N-jettiness for muon jet pairs in electroweak high-energy processes

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ABSTRACT: We study the $N$-jettiness in the electroweak high-energy process for the final muon jet pairs, $e^{-}e^{+} \rightarrow \mu^{+} \text{jet}+\mu^{-} \text{jet}$. Compared to QCD, the main difference is that there exist additional gauge nonsinglet contributions in the weak interaction, which make the factorization more elaborate. Especially the nonsinglet contributions arise due to the Block-Nordsieck violation in electroweak processes, which yields the Sudakov logarithms and the rapidity divergence. They change the evolution of the factorized parts considerably in the $N$-jettiness. There are two possible channels, initiated from the gauge bosons $WW \rightarrow \ell \mu \ell \mu$, and from the electrons $\ell_e \ell_e \rightarrow \ell \mu \ell \mu$, where $\ell$ denotes the weak doublet. The latter was discussed previously, and we complete the analysis by studying the first. The factorization for $WW \rightarrow \ell \mu \ell \mu$ can be proceeded in a similar way as in the factorization for $\ell_e \ell_e \rightarrow \ell \mu \ell \mu$, and the result exhibits a rich structure. The new ingredients in this study consist of the $W$ beam functions, and the complex color structure of the soft functions and the hard functions. The resummation of the large logarithms is performed by solving the renormalization group equations with respect to the renormalization scale and the rapidity scale. In the numerical analysis, we confine to the SU(2) weak gauge interaction, and the numerical results are presented for both channels at next-to-leading-logarithmic accuracy including the singlet and the nonsinglet contributions. The nonsinglet contributions turn out to be appreciable in the 2-jettiness.

KEYWORDS: jettiness, muon jet pairs in electroweak processes, Block-Nordsieck violation, rapidity divergence, resummation
# Introduction

In high-energy collisions, energetic particles are produced as collimated beams of particles, in the form of jets. The interplay between the constituent partons participating in the hard scattering and the resultant hadrons forming jets after hadronization is intertwined by the perturbative and the nonperturbative effects of quantum chromodynamics (QCD). Since the strong interaction is involved at every stage in the scattering process with disparate energy scales, it is difficult to disentangle the interlocking aspects of the strong interaction.

The proof of the factorization theorems in which the hard, collinear and soft parts are separated in high-energy scattering has been one of the biggest challenges in QCD. In a factorized process, the hard part describes the contribution from the large energy of
order $Q$. The collinear part takes care of the energetic, collinear particles, which include incoming beams and final-state jets. The soft part depicts the soft emissions interspersed between collinear directions.

Soft-collinear effective theory (SCET) [1–4] has set a new stage in proving the factorization for various jet observables in collider physics. SCET is an effective theory, in which collinear, soft modes are selected as the relevant degrees of freedom. The hard degrees of freedom are integrated out to yield the hard function. The collinear modes in different lightcone directions are decoupled, and by redefining the collinear fields with the soft Wilson lines, the soft sector is also decoupled at the Lagrangian level. As a result, the factorization can be established more transparently in SCET than in full QCD.

In SCET, the phase space is divided into the collinear and the soft regions. The invariant masses of their modes may be the same (SCET$_\text{II}$) or different (SCET$_\text{I}$) depending on the physics we are interested in. One of the important objectives in partitioning the phase space is to describe the physics with a single scale in each kinematic region, and perform the resummation of the large logarithms by solving the renormalization (RG) equation.

We step up a gear to consider what happens in electroweak processes at extremely high-energy of order $10$ TeV. Still, the effect of the strong interaction is obviously dominant at LHC. But some electroweak process can be observed in high-energy electron-positron colliders such as CEPC [5], ILC [6], FCC-ee [7], and CLIC [8]. As an example, the process $e^-e^+ \rightarrow \mu^-\mu^+$ can be described in analogy with the QCD process $q\bar{q} \rightarrow q\bar{q}'$. The incoming “partons” in an electron and a positron possess certain energy fractions, and they emit gauge bosons to be far off-shell before they participate in the hard scattering. This process is described by the weak beam functions. The energetic partons undergo a hard scattering and the final-state particles are observed in terms of jets, described by the jet functions. And the soft function describes the emission of soft particles between the collinear particles.

We may suspect that it is just another copy of QCD by merely replacing the gauge group, and nothing in particular can be gained. We claim that it is not true. The main difference lies in the fact that hadrons appear only as color singlets in QCD, but weak doublets such as the electrons or the neutrinos can be observed. Because only the color singlets contribute to the observables in QCD, the beam functions and the jet functions are obtained by taking the matrix elements of the relevant operators between the hadronic states (color singlets) or between the vacuum. However, both singlet and nonsinglet contributions are involved in all these factorized components in electroweak processes, and it makes the analysis more intriguing. We can cast this issue on the imaginary QCD, in which there is no confinement with free quarks and gluons. Then color nonsinglet contributions affect the jet observables and we can ask whether the factorization still works in a consistent fashion in the presence of these extra contributions. Of course, real QCD does not work that way, but we can ask the same question toward the weak interaction.

There are nonsinglet contributions in the weak interaction to the beam, jet, and soft functions as well as singlet contributions. More importantly, because the initial or final states can have fixed charges, Sudakov logarithms can arise from the non-cancellation of the electroweak logarithms, which is known as the Block-Nordsieck violation in electroweak
processes [9–12]. Note that the Sudakov logarithms from virtual and real contributions in QCD cancel in inclusive processes. Due to this cancellation, it yields the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution of the parton distribution functions (PDF) in QCD [13–16]. The non-cancellation of the Sudakov logarithms in weak interaction affects the renormalization behavior. For example, the nonsinglet PDF satisfies nontrivial RG equations due to the Sudakov logarithm, while the singlet PDF still satisfies the DGLAP equation. If we observe more exclusive, or differential quantities with kinematic constraints on the real emissions, there is only a partial cancellation between the virtual and real contributions, resulting in additional Sudakov logarithms. Exclusive jet cross sections, or jet shape observables such as $N$-jettiness are such examples. Then there appear Sudakov logarithms even in the singlet case.

Furthermore, there is the issue of rapidity divergence [17], which arises in SCET due to the dissection of the phase space. If we add the contributions of the factorized parts from all the phase spaces, the rapidity divergence cancels. It is why there is no rapidity divergence in QCD because the phase space is not divided. But aside from the fact that there is no rapidity divergence in the singlet contributions, the rapidity divergence in the electroweak nonsinglet contribution survives. It was first pointed out in ref. [18] that the electroweak nonsinglet PDF has the rapidity divergence. Because of the rapidity divergence, the double RG evolutions with respect to the rapidity scale, as well as the conventional renormalization scale should be solved for both the collinear functions and the soft functions.

In this paper, we consider the $N$-jettiness in electroweak processes, in which the Sudakov logarithms appear both in the singlet and the nonsinglet contributions, while the rapidity logarithm appears in the nonsinglet contributions. These result in part from the Block-Nordsieck violation, and in part from the kinematic constraint to extract the $N$-jettiness. The $N$-jettiness in electroweak processes at extreme high energies is analyzed for the scattering process in which the final dijets involve a muon and an anti-muon respectively. In order to address the main differences between QCD and the weak interaction, we consider the $SU(2)$ weak interaction only. The extension to the Standard Model is an interesting issue, but we will not pursue it here. It is enough to consider $SU(2)$ in considering the features from the nonsinglet contributions. And we select the 2-jettiness because it involves four lightlike directions, in which the dependence of the various lightcone directions appears in the hard and soft functions. It offers a nontrivial check to see the independence of the renormalization and the rapidity scales in the 2-jettiness.

Before we proceed, we need to point out that the electron here means the electron with a cloud of gauge bosons and all the possible leptons. It is analogous to the proton in QCD in the sense that the proton contains gluons and all the possible quarks (and antiquarks). The confusion may arise because there is no distinctive terminology to distinguish the electron cloud, and the electron itself, while a quark describes a parton in the proton in QCD. From now on, the electron $e$ corresponds to the proton in QCD, while the electron doublet $\ell_e$ describes the partons.

With this terminology in mind, we consider the 2-jettiness in the process $e^-e^+ \rightarrow \mu^-\text{-jet} + \mu^+\text{-jet}$. The final states consist of the dijets in which there is at least a muon and antimuon in each jet. In our previous paper [19], the factorization for the channel...
$\ell_e \bar{\tau}_e \to \ell_\mu \bar{\tau}_\mu$ has been analyzed to show the characteristics of the nonsinglet nature in the weak interaction. In order to analyze the 2-jettiness for the $\mu^-\mu^+$ jets, the remaining channel $WW \to \ell_\mu \bar{\tau}_\mu$ should be included, which is the main issue of the paper. In the factorization for this channel, we need the additional input of the gauge-boson beam functions, and the hard and soft functions associated with this channel. Especially the color structure in the soft and hard functions is more involved, which can be expressed in terms of the $3 \times 3$ matrices representing the space of the operators in the channel. After performing the computation at next-to-leading order (NLO), we estimate the contributions from the singlets and the nonsinglets with the resummed results at next-to-leading logarithmic accuracy (NLL) for these two channels. In contrast to the gluon PDF in QCD, the $W$ PDF may be suppressed compared to the electron PDF because the $W$ bosons are emitted perturbatively from the electron. But we keep the possibility of the appreciable $W$ PDF open and proceed. In this sense, the processes $e^+e^- \to WW$, $\ell_LW \to \ell_LW$ or $WW \to WW$, where a muon fragments from the $W$ boson in the final state, are regarded as the processes at higher orders.

The structure of the paper is as follows: In section 2, we construct the effective operators relevant to the channel $WW \to \ell_\mu \bar{\tau}_\mu$ in SCET. We choose a basis of operators proportional to $\delta^{ab}$, $if^{abc}$ and $d^{abc}$, where $f^{abc}$ and $d^{abc}$ are the structure constants of the $SU(N)$ group. This choice of the basis is significant because the symmetric structure constants $d^{abc}$ vanish in the $SU(2)$ weak interaction, and it is separated from the beginning. In section 3, the factorization theorem for $WW \to \ell_\mu \bar{\tau}_\mu$ is established. By including the previous result of the factorization for $\ell_e \bar{\tau}_e \to \ell_\mu \bar{\tau}_\mu$, the whole factorization for the muon-pair dijets is completed. We briefly review the rapidity divergence in SCET, and discuss how to regulate the rapidity divergence in section 4. Each factorized part is computed at NLO. In section 5, the beam functions and the PDFs for the gauge bosons with the matching coefficients are presented. For completeness, the muon semi-inclusive jet functions are quoted from ref. [19]. In section 6, the hard functions for $WW \to \ell_\mu \bar{\tau}_\mu$ are presented, based on the result from QCD [20], and the soft function for this process is given in section 7. In section 8, all the anomalous dimensions are collected, and the RG evolution of the $N$-jettiness is discussed. In section 9, we present a numerical analysis of the 2-jettiness near threshold for the $SU(2)$ gauge interaction. In section 10, we give a conclusion and an outlook. In appendix A, the detailed computation of the beam functions for the gauge bosons and the PDFs at NLO is explained. In appendix B, the tree-level soft matrices are listed.

2 Effective operators for $WW \to \ell_\mu \bar{\tau}_\mu$

The effective operators responsible for $WW \to \ell_\mu \bar{\tau}_\mu$ are given as

$$O^{\alpha\beta}_I = \bar{\ell}_3 T^a_I \gamma^\mu \ell_4 B^{a \alpha}_1 B^{b \beta}_2,$$

where the collinear-gauge invariant lepton fields $\ell_n$ and the gauge fields $B^\mu_\perp$ are defined as

$$\ell_n(x) = W^\dagger_n(x) \xi_n(x), \quad B^\mu_\perp(x) = \frac{1}{g} [W^\dagger_n(x) i D^\mu_n W_n(x)].$$
Here \( iD^\mu_{n_\perp} = \mathcal{P}^\mu_{n_\perp} + gA^\mu_{n_\perp} \) is the covariant derivative. The collinear Wilson line is given as
\[
W_n(x) = \mathbb{P} \exp \left( ig \int_{-\infty}^{0} ds \bar{\nu} \cdot A_n(x + s\bar{\nu}) \right) = \sum_{\text{perm}} \exp \left[ -\frac{g}{\bar{\nu} \cdot \mathcal{P}} \bar{\nu} \cdot A_n(x) \right],
\] (2.3)
where \( \mathbb{P} \) denotes the path ordering along the integration path.

There are three independent operators, and we choose the basis
\[
T_{ab}^I (I = 1, 2, 3) = \delta_{ab}, \quad T_{ab}^2 = i f^{abc} t^c, \quad T_{ab}^3 = d^{abc} t^c.
\] (2.4)
where \( t^a \) are the SU(\( N \)) generators. Another equivalent basis for \( N \geq 3 \) can be selected as \( \delta_{ab}, t^a t^b, \) and \( t^b t^a \). However, in SU(2) weak interaction, the bases \( \delta_{ab}, t^a t^b, \) and \( t^b t^a \) are no longer independent. In other words, \( d^{abc} = 0 \) in SU(2), and there are only two independent operators. In this case, we disregard \( T_{ab}^3 \).

The soft interactions are decoupled from the collinear fields by the field redefinition [4]
\[
\bar{\ell}^a_{n_\perp}(x) = Y^a \bar{\ell}^a_n(x), \quad B_{n_\perp}^{\mu a}(x) = Y^a B_{n_\perp}^{\mu a}(x) Y_n(x),
\] (2.5)
where the soft Wilson line \( Y_n(x) \) in the fundamental representation is given as
\[
Y_n(x) = \mathcal{P} \exp \left( ig \int_{-\infty}^{0} ds \bar{n} \cdot A_n(x + sn) \right) = \sum_{\text{perm}} \exp \left[ -\frac{g}{\bar{n} \cdot \mathcal{P}} \bar{n} \cdot A_n(x) \right].
\] (2.6)
For \( B_{n_\perp}^{\mu a} \), we can employ the adjoint representation \( \mathcal{Y} \) for the soft Wilson line [4] as
\[
B_{n_\perp}^{\mu a}(0) = \mathcal{Y}^a B_{n_\perp}^{\mu a}.
\] (2.7)
where the soft Wilson line \( \mathcal{Y} \) is obtained by replacing \( t^c \) by the generators \( \mathcal{T}^c \) in the adjoint representation with \( (\mathcal{T}^c)^{ab} = -i f^{cab} \). And the relation between \( \mathcal{Y} \) and \( Y \) is given as \( \mathcal{Y}^{ab} = \text{tr}[Y^a_1 b Y_n t^a] \). From now on, we use the fields after the decoupling and we drop the superscript (0) for simplicity.

With the field redefinition, the operators in eq. (2.1) are written as
\[
O_{I \mu}^{a \beta} = \bar{\ell}^a_{3L} \gamma_\mu Y^a_3 Y_{1L} \chi_{1L}^a \mathcal{Y}^{ab} \mathcal{B}_{2 \perp}^{a \beta}.
\] (2.8)
In terms of these operators, we can write the effective Lagrangian as
\[
\mathcal{L}_{\text{eff}} = -i \sum_I D^\mu_I O_{I \mu}^{a \beta} + \text{hermitian conjugate},
\] (2.9)
where \( D^\mu_I \) are the Wilson coefficients, which are obtained by integrating out the degrees of freedom of order \( Q \).

We consider the \( N \)-jettiness, which is defined as [21, 22]
\[
T_N = \sum_k \min \left\{ \frac{2q_i \cdot p_k}{\omega_i} \right\},
\] (2.10)
where \( i \) runs over 1, 2 for the beams, and 3, \( \cdots \), \( N + 2 \) for the final-state jets. Here \( q_i \) are the reference momenta of the beams and the jets with the normalization factors \( \omega_i = \pi_i \cdot q_i \), and \( p_k \) are the momenta of all the measured particles in the final state.

\[
q_{1,2}^\mu = \frac{1}{2} z_{1,2} E_{cm} n_{1,2}^\mu = \frac{1}{2} \omega_{1,2} n_{1,2}^\mu, \quad q_i^\mu = \frac{1}{2} \omega_i n_i^\mu, (i = 3, \cdots, N + 2). \tag{2.11}
\]

Here \( z_{1,2} \) are the momentum fractions of the beams. The lightcone vectors \( n_1 \) and \( n_2 \) for the beams are aligned to the \( z \) direction, \( n_1^\mu = (1, 0, 0, 1) \), \( n_2^\mu = (1, 0, 0, -1) \), and \( n_i \) are the lightcone vectors specifying the jet directions. The 2-jettiness from the channel \( WW \rightarrow \ell_\mu \bar{\ell}_\mu \) can be written as [23]

\[
\left( \frac{d\sigma}{dt_2} \right)_W = \frac{1}{2s} \int d^4x \frac{1}{X} \langle I|\mathcal{L}_{\text{eff}}(x)|X\rangle \langle X|\mathcal{L}_{\text{eff}}(0)|I\rangle \delta(t_2 - g(I, X)), \tag{2.12}
\]

where \(|I\rangle\) represents the initial state, \(|X\rangle\) denotes the final state, and the sum over \( X \) includes the phase space. The function \( g(I, X) \) extracts the jetness from the states \(|I\rangle\) and \(|X\rangle\). In SCET, the final states \(|X\rangle\) consist of the \( n_i \)-collinear states \(|X_i\rangle\) and the soft states \(|X_s\rangle\). Since the \( n_i \)-collinear particles do not interact with each other, and the soft particles are decoupled from the collinear sectors, the final states \(|X\rangle\) in the Hilbert space consist of the tensor product of the collinear states \(|X_i\rangle\) and the soft states \(|X_s\rangle\) as

\[
|X\rangle = |X_1\rangle \otimes |X_2\rangle \otimes |X_3\rangle \otimes |X_4\rangle \otimes |X_s\rangle. \tag{2.13}
\]

In order to obtain the 2-jettiness, we multiply \( O_{ij}^{ab} \) and the hermitian conjugate \( O_{ij}^{\dagger ab} \) with their respective Wilson coefficients and implement the measurement of the 2-jettiness. Since the result looks quite complicated, we first consider the product of the operators, for simplicity. It is written as

\[
\sum_{i,j} D_{Ja'\beta'}^\mu (O_{ij}^{a'b'}(x)) \dagger D_{I\alpha\beta}^\mu O_{ij}^{\mu}(0)
= \sum_{i,j} D_{Ja'\beta'}^\mu D_{Ia\beta}^\mu \left( \bar{\ell}_4 \mu_i j_3 \left( Y_3 \dagger T_{i j}^{a' b'} Y_3 \right)_{j_4 j_3} (\gamma_\mu)_{\mu_4 \mu_3} (\ell_3)_{\mu_3} Y_3^{a'} \bar{B}_1^{\mu'} B_2^{\mu'} \right)(x)
\times \left( \bar{\ell}_3 Y_3^{a'} \dagger T_{i k}^{a b} Y_3 \right)_{i k} (\gamma_\mu)_{\mu_3 \nu_3} (\ell_4)_{\nu_4} Y_4^{a} B_1^{a} B_2^{a} (0), \tag{2.14}
\]

where \( i_k, j_k \) are the gauge indices, while \( \mu_k, \nu_k \) are the Dirac indices. The measurement of the 2-jettiness, \( g(I, X) \), will be implemented after simplifying this expression.

We rearrange the lepton bilinears in each collinear direction as [18, 19]

\[
\left( \bar{\ell}_n(x) \right)_{\alpha} \left( \ell_n(y) \right)_{\beta} = (\mathcal{P}_L)^{\beta \alpha} \left[ \frac{1}{2N} \delta^{ij} \bar{\ell}_n(x) \frac{p_i}{2} \ell_n(y) + (t^c)_{ij} \bar{\ell}_n(x) \frac{p_c}{2} \ell_n(y) \right] = (\mathcal{P}_L)^{\beta \alpha} \sum_c k_c^i (T^c)^{ij} C_{ij}^\alpha(x, y), \tag{2.15}
\]

where \( \mathcal{P}_L = (1 - \gamma_5)/2 \) and we express the generators \((T^0)^{ij} = \delta^{ij}\) for the singlet and \((T^c)^{ij} = (t^c)^{ij}\) for the nonsinglets with the corresponding collinear operators

\[
C_{ij}^\alpha(x, y) = \bar{\ell}_n(x) \frac{p_i}{2} T^a \ell_n(y). \tag{2.16}
\]
The factors $k^m_W$ are defined as $k^0_W = 1/(2N)$ for the singlet and $k^m_W = 1$ for the nonsinglets.

The product of the collinear gauge fields can be written as

$$B_{n \perp}^\alpha(x) B_{n \perp}^\beta(y) = \frac{g^\alpha\beta}{D - 2} \left[ \frac{\delta^{ab}}{N^2 - 1} \left( \delta^{cd} B_{n \perp}^\mu(x) B_{n \perp \mu}^d(y) \right) + \frac{N d^{abc}}{N^2 - 4} \left( d^{de} B_{n \perp}^d(x) B_{n \perp \mu}^e(y) \right) \right]$$

$$\equiv g_{\perp}^{\alpha\beta} \sum_{m=0}^{N^2-1} k^m_W (G^m)^{ab} C^m_W (x,y),$$

(2.17)

where $D$ is the number of the spacetime dimensions. The factors $k^m_W$ are given as

$$k^m_W = \begin{cases} 1 & m = 0, \text{ (singlet)}, \\ \frac{1}{D - 2 N^2 - 1} & m = 1, \ldots, N^2 - 1, \text{ (nonsinglets)}, \\ \frac{1}{D - 2 N^2 - 4} & m = 0, \text{ (singlet)}. \end{cases}$$

(2.18)

and $(G^0)^{ab} = \delta^{ab}$ for the singlet and $(G^m)^{ab} = d^{abm}$ for the nonsinglets. Note that $d^{abm} = 0$ in SU(2), hence there are no nonsinglet beam functions for the gauge bosons. However, we keep these contributions for the general SU($N$) gauge theory, and put $N = 2$ later in the numerical analysis. And $C^m_W(x,y)$ are the operators, which are defined as

$$C^m_W(x,y) = (G^m)^{de} B_{n \perp}^{d\mu}(x) B_{n \perp \mu}^e(y).$$

(2.19)

Using these relations, eq. (2.14) can be written as

$$\sum_{IJ} D_{a'\beta}^{\mu} D_{Ia\beta}^{\mu} \text{Tr}(\mathcal{P}_L \Psi_L \gamma_{\mu} \Psi_R \gamma_{\mu}) g_{\perp}^{\alpha\alpha'} g_{\perp}^{\beta\beta'} \sum_{dfgh} \frac{g_{\perp}^{\gamma\delta}}{\delta^{fg}} k^I_W k^f_W k^d_W k^g_W C^f_I (x,0) C^g_B (0,x) C^d_J (x,0) C^h_J (0,x)$$

$$\times \left( \mathcal{Y}^{d'\delta'}(x) (G^r)^{cd'} (G^f)^{d'd'} \mathcal{Y}^{d'\delta'}(0) \right) \text{Tr} \left[ (T^d Y_4^\dagger T^g Y_3^\dagger Y_3(0)) \right].$$

(2.20)

This is the matrix element squared to be employed in expressing the $N$-jettiness, which is obtained by implementing the measurement of the $N$-jettiness $g(I,X)$ in eq. (2.12). In order to establish the factorization, we define the beam functions and the jet functions as the matrix elements of the collinear fields. The soft function consists of the soft Wilson lines, and the hard function is the combination of the Wilson coefficients of the operators, contracted with the appropriate Dirac structure.

### 3 Factorization of the muon dijet process

Though eq. (2.20) looks complicated, it can be rearranged to make the factorization look manifest. The hard function $H_{I,J}$ is obtained by combining the matching coefficients $D_{Ia\beta}^{\mu}$ contracted with the Dirac structure, and is written as

$$H_{I,J} = D_{Ia\beta}^{\mu} D_{Ia'\beta'}^{\mu} g_{\perp}^{\alpha\alpha'} g_{\perp}^{\beta\beta'} \text{tr}(\mathcal{P}_L \Psi_L \gamma_{\mu} \Psi_R \gamma_{\mu}).$$

(3.1)

Here we follow SCET$_1$, in which the hierarchy of the scales is given by $T^2 \sim M^2 \ll p_T^2 \sim Q T \ll Q^2$, where $M$ is the mass of the gauge boson. The $n$-collinear momentum scales as
\[ p_{\mu}^n = (\pi \cdot p, p_\perp, n \cdot p) \sim (Q, \sqrt{Q^2}, T), \] 
while the soft momentum scales as \( p_{\mu s}^n \sim (T, T, T). \) The \( N \)-jettiness probes the scale of order \( n \cdot p_n \sim n \cdot p_{\mu s} \sim T. \)

The collinear functions are obtained by taking the matrix elements of the operators \( C_W^a \) between the gauge bosons to yield the beam functions, and \( C_t^a \) between the vacuum to yield the jet function after inserting the measurement of the jettiness. The beam functions \( B_W^a(t, z, \mu, \nu) \) are defined as

\[ B_W^a(t, z, \mu, \nu) = -\omega \langle W | B_W^a(t + \omega n \cdot P) | \delta(\omega - \pi \cdot P) B_W^{d\mu}(G^a)^{d|W} \rangle, \tag{3.2} \]

where \( P^- = \bar{n} \cdot P \), and \( t = \omega T \). At tree level, the beam functions are normalized as \( B_W^{(0)}(t, z, \mu, \nu) = \delta(t)\delta(1 - z)(G^a). \)

As introduced in ref. [19], we define the semi-inclusive jet function as

\[ J_l^a(p^2, \mu, \nu) = \int \frac{d\ell}{\pi} \int \frac{d^2p}{\omega} \sum_X \text{Tr}(0 | \delta(n \cdot p + n \cdot P)\delta(2)(p_\perp + P_\perp) | X) \times (X | \delta(n \cdot P) \delta_L(0) | 0), \tag{3.3} \]

where the lepton \( l \) (muon or antimuon) is specified in the final state. It is normalized at tree level as \( J_l^{(0)}(p^2) = \delta(p^2)\text{Tr}(T^a P_1) \), where \( P_1 \) is the projection operator to the given lepton \( l \). For example, the projection operator for the muon is given by \( P_\mu = (1 - \sigma^3)/2 \) in \( \text{SU}(2) \), and \( P_\mu = (1 + \sigma^3)/2 \) for the muon neutrino.

The soft functions are given by the vacuum expectation values of the soft part in eq. (2.20), and are given by

\[ S_{lj}^{fgh}(T_s, \mu, \nu) = \langle 0 | \text{Tr} \left( T^g T^h Y_2 J^I \delta \left( \frac{G_f^{d\delta}}{c_c} \right) (x) \times \delta(T_s - \min \{|n_i \cdot P_X, 0\}) \right) \langle (G^d)^{d\delta} Y_1 J^d T^h Y_3 J^h Y_4 \rangle(0) \rangle. \tag{3.4} \]

The detailed procedure of the factorization is delineated in detail in ref. [19]. Here we present the result following the same procedure in the scheme of SCET. The factorization of the 2-jettiness from \( WW \rightarrow \ell \bar{\ell} \) is written as

\[
\left( \frac{d\sigma}{dT_2} \right)_W = \frac{1}{2s} \int \frac{dz_1}{z_1} \int \frac{dz_2}{z_2} \int dt_1 dt_2 \int \frac{d^4p_3}{(2\pi)^4} \frac{d^4p_4}{(2\pi)^4} (2\pi)^4 \delta(4) \left( \frac{\omega_1 n_1}{2} + \omega_2 n_2 + \omega_3 n_3 + \omega_4 n_4 \right) \sum_{I,J} H_{I,J} \frac{k^0_W}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \times B^I_1(t_1, z_1, \mu, \nu) B^J_2(t_2, z_2, \mu, \nu) J^{n_1}_{I}(p^2_{n_1}, \mu, \nu). \tag{3.5} \]

Combining all these ingredients, the 2-jettiness for the final \( \mu^- \mu^+ \) is obtained by adding the contribution from the initial particles \( WW \) and \( \ell \bar{\ell} \), and is factorized as

\[
\left( \frac{d\sigma}{dT_2} \right) = \frac{1}{2s} \int \frac{dz_1}{z_1} \int \frac{dz_2}{z_2} \int dt_1 dt_2 \int \frac{d^4p_3}{(2\pi)^3} \frac{d^4p_4}{(2\pi)^3} (2\pi)^3 \delta(4) \left( \frac{\omega_1 n_1}{2} + \omega_2 n_2 + \omega_3 n_3 + \omega_4 n_4 \right) \sum_{I,J} H_{I,J} \frac{k^0_W}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \times B^I_1(t_1, z_1, \mu, \nu) B^J_2(t_2, z_2, \mu, \nu) J^{n_1}_{I}(p^2_{n_1}, \mu, \nu) J^{n_2}_{I}(p^2_{n_2}, \mu, \nu) \tag{3.6} \]

\[
\times \sum \frac{k^0_W}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \frac{k^0_l}{m_0^2} \times B^I_1(t_1, z_1, \mu, \nu) B^J_2(t_2, z_2, \mu, \nu) J^{n_1}_{I}(p^2_{n_1}, \mu, \nu) J^{n_2}_{I}(p^2_{n_2}, \mu, \nu),
\]
and the corresponding hard and soft functions in each process should be employed.\textsuperscript{1} We refer to ref. [19] for the factorization in SCET\textsubscript{II} in which the PDFs are evolved from the scale of order $M$, and matched with the beam functions at the scale $\sqrt{s}$.

4 Rapidity divergence

We briefly explain the rapidity divergence and the method of extracting it by prescribing the rapidity regulators. The rapidity divergence arises because the collinear and the soft modes with the same invariant mass reside in disparate phase spaces. These modes lie on the same hyperbola in the $p^+p^-$ plane, and they are distinguished only by their rapidities. When the soft particles reach the collinear region, the momentum approaches the region $p^+p^- \to \infty$, with $p_\perp^2$ fixed. In this limit, an additional divergence is induced, which has different source from the UV and the IR divergences, hence the rapidity divergence cannot be handled with the dimensional regularization. The converse is also true. When the collinear particles reach the soft region, the zero-bin subtraction should be performed to avoid double counting. And the zero-bin contribution induces the rapidity divergence. However, no rapidity divergence shows up in the full theory because there is no separation of the phase space. It means that there may be rapidity divergence in the collinear and the soft sectors in SCET, but it is cancelled in the total contribution.

In spite of the cancellation, we need to regulate the rapidity divergence in each sector. There have been many methods to regulate the rapidity divergence [17, 24–30]. Recently, one of the authors has constructed the soft and collinear rapidity regulators in ref. [31], which correctly yields the angular dependence when different lightcone directions are involved. We will adopt that prescription to compute the rapidity divergence here. Let us describe how to prescribe the rapidity regulators in the collinear and the soft sectors.

In the $n$-collinear sector, the basic idea is to attach a rapidity regulator of the form $(\nu/\pi \cdot k)^\eta$ in the $n$-collinear Wilson line, where the rapidity divergence arises when $\pi \cdot k \to \infty$ with $k_\perp^2$ fixed. This prescription is employed in ref. [28], in which the collinear rapidity divergence appears as poles of $\eta$, and $\nu$ is the rapidity scale. The rapidity divergence is manipulated by modifying the region $\pi \cdot k \to \infty$. However, it also modifies the region $\pi \cdot k \to 0$, which is unwanted in the rapidity evolution. The undesirable divergence in this region is cancelled when the zero-bin subtraction is performed.

The construction of the soft rapidity regulator is different from that in ref. [28], and our choice yields the correct angular dependence of the lightlike directions. In terms of Feynman diagrams, the $n$-collinear Wilson line is obtained by summing over all the emissions of $n$-collinear gluons from the sources other than the $n$-collinear particle. For example, if we consider the back-to-back collinear current $\bar{\xi}_nW_nY_n^\dagger\gamma^\mu Y_nW_n^I\xi_n$, the $n$-collinear Wilson line is obtained by considering the emission of the $n$-collinear gluons from the $\pi$-collinear source, and it is exponentiated to produce the $n$-collinear Wilson line.

On the contrary, the soft Wilson line $Y_n$ is obtained by considering the emission of soft gluons from the $n$-collinear source. Note that the sources of the emitted gluons to

\textsuperscript{1}Because the contraction of the Dirac structure is different in two channels, we use $H^n_e = 16H$, where $H$ is the hard function for $\ell_e\bar{\tau}_e \to \ell_\mu\bar{\tau}_\mu$ in ref. [19].
construct the Wilson lines are different in the collinear case \((W_n)\) and in the soft case \((Y_n)\). If we require the consistency to treat the rapidity divergence in the collinear and the soft sectors, we should track the same source for the rapidity divergence. Therefore, the soft rapidity regulator in the soft Wilson line, corresponding to the \(n\)-collinear gluons in \(W_n\), should be attached to the \(\bar{n}\)-collinear source, that is, \(Y_n\). By generalizing the procedure, let us consider the collinear current \(\bar{\xi}_{n_1}W_{n_1\dagger}Y_{n_1}W_{n_2\dagger}\xi_{n_2}\), which is invariant under collinear and soft gauge transformations respectively. The modified collinear Wilson line \(W_{n_1}\) and the soft Wilson line \(Y_{n_2}\), with the rapidity regulators are given as

\[
W_{n_1} = \sum_{\text{perm}} \exp\left[ -g n_1 \cdot P \left( \frac{\nu}{|n_1 \cdot P|} n_1 \cdot A_{n_1} \right) \right],
\]

\[
Y_{n_2} = \sum_{\text{perm}} \exp\left[ -g n_2 \cdot P \left( \frac{\nu}{|n_2 \cdot P|} n_2 \cdot A_{n_2} \right) \right],
\]

where \(P\) is the operator extracting the momentum. The remaining Wilson lines \(W_{n_2}\) and \(Y_{n_1}\) can be obtained by switching \(n_1\) and \(n_2\). The point in selecting the rapidity regulator is to trace the same emitted gauge bosons both in the collinear and the soft sectors, which are eikonalized to produce the Wilson lines. Note that the rapidity divergences from \(W_{n_1}\) and \(Y_{n_2}\) have the same origin because the collinear and soft gauge bosons are emitted from the \(n_2\)-collinear quark for both of the Wilson lines.

When the rapidity divergence arises in the soft sector, the soft momentum \(k\), in the limit \(n_1 \cdot k \to \infty\) approaches \(k^\mu \approx (n_1 \cdot k)n_1^\mu/2\) and the soft rapidity regulator becomes

\[
\left( \frac{\nu}{n_2 \cdot k} \frac{n_1 \cdot n_2}{2} \right)^\eta \xrightarrow{n_1 \cdot k \to \infty} \left( \frac{\nu}{n_1 \cdot k} \right)^\eta, \tag{4.2}
\]

which has the same form as the collinear rapidity regulator for \(W_{n_1}\). Another pair possessing the same source of rapidity divergence is \(W_{n_2\dagger}\) and \(Y_{n_1\dagger}\).

## 5 Beam functions, PDF and semi-inclusive jet functions

The bare operator for the gauge-boson beam function is given as

\[
O_{W,\text{bare}}^{a}(t,\omega) = -\omega \theta(\omega) B_{n,\mu}^{a}(0) \delta(t + \omega n \cdot P) \delta(\omega - n \cdot P) B_{n,\mu}^{a}(0) (G^a)^{cd}. \tag{5.1}
\]

This operator is extended from QCD \cite{32} to include the nonsinglets. The beam function for the gauge boson is given by the matrix element of the renormalized operator \(O_{W}^{a}(t,\omega,\mu,\nu)\),

\[
B_{W}^{a}(t, x = \omega/P, \mu, \nu) = \langle W(P^-) | O_{W}^{a}(t, \omega, \mu, \nu) | W(P^-) \rangle
= -\omega \langle W | B_{n,\mu}^{a}(t + \omega n \cdot P) \delta(\omega - n \cdot P) B_{n,\mu}^{a}(0) (G^a)^{cd} | W \rangle. \tag{5.2}
\]

Note that there is no nonsinglet beam function for the SU(2) gauge group in the weak interaction because \(\delta^{abc} = 0\). However, we will proceed for the general SU(\(N\)) here. The definition of the singlet beam function for the gauge boson is the same as the gluon beam function in QCD \cite{32}. However, the authors in ref. \cite{32} have computed the gluon beam...
function with massless gluons, with no rapidity regulator. We compute the radiative corrections of the beam function at NLO with the nonzero gauge boson mass $M$, and the rapidity regulators.

The detailed computation of the beam functions at NLO is relegated to appendix A, and we present only the results here. The singlet and the nonsinglet functions in eq. (5.2) are denoted as $B_{W_s}^{(1)}(G^0)^{ab}$ and $B_{W_n}^{(1)}(G^n)^{bc}$ respectively by extracting the group theory factors. The singlet beam function for the gauge boson at NLO is given as

$$B_{W_s}^{(1)}(t,x,\mu) = \frac{\alpha}{2\pi} \left\{ \delta(1-x) \left[ C_A \left( \frac{2}{\epsilon^2} \delta(t) - \frac{2}{\epsilon} \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \right) \right] + \delta(t) \left( \frac{\beta_0}{2} \delta(1-x) + C_A P_{WW}(x) \right) \ln \frac{\mu^2}{\mathcal{L}_1 \left( \frac{t}{\mu^2} \right)} + C_A \delta(t) \left( P_{WW}(x) \ln \frac{1-x}{x} - \frac{\pi^2}{6} \delta(1-x) \right) + C_A \left[ \delta(1-x) \frac{2}{\mu^2} \mathcal{L}_1 \left( \frac{t}{\mu^2} \right) + P_{WW}(x) \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \right] + C_A \delta(t) \left[ \delta(1-x) \left( \frac{31}{18} - \frac{\pi^2}{9} - \frac{\pi}{2\sqrt{3}} \right) - \left( \frac{2(1-x)}{x} + 2x(1-x) + \frac{3x(1-x)}{2(1-x+x^2)} \right) \right] - P_{WW}(x) \ln(1-x+x^2) \right\},$$

(5.3) where $M$ is the mass of the gauge boson. Compared to QCD, the first three lines in eq. (5.3) are the same as those from QCD with the appropriate color factors after replacing $\ln \mu^2/M^2$ by the IR pole $1/\epsilon_{\text{IR}}$. The last two lines are the additional contributions due to the nonzero $M$. Here $P_{WW}(x)$ is the splitting function for $W \rightarrow W W$, which is the same as the splitting function $P_{gg}(x)$ for $g \rightarrow g g$, and is given by

$$P_{WW}(x) = 2 \mathcal{L}_0(1-x)x + 2 \theta(1-x) \left[ \frac{1-x}{x} + x(1-x) \right] = 2 \theta(1-x) \left[ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right],$$

(5.4) and the relation $2(1-x+x^2)/x = (1-x)P_{WW}(x)$ is used. The first term $\beta_0$ in the beta function is given by

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} \left( \frac{1}{2} n_f T_F - \frac{1}{3} n_s T_F \right),$$

(5.5) where $n_s$ is the number of complex scalar multiplets in the theory\(^2\), and the factor 1/2 in front of $n_f$ evokes the fact that only the left-handed fields contribute here.

The nonsinglet beam function for the gauge boson at NLO is given as

$$B_{W_n}^{(1)}(t,x,\mu,\nu) = B_{W_s}^{(1)}(t,x,\mu) + \frac{\alpha C_A}{4\pi} \left\{ 2 \delta(1-x) \left[ \delta(t) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \left( \frac{1}{\eta} + \ln \frac{\nu}{p^2} \right) - \frac{1}{\epsilon^2} \delta(t) + \frac{1}{\epsilon} \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) - \frac{1}{\mu^2} \mathcal{L}_1 \left( \frac{t}{\mu^2} \right) \right] - P_{WW}(x) \left( \delta(t) \ln \frac{\mu^2}{\mathcal{L}_1 \left( \frac{t}{\mu^2} \right)} + \delta(t) \left( P_{WW}(x) \ln \frac{x(1-x+x^2)}{1-x} + \frac{\pi^2}{6} \delta(1-x) \right) + 2 \frac{1-x}{x} + 2x(1-x) + \frac{3}{2(1-x+x^2)} \right\}. $$

(5.6)

\(^2\)In the standard model with the SU(2) weak interaction, there is one scalar multiplet, but in SU(N) gauge theory we need more multiplets to attain equal masses $M$ of the gauge bosons. We will not dwell on the detailed model building to achieve this here.
The result for the nonsinglet beam function in eq. (5.6) is new. Note that the nonsinglet beam function contains the rapidity divergence.

In our approach with SCET, the beam functions are convoluted with the jet, the soft and the hard functions. It corresponds to the case, in which the hierarchy of the scales satisfies $T^2 \sim M^2 \ll p_1^2 \sim QT \ll Q^2$. However, there can be another situation in which $p_1^2 \ll QT$. In this case, the previous collinear momentum in SCET, scaling as $(Q, \sqrt{QT}, T)$ is labeled as the hard-collinear momentum, and the corresponding degrees of freedom should be integrated out to yield SCET. The collinear momentum in SCET scales as $p_1^2 \sim (Q, T, T^2/Q)$. In this procedure the beam function is matched onto the PDF, with the appropriate matching coefficients.

In SCET, the PDF near the scale $T \sim M$ is evolved to the hard-collinear scale, and is matched to the beam functions. And the beam functions are evolved to the factorization scale. The two cases in SCET and SCET are considered in detail in ref [19]. We can deal with both cases, but for simplicity we consider the case of SCET only. Therefore the matching coefficients between the beam functions and the PDF below are derived to be used in SCET for completeness, but they will not be considered here.

The bare gauge-boson PDF operator is defined as

$$Q_W^{\text{bare}}(\omega) = -\omega \theta(\omega) B_{n,\mu}^c(0) \delta(\omega - \vec{p} \cdot \vec{P}) B_{n,\mu}^d(0) (Q^n)^{cd},$$

and the PDF for the gauge boson is given by the matrix element as

$$f_W^a(x = \omega/P^-, \mu, \nu) = \langle W(P^-) | Q_W^a(\omega, \mu, \nu) | W(P^-) \rangle.$$  

By performing the operator-product expansion of $Q_W^a(t, \omega, \mu, \nu)$, it can be expressed in terms of the operators $Q_W^b(\omega)$ with the matching coefficients $T_{ij}^{ab}$ as

$$Q_W^a(t, \omega, \mu, \nu) = \sum_{j, b} \int \frac{d\omega'}{\omega'} T_{ij}^{ab} (t, \omega', \mu) Q_W^b(\omega', \mu, \nu),$$

leading order in SCET. By taking the matrix elements, we obtain the relation between the beam function and the PDF as

$$B_W^a(t, x, \mu, \nu) = \sum_{j, b} \int_x^1 \frac{dx'}{x'} T_{ij}^{ab} (t, \frac{x}{x'}, \mu) f_j^b(x', \mu, \nu).$$

The matching coefficients $T_{ij}^{ab}$ describe the collinear initial-state radiation and can be computed perturbatively. Note that they are independent of the rapidity scale $\nu$. Here $i, j$ are the indices for particle species, and $a, b$ are the weak indices.

The singlet and the nonsinglet PDFs for the gauge bosons are denoted as $f_{Ws}(G^0)^{ab}$ and $f_{Ws}(G^a)^{bc}$ respectively. At NLO, the singlet PDF is given as

$$f_{W_s}^{(1)} = \frac{\alpha}{2\pi} \left\{ \left[ C_A P_{WW}(x) + \frac{\beta_0}{2} \delta(1-x) \right] \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right\}$$

$$- C_A \left[ P_{WW}(x) \ln(1-x+x^2) + 2 \frac{1-x}{x} + 2x(1-x) + 3 \frac{x(1-x)}{2} \right]$$

$$+ \delta(1-x) \left[ C_A \left( \frac{31}{18} - \frac{\pi^2}{2\sqrt{3}} - \frac{\pi^2}{9} \right) - \frac{2}{9} n_f T_F + \frac{1}{2} n_s T_F \left( -\frac{17}{9} + \frac{\pi}{\sqrt{3}} \right) \right].$$
The matching coefficient $\mathcal{I}_{WW}^{s(1)}$ is obtained by comparing eqs. (5.3) and (5.11) as

$$\mathcal{I}_{WW}^{s(1)} = \frac{\alpha C_A}{2 \pi} \left\{ \frac{2}{\mu^2} C_1 \left( \frac{t}{\mu^2} \right) \delta(1-x) + \frac{1}{\mu^2} C_0 \left( \frac{t}{\mu^2} \right) P_{WW}(x) \right. + \delta(t) \left[ P_{WW}(x) \ln \frac{1-x}{x} - \frac{\pi^2}{6} \delta(1-x) \right] \right\}. \tag{5.12}$$

It is the same as the matching coefficient for the gluon case in QCD except the group theory factor. (See eq. (2.14) in ref. [32].)

The nonsinglet PDF for the gauge boson at NLO is given as

$$f_{WW}^{n(1)} = f_{WW}^{s(1)} + \frac{\alpha}{2 \pi} C_A \left[ \delta(1-x) \left( \frac{2}{\eta} + 2 \ln \frac{\nu}{p^2} \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \right. - \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) P_{WW}(x) + 2 \frac{1-x}{x} + 2x(1-x) + \frac{3}{2} \frac{x(1-x)}{1-x+x^2} + P_{WW}(x) \ln(1-x+x^2) \right]. \tag{5.13}$$

The matching coefficient for the nonsinglet $\mathcal{I}_{WW}^{n(1)}$ is proportional to the singlet matching coefficient $\mathcal{I}_{WW}^{s(1)}$, and its relation is given by

$$\mathcal{I}_{WW}^{n(1)} = \frac{1}{2} \mathcal{I}_{WW}^{s(1)}. \tag{5.14}$$

It is interesting to note that the matching coefficients for the nonsinglets are proportional to those for the singlets both for the gauge bosons and for the leptons though the proportionality constant is different. $\mathcal{I}_{WW}^{n(1)} = - \mathcal{I}_{WW}^{s(1)}/(N^2 - 1)$. (See ref. [19].)

For completeness, we present the lepton beam functions from ref. [19]. We express the singlet and nonsinglet beam functions $B_s^0$ and $B_n^0$, in terms of $B_s$ and $B_n$ by extracting and separating the group theory factors as

$$B_s^0(t, x, M, \mu) = B_s(t, x, M, \mu) \text{Tr}(T^0 P_\ell), \quad B_n^0(t, x, M, \mu) = B_n(t, x, M, \mu) \text{Tr}(T^a P_\ell). \tag{5.15}$$

The bare beam functions $B_s$ and $B_n$ are given at NLO as

$$B_s^{(1)}(t, x, M, \mu) = \frac{\alpha C_F}{2 \pi} \left\{ \delta(t) \delta(1-x) \left( \frac{2}{\epsilon^2} + \frac{3}{2 \epsilon} + \frac{9}{4} - \frac{\pi^2}{2} \right) \right. \left. + \delta(t) \left[ P_{\ell\ell}(x) \ln \frac{\mu^2}{x^2 M^2} + (1 + x^2) C_1(1-x) - (1-x) \theta(x) \theta(1-x) \right] \right\}. \tag{5.16}$$

$$B_n^{(1)}(t, x, M, \mu) = B_s(t, x, M, \mu) \left[ -\frac{\alpha C_A}{4 \pi} \left\{ -2 \delta(t) \delta(1-x) \left[ \left( \frac{1}{\eta} + \ln \frac{\nu}{\omega} \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) - \frac{1}{\epsilon^2} + \frac{\pi^2}{12} \right] \right. \right. \left. + \delta(t) \left[ (1 + x^2) \ln \frac{\mu^2}{x^2 M^2} + (1 + x^2) C_1(1-x) - (1-x) \theta(x) \theta(1-x) \right] \right. \left. + \delta(1-x) \left[ -\frac{2}{\epsilon} \mu^2 C_0 \left( \frac{t}{\mu^2} \right) + \frac{2}{\mu^2} C_1 \left( \frac{t}{\mu^2} \right) \right] \right\}. \tag{5.17}$$
Here the splitting function $P_{\ell\ell}(x)$ for $\ell \to \ell W$ is the same as the quark splitting function $P_{qq}(x)$, and is given by

$$P_{\ell\ell}(x) = P_{qq}(x) = \mathcal{L}_0(1-x)(1+x^2) + \frac{3}{2} \delta(1-x) = \left[ \theta(1-x) \frac{1+x^2}{1-x} \right]_+. \quad (5.17)$$

The bare singlet and nonsinglet semi-inclusive jet functions $J_{s}^{0}$ and $J_{s}^{1}$, with the lepton $l$ in the final state, are given as [19]

$$J_{s}^{0}(p^2, M, \mu) = J_{s}(p^2, M, \mu) \text{Tr}(PT^0), \quad J_{s}^{1}(p^2, M, \mu) = J_{s}(p^2, M, \mu) \text{Tr}(P T^a), \quad (5.18)$$

where $P_l$ is the projection operator to the lepton $l$. The bare singlet and the nonsinglet jet functions at NLO are given as

$$J_{s}^{1}(p^2, M, \mu) = \frac{\alpha C_F}{2\pi} \left[ \delta(p^2) \left( \frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2} - \frac{\pi^2}{2} \right) - \left( \frac{2}{\epsilon} + \frac{3}{2} \right) \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{p^2}{\mu^2} \right) + \frac{2}{\mu^2} \mathcal{L}_1 \left( \frac{p^2}{\mu^2} \right) \right],$$

$$J_{s}^{1}(p^2, M, \mu) = J_{s}(p^2, M, \mu) + \frac{\alpha C_F}{2\pi} \left[ \delta(p^2) \left( \frac{1}{\eta} + \ln \frac{\mu^2}{M^2} \right) - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\nu}{\omega} + \ln \frac{\mu^2}{M^2} \ln \frac{\nu}{\omega} + \frac{3}{4} \ln \frac{\mu^2}{M^2} - \frac{5}{8} + \frac{\pi^2}{12} + \frac{1}{\epsilon} + \frac{3}{4} \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{p^2}{\mu^2} \right) - \frac{1}{\mu^2} \mathcal{L}_1 \left( \frac{p^2}{\mu^2} \right) \right]. \quad (5.19)$$

Like the beam functions, the nonsinglet jet function develops the rapidity divergence.

## 6 Hard function

The hard functions can be read off from those in QCD [20]. However, the bases in ref. [20] are $T_{1}^{i} = t^a t^b$, $T_{2}^{i} = t^b t^a$ and $T_{3}^{i} = \delta^{ab}$, while our bases consist of $T_{1} = \delta^{ab}$, $T_{2} = i f^{abc} t^c$ and $T_{3} = d^{abc} t^c$. The change of basis can be obtained by noting the relation

$$t^a t^b = \frac{1}{2} \left[ \frac{1}{N} \delta^{ab} + (if^{abc} + d^{abc}) t^c \right]. \quad (6.1)$$

Therefore the relation between $T_{1}^{i}$ and $T_{i}$ is given by

$$(T_{1}^{i} T_{2}^{i} T_{3}^{i}) = (T_{1} T_{2} T_{3}) A, \quad (6.2)$$

where the transformation matrix $A$ is given by

$$A = \begin{pmatrix} 1/2 N & 1/2 N & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}. \quad (6.3)$$

Then the sum of the operators $D_{i}^{j} O_{i}^{j}$ of the effective Lagrangian in eq. (2.9) has the relation $O_{i}^{j} D_{i}^{j} = O_{i} A_{i j} D_{j}^{j} = O_{j} D_{i}$, from which we obtain the relation $D_{i} = A_{i j} D_{j}^{j}$. Since the hard coefficients $H_{i j}$ are proportional to $D_{j}^{j}$, $H = AH' A^\dagger$.

In addition to the change of the basis, the appropriately adjusted color factors for SU(2), and the fact that only the left-handed leptons contribute to our hard functions should be implemented. With this in mind, the hard function can be written as

$$H_{i j}(s, t, u) = 4 g^{4} H^{(0)}_{i j} + 8 g^{4} \frac{\alpha}{4 \pi} H^{(1)}_{i j} + \cdots, \quad (6.4)$$
where $H_{jj}^{(0)}$ is the hard function at leading order (LO), and $H_{jj}^{(1)}$ at NLO. The Mandelstam variables $s$, $t$, $u$ are given by $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$, $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$, where $p_i$ are the partonic momenta. The variable $u$ is given by $u = -\omega_4 n_1 \cdot n_4 / 2 = -\omega_2 n_2 \cdot n_3 / 2$, and $s$, $t$ can be expressed accordingly.

The LO hard coefficients $H_{jj,\text{QCD}}^{(0)}$ from ref. [20] are given as

$$H^{(0)} = \frac{1}{s^2} \begin{pmatrix}
\frac{u}{t} (t^2 + u^2) & t^2 + u^2 & 0 \\
t^2 + u^2 & \frac{t}{u} (t^2 + u^2) & 0 \\
0 & 0 & 0
\end{pmatrix} , \quad (6.5)$$

and in our basis it becomes

$$H^{(0)}_{W} = \frac{1}{4s^2tu} \begin{pmatrix}
(t + u)^2 (t^2 + u^2) / N^2 & (u^4 - t^4) / N & (t + u)^2 (t^2 + u^2) / N \\
(u^4 - t^4) / N & (t - u)^2 (t^2 + u^2) & (t - u)^2 (t^2 + u^2) \\
(t + u)^2 (t^2 + u^2) / N & u^4 - t^4 & (t + u)^2 (t^2 + u^2)
\end{pmatrix} \quad (6.6)$$

The NLO hard functions $H_{jj}^{(1)}$ are given as

$$H_{11}^{(1)} = \frac{u(t^2 + u^2)}{t s^2} \left[(C_A + C_F)(L(s))^2 + V_1(s,t,u)L(s)\right] + \frac{tu}{s^2} W_1(s,t,u) + \frac{t^3}{s t^2} W_2(s,t,u),$$

$$H_{22}^{(1)} = \frac{t(t^2 + u^2)}{u s^2} \left[(C_A + C_F)(L(s))^2 + V_1(s,u,t)L(s)\right] + \frac{tu}{s^2} W_1(s,u,t) + \frac{t^3}{u s^2} W_2(s,u,t),$$

$$H_{12}^{(1)} = \frac{t^2 + u^2}{2 s^2} \left[-2(C_A + C_F)(L(s))^2 + V_1(s,t,u)L(s) + V_1(s,u,t)L(s)\right] + \frac{tu}{s^2} W_1(s,u,t) + \frac{t^2}{2 s^2} \left(W_1(s,t,u) + W_2(s,u,t)\right),$$

$$H_{13}^{(1)} = \frac{t^2}{2 s^2} V_2(s,t,u)L(s) + \frac{u^2}{2 s t} L(s) + \frac{t}{2 s} W_4(s,t,u) + \frac{u^2}{2 s} W_4(s,u,t),$$

$$H_{23}^{(1)} = \frac{t^2}{2 s^2} V_2(s,t,u)L(s) + \frac{u^2}{2 s} V_2(s,u,t)L(s) + \frac{t^2}{u s} W_4(s,t,u) + \frac{u^2}{2 s} W_4(s,u,t),$$

$H_{33}^{(1)} = 0$, and $H_{jj}^{(1)} = H_{jj}^{(1)*}$.

The functions $V_i$ and $W_i$ are given by

$$W_1(s,t,u) = (C_A - C_F) \left[(L(s) - L(t))^2 + \pi^2\right] + C_A - 8C_F + (7C_A + C_F) \frac{\pi^2}{6}, \quad (6.8)$$

$$W_2(s,t,u) = -\left[C_F - \frac{3}{u^3} - C_A \frac{t^3 + u^3 - 3u^3}{2u^3}\right] \left[(L(s) - L(t))^2 + \pi^2\right] + \left(2C_A \frac{t s}{u^2} + C_F \frac{s(2s - u)}{u^2}\right) (L(t) - L(s)) + C_F \frac{t - 7u}{u} - C_A \frac{t}{u} + (7C_A + C_F) \frac{\pi^2}{6},$$

$$W_3(s,t,u) = 2C_F - 2C_A - \frac{2t}{3s} (C_A - n_f),$$

$$W_4(s,t,u) = \frac{3u}{4t} \left(L(s) - L(u)^2\right)^2 - \left(L(s) - L(t)\right) \left(L(s) - L(u)\right) + \frac{3\pi^2 u^2}{2} \frac{u^2}{t},$$

$$V_1(s,t,u) = 3C_F - 2C_A \left(L(t) - L(s)\right) + \beta_0 \ V_2(s,t,u) = \left(L(s) - L(u)\right) + \beta_0 \ V_2(s,u,t).$$

The function $L(x)$ as a function of the Mandelstam variables is given by

$$L(t) = \ln \frac{-t}{\mu^2}, \quad L(u) = \ln \frac{-u}{\mu^2}, \quad L(s) = \ln \frac{s}{\mu^2} - i\pi. \quad (6.9)$$
In order to obtain $H_{1j}^{(1)}$ in our basis, we perform the transformation $H = AH' A^\dagger$.

7 Soft function

The soft function from $WW \rightarrow \ell_\mu \bar{\ell}_\mu$ is written in terms of $3 \times 3$ matrices in the basis of the operators in eq. (2.8). The computation for the virtual and the real contributions is the same as that in $\ell_\mu \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$, but the complexity comes from the group theory factors in the soft Wilson lines. [See eq. (3.4).] The number of the color factors to be computed is of the order of 6000 to NLO for $WW \rightarrow \ell_\mu \bar{\ell}_\mu$, while it is of order 1000 for $\ell_\mu \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$.

We have constructed a Mathematica package to compute all the group theory factors.

The relevant Feynman diagrams are shown in fig. 1. The virtual contributions in fig. 1(a) are obtained by contracting different soft Wilson lines on the same side of the unitarity cut. The contraction between the same Wilson lines vanishes because $n_i^2 = 0$.

The real contribution, shown in fig. 1(b) is obtained by contracting the soft Wilson lines across the unitary cut.

As in $\ell_\mu \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$, we decompose the soft function into the hemisphere and the non-hemisphere parts. We implement the hemisphere function by the constraint function $F$, with four independent labels, as [19]

$$F(k, \{q_i\}) = F_{ij,\text{hemi}}(k, \{q_i\}) + F_{ij,\text{ml}}(k, \{q_i\}) + F_{ij,\text{lm}}(k, \{q_i\}) + (i \leftrightarrow j),$$

(7.1)

where the hemisphere measurement function for the full hemisphere $q_j > q_i$ is given by

$$F_{ij,\text{hemi}}(k, \{q_i\}) = \theta(q_j - q_i) \delta(k - q_i).$$

(7.2)

Figure 1: Feynman diagrams for the emission of a soft gauge boson from the soft Wilson lines $Y$ or $Y'$. The vertical lines are the final-state cut. (a) The virtual contribution, (b) the real contribution. The gauge bosons attached to the same $i$ do not contribute because $n_i^2 = 0$. All the possible pairs of contractions with different $i$ and $j$ should be summed.
The indices \( l, m \) refer to the remaining lightcone directions when we contract the Wilson lines in the \( i \) and \( j \) directions. The non-hemisphere functions are given as

\[
F_{ij,ml}(k, \{q_i\}) = \theta(q_j - q_i) \theta(q_i - q_m) \theta(q_l - q_m) \left( \delta(k - q_m) - \delta(k - q_l) \right),
\]

\[
F_{ij,lm}(k, \{q_i\}) = \theta(q_j - q_i) \theta(q_i - q_l) \theta(q_m - q_l) \left( \delta(k - q_l) - \delta(k - q_i) \right),
\]

which are the non-hemisphere measurement function for the regions \( m \) and \( l \) respectively. Note that the constraint function \( F \) is constructed for the gauge boson emitted from the soft Wilson lines \( Y_{i1}(Y_{ij}) \) and \( Y_{j1}^{\dagger}(Y_{ij}) \). The hemisphere function for the \( i \) and \( j \) jet directions contains the collinear and the soft divergences. It is enough to focus on the hemisphere soft function, in which all the divergences reside. We can obtain the relevant anomalous dimensions at NLL.

We cite the hemisphere virtual contributions \( S^V_{ij,\text{hemi}} \) and the real contributions \( S^R_{ij,\text{hemi}} \) from ref. [19] as

\[
S^V_{ij,\text{hemi}} = \frac{\alpha}{2\pi} \delta(k) \left[ \frac{1}{\epsilon} - \frac{2}{\eta} \left( 1 + \ln \frac{\mu^2}{M^2} \right) - \frac{2 \ln n_{ij} \nu^2}{\mu^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{M^2} - \frac{\ln n_{ij} \nu^2}{\mu^2} \ln \frac{\mu^2}{M^2} - \frac{\pi^2}{12} \right],
\]

\[
S^R_{ij,\text{hemi}} = -\frac{\alpha}{2\pi} \delta(k) \left[ 2 \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) - \frac{2 \ln \nu^2}{\mu^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{M^2} + \frac{\nu^2}{\mu^2} + \frac{\pi^2}{6} \right]
\]

\[
+ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \frac{2}{\mu} L_0 \left( \frac{k}{\mu} \right) \theta(k < \sqrt{n_{ij} M}) + \frac{2}{k} \left( \frac{1}{\epsilon} + \ln \frac{n_{ij} \nu^2}{k^2} \right) \theta(k > \sqrt{n_{ij} M}) \right\},
\]

where \( n_{ij} = n_i \cdot n_j / 2 \). In terms of these contributions, the hemisphere soft function can be written as

\[
S_{\text{hemi}}(a_1, a_2, a_3, a_4) = \sum_{i \neq j} \left[ S^V_{ij}(a_1, a_2, a_3, a_4) S^V_{ij,\text{hemi}} + S^R_{ij}(a_1, a_2, a_3, a_4) S^R_{ij,\text{hemi}} \right].
\]  

The factors \( S^V_{ij}, S^R_{ij} \) in front of the virtual and real contributions represent the corresponding color factors, and they are represented in terms of the \( 3 \times 3 \) matrices for \( WW \rightarrow l_\mu l_\mu \). The indices \( a_i \) denote the color index in the presence of the nonsinglets from the originating \( i \)-th collinear particle (\( i = 1, 2 \) for the incoming particles, and \( i = 3, 4 \) for the outgoing particles in our convention). For example, the soft color matrix with all the singlets is given by \( S(0, 0, 0, 0) \) and the soft color matrix with the nonsinglet contributions from 1 and 3 is denoted as \( S(1, 0, 1, 0) \), etc. All the soft matrices at tree level are presented in appendix B.

8 Anomalous dimensions and RG evolution

The factorized hard, collinear and soft parts contain the logarithms which become small at their own characteristic scales. The typical hard scale is \( \mu_H \sim Q \), and the collinear scale is given by \( \mu_C \sim \sqrt{Q T} \sim \sqrt{Q M} \), while the soft scale is \( \mu_S \sim M \). However, if the common factorization scale \( \mu_F \) is away from these characteristic scales, the logarithms become so large that the perturbation theory breaks down. We can resum large logarithms by evolving
the factorized parts from their characteristic scales to the common factorization scale $\mu_F$ by solving the RG equations.

The nonsinglet contributions contain additional logarithms associated with the rapidity divergence. The characteristic collinear and soft rapidity scales are $\nu_C \sim Q$, $\nu_S \sim M$ respectively. We perform the double evolution with respect to the rapidity scale $\nu$, as well as the renormalization scale $\mu$. Because the order of the evolution is irrelevant [28], we first evolved with respect to $\nu$ from $\nu_C \sim Q$ for the collinear functions and from $\nu_S \sim M$ for the soft functions to the common factorization scale $\nu_F$. Then we evolve all the factorized functions with respect to $\mu$. The path of the evolution is shown in fig. 2. Note that the collinear scale starts at $\mu_c \sim \sqrt{Q M}$ in SCET$_I$, and we use the evolution of the beam functions from $\mu_C$ to the factorization scale $\mu_F$. However, if we employ SCET$_{II}$, in which we employ the PDF, the evolution of the PDF should start inside the scale $\mu_S \sim M$ to $\sqrt{Q M}$. Above the scale $\sqrt{Q M}$, the evolution of the beam functions should be employed after the matching.

### 8.1 Collinear functions

There are various collinear functions involved in the 2-jettiness. They are the beam functions for the leptons and the gauge bosons, the corresponding PDFs, and the semi-inclusive muon jet functions. Because of the crossing symmetry, the anomalous dimensions of the beam functions and the corresponding jet functions turn out to be the same.

It is convenient to express the $N$-jettiness in terms of the Laplace transforms, in which the factorization consists of the products instead of the convolution. The actual $N$-jettiness is obtained by taking the inverse Laplace transform, which will be discussed more in the

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**Figure 2**: The evolutions of the hard, collinear and soft functions start from their own characteristic scales $\mu_H$, ($\mu_C, \nu_C$), and ($\mu_S, \nu_S$) respectively. Since the order of the evolution is irrelevant, we first evolve the collinear and soft functions with respect to $\nu$, and then $\mu$. The path of the evolution is illustrated by the arrows.
numerical estimates. The Laplace transforms for the beam functions are written as
\[ \tilde{B}_i(\ln \omega_Q, z, M, \mu) = \int_0^\infty dk e^{-sk} B_i(\omega k, z, M, \mu), \]
\[ B_i(\omega k, z, M, \mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{sk} \tilde{B}_i(\ln e^{\gamma E_{QL}}, \mu, \mu), \]
with \( s = 1/(e^{\gamma E_{QL}}) \). Here \( Q_L \) is the scale introduced in performing the Laplace transform.

The Laplace transforms of the PDFs and the jet functions are defined similarly. Let us denote \( \tilde{f}_s \) and \( \tilde{f}_n \) as the singlet and the nonsinglet Laplace-transformed collinear functions, which can be either the beam functions, or the semi-inclusive muon jet functions. The singlet functions evolve with respect to the renormalization scale \( \mu \) only, and the RG equation is written as
\[ \frac{d}{d \ln \mu} \tilde{f}_s(\mu) = \tilde{\gamma}_{fs} \tilde{f}_s(\mu), \]
and the nonsinglet functions \( \tilde{f}_n \) satisfy the coupled RG equations, which are given as
\[ \frac{d}{d \ln \mu} \tilde{f}_n(\mu, \nu) = \tilde{\gamma}_n \tilde{f}_n(\mu, \nu), \quad \frac{d}{d \ln \nu} \tilde{f}_n(\mu, \nu) = \tilde{\gamma}_n \tilde{f}_n(\mu, \nu). \]
Here \( \tilde{\gamma}_{fs}, \tilde{\gamma}_n^\mu \) and \( \tilde{\gamma}_n^\nu \) are the anomalous dimensions of the singlets and the nonsinglets. The \( \mu \) and \( \nu \) anomalous dimensions for the beam functions of the gauge bosons in the Laplace transform are given as
\[ \tilde{\gamma}_{W_s}^\mu = 2C_A \Gamma_c \ln \frac{\mu^2}{Q_L \omega} - 2\gamma_W, \quad \tilde{\gamma}_{W_s}^\nu = 0, \]
\[ \tilde{\gamma}_{W_n}^\mu = \tilde{\gamma}_{W_s}^\mu - C_A \Gamma_c \ln \frac{\mu^2}{Q_L \nu}, \quad \tilde{\gamma}_{W_n}^\nu = C_A \Gamma_c \ln \frac{\mu}{M}, \]
where the non-cusp anomalous dimension at NLO is given as \( \gamma_W^{(1)} = -\beta_0 \alpha/(4\pi) \). The cusp anomalous dimension [33, 34] \( \Gamma_c(\alpha) \) can be expanded as
\[ \Gamma_c(\alpha) = \frac{\alpha}{4\pi} \Gamma_c^0 + \left( \frac{\alpha}{4\pi} \right)^2 \Gamma_c^1 + \cdots, \]
with
\[ \Gamma_c^0 = 4, \quad \Gamma_c^1 = \left( \frac{268}{9} - \frac{4}{3} \pi^2 \right) C_A - \frac{40n_f}{9}. \]
To NLL accuracy, the cusp anomalous dimension to two loops is needed.

The anomalous dimensions for the beam function of the gauge bosons in eq. (8.4), the beam function of the leptons and the semi-inclusive jet functions [19] can be written in a general form as
\[ \tilde{\gamma}_{is}^\mu = 2C_i \Gamma_c \ln \frac{\mu^2}{Q_L \omega} - 2\gamma_i, \quad \tilde{\gamma}_{in}^\mu = \tilde{\gamma}_{is}^\mu - C_A \Gamma_c \ln \frac{\mu^2}{Q_L \nu}, \]
\[ \tilde{\gamma}_{is}^\nu = 0, \quad \tilde{\gamma}_{in}^\nu = C_A \Gamma_c \ln \frac{\mu}{M}. \]
with \( i = \ell, W \), and the group theory factors are given as \( C_\ell = C_F, C_W = C_A \). The non-cusp anomalous dimensions at order \( \alpha \) are given by

\[
\gamma^{(1)}_\ell = -3C_F \frac{\alpha}{4\pi}, \quad \gamma^{(1)}_W = -\beta_0 \frac{\alpha}{4\pi}.
\]  

(8.8)

The evolution of the singlet collinear functions from eq. (8.2) can be written as

\[
\tilde{f}_{is}(\mu_F) = U_{is}(\mu_F, \mu_C) f_{is}(\mu_C),
\]

(8.9)

with \( i = \ell, W \), where the evolution kernel is given as

\[
U_{is}(\mu_F, \mu_C) = \exp \left[ 4C_i S_\Gamma(\mu_F, \mu_C) - 2C_i a_\Gamma(\mu_F, \mu_C) \ln \frac{\omega Q_L}{\mu_C^2} \right].
\]

(8.10)

The various quantities in eq. (8.10) are defined as

\[
\int_{\mu_C}^{\mu_F} \frac{d\mu}{\mu} \Gamma_c(\alpha) \ln \frac{\mu^2}{\omega Q_L} = 2S_\Gamma(\mu_F, \mu_C) - a_\Gamma(\mu_F, \mu_C) \ln \frac{\omega Q_L}{\mu_C^2}.
\]

(8.11)

where

\[
S_\Gamma(\mu, \mu_i) = \int_{\alpha(\mu_i)}^{\alpha(\mu)} \frac{d}\alpha \frac{\Gamma_c(\alpha)}{\beta(\alpha)} \int_{\alpha(\mu_i)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \quad a_f(\mu, \mu_i) = \int_{\alpha(\mu_i)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} f(\alpha),
\]

(8.12)

with \( f(\alpha) = \Gamma_c(\alpha) \) or any other noncusp anomalous dimensions such as \( \gamma_\ell \) or \( \gamma_W \).

On the other hand, the nonsinglet collinear functions satisfy the coupled RG equations, eq. (8.3). And the evolution from both RG equations are written as

\[
\tilde{f}_{in}(\mu_F, \nu_F) = U_{in}(\mu_F, \mu_C; \nu_F) V_{in}(\nu_F, \nu_C; \mu_C) \tilde{f}_{in}(\mu_C, \nu_C),
\]

(8.13)

where the evolution kernels are given as

\[
U_{in}(\mu_F, \mu_C; \nu_F) = U_{is}(\mu_F, \mu_C) \exp \left[ -2C_A S_\Gamma(\mu_F, \mu_C) + C_A a_\Gamma(\mu_F, \mu_C) \ln \frac{\nu_F Q_L}{\mu_C^2} \right],
\]

\[
V_{in}(\nu_F, \nu_C; \mu_C) = \exp \left[ C_A a_\Gamma(\mu_C, M) \ln \frac{\nu_F}{\nu_C} \right].
\]

(8.14)

### 8.2 Hard function

The RG equation for the hard function \( H_W \) in the channel \( WW \to \ell_\mu \bar{\ell}_\mu \) is written as

\[
\frac{d}{d\ln \mu} H_W = \Gamma_W^H H_W + H_W \Gamma_H^W.
\]

(8.15)

The relation between the anomalous dimensions \( \Gamma_H^W \) in our basis, and the anomalous dimensions \( \Gamma_H^{\text{ref.} [20]} \) is given by \( \Gamma_W^H = A \Gamma_W^{\text{ref.} [20]} A^{-1} \). Note that only the nondiagonal matrix is affected by this transformation.

The anomalous dimension matrix \([20, 35]\) can be generalized to include the channels \( WW \to \ell_\mu \bar{\ell}_\mu \) and \( ee \bar{e} \to \ell_\mu \bar{\ell}_\mu \). The anomalous dimensions \( \Gamma_i \ (i = W, e) \) of the hard functions \( H_i \) can be written as

\[
\Gamma_i^H(s, t, u) = \left[ \frac{\epsilon_H}{2} \Gamma_c(\alpha) L_i(\mu) + \gamma_H \right] 1 + \Gamma_c(\alpha) M_i(s, t, u),
\]

(8.16)
where \( c_H = n_\ell C_F + n_W C_A \) with \( n_\ell \) fermions and \( n_W \) gauge bosons involved in the channel. And \( \gamma_H = n_\ell \gamma_\ell + n_W \gamma_W \), with \( L_W(\mu) = L(t), L_\ell(\mu) = L(s) \). The matrices \( \mathbf{M}_W \) and \( \mathbf{M}_e \), in the two channels, are given by

\[
\mathbf{M}_W = \begin{pmatrix}
\frac{C_A + C_F}{2} n_{12} n_{34} & \frac{1}{2} C_A n_{12} n_{34} & \frac{1}{2} C_F n_{12} n_{34} & 0 \\
\ln \frac{n_{13} n_{24}}{n_{14} n_{23}} & \frac{1}{4} C_A n_{13} n_{24} + \frac{C_F}{2} \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} & 0 & \frac{3}{4} C_A n_{14} n_{23} + \frac{C_F}{2} \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} \\
0 & \frac{C_A}{4} n_{13} n_{24} & \frac{C_A n_{12} n_{34}}{n_{14} n_{23}} & -\frac{3}{4} C_A n_{14} n_{23} + \frac{C_F}{2} \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} \\
& & & \end{pmatrix},
\]

\[
\mathbf{M}_e = \begin{pmatrix}
2 C_F - C_A/2 & \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} - \frac{C_A}{2} \ln \frac{n_{12} n_{34}}{n_{14} n_{23}} & \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} & 0 \\
-C_F \left( C_F - C_A/2 \right) \ln \frac{n_{13} n_{24}}{n_{14} n_{23}} & \frac{C_A}{4} n_{12} n_{34} & 0 & \end{pmatrix}
\]

The evolution of the hard functions from the hard scale \( \mu_H \) to the factorization scale \( \mu_F \) is written in the form

\[
\mathbf{H}_i(\mu_F) = \Pi^i_H(\mu_F, \mu_H) \Pi^i_H(\mu_F, \mu_H) \mathbf{H}_i(\mu_H) \Pi^i_H(\mu_F, \mu_H),
\]

where \( \Pi^i_H(\mu_F, \mu_H) \) describes the evolution from the anomalous dimension, proportional to the identity matrix, while \( \Pi^i_H(\mu_F, \mu_H) \) is the evolution kernel from \( \mathbf{M}_i \). They are given as

\[
\Pi^i_H(\mu_F, \mu_H) = \exp \left[ -2 c_H S(\mu_F, \mu_H) + c_H a_T(\mu_F, \mu_H)L_i(\mu_H) + 2 a_H(\mu_F, \mu_H) \right],
\]

\[
\Pi^i_H(\mu_F, \mu_H) = \exp \left[ a_T(\mu_F, \mu_H) \mathbf{M}_i \right].
\]

### 8.3 Soft function

The RG equation for the soft function \( \tilde{S} \) is written as

\[
\frac{d}{d \ln \mu} \tilde{S} = \tilde{\Gamma}_{\tilde{S}}^{\mu} \tilde{S} + \tilde{S} \tilde{\Gamma}_{\tilde{S}}^{\mu},
\]

where \( \tilde{\Gamma}_{\tilde{S}}^{\mu} \) is the \( \mu \)-anomalous dimension matrix. The relation of the soft functions between our basis \( T_f \) and another basis \( T'_f \) is obtained by requiring that \( \text{tr}(\mathbf{H}' S') = \text{tr}(\mathbf{H} S) \), from which we obtain that \( S = (A^{-1})^i S^i A^{-1} \), and \( \tilde{\Gamma}_{\tilde{S}}^{\mu} = A \Gamma_{\tilde{S}}^{\mu} A^{-1} \).

At order \( \alpha \), eq. (8.20) is written as

\[
\frac{d}{d \ln \mu} \tilde{S}^{(1)} = \tilde{\Gamma}_{\tilde{S}}^{\mu(1)} \tilde{S}^{(0)} + \tilde{S}^{(0)} \tilde{\Gamma}_{\tilde{S}}^{\mu(1)},
\]

where \( \tilde{S}^{(1)} \) is the renormalized soft function. The \( \mu \)-soft anomalous dimensions can be extracted from the requirement that the sum of the anomalous dimensions from all the factorized parts should cancel. The \( \mu \)-soft anomalous dimensions at NLO are written as

\[
\tilde{\Gamma}_{iS}^{\mu(1)} = \left( \tilde{\Gamma}_{iH}^{(1)} + (c_H + \tilde{\gamma}_{iH}^{(1)} + \tilde{\gamma}_{iJ}^{(1)}) \otimes 1 \right),
\]

where \( \tilde{\gamma}_{iH}^{(1)} \) are the anomalous dimensions of the beam functions \( (i = \ell, W) \), and \( \tilde{\gamma}_{iJ}^{(1)} \) is that of the jet function. And the RG equation with respect to the rapidity scale \( \nu \) is written as

\[
\frac{d}{d \ln \nu} \tilde{S} = \tilde{\Gamma}_{\tilde{S}}^{\nu} \tilde{S}.
\]
The soft anomalous dimensions for $WW \to \ell_\mu \ell_\mu$ are given by

$$
\left( \Gamma^{k\mu}_S \right)_W = \Gamma_c \left[ \frac{c_H}{4} \ln \frac{Q_L^4}{n_{13} n_{24} \mu^4} + \frac{1}{2} (C_F - C_A) \ln \frac{n_{12}}{n_{34}} \right] - \Gamma_c M_W + \frac{k C_A}{2} \Gamma_c \ln \frac{\mu^2}{Q_L L'}
$$

$$
\left( \Gamma^{k\nu}_S \right)_W = -k C_A \Gamma_c \ln \frac{\mu}{M},
$$

where $k (k = 0, 2, 3, 4)$ is the number of the nonsinglets involved in the process.\(^3\) We can write the soft anomalous dimensions in a compact form including both channels $\ell e \ell e \to \ell_\mu \ell_\mu$ and $WW \to \ell_\mu \ell_\mu$, respectively. The Laplace transformed $\mu$ and $\nu$ soft anomalous dimensions can be written as

$$
\left( \hat{\Gamma}^{\mu}_S \right)_i = \Gamma_c \left[ \frac{c_H}{4} \ln \frac{Q_L^4}{\mu^4 N_i} + B_i \right] 1 - \Gamma_c M_i + \frac{k C_A}{2} \Gamma_c \ln \frac{\mu^2}{Q_L L'} 1,
$$

$$
\left( \hat{\Gamma}^{\nu}_S \right)_i = -k C_A \Gamma_c \ln \frac{\mu}{M},
$$

where $i = \ell$ or $W$. The quantities in eq. (8.25) are given as

$$
N_\ell = n_{12} n_{34}, \quad N_W = n_{13} n_{24}, \quad B_\ell = 0, \quad B_W = \frac{1}{2} (C_F - C_A) \ln \frac{n_{12}}{n_{34}}.
$$

Note that $B_W = 0$ for $n_{12} = n_{34}$ when the jets are back-to-back. The list of the anomalous dimensions for all the factorized parts is shown in table 1.

The evolution of the Laplace-transformed soft functions is written as

$$
\bar{S}_k^i (\mu_F, \nu_F) = \Pi^{\mu}_{Sk} (\mu_F, \mu_S; \nu_F, \nu_S) \Pi^i (\mu_F, \mu_S) \bar{S}_k^i (\mu_S, \nu_S) \Pi_S^i (\mu_F, \mu_S),
$$

where $\Pi^i$ is the evolution kernel from $M_i$ ($i = \ell, W$). And $\Pi^i_{Sk}$ is the kernel from the diagonal part, with $k$ nonsinglets. $\Pi^i_{Sk}$ involves a double evolution with respect to $\mu$ and $\nu$.\(^3\)

\(^3\)The case with $k = 1$ vanishes due to the color conservation, and only the even number of nonsinglets is allowed in SU(2).
the Born result, the uncertainties on the PDF or the beam functions can be somewhat reduced in the ratio.

In order to estimate the singlet and the nonsinglet contributions to the 2-jettiness, we confine ourselves to the threshold limit in which the two jets are emitted back-to-back near \( \theta = \pi/2 \) (perpendicular to the beam axis) at NLL. It not only simplifies the kinematics, but also is available from the current information obtained so far. We explain these points first, and present the numerical analysis accordingly.

Firstly, the anomalous dimensions depend on the angles among the light-like directions. [See, for example, eqs. (8.17) and (8.24).] Therefore the evolution of the 2-jettiness in eq (3.6) involves the integration over these angles. The numerical integration over the angles can be performed, but we would rather consider the differential 2-jettiness \( \frac{d\sigma}{dt} \) at \( \theta = \pi/2 \). It corresponds to the back-to-back jets, perpendicular to the beam directions.

Secondly, the information on the PDFs is not available, in contrast to QCD. That is, the information of the probability in finding a lepton with a certain fraction of the longitudinal momentum is lacking. We naively assume that the beam functions have the form \( B_i(t,x,M,\mu_C) = \delta(t)\delta(1-x) \) for \( i = \ell, W \) at tree level, which amounts to the PDFs with the form \( f_i(x) \propto \delta(1-x) \) at \( \mu_C \). Actually the detailed functional forms at tree level and even at higher orders are not known, hence it should be understood that we pick up the contribution from the region \( x = 1 \). These delta functions will be adopted in our analysis. If we consider the normalized 2-jettiness \( (d\sigma_1/dT_2 dt)/(d\sigma_1^{(0)}/dT_2 dt) \), where \( d\sigma_1^{(0)}/dT_2 dt \) is the Born result, the uncertainties on the PDF or the beam functions can be somewhat reduced in the ratio.

Finally, we have computed the hemisphere soft functions only at NLO, from which the anomalous dimensions can be obtained. The non-hemisphere part of the soft function is finite, and is needed for the computation beyond NLL. Here we confine to NLL accuracy such that only the hemisphere soft function is needed.

In summary, we consider the 2-jettiness near threshold and at NLL accuracy, and the jets are produced perpendicular to the beam directions. In this case, \( \omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega, t = (p_1 - p_3)^2 = -\omega\omega_3n_1 \cdot n_3/2 = -\omega^2(1 - \cos \theta)/2 \) by choosing \( n_1^d = (1, 0, 0, 1) \), \( n_2^d = (1, 0, 0, -1) \), \( n_3^d = (1, \sin \theta, 0, \cos \theta) \) and \( n_4^d = (1, -\sin \theta, 0, -\cos \theta) \).
With the SU(2) gauge interaction, the nonsinglet beam functions for the gauge bosons do not contribute \((d_{abc} = 0)\), and the beam functions for the leptons and the semi-inclusive muon jet functions are employed at NLL. The product of the hard and soft functions for \(WW \to \ell_\mu \ell_\mu\) is written as
\[
\sum_{IJ} H_{IJ}(\mu_F) S_{J}^{00 \ell_\mu}(\mu_F, \nu_F)
\]
\[
= \exp \left[ -4(C_A + C_F)S_\Gamma(\mu_F, \mu_H) - 2(C_A + C_F)a_\Gamma(\mu_F, \mu_H) \ln \frac{\mu_H^2}{\omega^2 n_{13}} \right]
+ 4 \left( a_{\gamma_W}(\mu_F, \mu_H) + a_{\gamma_\tau}(\mu_F, \mu_H) \right) \right]
\]
\[
\times \exp \left[ -4(C_A + C_F)S_\Gamma(\mu_F, \mu_S) + 2(C_A + C_F)a_\Gamma(\mu_F, \mu_S) \ln \frac{\mu_S^2}{\omega S n_{13}} \right]
\]
\[
\times \exp \left[ k \left( 2C_A S_\Gamma(\mu_F, \mu_S) + C_A a_\Gamma(\mu_F, \mu_S) \ln \frac{\mu_S^2}{Q_L \nu_F} - C_A a_\Gamma(\mu_S, M) \ln \frac{\nu_{FS}}{\nu_S} \right) \right]
\]
\[
\times \text{Tr} \left\{ H^{(0)} W \exp \left[ a_\Gamma(\mu_S, \mu_H) \mathbf{M}^t W \right] S_W^{(0)k} \exp \left[ a_\Gamma(\mu_S, \mu_H) \mathbf{M}^t W \right] \right\}.
\] (9.1)

Note that there should be an even number of nonsinglets in the jet functions.

Note that the inverse Laplace transform of the collinear and soft functions is given by
\[
\mathcal{L}^{-1}[Q_L^a] = \mathcal{L}^{-1}\left[ \left( \frac{1}{s \exp(\gamma_E)} \right)^a \right] = \frac{1}{T_2 \Gamma(a)} \left( \frac{T_2}{\exp(\gamma_E)} \right)^a = \frac{\exp \left( a \ln \frac{T_2}{\exp(\gamma_E)} \right)}{T_2 \Gamma(a)},
\] (9.2)

which is also valid for negative \(a\) [36, 37]. Therefore the differential 2-jettiness from \(WW \to \mu^- \mu^+\) is written as
\[
\frac{d\sigma_W}{dT_2 dt} = \frac{4\pi\alpha^2}{\omega^2} \sum_{k=0,2} \left( \frac{1}{6} \right)^2 \left( \frac{1}{4} \right)^{2-k} \left( \frac{1}{2} \right)^k \frac{1}{T_2 \Gamma(a_k) \exp(a_k \gamma_E)}
\]
\[
\times \exp \left[ -4(C_A + C_F)S_\Gamma(\mu_F, \mu_H) - 2(C_A + C_F)a_\Gamma(\mu_F, \mu_H) \ln \frac{\mu_H^2}{\omega^2 n_{13}} \right]
+ 4 \left( a_{\gamma_W}(\mu_F, \mu_H) + a_{\gamma_\tau}(\mu_F, \mu_H) \right) \right]
\]
\[
\times \exp \left[ 8(C_A + C_F)S_\Gamma(\mu_F, \mu_C) - 4(C_A + C_F)a_\Gamma(\mu_F, \mu_C) \ln \frac{\omega T_2}{\mu_C^2}
\right.
- 4 \left( a_{\gamma_W}(\mu_F, \mu_C) + a_{\gamma_\tau}(\mu_F, \mu_C) \right)
\]
\[
+ kC_A \left( -2S_\Gamma(\mu_F, \mu_C) + a_\Gamma(\mu_F, \mu_C) \ln \frac{\nu_F T_2}{\mu_C^2} + a_\Gamma(\mu_F, M) \ln \frac{\nu_F}{\nu_S} \right) \right]
\]
\[
\times \exp \left[ -4(C_A + C_F)S_\Gamma(\mu_F, \mu_S) + 2(C_A + C_F)a_\Gamma(\mu_F, \mu_S) \ln \frac{T_2^2}{\mu_S^2 n_{13}} \right]
\]
\[
+ kC_A \left( 2S_\Gamma(\mu_F, \mu_S) + a_\Gamma(\mu_F, \mu_S) \ln \frac{\mu_S^2}{\omega T_2} - a_\Gamma(\mu_S, M) \ln \frac{\nu_F}{\nu_S} \right) \right]
\]
\[
\times \text{Tr} \left\{ H^{(0)} W \exp \left[ a_\Gamma(\mu_S, \mu_H) \mathbf{M}^t W \right] S_W^{(0)k} \exp \left[ a_\Gamma(\mu_S, \mu_H) \mathbf{M}^t W \right] \right\},
\] (9.3)

where \(a_k\) is given by
\[
a_k = [4(C_A + C_F) - kC_A]a_\Gamma(\mu_C, \mu_S),
\] (9.4)
and \( n_{13} = (1 - \cos \theta)/2 \). And the numerical factors in front come from the product of the color factors \( k_i^a \) in eq. (3.6) and the muon projections in eq. (5.18).

The matrix \( M_W \) near threshold is given as

\[
M_W^{th} = \begin{pmatrix}
-(C_F + C_A) \ln n_{13} & \ln \frac{n_{13}}{1 - n_{13}} & 0 \\
2 \ln \frac{n_{13}}{1 - n_{13}} & C_A \ln \frac{1 - n_{13}}{C_F \ln n_{13}} & 1/2 (-3C_A + 8C_F) \ln \frac{n_{13}}{1 - n_{13}} \\
0 & C_A \ln \frac{1 - n_{13}}{1 - n_{13}} & C_A \ln \frac{1 - n_{13}}{1 - n_{13}} + C_F \ln n_{13}
\end{pmatrix}.
\]

The hard functions and the soft functions at LO are given as

\[
H_W^{(0)} = \frac{1}{n_{13}(1 - n_{13})} \begin{pmatrix}
1 - 2n_{13}(1 - n_{13}) & (1 - n_{13})^4 - n_{13}^4 & 1 - 2n_{13}(1 - n_{13}) \\
N & (1 - n_{13})^4 - n_{13}^4 & (1 - n_{13})^4 - n_{13}^4 \\
N & (1 - n_{13})^4 - n_{13}^4 & 1 - 2n_{13}(1 - n_{13})
\end{pmatrix},
\]

\[
S_W^{(0)0} = S^{(0)}(0, 0, 0, 0) = \begin{pmatrix}
2C_FC_A^2 & 0 & 0 \\
0 & C_FC_A^2 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
S_W^{(0)2} = S^{(0)}(0, 0, 1, 1) = \begin{pmatrix}
2C_FC_A & 0 & 0 \\
0 & C_A(C_F - C_A/2) & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Here \( (a_1 a_2 \cdots a_n) \) denotes \( \text{Tr}(t^{a_1}t^{a_2} \cdots t^{a_n}) \). All the other soft matrices at treel level are presented in appendix B.

Note that the hard and soft anomalous dimensions, hence the corresponding evolution kernels, depend on \( n_{13} = (1 - \cos \theta)/2 \), as well as the hard functions themselves. And we fix \( \theta = \pi/2 \) in the numerical estimation. It corresponds to the back-to-back jets perpendicular to the beam axis.

Similarly, we can write the differential 2-jettiness for the channel \( \ell_\mu \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu \). In contrast to the channel \( WW \rightarrow \ell_\mu \bar{\ell}_\mu \), there are nonsinglet contributions from the electron beam functions. The only constraint is that there should be even number of nonsinglet contributions as a whole. It is given as

\[
\frac{d\sigma_e}{dT_2 dt} = \frac{64\pi^2}{\omega^4} \sum_{k=0,2,4} \left( \frac{1}{4} \right)^k \frac{1}{T_2 \Gamma(a_k) \exp(a_k \gamma_E)}
\]

\[
\times \exp \left[ -8C_FS_F(\mu_F, \mu_H) - 4C_F \alpha_F(\mu_F, \mu_H) \ln \frac{\mu_H^2}{\omega^2} + 8\alpha_c(\mu_F, \mu_H) \right]
\]

\[
\times \exp \left[ 16C_FS_F(\mu_F, \mu_C) - 8C_F \alpha_F(\mu_F, \mu_C) \ln \frac{\omega T_2}{\mu_C^2} - 8\alpha_c(\mu_F, \mu_C) \right]
\]

\[
+ kC_A \left( -2S_F(\mu_F, \mu_C) + \alpha_F(\mu_F, \mu_C) \ln \frac{\mu_F^2 T_2}{\mu_C^2} + \alpha_F(\mu_F, M) \ln \frac{\mu_F^2}{\mu_C^2} \right)
\]

\[
\times \exp \left[ -8C_FS_F(\mu_F, \mu_S) + 4C_F \alpha_F(\mu_F, \mu_S) \ln \frac{T_2}{\mu_S^2} \right]
\]
Finally the soft function with $k$ and the soft functions can be found in ref. [19]. The tree-level soft factors are given as

$$k \left( 2S_F (\mu_F, \mu_S) + a_F (\mu_F, \mu_S) \ln \frac{\mu_S^2}{\mu_F^2} - a_F (\mu_F, M) \ln \frac{\mu_F}{\mu_S} \right)$$

$$\times \text{Tr} \left\{ H_e^{(0)} \exp \left[ a_F (\mu_S, \mu_H) M \right] S_e^{(0)k} \exp \left[ a_F (\mu_S, \mu_H) M \right] \right\}, \quad (9.7)$$

where we choose an even number of nonsinglets out of $T^e$, $T^f$, $T_g$ and $T_h$, which we denote as $k$. In this case, $a_k$ is given as

$$a_k = (8C_F - kC_A)a_F (\mu_C, \mu_S). \quad (9.8)$$

The numerical factor in front comes from the product of the color factors $k_i^n$ in eq. (3.6) and the muon projections in eq. (5.18).

Here all the matrices $M_e$, $H_e^{(0)}$ and $S^{(0)}$ are $2 \times 2$ matrices. The matrix $M_e$ at threshold is written as

$$M_e = \left( \begin{array}{cc} \left( 4C_F - C_A \right) \ln \frac{n_{13}}{1 - n_{13}} + C_A \ln (1 - n_{13}) & 2 \ln \frac{n_{13}}{1 - n_{13}} \\ -C_F \left( 2C_F - C_A \right) \ln \frac{n_{13}}{1 - n_{13}} & 0 \end{array} \right). \quad (9.9)$$

The hard function $H_e^{(0)}$ is given as

$$H_e^{(0)} = \left( \begin{array}{cc} (1 - n_{13})^2 & 0 \\ 0 & 0 \end{array} \right), \quad (9.10)$$

and the soft functions can be found in ref. [19]. The tree-level soft factors are given as follows: For the singlet with $k = 0$,

$$S_e^{(0)}(0,0,0,0) = \left( \begin{array}{cc} C_A C_F / 2 & 0 \\ 0 & C_A \end{array} \right), \quad (9.11)$$

and there are six possible nonsinglet soft functions with $k = 2$ as

$$S_e^{(0)}(1,1,0,0) = \frac{1}{2} \left( \begin{array}{cc} C_F - \frac{C_A}{2} & 0 \\ 0 & 2C_A \end{array} \right), \quad S_e^{(0)}(0,0,1,1) = \frac{1}{2} \left( \begin{array}{cc} C_F - \frac{C_A}{2} & 0 \\ 0 & 2C_A \end{array} \right), \quad (34),$$

$$S_e^{(0)}(1,0,1,0) = \frac{1}{2} \left( \begin{array}{cc} 2C_F - \frac{C_A}{2} & 0 \\ 1 & 0 \end{array} \right), \quad S_e^{(0)}(0,1,0,1) = \frac{1}{2} \left( \begin{array}{cc} 2C_F - \frac{C_A}{2} & 0 \\ 1 & 0 \end{array} \right), \quad (24),$$

$$S_e^{(0)}(1,0,0,1) = \frac{1}{2} \left( \begin{array}{cc} 2C_F - C_A & 0 \\ 1 & 0 \end{array} \right), \quad S_e^{(0)}(0,1,1,0) = \left( \begin{array}{cc} C_F - \frac{C_A}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right), \quad (23). \quad (9.12)$$

Finally the soft function with $k = 4$ is given by

$$S_e^{(0)}(1,1,1,1) = \frac{1}{2} \left( \begin{array}{cc} C_F - \frac{C_A}{2} & 0 \\ 1 & 0 \end{array} \right), \quad (1243) + \frac{1}{2} \left( \begin{array}{cc} C_F - \frac{C_A}{2} & 1 \\ 0 & 0 \end{array} \right), \quad (1342)$$

$$+ \left( \begin{array}{cc} (C_F - \frac{C_A}{2})^2 & C_F - \frac{C_A}{2} \\ C_F - \frac{C_A}{2} & 1 \end{array} \right) (12)(34) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad (13)(24). \quad (9.13)$$
Figure 3: The contributions to the 2-jettiness at NLL from the singlet (blue), the nonsinglet (red), and the total (green) contributions in (a) $WW \rightarrow \ell_\mu \bar{\ell}_\mu$, (b) $\ell_\mu \bar{\ell}_e \rightarrow \mu^- \mu^+$ near threshold. The bands show the theoretical uncertainties.

The plots of the 2-jettiness near threshold at $\theta = \pi/2$ with $50 \text{ GeV} \leq \mathcal{M} \leq 110 \text{ GeV}$ are shown in fig. 3. The singlet, the nonsinglet, and the total contributions are shown respectively with the red, blue, and green curves with the bands showing the theoretical uncertainties at NLL. By normalizing with the tree-level quantities, the uncertainties from the beam functions for the gauge bosons and the electrons in the initial state may be alleviated because there is a partial cancellation at higher orders.
In fig. 3 (a), the contribution to the jettiness from $WW \rightarrow \ell_\mu \bar{\ell}_\mu$ is shown, and the nonsinglet contribution is about 50% of the singlet contribution. It is expected because the nonsinglet contributions are more suppressed due to the additional evolution with respect to $\nu$. In fig. 3 (b), the corresponding jettines from $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$ is shown. It may look surprising that the nonsinglet contribution from $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$ becomes negative. The origin comes from eq. (9.8). When $k = 4$, $a_4$ becomes negative due to the different group theory factors, and the contribution from $k = 4$ is dominant, which causes the jettiness from the nonsinglets to be negative. However, it is not a worrisome problem because the nonsinglet contribution alone is not physical, but the total sum with the singlet contribution is physical. The physical jettiness, that is, the sum of the singlet and the nonsinglet contributions is positive in both cases. Still, the magnitude of the nonsinglet contribution is about 50% of the singlet contribution. Therefore the nonsinglet contribution in $WW \rightarrow \ell_\mu \bar{\ell}_\mu$ is enhanced about 1.5 times compared to the singlet contribution only, while the nonsinglet contribution in $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$ is diminished about by half due to the negative nonsinglet contribution.

Since we do not have information on the beam functions beyond the required accuracy, we do not try to estimate the total contribution to the 2-jettiness by adding the contributions from both channels. However, it is clear that the nonsinglet contributions contribute to the 2-jettiness in the weak interaction significantly in both channels, compared to QCD, in which there is a singlet contribution only.

10 Conclusions

We have analyzed the 2-jettiness in high-energy electroweak scattering in which muon and anti-muon dijets are observed. The underlying processes consist of $\ell_e \bar{\ell}_e, WW \rightarrow \ell_\mu \bar{\ell}_\mu$. The central issue in considering the 2-jettiness in weak interaction stems from the main difference between the electroweak and the QCD processes. That is, there exist gauge nonsinglets (electrons, muons or neutrinos) in weak interaction, while only the color singlets, or the hadrons are observed in strong interaction. This difference results in many interesting aspects, which do not appear in QCD, and exhibits more intricate structure of the factorization.

In inclusive quantities in QCD, the Sudakov logarithm from virtual corrections is cancelled by that from real contributions. For this reason, the PDF is free of the Sudakov logarithm and it satisfies the DGLAP evolution equation. However, if we consider exclusive processes, the phase space in the virtual contribution does not coincide with that in the real contribution. Therefore the residual Sudakov logarithm survives. As a result, the beam function with the small lightcone momentum induces the Sudakov logarithm and obeys a different evolution equation. This type of the Sudakov logarithm appears both in QCD and in weak interaction.

In weak interaction, additional Sudakov logarithms appear even in inclusive quantities because the observed particles include gauge nonsinglets. In this case, the group theory factors in the virtual contributions and in the real contributions are different. Therefore even for inclusive quantities, the sum of the virtual and real contributions does not cancel due to the different group theory factors and it brings out the Sudakov logarithm. It is
known as the Block-Nordsieck violation in electroweak processes. As a result, the PDF for nonsinglets contain the Sudakov logarithm and does not satisfy the DGLAP equation, but a different evolution equation. Furthermore, the factorization in weak interaction may be violated due to the Block-Nordsieck violation Glauber exchange between spectator partons may violate factorization when the weak charges of the final states are specified [38]. The possible breakdown of the factorization may start at order $\alpha^4$ of the magnitude $\sim \alpha^4 \ln^4(M^2/Q^2)$, and it is due to the fact that the group-theory factors for the exchange of two Glauber gauge bosons in different configurations across the unitarity cuts are different and the overall effects do not cancel. This should be considered seriously in ascertaining the factorization in electroweak interaction, but it is beyond the scope of this paper, and has not been considered here.

The nonsinglet beam functions contribute to the 2-jettiness, as well as the singlet contributions. Due to the Sudakov logarithm and the existence of the rapidity divergence, they obey coupled RG equations, and the evolution is distinct from that of the singlet beam functions. It is also interesting to note that the additional contribution to the nonsinglet contributions is proportional to $C_A$, whether it is the beam function (or the PDF) for the gauge bosons or for the lepton. And the matching coefficients which relate the PDF and the beam functions for the singlets and the nonsinglets are proportional to each other, both for the leptons and the gauge bosons. The nonsinglet soft functions get additional contributions, which are also proportional to $C_A$, and they also satisfy the coupled RG equations.

We have considered the singlet and the nonsinglet contributions to the 2-jettiness in both channels $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$ and $WW \rightarrow \ell_\mu \bar{\ell}_\mu$. The nonsinglet contributions in $WW \rightarrow \ell_\mu \bar{\ell}_\mu$ and $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$ are suppressed due to the additional evolution with respect to the rapidity scale, and they are about 50% of the singlet contributions in size. In $\ell_e \bar{\ell}_e \rightarrow \ell_\mu \bar{\ell}_\mu$, the nonsinglet contributions alone are negative. But the physical jettiness is the total sum with the singlet contributions, and it is positive. In summary, though the nonsinglet contributions are suppressed, the nonsinglet contribution plays a significant role in the 2-jettiness numerically, and possibly in other jet shape observables.

In conclusion, we have established a consistent factorization for the 2-jettiness including the nonsinglet contributions, and their contributions are numerically significant. This result is in contrast to QCD, in which there are only singlet contributions. If we have more information on the beam function (or PDF) for the weak gauge bosons and the electrons, the prediction can be more precise.

We admit that the 2-jettiness analyzed here is far from realistic experimental comparison because we confine the gauge group to SU(2). But the main focus here is to present clearly the difference between QCD and the weak interaction. And it turns out that the nonsinglet contributions in electroweak high-energy scattering is appreciable, in comparison to QCD. Of course, we have to choose the electroweak gauge group SU(2)×U(1), to be realistic. The phenomenology with the electroweak gauge group will be studied in the near future.
A NLO calculation of the gauge-boson beam function and PDF

A.1 Beam functions for the gauge bosons

The Feynman rules for the gauge boson $B_{n_1}^{\mu\alpha}$ get complicated due to the insertion of the rapidity regulator. The Feynman rules are shown in fig. 4. At NLO, the Feynman diagrams, contributing to the beam function, are presented in fig. 5. Figure 5 (f) represents the mixing between the fermion and the gauge boson, but it does not possess any UV divergence, hence the beam function does not have any mixing. In obtaining the final result, we take the limit of small mass $M$, and keep only the logarithmic terms in $M$. We also refer to fig. 4 for the Feynman diagrams of the PDFs with the caveat that $t$ is not measured for the PDFs.

Let us first consider the singlet matrix elements which are proportional to $(G^0)_{ab}$. Only the group theory factors differ in the nonsinglet matrix elements. The naive contribution from fig. 5(a) is written as

$$\tilde{M}_{(a)}(G^0)^{ab} = -\mu_2 c_\alpha c_\beta \int \frac{d^D \ell}{(2\pi)^D} (\ell^2 - M^2)^2 V_{3\alpha\nu\nu}^{bc} (-p, \ell, p - \ell)$$

$$\times B_{0}^{\mu}(\ell) B_{0\alpha\nu}(\ell) (2\pi) \delta \left((p - \ell)^2 - M^2\right) \delta\left(\ell - (p^+ + \ell^+)ight) \delta(t - \omega) \theta(\omega) \theta(t) \theta(p - \ell^-),$$

where we average over the possible $(D - 2)$ polarization, but not over the weak charges because we consider the beam function with a fixed weak charge.

$$\frac{1}{D - 2} \sum_{\text{pol}} \epsilon_\alpha \epsilon_\beta = -\frac{g_{\parallel\alpha\beta}}{D - 2}.$$  \hspace{1cm} (A.2)

And $V_{3\alpha\nu\nu}^{abc}(p_1, p_2, p_3)$ is the three-boson vertex, which is given as

$$V_{3\alpha\nu\nu}^{abc}(p_1, p_2, p_3) = g f^{abc} \left[ g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\beta}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu \right].$$  \hspace{1cm} (A.3)

$$\int_{4}^{8}$$

Figure 4: Feynman rules for $B_{n_1}^{\mu\alpha}$ with one and two gauge bosons.
Figure 5: Feynman diagrams for the beam functions and the PDFs. The wavy lines with solid lines denote collinear gauge bosons, and the solid lines are leptons. The mirror images of (b), (d) and (e) are omitted. For the beam functions, the virtuality $t$ and the longitudinal momentum fraction $z$ are measured, but only $z$ is measured for the PDFs.

Since $\tilde{M}_{(a)}$ is finite and we can put $\epsilon = \eta = 0$ and perform the integration. After some algebra, $\tilde{M}_{(a)}$ is given as

$$
\tilde{M}_{(a)} = -\frac{\alpha C_A}{4\pi} \frac{1}{x(t - x^2 M^2)^2} \left[ 2t(-2 + 2x - 3x^2 + 2x^3) + M^2 x(4 + 7x^2 + 2x^3) \right]
$$

$$
\rightarrow \frac{\alpha C_A}{2\pi} \left[ \left( \frac{2}{x} - 2x - 3x^2 + 2x^3 \right) \delta(t) \right. 
+ \left. 2 \left\{ \frac{1 - x}{x} + x(1 - x) + \frac{x}{2} \ln \frac{(1 - x)\mu^2}{x(1 - x + x^2) M^2} + \frac{1}{\mu^2} \right. 
\left. \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \right\} \right].
$$

(A.4)

In the small mass limit, it is regarded as the distribution in $t$. The coefficient of $\delta(t)$ is obtained by integrating $\tilde{M}_{(a)}$ over $t$. The remainder is determined by the fact that $\tilde{M}_{(a)}$ is independent of $\mu$. The zero-bin contribution is suppressed, hence neglected at leading order in SCET. The functions $\mathcal{L}_n(x)$ are defined as

$$
\mathcal{L}_n(x) \equiv \frac{\theta(x)\ln^n x}{x} = \lim_{\beta \to 0} \frac{\theta(x - \beta)\ln^n x}{x} + \delta(x - \beta) \frac{\ln^{n+1} \beta}{n+1}.
$$

(A.5)

The naive contribution $\tilde{M}_{(b)}$ from fig. 5(b) is given as

$$
\tilde{M}_{(b)}(G^0)^{ab} = \frac{\alpha C_A}{4\pi} \frac{\mu^2}{\mathcal{M}_S} \int \frac{d^D \ell}{(2\pi)^D} \frac{\eta^{\alpha\beta} B_{\alpha\beta}(p - \ell, -p)\delta(\ell - \omega)\delta(t + \omega(p^+ - \ell^+))}{D - 2}
$$

$$
\times (2\pi)\delta((p - \ell)^2 - M^2) V^{abc}_{\text{L}}(p, \ell - p, -\ell) \frac{1}{\ell^2 - M^2} B^{0\mu}_0(\ell) \theta(\omega)\theta(p^+ - \ell^+ - \ell^-) \theta(t)
$$

$$
= \frac{\alpha C_A}{4\pi} (G^0)^{ab} \frac{\mu^2 e^{\gamma_E} e}{\Gamma(1 - \epsilon)} \int d\ell^+ d\ell^- d\ell^1 d\ell_1^1 (\ell_1^1)^{-\epsilon} \ell^+ - p^- \ell^- - p^2 - M^2 \left( \frac{\mu^2}{\ell^-(p^- - \ell^-)} \right)^\eta
$$

- 31 -
In the limit of small $M$, the unknown functions $f_A$ with $k_A$ where $x$ is independent of $\mu$. In order to extract the coefficient for $\theta(x)\theta(1-x)$, we first look at the integral

$$
\frac{x(1+x)}{1-x} \int_{xM^2/(1-x)}^{\mu^2} \frac{dt}{t-x^2M^2} = \frac{x(1+x)}{1-x} \ln \frac{(1-x)(\mu^2 - M^2x^2)}{x(1-x + x^2)M^2},
$$

where $\mu$ is chosen to be an arbitrary scale. The right-hand side of eq. (A.10) can be expressed in terms of the dependence on $x$ as

$$
\frac{x(1+x)}{1-x} \ln \frac{(1-x)(\mu^2 - M^2x^2)}{x(1-x + x^2)M^2} = A\delta(1-x) + x(1+x)\mathcal{L}_0(1-x) \ln \frac{\mu^2}{x(1-x + x^2)M^2} + x(1+x)\mathcal{L}_1(1-x),
$$

where $A$ is a constant to be determined. Integrating the left-hand side of eq. (A.9) yields

$$
\int_0^k dx \frac{x(1+x)}{1-x} \ln \frac{(1-x)(\mu^2 - M^2x^2)}{x(1-x + x^2)M^2} = -\frac{9}{2} + \frac{5\sqrt{3}}{6} + \frac{9}{6} = 2 + \frac{5}{2} \ln \frac{\mu^2}{M^2} + \frac{\mu^2}{M^2},
$$

with $k = \mu^2/(\mu^2 + M^2)$, and integrating the right-hand side yields

$$
\int_0^1 dx \left[ A\delta(1-x) + x(1+x)\mathcal{L}_0(1-x) \ln \frac{\mu^2}{x(1-x + x^2)M^2} + x(1+x)\mathcal{L}_1(1-x) \right] = A - \frac{9}{2} + \frac{5\sqrt{3}}{6} = \frac{9}{2} \ln \frac{\mu^2}{M^2}.
$$

By comparing two quantities, we obtain $A = \ln^2(\mu^2/M^2)$.

As a result, we can write $\bar{M}_b$ as

$$
\bar{M}_b = \frac{C_A}{4\pi} \left\{ \delta(t) \left[ \delta(1-x) \ln \frac{\mu^2}{M^2} + x(1+x)\mathcal{L}_0(1-x) \ln \frac{\mu^2}{x(1-x + x^2)M^2} + x(1+x)\mathcal{L}_1(1-x) \right] + f_b(x, \mu_1) \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) + g_b(x, \mu_1) \frac{1}{\mu^2} \mathcal{L}_1 \left( \frac{t}{\mu^2} \right) \right\}.
$$

The unknown functions $f_b(x, \mu)$ and $g_b(x, \mu)$ are determined by the requirement that $\bar{M}_b$ is independent of $\mu$, that is, $d\bar{M}_b/d\ln \mu^2 = 0$. They are given as

$$
f_b(x, \mu) = 2\delta(1-x) \ln \frac{\mu^2}{M^2} + x(1+x)\mathcal{L}_0(1-x), \quad g_b(x) = 2\delta(1-x).
$$
The final result for $\tilde{M}_{(b)}$ is written as

$$
\tilde{M}_{(b)} = \frac{\alpha C_A}{4\pi} \left\{ \delta(t) \left[ \delta(1-x) \ln^2 \frac{\mu^2}{M^2} + x(1+x)\mathcal{L}_0(1-x) \ln \frac{\mu^2}{x(1-x+x^2)M^2} ight. \\
+ x(1+x)\mathcal{L}_1(1-x) \left. + \left(2\ln \frac{\mu^2}{M^2} \delta(1-x) + x(1+x)\mathcal{L}_0(1-x) \right) \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \\
+ 2\delta(1-x) \frac{1}{\mu^2} \mathcal{L}_1 \left( \frac{t}{\mu^2} \right) \right\}.
$$

(A.14)

From eq. (A.6), the zero-bin contribution $M_{(b)}^\varnothing$ can show up when $p - \ell$ or $\ell$ become soft. However, the contribution for the soft $\ell$ is suppressed and only the case with soft $p - \ell$ contributes. It is given, with $b^+ = t/\omega$, as

$$
M_{(b)}^\varnothing = \frac{\alpha C_A}{4\pi} \frac{\mu^2 e^\gamma e}{\Gamma(1-\epsilon)} \delta(1-x) \int_{M^2/b^+}^\infty \frac{d\ell^-}{b^+} \frac{2}{\ell^--(\ell-b^+ - M^2)^{-\epsilon}} \left( \frac{\nu}{\ell^-} \right)^\eta \\
= \frac{\alpha C_A}{2\pi} \delta(1-x) \frac{\mu^2 e^\gamma e}{\Gamma(1+\epsilon)} \left( \frac{\nu}{\omega M^2} \right)^\eta \left( \frac{\nu^2}{\ell^--\eta} \right) \\
= \frac{\alpha C_A}{2\pi} \delta(1-x) \left\{ \delta(t) \left[ \left( \frac{1}{\epsilon} + \ln \frac{\nu}{\ell^-} \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) - \frac{1}{\epsilon^2} + \frac{1}{2} \ln \frac{\mu^2}{M^2} + \frac{\pi^2}{12} \right] \\
+ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \right\}.
$$

(A.15)

The naive contribution $\tilde{M}_{(d)}$ is given as

$$
\tilde{M}_{(d)}(G^0)^{ab} = -\frac{\mu^2 e^\gamma e}{2(D-2)} \int \frac{d^D\ell}{(2\pi)^D} B_0^{\mu\nu}(\ell, p - \ell) V_{\delta a}^{\rho \sigma}(p, \ell - p, -\ell) \\
\times \frac{1}{\ell^2 - M^2} \frac{1}{(p - \ell)^2 - M^2} \delta(p^\lambda - \omega) \delta(t/\omega) \\
= g^2 (G^0)^{ab} C_A \frac{\mu^2 e^\gamma e}{2^{D-2} \mu_{MS}^2} \delta(1-x) \delta(t) \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 - M^2} \frac{1}{(p - \ell)^2 - M^2} \\
\times \left[ \frac{\ell^+ + p^-}{p^- - \ell^-} \left( \frac{\nu}{\ell^-} \right)^\eta + \frac{2p^- - \ell^-}{\ell^-} \left( \frac{\nu}{\ell^-} \right)^\eta \right] \\
= g^2 (G^0)^{ab} C_A \frac{\mu^2 e^\gamma e}{2^{D-2} \mu_{MS}^2} \delta(1-x) \delta(t) \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 - M^2} \frac{1}{(p - \ell)^2 - M^2} \\
\times \frac{\ell^+ + p^-}{(p - \ell)^2 - M^2} \frac{1}{p^- - \ell^-} \left( \frac{\nu}{p^- - \ell^-} \right)^\eta \\
= -\frac{\alpha C_A}{4\pi} (G^0)^{ab} \delta(1-x) \delta(t) \Gamma(\epsilon) \left( \frac{\mu^2}{M^2} \right)^\epsilon \left( \frac{\nu}{p^-} \right)^\eta \int_1^\infty dy \frac{1 + y}{(1-y)^{1+\eta}} (1 - y + y^2)^{-\epsilon}.
$$

(A.16)

The zero-bin contribution $M_{(d)}^\varnothing$ comes from the region in which $p - \ell$ becomes soft, while the contribution from the region in which $\ell$ becomes soft is suppressed. It is given as

$$
M_{(d)}^\varnothing = \frac{\alpha C_A}{4\pi} \delta(1-x) \delta(t) e^{\gamma e} \left( \frac{\mu^2}{M^2} \right)^\epsilon \left( \frac{\nu}{\omega M^2} \right)^\eta \Gamma(\epsilon + \eta/2) \Gamma(1+\eta/2) \int_0^\infty y^{-1+\eta}. 
$$

(A.17)

The net contribution $M_{(d)} = \tilde{M}_{(d)} - M_{(d)}^\varnothing$ is computed and the result is expanded near $\eta = 0$ first, and then near $\epsilon = 0$, and the result is given as

$$
M_{(d)} = \frac{\alpha C_A}{4\pi} \delta(1-x) \delta(t) \left[ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \left( \frac{2}{\eta} + 2 \ln \frac{\nu}{p^-} + 1 \right) + 2 - \frac{\pi}{\sqrt{3}} - \frac{\pi^2}{9} \right].
$$

(A.18)
In fig. 5(e), it involves the self-energy of the gauge boson and the blob contains the loops of the gauge bosons, the ghosts, the fermions and the scalar particles. The Feynman diagrams are shown in figs. 6 and 7. Contrary to QCD, we compute the self-energy corrections with the gauge boson mass \( M \) with the left-handed fermions only. And the additional contributions come from the interaction of the complex scalar multiplets.

The self-energy of the gauge boson can be written as

\[
\Pi(p^2)\delta^{ab}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right),
\]

and the field-strength renormalization \( Z \) is given as

\[
Z = \left(1 - \frac{d\Pi(p^2)}{dp^2}\bigg|_{p^2=M^2}\right)^{-1}.
\]

The field-strength renormalization \( Z_W \) from the gauge bosons and the ghost particles, common to all the SU\((N)\) gauge interactions, comes from fig. 6 (a) to (c). At order \( \alpha \), it is given as

\[
Z_W^{(1)} = \frac{\alpha C_A}{4\pi} \left[ \frac{5}{3} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2}\right) - \frac{5}{9} + \frac{\pi}{\sqrt{3}} \right].
\]

The contribution \( Z_f \) from the fermion loop in fig. 6 (d) at order \( \alpha \) is given by

\[
Z_f^{(1)} = \frac{\alpha}{4\pi} \frac{1}{2} n_f T_F \left[ -\frac{4}{3} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2}\right) - \frac{8}{9} \right].
\]

The factor 1/2 is to remind that only the left-handed fermions contribute.

The remaining contribution comes from the scalar particles. For the SU(2) gauge group, we follow the standard scalar fields in the Standard Model. However, for general SU\((N)\) gauge theories, the symmetry breaking pattern may depend on the model. We just assume that all the gauge bosons have a common mass \( M \), as in SU(2). The Feynman diagrams from the scalar particles are shown in fig. 7. Fig. 7(a) contains an UV divergence, and its contribution is denoted as \( Z_{sa} \) below. Fig. 7(b) depends on the structure of the complex scalar multiplets, to yield the same masses to all the gauge bosons in the SU\((N)\) gauge theory. However, since it is finite and does not contribute to the field-strength

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{Feynman diagrams for the self-energy of the gauge bosons from the loops of (a, b) the gauge bosons, (c) the ghost loops, and (d) the fermions.}
\end{figure}
Figure 7: Feynman diagrams for the self-energy of the gauge bosons from the scalar particles. (a) contains a divergence, (b) is finite, and (c) does not contribute to the field-strength renormalization.

renormalization, we neglect this. Note that this contribution cancels in obtaining the matching coefficients in eq. (5.12). And fig. 7(c) vanishes.

The field-strength renormalization $Z$ at order $\alpha$ is given as

$$Z^{(1)} = Z^{(1)}_W + Z^{(1)}_f + Z^{(1)}_{sa} \quad \text{(A.23)}$$

$$= \frac{\alpha}{4\pi} \left[ (\beta_0 - 2CA) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) + CA(-\frac{5}{9} + \frac{\pi}{\sqrt{3}}) - \frac{4}{9} n_f T_F + n_s T_F \left( -\frac{17}{9} + \frac{\pi}{\sqrt{3}} \right) \right],$$

where $\beta_0 = 11CA/3 - 2n_f T_F/3 - n_s T_F/3$ is the beta function at leading order.

Adding all the contributions, the singlet beam function $B_{W_s}^{(1)}$ for the gauge boson at NLO is written as

$$B_{W_s}^{(1)}(t, x, \mu) = M_{(a)} + 2(M_{(b)} - M^\emptyset_{(b)}) + M_{(d)} - M^\emptyset_{(d)} + Z^{(1)}(1-x)\delta(t)$$

$$= \frac{\alpha}{2\pi} \left\{ \delta(1-x) \left[ C_A \left( \frac{2}{\epsilon^2} \delta(t) - \frac{1}{\epsilon} \frac{1}{\mu^2} L_0\left( \frac{t}{\mu^2} \right) \right) + \frac{\beta_0}{2} \frac{1}{\epsilon} \right] \right.$$

$$+ \delta(t) \left( \frac{\beta_0}{2} \delta(1-x) + C_A P_{WW}(x) \right) \ln \frac{\mu^2}{M^2} + C_A \delta(t) \left( P_{WW}(x) \ln \frac{1-x}{x} - \frac{\pi^2}{6} \delta(1-x) \right)$$

$$+ C_A \left[ \delta(1-x) \frac{2}{\epsilon} \frac{1}{\mu^2} L_1\left( \frac{t}{\mu^2} \right) + P_{WW}(x) \frac{1}{\mu^2} L_0\left( \frac{t}{\mu^2} \right) \right]$$

$$+ C_A \delta(t) \left( \delta(1-x) \left( \frac{31}{18} - \frac{\pi^2}{9} - \frac{\pi}{2\sqrt{3}} \right) - \left( \frac{2(1-x)}{x} + 2x(1-x) + \frac{3}{2} \frac{x(1-x)}{1-x + x^2} \right) \right)$$

$$- P_{WW}(x) \ln(1-x + x^2) \right\} + \delta(1-x) \delta(t) \left( -\frac{2}{9} n_f T_F + \frac{1}{2} n_s T_F \left( -\frac{17}{9} + \frac{\pi}{\sqrt{3}} \right) \right). \quad \text{(A.24)}$$

For the nonsinglet beam functions, only the group theory factors are different. Extracting the nonsinglet matrix elements, proportional to $d^{abc}$, the matrix elements $M_{(a)}$, $M_{(b)}$ and $M^\emptyset_{(b)}$ in the singlet calculation are replaced by $M_{(a)}/2$, $M_{(b)}/2$ and $M^\emptyset_{(b)}/2$ respectively, while the remaining matrix elements are the same. Therefore, the nonsinglet beam
functions at NLO are given as

\[ B_{W_n}^{(1)}(t, x, \mu, \nu) = \frac{1}{2} \tilde{M}_a + \tilde{M}_b - M_{(b)} - 2(\tilde{M}_d - M_{(d)}) + Z^{(1)}(1 - x) \delta(t) \]

\[ = B_{W_s}^{(1)}(t, x, \mu) - \frac{1}{2} \left( \tilde{M}_a + 2(\tilde{M}_b - M_{(b)}) \right) \]

\[ = B_{W_s}^{(1)}(t, x, \mu) + \frac{\alpha C_A}{4\pi} \left[ 2\delta(1 - x) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \left( \frac{1}{\eta} + \ln \frac{\nu}{p} \right) - \frac{1}{\epsilon^2} \delta(t) + \frac{1}{\epsilon} \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) - \frac{1}{\mu^2} \mathcal{L}_1 \left( \frac{t}{\mu^2} \right) \right] - P_{WW}(x) \left( \delta(t) \ln \frac{\mu^2}{M^2} + \frac{1}{\mu^2} \mathcal{L}_0 \left( \frac{t}{\mu^2} \right) \right) + \delta(t) \left( P_{WW}(x) \ln \frac{x(1 - x + x^2)}{1 - x} + \frac{\pi^2}{6} \delta(1 - x) + \frac{1 - x}{x} + 2x(1 - x) + \frac{3}{2} \frac{x(1 - x)}{1 - x + x^2} \right) \right]. \]  

(A.25)

### A.2 PDF for the gauge bosons

We can compute the PDF either by computing the matrix elements in eq. (5.8), or by integrating the results of the beam functions with respect to the variable \( t \). This can be seen by comparing the definitions of the beam functions in eq. (5.2) and those of the PDFs in eq. (5.8). We adopt the second approach here.

We first consider the matrix elements for the singlet PDF. The naive contribution from fig. 5(a) can be obtained from eq. (A.4) by integrating with respect to \( t \), and it is given as

\[ \tilde{M}_a = \frac{\alpha C_A}{2\pi} \left[ 2 \left( \frac{1 - x}{x} + 3x - x^2 \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{(1 - x + x^2)M^2} \right) \right. \]

\[ + \frac{2(1 - x)}{x} + 2x(1 - x) + \frac{3}{2} \frac{x(1 - x)}{1 - x + x^2} \left. \right]. \]  

(A.26)

Integrating eq. (A.6) with respect to \( t \) yields \( \tilde{M}_b \) for the PDF. However, the correct rapidity divergence is obtained only after the zero-bin subtraction is performed. We present the true collinear contribution \( M_b = \tilde{M}_b - M_{(b)}^2 \) as

\[ M_b = \frac{\alpha C_A}{4\pi} \left[ -2 \left( \frac{1}{\eta} + \ln \frac{\nu}{p} \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) + x(1 + x) \mathcal{L}_0(1 - x) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2(1 - x + x^2)} \right) \right]. \]  

(A.27)

In a similar approach to computing \( M_b \), \( M_{(d)} \) can be obtained as

\[ M_{(d)} = \frac{\alpha C_A}{4\pi} \delta(1 - x) \left[ \left( \frac{2}{\eta} + 2 \ln \frac{\nu}{p} + 1 \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) + 2 - \frac{\pi}{\sqrt{3}} - \frac{\pi^2}{9} \right]. \]  

(A.28)

The wave function renormalization \( M_{(e)} \) is given by

\[ Z^{(1)} = \frac{\alpha}{4\pi} \left[ (\beta_0 - 2C_A) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) + C_A \left( -\frac{5}{9} + \frac{\pi}{\sqrt{3}} \right) - \frac{4}{9} n_f T_F + n_s T_F \left( -\frac{17}{9} + \frac{\pi}{\sqrt{3}} \right) \right]. \]  

(A.29)
Combining all the matrix elements, the singlet PDF at NLO is given as

$$f_{W_s}^{(1)}(x, \mu, \nu) = \frac{1}{2} M_a + 2(M_b + M_d) + Z^{(1)} \delta(1 - x)$$

$$= \frac{\alpha}{2\pi} \left\{ C_A \left( P_{WW}(x) + \frac{\beta_0}{2} \delta(1 - x) \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) \right.$$  

$$- C_A \left( P_{WW}(x) \ln(1 - x + x^2) + 2 \frac{1 - x}{x} + 2x(1 - x) + \frac{3}{2} \frac{x(1 - x)}{1 - x + x^2} \right)$$

$$+ \delta(1 - x) \left[ C_A \left( \frac{31}{18} - \frac{\pi}{2\sqrt{3}} - \frac{\pi^2}{9} - \frac{2}{9} n_f T_F + n_s T_F \left( -\frac{17}{9} + \frac{\pi}{\sqrt{3}} \right) \right) \right].$$  \hspace{1cm} (A.30)

The matrix elements for the nonsinglet PDFs have different group theory factors, and

the relation between the singlets and the nonsinglets is the same as that in the beam

functions. The nonsinglet PDFs at NLO are written as

$$f_{W_n}^{(1)}(x, \mu, \nu) = \frac{1}{2} M_a + 2(M_b - M_\theta) + Z^{(1)} \delta(1 - x)$$

$$= f_{W_s}^{(1)}(x, \mu) - \frac{1}{2} \left( \tilde{M}_a + 2(\tilde{M}_b - M_\theta) \right)$$

$$= f_{W_s}^{(1)}(x, \mu) + \frac{\alpha}{2\pi} \frac{C_A}{2} \left[ \delta(1 - x) \left( \frac{2}{\eta} + 2 \ln \frac{\nu}{p^2} \right) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) - \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} \right) P_{WW}(x) \right]$$

$$+ 2 \frac{1 - x}{x} + 2x(1 - x) + \frac{3}{2} \frac{x(1 - x)}{1 - x + x^2} + P_{WW}(x) \ln(1 - x + x^2) \right] \right].$$  \hspace{1cm} (A.31)

## B Soft matrix elements at tree level

When there are singlets only, the soft matrix is given by

$$S^{(0)}(0, 0, 0, 0) = \left( \begin{array}{cccc} N(N^2 - 1) & 0 & 0 & 0 \\ 0 & \frac{1}{2} N(N^2 - 1) & 0 & 0 \\ 0 & 0 & (N - 2)(N + 2)(N^2 - 1) & 2N \\ \end{array} \right).$$  \hspace{1cm} (B.1)

When there are 2 nonsinglets, the soft matrices are given as

$$S^{(0)}(1, 1, 0, 0) = (N - 2) \left( \begin{array}{ccc} 2(N + 2) & 0 & 0 \\ 0 & \frac{1}{2} (N + 2) & 0 \\ 0 & 0 & \frac{(N + 2)(N^2 - 12)}{2N^2} \\ \end{array} \right) \hspace{1cm} (12),$$

$$S^{(0)}(1, 0, 1, 0) = (N - 2) \left( \begin{array}{ccc} 0 & 0 & \frac{N + 2}{N} \\ 0 & \frac{1}{4} (N + 2) & \frac{1}{4} (N + 2) \\ \frac{N + 2}{N} & \frac{1}{4} (N + 2) & \frac{(N + 2)(N^2 - 12)}{4N^2} \\ \end{array} \right) \hspace{1cm} (13),$$

$$S^{(0)}(1, 0, 0, 1) = (N - 2) \left( \begin{array}{ccc} 0 & 0 & \frac{N + 2}{N} \\ 0 & \frac{1}{4} (N + 2) & -\frac{1}{4} (N + 2) \\ \frac{N + 2}{N} & -\frac{1}{4} (N + 2) & \frac{(N + 2)(N^2 - 12)}{4N^2} \\ \end{array} \right) \hspace{1cm} (14),$$

$$S^{(0)}(0, 1, 1, 0) = (N - 2) \left( \begin{array}{ccc} 0 & 0 & \frac{N + 2}{N} \\ 0 & \frac{1}{4} (N + 2) & -\frac{1}{4} (N + 2) \\ \frac{N + 2}{N} & -\frac{1}{4} (N + 2) & \frac{(N + 2)(N^2 - 12)}{4N^2} \\ \end{array} \right) \hspace{1cm} (23),$$
\[
S^{(0)}(0, 1, 0, 1) = (N - 2) \begin{pmatrix}
0 & 0 & \frac{N+2}{N} \\
0 & \frac{1}{4}(N + 2) & \frac{1}{4}(N+2)(N^2-12) \\
\frac{N+2}{N} & \frac{1}{4}(N + 2) & \frac{(N+2)(N^2-12)}{4N^2}
\end{pmatrix}, \quad (24),
\]
\[
S^{(0)}(0, 0, 1, 1) = \begin{pmatrix}
(N^2 - 1) & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & -\frac{(N-2)(N+2)}{2N^2}
\end{pmatrix}, \quad (34). \quad (B.2)
\]

Note that, with \(N = 2\), only the last term \(S^{(0)}(0, 0, 1, 1)\) survives, which correspond to the case with no nonsinglet beam functions and 2 jet functions.

For the case of 3 nonsinglets, they are given as
\[
S^{(0)}(1, 1, 1, 0) = (N - 2) \begin{pmatrix}
0 & \frac{N+2}{2N}((123) - (132)) & \frac{N^2-12}{2N}((123)+(132)) \\
\frac{N+2}{2N}((123) - (132)) & -\frac{N^2}{N^2}((123) - (132)) & \frac{N+2}{2N}((123)+(132)) \\
\frac{N^2-12}{2N}((123)+(132)) & \frac{N+2}{2N}((123)-(132)) & -\frac{N^2}{N^2}((123)-(132))
\end{pmatrix},
\]
\[
S^{(0)}(1, 1, 0, 1) = (N - 2) \begin{pmatrix}
0 & \frac{N+2}{2N}((124) - (142)) & \frac{N^2-12}{2N}((124)+(142)) \\
\frac{N+2}{2N}((124) - (142)) & -\frac{N^2}{N^2}((124) - (142)) & \frac{N+2}{2N}((124)+(142)) \\
\frac{N^2-12}{2N}((124)+(142)) & \frac{N+2}{2N}((124)-(142)) & -\frac{N^2}{N^2}((124)-(142))
\end{pmatrix}, \quad (B.3)
\]

For the case of 4 nonsinglets, they are given as
\[
S^{(0)}(1, 0, 1, 1) = (N - 2) \begin{pmatrix}
0 & 0 & \frac{N+2}{N}(143) \\
0 & -\frac{1}{2}((134)+(143)) & 0 \\
\frac{N+2}{N}(143) & 0 & -\frac{N^2-12}{2N^2}((134)+(143))
\end{pmatrix},
\]
\[
S^{(0)}(0, 1, 1, 1) = (N - 2) \begin{pmatrix}
0 & 0 & \frac{N+2}{N}(243) \\
0 & -\frac{1}{2}((234)+(243)) & 0 \\
\frac{N+2}{N}(243) & 0 & -\frac{N^2-12}{2N^2}((234)+(243))
\end{pmatrix}. \quad (B.3)
\]

For \(SU(2)\), all the soft matrices with an odd number of nonsinglets vanish.

Finally, when there are 4 nonsinglets, they are given as
\[
S^{(0)}(1, 1, 1, 1) = (N - 2) \begin{pmatrix}
\frac{2(N+2)}{2N^2}(12)(34) & \frac{N+2}{2N}((1243)-(2143)) & S_{13} \\
\frac{2(N+2)}{2N^2}(12)(34) & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix},
\]
where \(S_{ij}\) are given by
\[
S_{13} = \frac{N^2-12}{2N^2}N((1243)+(2143)) - 2(12)(34),
\]
\[
S_{22} = \frac{N}{4(N-2)}(14)(23) + (13)(24) - 2(1234) - 2(1423) - (1234) - (1243) - (1342) - (1432),
\]
\[
S_{23} = \frac{1}{4N^2(N-2)} \left[ 16((1342)-(1234)) + N(N^2-8)((13)(24)-(14)(23)) + N^2((1234)-(1243)-(1342)+(1432)) \right],
\]
\[– 38 –\]
\[
S_{31} = \frac{N^2 - 12}{2N^2(N - 2)} \left[ N \left( (1234) + (2134) \right) - 2(12)(34) \right],
\]
\[
S_{32} = \frac{1}{4N^2(N - 2)} \left[ 16 \left( (1243) - (1432) \right) + N(N^2 - 8) \left( (13)(24) - (14)(23) \right) + N^2 \left( (1234) - (1243) - (1342) + (1432) \right) \right],
\]
\[
S_{33} = \frac{1}{4N^3(N - 2)} \left[ -64(12)(34) + 16N \left( 2(1234) + 2(1243) + 2(1342) + 2(1432) \right) + (1324) + (1423) \right] + N^2(N^2 - 16) \left( (14)(23) + (13)(24) \right)
- N^3 \left( (1234) + (1243) + (1342) + (1432) - 2(1423) - 2(1324) \right). \tag{B.4}
\]

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