Abstract

We give new mechanisms for answering exponentially many queries from multiple analysts on a private database, while protecting differential privacy both for the individuals in the database and for the analysts. That is, our mechanism’s answer to each query is nearly insensitive to changes in the queries asked by other analysts. Our mechanism is the first to offer differential privacy on the joint distribution over analysts’ answers, providing privacy for data analysts even if the other data analysts collude or register multiple accounts. In some settings, we are able to achieve nearly optimal error rates (even compared to mechanisms which do not offer analyst privacy), and we are able to extend our techniques to handle non-linear queries. Our analysis is based on a novel view of the private query-release problem as a two-player zero-sum game, which may be of independent interest.

1 Introduction

Consider a tracking network that wants to sell a database of consumer data to several competing analysts conducting market research. The administrator of the tracking network faces many opposing constraints when deciding how to provide analysts with this data. For legal reasons, the privacy of the individuals contained in her database must be protected. At the same time, the analysts must be able to query the database and receive useful answers. Finally, the privacy of the queries made to the database must be protected, since the analysts are in competition and their queries may be disclosive of proprietary strategies.

This setting of analyst privacy was recently introduced in a beautiful paper of Dwork, Naor, and Vadhan [DNV12]. They showed that differentially private stateless mechanisms — which answer each query independently of the previous queries — can only give accurate answers when the number of queries is at most quadratic in the size of the database. This result rules out mechanisms that perfectly protect the privacy of the queries, while accurately answering exponentially many queries — answers must depend on the state, and hence on the previous queries. However, it turns out that mechanisms that offer a differential-privacy-like guarantee with respect to the queries are possible: Dwork, et al. [DNV12] give such a mechanism, with the guarantee that the marginal distribution on answers given to each analyst is differentially private with
respect to the set of queries made by all of the other analysts. Their mechanism is capable of answering exponentially many linear queries with error $O(1/n^{1/4})$, where $n$ is the number of records in the database. A linear query is a $(1/n)$-sensitive query of the form “What fraction of the individual records in the database satisfy some property $q$?”, so their mechanism gives non-trivial accuracy.

However, they note that their mechanism has several shortcomings. First, it does not promise differential privacy on the joint distribution over multiple analysts’ answers. Therefore, if multiple analysts collude, or if a single malicious analyst registers several false accounts with the mechanism, then the mechanism no longer guarantees query privacy. Second, their mechanism is less accurate than known, non-analyst private mechanisms — analyst privacy is achieved at a cost to accuracy. Finally, the mechanism can only answer linear queries, rather than general low-sensitivity queries.

In this paper, we address all of these issues. First, we consider mechanisms which guarantee one-query-to-many-analyst privacy: for each analyst $a$, the joint distribution over answers given to all other analysts $a' \neq a$ is differentially private with respect to the change of a single query asked by analyst $a$. This privacy guarantee is incomparable to that of Dwork, et al. [DNV12]: it is weaker, because we protect the privacy of a single query, rather than protecting the privacy of all queries asked by analysts $a' \neq a$. However, it is also stronger, because the privacy of one query from an analyst $a$ is preserved even if all other analysts $a' \neq a$ collude or register multiple accounts. Our first result is a mechanism in this setting, with error at most $\tilde{O}(1/\sqrt{n})$ for answering exponentially many linear queries. This error is optimal up to polylogarithmic factors, even when comparing to mechanisms that only guarantee data privacy.

We then extend our techniques to one-analyst-to-many-analyst privacy, where we require that the mechanism preserve the privacy of analyst when he changes all of his queries, even if all other analysts collude. Our second result is a mechanism in this setting, with error $\tilde{O}(1/n^{1/3})$. Although this error rate is worse than what we achieve for one-query-to-many-analyst privacy (and not necessarily optimal), our mechanism is still capable of answering exponentially many queries with non-trivial accuracy guarantees, while satisfying both data and analyst privacy.

These first two mechanisms operate in the non-interactive setting, where the queries from every analyst are given to the mechanism in a single batch. Our final result is a mechanism in the online setting that satisfies one-query-to-many analyst privacy. The mechanism accurately answers a (possibly exponentially long) fixed sequence of low-sensitivity queries. Although our mechanism operates as queries arrive online, it cannot tolerate adversarially chosen queries (i.e. it operates in the same regime as the smooth multiplicative weights algorithm of Hardt and Rothblum [HR10]). For linear queries, our mechanism gives answers with error at most $\tilde{O}(1/n^{2/5})$. For answering general queries with sensitivity $1/n$ (the sensitivity of a linear query), the mechanism guarantees error at most $\tilde{O}(1/n^{1/10})$.

When answering $k$ queries on a database $D \in \mathcal{X}^n$ consisting of $n$ records from a data universe $\mathcal{X}$, our offline algorithms run in time $\tilde{O}(n \cdot (|\mathcal{X}| + k))$ and our online algorithm runs in time $\tilde{O}(|\mathcal{X}| + n)$ per query. These running times are essentially optimal for mechanisms that answer more than $\omega(n^2)$ arbitrary linear queries [Ull12], assuming (exponentially hard) one-way functions exist.

1.1 Our Techniques

To prove our results, we take a novel view of private query release as a two player zero-sum game between a data player and a query player. For each element of the data universe $x \in X$, the data player has an action $a_x$. Intuitively, the data player’s mixed strategy will be his approximation of the true database’s distribution.

On the other side, for each query $q \in Q$, the query player has two actions: $a_q$ and $a_{\neg q}$. The two actions for each query allow the query player to penalize the data player’s play, both when the approximate answer to $q$ is too high, and when it is too low — the query player tries to play queries for which the data player’s
approximation performs poorly. Formally, we define the cost matrix by
\[ G(a_q, a_x) = q(x) - q(D) \]
and
\[ G(a_{¬q}, a_x) = q(D) - q(x), \]
where \( D \) is the private database. The query player wishes to maximize the cost, whereas the database player wishes to minimize the cost. We show that the value of this game is 0, and that any \( ρ \)-approximate equilibrium strategy for the database player corresponds to a database that answers every query \( q \in Q \) correctly up to additive error \( ρ \). Thus, given any pair of \( ρ \)-approximate equilibrium strategies, the strategy for the data player will constitute a database (a distribution over \( X \)) that answers every query to within error \( O(ρ) \).

Different privacy constraints for the private query release problem can be mapped into privacy constraints for solving two-player zero-sum games. Standard private linear query release corresponds to privately computing an approximate equilibrium, where privacy is preserved with respect to changing every cost in the game matrix by at most \( 1/n \). Likewise, query release while protecting one-query-to-many-analyst privacy corresponds to computing an approximate equilibrium strategy, where privacy is with respect to an arbitrary change in two rows of the game matrix — changing a single query \( q \) changes the payoffs for actions \( a_q \) and \( a_{¬q} \). Our main result can be viewed as an algorithm for privately computing the equilibrium of a zero-sum game while protecting the privacy of strategies of the players, which may be of independent interest.

To construct an approximate equilibrium, we use a well-known result: when two no-regret algorithms are played against each other in a zero-sum game, their empirical play distributions quickly converge to an approximate equilibrium. Thus, to compute an equilibrium of \( G \), we have the query player and the data player play against each other using no-regret algorithms, and output the empirical play distribution of the data player as the hypothesis database. We face several obstacles along the way.

First, no-regret algorithms maintain a state — a distribution over actions, which is not privacy preserving. (In fact, it is computed deterministically from inputs that may depend on the data or queries.) Previous approaches to private query release have addressed this problem by adding noise to the inputs of the no-regret algorithm.

In our approach, we crucially rely on the fact that sampling actions from the distributions maintained by the multiplicative weights algorithm is privacy preserving. Intuitively, privacy will come from the fact that the multiplicative weights algorithm does not adjust the weight on any action too aggressively, meaning that when we view the weights as defining a distribution over actions, changing the losses experienced by the algorithm in various ways will have a limited effect on the distribution over actions. We note that this property is not used in the private multiplicative weights mechanism of Hardt and Rothblum [HR10], who use the distribution itself as a hypothesis. Indeed, without the constraint of query privacy, any no-regret algorithm can be used in place of multiplicative weights [RR10, GRU12], which is not the case in our setting.

Second, sampling from the multiplicative weights algorithm is private only if the changes in losses are small. Intuitively, we must ensure that changing one query from one analyst does not affect the losses experienced by the data player too dramatically, so that samples from the multiplicative weights algorithm will indeed ensure query privacy. To enforce this requirement, we force the query player to play mixed strategies from the set of smooth distributions, which do not place too much weight on any single action. It is known that playing any no-regret algorithm, but projecting into the set of smooth distributions in the appropriate way (via a Bregman projection), will ensure no-regret with respect to any smooth distribution.
on actions. For comparison, no-regret is typically defined with respect to the best single action, which is not a smooth distribution. Thus, our regret guarantee is weaker.

The result of this simulation is an approximate equilibrium strategy for the data player, in the sense that it achieves approximately the value of the game when played against all but $s$ strategies of the query player, where $1/s$ is the maximum probability that the query player may assign to any action. This corresponds to a synthetic database, which we release to all analysts, that answers all but $s$ queries accurately. Then, since we choose $s$ to be small, we can answer the mishandled queries with the sparse vector technique [DNR+09, RR10, HR10] adding noise only $\tilde{O}(\sqrt{s/n})$ to these $s$ queries. The result is a nearly optimal error rate of $\tilde{O}(1/\sqrt{n})$.

Our techniques naturally extend to one-analyst-to-many-analyst privacy by making the actions of the query player correspond to entire workloads of queries, one for each analyst, where the query player picks analysts that have at least one query that has high error on the current hypothesis. Like before, a small number of analysts will have queries that have high error, which we handle with a separate private query release mechanism for each analyst.

Finally, we use these techniques to convert the private multiplicative weights algorithm of Hardt and Rothblum [HR10] into an online algorithm that preserves one-query-to-many-analyst privacy, and also answers arbitrary low-sensitivity queries. These last two extensions both give first-of-their-kind results, but at some degradation in the accuracy parameters: we do not obtain $O(1/\sqrt{n})$ error rate. We leave it as an open problem to achieve $\tilde{O}(1/\sqrt{n})$ error in these settings, or show that the accuracy cost is necessary.

1.2 Related Work

There is an extremely large body of work on differential privacy [DMNS06] that we do not attempt to survey. The study of differential privacy was initiated by a line of work [DN03, BDMN05, DMNS06] culminating in the definition by Dwork, Mcsherry, Nissim, and Smith [DMNS06], who also introduced the basic technique of answering low-sensitivity queries using the Laplace mechanism. The Laplace mechanism gives approximate answers to nearly $O(n^2)$ queries while preserving differential privacy.

A recent line of work [BLR08, DNR+09, DRV10, RR10, HR10, GHRU11, GRU12, HLM12] has shown how to accurately answer almost exponentially many queries usefully while preserving differential privacy of the data. Some of this work [RR10, HR10, GRU12] rely on no-regret algorithms — in particular, Hardt and Rothblum [HR10] introduced the multiplicative weights technique to the differential privacy literature, which we use centrally.

However, we make use of multiplicative weights in a different way from prior work in private query release — we simulate play of a two-player zero-sum game using two copies of the multiplicative weights algorithm, and rely on the fast convergence of such play to approximate Nash equilibrium [FS96]. We also rely on the fact that Bregman projections onto a convex set $K$ can be used in conjunction with the multiplicative weights update rule\(^1\) to achieve no regret with respect to the best element in the set $K$ [RS12]. Finally, we use that samples from the multiplicative weights distribution can be viewed as samples from the exponential mechanism of McSherry and Talwar [MT07], and hence are privacy preserving.

Our use of Bregman projections into smooth distributions is similar to its use in smooth boosting. Barak, Hardt, and Kale [BHK09] use Bregman projections in a similar way, and the weight capping used by Dwork, Rothblum, and Vadhan [DRV10] in their analysis of boosting for people can be viewed as a Bregman projection.

\(^1\)Indeed, they can be used in conjunction with any no-regret algorithm in the family of regularized empirical risk minimizers.
The most closely related paper to ours is the beautiful recent work of Dwork, Naor, and Vadhan [DNV12], who introduce the idea of analyst privacy. They show that any algorithm which can answer $\omega(n^2)$ queries to non-trivial accuracy must maintain common state as it interacts with many data analysts, and hence potentially violates the privacy of the analysts. Accordingly, they give a stateful mechanism which promises many-to-one-analyst privacy, and achieves per-query error $O(1/n^{1/4})$ for linear queries — their mechanism promises differential privacy on the marginal distribution of answers given to any single analyst, even when all other analysts change all of their queries. However, if multiple analysts collude, or if a single analyst promises differential privacy on the joint distribution of answers given to any single analyst, even when all other analysts change all of their queries. However, if multiple analysts collude, or if a single analyst can falsely register under many ids, then the privacy guarantees degrade quickly — privacy is not promised on the joint distribution on all analysts answers. Lifting this limitation, improving the error bounds and extending analyst privacy to non-linear queries, are all stated as open questions.

Finally, the varying notions of analyst privacy we use can be interpreted in the context of two-party differential privacy, introduced by McGregor, et al. [MMP+10]. If we consider a single analyst as one party, sending private queries to a second party consisting of the mechanism and all the other parties indirectly, the many-to-one-analyst privacy guarantee is equivalent to privacy of the first party’s view. Here, the privacy must be protected even if the second party changes its inputs arbitrarily, i.e., the other analysts change their queries arbitrarily.

On other hand, if we consider all but one analyst as the first party, sending queries to the mechanism and the remaining analyst, one-query-to-many-analyst privacy is equivalent to the first party’s view being private when the single analyst changes a query. One-analyst-to-many-analyst privacy is similar: the first party’s view must be private when the second party changes all of its queries.

2 Preliminaries

Differential Privacy and Analyst Differential Privacy Let a database $D \in \mathcal{X}^n$ be a collection of $n$ records (rows) \( \{x^{(1)}, \ldots, x^{(n)}\} \) from a data universe $\mathcal{X}$. Two databases $D, D' \in \mathcal{X}^n$ are adjacent if they differ only on a single row, which we denote by $D \sim D'$.

A mechanism $A : \mathcal{X}^n \rightarrow \mathcal{R}$ takes a database as input and outputs some data structure in $\mathcal{R}$. We are interested in mechanisms that satisfy differential privacy.

Definition 2.1. A mechanism $A : \mathcal{X}^n \rightarrow \mathcal{R}$ is $(\varepsilon, \delta)$-differentially private if for every two adjacent databases $D \sim D' \in \mathcal{X}^n$ and every subset $S \subseteq \mathcal{R}$,

$$\Pr[A(D) \in S] \leq e^\varepsilon \Pr[A(D') \in S] + \delta.$$ 

In this work we construct mechanisms that ensure differential privacy for the analyst as well as for the database. To define analyst privacy, we first define many-analyst mechanisms. Let $Q$ be the set of all allowable queries. The mechanism takes $m$ sets of queries $Q_1, \ldots, Q_m$ and returns $m$ outputs $Z_1, \ldots, Z_m$, where $Z_j$ contains answers to the queries $Q_j$. Thus, a many-analyst mechanism has the form $A : \mathcal{X}^n \times (\mathcal{Q})^m \rightarrow \mathcal{R}^m$. Given sets of queries $Q_1, \ldots, Q_m$, let $Q = \bigcup_{j=1}^m Q_j$ denote the set of all queries. In guaranteeing privacy even in the event of collusion, it will be useful to refer to the output given to all analysts other than some analyst $i$. For each $id \in [m]$ we write $A(D, Q)_{-id}$ to denote $(Z_1, \ldots, Z_{id-1}, Z_{id+1}, \ldots, Z_m)$, the output given to all analysts other than $id$.

Let $Q = Q_1, \ldots, Q_m$ and $Q' = Q'_1, \ldots, Q'_m$. We say that $Q$ and $Q'$ are analyst-adjacent if there exists $id^* \in [m]$ such that for every $id \neq id^*$, $Q_{id} = Q'_{id}$. That is, $Q \sim Q'$ are analyst adjacent if they differ only on the queries asked by one analyst. Intuitively, a mechanism satisfies one-analyst-to-many-analyst privacy
if changing all the queries asked by analyst id* does not significantly affect the output given to all analysts other than id*.

**Definition 2.2.** A many-analyst mechanism \( \mathcal{A} \) satisfies \((\varepsilon, \delta)\)-one-analyst-to-many-analyst privacy if for every database \( D \in X^n \), every two analyst-adjacent query sequences \( Q \sim Q' \) that differ only on one set of queries \( Q_{id}, Q'_{id} \), and every \( S \subseteq R^{m-1} \),

\[
Pr[\mathcal{A}(D, Q)_{-id} \in S] \leq e^\varepsilon Pr[\mathcal{A}(D, Q')_{-id} \in S] + \delta.
\]

Let \( Q = Q_1, \ldots, Q_m \) and \( Q' = Q'_1, \ldots, Q'_m \). We say that \( Q \) and \( Q' \) are query-adjacent if there exists id* such that for every id \( \neq \) id*, \( Q_{id} = Q'_{id} \) and \( |Q_{id} - Q'_{id}| \leq 1 \). That is, \( Q \sim Q' \) are query adjacent if they differ only on one of the queries. Intuitively, we say that a mechanism satisfies one-query-to-many-analyst privacy if changing one query asked by analyst id* does not significantly affect the output given to all analysts other than id*.

**Definition 2.3.** A many-analyst mechanism \( \mathcal{A} \) satisfies \((\varepsilon, \delta)\)-one-query-to-many-analyst privacy if for every database \( D \in X^n \), every two query-adjacent query sequences \( Q \sim Q' \) that differ only on one query in \( Q_{id}, Q'_{id} \), and every \( S \subseteq R^{m-1} \),

\[
Pr[\mathcal{A}(D, Q)_{-id} \in S] \leq e^\varepsilon Pr[\mathcal{A}(D, Q')_{-id} \in S] + \delta.
\]

In our proofs of both differential privacy and analyst privacy, we will often establish that for any \( D \sim D' \), the two distributions \( \mathcal{A}(D), \mathcal{A}(D') \) are such that with probability at least \( 1 - \delta \) over \( y \in_r M(D) \),

\[
\left| \ln \frac{Pr[\mathcal{A}(D) = y]}{Pr[\mathcal{A}(D') = y]} \right| \leq \varepsilon.
\]

This condition implies \((\varepsilon, \delta)\)-differential privacy [DRV10].

### 2.1 Queries and Accuracy

In this work we consider two types of queries: low-sensitivity queries and linear queries. Low-sensitivity queries are parameterized by \( \Delta \in [0, 1] \): a \( \Delta \)-sensitive query is any function \( q : X^n \rightarrow [0, 1] \) such that

\[
\max_{D \sim D'} |q(D) - q(D')| \leq \Delta.
\]

A linear query is a particular type of low-sensitivity query, specified by a function \( q : X \rightarrow [0, 1] \). We define the evaluation of \( q \) on a database \( D \in X^n \) to be

\[
q(D) = \frac{1}{n} \sum_{i=1}^{n} q(x^{(i)}),
\]

so a linear query is evidently \((1/n)\)-sensitive.

Since \( \mathcal{A} \) may output a data structure, we must specify how to answer queries in \( Q \) from the output \( \mathcal{A}(D) \). Hence, we require that there is an evaluator \( \mathcal{E} : R \times Q \rightarrow R \) that estimates \( q(D) \) from the output of \( \mathcal{A}(D) \). For example, if \( \mathcal{A} \) outputs a vector of “noisy answers” \( Z = \{q(D) + Z_q | q \in Q\} \), where \( Z_q \) is a random variable for each \( q \in Q \), then \( R = \mathbb{R}^Q \) and \( \mathcal{E}(Z, q) \) is the \( q \)-th component of \( Z \). Abusing notation, we write \( q(Z) \) and \( \mathcal{E}(\mathcal{A}(D), q) \) as shorthand for \( \mathcal{E}(Z, q) \) and \( \mathcal{E}(\mathcal{A}(D), q) \), respectively.

**Definition 2.4.** An output \( Z \) of a mechanism \( \mathcal{A}(D) \) is \( \alpha \)-accurate for query set \( Q \) if \( |q(Z) - q(D)| \leq \alpha \) for every \( q \in Q \). A mechanism is \((\alpha, \beta)\)-accurate for query set \( Q \) if for every database \( D \),

\[
Pr[\forall q \in Q, |q(\mathcal{A}(D)) - q(D)| \leq \alpha] \geq 1 - \beta,
\]

where the probability is taken over the coins of \( \mathcal{A} \).
2.2 Differential Privacy Tools

We will use a few previously known differentially private mechanisms. When we need to answer a small number of queries we will use the well-known Laplace mechanism [DMNS06], with an improved analysis from [DRV10].

**Lemma 2.5.** Let \( F = \{ f_1, \ldots, f_{|F|} \} \) be a set of \( \Delta \)-sensitive queries \( f_i : X^n \to [0,1] \), and let \( D \in X^n \) be a database. Let \( \epsilon, \delta \leq 1 \). Then the mechanism \( A_{Lap}(D, F) \) that outputs

\[
f_i(D) + \text{Lap}\left( \frac{\Delta \sqrt{8|F| \log(1/\delta)}}{\epsilon} \right)
\]

for every \( f_i \in F \) is:

1. \((\epsilon, \delta)\)-differentially private, and
2. \((\alpha, \beta)\)-accurate for any \( \beta \in (0,1) \) and \( \alpha = \epsilon^{-1} \Delta \sqrt{8|F| \log(1/\delta) \log(|F|/\beta)} \).

When we need to answer a large number of queries, we will use the multiplicative weights mechanism from [HR10], with an improved analysis from Gupta et al. [GRU12].

**Lemma 2.6.** Let \( F = \{ f_1, \ldots, f_{|F|} \} \) be a set of \((1/n)\)-sensitive linear queries, \( f_i : X^n \to [0,1] \). Let \( D \in X^n \) be a database. Then there is a mechanism \( A_{MW}(D, F) \) that is:

1. \((\epsilon, \delta)\)-differentially private, and
2. \((\alpha, \beta)\)-accurate for any \( \beta \in (0,1) \) and \( \alpha = O\left( \frac{\log^{1/4}|X| \sqrt{\log(|F|/\beta) \log(1/\delta)}}{\epsilon^{1/2} n^{1/2}} \right) \).

**Remark 2.7.** We use the above lemma as a black box, agnostic to the algorithm which instantiates these guarantees.

Our algorithms also use the private sparse vector algorithm. This algorithm takes as input a database and a set of low-sensitivity queries, with the promise that only a small number of the queries have large answers on the input database. Its output is a set of queries with large answers on the input database. Importantly for this work, the sparse vector algorithm (cf. [HR10, Rot11]) ensures the privacy of the input queries in a strong sense.

**Lemma 2.8.** Let \( F = \{ f_1, \ldots, f_{|F|} \} \) be a set of \( \Delta \)-sensitive functions, \( f_i : X^n \to [0,1] \). Let \( D \in X^n \) be a database, \( \alpha \in (0,1] \), \( k \in [|F|] \) such that

\[
|\{ i \mid f_i(D) \geq \alpha \}| \leq k.
\]

Then there is an algorithm \( A_{SV}(D, F) \) that

1. is \((\epsilon, \delta)\)-differentially private with respect to \( D \),
2. returns \( I \subseteq [|F|] \) of size at most \( k \) such that with probability at least \( 1 - \beta \),

\[
\left\{ i \mid f_i(D) \geq \alpha + \epsilon^{-1} \Delta \sqrt{8k \log(1/\delta) \log(|F|/\beta)} \right\} \subseteq I \subseteq \left\{ i \mid f_i(D) \geq \alpha \right\}.
\]
3. and is perfectly private with respect to the queries:
if \( \mathcal{F}' = \{ f_1, \ldots, f_j, \ldots, f_k \} \), then for every \( D \) and \( i \neq j \),
\[
\Pr [ i \in \mathcal{A}_{SV}(D, \mathcal{F}) ] = \Pr [ i \in \mathcal{A}_{SV}(D, \mathcal{F}') ] .
\]

We will also use the Composition Theorem of Dwork, Rothblum, and Vadhan [DRV10].

**Lemma 2.9.** Let \( \mathcal{A} : \mathcal{X}^* \to \mathcal{R}^T \) be a mechanism such that for every pair of adjacent inputs \( x \sim x' \), every \( t \in [T] \), every \( r_1, \ldots, r_{t-1} \in \mathcal{R} \), and every \( r_t \in \mathcal{R} \),
\[
\Pr [ A_t(x) = r_t \mid A_{1,\ldots,t-1}(x) = r_1, \ldots, r_{t-1} ] \leq e^{\varepsilon \delta} \Pr [ A_t(x') = r_t \mid A_{1,\ldots,t-1}(x') = r_1, \ldots, r_{t-1} ] + \delta_0
\]
for \( \varepsilon \leq 1/2 \). Then \( \mathcal{A} \) is \((\varepsilon, \delta)\)-differentially private for \( \varepsilon = \sqrt{8T \log(1/\delta)} + 2\varepsilon^2 T \) and \( \delta = \delta_0 T \).

### 2.3 Multiplicative Weights

Let \( A : \mathcal{A} \to [0, 1] \) be a measure over a set of actions \( \mathcal{A} \). We use \( |A| = \sum_{a \in \mathcal{A}} A(a) \) to denote the *density* of \( A \). A measure naturally corresponds to a probability distribution \( \tilde{A} \) in which
\[
\Pr [ \tilde{A} = a ] = A(a)/|A|
\]
for every \( a \in \mathcal{A} \). Throughout, we will use calligraphic letters \( \mathcal{A} \) to denote a set of actions, lower case letters \( a \) to denote the actions, capital letters \( A \) to denote a measure over actions, and capital letters with a tilde to denote the corresponding distributions \( \tilde{A} \). We will use the KL-divergence between two distributions, defined to be
\[
KL(\tilde{A} || \tilde{A}') = \sum_{a \in \mathcal{A}} \tilde{A}(a) \log \left( \frac{\tilde{A}(a)}{\tilde{A}'(a)} \right).
\]

Let \( L : \mathcal{A} \to [0, 1] \) be a loss function (losses \( L \)). Abusing notation, we can define \( L(A) = \mathbb{E} [ L(\tilde{A}) ] \). Given an initial measure \( A_1 \), we can define the multiplicative weights algorithm in Algorithm 1.

**Algorithm 1** The Multiplicative Weights Algorithm, \( MW_{\eta} \)

For \( t = 1, 2, \ldots, T \):
1. Sample \( a_t \leftarrow \tilde{A}_t \)
2. Receive losses \( L_t \) (may depend on \( A_1, a_1, \ldots, A_{t-1}, a_{t-1} \))
3. **Update:** For each \( a \in \mathcal{A} \):
   - Update \( A_{t+1}(a) = e^{-\eta L_t(a)} A_t(a) \) for every \( a \in \mathcal{A} \)

The following theorem about the multiplicative weights update is well-known.

**Theorem 2.10** (Multiplicative Weights). Let \( A_1 \) be the uniform measure of density 1, and let \( \{ a_1, \ldots, a_T \} \) be the actions obtained by \( MW_{\eta} \) with losses \( \{ L_1, \ldots, L_t \} \). Let \( A^* = 1_{a=a^*} \), for some \( a^* \in \mathcal{A} \), and \( \delta \in (0, 1] \). Then with probability at least \( 1 - \beta \),
\[
\mathbb{E}_{t \leftarrow \mathbb{R}[T]} [ L_t(a_t) ] \leq (1 + \eta) \mathbb{E}_{t \leftarrow \mathbb{R}[T]} [ L_t(A^*) ] + \frac{KL(\tilde{A^*} || \tilde{A}_1)}{\eta T} + \frac{4 \log(1/\beta)}{\sqrt{T}}
\]
\[
\leq \mathbb{E}_{t \leftarrow \mathbb{R}[T]} [ L_t(A^*) ] + \eta + \frac{\log |A|}{\eta T} + \frac{4 \log(1/\beta)}{\sqrt{T}}.
\]
We need to work with a variant of multiplicative weights that only produces measures $A$ of high density, which will imply that $A$ does not assign too much probability to any single element of $A$. To this end, we will apply (a special case of) the Bregman projection to the measures obtained from the multiplicative weights update rule.

**Definition 2.11.** Let $s \in (0, l]$. Given a measure $A$ such that $|A| \leq s$, let $A_s$ be the (Bregman) projection of $A$ into the set of density-$s$ measures, obtained by computing $c \geq 1$ such that $s = \sum_{a \in A} \min\{1, cA(a)\}$ and setting $A(a) = \min\{1, cM(a)\}$ for every $a \in A$. We call $s$ the density of measure $A$.

**Algorithm 2** The Dense Multiplicative Weights Algorithm, $DMW_{s, \eta}$

For $t = 1, 2, \ldots, T$:
- Let $A_t' = A_t \Gamma_s$, and sample $a_t \sim A_t'$
- Receive losses $L_t$ (may depend on $A_1, a_1, \ldots, A_{t-1}, a_{t-1}$)
- **Update:** For each $a \in A$:
  - Update $A_{t+1}(a) = e^{-\eta L_t(a)} A_t(a)$

Given an initial measure $A_1$ such that $|A_1| \leq s$, we can define the dense multiplicative weights algorithm in Algorithm 2. Note that we update the unprojected measure $A_t$, but sample $a_t$ using the projected measure $\Gamma_s A_t$. Observe that the update step can only decrease the density, so we will have $|A_t| \leq s$ for every $t$. Like before, given a sequence of losses $\{L_1, \ldots, L_T\}$ and an initial measure $A_1$ of density $s$, we can consider the sequence $\{A_1, \ldots, A_T\}$ where $A_{t+1}$ is given by the projected multiplicative weights update applied to $A_t, L_t$. The following theorem is known.

**Theorem 2.12.** Let $A_1$ be the uniform measure of density 1 and let $\{a_1, \ldots, a_T\}$ be the sequence of measures obtained by $DMW_{s, \eta}$ with losses $\{L_1, \ldots, L_T\}$. Let $A^* = 1_{a \in S^*}$ for some set $S^* \subseteq A$ of size $s$, and $\delta \in (0, 1]$. Then with probability $1 - \beta$,

$$
E_{t \sim \pi[T]} [L_t(A_t)] \leq (1 + \eta) E_{t \sim \pi[T]} [L_t(A^*)] + \frac{KL(\tilde{A}^* || \tilde{A}_1)}{\eta T} + \frac{4 \log(1/\beta)}{\sqrt{T}}
$$

$$
\leq E_{t \sim \pi[T]} [L_t(A^*)] + \eta + \frac{\log |A|}{\eta T} + \frac{4 \log(1/\beta)}{\sqrt{T}}.
$$

### 2.4 Regret Minimization and Two-Player Zero-Sum Games

Let $G : A_R \times A_C \to [0, 1]$ be a two-player zero-sum game between players (R)ow and (C)olumn, who take actions $r \in A_R$ and $c \in A_C$ and receive losses $G(r, c)$ and $-G(r, c)$, respectively. Let $\Delta(A_R), \Delta(A_C)$ be the set of measures over actions in $A_R$ and $A_C$, respectively. The well-known minimax theorem states that

$$
v := \min_{R \in \Delta(A_R)} \max_{C \in \Delta(A_C)} G(R, C) = \max_{C \in \Delta(A_C)} \min_{R \in \Delta(A_R)} G(R, C).
$$

We define this quantity $v$ to be the value of the game.

Freund and Schapire [FS96] showed that if two sequences of actions $\{r_1, \ldots, r_T\}, \{c_1, \ldots, c_T\}$ are “no-regret with respect to one another”, then $\tilde{r} = \frac{1}{T} \sum_{t=1}^T r_t$ and $\tilde{c} = \frac{1}{T} \sum_{t=1}^T c_t$ form an approximate equilibrium strategy pair. More formally, if

$$
\max_{c \in A_C} E_t[G(r_t, c)] - \rho \leq E_t[G(r_t, c_t)] \leq \min_{r \in A_R} E_t[G(r, c_t)] + \rho,
$$
then
\[ v - 2\rho \leq G(\tilde{r}, \tilde{c}) \leq v + 2\rho. \]

Thus, if Row chooses actions using the multiplicative weights update rule with losses \( L_t(r_t) = G(r_t, c_t) \) and Column chooses actions using the multiplicative weights rule with losses \( L_t(r_t) = -G(r_t, c_t) \), then each player’s distribution on actions converges to a minimax strategy. That is, if we play until both players have regret at most \( \rho \):
\[
\max_c G(\tilde{r}, c) \leq v + 2\rho \quad v - 2\rho \leq \min_r G(r, \tilde{c}).
\]

For query privacy in our view of query release as a two player game, Column must not put too much weight on any single query. Thus, we need an analogue of this result in the case where Column is not choosing actions according to the multiplicative weights update, but rather using the projected multiplicative weights update. In this case we cannot hope to obtain an approximate minimax strategy, since Column cannot play any single action with significant probability. However, we can define an alternative notion of the value of a game where Column is restricted in this way: let \( \Delta_s(\mathcal{A}_C) \) be the set of measures over \( \mathcal{A}_C \) of minimum density at least \( s \), and define

\[
v_s := \min_{R \in \Delta(\mathcal{A}_R)} \max_{C \in \Delta_s(\mathcal{A}_C)} G(R, C).
\]

Notice that \( v_s \leq v \), and \( v_s \) can be very different from \( v \).

**Theorem 2.13.** Let \( \{r_1, \ldots, r_T\} \in \mathcal{A}_R \) be a sequence of row-player actions, \( \{C_1, \ldots, C_T\} \in \Delta_s(\mathcal{A}_C) \) be a sequence of high-density measures over column-player actions, and \( \{c_1, \ldots, c_T\} \in \mathcal{A}_C \) be a sequence of column-player actions such that \( c_j \leftarrow_{\text{SM}} C_j \) for every \( t \in [T] \). Further, suppose that

\[
\mathbb{E}_t [G(r_t, c_t)] \leq \min_{R \in \Delta(\mathcal{A}_R)} \mathbb{E}_t [G(R, c_t)] + \rho \quad \text{and} \quad \mathbb{E}_t [G(r_t, c_t)] \geq \max_{C \in \Delta_s(\mathcal{A}_C)} \mathbb{E}_t [G(r_t, C)] - \rho.
\]

Then,
\[
v_s - 2\rho \leq G(\tilde{r}, \tilde{c}) \leq v + 2\rho.
\]

Moreover, \( \tilde{r} \) is an approximate min-max strategy with respect to strategies in \( \Delta_s(\mathcal{A}_C) \), i.e.,
\[
v_s - 2\rho \leq \max_{C \in \Delta_s(\mathcal{A}_C)} G(\tilde{r}, C) \leq v + 2\rho.
\]

**Proof.** For the first set of inequalities, we handle each part separately. For one direction,
\[
v_s = \min_{R \in \Delta(\mathcal{A}_R)} \max_{C \in \Delta_s(\mathcal{A}_C)} G(R, C)
\leq \max_{C \in \Delta_s(\mathcal{A}_C)} \mathbb{E}_t [G(r_t, C)] \leq \mathbb{E}_t [G(r_t, c_t)] + \rho
\leq \min_{R \in \Delta(\mathcal{A}_R)} \mathbb{E}_t [G(r_t, C)] + 2\rho = \min_{R \in \Delta(\mathcal{A}_R)} G(R, \tilde{c}) + 2\rho
\leq G(\tilde{r}, \tilde{c}) + 2\rho.
\]

The other direction is similar, starting with the fact that \( v = \max_{c \in \mathcal{C}} \min_{r \in \mathcal{R}} G(r, c) \).

For the second set of inequalities, we also handle the two cases separately. For the upper bound,
\[
\max_{C \in \Delta_s(\mathcal{A}_C)} \mathbb{E}_t [G(\tilde{r}, C)] \leq \mathbb{E}_t [G(r_t, c_t)] + \rho
\leq \min_{R \in \Delta(\mathcal{A}_R)} \mathbb{E}_t [G(r_t, C)] + 2\rho = \min_{R \in \Delta(\mathcal{A}_R)} G(R, \tilde{c}) + 2\rho
\leq v + 2\rho.
\]
For the lower bound,

\[ \max_{C \in \Delta_s(A_C)} G(\tilde{r}, C) \geq \mathbb{E}_t [G(\tilde{r}, \tilde{c})] \geq v_s - 2\rho \]

This completes the proof of the theorem.

**Corollary 2.14.** Let \( G : A_R \times A_C \rightarrow [0, 1] \). If the row player chooses actions \( \{r_1, \ldots, r_T\} \) by running \( \text{MW}_\eta \) with loss functions \( L_t(r) = G(r, c_t) \) and the column player chooses actions \( \{c_1, \ldots, c_T\} \) by running \( \text{DMW}_{s,\eta} \) with the loss functions \( L_t(c) = -G(r_t, c) \), then with probability at least \( 1 - \beta \),

\[ v_s - 2\rho \leq \max_{c \in C_s} G(\tilde{r}, c) \leq v + 2\rho, \]

for

\[ \rho = \eta + \max \{ \log |A_R|, \log |A_C| \} + \frac{4 \log(2/\beta)}{\eta T} + \frac{4 \log(2/\beta)}{\sqrt{T}}. \]

### 3 A One-Query-To-Many-Analyst Private Mechanism

#### 3.1 An Offline Mechanism for Linear Queries

We define our offline mechanisms for releasing linear queries in Algorithm 3.

#### 3.1.1 Accuracy Analysis

**Theorem 3.1.** The offline algorithm for linear queries is \((\alpha, \beta)\)-accurate for

\[ \alpha = \mathcal{O}\left( \frac{\sqrt{\log(|X| + |Q|)} \log(1/\delta) \log(|Q|/\beta)}{\varepsilon \sqrt{n}} \right). \]

**Proof.** Observe that the algorithm is computing an approximate equilibrium of the game \( G_D(x, q) = \frac{1 + q(D(x) - q(x))}{2} \). Let \( v, v_s \) be the value and constrained value of this game, respectively. First, we pin down the quantities \( v \) and \( v_s \).

**Claim 3.2.** For every \( D \), the value and constrained value of \( G_D \) is \( 1/2 \).

**Proof of Claim 3.2.** It’s clear that the value (and hence constrained value) is at most \( 1/2 \), because

\[ \min_{x} \max_{q} \frac{1 + q(D(x) - q(x))}{2} \leq \max_{q} \frac{1 + q(D(x) - q(x))}{2} = \frac{1}{2}. \]

Suppose we choose \( x \) such that \((1 + q(D(x) - q(x)))/2 < 1/2\) for some \( q \in Q \). Then, since the query \( q' = 1 - q \) is also in \( Q \), \((1 + q'(D(x) - q'(x)))/2 > 1/2\). But then \( \max_{q \in Q} (1 + q(D(x) - q(x)))/2 > 1/2 \), so the value of the game is at least \( 1/2 \).

For the constrained value, suppose we choose \( x \) such that \( \mathbb{E}_{q \sim Q} [(1 + q(D(x) - q(x)))/2] < 1/2 \) for some \( Q \in Q_s \). Then we can flip every query in \( Q \) to get a new distribution \( Q' \) such that \( \mathbb{E}_{q \sim Q} [(1 + q(D) - q(x))/2] > 1/2 \). So \( v_s \geq 1/2 \) as well.
Algorithm 3 Offline Mechanism for Linear Queries with One-Query-to-Many-Analyst Privacy

**Input:** Database $D \in X^n$ and sets of linear queries $Q_1, \ldots, Q_m$.

**Initialize:** Let $Q = \bigcup_{j=1}^m Q_j \cup -Q_j$, $D_0(x) = 1/|X|$ for every $x \in X$, $Q_0(q) = 1/|Q|$ for every $q \in Q$,

$$T = n \cdot \max \{ \log |X|, \log |Q| \}, \quad \eta = \frac{\varepsilon}{2 \sqrt{T \log(1/\delta)}}, \quad s = 12T$$

**DataPlayer:**

On input a query $\tilde{q}_t$, for each $x \in X$:

Update $D_t(x) = D_{t-1}(x) \cdot \exp \left( -\eta \left( \frac{1+\tilde{q}_t(D) - \tilde{q}_t(x)}{2} \right) \right)$

Choose $\tilde{x}_t \leftarrow_r \tilde{D}_t$ and send $\tilde{x}_t$ to **QueryPlayer**

**QueryPlayer:**

On input a data element $\tilde{x}_t$, for each $q \in Q$:

Update $Q_{t+1}(q) = Q_t(q) \cdot \exp \left( -\eta \left( \frac{1+q(D) - q(\tilde{x}_t)}{2} \right) \right)$

Let $P_{t+1} = \Gamma_s Q_{t+1}$

Choose $\tilde{q}_{t+1} \leftarrow_r \tilde{P}_{t+1}$ and send $\tilde{q}_{t+1}$ to **DataPlayer**

**GenerateSynopsis:**

Let $\hat{D} = (\hat{x}_1, \ldots, \hat{x}_T)$. By Corollary 2.14,

$$v_s - 2\rho \leq \max_{Q \in \Delta_s(Q)} \left( \frac{1}{2} \mathbb{E}_{q \leftarrow r \tilde{Q}} \left[ 1 + q(D) - q(\hat{D}) \right] \right) \leq v + 2\rho.$$

Applying Claim 3.2 and rearranging terms, with probability at least $1 - \beta/3$,

$$\max_{Q \in \Delta_s(Q)} \left( \mathbb{E}_{q \leftarrow r \tilde{Q}} \left[ q(D) - q(\hat{D}) \right] \right) = \max_{Q \in \Delta_s(Q)} \left( \mathbb{E}_{q \leftarrow r \tilde{Q}} \left[ \log |X|, \log |Q| \right] \right) \leq 4\rho$$

$$= 4 \left( \eta + \frac{\max \{ \log |X|, \log |Q| \}}{\eta T} + \frac{4 \log(2/\beta)}{\sqrt{T}} \right)$$

$$= O \left( \frac{\sqrt{\log(|X|) + |Q| \log(1/\delta) + \log(1/\beta)}}{\varepsilon \sqrt{n}} \right) = \alpha_{\hat{D}}.$$
We can now run the sparse vector algorithm (Lemma 2.5). With probability at least $1 - \beta / 3$, it will identify every query $q$ with error larger than $\alpha_D + \alpha_{SV}$ for

$$\alpha_{SV} = O \left( \frac{\sqrt{s \log(1/\delta)}}{\varepsilon n} \log(\frac{|Q|}{\beta}) \right).$$

Since there are at most $s$ such queries, with probability at least $1 - \beta / 3$, the Laplace mechanism (Lemma 2.8) answers these queries to within error

$$\alpha_{Lap} = O \left( \frac{\sqrt{s \log(1/\delta)}}{\varepsilon n} \log(\frac{s}{\beta}) \right).$$

Now, observe that in the final output, there are two ways that a query can be answered: either by $\hat{D}$, in which case its answer can have error as large as $\alpha_D + \alpha_{SV}$, or by the Laplace mechanism, in which case its answer can have error as large as $\alpha_{Lap}$. Thus, with probability at least $1 - \beta$, every query has error at most $\max\{\alpha_D + \alpha_{SV}, \alpha_{Lap}\}$. Substituting our choice of $s = 12T = O(n \log(|A| + |Q|))$ and simplifying, we conclude that the mechanism is $(\alpha, \beta)$-accurate for

$$\alpha = O \left( \frac{\sqrt{\log(|A| + |Q|) \log(1/\delta)}}{\varepsilon \sqrt{n}} \log(\frac{|Q|}{\beta}) \right).$$

\[ \square \]

### 3.1.2 Data Privacy

**Theorem 3.3.** Algorithm 3 satisfies $(\varepsilon, \delta)$-differential privacy for the data.

Before proving the theorem, we will state a useful lemma about the Bregman projection onto the set of high density measures (Definition 2.11).

**Lemma 3.4** (Projection Preserves Differential Privacy). Let $A_0, A_1 : A \to [0, 1]$ be two full-support measures over a set of actions $A$ and $s \in (0, |A|)$ be such that $|A_0|, |A_1| \leq s$ and $|\ln(A_0(a)/A_1(a))| \leq \varepsilon$ for every $a \in A$. Let $A_0' = \Gamma_s A_0$ and $A_1' = \Gamma_s A_1$. Then $|\ln(A_0'(a)/A_1'(a))| \leq 2\varepsilon$ for every $a \in A$.

**Proof of Lemma 3.4.** Recall that to compute $A' = \Gamma_s A$, we find a “scaling factor” $c > 1$ such that

$$\sum_{a \in A} \min\{1, cA(a)\} = s,$$

and set $A'(a) = \min\{1, cA(a)\}$. Let $c_0$ and $c_1$ be the scaling factors for $A_0'$ and $A_1'$ respectively. Assume without loss of generality that $c_0 \leq c_1$. First, observe that

$$\left| \ln \left( \frac{\min\{1, c_0 A_0(a)\}}{\min\{1, c_0 A_1(a)\}} \right) \right| \leq \left| \ln \left( \frac{A_0(a)}{A_1(a)} \right) \right| \leq \varepsilon,$$

for every $a \in A$. Second, observe that $c_1/c_0 \leq e^\varepsilon$. If this were not the case, then we would have $c_1 A_1(a) \geq c_0 A_1(a) e^\varepsilon \geq c_0 A_0(a)$ for every $a \in A$, with strict inequality for at least one $a$. But then,

$$\sum_{a \in A} \min\{1, c_1 A_1(a)\} > \sum_{a \in A} \min\{1, c_0 A_0(a)\} = s,$$
which would contradict the choice of $c_1$. Thus,
\[
\left| \ln \left( \frac{\min \{1, c_0 A_0(a)\}}{\min \{1, c_1 A_1(a)\}} \right) \right| \leq \left| \ln \left( \frac{\min \{1, c_0 A_0(a)\}}{\min \{1, c_0 A_0(a)\}} \right) \right| + \left| \ln \left( \frac{c_1}{c_0} \right) \right| \leq \varepsilon + \varepsilon = 2\varepsilon,
\]
for every $a \in \mathcal{A}$.

Now we prove the main result of this section.

**Proof of Theorem 3.3.** We focus on analyzing the privacy properties of the output $\hat{D} = (\hat{x}_1, \ldots, \hat{x}_T)$, the privacy of the final stage of the mechanism will follow from standard arguments in differential privacy. We will actually show the stronger guarantee that the sequence $v = (\hat{x}_1, \hat{q}_1, \ldots, \hat{x}_T, \hat{q}_T)$ is differentially private for the data. Fix a pair of adjacent databases $D_0 \sim D_1$ and let $V_0, V_1$ denote the distribution on sequences $v$ when the mechanism is run on database $D_0, D_1$ respectively. We will show that with probability at least $1 - \delta/3$ over $v = (\hat{x}_1, \hat{q}_1, \ldots, \hat{x}_T, \hat{q}_T) \leftarrow V_0$,

\[
\left| \ln \left( \frac{V_0(v)}{V_1(v)} \right) \right| \leq \frac{\varepsilon}{3},
\]

which is no weaker than $(\varepsilon/3, \delta/3)$-differential privacy. To do so, we analyze the privacy of each element of $v$, $\hat{x}_t$ or $\hat{q}_t$, and apply the composition analysis of Dwork, Rothblum, and Vadhan [DRV10]. Define $\varepsilon_0 = 2\eta T/n$.

**Claim 3.5.** For every $v$, and every $t \in [T]$,
\[
\left| \ln \left( \frac{V_0(\hat{x}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_{t-1}, \hat{q}_{t-1})}{V_1(\hat{x}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_{t-1}, \hat{q}_{t-1})} \right) \right| \leq \varepsilon_0.
\]

**Proof of Claim 3.5.** We can prove the statement by the following direct calculation.
\[
\left| \ln \left( \frac{V_0(\hat{x}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_{t-1}, \hat{q}_{t-1})}{V_1(\hat{x}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_{t-1}, \hat{q}_{t-1})} \right) \right| = \ln \left( \frac{\exp \left( \sum_{j=1}^{t-1} 1 + \hat{q}_j(D_0) - \hat{q}_j(\hat{x}_t) \right)}{\exp \left( \sum_{j=1}^{t-1} 1 + \hat{q}_j(D_1) - \hat{q}_j(\hat{x}_t) \right)} \right) = \frac{\eta}{2} \left| \sum_{j=1}^{t-1} 1 + \hat{q}_j(D_0) - \hat{q}_j(\hat{x}_t) \right| \leq \frac{\eta(t - 1)}{2n} \leq \frac{\eta T}{2n} \leq \varepsilon_0.
\]

**Claim 3.6.** For every $v$, and every $t \in [T]$,
\[
\left| \ln \left( \frac{V_0(\hat{q}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_t)}{V_1(\hat{q}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_t)} \right) \right| \leq \varepsilon_0.
\]
Proof of Claim 3.6. The sample \( \hat{q}_t \) is made according to \( \tilde{P}_t \), which is the distribution corresponding to the projected measure \( P_t \). First we’ll look at the unprojected measure \( Q_t \). Observe that, for any database \( D \) and query \( q \),

\[
Q_t(q) = \exp \left( -\left( \frac{\eta}{2} \right) \sum_{j=1}^{t-1} 1 + q(D) - q(\hat{x}_j) \right).
\]

Thus, if \( Q_0(q) \) is the measure we would have when database \( D_0 \) is the input, and \( Q_1(q) \) is the measure we would have when database \( D_1 \) is the input, then

\[
\left| \ln \left( \frac{Q_0(q)}{Q_1(q)} \right) \right| \leq \frac{\eta}{2} \sum_{j=1}^{t-1} \left| q_j(D_0) - q_j(D_1) \right| \leq \frac{\eta T}{2n},
\]

for every \( q \in Q \). Given that \( Q_0 \) and \( Q_1 \) satisfy this condition, Lemma 3.4 guarantees that the projected measures satisfy

\[
\left| \ln \left( \frac{P_0(q)}{P_1(q)} \right) \right| \leq \frac{\eta T}{n}.
\]

Finally, we note that if the above condition is satisfied for every \( q \in Q \), then the distributions \( \tilde{P}_0, \tilde{P}_1 \) satisfy

\[
\left| \ln \left( \frac{\tilde{P}_0(q)}{\tilde{P}_1(q)} \right) \right| \leq \frac{2\eta T}{n} \leq \epsilon_0,
\]

because the value of the normalizer also changes by at most a multiplicative factor of \( e^{\pm \eta T/n} \). We observe that \( V_b(\hat{q}_t | \hat{x}_1, \hat{q}_1, \ldots, \hat{x}_t) = \tilde{P}_b(\hat{q}_t) \) for \( b \in \{0, 1\} \), which completes the proof of the claim.

Now, the composition lemma (Lemma 2.9) (for \( 2T \)-fold composition) guarantees that with probability at least \( 1 - \delta/3 \),

\[
\left| \ln \left( \frac{V_0(v)}{V_1(v)} \right) \right| \leq \epsilon_0 \sqrt{4T \log(3/\delta)} + 4\epsilon_0^2 T,
\]

which is at most \( \epsilon/3 \) by our choice of \( \epsilon_0 \). This implies that \( \hat{D} \) is \((\epsilon/3, \delta/3)\)-differentially private.

We note that the sparse vector computation to find the \( s \) queries with large error is \((\epsilon/3, \delta/3)\)-differentially private, by our choice of parameters (Lemma 2.8), and the answers to the queries found by sparse vector are \((\epsilon/3, \delta/3)\)-differentially private for our choice of parameters (Lemma 2.5). The theorem follows from composition.

3.1.3 Query Privacy

Theorem 3.7. Algorithm 3 satisfies \((\epsilon, \delta)\)-one-query-to-many-analyst differential privacy.

Before proving query privacy of Algorithm 3, we will state a useful composition lemma. The lemma is a generalization of the “secrecy of the sample lemma” [KLN+11, DRV10] to the interactive setting. Consider the following game:

\footnote{We could improve the constants in our privacy analysis slightly by finding the queries with large error using sparse vector and answering them using the Laplace mechanism in one step. However, in our algorithm for achieving analyst-to-many privacy, we need to do the analogous steps separately, and thus we chose to present this way to maintain modularity.}
We also have that domain of 

\[ \mathcal{A} : \mathcal{U}^{*} \rightarrow \mathcal{R} \]

and a bit \( b \in \{0, 1\} \). Let \( D_{0} = \emptyset \).

For \( t = 1, \ldots, T \):

- The (randomized) adversary \( B(y_{1}, \ldots, y_{T}; r) \) chooses two distributions \( B_{t}^{0}, B_{t}^{1} \) such that \( SD(B_{t}^{0}, B_{t}^{1}) \leq \sigma \).
- Choose \( x_{t} \leftarrow_{R} B_{t}^{0} \) and let \( D_{t} = D_{t-1} \cup \{ x_{t} \} \).
- Choose \( y_{t} \leftarrow_{R} \mathcal{A}(D_{t}) \).

For a fixed mechanism \( \mathcal{A} \) and adversary \( \mathcal{B} \), let \( V^{0} \) be the distribution on \( (y_{1}, \ldots, y_{T}) \) when \( b = 0 \) and \( V^{1} \) be the distribution on \( (y_{1}, \ldots, y_{T}) \) when \( b = 1 \).

**Lemma 3.8.** If \( \varepsilon \leq 1/2 \) and \( T \sigma \leq 1/12 \), then with probability at least \( 1 - T \delta - \delta' \) over \( y = (y_{1}, \ldots, y_{T}) \leftarrow_{R} V^{0} \),

\[
\left| \ln \left( \frac{V^{0}(y)}{V^{1}(y)} \right) \right| \leq \varepsilon(T \sigma) \sqrt{2T \log(1/\delta')} + 30\varepsilon^{2}(T \sigma)T.
\]

(We prove this lemma in Appendix A.)

We also need another lemma about the Bregman projection onto the set of high-density measures (Definition 2.11)

**Lemma 3.9.** Let \( A_{0} : \mathcal{A} \rightarrow [0, 1] \) and \( A_{1} : \mathcal{A} \cup \{a^{*}\} \rightarrow [0, 1] \) be two full-support measures over their respective sets of actions and \( s \in (0, |\mathcal{A}|) \) be such that 1) \( |A_{0}|, |A_{1}| \leq s \) and 2) \( A_{0}(a) = A_{1}(a) \) for every \( a \in \mathcal{A} \). Let \( A_{0}' = \Gamma_{s}A_{0} \) and \( A_{1}' = \Gamma_{s}A_{1} \). Then \( SD(A_{0}', A_{1}') \leq 1/s \).

**Proof of Lemma 3.9.** Using the form of the projection (Definition 2.11), it is not hard to see that for \( a \neq a^{*} \), \( A_{0}'(a) \geq A_{1}'(a) \). For convenience, we will write \( A_{0}'(a^{*}) = 0 \) even though \( a^{*} \) is technically outside of the domain of \( A_{0}' \). We can now show the following.

\[
\sum_{a \in \mathcal{A} \cup \{a^{*}\}} |A_{0}'(a) - A_{1}'(a)| = |A_{0}'(a^{*}) - A_{1}'(a^{*})| + \sum_{a \neq a^{*}} |A_{0}'(a) - A_{1}'(a)|
\]

\[
\leq 1 + \sum_{a \neq a^{*}} |A_{0}'(a) - A_{1}'(a)|
\]

\[
= 1 + \sum_{a \neq a^{*}} A_{0}'(a) - A_{1}'(a) \quad (A_{0}'(a) \geq A_{1}'(a) \text{ for } a \neq a^{*})
\]

\[
= 1 + |A_{0}'| - (|A_{1}'| - |A_{1}'(a^{*})|) \leq 1 + |A_{0}'| - (|A_{1}'| - 1)
\]

\[
= 1 + s - (s - 1) = 2
\]

We also have that \( |A_{0}'| = |A_{1}'| = s \), so

\[
SD(\tilde{A}_{0}', \tilde{A}_{1}') = \frac{1}{2} \sum_{a \in \mathcal{A} \cup \{a^{*}\}} \left| \frac{A_{0}'(a)}{|A_{0}'|} - \frac{A_{1}'(a)}{|A_{1}'|} \right|
\]

\[
= \frac{1}{2s} \sum_{a \in \mathcal{A} \cup \{a^{*}\}} |A_{0}'(a) - A_{1}'(a)| \leq \frac{1}{s}.
\]
Now we can prove one-query-to-many-analyst privacy.

**Proof of Theorem 3.7.** Fix a database $D$. Consider two adjacent query sets $Q_0 \sim Q_1$ and, without loss of generality assume $Q_0 = Q_1 \cup \{q^*\}$ and that $q^* \in Q_{id}$ for some analyst $i$. We write the output to all analysts as $v = (\tilde{x}_1, \ldots, \tilde{x}_T, b_1, \ldots, b_{|Q|}, a_1, \ldots, a_{|Q|})$ where $\tilde{D} = \{\tilde{x}_1, \ldots, \tilde{x}_T\}$ is the database that is released to all analysts, $b_1, \ldots, b_{|Q|}$ is a sequence of bits that indicates whether or not $q_j(\tilde{D})$ is close to $q_j(D)$, and $a_1, \ldots, a_{|Q|}$ is a sequence of approximate answers to the queries $q_j(D)$ (or $\bot$, if $q_j(\tilde{D})$ is already accurate). We write $v_{-id}$ for the portion of $v$ that excludes outputs specific to analyst $id$’s queries. Let $V_0, V_1$ be the distribution on outputs when the query set is $Q_0$ and $Q_1$, respectively.

We analyze the three parts of $v$ separately. First we show that $\tilde{D}$, which is shared among all analysts, satisfies analyst privacy.

**Claim 3.10.** With probability at least $1 - \delta$ over the samples $\tilde{x}_1, \ldots, \tilde{x}_T \leftarrow \kappa V_0$,

$$\left| \ln \left( \frac{\mathbb{E}_0(\tilde{x}_1, \ldots, \tilde{x}_T)}{\mathbb{E}_1(\tilde{x}_1, \ldots, \tilde{x}_T)} \right) \right| \leq \varepsilon.$$ 

**Proof of Claim 3.10.** To prove the claim, we show how the output $\tilde{x}_1, \ldots, \tilde{x}_T$ can be viewed as the output of an instantiation of the mechanism analyzed by Lemma 3.8. For every $t \in [T]$ and $\tilde{q}_1, \ldots, \tilde{q}_{t-1}$, we define the measure $D_t$ over database items to be

$$D_t(x) = \exp \left( -\frac{\eta}{2} \sum_{j=1}^{t-1} 1 + \tilde{q}_j(D) - \tilde{q}_j(x) \right).$$

Notice that if we replace a single query $\tilde{q}_t$ with $\tilde{q}_t'$ and obtain the measure $D'_t$, then for every $x \in \mathcal{X}$,

$$\left| \ln \left( \frac{\tilde{D}_t(x)}{\tilde{D}'_t(x)} \right) \right| \leq \eta.$$ 

Thus we can view $\tilde{x}_t$ as the output of an $\eta$-differentially private mechanism $A_D(\tilde{q}_1, \ldots, \tilde{q}_{t-1})$, which fits into the framework of Lemma 3.8. (Here, $\tilde{x}_t$ plays the role of $y_t$ and $\tilde{q}_1, \ldots, \tilde{q}_{t-1}$ plays the role of $D_{t-1}$ in the description of the game, while the input database $D$ is part of the description of $A$.)

Now, in order to apply Lemma 3.8, we need to argue the distribution on samples $\tilde{q}_t$ when the query set is $Q_0$ is statistically close to the distribution on samples $\tilde{q}_t$ when the query set is $Q_1$. Fix any $t \in [T]$ and let $Q_0, Q_1$ be the measure $Q_t$ over queries maintained by the query player when the input query set is $Q_0, Q_1$, respectively. For $q \neq q^*$, we have

$$Q_0(q) = Q_1(q) = \exp \left( -\frac{\eta}{2} \sum_{j=1}^{t-1} 1 + q(D) - q(\tilde{x}_j) \right).$$

Additionally, we set $Q_0(q^*) = 0$ (for notational convenience), while $Q_1(q^*) \in (0, 1]$. Thus, if we let $T_0 = \Gamma_s Q_0$ and $T_1 = \Gamma_s Q_1$, we will have $SD(T_0, T_1) \leq 1/s$ by Lemma 3.9. Since the statistical distance is $1/s = 1/12T$, we can apply Lemma 3.8 to show that with probability at least $1 - \delta$,

$$\left| \ln \left( \frac{\mathbb{E}(\tilde{x}_1, \ldots, \tilde{x}_T)}{\mathbb{E}'(\tilde{x}_1, \ldots, \tilde{x}_T)} \right) \right| \leq \frac{\eta \sqrt{T \log(1/\delta)}}{8} + \frac{5\eta^2 T}{2} \leq \varepsilon. \quad (\eta = \varepsilon/(2\sqrt{T \log(1/\delta)}))$$

$\Box$
Now that we have shown \( \hat{D} \) satisfies \((\varepsilon, \delta)\)-one-query-to-many-analyst differential privacy, it remains to show that the remainder of the output satisfies perfect one-query-to-many-analyst privacy. Recall from the proof of Theorem 3.1 that \( \hat{D} \) will be accurate for all but \( s \) queries. That is, if we let \( \{f_j\}_{j \in [|Q|]} \) consist of the functions \( f_j(D) = |q_j(D) - q_j(\hat{D})| \), then
\[
|\{ j \mid f_j(D) \geq \alpha \}| \leq s,
\]
where \( \alpha \) is chosen as in Theorem 3.1. By Lemma 2.8, the sparse vector algorithm will release bits \( b_1, \ldots, b_{|Q|} \) (the indicator vector of the subset of queries with large error) such that for every \( j \in [|Q|] \), the distribution on \( b_j \) does not depend on any function \( f_{j'} \) for \( j' \neq j \). Thus, if \( z_{-a} \) contains all the bits of \( b_1, \ldots, b_{|Q|} \) that do not correspond to queries in \( Q_a \), then the distribution of \( z_{-id} \) does not depend on the queries asked by analyst \( id \), and thus \( z_{-id} \) is perfectly one-query-to-many analyst private. Finally, for each query \( q_j \) such that \( b_j = 1 \), the output to the owner of that query will include \( a_j = q_j(D) + z_j \) where \( z_j \) is an independent sample from the Laplace distribution. These outputs do not depend on any other query, and thus are perfectly one-query-to-many analyst private. This completes the proof of the theorem.

\[ \square \]

4 A One-Analyst-to-Many-Analyst Private Mechanism

4.1 An Offline Mechanism for Linear Queries

In this section we present an algorithm for answering linear queries that satisfies the stronger notion of one-analyst-to-many-analyst privacy. The algorithm is similar to Algorithm 3, but with two notable modifications.

First, instead of the “query player” of Algorithm 3, we will have an “analyst player” who chooses analysts as actions and is trying to find an analyst \( id \in [m] \) for which there is at least one query in \( Q_{id} \) with large error (recall that the queries are given to the mechanism in sets \( Q_1, \ldots, Q_m \)). That is, the analyst player attempts to find \( id \in [m] \) to maximize \( \max_{q \in Q_{id}} q(D) - q(\hat{D}) \).

Second, we will compute a database \( \hat{D} \) such that \( \max_{q \in Q_{id}} |q(D) - q(\hat{D})| \) is small for all but \( s \) analysts in the set \([m]\), rather than having the \( s \) mishandled queries in Algorithm 3. We can still use sparse vector to find these \( s \) analysts, however we can’t answer the queries with the Laplace mechanism, since each of the analysts may ask an exponential number of queries. However, since there are not too many analysts remaining, we can use \( s \) independent copies of the multiplicative weights mechanism (each run with \( \varepsilon' \approx \varepsilon/\sqrt{s} \)) to handle each analyst’s queries.

4.1.1 Accuracy Analysis

Theorem 4.1. Algorithm 4 is \((\alpha, \beta)\)-accurate for
\[
\alpha = \tilde{O}\left( \sqrt{\log(|X| + m)} \log |Q_{id}| \log(m/\beta) \log^{3/4}(1/\delta) \right). \varepsilon n^{1/3} \log^{3/4}(1/\delta). \varepsilon n^{1/3}
\]

Proof. As we discussed above, the algorithm is computing an approximate equilibrium of the game
\[
G_{D,Q}(x, id) = \max_{id \in [m]} \max_{q \in Q_{id}} \frac{1 + q(D) - q(x)}{2}.
\]
Algorithm 4 Offline Mechanism for Linear Queries with One-Analyst-to-Many-Analyst Privacy

**Input:** Database $D \in \mathcal{X}^n$, and $m$ sets of linear queries $Q_1, \ldots, Q_m$. For $id \in [m]$, let $Q_{id} = Q_{id} \cup -Q_{id}$.

**Initialize:** Let $D_0(x) = 1/|\mathcal{X}|$ for each $x \in \mathcal{X}$, $I_0(q) = 1/m$ for each $id \in [m]$,

$$T = n^{2/3} \max\{\log |\mathcal{X}|, m\}, \quad \eta = \frac{\sqrt{T \log(1/\delta)}}{2\epsilon}, \quad s = 12T.$$

**DataPlayer:**
On input an analyst $\hat{id}_t$, for each $x \in \mathcal{X}$, update:

$$D_t(x) = D_{t-1}(x) \cdot \exp \left( -\eta \max_{q \in Q_{\hat{id}_t}} \left( \frac{1 + \hat{q}_t(D) - \hat{q}_t(x)}{2} \right) \right)$$

Choose $\hat{x}_t \leftarrow_r \tilde{D}_t$ and send $\hat{x}_t$ to **AnalystPlayer**

**AnalystPlayer:**
On input a data element $\hat{x}_t$, for each $id \in I$, update:

$$I_{t+1}(id) = I_t(id) \cdot \exp \left( -\eta \max_{q \in Q_{id}} \left( \frac{1 + q(D) - q(\hat{x}_t)}{2} \right) \right)$$

Let $P_{t+1} = \Gamma_s I_{t+1}$.
Choose $\hat{id}_{t+1} \leftarrow_r P_{t+1}$ and send $\hat{id}_{t+1}$ to **DataPlayer**

**GenerateSynopsis:**
Let $\tilde{D} = (\tilde{x}_1, \ldots, \tilde{x}_T)$.
Run sparse vector on $\tilde{D}$, obtain a set of at most $s$ analysts:

$I_f = \{\hat{id}_1, \ldots, \hat{id}_s\} \subseteq [m]$

For each analyst $id \in I_f$, run $\mathcal{A}_{MW}(D, Q_{id})$ with parameters $\epsilon' = \frac{\epsilon}{10\sqrt{s \log(3s/\delta)}}$ and $\delta' = \frac{\delta}{3s}$.

Obtain a sequence of answers $\bar{a}_{id}$.
Output $\tilde{D}$ to all analysts.
For each id $\in [m] \setminus I_f$, output $\bar{a}_{id}$ to analyst $id$.

Let $v, v_s$ be the value and constrained value of this game, respectively. First we pin down the quantities $v$ and $v_s$.

**Claim 4.2.** For every $D, m, Q$, the value and constrained value of $G_{D,m,Q}$ is $1/2$.

The proof of this claim is omitted, but is nearly identical to that of Claim 3.2.

Let $\tilde{D} = \frac{1}{T} \sum_{t=1}^T \tilde{x}_t$. By Corollary 2.14,

$$v_s - 2\rho \leq \max_{I \in \Delta_s([m])} \mathbb{E}_{\hat{id} \leftarrow I} \left[ \max_{q \in Q_{\hat{id}}} \left( \frac{1 + q(D) - q(\tilde{D})}{2} \right) \right] \leq v + 2\rho. $$

19
Applying Claim 4.2 and rearranging terms, we have that with probability $1 - \beta/3$,

$$
\left| \max_{I \in \Delta_s([m])} \left( \mathbb{E}_{I \leftarrow \mathcal{R} \left[ \max_{q \in \mathcal{Q}_{id}} q(D) - q(\tilde{D}) \right] } \right) \right| = \max_{I \in \Delta_s([m])} \left( \mathbb{E}_{I \leftarrow \mathcal{R} \left[ \max_{q \in \mathcal{Q}_{id}} |q(D) - q(\tilde{D})| \right] } \right) \leq 4\rho
$$

$$= 4 \left( \frac{\eta + \max\{\log |\mathcal{X}|, \log m\}}{\eta T} + \frac{4\log(3/\beta)}{\sqrt{T}} \right)
$$

$$= O \left( \frac{\sqrt{\log(|\mathcal{X}| + m)\log(1/\delta) + \log(1/\beta)}}{\epsilon n^{1/3}} \right) := \alpha_{\tilde{D}}.
$$

The previous statement suffices to show that $\max_{q \in \mathcal{Q}_{id}} |q(D) - q(\tilde{D})| \leq \alpha_{\tilde{D}}$ for all but $s$ analysts $id \in [m]$. Otherwise, the uniform distribution over the analysts for which the error bound of $\alpha_{\tilde{D}}$ does not hold would be a distribution over analysts, contained in $\Delta_s([m])$ with expected error larger than $\alpha_{\tilde{D}}$.

Since there are at most $s$ such analysts we can run the sparse vector algorithm (Lemma 2.8), and, with probability at least $1 - \beta/3$, it will identify every analyst $id$ such that the maximum error over all queries in $\mathcal{Q}_{id}$ is larger than $\alpha_{\tilde{D}} + \alpha_{SV}$ for

$$\alpha_{SV} = O \left( \frac{s\log(1/\delta)\log(m/\beta)}{\epsilon n} \right).$$

There are at most $s$ such analysts. Thus, running the multiplicative weights mechanism (Lemma 2.6) independently for each of these analysts’ queries—with privacy parameters $\epsilon' = \Theta(\epsilon/\sqrt{s \log(s/\delta)})$ and $\delta' = \Theta(\delta/s)$—will yield answers such that, with probability $1 - \beta/3$, for every $id \in I'$,

$$\max_{q \in \mathcal{Q}_{id}} |q(D) - a_q| \leq O \left( \frac{s^{1/4}\log^{1/4} |\mathcal{X}|\sqrt{\log(s)|\mathcal{Q}_{id}|/\beta}\log^{3/4}(s/\delta)}{\sqrt{\epsilon n}} \right)
$$

$$\leq \tilde{O} \left( \frac{n^{1/6}\sqrt{\log(|\mathcal{X}| + m)\log(|\mathcal{Q}_{id}|/\beta)}\log^{3/4}(1/\delta)}{\sqrt{\epsilon n}} \right)
$$

$$\leq \tilde{O} \left( \frac{\sqrt{\log(|\mathcal{X}| + m)\log(|\mathcal{Q}_{id}|/\beta)}\log^{3/4}(1/\delta)}{n^{1/3}\sqrt{\epsilon}} \right) := \alpha_{MW}.
$$

Taking a union bound, observing that the maximum error on any query is $\max\{\alpha_{\tilde{D}} + \alpha_{SV}, \alpha_{MW}\}$, and simplifying, we get that the mechanism is $(\alpha, \beta)$-accurate for

$$\alpha = \tilde{O} \left( \frac{\sqrt{\log(|\mathcal{X}| + m)\log |\mathcal{Q}_{id}|\log(m/\beta)\log^{3/4}(1/\delta)}}{\epsilon n^{1/3}} \right).$$

\[\square\]

### 4.1.2 Data Privacy

**Theorem 4.3.** Algorithm 4 satisfies $(\epsilon, \delta)$-differential privacy for the data.

We omit the proof of this theorem, which follows that of Theorem 3.3 almost identically. The only difference is that in the final step, we need to argue that running $s$ independent copies of multiplicative weights with privacy parameters $\epsilon' = \Theta(\epsilon/\sqrt{s \log(s/\delta)})$ and $\delta' = \Theta(\delta/s)$ satisfies $(\epsilon/3, \delta/3)$-differential privacy, which follows directly from the composition properties of differential privacy (Lemma 2.9).
4.1.3 Query Privacy

In this section we prove query privacy for our one analyst to many analyst mechanism.

**Theorem 4.4.** Algorithm 4 satisfies \((\varepsilon, \delta)\)-one-analyst-to-many-analyst differential privacy.

**Proof.** Fix a database \(D\). Consider two adjacent sets of queries \(Q_0, Q_1\). Without loss of generality assume \(Q_0 = Q_{\text{id}_1} \cup \ldots \cup Q_{\text{id}_m}\) and \(Q_1 = Q_0 \cup Q_{\text{id}^*}\). That is \(Q_1\) is just \(Q_0\) with an additional set of queries \(Q_{\text{id}^*}\) added. We write the output to all analysts as \(v = (\bar{x}_1, \ldots, \bar{x}_T, b_1, \ldots, b_m, \bar{a}_1, \ldots, \bar{a}_m)\) where \(\bar{D} = \bar{x}_1, \ldots, \bar{x}_T\) is the database that is released to all analysts, \(b_1, \ldots, b_m\) is a sequence of bits that indicates whether or not \(q_j(\bar{D})\) is close to \(q_j(D)\) for every \(q \in Q_{\text{id}_d}\), and \(\bar{a}_1, \ldots, \bar{a}_m\) is a sequence consisting of the output of the multiplicative weights mechanism for every analyst \(\text{id} \in [m]\) and \(\bot\) for every other analyst. Let \(V_0, V_1\) be the distribution on outputs when the queries are \(Q_0\) and \(Q_1\), respectively.

The proof closely follows the proof of one-query-to-many-analyst privacy for Algorithm 3. Showing that the final two parts \(b, \bar{a}\) of the output are query private is essentially the same, so we will focus on proving that \(\bar{D}\) satisfies one-analyst-to-many-analyst privacy.

**Claim 4.5.** With probability at least \(1 - \delta\) over \(\bar{x}_1, \ldots, \bar{x}_T \leftarrow_{\text{r}} V_0\),

\[
\left| \ln \left( \frac{V_0(\bar{x}_1, \ldots, \bar{x}_T)}{V_1(\bar{x}_1, \ldots, \bar{x}_T)} \right) \right| \leq \varepsilon.
\]

**Proof of Claim 4.5.** To prove the claim, we show how the output \(\bar{x}_1, \ldots, \bar{x}_T\) can be viewed as the output of an instantiation of the mechanism analyzed by Lemma 3.8. Notice that for every \(t \in [T]\) and \(\text{id}_1, \ldots, \text{id}_{t-1}\), we can write the measure \(D_t\) over database items as

\[
D_t(x) = \exp \left( -\left( \eta/2 \right) \sum_{j=1}^{t-1} \max_{q \in Q_{\text{id}_j}} \frac{q_j(D) - \tilde{q}_j(x)}{1 + \tilde{q}_j(D) - \tilde{q}_j(x)} \right).
\]

If we replace a single analyst \(\text{id}_t\) with \(\text{id}_t'\), and obtain the measure \(D'_t\), then for every \(x \in X\),

\[
\left| \ln \left( \frac{D_t(x)}{D'_t(x)} \right) \right| \leq \eta.
\]

Thus we can view \(\bar{x}_t\) as the output of an \(\eta\)-differentially private mechanism \(A_D(\text{id}_1, \ldots, \text{id}_{t-1})\), which fits into the framework of Lemma 3.8. (Here, \(\bar{x}_t\) plays the role of \(y_t\) and \(\text{id}_1, \ldots, \text{id}_{t-1}\) plays the role of \(D_{t-1}\) in the description of the game, while the input database \(D\) is part of the description of \(A\).)

As before, we apply Lemma 3.8, to argue that the distribution on analysts \(\text{id}_t\) when the query set is \(Q_0\) is statistically close to the distribution on analysts \(\text{id}_t\) when the analyst set is \(Q_1\). The argument does not change significantly, thus we can apply Lemma 3.8 to show that with probability at least \(1 - \delta\),

\[
\left| \ln \left( \frac{V(\bar{x}_1, \ldots, \bar{x}_T)}{V'(\bar{x}_1, \ldots, \bar{x}_T)} \right) \right| \leq \frac{\eta \sqrt{T \log(1/\delta)}}{8} + \frac{5\eta^2 T}{2} \leq \varepsilon.
\]

\(\eta = \varepsilon/(2 \sqrt{T \log(1/\delta)})\)

As before, the remainder of the output satisfies perfect one-analyst-to-many-analyst privacy. This completes the proof of the theorem. \(\Box\)
5 A One-Query-to-Many-Analyst Private Online Mechanism

In this section, we present a mechanism that provides one-query-to-many-analyst privacy in an online setting. The mechanism can give accurate answers to any fixed sequence of queries that are given to the mechanism one at a time, rather than the typical setting of adaptively chosen queries.

The mechanism is similar to the online multiplicative weights algorithm of Hardt and Rothblum [HR10]. In their algorithm, a hypothesis about the true database is maintained throughout the sequence of queries. When a query arrives, it is classified according to whether or not the current hypothesis accurately answers that query. If it does, then the query is answered according to the hypothesis. Otherwise, the query is answered with a noisy answer computed from the true database and the hypothesis is updated using the multiplicative weights update rule.

The main challenge in making that algorithm query private is to argue that the hypothesis does not depend too much on the previous queries. We overcome this difficulty by “sampling from the hypothesis.” (recall that a database can be thought of as a distribution over the data universe). We must balance the need to take many samples — so that the database we obtain by sampling accurately reflects the hypothesis database, and the need to limit the impact of any one query on the sampled database. To handle both these constraints, we introduce batching — instead of updating every time we find a query not well-answer by the hypothesis, we batch together $s$ queries at a time, and do one update on the average of these queries to limit the influence of any single query.

A note on terminology: the execution of the algorithm takes place in several rounds, where each round processes one query. Rounds where the query is answered using the real database are called bad rounds; rounds that are not bad are good rounds. We will split the rounds into $T$ epochs, where the hypothesis $H_t$ is used during epoch $t$.

5.1 Accuracy

First, we sketch a proof that the online mechanism answers linear queries accurately. Intuitively, there are three ways that our algorithm might give an inaccurate answer, and we treat each separately. First, in a good round, the answer given by the hypothesis may be a bad approximation to the true answer. Second, in a bad round, the answer given may have too much noise. We address these two cases with straightforward arguments showing that the noise is not too large in any round.

The third way the algorithm may be inaccurate is if there are more than $R$ bad rounds, and the algorithm terminates early. We show that this is not the case using a potential argument: after sufficiently many bad rounds, the hypothesis $D_T$ and the sample $H_T$ will be accurate for all queries in the stream, and thus there will be no more bad rounds. The potential argument is a simple extension of the argument in Hardt and Rothblum [HR10] that handles the additional error coming from taking samples from $D_t$ to obtain $H_t$.

We will use the following tail bound on sums of Laplace variables.

**Lemma 5.1 ([GRU12]).** Let $X_1, \ldots, X_T$ be $T$ independent draws from Lap($2/\varepsilon$), and let $X = \sum_{t=1}^{T} X_t$. Then,

$$\Pr \left[ |X| > \frac{5\sqrt{T \log(2/\beta)}}{\varepsilon} \right] < \beta.$$  

**Theorem 5.2.** Algorithm 5 is $(\alpha, \beta)$-accurate for

$$\alpha = O \left( \frac{\log^{3/2}(k/\beta) \sqrt{\log |X| \log(1/\delta)}}{\varepsilon^{3/2} n^{2/5}} \right).$$
Algorithm 5 Analyst-Private Multiplicative Weights for Linear Queries

**Input:** Database $D \in \mathcal{X}^n$, sequence $q_1, \ldots, q_k$ of linear queries

**Initialize:** $D_0(x) = 1/|\mathcal{X}|$ for each $x \in \mathcal{X}$, $H_0 = D_0$, $U_0 = \emptyset$, $s_0 = s = \text{Lap}(2/\varepsilon)$, $t = 0$, $r = 0$,

\[
\eta = \frac{1}{n^{2/5}}, \quad s = \frac{128n^{2/5} \sqrt{\log |\mathcal{X}| \log (4k/\beta) \log (1/\delta)}}{\varepsilon},
\]

\[
\hat{n} = 32n^{4/5} \log (4k/\beta), \quad T = n^{4/5} \log |\mathcal{X}|, \quad R = 2sT,
\]

\[
\sigma = \frac{20000 \log^{3/4} |\mathcal{X}| \log^{1/4} (4k/\beta) \log^{5/4} (4/\delta)}{\varepsilon^{3/2} n^{2/5}},
\]

\[
\tau = \frac{80000 \log^{3/4} |\mathcal{X}| \log^{5/4} (4k/\beta) \log^{5/4} (4/\delta)}{\varepsilon^{3/2} n^{2/5}}.
\]

**AnswerQueries:**

While $t < T$, $r < R$, $i \leq k$, on input query $q_i$:

Let $z_i = \text{Lap}(\sigma)$

If $|q_i(D) - q_i(H_t) + z_i| \leq \tau$: Output $q_i(H_t)$

Else:

Let $u = \text{sgn}(q_i(H_t) - q_i(D) - z_i) \cdot q_i$, $U_t = U_t \cup \{u\}$

Output $q_i(D) + z_i$

Let $r = r + 1$

If $|U_t| > s_t$:

Let $(D_{t+1}, H_{t+1}) = \text{Update}(D_t, U_t)$

Let $U_{t+1} = \emptyset$, $s_{t+1} = s + \text{Lap}(2/\varepsilon)$

Let $t = t + 1$

Advance to query $q_{i+1}$

**Update:**

Input: distribution $D_t$, update queries $U_t = \{u_1, \ldots, u_{s_t}\}$

For each $x \in \mathcal{X}$:

Let $u_t(x) = \frac{1}{s_t} \sum_{j=1}^{s_t} u_j(x)$

Update $D_{t+1}(x) = \exp(-\alpha'/2)u_t(x)D_t(x)$

Normalize $D_{t+1}$

Let $H_{t+1}$ be $\hat{n}$ independent samples from $D_{t+1}$

Return: $(D_{t+1}, H_{t+1})$

We note that we can achieve a slightly better dependence on $k, |\mathcal{X}|, \frac{1}{\varepsilon}, \frac{1}{\beta}, \frac{1}{\delta}$ by setting the parameters a bit more carefully. See Section 5.4 for an intuitive picture of how to set the parameters optimally. We have made no attempt to optimize the constant factors in the algorithm.

**Proof.** First we show that, as long as the algorithm has not terminated early, it answers every query accurately.

**Claim 5.3.** Before the algorithm terminates, with probability $1 - \beta/4$, every query is answered with error at most $\tau + 6\sigma \log (3k/\beta)$

23
Proof of Claim 5.3. Condition on the event that $|z_i| \leq 6\sigma \log(4k/\beta)$ for every $i = 1, 2, \ldots, k$. A standard analysis of the tails of the Laplace distribution shows that this event occurs with probability at least $1 - \beta/3$.

First we consider bad rounds. In these rounds $q_i$ is answered with $q_i(D) + z_i$. Since we have assumed $|z_i|$ is not too large, all of these queries are answered accurately.

Now we consider good rounds. In these rounds we answer with $q_i(H_t)$, and we will only have a good round if $|q_i(D) - q_i(H_t) + z_i| \leq \tau$. Since we have assumed a bound on $|z_i|$, we can only have a good round if $|q_i(D) - q_i(H_t)| \leq \tau + 6\sigma \log(3k/\beta)$.

Now we must show that the algorithm does not terminate early. Recall that it can terminate early either because it hits a limit on the number of epochs, or because it hits a limit on the number of bad rounds. We will use a potential argument to show that there cannot be too many epochs. The number of bad rounds that is in epoch $t$ is a random variable $s_t$, and we will also show that with high probability, there are not too many bad rounds within the $T$ epochs.

Claim 5.4. With probability $1 - 3\beta/4$, the algorithm does not terminate before answering $k$ queries.

Proof of Claim 5.4. We will use a potential argument a la Hardt and Rothblum [HR10] on the sequence of databases $D_t$. The potential function will be

$$
\Phi_t = RE(D||D_t) := \sum_{x \in \mathcal{X}} D(x) \log(D(x)/D_t(x)).
$$

Elementary properties of the relative entropy function show that $\Phi_t \geq 0$ and $\Phi_0 = RE(D||D_0) \leq \log |\mathcal{X}|$. A lemma of Hardt and Rothblum expresses the potential decrease from the multiplicative weights update rule in terms of the error of the current hypothesis on the update query.

Lemma 5.5 ([HR10]).

$$
\Phi_{t-1} - \Phi_t \geq \eta (u_t(D) - u_t(D_{t-1})) - \eta^2/4.
$$

Since the potential function is bounded between 0 and $\log |\mathcal{X}|$, we can get a bound on the number of epochs by showing that the potential decreases significantly between most epochs. Given the preceding lemma, we simply need to show that the queries $u_1, u_2, \ldots$ have large (positive) error.

Recall that $u_t = \frac{1}{t} \sum_{u \in \mathcal{U}_t} u$. Also recall that if $u \in \mathcal{U}$ and $u = q_i$, then the reason $q_i$ is in $\mathcal{U}$ is because $q_i(D) - q_i(H_{t-1}) + z_i > \tau$. Similarly, if $u = -q_i$, then $q_i(D) - q_i(H_{t-1}) + z_i < -\tau$. We will focus on the first case where $q_i(D) - q_i(H_{t-1}) + z_i > \tau$, the other case will follow similarly. We can get a lower bound on $u(D) - u(D_{t-1})$ as follows.

$$
\begin{align*}
&u(D) - u(D_{t-1}) \\
\geq & (u(D) - u(H_{t-1}) + z_i - |z_i| - |q_i(H_{t-1}) - q_i(D_{t-1})|) \\
\geq & \tau - |z_i| - |q_i(H_{t-1}) - q_i(D_{t-1})|.
\end{align*}
$$

We need to show that the right-hand side of the final expression is large. We have already conditioned on the event that $|z_i| \leq 6\sigma \log(3k/\beta) \leq \tau/4$. Recall that $H_{t-1}$ is a collection of $\hat{n}$ samples from $D_{t-1}$. Thus a simple Chernoff bound (over the $\hat{n}$ samples) and a union bound (over the $k$ queries) shows that, with probability $1 - \beta/4$, for every $i \in [k]$, $|q_i(D_{t-1}) - q_i(H_{t-1})| \leq \sqrt{16 \log(3T/\beta)/\hat{n}} \leq \tau/4$.

Thus, with probability at least $1 - 2\beta/3$, for every $t$ and every $u \in \mathcal{U}_t$,

$$
u(D) - u(D_{t-1}) \geq \tau - \tau/4 - \tau/4 = \tau/2.
$$

24
Now,
\[ u_t(D) - u_t(D_{t-1}) = \frac{1}{s} \sum_{u \in U_t} u(D) - u(D_{t-1}) \geq \frac{\tau |U_t|}{2s}. \]

Conditioning on the event that all of the noise values \( z_i \) are small and all of the sampled hypotheses \( H_t \) are accurate for \( D_t \) on every query, we can calculate
\[
\Phi_{T'} - \Phi_0 \geq \frac{\eta T'}{2s} \left( \sum_{t \leq T'} |U_t| \right) - T' \eta^2
\]
\[
= \frac{\eta T'}{2s} \left( \sum_{t \leq T'} s_t \right) - T' \eta^2 = \frac{\eta T'}{2s} + \frac{\eta T'}{2s} \left( \sum_{t \leq T'} s_t \right) - T' \eta^2,
\]
where \( s_t \) is the value of the sample \( \text{Lap}(2/\varepsilon) \) used to compute \( s_t \) in the \( t \)-th epoch. Thus, applying Lemma 5.1 to \( S = \sum_{t \leq T'} s_t \), we have that with probability \( 1 - \beta/4 \),
\[
\Phi_{T'} - \Phi_0 \geq \frac{\eta T'}{2s} \left( \sum_{t \leq T'} s_t \right) - T' \eta^2 \geq \frac{\eta T'}{2s} - \frac{\eta T'}{2s} \varepsilon^{-1} \sqrt{T'} \log(20/\beta) - T' \eta^2.
\]

Now, noting that \( \tau/2 > 8\eta \) and simplifying,
\[
\Phi_{T'} - \Phi_0 \geq 2\eta^2 \tau T' - \eta^2 T' \geq \eta^2 T'.
\]

Thus, conditioning on all the events above, \( T' \leq \log |X'|/\eta^2 \leq n^{4/5} \log |X'| \). These events all occur together with probability at least \( 1 - 3\beta/4 \), and thus the algorithm does not terminate because it hits the limit of \( T \) epochs. Lastly, we need to show that the algorithm does not hit the limit of \( R \) bad rounds within those at-most \( T \) epochs. Notice that the number of bad rounds is at most
\[
\sum_{t=1}^{T} s_t = \sum_{t=1}^{T} s + S_t,
\]
where \( s_t \) is the sample of \( \text{Lap}(2/\varepsilon) \) used to compute \( s_t \). Applying Lemma 5.1 again we have
\[
\sum_{t=1}^{T} s + S_t \leq Ts + \sum_{t=1}^{T} S_t \leq Ts + 5\varepsilon^{-1} \sqrt{T} \log(2/\beta) \leq 2Ts = R.
\]

Thus the algorithm does not terminate due to having more than \( R \) bad rounds. Since the algorithm does not hit its limit of \( T \) epochs or \( R \) bad rounds, except with probability at most \( 3\beta/4 \), the claim is proven. \( \square \)

Combining the previous two claims proves the theorem. \( \square \)

### 5.2 Data Privacy

In this section we establish that our mechanism satisfies differential privacy. Our proof will rely on a modular analysis of interactive differentially private algorithms from Gupta, Roth, and Ullman [GRU12]. Although we have not presented our algorithm in their framework, the algorithm can easily be seen to fit, and thus we will state an adapted version of their theorem without proof.
Theorem 5.6 ([GRU12], Adapted). If Algorithm 5 experiences at most $R$ bad rounds, and the parameters are set so that $\sigma \geq \frac{1000\sqrt{R\log(4/\delta)}}{\epsilon n}$, then Algorithm 5 is $(\epsilon, \delta)$-differentially private.

Theorem 5.7. Algorithm 5 satisfies $(\epsilon, \delta)$-differential privacy.

Proof. The theorem follows directly from Theorem 5.6 and our choice of $R$. \hfill \square

5.3 Query Privacy

More interestingly, we show that this mechanism satisfies one-query-to-many-analyst privacy.

Theorem 5.8. Algorithm 5 is $(\epsilon, \delta)$-one-query-to-many-analyst private.

Proof. Fix the input database $D$ and the coins of the Laplace noise — we will show that for every value of the Laplace random variables, the mechanism satisfies analyst privacy. Consider any two adjacent sequences of queries $Q_0, Q_1$. Without loss of generality, we will assume that $Q = q_1, \ldots, q_k$ and $Q' = q^*, q_1, \ldots, q_k$. For notational simplicity, we assume that every query in $Q$ has a fixed index, regardless of the presence of $q^*$. More generally, we could identify each query in the sequence by a unique index (say, a long random string) that is independent of the other queries. We want to argue that the answers to all queries in $Q$ are private, but not that the answer to $q^*$ is private (if it is requested).

We will represent the answers to the queries in $Q$ by a sequence $\{(H_t, i_t)\}_{t \in [T]}$ where $H_t$ is the hypothesis used in the $t$-th epoch and $i_t$ is the index of the last query in that epoch (the one that caused the mechanism to switch to hypothesis $H_t$). Observe that for a fixed database $D$, Laplace noise, and sequence of queries $Q$, we can simulate the output of the mechanism for all queries in $Q$ given only this information — once we fix a hypothesis $H_t$, we can determine whether any query $q$ will be added to the update pool in this epoch. So once we begin epoch $t$ with hypothesis $H_t$, we have fixed all the bad rounds, and once we are given $i_t$, we have determined when epoch $t$ ends and epoch $t + 1$ begins. At this point, we fix the next hypothesis $H_{t+1}$ and continue simulating.

Formally, let $V_0, V_1$ be distribution over sequences $\{(H_t, i_t)\}$ when the query sequence is $Q_0, Q_1$, respectively. We will show that with probability at least $1 - \delta$, if $\{(H_t, i_t)\}_{t \in [T]}$ is drawn from $V_0$, then

$$\ln \left( \frac{V_0(\{(H_t, i_t)\})}{V_1(\{(H_t, i_t)\})} \right) \leq \epsilon.$$

Recall that $\mathcal{U}_t$ is the set of queries that are used to update the distribution $D_t$ to $D_{t+1}$. We will use $\mathcal{U}_{\leq t} = \bigcup_{j=0}^{t} \mathcal{U}_t$ to denote the set of all queries used to update the distributions $D_0, \ldots, D_t$. Notice that if $q^*$ does not get added to the set $\mathcal{U}_0$, then $V_0$ and $V_1$ will be distributed identically. Therefore, suppose $q^* \in \mathcal{U}_0$.

First we must reason about the joint distribution of the first component of the output.

Claim 5.9. For all $H_0, i_0$,

$$\ln \left( \frac{V_0(H_0, i_0)}{V_1(H_0, i_0)} \right) \leq \frac{\epsilon}{2}.$$

Proof of Claim 5.9. Since $H_0$ does not depend on the query sequence, it will be identically distributed in both cases. Once $H_0$ is fixed, we can determine whether a query $q$ will cause an update. Fix query $q_{i_0}$ and assume that it is the $s$-th update query in the sequence $q_1, \ldots, q_k$ and the $(s + 1)$-st update query in the sequence $q^*, q_1, \ldots, q_k$. Then $V_0(i_0 | H_0) = \Pr [s_0 = s]$ and $V_1(i_0 | H_0) = \Pr [s_0 = s + 1]$. By the basic properties of the Laplace distribution, $| \ln(V_0(i_0 | H_0) / V_1(i_0 | H_0)) | \leq \epsilon/100$. \hfill \square
Now we reason about the remaining components \((H_1, i_1), \ldots, (H_T, i_T)\).

**Claim 5.10.** For every \(H_0, i_0\), with probability at least \(1 - \delta\) over the choice of components \(v = (H_1, i_1, \ldots, H_T, i_T) \leftarrow \mathcal{R}(V_0 \mid v_{t-1})\), we have
\[
\left| \ln \left( \frac{V_0(v \mid H_0, i_0)}{V_1(v \mid H_0, i_0)} \right) \right| \leq \frac{\varepsilon}{2}.
\]

**Proof of Claim 5.10.** We will show that \(v\) is the \(\hat{n}T\)-fold composition of \((\varepsilon_0, 0)\)-differentially private mechanisms for suitable \(\varepsilon_0\). Fix a prefix \(v_{t-1} = H_0, i_0, \ldots, H_{t-1}, i_{t-1}\). Given this prefix, we can determine for any given sequence of queries \(q_1, \ldots, q_{i_{t-1}}\) or \(q^*, q_1, \ldots, q_{i_{t-1}}\) which queries are in the update set. Moreover, if \(\mathcal{U}_{<t}\) is the set of all update queries from the first query sequence, and \(\mathcal{U}'_{<t}\) is the set of all update queries from the second sequence, then \(\mathcal{U}_{<t} \Delta \mathcal{U}'_{<t} = q^*\).

Now consider the distribution of \(H_t\). Each sample in \(H_t\) comes from the distribution \(D_t\), which is either
\[
D_t(x) \propto \exp \left( -(\eta/s) \sum_{u \in \mathcal{U}_{<t}} u \right) \quad \text{or} \quad D'_t(x) \propto \exp \left( -(\eta/s) \frac{1}{s} \sum_{u \in \mathcal{U}'_{<t}} u \right).
\]

Given this, it is easy to see that for any \(x\) we have \(|\ln(D_t(x)/D'_t(x))| \leq 2\eta/s := \varepsilon_0\). Notice that once \(i_{t-1}\) and \(H_t\) are fixed, \(i_t\) depends only on the choice of \(s_t\) (the number of bad rounds to allow before updating the hypothesis), which is independent of the query sequence and thus incurs no additional privacy loss. Thus the only privacy loss comes from the \(\hat{n}\) samples in each of the \(T\) epochs, and the mechanism is a \(\hat{n}T\)-fold adaptive composition of \((\varepsilon_0, 0)\) differentially private mechanisms. A standard composition analysis (Lemma 2.9) shows that the components \(v\) are \((\varepsilon', \delta)\)-DP for \(\varepsilon' = \varepsilon_0 \sqrt{2nT \log(1/\delta) + 2\varepsilon_0^2 T} \leq \varepsilon/2\). This completes the proof of the claim.

Combining these two claims proves the theorem.

\[\square\]

### 5.4 Handling Arbitrary Low-Sensitivity Queries

We can also modify this mechanism to answer arbitrary \(\Delta\)-sensitive queries, albeit with worse accuracy bounds. As with our offline algorithms, we modify the algorithm to run the multiplicative weights updates over the set of databases \(X^n\) and adjust the parameters. When we run multiplicative weights over a support of size \(|X|^n\) (rather than \(|X|\)), the number of epochs increases by a factor of \(n\), which in turn affects the amount of noise we have to add to ensure privacy.

We will now sketch the argument, ignoring the parameters \(\beta\) and \(\delta\) for simplicity. In order to get convergence of the multiplicative weights distribution, we need to take \(T \approx \frac{n \log |X|}{\eta}\) and in order to ensure that \(H_t\) approximates \(D_t\) sufficiently well, we take \(\hat{n} \approx \sqrt{\frac{\log k}{\eta'}}\). Recall that to argue analyst privacy, we viewed the mechanism as being (essentially) the \(\hat{n}T\)-fold composition of \(\varepsilon_0\)-analyst private mechanisms, where \(\varepsilon_0 = \eta/s\). In order to get analyst privacy, we needed

\[
\frac{\eta}{s} \approx \frac{\varepsilon}{\sqrt{\hat{n}T}} \approx \frac{\varepsilon \eta^2}{\sqrt{n \log |X| \log^{1/4} k}}
\]

\[\Rightarrow s \gtrsim \frac{\sqrt{n \log |X| \log^{1/4} k}}{\varepsilon \eta}.
\]
Once we have set \( s \) (as a function of the other parameters) to achieve analyst privacy, we can work on establishing data privacy. As before, the number of bad rounds will be

\[
R \approx sT \approx \frac{n^{3/2} \log^{3/2} |X| \log^{1/4} k}{\varepsilon \eta^3}.
\]

Given this bound on the number of bad rounds, we need to set

\[
\sigma \approx \frac{\Delta \sqrt{R}}{\varepsilon} \approx \frac{\Delta n^{3/4} \log^{3/4} |X| \log^{1/8} k}{\varepsilon \eta^{3/2}}
\]

to obtain data privacy, and

\[
\tau \approx \sigma \log k \approx \frac{\Delta n^{3/4} \log^{3/4} |X| \log^{9/8} k}{\varepsilon \eta^{3/2}}
\]

to ensure that all the update queries truly have large error on the current hypothesis \( H_t \).

The final error bound will come from observing that \( \eta \) and \( \tau \) are both lower bounds on the error. The error is bounded below by \( \tau \) because that is the noise threshold set by the algorithm, and \( \tau \) must be larger than \( \eta \) or else we cannot argue that multiplicative weights makes progress during update rounds. Thus setting \( \eta = \tau \) will approximately minimize the error.

The final error bound we obtain (ignoring the parameters \( \beta \) and \( \delta \)) is

\[
O \left( \frac{\Delta^{2/5} n^{3/10} \log^{3/10} |X| \log^{9/20} k}{\varepsilon^{2/5}} \right)
\]

which gives a non-trivial error guarantee when \( \Delta \ll 1/n^{3/4} \).

6 Conclusions

We have shown that it is possible to privately answer many queries while also preserving the privacy of the data analysts even if multiple analysts may collude, or if a single analyst may register multiple accounts with the data administrator. In the one-query-to-many-analyst privacy for linear queries in the non-interactive setting, we are able to recover the nearly optimal \( \tilde{O}(1/\sqrt{n}) \) error bound achievable without promising analyst privacy. However, it remains unclear whether this bound is achievable for one-analyst-to-many-analyst privacy, or for non-linear queries, or in the interactive query release setting.

We have also introduced a novel view of the private query release problem as an equilibrium computation problem in a two-player zero-sum game. This allows us to encode different privacy guarantees by picking strategies of the different players and the neighboring relationship on game matrices (i.e., differing in a single row for analyst privacy, or differing by \( 1/n \) in \( \ell_{\infty} \) norm for data privacy). We expect that this will be a useful point of view for other problems. In this direction, it is known how to privately compute equilibria in certain types of multi-player games [KPRU12]. Is there a useful way to use this multi-player generalization when solving problems in private data release, and what does it mean for privacy?

6.1 Acknowledgments

We thank the anonymous reviewers, along with Jake Abernethy and Salil Vadhan, for their helpful suggestions.
References

[BDMN05] A. Blum, C. Dwork, F. McSherry, and K. Nissim. Practical privacy: the sulq framework. In \textit{PODS}, 2005.

[BHK09] Boaz Barak, Moritz Hardt, and Satyen Kale. The Uniform Hardcore Lemma via Approximate Bregman Projections. In \textit{SODA}, pages 1193–1200, 2009.

[BLR08] A. Blum, K. Ligett, and A. Roth. A learning theory approach to non-interactive database privacy. In \textit{STOC}, 2008.

[DMNS06] C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In \textit{TCC}, 2006.

[DN03] Irit Dinur and Kobbi Nissim. Revealing information while preserving privacy. In \textit{PODS}, 2003.

[DNR+09] C. Dwork, M. Naor, O. Reingold, G.N. Rothblum, and S. Vadhan. On the complexity of differentially private data release: efficient algorithms and hardness results. In \textit{STOC}, 2009.

[DNV12] Cynthia Dwork, Moni Naor, and Salil P. Vadhan. The Privacy of the Analyst and The Power of the State. In \textit{FOCS}, 2012.

[DRV10] Cynthia Dwork, Guy N. Rothblum, and Salil P. Vadhan. Boosting and Differential Privacy. In \textit{FOCS}, pages 51–60, 2010.

[FS96] Y. Freund and R.E. Schapire. Game theory, on-line prediction and boosting. In \textit{Proceedings of the ninth annual conference on Computational learning theory}, pages 325–332. ACM, 1996.

[GRU11] Anupam Gupta, Moritz Hardt, Aaron Roth, and Jonathan Ullman. Privately releasing conjunctions and the statistical query barrier. In \textit{STOC}, 2011.

[GRU12] A. Gupta, A. Roth, and J. Ullman. Iterative constructions and private data release. In \textit{TCC}, 2012.

[HR10] Moritz Hardt and Guy N. Rothblum. A Multiplicative Weights Mechanism for Privacy-Preserving Data Analysis. In \textit{FOCS}, pages 61–70, 2010.

[HRU12] Justin Hsu, Aaron Roth, and Jonathan Ullman. Differential privacy for the analyst via private equilibrium computation. \textit{CoRR}, abs/1211.0877, 2012.

[KLN+11] S.P. Kasiviswanathan, H.K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? \textit{SIAM Journal on Computing}, 40(3):793–826, 2011.

[KPRU12] M. Kearns, M.M. Pai, A. Roth, and J. Ullman. Mechanism design in large games: Incentives and privacy. \textit{arXiv preprint arXiv:1207.4084}, 2012.

[MMP+10] A McGregor, I Mironov, T Pitassi, O Reingold, K Talwar, and S Vadhan. The limits of two-party differential privacy. In \textit{Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on}, pages 81–90. IEEE, 2010.
First we restate the lemma. Consider the following process:

- Fix an $(\epsilon, \delta)$-differentially private mechanism $A : \mathcal{U}^* \rightarrow \mathcal{R}$ and a bit $b \in \{0, 1\}$. Let $D_0 = \emptyset$.
- For $t = 1, \ldots, T$
  - The (possibly randomized) adversary $B(y_1, \ldots, y_t; r)$ chooses two distributions $B^0_t, B^1_t$ such that $SD(B^0_t, B^1_t) \leq \sigma$.
  - Choose $x_t \leftarrow_R B^b_t$ and let $D_t = D_{t-1} \cup \{x_t\}$.
  - Choose $y_t \leftarrow_R A(D_t)$.

For a fixed mechanism $A$ and adversary $B$, let $V^0$ be the distribution on $(y_1, \ldots, y_T)$ when $b = 0$ and $V^1$ be the distribution on $(y_1, \ldots, y_T)$ when $b = 1$.

**Lemma A.1.** If $\epsilon \leq 1/2$ and $T\sigma \leq 1/12$, then with probability at least $1 - T\delta - \delta'$ over $y = (y_1, \ldots, y_T) \leftarrow_R V^0$,

$$|\ln \left( \frac{V^0(y)}{V^1(y)} \right)| \leq \epsilon(T\sigma)\sqrt{2T \log(1/\delta')} + 30e^2(T\sigma)T.$$  

**Proof.** Given distributions $B^0, B^1$ such that $SD(B^0, B^1) \leq \sigma$, there exist distributions $C^0, C^1, C$ such that $B^0 = \sigma C^0 + (1 - \sigma)C$ and $B^1 = \sigma C^1 + (1 - \sigma)C$. An alternative way to sample from the distribution $B^b$ is to flip a coin $c \in \{0, 1\}$ with bias $\sigma$, and if the coin comes up 1, sample from $C^b$, otherwise sample from $C$.

Consider a partial transcript $(r, y_1, \ldots, y_{t-1})$. Fixing the randomness of the adversary will fix the coins $c_1, \ldots, c_T$, which determine whether or not the adversary samples from $C^b_j$ or $C_j$ for $j \in [T]$. Let $w = \sum_{j=1}^T c_j$. Fixing the randomness of the adversary and $y_1, \ldots, y_{t-1}$ will also fix the distributions $C_j$ for $j \leq t$ and, in rounds for which $c_j = 0$, will fix the samples $x_j$ for $j \leq t$. If we let $D^0_t, D^1_t$ denote the database $D_t$ in the case where $b = 0, 1$, respectively, then we have

$$|D^0_t - D^1_t| \leq \sum_{j=1}^T c_j \leq \sum_{j=1}^T c_j = w.$$
Thus,
\[
\left| \ln \left( \frac{V_t^0(y_t, y_1, \ldots, y_{t-1})}{V_t^1(y_t, y_1, \ldots, y_{t-1})} \right) \right| \leq w \varepsilon,
\]
and
\[
\mathbb{E} \left[ \ln \left( \frac{V_t^0(y_t, y_1, \ldots, y_{t-1})}{V_t^1(y_t, y_1, \ldots, y_{t-1})} \right) \right] \leq w \varepsilon \min \{e^{w \varepsilon} - 1, 1\},
\]
where the expectation is taken over \(V_t^0| r, y_1, \ldots, y_{t-1}\).

Fix \( w \in \{0, \ldots, T\} \). Conditioning on any \( r \) such that \( \sum_{t=1}^T c_t = w \), we can apply Azuma’s inequality as in [DRV10] to obtain
\[
D_{\infty}^{T\delta+\delta'}(V^0|w||V^1|w) \leq w \varepsilon \sqrt{2T \log(1/\delta')} + w \varepsilon \min \{e^{w \varepsilon} - 1, 1\} T.
\]
Thus,
\[
D_{\infty}^{T\delta+\delta'}(V^0||V^1) \leq \sum_{w=1}^T \Pr \left[ w \right] \left( w \varepsilon \sqrt{2T \log(1/\delta')} + w \varepsilon \min \{e^{w \varepsilon} - 1, 1\} T \right)
= \sum_{w=1}^T \Pr \left[ w \right] w \varepsilon \sqrt{2T \log(1/\delta')} + \sum_{w=1}^T \Pr \left[ w \right] w \varepsilon \min \{e^{w \varepsilon} - 1, 1\} T. \tag{1}
\]

First, we consider the left sum in (1).
\[
\sum_{w=1}^T \Pr \left[ w \right] w \varepsilon \sqrt{2T \log(1/\delta')}
= \varepsilon \sqrt{2T \log(1/\delta')} \sum_{w=1}^T \binom{T}{w} \sigma^w (1-\sigma)^{T-w} w
= \varepsilon \sqrt{2T \log(1/\delta')} (T \sigma) \sum_{w=0}^{T-1} \binom{T-1}{w} \sigma^w (1-\sigma)^{T-1-w}
= \varepsilon \sqrt{2T \log(1/\delta')} (T \sigma)
\]
Now, we work on the right sum in (1).

\[
\sum_{w=1}^{T} \Pr[w] (w \in \{e^{w\varepsilon} - 1, 1\} \ T)
\]

\[
= \sum_{w=1}^{T} \binom{T}{w} \sigma^w (1 - \sigma)^{T-w} (w \in \{e^{w\varepsilon} - 1, 1\} \ T)
\]

\[
= (4\varepsilon^2 T) \sum_{w=1}^{1/\varepsilon} \binom{T}{w} \sigma^w (1 - \sigma)^{T-w} w + (\varepsilon T) \sum_{w=1/\varepsilon}^{T} \binom{T}{w} \sigma^w (1 - \sigma)^{T-w} w
\]

\[
\leq (4\varepsilon^2 T) \sum_{w=1}^{1/\varepsilon} (eT\sigma)^w w^2 + (\varepsilon T) \sum_{w=1/\varepsilon}^{T} (eT\sigma)^w w
\]

\[
\leq (4\varepsilon^2 T) \sum_{w=1}^{1/\varepsilon} (eT\sigma)^w + (\varepsilon T) \sum_{w=1/\varepsilon}^{T} (eT\sigma)^w \quad (w^2/w^w \leq 1 \text{ for } w \in \mathbb{N})
\]

\[
\leq 4\varepsilon^2 T (2eT\sigma) + 2(eT\sigma)^{-1/\varepsilon} \varepsilon T \leq 3\varepsilon^2 T \quad (eT\sigma \leq 1/4)
\]

\[
\leq 24\varepsilon^2 (T\sigma)T + 4\varepsilon^2 (T\sigma)T \leq 30\varepsilon^2 (T\sigma)T
\]

Combining our bounds for the left and right sums in (1) completes the proof. \qed