t \right] \\
+ \int_{t_0}^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_t \right] - \mathbb{E}\left[ \int_{t_0}^{t_0 + \rho} \frac{\lambda(s)}{\lambda(t_0 + \rho)} \langle f(s, P(s))z(s), z(s) \rangle ds | \mathcal{F}_t \right].
$$

(4.5)

We can write (4.4) and (4.5) into a unified form as

$$
\langle P(t)\sqrt{\rho}z(t), \sqrt{\rho}z(t) \rangle + \mathbb{E}\left[ \int_t^T \frac{\lambda(t_0 + \rho)}{\lambda(t)} \langle P(t_0 + \rho)z(t_0 + \rho), z(t_0 + \rho) \rangle I_{E_{\rho}}(s) ds | \mathcal{F}_t \right] \\
= \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi \sqrt{\rho}z(T), \sqrt{\rho}z(T) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))\sqrt{\rho}z(s), \sqrt{\rho}z(s) \rangle ds | \mathcal{F}_t \right] \\
- \rho \mathbb{E}\left[ \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))z(s), z(s) \rangle I_{E_{\rho}}(s) ds | \mathcal{F}_t \right], \quad \text{when } \rho \text{ is small.}
$$

(4.6)

\textit{Step 2: the case of } $\zeta(s) = \zeta_0$, $s \geq t_0$, for some $\zeta_0 \in L^{4\alpha}(\mathcal{F}_{t_0}, H)$. In this case, we denote the corresponding } $\sigma$ \textit{by } $\sigma^{t_0, \zeta_0}$.

Assume first that $\zeta_0 \in L^{4\alpha}(\mathcal{F}_{t_0}, V)$ and define the corresponding $z(t)$ as in Step 1. From the identity
we have that
\[
\sigma^t_0 \phi(t) = \{ E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T))|\mathcal{F}_t] - E[\frac{\lambda(T)}{\lambda(t)}(\xi(\sqrt{\rho}z(T), \sqrt{\rho}z(T))|\mathcal{F}_t] \\
+ \{ E[\int_t^T \frac{\lambda(s)}{\lambda(t)}(f(s, P(s))x(s), x(s))ds|\mathcal{F}_t] - E[\int_t^T \frac{\lambda(s)}{\lambda(t)}(f(s, P(s))\sqrt{\rho}z(s), \sqrt{\rho}z(s))ds|\mathcal{F}_t] \\
+ \{ E[\int_t^T \frac{\lambda(t+\rho)}{\lambda(t)}(P(t_0+\rho)z(t_0+\rho), z(t_0+\rho))I_{E^s}(s)ds|\mathcal{F}_t] - E[\int_t^T \frac{\lambda(s)}{\lambda(t)}(P(s)(\zeta_0, \zeta_0))I_{E^s}(s)ds|\mathcal{F}_t] \\
+ \{ (\langle P(t)\sqrt{\rho}z(t), \sqrt{\rho}z(t) \rangle - \langle P(t)x(t), x(t) \rangle) + \rho E[\int_t^T \frac{\lambda(s)}{\lambda(t)}(f(s, P(s))z(s), z(s))I_{E^s}(s)ds|\mathcal{F}_t] \\
=: I_1 + I_2 + I_3 + I_4 + I_5,
\]
when \( \rho \) is small.

We only provide the estimates for \( I_1, I_3 \) and \( I_5 \), the other terms can be handled in a similar manner. For notational simplicity, we use \( C_1 \) to denote a constant independent of \( \rho \), which may vary from line to line. For the \( I_1 \) term, let \( \alpha' \) be the Hölder conjugate of \( \alpha \), since \( \lambda \) is an exponential martingale, we have
\[
E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T))|\mathcal{F}_t] - (\xi(\sqrt{\rho}z(T), \sqrt{\rho}z(T))|\mathcal{F}_t] \\
\leq (E[\frac{\lambda(T)}{\lambda(t)}]|\mathcal{F}_t]) \frac{\alpha'}{\alpha} E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T)) - (\xi(\sqrt{\rho}z(T), \sqrt{\rho}z(T))|\mathcal{F}_t]) \frac{\alpha}{\alpha'} \\
\leq C_1 (E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T))] \frac{\alpha}{\alpha'} + (E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T))] \frac{\alpha'}{\alpha}) \frac{\alpha}{\alpha'}).
\]
Thus in virtue of Proposition 4.2, we obtain
\[
(E[I_1|\alpha]) \frac{\alpha'}{\alpha} \leq C_1 (E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T)] \frac{\alpha}{\alpha'} + (E[\frac{\lambda(T)}{\lambda(t)}(\xi(x(T), x(T))] \frac{\alpha'}{\alpha}) \frac{\alpha}{\alpha'}) \\
\leq C_1 \rho \frac{\alpha}{\alpha'}
= o_\rho(\rho).
\]

Now we consider the \( I_3 \) term. If \( t > t_0 \), it holds trivially that \( I_3 = 0 \) for \( \rho \) small enough. Now we assume \( t \leq t_0 \). Denote \( t_1 := t_0 + \rho \) for simplicity. Noting that \( z(t_1) = \frac{w(t_1) - w(t_0)}{\sqrt{\rho}} \zeta_0 \), then from the Itô’s isometry, we have
\[
E[\int_t^T \frac{\lambda(t_1)}{\lambda(t)}(P(t_0)z(t), z(t))I_{E^s}(s)ds|\mathcal{F}_t] = E[\int_t^T \frac{\lambda(t_0)}{\lambda(t)}(P(t_0)\zeta_0, \zeta_0)I_{E^s}(s)ds|\mathcal{F}_t].
\]
Thus,
\[
I_3 = \{ E[\int_t^T \frac{\lambda(t_1)}{\lambda(t)}(P(t_1)z(t_1), z(t_1))I_{E^s}(s)ds|\mathcal{F}_t] - E[\int_t^T \frac{\lambda(t_0)}{\lambda(t)}(P(t_1)z(t_1), z(t_1))I_{E^s}(s)ds|\mathcal{F}_t] \\
+ \{ E[\int_t^T \frac{\lambda(t_1)}{\lambda(t)}(P(t_1)z(t_1), z(t_1))I_{E^s}(s)ds|\mathcal{F}_t] - E[\int_t^T \frac{\lambda(t_0)}{\lambda(t)}(P(t_1)z(t_1), z(t_1))I_{E^s}(s)ds|\mathcal{F}_t] \\
+ \{ E[\int_t^T \frac{\lambda(t_0)}{\lambda(t)}(P(t_0)\zeta_0, \zeta_0)I_{E^s}(s)ds|\mathcal{F}_t] - E[\int_t^T \frac{\lambda(t_0)}{\lambda(t)}(P(t_0)\zeta_0, \zeta_0)I_{E^s}(s)ds|\mathcal{F}_t] \\
=: J_1 + J_2 + J_3.
\]
We only estimate \( J_2 \), and the other terms can be treated in the same way. Still denote by \( \alpha' \) the Hölder
conjugate of $\alpha$. Note that

$$
\mathbb{E}\left[\int_t^T \frac{\lambda(t)}{\lambda(t)} \langle (P(t_1)z(t_1), z(t_1)) - (P(t_0)z(t_1), z(t_1))(I_{E_\alpha}(s))ds|\mathcal{F}_t \right]
$$

$$
= \mathbb{E}\left[\int_t^T \frac{\lambda(t_0)w(t_1) - w(t_0)}{\sqrt{\lambda(t)}} \langle (P(t_1)\zeta_0, \zeta_0) - (P(t_0)\zeta_0, \zeta_0)(I_{E_\alpha}(s))ds|\mathcal{F}_t \right]
$$

$$
\leq \left(\mathbb{E}\left[\int_t^T \frac{\lambda(t_0)w(t_1) - w(t_0)}{\sqrt{\lambda(t)}} \langle \alpha' I_{E_\alpha}(s)ds|\mathcal{F}_t \right]\right)^{1/2} \left(\mathbb{E}\left[\int_t^T \langle (P(t_1)\zeta_0, \zeta_0) - (P(t_0)\zeta_0, \zeta_0)(I_{E_\alpha}(s))ds|\mathcal{F}_t \right]\right)^{1/2}
$$

$$
\leq C_1 \rho \mathbb{E}\left[\int_t^T \langle (P(t_1)\zeta_0, \zeta_0) - (P(t_0)\zeta_0, \zeta_0)(I_{E_\alpha}(s))ds|\mathcal{F}_t \right].
$$

Then by Proposition 2.10 we have

$$
(\mathbb{E}[|J_2|^\alpha])^{1/2} \leq C_1 \rho \mathbb{E}\left[\int_t^T \langle (P(t_1)\zeta_0, \zeta_0) - (P(t_0)\zeta_0, \zeta_0)(I_{E_\alpha}(s))ds|\mathcal{F}_t \right].
$$

Thus,

$$
(\mathbb{E}[|J_3|^\alpha])^{1/2} = \alpha \zeta_0(\rho).
$$

For the $I_5$ term, by a similar but simpler calculation,

$$
(\mathbb{E}[|J_5|^\alpha])^{1/2} \leq C_1 \rho^2 = \alpha \zeta_0(\rho).
$$

Therefore,

$$
(\mathbb{E}[|\sigma^{\zeta_0}(t)|^\alpha])^{1/2} = \alpha \zeta_0(\rho).
$$

An approximation argument gives the result for the case of $\zeta_0 \in L^{4\alpha}(\mathcal{F}_{t_0}, H)$. Indeed, for any $\delta > 0$, choose a $\zeta_0' \in L^{4\alpha}(\mathcal{F}_{t_0}, V)$ such that $\mathbb{E}[||\zeta_0 - \zeta_0'||_H^\alpha] \leq \delta$ and let $x'$ be the corresponding solution. Then

$$
\sigma^{\zeta_0, \zeta_0'}(t) = \{\sigma^{\zeta_0, \zeta_0'}(t) - \sigma^{\zeta_0, \zeta_0'}(t)\} + \sigma^{\zeta_0, \zeta_0'}(t)
$$

$$
= \left\{\mathbb{E}\left[\frac{\lambda(T)}{\lambda(t)} \langle \xi x(T), x(T)\rangle|\mathcal{F}_t \right] - \mathbb{E}\left[\frac{\lambda(T)}{\lambda(t)} \langle \xi x'(T), x'(T)\rangle|\mathcal{F}_t \right]\right\}
$$

$$
+ \left\{\mathbb{E}\left[\int_t^T \frac{\lambda(s)}{\lambda(t)} (f(s, P(s))x'(s), x'(s))ds|\mathcal{F}_t \right] - \mathbb{E}\left[\int_t^T \frac{\lambda(s)}{\lambda(t)} (f(s, P(s))x'(s), x'(s))ds|\mathcal{F}_t \right]\right\}
$$

$$
+ \left\{\mathbb{E}\left[\int_t^T \frac{\lambda(s)}{\lambda(t)} (P(s)|\zeta_0', \zeta_0', I_{E_\alpha}(s)|\mathcal{F}_t \right] - \mathbb{E}\left[\int_t^T \frac{\lambda(s)}{\lambda(t)} (P(s)|\zeta_0', \zeta_0', I_{E_\alpha}(s)|\mathcal{F}_t \right]\right\}
$$

$$
= K_1 + K_2 + K_3 + K_4 + \sigma^{\zeta_0, \zeta_0'}(t).
$$

We only give the calculation of $K_1$, and the terms $K_2, K_3, K_4$ can be estimated similarly. From a similar analysis as for $I_1$, we have for some constant $C_2$ independent of $\rho$ and $\zeta_0'$ that

$$
\left(\mathbb{E}[|K_1|^\alpha]\right)^{1/2} \leq C_2 \left(\mathbb{E}[||x(T) - x'(T)||_H^\alpha]\right)^{1/2} + \left(\mathbb{E}[||x'(T)||_H^\alpha]\right)^{1/2}
$$

$$
\leq C_2 \left(\mathbb{E}[||\zeta_0 - \zeta_0'||_H^\alpha]\right)^{1/2} \rho.
$$

Therefore,

$$
(\mathbb{E}[|\sigma^{\zeta_0, \zeta_0'}(t)|^\alpha])^{1/2} \leq C_2 \delta \rho + \alpha \zeta_0(\rho),
$$

which can be written as

$$
\frac{1}{\rho} (\mathbb{E}[|\sigma^{\zeta_0, \zeta_0'}(t)|^\alpha])^{1/2} \leq C_2 \delta \rho + \alpha \zeta_0(\rho).
Letting $\rho \to 0$ and utilizing the arbitrariness of $\delta$, we obtain

$$
(E[|\sigma^{t_0,\zeta(t_0)}(t)|^\alpha])^{\frac{1}{\alpha}} = o(\rho).
$$

**Step 3:** the general $\zeta$. Let $x^{t_0}$ be the solution of SEE (2.31) corresponds to $\zeta'$ satisfying $\zeta'(s) = \zeta(t_0)$, $s \geq t_0$, for each $t_0 \in [0, T]$. From the Lebesgue differentiation theorem (see also [H Theorem 2.2.9]), we have (for a.e. $t_0$)

$$
\frac{1}{\rho} \int_0^T \mathbb{E}[\|\zeta(s) - \zeta(t_0)\|^\alpha_{H}] I_{E_\rho}(s) ds = 0, \text{ as } \rho \to 0.
$$

From this we also get

$$
\frac{1}{\rho^{2\alpha}} \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t) - x^{t_0}(t)\|^\alpha_{H} \right] \leq C \frac{1}{\rho} \int_0^T \mathbb{E}[\|\zeta(s) - \zeta(t_0)\|^\alpha_{H}] I_{E_\rho}(s) ds = 0, \text{ as } \rho \to 0.
$$

Therefore,

$$
\int_0^T \mathbb{E}[\|\zeta(s) - \zeta(t_0)\|^\alpha_{H}] I_{E_\rho}(s) ds = o(\rho) \text{ and } \mathbb{E} \left[ \sup_{t \in [0,T]} \|x(t) - x^{t_0}(t)\|^\alpha_{H} \right] = o(\rho^{2\alpha}).
$$

Noting that

$$
\sigma(t) = \{\sigma(t) - \sigma^{t_0,\zeta(t_0)}(t)\} + \sigma^{t_0,\zeta(t_0)}(t)
$$

$$
= \{E[\lambda(T)\langle x(T), x(T)\rangle_{F_\rho}] - E[\lambda(T)\langle x^{t_0}(T), x^{t_0}(T)\rangle_{F_\rho}]\}
$$

$$
+ \{E[\int_0^T \lambda(s)\langle f(s, P(s))x(s), x(s)\rangle ds_{F_\rho}] - E[\int_0^T \lambda(s)\langle f(s, P(s))x^{t_0}(s), x^{t_0}(s)\rangle ds_{F_\rho}]\}
$$

$$
+ \{E[\int_0^T \lambda(s)\langle P(s)\zeta(t_0), \zeta(t_0)\rangle I_{E_\rho}(s) ds_{F_\rho}] - E[\int_0^T \lambda(s)\langle P(s)\zeta(s), \zeta(s)\rangle I_{E_\rho}(s) ds_{F_\rho}]\}
$$

$$
+ \{\langle P(t)x^{t_0}(t), x^{t_0}(t)\rangle - \langle P(t)x(t), x(t)\rangle\} + \sigma^{t_0,\zeta(t_0)}(t),
$$

we can deduce by a similar analysis as in Step 2 that

$$
(E[|\sigma(t)|^\alpha])^{\frac{1}{\alpha}} \leq (E[|\sigma(t) - \sigma^{t_0,\zeta(t_0)}(t)|^\alpha])^{\frac{1}{\alpha}} + (E[|\sigma^{t_0,\zeta(t_0)}(t)|^\alpha])^{\frac{1}{\alpha}} = o(\rho).
$$

\[\square\]

**Lemma 4.3** Suppose (H4) holds. For $\mu_1, \mu_2 \in L_{{F_\rho}}^{\infty}(0, T)$, define

$$
\tilde{A}(t) := A(t) + \mu_1(t)B(t) + \mu_2(t)I_d, \quad \tilde{B}(t) := B(t) + \mu_1(t)I_d
$$

and

$$
\lambda_1(t) := e^{\int_0^t [\mu_1(s) - \frac{1}{2} \mu_1(s)^2]ds + \mu_2(s)dw(s)}.
$$

Then

$$
L_{\tilde{A}, \tilde{B}}(t, s) = \frac{\lambda_1(s)}{\lambda_1(t)} L_{A,B}(t, s), \text{ for } 0 \leq t \leq s \leq T.
$$

**Proof.** For any $u \in L^2(F_\rho, H)$, $\{L_{A,B}(t, s)u\}_{t \leq s \leq T}$ solves the SEE (2.23) with initial value $u$. Then by Itô’s formula, we see that the process $\{\frac{\lambda_1(s)}{\lambda_1(t)} L_{A,B}(t, s)u\}_{t \leq s \leq T}$ is the solution of SEE (2.23) with unbounded operators $\tilde{A}$, $\tilde{B}$ and initial value $u$. Thus $\frac{\lambda_1(s)}{\lambda_1(t)} L_{A,B}(t, s)u = L_{\tilde{A}, \tilde{B}}(t, s)u$ and the proof is complete. \[\square\]
Proof of Theorem 2.23. According to Theorem 4.1, we have
\[
\langle P(t)x(t), x(t) \rangle + \sigma(t) = \mathbb{E}\left[ \frac{\lambda(T)}{\lambda(t)} \langle \xi(T), x(T) \rangle + \int_t^T \frac{\lambda(s)}{\lambda(t)} \langle f(s, P(s))x(s), x(s) \rangle ds \right]
\]
with \(\sigma\) and \(\lambda\) being defined by (4.1) and (4.2), respectively, and \(\sigma\) satisfying (2.34). This is in fact the explicit formula of the linear BSDE (2.33) with solution (\(\text{uniqueness of (2.33)}\)).

Proof of Theorem 2.23.

Now it remains to prove Proposition 4.2. We shall need an a priori estimate of SEEs when the non-homogeneous term \(a\) in the drift taking values in \(V^\ast\). It is worth to mention that if particularly \(a\) takes values in \(H\), we can in fact have a better version for such kind of estimate (see (2.25)).

Lemma 4.4 Assume (H4) holds. For any given \((a, b) \in L^2_{T}(t, T; V^\ast \times H)\) and \(z_0 \in L^2(t, T; H)\) with \(\alpha \geq 1\), denote by \(z\) the solution of
\[
\begin{align*}
\frac{dz(s)}{ds} &= [A(s)z(s) + a(s)]ds + [B(s)z(s) + b(s)]dw(s), \quad s \in [t, T], \\
z(t) &= z_0.
\end{align*}
\]
Then there is a constant \(C > 0\) depending on \(\delta, K\) and \(\alpha\) such that
\[
\mathbb{E}[\sup_{s \in [t, T]} \|z(s)\|_{H}^{2\alpha}] \leq C \mathbb{E}[\|z_0\|_{H}^{2\alpha}] + (\int_t^T \|a(s)\|_{V'}^\alpha ds)^\alpha + (\int_t^T \|b(s)\|_{H}^\alpha ds)^\alpha.
\]

Proof. The proof is a variant of the one for (2.25) in [3]. We only present the case of \(t = 0\), and the other cases can be proved in a similar way. By the coercivity condition,
\[
\|Bu\|_H \leq C(K)\|u\|_V, \text{ for } u \in V.
\]
Then,
\[
\begin{align*}
2\langle Az(t) + a(t), z(t) \rangle_s + \|Bz(t) + b(t)\|_H^2 &
\leq 2\langle Az(t), z(t) \rangle_s + \|Bz(t)\|_H^2 + 2\langle Bz(t), b(t) \rangle + \|b(t)\|_H^2 + 2\langle a(t), z(t) \rangle
\leq -\delta\langle z(t) \rangle_s + K\|z(t)\|_V^2 + C\|z(t)\|_V \|b(t)\|_H + \|b(t)\|_H^2 + 2\langle a(t), z(t) \rangle
\leq -\delta\langle z(t) \rangle_s + K\|z(t)\|_H^2 + \frac{\delta}{2}\|z(t)\|_V + C(\delta)\|b(t)\|_H^2 + C(\delta)\|a(t)\|_V^2
\leq C(\delta, K)(\|z(t)\|_H^2 + \|a(t)\|_V^2 + \|b(t)\|_H^2)
\end{align*}
\]
and
\[
\|(Bz(t) + b(t), z(t))\| \leq 2\|(Bz(t), z(t))\| + 2\|(b(t), z(t))\| \leq 2K^2\langle z(t) \rangle_s + 2\|(b(t))\|_H^2 + \|z(t)\|_H^2
\]
Let \(\epsilon > 0\) and \(\gamma > 0\) be undetermined. We have by the Hölder inequality and the Young’s inequality that
\[
\mathbb{E}\left[\int_0^T e^{-\gamma t}\|z(t)\|^{2(\alpha-1)}_H\|a(t)\|_V^\alpha dt\right] \leq \epsilon^2\mathbb{E}[\sup_{t \in [0, T]} e^{-\gamma t}\|z(t)\|^{2\alpha}_H] + C(\epsilon)\mathbb{E}[\int_0^T e^{-\gamma t}\|a(t)\|_V^\alpha dt] + C(\epsilon)\mathbb{E}[\int_0^T \|a(t)\|_V^\alpha dt],
\]

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and similarly,

\[ E \int_0^T e^{-\gamma t} \||z(t)||_{H}^{2(\alpha-1)} ||b(t)||_{H}^{2\alpha} dt \leq \epsilon^2 E[ \sup_{t \in [0,T]} e^{-\gamma t} \||z(t)||_{H}^{2\alpha} ] + C(\epsilon) E[ (\int_0^T ||b(t)||_{H}^{2\alpha} dt)^{\alpha} ]. \]

In the sequel of this proof, for the sake of notation simplicity, we use \( C_1 \) to denote a generic constant independent of \( \epsilon \) and \( \gamma \), which may be different from line to line. From the quasi-skew-symmetry condition, we can calculate

\[ E[ \sup_{t \in [0,T]} \int_0^t e^{-\gamma s} \||z(s)||_{H}^{2\alpha} (Bz(s) + b(s), z(s)) ds ] \]

\[ \leq C_1 E[ (\int_0^t e^{-2\gamma s} \||z(s)||_{H}^{4\alpha-4} (Bz(s) + b(t), z(t))^2 dt)^{\frac{1}{2}} ] \]

\[ \leq C_1 E[ \sup_{t \in [0,T]} e^{-\gamma t} \||z(s)||_{H}^{2\alpha} (\int_0^T e^{-\gamma t} (||z(t)||_{H}^{2\alpha} + ||z(t)||_{H}^{2\alpha-2} ||b(t)||_{H}^{2\alpha} dt)^{\frac{1}{2}} ] \]

\[ \leq \epsilon E[ \sup_{t \in [0,T]} e^{-\gamma t} ||z(t)||_{H}^{2\alpha} ] + \frac{C_1}{\epsilon} E[ \int_0^T e^{-\gamma t} (||z(t)||_{H}^{2\alpha} + ||z(t)||_{H}^{2\alpha-2} ||b(t)||_{H}^{2\alpha} dt) ] \]

\[ \leq C_1 E[ \sup_{t \in [0,T]} e^{-\gamma t} ||z(t)||_{H}^{2\alpha} ] + \frac{C_1}{\epsilon} E[ \int_0^T e^{-\gamma t} ||z(t)||_{H}^{2\alpha} dt ] + C(\epsilon) E[ (\int_0^T ||b(t)||_{H}^{2\alpha} dt)^{\alpha} ]. \]

Then applying Itô formula to \( e^{-\gamma t} ||z(t)||_{H}^{2\alpha} \), we obtain

\[ e^{-\gamma t} ||z(t)||_{H}^{2\alpha} + \gamma \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha} ds \]

\[ = ||z_0||_{H}^{2\alpha} + \alpha \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha-1} (2(Az(s) + a(s), z(s)) + ||Bz(s) + b(s)||_{H}^{2\alpha} ) ds \]

\[ + 2\alpha(\alpha-1) \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha-2} ||Bz(s) + b(s), z(s)||^2 ds \]

\[ + 2\alpha \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha} (Bz(s) + b(s), z(s)) dw(s) \]

\[ \leq ||z_0||_{H}^{2\alpha} + C_1 \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha-1} (||z(s)||_{H}^{2\alpha} + ||a(s)||_{H}^{2\alpha} + ||b(s)||_{H}^{2\alpha} ) ds \]

\[ + C_1 \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha-2} (||z(s)||_{H}^{2\alpha} + ||b(s)||_{H}^{2\alpha} ||z(s)||_{H}^{2\alpha} ) ds \]

\[ + 2\alpha \int_0^t e^{-\gamma s} ||z(s)||_{H}^{2\alpha} (Bz(s) + b(s), z(s)) dw(s) \]

Taking supremum and expectation on both sides, we get

\[ E[ \sup_{t \in [0,T]} e^{-\gamma t} ||z(t)||_{H}^{2\alpha} ] + \gamma E[ \int_0^T e^{-\gamma t} ||z(t)||_{H}^{2\alpha} dt ] \]

\[ \leq C_1 (\epsilon + \epsilon^2) E[ \sup_{t \in [0,T]} e^{-\gamma t} ||z(t)||_{H}^{2\alpha} ] + E[ ||z_0||_{H}^{2\alpha} ] + C(\epsilon) E[ (\int_0^T ||b(t)||_{H}^{2\alpha} dt)^{\alpha} ] \]

\[ + C(\epsilon) E[ (\int_0^T ||a(t)||_{H}^{2\alpha} )^{\alpha} dt ] + C(\epsilon) E[ (\int_0^T ||b(t)||_{H}^{2\alpha} dt)^{\alpha} ] \]

Choosing \( \epsilon \) small and \( \gamma \) large, we obtain

\[ E[ \sup_{t \in [0,T]} ||z(t)||_{H}^{2\alpha} ] \leq C E[ ||z_0||_{H}^{2\alpha} ] + (\int_0^T ||a(t)||_{H}^{2\alpha} dt)^{\alpha} + (\int_0^T ||b(t)||_{H}^{2\alpha} dt)^{\alpha} \].
Then according to Lemma 4.4, where are unbounded operators and Note that from the coercivity condition, is a nonlinear function.

We also note that and from the basic estimate of SEEs,

Combining the above analysis, we obtain the desired result.

We shall give a general result for possible future applications. We also note that the case of first-order equation has already proved in [5].

We impose the following assumptions. Given \( \beta \geq 2 \), we denote \( \delta(t) := y(t) - \sqrt{\rho} \eta(t) \) and have

\[
d\delta(t) = [A\delta(t) + \sqrt{\rho}A\eta(t)]dt + [B\delta(t) + \sqrt{\rho}B\eta(t)]dw(t), \quad \delta(t_0) = 0.
\]

Note that from the coercivity condition,

\[
\|Bu\|_H \leq C(K)\|u\|_V, \quad \text{for } u \in V.
\]

Then according to Lemma 4.4

\[
\mathbb{E}[\sup_{t \in [t_0, t_0 + \rho]} \|\delta(t)\|_H^{2\alpha}] \leq C\rho^\alpha \mathbb{E}[\int_{t_0}^{t_0 + \rho} \|A\eta(t)\|_V^2 dt] + (\int_{t_0}^{t_0 + \rho} \|B\eta(t)\|_H^2 dt)^\alpha
\]

\[
\leq \rho^\alpha \mathbb{E}[\int_{t_0}^{t_0 + \rho} \|\eta(t)\|_V^2 dt]^\alpha
\]

\[
= C\rho^{2\alpha - 1} \int_{t_0}^{t_0 + \rho} \mathbb{E}[\|\eta(t)\|_V^{2\alpha}] dt
\]

\[
\leq C \mathbb{E}[\|\zeta_0\|_V^{2\alpha}] \rho^{2\alpha}.
\]

We also note that

\[
\mathbb{E}[\sup_{t \in [0, t_0]} \|y(t) - \sqrt{\rho}z(t)\|_H^{2\alpha}] = 0,
\]

and from the basic estimate of SEEs,

\[
\mathbb{E}[\sup_{t \in [t_0 + \rho, T]} \|y(t) - \sqrt{\rho}z(t)\|_H^{2\alpha}] \leq C \mathbb{E}[\|y(t_0 + \rho) - \sqrt{\rho}\eta(t_0 + \rho)\|_H^{2\alpha}] \leq C \mathbb{E}[\|\zeta_0\|_V^{2\alpha}] \rho^{2\alpha}.
\]

Combining the above analysis, we obtain the desired result.

4.3 Proof of the \( L^\beta \)-estimate (3.6) of adjoint equations

We shall give a general result for possible future applications. We also note that the case of \( \beta = 2 \) for the first-order equation has already proved in [5].

We consider the following backward stochastic evolution equation (BSEE)

\[
\begin{cases}
-dp(t) = [M(t)p(t) + N(t)q(t) + f(p(t), q(t), t)]dt \\
p(T) = \xi,
\end{cases}
\]

where \( \xi \) is the terminal condition,

\( M : [0, T] \times \Omega \to \mathcal{L}(V, V^*) \), \( N : [0, T] \times \Omega \to \mathcal{L}(H, V^*) \)

are unbounded operators and

\( f : [0, T] \times \Omega \times H \times H \to H \)

is a nonlinear function.

Given \( \beta \geq 2 \), we denote by \( L_{L^\beta}^{1,\beta}(0, T; H) \) the space of \( H \)-valued progressively measurable processes \( y(\cdot) \) with norm

\[
\|y\|_{L_{L^\beta}^{1,\beta}(0, T; H)} = \left\{ \mathbb{E}[\{\int_0^T \|y(t)\|_H dt\}^\beta] \right\}^{1/\beta}.
\]

We impose the following assumptions.
For each \( u \in V \) and \( N(t, \omega)u \) are progressively measurable. There exist some constants \( \delta > 0 \) and \( K \geq 0 \) such that the following two assertions hold: for each \( (t, \omega) \in [0, T] \times \Omega \) and \( x \in V \),

1. Coercivity condition:
   \[
   2 \langle \mathcal{M}(t,x), x \rangle_p + \|N(t)x\|_H^2 \leq -\delta\|x\|_V^2 + K\|x\|_H^2 \text{ and } \|\mathcal{M}(t)x\|_V \leq K\|x\|_V;
   \]

2. For each \( (p, q) \in H \times H, f(\cdot, \cdot, p, q) \) are progressively measurable. \( f(\cdot, \cdot, 0, 0) \in L^{1, \beta}_{T}(0; H), \xi \in L^\beta(F_T, H), \) and
   \[
   \|f(t, p, q) - f(t, p', q')\|_H \leq K(\|p - p'\|_H + \|q - q'\|_H).
   \]

**Lemma 4.5** Assume the condition (A). If \( (p(\cdot), q(\cdot)) \) is the solution to BSEE \( \left[ \mathcal{L}, \mathcal{L} \right] \), then there exists some positive constant \( C \) depending on \( \delta \) and \( K \) such that

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|p(t)\|^\beta_H \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|_H^2 dt \right] + \mathbb{E}\left[ \int_0^T \|q(t)\|_H^2 dt \right] \leq C\left[ \mathbb{E}\left[ \|\xi\|^\beta_H \right] + \mathbb{E}\left[ \int_0^T \|f(t, 0, 0)\|_H dt \right] \right].
\]

**Proof.** In the proof, we use \( C > 0 \) to denote a generic constant that may change from line to line. Applying the Itô formula to \( \|p(t)\|^\beta_H \), we have

\[
\|p(t)\|^\beta_H + \int_t^T \|q(s)\|^2_H ds = \|\xi\|^\beta_H + 2\int_t^T \left[ \langle \mathcal{M}(s)p(s), p(s) \rangle_p + \langle N(s)q(s), p(s) \rangle_p \right] ds + \left( f(s, p(s), q(s)), p(s) \right)_H ds - 2\int_t^T \langle q(s), p(s) \rangle_H dw(s).
\]

Applying again the Itô formula to \( \|p(t)\|^\beta_H \), we get

\[
\|p(t)\|^\beta_H + \frac{1}{2} \int_t^T \|p(s)\|^{\beta-2}_H \|q(s)\|^2_H ds + \int_t^T \beta(\frac{\beta}{2} - 1)\|p(s)\|^{\beta-4}_H \langle p(s), q(s) \rangle_H^2 ds - \beta \int_t^T \|p(s)\|^{\beta-2}_H \langle p(s), q(s) \rangle_H^2 dw(s).
\]

Making use of the coercivity condition, we obtain for some undetermined \( \varepsilon > 0 \) that

\[
\|p(t)\|^\beta_H + \frac{1}{2} \int_t^T \|p(s)\|^{\beta-2}_H \|q(s)\|^2_H ds + \int_t^T \beta(\frac{\beta}{2} - 1)\|p(s)\|^{\beta-4}_H \langle p(s), q(s) \rangle_H^2 ds - \beta \int_t^T \|p(s)\|^{\beta-2}_H \langle p(s), q(s) \rangle_H^2 dw(s)
\]

\[
\leq \|\xi\|^\beta_H + \int_t^T \|p(s)\|^{\beta-2}_H \|q(s)\|^2_H + (1 + \varepsilon)(\|p(s)\|^{\beta-2}_H + K\|p(s)\|^{\beta-4}_H) + \frac{1}{1 + \varepsilon}\|q(s)\|^2_H
\]

\[
+ \frac{\varepsilon}{2}\|q(s)\|^2_H + C\varepsilon\|p(s)\|^2_H + \beta \sup_{s \in [0, T]} \|p(s)\|^{\beta-1}_H \int_t^T \langle f(s, 0, 0) \rangle_H ds
\]

\[
- \beta \int_t^T \|p(s)\|^{\beta-2}_H \langle p(s), q(s) \rangle_H dw(s)
\]

\[
\leq \|\xi\|^\beta_H + \int_t^T \|p(s)\|^{\beta-2}_H \|q(s)\|^2_H - (1 + \varepsilon)(\|p(s)\|^{\beta-2}_H + K\|p(s)\|^{\beta-4}_H) + C\varepsilon\|p(s)\|^2_H + \frac{1}{1 + \varepsilon}\|q(s)\|^2_H ds
\]

\[
+ \beta \sup_{s \in [0, T]} \|p(s)\|^{\beta-1}_H \int_t^T \langle f(s, 0, 0) \rangle_H ds - \beta \int_t^T \|p(s)\|^{\beta-2}_H \langle p(s), q(s) \rangle_H dw(s).
\]
Choose $\varepsilon$ small enough so that $(2\varepsilon K - (1 + \varepsilon)\delta) < 0$ and $\frac{1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}}{1 + \varepsilon} < 1$, we get

$$\|p(t)\|_{H}^{\beta} + \int_{t}^{T} \|p(s)\|_{H}^{-\delta} \|q(s)\|_{H}^{-\delta} ds + \int_{t}^{T} \|p(s)\|_{H}^{-\delta} \|p(s)\|_{H}^{-\delta} ds \leq C[\|\xi\|_{H}^{\beta} + \int_{t}^{T} \|p(s)\|_{H}^{\beta} ds$$

$$+ \sup_{s \in [0,T]} \|p(s)\|_{H}^{-\beta} \int_{t}^{T} \|f(s,0,0)\|_{H} ds - \beta \int_{t}^{T} \|p(s)\|_{H}^{-\delta} \langle p(s), q(s) \rangle_{H} dw(s).$$

Taking expectation on both sides, we obtain (from standard truncation techniques, the stochastic integral above can be assumed to be a martingale; see the proof of Theorem 4.4.4 in [32])

$$E[\|p(t)\|_{H}^{\beta}] + E[\int_{t}^{T} \|p(s)\|_{H}^{-\delta} \|q(s)\|_{H}^{-\delta} ds] \leq C E[\|\xi\|_{H}^{\beta} + \int_{t}^{T} \|p(s)\|_{H}^{\beta} ds + \sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta} \int_{0}^{T} \|f(t,0,0)\|_{H} dt].$$

Applying the Gronwall inequality, we obtain

$$E[\|p(t)\|_{H}^{\beta}] \leq C E[\|\xi\|_{H}^{\beta} + \sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta} \int_{0}^{T} \|f(t,0,0)\|_{H} dt].$$

Plugging this back into (4.11), we get

$$E[\|p(t)\|_{H}^{\beta}] + E[\int_{0}^{T} \|p(t)\|_{H}^{-\delta} \|q(t)\|_{H}^{-\delta} dt] \leq C E[\|\xi\|_{H}^{\beta} + \sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta} \int_{0}^{T} \|f(t,0,0)\|_{H} dt].$$

Then by the Young’s inequality, we obtain that for an undetermined $\delta > 0$ that

$$E[\|p(t)\|_{H}^{\beta}] + E[\int_{0}^{T} \|p(t)\|_{H}^{-\delta} \|q(t)\|_{H}^{-\delta} dt] \leq C E[\|\xi\|_{H}^{\beta}] + \delta E[\|p(t)\|_{H}^{\beta}] + C_{\beta} E[\int_{0}^{T} \|f(t,0,0)\|_{H} dt].$$

On the other hand, taking supremum and expectation on both sides of (4.11), we have

$$E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta}] \leq C E[\|\xi\|_{H}^{\beta}] + \int_{0}^{T} E[\|p(t)\|_{H}^{\beta} dt] + C E[\int_{0}^{T} \|f(t,0,0)\|_{H} dt]^{\beta} + \frac{1}{4} E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta}]$$

$$+ C E[\sup_{t \in [0,T]} \int_{t}^{T} \|p(s)\|_{H}^{-\delta} \|p(s), q(s)\|_{H} dw(s)].$$

$$\leq C E[\|\xi\|_{H}^{\beta}] + \int_{0}^{T} E[\|p(t)\|_{H}^{\beta} dt] + C E[\int_{0}^{T} \|f(t,0,0)\|_{H} dt]^{\beta} + \frac{1}{4} E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta}]$$

$$+ C E[\int_{0}^{T} \|p(t)\|_{H}^{-\delta} \|q(t)\|_{H}^{-\delta} dt]^{\beta}]$$

$$\leq C E[\|\xi\|_{H}^{\beta}] + \int_{0}^{T} E[\|p(t)\|_{H}^{\beta} dt] + C E[\int_{0}^{T} \|f(t,0,0)\|_{H} dt]^{\beta} + \frac{1}{4} E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta}]$$

$$+ C E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta} \int_{t}^{T} \|p(t)\|_{H}^{-\delta} \|q(t)\|_{H}^{-\delta} dt]^{\beta}]$$

$$\leq C E[\|\xi\|_{H}^{\beta}] + \int_{0}^{T} E[\|p(t)\|_{H}^{\beta} dt] + C E[\int_{0}^{T} \|f(t,0,0)\|_{H} dt]^{\beta} + \frac{1}{2} E[\sup_{t \in [0,T]} \|p(t)\|_{H}^{\beta}]$$

$$+ C E[\int_{0}^{T} \|p(t)\|_{H}^{-\delta} \|q(t)\|_{H}^{-\delta} dt]^{\beta}]$$

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Thus,
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|^\beta_H \right] \\
\leq C\mathbb{E}\left[ \|\xi\|^\beta_H \right] + \int_0^T \mathbb{E}\left[ \int_0^T \|f(t,0,0)\|_H dt \|^\beta_H \right] dt + \mathbb{E}\left[ \int_0^T \left( \|p(t)\|^\beta_H + C\delta \right) \|q(t)\|^\beta_H \right].
\]
(4.12)

Then plugging (4.11) into (4.12), we get
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|^\beta_H \right] \leq C\mathbb{E}\left[ \|\xi\|^\beta_H \right] + C\delta \mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|^\beta_H \right] + C\delta \mathbb{E}\left[ \int_0^T \|f(t,0,0)\|_H dt \|^\beta_H \right].
\]
Choosing \(\delta\) small enough, we get
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|^\beta_H \right] \leq C\mathbb{E}\left[ \|\xi\|^\beta_H \right] + \mathbb{E}\left[ \int_0^T \|f(t,0,0)\|_H dt \|^\beta_H \right].
\]
(4.13)

Next, taking \(\beta = 2\) in (4.13), we have
\[
\int_0^T \|q(s)\|^2_H ds + \int_0^T \|p(s)\|^2_H ds \leq C\|\xi\|^2_H + C \int_0^T \|p(s)\|^2_H ds
\\
+ C \sup_{s \in [0,T]} \|p(s)\|_H \int_0^T \|f(s,0,0)\|_H dt - C_2 \int_0^T \langle p(s), q(s) \rangle_H \, dw(s).
\]
Then
\[
\mathbb{E}\left[ \int_0^T \|q(t)\|^2_H \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H \right]
\leq C\mathbb{E}\left[ \|\xi\|^2_H \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H dt \right] + \mathbb{E}\left[ \int_0^T \langle p(s), q(s) \rangle_H \|f(t,0,0)\|_H dt \right]
\\
+ \mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|_H \int_0^T \|f(t,0,0)\|_H dt \right]
\leq C\mathbb{E}\left[ \|\xi\|^2_H \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H dt \right] + \mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|_H \right]
\\
+ \mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|_H \int_0^T \|f(t,0,0)\|_H dt \right]
\leq C\mathbb{E}\left[ \|\xi\|^2_H \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H dt \right] + \mathbb{E}\left[ \int_0^T \|f(t,0,0)\|_H dt \right] + \mathbb{E}\left[ \sup_{t \in [0,T]} \|p(t)\|_H \right]
\\
+ \frac{1}{2} \mathbb{E}\left[ \int_0^T \|q(t)\|^2_H dt \right]
\]
From this and (4.13), we get
\[
\mathbb{E}\left[ \int_0^T \|q(t)\|^2_H dt \right] + \mathbb{E}\left[ \int_0^T \|p(t)\|^2_H dt \right] \leq C\mathbb{E}\left[ \|\xi\|^2_H \right] + \mathbb{E}\left[ \int_0^T \|f(t,0,0)\|_H dt \right].
\]
This completes the proof. \(\square\)

On the other hand, the estimate \(\sup_{t \in [0,T]} \mathbb{E}[\|P(t)\|_H^\beta] < \infty\) for any \(\beta \geq 2\) follows trivially from the estimate (2.14) of the BSIE. Indeed, from (2.14) we have
\[
\|P(t)\|^\beta_H \leq C\mathbb{E}[\|\xi\|^\beta_H] + \mathbb{E}\left[ \int_0^T \|f(s,0,0)\|^2_H ds \right]_{\mathcal{F}_t}, \quad P\text{-}a.s.
\]
Taking expectation on both sides, we obtain

\[ E[\|P(t)\|_{L^2(H)}^2] \leq C E[\|\xi\|_{L^2(H)}^2] + \left( \int_0^T \|f(t,0)\|_{L^2(H)}^2 dt \right)^{\frac{1}{2}}, \text{ for each } t \in [0,T]. \]

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