On the evolution of a binary system with arbitrarily misaligned orbital and stellar angular momenta due to quasi-stationary tides

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ABSTRACT

We consider the evolution of a binary system interacting due to tidal effects without restriction on the orientation of the orbital, and where significant, spin angular momenta, and orbital eccentricity. We work in the low tidal forcing frequency regime in the equilibrium tide approximation. Internal degrees of freedom are fully taken into account for one component, the primary. In the case of the companion the spin angular momentum is assumed small enough to be neglected but internal energy dissipation is allowed for as this can be significant for orbital circularisation in the case of planetary companions. We obtain a set of equations governing the evolution of the orbit resulting from tidal effects. These depend on the masses and radii of the binary components, the form and orientation of the orbit, and for each involved component, the spin rate, the Coriolis force, the normalised rate of energy dissipation associated with the equilibrium tide due to radiative processes and viscosity, and the classical apsidal motion constant, $k_2$. These depend on stellar parameters with no need of additional assumptions or a phenomenological approach as has been invoked in the past. They can be used to determine the evolution of systems with initial significant misalignment of spin and orbital angular momenta as hypothesised for systems containing Hot Jupiters. The inclusion of the Coriolis force may lead to evolution of the inclination between orbital and spin angular momenta and precession of the orbital plane which may have observational consequences.

Key words: hydrodynamics - celestial mechanics - planetary systems: formation, planet -star interactions, stars: binaries: close, rotation, oscillations, solar-type
1 INTRODUCTION

Tidal interactions are important in close binary systems where they determine the rates of orbital circularisation, as well as synchronisation and alignment of the component spins with the angular momentum of the orbit (see Ogilvie 2014, for a review). Recent attention has focused on stars with planetary companions such as hot Jupiters where tides have been postulated to play an important role in shaping the system. Tidal interactions have been considered in a low tidal forcing equilibrium tide regime or alternatively in a regime where so called dynamical tides and the excitation of normal modes in one or both of the components is important.

In this paper we consider tidal interactions in the equilibrium tide regime in a binary system consisting of a primary component with spin angular momentum that is arbitrarily misaligned with that of the binary orbit. The companion is assumed to be compact and initially with no internal degrees of freedom though later this will be relaxed to allow for internal energy dissipation that can contribute to the circularisation of the orbit. This is important if the model is to be applied to cases with planetary mass companions.

Eggleton et al. (1998), hereafter EKH, derived the force and couple on a binary orbit that arises from dissipation of an assumed equilibrium tide under the imposed assumption that the rate of dissipation of energy is a positive definite function of the rate of change of the primary quadrupole tensor viewed in a frame rotating with the star. Coriolis forces were neglected. Their model was found to lead to results obtained under the assumption of a constant time lag between the tidal forcing and response without explicit reference to dissipative processes as implemented by Hut (1981). They also determined a way to connect the hypothesised relation between the dissipation rate and quadrupole tensor to a postulated turbulent viscosity. However, this was done by determining the velocity field from the continuity equation alone requiring an additional assumption connecting the radial and non radial components and so is incomplete. In addition, EKH only considered tidal dissipation in one primary component.

We calculate the response of the primary to tidal forcing from first principles in the low tidal forcing frequency limit. In the leading approximation the stellar configuration is fully adjusted under the action of tidal forces. Then, we include dissipative processes and Coriolis forces as next order corrections, to first order in the primary rotational frequency. In the case of the former we include both radiative and viscous effects noting that while they may dominate in the low viscosity
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Case radiative effects have not been considered previously in this context. Notably our approach removes any need for ad hoc assumptions about these processes such as connecting them with the behaviour of the quadrupole tensor and provides a complete form for the response displacement without the need for assumptions about unknown functions.

In common with previous treatments, centrifugal distortion and the toroidal component of the response displacement is neglected. The latter is potentially important in the inertial regime where the tidal forcing frequency is less than twice the rotation frequency on account of the possible excitation of inertial modes (e.g. Papaloizou & Pringle 1981; Papaloizou & Savonije 1997; Savonije & Papaloizou 1997; Ivanov & Papaloizou 2007, 2010). However, these may give only a minor contribution due to a weak overlap with the forcing potential. We use the calculated response to determine the effect of the tidal interaction on the orbit and go on to obtain equations governing the evolution of the orbit.

A qualitative difference between our results and those reported elsewhere (e.g. Eggleton et al. 1998, Barker & Ogilvie 2009) is the appearance of new terms determined by rotational effects. Unlike the standard terms due to dissipative processes in a star the new ones are readily calculated in detail. In addition to the orbital parameters, they are obtained from properties of the star, which are, in known in principal, being the stellar density distribution and angular velocity.

The new terms lead to precession of orbital plane, additional to that induced through orbital torques and stellar centrifugal distortion, as well as non-dissipative evolution of the angle between the orbital and spin angular momenta. Both effects are most prominent for binaries with sufficiently large companion masses and eccentricities. In particular, a typical change of the inclination angle over a timescale determined by apsidal precession is potentially significant provided the eccentricity is substantial, with optimal value $\sim 0.7$, the orbital period is sufficiently small, while mass ratio and primary angular velocity are sufficiently large. This could lead to observational consequences. Furthermore the effect of terms arising from the rotation should also be taken into account when studying the evolution on the tidal friction time scale of the system when the apsidal precession is non-uniform, say, due to the presence of a third perturbing body.

Before giving a complete plan of the paper we remark that those interested in the equations we derived governing the orbital evolution but who wish to avoid the lengthy derivations can, after reading the basic setup in Sections 2 and 2.2 skip to Sections 7.1 - 7.3 which contain the equations together with a summary account of the parameters involved.

The complete plan is as follows. In Section 2 we give some basic definitions and equations. In Section 2.2 we define the three coordinate systems we use to represent the dynamics of the binary
with misaligned orbital primary component spin angular momenta. One is a frame with origin at the centre of mass of the primary with vertical axis aligned with the conserved total angular momentum vector. A second, used to describe the orbit has vertical axis aligned with the orbital angular momentum vector, and the third, used to describe the primary has vertical axis aligned with its spin angular momentum vector. In Section 3 based on angular momentum conservation, we go on to derive equations governing the evolution of the angular momentum vectors that are determined by torques exerted between the orbit and primary that we go on to calculate.

The calculation begins with the specification of the perturbing tidal potential and its representation in terms of spherical harmonics in the orbit and stellar spin based coordinate systems in Sections 4 and 4.1. Transformation of the representation between the systems is facilitated with Wigner matrices. We go on to calculate the tidal response of the primary in the equilibrium tide limit in Section 5. The response which includes first order departures arising from Coriolis forces and dissipation is found in terms of the displacement associated with equilibrium tide in Sections 5.2 - 5.4

Having determined this, the induced torque acting on the star is obtained in Section 6 and the induced rate of change of orbital energy in Section 6.1. These are time averaged and reduced to closed form in Sections 6.2 - 6.5.1 with details given in appendix C.

Having obtained the time averaged torques and rate of change of orbital energy these are used in our set of equations for the determination of the orbital and spin evolution in Section 7 with expressions for the evolution of angular momentum vectors being given in Section 7.1 and expressions for the evolution of the semi-major axis and eccentricity in Section 7.2. The description is augmented by including energy dissipation in the companion under the assumption of negligible spin angular momentum in Section 7.3. This is an approximation appropriate for low mass planetary companions.

A discussion of the evolution of orbital parameters when only dissipative terms are included (thus, effects due to rotation are neglected) is given in Section 7.4, where we relate our results to those of EKH in the appropriate limit. The potential contribution of terms arising from rotational effects including the Coriolis force to the orbital evolution is then considered in Section 7.5. Finally, in Section 8 we review and discuss our results.
2 BASIC DEFINITIONS AND EQUATIONS

In this Section we describe the basic model setup and outline the objective of determining the tidal response and its using that to determine the exchange rates of angular momentum and energy between the non compact stellar component and the orbit. We go on to show how the evolution of the orbital and stellar angular momenta may be determined.

2.1 Basic model

For simplicity, we consider a binary system for which one of the components initially acts as a point mass, i.e. it has no internal degrees of freedom and said to be compact and described as the companion. Although later we shall relax this to allow for internal energy dissipation while still neglecting the internal angular momentum content. The other component, labelled the primary, possesses a distributed mass and a spin angular momentum which is unrestricted in comparison to the orbital angular momentum. Thus, both these angular momenta are allowed to evolve with the resultant total angular momentum being conserved.

We adopt a description of the system of interest by making use of quantities such as the tidal potential $U$ and the Lagrangian displacement vector $\xi$ associated with the non compact star etc. as in Ivanov & Papaloizou (2011). However, unlike that Paper we assume an elliptic orbit for the binary, and, accordingly, represent such quantities as Fourier series in time as in Ivanov & Papaloizou (2004). However, unlike these Papers, where it was assumed that only the f-mode amongst the spectrum of stellar normal modes is excited, here we consider the response of the star influenced by tides in the so-called quasi-static or equilibrium tide approximation. We note, however, that, formally, this approach can be shown to coincide with the one considered in those Papers after an appropriate redefinition of relevant quantities.

In addition to determining the response we aim to show that the evolution of the angular momentum vectors is fully determined by four simple governing equations following from the law of conservation of angular momentum provided the torques acting on the star as a result of tides have been determined. Calculation of the energy exchange with the orbit and application of the law of conservation of energy then enables a complete description of the evolution of the system once a prescription to determine the evolution of the orbital apsidal line is prescribed.
2.2 Coordinate system and notation

We introduce three reference frames. The first is a Cartesian coordinate system in a frame with origin at the centre of mass of the primary, and for which the direction of the conserved total angular momentum of the system, $J$, defines the $Z''$ axis. The corresponding $X''$ and $Y''$ axes are located in the orthogonal plane.

The second is a Cartesian frame such that the orbital angular momentum, $L$, defines the direction of the $Z'$ axis. This is inclined to the total angular momentum vector, $J$, with an inclination $i$ which need not be constant as the orbital angular momentum is not conserved.

The third $(X, Y, Z)$ coordinate system is defined as in Ivanov & Papaloizou (2011) with $z$-axis being directed along the direction of the stellar angular momentum vector, $S$. The azimuthal angle associated with both $J$ and $S$ measured in the $(X', Y', Z')$ system is $\pi/2 - \gamma$. The $Y$ axis lies in the orbital plane and defines the line of nodes as viewed in the $(X, Y)$ plane as in Ivanov & Papaloizou (2011). Note that the $X', Y'$, and $Y$ axes are coplanar as are the $Z, Z'$ and $Z''$ axes. For a Keplerian orbit with fixed orientation, the line of apsides can be chosen to coincide with the $X'$ axis. In this case the angle between this line and the $X'$ axis, which we shall more generally denote by $\varpi$, will simply be given by $\varpi = 0$. Note that the angle between the apsidal line and the $Y$ axis, being the line of nodes is quite generally given by $\varpi + \gamma - \pi/2$. The coordinate systems are illustrated in Fig. 1.

The coordinates $r' = (X', Y', Z')$ in the orbit frame are related to the coordinates $r = (X, Y, Z)$ in the stellar frame by

$$r' = R_0 r$$

where the rotation matrix is given by $R_0 \equiv R(0, \beta, \gamma)$. This corresponds to a rotation through an angle, $\gamma$, about the $Z$ axis followed by a rotation through an angle, $\beta$, about the $Y$ axis. Alternatively, it may be considered to be formed from a rotation through an angle $\beta$ about the $Y$ axis followed by a rotation through an angle $\gamma$ about the newly formed $Z'$ axis. Similarly, the coordinates $r'' = (X'', Y'', Z'')$ in the primary centred frame defined by the total angular momentum vector are related to the coordinates $r = (X, Y, Z)$ in the stellar frame through the relation

$$r'' = R_1 r$$

where in this case the rotation matrix $R_1$ is given by $R_1 = R(0, 0, \alpha)R(0, \beta - i, \gamma)$. Here we note that we apply an additional rotation through an arbitrary angle $\alpha$ about the newly defined $Z''$ axis, which is such that $\alpha + \gamma$ can be used to define an angle of precession of $S$ about $J$. We remark
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Figure 1. Illustration of the \((X, Y, Z)\) and \((X', Y', Z')\) coordinate systems together with the direction of the total angular momentum, which coincides with the \(Z''\) axis of a coordinate system that is fixed in the primary centred frame. Note that the \(X', Y',\) and \(Y\) axes are coplanar as are the \(Z, Z'\) and \(Z''\) axes. The angle between the angular momentum vectors \(L\) directed along the \(Z'\) axis and \(S\) directed along the \(Z\) axis is \(\beta\). The angle between \(L\) and the \(Z''\) axis directed along \(J\) is \(i\) and \(\delta = \beta - i\). The angle between the \(Y''\) axis and the \(Y\) axis is \(2\pi - \alpha\). The apsidal line, the location of pericentre and an orbital arc in its neighbourhood are shown.

that \(\delta = \beta - i\) is the angle of inclination between the spin angular momentum vector \(S\) and the total angular momentum vector, the latter being fixed in the primary centred frame. Thus, \(R_1\) is obtained from \(R_0\) by replacing \(\beta\) by \(\delta\) and \(\gamma\) by \(\gamma + \alpha\).

Recalling that the inverse of a rotation matrix is its transpose we find from (1) and (2) that

\[
r = R_1^T r'' \equiv R_2 r''
\]

where

\[
R_2 = R^T(0, \delta, \gamma + \alpha)
\]

corresponding to taking the inverse of the transform related to a rotation through an angle \(\gamma + \alpha\) about the \(Z\) axis followed by a rotation through an angle \(\delta\) about the \(Y\) axis. Definitions of parameters associated with the various coordinate systems and other symbols used later in the text are summarised in table 1.
From these relations we can also express the sines of $\beta$ and $\delta$, and $\alpha_v = 2\pi - \bar{\alpha}$ (see table I).

From the definition of $\beta$ given above it follows that

$$\cos \beta = \frac{(L \cdot S)}{LS},$$

(5)

where $L$ and $S$ are the magnitudes of $L$ and $S$, and

$$\cos i = \frac{(J \cdot L)}{JL},$$

(6)

where $J$ is the magnitude of $J$. We introduce the torque $T$ exerted on the star due to the tidal interaction. From the constancy of the total angular momentum $J = L + S$ it follows that in the primary centred frame

$$T = \dot{S} = -\dot{L},$$

(7)

where a dot denotes a time derivative, and, subsequently, indicates the time derivative of that quantity.

In addition, we also have

$$2J \cdot L = J^2 + L^2 - S^2$$

and

$$2L \cdot S = J^2 - L^2 - S^2.$$  

and, accordingly,

$$\cos \beta = \frac{J^2 - L^2 - S^2}{2LS} \quad \text{and} \quad \cos i = \frac{J^2 + L^2 - S^2}{2JL}.$$  

(8)

From these relations we can also express the sines of $\beta$, $i$ and $\delta$ in terms of $J$, $L$ and $S$. We obtain

$$\sin \beta = \frac{\sqrt{(J^2 - (L - S)^2)((L + S)^2 - J^2)}}{2LS}, \quad \text{and} \quad \sin i = \frac{\sqrt{(S^2 - (J - L)^2)((J + L)^2 - S^2)}}{2JL}.$$  

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together with
\[ \sin \delta = \frac{L}{S} \sin i = \frac{L}{J} \sin \beta. \]  \hfill (10)

We now make use of the rotation matrices \( R(0, \beta, \gamma) \) and
\[ R_1 = R(0, 0, \alpha) R(0, \beta - i, \gamma) \] defined in Section 2.2. These respectively enable the transformation of the components of any vector from representation in the stellar frame to the orbit frame and the stellar frame to the primary centred frame as indicated by equations (1) and (2). From these transformations and their inverses it is straightforward to express the components of \( L, S \) and the torque \( T = -\dot{L} \) in the primary centred frame in terms of \( L, S \) and the components of the torque in the frame associated with the star. We recall that by definition the components of \( L \) in the orbit frame are \((0, 0, L)\) and the components of \( S \) in the stellar frame are \((0, 0, S)\).

Thus the components of \( L \) in the primary centred frame, \((L^x'', L^y'', L^z'')\), are found to be given by
\[ L^x'' = L \cos \bar{\alpha} \sin i, \]  \[ L^y'' = -L \sin \bar{\alpha} \sin i \]  \[ L^z'' = L \cos i, \]  \hfill (11)
where \( \bar{\alpha} = \alpha + \gamma \). Similarly, the components of \( S \) in the primary centred frame, \((S^x'', S^y'', S^z'')\), are given by
\[ S^x'' = -S \cos \bar{\alpha} \sin \delta, \]  \[ S^y'' = S \sin \bar{\alpha} \sin \delta, \]  \[ S^z'' = S \cos \delta, \]  \hfill (12)
and the components of \( T \) in this frame, \((T^x'', T^y'', T^z'')\), are given by
\[ T^x'' = \sin \bar{\alpha} T^y + \cos \bar{\alpha} T^z, \]  \[ T^y'' = \cos \bar{\alpha} T^y - \sin \bar{\alpha} T^1, \]  \[ T^z'' = \sin \delta T^x + \cos \delta T^z, \]  \hfill (13)
where \( T^1 = \cos \delta T^x - \sin \delta T^z \)  \hfill (14)
with the components of \( T \) in the stellar frame being given by \((T^x, T^y, T^z)\).

From equation (7) together with equations (10) - (14) it is easy to obtain the following set of equations
\[ \frac{di}{dt} = \frac{1}{L} (-\cos \beta T^x + \sin \beta T^z), \]  \[ \frac{dL}{dt} = -\cos \beta T^z - \sin \beta T^x, \]  \[ \frac{d\bar{\alpha}}{dt} = \frac{T^y}{\sin i L} = \frac{J}{LS \sin \beta}, \]  \hfill (15)
and
\[ \frac{d\delta}{dt} = -\frac{T^x}{S}, \]  \[ \dot{S} = T^z. \]  \hfill (16)
Note that we use (10) to obtain the last equality in (15). Also note that it is easy to check that \( J^i = L^i + S^i \) are indeed first integrals of the set of equations (15) and (16). Using the first two equations of (15) together with (16) and the help of equations (8) and (10) it is straightforward to
obtain the evolution equation for angle $\beta$:

$$\frac{d\beta}{dt} = - \left( \frac{\cos \beta}{L} + \frac{1}{S} \right) T^x + \frac{\sin \beta}{L} T^z. \tag{17}$$

Finally, we remark that the angle $\bar{\alpha}$ was defined through making a right handed rotation when transforming from the $(X, Y, Z)$ system to the $(X'', Y'', Z'')$ system. Accordingly, we define $\alpha_r = 2\pi - \bar{\alpha}$, so that increasing $\alpha_r$ is associated with a right handed rotation from $(X'', Y'', Z'')$ to $(X, Y, Z)$. It may thus be used to describe precession of the stellar rotation axis in the conventional manner. Clearly $d\alpha_r/dt = -d\bar{\alpha}/dt$.

Together with the energy conservation law equations (15) and (16) form a complete set for our model. Note that an evolution equation for the angle $\gamma$ is absent. This is because, physically, only the angle $\bar{\alpha} = 2\pi - \alpha_r$ appears in the specification of the orientation of the angular momentum vectors in the primary centred frame. In addition, only the angle between the apsidal line and the projection of the stellar spin angular momentum vector onto the orbital plane, $\varpi + \gamma$ matters for the determination of the orbital evolution. The evolution of this angle should be found from other considerations which are not based on the law of angular momentum conservation. This evolution is determined, in general, from tidal interactions, stellar flattening, General Relativity and/or a presence of other perturbing bodies. We assume hereafter that this evolution is known. Also note that the components of the torque in the stellar frame, which we are going to calculate below do not depend on the angle $\alpha_r$ used to specify orbital precession with respect to the primary centred frame because it is ignorable in this context. Thus, the evolution equation for this angle can be considered separately from the others. This equation contains only a term proportional to $T^\varpi$ that is determined by tides. In addition to this term a standard contribution due to stellar flattening must be added, as discussed below. The remainder of the equations governing the evolution are determined by only two components of the torque in the stellar frame, $T^x$ and $T^z$.

4 THE PERTURBING TIDAL POTENTIAL

In this section we develop the standard quadrupole form of the tidal potential appropriate for a near Keplerian orbit with arbitrary orientation as a series of spherical harmonics in the orbit frame each term of which is expressed as a Fourier series. We transform the spherical harmonics expressed in the orbit frame to a representation in terms of spherical harmonics defined in the stellar frame using Wigner matrices. This is done to facilitate the calculation of the tidal response in the primary. A list of some of the parameters, variables and symbols associated with the developments in this Section is given in table 2.
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The perturbing potential, $U$, can be readily found in the orbit frame $(X', Y', Z')$, where a spherical coordinate system $(r, \theta', \phi')$ is defined in the usual way taking into account quadrupole terms only as

$$U = -\frac{GM_p r^2}{R^3} P_2(\cos \psi),$$

(18)

where $R$ is distance between the binary components, $P_2$ is the usual Legendre polynomial and

$$\cos \psi = \sin \theta' \cos(\Phi - \phi').$$

Here the orbit is taken to be in the $\theta' = \pi/2$ plane with its line of apsides located at $\phi' = \varpi$, with $\Phi$ being the azimuthal angle of the line joining the binary components. For convenience we shall measure both $\Phi$ and $\phi'$ from the $X'$ axis without loss of generality. Equation (18) may also written as

$$U = -\frac{GM_p r^2}{R^3} \left(\frac{4\pi}{5}\right) \sum_{m=0,|m|=2} Y_{2,m}(\theta', \phi') Y_{2,m}(\pi/2, 0) \exp(-im\Phi)$$

(19)

where, hereafter, the prime implies that only summations over $m = 0, \pm 2$ is performed, $Y_{2,m}(\theta', \phi')$, is the usual spherical harmonic.

4.1 Fourier development $U$ in terms of spherical harmonics defined in the stellar frame

For an eccentric Keplerian orbit $R$ is a periodic function of time with period $2\pi/n_o$ and $\Phi = n_o t + v(t) + \varpi$, where $v$ is a periodic function of time with period $2\pi/n_o$ and zero time average with $\varpi$ being the longitude of the apsidal line measured from the $X'$ axis. When as for a strictly Keplerian orbit this remains fixed, as remarked above we may choose the line to coincide with the $X'$ axis in which case $\varpi = 0$.

For slightly non Keplerian precessing orbits, the longitude of the apsidal line, $\varpi$, precesses at a rate given by $d\varpi/dt$, the latter being determined by the deviation of the time averaged background potential from proportionality to $1/r$. Accordingly we have $\varpi = \int_0^t (d\varpi/dt') dt' + \varpi_0$, where $\varpi_0$ is a constant reference angle, being the value of $\varpi$ at $t = 0$. When the rate of precession is independent of the precession phase, we have uniform precession with $\varpi$ increasing linearly with time. Then we have $\varpi = (d\varpi/dt) t + \varpi_0$. In addition, we have the Fourier expansion

$$\frac{\exp(-im(\Phi - \varpi)) a^3}{R^3} = \sum_{k=-\infty}^{k=\infty} \phi_{k,m} \exp(ikn_o t),$$

(20)

where $a$ is the orbital semi-major axis, with the Fourier coefficients being given by

$$\phi_{k,m} = \frac{n_o}{2\pi} \int \frac{\exp(-im(\Phi - \varpi)) a^3}{R^3} \exp(-ikn_o t) dt,$$

(21)

where the integral is taken over an orbital period $2\pi/n_o$. The coefficients, $\phi_{k,m}$, are related to the well known Hansen coefficients, $X_{k,m}$, (see e.g. Branham 1990, Laskar 2005),
through \( \phi_{k,m} = X_{k,-m}^{-3,-m} \), see Appendix A for more details. Practical prescriptions for calculating Hansen coefficients have been provided by many authors (e.g. Branham 1990, Laskar 2005). For small eccentricities a power law expansion in \( e \) developed from (A2) may be used. In this paper we have followed the notation of Ivanov & Papaloizou (2004), who give a useful prescription for calculating \( \phi_{k,m} \) for eccentricities \( e > 0.2 \).

In the stellar frame \((X,Y,Z)\) the associated spherical coordinate system is \((r,\theta,\phi)\). From the discussion given in Appendix A we can write the potential in terms of these in the form

\[
U = -GM_p \left( \frac{4\pi r^2}{5a^3} \right) \sum_{n=-2}^{n=2} Y_{2,n}(\theta,\phi) F_n(t) \equiv r^2 \sum_{n=-2}^{n=2} A_n Y_{2,n}(\theta,\phi),
\]

with \( A_n = -4\pi GM_p F_n/(5a^3) \). Here

\[
F_n(t) = \sum_{k=\infty}^{k=-\infty} \sum_{m=0,2} \phi_{k,m} D_{n,m}^{(2)} Y_{2,m}(\pi/2,0) \exp(i(kn_o t - m\omega))),
\]

where the coefficients (Wigner matrix elements), \( D_{n,m}^{(2)} \), are specified together with some of their relevant properties in Appendices A and E. We remark that

\[
D_{n,m}^{(2)} = \exp(-im\gamma) d_{n,m}^{(2)}(\beta)
\]

where \( d_{n,m}^{(2)} \) is an element of Wigner’s (small) d-matrix and is real, see e.g. Ivanov & Papaloizou 2011.

Note that \( F_n(t) \) is in general not simply harmonically varying in time. Making use of the Fourier expansion expressed by equation (20) it can be written in the alternative form

\[
F_n(t) = \frac{a^3}{R^3} \sum_{m=0,2} D_{n,m}^{(2)} Y_{2,m}(\pi/2,0) \exp(-im\Phi).
\]

We remark in addition that \( \phi_{-k,-m} = \phi_{k,m}^* \) and also that \( Y_{2,-m}(\theta,\phi) = (-1)^m Y_{2,m}^*(\theta,\phi) \) with \( D_{n,-m}^{(2)} = (-1)^{(n+m)} (D_{n,m}^{(2)})^* \), these relations together ensuring that the sum in (22) is real.

5 CALCULATION OF THE RESPONSE DISPLACEMENT

In this section we calculate the tidal response of the primary to the forcing tidal potential by solving for the Lagrangian displacement in the linear approximation. The aim is to find the density perturbation that will subsequently be used to find tidal torques and the rate of energy transfer from the orbit. A low frequency approximation is adopted for which the response consists of an equilibrium tide with corrections arising from coriolis forces, inertia and dissipative effects treated as small perturbations.

We shall assume that the effective stellar response is determined by a displacement of spheroidal
form appropriate to a spherically symmetric background state. We make the usual assumption that the rotational period of the star is much longer than the characteristic dynamical time scale which allows us in the first instance to regard stellar rotation as being small and adopt an expression for the Lagrangian displacement which is of spheroidal form which retains the angular dependence for each azimuthal wavenumber, \( n \). Thus, we write

\[
\xi = \sum_{n=-2}^{n=2} \xi_n, \tag{26}
\]

where

\[
\xi_n = \xi^e(r, \theta, t, n) \exp(in\phi)e_r + r\nabla_\perp(\xi^S(r, \theta, t, n) \exp(in\phi)), \tag{27}
\]

where \( e_r \) is unit vector in the radial direction. Note that, in general, \( \xi^e \) and \( \xi^S \) can depend on \( n \). If the response was that of a non rotating strictly spherically symmetric star the angular dependence is through a factor \( Y_{2,n} \). This extends to the case here to lowest order in the stellar angular velocity \( \Omega_r \). As higher order effects are treated by perturbation theory, modification of this dependence does not have to be considered. This procedure neglects any contribution from displacements of toroidal form and the excitation of related normal modes (Papaloizou & Pringle 1978).
Finally we remark that as $\xi$ is real we have $\xi^*_{-n} = \xi_n$.

To evaluate the response to tidal forcing we consider an equation of the generic form

$$\omega_f^2 \eta = \hat{L}(\eta) + \mathcal{L}(\eta) + \nabla \mathcal{U}$$  \hspace{1cm} (28)$$

where $\omega_f$ is the forcing frequency, the forcing potential

$$\mathcal{U} = r^2 A_{n,k} Y_{2,n}(\theta, \phi),$$  \hspace{1cm} (29)$$

where

$$A_{n,k} = -\sum_{m=0,\ldots,|k|} \frac{4\pi G M}{5a^3} \phi_{k,m} D_{n,m}^{(2)} Y_{2,m}(\pi/2, 0) \exp(-im\varpi).$$  \hspace{1cm} (30)$$

Here we consider a single term contributing to the sum in equation (22) (see Section 4.1 and equations (22) - (25)) and we consider the apsidal angle $\varpi$ to be sufficiently slowly varying that its variation may be neglected. We note that a term associated with a particular forcing frequency, $kn_o$, and azimuthal mode number, $n$, of the form, $\exp(ikn_o t) \eta$, where $\eta$ depends only on position will be contributed to the full Lagrangian displacement $\xi$ (see equation (26)). Such terms as well as those corresponding to different $n$ may be combined through linear superposition.

The linear operator, $\hat{L}$, is that applicable to an undisturbed spherical star for which a normal mode of oscillation, denoted by subscript, $k$, satisfies

$$\omega_k^2 \eta_k = \hat{L}(\eta_k).$$ \hspace{1cm} (31)$$

For a potential perturbation of the form (29) the frequency

$$\omega_f = kn_o + n\Omega_r$$ \hspace{1cm} (32)$$

is the Doppler shifted frequency as seen in the frame corotating with the star. The operator $\mathcal{L}$ contains the effects of stellar rotation and damping to lowest order and is considered to contribute a small perturbation (for more details see below).

### 5.1 Equilibrium response

For a spherical star

$$\hat{L}(\eta) \equiv \frac{1}{\rho} \nabla P' - \frac{\rho'}{\rho^2} \nabla P + \nabla \psi',$$ \hspace{1cm} (33)$$

where $P'$, $\rho'$ and $\psi'$ are respectively the pressure, density and the potential perturbations arising from self gravity. We have $\rho' = -\nabla \cdot (\rho \eta)$ and we set $P' = P'_{a} + P'_{na}$, where $P'_{a} = -\Gamma_1 P \nabla \cdot \eta - \eta \cdot \nabla P$ is the adiabatic component of the pressure perturbation and $P'_{na}$ is the non adiabatic part.
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The latter may be obtained from the energy equation, which may be written as

\[ P' + \Gamma_1 P \nabla \cdot \eta + \eta \cdot \nabla P = \frac{(\Gamma_3 - 1) \nabla F'}{i \omega_f}. \]  

Here \( \Gamma_1 \) and \( \Gamma_3 \) are the standard adiabatic exponents and \( F' \) is the perturbed energy flux, \( F \), which may contain contributions from both radiative and convective transport, and perturbation to the energy generation rate is neglected. Being second order in perturbations there is no contribution from viscous dissipation. Note that (34), after use of the equation of state, can be regarded as an equation for \( P' \) in terms of a specified \( \eta \) with there also being a dependence on forcing frequency. Hence there will be a dependence of \( P'_{na} \) on forcing frequency.

For \( L \) we set

\[ L(\eta) \equiv 2i \omega_f \Omega_r \hat{k} \times \eta + \frac{1}{\rho} \nabla P'_{na} + i \omega_f D_n(\eta). \]  

Here the first term on the right hand side arises from the Coriolis force, and the final term gives the effect of viscosity. Thus, the first term gives a non dissipative contribution and the others dissipative ones.

We develop a solution of (28) by considering firstly a solution for which both the term \( \propto \omega_f^2 \) and \( L \) are neglected and which applies in the low forcing frequency limit and so gives rise to an equilibrium tide, and we need not neglect self-gravity (Cowling approximation).

Setting \( \eta = \xi_{eq,n,k} \) in this case we see that it satisfies

\[ \hat{L}(\xi_{eq,n,k}) = -\nabla \hat{U}, \]  

where we have attached the subscripts \( n \) and \( k \) to denote that a response corresponds to azimuthal mode number \( n \) and forcing frequency \( kn_o \).

We remark that the form of the equilibrium tide mentioned above has been discussed in detail when the Cowling approximation applies in Terquem et al. (1998) and Bunting et al. (2019). In particular, the reader is referred to the discussion in Appendix D of the latter paper. In order to relax the Cowling approximation the potential \( \hat{U} \) should be replaced by \( \hat{U} + \hat{U}_{self} \), where \( \hat{U}_{self} \) is the contribution from self gravity. This satisfies the Poisson equation

\[ \nabla^2 \hat{U}_{self} = \left( \frac{4 \pi G}{g} \right) \frac{d\rho}{dr}(\hat{U} + \hat{U}_{self}), \]  

where \( g \) is the local acceleration due to gravity and \( G \) is the gravitational constant. We also note that in radiatively stratified regions the radial component of the equilibrium tide displacement is given by

\[ \xi_{eq,n,k} \cdot \hat{r} = -\frac{(\hat{U} + \hat{U}_{self})}{g}. \]
However, this changes in regions that are convectively neutral and so barotropic (see Bunting et al. 2019).

### 5.2 Finding the response for a given forcing frequency and value of \( n \).

We now return to our original equation (28) which we rewrite in the form

\[
\hat{L}(\eta) + \hat{L}_1(\eta) = -\nabla U,
\]

where \( \hat{L}_1(\eta) = \mathcal{L}(\eta) - \omega^2 \eta \).

We write \( \eta = \xi_{eq,n,k} + \xi_{eq,1,n,k} \), so that (39) implies that

\[
\hat{L}(\xi_{eq1,n,k}) = -\hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}),
\]

Noting that the tidal effects we are interested in are first order in \( \hat{L}_1 \) we remark that the term involving \( \xi_{eq1,n,k} \) on the right hand side of (40) can be regarded as second order. However, we shall retain it for now. In that case \( \xi_{eq1,n,k} \) is the equilibrium tide corresponding to an external force per unit mass, \( f = -\hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) \). It is important to note that in requiring \( \xi_{eq1,n,k} \) to be of spheroidal form we are discarding the toroidal component of \( f \) and hence neglecting toroidal and inertial modes governed by rotation. Thus, we make the implicit assumption that these do not play a significant role enabling us to retain the spheroidal component alone.

### 5.3 Calculation of the overlap integral

Our aim is to find the volume integral of the density perturbation response with the forcing potential as this is directly related to forces exerted on the star as a result of the tidal perturbation. We begin by using the self-adjoint property of \( \hat{L} \) to write

\[
\int \rho \xi_{eq1,n,k}^* \hat{L}(\xi_{eq,n,k}) dV = -\int \rho \xi_{eq1,n,k}^* \cdot \nabla U dV = -\left( \int \rho \xi_{eq,n,k}^* \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right)^*.
\]

From this it follows that

\[
\int \rho_{eq1,n,k}^* U dV = \left( \int \rho \xi_{eq,n,k}^* \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right)^*,
\]

where \( \rho_{eq1,n,k}^* \) is the density perturbation associated with \( \xi_{eq1,n,k} \). Thus,

\[
\mathcal{A}_{n,k} \int \rho_{eq1,n,k}^* r^2 Y_{2,n}(\theta, \phi) dV = \left( \int \rho \xi_{eq,n,k}^* \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right)^*.
\]

Similarly, if \( \rho_{eq,n,k}^* \) is the density perturbation associated with \( \xi_{eq,n,k} \) we have noting the right hand side of the following equation is real that

\[
\mathcal{A}_{n,k} \int \rho_{eq,n,k}^* r^2 Y_{2,n}(\theta, \phi) dV = -\int \rho \xi_{eq,n,k}^* \cdot \hat{L}(\xi_{eq,n,k}) dV.
\]
Setting \( \rho^*_n = \rho^*_q + \rho^*_e n, k \), it follows from the above results that

\[
A_{n,k} \int \rho^*_n r^2 Y_{2,n}(\theta, \phi) dV = \]

\[
- \int \rho \xi_{eq,n,k} \cdot \hat{L}(\xi_{eq,n,k}) dV \left( 1 - \frac{(\int \rho \xi_{eq,n,k} \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV)^*}{N_0 \omega_{eq}^2} \right). \tag{45}
\]

Here, the quantity multiplying \( A_{n,k} \) is what we define to be the overlap integral and

\[
N_0 = \int \rho |\xi_{eq,n,k}|^2 dV \quad \text{and} \quad \omega_{eq}^2 = \frac{\int \rho \xi_{eq,n,k} \cdot \hat{L}(\xi_{eq,n,k}) dV}{N_0}.
\tag{46}
\]

The latter squared frequency can be regarded as being obtained from the oscillation problem on having used \( \xi_{eq,n,k} \), which may be arbitrarily scaled, as a trial function. In fact, as the operator \( \hat{L} \) is for a spherical star and it has no explicit frequency dependence, when each of the quantities \( \xi_{eq,n,k} \) are scaled appropriately, \( N_0 \) and \( \omega_{eq} \) are independent of \( n \) and \( k \). However, this is not the case for \( \hat{L}_1 \) and \( \xi_{eq1,n,k} \).

Making use of equation (44) and with the help of equations (46) and (47) equation (45) may be rewritten as

\[
\int \rho^*_n r^2 Y_{2,n}(\theta, \phi) dV = \]

\[
- A_{n,k} \left| \int \rho^*_n r^2 Y_{2,n}(\theta, \phi) dV \right|^2 \left( 1 - \frac{(\int \rho \xi_{eq,n,k} \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV)^*}{N_0 \omega_{eq}^2} \right). \tag{48}
\]

We remark that the expression on the right hand is invariant to scaling the various displacements by an arbitrary complex constant. This could be chosen to provide a normalisation such that \( N_0 = 1 \). To proceed further we note that equation (28) together with the self-adjoint property of \( \hat{L} \) imply that the quantity

\[
\frac{1}{2} \omega_f I \left( \int \rho \eta \cdot \hat{L}_1(\eta) dV \right) = \frac{1}{2} \omega_f I \left( \int \rho \eta \cdot L(\eta) dV \right) = \Gamma \omega_f^2 \int \rho |\eta|^2 dV, \tag{49}
\]

where \( I \) denotes that the imaginary part is to be taken, represents the rate of energy dissipation associated with the displacement \( \eta \). Equation (49) also defines the quantity \( \Gamma \), which is the ratio of this dissipation rate to four times the kinetic energy associated with the disturbance. Introduced in this way it would correspond to the decay rate \( \Gamma \) were \( \eta \) a normal mode of the system with eigenfrequency \( \omega_f \). In addition, recalling that for \( \eta = \xi_{eq,n,k} + \xi_{eq1,n,k} \) and making use of the fact

\[1\] Clearly, \( \Gamma \) is the decay rate, as assumed here, only for a stable mode, it is the growth rate for an unstable mode.
that $\hat{L}$ is self-adjoint, equation (40) implies that

$$\mathcal{I} \left( \int \rho \eta^* \cdot \hat{L}_1(\eta) dV \right) = \mathcal{I} \left( \int \rho \xi_{eq,n,k}^* \cdot (\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right).$$  \hfill (50)

Similarly, we write

$$\frac{1}{2} \omega_f \mathcal{R} \left( \int \rho \xi_{eq,n,k}^* \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right) = \mathcal{X} \int \rho |\eta|^2 dV$$  \hfill (51)

with $\mathcal{R}$ denoting that the real part is to be taken and which may be regarded as defining $\mathcal{X}$. To evaluate this we make use of (35) and (39) which define $\hat{L}_1$ and, noting that in evaluating the integrals in (51) we may consider only lowest order quantities, we accordingly neglect $\xi_{eq1,n,k}$. In addition, we neglect contributions from the nonadiabatic dissipative terms, which we assume to be much less than those retained. Thus, we find that

$$\frac{1}{2} \omega_f \mathcal{R} \left( \int \rho \xi_{eq,n,k}^* \cdot \hat{L}_1(\xi_{eq,n,k} + \xi_{eq1,n,k}) dV \right) = \omega_f^2 \left( \frac{\int \rho \Omega_r \xi_{eq,n,k}^* \cdot (\hat{k} \times \xi_{eq,n,k}) dV}{\int \rho |\xi_{eq,n,k}|^2 dV} - \frac{\omega_f}{2} \right) \equiv \mathcal{X} = -n \omega_f^2 (\beta_* - 1) \Omega_r - \omega_f^3 / 2$$  \hfill (52)

which can be regarded as defining the quantity $\beta_*$. Making use of (49) - (52) we may write (48) in the form

$$\int \rho_{n,k}^* r^2 Y_{2,n}(\theta, \phi) dV = \nonumber$$

$$\nonumber - \mathcal{A}_{n,k}^* \left[ \frac{\int \rho_{eq,n,k}^* r^2 Y_{2,n}(\theta, \phi) dV}{N_0 \omega_{eq}^2} \right] \left( 1 - \frac{2\omega_f (-n \beta_* \Omega_r - i\Gamma) - k^2 n_o^2 - n^2 \Omega_r^2}{n_0 \omega_{eq}^2} \right)$$  \hfill (53)

which gives an expression for the complex conjugate of the overlap integral. Here we have again made the approximation that

$$\int \rho |\eta|^2 dV = N_0$$  \hfill (54)

which is expected to be valid when the norm of $\xi_{eq1,n,k}$ is much less than that of $\xi_{eq,n,k}$ as assumed. However, in passing we remark that the retention of $\xi_{eq1,n,k}$ when considering dissipative terms allows its gradient to become significant when dissipative processes are concerned. In addition, we remark that

$$-n (\beta_* - 1) \Omega_r = \frac{\int \rho \Omega_r \xi_{eq,n,k}^* \cdot (\hat{k} \times \xi_{eq,n,k}) dV}{\int \rho |\xi_{eq,n,k}|^2 dV}$$  \hfill (55)

gives the frequency shift produced by the Coriolis force as seen in the rotating frame for a putative normal mode with associated eigenfunction $\xi_{eq,n,k}$ (Christensen-Dalsgaard 1998) and importantly $\beta_*$ does not depend on $n$ (see next Section 5.4.2).
5.4 Relating the response to the tidal forcing to the overlap integral associated with the equilibrium tide

Taking the complex conjugate of (53) we obtain

\[ \int \rho_{n,k} r^2 Y_{2,n}^*(\theta, \phi) dV = -A_{n,k} \times \]
\[ \frac{\left( \int \rho_{eq,n,k} r^2 Y_{2,n}^*(\theta, \phi) dV \right) \left( \int \rho_{eq,n,k}^* r^2 Y_{2,n}(\theta, \phi) dV \right)}{N_0 \omega_{eq}^2} \left( 1 - \frac{2 \omega_f (-n \beta_\ast \Omega_r + i \Gamma) - k^2 n_o^2 - n^2 \Omega_r^2}{\omega_{eq}^2} \right) \].

(56)

Recalling that we can arbitrarily scale \( \xi_{eq,n,k} \) when evaluating the right hand side of (56) we now do this so as to ensure \( N_0 = 1 \) and that after specifying the angular dependence of the spheroidal associated decomposition functions through (see equation (27))

\[ \xi^r(r, \theta, t, n) \exp(in \phi) = \xi^r(r) Y_{2,n} \exp(in_o t), \]
and

\[ \xi^S(r, \theta, \phi, n) \exp(in \phi) = \xi^S(r) Y_{2,n} \exp(in_o t) \]
(57)

the associated functions \( \xi^r(r) \) and \( \xi^S(r) \) are real. As these apply to a spherically symmetric background they are in addition independent of \( k \) and \( n \). Then we can write (56) as

\[ \int \rho_{n,k} r^2 Y_{2,n}(\theta, \phi) dV = \]
\[ -A_{n,k} \left( \int \rho_{eq,n,k} r^2 Y_{2,n}(\theta, \phi) dV \right) Q_{eq} \left( 1 - \frac{2 \omega_f (-n \beta_\ast \Omega_r + i \Gamma) - k^2 n_o^2 - n^2 \Omega_r^2}{\omega_{eq}^2} \right) \],

(58)

where

\[ Q_{eq} = \int \rho_{eq,n,k} r^2 Y_{2,n}(\theta, \phi) dV = -\left( \nabla \cdot (\rho \xi_{eq,n,k}^*) \right) r^2 Y_{2,n}(\theta, \phi) dV = 2 \int dr \rho r^3 (\xi^r + 3 \xi^S) \]
(59)

is the overlap integral evaluated using the normalised displacement \( (N_0 = 1) \) which as the subscript indicates corresponds to \( \xi_{eq,n,k} \). In evaluating this we have made use of the decomposition, given by (27) with (57), applied to \( \xi_{eq,n,k} \), and note that there is in fact no dependence of this on \( k \), that the \( n \) dependence is only through the spherical harmonics, and that the factor \( \exp(i n_o t) \) may be dropped from perturbations as a result of cancelation. Note that if we restore the normalisation factor, \( N_0 \), evaluated with the equilibrium tide, we see that \( Q_{eq} \equiv Q_{eq} / \sqrt{N_0} \) has dimensions \( \sqrt{M_\ast R_\ast} \), with \( M_\ast \) and \( R_\ast \), respectively, being the mass and radius of the primary.

By inspection of the way the density perturbations appear in (58) and relating them back to associated displacements we may make the identification

\[ \xi_{n,k} = D_{n,k} \xi_{eq,n,k} \] where \( D_{n,k} = -A_{n,k} \frac{Q_{eq}}{\omega_{eq}^2} \left( 1 - \frac{2 \omega_f (-n \beta_\ast \Omega_r + i \Gamma) - k^2 n_o^2 - n^2 \Omega_r^2}{\omega_{eq}^2} \right) \).

(60)
Making the time dependence explicit and summing over \( k \) we may write

\[
\xi_n = \xi_{eq,n} \sum_{k=-\infty}^{\infty} D_{n,k} \exp(ik\eta_0 t) \tag{61}
\]

where we have removed the subscript \( k \) from \( \xi_{eq,n,k} \) as there is no such dependence for this quantity. We write equation (60) in a more compact form

\[
D_{n,k} = -Q_{eq} R_{n,k} A_{n,k} \tag{62}
\]

Recalling that \( \omega_f = k\eta_0 + n\Omega_r \), we have \( \tilde{\Gamma} = \Gamma/\omega_{eq}, \tilde{\Omega} = n\sigma/\omega_{eq} \) and \( \sigma = \Omega_r/(\lambda\eta_0) \) with \( \lambda = 2\beta_*/(2\beta_* + 1) \).

5.4.1 Evaluating the response

Equation (62) relates the Fourier expansion coefficient \( D_{n,k} \) of the displacement to the corresponding Fourier expansion coefficient in the forcing potential \( A_{n,k} \) through the factor \(-Q_{eq} R_{n,k}\). This consists of a sum of terms \( \propto \delta_{n,k} \) and \( \propto k^2 \tilde{\Omega}^2 \). The latter does not depend on \( n \), and it can be verified after summing over \( k \) that it produces a density response with the same angular dependence as \( \partial^2 U/\partial t^2 \), as it can be written as a product of this and a function of \( r \). Given that it is periodic in time this response can readily be shown to produce no time-averaged torques or energy dissipation after time averaging over an orbit. Accordingly, as it has no secular consequences we shall neglect this term from now on.

5.4.2 Reduction of the integral determining \( \beta_* \) and discussion of the related response

Here we discuss the contribution to the response from terms \( \propto \beta_* \) that are also independent of \( k \). Evaluating the integral equation (55) by making use of the decomposition, given by (27) with (57), applied to \( \xi_{eq,n,k} \) as above, we obtain

\[
-n(\beta_* - 1)\Omega_r = \frac{\int \rho r \xi^r_{eq,n,k} \cdot (\hat{k} \times \xi_{eq,n,k}) dV}{\int \rho |\xi_{eq,n,k}|^2 dV} = n\Omega_r \int r^2 \rho(2\xi^r \xi^S + (\xi^S)^2) dr, \tag{63}
\]

where we have scaled the displacement to be normalised such that

\[
N_0 = \int r^2 \rho((\xi^r)^2 + 6(\xi^S)^2) dr = 1. \tag{64}
\]

Given that \( \xi^r \) and \( \xi^S \) do not depend on \( n \), we confirm from (63) that neither does \( \beta_* \) (see eg. Christensen-Dalsgaard 1998).

We now discuss the physical form of the response induced by the contribution of terms \( \propto \beta_* \).
that depend on \( n \) but are independent of \( k \) to the Fourier expansion coefficients \( D_{n,k} \) of the response displacement corresponding to the Fourier expansion coefficient in the forcing potential \( A_{n,k} \). These terms arise from the density response resulting from the Coriolis force produced from the equilibrium tide. In particular from (62) we see that for a given Fourier component such a term contributes 

\[-2n^2\beta_\lambda \sigma^2 \tilde{\Omega}^2 = n^2 \sigma \delta_3,\]

so defining \( \delta_3 \), to \( \delta_{n,k} \). As there is no dependence on \( k \), on account of the factor \( n^2 \), one can see that the angular response to each Fourier component of the tidal potential will be misaligned with the angular dependence of that component in a way that depends on \( \beta \). In fact, when \( U \) is restricted to a particular Fourier component in time, the angular dependence of the density response associated with these terms is \( \propto \partial^2 U / \partial \phi^2 \). Thus because \( U \) is not restricted to a single spherical harmonic, but rather a linear combination of them in a misaligned system, non zero torques that are directed in the plane perpendicular to the rotation axis may occur after time averaging. Such torques tend to cause a precession of the angular momentum vectors and because the magnitude of the orbital angular momentum is affected but not the orbital energy (as can be inferred from the above discussion of the dependence on \( \phi \)) so is the orbital eccentricity. Important in this regard is that the direction of such torques depends on the orientation of the orbit in its plane and hence, \( \varpi \). This is unlike the situation for the standard precessional torque resulting from rotational distortion considered in appendix D. This is because in that case the time averaged tidal potential is axisymmetric in the orbital plane which is a special feature of its quadrupole form that would not occur in more general cases.

We remark that it is implied in the analysis presented in this Paper that both \( \tilde{\Gamma} \ll 1 \) and \( \tilde{\Omega} \ll 1 \). On the other hand, \( \sigma \) can be order of unity. It is convenient to represent \( \delta_{n,k} \) as

\[
\delta_{n,k} = i(\delta_1 k + \delta_2 n) + (kn + n^2 \sigma)\delta_3, \quad \text{where} \quad \delta_1 = 2\tilde{\Gamma}\tilde{\Omega}, \quad \delta_2 = \lambda \sigma \delta_1, \quad \text{and} \quad \delta_3 = -2\beta_\lambda \lambda \sigma \tilde{\Omega}^2.
\]

(65)

We remark that the \( \delta_i \) have been defined such that they are independent of \( k \) and \( n \).

6 FINDING THE INDUCED TORQUE ACTING ON THE STAR

In this Section we use the tidal response calculated in the previous Sections to evaluate the components of the torque acting on the primary by performing the appropriate volume integrals. On account of the decomposition of the forcing tidal potential, this naturally leads to results expressed as double sums over terms arising from each Fourier component of the contribution for each azimuthal mode number, which are then time averaged. For the particular form of the response we calculate, this summation can be performed by use of Parseval’s theorem leading to expressions in
closed form. Some of the parameters and symbols that occur in the specification and calculation of these quantities are listed in Table 3.

Table 3. Table of some parameters, variables and symbols occurring in the calculation of the components of the torque and rate of change of orbital energy.

| Symbol       | Definition                                                                 |
|--------------|-----------------------------------------------------------------------------|
| \( \Phi \)  | This is the operator \( \mathbf{r} \times \nabla \). Thus \( \Phi \equiv \mathbf{r} \times \nabla \) |
| \( \Phi^+, \Phi^- \) | These give the components of the operator \( \Phi \) in the \((X, Y, Z)\) Cartesian coordinate system through \( \Phi = ((\Phi^+ - \Phi^-)/\sqrt{2}, -i(\Phi^+ + \Phi^-)/\sqrt{2}, \Phi^0) \) (see equation (B5)) |
| \( W^{(j)}_{n_1, n_2, m}(\beta) \) | Quantity appearing in the expressions for tidal torque components. It is constructed from Wigner small \( d \) matrix elements and is given by equation (C8) |

Substituting the displacement given by (26) the torque acting on the star, \( T \), is readily calculated. Working in the star’s frame we have

\[
T = - \int_V \rho' \mathbf{r} \times \nabla U dV
\]

where \( \rho' = -\nabla \cdot (\rho \mathbf{\xi}) \) is the density response and the integral is taken over the volume of the star. We denote the operator \( \mathbf{r} \times \nabla \equiv \hat{\Phi} \). The components in the \((X, Y, Z)\) coordinate system are

\[
\hat{\Phi} = \left( -\cot \theta \cos \phi \frac{\partial}{\partial \phi} - \sin \phi \frac{\partial}{\partial \theta} , -\cot \theta \sin \phi \frac{\partial}{\partial \phi} + \cos \phi \frac{\partial}{\partial \theta} , \frac{\partial}{\partial \phi} \right)
\]

and the torque is given by

\[
T = \int_V \nabla \cdot (\rho \mathbf{\xi}^*) \hat{\Phi} U dV,
\]

where we remark that because \( \mathbf{\xi} \) is real we may take its complex conjugate in (68). Using equation (26) for \( \mathbf{\xi} \), after some algebra (see Appendix B) we obtain the components of the torque expressed in terms of the equilibrium tide components for a given \( n \) specified by equation (60) in the form

\[
T^z = - \sum_{n=0}^{n=2} 2R \left[ \left( \frac{4\pi GM_p}{5a^3} \right) \int_V \nabla \cdot (\rho \mathbf{\xi}^*_n) \left( \text{inr}^2 Y_{2,n}(\theta, \phi) F_n(t) \right) dV \right],
\]

\[
\frac{5a^3 T^x}{2\pi GM_p} = \sum_{n=0}^{n=2} (\delta^m_0 - 2) R \left[ \int_V \nabla \cdot (\rho \mathbf{\xi}^*_{n+1}) \left( \text{ir}^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right. \\
+ \left. \int_V \nabla \cdot (\rho \mathbf{\xi}^*_{n-1}) \left( \text{ir}^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right]
\]
and
\[
\frac{5a^3 T_w}{2 \pi G M_p} = \sum_{n=0}^{n=2} \left(2 - \delta_0^n\right) \mathcal{R} \left[ \int_V \nabla \cdot \left( \rho \xi_{n-1}^* \right) \left( r^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right.
\]
\[- \left. \int_V \nabla \cdot \left( \rho \xi_{n+1}^* \right) \left( r^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right],
\]
(71)
where \( \delta_0^n \) is the Kronecker \( \delta \).

We comment that when calculating the \( X \) and \( Y \) components of the torque, the above equations imply that the azimuthal mode number of a significant response has to differ from that of the original forcing potential by \( \pm 1 \). Apart from this, the expressions consist of contributions that are similar in form to that given by equation (69) for the component of the torque in the \( Z \) direction.

6.1 The rate of change of orbital energy

Working in the stellar frame the rate of change of orbital energy is given by
\[
\frac{dE_{\text{orb}}}{dt} = -\int_V \rho' \left( \frac{\partial U}{\partial t} \right) dV.
\]
(72)
Note that, by taking the time derivative of (22) we obtain
\[
\frac{\partial U}{\partial t} = -G M_p r^2 \left( \frac{4\pi}{5a^3} \right) \sum_{n=-2}^{n=2} Y_{2,n}(\theta, \phi) \frac{\partial F_n(t)}{\partial t}.
\]
(73)
In addition, from (23), making the assumption that the orbital elements apart from \( \varpi \) are fixed, we infer that
\[
\frac{\partial F_n(t)}{\partial t} = \sum_{m=0, -2}^{k=\infty} \sum_{m=-k}^{k=\infty} \left( i k n_o - m \varpi / dt \right) \phi_{k,m} D_{n,m}^{(2)}(\pi/2, 0) \exp(i(k n_o t - m \varpi))
\]
(74)
Supposing initially that only the terms in the sum (73) corresponding to a particular pair of values \( \pm n \) are retained, from (72) - (74) we obtain
\[
\frac{dE_{\text{orb}}}{dt} = -G M_p \left( \frac{4\pi}{5a^3} \right) \int_V r^2 \left( \nabla \cdot (\xi_n^*) \right) \left( Y_{2,n}(\theta, \phi) \frac{\partial F_n(t)}{\partial t} \right) dV + cc,
\]
(75)
where \( cc \) denotes the complex conjugate. The first term on the right hand side of the expression (75) is such that when \( n \to -n \) the complex conjugate is obtained. Thus, when a sum of these terms over \( n \) is made, the result is real. As above we may consider only \( n \geq 0 \) and write the total rate of change of orbital energy as
\[
\frac{dE_{\text{orb}}}{dt} = -G M_p \left( \frac{4\pi}{5a^3} \right) \sum_{n=0}^{n=2} \mathcal{R} (2 - \delta_0^n) \left[ \int_V r^2 \left( \nabla \cdot (\xi_n^*) \right) \left( Y_{2,n}(\theta, \phi) \frac{\partial F_n(t)}{\partial t} \right) dV \right].
\]
(76)
The integral in the above expression (76) is of a similar form to that found in the expression for \( T_z \) given by equation (69). The latter involves \( F_n(t) \), while the former involves its time derivative.
6.2 Reduction of the torque and rate of energy exchange integrals

Now we substitute our expression for the response displacement associated with a particular \( n \) given by equation (61) into the expressions for the components of the torques given by (69) - (71) and the rate of change of orbital energy given by (76). Using the expression for the displacement given by (27) and the decomposition for the equilibrium tide response given by (57) we have

\[
\nabla \cdot (\rho \xi_{eq,n}) = \overline{R} Y_{2,n}, \quad \text{where} \quad \overline{R} = \frac{1}{r^2} \frac{d}{dr} (r^2 \rho \xi^r) - \frac{6 \rho}{r} \xi^S.
\]

(77)

Here we imply the known properties of spherical harmonics to obtain the second equality.

We then make use of (23) to allow us to eliminate the \( F_n \) in favour of expressing the result in terms of the coefficients \( A_n \), perform the integration over the volume of the star by parts and the summation over \( n \) to obtain

\[
T^z = 2Q_{eq} \mathcal{I} (A_1 b_1^* + 2b_2^* A_2),
\]

(78)

where \( \mathcal{I} \) denotes that the imaginary part is to be taken and (see equation (61))

\[
b_n = \sum_{k=-\infty}^{\infty} D_{n,k} \exp(ikn_o t),
\]

(79)

where we have implied that \( Q_{eq} \) is real. Similarly we find

\[
T^x = 2Q_{eq} \mathcal{I} \left( A_1 b_2^* + b_1^* A_2 + \sqrt{3/2} \left( \frac{1}{2} (b_1^* A_0 - b_1 A_0) + b_0^* A_1 \right) \right),
\]

(80)

and

\[
T^y = -2Q_{eq} \mathcal{R} \left( A_1 b_2^* - b_1^* A_2 + \sqrt{3/2} \left( \frac{1}{2} (b_1^* A_0 + b_1 A_0) - b_0^* A_1 \right) \right),
\]

(81)

From the above two equations it follows that

\[
T^x - iT^y = 2Q_{eq} i (A_1^* b_2 - b_1^* A_2 + \sqrt{3/2} (b_1 A_0 - b_0^* A_1),
\]

(82)

We remark that we have made use of the fact that \( A_{-n}^* = (-1)^n A_n \) and \( b_{-n}^* = (-1)^n b_n \) with the former implying that \( A_0 \) is real. Finally, the above process can be applied to (76) to determine the rate of change of orbital energy as

\[
\frac{dE_{orb}}{dt} = -2Q_{eq} \mathcal{R} \left( b_1^* \frac{dA_1}{dt} + b_2^* \frac{dA_2}{dt} + \frac{1}{2} b_0^* \frac{dA_0}{dt} \right).
\]

(83)

6.3 Fourier decomposition of the \( A_n(t) \)

Recalling that \( A_n = -4\pi G M_p F_n/(5a^3) \) and using equations (22), (23) and (30) we may write

\[
A_n(t) = \sum_{k=-\infty}^{\infty} A_{n,k} \exp(ikn_o t).
\]

(84)
We shall suppose that the orbit has an extremely slowly varying or fixed apsidal line so that it can be regarded as being constant when averaging over the fast orbital time scale. Even so the apsidal rotation rate may be rapid compared to the rate of tidal evolution.

6.4 Time averages of relevant quantities

6.4.1 The case with non zero apsidal precession

From the above analysis we can evaluate quantities such as the time average of the products $b_{n_1}^*(t)A_{n_2}(t)$. With the help of the expansions (79) and (84) as well as (62) we find that when the longitude of pericentre varies sufficiently slowly that can be taken to be constant during the time averaging process, we find

$$\langle b_{n_1}^*(t)A_{n_2}(t) \rangle = \sum_{k=-\infty}^{k=\infty} A_{n_2,k} D_{n_1,k}^* = - \sum_{k=-\infty}^{k=\infty} Q_{eq} R_{n_1,k} A_{n_2,k}^* A_{n_1,k},$$

(85)

where the angled brackets denote time averaging. Corresponding to this we have

$$\langle b_{n_1}^*(t)dA_{n_2}(t)/dt \rangle = - \sum_{k=-\infty}^{k=\infty} ikn_o Q_{eq} A_{n_2,k}^* A_{n_1,k}^* R_{n_1,k},$$

(86)

6.4.2 First order departures from the equilibrium tide

Assuming that corrections to an equilibrium tide are small, $\delta_{n,k}$ is also small in magnitude. Thus, terms only up to first order in this quantity are retained recalling that

$$R_{n,k} = \omega_{eq}^{-2} (1 - \delta_{n,k}).$$

(87)

Accordingly, an expression such as for example (85), which is needed in order to evaluate the torque components becomes

$$\langle b_{n_1}^*(t)A_{n_2}(t) \rangle = - \sum_{k=-\infty}^{k=\infty} Q_{eq} \omega_{eq}^2 (A_{n_2,k} A_{n_1,k}^* (1 + i(\delta_1 k + \delta_2 n_1) - (kn_1 + (n_1)^2 \sigma) \delta_3)).$$

(88)

6.5 Expressions for the components of the torque in closed form

The torque components and rate of orbital energy change given by equations (78) - (83), that are required to enable calculation of the evolution of orbital elements and stellar spin are expressed in terms of time averages of coefficient products, that can be expressed as infinite sums as exemplified in equations (86) - (88). As the terms in these summations are quadratic in $k$ these summations can be performed by making use of sum rules obtained with help of Parseval’s theorem applied to the
Fourier coefficients occurring in the expansion of the perturbing potential as specified in Appendix C.

We should emphasise that, the property whereby the terms in the summations involve powers of $k$ less than 2 comes about from the assumed constancy of the decay rate $\Gamma$, which has to be independent of forcing frequency. While this is correct for standard viscosity (see Ivanov & Papaloizou 2004), it is not true for radiative diffusion. In such a case one cannot readily take advantage of the sum rules and must employ infinite summations to determine the evolution of the system. Alternatively, a constant average value of $\Gamma$ could be assumed. It can be assumed that has been adopted in what follows below.

6.5.1 Evaluation of time averaged torques and rate of change of orbital energy

These quantities are evaluated in appendix C with help of the integrals with respect to time obtained there, that are used to obtain sum rules, that can be used to evaluate the sums of the form specified in (86) and (88) that are needed to evaluate time averaged torques and rate of change of orbital energy. The latter are determined in appendix C where the reader interested in the details is referred. The results for the torque components given there are

$$T^z = T_\star \left( 2 \delta_1 \cos \beta \phi_1 - \delta_2 (1 - e^2)^{3/2} \left( (1 + \cos^2 \beta) \phi_2 - \sin^2 \beta \cos 2 \hat{\omega} \phi_3 \right) \right)$$

and

$$T \equiv T^x - i T^y =$$

$$T_\star \sin \beta ((2 \delta_1 - i \delta_3) \phi_1 - (1 - e^2)^{3/2} (\delta_2 - i \sigma \delta_3) ((\phi_2 + \phi_3 \cos (2 \hat{\omega})) \cos \beta - i \sin (2 \hat{\omega}) \phi_3)),$$

and the change of orbital energy is given by

$$\frac{dE_{\text{orb}}}{dt} = \dot{E}_\star \left( \delta_2 \phi_1 \cos \beta - \frac{\delta_1}{(1 - e^2)^{3/2}} \phi_4 \right).$$

Here

$$T_\star = \frac{6\pi}{5} \left( \frac{GM_p Q_{eq}}{a^3 (1 - e^2)^{3/2} \omega_{eq}} \right)^2 = \frac{3k_2 q^2}{1 + q} \left( \frac{R_\star^5}{a^5} \right) \frac{M_p n_0^2 a^2}{(1 - e^2)^6}$$

and

$$\dot{E}_\star = 2 n_o T_\star,$$

respectively, represent typical values of the torque and rate of change of energy. Note that in the second equality we have used equation (D7) in Appendix D with $N_0 = 1$ to relate $T_\star$ to the apsidal motion constant. The quantities $\phi_1, \phi_2, \phi_3,$ and $\phi_4$ are functions of the eccentricity. They are specified both in table 3 and appendix C.
7 DETERMINATION OF THE ORBITAL AND SPIN EVOLUTION

In this Section we use the time averaged torque and rate of orbital energy change obtained above and with the help of results in appendix [C] to obtain equations governing the orbital and spin evolution that depend only on parameters required to specify them and quantities intrinsic to the primary star. We separately incorporate the standard precession of its spin axis induced by rotational flattening that is otherwise not included in our discussion. In addition we incorporate effects arising from dissipation in the compact companion under the assumption that it can only contain negligible angular momentum and thus instantaneously adjusts its spin so as to attain a condition of alignment with the orbit and net zero torque.

7.1 Expressions for the evolution of angular momentum vectors

The rate of change of the absolute values of orbital and spin angular momentum vectors and the angles determining their orientation with respect to the primary centred coordinate system follow from equations (15) and (16) after substitution of the components of the torque obtained from equations (C23) and (C24). Proceeding in this way we obtain the rates of change of the orientation specifying angles in the form

\[
\frac{di}{dt} = -(1 - e^2)^{3/2} \sin \beta \frac{T_x}{L} (\sigma \delta_3 \phi_3 \cos \beta \sin 2 \hat{\omega} + \delta_2 (\phi_2 - \phi_3 \cos 2 \hat{\omega})),
\]

\[
\frac{d\delta}{dt} = -\frac{T_x}{S} \sin \beta \left(2 \delta_1 \phi_1 - (1 - e^2)^{3/2} (\delta_2 \cos \beta (\phi_2 + \phi_3 \cos 2 \hat{\omega}) + \sigma \delta_3 \phi_3 \sin 2 \hat{\omega}) \right),
\]

\[
\frac{d\alpha_r}{dt} = -\frac{J T_x}{S L} \left( \delta_3 \phi_1 - (1 - e^2)^{3/2} (\sigma \delta_3 \cos \beta (\phi_2 + \phi_3 \cos 2 \hat{\omega}) + \delta_2 \phi_3 \sin 2 \hat{\omega}) \right) + \frac{1}{3} (1 - e^2)^{9/2} \frac{1 + q}{q} \sigma^2 \cos \beta
\]

and we note that the rate of change of the angle of inclination between the spin and orbital angular momenta is

\[
\frac{d\beta}{dt} = \frac{di}{dt} + \frac{d\delta}{dt}.
\]

The rate of change of the magnitudes of the orbital and spin angular momenta are given by

\[
\frac{dL}{dt} = -2 T_x (\delta_1 \phi_1 - (1 - e^2)^{3/2} \delta_2 \phi_2 \cos \beta) - (1 - e^2)^{3/2} T_x \sigma \delta_3 \phi_3 \sin^2 \beta \sin 2 \hat{\omega},
\]

and

\[
\frac{dS}{dt} = T^z,
\]

where \(T^z\) is given by equation (C23).

Note that in addition to the torque component \(T^y\) that has been incorporated in equation (95).
(see equation (15)) we have also included an additional torque component \( T_{SF}^y \) arising from the effect of stellar flattening due to rotation. It is calculated in a form convenient for our purposes in Appendix D (see equation (D9)). It is represented in (95) as the last term and has the factor \((1 + q)/q\), where \( q = M_p/M_\ast \) is the mass ratio. While our 'standard' torque component \( T_y \) is proportional to stellar rotational frequency \( \Omega_r \), \( T_{SF}^y \) is proportional to the square of \( \Omega_r \).

In addition, we recall that \( J \) is the conserved total angular momentum of the system, while \( i \) and \( \delta \) are, respectively, the angles of inclination between this and the orbital and spin angular momenta.

The quantities \( \phi_i \) are given by equations (C27)-(C30) with \( T_\ast \) and \( \dot{E}^\ast \) being given by equation (C26).

7.2 Evolution of the semi-major axis and eccentricity

For Keplerian orbits the relationship between the rate of change of the semi-major axis and the rate of change of orbital energy is given by

\[
\frac{da}{dt} = \frac{2a^2}{GM_pM_\ast} \frac{dE_{orb}}{dt} = \frac{2a^2}{GM_pM_\ast} \dot{E}_\ast \left( \delta_2 \phi_1 \cos \beta - \frac{\delta_1 \phi_4}{(1 - e^2)^{3/2}} \right),
\]

where we have used the expression for \( dE_{orb}/dt \) given by equation (C25). The rate of change of the orbital eccentricity is given in terms of the rates of change of orbital angular momentum and energy by

\[
\frac{de}{dt} = \frac{a(1 - e^2)}{GM_pM_\ast e} \left( \frac{dE_{orb}}{dt} - \frac{dL}{dt} \sqrt{G(M_p + M_\ast)} \right).
\]

Substituting (C25) and (97) in (100) we obtain

\[
\dot{e} = -\frac{3ae(1 - e^2)^{-1/2} \dot{E}_\ast}{GM_pM_\ast} (3\delta_1 \phi_5 - \frac{11}{6}\delta_2 \phi_6 (1 - e^2)^{3/2} \cos \beta) - \\
\frac{3ae(1 + e^2/6)(1 - e^2)^2 \dot{E}_\ast}{4GM_pM_\ast} \sigma_3 \sin^2 \beta \sin 2\dot{\varpi},
\]

where we make use of equation (C29) to obtain \( \phi_3 \) and

\[
\phi_5 = (\phi_4 - (1 - e^2)\phi_1)/(9e^2) = 1 + \frac{15}{4}e^2 + \frac{15}{8}e^4 + \frac{5}{64}e^6,
\]

and

\[
\phi_6 = \frac{2(\phi_4 - (1 - e^2)\phi_2)}{11e^2} = 1 + \frac{3}{2}e^2 + \frac{1}{8}e^4.
\]
7.3 Incorporating tidal dissipation in the companion

So far we have regarded the companion of mass, $M_p$, as acting like a point mass. However, when it represents a giant planet, tides are expected to be significant in leading to orbital circularisation (e.g. Ivanov & Papaloizou 2007). During this process it is still possible to neglect its angular momentum content with the consequence that the evolution of the angular momentum vectors described above is unaffected. We further assume that the evolution of the companion spin angular momentum rapidly adjusts so that a zero torque applies. However, energy dissipation still occurs which can lead to eccentricity damping. In order to apply the zero torque condition we assume the companions spin is aligned with the orbital angular momentum and apply forms of equations (97) - (98) and (C25), respectively, governing the spin up torque and rate of energy dissipation adapted to apply to the companion. To do this we interchange $M_p$ and $M_*$ and signify that quantities apply to the companion by adding a subscript, $p$, so that e.g. $\delta_i \rightarrow \delta_{i,p}$. Thus, with assumed spin-orbit alignment and zero spin up torque (98) implies that $\delta_{1,p} \phi_1 = (1 - e^2)^{3/2} \delta_{2,p} \phi_2$. Using this together with the adapted form of equation (C25) gives

$$\left(\frac{dE_{\text{orb}}}{dt}\right)_p = -\dot{E}_{*,p} \frac{\delta_{1,p} e^2 \phi_7}{(1 - e^2)^{3/2} \phi_2}, \quad (104)$$

where

$$\phi_7 = \frac{\phi_2 \phi_4 - \phi_1^2}{e^2} = \frac{7}{2} + \frac{45}{4} e^2 + 28 e^4 + \frac{685}{64} e^6 + \frac{255}{128} e^8 + \frac{25}{512} e^{10} \quad (105)$$

with

$$\dot{E}_{*,p} = 2n_o T_{*,p} \quad \text{and} \quad T_{*,p} = \frac{6\pi}{5} \left(\frac{G M_* Q_{eq,p}}{a^3 (1 - e^2)^3 \omega_{eq,p}}\right)^2 \quad (106)$$

and we note that $\delta_{1,p} = 2\Gamma_p n_o / \omega_{eq,p}$.

We can now add $(dE_{\text{orb}}/dt)_p$ found above to $dE_{\text{orb}}/dt$ in equations (99) and (100) in order to find the effect of the companion on the orbital evolution. Only the semi-major axis and eccentricity are affected. Equation (99) for the rate of change of the semi-major axis becomes

$$\frac{da}{dt} = \frac{2a^2}{GM_p M_*} \left(\dot{E}_* \left(\delta_2 \phi_1 \cos \beta - \frac{\delta_1 \phi_4}{(1 - e^2)^{3/2}}\right) - \dot{E}_{*,p} \frac{\delta_{1,p} e^2 \phi_7}{(1 - e^2)^{3/2} \phi_2}\right) \quad (107)$$

Similarly, equation (101) for the rate of change of the eccentricity becomes

$$\dot{e} = -\frac{3ae(1 - e^2)^{-1/2}}{GM_p M_*} \left(\dot{E}_* \left(3 \delta_1 \phi_5 - \frac{11}{6} \delta_2 \phi_6 (1 - e^2)^{3/2} \cos \beta\right) + \dot{E}_{*,p} \frac{\delta_{1,p} \phi_7}{3 \phi_2}\right) - \frac{3ae(1 + e^2/6)(1 - e^2)^2 \dot{E}_*}{4 GM_p M_*} \sigma_3 \sin^2 \beta \sin 2\hat{\omega}. \quad (108)$$

The evolution equations (99)-(100) or (101) form a complete set for the system in which tides in the companion play no role. Tidal energy dissipation can be included under the assumption
that the spin angular momentum may be neglected provided equation (99) is replaced by equation (107), and equation (101) by equation (108). In both cases an equation for the evolution of the orientation of the apsidal line characterised by the angle \( \hat{\varpi} \) should be specified. The contribution of tidal distortion to this is readily estimated by noting that this is produced by a deformation that follows the perturbing star and so it does not depend on the inclination of the orbit or spin axis if the unperturbed star is approximated as being spherical. Accordingly, in the first instance taking into account the distortion of the primary by the companion, we apply the standard theory (e.g. Sterne 1939) which gives

\[
\frac{d\hat{\varpi}}{dt} = \frac{d\varpi}{dt} = 15k_2 n_0 \frac{M_p R_p^5}{M_\star (a(1-e^2))^5} \phi_6. \tag{109}
\]

In writing the first equality in (109) we recall that the orbital and spin angular momenta precess together about the fixed total angular momentum vector with the angle \( \gamma \) defined in Section 2.2 remaining constant. In addition, \( \phi_6 \) is given by eq. (103) and the apsidal motion constant \( k_2 \) can be related to the parameters in our formalism through equation (D7) of Appendix D which gives

\[
2k_2 = \frac{4\pi GQ_{eq}^2}{5N_0 \omega_{eq}^2 R_\star^5}. \tag{110}
\]

As this does not depend on dissipation it leads to an evolution time scale that will be much shorter than processes that do. Assuming that the effect of tidal distortion of the companion by the primary may be important in some circumstances we note that it can be taken into account by modifying (109) in the form

\[
\frac{d\hat{\varpi}}{dt} = \frac{d\varpi}{dt} = 15k_2 n_0 \frac{M_p R_p^5}{M_\star (a(1-e^2))^5} (1 + F_{cp}) \phi_6, \tag{111}
\]

where

\[
F_{cp} = \frac{k_{2,p} M_\star^2 R_{\star,p}^5}{k_2 M_p^2 R_\star^5}, \tag{112}
\]

with \( k_{2,p} \) and \( R_{\star,p} \), respectively, being the apsidal motion constant and radius of the companion. We remark that non dissipative contributions arising from \( \delta_3 \) generally lead to significantly smaller effects and this is discussed in Section 7.5. It is possible that other effects such as perturbations due to other orbiting bodies are more important and these may be added in. Note too that equation (94) for \( \alpha_r \) decouples from the others in that it is ignorable when solving them for the evolution of the system. It can be integrated once that has been determined.

Focusing on the primary while noting that a parallel discussion applies to quantities associated with the companion, we recall that coefficients \( \delta_i \), \( (i = 1, 3) \), entering the above equations are defined through equation (65) and also set out in table 2 as \( \delta_1 = 2\Gamma\bar{\Omega}, \delta_2 = \lambda\sigma\delta_1 \) and \( \delta_3 = \)
On the evolution of a binary system due to quasi-stationary tides

\[-2\beta_*\lambda\sigma\tilde{\Omega}^2,\] where \(\tilde{\Gamma} = \Gamma/\omega_{eq}, \tilde{\Omega} = n_o/\omega_{eq}\) and \(\sigma = \Omega_r/(\lambda n_o)\). with \(\lambda = \beta_*/(\beta_* + 1/2)\), is the product of \(\lambda^{-1}\) and the ratio of the stellar rotation frequency to the orbital mean motion.

The coefficient \(\beta_*\) is defined in equations (52), (55) and (63) with the influence of terms \(\propto \delta_3\sigma\), which is \(\propto \beta_*\), on the tidal response being discussed in Section 5.4.2. We remark that \(\beta_*\) is expected to be smaller than, but of the order of unity. The quantity \(2\tilde{\Gamma}\) defined through equation (49) represents the ratio of the rate of energy dissipation to twice the kinetic energy associated with the tidally excited disturbance, which takes the form of an equilibrium tide. It would be the decay rate of this disturbance were it to be a normal mode. The quantities \(\omega_{eq}\), and \(Q_{eq}\) defined by equations (47) and (59), respectively, represent a putative normal mode frequency and overlap integral associated with this disturbance.

A distinctive feature of the evolution equations is that they contain two types of terms - those proportional to \(\delta_1\) and \(\delta_2\), which are, in turn, proportional to the decay rate \(\Gamma\), when tides in the companion are included, \(\delta_{1,p}\) which is proportional to \(\Gamma_p\) is also involved, and those proportional to \(\delta_3\), which are independent of the dissipation rate and are determined by rotational effects. We respectively describe these terms as dissipative and rotational, and consider them in turn below.

However, we note in passing that apart from in equation (95) for \(\alpha_r\), which decouples from the rest, \(\delta_3\) appears together with a factor \(\sin 2\hat{\omega}\) or \(\cos 2\hat{\omega}\). If the apsidal line of orbit precesses at a uniform rate with a period short enough compared to other time scales these quantities can be dealt with following an averaging approach to determine the role of terms \(\propto \delta_3\) in character of the orbital evolution. If the precession is not uniform and/or the inclination angle \(\beta\) has significant variations these quantities can have a non zero time average even when the precession is fast, thus the value of \(\delta_3\) may affect the orbital evolution. This issue is discussed further in Section 7.5 where we estimate conditions under which tidally driven apsidal precession is rapid enough for averaging to be valid.

### 7.4 The evolution of orbital parameters when only dissipative terms are included

In this Section we formally neglect the contribution of rotational terms by setting \(\delta_3\) to zero in the evolution equations given above. We shall also neglect tidal effects arising from the companion. The set of equations so obtained can then be compared with those of EKH. These sets can be seen to be formally equivalent provided we take into account that EKH use different angles to charac-
terise their coordinate systems\(^3\) and relate the quantities defining the corresponding evolutionary timescale in our model to the 'tidal friction' timescale, \(t_{TF}\), adopted in EKH through

\[
t_{TF} = \frac{\mu a^2 \omega_{eq}^2}{8(1 - e^2)^2 T_\ast} \tilde{\Gamma}^{-1},
\]

where \(\mu = M_p M_\ast / (M_p + M_\ast)\) is reduced mass. It is instructive to substitute the explicit expression for the typical torque \(T_\ast\) given by equation (C26) in (113). We then express \(Q_{eq}\) and \(\omega_{eq}\) in terms of natural units by introducing the dimensionless quantities

\[
\tilde{Q} = \frac{Q_{eq}}{M_{1/2} R_\ast} \quad \text{and} \quad \tilde{\omega} = \tau_s \omega_{eq},
\]

where \(\tau_s = \sqrt{R_\ast^3 / (G M_\ast)}\) with \(R_\ast\) being the radius of the primary star. We then have

\[
t_{TF} = \frac{5}{48 \pi} \tilde{\Gamma}^{-1} \tilde{Q}^{-2} \tilde{\omega}^4 \frac{1}{q(1 + q)} \left( \frac{a}{R_\ast} \right)^8.
\]

We now use equation (72) of EKH to relate this to quantities characterising their model and intrinsic to it, \(\sigma_{EKH}\) determining the magnitude of the rate of energy dissipation and dimensionless parameter \(Q_{EKH}\) characterising density distribution in the primary. Doing this we find how these quantities relate to \(\tilde{\Gamma}, \tilde{Q}\) and \(\tilde{\omega}\), thus obtaining

\[
\left( \frac{Q_{EKH}}{1 - Q_{EKH}} \right)^2 \sigma_{EKH} = \frac{16 \pi}{15} \tilde{\Gamma} \tilde{Q}^2 M_\ast R_\ast^2 \tilde{\omega}^4.
\]

### 7.5 Contribution of the rotational terms to the orbital evolution

In this Section we consider the effect of terms in the tidal response that are \(\propto \delta_3\). In doing this we neglect tides in the companion and dissipative effects in the primary. The tidal interaction then conserves orbital energy and consists of an interaction between the spin and orbital angular momenta characteristically leading to changes in their mutual inclination accompanied by their precession around the total angular momentum vector.

Thus we consider the evolution of orbital parameters due to the presence of rotational terms proportional to \(\delta_3\) formally setting \(\delta_1 = \delta_{1,p} = \delta_2 = 0\) in the evolutionary equations above. It is important to stress that this approximation may be adequate for sufficiently short time intervals, since the dimensionless parameters \(\tilde{\Gamma}^{-1}\) and \(\tilde{\Gamma}_p^{-1}\) which determine the timescale of evolution due to the presence of non conservative effects are expected to be quite large.

When \(\delta_1, \delta_{1,p}\) and \(\delta_2\) are set to zero the orbital energy is conserved and the \(Z\) component of torque \(T_x = 0\). In this case from equation (98) it follows that the absolute value of rotational

\(^3\) EKH use the angles \(\eta\) and \(\chi\) to characterise inclination and rotation of the orbital plane with respect to an inertial coordinate system, respectively, and the angle \(\psi\) to characterise position of the apsidal line. It turns out that they are related to the angles used in this Paper as \(\eta = i, \chi = -\alpha - \pi/2 \equiv \alpha_r - \pi/2\) and \(\psi = \dot{\omega} - \pi/2\).
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angular momentum, $S$, is an integral of motion. Also, from the second equation in the set (15) and (17) it follows that there is additional integral of motion

$$ I = \frac{L^2}{2S} - L \cos \beta, \quad (116) $$

which is valid for any form of $T_x$.

Since eccentricity $e$ depends only on $L$ when the orbital energy, and, accordingly, semi-major axis $a$ are fixed, from equations (17) and (C24) it follows that the evolution equation for $\beta$ is a function of $\beta$, $e$ and $\dot{\omega}$ when $a$ and $S$ are fixed. Therefore, if the form of $\dot{\omega}$ is known as a function of time, the evolution of all orientation specifying angles is reduced to finding solution of only one first order ordinary differential equation. We assume below that the rate of change of $\dot{\omega}$ under classical tidal distortion is given by equation (111). We now consider the contribution of perturbations arising from $\delta_3$ with other $\delta_i = 0$ in the limits of large and small $S$ which as indicated above is constant under these conditions. In the limit of large $S$, the spin axis coincides with the total angular momentum vector and the angle $\delta = 0$, with $i \equiv \beta$. In this case $L \cos i$ is a constant of the motion and the system is Hamiltonian with two degrees of freedom. Accordingly, we may write

$$ \frac{dL}{dt} = -\frac{\partial R_H}{\partial \dot{\omega}} \quad (117) $$

where

$$ R_H = -(1 - e^2)^{3/2}T_x\sigma_3 \phi_3 \sin^2 \beta \cos 2\dot{\omega}. \quad (118) $$

As the system is Hamiltonian, we have (see e.g. Murray & Dermott 2012)

$$ \frac{d\dot{\omega}}{dt} = \frac{d\omega}{dt} = \frac{\sqrt{1 - e^2}}{\mu e n_o a^2} \frac{\partial R_H}{\partial e} \quad (119) $$

after using this to find the contribution to the apsidal line advance rate arising from $\delta_3$ and adding it to the tidal contribution given by (111) we obtain

$$ \frac{d\dot{\omega}}{dt} = 15k_2 n_o \frac{M_p R_s^5}{M_s(a(1 - e^2))} \left( (1 + F_{cp})\phi_6 - \phi_8 \sigma_3 \sin^2 \beta \cos 2\dot{\omega} \right) \quad (120) $$

where $\phi_8 = (12 + 46e^2 + 5e^4)/20$. We note that corrections to apsidal advance rate arising from the term $\propto \delta_3$ (see table 2 for its definition) will be small for slow subcritical rotation.

We may also consider the limit of small $S$, normally the one of physical interest in the same way. In the extreme limit of that case the orbital angular momentum coincides with the total angular momentum vector, $i = 0$, while the angle $\delta \equiv \beta$ determines the orientation of the spin. Regarding this as a given function of time, although the system, being non autonomous is not strictly Hamiltonian, equations (117) and (119) still apply with $R_H$ playing the role of a disturbing function. Hence we again obtain (120). But note that although the orbital plane approaches
coplanarity with the plane with normal to the total angular momentum vector that defines the primary centred frame, the apsidal line is measured relative to a line that asymptotically rotates (precesses) with angular velocity \( d\alpha_r/dt \). Thus, if \( \dot{\omega} \) is instead measured relative to a fixed line in the primary centred frame, we have

\[
\frac{d\dot{\omega}}{dt} = \frac{d\alpha_r}{dt} + 15k_2n_0 \frac{M_p R_*^5}{M_* (a(1-e^2))^{3/2}} \left( \phi_6 (1 + F_{c,p}) - \phi_8 \delta_3 \sin^2 \beta \cos 2(\dot{\omega} - \alpha_r) \right),
\]

(121)

Returning to the equivalent equation (120) we see that the correction arising from \( \delta_3 \) is of order \( (\Omega_r/\omega_{eq})^2 \), which is assumed to be small and in fact comparable to neglected effects arising from rotational distortion. It also follows that a time average of \( \sin 2\dot{\omega} \) over a precession cycle is zero and that of \( \cos 2\dot{\omega} \) is of order \( (\Omega_r/\omega_{eq})^2 \). Thus, terms involving these and \( \delta_3 \) will give negligible contributions to (121) if the evolution of the system occurs on a time scale significantly longer than an apsidal rotation period and no other processes affect the evolution of \( \dot{\omega} \).

We now consider the evolution of the inclination angles in more detail in the limit \( S/L \ll 1 \) when the absolute value of orbital angular momentum is also conserved to within a variation of order \( S/L \). In this case equation (17) is reduced to

\[
\dot{\beta} = -\frac{T^x}{S} \quad \text{where} \quad T^x = T_*(1 - e^2)^{3/2} \sigma \delta_3 \phi_3 \sin \beta \sin 2\dot{\omega},
\]

(122)

which can be readily integrated to give

\[
\ln \left( \frac{1 - \cos \beta}{1 + \cos \beta} \right) = \frac{1}{T_\beta} \int dt \sin 2\dot{\omega}, \quad T_\beta = -\frac{S}{2(1 - e^2)^{3/2} \sigma \delta_3 \phi_3 T_*},
\]

(123)

and we recall that \( \delta_3 < 0 \) and, therefore, \( T_\beta \) is a positive quantity.

Now let us assume that the apsidal precession is uniform and \( \dot{\omega} = \omega_\alpha t \), where we set an initial value of apsidal angle to zero without loss of generality. Then, we have from (123)

\[
\tan \frac{\beta}{2} = \tan \frac{\beta_0}{2} \exp \left( \frac{1 - \cos 2\omega_\alpha t}{4T_\beta \omega_\alpha} \right),
\]

(124)

where \( \beta_0 = \beta(t = 0) \). In this case \( \beta \) changes periodically, with the period of change being one half the period of apsidal precession. A typical amplitude of change is inversely proportional to the product \( T_\beta \omega_\alpha \).

It is instructive to represent \( T_\beta^{-1} \) in terms of natural units in the form

\[
T_\beta^{-1} = -\frac{6Lk_2}{S} \frac{\phi_3 \sigma \delta_3}{(1 - e^2)^{3/2} \sigma a^5 n_0} M_p R_*^5 M_* a^5 n_0,
\]

(125)

where we have made use of the expression for \( k_2 \) given by (D7) in Appendix D. Equations (109) and (125) state that \( T_\beta^{-1} \) is the product of the classical apsidal advance rate induced by the com-
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Figure 2. The value of ‘effective’ period $P_{\text{eff}}$ defined in eq. (127) as a function of eccentricity $e$, for a given value of $f_\beta$. Dotted, solid and dashed curves correspond to $f_\beta = 0.1$, 1 and 10, respectively.

The situation where $f_\beta$ is significant may be of special importance, since it could lead to observational consequences and in this context note that relatively small variations may be noticeable in eclipsing or transiting systems. To further examine conditions leading to such situation we rewrite

\[ f_\beta = \frac{4}{5} \left( \beta_* + \frac{1}{2} \right) \frac{\phi_3}{\phi_0} \frac{\Omega^2 R_*^3}{\tilde{\omega} GM_*} \frac{L}{S}, \]  

where $\tilde{\omega} = \omega_{\text{eq}} \sqrt{R_*^3 / GM_*}$ is expected to be order of unity. In order for averaging to be valid and a resulting small change in $\beta$ we require that $f_\beta \ll 1$ which will be violated for sufficiently small $S$ or moment of inertia of the primary. In that case all terms involving $\delta_3$, apart from in the equation for $d\tilde{\omega}/dt$, should be retained.

More generally the variation of $\beta$ over one half of an apsidal precession period is very roughly $\sim f_\beta / 2$ radians.

4 Let us stress that here we consider only tidal precession induced by tidally deformed primary. However, in some realistic situations the contribution of the companion could be potentially dominant, see e.g. the corresponding analysis in Ragozzine & Wolf (2009) for Hot Jupiter systems.
Figure 3. $f_\beta$ defined in eq. $^{(129)}$ as a function of eccentricity. Curves with smaller (larger) values of the argument correspond to the nominal solar model (the model of Kepler 91). Solid, dashed and dotted curves correspond to $P_r/P_{orb} = 1$, 10 and 0.1.

$^{(126)}$ in the form

$$f_\beta = 1.9 \frac{\phi_3}{\phi_6} \sqrt{1 - e^2} \left( \frac{1 \text{day}}{P_{eff}} \right)^{2/3}, \quad \text{where}$$

$$P_{eff} = \frac{(1 + q)^{1/2}}{q^{3/2}} \left( \frac{I_{0.1} \omega^2}{(\beta_\ast + 1/2)} \right)^{3/2} \left( \frac{\bar{\rho}_\ast}{\bar{\rho}_\odot} \right)^{1/2} \left( \frac{P_r}{P_{orb}} \right)^{3/2} P_{orb}. \quad (127)$$

Here $P_{orb}$ is orbital period, $I_{0.1} = I/(0.1 M_\ast R_\ast^2)$ with $I$ being primary’s moment of inertia, $\bar{\rho}_\ast$ and $\bar{\rho}_\odot$ are respectively the mean densities of the primary and of the Sun.

From $^{(127)}$ it follows that

$$\frac{P_{eff}}{1 \text{day}} = \left( \frac{1.9 \phi_3 \sqrt{1 - e^2}(1 + q)^{2/3}}{f_\beta \phi_6} \right)^{3/2}. \quad (128)$$

A plot of $P_{eff}/(1 \text{day})$ as a function of $e$ for $q = 1$ and fixed $f_\beta = 0.1$, 1 and 10 is shown in Fig. 2. Accordingly, we infer that rather large eccentricities $e \sim 0.7$ are needed to produce significant values of $f_\beta$. From $^{(127)}$ it also follows that in order to have significant variations of $\beta$ for a realistic orbital period a system should have in addition, relatively large companion masses and small rotation periods. Also, stars with large central condensations corresponding to smaller $I_{0.1}$ and small mean densities are favoured.
To illustrate the dependency of $f_\beta$ on other parameters we represent it in the form

$$f_\beta = f_\ast \left( \frac{1 \text{day}}{P_{\text{orb}}} \right)^{2/3},$$

(129)

where an expression for $f_\ast$ directly follows from (127). In Fig. 3 we plot $f_\ast$ as a function of $e$ for $q = 1$, $\beta_\ast = 0.5$, $\bar{\omega} = 1$ and three values of $P_r/P_{\text{orb}}$ for two stellar models. One is our 'nominal' solar model with $I_{0.1} = 1$ and $\bar{\rho}_\ast/\bar{\rho}_\odot = 1$, while the other represents a model of an evolved star used in Chernov, Ivanov & Papaloizou (2017) to model the exoplanetary system Kepler 91. In this model $I_{0.1} \approx 1.4$ and $\bar{\rho}_\ast/\bar{\rho}_\odot \approx 5.2 \cdot 10^{-3}$. One can see from this Figure that the use of the star with smaller average density results in larger values of $f_\beta$ as expected. Also, smaller ratios $P_r/P_{\text{orb}}$ (and, accordingly, larger ratios of rotational to orbital frequencies) are favoured. Finally, we illustrate the result of solution of equation (124) for $f_\beta = 1, 10$ and 0.1 in Fig. 4. One can see from this Fig. that a typical change of $\beta$ over half an apsidal period gets larger with increase of $f_\beta$ as expected.

Now let us consider the evolution of the precessional angle $\alpha_r$. The corresponding equation directly follows from equation (95), where we set $\delta_1 = \delta_2 = 0$ and use the explicit expression for
\[ \frac{d\alpha_r}{dt} = \frac{JT_s}{SL} \sigma \left( (2\beta_s + 1)\tilde{\Omega}^2(\phi_1 - (1 - e^2)^{3/2}\sigma \cos \beta(\phi_2 + \phi_3 \cos 2\hat{\omega})) + \frac{1}{3}(1 - e^2)^{9/2} \frac{1+q}{q} \sigma \cos \beta \right) \]  

(130)

As discussed above, when the apsidal precession is uniform and on a timescale much smaller than \( T_{\beta} \), the term proportional to \( \phi_3 \) can be neglected. Then the angle \( \beta \) and the eccentricity \( e \) may be considered as constants so that the evolution of \( \alpha_r \) has the character of uniform precession.

It is of interest to estimate the condition under which the new term proportional to \( 2\beta_s + 1 \) is larger than the standard term responsible for precession due to rotational flattening of the star. The corresponding condition readily follows from (130), in which, in the assumed limit of small \( \Omega_r \), we neglect the contribution of the term proportional to \( \phi_2 \) and insert the expression for \( \phi_1 \). It is found to be

\[ \sigma < \frac{(6\beta_s + 3)q}{(1 + q)(1 - e^2)^{9/2} \cos \beta} \left( \frac{n_0}{\omega_{eq}} \right)^2 \left( 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{516}{16}e^6 \right). \]  

(131)

This can also be expressed as

\[ \frac{\Omega_r}{n_o} < \left( \frac{M_p R_p^3}{M_s R_{peri}^3} \right) (6\beta_s + 3) \left( 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6 \right) \left( \frac{GM_s}{R_s^2 \omega_{eq}^2} \right). \]  

(132)

where \( R_{peri} \) is the pericentre distance.

From (132) it follows that the new term prevails when either rotation and, accordingly, \( \Omega_r \), is quite small and/or when either eccentricity or mass ratio is large. While all factors in brackets apart from the first are arguably of order unity unless the eccentricity is close to unity or \( \beta \) is close to \( \pi/2 \), the first factor is usually small, especially for planetary mass ratios. Thus unless \( \beta \) is very close to \( \pi/2 \), the new term could only be significant for eccentricities that are extremely close to unity in that case. However, a qualitatively different analysis based on a sequence of parabolic encounters and the possibility of tidal capture may be more appropriate in that limit, which is beyond the scope of this paper.

8 DISCUSSION

In this paper we have considered the tidal interactions of a binary system consisting of a primary component and a compact companion. The primary component has spin angular momentum which is misaligned with the orbital angular momentum. There are no restrictions on the mutual inclination of these vectors or on the orbital eccentricity. The companion is initially assumed to have no internal degrees of freedom but at a later stage we allow for internal energy dissipation as would be important for a planetary mass companion. This enables us to consider the evolution of systems
such as those containing hot Jupiters in orbits with angular moments significantly misaligned with the primary spin axis.

We derived equations governing the evolution of the angular momentum vectors in Section 3 and calculated the tidal response of the primary in the equilibrium tide limit in Section 5. Effects arising from both dissipation and rotation were considered. The torque acting between the primary and orbit was determined in Section 6 and the rate of change of orbital energy in Section 6.1, being reduced to a final closed form in Sections 6.2 - 6.5.1.

Using the above results we obtained equations enabling the determination of the orbital and spin evolution in Sections 7 and 7.1. Energy dissipation in the companion under the assumption of negligible spin angular momentum is incorporated into the equations governing the evolution of the system in Section 7.3. This can be important for orbital circularisation for planetary mass companions. The parameters occurring in these equations is also reviewed in these Sections. Apart from the masses and radii of the binary components, the rotation rate of the primary, and parameters specifying the form and orientation of the orbit, the system of equations depends, for each component, on $\Gamma$, being the ratio of the rate of energy dissipation to four times the kinetic energy associated with the disturbance in the form of the equilibrium tide (see equation (49)), and the quantities $\omega_{eq}$ and $Q_{eq}$ defined by equations (47) and (59) representing a putative normal mode frequency and overlap integral associated with the equilibrium tide, respectively. But note that as an alternative to the last two parameters use can be made of the classical apsidal motion constant, $k_2$, see equation D7. There is also a dependence on the dimensionless quantity $\beta_*$, which is associated with rotational effects and the Coriolis force acting on the primary and is defined through equations (52) and (55) (see also the discussion in Section 5.4.2). We remark that Ivanov & Papaloizou (2004) assumed only the $l = 2$ $f$ mode contributed to the tidal response.

This corresponds to replacing the displacement associated with the equilibrium tide by that associated with the $f$ mode in our analysis while $\Gamma$ is replaced by the decay rate of this mode. But note that the forcing frequency dependence of $\Gamma$ in the case of radiative damping requires that this has been assumed to be the frequency of the $f$ mode in that scheme. A simple estimate based on equation (49) indicates $\Gamma$ varies as the inverse of the square of the forcing frequency and so may be significantly underestimated by that substitution.

It is also important to realise that the closed form of equations we obtain is valid only if the quantity $\Gamma$ is independent of tidal forcing frequency. This would be the case for a standard Navier Stokes viscosity. However, as we indicated a contribution from radiative damping should also be considered and this is frequency dependent. In order to obtain our equations in their closed
form, an average value of $\Gamma$ has to be adopted. In the first instance we would suggest the value corresponding to the angular velocity at periastron $\sim (1 + e)^{1/2}(1 - e)^{-3/2}n_o$ is appropriate as this location is where most of the tidal interaction takes place. However, dealing with the frequency dependence associated with radiative damping as well as relaxing the equilibrium tide assumption to incorporate dynamical effects, which could be important for determining where most energy dissipation takes place, is an important issue when considering planetary mass objects such as Hot Jupiters, which should be a subject for future investigation.

Our equations for the tidal evolution include both dissipative terms and those associated with the Coriolis force and the dependence of forcing frequency on angular velocity on account of the Doppler effect, which we describe as rotational terms. If the latter are dropped we may relate our results to those of EKH in the appropriate limit as done in Section 7.4. There we were able to link the parameters in our equations to theirs thus showing how their parameter which was assumed to relate the energy dissipation rate to the rate of change of the quadrupole tensor associated with the primary can be connected to the energy dissipation rate occurring in the proper solution of the tidal response problem.

The contribution of terms arising from rotation to the orbital evolution not included in EKH takes two forms that were considered in Section 7.5. The first is through effects on the evolution of the orbital parameters and orientation specifying angles through terms $\propto (\beta_{\ast} + 1/2)$ and sines and cosines of the longitude of periapse. These can be dealt with by an averaging procedure if the advance of the apsidal line is uniform and on a time scale shorter than the time scale of the tidal evolution. In general, the influence of other perturbing bodies may have to be included to determine if this is appropriate. However, we estimated a condition for this to hold when the apsidal line advance was determined purely by tidal effects in the form $f_{\beta} \ll 1$, see equations (126-128).

The situation when this is not satisfied is of special interest since it may lead to significant variations of the inclination angle over a half of an apsidal line precession period and, therefore, in a system with appropriate parameters this may be directly observable. Under the assumption of uniform apsidal precession induced by tides in the primary the corresponding condition for this can be obtained from equation (128) (see also Fig. 2). From this it follows that in order to have a significant effect the mass ratio and primary angular velocity should be sufficiently large, while orbital period should be sufficiently small. An optimal value of eccentricity is close to 0.7. Also, centrally condensed stars with small mean densities are favoured and, clearly, spin and orbital
angular momenta should be misaligned. It should be interesting to investigate whether there are observed systems with such properties.

The other phenomenon that is affected is the precession of the spin and orbital angular momentum vectors about the total conserved angular momentum vector. A new rotational term contributes a precession rate that is linear in the primary’s angular velocity. This was compared with the classical precession rate driven by centrifugal distortion, which is proportional to the square of the angular velocity. Formally, the new term dominates for sufficiently small rotational frequency estimated in equation (131). This indicates that a small angular velocity is needed unless the mass ratio is around unity and/or the eccentricity is close to unity. Note that that in order to have the new term dominate orbital precession it is necessary to have large eccentricities and/or a small angular velocity. The latter is the opposite of what is needed to produce significant variations in the inclination angle discussed in Section 7.5. However, in both cases sufficiently large mass ratios are favoured. Further work is necessary to investigate these features more fully.

There is a simple physical explanation for these new non-dissipative effects. Namely, in the absence of rotational terms (and, of course, neglecting dissipation) tidal bulge is aligned with the direction to perturbing body. When the rotation axis is misaligned with respect to orbital angular momentum, the presence of rotational terms breaks that alignment, thus producing torques, which lead to the evolution of the corresponding orbital elements.

Also, it is important to stress that we have neglected a contribution of toroidal displacements to perturbation of the star due to tides. Although it may be small due to relative smallness of the appropriate overlap integrals, this contribution should be separately analyzed. A convenient framework for such an analysis would be the self-adjoint approach to the problem of tidal excitation of normal modes of any kind in a rigidly rotating star put forward in [Papaloizou & Ivanov (2005)] and [Ivanov & Papaloizou (2007)].

It is interesting to note that a different approach has been used to describe tidal evolution of ’elastic’ bodies (say, rocky planets) for inclined systems (see [Kaula (1964)] and e.g. [Boue & Efroimsky (2019)] and references therein for a recent development). It would be of interest to establish how this approach and the formalism developed in this Paper are related to each other.

Finally, it is important to note that in order to find a complete self-consistent set of equations for the evolution of the orbital parameters on the tidal timescale one must consider dynamical tides in addition to quasi-stationary ones. This Paper provides a convenient foundation for such a study. In future work we will undertake to generalise the treatment of dynamical tides in the
so-called regime of moderately large viscosity developed in [Ivanov et al. (2013)] for the aligned case to tidally interacting systems with orbital and spin axes arbitrarily misaligned.

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: FOURIER DEVELOPMENT \( U \) IN TERMS OF SPHERICAL HARMONICS DEFINED IN THE STELLAR FRAME

We begin with the Fourier expansion given in Section 4.1 in the form

\[
\frac{\exp(-i n (\Phi - \omega)) a^3}{R^3} = \sum_{k=\infty}^{k=-\infty} \phi_{k,n} \exp(ik_n t), \tag{A1}
\]

The Fourier coefficients, \( \phi_{k,m} \), are related to the well known Hansen coefficients \( X^{q,m}_k \) through \( \phi_{k,m} = X^{-3,-m}_k \). A formal expression for \( X^{q,m}_k \) is given by

\[
X^{q,m}_k = (1 + \beta^2)^{-q-1} \sum_{p=-\infty}^{p=\infty} J_p(je) H^{q,m}_{k,p}, \tag{A2}
\]

where \( \beta = (1 - \sqrt{1 - e^2})/e \), \( J_p \) denotes the Bessel function and

\[
H^{q,m}_{k,p} = \frac{(-\beta^2)^{k-p-m} \Gamma(q + 2 - m)}{\Gamma(k - p - m + 1) \Gamma(q + 2 - k + p)!} {}_2F_1(k - p - q - 1, -m - q - 1, j - p - m + 1, \beta^2), \tag{A3}
\]

for \( j > p + m \), and

\[
H^{q,m}_{j,p} = \frac{(-\beta^2)^{-j+p+m} \Gamma(q + 2 + m)}{\Gamma(-j + p + m + 1) \Gamma(q + 2 + j - p)!} {}_2F_1(-j + p - q - 1, m - q - 1, -j + p + m + 1, \beta^2), \tag{A4}
\]

for \( j < p + m \), and \( {}_2F_1(\alpha, \beta, \gamma, z) \) denotes the hypergeometric function (see Abramowitz & Stegun 1964 for the definition and properties).
Practical prescriptions for calculating Hansen coefficients have been provided by many authors (e.g. Branham 1990, Laskar 2005). For small eccentricities a power law expansion in \( e \) developed from (A2) may be used. In this paper we have followed the notation of Ivanov & Papaloizou (2004) who give a useful prescription for calculating \( \phi_{k,m} \) for eccentricities \( e > 0.2 \). Inserting the expansion (A1) into the expression (19) for \( U \), given in Section 4.1 we obtain in a form appropriate to the orbit frame

\[
U = -GM_p \left( \frac{4\pi r^2}{5a^3} \right) \sum_{n=0,|2|}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{k,n} Y_{2,m}(\theta', \phi') Y_{2,m}(\pi/2,0) \exp(ikn_o t - m\varpi) \tag{A5}
\]

### A1 The potential expressed in the stellar frame

In the stellar frame \((X, Y, Z)\) the associated spherical coordinate system is \((r, \theta, \phi)\). Spherical harmonics in the orbit frame are connected to those in the stellar frame through a relation of the form

\[
Y_{2,m}(\theta', \phi') = \sum_{n=-2}^{n=2} D_{n,m}^{(2)}(0, \beta, \gamma) Y_{2,n}(\theta, \phi), \tag{A6}
\]

where the coefficients (Wigner matrix elements) \( D_{n,m}^{(2)} \), being determined by the rotation defined by equation (1), depend on the orientation specifying angles \( \beta \) and \( \gamma \). It is important to note that the arguments of \( D_{n,m}^{(2)} \), hereafter taken as read, have the form they do because of our particular choice that the coordinate system \((X', Y', Z')\) is connected to the \((X, Y, Z)\) system by rotation through an angle \( \gamma \) followed by an angle \( \beta \) about fixed \( Z \) and \( Y \) axes, respectively (see Fig. 1). Accordingly, we have

\[
D_{n,m}^{(2)} = \exp(-im\gamma) d_{n,m}^{(2)}(\beta) \tag{A7}
\]

where \( d_{n,m}^{(2)} \) is an element of Wigner’s (small) d-matrix and is real (see Ivanov & Papaloizou 2011). Thus we have the perturbing potential expressed in terms of coordinates in the stellar frame

\[
U = -GM_p \left( \frac{4\pi r^2}{5a^3} \right) \sum_{n=-2}^{n=2} Y_{2,n}(\theta, \phi) \sum_{m=0,|2|}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{k,m} D_{n,m}^{(2)} Y_{2,m}(\pi/2,0) \exp(ikn_o t - m\varpi) \tag{A8}
\]

For convenience we write (A8) in the more compact form

\[
U = -GM_p \left( \frac{4\pi r^2}{5a^3} \right) \sum_{n=-2}^{n=2} Y_{2,n}(\theta, \phi) F_n(t) \equiv r^2 \sum_{n=-2}^{n=2} A_n Y_{2,n}(\theta, \phi), \tag{A9}
\]

with \( A_n = -4\pi GM_p F_n / (5a^3) \) and

\[
F_n(t) = \sum_{m=0,|2|}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{k,m} D_{n,m}^{(2)} Y_{2,m}(\pi/2,0) \exp(i(kn_o t - m\varpi)) \tag{A10}
\]
Making use of the Fourier expansion given by equation (A1) we have the alternative expression

\[ F_n(t) = \frac{a^3}{R^3} \sum_{n=0,[0]} D^{(2)}_{n,m} Y_{2,m}(\pi/2, 0) \exp(-im\Phi) \]  

(A11)

APPENDIX B: CALCULATION OF THE TORQUE IN TERMS OF THE INDUCED LAGRANGIAN DISPLACEMENT

The torque acting on the star, \( T \), is given by

\[ T = -\int_V \rho' \mathbf{r} \times \nabla U dV \]  

(B1)

where \( \rho' = -\nabla \cdot (\rho \xi) \) is the density response and the integral is taken over the volume of the star. The components of the operator \( \mathbf{r} \times \nabla \equiv \hat{\Phi} \) in the \( (X,Y,Z) \) coordinate system are

\[ \hat{\Phi} = \left( -\cot \theta \cos \phi \frac{\partial}{\partial \phi} - \sin \phi \frac{\partial}{\partial \theta}, -\cot \theta \sin \phi \frac{\partial}{\partial \phi} + \cos \phi \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right) \]  

(B2)

We find it convenient to introduce the so-called cyclic coordinates \( X^\pm = \pm \frac{1}{\sqrt{2}}(X \mp iY) \), \( X^0 = Z \)

and utilise the known expressions for the action of the above operator on spherical harmonics expressed in them. The components of \( \hat{\Phi} \) in these coordinates are given by

\[ \hat{\Phi}^+ = \frac{1}{\sqrt{2}} \left( \cot \theta e^{-i\phi} \frac{\partial}{\partial \phi} + i e^{-i\phi} \frac{\partial}{\partial \theta} \right) \]  

(B3)

and

\[ \hat{\Phi}^- = \frac{1}{\sqrt{2}} \left( -\cot \theta e^{i\phi} \frac{\partial}{\partial \phi} + i e^{i\phi} \frac{\partial}{\partial \theta} \right) \]  

(B4)

with

\[ \hat{\Phi}^0 = \frac{\partial}{\partial \phi} \]  

(B5)

In terms of these quantities, the components in the \( (X,Y,Z) \) coordinate system can be written as

\[ \hat{\Phi} = \left( \frac{1}{\sqrt{2}} (\hat{\Phi}^- - \hat{\Phi}^+) , -\frac{1}{\sqrt{2}} (\hat{\Phi}^+ + \hat{\Phi}^-) , \hat{\Phi}^0 \right) \]  

(B6)

Expressing \( \rho' \) in terms of \( \xi \), (B1) may be written

\[ T = \int_V \nabla \cdot (\rho \xi)(\mathbf{r} \times \nabla U) dV = \int_V \nabla \cdot (\rho \xi\hat{\Phi}) U dV, \]  

(B7)

where we remark that because \( \xi \) is real we may take its complex conjugate in (B7). Considering only the contribution of the terms with the pair of values \( \pm n \) for a specified \( n \) in the sum (22) for the forcing potential and then summing over \( n \geq 0 \) (it turns out that there is no contribution from \( n = 0 \) in the case that follows), we obtain from equations (B1), (22), (B5), (B6) and (B7) the \( Z \) component of \( T \) as

\[ T^z = -2 \sum_{n=0}^{n=2} \mathcal{R} \left[ \left( \frac{4\pi GM_p}{5a^3} \right) \int_V \nabla \cdot (\rho \xi_n^*) \left( inr^2Y_{2,n}(\theta, \phi)F_n(t) \right) dV \right], \]  

(B8)
where we recall that $R$ here denotes that the real part is to be taken. We remark that on account of the form of the forcing potential given by equation (22) and, accordingly, the response to it being real (see the discussion just below that equation), the expressions in the summation associated with a particular $n$, whose real part is to be taken to give (B8), along with similar expressions occurring in the context of other torque components below, are such that for $n \to -n$ the complex conjugate is obtained. These contributions must be added in order to get the final total torque component, which is therefore, accordingly, real. Thus, we take twice the real part of the expression for $n > 0$.

To obtain the total torque component, we then sum over only positive values of $n$.

Turning to the $X$ and $Y$ components of the torque, with the help of (B6) and (B7) we obtain

$$T_x = -\frac{4\pi GM_p}{5a^3} \sum_{n=-2}^{n=2} \int_V \nabla \cdot \left( \rho \left( \sum_{n'=n}^{n'=2} \xi_{n'}^* \right) \right) \left( \frac{1}{\sqrt{2}}(\hat{\Phi}^- - \hat{\Phi}^+)r^2Y_{2,n}(\theta, \phi)F_n(t) \right) dV , \quad (B9)$$

and

$$T_y = \frac{4\pi GM_p}{5a^3} \sum_{n=-2}^{n=2} \int_V \nabla \cdot \left( \rho \left( \sum_{n'=n}^{n'=2} \xi_{n'}^* \right) \right) \left( \frac{i}{\sqrt{2}}(\hat{\Phi}^+ + \hat{\Phi}^-)r^2Y_{2,n}(\theta, \phi)F_n(t) \right) dV , \quad (B10)$$

We now take note of the known result that for general $j$ we have

$$\hat{\Phi}^\pm Y_{j,n} \equiv \Phi^\pm_{j,n} = \pm \frac{i}{\sqrt{2}}Y_{j,n\pm1}\sqrt{(j \mp n)(j \pm n + 1)} \quad (B11)$$

and of course in our case we need to specify $j = 2$ and this is implied from now on below. Making use of (B11) to express the action of the components of the operator $\hat{\Phi}$ in equations (B9) and (B10), these then become

$$T_x = -\left( \frac{2\pi GM_p}{5a^3} \right) \sum_{n=-2}^{n=2} \int_V \nabla \cdot \left( \rho \left( \sum_{n'=n}^{n'=2} \xi_{n'}^* \right) \right) \left( i^2(Y_{j,n+1}\sqrt{(j-n)(j+n+1)} + Y_{j,n-1}\sqrt{(j+n)(j-n+1)})F_n \right) dV , \quad (B12)$$

and

$$T_y = \left( \frac{2\pi GM_p}{5a^3} \right) \sum_{n=-2}^{n=2} \int_V \nabla \cdot \left( \rho \left( \sum_{n'=n}^{n'=2} \xi_{n'}^* \right) \right) \nabla \left( r^2(Y_{j,n-1}\sqrt{(j+n)(j-n-1)} - Y_{j,n+1}\sqrt{(j-n)(j+n+1)})F_n(t) \right) dV . \quad (B13)$$

Remarking that the azimuthal mode number associated with $\xi_{n'}$ is $n'$ and this has to be the same as that associated with the spherical harmonic with which it combines, we see that (B12) and (B13) may be also be written in the form
On the evolution of a binary system due to quasi-stationary tides

\( T^x = - \left( \frac{2\pi GM_p}{5a^3} \right) \sum_{n=-2}^{n=2} \left[ \int_V \nabla \cdot (\rho \xi_{n+1}^*) \left( ir^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right. \\
+ \left. \int_V \nabla \cdot (\rho \xi_{n-1}^*) \left( ir^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right] (B14) \\

and

\( T^y = \left( \frac{2\pi GM_p}{5a^3} \right) \sum_{n=-2}^{n=2} \left[ \int_V \nabla \cdot (\rho \xi_{n-1}^*) \left( r^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right. \\
- \left. \int_V \nabla \cdot (\rho \xi_{n+1}^*) \left( r^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right] (B15) \\

Note that if \( n \to -n \) in each of the terms in the above sums, as remarked above the complex conjugates of the expressions are obtained. Thus, total torque components are obtained by taking twice the real parts and summing over \( n > 0 \) and then adding in the contribution for \( n = 0 \) which can be seen to be real. We thus write the torque components in the form

\( T^x = - \sum_{n=0}^{n=2} 2 \mathcal{R} \left[ \left( \frac{4\pi GM_p}{5a^3} \right) \int_V \nabla \cdot (\rho \xi_n^*) \left( ir^2 Y_{2,n} \phi F_n(t) \right) dV \right], \quad (B16) \)

\( \frac{5a^3 T^x}{2\pi GM_p} = \sum_{n=0}^{n=2} (\delta_0^n - 2) \mathcal{R} \left[ \int_V \nabla \cdot (\rho \xi_{n+1}^*) \left( ir^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right. \\
+ \left. \int_V \nabla \cdot (\rho \xi_{n-1}^*) \left( ir^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right] (B17) \)

and

\( \frac{5a^3 T^y}{2\pi GM_p} = \sum_{n=0}^{n=2} (2 - \delta_0^n) \mathcal{R} \left[ \int_V \nabla \cdot (\rho \xi_{n-1}^*) \left( r^2 Y_{j,n-1} \sqrt{(j+n)(j-n+1)} F_n \right) dV \right. \\
- \left. \int_V \nabla \cdot (\rho \xi_{n+1}^*) \left( r^2 Y_{j,n+1} \sqrt{(j-n)(j+n+1)} F_n \right) dV \right], \quad (B18) \)

where \( \delta_0^n \) is the Kronecker \( \delta \).

We comment that when calculating the \( X \) and \( Y \) components of the torque, the above equations imply that the azimuthal mode number of a significant response has to differ from that of the original forcing potential by \( \pm 1 \). Apart from this, the expressions consist of contributions that are similar in form to that given by equation (B8) for the component of the torque in the \( Z \) direction.
APPENDIX C: USEFUL SUM RULES, TIME INTEGRALS TAKEN AROUND THE ORBIT AND THEIR USE TO CALCULATE TIME AVERAGED COMPONENTS OF THE TORQUE AND THE RATE OF CHANGE OF ORBITAL ENERGY

We find it convenient to make use of several sum rules derived from Parseval’s theorem when evaluating sums of the form (88). To obtain these we recall the Fourier expansion (20) defining $\phi_{kn}$, which may also be written in the form

$$F_n(t) = \frac{\exp(-in(\Phi - \varpi))a^3}{R^3} = \sum_{k=-\infty}^{\infty} \phi_{k,n} \exp(ikn_o t), \quad (C1)$$

With the help of equation (30) we then reconstruct (84) in the form

$$A_n(t) = \sum_{k=-\infty}^{\infty} A_{n,k} \exp(ikn_o t) = -\frac{4\pi GM_p}{5a^3} \sum_{m=0,|2|} F_n D_{n,m}^{(2)} Y_{2,m}(\pi/2,0) \exp(-im\varpi), \quad (C2)$$

By considering the time derivative we also have

$$\frac{dA_n(t)}{dt} = in_o \sum_{k=-\infty}^{\infty} k A_{n,k} \exp(ikn_o t), \quad (C3)$$

From these two expressions we obtain the sum rules

$$\frac{n_o}{2\pi} \int A_{n_1}^*(t) A_{n_2}(t) dt = \sum_{k=-\infty}^{\infty} A_{n_1,k}^* A_{n_2,k}, \quad (C4)$$

$$\frac{n_o}{2\pi} \int A_{n_1}^*(t) \frac{dA_{n_2}(t)}{dt} dt = in_o \sum_{k=-\infty}^{\infty} k A_{n_1,k}^* A_{n_2,k}, \quad (C5)$$

$$\frac{n_o}{2\pi} \int \frac{dA_{n_1}(t)}{dt} \frac{dA_{n_2}(t)}{dt} dt = n_o^2 \sum_{k=-\infty}^{\infty} k^2 A_{n_1,k}^* A_{n_2,k}, \quad (C6)$$

Then, making use of (C4) – (C6) together with (C2) we readily obtain the sum rules

$$\sum_{k=-\infty}^{\infty} A_{n_1,k}^* A_{n_2,k} = \left(\frac{4\pi GM_p}{5}\right)^2 \sum_{m=-4}^{m=4} \exp(-im(\varpi + \gamma)) W_{n_1,n_2,m}^{(0)} \frac{n_o}{2\pi} \int \cos(m(\Phi - \varpi)) R^{-6} dt, \quad (C7)$$

where

$$W_{n_1,n_2,m}^{(j)} = \sum_{n=\max(-2,-2-m)}^{n=\min(2,2-m)} (n + m/2)^j d_{n_2,n+m}^{(2)} d_{n_1,n}^{(2)} Y_{2,n+m}(\pi/2,0) Y_{2,n}(\pi/2,0), \quad (C8)$$

has the property, as can be verified directly, while noting that only even values of $n$ and $m$ give
non zero contributions to the sum and all the factors in (C8) are real, that $W_{n_1,n_2,m}^{(j)} = W_{n_2,n_1,-m}^{(j)}$.

$$\sum_{k=-\infty}^{k=\infty} k A_{n_1,k}^{*} A_{n_2,k} = -\left(\frac{4\pi GM_p}{5}\right)^2 \sqrt{G(M_p + M_s)a(1 - e^2)} \times$$

$$\sum_{m=-4}^{m=4} \exp(-im(\varpi + \gamma)) W_{n_1,n_2,m}^{(1)} \int \frac{dt \cos(m(\Phi - \varpi))}{R^8},$$

$$\sum_{k=-\infty}^{k=\infty} k^2 A_{n_1,k}^{*} A_{n_2,k} = \left(\frac{4\pi GM_p}{5}\right)^2 \frac{1}{2\pi n_o} \times$$

$$\sum_{m=-4}^{m=4} \exp(-im(\varpi + \gamma)) \int \left(W_{n_1,n_2,m}^{(0)} p_0 + \left(W_{n_1,n_2,m}^{(2)} - \frac{m^2}{4} W_{n_1,n_2,m}^{(0)}\right) p_2\right) \cos(m(\Phi - \varpi))dt,$$

with

$$p_0 = \frac{d}{dt} \left(\frac{3dR/dt}{R^7}\right) + \frac{9(dR/dt)^2}{R^8},$$

$$p_2 = \frac{G(M_p + M_s)a(1 - e^2)}{R^{10}}.$$ (C11)

C1 Useful time integrals taken around the orbit

For standard Keplerian elliptical orbits we also make use of the time integrals taken around the orbit for $n$ equal to a positive integer $> 1$.

$$\int \frac{dt}{R^{2n}} = \frac{2\pi(2n - 2)!}{(a(1 - e^2))^{2n-2} \sqrt{G(M_p + M_s)a(1 - e^2)}} \sum_{k=0}^{n-1} \frac{e^{2k}}{(2n - 2 - 2k)!k!2^{2k}}.$$ (C12)

$$\int \frac{\cos(2(\Phi - \varpi))dt}{R^{2n}} = \frac{2\pi(2n - 2)!}{(a(1 - e^2))^{2n-2} \sqrt{G(M_p + M_s)a(1 - e^2)}} \sum_{k=1}^{n-1} \frac{e^{2k}}{(2n - 2 - 2k)!(k + 1)!(k - 1)!2^{2k+1}}.$$ (C13)

and

$$\int \frac{R^2 dt}{R^{2n}} = \frac{2\pi e^2(2n - 2)! \sqrt{G(M_p + M_s)a(1 - e^2)}}{(a(1 - e^2))^{2n}} \sum_{k=0}^{n-1} \frac{e^{2k}}{(2n - 2 - 2k)!k!(k + 1)!2^{2k+1}}.$$ (C14)

which can be used to complete the evaluation of integrals involving quantities such as $p_0$ and $p_2$.

By making use of the sum rules given by equations (C7) - (C10) together with the time integrals taken around the orbit expressed by equations (C12) - (C14) we can derive expressions for the time averages of the components of the torque given by equations (78) - (81), expressed as combinations of infinite sums by making use of equations (86) - (88), in closed form. In a similar way we may find the rate of change of orbital energy given by (83) with the help of the time derivative of (84).
Thus the $Z$ component of the torque is given by
\[ T^z = (4\pi Q_{eq} GM_p/(5\omega_{eq}))^2 \times \]
\[ \left( \delta_1 \frac{\sqrt{G(M_p + M_*)a(1 - e^2)}}{\pi} \sum_{m=0}^{m=4} (W_{1,1,m}^{(1)} + 2W_{2,2,m}^{(1)})(2 - \delta_0^m) \cos(m\hat{\omega}) \int \cos(m(\Phi - \varpi)) \frac{dt}{R^8} \right) - \delta_2 \left( n_o/\pi \right) \sum_{m=0}^{m=4} (W_{1,1,m}^{(0)} + 4W_{2,2,m}^{(0)})(2 - \delta_0^m) \cos(m\hat{\omega}) \int \cos(m(\Phi - \varpi)) \frac{dt}{R^6} \right), \tag{C15} \]
where $e$ is the orbital eccentricity and $\hat{\omega} = \varpi + \gamma$. Here we remark that only even values of $m$ contribute to the sums.

Similarly, working with $T = T^x - iT^y$ as given by (82) we find
\[ T = -n_o Q_{eq}^2 / (\pi \omega_{eq}^2) (4\pi GM_p/5)^2 \left( (\delta_2 - i\sigma\delta_3) \int (f_1 + if_2) R^{-6} dt - \frac{\sqrt{G(M_p + M_*)a(1 - e^2)}}{2n_o} (2\delta_1 - i\delta_3) \int (f_3 + if_4) R^{-8} dt \right), \tag{C16} \]
where
\[ f_1 = \sum_{m=0}^{m=4} \left( \frac{3}{2} (W_{1,2,m}^{(0)} + W_{2,1,m}^{(0)}) + \sqrt{3/8} (W_{1,0,m}^{(0)} + W_{0,1,m}^{(0)}) \right) \left( 2 - \delta_0^m \right) \cos(m\hat{\omega}) \cos(m(\Phi - \varpi)), \]
\[ f_2 = \sum_{m=0}^{m=4} \left( 3(W_{2,1,m}^{(0)} - W_{1,2,m}^{(0)}) - \sqrt{3/2} (W_{1,0,m}^{(0)} - W_{0,1,m}^{(0)}) \right) \sin(m\hat{\omega}) \cos(m(\Phi - \varpi)), \]
\[ f_3 = \sum_{m=0}^{m=4} \left( W_{1,2,m}^{(1)} + W_{2,1,m}^{(1)} + \sqrt{3/2} (W_{1,0,m}^{(1)} + W_{0,1,m}^{(1)}) \right) \left( 2 - \delta_0^m \right) \cos(m\hat{\omega}) \cos(m(\Phi - \varpi)), \]
and
\[ f_4 = 2 \sum_{m=0}^{m=4} \left( W_{2,1,m}^{(1)} - W_{1,2,m}^{(1)} - \sqrt{3/2} (W_{0,1,m}^{(1)} - W_{1,0,m}^{(1)}) \right) \sin(m\hat{\omega}) \cos(m(\Phi - \varpi)). \]

The rate of change of orbital energy is in turn given by
\[ \frac{dE_{orb}}{dt} = \frac{n_o Q_{eq}^2}{\pi \omega_{eq}^2} \left( \frac{4\pi GM_p}{5} \right)^2 \left( \frac{\delta_2 \sqrt{G(M_p + M_*)a(1 - e^2)}}{2n_o} \int f_5 R^{-8} dt - \delta_1 \left( 1/n_o \right) f_6 \right), \tag{C17} \]
where
\[ f_5 = \sum_{m=0}^{m=4} \left( W_{1,1,m}^{(1)} + 2W_{2,2,m}^{(1)} \right) \left( 2 - \delta_0^m \right) \cos(m\hat{\omega}) \cos(m(\Phi - \varpi)), \]
and
\[ f_6 = \sum_{m=0}^{m=4} \phi \left( \left( W_{2,2,m}^{(2)} + W_{1,1,m}^{(2)} + \frac{1}{2} W_{0,0,m}^{(2)} \right) - \frac{m^2}{4} (W_{2,2,m}^{(0)} + W_{1,1,m}^{(0)} + \frac{1}{2} W_{0,0,m}^{(0)}) \right) P_2 + \left( W_{0,0,m}^{(0)} + W_{1,1,m}^{(0)} + W_{0,0,m}^{(0)}/2 \right) P_0 \right) \left( 2 - \delta_0^m \right) \cos(m\hat{\omega}) \cos(m(\Phi - \varpi)) dt, \tag{C18} \]
where
\[ P_0 = \frac{d}{dt} \left( \frac{3dR/dt}{R^7} \right) + \frac{9(dR/dt)^2}{R^8}, \quad \text{and} \quad P_2 = \frac{G(M_p + M_*)a(1 - e^2)}{R^{10}}. \tag{C19} \]
C1.1 Torque and rate of change of orbital energy in terms of the inclination $\beta$

The components of the Wigner $d$ matrix are given in Appendix E and evaluation of relevant $W_{n_1,n_2}^{(j)}$ in Appendix F. Using results provided there the $Z$ component of the torque is found to be

$$T^z = \frac{6}{5} (Q_{eq} GM_p/\omega_{eq})^2 \left( \delta_1 \sqrt{G(M_p + M_*)a(1 - e^2)} \cos \beta \oint R^{-8} dt \right. $$

$$- \delta_2 \left( \frac{n_o}{2} \right) \left( (1 + \cos^2 \beta) \oint R^{-6} dt - \sin^2 \beta \cos 2\tilde{\omega} \int \cos(2(\Phi - \tilde{\omega}))R^{-6} dt \right) \right). \quad (C20)$$

Similarly, from Appendix F we find $f_1 = 15 \sin \beta \cos \beta (1 + \cos(2\tilde{\omega}) \cos(2(\Phi - \tilde{\omega})))/(16\pi)$, $f_2 = -15 \sin \beta \sin(2\tilde{\omega}) \cos(2(\Phi - \tilde{\omega}))/ (16\pi)$, $f_3 = 15 \sin \beta/(8\pi)$, and $f_4 = 0$. Accordingly,

$$T = -\frac{3n_o(GM_pQ_{eq})^2 \sin \beta}{5\omega_{eq}^2} \times \left( \delta_2 - i\sigma \delta_3 \right) \left( \frac{(\cos \beta \cos(2\tilde{\omega}) - i \sin(2\tilde{\omega})) \cos(2(\Phi - \tilde{\omega})) + \cos \beta)}{R^6} \int \frac{2}{R^8 dt} \right) \left( 2\delta_1 - i\delta_3 \right) \left( \frac{5}{8\pi} P_0 + \frac{15}{8\pi} P_2 \right). \quad (C21)$$

In addition, we find that $f_5 = 15 \cos \beta/(8\pi)$ and $f_6 = (5P_0 + 15P_2)/(8\pi)$, and thus the rate of change of orbital energy is given by

$$\frac{dE_{\text{orb}}}{dt} = \frac{n_o Q_{eq}^2}{\pi \omega_{eq}^2} \left( \frac{4\pi GM_p}{5} \right)^2 \delta_2 \sqrt{G(M_p + M_*)a(1 - e^2)} \frac{15}{8\pi} \cos \beta \int R^{-8} dt $$

$$- \delta_1 (1/n_o) \int \left( \frac{5}{8\pi} P_0 + \frac{15}{8\pi} P_2 \right) dt. \quad (C22)$$

C1.2 Evaluation of the time integrals around the orbit

The evaluation of the above quantities is completed with help of the integrals calculated above. Making use of (C12) - (C14) after taking note of (C19) to evaluate the integrals in (C20) - (C22), we obtain

$$T^z = T_* \left( 2\delta_1 \cos \beta \phi_1 - \delta_2 (1 - e^2)^{3/2} \left( (1 + \cos^2 \beta) \phi_2 - \sin^2 \beta \cos 2\tilde{\omega} \phi_3 \right) \right) \quad \text{and} \quad (C23)$$

$$T = T_* \sin \beta((2\delta_1 - i\delta_3) \phi_1 - (1 - e^2)^{3/2}(\delta_2 - i\sigma \delta_3) \left( \phi_2 + \phi_3 \cos(2\tilde{\omega}) \right) \cos \beta - i \sin(2\tilde{\omega}) \phi_3), \quad (C24)$$

The change of orbital energy is given by

$$\frac{dE_{\text{orb}}}{dt} = \dot{E}_* \left( \delta_2 \phi_1 \cos \beta - \frac{\delta_1}{(1 - e^2)^{3/2}} \phi_4 \right). \quad (C25)$$

Here

$$T_* = \frac{6\pi}{5} \left( \frac{GM_pQ_{eq}}{a^3(1 - e^2)^{3/2} \omega_{eq}} \right)^2 = \frac{3kq^2}{1 + q} \left( \frac{R_5^5}{a^5} \right) \frac{M_* n_o^2 a^2}{(1 - e^2)^6} \quad \text{and} \quad \dot{E}_* = 2n_o T_* \quad (C26)$$
respectively, represent typical values of the torque and rate of change of energy. Note that in the
second equality we have used equation (D7) in Appendix D with $N_0 = 1$ to relate $T_*$
to the apsidal motion constant. In addition,

\begin{align}
\phi_1 &= 1 + \frac{15}{2} e^2 + \frac{45}{8} e^4 + \frac{5}{16} e^6, \\
\phi_2 &= 1 + 3 e^2 + \frac{3}{8} e^4, \\
\phi_3 &= \frac{3}{2} e^2 + \frac{1}{4} e^4 \text{ and} \\
\phi_4 &= 1 + \frac{31}{2} e^2 + \frac{255}{8} e^4 + \frac{185}{16} e^6 + \frac{25}{64} e^8.
\end{align}

**APPENDIX D: PRECESSION DUE TO SECOND ORDER ROTATIONAL DISTORTION: A COMPARISON WITH THE FIRST ORDER CONTRIBUTION**

We here compare the component of the torque in the $Y$ direction that is $\propto \beta_* + 1/2$ and first order in the stellar rotation frequency to the conventional precessional torque generated by the coupling of the tidal forcing to the density perturbation produced by centrifugal distortion. The action of these two torques, which may be added together causes precession of the rotation axis but does not play any role in the dissipative tidal interaction.

We consider without loss of generality the simplified circumstance of an eccentric orbit inclined by an angle $\beta$ to the $(X,Y)$ plane with both the line of nodes and the line of apsides coinciding with the $Y$ axis in the stellar frame. But note that it is not difficult to show that the time averaged component of the potential arising from the companion that we require is independent of the longitude of the apsidal line. The effective perturbing potential arising from the companion that we need is the time averaged quadrupole component with $m = 1$ as it is this component that gives rise to a precessional torque to lowest order in the tidal perturbation. After performing a multipole expansion and taking a time average, this is given by

$$U = \frac{3GM_p}{2a(1-e^2)^{3/2}} \left( \frac{r}{a} \right)^2 \sin \beta \cos \beta \sin \theta \cos \theta \cos \phi$$

(D1)

The interaction with the axisymmetric density distribution resulting from rotational distortion produces a torque with $Y$ component

$$T_{SF}^y = - \int \rho (r \times \nabla U) y dV = - \frac{3\pi GM_p}{2a^3(1-e^2)^{3/2}} \sin \beta \cos \beta \int \rho (r, \theta) (3 \cos^2 \theta - 1) \sin \theta \theta^4 d\theta dr$$

(D2)

or equivalently noting that $\rho(r, \theta)$ can be replaced by $\rho'(r, \theta)$ being the response to the perturb-
ing potential (D1). In the case discussed here this is also equal to the equilibrium tide response determined by equation (36), thus \( \rho'(r, \theta) \equiv \rho'_{eq} \).

\[
T_{SF}^y = -\frac{3GM_p}{2a^3(1 - e^2)^{3/2}} \sqrt{\frac{4\pi}{5}} \sin \beta \cos \beta \int \rho'_{eq} r^2 Y_{2,0}(\theta) dV
\]  

(D3)

The centrifugal distorting potential is

\[
U = \frac{\Omega^2}{3} \sqrt{\frac{4\pi}{5}} r^2 Y_{2,0}(\theta) - \frac{\Omega^2_r r^2}{3}.
\]  

(D4)

The second spherically symmetric term plays no role in determining the torques acting and so may be dropped. Aligning with the notation in Section 5 (see e.g. equation (29)) we write

\[
U = A_0 r^2 Y_{2,0}(\theta) \quad \text{where} \quad A_0 = \frac{\Omega^2_r}{3} \sqrt{\frac{4\pi}{5}}.
\]  

(D5)

We then determine the equilibrium tide displacement response \( \xi_{eq} \) using the formalism of Section 5. Thus, we use equation (36) with the above forcing potential. From the appropriately adapted (53), noting that in this case the operator \( \hat{L}_1 \equiv 0 \), one finds that

\[
A_0 \int r^2 Y_{2,0}(\theta) \rho'_{eq} dV \frac{N_0 \omega^2_{eq}}{1/2} = -1
\]  

(D6)

We note in passing that the integral in the above equation is proportional to the gravitational potential perturbation at the stellar surface and the ratio of this to the forcing potential there is defined to be twice the apsidal motion constant \( k_2 \). Accordingly, we have

\[
2k_2 = \frac{4\pi GQ^2_{eq}}{5N_0 \omega^2_{eq} R^5_\ast} = \frac{4\pi Q^2}{5} \left( \frac{GM_\ast}{R^5_\ast \omega^2_{eq}} \right),
\]  

(D7)

where \( Q_{eq} \) has been expressed in terms of its dimensionless form \( \tilde{Q} \) with \( N_0 \) chosen to be unity in the second expression on the right.

Using (D6) together with (D3) we obtain

\[
T_{SF}^y = \frac{3A_0 GM_p}{2a^3(1 - e^2)^{3/2} \omega^2_{eq} N_0} \sqrt{\frac{4\pi}{5}} \sin \beta \cos \beta \left( \int \rho'_{eq} r^2 Y_{2,0}(\theta) dV \right)^2.
\]  

(D8)

In this form we remark that \( \xi_{eq} \) can be rescaled (normalised) such that \( N_0 = 1 \). Noting that last term in brackets on the right hand side of (D8) is \( Q^2_{eq} \) evaluated with \( \xi_{eq} \) (see equation (59)) we obtain

\[
T_{SF}^y = \frac{2\pi GM_p \Omega^2 \tilde{Q}^2_{eq}}{5a^3(1 - e^2)^{3/2} \omega^2_{eq}} \sin \beta \cos \beta.
\]  

(D9)

This torque can be added to \( T^y \) in the third equation in the set (15) which reads

\[
d\alpha/dt = -d\alpha_r/dt = T^y J/(LS \sin \beta)
\]

in order to include the effects of second order centrifugal distortion.
APPENDIX E: ELEMENTS OF THE WIGNER \( d \) MATRIX

These can all be obtained from (see e.g. Khersonskii, Moskalev & Varshalovich (1988))

\[
\begin{align*}
    d_{2,2} &= \frac{1}{4}(1 + \cos \beta)^2, \\
    d_{2,1} &= -\frac{1}{2} \sin \beta (1 + \cos \beta), \\
    d_{2,0} &= \sqrt{\frac{3}{8}} \sin^2 \beta, \\
    d_{2,-1} &= -\frac{1}{2} \sin \beta (1 - \cos \beta), \\
    d_{2,-2} &= \frac{1}{4}(1 - \cos \beta)^2, \\
    d_{1,1} &= \frac{1}{2}(2 \cos^2 \beta + \cos \beta - 1), \\
    d_{1,0} &= -2 \sqrt{\frac{3}{8}} \sin \beta \cos \beta, \\
    d_{1,-1} &= \frac{1}{2}(-2 \cos^2 \beta + \cos \beta + 1), \\
    d_{0,0} &= \frac{1}{2}(3 \cos^2 \beta - 1). 
\end{align*}
\]

(E1)

Note that for ease of notation we have dropped the superscript 2 and that components not listed can be obtained from those listed by making use of the relations \( d_{n_1,n_2}(\beta) = (-1)^{n_1-n_2}d_{-n_1,-n_2}(\beta) \) and \( d_{n_1,n_2}(\beta) = d_{n_2,n_1}(-\beta) \).

APPENDIX F: QUANTITIES REQUIRED FOR TORQUE AND RATE OF CHANGE OF ORBITAL ENERGY EVALUATION

We here give explicit expressions for some combinations of Wigner matrix elements that are required for evaluation of torques and the rate of change of orbital energy.

F0.1 Terms associated with \( m = 4 \)

For \( m = 4 \), we note that from (C8) we have

\[
W^{(0)}_{n_1,n_2,4} = \frac{15}{32\pi} d_{n_2,2} d_{n_1,-2},
\]

(F1)

while for \( j > 0 \), \( W^{(j)}_{n_1,n_2,4} = 0 \) identically. From (F1) we find

\[
W^{(0)}_{1,1,4} = -\frac{15}{128\pi} \sin^4 \beta, \quad W^{(0)}_{2,2,4} = -W^{(0)}_{1,1,4}/4 \quad \text{and} \quad W^{(0)}_{0,0,4} = -3W^{(0)}_{2,2,4}/2.
\]
In addition we find
\[ W_{2,1,2}^{(0)} = -\frac{1}{4} \sqrt{\frac{8}{3}} W_{1,0,4}^{(0)} = \frac{15}{256\pi} \sin^3 \beta (1 - \cos \beta) \quad \text{and} \]
\[ W_{1,2,4}^{(0)} = -\frac{1}{4} \sqrt{\frac{8}{3}} W_{0,1,4}^{(0)} = -\frac{15}{256\pi} \sin^3 \beta (1 + \cos \beta). \]
Using the above results it is readily found that through a series of cancellations there are no contributions to \( f_1 - f_6 \) and hence the orbital evolution equations proportional to \( \cos(4\pi) \) or \( \sin(4\pi) \).

**F0.2 Terms associated with \( m = 2 \)**

For \( m = 2 \), we note that from (C8) we have
\[ W_{n_1,n_2,2}^{(j)} = -\frac{5}{16\pi} \sqrt{\frac{3}{2}} (-1)^j d_{n_2,0} d_{n_1,-2} + d_{n_2,2} d_{n_1,0}. \]
(F2)
Thus, coefficients with \( j = 2 \) are identical to those with \( j = 0 \) and so need not be considered separately. From (F2) we obtain considering first \( j = 1 \)
\[ W_{1,1,2}^{(1)} = -2W_{2,2,2}^{(1)} = \frac{15}{32\pi} \sin^2 \beta \cos \beta, \]
\[ W_{1,0,2}^{(1)} = -\sqrt{\frac{2}{3}} W_{2,1,2}^{(1)} = \frac{5}{32\pi} \sqrt{\frac{3}{8}} \sin \beta (1 - \cos \beta)(1 + 3 \cos \beta) \quad \text{and} \]
\[ W_{0,1,2}^{(1)} = -\sqrt{\frac{2}{3}} W_{1,2,2}^{(1)} = \frac{5}{32\pi} \sqrt{\frac{3}{8}} \sin \beta (1 + \cos \beta)(1 - 3 \cos \beta). \]
Considering the case \( j = 0 \), from (F2) we find
\[ W_{1,1,2}^{(0)} = \frac{15}{32\pi} \sin^2 \beta \cos^2 \beta, \quad W_{2,2,2}^{(0)} = -\frac{15}{128\pi} \sin^2 \beta (1 + \cos^2 \beta), \quad W_{0,0,2}^{(0)} = -\frac{15}{64\pi} \sin^2 \beta (3 \cos^2 \beta - 1), \]
\[ W_{1,0,2}^{(0)} = \frac{5}{32\pi} \sqrt{\frac{3}{8}} \sin \beta (1 - \cos \beta)(6 \cos^2 \beta + 3 \cos \beta - 1), \]
\[ W_{0,1,2}^{(0)} = \frac{5}{32\pi} \sqrt{\frac{3}{8}} \sin \beta (1 + \cos \beta)(1 + 3 \cos \beta - 6 \cos \beta^2), \]
\[ W_{2,1,2}^{(0)} = -\frac{15}{128\pi} \sin \beta (1 + \cos^2 \beta - 2 \cos^3 \beta) \quad \text{and} \]
\[ W_{1,2,2}^{(0)} = \frac{15}{128\pi} \sin \beta (1 + \cos \beta)(1 - \cos \beta + 2 \cos^2 \beta). \]
Using the above results the contributions to \( f_1 - f_6 \) and hence the orbital evolution equations that are \( \propto \cos 2\pi \) and \( \sin 2\pi \) are readily found.

**F0.3 Terms associated with \( m = 0 \)**

For \( m = 0 \), from (C8) we have
\[ W_{n_1,n_2,0}^{(0)} = \frac{15}{32\pi} (d_{n_2,2} d_{n_1,2} + d_{n_2,2} d_{n_1,-2}) + \frac{5}{16\pi} d_{n_2,0} d_{n_1,0}, \]
\[ W_{n_1,n_2,0}^{(1)} = \frac{15}{16\pi} (d_{n_2,2} d_{n_1,2} - d_{n_2,2} d_{n_1,-2}) \quad \text{and} \quad W_{n_1,n_2,0}^{(2)} = \frac{15}{8\pi} (d_{n_2,2} d_{n_1,2} + d_{n_2,2} d_{n_1,-2}). \]
The above expressions lead to

\[ W_{1,1,0}^{(0)} = \frac{15}{64\pi} \sin^2 \beta (1 + 3 \cos^2 \beta), \quad W_{1,0,0}^{(0)} = \frac{5}{32\pi} \sqrt{\frac{3}{8}} \sin \beta \cos \beta (5 - 9 \cos^2 \beta), \]

\[ W_{1,2,0}^{(0)} = \frac{15}{128\pi} \sin \beta \cos \beta (1 + 3 \cos^2 \beta), \quad W_{2,2,0}^{(0)} = \frac{15}{256\pi} (3 + 2 \cos^2 \beta + 3 \cos^4 \beta), \]

\[ W_{0,0,0}^{(0)} = \frac{5}{128\pi} (11 - 3 \cos^2 \beta - 27 \sin^2 \beta \cos^2 \beta), \quad W_{2,0,0}^{(1)} = \frac{15}{32\pi} \cos \beta (1 + \cos^2 \beta), \]

\[ W_{1,1,0}^{(1)} = \frac{15}{16\pi} \sin^2 \beta \cos \beta, \quad W_{1,0,0}^{(1)} = \frac{15}{16\pi} \sqrt{\frac{3}{8}} \sin^3 \beta, \quad W_{2,1,0}^{(1)} = \frac{15}{64\pi} \sin \beta (1 + 3 \cos^2 \beta), \]

\[ W_{1,1,0}^{(2)} = \frac{15}{16\pi} \sin^2 \beta (1 + \cos^2 \beta), \quad W_{0,0,0}^{(2)} = \frac{45}{32\pi} \sin^4 \beta, \quad \text{and} \quad W_{2,2,0}^{(2)} = \frac{15}{64\pi} (1 + \cos^4 \beta + 6 \cos^2 \beta). \]

From these one can verify the identities

\[ W_{2,2,0}^{(0)} + W_{1,1,0}^{(0)} + W_{0,0,0}^{(0)} / 2 = \frac{5}{8\pi}, \]

\[ W_{2,2,0}^{(2)} + W_{1,1,0}^{(2)} + W_{0,0,0}^{(2)} / 2 = \frac{15}{8\pi}, \]

and

\[ 2W_{2,2,0}^{(1)} + W_{1,1,0}^{(1)} = \frac{15}{8\pi} \cos \beta. \]

Recalling the symmetry condition that \( W_{n_1,n_2,0}^{(j)} = W_{n_2,n_1,0}^{(j)} \), the above coefficients expressed in terms of \( \beta \) can be used to evaluate the contribution of terms with \( m = 0 \) to the quantities \( f_1 - f_6 \) and thus the components of the torque and the rate of change of orbital energy.