ON THE BERGMAN PROJECTION AND KERNEL IN PERIODIC PLANAR DOMAINS

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ABSTRACT. We study Bergman kernels \( K_\Pi \) and projections \( P_\Pi \) in unbounded planar domains \( \Pi \), which are periodic in one dimension. In the case \( \Pi \) is simply connected we write the kernel \( K_\Pi \) in terms of a Riemann mapping \( \varphi \) related to the bounded periodic cell \( \varpi \) of the domain \( \Pi \). We also introduce and adapt to the Bergman space setting the Floquet transform technique, which is a standard tool for elliptic spectral problems in periodic domains. We investigate the boundedness properties of the Floquet transform operators in Bergman spaces and derive a general formula connecting \( P_\Pi \) to a projection on a bounded domain. We show how this theory can be used to reproduce the above kernel formula for \( K_\Pi \). Finally, we consider weighted \( L^p \)-estimates for \( P_\Pi \) in periodic domains.

1. INTRODUCTION.

In this paper we consider Bergman spaces \( A^p(\Pi) \) and the Bergman projection \( P_\Pi \) on planar domains \( \Pi \subset \mathbb{C} \), which have the special geometry of being periodic in one direction; the domain is obtained as the union of infinitely many translated copies of the bounded periodic cell \( \varpi \subset \mathbb{C} \). We aim to work out the basic concepts of the Floquet transform theory in the context of operator theory in Bergman spaces of periodic domains. If the periodic domain \( \Pi \) is considered just as a domain in the two dimensional real Euclidean space, the Floquet-transform, acting in appropriate Sobolev spaces, is a standard tool for example in spectral elliptic boundary value problems, see e.g. [13]. Here, we will consider the mapping properties of the Floquet-transform in Bergman spaces. The main goal is to study the connection of the Bergman projection and kernel on \( \Pi \) with certain kernels related with the periodic cell especially in the simply connected case.

Let us fix some notation. Given a domain \( \Omega \) in the complex plane \( \mathbb{C} \), we denote by \( L^2(\Omega) \) the usual Lebesgue-Hilbert space with respect to the (real) area measure \( dA \) and by \( A^2(\Omega) \) the corresponding Bergman space, which is the subspace consisting of analytic functions. It is a consequence of the Cauchy integral formula that the norm topology of \( A^2(\Omega) \) is stronger than the topology of the uniform convergence on compact subsets, and this implies that the Bergman space is always a closed subspace of \( L^2(\Omega) \), hence it is a Hilbert space and in particular complete. We denote by \( P_\Omega \) the orthogonal projection from \( L^2(\Omega) \) onto \( A^2(\Omega) \). It can always written by the help of the Bergman kernel \( K_\Omega : \Omega \times \Omega \to \mathbb{C} \),

\[
P_\Omega f(z) = \int_\Omega K_\Omega(z, w)f(w)dA(w)
\]

and the kernel has the properties that \( K_\Omega(z, \cdot) \in L^2(\Omega) \) for all \( z \) and \( K(z, w) = \overline{K(w, z)} \) for all \( z, w \in \Omega \). See e.g. [10] for a proof of these assertions.
We denote the periodic cell by \( \varpi \) and assume that \( \varpi \subset [0, 1] - M, M \subset \mathbb{R}^2 \cong \mathbb{C} \) for some \( M > 0 \) and that its intersection with the axis \( \{ z : \text{Re } z = \frac{1}{2} \pm \frac{1}{2} \} \) coincides with \( \{ \frac{1}{2} \pm \frac{1}{2} \} \times (a, b) =: J_\pm \) for some real numbers \( b > a \). We denote the translates of \( \varpi \) by \( \varpi_m = \varpi + m \), where \( m \in \mathbb{Z} \subset \mathbb{C} \), and then define the periodic domain \( \Pi \) as the interior of the set

\[
\bigcup_{m \in \mathbb{Z}} \text{cl}(\varpi_m).
\]

where \( \text{cl}(A) \) denotes the closure of the set \( A \). Moreover, it is assumed that \( \varpi \) and \( \Pi \) are Lipschitz domains such that the boundaries \( \partial \varpi \) and \( \partial \Pi \) are in addition piecewise smooth: more precisely, \( \partial \varpi \) consists of finitely many parametrized curves \( \gamma : [0, 1] \to \mathbb{C} \) where \( \gamma \) is continuously differentiable on \([0, 1]\). Note that the Lipschitz assumption excludes cusps both in \( \varpi \) and \( \Pi \). Consequently, \( \partial \varpi \) is a Jordan curve. This implies that \( \varpi \) is a Carathéodory domain so that by an old result of Carleman, polynomials form a dense subspace of the Bergman space \( A^2(\varpi) \) (see also the introduction in the paper [2]). One more technical, geometric assumption \((A)\) will be posed in Section 2. Note also that \( \Pi \) is simply connected, if and only \( \varpi \) is; in the case \( \varpi \) is multiply connected, the periodic domain is of course no more finitely connected.

Our results are as follows. In Section 2 we consider simply connected \( \Pi \), and in Proposition 2.3 we present a general formula for the kernel \( K_{\Pi} \) in terms of a Riemann mapping \( \varphi \) related to the periodic cell \( \varpi \). This is actually a direct consequence of the fact that it is easy to write the Riemann mapping from \( \Pi \) onto the strip, where the kernel has an explicit formula. We consider cases of domains with polygonal boundaries with explicit Schwartz-Christoffel-type formulas for the Riemann mapping.

In Section 3 we present the general definition of the Floquet transform \( F \) for functions of real variables in periodic planar domains. Briefly, \( F \) transforms functions defined on the periodic domain \( \Pi \) into functions on \( \varpi \) depending on the so-called Floquet variable or quasimomentum \( \eta \in [-\pi, \pi] \). We observe that the definition works nicely also in the case of analytic functions, and in particular in Theorem 3.6 we prove the isometry of \( F \) between the space \( A^2(\Pi) \) and the corresponding parametrized Bergman-type space \( L^2(-\pi, \pi; A^2_\eta(\varpi)) \) on the periodic cell.

In Section 4 we apply the Floquet transform techniques to formulate a general formula involving the projection \( P_{\eta} \) and \( \eta \)-dependent Bergman-type projections \( P_{\eta} \) related to the periodic cell \( \varpi \). In contrast to Section 2 this consideration is made for general, not necessarily simply connected domains.

Section 5 contains some elementary preparations for Section 6 where we derive a presentation for the projections \( P_{\eta} \). This formula is applied in Section 4 to reproduce the kernel formula for \( P_{\eta} \) in the simply connected case by using Floquet transform techniques. We also give an application of the kernel formula to weighted \( L^p \)-estimates.

As for the notation used in this paper, we let \([x]\) be the integer part of the number \( x \in \mathbb{R} \), i.e. the largest integer \( y \) with \( y \leq x \). We write \( \mathbb{R}^+ = \{ x + iy \in \mathbb{C} : x > 0 \} \). If \( a \in \mathbb{C} \) and \( r > 0 \), \( B(a, r) \) denotes the open Euclidean disc with center \( a \) and radius \( r \). Given a domain \( \Omega \subset \mathbb{C} \) we denote by \( \| f \|_\Omega \) the norm of \( L^2(\Omega) \), which is the \( L^2 \)-space with respect to the (real) area measure. If \( H \) is a Hilbert space, its inner product is denoted by \( (f|g)_H \), where \( f, g \in H \). We recall that if \( \varphi \) is a conformal
mapping from the domain $\Omega \subset \mathbb{C}$ onto the domain $\Omega'$, then the $|\varphi'|^2$ is the Jacobian of the coordinate transform $\varphi$ in the area integrals.

2. Bergman kernels and conformal mappings on periodic domains.

If $\Omega$ and $\Omega'$ are conformally equivalent domains in $\mathbb{C}$ and $\varphi : \Omega' \to \Omega$ is a conformal mapping and if $K(z, w)$ is the Bergman kernel of the domain $\Omega'$, then the Bergman kernel of the domain $\Omega$ can be got from the formula

$$K_\Omega(z, w) = K(\varphi(z), \varphi(w))|\varphi'(z)\varphi'(w)|,$$

see for example [1], formula (1), or [10], Proposition 2.7. We refer to the papers [4], [6], [7] for general constructions of conformal mappings onto periodic domains, but we will here construct one which leads to a useful formula of the Bergman kernel.

From now on we assume in this section that $\Pi$ and $\varpi$ are simply connected, denote $E(z) := e^{i2\pi z}$ and define the domain $D := \{E(z) : z \in \varpi \cup J_+ \cup J_-\}$, which as a consequence of the geometric assumptions in Section 1 is doubly connected. Consequently, there exists a conformal mapping $\phi$ from $D$ onto the annulus $A = \{z : 1/\rho < |z| < \rho\}$ for a uniquely determined $\rho > 1$ (see for example [7], [9], p. 362, the discussion in [19]). We also denote

$$D_l = \{E(z) : z \in \varpi\} = D \setminus (D \cap \mathbb{R}^+) \quad \text{and} \quad A_l = \phi(D_l).$$

We next observe that the function $z \mapsto (i2\pi)^{-1}\log(E(z))$ can be extended as an analytic function to the periodic domain $\Pi$. This is done by defining the branch cut of log on the annulus $A$ on the curve $\Gamma := \phi(\partial D \cap \mathbb{R}^+) = \phi(E(J_+)) = \phi(E(J_-))$. In order to describe this in detail and for the sake of simplicity, we pose our final additional geometric assumption, which obviously could be made weaker.

(A) We assume that $\Gamma$ is contained, for some $0 < \delta < 1$, in the sector $\{z \in \mathbb{C} : \delta < \arg z < \pi - \delta\}$.

**Remark 2.1.** It is plain that (A) is satisfied, if the mapping $\phi$ distorts the arguments of $z \in D \cap \mathbb{R}^+$ at most by $\pm(\pi/2 - \delta)$ for some fixed $\delta > 0$. One then achieves (A) by applying a rotation of the annulus $A$, and redefining $\phi$ accordingly. This holds for example, if $\Pi$ is a small 1-periodic perturbation of a horizontal strip. However, we conjecture that assumption (A) is completely unnecessary for the results of this paper.

Let us denote $\varpi_+ = \varpi \cap \{z \in \mathbb{C} : \text{Re} z \geq 1/2\}$ and $\varpi_- = \varpi \setminus \varpi_+$. If $\tilde{z} \in \Gamma$, then it has two preimages $\tilde{z}_\pm \in J_\pm$ with $\phi(E(\tilde{z}_\pm)) = \tilde{z}$. We consider a disc $B(\tilde{z}, r) \subset A$, where $r = r(\tilde{z})$ is so small that both sets $B(\tilde{z}, r) \cap \phi(E(\varpi_-))$ are connected, and we define $\arg z \in [0, 3\pi]$ such that

$$\arg z < \pi \quad \text{for} \quad z \in B(\tilde{z}, r) \cap \phi(E(\varpi_-)) \quad \text{and} \quad \arg z > 2\pi \quad \text{for} \quad z \in B(\tilde{z}, r) \cap \phi(E(\varpi_+)).$$

It is clear that

$$\lim_{z \to \tilde{z}} \arg z = 2\pi + \lim_{z \to \tilde{z}} \arg z.$$

The logarithm is defined accordingly in $\phi(E(\varpi))$. 


Lemma 2.2. The mapping

\begin{equation}
\varphi(z) = \frac{1}{i2\pi} \log \left( \phi(e^{i2\pi z}) \right) + \text{Re } z
\end{equation}

is conformal from \( \Pi \) onto the strip \( S = (-\infty, \infty) \times (-\pi, \pi) \).

Proof. On \( \varpi \ni z \) we have \( \varphi(z) = \frac{1}{i2\pi} \log \left( \phi(e^{i2\pi z}) \right) \), which is a conformal mapping, and we also have for every \( m \in \mathbb{Z} \),

\[ \varphi(\varpi_m) = \varphi(\varpi) + m \quad \text{with} \quad \varphi(\varpi_m) \cap \varphi(\varpi_n) = \emptyset, \text{ if } m \neq n. \]

It remains to observe that \( \varphi \) is continuous on \( \Pi \), by the explained branch cut and the defining formula (2.4), and that \( \varphi \) is a surjection, since \( \phi \circ \varphi \) is a surjection onto \( \mathcal{A} \). \( \square \)

Combining this lemma with formula (2.1) yields a kernel formula for the periodic domain \( \Pi \). We denote \( \text{sech} \ z = 2/(e^z + e^{-z}) \), which is the hyperbolic secant.

Proposition 2.3. We have

\begin{equation}
K_{\Pi}(z, w) = \tilde{K}_{\Pi}(z, w) \frac{\pi^3}{4(\log \rho)^2} \text{sech}^2 \left( \frac{\pi^2(\varphi(z) - \varphi(w))}{2 \log \rho} \right),
\end{equation}

where

\begin{equation}
\tilde{K}_{\Pi}(z, w) = e^{i2\pi(z-\varphi(z)-\varphi(w))} \phi'(e^{i2\pi z}) \phi'(e^{i2\pi w}).
\end{equation}

Proof. A conformal mapping from the strip \( \Sigma = (-\infty, \infty) \times (-\pi, \pi) \) onto the upper half-plane is given by \( z \mapsto i e^{z/2} \), and the kernel of the upper half-plane is \(-1/(\pi(z - \varpi)^2)\). By (2.1) one obtains

\begin{equation}
K_{\Sigma}(z, w) = \frac{1}{16\pi} \text{sech}^2 \left( (z - \bar{w})/4 \right)
\end{equation}

Hence, (2.5) follows by applying (2.1) to the conformal mapping \( z \mapsto 2\pi^2 \varphi(z)/\log \rho \) from \( \Pi \) onto \( \Sigma \) and observing that \( \phi(e^{i2\pi z}) = e^{i2\pi \varphi(z)} \), see (2.4). \( \square \)

We will show later, among other things, how this formula also follows from the Floquet transform theory. We now mention a concrete example, where the boundary of \( \Pi \) consists of straight line segments: in this case, slightly more can be said about (2.5), namely, the inverse \( \psi \) of the mapping \( \phi \) can be described up to some point, see [4], [6]. We assume that \( \partial \Pi \cap R \), where \( R = \{ z : 0 < \text{Re } z < 1 \} \) is a strip, consists of two polygonal lines \( P_j, j = 0, 1 \) with \( n_j \in \mathbb{N} \) edges and vertices at the points \( z_k^{(j)} \), \( k = 1, \ldots, n_j \), and we have \( \text{Re } z_1^{(0)} = \text{Re } z_1^{(1)} = 1, \text{Re } z_{n_0}^{(0)} = \text{Re } z_{n_1}^{(1)} = 0 \). We denote the interior angle at \( z_k^{(j)} \) by \( \alpha_k^{(j)} \), where

\begin{equation}
\alpha_k^{(j)} = \pi (\beta_k^{(j)} + 1) \quad \text{and} \quad \sum_{k=1}^{n_0} \beta_k^{(0)} = \sum_{k=1}^{n_1} \beta_k^{(0)} = 0.
\end{equation}

According to [6], (8.14), the inverse \( \psi = \phi^{-1} : \mathcal{A} \to \mathcal{D} \) of the conformal mapping \( \phi \) can be defined by the formula

\begin{equation}
\psi(z) = \frac{1}{A} \oint_{1} \prod_{j=0}^{n_j} \prod_{k=1}^{n_k} \left( P \left( \frac{\zeta}{\zeta_k^{(j)}}, \rho \right) \right) \frac{\beta_k^{(j)}}{\zeta} d\zeta
\end{equation}
where $A \in \mathbb{C}$ is a constant, the points $a_k^{(j)} \in \partial A$ are the images of the vertices $z_k^{(j)}$ and $P : A \to \mathbb{C}$ is by (8.5) of [4] the mapping

\begin{equation}
    P(\zeta, \rho) = (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^{2k}\zeta)(1 - \rho^{2k}\zeta^{-1}).
\end{equation}

However, as well-known, calculating the points $a_k^{(j)}$ or the function $\phi$ explicitly is in general impossible.

### 3. Floquet transform in Bergman spaces.

We next introduce the operator theoretic tools. Since the origin of the Floquet transform, denoted here by $\mathcal{F}$, is in the field of partial differential equations for functions of real variables, we put aside for a moment the complex variable, identify $\mathbb{C}$ with the plane $\mathbb{R}^2$, and denote the variable by $(x_1, x_2)$ instead of $z$; the functions are still complex valued in the following. We refer to [11], [12] for general expositions, [13] for applications to elliptic PDE’s and e.g. [3] for the contributions of the author.

The Floquet transform with respect to the variable $x_1$ for functions $f \in L^2(\Pi)$ is given by

\begin{equation}
    \mathcal{F}f(x_1, x_2, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-im\eta} f(x_1 + m, x_2),
\end{equation}

where $(x_1, x_2) \in \varpi$ and $\eta \in [-\pi, \pi)$ is the so called Floquet parameter, or quasi-momentum. As is well known (see Theorem 2.2.5 in [11] or Theorem 4.2 in [12]), the Floquet transform establishes an isometric isomorphism (isometric linear homeomorphism) between $L^2$-spaces,

\begin{equation}
    \mathcal{F} : L^2(\Pi) \to L^2(-\pi, \pi; L^2(\varpi)),
\end{equation}

where $L^2(-\pi, \pi; X)$ is the space of vector valued, Bochner-$L^2$ integrable functions on $[-\pi, \pi]$ with values in the Banach space $X$, and it is endowed with the norm

\begin{equation}
    \|g\|_{L^2(0,2\pi;B)} = \left( \int_{-\pi}^{\pi} \|f(\eta)\|_X d\eta \right)^{1/2}.
\end{equation}

In particular, the series (3.1) converges in the space $L^2(-\pi, \pi; L^2(\varpi))$ for every $f \in L^2(\Pi)$, and thus it also converges for example pointwise for almost every $\eta$ and $(x_1, x_2)$.

The inverse transform is also explicitly given by the formula

\begin{equation}
    \mathcal{F}^{-1}g(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(x_1+\eta)} g(x_1 - [x_1], x_2, \eta) d\eta.
\end{equation}

where $g \in L^2(-\pi, \pi; L^2(\varpi))$ and $[x]$ denotes the integer part of a real number $x$.

In the literature there is also another variation of the definition of the Floquet transform: if $f \in L^2(\Pi)$, one can also define it as

\begin{equation}
    \mathcal{F}f(x_1, x_2, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-im\eta(x_1+m)} f(x_1 + m, x_2),
\end{equation}

where \((x_1, x_2) \in \varpi\) and \(\eta \in [-\pi, \pi]\). This differs from \((3.5)\) only by a factor, which is a non-constant, unimodular, smooth function. In this case the inverse transform is defined on \(L^2(-\pi, \pi; L^2(\varpi))\) by

\[
(3.6) \quad \mathcal{F}^{-1}g(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ix_1\eta} g(x_1 - [x_1, x_2, \eta]) d\eta,
\]

The isometry \((3.2)\) holds also for \(\mathcal{F}\).

**Remark 3.1.** We list some very elementary facts, some of which are consequences of the isomorphism relation \((3.2)\) and will be needed throughout the paper.

(i) The space \(L^2(-\pi, \pi; L^2(\varpi))\) can be identified in a canonical way with a \(L^2\)-space defined on a bounded Lipschitz domain of \(\mathbb{R}^3\), namely the space \(L^2([-\pi, \pi] \times \varpi)\), see [8], Prop. 1.2.24.

(ii) If a sequence \((f_n)_{n=1}^{\infty}\) converges in \(L^2(\Pi)\), then, by definition, the sequence \((Ff_n)_{n=1}^{\infty}\) converges in \(L^2(-\pi, \pi; L^2(\varpi))\), and consequently, there is a subsequence \((Ff_{nk} (\cdot, \eta))_{k=1}^{\infty}\) which converges in \(L^2(\varpi)\) for almost all \(\eta \in [-\pi, \pi]\). See [16], Theorem 3.12 and its vector valued version as explained after Definition 1.2.15. of [8]. Apparently, this does not need to happen for all \(\eta\); a relevant case will be considered in Example 3.3.

Although we will not need Sobolev-spaces in the sequel, we mention them briefly in order to explain the quasiperiodic boundary condition, which is essential later. Given a domain \(\Omega \subset \mathbb{R}^2\), let \(H^1(\Omega)\) denote the standard Sobolev-Hilbert space of complex valued \(L^2(\Omega)\)-functions \(u\) which have weak partial derivatives of the first order belonging to the space \(L^2(\Omega)\). According to the Sobolev embedding theorem, the space \(H^1(\Omega)\) embeds into the space \(C(\text{cl}(\Omega))\) of continuous functions in the closure of \(\Omega\) for example in the case \(\Omega\) is a bounded Lipschitz domain. In particular, the functions in \(H^1(\Omega)\) have well-defined, continuous boundary values in \(\partial \Omega\). The Floquet transform \(F\) is also an isomorphism from the Sobolev space \(H^1(\Pi)\) onto \(L^2(-\pi, \pi; H^1_\eta(\varpi))\), where \(H^1_\eta(\varpi)\) consists of Sobolev functions \(u\) with the quasiperiodic boundary conditions

\[
(3.7) \quad u(1, x_2) = e^{im} u(0, x_2), \quad a < x_2 < b
\]

and \(L^2(-\pi, \pi; H^1_\eta(\varpi))\) is defined as the closed subspace of \(L^2(-\pi, \pi; H^1(\varpi))\) such that \(f(\cdot, \eta) \in H^1_\eta(\varpi)\) for a.e. \(\eta\). (If the other version \(F\) of the Floquet transform were used, one would replace \(H^1_\eta(\varpi)\) by \(H^1_{\text{per}}(\varpi)\), where instead of \((3.7)\) the periodic boundary conditions, i.e. \((3.7)\) with \(\eta = 0\), hold.)

The terminology for the Floquet transform varies from source to source: it may be called the Floquet-Bloch, Bloch or Zak transform. In the Russian school of function spaces it is also known as the Gelfand transform, which is not to be mixed with the Gelfand transform of commutative Banach algebra theory. In the theory of spectral elliptic boundary problems, the transform \(F\) is used to convert a problem in the domain \(\Pi\) into another one in the cell \(\varpi\), containing quasiperiodic boundary conditions. Here, we instead wish to study the Bergman kernel in \(\Pi\) with the help of some kernels in \(\varpi\).

We proceed to work in Bergman spaces of analytic functions. The transform \((3.3)\), leading to periodic boundary condition, is not useful as such for the study of analytic function spaces, since \(FF\) is not analytic even if \(f\) happens to be. Hence, we will use...
the transform (3.1), which is written using the complex variable, for \( f \in A^2(\Pi) \),

\[
(3.8) \quad Ff(z, \eta) = \hat{f}(z, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-im \eta} f(z + m), \quad z \in \mathbb{C}, \quad \eta \in [-\pi, \pi].
\]

We note that \( F \) is a continuous (isometric) mapping from \( A^2(\Pi) \) into \( L^2(-\pi, \pi; L^2(\mathbb{C})) \).

The series in (3.8) converges in \( L^2(-\pi, \pi; L^2(\mathbb{C})) \), thus also pointwise for a.e. \( \eta, z \), and also in \( L^2(\mathbb{C}) \) for a.e. \( \eta \). If \( g \in L^2(-\pi, \pi; L^2(\mathbb{C})) \), we also denote

\[
(3.9) \quad F^{-1}g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i \text{Re} \eta} g(z - [\text{Re} z], \eta) \, d\eta, \quad z \in \Pi.
\]

The fact that formula (3.9) indeed gives the inverse of \( F \) will be established only in Theorem 3.6, since we first need to specify the proper function spaces. As for the motivation of the next proposition, we mention that the boundary condition (3.7) has to be taken into account in the subsequent considerations even though Bergman space functions in general do not have well defined boundary values.

**Proposition 3.2.** (i) If \( f \in A^2(\Pi) \subset L^2(\Pi) \), then for almost all \( \eta \in [-\pi, \pi] \), the function \( z \mapsto Ff(z, \eta) \) is analytic in \( \mathbb{C} \ni z \).

(ii) There is a dense subspace \( X \) of \( A^2(\Pi) \), the elements \( f \) of which have the property that for all \( \eta \in [-\pi, \pi] \), the function \( Ff(\cdot, \eta) \) is analytic in \( \mathbb{C} \) and has an analytic extension to a neighborhood of the closure of \( \mathbb{C} \) in \( \Pi \).

For an arbitrary \( f \in A^2(\Pi) \), the analyticity of \( Ff(\cdot, \eta) \) does not necessarily hold for all \( \eta \), see Example 3.3 below.

**Proof.** (i) Let \( 0 \neq f \in A^2(\Pi) \) and \( \varepsilon > 0 \), and denote

\[
(3.10) \quad \varphi_{\varepsilon}(z) = e^{-e\varepsilon z^2}
\]

Then, we have \( \varphi_{\varepsilon}f \to f \) in \( A^2(\Pi) \), as \( \varepsilon \to 0 \). (The proof of this can be obtained by choosing a large enough compact subset \( K \) such that the \( L^2 \)-norm of the restriction of \( f \) to \( \Pi \setminus K \) is small, and then observing that \( \varphi_{\varepsilon} \) converges to the constant function 1 uniformly on \( K \cap \Pi \).

We now consider the domain \( \mathbb{C} \) which is the interior of \( \text{cl}(\mathbb{C} \cup \mathbb{C}_{-1} \cup \mathbb{C}_{1}) \subset \Pi \). Let us fix \( z_0 \in \mathbb{C} \) and a small enough open disc \( B(z_0, \varrho) \), \( \varrho > 0 \), such that \( \text{cl}(B(z_0, 3\varrho)) \subset \mathbb{C} \). The Cauchy integral formula implies the estimate \( |f(z)| \leq C\varrho^{-1}\|f\|_{\mathbb{C}} \) for all \( z \in B(z_0, \varrho) \). Thus, applying a translation, or the same argument in all sets \( \mathbb{C} \mbox{ plus } m, m \in \mathbb{Z} \), we get

\[
(3.11) \quad \sup_{z \in \mathbb{C}} |f(z + m)| = \sup_{z \in \mathbb{C}} |f(z)| \leq \frac{C}{\varrho} \|f\|_{\mathbb{C}} \leq \frac{C}{\varrho} \|f\|_{\Pi},
\]

for all \( z \in B(z_0, \varrho) \). Moreover, for a fixed \( \varepsilon < 1 \) we have the estimate (see the choice of the domain \( \mathbb{C} \) for the number \( M \))

\[
(3.12) \quad |\varphi_{\varepsilon}(z)| = e^{-e\varepsilon z^2 + \varepsilon \varrho^2} \leq e^{M^2} e^{-e\varepsilon z^2} \Rightarrow \sup_{z \in \mathbb{C}} |\varphi_{\varepsilon}(z + m)| \leq C e^{-e|m|^2}
\]

for all \( m \in \mathbb{Z} \). As a consequence of these estimates, for a fixed \( \varepsilon \) the series in

\[
(3.13) \quad F(\varphi_{\varepsilon}f)(z, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-im \eta} \varphi_{\varepsilon}(z + m) f(z + m)
\]
converges uniformly in the disc \( B(z_0, \rho) \) and thus defines an analytic function there. Since \( z_0 \in \varpi_U \) was arbitrary, we infer that (3.13) thus defines an analytic function in \( \varpi_U \).

Since the Floquet transform is an isometry from \( L^2(\Pi) \) onto \( L^2(-\pi, \pi; L^2(\varpi)) \) and \( \varphi_\varepsilon f \to f \) in \( L^2(\Pi) \), we see that \( F(\varphi_\varepsilon f) \) converges to \( Ff \) in \( L^2(-\pi, \pi; L^2(\varpi)) \) as \( \varepsilon \to 0 \). This implies, in view of the definition of the vector valued norm (see Remark 3.1(ii)), that there is a decreasing sequence \( (\varepsilon_k)_{k=1}^\infty \) with \( 0 < \varepsilon_k \to 0 \) and

\[
\lim_{k \to \infty} F(\varphi_{\varepsilon_k} f)(\cdot, \eta) = Ff(\cdot, \eta) \quad \text{in } L^2(\varpi)
\]

for almost all \( \eta \in [-\pi, \pi] \), and since \( F(\varphi_\varepsilon f)(\cdot, \eta) \) is analytic, by (3.13), the convergence in (3.14) happens in \( A^2(\varpi) \) for almost all \( \eta \in [-\pi, \pi] \). This completes the proof that \( Ff \) is analytic in \( \varpi \) for almost all \( \eta \in [-\pi, \pi] \).

(ii) The dense subspace \( X \) can be defined to consist of all functions \( \varphi_\varepsilon f \), where \( f \in A^2(\Pi) \); see just below (3.10). The result follows from the observations around (3.13). \( \square \)

In (i) of the previous proposition, the analyticity does not need to hold for exactly all \( \eta \), as shown in the next remark.

**Example 3.3.** Consider the simple case that \( \Pi \) is the strip \( (-\infty, \infty) \times (\frac{1}{2}, \frac{3}{2}) \), corresponding to \( \varpi = (0, 1) \times (\frac{1}{2}, \frac{3}{2}) \). Then, the function \( f(z) = 1/z \) is analytic on \( \Pi \) and belongs to \( A^2(\Pi) \). Moreover, we have \( |z| \leq 1 \) for all \( z = x + iy \in \varpi \), so that the triangle inequality implies for \( m \in \mathbb{Z}, m \neq 0, \pm 1 \), and \( z \in \varpi \),

\[
|f(z + m) - \frac{1}{m}| = \left| \frac{x + iy}{(x + iy + m)m} \right| \leq \frac{1}{(|m| - 1)|m|} < \frac{1}{(|m| - 1)^2}.
\]

Thus for every \( z \in \varpi \) and \( \eta = 0 \), the series

\[
\sum_{m \in \mathbb{Z}} e^{-im\eta} f(z + m) = \sum_{m \in \mathbb{Z}} \left( \frac{1}{m} + (f(z + m) - \frac{1}{m}) \right)
\]

diverges by the triangle inequality and (3.15), since \( \sum_m 1/m \) diverges. Thus \( Ff \) is not a well defined analytic function for this particular value of \( \eta \).

In order to treat the Bergman projection we will need the unitarity of certain operators, and this requires a selection of carefully defined function spaces. These are given in the next.

**Definition 3.4.** 1°. Given the number \( \eta \in [-\pi, \pi] \), we denote by \( A^2_{\eta, \text{ext}}(\varpi) \) the subspace of \( A^2(\varpi) \) consisting of functions \( f \) which can be extended as analytic functions to a neighborhood in the domain \( \Pi \) of the set \( \text{cl}(\varpi) \cap \Pi \) and satisfy the boundary condition (3.7), i.e.

\[
f(1 + iy) = e^{i\eta} f(0 + iy) \quad \text{for all } a < y < b.
\]

We define the space \( A^2_{\eta}(\varpi) \) as the closure of \( A^2_{\eta, \text{ext}}(\varpi) \) in \( A(\varpi) \). We also denote by \( P_{\eta} \) the orthogonal projection from \( L^2(\varpi) \) onto the subspace \( A^2_{\eta}(\varpi) \).

This definition will be motivated by the isomorphy property of \( F \) in Theorem 3.6 see also the remark before (3.7) and (ii) of Proposition 3.2. Formula (3.13) gives plenty of examples of functions belonging to the subspace \( A^2_{E,\eta}(\varpi) \). In general, the functions belonging to \( A^2(\varpi) \) may not have properly defined boundary values on \( \partial \varpi \) so that the condition (3.16) cannot be posed directly.
2°. We denote by \( L^2(\pi, \pi; A^2_\eta(\omega)) \) the subspace of \( L^2(\pi, \pi; A^2(\omega)) \) consisting of functions \( f \) such that the function \( z \mapsto f(z, \eta) \) belongs to \( A^2_\eta(\omega) \) for a.e. \( \eta \in [-\pi, \pi] \). Since \( A^2_\eta(\omega) \) is by definition, for every \( \eta \), a closed subspace of \( A^2(\omega) \) and thus of \( L^2(\omega) \), we obtain that \( L^2(\pi, \pi; A^2_\eta(\omega)) \) is a closed subspace of \( L^2(\pi, \pi; L^2(\omega)) \) (follows by using Remark 3.2(ii)).

3°. We define the subspace

\[
\mathcal{H}_\eta := L^2\left(-\pi, \pi; A^2_{\eta, \text{ext}}(\omega)\right)
\]

of \( L^2(-\pi, \pi; A^2_\eta(\omega)) \) which consists of functions \( g \) such that the mapping \( z \mapsto g(z, \eta) \) belongs to \( A^2_{\eta, \text{ext}}(\omega) \) for a.e. \( \eta \).

It is good to keep in mind that \( L^2(-\pi, \pi; A^2_\eta(\omega)) \) and \( \mathcal{H}_\eta \) are not vector valued \( L^2 \)-spaces (since the space \( A^2_\eta(\omega) \) depends on \( \eta \)) but they only have the structure of a Banach vector bundle, see for example Section 1.3 of [11]. This fact complicates the proof of Theorem 3.6 since simple functions of \( L^2(-\pi, \pi; A^2(\omega)) \) are not contained in these subspaces. Thus, the following fact needs to be proven.

**Lemma 3.5.** The subspace \( \mathcal{H}_\eta \) is dense in \( L^2(-\pi, \pi; A^2_\eta(\omega)) \)

Proof. Let us define the Bochner space \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \), where \( A^2_{\text{per}}(\omega) \) equals the space \( A^2_\eta(\omega) \) with \( \eta = 0 \), corresponding to the periodic boundary condition. We also denote by \( A^2_{\text{per, ext}}(\omega) \subset A^2_{\text{per}}(\omega) \) the space \( A^2_{\eta, \text{per}}(\omega) \) with \( \eta = 0 \). Finally, we define the subspace \( L^2\left(-\pi, \pi; A^2_{\text{per, ext}}(\omega)\right) \) of \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \) consisting of those functions \( g \) which have values in \( A^2_{\text{per, ext}}(\omega) \) for almost every \( \eta \).

By the theory of Bochner spaces, see Lemma 1.2.19 in [8], a dense subspace of \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \) is formed by simple functions

\[
(3.17) \quad \sum_j \chi_j(\eta)g_j(z),
\]

where \( \chi_j \) is the characteristic function of some measurable subset of \( [-\pi, \pi] \), \( g_j \in A^2_{\text{per}}(\omega) \) and the sum contains only finitely many terms. Moreover, since \( A^2_{\text{per, ext}}(\omega) \) is dense in \( A^2_{\text{per}}(\omega) \) by definition, a dense subspace of \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \) is formed by functions (3.17), where every \( g_j \) belongs to \( A^2_{\text{per, ext}}(\omega) \). This shows that \( L^2\left(-\pi, \pi; A^2_{\text{per, ext}}(\omega)\right) \) is dense in \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \).

The assertion of the lemma now follows from the facts that the mapping \( f(z, \eta) \mapsto e^{i\eta z}f(z, \eta) \) is an isomorphism from \( L^2\left(-\pi, \pi; A^2_{\text{per}}(\omega)\right) \) onto \( L^2\left(-\pi, \pi; A^2_\eta(\omega)\right) \) and from \( L^2\left(-\pi, \pi; A^2_{\text{per, ext}}(\omega)\right) \) onto \( \mathcal{H}_\eta \). The proofs of these isomorphisms are left to the reader. \( \square \)

We are finally prepared to prove the main result of this section.

**Theorem 3.6.** Floquet transform \( \mathcal{F} \) maps \( A^2(\Pi) \) onto \( L^2(-\pi, \pi; A^2_\eta(\omega)) \). Its inverse \( \mathcal{F}^{-1} : L^2(-\pi, \pi; A^2_\eta(\omega)) \rightarrow A^2(\Pi) \) is given by the formula (3.9). Moreover, \( \mathcal{F} \) preserves the inner product and is thus a unitary operator.

Proof. We recall that \( \mathcal{F} \) is an isometric isomorphism from \( L^2(\Pi) \) onto \( L^2(-\pi, \pi; L^2(\omega)) \) and first show that it maps the space \( A^2(\Pi) \) onto \( L^2(-\pi, \pi; A^2_\eta(\omega)) \). Let the dense subspace \( X \subset L^2(\Pi) \) be as in Proposition 3.2(ii) and assume \( f \in X \) so that for every \( \eta \), \( \mathcal{F}f(\cdot, \eta) \) is an analytic function of \( z \) on \( \omega \) and also has an analytic extension to a neighborhood of the closure of \( \omega \), see Definition 3.4 Thus \( \mathcal{F}f(\cdot, \eta) \) also has well
defined values on the lateral boundaries of \( \varpi \). Moreover, given \( \eta \in [-\pi, \pi] \), the function \( F_f(\cdot, \eta) \) satisfies the quasiperiodic boundary condition (3.16), since

\[
F_f(1 + iy, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-in(m-1)} f(1 + iy + (m - 1)) = e^{in\eta} F_f(iy, \eta).
\]

Hence, \( F \) maps the dense subspace \( X \) into \( L^2(-\pi, \pi; A^2_{\eta}(\varpi)) \), and since \( F \) is an isometry and \( L^2(-\pi, \pi; A^2_{\eta}(\varpi)) \) is complete, it also maps \( A^2(\Pi) \) into \( L^2(-\pi, \pi; A^2_{\eta}(\varpi)) \).

To see that \( F \) is surjective we first consider a function \( g \in \mathcal{H}_\eta \). Clearly, for all \( m \in \mathbb{Z} \) and a.e. \( \eta \in [-\pi, \pi] \), the function

\[
G_{\eta,m} : \varpi_m \to \mathbb{C}, \quad G_{\eta,m}(z) = e^{in\eta} g(z - m, \eta),
\]

is analytic on the subdomain \( \varpi_m, m \in \mathbb{Z} \) (see (1.2)). Moreover, by Definition 3.4 it has an analytic extension, still denoted by \( G_{\eta,m} \), to a neighborhood of the closure of \( \varpi_m \) in \( \Pi \). Since \( g(\cdot, \eta) \) satisfies the boundary condition (3.16), we get for all \( m \in \mathbb{Z} \), all \( z = m + 1 + iy \) with \( y \in [a, b] \),

\[
G_{\eta,m}(z) = \lim_{\varepsilon \to 0^+} G_{\eta,m}(m + 1 - \varepsilon + iy) = \lim_{\varepsilon \to 0^+} e^{in\varepsilon} g(1 + iy - \varepsilon, \eta) = e^{in\varepsilon} g(1 + iy, \eta) = \lim_{\varepsilon \to 0^+} e^{in\varepsilon} g(iy + \varepsilon, \eta) = \lim_{\varepsilon \to 0^+} G_{\eta,m+1}(m + 1 + \varepsilon + iy) = G_{\eta,m+1}(m + 1 + iy) = G_{\eta,m+1}(z).
\]

In other words, \( G_{\eta,m} \) and \( G_{\eta,m+1} \) are two analytic functions with overlapping domains of definition and they coincide on the line segments \( J_m = \{ m + 1 \} \times [a, b] \). Thus, \( G_{\eta,m} \) and \( G_{\eta,m+1} \) coincide in their common domain of definition, and we can define the analytic function \( G_\eta \) on \( \Pi \) by setting \( G_\eta(z) = G_{\eta,m}(z) \) for \( z \) belonging to the closure of \( \varpi_m \) in \( \Pi \).

Comparing (3.18) with (3.9) we see that

\[
F^{-1} g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} G_\eta(z) d\eta,
\]

which is an analytic function in \( \Pi \), since \( G_\eta \) is. Hence, \( F^{-1} g \in A^2(\Pi) \). Then, we have \( FF^{-1} g = g \) for all \( g \in \mathcal{H}_\eta \subset L^2(-\pi, \pi; L^2(\varpi)) \), by the general Floquet inversion formula (3.5). We conclude that image of \( F \) contains the subspace \( \mathcal{H}_\eta \), which is dense in \( L^2(-\pi, \pi; A^2_{\eta}(\varpi)) \) by Lemma 3.3. This implies the surjectivity, since \( F \) is an isometry and thus bounded from below with respect to the \( L^2 \)-norms.

It is well known that the Floquet transform preserves the inner product. This can also be seen directly, since for \( f, g \in L^2(\Pi) \) and \( H := L^2(-\pi, \pi; L^2(\varpi)) \) there holds

\[
(Ff, Fg)_H = \sum_{m,n \in \mathbb{Z}} \int_{\varpi} \int_{-\pi}^{\pi} e^{-in\eta} f(z + m) e^{in\eta} g(z + n) dz d\eta
= \sum_{m \in \mathbb{Z}} \int_{\varpi} f(z + m) g(z + m) dz = \int_{\Pi} f(z) g(z) dz = (f, g)_{L^2(\varpi)}. \quad \square
\]
4. General kernel formula for the periodic domain.

Our purpose is to show how the Bergman projection in \( \Pi \) can be presented with the help of an orthogonal projection in the periodicity cell \( \varpi \) by using the Floquet transform.

**Definition 4.1.** For all \( \eta \in [−\pi, \pi] \) we denote by \( P_\eta \) the orthogonal projection from \( L^2(\varpi) \) onto \( A^2_\eta(\varpi) \); see Definition 3.4.1° for the notation.

The well-known proof for the existence of Bergman kernels, see e.g. [10], p.1060, applies also in this case and implies that \( P_\eta \) can be written with the help of the integral kernel \( K_\eta : \varpi \times \varpi \to \mathbb{C} \)

\[
P_\eta f(z) = \int_\varpi K_\eta(z, w) f(w) dA(w),
\]

where \( K_\eta(z, \cdot) \in L^2(\varpi) \) for all \( z \in \varpi \). In Section 4 we will calculate the kernel \( K_\eta \) in some cases. We also define an operator \( \mathcal{P} \) by denoting, for all \( f \in L^2(−\pi, \pi; L^2(\varpi)) \),

\[
\mathcal{P} f(z, \eta) = (P_\eta f(\cdot, \eta))(z), \quad z \in \varpi, \quad \eta \in [−\pi, \pi].
\]

It follows directly from the definitions of the spaces and their norms that this is a bounded operator from \( L^2(−\pi, \pi; L^2(\varpi)) \) into \( L^2(−\pi, \pi; A^2_\eta(\varpi)) \). More precisely, the following holds.

**Lemma 4.2.** The operator \( \mathcal{P} \) is the orthogonal projection from \( L^2(−\pi, \pi; L^2(\varpi)) \) onto \( L^2(−\pi, \pi; A^2_\eta(\varpi)) \).

Proof. Since \( P_\eta \) is the orthogonal projection \( L^2(\varpi) \) onto \( A^2_\eta(\varpi) \), we have \( \mathcal{P}^2 = \mathcal{P} \) and \( \mathcal{P} f = f \) for every \( f \in L^2(−\pi, \pi; A^2_\eta(\varpi)) \). Thus, \( \mathcal{P} \) is a projection operator between the given spaces, and it suffices to observe that it is also self-adjoint. With \( H := L^2(−\pi, \pi; L^2(\varpi)) \) and \( f, g \in H \) we obtain, by the self-adjointness of \( P_\eta \),

\[
\langle \mathcal{P} f | g \rangle_H = \int_\varpi \int_{−\pi}^{\pi} P_\eta f(\cdot, \eta) g(\cdot, \eta) d\eta dA = \int_\varpi \int_{−\pi}^{\pi} f(\cdot, \eta) \overline{P_\eta g(\cdot, \eta)} d\eta dA = \langle f | \mathcal{P} g \rangle_H. \quad \square
\]

We can now describe the general relation of the orthogonal projections in the bounded domains and the full domain \( \Pi \).

**Theorem 4.3.** The Bergman projection \( P_\Pi \) from \( L^2(\Pi) \) onto \( A^2(\Pi) \) can be written as

\[
\mathcal{F}^{-1} \mathcal{P} \mathcal{F} f(z) = \frac{1}{\sqrt{2\pi}} \int_{−\pi}^{\pi} e^{i [\text{Re} \eta]} (P_\eta \hat{f}(\cdot, \eta))(z - [\text{Re} \eta]) d\eta
\]

\[
= \frac{1}{2\pi} \int_{\Pi} \int_{−\pi}^{\pi} e^{in[\text{Re} \eta] - [\text{Re} w]} K_\eta(z - [\text{Re} \zeta], w - [\text{Re} \omega]) f(w) d\eta dA(w)
\]

(4.3)

Proof. The operator \( \mathcal{F}^{-1} \mathcal{P} \mathcal{F} \) is self-adjoint, since \( \mathcal{F} \) preserves the inner product (Theorem 3.6), and also clearly a projection, since \( \mathcal{P} \) is. So, this operator must be the orthogonal projection from \( L^2(\Pi) \) onto \( A^2(\Pi) \), in view of Theorem 3.6. There remains to verify the rest of the formula (1.3).
The series (3.8) for $Ff$ converges in $L^2(\varpi)$ for almost all $\eta \in [-\pi, \pi]$, and since $P_\eta$ is a bounded operator, we can write
\begin{equation}
(4.4) \quad \mathcal{P}Ff(z; \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-i\eta m} \int_{\varpi} K_\eta(z, w) f(w + m) dA(w), \quad z \in \varpi,
\end{equation}
for almost all $\eta$. On the other hand, the series (3.8) for $Ff$ converges also in $L^2(-\pi, \pi; L^2(\varpi))$, hence the series (4.4) converges in this space, too. Since $F^{-1}$ is a bounded operator $L^2(-\pi, \pi; L^2(\varpi)) \to L^2(\Pi)$, the result follows from the defining formula for $F^{-1}$ in (3.9),
\begin{align*}
F^{-1}\mathcal{P}Ff(z) &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{\varpi} e^{i\eta [\Re(z) - m]} K_\eta(z - \Re(z), w) f(w + m) d\eta dA(w) \\
&= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{\varpi} e^{i\eta [\Re(z) - Re(z)]} K_\eta(z - \Re(z), w - \Re(z)) f(w) d\eta dA(w) \\
&= \frac{1}{2\pi} \int_{\Pi} e^{i\eta [\Re(z) - Re(z)]} K_\eta(z - \Re(z), w - \Re(z)) f(w) d\eta dA(w). \quad \Box
\end{align*}

We will show in Section 3 that if the periodic cell $\varpi$ is simply connected, the kernel $K_\eta$ can be constructed by applying a certain canonical Riemann conformal mapping of doubly connected domains and derive again the kernel formula of Section 2 in this way.

5. Weights, domains and projections.

In this section we collect some basic facts concerning the Bergman projection in domains and weighted spaces. These results are known at least to experts in this area, but we give some proofs for the sake of the completeness of the presentation.

Let $\Omega \subset \mathbb{C}$ be a domain. By a weight on $\Omega$ we mean a continuous function $V : \Omega \to \mathbb{R}^+ = (0, \infty)$, and we denote by $L^2_V(\Omega)$ the weighted $L^2$-space on $\Omega$ with norm and inner product
\begin{equation}
(5.1) \quad \|f\|_{\Omega, V}^2 = \int_{\Omega} |f|^2 V \, dA, \quad (f|g)_{\Omega, V} = \int_{\Omega} f\overline{g} V \, dA,
\end{equation}
where $f, g \in L^2_V(\Omega)$. We also denote by $A^2_V(\Omega)$ the subspace of $L^2_V(\Omega)$ consisting of analytic functions. The proof of the unweighted case applies here, too, and shows that $A^2_V(\Omega)$ is a closed subspace. Our weight functions will mostly be of the form
\begin{equation}
(5.2) \quad V(z) = |v(z)|^2 = v(z)v(\overline{z}),
\end{equation}
where $v : \Omega \to \mathbb{C} \setminus \{0\}$ is analytic.

We will use the following observations.

Lemma 5.1. Let $\Omega$ and $\Omega'$ be conformally equivalent domains and let $\varphi$ be a conformal mapping from $\Omega$ onto $\Omega'$. Then, the composition operator $I : f \mapsto f \circ \varphi$ is a unitary isomorphism from the space $L^2_V(\Omega')$ onto $L^2(\Omega)$ and also from $A^2_V(\Omega')$ onto $A^2(\Omega)$, where the weight is $V(z) = |\varphi'(z)|^2$ with $\psi = \varphi^{-1} : \Omega' \to \Omega$. 
If $X$ is a closed subspace of $L^2_{\nu}(\Omega')$ and $P_X$ is the orthogonal projection from $L^2_{\nu}(\Omega')$ onto $X$, then the orthogonal projection $P$ from $L^2(\Omega)$ onto $I(X) \subset L^2(\Omega)$ is given by $P = IP_XI^{-1}$. If the function $K_z \in L^2(\Omega' \times \Omega')$ is the integral kernel of $P_X$, then the kernel of $P$ is given by
\begin{equation}
K(z, w) = K_X(\varphi(z), \varphi(w))|\varphi'(w)|^2, \quad z, w \in \Omega.
\end{equation}

Proof. The unitary isomorphism property of $I$ is a direct consequence of the definitions and the fact that $V$ is the Jacobian of the conformal transform $\psi$. Note that the inverse of the operator $I$ is given by $I^{-1} : L^2(\Omega) \rightarrow L^2(\Omega')$, $f \mapsto f \circ \psi$.

Next, it is straightforward to see that the mapping $P = IP_XI^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is selfadjoint (since the adjoint of $I$ is $I^{-1}$ by the unitarity), a projection ($P^2 = P$), and maps $L^2(\Omega)$ onto $I(X)$. Hence, it is the claimed orthogonal projection. Finally, given $f \in L^2(\Omega)$ we write
\begin{equation}
Pf(z) = IP_XI^{-1}f(z) = \int_{\Omega'}K_X(\varphi(z), w)f(\varphi^{-1}(w))dA(w)
= \int_{\Omega}K_X(\varphi(z), \varphi(w))f(w)|\varphi'(w)|^2dA(w).
\end{equation}

6. Simply connected periodic domain.

In the following we construct the kernel for the operator $P_\eta$, assuming eventually that $\Pi$ is simply connected and satisfied assumption (A) of Section 2. Due to the quasiperiodic boundary condition in the definition of the target space $A^2_\eta(\varpi)$, the Riemann mapping from $\varpi$ onto $\mathbb{D}$ is not useful but instead we will use the doubly connected exponential image of $\varpi$ and its Riemann map to an annulus. In the next section we will rewrite the formula of Theorem 4.3 of the Bergman projection $P_\Pi$ with the help of the mentioned Riemann map of the bounded domain and show how formula (5.3) follows from these considerations.

We proceed to the construction of the kernel $K_\eta$ of the orthogonal projection $P_\eta : L^2(\varpi) \rightarrow A^2_\eta(\varpi)$ for any $\eta \in [-\pi, \pi]$, see Definition 5.1. This will be accomplished in several steps, where we also employ certain classical methods in complex analysis. We will use the notation of Section 2 and recall that the exponential map $E : z \mapsto e^{i2\pi z}$ maps the set $\varpi \cup J_+ \cup J_-$ onto the domain $\mathcal{D}$, which is contained in an annulus so that there exist numbers $0 < \rho_0 < \rho_1$ with
\begin{equation}
\mathcal{D} \subset \{z \in \mathbb{C} : \rho_0 < |z| < \rho_1\}.
\end{equation}

The domain $\mathcal{D}$ is doubly connected, if $\varpi$ is simply connected. Let $\mathcal{D}_1 = E(\varpi) \subset \mathcal{D}$ be as in (2.2). We will soon employ the branch cut of the logarithm as defined in Section 2 but we need to agree that when considered in $\mathcal{D}$, the branch cut of the logarithm happens on $\mathcal{D} \cap \mathbb{R}^+$ so that $\text{Im} \log z \in (0, 2\pi)$ for $z \in \mathcal{D}_1$ and log is analytic on $\mathcal{D}$. Let us denote $L = E^{-1} : \mathcal{D}_1 \rightarrow \varpi$, $\tilde{L}(z) = (i2\pi)^{-1}\log z$.

We define for all $\eta \in [-\pi, \pi]$ the weight function
\begin{equation}
W(z) = |L'(z)|^2 = \frac{1}{4\pi^2|z|^2}, \quad z \in \mathcal{D},
\end{equation}
and consider the weighted Bergman space $A^2_W(\mathcal{D}_1)$ defined below (5.1) (restricting the weight $W$ onto $\mathcal{D}_1$, of course). The Bergman space $A^2_W(\mathcal{D})$ can be considered as a subspace $A^2_W(\mathcal{D}_1)$, which consists of functions that can be extended from $\mathcal{D}_1$ to
the full domain $\mathcal{D}$ as analytic functions. Since $A^2_W(\mathcal{D})$ is complete, it is a closed subspace of $A^2_W(\mathcal{D})$.

**Lemma 6.1.** (i) The operator $I_1 : f \mapsto f \circ L$ is a unitary isomorphism $A^2(\varpi) \to A^2_W(\mathcal{D})$ and $L^2(\varpi) \to L^2_W(\mathcal{D})$.
(ii) The operator $I_1$ maps the space $A^2_\eta(\varpi)$ onto $A^2_{W,\eta}(\mathcal{D})$, which is the closure of the subspace

$$
A^2_{W,\eta,\text{ext}}(\mathcal{D}) = \left\{ z^{\eta/(2\pi)} g : g \in A^2_W(\mathcal{D}) \right\}
$$

in $A^2_W(\mathcal{D})$.

Proof. The claim (i) follows by applying Lemma 5.1 to the inverse operator $I_1^{-1} : f \mapsto f \circ E$. As for the claim (ii), given an arbitrary $g \in A^2_W(\mathcal{D})$, the function

$$
f := (z^{\eta/(2\pi)}) g \circ E = : h \circ E \in L^2(\varpi),
$$

belongs to the subspace $A^2_{\eta,\text{ext}}(\varpi)$ of $A^2_\eta(\varpi)^2$ (see Definition 3.4) and we have $I_1 f = h$, where $h$ is an arbitrary element of $A^2_{W,\eta,\text{ext}}$. Thus, the operator $I_1$ is surjection onto $A^2_{W,\eta,\text{ext}}$ and consequently also onto $A^2_{W,\eta}$, by (i) and the closedness of $A^2_\eta(\varpi)$ in $A^2(\varpi)$.

On the other hand, in view of Definition 3.4, we consider a function $f \in A^2_{\eta,\text{ext}}(\varpi)$ and thus has well defined continuous boundary values satisfying (3.16). The function $f \circ L \in A^2_W(\mathcal{D})$ has a discontinuity at the slit of $\mathcal{D}$, more precisely

$$
\lim_{y \to 0^-} f \circ L(x + iy) = e^{i\eta} \lim_{y \to 0^+} f \circ L(x + iy) \quad \text{for all } x \in \mathcal{D} \cap \mathbb{R}^+.
$$

Hence, the function $g = z^{-\eta/(2\pi)} f \circ L(z)$ has a continuous extension to $\mathcal{D}$, which makes it into an analytic function in $\mathcal{D}$. Moreover, since the multiplier $z^{-\eta/(2\pi)}$ is a bounded function, we also have $g \in L^2_W(\mathcal{D})$ and thus $g \in A^2_W(\mathcal{D})$, hence, we obtain $I_1 f = z^{\eta/(2\pi)} g \in A^2_{W,\eta,\text{ext}}(\mathcal{D})$. Since, by definition, $A^2_\eta(\varpi)$ is the closure of $A^2_{\eta,\text{ext}}(\varpi)$ in $A^2(\varpi)$ and $I_1$ is an isometry, we obtain that $I_1$ maps $A^2_\eta(\varpi)$ into $A^2_{W,\eta}(\mathcal{D})$. $\Box$

From now on we will additionally assume that $\Pi$ and $\varpi$ are simply connected (thus $\mathcal{D}$ is doubly connected). We recall from Section 2 the conformal mapping

$$
\phi : \mathcal{D} \to \mathcal{A} = \{ z \in \mathbb{C} : 1/\rho < |z| < \rho \}
$$

where the number $\rho > 1$ is uniquely determined by $\mathcal{D}$. Let us also agree that when considered on $\mathcal{A}$, the logarithm is defined as in Section 2. Note that our assumptions on the geometry of $\Pi$ and $\varpi$, (1.2), do not imply smoothness of the boundary $\partial \mathcal{D}$ of $\mathcal{D}$. Hence, $\partial \mathcal{D}$ may include corners for example. This has the consequence that the derivative $\phi'$ does not need to be bounded or bounded away from zero. See for example [15, 17, 18, 1, 20].

We define on $\mathcal{A}$ the weight

$$
V(z) = W(\psi(z)) |\psi'(z)|^2, \quad z \in \mathcal{A},
$$

where $\psi = \phi^{-1} : \mathcal{A} \to \mathcal{D}$ is the inverse map. According to (6.2),

$$
V(z) = \frac{|(L \circ \psi)'(z)|^2}{4\pi^2 |\psi(z)|^2} = v(z) \overline{v(z)} \quad \text{with } v(z) = \frac{1}{2\pi} \frac{\psi'(z)}{\overline{\psi(z)}}.
$$
where \( z \in \mathcal{A} \). Moreover, we denote by \( A^2_{V,\eta,\text{ext}}(\mathcal{A}_i) \) the subspace of all functions of the form

\[
(6.9) \quad z^{n/(2\pi)}g, \quad \text{where } g \in A^2_{V}(\mathcal{A}),
\]

and by \( A^2_{V,\eta}(\mathcal{A}_i) \) the closure of \( A^2_{V,\eta,\text{ext}}(\mathcal{A}_i) \) in \( A^2_{V}(\mathcal{A}_i) \). We recall that the functions belonging to \( A^2_{V,\eta}(\mathcal{A}_i) \) may be discontinuous on the curve \( \Gamma = \phi(D \cap \mathbb{R}^+) \), see Section 2. Finally, we denote by \( I_2 \) the composition operator \( I_2 : f \mapsto f \circ \psi \).

**Lemma 6.2.** (i) The operator \( I_2 \) is a unitary isomorphism \( L^2_{W}(D) \to L^2_{V}(\mathcal{A}) \).
(ii) The operator \( I_2 \) maps the space \( A^2_{V,\eta}(\mathcal{D}_i) \) onto \( A^2_{V,\eta}(\mathcal{A}_i) \) for every \( \eta \in [-\pi, \pi] \).

As a consequence, the composition operator

\[
(6.10) \quad I_2I_1 : f \mapsto f \circ L \circ \psi
\]

is a unitary isomorphism from \( L^2(\varpi) \) onto \( L^2_{V}(\mathcal{A}) \) and from \( A^2_{\eta}(\varpi) \) onto \( A^2_{V,\eta}(\mathcal{A}_i) \).

Proof. Here, the first assertion is just a variant of Lemma 5.1 with a similar proof. To prove (ii) we note that due to choice of the branch cut on \( \mathcal{A} \) (see (2.3)) and (6.1), the function \( F_\eta(z) = (\phi(z)/z)^{\eta/(2\pi)} \) is analytic, bounded and bounded away from zero on \( D \) for all \( \eta \in [-\pi, \pi] \), and the same is thus true also for the function

\[
(\psi(z)/z)^{\eta/(2\pi)} = \frac{1}{F_\eta \circ \psi(z)} \quad \text{on } \mathcal{A}.
\]

Given \( f = z^{n/(2\pi)}g \in A^2_{W,\eta,\text{ext}}(\mathcal{D}_i) \) with \( g \in A^2_{W}(\mathcal{D}) \) as in (6.3) we thus have

\[
(6.11) I_2f(z) = \psi(z)^{\eta/(2\pi)}g(\psi(z)) = z^{\eta/(2\pi)}(\psi(z)/z)^{\eta/(2\pi)}g(\psi(z)) = z^{\eta/(2\pi)}h(z)
\]

where

\[
\|h\|_{D,V}^2 = \int_{\mathcal{A}} \left| \frac{\psi(z)}{z} \right|^{\eta/(2\pi)} \left| g(\psi(z)) \right|^2 \frac{|\psi'(z)|^2}{4\pi^2|\psi(z)|^2} dA(z)
\]

\[
\leq C \int_{\mathcal{A}} |g \circ \psi| \frac{|\psi'|^2}{4\pi^2|\psi|^2} dA = C \int_D |g|^2 W dA.
\]

Hence, \( h \in A^2_{V}(\mathcal{A}) \), and \( I_2 \) maps \( A^2_{W,\eta,\text{ext}}(\mathcal{D}_i) \) into \( A^2_{V,\eta,\text{ext}}(\mathcal{A}_i) \). The converse relation

\[
I_2^{-1}(A^2_{V,\eta,\text{ext}}(\mathcal{A}_i)) \subset A^2_{W,\eta,\text{ext}}(\mathcal{D}_i)
\]

can be proved in the same way, starting by \( f = z^{\eta/(2\pi)}g \in A^2_{W,\eta,\text{ext}}(\mathcal{D}_i) \) with \( g \in A^2_{W}(\mathcal{D}) \) and using \( \phi \) instead of \( \psi \).

The proof is completed by observing that the operator \( I_2 : A^2_{W,\eta}(\mathcal{D}) \to A^2_{V,\eta}(\mathcal{A}_i) \) as well as its inverse are bounded and by taking into account the densities of \( A^2_{W,\eta,\text{ext}}(\mathcal{D}_i) \) and \( A^2_{V,\eta,\text{ext}}(\mathcal{A}_i) \) in these spaces. \( \Box \)

**Lemma 6.3.** An orthonormal basis in \( A^2_{V,\eta}(\mathcal{A}_i) \) is formed by the functions

\[
(6.13) \quad f_{n,\eta}(z) = C_{n,\eta} \frac{z^{n+\eta/(2\pi)}}{v(z)}, \quad n \in \mathbb{Z},
\]
where \( C_{n,\eta} \) are the normalization constants,

\[
C_{n,\eta}^{-2} = \left( \frac{2\pi}{2(n+1)+\eta/\pi} \right)^2 V(z) dA(z) = \int_0^\rho \int_0^{2\pi} v^{2n+\eta/\pi+1}drd\theta
\]

\[
= \frac{2\pi}{2(n+1)+\eta/\pi} (\rho^{2(n+1)+\eta/\pi} - \rho^{-2(n+1)-\eta/\pi}).
\]

(6.14)

Proof. The mutual orthogonality of the functions (6.13) follows from the usual orthogonality relation of the monomials \( z^n \) and the fact that the factor \( 1/v \) is cancelled by the weight in the inner product of \( A_\nu^2(A) \). Indeed, given \( n, m \in \mathbb{Z} \) and \( \eta \in [-\pi, \pi] \) we have

\[
\int_A f_{n,\eta} f_{m,\eta} V dA = C_{n,\eta} C_{m,\eta} \int_A z^{n+\eta/(2\pi)} z^{m+\eta/(2\pi)} dA
\]

where \( z^{\eta/(2\pi)} = (re^{i\theta})^{\eta/(2\pi)} = r^{\eta/(2\pi)} e^{i\theta(\eta/(2\pi)+\eta(\theta))} \) for some \( \ell(\theta) \in \mathbb{Z} \), hence, the integral in (6.15) reads in polar coordinates as

\[
\int_0^\rho \int_0^{2\pi} e^{i\theta(n-m+\eta/(2\pi)+\eta(\theta)-\eta/(2\pi) - \eta(\theta))} drd\theta
\]

which is null unless \( n = m \).

As for the completeness of the orthonormal sequence (6.13), an arbitrary \( h \in A_\nu^2(A) \) can be approximated in the Bergman space \( A_\nu^2(A) \) by a linear combination of the functions \( z^n, n \in \mathbb{Z} \), since these form an orthonormal basis of \( A_\nu^2(A) \), after a proper normalization. Since the mapping \( f \mapsto v^{-1}f \) is an isometry from \( A_\nu^2(A) \) onto \( A_\nu^2(A) \), we find that an arbitrary \( g \in A_\nu^2(A) \) can be approximated in \( A_\nu^2(A) \) by a linear combination of the functions \( v^{-1}z^n, n \in \mathbb{Z} \). In view of (6.9), the linear combinations of the functions (6.13) are dense in \( A_{w,\eta,ext}^2(A) \) and thus the system (6.13) is complete in \( A_{w,\eta}^2(A) \). \( \Box \)

We now construct the kernel function \( K_{\eta}(z, w) \), see (4.11). Due to Lemma 6.3 the kernel \( K_{\eta,A} \) of the orthogonal projection \( f \mapsto \int_A K_{\eta,A}(\cdot, w)f(w)dA(w) \) from \( L_\nu^2(A) \) onto \( A_{\nu,\eta}^2(A) \) is

\[
K_{\eta,A}(z, w) = \sum_{n \in \mathbb{Z}} f_{n,\eta}(z) \overline{f_{n,\eta}(w)} V(w),
\]

where the weight function \( V \) comes from the inner product of \( L_\nu^2(A) \). We have, by (6.10), \( L_\nu^{-1} L_\nu^{-1} f = f \circ \phi \circ E \) hence, Lemma 5.1 and (6.10) imply that the kernel \( K_{\eta} \) of the orthogonal projection from \( L_\nu^2(\mathbb{C}) \) onto \( A_{\eta}^2(\mathbb{C}) \) can be written as

\[
K_{\eta}(z, w) = K_{\eta,A}(\phi \circ E(z), \phi \circ E(w)) \left| (\phi \circ E)'(w) \right|^2
\]

\[
= \sum_{n \in \mathbb{Z}} C_{n,\eta}^2 \phi(e^{i2\pi z})^{n+\eta/(2\pi)} \frac{\overline{\phi(e^{i2\pi w})}^{n+\eta/(2\pi)}}{v(\phi(e^{i2\pi z})) \overline{v(\phi(e^{i2\pi w}))}}, \quad z, w \in \mathbb{C},
\]

(6.17)

where we used (6.8). Moreover, we have

\[
v(\phi(e^{i2\pi z})) = \frac{1}{2\pi} \frac{e^{-i2\pi z}}{\phi'(e^{i2\pi z})}
\]
so that denoting
\begin{equation}
\tilde{K}(z, w) = 4\pi^2 e^{i2\pi(z-w)} \phi'(e^{i2\pi z}) \overline{\phi'(e^{i2\pi w})}
\end{equation}
the kernel becomes
\begin{equation}
K_\eta(z, w) = \tilde{K}(z, w) \sum_{n \in \mathbb{Z}} C_{\eta^{n}} \phi(e^{i2\pi z})^{n+\eta/(2\pi)} \overline{\phi(e^{i2\pi w})}^{n+\eta/(2\pi)}, \quad z, w \in \mathcal{W}.
\end{equation}

Next, we recall from Lemma 2.2 the conformal mapping
\begin{equation}
\varphi(z) = \frac{1}{i2\pi} \log \left( \phi(e^{i2\pi z}) \right) + [\text{Re } z] \quad \text{with } \phi(e^{i2\pi z}) = e^{i2\pi \varphi(z)}.
\end{equation}
Taking into account (6.14) we now write (6.19) as
\begin{equation}
K_\eta(z, w) = \tilde{K}(z, w) \sum_{n \in \mathbb{Z}} \frac{2n + \eta/\pi}{2\pi \left( \rho^{2n+\eta/\pi} - \rho^{-2n-\eta/\pi} \right)} e^{i(2\pi(n-1)+\eta)(\varphi(z)-\varphi(w))}.
\end{equation}
By (6.20), we have $|e^{i2\pi \varphi(z)}| = |\phi(e^{i2\pi z})|$ for all $z \in \mathcal{W}$, and since $\phi(e^{i2\pi z}) \in \mathcal{A}$, we have
\begin{equation}
1/\rho < |e^{i2\pi \varphi(z)}| < \rho,
\end{equation}
which implies that the series (6.21) converges absolutely and uniformly on all compact subsets of $\mathcal{W}$. We arrive at the main result of this section.

**Theorem 6.4.** Let the domain $\mathcal{W}$ (equivalently, $\Pi$) be simply connected and satisfy assumption (A) of Section 2, and let $\eta \in [-\pi, \pi]$. The kernel $K_\eta$ of the projection from $L^2(\mathcal{W})$ onto $A^2_\eta(\mathcal{W})$, see (4.1), is given by formulas (6.18), (6.21).

7. Applications.

In order to demonstrate the potential applicability of Theorem 6.4 we derive the kernel formula (6.21) from it. To this end we combine the kernel formula (6.21) with (6.13) in Theorem 4.3
\begin{equation}
K_\Pi(z, w) = \frac{\tilde{K}(z, w)}{2\pi} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} \frac{2n + \eta/\pi}{2\pi \left( \rho^{2n+\eta/\pi} - \rho^{-2n-\eta/\pi} \right)} \times e^{i\eta \left( [\text{Re } z] - [\text{Re } w] \right) + i(2\pi(n-1)+\eta)(\varphi(z-[\text{Re } z]) - \varphi(w-[\text{Re } w]))} d\eta
\end{equation}
\begin{equation}
= \frac{\tilde{K}(z, w)}{4\pi^2} e^{-i2\pi \varphi(z) - \overline{\varphi(w)}} \frac{4\pi}{\rho^{2\pi t} - \rho^{-2\pi t}} e^{i2\pi t(\varphi(z) - \overline{\varphi(w)})} dt,
\end{equation}
where we also made the summation and integration over $[-\pi, \pi]$ into an integration over the real line (with $t = \eta/(2\pi)$) and used $\varphi(z-[\text{Re } z]) = \varphi(z) - [\text{Re } z]$ (see (6.20)) and similarly for the variable $w$. There holds the integral formula (Fourier transform)
\begin{equation}
\int_{-\infty}^{\infty} \frac{t}{2} \text{csch} (at) e^{-ist} dt = \int_{-\infty}^{\infty} \frac{t}{e^{at} - e^{-at}} e^{-ist} dt = \frac{1}{4a^2} \pi^a \text{sech}^2 \left( \frac{\pi s}{2a} \right), \quad s \in \mathbb{R},
\end{equation}
where $a > 0$ is a parameter and csch denotes the hyperbolic cosecant; the identity (7.2) can be got from the Fourier cosine transform given in the table 1.9(18) of
since the transformed function is even. We apply (7.2) with \( a = 2 \log \rho \) and \( s = 2\pi(\varphi(z) - \varphi(w)) \). The equations (6.18), (7.1) yield formula (2.5):

**Corollary 7.1.** If the periodic domain \( \Pi \), (1.2), is simply connected and satisfies the assumption (A) of Section 2 then its Bergman kernel equals

\[
K_\Pi(z, w) = \tilde{K}_\Pi(z, w) \frac{\pi^3}{4(\log \rho)^2} \text{sech}^2 \left( \frac{\pi^2(\varphi(z) - \varphi(w))}{2 \log \rho} \right),
\]

where

\[
\tilde{K}_\Pi(z, w) = e^{i2\pi(z-w)}(e^{i2\pi z})' \phi' \left( e^{i2\pi z} \right),
\]

\( \phi : \mathcal{D} \to \mathcal{A} \) is the Riemann mapping between doubly connected domains \( \mathcal{D} = \left\{ e^{i2\pi z} : z \in \mathbb{W} \right\} \) and the annulus \( \mathcal{A} = \left\{ w : \rho^{-1} < w < \rho \right\}, \rho > 1 \), and \( \varphi : \Pi \to \mathbb{C} \) is given in (6.20).

Note that in the case \( \Pi \) is the strip \( \Sigma = (-\infty, \infty) \times (-\pi, \pi) \), see the proof of Proposition 2.3, the mapping \( \phi \) is the identity, and obviously formula (7.3) – (7.4), with \( \log \rho = \frac{2\pi}{\rho} \), boils down to (2.7).

The invariance of the kernel under the mapping

\[
(z, w) \mapsto (z + m, w + m), \quad m \in \mathbb{Z},
\]

follows from (6.20). We also have the following estimate for the kernel; estimates for arbitrary \((z, w) \in \Pi \times \Pi\) follow by applying the mapping (7.5).

**Corollary 7.2.** Let \( \Pi \) be as in Corollary 7.1 and assume in addition that there exists a constant \( C > 0 \) such that

\[
\frac{1}{C} \leq \sup_{w \in \mathcal{D}} |\phi'(w)| \leq C.
\]

There exist constants \( 0 < C_1 < C_2 \) such that

\[
C_1 e^{-\pi^2 n/(2 \log \rho)} \leq |K_\Pi(z, 0)| \leq C_2 e^{-\pi^2 n/(2 \log \rho)}
\]

for all \( n \in \mathbb{N} \) and all \( z \in \Pi \) with \( n - 1 \leq |z| \leq n \).

Proof. We have for all \( z \in \Pi \)

\[
\exp \left( \frac{\pi^2 \varphi(z)}{2 \log \rho} \right) = \exp \left( \frac{\pi^2 \varphi(z - \text{Re} z)}{2 \log \rho} \right) \exp \left( \frac{\pi^2 |\text{Re} z|}{2 \log \rho} \right),
\]

and on the right hand side the modulus of the first factor is bounded and bounded from below by a positive constant, since \( \varphi \) is a maps \( \mathbb{W} \) onto a bounded domain. The same holds for the factor \( \tilde{K}_\Pi \), by the assumption (7.6). The result follows from the properties of \( \text{sech} \).

**Remark 7.3.** Condition (7.6) holds, if the boundary of the doubly connected domain \( \mathcal{D} \), equivalently, the boundary of \( \Pi \), is for example \( C^2 \)-smooth. There are non-smooth domains \( \mathcal{D} \) such that for some boundary point \( \zeta \in \partial \mathcal{D} \) we have \( \lim_{z \to \zeta} |\phi'(\zeta)| = \infty \); see the references mentioned below (6.6). In this case the decay (7.7) of course fails, since (7.3), (7.4) imply \( \sup_{z \in \mathbb{W}} |K_\Pi(z, 0)| = \infty \) for all \( m \in \mathbb{Z} \).

The following decay estimate for the kernel does not depend on the geometry of the periodic cell \( \mathbb{W} \).
Corollary 7.4. Let $\Pi$ be as in Corollary 7.1 and let $G \subset \varpi$ be a compact subset. Then, there exist constants $0 < C_1 < C_2$ such that

$$C_1 e^{-\pi^2|n|/(2\log \rho)} \leq |K(z,0)| \leq C_2 e^{-\pi^2|n|/(2\log \rho)}$$

for all $n \in \mathbb{Z}$ and all $z \in \Pi$ such that $z - n \in G$.

Remark 7.5. If the periodic cell $\varpi$ were doubly connected, the entire domain $\Pi$ would be infinitely connected, and in principle the above method yields a representation for the Bergman kernel $K_\Pi$ in terms of Riemann mappings between finitely instead of infinitely connected domains. However, obtaining the concrete formula (7.3)–(7.4) required the use of a quite explicit, simple sequence of orthogonal functions on the annulus and an exact calculation of their norms. The author does not know a good enough example of such sequences for triply and higher connected domains.

We finally prove a simple boundedness result for the Bergman projection with respect to certain weighted $L^p$-norms. Let us consider continuous weights $W : \Pi \to \mathbb{R}^+$ which only depend on the real part of the variable $z \in \Pi$. We assume that there are constants $a, C > 0$ and $0 < b < 1$ such that for all $x \in \mathbb{R}, n \in \mathbb{Z}$

$$\frac{1}{C} W(x) e^{-a|n|^b} \leq |W(x + n)| \leq CW(x) e^{a|n|^b}. \tag{7.9}$$

Theorem 7.6. Let $\Pi$ be as in Corollary 7.2. Let $W : \Pi \to \mathbb{R}^+$ be a weight as above and $1 \leq p < \infty$. Then, the projection operator $P_\Pi : L^p_W(\Pi) \to L^p_W(\Pi)$ is bounded.

Proof. We apply the Schur test, [21], Theorem 3.6., with the constant test function $h(z) \equiv 1$; note that we take $\tilde{K}_\Pi(z,w)W(w)^{-1}$ for the kernel $K$ in the reference. By also taking into account that $|\tilde{K}_\Pi|$ is function bounded by a constant $C > 0$ in $\Pi \times \Pi$, we obtain from (7.7), (7.9) for all $z = x + iy \in \Pi$

$$\int_{\Pi} |K_\Pi(z,w)| W(w)^{-1} W(z) dA(z)$$

$$\leq C \sum_{n \in \mathbb{N}} \int_{n-1 \leq |x - \xi| \leq n} \exp \left( -\frac{\pi^2 n}{2 \log \rho} \right) \exp (an^b) W(x)^{-1} W(x) dA(w)$$

$$\leq C' \sum_{n \in \mathbb{N}} \exp \left( -\frac{\pi^2 n}{2 \log \rho} + an^b \right)$$

$$\leq C''. \tag{7.10}$$

The second integral $\int_{\Pi} |K_\Pi(z,w)| dA(w)$ in the Schur test is also bounded by a constant; the estimation is easier. $\square$
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