LOCAL POINTS ON QUADRATIC TWISTS OF $X_0(N)$

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Abstract. Let $X^d(N)$ be the quadratic twist of the modular curve $X_0(N)$ through the Atkin-Lehner involution $w_N$ and a quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. The points of $X^d(N)(\mathbb{Q})$ are precisely the $\mathbb{Q}(\sqrt{d})$-rational points of $X_0(N)$ that are fixed by $\sigma \circ w_N$, where $\sigma$ is the generator of $\text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$. Ellenberg [4] asked the following question:

For which $d$ and $N$ does $X^d(N)$ have rational points over every completion of $\mathbb{Q}$?

Given $(N, d, p)$ we give necessary and sufficient conditions for the existence of a $\mathbb{Q}_p$-rational point on $X^d(N)$, whenever $p$ is not simultaneously ramified in $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{-N})$, answering Ellenberg’s question for all odd primes $p$ when $(N, d) = 1$. The main theorem yields a population of curves which have local points everywhere but no points over $\mathbb{Q}$; in several cases we show that this obstruction to the Hasse Principle is explained by the Brauer-Manin obstruction.

1. Introduction

Let $N = p_1 \cdots p_r$ be a positive, square-free integer. The modular curve $Y_0(N)$ is a moduli space of tuples $(E, C)$, where $E$ is an elliptic curve and $C$ is a cyclic subgroup of order $N$ in $E[N]$. Equivalently, any point of $Y_0(N)$ corresponds to $(E, \phi)$ where $\phi$ is a cyclic $N$-isogeny of $E$. A projective smooth curve $X_0(N)$ is obtained by adding $2r$ cusps to $Y_0(N)$.

The Atkin-Lehner involution $w_N$ of $Y_0(N)$ sends $(E, C)$ to the pair $(E/C, E[N]/C)$. Equivalently, in terms of isogenies, $w_N : (E, \phi) \mapsto (E', \hat{\phi})$ where $\phi : E \to E'$ and $\hat{\phi}$ is the dual isogeny. The action of the rational map $w_N$ extends to $X_0(N)$ such that it permutes the cusps.

A celebrated theorem of Mazur [12] and its extensions by Kenku and Momose, give much more information about the rational points of $X_0(N)$.

Theorem: [Mazur, Kenku-Momose] $X_0(N)(\mathbb{Q})$ consists of only cusps when $N > 163$.

This result is proved by Mazur for prime levels of $N$ and generalized to square-free integers by Kenku and Momose.

Let $K := \mathbb{Q}(\sqrt{d})$, $\sigma$ be the generator of $\text{Gal}(K/\mathbb{Q})$ and $N$ be a square-free integer. The twist $X^d(N)$ of $X_0(N)$ is constructed by etale
descent from $X_0(N)/K$. It is a smooth proper curve over $\mathbb{Q}$, isomorphic to $X_0(N)$ over $K$ but not over $\mathbb{Q}$. The action of $\sigma$ is ‘twisted’ on $X^d(N)$, meaning that $\mathbb{Q}$-rational points of $X^d(N)$ are naturally identified with the $K$-rational points of $X_0(N)$ that are fixed by $\sigma \circ w_N$. We are interested in those points. However, the existence of rational points in this case is not immediate, as it is for $X_0(N)$. Since cusps are interchanged by $w_N$, they are not rational anymore.

Like $X_0(N)$, the twisted curve $X^d(N)$ is a moduli space. Rational points on $X^d(N)$ parametrize a special class of elliptic curves, called quadratic $\mathbb{Q}$-curves of degree $N$. A $\mathbb{Q}$-curve is an elliptic curve which is isogenous to all of its Galois conjugates, and appears in many interesting questions, such as questions about ‘twisted’ Fermat equations. More details about these results and in general about $\mathbb{Q}$-curves, as well as related questions can be found in Ellenberg’s survey article [4].

In general, there is no algorithm to follow when trying to show existence or non-existence of a rational point on a variety. One of the first things to check is the existence of adelic points. If a curve over $\mathbb{Q}$ fails to have a $\mathbb{Q}_p$-point for some $p$, then there is no rational point on that curve. This gives rise to the main question of the paper, which was originally stated as Problem A by Ellenberg in [4]:

**Question [Ellenberg]:** For which $d$ and $N$ does $X^d(N)$ have rational points over every completion of $\mathbb{Q}$?

The main theorem of this paper gives an answer to this question under the assumption that no prime is ramified in both $K$ and $\mathbb{Q}(\sqrt{-N})$.

**Theorem 1.1.** Let $p$ be a prime, $N$ square-free integer, $K$ a quadratic field. Then

1. $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ for all $p$ that split in $K$ and for $p = \infty$. (Proposition 3.7)
2. $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if $p$ is inert in $K$ and not dividing $N$. (Theorem 4.18)
3. For all odd $p$ inert in $K$ and dividing $N$, $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if and only if
   - (a) $N = p\Pi q_i$ where $p \equiv 3$ modulo 4 and $q_i \equiv 1$ modulo 4 for all $i$ or
   - (b) $N = 2p\Pi q_i$ where $p \equiv 3$ modulo 4 and $q_i \equiv 1$ modulo 4 for all $i$. (Theorem 4.8)
4. If 2 is inert in $K$ and divides $N$, $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if and only if $N = 2\Pi q_i$ where $q_i \equiv 1$ modulo 4 for all $i$. (Theorem 4.9)
5. For all $p$ ramified in $K$ and not dividing $N$, $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if and only if $p$ is in the set $S$ defined in Proposition 5.3. (Theorem 5.9)

**Organization of the paper:** In Section 2 we give an overview of the previous results and where our result fits in the general scheme.
We prove under the assumption that no prime is ramified in both $K$ and $Q(\sqrt{-N})$, the previous results on this problem by Clark, Gonzalez, Quer, and Shih are implied by Theorem 1.1. We also answer a question of Clark raised in [1].

We draw on a number of different techniques to handle the cases in Theorem 1.1. In Section 3, we deal with the case $p = \infty$. In Section 4, we handle the case when $p$ is inert in $K$. For instance, in this section we use Hensel’s Lemma when $p | N$. On the other hand, in Section 5 we construct a $Q_p$-point using the theory of CM elliptic curves.

In Section 6, we give some examples that illustrate Theorem 1.1 and compare our result with previous results. In Section 7, we give ideas about further directions, give examples of genus 2 curves that violate Hasse principle, and show that these violations are explained by the Brauer-Manin obstruction.

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2. Relation with Previous Work

In the case of conics there is a rational point if and only if there is a local point for every completion of $Q$. Moreover if a conic has a rational point then it has many others, since it is isomorphic to $\mathbb{P}^1$. Hence for the case of conics i.e. when genus of $X^d(N)$ is zero, we have a complete answer to this question due to the work of Shih [23], Gonzalez and Quer given in [18] based on the earlier work of Hasegawa [9]. The proof is based on a special parametrization of the $j$-invariants of these curves and some Hilbert symbol computations. Hence Gonzalez and Quer give the following complete list in the case of genus 0.

Theorem 2.1 (Gonzalez,Quer,Shih). Using the notation above:

- When $N = 2, 3, 7$: $X^d(N)(Q)$ is infinite for any quadratic field $Q(\sqrt{d})$.
- When $N = 5$: $X^d(N)(Q)$ is infinite if and only if $d$ is of the form $m$ or $5m$ where $m$ is a square-free integer each of whose prime divisors are quadratic residues modulo 5.
- When $N = 6$: $X^d(N)(Q)$ is infinite if and only if $d$ is of the form $m$ or $6m$ where $m$ is a square-free integer such that 2 is a quadratic residue modulo each prime divisor of $m$.
- When $N = 10$: $X^d(N)(Q)$ is infinite if and only if $d$ is of the form $m$ or $10m$ where $m$ is a square-free integer each of whose prime divisors are quadratic residues modulo 5.
• When \( N = 13 \) : \( X^d(N)(\mathbb{Q}) \) is infinite if and only if \( d \) is of the form \( m \) or \( 13m \) where \( m \) is a square-free integer each of whose prime divisors are quadratic residues modulo 13.

Note that since \( X^d(N) \) and \( X_0(N) \) are isomorphic over \( K \) they are geometrically the same in particular they have the same genus. Therefore the cases that we know the answer completely corresponds to the values \( N = 2, 3, 5, 6, 7, 10 \) and 13.

For \( N = 2, 3, 7, \) since the class number of \( \mathbb{Z}[\sqrt{-N}] \) is 1, any \( w_N \)-fixed point of \( X_0(N) \) is defined over \( \mathbb{Q} \), hence gives a point in \( X^d(N)(\mathbb{Q}) \) for any \( d \). This is another way of saying the first part of Theorem \( 2.1 \).

Now we’ll derive the other parts of Theorem \( 2.3 \) using Theorem \( 1.1 \) for relatively prime \( N \) and \( d \).

**Corollary 2.2.** Let \( N, d \) be square-free integers such that there is no prime \( p \) that is ramified in both \( \mathbb{Q}(\sqrt{-N}) \) and \( \mathbb{Q}(\sqrt{d}) \). Then Theorem \( 2.1 \) can be derived from Theorem \( 1.1 \).

**Proof.** Since we are dealing with the conics having a \( \mathbb{Q} \)-rational point is equivalent to have \( \mathbb{Q}_p \) points for every prime \( p \). By Proposition \( 3.1 \) \( X^d(N)(\mathbb{R}) \neq \emptyset \) for any \( N \) and \( d \) hence we only need to check the finite primes. Let \( d = \pm \prod p_i \) be the prime decomposition of \( d \).

- **\( N = 5 \):** By Theorem \( 1.1 \) part 5, \( X^d(5)(\mathbb{Q}_{p_i}) \neq \emptyset \) if and only if there is a prime of \( \mathbb{Q}(j(\sqrt{-5})) \) lying over \( p \) with inertia degree 1. Note that class number of \( \mathbb{Z}[\sqrt{-5}] \) is 2 and it is the maximal order of \( M := \mathbb{Q}(\sqrt{-5}) \). Hilbert class field of \( M \) is \( \mathbb{Q}(\sqrt{5}, i) \) hence \( \mathbb{Q}(j(\sqrt{-5})) \) is \( \mathbb{Q}(\sqrt{5}) \), since \( j(\sqrt{-5}) \) is real. Therefore \( X^d(5)(\mathbb{Q}_{p_i}) \neq \emptyset \) if and only if \( \left( \frac{2}{p_i} \right) = 1 \).

  Since \( \left( \frac{2}{p} \right) = \left( \frac{2}{5} \right) = 1, \left( \frac{2}{7} \right) = 1, \left( \frac{2}{2} \right) = 1, \left( \frac{2}{3} \right) = 0 \).

  For all other primes \( q \), \( X^d(5)(\mathbb{Q}_q) \neq \emptyset \) by first and second parts of Theorem \( 1.1 \).

  The case \( N = 13 \) is quite similar to the case \( N = 5 \), since corresponding Hilbert class field is \( \mathbb{Q}(\sqrt{13}, i) \) and they are both 1 mod 4.

- **\( N = 6 \):** Hilbert class field of \( \mathbb{Q}(\sqrt{-6}) \) is \( \mathbb{Q}(\sqrt{-3}, \sqrt{2}) \) and \( \mathbb{Q}(j(\sqrt{-6})) \) is \( \mathbb{Q}(\sqrt{2}) \). Therefore using Theorem \( 1.1 \) part 5, \( X^d(6)(\mathbb{Q}_{p_i}) \) has \( \mathbb{Q}_{p_i} \)-rational points if and only if \( \left( \frac{2}{p_i} \right) = 1 \).

  If 3 is inert in \( K \) or splits in \( K \) then \( X^d(6)(\mathbb{Q}_3) \neq \emptyset \) by parts 1 and 3 of Theorem \( 1.1 \).

  If 2 splits in \( K \) then \( X^d(6)(\mathbb{Q}_2) \neq \emptyset \) by part 1 of Theorem \( 1.1 \).

  And 2 can not be inert in \( K \), since \( \left( \frac{2}{p} \right) = 1 \) for all \( p_i \).

  For all other primes \( q \), \( X^d(6)(\mathbb{Q}_q) \neq \emptyset \) by first and second parts of Theorem \( 1.1 \).
• \(N = 10\): Hilbert class field of \(\mathbb{Q}(\sqrt{-10})\) is \(\mathbb{Q}(\sqrt{-2}, \sqrt{5})\) and \(\mathbb{Q}(\sqrt{10})\) is \(\mathbb{Q}(\sqrt{5})\).

Therefore using Theorem 1.1 part 5, \(X^d(10)(\mathbb{Q}_{p_i}) \neq \emptyset\) if and only if \((\frac{5}{p_i}) = 1\).

Since \((\frac{5}{p_i}) = (\frac{p_i}{5}) = 1\), \((\frac{d}{5}) = 1\), \(X^d(10)(\mathbb{Q}_5) \neq \emptyset\).

If 2 is inert or 2 splits in \(K\) then \(X^d(10)(\mathbb{Q}_2) \neq \emptyset\) by parts 1 and 4 of Theorem 1.1.

For all other primes \(q, X^d(10)(\mathbb{Q}_q) \neq \emptyset\) by first and second parts of Theorem 1.1.

\[\square\]

Another result along these lines, which is a necessary condition for the existence of degree-\(N\) \(\mathbb{Q}\)-curves, was given in [18]:

**Theorem 2.3.** ([18], Theorem 6.2) Assume that there exists a quadratic \(\mathbb{Q}\)-curve of degree \(N\) defined over some quadratic field \(K\). Then every divisor \(N_1|N\) such that

\[N_1 \equiv 1(\text{mod } 4) \text{ or } N_1 \text{ is even and } N/N_1 \equiv 3(\text{mod } 4)\]

is a norm of the field \(K\).

Proof of this theorem is analytic, by constructing some functions on \(X_0(N)\) with rational Fourier coefficients and studying the action of the involution \(w_N\) on them. We’ll take a more algebraic approach and given any square-free, relatively prime \(d\) and \(N\) show that Theorem 1.1 implies Theorem 2.3 in the following two corollaries.

Recall that saying that ‘\(N_1\) is a norm in \(K\)’ is equivalent to say that \((N_1,d) = 1\) where \((-,-)\) denotes the Hilbert symbol. Moreover \((N_1,d) = 1\) if and only if all local Hilbert symbols, \((N_1,d)_p = 1\) for all primes \(p\). In other words, \((N_1,d) = -1\) if and only if there is a prime \(p\) such that \((N_1,d)_p = -1\). Therefore, Theorem 2.3 gives a condition on the existence of local points.

The local Hilbert symbol is given by the following formula [21]:

- \(p = \infty\): \((a,b)_\infty = 1\) if and only if \(a\) or \(b\) is positive.
- \(p \neq 2\): \((a,b)_p = (-1)^{\alpha\beta\epsilon(p)} \left(\frac{a}{p}\right)^\beta \left(\frac{b}{p}\right)^\alpha\)
- \(p = 2\): \((a,b)_2 = (-1)^{\epsilon(u)\epsilon(v)+\alpha w(u)+\beta w(v)}\)

where \(a = up^\alpha, b = vp^\beta, u, v\) are \(p\)-adic units in \(\mathbb{Q}_p\) and \(\epsilon(u), w(u)\) denote the class modulo 2 of \(\frac{u-1}{2}\) and of \(\frac{u^2-1}{8}\), respectively. Therefore if \((N_1,d)_p = -1\) for some prime \(p\) then \(p\) divides \(N_1\) or \(d\). Since \(\Pi_p(a,b)_p = 1\), one can deduce that \((N_1,d)_p = -1\) for some odd prime \(p\) that is dividing \(N_1\) or \(d\).

**Corollary 2.4.** Let \(N\) be an odd square-free integer such that there exists a divisor \(N_1\) of \(N\), \(N_1 \equiv 1(\text{mod } 4)\) and \((N_1,d)_p = -1\) for some \(p\). Then \(X^d(N)(\mathbb{Q}_p) = \emptyset\).
Proof. Say $p|N_1$. Since $(N_1,d)_p = \left( \frac{d}{p} \right) = -1$, $p$ is inert in $K$. Since $N_1 \equiv 1 \mod 4$, either $p \equiv 1 \mod 4$ or $p \equiv 3 \mod 4$ and there is another divisor $p'$ of $N_1$ that is also congruent to $3 \mod 4$. If $p \equiv 1 \mod 4$, then $X^d(N)(\mathbb{Q}_p) = \emptyset$ and if $p \equiv 3 \mod 4$, then there are at least 2 primes dividing $N_1$ that are congruent to $3 \mod 4$, hence $X^d(N)(\mathbb{Q}_p) = \emptyset$, by part 3 of Theorem 1.1.

Say $p|d$. Since $(N_1,d)_p = \left( \frac{N_1}{p} \right) = -1$, $p$ is inert in $\mathbb{Q}(\sqrt{N_1})$. Let $H_0$ denote the ring class field of the order $\mathbb{Z}[\sqrt{N}]$. Then $H_0 = \mathbb{Q}(\sqrt{-N},j(\sqrt{-N}))$ and $H_0 \cap \mathbb{R} = \mathbb{Q}(j(\sqrt{-N}))$ by class field theory. Since $N_1 \equiv 1 \mod 4$, $\mathbb{Q}(\sqrt{N_1})$ lies in the genus field of $\mathbb{Q}(\sqrt{-N})$, hence $\mathbb{Q}(\sqrt{N_1}) \subseteq \mathbb{Q}(j(\sqrt{-N}))$. This shows that there is no prime of $\mathbb{Q}(j(\sqrt{-N}))$ lying above $p$ with residue degree 1, thus $X^d(N)(\mathbb{Q}_p) = \emptyset$ by part 5 of Theorem 1.1. □

Corollary 2.5. Let $N$ be an even square-free integer such that there exists an even divisor $N_1$ of $N$, $N/N_1 \equiv 1 \mod 4$ and $(N_1,d)_p = -1$ for some $p$. Then $X^d(N)(\mathbb{Q}_p) = \emptyset$.

Proof. Say $p|N_1$. Since $(N_1,d)_p = \left( \frac{d}{p} \right) = -1$, $p$ is inert in $K$ and since $\prod_{\nu}(a,b)_\nu = 1$, we can assume that $p$ is odd. By part 3 of Theorem 1.1 $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if and only if $N = 2p\Pi_iq_i$ such that $p$ is 3 mod 4 and $q_i$ is 1 mod 4. Therefore $N_1 = 2p\Pi_iq_i$ for some $q_i$ congruent to 1 mod 4, this contradicts to the assumption that $N/N_1 \equiv 3 \mod 4$, hence $X^d(N)(\mathbb{Q}_p) = \emptyset$.

Say $p|d$, then $(N_1,d)_p = \left( \frac{N_1}{p} \right) = -1$. Again we can assume $p$ is odd. Let $H_0$ denote the ring class field of the order $\mathbb{Z}[\sqrt{N}]$. Then $H_0 = \mathbb{Q}(\sqrt{-N},j(\sqrt{-N}))$ and $H_0 \cap \mathbb{R} = \mathbb{Q}(j(\sqrt{-N}))$ by class field theory. If $N$ has odd number of prime factors that is congruent to $3 \mod 4$ then $N_1$ has even number of prime factors that is congruent to $3 \mod 4$ and genus field of $\mathbb{Q}(\sqrt{-N})$ includes $\sqrt{2}, \sqrt{-l_i}, \sqrt{q_i}$ where $l_i$ and $q_i$ are prime factors of $N$ that are 3 and 1 mod 4 respectively. If $N$ has even number of prime factors that is congruent to $3 \mod 4$ then $N_1$ has odd number of prime factors that is congruent to $3 \mod 4$ and genus field of $\mathbb{Q}(\sqrt{-N})$ includes $\sqrt{2}, \sqrt{-l_i}, \sqrt{q_i}$. In either case, the genus field includes $\mathbb{Q}(\sqrt{N_1})$. And since $H_0 \cap \mathbb{R} = \mathbb{Q}(j(\sqrt{-N}))$, $\mathbb{Q}(\sqrt{N_1})$ lies in $\mathbb{Q}(j(\sqrt{-N}))$, where $H_0$ is ring class field of $\mathbb{Z}[\sqrt{-N}]$. Hence every prime of $\mathbb{Q}(\sqrt{-N})$ that is lying above $p$ has a residue degree greater than 1, therefore $X^d(N)(\mathbb{Q}_p) = \emptyset$ by part 5 of Theorem 1.1. □

Corollaries 2.3 and 2.5 imply Theorem 2.3.

Another result about existence of local points on $X^d(N)$ is given by Clark in [1]. Generalizing the techniques that is used in this proof we
prove part 3 of Theorem 1.1. As a result, the following theorem follows from part 3 of Theorem 1.1.

**Theorem 2.6.** Let \( N \) be a prime number congruent to 1 mod 4, \( d = p^* = (-1)^{\frac{N-1}{p}}p \) where \( p \) is a prime different than \( N \) and \( \left( \frac{N}{p} \right) = -1 \). Then \( X^d(N)(\mathbb{Q}_N) = \emptyset \).

In the same paper ([1](#)), it was asked whether or not \( p \) and \( N \) were the only primes that \( X^d(N) \) fails to have local points. We prove that the answer is ‘yes’ in Corollary 5.13.

**Corollary:** Let \( N \) be a prime congruent to 1 mod 4, \( p \) be an odd prime such that \( \left( \frac{N}{p} \right) = -1 \) and \( d = p^* \) then

1. \( X^d(N)(\mathbb{Q}_N) = \emptyset \)
2. \( X^d(N)(\mathbb{Q}_p) = \emptyset \)
3. \( X^d(N)(\mathbb{Q}_\ell) \neq \emptyset \) for any other prime \( \ell \) different than \( p \) and \( N \).

3. **Real Points**

We will keep the same notation as in the previous sections. Given a square-free integer \( N \), a quadratic number field \( K := \mathbb{Q}(\sqrt{d}) \) and a prime \( p \) what can be said about \( X^d(N)(\mathbb{Q}_p) \), where \( X^d(N) \) is the twist of \( X_0(N) \) with \( w_N \)? and \(< \sigma > := \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \). Since we are dealing with local points, by abuse of notation we regard \( \sigma \) as the generator of the extension \( K_\nu \) over \( \mathbb{Q}_p \), where \( \nu \) is a prime of \( K \) lying over \( p \). Let \( k \) be the residue field and \( R \) be the valuation ring of \( K_\nu \).

We start with the real points of \( X^d(N) \):

**Proposition 3.1.** \( X^d(N)(\mathbb{R}) \neq \emptyset \).

**Proof.** If \( K \) is an imaginary quadratic field, \( K_\nu = \mathbb{C}, \mathbb{Q}_\infty = \mathbb{R} \) and \( \sigma \) is complex conjugation. Say \( E \) has CM with the full ring of integers of \( \mathbb{Q}(\sqrt{-N}) \). Then \( E \) corresponds to the lattice \( a \) \([1, \sqrt{-N}] \) or \( b \) \([1, (1 + \sqrt{-N})/2] \) and \( E \) induces a fixed point, \( P \) of \( w_N \), in \( X_0(N)(\mathbb{C}) \).

Now for \( a \), conjugate of \([1, \sqrt{-N}] \) is \([1, -\sqrt{-N}] \). And the matrix \[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
sends the first basis to the second.

For \( b \), conjugate of \([1, (1 + \sqrt{-N})/2] \) is \([1, (1 - \sqrt{-N})/2] \) and the matrix \[
\begin{pmatrix}
1 & 0 \\
1 & -1
\end{pmatrix}
\]
sends the first basis to the second.

Since these matrices are invertible in \( \mathbb{Z} \), they both induce the same elliptic curve i.e. \( P \) is a real point of \( X^d(N) \) that is fixed by \( w_N \), hence induces a point of \( X^d(N)(\mathbb{R}) \).

If \( K \) is real quadratic then \( \sigma \) induces trivial map and same argument above shows that \( X^d(N)(\mathbb{R}) \neq \emptyset \). \(\square\)
Now let’s assume $p$ splits in $K$ then a copy of $K$ is in $Q_p$. Since $X_0(N)$ and $X^d(N)$ are isomorphic over $K$ and $X_0(N)(Q_p)$ is non-empty, $X^d(N)(Q_p)$ is non-empty.

Therefore, $X^d(N)$ might fail to have $p$-adic points only for finite primes that are inert or ramified in $K$.

4. PRIMES THAT ARE INERT IN $Q(\sqrt{-N})$

For the two main cases that we are dealing—the inert case and ramified case—we will be using different tools. For the inert case, since we have the notion of etale descent for $X_0(N)/R$, existence or non-existence of local points will be showed by checking the existence of points over the corresponding special fiber. For this, the following version of Hensel’s Lemma will be used:

**Lemma 4.1.** Let $K$ be a complete local ring, $R$ its valuation ring and $k$ its residue field. Let $X$ be a regular scheme over $S := \text{Spec}(R)$ and $f : X \to S$ a proper flat morphism. Say $X_\eta := X \times_S \text{Spec}(K)$ is the generic fiber and $X_o := X \times_S \text{Spec}(k)$, the special fiber. Then each $K$-rational point of $X_\eta$ corresponds to a smooth $k$-rational point of $X_o$.

We will also use the following theorems of Deuring:

**Theorem 4.2.** Let

- $\tilde{E}$ be an elliptic curve over a number field $L$ that has CM by $Q(\sqrt{-N})$;
- $p$ be a rational prime;
- $\beta$ be a prime of $L$ lying over $p$ such that $\tilde{E}$ has good reduction over $\beta$ and
- $E$ denote the reduction of $\tilde{E}$.

Then $E$ is supersingular if and only if $p$ is ramified or inert in $Q(\sqrt{-N})$.

**Theorem 4.3.** (Deuring Lifting Theorem, see [10] Theorem 14) Let $E$ be an elliptic curve over a finite field $k$ of characteristic $p$, and let $\alpha$ be an element of $\text{End}(E)$. Then there exists:

- an elliptic curve $\tilde{E}$ over a number field $B$
- an endomorphism $\tilde{\alpha} \in \text{End}(\tilde{E})$ and
- a place $\beta$ of $B$ lying over $p$

such that

- the reduction of $\tilde{E}$ mod $\beta$ is $E$
- the reduction of $\tilde{\alpha}$ mod $\beta$ is $\alpha$ and
- $|k| = p^f$ where $f$ is the inertia degree of $\beta$ over $p$.

When $p$ is inert in $Q(\sqrt{d})$, $\sigma$ induces the non-trivial map (Frobenius) on $\text{Gal}(k/F_p)$, where $k = F_p^{2^f}$. We have different cases according to $p|N$ or not.
4.1. $p$ dividing the level. Say $p \mid N$, $\nu$ the prime of $K$ lying over $p$ and $R$ is the ring of integers of the localization $K_{\nu}$.

In Mazur[12] and Deligne-Rapaport[3] there is a model of $X_0(N)/\mathbb{Z}_p$ whose special fiber $X_0(N)_{\mathbb{F}_p}$ is 2 copies of $X_0(N/p)_{\mathbb{F}_p}$ glued along supersingular points twisted by first power Frobenius. The Atkin-Lehner involution $w_N$ interchanges the two branches and Frobenius stabilizes each branch, see Figure 1. A regular model $\tilde{X}_0(N)$ can be obtained by blowing-up $(|\text{Aut}(E,C)|/2 - 1)$-many times at each supersingular point. Actions of Frobenius and $w_N$ extends to regularization as well.

\[ \bar{x} = \{ x, x^{\text{frob}} \} \]

**Figure 1.** Special fiber of $X_0(N)/\mathbb{Z}_p$

Since $p$ is ramified in $\mathbb{Q}(\sqrt{-N})$, by Theorem 4.2 any $w_N$-fixed point is supersingular. Say $x$ is a $w_N$-fixed supersingular(hence singular) point of $X_0(N)_{\mathbb{F}_p}$. To have a regular model, we need to blow-up at each supersingular point $(|\text{Aut}(E,C)|/2 - 1)$-times where $(E,C)$ is on $X_0(N/p)_{\mathbb{F}_p}$. To keep track of the different schemes, we need to introduce some notation. As introduced in the beginning, $X_0(N)$ denotes the model of $X_0(N)$ over $\mathbb{Z}_p$(not necessarily regular). Let $\tilde{X}_0(N)$ denote the regularization of $X_0(N)$ after blow-ups, and $\tilde{X}_0(N)_{\mathbb{F}_p}$ be its special fiber.

We can define a model of $X^d(N)$ over $\mathbb{Z}_p$ as a descent to $\text{Spec}\mathbb{Z}_p$ of $X_0(N)_{x_{\text{Spec}\mathbb{Z}_p}}$Spec$R$ by a descent datum twisted by $w_N$. Note that extension $R/\mathbb{Z}_p$ is Galois since $p$ is inert in $K$. This model will be denoted as $X^d(N)_{/\mathbb{Z}_p}$. Our aim is to use Hensel’s Lemma(Lemma 4.1) to make conclusions about $\mathbb{Q}_p$-rational points of the generic fiber of $X^d(N)_{/\mathbb{Z}_p}$. In order to do this, we must first show that $X^d(N)_{/\mathbb{Z}_p}$ is regular.

**Proposition 4.4.** Let $R \hookrightarrow S$ be a flat extension of local rings. If $S$ is regular than so is $R$.

**Proof.** Matsumura, Commutative Algebra, page 155, Theorem 51. \qed

**Proposition 4.5.** $X^d(N)_{/\mathbb{Z}_p}$ is a regular model of $X^d(N)$. 

Proof. By definition of etale descent, $\mathcal{X}^d(N)$ and $\tilde{\mathcal{X}}_0(N)$ are isomorphic over $R$. Since $R$ is an unramified extension of $\mathbb{Z}_p$, $\tilde{\mathcal{X}}_0(N) \times_{\mathbb{Z}_p} R$ is regular, hence $\mathcal{X}^d(N) \times_{\mathbb{Z}_p} R$ is also regular. By Proposition 4.4 $\mathcal{X}^d(N)/\mathbb{Z}_p$ is regular.

Now we will give a necessary condition for the existence of a smooth point on $X_0^d(N)(\mathbb{F}_p)$.

**Proposition 4.6.** There exists a smooth point on $X_0^d(N)(\mathbb{F}_p)$ if and only if there is a point on $X_0(N/p)(\mathbb{F}_p)$ corresponding to a supersingular elliptic curve with a degree-4 automorphism.

**Proof.** As explained above, $X_0^d(N)$ is generic fiber of $\mathcal{X}^d(N)/\mathbb{F}_p$, which is etale descent of $\mathcal{X}_0(N)/\mathbb{F}_p$ by $R/\mathbb{Z}_p$. Since $w_N$ interchanges the branches of $X_0(N)/\mathbb{F}_p$, $\mathbb{F}_p$-rational points on the special fiber $X_0^d(N)/\mathbb{F}_p$ are coming from supersingular points of $X_0(N)/\mathbb{F}_p$, which are all singular. And we must keep in mind that at each singular(hence supersingular) point we have $(|\text{Aut}(E,C)|/2 - 1)$-many exceptional lines. Note that automorphism group of an elliptic curve over a field of characteristic $\ell$ is $\mu_2, \mu_4$ or $\mu_6$ if $\ell$ is not 2 or 3 where $\mu_s$ denotes the group of primitive $s$-th roots of unity. If $\ell = 2$ or 3 and $E$ is the unique supersingular elliptic curve in characteristic $\ell$ then $\text{Aut}(E)$ is $C_3 \times \{\pm 1, \pm i, \pm j, \pm k\}$ or $C_3 \rtimes C_4$ respectively, where $C_m$ denotes the cyclic group of order $m$.

Therefore if $|\text{Aut}(P)| = 4n$ for $n > 1$, there is an element of order 4 in $\text{Aut}(P)$ and the number of blow-ups is $2n - 1$ which is odd. Since we have odd number of exceptional lines, then there is one line $L/\mathbb{F}_p$ that is fixed by the action of $w_N$(see the second column of Figure 2). On this line $L$ the points $A$ and $B$ are singular and fixed by $w_N \circ \sigma$ but these are not the only fixed points. The action of $\sigma \circ w_N$ on zeroth, first and second cohomology of $L$ has traces 1, 0 and $p$ respectively. Then by Lefschetz fixed point theorem, there is a smooth $w_N \circ \sigma$-point on this exceptional line $L$. Therefore if we have a supersingular point with a degree-4 automorphism then there is a smooth point on $X^d(N)(\mathbb{F}_p)$.

For the reverse direction, say there is no such supersingular point $P$ with a degree-4 automorphism. If $|\text{Aut}(P)|$ is 2, then $\mathcal{X}_0(N)$ is already regular but all $\mathbb{F}_p$-rational points of $X^d(N)$ are singular.

If $|\text{Aut}(P)| = 6$, then we replace this point by 2 exceptional lines over $\mathbb{F}_p$ and $\sigma \circ w_N$ interchanges these lines. Each of these exceptional lines cuts one of the branches and also the other exceptional line once. Call the intersection point of these lines as $x$, then $\sigma(x) = x$ and it is the only point fixed by the action of $\sigma$ on these lines. Then $w_N(x) = w_N(\sigma(x)) = \sigma(w_N(x))$, hence $w_N(x)$ is also fixed by $\sigma$ i.e. $w_N(x) = x = \sigma(x)$. Thus $x$ induces an $\mathbb{F}_p$-rational point of $X^d(N)$. However $x$ is a singular point. For a picture of this situation we refer to the table at the end of this section. □
Using Proposition 4.6 and Hensel’s Lemma we get the following:

**Corollary 4.7.** There exists a point on $X^d(N)(\mathbb{Q}_p)$ if and only if there is a supersingular point with a degree-4 automorphism.

**Theorem 4.8.** Let $N$ be a square-free positive integer and $p$ be an odd prime that is inert in $K$ and dividing $N$ then $X^d(N)(\mathbb{Q}_p) \neq \emptyset$ if and only if $p$ is $3 \mod 4$ and $N$ is of the form:

1. $p\Pi_iq_i$ such that all $q_i \equiv 1 \mod 4$ or
2. $2p\Pi_iq_i$ such that all $q_i \equiv 1 \mod 4$.

**Proof.** By Corollary 4.7 above, a local point exists if and only if there is a supersingular point $(E, C)$ on $X_0(N)/\mathbb{F}_p$ with automorphism group divisible by 4. In particular we want $j = 1728$ to be a supersingular $j$-invariant over characteristic $p$.

In order to have such a point, $\mathbb{Z}/4\mathbb{Z}$ must inject into $(\mathbb{Z}/q_i\mathbb{Z})^*$ for every odd prime divisor $q_i$ of $N$ that is not $p$. Therefore if $N$ is odd, $N = p\Pi_iq_i$ such that $p \equiv 3 \mod 4$ and all $q_i \equiv 1 \mod 4$.

Since the degree-4 automorphism $[i]$ sends a 2-torsion point $(x, 0)$ of $E_{1728} : y^2 = x^3 + x$ to $(-x, 0)$, $[i]$ fixes the cyclic-2 subgroup $<(0, 0)>$ of $E_{1728}[2]$. Hence if $N$ is even and there is a supersingular point on $X_0(N)/\mathbb{F}_p$ with automorphism group divisible by 4, then $N = 2p\Pi_iq_i$ such that $p \equiv 3 \mod 4$ and all $q_i \equiv 1 \mod 4$.

Conversely, if $N$ is one of the given form then 1728 is a supersingular $j$-invariant. Let $E_{1728}$ be the elliptic curve over $\mathbb{F}_p$ of which $j$ invariant is 1728. Then $[i]$ is in $Aut(E_{1728})$ and acts on $E_{1728}[\Pi_iq_i]$. The automorphism $[i]$ stabilizes a cyclic-$\Pi_iq_i$ subgroup if and only if $[i]$ stabilizes cyclic-$q_i$ subgroups of $E_{1728}[q_i] = \mathbb{Z}/q_i\mathbb{Z} \times \mathbb{Z}/q_i\mathbb{Z}$ for all $i$. The automorphism $[i]$ can be seen as an element of $GL_2(\mathbb{F}_{q_i})$ and it stabilizes a cyclic subgroup of order $q_i$ if and only if $[i]$ has eigenvalues defined over $\mathbb{F}_p$. If $q_i$ is odd then minimal polynomial of $[i]$ is $x^2 + 1$, this is equivalent to say that $q_i \equiv 1 (\mod 4)$ for all $i$. If $q_i = 2$ then minimal polynomial of $[i]$ is $x + 1$ and $[i]$ fixes the cyclic-2 subgroup $<(0, 0)>$ of $E_{1728}[2]$. \hfill \Box

For $p = 2$ inert in $K$ and $N$ even we get the following result:

**Theorem 4.9.** If 2 is inert in $K$ and dividing $N$ then $X^d(N)(\mathbb{Q}_2) \neq \emptyset$ if and only if $N = 2\Pi_iq_i$ such that $q_i \equiv 1 \mod 4$.

**Proof.** Over $\mathbb{F}_2$, 1728 is the only supersingular $j$-invariant and $|Aut(E_{1728})| = 24$. Say $q_i$ is a prime dividing $N/2$. Since $N$ is square-free, $q_i$ is odd.

By Corollary 4.7 above, a $\mathbb{Q}_2$-point exists if and only if there is a supersingular point $(E_{1728}, C)$ on $X_0(N)/\mathbb{F}_2$ with automorphism group divisible by 4. In order to have such a point, $\mathbb{Z}/4\mathbb{Z}$ must inject into $(\mathbb{Z}/q_i\mathbb{Z})^*$ for every odd prime divisor $q_i$ of $N/2$, hence $N = 2\Pi_iq_i$ such that $q_i \equiv 1 \mod 4$ for all $i$. \hfill \Box
For the converse, the argument is the same as in the corresponding part of Theorem 4.8.

\[
\begin{array}{|c|c|c|}
\hline
\text{Over } \mathbb{Q}_p \text{ or } K_v \text{ such that} & \text{Singularity type is } A_{k-1} \\
K_v/Q_p \text{ is unramified} & \text{where } k = h/2 \\
\hline
\text{no smooth point} & h = 2, A_0 & h = 4, A_1 \\
\exists \text{ point} & h = 6, A_2 & (p = 2) \\
\text{no smooth point} & h = 8, A_3 & \\
\hline
\end{array}
\]

Figure 2. Blow-ups

4.2. \textbf{p does not divide }N. We will be using the same notations introduced in the previous subsection, in particular \(X^d(N)/\mathbb{Z}_p\) denotes etale descent \(X_0(N)\) from \(R\) to \(\mathbb{Z}_p\). If \(p\) is not dividing \(N\) then the following models are smooth, in particular regular:

- \(X_0(N)/\mathbb{Z}_p\),
- \(X_0(N) \times_{\mathbb{Z}_p} R\) (since \(R/\mathbb{Z}_p\) is unramified) and
- \(X^{d}(N)/\mathbb{R} := X^d(N) \times_{\mathbb{Z}_p} R\) (since \(X^{d}(N)/\mathbb{R}\) is isomorphic to \(X_0(N) \times_{R} \mathbb{R}\) )

By Proposition \(\square\), \(X^d(N)/\mathbb{Z}_p\) is also regular. We will construct a point on the special fiber \(X^d(N)(\mathbb{F}_p)\) using theory of CM curves and then by Hensel’s Lemma we will be done.

Let \(\Sigma_N\) be the set of tuples \((E, C)\) such that \(E\) is a supersingular elliptic curve over characteristic \(p\) and \(C\) is cyclic group of order \(N\). We start by studying the action of the involution \(w_N \circ \sigma\) on \(\Sigma_N\).

**Definition 4.10.** Let \(B\) be the unique quaternion algebra over \(\mathbb{Q}\) that is ramified only at \(p\) and infinity. An order \(O\) of \(B\) is of level \(N\) if

- \(q \neq p, O_q = O \otimes \mathbb{Z} \mathbb{Z}_q \cong \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ N\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix}\)
- \(O_p \cong \{ \alpha \beta \overline{\beta} \overline{\alpha} \mid \alpha, \beta \in R\}\) where \(R\) is the ring of integers of the unique unramified quadratic extension of \(\mathbb{Q}_p\).
Proposition 4.11. Let $B := \text{End}(E) \otimes \mathbb{Z} \mathbb{Q}$, $E$ is a supersingular elliptic curve over $\mathbb{F}_p^2$ and $C$ a cyclic subgroup of $E$ of order $N$. Then $B$ is the unique quaternion algebra over $\mathbb{Q}$ which is ramified only at $p$ and $\infty$, $\text{End}(E)$ is a maximal order in $B$ and $\text{End}(E, C)$ is an Eichler order of level $N$.

Proof. For $B$ being the claimed quaternion algebra and $\text{End}(E)$ its maximal order we refer to Silverman \cite{Silverman} Chapter 5. Assuming this let $(E, C)$ be an element of the set $\Sigma_N$. The map $\pi : E \to E/C$ induces an isomorphism between $B = \text{End}(E) \otimes \mathbb{Q}$ and $\text{End}(E/C) \otimes \mathbb{Q}$ via the map $f \mapsto \pi f \pi^{-1}$. Since $\text{End}(E/C)$ is a maximal order in the latter, its image is a maximal order in the former. Moreover the intersection of these two maximal orders is $\text{End}(E, C)$, hence an Eichler order. \qed

It’s important to understand what kind of maps are in $R = \text{End}(E, C)$. Since we have an algebraic way to study these rings, properties of endomorphisms of $(E, C)$ can be also understood by studying orders embedded in $R$.

Definition 4.12. Let $L$ be an imaginary quadratic number field, $O$ an order of $L$, $\alpha : L \rightarrow B$ an algebra embedding such that $\alpha(L) \cap R = \alpha(O)$ where $R$ is an Eichler order of level $N$ in $B$ as above. Then the pair $(R, \alpha)$ is called as an optimal embedding of $O$.

The following theorem of Eichler states conditions for existence of an optimal embedding:

Proposition 4.13. Given an $R$, $B$ as above and $L = \mathbb{Q}(\sqrt{M})$, an optimal embedding $(R, \alpha)$ of an order $O$ of $L$, exists if and only if

- $M < 0$, $p$ is inert or ramified in $L$ and $p$ is relatively prime to the conductor of $O$ and
- $q$ splits or ramifies in $O$ for every $q$ dividing $N$.

For any $q | N$ and $q' := N/q$, let $w_q$ be the Atkin-Lehner operator that sends $(E, C) \mapsto (E/q'C, E[q] + C/q'C)$ where $E[q]$ is the kernel of multiplication by $q$. Each $w_i$ is an involution and $w_i \circ w_j = w_j \circ w_i = w_{ij}$ for every coprime $i, j$. We have another operator, Frobenius acting on the set $\Sigma_N$ and remember that $\sigma$ is acting as Frobenius in the inert case. The following result shows that Frobenius also can be seen as an Atkin-Lehner operator on the set $\Sigma_N$.

Theorem 4.14. (see Chapter V, Section 1 of \cite{Silverman}) The involution $w_p$ permutes the two components of $X_0(Np)_{\mathbb{F}_p}$. It acts on the set of singular points of $X_0(Np)_{\mathbb{F}_p}$ as the Frobenius morphism $x \mapsto x^p$.

Let $\psi$ be a map from $X_0(N)_{\mathbb{F}_p}$ to $X_0(Np)_{\mathbb{F}_p}$, an isomorphism onto one of the two components. The map $\psi$ takes the supersingular locus of $X_0(N)_{\mathbb{F}_p}$ to the supersingular locus of $X_0(Np)_{\mathbb{F}_p}$. The set $\Sigma_N$ defined
in the beginning of the section is the supersingular locus of $X_0(N)_{/ \mathbb{F}_p}$. The Atkin-Lehner operator $w_{Np}$ acts on $X_0(Np)$, in particular acts on $\psi(\Sigma_N)$. When we say the action of $w_{Np}$ on $\Sigma_N$, we actually mean the action of $w_{Np}$ on $\psi(\Sigma_N)$.

**Corollary 4.15.** Given $B$ there is an embedding of $\mathbb{Z}[\sqrt{-pN}]$ into some Eichler order $R$ of level $N$.

**Proof.** Using the first part of Proposition 4.14, quadratic imaginary field $L = \mathbb{Q}(\sqrt{-pN})$ embeds in $B$ since $p$ is ramified in $\mathbb{Q}(\sqrt{-pN})$ and the embedding is optimal for the maximal order of $\mathbb{Q}(\sqrt{-pN})$ since any prime $q$ dividing $N$ is ramified in $\mathbb{Z}[\sqrt{-pN}]$. If the maximal order $O_L$ of $L$ is $\mathbb{Z}[\sqrt{-pN}]$ we are done. Say $\mathbb{Z}[\sqrt{-pN}]$ is of conductor 2 in $O_L$ then $\alpha(\mathbb{Z}[\sqrt{-pN}]) \subset \alpha(O_L) = \alpha(L) \cap R$, hence $\mathbb{Z}[\sqrt{-pN}]$ embeds in $R$.

**Corollary 4.16.** If there is an embedding $\alpha$ of $\mathbb{Z}[\sqrt{-pN}]$ into $R$ for some Eichler order $R = \text{End}(E,C)$ of level $N$ then there is a fixed point of $w_{Np}$ in $\Sigma_N$.

**Proof.** By assumption there exists an element whose square is $-pN$ in $R$ i.e. an endomorphism of degree $Np$ of $(E,C)$, in particular an endomorphism of degree $Np$, say $f$, of $E$. Using Deuring’s Lifting Theorem (Theorem 4.3) this endomorphism and $E$ can be lifted to characteristic 0 i.e. $(\tilde{E}, \text{ker}(\tilde{f}))$ ($E$ and $f$ lifted to characteristic 0) is in $X_0(Np)(\mathbb{Q})$ and is fixed by $w_{Np}$. The reduction of this point mod $p$ is a supersingular point on $X_0(Np)_{/ \mathbb{F}_p}$ that is fixed by $w_{Np}$ since $E$ has no $p$-torsion $|\text{ker}(f)| = N$, $(E, \text{ker}(f))$ is identified with a point in $\Sigma_N$ i.e. is in $\psi(\Sigma_N)$.

**Example 4.17.** Let $p = 7$ and $N = 5$. Since $(\frac{-5}{7}) = 1, $by Theorem 4.2, reduction of any elliptic curve which has CM by $\mathbb{Q}(\sqrt{-5})$ over $p$ is ordinary. Hence there isn’t any $w_5$-fixed point in $\Sigma_5$ i.e. there is no optimal embedding of $\mathbb{Z}[\sqrt{-5}]$ into any $R$ where $R$ is an Eichler order of level 5 in the quaternion algebra $\mathbb{Q}_{7,\infty}$. In fact, there isn’t any embedding of $\mathbb{Q}(\sqrt{-5})$ into $\mathbb{Q}_{7,\infty}$ since 7 splits in $\mathbb{Q}(\sqrt{-5})$ the localization of $\mathbb{Q}(\sqrt{-5})$ at the primes lying above 7, is not even a field.

**Theorem 4.18.** If $p$ is inert in $\mathbb{Q}(\sqrt{d})$ and $p \nmid N$ then $X^d(N)(\mathbb{Q}_p) \neq \emptyset$.

**Proof.** By Corollaries 4.15 and 4.16, there is a point $x \in \Sigma_N$ such that $w_{Np}(x) = x$. Since $(p, N) = 1$ we have $w_N \circ w_p(x) = w_{Np}(x) = x$. By Theorem 4.14, $w_p$ acts as $\text{frob}_p$ on the set $\Sigma_N$, hence $w_N \circ \text{frob}_p(x) = x$. By theory of etale descent this gives a point in $X^d(N)(\mathbb{F}_p)$ and since $p \nmid N$, we have smooth model, by Hensel’s Lemma (Lemma 4.1) we are done.
Let $p$ be a prime that is ramified in the quadratic field $K$ but not in $\mathbb{Q}(\sqrt{-N})$. Let $\nu$ be the prime of $K$ lying over $p$, and $R$ be the ring of integers of $K_{\nu}$. Note that we don’t have a good model for $X^d(N)$ over $\mathbb{R}$, since $R/\mathbb{Z}_p$ is not etale. By assumption, $p \nmid N$. Then by a well-known theorem of Igusa, $\mathcal{X}_0(N)$ is a smooth $\mathbb{Z}[1/N]$-scheme, hence for any $p \nmid N$ the special fiber of $\mathcal{X}_0(N) \to \text{Spec}(R)$ is smooth over the residue field $R/\nu$.

Since $p$ is ramified, the residue field $R/\nu$ is $\mathbb{F}_p$, and the induced action of $\sigma$ on the residue field is trivial. In this setting, our approach will be to produce points on $X_0(N)(\mathbb{Q}_p)$ which are fixed by $w_N$ and which are thus CM points. Note that such points are $\mathbb{Q}_p$-rational points of $X^d(N)$. The main tool in showing the existence of such points is Deuring’s Lifting Theorem which is stated as Theorem 4.3. This theorem allows us to lift $w_N$-fixed points of $X_0(N)(\mathbb{F}_p)$ to $w_N$-fixed points of $X_0(N)(\mathbb{Q}_p)$, as Proposition 5.2 demonstrates. Before stating the proposition we need to recall the following facts about CM elliptic curves.

If $E$ corresponds to a fixed point of $w_N$ on $X_0(N)(\mathbb{Q})$ then $E$ has an endomorphism whose square is $[-N]$. Hence $\text{End}(E)$ contains a copy of $\mathbb{Z}[\sqrt{-N}]$ and can be embedded in $\mathbb{Z}[\mathbb{Z}+\sqrt{-D}]$ where $D$ is the discriminant of the CM field $M := \mathbb{Q}(\sqrt{-N})$. If $N \equiv 1$ or $2$ mod $4$ then these two orders are the same, hence $\text{End}(E)$ is the maximal order of $M$. Otherwise, $\text{End}(E)$ is an order of conductor $2$ in the maximal order. Throughout the section we will use the following notation:

- $O := \mathbb{Z}[\sqrt{-N}]$
- $h$ denotes the class number of $O$
- $E$ is an elliptic curve such that $\text{End}(E)$ contains $\mathbb{Z}[\sqrt{-N}]$
- $H_O$ is the ring class field of $O$

Recall that by theory of CM, we have $h$-many elliptic curves which has CM by $O$ and their $j$-invariants are all conjugate.

**Proposition 5.1.** Let $E$ be an elliptic curve over a number field $B$ and $E$ has an endomorphism $\alpha_0$ whose square is $[-N]$. Then $(E, \ker(\alpha_0))$ is a $w_N$-fixed point on $X_0(N)(B)$.

**Proof.** By definition $(E, \ker(\alpha_0))$ is a $w_N$-fixed point of $X_0(N)(\mathbb{Q})$ and $E$ is defined over $B$, $\alpha_0$ is defined over $B(\sqrt{-N})$. Let $\phi$ be the generator of $\text{Gal}(B(\sqrt{-N})/B)$ then $\ker(\alpha_0)^\phi = \ker(\pm \alpha_0) = \ker(\alpha_0)$ since the only endomorphisms of $E$ whose square is $[-N]$ are $\pm \alpha_0$. Therefore $\ker(\alpha_0)$ is defined over $B$ as well. \qed

**Proposition 5.2.** Let $p$ be an odd prime. Any $w_N$-fixed point on $X_0(N)(\mathbb{F}_p)$ is the reduction of a $w_N$-fixed $\mathbb{Q}_p$-rational point on the
generic fiber of $X_0(N)$. Conversely, a $w_N$-fixed point on the generic fiber reduces to a $w_N$-fixed point on $X_0(F_p)$.

Proof. Say there is a $w_N$-fixed point on $X_0(N)(F_p)$. This point corresponds to an elliptic curve $E$ over $F_p$ which has an endomorphism $\alpha$ whose square is $[-N]$. By Theorem 4.3 we have the following:

- An elliptic curve $\tilde{E}$ over a number field $B$,
- an endomorphism $\alpha_0$ whose square is $[-N]$ and
- a prime $\beta$ over $p$ with residue degree 1.

such that
- Reduction of $\tilde{E}$ mod $\beta$ is $E$ and
- reduction of $\alpha_0$ mod $\beta$ is $\alpha$.

By Proposition 5.1, it is enough to check whether or not $B$ can be embedded in $Q_p$. Since we have $f(\beta|p) = 1$ it suffices to check that $B$ is unramified at $p$.

- Say $O = Z[\sqrt{-N}]$ is the maximal order of $Q(\sqrt{-N})$ then $H_O$ is the Hilbert class field hence $H_O/Q(\sqrt{-N})$ is unramified. By assumption $p$ is unramified in $Q(\sqrt{-N})$ hence $p$ is unramified in $H_O/Q$. Therefore $p$ is unramified in $B/Q$ since $B = Q(j(E))$ lies inside $H_O$.
- Say $O = Z[\sqrt{-N}]$ is order of conductor 2 in $O' = Z[\frac{1+\sqrt{-N}}{2}]$. Any odd prime (that is not lying over 2) is unramified in $H_O/Q(\sqrt{-N})$. Then the above argument shows that any $p > 2$ is unramified in $B/Q$.

Therefore $e(\beta|p) = 1$ and $B_\beta \cong Q_p$. Conversely, since the fixed locus of $w_N$ is proper, a $w_N$-fixed point on $X_0(N)(Q_p)$ reduces to a $w_N$-fixed point on $X_0(N)(F_p)$. \[\square\]

Proposition 5.2 shows that if $X_0(N)(F_p)$ contains a smooth $w_N$-fixed point, then $X^d(N)(Q_p)$ is nonempty. We now show the converse.

**Proposition 5.3.** Let $x$ be a point of $X_0(N)(K_\nu)$ such that $w_N(x^\sigma) = x$ then $x$ reduces to a $w_N$-fixed point on the special fiber of $X_0(N)/R$.

**Proof.** Note that $\sigma$ is not a morphism of $Spec(R)$-schemes. We define the map $\tilde{\sigma} : X_0(N) \to X_0(N)$ using the following diagram:

$$
\begin{array}{ccc}
Spec(R) & \xrightarrow{\sigma} & Spec(R) \\
\downarrow & & \downarrow \\
X_0(N) & \xrightarrow{\tilde{\sigma}} & X_0(N)
\end{array}
$$

Since $K_\nu/Q_\nu$ is ramified, $\sigma$ induces the trivial action on the residue field $R/\nu$, therefore $\sigma$ induces trivial action on the special fiber:
We now add to the picture the Atkin-Lehner involution \( w_N \) which is a morphism of \( \text{Spec}(R) \)-schemes. Every point on \( X_0(N)(K_\nu) \) extends to a morphism \( \phi : \text{Spec}(R) \to X_0(N) \) by properness, and if the point of \( X_0(N)(K_\nu) \) is fixed by \( w_N \circ \sigma \) then the morphism \( \phi \) is preserved under composition with \( w_N \circ \hat{\sigma} \). To be more precise, let \( x \) be a point in \( X_0(N)(K_\nu) \) such that \( w_N \circ \hat{\sigma}(x) = x \). By properness, \( x = \phi \circ i \) where is the injection \( i : \text{Spec}(K_\nu) \to \text{Spec}(R) \), then \( w_N \circ \hat{\sigma} \circ \phi \circ i = \phi \circ i \), hence \( w_N \circ \hat{\sigma} \circ \phi = \phi \).

And the diagram shows that the restriction of \( \phi \) to the special fiber \( \tilde{p} : \text{Spec}(R/\nu) \to X_0(N) \times_R \text{Spec}(R/\nu) \) is a \( w_N \)-fixed point on \( X_0(N)(\mathbb{F}_p) \).

\[ \square \]

In fact the Proposition 5.3 is true even if \( p \) is ramified in \( \mathbb{Q}(\sqrt{-N}) \), in order to have a \( K_\nu \) rational \( w_N \circ \sigma \)-fixed point there must be a \( w_N \)-fixed \( \mathbb{F}_p \)-rational point of \( X_0(N) \). However the converse can not be concluded using Proposition 5.2. Since if \( p \) is ramified in \( \mathbb{Q}(\sqrt{-N}) \), \( p \) is ramified in \( H/\mathbb{Q} \), it is not immediately clear how \( B \) ramifies at primes over \( p \). Nonetheless, we have the following result for any prime
Let \( X \) be a prime ramified in \( K \), without any restriction on the decomposition of \( p \) in \( \mathbb{Q}(\sqrt{-N}) \):

**Proposition 5.4.** Let \( p \) be a prime ramified in \( K \). If \( X^d(N) \mathbb{Q}_p) \neq \emptyset \) then there is a \( w_N \)-fixed point in \( X_0(N)(F_p) \).

It remains to determine when there are \( w_N \)-fixed points of \( X_0(N)(F_p) \). Let \( S_N \) be the set of primes \( p \) such that there is a \( w_N \) fixed, \( F_p \)-rational point on the special fiber of \( X_0(N)/\mathcal{R} \). In Proposition 5.5 we will describe the set \( S_N \) explicitly as a Chebotarev set. In addition to the notation introduced in the beginning of the section, \( B \) denotes \( \mathbb{Q}(j(O)) \) where \( j(O) \) is \( j \)-invariant of the order \( O = \mathbb{Z}[\sqrt{-N}] \) and \( M := \mathbb{Q}(\sqrt{-N}) \).

**Proposition 5.5.** Let \( p \) be an odd prime and \( \mathcal{P} \) be a prime of \( \mathcal{M} \) lying over \( p \). Then \( p \) is in \( S_N \) if and only if there exists a prime \( \nu \) of \( B \) lying over \( p \) such that \( f(\nu|p) = 1 \) and \( \mathcal{P} \) totally splits in \( H/\mathcal{M} \).

**Proof.** We know that, \( H/\mathcal{M} \) is an abelian extension with Galois group \( G \) isomorphic to the ideal class group of \( \mathcal{M} \) and \([H : \mathcal{M}] = [B : \mathbb{Q}] \). The extension \( H/\mathbb{Q} \) is Galois with Galois group \( G \cong \mathbb{Z}/2\mathbb{Z} \) and the \( \mathbb{Z}/2\mathbb{Z} \)-fixed subfield of \( H \) is \( B \).

We want to know for which primes there is a \( w_N \)-fixed point on \( X_0(N)(F_p) \). We have shown in Proposition 5.2 that this is equivalent to the presence of a \( w_N \)-fixed point on \( X_0(N)(\mathbb{Q}_p) \).

Let \( P \) be a \( w_N \)-fixed point of \( X_0(N) \), defined over \( B \). Then \( P \) reduces to an \( F_p \)-point on the special fiber if and only if it is fixed by Frobenius. Recall that since \( B/\mathbb{Q} \) is unramified at \( p \), Frobenius acts on \( B \). Hence we should find out for which \( p \), there exists a prime \( \nu \) of \( B \) such that \( f(\nu|p) = 1 \).

Let \( \pi_p \) be the Frobenius at \( p \). The map \( \pi_p \) gives a conjugacy class \( \lambda \) in \( \text{Gal}(H/\mathbb{Q}) \) via Artin symbol.

Now we need to find the conjugacy classes of \( G \times \mathbb{Z}/2\mathbb{Z} \). Let \( g_i \in G \) and \( a, b \in \mathbb{Z}/2\mathbb{Z} \), then \((g_1, a) \ast (g_2, b) = (g_1 + \phi_0(g_2), a + b) \) where \( \phi_1(g) = -g \) and \( \phi_0 \) is identity for all \( g \in G \).

Let \((g, a) \) be in \( G \times \mathbb{Z}/2\mathbb{Z} \). If \( a = 0 \), \((g, a) \) is only conjugate to itself and \((-g, a) \). If \( a = 1 \) then \((g, a) \) is conjugate to \((-2x + g, a) \) for some \( x \in G \).

The conjugacy classes of \( G \times \mathbb{Z}/2\mathbb{Z} \) are thus as follows:

1. \(
\{(g, 0)\} \text{ one for each } g \in G[2] \).
2. \(
\{(g, 0), (-g, 0)\} \text{ one for each } g \in G - G[2] \).
3. \(
\{(g + 2x, 1) | x \in G\} \text{ one for each representative } g \text{ of } G/2G \).

Let \( \pi_p \) be the conjugacy class in \( \text{Gal}(H/\mathbb{Q}) \) given by a Frobenius at \( p \). A prime \( \nu \) of \( B \) over \( p \) has \( f(\nu|p) = 1 \) if and only if the conjugacy class \( \pi_p \) contains an element of the form \((0, y) \) for some \( y \) in \( \mathbb{Z}/2\mathbb{Z} \).
Hence, only allowed conjugacy classes are the trivial class and one of the classes of type (3).

Hence \( p \) is in \( S_N \) if and only if \( \pi_p \) contains an automorphism which fixes \( B \), which is equivalent to say that \( \mathcal{P} \) totally splits in \( H/M \).

\[ \square \]

Remark 5.6. Note that if \( p \) splits in \( M/Q \) then there are 2 primes of \( M \) lying over \( p \). If \( \mathcal{P} \) of \( M \) lying over \( p \) splits totally in \( H/M \) then \( p \) splits totally in \( H/Q \), hence it doesn’t matter which prime of \( M \) lying over \( p \) we take.

Remark 5.7. Proposition 5.5 determines for which \( p \) the field of definition of an elliptic curve whose endomorphism ring contains \( \mathbb{Z}[\sqrt{-N}] \) embeds into \( Q_p \). Then using Proposition 5.1 we get a \( w_N \)-fixed \( Q_p \)-rational point of \( X_0(N) \).

We have thus established a complete criterion for the non-emptiness of \( X^d(N)(Q_p) \), where \( p \) is an odd prime ramified in \( K \) but not in \( Q(\sqrt{-N}) \).

For \( Q_2 \)-points, we can argue as follows. Let \( d \) and \( N \) be square-free integers such that \( d \equiv 2, 3 \) and \( -N \equiv 1 \mod 4 \). Over \( F_2 \) there are 2 elliptic curves one with endomorphism ring \( \mathbb{Z}[\sqrt{\frac{1+i+j+k}{2}}] \), the ordinary one, and the other supersingular one whose endomorphism ring is Hurwitz quaternions, \( B(Z) := Z + iZ + jZ + \frac{1+i+j+k}{2}Z \). It is a maximal order in the quaternion algebra ramified only at 2 and infinity. Hence, if \( w_N \)-fixed point \((E, C)\) of \( X_0(N)(F_2) \) is ordinary - in particular \( N = 7 \)- then \( E \) can be lifted to an elliptic curve over a number field \( B \) that has complex multiplication by the maximal order of \( Q(\sqrt{-7}) \) by Theorem 4.3. If \( w_N \)-fixed point \((E, C)\) of \( X_0(N)(F_2) \) is the supersingular one, then the maximal order of \( Q(\sqrt{-N}) \) embeds in \( \text{End}(E) \) since the local order \( B(Z) \otimes Z_2 \) contains all elements of \( B(Q) \otimes Q_2 \) with norm in \( Z_2 \). Therefore by Theorem 4.3 \( E \) can be lifted to an elliptic curve over a number field \( B \) that has complex multiplication by the maximal order of \( Q(\sqrt{-N}) \). Hence we proved the following lemma:

**Lemma 5.8.** Let \((E, C)\) be a \( w_N \)-fixed point of \( X_0(N)(F_2) \). Then \( E \) can be lifted to an elliptic curve \( \tilde{E} \) over a number field \( B \) such that \( \tilde{E} \) has complex multiplication by the maximal order of \( Q(\sqrt{-N}) \).

Let \( \tilde{E} \) has CM by the maximal order of \( Q(\sqrt{-N}) \). Since 2 is unramified in \( Q(\sqrt{-N}) \) and the Hilbert class field is an unramified extension of \( Q(\sqrt{-N}) \), 2 is unramified in \( B/Q \). This implies that \( B_\nu \) embeds in \( Q_2 \) where \( \nu \) is a prime of \( B \) lying over 2. Therefore \( \tilde{E} \) induces a point in \( X^d(N)(Q_2) \). Hence we conclude that \( X^d(N)(Q_2) \neq \emptyset \) if and only if there is a \( w_N \)-fixed point on \( X_0(N)(F_2) \) and such a point exists if and only if \( p \) is in the set \( S_N \) defined above, exactly as in the case of odd primes. This gives us the following result:
**Theorem 5.9.** Let $p$ be a prime ramified in $\mathbb{Q}(\sqrt{d})$ and $N$ a squarefree integer such that $p$ is unramified in $\mathbb{Q}(\sqrt{-N})$. Then $X(\mathbb{Q}_p) \neq \emptyset$ if and only if $p$ is in the set $S_N$ defined in Proposition 5.5.

**Definition 5.10.** A set of rational primes $S$ is a Chebotarev set if there is some finite normal extension $L/\mathbb{Q}$ such that $p$ is in $S$ if and only if the Artin symbol of $p$ is in some specified conjugacy class or union of conjugacy classes of the Galois group $\text{Gal}(L/\mathbb{Q})$.

The density of a Chebotarev set is well-defined by Chebotarev density theorem, and the set of primes $S_N$ defined in Proposition 5.5 is a Chebotarev set with density $\frac{|2G|+1}{2|G|}$.

**Theorem 5.11** (Serre, Theorem 2.8 in [22]). Let $0 < \alpha < 1$ be Frobenius density of a set of primes $S$ and $N_S(X) := \{n \leq X : n = \Pi_{p \in S} p_i \in S\}$. Then $|N_S(X)| = c_S \frac{X}{\log^{1-\alpha} X} + O \left( \frac{X}{\log^{2-\alpha} X} \right)$ for some positive constant $c_S$.

**Corollary 5.12.** Given $N$ let $A$ be the set of squarefree integers $d$ in $[1 \cdot \cdots X]$ such that $(d, N) = 1$ and $X^d(N)(\mathbb{Q}_N)$ is nonempty for all primes $p$. Then $|A| = M_{S_N} \frac{X}{\log^{1-\alpha} X} + O \left( \frac{X}{\log^{2-\alpha} X} \right)$ where $\alpha = \frac{|2G|+1}{2|G|}$ is the density of $S_N$ and $M_{S_N}$ a positive constant.

**Proof.** Follows from Theorem 5.9 combined with Theorem 5.11.

The following corollary gives an answer to a question of Clark asked in [1].

**Corollary 5.13.** Let $N$ be a prime congruent to 1 mod 4, $p$ be an odd prime such that $(N/p) = -1$ and $d = p^*$ then

1. $X^d(N)(\mathbb{Q}_N) = \emptyset$
2. $X^d(N)(\mathbb{Q}_p) = \emptyset$
3. $X^d(N)(\mathbb{Q}_\ell) \neq \emptyset$ for any other prime $\ell$ different than $p$ and $N$.

**Proof.** The first conclusion was also given in [1] and can be seen as a result of Theorem 4.8. For the second conclusion we proceed as follows:

Let $M := \mathbb{Q}(\sqrt{-N})$. Since $N \equiv 1 \mod 4$, the genus field of $M$ is $\mathbb{Q}(i, \sqrt{-N})$. Note that since $-N \equiv 3 \mod 4$, ring class field of $\mathbb{Z}[\sqrt{-N}]$ is the Hilbert class field. Let $B := Q(j(\mathbb{Z}[\sqrt{-N}]))$. Since $j(\mathbb{Z}[\sqrt{-N}])$ is real, $B \cap \mathbb{Q}(i, \sqrt{-N})$ is $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{-N})$. Say $B \cap Q(i, \sqrt{-N})$ is $\mathbb{Q}$, this implies that the class number of $\mathbb{Z}[\sqrt{-N}]$ is 1, contradiction.

By Theorem 5.9 and Lemma 5.5, $X^d(N)(\mathbb{Q}_p) = \emptyset$ if and only if $p \notin S_N$. Since $(N/p) = -1$ this is equivalent to say that for all primes $\nu$ of $B$ lying over $p$, $f(\nu|p) > 1$.

The third conclusion can be derived from Theorem 4.18.
6. Examples

Example 6.1. This example deals with the case \( N = 7*19 \), \( p = 19 \) and \( d = 3 \). According to Theorem 2.3, since \((7*19, 3) = -1\), \( X^d(7*19)(\mathbb{Q}) = \emptyset \). Since \((\frac{7}{19}) = -1 \) and 19 divides \( N \) we are in the first case of Section 4. Using the related theorem in this case (Theorem 4.8), we can in fact see that \( X^d(7*19)(\mathbb{Q}_{19}) = \emptyset \).

To explain this more concretely, we use the idea of the proof Theorem 4.8. Using the genus formula,

\[
g(N) = 2g(N/p) + n - 1
\]

where \( n \) is the number of supersingular points on \( X_0(N)_{\overline{\mathbb{F}}_p} \), we see that there are 12 supersingular points on \( X_0(19*7)_{\overline{\mathbb{F}}_{19}} \). And there are 2 supersingular j-invariants over \( \overline{\mathbb{F}}_{19} \), one of them is 1728 (since 19 is congruent 3 mod 4) and the other one is 7. The elliptic curve \( E \) with j-invariant 1728, has 8 order-7 subgroups namely \( C_1, ..., C_8 \). The extra automorphism \([\bar{i}] \), where \( i \) is the fourth root of unity) sends \( C_k \) to \( C_{9-k} \). Hence we get 4 supersingular points on \( X_0(7)_{\overline{\mathbb{F}}_{19}} \) each with automorphism group isomorphic to \( \{\pm 1\} \); namely \( P_1 = (E, C_1), P_2 = (E, C_2), P_3 = (E, C_3), P_4 = (E, C_4) \). For the elliptic curve with j-invariant 7, we get 8 many order 7 groups each with the smallest possible automorphism group. Hence we get 8 supersingular points, satisfying the total number 12 found above. Since the number of blow-ups is 0, the model of \( X_0(7*19)_{\overline{\mathbb{F}}_{19}} \) is regular and the only \( \overline{\mathbb{F}}_{19} \)-rational points are the singular ones, hence \( X^d(7*19)(\mathbb{Q}_{19}) = \emptyset \), this is also consistent with the observation that \((7*19, 3)_{19} = -1\).

Since \((7*19, 3)_{19} = -1\), there should be at least one more local Hilbert symbol that is negative, in this case it is \((7*19, 3)_7 = -1\). Now we will do the same operations as above, this time for \( X_0(7*19)_{\mathbb{F}_7} \). Using the genus formula, there are 10 supersingular points on \( X_0(7*19)_{\mathbb{F}_7} \) and there is one supersingular elliptic curve over characteristic 7, the one with j-invariant 1728. Let \( E \) be the elliptic curve with j invariant 1728. There are 20 tuples like \((E, C_i)\) and each has an automorphism group equal to \( \{\pm 1\} \) or primitive fourth roots of unity. Since there are 10 supersingular points, all has the smallest automorphism group i.e. \( X^d(7*19)(\mathbb{Q}_7) = \emptyset \) using the same argument above. This also can be deduced directly from Theorem 4.8.

Example 6.2. Let \( d = 5 \) and \( N = 29 \). According to Theorem 2.3 since \((5, 29) = 1\), we have the necessary condition for the existence of a \( \mathbb{Q} \)-curve of degree 29 over \( K = \mathbb{Q}(\sqrt{5}) \), but it is not guaranteed. Note that this case is not covered by Theorem 2.6.

If we use Theorem 5.9 for the ramified prime 5, we see that \( X^5(\mathbb{Q}_5) \) is empty, hence there is no \( \mathbb{Q} \)-curve of degree 29 over \( K \). In order to \( X^5(\mathbb{Q}_5) \) to be non-empty, 5 should split in \( H/\mathbb{Q}(\sqrt{-29}) \), where \( H \) is the
Hilbert class field of $\mathbb{Q}(\sqrt{-29})$. However the prime $P|5$ of $\mathbb{Q}(\sqrt{-29})$, decomposes as $P_1P_2$ where inertia degree of $P_i$ is 3, hence $X^5(\mathbb{Q}_5) = \emptyset$. Note that $X^5(\mathbb{Q}_p)(29) \neq \emptyset$ for any other prime $p$ different than 5 using Theorem [13].

7. Further Directions

Using Theorem [13] we can produce lots of rational curves which have local points everywhere. One natural follow-up question would be asking about the $\mathbb{Q}$-rational points. There is an answer to this question in the case of imaginary quadratic fields $K$ and when $N$ is inert in $K$ implied by the following theorem of Mazur ([13]):

**Theorem 7.1.** If $K$ is a quadratic imaginary field, and $N$ is a sufficiently large prime which is inert in $K$, then $X_0(N)(K)$ is empty. In particular, there are no $\mathbb{Q}$-curves over $K$ of degree $N$.

When $N$ splits in $K$ and $K$ is imaginary quadratic or $N$ is inert in $K$ and $K$ is real quadratic field, using the formula of Weil given in [11], every cuspform associated with a quotient of jacobian of $X^d(N)$ has odd functional equation. Thus, conjecturally none of these quotients have Mordell-Weil rank 0 and we cannot hope to apply Mazur’s techniques. The future plan is to prove a result about existence of rational points on $X^d(N)$ where $K$ is a real quadratic field and $N$ splits in $K$ using Mazur’s techniques.

Another direction to go is understanding the reasons of violations to Hasse principle. Say for some $d$ and $N$, $X^d(N)$ has local points for every $p$ but no global points, hence it violates Hasse principle. What is the reason for that? One natural guess would be Brauer-Manin obstruction. It is a folklore conjecture that for curves, ‘Under the assumption that $Sha(J)$ is finite, Brauer-Manin obstruction is the only obstruction to Hasse principle’; which is a theorem in the following cases:

- **(Manin)** If $C$ is proper, smooth of genus 1 and $Sha(J)$ is finite ([5])
- **(Scharaschkin)** If $C$ is proper, smooth, has a rational divisor class of degree 1 and $J(\mathbb{Q})$ and $Sha(J)$ are finite ([20])
- **(Skorobogatov)** If $C$ is proper, smooth, $Sha(J)$ is finite and $C$ has no rational divisor class of degree 1 ([25])

For some specific examples, we will use the following idea of Scharaschkin ([20]) to see if violation to Hasse principle can be explained by Brauer-Manin obstruction.

Given a smooth, projective, geometrically integral curve $C$ over $\mathbb{Q}$ and a finite set $S$ of good primes, if images of $red$ and $inj$ in the following map do not intersect then $C(\mathbb{Q}) = \emptyset$. 
In order to apply Scharaschkin’s technique, one needs an equation of the curve \(C\), generators of \(Jac(\mathbb{Q})\) and also existence of a \(\mathbb{Q}\)-rational degree one divisor class. In the case of quadratic twists of \(X_0(N)\), if the curve is hyperelliptic and \(w_N\) is the hyperelliptic involution then finding the equation of the twist is easy. Moreover there exists relatively simple equations of \(X_0(N)\) given by Galbraith in [6] that makes the computations feasible. Such equations for hyperelliptic modular curves is first given by Gonzalez [7], see also the works of Murabayashi [15] and Hasegawa [8]. It’s a result of Ogg ([16]) that automorphism group of \(X_0(N)\) is \(\{1, w_N\}\) for prime \(N\) such that \(N \neq 2, 3, 5, 7, 13, 37\). Then the equation of the twist \(X^d(N)\) is \(dy^2 = f_{2g+2}(x)\) where \(g\) is the genus of the curve and \(f_m\) is a degree \(m\) polynomial.

Since for genus 1 the claim is already proved, we’ll restrict to the cases \(g \geq 2\) and we want an hyperelliptic curve with \(w_N = -1\). Smallest such \(N\) is 23 and \(X_0(23)\) is given by \((x^3 - x + 1)(x^3 - 8x^2 + 3x - 7)\) in [6]. The following computations are done using the computer package MAGMA.

**Example 7.2.** Let \(N = 23\). We will study the twists of \(X_0(23)\) for all primes \(d\) between -300 and 300. There are 124 primes between -300 and 300. The twist is given by the equation \(y^2 = d(x^3 - x + 1)(x^3 - 8x^2 + 3x - 7)\). Let \(a_1, a_2, a_3\) be roots of \(x^3 - x + 1\), and \(P_i = (a_i, 0)\) be the corresponding points on \(X^d(N)\) then \(D = [P_1 + P_2 + P_3 - \infty_1 - \infty_2]\) is a rational divisor of degree one on \(X^d(N)\). (Similar construction is in [5]).

1. Let \(|d|\) be prime different then 23 and between -300 and 300, \(X^d(N)\) has local points everywhere for 39 values of \(d\). Among the 39 values of such \(d\), for 10 of them \(X^d(N)\) has points with small height (less than 1000) when we eliminate these, we are left with the set: \{-283, -271, -263, -251, -227, -223, -211, -199, -191, -83, -59, 17, 37, 53, 61, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 229, 241, 281, 293\}

2. The remaining thing we need in order to apply Scharaschkin’s technique are the generators of \(J^d(\mathbb{Q})\) where \(J^d\) is the Jacobian of \(X^d(N)\). This seems to be the hardest thing to do. First lets consider the only one that we were able to apply Scharaschkin’s technique.

Say \(d = 17\): Let \(C\) be \(X^{17}(23)\) and \(J^{17}\) be its jacobian. It can be computed that \(J^{17}\) has no nontrivial torsion and rank of \(J^{17}(\mathbb{Q})\) is less than or equal to 2. After a short search we come
up with the generators of $J^{17}(\mathbb{Q})$, given by $D_1 = \langle x^2 + 3, 17*x - 34, 2 \rangle$ and $D_2 = \langle x^2 - 3/4 * x + 5/8, 153/16 * x - 17/32, 2 \rangle$. The notation means that $D_1 = [P_1 + P_1 - \infty_1 - \infty_2]$ with $P_1 = (a, 17 * a - 34)$ where $a$ is one of the roots of $x^2 + 3$ and $P_1 = (\bar{a}, 17 * \bar{a} - 34)$. Similarly for $D_2$.

To apply Scharaschkin’s idea, we need a set of primes $S$, in our case $S$ will be $\{13\}$.

Let $\tilde{C}, J^{17}, \tilde{D}, \tilde{D}$ be the reductions of $C, J^{17}, D_1, D_2$ modulo $13$. Order of $D_1$ and $D_2$ are 11 in $\tilde{J}^{17}(\mathbb{F}_{13})$.

Say $P$ is in $C(\mathbb{Q})$ then the reduction $\tilde{P}$ in $\tilde{C}(\mathbb{F}_{13})$ = \{(1 : 1 : 0), (1 : 12 : 0), (3 : 11 : 1), (3 : 2 : 1), (6 : 10 : 1), (6 : 5 : 1), (7 : 3 : 1), (7 : 10 : 1)\} and $[\tilde{P} + \tilde{P} - \infty_1 - \infty_2]$ will be in $\tilde{J}^{17}(\mathbb{F}_{13})$.

The points in $\tilde{J}^{17}(\mathbb{F}_{13})$ are given by $n_1$ times $\tilde{D}_1 = [(6, 3) + (7, 7) - \infty_1 - \infty_2]$ and $n_2$ times $\tilde{D}_2 = [(a, 12a + 8) + (b, 12b + 8) - \infty_1 - \infty_2]$ $(a, b$ are roots of $x^2 - 3/4 * x + 5/8$ where $n_1, n_2$ are between 0 and 10. If we construct all these linear combinations we see that none of these is of the form $[\tilde{P} + \tilde{P} - \infty_1 - \infty_2]$ for $\tilde{P}$ in $\tilde{J}^{17}(\mathbb{F}_{13})$, hence there is no $P$ in $C(\mathbb{Q})$.

This example shows that, the twisted modular curve $X^{17}(23)$ has local points everywhere but no global points, hence it violates Hasse principle and this violation can be explained by Brauer-Manin obstruction.

This example is also interesting in the sense that 23 is inert in the quadratic field $\mathbb{Q}(\sqrt{17})$. Then, using the formula of Weil given in [11], every cusps form associated with a quotient of jacobian of $X^{17}(23)$ has odd functional equation. Thus, conjecturally none of these quotients have Mordell-Weil rank 0 and Mazur’s methods cannot be applied. In fact, the jacobian of $X^{17}(23)$ is simple since $J^{17}(23)$ is an abelian surface with rank 2 and therefore its only notrivial quotients are the elliptic ones. However, the $q$-expansion of the corresponding newforms of level 23 has conjugate coefficients in $\mathbb{Q}(\sqrt{-5})$ (see [20]) hence there is only one isogeny class, therefore $J^{17}(23)$ is simple.

If we continue our search for the same level, $N = 23$, we get some more $d$ values such that corresponding twists fails to have rational points.

**Example 7.3.** (continued from above) For $d = 173, -211, 101, -59, -223$ the rank of $Jac(\mathbb{Q})$ is 0 since the 2-Selmer group is trivial. Moreover, the torsion part of $Jac(\mathbb{Q})$ is also trivial in all these cases. Say there exists $P \in C(\mathbb{Q})$ where $C$ is the twist $X^d(23)$, then $[P] - D$, where $D$ is as above, is in $J(\mathbb{Q})$. It can be proved that $D$ is not equivalent to $[P]$ for any point $P$ hence $[P] - D$ is nonzero, contradiction. Therefore $X^d(23)(\mathbb{Q}) = \emptyset$ for $d = 173, -211, 101, -59, -223$. 
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