Perturbations to $\mu - \tau$ symmetry in neutrino mixing

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Abstract

Many neutrino mixing scenarios that have $\mu - \tau$ symmetry with $\theta_{13} = 0$ are in disagreement with recent experimental results that indicate a nonzero value for $\theta_{13}$. We investigate the effect of small perturbations on Majorana mass matrices with $\mu - \tau$ symmetry and derive analytic formulae for the corrections to the mixing angles. We find that since $m_1$ and $m_2$ are nearly degenerate, $\mu - \tau$ symmetry mixing scenarios are able to explain the experimental data with about the same size perturbation for most values of $\theta_{12}$. This suggests that the underlying unperturbed mixing need not have $\theta_{12}$ close to the experimentally preferred value. One consequence of this is that a new class of models with $\mu - \tau$ symmetry is possible, with unperturbed $\theta_{12}$ equal to zero or $90^\circ$ for arbitrary unperturbed $\theta_{13}$. 
Of the numerous neutrino mixing scenarios discussed in the literature \[1\], several have \(\mu - \tau\) symmetry, such as tri-bimaximal mixing (TBM) \[2\], bimaximal mixing (BM) \[3\], hexagonal mixing (HM) \[4\] and scenarios of \(A_5\) mixing \[5\]. In these scenarios, \(\theta_{23} = 45^\circ\), \(\theta_{13} = 0\), and only \(\theta_{12}\) depends on the particular model. Tri-bimaximal mixing is most popular because the value of \(\theta_{12}\) predicted by TBM is close to that preferred by the current experimental data. However, the latest results from the T2K \[6\], MINOS \[7\], and Double Chooz \[8\] experiments suggest a nonzero value of \(\theta_{13}\), and the recent Daya Bay \[9\] and RENO \[10\] experiments find \(\theta_{13} \neq 0\) at the \(5.2\sigma\) and \(4.9\sigma\) level, respectively. Various corrections may reconcile such models with nonzero \(\theta_{13}\) \[1\]. In this Letter we consider small perturbations acting on Majorana mass matrices with \(\mu - \tau\) symmetry and estimate the size of perturbations required to explain the experimental data.

We find that for \(\mu - \tau\) symmetries with almost any initial value of \(\theta_{12}\) (i.e., before the perturbation), the minimal size of the perturbations needed to bring the model in agreement with experimental data varies by only about 20%. The reason is that the \(\theta_{12}\) correction depends only on the ratio of perturbation terms and not on their absolute size, and the overall size of the perturbation is determined by the corrections to \(\theta_{13}\) and \(\theta_{23}\), which are relatively small. We also show that a new category of models with \(\mu - \tau\) symmetry, \(\theta_{23} = 45^\circ\), \(\theta_{12} = 0\) or \(90^\circ\), and arbitrary \(\theta_{13}\), can also fit the data with small perturbations.

We start with the mass matrix for Majorana neutrinos

\[
M = U^* M^{\text{diag}} U^\dagger,
\]

(1)

where \(M^{\text{diag}} = \text{diag}(m_1, m_2, m_3)\), \(U\) is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix \[11\] (without the multiplicative diagonal matrix of Majorana phases), and we work in the basis in which the charged lepton mass matrix is diagonal. The masses \(m_2\) and \(m_3\) are complex and \(m_1\) can be taken to be real and non-negative.

The general condition describing \(\mu - \tau\) symmetry (also sometimes called \(\mu - \tau\) universality) is \[12\]

\[
|U_{\mu i}| = |U_{\tau i}|, \text{ for } i = 1, 2, 3.
\]

(2)

From the standard form of the mixing matrix these conditions are equivalent to

\[
\theta_{23} = 45^\circ, \quad \text{Re}(\cos\theta_{12}\sin\theta_{12}\sin\theta_{13}e^{i\delta}) = 0.
\]

(3)

Hence, there are four classes of \(\mu - \tau\) symmetry: (a) \(\theta_{23} = 45^\circ, \theta_{13} = 0\); (b) \(\theta_{23} = 45^\circ, \theta_{12} = 0\); (c) \(\theta_{23} = 45^\circ, \theta_{12} = 90^\circ\); (d) \(\theta_{23} = 45^\circ, \delta = \pm 90^\circ\). Class (a) contains models with tri-bimaximal, bimaximal, hexagonal, and \(A_5\) symmetries, while class (d) includes tetramaximal symmetry \[13\]. Classes (b) and (c) have not been studied before because the unperturbed \(\theta_{12}\) angle is far from the experimentally preferred value, but, as we show below, small perturbations can have a large effect on \(\theta_{12}\), and therefore these models should not be ignored.
Class (a): $\theta_{23}^{a} = 45^\circ$, $\theta_{13}^{a} = 0$

We first examine the effect of small perturbations on models in class (a). The initial (unperturbed) mixing matrix can be written as

$$U_{0} = \begin{pmatrix} \cos \theta_{12}^{0} & \sin \theta_{12}^{0} & 0 \\ -\sin \theta_{12}^{0} & \cos \theta_{12}^{0} & 1 \\ \sin \theta_{13}^{0} & \cos \theta_{13}^{0} & 1 \end{pmatrix},$$  \hspace{1cm} (4)

and the initial mass matrix is

$$M_{0} = U_{0}^{*} M_{0}^{\text{diag}} U_{0}^{\dagger} = \begin{pmatrix} m_{1}^{2} c_{12}^{2} + m_{2}^{0} s_{12}^{2} & \frac{(m_{3}^{0} - m_{2}^{0}) s_{12} c_{12}}{\sqrt{2}} & \frac{(m_{3}^{0} - m_{2}^{0}) s_{12} c_{12}}{\sqrt{2}} \\ \frac{(m_{3}^{0} - m_{2}^{0}) s_{12} c_{12}}{\sqrt{2}} & \frac{1}{2} (m_{3}^{0} + m_{2}^{0} c_{12}^{2} + m_{1}^{0} s_{12}^{2}) & \frac{1}{2} (m_{3}^{0} - m_{2}^{0} c_{12}^{2} - m_{1}^{0} s_{12}^{2}) \\ \frac{(m_{3}^{0} - m_{2}^{0}) s_{12} c_{12}}{\sqrt{2}} & \frac{1}{2} (m_{3}^{0} - m_{2}^{0} c_{12}^{2} - m_{1}^{0} s_{12}^{2}) & \frac{1}{2} (m_{3}^{0} + m_{2}^{0} c_{12}^{2} + m_{1}^{0} s_{12}^{2}) \end{pmatrix},$$ \hspace{1cm} (5)

where $M_{0}^{\text{diag}} = \text{diag}(m_{1}^{0}, m_{2}^{0}, m_{3}^{0})$, and $c_{jk}$, $s_{jk}$ denotes $\cos \theta_{jk}$ and $\sin \theta_{jk}$ respectively. Under a small perturbation the final (resultant) mass matrix can be written as

$$M = U_{0}^{*} M_{0}^{\text{diag}} U_{0}^{\dagger} + E,$$ \hspace{1cm} (6)

where the perturbation matrix $E$ has the general form

$$E = M - M_{0} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix}.$$ \hspace{1cm} (7)

Treating the three masses as eigenvalues of the mass matrix with each column of the mixing matrix as the corresponding eigenvector, we can use traditional perturbation methods to find the corrections to the three angles and three masses. From experiment we know that $m_{1}$ and $m_{2}$ are nearly degenerate, so that degenerate perturbation theory with $|\delta m_{21}^{0}| \ll |\delta m_{31}^{0}|$ and $|\epsilon_{ij}| < |m_{0}^{0}|$ (where $\delta m_{ij}^{0} = m_{j}^{0} - m_{i}^{0}$, and the index $k$ denotes the heaviest eigenstate), can be used. For simplicity, we assume $M_{0}$ and $E$ are real and employ the following notation:

$$\epsilon_{1} = \epsilon_{11}, \hspace{0.5cm} \epsilon_{2} = \epsilon_{12} + \epsilon_{13}, \hspace{0.5cm} \epsilon_{3} = \epsilon_{12} - \epsilon_{13}, \hspace{0.5cm} \epsilon_{4} = \epsilon_{22} + \epsilon_{33} + 2\epsilon_{23},$$
$$\epsilon_{5} = \epsilon_{22} - \epsilon_{33}, \hspace{0.5cm} \epsilon_{6} = \epsilon_{22} + \epsilon_{33} - 2\epsilon_{23} - 2\epsilon_{11}.$$ \hspace{1cm} (8)

We find the first order corrections to the three masses to be

$$\delta m_{i}^{(1)} = \frac{1}{4} \left[ 4\epsilon_{1} + \epsilon_{6} \pm \sqrt{8\epsilon_{3}^{2} + \epsilon_{6}^{2} + 4(\delta m_{21}^{0})^{2} + 4\epsilon_{6}(\sqrt{2}\epsilon_{3} \sin \theta_{12}^{0} + \epsilon_{6} \cos \theta_{12}^{0})} \right],$$ \hspace{1cm} (9)

where the plus sign is for $i = 1$ and the minus sign is for $i = 2$, and

$$\delta m_{3}^{(1)} = \frac{1}{2}\epsilon_{4}.$$ \hspace{1cm} (10)
Table 1: Best-fit values and 2σ ranges of the oscillation parameters [14] used to find the $\epsilon_{ij}$, with $\delta m^2 \equiv |m_2|^2 - m_1^2$ and $\Delta m^2 \equiv |m_3|^2 - (m_1^2 + |m_2|^2)/2$.

| Hierarchy | $\theta_{12}^{(\circ)}$ | $\theta_{13}^{(\circ)}$ | $\theta_{23}^{(\circ)}$ | $\delta m^2 (10^{-6}\text{eV}^2)$ | $|\Delta m^2| (10^{-3}\text{eV}^2)$ |
|-----------|-----------------|----------------|----------------|-----------------|-----------------|
| Normal    | $33.6^{+2.1}_{-2.0}$ | $8.9^{+0.9}_{-0.9}$ | $38.4^{+3.6}_{-3.3}$ | $7.54^{+0.36}_{-0.39}$ | $2.43^{+0.12}_{-0.16}$ |
| Inverted  | $33.6^{+2.1}_{-2.0}$ | $9.0^{+0.8}_{-1.0}$ | $38.8^{+0.3}_{-2.3} \oplus 47.5 - 53.2$ | $7.54^{+0.36}_{-0.39}$ | $2.42^{+0.11}_{-0.16}$ |

The first order corrections to the mixing angles are

$$
\delta \theta_{12}^{(1)} = \frac{1}{2} \arctan \frac{2\sqrt{2} \epsilon_3 \cos 2 \theta_{12}^0 - \epsilon_6 \sin 2 \theta_{12}^0}{2\sqrt{2} \epsilon_3 \sin 2 \theta_{12}^0 + \epsilon_6 \cos 2 \theta_{12}^0 + 2 \delta m_{0_{21}}^0},
$$

$$
\delta \theta_{23}^{(1)} = \frac{\epsilon_5 s_{12}^2 - \sqrt{2} \epsilon_3 s_{12} c_{12}^2}{2 \delta m_{0_{31}}^0} + \frac{\epsilon_5 c_{12}^2 + \sqrt{2} \epsilon_3 s_{12} c_{12}^2}{2 \delta m_{0_{32}}^0},
$$

$$
\delta \theta_{13}^{(1)} = \frac{\sqrt{2} \epsilon_2 c_{12}^2 - \epsilon_5 s_{12} c_{12}^2}{2 \delta m_{0_{31}}^0} + \frac{\sqrt{2} \epsilon_2 s_{12}^2 + \epsilon_5 s_{12} c_{12}^2}{2 \delta m_{0_{32}}^0},
$$

and the second order correction to $\theta_{12}$ is

$$
\delta \theta_{12}^{(2)} = -\frac{\sqrt{2} \epsilon_2 \epsilon_5 \cos 2(\theta_{12}^0 + \delta \theta_{12}^{(1)}) + (\epsilon_2^2 - \epsilon_5^2/2) \sin 2(\theta_{12}^0 + \delta \theta_{12}^{(1)})}{4 \delta m_{0_{21}}^0 \delta m_{0_{32}}^0}.
$$

Imposing $|\delta m_{0_{21}}^0| \ll |\delta m_{0_{31}}^0|$, the expressions for $\delta \theta_{23}^{(1)}$ and $\delta \theta_{13}^{(1)}$ simplify to

$$
\delta \theta_{23}^{(1)} \simeq \frac{\epsilon_5}{2 \delta m_{0_{31}}^0}, \quad \delta \theta_{13}^{(1)} \simeq \frac{\sqrt{2} \epsilon_2}{2 \delta m_{0_{31}}^0}.
$$

We note that while $\delta \theta_{23}^{(1)}$ and $\delta \theta_{13}^{(1)}$ are suppressed by a factor of order $\epsilon_j / \delta m_{0_{31}}^0$, to leading order $\delta \theta_{12}$ depends only on ratios of linear combinations of $\epsilon_3$, $\epsilon_6$ and $\delta m_{0_{21}}^0$ (which is $\mathcal{O}(\epsilon_{ij})$). Therefore large corrections to $\theta_{12}$ are possible even for small corrections to $\theta_{23}$ and $\theta_{13}$.

A recent global three-neutrino fit [14] yields the parameter values in Table 1. We have done a numerical search to find perturbed mass matrices that give the oscillation parameters and which have small perturbations. In our search, we first fix $\theta_{23}^0 = 45^\circ$ and $\theta_{13}^0 = 0$, consistent with $\mu - \tau$ symmetry, and choose a particular value for $\theta_{12}^0$ and the magnitude of $m_1$ for the normal hierarchy (or $m_3$ for the inverted hierarchy). The global fit in Table 1 then defines the magnitudes of the other two final masses and the three final mixing angles (since $\theta_{13}^0 = 0$, the initial Dirac phase does not matter).

We characterize the size of the perturbation as the root-mean-square (RMS) value of the perturbations, i.e.,

$$
\epsilon_{RMS} = \sqrt{\frac{\sum_{i,j=1}^3 |M_{ij} - M_{0ij}|^2}{9}},
$$

(16)
Table 2: Top half: values of the perturbations (in $10^{-3}$ eV) that give the best-fit parameters in Table 1 and have the minimum $\epsilon_{RMS}$ for the given $\theta_{12}^0$, for the normal hierarchy and $m_1 = 0$. Bottom half: representative values that fit the experimental data within $2\sigma$ and for which all $\epsilon_{ij}$ have a similar magnitude (with $m_1^0 = 0$, $m_2^0 = 0.0054$ eV, $m_3^0 = 0.0595$ eV, $m_1 = 0.0072$ eV, $\delta = 180^\circ$ and all other phases equal to 0).

| $\theta_{12}^0$ ($^\circ$) | $\epsilon_{11}$ | $\epsilon_{12}$ | $\epsilon_{13}$ | $\epsilon_{22}$ | $\epsilon_{23}$ | $\epsilon_{33}$ | $\epsilon_{RMS}$ |
|--------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 60                       | -3.05          | -3.50          | -5.99          | -2.72          | -1.52          | 5.77           | 4.10           |
| 45 (BM)                  | -1.32          | -4.74          | -4.74          | -3.58          | -0.66          | 4.90           | 3.79           |
| 35.3 (TBM)               | 0.32           | -4.66          | -4.82          | -4.40          | 0.16           | 4.08           | 3.74           |
| 30 (HM)                  | 1.07           | -4.31          | -5.18          | -4.78          | 0.54           | 3.71           | 3.79           |
| 0                        | 0.00           | -1.38          | -8.11          | -4.24          | 0.00           | 4.24           | 4.36           |
| 60                       | 5.41           | -4.17          | -4.52          | -5.00          | -9.94          | 3.36           | 6.14           |
| 45 (BM)                  | 6.76           | -4.43          | -4.26          | -5.67          | -9.27          | 2.69           | 6.08           |
| 35.3 (TBM)               | 7.66           | -4.32          | -4.37          | -6.12          | -8.82          | 2.24           | 6.08           |
| 30 (HM)                  | 8.11           | -4.17          | -4.52          | -6.35          | -8.59          | 2.01           | 6.09           |
| 0                        | 9.46           | -2.52          | -6.17          | -7.02          | -7.92          | 1.34           | 6.28           |

where $i$ and $j$ sum over neutrino flavors. Hence, $\epsilon_{RMS}$ is determined by the following quantities: three initial masses, two initial Majorana phases, two final Majorana phases and one final Dirac phase. We scan over these quantities with all phases taken to be either 0 or $180^\circ$ to find the minimum value of $\epsilon_{RMS}$ for a given $\theta_{12}^0$. We follow the same procedure for classes (b) and (c) below. For class (d), all values of the phases are allowed.

We show the perturbations that give the smallest $\epsilon_{RMS}$ for the normal hierarchy, $m_1 = 0$ and several values of $\theta_{12}^0$ in Table 2. It is clear that the sizes of $\epsilon_{RMS}$ are approximately the same regardless of the value of $\theta_{12}^0$; we find that the smallest $\epsilon_{RMS}$ for each $\theta_{12}^0$ varies by at most 17% for the examples shown. This can be explained by the perturbation results derived above as follows. From Eq. (8) we have $\epsilon_{RMS} = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \frac{1}{2}\epsilon_5^2} + \frac{1}{3}(2\epsilon_1 + \epsilon_6)^2/3$; since $m_3 \gg m_1, m_2$ for the normal hierarchy with $m_1 = 0$ eV and the first order perturbations of the three masses are much smaller than $m_3$, we can assume $\delta m_{31}^0 \approx m_3^0 \approx m_3 \approx \sqrt{\Delta m^2} = 0.0493$ eV. Then from Eq. (15) we know that in order to get the correction $\delta \theta_{23} = -6.6^\circ$ and $\delta \theta_{13} = 8.9^\circ$ for any value of $\theta_{12}^0$, we need $\epsilon_5 = -0.0114$ eV and $\epsilon_2 = 0.0108$ eV, so that $\sqrt{\epsilon_3^2 + \epsilon_6^2/2}/3 = 0.00449$ eV, which is already close to the $\epsilon_{RMS}$ values found in Table 2. The small discrepancy can be explained by the perturbation of the three masses and other $\epsilon$’s. Hence, we can say that the size of the perturbation mainly comes from the corrections to $\theta_{23}$ and $\theta_{13}$. From Eq. (11) we know that the correction to $\theta_{12}$ is determined by the relative ratio of $\epsilon_3$ to $\epsilon_6$ and the actual size of the perturbation does not matter. This means that we can have large corrections for $\theta_{12}$ with a (relatively) small perturbation.

We note that initial values of $\theta_{12}$ on the “dark side” ($\theta_{12}^0 > 45^\circ$ and $m_1^0 < m_2^0$) can also fit the data with perturbations that are similar in magnitude to those needed for tri-bimaximal mixing (see the entry for $\theta_{12}^0 = 60^\circ$ in Table 2).
Table 3: Top half: same as Table 2 except for the inverted hierarchy and $m_3 = 0$. Bottom half: same as Table 2 except for the inverted hierarchy and $m_1^0 = 0.05$ eV, $m_2^0 = 0.052$ eV, $m_3^0 = 0$, and all phases equal to 0.

| $\theta_{12}$ (°) | $\epsilon_{11}$ | $\epsilon_{12}$ | $\epsilon_{13}$ | $\epsilon_{22}$ | $\epsilon_{23}$ | $\epsilon_{33}$ | $\epsilon_{RMS}$ |
|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 60 (BM)           | -0.47           | -5.29           | -5.28           | 5.17            | 0.03            | -5.12           | 4.28            |
| 35.3 (TBM)        | 0.16            | -5.23           | -5.36           | 5.07            | 0.08            | -5.22           | 4.28            |
| 0                 | 0.00            | -4.47           | -6.12           | 5.15            | 0.00            | -5.15           | 4.32            |
|                   | -3.56           | -4.89           | -5.27           | 5.95            | 2.67            | -3.92           | 4.49            |
| 45 (BM)           | -3.06           | -4.98           | -5.18           | 5.70            | 2.92            | -4.17           | 4.47            |
| 35.3 (TBM)        | -2.73           | -4.94           | -5.22           | 5.54            | 3.08            | -4.34           | 4.46            |
| 30 (HM)           | -2.56           | -4.89           | -5.27           | 5.45            | 3.17            | -4.42           | 4.46            |
| 0                 | -2.06           | -4.28           | -5.88           | 5.20            | 3.42            | -4.67           | 4.50            |

In the top half of Table 2, $\epsilon_{11}$ and $\epsilon_{23}$ are much smaller than the other $\epsilon_{ij}$ for some values of $\theta_{12}$. We have checked that if these values are set to zero, the experimental constraints can still be satisfied at the 2σ level without a large change in the nonzero parameters. Therefore if some perturbations are exactly zero due to symmetries, the resulting mass matrix can still fit the experimental data with small perturbations.

For the inverted hierarchy, some representative sets of $\epsilon_{ij}$ that give the minimum $\epsilon_{RMS}$ are shown in Table 3 for $m_3 = 0$. The minimum $\epsilon_{RMS}$ as a function of $\theta_{12}$ varies only by about 1% in this case, i.e., the minimum $\epsilon_{RMS}$ varies with $\theta_{12}$ even less for the inverted hierarchy than for the normal hierarchy.

Clearly, if perturbations are large enough that tri-bimaximal mixing can explain the experimental data, then other $\mu - \tau$ mixing scenarios, such as bimaximal, hexagonal mixing and $A_5$ mixing, can also explain the experimental data with about the same size perturbation. Hence, tri-bimaximal mixing has no special position among the $\mu - \tau$ symmetry mixing scenarios when a perturbation is required to fit the experimental data. Also, it is possible for all the perturbations to have a similar magnitude and still give the oscillation parameters within their 2σ ranges; see the bottom half of Tables 2 and 3.

We also varied the size of the final masses by changing the value of $m_1$ in the normal hierarchy and $m_3$ in the inverted hierarchy. We find that the minimum $\epsilon_{RMS}$ decreases as the size of the final masses increases for both the normal and inverted hierarchies. For the quasi-degenerate hierarchy (in which the magnitude of the absolute masses is larger than $\sqrt{\Delta m^2}$) the size of the perturbation can be very small. This can be explained by the perturbation equations: since $\delta m_{31}^0 \approx m_3 - m_1 \approx \Delta m^2/(m_3 + m_1)$ for small perturbations, and $\Delta m^2$ is fixed by experimental data, then $\delta m_{31}^0$ will decrease if the masses increase, and similarly for $\delta m_{32}^0$. Then Eqs. (12) and (13) show that in order to get the same corrections for $\theta_{13}^0$ and $\theta_{23}^0$, the size of the perturbation should also decrease.
Classes (b) and (c): $\theta_{23}^0 = 45^\circ, \theta_{12}^0 = 0$ or $90^\circ$

For class (b) ($\theta_{23}^0 = 45^\circ, \theta_{12}^0 = 0$), since the Dirac phase is irrelevant, the initial mixing matrix and mass matrix can be written as

$$U_0 = \begin{pmatrix} \cos \theta_{13}^0 & 0 & \sin \theta_{13}^0 \\ -\sin \theta_{13}^0 \sqrt{2} & 1 & -\cos \theta_{13}^0 \sqrt{2} \\ \sin \theta_{13}^0 \sqrt{2} & 1 & -\cos \theta_{13}^0 \sqrt{2} \end{pmatrix},$$  \quad (17)

and the initial mass matrix is

$$M_0 = U_0^\dagger M_0^{\text{diag}} U_0^\dagger = \begin{pmatrix} m_1^0 c_{13}^2 + m_3^0 s_{13}^2 & (m_3^0 - m_1^0) s_{13} c_{13} & (m_3^0 - m_2^0) s_{13} c_{13} \\ (m_3^0 - m_1^0) s_{13} c_{13} & \frac{1}{2}(m_1^0 + m_3^0 c_{13}^2 + m_1^0 s_{13}^2) - \frac{1}{2}(m_2^0 + m_3^0 c_{13}^2 + m_1^0 s_{13}^2) \\ (m_3^0 - m_2^0) s_{13} c_{13} & \frac{1}{2}(m_2^0 + m_3^0 c_{13}^2 + m_1^0 s_{13}^2) - \frac{1}{2}(m_1^0 + m_3^0 c_{13}^2 + m_1^0 s_{13}^2) \end{pmatrix}.$$  \quad (18)

If we redefine the phase of the wavefunction $\psi_3$ to $-\psi_3$, or change the initial angle $\theta_{23}^0$ from $45^\circ$ to $135^\circ$ and switch the indices 2 and 3, then the mass matrix in Eq. (18) is exactly the same as that in Eq. (17).

For the above initial mass matrix, corrections must shift $\theta_{12}$ from 0 to $33.6^\circ$, and $\theta_{13}$ from the initial arbitrary angle to $9.0^\circ$. We used the same scan procedure as before and searched for the minimum $\epsilon_{\text{RMS}}$ for various values of $\theta_{13}^0$ (see Table 4). We find that for $\theta_{13}^0 < 20^\circ$, the data can be explained with about the same size perturbation as was found for class (a). For example, when $\theta_{13}^0 = 0$ for class (b), the initial mass matrix is the same as $\theta_{12}^0 = 0$ for class (a), and therefore the minimum $\epsilon_{\text{RMS}}$ is also the same. In particular, when $\theta_{13}^0$ is close to $9.0^\circ$ in class (b), the minimum $\epsilon_{\text{RMS}}$ is even smaller than the minimum value for class (a) because the correction to $\theta_{13}$ is smaller in this case. Although the correction to $\theta_{12}$ is large, it does not affect the size of the perturbation too much because its size is mainly determined by the corrections to $\theta_{13}$ and $\theta_{23}$, as noted before. However, for $\delta\theta_{13}$ greater than about $20^\circ$, the size of the perturbation required to fit the data becomes larger since $\theta_{13}$ must change by more than $10^\circ$.

For class (c) ($\theta_{23}^0 = 45^\circ, \theta_{12}^0 = 90^\circ$), we find that switching $m_1^0$ with $m_2^0$ makes the initial mass matrix the same as the initial mass matrix of class (b). Since we scan all possible values of $m_1^0$ and $m_2^0$, the minimum $\epsilon_{\text{RMS}}$ for a given $\theta_{13}^0$ for class (c) is the same as for class (b).

Class (d): $\theta_{23}^0 = 45^\circ, \delta^0 = \pm 90^\circ$

If we fix $\theta_{23}^0 = 45^\circ, \delta^0 = \pm 90^\circ$ and vary both $\theta_{12}^0$ and $\theta_{13}^0$, this category includes mixing scenarios such as the tetramaximal mixing pattern ($T^4M$) [13], and the correlative mixing pattern with $\delta = \pm 90^\circ$ [15]. For $\theta_{13}^0 < 20^\circ$ and $\theta_{12}^0 \leq 45^\circ$, the smallest $\epsilon_{\text{RMS}}$ for the normal hierarchy (with $m_1 = 0$) varies from $2.29 \times 10^{-3}$ eV to $5.26 \times 10^{-3}$ eV, where the minimum value occurs at $\theta_{13}^0 = 9^\circ$ and $\theta_{12}^0 = 32^\circ$, and the maximum value occurs at $\theta_{13}^0 = 20^\circ$ and
Table 4: Top half: same as Table 2, except for class (b) ($\theta_{12}^0 = 0$). Bottom half: same as Table 2, except for class (b).

| $\theta_{13}^0$ (°) | $\epsilon_{11}$ | $\epsilon_{12}$ | $\epsilon_{13}$ | $\epsilon_{22}$ | $\epsilon_{23}$ | $\epsilon_{33}$ | $\epsilon_{RMS}$ |
|----------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0                    | 0.00           | -1.38          | -8.11          | -4.24          | 0.00           | 4.24           | 4.36           |
| 5                    | 0.48           | 1.44           | -5.28          | -4.48          | -0.24          | 4.00           | 3.27           |
| 10                   | -0.44          | 4.21           | -2.52          | -4.02          | 0.22           | 4.46           | 3.06           |
| 15                   | -2.64          | 6.59           | -0.14          | -2.92          | 1.32           | 5.56           | 3.90           |
| 20                   | -5.85          | 8.30           | 1.57           | -1.32          | 2.93           | 7.17           | 5.24           |
| 0                    | 9.46           | -2.52          | -6.17          | -7.02          | -7.92          | 1.34           | 6.28           |
| 5                    | 9.01           | 1.13           | -2.52          | -6.80          | -7.69          | 1.56           | 5.41           |
| 10                   | 7.66           | 4.67           | 1.02           | -6.12          | -7.02          | 2.24           | 5.22           |
| 15                   | 5.47           | 8.00           | 4.35           | -5.03          | -5.93          | 3.33           | 5.80           |
| 20                   | 2.50           | 11.00          | 7.35           | -3.54          | -4.44          | 4.82           | 6.92           |

$\theta_{12}^0 = 0$. Therefore small perturbations can fit the experimental data for a wide range of $\theta_{12}$ and $\theta_{13}$ for class (d).

In summary, we studied small perturbations to Majorana mass matrices with $\mu - \tau$ symmetry that yield experimentally preferred oscillation parameters. We find that the size of the perturbations (which decreases as the neutrino mass scale is increased), is mainly determined by the corrections to $\theta_{23}$ and $\theta_{13}$, and that small perturbations can give a very large correction to $\theta_{12}$ because to first order, the $\theta_{12}$ correction depends only on the ratio of perturbation terms and not on their absolute size. Hence, most mixing scenarios with $\mu - \tau$ symmetry can explain the experimental data with perturbations of similar magnitude, and tri-bimaximal mixing has no special place among scenarios with $\mu - \tau$ symmetry. We also find that slightly perturbed $\mu - \tau$ symmetric models with $\theta_{12} = 0$ or 90° are viable for $\theta_{13} < 20°$.

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