The Critical Point of Unoriented Random Surfaces with a Non-Even Potential

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Abstract

The discrete model of the real symmetric one-matrix ensemble is analyzed with a cubic interaction. The partition function is found to satisfy a recursion relation that solves the model. The double scaling-limit of the recursion relation leads to a Miura transformation relating the contributions to the free energy coming from oriented and unoriented random surfaces. This transformation is the same kind as found with a quartic interaction.
1. Introduction

Matrix models have been used to study the formulation of non-perturbative two-dimensional Euclidean quantum gravity (pure or coupled with conformal matter with $c \leq 1$) following the seminal work of references [1], [2], [3]. These models may also be viewed as models of random surfaces or zero-dimensional string theory. The partition function of the Hermitian matrix ensemble is identified as the theory of oriented surfaces. This connection is established by identifying the dual to the Feynman diagrams of the matrix model with random discretizations of two-dimensional surfaces of arbitrary genus. The technique of the double-scaling limit allows us to recover the continuum limit of the theory [1], [2], [3].

Many other one-matrix models have been solved following the tools developed for the Hermitian ensemble [4], [5], [6]. In this paper we shall consider the real symmetric one-matrix model [7], [8], [9], [10] introduced to describe theories with unoriented random surfaces. In these treatments, the potential in the matrix model was for simplicity taken to be even. When this is the case, the orthogonal polynomials suitable to solve the model happen to have a definite parity, thereby providing a computational simplification similar to the one appearing in the standard Hermitian model [11]. In this letter we shall address the problem of a non-even potential which we solve for the discrete model with a cubic interaction in sections 2, 3, 4.

This study is motivated by the equivalent problem taking place in the Hermitian ensemble where it is found [12], [13] that the continuum limit of the model with a cubic potential leads to a string susceptibility satisfying the Painlevé I equation, as it happens with a quartic potential. In section 5 the double-scaling limit of a real symmetric matrix ensemble with a cubic term is computed and compared with the quartic potential case of [7], [9].

Our starting point is the partition function for the real symmetric $2N \times 2N$ matrices $M^T = M$,

$$\mathcal{Z}_{2N} = e^{\mathcal{F}_{2N}} = \int \prod_{0 \leq i \leq j \leq 2N-1} [dM^j_i] e^{-\frac{1}{T} \text{tr}(V(M))}$$  \hspace{1cm} (1)
with the matrix potential given by

\[ V(M) = \frac{g_2}{2} M^2 + \frac{g_3}{3} M^3 \]  

(2)

The identification of (1) as a theory of unoriented random surfaces comes from the perturbation expansion of the interaction term in (1), (2) around the gaussian ensemble \( g_3 = 0 \). The propagator is

\[ < M^i_j M^a_m >_{0} = g^{-1}_2 (\delta_{im} \delta_{jn} + \delta_{i}^{a} \delta_{j}^{m}) \]  

(3)

The first term in (3) represents the “usual” propagator in the double line notation of the Feynman diagrams, while the second one represents a “twisted” propagator. The latter contribution amounts to the loss of orientability of the triangulated surface associated to the dual of a Feynman graph\(^1\). Each connected vacuum diagram is weighed by a factor of \( N^\chi \), where \( \chi \) is the Euler characteristic of the surface, which now can take both even and odd values. Thus, all kinds of non-orientable surfaces contribute to the sum of the random surfaces described by (3) [7], [9],

\[ Z_{2N} = N^2 Z_{\text{sphere}} + N^1 Z_{\text{RP}^2} + N^0 (Z_{\text{torus}} + Z_{\text{Klein bottle}}) + \cdots \]  

(4)

2. The Method of Skew Orthogonal Polynomials

The partition function of non-orientable pure gravity is computed with the help of an appropriate set of polynomials in a similar way to what happens in the case of the Hermitian matrix model. Thus, when we integrate over “angles”, i.e., over the real orthogonal group with the action \( M = OXO^{-1} = OXO^T \) in (1), it is possible to express the partition function in terms of the eigenvalues \( X = \text{diag}(x_1, \ldots, x_{2n}) \) of the matrix \( M \). The result is [14]

\[ Z_{2N} = K \times \int | \prod_{i=1}^{2N} dx_i | \Delta(x) | e^{-\frac{g}{2} \sum_{i=1}^{2N} V(x_i)} \]  

(5)

\(^1\)To be more precise, in the continuum limit it is also needed that both contributions in (3) have the same positive weight [14].
where \( V(x) \) is given by (2). The main new feature is the appearance of the Vandermonde determinant \( \Delta(x) = \prod_{i<j}(x_i - x_j) \) in absolute value and raised to the power of one, rather than the usual factor \( \Delta^2(x) \) showing up in the Hermitian matrix model. This is a major difference between both models as far as the computation of (3) is concerned. From the symmetry properties of (3) we may write [14]

\[
Z_{2N} = K(2N)! \int_{R(-\infty,x_1,...,x_{2N},\infty)} \prod_{i=1}^{2N} e^{-\beta/2 \sum_{i=1}^{2N} V(x_i)} \times [\det(x_j^{i-1})]_{i,j=1,...,2N}
\]

where the region of integration \( R \) is \(-\infty < x_1 \leq \ldots \leq x_{2N} < \infty\). Note that the absolute value in (3) is no longer required. As usual, it is convenient to introduce monic polynomials \( R_i(x) \) in order to rewrite the Vandermonde determinant as

\[
\Delta(x) = [\det(x_j^{i-1})]_{i,j=1,...,2N} = [\det R_{i-1}(x_j)]_{i,j=1,...,2N}
\]

(7)

We want to solve the partition function \( Z_{2N} \) in terms of quantities related to these monic polynomials. To do that, we have to chose the \( R_i(x) \) as skew orthogonal polynomials with respect to the following anti-symmetric scalar product [10]

\[
<R_i, R_j>_{R} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \epsilon(x-y) e^{-\beta/2[V(x)+V(y)]} \equiv \tau_{i/2} z_{ij}
\]

where the only non-zero values of \( z_{ij} \) are

\[
z_{2i,2i+1} = -z_{2i+1,2i} = 1
\]

(9)

and

\[
\epsilon(y-x) = \begin{cases} 
\frac{1}{2} & \text{if } y > x \\
\frac{1}{2} & \text{if } y < x 
\end{cases}
\]

(10)
Using Mehta’s method [14] of integrating over alternate variables, we carry out the integration over the odd variables $x_1, x_3, \ldots, x_{2N-1}$ first and then over the remaining even variables. The final result is

$$Z_{2N} = K(2N)! \prod_{k=0}^{N-1} r_k$$

(11)

So far the problem is formally solved. In order to make the solution explicit, the strategy consists of relating the set of skew orthogonal polynomials $R_i$ with the well-known set of orthogonal polynomials solving the Hermitian one-matrix model. Thus, let us introduce orthogonal polynomials $C_i$ with respect to the following symmetric scalar product

$$< C_i, C_j >_C = \int_{-\infty}^{\infty} dx e^{-\beta V(x)} C_i(x) C_j(x)$$

$$\equiv \hbar_i \delta_{ij}$$

(12)

These polynomials are also monic and satisfy a two-step recursion relation

$$xC_i = C_{i+1} + \sigma_i C_i + \rho_i C_{i-1}$$

(13)

The coefficient $\sigma_i$ takes into account that we are dealing with a non-even potential (2). In addition, it is possible to derive equations for $\rho_i, \sigma_i$ from the relations $< C_i, C_i' >_C = 0$ and $< C_{i-1}, C_i' >_C = i\hbar_{i-1}$. This yields to

$$< C_i, \beta V' C_i >_C = 0$$

(14)

$$< C_{i-1}, \beta V' C_i >_C = i\hbar_{i-1}$$

(15)

Now let us express the set of polynomials $R_i$ as a linear combination of the set $C_i$.

$$R_i(x) = C_i(x) + a_{i,i-1} C_{i-1}(x) + a_{i,i-2} C_{i,i-2}(x) + a_{i,i-3} C_{i,i-3}(x)$$

(16)

In the next section we shall see that the combination (16) ends at $C_{i-3}$ because the order of the potential $V(x)$ is 3.
Neither $R_i$ nor $C_i$ have a definite parity anymore due to the cubic term in $V(x)$. Then, it is no longer possible to split the relation (16) into two, one for the set $R_{2i}$ and another for $R_{2i-1}$, as occurred when dealing with even potentials [10], [7]. This amounts to an extra complication in solving (16). At the end of the computation of the coefficients $a_{i,j}$, we shall make such a splitting to notice the appearance of mixed terms between both sets.

To proceed further, we have to relate the two scalar products introduced in (8) and (12). Given two functions $f$, $g$ it is easy to prove, upon integration by parts, the following essential relation

$$< f, g >_C = -\frac{1}{2} < \beta V' f, g >_R + < f', g >_R$$

(17)

This equation will give us enough relations to find out $a_{i,j}$, provided we know the action of $V'(x)$ and $\frac{d}{dx}$ on the set of orthogonal polynomials $C_i$. In fact, computing $< C_{i-j}, R_i >_C$, $j = 0, 1, 2, 3$ in (16) with the aid of (17), it is possible to set up the following relations for the unknown variables $a_{i,j}$

$$a_{i,i-j} = < C_{i-j}, R_i >_C$$

$$= -\frac{1}{2} < \beta V' C_{i-j}, R_i >_R + < C'_{i-j}, R_i >_R$$

(18) $j = 0, 1, 2, 3 \forall i.$

For the action of $V'(x)$ on $C_i$, using (3) and (13), we have

$$V'(x)C_i(x) = g_3 C_{i+2} + [g_3(\sigma_{i+1} + \sigma_i) + g_2] C_{i+1}$$

$$+ [g_3(\rho_{i+1} + \sigma_i^2 + \rho_i) + g_2 \sigma_i] C_i$$

$$+ [g_3 \rho_i(\sigma_i + \sigma_{i-1}) + g_2 \rho_i] C_{i-1} + g_3 \rho_i \rho_{i-1} C_{i-2}$$

$$\equiv \sum_{l=-2}^{l=+2} \nu_{i+l} C_{i+l}(x)$$

(19)

It is possible to simplify this expression using the discrete string equations (14) and (15). In fact, the following expressions for the string equations are readily obtained

$$\beta \nu_{i,i-1} = i = \beta [g_3 \rho_i(\sigma_i + \sigma_{i-1}) + g_2 \rho_i]$$

(20)
\[ v_{i,i} = 0 = g_3(\rho_{i+1} + \sigma_i^2 + \rho_i) + g_2 \sigma_i, \]  

From (19) and (20) we can also express the coefficient \( v_{i,i+1} \) in a more tractable way

\[ v_{i,i+1} = \frac{v_{i+1,i}}{\rho_{i+1}} = \frac{i + 1}{\beta \rho_{i+1}} \]  

(22)

On the other hand, the action of \( d/dx \) on \( C_i \) is given by

\[ \frac{dC_i}{dx} = \sum_{j=0}^{i-1} <C_j, \beta V' C_i >_C C_j \]  

(23)

which is obtained by partial integration on \(<C_j, \frac{dC_i}{dx} >_C \) with \( j < i \). As \( V(x) \) has degree 3, this expression turns out simply to be

\[ \frac{dC_i}{dx} = \beta v_{i,i-2} C_{i-2} + \beta v_{i,i-1} C_{i-1} \]  

(24)

3. Recursion Relation for the Real Symmetric One-Matrix Model

Due to the lengthy calculations involved for establishing and solving the equations (18), it is convenient to outline the main steps needed to achieve the final result as follows,

i) Choose one of the four equations in (18) \( (j = 0, 1, 2, 3) \).

ii) Express \( V'(x)C_{i-j} \) and \( C'_{i-j} \) in terms of the orthogonal polynomials \( C_i \). To do this, use (15)-(22) and (24).

iii) Express the \( C_i \) polynomials of step ii) in terms of skew orthogonal polynomials. To do this, it is required to invert the relations (16).

iv) Use the skew-orthogonality relations (8) and (9) to solve the right hand side of (18).

Moreover, to carry out the above program we shall make repeatedly use of the following essential observations,

a) As \( R_i = C_i + \text{order } (C_{i-1}) \), then \( C_i = R_i + \text{order } (R_{i-1}) \).

b) From the skew-orthogonality relations (8), (9) we have
\[ < R_{2i}, R_{2i+1} >_R = r_i = -< R_{2i+1}, R_{2i} >_R \]  

and the rest of scalar products are zero.

c) The polynomial \( R_{2i} \) is skew orthogonal to any polynomial of degree less than or equal to \( 2i \).

The polynomial \( R_{2i+1} \) is skew orthogonal to any polynomial of degree less than \( 2i \). Symbolically,

\[ R_{2i} \text{ skew} \perp R_i, R_{2i-1}, \ldots \]

\[ R_{2i+1} \text{ skew} \perp R_{2i-1}, \ldots \]  

To be consistent, we need to show that the relations (16) actually stop at \( C_{i-3} \). Namely, let us assume that a term, say, \( C_{i-4} \) is present. Then, the term in the r.h.s. of (18) with the highest \( C_i \) is \( < C_{i-4+2}, R_i >_R \). This follows from (19) and (23). Now using observations a) and c) we easily get

\[ < C_{i-2}, R_i >_R = 0. \]

We shall first work out the equations (18) in the order \( j = 3, 2, 1 \). The expressions for \( a_{i,i-3}, a_{i,i-2}, a_{i,i-1} \) thus obtained will be replaced in equation (18) for \( j = 0 \). This will turn out to be the desired recursion relation that solves the discrete model.

Equation (18) \( j = 3 \).

\[ a_{i,i-3}h_{i-3} = -\frac{\beta}{2} v_{i-3,i-1} < R_{i-1}, R_i >_R \]  

It is convenient to split the indices into even and odd. If \( i = 2k \), then as \( < R_{2k-1}, R_{2k} >_R = 0 \), one deduces

\[ a_{2k,2k-3} = 0 \]  

If \( i = 2k + 1 \), as \( < R_{2k}, R_{2k+1} >_R = r_k \) and \( v_{i-3,i-1} = g_3 \), one deduces

\[ a_{2k+1,2k-2}h_{2k-2} = -\frac{\beta}{2} g_3 r_k \]  

Equation (18) \( j = 2 \).
It is convenient to start with

\[ a_{i,i-2}h_{i-2} = -\frac{\beta}{2}v_{i-2,i} < C_i, R_i >_R \]

After some algebra we get

\[ a_{i,i-2}h_{i-2} = -\frac{\beta}{2}\left[v_{i-1,i}a_{i,i-1} + v_{i-2,i-1}\right] < R_{i-1}, R_i >_R \]  

If \( i = 2k \) then,

\[ a_{2k,2k-2} = 0 \]  

If \( i = 2k + 1 \) then,

\[ a_{2k+1,2k-1}h_{2k-1} = -\frac{\beta}{2}\left[g_3a_{2k+1,2k} + \frac{2k}{\beta \rho_{2k}}\right] r_k \]

and \( a_{2k+1,2k} \) is to be determined.

Equation (18) \( j = 1 \).

Applying again the procedure previously described, we obtain

\[ \frac{2}{\beta}a_{i,i-1}h_{i-1} = -v_{i-1,i+1} < R_{i+1}, R_i >_R \]

\[ -\left[v_{i-1,i+1}(a_{i+1,i}a_{i,i-1} - a_{i+1,i-1}) + v_{i-1}a_{i,i-1}\right] < R_{i-1}, R_i >_R \]  

Where we have kept in mind that \( v_{i-1,i-1} = 0 \) due to the string equation (21). Notice that in (34) appears \( a_{i+1,i-1} \), which is of the same type described by the previous case (31). Replacing the value of \( a_{i+1,i-1} \) in (34) we shall obtain an equation for \( a_{i,i-1} \) solely. However, the analysis is simplified if we first make the splitting of indices into even and odd.

If \( i = 2k \), then \( < R_{i-1}, R_i >_R = 0 \) and

\[ \frac{2}{\beta}a_{2k,2k-1}h_{2k-1} = g_3r_k \]  

If \( i = 2k + 1 \), then \( < R_{i+1}, R_i >_R = 0 \) and

\[ \frac{2}{\beta}a_{2k+1,2k}h_{2k} = -g_3a_{2k+1,2k}[a_{2k+2,2k+1} + \frac{2k + 1}{\beta \rho_{2k+1}}] r_k \]  

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This equation admits two solutions. We shall take the simplest one,
\[
a_{2k+1,2k} = 0
\]
(37)

The case \(a_{2k+1,2k} \neq 0\) is considered in Appendix I. Now we can replace (37) into (33) to obtain the value of \(a_{2k+1,2k-1}\),
\[
a_{2k+1,2k-1} = -\frac{k}{\rho_{2k}} \frac{r_k}{h_{2k-1}}
\]
(38)

Now we have the solution of all the coefficients relating both sets of polynomials in (16). With this solution, it is convenient to split the relation (16) into polynomials with even and odd indices to see the picture that emerges from the previous calculations. Therefore,
\[
R_{2k} = C_{2k} + a_{2k,2k-1}C_{2k-1}
\]
(39)
\[
R_{2k+1} = C_{2k+1} + a_{2k+1,2k-1}C_{2k-1} + a_{2k+1,2k-2}C_{2k-2}
\]
(40)

where the coefficients are given by (35), (37) and (40). In this way, we observe that there are odd contributions to \(R_{2k}\) as well as even ones to \(R_{2k+1}\). This is the new effect of dealing with a non-even potential, causing the polynomials \(R_{2k}, R_{2k+1}\) not to have a definite parity anymore.

Now we are in position to set up the recursion relation between the relevant quantities \(r_k, h_k\) that ultimately solves our model. This is achieved with equation (18) for \(j = 0\). The main steps of this lengthy calculation is given in Appendix II with the following result,
\[
h_{2k} = \left[-\frac{\beta^2}{4} g_3^2 \frac{r_{k+1}}{h_{2k+1}} + \frac{2k+1}{2\rho_{2k+1}}\right] r_k
\]
(41)

The equation (18) for \(j = 0\) are actually two relations, depending on whether the index \(i\) is even or odd, as was done in the previous cases. This means that the system of equations is constrained, for there is one more equation that unknown variables. It is worth noticing that either equation (18) for \(j = 0\) leads to the same solution (see Appendix II), thereby providing
a proof of the consistency of our solution. In addition, if we set \( g_3 = 0 \) we should end up with the solution of the gaussian model \([10],[4]\), as it happens

\[
h_{2k} = \frac{\beta}{2} r_k
\]

for \( \rho_i = i/\beta \) in the gaussian case.

4. The Partition Function of the Model

So far, we have related the calculation of the partition function for the real symmetric one-matrix model to the solution of the Hermitian ensemble. Once the norms of the orthogonal polynomials \( C_i \) are determined from the discrete string equations \((\text{20}), (\text{21})\), then the quantities \( r_k \) are in principle computed from \((\text{41})\) yielding the solution of the partition function \((\text{11})\). In fact, the recursion relation \((\text{11})\) obtained for \( r_k \) amounts to a recursion relation between partition functions at different \( N \). From \((\text{11})\) it is possible to rewrite \( r_k \) in the form

\[
r_k = \frac{1}{(2k + 2)(2k + 1)} \times \frac{Z_{2k+2}}{2k}
\]

Upon substitution of \((\text{43})\) in \((\text{11})\) we readily obtain the following recursion relation for the partition function,

\[
\frac{\beta^2}{4} g_3^2 Z_{2k+4} - \frac{1}{2} (2k + 4)(2k + 3)(2k + 1) h_{2k} Z_{2k+2} + \frac{(2k + 4)!}{(2k)!} h_{2k+1} h_{2k} Z_{2k} = 0
\]

(44)

When the continuum limit of the model is to be taken, we shall see that it is convenient to introduce the following quantities related to the “norms” \( r_k \) by

\[
W_k = \beta \frac{r_k}{h_{2k}}
\]

and the partition function \((\text{11})\) is determined by the quantities \( W_k \) and \( h_{2k} \) as

\[
Z_{2N} = K \times (2N)! \beta^{-(N-1)} \prod_{k=0}^{N-1} W_k h_{2k}
\]

(46)
Hence, the free energy takes the form

\[ F_{2N} = \ln K_N + \sum_{k=0}^{N-1} \ln h_{2k} + \sum_{k=0}^{N-1} W_{2k} \] (47)

According to [7],[9], the second term in \( F_{2N} \) represents the contribution of the orientable surfaces to the free energy, for it entirely involves the constants related to the Hermitian matrix model which describes only oriented surfaces. On the other hand, the constants \( W_k \) amounts to the contribution of non-orientable surfaces to the free energy. This is made more plausible by noticing that the odd powers of \( (1/N) \) in \( F_{2N} \) come precisely from \( W_k \) [7].

5. The Continuum Limit

We now want to take the continuum limit of our solution (41) to describe the critical point of non-orientable random surfaces so, we need to borrow the results concerning the double-scaling limit for the Hermitian model with a cubic interaction.

The scaling limit of the string equations (20),(21) goes as follows. First, we take the planar limit \((N \to \infty, \text{with } N/\beta \to 1)\) of (20),(21) at \( i = 2N \). Therefore, it is convenient to introduce the functions

\[ V_1(\rho, \sigma) = 2\rho + g_3\rho\sigma = 1 \] (48)

\[ V_2(\rho, \sigma) = 2\sigma + g_3(2\rho + \sigma^2) = 0 \] (49)

where we have normalized \( g_2 = 4 \) without loss of generality. Then, the critical values of \( g_{3c} \) and \( \rho_c \) are defined by

\[ \left. \frac{dV_1(\rho, \sigma)}{d\rho} \right|_c = 0 \] (50)

subject to

\[ \left. \frac{dV_2(\rho, \sigma)}{d\rho} \right|_c = 0 \] (51)

The analysis of these conditions leads to the following critical values that, for future convenience, we recast as
\[ \rho_c^2 = \frac{3}{4} \quad \text{and} \quad g_{3c}^2\rho_c = \frac{4}{3} \]  

(52)

Second, we insert the values (52) in the discrete string equations (20), (21) at \( i = 2N \) and consider the following scaling ansatz \([3, 7, 12]\):

\[ \rho_{2N+k} = \rho_c[1 - \delta^2 f(t_k)] \]  

(53)

\[ \sigma_{2N+k} = -\delta^2 s(t_k) \]  

(54)

where

\[ \delta = \beta^{-1/5} \to 0 \]  

(55)

\[ t_k = (2\beta - 2N - k)\delta \]  

(56)

The double-scaling limit is achieved when \( \delta \to 0 \) and \( \frac{\beta}{N} \to 1 \) while \( t = t_0 \) is held fixed. Following reference \([12]\), it is appropriate to introduce two auxiliary scaling functions \( f_{\pm}(t_k) \) defined in terms of \( \rho(t_k) \) and \( \sigma(t_k) \) by

\[ f_{\pm}(t_k) = f(t_k) \pm s(t_k) \]  

(57)

When dealing with non-even potentials, the key point to obtain the differential equations for \( f_+ \) and \( f_- \) in the limit \( \delta \to 0 \) is to express \( f(t_k) \) and \( s(t_k) \) again as a perturbative series in the \( \delta \) parameter:

\[ f(t) = f_0(t) + f_1(t)\delta + f_2(t)\delta^2 + \cdots \]  

(58)

with

\[ f_0(t) = \frac{1}{2}\rho_-(t) \]  

(59)

\(^2\)Let us notice that this is the same ansatz as is \([12]\) but in a different notation.

\(^3\)I am very grateful to S. Dalley for explaining this point to me and drawing my attention to ref. \([12]\) during the course of this work.

\(^4\)We refer the reader to ref. \([12]\) for details.
and

\[ s(t) = s_0(t) + s_1(t)\delta + s_2(t)\delta^2 + \cdots \tag{60} \]

As far as the leading behaviour of the partition function is concerned, we only need to know about the differential equations satisfied the first terms in the expressions (58), (60). These are found to satisfy \[12\] \[13\] (up to trivial rescaling) the Painlevé I equation

\[ f_0^2 - \frac{1}{3} f_0'' = t \tag{61} \]
supplemented by the equation

\[ \rho_{+|\delta=0} = f_0 + s_0 = 0 \tag{62} \]

This completes the analysis of the Hermitian model with a cubic potential.

To begin with the continuum limit of the real symmetric model, we rewrite the recursion relation at \( i = N \) in the following form

\[ \frac{g_3^2}{4} W_{N+1} W_N \rho_{2N+2} \rho_{2N+1} - \frac{2N + 1}{2\beta} W_N + \rho_{2N+1} = 0 \tag{63} \]

where \( W_N \) is given by (45). This is a quadratic equation in the unknown variable \( W \). The spherical limit of (63) is achieved by taking \( N, \beta \to \infty \) with \( N/\beta \) finite and assuming \( W_N \to W_c, \rho_N \to \rho_c \). Tuning \( g_3 \) and \( \rho \) to their critical values (52) previously found, we observe that (63) has a unique root \( W_c \) given by

\[ W_c = 2\rho_c \tag{64} \]

Let us notice though that \( \rho_c \) can take two possible values according to (52). So, the critical behaviour of the Hermitian case leads also to critical behaviour in the symmetric model.

The equation (63) is similar to the ones appearing in the analysis of the real symmetric ensemble with a quartic potential \[7\], \[9\] and in the simplectic ensemble \[9\]. The technical difference is that our equation is quadratic in the
unknown variable $W$ while those in the forementioned references are cubic. Therefore, we shall use the same scaling ansatze for $W$ as in [5], namely

$$W_{N+k} = W_c e^{\delta \omega(t_k)}$$

(65)

where $\delta = \beta^{-1/5}$, and $W_c, t_k$ are given by (34), (56) respectively. The novelty of working with a non-even potential is that $\omega(t_k)$, with $t \equiv t_0$, has to be expanded in a series of $\delta$

$$\omega(t) = \omega_0(t) + \omega_1(t)\delta + \omega_2(t)\delta^2 + \cdots$$

(66)

in agreement with the procedure developed in (58) for the functions $\rho_N$. As far as the leading behaviour of the partition function is concerned, we are only interested in the differential equation obeyed by the term $\omega_0$ in (66). The way to proceed is to insert the ansatze (53) and (65) in the recursion relation (63) and to expand in $\delta$ with the aid of (56). Then, we solve for the $\omega'$s in order by order until we obtain a differential equation that determines $\omega_0$. This turns out to be,

$$3f_0 = \omega_0^2 - 2\omega_0'$$

(67)

We can thought of (67) as an inhomogeneous ordinary differential equation yielding the solution for the unorientable contribution $\omega_0$ to the free energy once the orientable contribution $f_0$ is known after solving the Painlevé I equation (61). This is the same kind of differential equation that appears in the analysis of the real symmetric ensemble carried out with a quartic potential [7], [9]. To be precise, the general solution found out with a quartic interaction also includes in (67) a term of the type $ce\int_0^t \omega_0(s)ds$, where $c$ is a constant of integration. Our solution corresponds to the choice $c = 0$. This $c$-term corresponds to different non-perturbative contributions to $\omega_0$ but, as far as the finite genus contributions is concerned, there is no difference of information among the possible choices of $c$. Thus, the asymptotics series of $\omega_0$ is $c$-independent [7] and is given by

\footnote{Up to trivial rescalings of $f_0$, $\omega_o$ and $t$ as in (61).}
\[ \omega_0 = \sum_0^\infty \omega_n^{(0)} t^{-(5n-1)/4} \]  

(68)

where now there are even and odd contributions to the free energy as an expansion of integer powers of $1/N$. It is also straightforward to rewrite the expression of the free energy $\phi$ \cite{1}, \cite{3} in the presence of a cubic interaction,

\[ \phi = f_0 + \omega' \]  

(69)

Let us finally remark that when $c = 0$, the mapping between $\omega_0$ and $f_0$ turns out to be an ordinary Miura transformation \cite{1} \cite{15}.

6. Concluding Remarks

There is a good understanding of the features exhibited by the Hermitian ensemble in its several versions such as the one-matrix model, multimatrix extensions \cite{16} \cite{17} and the Hermitian matrix model in the presence of an external field (the Kontsevich model) \cite{18}, \cite{19}, \cite{20}. However, if we exchange the Hermitian ensemble with the real symmetric ensemble in the aforementioned models, our knowledge decreases drastically. Even in the simplest case of a real symmetric one-matrix model, our understanding is not that good. This makes it quite interesting to check the properties of the real symmetric matrices in comparison with the equivalent properties for the Hermitian matrices, as has been done in this letter for the case of pure gravity.

Following this philosophy, it would be quite interesting to know whether the partition function of the real symmetric model is the $\tau$-function of a certain hierarchy of integrable differential equations. The Miura transformation \cite{17} might be helpful for this purpose. Unfortunately, this is not the same Miura transformation relating the KdV hierarchy associated with $(\partial^2 + f_0)$ to its partner known as the mKdV hierarchy \cite{1}.

There is a natural extension of the present work to check whether or not there exits a doubling of the ordinary differential equation \cite{67} when a more general non-even potential is considered \cite{21}. The simplest case in which this feature should appear is with a potential like \[ V(x) = g_2 x^2 + g_3 x^3 + g_4 x^4, \] as it happens in the Hermitian model \cite{11} \cite{12}. 

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Appendix I

Let us analyze the other possibility for the equation (36). If $a_{2k+1,2k} \neq 0$, we end up with the following solution for the element $a_{2k+2,2k+1}$,

$$a_{2k+2,2k+1} = -\frac{2}{\beta g_3 h_{2k}^2} - \frac{2k + 1}{\beta \rho_{2k+1}}$$  \hspace{1cm} (70)

But let us notice that this is precisely an element of the type $a_{2k,2k-1}$ already determined in equation (35). Thus, comparing the equations (70) and (35) we obtain now a recursion relation between $r_k$ and $h_{2k}$

$$h_{2k} = -\frac{g_3}{2} [\beta^2 g_3 \frac{r_{k+1}}{h_{2k+1}} + \frac{2k + 1}{\rho_{2k+1}}] r_k$$  \hspace{1cm} (71)

However, $a_{2k+1,2k}$ have not been determined yet. To do this, we have to use the remainig equation (18) for $j = 0$. In this way, the roles of equations (18) $j = 1$ and $j = 0$ have been exchanged.

It is convenient again to split indices in (18) $j = 0$ in evens and odds. If $i = 2k$, as $< R_{2k-1}, R_{2k} > R = 0$ and $< R_{2k+1}, R_{2k} > = -r_k$, we have

$$h_{2k} = \frac{\beta}{2} [g_3 a_{2k+2,2k+1} - \frac{2k + 1}{\beta \rho_{2k+1}}](-r_k)$$  \hspace{1cm} (72)

and this determines the last coefficient that we were left with being

$$a_{2k+2,2k+1} = -\frac{2}{g_3 \beta} \frac{h_{2k}}{r_k} + \frac{2k + 1}{g_3 \beta \rho_{2k+1}}$$  \hspace{1cm} (73)

Now, comparing the equations (70) an (73) we draw the conclusion that the possibility we are dealing with is only consistent iff the coupling constant takes the value
\[ g_3 = -1 \tag{74} \]

In addition we have still to impose the relation coming from equation (18) \( j = 0 \) when the indices are odd. Unlike the case of the simple solution (37), now this relation takes an extremely long and cumbersome form due to the fact that most of the \( a \)-coefficients do not vanish. Nevertheless, as the coupling constant is absolutely fixed by (74), the possibility (70), if consistent, would not be relevant as far as the continuum limit is concerned.

**Appendix II**

To obtain the recursion relation (41) solving the model, we start with equation (18) for \( j = 0 \). After some algebra involving the use of the observations described in section 3, we arrive at

\[ h_i = -\beta \frac{2}{3} v_{i,i+2} C_{i+2,R} < R_{i+2} > - \beta \frac{2}{3} v_{i,i+1} C_{i+1,R} < R_{i+1} > R + \beta \frac{2}{3} v_{i,i-1} C_{i-1,R} < R_{i-1} > R \tag{75} \]

Using (16) to invert the relation between \( R_i \) and \( C_i \), we find

\[ h_i = -\beta \frac{2}{3} v_{i,i+2} a_{i+2,i,i+1} - v_{i,i+1} ] < R_{i+1} > R + \beta \frac{2}{3} \{ v_{i,i+2} \{ -(a_{i+2,i+1} a_{i+1,i} - a_{i+2,i}) a_{i,i-1} + (a_{i+2,i+1} a_{i+1,i-1} - a_{i+2,i-1}) \} + v_{i,i+1} (a_{i+1,i,i+1} - a_{i+1,i-1}) - v_{i,i-1} \} < R_{i-1} > R \tag{76} \]

It is useful again to split the indices into even and odd. If \( i = 2k \), as \( < R_{2k-1}, R_{2k} > R = 0 \) and \( < R_{2k+1}, R_{2k} > R = -r_k \), it follows

\[ a_{2k+2,2k+1} = \frac{\beta}{2} g_3 \frac{r_{k+1}}{h_{2k+1}} \tag{77} \]

Replacing the value of \( a_{2k+2,2k+1} \) from (35) we obtain the desired result (41) for the recursion relation,

\[ h_{2k} = [ -\beta^2 \frac{3}{4} g_3^2 \frac{r_{k+1}}{h_{2k+1}} + \frac{2k + 1}{2r_{2k+1}} ] r_k \tag{78} \]
If \( i = 2k + 1 \), the analysis in principle is more cumbersome. Now we have
\[
< R_{2k+2}, R_{2k+1} >_R = 0 \quad \text{and} \quad < R_{2k}, R_{2k+1} >_R = r_k.
\]
Fortunately, using the equations (28), (32) and (37) it easy to see that all the constants \( a's \) entering the equation (76) are now zero, except for
\[
a_{i+2,i-1} = a_{2k+3,2k} = -\frac{\beta^2}{4} g_3 \frac{r_{k+1}}{h_{2k}}.
\]
Then, after using (19), the equation takes the form
\[
h_{2k+1} = \left[ -\frac{\beta^2}{4} g_3 \frac{r_{k+1}}{h_{2k}} + \frac{2k+1}{2} \right] r_k \tag{79}
\]
Dividing (79) by \( \rho_{2k+1} = \frac{h_{2k+1}}{h_{2k}} \) we achieve precisely the equation (78) again.

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