Connectedness of levels for moment maps on various classes of loop groups

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Dedicated to Jost-Hinrich Eschenburg on his sixtieth birthday

Abstract. The space $\Omega(G)$ of all based loops in a compact simply connected Lie group $G$ has an action of the maximal torus $T \subset G$ (by pointwise conjugation) and of the circle $S^1$ (by rotation of loops). Let $\mu : \Omega(G) \to (t \times i\mathbb{R})^*$ be a moment map of the resulting $T \times S^1$ action. We show that all levels (that is, pre-images of points) of $\mu$ are connected subspaces of $\Omega(G)$ (or empty). The result holds if in the definition of $\Omega(G)$ loops are of class $C^\infty$ or of any Sobolev class $H^s$, with $s \geq 1$ (for loops of class $H^1$ connectedness of regular levels has been proved by Harada, Holm, Jeffrey, and the author in [3]).

1. Introduction

Let $G$ be a compact simply connected Lie group and $T \subset G$ a maximal torus. The based loop group of $G$ is the space $\Omega(G)$ consisting of all smooth maps $\gamma : S^1 \to G$ with $\gamma(1) = e$. The assignments

$$T \times \Omega(G) \to \Omega(G), \quad (t, \gamma) \mapsto [S^1 \ni z \mapsto t \gamma(z) t^{-1}]$$

and

$$S^1 \times \Omega(G) \to \Omega(G), \quad (e^{i\theta}, \gamma) \mapsto [S^1 \ni z \mapsto \gamma(ze^{i\theta}) \gamma(e^{i\theta})^{-1}]$$

define an action of $T \times S^1$ on $\Omega(G)$. In fact, the latter space is an infinite dimensional smooth symplectic manifold and the action of $T \times S^1$ is Hamiltonian. Let

$$\mu : \Omega(G) \to (t \oplus i\mathbb{R})^*$$

denote a moment map, where $t := \text{Lie}(T)$ and $i\mathbb{R} = \text{Lie}(S^1)$. Atiyah and Pressley [1] extended the celebrated convexity theorem of Atiyah and Guillemin-Sternberg and showed that the image of $\mu$ is the convex hull of its singular values. Their proof’s idea is to determine first the image under $\mu$ of the subspace $\Omega_{\text{alg}}(G) \subset \Omega(G)$ whose elements are restrictions of algebraic maps from $\mathbb{C}^*$ to the complexification $G^\mathbb{C}$ of $G$: they notice that $\mu(\Omega_{\text{alg}}(G))$ is a closed subspace of $(t \oplus i\mathbb{R})^*$; since $\Omega_{\text{alg}}(G)$ is dense in $\Omega(G)$ (by a theorem of Segal), they deduce from the continuity of $\mu$ that

$$\mu(\Omega(G)) = \mu(\Omega_{\text{alg}}(G)).$$

The goal of this paper is to extend to $\Omega(G)$ the well-known result which says that all levels of the moment map arising from a Hamiltonian torus action on a compact symplectic manifold are connected. That is, we will prove the following theorem.

Theorem 1.1. For any $a \in \mu(\Omega(G))$, the pre-image $\mu^{-1}(a)$ is a connected topological subspace of $\Omega(G)$.

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Remarks. 1. A version of Theorem 1.1 has been proved in [3]. More specifically, instead of \( \Omega(G) \) the authors consider there the space \( \Omega_1(G) \) of all loops \( S^1 \to G \) of Sobolev class \( H^1 \). They prove that all regular levels of \( \mu : \Omega_1(G) \to (t \oplus \mathbb{R})^* \) are connected. It is not obvious how to adapt that proof for singular levels or/and for loops of class \( C^\infty \).

2. One can easily see that our proof of Theorem 1.1 works for \( \mu : \Omega_1(G) \to (t \oplus \mathbb{R})^* \) (and even loops of Sobolev class \( H^s \), with \( s \geq 1 \)) as well. In other words, we can prove that all levels of \( \mu : \Omega_1(G) \to (t \oplus \mathbb{R})^* \) are connected topological subspaces of \( \Omega_1(G) \). We decided to deal here with \( \Omega(G) \) (smooth loops) rather than \( \Omega_1(G) \) because the former is discussed in detail in our main reference [11], and the reader can make the connections directly.

We will give here an outline of the paper. In section 2 we present basic notions and results concerning loop groups. In section 3 we define the key ingredient of the proof of Theorem 1.1. This is a certain Geometric Invariant Theory (shortly G.I.T.) quotient of \( \Omega(G) \) with respect to the complexification \( T^C \times \mathbb{C}^* \) of \( T \times S^1 \). To define this quotient, we face the difficulty that the \( S^1 \) action on \( \Omega(G) \) mentioned above does not extend to a \( \mathbb{C}^* \) action (only the \( T \) action extends canonically to a \( T^C \) action). However, for any \( \gamma \in \Omega(G) \) there is a natural way to define the loop \( u\gamma \) for any \( u \in \mathbb{C} \) which is contained in the exterior of a disk with center at 0 and radius smaller than 1 (which depends on \( \gamma \)); if \( |u| = 1 \) then \( u\gamma \) is given by the \( S^1 \) action on \( \Omega(G) \) defined above. By putting \( \gamma \sim gu\gamma \), where \( u \) is as before and \( g \in T^C \) arbitrary, we obtain an equivalence relation \( \sim \) on \( \Omega(G) \). The G.I.T. quotient mentioned before is \( A/ \sim \), where \( A \) consists of all elements of \( \Omega(G) \) which are equivalent to elements of \( \mu^{-1}(a) \). The main result of section 3 is Proposition 3.5, which says that the natural map \( \mu^{-1}(a)/(T \times S^1) \to A/ \sim \) is bijective (the idea of the proof belongs to Kirwan, see [3, chapter 7]). In section 4 we note that the image of \( (\mu^{-1}(a) \cap \Omega_\text{alg}(G))/(T \times S^1) \) under the map above is \( (A \cap \Omega_\text{alg}(G))/ \sim \). The former space is connected (by a result of [3]) and we prove that the latter is dense in \( A/ \sim \) (see Proposition 4.2). Consequently, \( A/ \sim \) is a connected topological subspace of \( \Omega(G)/ \sim \). We deduce that \( \mu^{-1}(a)/T \times S^1 \) is connected. Hence \( \mu^{-1}(a) \) is connected as well.

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2. Notions of loop groups

In this section we collect results about loop groups which will be needed later. The details can be found in Pressley and Segal [11] and/or Atiyah and Pressley [1].

Like in the introduction, \( G \) is a compact semisimple simply connected Lie group. We denote by \( L(G) \) the space of all smooth maps \( S^1 \to G \) (call them loops). The obvious multiplication makes it into a Lie group. By \( \Omega(G) \) we denote the space of all loops which map 1 ∈ \( S^1 \) to the unit \( e \) of \( G \). It can be naturally identified with the homogeneous space \( L(G)/G \). In fact, the presentation of \( \Omega(G) \) which is most appropriate for our goals is

\[
\Omega(G) = L(G^C)/L^+(G^C).
\]

Here \( G^C \) is the complexification of \( G \) and \( L(G^C) \) the set of all (smooth) loops \( \alpha : S^1 \to G^C \); by \( L^+(G^C) \) we denote the subgroup of \( L(G^C) \) consisting of all \( \alpha \) as above which extend
holomorphically for $|\zeta| \leq 1$ (this notion is explained in detail at the beginning of the next section). Since $L(G^C)$ is a complex Lie group and $L^+(G^C)$ a complex Lie subgroup, equation (1) shows that the manifold $\Omega(G)$ has a complex structure. More precisely, the complex structure $J_x$ at a point $x \in \Omega(G)$ is induced by the multiplication by $i$ in the tangent space $T_x L(G^C)$, where $\alpha \in L(G^C)$ is such that $x = \alpha L^+(G^C)$.

Let us embed $G$ into some special unitary group $SU(N)$. We consider the Hilbert space $H := L^2(S^1, \mathbb{C}^N)$ and the corresponding “Grassmannian” $Gr(H)$. The latter consists of all closed vector subspaces of $H$ which satisfy certain supplementary properties; it turns out that $Gr(H)$ can be equipped with a Kähler (Hilbert) manifold structure (the details can be found in [11, chapter 7]). An important subspace of $Gr(H)$ is $Gr_0(H)$. For the goals of our paper it is sufficient to mention that $Gr_0(H)$ contains $H^+$, which is the closed vector subspace of $H$ spanned by $S^1 \ni z \mapsto z^k v$, with $k \geq 0$ and $v \in \mathbb{C}^N$. Also, the connected component of $H^+$ in $Gr_0(H)$ consists of all vector subspaces $W$ of $H$ for which there exists $n \geq 0$ such that

$$z^n H^+ \subset W \subset z^{-n} H^+$$

and

$$\dim[(z^{-n} H^+)/W] = \dim[W/(z^n H^+)].$$

In other words, if $\mathcal{G}_n$ denotes the subspace of all $W$ which satisfy the last two equations, then the connected component of $H^+$ in $Gr_0(H)$ is $\bigcup_{n \geq 0} \mathcal{G}_n$. It is important to note that via the map

$$\mathcal{G}_n \ni W \mapsto W/z^n H^+,$$

the space $\mathcal{G}_n$ can be identified with the Grassmannian $Gr_{nN}(\mathbb{C}^{2nN})$ of all vector subspaces of dimension $nN$ in

$$\mathbb{C}^{2nN} = z^{-n} H^+/z^n H^+. \tag{2}$$

Also note that we have the chain of inclusions

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots. \tag{3}$$

Less obvious is the fact that for any $n \geq 0$, the canonical symplectic structure on the Grassmannian $\mathcal{G}_n$ makes this space into a symplectic submanifold of $Gr(H)$. The role of the above construction is revealed by the following result.

**Proposition 2.1.** (a) The map

$$\Omega(G) \to Gr(H), \ \gamma \mapsto \gamma H^+ \tag{4}$$

is an embedding, which induces on $\Omega(G)$ a structure of symplectic manifold. Together with the complex structure $J$ defined above, this makes $\Omega(G)$ into a Kähler manifold.

(b) The image of $\Omega_{\text{alg}}(G)$ (see the introduction) under the embedding (7) is contained in $\bigcup_{n \geq 0} \mathcal{G}_n$.

Based on point (b), we identify $\Omega_{\text{alg}}(G)$ with a subspace of $\bigcup_{n \geq 0} \mathcal{G}_n$. The inclusions (3) induce the filtration

$$\Omega_{\text{alg}}(G) = \bigcup_{n \geq 0} \Omega_n, \ \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots,$$
where
\[ \Omega_n := \Omega_{\text{alg}}(G) \cap \mathcal{G}_n. \]
The space \( \Omega_n \) is a closed subvariety of the Grassmannian \( \mathcal{G}_n \). We refer to the topology on \( \Omega_{\text{alg}}(G) \) induced by the filtration above as the \textit{direct limit topology}. There is another natural topology on \( \Omega_{\text{alg}}(G) \), namely the subspace topology, induced by the inclusion \( \Omega_{\text{alg}}(G) \subset \Omega(G) \).

The following proposition can be proved with the same arguments as Proposition 2.1 of [3] (the result is also mentioned in [2, section 2]).

**Proposition 2.2.** The direct limit topology on \( \Omega_{\text{alg}}(G) \) is finer than the subspace topology.

Let us consider again the \( T \times S^1 \) action on \( \Omega(G) \) described at the beginning of the paper, and the corresponding moment map \( \mu : \Omega(G) \to (t \oplus i\mathbb{R})^* \). This is uniquely determined up to an additive constant, which will be made more precise momentarily (a standard moment map is described explicitly in [1, section 3], but we will not need that expression here). For the moment, we would like to deduce from Proposition 2.2 a result which will be useful later. Namely, let us take \( a \in \mu(\Omega(G)) \); by [3, Proposition 3.4], \( \mu^{-1}(a) \cap \Omega_{\text{alg}}(G) \) is a connected subspace of \( \Omega_{\text{alg}}(G) \) with respect to the direct limit topology. We deduce:

**Proposition 2.3.** For any \( a \in \mu(\Omega_{\text{alg}}(G)) \), the space \( \mu^{-1}(a) \cap \Omega_{\text{alg}}(G) \) is a connected topological subspace of \( \Omega(G) \).

There is also an action of \( T \times S^1 \) on each \( \mathcal{G}_n, n \geq 0 \), which can be described as follows. We fix a basis, say \( b_1, \ldots, b_N \), of \( \mathbb{C}^N \), and consider the induced basis \( z^k b_j, -n \leq k \leq n - 1, 1 \leq j \leq N \), of \( \mathbb{C}^{2nN} \) (see equation (2)). The action of \( T \) on \( \mathcal{G}_n \) is induced by
\[ t.(z^k b_j) := z^k (tb_j), \]
for any \( t \in T \) and \( k, j \) as above; the action of \( S^1 \) is induced by
\[ e^{i\theta} (z^k b_j) := (e^{i\theta} z)^k b_j = z^k e^{ik\theta} b_j \]
for all \( e^{i\theta} \in S^1 \). This \( T \times S^1 \) action is the restriction of an obvious \( T^C \times \mathbb{C}^* \) action: namely, in equation (4) we take \( t \in T^C \) and in equation (6) we replace \( e^{i\theta} \) by an arbitrary element of \( \mathbb{C}^* \). The \( T^C \times \mathbb{C}^* \) action turns out to be linear with respect to the Plücker embedding of \( \mathcal{G}_n \) (see [1, section 4]). Thus, the \( T \times S^1 \) action is Hamiltonian. We pick
\[ \mu_n : \mathcal{G}_n \to (t \oplus i\mathbb{R})^* \]
a moment map, which is again uniquely determined up to an additive constant. We can arrange the constants in such a way that if \( m < n \) then
\[ \mu_n|\mathcal{G}_m = \mu_m. \]
The reason is that \( \mathcal{G}_m \) is a \( T \times S^1 \)-invariant symplectic submanifold of \( \mathcal{G}_n \). We obtain the map \( \tilde{\mu} : \bigcup_{n \geq 0} \mathcal{G}_n \to (t \oplus i\mathbb{R})^* \) such that \( \tilde{\mu}|_{\mathcal{G}_n} = \mu_n \), for all \( n \geq 0 \). The map \( \tilde{\mu} \) is uniquely determined up to an additive constant. The following proposition relates the moment maps \( \mu \) and \( \tilde{\mu} \).

**Proposition 2.4.** We can choose \( \mu \) and \( \tilde{\mu} \) such that
\[ \mu|_{\Omega_{\text{alg}}(G)} = \tilde{\mu}|_{\Omega_{\text{alg}}(G)}. \]
Proof. The idea of the proof is that there exists a submanifold $Gr_\infty(H)$ of $Gr(H)$ acted on smoothly by $T \times S^1$ and such that

- $Gr_0(H) \subset Gr_\infty(H)$ and the inclusion is $T \times S^1$ equivariant
- there exists $\mu : Gr_\infty(H) \to t \oplus i\mathbb{R}$ which is a moment map for the $T \times S^1$ action
- the image of $\Omega(G)$ under the inclusion \([\Pi]\) is contained in $Gr_\infty(H)$.

It is worth noticing that $Gr(H)$ does not admit a smooth action of $T \times S^1$; only its subspace $Gr_\infty(H)$ does (see [11, section 7.6]). This is why we need to use the latter space in our proof.

We deduce that $\hat{\mu}|_{\Omega(G)}$ differs from $\mu$ by a constant; the same can be said about $\hat{\mu}|_{\mathcal{G}_n}$ and $\mu_n$, for any $n \geq 0$. The result follows. \(\square\)

Let us consider again the $T^\mathbb{C} \times \mathbb{C}^*$ action on $\mathcal{G}_n$ defined above. Any of the inclusions $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ is equivariant. Thus, we have an action of $T^\mathbb{C} \times \mathbb{C}^*$ on $\bigcup_{n \geq 0} \mathcal{G}_n$. The same group acts on $\Omega_{\text{alg}}(G)$, as follows. We take into account that

\begin{equation}
\Omega_{\text{alg}}(G) = L_{\text{alg}}(G^\mathbb{C})/L_{\text{alg}}^+(G^\mathbb{C})
\end{equation}

where $L_{\text{alg}}(G^\mathbb{C})$ is the space of all algebraic maps $\alpha : \mathbb{C}^* \to G^\mathbb{C}$ and $L_{\text{alg}}^+(G^\mathbb{C})$ the subgroup consisting of those $\alpha$ which can be extended holomorphically to $\mathbb{C}$. Then the action we are referring to is

\begin{equation}
T^\mathbb{C} \times \mathbb{C}^* \times \Omega_{\text{alg}}(G) \ni (g, u, \alpha L_{\text{alg}}^+(G^\mathbb{C})) \mapsto [\mathbb{C}^* \ni \zeta \mapsto g\alpha(u\zeta)]L_{\text{alg}}^+(G^\mathbb{C}).
\end{equation}

The following result will be needed later.

**Proposition 2.5.** The inclusion $\Omega_{\text{alg}}(G) \subset \bigcup_{n \geq 0} \mathcal{G}_n$ defined in Proposition 2.1 (b) is $T^\mathbb{C} \times \mathbb{C}^*$ equivariant.

Proof. Take $\gamma \in \Omega_{\text{alg}}(G)$, which is of the form

$$S^1 \ni z \mapsto \gamma(z) = \sum_{-k_0 \leq k \leq k_0} A_k z^k,$$

where $k_0 \geq 0$. Here $A_k$ are $N \times N$ matrices with entries in $\mathbb{C}$. The subspace $\gamma H_+$ of $H$ has the property that

$$z^n H_+ \subset \gamma H_+ \subset z^{-n} H_+,$$

for some $n \geq 0$. Any element $v$ of $H_+$ has a Fourier expansion of the form $v = \sum_{m \geq 0} v_m z^m$, where $v_m \in \mathbb{C}^N$, for all $m \geq 0$. Then

$$\gamma v = \sum_{m \geq 0, k \in \mathbb{Z}} (A_k v_m) z^{k+m}.$$

The corresponding element of $(\gamma H_+)/z^n H_+$ is

$$[\gamma v] = \gamma v \mod z^n H_+ = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} (A_k v_m) z^{k+m}.$$

In this sum we have $m \leq n-k-1 \leq n+k_0-1$. Thus, to describe all elements of $\gamma H_+/z^m H_+$, it is sufficient to take $v$ of the form

$$v = \sum_{0 \leq m \leq n+k_0-1} v_m z^m.$$
If \( t \in T_C \), then
\[
t[\cdot \gamma v] = \sum_{m \geq 0, -k_0 \leq k_0, k + m \leq n - 1} t(A_k v_m) z^{k+m} = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k + m \leq n - 1} (tA_k) v_m z^{k+m} = [(t, \gamma) v].
\]
Consequently, \( t.(\gamma H) = (t, \gamma) H \). If \( u \in C^* \), then
\[
u[\cdot \gamma v] = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k + m \leq n - 1} (A_k v_m) (uz) z^{k+m} = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k + m \leq n - 1} (u^k A_k u^m v_m) z^{k+m}.
\]
This is the same as \([(u, \gamma) \tilde{v}]\), where
\[
\tilde{v} = \sum_{0 \leq m \leq n + k_0 - 1} u^m v_m z^m.
\]
Consequently, \( u.(\gamma H) = (u, \gamma) H \). □

Finally, let us pick \( B \subset G^C \) a Borel subgroup with \( T \subset B \). The presentation of \( \Omega_{\text{alg}}(G) \) allows us to define on the latter space a natural action of the group
\[
\mathcal{B}_+ := \{ \alpha \in L^+_\text{alg}(G^C) : \alpha(0) \in B \}
\]
on \( \Omega_{\text{alg}}(G) \). The orbit decomposition is
\[
\Omega_{\text{alg}}(G) = \bigcup_{\lambda \in \hat{T}} C_\lambda,
\]
where the union is disjoint and
\[
C_\lambda := \mathcal{B}_+ \lambda
\]
is called a Bruhat cell. Here \( \hat{T} \) denotes the lattice of group homomorphisms \( S^1 \rightarrow T \). The space \( C_\lambda \) is really a (finite dimensional) cell, being homeomorphic to \( C^r \) for some \( r \). In this paper, by \( \overline{C_\lambda} \) we will always mean the closure of \( C_\lambda \) in the direct limit topology (see above). The following property of the Bruhat cells will be needed later.

**Proposition 2.6.** For any \( \lambda \in \hat{T} \), there exists \( n \geq 1 \) such that \( \overline{C_\lambda} \) is contained in \( G_n \) as a \( T^C \times C^* \)-invariant closed subvariety.

This can be proved as follows. There exists \( n \geq 1 \) such that \( C_\lambda \subset G_n \), because \( \mathcal{B}_+ \) leaves each \( \Omega_k \), \( k \geq 0 \), invariant (see [6] Lemma 3.3.2]). The space \( C_\lambda \) is a locally Zariski closed subspace of \( \Omega_{\text{alg}}(G) \) (see [8] Proposition 2.13 and Theorem 3.1]), thus also of \( \Omega_n \) and of \( G_n \). Consequently, the closures of \( C_\lambda \) in the Zariski, respectively differential topology of \( G_n \) are equal.

Another result concerning the Bruhat cells is the following proposition (cf. [7] section 1], see also [3] proof of Proposition 3.4]).

**Proposition 2.7.** For any \( \lambda_1, \lambda_2 \in \hat{T} \) there exists \( \lambda \in \hat{T} \) such that
\[
C_{\lambda_1} \subset \overline{C_\lambda} \text{ and } C_{\lambda_2} \subset \overline{C_\lambda}.
\]
Consequently, for any \( x, y \in \Omega_{\text{alg}}(G) \) there exists \( \lambda \in \hat{T} \) such that both \( x \) and \( y \) are in \( \overline{C_\lambda} \).
3. The equivalence relation ~

We begin with the following definition. Take $0 < r \leq 1$. We say that a free loop $S^1 \to G^\mathbb{C}$ extends holomorphically for $|\zeta| \geq r$ if it is the restriction of a map
\[
\alpha : \{ \zeta \in \mathbb{C} \cup \infty : |\zeta| \geq r \} \to G^\mathbb{C}
\]
which is continuous, holomorphic on $\{ \zeta \in \mathbb{C} \cup \infty : |\zeta| > r \}$ and smooth on $\{ \zeta \in \mathbb{C} : |\zeta| = r \}$: the same terminology is adopted if we take $r \geq 1$ and replace “$>$” and “$>$” by “$\leq$”, respectively “$<$” (and also $\mathbb{C} \cup \infty$ by $\mathbb{C}$).

Let $L^-(G^\mathbb{C})$ denote the subspace of $L(G^\mathbb{C})$ consisting of those $\alpha$ which extend holomorphically for $|\zeta| \geq 1$ in the sense of the definition above. One knows that any $\alpha \in L(G^\mathbb{C})$ can be written as
\[
\alpha = \alpha_- \lambda \alpha_+,
\]
where $\alpha_- \in L^-(G^\mathbb{C})$, $\alpha_+ \in L^+(G^\mathbb{C})$, and $\lambda$ is a group homomorphism $S^1 \to T$ (see [11, Theorem 8.1.2]). By using the presentation (1), the elements of $\Omega(G)$ are cosets of the form $\alpha_- \lambda L^+(G^\mathbb{C})$, where $\alpha_-$ and $\lambda$ are as above. The following lemma will be used later.

**Lemma 3.1.** Take $\alpha_-, \beta_-$ in $L^-(G^\mathbb{C})$ and $\lambda, \mu : S^1 \to T$ group homomorphisms such that
\[
\alpha_- \lambda L^+(G^\mathbb{C}) = \beta_- \mu L^+(G^\mathbb{C}).
\]
Let $r$ be a strictly positive real number.

(a) Assume that $r < 1$. If $\alpha_-$ extends holomorphically for $|\zeta| \geq r$ then $\beta_-$ extends holomorphically for $|\zeta| \geq r$ as well.

(b) Assume that $r \geq 1$ or $r < 1$ and $\alpha_-$ extends holomorphically for $|\zeta| \geq r$. For any $u \in \mathbb{C}^\ast$ with $|u| \geq r$ we have
\[
[S^1 \ni z \mapsto \alpha_-(uz)] \lambda L^+(G^\mathbb{C}) = [S^1 \ni z \mapsto \beta_-(uz)] \mu L^+(G^\mathbb{C}).
\]

**Proof.** We have
\[
(9) \quad \alpha_- \lambda = \beta_- \mu \alpha_+,
\]
where $\alpha_+ \in L^+(G^\mathbb{C})$.

(a) The loops $\lambda$ and $\mu$ are one-parameter subgroups in $T$, thus they have obvious (holomorphic) extensions to group homomorphisms $\mathbb{C}^\ast \to T^\mathbb{C}$. From (9) we deduce that $\beta_-$ is the restriction of a function holomorphic on the annulus
\[
\{ \zeta \in \mathbb{C} : r < |\zeta| < 1 \}
\]
and continuous on the closure of this space. Consequently, the map $\xi \mapsto \beta_-(\xi)$ extends holomorphically for $|\xi| \leq \frac{1}{r}$, that is, $\beta_-$ extends holomorphically for $|\zeta| \geq r$. Indeed, let us consider again the embedding $G \subset SU(N)$, as in section 2. The resulting embedding $G^\mathbb{C} \subset \text{Mat}^{N \times N}(\mathbb{C})$ is holomorphic. We use the following claim:

**Claim.** If $f : \{ \xi \in \mathbb{C} : |\xi| \leq \frac{1}{r} \} \to \mathbb{C}$ is a continuous function which is holomorphic on $\{ \xi \in \mathbb{C} : |\xi| < \frac{1}{r}, |\xi| \neq 1 \}$, then $f$ is holomorphic on $\{ \xi \in \mathbb{C} : |\xi| < \frac{1}{r} \}$.

This can be proved by comparing the Laurent series of $f$ on $\{ \xi \in \mathbb{C} : |\xi| < 1 \}$, respectively $\{ \xi \in \mathbb{C} : 1 < |\xi| < \frac{1}{r} \}$. The series are equal, since the coefficients of both of them are
equal to $\frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\xi^k} d\xi$, $k \in \mathbb{Z}$ (by a uniform continuity argument). Thus, the radius of convergence of the first of the two series (which is actually the Taylor series of $f$ around 0) is at least equal to $\frac{1}{r}$. The claim is proved.

(b) From equation (9) we deduce that $\alpha_+$ extends holomorphically to $\mathbb{C}$. The reason is that the entries of the $N \times N$ matrix $\alpha_+ = \mu^{-1} \beta_+ \alpha_\lambda$ are $\mathbb{C}$-valued functions which are continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \{ \xi \in \mathbb{C} : |\xi| = 1 \}$; by the same argument as in the claim above, they are holomorphic on the whole $\mathbb{C}$. Again from equation (9), we deduce that

$$\alpha_-(uz)\lambda(uz) = \beta_-(uz)\mu(uz)\alpha_+(uz),$$

for all $z \in S^1$. The map $S^1 \ni z \mapsto \alpha_+(uz)$ is in $L^+(G^C)$. We only need to notice that

$$\lambda(uz) = \lambda(z)\lambda(u), \quad \mu(uz) = \mu(z)\mu(u).$$

Definition 3.2. (a) Take $x \in \Omega(G)$ and $u \in \mathbb{C}^*$. We say that the pair $(u, x)$ is admissible if

- $|u| \geq 1$

or

- $|u| < 1$ and $x = \alpha_\lambda L^+(G^C)$, where $\alpha_- \in L^{-}(G^C)$ extends holomorphically for $|\xi| \geq |u|$ and $\lambda : S^1 \to T$ is a group homomorphism.

If $(u, x)$ is as above and $g \in T^C$, then

$$gux := g[S^1 \ni z \mapsto \alpha_-(uz)]\lambda L^+(G^C)$$

is an element of $\Omega(G)$.

(b) Take $x, y \in \Omega(G)$. We say that

$$x \sim y$$

if there exist $u \in \mathbb{C}^*$ and $g \in T^C$ such that $(u, x)$ is an admissible pair and $y = gux$.

Remark. We can also express $gux$ as

$$gux := g[S^1 \ni z \mapsto (\alpha_-\lambda)(uz)]L^+(G^C),$$

because $\lambda$ is a group homomorphism $\mathbb{C}^* \to T^C$.

Note that by Lemma 3.1, the definition of $gux$ in part (a) is independent of the choice of the representative $\alpha_-\lambda$ of $x \in L(G^C)/L^+(G^C)$. The following lemma shows that $\sim$ is an equivalence relation.

Lemma 3.3. (a) If $x \in \Omega(G)$, $u \in \mathbb{C}^*$ and $g \in T^C$ such that $(u, x)$ is admissible, then $(u^{-1}, gux)$ is admissible and we have

$$g^{-1}u^{-1}(gux) = x.$$

(b) If $x \in \Omega(G)$, $u_1, u_2 \in \mathbb{C}^*$, and $g_1, g_2 \in T^C$ such that $(u_1, x)$ and $(u_2, g_1u_1x)$ are admissible, then $(u_1u_2, x)$ is admissible and

$$(g_1g_2)(u_1u_2)x = g_2u_2(g_1u_1x).$$
Lemma 3.4. Section 1.

Proof. (a) We can assume that \( g = 1 \). We write \( x = \alpha_-\lambda L^+(G^C) \). Assume first that \(|u| \geq 1\). The loop \( S^1 \ni z \mapsto \alpha_-(uz) \) extends holomorphically for \(|\zeta| \geq \frac{1}{|u|} \) by \( \zeta \mapsto \alpha_-(u\zeta) \). The case \(|u| < 1\) is even easier to analyze. Verifying that \( u^{-1}(ux) = x \) is equally easy.

(b) We can assume that \( g_1 = g_2 = 1 \). Again we write \( x = \alpha_-\lambda L^+(G^C) \). It is sufficient to analyze the case when \(|u_1u_2| < 1\). Thus, at least one of the numbers \(|u_1|\) and \(|u_2|\) is strictly less than 1. We distinguish the following two cases.

Case 1. \(|u_2| < 1\). The loop \( S^1 \ni z \mapsto \alpha_-(u_1z) \) is well defined and extends holomorphically for \(|\xi| \geq |u_2|\). Let \( \hat{\alpha} : \{\xi \in \mathbb{C} \cup \infty : |\xi| \geq |u_2|\} \rightarrow G^C \) be an extension of this loop. The map \( \hat{\alpha} : \{\zeta \in \mathbb{C} \cup \infty : |\zeta| \geq |u_1u_2|\} \rightarrow G^C \) given by

\[
\hat{\alpha}(\zeta) = \begin{cases} 
\hat{\alpha}(u_1^{-1}\zeta), & \text{if } |u_1u_2| \leq |\zeta| \leq |u_1| \\
\alpha_-(\zeta), & \text{if } |u_1| \leq |\zeta| 
\end{cases}
\]

is the desired extension of \( \alpha_- \) for \(|\zeta| \geq |u_1u_2| \) (note that \( \hat{\alpha} \) is holomorphic on \(|\zeta| > |u_1u_2| \), since it is continuous and is holomorphic on the complement of the circle \( \{\zeta \in \mathbb{C} : |\zeta| = |u_1|\} \)).

Case 2. \(|u_2| \geq 1\). This implies \(|u_1| < 1\). We notice that the pair \((u_1, u_2x)\) is admissible: indeed, by hypothesis, the loop \( S^1 \ni z \mapsto \alpha_-(u_2z) \) extends holomorphically for \(|u_2\zeta| \geq |u_1|\), hence also for \(|\zeta| \geq |u_1|\). The pair \((u_2, x)\) is admissible too. From the result proved in case 1 we deduce that \((u_1u_2, x)\) is admissible.

The equation \( u_2(u_1x) = (u_1u_2)x \) is straightforward. \( \square \)

The following result relates the equivalence relation \( \sim \) to the \( T \times S^1 \) action on \( \Omega(G) \) (see section 1).

Lemma 3.4. Take \( \gamma \in \Omega(G) \). If \( \theta \in \mathbb{R} \), then the pair \((e^{i\theta}, \gamma)\) is admissible. If \( t \in T \), then the loop \( te^{i\theta}\gamma \) given by Definition 3.3 (b) can be expressed as

\[ te^{i\theta}\gamma = t\gamma^\theta t^{-1}. \]

Here the right-hand side is given by

\[ (t\gamma^\theta t^{-1})(z) = t\gamma(ze^{i\theta})\gamma(e^{i\theta})^{-1}t^{-1}, \]

for all \( z \in S^1 \).

Proof. There exist \( \alpha_- \in L^-(\mathbb{C}) \) and \( \lambda : S^1 \rightarrow T \) a group homomorphism such that the image of \( \gamma \) under the isomorphism \( {}^1 \mathbb{L} \) is \( \alpha_-\lambda L^+(\mathbb{C}) \). This means that

\[ \alpha_- \lambda = \gamma \alpha_+, \]

for some \( \alpha_+ \in L^+(\mathbb{C}) \). We deduce that for any \( z \in S^1 \) we have

\[ [t\alpha_-(ze^{i\theta})\lambda(z)]\lambda(e^{i\theta}) = [t\gamma^\theta(z)t^{-1}]t\gamma(e^{i\theta})\alpha_+(ze^{i\theta}). \]

In other words, via the isomorphism \( {}^1 \mathbb{L} \), to \( t\gamma^\theta t^{-1} \) corresponds the coset of

\[ t[S^1 \ni z \mapsto \alpha_-(ze^{i\theta})] \lambda, \]

which is the same as \( te^{i\theta}(\alpha_-\lambda L^+(\mathbb{C})) \). \( \square \)
We now denote by $A$ the set of all $x \in \Omega(G)$ with $x \sim y$, for some $y \in \mu^{-1}(a)$. We are interested in the quotient space $A/\sim$ and the (natural) map $\mu^{-1}(a)/(T \times S^1) \to A/\sim$ which assigns to the coset of $x \in \mu^{-1}(a)$ the equivalence class of $x$. By Lemma 3.4 this map is well defined.

**Proposition 3.5.** The natural map

$$\mu^{-1}(a)/(T \times S^1) \to A/\sim$$

is bijective.

**Proof.** Only the injectivity has to be proved. We have to show that if $x, y \in \mu^{-1}(a)$ with $x \sim y$, then $y = t e^{i\beta} x$, where $(t, e^{i\beta}) \in T \times S^1$. By Definition 3.2 we have

$$y = gux$$

for some $u \in \mathbb{C}^*$ and $g \in T^\mathbb{C}$. We write $g = \exp(w_1) \exp(iw_2)$ and $u = e^{i\alpha} e^{-\beta}$, where $w_1, w_2 \in \mathbb{t}$ and $\alpha, \beta \in \mathbb{R}$ (here we see $-\beta$ as $i(\beta)$). Since the pair $(u, x)$ is admissible and $|u| = |e^{-\beta}|$, the pair $(e^{-\beta}, x)$ is admissible too. By Lemma 3.3 (b) we have

$$y = \exp(w_1) e^{i\alpha} (\exp(iw_2) e^{-\beta} x).$$

Thus, it is sufficient to assume that

$$y = \exp(iw_2) e^{-\beta} x.$$

Moreover, without loss of generality we assume that

$$\beta \geq 0,$$

because if contrary we write $x = \exp(-iw_2) e^{\beta} y$ (by Lemma 3.3 (a)). Let us consider the function $h : [0, 1] \to \mathbb{R}$,

$$h(s) = \left[ \mu(\exp(isw_2) e^{-s\beta} x) - a \right] (w_2, i\beta),$$

where $0 \leq s \leq 1$. Notice that $h(s)$ is well defined for any $s$ with $0 \leq s \leq 1$: indeed, the pair $(e^{-\beta}, x)$ is admissible hence, because $e^{-s\beta} \geq e^{-\beta}$, the pair $(e^{-s\beta}, x)$ is admissible too. Since $\mu(x) = \mu(y)$, we have $h(0) = h(1) = 0$. Consequently, there exists $s_0$ in the interval $(0, 1)$ such that $h'(s_0) = 0$. We denote

$$(10) \quad x_0 := \exp(is_0 w_2) e^{-s_0 \beta} x.$$ 

**Claim.** We have

$$\frac{d}{ds} \bigg|_{s_0} \exp(isw_2) e^{-s\beta} x = J_{x_0} ((w_2, i\beta), x_0),$$

where $J_{x_0}$ is the complex structure at $x_0$ (see section 2) and

$$(w_2, i\beta), x_0 := \frac{d}{ds} \left[ \exp(sw_2) e^{is\beta} x_0 \right]$$

arising from the infinitesimal action of $T \times S^1$ on $\Omega(G)$.

The claim can be proved as follows. Write $x_0 = \alpha_- \lambda L^+(G^\mathbb{C})$, where $\alpha_- \in L^-(G^\mathbb{C})$ and $\lambda : S^1 \to T$ is a group homomorphism. By using Lemma 3.3 and the remark following Definition 3.2 we have

$$\exp(isw_2) e^{-s\beta} x = \exp(i(s - s_0)w_2) e^{-(s - s_0)\beta} x_0 = \exp(i(s - s_0)w_2) (\alpha_- \lambda)_{-(s - s_0)} L^+(G^\mathbb{C}).$$
Here we have denoted
\[(\alpha_{-\lambda})_{-(s-s_0)}(z) := (\alpha_{-\lambda})(e^{-(s-s_0)\beta}z)\]
for all \(s\) in the interval \((0,1)\) and all \(z \in S^1\). By the definition of the complex structure \(J\) (see section 2), it is sufficient to prove that
\[
(11) \quad \frac{d}{ds}_{s_0} \left[ \exp(i(s-s_0)w_2)(\alpha_{-\lambda})_{-(s-s_0)} \right] = i \frac{d}{ds}_{s_0} \left[ \exp(sw_2)(\alpha_{-\lambda})_{is} \right],
\]
where
\[(\alpha_{-\lambda})_{is}(z) := (\alpha_{-\lambda})(e^{is\beta}z)\]
for all \(s \in \mathbb{R}\) and all \(z \in S^1\). By using the Leibniz rule, the left-hand side of (11) is
\[
\frac{d}{ds}_{s_0} \left[ \exp(isw_2)(\alpha_{-\lambda})_{-s} \right] = i \frac{d}{ds}_{s_0} \left[ \exp(sw_2) \right] (\alpha_{-\lambda})_{-s} + i \frac{d}{ds}_{s_0} \left[ (\alpha_{-\lambda})_{is} \right].
\]
Here we have used that
\[
\frac{d}{ds}_{s_0} \left[ \exp(isw_2) \right] = i w_2 = i \frac{d}{ds}_{s_0} \left[ \exp(sw_2) \right]
\]
and also that
\[
\frac{d}{ds}_{s_0} (\alpha_{-\lambda})(e^{-s\beta}z) = i \frac{d}{ds}_{s_0} (\alpha_{-\lambda})(e^{is\beta}z),
\]
for all \(z \in S^1\) (the last equation follows from the fact that \(\alpha_{-\lambda}\) is holomorphic on the exterior of a closed disk with center at 0 and radius strictly smaller than 1). The claim is proved.

From the claim we deduce as follows:
\[
h'(s_0) = (d\mu)_{x_0} \left( \frac{d}{ds}_{s_0} \left[ \exp(isw_2)e^{-s\beta}x \right] \right)(w_2, i\beta)
\]
\[
= \omega_{x_0} \left( \frac{d}{ds}_{s_0} \left[ \exp(isw_2)e^{-s\beta}x \right] \right), (w_2, i\beta).x_0)
\]
\[
= \omega_{x_0} (J_{x_0}((w_2, i\beta).x_0), (w_2, i\beta).x_0)
\]
\[
= \langle (w_2, i\beta).x_0, (w_2, i\beta).x_0 \rangle,
\]
where \(\omega\) denotes the symplectic form and \(\langle \ , \ \rangle\) the Kähler metric on \(\Omega(G)\) (see Proposition 2.1). We deduce that
\[
(w_2, i\beta).x_0 = 0
\]
which, according to the claim above, implies that
\[
\frac{d}{ds}_{s_0} \exp(isw_2)e^{-s\beta}x_0 = 0.
\]
From this we deduce that
\[
(12) \quad \exp(isw_2)e^{-s\beta}x_0 = x_0
\]
Indeed, by using Lemma 3.3 we deduce that for any $s \geq 1$ we have
\[
\frac{d}{ds} |_{s=1} \exp(isw_2)e^{-s\beta}x_0 = \frac{d}{ds} |_{s=1} (\exp(isw_2)e^{-s\beta}) \exp(i(s-1)w_2)e^{-(s-1)\beta}x_0
\]
\[
= d(\exp(isw_2)e^{-s\beta})x_0(\frac{d}{ds}_{|s=1} (\exp(isw_2)e^{-s\beta}x_0))
\]
\[
= 0.
\]
Here we have used the (differential of the) map $\exp(isw_2)e^{-s\beta}: \Omega(G) \to \Omega(G)$ given by
\[
\gamma \mapsto \exp(isw_2)e^{-s\beta}\gamma,
\]
which is well defined, since $e^{-s\beta} \geq 1$.

By Lemma 3.3 (a), equation (12) implies that the pair $(e^{-s\beta}, x_0)$ is admissible for any $s \geq 0$; moreover, equation (12) holds for all $s \geq 0$ as well. We make $s = -s_0$ in (12) and deduce $x = x_0$; then we make $s = 1 - s_0$ and deduce $y = x_0$. We conclude
\[
x = y
\]
and the proof is finished.

**Remark.** Let $M$ be a compact Kähler manifold acted on by a complex Lie group $G$, which is the complexification of a compact Lie group $K$, in such a way that the action of $K$ on $M$ is Hamiltonian. Kirwan has proved that if $x, y \in M$ have the same image under the moment map and are on the same $G$ orbit, then they are on the same $K$-orbit (see Lemma 7.2). We have used above the idea of her proof. Kirwan’s result cannot be used directly in our context: first, $\Omega(G)$ is not a compact manifold; second, and most importantly, the $T \times S^1$ action on $\Omega(G)$ does not extend in any reasonable way to a $T^\mathbb{C} \times \mathbb{C}^*$ action. We are substituting this action by the equivalence relation $\sim$.

4. CONNECTEDNESS OF $A/\sim$ AND OF $\mu^{-1}(a)$

We start with the following proposition.

**Proposition 4.1.** (a) If $x \in \Omega_{\text{alg}}(G)$ then the pair $(u, x)$ is admissible (in the sense of Definition 3.2) for any $u \in \mathbb{C}^*$. The map
\[
T^\mathbb{C} \times \mathbb{C}^* \times \Omega_{\text{alg}}(G) \to \Omega_{\text{alg}}(G), \ (g, u, x) \mapsto gux
\]
is the action of $T^\mathbb{C} \times \mathbb{C}^*$ on $\Omega_{\text{alg}}(G)$ defined in section 2 (see equation (8)).

(b) The image of $(\mu^{-1}(a) \cap \Omega_{\text{alg}}(G))/(T \times S^1)$ under the map in Proposition 3.5 is $(A \cap \Omega_{\text{alg}}(G))/\sim$. The latter space is a connected topological subspace of $\Omega(G)/\sim$.

**Proof.** Point (a) follows from equations (7) and (8) and the remark following Definition 3.2. To prove the first assertion of (b), we only need to note that if $x \in \Omega_{\text{alg}}(G)$ and $y \in \Omega(G)$ such that $x \sim y$, then $y \in \Omega_{\text{alg}}(G)$. To prove the second assertion of (b), we note that the natural map
\[
(\mu^{-1}(a) \cap \Omega_{\text{alg}}(G))/(T \times S^1) \to \Omega(G)/\sim
\]
is continuous. We use Proposition 2.3.

The key result of this section is

**Proposition 4.2.** The subspace \((A \cap \Omega_{\text{alg}}(G))/\sim\) of \(A/\sim\) is dense (both spaces have the topology of subspace of \(\Omega(G)/\sim\)).

Combined with Proposition 4.1 (b), this implies

**Corollary 4.3.** The space \(A/\sim\) is a connected topological subspace of \(\Omega(G)/\sim\).

In turn, this implies the main result of the paper, as follows.

**Proof of Theorem 1.1.** The natural map

\[
\mu^{-1}(a)/(T \times S^1) \to \Omega(G)/\sim
\]

is continuous, one-to-one, and its image is \(A/\sim\) (by Proposition 3.5). Since \(A/\sim\) is connected (see the previous corollary), we deduce that \(\mu^{-1}(a)/(T \times S^1)\) is connected as well. Consequently, \(\mu^{-1}(a)\) is a connected topological subspace of \(\Omega(G)\). \(\square\)

The rest of the section is devoted to the proof of Proposition 4.2. First, if \(\lambda \in \check{T}\), we say that a point \(x \in C_\lambda\) is \((\mu - a)\)-semistable if

\[
(T^C \times \mathbb{C}^*)x \cap (\mu^{-1}(a) \cap C_\lambda) \neq \emptyset.
\]

Here the closure is taken in \(\Omega_{\text{alg}}(G)\) with respect to the direct limit topology. We may assume that \(C_\lambda\) is contained in the Grassmannian \(G_n\) as a \(T^C \times \mathbb{C}^*\)-invariant closed subvariety (see Proposition 2.6). Then \(x\) is \((\mu - a)\)-semistable if and only if it is \((\mu_n - a)\)-semistable in the usual sense, that is, if

\[
(T^C \times \mathbb{C}^*)x \cap (\mu_n^{-1}(a) \cap C_\lambda) \neq \emptyset
\]

(see for instance [5, chapter 7]). This follows immediately from the fact that \(\mu\) and \(\mu_n\) coincide on \(C_\lambda\), by Proposition 2.4. We denote by \(C_\lambda^{ss}\) the set of all semistable points in \(C_\lambda\). We also consider the set \(G_n^{ss}\) of all \((\mu_n - a)\)-semistable points in \(G_n\). We have

\[
C_\lambda^{ss} = C_\lambda \cap G_n^{ss}.
\]

The following description of the semistable set of \(C_\lambda\) will be needed later.

**Lemma 4.4.** We have

\[
A \cap C_\lambda = C_\lambda^{ss}.
\]

**Proof.** By Proposition 4.1 (b), we have

\[
A \cap \Omega_{\text{alg}}(G) = (T^C \times \mathbb{C}^*)\mu^{-1}(a) \cap \Omega_{\text{alg}}(G).
\]

Consequently, a point \(x \in \Omega(G)\) is in \(A \cap C_\lambda\) if and only if \(x \in [(T^C \times \mathbb{C}^*)\mu^{-1}(a)] \cap C_\lambda\). The latter set is obviously equal to \((T^C \times \mathbb{C}^*)\mu^{-1}(a) \cap C_\lambda\), which is the same as \(C_\lambda^{ss}\) (by [5] Theorems 7.4 and 8.10], applied for the Grassmannian \(G_n\) which contains \(C_\lambda\) as a \(T^C \times \mathbb{C}^*\)-invariant closed subvariety, as indicated above). \(\square\)
We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. We show that in any open subset $V$ of $A/\sim$ there exists an element of $(A \cap \Omega_{\text{alg}}(G))/\sim$. Since $A/\sim$ is equipped with the topology of subspace of $\Omega(G)/\sim$, we can write

$$V = (A/\sim) \cap (U/\sim) = (A \cap U)/\sim.$$  

Here $U$ is an open subspace of $\Omega(G)$ with the property that for any $x \in U$, we have

$$\{y \in \Omega(G) : y \sim x\} \subset U.$$

The subspace $U \cap \Omega_{\text{alg}}(G)$ of $\Omega_{\text{alg}}(G)$ is open in the direct limit topology (because the direct limit topology on $\Omega_{\text{alg}}(G)$ is finer than the subspace topology, see Proposition 2.2), and non-empty (because $\Omega_{\text{alg}}(G)$ is dense in $\Omega(G)$, see [11, section 3.5]). For any $x \in U \cap \Omega_{\text{alg}}(G)$ we have

$$\lambda \in \mathbb{T}$$

which follows from Proposition 4.1 (a). There exists $\lambda \in \mathbb{T}$ such that $C_\lambda \cap U \neq \emptyset$ and $\mu^{-1}(a) \cap C_\lambda \neq \emptyset$. Indeed, we can pick $x \in \Omega_{\text{alg}}(G) \cap U$ (the intersection is non-empty, see above) and $y \in \Omega_{\text{alg}}(G) \cap \mu^{-1}(a)$ (the intersection is non-empty, since $a \in \mu(\Omega(G)) = \mu(\Omega_{\text{alg}}(G))$); by Proposition 2.7, there exists $\lambda \in \mathbb{T}$ such that $x$ and $y$ are both in $C_\lambda$.

Claim. If $\lambda \in \mathbb{T}$ is as above, then $C_\lambda^{\text{ss}}$ is a dense subspace of $C_\lambda$ (here $C_\lambda$ is equipped with the direct limit topology it inherits from $\Omega_{\text{alg}}(G)$).

To prove the claim, we consider again a Grassmannian $G_n$ which contains $C_\lambda$ as a $T^C \times \mathbb{C}^*$-invariant closed subvariety. By the main theorem of [4], there exists on $G_n$ a $T^C \times \mathbb{C}^*$-invariant very ample line bundle $L$ such that $G_n^{\text{ss}} = G_n^{\text{ss}}(L)$. The latter space consists of all $L$-semistable points in $G_n$, that is points $x \in G_n$ such that there exists $k \geq 1$ and $s : X \rightarrow L^\otimes k$ equivariant holomorphic section with $s(x) \neq 0$ (cf. e.g. [10]). Consequently, $G_n^{\text{ss}}$ is a Zariski open subspace of $G_n$. Since $C_\lambda^{\text{ss}} = C_\lambda^{\text{ss}} \cap C_\lambda$, we deduce that $C_\lambda^{\text{ss}}$ is a Zariski open subspace of $C_\lambda$. The space $C_\lambda^{\text{ss}}$ is non-empty, since $\mu^{-1}(a) \cap C_\lambda \subset C_\lambda^{\text{ss}}$. Thus $C_\lambda^{\text{ss}}$ is dense in $C_\lambda$ with respect to the usual differential topology on the latter space: this can be deduced by using [9, Theorem 2.33] for $C_\lambda$, which is an irreducible projective variety (cf. [8, p. 360]).

From the claim we deduce that the intersection $C_\lambda^{\text{ss}} \cap U$ is non-empty (since $C_\lambda \cap U$ is a non-empty subspace of $C_\lambda$ which is open with respect to the direct limit topology). By Lemma 4.4 we have

$$C_\lambda^{\text{ss}} \cap U = A \cap C_\lambda \cap U,$$

thus

$$U = A \cap \Omega_{\text{alg}}(G) \neq \emptyset.$$

By equation (16), the quotient $(U \cap A \cap \Omega_{\text{alg}}(G))/\sim$ is a (non-empty) subspace of $\Omega(G)/\sim$. It is contained in both $V = (U \cap A)/\sim$ and $(A \cap \Omega_{\text{alg}}(G))/\sim$. Consequently, the intersection $V \cap [(A \cap \Omega_{\text{alg}}(G))/\sim]$ is non-empty. This finishes the proof.

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