Fourier-Mukai Partners of a $K3$ Surface and the Cusps of its Kahler Moduli

Shouhei Ma
Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan
E-mail: sma@ms.u-tokyo.ac.jp

Abstract
Using lattice theory, we establish a one-to-one correspondence between the set of Fourier-Mukai partners of a projective $K3$ surface and the set of 0-dimensional standard cusps of its Kahler moduli. We also study the relation between twisted Fourier-Mukai partners and general 0-dimensional cusps, and the relation between Fourier-Mukai partners with elliptic fibrations and certain 1-dimensional cusps.

1 Introduction

The excellent work of Mukai ([9], [10]) and Orlov ([12]) opened a way to study derived equivalence for $K3$ surfaces via their periods. One of their conclusions is that two projective $K3$ surfaces are derived equivalent if and only if there exists an isometry between their Mukai lattices preserving the periods. This theorem can be viewed as a generalization of the global Torelli theorem for $K3$ surfaces. Let $FM(S)$ be the set of isomorphism classes of Fourier-Mukai (FM) partners of a projective $K3$ surface $S$, i.e., $K3$ surfaces derived equivalent to $S$. As an application of Mukai-Orlov’s theorem, Hosono-Lian-Oguiso-Yau ([5]) gave a counting formula for $\#FM(S)$.

In this paper, we construct a bijection between $FM(S)$ and the set of embeddings of the hyperbolic plane $U$ into the lattice $\widetilde{NS}(S) := NS(S) \oplus U$ up to the action of a certain finite-index subgroup $\Gamma_S \subset O(\widetilde{NS}(S))$. Then, by considering the isotropic vector $(0, 1) \in U$, we obtain 0-dimensional standard cusps of the modular variety $\Gamma_S^+ \backslash \Omega^+_{\widetilde{NS}(S)}$ associated with $\Gamma_S^+$ and $\widetilde{NS}(S)$ ([11], [12]). Let $\Gamma_S^+ \backslash I^1(\widetilde{NS}(S))$ be the set of 0-dimensional standard cusps of $\Gamma_S^+ \backslash \Omega^+_{\widetilde{NS}(S)}$. We shall prove the following.

**Theorem 1.1** (Theorem 3.6). There exists a bijective map

$$\mu_0 : FM(S) \rightarrow \Gamma_S^+ \backslash I^1(\widetilde{NS}(S)).$$
In particular,
\[ \# \text{FM}(S) = \# \{ \text{0-dimensional standard cusps of } \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \}. \]

Of course, not every 0-dimensional cusp of \( \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \) is standard. It turns out that the existence of non-standard cusps corresponds to the existence of twisted FM-partners of \( S \). Let \( \text{FM}^{d}(S) \) be the set of isomorphism classes of twisted FM-partners \((S', \alpha')\) of \( S \) with \( \text{ord}(\alpha') = d \), and let \( \Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) \) be the set of 0-dimensional cusps \([l]\) of \( \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \) with \( \text{div}(l) = d \). Introducing certain quotient sets \( \text{FM}^{d}(S), r(\Gamma_{S}^{+}) \setminus \text{I}^{d}(\overline{A N S(S)}) \) of \( \text{FM}^{d}(S) \), \( \Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) \) respectively (see Sect.3.2 for the definitions), and using Huybrechts-Stellari’s solution of Căldăraru conjecture (\[8\]), we shall prove the following.

**Theorem 1.2 (Theorem 3.10).** There exist a map
\[ \nu_{0} : \Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) \longrightarrow \text{FM}^{d}(S) \]
and a bijective map
\[ \xi_{0} : r(\Gamma_{S}^{+}) \setminus \text{I}^{d}(\overline{A N S(S)}) \xrightarrow{\sim} \text{FM}^{d}(S) \]
which fit in the following commutative diagram:
\[
\begin{array}{ccc}
\Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) & \xrightarrow{\nu_{0}} & \text{FM}^{d}(S) \\
\downarrow p & & \downarrow \pi \\
r(\Gamma_{S}^{+}) \setminus \text{I}^{d}(\overline{A N S(S)}) & \xrightarrow{\xi_{0}} & \text{FM}^{d}(S).
\end{array}
\]

Via Theorem 1.1 and 1.2, we can obtain informations about the abstract set \( \text{FM}^{d}(S) \) by studying the 0-dimensional cusps of the modular variety \( \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \).

Besides 0-dimensional cusps, \( \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \) has also 1-dimensional cusps. Let \( \text{FM}_{dI}(S) \) be the set of isomorphism classes of pairs \((S', L')\), where \( S' \in \text{FM}(S) \) and \( L' \in \text{Pic}(S') \) is the line bundle associated to a smooth elliptic curve on \( S' \). In the same manner as in the case of 0-dimensional standard cusps, we relate \( \text{FM}_{dI}(S) \) to the set \( \Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) \) of those 1-dimensional cusps of \( \Gamma_{S}^{+} \setminus \Omega_{S}^{+}(\overline{N S(S)}) \) whose closure contains a 0-dimensional standard cusp. The map \( \mu_{1} \) from \( \text{FM}_{dI}(S) \) to the set of such 1-dimensional cusps is surjective but not injective in general. For example, for generic \( K3 \) surface \( S \) with \( NS(S) = U(r) \) (\( r > 2 \)), we will calculate \( \mu_{1} \) explicitly to conclude that \( \mu_{1} \) is far from being injective (Example 4.13). On the other hand, if \( \text{det}NS(S) \) is square-free, then \( \mu_{1} \) is bijective (Corollary 4.14). It turns out that \( \text{FM}_{dI}(S) \) carries informations about the compactifications of 1-dimensional cusps which belong to \( \Gamma_{S}^{+} \setminus \text{I}^{d}(\overline{N S(S)}) \). We shall observe that if two elliptic \( K3 \) surfaces \((S, L) \) and \((S', L') \) give the same 1-dimensional cusp, there exists a coherent sheaf on \( S \times S' \) via which \( L \)
gives rise to $L'$. (For precise statement see Proposition \[4.12\] and the remark after it.)

According to Bridgeland’s results ([2], [3]), $\Gamma^+_S \setminus \Omega^+_S$ is biholomorphic to a natural quotient of the space of certain stability conditions on $D^b(S)$, after removing some divisors on $\Omega^+_S$. That is, we have a canonical isomorphism (Proposition \[5.1\])

$$\text{Aut}^l(D^b(S) \setminus \text{Stab}^l(S)/GL_2^+(\mathbb{R}) \simeq \Gamma^+_S \setminus \Omega^+_S - \bigcup_{\delta \in \Delta(S)} \delta^+.\)$$

Hence Theorem 1.1 asserts that $\text{FM}(S)$ is identified with the 0-dimensional standard cusps of the Baily-Borel compactification of $\text{Aut}^l(D^b(S) \setminus \text{Stab}^l(S)/GL_2^+(\mathbb{R})$. However, the lattice theoretic approach employed in this paper does not explain any intrinsic reason for the correspondence between (twisted) FM-partners and 0-dimensional cusps. It may be interesting to understand the correspondence in more derived-categorical way.

This paper is organized as follows. In Sect.2.1, we recall some facts about even lattices and their discriminant forms, following Nikulin ([11]). Sect.2.2 is devoted to the study of isotropic elements of an even lattice. In Sect.2.3, we recall the Baily-Borel compactification of an arithmetic quotient of a type IV symmetric domain, following [11], [13], [15]. In Sect.3.1, we prove Theorem 1.2. In Sect.4.1, we define $\text{FM}_{\ell}(S)$ and relate it to primitive embeddings of $U + \mathbb{Z}l$ into $\tilde{\text{NS}}(S)$. In Sect.4.2, we study the relation between $\text{FM}_{\ell}(S)$ and certain 1-dimensional cusps of $\Gamma^+_S \setminus \Omega^+_S$. In Sect.5, we observe the action of $\Gamma^+_S$ on $\Omega^+_S$, and we also describe $\Gamma^+_S \setminus \Omega^+_S$ in terms of the space of stability conditions.

**Notation 1.3.** By an **even lattice**, we mean a free $\mathbb{Z}$-module $L$ of finite rank equipped with a non-degenerate symmetric bilinear form $(,) : L \times L \rightarrow \mathbb{Z}$ satisfying $(l,l) \in 2\mathbb{Z}$ for all $l \in L$. Denote by $\text{rk}(L)$ and $\text{sign}(L)$ the rank and the signature of $L$, respectively. For a lattice $L$ and a field $\mathbb{K}$, $L \otimes \mathbb{K}$ denotes the $\mathbb{K}$-vector space $L \otimes \mathbb{K}$. For two lattices $L$ and $M$, $L + M$ is the lattice defined as the orthogonal direct sum of $L$ and $M$, while $L + M$ denotes the direct sum of the $\mathbb{Z}$-modules underlying $L$ and $M$. The projection from $L + M$ to $L$ is denoted by $\text{pr}_L : L + M \rightarrow L$. The group of isometries of $L$ is denoted by $O(L)$. For an element $l \in L$, we define the positive integer $\text{div}(l)$ to be the generator of the ideal $(x,l) \subseteq \mathbb{Z}$. A sublattice $M \subseteq L$ is called **primitive** if $L/M$ is a free $\mathbb{Z}$-module. For (possibly degenerate) lattices $L$ and $M$, we denote by $\text{Emb}(L,M)$ the set of primitive embeddings of $L$ into $M$. A sublattice $M \subseteq L$ is called **isotropic** if $(x,y) = 0$ for all $x, y \in M$. We denote by $I_r(L)$ the set of primitive isotropic sublattices of $L$ of rk $= r$. A non-zero element $l \in L$ is called isotropic (resp. primitive) if $\mathbb{Z}l$ is isotropic (resp. primitive). We denote $I^d(L) := \{l \in L | l \text{ is primitive }, (l,l) = 0, \text{div}(l) = d\}$.

By a $K3$ surface, we mean a projective $K3$ surface over $\mathbb{C}$. For a $K3$ surface $S$, we denote by NS($S$) (resp. $T(S)$) the Neron-Severi (resp. transcendental)
Thus the correspondence \( \gamma \lambda \) here the equality holds if and only if \( l \) every prime number \( U \) is the even indefinite unimodular lattice \( A \) lattice \( L \) instead of \( \text{Proposition 2.1} \) frequently in this paper: \( f \) mainly \( M \), \( A \) of the Abelian group \( \text{Lemma 2.3} \). Note that if an even unimodular lattice \( \gamma \) isometry classes of lattices isogenus to \( O \) since \( \text{Proposition 2.2} \), \( \{ \gamma \} \), \( \Lambda \) \( O \) \( L \) \( L \) \( \), and \( \text{sign}(L) \) \( (\gamma) \) \( L \) \( L \) \( S \), \( \gamma \) \( L \) \( Z \), \( Z \), \( Z \), \( 8 \), \( 3 \), \( 2 \), \( 1 \), \( e, e \) \( f, f \) \( 1 \), \( (e, f) = 1 \), \( (e, e) = (f, f) = 0 \). Note that if an even unimodular lattice \( L \) is embedded in another even lattice \( M \), \( L \) must be an orthogonal summand of \( M \).

**Lemma 2.3.** Let \( L \) and \( M \) be even lattices. If there is an isometry \( \varphi : L \oplus U \simeq M \oplus U \) with \( \varphi(f) = f \), then the composition \( pr_M \circ (\varphi|_L) : L \hookrightarrow \varphi(f) \perp = M \oplus \mathbb{Z}f \rightarrow M \) is an isometry.

**Proof.** Since \( L \oplus \mathbb{Z}f = f \perp \simeq \varphi(f) \perp = M \oplus \mathbb{Z}f \), we can write \( \varphi(l) = m + \alpha f \) for each \( l \in L \), where \( m \in M \) and \( \alpha \in \mathbb{Z} \). Then \( (l, l) = (\varphi(l), \varphi(l)) = (m, m) \). Thus the correspondence \( \varphi(l) \mapsto m \) gives an embedding of lattice \( L \hookrightarrow M \). By the inclusions \( L \subseteq M \subseteq M^\vee \subseteq L^\vee \), we have \( |A_M| = |M/M^\vee| \leq |L/L^\vee| = |A_L| \). Here the equality holds if and only if \( L \simeq M \) by \( pr_M \circ (\varphi|_L) \). Since \( A_L \simeq A_L \oplus U \simeq A_M \oplus U \simeq A_M \), we get \( |A_M| = |A_L| \), so that \( L \simeq M \).

\[ \blacksquare \]
2.2 Primitive isotropic vectors

Let $L$ be an even lattice possessing a primitive isotropic vector $l$. We shall study some properties of $L$ related to $l$.

**Proposition 2.4.** Let $l \in I^1(L)$ be an arbitrary element. Then there is a canonical bijection 
\[ O(L) \setminus I^1(L) \simeq \mathcal{G}(l^+ / \mathbb{Z}l). \]

**Proof.** For $l \in I^1(L)$, there exists $m' \in L$ with $(l, m') = 1$. Setting $m := m' - \frac{(m', m')}{2} l$, we have $(l, l) = (m, m) = 0$ and $(l, m) = 1$. Hence there is an embedding $\varphi : U \to L$ with $\varphi(f) = l$. Since the lattice $U$ is unimodular, we have 
\[ L = \varphi(U) \oplus \varphi(U)^\perp \simeq \varphi(U) \oplus (l^+ / \mathbb{Z}l) \]
so that $A_L \simeq A_{l^+ / \mathbb{Z}l}$ for each $l \in I^1(L)$. In particular, $\mathcal{G}(l^+ / \mathbb{Z}l)$ is independent of the choices of $l \in I^1(L)$.

We define the map $\mu : O(L) \setminus I^1(L) \to \mathcal{G}(l^+ / \mathbb{Z}l)$ by $\mu(l') := (l')^+ / \mathbb{Z}l'$. If there is an isometry $\gamma : l_1^+ / \mathbb{Z}l_1 \simeq l_2^+ / \mathbb{Z}l_2$, we can extend $\gamma$ to the isometry $\tilde{\gamma} \in O(L)$ with $\tilde{\gamma}(l_1) = l_2$. Therefore $l_1$ and $l_2$ are $O(L)$-equivalent so that $\mu$ is injective. Given a lattice $K \in \mathcal{G}(l^+ / \mathbb{Z}l)$, we have an isometry $\Psi : K \oplus U \to L$ by Proposition 2.2 which gives $\Psi(f) \in I^1(L)$. Since $\mu(\Psi(f)) = K$, $\mu$ is surjective. \qed

**Proposition 2.5.** For a primitive isotropic vector $l \in I^4(L)$, we have the equality $d^2 \cdot \#(D_{l^+ / \mathbb{Z}l}) = \#(D_L)$.

**Proof.** The free $\mathbb{Z}$-module $\tilde{L} := \langle L, \frac{1}{2} \rangle \subset L'$ is an even overlattice of $L$. Since $\frac{1}{2} \in I^1(\tilde{L})$, we have $\tilde{L} \simeq U \oplus (l^+ / \mathbb{Z}l)$ so that $D_{\tilde{L}} \simeq D_{l^+ / \mathbb{Z}l}$. If we set $H := \tilde{L} / L \subset D_L$, then $H$ is an isotropic cyclic group of order $d$ satisfying $D_L \simeq H^+ / H$. Hence $\#(D_L) = d^2 \cdot \#(D_{l^+ / \mathbb{Z}l}) = d^2 \cdot \#(D_{l^+ / \mathbb{Z}l})$. \qed

**Corollary 2.6.** If $\det(L)$ is square-free, every primitive isotropic vector $l \in L$ satisfies $\text{div}(l) = 1$.

Let $l \in I^1(L)$. Choose an element $m \in L$ so that $(m, m) = 0$ and $(l, m) = 1$. Then we define $L_0 := \mathbb{Z}l + \mathbb{Z}m \simeq U$ and $L_1 := L_0^+ \simeq l^+ / \mathbb{Z}l$. We can describe the group 
\[ O(L)_1 := \{ \gamma \in O(L) | \gamma(l) = l \} \]
in terms of the free Abelian group $L_1$ and the group $O(L_1)$ as follows. For an element $v \in L_1$, define the isometry $T_v \in O(L)_1$ by 
\begin{align*}
T_v(l) &= l, \\
T_v(m) &= m + v - \frac{1}{2} (v, v) l, \\
T_v(v') &= v' - (v', v) l, \quad v' \in L_1.
\end{align*}
(1)

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For \( l \in I^1(L) \), we regard \( L_1 \subset O(L)^I \) by the correspondence \( v \mapsto T_v \) and regard \( O(L_1) \subset O(L)^I \) by the correspondence \( g \mapsto \text{id}_{L_0} \oplus g \). Then

\[
O(L)^I \simeq O(L_1) \ltimes L_1.
\]

Moreover, the subgroup \( L_1 \subset O(L)^I \) acts trivially on \( D_L \simeq D_{L_1} \).

**Proof.** Take an isometry \( \gamma \in O(L)^I \). Since \( \gamma(l^\perp) = l^\perp \) and \( l^\perp = \mathbb{Z}l \oplus L_1 \), \( \gamma \) induces the isometry \( pr_{L_1} \circ (\gamma|_{L_1}) \in O(L_1) \). The correspondence \( \gamma \mapsto pr_{L_1} \circ (\gamma|_{L_1}) \) induces the homomorphism \( \pi : O(L)^I \to O(L_1) \). Then \( \pi \) is surjective because \( \pi(\text{id}_{L_0} \oplus g) = g \) for \( g \in O(L_1) \). The correspondence \( g \mapsto \text{id}_{L_0} \oplus g \) is a section of \( \pi \).

We prove that Ker(\( \pi \)) = \( L_1 \subset O(L)^I \). The inclusion Ker(\( \pi \)) \( \supset L_1 \) is apparent. For \( \gamma \in \text{Ker}(\pi) \), we can write \( \gamma(m) = m + v + \alpha l \) for some integer \( \alpha \in \mathbb{Z} \) and some vector \( v \in L_1 \). Since \( \gamma(m),\gamma(m) = (m,m) = 0 \), we can determine \( \alpha \) as \( \alpha = -\frac{(v,v)}{2} \). On the other hand, if we take \( v' \in L_1 \), we can write \( \gamma(v') = v' + \beta(v')l \) for some integer \( \beta(v') \in \mathbb{Z} \). Since \( \gamma(v'),\gamma(m) = (v',m) = 0 \), we have \( \beta(v') = -(v,v') \). Hence we have \( \gamma = T_v \). The claim that \( r_L(L_1) = \{ \text{id} \} \) is obvious. \( \square \)

### 2.3 The Baily-Borel compactification

In this subsection, we recall the Baily-Borel compactification of an arithmetic quotient of a type IV symmetric domain \([11, 13]\), following \([15]\). Although the Baily-Borel compactifications are defined for general arithmetic groups, we restrict ourselves to finite-index subgroups of \( O(L)^+ \) containing \( \{ \pm \text{id} \} \).

Let \( L \) be an even lattice of sign(\( L \)) = \( (2, b^-) \). Set

\[
\Omega_L := \{ \omega \in P(L) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \},
\]

which has two connected components \( \Omega_L^+ \) and \( \Omega_L^- \) exchanged by the complex conjugation on \( \Omega_L \). Denote by \( O(L)^+ \subset O(L) \) the subgroup of index at most 2 which consists of isometries preserving \( \Omega_L^+ \). By associating the oriented two-plane \( \Re \omega \oplus \Im \omega \) to \( \omega \in \Omega_L \), we have the isomorphism

\[
\Omega_L \simeq \{ (E, \tau) \mid E \subset L_\mathbb{R} \text{ is a positive-definite two-plane, } \tau \text{ is an orientation of } E \}.
\]

The right hand side of (2) is an open subset of the oriented Grassmannian. Choosing a connected component of \( \Omega_L \), we can attach the orientation to every positive-definite two-plane in \( L_\mathbb{R} \). A choice of a component of \( \Omega_L \) is sometimes called an orientation of \( L \).

Regarding \( \Omega_L^+ \) as an open subset of \( Q := \{ \omega \in P(L) | (\omega, \omega) = 0 \} \), we can decompose its boundary in \( Q \) as follows:

\[
\partial \Omega_L^+ = \bigcup_I B_I,
\]

where
where \( I \) is an isotropic subspace of \( L \) and \( B_I \) is the interior of \( \mathbb{P}(I_c) \cap \overline{\Omega_I} \). Each \( B_I \) is biholomorphic to the upper half-plane \( \mathbb{H} \) or a point. The set \( B_I \) is called a rational boundary component if \( I = E \) for some isotropic sublattice \( E \) of \( L \). Thus there is a canonical identification

\[
I_{i+1}(L) = \{ \text{i-dimensional rational boundary components of } \Omega_I \}
\]

for \( i = 0, 1 \), and \( \Omega_I \) has no higher dimensional rational boundary components.

Now assume that we are given a finite-index subgroup \( \Gamma \subset O(L)^+ \) with \( \{ \pm \text{id} \} \subset \Gamma \). The Baily-Borel compactification of \( \Gamma \backslash \Omega_I^+ \) is set-theoretically the set

\[
\Gamma \backslash \Omega_I^+ = \Gamma \backslash \left( \Omega_I^+ \cup \bigcup_{I_1(L)} B_{E_{I_1}} \cup \bigcup_{I_2(L)} B_{E_{I_2}} \right)
\]

where \( \Gamma \backslash \Omega_I^+ = \{ \gamma \in \Gamma \mid \gamma(E) = E \} \) for \( E \in I_2(L) \). It turns out that \( \Gamma \backslash \Omega_I^+ \) is a normal projective variety. The boundary components of \( \Gamma \backslash \Omega_I^+ \) are often called cusps. We have the canonical identification

\[
\Gamma \backslash I_{i+1}(L) = \{ \text{i-dimensional cusps of } \Gamma \backslash \Omega_I^+ \}.
\]

**Definition 2.8.** A 0-dimensional cusp corresponding to \( [Zl] \) with \( \text{div}(l) = 1 \) is called a standard cusp.

By our assumption that \( \Gamma \subset O(L)^+ \), the notion of standardness is well-defined. Since \( \{ \pm \text{id} \} \subset \Gamma \), the set of 0-dimensional standard cusps of \( \Gamma \backslash \Omega_I^+ \) is identified with \( \Gamma \backslash \Omega_I^+ \).

For a rank 2 primitive isotropic sublattice \( E \in I_2(L) \), the compact curve

\[
\left( \Gamma \backslash \Omega_I^+ \right) \cup \left( \Gamma \backslash \bigcup_{\gamma \in \Gamma \backslash \Omega_I^+} \bigcup_{l \in \gamma(E)} [Cl] \right)
\]

is obtained from the curve

\[
\left( \Gamma \backslash \Omega_I^+ \right) \cup \left( \Gamma \backslash \bigcup_{l \in \mathbb{E}} [Cl] \right)
\]

by identifying \( \Gamma \)-equivalent points in \( \bigcup_{l \in \mathbb{E}} [Cl] \). The curve \((4)\) is the canonical compactification of \( \Gamma \backslash B_E \simeq \mathbb{H} \). When there exists an embedding \( \varphi : U \hookrightarrow L \), we can write \( L = \varphi(U) \oplus L_\varphi \) for the lattice \( L_\varphi := \varphi(U)^+ \cap L \) of \( \text{sign}(L_\varphi) = (1, b^- - 1) \). Assume that we have chosen an orientation, say \( \Omega_I^+ \), of \( L \). Then we can choose the connected component \( L_\varphi^+ \) of the open set \( \{ v \in (L_\varphi)_R, \langle v, v \rangle > 0 \} \) so that for each vector \( v \in L_\varphi^+ \) the oriented two-plane \( \mathbb{R} \varphi(v + f) \oplus \mathbb{R} v \) belongs to \( \Omega_I^+ \). If there is another embedding \( \varphi' : U \hookrightarrow L \) with \( \varphi(f) = \varphi'(f) \), the projection \( \gamma_{\varphi, \varphi'} := (pr_{L_\varphi})|_{L_\varphi} : L_\varphi \rightarrow L_{\varphi'} \) is an isometry (Lemma 2.3). Then it can be immediately checked that \( \gamma_{\varphi, \varphi'} \) maps the cone \( L_\varphi^+ \) to the cone \( L_{\varphi'}^+ \).
3 FM(S), FM\(^d\)(S) and the 0-dimensional cusps of \(\Gamma^+_S \setminus \Omega^+_{NS(S)}\)

3.1 0-dimensional standard cusps and FM-partners

Let \(S\) be a K3 surface. We denote by \(D^b(S)\) the bounded derived category of \(S\). A K3 surface \(S'\) is called a Fourier-Mukai (FM) partner of \(S\) if \(D^b(S) \simeq D^b(S')\) as triangulated categories. Let FM(S) be the set of isomorphism classes of FM-partners of \(S\). By the results of Mukai and Orlov ([9], [10], [12]), FM(S) can be studied via the period of \(S\). We denote by \(\bar{H}(S, Z)\) the total cohomology group \(H^*(S, Z)\) equipped with the Mukai pairing

\[
\left( (r_1, l_1, s_1), (r_2, l_2, s_2) \right) := (l_1, l_2) - (r_1, s_2) - (s_1, r_2)
\]

where \(r_i \in H^0(S, Z), l_i \in H^2(S, Z), s_i \in H^4(S, Z)\). Since \(H^2(S, Z) \simeq \Lambda_{K3}\) as a lattice, \(\bar{H}(S, Z) \simeq H^2(S, Z) \oplus U \simeq \Lambda_{K3}\) as a lattice. We fix an isometry \(H^0(S, Z) \oplus H^4(S, Z) \simeq U\) by identifying \((1, 0, 0)\) with \(e\), and \((0, 0, -1)\) with \(f\). The lattice \(\bar{H}(S, Z)\) inherits the weight-two Hodge structure from \(H^2(S, Z)\). If we denote by \(\omega_S \in H^2(S, \mathbb{C})\) the period of \(S\), then

\[
\omega_S^1 \cap \bar{H}(S, Z) = \tilde{NS}(S) \simeq NS(S) \oplus U.
\]

It is obvious that \(T(S) = \tilde{NS}(S)^\perp \cap \bar{H}(S, Z)\). Let \(NS(S)^\perp\) be the positive cone, i.e., the component of \(\{ v \in NS(S)_{\mathbb{R}}, (v, v) > 0 \}\) containing ample classes. We choose an orientation, say \(\Omega^+_{NS(S)}\), of \(\tilde{NS}(S)\) so that for a vector \(v \in NS(S)^\perp\) the oriented positive-definite two-plane \(\mathbb{R}(1, 0, -1) \oplus \mathbb{R}(0, v, 0)\) belongs to \(\Omega^+_{NS(S)}\).

**Theorem 3.1** ([10], [12]). Let \(S\) and \(S'\) be K3 surfaces. Then \(D^b(S) \simeq D^b(S')\) if and only if there exists a Hodge isometry \(\bar{H}(S, Z) \simeq \bar{H}(S', Z)\).

Thanks to this theorem, we are able to obtain every member of FM(S) by an embedding of \(U\) into \(\tilde{NS}(S)\).

**Lemma 3.2.** For an embedding \(\varphi \in \operatorname{Emb}(U, \tilde{NS}(S))\), there exists a unique (up to isomorphism) K3 surface \(S_\varphi \in \operatorname{FM}(S)\) such that \(H^2(S_\varphi, Z)\) is Hodge isometric to \(\Lambda_\varphi := \varphi(U)^\perp \cap \bar{H}(S, Z)\). The correspondence \(\varphi \mapsto S_\varphi\) gives a surjection

\[
\operatorname{Emb}(U, \tilde{NS}(S)) \twoheadrightarrow \operatorname{FM}(S).
\]

**Proof.** The lattice \(\Lambda_\varphi\) is an even unimodular lattice of signature \((3, 19)\), hence is isometric to the K3 lattice \(\Lambda_{K3}\). Since \(T(S) \subset \Lambda_\varphi\), \(\Lambda_\varphi\) inherits the period from that of \(T(S)\). By the surjectivity of period map ([16], [17]), there is a K3 surface \(S_\varphi\) such that \(H^2(S_\varphi, Z)\) is Hodge isometric to \(\Lambda_\varphi\). By the Torelli theorem ([14], [17]), \(S_\varphi\) is unique up to isomorphism. Since

\[
\bar{H}(S_\varphi, Z) \simeq U \oplus \Lambda_\varphi \simeq \varphi(U) \oplus \Lambda_\varphi = \bar{H}(S, Z)
\]
is an isometry of lattices preserving the periods, $S_\varphi \in FM(S)$ by Theorem 3.4. Conversely, if $S' \in FM(S)$, there is a Hodge isometry $\Phi : \tilde{H}(S', \mathbb{Z}) \cong \tilde{H}(S, \mathbb{Z})$ which induces a Hodge isometry

$$H^2(S', \mathbb{Z}) \cong \tilde{H}((H^0(S', \mathbb{Z}) + H^1(S', \mathbb{Z})) \cap \tilde{H}(S, \mathbb{Z}).$$

It follows that $S' \cong S_{\tilde{\Phi}((H^0(S', \mathbb{Z}) + H^1(S', \mathbb{Z})) \cap \tilde{H}(S, \mathbb{Z}))}$ by the Torelli theorem.

By the construction of $S_\varphi$, we can identify $T(S_\varphi) = T(S)$, and $NS(S_\varphi) = \varphi(U)^{-1} \cap NS(S)$. We shall introduce an equivalence relation on $Emb(U, NS(S))$ to make the above correspondence bijective.

**Definition 3.3.** Set

$$\Gamma_S := r^{-1}_{NS(S)}(\lambda \circ r_T(O_{Hodge}(T(S)))),$$

where $O_{Hodge}(T(S))$ is the group of Hodge isometries of $T(S)$, $r_{NS(S)} : O(NS(S)) \to O(A_{NS(S)})$ and $r_T : O(T(S)) \to O(A_T(S))$ are the natural homomorphisms, and the isomorphism $\lambda : O(A_T(S)) \cong O(A_{NS(S)})$ is induced from the isometry $(A_T(S), -q) \cong (A_{NS(S)}, q)$ (cf. Proposition 2.1).

By the identifications $NS(S) = NS(S')$ and $T(S) = T(S')$, we have $\Gamma_S = \Gamma_{S'}$. If we denote by $O_{Hodge}(\tilde{H}(S, \mathbb{Z}))$ the group of Hodge isometries of $\tilde{H}(S, \mathbb{Z})$, then $\Gamma_S$ is the image of the natural homomorphism

$$O_{Hodge}(\tilde{H}(S, \mathbb{Z})) \to O(NS(S))$$

by Proposition 2.1.

In generic case, the period $\omega_S$ is not contained in any eigenspace of any $\varphi \in O(T(S)) - \{\pm id\}$ so that $O_{Hodge}(T(S)) = \{\pm id\}$. By the inclusions

$$O(NS(S))_0 \times \{\pm id\} \subseteq \Gamma_S \subseteq O(NS(S)),$$

$\Gamma_S$ is a finite-index subgroup of $O(NS(S))$. Then $\Gamma_S$ acts on $Emb(U, NS(S))$ from left as $\gamma(\varphi) := \gamma \circ \varphi$ for $\gamma \in \Gamma_S$ and $\varphi \in Emb(U, NS(S))$.

**Proposition 3.4.** The correspondence $\varphi \mapsto S_\varphi$ of Lemma 3.2 induces the bijection

$$\Gamma_S \setminus Emb(U, NS(S)) \cong FM(S).$$

**Proof.** It suffices to show that $S_{\varphi_1} \cong S_{\varphi_2}$ if and only if there exists an isometry $\gamma \in \Gamma_S$ such that $\varphi_1 = \gamma \circ \varphi_2$. By the Torelli theorem, the former condition is equivalent to the existence of a Hodge isometry $\Phi : H^2(S_{\varphi_2}, \mathbb{Z}) \cong H^2(S_{\varphi_1}, \mathbb{Z})$.

Given such $\Phi$, we get a Hodge isometry $\tilde{\Phi} : \tilde{H}(S_{\varphi_2}, \mathbb{Z}) \cong \tilde{H}(S_{\varphi_1}, \mathbb{Z})$ by adding $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U) \cong \varphi_1(U)$. Then, by Proposition 2.1 (2), the Hodge isometry $\Phi|_{T(S)} : T(S) \cong T(S)$ and the isometry $\tilde{\Phi}|_{NS(S)} : NS(S_{\varphi_2}) \oplus$
\( \varphi_2(U) \cong NS(S_{\varphi_1}) \oplus \varphi_1(U) \) are compatible on the discriminant form. It follows that \( \Phi|_{NS(S)} \in \Gamma_S \). By the construction \( \varphi_2 = (\Phi|_{NS(S)}) \circ \varphi_2 \).

Conversely, if there exists \( \gamma \in \Gamma_S \) such that \( \varphi_1 = \gamma \circ \varphi_2 \), we can find a Hodge isometry \( g : T(S) \cong T(S) \) which is compatible with \( \gamma \) on the discriminant group.

Then, again by Proposition 2.1 (2), \( \gamma + g \) extends to \( \Phi \in O_{Hodge}(\overline{H}(S, \mathbb{Z})) \), which gives a Hodge isometry between \( H^2(S_{\varphi_2}, \mathbb{Z}) = \varphi_2(U)^{\perp} \cap \overline{H}(S, \mathbb{Z}) \) and \( H^2(S_{\varphi_1}, \mathbb{Z}) = \varphi_1(U)^{\perp} \cap \overline{H}(S, \mathbb{Z}) \).

We now turn to the 0-dimensional standard cusps. Let \( \Omega^+_{NS(S)} \) be the bounded symmetric domain associated with the lattice \( \overline{NS}(S) \) (cf. Section 2.3). Denote by \( \Gamma_S^+ \) the arithmetic group \( \Gamma_S \cap O(\overline{NS}(S))^+ \). Since the isometry \( \text{id}_{NS(S)} \oplus -\text{id}_U \in O(\overline{NS}(S)) \) exchanges \( \Omega^+_{NS(S)} \) and \( \Omega^-_{NS(S)} \), we know that \( \Gamma_S^+ \) is of index 2 in \( \Gamma_S \). Recall from Section 2.3 the canonical identification

\[
\Gamma_S^+ \backslash I^1(\overline{NS}(S)) = \{ \text{0-dimensional standard cusps of } \Gamma_S^+ \backslash \Omega^+_{NS(S)} \}.
\]

**Lemma 3.5.** The natural surjection

\[
\Gamma_S^+ \backslash I^1(\overline{NS}(S)) \longrightarrow \Gamma_S \backslash I^1(\overline{NS}(S))
\]

is bijective.

**Proof.** For \( l \in I^1(\overline{NS}(S)) \), there exist an embedding \( \varphi \in \text{Emb}(U, \overline{NS}(S)) \) such that \( l \in \varphi(U) \). Then the isometry \( \text{id}_{\varphi(U)^{\perp}} \oplus -\text{id}_{\varphi(U)} \in O(\overline{NS}(S))_0 \) preserves \( \mathbb{Z}l \) and interchanges \( \Omega^+_{NS(S)} \) with \( \Omega^-_{NS(S)} \). Therefore \( \Gamma_S^+ \cdot l = \Gamma_S \cdot l \).

**Theorem 3.6.** Let \( \{ e, f \} \) be the canonical basis of the lattice \( U \). The map

\[
\mu_0 : \text{FM}(S) \ni [S_{\varphi}] \longrightarrow [\varphi(f)] \in \Gamma_S \backslash I^1(\overline{NS}(S))
\]

is bijective.

**Proof.** The map \( \mu_0 \) is well-defined by Proposition 3.4. Since \( l \in I^1(\overline{NS}(S)) \) induces an embedding \( \varphi \in \text{Emb}(U, \overline{NS}(S)) \) with \( \varphi(f) = l \), \( \mu_0 \) is surjective.

If \( [\varphi_1(f)] = [\varphi_2(f)] \), then by identifying \( \varphi_1(\mathbb{Z}f) = H^4(S_{\varphi_1}, \mathbb{Z}) \), the isometry \( \text{id}_{H^2(S, \mathbb{Z})} \) induces the Hodge isometry \( \Phi : \overline{H}(S_{\varphi_1}, \mathbb{Z}) \cong \overline{H}(S_{\varphi_2}, \mathbb{Z}) \) with \( \Phi(H^4(S_{\varphi_1}, \mathbb{Z})) = H^4(S_{\varphi_2}, \mathbb{Z}) \). Now Lemma 2.3 implies that \( \Pr H^2(S_{\varphi_2}, \mathbb{Z}) \circ (\Phi|_{H^2(S_{\varphi_1}, \mathbb{Z})}) \) gives a Hodge isometry between \( H^2(S_{\varphi_1}, \mathbb{Z}) \) and \( H^2(S_{\varphi_2}, \mathbb{Z}) \), since \( \Pr H^2(S_{\varphi_2}, \mathbb{Z}) \) is identity on \( T(S_{\varphi_2}) = \Phi(T(S_{\varphi_1})) \).

In this way, we have

\[
\# \text{FM}(S) = \# \{ \text{0-dimensional standard cusps of } \Gamma_S^+ \backslash \Omega^+_{NS(S)} \}.
\]
3.2 General 0-dimensional cusps and twisted FM-partners

In this subsection, using Huybrechts-Stellari’s solution of Căldăraru conjecture ([8]), we study the relation between general 0-dimensional cusps of $\Gamma^+_S \backslash \Omega^+_{N S(S)}$ and twisted Fourier-Mukai partners of $S$. For twisted K3 surfaces, see [7].

Let $(S', \alpha')$ be a twisted K3 surface. By the natural isomorphism

$$Br(S') \cong \text{Hom}(T(S'), \mathbb{Q}/\mathbb{Z}),$$

we identify the twisting $\alpha' \in Br(S')$ with a surjective homomorphism $\alpha' : T(S') \to \mathbb{Z}/d\mathbb{Z}$ where $d = \text{ord}(\alpha')$. Then $\ker(\alpha')$ is denoted by $T(S', \alpha')$.

A twisted K3 surface $(S', \alpha')$ is called a twisted Fourier-Mukai partner of $S$ if there is an equivalence $D^b(S', \alpha') \cong D^b(S)$. By the result of Canonaco-Stellari ([4]), the equivalence is of Fourier-Mukai type. We define

$$\text{FM}^d(S) := \left\{(S', \alpha') : \text{twisted K3 surface}, D^b(S', \alpha') \cong D^b(S), \text{ord}(\alpha') = d\right\}/\cong.$$

It is obvious that $\text{FM}^1(S) = \text{FM}(S)$.

**Proposition 3.7.** There exists a map

$$\nu_0 : \Gamma^+_S \backslash I^d(\tilde{N}S(S)) \to \text{FM}^d(S).$$

If $d = 1$, we have $\nu_0 = \mu_0^{-1}$ on $\Gamma^+_S \backslash I^1(\tilde{N}S(S))$.

**Proof.** For $l \in I^d(\tilde{N}S(S))$, we define

$$\tilde{M}_l := \left\langle \frac{l}{d}, \tilde{N}S(S) \right\rangle \subset \tilde{N}S(S)^\vee,$$

which is an even overlattice of $\tilde{N}S(S)$. Via the isometry

$$\lambda := \lambda_{\tilde{H}(S, \mathbb{Z})} : (A_{\tilde{N}S(S)}, q) \cong (A_{T(S)}, -q),$$

we have an isotropic element $\lambda(\frac{l}{d}) \in A_{T(S)}$ of order $d$. Set

$$T_l := \left\langle \lambda(\frac{l}{d}), T(S) \right\rangle \subset T(S)^\vee.$$

We have a surjective homomorphism

$$\alpha_l : T_l \to \mathbb{Z}/d\mathbb{Z} \text{ with } \ker(\alpha_l) = T(S), \quad \alpha_l(\lambda(\frac{l}{d})) = 1.$$

Since the isometry $\lambda$ induces the isometry $\tilde{\lambda} : (A_{\tilde{M}_l}, q) \cong (A_{T_l}, -q)$, we have an embedding $\tilde{M}_l \oplus T_l \hookrightarrow \tilde{\Lambda}_{K3}$ with both $\tilde{M}_l$ and $T_l$ embedded primitively. Since $\frac{l}{d} \in I^1(\tilde{M}_l)$, there exists an embedding $\varphi : U \hookrightarrow \tilde{M}_l$ with $\varphi(f) = \frac{l}{d}$. The orthogonal complement $\Lambda_{\varphi} := \varphi(U)^\perp \cap \tilde{\Lambda}_{K3}$ is isometric to the K3 lattice $\Lambda_{K3}$ and has the period induced from $T_l$. Moreover, via the orientation of $\tilde{M}_l$...
Proposition 2.2, after composing $\gamma$.

Since the natural homomorphism $r : T|_{T(S)} \to \mathbb{Z}/d\mathbb{Z}$ by $\Phi|_{T(S)}$, we obtain a surjective homomorphism $\alpha : T(S) \to \mathbb{Z}/d\mathbb{Z}$. Thus we constructed a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$.

If there are another $K3$ surface $S_\varphi'$ and a Hodge isometry $\Phi' : H^2(S_\varphi', \mathbb{Z}) \cong H^2(S_\varphi, \mathbb{Z})$ such that $\Phi'(NS(S_\varphi')^+) = M^+_\varphi$. Pulling back the homomorphism $\alpha_\varphi : T|_{T(S)} \to \mathbb{Z}/d\mathbb{Z}$ by $\Phi'|_{T(S)}$, we obtain a twisted $K3$ surface $(S_\varphi', \alpha_\varphi')$. Hence we constructed a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$ from an embedding $\varphi$.

If we take another embedding $\varphi' : U \hookrightarrow \tilde{M}_l$ with $\varphi'(f) = \frac{1}{q}$, it follows from Lemma 2.3 and Section 2.3 that the projection defines an isometry $\Lambda_\varphi \cong \Lambda_\varphi'$, which is identity on $T_l$ and maps the cone $M^+_\varphi$ to the cone $M^+_\varphi'$. So we have a Hodge isometry $\Phi : H^2(S_\varphi, \mathbb{Z}) \cong H^2(S_\varphi', \mathbb{Z})$, and $\Phi'(NS(S_\varphi')^+) = NS(S_\varphi')^+$ and $(\Phi'|_{T(S)})^*\alpha_\varphi = \alpha_\varphi$. As above, we obtain an isomorphism $(S_\varphi, \alpha_\varphi) \cong (S_\varphi', \alpha_\varphi')$. Hence we constructed a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$ from an embedding $\varphi$.

Next we consider the action of the group $\Gamma^+_S$. For an isometry $\gamma \in \Gamma^+_S$, set $l' := \gamma(l)$. The isometry $\gamma$ extends to the isometry $\gamma : M_l \cong M_{l'}$. Take an embedding $\varphi : U \hookrightarrow \tilde{M}_l$ with $\varphi(f) = \frac{1}{q}$ and set $\varphi' := \gamma \circ \varphi : U \hookrightarrow \tilde{M}_{l'}$, which satisfies $\varphi'(f) = \frac{1}{q}$. On the other hand, there exists a Hodge isometry $\varphi' : U \hookrightarrow \tilde{M}_l$ with $\varphi'(f) = \frac{1}{q}$. The orientation $\gamma \circ r(\gamma) = r(g) \circ \lambda$. Since $r(g)(\lambda(\frac{1}{q})) = \lambda(\frac{1}{q})$, $g$ extends to the Hodge isometry $\tilde{g} : T_l \cong T_{l'}$ with $\tilde{g}^*\alpha_\gamma = \alpha_l$. Because we have $\lambda \circ r(\gamma) = r(g) \circ \lambda$, the isometry $\gamma \oplus \tilde{g}$ extends to the Hodge isometry $\gamma \oplus \varphi^{-1} \oplus \Phi : \varphi(U) \oplus \Lambda_\varphi \cong \varphi'(U) \oplus \Lambda_{\varphi'}$ with $(\Phi|_{T_l})^*\alpha_\gamma = \alpha_l$.

Since $\gamma$ preserves the orientation, we have $\Phi(M^+_\varphi) = M^+_\varphi'$. In this way we obtain a Hodge isometry $\Phi : H^2(S_l, \mathbb{Z}) \cong H^2(S_{l'}, \mathbb{Z})$ with $\Phi(NS(S_l)^+) = NS(S_{l'})^+$ and $(\Phi|_{T(l)})^*\alpha_{l'} = \alpha_l$. As above we have $(S_l, \alpha_l) \cong (S_{l'}, \alpha_{l'})$.

Finally, we see that $D^b(S_l, \alpha_l) \cong D^b(S)$. Let $B_l \in H^2(S_l, \mathbb{Q})$ be a B-field lift of $\alpha_l \in Br(S_l)$. Since we have a Hodge isometry $T(S) \cong T(S_l, B_l) \subset \tilde{H}(S_l, B_l, \mathbb{Z})$, the lattice $\tilde{N}S(S_l, B_l) := T(S_l, B_l)^{\perp} \cap \tilde{H}(S_l, B_l, \mathbb{Z})$ is isogenous to $\tilde{N}S(S)$, so we have an isometry $\tilde{N}S(S) \cong \tilde{N}S(S_l, B_l)$ by Proposition 2.4. Hence there is a Hodge embedding $\gamma \oplus g : \tilde{N}S(S) \oplus T(S) \hookrightarrow \tilde{H}(S_l, B_l, \mathbb{Z})$. Since the natural homomorphism $r : O(\tilde{N}S(S)) \to O(A_{\tilde{N}S(S)})$ is surjective by Proposition 2.4 after composing $\gamma$ with an element of $O(\tilde{N}S(S))$ (and with
id_{NS(S)} + \text{id}_U \in O(\widetilde{NS(S)})_0 \text{ if necessary}, \gamma \oplus g \text{ extends to an orientation-preserving Hodge isometry } H(S, \mathbb{Z}) \simeq H(S_1, B_1, \mathbb{Z}). \text{ Then our claim follows from Huybrechts-Stellari’s theorem } [8].

The proof of the assertion that \nu_0 = \mu_0^{-1} \text{ for } d = 1 \text{ can be left to the reader.} \qquad \square

**Definition 3.8.** Set

$$\mathcal{F}M^d(S) := FM^d(S)/\sim,$$

where \( (S_1, \alpha_1) \sim (S_2, \alpha_2) \) if there exists a Hodge isometry \( g : T(S_1) \overset{\sim}{\rightarrow} T(S_2) \) with \( g^* \alpha_2 = \alpha_1 \). Since \( (S_1, \alpha_1) \simeq (S_2, \alpha_2) \) implies \( (S_1, \alpha_1) \sim (S_2, \alpha_2) \), the set \( \mathcal{F}M^d(S) \) is well-defined. Denote by \( \pi : FM^d(S) \rightarrow \mathcal{F}M^d(S) \) the quotient map.

**Definition 3.9.** Set

$$I^d(A_{\widetilde{NS(S)}}) := \left\{ x \in A_{\widetilde{NS(S)}} \mid q_{\widetilde{NS(S)}}(x) = 0 \in \mathbb{Q}/2\mathbb{Z}, \text{ ord}(x) = d \right\}.$$

Since \( \widetilde{NS(S)} = NS(S) \oplus U \), the map

$$p : I^d(\widetilde{NS(S)}) \longrightarrow I^d(A_{\widetilde{NS(S)}}), \quad l \mapsto \frac{l}{d}$$

is surjective by Proposition 4.1.1 of [15].

**Theorem 3.10.** There exists a bijective map \( \xi_0 : r(\Gamma_S^+) \backslash I^d(A_{\widetilde{NS(S)}}) \rightarrow \mathcal{F}M^d(S) \) which fits in the following commutative diagram.

$$
\begin{array}{ccc}
\Gamma_S^+ \backslash I^d(\widetilde{NS(S)}) & \longrightarrow & FM^d(S) \\
p \downarrow & & \downarrow \pi \\
r(\Gamma_S^+) \backslash I^d(A_{\widetilde{NS(S)}}) & \longrightarrow & \mathcal{F}M^d(S).
\end{array}
$$

**Proof.** We define \( \xi_0 \) by \( \pi \circ \nu_0 \circ p^{-1} \). For \( x = \frac{l}{d} \in I^d(A_{\widetilde{NS(S)}}) \) we have an isotropic cyclic subgroup \( \lambda(\langle x \rangle) \subset A_{T(S)} \) of order \( d \) and its generator \( \lambda(x) \in A_{T(S)} \), which define an overlattice \( T_x \supset T(S) \) and a surjective homomorphism \( \alpha_x : T_x \rightarrow \mathbb{Z}/d\mathbb{Z} \) with \( \text{Ker}(\alpha_x) = T(S), \alpha_x(\lambda(x)) = 1 \in \mathbb{Z}/d\mathbb{Z} \). Here we write \( \lambda \) for \( \lambda_{H(S, \mathbb{Z})} : (A_{\widetilde{NS(S)}}, q) \overset{\sim}{\rightarrow} (A_{T(S)}, -q) \). The isotropic subgroup \( \langle x \rangle \subset A_{\widetilde{NS(S)}} \) defines an overlattice \( \widetilde{M}_x = \left\langle \frac{l}{d}, \widetilde{NS(S)} \right\rangle \supset \widetilde{NS(S)} \). Similarly as the construction of \( \nu_0 \) in Proposition 3.7, \( \lambda \) induces an isometry \( \lambda : (A_{\widetilde{M}_x}, q) \simeq (A_{T_x}, -q) \), which gives an embedding \( \widetilde{M}_x \rightarrow \mathbb{A}_{K3} \). Then the isotropic vector \( \frac{l}{d} \in I^1(\widetilde{M}_x) \) defines the twisted K3 surface \( (S_1, \alpha_x) \in FM^d(S) \).

Since the definitions of \( T_x \) and \( \alpha_x \) depend only on \( x \in I^d(A_{\widetilde{NS(S)}}) \), the equivalence class \( [(S_1, \alpha_x)] \in \mathcal{F}M^d(S) \) is independent of the choices of \( l \in I^d(\widetilde{NS(S)}) \) with \( x = \frac{l}{d} \).

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Take an isometry $\gamma \in \Gamma^+_S$ and set $x' := r(\gamma)(x)$. If we choose an isotropic vector $l \in I^d(\tilde{NS}(S))$ with $\frac{l}{d} = x$, then $\gamma(\frac{l}{d}) = x' \in \tilde{A}_{\tilde{NS}(S)}$. Now we have $(S_1, \alpha_1) \simeq (S_{\gamma(l)}, \alpha_{\gamma(l)})$ so that $[(S_1, \alpha_1)] = [(S_{\gamma(l)}, \alpha_{\gamma(l)})]$. Hence the map $\xi_0$ is well-defined.

We prove the surjectivity of $\xi_0$. For a twisted FM partner $(S', \alpha') \in \text{FM}^d(S)$, we have a Hodge isometry $T(S) \simeq T(S', \alpha')$ and an inclusion $T(S') \subset T(S)$ with $\text{Ker}(\alpha') = T(S', \alpha')$. Setting $y := (\alpha')^{-1}(1) \in A_{T(S)} \simeq A_{T(S', \alpha')}$, where $1 \in \mathbb{Z}/d\mathbb{Z}$ is the generator. Then we have an isotropic element $x := \lambda^{-1}(y) \in I^d(\tilde{A}_{\tilde{NS}(S)})$. By the construction of $T_x$ and $\alpha_x$, we have a Hodge isometry $(T_x, \alpha_x) \simeq (T(S'), \alpha')$ so that $[(S_1, \alpha_1)] = [(S', \alpha')] \in \text{FM}^d(S)$, where $l \in I^d(\tilde{NS}(S))$ is such that $\frac{l}{d} = x$.

Finally, we prove the injectivity of $\xi_0$. Suppose we have $[(S_1, \alpha_1)] = [(S_1', \alpha_1')] \in \text{FM}^d(S)$. Since there exists a Hodge isometry $\tilde{g} : T(S_1) \xrightarrow{\cong} T(S_1')$ with $\tilde{g}^* \alpha_1' = \alpha_1$, $\tilde{g}$ maps $T(S) = \text{Ker}(\alpha_1) \subset T(S_1)$ to $T(S) = \text{Ker}(\alpha_1') \subset T(S_1')$ isometrically. Setting $g := \tilde{g}|_{T(S)} \in O_{\text{Hodge}}(T(S))$, we have $r(g)(\lambda(l)) = \lambda(l').$ Hence $\lambda^{-1} \circ r(g) \circ \lambda \in r(\Gamma_S) = r(\Gamma^+_S)$ maps $\frac{l}{d}$ to $\frac{l'}{d}$. \hfill \Box

The set $r(\Gamma^+_S)\backslash I^d(\tilde{A}_{\tilde{NS}(S)})$ is relatively easy to calculate.

**Corollary 3.11.** If there is an embedding $U \hookrightarrow NS(S)$, then $\nu_0 : \Gamma^+_S \backslash I^d(\tilde{NS}(S)) \to \text{FM}^d(S)$ is bijective. In particular, we have the formula

$$\#\text{FM}^d(S) = \#\{\text{0-dimensional cusp } [l] \text{ of } \Gamma^+_S \backslash \Omega_{\tilde{NS}(S)}^+ \text{ with } \text{div}(l) = d\}.$$  

**Proof.** Since there is an embedding $U \oplus U \hookrightarrow \tilde{NS}(S)$, $p$ is bijective by Proposition 4.1.3 of [15]. On the other hand, we have an embedding $U \oplus U \hookrightarrow \tilde{NS}(S) \subset \tilde{M}_x$ so that Proposition 2.2 implies that $\pi$ is bijective. \hfill \Box

In this way, we can obtain informations about $\text{FM}^d(S)$ by studying 0-dimensional cusps of $\Gamma^+_S \backslash \Omega_{\tilde{NS}(S)}^+$ and vice versa.

## 4 FM\(_{ell}\)(S) and certain 1-dimensional cusps of $\Gamma^+_S \backslash \Omega_{\tilde{NS}(S)}^+$

Now we study the relation between FM-partners with elliptic fibrations and 1-dimensional cusps containing 0-dimensional standard cusps in their closures.

### 4.1 FM-partners with elliptic fibrations

**Definition 4.1.** Define $\text{FM}_{ell}(S) := \{(S', \mathcal{O}_{S'}(C')) \mid S' \in \text{FM}(S), C' \text{ is a smooth elliptic curve on } S'\} / \simeq$

where $(S_1, \mathcal{O}_{S_1}(C_1)) \simeq (S_2, \mathcal{O}_{S_2}(C_2))$ if there exists an isomorphism $\varphi : S_1 \simeq S_2$ such that $\varphi^* \mathcal{O}_{S_2}(C_2) \simeq \mathcal{O}_{S_1}(C_1)$. In other words, $(S_1, \mathcal{O}_{S_1}(C_1)) \simeq (S_2, \mathcal{O}_{S_2}(C_2))$ if they are isomorphic as elliptic $K3$ surfaces.
For an even lattice $L$, let $I(L) = \cup_d I^d(L)$ be the set of primitive isotropic vectors in $L$.

**Lemma 4.2.** By associating to $O_S(C')$ its class in $NS(S')$, there exists a one-to-one correspondence between $FM_{ell}(S)$ and the set

$$\left\{ (S', l') \mid S' \in FM(S), l' \in I(NS(S')) \right\} \cong .$$

In [2], $(S_1, l_1) \simeq (S_2, l_2)$ if there exists a Hodge isometry $\Phi : H^2(S_1, \mathbb{Z}) \simeq H^2(S_2, \mathbb{Z})$ with $\Phi(l_1) = l_2$.

**Proof.** It is known (Corollary 6.1 of [14]) that every primitive isotropic element of $NS(S)$ can be transformed by the action of $\{ \pm id \} \times W(S)$ to the class of a smooth elliptic curve. Thus, each element of the set (3) is represented by the divisor class of a smooth elliptic curve. Let $l_1 \in NS(S_1), l_2 \in NS(S_2)$ be the classes of smooth elliptic curves. By the Torelli theorem, it suffices to show that, if there is a Hodge isometry $\Phi : H^2(S_1, \mathbb{Z}) \simeq H^2(S_2, \mathbb{Z})$ with $\Phi(l_1) = l_2$, then there is an effective Hodge isometry $\Phi' : H^2(S_1, \mathbb{Z}) \simeq H^2(S_2, \mathbb{Z})$ with $\Phi'(l_1) = l_2$. Let $\mathcal{A}(S)_{\mathbb{R}}$ be the cone in $NS(S)_{\mathbb{R}}$ generated by the ample classes of $S$ over $\mathbb{R}_{>0}$. Since $l_2$ is contained in the boundary of the positive cone $NS(S_2)^+$, we have $\Phi(\mathcal{A}(S_1)_{\mathbb{R}}) \subset NS(S_2)^+$. Then we have two chambers $\Phi(\mathcal{A}(S_1)_{\mathbb{R}})$ and $\mathcal{A}(S_2)_{\mathbb{R}}$ in $NS(S_2)^+$, and $\Phi(l_1) = l_2$ is contained in the closures of both chambers. If we take $x_1 \in \Phi(\mathcal{A}(S_1)_{\mathbb{R}})$ and $x_2 \in \mathcal{A}(S_2)_{\mathbb{R}}$, only finite number of walls $\{ W_i \}_{i=1}^N$ intersect with the segment $x_1x_2$, and each $W_i$ must passes through $l_2$. Writing $W_i \cap x_1x_2 = t_ix_2 + (1-t_i)x_1$, we may assume that $0 < t_1 < \cdots < t_N < 1$. If we denote by $s_{W_i}$ the reflection with respect to $W_i$, then $s_{W_N} \circ \cdots \circ s_{W_1}$ maps $\Phi(\mathcal{A}(S_1)_{\mathbb{R}})$ to $\mathcal{A}(S_2)_{\mathbb{R}}$ and fixes $l_2$. Hence $s_{W_N} \circ \cdots \circ s_{W_1} \circ \Phi$ gives the desired effective Hodge isometry $\Phi'$.

In what follows, we identify the two sets in Lemma 4.2. Let $\mathbb{Z}l$ be a degenerate lattice of rank 1, and consider the degenerate lattice $U \oplus \mathbb{Z}l = Ze + Zf + \mathbb{Z}l$ of rank 3, where $(e, f) = 1, (e, l) = (f, l) = (l, l) = (e, e) = (f, f) = 0$. The proof of the following theorem is parallel to that of Lemma 4.2 and Proposition 3.4.

**Theorem 4.3.** For an embedding $\varphi \in \text{Emb}(U \oplus \mathbb{Z}l, \overline{NS}(S))$ we have $\varphi(l) \in \varphi(U)^{\perp} \cap \overline{NS}(S) = NS(S_\varphi)$ so that we get $[(S_\varphi, \varphi(l))] \in FM_{ell}(S)$. Then the correspondence

$$\Gamma_S \times \text{Emb}(U \oplus \mathbb{Z}l, \overline{NS}(S)) \ni [\varphi] \mapsto [(S_\varphi, \varphi(l))] \in FM_{ell}(S)$$

is a bijection.

**Corollary 4.4.** The following counting formula for $FM_{ell}(S)$ holds:

$$\#(FM_{ell}(S)) = \# \left( \bigcup_{[M]} \bigcup_{[k]} r_T(S)(O_{Hodge}(T(S))) \setminus O(A_M)/r_M(O(M)^k) \right)$$

where $[M]$ runs over $G(NS(S))$ and $[k]$ runs over $O(M) \setminus I(M) = O(M) \setminus I_1(M)$. 

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Proof. We have
\[
\Gamma_S \setminus \text{Emb}(U \oplus \mathbb{Z}l, \widetilde{NS}(S)) = \bigsqcup \Gamma_S \setminus \left\{ \varphi \in \text{Emb}(U \oplus \mathbb{Z}l, \widetilde{NS}(S)) \mid \varphi(U)^\perp \simeq M, [\varphi(l)] = [k] \right\}.
\]
Since \(O(M \oplus U)\) acts transitively on each set \(\{ \varphi \in \text{Emb}(U \oplus \mathbb{Z}l, \widetilde{NS}(S)) \mid \varphi(U)^\perp \simeq M, [\varphi(l)] = [k] \}\) with stabilizer \(O(M)^k\), the last line can be written as
\[
= \bigsqcup \Gamma_S \setminus O(M \oplus U)/O(M)^k
= \bigsqcup \Gamma_S \setminus r_{T(S)}(O_{\text{Hodge}}(T(S)))\setminus O(A_M)/r_M(O(M)^k),
\]
where we used Proposition 2.2 and the inclusion \(O(M \oplus U)_0 \subset \Gamma_S\) in the last equality. \(\Box\)

**Definition 4.5.** Define
\[
\text{FM}_{\text{ell,sec}}(S) := \left\{ [(S', O_{S'}(C'))] \in \text{FM}_{\text{ell}}(S) \mid [C'] \in NS(S'), \text{div}([C']) = 1 \right\}.
\]
Note that \(\text{div}([C']) = 1\) if and only if the elliptic fibration associated to \(O_{S'}(C')\) has a section. If \(S\) is an elliptic K3 surface admitting a section, then \(\text{FM}(S) = \{S\}\) by Corollary 2.7 of [5]. Hence \(\text{FM}_{\text{ell,sec}}(S)\) (if not empty) is exactly the set of isomorphism classes of elliptic fibrations on \(S\) admitting a section.

**Corollary 4.6.** The following equality holds:
\[
\#(\text{FM}_{\text{ell,sec}}(S)) = \# \left( \bigsqcup_{L \in G(l^+/\mathbb{Z}l)} r_{T(S)}(O_{\text{Hodge}}(T(S)))\setminus O(A_L)/r_L(O(L)) \right),
\]
where \(l\) is an arbitrary standard isotropic vector in \(\widetilde{NS}(S)\), and \(A_{\widetilde{NS}(S)}\) is identified with \(A_L\) by the isometry \(\widetilde{NS}(S) \simeq U \oplus l\).

**Proof.** By Proposition 2.4, \(O(\widetilde{NS}(S))\setminus l^1(\widetilde{NS}(S))\) is identified with \(G(l^+/\mathbb{Z}l)\). On the other hand, \(r_{\widetilde{NS}(S)}(O(\widetilde{NS}(S))^1) \simeq r_L(O(l^+/\mathbb{Z}l))\) by Proposition 2.2. Now the assertion follows from Corollary 4.4. \(\Box\)

### 4.2 FM_{ell}(S) and certain 1-dimensional cusps

**Definition 4.7.** Set
\[
I^e_2(\widetilde{NS}(S)) := \left\{ E \in I_2(\widetilde{NS}(S)) \mid \text{there is } e \in \widetilde{NS}(S) \text{ with } (e, E) = \mathbb{Z} \right\}.
\]
If we take \(f, l \in E\) so that \((e, f) = 1\) and \(\text{Ker}(e, \cdot)|_E = \mathbb{Z}l\), then \(E + Ze = \mathbb{Z}l + Zf + Ze \simeq U \oplus \mathbb{Z}l\). We may assume that \(e\) in (6) is taken to be isotropic.
Lemma 4.8. Let $E \in I^t_2(\widetilde{NS}(S))$.

1. If $f, f' \in E$ satisfy the equalities $(e, f) = (e, f') = 1$, then
   $$(S_{ZZ+zf}, l) \simeq (S_{ZZ+zf'}, l).$$

2. If $e, e' \in \widetilde{NS}(S)$ satisfy the relations $(e, E) = (e', E) = \mathbb{Z}$ and $Ker(e, \cdot)|_E = Ker(e', \cdot)|_E = \mathbb{Z}l$, then
   $$(S_{ZZ+zf}, l) \simeq (S_{ZZ+zf'}, l).$$

Here $f' \in E$ is an arbitrary element satisfying $(e', f') = 1$.

3. If $e, e' \in \widetilde{NS}(S)$ satisfy $(e, f) = (e', f) = 1$ for some $f \in E$, then
   $$(S_{ZZ+zf}, l) \simeq (S_{ZZ+zf'}, l').$$

Here $l$ (resp. $l'$) is a generator of $Ker(e, \cdot)|_E$ (resp. $Ker(e', \cdot)|_E$).

Proof. (1) Since $l \in (Z + Zf)^\perp \cap (Z + Zf')^\perp$, Lemma 2.3 tells that the projection from $(Z + Zf)^\perp$ to $(Z + Zf')^\perp$ gives a Hodge isometry $H^2(S_{ZZ+zf}, \mathbb{Z}) \simeq H^2(S_{ZZ+zf'}, \mathbb{Z})$, which is identity on $\mathbb{Z}l$.

(2) Both $\{l, f\}$ and $\{l, f'\}$ are basis of $E$, so that $f' = f + \alpha l$ for some integer $\alpha \in \mathbb{Z}$. Hence $(e, f) = (e, f') = 1$, and $(S_{ZZ+zf}, l) \simeq (S_{ZZ+zf'}, l) \simeq (S_{ZZ(\pm e)zf'}, l)$ by (1). On the other hand, $(S_{ZZ(\pm e)zf'}, l) \simeq (S_{ZZ'zf'}, l)$ by Lemma 2.3.

(3) As in the proof of (2), $l' = \pm l + \beta f$ for some integer $\beta \in \mathbb{Z}$. If we project $l' \in (Ze' + Zf)^\perp$ to $(Z + Zf)^\perp$, the image of $l'$ is given by $\pm l$. Now the claim follows from Lemma 2.3.

Now we relate $FM_{ell}(S)$ to $\Gamma^+_S \setminus I^t_2(\widetilde{NS}(S))$, the set of 1-dimensional cusps of $\Gamma^+_S \setminus \Omega^+_S(\widetilde{NS}(S))$ whose closures contain 0-dimensional standard cusps, via Theorem 4.3. Similarly as Lemma 4.8, the projection

$$\Gamma^+_S \setminus I^t_2(\widetilde{NS}(S)) \longrightarrow \Gamma_S \setminus I^t_2(\widetilde{NS}(S))$$

is bijective.

Definition 4.9. Define the map $\mu_1 : FM_{ell}(S) \longrightarrow \Gamma_S \setminus I^t_2(\widetilde{NS}(S))$ by

$$\mu_1((S_\varphi, \varphi(l))) := [Z \varphi(f) \oplus Z \varphi(l)].$$

By the definition of $I^t_2(\widetilde{NS}(S))$, $\mu_1$ is surjective. It follows that

$$\#(FM_{ell}(S)) \geq \# \left\{ 1 \text{-dimensional cusp of } \Gamma^+_S \setminus \Omega^+_S(\widetilde{NS}(S)) \text{ whose closure contains a 0-dimensional standard cusp } \right\}.$$

Proposition 4.10. The map

$$\mu_1 : \mu_1^{-1}(\mu_1(FM_{ell, sec}(S))) \longrightarrow \mu_1(FM_{ell, sec}(S))$$

is bijective. In particular, we have $\mu_1^{-1}(\mu_1(FM_{ell, sec}(S))) = FM_{ell, sec}(S)$. 

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Proposition 4.10, while the last assertion is Corollary 2.7 of [5]. The assertions except the last one are consequences of Corollary 2.6 and Corollary 4.11.

Proof. Take \( |E| = \mu_1((S_{ze+zf}, l)) \in \mu_1(\text{FM}_{\ell, sec}(S)) \). We can find an element \( m \in (\mathbb{Z}e + zf)^1 \cap N\overline{S}(S) \) with \((m, m) = 0, (l, m) = 1\). Write \( U_1 := \mathbb{Z}e + zf \) and \( U_2 := \mathbb{Z}l + zm \). Let \( e', f', l' \in \overline{N}S(S) \) be such that \( |E| = \mu_1((S_{ze'zf}, l')) \). We may assume that \( E = zf + zl = zf' + zl' \). Then we can write \( f' = \alpha l + \beta l' = \gamma f + \delta l \) with \( \alpha \delta - \beta \gamma = \pm 1 \). Since \((\pm(\delta e - \gamma m), f') = 1\) and \((\pm(\delta e - \gamma m), l') = 0\), we get \((S_{\pm(\delta e - \gamma m)} + zf', l') \simeq (S_{ze'zf}, l') \) by Lemma 4.8 (2). If we set \( e'' := \pm(\delta e - \gamma m) \) and \( m'' := \pm(\beta e + cm) \), then the correspondence \( e \mapsto e'', f \mapsto f', l \mapsto l', m \mapsto m' \) gives an isometry \( \varphi \in O(U_1 \oplus U_2) \). The isometry \( \tilde{\varphi} := \varphi \oplus \text{id}(U_1 \oplus U_2) \) is identity on \( A_{\overline{NS}(S)} \simeq A(U_1 \oplus U_2)^{\perp} \), so that \( \tilde{\varphi} \in \Gamma_S \). It follows that

\[
(S_{ze+zf}, l) \simeq (S_{ze'zf}, l') \simeq (S_{ze'zf}, l').
\]

Corollary 4.11. If \( \det NS(S) \) is square-free, then every 0-dimensional cusp of \( \Gamma^+_S \backslash \Omega^+_{\overline{NS}(S)} \) is standard. Every elliptic fibration on \( S' \in \text{FM}(S) \) (if exists) has a section. Therefore, \( \mu_1 \) is bijective in this case. If \( \text{rk}(NS(S)) \geq 3 \) in addition, \( \Gamma^+_S \backslash \Omega^+_{\overline{NS}(S)} \) has exactly one 0-dimensional cusp corresponding to \( S \).

Proof. The assertions except the last one are consequences of Corollary 2.6 and Proposition 4.10, while the last assertion is Corollary 2.7 of [5].

However, the map \( \mu_1 \) is not injective in general. By Lemma 4.8 (3), it is easily seen that

\[
\mu^{-1}_1(\{E\}) \simeq \Gamma^+_S \backslash \left\{ f \in E \mid \text{div}(f) = 1 \right\}
\]

for \( E \in I^+_S(\overline{NS}(S)) \), where \( \Gamma^+_S = \{ \gamma \in \Gamma_S \mid \gamma(E) = E \} \). The right hand side of (7) is the set of standard cusps appearing in the canonical compactification of the curve \( \Gamma^+_S \backslash B_E \simeq \Gamma^+_S \backslash H \) (cf. (1)). From the identification (7), we observe the following two facts. Firstly, for two elliptic \( K3 \) surfaces \((S_1, l_1)\) and \((S_2, l_2)\) with \( S_1 \not\simeq S_2, \mu_1((S_1, l_1)) = \mu_1((S_2, l_2)) \) if and only if the two distinct standard cusps \( \mu_0(S_1) \) and \( \mu_0(S_2) \) are connected by the 1-dimensional cusp \( \mu_1((S_1, l_1)) \).

Secondly, fixing an elliptic \( K3 \) surface \((S, l)\), we have

\[
\# \left\{ ([S', l']) \in \text{FM}_{\ell}(S) \mid \mu_1((S', l')) = \mu_1((S, l)) \right\} = \text{mult}_{\mu_0(S)}(\overline{\mu_1((S, l))}), \tag{8}
\]

where \( \overline{\mu_1((S, l))} \) is the closure of the 1-dimensional cusp \( \mu_1((S, l)) \) in \( \Gamma^+_S \backslash \Omega^+_{\overline{NS}(S)} \), and \( \text{mult}_x(C) \) is the multiplicity of the curve \( C \) at the point \( x \). Hence \( \text{FM}_{\ell}(S) \) carries informations about canonical compactifications of certain 1-dimensional cusps themselves.

We remark that, by Proposition 3.7 and Theorem 4.3 an elliptic fibration on an FM-partner \( S' \in \text{FM}(S) \) gives rise to a twisted FM-partner of \( S \). If

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the elliptic fibration has a section, then $S' \simeq S$, and the resulting twisted FM-partner is also isomorphic to $S$. In particular, untwisted FM-partner except $S$ itself can never be obtained in this way.

When two elliptic $K3$ surfaces are connected by a 1-dimensional cusp, there certainly exists a geometric (but not so direct) relation between them. Let $L \in \text{Pic}(S)$ be the line bundle associated to an elliptic fibration, and denote by $l \in NS(S)$ the class of $L$. Define the isotropic lattice $E := \mathbb{Z}l \oplus H^4(S, \mathbb{Z})$. Assume we are given a hyperbolic plane $\mathbb{Z}e' + \mathbb{Z}f' \subset NS(S)$ such that $f' \in E$ and $(S, l) \not= (S_{\mathbb{Z}e'+\mathbb{Z}f'}, l')$. Here $l' \in E$ is a generator of $\text{Ker}(\cdot, e')|_E$. For brevity, we write $S'$ instead of $S_{\mathbb{Z}e'+\mathbb{Z}f'}$. The situation is that, two non-isomorphic elliptic $K3$ surfaces $(S, l)$ and $(S', l')$ are connected by the boundary curve corresponding to $E$. Let $\pi_1 : S \times S' \to S$ and $\pi_2 : S \times S' \to S'$ be the projections. For a coherent sheaf $\mathcal{E}$ on $S \times S'$, we can associate the Mukai vector $v_\mathcal{E} := ch(\mathcal{E}) \sqrt{td(S \times S')} \in H^*(S \times S', \mathbb{Z})$. The following proposition gives a way to obtain $l \in NS(S)$ from $l' \in NS(S')$.

**Proposition 4.12.** (1) There is a coherent sheaf $\mathcal{E}$ on $S \times S'$ such that the cohomological FM transform 
\[
\Phi_S^H : \tilde{H}(S', \mathbb{Z}) \to \tilde{H}(S, \mathbb{Z}), \quad a \mapsto \pi_{1*}(v_\mathcal{E} \wedge \pi_2^*a)
\]
is a Hodge isometry with $\Phi_S^H(H^4(S', \mathbb{Z})) = \mathbb{Z}e'$, $\Phi_S^H(H^0(S', \mathbb{Z})) = \mathbb{Z}f'$. By the construction of $S'$, we may assume that $\Phi_S^H$ gives the identification $H^2(S', \mathbb{Z}) \simeq (\mathbb{Z}e'+\mathbb{Z}f')^\perp \cap \tilde{H}(S, \mathbb{Z})$. (In fact, $\mathcal{E}$ induces an equivalence $D^b(S') \simeq D^b(S)$.)

(2) Let $l' \in \text{Pic}(S')$ be the line bundle representing $l' \in NS(S')$. Then the line bundle 
\[
\det(R\pi_{1*}(\mathcal{E} \otimes \pi_2^*l')) \otimes \det(R\pi_{1*}\mathcal{E})^{-1} \in \text{Pic}(S)
\]
is a multiple of $L$.

**Proof.** (1) Firstly, we claim that the $H^0(S, \mathbb{Z})$ component of $e'$ is not 0. Assume $e' = (0, m, s)$. Since $f' \in E = \mathbb{Z}l \oplus H^4(S, \mathbb{Z})$, we can write $f'$ degreewise as $f' = (a, \alpha, t)$ for some integer $\alpha$. Then we have $1 = (e', f') = a(m, l)$. It follows that $\text{div}(l) = 1$, so we have $(S, l) \simeq (S', l')$ by Proposition 10. (Or Proposition 10.10 of [16], there exists a coherent sheaf $\mathcal{E}'$ on $S \times S'$ such that the cohomological FM transform $\Phi_{S'}^H : \tilde{H}(S', \mathbb{Z}) \to \tilde{H}(S, \mathbb{Z})$ is a Hodge isometry with $\Phi_{S'}^H(H^4(S', \mathbb{Z})) = \mathbb{Z}e'$. Replacing $(e', f')$ by $(-e', -f')$ if necessary, we may assume that the positive generator $v'$ of $H^4(S', \mathbb{Z})$ satisfies $\Phi_{S'}^H(v') = e'$. Since $(e', f') = 1$, the $H^0(S', \mathbb{Z})$ component of $(\Phi_{S'}^H)^{-1}(f')$ is equal to $-1$. Then, as $(\Phi_{S'}^H)^{-1}(f')$ is isotropic, $(\Phi_{S'}^H)^{-1}(f') = -(1, [M], [M]^0_{-1}) = -ch(M)$ for some line bundle $M \in \text{Pic}(S')$. Setting $\mathcal{E} := \mathcal{E}' \otimes \pi_2^*M$, we have $\Phi_{S'}^H(v') = \Phi_{S'}^H(v' \wedge ch(M)) = \Phi_{S'}^H(v') = e'$ and $\Phi_{S'}^H(1) = \Phi_{S'}^H(ch(M)) = -f'$. 

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that connected component of generic member of the moving part.

Example 4.13. To obtain an elliptic curve on $S$ the linear system $R\pi_1E \otimes \pi_2L$ and $(\det(R\pi_1E \otimes \pi_2L)) \otimes \det(R\pi_1E)^{-1}$ is the $H^2(S, \mathbb{Z})$ component of $\Phi_H^{(1,0,1)}$. Hence,

$$c_1\left(\det(R\pi_1E)\right) = \text{the } H^2(S, \mathbb{Z}) \text{ component of } \Phi_H^{(1,0,1)}.$$ 

Since the identification $H^2(S', \mathbb{Z}) \simeq (\mathbb{Z}e' + \mathbb{Z}f') \cap \tilde{H}(S, \mathbb{Z})$ is given by $\Phi_H$, $\Phi_H^{(l', l)}$ belongs to $E = \mathbb{Z}l \oplus H^4(S, \mathbb{Z})$. Hence the $H^2(S, \mathbb{Z})$ component of $\Phi_H^{(l', l)}$ lies in $\mathbb{Z}l$. 

Note that the statement of Proposition 4.12 is symmetric with respect to $(S, l)$ and $(S', l')$. In this way, we can construct $L'$ (resp. $L$) from $L$ (resp. $L'$) via a certain sheaf on $S \times S'$. (Of course, even if a sheaf $\mathcal{E}$ on $S \times S'$ induces a Hodge isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(S', \mathbb{Z})$, it is not true in general that the first Chern class of the line bundle $\det(R\pi_2\mathcal{E} \otimes \pi_1L) \otimes \det(R\pi_2\mathcal{E})^{-1}$ is isotropic.) To obtain an elliptic curve on $S'$, we first eliminate the fixed components of the linear system $\det(R\pi_2\mathcal{E} \otimes \pi_1L) \otimes \det(R\pi_2\mathcal{E})^{\mp 1}$ and then we take a connected component of generic member of the moving part.

We give an example for which $\mu_1$ is not injective.

Example 4.13. Let $S$ be a K3 surface with $NS(S) \simeq U(r), r > 2$, and denote by $r = \prod_{i=1}^{\tau(r)} \mu_i^{e_i}$ the prime decomposition of $r$. Assume that $S$ is generic so that $O_{Hodge}(T(S)) = \{\pm id\}$. Then the followings hold:

1. $G(U(r)) = \{U(r)\}$.
2. $\#(\text{FM}(S)) = 2^{\tau(r) - 2}\varphi(r)$, where $\varphi$ is the Euler function.
3. $S$ has two non-isomorphic elliptic fibrations, whose image by $\mu_1$ are different 1-dimensional cusps.
4. $\#(\text{FM}_{\text{crit}}(S)) = 2^{\tau(r) - 1}\varphi(r)$.
5. For $E \in I^1(S)$, $\#\mu^{-1}(E) = \frac{\varphi(r)}{2}$.
6. $\#\left(\Gamma_S \setminus I^1_2(S)\right) = 2^{\tau(r)}$. 

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Proof. Write $U(r) = \mathbb{Z}l + \mathbb{Z}m$ with $(l, l) = (m, m) = 0$, $(l, m) = r$. Then $A_{U(r)} = \langle \frac{1}{r}, \frac{m}{r} \rangle \simeq (\mathbb{Z}/r\mathbb{Z}) \oplus (\mathbb{Z}/r\mathbb{Z})$ with

$$\left( \frac{l}{r}, \frac{m}{r} \right) \equiv 0 \mod 2\mathbb{Z}, \quad \left( \frac{l}{r}, \frac{m}{r} \right) \equiv \frac{1}{r} \mod \mathbb{Z}. $$

We have $O(U(r)) = \{ \text{id}, -\text{id}, \iota, -\iota \} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, where $\iota(l) = m, \iota(m) = l$. Therefore $O(U(r))^\dagger = O(U(r))^m = \{ \text{id} \}$. Let $L$ be an even lattice with sign($L$) = (1, 1), $A_L \simeq \langle \frac{1}{l}, \frac{m}{m} \rangle$. Given a basis $\{l, m\}$ of $L^\vee$, we have $rl, rm \in L$. The inclusions $\mathbb{Z}rl + \mathbb{Z}rm \subset L \subset L^\vee = \mathbb{Z}l + \mathbb{Z}m$ imply that $\mathbb{Z}rl + \mathbb{Z}rm = L$. Thus the Gram matrix of $L$ with respect to the basis $\{rl, rm\}$ is divisible by $r$. Then $|\det(L(\frac{1}{l}))| = #(A_L)/r^2 = 1$, $\text{sign}(L(\frac{1}{l})) = (1, 1)$, and $L(\frac{1}{l})$ is even so that $L(\frac{1}{l})$ must be isometric to $U$.

(2) To calculate $\#\text{FM}(S)$, we shall calculate $O(A_{U(r)})$.

Claim 4.14. $O(A_{U(r)}) \simeq \prod_{r=1}^{(r)} O(A_{U(p_r^*)})$. Thus $\#O(A_{U(r)}) = 2^{(r)}\varphi(r)$.

With respect to the basis $\{\frac{1}{l}, \frac{1}{m}\}$ of $(\mathbb{Z}/r\mathbb{Z}) \oplus (\mathbb{Z}/r\mathbb{Z})$,

$$O(A_{U(r)}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/r\mathbb{Z}) \mid ad + bc \equiv 1 \mod r, \ ab \equiv cd \equiv 0 \mod r \right\}$$

$$= \prod_{i=1}^{(r)} \left\{ \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL_2(\mathbb{Z}/p_r^i\mathbb{Z}) \mid ad + bc - 1 \equiv 0 \mod p_r^i \right\}$$

$$= \prod_{i=1}^{(r)} O(A_{U(p_r^i)}),$$

where the second equality follows from the Chinese Remainder theorem. Direct calculations show that $O(A_{U(p_r^i)}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{Z}/p_r^i\mathbb{Z})^\times \right\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{Z}/p_r^i\mathbb{Z})^\times \right\},$ so that $\#O(A_{U(p_r^i)}) = 2\varphi(p_r^i)$. Since $\varphi(r) = \prod \varphi(p_r^i)$, the second assertion follows.

Now (2) follows from (1), Claim 4.2, and the counting formula for $\#\text{FM}(S)$.

(3) $NS(S)$ has exactly four primitive isotropic elements $\{\pm l, \pm m\}$. We may assume that $l, m$ are classes of effective divisors. Since $W(S) = \{ \text{id} \}$, both of $l, m$ are classes of elliptic fibrations by the first part of the proof of Lemma 172. The only isometry of $NS(S)$ mapping $l$ to $m$ is $\iota$, which is not compatible with any element of $O_{\text{Hodge}}(T(S)) = \{ \pm \text{id} \}$ on the discriminant group. Therefore $(S, l)$ and $(S, m)$ are not isomorphic. If there exists $\gamma \in \Gamma_S$ which sends $\mu_1((S, l)) = H^4(S, \mathbb{Z}) \oplus \mathbb{Z}l$ to $\mu_1((S, m)) = H^4(S, \mathbb{Z}) \oplus \mathbb{Z}m$, $\gamma$ maps $\langle \frac{1}{l} \rangle$ to $\langle \frac{1}{m} \rangle$ on $A_{U(r) \oplus U} \simeq A_{U(r)}$, which does not coincide with $\{ \pm \text{id} \}$.

The assertion (4) follows from (2) and (3).

(5) By (1), for each $E \in \mathcal{L}_r^1(U(r) \oplus U)$, we can find a basis $\{l_1, m_1, e_1, f_1\}$ of $U(r) \oplus U$ such that $E = \mathbb{Z}f_1 + \mathbb{Z}l_1$ and $(l_1, l_1) = (m_1, m_1) = (e_1, e_1) = (f_1, f_1) = (l_1, e_1) = (l_1, f_1) = (m_1, e_1) = (m_1, f_1) = 0, (l_1, m_1) = r, (e_1, f_1) = 1.$
Take integers $\alpha, \beta \in \mathbb{Z}$ so that $\beta$ and $r\alpha$ are coprime to each other. Then there exist $\gamma, \delta \in \mathbb{Z}$ such that $\beta\delta + r\alpha\gamma = 1$. Set $f_2 := \alpha l_1 + \beta f_1 \in E, e_2 := \gamma m_1 + \delta e_1, l_2 := \delta l_1 - r\gamma f_1 \in E, m_2 := \beta m_1 - r\alpha e_1$. We have an isometry $\varphi(\alpha, \beta, \gamma, \delta) \in O(U \oplus U(r))E$ which maps $e_1(\text{resp. } f_1, l_1, m_1)$ to $e_2(\text{resp. } f_2, l_2, m_2)$.

On the discriminant group $A(U(r)) \oplus U(r), \varphi(\alpha, \beta, \gamma, \delta)(\frac{m_1}{r}) = \beta \frac{m_1}{r}, \varphi(\alpha, \beta, \gamma, \delta)(\frac{1}{r}) = \frac{\delta}{2}$. Note that the image of $\alpha + \gamma m \in \mathbb{Z}/r\mathbb{Z}$ is uniquely determined by $\alpha, \beta$ as above. Conversely, given $f_2 = \alpha l_1 + \beta f_1 \in E$ with $\text{div}(f_2) = 1$, we can find $e_2 = \gamma m_1 + \delta e_1 + \epsilon f_1 + \zeta l_1$ with $(f_2, e_2) = 1, (e_2, e_2) = 0$. We have $\beta\delta + r\alpha\gamma = 1$. Then Lemma 2.8 assures us that we are allowed to take $e_2' := \gamma m_1 + \delta e_1$ instead of $e_2$. In this case, $l_2 = \delta l_1 - r\gamma f_1$. Setting $m_2 := \beta m_1 - r\alpha e_1$, we get $\varphi(\alpha, \beta, \gamma, \delta) \in O(NS(S))$ which maps $f_1(\text{resp. } e_1, l_1, m_1)$ to $f_2(\text{resp. } e_2', l_2, m_2)$. Therefore, each $Z\in \mathbb{Z}/Z$ can be written as $\varphi(\alpha, \beta, \gamma, \delta)(Z\epsilon_1 + Zf_1 + Zl_1)$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with $\beta\delta + r\alpha\gamma = 1$. Recall that $\varphi(\alpha, \beta, \gamma, \delta) \in \Gamma S \cdot \varphi(\alpha', \beta', \gamma', \delta')$ if and only if $rU \cap U(\varphi(\alpha, \beta, \gamma, \delta)) = \pm rU \cap U(\varphi(\alpha', \beta', \gamma', \delta'))$. Since $rU \cap U(\varphi(\alpha, \beta, \gamma, \delta))$ is expressed as the matrix $\left( \begin{array}{cc} \beta & 0 \\ 0 & \delta \end{array} \right)$, we see that $\Gamma S \varphi(\alpha, \beta, \gamma, \delta) = \Gamma S \varphi(\alpha', \beta', \gamma', \delta')$ if and only if $\beta = \pm \beta', \delta = \pm \delta'$ mod $r\mathbb{Z}$. It follows that

$$
\mu^{-1}(E) \simeq \{[\beta, [\gamma] \in Z/rZ | [\beta][\gamma] \equiv 1 \in Z/rZ\}/\{id\} \\
\simeq \left(\mathbb{Z}/r\mathbb{Z}\right)^{\times}/\{id\}.
$$

The assertion (6) follows from (4) and (5).

\[\square\]

5 Remarks on $\Gamma^{+}_S \backslash \Omega^{+}_{NS(S)}$

In this section, we mention two remarks on the modular variety $\Gamma^{+}_S \backslash \Omega^{+}_{NS(S)}$.

For $\varphi \in \text{Emb}(U, \tilde{NS}(S))$, the splitting $\tilde{NS}(S) = NS(S, \varphi) \oplus \varphi(U)$ induces the realization of $\Omega^{+}_{NS(S)}$ as a tube domain

$$
\iota_{\varphi} : NS(S, \varphi)R + \sqrt{-1}NS(S, \varphi)^+ \xrightarrow{\simeq} \Omega^{+}_{NS(S)}, \quad y \mapsto C\left(\varphi(e) + y - \frac{1}{2}(y, y)\right).
$$

Let us observe briefly the action of $O(\tilde{NS}(S))^\varphi(f) \cap \Gamma^{+}_S$ on the tube domain $NS(S, \varphi)R + \sqrt{-1}NS(S, \varphi)^+$. By Proposition 2.7 we have

$$
O(\tilde{NS}(S))^\varphi(f) \simeq O(NS(S, \varphi)) \ltimes NS(S, \varphi),
$$

and the action of $NS(S, \varphi)$ is identity on the discriminant group. Take an element $\alpha + \sqrt{-1}\omega \in NS(S, \varphi)R + \sqrt{-1}NS(S, \varphi)^+$. According to the equations (11), $m \in NS(S, \varphi)$ and $g \in O(NS(S, \varphi))$ act as

$$
m(\alpha + \sqrt{-1}\omega) = \alpha + m + \sqrt{-1}\omega, \\
g(\alpha + \sqrt{-1}\omega) = g(\alpha) + \sqrt{-1}g(\omega).
$$
There is a normal subgroup $W(S_\varphi) \ltimes NS(S_\varphi) \subset O(\widetilde{NS}(S))^{\varphi(f)} \cap \Gamma_S^+$, where $W(S_\varphi)$ is the Weyl group of $NS(S_\varphi)$. Since $\mathcal{A}(S_\varphi)_{\mathbb{R}}$ is the fundamental domain for the action of $W(S_\varphi)$ on $NS(S_\varphi)^+$,

$$\left(NS(S_\varphi)_{\mathbb{R}}/NS(S_\varphi)\right) + \sqrt{-1}\mathcal{A}(S_\varphi)_{\mathbb{R}}$$

(9)

is the fundamental domain for the action of $W(S_\varphi) \ltimes NS(S_\varphi)$ on $NS(S_\varphi)_{\mathbb{R}} + \sqrt{-1}NS(S_\varphi)^+$. Hence the fundamental domain for the action of $O(\widetilde{NS}(S))^{\varphi(f)} \cap \Gamma_S^+$ on $NS(S_\varphi)_{\mathbb{R}} + \sqrt{-1}NS(S_\varphi)^+$ is a quotient of (9).

Next, we explain $\Gamma_S^+ \backslash \Omega^+_{\widetilde{NS}(S)}$ in connection with $Stab(S)$, the space of numerical locally finite stability conditions on $D^b(S)$, following Bridgeland ([2], [3]). The space $Stab(S)$ admits a right action of $GL_2^+(\mathbb{R})$, the universal covering of $GL_2^+(\mathbb{R})$, and a left action of $Aut(D^b(S))$. These two actions commute. By definition, the central charge of $\sigma \in Stab(S)$ takes the form

$$Z : E \mapsto \left(\pi(\sigma), ch(E)\sqrt{Td(S)}\right), E \in K(D^b(S))$$

for some vector $\pi(\sigma) \in \widetilde{NS}(S)_C$. The correspondence $\sigma \mapsto \pi(\sigma)$ defines a continuous map $\pi : Stab(S) \rightarrow \widetilde{NS}(S)_C$. Set

$$P(S) := \left\{ \omega \in \widetilde{NS}(S)_C \mid \mathbb{R}Re\omega + \mathbb{R}Im\omega \text{ is a positive-definite two-plane} \right\}.$$

Via the isomorphism ([2], $P(S)$ is a principal $GL_2^+(\mathbb{R})$-bundle over $\Omega_{\widetilde{NS}(S)}$ by the projection

$$p : P(S) \rightarrow \Omega_{\widetilde{NS}(S)}, \quad \omega \mapsto \mathbb{R}Re\omega + \mathbb{R}Im\omega.$$

Here the right action of $GL_2^+(\mathbb{R})$ is given by

$$\omega \cdot g := (aRe\omega + bIm\omega) + \sqrt{-1}(cRe\omega + dIm\omega),$$

where $\omega \in P(S)$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \in GL_2^+(\mathbb{R})$. Each tube domain realization $\iota_\varphi$ induces a section $\Omega_{\widetilde{NS}(S)} \rightarrow P(S)$. Denote by $P^+(S) \subset P(S)$ the component lying over $\Omega_{\widetilde{NS}(S)}^+$, and set

$$\Delta(S) := \left\{ \delta \in \widetilde{NS}(S) \mid \langle \delta, \delta \rangle = -2 \right\}.$$

As an immediate corollary of Bridgeland’s results, we have the following description of the modular variety $\Gamma_S^+ \backslash \Omega_{\widetilde{NS}(S)}^+$:

**Proposition 5.1.** There is a connected component $Stab^+(S) \subset Stab(S)$ and a subgroup $Aut^+(D^b(S)) \subset Aut(D^b(S))$ such that $p \circ \pi$ induces the isomorphism

$$Aut^+(D^b(S)) \backslash Stab^+(S)/GL_2^+(\mathbb{R}) \approx \Gamma_S^+ \backslash \left(\Omega_{\widetilde{NS}(S)}^+ - \bigcup_{\delta \in \Delta(S)} \delta^\perp\right).$$
Proof. According to [2] and [3], \( \pi \) induces the isomorphism
\[
\text{Aut}^l(D^b(S)) \setminus \text{Stab}^l(S) \cong \Gamma_S^+ \setminus \left( P^+(S) - \bigcup_{\delta \in \Delta(S)} \delta^\perp \right).
\]

Since \( \pi \) is \( GL_2^+ (\mathbb{R}) \)-equivariant, we have
\[
\text{Aut}^l(D^b(S)) \setminus \text{Stab}^l(S)/GL_2^+ (\mathbb{R}) \cong \Gamma_S^+ \setminus \left( \Omega^+_N(S(S)) - \bigcup_{\delta \in \Delta(S)} \delta^\perp \right).
\]

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