WEAK$^*$ SEQUENTIAL CLOSURES IN BANACH SPACE THEORY AND THEIR APPLICATIONS

M.I. Ostrovskii

§1 INTRODUCTION

Let $X$ be a (real or complex) Banach space, its dual Banach space will be denoted by $X^*$. We use standard notation and terminology of Banach space theory, see J. Lindenstrauss and L. Tzafriri [LT]. By a subspace we mean a linear, but not necessarily closed, subspace. We also assume some knowledge of general topology and ordinal numbers, see P. S. Aleksandrov [A].

Definition 1.1. Let $A$ be a subset of $X^*$. The set of all limits of weak$^*$-convergent sequences in $A$ is called the weak$^*$ sequential closure of $A$ and is denoted by $A_{(1)}$.

S. Banach asked the following question (see [Maz]).

**Question.** Let $X$ be separable Banach space and $A$ be a subspace of $X^*$. Whether $(A_{(1)})_{(1)} = A_{(1)}$?

This question was answered in negative by S. Mazurkiewicz [Maz]. The result of S. Mazurkiewicz makes it natural to introduce the following definition. (It was done by S. Banach [B2, p. 208, 213]. S. Banach used the term “dérivé faible”.)

**Definition 1.2.** For an ordinal $\alpha > 1$ the weak$^*$ sequential closure of order $\alpha$ of $A$ is the set

$$A_{(\alpha)} = \bigcup_{\beta < \alpha} (A_{(\beta)})_{(1)}.$$ 

Weak$^*$ sequential closures were studied by S. Banach in his book [B2] (see, also, [B3] and [B4]). He proved the following results.

**Theorem 1.3** [B2, p. 124]. Let $X$ be a separable Banach space and let $A$ be a subspace in $X^*$. Then $A = A_{(1)}$ if and only if for every $f \in X^* \setminus A$ there exists $x \in X$ such that $f(x) = 1$ but $a(x) = 0$ for every $a \in A$.

In modern terminology this result can be stated as: a subspace in the dual of a separable Banach space is weak$^*$ closed if and only if it is weak$^*$ sequentially closed.

1991 Mathematics Subject Classification. 46B10, 46B03, 54A20, 47G10.

*Key words and phrases.* Banach space, weak$^*$ sequential closure, total subspace.

Typeset by A4S-TEX
Theorem 1.4 [B2, p. 213]. Let $X$ be a separable Banach space and let $A$ be a subspace in $X^*$. A necessary and sufficient condition for $A^{(1)} = X^*$ is that there exists a number $M > 0$ such that, for each $x \in X$, the subspace $A$ contains a functional $f$ satisfying the conditions

$$||f|| \leq M \quad \text{and} \quad |f(x)| = ||x||.$$

Theorem 1.5 [B2, pp. 213 and 124]. Let $X$ be a separable Banach space and let $A$ be a subset in $X^*$. Then there exists a countable ordinal $\alpha$ such that $A^{(\alpha)} = A^{(\alpha+1)}$.

Theorem 1.6 [B2, p. 209]. For every positive integer $n$ there exists a subspace $A$ in $(c_0)^* = l_1$ such that $A^{(n)} \neq A^{(n+1)}$.

The book [B2] also contains the following statement:

Statement 1.7 [B2, p. 213]. For every countable ordinal $\alpha$ there exists a subspace $A$ in $(c_0)^* = l_1$ such that $A^{(\alpha)} \neq A^{(\alpha+1)}$.

This result was not proved in [B2]. At this point S.Banach referred to a paper that never appeared. Unfortunately the most comprehensive survey on developments initiated in Banach’s book (I mean the survey by A.Pelczyński and Cz.Bessaga [PB]) does not contain any comments on this statement. To the best of my knowledge the first proof of this statement was given by O.C.McGehee [McG].

It is natural to suppose that the reason for studying weak* sequential closures by S.Banach and S.Mazurkiewicz was the lack of acquaintance of S.Banach and his school with concepts of general topology. Although the name “General topology” was introduced later, the subject did already existed, see F.Hausdorff [H], Alexandroff-Urysohn [AU] and A.Tychonoff [Ty]. It is also worth mentioning that J.von Neumann [N, p. 379] had already introduced the notion of a weak topology.

Using the notions of a topological space and the Tychonoff theorem, more elegant treatment of weak and weak* topologies, and the duality of Banach spaces was developed by L.Alaoglu [Al1], [Al2], N.Bourbaki [Bou] and S.Kakutani [Kak]. See N.Dunford & J.T.Schwartz [DS, Sections V.3–V.6] for a well-organized presentation of this topic.

Nevertheless, an “old-fashioned” treatment of S.Banach still attracts attention. It happens because the “sequential” approach is very useful in several contexts. Now we would like to mention some of them.

In Harmonic Analysis weak* sequential closures lead to a very useful and natural classification of sets of uniqueness. This fact was noticed by I.I.Piatetski-Shapiro ([P1], [P2]). Since then the classification of sets of uniqueness was repeatedly used to prove results on sets of uniqueness using Banach space techniques and results. We are not going to discuss this topic in any detail. We refer to A.Kechris and A.Louveau [KL], A.Kechris, A.Louveau and V.Tardivel [KLT] and to R.Lyons [Ly].

Ill-posed problems. V.A.Vinokurov, Yu.I.Petunin and A.N.Plichko ([VPP1] and [VPP2]) proved that a very important in the theory of ill-posed problems class of regularizable linear operators can be characterized in terms of weak* sequential closures (at least in some important cases). For further results in this direction see
the systematic exposition in the book of Yu.I.Petunin and A.Plichko [PP] and a more recent paper [O2].

J.Saint-Raymond [S] and A.N.Plichko [Pl1], [Pl4] proved that the Borel and Baire classification of inverses of continuous injective linear operators can be described in terms of weak∗ sequential closures (we shall discuss this result in more detail in §3).

S.Dierolf and V.B.Moscatelli [DM] found connections with the structure theory of Fréchet spaces. See E.Behrends, S.Dierolf and P.Harmand [BDH], G.Metafune and V.B.Moscatelli [MM1], [MM2], V.B.Moscatelli [M2] and the present author [O9] for further results in this direction.

A.Plichko [Pl3, Theorem 4] used weak∗ sequential closures to solve a problem on universal Markushevich bases in Banach spaces posed by N.J.Kalton [Ka, Problem 1, p. 187].

In connection with these applications of weak∗ sequential closures we find it natural and useful to give an account on the present state of the study of weak∗ sequential closures initiated by the results of S.Banach and S.Mazurkiewicz.

Our main purpose is to study the properties of weak∗ sequential closures of total subspaces in the dual spaces of separable Banach spaces (because it is the case that is the most important for all listed applications).

Recall that a subset $M$ of the dual space $X^*$ is called total if for every $0 \neq x \in X$ there exists $f \in M$ such that $f(x) \neq 0$.

Before we turn to the main purpose we would like to make some remarks on non-separable case.

1. It is clear that for reflexive spaces weak∗ sequential closures coincide with weak sequential closures. It turns out that reflexive spaces are not the only spaces with this property. A.Grothendieck [Gr, p. 168] found nonreflexive spaces $X$ such that every weak∗ convergent sequence in $X^*$ is weakly convergent. (Now such spaces $X$ are called Grothendieck spaces.) It is easy to see that nonreflexive Grothendieck spaces are non-separable. See J.Diestel and J.J.Uhl, Jr. [DU, p. 179] for a survey on Grothendieck spaces and J.Bourgain [Bo], R.Haydon [Ha], S.S.Khurana [Kh] and M.Talagrand [Ta] for more recent results. (Additional references can be found using MathSciNet.)

It is easy to verify that for convex subsets in the duals of Grothendieck spaces the weak∗ sequential closure coincide with the norm closure. B.V.Godun [G1, Proposition 3] observed that the following converse to this statement is true: if $A = A_{(1)}$ for every closed subspace in $X^*$, then $X$ is a Grothendieck space.

2. One of the natural and important questions about weak∗ sequential closures is: when is the second dual $X^{**}$ of a Banach space $X$ equal to the weak∗ sequential closure of the canonical image of $X$ in $X^{***}$? For separable spaces this question was answered by E.Odell and H.P.Rosenthal [OR]. They proved the following theorem.
**Theorem 1.8.** The second dual of a separable Banach space \( X \) coincide with the weak∗ sequential closure of the canonical image of \( X \) if and only if \( X \) does not contain a subspace isomorphic to \( l_1 \).

J.Diestel [D, Chapter XIII] presents a proof of Theorem 1.8 with all necessary preliminaries. The papers R.J.Fleming [Fl], R.J.Fleming, R.D.McWilliams and J.R.Retherford [FMR], A.Grothendieck [Gr], R.D.McWilliams [McW1], [McW2] and H.P.Rosenthal [R1], [R2], [R3] contain preceding and relevant results.

**§2 Existence of total subspaces with long chains of strictly increasing weak∗ sequential closures**

**Definition 2.1.** Let \( A \) be a subset in \( X^* \). The least ordinal \( \alpha \) for which \( A(\alpha) = A(\alpha+1) \) is called the order of \( A \). (We use the convention \( A(0) := A \). Hence the order of any weak∗ closed subset in \( X^* \) is equal to 0.)

Theorem 1.5 implies that for separable \( X \) the order of any subset in \( X^* \) is a countable ordinal.

This section is devoted to the following question.

Let \( X \) be a separable Banach space. What are the possible orders of total subspaces \( M \subset X^* \)?

To answer this question we need

**Definition 2.2.** A Banach space \( X \) is called quasi-reflexive if \( \dim(X^{**}/\pi(X)) < \infty \), where \( \pi : X \to X^{**} \) is the canonical embedding and \( X^{**}/\pi(X) \) is the quotient space.

It is worth mentioning that classical Banach spaces (\( L_p, H_p, C(K) \)) are either reflexive or non-quasi-reflexive. The first example of a non-reflexive quasi-reflexive space was given by R.C.James [Jam]. (This example is discussed in [LT] (p. 25).) The term “quasi-reflexive space” was introduced by P.Civin and B.Yood [CY], their paper contains a systematic study of quasi-reflexive spaces.

**Theorem 2.3.** (1) If \( X \) is a non-quasi-reflexive separable Banach space, then for every countable ordinal \( \alpha \) there exists a total subspace \( \Gamma \subset X^* \) of order \( \alpha+1 \), that is
\[
\Gamma \subset \Gamma(1) \subset \Gamma(2) \subset \cdots \subset \Gamma(\alpha) \subset \Gamma(\alpha+1) = X^*,
\]
where all inclusions are proper.

(2) If \( X \) is a quasi-reflexive separable Banach space, then a total subspace \( \Gamma \subset X^* \) is of order 1 if \( \Gamma \neq X^* \) and of order 0 if \( \Gamma = X^* \).

**Remark 2.4.** (a) For a separable Banach space \( X \) the whole dual space \( X^* \) is the only total subspace of order 0. A dense subspace \( \Gamma \subset X^* \) satisfying \( \Gamma \neq X^* \) is a total subspace of order 1. Hence in the statement of Theorem 2.3 \( \alpha \) may be equal not only to a usual ordinal number, but also to 0 or \(-1\).

(b) By Theorem 1.5 the order of a subspace in the dual of a separable Banach space is a countable ordinal. B.V.Godun [G2, Lemma 1] observed that the order of a subspace in the dual of a separable Banach space cannot be equal to a countable limit ordinal. (Recall that an ordinal \( \gamma \) is said to be limit if it cannot be written in the form \( \alpha + 1 \). We consider 0 and 1 as non-limit ordinals.)
(c) Hence Theorem 2.3 gives a complete answer to the question above.

Remark 2.5. Using the fact that a subspace $M$ in $X^*$ can be identified in a canonical way with the total subspace of the dual of the quotient $X/(M^\perp)$ and the fact that quotients of quasi-reflexive spaces are quasi-reflexive (see [CY]) it is easy to show that any subspace in the dual of a quasi-reflexive separable Banach space has order 1 (if it is not weak$^*$ closed) and order 0 (if it is weak$^*$ closed).

The first statement of Theorem 2.3 was proved by the present author in [O1], the second was known earlier (see below). Theorem 2.3 (1) has a rather long history and many partial results were known prior the publication of [O1]. The history started with the results of S.Mazurkiewicz and S.Banach mentioned in §1 (see Theorems 1.5, 1.6 and Statement 1.7). To the best of my knowledge the Banach’s proof of Statement 1.7 did not exist and the first example of a Banach space whose dual contain subspaces of any non-limit countable order was given by D.Sarason (see [S1] and [S3]). In these papers he constructed such subspaces in $H_\infty$. Soon afterwards O.C.McGehee [McG] proved that the same is true for $X = c_0$ thus proving the Banach’s Statement 1.7. In his proof he identifies $l_1$ with the space of absolutely convergent Fourier series and uses a quite nontrivial amount of Harmonic Analysis. A bit later D.Sarason [S2] found total subspaces of all possible orders in $l_\infty$. D.Sarason’s proofs in all of the mentioned paper use some tools from Complex Analysis. The subspaces constructed by O.C.McGehee [McG] and D.Sarason [S1] are also total.

Remark 2.6. Both O.C.McGehee and D.Sarason consider complex spaces. But their results imply similar results for the real spaces $c_0$ and $l_1$ also. We discuss this observation in the case $X = c_0$, the case $X = l_1$ can be considered in the same way. Observe that the complex space $c_0$ considered as a real space is isomorphic to the real space $c_0$. Let $M$ be the total subspace in the dual of complex $c_0$ of order $\gamma$ constructed by O.C.McGehee. Each $f \in M$ has a unique representation as $f = f_1 + if_2$, where $f_1$ and $f_2$ are real-valued $\mathbb{R}$–linear functionals. The set $L := \{f_1 : f \in M\}$ is a linear subspace of the dual of complex $c_0$ considered as a real space. Using the well-known trick due to Bohnenblust-Sobczyk-Soukhomlinoff (see [DS, pp. 63–64]) or the explicit representation of $L$ it is easy to show that $L$ is a total subspace in the dual of $c_0$ considered as a real space and the order of $L$ coincide with the order of $M$.

J.Dixmier [Di] introduced a very useful in the present context notion of a characteristic of a subspace in a dual Banach space and found several equivalent definitions of it. One of the equivalent definitions is:

Definition 2.7. Let $M$ be a subspace in $X^*$. The characteristic of $M$ is defined by

$$r(M) := \inf_{0 \neq x \in X} \sup_{||f|| \leq 1} \frac{||f(x)||}{||x||}.$$  

It is easy to see that Theorems 1.3 and 1.4 imply the following result.

Corollary 2.8. The characteristic of a total subspace in the dual of a separable Banach space is $> 0$ if and only if the order of the subspace is $\leq 1$.

J.Dixmier observed that the construction of S.Mazurkiewicz gives an example of a total subspace in $c_0^*$ with zero characteristic. He constructed a similar example in...
J. Dixmier’s construction is based on the following result.

**Theorem 2.9** [Di, p. 1064]. Let $X$ be a Banach space and let $\Gamma$ be a subspace in $X^*$. Then

$$r(\Gamma) = \inf \left\{ \frac{||x + z||}{||x||} : x \in \pi(X), x \neq 0, z \in \Gamma^\perp \right\},$$

where $\Gamma^\perp := \{ z \in X^{**} : z(x^*) = 0 \text{ for every } x^* \in \Gamma \}$.

Joining this result with Banach’s Theorem 1.4 we get the second statement of Theorem 2.3.

In fact, let $X$ be quasi-reflexive and $\Gamma \subset X^*$ be total. By definition of a total subspace it implies $\Gamma^\perp \cap \pi(X) = \{0\}$. The definition of a quasi-reflexive space immediately implies that $\Gamma^\perp$ is finite-dimensional. Using Theorem 2.9 and the compactness argument we get $r(\Gamma) > 0$. Jointly with Corollary 2.8 it proves the second part of Theorem 2.3.

The study of total subspaces with zero characteristic was continued by Yu.I. Petunin [P] who found a wide class of Banach spaces whose duals contain total subspaces of characteristic zero, observed that it cannot happen for quasi-reflexive spaces and asked whether such subspaces exist for every non-quasi-reflexive space. W.J. Davis and J. Lindenstrauss [DL] answered this question in the affirmative. Joining their result with the mentioned result of Yu.I. Petunin [P] we get the following theorem.

**Theorem 2.10.** $X^*$ contains a total subspace with characteristic zero if and only if $X$ is non-quasi-reflexive.

A. Plichko [Pl5] found another proof of this result, see also [Pl2].

W.J. Davis and W.B. Johnson [DJ] found a very important characterization of non-quasi-reflexive spaces.

**Theorem 2.11** [DJ, p. 362]. A Banach space $X$ is non-quasi-reflexive if and only if $X$ contains a bounded away from 0 basic sequence $\{z_n\}_{n=0}^\infty$ such that the set

$$\{ ||\sum_{i=j}^k z_{i(i+1)/2+j}||_{j=0, k=j}^\infty \}$$

is bounded.

**Remark 2.12.** The difficult part of this theorem is the fact that each non-quasi-reflexive space contains such basic sequence.

Using Theorem 2.11 B.V. Godun [G1] proved that for every $n \in \mathbb{N}$ the dual of every non-quasi-reflexive separable Banach space contain a total subspace $\Gamma$ satisfying $\Gamma(n) \neq \Gamma(n+1)$. His argument can be used to show that there exists a total subspace $\Gamma$ satisfying

$$\Gamma(\omega) \neq \Gamma(\omega+1),$$

where $\omega$ is the first infinite ordinal.
Being unaware of the B.V.Godun’s result V.B.Moscatelli (see [Op]) posed a problem on existence of subspaces satisfying (*) and rediscovered the B.V.Godun’s result (with a bit different proof) in [M1].

B.V.Godun [G2] made an attempt to prove similar results for larger cardinals, but the inductive argument in [G2] does not work for infinite ordinals.

The final step in proving the first statement of Theorem 2.3 was made by the present author in [O1]. The argument of [O1] gives a new proof of the result even for $c_0$. Reading [O1] it may be useful to look at the special case of $c_0$ first. The approach of [O1] is such that using the W.J.Davis-W.B.Johnson characterization of non-quasi-reflexive spaces (Theorem 2.11), the proof for $c_0$ can be easily transferred to any non-quasi-reflexive space. (It is far from being clear whether the McGehee’s proof can be transferred.)

The paper [O1] is available in English, Russian and Ukrainian. Since no simplifications of the original proof have been found since the publication of [O1] in 1987 I have decided not to reproduce it here.

As we have already observed Theorem 2.3 gives a complete answer to the question posed at the begining of §2. But it is not the end of the story. The present author generalized Theorem 2.3 in two different directions. One of the directions is discussed in §3 (see Theorem 3.8). The other can be found in [O7]. The present author [O3] studied the isomorphic structure of total subspaces of given order. More recently, results of this type attracted attention of A.J.Humphreys and S.G.Simpson [HS], whose main interest was foundations of mathematics. They found another proof of the result of O.C.McGehee. Their proof does not use Harmonic Analysis. They use the notion of a well-founded tree. Their proof is a kind of a dual of the proof in [O1] (if we apply the latter to $X = c_0$). The use of well-founded trees is quite natural in this context (and is implicit in [O1]). Nevertheless I do not feel that their proof is more elementary than the proof in [O1].

It seems that the question posed at the begining of this section has not been studied in any detail for general Banach spaces. It is known that Theorems 1.3–1.5 are not valid for non-separable spaces. J.Dixmier [Di] found natural and useful analogues of Theorems 1.3 and 1.4 for general Banach spaces; these analogues are not in terms of weak* sequential closures.

See T.Banakh, T.Dobrowolski and A.Plichko [BDP] and V.P.Fonf [Fo] for applications of Theorem 2.3 in contexts that are not mentioned in this survey. See D.Sarason [S4] for applications of weak* sequential closures to certain problems about the invariant subspaces of normal operators on a Hilbert space.

The papers A.A.Albanese [Alb], V.B.Moscatelli [M2] and the present author [O4], [O5], [O6], [O7] and [O10] contain some more results on classification of total subspaces of characteristic zero.

The papers J.M.Anderson and J.E.Jayne [AJ], J.E.Jayne [Jay] and F.Jellett [Jel] contain generalizations of Sarason’s results in another direction. These authors study not the weak* sequential closures, but the sets of pointwise limits of a given space of functions.
§3 Borel and Baire classification of linear operators

Let $X, Y$ be separable Banach spaces (in fact, only the fact that $X$ is separable is important). Let $T : X \to Y$ be an injective continuous linear operator. Then $T^{-1} : T(X) \to X$ is a well-defined linear operator. This operator is discontinuous if for some sequence $\{x_n\}_{n=1}^\infty \subset X$ we have $\lim_{n \to \infty} \frac{||T x_n||}{||x_n||} = 0$. On the other hand the well-known Suslin theorem (one-to-one continuous image of a Borel subset of a complete separable metric space is a Borel set, see [Kur, p. 487]) implies that $T^{-1}$ is a Borel map. In this section we consider the question: what is the Borel class of $T^{-1}$?

Let us recall necessary definitions (see [KL, Chapter IV], [Kur, §30, §31]). Let $X$ be a metric space. Let $B$ be the smallest collection of subsets of $X$ that

(a) contains all open subsets;

(b) is closed under the operations of complementation, countable union and intersection.

Sets from $B$ are called Borel sets.

We consider the following hierarchy of Borel sets.

For every countable ordinal $\alpha$ we define multiplicative and additive classes $\alpha$ in the following way.

(a) $\alpha = 0$
- Multiplicative class 0 is the collection of all closed subsets of $X$.
- Additive class 0 is the collection of all open subsets of $X$.

(b) For $\alpha \geq 1$
- Multiplicative class $\alpha$ is the collection of all countable intersections $\cap_{n=1}^\infty P_n$, where $P_n$ belongs to some additive class $\alpha_n$ with $\alpha_n < \alpha$.
- Additive class $\alpha$ is the collection of all countable unions $\cup_{n=1}^\infty P_n$, where $P_n$ belongs to some multiplicative class $\alpha_n$ with $\alpha_n < \alpha$.

It is easy to see that each Borel set belongs to some of the additive (or multiplicative) classes.

A map $f : X \to Y$ between metric spaces is said to be of Borel class $\alpha$ if the set $f^{-1}(F)$ is a Borel set of multiplicative class $\alpha$ for every closed subset $F \subset Y$.

We recall one more natural classification of maps between metric spaces [Kur, §31.IX].

Let $X$ and $Y$ be metric spaces. The family of analytically representable maps from $X$ to $Y$ is defined to be the smallest family of maps from $X$ to $Y$ which contains

1) all continuous maps;
2) the limits of convergent sequences of maps belonging to it.

This family is representable as a union $\cup_{\alpha \in \Omega} \Phi_\alpha$, where $\Omega$ is the set of all countable ordinals and $\Phi_\alpha$ are defined in the following way.

1. The class $\Phi_0$ is the set of all continuous maps.
2. The class $\Phi_\alpha (\alpha > 0)$ consists of all maps which are limits of convergent sequences of maps belonging to $\cup_{\xi < \alpha} \Phi_\xi$. 
The class $\Phi_\alpha$ is called Baire class $\alpha$.

It is known (see S.Banach [B1], K.Kuratowski [Kur, §31], S.Rolewicz [R]) that if $Y$ is a separable Banach space, and $\alpha$ is finite then the Borel class $\alpha$ coincides with the Baire class $\alpha$. If $\alpha$ is infinite then the Borel class $\alpha + 1$ coincide with the Baire class $\alpha$.

V.A.Vinokurov [V] proved that for arbitrary Banach space $Y$ the class of all regularizable maps from $X$ to $Y$ coincide with the Baire class 1. (Regularizability is a very important concept in the theory of ill-posed problems. We are not going to define it here, see Yu.I.Petunin and A.N.Plichko [PP] or V.A.Vinokurov [V].)

See R.D.Mauldin [Mau] for an interesting exposition of results on Borel sets and Baire classes of real-valued functions.

Now we return to the situation described at the begining of this section.

**Theorem 3.1.** Let $X, Y$ be Banach spaces. Let $X$ be separable and let $T : X \rightarrow Y$ be an injective continuous linear operator. Then the map $T^{-1} : T(X) \rightarrow X$ is of Borel class $\alpha$ if and only if the subspace $T^*Y^*$ is of order $\alpha$ (in the sense of Definition 2.1).

**Remark 3.2.** This result was proved by J.Saint-Raymond [S, Corollaries 42 and 45]. Somewhat later A.Plichko independently found a different proof of it (see the summary [Pli1]). Because of some errors in the Plichko’s proof its publication was delayed. Eventually it was published in [Pli4]. Writing [Pli4] A.Plichko was already aware of J.Saint-Raymond’s paper [S].

To apply results on the existence of total subspaces of given orders to show the existence of operators with inverses in given Borel (Baire) classes we need the following result (folk-lore).

**Lemma 3.3.** Let $X$ be a separable Banach space, let $M$ be a closed total subspace in $X^*$ and let $Y$ be an infinite-dimensional Banach space. Then there exists an injective continuous linear operator $T : X \rightarrow Y$ such that $T^*Y^* \subset M$ and $(T^*Y^*)_1 = M_1$.

**Proof.** Since $X$ is separable, then $B(X^*) = \{x^* \in X^* : \|x^*\| \leq 1\}$ is a metrizable separable topological space in the weak* topology. Let $\{m_i\}_{i=1}^\infty$ be a weak* dense sequence in $B(M) = \{x^* \in M : \|x^*\| \leq 1\}$. Let $\{y_i\}_{i=1}^\infty$ be a basic sequence in $Y$ satisfying $\|y_i\| \leq 1$ ($i \in \mathbb{N}$) (see [LT, p. 4]).

Let $T : X \rightarrow Y$ be defined by

$$Tx = \sum_{i=1}^\infty \frac{1}{2^i} m_i(x)y_i.$$ 

It is easy to verify that

$$T^*y^* = \sum_{i=1}^\infty \frac{1}{2^i} y^*(y_i)m_i.$$ 

From here it is clear that $T^*Y^* \subset M$ and $m_i \subset T^*Y^*$. The lemma follows. □

Using Theorem 2.3, Theorem 3.1 and Lemma 3.3 we get
Corollary 3.4. Let $X$ be a separable non-quasi-reflexive Banach space and let $Y$ be an infinite-dimensional Banach space. Then for every countable ordinal $\alpha \geq 1$ there exists an injective continuous linear operator $T : X \to Y$ such that the map $T^{-1} : T(X) \to X$ is of Borel class $\alpha$, but is not of Borel class $\beta$ for any $\beta < \alpha$.

If $X$ and $Y$ are “concrete” spaces, it is natural to ask: whether the operator $T$ can be chosen from some “natural” class of operators? We consider one question of this type, many other can be treated in a similar way.

Question. Let $F$ and $G$ be Banach function spaces on $[0,1]$ and $T : F \to G$ be an injective linear integral operator with an analytic kernel. What Borel class can the map $T^{-1} : T(F) \to F$ belong to?

Let us give some clarifications.

1. By an analytic kernel we mean a map $K : [0,1] \times [0,1] \to \mathbb{C}$ with the following property: For some open sets $\Gamma$ and $\Delta$ of $\mathbb{C}$ such that $[0,1] \subset \Gamma$ and $[0,1] \subset \Delta$, there exists an analytic continuation of $K$ to $\Gamma \times \Delta$. Diminishing $\Gamma$ if necessary we may assume that the functions are bounded on $\Gamma$.

2. If the spaces are real then we consider map $K$ taking real values on $[0,1] \times [0,1]$.

3. We assume that $F$ and $G$ are infinite-dimensional and that $F$ is a separable Banach function space on the closed interval $[0,1]$ continuously and injectively embedded into $L_1((0,1))$.

Remark 3.5. It is easy to see that if $F$ and $G$ are infinite-dimensional function spaces and there exists an injective integral operator with an analytic kernel $T : F \to G$, then $G$ should contain an infinite-dimensional subspace consisting of functions having analytic continuations to some open subset $\Gamma \subset \mathbb{C}$ satisfying $\Gamma \supset [0,1]$. Using the well-known argument (see [LT, p. 4]) it is easy to see that in such a case $G$ contains a basic sequence $\{y_i\}_{i=1}^\infty$ consisting of functions satisfying 1) $\|y_i\|_G \leq 1$ and $\sup_{z \in \Gamma} |\bar{y}_i(z)| \leq 1$, where $\bar{y}_i$ is an analytic continuation of $y_i$.

Let $\Delta$ be a bounded open subset of the complex plane such that $\Delta \supset [0,1]$. Let $f : \Delta \to \mathbb{C}$ be a bounded analytic function on $\Delta$. (If we consider real spaces then we assume in addition that $f$ takes real values on the real axis.) Function $f$ generates a continuous functional on $L_1((0,1))$ by means of the formula

$$\langle f, x \rangle = \int_0^1 f(t)x(t)dt.$$ 

Since $F$ is continuously embedded into $L_1((0,1))$ then $f$ generates a continuous functional on $F$. Let us denote by $U$ the subspace of $F^*$ consisting of all functionals of this type. Observe that $U$ is a total subspace of $F^*$. It follows from the following facts: 1) $F$ is injectively embedded into $L_1((0,1))$; 2) the set of functionals generated by polynomials is a total subspace of $(L_1((0,1)))^*$.

We need the following variant of Lemma 3.3.

Proposition 3.6. Let $\alpha \geq 1$ be a countable ordinal. If $U$ contains a total subspace $L$ of order $\alpha$ (as a subspace of $F^*$) and $G$ satisfies the condition from Remark 3.5, then there exists a linear integral operator $T : F \to G$ with an analytic kernel such that $T^{-1} : T(F) \to F$ is in Borel class $\alpha$, but is not in Borel class $\beta$ for any $\beta < \alpha$.

Proof. We follow the proof of Lemma 3.3. Let $\{y_i\}_{i=1}^\infty \subset G$ be the sequence from Remark 3.5 and let $\{m_i\}_{i=1}^\infty$ be a weak* dense sequence in $\{l \in L : \|l\| \leq 1\}$. Let
\[ a_i = \sup_{z \in \Delta} |m_i(z)| \] (recall that \( U \) and, hence \( L \) consists of functionals generated by bounded analytic functions on \( \Delta \).)

Let \( T : F \to G \) be defined by

\[ Tx = \sum_{i=1}^{\infty} \frac{1}{2^i a_i} m_i(x) y_i. \]

(We may and shall assume that none of \( m_i \)'s is zero and hence none of \( a_i \)'s is zero.)

Observe that \( T \) is an integral operator of the required type.

The rest of the proof is as in Lemma 3.3 (the only difference is that we have \( T^*G^* \subset \text{cl}(L) \) instead of \( T^*G^* \subset L \), but it does not cause any problems). \( \square \)

**Remark 3.7.** Proposition 3.6 is taken from [O8]. The idea goes back to L.D. Menikh [Men].

To apply Proposition 3.6 we need a condition under which \( U \) contains a subspace of order \( \alpha \). One such condition was found by the present author in [O8]. As is natural in such degree of generality, it is a condition for existence of subspaces of large orders only.

**Theorem 3.8.** Let \( F \) be a separable Banach space and \( U \) be a total subspace in \( F^* \). Consider elements of \( F \) as functionals on \( U \). We get a subspace in \( U^* \). Denote it by \( W \). If \( \text{cl}(W) \) is of infinite codimension in \( U^* \), then for every countable ordinal \( \alpha \) there exists a total subspace \( L \) of \( U \) whose order is \( \geq \alpha \).

The proof of this theorem uses a ramification of the approach of [O1] to the proof of Theorem 2.3. It is too technical to be presented here (see [O7] and [O8]).

It is natural to ask: how can we check whether the condition of Theorem 3.8 is satisfied for a given space? For many classical spaces the answer follows from the following observation: if \( \text{cl}(U) \) contains a subspace isomorphic to \( l_1 \), then \( U^* \) is non-separable and hence \( \text{cl}(W) \) (it is obviously a separable space) is of infinite codimension in it. For classical non-reflexive Banach spaces (e.g. \( C(0,1) \), \( L_1(0,1) \)) it is not difficult to find such isomorphic copies of \( l_1 \).

**Remark 3.9.** As a corollary of Theorem 3.8 and Proposition 3.6 we get a generalization of the results of L.D. Menikhes [Men] and A. Plichko [Pl2] (see Remark 3 in [O8]). L.D. Menikh [Men] proved that there exists an integral operator from \( C(0,1) \) to \( L_2(0,1) \) with infinitely differentiable kernel and nonregularizable (=not of the Borel class 1) inverse. A.N. Plichko [Pl2] constructed operators with these properties for wide classes of function spaces.
References

[AJ] J.M. Anderson and J.E. Jayne, The sequential stability index of a function space, Mathematika 20 (1973), 210–213.

[A1] L. Alaoglu, Weak topologies of normed linear spaces, Annals of Math. (2) 41 (1940), 252–267.

[A2] L. Alaoglu, Weak convergence of linear functionals (abstract), Bull. Amer. Math. Soc. 44 (1938), 196.

[Al] A.A. Albanese, On total subspaces in duals of spaces of type $C(K)$ or $L^1$, Proc. Roy. Irish Acad., Sect. A 93 (1993), 43–47.

[A] P.S. Alexandrov, Introduction to the set theory and general topology, “Nauka”, Moscow, 1977 (Russian); German translation: P.S. Alexandroff, Einführung in die Mengenlehre und in die allgemeine Topologie, Hochschulbcher für Mathematik, 85, VEB Deutscher Verlag der Wissenschaften, Berlin, 1984.

[AU] P. Alexandrov and P. Urysohn, Zur Theorie der topologischen Räume, Math. Annalen 92 (1924), 258–266.

[B1] S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, Fund. Math. 17 (1931), 283–295; Reprinted with commentary in: S. Banach, Oeuvres, v. I, Warsaw, PWN-Éditions Scientifiques de Pologne, 1967, pp. 207–217.

[B2] S. Banach, Théorie des opérations linéaires, (This edition was reprinted by Chelsea Publishing Company, without proper reference. It was also reprinted in [B5]), Monografje Matematyczne, Warszawa, 1932.

[B3] S. Banach, Kurs funktsional’nego analiza (A course of functional analysis), (A Ukrainian translation of [B2], with some alterations. Alterations are not commented and it is not clear who made them), Radyans’ka Shkola, Kyiv, 1948. (Ukrainian)

[B4] S. Banach, Theory of linear operations, North-Holland, Amsterdam New York Oxford Tokyo, 1987.

Remark on [B4]. The English translation of the Banach’s book was published when many new results on weak* sequential closures were available. Since these results are closely connected to the results of the appendix to [B4] it was natural to mention them in the English translation. Unfortunately it was not done. Also all Banach’s footnotes and, in particular, majority of his references were removed. So the history of the subject cannot be properly understood without looking at [B2] or [B3].

[B5] S. Banach, Oeuvres, vol. II, PWN-Éditions Scientifiques de Pologne, Warsaw, 1979.

[BDP] T. Banak, T. Dobrowolski and A. Plichko, The topological and Borel classification of operator images, Dissert. Math. 387 (2000), 37–52.

[BDH] E. Behrends, S. Dierolf and P. Harmand, On a problem of Bellenot and Dubinsky, Math. Ann. 275 (1986), 337–339.

[Bou] N. Bourbaki, Sur les espaces de Banach, C. R. Acad. Sci. Paris 206 (1938), 1701–1704.

[Bo] J. Bourgain, $H^\infty$ is a Grothendieck space, Studia Math. 75 (1983), 193–216.

[CY] P. Civin and B. Yood, Quasi-reflexive spaces, Proc. Amer. Math. Soc. 8 (1957), 906–911.

[DJ] W. J. Davis and W. B. Johnson, Basic sequences and norming subspaces in non- quasi-reflexive Banach spaces, Israel J. Math. 14 (1973), 353–367.

[DL] W. J. Davis and J. Lindenstrauss, On total non-norming subspaces, Proc. Amer. Math. Soc. 31 (1972), 109–111.

[DM] S. Dierolf and V. B. Moscatelli, A note on quasi-jections, Functiones et approximatio 17 (1987), 131–138.

[D] J. Diestel, Sequences and series in Banach spaces, Graduate Texts in Mathematics, 92, Springer-Verlag, New York-Berlin, 1984.

[DU] J. Diestel and J. J. Uhl, Jr., Vector measures, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[Di] J. Dixmier, Sur un théorème de Banach, Duke Math. J. 15 (1948), 1057–1071.

[DS] N. Dunford and J. T. Schwartz, Linear Operators: General Theory, Pure and Applied Mathematics, vol. 7, Interscience, New York, 1958.

[Fl] R. J. Fleming, Weak*-sequential closure and characteristic of subspaces of conjugate Banach spaces, Studia Math. 26 (1966), 307–313.

[FMR] R. J. Fleming, R. D. McWilliams and J. R. Retherford, On $w^*$-sequential convergence, type $P^*$ bases and reflexivity, Studia Math. 25 (1965), 325–332.
REFERENCES

[Fo] V.P.Fonf, Two theorems on quasireflexive Banach spaces, Ukrain. Mat. Zh. 42 (1990), no. 2, 276–279 (Russian); English transl. in Ukrainian Math. J. 42 (1990), no. 2, 245–247.

[G1] B.V.Godun, Weak∗ derived sets of set of linear functionals, Mat. Zametki 23 (1978), 607–616 (Russian); English transl. in Math. Notes 23 (1978), 333–338.

[G2] Weak∗ derived sets of transfinite order of sets of linear functionals, Sib. Mat. Zh. 18 (1977), no. 6, 1289–1295 (Russian); English transl. in Siberian Math. J. 18 (1977), 913–917.

[Gr] A.Grothendieck, Sur les applications linéaires faiblement compactes d’espaces du type C(K), Canadian J. Math. 5 (1953), 129–173.

[Ha] R.Haydon, A nonreflexive Grothendieck space that does not contain l∞, Israel J. Math. 40 (1981), 65–73.

[KL] A.S.Kechris and A.Louveau, Descriptive set theory and the structure of sets of uniqueness, Cambridge University Press, 1987.

[KLTT] A.S.Kechris, A.Louveau and V.Tardivel, The class of synthesizeable pseudomeasures, Illinois J. Math. 35 (1991), 107–146.

[Kh] S.S.Khurana, Grothendieck spaces, Illinois J. Math. 22 (1978), 79–80.

[Kur] K.Kuratowski, Topology, vol. I, Academic Press and PWN, New York and Warsaw, 1966.

[LT] J. Lindenstrauss and L.Tzafriri, Classical Banach spaces I, Sequence spaces, Springer-Verlag, Berlin, 1977.

[Ly] R.Lyons, A new type of sets of uniqueness, Duke Math. J. 57 (1988), 431–458.

[Mau] R.D.Mauldin, Baire functions, Borel sets, and ordinary function systems, Advances in Math. 12 (1974), 418–450.

[Maz] S.Mazurkiewicz, Sur la dérivée faible d’un ensemble de fonctionnelles linéaires, Studia Math. 2 (1930), 68–71.

[McG] O.C.McGehee, A proof of a statement of Banach about the weak∗ topology, Michigan Math. J. 15 (1968), 135–140.

[McW1] R.D.McWilliams, Iterated w∗-sequential closure of a Banach space in its second conjugate, Proc. Amer. Math. Soc. 16 (1965), 1195–1199.

[McW2] On certain Banach spaces which are w∗-sequentially dense in their second duals, Duke Math. J. 37 (1970), 121–126.

[Men] L.D.Menikhes, On the regularizability of mappings inverse to integral operators, Doklady AN SSSR 241 (1978), no. 2, 282–285 (Russian); English transl. in Soviet Math. Dokl 19 (1978), 838–841 (1979).

[M1] V.B.Moscatelli, On strongly non-norming subspaces, Note Mat. 7 (1987), 311–314.
[M2] M.I.Ostrovskii, Strongly nonnorming subspaces and prequojections, Studia Math. 95 (1990), 249-254.

[N] J. von Neumann, Zur Algebra der Funktionaloperationen und Theorie der Normalen Operatoren, Math. Annalen 102 (1929-1930), 370–427; Reprinted in: J.von Neumann, Collected works, v. II, Pergamon Press, New York, 1961, pp. 86–143.

[OR] E.Odell and H.P.Rosenthal, A double-dual characterization of separable Banach spaces containing $l_1$, Israel J. Math. 20 (1975), 375–384.

[Op] Open Problems, Presented at the Ninth Seminar (Poland - GDR) on Operator Ideals and Geometry of Banach Spaces, Forschungsberichte Friedrich–Schiller–Universität, Jena, N/87/28 Georegental, April, 1986, Communicated by A.Pietsch, 1987.

[O1] M.I.Ostrovskii, $w^*$-derived sets of transfinite order of subspaces of dual Banach spaces, Dokl. Akad. Nauk Ukrain. SSR 1987, no. 10, 9–12 (Russian, Ukrainian); An English version of this paper can be found on the web at http://front.math.ucdavis.edu/

[O2] , On the problem of regularizability of the superpositions of inverse linear operators, Teor. Funktsii, Funktsional. Anal. i Prilozhen. 55 (1991), 96–100 (Russian); English transl. in J. Soviet Math. 59 (1992), 652–655.

[O3] , The structure of total subspaces of dual Banach spaces, Teor. Funktsii, Funktsional. Anal. i Prilozhen. 58 (1992), 60–69 (Russian); English transl. in J. Math. Sci. 85 (1997), 2188–2193.

[O4] , Total subspaces in dual Banach spaces which are not norming over any infinite-dimensional subspace, Studia Math. 105 (1993), no. 1, 37–49.

[O5] , On the classification of total subspaces of dual Banach spaces, C. r. Acad. bulg. Sc. 45 (1992), no. 7, 9–10.

[O6] , Characterizations of Banach spaces which are completions with respect to total nonnorming subspaces, Arch. Math. 60 (1993), 349–358.

[O7] , Total subspaces with long chains of nowhere norming weak$^*$ sequential closures, Note Mat. 13 (1993), 217–227.

[O8] , A note on analytical representability of mappings inverse to integral operators, Matematicheskaya Fizika, Analiz i Geometriya 1 (1994), no. 3/4, 513–519 (Russian); MR98k:47064; An English version of this paper can be found on the web at http://front.math.ucdavis.edu/

[O9] , On prequojections and their duals, Revista Mat. Univ. Complutense Madrid 11 (1998), 59–77.

[O10] , Completions with respect to total nonnorming subspaces, Matematicheskaya Fizika, Analiz i Geometriya 6 (1999), no. 3/4, 317–322.

[PB] A.Pelczyński and Cz. Bessaga, Some aspects of the present theory of Banach spaces, in: S.Banach, Oeuvres, vol. II, PWN-Éditions Scientifiques de Pologne, Warsaw, 1979, pp. 221–302; Reprinted in [B4], pp. 161–237.

[P] Yu.I.Petunin, Conjugate Banach spaces containing subspaces of zero characteristic, Dokl. Akad. Nauk SSSR 154 (1964), 527–529 (Russian); English transl. in Soviet Math. Dokl. 5 (1964), 131–133.

[PP] Yu.I.Petunin and A.N.Plichko, The theory of characteristic of subspaces and its applications, “Vyshcha Shkola”, Kiev, 1980. (Russian)

[P1] I.I.Piatetski-Shapiro, On the problem of the uniqueness of the expansion of a function in a trigonometric series, Moskov. Gos. Univ. Uč. Zap. Mat. 155(5) (1952), 54–72. (Russian)

[P2] , Supplement to the work “On the problem of uniqueness of expansion of a function in a trigonometric series”, Moskov. Gos. Univ. Uč. Zap. Mat. 165(7) (1954), 79–97. (Russian)

[P11] A.Plichko, Weak$^*$ sequential closures and B-measurability of mappings inverse to linear continuous operators in WCG-spaces (summary), Sibirsk. Mat. Zh. 22 (1981), no. 6, 217. (Russian)

[P2] , Non-norming subspaces and integral operators with non-regularizable inverses, Sibirsk. Mat. Zh. 29 (1988), no. 4, 208-211 (Russian); English transl. in Siberian Math. J. 29 (1988), 687–689.

[P3] , On bounded biorthogonal systems in some function spaces, Studia Math. 84 (1986), 25–37.
REFERENCES

[Pl4] ______, Decomposition of Banach space into a direct sum of separable and reflexive subspaces and Borel maps, Serdica Math. J. 23 (1997), 335-350.

[Pl5] ______, A criterion for the quasireflexivity of a Banach space, Dopovidi Akad. Nauk Ukrain. RSR Ser. A, 1974, 406–408. (Ukrainian)

[R] S.Roelwick, On the inversion of non-linear transformations, Studia Math. 17 (1958), 79–83.

[R1] H.P.Rosenthal, A characterization of Banach spaces containing $l_1$, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411–2413.

[R2] ______, Some recent discoveries in the isomorphic theory of Banach spaces, Bull. Amer. Math. Soc. 84 (1978), 803–831.

[R3] ______, A characterization of Banach spaces containing $c_0$, J. Amer. Math. Soc. 7 (1994), 707–748.

[S] J.Saint-Raymond, Espaces a modèle séparable, Annales Inst. Fourier (Grenoble) 26 (1976), no. 3, 211–256.

[S1] D.Sarason, On the order of a simply connected domain, Michigan Math. J. 15 (1968), 129–133.

[S2] ______, A remark on the weak-star topology of $l^\infty$, Studia Math. 30 (1968), 355–359.

[S3] ______, Weak-star generators of $H^\infty$, Pacific J. Math. 17 (1966), 519–528.

[S4] ______, Weak-star density of polynomials, J. Reine Angew. Math. 252 (1972), 1–15.

[Ta] M.Talagrand, Un nouveau $C(K)$ qui possède la propriété de Grothendieck, Israel J. Math. 37 (1980), 181–191.

[Ty] A.Tychonoff, Über die topologische Erweiterung von Räumen, Math. Annalen 102 (1929), 544-561.

[V] V.A.Vinokurov, Regularizability and analytical representability, Doklady AN SSSR 220 (1975), no. 2, 269–272 (Russian); English transl. in Soviet Math. Dokl. 16 (1975), 52–55.

[VPP1] V.A.Vinokurov, Yu.I.Petunin and A.N.Plichko, Conditions for measurability and regularizability of mappings inverse to linear continuous mappings, Doklady AN SSSR 220 (1975), no. 3, 509–511 (Russian); English transl. in Soviet Math. Dokl. 16 (1975), 97–99.

[VPP2] ______, Measurability and regularizability of mappings that are inverses of continuous linear operators, Mat. Zametki 26 (1979), no. 4, 583–591 (Russian); English transl. in Math. Notes 26 (1979), 781–785.

DEPARTMENT OF MATHEMATICS, THE CATHOLIC UNIVERSITY OF AMERICA, WASHINGTON, D.C. 20064, USA

E-mail address: ostrovskii@cua.edu