ON THE CLASSIFICATION OF SELF-DUAL [20, 10, 9] CODES OVER GF(7)

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Abstract. It is shown that there is a unique self-dual [20, 10, 9] code C over GF(7) such that the lattice obtained from C by Construction A is isomorphic to the 20-dimensional unimodular lattice $D_{20}^+$ up to equivalence. This is done by converting the classification of such self-dual codes to that of skew-Hadamard matrices of order 20.

1. Introduction

Let GF($p$) be the finite field of order $p$, where $p$ is prime. As described in [15], self-dual codes are an important class of linear codes for both theoretical and practical reasons. For $p \equiv 1 \pmod{4}$, a self-dual code of length $n$ over GF($p$) exists if and only if $n$ is even, and for $p \equiv 3 \pmod{4}$, a self-dual code of length $n$ over GF($p$) exists if and only if $n \equiv 0 \pmod{4}$. It is a fundamental problem to classify self-dual codes over GF($p$) and determine the largest minimum weight among self-dual codes over GF($p$) for a fixed length. Much work has been done towards classifying self-dual codes over GF($p$) and determining the largest minimum weight among self-dual codes of a given length over GF($p$) for $p = 2$ and 3 (see [15]).

Self-dual codes over GF(7) have been classified for lengths up to 12 (see [7]), and the largest minimum weight $d_7(n)$ among self-dual codes of length $n$ over GF(7) has been determined for $n \leq 28$ (see [5, Table 2]). For example, it is known that $d_7(20) = 9$. Some self-dual [20, 10, 9] codes over GF(7) can be found in [4, Table 6] and [5, Table 7]. As described in [10], it is an open problem to determine whether such a code is unique or not.

There are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [2, Table 16.7]). The lattice $D_{20}^+$ is one of

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the 20-dimensional unimodular lattices having minimum norm 2. Let $A_7(C)$ denote the unimodular lattice obtained from a self-dual code $C$ over GF(7) by Construction A.

In this paper, we convert the classification of self-dual [20, 10, 9] codes $C$ over GF(7) such that $A_7(C)$ is isomorphic to $D_{20}^+$ to that of skew-Hadamard matrices of order 20. The main aim of this paper is to give the following partial classification of self-dual [20, 10, 9] codes over GF(7).

**Theorem 1.** Up to equivalence, there is a unique self-dual [20, 10, 9] code $C$ over GF(7) such that $A_7(C)$ is isomorphic to $D_{20}^+$.

All computer calculations in this paper were done by Magma [1].

2. Preliminaries

In this section, we give definitions and notions on self-dual codes, unimodular lattices and skew-Hadamard matrices. Some basic facts on these subjects are also provided.

2.1. Self-dual codes. An $[n, k]$ code $C$ over GF($p$) is a $k$-dimensional subspace of GF($p$)$^n$. The value $n$ is called the length of $C$. The weight $wt(x)$ of a vector $x \in$ GF($p$)$^n$ is the number of non-zero components of $x$. A vector of $C$ is called a codeword of $C$. The minimum non-zero weight of all codewords in $C$ is called the minimum weight of $C$ and an $[n, k]$ code with minimum weight $d$ is called an $[n, k, d]$ code. The weight enumerator $W(C)$ of $C$ is given by $W(C) = \sum_{i=0}^{n} A_i y^i$, where $A_i$ is the number of codewords of weight $i$ in $C$. The dual code $C^\perp$ of $C$ is defined as

$$C^\perp = \{x \in$ GF($p$)$^n \mid x \cdot y = 0 \text{ for all } y \in C\},$$

under the standard inner product $x \cdot y$. A code $C$ is called self-dual if $C = C^\perp$. Two codes $C$ and $C'$ are equivalent if there exists a $(1, -1, 0)$-monomial matrix $M$ with $C' = \{cM \mid c \in C\}$.

2.2. Unimodular lattices. An $n$-dimensional (Euclidean) lattice is a discrete subgroup of rank $n$ in $\mathbb{R}^n$. A lattice $L$ is unimodular if $L = L^*$, where the dual lattice $L^*$ is defined as

$$L^* = \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\},$$

under the standard inner product $(x, y)$. The norm $\|x\|^2$ of a vector $x \in \mathbb{R}^n$ is $(x, x)$. The minimum norm of $L$ is the smallest norm among all nonzero vectors of $L$. Two lattices $L$ and $L'$ are isomorphic, denoted $L \cong L'$, if there exists an orthogonal matrix $A$ with $L' = \{xA \mid x \in L\}$.
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Let \(C\) be a code of length \(n\) over GF\((p)\) and let \(\varepsilon_1, \ldots, \varepsilon_n\) be an orthogonal basis of \(\mathbb{R}^n\) satisfying \((\varepsilon_i, \varepsilon_j) = p\delta_{i,j}\), where \(\delta_{i,j}\) is the Kronecker delta. Then we define the lattice \(A_p(C)\) obtained from \(C\) by Construction \(A\) as

\[
A_p(C) = \left\{ \frac{1}{p} \sum_{i=1}^{n} x_i \varepsilon_i \mid x = (x_1, \ldots, x_n) \in \mathbb{Z}^n, x \mod p \in C \right\}.
\]

It is known that \(A_p(C)\) is unimodular if and only if \(C\) is self-dual. A set \(\{f_1, \ldots, f_n\}\) of \(n\) vectors \(f_1, \ldots, f_n\) in an \(n\)-dimensional lattice \(L\) with \((f_i, f_j) = k\delta_{i,j}\) is called a \(k\)-frame of \(L\). Clearly, if a unimodular lattice \(L\) contains a \(p\)-frame, then there is a self-dual code \(C\) over GF\((p)\) with \(A_p(C) \cong L\) (see [6]).

Let \(C\) be a self-dual \([20,10,d]\) code over GF\((7)\) with \(d \in \{8,9\}\). Then it is easy to see that \(A_7(C)\) has minimum norm 2. It is known that there are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [2, Table 16.7]). These 12 lattices have distinct components (see [2, Table 16.7] for the components). The 20-dimensional unimodular lattice \(D_{20}^+\) with minimum norm 2 has component \(D_{20}\). The lattices \(D_{20}\) and \(D_{20}^+\) are defined as follows:

\[
D_{20} = \left\{ \sum_{i=1}^{20} \alpha_i e_i \mid (\alpha_1, \ldots, \alpha_{20}) \in \mathbb{Z}^{20}, \sum_{i=1}^{20} \alpha_i \equiv 0 \pmod{2} \right\},
\]

\[
D_{20}^+ = \langle D_{20}, 1 \rangle,
\]

where \(e_i = (\delta_{1,i}, \ldots, \delta_{20,i})\) \((1 \leq i \leq 20)\) and 1 denotes the all-one vector. Note that \(D_{20}\) is the even sublattice of \(D_{20}^+\), that is, the sublattice consisting of vectors of even norm in \(D_{20}^+\).

2.3. Skew-Hadamard matrices. A Hadamard matrix of order \(n\) is an \(n \times n\) \((1, -1)\)-matrix \(H\) such that \(HH^\top = nI\), where \(I\) is the identity matrix and \(H^\top\) denotes the transposed matrix of \(H\). It is well known that the order \(n\) is necessarily 1, 2, or a multiple of 4. Two Hadamard matrices \(H\) and \(K\) are said to be equivalent if there are \((1, -1, 0)\)-monomial matrices \(P\) and \(Q\) with \(K = PHQ\). All Hadamard matrices of orders up to 32 have been classified (see [8, Chap. 7] for orders up to 28 and [9] for order 32, see also [17]).

A Hadamard matrix \(H\) of order \(n\) is called a skew-Hadamard matrix if \(H + H^\top = 2I\). Skew-Hadamard matrices are a class of Hadamard matrices, which has been widely studied (see e.g., [3], [11]). The numbers of inequivalent skew-Hadamard matrices of orders 4, 8, 12, 16, 20, 24 are 1, 1, 1, 3, 2, 11, respectively [11]. As an example, we give two inequivalent skew-Hadamard matrices \(S_1\) and \(S_2\) of order 20 in Figure 1, where
we use $+, -$ to denote 1, $-1$, respectively. Note that $S_1$ is equivalent to the Paley Hadamard matrix. Moreover, we have verified by Magma that $S_2$ is equivalent to {\tt had.20.tonchev} in [17].

\[
S_1 = 
\begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{pmatrix}
\]

\[
S_2 = 
\begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
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- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
- & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{pmatrix}
\]

\textbf{Figure 1.} Skew-Hadamard matrices of order 20

The following lemma can be proved in the same manner as [13, Lemma 3].
Lemma 2. Let \( F \) be a square matrix all of whose entries are integers. If \( FF^\top = kI \) and \( p \) is a prime divisor of \( k \) such that \( p^2 \nmid k \), then \( F \) generates a self-dual code over \( \text{GF}(p) \).

Hence, the code over \( \text{GF}(7) \) generated by the row vectors of \( H + 2I \) is self-dual, where \( H \) is a skew-Hadamard matrix of order 20.

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1, which is the main result of this paper.

Lemma 3. Let \( C \) be a self-dual \([20, 10, 9]\) code over \( \text{GF}(7) \). If \( \xi \in A_7(C) \) and \( ||\xi||^2 = 2 \), then

\[
|\{i \mid 1 \leq i \leq 20, \ |(\xi, \varepsilon_i)| \geq 2\}| \leq 1.
\]

Proof. Write

\[
\xi = \frac{1}{7} \sum_{i=1}^{20} x_i \varepsilon_i, \ x = (x_1, \ldots, x_n) \in \mathbb{Z}^{20}, \ x \mod 7 \in C.
\]

Since for each \( j \in \{1, \ldots, 20\} \),

\[
x_j^2 = 7 ||\frac{1}{7} x_j \varepsilon_j||^2 
\leq 7 ||\frac{1}{7} \sum_{i=1}^{20} x_i \varepsilon_i||^2 
= 7 ||\xi||^2 
= 14,
\]

we have

\[
(1) \quad x_j \equiv 0 \pmod{7} \iff x_j = 0 \iff (\xi, \varepsilon_j) = 0.
\]

Set

\[
a_1 = |\{i \mid 1 \leq i \leq 20, \ |(\xi, \varepsilon_i)| = 1\}|, 
\]

\[
a_2 = |\{i \mid 1 \leq i \leq 20, \ |(\xi, \varepsilon_i)| \geq 2\}|.
\]

Then by (1) we have

\[
a_1 + a_2 = \text{wt}(x) \geq 9,
\]

and we have

\[
a_1 + 4a_2 \leq \sum_{i=1}^{20} (\xi, \varepsilon_i)^2 
= 7 ||\xi||^2
\]
Thus $a_2 \leq \frac{2}{3}$, and hence $a_2 \leq 1$. 

**Proposition 4.** Let $C$ be a self-dual $[20, 10, 9]$ code over $GF(7)$ with $A_7(C) \cong D_{20}^+$. Then there exists a skew-Hadamard matrix $H$ of order 20 such that $C$ is generated by the row vectors of $H + 2I$ over $GF(7)$.

**Proof.** Let $\Psi : A_7(C) \to D_{20}^+$ be an isomorphism. Since $\|\Psi(\varepsilon_j)\|^2 = \|\varepsilon_j\|^2 = 7$ is odd, $\Psi(\varepsilon_j) \not\in D_{20}$. Thus $\Psi(\varepsilon_j) \in \frac{1}{2}1 + D_{20} \subset \frac{1}{2}(1 + 2\mathbb{Z})^{20}$, and hence there exist odd integers $f_{i,j}$ such that

$$\Psi(\varepsilon_j) = \frac{1}{2} \sum_{i=1}^{20} f_{i,j}e_i.$$ 

Let $F$ denote the $20 \times 20$ matrix whose $(i, j)$ entry is $f_{i,j}$. Then $F^\top F = 28I$. In particular,

$$\sum_{h=1}^{20} f_{h,i}^2 = 28.$$ 

Since $f_{h,i}$ are odd integers, we see that there exists a unique $h_i$ such that $f_{h,i} = \pm 3$.

We claim that the mapping $i \mapsto h_i$ is a bijection from $\{1, \ldots, 20\}$ to itself. Indeed, suppose, for example, $h_1 = h_2 = 1$. Replacing $\varepsilon_1, \varepsilon_2$ by $-\varepsilon_1, -\varepsilon_2$, respectively, if necessary, we may assume $f_{1,1} = f_{1,2} = 3$. Since $f_{h,1} = \pm 1$ and $f_{h,2} = \pm 1$ for all $h \in \{2, \ldots, 20\}$ and

$$0 = \sum_{h=1}^{20} f_{h,1}f_{h,2}$$

$$= 9 + \sum_{h=2}^{20} f_{h,1}f_{h,2}$$

$$= 9 + |\{h \mid 2 \leq h \leq 20, \ f_{h,1} = f_{h,2}\}|$$

$$- |\{h \mid 2 \leq h \leq 20, \ f_{h,1} = -f_{h,2}\}|$$

$$= 9 + |\{h \mid 2 \leq h \leq 20, \ f_{h,1} = f_{h,2}\}|$$

$$- (19 - |\{h \mid 2 \leq h \leq 20, \ f_{h,1} = f_{h,2}\}|)$$

$$= -10 + 2|\{h \mid 2 \leq h \leq 20, \ f_{h,1} = f_{h,2}\}|,$$

we see that there exists $h \in \{2, \ldots, 20\}$ such that $f_{h,1} = f_{h,2}$. Set $\xi = \Psi^{-1}(\varepsilon_1 + f_{h,1}e_h)$. Then $\|\xi\|^2 = \|e_1 + f_{h,1}e_h\|^2 = 2$. However, for $i = 1, 2$, we have

$$\langle \xi, \varepsilon_i \rangle = \langle \Psi(\xi), \Psi(\varepsilon_i) \rangle$$
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\[
= (e_1 + f_{h,1} e_h, \frac{1}{2} \sum_{j=1}^{20} f_{j,i} e_j)
\]

\[
= \frac{1}{2} (f_{1,i} + f_{h,1} f_{h,i})
\]

\[
= 2.
\]

This contradicts Lemma 3, and completes the proof of the claim.

Now we may assume without loss of generality

\[
f_{h,i} = \begin{cases} 
3 & \text{if } h = i, \\
\pm 1 & \text{otherwise.}
\end{cases}
\]

Set \(H = F - 2I\). Then all the entries of \(H\) are \(\pm 1\), and the diagonal entries are 1.

We claim \(H + H^\top = 2I\). To prove this, we need to show \(f_{h,i} + f_{i,h} = 0\) for \(1 \leq h < i \leq 20\). Suppose \(f_{h,i} = f_{i,h}\) for some \(1 \leq h < i \leq 20\). Set \(\xi = \Psi^{-1} (e_h + f_{i,h} e_i)\). Then \(\|\xi\|^2 = \|e_h + e_i\|^2 = 2\), and

\[
(\xi, e_i) = (\Psi(\xi), \Psi(e_i))
\]

\[
= (e_h + f_{i,h} e_i, \frac{1}{2} \sum_{j=1}^{20} f_{j,i} e_j)
\]

\[
= \frac{1}{2} (f_{h,i} + f_{i,h} f_{i,i})
\]

\[
= 2f_{i,h}.
\]

Similarly,

\[
(\xi, e_h) = (\Psi(\xi), \Psi(e_h))
\]

\[
= (e_h + f_{i,h} e_i, \frac{1}{2} \sum_{j=1}^{20} f_{j,h} e_j)
\]

\[
= \frac{1}{2} (f_{h,h} + f_{i,h}^2)
\]

\[
= 2.
\]

These contradict Lemma 3, and complete the proof of the claim.

Since

\[
H^\top H = (F^\top - 2I)(F - 2I)
\]

\[
= 28I - 2(H^\top + H + 4I) + 4I
\]

\[
= 20I,
\]

\(H\) is a Hadamard matrix.
Finally, since
\[ D_{20}^2 \ni 2e_i \]
\[ = \frac{1}{14} \sum_{h=1}^{20} 28 \delta_{h,i} e_h \]
\[ = \frac{1}{14} \sum_{h=1}^{20} \sum_{j=1}^{20} f_{i,j} f_{h,j} e_h \]
\[ = \frac{1}{7} \sum_{j=1}^{20} f_{i,j} \frac{1}{2} \sum_{h=1}^{20} f_{h,j} e_h \]
\[ = \frac{1}{7} \sum_{j=1}^{20} f_{i,j} \Psi(\varepsilon_j) \]
\[ = \Psi(\frac{1}{7} \sum_{j=1}^{20} f_{i,j} \varepsilon_j), \]
we have
\[ \frac{1}{7} \sum_{j=1}^{20} f_{i,j} \varepsilon_j \in A_7(C). \]
Thus the \( i \)-th row of \( F = H + 2I \) belongs to \( C \). The fact that \( F \) generates the self-dual code \( C \) follows from Lemma 2. \( \square \)

We say that skew-Hadamard matrices \( H \) and \( H' \) of order \( n \) are skew-Hadamard equivalent if there exists a \((1, -1, 0)\)-monomial matrix \( P \) with \( PHP^\top = H' \). Let \( H \) and \( H' \) be skew-Hadamard matrices of order 20. Let \( C(H) \) denote the code over \( GF(7) \) generated by the row vectors of \( H + 2I \). If \( H \) and \( H' \) are skew-Hadamard equivalent, then \( C(H) \) and \( C(H') \) are equivalent. By Proposition 4, we can convert the classification of self-dual \([20, 10, 9]\) codes \( C \) over \( GF(7) \) with \( A_7(C) \cong D_{20}^2 \) to that of skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence. The existence of a skew-Hadamard matrix of order \( n \) is equivalent to the existence of a doubly regular tournament of order \( n - 1 \) \([16]\). It is known that there are two doubly regular tournaments of order 19, up to isomorphism (see \([12]\)). This implies that there are two skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence. Indeed, let \( H \) be a skew-Hadamard matrix of order 20 and let \( D \) be the diagonal matrix whose diagonal entries are the first row of \( H \). Then
\[ DHD = \begin{pmatrix} \frac{1}{14} & 1 \\ -1^\top & M \end{pmatrix}. \]
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Here the $19 \times 19$ $(1,0)$-matrix $(M + J)/2 - I$ is the adjacency matrix of a doubly regular tournament of order 19, where $J$ is the $19 \times 19$ all-one matrix. Hence, isomorphic doubly regular tournaments of order 19 give skew-Hadamard matrices of order 20, which are skew-Hadamard equivalent. The matrices $S_1$ and $S_2$ in Figure 1 give the two skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence.

We have verified by MAGMA that the two self-dual codes $C(S_1)$ and $C(S_2)$ have the following weight enumerators:

$$W(C(S_1)) = 1 + 6840y^9 + 47880y^{10} + 200640y^{11} + 957600y^{12} + 3625200y^{13} + 10766160y^{14} + 25701984y^{15} + 48495600y^{16} + 68276880y^{17} + 68299680y^{18} + 43155840y^{19} + 12940944y^{20},$$

$$W(C(S_2)) = 1 + 1080y^8 + 5040y^9 + 40320y^{10} + 215760y^{11} + 977040y^{12} + 3571200y^{13} + 10751040y^{14} + 25814304y^{15} + 48431880y^{16} + 68208840y^{17} + 68299680y^{18} + 43106160y^{19} + 12949584y^{20},$$

respectively. In particular, $C(S_1)$ is a $[20,10,9]$ code, while $C(S_2)$ has minimum weight 8. By Proposition 4, $C(S_1)$ is a unique self-dual $[20,10,9]$ code $C$ over GF(7) with $A_7(C) \cong D_{20}^+$. This completes the proof of Theorem 1.

4. SOME OTHER CONSTRUCTIONS OF SELF-DUAL $[20,10,9]$ CODES

Finally, in this section, we investigate some other constructions of self-dual $[20,10,9]$ codes over GF(7). In the above classification, we employed $(1,-1,0)$-monomial matrices in the definition for equivalence of codes. In some earlier work, a weaker equivalence is used. We say that two codes $C$ and $C'$ over GF(7) are monomially equivalent if there exists a monomial matrix $M$ over GF(7) with $C' = \{cM \mid c \in C\}$. Clearly, if $C$ and $C'$ are equivalent, then they are monomially equivalent.

Note that $D_{20}^+$ is the unique 20-dimensional unimodular lattice with minimum norm 2 and kissing number 760. For a given self-dual $[20,10,9]$ code $C$ over GF(7), one can determine whether $A_7(C)$ is isomorphic to $D_{20}^+$ or not, by computing the kissing number of $A_7(C)$ by MAGMA. If $A_7(C)$ is isomorphic to $D_{20}^+$, then by Theorem 1, we have that $C$ is equivalent to $C(S_1)$.

• Some self-dual $[20,10,9]$ codes over GF(7) were constructed in [4, Table 6] and [5, Table 7] as double circulant codes and
quasi-twisted self-dual codes, respectively (see [5] for the construction). It was verified by Magma that all double circulant self-dual \([20, 10, 9]\) codes and all quasi-twisted self-dual \([20, 10, 9]\) codes are monomially equivalent [5, Section 4.3].

By exhaustive search, we have verified that \(A_7(C) \cong D_{20}^+\) for all double circulant self-dual \([20, 10, 9]\) codes \(C\). Also, we have verified that \(A_7(C) \cong D_{20}^+\) for all quasi-twisted self-dual \([20, 10, 9]\) codes \(C\).

- Let \(A\) and \(B\) be \(5 \times 5\) circulant (resp. negacirculant) matrices. A \([20, 10]\) code over GF(7) with the following generator matrix

\[
\left( \begin{array}{cc}
I & A \\
-B^\top & A^\top
\end{array} \right)
\]

is called a four-circulant (resp. four-negacirculant) code. By exhaustive search, we have verified that \(A_7(C) \cong D_{20}^+\) for all four-circulant self-dual \([20, 10, 9]\) codes \(C\). Also, we have verified that \(A_7(C) \cong D_{20}^+\) for all four-negacirculant self-dual \([20, 10, 9]\) codes \(C\).

- Let \(C\) be a self-dual code of length 20 over GF(7). Let \(x\) be a vector with \(x \cdot x = 0\). Then \(C_x = \langle C \cap \langle x \rangle^\perp, x \rangle\) is a self-dual code over GF(7). By exhaustive search, we have verified that \(A_7(C(S_1)_x) \cong D_{20}^+\) for all \(x\) with \(x \cdot x = 0\).

Moreover, our extensive search failed to discover a self-dual \([20, 10, 9]\) code \(C\) over GF(7) with \(A_7(C) \not\cong D_{20}^+\). We are lead to conjecture that \(C(S_1)\) is a unique self-dual \([20, 10, 9]\) code over GF(7).

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