Recurrent words with constant Abelian complexity

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Abstract

We prove the non-existence of recurrent words with constant Abelian complexity containing 4 or more distinct letters. This answers a question of Richomme et al.

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1 Introduction

One of the central notions in combinatorics on words is that of the subword complexity of an infinite word. Richomme, Saari, and Zamboni [3] have recently begun a systematic study of the Abelian analogue of the subword complexity of infinite words. In this paper we resolve one of the open problems from their study.

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by showing the non-existence of recurrent words with constant Abelian complexity containing 4 or more distinct letters.

Let $\Sigma$ be a finite alphabet and let $\Sigma^*$ be the set of all finite words over the alphabet $\Sigma$. Consider the equivalence relation $\sim$ on $\Sigma^*$, defined by

$$u \sim v \text{ if } u \text{ is an anagram of } v.$$ 

Thus $1232 \sim 2132$. We write $[u]$ for the equivalence class of $u$ under $\sim$. For example, $[121] = \{112, 121, 211\}$. We call $[u]$ an Abelian word. If $u$ is a factor of a word $w$, we call $[u]$ an Abelian factor of $w$. The length of an Abelian factor is the length of any one of its representatives.

If $w$ is an infinite word, the subword complexity function of $w$ is the function $f : \mathbb{N} \to \mathbb{N}$, where for $m = 1, 2, \ldots$, the value of $f(m)$ is the number of factors of $w$ of length $m$. Similarly, the Abelian complexity function of $w$ is the function $\tilde{f}(m) : \mathbb{N} \to \mathbb{N}$, where for $m = 1, 2, \ldots$, the value of $\tilde{f}(m)$ is the number of Abelian factors of $w$ of length $m$.

An infinite word $w = w_0w_1\cdots$, where $w_i \in \Sigma$ for $i = 0, 1, 2, \ldots$, is ultimately periodic if there exist a non-negative integer $c$ and a positive integer $p$ such that $w_i = w_{i+p}$ for all $i \geq c$. A classical result of Morse and Hedlund \[2\] shows that an infinite word $w$ is ultimately periodic if and only if its complexity function $f$ is eventually constant. If $w$ is not ultimately periodic, then $f(m) \geq m + 1$ for all $m$.

The well-studied Sturmian words are precisely the aperiodic words of minimal complexity (i.e., those words for which $f(m) = m + 1$ for all $m \geq 1$). Coven and Hedlund \[1\] showed that any Sturmian word has constant Abelian complexity. In particular, for any Sturmian word, one has $\tilde{f}(m) = 2$ for all $m \geq 1$.

Sturmian words are necessarily over a binary alphabet; it is therefore natural to ask if over an $n$-letter alphabet, where $n \geq 3$, there is an infinite word $w$ with Abelian complexity function $\tilde{f}(m) = n$ for all $m \geq 1$. Without further qualification, this question is not very interesting, as one easily sees that the word $123\cdots(n-1)nnnnn\cdots$

over the alphabet $\{1, 2, \ldots, n\}$ has exactly $n$ Abelian factors of each length $m \geq 1$.

This observation leads us to the following definition. We say that a word $w$ is recurrent if every factor of $w$ occurs infinitely often in $w$. Any Sturmian word is recurrent, so such words provide examples of recurrent words with constant Abelian complexity over a binary alphabet. Richomme, Saari, and Zamboni showed that there are recurrent words over a 3-letter alphabet with exactly 3
Abelian factors of each length $m \geq 1$, thereby answering a question of Rauzy.
They also posed a question of their own, namely, “Does there exist a recurrent
word over a 4-letter alphabet with exactly 4 Abelian factors of each length?” They
also conjectured that the answer to the question should be “no”. We show that
this is indeed the case. Moreover, our main result also applies to alphabets of size
greater than 4.

**Theorem 1** Let $n \geq 4$ be an integer. There is no recurrent word over an $n$-letter
alphabet with exactly $n$ Abelian factors of each length $\geq 1$.

**2 Proof of Theorem [1]**

Fix a positive integer $n \geq 4$. Let $\Sigma$ be the alphabet $\{1, 2, 3, \ldots, n\}$. Let $w$ be
a finite or infinite word. Consider the graph $G$ with vertex set $\Sigma$, and an edge
$ij$ whenever at least one of $ij$ and $ji$ is a factor of $w$. Note that $G$ may contain
loops, but not multiple edges. From now on suppose that $w$ is a fixed recurrent
word, having constant Abelian complexity $n$.

**Lemma 2** Graph $G$ consists of a spanning tree and one additional edge. Thus $G$
contains a unique cycle $C$ (which is possibly a loop).

**Proof:** Since $w$ has Abelian complexity $n$, it contains all $n$ letters. It follows
that $G$ must be connected. This implies that $G$ contains a spanning tree. The
spanning tree contains $n - 1$ edges. Since the factors of $w$ of length 2 represent
exactly $n$ Abelian words, $G$ contains exactly $n$ edges. □

Let $b \in \Sigma$. Define $T(b) = \{[abc] : abc$ is a factor of $w$ for some $a, c \in \Sigma\}$. We
call an element of $T(b)$ a **triple associated with** $b$. Since $w$ is recurrent, every
letter of $\Sigma$ occurs in $w$ as the middle letter of at least one factor of length 3. This
means that each letter of $\Sigma$ has at least one triple associated with it.

**Lemma 3** Suppose that $a \neq b$ but triple $[abc]$ is associated with both $a$ and $b$.
Then exactly one of the following occurs:

1. $abc$ is a triangle in $G$

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1For definiteness of notation, let us say that we never call $a$ a neighbour of itself; however,
we will count a loop based at $a$ as contributing 1 to the degree of $a$. Thus the degree of a vertex
$a$ in $G$ will be the number of distinct neighbours of $a$, plus the number of loops based at $a$; since
we do not allow multiple edges, the number of loops based at $a$ is 0 or 1.
2. \(a = c\) and \(G\) contains the loop \(aa\).

3. \(b = c\) and \(G\) contains the loop \(bb\).

**Proof:** Since \(abc\) is associated to \(b\), at least one of \(abc\) and \(cba\) is a factor of \(w\). It follows that \(ab\) and \(bc\) are edges of \(G\). Since \(abc\) is a triple associated to \(a\), at least one of \(bac\) and \(cab\) is a factor of \(w\), so that \(ca\) is also an edge of \(G\). If \(a, b\) and \(c\) are distinct, then \(G\) contains triangle \(abc\). If two of them are the same, then since \(a \neq b\), one of \(bc\) and \(ca\) is a loop. □

**Lemma 4** Suppose that \(b, c\) and \(d\) are distinct neighbours of \(a\) in \(G\). Then \(|T(a)| \geq 2\).

**Proof:** Since \(b\) is a neighbour of \(a\), then either \(ba\) or \(ab\) is a factor of \(w\). By recurrence, either \([bax]\) or \([xab]\) will therefore be a factor of \(w\) for some \(x \in \Sigma\), and \([xb]\) is a triple associated with \(a\). Similarly, \(a\) must have associated triples \([cay]\), \([daz]\) for some letters \(y, z \in \Sigma\). If \([bax] \neq [cay]\), then we are done. Otherwise, \(x = c\) and \(y = b\) so that \([daz] \neq [bax]\). □

**Lemma 5** Suppose that \(a\) has distinct neighbours \(b\) and \(c\) in \(G\). Either \([bac] \in T(a)\) or \(|T(a)| \geq 2\).

**Proof:** One of \(ab\) and \(ba\) is a factor of \(G\). Suppose \(ab\) is a factor of \(w\). (The other case is similar). By recurrence, \(w\) has a factor \(xab\) for some \(x\). If \(x = c\), then \([bac]\) is associated to \(a\), and we are done. Otherwise, \([xab] \in T(a)\), and \(T(a)\) also includes a triple involving \(c\). □

**Case 1: Cycle \(C\) is an \(m\)-cycle, \(m \geq 4\).**

By Lemma 2, \(C\) is the unique cycle in \(G\). This implies that \(G\) contains no loops or triangles, so that triples associated with distinct vertices are distinct by Lemma 3. At least one triple is associated with each of the \(n\) vertices of \(G\). Since the Abelian complexity of \(w\) is exactly \(n\), the total number of triples associated with the vertices of \(G\) is \(n\). We conclude that \(|T(a)| = 1\) for each \(a \in \Sigma\). From Lemma 4, we conclude that each vertex of \(C\) has degree exactly 2, so that \(C\) is a connected component of \(G\). Since \(G\) is connected, \(G = C\) is an \(n\)-cycle. Without loss of generality let the vertices be connected in the natural order 123 \(\cdots n\) 1. By Lemma 5, we conclude that the triples associated with the vertices of \(G\) are \([123], [234], [345], \ldots, [(n - 2)(n - 1)n], [(n - 1)n1], [n12]\). Since \(w\) must be walked
on \(G\) respecting the possible triples, we conclude that \(w\) is a suffix of \((123 \cdots n)\omega\) or of \((n \cdots 321)\omega\) and thus has period \(n\). However, this means that \(w\) contains exactly one factor of length \(n\), up to anagrams. This is a contradiction.

**Case 2: Cycle \(C\) is a loop.**

Without loss of generality, let the loop edge be 11. At least one triple is associated with each of the \(n\) vertices of \(G\). A triple of the form \([111]\) could only ever be associated to 1. Also, \(b, c\) are neighbours of 1 and \([b1c]\) is associated to \(b\), then \(1bc\) or \(cb1\) is a factor of \(w\). This implies that \(bc\) is an edge of \(G\) so that \(ibc\) is a triangle (if \(b \neq c\)) or \(bc\) is a loop (if \(b = c\)). Since 11 is the only cycle in \(G\) by Lemma 2, this is impossible. It follows that triples of the form \(111\) or \(b1c\) where \(b\) and \(c\) are neighbours of 1 can only ever be associated to 1. It now follows that 1 can be associated to at most a single triple of the form \(111\) or \(b1c\) where \(b, c\) are neighbours of 1; if 1 is associated to two such triples \(T_1\) and \(T_2\), then each of the \(n - 1\) other vertices of \(G\) is associated to a triple, and these triples are distinct from \(T_1\) and \(T_2\), and from each other by Lemma 3. Then, however, we have at least \(n + 1\) distinct triples, violating the Abelian complexity of \(w\).

We make cases based on whether 1 is associated to a triple of the form \(111\) or \(b1c\) where \(b\) and \(c\) are neighbours of 1.

**Case 2a: Vertex 1 is associated to a triple of the form \([111]\).**

Each vertex of \(G - \{1\}\) is associated to some triple other than \([111]\), and those triples are distinct from each other and from \([111]\). Let \(b\) be a neighbour of 1. At least one of \(bl\) and \(lb\) is a factor of \(w\). Since 1 is not associated to any triple \([b1c]\) where \(b\) and \(c\) are neighbours of 1, it follows that \(b1l\) or \(l1b\) must be a factor of \(w\). Since the Abelian complexity of \(w\) is \(n\), we conclude that \([1lb] = [lb1]\) is the unique triple associated with \(b\). From Lemma 5, it follows that 1 is the only neighbour of \(b\). Graph \(G\) is therefore the star with center 1; the edges of \(G\) are precisely \(E(G) = \{lk : 1 \leq k \leq n\}\).

Let \(m\) be least such that \(w\) has a factor \(dl^me\) where \(d, e \neq 1\). Without loss of generality, say that \(21^m3\) is a factor of \(w\). Let \(b\) be any vertex of \(G - \{1\}\). Since \(1b\) is an edge of \(G\), \(w\) has a factor \(bl\) or \(lb\), hence a factor \(1^2bl^m\) or \(1^mbl^2\). (Recall that 1 is the only neighbour of \(b\) in \(G\).) It follows that up to anagrams, the \(n\) factors of \(w\) of length \(m + 3\) are \(121^m3, 1^m21^2, 1^m31^2, 1^m41^2, \ldots, 1^mn1^2\). In particular, \(w\) has no factor \(1^{m+2}\), and in any factor of the form \(bl^kc\) with \(b, c \neq 1\) and \(k \leq m\) we must have \(\{b, c\} = \{2, 3\}\) and \(k = m\).
Now consider the shortest factor of \( w \) containing a letter from \( \{2, 3\} \) and a letter from \( \{4, 5, \ldots, n\} \). By our last remark, it must have the form \( b1^k c \) or \( c1^k b \) where \( b \in \{2, 3\}, \ c \in \{4, 5, \ldots, n\} \) and \( k \geq m + 1 \). Since \( w \) has no factor \( 1^{m+2} \), \( k = m + 1 \), and we have found an \((n + 1)^{st}\) Abelian factor of \( w \). This is a contradiction.

**Case 2b: Vertex 1 is associated to exactly one triple of the form \([b1c]\) where \( b \) and \( c \) are neighbours of 1.**

In this case, \([111]\) is not associated with 1; i.e., 111 is not a factor of \( w \). Each vertex of \( G - \{1\} \) is again associated to some triple other than \([b1c]\), and these triples are distinct from each other. Let \( d \) be a neighbour of 1 other than \( b \) or \( c \). Vertex 1 cannot be associated to another triple \([d1e]\) where \( e \) is a neighbour of 1. Therefore, at least one of \( d11 \) and \( 11d \) is a factor of \( w \). It follows that \([11d] = [1d1]\) is the unique triple associated with \( d \). We conclude that 1 is the only neighbour of \( d \); viz., \( d \) is a leaf. We see also that (except possibly once at the beginning of \( w \)) \( d \) always appears in \( w \) in the context 111d11.

Now consider the shortest factor of \( w \) containing a letter from \( \{b, c\} \) and a neighbour of 1 other than \( b \) or \( c \). By our last remark, and relabeling \( b \) and \( c \) if necessary, this factor must have the form \( c1^k d \) or \( d1^k c \), \( k \geq 2 \), \( d \) some neighbour of 1 other than \( b \) and \( c \). If \([11c]\) is not associated to \( c \), then \([11c]\) and \([b1c]\) are associated only to 1, and we can count \( n + 1 \) distinct triples associated to vertices of \( G \). This violates the Abelian complexity of \( G \). We conclude that \([11c]\) must be the unique triple associated with \( c \), and \( c \) is a leaf.

**Case 2bi: Vertex \( b \) is a leaf.**

In this case, each neighbour of 1 is a leaf; graph \( G \) is the star with center 1. The edges of \( G \) are precisely \( E(G) = \{1k : 1 \leq k \leq n\} \). Let \( m \) be least such that \( w \) has a factor \( x1^md \) or \( d1^mx \) where \( x \in \{b, c\}, d \notin \{1, b, c\} \). Since \([b1c]\) is the unique triple associated only with 1, \( m \geq 2 \). On the other hand 111 is not a factor of \( w \), so \( m = 2 \). Without loss of generality, assume that \( b = 2, c \neq 3 \), and 2113 or 3112 is a factor of \( w \). Since 1 is the only neighbour of 2, it follows that 12113 or 31121 is a factor of \( w \). We have already seen that 11d11 is a factor of \( w \) if \( d \neq 1, b, c \). It follows that, up to anagrams, the following \( n + 1 \) factors of length 4 appear in \( w \):

\[
121c, 2113, 1121, 1131, 1141, \ldots, 11n1.
\]

This is a contradiction.
Case 2bii: Vertex $b$ has degree at least 2.

Since our Abelian complexity is $n$, and triple $b1c$ is associated only with 1, for any vertex $d$ of $G - \{1\}$, $|T(d)| = 1$. By Lemma 4, $\deg(d) \leq 2$. We may therefore assume that the edges of $G$ are

11, 1n, 1(n-1), 1(n-2), \ldots 1(r+1), 1r, r(r-1), (r-1)(r-2), (r-2)(r-3), \ldots, 32

and the triples are

$[11n], [11(n-1)], [11(n-2)], \ldots$

$[11(r+1)], (r+1)1r, [1r(r-1)], [r(r-1)(r-2)], \ldots, [432], [323].$

(We have $c = r+1$, $b = r$.) It follows that up to anagrams, $w$ has the $n$ length 4 factors

$11n1, 11(n-1)1, \ldots, 11(r+1)1$ \hspace{1cm} ($n - r$ factors)

$1(r+1)1r, (r+1)1r(r-1)$ \hspace{1cm} (2 factors)

$1r(r-1)(r-2), r(r-1)(r-2)(r-3), \ldots, 5432, 4323$ \hspace{1cm} ($r - 2$ factors).

Now however, consider the shortest factor of $w$ containing letters from both $\{n, n-1, \ldots, r-2\}$ and $\{r+1, r\}$. This must have the form $x1^ky$ or $y1^kx$ where $x \in \{n, n-1, \ldots, r-2\}$ and $y \in \{r+1, r\}$. Since $[b1c]$ is the unique triple associated only to 1, we cannot have $k = 1$. Since 111 is not a factor of $w$, we must have $k = 2$. This gives an $(n+1)^{st}$ length 4 Abelian word in $w$, namely $x11y$. This is a contradiction.

Case 2c: Every triple associated with 1 has the form $[11b]$, $b \neq 1$.

In this case, $w$ has no factors 111 or $b1c$ with $b, c \neq 1$. If $b$ is any neighbour of 1 therefore, either $b11$ or $11b$ is a factor of $w$. If 1 has no non-leaf neighbour, $G$ is a star centered at 1; for $2 \leq k \leq n$, the only length three factors of $w$ containing $k$ are among $11k, 1k1$ and $11k$. The triples of $G$ are precisely those of the form $[11k], k \neq 1$, and $G$ has only $n - 1$ triples. This is a contradiction.

Therefore, let $b$ be a non-leaf neighbour of 1, let $b' \neq 1$ be a neighbour of $b$. The shortest factor of $w$ containing 1 and $b'$ must be $1bb'$ or $b'1b$, so $[1bb']$ is a triple associated with $b$. Now every vertex of $G - \{1\}$ has at least one triple associated with it, and all such associated triples must be distinct. Moreover, $b$ has triples $[11b]$ and $[1bb']$ associated with it. We have now listed $n$ distinct triples associated with the vertices of $G - \{1\}$. If 1 had another non-leaf neighbour $c \neq b$, then an
\((n+1)^{st}\) triple \([1cc']\) would be associated to \(c\). Since this is impossible, it follows that \(b\) is the only non-leaf neighbour of 1.

Without loss of generality, let the neighbours of 1 be exactly \(2, 3, \ldots r = b\), and let \(r + 1\) be a neighbour of \(r\). The \(r\) triples \([112], [113], \ldots [11r], [1r(r + 1)]\) will be associated to vertices 1, 2, \ldots, \(r\), while the triples associated with vertices \((r + 1), (r + 2), \ldots, n\) must be distinct from these and from each other. This means that exactly one triple is associated to each of vertices \((r + 1), (r + 2), \ldots, n\), so that by Lemma \(\ref{lem:degree}\) they each have degree at most 2. Without loss of generality we may thus assume that the edges of \(G\) are

\[11, 12, 13, \ldots\]

\[1r, r(r + 1), (r + 1)(r + 2), (r + 2)(r + 3), \ldots, (n - 1)n\]

some \(r, 1 < r \leq n\). The \(n\) triples associated to vertices of \(G\) must thus be precisely

\[ [112], [113], [114],\]

\[ [11r], [1r(r + 1)], [r(r + 1)(r + 2)], \ldots, [(n - 2)(n - 1)n], [(n - 1)n(n - 1)]. \]

For \(2 \leq k \leq r - 1\), the only neighbour of vertex \(k\) is vertex 1. Since \(w\) has no factors 111 or \(bc\) with \(b, c \neq 1\), it follows that \(w\) has a factor 1111 for \(2 \leq k \leq r - 1\). In addition to these \(r - 2\) factors of length 4, the specification of triples forces \(w\) to have (up to reversal) the \(n - r + 1\) factors

\[11r(r + 1), 1r(r + 1)(r + 2), r(r + 1)(r + 2)(r + 3), (r + 1)(r + 2)(r + 3)(r + 4),\]

\[\ldots, (n - 3)(n - 2)(n - 1)n, (n - 2)(n - 1)n(n - 1).\]

In addition, since 11r or r11 is a factor of \(w\), so is a word \(c11r\) or \(r11c\), where \(c\) is some neighbour of 1 in \(G\). This brings the count of Abelian factors of length 4 to \((r - 2) + (n - r + 1) + 1 = n\). Suppose now that \(d\) is a neighbour of 1 other than \(r\) and \(c\). Then \(w\) contains a factor \(d11\) or \(1ld\), hence a word \(d1le\) or \(e11d\), where \(e\) is a neighbour of 1. This brings the number of length 4 Abelian factors of \(w\) to \(n + 1\), which is a contradiction. It follows that the only neighbours of 1 are \(r\) and \(c\). (Note that perhaps \(r = c\).) The length 4 Abelian factors of \(w\) are thus

\[ c11r, [11r(r + 1)], [1r(r + 1)(r + 2)], [r(r + 1)(r + 2)(r + 3)], [(r + 1)(r + 2)(r + 3)(r + 4)], \ldots,\]

\[ [(n - 3)(n - 2)(n - 1)n], [(n - 2)(n - 1)n(n - 1)]. \]
In the case that $c \neq r$, this forces $w$ to be a suffix of 

$$\begin{align*}
&\quad (c1r(r+1)(r+2)(r+3)(r+4)\cdots \\
&\quad (n-2)(n-1)n(n-1)(n-2)\cdots (r+4)(r+2)(r+1)r^{11}\omega ,
\end{align*}$$

and $w$ is periodic, with period $2n$. However, this means that $w$ contains exactly one factor of length $2n$, up to anagrams, which is a contradiction.

In the case that $c = r$, this forces $w$ to be a suffix of 

$$\begin{align*}
&\quad (r(r+1)(r+2)(r+3)(r+4)\cdots \\
&\quad (n-2)(n-1)n(n-1)(n-2)\cdots (r+4)(r+2)(r+1)r^{11}\omega ,
\end{align*}$$

and again $w$ is periodic, with a contradiction.

### Case 3: Cycle $C$ is a 3-cycle.

Let the vertices of $C$ be $a, b, c$. By Lemma 3, the only triple which can be associated with more than one vertex is $[abc]$. Each vertex of $G - \{a, b, c\}$ is associated with some triple, and these must all be distinct. This accounts for $n - 3$ triples. Since $G$ is connected, $w$ must contain some factor of the form $xab, xba, xbc, xcb, xca$ or $xac$, for some $x \notin \{a, b, c\}$. Suppose without loss of generality that $[xab]$ is associated to $a$ for some $x \notin \{a, b, c\}$. Then $a$ has degree at least 3, so that $|T(a)| \geq 2$ by Lemma 4. Since $|T(b)|, |T(c)| \geq 1$ but the total number of distinct triples associated to vertices of $G$ is $n$, at least two of $a, b$ and $c$ have an associated triple in common. That triple must be $[abc]$. So far, we have found that $[xab], [abc] \in T(a) \cup T(b) \cup T(c)$.

### Case 3a: $T(a) \cup T(b) \cup T(c) = \{[abc], [bax]\}$.

The shortest factor of $w$ starting with $x$ and ending in one of $b$ or $c$ will be $xab$. (Such a factor exists because $w$ is recurrent.) Let $uxab$ be a prefix of $w$. As the only triple associated to $b$ is $[abc]$, $w$ has $uxabc$ as a prefix. Again, $T(c) = \{[abc]\}$, so $uxabca$ is a prefix of $w$. The only triple in $T(a)$ having $c$ as one of its letters is $[abc]$, so $uxabcab$ is a prefix of $w$. Continuing in this way, we find that $w = ux(abc)^{\omega}$. This is impossible, since $x$ must appear in $w$ infinitely often, by recurrence.
Case 3b: $|T(a) \cup T(b) \cup T(c)| = 3$.

The argument of Case 3a can still be applied if we add to $T(a) \cup T(b) \cup T(c)$ another triple from $bab$, $cbc$ or $aca$; none of these triples allows us to break the circular order $a - b - c - a$ on $\{a, b, c\}$ which commences with $xab$. Similarly, adding to $T(a) \cup T(b) \cup T(c)$ a triple $[ybc]$ where $y \neq \{a, c\}$ is a neighbour of $b$ would lead to the same contradiction. Again a triple $[yca]$ where $y \neq \{b, a\}$ is a neighbour of $c$, or a triple $[yab]$ where $y \neq \{b, c\}$ is a neighbour of $a$ leads to a contradiction.

We may therefore assume $w$ contains a factor $aba$, $bcb$, $cac$, or a factor of the form $aby$, $bcy$ or $cay$, $y \notin \{a, b, c\}$.

Since $T(a) \cup T(b) \cup T(c)$ contains three distinct triples, and a distinct triple is associated to each of the $n - 3$ vertices of $G - \{a, b, c\}$, we deduce that each vertex of $G - \{a, b, c\}$ is only associated with a single triple, and thus has degree at most 2 by Lemma 4. We recall that $G$ contains exactly one cycle. Graph $G$ therefore consists of the triangle $abc$, together with 1 or more paths radiating from its vertices.

Case 3bi: Word $w$ contains a factor $aba$, $bcb$, $cac$.

In this case, the only vertex of $G - \{a, b, c\}$ which is adjacent to any of $a$, $b$ and $c$ is $x$. Graph $G$ consists of triangle $abc$ together with a single path adjacent to $a$. Relabel $x = a_1$, and let the edges of $G$ be

$$ab, bc, ca, aa_1, a_1a_2, a_2a_3, \ldots, a_{r-1}a_r$$

where $r = n - 3$.

By Lemma 5 the triples of $G$ are precisely

$$[baa_1], [aa_1a_2], [a_1a_2a_3], [a_2a_3a_4], \ldots, [a_{r-2}a_{r-1}a_r], [a_{r-1}a_1a_{r-1}], [abc], T$$

where $T$ is one of $[aba]$, $[bcb]$ and $[cac]$.

The only two triples containing both $a$ and $a_1$ are $[baa_1]$ and $[aa_1a_2]$. We conclude that $w$ contains Abelian factor $[aba_1a_2]$. Reasoning similarly, we find that for $r \geq 4$, the following $n - 2$ length 4 Abelian factors must be in $w$:

$$[baa_1a_2], [aa_1a_2a_3], [a_1a_2a_3a_4], \ldots, [a_{r-3}a_{r-2}a_{r-1}a_r], [a_{r-2}a_{r-1}a_1a_{r-1}].$$

(The stipulation $r \geq 4$ is only for notational convenience. If $r = 3$, let $a_0 = a$, and the 3 length 4 Abelian factors are $[baa_1a_2], [aa_1a_2a_3], [a_1a_2a_3a_2]$. If $r = 2$, let

\[2\text{If }r = 1, \text{ we use the convention } a_0 = a_2 = a.\]
Now consider a factor $v$ of $w$ of the form $a_1\{a, b, c\}^*a_1$, containing $ac$ or $ca$ as a factor. Such a $v$ exists by recurrence. Since the only triple joining $abc$ and $G - \{a, b, c\}$ is $[baa_1]$, $v$ can be written in the form $a_1abv_1baa_1$ where $v_1 \in \{a, b, c\}^*$ and $ac$ or $ca$ appears in $v_1$. The circular order of $abv_1ba$ changes exactly once, from $a - b - c - a$ to $a - c - b - a$, at triple $T$. Thus $v_1$ cannot both begin and end with $a$, lest $aba$ appear twice in $abv_2ba$. Thus $v_1$ must either begin or end with $c$, so that $a_1abc$ or $cbaa_1$ is a factor of $w$, yielding Abelian factor $[a_1abc]$ in either case. Notice that we have shown that $abv_1ba$ cannot both begin and end with a palindrome. It thus follows that $\{tz\}$ is also an Abelian factor of $w$, where $t \in T$ is a palindrome and $z$ is the letter of $\{a, b, c\}$ not appearing in $t$.

Suppose now that $v_1$ begins with $a$. The case where $v_1$ ends in $a$ is similar. Then $a_1aba$ is a factor of $w$, and we have enumerated all $n$ length 4 Abelian factors of $w$: the $n - 3$ previously listed, plus $[a_1abc], [tz] = [abc]$ and $[a_1aba]$. The circular order of $\{a, b, c\}$ in $abv_1ba$ changes exactly once (with $aba$), so that $abv_1ba \in aba(cba)^*$. It follows that $bacba$ is a suffix of $abv_1ba$, and $w$ also contains Abelian factor $[bacb]$. This is a contradiction. We conclude that $v_1$ cannot begin or end with $a$, and hence must begin and end with $c$. Since $[ca]$ is an Abelian factor of $v$, we cannot have $abv_1ba = abcba$.

Thus far, $w$ has Abelian factors $[a_1abc]$ and $[tz]$ in addition to the $n - 3$ length 4 Abelian factors previously listed. Let $y$ be the central letter of palindrome $t$ and write $abv_1ba = v_2yv_3$ where $v_2y$ is a prefix of $(abc)^\omega$ and $yv_3$ is a suffix of $(cba)^\omega$. We must have $|v_2| \equiv |v_3| \pmod{3}$. Also, $|v_2|, |v_3| \geq 2$. Suppose that $|v_2| > |v_3|$. Then $|v_2| \geq |v_3| + 3 \geq 5$. In this case, $abv_1$ has a prefix $abcabc$, and $w$ contains Abelian factors $[abc], [bcac], [cabc]$. One of these is $[tz]$, but this still gives $n + 1$ length 4 Abelian factors of $w$, which is impossible. We similarly rule out $|v_2| < |v_3|$. Note that we may also assume that $|v_2| \leq 4$. Since $|abv_1ba| > 5$, we find that $3 \leq |v_2| = |v_3| \leq 4$. If $|v_2| = 3$, then $abv_1ba = abacacba$, $t = cac$ and $[abca]$ and $[bcac]$ are Abelian factors of $w$. We have now specified all length 4 Abelian factors of $w$; none of these is $[a_1t]$, the central letter in $t$ is not $c$ and the set of length 4 Abelian factors of $w$ turns out to be determined by $t$ and $|v| = 2|v_2| + 3$. Similarly, if $|v_2| = 4$, then $abv_1ba = abcabacba$, $t = aba$ and $[abca]$ and $[bcab]$ are Abelian factors of $w$. Again the set of all length 4 Abelian factors of $w$ is determined by $t$ and $|v|$, none of the Abelian factors is $[a_1t]$ and $c$ is not the central letter in $t$.

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3Throughout, when we say “palindrome” we mean one of the three palindromes $aba, bcb, cac$. 

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Since the two different possible lengths for $v$ give different sets of Abelian factors in $w$, it follows that $w$ contains exactly one factor $v$ of the form $a_1\{a, b, c\}^*a_1$ containing $[ac]$ as an Abelian factor. Now let $v'$ be any factor of $w$ of the form $a_1\{a, b, c\}^+a_1$. Word $v'$ must have prefix $a_1ab$ and suffix $baa_1$. However, since $[a_1t]$ is not an Abelian factor of $w$, $v'$ cannot have $a_1aba$ as a prefix or $abaa_1$ as a suffix. Word $v'$ therefore has $a_1abc$ as a prefix and $cbaa_1$ as a suffix. Again, $v' \neq a_1abcbaa_1$, since the central letter of $t$ is not $c$. We deduce that $v'$ has prefix $a_1abca$ or suffix $acbaa_1$, and must contain $[ac]$ as an Abelian factor. In summary, $w$ contains exactly one factor of the form $v = a_1\{a, b, c\}^*a_1$. If $r = 1$, this shows that $w$ is periodic, giving a contradiction. If $r \geq 2$, our earlier specification of the $n – 3$ triples of $w$ along the path $baa_1 \cdots a_r$ shows that $w$ contains a single factor of the form $a_1(\Sigma – \{a, b, c\})^*a_1$, namely $a_1a_2 \cdots a_{r-1}ra_{r-1} \cdots a_2a_1$. Since $aa_1a$ is not a factor of $w$, we again deduce that $w$ is periodic, giving a contradiction.

Case 3bii: Word $w$ contains a factor $aby$, $bcy$ or $cay$, $y \notin \{a, b, c\}$.

We consider first the case where $w$ contains a factor $aby$, $y \notin \{a, b, c\}$. Since $b, c$ and $x$ are neighbours of $a$, and $abc$ is the only cycle in $G$, we cannot have $y = x$. Graph $G$ consists of triangle $abc$ together with two paths adjacent to $a$ and $b$. Relabel $x = a_1$, $y = b_1$ and let the edges of $G$ be

$$ab, bc, ca, a_1a_2, a_2a_3, \ldots, a_{r-1}a_r, bb_1, b_1b_2, \ldots, b_{s-1}b_s$$

where $r + s = n – 3$. By Lemma 5, the $n$ triples of $G$ are

$$[baa_1], [aa_1a_2], [a_1a_2a_3], [a_2a_3a_4], \ldots, [a_{r-2}a_{r-1}a_r], [a_{r-1}a_ra_{r-1}], [abc]$$

and

$$[abb_1], [bb_1b_2], [b_1b_2b_3], [b_2b_3b_4], \ldots, [b_{s-2}b_{s-1}b_s], [b_{s-1}b_sb_{s-1}].$$

The following $n – 3$ length 4 Abelian factors must be in $w$:

$$[baa_1a_2], [aa_1a_2a_3], [a_1a_2a_3a_4], \ldots, [a_{r-3}a_{r-2}a_{r-1}a_r], [a_{r-2}a_{r-1}a_ra_{r-1}]$$

and

$$[abb_1b_2], [bb_1b_2b_3], [b_1b_2b_3b_4], \ldots, [b_{s-3}b_{s-2}b_{s-1}b_s], [b_{s-2}b_{s-1}b_sb_{s-1}].$$

Let $v$ be a shortest factor of $w$ of the form $\{a_1, b_1\}uc$. A prefix of $v$ must be $a_1ab$ or $b_1ba$. Suppose $v$ has prefix $a_1ab$. Then $v \in a_1ab\{a, b, c\}^*c$. Letters $a, b, c$
must have circular order \( a - b - c - a \) in \( v \), since there are no palindromes in \( v \) to change the circular order, and \( v \) starts \( a_1abc \). Let \( p \) be a prefix of \( w \) of the form \( qa_1(abc)^j \) with \( j \) as large as possible. If \( j = 2 \), then \( abcabc \) is a factor of \( w \), so that \( w \) contains 4 more Abelian factors:

\[
[a_1abc], [abca], [bcab], [cabc].
\]

This is impossible, since then \( w \) has \( n + 1 \) distinct length 4 Abelian factors. It follows that \( j = 1 \), and \( qa_1abcabb_1 \) is a prefix of \( w \). (Recall that the only triples associated with \( a, b \) or \( c \) are \([abc], [a_1ab], [abb] \).) Now, however, \( w \) contains Abelian factors

\[
[a_1abc], [abca], [bcab], [cabb_1],
\]

again giving a contradiction.

Now consider the case where \( w \) contains a factor \( bcy \), \( y \not\in \{a, b, c\} \). Since \( abc \) is the only cycle in \( G \), \( y \) is not a neighbour of \( b \) or \( a \). Since \( ac \) is an edge of \( G \), either \( ac \) or \( ca \) is a factor of \( w \). Suppose \( ac \) is a factor of \( w \). (The other case is similar.) Recall that the only triples associated with one of \( a, b \) or \( c \) are \([abc], [xab] \) and \([bcy] \). The only one of these containing both \( a \) and \( c \) is \([abc] \). If \( ac \) is a factor of \( w \), then it must therefore be preceded and followed by \( b \), and occurs in the context \( bacb \). Since neither of \([xab] \) and \([bcy] \) is associated with \( b \), \( cb \) is followed by \( a \). Again, \( ba \) is preceded by \( c \), so \( ac \) occurs in the context \( cbacba \). Since \( w \) is recurrent, it cannot have \((cba)^w \) as a suffix. It follows that \( w \) must have a factor \( cbacbx \). This implies that \([cabc], [bcab], [acba], [cbax] \) are length 4 Abelian factors of \( w \). As in previous cases, the paths attached to vertices \( a \) and \( c \) of triangle \( abc \) furnish another \( n - 3 \) distinct length 4 Abelian factors, giving a contradiction.

The final case occurs when \( w \) contains a factor \( cay \), \( y \not\in \{a, b, c\} \). In this case \( G \) consists of a triangle with two disjoint paths attached at \( a \). In the usual way, we find \( n - 3 \) length 4 Abelian factors of \( w \), each containing at least two path vertices (i.e. vertices of \( G - \{a, b, c\} \)). If \( abcabc \) or \( cbacba \) were a factor of \( w \), \( w \) would then contain 4 additional length 4 Abelian factors \([abca], [bcab], [cabc] \) and \([bcay] \), giving a contradiction. We therefore conclude that the only factor of \( w \) of the form \( x\{a, b, c\}^*y \) is \( xabcay \), and the only factor of \( w \) of the form \( y\{a, b, c\}^*x \) is \( xabc \). Thus, if \( L_1 \) is the leaf at the end of the path starting with \( a - x \) and \( L_2 \) is the leaf at the end of the path starting with \( a - y \), \( w \) has only one factor of the form \( L_1(\Sigma - \{L_1, L_2\}^*)L_2 \), and only one factor of the form \( L_2(\Sigma - \{L_1, L_2\})^*L_1 \), so that \( w \) is periodic, oscillating between \( L_1 \) and \( L_2 \). The periodicity of \( w \) gives a contradiction. □
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