Hopf Algebroids and quantum groupoids

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Abstract

We introduce the notion of Hopf algebroids, in which neither the total algebras nor the base algebras are required to be commutative. We give a class of Hopf algebroids associated to module algebras of the Drinfeld doubles of Hopf algebras when the \( R \)-matrices act properly. When this construction is applied to quantum groups, we get examples of quantum groupoids, which are semi-classical limits of Poisson groupoids. The example of quantum \( sl(2) \) is worked out in details.

Contents

1 Introduction 1
2 Bi-algebroids 6
3 Two examples of bi-algebroids: \( A \otimes A^\text{op} \) and \( \text{End}_k(A) \) 11
4 Hopf algebroids 14
5 The Hopf algebroid \( V \# A \) 15
6 The Hopf algebroid \( H(A^\ast) \) 25
7 An example of a quantum groupoid associated to \( SL_q(2) \) 26

1 Introduction

The notion of commutative Hopf algebroids was given in [Ra]. It is modeled on the space of functions on a groupoid. In this definition, both the algebra for the total space and the
algebra for the base space are commutative. The commutativity of these algebras plays a crucial role in this definition.

It is natural to try to replace the algebras for both the total space and the base space by non-commutative algebras. In [Ma], Maltsiniotis gives a definition of Hopf algebroids, where the base algebras are still required to be commutative and their images under the source and target maps are required to lie in the centers of the total algebras. See also [B-M] for a study of “quasi-quantum groupoids”, where the base algebras are again commutative.

In this paper, we formulate a definition of Hopf algebroids where none of the algebras are required to be commutative. Like in [Ma], we use the term Hopf algebroids in general and reserve the term “quantum groupoids” for the examples associated to quantum groups. A class of examples associated to module algebras of the Drinfeld doubles of Hopf algebras is described in Theorem 5.1. An example constructed from the quantum group $SL_q(2)$ when $q$ is a root of unity is described in details.

Our work is motivated by the notion of Poisson groupoids in Poisson geometry [We2]. Roughly speaking, we think of Hopf algebroids as “quantizations” of Poisson groupoids, just like quantum groups are quantizations of Poisson groups. Our main example (see Theorem below) describes the direct Hopf analog of some Poisson groupoids associated to Poisson Lie groups.

Our definition reduces to the one given by Maltsiniotis in [Ma] when the base algebra is commutative and when its images under the source and target maps lie in the center of the total algebra. The semi-classical limits of the quantum groupoids given by Maltsiniotis are Poisson groupoids with the zero Poisson structure on the base spaces, while those by our definition have arbitrary Poisson groupoids as their semi-classical limits.

We would also like to point out that our definition of a Hopf algebroid is not the same as the one given in [Va], either, where Vainerman studies the Hopf analog of transformation groupoids. The semi-classical limit of his example does not give rise to a Poisson groupoid. The main difference between our example and the one in [Va] is that he uses a direct product algebra structure while we use a smash product algebra structure.

For any algebra $A$ over a field $k$, we can associate to it the bi-algebroid $\text{End}_k(A)$ over $A$, and the Hopf algebroid $A \otimes A^{op}$ over $A$. The latter is obviously the Hopf analog of the coarse
groupoid structure on $X \times X$ over any space $X$. Our main examples of Hopf algebroids, however, are the following:

**Theorem** Let $A$ be a Hopf algebra and let $D(A)$ be the Drinfeld double of $A$. Let $V$ be a left $D(A)$-module algebra. Assume that the $R$-matrix of $D(A)$ acts on $V \otimes V$ in the following way:

$$m_V^{\text{op}} \circ R = m_V,$$

where $m_V : V \otimes V \to V$ is the product on $V$ and $m_V^{\text{op}}$ is its opposite. Then there is a Hopf algebroid structure over $V$ on the smash product algebra $H = V \# A$, where the smash product is constructed using the left action of $A$ on $V$ that is the restriction to $A$ of the action of $D(A)$ on $V$.

One example of such a $V$ is the dual Hopf algebra $A^*$ of $A$. The smash product algebra $H = A^* \# A$ is, by definition, the Heisenberg double of $A^*$, and we denote it by $H(A^*)$. Thus, the above Theorem says that there is a Hopf algebroid structure over $A^*$ on the Heisenberg double $H(A^*)$.

In formulating the definition, we use the following simple dictionary of Poisson geometry and the theory of algebras (see [Lu2] for more details):

- Poisson manifold $P$
- associative algebra $V$
- Poisson morphisms
- algebra homomorphisms
- Poisson submanifolds of $P$
- two-sided ideas of $V$
- coisotropic submanifolds of $P$
- one-sided ideas of $V$
- Poisson group $G$
- Hopf algebra $A$
- Poisson action of $G$ on $P$ $A$-co-module algebra $V$
- dual Poisson group $G^*$
- dual Hopf algebra $A^*$
- Semidirect product Poisson structure on $P \times G^*$
- smash product on $V \otimes A^*$
- Poisson double $(D = G \bowtie G^*, \pi_-)$
- dual of Drinfeld double $D(A)$
- symplectic double $(D = G \bowtie G^*, \pi_+)$
- Heisenberg double $H(A)$

We now recall the definition of a Poisson groupoid. A **groupoid** over a set $P$ is a set $X$ together with

1) surjections $\alpha, \beta : X \to P$ (called the source and target maps respectively);
2) \( m : X_2 \rightarrow X \) (called the multiplication), where 
\[ X_2 := \{(x, y) \in X \times X| \beta(x) = \alpha(y)\}, \]
such that \( \alpha(m(x, y)) = \alpha(x) \) and \( \beta(m(x, y)) = \beta(y) \);
3) an injection \( \epsilon : P \rightarrow X \) (identities) such that \( \beta \epsilon = \alpha \epsilon = id_P \);
4) a bijection \( \iota : X \rightarrow X \) (inversion).

These maps must satisfy

1) associative law: \( m(m(x, y), z) = m(x, m(y, z)) \) (if one side is defined, so is the other);
2) identities: for each \( x \in X, (\epsilon(\alpha(x)), x) \in X_2, (x, \epsilon(\beta(x))) \in X_2 \) and \( m(\epsilon(\alpha(x)), x) = m(x, \epsilon(\beta(x))) = x \);
3) inverses: for each \( x \in X, (x, \iota(x)) \in X_2, (\iota(x), x) \in X_2, m(x, \iota(x)) = \epsilon(\alpha(x)), \) and \( m(\iota(x), x) = \epsilon(\beta(x)) \).

A groupoid \( P \) over \( X \) is called a Lie groupoid if both \( X \) and \( P \) are smooth manifolds and if all the structure maps are smooth. We also require that both \( \alpha \) and \( \beta \) be submersions.

A Poisson groupoid \( \text{[We2]} \) is a Lie groupoid \( P \) over \( X \) together with a Poisson structure such that the graph of the groupoid multiplication, i.e., the set
\[
\{(m(x, y), x, y) : \beta(x) = \alpha(y)\} \subset X \times X \]
is a coisotropic submanifold of \( X \times \bar{X} \times \bar{X} \), where \( \bar{X} \) denotes \( X \) equipped with the negative Poisson structure. When the Poisson structure on \( X \) is nondegenerate, we say that \( X \) is a symplectic groupoid \( \text{[We1]} \) over \( P \).

We summarize the properties of Poisson groupoids in the following proposition. See \( \text{[We2]} \) for the proofs. This proposition explains some of the requirements we put in our definition of a Hopf algebroid.

**Proposition 1.1** Let \( X \) be a Poisson groupoid over \( P \) with structure maps \( \alpha, \beta, m, \epsilon \) and \( \iota \). Then

1) there is a unique Poisson structure on \( P \) such that \( \alpha \) is a Poisson map and \( \beta \) is an anti-Poisson map. Consequently, the submanifold
\[
\{(x, y) : \beta(x) = \alpha(y)\} \subset X \times X
\]
is a coisotropic submanifold of \( X \times X \);
2) the antipode map \( \iota \) is an anti-Poisson isomorphism for \( X \);
3) the map \( \epsilon : P \rightarrow X \) embeds \( P \) into \( X \) as a coisotropic submanifold.
Our main example of a Hopf algebroid is a direct Hopf analog of the following theorem on Poisson Lie groups [Lu1].

**Theorem 1.2** Let $P$ be a Poisson manifold. Let $G$ be a Poisson Lie group, $G^*$ its dual Poisson Lie group and $(D, \pi_-)$ the Poisson double of $G$ and $G^*$. Assume that

1) there exists a right Poisson action $\sigma$ of $(D, \pi_-)$ on $P$: 
$$\sigma: P \times D \rightarrow P$$

2) the Poisson structure on $P$ is determined by the action $\sigma$ in the sense that the Poisson bracket of two functions $\phi_1$ and $\phi_2$ on $P$ is given by
$$\{\phi_1, \phi_2\} = (\sigma_{\phi_1}, \sigma'_{\phi_2})$$

where $(\cdot, \cdot)$ denotes the pairing between the Lie algebra $\mathfrak{g}$ of $G$ and the Lie algebra $\mathfrak{g}^*$ of $G^*$, and $\sigma_{\phi_1}$ and $\sigma'_{\phi_2}$ are the $\mathfrak{g}$ and $\mathfrak{g}^*$-valued functions on $P$ respectively given by
$$\sigma_{\phi_1}(X)(p) = \left. \frac{d}{dt} \right|_{t=0} \phi_1(p \cdot \exp tX), \quad X \in \mathfrak{g}$$
$$\sigma'_{\phi_2}(\xi)(p) = \left. \frac{d}{dt} \right|_{t=0} \phi_2(p \cdot \exp t\xi), \quad \xi \in \mathfrak{g}^*.$$ 

Then the manifold $P \times G^*$, together with the transformation groupoid structure over $P$ defined by the restricted $G^*$ action on $P$ and the semi-Poisson structure defined by the restricted action of $G$ on $P$, becomes a Poisson groupoid over $P$.

Throughout this paper, vector spaces are assumed to be over the ground field $k$, and tensor products are over $k$ unless otherwise indicated. Algebras are assumed to have identity elements, and algebra morphisms are assumed to preserve identity elements.

The paper is organized as follows. In Section 2 we give the definition of a bi-algebroid. We show in Section 3 that for any finite dimensional algebra $A$, the space $\text{End}_k(A)$ has a natural bi-algebroid structure over $A$. Moreover, for any other bi-algebroid $H$ over $A$, there is a bi-algebroid morphism from $H$ to $\text{End}_k(A)$. Another example of a bi-algebroid over $A$ is $A \otimes A^{op}$. In Section 4, we give the definition of a Hopf algebroid. The main examples are described in Theorem 5.1 in Section 5. The Hopf algebroid structure on the Heisenberg double $H(A^*)$ of a Hopf algebra $A^*$ is described in details in Section 6. Finally, we give the example related to the quantum group $SL_q(2)$ at a root of unity in Section 7.
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2 Bi-algebroids

Definition 2.1 A bi-algebroid consists of the following data:

1) an algebra $H$ called the total algebra;
2) an algebra $A$ called the base algebra;
3) the source map: an algebra homomorphism $\alpha \colon A \to H$ and the target map: an algebra anti-homomorphism $\beta \colon A \to H$

such that the images of $\alpha$ and $\beta$ commute in $H$, i.e., $\forall a, b \in A$,

$$\alpha(a)\beta(b) = \beta(b)\alpha(a). \quad \text{(1)}$$

There is then a natural $(A, A)$-bimodule structure on $H$ given by

$$\lambda : H \otimes H \to H : a \otimes h \mapsto \alpha(a)h \quad \text{(2)}$$
$$\rho : H \otimes A \to H : h \otimes a \mapsto \beta(a)h. \quad \text{(3)}$$

Using this bimodule structure, we can form the $(A, A)$-bimodule product $H \otimes_A H$ of $H$ with itself. As a vector space over $k$, it is simply the quotient of $H \otimes H$ by the subspace

$$I_2 := \{\beta(a)h_1 \otimes h_2 - h_1 \otimes \alpha(a)h_2 = (\beta(a) \otimes 1 - 1 \otimes \alpha(a))(h_1 \otimes h_2) : h_1, h_2 \in H, a \in A\}. \quad \text{(4)}$$

Notice that in this case, $I_2$ is in fact the right ideal of $H \otimes H$ generated by the subset

$$\{\beta(a) \otimes 1 - 1 \otimes \alpha(a) : a \in A\}.$$
We will still denote elements of $H \otimes_A H$ by $h_1 \otimes h_2$. The $(A, A)$-bimodule structure on $H \otimes_A H$ is now given by

$$A \otimes (H \otimes_A H) \to H \otimes_A H : a \otimes (h_1 \otimes h_2) \mapsto \alpha(a) h_1 \otimes h_2$$

and

$$(H \otimes_A H) \otimes A \to H \otimes_A H : (h_1 \otimes h_2) \otimes a \mapsto h_1 \otimes \beta(a) h_2.$$ We can then form the $(A, A)$-bimodule product of $H \otimes A H$ with $H$ to get the triple product $H \otimes_A H \otimes A H$. In general, we can form the $n$’th $(A, A)$-bimodule product of $H$ with itself. It is the quotient of $H \otimes^n$ by the right ideal $I_n$ generated by the subspace

$$\{1 \otimes \cdots \otimes 1 \otimes \beta(a_i) \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes \alpha(a_i) \otimes 1 \otimes \cdots \otimes 1, i = 1, \ldots, n-1, a_i \in A\}. \quad (5)$$

4) the co-product: an $(A, A)$-bimodule map

$$\Delta : H \to H \otimes_A H : h \mapsto h_{(1)} \otimes h_{(2)}$$

with $\Delta(1) = 1 \otimes 1$ satisfying the co-associativity:

$$(\Delta \otimes_A id_H) \Delta = (id_H \otimes_A \Delta) \Delta : H \to H \otimes_A H \otimes_A H.$$ The co-product $\Delta$ and the algebra structure on $H$ are required to be compatible in the sense that the kernel of the following map

$$\Phi : H \otimes H \otimes H \to H \otimes_A H : h_1 \otimes h_2 \otimes h_3 \mapsto (\Delta h_1)(h_2 \otimes h_3) \quad (6)$$

is a left ideal of $H \otimes H^{op} \otimes H^{op}$, where $H^{op}$ denotes $H$ with the opposite product. Here we are using the fact that $H \otimes H$ acts on $H \otimes_A H$ from the right by right multiplications;

5) the co-unit map: an $(A, A)$-bimodule map

$$\epsilon : H \to A$$

satisfying

$$\epsilon(1_H) = 1$$

(it follows then that $\epsilon \beta = \epsilon \alpha = id_A$) and

$$\lambda(\epsilon \otimes id_H) \Delta = \rho(id_H \otimes \epsilon) \Delta = id_H : H \to H, \quad (7)$$
where \( \lambda \) and \( \rho \) are respectively the left and right \( A \)-module maps for \( H \) given by (3) and (2). Notice that two maps on the left hand side of the above equation are well-defined. It is also required to be compatible with the algebra structure on \( H \) in the sense that the kernel of \( \epsilon \) is a left ideal of \( H \).

The following Lemma shows that when \( A \) is the 1-dimensional algebra \( k \), our definition of a bi-algebroid over \( k \) is reduced to that of a bi-algebra over \( k \).

**Lemma 2.2** Let \( A \) and \( B \) be two algebras. A linear map \( f : A \to B \) mapping \( 1_A \) to \( 1_B \) is an algebra homomorphism if and only if the kernel of the map

\[
F : A \otimes B \to B : a \otimes b \mapsto f(a)b
\]

is a left ideal of \( A \otimes B^{op} \). In this case, the left ideal is generated by the subspace \( \{ a \otimes 1 - 1 \otimes f(a) : a \in A \} \).

**Proof.** Assume first that \( f \) is an algebra morphism. Then it is easy to see that \( \ker F \) is a left ideal of \( A \otimes B^{op} \). Moreover, if \( \sum a_i \otimes b_i \in \ker F \), then

\[
\sum a_i \otimes b_i = \sum (1 \otimes b_i)(a_i \otimes 1 - 1 \otimes f(a_i)) \in A \otimes B^{op}.
\]

Thus \( \ker F \) coincides with the left ideal generated by the given subspace. Conversely, assume that \( \ker F \) is a left ideal of \( A \otimes B^{op} \). Then for any \( a_1, a_2 \in A \), since

\[
a_2 \otimes 1 - 1 \otimes f(a_2) \in \ker F,
\]

we know that

\[
a_1 a_2 \otimes 1 - a_1 \otimes f(a_2) = (a_1 \otimes 1)(a_2 \otimes 1 - 1 \otimes f(a_2)) \in \ker F.
\]

Thus \( f(a_1 a_2) = f(a_1)f(a_2) \). Hence \( f \) is an algebra homomorphism.

Q.E.D.

Assume that \( H \) is a bi-algebroid over the algebra \( A \) with structure maps \( \alpha, \beta, \Delta \) and \( \epsilon \). Then since \( \epsilon \alpha = id_A \), the map \( \epsilon \) is subjective, so we can identify \( A \) with the quotient space \( H/\ker \epsilon \). Since \( \ker \epsilon \subset H \) is a left ideal of \( H \), there is an induced left action of \( H \) on \( A \).
Since $\alpha : A \to H$ is a section of $\epsilon$, we can identify $A$ with the subspace $\alpha(A) \subset H$. The left action of $H$ on $A$ can then be explicitly written down as

$$T_1 : H \to \text{End}_k(A) : h(a) := \epsilon(h\alpha(a)).$$ \hfill (8)

**Definition 2.3** We call the action $T_1$ given by (8) the left action of $H$ on $A$ associated to the bi-algebroid structure on $H$ over $A$.

**Proposition 2.4** The action $T_1$ given by (8) has the following properties:

1) The composition $T_1 \circ \alpha$ (resp. $T_1 \circ \beta$): $A \to \text{End}_k(A)$ gives the left (resp. right) action of $A$ on itself by left (resp. right) multiplications;

2) $\epsilon(h) = h(1_A)$, $\forall h \in H$;

3) The map $T_2 : H \otimes H \to \text{Hom}_k(A \otimes A, A)$ given by

$$T_2(h_1 \otimes h_2)(a \otimes b) = h_1(a)h_2(b)$$

induces a well-defined map, still denoted by $T_2$, from $H \otimes A$ to $\text{Hom}_k(A \otimes A, A)$. We have

$$h(ab) = (T_2 \circ \Delta)(h)(a \otimes b) = h_{(1)}(a)h_{(2)}(b), \quad \forall h \in H, \ a, b \in A. \hfill (9)$$

**Proof.** Statement 1) follows from the fact that $\epsilon : H \to A$ is a $(A, A)$-bimodule map, i.e., for all $a \in A$ and $h \in H$,

$$\epsilon(\alpha(a)h) = a\epsilon(h), \quad \epsilon(\beta(a)h) = \epsilon(h)a.$$  

Statement 2) follows from the definition of the action $T_1$. Since $T_2$ maps elements in the right ideal $I_2$ given by (8) to zero, it induces a well-defined map from $H \otimes A$ to $\text{Hom}_k(A \otimes A, A)$. To see the second part of statement 3), we first prove the following:

$$h\alpha(a) = \alpha(h_{(1)}(a))h_{(2)}, \quad \forall a \in A, h \in H. \hfill (10)$$

Since $\Delta(1) = 1 \otimes 1$ and since $\Delta$ is a left $A$-module map, we have

$$\Delta(\alpha(a)) = \alpha(a) \otimes 1, \quad a \in A.$$

Thus

$$\alpha(a) \otimes 1 \otimes 1 - 1 \otimes \alpha(a) \otimes 1 \in \ker \Phi,$$
where $\Phi$ is given by (6). Since $\ker \Phi$ is a left ideal in $H \otimes \bar{H} \otimes \bar{H}$, we have

$$h\alpha(a) \otimes 1 \otimes 1 - h \otimes \alpha(a) \otimes 1 \in \ker \Phi, \quad h \in H.$$ 

Hence

$$\Delta(h\alpha(a)) = h_{(1)} \alpha(a) \otimes h_{(2)}.$$ 

Applying (7), we get

$$\alpha(\epsilon(h_{(1)} \alpha(a))) h_{(2)} = h\alpha(a).$$ 

But

$$\epsilon(h_{(1)} \alpha(a)) = h_{(1)}(a)$$

by definition. Therefore we get (10). Consequently, we have

$$h(ab) = (h\alpha(a))(b) = h_{(1)}(a) h_{(2)}(b).$$

Q.E.D.

**Remark 2.5** Note how the condition that $\ker \Phi$ be a left idea in $H \otimes \bar{H} \otimes \bar{H}$ and the compatibility condition of $\epsilon$ and $\Delta$ are used in the above proof.

**Remark 2.6** Apart from the right $A$-module structure $\rho$ on $H$ given by (3), there is another right $A$-module structure on $H$ given by

$$H \otimes A \longrightarrow H : h \otimes a \longmapsto h\alpha(a).$$

Equation (10) should be thought of as expressing this right $A$-module structure in terms of the left $A$-module structure $\lambda$ as defined in (3). Similarly, we have

$$h\beta(a) = \beta(h_{(2)}(a)) h_{(1)}, \quad \forall a \in A, \quad h \in H,$$

which expresses the left $A$ module structure on $H$ given by

$$A \otimes H \longrightarrow H : a \otimes h \longmapsto h\beta(a)$$

in terms of the right $A$-module map $\rho$.

We think of (3) as the “product” rule for the action $T_1$ of $H$ on $A$ with respect to the algebra structure of $A$. Thus, roughly speaking, a bi-algebroid over $A$ is an algebra which acts on $A$ in such a way that 1) it contains as part of it the left and right actions of $A$ on itself by left and right multiplications, and 2) it obeys a product rule specified by the co-product of the bi-algebroid.
3 Two examples of bi-algebroids: $A \otimes A^{op}$ and $\text{End}_k(A)$

We now give two examples of bi-algebroids.

**Example 3.1** Let $A$ be an arbitrary algebra and let $H = A \otimes A^{op}$, i.e., $H$ is the direct product algebra of $A$ and its opposite. Then there is a bi-algebroid structure on $H$ over $A$ with the following structure maps:

1) the source map $\alpha : A \to H : a \mapsto a \otimes 1$, and the target map $\beta : A \to H : a \mapsto 1 \otimes a$;
2) the co-product

\[ \Delta : H \to H \otimes_A H : a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b). \]

3) the co-unit

\[ \epsilon : H \to A : a \otimes b \mapsto ab. \]

It is easy to check that these maps indeed define a Hopf algebroid structure on $H = A \otimes A^{op}$ over $A$. The associated left action of $H$ on $A$ is given by

\[ T_1 : H \to \text{End}_k(A) : T_1(a \otimes b)(c) = abc. \]

**Remark 3.2** This example is modeled on the Poisson groupoid structure on $P \times \hat{P}$ for any Poisson manifold (see [We2]), which is in turn a Poisson version of the coarse groupoid structure on $X \times X$ over any space $X$. Thus we call $A \otimes A^{op}$ the “coarse Hopf algebroid” of $A$.

**Remark 3.3** We will show in Section 4 that in addition to the bi-algebroid structure, there is actually a Hopf algebroid structure on $A \otimes A^{op}$.

**Example 3.4** Let $A$ be any finite dimensional algebra over $k$. Let $H = \text{End}_k(A)$ be the algebra of $k$-linear maps from $A$ to itself. We will show that there is a bi-algebroid structure on $H$ over $A$. We need the following Lemma.

**Lemma 3.5** Let $A$ and $H$ be as above. For any $a \in A$, let $\alpha(a) \in H$ and $\beta(a) \in H$ be respectively the maps from $A$ to itself defined by the left and right multiplications by $a$. Then the right ideal $I_2$ of $H \otimes H$ given by $[4]$ coincides with the kernel of the map

\[ T_2 : H \otimes H \to \text{Hom}_k(A \otimes A, A) : T_2(h_1 \otimes h_2)(a \otimes b) = h_1(a)h_2(b). \]
Thus we get a natural identification between $H \otimes_A H$ and the space $\text{Hom}_k(A \otimes A, A)$. In general, for any positive integer $n$, the map

$$T_n : H^\otimes n \rightarrow \text{Hom}_k(A^\otimes n, A) : T_n(h_1 \otimes \cdots \otimes h_n)(a_1 \otimes \cdots \otimes a_n) = h_1(a_1) \cdots h_n(a_n)$$

gives a natural identification of the $(A, A)$-bimodules $H \otimes_A \cdots \otimes_A H$ and $\text{Hom}_k(A^\otimes n, A)$.

**Proof.** Clearly, $I_2 \subset \ker T_2$. Conversely, let $\{a_s\}$ be a basis of $A$ and let $\{x_s\}$ be its dual basis for $A^*$. For each $s = 1, 2, \ldots, \dim A$, let $h_s \in \text{End}_k(A)$ be defined by

$$h_s(a) = (a, x_s)1_A.$$

Then any $\sum_i h_i' \otimes h_i'' \in \ker T_2$ can be written as

$$\sum_i h_i' \otimes h_i'' = \sum_i \sum_{s=1}^{\dim A} \left( \beta(h_i'(a_s)) \otimes 1 - 1 \otimes \alpha(h_i'(a_s)) \right) (h_s \otimes h_i'').$$

Thus $\sum_i h_i' \otimes h_i'' \in I_2$. Hence $I_2 = \ker T_2$. For a general $n \geq 2$, recall that the space $H \otimes_A H \otimes_A \cdots \otimes_A H$ is the quotient of $H^\otimes n$ by the right ideal $I_n$ given in (B). Clearly, $I_n \subset \ker T_n$. Conversely, any $\sum_i h_i' \otimes h_i'' \otimes \cdots \otimes h_i^{(n)} \in \ker T_n$ can be written as

$$\sum_i h_i' \otimes h_i'' \otimes \cdots \otimes h_i^{(n)} = \sum_i \sum_{j=1}^{n-1} \sum_{1 \leq s_1, \ldots, s_j \leq \dim A} (h_{j; s_1, \ldots, s_j})(h_{s_1} \otimes \cdots \otimes h_{s_j} \otimes h_i^{(j+1)} \otimes \cdots \otimes h_i^{(n)}),$$

where for $1 \leq s_1, \ldots, s_j \leq \dim A$,

$$h_{j; s_1, \ldots, s_j} = \frac{1 \otimes \cdots \otimes 1}{j-1} \left( \beta(h_i'(a_{s_1})h_i''(a_{s_2}) \cdots h_i^{(j)}(a_{s_j})) \otimes 1 \right) \otimes 1 \otimes \cdots \otimes 1 - \frac{1 \otimes \cdots \otimes 1}{j-1} \left( 1 \otimes \alpha(h_i'(a_{s_1})h_i''(a_{s_2}) \cdots h_i^{(j)}(a_{s_j})) \right) \otimes 1 \otimes \cdots \otimes 1 \in I_n.$$

Thus

$$\sum_i h_i' \otimes h_i'' \otimes \cdots \otimes h_i^{(n)} \in I_n.$$ 

Hence $I_n = \ker T_n$. Consequently, the space $H \otimes_A H \otimes_A \cdots \otimes_A H$ can be identified with the space $\text{Hom}_k(A^\otimes n, A)$. Since $T_n$ is an $(A, A)$-bimodule map, this is an identification of $(A, A)$-bimodules.
We now define the bi-algebroid structure on $End_k(A)$ over $A$. Define

$$\Delta : H \rightarrow H \otimes_A H \cong Hom_k(A \otimes A, A)$$

by

$$\Delta(h)(a \otimes b) = h(ab),$$

and

$$\epsilon : H \rightarrow A$$

by

$$\epsilon(h) = h(1_A).$$

Then it is straightforward to check that the maps $\alpha, \beta, \Delta$ and $\epsilon$ define a bi-algebroid structure on $H = End_k(A)$ over $A$. Here we only prove the compatibility of the product and the co-product on $H = End_k(A)$. Identify

$$H \otimes_A H \cong Hom_k(A \otimes A, A)$$

by the map $T_2$. The map $\Phi$ given in (6) now becomes

$$\Phi : H \otimes H \otimes H \rightarrow Hom_k(A \otimes A, A) : \Phi(h_1 \otimes h_2 \otimes h_3)(a \otimes b) = h_1(h_2(a)h_3(b)).$$

Let $\{a_s\}$ be a basis of $A$, and let $\{x_s\}$ be the dual basis for $A^*$. Then the algebra structure on $A$ corresponds to the element

$$m = a_s a_t \otimes x_s \otimes x_t \in A \otimes A^* \otimes A^*$$

under the natural identification of $A \otimes A^* \otimes A^*$ with $Hom_k(A \otimes A, A)$. The kernel of $\Phi$ is now the annihilator in $H \otimes H^{op} \otimes H^{op}$ of the element $m \in A \otimes A^* \otimes A^*$ with respect to the left representation of $H \otimes H^{op} \otimes H^{op}$ on $A \otimes A^* \otimes A^*$, where $H^{op}$ acts on $A^*$ by dualizing the action of $H = End_k(A)$ on $A$. Hence $ker \Phi$ is a left ideal in $H \otimes H^{op} \otimes H^{op}$.

**Definition 3.6** A bi-algebroid morphism from a bi-algebroid $(H_1, A_1, \alpha_1, \beta_1, \Delta_1, \epsilon_1)$ to another bi-algebroid $(H_2, A_2, \alpha_2, \beta_2, \Delta_2, \epsilon_2)$ consists of an algebra morphism $T : H_1 \rightarrow H_2$ and an algebra morphism $t : A_1 \rightarrow A_2$ which commute with all the structure maps.
Our previous discussion leads immediately to the following proposition:

**Proposition 3.7** Assume that $A$ is a finite dimensional algebra. Then for any bi-algebroid $H$ over $A$, there is a bi-algebroid morphism from $H$ to $End_k(A)$ given by the map $T_1$ defined by (8) and the identity map on $A$.

**Example 3.8** When $A$ is a finite-dimensional simple algebra, the map

$$T_1 : A \otimes A^{op} \to End_k(A) : T_1(a \otimes b)(c) = acb$$

defines an algebra isomorphism. Thus in this case, the two bi-algebroids $A \otimes A^{op}$ and $End_k(A)$ over $A$ are isomorphic.

## 4 Hopf algebroids

Naturally, Hopf algebroids should be bi-algebroids with antipodes.

**Definition 4.1** A Hopf algebroid is a bi-algebroid $H$ over an algebra $V$ with structure maps $\alpha, \beta, \Delta$ and $\epsilon$ together with a bijective map $\tau : H \to H$, called the antipode map, which has the following properties:

1) $\tau$ is an algebra anti-isomorphism for $H$;
2) $\tau \beta = \alpha$;
3) $m_H(\tau \otimes id) \Delta = \beta \epsilon \tau : H \to H$, where $m_H$ denotes the multiplication of $H$;
4) there exists a linear map $\gamma : H \otimes H \to H \otimes A H$ with the following properties:
   4a) $\gamma$ is a section for the natural projection $p : H \otimes H \to H \otimes A H$;
   4b) the following identity holds:

$$m_H(id \otimes \tau) \gamma \Delta = \alpha \epsilon : H \to H.$$  

**Remark 4.2** 1) The map $m_H(\tau \otimes id) \Delta$ in the formula in 3) is well-defined as a map from $H$ to $H$, but the map $m_H(id \otimes \tau) \Delta$ is not well-defined. This is why we need to require the existence of the section $\gamma$.

2) In general, $\epsilon \tau \neq \epsilon$, as can be seen in Example 4.4.

3) In [Ma], Maltsiniotis studies the case when $A$ is commutative and when the images of $V$ under $\alpha$ and $\beta$ lie in the center of $H$. In this case, his definition coincides with ours.
Proposition 4.3  There exists an algebra automorphism $\theta$ for $V$ such that

$$\tau \alpha = \beta \theta. \quad (13)$$

**Proof**  We know that

$$\Delta \alpha(v) = \alpha(v) \otimes 1.$$  

Apply the identity $m_H(\tau \otimes \text{id})\Delta = \beta \epsilon \tau$ to $\alpha(v)$. We get

$$\tau(\alpha(v)) = \beta \epsilon \tau \alpha(v).$$

Set

$$\theta(v) = \epsilon \tau \alpha(v) \in V. \quad (14)$$

Then

$$\tau(\alpha(v)) = \beta(\theta(v)).$$

Since $\tau$ is an algebra anti-isomorphism for $H$ and since $\beta$ is injective, we know that $\theta$ is an algebra automorphism for $V$.

 Q.E.D.

**Example 4.4**  For any algebra $A$, let $H = A \otimes A^{op}$ be the bi-algebroid over $A$ as described in Example 3.1. Define $\tau : H \to H$ by

$$\tau(a \otimes b) = b \otimes a.$$  

Then $H = A \otimes A^{op}$ is a Hopf algebroid over $A$ with antipode $\tau$. The induced algebra isomorphism $\theta$ for $A$ in this example is the identity map.

**5  The Hopf algebroid $V \# A$**

We now give a class of Hopf algebroids. They should be considered as the Hopf version of transformation groupoids.

**Theorem 5.1**  Let $A$ be a Hopf algebra and let $D(A)$ be the Drinfeld double of $A$. Let $V$ be a left $D(A)$-module algebra. Assume that the $R$-matrix of $D(A)$ acts on $V \otimes V$ in the following way:

$$m_V^{op} \circ R = m_V, \quad (15)$$
where \( m_V : V \otimes V \to V \) is the product on \( V \) and \( m_V^{\text{op}} \) is its opposite. Then there is a Hopf algebroid structure over \( V \) on the smash product algebra \( H = V \# A \), where the smash product is constructed using the left action of \( A \) on \( V \) that is the restriction to \( A \) of the action of \( D(A) \) on \( V \).

The rest of this section is devoted to the explanation and the proof of this theorem.

We first recall the definition of the Drinfeld double \( D(A) \) of a Hopf algebra \( A \). As a vector space, \( D(A) \) is isomorphic to the tensor product space \( A^* \otimes A \). As a coalgebra, it has the direct product coalgebra structure of \( A^* \) and \( A \), where \( A^* \) is \( A^* \) with the opposite co-product. Its algebra structure is defined as follows: for \( x \otimes a \) and \( y \otimes b \) in \( D(A) \), their product in \( D(A) \) is given by

\[
D(A) \ni (x \otimes a)(y \otimes b) = x (a_{(1)} \triangleright y_{(2)}) \otimes (a_{(2)} \triangleleft y_{(1)}) b,
\]

where

\[
A \otimes A^* \to A^* : a \otimes x \mapsto a \triangleright x = a_{(1)} \rightarrow x \leftarrow S^{-1}(a_{(2)})
\]

is the left co-adjoint representation of \( A \) on \( A^* \) and

\[
A \otimes A^* \to A : a \otimes x \mapsto a \triangleleft x = S^{-1}(x_{(1)}) \rightarrow a \leftarrow x_{(2)}
\]

is the right co-adjoint representation of \( A^* \) on \( A \). Here we are using the standard notion \( \rightarrow \) and \( \leftarrow \) to denote the left and right representations of \( A \) and \( A^* \) on each other given as follows: for \( a \in A \) and \( x \in A^* \),

\[
a \rightarrow x = x_{(1)} \langle x_{(2)}, a \rangle \quad (16)
\]

\[
x \leftarrow a = x_{(2)} \langle a, x_{(1)} \rangle. \quad (17)
\]

The antipode map \( S_{D(A)} \) of \( D(A) \) is given by

\[
S_{D(A)}(x \otimes a) = S(a)S^{-1}(x).
\]

Notice that the natural inclusions

\[
A \hookrightarrow D(A) : a \mapsto 1 \otimes a
\]

\[
A^* \hookrightarrow D(A) : x \mapsto x \otimes 1
\]
are Hopf algebra morphisms.

The $R$-matrix for $D(A)$ is given by

$$R = (1 \otimes a_t) \otimes (x_t \otimes 1) \in D(A) \otimes D(A)$$  \hspace{1cm} (18)

where $\{a_t\}$ is a basis for $A$, and $\{x_t\}$ is its dual basis for $A^*$. Condition (15) now reads as

$$x_t(u) \ a_t(v) = vu, \quad \forall u, v \in V.$$  \hspace{1cm} (19)

We also recall that $V$ is said to be a left $D(A)$-module algebra if the action of $D(A)$ on $V$ satisfies

$$d(uv) = d_1(u) \ d_2(v), \quad \forall d \in D(A), \ u, v \in V.$$  

In particular, since the inclusion

$$A \rightarrow D(A) : a \mapsto 1 \otimes a$$

is a Hopf algebra inclusion, we can restrict the action to an action of $A$ on $V$, making $V$ into a left $A$-module algebra. The smash product algebra $H = V \# A$ is, by definition, the algebra structure on the vector space $V \otimes A$, whose elements are now denoted by $v \# a$, given by

$$(v \# a)(u \# b) = va(1)(u) \# a(2)b.$$  

This algebra is defined in such a way that the map

$$T_1 : H = V \# A \rightarrow \text{End}_k(V) : T_1(v \# a)(u) = va(u)$$  \hspace{1cm} (20)

defines a left representation of $H$ on $V$.

Before describing the Hopf algebroid structure on $H = V \# A$ over $V$, we first give an example of this Theorem.

**Example 5.2** The following left action of $D(A)$ on $A^*$ makes $A^*$ into a left $D(A)$-module algebra:

$$D(A) \ni x \otimes a : y \mapsto x(2) (a \rightarrow y) S^{-1}(x(1)), \quad y \in A^*.$$  \hspace{1cm} (21)
Moreover, for this action, Condition (15) is satisfied. Indeed, we have, for any $x, y \in A^*$,
\[
x_t(y)a_t(x) = (x_t)_2yS^{-1}(x_t)_1(a_t \rightarrow x)
\]
\[
= x_syS^{-1}(x_t)((a_t a_s) \rightarrow x)
\]
\[
= x_syS^{-1}(x_t)x_{(1)}(x_{(2)}, a_t)\langle x_{(3)}, a_s\rangle
\]
\[
= x_{(3)}yS^{-1}(x_{(2)})x_{(1)}
\]
\[
= xy.
\]

Notice that as a Hopf subalgebra of $D(A)$, $A^{*_{coop}}$ acts on $A^*$ from the left by
\[
A^{*_{coop}} \otimes A^* \rightarrow A^* : x \otimes y \mapsto ad_x y =: x_{(2)} y S^{-1}(x_{(1)}).
\] (22)

It makes $A^*$ into a left $A^{*_{coop}}$-module algebra. We call it the **left adjoint representation** of $A^{*_{coop}}$ on $A^*$.

As a Hopf subalgebra of $D(A)$, $A$ acts on $A^*$ by
\[
A \otimes A^* \rightarrow A^* : a \otimes x \mapsto a \rightarrow x = x_{(1)}\langle x_{(2)}, a\rangle.
\] (23)

It makes $A^*$ into a left $A$-module algebra. We call it the **left regular representation** of $A$ on $A^*$. The corresponding smash product algebra $H = A^* \# A$ is, by definition, the Heisenberg double of $A^*$. Thus our theorem describes a Hopf algebroid structure over $A^*$ on its Heisenberg double.

Returning to Theorem 5.1, we now describe the Hopf algebroid structure on $H = V \# A$.

- The source map is defined by
  \[
  \alpha : V \rightarrow H : v \mapsto v \# 1.
  \]
  It is clearly an algebra homomorphism.

- To define the target map $\beta : V \rightarrow H$, we first restrict the action of $D(A)$ on $V$ to a left action of $A^*$ on $V$. Since the inclusion
  \[
  A^{*_{coop}} = (A^{op})^* \rightarrow D(A) : x \mapsto x \otimes 1
  \]
is a Hopf algebra inclusion, \( V \) becomes a left \( A^{*\text{coop}} \cong (A^{op})^* \)-module algebra. The corresponding co-module map, which will be the target map so we denote it by \( \beta \):

\[
\beta : V \rightarrow V \otimes A^{op} : \beta(v) = x_t(v) \otimes a_t
\]

is an algebra homomorphism from \( V \) to the direct product algebra \( V \otimes A^{op} \).

**Lemma 5.3** As a map from \( V \) to the algebra \( H = V \# A \), the map \( \beta \) has the following properties:

1) \( \alpha(v)\beta(u) = \beta(u)\alpha(v) \) for all \( u, v \in V \). In fact, this is equivalent to Condition (15);
2) It is an algebra anti-homomorphism from \( V \) to \( H = V \# A \).

**Proof** 1) For \( u, v \in V \), we have, on the one hand,

\[
\alpha(v)\beta(u) = (v\#1)(x_t(u)\#a_t) = vx_t(u)\#a_t,
\]

and on the other hand,

\[
\beta(u)\alpha(v) = (x_t(u)\#a_t) (v\#1) = x_t(u)(a_t)(1)(v)(a_t)(2) = (x_s x_t)(u) \ a_s(v)\#a_t = x_s(x_t(u)) \ a_s(v)\#a_t.
\]

By Condition (15), we have

\[
\beta(u)\alpha(v) = vx_t(u)\#a_t = \alpha(v)\beta(u).
\]

It is easy to see that conversely 1) implies Condition (15).

2) Using Condition (15) again, we have, for all \( u, v \in V \),

\[
\beta(u)\beta(v) = (x_t(u)\#a_t) (x_s(v)\#a_s) = x_t(u)(a_t)(1) (x_s(v))\#(a_t)(2)a_s = (x_v x_\xi)(u) \ a_v \ (x_s(v))\#a_\xi a_s = x_s(x_t(u)) \ a_s(v)\#a_s = (x_t)(2)(v)(x_t)(1)(u)\#a_t = x_t(vu)\#a_t = \beta(vu).
\]

Thus \( \beta \) is an algebra anti-homomorphism from \( V \) to \( H = V \# A \).
• The coproduct: The co-product should be a map from $H$ to $H \otimes_A H$. In our case, the latter is the quotient space of $H \otimes H$ by the subspace spanned by the following subset:

$$\{ \beta(v)(v_1 \# a_1) \otimes (v_2 \# a_2) - (v_1 \# a_1)(v v_2 \# a_2) : v, v_1, v_2 \in V, a_1, a_2 \in A \}.$$ 

Thus each element in $H \otimes_A H$ is uniquely represented by an element of the form $\sum (v_i \# a_i) \otimes (1 \# b_i)$. In other words, we have a map

$$\gamma : H \otimes_A H \rightarrow H \otimes H : (v_1 \# a_1) \otimes (v_2 \# a_2) \mapsto \beta(v_2)(v_1 \# a_1) \otimes (1 \# a_2) \quad (24)$$

which is a section for the natural projection from $H \otimes H$ to $H \otimes_A H$.

Define $\Delta : H \rightarrow H \otimes_A H$ by

$$\Delta(v \# a) = (v \# a(1)) \otimes (1 \# a(2)). \quad (25)$$

The associativity of the co-product for $A$ implies that $\Delta$ is co-associative. It is also easy to see that $\Delta$ is a $(V, V)$-bimodule map. It remains to prove the compatibility of the co-product and the product of $H$.

We first observe that

$$\gamma(H \otimes_A H) = \{ h \otimes (1 \# b) : h \in H, b \in A \}$$

is a subalgebra of $H \otimes H$. Moreover, the map

$$\gamma \circ \Delta : H \rightarrow \gamma(H \otimes_A H)$$

is an algebra homomorphism. Thus the following defines a left $H$-module structure on $\gamma(H \otimes_A H)$:

$$(v \# a) \cdot (h \otimes 1 \# b) = \Delta(v \# a)(h \otimes 1 \# b) = (v \# a(1))h \otimes (1 \# a(2))b.$$ 

On the other hand, by identifying $\gamma(H \otimes_A H)$ with $H \otimes_A H$, the right $(H \otimes H)$-module structure on $H \otimes_A H$ now becomes the following right $(H \otimes H)$-module structure on $\gamma(H \otimes_A H)$:

$$(h \otimes 1 \# b) \cdot (h_1 \otimes (v_2 \# a_2)) = \beta(a(2)(v_2))h h_1 \otimes (1 \# b(2))a(2).$$
These two module structures on $\gamma(H \otimes_A H)$ commute. Indeed, let $v\#a$, $h_1$, $h_2 = v_2\#a_2 \in H$. We have

\[
((v\#a) \cdot (h \otimes 1\#b)) \cdot (h_1 \otimes h_2) = \beta \left( a_2(b_1(v_2)) \right) (v\#a_1) hh_1 \otimes (1\#a_3b_2a_2) = v\beta \left( a_2(b_1(v_2)) \right) (1\#a_1) hh_1 \otimes (1\#a_3b_2a_2) = v(1\#a_2) \beta(b_1(v_2)) hh_1 \otimes (1\#a_3b_2a_2) = (v\#a) \cdot ((h \otimes 1\#b) \cdot (h_1 \otimes h_2)).
\]

We used 1) in Lemma 5.3 and the following fact in deriving the above identities:

**Lemma 5.4** For any $u \in V$ and $a \in A$,

\[
\beta(a_2(u))(1\#a_1) = (1\#a) \beta(u).
\] (26)

**Proof of Lemma 5.4.** By the definition of the map $\beta$, we have

\[
\beta(a_2(u))(1\#a_1) = x_s(a_2(u)) \# a_s a_1 \in V \otimes A
\]

and

\[
(1\#a) \beta(u) = a_1(x_s(u)) \# a_2 a_s \in V \otimes A.
\]

Regarding both elements as in $Hom_k(A^*, V)$, we see that (26) is equivalent to

\[
x_1(a_2(u)) a_1 = a_1 x_2(a_2(u)), \quad \forall a \in A, x \in A^*.
\] (27)

**Q.E.D.**

We now return to the proof of the compatibility of the co-product $\delta$ and the product of $A$. We need to show that the kernel of the map

\[
\Phi : H \otimes H \otimes H \rightarrow H \otimes_A H : h \otimes h_1 \otimes h_2 \mapsto (\Delta h)(h_1 \otimes h_2)
\]
is a left ideal in $H \otimes H^{op} \otimes H^{op}$. Let $h \otimes h_1 \otimes h_2 \in H \otimes H \otimes H$ and let $\sum h_i \otimes h_i' \otimes h_i'' \in \ker \Phi$. Then

$$\Phi(\sum h_i \otimes h_i' \otimes h_i'' \otimes h_1 \otimes h_2) = \sum (\gamma \circ \Delta)(hh_i) \cdot (h_i' \otimes h_i'' \otimes h_1 \otimes h_2)$$

$$= \sum (\Delta(h) \cdot (\gamma \circ \Delta)(h_i)) \cdot (h_i' \otimes h_i'' \otimes h_1 \otimes h_2)$$

$$= \sum \Delta(h) \cdot \left( (\gamma \circ \Delta)(h_i) \cdot (h_i' \otimes h_i'' \otimes h_1 \otimes h_2) \right)$$

$$= \Delta(h) \cdot \left( \Phi(\sum h_i \otimes h_i' \otimes h_i'' \otimes h_1 \otimes h_2) \right)$$

$$= 0.$$

Here we have just used the fact that the left $H$-module structure and the right $H \otimes H^{op}$-module structure on $\gamma(H \otimes A)$ commute. Hence $\ker \Phi$ is a left ideal of $H \otimes H^{op} \otimes H^{op}$.

• The co-unit: define

$$\epsilon : H \rightarrow V : \epsilon(v \# a) = \epsilon(a)v. \quad (28)$$

Then it satisfies all the requirements for a co-unit. For example, the kernel of $\epsilon$ is the space of annihilators in $H$ of the identity element $1_v$ of $V$ under the left action $T_1$ of $H$ on $V$. Therefore it is a left ideal of $H$.

We have thus obtained a bi-algebroid structure on the smash-product algebra $H = V \# A$ over the algebra $V$. The associated left action of $H$ on $V$ is the one given by (21).

• The antipode: introduce a special element $d_0 \in D(A)$ given by

$$d_0 = S^2(a_t)x_t \in D(A). \quad (29)$$

We now define the antipode map $\tau$ by

$$\tau : H \rightarrow H : \tau(v \# a) = (1 \# S(a)) \beta(d_0(v)). \quad (30)$$

**Lemma 5.5** The element $d_0 \in D(A)$ has the following properties:

1) It is invertible in $D(A)$ and $d_0^{-1} = S^{-1}(a_t)x_t$. 

2) For any $x \otimes a \in D(A)$ we have

$$S^2_{D(A)}(x \otimes a) = d_0(x \otimes a)d_0^{-1},$$

where $S_{D(A)}$ is the antipode map for $D(A)$ given by $S_{D(A)}(x \otimes a) = S(a)S^{-1}(x)$.

3) $\Delta d_0 = (R^{21}R)(d_0 \otimes d_0)$, where $R$ is the $R$-matrix for $D(A)$ given by (18) and $R^{21}$ is its flip, i.e.,

$$R^{21} = (x_t \otimes 1) \otimes (1 \otimes a_t) \in D(A) \otimes D(A).$$

4) $d_0$ acts as an algebra isomorphism on $V$.

**Proof.** The first three properties of $d_0$ were proved in [Mt] [Dr]. We only prove 4). Condition (15) implies, on the one hand, $m^{op}_V = m_V \circ R^{-1}$, and on the other hand, $m^{op}_V = m_V \circ R^{21}$. Therefore, we have

$$m_V = m_V \circ (R^{21}R).$$

Hence, using the “product rule” for the action of $D(A)$ on $V$, we get, for any $u, v \in V$,

$$d_0(uv) = m_V \Delta d_0(u \otimes v) = m_V \circ R^{21}R(d_0(u) \otimes d_0(v)) = d_0(u)d_0(v).$$

Thus $d_0$ acts as an isomorphism on $V$.

Q.E.D.

**Lemma 5.6**

1) $\tau$ is an algebra anti-isomorphism for $H$, and $\tau^{-1} : H \to H$ is given by

$$\tau^{-1}(v \# a) = (1 \# S^{-1}(a)) \beta(v) = (1 \# S^{-1}(a)) (x_t(v) \# a_t).$$

2) $\tau \beta = \alpha$ and $\tau \alpha = \beta d_0$.

3) $m_H(\tau \otimes id) \Delta = \beta \epsilon \tau$.

4) $m_H(id \otimes \tau) \gamma \Delta = \alpha \epsilon$, where $\gamma : H \otimes_A H \to H \otimes H$ is given by (24).

**Proof.** 1) To prove that $\tau$ is an algebra anti-isomorphism, it is sufficient to prove the following: for any $v \in V$ and $a \in A$:

$$\tau((1 \# a)(v \# 1)) = \tau(v \# 1) \tau(1 \# a).$$

23
Now

\[ \text{lhs} = \tau(a_{(1)}(v) \# a_{(2)}) \]
\[ = (1 \# S(a_{(2)})) \beta(d_0 a_{(1)}(v)) \]
\[ = \beta(S(a_{(2)})d_0 a_{(1)}(v)) S(a_{(3)}). \]

Using (31), we have

\[ S(a_{(2)})d_0 a_{(1)} = S(a_{(2)})S^2(a_{(1)})d_0 = \epsilon(a_{(1)})d_0. \]

Thus

\[ \text{lhs} = \beta(d_0(v)) S(a) = \text{rhs}. \]

The proof of 3) uses Identity (26). The proof of \( \tau a = \beta d_0 \) is trivial, and so is the proof of 4). We now give the proof of \( \tau \beta = \alpha \). Let \( v \in V \) be arbitrary. Then

\[ \tau \beta(v) = \tau(x_t(v) \# a_t) \]
\[ = (1 \# S(a_t)) \beta(d_0 x_t(v)) \]
\[ = (1 \# S(a_t)) ((x_s d_0 x_t)(v) \# a_s) \]
\[ = (S(a_\eta)x_s d_0 x_\xi x_\eta)(v) \# S(a_\xi)a_s. \]

By (31),

\[ d_0 x_\xi = S^2(x_\xi)d_0. \]

Moreover,

\[ x_s S^{-2}(x_\xi) \otimes S(a_\xi)a_s = 1 \otimes 1 \in A^* \otimes A. \]

This can be seen by pairing both sides with arbitrary elements \( a \otimes x \in A \otimes A^* \). Thus

\[ \tau \beta(v) = (s(a_\eta)d_0 x_\eta)(v) \# 1 \]
\[ = (S(a_\eta)S^{-2}(x_\eta)d_0)(v) \# 1. \]

But

\[ S(a_\eta)S^{-2}(x_\eta) = S^2_{D(A)}(S^{-1}(a_\eta)x_\eta) = S^2_{D(A)}(d_0^{-1}) = d_0^{-1}. \]

Hence \( \tau \beta(v) = v \# 1 = \alpha(v) \).

Q.E.D.
We have finished the proof of Theorem 5.1.

**Corollary 5.7** The algebra automorphism \( \theta \) on \( V \) as defined in Proposition 4.3 is given by the action of \( d_0 \) on \( V \).

The proof of the following Proposition is straightforward.

**Proposition 5.8** Let \( V, A \) and \( H = V \# A \) be as in Theorem 5.1. The left representation of \( H \) on \( V \) associated to the Hopf algebroid structure on \( H \) (see Definition 2.3) is the one given by (20).

### 6 The Hopf algebroid \( H(A^*) \)

In this section, we study in more details the Hopf algebroid \( H(A^*) \) for any finite dimensional Hopf algebra \( A \).

The representation \( T_1 \) of \( H(A^*) \) on \( A^* \) defined by (20) now becomes

\[
T_1 : H(A^*) : \rightarrow \text{End}_k(A^*) : T_1(x\#a)(y) = x(a \rightarrow y).
\]

The following Proposition says that \( T_1 \) is an algebra isomorphism from \( H(A^*) \) to \( \text{End}_k(A^*) \).

**Proposition 6.1** For any \( \phi \in \text{End}_k(A^*) \),

\[
T_1 \left( \phi(x_s)x_t \# S^{-1}(a_t)a_s \right) = \phi,
\]

where \( \{a_t\} \) is a basis for \( A \) and \( \{x_t\} \) is its dual basis for \( A^* \).

**Proof.** For any \( x \in A^* \),

\[
T_1 \left( \phi(x_s)x_t \# S^{-1}(a_t)a_s \right)(x) = \phi(x_s)x_t \left( S^{-1}(a_t)a_s \rightarrow x \right)
\]

\[
= \phi(x_s)x_t x_{(1)} \langle S^{-1}(a_t), x_{(2)} \rangle \langle a_s, x_{(3)} \rangle
\]

\[
= \phi(x_{(3)})S^{-1}(x_{(2)})x_{(1)}
\]

\[
= \phi(x).
\]

Q.E.D.
Corollary 6.2 The map
\[ D(A) \to H(A^*) : a \# x \mapsto x(a \to x_t)S^{-1}(x_{(1)})x_s \# S^{-1}(a_s)a_t \] (34)
is an algebra homomorphism.

The structure maps for the Hopf algebroid \( H(A^*) \) are now given by
1) \( \alpha : A^* \to H(A^*) : x \mapsto x\#1; \)
2) \( \beta : A^* \to H(A^*) : x \mapsto x_sxx_t \# S^{-1}(a_t)a_s; \)
3) \( \epsilon : H(A^*) \to A^* : x\#a \mapsto \epsilon(a)x; \)
4) \( \Delta : H(A^*) \to H(A^*) \otimes A^* H(A^*) : x\#a \mapsto x\#a_{(1)} \otimes 1\#a_{(2)}; \)
5) \( \tau : H(A^*) \to H(A^*) : x\#a \mapsto (S(a) \to (x_sS^{-2}(x))) x_t \# S^{-1}(a_t)a_s. \)

7 An example of a quantum groupoid associated to \( SL_q(2) \)

Let \( d > 1 \) be an odd integer. Let \( q \) be a root of unity of order \( d \). Let \( A \) be the Hopf algebra with two generators \( E \) and \( K \) and relations:
\[
K = q^2EK, \quad K^d = 1, \quad E^d = 0;
\]
\[
\Delta K = K \otimes K, \quad \Delta E = 1 \otimes E + E \otimes K;
\]
\[
S(K) = K^{-1}, \quad S(E) = -EK^{-1};
\]
\[
\epsilon(K) = 1, \quad \epsilon(E) = 0.
\]

It is easy to see that the set \( \{E^mK^n\}_{m,n=0,1,...,d-1} \) is a basis for \( A \), so \( \dim A = d^2 \).

The dual Hopf algebra \( A^* \) of \( A \) is the Hopf algebra with two generators \( \kappa \) and \( \eta \) and relations
\[
\kappa \eta = q^{-2}\eta \kappa, \quad \kappa^d = 1, \quad \eta^d = 0;
\]
\[
\Delta \kappa = \kappa \otimes \kappa, \quad \Delta \eta = \eta \otimes 1 + \kappa \otimes \eta;
\]
\[
S(\kappa) = \kappa^{-1}, \quad S(\eta) = -\kappa^{-1}\eta;
\]
\[
\epsilon(\kappa) = 1, \quad \epsilon(\eta) = 0.
\]

Again, the set \( \{\eta^i\kappa^j\}_{i,j=0,1,...,d-1} \) forms a basis for \( A^* \), with the pairing between \( A \) and \( A^* \) given by
\[
\langle \eta^i\kappa^j, E^mK^n \rangle = \delta_{mi}(i)q^2lq^2j(i+n), \quad (35)
\]
where
\[(i)_{q^2} = \frac{q^{2i} - 1}{q^2 - 1}\]
and
\[(i)_{q^2}! = (i)_{q^2} (i - 1)_{q^2} \cdots (2)_{q^2} (1)_{q^2}.\]

By definition,
\[(0)_{q^2} = 0, \quad (0)_{q^2}! = 1.\]

In particular,
\[\langle \kappa, E^m \rangle = \delta_{m0}, \quad \langle \kappa, K^n \rangle = q^{2n},\]
\[\langle \eta, E^m \rangle = \delta_{m1}, \quad \langle \eta, K^n \rangle = 0.\]

The Drinfeld Double \(D(A)\) of \(A\) has four generators \(E, K, \eta\) and \(\kappa\) with the following relations:
\[
\begin{align*}
K^d &= 1, \quad \kappa^d = 1, \quad E^d = 0, \quad \eta^d = 0; \\
KE &= q^2 \kappa, \quad \kappa \eta = q^{-2} \eta \kappa; \\
K \kappa &= \kappa K, \quad E \kappa = q^{-2} \kappa E, \\
K \eta &= q^{-2} \eta K, \quad E \eta = q^{-2} (-1 + \eta E + \kappa K); \\
S(K) &= K^{-1}, \quad S(\kappa) = \kappa^{-1}, \quad S(E) = -EK^{-1}, \quad S(\eta) = -\eta \kappa^{-1}; \\
\Delta K &= K \otimes \kappa, \quad \Delta \kappa = \kappa \otimes \kappa, \\
\Delta E &= 1 \otimes E + E \otimes K, \quad \Delta \eta = 1 \otimes \eta + \eta \otimes \kappa; \\
\epsilon(K) &= \epsilon(\kappa) = 1, \quad \epsilon(E) = \epsilon(\eta) = 0.
\end{align*}
\]

The \(R\)-matrix for \(D(A)\) is given by
\[
R = \frac{1}{d} \sum_{0 \leq m, n, j \leq d-1} \frac{1}{(m)_{q^2}} q^{-2j(m+n)} E^m K^n \otimes \eta^m \kappa^j \in D(A) \otimes D(A). \quad (36)
\]

The left regular representation of \(A\) on \(A^*\) (see Example 5.2) is given by
\[
\begin{align*}
K &\rightarrow (\eta^i \kappa^j) = q^{2j} \eta^i \kappa^j \\
E &\rightarrow (\eta^i \kappa^j) = (i)_{q^2} q^{2(j-i+1)} \eta^i \kappa^j.
\end{align*}
\]
In particular,
\[ K \rightarrow \kappa = q^2 \kappa, \quad K \rightarrow \eta = \eta; \]
\[ E \rightarrow \kappa = 0, \quad E \rightarrow \eta = \kappa. \]
This makes \( A^* \) into a left \( A \)-module algebra.

The left adjoint representation of \( A^{*\text{coop}} \) on \( A^* \) (see Example 5.2) is given by
\[
\mathrm{ad}_\kappa(\eta^i \kappa^j) = q^{-2i} \eta^i \kappa^j \\
\mathrm{ad}_\eta(\eta^i \kappa^j) = (1 - q^{-2j}) \eta^{i+1} \kappa^{j-1}.
\]
In particular,
\[
\mathrm{ad}_\kappa \kappa = \kappa, \quad \mathrm{ad}_\kappa \eta = q^{-2} \eta; \\
\mathrm{ad}_\eta \kappa = (1 - q^{-2}) \eta, \quad \mathrm{ad}_\eta \eta = 0.
\]
This action makes \( A^* \) into a left \( A^{*\text{coop}} \)-module algebra.

The left regular representation of \( A \) on \( A^* \) and the left adjoint representation of \( A^{*\text{coop}} \) on \( A^* \) combine to give a left action of \( D(A) \) on \( A^* \) (see Example 5.2). This makes \( A^* \) into a left \( D(A) \)-module algebra. Thus by Theorem 5.1, there is a Hopf algebroid structure over \( A^* \) on the Heisenberg double of \( A^* \).

The Heisenberg double \( H(A^*) = A^* \# A \) of \( A^* \) is the algebra with four generators, again denoted by \( E, K, \kappa \) and \( \eta \) with relations:
\[
KE = q^2 EK, \quad K^d = 1, \quad \kappa^d = 0; \\
\kappa \eta = q^{-2} \eta \kappa, \quad \kappa^d = 1, \quad \eta^d = 0; \\
K \kappa = q^2 \kappa K, \quad K \eta = \eta K, \quad E \kappa = \kappa E, \quad E \eta = \eta E + \kappa K.
\]
It has a natural left representation \( T_1 \) on \( A^* \) given by \( \{k, \eta\} \). Namely, under \( T_1 \), \( A \) acts on \( A^* \) by the left regular representatin and \( A^* \) acts on \( A^* \) by left multiplications.

The Hopf algebroid structure on \( H(A^*) \) over \( A^* \) is the following:
1) The source map
\[
\alpha : A^* \rightarrow H(A^*) = A^* \# A : \quad \alpha(k) = k \# 1 \\
\alpha(\eta) = \eta \# 1;
\]
2) The target map
\[ \beta : A^* \longrightarrow H(A^*) = A^* \# A : \quad \beta(k) = k + (1 - q^{-2}) \eta EK^{-1} \]
\[ \beta(\eta) = \eta K^{-1}. \]

3) Identify the space \( H \otimes_A H \) with the space spanned by the set
\[ \{ \eta^i \kappa^j E^m K^n \otimes E^r K^s : i, j, m, n, r, s = 0, \ldots, d - 1 \}, \]
then the co-product on \( H(A^*) \) is given by
\[ \Delta : H(A^*) \longrightarrow H \otimes_A H : \quad \eta^i \kappa^j E^m K^n \mapsto \sum_r \left( \begin{array}{c} m \\ r \end{array} \right) q^2 \eta^i \kappa^j E^m K^n \otimes E^r K^{m+n-r}. \]

4) The antipode map
\[ \tau : H(A^*) \longrightarrow H(A^*) : \quad \tau(k) = q^2 k + (q^2 - 1) \eta EK^{-1} \]
\[ \tau(\eta) = \eta K^{-1} \]
\[ \tau(K) = K^{-1} \]
\[ \tau(E) = -EK^{-1}. \]

5) The co-unit map
\[ \epsilon : H(A^*) = A^* \# A \longrightarrow A^* : \quad \epsilon(\eta^i \kappa^j E^m K^n) = \delta_{m0} \eta^i \kappa^j. \]

6) The induced algebra automorphism \( \theta \) on \( A^* \) as defined in Proposition 4.3 is given by
\[ \theta = q^2 S^{-2} : \quad \theta(k) = q^2 k \]
\[ \theta(\eta) = \eta. \]

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