Strong consistent model selection for general causal time series

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Abstract: We consider the strongly consistent question for model selection in a large class of causal time series models, including AR($\infty$), ARCH($\infty$), TARCH($\infty$), ARMA-GARCH and many classical others processes. We propose a penalized criterion based on the quasi likelihood of the model. We provide sufficient conditions that ensure the strong consistency of the proposed procedure. Also, the estimator of the parameter of the selected model obeys the law of iterated logarithm. It appears that, unlike the result of the weak consistency obtained by Bardet et al. [2], a dependence between the regularization parameter and the model structure is not needed.

Keywords: Model selection, strong consistency, causal processes, quasi-maximum likelihood estimation, penalized contrast.

1 Introduction

We consider a general class of autoregressive time series in a semiparametric framework. Let $M, f : \mathbb{R}^N \rightarrow \mathbb{R}$ be two measurable functions and $(\xi_t)_{t \in \mathbb{Z}}$ a sequence of centered independent and identically distributed (iid) random variables satisfying $\text{var}(\xi_0) = 1$. Consider the class of affine causal models,

Class $\mathcal{AC}(M, f)$: A process $X = (X_t)_{t \in \mathbb{Z}}$ belongs to $\mathcal{AC}(M, f)$ if it satisfies:

$$X_t = M((X_{t-i})_{i \in \mathbb{N}^*}) \xi_t + f((X_{t-i})_{i \in \mathbb{N}^*}) \text{ for any } t \in \mathbb{Z}. \quad (1)$$

The existence of a stationary and ergodic solution of the class $\mathcal{H}$ has been studied by Bardet and Wintenberger (2009) as a particular case of models considered in Doukhan and Wintenberger (2008). Bardet and Wintenberger (2009) and Bardet et al. (2017) carried out the inference question in the semiparametric setting in the class $\mathcal{AC}(M, f)$, whereas Bardet et al. (2012), Kengne (2012), Bardet and Kengne (2014) focussed on the change-point problem in this class. Numerous classical time series models belongs to the class $\mathcal{H}$: for instance AR($\infty$), ARCH($\infty$), TARCH($\infty$), ARMA-GARCH or APARCH processes.

\footnotesize
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\end{itemize}
Consider a trajectory \((X_1, \ldots, X_n)\) of a process \(X = (X_t)_{t \in \mathbb{Z}}\) that belongs to \(\mathcal{AC}(M^*, f^*)\) where \(M^*\) and \(f^*\) are unknown. We consider a finite collection \(M\) of affine causal models, where the true model \(m^* \in M\) corresponds to \(M^*\) and \(f^*\). Our main aim is to select a model \(\hat{m}\) (among the collection \(M\)) which is ”close” to \(m^*\) for large \(n\).

We focus on the semiparametric framework and assume that the distribution of \(\xi_0\) is unknown and that the functions \(f\) and \(M\) are known up to a parameter \(\theta \in \Theta\), where \(\Theta\) is a compact subset of \(\mathbb{R}^d\) (\(d \in \mathbb{N}\)). That is, the model \(m^*\) corresponds to the true parameter \(\theta^* \in \Theta\) and the process \(X\) belongs to \(\mathcal{AC}(M_{\theta^*}, f_{\theta^*})\). In the sequel:

- each model \(m \in M\) is considered as a subset of \(\{1, \ldots, d\}\) and denote by \(|m|\) the dimension of the model, typically, \(|m| = \#(m)\);
- for \(m \in M\), the parameter space of \(m\) is \(\Theta(m) = \{(\theta_i)_{1 \leq i \leq d} \in \Theta, \theta_i = 0 \text{ if } i \notin m\}; \theta(m)\) is the parameter vector associated to \(m\);
- the collection \(M\) is considered as a subset of the power set of \(\{1, \ldots, d\}\), i.e. \(M \subset \mathcal{P}\{1, \ldots, d\}\).

Therefore, for any model \(m \in M\), \(m \in \mathcal{AC}(M_\theta, f_\theta)\) when \(\theta \in \Theta(m)\). Also, we could consider hierarchical as well as exhaustive families of models.

For instance, assume that \((X_1, \ldots, X_n)\) is generated from a GARCH\((p^*, q^*)\) process. The collection \(M\) could be a family of ARMA\((p, q)\)-GARCH\((p', q')\) with \((p, q, p', q') \in \{0, 1, \ldots, p_{max}\} \times \{0, 1, \ldots, q_{max}\} \times \{0, 1, \ldots, p'_{max}\} \times \{0, 1, \ldots, q'_{max}\}\) where \(p_{max}, q_{max}, p'_{max}, q'_{max}\) are the fixed upper bounds of the orders, assumed to satisfy \(p^* \leq p_{max}\) and \(q^* \leq q_{max}\). Therefore, consider \(\Theta\) as a compact subset of \(\mathbb{R}^{p_{max}+q_{max}} \times (0, \infty) \times (0, \infty)^{p'_{max}+q'_{max}}\). Thus, a model \(m\) is a subset of \(\{1, \ldots, p_{max} + q_{max} + p'_{max} + q'_{max} + 1\}\) and its parameter space is \(\Theta(m) = \{(\theta_i)_{1 \leq i \leq d} \in \Theta, \theta_i = 0 \text{ if } i \notin m\}\).

The model selection problem for time series has already been considered by several authors; we refer to the book of McQuarrie and Tsai (1998), the monograph of Rao and Wu (2001), the recent review paper of Ding et al. (2018), the recent works of Hsu et al. (2019) and the references therein for an overview on this topic. Hannan (1980) and Hannan and Deistler (2012) provided general conditions for strong consistency of the order estimator of an ARMA and ARMAX model. Resende and Dorea (2016) proposed the efficient determination criterion (introduced by Zhao et al. (2001) for the strongly consistent estimation of the order of multiple Markov chains) for model selection in a general class of multivariate time series. They established the strong consistency of the procedure under some conditions which may seem a bit strong for some applications, for instance, the existence of the third order derivative of the contrast function (likelihood), the existence of moments of order 16 for the BEKK-GARCH model. Recently, Bardet et al. (2020) addressed the model selection question in the class of model \(\mathcal{AC}(M_\theta, f_\theta)\). They proposed a procedure based on the quasi likelihood of the model and provided sufficient conditions that ensure the weak consistency of the selected model.

In this new contribution, we focus on the model selection in the class of model \(\mathcal{AC}(M_\theta, f_\theta)\) with a penalized contrast which is based on the Gaussian quasi likelihood of the model.

(i) Under the assumptions that \(E[|\xi_0|^r] < \infty\) with \(r > 4\), the functions \(\theta \mapsto f_\theta\), \(M_\theta\) are twice times continuously differentiable on \(\Theta\) and satisfy some Lipschitz-type properties, we establish the strong consistency of the proposed procedure.
(ii) We show that the quasi maximum likelihood estimator (QMLE) of the selected model obeys the law of iterated logarithm.

The rest of the paper is structured as follows. In Section 2, we set some notations, assumptions and define the model selection criterion. The main results are provided in Section 3 whereas Section 4 is devoted to a concluding remarks. Section 5 focuses on the proofs of the main results.

2 Assumptions and the model selection criterion

2.1 Assumptions on the class of models $\mathcal{AC}(M_\theta, f_\theta)$

In the sequel, we will use the norms:

1. $\| \cdot \|$ applied to a vector denotes the Euclidean norm of the vector;
2. for any compact set $K \subseteq \mathbb{R}^d$ and for any $g : \Theta \to \mathbb{R}^p$, $\|g\|_K = \sup_{\theta \in \Theta}(\|g(\theta)\|)$.

Throughout the sequel, we assume that the functions $\theta \mapsto M_\theta$ and $\theta \mapsto f_\theta$ are twice times continuously differentiable on $\Theta$. Also, we will use $H_\theta = M_\theta^2$ and for any function $g_\theta$ which is $i$ times differentiable on $\Theta$, we set $\partial_i g_\theta = \partial^i g_\theta / \partial \theta^i$. Let us consider the following assumptions for any compact set $K \subseteq \Theta$, $i = 0, 1, 2$ and $\Psi_\theta = \partial_0 f_\theta$ or $\partial_0 M_\theta$:

**Assumption A($\Psi_\theta, K$):** for any $x \in \mathbb{R}^N$, the function $\theta \mapsto \Psi_\theta(x)$ is continuous on $\Theta$ with $\|\Psi_\theta(0)\|_\Theta < \infty$ and there exists a sequence of non-negative real numbers $(\alpha_k(\Psi_\theta, K))_{k \geq 1}$ satisfying $\sum_{k=1}^{\infty} \alpha_k(\Psi_\theta, K) < \infty$ such that:

$$\|\Psi_\theta(x) - \Psi_\theta(y)\|_K \leq \sum_{k=1}^{\infty} \alpha_k(\Psi_\theta, K)|x_k - y_k| \text{ for all } x, y \in \mathbb{R}^N.$$ 

In the sequel we refer to the particular case of the non linear ARCH($\infty$) (NLARCH($\infty$), see Bardet and Wintenberger(2009)) processes define when $f_\theta = 0$. In this case, we consider the following assumption for $i = 0, 1, 2$:

**Assumption A($\partial_0, H_\theta, K$):** for any $x \in \mathbb{R}^\infty$, the function $\theta \mapsto \partial_0 H_\theta(x)$ is continuous on $\Theta$ with $\|\partial_0 H_\theta(0)\|_\Theta < \infty$ and there exists a sequence of non-negative real numbers $(\alpha_k(\partial_0, H_\theta, K))_{k \geq 1}$ satisfying $\sum_{k=1}^{\infty} \alpha_k(\partial_0, H_\theta, K) < \infty$ such that:

$$\|\partial_0 H_\theta(x) - \partial_0 H_\theta(y)\|_K \leq \sum_{k=1}^{\infty} \alpha_k(\partial_0, H_\theta, K)|x_k^2 - y_k^2| \text{ for all } x, y \in \mathbb{R}^N.$$ 

Then define the set:

$$\Theta(r) = \left\{ \theta \in \mathbb{R}^d, A(f_\theta, \{\theta\}) \text{ and } A(M_\theta, \{\theta\}) \text{ hold with } \sum_{k=1}^{\infty} \alpha_k(f_\theta, \{\theta\}) + \|\xi_0\|_r \sum_{k=1}^{\infty} \alpha_k(M_\theta, \{\theta\}) < 1 \right\}$$

$$\cup \left\{ \theta \in \mathbb{R}^d, f_\theta = 0 \text{ and } A(H_\theta, \{\theta\}) \text{ holds with } \|\xi_0\|_r^2 \sum_{k=1}^{\infty} \alpha_k(H_\theta, \{\theta\}) < 1 \right\} \quad (2)$$

The above Lipschitz-type conditions are classical when studying the existence of a stationary and ergodic solution of such model, see Doukhan and Wintenberger (2008). In the case of the class $\mathcal{AC}(M_\theta, f_\theta)$, if $\theta \in \Theta(r)$,
then there exists a unique, causal, stationary and ergodic solution $X = (X_t)_{t \in \mathbb{Z}} \in AC(M_\theta, f_\theta)$ with a finite moment of order $r$, see Bardet and Wintenberger (2009).

The following assumptions are useful in the study of the asymptotic behavior of the QLME.

**Assumption D(\(\Theta\)):** \(\exists b > 0\) such that \(\inf_{\theta \in \Theta} (H_0(x)) \geq b\) for all \(x \in \mathbb{R}^N\).

**Assumption Id(\(\Theta\)):** For all \((\theta, \theta') \in \Theta^2\),

\[
\left(f_\theta(X_0, X_{-1}, \cdots) = f_{\theta'}(X_0, X_{-1}, \cdots) \text{ and } M_\theta(X_0, X_{-1}, \cdots) = M_{\theta'}(X_0, X_{-1}, \cdots) \text{ a.s.} \right) \Rightarrow \theta = \theta'.
\]

**Assumption Var(\(\Theta\)):** For all \(\theta \in \Theta\), one of the families \((\frac{\partial f_\theta}{\partial \theta^i}(X_0, X_{-1}, \cdots))_{1 \leq i \leq d}\) or \((\frac{\partial h_\theta}{\partial \theta^i}(X_0, X_{-1}, \cdots))_{1 \leq i \leq d}\) is a.s. linearly independent.

In the following assumption, we make the convention that if \(\{A(t)\}_{t \in \mathbb{Z}} \subseteq \mathcal{M}\) is defined for \(k \geq 1\), then there exists a unique causal, stationary and ergodic solution \(X_\theta\) with \(\inf_{\theta \in \Theta} (H_\theta(x)) = 0\) and if \(\mathcal{A}(\theta, H_\theta, \Theta)\) holds, then \(\alpha_k(\partial_{\theta^i} H_\theta, \Theta) = 0\). For \(\ell = 0, 1, 2\),

**Assumption K(\(\Theta\)):** there exists \(r > 4\) such that \(\theta^* \in \Theta(r) \cap \Theta\) and for \(i = 0, \cdots, \ell\), \(A(\partial_{\theta^i} f_\theta, \Theta), A(\partial_{\theta^i} M_\theta, \Theta), \) \(A(\partial_{\theta^i} H_\theta, \Theta)\) hold with

\[
\sum_{k \geq 1} \frac{1}{\sqrt{k \log \log k}} \sum_{j \geq k} \sum_{i=0}^{\ell} \alpha_j(\partial_{\theta^i} f_\theta, \Theta) + \alpha_j(\partial_{\theta^i} M_\theta, \Theta) + \alpha_j(\partial_{\theta^i} H_\theta, \Theta) < \infty.
\]

These aforementioned assumptions hold for many classical models, including AR(\(\infty\)), ARCH(\(\infty\)), TARCH(\(\infty\)) type processes, see for instance Bardet and Wintenberger (2009), Bardet et al. (2012), Kengne (2012). In the case of assumption K(\(\Theta\)), let us consider for \(\ell = 0, 1, 2\):

1. The geometric case: \(\sum_{i=0}^{\ell} \alpha_j(\partial_{\theta^i} f_\theta, \Theta) + \alpha_j(\partial_{\theta^i} M_\theta, \Theta) + \alpha_j(\partial_{\theta^i} H_\theta, \Theta) = \mathcal{O}(a^j)\) for some \(a \in [0, 1]\). In this case, assumption K(\(\Theta\)) holds.

2. The Riemannian case: \(\sum_{i=0}^{\ell} \alpha_j(\partial_{\theta^i} f_\theta, \Theta) + \alpha_j(\partial_{\theta^i} M_\theta, \Theta) + \alpha_j(\partial_{\theta^i} H_\theta, \Theta) = \mathcal{O}(j^\gamma)\) with \(\gamma > 0\). If \(\gamma > 3/2\), then K(\(\Theta\)) holds.

### 2.2 The model selection criterion

Consider a model \(m \in \mathcal{M}\) and the class \(AC(M_\theta, f_\theta)\) for \(\theta \in \Theta(m) \subset \Theta \subset \mathbb{R}^d\). Assume that a trajectory \((X_1, \ldots, X_n)\) is observed. The conditional Gaussian quasi (log)likelihood (up to a constant) \(L_n\) is defined for all \(\theta \in \Theta(m)\) by,

\[
L_n(\theta) := -\frac{1}{2} \sum_{i=1}^{n} q_i(\theta), \text{ with } q_i(\theta) := \frac{(X_i - f^i_{\hat{\theta}})^2}{H^i_{\hat{\theta}}} + \log(H^i_{\hat{\theta}})
\]

where \(f^i_{\hat{\theta}} := f_\theta(X_{i-1}, X_{i-2}, \cdots), M^i_\theta := M_\theta(X_{i-1}, X_{i-2}, \cdots)\) and \(H^i_{\theta} = (M^i_{\hat{\theta}})^2\). Since \(L_n(\theta)\) depends on \((X_t)_{t \leq 0}\) which are not observed, it is common practice (see [5], [11], [13]) to consider the approximated quasi (log)likelihood given (up to a constant) for all \(\theta \in \Theta(m)\) by

\[
\widehat{L}_n(\theta) := -\frac{1}{2} \sum_{i=1}^{n} \tilde{q}_i(\theta), \text{ with } \tilde{q}_i(\theta) := \frac{(X_i - f^i_{\hat{\theta}})^2}{H^i_{\hat{\theta}}} + \log(\tilde{H}^i_{\hat{\theta}})
\]
where \( \hat{f}_t = f_\theta(X_{t-1}, X_{t-2}, \ldots, X_1, 0, \ldots) \), \( \hat{M}^t_\theta = M_\theta(X_{t-1}, X_{t-2}, \ldots, X_1, 0, \ldots) \), \( \hat{H}^t_\theta = (\hat{M}^t_\theta)^2 \). Note that, the "best" parameter associated to the model \( m \) is defined by,

\[
\theta^*(m) = \arg\min_{\theta \in \Theta(m)} \mathbb{E}[q_0(\theta)].
\]

According to [2], \( \theta^*(m) \) exists and it is unique under \( \text{Id}(\Theta(m)) \). When \( m = m^* \), we have \( \theta^*(m^*) = \theta^* \). For any \( m \in M \), the QMLE of \( \theta^*(m) \) is given by,

\[
\hat{\theta}(m) = \arg\max_{\theta \in \Theta(m)} \hat{L}_n(\theta).
\]

The selection of the "best" model \( \hat{m} \) among the collection \( M \) is performed by minimizing the penalized contrast

\[
\hat{C}(m) = -2\hat{L}_n(\hat{\theta}(m)) + |m|\kappa_n,
\]

that is

\[
\hat{m} = \arg\min_{m \in M} \hat{C}(m),
\]

where

- \( (\kappa_n)_n \) is the sequence of the regularization parameter (possibly data-dependent) that will be used to calibrate the penalty term;

- \( |m| \) is the dimension of the model \( m \), typically, the cardinal of \( m \) (considered as a subset of \( \{1, \ldots, d\} \)), which is also the number of the estimated components of \( \theta \) (the others are fixed to zero).

## 3 Asymptotic results

Recall that, when the model is correctly specified, Bardet and Wintenberger (2009) have established the consistency and the asymptotic normality of \( \hat{\theta}(m^*) \). The following proposition shows that the estimator \( \hat{\theta}(m^*) \) obeys the law of iterated logarithm.

**Proposition 3.1** Let \((X_1, \ldots, X_n)\) be a trajectory of a process \( X \) belonging to \( \mathcal{AC}(M_\theta^*, f_\theta^*) \) where \( \theta^* \in \Theta(r) \cap \Theta \subset \mathbb{R}^d \) with \( r > 4 \). Assume that \( D(\Theta), \text{Id}(\Theta), \text{Var}(\Theta), K_2(\Theta) \) hold. Then,

\[
\hat{\theta}(m^*) - \theta^* = O\left(\sqrt{\frac{\log \log n}{n}}\right) \quad \text{a.s.}
\]

The following theorem provides sufficient conditions that ensure the strong consistency of the model selection procedure.

**Theorem 3.1** Let \((X_1, \ldots, X_n)\) be a trajectory of a process \( X \) belonging to \( \mathcal{AC}(M_\theta^*, f_\theta^*) \). Under the assumptions of Proposition 3.1 and if \( \kappa_n / \log \log n \xrightarrow{n \to \infty} \infty \) and \( \kappa_n / n \xrightarrow{n \to \infty} 0 \), then

\[
\hat{m} \xrightarrow{\text{a.s.}} m^*.
\]
Remark 3.1  1. This result, besides it is stronger than those obtained by Bardet et al. (2020), do not impose any condition on the dependence between the regularization parameter $\kappa_n$ and the Lipschitz-type coefficients $\alpha_j(\partial_\theta f_\theta, \Theta)$, $\alpha_j(\partial_\theta M_\theta, \Theta)$, $\alpha_j(\partial_\theta H_\theta, \Theta)$ as have been set by these authors. For instance, in the Riemannian case with $\sum_{i=0}^t \alpha_j(\partial_\theta f_\theta, \Theta) + \alpha_j(\partial_\theta M_\theta, \Theta) + \alpha_j(\partial_\theta H_\theta, \Theta) = O(j^\gamma)$ for some $3/2 < \gamma < 2$, the BIC ($\kappa_n = \log n$) is strongly consistent from this Theorem while the result of Bardet et al. (2020) can not assure the weak consistency of the BIC.

2. Hannan and Deistler (2012) have considered the estimation of the order of an ARMAX (including ARMA), where the contrast $C$ is based on the Gaussian likelihood of the model. Under the condition $\kappa_n/n \rightarrow 0$, they have established that there exists a constant $c_1 > 0$ such that, if $\lim \inf \kappa_n/(2 \log \log n) > c_1$, then the estimator of the order is strongly consistent. From the proof of Theorem 3.1, one can see that such result holds for the general class of model considered here; that is, we can find a constant $c_2 > 0$ such that if $\lim \inf \kappa_n/\log \log n > c_2$, then $\hat{m} \overset{a.s.}{\rightarrow} \infty$.

The next corollary show that the estimator of the parameter of the selected model $\hat{\theta}(\hat{m})$ obeys the law of iterated logarithm.

Corollary 3.1 Let $(X_1, \ldots, X_n)$ be a trajectory of a process $X$ belonging to $\mathcal{AC}(M_{\theta*}, f_{\theta*})$. Under the assumptions of Theorem 3.1,

$$\hat{\theta}(\hat{m}) - \theta^* = O\left(\sqrt{\frac{\log \log n}{n}}\right) \text{ a.s. } .$$

(10)

4 Concluding remarks

This paper focuses on the model selection in a large class of causal time series models in a semiparametric framework. The strong consistency of an estimator based on a penalized quasi likelihood contrast is established, under some classical conditions on the regularization parameters $\kappa_n$.

For the estimation of the order of an ARMAX model, Hannan and Deistler (2012) have established that, there exists a constant $c_0$ such that, if $\lim \sup \kappa_n/(2 \log \log n) < c_0$ then the strong consistency of the estimator of the order fails. A topic of a future works could be to investigate if such result is applied to the general class of model considered here or to derive an upper bound of $\kappa_n$ for which the strong consistency fails.

Another extension of this works is to carry out the model selection problem in the class $\mathcal{AC}(M_\theta, f_\theta)$ with a procedure based on a non Gaussian (for instance Laplacian, see Bardet et al. (2017)) quasi likelihood, for the purpose of reducing the order moment imposed on the process.

5 Proofs of the main results

The following lemma will be useful in the sequel. The proof is carried out by going along similar lines as in Lemma 2 of [2] by using Corollary 1 of [12]; so, it is then omitted.

Lemma 5.1 Let $X \in \mathcal{AC}(M_\theta, f_\theta)$ and $\Theta \subseteq \Theta(r)$ with $r > 4$. Assume that the assumptions $D(\Theta)$ and $K_1(\Theta)$ hold. Then,

$$\frac{1}{\sqrt{n \log \log n}} \left\| \frac{\partial \hat{L}_n(\theta)}{\partial \theta} - \frac{\partial L_n(\theta)}{\partial \theta} \right\| \Theta \overset{a.s.}{\rightarrow} 0. \text{ (11)}$$


Proof of Proposition 3.1 According to [5], it holds that \( \hat{\theta}(m^*) \xrightarrow{n \to \infty} \theta^* \). Also, since \( \theta^* \in \Theta(m^*) \cap \hat{\Theta} \), we get \( \frac{\partial L_n(\hat{\theta}(m^*))}{\partial \theta} = 0 \) for \( n \) large enough. Thus, for any \( i = 1, \ldots, |m^*| \), the Taylor expansion of \( \frac{\partial L_n}{\partial \theta_i} \) implies

\[
0 = \frac{\partial L_n(\hat{\theta}(m^*))}{\partial \theta_i} = \frac{\partial L_n(\theta^*)}{\partial \theta_i} + \frac{\partial^2 L_n(\hat{\theta}_i(m^*))}{\partial \theta \partial \theta_i} (\hat{\theta}(m^*) - \theta^*),
\]

where \( \hat{\theta}_i(m^*) \) lies between \( \hat{\theta}(m^*) \) and \( \theta^* \). Therefore,

\[
\sqrt{\frac{n}{\log n}} (\hat{\theta}(m^*) - \theta^*) = \frac{2}{\sqrt{n \log n}} \hat{F}_n^{-1}(m^*) \frac{\partial L_n(\theta^*)}{\partial \theta} \text{ where } \hat{F}_n(m^*) = -2 \left( \frac{\partial^2 L_n(\hat{\theta}_i(m^*))}{\partial \theta \partial \theta_i} \right)_{i \in m^*}. \tag{12}
\]

Note that, by dealing with the first (stationary) regime in the Corollary 6.1 of [4] and since \( \hat{\theta}_i(m^*) \xrightarrow{n \to \infty} \theta^* \) for \( i = 1, \ldots, |m^*| \), we get

\[
\hat{F}_n(m^*) \xrightarrow{n \to \infty} F(\theta^*, m^*) \text{ where } F(\theta^*, m^*) = \left( \mathbb{E} \left[ \frac{\partial^2 q_0(\theta^*)}{\partial \theta_i \partial \theta_j} \right] \right)_{i, j \in m^*}. \tag{13}
\]

Since \( F(\theta^*, m^*) \) is invertible (see [5]), then for \( n \) large enough and with a sufficiently large probability, the matrix \( \hat{F}_n(m^*) \) is invertible. We have from Lemme [5.1] [12] and [13],

\[
\sqrt{\frac{n}{\log n}} (\hat{\theta}(m^*) - \theta^*) = \frac{2}{\sqrt{n \log n}} \hat{F}_n^{-1}(m^*) \frac{\partial L_n(\theta^*)}{\partial \theta} + o(1) \text{ a.s.} \tag{14}
\]

We have,

\[
\frac{\partial L_n(\theta^*)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial q_i(\theta^*)}{\partial \theta}.
\]

Denote for all \( t \in \mathbb{Z}, F_t = \sigma(X_t, X_{t-1}, \ldots) \) the \( \sigma \)-field generated by the whole past at time \( t \). Then, \( \left( \frac{\partial q_i(\theta^*)}{\partial \theta}, F_t \right) \) is a stationary ergodic square integrable martingale difference process (see [5]). Therefore, from the law of iterative logarithm for martingales (see [16] [17]), we get,

\[
\frac{1}{\sqrt{n \log n}} \frac{\partial L_n(\theta^*)}{\partial \theta} = \mathcal{O}(1) \text{ a.s.} \tag{15}
\]

Thus, the proposition follows from [13], [14] and [15].

\[\blacksquare\]

Proof of Theorem 3.1

1. Let \( m \in M \) such as \( m \geq m^* \). We have,

\[
\frac{1}{\log n} \left( \hat{C}(m^*) - \hat{C}(m) \right) = \frac{2}{\log n} \left( \hat{L}_n(\hat{\theta}(m)) - \hat{L}_n(\hat{\theta}(m^*)) \right) - \frac{\kappa_n}{\log n} (|m| - |m^*|). \tag{16}
\]

Let us establish that

\[
\frac{1}{\log n} \left( \hat{L}_n(\hat{\theta}(m)) - \hat{L}_n(\hat{\theta}(m^*)) \right) = \mathcal{O}(1) \text{ a.s.} \tag{17}
\]

Since \( \theta^* \in \Theta(m) \cap \hat{\Theta} \) and \( \hat{\theta}(m) \xrightarrow{n \to \infty} \theta^* \), then \( \frac{\partial \hat{L}_n(\hat{\theta}(m))}{\partial \theta} = 0 \) for \( n \) large enough. Therefore, from the Taylor expansion of \( \hat{L}_n \), we can find \( \bar{\theta}(m) \) between \( \hat{\theta}(m) \) and \( \theta^* \) such that

\[
\hat{L}_n(\bar{\theta}(m)) - \hat{L}_n(\theta^*) = \frac{1}{2} (\hat{\theta}(m) - \theta^*)' \frac{\partial^2 \hat{L}_n(\hat{\theta}(m))}{\partial \theta^2} (\hat{\theta}(m) - \theta^*). \tag{18}
\]
Also, for any \(i = 1, \ldots, |m|\), we can find \(\hat{\theta}_i(m)\) between \(\hat{\theta}(m)\) and \(\theta^*\) such that, for \(n\) large enough,

\[
0 = \frac{\partial \hat{L}_n(\hat{\theta}(m))}{\partial \theta_i} = \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta_i} + \frac{\partial^2 \hat{L}_n(\hat{\theta}_i(m))}{\partial \theta \partial \theta_i} (\hat{\theta}(m) - \theta^*).
\]

Hence,

\[
\hat{\theta}(m) - \theta^* = \frac{2}{n} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta_i} \frac{\hat{L}_n(\theta^*)}{\partial \theta_i} \text{ where } \hat{F}_n(m) = -2 \left( \frac{\partial^2 \hat{L}_n(\hat{\theta}_i(m))}{\partial \theta \partial \theta_i} \right)_{i \in m}.
\]  

(19)

Since \(\hat{\theta}(m), \bar{\theta}(m), \hat{\theta}_i(m) \xrightarrow{a.s.} \theta^*\) for \(i = 1, \ldots, |m|\), in this case of overfitting, the same arguments as in the proof of Proposition 3.1 lead to

\[
\hat{F}_n(m) \xrightarrow{a.s.} n \rightarrow \infty F(\theta^*, m) \text{ and } -\frac{2\partial^2 \hat{L}_n(\bar{\theta}(m))}{\partial \theta^2} \xrightarrow{a.s.} n \rightarrow \infty F(\theta^*, m) \text{ where } F(\theta^*, m) = \left( \mathbb{E} \left[ \frac{\partial^2 q_0(\theta^*)}{\partial \theta_i \partial \theta_j} \right] \right)_{i,j \in m}.
\]

(20)

For the overfitted model \(m\), one can deduce from [5] that \(F(\theta^*, m)\) is invertible, thus for \(n\) large enough and with a sufficiently large probability, the matrix \(\hat{F}_n\) is invertible. From (18), (19), (20) and Lemma 5.1 it holds that

\[
\frac{1}{\log \log n} \left( \hat{L}_n(\hat{\theta}(m)) - \hat{L}_n(\hat{\theta}(m^*)) \right)
\]

\[
= \frac{2}{n^2 \log \log n} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta^2} \hat{F}_n^{-1}(m) \frac{\partial^2 \hat{L}_n(\bar{\theta}(m))}{\partial \theta^2} \hat{F}_n^{-1}(m) \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta}
\]

\[
= -\left( \frac{1}{\sqrt{n \log \log n}} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta^2} \hat{F}_n^{-1}(m) \left( -\frac{2\partial^2 \hat{L}_n(\bar{\theta}(m))}{\partial \theta^2} \right) \hat{F}_n^{-1}(m) \left( \frac{1}{\sqrt{n \log \log n}} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta} \right) \right)
\]

\[
= \left( \frac{1}{\sqrt{n \log \log n}} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta^2} + o(1) \right) O(1) \left( \frac{1}{\sqrt{n \log \log n}} \frac{\partial \hat{L}_n(\theta^*)}{\partial \theta} + o(1) \right) \text{ a.s.}.
\]

(21)

Thus, according to \((21)\) and \((15)\), \((17)\) follows.

Therefore, since \(\kappa_n / \log \log n \xrightarrow{n \rightarrow \infty} \infty\) and \(|m| > |m^*|\), then \((16)\) and \((17)\) lead to

\[
\lim_{n \rightarrow \infty} \frac{1}{\log \log n} (\hat{C}(m^*) - \hat{C}(m)) = -\infty \text{ a.s.}.
\]

(22)

This implies,

\[
\hat{C}(m) - \hat{C}(m^*) > 0 \text{ a.s. for large } n.
\]

(23)

2. Let \(m \in \mathcal{M}\) such that \(m \nless m^*\). We have,

\[
\frac{1}{n} (\hat{C}(m^*) - \hat{C}(m)) = \frac{1}{n} \left( \hat{L}_n(\hat{\theta}(m)) - \hat{L}_n(\hat{\theta}(m^*)) \right) - \frac{\kappa_n}{n} (|m| - |m^*|).
\]

(24)

For all \(\theta \in \Theta\), denote \(L(\theta) = -\frac{1}{2} \mathbb{E}[q_0(\theta)]\). According to the proof of Theorem 3.1 of [4], we get

\[
\frac{1}{n} \left( \hat{L}_n(\hat{\theta}(m)) - \hat{L}_n(\hat{\theta}(m^*)) = L(\theta^*(m)) - L(\theta^*) + o(1) \text{ a.s.}.
\]

Note that, from [3], the function \(L : \Theta \rightarrow \mathbb{R}\) has a unique maximum at \(\theta^*\). Since \(m \nless m^*\), it holds that \(\theta^* \notin \Theta(m)\). Hence, \(L(\theta^*(m)) - L(\theta^*) < 0\) a.s.. Thus, according to \((24)\) and since \(\kappa_n / n \xrightarrow{n \rightarrow \infty} 0\), we get

\[
\lim_{n \rightarrow \infty} \frac{1}{n} (\hat{C}(m^*) - \hat{C}(m)) < 0 \text{ a.s. and } \hat{C}(m) - \hat{C}(m^*) > 0 \text{ a.s. for large } n.
\]

(25)

Thus, the strong consistency of \(\hat{m} = \arg \min_{m \in \mathcal{M}} \hat{C}(m) = \arg \min_{m \in \mathcal{M}} (\hat{C}(m) - \hat{C}(m^*))\) follows from \((23)\) and \((25)\).
Proof of Corollary 3.1

According to the proof of Theorem 3.1 (equations (23) and (25)) it holds that \( \hat{m} = m^* \) a.s. for large \( n \). Thus, the corollary follows from Proposition 3.1.

References

[1] Bardet, J.-M., Boularouk, Y., and Djaballah, K. Asymptotic behavior of the laplacian quasi-maximum likelihood estimator of affine causal processes. Electronic journal of statistics 11, 1 (2017), 452–479.

[2] Bardet, J.-M., Kamila, K. and Kengne, W. Consistent model selection criteria and goodness-of-fit test for common time series models. Electronic Journal of Statistics 14, (2020), 2009–2052.

[3] Bardet, J.M. and Kengne, W. Monitoring procedure for parameter change in causal time series. Journal of Multivariate Analysis 125, (2014), 204-221.

[4] Bardet, J.-M., Kengne, W., and Wintenberger, O. Detecting multiple change-points in general causal time series using penalized quasi-likelihood. Electronic journal of statistics 6 (2012), 435–477.

[5] Bardet, J.-M., and Wintenberger, O. Asymptotic normality of the quasi-maximum likelihood estimator for multidimensional causal processes. The Annals of Statistics 37, 5B (2009), 2730–2759.

[6] Ding, J., Tarokh, V., and Yang, Y. Model selection techniques: An overview. IEEE Signal Processing Magazine 35, 6 (2018), 16–34.

[7] Doukhan, P., and Wintenberger, O. Weakly dependent chains with infinite memory. Stochastic Processes and their Applications 118, 11 (2008), 1997–2013.

[8] Hannan, E. J. The estimation of the order of an ARMA process. The Annals of Statistics 8, 5 (1980), 1071–1081.

[9] Hannan, E. J. and Deistler, M. The statistical theory of linear systems. SIAM (2012).

[10] Hsu, H.-L., Ing, C.-K., and Tong, H. On model selection from a finite family of possibly misspecified time series models. The Annals of Statistics 47, 2 (2019), 1061–1087.

[11] Kengne W. Testing for parameter constancy in general causal time-series models. J. Time Ser. Anal. 33, (2012), 503-518.

[12] Kounias, E., and Weng, T. An inequality and almost sure convergence. The Annals of Mathematical Statistics 40, 3 (1969), 1091–1093.

[13] McQuarrie, A., and Tsai, C. Regression and Time Series Model Selection. World Scientific Pub Co Inc, 1998.

[14] Rao, C. R. and Wu, Y. On model selection. IMS Lecture Notes-Monograph Series 38 (2001), 1–64.
[15] Resende, P. A. A. and Dorea, C. C. Y. Model identification using the efficient determination criterion. *Journal of Multivariate Analysis* 150, (2016), 229–244.

[16] Stout, W. F. The Hartman-Wintner law of the iterated logarithm for martingales. *The Annals of Mathematical Statistics* 41 (1970), 2158–2160.

[17] Stout, W. F. Almost sure convergence. *Academic press* (1974).

[18] Zhao, L. C., Dorea, C. C. Y. and Gonçalves, C. R. On determination of the order of a Markov chain. *Statistical inference for stochastic processes* 4 (2001), 273–282.