Abstract. Continuous first-order logic is used to apply model-theoretic analysis to analytic structures (e.g. Hilbert spaces, Banach spaces, probability spaces, etc.). Classical computable model theory is used to examine the algorithmic structure of mathematical objects that can be described in classical first-order logic. The present paper shows that probabilistic computation (sometimes called randomized computation) can play an analogous role for structures described in continuous first-order logic.

The main result of this paper is an effective completeness theorem, showing that every decidable continuous first-order theory has a probabilistically decidable model. Later sections give examples of the application of this framework to various classes of structures, and to some problems of computational complexity theory.

1. Introduction

Continuous first-order logic was introduced in [3, 1] as a model-theoretic context sufficient to handle stability theory for so-called “metric structures.” These are many-sorted structures in which each sort is a complete metric space of finite diameter. Key examples include Hilbert spaces, Banach spaces, probability spaces, and probability spaces with a distinguished automorphism.

For classical first-order model theory, there is a meaningful sense of computation and effectiveness: In a group, for instance, we have a reasonable algorithmic understanding of a group if the set of triples constituting the Cayley table (equivalently, the set of words equal to the identity element) is decidable. Of course, there are often still many algorithmic unknowns in the group, such as the conjugacy problem and the isomorphism problem [9]. The aim of the present paper is to provide a similar framework for continuous first-order logic.

The framework suggested is probabilistic computation. This model of computation has seen wide use in complexity theory [14, 27], and there is some room for hope that an understanding of the relationship between continuous and classical first-order logic might yield insights into the relationship between probabilistic and deterministic computation. Section 6 gives reasons for such hope.

Not to get carried away in speculation, though, it is still cause for contentment that a way can be found to meaningfully talk about algorithmic information in the context of metric structures. The impossibility of finding an algorithm to solve arbitrary Diophantine equations (see [19]), the relationship of isoperimetric functions to the word problem [24, 8], and much more depend on a notion of computation adequate to the context of countable rings (in the case of Diophantine
equations) and groups (in the case of the word problem). Some preliminary results on specific metric structures, given in Section 5 will suggest that there is ground for fruitful research in the effective theory of these structures.

A key argument that probabilistic computation is the right algorithmic framework for this context is that it admits an effective completeness theorem. The classical theorem is this.

**Theorem 1.1 (Effective Completeness Theorem).** A (classically) decidable theory has a (classically) decidable model.

A full proof of this result may be found in [12], but it was known much earlier, at least to Millar [20]. The main theoretical contribution of the present paper will be to interpret the terms of this theorem in such a way as to apply to a continuous first-order theory and probabilistic computation. The main result of the present paper is the proof in Section 4 of the theorem naturally corresponding to Theorem 1.1.

Section 2 will describe the syntax and semantics for continuous first-order logic. The reader familiar with [2] or [1] will find nothing new in Section 2, except a choice of a finite set of logical connectives (no such choice is yet canonical in continuous first-order logic). Section 3 will define probabilistic Turing machines and the class of structures they compute. Section 4 will contain the proof of the main result. Section 5 will contain some examples, exhibiting different aspects of the information which is conveyed by the statement that a certain structure is probabilistically computable, and in Section 6 we will conclude with some remarks on time complexity of structures.

2. Continuous First-Order Logic

We will, in keeping with the existing literature on continuous first-order logic, adopt the slightly unusual convention of using 0 as a numerical value for True (or acceptance) and 1 as a numerical value for False (or rejection). The authors of [3] chose this convention to emphasize the metric nature of their logic. Continuous first-order logic is an extension of Lukasiewicz propositional logic. The following definitions are from [2].

2.1. Semantics.

**Definition 2.1.** A continuous signature is an object of the form \( \mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n) \) where

1. \( \mathcal{R} \) and \( \mathcal{F} \) are disjoint and \( \mathcal{R} \) is nonempty, and
2. \( n \) is a function associating to each member of \( \mathcal{R} \cup \mathcal{F} \) its arity
3. \( \mathcal{G} \) has the form \( \{ \delta_{s,i} : \{0, 1\} \rightarrow \{0, 1\} : s \in \mathcal{R} \cup \mathcal{F} \text{ and } i < n_s \} \)

Members of \( \mathcal{R} \) are called relation symbols, and members of \( \mathcal{F} \) function symbols. We now define the class of structures.

**Definition 2.2.** Let \( \mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n) \) be a continuous signature. A continuous \( \mathcal{L} \)-pre-structure is an ordered pair \( \mathfrak{M} = (M, \rho) \), where \( M \) is a non-empty set, and \( \rho \) is a function on \( \mathcal{R} \cup \mathcal{F} \) such that

1. To each function symbol \( f \), the function \( \rho \) assigns a mapping \( f^\mathfrak{M} : M^{n(f)} \rightarrow M \)
Definition 2.3. A continuous weak \( \mathcal{L} \)-structure is a continuous \( \mathcal{L} \)-pre-structure such that \( \rho \) assigns to \( d \) a metric.

Since we are concerned here with countable structures (i.e. those accessible to computation), we will not use the stronger notion of a continuous \( \mathcal{L} \)-structure common in the literature, which requires that \( \rho \) be assigned to a complete metric. However, it is possible, given a continuous weak structure (even a pre-structure), to pass to a completion \([2]\).

Definition 2.4. Let \( V \) denote the set of variables, and let \( \sigma : V \rightarrow M \). Let \( \varphi \) be a formula.

1. The interpretation under \( \sigma \) of a term \( t \) (written \( \mathfrak{M}(t^{\mathfrak{M}}, \sigma) \)) is defined by replacing each variable \( x \) in \( t \) by \( \sigma(x) \).
2. Let \( \varphi \) be a formula. We then define the value of \( \varphi \) in \( \mathfrak{M} \) under \( \sigma \) (written \( \mathfrak{M}(\varphi, \sigma) \)) as follows:
   \( \mathfrak{M}(P(l), \sigma) := P^{d_{\mathfrak{M}}}(l^{\mathfrak{M}}, \sigma) \)
   \( \mathfrak{M}(\alpha \lor \beta, \sigma) := \max(\mathfrak{M}(\alpha, \sigma) - \mathfrak{M}(\beta, \sigma), 0) \)
   \( \mathfrak{M}(\neg \alpha, \sigma) := 1 - \mathfrak{M}(\alpha, \sigma) \)
   \( \mathfrak{M}(\frac{1}{2} \alpha, \sigma) := \frac{1}{2} \mathfrak{M}(\alpha, \sigma) \)
   \( \mathfrak{M}(\sup x_{\alpha}, \sigma) := \sup_{a \in M} \mathfrak{M}(\alpha, \sigma^{a}_{x}) \)

3. We write \( \mathfrak{M}(\varphi, \sigma) \models \varphi \) exactly when \( \mathfrak{M}(\varphi, \sigma) = 0 \).

Of course, if \( \varphi \) has no free variables, then the value of \( \mathfrak{M}(\varphi, \sigma) \) is independent of \( \sigma \).

2.2. Syntax.

Definition 2.5. Let \( S_{0} \) be a set of distinct propositional symbols. Let \( S \) be freely generated from \( S_{0} \) by the formal binary operation \( \lor \) and the unary operations \( \neg \) and \( \frac{1}{2} \). Then \( S \) is said to be a continuous propositional logic.

We now define truth assignments for continuous propositional logic.

Definition 2.6. Let \( S \) be a continuous propositional logic.

1. if \( v_{0} : S_{0} \rightarrow [0, 1] \) is a mapping, we can extend \( v_{0} \) to a unique mapping \( v : S \rightarrow [0, 1] \) by setting
   \( v(\varphi \lor \psi) := \max(v(\varphi) - v(\psi), 0) \)
   \( v(\neg \varphi) := 1 - v(\varphi) \)
   \( v(\frac{1}{2} \varphi) = \frac{1}{2} v(\varphi) \)
We say that \( v \) is the **truth assignment** defined by \( v_0 \).

(2) We write \( v \models \Sigma \) for some \( \Sigma \subseteq \mathcal{S} \) whenever \( v(\varphi) = 0 \) for all \( \varphi \in \Sigma \).

Roughly, \( \varphi \models \psi \) has the sense of \( \psi \rightarrow \varphi \). We can also, of course, define \( \varphi \land \psi \) as \( \varphi \models (\varphi \land \psi) \), and \( \varphi \lor \psi \) via deMorgan’s law. We can also define something resembling equivalence, \(|\varphi \land \psi| = (\varphi \models \psi) \lor (\psi \models \varphi) \). Lukasiewicz propositional logic is the fragment of this logic which does not involve \( \frac{1}{2} \).

To make a first-order predicate variant of this logic, we use \( \sup \) in the place of \( \forall \) and \( \inf \) in the place of \( \exists \) (with the obvious semantics, as will be described in what follows). We typically also include a binary function \( d \), whose standard interpretation is generally a metric. Now we give the syntactic axioms for continuous first-order logic:

\[
\begin{align*}
(A1) \quad & (\varphi \models \psi) \models \varphi \\
(A2) \quad & ((\chi \models \varphi) \models (\chi \models \psi)) \models (\psi \models \varphi) \\
(A3) \quad & (\varphi \models (\varphi \models \psi)) \models (\psi \models (\psi \models \varphi)) \\
(A4) \quad & (\varphi \models \psi) \models (\neg \psi \models \neg \varphi) \\
(A5) \quad & \frac{1}{2} \varphi \models (\varphi \models \frac{1}{2} \varphi) \\
(A6) \quad & (\varphi \models \frac{1}{2} \varphi) \models \frac{1}{2} \varphi. \\
(A7) \quad & (\sup_x \varphi \models \sup_x \varphi) \models \sup_x (\psi \models \varphi) \\
(A8) \quad & \sup_x \varphi \models \psi \models \varphi \text{ where no variable in } t \text{ is bound by a quantifier in } \varphi. \\
(A9) \quad & \sup_x \varphi \models \varphi, \text{ wherever } x \text{ is not free in } \varphi. \\
(A10) \quad & d(x, x) \\
(A11) \quad & d(x, y) = d(y, x) \\
(A12) \quad & d(x, z) = d(y, x) = d(y, z) \\
(A13) \quad & \text{For each } f \in \mathcal{F}, \text{ each } \epsilon \in (0, 1], \text{ and each } r, q \in \mathcal{D} \text{ with } r > \epsilon \text{ and } q < \delta_{f,i}(\epsilon), \text{ the axiom } (q \models d(z, w)) \land (d(f(\bar{x}, z, y), f(\bar{x}, w, y)) = r), \text{ where } lh(\bar{x}) + lh(\bar{y}) = n_f - 1. \\
(A14) \quad & \text{For each } P \in \mathcal{R}, \text{ each } \epsilon \in (0, 1], \text{ and each } r, q \in \mathcal{D} \text{ with } r > \epsilon \text{ and } q < \delta_{P,i}(\epsilon), \text{ the axiom } (q \models d(z, w)) \land (d(P(\bar{x}, z, y), P(\bar{x}, w, y)) = r), \text{ where } lh(\bar{x}) + lh(\bar{y}) = n_P - 1.
\end{align*}
\]

Axioms A1–A4 are those of Lukasiewicz propositional logic, and axioms A5–A6 are those of continuous propositional logic. Axioms A7–A9 describe the role of the quantifiers. Axioms A10–A12 guarantee that \( d \) is a pseudometric, and axioms A13–A14 guarantee uniform continuity of functions and relations. We write \( \Gamma \models Q \varphi \) whenever \( \varphi \) is provable from \( \Gamma \) in continuous first-order logic. Where no confusion is likely, we will write \( \Gamma \models \varphi \).

### 3. Probabilistically Computable Structures

If \( M \) is a Turing machine, we write \( M^\omega(n) \) for the result of applying \( M \) to input \( n \) with oracle \( x \). Excepting the polarity change to match the conventions above, the following definition is standard; it may be found, for instance, in [27].

**Definition 3.1.** Let \( 2^\omega \) be the set of infinite binary sequences, with the usual Lebesgue probability measure \( \mu \).

1. A **probabilistic Turing machine** is a Turing machine equipped with an oracle for an element of \( 2^\omega \), called the *random bits*, with output in \( \{0, 1\} \).
2. We say that a probabilistic Turing machine \( M \) **accepts** \( n \) with probability \( p \) if and only if \( \mu\{x \in 2^\omega : M^\omega(n) \downarrow = 0\} = p \).
(3) We say that a probabilistic Turing machine $M$ rejects $n$ with probability $p$ if and only if $\mu\{x \in 2^\omega : M^x(n) \downarrow = 1\} = p$.

**Definition 3.2.** Let $L$ be a computable continuous signature. Let $M$ be a continuous $L$-structure. Let $L(M)$ be the expansion of $L$ by a constant $c_m$ for each $m \in M$ (i.e. a unary predicate $c_m \in R$ where $c_m(x) := d(x,m)$). Then the continuous atomic diagram of $M$, written $D(M)$ is the set of all pairs $(\varphi, p)$, where $\varphi$ is a quantifier-free (i.e. sup- and inf-free) sentence in $L(M)$ and $M(\varphi, \sigma) = p$. The continuous elementary diagram $D^*(M)$ is the same, except that $\varphi$ is not required to be quantifier-free.

Note that the definition is independent of $\sigma$, since a sentence has no free variables.

**Definition 3.3.** We say that a continuous pre-structure $M$ is probabilistically computable (respectively, probabilistically decidable) if and only if there is some probabilistic Turing machine $T$ such that, for every pair $(\varphi, p) \in D(M)$ (respectively, $D^*(M)$) the machine $T$ accepts $\varphi$ with probability $p$.

Suppose $T$ is a deterministic machine (i.e. one that makes no use of its random bits; a classical Turing machine) and $M$ a classical first-order structure. Then this definition corresponds exactly to the classical definition of a computable structure.

We cannot do entirely without the probabilistic machines (that is, we cannot thoroughly understand probabilistically computable structures using only classical Turing machines), as the following result shows.

**Lemma 3.4** (No Derandomization Lemma). There is a probabilistically computable weak structure $M$ such that the set $\{(\varphi, p) \in D(M) : p \in D\}$ is not classically computable.

**Proof.** Let $U$ be a computably enumerable set, and let $S$ be the complement of $U$. We first construct a probabilistically computable function $f$ such that

$$P(f^x(x) = 0) = \frac{1}{2}$$

if and only if $x \in S$. At stage $t$, if $x \in U_t$, pick two strings $\sigma_t, \tau_t$ of length $t + 2$ such that $f_t(x)$ does not halt with random bits $\sigma_t$ or $\tau_t$. We define the function

$$f_{t+1} := f_t \cup \{f^\sigma(x) = 0, f^\tau(x) = 1\}.$$  

On the other hand, if $x \in U_t$, then we arrange that $f^\sigma_t(x) = 0$ for all $\sigma$ of length at most $t + 2$ where $f^\sigma_t(x)$ does not halt. Let $f = \bigcup_{t \in \omega} f_t$. Now if $x \in S$, we never see $x \in U_t$, so $f(x) = 0$ with probability $\frac{1}{2}$. Otherwise, there is some $t$ such that $x \in U_t - U_{t-1}$, and then $f(x) = 0$ with probability $1 - \sum_{i=2}^t 2^{-i} > \frac{1}{2}$.

Now we let $M$ be the structure $(\omega, f)$, where $f$ is interpreted as a unary predicate in the obvious way, and $d$ is the discrete metric. If we could decide membership in $\{(\varphi, p) \in D(M) : p \in D\}$ with a classical Turing machine, then we could also decide membership in $U$. □

Of course, this same argument could work for any other uniformly computable set of reals in place of $D$. To some extent, though, we could do without the probabilistic machines. The following results show that the statement of the No Derandomization Lemma is the strongest possible.
Proposition 3.5. For any probabilistically computable pre-structure \( \mathcal{M} \), there is some (classically) computable function \( f \), monotonically increasing in the second variable, and some (classically) computable function \( g \), monotonically decreasing in the second variable, such that for any pair \( (\varphi, p) \in D(\mathcal{M}) \), we have \( \lim_{s \to \infty} f(\varphi, s) = p \) and \( \lim_{s \to \infty} g(\varphi, s) = p \).

Proof. Let \( \mathcal{M} \) be computed by the probabilistic Turing machine \( T_\mathcal{M} \). Let \( (\sigma_s)_{s \in \omega} \) be an effective list of all strings in \( 2^{< \omega} \). Now we define

\[
\begin{align*}
    f(\varphi, s) &:= \sum_{i \leq s} \left( T_{\mathcal{M}}^\sigma_i(\varphi) \cdot 2^{-lh(\sigma_i)} \right), \\
    g(\varphi, s) &:= \sum_{i \leq s} \left( T_{\mathcal{M}}^\sigma_i(\varphi) \cdot 2^{-lh(\sigma_i)} \right).
\end{align*}
\]

The definition of \( g \) is symmetric. These clearly have the correct properties. \( \Box \)

Functions of the same form as \( f \) and \( g \) are often seen in classical computable model theory \([15, 13, 6, 7]\).

Corollary 3.6. For any probabilistically computable pre-structure \( \mathcal{M} \), the set of pairs, \( (\varphi, p) \in D(\mathcal{M}) \) is the complement of a (classically) computably enumerable set.

Proof. Follows immediately from Proposition 3.5. \( \Box \)

These limitations notwithstanding, the definition via probabilistic machines gives a more natural continuity with the established literature on continuous first-order model theory \([1]\). In addition, this definition is in any case not dispensable when, for instance, time complexity of computation is at issue (see Section 6).

4. Effective Completeness

Theorem 1.1 is an important piece of evidence that classical Turing computation (or any of the many equivalent concepts) is properly synchronized with classical first-order logic. In particular, it asserts that under the minimal, obviously necessary hypotheses, a classical first-order theory has a model which can be represented by a classical computation. The aim of the present section is to prove a similar result for continuous first-order logic and probabilistic computation. The following analogue to the classical concept of the decidability of a theory was proposed in \([2]\).

Definition 4.1. Let \( \mathcal{L} \) be a continuous signature and \( \Gamma \) a set of formulas of \( \mathcal{L} \).

(1) We define

\[ \varphi^\circ_T := \sup \{ \mathcal{M}(\varphi, \sigma) : (\mathcal{M}, \sigma) \models \Gamma \} \].

(2) If \( T \) is a complete continuous first-order theory, we say that \( T \) is \emph{decidable} if and only if there is a (classically) computable function \( f \) such that \( f(\varphi) \) is an index for a computable real number equal to \( \varphi^\circ_T \).

Theorem 4.2. Let \( T \) be a decidable continuous first-order theory. Then there is a probabilistically decidable continuous pre-structure \( \mathcal{M} \) such that \( \mathcal{M} \models T \).

Proof. The construction of a model \( \mathcal{M} \) is given in \([2]\), by an analogue of Henkin’s method. Our model will be essentially the same, except that some care must be taken with effectiveness. The principal content of the theorem consists in showing that this structure is probabilistically decidable. We will define a probabilistic Turing machine which, for any formula \( \varphi \), accepts \( \varphi \) with probability \( \mathcal{M}(\varphi) \).
We begin by adding Henkin witnesses. Let $D$ denote the dyadic numbers in the interval (i.e. those of the form $\frac{k}{2^n}$ for $k, n \in \mathbb{N}$).

**Definition 4.3.** Let $\Gamma$ be a set of formulae. Then $\Gamma$ is said to be *Henkin complete* if for every formula $\varphi$, every variable $x$, and every $p < q \in D$, there is a constant $c$ such that

$$(\sup_x \varphi \dashv q) \land (p \dashv \varphi[c/x]) \in \Gamma.$$ 

**Lemma 4.4.** We can effectively extend $T$ to a consistent set $\Gamma$ of formulae which is Henkin complete.

**Proof.** Let $L_0 = L$. For each $n$, let $L_{n+1}$ be the result of adding, for each formula $\varphi$ in $L_n$, and for each $x, p, q$ as in the previous definition, a new constant $c_{(\varphi, x, p, q)}$. We can also extend the theory $T$, beginning with $\Gamma_0 = T$. For each $n$, the set $\Gamma_{n+1}$ is produced by adding to $T$, for each formula $\varphi$ in $L_n$ and each $x, p, q$ as in the previous definition, the formula $(\sup_x \varphi \dashv q) \land (p \dashv \varphi[c_{(\varphi, x, p, q)}/x])$. Let $\Gamma = \bigcup_n \Gamma_n$.

The consistency of $\Gamma$ is demonstrated in [2].

Note that this construction is in every way effective. In particular, there is a (classically) computable function which will, given $p, q \in D$ and G"{o}del numbers for $\varphi$ and $x$, give us a G"{o}del number for $c_{(\varphi, x, p, q)}$. Moreover, the set $\Gamma$ is (classically) computable.

We write $L^* = \bigcup_n L^n$, and $C = \{c_{(\varphi, x, p, q)}\}$.

**Lemma 4.5.** We can effectively extend $\Gamma$ to a consistent set $\Delta^0$ such that for all formulae $\varphi, \psi$ in $L^*$ we have $\varphi \dashv \psi \in \Delta$ or $\psi \dashv \varphi \in \Delta^0$.

**Proof.** We set $\Delta^0 = \Gamma$. At stage $s+1$, for each pair $\psi, \varphi$ of sentences from $L^*$ such that neither $\psi \dashv \varphi$ nor $\varphi \dashv \psi$ is in $\Delta_s$, we proceed as follows. Let $\theta$ be the conjunction of all elements of $\Delta_s$, and let $\bar{c}$ be the constants from $C$ which occur in $(\psi \dashv \varphi) \lor \theta$. We then check (effectively, since the theory is decidable), whether $(\forall \bar{x}( (\psi \dashv \varphi) \lor \theta) (\bar{x}/\bar{c})^0_T = 0$. If so, then we add $\psi \dashv \varphi$ to form $\Delta_{s+1}$. Otherwise, we do so with $\varphi \dashv \psi$. Now $\Delta^0 = \bigcup_s \Delta_s$ is as required. That this extension is consistent is established in [2].

**Definition 4.6.** Let $\Delta$ be a set of formulas. We say that $\Delta$ is *maximal consistent* if $\Delta$ is consistent and for all formulae $\varphi, \psi$ we have

1. If $\Delta \vdash \varphi \vdash 2^{-n}$ for all $n$, then $\varphi \in \Delta$, and
2. $\varphi \dashv \psi \in \Delta$ or $\psi \dashv \varphi \in \Delta$.

Now let $\Delta^0 = \bigcup_s \Delta_s$, and

$$\Lambda = \{\varphi : \forall n[\delta^0 \vdash \varphi \vdash 2^{-n}]\}.$$ 

Now $\Delta = \Delta^0 \cup \Lambda$ is maximal consistent, by construction of $\Delta^0$. Let $M$ be the model of $T$ whose universe is the set of closed terms in $C$, as in [2].

We now define the probabilistic Turing machine $G$ which will witness that $M$ is probabilistically computable. We set $K^A_0 = K^R_0 = A_0 = R_0 = \emptyset$. We define the functions $E(S) = \{\sigma \geq \tau : \tau \in S\}$ and

$$P(S) = \sum_{\sigma \in S} \frac{1}{2^{h(\sigma)}}.$$
At stage \(s\), if \(\Delta_s \vdash \varphi \not\models \frac{k}{2^n}\), then we will arrange that \(G\) accepts \(\varphi\) with probability at least \(1 - \frac{k}{2^s}\). If \(K_s^A = \emptyset\), then we find \(2^n - k\) nodes \(\sigma_1, \ldots, \sigma_{2^n - k}\) of length \(n\) in \(2^{<\omega} \setminus E(K_s^R)\), and let \(K_{s+1}^A = \{\sigma_1, \ldots, \sigma_{2^n - k}\}\). If \(K_s^A\) is nonempty and \(P(K_s^A) \geq 1 - \frac{k}{2^n}\), then we do nothing with \(K_s^A\). If \(K_s^A\) is nonempty and \(P(K_s^A) < 1 - \frac{k}{2^n}\), then we find some set \(\Sigma\) of elements of \(2^{<\omega} - E(K_s^R)\) with length \(n\) so that \(P(K_s^A \cup \Sigma) = 1 - \frac{k}{2^n}\), and let \(K_s^A = K_s^A \cup \Sigma\).

If \(\Delta_s \vdash \frac{k}{2^n} \models \varphi\) then we will arrange that \(G\) rejects \(\varphi\) with probability at least \(\frac{k}{2^n}\). If \(K_s^R = \emptyset\), then we find \(k\) nodes \(\sigma_1, \ldots, \sigma_k\) of length \(n\) in \(2^{<\omega} \setminus E(K_s^A)\), and let \(K_{s+1}^R = \{\sigma_1, \ldots, \sigma_k\}\). If \(K_s^R\) is nonempty and \(P(K_s^R) \geq \frac{k}{2^n}\), then we do nothing with \(K_s^R\). If \(K_s^R\) is nonempty and \(P(K_s^R) < \frac{k}{2^n}\), then we find some set \(\Sigma\) of elements of \(2^{<\omega} - E(K_s^A)\) with length \(n\) so that \(P(K_s^R \cup \Sigma) = \frac{k}{2^n}\), and let \(K_{s+1}^R = K_s^R \cup \Sigma\).

At this point, it is necessary to verify that certain aspects of the construction described so far are actually possible. In particular, we need to show that when we search for elements of \(2^{<\omega} \setminus E(K_s^A)\), for instance, there will be some. Now if \(E(K_s^A)\) contains more than \(2^n - k_1\) elements, we must have \(P(K_s^A) < 1 - \frac{k}{2^n}\), so that we must have had \(\Delta_s \vdash \varphi \not\models \frac{k}{2^n}\) (Note that if \(\Delta_s \vdash \varphi \models p\) and \(q > p\), then also \(\Delta_s \vdash \varphi \not\models q\)).

**Lemma 4.7.** If there is some \(s\) such that \(\Delta_s \vdash \varphi \not\models \frac{k}{2^n}\) and \(\Delta_s \vdash \frac{k}{2^n} \models \varphi\), then \((1 - \frac{k}{2^n}) + \frac{k}{2^n} \leq 1\).

**Proof.** Suppose not. Then \(2^n - k_1 + k > 1\), so that \(k - k_1 > 0\) and \(k > k_1\). However, we also have \(\frac{k}{2^n} - \frac{k}{2^n} = 0\), so that \(k_1 \geq k\), a contradiction. \(\square\)

The situation for finding elements of \(2^{<\omega} - E(K_s^R)\) is symmetric.

Returning to the construction, at stage \(s\), we will add more instructions. We will guarantee that for any \(\sigma \in E(K_s^A)\), we will have \(G^*(\varphi) \downarrow = 0\), and for any \(\sigma \in E(K_s^R)\) we will have \(G^*(\varphi) \downarrow = 1\).

Let \(\varphi\) be a sentence in \(\overline{L}\), and suppose \(M(\varphi) = p\). We need to show that \(G\) accepts \(\varphi\) with probability \(p\). Since \(\Delta\) is maximal consistent, for each \(q_0, q_1 \in D\) with \(q_0 \leq M(\varphi) \leq q_1\), there was some \(s\) for which \(\Delta_s \vdash \varphi \models q_1\) and for which \(\Delta_s \vdash \varphi \not\models q_0\) and at that stage, we ensured that \(G\) would accept \(\varphi\) with probability between \(q_0\) and \(q_1\). Since this is true for all \(q_0 \leq p \leq q_1 \in D\), it must follow that \(G\) accepts \(\varphi\) with probability \(p\).

\(\square\)

We can strengthen Theorem 4.2 to produce a continuous weak structure if we require \(T\) to be complete.

**Definition 4.8.** Let \(M\) be a continuous \(\mathcal{L}\)-structure. We write \(Th(M)\) for the set of continuous \(\mathcal{L}\)-sentences \(\varphi\) such that \(M(\varphi) = 0\). We say that \(T\) is complete if \(T = Th(M)\) for some \(M\).

**Corollary 4.9.** Let \(T\) be a complete decidable continuous first-order theory. Then there is a probabilistically decidable weak structure \(M\) such that \(M \models T\).

**Proof.** If the signature has no metric, then Theorem 4.2 suffices. Otherwise, we note that \(T\) must contain the sentence \(\sup_{x,y} (\varphi(x, y) \models d(x, y))\), so that when we apply Theorem 4.2 the function \(d_M\) is a metric on \(M\). \(\square\)
5. Examples

A full treatment of each of the following classes of examples suggests a paper—or many papers—of its own. However, in each case some suggestion is given of the kind of data given by the assumption that an element of the class is probabilistically computable.

5.1. Hilbert Spaces. A pre-Hilbert space over a topological field $F$ is a vector space with an inner product meeting all requirements of being a Hilbert space except perhaps that it may not be complete with respect to the norm. The authors of [1] identify a pre-Hilbert space $H$ with the many-sorted weak structure

$$\mathcal{M}(H) = ((B_n(H) : n \geq 1), 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r \in F}, +, -, \langle \cdot, \cdot \rangle)$$

where $B_n(H)$ is the closed ball of radius $n$, the map $I_{mn}$ is inclusion of $B_m$ in $B_n$, each $\lambda_r$ is a scaling function, $+$ and $-$ are the standard vector operations, and $\langle \cdot, \cdot \rangle$ is the inner product. We also write $||x||$ for $\sqrt{\langle x, x \rangle}$, and the structure has a metric given by $d(x, y) = ||x - y||$. Of course, the normal case is to let $F = \mathbb{R}$, but this is not necessary. Now the pre-Hilbert space $H$ is clearly a continuous weak structure in the obvious signature. A true Hilbert space is a continuous structure.

**Theorem 5.1** ([1]). There is a continuous first order theory, IHS, such that the following hold:

1. IHS is (classically) computably axiomatized
2. IHS is complete
3. Any two models of IHS with the same infinite cardinality are isomorphic.
4. The continuous first-order structures $\mathcal{M}$ that satisfy IHS are exactly the infinite-dimensional Hilbert spaces.
5. IHS admits quantifier elimination.
6. IHS is $\omega$-stable.

Since IHS is computably axiomatizable and complete, it must also be decidable. Consequently, Theorem [4.9] shows that there must exist some probabilistically computable weak structure satisfying IHS. Clearly one challenge of handling Hilbert spaces from a computational viewpoint is the essential uncountability of a true Hilbert space. However, computational scientists are generally undaunted by this feature, being satisfied with approximations in place of true limits of Cauchy sequences. We will adopt a similar approach. The following result is well known (see [17] for a proof).

**Lemma 5.2.** For any $p$, the space $L^p(\mathbb{R})$ is separable (i.e. has a countable dense subset).

Let $F$ be a countable topological field dense in $\mathbb{R}$, let $H$ be a separable Hilbert space over the reals, and let $D$ be a countable dense subspace of $H$. We will write $H^F$ for the restriction of $D$ to the language which includes scalars only from $F$. It is well known that $H$ must have a countable orthonormal basis (see, for instance, [22]).

**Proposition 5.3.** If $H^F$ is probabilistically computable, then there is a probabilistic Turing machine which will, given a probabilistically computable countable basis for $H^F$, produce a probabilistically computable orthonormal basis for $H^F$. 
Proof. The proof is precisely an implementation of the Gram–Schmidt process. Let \((u_i)_{i \in \omega}\) be a probabilistically computable countable basis. Set \(v_1 := u_1\). For \(s \geq 2\), at stage \(s\), we set

\[
v_s := u_s - \sum_{i=1}^{s-1} \frac{\langle u_s, v_i \rangle}{||v_i||} v_i.
\]

Now \((v_i)_{i \in \omega}\) is an orthogonal basis for \(H^F\). We can normalize to an orthonormal basis \((\tilde{v}_i)_{i \in \omega}\) by setting \(\tilde{v}_i := \frac{v_i}{||v_i||}\).

\[\square\]

5.2. Banach Spaces and Banach Lattices. Certainly, the framework of Section 5.1 is sufficient, without the inner product, to account for any normed vector space as a continuous pre-structure (and thus any Banach space as a continuous structure).

One also sometimes sees some Banach spaces with the additional structure of a lattice included. In such a structure, \(f \vee g\) is the pointwise minimum of \(f\) and \(g\), and \(f \wedge g\) is the pointwise maximum. Such an approach is taken in [25, 18] and also in [1].

Fixed point theorems for operators on Banach spaces or appropriate subsets of them are an important technique for many non-linear differential equations [10, 11, 23, 22]. It is well-known that the classical Brouwer fixed point theorem is not effectively true (that is, roughly, if the convex hull \(C\) of a nonempty finite set of points in \(\mathbb{R}^n\) and a continuous function \(f : C \rightarrow C\) are given in a computable way, there may still be no algorithm to find a fixed point for \(f\); see [26]). Since the countable spaces we are working with are not complete, we cannot hope for an exact version of the Banach fixed point theorem. However, the following approximate version does hold (the classical proof is described in [10]; the proof below is a slight modification).

**Proposition 5.4** (Effective Banach Fixed Point Theorem). There is an effective procedure which will, given \(\epsilon > 0\), a probabilistically computable normed vector space \(X\) and a computable nonlinear mapping \(A : X \rightarrow X\) such that

\[d(A(u) - A(v)) \leq \gamma d(u, v)\]

for some \(\gamma < 1\), produce an element \(x^*\) such that \(d(x^*, A(x^*)) < \epsilon\).

**Proof.** Fix some \(u_0 \in X\), and let \(i > \log_{\gamma} \frac{\epsilon}{\gamma d(A(u_0), u_0)}\). Such an \(i\) can be found effectively by standard approximations. Write \(u_k\) for \(A^k(u_0)\). Then

\[d(A(u_{i+1}), u_{i+1}) = d(A(u_{i+1}), A(u_i)) \leq \gamma d(u_{i+1}, u_i) \leq \cdots \leq \gamma^i d(A(u_0), u_0) < \epsilon.\]

Let \(u^* = u_{i+1}\), and the result holds. \(\square\)

This shows that approximate weak solutions to certain differential equations (for instance, some of reaction-diffusion type, see [10]) can be found effectively in probabilistically computable structures. The key insight of this fixed point result is this. The traditional view of effective model theory has asked whether a particular theorem is “effectively true.” By contrast, the important question for applications is more typically whether the theorem is “effectively nearly true.” That is, approximations are good enough, and often all that is necessary. For solutions of differential equations modeling applications, for instance, practitioners often “would rather have an accurate numerical solution of the correct model than an explicit solution of the wrong model. Explicit solutions are so rare that fast accurate numerical
analysis is essential" [28]. While it is not new to observe that classically unsolvable problems can be effectively approximated, the framework of probabilistically computable structures in continuous first-order logic is one that calls attention to the possibility of approximate solutions, rather than to the impossibility of exact ones.

5.3. Probability Spaces. Let $\mathcal{X} = (X, \mathcal{B}, \mu)$ be a probability space. We say that $B \in \mathcal{B}$ is an atom if $\mu(B) > 0$ and there is no $B' \in \mathcal{B}$ with $B' \subseteq B$ and $0 < \mu(B') < \mu(B)$. We say that $\mathcal{X}$ is atomless if and only if $\mathcal{B}$ contains no atoms. Let $\hat{\mathcal{B}}$ be the quotient of $\mathcal{B}$ by the relation $B_1 \sim B_2$ if and only if $\mu(B_1 \triangle B_2) = 0$.

The authors of [1] identified $\mathcal{X}$ with the structure $\left(\hat{\mathcal{B}}, 0, 1, \cap, \cup, \mu\right)$ with the metric $d(A, B) = \mu(A \triangle B)$.

Theorem 5.5 ([1]). There is a continuous first-order theory APA such that

1. APA is finitely axiomatizable.
2. APA is complete.
3. APA admits quantifier elimination.
4. The continuous pre-structures satisfying APA are exactly the atomless probability spaces, represented as above.

Since APA is finitely axiomatizable and complete, it is also decidable. Thus, Theorem 4.9 gives us a probabilistically decidable model. Of course, any separable probability structure can be approximated by a countable dense set.

A standard issue in effective model theory is whether two isomorphic structures must be isomorphic via a computable function. The following result shows that the answer for probability structures is affirmative.

Proposition 5.6. Let $\mathcal{B}$ and $\mathcal{C}$ be isomorphic, probabilistically computable atomless probability structures with universes $\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$, respectively. Then there is a (classically) computable function witnessing the isomorphism.

Proof. The isomorphism is constructed by a standard back-and-forth argument. Suppose that $f : \mathcal{B} \rightarrow \mathcal{C}$ is a finite partial isomorphism, and that $x \in \mathcal{B} - \text{dom}(f)$. We wish to find some $y \in \mathcal{C}$ such that $f \cup \{(x, y)\}$ is still a partial isomorphism. Without loss of generality, we may assume that $x$ is not in the substructure of $\mathcal{B}$ generated by $\text{dom}(f)$. Let $a_0, \ldots, a_n$ be the atoms of the substructure of $\mathcal{B}$ generated by $\text{dom}(f)$. We may assume, without loss of generality, that each $a_i$ is in $\text{dom}(f)$. Now the isomorphism type of $x$ is determined by the values $\mu(x \cap a_i)$. Since $\mathcal{B}$ is atomless, there is an element $y \in \mathcal{C}$ such that for each $i$ we have $\mu(\mathcal{B})(x \cap a_i) = \mu(\mathcal{C})(y \cap f(a_i))$, and since $\mathcal{B}$ and $\mathcal{C}$ are probabilistically computable, we can effectively find this $y$. The extension to a new element of $\mathcal{C}$ is entirely symmetric. The union of the partial isomorphisms constructed in this way will be a computable function, and will be an isomorphism from $\mathcal{B}$ to $\mathcal{C}$. □

5.4. Probability Spaces with a Distinguished Automorphism. A standard sort of enrichment in stability theory is to expand a known structure by adding a new function symbol to define a new function, and to specify that this function be generic. Fix an interval $I$ under the Lebesgue measure $\lambda$, and let $\mathcal{L}$ be the algebra of measurable sets. Let $G$ denote the group of measure preserving automorphisms of $(I, \mathcal{L}, \lambda)$, modulo the relation of almost everywhere agreement. Let $\tau \in G$. Now
\( \tau \) induces an automorphism on \((\hat{L}, 0, 1, \cdot, \cap, \cup, \lambda)\) in a straightforward way (see [4][1]). We say that \( \tau \in G \) is \textit{aperiodic} if for every positive integer \( n \) we have
\[
\lambda\{x \in I : \tau^n(x) = x\} = 0.
\]
To have a countable structure of this type, we could take a countable dense subset \( I' \subseteq I \), and for \( X \subseteq I' \), set \( \lambda(X) = \lambda(\text{cl}(X)) \).

In [4] and [1], an axiomatization is given for the theory of atomless probability spaces with a distinguished aperiodic automorphism. This theory is complete and admits elimination of quantifiers. The authors of [4] show that entropy arises as a model-theoretic rank.

The result below partially describes the degree of algorithmic control we can expect on iterations of such an automorphism. Before stating the result, though, a probabilistic analogue to computable enumerability should be given:

**Definition 5.7.** We say that a set is probabilistically computably enumerable if and only if there is some probabilistic Turing machine \( M \) such that
- If \( x \in S \), then for any \( q < 1 \) the machine \( M \) accepts \( x \) with probability at least \( q \), and
- If \( x \notin S \), then there is some \( q < 1 \) such that \( M \) accepts \( x \) with probability at most \( q \).

In particular (and especially in light of the time complexity considerations in Section 6), if we specify an error tolerance \( q \), there is some \( s \) such that \( M(s, x) \) is below \( q \) whenever \( x \in S \), and (assuming the tolerance is sufficiently small) no such \( s \) otherwise.

**Theorem 5.8.** Let \( \mathcal{I} = (\hat{L}, 0, 1, \cdot, \cap, \cup, \lambda, \tau) \) be a probabilistically computable probability structure based on a dense subset of the unit interval, with a measure-preserving transformation \( \tau \). Let \( A \subseteq I \) be a set of positive measure, defined without quantifiers in continuous first-order logic. Write \( \mathfrak{A} \) for the set \( \bigcup_{n \in \omega} \tau^n(A) \). Then for any isomorphism \( f : \mathcal{I} \to \mathcal{J} \) to a probabilistically computable structure \( \mathcal{J} \), the set \( f(\mathfrak{A}) \) is probabilistically computably enumerable.

**Proof.** Toward part 1, note that \( \mathfrak{A} \) is defined by the infinitary disjunction
\[
\varphi(x) = \bigvee_{n \in \omega} \tau^{-n}(x) \in \mathcal{A}
\]
and that the set \( \mathcal{A} \) is defined by a quantifier-free continuous first-order formula. The isomorphism \( f \) must preserve satisfaction of \( \varphi \) — that is, \( \mathcal{I} \models \varphi(x) \) if and only if \( \mathcal{J} \models \varphi(f(x)) \). Now let \( M \) be a probabilistic Turing machine such that \( M(x, s) \) is the minimum value of \( \bigwedge_{n \leq k} \tau^{-n}(x) \in \mathcal{A} \), where \( k \) ranges over all numbers less than or equal to \( s \). Then \( M \) witnesses that \( f(\mathfrak{A}) \) is probabilistically computably enumerable. \( \Box \)

6. Time Complexity of Structures

One of the most important applications of probabilistic Turing machines is their role in computational complexity theory (see [16][21][14]). Let \( P \) be some decision problem. We say that \( Q \) is of class RP if and only if there is a probabilistic Turing machine \( M_Q \), halting in time polynomial in the length of the input, such that if
$x \in Q$, then $M_Q$ accepts $x$ with probability at least $\frac{3}{4}$, and if $x \notin Q$, then $M_Q$ rejects $x$ with probability 1. This class has the property that $P \subseteq \text{RP} \subseteq \text{NP}$.\footnote{In both this and the succeeding paragraph, the particular fraction $\frac{1}{2}$ is not critical. Using a so-called “Amplification Lemma,” any fraction above and bounded away from $\frac{1}{2}$ will do [27].}

Another complexity class of interest is the class BPP. We say that $Q$ is of class BPP if and only if there is a probabilistic Turing machine $M_Q$, halting in time polynomial in the length of the input, such that if $x \in Q$, then $M_Q$ accepts $x$ with probability at least $\frac{3}{4}$, and if $x \notin Q$, then $M_Q$ rejects $x$ with probability at least $\frac{1}{4}$.

Here we know that $\text{RP} \subseteq \text{BPP} \subseteq \Sigma^p_2 \cap \Pi^p_2$.

**Definition 6.1** (Cenzer–Remmel [8]). Let $A$ be a computable structure. We say that $A$ is uniformly polynomial time if the atomic diagram of $A$ is a polynomial time set.

Clearly, $A \in P$ if and only if the structure $(\omega, A)$ is polynomial time. Also, for any polynomial time structure $M$ and any quantifier-free definable $A \subseteq M^n$, we have $A \in P$. We can extend Definition 6.1 in a routine way for probabilistically computable structures.

**Definition 6.2.** We say that a probabilistically computable structure is polynomial time if and only if there is some probabilistic Turing machine $T$ such that, for every pair $(\varphi, p) \in D(M)$ the machine $T$ halts in polynomial time and accepts $\varphi$ with probability $p$.

Now we can characterize the members of BPP in terms of continuous weak structures.

**Theorem 6.3.** The class BPP can be identified with the class of quantifier-free definable sets in polynomial time probabilistically computable structures in the following way:

1. Let $A \in \text{BPP}$ be a subset of $\omega$. Then there is a polynomial time probabilistically computable weak structure $M$ and a polynomial time computable function $f : \omega \to M$ such that there is a quantifier-free formula $\varphi(x)$ such that $\varphi(x) \leq \frac{1}{4}$ for $x \in f(A)$ and $\varphi(x) \geq \frac{3}{4}$ for $x \in M - f(A)$.

2. Let $M$ be a polynomial time probabilistically computable weak structure, and let $A, B$ be quantifier-free disjoint definable subsets of $M^n$, where

$$\inf \{d(x, y) : x \in A, y \in B\} > 0$$

and $A \cup B$ is classically computably enumerable. Then $A$ and $B$ are each of class BPP.

**Proof.** Toward the first point, let $M$ be a probabilistic Turing machine witnessing that $A \in \text{BPP}$. We let $M$ be the structure $(\omega, A)$, where $A$ is a unary predicate and $A(x)$ is the probability that $M$ accepts $x$. We give $M$ the discrete metric.

For the second point, let $A$ be defined by $\varphi(\bar{x})$, and $B$ by $\psi(\bar{x})$. Now for $\bar{a} \in A \cup B$, to check whether $\bar{a} \in A$, we compute $M(\varphi(\bar{a}) \wedge \psi(\bar{a}))$. The computation runs in polynomial time, and $\bar{a}$ is accepted with probability at least $\frac{1}{2} + \inf \{d(\bar{a}, y) : y \in B\}$ when $\bar{a} \in A$ and with probability at most $\frac{1}{2} - \inf \{d(\bar{a}, y) : y \in B\}$ when $\bar{a} \in B$. \hfill \Box
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