A systematic construction of integrable delay-difference and delay-differential analogues of soliton equations

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Abstract. We propose a systematic method for constructing integrable delay-difference and delay-differential analogues of known soliton equations such as the Lotka-Volterra, Toda lattice, and sine-Gordon equations and their multi-soliton solutions. It is carried out by applying a reduction and delay-differential limit to the discrete KP or discrete two-dimensional Toda lattice equations. Each of the delay-difference and delay-differential equations has the \(N\)-soliton solution, which depends on the delay parameter and converges to an \(N\)-soliton solution of a known soliton equation as the delay parameter approaches 0.

Keywords: delay-differential equations, delay-difference equations, soliton equations, integrable systems, multi-soliton solutions

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1. Introduction

Delay-differential equations have been used as mathematical models in various fields of science and engineering such as traffic flow, population dynamics, nonlinear optics, fluid mechanics and infectious disease \([1,2]\). Because of their importance in various fields, the mathematical properties of delay differential equations have been actively studied. For example, exact solutions of delay differential equations play an important role in the study of traffic flow \([3-6]\).

In recent years, there has been active research on integrability of delay differential equations. Quispel \textit{et al} obtained a delay-differential equation which has a continuum limit to the first Painlevé equation \([7]\). This delay-differential equation was derived by a similarity reduction of the Lotka-Volterra (LV) equation, which is an integrable differential-difference equation. Levi and Winternitz also obtained delay-differential equations as reductions based on the symmetry group of the two-dimensional Toda lattice (2DTL) \([8]\). In addition, Grammaticos \textit{et al} \([9,10]\) introduced delay-differential Painlevé equations by the method blending the Painlevé analysis with the singularity confinement. After their monumental
works, several important studies have revealed mathematical properties of integrable (ordinary) delay-differential equations [11–17].

On the other hand, there are few examples of integrable partial delay-differential equations (the word “partial” means including multi independent variables). Villarroel and Ablowitz found the Lax pair for a delay-differential analogue of the 2D TL equation and established the inverse scattering formalism [18]. Recently, we constructed the \( N \)-soliton solution of the delay-differential analogues of the 2D TL equation [19]. Although there have been such works, research on integrable partial delay-differential equations is not fully developed.

In this paper, we propose a systematic method for constructing integrable delay-difference and delay-differential analogues of soliton equations and their multi-soliton solutions. Our construction starts from the discrete KP equation (or discrete 2DTL equation) and uses reduction and delay-differential limit. As examples of our method, we present delay-difference and delay-differential analogues of the LV, Toda lattice (TL), and sine-Gordon (sG) equations and their exact \( N \)-soliton solutions.

This paper is organized as follows. In the rest of this section, we show a systematic method to construct integrable delay-difference and delay-differential analogues of soliton equations. In sections 2, 3, and 4, we construct delay-difference and delay-differential analogues of the LV, TL, and sG equations and their \( N \)-soliton solutions. Section 5 is devoted to conclusions.

1.1. A method to construct delay-analogues of soliton equations

In the rest of this section, we show how to construct integrable delay-difference and delay-differential analogues of soliton equations and their \( N \)-soliton solutions. The first step of our method is to obtain a discrete equation by a reduction of the discrete KP equation [20, 21]

\[
a(b - c)f_{n+1,m,k}f_{n,m+1,k+1} + b(c - a)f_{n,m+1,k}f_{n+1,m,k+1} + c(a - b)f_{n,m,k+1}f_{n+1,m+1,k} = 0
\]

or the discrete 2DTL equation [20, 22]

\[
abf_{k+1,n+1,m}f_{k-1,n,m+1} + f_{k,n+1,m}f_{k,n,m+1} - (1 + ab)f_{k,n+1,m+1}f_{k,n,m} = 0,
\]

where \( a, b, c \) are real constants. Here we include a free parameter \( \alpha \) to the reduction condition such as (6). By applying the reduction, we obtain a discrete equation which depends on the free parameter \( \alpha \), the discrete variable \( n, m \), and the time-lattice parameter \( \delta \) which is defined by the parameters \( a, b, c \) such as (7). Note that the dependency on \( k \) is removed by the reduction, and the parameter \( \alpha \) appears as shifts of \( m \) (such as \( f_{m+\alpha} \)). This discrete equation can be considered as a delay-difference soliton equation.

Then, we apply the delay-differential limit

\[
\delta \to 0, \quad m\delta = t, \quad \alpha\delta = \tau = \text{const.}
\]

(3)

to the delay-difference soliton equation, where \( t \) is the continuous time variable and \( \tau \) is the delay parameter. This limit yields a delay-differential equation which includes the delay
An important point of this process is that we can obtain explicit $N$-soliton solutions of the delay-difference and delay-differential soliton equations. It is carried out by reduction and delay-differential limit to the $N$-soliton solution of the discrete KP or discrete 2DTL equations. In addition, we can obtain a known soliton equation and its $N$-soliton solution as $\tau \to 0$ if we properly determine the reduction at the first step of our method. Here the soliton equation which is obtained as $\tau \to 0$ depends on the reduction. Therefore, considering various reductions at the first step of our method, we can construct various delay-difference and delay-differential analogues of soliton equations.

### 2. An integrable delay Lotka-Volterra equation

In this section, we show the construction of the delay-difference and delay-differential analogue of the LV equation and their $N$-soliton solutions. We show the detail of our method through this section.

First, we consider the discrete KP equation (1) and its $N$-soliton solution in the Gram determinant form [23]:

$$f_{n,m,k} = \det \left( \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} = \prod_{i=1}^{N} \phi_i \det \left( \frac{\phi_j \psi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} \quad (4)$$

This Gram determinant form of the $N$-soliton solution can be rewritten as

$$f_{n,m,k} = \sum_{I \subset \{1, \ldots, N\}} \prod_{i \in I} \phi_i \prod_{i < j, i,j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \quad (5)$$

where

$$\Phi_i(n, m, k) = \beta_i (1 - a p_i)^n (1 - b p_i)^m (1 - c p_i)^k$$

$$\psi_i(n, m, k) = \gamma_i (1 - a q_i)^n (1 - b q_i)^m (1 - c q_i)^k$$

Here $\sum_{I \subset \{1, \ldots, N\}}$ is taken to be summed over all possible combinations in the set $\{1, \ldots, N\}$.

Now, we apply the following reduction condition to the discrete KP equation (1):

$$f_{n,m,k+1} = f_{n+2,m+a,k} \quad (6)$$

where the free parameter $\alpha$ is a fixed real value considered as the delay. Setting $c = -a$ and $\delta = 2b/(a - b)$, we obtain the following discrete bilinear equation by the reduction of the discrete KP equation (1):

$$(1 + \delta) f_{n}^{m+1} + f_{n-1}^{m} - \delta f_{n+1}^{m+1} - f_{n}^{m+1} f_{n-1}^{m+1} = 0 \quad (7)$$

where $f_{n}^{m} \equiv f_{n,m,k}$. More precisely, applying the reduction condition (6) to equation (1), we have made the index $k + 1$ of $f$ become $k$. Then we can omit the variable $k$ since the iterations...
of $k$ vanish. We can rewrite the bilinear equation (7) as follows by using Hirota’s D-operators:

$$2 \sinh \left( \frac{D_m}{2} \right) \sinh \left( \frac{D_n + \alpha D_m}{2} \right) - 2 \delta \sinh \left( \frac{D_n - D_m}{2} \right) \sinh \left( D_n + \frac{\alpha D_m}{2} \right) f_n^m \cdot f_n^m = 0.$$  

(8)

Here Hirota’s D-operators are defined by

$$D^l g(t) \cdot h(t) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right)^l g(t) h(s) \bigg|_{s=t}, \quad e^{D_m} g_m \cdot h_m = g_{m+1} h_{m-1}.$$  

(9)

Next we consider the reduction of the $N$-soliton solution. By applying the constraint

$$\left( \frac{1 - a q_i}{1 - a p_i} \right)^2 \left( \frac{1 - b q_i}{1 - b p_i} \right)^\alpha = \frac{1 - c q_i}{1 - c p_i}$$

(10)

to the $N$-soliton solution (4) and (5), we obtain

$$\Phi_i(n, m, k + 1) = \beta_i \gamma_i \left( \frac{1 - a q_i}{1 - a p_i} \right)^n \left( \frac{1 - b q_i}{1 - b p_i} \right)^m \left( \frac{1 - c q_i}{1 - c p_i} \right)^{k+1}$$

$$= \beta_i \gamma_i \left( \frac{1 - a q_i}{1 - a p_i} \right)^{n+2} \left( \frac{1 - b q_i}{1 - b p_i} \right)^{m+\alpha} \left( \frac{1 - c q_i}{1 - c p_i} \right)^k$$

(11)

thus the reduction condition (6) is satisfied. Setting $c = -a$, $\delta = 2b/(a - b)$, $k = 0$, and replacing $p_i$ and $q_i$ by $(-2p_i - 1)/a$ and $(-2q_i - 1)/a$ respectively, we obtain the $N$-soliton solution of the bilinear equation (7) by the reduction to (4) and (5):

$$f_n^m = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} = \sum_{I \subset \{1, \ldots, N\}} \prod_{i \in I} \Phi_i \prod_{i < j, i, j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)},$$

(12)

$$\Phi_i(n, m) = \beta_i \gamma_i \left( \frac{1 + q_i}{1 + p_i} \right)^n \left( \frac{1 + \delta + \delta q_i}{1 + \delta + \delta p_i} \right)^m,$$

$$\frac{q_i}{p_i} = \left( \frac{1 + q_i}{1 + p_i} \right)^2 \left( \frac{1 + \delta + \delta q_i}{1 + \delta + \delta p_i} \right)^\alpha.$$

The delay parameter $\alpha$ appears in the bilinear equation (7) as shifts of the discrete time variable $m$. We remark that the bilinear equation (7) and the $N$-soliton solution (12) in the case of $\alpha = 0$ (thus $p_i = 1/q_i$) are known as the bilinear equation of the discrete LV equation and its $N$-soliton solution [24, 25]. Thus equation (7) can be seen as the bilinear equation of the delay-difference analogue of the LV equation.

By the dependent variable transformation

$$u^m_n = \frac{f_n^{m+\alpha} f_n^{m+1}}{f_n^{m+\alpha} f_n^{m+1}},$$

(13)

the bilinear equation (7) is transformed into the nonlinear delay-difference equation

$$\frac{u_n^{m+\alpha+1} u_n^{m-1}}{u_n^{m+\alpha} u_n^{m+1}} = \frac{(1 + \delta u_n^{m+\alpha})(1 + \delta u_n^{m+1})}{(1 + \delta u_n^{m+\alpha})(1 + \delta u_n^{m+1})},$$

(14)
which is the delay-difference analogue of the LV equation. If $\alpha = 0$, equation (14) is just a division of the discrete LV equation [24]:

$$
\frac{u_{n+1}^m}{u_n^m} = \frac{1 + \delta u_{n+1}^m}{1 + \delta u_{n+1}^m}, \quad \frac{u_{n-1}^m}{u_{n-1}^m} = \frac{1 + \delta u_{n-2}^m}{1 + \delta u_{n-2}^m}.
$$

(15)

Now, we apply the delay-differential limit

$$
\delta \to 0, \quad m\delta = t, \quad \alpha \delta = 2\tau,
$$

(16)

where $\tau$ is a constant value called the delay parameter. The delay-differential limits of (7) and (12) are calculated respectively as follows:

$$
D_t f_n(t) \cdot f_{n-1}(t) - f_{n+1}(t) f_{n-2}(t) + f_n(t) f_{n-1}(t) = 0,
$$

(17)

$$
f_n(t) = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i q_j} \right)_{1 \leq i, j \leq N} = \sum_{I \subset \{1, \ldots, N\}} \prod_{i \in I} \Phi_i \prod_{i < j, i, j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)},
$$

(18)

$$
\Phi_i(n, t) = \beta_i \gamma_i \left( \frac{1 + q_i}{1 + p_i} \right)^n e^{(q_i - p_i) t},
$$

$$
\frac{q_i}{p_i} = \left( \frac{1 + q_i}{1 + p_i} \right)^2 e^{2\tau (q_i - p_i)}.
$$

Here the bilinear equation (17) can be rewritten as

$$
\left( D_t \sinh \left( \frac{D_n}{2} + \tau D_t \right) - 2 \sinh \left( \frac{D_n}{2} \right) \sinh \left( D_n + \tau D_t \right) \right) f_n(t) \cdot f_n(t) = 0.
$$

(19)

The bilinear equation (17) is a delay differential-difference equation which includes the delay $\tau$ as shifts of the continuous time variable $t$. Putting $\tau = 0$ (thus $p_i = 1/q_i$), we can easily check that equation (17) becomes the bilinear equation of the LV equation and (18) becomes the $N$-soliton solution of the LV equation [26]. Thus we claim that equation (17) is the bilinear equation of the integrable delay LV equation and the bilinear equation (7) is the fully discrete analogue of (17).

**Remark 2.1.** The last relation of (18), which is the dispersion relation, is rewritten as

$$
R(p_i) = R(q_i), \quad R(p) = \frac{p e^{-2\tau p}}{(1 + p)^2}.
$$

(20)

According to the graph of $R(p)$, it is easily seen that there exist $p_i$ and $q_i$ satisfying $R(p_i) = R(q_i)$ and $p_i \neq q_i$.

**Remark 2.2.** We remark that soliton solutions of (17) are also obtained by Hirota’s direct method [27], which does not require any reduction. We show a few examples of deriving soliton solutions without using the soliton solutions of the discrete KP equation. Let $f_n(t)$ be 1-soliton or 2-soliton with the perturbation parameter $\epsilon$:

$$
f_n(t) = 1 + \epsilon P_1^n e^{Q_1 t + \xi_0},
$$

(21)

$$
f_n(t) = 1 + \epsilon P_1^n e^{Q_1 t + \xi_0} + \epsilon P_2^n e^{Q_2 t + \xi_0} + \epsilon^2 a_{12} (P_1 P_2)^n e^{(Q_1 + Q_2)t + \xi_0 + \xi_0}.
$$

(22)
Substituting them into the bilinear equations (19) and assuming that each of the orders of $\epsilon$ vanishes, we have the following conditions.

$$F(P_i, Q_i) = 0 \quad (i = 1, 2), \quad a_{12} = \frac{F(P_1/P_2, Q_1 - Q_2)}{F(P_1 P_2, Q_1 + Q_2)},$$

where the function $F$ is defined by

$$F(P, Q) = Q \left( \sqrt{P} e^{\tau Q} - \frac{1}{\sqrt{P}} e^{-\tau Q} \right) - \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right) \left( P e^{\tau Q} - \frac{1}{P} e^{-\tau Q} \right).$$

By computations, we can check that the solutions (21) and (22) are equivalent to the case of $N = 1, 2$ in the $N$-soliton solution (18). As we can see from this example, we can use Hirota’s direct method even if the bilinear equation includes some delays.

Let us move the discussion to the nonlinear form of the delay LV equation. Via the dependent variable transformation

$$u_n(t) = \frac{f_{n+1}(t + \tau)f_{n-2}(t - \tau)}{f_n(t + \tau)f_{n-1}(t - \tau)},$$

the bilinear equation (17) is transformed into the nonlinear delay-differential equation

$$\frac{d}{dt} \log u_{n-1}(t - \tau) = u_{n+1}(t + \tau) - u_n(t + \tau) - u_{n-1}(t - \tau) + u_{n-2}(t - \tau),$$

which is the nonlinear form of (17). If $\tau = 0$, equation (26) is just a subtraction of the following nonlinear forms of the LV equation (26):

$$\frac{d}{dt} \log u_n(t) = u_{n+1}(t) - u_n(t), \quad \frac{d}{dt} \log u_{n-1}(t) = u_n(t) - u_{n-2}(t).$$

Next, we derive the bilinear form of the delay LV equation (17) and the $N$-soliton solution (18) directly from the semi-discrete KP equation (28)

$$ac D_t f_n^k(t) \cdot f_n^{k+1}(t) - (a - c)(f_n^{k+1}(t)f_n^k(t) - f_n^{k+1}(t)f_n^k(t)) = 0.$$  

We remark that the bilinear equation (28) and its solutions are derived by the continuum limit $b \to 0, mb = t$ of the discrete KP equation (1) and its solutions.

The Wronskian solution of the semi-discrete KP equation (28) is given as follows (28):

$$f_n^k(t) = \begin{vmatrix} \phi_1(n, k, t) & \phi_1^{(l)}(n, k, t) & \cdots & \phi_1^{(N-1)}(n, k, t) \\ \phi_2(n, k, t) & \phi_2^{(l)}(n, k, t) & \cdots & \phi_2^{(N-1)}(n, k, t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n, k, t) & \phi_N^{(l)}(n, k, t) & \cdots & \phi_N^{(N-1)}(n, k, t) \end{vmatrix},$$

$$\frac{\phi_i(n, k, t) - \phi_i(n, k - 1, t)}{\alpha} = \frac{\partial \phi_i(n, k, t)}{\partial t}, \quad \frac{\phi_i(n, k, t) - \phi_i(n, k - 1, t)}{\beta} = \frac{\partial \phi_i(n, k, t)}{\partial t},$$

$$\phi_i^{(l)}(n, k, t) \equiv \frac{\partial^l \phi_i(n, k, t)}{\partial t^t}.$$

The $N$-soliton solution of the semi-discrete KP equation is given by choosing the elements in the Wronskian solution as

$$\phi_i(n, k, t) = (1 - a_p) - n(1 - c_p) - k e^{p_it + p_{0i}} + (1 - a_q) - n(1 - c_q) - k e^{q_it + q_{0i}}.$$  

$$(i = 1, 2, \ldots, N)$$
Now, we can obtain the bilinear equation \((17)\) by applying the reduction condition
\[
f_{n}^{k+1}(t) \simeq f_{n+2}^{k}(t + 2\tau)
\] (31)
and setting \(a = 2, c = -2\) to the semi-discrete KP equation \((28)\). Here, the relation \(g_{n}^{k}(t) \simeq h_{n}^{k}(t)\) is defined by
\[
g_{n}^{k}(t) = (C_{0}C_{1}^{n}C_{2}^{k}e^{C_{l}t}) h_{n}^{k}(t), \quad C_{l} = \text{const.} \quad (l = 0, 1, 2, 3)
\] (32)
To realize this reduction condition \((C_{1} = C_{2} = 1, C_{3} = 0)\) for the \(N\)-soliton solutions, we can apply the constraint
\[
(1 - ap_{i})^{-2}(1 - cq_{i})e^{2\tau p_{i}} = (1 - aq_{i})^{-2}(1 - cq_{i})e^{2\tau q_{i}}
\] (33)
to \((30)\). Replacing \(p_{i}\) and \(q_{i}\) by \((-2p_{i} - 1)/a\) and \((-2q_{i} - 1)/a\) respectively and setting \(a = 2, c = -2\), we obtain the \(N\)-soliton solution of the delay LV equation in the Wronskian form
\[
f_{n}(t) = \begin{vmatrix}
\phi_{1}(n, t) & \phi_{1}^{(1)}(n, t) & \cdots & \phi_{1}^{(N-1)}(n, t) \\
\phi_{2}(n, t) & \phi_{2}^{(1)}(n, t) & \cdots & \phi_{2}^{(N-1)}(n, t) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{N}(n, t) & \phi_{N}^{(1)}(n, t) & \cdots & \phi_{N}^{(N-1)}(n, t)
\end{vmatrix},
\] (34)
\[
\phi_{i}(n, t) = (1 + p_{i})^{-n}e^{-p_{i}t + \zeta_{0i}} + (1 + q_{i})^{-n}e^{-q_{i}t + \eta_{0i}}, \quad (i = 1, 2, \ldots, N)
\]
\[
q_{i} = \left(1 + q_{i} \right) \left(1 + \frac{1}{p_{i}} \right)^{2} e^{2\tau(q_{i} - p_{i})}.
\]
**Remark 2.3.** We can also obtain the delay LV equation from the Bäcklund transformation (BT) of the 2DTL equation \([27, 29]\)
\[
\mu D_{t} f_{n}^{k-1}(t) \cdot f_{n+1}^{k}(t) - f_{n+1}^{k-1}(t) f_{n+1}^{k}(t) + f_{n+1}^{k-1}(t) f_{n}^{k}(t) = 0.
\] (35)
The following derivation is much easier than the above derivation from the semi-discrete KP equation.

The \(N\)-soliton solution in the Wronskian (Casorati determinant) form is given as
\[
f_{n}^{k}(t) = \begin{vmatrix}
\phi_{1}(n, k, t) & \phi_{1}^{(1)}(n, k, t) & \cdots & \phi_{1}^{(N-1)}(n, k, t) \\
\phi_{2}(n, k, t) & \phi_{2}^{(1)}(n, k, t) & \cdots & \phi_{2}^{(N-1)}(n, k, t) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{N}(n, k, t) & \phi_{N}^{(1)}(n, k, t) & \cdots & \phi_{N}^{(N-1)}(n, k, t)
\end{vmatrix},
\] (36)
\[
\phi_{i}(n, k, t) = p_{i}^{-n}e^{-p_{i}t + \zeta_{0i}} + q_{i}^{-n}e^{-q_{i}t + \eta_{0i}}, \quad (i = 1, 2, \ldots, N)
\]
Applying the reduction condition
\[
f_{n}^{k-1}(t) \simeq f_{n+2}^{k}(t + 2\tau)
\] (37)
and setting \(\mu = -1\), we obtain the bilinear equation \((17)\) and its \(N\)-soliton solution \((34)\).
3. An integrable delay Toda lattice equation

In this section, we construct a delay differential-difference equation that should be called an integrable delay TL equation by using our method.

We first apply the reduction condition

$$f_{n,m,k+1} = f_{n+1,m+1+\alpha,k}$$

(38)

to the discrete KP equation (1) and its $N$-soliton solution (4). The independent variables $n$ and $m$ are considered to be the discrete space variable and discrete time variable respectively, and the parameter $\alpha$ is a delay parameter. Replacing $p_i$ and $q_i$ by $(1-p_i)/c$ and $(1-q_i)/c$ respectively and setting $\alpha\delta = (a-c)/a$, $\delta = b/(b-c)$, we have

$$f_{n,m+1+\alpha} f_{m-1} - \alpha\delta^2 f_{n+1} f_{n-1} - (1 - \alpha\delta^2) f_{n+1} = 0,$$

(39)

$$f_n = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} = \prod_{i \in I} \prod_{i<j, i,j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)},$$

(40)

$$\Phi_i = \beta_i \gamma_i \left( \frac{q_i - \alpha\delta}{p_i - \alpha\delta} \right)^{n} \left( \frac{1 - \delta q_i}{1 - \delta p_i} \right)^{m},$$

$$\frac{q_i}{p_i} = \frac{q_i - \alpha\delta}{p_i - \alpha\delta} \left( \frac{1 - \delta q_i}{1 - \delta p_i} \right)^{1+\alpha}.$$

The bilinear equation (39) is rewritten as

$$\left( 2 \sinh \left( \frac{D_m}{2} \right) \sinh \left( \frac{D_m + \alpha D_m}{2} \right) - 2\alpha\delta^2 \sinh \left( \frac{D_m}{2} \right) \sinh \left( \frac{D_m + \alpha D_m}{2} \right) \right) f_n f_m = 0,$$

(41)

We can consider that equation (39) is the bilinear equation of the delay-difference analogue of the TL equation. We call this the delay discrete TL equation.

By the dependent variable transformation

$$1 + V_n^m = f_{n+1} f_{n-1} f_n,$$

(42)

we can transform (39) into the nonlinear delay-difference equation

$$\frac{(1 + V_n^{m+1+\alpha})(1 + V_n^{m-1})}{(1 + V_n^{m+1})(1 + V_n^m)} = \frac{(1 + \alpha\delta^2 V_n^{m+1})(1 + \alpha\delta^2 V_n^{m-1})}{(1 + \alpha\delta^2 V_n^{m+1})(1 + \alpha\delta^2 V_n^m)},$$

(43)

which is the delay discrete TL equation.

Now, we apply the delay-differential limit

$$\delta \to 0, \quad m\delta = t, \quad \alpha\delta = 2\tau$$

(44)

to (39) and (40), where $t$ is the continuous time variable and $\tau$ is the delay parameter. Then we obtain

$$D_t f_n(t+\tau) \cdot f_n(t-\tau) - 2\tau(f_{n+1}(t+\tau)f_{n-1}(t-\tau) - f_n(t+\tau)f_n(t-\tau)) = 0$$

(45)
and
\[ f_n(t) = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} = \sum_{I \subset \{1, \ldots, N\}} \prod_{i \in I} \prod_{i < j, i, j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \cdot (46) \]

\[ \Phi_i = \beta_i \gamma_i \left( \frac{q_i - 2\tau}{p_i - 2\tau} \right)^n e^{(p_i - q_i)t}, \]

\[ q_i = \frac{q_i - 2\tau}{p_i - 2\tau} e^{2\tau(p_i - q_i)}. \]

Here the bilinear equation (45) is equivalent to
\[ \left( D_t \sinh (\tau D_t) - 4\tau \sinh \left( \frac{D_n}{2} \right) \sinh \left( \frac{D_n}{2} + \tau D_t \right) \right) f_n(t) \cdot f_n(t) = 0. \]  

Calculating the limit of this equation as \( \tau \to 0 \), we obtain the bilinear TL equation [30]
\[ \left( D_t^2 - 4 \sinh^2 \left( \frac{D_n}{2} \right) \right) f_n(t) \cdot f_n(t) = 0. \]  

The last relation of (46) is described by \((p_i + 1/p_i - q_i - 1/q_i)\tau + O(\tau^2) = 0\), thus we have \(p_i = 1/q_i\) as \(\tau \to 0\). Therefore, we can check that the limit of (46) as \(\tau \to 0\) is the \(N\)-soliton solution of the TL equation [27, 30, 31]. Thus we claim that equation (45) is the bilinear equation of the delay TL equation.

**Remark 3.1.** If we use the relation \(\delta = (a - c)/a\) instead of the above one \(\alpha \delta = (a - c)/a\), we obtain the bilinear equation
\[ f_n^{m+1+\alpha} f_n^{m-1} - \delta^2 f_n^{m+1} f_n^{m-1} - (1 - \delta^2) f_n^{m+\alpha} f_n^m = 0 \]

instead of the delay discrete TL equation (39). This equation can be considered more natural than (39), because we can obtain the discrete TL equation from it by putting \(\alpha = 0\) [30]. However it does not yield a good delay-differential equation, because the order \(O(\delta^2)\) vanishes in the delay-differential limit. On the other hand, the bilinear equation (39) yields the good delay-differential equation (45), which should be called a delay TL equation.

We present the nonlinear form of the bilinear equation of the delay TL equation (45) under the dependent variable transformation
\[ 1 + V_n(t) = \frac{f_{n+1}(t + \tau) f_{n-1}(t - \tau)}{f_n(t + \tau) f_n(t - \tau)}. \]  

By using this transformation, we can transform (45) into the delay-differential equation
\[ \left( \frac{1}{2\tau} \right) \frac{d}{dt} \log \frac{1 + V_n(t + \tau)}{1 + V_n(t - \tau)} = V_{n+1}(t + \tau) + V_{n-1}(t - \tau) - V_n(t + \tau) - V_n(t - \tau), \]  

which is the nonlinear form of (45). The limit of (50) as \(\tau \to 0\) is the nonlinear form of the TL equation [27, 30, 31]:
\[ \frac{d^2}{dt^2} \log(1 + V_n(t)) = V_{n+1}(t) - 2V_n(t) + V_{n-1}(t). \]  

We can also obtain the above result from the BT of the 2DTL equation (35):
\[ \mu D_t f_n^{k-1}(t) \cdot f_n^{k}(t) + f_n^{k}(t) f_{n+1}^{k}(t) + f_n^{k-1}(t) f_{n+1}^{k}(t) = 0. \]  

Therefore, we can check that the limit of (46) as \(\tau \to 0\) is the \(N\)-soliton solution of the TL equation [27, 30, 31]. Thus we claim that equation (45) is the bilinear equation of the delay TL equation.
Applying the reduction condition
\[ f_n^{k-1}(t) \simeq f_{n+1}^k(t + 2\tau) \]  
and setting \( \mu = -1/(2\tau) \), we obtain the bilinear equation (45) and its \( N \)-soliton solution
\[ f_n(t) = \begin{vmatrix} \phi_1(n, t) & \phi_1^{(1)}(n, t) & \ldots & \phi_1^{(N-1)}(n, t) \\ \phi_2(n, t) & \phi_2^{(1)}(n, t) & \ldots & \phi_2^{(N-1)}(n, t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n, t) & \phi_N^{(1)}(n, t) & \ldots & \phi_N^{(N-1)}(n, t) \end{vmatrix}, \]  
\[ \phi_i(n, t) = (2\tau + p_i)^{-n}e^{-p_i t + \xi_i t} + (2\tau - q_i)^{-n}e^{-q_i t + \eta_i t}, \quad (i = 1, 2, \ldots, N) \]  
\[ q_i = \frac{2\tau + q_i}{2\tau + p_i} e^{2\tau(t - p_i)} , \]
which leads to (46) by replacing \( p_i \) and \( q_i \) by \(-p_i\) and \(-q_i\). This construction of the delay TL equation does not require the delay-differential limit.

4. An integrable delay sine-Gordon equation

In this section, we find an integrable delay sG equation by the process similarly to the previous sections. It is a delay partial differential equation which can be obtained simply.

We consider the bilinear equation of the discrete 2DTL equation [20, 22]
\[ ab f_{k+1,n+1,m} f_{k-1,n,m+1} + f_{k,n+1,m} f_{k,n,m+1} - (1 + ab) f_{k,n+1,m+1} f_{k,n,m} = 0 , \]

where \( k \) is the discrete space variable, and \( n, m \) are the discrete time variables. The \( N \)-soliton solution of (55) is given as follows [22]:
\[ f_{k,n,m} = \det \left( \delta_{ij} + \frac{\phi_j \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} \]
\[ = \det \left( \delta_{ij} + \frac{\phi_j \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} = \sum_{i \in \{1, \ldots, N\}}^{1 \leq i, j \leq N} \prod_{i < j} \Phi_i \prod_{i, j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} , \]
\[ \phi_i = \beta_i p_i^k (1 - a p_i)^{-n} (1 + b/p_i)^{-m} , \quad \psi_i = \gamma_i q_i^{-k} (1 - a q_i)^{n} (1 + b/q_i)^{m} , \]
\[ \Phi_i = \beta_i \gamma_i \left( \frac{p_i}{q_i} \right)^k \left( \frac{1 - aq_i}{1 - ap_i} \right)^n \left( \frac{1 + b/q_i}{1 + b/p_i} \right)^m . \]
Applying the reduction condition
\[ f_{k+1,n-\alpha,m+\beta} = f_{k-1,n+\alpha,m+\beta} , \]  
and setting
\[ f_n^m \equiv f_{k,n,m} = f_{k-2,n+2\alpha,m+2\beta} , \quad g_n^m \equiv f_{k+1,n-\alpha,m-\beta} = f_{k-1,n+\alpha,m+\beta} , \quad a = b = \delta \]
to (55) and (56) respectively, we obtain
\[ (1 + \delta^2) f_{n+1}^{m+1} f_n^m - f_{n+1}^m f_n^{m+1} - \delta^2 g_{n+1\alpha} g_{n-\alpha}^{m+1\beta} = 0 , \]
\[ (1 + \delta^2) g_{n+1}^m g_n^m - g_{n+1}^m g_n^{m+1} - \delta^2 f_{n+1\alpha} f_{n-\alpha}^{m+1\beta} = 0 \]
and

\[ f_n^m = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N}, \quad \Phi_i = \beta_i \gamma_i \left( \frac{1 - \delta q_i}{1 - \delta p_i} \right)^n \left( \frac{1 + \delta / q_i}{1 + \delta / p_i} \right)^m, \]

To construct a delay-difference analogue of the sG equation, we take \( g_n^m \) to be the complex conjugate of \( f_n^m \). Considering the regularity conditions \( f_n^m \neq 0 \), the \( N \)-soliton solution is given as

\[ f_n^m = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N}, \quad g_n^m = \det \left( \delta_{ij} - \frac{\Phi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N}, \]

\[ \Phi_i = \sqrt{-\mu_i} \left( \frac{1 - \delta q_i}{1 - \delta p_i} \right)^n \left( \frac{1 + \delta / q_i}{1 + \delta / p_i} \right)^m, \]

where \( \mu_i \) is a real constant.

Setting \( f_n^m = F_n^m + \sqrt{-1} G_n^m \) and \( g_n^m = F_n^m - \sqrt{-1} G_n^m \), we can rewrite the bilinear equations (59) and (60) as

\[ \left( 2 \sinh \left( \frac{D_n}{2} \right) \sinh \left( \frac{D_m}{2} \right) + 2 \delta^2 \sinh \left( \frac{D_n}{2} + \frac{\alpha D_n}{2} + \frac{\beta D_m}{2} \right) \sinh \left( \frac{D_m}{2} - \frac{\alpha D_n}{2} - \frac{\beta D_m}{2} \right) \right) = 0, \]

\[ \left( 2 \sinh \left( \frac{D_n}{2} \right) \sinh \left( \frac{D_m}{2} \right) + 2 \delta^2 \cosh \left( \frac{D_n}{2} + \frac{\alpha D_n}{2} + \frac{\beta D_m}{2} \right) \cosh \left( \frac{D_m}{2} - \frac{\alpha D_n}{2} - \frac{\beta D_m}{2} \right) \right) = 0. \]

When \( \alpha = \beta = 0 \), we can find that the bilinear equations (59), (60) (and also (63), (64)) and the \( N \)-soliton solution (62) are actually equivalent to the bilinear equations of the discrete sG equation and their \( N \)-soliton solution [32].

To construct a nonlinear form of the delay-difference analogue of the sG equation, we consider the dependent variable transformation

\[ f_n^m = \exp \left( \frac{F_n^m}{4} + \sqrt{-1} \frac{G_n^m}{4} \right), \quad g_n^m = \exp \left( \frac{\rho_n^m}{4} - \sqrt{-1} \frac{\theta_n^m}{4} \right), \]

which is equivalent to

\[ \theta_n^m = 2 \sqrt{-1} \log \frac{g_n^m}{f_n^m} = 4 \tan^{-1} \frac{G_n^m}{F_n^m}, \quad \rho_n^m = 2 \log f_n^m g_n^m = 2 \log ((F_n^m)^2 + (G_n^m)^2). \]

By using this transformation, we can transform the bilinear equations (59) and (60) into the nonlinear delay-difference equation

\[ \sin \left( \frac{\theta_{n+1}^m + \theta_n^m + \theta_{n+1}^m + \theta_n^m}{4} \right) \]
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\[ \equiv \delta^2 \exp \left( \frac{\rho_{n+1}^{\alpha} + \rho_{n-\alpha}^{m+1-\beta} - \rho_{n+1}^{m} - \rho_{n}^{m+1}}{4} \right) \sin \left( \frac{\theta_{n+1}^{\alpha} + \theta_{n-\alpha}^{m+1-\beta} + \theta_{n+1}^{m} + \theta_{n}^{m}}{4} \right) \]

(67)

\[ \sinh \left( \frac{\rho_{n+1}^{m} + \rho_{n-\alpha}^{m+1} - \rho_{n+1}^{m} - \rho_{n}^{m}}{4} \right) = \delta^2 \exp \left( \frac{-\rho_{n+1}^{m} - \rho_{n+1}^{m+1} + \rho_{n}^{m}}{4} \right) \]

\[ - \delta^2 \exp \left( \frac{\rho_{n+1}^{\alpha} + \rho_{n-\alpha}^{m+1-\beta} - \rho_{n+1}^{m} - \rho_{n}^{m}}{4} \right) \cos \left( \frac{\theta_{n+1}^{\alpha} + \theta_{n-\alpha}^{m+1-\beta} + \theta_{n+1}^{m} + \theta_{n}^{m}}{4} \right) \]

\[ - \delta^4 \exp \left( \frac{\rho_{n+1}^{\alpha} + \rho_{n-\alpha}^{m+1-\beta} - \rho_{n+1}^{m} - \rho_{n}^{m}}{4} \right) \sin \left( \frac{\rho_{n+1}^{\alpha} + \rho_{n-\alpha}^{m+1-\beta} + \theta_{n+1}^{m} + \theta_{n}^{m}}{4} \right) \]

(68)

which is the delay-difference analogue of the sG equation. Equation (67) in the case of \( \alpha = \beta = 0 \) is the discrete sG equation [32]

\[ \sin \left( \frac{\theta_{n+1}^{m} + \theta_{n-\alpha}^{m+1} - \theta_{n+1}^{m} - \theta_{n}^{m}}{4} \right) = \delta^2 \sin \left( \frac{\theta_{n+1}^{m} + \theta_{n+1}^{m+1} + \theta_{n}^{m}}{4} \right) \]

(69)

Now, let us apply the delay-differential limit \( \delta \to 0 \), \( n\delta = x \), \( \alpha \delta = \xi \), \( m\delta = y \), \( \beta \delta = \eta \)

(70)
to the bilinear equations (59), (60) and the \( N \)-soliton solution (62). Here \( x, y \) are the continuous variables, and \( \xi, \eta \) are the delay parameters. Consequently we obtain the bilinear equations

\[ D_x D_y f(x, y) \cdot f(x, y) + 2(f(x, y) f(x, y) - g(x + \xi, y + \eta)g(x - \xi, y - \eta)) = 0, \]

\[ D_x D_y g(x, y) \cdot g(x, y) + 2(g(x, y) g(x, y) - f(x + \xi, y + \eta)f(x - \xi, y - \eta)) = 0, \]

(71)

(72)

and the \( N \)-soliton solution

\[ f(x, y) = \det \left( \delta_{ij} + \frac{\Phi_j}{p_i - q_j} \right) \]

\[ g(x, y) = \det \left( \delta_{ij} - \frac{\Phi_j}{p_i - q_j} \right) \]

(73)

\[ \Phi_i = \sqrt{-1} \mu_i \exp \left( (p_i - q_i) x - \left( \frac{1}{p_i} - \frac{1}{q_i} \right) y \right), \]

\[ \frac{p_i}{q_i} = - \exp \left( \xi(p_i - q_i) - \eta \left( \frac{1}{p_i} - \frac{1}{q_i} \right) \right). \]

Setting \( f(x, y) = F(x, y) + \sqrt{-1} G(x, y) \) and \( g(x, y) = F(x, y) - \sqrt{-1} G(x, y) \), we can rewrite (71) and (72) with

\[ \left( D_x D_y - 4 \sinh^2 \left( \frac{\xi D_x + \eta D_y}{2} \right) \right) (F \cdot F - G \cdot G) = 0, \]

\[ \left( D_x D_y + 4 \cosh^2 \left( \frac{\xi D_x + \eta D_y}{2} \right) \right) F \cdot G = 0. \]

(74)

(75)

In the case of \( \xi = \eta = 0 \), the bilinear equations (71), (72) (and also (74), (75)) and the \( N \)-soliton solution (73) lead to the sG equation and their \( N \)-soliton solution [32–34].

To construct a nonlinear form of the delay-differential analogue of the sG equation, we consider the dependent variable transformation

\[ f(x, y) = \exp \left( \frac{\rho(x, y)}{4} + \sqrt{-1} \frac{\phi(x, y)}{4} \right), \]

\[ g(x, y) = \exp \left( \frac{\rho(x, y)}{4} - \sqrt{-1} \frac{\phi(x, y)}{4} \right) \]

(76)
which is equivalent to
\[ \theta(x, y) = 2\sqrt{-1}\log\frac{g}{f} = 4\tan^{-1}\frac{G}{F}, \quad \rho(x, y) = 2\log fg = 2\log(F^2 + G^2). \] (77)

By using this transformation, we can transform the bilinear equations (71) and (72) into the nonlinear delay-differential equation
\[ \frac{\partial^2}{\partial x \partial y} \theta(x, y) = -4\exp\left(\frac{\rho(x + \xi, y + \eta) + \rho(x - \xi, y - \eta) - 2\rho(x, y)}{4}\right) \times \sin\left(\frac{\theta(x + \xi, y + \eta) + \theta(x - \xi, y - \eta) + 2\theta(x, y)}{4}\right), \] (78)
\[ \frac{\partial^2}{\partial x \partial y} \rho(x, y) = -4 + 4\exp\left(\frac{\rho(x + \xi, y + \eta) + \rho(x - \xi, y - \eta) - 2\rho(x, y)}{4}\right) \times \cos\left(\frac{\theta(x + \xi, y + \eta) + \theta(x - \xi, y - \eta) + 2\theta(x, y)}{4}\right), \] (79)
which is the delay-differential analogue of the sG equation. In the case of \( \xi = \eta = 0 \), the above nonlinear equation leads to the sG equation [32, 33]:
\[ \frac{\partial^2}{\partial x \partial y} \theta(x, y) = -4\sin\theta(x, y). \] (80)

We can obtain the above delay-differential analogue of the sG equation by a reduction of the 2DTL equation. The bilinear equation of the 2DTL equation
\[ D_x D_y f_k(x, y) \cdot f_k(x, y) + 2(f_k(x, y)f_k(x, y) - f_{k+1}(x, y)f_{k-1}(x, y)) = 0 \] (81)
has the following \( N \)-soliton solution [22, 27]:
\[ f_k(x, y) = \det \left( \delta_{ij} + \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} = \det \left( \delta_{ij} + \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N} = \prod_{i \in I} \prod_{i<j, i,j \in I} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}, \] (82)
\[ \phi_i = \beta_i p_i^k e^{p_i x - p_i^{-1} y}, \quad \psi_i = \gamma_i q_i^{-k} e^{-q_i x + q_i^{-1} y}, \]
\[ \Phi_i = \beta_i \gamma_i \left( \frac{p_i}{q_i} \right)^k \exp \left( (p_i - q_i)x - \left( \frac{1}{p_i} - \frac{1}{q_i} \right)y \right), \]
where \( p_i, q_i, \beta_i, \gamma_i \) are constants. We apply the reduction condition
\[ f_{k+1}(x - \xi, y - \eta) = f_{k-1}(x + \xi, y + \eta) \] (83)
and set
\[ f(x, y) \equiv f_k(x, y) = f_{k-2}(x + 2\xi, y + 2\eta), \] (84)
\[ g(x, y) \equiv f_{k+1}(x - \xi, y - \eta) = f_{k-1}(x + \xi, y + \eta). \] (85)

Then we obtain the bilinear equations
\[ D_x D_y f(x, y) \cdot f(x, y) + 2(f(x, y)f(x, y) - g(x + \xi, y + \eta)g(x - \xi, y - \eta)) = 0, \] (86)
\[ D_x D_y g(x, y) \cdot g(x, y) + 2(g(x, y)g(x, y) - f(x + \xi, y + \eta)f(x - \xi, y - \eta)) = 0. \] (87)
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For the $N$-soliton solution, the constraint

$$\left(\frac{p_i}{q_i}\right)^2 = \exp\left(2\xi(p_i - q_i) - 2\eta\left(\frac{1}{p_i} - \frac{1}{q_i}\right)\right)$$

provides the reduction condition (83). The constraint (88) leads to

$$\frac{p_i}{q_i} = \pm \exp\left(\xi(p_i - q_i) - \eta\left(\frac{1}{p_i} - \frac{1}{q_i}\right)\right).$$

To take $g(x, y)$ to be the complex conjugate of $f(x, y)$, we can choose

$$\frac{p_i}{q_i} = -\exp\left(\xi(p_i - q_i) - \eta\left(\frac{1}{p_i} - \frac{1}{q_i}\right)\right)$$

and $\beta_i\gamma_i = \sqrt{-1}\mu_i$, where $\mu_i$ is a real constant. Thus we obtain the delay-differential analogue of the sG equation and its $N$-soliton solution from the 2DTL equation. This construction of the delay sG equation does not require the delay-differential limit.

5. Conclusions

We have presented the systematic method to construct delay-difference and delay-differential analogues of soliton equations and their $N$-soliton solutions.

Our construction starts from the discrete KP equation (or discrete 2DTL equation) and uses reduction and delay-differential limit. As examples, we have obtained the delay-difference and delay-differential analogues of the LV, TL, and sG equations and their $N$-soliton solutions. We have also presented another construction of delay-differential analogues of soliton equations starting from the semi-discrete KP, BT of 2DTL, and 2DTL equations without applying a delay-differential limit. In the construction of them, the important thing is to integrate discrete variables with continuous variables by reduction.

In this paper, we have not discussed Lax pairs and conserved quantities of the delay soliton equations and the relationship to the delay-differential Painlevé equations. These problems remain to be revealed in future studies.

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