Exact Solutions of the Relativistic Many-Body Problem

Domingo J. Louis-Martinez

Science One Program and
Department of Physics and Astronomy,
University of British Columbia
Vancouver, Canada

Abstract

Exact solutions of the relativistic many-body problem are presented.
In this paper we present exact solutions of the relativistic many-body problem in action at a distance electrodynamics[1].

Let us consider the Fokker action:

$$S = \sum_i m_i c^2 \int d\tau_i \eta_{\alpha\beta} \dot{z}_i^\alpha \dot{z}_i^\beta + \sum_i e_i \sum_{j \neq i} e_j \int \int d\tau_i d\tau_j \eta_{\alpha\beta} z_i^\alpha z_j^\beta \delta ((z_i - z_j)^2)$$ \hspace{1cm} (1)

In (1), \(m_i\) and \(e_i\) (\(i = 1, 2, ..., N\)) are the mass and electric charge of particle \(i\), \(z_i^\mu\) its world line, \(\dot{z}_i^\mu\) its four-velocity, \(\tau_i\) its proper time and \(c\) the speed of light. The metric tensor: \(\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)\). The Dirac delta function in (1) accounts for the interactions propagating at the speed of light forward and backward in time[1].

The Minkowski equations of motion for \(N\) interacting relativistic electric point-charges can be found from the action (1) using the variational principle[2]. We find:

$$m_i \ddot{z}_i^\mu = \sum_{s=-,+} K_i^{(s)\mu}$$ \hspace{1cm} (2)

where,

$$K_i^{(s)\mu} = e_i e_j \sum_{j \neq i} \frac{\dot{z}_i^\nu \dot{z}_j^\sigma}{|z_i^{(s)} - z_j^{(s)}|^3} \left[ z_i^{(s)} \ddot{z}_j^{(s)} - z_j^{(s)} \ddot{z}_i^{(s)} \right]$$

Here, \(z_j^{(s)} \equiv z_j^{(s)}(\tau_j^{(i,s)}), s = -, +,\) are the two roots of the equation:

$$\left( z_i(\tau_i) - z_j(\tau_j^{(i,s)}) \right)^2 = 0$$ \hspace{1cm} (4)

In this paper we show how the equations of motion (2,3) can be solved exactly, for any number of particles. In the solutions presented here, the particles follow concentric uniform circular orbits:

$$z_i^\mu = (ct, r_i \cos(\omega t + \phi_i), r_i \sin(\omega t + \phi_i), 0)$$ \hspace{1cm} (5)
Consider a particle $i$ at time $t$ in a reference frame $K$. Assume a signal travelling at the speed of light is emitted from particle $j$ and reaches $i$ at time $t$.

$t - t_j^{(i,-)}$ is the time it takes for a signal to travel forward in time at the speed of light from particle $j$ to particle $i$ in $K$.

$i_j^{(i,+)} - t$ is the time it takes for a signal to travel backward in time at the speed of light from particle $j$ to particle $i$ in $K$.

The four-vector force acting on charge $i$ depends on the state of motion of particle $i$ at time $t$ and, to account for the delay in the transmission of the interactions, on the states of motion of the remaining $N - 1$ particles at the past and future times $t_j^{(i,s)}(j \neq i, s = -, +)$.

In action-at-a-distance electrodynamics, the interactions carry energy and momentum to and from the particles and may simulate a field between them. However, in the absence of electrically charged particles outside the system, this fictitious field cannot carry energy or momentum into or away from the system. If there are no other electrically charged particles in the universe, the total energy and total momentum of a system of point particles are conserved (assuming that only electromagnetic interactions take place). In this paper we present exact formulas for the total mass and total angular momentum (in the center-of-momentum frame) for $N$ interacting relativistic point charges moving in concentric uniform circular orbits.

Consider the tangential and radial components of the net force acting on each particle. Assuming the angular velocity $\omega$ to be constant, we find that the tangential component of the net force acting on each charge $i$ ($i = 1, 2, ..., N$) vanishes:

\[
\sum_{j \neq i} e_j r_j \left[ \left( 1 - \frac{(r_i^2 + r_j^2 - r_i r_j \cos(\theta_{ij})) \omega^2}{c^2} \right) \sin(\theta_{ij}) - \frac{R_{ij}^{ret} \omega}{c} \left( \cos(\theta_{ij}) - \frac{r_i r_j \omega^2}{c^2} \right) \right] + \left( 1 - \frac{(r_i^2 + r_j^2 - r_i r_j \cos(\eta_{ij})) \omega^2}{c^2} \right) \sin(\eta_{ij}) + \frac{R_{ij}^{adv} \omega}{c} \left( \cos(\eta_{ij}) - \frac{r_i r_j \omega^2}{c^2} \right) = 0 \tag{6}
\]

where,
\[ \theta_{ij} = \phi_i - \phi_j + \frac{\omega R_{ij}^{ret}}{c} \]  
(7)

\[ \eta_{ij} = \phi_i - \phi_j - \frac{\omega R_{ij}^{adv}}{c} \]  
(8)

and,

\[ R_{ij}^{ret} = c \left( t - t_j^{(i,-)} \right) \]  
(9)

\[ R_{ij}^{adv} = c \left( t_j^{(i,+)} - t \right) \]  
(10)

In other words, the tangential component of the net retarded force acting on particle \( i \) cancels with the tangential component of the net advanced force acting on \( i \).

The net work done by the electric force acting on each charge \( i \) \((i = 1, 2, ..., N)\) is zero.

Notice that from simple geometrical considerations, \( R_{ij}^{ret} \) and \( R_{ij}^{adv} \) are determined by the equations:

\[ (R_{ij}^{ret})^2 = r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_{ij}) \]  
(11)

\[ (R_{ij}^{adv})^2 = r_i^2 + r_j^2 - 2r_i r_j \cos(\eta_{ij}) \]  
(12)

For the radial component of the net force acting on charge \( i \) we obtain:

\[
\frac{e_i}{2} \sum_{j \neq i} e_j \left[ \frac{(r_i - r_j \cos(\theta_{ij}))(1 - r_i^2 \omega^2)(1 - r_j^2 \omega^2)}{(R_{ij}^{ret} - \frac{r_i r_j \omega}{c} \sin(\theta_{ij}))^3} + \frac{\omega}{c} \left( -r_j \sin(\theta_{ij}) + \frac{r_i \omega}{c} R_{ij}^{ret} \right) \right] \\
+ \frac{(r_i - r_j \cos(\eta_{ij}))(1 - r_i^2 \omega^2)(1 - r_j^2 \omega^2)}{(R_{ij}^{adv} + \frac{r_i r_j \omega}{c} \sin(\eta_{ij}))^3} + \frac{\omega}{c} \left( r_j \sin(\eta_{ij}) + \frac{r_i \omega}{c} R_{ij}^{adv} \right) \right] \\
= - \frac{m_i r_i \omega^2}{\left(1 - \frac{r_i^2 \omega^2}{c^2}\right)^{\frac{3}{2}}} \]  
(13)
The total mass of the system can be found following the method of Landau and Lifshitz [3]. We find:

\[ M = \sum_{i=1}^{N} m_i \left(1 - \frac{r_i^2 \omega^2}{c^2}\right)^{\frac{1}{4}} \]  

(14)

As a conjecture, I present here a formula for the total angular momentum of the system (in the center-of-momentum inertial reference frame):

\[ L = -\frac{1}{4\omega} \sum_{i} \sum_{j \neq i} e_i e_j \left[ \frac{\left(1 - \frac{r_i r_j \omega^2}{c^2} \cos(\theta_{ij})\right)}{\left(R_{ij}^{ret} - \frac{r_i r_j \omega}{c} \sin(\theta_{ij})\right)} + \frac{\left(1 - \frac{r_i r_j \omega^2}{c^2} \cos(\eta_{ij})\right)}{\left(R_{ij}^{adv} + \frac{r_i r_j \omega}{c} \sin(\eta_{ij})\right)} \right] \]

(15)

Formula (15) gives the correct expression for the total angular momentum of N interacting point-charges in uniform circular motion if the velocities of all the particles are small compared with the velocity of light (I have checked this up to terms of second order, \(\frac{v^2}{c^2}\)).

For \(N = 2\), Eqs(6,13,14,15) reduce to the exact solutions found by Schild [4] for the electromagnetic two-body problem.

From (7,8,11,12) it follows that:

\[ R_{ji}^{adv} = R_{ij}^{ret} \]  

(16)

\[ \eta_{ji} = -\theta_{ij} \]  

(17)

Therefore, formula (15) can be written in a more compact form.

It is also not difficult to prove that Eqs (6) are not all independent. Indeed, from (16,17) it follows that:

\[
\sum_{i} \sum_{j \neq i} (e_i r_i)(e_j r_j) \left[ \left(1 - \frac{r_i^2 + r_j^2 - r_i r_j \cos(\theta_{ij})}{c^2} \right) \frac{\sin(\theta_{ij})}{\left(R_{ij}^{ret} - \frac{r_i r_j \omega}{c} \sin(\theta_{ij})\right)^3} \right] + \\
\left[ \left(1 - \frac{r_i^2 + r_j^2 - r_i r_j \cos(\eta_{ij})}{c^2} \right) \frac{\sin(\eta_{ij})}{\left(R_{ij}^{adv} + \frac{r_i r_j \omega}{c} \sin(\eta_{ij})\right)^3} \right] \left(1 - \frac{r_i r_j \omega^2}{c^2} \cos(\eta_{ij}) - \frac{r_i r_j \omega}{c} \sin(\eta_{ij})\right) \]

(18)
identically vanishes (since the expression inside the bracket is antisymmetric).

Eqs (6.13) completely determine the \(2N - 1\) unknowns \(r_i (i = 1, 2, ..., N)\), \(\phi_i - \phi_{i-1}\)
\((i = 2, ..., N)\), as functions of the angular velocity \(\omega\).

Let us now consider two simple classes of solutions for \(N\) interacting point particles
moving along circular orbits:

Regular polygonal solutions with a nucleus:

Assume all the masses are equal \((m_i = m)\) and all charges are equal \((e_i = -e) (i = 1, 2, ..., N)\). Assume there is a central mass \(m_0\) with electric charge \(e_0 = Ne\).

The charges \(e_i (i = 1, 2, ..., N)\) are located at the vertices of a regular polygon:

\[
\phi_{j+1} - \phi_j = \frac{2\pi}{N}
\]

\[
\frac{m\omega^2 r^3}{e^2 \left(1 - \frac{r^2 \omega^2}{c^2}\right)^{\frac{3}{2}}} = \frac{N}{n} - \sum_{k=1}^{N-1} \left((1 + \frac{r^4 \omega^4}{c^4})(1 - \cos \gamma_k) + \frac{r^2 \omega^2}{c^2} \sin^2 \gamma_k - \frac{4 \omega}{c} \left(1 + \frac{r^2 \omega^2}{c^2}\right) \sin \gamma_k\right) (l_k r - \frac{r \omega}{c} \sin \gamma_k)^{\frac{3}{2}}
\]

where,

\[
\gamma_k = \frac{2\pi}{N} k + \frac{\omega l_k}{c}
\]

\[
l_k = \sqrt{2} r \left(1 - \cos (\gamma_k)\right)^{\frac{3}{2}}
\]

\((k = 1, 2, ..., N - 1)\)

For the total mass and total angular momentum we find:

\[
M = m_0 + N m \left(1 - \frac{r^2 \omega^2}{c^2}\right)^{\frac{1}{2}}
\]

\[
L = \frac{N^2 e^2}{\omega r} \left[1 - \frac{1}{2N} \sum_{k=1}^{N-1} \left(\frac{1 - \frac{r^2 \omega^2}{c^2} \cos \gamma_k}{l_k r - \frac{r \omega}{c} \sin \gamma_k}\right)\right]
\]

Regular polygonal solutions without a nucleus:
Assume all the masses are equal, \( m_i = m \), and the charges all have the same magnitude. Half of the particles are positively charged and the other half negatively charged: \( e_i = (-1)^i e \), \( i = 1, 2, \ldots, N \). Here we assume the number of particles \( N \) is even. The charges are located at the vertices of a regular polygon, with alternating signs (nearest neighbours attract each other, as their charges have opposite signs). We find for this class of solutions:

\[
\phi_{j+1} - \phi_j = \frac{2\pi}{N} \tag{25}
\]

\[
\frac{m\omega^2 r^3}{e^2 \left(1 - r^2\frac{\omega^2}{c^2}\right)^2} = \sum_{k=1}^{N-1} (-1)^{k+1} \frac{(1 + r^4\frac{\omega^4}{c^4})(1 - \cos \gamma_k) + r^2\frac{\omega^2}{c^2} \sin \gamma_k - \frac{l_k\omega}{e} \left(1 + r^2\frac{\omega^2}{c^2}\right) \sin \gamma_k}{\left(l_k - \frac{r\omega}{e} \sin \gamma_k\right)^3} \tag{26}
\]

where \( \gamma_k \) and \( l_k \) \((k = 1, \ldots, N - 1)\) are determined by the relations (21, 22).

For the total mass and total angular momentum we find:

\[
M = Nm \left(1 - \frac{r^2\omega^2}{c^2}\right)^{\frac{1}{2}} \tag{27}
\]

\[
L = \frac{Ne^2}{2\omega r} \sum_{k=1}^{N-1} (-1)^{k+1} \frac{1 - r^2\frac{\omega^2}{c^2} \cos \gamma_k}{\left(l_k - \frac{r\omega}{e} \sin \gamma_k\right)} \tag{28}
\]

References

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