Finding patterns within perturbative approximation in QCD and indirect relations

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We make an extensive use of BLM approach to study details of predicting higher order corrections based on the approach. This way we are able to test two procedures to improve the prediction process. Beside the main line of the two procedures it is found out that overall normalization could change BLM patterns effectively. Finally we try to find out whether a BLM pattern is sufficient for a prediction or not, and how one should use such a pattern.

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I. INTRODUCTION

Optimization procedures provide a good framework in investigating the behavior of perturbative descriptions of gauge field theories' observables. Perturbative analysis of these theories encounters the problem of scheme-scale ambiguity. An appropriate approach to this problem should give warnings in cases where it should not be applied, or we should have a mechanism to check the validity. BLM approach \[1\] contains such a mechanism, it is not applicable in analyzing processes containing gluon-gluon interactions in leading order since we should not absorb all the flavor dependent part into the running coupling constant in these cases. If we use the approach to resolve scheme ambiguity and find physical schemes, such processes will automatically be eliminated. Strange perturbative series would be the outcome of using them as the physical scheme. The mathematical form of perturbative series suggests a kind of symmetry and relation between observables but without considering optimization processes the real hidden relation can not be revealed. BLM approach is particularly able to reveal commensurate scale relations \[2\] between observables. This flexibility of BLM approach can provide us with very simple relations such as the generalized Crewther relation \[3\] which can be used to test some serious aspects of QFT such as the violation of conformal symmetry due to the renormalization procedure.

One can use optimization procedures to predict higher-order coefficients. Clearly any kind of prediction strongly assumes existence of patterns. A pattern at least must not have mathematical defects. However the quality of a proposed pattern is usually determined through the comparison of the predicted and exact coefficients. For a flexible approach this comparison could be a guideline in improving the approach. As a matter of fact what should be followed here rather than improving an approach is finding patterns. Fortunately we do not have many choices for the first step. The first step towards these aims is to make the prediction rebuild exact values.

This way we encounter a new face of the observable. This mathematical adaptation can be interpreted as isolating parts of the observable. These isolated parts carry the physical considerations which supports the optimization procedure as parts of them. The set of isolated parts can be called a pattern. However this is not the only way to find patterns. Regarding the approach extension and improvement, we introduce biased BLM approaches. Apparently a normal BLM approach cannot correctly analyze gluon-gluon interactions in leading order. Here the suggestion of finding BLM generated parts(the isolated parts generated through BLM prediction comparison) inside the observable and using the new obtained parameters instead of the original ones constitutes an immediate biasing. Also we introduce a second-class biasing where the leading-order is totally dismissed from prediction.

From a more illuminative point of view, we check the potential of the approach in revealing a pattern for the observable. In the case of BLM this has been preferred to finding a way to separate gluon-gluon interactions from the vacuum polarizations. If the approach can be adjusted to find a satisfying pattern, it has jumped over the obstacle.

Following the route of finding patterns will lead us through a pile of equations to a simple sufficient condition for the relations that connects observables. The condition is a sufficient one if someone wants to get new results from prediction through a second observable.

For the convenience of the reader: firstly in this way of finding patterns and biasing we deal with systems of equations that are built upon comparisons. These comparisons may aim different types of parameters in the pattern such as the perturbative series coefficients or solely the local symmetry parameters. Secondly we make notices anywhere one may find himself in a state to perform a selection of equations or parameters such as when dealing with unequal number of equations and parameters.

In the beginning of Sec.II we will discuss primary aspects of prediction. Deviation pattern approach is introduced in Sec.IIA accompanied by its details and uses for Higgs boson decay widths, Bjorken sum rule and Adler function and also some complementary discussions on prediction. In this section we also consider patterns developed from the combination of different patterns. As
we go further we will try take advantage of a second ob-
serveable for finding patterns in Sec. II B and introduce
indirect relations.

II. THE PROCEDURES

We consider a RG-invariant perturbative expansion
\[ R(Q^2) = r_0 + r_1 a_s(Q^2) + \sum_{i=0}^{\infty} r_i a_i(Q^2) \]
where \( a_s = \frac{\alpha_s}{\pi} \) (\( \alpha_s \) is the RG-invariant effective coupling constant) and
\( r_i = \sum_{k=0}^{\infty} r_{ik} n_f^k \) for any effective charge or MS-like
scheme. Predicting any coefficient \( r_i \) using BLM ap-
proach results in a series of number of flavors \( n_f \) for that
order that depends on all the lower order coefficients. For
instance the predicted third order coefficient is
\[
r^3_{\text{pre}} = \frac{121 C_F^2 r_{21}^2}{16 T_F} - \frac{17 C_A^2}{8 T_F} r_{21}^2 - \frac{11 C_A r_{20} r_{21}}{2 T_F} r_1 + \left(2 \frac{r_{20} r_{21}}{r_1} + \frac{5}{4} r_{21} C_A + \frac{3}{4} r_{21} C_F\right) n_f + \frac{r_{21}^2 n_f^2}{r_1}.
\]

Here we have used the 4-dimensional \( \beta \)-function of
the massless MS-like schemes. The \( n_f \)th order predicted co-
efficient \( r^3_{\text{pre}} = \sum_{k=0}^{n_f-1} r^3_{\text{pre}} n_f^k \) appears when re-expanding
the BLM scale-fixed \( (n-1) \)th order series
\[ R = r_0 + r_1 a(Q^2) + r_2 a^2(Q^2) + \ldots + r_{n-1} a^{n-1}(Q^2). \]

\( Q^* \) is to absorb all vacuum polarization. Re-expanding
the above series happens through the \( n \)th order running of
the couplant:
\[
a(Q^{2'}) = a(Q^2) - a^2(Q^2) \beta_0 \ell + a^3(Q^2) (\beta_0^2 \ell^2 - \beta_1 \ell) + a^4(Q^2) (-\beta_0^3 \ell^3 + \frac{5}{2} \beta_0 \beta_1 \ell^2 - \beta_2 \ell) + \ldots + O(a^{n+1}),
\]
where \( \ell = \ln(Q'^2/Q^2) \) and it may occur in multiple steps
with appropriate \( \ell \)s for a multi-scale version of the BLM
scale-fixed series. We will discuss more details of BLM
extensions later.

One more preliminary review on overall normalization
and Casimir operators is needed before going on to
the procedures. Local gauge invariance constructs Yang-
Mills theories i.e., non-Abelian gauge theories. Such lo-
cal symmetries suites well to compact semi-simple Lie
algebras. For such an algebra one can find the overall
normalization \[4, 6] that can determine the structure of
the algebra and its representations. The tensor prod-
uct of generators of the color \( SU(N) \)s of two particles,
\( T_A^a T_B^b \), gives the taste of color interaction between them.
This operator is surprisingly an invariant of the tensor
product space. The Casimir operator \( T_2 \), where \( T_08 \)s are
generators of the product space, related to this inter-
action is a number depending on the dimension of the
representation. \( C_A \) and \( C_F \) are quadratic Casimir op-
erators of the adjoint and fundamental representations
transforming gauge and fermion fields. The representa-
tion’s generators scale factor i.e., the trace normalization
of the representation is \( T_F \) for the fundamental re-
presentation. Obviously setting \( N_A \), the number of gener-
ators of the group, \( C_F, T_F, \) and \( d_{\gamma}^{\delta bc} \), the higher order
invariants, to 1 and the parameters of the adjoint
representation to 0 will separate the \( U(1) \) factor of the
group. The dynamics of the quenched approximation of
the non-Abelian part is revealed in the limit \( n_f = 0 \) for
all physical energies while the conformal invariant part-
er of the symmetry group sits where the \( \beta \)-function is
0.

A. Deviation

If BLM prediction pattern is some good pattern for
the perturbative series, one can rewrite the whole se-
ries in terms of just \( r_1 \) and \( r_2 \) coefficients. The quality
of the pattern will depend on these parameters beside
the BLM approach itself or truthfully the mathematical
employment used for it. Anyway if BLM approach is
able to discover the pattern, it will not be so surprising
if the original \( r_1 \) and \( r_2 \) do not be the ones that develop
the pattern. A straightforward step would be to bor-
row the \( r_1 \) and \( r_2 \) that form the real \( r_3 \) since it is the
most important part of the pattern. Yet the procedure
would be flexible enough to be able to handle exceptional
but important cases such as the ones with large higher-
order corrections. We could discuss the situation also as
if the weights of \( r_1 \) and \( r_2 \) parts should be other values in
Eq. (1) or other combinations of weights of lower-order
coefficients are appropriate.

In general a BLM pattern contains several coupled
equations corresponding to all we know about the per-
turbative series. These equations are acquired by com-
paring the predicted coefficient and the exact ones. For
instance when one knows \( r_2 \) in addition to \( r_3 \) and wants
adapt the prediction pattern so that all the details are
involved in it, he will notice that \( r^4_{\text{pre}} \) depends on all the 6
parts of lower order coefficients \{ \( r_1, r_20, r_21, r_30, r_31, r_32 \) \}. But
the 4th order comparison \( r^4_{\text{pre}} = r_4 \) yields 4 equa-
tions. Considering the 3rd order comparison, we see that
7 equations are controlled by just 6 variables. This
indicates that one of these 7 parts is generated automatic-
ly through the adaptation of the other parts. The
more information, the more automatic generation. So
here one is forced to do a selection of equations consid-
ering the importance of the corresponding comparison.
For instance one may find out that \( r^4_{\text{pre}} \) plays the least
important role in the deviation of \( r^3_{\text{pre}} \) from \( r_3 \), the com-
parison \( r^4_{\text{pre}} = r_4 \) will be easily removed from the list
of equations. The same works for other selections of pa-
rameters too. Deviation Pattern procedure obtains new
parameters through a selection of comparisons and then
replaces the original parameters with the new ones to
build the higher-order coefficients.

It is possible to involve Adler transformation into
the above procedure by performing the prediction or com-
parison for the corresponding Adler function. Some notes
The reverse transformation rebuilds $R(s)$ as follows:

$$R(s) = \frac{i}{2\pi} \int_{s-i\epsilon}^{s+i\epsilon} dz \frac{D_{pt}(-z)}{z}.$$

One may use the RG-invariant perturbative approximation of the Adler $D$-function $D_{pt}(Q^2) = \sum_{n} d_{n} \alpha_{n}(Q^2)$ in the reverse transformation. This will generate the time-like RG-invariant effective coupling $\tilde{\alpha}(s)$ and a set of pipitized functions. Certainly we deal with the analytical continuations $[2]$ of Euclidean perturbative expansions when using the reverse transformation. This results in pipitized expressions $[3, 10]$. The non-power sets of pipitized functions and analyticized functions replace time-like and space-like couplings which regenerate the perturbative series in terms of a self-consistent scheme.

1. Higgs decay widths

As an important and interesting illustration of the Deviation Pattern procedure (DPA), we first consider Higgs boson decays $[10]$ into bottom quarks $[11, 12]$ and gluons $[13]$. The hadronic decay width of the Higgs boson seems to have large QCD corrections to the Born approxima-

tion. Considering $M_H >> 2m_q$ results in

$$\Gamma(H \rightarrow q\bar{q}) = \frac{3G_F}{4\sqrt{2}\pi M_H} m_q^2 \bar{R}(s)$$

for the decay width, where the term to the left of $\bar{R}$ is $\Gamma_{Born}(H \rightarrow q\bar{q})$. The absorptive part of the corresponding two point correlator $\bar{R}$ generates the QCD corrections. Gluonic decay width takes the following form based on the fact that top quarks plays the main role in coupling of gluons to Higgs boson,

$$\Gamma(H \rightarrow gg) = \frac{\sqrt{2}G_F}{M_H} C_1^2\Im \Pi_{GG}^{GG}(q^2),$$

where $C_1$ carries the top mass dependence and $\Pi_{GG}^{GG}$ is the induced vacuum polarization by the renormalized gluon operator. The gluonic decay width is factorized by the $K$-factor, $\Gamma(H \rightarrow gg) = K\Gamma_{Born}(H \rightarrow gg)$.

Now we are in a position that we should consider generalizations of the BLM approach. For a historical overview of the extensions to the approach see $[14]$ and references therein. Our emphasis here is on the single-scale extension $[15]$ used in $[8]$ and the multi-scale one developed in $[2]$. Scale or scales are in need of $n(n-1)/2$ parameters to be able to absorb all vacuum polarization insertions in the $n$th order truncated series. The single-scale extension uses these parameters at once. However the multi-scale version takes them step by step with a specific theoretical consideration. Categorizing corrections forms the basis of some of the other generalizations such as $[14]$ also.

Taking advantage of the single-scale $\ell$ results in

$$r_4^{pre} = \frac{187}{32} C_A r_{31}^2 - \frac{17}{8} C_F r_{31} + \frac{203}{1152} C_A^3 r_{21} + \frac{51}{32} C_F^2 r_{21} - \frac{121}{8} C_A r_{20} r_{32} - \frac{3993}{64} C_A r_{31} r_{32} - \frac{11}{2} C_A r_{20} r_{31}$$

$$- \frac{5}{8} C_A r_{20} r_{31} + \frac{1}{4} C_F r_{21} - \frac{1}{2} C_F r_{21} + \frac{1}{2} C_A r_{21} r_{31} - \frac{173}{32} C_A r_{21} r_{31} - \frac{5}{2} C_A r_{20} r_{31}$$

$$- \frac{3}{2} C_F r_{20} r_{21} + \frac{121}{16} C_A r_{21} r_{31} + \frac{165}{32} C_A r_{21} r_{31} + \frac{2 r_{20} r_{31}}{r_1} - \frac{1}{2} r_{20} r_{31} + \frac{3}{2} r_{20} r_{31} |n_f$$

$$+ \frac{5}{4} C_A + \frac{3}{4} C_F r_{32} - \frac{79}{288} C_A T_F + \frac{11}{48} C_F T_F r_{21} - \frac{5}{8} C_A - \frac{3}{8} C_F r_{31} + \frac{11}{48} r_{31} r_{20} - \frac{3}{2} r_{20} r_{32} + \frac{2 r_{20} r_{32}}{r_1}$$

$$+ \left(\frac{r_{31}^3}{r_1^3} + \frac{2 r_{20} r_{32}}{r_1}\right) n_f,$$

as the predicted 4th order coefficient. Strange thing is the appearance of $T_F$ in the denominator of Eq. (1) and Eq. (2). This is even more unusual for the multi-scale based prediction where the common factor $r_{4}^{pre}(4T_F r_{20} + 11C_A r_{21}) T_F$ sits as the denominator of the whole $r_4^{pre}$. Apparently in both cases $r_4^{pre}$ does not posses a nor-
nal coefficient structure. A normal structure is constructed through a combination of β-coefficients multiplications. For re-expansion we are using a version of ℓ which contains n(n−1)/2 parameters for the nth order series. One suggestion might be to add n extra parameters to ℓ or ℓs when re-expanding to the (n + 1)th order series and see whether a standard structure of r₄ᵖʳᵉ could determine these parameters. Perturbative series parameters {r₁, r₂, r₃} presented in Eq. (1) and Eq. (2) intrinsically contains local symmetry group parameters {Cₐ, Cᵣ, Tᵣ, . . .}. So before following the above suggestion, relations must be rewritten in terms of new series parameters which are group parameters independent. However we will get around this and take advantage of the single-scale extension in the following predictions.

In this way the normal BLM prediction generates the following series as the 4th order coefficient of Π comparing to the calculated one:

\[ \hat{r}_4^{pr} = 1103.4 - 268.480n_f + 10.67n_f^2 - 0.046n_f^3, \]
\[ \hat{r}_4 = 39.354 - 220.943n_f + 9.685n_f^2 - 0.020n_f^3. \]

If we perform comparison and prediction for the corresponding Adler function, we will have:

\[ \hat{r}_4^{pr} = -1627.6 + 191.328n_f - 11.41n_f^2 + 0.245n_f^3, \]

which indicates an improvement.

The normal prediction for the 4th order coefficient of the K factor reads:

\[ g_3^{pr} = 2219.878 - 901.698n_f + 53.524n_f^2 - 0.516n_f^3, \]
\[ g_3 = 3372.073 - 866.588n_f + 48.088n_f^2 - 0.538n_f^3. \]

Performing comparison and prediction for the corresponding Adler function improves the result:

\[ g_3^{pr} = 3783.694 - 792.353n_f + 42.653n_f^2 - 0.581n_f^3, \]

whereas preventing the first term of the perturbative series affect the prediction during re-expansion (a biased BLM prediction [13]) alongside the above procedure results in a better behavior for higher number of flavors:

\[ g_3^{pr} = 533.831 - 189.814n_f + 15.738n_f^2 - 0.318n_f^3. \]

Biased prediction refers to this kind of biasing.

Briefly normal BLM prediction for both the decays results in patterns whose deviations from the calculated coefficients do not have serious n_f dependences. Special for Π it is so clear that the major part of deviation is due to the first part of 4th order coefficient. In the case of K factor we have two large parts. The deviation is shared among both of them, however again the first part plays the main role. DPA will make \( \hat{r}_4^{pr} \) recover \( \hat{r}_4 \) for higher values of n_f, whilst this happens for \( g_3^{pr} \) when DPA is utilized by a second-class biasing.

If we consider the ratio \( a_4(Q^2) = \frac{\hat{r}_4}{\hat{r}_4} \), a normal BLM prediction will indicate the same behavior as before. This is whilst a biased one shows a tendency between the predicted and calculated coefficients for higher values of n_f. It is more evident in the β-representation of the coefficients:

\[ r_4^{βi} = 81.04 - 259.09β_0 + 25.09β_0^2 - 15.92β_0^3 + 4.03β_2, \]
\[ r_4 = 120.0 - 261.26β_0 + 75.95β_0^2 + 12.96β_0^3 + 4.03β_2. \]

The genuine β-representation and seeBLM are developed in [14] but here we have used a different combination of β-coefficients for the β-representation. As we go further and employ DPA, normal prediction will be closer to the calculated result. So the first term in its β-representation is near \( r_4 \)’s. At the same time the biased prediction decreases its slope and reaches \(-54.5961\) for \( n_f = 6 \) in comparison to \(-382.9029\) of the last step while the calculated value is \(-37.2079\). Taking advantage of the Adler function makes the biased prediction get away from \( r_4 \) but the normal prediction continues decreasing its slope and this time gets really close to \( r_4 \) as \( n_f \) increases:

\[ r_4^{pr} = 86.31 - 112.75β_0 + 29.87β_0^2 - 2.49β_0^3 + 4.61β_2, \]

which reads \(-35.2114\) for \( n_f = 6 \).

If we establish the ratio \( a_4(Q^2) = \frac{\hat{r}_4}{\hat{r}_4} \), normal prediction will generate a result diverging from the calculated coefficient as \( n_f \) increases. As we use DPA it suddenly changes its orientation towards the calculated result. Utilizing DPA with the Adler transformation will make them behave more similarly.

2. Adler function and Bjorken sum rule

The vacuum polarization induced by vector current \( j_μ = \sum_ν \bar{ψ}_νγ_μψ_ν \) is the time-ordered correlation function

\( (q_μq_ν - q^2g_μν)Π(Q^2) = i \int d^4xe^{iq.x}⟨0 | T[j_μ(x)j_ν(0)] | 0⟩ \),

where \( Q^2 = -q^2 \). The Adler function corresponding to the current takes form through the scalar correlator \( Π(Q^2) \),

\[ D(Q^2) = -12π^2Q^2 \frac{d}{dQ^2}Π(Q^2). \]

Integration over polarized proton and neutron structure functions \( g_1^{p,n} \) is related to the Bjorken polarized sum rule quantity \( d_{Bj} \) through

\[ Γ_{1}^{p−n} = \int_0^1 dx(g_1^p(x,Q^2) - g_1^n(x,Q^2)) \]
\[ = \frac{g_A}{6}(1 - d_{Bj}(Q^2)) + \sum_{j=2}^∞ \frac{μ_{2j}^{p−n}(Q^2)}{Q^{2j−2}}, \]

where \((1 - d_{Bj})\) denotes the coefficient function \( C_{Bj,p} \) and \( g_A \) is the charge corresponding to the axial vector current of the nucleon. Recently the coefficient function \( C_{Bj,p}(Q^2) \) and the non-singlet component of the Adler function \( D^{NS}(Q^2) \) [17] have been calculated to
order \(\alpha_s^4\). They are based on the constraints due to the special form of the generalized Crewther relation 
\[ D(a_s)C_{Bjp}^\beta(a_s) = 1 + \sum_{n \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n(a_s). \]  

Adapting the old and new structures of CR reveals the structure of \(P_n\) coefficients.

Prediction results for \(C_{Bjp}^\beta\) and \(D^{NS}\) are summarized in Table I. For \(C_{Bjp}^\beta\) it seems that a major part of deviation for the normal prediction is due to the first term. But unlike the case of Higgs decay into bottom quarks the gap between prediction and calculation is not so large. It is whilst \(D^{NS}\) is completely different. The prediction and calculated results may seem really close to each other but their \(\beta\)-representations are really far. There seems to be a similarity between \(D^{NS}\)’s and Higgs decay into gluons width’s deviation patterns, both of them have extremum near \(n_f = 3\). DPA makes an improvement in \(C_{Bjp}^\beta\) prediction for higher values of \(n_f\) and provides a slight improvement in the \(\beta\)-representation. A good improvement in the first term of \(\beta\)-representation for \(D^{NS}\) is visible in its DPA prediction. At last taking advantage of Adler transformation will not improve \(C_{Bjp}^\beta\) prediction.

The results of predictions for the ratio \(\frac{D^{NS}(Q^2)}{C_{Bjp}(Q^2)}\) could be illustrated in Table III. The deviation pattern of the ratio does not contain any extremum and is decreasing for higher values of \(n_f\). DPA prediction is good for \(n_f = 6\) and making use of Adler transformation will not help. It seems that \(C_{Bjp}^\beta\) in this ratio affects parts of \(D^{NS}\) for which the \(D^{NS}\) pattern was weakened while employing DPA. It makes the new predictions better for \(n_f = 6\).

Table III shows the results for the ratio \(\frac{C_{Bjp}^\beta(Q^2)}{D^{NS}(Q^2)}\).

### TABLE I. \(C_{Bjp}^\beta\) and \(D^{NS}\) prediction results.

| \(C_{Bjp}^\beta\) | \(n_f^0\) | \(n_f^1\) | \(n_f^2\) | \(n_f^3\) | \(D^{NS}\) | \(n_f^0\) | \(n_f^1\) | \(n_f^2\) | \(n_f^3\) |
|---|---|---|---|---|---|---|---|---|---|
| \(c_{Bjp}^{pr-}\) | -265.4 | 95.26 | 5.94 | 0.08 | 362.1 | -99.08 | 5.04 | -0.05 |
| \(c_{Bjp}^{pr+1}\) | -261.3 | 67.71 | -3.62 | 0.04 | 317.2 | -92.36 | 4.76 | -0.03 |
| \(c_{Bjp}^{pr+2}\) | -20.89 | 26.16 | -1.23 | -0.01 | 407.4 | -103.3 | 5.63 | -0.03 |
| \(c_{Bjp}^{exact}\) | -479.4 | 123.4 | 7.69 | 0.10 |

\(a\) DPA. 
\(b\) DPA utilized by Adler function.

### TABLE II. \(\frac{D^{NS}}{C_{Bjp}}\) prediction results.

| \(\frac{D^{NS}}{C_{Bjp}}\) | \(n_f^0\) | \(n_f^1\) | \(n_f^2\) | \(n_f^3\) | 1st term of the \(\beta\)-rep. |
|---|---|---|---|---|---|
| \(d_{n_f}^{pr-}\) | 1925.9 | -463.57 | 24.11 | -0.2872 | 239.47 |
| \(d_{n_f}^{pr+1}\) | 1409.6 | -332.47 | 16.05 | -0.1302 | 397.27 |
| \(d_{n_f}^{pr+2}\) | 111.29 | -140.82 | 7.128 | -0.0043 | 397.27 |
| \(d_{n_f}^{exact}\) | 2444.4 | -572.69 | 31.02 | -0.3415 | 595.05 |

### Table III. \(\frac{C_{Bjp}^\beta(Q^2)}{D^{NS}(Q^2)}\) prediction results.

| \(\frac{C_{Bjp}^\beta}{D^{NS}}\) | \(n_f^0\) | \(n_f^1\) | \(n_f^2\) | \(n_f^3\) | 1st term of the \(\beta\)-rep. |
|---|---|---|---|---|---|
| \(c_{CD}^{pr-}\) | -54.66 | 33.49 | -2.26 | 0.0319 | -50.64 |
| \(c_{CD}^{pr+1}\) | -101.41 | 30.14 | -1.67 | 0.0153 | -66.90 |
| \(c_{CD}^{pr+2}\) | -37.55 | 18.99 | -1.02 | 0.0028 | -66.90 |
| \(c_{CD}^{exact}\) | -172.83 | 45.62 | -3.03 | 0.0379 | -150.66 |

Existence of \(D^{NS}\) in this ratio absorbs the converging behavior of DPA prediction for higher values of \(n_f\). This converts it to a general improvement along all values of \(n_f\). The corresponding results for the biased BLM prediction are tabulated in Table IV. This indicates real improvements in the \(\beta\)-representation.

### Table IV. \(\frac{C_{Bjp}^\beta}{D^{NS}}\) biased prediction results.

| \(\frac{C_{Bjp}^\beta}{D^{NS}}\) | \(n_f^0\) | \(n_f^1\) | \(n_f^2\) | \(n_f^3\) | 1st term of the \(\beta\)-rep. |
|---|---|---|---|---|---|
| \(c_{CD}^{pr-}\) | 52.07 | 13.64 | -1.40 | 0.0244 | -72.32 |
| \(c_{CD}^{pr+1}\) | 173.65 | -21.23 | 0.41 | 0.0 | -140.29 |
| \(c_{CD}^{pr+2}\) | 461.62 | -73.41 | 3.19 | -0.0409 | -140.29 |
| \(c_{CD}^{exact}\) | -172.83 | 45.62 | -3.03 | 0.0379 | -150.66 |

### 3. Quadratic Casimir invariants

Each part in the first term of a normal BLM pattern Eq. (2) contains at least one factor \(C_A\). So for the case of Higgs decay into bottom quarks setting \(C_A = 0\) improves the normal prediction significantly along all values of \(n_f\) specifically for lower ones.

Working on the ratio \(a_s(Q^2)\frac{R}{K}\) and using \(\{r_1, r_2\}\) that came out of DPA, one will find out the sensitivity of the \(n_f^1\) part of the 4th order prediction to \(C_A\) in such a way that setting \(C_A = 0, C_F = 1\), and \(T_F = 1\) results in a 0.04 relative error in predicting this part.

For the ratio \(a_s(Q^2)\frac{K}{R}\) an interesting improvement occurs when choosing the canonical i.e., conventional form of the quadratic Casimir invariants for \(SU(2)\). More freedom is obtained by departing from the conventional choice to the overall normalization \[22\] where parameters \(b\) and \(N\) take control of the normalization. This also indicates that \(SU(2)\) is a better choice for this ratio.

Adapting the first part of the 4th order coefficient of \(C_{Bjp}^\beta\) by using the overall normalization indicates that \(SU(2)\) improves the \(n_f^1\) part and \(SU(4)\) improves the \(n_f^1\) part in comparison to \(SU(3)\) while \(SU(3)\) still provides a better combination.
A very exciting result is obtained when performing the last procedure for $D^{NS}$. In this case $SU(2)$ gives us such a combination of $n_f^1$ and $n_f^2$ parts that makes the BLM pattern fit to the calculated terms in the whole region of $n_f$. It is in spite of the slight separation occurred in all the previous cases for higher values of $n_f$.

One should pay attention to these parameters of gauge symmetry because their effects are noticeable on the problem and especially on an approach which seems to be related to the concept of conformal symmetry.

B. Second observable

In the previous section, we tried to find parts of the observable that generate a BLM pattern. It is possible to write the observable in a complete BLM pattern form in terms of an appropriate effective charge. This means an absolute freedom in choosing the first two coefficients $e_1$, $e_2$ or equivalently $b_1$, $b_2$:

$$R = e_1 a_2 + e_2 a_2^2 + e_3 a_2^3 + e_4 a_2^4 + \ldots, \quad a_2 = b_1 a + b_2 a^2 + b_3 a^3 + b_4 a^4 + \ldots.$$ 

As in Eqs. (12) we will have the following relations for $e_i$

$$e_3 = \frac{121 C_A^2 e_2^3}{16 T_F^3} \frac{e_2}{e_1} - \frac{17 C_A^2 e_2 e_32}{8 T_F e_2} - \frac{11 C_A^2 e_2 e_32}{2 T_F e_2} - \frac{11 C_A e_2 e_32}{8 T_F e_1},$$
$$e_4 = \frac{C_A}{e_1 T_F} \left[ -\frac{11}{2} e_2 e_32 \frac{T_F}{e_32} + \frac{33}{4} e_2 e_32 C_F - \frac{33}{4} e_2 e_32 C_F \right],$$
$$+ \left[ \frac{627 C_A^2}{32 e_1} - \frac{32}{32 e_32} \right] + \frac{C_A}{e_32 e_4} + \frac{187}{12} e_32 C_F - \frac{17}{32} e_32 C_F + \frac{C_A}{e_32 e_32} \frac{1089}{64} e_32 C_F - \frac{363}{16} e_32 C_F + \ldots,$$

so there is an infinite number of observables in terms of which a BLM pattern is obtained for $R$.

A very first choice for $e_1$, $e_2$ would be $r_1$, $r_2$. For the predicted result suffers the same problem as the normal BLM prediction i.e., a large deviation due to the first term. An impossible state would be the generation of both $e$ and $b$ through BLM pattern.

Obviously we cannot use strict constraints to obtain some $b_1$, $b_2$ because these cannot be partners to the first constraint due to the BLM pattern. For instance if $a_2$ is determined as the Adler function corresponding to $R$, $e_1$ will not be able to normalize $R$ appropriately. It is enslaved to take $r_1$ while it must play the role of an important building block in the BLM pattern at the same time. Generating all $b$ coefficients through Adler function except $b_1$ and $b_2$ would be a much more flexible way which reduces the infinite set of $a_2$ observables to a set of few $a_2$s.

Every $a_2$ observable connects to $R$ through a specific channel. Temporarily we put the BLM pattern constraint aside. If we consider $a_2$ to be the Adler function corresponding to $R$, a very simple dependence of the 4th order coefficient of $R$ to both $e_4$ and $b_4$ is obtained. So either one can predict both $e_4$ and $b_4$ or just predict $e_4$ and use Adler transformation for $b_4$ at the same time. Performing the first choice for $R$ results in a pattern crossing the exact pattern around $n_f = 5$ and for the second choice a crossing at $n_f = 2$.

A more reasonable step would be to start with a specific type of relation between two observables and then equip the situation with a BLM pattern. Of course it would be a waste of time if this ends to direct relations where we encounter observables whose predictions are equivalent. What we mean by a direct relation is a kind of linear relation between two variables. The situation is clarified later. For now we should start with a Crewther like relation between two observables $R$ and $a_2$:

$$[r_0 + r_1 a_s + \sum_{i=2}^{\infty} r_i a_s^i] [r_0 - r_1 a_s + \sum_{i=2}^{\infty} b_i a_s^i] = C + \frac{\beta_3 K a}{a},$$

where for the first observable we have $R = r_1 a_s + \sum_{i=3}^{\infty} r_i a_s^i$, and for the second one we have $a_2 = -r_1 a_s + \sum_{i=3}^{\infty} b_i a_s^i$. $\beta_3$ is the famous $\beta$-function $-\sum_{i=0}^{\infty} \beta_i a^{i+1}$, and $K = \sum_{i=1}^{\infty} K_i a_i$. The $\{r_0...r_1\}$ and $\{K_0...K_1\}$ determines any coefficient $b_i$ in a special manner, considering all known coefficients prior to $r_i$ and $K_{i-1}$,

$$b_{ij} = -r_{ij} - \frac{11 C_A}{12 r_0} K_{(i-1)j} + C_{ij}, \quad j < i - 1,$$
$$b_{i(i-1)} = -r_{i(i-1)} - \frac{T_F}{3 r_0} K_{(i-1)(i-2)}.$$

Despite the fact that $R$ does not contain $r_0$, it plays a crucial role in normalizing $r_1$ coefficients in Eq. (9). To involve $r_0$ in the problem, one should perform prediction one order backward and normalize the result with the appropriate $r_0$. Finally this will make $e_i$ coefficient be determined in terms of $\{r_1...r_{i-1}\}$. Taking $K$ coefficients as the free variables, Eq. (9) reduces into the very simple direct relation $b_i = -r_i + f_i(n_f)$. The result comes out of such a relation suffers the same problem as a normal prediction for $r_i$ since $b_i$ and $r_i$ can exchange roles. The same is true about predicting $r_i$ through $e_i$. Although we are able to determine $e_i$ prior to $r_i$, using Eq. (9), at the same time $e_i$ has a direct relation with $r_i$ i.e., $e_i = c_i r_i + h_i(n_f)$. So if anything is to give us real different results, it must have an indirect relation with $r_i$:

$$b_{ij} = \sum_{j=0}^{i-1} [b_{ij} r_{ij} + m_{ij} K_{(i-1)(j-1)}] + C_{ij},$$
$$e_i = \sum_{j=0}^{i-1} e_{ij} r_{ij} + h_i(n_f).$$
We have done a kind of converting such direct relations into indirect ones in the past when we tried to interpret DPA as changing weights of \( r_{ij} \) coefficients in the normal BLM pattern. Obviously Eq. (6) is also a kind of indirect relation between \( r_i \) and \( K_{i-1} \). Following this line, it is more appropriate to rewrite Eq. (6) in terms of these independent polynomials in Eq. (5): 

\[
[r_0 + r_1 a + (r_{20} + r_{21} \beta_0) a^2 + \sum_{i=3}^{\infty} r_i a^i] [r_0 - r_1 a + (b_{20} + 2b_{21} \beta_0) a^2 + \sum_{i=3}^{\infty} b_i a^i] = C + \frac{\beta_2 K}{a^2},
\]

where we’ve written all the coefficients in the \( \beta \)-representation so \( r_i = r_{(i-1)(i-2)} \beta_{i-2} + \sum_{j=0}^{i-1} r_{ij} \beta_j \). The intrinsic property of this constraint would be to eliminate \( \{K_0, K_1, K_{31}, K_{32}, K_{34}\} \) from the pattern. It is possible to use Eq. (7) alongside Eq. (5). When \( \frac{\beta_2 K}{a^2} \) get involved in Eq. (7), an oversimple indirect relation appears. The indirectness comes from the absence of \( K_i \) parts in \( b_{i0} \):

\[
\begin{align*}
  b_{i0} &= -r_{i0} + D_{ij}, \\
  b_{ij} &= -r_{ij} - \frac{K_{i(j-1)}}{r_0} + E_{ij} \quad 0 < j < i - 1, \\
  b_{i(i-1)} &= -r_{i(i-1)} - \frac{K_{i(i-2)}}{r_0}.
\end{align*}
\]

As in Eq. (6) we combined all lower order coefficients’ dependences in constant parts \( D_{ij} \) and \( E_{ij} \). Again prediction will depend on \( r_0 \) which should be resolved like before. 

One could rewrite Eq. (6) in terms of the coefficients of the \( \beta \)-independent polynomials \( P_n \) in Eq. (5) using the relations in (21). The idea in (21) strongly reveals existence of flavor-independent polynomials \( P_n \). Conformal symmetry breaking totally happens through \( \beta \)-function in this way. So it is more appropriate to rewrite Eq. (6) in terms of these polynomial coefficients, even though this does not affect the directness of Eq. (6).

\[ \text{III. CONCLUSIONS} \]

Flexibility of any optimization prescription in perturbative analysis is the key to check the possibility of improving the prediction by taking deviation of lower order predictions into account i.e., finding patterns for the perturbative series. Taking DPA as finding parts of the observable that generates the appropriate BLM pattern for each order might help us in categorizing observables and their responses to the prediction respecting their deviation patterns. But it could not be fully utilized to improve the prediction since for any order \( n \) we will suffer lack of information about \((n-1)\) parameters. In fact the effect of the symmetry group parameters on the BLM pattern is much more noticeable where the overall normalization plays a key role. A reasonable way to improve predicted patterns is to perform predictions based on indirect relations. Observables are connected in a direct channel when they are expressed in terms of each other. In this case even specific insightful constraints such as the Crewther relation does not guarantee a simple indirect connection between observables. As a note for the reader, clearly we are not referring to the physical content of Crewther relation but just the mathematical form of it as a connection channel. So finding a mother constraint that produces elegant indirect relations is the challenging problem in the context of finding patterns.

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