Plane Waves and Spacelike Infinity

Donald Marolf* and Simon F. Ross†

* Physics Department, Syracuse University, Syracuse, New York 13244 USA
† Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, South Road, Durham DH1 3LE UK

Abstract: In an earlier paper, we showed that the causal boundary of any homogeneous plane wave satisfying the null convergence condition consists of a single null curve. In Einstein-Hilbert gravity, this would include any homogeneous plane wave satisfying the weak null energy condition. For conformally flat plane waves such as the Penrose limit of $AdS_5 \times S^5$, all spacelike curves that reach infinity also end on this boundary and the completion is Hausdorff. However, the more generic case (including, e.g., the Penrose limits of $AdS_4 \times S^7$ and $AdS_7 \times S^4$) is more complicated. In one natural topology, not all spacelike curves have limit points in the causal completion, indicating the need to introduce additional points at ‘spacelike infinity’—the endpoints of spacelike curves. We classify the distinct ways in which spacelike curves can approach infinity, finding a two-dimensional set of distinct limits. The dimensionality of the set of points at spacelike infinity is not, however, fixed from this argument. In an alternative topology, the causal completion is already compact, but the completion is non-Hausdorff.

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*E-mail: marolf@physics.syr.edu
†E-mail: S.F.Ross@durham.ac.uk
1. Introduction

The understanding of the asymptotic structure of spacetimes plays an important role in many problems in both classical and quantum gravity. In the past, attention was primarily concentrated on the formulation of suitable notions of asymptotic flatness and the exploration of their consequences. There has, however, recently been a renewal of interest in the careful investigation of the asymptotic structure of other solutions in the context of the holographic description of string theory. In particular, the proposed duality between string theory on a plane wave background and $\mathcal{N} = 4$ SYM [1] has made it important to understand the structure of these backgrounds.

A powerful technique for studying the asymptotic structure is to construct a suitable completion of the spacetime, adjoining some ideal points representing the asymptotic behaviour. An elegant method of constructing such a completion based on the causal structure was developed in [2, 3, 4, 5, 6]. This technique was applied to smooth homogeneous plane waves in [8], where it was shown that the causal boundary, as defined in [2, 3, 4, 5], of any homogeneous plane wave satisfying the null convergence condition\(^1\) is a single null curve. This generalised a result previously obtained for the special case of the maximally symmetric (attractive) plane wave in [7].

\(^1\)This is a purely geometric condition which does not refer to the dynamics of the theory. In the interesting special case of Einstein-Hilbert gravity, it is equivalent to the weak null energy condition.
Consideration of the plane wave and other examples has exposed some defects in the approach to defining the causal completion in terms of a quotient adopted in [2, 4, 5]. In [6], a new definition of the causal completion $\bar{M}$ for a general spacetime $M$ in terms of IP-IF pairs was proposed, and two new candidate topologies were introduced on this completion\(^2\). Neither of these topologies is completely satisfactory, but they represent a net improvement on previous proposals. This new definition was applied to the homogeneous plane waves in [6], and we found that it reproduces the results previously announced in [8].

In this paper, we will extend our previous investigations of the asymptotic structure, applying the topologies defined in [6] to investigate curves that approach infinity along spacelike directions. We will show that taking limits of past and future sets along spacelike curves generically leads to complicated behavior. In addition to learning more about the asymptotic structure of the plane wave spacetime, we hope that the explicit application of the topologies may help us to better understand the differences between the two definitions advanced in [6].

In one of these topologies, known as $\bar{T}$, most spacelike curves in these plane waves do not have limit points, even when we attach the causal boundary to the spacetime. Hence, the causal completion of the spacetime is non-compact. If we wanted to obtain a truly compact completion of the spacetime, we would need to adjoin some additional ideal points reachable only by spacelike curves. Such points are said to constitute spacelike infinity.

This is in fact a familiar situation. In the conformal compactification of Minkowski space, there is a single point $i^0$ in the boundary which is reachable only by spacelike curves (see figure 1). If we apply the causal completion technique to Minkowski space, on the other hand, this point will not be a part of $\bar{M}$, because no timelike curve reaches it. Hence, the causal completion is non-compact, and if we want to recover the usual conformal completion, we have to add in the point $i^0$ 'by hand'.

\(^2\)Also see [6] for comments on the relation of this scheme to the recent proposal of [9] for constructing Penrose-like diagrams based only on the causal structure. While this latter construction is far from unique, it may also be of interest in the planewave context.
The consideration of spacelike infinity shows that the asymptotic structure is not the same for all plane waves. By a detailed study of general sequences that approach infinity we show below that the homogeneous plane waves fall into three classes.

1. The maximally symmetric attractive homogeneous plane wave. An example is the BFHP plane wave [10] which arises from a Penrose limit of $\text{AdS}_5 \times S^5$ and has been of much recent interest in string theory, beginning with the work of BMN [1].

2. Homogeneous plane waves violating the null convergence condition. These are unphysical in, e.g., Einstein-Hilbert gravity.

3. All other homogeneous plane waves, including many [11, 12, 13, 14, 15, 16, 17, 18] of interest to string theory, such as those arising from Penrose limits of $\text{AdS}_4 \times S^7$ [12]. Here the causal boundary matches that of case 1, but using $\bar{T}$ spacelike infinity turns out to be larger in the sense described below.

The first case is conformally flat and is readily analyzed by conformal embedding inside a simple globally hyperbolic spacetime (in practice, the Einstein static universe $S^n \times \mathbb{R}$). In this context, it is clear from the results of [7] that all spacelike curves end on the null curve which forms the causal boundary. The same observations are readily seen to be true in the causal completion, using either $\bar{T}$ or the alternate topology $\bar{T}_{\text{alt}}$ also introduced in [6].

Case 2 may be discarded. Thus, the real interest is in case 3. This third case contains all smooth homogeneous plane waves that fail to be conformally flat. As a result, the Weyl tensor does not vanish for such spacetimes. Furthermore, since they are homogeneous, the Weyl tensor cannot vanish even asymptotically, and these spacetimes cannot be conformally embedded into a compact region of a smooth manifold, so the boundary of such spacetimes cannot even in principle be addressed by the conformal method of Penrose [19].

In this work, we will show that unlike case 1, the causal completion $(\bar{M}, \bar{T})$ in case 3 does not yet contain the limits of all spacelike curves. That is, the causal completion for these more general plane waves is non-compact. If we wish to construct a truly compact completion such as arises from the process of conformal embedding in case 1, we need to adjoin additional points at spacelike infinity. We will classify the distinct ways in which spacelike curves can approach infinity, finding that there is a two-dimensional set of distinct limits. Unfortunately, this does not directly imply that we should attach a two-dimensional spacelike boundary. The subtlety is that distinct limits can arise either from their being different points at spatial infinity, or from approaching the same point in different ways. We will not therefore not be able to discuss the construction of the extended completion in any generality. It may be difficult to give a satisfactory definition of such a completion in the generic case, although we feel that it is clear that one should exist in sufficiently restrictive circumstances; however, this may require the use of more information than just the causal information that our construction of $\bar{M}$ is based on.

\footnote{We thank Gary Horowitz for this argument: The infinite conformal rescaling would require the Weyl tensor of the spacetime in which we embed to diverge.}
The discussion above holds for the primary topology \( \mathcal{T} \) introduced in \( \mathbb{R} \); but \( \mathbb{R} \) also introduces an alternate topology \( \mathcal{T}_{\text{alt}} \), in which more sequences converge. We discuss this topology briefly in section 4.4. It yields identical results for case 1, while the causal completion of case 3 becomes compact. On the other hand, in this case \( \bar{M} \) ceases to satisfy the Hausdorff \((T_2)\) separation axiom. We therefore regard the use of \( \mathcal{T} \) as somewhat more satisfactory than that of \( \mathcal{T}_{\text{alt}} \), since the failure of compactness has a fairly simple physical interpretation in terms of spacelike infinity, while we can associate no obvious physical significance with the failure of Hausdorffness in \( \mathcal{T}_{\text{alt}} \).

We begin in section 3 with a summary of useful results from \( \mathbb{R} \). In section 3, we consider a specific spacelike curve, and show that it has no limit point in the causal completion \( (\bar{M}, \mathcal{T}) \). Thus, \( (\bar{M}, \mathcal{T}) \) is not compact. In section 4, we study the asymptotic behavior of arbitrary sequences (focusing on those along spacelike curves); this data will be used to discuss the construction of spacelike infinity, and to show that \( \bar{M} \) is compact in the alternate topology \( \mathcal{T}_{\text{alt}} \). We then discuss going to infinity along spacelike curves in terms of the causal structure of the spacetime in section 5. Finally, we conclude with some discussion in section 6.

2. Preliminaries

The homogeneous plane waves are those solutions \([20, 21, 22, 23, 24, 25]\) to theories incorporating Einstein-Hilbert gravity for which the metric takes the form

\[
d s^2 = -2 dx^+ dx^- - (\mu_1^2 x_1^2 + \ldots + \mu_j^2 x_k^2 - m_1^2 y_1^2 - \ldots - m_n^2 y_n^2)(dx^+)^2 + dx^i dx^i + dy^a dy^a, \tag{2.1}
\]

in a global coordinate system \((x^\pm, x^i, y^a)\) where each coordinate ranges over \((0, \infty)\). We order the \(x_i\) so that \(\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\). Such spacetimes satisfy the null convergence condition when \(\sum_i \mu_i^2 - \sum_a m_a^2\) is non-negative. Thus, we include the case \(k = n\) (with no \(y^a\) directions) but require \(k \geq 1\).

It was shown in \([8, 6]\) that the causal completion of this spacetime has a null curve of ideal points. Let us call the boundary points \(\bar{x}_\pm\), where we have labeled these points with the null coordinate \(x^\pm\) used above. Here \(x^\pm\) takes values in the extended real line \([-\infty, +\infty]\), which includes the endpoints \(\pm \infty\) and has the topology of the closed interval \([0, 1]\). To understand the way that these so-called ideal points are attached, let us recall from \([8]\) that a point \(p \in \bar{M}\) lies to the past of \(\bar{P}_{\mu_1}^+\) (in the sense that they can be connected by a timelike curve) if and only if \(x^+(p) < x_0^+\). Similarly, \(p \in \bar{M}\) lies to the future of \(\bar{P}_{\mu_1}^+\) if and only if \(x^+(p) > x_0^+ + \pi/\mu_1\), where \(\mu_1\) is the greatest value of \(\mu_i\). Together with the points \(p \in \bar{M}\), these \(\bar{P}_{\mu_1}^+\) constitute the causal completion \(\bar{M}\).

In our analysis below, we will also make use of the characterization of the future (and past) light cones of interior points \(p \in \bar{M}\). This is most simply displayed in terms of the future \(I^+(0)\) of the origin \((x = 0, y = 0, x^+ = 0, x^- = 0)\). From \([8]\), \(x \in I^+(0)\) if \(x^+ > 0\) and either of the following two conditions hold:

\[
2x^- > \sum_i \mu_i x_i^2 \cot(\mu_i x^+) + \sum_a m_a y_a^2 \coth(m_a x^+) \quad \text{or} \quad x^+ > \pi/\mu_1. \tag{2.2}
\]
We will find it useful below to have an explicit characterization of when any point \( \hat{x} \in M \) lies to the past of an arbitrary point \( x \). This can be obtained from the light cone of the origin by using the symmetries of the plane wave spacetime to translate the origin to \( \hat{x} \). Two of the symmetries are simply translations in \( x^+ \) and \( x^- \). However, the symmetries that change the values of \( x^i, y^a \) are more subtle. From e.g. [20], a symmetry that moves \( x^i = 0, y^a = 0 \) to \( x^i = \hat{x}^i, y^a = \hat{y}^a \) takes the form

\[
\begin{align*}
x^i &\rightarrow x^i + \hat{x}^i \cos(\mu_i x^+) \\
y^a &\rightarrow y^a + \hat{y}^a \cosh(m_a x^+) \\
x^- &\rightarrow x^- - \frac{1}{2} \sum_i \mu_i \hat{x}^i \sin(\mu_i x^+) [x^i + \hat{x}^i \cos(\mu_i x^+)] \\
&+ \frac{1}{2} \sum_a m_a \hat{y}^a \sinh(m_a x^+) [y^a + \hat{y}^a \cosh(m_a x^+)].
\end{align*}
\] (2.3)

As a result, we see that \( \hat{x} \) lies to the past of \( x \) exactly when \( x^+ > \hat{x}^+ \) and either

\[
2(x^- - \hat{x}^-) + \sum_i \mu_i \hat{x}^i x^i \sin(\mu_i [x^+ - \hat{x}^+]) - \sum_a m_a \hat{y}^a y^a \sinh(m_a [x^+ - \hat{x}^+]) \\
> \sum_i \mu_i (x^i - \cos(\mu_i [x^+ - \hat{x}^+]) \hat{x}^i)^2 \cot(\mu_i [x^+ - \hat{x}^-]) \\
+ \sum_a m_a (y^a - \cosh(m_a [x^+ - \hat{x}^+]) \hat{y}^a)^2 \coth(m_a [x^+ - \hat{x}^+]) \\
or \quad x^+ - \hat{x}^+ > \pi/\mu_1. \quad (2.4)
\]

Similarly, \( \hat{x} \) lies to the future of \( x \) exactly when \( x^+ < \hat{x}^+ \) and either

\[
2(x^- - \hat{x}^-) + \sum_i \mu_i \hat{x}^i x^i \sin(\mu_i [x^+ - \hat{x}^+]) - \sum_a m_a \hat{y}^a y^a \sinh(m_a [x^+ - \hat{x}^+]) \\
< \sum_i \mu_i (x^i - \cos(\mu_i [x^+ - \hat{x}^+]) \hat{x}^i)^2 \cot(\mu_i [x^+ - \hat{x}^-]) \\
+ \sum_a m_a (y^a - \cosh(m_a [x^+ - \hat{x}^+]) \hat{y}^a)^2 \coth(m_a [x^+ - \hat{x}^+]) \\
or \quad \hat{x}^+ - x^+ > \pi/\mu_1. \quad (2.5)
\]

Luckily these rather cumbersome expressions will simplify immediately to (2.2) and the corresponding past light cone in the limits considered below.

3. Non-compactness of \( \bar{M} \)

We would like to show that the causal completion \( \bar{M} \) of the plane wave (2.1) is non-compact in the topology \( \mathcal{T} \) introduced in [3]. To do so, we need only show that there is some infinite sequence of points in \( \bar{M} \) which does not have a limit point, as compactness implies the existence of a limit point for every infinite sequence. In this section we consider the common special case where there are at least two harmonic oscillator directions; i.e., where we have both \( \mu_1 \) and \( \mu_2 \). A similar counter-example can be found when \( \mu_1 \) represents the only harmonic oscillator direction. We will not present the details as the argument is similar.
Thus, there are infinite discrete subsets of this $\bar{\mathcal{S}}$ which contains all the ideal points. Since $\bar{\mathcal{S}}$ is not conformally flat, $\bar{\mathcal{S}}$ is not compact. For the present application to homogeneous plane wave spacetimes, all we need to know is that an ideal point $\bar{\mathcal{P}}_{\pm}$ of the plane wave will lie in $L^\pm(\bar{\mathcal{S}})$ if and only if its future (past) is a subset of $I^\pm(\bar{\mathcal{S}})$, and that $\bar{\mathcal{S}}$ is always contained in $L^\pm(\bar{\mathcal{S}})$. This is a direct consequence of theorem 8 of [6] and the observation that $(\bar{\mathcal{M}}, \bar{\mathcal{T}})$ is causally continuous (see appendix).

Consider the set $\bar{\mathcal{S}}$ consisting of the spacelike curve defined by $x^+ = x^- = 0, y_a = 0$, and $x_i = 0$ for $i \neq 2$. (Since we only need to produce one counter-example to disprove compactness, we have considered a particularly simple case. We will discuss the behaviour of more general curves in the next section.) There are discrete subsets of this curve which do not have any point of $\bar{\mathcal{M}}$ as a limit point in the topology of $\bar{\mathcal{M}}$ (for example, $\{x : x_2 = n, n \in \mathbb{Z}\}$). Since $\bar{\mathcal{M}}$ is homeomorphic to its image in $\bar{\mathcal{M}}$, if such a subset of $\bar{\mathcal{S}}$ has a limit point in $\bar{\mathcal{M}}$, it can only be an ideal point.

To exclude this possibility, let us consider the past and future of this curve. By (2.4), the past of $\bar{\mathcal{S}}$ is

$$I^-(\bar{\mathcal{S}}) = \{x : x^+ < 0 \text{ and either } x^+ < -\pi/\mu_1 \text{ or } \exists z \text{ such that } 2x^- + \sum_{i \neq 2} \mu_i x_i^2 \cot(-\mu_i x^+) + \sum_{a} m_a y_a^2 \coth(-m_a x^+) \leq \mu_2(x_2 - z \cos[\mu_2 x^+])^2 \cot(\mu_2 x^+) - \mu_2 x_2 z \sin(\mu_2 x^+)\}.$$  

In particular, we see that $\{x : x^+ < x_0^+\}$ is a subset of $I^-(\bar{\mathcal{S}})$ when $x_0^+ \leq -\pi/2\mu_2$, as when $x^+ < -\pi/(2\mu_2)$ we can make the RHS of the inequality arbitrarily positive by taking $z$ large so that the quadratic term dominates. Similarly, the future of $\bar{\mathcal{S}}$ is, from (2.3),

$$I^+(\bar{\mathcal{S}}) = \{x : x^+ > 0 \text{ and either } x^+ > \pi/\mu_1 \text{ or } \exists z \text{ such that } +2x^- - \sum_{i \neq 2} \mu_i x_i^2 \cot(\mu_i x^+) - \sum_{a} m_a y_a^2 \coth(m_a x^+) \geq \mu_2(x_2 - z \cos[\mu_2 x^+])^2 \cot(\mu_2 x^+) - \mu_2 x_2 z \sin(\mu_2 x^+)\},$$

so $\{x : x^+ > x_0^+\}$ is a subset of $I^+(\bar{\mathcal{S}})$ for $x_0^+ \geq \pi/(2\mu_2)$. Thus, the ideal points which lie in $L^-(\bar{\mathcal{S}})$ are $\bar{\mathcal{P}}_{\pm}$, for $x^+ \leq -\pi/(2\mu_2)$, while the ideal points which lie in $L^+(\bar{\mathcal{S}})$ are $\bar{\mathcal{P}}_{\pm}$, for $x^+ \geq \pi/(2\mu_2) - \pi/\mu_1$. Since we assume $\mu_1 > \mu_2$, there are no ideal points which lie in $L^+(\bar{\mathcal{S}}) \cap L^-(\bar{\mathcal{S}})$. Hence, $[\bar{\mathcal{M}} \setminus L^+(\bar{\mathcal{S}})] \cup [\bar{\mathcal{M}} \setminus L^-(\bar{\mathcal{S}})]$ is an open set in the topology of $[\bar{\mathcal{M}}]$, which contains all the ideal points. Since $\bar{\mathcal{S}} \subset L^+(\bar{\mathcal{S}}) \cap L^-(\bar{\mathcal{S}})$, the spacelike curve never enters this open set, so no subset of $\bar{\mathcal{S}}$ can have any of the ideal points as a limit point. Thus, there are infinite discrete subsets of this $\bar{\mathcal{S}}$ which have no point of $\bar{\mathcal{M}}$ as a limit point.

Hence, if the homogeneous plane wave $\bar{\mathcal{M}}$ satisfies the null convergence condition and is not conformally flat, $\bar{\mathcal{M}}$ is not compact.

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Note that in the maximally symmetric case, where $\mu_1 = \mu_2 = \mu$, the ideal point $\bar{\mathcal{P}}_{-\pi/(2\mu)}$ will lie in this set; this is in fact the limit point of this spacelike curve in this case.
4. Limits of spacelike curves for the plane waves

We have seen that there are spacelike curves in a general plane wave which do not have any point of \( \bar{M} \) as an endpoint in the topology \( \bar{T} \). In this section, we will classify the different ways in which a spacelike curve can go to infinity in terms of the causal structure. We will apply this classification to show that \( \bar{M} \) is compact in the topology \( \bar{T}_{alt} \). In the subsequent section, we will discuss what this information tells us about the points at spacelike infinity we need to add if we want to form a compact set \( \tilde{M} \) in the topology \( \bar{T} \).

4.1 Limiting pasts and futures

We want to study the asymptotic behaviour of general sequences of points in terms of the causal structure. One technical complication arises from the fact that some spacelike curves never leave a compact region of the spacetime. Thus, the past and future sets along a generic curve will not converge. Recall, however, that our goal is to add boundary points at spacelike infinity to construct a compact space \( \bar{M} \supset M \). This requires every sequence of points \( \{x_n\} \subset M \) to have a limit point in \( \bar{M} \), but not that every sequence converge. Note that we need only construct a limit point for each sequence which fails to have one in \( M \) itself. It turns out to be simplest to directly consider such sequences instead of spacelike curves per se.

Since our approach to the construction of the completion \( \bar{M} \) was based on the causal structure, defining points of \( \bar{M} \) by identifying the spacetime regions to their past and future, it seems natural to attempt to classify the limits by associating some past and future sets with them. For limits where we go to infinity along a timelike curve, there is a very straightforward identification; the past of the endpoint is naturally identified with the past of the whole curve. This is how the ideal points were defined in \[6\]: we defined an ideal point by saying \( I^+(\bar{P}) = I^+[\gamma] \) for any curve \( \gamma \) for which \( \bar{P} \) should provide a past endpoint, and similarly \( I^-(\bar{P}) = I^-[\gamma'] \) for any \( \gamma' \) for which it provides a future endpoint. The past and future sets \( I^\pm(\bar{P}) \) associated with points on the causal boundary are thus constructed directly from knowledge of the causal relations in the original spacetime \( M \). The ideal point is then defined in terms of its future and past; that is, formally as a pair \( (P, P^*) \) of past- and future-sets \( P, P^* \).

Spacelike curves are more subtle; the past and future of an endpoint is generally not the past or future of a curve. We would nonetheless like to associate limiting past and future sets with sequences which go to infinity along spacelike curves; in our approach, this seems a natural way to label distinct limits. In \[6\], a more general notion of the limit of a sequence of past and/or future sets was defined\(^5\) (not necessarily timelike or spacelike).

We can use this to define the limiting past or future for some sequence \( s = \{x_n\} \subset M \). We say that the past sets \( I^-(x_n) \subset M \) converge to the open past set \( I^-(\lim s) \subset M \) if and only if

1. Given \( x \in I^-(\lim s) \), there exists an integer \( N \) such that \( x \in I^-(x_n) \) for all \( n > N \), and

\(^5\)The definition given here differs slightly from that of \[6\], but the two are equivalent for strongly causal spacetimes \( M \) such as the homogeneous plane waves.
Figure 2: In the spacetime constructed by removing the shaded region from 1 + 1-dimensional Minkowski space, the sets \( I^- (x) \) for \( x \in \gamma \) do not approach \( I^- (y) \).

2. Given \( x \notin Cl[I^- (\lim s)] \), there exists an integer \( N \) such that, \( x \notin Cl[I^- (x_n)] \) for all \( n > N \), and similarly for open future sets \( I^+(x_n) \) converging to some future set \( I^+(\lim s) \). Here \( Cl \) denotes the closure in \( M \).

If there is a point \( \bar{P} \in \bar{M} \) such that \( I^\pm (\lim s) = I^\pm (\bar{P}) \), then theorem 8 of [6] guarantees that the sequence \( s \subset M \) will have \( \bar{P} \) as a limit point in the topology \( \bar{T} \). It seems reasonable to regard the sequence \( s \) as having sensible limiting behaviour in a causal sense under more relaxed conditions, however. We will therefore say that a sequence \( s \subset M \) converges in \( \bar{M} \) when both limits \( I^\pm (\lim s) \) exist. Similarly, we will say that a sequence \( s \) has a limit point in \( \bar{M} \) when it has a convergent subsequence. This gives us some information about the topology and point-set structure of \( \bar{M} \): we are requiring that \( \bar{M} \) associate some endpoint with any sequence such that the sets \( I^\pm (\lim s) \) exist.

We now need to address the physical interpretation of the limiting past and future sets defined above. Note first that in a general spacetime, the past or future of the endpoint need not be simply related to these limiting past and future sets. See figure 2 for an example. The assumption that the past (future) of an endpoint is a limit of pasts (futures) of points in the interior of a curve is known as causal continuity [27]. Thus, one can generally determine the past- and future-sets associated with an endpoint of a spacelike curve from the data intrinsic to the curve only if the spacetime is causally continuous. Thus, although we require that any sequence \( s \) such that \( I^\pm (\lim s) \) exist has an endpoint in \( \bar{M} \), we do not want to assume that the \( I^\pm (\lim s) \) determine the past and future of this endpoint.

Happily, it is straightforward to show (see appendix) that the planewave spacetime \( \bar{M} \) equipped with the causal boundary of [5] and the topology \( \bar{T} \) is causally continuous.\(^6\)

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\(^6\)Note that \( \bar{M} \) is also weakly distinguishing, meaning that if \( \bar{P}, \bar{Q} \in \bar{M} \) have identical futures (\( I^+ (\bar{P}) = I^+ (\bar{Q}) \)) and pasts (\( I^- (\bar{P}) = I^- (\bar{Q}) \)) then they are the same point (\( \bar{P} = \bar{Q} \)).
Thus, we can conclude that if $I^\pm(\lim s)$ are not the past and future of some point in $\bar{M}$, the sequence $s$ will converge to no point of $\bar{M}$. Thus, we need to extend $\bar{M}$ to construct a bigger completion $\bar{M}$ if there are sequences $s$ such that $I^\pm(\lim s)$ exist, but are not the past and future of some point in $\bar{M}$ (however, this will not tell us what points we need to add). We will therefore attempt to classify spacelike sequences for which $I^\pm(\lim s)$ exist.

Let us now apply this general approach to the plane waves. Our approach will be to identify a particular class $C$ of sequences which will determine the structure of $\bar{M}$lyzed along with those in $\bar{M}$. This class contains the 'exceptional' sequences which for technical reasons cannot be ana-

It is most transparent to present $C$ in several stages. First, given a sequence of points $\{x_n\} \subset M$, we would like to have some control over the behavior of the coordinate sequences $\{x_n^i\}, \{x_n^{-}\}, \{x_n^{0}\}, \{y_n^{0}\}$. Thus, we first restrict to the class $C_0$ defined as follows:

**Definition 1** $s = \{x_n\} \in C_0$ if and only if each corresponding sequence of coordinates $\{x_n^i\}, \{x_n^{-}\}, \{x_n^{0}\}, \{y_n^{0}\}$ converges in the extended real line $[-\infty, +\infty]$.

Since the extended real line has the topology of $[0,1]$ and in particular is compact, any sequence of points in $M$ contains a subsequence in $C_0$. Our preferred class of sequences $C$ will be a subclass of $C_0$.

We present $C$ in terms of three classes, $C_M, C_1, C_{\text{except}}$ defined as follows:

**Definition 2** $s = \{x_n\} \in C_M$ if and only if $s$ has a limit point in $M$.

Thus, $C_M$ dispenses with the cases already understood.

**Definition 3** $s = \{x_n\} \in C_1$ if and only if $s \in C_0 \setminus C_M$ and each corresponding sequence of ratios $\frac{(x_n^i)^2}{x_n^a}, \frac{(y_n^0)^2}{x_n^2}$ converges to a finite real number: $\frac{(x_n^i)^2}{x_n^a} \to c_i \in \mathbb{R}, \frac{(y_n^0)^2}{x_n^2} \to c_a \in \mathbb{R}$.

This is in some sense the generic class of sequences, which will determine the structure of spacelike infinity.

**Definition 4** $\{x_n\} \in C_{\text{except}}$ if and only if $s \in C_0 \setminus C_M$, each corresponding sequence of ratios of pairs of transverse coordinates $\frac{x_n^i}{x_n^a}, \frac{y_n^0}{x_n^2}, \frac{y_n^a}{x_n^2}, \frac{y_n^0}{x_n^2}$ converges in the extended real line $[-\infty, +\infty]$, and at least one of the sequences $\frac{(x_n^i)^2}{x_n^a}, \frac{(y_n^0)^2}{x_n^2}$ diverges to $\pm\infty$.

This class contains the ‘exceptional’ sequences which for technical reasons cannot be analyzed along with those in $C_1$. However, the study of sequences in $C_{\text{except}}$ has the same flavor and yields the same set of limit points. Again, the compactness of the extended real line guarantees that any sequence in $C_0$ has a subsequence in $C_M, C_1, \text{or } C_{\text{except}}$.

Finally, it is useful to further subdivide $C_1$ into two parts $C_1^\pm$ based on the behavior of $x_n^-:

**Definition 5** $s = \{x_n\} \in C_1^+$ if and only if $s \in C_1$ and $x_n^- > 0$ for all $n$.

**Definition 6** $s = \{x_n\} \in C_1^-$ if and only if $s \in C_1$ and $x_n^- < 0$ for all $n$. 

---
Since any sequence \( \{x_n\} \in C_1 \) must contain infinitely many points for which \( x_n^- \neq 0 \), it contains a subsequence which lies in \( C_1^\pm \). Thus we may take \( C \) to be \( C_M \cup C_1^+ \cup C_1^- \cup C_{\text{except}}^\pm \). Note that the long inequalities \((2.4),(2.5)\) differ only by the direction of the inequalities. Though we will consider the two classes \( C_1^\pm \) separately, their treatment is identical as they are simply related by time reversal.

4.2 Convergence of sequences in \( C_1^\pm \)

Let us now consider a sequence \( s = \{x_n\} \in C_1^+ \). We will show that each such sequence is convergent and that the sets \( I^\pm(\lim s) \) take the form

\[
\begin{align*}
I^-&(\lim s) = \{x : x^+ < \hat{x}^+ - \delta^-\}, \\
I^+(\lim s) & = \{x : x^+ > \hat{x}^+ + \delta^+\},
\end{align*}
\]

where \( x_n^+ \to \hat{x}^+ \in [-\infty, +\infty] \). The parameter \( \delta^- \) is the smaller of \( \pi/\mu_1 \) and the smallest positive solution of

\[
2 = \sum_i \mu_i c_i \cot(\mu_i \delta^-) + \sum_a m_a c_a \coth(m_a \delta^-). \tag{4.2}
\]

Similarly, \( \delta^+ \) is the smaller if \( \pi/\mu_1 \) and the smallest positive solution of

\[
-2 = \sum_i \mu_i c_i \cot(\mu_i \delta^+) + \sum_a m_a c_a \coth(m_a \delta^+). \tag{4.3}
\]

To derive this result, notice that since \( s \in C_0 \) but \( s \notin C_M \), at least one of the coordinate sequences \( \{x_n^+\}, \{x_n^-\}, \{y_n^a\} \) must approach \( \pm \infty \). If this is \( x_n^+ \to +\infty \), then it is clear that for any \( \hat{x} \) we have, for large enough \( n \), \( x_n^+ - \hat{x}^+ > \pi/\mu_1 \). Thus, \( \hat{x} \) is in the past of all \( x_n^+ \) with sufficiently large \( n \). Since the plane waves contain no closed causal curves, it also follows that \( \hat{x} \) is not in the future of such \( x_n \). Thus \( s \) converges and \( I^- (\lim s) = M, I^+ (\lim s) = \emptyset \) in agreement with the statement above. Similarly, if \( x_n^+ \to -\infty \), we have \( I^+ (\lim s) = M, I^- (\lim s) = \emptyset \).

Let us now consider the case \( x_n^+ \to \hat{x}^+ \) with finite \( \hat{x}^+ \). Here one of the other coordinate sequences \( \{x_n^-, x_n^+, y_n^a\} \) must diverge. Since the \( c_i, c_a \) are finite and \( s \in C_1^+ \) we must have \( x_n^- \to +\infty \). Thus, for large enough \( n \), \( \Delta_n^a = x_n^- - \hat{x}^- \) is positive and we may divide the inequality \((2.4)\) by \( \Delta_n^- \) to yield

\[
\begin{align*}
2 > & \sum_i \mu_i \frac{(x_n^i)^2}{x_n^-} \cot(\mu_i [x_n^+ - \hat{x}^+]) + \sum_a m_a \frac{(y_n^a)^2}{x_n^-} \coth(m_a [x_n^+ - \hat{x}^+]) \\
& + O(y_n^a/x_n^-) + O(1/x_n^-) \text{ or } x_n^+ - \hat{x}^+ > \pi/\mu_1. \tag{4.4}
\end{align*}
\]

Note that since some coordinate must diverge, the suppressed terms are subleading and can be neglected for large \( n \). Taking the limit \( n \to \infty \), we thus arrive at the result \((4.1)\) with \( \delta^- \) defined by \((4.2)\).

On the other hand, the future condition \((2.5)\) differs from \((2.4)\) only by the direction of the inequalities while the definition \((4.1)\) of \( \delta^+ \) involves a single extra sign. As a result, \( I^+ (\lim s) \) is given by \((4.1)\) with \( \delta^+ \) defined by \((4.3)\).
This establishes the convergence of \( s \in C_1^+ \). Sequences in the class \( C_1^- \) behave in much the same way, with the only difference being that \( \Delta_n^- \) has the opposite sign. Thus we find that any sequence \( s \in C_1^- \) is convergent and that the limit again yields sets of the form (4.1) where \( \delta^- \) is the smaller of \( \pi/\mu_1 \) and the solution of
\[
-2 = \sum_i \mu_i |c_i| \cot(\mu_i \delta^-) + \sum_a m_a |c_a| \coth(m_a \delta^-).
\] (4.5)
and \( \delta^+ \) is the smaller of \( \pi/\mu_1 \) and the solution of
\[
2 = \sum_i \mu_i |c_i| \cot(\mu_i \delta^+) + \sum_a m_a |c_a| \coth(m_a \delta^+).
\] (4.6)

### 4.3 Limits of \( C_{\text{except}} \)

Let us now consider the class \( C_{\text{except}} \) of exceptional sequences. As before, if \( x^+ \to \pm \infty \) it is easy to find the limit, so we focus on the case \( x^+ \to \hat{x}^+ \) with finite \( \hat{x}^+ \). Since all ratios of the form \( x_i^+/x_j^-, x_i^-/x_j^-, y_i^+/y_j^-, y_i^-/y_j^- \), converge in the extended real line \([-\infty, +\infty]\), we may identify the most rapidly growing transverse coordinate, which we call \( z \). Note that since at least one of the ratios \( \frac{(x_i^+)^2}{x_n^-}, \frac{(y_i^-)^2}{y_n^-} \) diverges to \( \pm \infty \), \( z^2 \) diverges more rapidly than \( x^- \). Let us therefore divide the inequalities (2.4) and (2.5) by \( z^2 \) and take the limit of large \( n \) to find that \( \hat{x} \) lies to the past of \( x_n \) for large \( n \) when \( \hat{x}^+ > \hat{x}^+ \) and
\[
0 > \sum_i \mu_i \lim_{n \to \infty} (x_i^+/z_n)^2 \cot(\mu_i [\hat{x}^+ - \hat{x}^+]) + \sum_a m_a \lim_{n \to \infty} (y_n^-/z_n)^2 \coth(m_a [\hat{x}^+ - \hat{x}^+])
\]

\[\text{or } \hat{x}^+ - \hat{x}^+ > \frac{\pi}{\mu_1}.\] (4.7)

Similarly, \( \hat{x} \) lies to the future of \( x_n \) for large \( n \) exactly when \( \hat{x}^+ < \hat{x}^+ \) and
\[
0 < \sum_i \mu_i \lim_{n \to \infty} (x_i^+/z_n)^2 \cot(\mu_i [\hat{x}^+ - \hat{x}^+]) + \sum_a m_a \lim_{n \to \infty} (y_n^-/z_n)^2 \coth(m_a [\hat{x}^+ - \hat{x}^+])
\]
\[\text{or } \hat{x}^+ - \hat{x}^+ > \frac{\pi}{\mu_1}.\] (4.8)

Thus, in this case we again find that any \( s \in C_{\text{except}} \) converges and that \( I^\pm(\lim s) \) are given by (4.3), though this time \( \delta^+ = \delta^- \) and this parameter is the smaller of \( \pi/\mu_1 \) and the smallest positive solution of
\[
0 = \sum_i \mu_i \lim_{n \to \infty} (x_i^+/z_n)^2 \cot(\mu_i \delta^-) + \sum_a m_a \lim_{n \to \infty} (y_n^-/z_n)^2 \coth(m_a \delta^-).\] (4.9)

We have seen that for appropriate \( \delta^\pm \) we always have
\[
I^-(\lim s) = I^-(P_{x_n^+}^{\delta^-}),
I^+(\lim s) = I^+(P_{x_n^+}^{\delta^+ - \pi/\mu_1}).\] (4.10)

Note, however, that in general we will not have \( \delta^+ + \delta^- = \pi/\mu_1 \), so that these sets will not be the past and future of the same point in \( M \). This is seen explicitly in the example considered in section 4.
We should note again that in the special case where \( \mu_i = \mu_1 \) and \( k = n \) (so that there are no \( m_a \) terms in (2.1)), no additional points are needed. This is because the left-hand side of (4.3) changes sign under \( \delta^+ \to \pi/\mu_1 - \delta^+ \). Since this transformation preserves the range \([0, \pi/\mu_1]\) of valid \( \delta^+ \), and since (4.3) and (4.2) agree on the left hand side and differ only by a sign on the right, this transformation maps \( \delta^+ \to \delta^- \). For the (attractive) conformally invariant plane wave, we have identically \( \delta^+ + \delta^- = \pi/\mu_1 \) and in this case the sequence \( \{x_n\} \) converges to the ideal point \( \bar{P}_{x_0^+ - \delta^-} \). For such plane waves, the space \( \bar{M} \) is already compact without the addition of further points at spacelike infinity. This is exactly what one would expect from the conformal diagram provided in [7].

In the more general case, one might ask whether the ideal point \( \bar{P}_{x_0^+ + \delta^+} \) associated with the limiting future can ever precede the ideal point \( \bar{P}_{x_0^+ - \delta^-} \) associated with the limiting past. This would happen only if \( \delta^+ + \delta^- < \pi/\mu_1 \). That this does not occur can be seen by adding together (4.3) and (4.2). The result is:

\[
\sum_i \mu_i |c_i| [\cot(\mu_i \delta^-) + \cot(\mu_i \delta^+)] + \sum_a m_a |c_a| [\coth(m_a \delta^-) + \coth(m_a \delta^+)] = 0. \tag{4.11}
\]

Now, the cotangent function is monotonically decreasing on \((0, \pi)\), and is antisymmetric about \( \pi/2 \). Suppose for a moment that \( \delta^+ \geq \delta^- \). Then for any term \([\cot(\mu_i \delta^-) + \cot(\mu_i \delta^+)]\) to be negative or zero, we must have \( \mu_i \delta^+ - \pi/2 \geq \pi/2 - \mu_i \delta^- \). Thus, \( \delta^+ + \delta^- \geq \pi/2 \mu_i \geq \pi/\mu_1 \). In fact, the boundary value \( \delta^+ + \delta^- = \pi/\mu_1 \) is achieved only when \( c_0 = 0 \) and \( c_i = 0 \) for \( \mu_i \neq \mu_1 \). The same is true in the remaining case \( \delta^+ \leq \delta^- \) due to the symmetry of (1.11) under exchange of \( \delta^+ \) and \( \delta^- \). An identical argument can be made for sequences in \( C_{except} \).

Finally, note that both \( \delta^\pm \) lie in the range \([0, \pi/\mu_1]\). Thus, \( 0 < \delta^+ + \delta^- - \pi/\mu_1 \leq \pi/\mu_1 \).

### 4.4 \( \bar{M} \) is compact in the alternate topology \( \bar{T}_{\text{alt}} \)

In addition to the topology \( \bar{T} \) addressed thus far, an alternate topology \( \bar{T}_{\text{alt}} \) was also introduced in [3]. All that we need know about this topology is captured in theorem 16 of [3], which states that if \( I^+(x_n) \to I^+(\bar{P}) \) for a sequence \( \{x_n\} \subset M \) and a point \( \bar{P} \in \bar{M} \) with \( I^-(\bar{P}) \neq \emptyset \), then \( \{x_n\} \) converges to \( \bar{P} \). Similarly, if \( I^-(x_n) \to I^-(\bar{P}) \) for a sequence \( \{x_n\} \subset M \) and a point \( \bar{P} \in \bar{M} \) with \( I^+(\bar{P}) \neq \emptyset \), then again \( \{x_n\} \to \bar{P} \) converges to \( \bar{P} \).

As a result, we can read off the convergence of various classes of sequences directly from the results of this section. In fact, together with the observation that \( \delta^+ + \delta^- \) is bounded, equation (4.11) tells us that each sequence in the class \( C \) converges to at least one point in \( \bar{M} \). Thus, \( \bar{M} \) is already compact in the topology \( \bar{T}_{\text{alt}} \). On the other hand, we see that most of these sequences converge to two distinct points \( \bar{P}_{x_0^+ - \delta^-} \) and \( \bar{P}_{x_0^+ + \delta^- - \pi/\mu_1} \), since in general \( \delta^+ + \delta^- \neq \pi/\mu_1 \). Thus \( \bar{M}, \bar{T}_{\text{alt}} \) is not Hausdorff.

### 5. Spacelike infinity

In the last section, we saw that we could characterise different ways of going to infinity along spacelike directions in terms of the limiting past and future sets, using a definition of the limit of a sequence of past or future sets introduced in [3]. In the topology \( \bar{T} \), these spacelike sequences have no limit points in \( \bar{M} \). In defining a larger completion \( \bar{M} \supset \bar{M} \supset M \)
Figure 3: Points to the left of the solid line define a subset of 1+1 Minkowski space for which spacelike curves approaching $i^0$ give nontrivial limiting past and future sets.

by adding points at spacelike infinity by hand, we need to ask how these limiting past and future sets are related to the points we add.

First, we should stress that we can never interpret the limiting past and future sets $I^\pm(\lim s)$ defined in the previous section as giving the chronological past and future of some point at spacelike infinity. By definition, there are no timelike curves which approach a point at spacelike infinity from either future or past. Points at spacelike infinity are therefore spacelike separated from all points in the interior of the spacetime, and their pasts and futures in $M$ are correspondingly empty.

Given that, it may seem surprising that a sequence of points which is meant to be converging to a point at spacelike infinity can have non-trivial limiting past or future-sets. However, this is just a failure of causal continuity in $\tilde{M}$. As explained previously in section 4, if a spacetime is not causally continuous, there is no necessary connection between the limiting past and future sets defined by some convergent sequence and the past and future sets associated with the endpoint of that sequence. A simpler example where this feature is more readily apparent is shown in figure 3. In this example $I^+(\lim s) = I^+(\bar{P})$, while $I^-(\lim s) = I^-(\bar{Q})$. Nonetheless, the sequence $s = \{x_n\}$ converges to the point $i^0$ at spacelike infinity, for which it is clear that $I^+(i^0) = \emptyset$, $I^-(i^0) = \emptyset$. Somehow in the plane wave this ‘corner’ phenomenon occurs for a continuum of such pairs $\bar{P}, \bar{Q}$ on the causal boundary.

Thus, we cannot interpret $I^\pm(\lim s)$ as the past and future of the endpoint we add. However, we would like to ask if we can interpret them as *labeling* the endpoints we need to add. In general, even this fails to be true. To see this, consider two simple 2+1-dimensional examples constructed from the example in figure 3. First, let us consider an example where

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7Thus, we learn that if we can define a compact $\tilde{M}$ by adding points at spacelike infinity, this $\tilde{M}$ will not be causally continuous. This does not contradict the result in the appendix, which only concerns the previously-defined causal completion $\tilde{M}$. 

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part of the boundary of our 2 + 1 dimensional spacetime is constructed by rotating the boundary in figure 3 about a vertical axis through \( i^0 \) (through \( \pi/4 \), say). This portion of the boundary then still contains a single point at spacelike infinity \( i^0 \). However, we now have a one-dimensional family of distinct spacelike limits approaching this point, labeled by the limiting past and future sets \( P_\alpha, Q^*_\alpha \) which are the images of the original \( P \) and \( Q^* \) under rotation through an angle \( \alpha \). Hence, spacelike limits with different \( I^\pm(\lim s) \) can approach the same point.

Next, consider an example where part of the boundary of our 2 + 1 dimensional spacetime is constructed by rotating the boundary in figure 3 about a vertical axis through \( \bar{P}, \bar{Q} \) (through \( \pi/4 \), say). This now contains a one-dimensional family of points at spacelike infinity \( i^0_\alpha \). However, any spacelike sequence approaching any one of these points will have \( I^{-}(\lim s) = P, I^{+}(\lim s) = Q^* \). Hence, spacelike limits with the same \( I^\pm(\lim s) \) can approach different points.

Finally, a 2 + 1 dimensional example with the naïve one-to-one relationship between \( I^\pm(\lim s) \) and the points at spacelike infinity can easily be constructed by taking the product of figure 3 with a real line. In this case, we have one-dimensional families both of \( \bar{P}_z, \bar{Q}_z \) and \( i^0_z \).

Thus, it seems reasonable to use the limiting past and future sets \( I^\pm(\lim s) \) associated with a sequence \( s \) to determine if \( s \) should become convergent in \( \tilde{M} \), but perhaps not to determine if different sequences have the same or different endpoints. Unfortunately, no technology has been developed to address this latter question within the causal approach.

6. Discussion

The investigation of the asymptotic structure of plane wave spacetimes is interesting both because of its potential to teach us more about the holographic relation between string theory and field theory, and because it presents a challenging case which pushes the causal boundary technology to its limits. In this paper, we have shown that the causal completion constructed in \([3]\) is non-compact in the topology \( \tilde{T} \) introduced in \([3]\). Thus, to obtain a compact completion, we need to add additional boundary points at spacelike infinity.

Understanding the limits in spacelike directions seems important, both because it enables a closer comparison with the more familiar conformal compactification technique, and because points at spacelike separation can play an important role in the definition of a duality between quantum theories. We have therefore investigated the classification of different spacelike sequences in terms of the causal structure—how the past and future sets behave as we go to infinity along the sequence. We used this information to show that in the alternate topology \( \tilde{T}_{alt} \) of \([3]\), the resulting causal completion \( \tilde{M} \) is already compact; however, it is also non-Hausdorff.

We have attempted to relate the classification of spacelike limits in terms of the causal structure to some characterization of the points at spacelike infinity that need to be added to the spacetime. However, it is not clear to what extent this determines the precise set of points that should be added at spacelike infinity. Thus, the determination of the appropriate points at spacelike infinity for the general homogeneous plane waves would
require further work, and will probably involve additional technical choices such as appeared in the definition of the topologies in \[6\].

It would be interesting to see if one can extend the causal structure on \(\tilde{M}\) to a causal structure on the extended compact completion \(\tilde{\tilde{M}}\), once one had a definition of that set. In \[6\], the chronology relation was defined by stating that \(\bar{P}\) is to the future of \(\bar{Q}\) if and only if \(I^+(\bar{Q}) \cap I^-(\bar{P}) \neq \emptyset\). This definition is naturally extended by saying that points at spacelike infinity have no chronological relations, as they are associated with empty past and future sets. Note that this implies that if \(\tilde{M}\) contains more than one point at spacelike infinity, it will fail to be weakly distinguishing in this natural chronology. It is much less clear how one would extend the lightlike (causality) relation. The example of the conformal compactification of Minkowski space shows that there may be causality relations which involve points at spacelike infinity, but it is not clear how these should be reconstructed. An answer to this question might also clarify precisely what set of points should be added at spacelike infinity.

It would be very interesting to relate this discussion to the dual field theory for string theory on these backgrounds. In particular, it would be of interest to determine which of the two topologies used here is most directly associated with properties of the dual gauge theory.

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**A. Causal Continuity of \((\tilde{M}, T)\)**

In this appendix we show that \((\tilde{M}, T)\) is causally continuous and, furthermore, that the topology \(T\) on \(\tilde{M}\) can be characterized by the property that a sequence \(\{x_n\} \subset \tilde{M}\) converges to \(\bar{P}\) if and only if \(\lim I^+(x_n) = I^+(\bar{P})\) and \(\lim I^-(x_n) = I^-(\bar{P})\). Note that here we allow any sequence in \(\tilde{M}\), whereas the restriction \(\{x_n\} \subset M\) was imposed in the main text. For \(x_n \notin M\) we define \(I^\pm(x_n)\) to be \(I^\pm_C(x_n) \cap M\), where \(I^\pm_C\) is the chronology defined on \(\tilde{M}\) in \[6\]: \(\bar{Q} \in I^+_C(\bar{P})\) if and only if \(I^+(\bar{Q}) \cap I^-(\bar{P}) \neq \emptyset\).

The concept of causal continuity was introduced in \[27\] for manifolds, and in fact several equivalent definitions were given. Since \(\tilde{M}\) is not a manifold, we have somewhat less structure here. Thus, it is no longer clear that the definitions in \[27\] are equivalent. We take one of these properties, inner and outer continuity of \(I^\pm_C\), to define causal continuity in the present context.

Recall \[27\] that a function \(F\) which maps points in \(\tilde{M}\) to subsets of \(\tilde{M}\) is called *inner continuous* if, for any \(\bar{P} \in \tilde{M}\) and any compact set \(C \subset F(\bar{P})\) there is an open neighborhood \(U\) of \(\bar{P}\) such that \(C \subset F(u)\) for all \(u \in U\). Similarly, \(F\) is *outer continuous* if, for any \(\bar{Q} \in \tilde{M}\) and any compact set \(K \subset M \setminus C\llbracket F(\bar{Q})\rrbracket\), there is a neighborhood \(V\) of \(\bar{Q}\) such that \(K \subset \tilde{M} \setminus C\llbracket F(v)\rrbracket\) for all \(v \in V\). Here \(C\llbracket\) denotes the topological closure in \(\tilde{M}\).
We now show the following results:

**Theorem 1** $I^C$ is inner continuous with respect to $\bar{T}$ for any causal completion $\bar{M}$ of any strongly causal spacetime $M$.

**Proof:** Consider $\bar{P} \in \bar{M}$ and a compact set $C \subset I^C_{\bar{C}}(\bar{P})$. Let $U = \cup_{1^{-}(\bar{Q})\supset C} I^+_{C}(\bar{Q})$. Recall that any $I^C_{\bar{C}}(\bar{Q})$ is open by Lemma 3 of [6]. Thus $U$ is open, and clearly $C \subset I^C_{\bar{C}}(u)$ for all $u \in U$. Thus we need only show $\bar{P} \in U$.

Now, $\bar{P}$ is connected to each point of $C$ by a timelike curve on which there exist intermediate points $\bar{R}$. But since $C$ is compact it must be contained in $I^C_{\bar{C}}(\bar{R})$ for a finite collection of such points $\bar{R}_i$. Since each such $\bar{R}_i$ can signal $\bar{P}$ via a timelike curve in $M$, there in fact must be some $r \in M$ with $\bar{P} \in I^C_{\bar{C}}(r)$ that they can all signal as well. This $r$ will have $I_{\bar{C}}(r) \supset C$, so $\bar{P} \in I^C_{\bar{C}}(r) \subset U$. Thus $I^C_{\bar{C}}$ is inner continuous. □

**Theorem 2** $I^C_{\bar{C}}$ is outer continuous with respect to $\bar{T}$ for the causal completion $\bar{M}$ of a homogeneous plane wave satisfying the null convergence condition.

**Proof:** It is useful to first note that, for such $\bar{M}$, a point $\bar{Q} \in L^- (\bar{P})$ if any only if $\bar{P} \in L^-(\bar{Q})$, where $L^\pm$ are defined in [6]. Furthermore, these are the closures of the sets $I^+_{\bar{C}}(\bar{P}), I^C_{\bar{C}}(\bar{Q})$. We also recall from [6] that $\bar{R} \in L^- (\bar{Q})$ implies $I^C_{\bar{C}}(\bar{R}) \subset L^- (\bar{Q})$.

Now consider any $\bar{Q} \in \bar{M}$ and any compact $K \subset M \setminus L^- (\bar{Q})$ and note that the boundary point $\bar{P}_{-\infty}$ lies in $I^C_{\bar{C}}(\bar{Q}) \subset L^- (\bar{Q})$. As a result, it does not lie in $K$. Furthermore, note that there is a past-directed timelike curve $\gamma_{\bar{R}}$ through $M$ from any point $\bar{R} \in \bar{K}$ to $\bar{P}_{-\infty}$. Since $\bar{R} \notin L^- (\bar{Q})$, we have $I^- (\bar{R}) \subset L^- (\bar{Q})$ from the definition of $L^-$ in [6]. Thus, some points on $\gamma_{\bar{R}}$ will lie in $\bar{M} \setminus L^- (\bar{Q})$ and we may associate each $\bar{R} \in \bar{K}$ with some $\bar{R}' \in \gamma_{\bar{R}} \setminus L^- (\bar{Q})$. Now, $K \subset \cup I^C_{\bar{C}}(\bar{R}')$ so in fact $K \subset \cup_i I^C_{\bar{C}}(\bar{R}_i')$ for a finite subcollection $\{\bar{R}_i'\}_{i}$ since $K$ is compact.

Recall that $L^+ (\bar{R}_i') \supset I^C_{\bar{C}}(\bar{R}_i')$ and let $V = \bar{M} \setminus [\cup_i I^+ (\bar{R}_i')]$. The set $V$ is open since the collection $\{\bar{R}_i'\}$ is finite. Since $\bar{R}_i' \notin L^- (\bar{Q})$, it is clear that $\bar{Q} \in V$. Note that any $v \in V$ must have $L^- (v) \cap K = \emptyset$, else some $\bar{R}_i'$ would lie in $L^- (v)$ and $v$ would lie in $L^+ (\bar{R}_i')$. Thus $I^C_{\bar{C}}$ is outer continuous. □

**Theorem 3** For homogeneous plane wave spacetimes satisfying the null convergence condition, a sequence $\{x_n\} \subset \bar{M}$ converges to $\bar{P}$ in $\bar{T}$ if and only if $\lim I^C_{\bar{C}}(x_n) = I^C_{\bar{C}}(\bar{P})$ and $\lim I^C_{\bar{C}}(x_n) = I^C_{\bar{C}}(\bar{P})$.

**Proof:** Theorem 8 of [6] shows that convergence of the past and future sets $\lim I^C_{\bar{C}}(x_n)$ implies convergence in $\bar{T}$ when $\{x_n\} \subset \bar{M}$. For homogeneous plane waves satisfying the null convergence condition, the case where boundary points appear in this sequence is easily handled by inspection.

For the converse, suppose that $\{x_n\} \subset \bar{M}$ converges to $\bar{P}$ in $\bar{T}$. Consider any $\bar{Q} \in I^-(\bar{P})$ and note that $\{\bar{Q}\}$ is compact. Then from theorem 1 we see that $\bar{Q} \in I^C_{\bar{C}}(x_n)$ for sufficiently large $n$. Similarly, consider any $\bar{R} \notin Cl [I^C_{\bar{C}}(\bar{P})]$ and note that $\{\bar{Q}\}$ is compact. Then from theorem 2 we find that $\bar{Q} \in I^C_{\bar{C}}(x_n)$ for sufficiently large $n$. □

Thus, we may characterize the topology $\bar{T}$ by theorem 3.
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