Higher-order coefficients of the Williams series expansion for near crack-tip fields and their extraction from FEM experiments and digital photoelasticity method

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Abstract. Multi-parameter stress field near the Mode I crack tip in an isotropic linear elastic material is presented. The coefficients of the Williams series expansion of the stress field in the vicinity of the Mode I crack in the isotropic linear elastic material are obtained by the digital photoelasticity technique and by finite element analysis. The main objective of this study is to determine first fifteen coefficients with good accuracy and to compare the numerical results with the analytical solution for the infinite plate with the central crack. It is shown that the coefficients of the higher-order terms of the Williams series expansion are extracted with good accuracy. Comparison of the coefficients of the Williams series known from the theoretical solution for an infinite plate with the central crack and the coefficients extracted from the photoelasticity method and from finite element analysis is given. The FEM analysis is performed for a series of cracked specimens with a small crack and for a series of cracked specimens considered in the phototelasticity experiments. It is shown significant importance of higher-order terms in the Williams series expansion.

1. Introduction

Photoelasticity is a whole field experimental technique to obtain stress fields in both 2-D and 3-D elasticity problems [1-15]. Digital photoelasticity has rapidly progressed in the last few years and has matured into an industry-friendly technique. Recently there has been a lot of works devoted to various aspects of the method and its applications [1-15].

Thus, in [1] closely spaced asymmetric cracks in a biaxial cruciform specimen interacting at different orientations is studied by this approach allowing to the authors to capture isochromatic fringe patterns under biaxial loading conditions with varying biaxial ratios from 0 to 1, zero indicating uniaxial loading and one indicating equi-biaxial loading. Empirical relations to estimate the normalized SIF under different biaxial ratios are also given. In [2] digital photoelasticity is used for the experimental evaluation of the strain intensity factor for a rigid line inclusion embedded in an elastic matrix.

It is worth to note that the procedure of the photoelasticity method is either improving now. Thus, nuances of theoretical calibration table for fringe pattern demodulation is studied in [3]. In white light photoelasticity, the fringe order at a point of interest is resolved by comparing the colour data with a calibration table recorded experimentally. Recently, some researchers have proposed the idea of using a theoretical calibration table if the application image is normalized.
In [3] a systematic study is carried out on the applicability of the theoretical calibration table for fringe demodulation. A novel normalization methodology is developed to improve the dynamic range in both low and high fringe density areas. It is demonstrated in [3] that while applying normalization for stretching the dynamic range, one has to use only an experimentally recorded calibration table that is normalized.

In [4] colour adaptation techniques are useful in extending the method of twelve fringe photoelasticity to solve problems in industries as one can minimise the generation of specific calibration tables for each experiment. In [4] the applicability of two colour transfer strategies is explored, of which one is selected for implementation. A comparative study on the performance of the novel colour transfer method with the existing colour adaptation techniques is carried out.

In [5] a method to determine the stress distribution by means of phase shifting and a modified shear-difference is proposed. A method to reduce integration error in the shear difference scheme is proposed and compared to the existing methods; the integration error is reduced when using theoretical photoelastic parameters to calculate the stress component with the same points. Results show that the proposed approach provides a complete solution for determining the full-field stresses in photoelastic models.

In [6] photoelasticity method is used to study experimentally the Williams series expansion of the stress and displacement fields in the vicinity of the crack tip in isotropic linear elastic plates under Mixed Mode loading. The distribution of the isochromatic fringe patterns is employed for obtaining the stress field near the crack tip by the use of the Williams asymptotic expansion for various classes of the experimental specimens. The higher order terms of the Williams series expansion are taken into account and the coefficients of the higher order terms are experimentally obtained.

Digital photoelasticity method is used in [7] to analyze experimentally the Williams series expansion of the stress and displacement fields in the vicinity of the crack tip in isotropic linear elastic plates under Mixed Mode loading. The multi-parameter description of the stress field in the neighborhood of the crack tips of two interacting cracks based on the photoelastic study and finite element analyses is given in [8]. Digital photoelasticity method is used for obtaining the isochromatic and isoclinic patterns in the plate with two collinear horizontal and two inclined cracks in anisotropic linear elastic material. Stress intensity factors, T-stresses and coefficients of the higher order terms in the multi-parameter Williams series expansion are experimentally determined.

The paper [9] proposes the algorithm for the determination of the coefficients of the Williams series expansion in notched semidisks with different angles of the notch. The algorithm is based on the experimental procedure of the photoelasticity method and the finite element analysis. The study [10] is aimed at theoretical, experimental and computational determination of the coefficients in crack tip asymptotic expansions for a wide class of specimens under mixed mode loading conditions. Multiparametric presentation of the stress filed near the crack tips for a wide class of specimens is given. Theoretical, experimental and computational results obtained in this research show that the isochromatic fringes in the vicinity of the crack tip require to keep the higher order stress terms in the asymptotic expansion of the stress field around the crack tip since the contribution of the higher order stress terms (besides the stress intensity factors and the T-stress) is not negligible in the crack tip stress field. It is shown that at large distances from the crack tips the effect of the higher order terms of the Williams series expansion becomes more considerable. In [12] digital photoelasticity technique is used to estimate the crack tip fracture parameters for different crack configurations. With the use of digital photoelasticity pixel-wise availability of both isoclinic and isochromatic data have been exploited for SIF estimation.

In the present study the plate with the central crack is considered by two approaches: digital photoelasticity method and finite element calculations. The present study aims at improving the determination procedure for obtaining higher-order coefficients of the Williams series expansion and at develop the approach to find the higher-order coefficients with high accuracy.
2. The experimental procedure

The main goal of this paper is the experimental determination of higher-order coefficients of the Williams series expansion. Polar coordinates \(r, \theta\) are introduced and centered at the crack tip. With reference to the polar coordinates

\[
\sigma_j(r, \theta) = \sum_{m=1}^{\infty} \sum_{k=-\infty}^{\infty} a_k^n r^{k/2-1} f_{m,j}^{(k)}(\theta),
\]

where index \(m\) is associated to the fracture mode; \(a_k^n\) are coefficients related to the geometric configuration, load and mode; \(f_{m,j}^{(k)}(\theta)\) are angular functions depending on stress components and mode. Analytical expressions for angular eigenfunctions \(f_{m,j}^{(k)}(\theta)\) are available [16,17]:

\[
\begin{align*}
    f_{1,11}^{(1)}(\theta) &= (k / 2) \left[ (2 + k / 2 + (-1)^s) \cos(k / 2 - 1) \theta - (k / 2 - 1) \cos(k / 2 - 3) \theta \right], \\
    f_{2,22}^{(1)}(\theta) &= (k / 2) \left[ (2 - k / 2 - (-1)^s) \cos(k / 2 - 1) \theta + (k / 2 - 1) \cos(k / 2 - 3) \theta \right], \\
    f_{2,12}^{(1)}(\theta) &= (k / 2) \left[ -(k / 2 + (-1)^s) \sin(k / 2 - 1) \theta + (k / 2 - 1) \sin(k / 2 - 3) \theta \right], \\
    f_{2,21}^{(1)}(\theta) &= -(k / 2) \left[ (2 + k / 2 - (-1)^s) \sin(k / 2 - 1) \theta - (k / 2 - 1) \sin(k / 2 - 3) \theta \right], \\
    f_{2,11}^{(1)}(\theta) &= -(k / 2) \left[ (2 - k / 2 + (-1)^s) \sin(k / 2 - 1) \theta + (k / 2 - 1) \sin(k / 2 - 3) \theta \right].
\end{align*}
\]

The multi-parameter fracture mechanics concept consists in the idea that the crack-tip stress field is described by means of the Williams expansion (1). In this work central crack in an infinite plane medium is considered. Analytical determination of coefficients in crack-tip expansion for a finite crack in an infinite plane medium is given in [16]:

\[
a_{2n+1}^{1} = (-1)^{n+1} \frac{(2n)\sigma_{22}^2}{2^{3n+1/2} (n!)^2 (2n-1)a^{a-1/2}}, \quad a_2^1 = -\sigma_{22}^2 / 4, \quad a_{2k}^1 = 0
\]

for Mode I crack;

\[
a_{2n+1}^{2} = (-1)^{n+1} \frac{(2n)\sigma_{22}^2}{2^{3n+1/2} (n!)^2 (2n-1)a^{a-1/2}}, \quad a_{2k}^2 = 0
\]

for Mode II crack.

The analytical presentation of higher-order coefficients (4) and (5) caused the choice of the type of the experimental specimen for both finite element simulation and for photoelasticity observations. The experimental equipment is shown in figure 1. The coefficients \(a_k^n\) in (1) for the experimental specimen shown in figures 2-3 are unknowns and should be determined from the photoelasticity method and from numerical FEM calculations. The isochromatic fringe patterns are shown in Figures 2-3. To accurately collect the data from the fringes the skeleton of the fringe is identified first. Fringe skeletons for different loads are shown in figures 4-6. Global fringe thinning algorithm proposed by Ramesh and Promod [18] which uses the intensity information for the location of the fringe skeleton that is demonstrated to be the best methodology for fringe thinning [1,19] is used to thin the fringes.

The data which are the positional coordinates \((r, \theta)\) and the corresponding fringe order value \(N\) along the fringe contours are collected. Then, one can use the approach described in [12] and present Eqn. (1) in the matrix form as

\[a = (C^T C)^{-1} C^T \sigma.\]

The coefficients are estimated by minimizing the objective function:
\[ J(a) = \frac{1}{2}(\sigma - Ca)^T (\sigma - Ca). \]

The objective function is of quadratic form for stress expression in terms of unknown parameters. The stress optic law relates the fringe order \( N \) and the in-plane principal stresses \( \sigma_1, \sigma_2 \) as

\[ \frac{N f_\sigma}{t} = (\sigma_1 - \sigma_2) \]

where \( f_\sigma \) is the material stress fringe and \( t \) is the thickness of the specimen.

**Figure 1.** Experimental setup of transmission photoelasticity.

**Figure 2.** Isochromatic images for 25 kg, 50 kg, 75 kg and 80 kg.

**Figure 3.** Isochromatic images for 85 kg, 90 kg, 100 kg and 125 kg.
For a plane stress problem, the stress components $\sigma_{11}, \sigma_{22}$ and $\sigma_{12}$ are related to the principal stress as

$$
\sigma_1, \sigma_2 = \left( \sigma_{11} + \sigma_{22} \right) / 2 \pm \sqrt{\left( \sigma_{11} - \sigma_{22} \right)^2 / 4 + \sigma_{12}^2}
$$

(7)

According to the classical paper [19] one can introduce function defined for the $k$ th data point as follows

$$
g_k = \left( \left( \sigma_{11} - \sigma_{22} \right) / 2 \right)_k^2 + \left( \sigma_{12} \right)_k^2 - \left( N_k f_0 / t \right)_k^2.
$$

(8)

If Eqn. (1) is substituted into (8) then Eqn. (8) is non-linear in terms of the unknowns $a_1^i, a_2^i, \ldots, a_M^i$, where $M$ is the number of Mode I parameters [19]. If initial estimates for $a_1^i, a_2^i, \ldots, a_M^i$ are made and substituted into (8) it is possible that $g_k$ is not equal to zero since the estimates may not be accurate. To correct the estimates, a series of iterative equations based on a Taylor series expansion of $g_k$ is written as
\[
(g_k)_{i+1} = (g_k) + \frac{\partial g_k}{\partial a_1} (\Delta a_1) + \frac{\partial g_k}{\partial a_2} (\Delta a_2) + \ldots + \frac{\partial g_k}{\partial a_M} (\Delta a_M) \ldots
\]  

where the subscript “i” refers to the \(i\)th iteration step and \(\Delta a_1, \Delta a_2, \ldots, \Delta a_M\) denote the corrections to the previous estimates of \(a_1, a_2, \ldots, a_M\). Corrections are determined such that \((g_k)_{i+1} = 0\) and thus Eqn. (9) gives

\[
-(g_k) = \frac{\partial g_k}{\partial a_1} (\Delta a_1) + \frac{\partial g_k}{\partial a_2} (\Delta a_2) + \ldots + \frac{\partial g_k}{\partial a_M} (\Delta a_M) \ldots
\]

Applying the iteration scheme to \(L\) data points results in an over determined set of linear equations in terms of the unknown coefficients \(\Delta a_1, \Delta a_2, \ldots, \Delta a_M\). The results are tabulated in Table 1.

3. Extraction the coefficients of the Williams series expansion from the FEM analysis
In this work, 2D finite element analysis of plates with central crack is carried out using ABAQUS software. To estimate SIF, T-stress and higher-order terms and verify the experimental results obtained FEM calculations have been employed. The analysis is done with 8-noded strain elements. The quarter point element is used to capture square root singularity at the crack tip. Several cracked plates have been considered. First, the cracked plate with small crack has been considered to compare numerical results with the analytical solution (4). Then the same configuration as in the photoelasticity experiments has been modelled. The results of FEM analysis are shown in figures 7-17. Figure 7 shows distribution of the von Mises stress intensity (left), stress component \(\sigma_{11}\) (center), stress component \(\sigma_{22}\) (right). Figures 8-17 show distribution of the stress component \(\sigma_{11}\) at different distances from the crack tip and compare the FEM distributions and the asymptotic solutions with different numbers of the terms. It is clearly seen that the larger distance from the crack tip the more terms in the asymptotic solution have to be kept.

Figure 7. FEM simulations: distribution of the von Mises stress intensity (left), stress component \(\sigma_{11}\) (center), stress component \(\sigma_{22}\) (right).

Figure 8. Distribution if the stress component \(\sigma_{11}\) at the distance from the crack tip \(r = 0.033 cm\): the red curve shows the one-term asymptotic solution (left), the two-term asymptotic solution (right), the blue points show the FEM solution.
Figure 9. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.033cm$: the red curve shows the three-term asymptotic solution, the blue points show the FEM solution.

Figure 10. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.059cm$: the red curve shows the two-term asymptotic solution (left), the seven-term asymptotic solution (right), the blue points show the FEM solution.

Figure 11. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.199cm$: the red curve shows the one-term asymptotic solution (left), the two-term asymptotic solution (right).

Figure 12. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.199cm$: the red curve shows the three-term asymptotic solution (left), the five-term asymptotic solution (right).
Figure 13. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.199cm$: the red curve shows the seven-term asymptotic solution, the blue points show the FEM solution.

Figures 9-17 accentuate the important implications of the coefficients of the Williams series expansion for the stress and strain field description in the vicinity of the crack tip.

Figure 14. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.399cm$: the red curve shows the one-term asymptotic solution (left), the two-term asymptotic solution (right).

Figure 15. Distribution if the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.399cm$: the red curve shows the three-term asymptotic solution (left), the five-term asymptotic solution (right).

The results for the specimen shown in figure 2-3 are tabulated in Table 1. The results for the cracked plate with the small crack are tabulated in Table 2 where the coefficients of the Williams series expansion are shown. The second column shows the results of comparison of these values with the theoretical solution (4) obtained for the infinite plate.
Figure 16. Distribution of the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.399\,cm$: the red curve shows the seven-term asymptotic solution (left), and the nine-term asymptotic solution (right).

Figure 17. Distribution of the stress component $\sigma_{11}$ at the distance from the crack tip $r = 0.399\,cm$: the red curve shows the eleven-term asymptotic solution (left), and the thirteen-term asymptotic solution (right).

Table 1. Coefficients of the Williams series expansion

| Coefficients obtained by the photoelasticity method | Coefficients obtained by the FEM analysis |
|---------------------------------------------------|------------------------------------------|
| $a_1^0 = 72.528\left( \frac{\kappa \Gamma}{cm^{3/2}} \right)$ | $a_1^0 = 72.527\left( \frac{\kappa \Gamma}{cm^{3/2}} \right)$ |
| $a_2^0 = -27.503\left( \frac{\kappa \Gamma}{cm^3} \right)$ | $a_2^0 = -27.516\left( \frac{\kappa \Gamma}{cm^3} \right)$ |
| $a_3^1 = 21.506\left( \frac{\kappa \Gamma}{cm^{5/2}} \right)$ | $a_3^1 = 20.163\left( \frac{\kappa \Gamma}{cm^{5/2}} \right)$ |
| $a_4^1 = -0.337\left( \frac{\kappa \Gamma}{cm^3} \right)$ | $a_4^1 = -0.302\left( \frac{\kappa \Gamma}{cm^3} \right)$ |
| $a_5^1 = -2.844\left( \frac{\kappa \Gamma}{cm^{7/2}} \right)$ | $a_5^1 = -2.757\left( \frac{\kappa \Gamma}{cm^{7/2}} \right)$ |
| $a_6^1 = -0.819\left( \frac{\kappa \Gamma}{cm^4} \right)$ | $a_6^1 = -0.998\left( \frac{\kappa \Gamma}{cm^7/2} \right)$ |
| $a_7^1 = 0.965\left( \frac{\kappa \Gamma}{cm^{9/2}} \right)$ | $a_7^1 = 0.712\left( \frac{\kappa \Gamma}{cm^{9/2}} \right)$ |
| $a_8^1 = 0.025\left( \frac{\kappa \Gamma}{cm^5} \right)$ | $a_8^1 = 0.019\left( \frac{\kappa \Gamma}{cm^5} \right)$ |
| $a_9^1 = -0.340\left( \frac{\kappa \Gamma}{cm^{11/2}} \right)$ | $a_9^1 = -0.315\left( \frac{\kappa \Gamma}{cm^{11/2}} \right)$ |
\[ a_{10} = -0.017 \left( \kappa \Gamma / \text{cm}^6 \right) \quad a_{10} = -0.016 \left( \kappa \Gamma / \text{cm}^6 \right) \]
\[ a_{11} = 0.098 \left( \kappa \Gamma / \text{cm}^{13/2} \right) \quad a_{11} = 0.077 \left( \kappa \Gamma / \text{cm}^{13/2} \right) \]
\[ a_{12} = 0.019 \left( \kappa \Gamma / \text{cm}^7 \right) \quad a_{12} = 0.012 \left( \kappa \Gamma / \text{cm}^7 \right) \]
\[ a_{13} = 0.056 \left( \kappa \Gamma / \text{cm}^{15/2} \right) \quad a_{13} = 0.050 \left( \kappa \Gamma / \text{cm}^{15/2} \right) \]
\[ a_{14} = 0.008 \left( \kappa \Gamma / \text{cm}^8 \right) \quad a_{14} = 0.0007 \left( \kappa \Gamma / \text{cm}^8 \right) \]
\[ a_{15} = 0.018 \left( \kappa \Gamma / \text{cm}^{17/2} \right) \quad a_{15} = 0.014 \left( \kappa \Gamma / \text{cm}^{17/2} \right) \]

**Table 2.** Coefficients of the Williams series expansion for the specimen with small crack under load 100 kg

| Coefficients | Error |
|--------------|-------|
| \( a_1 \) = 50.059 \left( \kappa \Gamma / \text{cm}^{3/2} \right) | 0.11% |
| \( a_2 \) = -24.972 \left( \kappa \Gamma \right) | 0.11% |
| \( a_3 \) = 24.943 \left( \kappa \Gamma / \text{cm}^{5/2} \right) | 0.12% |
| \( a_4 \) = -6.236 \left( \kappa \Gamma / \text{cm}^{7/2} \right) | 0.18% |
| \( a_5 \) = 3.112 \left( \kappa \Gamma / \text{cm}^{9/2} \right) | 0.20% |
| \( a_6 \) = -1.951 \left( \kappa \Gamma / \text{cm}^{11/2} \right) | 0.21% |
| \( a_{11} \) = 1.361 \left( \kappa \Gamma / \text{cm}^{13/2} \right) | 0.28% |
| \( a_{13} \) = -1.056 \left( \kappa \Gamma / \text{cm}^{15/2} \right) | 0.31% |
| \( a_{15} \) = 0.786 \left( \kappa \Gamma / \text{cm}^{17/2} \right) | 0.40% |

**4. Conclusions**

Multi-parameter stress field with higher – order terms near the crack tip is found. In the paper the coefficients of the Williams series expansion of the stress field in the vicinity of the crack in an isotropic linear elastic material are obtained by the digital photoelasticity technique and by FEM analysis. The coefficients of the higher-order terms of the series expansion are extracted with good accuracy. Comparison of the coefficients of the Williams series known from the theoretical solution for an infinite plate with the central crack and the coefficients extracted from the photoelasticity method and from finite element analysis is given. The FEM analysis is performed for a series of cracked specimens with a small crack and for a series of cracked specimens considered in the phototelasticity experiments.

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