SCATTERING AND STRONG INSTABILITY OF THE STANDING WAVES FOR DIPOLAR QUANTUM GASES

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ABSTRACT. This paper concerns the nonlinear Schrödinger equation which describes the dipolar quantum gases. When the energy plus mass is lower than the mass of the ground state, we find we can use the kinetic energy and mass of the initial data to divide the subspace into two parts. If the initial data are in one of the parts, the solutions exist globally. Moreover, by using the Kening-Merle roadmap method, we find that these solutions will scatter. If initial data are in the other part, the solutions will collapse. And hence, the standing waves are strong unstable.

1. Introduction. In 1995, the Bose-Einstein condensation in dilute gases was observed for the first time in experiment. Since then, the dipole-dipole interactions between the particles, which are both long-range and non-isotropic, attract more and more attention in atomic gases. It was not until 2000 that Yi and You introduce a pseudo-potential appropriate to describe the dipolar interactions between particles [23]. Following their results one describes the corresponding Bose-Einstein condensation in dilute gases by the following nonlinear Schrödinger equation:

\[ i\hbar \partial_t \varphi = -\frac{\hbar^2}{2m} \Delta \varphi + V(x)\varphi + \frac{4\pi \hbar^2 \ell a}{m} |\varphi|^2 \varphi + d^2 (K * |\varphi|^2)\varphi, \quad t \in \mathbb{R}, x \in \mathbb{R}^3, \] (1)

where \( \varphi(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C} \) is the wave function, \( \ell \in \mathbb{N} \) denotes the number of the particles, \( m \) is the mass of the individual particle with scattering length \( a \in \mathbb{R} \) and the parameter \( d^2 \) denotes the strength of the dipole moment. The potential \( V(x) \) describes the electromagnetic trap for the condensate. And the dipole-interaction kernel

\[ K(x) = \frac{1 - 3 \cos^2 \theta}{|x|^3}, \]

where \( \theta = \theta(x) \) is the angle between \( x \in \mathbb{R}^3 \) and a fixed dipole axis \( n \in \mathbb{R}^3 \).

In this paper, we consider the case where the electromagnetic trap vanishes (i.e., \( V(x) = 0 \)). For convenience, we consider the following form of Eq.(1) with initial
data $\varphi_0(x) \in H^1(\mathbb{R}^3)$:

$$
\begin{cases}
    i\partial_t \varphi = -\frac{1}{2} \Delta \varphi + \lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi, \\
    \varphi(0, x) = \varphi_0, \quad x \in \mathbb{R}^3,
\end{cases}
$$

(2)

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are two given numbers.

Because the dipole interaction kernel crucially influences the dynamics of the condensate and stability, Eq.(2) has received much attention in physics (such as [10, 21]) and mathematics (such as [1, 2, 5, 6, 18, 19]). Mathematically, we are interested in the dynamics and stability of this system. More precisely, Carles, Markowich and Sparber in [6], where the local well-posedness of Eq.(2) was analyzed, and mass, and how to give the sufficient conditions for scattering in terms of the kinetic energy and mass? These motivate the current study.

As we mentioned before, the first rigorous mathematical study of Eq.(2) is by Carles, Markowich and Sparber in [6], where the local well-posedness of Eq.(2) was given.

**Proposition 1.** ([6]) For $\lambda_1, \lambda_2 \in \mathbb{R}$, $\varphi \in H^1(\mathbb{R}^3)$, there exists a $T$, depending only on $\|\varphi\|_{H^1(\mathbb{R}^3)}$, such that Eq.(2) has a unique maximal solution

$$
\varphi \in \{ \varphi \in C([0, T]; H^1(\mathbb{R}^3)); \varphi, \nabla \varphi \in C([0, T], L^2(\mathbb{R}^3)) \cap L^4([0, T], L^4(\mathbb{R}^3)) \}. 
$$

It is maximal in the sense that if $T^* < \infty$, then

$$
\|\nabla \varphi\|_{L^2} \to \infty \quad \text{as} \quad t \to T^*.
$$

Moreover, the mass $M(\varphi)$ (the square of $L^2$ norm of the solution $u$) and the energy $E(\varphi)$ are conserved for $t \in [0, T^*)$. Namely,

$$
M(\varphi(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\varphi|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} |\varphi_0|^2 dx = M(\varphi_0)
$$

(3)

and

$$
E(\varphi(t)) := \frac{1}{2} E_0(\varphi(t)) - \frac{1}{2} P(\varphi(t)) = E(\varphi_0),
$$

(4)

where $E_0(\varphi(t)) := \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx$ and $P(\varphi(t)) := -\lambda_1 \int_{\mathbb{R}^3} |\varphi|^4 dx - \lambda_2 \int_{\mathbb{R}^3} (K * |\varphi|^2)|\varphi|^2 dx$.

Obviously, Eq.(2) also has several invariances, such as scaling invariance and Galilean invariance and so on. It is worth mentioning the scaling invariance: if $\varphi(\mu t, \mu x)$ is the solution of Eq.(2), then

$$
\varphi_\mu(t, x) = \mu \varphi(\mu^2 t, \mu x)
$$

is also solution of Eq.(2). Furthermore, this invariance shows $\|\varphi_\mu\|_{H^{1/2}} = \|\varphi\|_{H^{1/2}}$. The other form of the solutions for Eq.(2) is $\varphi(t, x) = e^{i\omega t} Q_\omega(x)$, which is called standing wave solution of Eq.(2). Here $Q_\omega(x)$ is the ground state solution of the following nonlinear elliptic equation

$$
-\frac{1}{2} \Delta Q_\omega(x) + \omega Q_\omega(x) + \lambda_1 |Q_\omega(x)|^2 Q_\omega(x) + \lambda_2 (K * |Q_\omega(x)|^2) Q_\omega(x) = 0,
$$

(5)
which is obtained by minimizing an associated energy functional. According to [1] and [22], when $\lambda_1$ and $\lambda_2$ satisfy
\[
\lambda_1 < \begin{cases} 
\frac{8}{3} \pi \lambda_2, & \text{if } \lambda_2 > 0; \\
\frac{4}{3} \pi \lambda_2, & \text{if } \lambda_2 < 0,
\end{cases}
\]
(6)
such minimizers can be obtained by considering the corresponding Weinstein functional
\[
\mathcal{G}(u) = \frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |u|^2 \, dx \right)^{\frac{1}{2}} - \lambda_1 \int_{\mathbb{R}^3} |u|^4 \, dx - \lambda_2 \langle K * |u|^2, |u|^2 \rangle,
\]
(7)
where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^3)$. This problem was discussed by Antonelli and Sparber in [1], in which the existence of the solitary waves $Q_{\omega}(x)$ was proved. Moreover, by setting the dipole axis $n = (0, 0, 1)$, they showed $Q_{\omega}(x)$ is radially symmetric in the $x_1$, $x_2$-plane, axially symmetric with respect to the $x_3$-axis and $Q_{\omega}(x)$ is exponentially decaying, nonnegative and belongs to $H^s$ for all $s > 1$. In fact, if we denote the scaling $Q_{\omega}(x) = \sqrt{2\omega} Q_{\sqrt{2\omega}x}$, then $Q(x)$ satisfies the following elliptic equation
\[
-\frac{1}{2} \Delta Q(x) + \frac{1}{2} Q(x) + \lambda_1 |Q(x)|^2 Q(x) + \lambda_2 (K * |Q(x)|^2) Q(x) = 0.
\]
(8)
Of course, $Q(x)$ is also the minimizer of the Weinstein functional (7), and hence $Q(x)$ is the ground state of Eq.(8).

Before stating our main results, we recall some related terminology. For the focusing case, Kenig and Merle[16] introduced roadmap ideas to prove the scattering of a focusing nonlinear Schrödinger equation in the radial case by a contradiction argument. Then based on these, Holmer and Roudenko[13] gave the asymptotic dynamics for the 3D cubic nonlinear Schrödinger equation, but still in radial case. More precisely, they applied the mass and energy product to control the initial data to obtain the dichotomy criterion. And hence by the idea of [16] the scattering result was obtained. Soon, the results are extended to non-radial case by Duyckaerts, Holmer and Roudenko in [9]. For the defocusing case, there are some papers concerning with asymptotic dynamics of the 3D cubic Schrödinger equations. The scattering in the space $H^1(\mathbb{R}^3)$ was proved in [12, 4] by using the Morawetz inequality. In [8], Colliander, Keel, Staffilani, Takaoka and Tao discovered a new interaction Morawetz inequality which is a more simple way to deal with this problem for a rough solution.

In this paper, we will discuss the Cauchy problem (2) and establish global existence and scattering results. We also obtain the finite time blowup and prove strong instability of the standing waves. It is remarkable that the dipole term has no definite sign. To overcome that, the cubic term need to be considered together with the dipolar term, and we give two conditions that are directly relevant to the stable regime and unstable regime of this system. Another motivation for studying Eq.(2) also lies in the influence of the dipolar nonlinearity, which makes the radial symmetry broken. In the unstable regime, by using the roadmap method in [16] and combining with the idea in [15], we show the scattering and strong instability for the quantum gases. The difference from these works in [9, 13, 16] is that we use the kinetic energy and mass to divide a subspace of $H^1(\mathbb{R}^3)$, which restricted by its energy plus mass, into two parts. Moreover, one part of the subspace has the boundary between two parts. This is distinguished from the previous studies.
of sharp threshold for global existence and blowup [9, 13]. For the one part of
the subspace, we use the variational properties and compactness of the equation
to prove the main results. More specifically, we analyze a special functional by
variational method. And then, we start our argument from small data result. By
the concentration compactness principle and Morawetz estimate for data of critical
size, we get the rigidity theorem. Finally, we prove our result by using reduction
to absurdity. For the strong instability, although it was discussed in [14], here we
apply a quite different method from the cross-constrain problem in [14]. In fact, we
only use the functional associated with the kinetic energy plus mass to derive the
blowup result and hence the strong instability.

Our main result addresses the question of global existence, scattering and blowup
for Eq.(2) in the energy space in the unstable regime:

**Theorem 1.1.** Let $\lambda_1$ and $\lambda_2$ satisfy (6) and $\varphi_0 \in H^1(\mathbb{R}^3)$ satisfy $I(\varphi_0) := (\mathcal{E} + \mathcal{M})(\varphi_0) < \|Q\|_{L^2}^2/2$.

1. If $(\frac{1}{2}\mathcal{E}_0 + \mathcal{M})(\varphi_0) < \|Q\|_{L^2}^2$, the corresponding solution $\varphi$ of Eq.(2) globally exists and scatters. More precisely, $I = \mathbb{R}$ and there exist $\varphi_{\pm}$ such that
   \[ \|U(-t)\varphi(t) - \varphi_{\pm}\|_{H^1} \to 0 \quad (t \to \pm\infty). \]

2. If $(\frac{1}{2}\mathcal{E}_0 + \mathcal{M})(\varphi_0) \geq \|Q\|_{L^2}^2$ and $|x|\varphi_0 \in L^2(\mathbb{R}^3)$, the corresponding solution $\varphi$
of Eq.(2) blows up in finite time.

In this paper, we only discuss the scattering results for positive times. The
results for negative times can be obtained by the equation’s time reversibility. At
the same time, from the blowup results in Theorem 1.1 (2), the strong instability
of the standing waves are obtained.

**Corollary 1.** Suppose $\lambda_1$ and $\lambda_2$ satisfy (6), then the standing wave $\varphi(t, x) = e^{it}Q(x)$ is $H^1(\mathbb{R}^3)$ strongly unstable.

Obviously, in [2, 6], they discussed the Eq.(1) with confining potential. But in
this paper, Eq.(1) without potential is studied. It is this potential function that
leads to very different results. More specifically, in [6], the authors found that if
$\lambda_1 \geq \frac{4\pi}{\sqrt{3}}\lambda_2 > 0$, the standing waves are orbitally stable for any positive mass. As
for reference [2], for $\lambda_1 \geq 0, -\frac{1}{4}\lambda_1 \leq \lambda_2 \leq \lambda_1$, they showed that the existence of the
ground state and global solutions. And the nonexistence of the ground state and
finite time blowup were obtained with $\lambda_1 < 0, \lambda_2 < -\frac{1}{4}\lambda_1$ or $\lambda_2 > \lambda_1 > 0$. But
from [1] and Corollary 1, we know that if the coefficients in Eq.(2) satisfy (6), there
exists standing waves, which are strongly unstable.

This paper is organized as follows. The Strichartz estimates and some properties
of the dipole kernel are given in Section 2. In Section 3, we show the local theory of
Eq.(2). The global bound in $H^1$ and blowup phenomenon including the properties
of the ground states are discussed in Sections 4. In section 5, the compactness
of Eq.(2) are obtained. In the final section, we show the scattering may occur in
energy space.

2. Strichartz estimates and some properties of the dipole kernel. At the
beginning, we recall some relevant Strichartz estimates and the properties of the
dipole kernel. Throughout this paper, $C$ denotes various positive constants and
$H^1(\mathbb{R}^3) = W^{1,2}(\mathbb{R}^3)$ is the standard Sobolev space with the norm $\|\varphi\|_{H^1}^2 = \int_{\mathbb{R}^3}(|\nabla \varphi|^2 + |\varphi|^2)dx$. And we use $\| \cdot \|_{L^p}$ to denote the norm of $L^p(\mathbb{R}^3)$ and
$\Sigma := \{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2|u|^2dx < \infty \}$. For simplicity, here and hereafter, we
will often use $p'$ to denote the Hölder conjugate exponents with respect to the real number $p$.

2.1. Strichartz estimates. In this subsection, we will recall some tools in [13]. We say that $(q, r)$ is $\dot{H}^s$ Strichartz admissible (in 3 dimension) if $\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s$. Denote
\[
\|u\|_{S(\mathbb{R})} = \sup \|u\|_{L^q_t L^s_x}, \quad \|u\|_{S'(\mathbb{R})} = \inf \|u\|_{L^{q'}_t L^{s'}_x},
\]
where the sup and the inf is taken over all $L^2$-admissible pairs $(q, r)$. $2 \leq r \leq 6$, $2 \leq q \leq \infty$;
\[
\|u\|_{X(\mathbb{R})} = \sup \|u\|_{L^q_t L^s_x},
\]
where the sup is taken over all $\dot{H}^{1/2}$-admissible pairs $(q, r)$ and $3 \leq r < 6$, $4 < q \leq \infty$. In particular, we will use the following Strichartz space
\[
W(\mathbb{R}) := L^2_t L^\infty_x \cap L^4_t L^4_x.
\]

To indicate a restriction to a time subinterval $I \subset \mathbb{R}$, we will write $S(I)$, $X(I)$, $W(I)$ and $L^q_t L^s_x(I)$ respectively. We denote the group
\[
U(t) = e^{\frac{2 t}{n} \Delta},
\]
which generates the time-evolution for the linear problem. In what follows, we will recall some useful estimates. At first, the $L^p - L^p$ estimate is given in the following lemma.

Lemma 2.1. ([13]) For the Schrödinger propagator $U(t)$, we have
\[
\|U(t)u\|_{L^p} \leq C|t|^{-\frac{3}{2}} \|u\|_{L^{p'}}, \quad 2 \leq p \leq \infty.
\]

Now, we recall the Strichartz estimates in the next lemma.

Lemma 2.2. ([13]) Let $(q, r)$ be $L^2$-admissible pair (i.e., $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$). Then
(i)
\[
\|U(t)u(t, x)\|_{L^q_t L^s_x} \leq C \|u\|_{L^2};
\]
(ii)
\[
\left\| \int_0^t U(t - \tau)f(\tau, \cdot) d\tau \right\|_{L^q_t L^s_x} \leq C \|f\|_{L^{p'}_t L^{s'}_x},
\]
where $(p, s)$ is $L^2$-admissible pair.

Thus, by the Sobolev embedding, we have
\[
\|U(t)u(t, x)\|_{X(\mathbb{R})} \leq \|u\|_{H^{1/2}},
\]
and
\[
\left\| \int_0^t U(t - \tau)f(\tau, \cdot) d\tau \right\|_{X(\mathbb{R})} \leq C \|D^{1/2}f\|_{L^{p'}_t L^{s'}_x},
\]
where $(p, s)$ is $L^2$-admissible pair. It follows from the Kato inhomogeneous Strichartz estimate that
\[
\left\| \int_0^t U(t - \tau)f(\tau, \cdot) d\tau \right\|_{X(\mathbb{R})} \leq C \|f\|_{L^{p'}_t L^{s'}_x},
\]
where $(q, r)$ is $\dot{H}^{-\frac{1}{2}}$-admissible pair, $3^+ < r < 6^-$, $\frac{4}{3^+} < q < 2^-$ is an arbitrarily preselected and fixed number $> 3$ and similarly for $6^-$, $\frac{4}{3^+}$ and $3^+$. 


2.2. Some properties of the dipole kernel. To simplify notations, without restriction of generality, we assume that \( n = (0, 0, 1) \). Then the expression of \( K(x) \) is

\[
K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^3}, \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

Now given a function \( u \in S(\mathbb{R}^3) \), we denote the Fourier transform of \( u \) by

\[
\hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) dx \quad \text{for } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]

Then in [6], it is computed that

\[
\hat{K} = \frac{4\pi}{3} \left(3 \cos^2 \Theta - 1\right) = \frac{4\pi}{3} \left(\frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2}\right) = \frac{4\pi}{3} \left(3 \left|\xi_3\right|^2 - 1\right). \quad (9)
\]

Here \( \Theta \) stands for the angle between \( \xi \) and the dipole axis \( n = (0, 0, 1) \).

Then we cite the following lemma from [6].

**Lemma 2.3.** ([6]) The operator \( \hat{K} : u \mapsto K * u \) can be extended as a continuous operator on \( L^p(\mathbb{R}^3) \) for all \( 1 < p < \infty \).

For the proof of Lemma 2.3, the reader is recommended to refer to [6].

3. Local theory.

**Proposition 2.** (Small data theory) Let \( \varphi_0 \in H^1(\mathbb{R}^3) \) and \( \|\varphi_0\|_{H^{1/2}} \leq A \). There exists \( \eta = \eta(A) > 0 \) such that if \( \|U(t)\varphi_0\|_{W(\mathbb{R})} \leq \eta \), then there exists a unique solution \( u \) to Eq. (2) on \( \mathbb{R} \times \mathbb{R}^3 \) with

\[
\| \varphi \|_{W(\mathbb{R})} \leq 2\|U(t)\varphi_0\|_{W(\mathbb{R})},
\]

\[
\| D^{3/2} \varphi \|_{S(\mathbb{R})} \leq 2C \|\varphi_0\|_{H^{1/2}}.
\]

**Proof.** From Duhamel’s formulation, we have

\[
\varphi(t) = U(t)\varphi_0 - \int_0^t U(t - \tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) d\tau.
\]

Hence, we denote

\[
\Phi_{\varphi_0}(v) = U(t)\varphi_0 + Y_1(t) + Y_2(t),
\]

where \( Y_1(t) = -\lambda_1 \int_0^t U(t - \tau)|v|^2 v(\tau) d\tau, \ Y_2(t) = -\lambda_2 \int_0^t U(t - \tau)(K * |v|^2) v(\tau) d\tau. \)

Now assume that \( B = \{ v : \|v\|_{W(\mathbb{R})} \leq 2\|U(t)\varphi_0\|_{W(\mathbb{R})}, \| D^{3/2} v \|_{S(\mathbb{R})} \leq 2C \|\varphi_0\|_{H^{1/2}} \} \). The Leibniz rule for fractional derivatives [17], Strichartz estimates and Sobolev embedding yield:

\[
\| \Phi_{\varphi_0}(v) \|_{W(\mathbb{R})} \leq \|U(t)\varphi_0\|_{W(\mathbb{R})} + \|Y_1\|_{W(\mathbb{R})} + \|Y_2\|_{W(\mathbb{R})}
\]

\[
\leq \|U(t)\varphi_0\|_{W(\mathbb{R})} + C|\lambda_1|\| D^{3/2} (|v|^2 v) \|_{L^{\frac{5}{3}} L^{\frac{6}{5}}}
\]

\[
+ C|\lambda_2|\| D^{3/2} (K * |v|^2) v \|_{L^{\frac{5}{3}} L^{\frac{6}{5}}} \quad (10)
\]

\[
\leq \|U(t)\varphi_0\|_{W(\mathbb{R})} + C(|\lambda_1| + |\lambda_2|)\|v\|_{L^2}^2 \|D^{3/2} v\|_{L^{\frac{5}{3}} L^{\frac{6}{5}}}
\]

\[
\leq \|U(t)\varphi_0\|_{W(\mathbb{R})} + C(|\lambda_1| + |\lambda_2|)\|v\|^2_{W(\mathbb{R})} \|D^{3/2} v\|_{S(\mathbb{R})}
\]
and
\[ |D^{\frac{3}{2}}\Phi_{\varphi_0}(v)|_{L^2}\leq|U(t)D^{\frac{1}{2}}\varphi_0|_{L^2} + |D^{\frac{1}{2}}Y_1|_{L^2} + |D^{\frac{1}{2}}Y_2|_{L^2} \]
\[ \leq|\varphi_0|_{L^{1/2}} + C|\lambda_1||v||L^1_{\epsilon}L^\frac{1}{2} \parallel D^{\frac{1}{2}}v||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ + C|\lambda_2||D^{\frac{1}{2}}([K \ast |v|^2]v)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ \leq|\varphi_0|_{L^{1/2}} + (|\lambda_1| + |\lambda_2|)|\varphi_0||L^1_{\epsilon}L^\frac{1}{2} \parallel D^{\frac{1}{2}}v||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ \leq|\varphi_0|_{L^{1/2}} + C(|\lambda_1| + |\lambda_2|)||v||^2_{W(R)}||D^{\frac{1}{2}}v||_{L^\frac{1}{2}L^\frac{1}{2}}. \] (11)

Choose \( \eta \leq \min \left\{ \frac{1}{\sqrt{24(|\lambda_1| + |\lambda_2|)^2}}, \frac{1}{24(|\lambda_1| + |\lambda_2|)^2CA} \right\} \). Thus, we find \( \Phi_{\varphi_0} : B \to B \) and \( \Phi_{\varphi_0} \) is also a contraction on \( B \). Indeed, for \( v_1, v_2 \in B \), we have, from Strichartz estimates and Leibniz rule for fractional derivatives [17],
\[ ||\Phi_{\varphi_0}(v_1) - \Phi_{\varphi_0}(v_2)||_{W(R)} \]
\[ \leq C|\lambda_1||D^{\frac{1}{2}}(v_1^2v_1 - |v_2|^2v_2)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ + C|\lambda_2||D^{\frac{1}{2}}([K \ast |v_1|^2]v_1 - [K \ast |v_2|^2]v_2)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ \leq |\lambda_1||D^{\frac{1}{2}}(v_1^2 - |v_2|^2)v_1||_{L^\frac{1}{2}L^\frac{1}{2}} + |\lambda_1||D^{\frac{1}{2}}(|v_1|^2v_1 - v_1)v_1||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ + |\lambda_2||D^{\frac{1}{2}}([K \ast (|v_1|^2 - |v_2|^2)]v_1||_{L^\frac{1}{2}L^\frac{1}{2}} + |\lambda_2||D^{\frac{1}{2}}([K \ast |v_2|^2]v_1 - v_2)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ \leq C(|\lambda_1| + |\lambda_2|)(||U(t)v_0||^2_{W(R)}||D^{\frac{1}{2}}(v_1 - v_2)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ + ||U(t)v_0||_{W(R)}||\varphi_0||_{L^{1/2}}||v_1 - v_2||_{S(R)}. \] (12)

Similarly, we also get
\[ ||D^{\frac{3}{2}}[\Phi_{\varphi_0}(v_1) - \Phi_{\varphi_0}(v_2)||_{L^2} \]
\[ \leq C(|\lambda_1| + |\lambda_2|)(||U(t)v_0||^2_{W(R)}||D^{\frac{1}{2}}(v_1 - v_2)||_{L^\frac{1}{2}L^\frac{1}{2}} \]
\[ + ||U(t)v_0||_{W(R)}||\varphi_0||_{L^{1/2}}||v_1 - v_2||_{S(R)}. \] (13)

And hence this completes the proof of the proposition.

\[ \square \]

Proposition 3. (Long time perturbation theory) Let \( \hat{\varphi} \in L^\infty H^1_x \) be an approximate solution to Eq. (2) in the sense that
\[ i\hat{\varphi}_t = -\frac{1}{2}\Delta\hat{\varphi} + \lambda_1|\hat{\varphi}|^2\hat{\varphi} + \lambda_2(K \ast |\hat{\varphi}|^2)\hat{\varphi} + \varepsilon, \]
for some function \( \varepsilon \). Assume that
\[ ||\hat{\varphi}||_{W(R)} \leq L \]
\[ ||U(t)(\varphi_0 - \hat{\varphi_0})||_{W(R)} \leq \varepsilon, \]
\[ ||\varepsilon||_{L^\frac{1}{2}L^\frac{1}{2}} \leq \varepsilon, \]
for positive constant \( L \gg 1 \) and small constant \( \varepsilon \leq \varepsilon_0 \), where \( \varepsilon_0 = \varepsilon_0(L) \ll 1 \). Then, there exists a constant \( C = C(L) \gg 1 \) and a solution \( \varphi \) to Eq. (2) on \( \mathbb{R} \times \mathbb{R}^3 \) with initial data \( \varphi_0 \) satisfying
\[ ||\varphi||_{W(R)} \leq C(L). \]
It is worth mentioning that the proof of Proposition 2.3 in [13] remains valid here. Therefore, for the sake of brevity, we omit the proof of Proposition 3.

4. Global bound and blowup phenomenon in energy space.

4.1. Properties of ground state. At the beginning, we recall the refined Gagliardo-Nirenberg inequality which was obtained in [1].

**Lemma 4.1.** (Refined Gagliardo-Nirenberg inequality)[1] Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) be such that condition (6) holds. Then, for any \( u \in H^1 \), there exists a constant \( C > 0 \) such that

\[
P(u) \leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |u|^2 dx \right)^{\frac{1}{2}}
\]

(14)

where the optimal constant \( C = C_* \) is given by \( C_* = \frac{P(Q_\omega)}{\|\nabla Q_\omega\|_{L^2}^2 \|Q_\omega\|_{L^2}^2} \) with \( Q_\omega \) being the minimizer of Weinstein functional (7) and also the solution of Eq.(5) for \( \omega > 0 \).

From the introduction, we know if \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfy (6), \( Q(x) \) is ground state of Eq.(8). In fact, a direct computation leads to the following:

\[
\|\nabla Q_\omega\|_{L^2}^2 = \sqrt{2\omega}\|\nabla Q\|_{L^2}^2, \quad \|Q_\omega\|_{L^2}^2 = \frac{1}{\sqrt{2\omega}}\|Q\|_{L^2}^2, \quad P(Q_\omega) = \sqrt{2\omega}P(Q),
\]

which implies \( G(Q_\omega) = G(Q) \). Furthermore, from [14], we have:

\[
\|\nabla Q\|_{L^2}^2 = 3\|Q\|_{L^2}^2 = 3P(Q).
\]

It follows \( C_* = \frac{2}{\sqrt{3}\|\nabla Q\|_{L^2}^2} = \frac{2}{3\sqrt{3}\|Q\|_{L^2}^2} \).

4.2. Global bound in energy space. From the properties of ground state, we obtain the following proposition.

**Proposition 4.** Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfy (6), \( \Theta := \{ u \in H^1(\mathbb{R}^3) : I(u) := (\mathcal{E} + \mathcal{M})(u) < \|Q\|_{L^2}^2, (\frac{1}{6} \mathcal{E}_0 + \mathcal{M})(u) < \|Q\|_{L^2}^2 \} \) and \( \Lambda := \{ u \in H^1(\mathbb{R}^3) : I(u) < \|Q\|_{L^2}^2 < (\frac{1}{6} \mathcal{E}_0 + \mathcal{M})(u) \} \). Then \( \Theta \) and \( \Lambda \) are invariant evolution flows generated by Eq.(2), i.e., if \( \varphi_0 \in \Theta \) (or \( \Lambda \)), then the solution \( \varphi \) of the Eq.(2) corresponding to the initial datum \( \varphi_0 \) still satisfies \( \varphi(t) \in \Theta \) (or \( \Lambda \)) for all \( t \in [0, T) \).

**Proof.** Let \( \varphi_0 \in \Theta \). Then by the conservation of the mass and energy, we have \( I(\varphi(t)) < \|Q\|_{L^2}^2 \) for \( t \in [0, T) \). Now, we will prove for \( t \in [0, T) \),

\[
\frac{1}{6} \|\nabla \varphi(t_1)\|_{L^2}^2 + \frac{1}{2} \|\varphi(t_1)\|_{L^2}^2 - \frac{C_*}{2} \|\nabla \varphi(t_1)\|_{L^2}^2 \geq \frac{1}{2} \|\nabla \varphi(t_1)\|_{L^2}^2 + \frac{1}{2} \|\varphi(t_1)\|_{L^2}^2
\]

\[
- \frac{\sqrt{3}C_*}{2} \|\nabla \varphi(t_1)\|_{L^2}^2 \left( \frac{1}{6} \|\nabla \varphi(t_1)\|_{L^2}^2 + \frac{1}{2} \|\varphi(t_1)\|_{L^2}^2 \right)
\]

\[
\geq \frac{1}{2} \|\nabla \varphi(t_1)\|_{L^2}^2 + \frac{1}{2} \|\varphi(t_1)\|_{L^2}^2 - \frac{1}{3} \|\nabla \varphi(t_1)\|_{L^2}^2
\]

\[
= \|Q\|_{L^2}^2,
\]

which is a contradiction. Hence \( \frac{1}{6} \|\varphi(t)\|_{L^2}^2 + \frac{1}{2} \|\varphi(t)\|_{L^2}^2 < \|Q\|_{L^2}^2 \) for \( t \in [0, T) \).

By the similar argument, we get that if the initial data \( \varphi(t_0, \cdot) \in \Lambda \), then the corresponding solution \( \varphi(t, \cdot) \in \Lambda \). \( \square \)
Theorem 4.2. (Global bound in $H^1$) If $\varphi_0 \in \Theta$ and $\lambda_1, \lambda_2$ satisfy (6), then the solution $\varphi(t, x)$ of Eq.(2) corresponding to $\varphi_0$ exists globally in time and satisfies
$$\|\varphi\|_{H^1}^2 < 6\|Q\|_{L^2}^2.$$ 

Proof. From Proposition 4, we know that $\varphi_0 \in \Theta$ implies $\varphi(t, x) \in \Theta$. Thus,
$$\frac{1}{6}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2 < \|Q\|_{L^2}^2.$$ 
Hence, $\|\varphi\|_{H^1}^2 < 6\left(\frac{1}{6}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2\right) < 6\|Q\|_{L^2}^2$. Hence, we have the solution $\varphi(t, x)$ of Eq.(2) exists globally. \hfill $\square$

4.3. Some variational estimates. In order to get the scattering in the unstable regime, we also need to study the properties of the invariant evolution flows.

Lemma 4.3. (Comparability of $H^1$ norm and energy-mass) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy (6). If $\varphi_0 \in \Theta$, then the corresponding solution $\varphi(t, x)$ with maximal life-span $I$ satisfies
$$\frac{1}{6}\|\varphi(t)\|_{H^1}^2 \leq \mathcal{I}(\varphi(t)) \leq \frac{1}{2}\|\varphi(t)\|_{H^1}^2, \quad \forall t \in I. \quad (17)$$

Proof. We claim that (6) implies $\mathcal{P}$ is nonnegative. In fact, in the view of (9), we have $\tilde{K}(\xi) \in \left[-\frac{4\pi}{3}, \frac{2\pi}{3}\right]$. Denote $\tilde{\rho}(\xi) = |\tilde{\varphi}|^2(\xi)$. Thus, by Plancherel’s theorem and (6), we have if $\lambda_2 > 0$,
$$\mathcal{P}(\varphi(t)) = -\int_{\mathbb{R}^3} (\lambda_1 \tilde{\rho}^2(\xi) + \lambda_2 \tilde{K}(\xi) \tilde{\rho}^2(\xi)) d\xi \geq -\int_{\mathbb{R}^3} (\lambda_1 + \frac{8\pi}{3} \lambda_2) \tilde{\rho}^2(\xi) d\xi > 0.$$ 
And if $\lambda_2 < 0$, we get
$$\mathcal{P}(\varphi(t)) = -\int_{\mathbb{R}^3} (\lambda_1 \tilde{\rho}^2(\xi) + \lambda_2 \tilde{K}(\xi) \tilde{\rho}^2(\xi)) d\xi \geq -\int_{\mathbb{R}^3} (\lambda_1 - \frac{4\pi}{3} \lambda_2) \tilde{\rho}^2(\xi) d\xi > 0.$$ 
Hence, the second inequality is directly obtained by the definition of $\mathcal{I}(\varphi)$. Now we turn to prove the first inequality. Since $\varphi_0 \in \Theta$, by Proposition 4, we have
$$\frac{1}{6}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2 < \|Q\|_{L^2}^2.$$ 
Thus, by Young’s inequality, we get
$$\mathcal{I}(\varphi(t)) \geq \frac{1}{2}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2 - \frac{C_*}{2}\|\nabla \varphi(t)\|_{L^1}^2 \|\varphi(t)\|_{L^2}$$
$$\geq \frac{1}{2}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2 - \frac{\sqrt{3}C_*}{2}\|\nabla \varphi(t)\|_{L^2}^2 \left(\frac{1}{6}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2\right)$$
$$\geq \frac{1}{2}\|\nabla \varphi(t)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t)\|_{L^2}^2 - \frac{1}{3}\|\nabla \varphi(t)\|_{L^2}^2$$
$$\geq \frac{1}{6}\|\varphi(t)\|_{H^1}^2.$$ 
Hence, the proof is complete. \hfill $\square$

It is worth to mention that Proposition 4.3 is needed to prove Theorem 6.1 in Section 6. Now, we introduce a lemma which will be used in Section 5 and Section 6.

Lemma 4.4. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy (6) and $u \in \Theta \setminus \{0\}$. Then there exists a $\sigma = \sigma(\mathcal{M}, \mathcal{E}) > 0$, such that $\mathcal{H}(u) := \frac{1}{2}\mathcal{E}(u) - \frac{3}{4}\mathcal{P}(u) \geq \sigma$.

Proof. On the one hand, by the definition of $\mathcal{E}(u)$ and $\mathcal{H}(u)$, it is easy to show that
$$\mathcal{P}(u) = 4[\mathcal{E}(u) - \mathcal{H}(u)]$$
and
$$\|\nabla u\|_{L^2}^2 = 6[\mathcal{E}(u) - \frac{2}{3}\mathcal{H}(u)].$$
Lemma 4.5. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy (6). If $u \in \Lambda$, then $\mathcal{M}(u) + \frac{1}{4} \mathcal{P}(u) > \|Q\|_{L^2}^2$.

Proof. Let $u_\mu = e^{2\mu}u(e^\mu x)$ for $\mu > 0$. Thus

$$
\mathcal{I}(u_\mu) = \frac{1}{2} e^{2\mu} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 - \frac{1}{2} e^{3\mu} \mathcal{P}(u).
$$

If $u \in \Lambda$, then $\mathcal{H}(u) < 0$. It follows that there exists $\mu_0 > 0$ such that $\mathcal{H}(u_{\mu_0}) = 0$ and $\mathcal{H}(u_{\mu}) < 0$ for $\mu_0 < \mu < 0$. Hence, $\mathcal{I}(u_{\mu_0}) \geq \|Q\|_{L^2}^2$.

Furthermore, direct computations show that

$$
\frac{d}{d\mu} \mathcal{I}(u_\mu) = e^{2\mu} \|\nabla u\|_{L^2}^2 - \frac{3}{2} e^{3\mu} \mathcal{P}(u) = 2 \mathcal{H}(u_\mu),
$$

$$
\frac{d^2}{d\mu^2} \mathcal{I}(u_\mu) = 2e^{2\mu} \|\nabla u\|_{L^2}^2 - \frac{9}{2} e^{3\mu} \mathcal{P}(u) < 2 \frac{d}{d\mu} \mathcal{I}(u_\mu).
$$

Now, integrating $\frac{d^2}{d\mu^2} \mathcal{I}(u_\mu) < 2 \frac{d}{d\mu} \mathcal{I}(u_\mu)$ on both sides over $(\mu_0, 0)$, we obtain

$$
\mathcal{H}(u) < \mathcal{I}(u) - \mathcal{I}(u_{\mu_0}) \leq \mathcal{I}(u) - \|Q\|_{L^2}^2.
$$

Thus, we get $\mathcal{M}(u) + \frac{1}{4} \mathcal{P}(u) > \|Q\|_{L^2}^2$. \hfill \square

4.4. Strong instability of the standing waves.

Lemma 4.6. ([20]) Let $|x|\varphi \in L^2(\mathbb{R}^3)$, $\varphi \in H^1(\mathbb{R}^3)$. One has

$$
\int_{\mathbb{R}^3} |\varphi|^2 \, dx \leq \frac{2}{3} \left( \int_{\mathbb{R}^3} |x\varphi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}}.
$$

Lemma 4.7. (Virial identity)([6]) Let $\varphi(t, x)$ be a solution of the Eq. (2). If $y(t) := \int_{\mathbb{R}^3} |x|^2 |\varphi|^2 \, dx$, then

$$
y'(t) = 2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx + 3\lambda_1 \int_{\mathbb{R}^3} |\varphi|^4 \, dx + 3\lambda_2 \int_{\mathbb{R}^3} (K * |\varphi|^2) |\varphi|^2 \, dx.
$$

For the proof of this lemma, we refer the reader to see [6] for details.
Proposition 5. If $\varphi_0 \in \Lambda \cap \Sigma$ and $\lambda_1, \lambda_2$ satisfy (6), then the solution $\varphi(t,x)$ corresponding to $\varphi_0$ of Eq.(2) blows up in a finite time.

Proof. By Proposition 4, we have

$$\ddot{\psi}(t) = 6\left\{ I(\varphi(t, \cdot)) - \frac{1}{6} E_0(\varphi(t, \cdot)) + M(\varphi(t, \cdot)) \right\}$$

$$\leq 6[I(\varphi_0) - \|Q\|_{L^2}^2]$$

$$:= -\tilde{\delta} < 0.$$  

Thus, according to [7, 11] and Lemma 4.6, we show the solution $\varphi(t,x)$ blows up in a finite time. 

The Proof of Corollary 1:

Proof. For any $\delta > 0$, we can take $\mu > 1$ such that $\|\mu Q - Q\|_{H^1} < \delta$. Note that $\frac{1}{6} \|\nabla Q\|_{L^2}^2 + \frac{1}{2} \|Q\|_{L^2}^2 = \|Q\|_{L^2}^2$. Thus, for every $\mu > 1$, $I(\mu Q) < \|Q\|_{L^2}^2$ and $\frac{1}{6} \|\nabla(\mu Q)\|_{L^2}^2 + \frac{1}{2} \|\mu Q\|_{L^2}^2 > \|Q\|_{L^2}^2$. Hence, $\mu Q \in \Lambda$. Meanwhile, $Q$ is the ground state of Eq.(8) and exponentially decays at infinity, we know that $\mu Q \in \Lambda \cap \Sigma$. By Proposition 5, we get the strong instability of the standing waves for Eq.(2).

5. Compactness in unstable regime.

5.1. Linear profile decomposition.

Proposition 6. (Linear profile decomposition) Let $\{\varphi_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^3)$. Then, there exists a subsequence of $\{\varphi_n\}_{n=1}^{\infty}$ (still denoted by $\{\varphi_n\}_{n=1}^{\infty}$), functions $\{\psi_j\} \in H^1$, time sequence $\{t_j\}_{n=1}^{\infty}$ and spatial sequences $\{x_j\}_{n=1}^{\infty}$ such that for every $J \geq 1$,

$$\varphi_n = \sum_{j=1}^{J} U(-t_j) \psi_j(x - x_j) + \gamma_n(x)$$

with $\gamma_n \in H^1$ and for any $\dot{H}^{1/2}$-admissible pair $(q,r)$

$$\lim_{J \to \infty} \lim_{n \to \infty} \|U(t)\gamma_n\|_{L^q_t L^r_x} = 0,$$  

(21)

Moreover, we have the orthogonality

$$\forall 1 \leq l \neq j \leq J, \quad \lim_{n \to \infty} |t_j - t_l| + |x_j - x_l| = \infty.$$  

(22)

For $J \geq 1$,

$$\|\varphi_n\|_{L^q_t L^r_x}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^q_t L^r_x}^2 + \|\gamma_n\|_{L^q_t L^r_x}^2 + o_n(1).$$  

(23)

and

$$\|\varphi_n\|_{H^1}^2 = \sum_{j=1}^{J} \|\psi_j\|_{H^1}^2 + \|\gamma_n\|_{H^1}^2 + o_n(1).$$  

(24)

By the similar proof in [13], we have
Lemma 5.1. In the situation of Proposition 6, we get
\[ I(\varphi_n) = \sum_{j=1}^{J} I(U(-t_n^j)\psi^j) + I(\gamma_n^j) + o_n(1), \]
and
\[ H(\varphi_n) = \sum_{j=1}^{J} H(U(-t_n^j)\psi^j) + H(\gamma_n^j) + o_n(1), \]
where \( H(\varphi) := \frac{1}{2}E_0(\varphi) - \frac{3}{4}P(\varphi). \)

Lemma 5.2. Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfy (6) and \( \varphi_n \) be as in Proposition 6. If \( \varphi_n \in \Theta \), then \( U(-t_n^j)\psi^j \in \Theta \) for all \( j \in \mathbb{N} \).

Proof. If \( \varphi_n \in \Theta \setminus \{0\} \), by Lemma 5.1, we get for \( 1 \leq j \leq J \),
\[ I(U(-t_n^j)\psi^j) < \|Q\|_{L^2}^2. \]
Now we prove \( \frac{1}{6}\|\nabla U(-t_n^j)\psi^j\|_{L^2}^2 + \frac{1}{2}\|U(-t_n^j)\psi^j\|_{L^2}^2 < \|Q\|_{L^2}^2 \) for \( 1 \leq j \leq J \). If otherwise, there exists a \( 1 \leq j_0 \leq J \) such that \( \frac{1}{6}\|\nabla U(-t_n^j)\psi^j\|_{L^2}^2 + \frac{1}{2}\|U(-t_n^j)\psi^j\|_{L^2}^2 \geq \|Q\|_{L^2}^2 \). Hence, by the nonnegative of \( \frac{1}{6}\|\nabla U(-t_n^j)\psi^j\|_{L^2}^2 + \frac{1}{2}\|U(-t_n^j)\psi^j\|_{L^2}^2 \), we get
\[ \|Q\|_{L^2}^2 \leq \sum_{j=1}^{J} \left[ \frac{1}{6}\|\nabla U(-t_n^j)\psi^j\|_{L^2}^2 + \frac{1}{2}\|U(-t_n^j)\psi^j\|_{L^2}^2 \right] \]
\[ = \sum_{j=1}^{J} \left[ I(U(-t_n^j)\psi^j) - \frac{2}{3}H(U(-t_n^j)\psi^j) \right]. \]

By Lemma 5.1, for all \( \epsilon > 0 \), there exists \( N_0 \) such that for \( n \geq N_0 \) we have
\[ I(\varphi_n) - \sum_{j=1}^{J} I(U(-t_n^j)\psi^j) - I(\gamma_n^j) \geq -\epsilon, \]
and
\[ H(\varphi_n) - \sum_{j=1}^{J} H(U(-t_n^j)\psi^j) - H(\gamma_n^j) \leq \frac{5}{2}\epsilon. \]
Therefore, by Lemma 4.4, it follows that
\[ \sum_{j=1}^{J} \left[ I(U(-t_n^j)\psi^j) - \frac{2}{3}H(U(-t_n^j)\psi^j) \right] \]
\[ \leq \left[ I(\varphi_n) - I(\gamma_n^j) - \epsilon \right] - \frac{2}{3} \left[ H(\varphi_n) - H(\gamma_n^j) - \frac{5}{2}\epsilon \right] \]
\[ = I(\varphi_n) - \frac{2}{3}H(\varphi_n) - \frac{1}{6}\|\gamma_n^j\|_{L^2}^2 + \frac{1}{2}\|\gamma_n^j\|_{L^2}^2 + \frac{2\epsilon}{3} \]
\[ \leq I(\varphi_n) - \frac{2}{3}\sigma + \frac{2}{3}\epsilon. \]
From Lemma 4.4, by choosing \( \epsilon > 0 \) such that \( \sigma > \epsilon \), we obtain that
\[ \sum_{j=1}^{J} \left( I(U(-t_n^j)\psi^j) - \frac{2}{3}H(U(-t_n^j)\psi^j) \right) < \|Q\|_{L^2}^2, \]
which contradict to (25).

If \( \varphi_n \equiv 0 \), obviously, the result holds. Thus, the proof is complete. \( \square \)
5.2. Existence of critical element. In this subsection, the existence of the critical element will be proved. Now we begin with the following definition:

**Definition 5.3.** We say that $(SC)(\varphi_0)$ holds if $\varphi$ is the corresponding solution to Eq.(2) in $H^1(\mathbb{R}^3)$ with maximal interval of existence $I$, then we have $I = (-\infty, +\infty)$ and $\|\varphi\|_{W(\mathbb{R})} < \infty$.

By (17), if we confine $I(\varphi_0) < \frac{1}{3}t^2$ and $\frac{1}{6}\|\nabla \varphi_0\|_{L^2}^2 + \frac{1}{6}\|\varphi_0\|_{L^2}^2 < \|Q\|_{L^2}^2$, then

$$\|\varphi_0\|_{H^1/2}^2 \leq \|\varphi_0\|_{L^2} \|\nabla \varphi_0\|_{L^2} \leq \frac{1}{2}\|\varphi_0\|_{H^1}^2 < 3I(\varphi_0) < \eta^2.$$  \hspace{1cm} (26)

Thus, by the Proposition 2, Lemma 2.2 and Sobolev embedding, we obtain that if $\varphi_0 \in \Theta$ satisfies $I(\varphi_0) < \frac{1}{3}\eta^2$, $\frac{1}{6}\|\nabla \varphi_0\|_{L^2}^2 + \frac{1}{6}\|\varphi_0\|_{L^2}^2 < \|Q\|_{L^2}^2$, then $(SC)(\varphi_0)$ holds. Moreover, we get that there exists a critical number $I_0$ such that $0 < \frac{1}{3}\eta^2 < I_0$.

For $\varphi \in \Theta$, (i) if $I(\varphi_0) \geq I_0$, then $\|\varphi\|_{W(\mathbb{R})} = \infty$; (ii) if $I(\varphi_0) < I_0$, then $(SC)(\varphi_0)$ holds. Obviously, if $I_0 = \|Q\|_{L^2}$, then Theorem 2 (1) holds.

Therefore, we will show that there exists a critical element $\varphi_c$ whose energy plus mass equal to the critical number $I_c$.

**Proposition 7.** *(Existence of the critical element)* There exists a global ($T = +\infty$) solution $\varphi_c \in \Theta$ to Eq.(2) with initial data $\varphi_{c,0}$ such that $I(\varphi_c) = I_c$, and $\|\varphi_c\|_{W(\mathbb{R})} = \infty$.

**Proof.** By the discussion above, there exists a sequence of solutions $\varphi_n$ to Eq.(2) corresponding to the initial data $\varphi_{n,0} \in \Theta$ such that

$$I(\varphi_{n,0}) \rightarrow I_c, \quad \limsup_{n \rightarrow \infty} \|\varphi_n\|_{W(\mathbb{R})} = +\infty.$$ 

By Proposition 6, we have

$$\varphi_{n} = \sum_{j=1}^{j} U(-t_n^j)\psi^j(-x_n^j) + \gamma_n^j.$$ 

Next, we will show the existence of the critical solution by a contradiction argument.

**Case 1.** There are more than one $\psi^j \neq 0$. From the proof of Lemma 5.1, by passing to the subsequence (still denoted by $t_n^j$), we get that $|t_n^j| \rightarrow \infty$ or $t_n^j$ converges to a finite number. **Case (a):** If $t_n^j \rightarrow \infty$, then denote $\phi^j$ as the maximal life-span solution to Eq.(2) which scatters to $U(t)\psi^j$ as $t \rightarrow \infty$. Denote

$$\Psi_n^j(t, x) = \sum_{j=1}^{J} \phi^j(t - t_n^j, x - x_n^j).$$

Thus, $\Psi_n^j$ satisfies the equation

$$i\Psi_n^j + \Delta \Psi_n^j + \lambda_1 |\Psi_n^j|^2 \Psi_n^j + \lambda_2 (K \ast |\Psi_n^j|^2) \Psi_n^j = e_n$$

where $e_n = \lambda_1 |\Psi_n^j|^2 \Psi_n^j + \lambda_2 (K \ast |\Psi_n^j|^2) \Psi_n^j - \lambda_1 \left( \sum_{j=1}^{J} |\phi^j|^2 \phi_n^j \right) - \lambda_2 \left( \sum_{j=1}^{J} (K \ast |\phi_n^j|^2) \phi_n^j \right)$ and $\phi_n^j = \phi^j(t - t_n^j, x - x_n^j)$. Applying Proposition 6, we know that there exists a $J_2$ such that $\sum_{j>j_2} \|\psi^j\|_{H^{1/2}} \leq A$. Hence, by Proposition 2, $\sum_{j>j_2} \|\psi^j\|_{W(\mathbb{R})} \leq 2\eta(A)$. Now consider $1 \leq j \leq J_2$. From Lemma 5.1 and Lemma 5.2, we have

$$I(\phi^j(-t_n^j)) < I_c \text{ and } \frac{1}{6}\|\nabla \phi^j(-t_n^j)\|_{L^2}^2 + \frac{1}{2}\|\phi^j(-t_n^j)\|_{L^2}^2 < \|Q\|_{L^2}^2.$$
It follows from the definition of $\mathcal{I}_c$ that $\|\phi^j(0, x - x_n^j)\|_{W(R)} \leq C(\mathcal{I}_c)$. Hence,

$$\sum_{j=1}^{\infty} \|\phi^j(t - t_n^j, x - x_n^j)\|_{W(R)} \leq C(\mathcal{I}_c).$$

This implies that

$$\lim_{j \to \infty} \limsup_{n \to \infty} \|\Psi_n^j\|_{W(R)} \leq C(\mathcal{I}_c).$$

Moreover, we have

$$\left\| U(-t)(\Psi_n^j(0) - \varphi_{0,n}) \right\|_{W(R)} = \left\| U(-t) \left( \Psi_n^j(0) - \sum_{j=1}^{J} U(-t_n^j) \psi^j - \gamma_n^j \right) \right\|_{W(R)}$$

$$\leq \sum_{j=1}^{J} \left\| U(-t) \left( \phi^j(-t_n^j) - U(-t_n^j) \psi^j \right) \right\|_{W(R)} + \|U(-t)\gamma_n^j\|_{W(R)}$$

Taking the superior limit and limit when $n$ and $J$ go to infinity respectively, and applying (21), we obtain

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|U(-t)(\Psi_n^j(0) - \varphi_{0,n})\|_{W(R)} = 0.$$ 

Now, we estimate the error term.

$$\|e_n\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} \leq \left\| \lambda_1 \sum_{j=1}^{J} \phi_n^{j2} \left( \sum_{j=1}^{J} \phi_n^{j2} \right) - \lambda_1 \left( \sum_{j=1}^{J} |\phi_n^{j2}|^2 \phi_n^j \right) \right\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}}$$

$$+ \left\| \lambda_2 (K * \sum_{j=1}^{J} \phi_n^{j2}) (\sum_{j=1}^{J} \phi_n^j) - \lambda_2 \left[ \sum_{j=1}^{J} (K * |\phi_n^{j2}|^2) \phi_n^j \right] \right\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}}$$

$$\leq \left\| \left( \sum_{j,k,l} |\phi^j \phi_n^k \phi_n^l| \right)_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} \right\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} + \left\| \left( \sum_{j,k,l} (K * |\phi_n^{j2}|^2) \phi_n^l \right)_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} \right\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}}$$

where not all of $j, k$ and $l$ are same. For the cubic term, without loss of generality, suppose $j \neq k$, and hence $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty$ as $n \to +\infty$. Thus, it follows that

$$\|\phi_n^j \phi_n^k \phi_n^l\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} \leq \|\phi_n^j \phi_n^k\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}} \|\phi_n^l\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}}$$

$$= \|\phi(t - t_n^j, x - x_n^j)\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}} \|\phi(t - t_n^k, x - x_n^k)\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}} \to 0.$$ 

Now, we consider the dipolar term. If $j \neq k$, by the property of the dipole kernel, we have

$$\|K * \phi_n^j \phi_n^j\phi_n^k\|_{L_t^{\frac{8}{3}}L_x^{\frac{4}{3}}} \leq C\|\phi_n^j \phi_n^k\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}} \|\phi_n^l\|_{L_t^{\frac{4}{3}}L_x^{\frac{8}{3}}} \to 0.$$
If \( j = k \neq l \), by orthogonality and by doing a change of variables,
\[
\| (K * |\phi_n'^1|^2)(t^l_n, x - x_n^l)\phi_n^l(t^l_n, x - x_n^l) \|_{L^2_t L^\frac{4}{3}}^\frac{8}{5} \\
= \| (K * |\phi_n'^1|^2)(t, x)\phi_n^l(t - t^l_n, x - (x_n^l - x_n^l)) \|_{L^2_t L^\frac{4}{3}}^\frac{8}{5} \to 0.
\]
All of these imply
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| e_n \|_{L^2_t L^\frac{4}{3}}^\frac{8}{5} = 0.
\]
Thus, applying the long time perturbation theory, we have \( \| \phi_n \|_{W(\mathbb{R})} < C(\mathcal{I}_c) \).

**Case (b).** If \( t_n^1 \) converges to a finite number, without loss of generality, we may assume that \( t^1 \equiv 0 \) and thus \( t_n^1 \equiv 0 \). Now, define the \( \phi^j \) as the maximal life-span solution to Eq.(2) corresponding to the initial data \( \phi^j(0) = \psi^j \). It follows from Lemma 5.2 and Theorem 4.2 that \( \phi^j \) exists globally in \( H^1 \). By the similar argument above with \( t_n^1 \equiv 0 \), we have \( \| \phi_n \|_{W(\mathbb{R})} < C(\mathcal{I}_c) \). Hence, Case 1 does not hold.

**Case 2.** \( \psi^1 \neq 0 \) and \( \psi^j = 0 \) for all \( j \geq 2 \). There are still two cases to consider.

**Case (a).** \( t_n^1 \to \pm \infty \). Thus,
\[
\lim_{n \to \infty} \| \phi^1(-t_n^1) - U(-t_n^1)\psi^1 \|_{H^1} = 0.
\]
Thus, from the proof of Lemma 2.1 in [9] and the Sobolev embedding, we can rewrite
\[
\varphi_{0,c} = \phi^1(-t_n^1) + \gamma_{\infty}^1.
\]
with \( \limsup_{n \to \infty} \| U(t)\varphi_{0,c}^1 \|_{W(\mathbb{R})} = 0 \). Moreover, \( \mathcal{I}(\phi^1(-t_n^1)) \leq \mathcal{I}_c \) and \( \frac{1}{2} \| \phi^1(-t_n^1) \|_{L^2}^2 + \epsilon \| \phi^1(-t_n^1) \|_{L^2}^2 < \| Q \|_{L^2}^2 \). Hence, let \( \varphi_{0,c} = \phi^1(-t_n^1) \) and \( \varphi_c \) be the corresponding solution of Eq.(2). Furthermore, \( \mathcal{I}(\psi^1) = \mathcal{I}_c \). If otherwise, by the definition of \( \mathcal{I}_c \), we obtain \( \| \phi^1 \|_{W(\mathbb{R})} < \infty \). Thus, by the similar argument in Case 1, it is easy to show that \( \| \varphi_c \|_{W(\mathbb{R})} < \infty \), which is a contradiction. Thus, we have \( \| \varphi_c \|_{W(\mathbb{R})} = \infty \).

At the same time, \( \varphi_c \in \Theta \), \( \mathcal{I}(\varphi_c) = \mathcal{I}_c \). **Case (b).** \( t_n^1 \) converges to a finite number. Thus, as we mentioned before, \( t_n^1 \equiv 0 \) and \( \phi^1 \) is global solution in \( H^1 \). By the same discussion in Case 2 of Case (a), we have \( \mathcal{I}(\psi^1) = \mathcal{I}_c \). Thus, \( \varphi_c = \phi^1 \) and \( \varphi_{0,c} = \psi^1 \).

### 5.3. Some properties of the critical element.

**Proposition 8.** (Precompactness of the critical element) If \( \varphi_c \) is the critical solution, then there exists a continuous \( x(t) : \mathbb{R} \to \mathbb{R}^3 \) such that \( \mathcal{C} = \{ \varphi_c(t, \cdot - x(t)) : t \in [0, +\infty) \} \subset H^1 \) is precompact in \( H^1 \).

**Proof.** Applying the linear profile decomposition on \( \varphi_c(t_n) \) and by the argument of Case 1 in the proof Proposition 7, we get
\[
\varphi_c(t_n) = U(-t_n^1)\psi^1(\cdot - x_n^1) + \gamma_{\infty}^1,
\]
\[
\lim_{n \to \infty} \mathcal{I}(U(-t_n^1)) = \mathcal{I}_c \text{ and } \lim_{n \to \infty} \| U(t)\gamma_{\infty}^1 \|_{H^1} = 0.
\]
For \( t_n^1 \), by passing to a subsequence (still denoted by \( t_n^1 \)) such that \( t_n^1 \to t^* \). Thus, \( t^* \) must be a finite number. In fact, if \( t^* \to -\infty \), then
\[
\| U(t)\varphi_c(t_n) \|_{W([0, +\infty))} \leq \| U(t - t_n^1)\psi^1(\cdot - x_n^1) \|_{W([0, +\infty))} + \| \gamma_{\infty}^1 \|_{W([0, +\infty))}.
\]
Obviously,
\[
\lim_{n \to +\infty} \| U(t - t_n^1)\psi^1(\cdot - x_n^1) \|_{W([0, +\infty))} = \lim_{n \to +\infty} \| U(t)\psi^1(\cdot - x_n^1) \|_{W([-t_n^1, +\infty))} = 0.
\]
Combining with \( \|U(t)\gamma_n\|_{W(\mathbb{R})} \leq \frac{1}{2}\eta \), we have
\[
\lim_{n \to +\infty} \|U(t)\varphi_c(t_n)\|_{W([0, +\infty)}) = 0,
\]
which is a contradiction to small data theory.

Now, suppose \( t \to +\infty \), the similarly argument shows that
\[
\|U(t)\varphi_c(t_n)\|_{W((-\infty, 0])} \leq \frac{1}{2}\eta.
\]
It follows from the small data theory that
\[
\|\varphi_c\|_{W((-\infty, t_n])} \leq \eta.
\]
Let \( t_n \to +\infty \) as \( n \to +\infty \) in above inequality, we have \( \|\varphi_c\|_{W(\mathbb{R})} \leq \eta \). It is a contradiction with the definition of \( \varphi_c \). Hence, \( t_n \to t^* < \infty \). It follows that
\[
\varphi_c(t_n) = \phi^1(\cdot - t_n^1, \cdot - x_n^1) + \gamma_n.
\]
Thus,
\[
\|\varphi_c(t_n, x-x_0) - \varphi_c(t_k)\|_{H^1} \leq \|\phi^1(t_n - t_k^1, x-x_k^1 + x_k^1 - x_n^1) - \phi^1(t_n - t_k^1, x-x_k^1)\|_{H^1}.
\]
Taking \( n, k \) sufficient large and by the argument of Appendix A in [9], we obtain \( \varphi_c(t_n, \cdot - x(t)) \) converges in \( H^1 \).

**Proposition 9.** (Zero momentum) Let \( \varphi(t, x) \) be as in Proposition 7 and \( \mathcal{C} = \{\varphi(t, \cdot - x(t)) : t \in [0, +\infty)\} \subset H^1 \) is precompact in \( H^1 \) for some continuous function \( x(\cdot) \), then \( \mathcal{Z}(\varphi) := \text{Im} \int_{\mathbb{R}^3} \varphi \nabla \varphi dx = 0 \).

The readers are referred to Proposition 4.1 of [9] for a detailed proof of the above proposition.

**Proposition 10.** (Uniform localization) Let \( \varphi(t, x) \) be a solution to Eq.(2) such that
\[
\mathcal{C} = \{\varphi(t, \cdot - x(t)) : t \in [0, +\infty)\}
\]
is precompact in \( H^1 \). Then for each \( \epsilon > 0 \), there exists \( R_0(\epsilon) > 0 \) so that
\[
\int_{|x+x(t)| \geq R_0(\epsilon)} (|\nabla \varphi|^2 + |\varphi|^2) dx \leq \epsilon, \quad \forall t > 0.
\]
(27)

The proposition can be proved in a similar fashion of Corollary 3.3 in [9] and we omit the details.

By Proposition 9 and 10, we can get the following proposition.

**Proposition 11.** (Control over \( x(t) \)) Let \( \varphi \) be a solution to Eq.(2) define on \( [0, +\infty) \) such that \( \mathcal{Z}(\varphi) = 0 \) and \( \mathcal{C} \) be as in Proposition 10. If \( \mathcal{C} \) is precompact in \( H^1 \) for some continuous function \( x(\cdot) \), then \( x(t) = o(t) \) as \( t \to \infty \).

The proof of this proposition is very similar to Lemma 5.1 in [9]. So we omit the it here.

6. Scattering in energy space.

**Theorem 6.1.** (Rigidity Theorem) Let \( \varphi_0 \in \Theta \) with \( \mathcal{Z}(\varphi_0) = 0 \). If there exists \( x(t) \in \mathbb{R}^3 \) such that \( \mathcal{C} = \{\varphi_c(t, \cdot - x(t)) : t \in [0, +\infty)\} \) is precompact in \( H^1 \). Then \( \varphi_0 \equiv 0 \).
Proof. Let $\chi(x) \in C_0^\infty(\mathbb{R}^3)$ be a radial bump function with $\chi(x) = |x|^2$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. And for $R > 0$, denote $z_R(t) = \int_{\mathbb{R}^3} |R^2(x/R)|^2 \varphi(t,x)^2 dx$.

A direct calculation shows that
\[
\dot{z}(t) = \text{Im} \int_{\mathbb{R}^3} R \bar{\varphi}(t,x) \nabla \chi(x/R) \cdot \nabla \varphi(t,x) dx.
\]

Thus, by Cauchy-Schwarz inequality and Lemma 4.3, we have
\[
|\dot{z}(t)| \leq CR \|\varphi\|_{L^2} \|\nabla \varphi\|_{L^2} \leq CR(\|\varphi\|_{L^2}^2 + \|\nabla \varphi\|_{L^2}^2) \leq CR\mathcal{I}(\varphi(t)).
\]

Hence, we get the local virial identity,
\[
\dot{z}(t) = \sum_{j,k} \int_{\mathbb{R}^3} \partial_j \chi(x/R) \partial_j \varphi \partial_k \bar{\varphi} dx - \int_{\mathbb{R}^3} \frac{1}{4R^2} \Delta^2 \chi(x/R) |\varphi|^2 dx \]
\[
+ \frac{1}{2} \lambda_1 \int_{\mathbb{R}^3} \Delta \chi(x/R) |\varphi|^4 dx - \frac{1}{2} \lambda_2 \int_{\mathbb{R}^3} R \nabla \chi(x/R) (\nabla K * |\varphi|^2) |\varphi|^2 dx
\]
\[
= 2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx + 3 \lambda_1 \int_{\mathbb{R}^3} |\varphi|^4 dx - \lambda_2 \int_{\mathbb{R}^3} (|\nabla K \cdot x| + |\varphi|^2) |\varphi|^2 dx
\]
\[
+ B_1(\varphi(t)) + B_2(\varphi(t)),
\]

where
\[
B_1(\varphi(t)) = \sum_{j} \int_{\mathbb{R}^3} (\partial_j^2 \chi(x/R) - 2) |\partial_j \varphi|^2 dx + \sum_{j \neq k} \int_{\mathbb{R}^3} \partial_j \chi(x/R) \partial_j \varphi \partial_k \bar{\varphi} dx
\]

\[- \int_{\mathbb{R}^3} \frac{1}{4R^2} \Delta^2 \chi(x/R) |\varphi|^2 dx + \frac{1}{2} \lambda_1 \int_{\mathbb{R}^3} (\Delta \chi(x/R) - 6) |\varphi|^4 dx
\]

and
\[
B_2(\varphi(t)) = \frac{1}{2} \lambda_2 \int_{\mathbb{R}^3} ((R \nabla \chi(x/R) - 2x) \cdot \nabla K) * |\varphi|^2 |\varphi|^2 dx.
\]

Obviously, by the definition of $\chi$, we have the bound
\[
|B_1(\varphi(t))| \leq C \int_{|x| > R} (|\nabla \varphi(t)|^2 + \frac{1}{R^2} |\varphi(t)|^2 + \frac{1}{2} \lambda_1 |\varphi(t)|^4) dx.
\]

Now, we turn to $B_2(\varphi(t))$.
\[
B_2(\varphi(t)) \leq -\frac{1}{2} \lambda_2 \left\{ \int_{\Omega_1} [R(\nabla \chi(x/R) - \nabla \chi(x/R)) - 2(x - y)] \cdot \nabla K(x - y) \varphi(y)^2 |\varphi(x)|^2 dx
\]
\+
\int_{\Omega_2} [R(\nabla \chi(x/R) - \nabla \chi(x/R)) - 2(x - y)] \cdot \nabla K(x - y) \varphi(y)^2 |\varphi(x)|^2 dx
\]
\+
\int_{\Omega_3} [R(\nabla \chi(x/R) - \nabla \chi(x/R)) - 2(x - y)] \cdot \nabla K(x - y) \varphi(y)^2 |\varphi(x)|^2 dx
\]
\+
\int_{\Omega_4} [R(\nabla \chi(x/R) - \nabla \chi(x/R)) - 2(x - y)] \cdot \nabla K(x - y) \varphi(y)^2 |\varphi(x)|^2 dx \right\}
\]
Thus, we define 1

Then, integral above inequality over \( [t_0, t_1] \), we get

By the conservation of the mass and energy, we get a contradiction unless \( \varphi_0 = 0 \) by using Proposition 11 and letting \( t_1 \to \infty \).

**Proposition 12.** \((H^1 \text{ scattering}) \) Let \( \varphi_0 \in H^1(\mathbb{R}^3) \) and \( \varphi(t) \) be the corresponding solution of Eq. (2) exists globally. If \( \|\varphi\|_{L^4(\mathbb{R})} < \infty \) and \( \sup_{t \in [0, t_1]} \|\varphi(t)\|_{L^4} \leq D \), then \( \varphi(t) \) scatters in \( H^1(\mathbb{R}^3) \) as \( t \to \infty \), in the sense that there exists unique \( \varphi_+ \in H^1(\mathbb{R}^3) \) such that

\[
\lim_{t \to \infty} \|\varphi(t) - U(t)\varphi_+\|_{H^1} = 0. \tag{29}
\]

**Proof.** For \( 0 < t < \infty \), we define

\[
\varphi_+(t) := \varphi_0 - i \int_0^t U(t - \tau)(\lambda_1|\varphi|^2\varphi + \lambda_2(K*|\varphi|^2)\varphi)\,d\tau.
\]
By Strichartz estimate, we have \( \varphi_+ \in H^1 \). Now, we will show that \( \varphi_+(t) \) converges in \( H^1 \) as \( t \to \infty \). For \( 0 < t' < t \), we get

\[
\|\varphi_+(t) - \varphi_+(t')\|_{H^1} = \left\| \int_{t'}^t U(-\tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) d\tau \right\|_{H^1}
\]

\[
\leq C \int_{t'}^t \| U(-\tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) \|_{L^\infty_t H^1([t', t] \times \mathbb{R}^3)} d\tau
\]

\[
\leq C |\lambda_1| \| \varphi^2 (1 + |\nabla|) \varphi \|_{L^1_t L^1_x([t', t] \times \mathbb{R}^3)}
\]

\[
+ |\lambda_2| \| (K * |\varphi|^2) (1 + |\nabla|) \varphi \|_{L^1_t L^1_x([t', t] \times \mathbb{R}^3)}
\]

\[
\leq C(|\lambda_1| + |\lambda_2|) \| \varphi \|^{2}_{L^\infty_t L^2_x([t', t] \times \mathbb{R}^3)} \| (1 + |\nabla|) \varphi \|_{L^\infty_t L^2_x}
\]

\[
\leq C(|\lambda_1| + |\lambda_2|) \| \varphi \|^{2}_{W^{1, \infty}_t([t', t] \times \mathbb{R}^3)} \| \varphi \|_{L^\infty_t H^1_x}
\]

Since \( \| \varphi \|_{W^1(\mathbb{R})} < \infty \), we see that for \( \varepsilon > 0 \) there exists \( T_\varepsilon > 0 \) such that \( \| \varphi(t) - \varphi_+(t') \|_{H^1} \leq \varepsilon \) for any \( t, t' > T_\varepsilon \). Hence, we get \( \varphi_+(t) \) converges in \( H^1 \) as \( t \to \infty \) to some function \( \varphi_+ \). Thus, Strichartz estimate imply that

\[
\varphi_+ := \varphi_0 - i \int_0^\infty U(-\tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) d\tau.
\]

In fact, \( \varphi_+ \) is the scattering state in the positive time direction.

Next, we show that the linear evolution \( U(t)\varphi_+ \) approximates \( \varphi(t) = U(t)\varphi_0 - i \int_0^t U(t - \tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) \) in \( H^1 \) as \( t \to \infty \). Indeed, we have

\[
\varphi(t) - U(t)\varphi_+ = i \int_t^{t+\infty} U(t - \tau)(\lambda_1 |\varphi|^2 \varphi + \lambda_2 (K * |\varphi|^2) \varphi)(\tau) d\tau. \tag{30}
\]

It follows that

\[
\| \varphi(t) - U(t)\varphi_+ \|_{H^1} \leq C |\lambda_1| \| \varphi^2 (1 + |\nabla|) \varphi \|_{L^1_t L^1_x([t, t+\infty) \times \mathbb{R}^3)}
\]

\[
+ |\lambda_2| \| (K * |\varphi|^2) \|_{L^1_t L^1_x([t, t+\infty) \times \mathbb{R}^3)}
\]

\[
\leq C(|\lambda_1| + |\lambda_2|) \| \varphi \|^{2}_{L^\infty_t L^2_x([t, t+\infty) \times \mathbb{R}^3)} \| (1 + |\nabla|) \varphi \|_{L^\infty_t L^2_x}
\]

\[
\leq C(|\lambda_1| + |\lambda_2|) \| \varphi \|^{2}_{W^{1, \infty}_t([t, t+\infty))} \| \varphi \|_{L^\infty_t H^1_x}
\]

Thus, send \( t \to +\infty \) in the above inequality and (29) holds in the positive direction. Thus, the proof is complete. \( \square \)

In order to prove Theorem 1.1(1), from Theorem 4.2 and Proposition 12, we only need to show that the solution corresponding \( \varphi_0 \in \Theta \) satisfies \( \| \varphi \|_{W^1(\mathbb{R})} < \infty \). This means we need to prove (SC)(\( \varphi_0 \)) holds. On the one hand, we assume that (SC)(\( \varphi_0 \)) does not holds. It implies that there exists the critical element (see Propositions 7). Thus, Proposition 8 and 9 hold. On the other hand, the rigidity theorem (see Theorem 6.1) shows such critical element exists unless it is identically zero. This is
a contradiction. Hence, we have \((SC)(\varphi_0)\) holds. Then, Theorem 1.1(1) have been proved.

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