Perturbative and Non-Perturbative Partial Supersymmetry Breaking: \( N = 4 \rightarrow N = 2 \rightarrow N = 1 \)

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ABSTRACT

We show the existence of a supersymmetry breaking mechanism in string theory, where \( N = 4 \) supersymmetry is broken \emph{spontaneously} to \( N = 2 \) and \( N = 1 \) with moduli dependent gravitino masses. The spectrum of the spontaneously broken theory with lower supersymmetry is in one-to-one correspondence with the spectrum of the heterotic \( N = 4 \) string. The mass splitting of the \( N = 4 \) spectrum depends on the compactification moduli as well as the three \( R \)-symmetry charges. We also show that, in string theory, chiral theories can be obtained after spontaneous breaking of extended supersymmetry. This was impossible at the level of field theory.

In the large moduli limit a restoration of the \( N = 4 \) supersymmetry is obtained. As expected the graviphotons and some of the gauge bosons become massive in \( N = 1 \) vacua. At some special points of the moduli space some of the \( N = 4 \) states with non-zero winding numbers and with spin 0 and 1/2 become massless chiral superfields of the unbroken \( N = 1 \) supersymmetry. Such vacua have a dual type II description, in which there are magnetically charged states with spin 0 and 1/2 that become massless. The heterotic-type II duality suggests some novel non-perturbative transitions on the type II side. Such transitions do not seem to have a geometric interpretation, since they relate type II vacua with symmetric worldsheet structure to asymmetric ones. The heterotic interpretation of such a transition is an ordinary higgsing of an \( SU(2) \) factor.

In the case of \( N = 4 \rightarrow N = 2 \), the perturbative \( N = 2 \) prepotential is determined by the perturbative \( N = 4 \) BPS states. This observation permits us to suggest a method to determine the exact non-perturbative prepotential of the effective \( N = 2 \) supergravity using the shifted spectrum of the non-perturbative BPS states of the underlying \( N = 4 \) theory.
1 Introduction

When a local symmetry is spontaneously broken, the physical states can be classified in terms of the unbroken phase spectrum and in terms of a well-defined mass splitting given in terms of vacuum expectation values of some fields, weighted by the charges of the broken symmetry. In the case of gauge symmetry breaking, the fields with non-zero vev’s are physical scalar fields, while in the case of supersymmetry breaking they are auxiliary fields. In extended supersymmetric theories (local or global), the supersymmetric vacua are degenerate, with zero vacuum energy for any vev of the moduli fields \((S,T)\). For instance, in the case of \(N=4\) supergravity based on a gauge group \(U(1)^6 \times G\), the space of the moduli fields is given in terms of \(2 + 6r\) physical scalars, which are coordinates of the coset space 
\[
\left[ \frac{SU(1,1)}{U(1)} \right]_S \times \left[ \frac{SO(6,r)}{SO(6) \times SO(r)} \right]_T
\]
(1.1)

\(r\) is the rank of the gauge group \(G\).

In an arbitrary point of the moduli space the gauge symmetry \(G\) is broken down to \(U(1)^r\) while at some special points of the moduli space the gauge symmetry is extended to some non-Abelian gauge group of the same rank due to the presence of some extra gauge multiplets that become massless at the special points of the moduli space.

In the heterotic \(N=4\) superstring solution obtained by \(T^6\) compactification of the ten dimensional superstring, the rank of the group \(r\) has a fixed value, \(r = 22\). In an arbitrary point of the moduli space the gauge group is \(U(1)^r\) and in special points the symmetry is extended as in field theory. There is however a fundamental difference between the field theory Higgs phenomenon and the string theory one. Indeed, if in an \(N=4\) field theory the gauge group is \(G = U(1)^6 \times SO(32)\) at any given point of the moduli space, then at any other point the remaining gauge symmetry \(G_T\) is always a subgroup of \(G\) with smaller dimensionality \(\dim(G_T) \leq \dim(G)\). On the contrary, in the string Higgs phenomenon, owing to the existence of winding states, we can connect gauge groups which are not subgroups of a larger gauge group. For instance, it is possible to connect continuously \(G = U(1)^6 \times SO(32)\) with \(G = U(1)^6 \times E_8 \times E_8\), as well as with the most symmetric one of the same rank, namely \(G = SO(44)\). Indeed, starting from a ten dimensional \(N=1\) supergravity theory with \(G = SO(32)\) or \(G = E_8 \times E_8\) after compactification in four dimensions the only possible \(N=4\) supergravity effective theories are based either to \(G = U(1)^6 \times SO(32)\) or \(G = U(1)^6 \times E_8 \times E_8\) (and their subgroups obtained with Higgs phenomenon). In string theory the gauge group can be further extended due to the existence of extra gauge bosons with non-zero winding numbers, which can become massless in special points of the moduli space.

When some auxiliary fields of the supergravity theories have a non-vanishing vev, some (or all) of the supersymmetries are spontaneously broken \([3]-[4]\). There is a consistent class of \(N=1, 2\) and \(N=4\) models defined in flat space-time in which all supersymmetries are broken or partially broken \([3]-[11]\). The most interesting case for our purposes is that in which there is one of the supersymmetries left unbroken. In that case we know that it is possible, in general, to have chiral representations of matter scalar multiplets which can describe the quarks and leptons of the supersymmetric standard model. All previous examples about the partial breaking of \(N=2\) to \(N=1\) supersymmetry was done at the field theory level \([12]\). In this work we will first show the extension of the partial spontaneous breaking at the perturbative string level and then we will generalize our result to the non-perturbative level using as a tool the heterotic–type II string duality of
the N=4 four dimensional superstrings [13, 14, 15].

In the process we will present evidence that non-perturbatively string ground states are of the spontaneously broken kind, the massive gravitini sometimes being solitons. We will also be able to find evidence for some novel non-perturbative transition in string theory between symmetric (geometric) and asymmetric (non-geometric) compactifications.

Moreover we construct examples of string ground states with spontaneously broken N=2→N=1 supersymmetry and chiral N=1 spectrum. This shows that, unlike field theory, chirality can appear during spontaneous breaking of extended supersymmetry in string theory.

The structure of the present paper is as follows: In section 2 we review the perturbative N=4 spectrum of string theories. Based on N=4 supergravity, we describe the (non-perturbative) BPS mass formula and analyse its moduli dependence.

In section 3 we describe heterotic ground states with N = 4 → N = 2 spontaneously broken supersymmetry. We give two complementary descriptions: one in terms of specific freely acting orbifolds and another in terms of generalized Lorentzian boosts of the unbroken theory at a special value in moduli space. We analyse and compare the behaviour of thresholds in such ground states to those of conventional $K_3 \times T^2$ compactifications.

In section 4 we discuss the spectrum of perturbative BPS states for heterotic ground-states with N = 4 → N = 2 spontaneously broken supersymmetry. We calculate the BPS multiplicities and by analysing the BPS mass formula we describe the points in moduli space where BPS states become massless.

In section 5 we describe (partial) spontaneous N = 4 → N = 1 supersymmetry breaking. We construct as an example a heterotic model that realizes this breaking pattern, and calculate its thresholds. Similarly in section 6 we describe spontaneous N = 2 → N = 1 supersymmetry breaking and provide as an example a heterotic ground state that realizes this possibility with chiral massless spectrum. Its gauge thresholds are also computed.

In section 7 we construct and analyse the type II duals of the heterotic models with spontaneously broken N = 4 → N = 2 spacetime supersymmetry. Evidence is presented for non-geometric non-perturbative transitions in the type II side which correspond to the ordinary Higgs effect on the heterotic side.

Finally in section 8 we present a conjecture on the non-perturbative BPS spectrum on the string ground states with N = 4 → N = 2 described in this paper.

2 Perturbative and Non-Perturbative N = 4 Mass Spectrum

Our starting point is a four dimensional heterotic N = 4 superstring solution. From the world-sheet viewpoint these theories are constructed by the following left- and right-moving degrees of freedom:

- Four left-moving non-compact super-coordinates, $X^\mu, \Psi^\mu$
- Six left-moving compactified supercoordinates, $\Phi^I, \Psi^I$
- The left-moving super-reparametrization ghosts, $b, c$ and $\beta, \gamma$
- Four right-moving coordinates, $\bar{X}^\mu$
• Six right-moving compactified coordinates, $\Phi^I$

• 32 right-moving fermions, $\bar{\Psi}^A$

• The right-moving reparametrization ghosts, $\bar{b}, \bar{c}$

In order to obtain consistent (without ghosts) $N = 4$ solution the left-moving fermions $\Psi^\mu$, $\Psi^I$ and the $\beta, \gamma$ ghosts must have the same boundary conditions. In that case the global existence of the left-moving spin-3/2 world-sheet supercurrent

$$T_F = \Psi^\mu \partial X^\mu + \Psi^I \partial \Phi^I$$

implies periodic boundary conditions for the compact and non-compact left-moving coordinates, $\Phi^I$, $X^\mu$. Modular invariance implies the right-moving coordinates $\bar{\Phi}^I$, $\bar{X}^\mu$ to be periodic as well. The solution with $G = U(1)^6 \times SO(32)$ is when the right-moving fermions $\bar{\Psi}^A$ have the same boundary conditions (periodic or antiperiodic), while the solution with $G = U(1)^6 \times E_8 \times E_8$ is when the $\bar{\Psi}^A = (\bar{\Psi}^{A_1}, \bar{\Psi}^{A_2})$ are in two groups of sixteen with the same boundary conditions. Starting either from the $G = U(1)^6 \times E_8 \times E_8$ solution or from the $G = U(1)^6 \times SO(32)$ we can obtain all others by deforming the momentum lattice of compactified bosons together with the charge lattice of the 32 fermions $\bar{\Psi}^A$.

The partition function of the heterotic $N = 4$ solutions in a generic point of the moduli space is well known and has the following expression:

$$Z(\tau, \bar{\tau}) = \frac{1}{\tau_2 |\eta|^4} \frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha + \beta + \alpha \beta} \frac{\eta^4(\tau)}{\eta^4(\tau)} \frac{\Gamma_{(6,22)}(\tau, \bar{\tau})}{\eta^6 \bar{\eta}^{22}}$$

(2.2)

where $\Gamma_{(6,22)}(\tau, \bar{\tau})$ denotes the partition function due to the compactified coordinates $\Phi^I$, $\bar{\Phi}^A$ and due to the sixteen right-moving $U(1)$ currents constructed with the fermions $\bar{\Psi}^I$

$$J^k = \bar{\Psi}^{2k} \bar{\Psi}^{2k}, \quad k = 1, 2, \cdots, 16. \quad (2.3)$$

$\Gamma_{(6,22)}(\tau, \bar{\tau})$ has in total $6 \times 22$ moduli parameters which correspond to $(1,1)$ marginal deformations of the world-sheet action:

$$\delta S^{2d} = \delta T_{IJ} \partial \Phi^I \partial \bar{\Phi}^J + Y^k_I \partial \Phi^I \bar{J}^k \quad (2.4)$$

In terms of the six dimensional backgrounds of the compactified space, the $T_{IJ}$ moduli are related to the internal background metric $G_{IJ}$ and the internal antisymmetric tensor $B_{IJ}$; $T_{IJ} = G_{IJ} + B_{IJ}$. The $Y^k_I$ moduli are the six dimensional internal gauge fields backgrounds which belong in the Cartan subalgebra of the ten-dimensional gauge group (either $E_8 \times E_8$ or $SO(32)$). From the four dimensional viewpoint the moduli $T_{IJ}$ and $Y^k_I$ correspond to the vev’s of massless scalar fields, members of the $N = 4$ vector supermultiplets.

The explicit form of the $N = 4$ heterotic partition function $\Gamma_{SO(32)}^{(6,22)}[T_{IJ}, Y^k_I]$ is:

$$\Gamma_{SO(32)}^{(6,22)}(T, Y)(\tau, \bar{\tau}) = \left( \frac{\text{det} G} {\tau_2^3} \right)^3 \sum_{m^I, n^I} \exp \left[ - \pi \frac{T_{IJ}(m^I + \tau n^I)(m^I + \bar{\tau} n^I)} {\tau_2} \right]$$

$$\times \frac{1}{2} \sum_{\gamma, \delta} \prod_{k=1}^{16} \exp \left[ \frac{i\pi}{4}(n^I Y^k_I Y^k_J m^J + 2 \delta Y^k_I n^I) \right] \tilde{\bar{\eta}} \left[ \frac{Y^k_I}{\delta + m^J} \right](\bar{\tau}) \quad (2.5)$$

When all $Y$–moduli are zero ($Y^k_I = 0$) then the gauge group is extended from $G = U(1)^{22}$ to $G = U(1)^6 \times SO(32)$. 

An alternative representation of $\Gamma_{(6,22)}^{E_8 \times E_8} (T, Y) (\tau, \bar{\tau})$ is the one in which, for $Y^k_I = 0$, the extended gauge symmetry is $G = U(1)^6 \times E_8 \times E_8$ instead of $U(1)^6 \times SO(32)$:

$$\Gamma_{(6,22)}^{E_8 \times E_8} (T, Y) (\tau, \bar{\tau}) = \frac{(\text{det} G) \bar{\tau}}{\pi^2} \sum_{m^I, n^I} \exp \left[ -\pi T_{IJ} \frac{(m^I + \tau n^I)(m^I + \tau n^I)}{\bar{\tau}} \right]$$

$$\times \frac{1}{2} \sum_{\gamma, \delta} \prod_{k=1}^8 \exp \left[ \frac{i \pi}{4} (n^I Y^k_I Y^k_J m^J + 2 \delta_1 Y^k_I n^I) \right] \bar{\theta} \left[ \frac{(1+n^I Y^k_I)}{\delta_1 + m^I Y^k_I} \right] (\bar{\tau})$$

$$\times \frac{1}{2} \sum_{\gamma, \delta} \prod_{k=9}^{16} \exp \left[ \frac{i \pi}{4} (n^I Y^k_I Y^k_J m^J + 2 \delta_2 Y^k_I n^I) \right] \bar{\theta} \left[ \frac{(1+n^I Y^k_I)}{\delta_2 + m^I Y^k_I} \right] (\bar{\tau})$$

Both the $SO(32)$ and $E_8 \times E_8$ representations are connected continuously by marginal deformations with the $G = SO(44)$ maximal gauge symmetry point:

$$\Gamma_{(6,22)}^{SO(44)} (\tau, \bar{\tau}) = \frac{1}{2} \sum_{\gamma, \delta} \bar{\theta} 6 \left[ \frac{\bar{\gamma}}{\gamma} \right] (\tau) \bar{\theta} 22 \left[ \frac{\bar{\gamma}}{\gamma} \right] (\bar{\tau})$$

Another useful representation of the $\Gamma_{(6,22)}$ is that of the lorentzian left- and right-momentum even self-dual lattice. This representation is obtained by performing Poisson resummation on $m^I$ using either $\Gamma_{(6,22)}^{SO(32)} (T, Y)$ or $\Gamma_{(6,22)}^{E_8 \times E_8} (T, Y)$:

$$\Gamma_{(6,22)} (P_I, \bar{P}_I, Q^k) = \sum_{P_I, \bar{P}_I, Q^k} \exp \left[ \frac{i \pi}{2} P_I G^{IJ} P_J - \frac{i \pi}{2} \bar{P}_I \bar{G}^{IJ} \bar{P}_J - i \pi \bar{\tau} \hat{Q}^k \hat{Q}^k \right]$$

with

$$\frac{1}{2} P_I G^{IJ} P_J - \frac{1}{2} \bar{P}_I \bar{G}^{IJ} \bar{P}_J - \hat{Q}^k \hat{Q}^k = \text{even integer}$$

In the above equations $G^{IJ}$ is the inverse of $G_{IJ}$; the lattice momenta $P_I$, $\bar{P}_I$, and the left charges $\hat{Q}^k$ are given in terms of the moduli parameters $G_{IJ}$, $B_{IJ}$ and $Y^k_I$ and in terms of the charges $(m_I, n^I, Q^k)$:

$$P_I = m_I + Y^k_I Q^k + \frac{1}{2} Y^k_I Y^k_J n^J + B_{IJ} n^J + G_{IJ} n^J$$

$$\bar{P}_I = m_I + Y^k_I \bar{Q}^k + \frac{1}{2} Y^k_I Y^k_J n^J + B_{IJ} n^J - G_{IJ} n^J$$

$$\hat{Q}^k = Q^k + Y^k_I n^J$$

All $N=4$ heterotic solutions are defined in terms of the vacuum expectation values of the moduli fields $(T_{IJ}, Y^k_I)$ and thus different solutions are connected to each other by a string-Higgs phenomenon. At some special points of the moduli space, we have extensions of the gauge group as in the effective $N = 4$ supergravity theories. In string theories, a further extension can occur due to the non-zero winding charges $(n^I)$ which can become massless in special points of the moduli space. Thus, in string theory, a large class of disconnected $N = 4$ supergravities are continuously related to one another due to the existence of the winding states. This precise fact is the origin of the perturbative string unification of all interactions in string theories.
There is another way of writing the left and right conformal weights for a (6,22) lattice. Introduce the $28 \times 28$ matrices

$$L = \begin{pmatrix} 0 & 1_6 & 0 \\ 1_6 & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix}$$

(2.11)

which is the $O(6,22)$ invariant metric, and

$$M = \begin{pmatrix} G^{-1} & G^{-1}C & G^{-1}Y^t \\ C^t G^{-1} & G + C^t G^{-1}C + Y^t Y & C^t G^{-1}Y^t + Y^t \\ Y G^{-1} & Y G^{-1}C + Y & 1 + Y G^{-1}Y^t \end{pmatrix}$$

(2.12)

with

$$C_{IJ} = B_{IJ} + \sum_k \frac{1}{2} Y_k^I Y_k^J$$

(2.13)

Notice that $M$ is a symmetric element of $O(6,22)$ so that $M'LM = MLM = L$, and that its inverse reads

$$M^{-1} = LML = \begin{pmatrix} G + C^t G^{-1}C + Y^t Y & C^t G^{-1} & -(C^t G^{-1} + 1)Y^t \\ G^{-1}C & G^{-1} & -G^{-1}Y^t \\ -Y(G^{-1}C + 1) & -YG^{-1} & 1 + Y G^{-1}Y^t \end{pmatrix}$$

(2.14)

Introduce also the 28-vector of charges

$$\vec{a} \equiv (\vec{m}, \vec{n}, \vec{Q})$$

(2.15)

Then

$$p_L^2 \equiv P_I G^{IJ} P_J = a_i (M + L)_{ij} a_j , \quad P_R^2 \equiv \bar{P}_I G^{IJ} \bar{P}_J + 2 \bar{Q}^k \bar{Q}^k = a_i (M - L)_{ij} a_j$$

(2.16)

Thus

$$\Gamma_{(6,22)} = \sum_{\vec{a} \in \mathcal{E}_{6,22}} q^{\vec{a}^T (M+L)\vec{a}} \bar{q}^{\bar{\vec{a}}^T (M-L)\bar{\vec{a}}}$$

(2.17)

where the lattice $\mathcal{E}_{6,22}$ is the even self-dual lattice $\mathcal{E}_{1,1}^6 \otimes \mathcal{E}_{16}$ and $\mathcal{E}_{16}$ is the Spin(32)/$\mathbb{Z}_2$ lattice. $\mathcal{E}_{16}$ can be described by the roots and spinor weight of SO(32). Let $u_i$ $i = 1, 2, \ldots, 16$ be an orthonormal basis, $u_i \cdot u_j = \delta_{ij}$. $u_i$ is a 16-dimensional vector with 1 in the $i$-th entry and 0 elsewhere. The SO(32) roots are:

$$f_i = u_i - u_{i+1} , \quad i = 1, 2, \ldots, 15 , \quad f_{16} = u_{15} + u_{16}$$

(2.18)

The spinor weight is $f_s = -(\sum_{i=1}^{16} u_i)/2$. We have

$$f_i \cdot f_j = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$$

(2.19)

$$f_i \cdot f_s = -\delta_{i,16} , \quad f_s \cdot f_s = 2$$

(2.20)

Then an arbitrary lattice vector can be written as

$$l = \sum_{i=1}^{16} n_i \ (f_i + \alpha f_s), \quad \alpha = 0, 1$$

(2.21)
The low energy effective N=4 supergravity is manifestly invariant under the full O(6,22) group, acting on the fields in the following way
\[ M \rightarrow \Omega M \Omega^T, \quad F_{\mu\nu} \rightarrow \Omega F_{\mu\nu} \quad (2.22) \]
where \( \Omega \in O(6,22) \), i.e. \( \Omega^T \Omega = I \). The full string theory is invariant under the discrete subgroup O(6,22,Z), and the (electric) charges transform as
\[ a_i \rightarrow \Omega_{ij} a_j \quad (2.23) \]

Furthermore the equations of motion and Bianchi identities are left invariant by the \( SL(2,R) \) transformation
\[ S \rightarrow \frac{aS + b}{cS + d}, \quad M \rightarrow M, \quad F_{\mu\nu} \rightarrow (c \text{Re} S + d) F_{\mu\nu}^i + c \text{Im} S (ML)_{ij} * F_{\mu\nu}^j \quad (2.24) \]
with \( ad - bc = 1 \). In particular, the transformation \( S \rightarrow -1/S \) interchanges electric and magnetic charges. It has been conjectured \([16]-[18]\), that a discrete subgroup \( SL(2,Z) \) of this continuous symmetry of the equations of motion of the effective theory is a (non-perturbative) symmetry of the full theory. For this to be true we will have to include in the theory states that carry both electric and magnetic charges. Magnetic or dyonic states are non-perturbative since the full perturbative heterotic spectrum is electrically charged.

Following references \([17, 19]-[22]\) let us parametrize the electric and magnetic charges in terms of the integer-valued 28-vectors \( \vec{\alpha}, \vec{\beta} \) and the moduli as follows:
\[ \vec{Q}_e = \frac{1}{\sqrt{2} \text{Im} S} M(\vec{\alpha} + \text{Re} S \vec{\beta} \), \quad \vec{Q}_m = \frac{1}{\sqrt{2}} L \vec{\beta} \quad (2.25) \]
This parametrization incorporates automatically the Dirac-Schwinger-Zwanziger-Witten quantization condition for dyons with a \( \theta \)-angle. The BPS mass formula can then be expressed in two equivalent ways
\[ M_{BPS}^2 = \frac{4 \text{Im} S}{\alpha^t M_+ \alpha} \left[ Q_e^t M_+ Q_e + Q_m^t M_+ Q_m + 2 \sqrt{(Q_e^t M_+ Q_e)(Q_m^t M_+ Q_m) - (Q_e^t M_+ Q_m)^2} \right] \]
\[ = \frac{1}{4 \text{Im} \tau} (\alpha^t + S \beta^t) M_+ (\alpha + \bar{S} \beta) + \frac{1}{2} \sqrt{(\alpha^t M_+ \alpha)(\beta^t M_+ \beta) - (\alpha^t M_+ \beta)^2} \quad (2.26) \]
with \( M_+ = M + L \) and \( \tilde{M}_+ = LM_+ L \). The square-root factor in the above expressions is proportional to the difference of the two central charges squared: depending on whether this vanishes or not, the representation preserves 1/2 or 1/4 of the supersymmetries, and is thus either short or intermediate. For perturbative BPS states of the heterotic string, \( \vec{\beta} = 0 \). Thus they belong to short multiplets. Their mass reads
\[ M_{BPS,\text{pert}}^2 = \frac{1}{4 \text{Im} S} \alpha^t M_+ \alpha = \frac{1}{4 \text{Im} S} \vec{p}_L^2 \quad (2.27) \]
The factor of \( \text{Im} S \) is there because masses are measured in units of \( M_{\text{Planck}} \).

The BPS mass formula is manifestly invariant under \( O(6,22,Z) \) acting on the fields as in \( (2.22) \) and on the charge vectors as
\[ \alpha \rightarrow \Omega \alpha, \quad \beta \rightarrow \Omega \beta, \quad (2.28) \]
It is also invariant under $SL(2, Z)_S$ acting on the fields as in (2.24) and on the vectors as
\[
\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}
\] (2.29)

It can be checked that (2.26) is $SL(2, Z)_S$ invariant. The spectrum of N=4 heterotic string theory is mapped to that of N=4 type II string theory under a transformation which is an avatar of string-string duality in 6-d [14]. The two low energy field theories are generically distinct, but they coincide when the R-R gauge fields are set to zero.

We will analyse the BPS mass formula in a subspace of the full (6,22) moduli space. In particular we will keep the 4 real moduli of a 2-torus as well as the 16 Wilson-line moduli which is an avatar of string-string duality in 6-d [14]. The two low energy field theories are generically distinct, but they coincide when the R-R gauge fields are set to zero.

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\[
\begin{align*}
G &= \frac{T_2 - \frac{1}{2U_2} \sum_i (\text{Im} W_i)^2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \\
B &= \left( T_1 - \frac{\sum_i \text{Re} W_i \text{Im} W_i}{2U_2} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{align*}
\] (2.30)

Define $e^{-K} = T_2 U_2 - \frac{1}{2} \sum_i \text{Im} W_i^2$. $K$ is the Kähler potential for the moduli.

The perturbative part (electric charges only) becomes the well-known SO(2,18) invariant mass formula
\[
M^2_{\text{pert}} = \frac{1}{4 S_2 \left( T_2 U_2 - \frac{1}{2} \sum_i \text{Im} W_i^2 \right)} \left| -m_1 U + m_2 \bar{T} n_1 + (T U - \frac{1}{2} \sum_i W_i^3) n_2 + W_i q_i^2 \right|^2
\] (2.32)

The quantity under the square root in (2.20) is a perfect square,
\[
\sqrt{\alpha \cdot M_+ \cdot \bar{\alpha} (\beta \cdot M_+ \cdot \bar{\beta}) - (\alpha \cdot M_+ \cdot \beta)^2} = |\alpha \cdot F \cdot \beta|
\] (2.33)

Then (we assume for the moment that the number in the absolute value is positive),
\[
M_+ + i F = e^K R
\] (2.34)

where $R$ is the following complex matrix
\[
R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}
\] (2.35)

$R_{11}$ is a $4 \times 4$ matrix given by
\[
R_{11} = \begin{pmatrix} |U|^2 & -U & -U \bar{T} & U \left( \frac{1}{2} \sum_i \bar{W}_i^2 - \bar{T} \bar{U} \right) \\ -\bar{U} & 1 & \bar{T} & \bar{T} \left( \bar{T} \bar{U} - \frac{1}{2} \sum_i \bar{W}_i^2 \right) \\ -U \bar{T} & \bar{T} & |T|^2 & T \left( \bar{T} \bar{U} - \frac{1}{2} \sum_i \bar{W}_i^2 \right) \\ U \left( \frac{1}{2} \sum_i W_i^2 - T U \right) & T U - \frac{1}{2} \sum_i W_i^2 & T \left( T U - \frac{1}{2} \sum_i W_i^2 \right) & |T U - \frac{1}{2} \sum_i W_i^2|^2 \end{pmatrix}
\] (2.36)

$R_{12}$ is a $4 \times 24$ matrix,
\[
R_{12} = \begin{pmatrix} \cdots & -U \bar{W}_i & \cdots \\ \cdots & \bar{W}_i & \cdots \\ \cdots & T W_i & \cdots \\ \cdots & (T U - \frac{1}{2} \sum_i W_i^2) W_i & \cdots \end{pmatrix}
\] (2.37)
while
\[ R_{21} = \overline{R}_{12} \]  
(2.38)

and finally \( R_{22} \) is a \( 24 \times 24 \) matrix
\[ R_{22,ij} = W_i \overline{W}_j \]  
(2.39)

The matrix \( R \) satisfies \( R^t = \overline{R} \) which implies (as it should) that \( M_+ \) is symmetric and \( F \) is antisymmetric.

Then the BPS formula can be cast in the form (when \( \alpha \cdot F \cdot \beta \) is positive)
\[ M_{BPS,+}^2 = \frac{\left( \alpha + S\beta \right) \cdot R \cdot \left( a + \overline{S}\beta \right)}{4 S_2 \left( T_2 U_2 - \frac{1}{2} \sum_i \text{Im} W_i^2 \right)} \]  
(2.40)

The matrix \( R \) has also the property:
\[ R = (-U, 1, T, TU - \frac{1}{2} \sum_i W_i^2, W_i) \otimes \left( \begin{array}{c} -\overline{U} \\ 1 \\ \overline{T} \\ \overline{TU} - \frac{1}{2} \sum_i \overline{W}_i^2 \\ \overline{W}_i \end{array} \right) \]  
(2.41)

which implies that \( \det R = 0 \) and
\[ M_{BPS,+}^2 = \frac{1}{4 S_2 \left( T_2 U_2 - \frac{1}{2} \sum_i \text{Im} W_i^2 \right)} \left| -m_1 U + m_2 + T n_1 + (TU - \frac{1}{2} \sum_i W_i^2) n_2 + W_i q^i + S[-\overline{m}_1 U + \overline{m}_2 + T \overline{n}_1 + \overline{n}_2 (TU - \frac{1}{2} \sum_i \overline{W}_i^2) + \overline{q}^i \overline{W}_i] \right|^2 \]  
(2.42)

Again, in the negative case \( S \to \overline{S} \).

Now the \( O(2,18,Z) \) (T-duality) transformations are:
\[ T \to T + 1, \quad W_i \to W_i \]  
(2.43)
\[ U \to U + 1, \quad W_i \to W_i \]  
(2.44)
\[ T \leftrightarrow U, \quad W_i \to W_i \]  
(2.45)
\[ T \to -\frac{1}{T}, \quad U \to U - \sum_i \frac{W_i^2}{2T}, \quad W_i \to \frac{W_i}{T} \]  
(2.46)
\[ U \to -\frac{1}{U}, \quad T \to T - \sum_i \frac{W_i^2}{2U}, \quad W_i \to \frac{W_i}{U} \]  
(2.47)
\[ W_i \to W_i + a_i U, \quad U \to U, \quad T \to T + a_i W_i + \frac{1}{2} \sum_i a_i^2 U \]  
(2.48)
\[ W_i \to W_i + a_i, \quad U \to U, \quad T \to T \]  
(2.49)

The mass formula here is not invariant under heterotic-type II duality. The reason is that the Wilson lines we turned on correspond to R-R gauge fields in the type II side. The mass formula in this case is mapped to the type II BPS formula.
To go further we will set $W_i = 0$. We obtain

$$M_{BPS,+}^2 = \frac{|-m_1 U + m_2 + T(n_1 + n_2 U) + S[-\tilde{m}_1 U + \tilde{m}_2 + T(\tilde{n}_1 + \tilde{n}_2 U)]|^2}{4 \, U_2 T_2 S_2}$$  \hspace{1cm} (2.50)$$

$$= \frac{|-m_1 U + m_2 + T(n_1 + n_2 U) + S(-\tilde{m}_1 U + \tilde{m}_2) + ST(\tilde{n}_1 + \tilde{n}_2 U)|^2}{4 \, U_2 T_2 S_2}$$

Cast in this form, (2.50) is invariant under $S \leftrightarrow T$ interchange:

$$T \leftrightarrow S \hspace{0.5cm}, \hspace{0.5cm} \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix} \rightarrow \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}, \hspace{0.5cm} \begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix} \rightarrow \begin{pmatrix} -n_2 \\ n_1 \\ \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}$$  \hspace{1cm} (2.51)$$

This is precisely the transformation implied by string-string duality at the action level, as we will see in detail in another section. Note also that for generic moduli the heterotic-type II duality transformation on electric and magnetic charges is still given by (2.51), all other charges being invariant, while the map on the moduli themselves becomes more complicated.

If the quantity in the absolute value is negative we obtain (2.50) with $S \rightarrow \tilde{S}$

$$M_{BPS,-}^2 = \frac{|-m_1 U + m_2 + T(n_1 + n_2 U) + \tilde{S}[-\tilde{m}_1 U + \tilde{m}_2 + T(\tilde{n}_1 + \tilde{n}_2 U)]|^2}{U_2 T_2 S_2}$$  \hspace{1cm} (2.52)$$

$$= \frac{|m_1 U - m_2 - T(n_1 + n_2 U) - \tilde{S}[-\tilde{m}_1 U + \tilde{m}_2 - T(\tilde{n}_1 + \tilde{n}_2 U)]|^2}{U_2 T_2 S_2}$$

This can be obtained from (2.50) by changing the signs of electric and magnetic charges as well as $S_1$. This formula looks somewhat peculiar. We will see however that for Type II perturbative states that break $3/4$ of the supersymmetry, it correctly describes their spectrum. Also (2.52) is $SL(2, Z)_S$ invariant, but also invariant now under $T \leftrightarrow \tilde{S}$ interchange.

Although the mass formula for non-perturbative BPS states is understood we do not know a priori the multiplicities of all these states. From the $N = 4$ heterotic string we know the multiplicities when $\beta_1 = 0$. Using $SL(2, Z)$ we also know the multiplicities of all states with $\alpha \cdot \beta = 0$. To go further and learn more about the states with $\alpha \cdot \beta \neq 0$ (namely intermediate multiplets) it is necessary to go beyond the string picture and learn more about the non-perturbative structure of the theory. The heterotic string on $T^6$ is supposed to be equivalent in the strong coupling limit with the type II theory compactified on $K_3 \times T^2$. Moreover, there is a hypothetical 11-d theory (M-theory) that includes the non-perturbative dynamics of type IIA theory [13]. Thus compactification of M-theory on $K_3 \times T^3$ contains all the relevant non-perturbative information about the heterotic $N=4$ theory. This idea led to a conjecture on the multiplicities of dyonic BPS states in the 4-d $N=4$ theory [21]. This will be an important input, for our non-perturbative analysis of the spontaneously broken $N=4$ theory.

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*This was first observed at the level of the action in [13]. More indications for the string-string duality conjecture were provided in [14].

†Similar observations were made in [22].
partial spontaneous breaking of supersymmetry

One of the defining characteristics of the $N=4$ theories is that the states are classified by their transformation properties under the $R$-symmetry group which, for $N=4$ supersymmetry, is $G_R=SU(4) \sim SO(6)$. In the gravitational multiplet the gravitinos are in 4 representation of $G_R$, the graviphotons are in 6, while the graviton, the dilaton and the antisymmetric tensor field are singlets. The degrees of freedom of a massless $N=4$ vector multiplet are also in definite representations of $G_R$: the scalars are in 6, the gauginos are in 4, while the gauge bosons are singlets. In the heterotic string, $G_R$ is constructed in terms of the six left-moving compactified supercoordinates, $(\Phi^I, \Psi^I)$. The world-sheet fermion bilinears $\Psi^I \Psi^J$ form an $SO(6)_{k=1}$ Kac–Moody algebra. In the light-cone picture, the full spectrum of the theory is classified in representations of $SO(6)_{k=1}$ and in terms of the $U(1)_0$ helicity charge $q^0 = \oint j^0$, $j^0 = \Psi^\mu \Psi^\nu$, $\mu, \nu = 3, 4$. In the $N=4$ spectrum the three internal helicity charges $q^i = \oint j^i$, $j^i = \Psi^{2k-1} \Psi^{2k}$, $k = 1, 2, 3$ and $q^0$ are all simultaneously integers for space-time bosonic states and simultaneously half-integers for the fermionic states:

$q^i = \text{half-integers for spacetime fermions}$

$q^i = \text{integers for spacetime bosons.}$

Furthermore all physical states have odd total $q^i$ charge (GSO-projection)

$$q^0 + q^1 + q^2 + q^3 = \text{odd integer.}$$

(3.1)

The last condition remains valid for supersymmetric solutions with less than four supersymmetries. In order to have a lower number of supersymmetries, the $q^i$’s must not be simultaneously integers or half-integers. It is then necessary to modify the world-sheet action $S^{2d}$, adding background fields that can change the individual values of the $q^i$’s, keeping however their total $q^i$ charge:

$$\Delta S^{2d} = \int dz d\bar{z} \, F^a_{IJ} (\Psi^I \Psi^J - \Phi^I \partial \Phi^J) \, \bar{J}^a,$$

(3.3)

where $\bar{J}^a$ denotes any dimension (0,1) operator. The part of the left-moving operator ($\Phi^I \partial \Phi^J$) is necessary to ensure the $N=(1,0)$ super-reparametrization of the 2-d action. From a higher-dimensional point of view, the $F^a_{IJ}$ denote non-trivial gauge or gravitational ($R^{(KL)}_{IJ}$) field backgrounds. In four dimensions they give rise to non-vanishing auxiliary fields. The permitted values of $F^a_{IJ}$ ($R^{(KL)}_{IJ}$) are not arbitrary. Only those for which

$$U_L(F) = \exp \left[ \int dz \, F^a_{IJ} (\Psi^I \Psi^J - \Phi^I \partial \Phi^J) \right]$$

(3.4)

commutes with the 2-d supercurrent ($T_F = \Psi^\mu \partial \Phi^\mu + \Psi^I \partial \Phi^I$) are allowed. This restriction generates a quantization of the permitted $F^a_{IJ}$ ($R^{(KL)}_{IJ}$) backgrounds.

A partial $N=4 \rightarrow N=2$ breaking is possible when $F_{3,4} = -F_{5,6} = H$ is not zero (self-duality condition). Indeed, in that case the $q^2$ and $q^3$ charges are shifted, preserving
the total $q^i$ charge. In order to define the full deformation of the spectrum it is necessary to find a representation of the partition function in which the bosonic charges

$$Q_2^B = \oint dz \Phi^3 \overset{\leftrightarrow}{\partial} \Phi^4 \quad \text{and} \quad Q_3^B = \oint dz \Phi^5 \overset{\leftrightarrow}{\partial} \Phi^6$$

(3.5)

are well defined. As a starting point we fermionize the four internal bosonic coordinates

$$\partial \Phi^I = y^I w^I \quad \text{and} \quad \bar{\partial} \Phi^I = \bar{y}^I \bar{w}^I, \quad I = 3, 4, 5, 6.$$  

(3.6)

In this representation the two dimensional supercurrent is [4],

$$T_F = \Psi^\mu \overset{\leftrightarrow}{\partial} \Phi^\mu + \sum_{I=1}^{2} \Psi^I \overset{\leftrightarrow}{\partial} \Phi^I + \sum_{I=3}^{6} \Psi^I y^I w^I.$$  

(3.7)

We will now perform the following $Z_4$ transformation:

$$\Psi^3 \rightarrow \Psi^4, \quad y^3 \rightarrow y^4, \quad \Psi^5 \rightarrow -\Psi^6, \quad y^5 \rightarrow -y^6,$$

$$\Psi^4 \rightarrow -\Psi^3, \quad y^4 \rightarrow -y^3, \quad \Psi^5 \rightarrow \Psi^6, \quad y^5 \rightarrow y^6,$$

$$w^3 \rightarrow w^4, \quad w^4 \rightarrow w^3, \quad w^5 \rightarrow w^6, \quad w^6 \rightarrow w^5,$$

(3.8)

which leaves (3.7) invariant. The above transformation corresponds to a $\pi/2$ rotation on the complex fermion basis:

$$\chi_1 \rightarrow e^{2i\pi\phi} \chi_1, \quad \chi_2 \rightarrow e^{-2i\pi\phi} \chi_2, \quad Y_1 \rightarrow e^{2i\pi\phi} Y_1, \quad Y_2 \rightarrow e^{-2i\pi\phi} Y_2$$

(3.9)

$$W_+ \rightarrow W_+, \quad W_- \rightarrow e^{4i\pi\phi} W_-$$

(3.10)

where

$$\chi_1 = \frac{\Psi^3 + i\Psi^4}{\sqrt{2}}, \quad \chi_2 = \frac{\Psi^5 + i\Psi^6}{\sqrt{2}}, \quad Y_1 = \frac{y^3 + iy^4}{\sqrt{2}}, \quad Y_2 = \frac{y^5 + iy^6}{\sqrt{2}}$$

(3.11)

$$W_+ = \frac{(w^3 + w^4) + i(w^5 + w^6)}{2}, \quad W_- = \frac{(w^3 - w^4) + i(w^5 - w^6)}{2}.$$  

(3.12)

Similarly for the right-moving degrees of freedom ($\bar{\Psi}^I, \bar{y}^I, \bar{w}^I, \quad I = 3, 4, 5, 6$). The above transformation is a symmetry only if the rotation angle is a multiple of $\pi/2$ or $\phi = k/4$, with $k$ integer.

Observe that with the help of the world-sheet fermions we can classify the $N = 4$ string spectrum in terms of a left and a right $U(1)$ charges $Q_L = \oint j_L$ and $Q_R = \oint j_R$, where

$$j_L = \chi_1 \chi_1^\dagger - \chi_2 \chi_2^\dagger + Y_1 Y_1^\dagger - Y_2 Y_2^\dagger + 2W_- W_+^\dagger,$$

$$Q_L = q_{x_1} - q_{x_2} + q_{y_1} - q_{y_2} + 2q_{w_-}$$

(3.13)

and

$$j_R = \bar{\chi}_1 \bar{\chi}_1^\dagger - \bar{\chi}_2 \bar{\chi}_2^\dagger + \bar{Y}_1 \bar{Y}_1^\dagger - \bar{Y}_2 \bar{Y}_2^\dagger + 2\bar{W}_- \bar{W}_+^\dagger,$$

$$Q_R = q_{\bar{x}_1} - q_{\bar{x}_2} + q_{\bar{y}_1} - q_{\bar{y}_2} + 2q_{\bar{w}_-}$$

(3.14)

We are now in a position to switch on non-vanishing $F^a_{IJ}$ by performing a boost among the fermionic charge lattice and the $\Gamma_{(2,n)}$ lattice:

$$q_{x_1} \rightarrow q_{x_1} + h_in^i, \quad q_{x_2} \rightarrow q_{x_2} - h_in^i, \quad q_{\bar{x}_1} \rightarrow q_{\bar{x}_1} + h_{\bar{n}}n^i, \quad q_{\bar{x}_2} \rightarrow q_{\bar{x}_2} - h_{\bar{n}}n^i$$

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\[ q_{Y_1} \to q_{Y_1} + h_i n^i, \quad q_{Y_2} \to q_{Y_2} - h_i n^i, \quad q_{Y_1} \to q_{Y_1} + h_i n^i, \quad q_{Y_2} \to q_{Y_2} - h_i n^i \]
\[ q_{W_-} \to q_{W_-} + 2h_i n^i, \quad q_{W_+} \to q_{W_+} \]
\[ P^L_i(h_i) = P^L_i - h_i(Q_L - Q_R), \quad P^R_i(h_i) = P^R_i - h_i(Q_L - Q_R) \]
with
\[ P^L_i = m_i + Y_i^a Q^a + \frac{1}{2} Y_i^a Y_j^a n^j + B_{ij} n^j + G_{ij} n^j \]
\[ P^R_i = m_i + Y_i^a Q^a + \frac{1}{2} Y_i^a Y_j^a n^j + B_{ij} n^j - G_{ij} n^j \]
\[ Y_i^a i = i, 2, a = 1, 2, \ldots, 18 \text{ are the Wilson-line moduli of the } \Gamma(2, 18) \text{ lattice.} \]

Owing to the non-zero \( h_i \) shift, two of the \( N = 4 \) gravitini become massive, with mass proportional to \( |q_{x_1} - q_{x_2}| \). The \( N = 4 \) gravitini have vanishing \( m_i, n^i, q_{Y_1}, q_{Y_2}, q_{W_1}, q_{W_2} \) charges. The two of them remain massless since \( |q_{x_1} - q_{x_2}| = 0 \), while the other two become massive since \( |q_{x_1} - q_{x_2}| = 1 \):
\[ (m_{3/2}^2)_{1, 2} = 0, \quad (m_{3/2}^2)_{3, 4} = \frac{|F|^2}{4 \text{Im} T \text{Im} U}, \]
with \( F = h_1 + U h_2, T \) and \( U \) being the usual complex moduli of the \( \Gamma_{(2, 2)} \) lattice.

The global existence of the supercurrent implies in this case the quantization condition: \( 4h_i = \text{integer} \). The \( N = 2 \) partition function \( Z^{4 \to 2}(F) \) is obtained from that of \( N = 4 \) by shifting the lattice momenta \( P_i \) and the \( R \)-charges \( q_i \) as above. Performing a Poisson resummation on \( m_i \), we obtain the following expression:
\[ \gamma = 2h_i n^i, \quad \delta = 2h_i m^i, \quad F = h_1 + U h_2 \]
\[ Z^{4 \to 2}(F) = \frac{(\tau_2)^{-1}}{\eta^2 \bar{\eta}^2} \sum_{m^i, n^i, \gamma, \delta} \frac{1}{2} (-)^{\alpha + \beta + \gamma + \delta} \frac{\vartheta^2[\alpha]}{\eta^2} \frac{\vartheta[\beta + \gamma]}{\eta} \frac{\vartheta[\beta - \delta]}{\eta} \frac{\Gamma_{(2, 2)}[n^i]}{|\eta|^4} Z_{(4, 4)}[\gamma] \times \]
\[ \times \sum_{\alpha, \beta} \frac{\vartheta^6[\alpha]}{\eta^6} \frac{\vartheta[\alpha + \beta]}{\eta} \frac{\vartheta[\alpha - \beta]}{\eta} \sum_{\epsilon, \zeta} \frac{1}{2} \vartheta^8[\epsilon] \frac{\vartheta^8[\zeta]}{\eta^8}, \]
where
\[ \Gamma_{(2, 2)}[n^i] = \frac{\sqrt{\det G}}{\tau_2} \exp \left[ -\pi G_{ij} \frac{(m^i + n^i \tau)(m^j + n^j \tau)}{\tau_2} + 2i \pi B_{ij} m^i n^j \right] \]
and
\[ Z_{(4, 4)}[\gamma] = \frac{1}{2} \sum_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta} \frac{\vartheta[\alpha + 2 \gamma]}{\eta} \frac{\vartheta[\alpha - \gamma]}{\eta} \left. \right| \frac{\vartheta[\alpha + \gamma]}{\eta} \frac{\vartheta[\alpha + \gamma]}{\eta} \frac{\vartheta[\alpha - \gamma]}{\eta} \left. \right| \]
When \( h_i = 0 (\gamma, \delta = 0) \), \( Z^{4 \to 2}(F = 0) \) corresponds to the \( N = 4 \) heterotic string solution based on a gauge group \( U(1) \times U(1) \times SO(8) \times E_8 \times E_8 \); the \( SO(8) \) gauge group factor corresponds to the extended symmetry of the \( \Gamma_{(4, 4)} \) lattice at the fermionic point
\[ Z_{(4, 4)}[0] = \frac{1}{2} \sum_{\alpha, \beta} \left| \frac{\vartheta[\alpha]}{\eta} \right|^8 \]
The sum over \( m^i \) and \( n^i \) gives rise to the \( \Gamma(2, 2) \) lattice at an arbitrary point of the moduli space:

\[
\sum_{m^i, n^i} \Gamma_{(2,2)}[m^i] = \Gamma_{(2,2)}[T, U].
\] (3.22)

When \( h_i \neq 0 \) (\( \gamma, \delta \) = \((2h_i n^i, 2h_i m^i)\)), then the \( N = 4 \) supersymmetry is spontaneously broken to \( N = 2 \) and the gauge group is reduced to \( U(1)^2 \times E_7 \times E_8 \), as in orbifold models.

The important difference between the \( N = 2 \) model described above and the orbifold models \( \mathbb{Z} \) of order \( N \) is in the parameters \( \gamma \) and \( \delta \), which appear as arguments in \( \vartheta \)-functions. In the model in which some of the \( N = 4 \) the supersymmetries are broken spontaneously, \( \gamma = 2h_i n^i \) and \( \delta = 2h_i m^i \) are not independent but are given in terms of the \( h_i \) and in terms of the charges \( n^i, m^i \) of the \( \Gamma(2, 2)[m^i] \) lattice. In the standard symmetric orbifolds of order \( N \), the arguments \( \gamma \) and \( \delta \) (\( \gamma = 2l/N \) and \( \delta = 2k/N \) with \( l, k = 0, 1, \ldots, N-1 \)) are independent arguments; their summation gives rise to the orbifold projections and to some additional states in the twisted sector:

\[
Z_{\text{orb}}^{N=2} = \frac{1}{\tau_2|\eta|^4} \sum_{\gamma, \delta} \frac{1}{N} \Gamma_{(2,2)}[\gamma, \delta] \frac{\vartheta^2\bar{\vartheta}^2\vartheta^4\bar{\vartheta}^4}{\eta^2} \frac{\vartheta^{3+\gamma}\vartheta^{3+\delta}}{\eta} \frac{\vartheta^{3-\gamma}\vartheta^{3-\delta}}{\eta} Z_{Z(4,4)\lambda}[\eta]|\eta|^4.
\] (3.23)

In the language of orbifolds, the spontaneously broken theory, \( Z^{4 \to 2} \), corresponds to a freely acting orbifold. The possible (left–right symmetric) rotations we can use are of the \( Z_N \) type with \( N = 2, 3, 4, 6 \). The model described above corresponds to \( N = 4 \). The quantization condition becomes

\[
N \ h_i \ = \ \text{integer}
\] (3.24)

The \( mod 2 \) periodicity properties of the \( \vartheta \)-functions in the arguments,

\[
\vartheta_{[a+2k]}^\alpha = \vartheta_{[a]}^\alpha e^{i\pi a},
\] (3.25)

give us the possibility to write \( Z^{4 \to 2} \) in terms of the orbifold language. First we redefine the lattice charges \( n^i = N\tilde{n}^i + \gamma^i \) and \( m^i = N\tilde{m}^i + \delta^i \). This redefinition makes the arguments of the \( \vartheta \)-functions independent of \( \tilde{n}^i \) and \( \tilde{m}^i \); they depend only on \( \gamma^i = 2h_i \gamma^i \) and \( \delta^i = 2h_i \delta^i \).

We can perform a Poisson resummation on \( \tilde{m}^i \) to obtain the following expression for \( Z^{4 \to 2} \)

\[
Z^{4 \to 2}(F) = \frac{1}{\tau_2|\eta|^4} \frac{1}{N} \sum_{\gamma, \delta} \frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha+\beta+\alpha\beta} \frac{\vartheta^2\bar{\vartheta}^2\vartheta^4\bar{\vartheta}^4}{\eta^2} \frac{\vartheta^{3+\gamma}\vartheta^{3+\delta}}{\eta} \frac{\vartheta^{3-\gamma}\vartheta^{3-\delta}}{\eta} Z_{Z(4,4)\lambda}[\eta]|\eta|^4.
\] (3.26)

where

\[
\Gamma_{(2,2)\lambda}[\gamma^i] = \sum \exp[i\pi 2\delta^i \tilde{m}_i/N] + i\pi \tau \frac{1}{2} P^L_i g^{ij} P^L_j - i\pi \tau \frac{1}{2} P^L_i g^{ij} P^L_j
\] (3.27)

and

\[
P^L_i = \tilde{m}_i + (\tilde{n}^j + \frac{\gamma^j}{N})G_{ij} \quad \text{and} \quad P^R_i = \tilde{m}_i - (\tilde{n}^j + \frac{\gamma^j}{N})G_{ij}
\] (3.28)
The connection of \( \mathbb{Z}^{4 \to 2} \) with the freely acting orbifolds gives us the way to switch all the moduli of the \( Z_{(4,4)} \) and thus move out of the extended symmetry of the \( SO(8) \). This extension can be done by replacing \( Z_{(4,4)[6]} \) which was defined at the fermionic point by

\[
Z_{(4,4)[6]} = \frac{\Gamma_{4,4}[T_{IJ}]}{|\eta(\tau)|^8},
\tag{3.29}
\]

while for \((\gamma, \delta) \neq (0, 0)\)

\[
Z_{(4,4)[\delta]} = Z_{(4,4)[\delta]}^{\text{twist}} = \frac{1}{2} \sum_{a, b} \left| \frac{\vartheta_{[a+b]}^\delta}{\eta} - \frac{\vartheta_{[a+2b]}^\delta}{\eta} - \frac{\vartheta_{[b+\delta]}^\delta}{\eta} - \frac{\vartheta_{[b-\delta]}^\delta}{\eta} \right|^2.
\tag{3.30}
\]

The \((\gamma, \delta) \neq (0, 0)\) part is the same at any point of the moduli \( T_{IJ} \) and is equal to the twisted part of the corresponding orbifold partition function,

\[
Z_{(4,4)[2k/N]}^{\text{twist}} = 16 \sin^4 \left[ \frac{\pi \Lambda(k, k')}{N} \right] \left| \frac{\eta^2}{\vartheta_{[1+2k/N]} - \vartheta_{[1-2k/N]}^2} \right|^4,
\tag{3.31}
\]

where \( \Lambda(k, k') = \Lambda(k', k) \) is:

- 1 for all non-trivial sectors for \( N = 2 \)
- 1 for all nontrivial sectors for \( N = 3 \)
- 2 for the sectors \((k, k') = (0, 2), (2, 0), (2, 2)\) and 1 for the rest for \( N = 4 \).
- 3 for the sectors \((0, 3), (3, 0), (3, 3), 2 \) for \((0, 2), (2, 0), (0, 4), (4, 0), (2, 2), (2, 4), (4, 2), (4, 4)\) and 1 otherwise for \( N = 6 \).

The models described above are special cases of a general class of models having the interpretation of freely acting orbifolds of the \( N=4 \) heterotic string theory. They are obtained in the following way. Consider \( \Gamma(6, 22) \) as in (2.17) and set the appropriate moduli to special values so that it factorizes as

\[
\Gamma_{6,22} \to \Gamma_{2,18} \Gamma_{4,4}
\tag{3.32}
\]

Now consider the orbifold that acts as a \( Z_N \) rotation on \( \Gamma_{4,4} \) and a translation by an \( N \)-th lattice vector \( \varepsilon/N \) with \( \varepsilon = (\tilde{e}_L, \tilde{e}_R, \tilde{e}_T) \), on \( \Gamma_{2,18} \). \( \tilde{e}_{L,R} \) are two-dimensional vectors while \( \tilde{e}_T \) is a sixteen-dimensional vector. The twisted blocks of the \( (4,4) \) piece are given by \( \Gamma_{4,4}[0] \) = \( \Gamma_{4,4} \) and

\[
\Gamma_{4,4}^{[h]} = 16 \sin^4 \left[ \frac{\pi \Lambda(h, g)}{N} \right] \left| \frac{\eta^2}{\vartheta_{[1+2h]} - \vartheta_{[1-2h]}^2} \right|^2 \text{ for } (h, g) \neq (0, 0)
\tag{3.33}
\]

with \( h, g = 1/N, \cdots, (N - 1)/N \). Similarly for the \( (2,18) \) piece we obtain

\[
\Gamma_{2,18}^{[h]} = \frac{1}{N} \sum_{a \in \mathcal{E}_{2,18} + h \varepsilon} e^{2 \pi i a \cdot \eta} q^{\frac{1}{2} a^T(M+L)a} q^{-\frac{1}{2} a^T(M-L)a} \text{ for } (h, g) \neq (0, 0)
\tag{3.34}
\]

Due to the accompanying translation, this is a freely acting orbifold.
The partition function can thus be written as

\[ Z_{\epsilon}^{1\rightarrow 2} = \frac{1}{\tau_2 |\eta|^4} \frac{1}{2\mathbb{N}} \sum_{\alpha,\beta,h,g} (-)^{\alpha+\beta+\alpha\beta} \frac{\vartheta^2[\alpha]}{\eta^2} \frac{\vartheta^{[\alpha+2\beta]}[\beta+2\eta]}{\eta} \frac{\vartheta^{[\alpha-2\beta]}[\beta-2\eta]}{\eta} \frac{\Gamma_{\epsilon}^{[h]}[\Gamma_{\epsilon}^{[h]}]}{\eta^6 \bar{\eta}^{22}} \]  

(3.36)

Modular invariance constrains the norm of \( \epsilon \):

\[ \epsilon^2 \equiv 2 \bar{\epsilon}_L \cdot \epsilon_R - \bar{\epsilon} \cdot \bar{\epsilon} \in 2\mathbb{Z} \]  

(3.37)

From the modular properties

\[ \tau \rightarrow \tau + 1, \quad \Gamma_{4,4}[h] \rightarrow \Gamma_{4,4}[-h], \quad \Gamma_{2,18}[h] \rightarrow e^{2\pi i h^2 \epsilon^2} \Gamma_{2,18}[-h] \]  

(3.38)

\[ \tau \rightarrow -\frac{1}{\tau}, \quad \Gamma_{4,4}[g] \rightarrow \Gamma_{4,4}[-g], \quad \Gamma_{2,18}[-g] \rightarrow e^{-2\pi i g^2 \epsilon^2} \Gamma_{2,18}[-g] \]  

(3.39)

and

\[ \Gamma_{2,18}^{[h+1]} = e^{-2\pi i g^2 \epsilon^2} \Gamma_{2,18}^{[h]}, \quad \Gamma_{2,18}^{[g]} = \Gamma_{2,18}^{[g]} \]  

(3.40)

we obtain that

\[ \frac{\epsilon^2}{2} = 1 \mod \mathbb{N}^2 \]  

(3.41)

Moreover different lattice shifts do not always give different models since

\[ \Gamma_{2,18}^{[\epsilon+\mathbb{N} \epsilon']} = e^{-2\pi i \mathbb{N} g h} \epsilon \cdot \epsilon' \Gamma_{2,18}^{[\epsilon]} \]  

(3.42)

The two types of constructions we have presented have complementary features. In the first approach, i.e. using a specific generalized boost at the fermionic point, it is evident that there is a one-to-one correspondence of states between the original N=4 supersymmetric theory and the final spontaneously broken \( N = 4 \rightarrow N = 2 \) theory. This is what should be expected during spontaneous symmetry breaking. In the second, freely acting orbifold approach, we have a clear geometrical intuition about the spontaneously broken theory, which will be very useful to identify the type II dual.

We should remark here on the fate of T-duality. Factorization of the (6,22) lattice gives a (2,18) lattice associated with the vector multiplets, with original \( SO(2,18,\mathbb{Z}) \) invariance. There is also of (4,4) lattice associated with the neutral hypermultiplets. Its geometry, \( SO(4,4)/SO(4) \times SO(4) \) as well as \( O(4,4,\mathbb{Z}) \) are exact, since no perturbative or non-perturbative corrections can modify them. However, the discrete \( O(2,18,\mathbb{Z}) \) symmetry is already broken by the lattice shift \( \epsilon \). Let \( \Omega \) be an \( O(2,18,\mathbb{Z}) \) matrix. Then the shifted lattice sum transforms as

\[ \Gamma_{2,18}^{[\epsilon]}(T_i) \rightarrow \Gamma_{2,18}^{[\epsilon]}(T_i \Omega) \]  

(3.43)

where \( T_i^{\Omega} \) are the standard transformed \( (2,18) \) moduli. The duality symmetry for a given groundstate (given \( \epsilon \)) is the \( O(2,18,\mathbb{Z}) \) subgroup that leaves \( \epsilon \) invariant up to even shifts on the lattice. Broken transformations move us in the space of \( \epsilon \). However, \( \epsilon^2 \) is \( O(2,18,\mathbb{Z}) \) invariant, and models with different \( \epsilon^2 \) cannot be related by “broken” \( O(2,18,\mathbb{Z}) \) transformations.

We can also give here the general mass formula for the massive gravitini. Inspection of the standard N=4 gravitini vertex operators shows that two of them are invariant while the other two transform, one with a phase \( e^{2\pi i / \mathbb{N}} \) and the other with \( e^{-2\pi i / \mathbb{N}} \). In order for them to survive in the spectrum they have to pair up with a state of the (2,18) lattice carrying momentum \( p = (\vec{m}; \vec{n}, \vec{\eta}) \) but no oscillators (these will shift the mass to the Planck scale).
Since such a lattice state picks up a phase $e^{2\pi i \varepsilon / N}$ one of the two massive gravitini will have momentum $p_1$ with the property that $p_1 \cdot \varepsilon = 1 \mod N$ while the other $p_2$ with $p_2 \cdot \varepsilon = -1 \mod N$. The mass formulae given in (3.17) are special cases of the above.

Thus the mass of the gravitini are given by the holomorphic (2,18) mass formula (2.32). However there are several lattice vectors that satisfy the above constraints. The ones that have the smallest mass are the gravitini whereas the rest are Kaluza-Klein states of the usual or massive gravitini. However, it is true that the statement of lowest mass depends on where we are in the moduli space. In fact there are explicit examples of models where supersymmetry is restored in different boundaries of moduli space, and it can be checked that the gravitini that become massless there are different for the two boundaries.

There is a related model with spontaneously broken $N=4 \rightarrow N=2$ supersymmetry and a much smaller vector moduli space [24]. It can be obtained by accompanying the $Z_2$ freely acting orbifold projection described above with an operation that changes the sign of the $E_8 \times E_8$ lattice. This gives a model with three vector multiplets (plus the graviphoton).

There is an essential difference between the models with spontaneous breaking of the $N = 4 \rightarrow N = 2$ and the standard $N = 2$ orbifold models.

- First, in the spontaneously broken case, one expects an effective restoration of the $N = 4$ supersymmetry in a corner of the moduli space $T, U$, where the two massive gravitinos become light, $m_{3/2} \rightarrow 0$.

- Second, in the standard orbifolds there is no restoration of the $N = 4$ supersymmetry at any point of the moduli space.

If there is an effective restoration of the $N = 4$ supersymmetry in the spontaneously broken case, then one must find zero higher-genus corrections to the coupling constants of the theory in the $N = 4$ restoration limit $m_{3/2} \rightarrow 0$. This restoration phenomenon has been checked in ref. [24] where the one-loop corrections of the coupling constants were performed for a class of $Z_2$ models based on $E_8 \times E_7 \times SU(2) \times U(1)^2$ gauge group. A more detailed discussion of the general heterotic models and their type II duals will appear in ref. [24]. Here I will restrict myself to the case of $Z_2$ freely acting orbifolds with $F = h_1 + Uh_2 = 1/2$. The $m_{3/2} \rightarrow 0$ limit in this class of models corresponds to the corner of the moduli space $\text{Im} T, \text{Im} U \rightarrow \infty$, which implies an effective decompactification of one of the two coordinates of $\Gamma_{2,2}(T, U)$, $(R_1 \rightarrow \infty$ and $R_2$ arbitrary; $\text{Im} T \sim R_1 R_2$, $\text{Im} U \sim R_1 / R_2$). In this limit, $T, U \rightarrow \infty$, one expects vanishing corrections to the coupling constants due to the effective $N = 4$ restoration. Using the explicit results of ref. [24],

$$\Delta_{\text{free}}^{(8,7)} = \frac{16\pi^2}{g_{E_8}^2} - \frac{16\pi^2}{g_{E_7}^2} = \delta b \log \left[ \frac{\mu^2}{M_5^2} \text{Im} T \text{Im} U |\vartheta_4(T) \vartheta_4(U)|^4 \right] +$$

$$+ \left( \delta b - \delta \bar{b} \right) \delta [T, U]$$

(3.44)

where, $\bar{b}_i$ are the massless $\beta$-functions of this model, $b_i$ are the massless $\beta$-functions of the standard $Z_2$ orbifold and

$$\delta [T, U] = \int_F \frac{d^2 \tau}{\tau_2} \sum_{h, g} \Gamma_{2,2}[h] \bar{\sigma}[h]$$

(3.45)

with

$$\bar{\sigma}[h] = \frac{e^{2\pi i (g+h)}}{16} \frac{\bar{\eta}^{12(1+2h)}}{\eta^{12}}, \quad \sum_{h, g} \bar{\sigma}[h] = 3$$

(3.46)
When $T$ and $U$ are large, $\text{Im} T \text{Im} U \gg 1$, due to the asymptotic behaviour of $\vartheta_4(T) = 1 + O(e^{-i\pi T})$:

$$\Delta_{(8,7)}^{\text{free}} \to \delta b \log (\text{Im} T \text{Im} U) + O(1/\text{Im} T \text{Im} U).$$  (3.47)

The logarithmic contribution is an artefact due to the infrared divergences. In fact by turning on Wilson lines appropriately (e.g. small Higgs vev’s of the vector multiplets), we can arrange that there are no charged states with masses $\mu^2_{W} \sim |W|^2/\text{Im} T \text{Im} U$ below $m^2_{3/2}$. In this case the logarithmic term becomes:

$$\delta b \log (\mu^2 \text{Im} T \text{Im} U) \to \delta b \log \frac{\mu^2_{W}}{m^2_{3/2} + \mu^2_{W}} \sim O\left(\frac{m^2_{3/2}}{\mu^2_{W}}\right);$$  (3.48)

the logarithmic divergence thus disappears and the thresholds vanish, which shows the restoration of the $N = 4$ supersymmetry in the light massive gravitino limit as expected. In the calculation of individual couplings, there is an extra contribution $Y(T, U)$, which is “universal” for $g_{E_8}$ and $g_{E_7}$; the explicit calculation in [25], [26] shows that $Y(T, U)$ behaves like

$$Y(T, U) \to \frac{m^2_{3/2}}{M^2_s} \text{ as } m^2_{3/2} \to 0.$$  (3.49)

Thus individual couplings also vanish in the limit $m^2_{3/2} \to 0$.

In the standard orbifold with $N = 2$ space-time supersymmetry, the corrections to the coupling constants have a different behaviour for $T, U \gg 1$ [27]:

$$\Delta_{(8,7)}^{\text{orb}} = \delta b \log \left[\mu^2 \text{Im} T \text{Im} U |\eta(T)\eta(U)|^4\right].$$  (3.50)

When $T, U$ is large, $\text{Im} T \text{Im} U \gg 1$, [28]

$$\Delta_{(8,7)}^{\text{orb}} \to \delta b \left[\frac{\pi}{3} (\text{Im} T + \text{Im} U) + \log \frac{|W|^2}{M^2_s}\right] + \text{finite terms.}$$  (3.51)

while the universal piece blows up linearly with the volume [29, 28].

So, in the standard orbifolds, the correction to the coupling constants grows linearly with the five-dimensional volume. This shows that the $N = 2$ supersymmetry is “not extended” in the decompactification limit $R_1 \to \infty$. On the other hand there is an extension of the supersymmetry in the freely acting orbifold case.

In the opposite limit $\text{Im} T \text{Im} U \to 0$, the situation is different:

i) In the freely acting orbifold the two massive gravitinos become superheavy: $m^2_{3/2} \to \infty$ in the limit $\text{Im} T \text{Im} U \to 0$.

ii) In the standard orbifold, thanks to the duality symmetry $R_i \to 1/R_i$ the behaviour $T, U \to 0$ is identical to the dual model with $T’ = -1/T$, $U’ = -1/U \to \infty$ and thus

$$\Delta_{(8,7)}^{\text{orb}}(T, U, W) = \Delta_{(8,7)}^{\text{orb}}(T’, U’, W’) \to \delta b \left[\frac{\pi}{3} (\text{Im} T’ + \text{Im} U’) + \log \frac{|W’|^2}{M^2_s}\right] + \text{finite} \quad (3.52)$$

In the freely acting orbifolds, the $SO(2, 2; Z)$ duality symmetry is reduced to a smaller subgroup due to the $Z_2$ action on the lattice. Thus one expects non-restoration of the $N = 4$ supersymmetry in this limit ( $T, U \to 0 m^2_{3/2} \to \infty$). Using

$$\text{Im} T |\vartheta_4(T)|^4 = \text{Im} T’ |\vartheta_2(T’)|^4, \quad T’ = -\frac{1}{T}.$$  (3.53)

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we obtain in the limit
\[ \Delta_{(8,7)}^{\text{free}} \to \delta \bar{b} \left[ \frac{2}{3} (\text{Im } T + \text{Im } U') + \log \frac{|W'|^2}{M_s^2} \right] + \text{finite terms.} \] (3.54)

Note that $\delta \bar{b}$ has disappeared.

It is interesting to observe that the $m_{3/2} \to \infty$ limit \[25\] of the freely acting orbifolds corresponds to a corner in the moduli space of $T, U$ where the two classes of theories (the freely and non-freely acting orbifolds) “touch” each other. Both theories are effectively five-dimensional. Thus the five-dimensional standard $N = 2$ orbifolds can be viewed as an $m_{3/2} \to \infty$ limit of some spontaneously broken $N = 4$ models.

4 BPS States in $N = 4 \to N = 2$ ground states

Let us consider the interesting question concerning the BPS spectrum of the theories where $N = 4$ is spontaneously broken to $N=2$. In the original heterotic $N=4$ theory, there are only short multiplets in the perturbative spectrum. Their multiplicities can be easily counted using helicity supertrace formulae \[30\]. In particular, the supertrace of helicity to the power four counts exactly the multiplicities\[‡\] of $N=4$ short (massless or massive) multiplets. Introduce the helicity generating partition function

\[ Z_{N=4}^{\text{het}}(v, \bar{v}) = \text{Str}[g^L_0 \bar{q}^L_0 e^{2\pi i v \lambda_R - 2\pi i \bar{v} \lambda_L}] = \frac{1}{2} \sum_{\alpha} (-1)^{\alpha + \beta + \gamma} \frac{\theta_3^{[\alpha]}(v) \theta_3^{[\beta]}(v)}{\eta^{12} \bar{\eta}^{24}} \xi(v) \bar{\xi}(\bar{v}) \frac{\Gamma_{6,22}}{\text{Im} \tau} = \frac{\vartheta_4^{1}(v/2) \xi(v) \bar{\xi}(\bar{v}) \Gamma_{6,22}}{\text{Im} \tau} \] (4.1)

The physical helicity in closed string theory $\lambda$ is a sum of the left helicity $\lambda_L$ and the right helicity $\lambda_R$

\[ \xi(v) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^{2\pi i v})(1 - q^n e^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\vartheta_4^{1}(v)}{\vartheta_4^{1}(v)} \xi(v) = \xi(-v) \] (4.2)

counts the contributions to the helicity due to the world-sheet bosons. If we define

\[ Q = \frac{1}{2\pi i} \frac{\partial}{\partial v}, \quad \bar{Q} = -\frac{1}{2\pi i} \frac{\partial}{\partial \bar{v}} \] (4.3)

then

\[ B_4 = \langle \lambda^4 \rangle = (Q + \bar{Q})^4 Z_{N=4}^{\text{het}}(v, \bar{v})|_{v=\bar{v}=0} = \frac{3}{2} \Gamma_{6,22} \bar{\eta}^{24} \] (4.4)

The numerator provides the mass formula while the denominator $1/\bar{\eta}^{24}$ provides the multiplicities. More precisely define

\[ \frac{1}{\eta^{24}} = \frac{1}{q} + \sum_{n=1}^{\infty} d(n) q^n = \frac{1}{q} + 24 + 324q + O(q^2) \quad \] (4.5)

\[ \text{‡More accurately this is an “index”, namely, the difference between integer spin minus half-integer spin multiplicities.} \]
Then at the mass level, $M^2 = \frac{1}{2}p_L^2$, with

$$\frac{1}{2}q_e^2 \equiv \vec{m} \cdot \vec{n} - \vec{Q} \cdot \vec{Q}/2$$  \hspace{1cm} (4.6)$$

the multiplicity is $d(q_e^2)$. This multiplicity formula was generalized to dyonic states with electric charges, also by magnetic charges $\vec{m}', \vec{n}', \vec{Q}'$. Define

$$\frac{1}{2}q_m^2 = \vec{m}' \cdot \vec{n}' - \vec{Q}' \cdot \vec{Q}'/2$$  \hspace{1cm} (4.7)

and consider the genus-2 $\eta$ function

$$\eta \left[ \begin{array}{cc} T & V \\ V & U \end{array} \right]^{-24} = \sum_{k_1,k_2,k_3} d(k_1,k_2,k_3) e^{2\pi i(k_1T+k_2U+k_3V)}$$  \hspace{1cm} (4.9)$$

Then, the non-perturbative multiplicities are given by $d(q_e^2/2, q_m^2/2, q_e \cdot q_m)$. In general we expect $B_4$ for heterotic $N=4$ groundstates to transform as

$$\tau \rightarrow \tau + 1 : B_4 \rightarrow B_4 \hspace{0.5cm}, \hspace{0.5cm} \tau \rightarrow -\frac{1}{\tau} : B_4 \rightarrow \tau^4 B_4$$  \hspace{1cm} (4.10)$$

Consider now the spontaneously broken $N = 4 \rightarrow N = 2$ theories, and for simplicity we will restrict to the $Z_2$ case. Massless multiplets $M_0^\lambda$ have the following helicity content

$$\pm \left( \lambda \pm \frac{1}{2} \right) + 2(\pm \lambda)$$  \hspace{1cm} (4.11)$$

$M_0^0$ is the hypermultiplet, $M_0^{1/2}$ is the vector multiplet while $M_0^{3/2}$ is the supergravity multiplet. The massive BPS multiplets have the following $SO(3)$ spin content

$$M^j : [j] \otimes ([1/2] + 2[0])$$  \hspace{1cm} (4.12)$$

and contain $2(2j + 1)$ bosonic and an equal number of fermionic states. Finally the generic long massive multiplet has the following $SO(3)$ content

$$L^j : [j] \otimes ([1] + 4[1/2] + 5[0])$$  \hspace{1cm} (4.13)$$

In the $N=2$ case, only the short (BPS) multiplets are picked up by the supertrace of helicity squared, $B_2 = \langle \lambda^2 \rangle$. We have

$$B_2(M_0^\lambda) = (-1)^{2\lambda} \hspace{0.5cm}, \hspace{0.5cm} B_2(M^j) = (-1)^{2j+1}(2j + 1)/2 \hspace{0.5cm}, \hspace{0.5cm} B_2(L^j) = 0$$  \hspace{1cm} (4.14)$$

A direct computation along the lines of [30] gives

$$\tau_2 B_2 = \tau_2 \langle \lambda^2 \rangle = \Gamma_{2,18} \frac{\bar{\gamma}_2^2 \bar{\gamma}_4^2}{\bar{\eta}^{24}} - \Gamma_{2,18} \frac{\bar{\eta}_2^2 \bar{\eta}_4^2}{\eta^{24}} - \Gamma_{2,18} \frac{\bar{\eta}_2 \bar{\eta}_4}{\eta^{24}}$$  \hspace{1cm} (4.15)$$

$$= \frac{\Gamma_{2,18}[0][1]}{2} F_1 - \frac{\Gamma_{2,18}[0][1]}{2} F_1 - \frac{\Gamma_{2,18}[1][1]}{2} F_1 - \frac{\Gamma_{2,18}[1][1]}{2} F_1 - \frac{\Gamma_{2,18}[0][0]}{2} F_1 - \frac{\Gamma_{2,18}[0][0]}{2} F_1$$

$$= \frac{\Gamma_{2,18}[0][1]}{2} F_1 - \frac{\Gamma_{2,18}[0][1]}{2} F_1 - \frac{\Gamma_{2,18}[1][1]}{2} F_1 - \frac{\Gamma_{2,18}[1][1]}{2} F_1 - \frac{\Gamma_{2,18}[0][0]}{2} F_1 - \frac{\Gamma_{2,18}[0][0]}{2} F_1$$
with
\[ F_1 = \frac{\partial_2^2 \partial_4^2}{\eta^{24}}, \quad \bar{F}_\pm = \frac{\partial_2^2 (\partial_2^2 \pm \partial_4^2)}{\eta^{24}} \] (4.16)

For all \( N=2 \) heterotic groundstates \( B_2 \) transforms as
\[ \tau \to \tau + 1 : \quad B_2 \to B_2, \quad \tau \to -\frac{1}{\tau} : \quad B_4 \to \tau^2 B_4 \] (4.17)

All functions \( \bar{F}_i \) have positive coefficients and have the generic expansions
\[ F_1 = \frac{1}{q} + \sum_{n=0}^{\infty} d_1(n)q^n = \frac{1}{q} + 16 + 156q + O(q^2) \] (4.18)
\[ F_+ = \frac{8}{q^{3/4}} + q^{1/4} \sum_{n=0}^{\infty} d_+(n)q^n = \frac{8}{q^{3/4}} + 8q^{1/4}(30 + 481q + O(q^2)) \] (4.19)
\[ F_- = \frac{32}{q^{3/4}} + q^{3/4} \sum_{n=0}^{\infty} d_-(n)q^n = \frac{32}{q^{3/4}} + 32q^{3/4}(26 + 375q + O(q^2)) \] (4.20)

Also the lattice sums \( (\Gamma_{2,18^0[0]} \pm \Gamma_{2,18^1[1]})/2 \) have positive multiplicities. Overall plus signs correspond to vector-like multiplets, minus signs to hyper-like multiplets. The contribution of the generic massless multiplets is given by the constant coefficient of \( F_1 \) and agrees with what was expected: \( 16=20-4 \) since we have the supergravity multiplet and 19 vector multiplets contributing 20 and 4 hypermultiplets contributing \(-4\). Turning off all the Wilson lines and restoring the \( E_7 \times E_8 \) group, the above result becomes
\[ \langle \lambda^2 \rangle = \Gamma_{2,2[0]} \frac{\partial_2^2 \partial_4^2 (\bar{\partial}_2^4 + \bar{\partial}_4^4) E_4}{2\eta^{24}} - \Gamma_{2,2[1]} \frac{\partial_2^2 \partial_4^2 (\bar{\partial}_2^4 + \bar{\partial}_4^4) E_4}{2\eta^{24}} - \Gamma_{2,2[1]} \frac{\partial_2^2 \partial_4^2 (\bar{\partial}_2^4 - \bar{\partial}_4^4) E_4}{2\eta^{24}} \] (4.21)

Let us analyse the BPS mass formulae associated with \( [3,23] \) with \( N = 2 \). We will use the notation of the previous chapter with the \( (2,18) \) shift vector \( \varepsilon = (\bar{\varepsilon}_L, \bar{\varepsilon}_R, \bar{\zeta}) \) satisfying the constraint \( (3.44) \). Using the results of section 2 we can write the mass formulae associated to the lattice sums above. For \( h = 0 \) the mass formula is
\[ M^2 = \frac{|-m_1 U + m_2 + T n_1 + (TU - \frac{1}{2} W^2) n_2 + \bar{W} \cdot \bar{Q}|^2}{4 S_2 (T_2 U_2 - \frac{1}{2} \text{Im} W^2)} \] (4.22)

where \( \bar{W} \) is the sixteen-dimensional complex vector of Wilson lines. When the integer
\[ \rho = \bar{m} \cdot \bar{\varepsilon}_R + \bar{n} \cdot \bar{\varepsilon}_L - \bar{Q} \cdot \bar{\zeta} \] (4.23)
is even these states are vector multiplet-like with multiplicity function \( d_1(s) \) of \( (4.18) \) and
\[ s = \bar{m} \cdot \bar{n} - \frac{1}{2} \bar{Q} \cdot \bar{Q} \] (4.24)

and when \( \rho \) is odd these states are hypermultiplet-like with multiplicities \( d_1(s) \). In the \( h = 1 \) sector the mass formula is
\[ M^2 = \frac{|-(m_1 + \frac{\bar{e}_1^1}{2}) U + (m_2 + \frac{\bar{e}_2^1}{2}) + T(n_1 + \frac{\bar{e}_1^1}{2}) + (TU - \frac{1}{2} W^2)(n_2 + \frac{\bar{e}_2^2}{2}) + \bar{W} \cdot (\bar{Q} + \bar{\zeta})|^2}{4 S_2 (T_2 U_2 - \frac{1}{2} \text{Im} W^2)} \] (4.25)
The states with $\rho$ even are hypermultiplet-like with multiplicities $d_+(s')$ with

$$s' = \left( \vec{m} + \frac{\varepsilon_L}{2} \right) \cdot \left( \vec{n} + \frac{\varepsilon_R}{2} \right) - \frac{1}{2} \left( \vec{Q} + \frac{\zeta}{2} \right) \cdot \left( \vec{Q} + \frac{\zeta}{2} \right)$$

(4.26)

while the states with $\rho$ odd are hypermultiplet-like with multiplicities $d_-(s')$.

Let us here consider symmetry enhancement in the presence of shift vectors. For simplicity we will set the Wilson lines to zero and ignore the charged sector. The case with non-zero Wilson lines is straightforward but more involved. Let us first consider the untwisted sector ($h=0$). According to the above analysis the masses are given by the unshifted mass formula (4.22) and they are vector multiplets when $\rho$ is even and hypermultiplets when $\rho$ is odd. Now the points where the standard $(2,2)$ mass vanishes are well known. They are $O(2,2,Z)$ images of $T = U$. In the unshifted case, $O(2,2,Z)$ is a symmetry and up to it there is a single enhanced symmetry point, namely $T = U$. When there are non-trivial shifts involved, the T-duality group is smaller as we discussed in the previous section, and there is a finite little group $G^S$ which acts non-trivially on the moduli. Then the points where we obtain massless states are images of $T = U$ under $G^S$. We will pick $T = U$ the other points can be obtained by duality. There are two configurations with zero mass given by $m_1 = n_1 = \pm 1$, all the rest being zero. For both states $|\rho| = |\varepsilon_L^1 + \varepsilon_R^L|$. Depending on it being even or odd these states are either vector multiplets that enhance the gauge group $U(1)^2 \rightarrow SU(2) \times U(1)$ or hypermultiplets charged under one of the $U(1)$’s.

Let us now look for states becoming massless in the twisted ($h = 1$) sector. (4.16) implies that massless states from the twisted sector are always hypermultiplets. At $T = U$ a potentially massless state must have $m_1 + \varepsilon_L^1/2 = n_1 + \varepsilon_R^1/2$ and $\varepsilon_L^2, \varepsilon_R^2$ even integers. If $\rho$ is even then for the states to be massless they must satisfy $s' = 3/4$. From (4.19) we deduce that in such a case there will be 8 massless hypermultiplets. If $\rho$ is odd, then the state will be massless if $s' = 1/4$ and (4.20) implies that there will be 32 massless hypermultiplets.

5 \hspace{1cm} $N = 4 \rightarrow N = 1$ Spontaneous Supersymmetry Breaking

Using the connection between the freely acting orbifolds and the spontaneous breaking $N = 4 \rightarrow N = 2$, we can proceed to further break the supersymmetry to $N = 1$. We will restrict ourselves to the case where the possible quantized parameters are of order $N=2$, $2|h^i| = 1$. In that case the spontaneously broken $N = 4 \rightarrow N = 1$ theory is related to $Z^2 \times Z^2$ freely acting orbifolds; the $Z^2 \times Z^2$ acts simultaneously as a rotation on the coordinates $\Phi^I$, $\Phi'^I$ and $\Psi^I$, $\Psi'^I$ of the two complex planes and as a translation on the third complex plane $\Phi^K$. Denoting by $\Phi_A$, $A = 1, 2, 3$, the complex internal coordinates and by $\chi_A$, $A = 1, 2, 3$, the three complex fermionic world-sheet superpartners, the non-trivial actions of the orbifold are:

1) $\Phi_1 \rightarrow \Phi_1 + 2\pi h_1$, \hspace{0.5cm} $(\Phi_2, \chi_2) \rightarrow e^{2\pi i h_1} (\Phi_2, \chi_2)$, \hspace{0.5cm} $(\Phi_3, \chi_3) \rightarrow e^{-2\pi i h_1} (\Phi_3, \chi_3)$.

2) $\Phi_2 \rightarrow \Phi_2 + 2\pi h_2$, \hspace{0.5cm} $(\Phi_1, \chi_1) \rightarrow e^{2\pi i h_2} (\Phi_1, \chi_1)$, \hspace{0.5cm} $(\Phi_3, \chi_3) \rightarrow e^{-2\pi i h_2} (\Phi_3, \chi_3)$.
3) \( \Phi_3 \rightarrow \Phi_3 + 2 \pi h_3 \), \( (\Phi_1, \chi_1) \rightarrow e^{2 \pi i h_3} (\Phi_1, \chi_1) \), \( (\Phi_2, \chi_2) \rightarrow e^{-2 \pi i h_3} (\Phi_2, \chi_2) \).

In order to obtain the partition function and define the theory, we need to introduce the “shifted” and “twisted” characters of the three complex coordinates. We denote by \((\gamma_A, \delta_A)\) the translation shifts and by \((H_A, G_A)\) the rotation twists. These orbifold blocks are derived in Appendix A.

We are now in a position to construct \( N = 4 \rightarrow N = 1 \) models. We will display below the partition function of a model with one unbroken and three spontaneously broken supersymmetries, \( N = 4 \rightarrow N = 1 \) (the unbroken gauge group of this example is \( E_8 \times E_6 \times U(1)^2 \):

\[
Z^{4 \rightarrow 1} = \frac{1}{\tau_2 |\eta|^4} \frac{1}{4} \sum_{h_{1,2}} Z_1^{[h_1; h_2]} Z_2^{[h_1 + h_2; g_1 + g_2]} Z_3^{[h_1 + h_2; g_1 + g_2]} \frac{1}{2} \sum_{\epsilon, \zeta} \frac{\bar{h}^{[\epsilon]} [\zeta]}{g^{[\epsilon]} [\zeta]} \tag{5.1}
\]

The massless chiral multiplets (appart from the universal ones) are the following \((E_8, E_6, U(1), U(1))\)

- One \((1, 27, 1/2, 1/2)\) + c.c.
- One \((1, 27, -1/2, 1/2)\) + c.c.
- One \((1, 27, 0, -1)\) + c.c.
- One \((1, 1, 1, -1/2, 2, 3/2)\) + c.c.
- One \((1, 1, 1, +1, 2, 3/2)\) + c.c.
- One \((1, 1, +1, 0)\) + c.c.

The spectrum is non-chiral.

It is easy to see that the partition function \(Z^{4 \rightarrow 1}\) can be decomposed in four sectors (we write \( g_3 = -(g_1 + g_2)\), \( h_3 = -(h_1 + h_2)\)):

- **The** \( N = 4 \) sector, with no rotations or translations in all three complex planes ((\( h_A, g_A \) = \( (0, 0) \))
- **Three** \( N = 2 \) sectors, with a non-zero translation in one of the complex planes and opposite non-zero rotations in the remaining two complex planes.

The contribution to the partition function of the \( N = 4 \) sector is one quarter of the \( N = 4 \) partition function with lattice momenta in the reduced \( \Gamma (2, 2)^3 \) lattice. The contribution of the other three \( \Gamma = 2 \) sectors are equal sector by sector to the corresponding \( N = 4 \rightarrow N = 2 \) partition function divided by a factor of 2. The untwisted complex plane lattice momenta correspond to the shifted \( \Gamma (2, 2) \) \([A]_3\) lattice. The moduli-dependent corrections to the gauge couplings can be easily determined by combining the results of the individual \( N = 2 \) sectors:

\[
\frac{16 \pi^2}{g_{E_8}^2} - \frac{16 \pi^2}{g_{E_8}^2} = \Delta_{(8,6)} = \frac{1}{2} \sum_{A=1}^{3} \Delta_{(8,7)}^A, \tag{5.2}
\]

where the expressions of the \( \Delta_{(8,7)}^A \) are given in (3.33).

As we mentioned in the \( N = 4 \rightarrow N = 2 \) spontaneous breaking, one expects a restoration of the \( N = 4 \) supersymmetry in the limit in which the massive gravitini become massless; in order to prove the \( N = 4 \) restoration in the \( N = 4 \rightarrow N = 1 \) defined above as
a \( Z^2 \times Z^2 \) freely acting orbifold, we need to identify the three massive gravitini and express their masses in terms of the moduli fields and the three \( R \)-symmetry charges \( q_i \) (\( i = 1, 2, 3 \)):

\[
m^2_{3/2}(q_i) = \frac{|q_2 - q_3|^2}{4 \text{Im} T_1 \text{Im} U_1} + \frac{|q_3 - q_1|^2}{4 \text{Im} T_2 \text{Im} U_2} + \frac{|q_1 - q_2|^2}{4 \text{Im} T_3 \text{Im} U_3} \tag{5.3}
\]

with \( |q_0 + q_1 + q_2 + q_3| = 1 \) and \( |q_i| = |q_0| = \frac{1}{2} q_0 \) being the left-helicity charge. Using the above expression, one finds the desired result:

\[
\begin{align*}
(m^2_{3/2})_1 &= \frac{1}{4 \text{Im} T_2 \text{Im} U_2} + \frac{1}{4 \text{Im} T_3 \text{Im} U_3}, \\
(m^2_{3/2})_2 &= \frac{1}{4 \text{Im} T_3 \text{Im} U_3} + \frac{1}{4 \text{Im} T_1 \text{Im} U_1}, \\
(m^2_{3/2})_3 &= \frac{1}{4 \text{Im} T_1 \text{Im} U_1} + \frac{1}{4 \text{Im} T_2 \text{Im} U_2},
\end{align*}
\tag{5.4}
\]

and \( (m^2_{3/2})_0 = 0 \).

The three massive gravitini become massless in the decompactification limit \( \text{Im} T_I \text{Im} U_I \to \infty, \ I = 1, 2, 3, \) with ratios \( \text{Im} T_I / \text{Im} U_I \) fixed. Thus the full restoration of the \( N = 4 \) effectively takes place in seven dimensions. Partial restoration of an \( N = 2 \) supersymmetry can happen in six dimensions when \( \text{Im} T_I \text{Im} U_I \to \infty, \ I = 1, 2; \) in this limit \( (m^2_{3/2})_0 = 0 \) and \( (m^2_{3/2})_3 \to 0 \).

## 6 \ N=2 \to N=1 \ spontaneous \ SUSY \ breaking

Using similar techniques as before, it is possible to construct \( N=2 \) models with one of the supersymmetries to be spontaneously broken, \( N=2 \to N=1 \). In this class of models the restoration of \( N=2 \) takes place in six dimensions. No further restoration of supersymmetry is possible. Examples can be obtained as in \( (T^2 \otimes K_3) / Z_f^2 \) orbifold compactification in which the \( Z_f^2 \) is freely acting (this is known as the Enriques involution of \( K_3 \)). Moreover, as we will see chirality can be present in the \( N=1 \) groundstate. A representative example of this class of models is the one in which the \( K_3 \) compactification is chosen to be at the orbifold point \( T^4 / Z_2^\nu \sim K_3 \) (we denote by \( Z_2^\nu \) the orbifold group and by \( Z_f^2 \) that which corresponds to the freely acting orbifold). We will give below the partition function that corresponds to this construction. From the explicit expression we can directly verify the effective restoration of \( N = 2 \) supersymmetry in the large-volume limit of \( K_3 \). Using the \( Z_2^\nu \times Z_f^2 \) orbifold notation, the partition function of the \( (T^2 \otimes T^4 / Z_2^\nu) / Z_f^2 \) model is:

\[
Z^{2 \to 1} = \frac{1}{\tau_2 |\eta|^4} \sum_{h_f, g_f, h_o, g_o} Z_1^{[\emptyset; h_f; g_f; 0; h_o; g_o]} Z_2^{[h_f; h_o; g_f; g_o]} Z_3^{[h_{f'}; -h_{f''}; h_{o'}; -g_{o''}]} \frac{1}{2} \sum_{\varepsilon, \zeta} \frac{\bar{\gamma}^{[\eta]} \bar{\eta}^{[\varepsilon]} \bar{\zeta}^{[\eta]}}{\bar{\eta}^{[\zeta]} \bar{\gamma}^{[\eta]}} \tag{6.1}
\]

\[
\frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha + \beta} \frac{\eta^{[\alpha]} \eta^{[\alpha + h_f]} \eta^{[\beta + g_f]} \eta^{[\alpha + h_o]} \eta^{[\beta - h_f - h_o]} \eta^{[\beta - g_f - g_o]}}{\eta^{[\zeta]} \eta^{[\eta]}} \frac{1}{2} \sum_{\alpha, \beta} \frac{\bar{\gamma}^{[\eta]} \bar{\eta}^{[\alpha + h_f]} \bar{\eta}^{[\beta + g_f]} \bar{\gamma}^{[\alpha + h_o]} \bar{\eta}^{[\beta - h_f - h_o]} \bar{\eta}^{[\beta - g_f - g_o]}}{\bar{\eta}^{[\zeta]} \bar{\eta}^{[\bar{\gamma}]} \bar{\zeta}^{[\eta]}}
\]

In the above expression, the parameters \((h_f, g_f)\) and \((h_o, g_o)\) correspond to \( Z_f^2 \) and \( Z_2^\nu \) respectively. The unbroken gauge group of this model is the \( E_8 \otimes E_6 \otimes U(1)^2 \). Switching on continuous or discrete Wilson lines, we can construct a large class of models with different
gauge group but with a universal behaviour with respect to the $N = 2$ restoration at the large moduli limit; the massive gravitino of the broken $N = 2$ becomes massless when $(\text{Im } T_2 \text{ Im } U_2$ and $\text{Im } T_3 \text{ Im } U_3$ large).

$$ (m^2_{3/2} )_1 = \frac{1}{4 \text{ Im } T_2 \text{ Im } U_2} + \frac{1}{4 \text{ Im } T_3 \text{ Im } U_3}, \quad (m^2_{3/2} )_0 = 0. \quad (6.2) $$

The massless spectrum coming from the untwisted sectors is non-chiral. However, here we do obtain chiral fermions from the twisted sectors. In particular we have 16 copies of the 27 of $E_6$. This implies that in string theory, unlike field theory, chirality can appear after spontaneous breaking of extended supersymmetry ($N=2$ in our example above). Moreover, we can vary the supersymmetry breaking scale without breaking the gauge group with chiral representations ($E_6$ here). It is a very interesting open problem if one can produce a chiral spectrum in the spontaneous breaking of $N = 4 \rightarrow N = 1$. We have no concrete example of this but also no counter-argument either.

An easy way to view this groundstate is as an orbifold of the original $N = 4$ theory by the following non-trivial $Z_2 \times Z_2$ elements: $(1, r, r)$, $(r, rt, t)$, $(r, t, rt)$, $(r$ stands for “π-rotation” and $t$ for half-lattice translation); $(1, r, r)$ has four fixed planes while the others have none. Because of the $N = 2$ restoration phenomenon, we expect that the only non-vanishing corrections to the gauge coupling constants are those that correspond to the $N = 2$ sector with $(h_f, g_f) = (0, 0)$ and $(h_o, g_o) \neq (0, 0)$. Indeed in this sector the $Z^2_2$ acts trivially on the $\Gamma(2, 2)(T_1, U_1)$ lattice as in the usual orbifolds. On the other hand, in the remaining two $N = 2$ sectors,

i) $(h_o, g_o) = (0, 0), (h_f, g_f) \neq (0, 0)$

ii) $(h_o, g_o) + (h_f, g_f) = (0, 0), (h_f, g_f) \neq (0, 0)$.

In both sectors the corresponding $Z^2_2$ acts without fixed points because of the simultaneous non-trivial shift $(h_f, g_f)$ on the corresponding $\Gamma(2, 2)(T_A, U_A)$, $A = 2, 3$, lattice.

The moduli-dependent corrections to the gauge couplings can be easily determined by combining the results of the individual $N = 2$ sectors:

$$ \Delta_{(8, 6)} = \frac{16 \pi^2}{g_{E_6}^2} - \frac{16 \pi^2}{g_{E_6}^2} = \frac{1}{2} \left( \Delta_{(8, 7)}^1 + \Delta_{(8, 7)}^2 + \Delta_{(8, 7)}^3 \right), \quad (6.3) $$

where the $\Delta_{(8, 7)}^A$ are the threshold corrections of the three $N = 2$ sectors:

$$ \Delta_{(8, 7)}^1 = (b^1_8 - b^1_t) \log \left[ |\mu|^2 \text{Im } T_1 \text{Im } U_1 |\eta(T_1)\eta(U_1)|^4 \right] \quad (6.4) $$

$$ \rightarrow (b^1_8 - b^1_t) \left[ \frac{\pi}{3} (\text{Im } T_1 + \text{Im } U_1) + \log |\mu|^2 \text{Im } T_1 \text{Im } U_1 \right] $$

which corresponds to the threshold corrections of the standard orbifolds.

On the other hand $\Delta_{(8, 7)}^A$ for $A = 2, 3$ will correspond to the threshold corrections of freely acting orbifolds which have different behaviour in the large-moduli limit:

$$ \Delta_{(8, 7)}^A = (b^A_8 - b^A_t) \log \left[ |\mu|^2 \text{Im } T_A \text{Im } U_A \right] + (b^A_8 - b^A_t) \log \left[ |\varphi_A(T_A)\varphi_A(U_A)|^4 \right] \quad (6.5) $$

$$ \rightarrow (b_8 - b_t) \log |\mu|^2 \text{Im } T_A \text{Im } U_A. $$

Modulo the artificial sub-leading logarithmic contribution (due to the infrared divergences), the moduli contribution of the second and third plane $T_A, U_A, A = 2, 3$, is
exponentially suppressed due to the asymptotic behaviour of $\vartheta_4(T_A), \vartheta_4(U_A)$ for large $T_A$ and $U_A$, $\vartheta_4(T_A) = 1 + \mathcal{O}(e^{-i\pi T_A})$.

There is a class of such models obtained from $N = 2$ $Z_2$ orbifold compactifications by using $D_4$ type symmetries that act on the twist fields as well as the lattice.

7 Type II duals of heterotic groundstates with Spontaneously Broken Supersymmetry

The heterotic string compactified on $T^4$ with $N=2$ (6-d) spacetime supersymmetry has been conjectured to be dual to type II theory compactified on $K_3$ [13, 14]. This duality changes the sign of the dilaton, dualizes the field strength of the antisymmetric tensor and leaves the (4,20) gauge fields $A^I_{\mu}$, the (4,20) moduli matrix $M_{4,20}$ and the Einstein metric invariant. Obviously this duality descends to 4-d upon compactifying both theories on an extra $T^2$. In four dimensions there are four extra gauge fields, two coming from the metric $A^i_{\mu}$ whose charges are the momenta of the $T^2$ and two coming from the antisymmetric tensor $B_{i,\mu}$ whose charges are the winding numbers of the $T^2$. Moreover we have three extra scalars from the components of the metric on $T^2$, $G_{ij}$ and one from the antisymmetric tensor $B_{ij}$. There are also $2 \times 24$ extra scalars, $Y^i_I$ coming from the 6-d gauge bosons. Moreover we can dualize in four dimensions the antisymmetric tensor to an axion field $A^i$.

If we denote heterotic variables by unprimed names and type II ones by primed names then the heterotic-type II duality in four dimensions implies that

\begin{align}
\exp(-\phi) &= \sqrt{\det G'_{ij}}, \quad \exp(-\phi') = \sqrt{\det G_{ij}} \quad (7.1) \\
\frac{G_{ij}}{\sqrt{\det G_{ij}}} &= \frac{G'_{ij}}{\sqrt{\det G'_{ij}}}, \quad A'^i_{\mu} = A^i_{\mu} \quad (7.2) \\
\exp(-\phi) g_{\mu\nu} = \exp(-\phi') g'_{\mu\nu} \rightarrow g^E_{\mu\nu} = g'^E_{\mu\nu} \quad (7.3) \\
M'_{4,20} = M_{4,20}, \quad A^I_{\mu} = A'^I_{\mu}, \quad Y^i_I = Y'^i_I \quad (7.4)
\end{align}

Moreover it effects an electric-magnetic duality transformation on the $B^i_{\mu}$ gauge fields

\begin{align}
A = \frac{1}{2} \varepsilon^{ij} B'_{ij}, \quad A' = \frac{1}{2} \varepsilon^{ij} B_{ij} \quad (7.5)
\end{align}

On the electric and magnetic charges it acts as in (2.51) on the $T^2$ charges and leaves the rest invariant.

For the configurations of moduli we are interested in our paper, namely the factorization $(6,22) \rightarrow (2,18) \times (4,4)$, using [2.30], [2.31] we proceed as follows. In the case of the heterotic string the complex moduli $S, T, U, \vec{W}$ are defined in terms of the $\sigma$-model data as described in section 2. However for the type II string the situation is different. A careful analysis of the tree level action shows that there is an analogue of the Green–Schwarz term
\[ B \wedge F \wedge F \] at tree level \[ \text{§} \]. This term changes at tree level the definition of the type II \( S' \) field. There is an analogous phenomenon which changes also at tree level the definition of the \( T' \) field. The correct formulae read:

\[ S' = A' - \frac{1}{2} Y_1 Y_1' + \frac{U_1}{2} Y_2 Y_2' + i(e^{-\phi'} + \frac{U_2}{2} Y_2 Y_2') \]  \( (7.7) \)

\[ T' = \sqrt{\det G_{ij}} + i B' \]  \( (7.8) \)

where as usual

\[ \frac{1}{\sqrt{\det G_{ij}}} G'_{ij} = \frac{1}{U_2} \left( \begin{array}{c|c} U_1 & \hline U_1 & |U|^2 \end{array} \right) \]  \( (7.9) \)

Thus (7.1)–(7.5) translate to

\[ U = U' \quad , \quad \bar{W} = \bar{W}' \]  \( (7.10) \)

\[ S = T' \quad , \quad T = S' \]  \( (7.11) \)

Let us indicate how the N=4 heterotic-type II duality works at the level of our restricted \((2,18)\) BPS formula given in \((2.42)\). Let us start from the heterotic string not necessarily weakly coupled. We would like however to end up and compare with the weakly coupled type II string. Thus we must take the limit \( T_2 \) large in the mass formula and keep light states

\[ M^2_{\text{het}} = \frac{|-m_1 + m_2 U + \bar{W} \cdot \bar{Q} + S(-\bar{m}_1 U + \bar{m}_2 + \bar{W} \cdot \bar{Q})|^2}{4 S_2(T_2 U_2 - (\bar{W}_2)^2/2)} \]  \( (7.12) \)

Using type II variables from \((7.11)\) we can write \((7.12)\) as

\[ M^2_{\text{pert-II}} = \frac{|-m_1 + m_2 U' + \bar{W}' \cdot \bar{Q} + T'(-\bar{m}_1 U' + \bar{m}_2 + \bar{W}' \cdot \bar{Q})|^2}{4 \left( S'_2 - \frac{\bar{W}'_2}{2} \right) T'_2 U'_2} \]  \( (7.13) \)

This gives the almost correct tree-level type II mass formula in the large \( T'_2 \) limit, taking into account \((7.7)\) and the duality map \((2.51)\). In type II perturbation theory there are no charged states coupled to the Wilson lines. However such states seems to appear in the perturbative formula. The reason is that although such states are not visible in type II perturbation theory their mass is not suppressed in perturbation theory. This is similar to what is expected to happen in conifold transitions \([32]\).

Orbifolding both sides by the same freely acting symmetry we will obtain new dual groundstates, due to the adiabatic argument of \([24]\). Thus we would like to identify the duals of the heterotic models constructed in the previous sections with spontaneously broken supersymmetry.

For concreteness we will go to the \( Z_2 \) submanifold of \( K_3 \) where the conformal field theory is explicit and we will map directly the heterotic to the type II string. The type II partition function on \( K_3 \times T^2 \) at the orbifold point is

\[ Z^{II}_{N=4} = \frac{1}{6} \sum_{\alpha,\beta=0} \sum_{\alpha',\beta'=0} \sum_{h,g=0} (-1)^{\alpha+\beta+\alpha'\beta'} \frac{\varphi^{2[a]}}{[\beta' + g]} \frac{\varphi^{[\alpha+h]}}{[\beta-g]} \times \]  \( (7.14) \)

\[ \text{§This appears at one loop at the heterotic side for both 4-d descendants of } B \wedge F^4 \text{ and } B \wedge R^4. \text{ The } B \wedge R \wedge R \text{ term appears at one loop in the type II side} \[31]. \]
Let us look at the massless bosonic spectrum and match it to that of the N=4 heterotic string. Consider first the NS-NS sector ($\alpha = \bar{\alpha} = 0$). In the untwisted sector ($h=0$) there are 32 degrees of freedom corresponding to the graviton, 2 scalars (axion-dilaton), 4 vectors, and another 20 scalars (the $\Gamma_{2,2}$ and $Z_{4,4}^{\text{twist}}$ moduli). Two of the gauge bosons are graviphotons while the other two belong to U(1) vector multiplets. Thus these four gauge bosons have lattice signature (2,2). Similarly the (2,2) moduli belong to these two vector multiplets while the (4,4) moduli are in multiplets with vectors coming from the R-R untwisted sector. In the twisted sector ($h=1$) there are 16 $Z_2$ invariant ground states in the $T^4/Z_2$ part: $H^I$. There are in total $4 \times 16$ massless states, all of them scalars belonging to vector multiplets along with vectors coming from the R-R twisted sector.

In the R-R ($\alpha = \bar{\alpha} = 1$) untwisted sector there are 32 physical degrees of freedom. These correspond to 8 vectors and 16 scalars. The vectors have lattice signature (4,4) and four of them are graviphotons while the other four are in vector multiplets. The sixteen scalars complete the six vector multiplets.

Finally in the R-R, twisted sector, there are $4 \times 16$ massless states corresponding to 16 vectors and 32 scalars.

Here the gauge group is composed of U(1)’s which implies that we are at a generic point in the space of Wilson lines. The perturbative spectrum is charged under two of the graviphotons and two of the other gauge bosons with charges given by $p_L, p_R$ of the $T^2$.

Consider now the freely acting orbifold groundstates on the heterotic side that consisted of a rotation on the (4,4) part of the lattice and a (2,18) translation. Again for simplicity we focus on the $Z_2$ case. The $Z_2$ rotation on the (4,4) part changes on the type II side the sign of the massless states which come from the untwisted R-R sector as well as the scalars coming from the twisted NS-NS sector. The effect of the (2,18) translation $\varepsilon = (\vec{\varepsilon}_L; \varepsilon_R, \vec{\zeta})$ is to give phases to massive charged states, but has no effect on the massless spectrum. Thus at the massless level the NS-NS twisted and R-R untwisted sectors have to be projected out. The projection in the type II case which has the same effect as the (4,4) rotation in the heterotic side is a combination of $(-1)^F_R$, which changes the sign of the right-moving Ramond sector, and the symmetry transformation $e$ described in Appendix A, which acts on the twisted ground states of the orbifold with a minus sign and is inert on anything else.

The $\vec{\zeta}$ translation vector does not act in the perturbative type II string since the perturbative spectrum does not contain states charged under the 16 gauge bosons coming from the R-R twisted sector. However it will act on non-perturbative D-brane states carrying R-R charges. Finally the phase coming from the translation of the (2,2) piece is

$$(-1)^{\vec{m} \cdot \varepsilon_R + \vec{n} \cdot \varepsilon_L} \ (7.15)$$

in the heterotic side. Under the type II-heterotic map (2.51) it becomes in the heterotic side

$$(-1)^{\vec{m} \cdot \varepsilon_R + \vec{n} \times \varepsilon_L} \ (7.16)$$

where $\vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1$. Thus the $\varepsilon_L$ translation acts on the type II side on the magnetically charged states of the momentum-gauge fields of the 2-torus and thus is also not visible in type II perturbation theory. The type II duals have 20 vector multiplets and 4.
Thus, at the perturbative type II level the partition function for the models dual to the heterotic ones is

\[
Z_{N=4-2}^{II} = \frac{1}{16} \sum_{\alpha,\beta=0}^{\text{T}} \sum_{h,g,h_g=0}^{\text{T}} (-1)^{\alpha+\beta+\alpha g+\beta g} \frac{\eta^{2(\alpha+\beta)}}{\eta^{(\alpha+\beta)}} \times (7.17)
\]

Here the reader might have noticed a potential puzzle. Consider a heterotic ground state that contains a translation with \( \vec{m} \). In such a groundstate, in the limit \( \text{Im} T \to 0 \) \( N=2 \) supersymmetry is restored to \( N=4 \). Alternatively speaking \( m_{3/2} \sim \text{Im} T \). Thus, in weakly-coupled heterotic string, we take \( S \to \infty \) and also \( T \to 0 \). According to our duality map described above, there is no perturbative shift of the type II side. Thus, in perturbation theory the type-II ground-state does not look like a spontaneously broken \( N=4 \) groundstate. However a look at (7.16) is enough to convince us that there are two gravitini, with \( m_{3/2} \sim \text{Im} S' \) which are light in the strong coupling region of the type II theory and certainly not visible in the weak coupling type II perturbation theory.

A similar phenomenon can happen in reverse. Consider a freely acting orbifold of the type II (\( N=4 \)) side as in (7.17), where the (2,2) lattice translation acts on the windings of the 2-torus with the phase \( (-1)^{\vec{m} \cdot \vec{n}} \). This is modular invariant on the type II side. On the heterotic side the shift of the 2-torus becomes non-perturbative via the heterotic-type II map \( (2.51) \), \( (-1)^{\vec{m} \cdot \vec{x}} \). Thus, in heterotic perturbation theory, we only see the \( Z_2 \) rotation of the (4,4) torus. As it stands the heterotic ground state is not modular invariant. An extra shift in the gauge lattice is needed (not visible on the type II side). Thus the perturbative heterotic ground state has a \( K_3 \times T^2 \) structure (at the \( Z_2 \) orbifold point) and the supersymmetry \( N = 4 \to N = 2 \) is explicitly broken in perturbation theory. Turning on all Wilson lines we find that the generic massless spectrum has 19 vector multiplets (including the dilaton) and 4 hypermultiplets. Moreover the \( SL(2, Z) \) is broken to \( \Gamma^- (2) \) as can be easily seen by following the fate of \( T \)-duality of the type II dual.

This brings us to analyse the following issue. It is widely believed that \( K_3 \times T^2 \) compactifications of the heterotic string (with \( N=2 \) spacetime supersymmetry) have type II duals. In the cases that have been studied \( (33, 35) \) the type II duals are CY (symmetric) compactifications. One of the spacetime supersymmetries comes from the left-moving sector while the other comes from the right-moving sector. Moreover, it has been argued \( (38) \) that the CY manifold must be a \( K_3 \) fibration. Let us consider the question whether the \( K_3 \times T^2 \) compactification with the standard embedding of the gauge group into \( E_8 \), described by the above orbifold has a type II dual that is a \( K_3 \)-fibration. Such a ground state has generically 19 vector multiplets and 4 hypermultiplets. At the orbifold point with zero Wilson lines the gauge group is \( E_8 \times E_7 \times SU(2) \times U(1)^3 \). The \( K_3 \)-fibered Calabi-Yau must have \( (h_2, h_1) = (19, 3) \). It turns out that such a manifold does not exist \( (37) \). Our previous argument strongly suggests that the correct type II dual of this heterotic compactification is the asymmetric type II groundstate described above where both supersymmetries come from one side. In particular this type II “compactification” does not have a geometrical interpretation. The story becomes more intriguing once we first go to the enhanced symmetry point and subsequently higgs the \( SU(2) \). And there is
a series of candidate $K_3$-fibrations in the list of \[37\] describing a sequence of ground states, obtained from the original one by sequential Higgsing.\[\] We have the following sequence

\[(h_{21}, h_{11}) = (18, 64) \rightarrow (17, 83) \rightarrow (16, 100) \rightarrow (15, 115) \rightarrow (14, 166) \rightarrow (13, 229) \rightarrow (12, 318) \rightarrow (11, 491)\]

These correspond to the cascade breaking \[33, 35\]

\[E_7 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 0\]

This strongly suggests that the higgsing of the SU(2) on the heterotic side corresponds to a non-perturbative transition between the original asymmetric type II vacuum to a symmetric one described by the $(18, 64)$ $K_3$-fibration. In reverse, this CY manifold should have a singular limit where an SU(2) symmetry appears. At this point a new region of moduli space is opening where there is no longer any geometrical interpretation, the ground state being described by an asymmetric CFT. In some respects this looks like the conifold transition but its interpretation seems to be even more exotic. It would be very interesting to quantitatively test this picture.

Another comment concerns the fate of the $SL(2, Z)_S$ electric-magnetic duality symmetry of the original N=4 theory, in the spontaneously broken phase. It is known that in the N=4 case $SL(2, Z)_S$ is a corollary of heterotic-type II duality, since the $T$-duality of type II translates into the S-duality of the heterotic theory. Let us investigate what remains of the perturbative $T$ duality in the broken type II vacuum. We have argued above that the 2-torus on the type II side gets a (perturbative) shift $(\vec{0}; \vec{\varepsilon}_R)$ which amounts to the phase $(-1)^{\vec{m} \cdot \vec{\varepsilon}_R}$. The $SL(2, Z)_T$ acts on the 2-torus charges as the set of matrices

\[
SL(2, Z)_T : \begin{pmatrix} \vec{m} \\ \vec{n} \end{pmatrix} \rightarrow \begin{pmatrix} a & b i\sigma^2 \\ -c i\sigma^2 & d \end{pmatrix} \begin{pmatrix} \vec{m} \\ \vec{n} \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in Z
\]

There are two subgroups of $SL(2, Z)$ that are relevant in this paper. One is $\Gamma^+(2)$ defined by $b$ even in (7.18), the other one is $\Gamma^-(2)$ defined by $c$ even in (7.18). Thus when $\vec{\varepsilon}_R \neq \vec{0}$, $SL(2, Z)_T$ is broken to $\Gamma^+(2)_T$. Thus, the S-duality group of these ground states is $\Gamma^+(2)_S$.

In the above discussion, it is obvious that there are non-perturbative ambiguities in the translation related projections. The most general projection conceivable is determined by our “electric” translation vector $\varepsilon$, but simultaneously by a “magnetic” translation vector $\tilde{\varepsilon}$ whose effects are not visible in the perturbative spectrum. Parts of these translations are never perturbatively visible either on the heterotic or on the type II side. We will comment more on this issue in the next section.

One more remark is in order about the type II duals described above. Inspection shows that all of the N=2 spacetime supersymmetry comes from the left side. Consequently, in these models the S field is in a vector multiplet \[24\]. Thus, like in the heterotic side, the vector-moduli space gets corrections while the hypermultiplet moduli space does not. At generic Wilson lines this class of models has a massless spectrum which consists, apart from the supergravity and the dilaton vector multiplet, of eighteen vector multiplets and four neutral hypermultiplets (the moduli of the 4-torus). The non-perturbatively exact hypermultiplet quaternionic manifold is $SO(4,4)/SO(4) \times SO(4)$. The exactness of the hypermultiplet moduli space restricts the orbifolding possibilities on the type II side to the ones described in (7.17).

\footnote{The final four were already found in \[33\] based on the list given in \[34\].}
The observations made above suggest the intriguing possibility that all heterotic groundstates with \( N=2 \) (or even less) spacetime supersymmetry in four dimensions always have massive gravitini in the full non-perturbative spectrum. All such groundstates would have the features of a spontaneously broken \( N=4 \) theory once non-perturbative corrections are taken into account.

8 Non-perturbative BPS spectrum in partially broken SUSY \( N = 4 \to N = 2 \)

Following the philosophy of [23] we can make a conjecture for the non-perturbative multiplicities of BPS states in the groundstates we discussed with spontaneously broken \( N = 4 \to N = 2 \) supersymmetry. This consists in generalizing the perturbative multiplicity functions (4.18)-(4.20) \( F_i \) to genus-2 forms. First we rewrite \( F_i \) in a more convenient form:

\[
F_1 = \frac{1}{\eta^2} \chi^0 \chi^1, \quad F_\pm = \frac{1}{\eta^2} \left( \chi^0 \mp \chi^1 \right)
\]

where \( \chi^0 \) are given in terms of the characters of four twisted 2d right-moving bosons:

\[
\chi^0 = \frac{4(-)^h \bar{\eta}^6}{\vartheta_{[1+h]} \vartheta_{[1-h]}},
\]

where in the above equation \((h, g) \neq (0, 0)\). We can extend the validity of \( \chi^0 \) for all \((h, g)\) sectors using identities between right-moving, bosonic and fermionic, “twisted” characters:

\[
\chi^0 = \frac{1}{8 \bar{\eta}^6} \sum_{a,b} (-)^h \bar{\eta}^{a+h} \bar{\vartheta}_{[a+h]} \bar{\vartheta}_{[a-h]} \vartheta_{[1+h]} \vartheta_{[1-h]}.
\]

In this expression, the absence of the \((h, g) = (0, 0)\) sector is due to the vanishing of the odd-spin structures \((\bar{\vartheta}_{[1]}\) terms). At genus-2 \( h \) and \( g \) become \( h = (h, \bar{h}) \) and \( g = (g, \bar{g}) \) in correspondence with the “electric” and “magnetic” charge shifts. The generalization in genus-2 of the twisted characters consists in promoting the various \( \vartheta \)-functions with characteristics to their genus-2 form:

\[
\bar{\vartheta}_{[a+h]} (\tau) \to \bar{\vartheta}_{[a+h]} (\bar{\tau}_{ij}).
\]

Then, the proposed non-perturbative multiplicities will be generated by the genus-2 functions:

\[
F_{[h \quad g]} = \Phi(\bar{\tau}_{ij}) \chi_{[h \quad g]} (\bar{\tau}_{ij}),
\]

where \( \Phi(\bar{\tau}_{ij}) \) is the \( N = 4 \) multiplicity function and \( \chi_{[h \quad g]} (\bar{\tau}_{ij}) \) are the genus-2 analogues of the genus-1 “twisted” characters \( \chi_{[h \quad g]} (\tau) \) defined above.

Using the genus-2 multiplicity functions, we can construct weighted free-energy super-traces, which extend at the non-perturbative level the same perturbative quantities, e.g. the moduli dependence of the gauge and gravitational couplings. We define by \( \mathcal{L}^D \) the following quantity:

\[
\mathcal{L}^D = \int_c [dt \prod dX^{ij}] \sum_{h_i, g_i, q_{ij}} D(\tau^{ij}) F_{[h \quad g]} (\bar{\tau}_{ij}) \exp \left[ -2i\pi \Re \tau^{ij} (q_i + \bar{q}_i) \cdot (q_j + \bar{q}_j) \right]
\]
× exp \[−π t M_{BPS}^2(S; \vec{q}_i, \vec{ε}_i) \], \hspace{1cm} (8.6)

where \(M_{BPS}^2(S; \vec{q}_i, \vec{ε}_i)\) stands for the non-perturbative mass formula (2.42) with shifted charges; \(M_{BPS}^2\) depends on the shifted “electric” and “magnetic” charges, the moduli \(T, U, \text{ and } \vec{W}\) as well as the dilaton–axion modulus field \(S\). The “period” matrix \(τ^{ij}\) in eq. (8.7), is constructed in terms of the parameters \(t, X^{ij}\) and \(S\) in the following way:

\[
t = \sqrt{\det(τ^{ij})}, \quad X^{ij} = \text{Re} \, τ^{ij}, \quad \text{and} \quad \frac{τ^{ij}}{\sqrt{\det τ^{ij}}} = \frac{1}{\text{Im}S} \left( \begin{array}{cc} 1 & \text{Re}S \\ \text{Re}S & |S|^2 \end{array} \right). \hspace{1cm} (8.7)
\]

The integration on \(X^{ij}\) in the domain \([-1/2, +1/2]\) would give rise to the non-perturbative matching conditions (4.7). The relevant multiplicities are generated by the functions \(F[\vec{h} \mid \vec{g}]\). This is a suggestive argument, and stands on a similar footing with the analogous \(τ_1\) integration in the perturbative string. However we suggest that, like in the string case, the correct integration domain is the genus-2 fundamental region. Thus we expect that the integration over \(t\) (in the fundamental domain of genus-2 with \(S\) fixed) gives rise to the non-perturbative quantity \(\mathcal{L}_D[S; T, U, \vec{W}]\) in terms of all moduli, \(S\) included.

The kernel \(D\) is the non-perturbative analogue of a product of charge operators. In the perturbative string, this is given by a product of right-moving lattice vectors and contains also a “back-reaction” term \([38]\). There is an analogue of “right-moving” charges in the non-perturbative case when we also include the magnetic charges. The charge sum for the overall trace can be written in the perturbative case as a \(\bar{τ}\) derivative, which would generalize in the non-perturbative treatment to the \(\partial_{\bar{τ}_{11}} + \partial_{\bar{τ}_{22}}\). The “back-reaction” term can be fixed since it has to restore the modular properties of the \(\bar{τ}^{ij}\) derivatives.

The physical interpretation of the summation over the “magnetic” charges reproduces the Euclidean space-time instanton corrections to the couplings.

The determination of the non-perturbative effective coupling constants (the gravitational one included) defines the non-perturbative prepotential of the \(N = 2\) effective theory. Therefore, the knowledge of \(\mathcal{L}_D^P\) determines at the non-perturbative level the \(N = 2\) low-energy effective supergravity, which includes terms up to two derivatives.

9 Outlook

We have demonstrated the existence of partial spontaneous supersymmetry breaking in string theory, and gave several concrete examples in both the heterotic and type II theories. We have studied the issue of restoration of supersymmetry, at the classical and perturbative level. We have further analysed the consequences of heterotic–type II duality valid for the \(N = 2\) models we presented. We have pointed out that in the dual theories the \(N = 4 \rightarrow N = 2\) supersymmetries may look explicitly broken in their perturbation theory. This was also corroborated by our conjecture on the full non-perturbative structure of their effective theories. In some cases we can predict some novel non-perturbative (non-geometric) transitions between vacua of the type II string with (2,0) and (1,1) space-time supersymmetry.

An analysis of the perturbative BPS states of strings, with supersymmetry spontaneously broken \(N = 4 \rightarrow N = 2\), and the underlying duality structure permit us to conjecture the full non-perturbative form of the effective field theory. This conjecture
needs to be elaborated and tested in the context of explicit models. This will be the subject of future analysis.

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Appendix A: Orbifold Blocks

In this appendix we will derive various $Z^2$ orbifold blocks relevant for the partition functions of ground states with spontaneously broken supersymmetry. Consider the 2-torus lattice sum

$$\Gamma_{2,2}(T, U) = \sum_{(\vec{m}; \vec{n}) \in (Z^2; Z^2)} q^{\frac{1}{4} p_L^2} q^{\frac{1}{4} p_R^2}$$

Let us first consider the blocks of the orbifold generated by a non-trivial $Z^2$ translation given by one-half the lattice vector $\varepsilon = (\vec{\varepsilon}_L, \vec{\varepsilon}_R)$, whose components are composed of zeros and ones. Under such a translation, the $U(1)$ current oscillators are invariant while the ground states transform as:

$$|\vec{m}; \vec{n}\rangle \rightarrow e^{\pi i (\vec{\varepsilon}_L \cdot \vec{n} + \vec{\varepsilon}_R \cdot \vec{m})} |\vec{m}; \vec{n}\rangle$$

Then the projected partition function is

$$Z_{2,2}^{\varepsilon, t}[1] = \frac{1}{|\eta|^4} \sum_{(\vec{m}; \vec{n}) \in (Z^2; Z^2)} e^{\pi i (\vec{\varepsilon}_L \cdot \vec{n} + \vec{\varepsilon}_R \cdot \vec{m})} q^{\frac{1}{4} p_L^2} q^{\frac{1}{4} p_R^2}$$

where $t$ stands for the translation element satisfying $t^2 = 1$. In the twisted sector states are in one-to-one correspondence with lattice states with $(\vec{m}; \vec{n})$ shifted by $\varepsilon/2$. Thus,

$$Z_{2,2}^{\varepsilon, t}[t] = \frac{1}{|\eta|^4} \sum_{(\vec{m}; \vec{n}) \in (Z^2; Z^2) + \varepsilon/2} q^{\frac{1}{4} p_L^2} q^{\frac{1}{4} p_R^2}$$

where $t$ stands for the translation element satisfying $t^2 = 1$. In the twisted sector states are in one-to-one correspondence with lattice states with $(\vec{m}; \vec{n})$ shifted by $\varepsilon/2$. Thus,

$$Z_{2,2}^{\varepsilon, t}[g] = \frac{1}{|\eta|^4} \sum_{(\vec{m}; \vec{n}) \in (Z^2; Z^2) + h\varepsilon/2} e^{\pi i g (\vec{\varepsilon}_L \cdot \vec{n} + \vec{\varepsilon}_R \cdot \vec{m})} q^{\frac{1}{4} p_L^2} q^{\frac{1}{4} p_R^2}$$

Similarly, denote by $r$ a $Z_2$ rotation ($r^2 = 1$) that acts in the standard way:

$$r : a^i_m \rightarrow -a^i_m , \bar{a}^i_m \rightarrow -\bar{a}^i_m , |\vec{m}; \vec{n}\rangle \rightarrow | -\vec{m}; -\vec{n}\rangle$$
We have as usual
\[ Z^{\text{twist}}_{2,2}[^{[h]}_g] = \Gamma_{2,2}[^{[h]}_{ro}] = 4 \frac{|\eta|^2}{|\vartheta[1+g]|^2} \quad \text{(A.8)} \]

There are four ground states in the \( r\)-twisted sector, the twist fields \( H^{ij}, i, j = 0, 1 \) which are in one-to-one correspondence with the fixed points of the \( r \) action on the 2-torus located at the half-period points. The twist fields are invariant under the rotation \( r \) but do transform under the translation \( t_\varepsilon \). To determine the transformation properties let us write \( \varepsilon_L = (\varepsilon^1_L, \varepsilon^2_L) \) and \( \varepsilon_R = (\varepsilon^1_R, \varepsilon^2_R) \) and define the matrices
\[ T^1_\varepsilon = \begin{pmatrix} (-1)^{\varepsilon^1_L} (1 - \varepsilon^1_R) & (-1)^{\varepsilon^1_L} \varepsilon^1_R \\ \varepsilon^1_R & 1 - \varepsilon^1_R \end{pmatrix}, \quad T^2_\varepsilon = \begin{pmatrix} (-1)^{\varepsilon^2_L} (1 - \varepsilon^2_R) & (-1)^{\varepsilon^2_L} \varepsilon^2_R \\ \varepsilon^2_R & 1 - \varepsilon^2_R \end{pmatrix} \quad \text{(A.9)} \]

Then
\[ t_\varepsilon : H^{ij} \to \sum_{k,l} T^1_{ik} T^2_{jl} H^{kl} \quad \text{(A.10)} \]

There is another \( Z_2 \) transformation which we will denote by \( e \) which acts trivially on all torus states but has a non-trivial action on the twist fields:
\[ e : H^{ij} \to -H^{ij} \quad \text{(A.11)} \]

Consider also the element \( rt \) which is a product of a rotation and translation, \( (rt)^2 = 1 \). In terms of its action on the spectrum it is essentially the same as the element \( r \). The 4 twist fields are now located at the half-periods shifted by \( \varepsilon_R/4 \):
\[ \Gamma_{2,2}[^{[r]}_{(rt)\varphi}] = \Gamma_{2,2}[^{[r]}_{r\varphi}] \quad \text{(A.12)} \]

Let us now consider the remaining blocks. We have
\[ Z^\varepsilon_{2,2,[r]} = 0 \quad \text{(A.13)} \]

since the \( r \) projection gets contributions only from the states with \( \vec{m} = \vec{n} = \vec{0} \) while the \( t \) twist produces states with \( (\vec{m}; \vec{n}) \neq (\vec{0}; \vec{0}) \) as can be seen from (A.4). We also have
\[ Z^\varepsilon_{2,2,[l]} = 0 \quad \text{(A.14)} \]

consistent with modular invariance and (A.13), due to the transformation properties of the twisted ground states (A.10)
\[ Z^\varepsilon_{2,2,[r]} = 0 \quad \text{(A.15)} \]

since the twisted ground states of \( rt \) are isomorphic to those of \( r \) but localized at quarter points on the torus. The rotation acts by interchanging them giving zero trace. Also
\[ Z^\varepsilon_{2,2,[lt]} = 0 \quad \text{(A.16)} \]

since the translation projection acts non-trivially on the twist fields.

Finally
\[ Z^\varepsilon_{2,2,[rt]} = Z^\varepsilon_{2,2,[rt]} = 0 \quad \text{(A.17)} \]

We can summarize the above in the following way:
\[ Z^\varepsilon_{2,2}|_{h;H} = Z^\varepsilon_{2,2}|_{t^h;H} \]  
(A.18)

with

\[ Z^\varepsilon_{2,2}|_{h;0} = Z^\varepsilon_{2,2}|_{t^h;0} \]  
(A.19)

\[ Z^\varepsilon_{2,2}|_{h;g} = Z^\varepsilon_{2,2}|_{t^h;g} = Z^\varepsilon_{2,2}|^{\text{twist}}_{t^h;g} \]  
(A.20)

\[ Z^\varepsilon_{2,2}|_{h;H} = 0 \text{, otherwise} \]  
(A.21)

In order to construct N=1 models in sections 5,6 the modular properties of the blocks above are important. We will assume that \( \vec{\varepsilon}_R = 0 \) since this is the case of interest. Then

\[ \tau \to \tau + 1 : \quad Z^\varepsilon_{2,2}|_{h;g;G} \to Z^\varepsilon_{2,2}|_{h;H+g;G+H} \]  
(A.22)

\[ \tau \to -\frac{1}{\tau} : \quad Z^\varepsilon_{2,2}|_{g;G} \to Z^\varepsilon_{2,2}|_{h;H} \]  
(A.23)
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