SIMPLE HOMOTOPY TYPES OF HOM-COMPLEXES, NEIGHBORHOOD COMPLEXES, LOVÁSZ COMPLEXES, AND ATOM CROSSCUT COMPLEXES

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Abstract. In this paper we provide concrete combinatorial formal deformation algorithms, namely sequences of elementary collapses and expansions, which relate various previously extensively studied families of combinatorially defined polyhedral complexes.

To start with, we give a sequence of elementary collapses leading from the barycentric subdivision of the neighborhood complex to the Lovász complex of a graph. Then, for an arbitrary lattice $L$ we describe a formal deformation of the barycentric subdivision of the atom crosscut complex $\Gamma(L)$ to its order complex $\Delta(\bar{L})$. We proceed by proving that the complex of sets bounded from below $J(L)$ can also be collapsed to $\Delta(\bar{L})$.

Finally, as a pinnacle of our project, we apply all these results to certain graph complexes. Namely, by describing an explicit formal deformation, we prove that, for any graph $G$, the neighborhood complex $N(G)$ and the polyhedral complex $\text{Hom}(K_2, G)$ have the same simple homotopy type in the sense of Whitehead.

1. Introduction.

Motivation for the research presented in this paper came from the quest for better understanding of the relationship between neighborhood complexes and Hom-complexes.

Originally, neighborhood complexes were introduced and used by Lovász, see [Lo78], to attack the Kneser Conjecture, as well as to provide some of the first nontrivial algebro-topological lower bounds for chromatic numbers of graphs. After an active period of research and several attempts at the generalizations of the neighborhood complexes, the so-called Hom-complexes were introduced, again by Lovász. We refer the reader to the survey article [Ko05a].

Hom-complexes depend on two parameters, both of them graphs. One of the motivations for introducing these gadgets was the fact that the polyhedral complex $\text{Hom}(K_2, G)$ turned out to be homotopy equivalent to the simplicial complex $N(G)$, for any graph $G$, see e.g., [BK03b, Proposition 4.2] for an argument. We remark, that all known proofs of this fact are in a way nonconstructive, making use of statements like Quillen’s Fiber Lemma.

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One of the classical views of topology is combinatorial, using “moves” between cell complexes called elementary collapses and elementary expansions, see e.g., [Al30] for a prototypical approach. These ideas were further developed and reached their maturity in the work of Whitehead, see e.g., [Wh39].

The suggested modus operandi would be to, instead of looking for continuous homotopies, construct a discrete object: a so-called formal deformation. The natural question of whether two homotopy equivalent spaces would necessarily be connected by a formal deformation turned out to have a negative answer, and as a result an exciting and important theory of simple homotopy type and Whitehead torsion ensued.

Precisely this circle of ideas has been the driving force behind this article. As a consequence, we were able to find explicit formal deformations between various combinatorially defined simplicial complexes. These sequences, when concatenated, prove that $\text{Hom}(K_2, G)$ and $\mathcal{N}(G)$ have the same simple homotopy type, for any graph $G$.

In the process of constructing these formal deformations we had to revise and upgrade several central results from Topological Combinatorics, these are Theorems 4.4 and 5.2. Classically these results would just conclude the existence of a deformation retraction, or, even sometimes only the existence of a homotopy equivalence, see [Bj95]. Here, both in Theorem 4.4 and in Theorem 5.2, we provide an explicitly described algorithmic sequence of collapses, and expansions (expansions are only required in the second theorem).

Furthermore, we needed to consider a complex, which does not seem to have appeared before. For any finite lattice we construct a simplicial complex of bounded below sets of elements: its vertices are elements of $\bar{\mathcal{L}}$ and $S \subseteq \bar{\mathcal{L}}$ is a simplex if and only if $S$ has lower bound different from $\hat{0}$. It turned out that this complex collapses onto the order complex $\Delta(\bar{\mathcal{L}})$, and that this sequence of collapses can be described algorithmically.

Finally, in the last section we bring all these results into play in order to construct the promised formal deformation between the studied graph complexes. It is well known, see [Co73], that in a formal deformation all the expansions can be carried out first, followed by the collapses, however we have chosen to present our formal deformation as in (7.1), since it passes through several complexes, which appear to be of interest in their own.

Unfortunately, our formal deformation is still rather complicated. Finding a simpler natural formal deformation from $\text{Hom}(K_2, G)$ to $\mathcal{N}(G)$ remains a challenging task.

2. Notations.

We start by recalling some notations. For a partially ordered set $P$ we let $\Delta(P)$ denote its order complex (also known as the nerve of the corresponding category), that is the simplicial complex whose set of vertices is the set of elements of $P$, and whose set of simplices is the set of chains (ordered subsets) of $P$.

Let $P$ be an arbitrary partially ordered set. For any subset $S \subseteq P$, we let $P[S]$ denote the induced partial order on $S$. We let $P_{\text{op}}$ denote the poset whose set of elements is the same as that of $P$, but whose partial order is the reverse of the partial order of $P$. Note that for arbitrary poset $P$ we have $\Delta(P) = \Delta(P_{\text{op}})$. The minimal, resp. maximal, element of $P$ (if it has one), is denoted by $\hat{0}$, resp. $\hat{1}$. In
Deformation algorithm from a concatenation of two steps. Any CW complex $X$ of our results we describe here an explicit formal deformation leading from $X$ to $sd\left(\left(X,\sigma\right)\right)$. Stellar subdivisions, it is enough to find a formal deformation leading from the complex $Y$ of elementary collapses and elementary expansions leading from a complex $X$ to $Bd\left(\left(X,\sigma\right)\right)$. For an arbitrary lattice $L$, and a subset $S \subseteq L$, we let $\wedge S$ denote the common meet of all the elements in $S$, and, analogously, we let $\vee S$ denote the common join of all the elements in $S$.

3. Barycentric and stellar subdivisions.

For an arbitrary CW complex $X$, we let $F(X)$ denote its face poset: the partially ordered set whose elements are all nonempty cells of $X$, and whose partial order is given by the cell inclusion. When $X$ is a regular CW complex, we let $Bd\left(X\right)$ denote its barycentric subdivision. Clearly, we have $Bd\left(X\right) = \Delta\left(F\left(X\right)\right)$.

For a simplicial complex $X$, and an arbitrary simplex $\sigma \in X$, let $lk_X\sigma$ denote the link of $\sigma$ in $X$, let $st_X\sigma$ denote the closed star of $\sigma$ in $X$. Furthermore, let $sd\left(X,\sigma\right)$ denote the stellar subdivision of $X$ at $\sigma$. The effect that the stellar subdivision has on the face poset $F(X)$ is a special case of the combinatorial blowup in a lattice: $F(sd\left(X,\sigma\right)) \cup \{0\} = \mathrm{Bl}_{\sigma}(F(X) \cup \{0\})$, see [FK04, Proposition 4.9].

It is a classical fact that the barycentric subdivision can be represented as a sequence of stellar subdivisions: simply take a reverse linear extension of $F(X)$ and perform stellar subdivisions of the corresponding simplices in this order. Combinatorially, using the terminology of [FK04], this corresponds to taking the whole poset $F(X)$ as a building set. The above mentioned fact can then be seen as a special case of [FK04, Theorem 3.4].

When $Y$ is a simplicial subcomplex of $X$, we say that $X$ collapses onto $Y$ if there exists a sequence of elementary collapses leading from $X$ to $Y$; in this case we write $X \searrow Y$ (or, equivalently, $Y \nearrow X$); we refer the reader to [Cö73, §4, p. 14], for the definition of the elementary collapse for an arbitrary finite CW pair. The reverse of an elementary collapse is called an elementary expansion. A sequence of elementary collapses and elementary expansions leading from a complex $X$ to the complex $Y$ is called a formal deformation. If such a sequence exists, then the simplicial complexes $X$ and $Y$ are said to have the same simple homotopy type, see [AE30, Cö73, Wh39].

It is well-known, see e.g., [Cö73, §25, Statement (25.1)], that a subdivision of any CW complex $X$ has the same simple homotopy type as $X$. For completeness of our results we describe here an explicit formal deformation from $X$ to $Bd X$.

To start with, since the barycentric subdivision can be represented as a sequence of stellar subdivisions, it is enough to find a formal deformation leading from $X$ to $sd\left(X,\sigma\right)$, for an arbitrary simplex $\sigma \in X$. One choice of such deformation is a concatenation of two steps.

Deformation algorithm from $X$ to $sd\left(X,\sigma\right)$.

Step 1. Add a cone over $st_X\sigma$. More precisely, consider a new simplicial complex $X'$, such that $V(X') = V(X) \cup \{v\}$, $X$ is an induced subcomplex of $X'$, and $lk_{X'}v = st_X\sigma$.

Step 2. Delete from $X'$ all the simplices containing $\sigma$. From these steps, by taking barycentric and stellar subdivisions, we get our desired formal deformation.

This completes the proof of Theorem 1.

\[\blacksquare\]
Since \( st(X)(\sigma) \) is a cone, in particular collapsible, the Step 1 can be performed as a sequence of elementary expansions. Furthermore, the Step 2 can be performed as a sequence of elementary collapses as follows. The set of the simplices which are to be deleted can be written as a disjoint union of sets \( A \) and \( B \), where \( B \) is the set of all simplices which contain both \( \sigma \) and \( v \). Clearly, adding \( v \) to a simplex is a bijection \( \mu : A \to B \). Let \( \{\tau_1, \ldots, \tau_t\} \) be a reverse linear extension order on \( A \), then \( \{\tau_1, \mu(\tau_1), \ldots, \tau_t, \mu(\tau_t)\} \) is an elementary collapsing sequence.

Finally, we see that performing Steps 1 and 2, in this order, will yield a stellar subdivision of \( X \) at \( \sigma \), and therefore our description is completed.

4. Collapsing the neighborhood complex of a graph onto its Lovász complex.

The next theorem is a specialization of [Ko05b Theorem 3.1(b)] to the case of the finite posets.

Theorem 4.1. [Ko05b]. Let \( P \) be a finite poset, and let \( \varphi : P \to P \) be a monotone map. Assume \( P \supseteq Q \supseteq \text{Fix} \varphi \), then \( \Delta(P) \) collapses onto \( \Delta(Q) \).

Let \( G \) be an arbitrary undirected graph. We let \( V(G) \) denote the set of vertices of \( G \). For any \( v \in V(G) \), we let \( N(v) \) denote the set of all neighbors of \( v \), i.e., \( N(v) = \{w \in V(G) \mid (w, v) \in E(G)\} \). Furthermore, for any subset \( S \subseteq V(G) \), we set \( N(S) := \bigcap_{v \in S} N(v) \), i.e., \( N(S) \) denotes the set of common neighbors of all the vertices in \( S \).

In [Lo78], Lovász has introduced the following class of simplicial complexes, in order to study the topological obstructions to graph colorings.

Definition 4.2. For an arbitrary graph \( G \), let \( \mathcal{N}(G) \) be the simplicial complex, whose set of vertices consists of all nonisolated vertices of \( G \), and whose set of simplices consists of all subsets \( S \subseteq V(G) \), such that the vertices in \( S \) have a common neighbor, i.e., such that \( N(S) \neq \emptyset \).

These complexes have been studied fairly extensively, see e.g., [CsH, Ziv04].

Note that \( N \) induces an order-reversing map \( N : \mathcal{F}(\mathcal{N}(G)) \to \mathcal{F}(\mathcal{N}(G)) \), in particular \( N^2(A) \supseteq A \), for any \( A \subseteq V(G) \). It can also be seen that \( N^3 = N \).

Definition 4.3. For an arbitrary graph \( G \), the complex \( \Delta(N(\mathcal{F}(\mathcal{N}(G)))) \) is called the Lovász complex of \( G \) and is denoted by \( \mathcal{L}(G) \).

One property, which distinguishes the Lovász complex as an interesting object of study, is that it possesses a natural \( \mathbb{Z}_2 \)-action, induced by the map \( N \). Indeed, we see that

\[
N(\mathcal{F}(\mathcal{N}(G))) \supseteq N^2(\mathcal{F}(\mathcal{N}(G))) \supseteq N^3(\mathcal{F}(\mathcal{N}(G))) = N(\mathcal{F}(\mathcal{N}(G))),
\]

hence \( N(\mathcal{F}(\mathcal{N}(G))) = N^2(\mathcal{F}(\mathcal{N}(G))) \). It is an easy check that when the graph \( G \) has no loops, this action is free, and so in this case \( \mathcal{L}(G) \) has a natural structure of \( \mathbb{Z}_2 \)-space.

It is well-known that for any graph, its neighborhood complex and its Lovász complex are homotopy equivalent. The next proposition strengthens this result.

Theorem 4.4. The simplicial complex \( \text{Bd}(\mathcal{N}(G)) \) collapses onto the simplicial complex \( \mathcal{L}(G) \). In particular, \( \mathcal{N}(G) \) and \( \mathcal{L}(G) \) have the same simple homotopy type.
Figure 4.1. A graph $G$ and its Lovász complex.

**Proof.** Define the map $\varphi : \mathcal{F}(\mathcal{N}(G)) \to \mathcal{F}(\mathcal{N}(G))$, by simply setting $\varphi := N^2$. From our previous comments, it is clear that $\varphi$ is an order-preserving map, and that $A \leq \varphi(A)$, for any $A \subseteq V(G)$. Note, that it is also true that $\varphi^2 = \varphi$, but we do not need this additional fact.

We conclude that $\varphi$ is an ascending map, and hence, by the Theorem 4.1 we obtain that the simplicial complex $\Delta(\mathcal{F}(\mathcal{N}(G))) = \text{Bd}(\mathcal{N}(G))$ collapses onto the simplicial complex $\Delta(\varphi(\mathcal{F}(\mathcal{N}(G)))) = \Delta(\mathcal{N}(\mathcal{F}(\mathcal{N}(G)))) = L(G)$. □

**Remark 4.5.** By the discussion in Section 3, the Theorem 4.4 allows us to construct an explicit formal deformation from $\mathcal{N}(G)$ to $L(G)$.

5. Simple homotopy type of crosscut complexes.

Crosscut complexes play a prominent role in Topological Combinatorics, e.g., see the survey \[1395\].

**Definition 5.1.** Let $\mathcal{L}$ be a lattice, the **atom crosscut complex** $\Gamma(\mathcal{L})$ associated to $\mathcal{L}$ is a simplicial complex defined as follows:

- the set of vertices of $\Gamma(\mathcal{L})$ is equal to the set of atoms of $\mathcal{L}$, in other words, $V(\Gamma(\mathcal{L})) = A(\mathcal{L})$;
- the subset $\sigma \subseteq A(\mathcal{L})$ is a simplex in $\Gamma(\mathcal{L})$ if and only if the join of elements in $\sigma$ is not equal to 1.

Recall that a lattice $\mathcal{L}$ is called **atomic**, if all elements of $\mathcal{L}$ can be represented as joins of atoms. For an arbitrary lattice $\mathcal{L}$, let $\mathcal{L}_a$ denote the sublattice consisting of 0, and of all the elements which are joins of atoms.

**Theorem 5.2.** Let $\mathcal{L}$ be an arbitrary finite lattice.

(a) If $\mathcal{L}$ is atomic, then the simplicial complex $\text{Bd}(\Gamma(\mathcal{L}))$ collapses onto the simplicial complex $\Delta(\bar{\mathcal{L}})$.

(b) In the general case, both $\text{Bd}(\Gamma(\mathcal{L}))$ and $\Delta(\bar{\mathcal{L}})$ collapse onto the simplicial complex $\Delta(\bar{\mathcal{L}}_a)$.

In both cases we conclude that the simplicial complexes $\Gamma(\mathcal{L})$ and $\Delta(\bar{\mathcal{L}})$ have the same simple homotopy type.
Proof. Assume first that $L$ is atomic. Define a map $\varphi : \mathcal{F}(\Gamma(L)) \to \mathcal{F}(\Gamma(L))$ as follows: a simplex $\sigma$ is mapped to $A(L)_{\leq \sigma}$. To start with, the map $\varphi$ is well-defined, since $\bigvee A(L)_{\leq \sigma} \leq \bigvee \sigma < 1$, also, clearly $\varphi(\sigma) \supseteq \sigma$. Furthermore, $\varphi$ is order-preserving, since if $\tau \supseteq \sigma$, then $\bigvee \tau \geq \bigvee \sigma$, implying $A(L)_{\leq \tau} \supseteq A(L)_{\leq \sigma}$. We remark, that $\bigvee \sigma \geq \bigvee \varphi(\sigma) \geq \bigvee \sigma$, hence $\bigvee \sigma = \bigvee \varphi(\sigma)$, and therefore $\varphi(\varphi(\sigma)) = \varphi(\sigma)$; however we do not need the latter fact for our argument.

From the discussion above we see that $\varphi$ is a monotone map, and hence, by the Theorem 5.1 we conclude that the simplicial complex $\Delta(F(\Gamma(L))) = \text{Bd} (\Gamma(L))$ collapses onto the simplicial complex $\Delta(F(\Gamma(L))))$. On the other hand, since the lattice is atomic, we have $\varphi(F(\Gamma(L))) = L$, and so, as desired, the simplicial complex $\text{Bd} (\Gamma(L))$ collapses onto the simplicial complex $\Delta(L)$.

Now, remove the assumption that $L$ is atomic, and consider the general case. By the argument above we see that $\text{Bd} (\Gamma(L))$ collapses onto $\Delta(L_a)$. On the other hand, it is not difficult to check that the order-preserving map $\psi : L \to L$ mapping $x$ to the join of the elements of $\mathcal{A}(L)_{\leq x}$, and mapping 0 to itself, is a descending map. Its image is precisely $L_a$. \hfill \Box

Remark 5.3. Again, by the discussion in Section 3, the Theorem 5.2 can be used to construct an explicit formal deformation from $\Gamma(L)$ to $\Delta(L)$.

Due to its general nature, the Theorem 5.2 has many applications. Let us mention one of them.

Definition 5.4. Let $n$ be any natural number, and let $DG_n$ be the simplicial complex of all disconnected graphs on $n$ labeled vertices. In other words, the vertices of $DG_n$ are all pairs $(i, j)$, with $i < j$, $i, j \in [n]$, i.e., all possible edges of a graph on labeled $n$ vertices; and simplices of $DG_n$ are all collections of edges which form a graph with at least 2 connected components.

Recall, that for an arbitrary natural number $n$, $\Pi_n$ denotes the partition lattice: the poset consisting of all set partitions of the set $\{1, \ldots, n\}$, partially ordered by partition refinement.

Corollary 5.5. The simplicial complex $\text{Bd} (DG_n)$ collapses onto $\Delta(\Pi_n)$.

Proof. A direct check yields $DG_n = \Gamma(\Pi_n)$, hence the result follows from the Theorem 5.2. \hfill \Box

We remark that the complex $DG_n$ appeared in the work of Vassiliev on knot theory. [Va93], whereas $\Delta(\Pi_n)$ encodes the geometry of the braid arrangement by means of the Goresky-MacPherson theorem, see [GM92].

Recall, that for an arbitrary lattice $L$, a crosscut is a subset $C \subseteq \hat{L}$, such that:

- $C$ is an antichain (a set of mutually incomparable elements);
- $C$ is saturated in the following sense: for any chain $\gamma$ of $L$ there exists an element $x \in C$, such that $\gamma \cup \{x\}$ is again a chain.

Generalizing the Definition 5.1 the crosscut complex $\Gamma(C, L)$ associated to the crosscut $C$ is a simplicial complex defined as follows:

- the set of vertices of $\Gamma(C, L)$ is equal to the set $C$;
- the subset $\sigma \subseteq C$ is a simplex in $\Gamma(C, L)$ if and only if either the join of the elements in $\sigma$ is not equal to $\hat{1}$, or the meet of the elements in $\sigma$ is not equal to 0.
The set of atoms is a special case of a crosscut, and the atom crosscut complex is a special case of the crosscut complex.

Naturally, a crosscut \( C \) divides the lattice \( \mathcal{L} \) into two parts

\[
\mathcal{L}_{\geq C} = \{ x \in \mathcal{L} \mid x \geq s, \text{ for some } s \in C \},
\]

and

\[
\mathcal{L}_{\leq C} = \{ x \in \mathcal{L} \mid x \leq s, \text{ for some } s \in C \},
\]

which intersect in \( C \). Let \( \mathcal{L}_C \) be the subposet consisting of \( \hat{0}, \hat{1} \), and of all joins and meets of the elements of the crosscut \( C \). Let \( \varphi : \mathcal{L} \to \mathcal{L} \) be a map defined as follows:

\[
\varphi(x) = \begin{cases} 
\bigvee C_{\leq x}, & \text{if } x \in \mathcal{L}_{\geq C}; \\
\bigwedge C_{\geq x}, & \text{if } x \in \mathcal{L}_{\leq C}.
\end{cases}
\]

We can see that \( \varphi \) is order-preserving. The only nontrivial case to be checked is when \( x \geq y, x \in \mathcal{L}_{\geq C}, \) and \( y \in \mathcal{L}_{\leq C} \). Since in this case \( x \geq y \) is a chain, there must exist an element \( z \in C \), such that \( \{x, y, z\} \) is also a chain. Obviously, we must have \( x \geq z \geq y \). Since \( z \in C_{\leq x} \cap C_{\geq y} \), we conclude that \( z \geq \varphi(y) \), and \( \varphi(x) \geq z \), hence \( \varphi(x) \geq \varphi(y) \).

It is also easy to check that \( \varphi \) is a monotone map, namely \( \varphi(x) \leq x \), if \( x \in \mathcal{L}_{\geq C} \), and \( \varphi(x) \geq x \), if \( x \in \mathcal{L}_{\leq C} \). Furthermore, the image of \( \varphi \) is precisely \( \mathcal{L}_C \). By Theorem 4.1 we see that \( \Delta(\bar{\mathcal{L}}) \) collapses onto \( \Delta(\bar{\mathcal{L}}_C) \).

Interestingly, Sonja Ćukić has remarked that in general the simplicial complex \( \text{Bd}(\Gamma(C, \mathcal{L})) \) does not have to collapse onto the simplicial complex \( \Delta(\bar{\mathcal{L}}_C) \), [Cu05].

We conclude this section by conjecturing that the weak version of Theorem 5.2 is still true in general.

**Conjecture 5.6.** For an arbitrary lattice \( \mathcal{L} \) and an arbitrary crosscut \( C \), the simplicial complex \( \text{Bd}(\Gamma(C, \mathcal{L})) \) and the simplicial complex \( \Delta(\bar{\mathcal{L}}_C) \) have the same simple homotopy type.

Together with our previous observations, this conjecture can equivalently be formulated as:

**Conjecture 5.7.** For an arbitrary lattice \( \mathcal{L} \) and an arbitrary crosscut \( C \), the simplicial complex \( \Gamma(C, \mathcal{L}) \) and the simplicial complex \( \Delta(\bar{\mathcal{L}}_C) \) have the same simple homotopy type.

6. **Collapsing the complex of sets bounded from below onto the order complex.**

We start by defining a combinatorial gadget, which provides a convenient language for describing sequences of elementary collapses.

**Definition 6.1.** Let \( P \) be a poset with the covering relation \( \succ \).

- We define a **partial matching** on \( P \) to be a set \( \Sigma \subseteq P \), and an injective map \( \mu : \Sigma \to P \setminus \Sigma \), such that \( \mu(x) \succ x \), for all \( x \in \Sigma \).
- The elements of \( P \setminus (\Sigma \cup \mu(\Sigma)) \) are called **critical**. We let \( \mathcal{C}(P, \mu) \) denote the set of critical elements.
- Additionally, such a partial matching \( \mu \) is called **acyclic** if there exists no sequence of distinct elements \( x_1, \ldots, x_t \in \Sigma \), where \( t \geq 2 \), satisfying \( \mu(x_1) \succ x_2, \mu(x_2) \succ x_3, \ldots, \mu(x_t) \succ x_1 \).
The partial acyclic matchings and elementary collapses are closely related, as the next proposition shows.

**Proposition 6.2.** Let $\Delta$ be a regular CW complex and $\Delta'$ a subcomplex of $\Delta$, then the following are equivalent:

a) there is a sequence of elementary collapses leading from $\Delta$ to $\Delta'$;
b) there is a partial acyclic matching on the poset $F(\Delta)$ with the set of critical cells being exactly $F(\Delta')$.

**Proof.** See [Ko02, Proposition 5.4]. □

We remark that the implication b)$\Rightarrow$a) is a special case of a more general result proved by R. Forman, see [Fo98].

There is a number of constructions associating a simplicial complex to a poset (or more generally, to a category), here is one which works for lattices.

**Definition 6.3.** Let $\mathcal{L}$ be an arbitrary finite lattice. We define $\mathcal{J}(\mathcal{L})$ be the simplicial complex whose set of vertices is equal to the set of elements of $\mathcal{L}$, and whose simplices are all subsets $S \subseteq \mathcal{L}$ which have a nontrivial lower bound, i.e., such that $\wedge S \neq 0$.

Clearly, the simplicial complex $\mathcal{J}(\mathcal{L})$ contains $\Delta(\mathcal{L})$ as a subcomplex. It turns out that much more is true.

**Theorem 6.4.** Let $\mathcal{L}$ be an arbitrary finite lattice, then $\mathcal{J}(\mathcal{L}) \setminus \Delta(\mathcal{L})$.

**Proof.** As the centerpiece of the argument we define the following partial acyclic matching on $F(\mathcal{J}(\mathcal{L}))$. Let $S$ be an arbitrary simplex of $\mathcal{J}(\mathcal{L})$. Assume that $F(\mathcal{J}(\mathcal{L}))[S]$ is not a chain. Set $t := |S|$, and let $\{a_1, a_2, \ldots, a_t\}$ be a linear extension of $F(\mathcal{J}(\mathcal{L}))[S]$, i.e., if $1 \leq i < j \leq t$, then $a_i \not\geq a_j$. Let $k(S)$ be the maximal index, $1 \leq k(S) \leq t$, such that $a_1 < a_2 < \cdots < a_{k(S)}$, and $a_{k(S)} < a_i$, for all $k(S)+1 \leq i \leq t$, see Figure 6.1. If $S$ has no minimal element, then we set $k(S) := 0$. Set $a(S) := a_{k(S)+1} \land \cdots \land a_t$. Since $F(\mathcal{J}(\mathcal{L}))[S]$ is not a chain, we have $k(S) \leq t - 2$, and hence $a(S)$ is well-defined.

Let $\Sigma$ be the set of all subsets $S \subseteq \mathcal{L}$, such that $F(\mathcal{J}(\mathcal{L}))[S]$ is not a chain, and such that $a(S) \not\in S$. For $S \in \Sigma$ define $\mu(S) := S \cup \{a(S)\}$, again see Figure 6.1. Clearly, $\mu$ defines a partial matching, and, since for any $S \in \Sigma$ we have $a(\mu(S)) = a(S)$, we see that the set $\mu(\Sigma) \cup \Sigma$ consists of all subsets $S \subseteq \mathcal{L}$, such that $F(\mathcal{J}(\mathcal{L}))[S]$ is not a chain. Consequently, the set of critical elements $\mathcal{C}(\mathcal{J}(\mathcal{L})), \mu)$ consists of all chains $S \in F(\Delta(\mathcal{L}))$.

Let us see that the partial matching $\mu$ is acyclic. Assume there exists a sequence $S_1, \ldots, S_t \in \Sigma$, where $t \geq 2$, such that $\mu(S_1) \succ S_2, \mu(S_2) \succ S_3, \ldots, \mu(S_t) \succ S_1$. Let again $\{a_1, a_2, \ldots, a_t\}$ be a linear extension of $F(\mathcal{J}(\mathcal{L}))[S_1]$, as above. By the definition of covering relations, and, since $S_2 \neq S_1$, we have $S_2 = \mu(S_1) \setminus \{a_1\}$, for some $1 \leq i \leq t$. If $1 \leq i \leq k(S_1)$, then $a(S_2) = a(S_1)$, which, together with $S_1 = \mu(S_1) \setminus \{a(S_1)\}$, implies $a(S_2) \in S_2$, and hence $S_2 \in \mu(\Sigma)$, giving a contradiction.

Finally, the only option left is that $k(S_1)+1 \leq i \leq t$, in which case $a(S_2) \geq a(S_1)$, since the join is taken over a set, where each element is larger than $a(S_1)$. If the equality $a(S_2) = a(S_1)$ holds, then $S_2 \in \mu(\Sigma)$, again giving a contradiction. Thus we have shown that a strict inequality must hold: $a(S_2) > a(S_1)$.

Analogously, we can prove that $a(S_{i+1}) > a(S_i)$, for all $1 \leq i \leq t - 1$, and that $a(S_1) > a(S_t)$, which, when combined together, yields a contradiction to the
assumption that the matching is not acyclic. By Proposition 6.2 we see that the acyclic matching \( \mu \) provides a sequence of elementary collapses leading from \( \mathcal{F}(\mathcal{L}) \) to \( \Delta(\overline{\mathcal{L}}) \).

We invite the interested reader to see what the statement of the Theorem 6.4 translates to for their favorite lattice \( \mathcal{L} \).

7. Application to graph complexes.

Let \( G \) and \( T \) be two undirected graphs. Recall that the set map \( \varphi : V(G) \to V(T) \) is called a graph homomorphism from \( G \) to \( T \) if, for any pair of vertices \( x, y \in V(G) \), such that \( (x, y) \in E(G) \), we have \( (\varphi(x), \varphi(y)) \in E(T) \).

**Definition 7.1.** For arbitrary undirected graphs \( T \) and \( G \), we let \( \text{Hom}(T, G) \) denote the polyhedral complex whose cells are indexed by all functions \( \eta : V(T) \to 2^{V(G)} \setminus \{\emptyset\} \), such that for any \( (x, y) \in E(T) \), we have \( \eta(x) \times \eta(y) \subseteq E(G) \).

The closure of a cell \( \eta \) consists of all cells indexed by functions \( \tilde{\eta} : V(T) \to 2^{V(G)} \setminus \{\emptyset\} \), which satisfy \( \tilde{\eta}(v) \subseteq \eta(v) \), for all \( v \in V(T) \).

We note that the set of vertices of \( \text{Hom}(T, G) \) coincides with the set of all graph homomorphisms from \( T \) to \( G \), so the polyhedral complex \( \text{Hom}(T, G) \) may be thought of as an appropriate topologization of this set.

The \( \text{Hom} \)-complexes were introduced by Lovász, and recently studied in a series of papers, see [BK03a, BK03b, BK04, CK04a, CK04b, Ko04, Ko05a, Ziv04], in connection with topological obstructions to graph colorings.

For the case \( T = K_2 \), the Definition 7.1 can be restated somewhat more directly. Recall that, for arbitrary \( A, B \subseteq V(G) \), \( A, B \neq \emptyset \), we call the pair \((A, B)\) a complete bipartite subgraph of \( G \), if for any \( x \in A, y \in B \), we have \((x, y) \in E(G)\), i.e., \( A \times B \subseteq E(G) \). Let \( \Delta^{V(G)} \) be the simplex whose set of vertices is \( V(G) \), in particular, the faces of \( \Delta^{V(G)} \) can be identified with the subsets of \( V(G) \).

Clearly, \( \Delta^{V(G)} \times \Delta^{V(G)} \) is a polyhedral complex, whose cells are direct products of two simplices. \( \text{Hom}(K_2, G) \) can be identified as the subcomplex of \( \Delta^{V(G)} \times \Delta^{V(G)} \).
defined by the following condition: $\sigma \times \tau \in \text{Hom}(K_2, G)$ if and only if $(\sigma, \tau)$ is a complete bipartite subgraph of $G$.

We are now ready to formulate one of the main results of this paper.

**Theorem 7.2.** For an arbitrary graph $G$, the neighborhood complex $\mathcal{N}(G)$ and the polyhedral complex $\text{Hom}(K_2, G)$ have the same simple homotopy type.

**Proof.** Set $P := \mathcal{F}^{\text{op}}(\text{Hom}(K_2, G)) \cup \{\emptyset, \hat{1}\}$. As was mentioned before, $P$ is a lattice, and $\Delta(P) = \text{Bd} \, \text{Hom}(K_2, G)$. By the Theorem 5.2(b), we see that both simplicial complexes $\text{Bd} \, \text{Hom}(K_2, G)$ and $\text{Bd} \, \Gamma(P)$ collapse onto the simplicial complex $\Delta(P_a)$.

**Description of $\Gamma(P)$.** The vertices of $\Gamma(P)$ are all the pairs $(A, B)$, $A, B \subseteq V(G)$, such that $N(A) = B$, and $N(B) = A$. These can be indexed with the simplices $A \in \mathcal{N}(G)$, $A \in \text{Im} \, N$, which is the same as to take the elements of $N(\mathcal{F}(\mathcal{N}(G)))$, or the vertices of $\Delta(N(\mathcal{F}(\mathcal{N}(G)))) = \mathcal{L}o(G)$.

The simplices of $\Gamma(P)$ are all sets of pairs $\{(A_1, B_1), \ldots, (A_t, B_t)\}$, such that $\bigcap_{i=1}^t A_i \neq \emptyset$, and $\bigcap_{i=1}^t B_i \neq \emptyset$. Since $N(A) \cap N(B) = N(A \cup B)$, for arbitrary subsets $A, B \subseteq V(G)$, and since $B_i = N(A_i)$, for $1 \leq i \leq t$, the second condition amounts to saying that $N(\bigcup_{i=1}^t A_i) \neq \emptyset$.

Let $\mathcal{L}$ denote the poset of all $A \in \mathcal{N}(G)$, $A \in \text{Im} \, N$, ordered by inclusion, with a minimal and a maximal elements attached. Clearly, $\Delta(\mathcal{L}) = \mathcal{L}o(G)$. From the description of $\Gamma(P)$ above, we see that $\Delta(\mathcal{L})$ is a subcomplex of $\Gamma(P)$. On the other hand, by the Theorem 5.3, the simplicial complex $\mathcal{F}(\mathcal{L})$ collapses onto $\Delta(\mathcal{L})$.

Let $\mu$ be the acyclic matching from the proof of the Theorem 6.4 which gives the collapsing sequence. We claim that the restriction of $\mu$ to $\mathcal{F}(\Gamma(P))$ is again an acyclic matching. Since $\mathcal{F}(\Gamma(P))$ is a lower ideal in $\mathcal{F}(\mathcal{L})$, the only thing which has to be checked is that if $S \in \mathcal{F}(\Gamma(P)) \cap \Sigma$, then $\mu(S) \in \mathcal{F}(\Gamma(P))$; here $\Sigma$ is as in the proof of the Theorem 6.4.

Assume that $S = \{A_1, \ldots, A_t\}$, where the sets are listed in the linear extension order, i.e., if $1 \leq i < j \leq t$, then $A_i \nsubseteq A_j$. Let $a(S)$ be the subset of $V(G)$ defined as in the proof of the Theorem 6.4. Clearly, $a(S) \subseteq A_i$, this implies that $a(S) \cup \bigcup_{i=1}^t A_i = \bigcup_{i=1}^t A_i$, and therefore, the set of pairs

$$\mu(S) = \{(A_1, B_1), \ldots, (A_t, B_t), (a(S), N(a(S)))\}$$

**Figure 7.1.** The poset $\bar{P}_a$, for $P = \mathcal{F}^{\text{op}}(\text{Hom}(K_2, G)) \cup \{\emptyset, \hat{1}\}$. 

\[ \text{Description of } \Gamma(P). \text{ The vertices of } \Gamma(P) \text{ are all the pairs } (A, B), A, B \subseteq V(G), \text{ such that } N(A) = B, \text{ and } N(B) = A. \text{ These can be indexed with the simplices } \mathcal{N}(G), \text{ which are the same as to take the elements of } N(\mathcal{F}(\mathcal{N}(G))). \text{ Or the vertices of } \Delta(N(\mathcal{F}(\mathcal{N}(G)))) = \mathcal{L}o(G). \text{ The simplices of } \Gamma(P) \text{ are all sets of pairs } \{(A_1, B_1), \ldots, (A_t, B_t)\}, \text{ such that } \bigcap_{i=1}^t A_i \neq \emptyset \text{ and } \bigcap_{i=1}^t B_i \neq \emptyset. \text{ Since } N(A) \cap N(B) = N(A \cup B), \text{ for arbitrary subsets } A, B \subseteq V(G), \text{ and since } B_i = N(A_i), \text{ for } 1 \leq i \leq t, \text{ the second condition amounts to saying that } N(\bigcup_{i=1}^t A_i) \neq \emptyset. \]
is a simplex of $\Gamma(P)$.

We conclude that the restriction of $\mu$ to $F(\Gamma(P))$ gives a collapsing sequence from $\Gamma(P)$ to $L_0(G)$.

Let us summarize our findings in the following concatenation of sequences of collapses and expansions:

\begin{equation}
\text{Bd} \text{Hom}(K_2, G) \searrow \Delta(\tilde{P}_2) \nearrow \text{Bd} \Gamma(P), \quad \Gamma(P) \searrow L_0(G) \nearrow \text{Bd} \mathcal{N}(G),
\end{equation}

where the first two sequences are given by the Theorem 5.2(b), the third sequence is given by the restriction of the acyclic matching $\mu$ as above, and the fourth sequence is given by the Theorem 4.4.

The discussion in Section 3 implies now that the polyhedral complex of all bipartite subgraphs of $G$, $\text{Hom}(K_2, G)$, and the neighborhood complex $\mathcal{N}(G)$, have the same simple homotopy type, and yields an explicit formal deformation between these two complexes.

\textbf{Remark 7.3.} All 4 sequences of collapses and expansions can be nondegenerate. Figures 4.1 and 7.1 show an example of a graph which satisfies this.

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