Degenerated ground-states in a spin chain with pair interactions: a characterization by symbolic dynamics

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Abstract. In this paper we study a class of one-dimensional spin chain having a highly degenerated set of ground-state configurations. The model consists of spin chain having infinite-range pair interactions with a given structure. We show that the set of ground-state configurations of such a model can be fully characterized by means of symbolic dynamics. Particularly we found that the set ground-state configurations define what in symbolic dynamics is called sofic shift space. Finally we prove that this system has a non-vanishing residual entropy (the topological entropy of the shift space), which can be exactly calculated.

Keywords: rigorous results in statistical mechanics, solvable lattice models
1. Introduction

It has long been known that several models of spin systems might have a highly degenerated set of ground-state configurations at a critical magnetic field [1]. One of the earlier examples of the occurrence of such a phenomenon was given by Domb [1], who found that in a one-dimensional spin chain with nearest-neighbor interactions at a critical magnetic field the entropy does not vanish at zero temperature [1]. Bonner and Fisher [2] proved later that this system exhibits a large degeneracy of the ground state configurations, and particularly they found that the set of the configurations minimizing the energy grows exponentially with the size of the system. Within the language of thermodynamical formalism, this classical example is known to have a ground-state described by the so-called golden mean shift [3]. The importance of this phenomenon is due to the fact that it violates the third law of thermodynamics in the sense that the entropy of the system must vanish at the zero absolute temperature. The occurrence of such a phenomenon, in spin systems particularly [4–8], led to some authors reexamining such a basic principle [9–11]. Nowadays it is accepted that a thermodynamical system can have a non-zero entropy at zero absolute temperature, which is referred to as residual entropy [11]. In fact, the residual entropy is related directly to the degeneracy of the set ground-state configurations of the system. Within the context of the thermodynamic formalism, it is known that a spin system on a regular lattice can be viewed as a symbolic dynamical system endowed with a certain function characterizing the interactions among spins. Within this setting, we will call ground-state the set of all configurations that minimize the energy of the system, or, in other words the minimal compact set containing the support of the zero-temperature limit of the Gibbs measure of the system (if such a limit does exists). Within the language of symbolic dynamics, the ground-state as defined here can be seen as a shift space, i.e. as a subset of the set of all spin configurations that is invariant under the shift mapping [12].

Spin systems having finite-range two-body interactions have in general a ground-state which turns out to be a finite union of subshifts of finite type [12, 13]. In general,
spin systems might have ground-states which are not necessarily subshifts of finite type. For example, in [14] it has been proved that a discrete spin system with infinite-range four-body interactions has a non-periodic ground-state. Indeed the authors of [15] proved that such a ground-state is a Thue–Morse subshift whose residual entropy is zero. More generally one can build spin systems, with an artificial form of spin interactions, whose ground-states can be actually any subshift space [16], but with spin interactions that are not, necessarily, physically realistic.

It is also worth mentioning some works also related to ours. Recently, Bruin and Leplaideur [17, 18] have studied the so-called freezing phase transitions in one-dimensional regular lattices. These kinds of transitions are characterized by the existence of a finite critical inverse temperature $\beta_c$ at which the system reaches the ground-state. Above $\beta_c$ the system no longer changes thus maintaining itself at the ground-state. Particularly interesting is the fact that the ground-state of the system reached at a finite temperature is a minimal subshift such as the one obtained by the Fibonacci substitution or the Thue–Morse subshift.

In this work we provide an example of a spin system with infinite-range pair interactions having a ground-state which turns out to be a strictly sofic subshift. The system that we introduce here and its ground-state has some interesting properties. Particularly, our model has two-body (arbitrarily fast decaying) interactions. The set of ground-state configurations can be fully characterized through symbolic dynamics techniques, and the residual entropy can be exactly determined. Recall that a sofic subshift is defined as a factor of a subshift of finite type [19]. Thus any subshift of finite type is sofic. A strictly sofic subshift is therefore a sofic subshift that is not of finite type. The main difference between strictly sofic subshift and a subshift of finite type is that the later can be characterized by a finite set of forbidden words, while the former is characterized by an infinite set of forbidden words [20].

This work is organized as follows. In section 2 we state the model, we give some basic definitions on symbolic dynamics and state the notation that we will use throughout this work. In section 3 we state the main results of this work and finally in section 4 we give the proof of our results.

2. Setting and generalities

2.1. Preliminary concepts

In this section we introduce some concepts and notations of symbolic dynamics, which will be used throughout this work. All the definitions and well-known results stated here about theory on symbolic dynamics can be found in standard textbooks on the subject, such as [20, 21]. First let $\mathcal{A}$ be a finite set, to which we will refer to as alphabet. We denote by $\mathcal{A}^\mathbb{Z}$ the set of all infinite symbolic sequences made up from elements of the alphabet $\mathcal{A}$. We will also refer to $\mathcal{A}^\mathbb{Z}$ as the full shift. An infinite symbolic sequence $x \in \mathcal{A}^\mathbb{Z}$ is also written as $x = \ldots x_{-1}x_0x_1\ldots$, where each $x_j$ is an element from the alphabet $\mathcal{A}$. Each $x_j$ in $x$ is referred to as a coordinate of $x$ or as the $j$th coordinate of $x$ if we would like to emphasize its location along $x$. The $j$th coordinate of the symbolic sequence $x$ is alternatively written as $(x)_j := x_j$. We call $\mathcal{A}^\mathbb{N}$ the set of all finite symbolic sequences...
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made up of \( n \) symbols. A finite string \( a = a_0a_1 \ldots a_{n-1} \in \mathcal{A}^n \) will also be referred to as a \textit{word} or as a \textit{block} of size \( |a| = n \). If \( x \in \mathcal{A}^\mathbb{Z} \) is a symbolic sequence we denote the finite string \( x_jx_{j+1} \ldots x_{j+n-1} \) as \( x_{j+n-1}^{j+1} \). Clearly \( x_{j+n-1}^{j+1} \) is an element of \( \mathcal{A}^n \). We say that a word \( a \in \mathcal{A}^n \) \textit{occurs} in \( x \in \mathcal{A}^\mathbb{Z} \) if there is a \( j \in \mathbb{Z} \) such that \( x_{j+n-1}^{j+1} = a \). We also say that \( x \) has as \textit{prefix} \( a \) if \( a \) occurs in \( x \) for \( j = 0 \). The \textit{concatenation} of two words \( a \in \mathcal{A}^n \) and \( b \in \mathcal{A}^m \) is an operation, denoted as a \textit{multiplication} between words, that gives a new word \( c := ab \) in the set \( \mathcal{A}^{n+m} \) for any \( n, m \in \mathbb{N} \). It is clear that this ‘multiplication’ of words is not commutative, since in general \( ab \neq ba \). The \textit{exponentiation} of words should be understood as a concatenation of a word with itself, i.e. if \( a \in \mathcal{A}^0 \) then \( a^m = aaa \cdots a \) (\( m \) times) is a word in the set \( \mathcal{A}^m \). The exponentiation of a word to the 0th power, \( a^0 \), will be defined as the empty word by convenience.

Given a word \( a \in \mathcal{A}^n \) a word of size \( n \) we define a \textit{cylinder set} (or simply a \textit{cylinder}) \([a]\) as the subset of \( \mathcal{A}^\mathbb{Z} \) containing all the symbolic sequences having as prefix the word \( a \), i.e.

\[
[a] := \{ x \in \mathcal{A}^\mathbb{Z} : x_0^{n-1} = a \}.
\]

The collection of all cylinder sets forms a basis for a topology for \( \mathcal{A}^\mathbb{Z} \) which makes \( \mathcal{A}^\mathbb{Z} \) a compact space. Actually, the cylinder sets are closed and open sets within this topology.

The \textit{shift mapping} \( T : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \) is a function that shifts the sequence to the left with respect to the index. In other words, if \( x \in \mathcal{A}^\mathbb{Z} \) is a point on the full shift, then \( T(x) \) is another point on \( \mathcal{A}^\mathbb{Z} \) such that \( (T(x))_j = x_{j+1} \) for all \( j \in \mathbb{Z} \).

If \( Y \subseteq \mathcal{A}^\mathbb{Z} \), we say that \( (Y, T) \) is a shift space if it is invariant and closed set under the shift mapping \( T \). The full shift \( \mathcal{A}^\mathbb{Z} \) itself is a shift space. A shift space (or subshift) \( Y \) can also be defined through a collection of \textit{forbidden words} in the sequences contained in \( Y \). We denote the set of forbidden words for \( Y \) as \( F(Y) \). In particular, for the full shift \( \mathcal{A}^\mathbb{Z} \), the set of forbidden words is the empty set. Those shift spaces having a finite collection of forbidden words are said to be of \textit{finite type}. The class of shift spaces that we will consider here are known as \textit{sofic shifts}. A shift space \( Y \) is said to be \textit{sofic} if there is a shift of finite type \( \hat{Y} \) and if there is an onto mapping \( \alpha : \hat{Y} \to Y \) such that the 0th coordinate of \( \alpha(a) \) depends on a finite number of coordinates of \( a \), for all \( a \in \hat{Y} \), and \( \alpha \circ \hat{T} = T \circ \alpha \) where \( \hat{T} \) and \( T \) stands for the shift mapping on \( \hat{Y} \) an \( Y \) respectively. Then, a sofic shift is \textit{factor} of a shift of finite type. It can be show that sofic shifts can also be characterized by an infinite set of forbidden words.

A different way to characterize a shift space is through the concept of \textit{language}. Let \( Y \) be a subshift space, then, the \textit{language} \( \mathcal{B}(Y) \) for \( Y \) is the collection of all the words occurring in any element of \( Y \). Particularly we denote by \( \mathcal{B}_n(Y) \) the set of all the words of size \( n \) occurring in any element of \( Y \). In this way we can say that \( \mathcal{B}(Y) = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n(Y) \). Finally, a different way to characterize a sofic subshift is through the concept of \textit{follower sets} [20]. Given a word \( a \in \mathcal{B}_n(Y) \) in the language of a shift space \( Y \), the \textit{follower set} for \( a \) is the set \( \mathcal{F}(a) \subseteq \mathcal{B}(Y) \) containing all the words \( b \in \mathcal{B}(Y) \) such that the \( ab \) is contained in the language, i.e. \( ab \in \mathcal{B}(Y) \). Let \( \mathcal{S}(Y) \) be the family of all the different follower sets for word in \( \mathcal{B}(Y) \). It is known that a subshift \( Y \) is sofic if and only if \( \mathcal{S}(Y) \) is finite.

**Example 2.1.** To exemplify the concepts given above let us consider the set \( X_F \subset \{0, 1\}^\mathbb{Z} \) defined as follows,
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\[ X_F := \{ x \in \{0, 1\}^\mathbb{Z} : x_i x_{i+1} \neq 11 \}. \]

The set \( X_F \) is a simple example of a subshift of finite type. This subshift is called the golden mean shift (or Fibonacci shift) because it has a close relation with the Fibonacci numbers and the golden mean. The Fibonacci shift is clearly of finite type because the set of forbidden words is finite and contains only the word 11, \( F(X_F) = \{11\} \). It can be shown that the cardinality of the sets of admitted words is \( \#B_n(X_F) = F_{n+1} \) where \( F_n \) is the sequence of Fibonacci numbers, i.e. \( F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5 \ldots \). On the other hand, let us consider the set \( Y_e \subset \{0, 1\}^\mathbb{Z} \) defined as the set of all binary sequences so that between any two 1s there are an even number of 0s. The set \( Y_e \) is known as the even shift. The even shift can be characterized by a set of infinite forbidden words given by \( F(Y_e) := \{10^{2n+1} : n \in \mathbb{N}\} \). It can be show also that \( Y_e \) has only three different follower sets, which makes it a strict sofic subshift [20].

2.2. The spin chain model with a sofic ground-state

In statistical mechanics an interaction potential among spins on a one-dimensional lattice is a family of shift-invariant functions \( \Phi = (\Phi_\Lambda)_{\Lambda \subset \mathbb{Z}} \) indexed by finite subsets \( \Lambda \) of \( \mathbb{Z} \) [22]. On the other hand, within the thermodynamical formalism, is a potential which describes the interactions on a given spin system. A potential is a function \( \psi : \Sigma \to \mathbb{R} \) on the set of all configurations of infinite spin chains \( \Sigma := \{+, -\}^\mathbb{Z} \) to the real line. Of course, these two concepts are related each other [22]. Given a family of interactions on a one-dimensional spin system \( \Phi \), the potential \( \psi \) is obtained as follows,

\[ \psi = \sum_{\Lambda \ni 0} \frac{\Psi_\Lambda}{\#\Lambda} \]

where \( \#\Lambda \) stands for the cardinality of \( \Lambda \). In this work we will adopt the last point of view, i.e. our spin system will be determined by a potential function in the sense of the thermodynamical formalism [23].

Let us denote by \( \sigma = (\ldots \sigma_1 \sigma_0 \sigma_1 \ldots) \) an element of \( \Sigma \). Thus, \( \sigma \in \Sigma \) represents the infinitely long spin chain and we can think of every coordinate of \( \sigma \) as a spin variable. We assume that every spin interacts with the rest of spins via a set of symmetric pair interactions. An infinite-range pair interaction potential on this spin chain can be defined as a function on the set of all spin configurations to the real line, \( \psi : \Sigma \to \mathbb{R} \), which can be written with certain generality as

\[ \psi(\sigma) := H\sigma_0 + \sum_{j=1}^{\infty} K(j)\sigma_0\sigma_j. \] (1)

The first term in the above equation, \( H\sigma_0 \) represents the interaction of the zeroth spin with an external magnetic field. Here \( K(j) \) is the coupling constant for the interaction between the 0th spin and the \( j \)th spin on the chain, or more generally, \( K(j) \) represents the interaction between any two spins separated by \( j \) spin lattice sites. Notice that the summation only runs on \( \mathbb{N} \) by symmetry of the interactions; once we take into account the interaction of the 0th with the \( j \)th spin, it is unnecessary to take into account the interaction of the \( j \)th with the 0th spin, since when we calculate the total energy of a
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The summability of the interactions is important because it implies that $\psi$ depends continuously on $\sigma$ when $\{+1, -1\}$ is given the discrete topology and $\Sigma$ the product topology [24].

A **coboundary** is a function $f : \Sigma \to \mathbb{R}$ such that there is a function $g : \Sigma \to \mathbb{R}$ for which, $f = g \circ T$. It is know that two potentials which differ in a coboundary are physically equivalent in the sense that they define the same Gibbs measure and have the same set of ground-state configurations [23–26]. It is said that two potentials that differs in a coboundary are **cohomologous**.

We can use the concept of coboundary to simplify our problem by means of a potential on a different shift space with equivalent ground-state. First, recall that the spin variables $\sigma$ along the infinite spin chain can take the values $+1$ and $-1$. Let us define new variables $x_i$ taking values in the set $\{0, 1\}$ through the transformation

$$x_i = r(\sigma_i) := \frac{\sigma_i + 1}{2}. \tag{3}$$

It is clear that the symbol ‘0’ corresponds to the spin state $-1$ and the symbol ‘1’ corresponds to the spin state $+1$ of the spin variable. The above transformation on a single symbol (or a single coordinate) can be extended coordinate-wise to the whole spin chain. This extended transformation, which we will denote by $R : \{-1, +1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$, establishes an isomorphism between $\Sigma$ and $\{0, 1\}^\mathbb{Z}$. If we introduce this ‘change of variable’ in the potential (1) we obtain a new potential $\phi : \{0, 1\}^\mathbb{Z} \to \mathbb{R}$ on the full shift $X := \{0, 1\}^\mathbb{Z}$. Thus, given a symbolic sequence $\mathbf{x} = \cdots x_{-1}x_0x_1 \cdots \in X$, the potential $\tilde{\phi}(\mathbf{x})$ is defined as,

$$\tilde{\phi}(\mathbf{x}) := \psi \circ R^{-1}(\mathbf{x}), \tag{4}$$

which is explicitly given by,

$$\tilde{\phi}(\mathbf{x}) = Hr^{-1}(x_0) + \sum_{j=1}^{\infty} K(j)r^{-1}(x_0)r^{-1}(x_j),$$

$$= H(2x_0 - 1) + \sum_{j=1}^{\infty} K(j)(2x_0 - 1)(2x_j - 1),$$

$$= 2Hx_0 - H + \sum_{j=1}^{\infty} K(j)(4x_0x_j - 2(x_0 + x_j) + 1). \tag{5}$$

Performing some additional calculations we obtain,

$$\tilde{\phi}(\mathbf{x}) = 2Hx_0 - H + \sum_{j=1}^{\infty} 4K(j)x_0x_j - 2K_0x_0 - 2 \sum_{j=1}^{\infty} K(j)x_j + K_0,$$

or equivalently

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\[ \hat{\phi}(x) = 2(H - K_0)x_0 + \sum_{j \in \mathbb{Z}} 4K(j)x_0x_j - 2 \sum_{j \in \mathbb{Z}} K(j)x_j + (K_0 - H), \]

where \( K_0 \) is a constant defined as

\[ K_0 := \sum_{j=1}^{\infty} K(j). \]

Now, let us define a new potential \( \phi \) which will be physically equivalent to the potential \( \hat{\phi} \). These two potentials differ in a coboundary as follows,

\[ \phi = \tilde{\phi} + \xi - \xi \circ T \] (6)

where \( \xi : X \to \mathbb{R} \) is a continuous and real valued function on \( X \). By convenience, we chose the function \( \xi \) such that it can be written as,

\[ \xi(x) := 2 \sum_{j=0}^{\infty} L(j)x_j, \]

where

\[ L(j) := -\sum_{n=j+1}^{\infty} K(n). \]

Notice that this function defines the coboundary,

\[ \xi(x) - \xi \circ T(x) = 2L(0)x_0 + \sum_{j=1}^{\infty} 2(L(j) - L(j-1))x_j, \]

\[ = -2K_0x_0 + \sum_{j=1}^{\infty} 2K(j)x_j. \] (7)

where we have identified \( L(0) \) with the constant \( K_0 \). Thus, if we add this coboundary to the potential \( \tilde{\phi} \) we obtain,

\[ \phi(x) := 2(H - 2K_0)x_0 + (K_0 - H) + \sum_{j=1}^{\infty} 4K(j)x_0x_j. \] (8)

If we chose the magnetic field \( H \) at the critical value \( H_c := 2K_0 \), we have that the potential becomes

\[ \phi(x) = -\frac{H_c}{2} + \sum_{j=1}^{\infty} J(j)x_0x_j, \] (9)

where we have defined the coupling constant \( J(j) \) as \( J(j) := 4K(j) \). This result means that the original spin model at the critical magnetic field \( H_c \) is physically equivalent to a symbolic chain with lattice variables having one of the two possible states, ‘0’ or ‘1’, with interactions given by the potential (9). We should also emphasize that the constant term \( H_c/2 \) in \( \phi \) can be neglected since any constant term added to a given potential defines the same Gibbs measure as the original potential thus having the same set of ground-state configurations. In this way, two potentials which differ in a constant can be considered as physically equivalent.
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It is clear that in order for the potential $\phi$ to be continuous it is necessary to verify the summability condition on $\phi$. Indeed, a direct calculation shows that the summability for $\psi$ (which states that $s(\psi) < \infty$) implies the summability for $\phi$. However, in order for the coboundary to be continuous we also need to verify the summability condition for $\xi$. The last condition leads us to impose that the coupling constants $J(j)$ also satisfy,

$$
\sum_{j=0}^{\infty} L(j) \leq \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} |K(k)| = \frac{1}{4} \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} |J(k)| < \infty.
$$

It is clear that the above condition automatically implies that $s(\psi) < \infty$, by which we will only need to assume the last one without loss of generality. We state this condition explicitly below.

**Condition 1.** We consider potentials $\phi : \{0, 1\}^Z \to \mathbb{R}$, defined as

$$
\phi(x) = \sum_{j=1}^{\infty} J(j)x_0x_j,
$$

satisfying that

$$
\sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} |J(k)| < \infty.
$$

It is important to mention that, despite what we originally consider spin systems with site variables in $\{+1, -1\}$, which we transformed into a lattice with site variables in $\{0, 1\}$, it is also natural to consider the last setting. This is because many physical systems have natural site variables in $\{0, 1\}$ from the beginning. For instance, a system of particles moving on a regular lattice (a lattice gas model), the ‘0’ stands for ‘empty’ site and the ‘1’ stands for ‘filled’ site (occupied by one particle). In this case the condition 1 can be replaced by the summability condition on the order for the ground-state to be well defined as the support of a Gibbs measure at zero temperature.

3. Main result

In the rest of this work we will denote by $X$ the binary full shift, $X := \{0, 1\}^Z$. Let $\phi : X \to \mathbb{R}$ be a potential on $X$. Given $p \in \mathbb{N}$ and $x \in X$ we define $S_p\phi(x)$ as the partial ergodic sum,

$$
S_p\phi(x) := \sum_{j=0}^{p-1} \phi \circ T^j(x).
$$

If $\phi(x)$ is interpreted as the interaction energy of the 0th spin with all the other spins in the chain, then, the function $S_p\phi(x)$ can be interpreted as the total energy of a block of spins $x_0x_1 \ldots x_{p-1}$ of size $p$. In this way we can think of $S_p\phi$ as the equivalent to what in statistical physics is called the *Hamiltonian* of a block of spins of size $p$. In the same line of interpretation, we can say that $S_p\phi(x)/p$ corresponds to the mean energy per spin in the referred block of spins.

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Let us also call \( \text{Per}_p(X) \subset X \) to the set of all the periodic points \( x \in X \), under \( T \), of period \( p \). This set is equivalent to the set of all infinite sequences which can be seen as a block of size \( p \) infinitely repeated along the symbolic sequence,

\[
\text{Per}_p := \{ x \in X : T^p(x) = x \}.
\]

We also define the \textit{minimal mean energy per spin} \( \phi_{\text{min}} \) as \([12, 27]\),

\[
\phi_{\text{min}} := \inf \left\{ \min \left\{ \frac{S_p \phi(x)}{p} : x \in \text{Per}_p(X) \right\} : p \in \mathbb{N} \right\}.
\]

Now we proceed to define what we will call ground-state in the sense described above.

**Definition 3.1.** Let \( \phi : X \to \mathbb{R} \) be a potential on the full shift \( X \). The ground-state of the potential \( \phi \) (also referred to as the \( \phi \)-minimizing subshift) \( \mathcal{X}(\phi) \), is defined as

\[
\mathcal{X}(\phi) := \text{clos} \left\{ \bigcup_{p \in \mathbb{N}} \{ x \in \text{Per}_p(X) : S_p \phi(x) = p \phi_{\text{min}} \} \right\}.
\]

It is clear from the above definition that the \( \phi \)-minimizing sets are \( T \)-invariant, and then, by the closure action, shift spaces. With this definition of ground-states we can establish the following result.

**Theorem 3.1.** Let \( \phi : X \to \mathbb{R} \) be a potential on the full shift \( X \) defined as

\[
\phi(x) := \sum_{j=1}^{\infty} J(j) x_0 x_j,
\]

satisfying condition 1, with coupling constants of the form,

\[
J(j) := \begin{cases} 0 & j = kq \text{ for all } k \in \mathbb{N} \\ \text{positive} & \text{otherwise.} \end{cases}
\]

for some \( q \in \mathbb{N} \). Then, the ground-state for \( \phi \) is given by,

\[
\mathcal{X}(\phi) = \text{clos} \left( \bigcup_{n \in \mathbb{N}} \mathcal{Y}^{(q)}_n \right),
\]

i.e. \( \mathcal{X}(\phi) = \mathcal{Y}^{(q)} \). Here the set \( \mathcal{Y}^{(q)}_n \) is defined as,

\[
\mathcal{Y}^{(q)}_n := \{ x \in \text{Per}_n(X) : x_{j + kq + l} = 10^{kq+l-1}, \text{ for } 1 \leq l \leq q-1, \forall k \in \mathbb{N}_0 \text{ and } \forall j \in \mathbb{Z} \}.
\]

The above theorem states that the model we propose has a highly degenerated ground-state and gives its explicit form by characterizing the corresponding shift space through an infinite set of forbidden words. Actually, each set \( \mathcal{Y}^{(q)}_n \) can be seen as defined through a set of ‘forbidden words’. The forbidden words turn out to be of the form \( 10^m \) for all \( m \) such that \( m = kq + l - 1 \) with \( 1 \leq l \leq q-1 \) and \( k \in \mathbb{N}_0 \). Thus, taking the closure of the union of all the sets \( \mathcal{Y}^{(q)}_n \) we obtain a shift space with a set forbidden
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words which turns out to be infinite. Actually, the set of forbidden words for $Y(q)$ can be written as

$$ F(Y_n(q)) = \{10^{kq+m-1} : \forall k \in \mathbb{N}_0, \forall 1 \leq m \leq q - 1 \}. $$

Moreover, we will see below that the set of forbidden words can be seen, alternatively, as

$$ \tilde{F}(Y_n(q)) = \{1a1 : \forall a \in A^{kq+m-1}, \forall k \in \mathbb{N}_0, \forall 1 \leq m \leq q - 1 \}. $$

Then, it becomes clear that the resulting ground-state is not of finite type because its set of forbidden words is not finite. However, this is not enough to say that the ground-state is sofic. Actually, to state the sofic property for $X(\phi)$ we need to characterize explicitly the (finite) family of follower sets.

**Theorem 3.2.** The ground-state $X(\phi) = Y(q)$ is sofic and all its follower sets are given by

$$ \mathcal{F}(0^q) := B(Y(q)), $$
$$ \mathcal{F}(0^m10^{q-m-1}) := \bigcup_{n \in \mathbb{N}} C_{n,m}^{(q)}, \text{ for } 0 \leq m \leq q - 1 $$

where the sets $C_{n,m}^{(q)}$ are defined as

$$ C_{n,m}^{(q)} := \{a = a_0a_1a_2 \ldots a_{n-1} \in B_n(Y(q)) : a_i = 0 \forall i \neq kq + m, \forall 0 \leq k \leq \lfloor n/q \rfloor \}. $$

The family of follower sets allows us to build up the corresponding right-resolving presentation for the sofic subshift $X(\phi) = Y_n(q)$. Then, the corresponding graph associated to the subshift $Y(q)$ allows us to calculate explicitly the topological entropy of the system [20].

**Corollary 3.1.** The residual entropy of the ground-state $X(\phi)$ is $h_{\text{top}}(X(\phi)) = \log(2)/q$.

Finally it is particularly important to observe what happens in the case $q = 1$. As we see from the definition of $Y(q)$, the set of forbidden words is given by

$$ F(Y_n(q)) = \{10^{kq+m-1} : \forall k \in \mathbb{N}_0, \forall 1 \leq m \leq q - 1 \} $$

from which it follows that, for $q = 1$, no word can be in $F(Y_n(q))$ implying that $F(Y_n(q)) = \emptyset$. Thus, it follows that $Y(q) = X$ for $q = 1$. This is consistent with the fact that, for the case $q = 1$ the potential reduces to $\phi = 0$, whose ground-state is the full shift because any configuration minimized the mean energy. Therefore the corresponding topological entropy is the one of the full shift, i.e. $h_{\text{top}}(X(\phi)) = \log(2)$.

4. Proof of the main result

Before giving the proof of the main theorem, let us state two results which will allow us to understand better the structure of the subshift $Y(q)$ as well as the ground-state for $\phi$. 

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Lemma 4.1. The minimal mean energy per spin for $\phi$ is $\phi_{\min} = 0$.

Proof. Notice that if we take $x \in \text{Per}_n(X)$, then $\phi_{\min} \leq S_0\phi(x)/n$. Take $x = 0^\infty$. Then, it is clear that $\phi_{\min} \leq 0$. On the other hand it is clear that $\phi_{\min} \geq \min\{\phi(x) : x \in X\}$. However we have that $\min\{\phi : x \in X\} = 0$. This implies the result.

The above lemma states that the mean energy per spin for the ground-state associated to $\phi$ is zero. Therefore, all the $n$-periodic points $x$ belonging to the ground-state should satisfy that $S_0\phi(x) = 0$. This fact will be used below for the proof of the main theorem.

Lemma 4.2. If $n$ is not a multiple of $q$, then $Y^{(q)}_n = \{0^\infty\}$.

Proof. To prove this statement, first notice that $0^\infty \in Y^{(q)}$, since in $0^\infty$ does not occur any word of the type $10^k + l - 11$. Thus, $\{0^\infty\} \subseteq Y^{(q)}$. On the other hand, let us assume that there is a periodic point $x \in Y^{(q)}_n$ with period $n = k_0q + m_0$, with $1 \leq m_0 \leq q - 1$, for which there is a $j_0 \in Z$ such that $x_{j_0} = 1$. By periodicity we have that $x_{j_0 + k_0q} = 1$. Then notice that $x$ contains a word of the form $1a_01$ with $|a_0| = k_0q + m_0 - 1$ for $1 \leq m_0 \leq q - 1$, specifically, $x_{j_0 + k_0q + m_0} = 1a_01$. But, by definition of $Y^{(q)}_n$, no word of the form $10^k + l - 11$ occurs in $x$ for $1 \leq l \leq q - 1$ and all $k \in N_0$. This means that $a_0$ cannot be a word of the form $0^{k_0q + m_0 - 1}$ since it is ‘forbidden’ for the elements of $Y^{(q)}_n$. Thus, there would be an integer $j_1$, with $1 < j_1 < k_0q + m_1 - 1$, such that $x_{j_1} = 1$. Then we have two words occurring in $x$ of the form $1a_11$ and $1a'_1$. Assume for the moment that the two resulting words are such that $|1a_1| = kq - 1$ and $|1a'_1| = k'q - 1$ for some $k, k' \in N_0$ then, the word $1a_01 = 1a_1a'_1$ would have a size $|1a_1| = kq - 1$ with $k = k + k'$ which clearly contradict the hypothesis. Then, at least one of these words should have the form $1a_11$ with $|a_1| = k_1q + m_1 - 1$ for $1 \leq m_1 \leq q - 1$. By the same argument as above, it is not possible that $a_1 = 0^{k_1q + m_1 - 1}$ and thus, there is a $j_2$, with $1 < j_2 < k_1q + m_1$, such that $x_{j_2} = 1$. Then, using the same argument as above, there is a word of the form $1a_21$ occurring in $x$. Repeating this argument successively we obtain a sequence of words $a_0, a_1, a_2, \ldots$ such that $|a_0| > |a_1| > |a_2| > \ldots$ Since all these words have a size of the form $kq + m_1 - 1$ for some $k \in N_0$ and some $1 \leq m_1 \leq q - 1$, then, the smallest word $a_n$ we can obtain from this procedure is $|a_n| = 0$. This means that the word $a_n$ is the empty symbol, and therefore the word $1a_n1 = 11$ occurs in $x$. This is impossible because the word $11$ is forbidden in the set $Y^{(q)}_n$ (take $k = 0$ and $m = 1$ in $10^kq + m - 1$).

Now we give a proof of the theorem 3.1.

Proof of theorem 3.1. First let us prove that $Y^{(q)}_n \subseteq \overline{X}(\phi)$. This is equivalent to proving that $Y^{(q)}_n \subseteq \overline{X}(\phi)$ for all $n \in N$ since the closure of the union of all the $Y^{(q)}_n$ should also be a subset of $\overline{X}(\phi)$ because the latter is closed.

First let us prove that $Y^{(q)}_n \subseteq \overline{X}(\phi)$ when $n$ is not a multiple of $q$. In this case, as stated in the above lemma, $Y^{(q)}_n$ has only one element $Y^{(q)}_n = \{0^\infty\}$. But $0^\infty$ is a configuration that trivially minimize the energy, since $\phi(0^\infty) = 0$.

Now let us consider the case $n = kq$, for some $k \in N$, and take an element $x \in Y^{(q)}_n$. We need to prove that $S_0\phi(x) = 0$ in order for $x$ to belong to the ground-state. Then, assume that $S_0\phi(x) > 0$. If this is the case we would have that

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\[ S_n \phi(x) = \sum_{j=0}^{n-1} \phi \circ T^j(x) > 0. \]

Recall that the function \( \phi \) is non-negative, i.e. \( \phi(x) \geq 0 \) for all \( x \in X \). Thus, we can say that there is a \( j_0 \in \mathbb{Z} \) such that \( \phi \circ T^{j_0}(x) > 0 \). Then,

\[ \phi \circ T^{j_0}(x) = \sum_{i=1}^{\infty} J(i)x_{j_0}x_{j_0+i} > 0. \]

From the above we have necessarily that \( x_{j_0} = 1 \), if not, we would have that \( \phi \circ T^{j_0}(x) = 0 \). Moreover, we also have that there is an \( i' \in \mathbb{N} \) such that \( x_{j_0+i'} = 1 \), in order to satisfy the above inequality. This integer \( i' \) cannot be chosen as a multiple of \( q \), because in this case we would have that \( \phi \circ T^{j_0}(x) = 0 \) since \( J(pq) = 0 \) for all \( p \in \mathbb{N} \) by definition. Then, such an integer can be written as \( i' = k_0q + m_0 \) for some \( 1 \leq m_0 \leq q - 1 \) and some \( k_0 \in \mathbb{N}_0 \). This means that a word of the form \( 1a_01 \) occurs in \( x \) with \( |a_0| = k_0q + m_0 - 1 \). Specifically we have that \( x_{k_0 + k_0q + m_0} = 1a_01 \). However, the word \( a_0 \) cannot be \( 0^{k_0 + m_0} \) because it is forbidden by hypothesis (just recall that \( x \in Y_n^q \)). Then we follow a similar argument to prove lemma 4.2. Since \( a_0 \) cannot be \( 0^{k_0 + m_0} \) then there is a \( j_1 \) such that \( (a_0)_{j_1} = 1 \), which implies that a word of the form \( 1a_1 \) occurs in \( x \) with \( |a_1| = k_0q + m_0 - 1 \). Specifically we have that \( x_{k_0 + k_0q + m_0} = 1a_1 \). However this is impossible, by definition of \( Y_n^q \). This proves by contradiction that necessarily \( Y_n^q \subset X(\phi) \).

Now let us prove that \( X(\phi) \subset Y_n^q \). As in the preceding case, it is enough to prove that if a \( n \)-periodic point belongs to \( X(\phi) \) then it belongs to \( Y_n^q \).

First let us consider the case in which \( n \) is not a multiple of \( q \), i.e. that \( n = kq + m \) for some \( k \in \mathbb{N}_0 \) and some \( 1 \leq m \leq q - 1 \). We know from lemma 4.2 that in this case \( Y_n^q \) has only one element. Then we need to prove that the unique \( n \)-periodic point belonging to \( X(\phi) \) is \( 0^\infty \). Assume that it is not the case and that therefore there is a \( x \in X(\phi) \) such that \( x_j = 1 \) for some \( j \in \mathbb{Z} \). By periodicity of \( x \) we can chose \( j \) to be such that \( 0 \leq j \leq n \). Moreover, by the periodicity of \( x \) we also have that \( x_{j+kq+m} = 1 \) for some \( k \in \mathbb{N}_0 \) and some \( 1 \leq m \leq q - 1 \). This means that a word of the form \( 1a1 \) occurs in \( x \) with \( |a| = kq + m - 1 \). Then notice that

\[ S_n \phi(x) = \sum_{i=0}^{n-1} \phi \circ T^i(x) > 0, \]

for some \( 0 \leq j < n \)

which in turns implies that

\[ S_n \phi(x) = \sum_{i=1}^{\infty} J(i)x_jx_{j+i} \geq J(kq + m) > 0, \]

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Degenerated ground-states in a spin chain with pair interactions: a characterization by symbolic dynamics by choosing \( i = kq + m \). The latter inequality contradicts the hypothesis that \( x \in X(\phi) \). This proves that the only \( n \)-periodic point in \( X(\phi) \) is \( 0^\infty \).

Now let us take a \( n \)-periodic point \( x \in X(\phi) \) with \( n = k \tilde{q} \) for some \( \tilde{k} \in \mathbb{N} \). Assume that \( x \) does not belong to \( Y_n^{(q)} \). The latter means that a word of the form \( 10^{kq+l-1} \) occurs in \( x \) for some \( k \in \mathbb{N}_0 \) and some \( 1 \leq l \leq q - 1 \). We do not lose generality if we assume that \( x_0^{kq+l-1} = 10^{kq+l-1} \) (we can apply the shift mapping successively until \( 10^{kq+l-1} \) be a prefix of the shifted sequence and then redefine \( x \)). Then note that

\[
S_n \phi(x) = \sum_{j=0}^{n-1} \phi \circ T^j(x) > \phi(x) = \sum_{i=1}^{\infty} J(i)x_0x_i > J(kq + l) > 0.
\]

However, the above inequality contradicts the hypothesis which says that \( x \in X(\phi) \) and therefore \( S_n \phi(x) \) should be strictly zero. This completes the proof. \( \square \)

**Remark 4.1.** We should observe that in the proof given above for theorem 3.1 we obtained an additional result concerning the set of forbidden words. Indeed, to prove that a word of the form \( 10^{kq+m-1} \) does not appear in the ground-state we needed to prove that, actually, a word of the form \( a_{11} \) is forbidden for any \( a \in X \) with \( |a| = kq + m - 1 \). Thus, the shift space \( Y^{(q)} \) can be expressed either, in terms of the set of forbidden words defined as

\[
F(Y^{(q)}) := \{10^{kq+m}1 \in X : k \in \mathbb{N}_0 \text{ and } 1 \leq m \leq q - 1\},
\]

or in terms of the set \( \tilde{F}(Y^{(q)}) \) of forbidden words defined as,

\[
\tilde{F}(Y_n^{(q)}) = \{1a_1 : \forall a \in A^{kq+m-1}, \forall k \in \mathbb{N}_0, \forall 1 \leq m \leq q - 1\}.
\]

Now we proceed to prove theorem 3.2. We first establish the follower sets for all the admitted words of size \( q \). Next, we prove that the follower sets of any other admitted word in \( Y_n^{(q)} \) must be necessarily one of the follower sets of the admitted words of size \( q \).

**Proof of theorem 3.2.** We now have that the ground-state for \( \phi \) is the subshift \( Y^{(q)} \) defined above. Notice that there are exactly \( q + 1 \) admitted words of size \( q \). This is easily seen because the forbidden words up to of size \( q \) are \( 11, 101, \ldots, 10^{q-2}1 \). Then, it follows that admitted words of size \( q \) cannot contain two ones. This implies that the set of admitted words of size \( q \) is,

\[
B_q(Y^{(q)}) = \{0^q\} \cup \{0^{q-m-1}10^m : 0 \leq m \leq q - 1\}.
\]

First we will prove that the follower sets corresponding to these words are given by equation (10). Let us start with the follower sets of \( 0^q \). Let \( n \in \mathbb{N}_0 \), then it is easy to see that any admitted word \( a \) of size \( n \) in \( Y^{(q)} \) is such that \( 0^qa \) is also an admitted word. This is because the concatenation \( 0^qa \) does not generates a forbidden words, because forbidden words \( Y^{(q)} \) are all of the form \( 10^{kq+m}1 \) for all \( k \in \mathbb{N}_0 \) and all \( 1 \leq m \leq q - 1 \) (there are no ‘ones’ in \( 0^q \) giving rise to forbidden word after the concatenation). This implies that \( B_q(Y^{(q)}) \subset F(0^q) \) for all \( n \in \mathbb{N}_0 \). The latter immediately implies that \( F(0^q) = B(Y^{(q)}) \) since by definition every follower set is a subset of \( B(Y^{(q)}) \).
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Now we will prove that
\[ \mathcal{F}(0^n10^{q-m-1}) = \bigcup_{n \in \mathbb{N}} \mathcal{C}_{n,m}^{(q)} \]

where the sets \( \mathcal{C}_{n,m}^{(q)} \) are given by,
\[ \mathcal{C}_{n,m}^{(q)} := \{ \mathbf{a} = a_0a_1a_2 \ldots a_{n-1} \in \mathcal{B}(Y^{(q)}) : a_i = 0 \ \forall \ i \neq kq + m, \ \forall \ 0 \leq k \leq \lfloor n/q \rfloor \} \).

First let us show that \( \mathcal{C}_{n,m}^{(q)} \subset \mathcal{F}(0^n10^{q-m-1}) \) for every \( n \in \mathbb{N} \) and for every \( 0 \leq m \leq q-1 \). Take an element \( \mathbf{a} \in \mathcal{C}_{n,m}^{(q)} \) and assume that the word \( 0^n10^{q-m-1}\mathbf{a} \) give rise some forbidden word. Since \( \mathbf{a} \) does not contain forbidden words, the only possibility is that the forbidden word arises due to the presence of the ‘1’ in \( 0^n10^{q-m-1} \). This means that the forbidden word should be of the form \( \mathbf{w} = 10^{q-m-1}a_0a_1a_2 \ldots a_{j-1}a_j \) for some \( 1 \leq j \leq n \). If this is the case, then it is clear that \( a_j = 1 \) and \( a_i = 0 \) for all \( 0 \leq i \leq j-1 \). Thus, the word \( \mathbf{w} \) takes the form,
\[ \mathbf{w} = 10^{q-m-1}a_0a_1a_2 \ldots a_{j-1}a_j = (10^{q-m-1})(0^j1) = 10^{q-m+j-1}1. \]

Since \( \mathbf{w} \) is forbidden, it is clear that \( q - m + j - 1 \) should be of the form \( k'q + l - 1 \), for some \( k' \in \mathbb{N}_0 \) and some \( 1 \leq l \leq q-1 \). Next have that \( q - m + j - 1 = k'q + l - 1 \) implies that
\[ j = (k' - 1)q + m + l. \] \hspace{1cm} (12)

On the other hand, since \( \mathbf{a} \) belongs to \( \mathcal{C}_{n,m}^{(q)} \) hence we have that \( a_i = 0 \ \forall \ i \neq kq + m, \ \forall \ 0 \leq k \leq \lfloor n/q \rfloor \). Due to the fact that \( a_j = 1 \), we have that there is a \( 0 \leq k \leq \lfloor n/q \rfloor \) such that \( j = kq + m \). However, in view of equation (12) we see that \( l = 0 \) contradicts the hypothesis that \( \mathbf{w} = 10^{q-m+j-1}1 = 10^{k'q+l-1}1 \) is a forbidden word. This proves that \( \mathcal{C}_{n,m}^{(q)} \subset \mathcal{F}(0^n10^{q-m-1}) \).

Now let us prove the reverse inclusion, \( \mathcal{F}(0^n10^{q-m-1}) \subset \mathcal{C}_{n,m}^{(q)} \). Take an element \( \mathbf{a} \in \mathcal{F}(0^n10^{q-m-1}) \) and assume that it is not contained in \( \mathcal{C}_{n,m}^{(q)} \). Since \( \mathbf{a} \notin \mathcal{C}_{n,m}^{(q)} \), we have that \( a_i = 1 \) for some \( i \neq kq + m \) with \( 0 \leq k \leq \lfloor n/q \rfloor \). But the concatenated word \( 0^n10^{q-m-1}\mathbf{a} \) has a word \( \mathbf{w} \) of the form
\[ 10^{q-m-1}a_0a_1a_2 \ldots a_{i-1}a_i = 10^{q-m-1}a_0a_1a_2 \ldots a_{i-1}1. \]

Because the size of the word \( 0^{q-m-1}a_0a_1a_2 \ldots a_{i-1} \) is \( q - m + i - 1 \) we have that, in order for \( \mathbf{w} \) be admitted, we require that \( q - m + i - 1 = kq - 1 \) for some \( k \in \mathbb{N} \). The latter implies that \( i = (k - 1)q + m \). This is not possible by hypothesis, since we assumed that \( \mathbf{a} \notin \mathcal{C}_{n,m}^{(q)} \), and hence \( i \neq kq + m \) for any \( 0 \leq k \leq \lfloor n/q \rfloor \).

Next we prove that given any other word in the language \( \mathcal{B}(Y^{(q)}) \) has a follower set that coincides with one of the already given for the admitted words of size \( q \).

Let \( \mathbf{a} = a_0a_1a_2 \ldots a_n \) be an admitted word, i.e. \( \mathbf{a} \in \mathcal{B}(Y^{(q)}) \). We need to prove that the follower set \( \mathcal{F}(\mathbf{a}) \) is one of the follower sets already described for the words of size \( q \).
If the word $a$ turns out to be $0^{n+1}$, it is clear that all the words in the language of $Y^\emptyset$ can be a follower of $a$. This means that $\mathcal{F}(0^n) = \mathcal{F}(0^n) = \mathcal{B}(Y^{\emptyset})$.

Now, let us consider the case in which there is a $j^*$ such that $a_{n-j^*} = 1$ for some $0 \leq j^* \leq n$. Without loss of generality, we can write $j^* = k^*q - m^* - 1$ for some $0 \leq m^* \leq q - 1$ and some $k^* \in \mathbb{N}_0$. We will prove that the follower set for $a$, $\mathcal{F}(a)$ coincides with $\mathcal{F}(0^{m^*+m^*+1})$. To prove the latter assume that the word $ab$ contains a forbidden word for some $b \in \mathcal{F}(0^{m^*+m^*+1})$. We can assume without loss of generality that the $(n - j^*)$th coordinate of $a$ and the $i^*$th coordinate of $b$, are the symbols that generate the forbidden word in the concatenation $ab$, i.e.

$$a_{n-j^*}^n b_{i^*} = 1w1,$$

where,

$$w := a_{n-j^*+1}a_{n-j^*+2} \ldots a_nb_{i^*-1}.$$

Since $1w1$ is forbidden, then, according to remark 4.1, we have that the size of $w$, which is given by $|w| = j^* + i^*$, should be of the form $kq + m - 1$ for some $k \in \mathbb{N}_0$ and some $1 \leq m \leq q - 1$. This means that $j^* + i^* = k^*q - m^* - 1$ for some $0 \leq k^* \leq \lfloor n/q \rfloor$. This last condition together with the requirement (13) implies that $m = 0$, which is not possible by hypothesis. This implies that $b$ is necessarily a follower of $a$, which shows that $\mathcal{F}(0^{m^*+m^*+1}) \subset \mathcal{F}(a)$.

Now, take an element $c \in \mathcal{F}(a)$. We will prove that $c \in \mathcal{F}(0^{m^*+m^*+1})$ if, as above, $a$ is such that $a_{n-j^*} = 1$ for some $0 \leq j^* \leq n$, with $j^* = k^*q - m^* - 1$ for some $0 \leq m^* \leq q - 1$ and some $k^* \in \mathbb{N}_0$. If there is another coordinate of $a$ is such that $a_{n-j^*} = 1$, for some $j^* > j^*$ then notice that the word $1a_{n-j^*+1}^n$ is not forbidden, which means that $|a_{n-j^*+1}^{n-j^*+1}| = j^* - j^* - 1$ should be necessarily of the form $k^*q - 1$. This implies that $j^* - j^* - 1 = j^* - k^*q + m^* = k^*q - 1$, or equivalently $j^* = (k^* - k^*)q - m^* - 1$.

A similar result is obtained under the contrary assumption, $j^* < j^*$. This result implies that the choice of $m^*$ is unique, independently of which coordinate is equal to one along $a$.

Since $c$ does not belong to $\mathcal{F}(0^{m^*+m^*+1})$ by hypothesis, we have that $c_{kq+m^*+l} = 1$ for some $k \in \mathbb{N}_0$ and some $1 \leq l \leq q - 1$. But the word $c$ is a follower of $a$, then $ac$ does not contain any forbidden word. Since $a_{n-j^*} = 1$ and $c_{kq+m^*+l} = 1$ we need to verify that the word

$$a_{n-j^*}^n c_{kq+m^*+l} = 1w1$$

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is not forbidden. In this case we have defined $w$ as $a_{n-j+1}c_{0}^{kq+m^1+l-1}$. In order for $1w1$ to be admitted we need at least that $|w| = kq + m^1 + l + j^*$ be of the form $\tilde{k}q - 1$ for some $\tilde{k} \in \mathbb{N}$. This condition is equivalent to

$$kq + m^1 + l + j^* = (k - k^*)q + l - 1 = \tilde{k}q - 1.$$ 

The latter implies that $l = 0$ or $l = k'q$ for some $k \in \mathbb{N}$. But this is impossible, since we assumed that $1 \leq l \leq q - 1$. This proves that $c$ belongs to $\mathcal{F}(0^{m^1}1^{m^1-m^1-1})$. □

Finally we calculate the topological entropy as follow. It is known that the follower sets of a given sofic subshift allows us to build up a right-resolving presentation. Actually, there is a standard way to do this [20]. The vertices of the graph are represented by the follower sets given in theorem 3.2. To label the vertex of the graph we will use the following short-hand notation,

$$C_m := \mathcal{F}(0^{m}1^{m-1}) \quad \text{for} \quad 1 \leq m \leq q,$$

$$C_0 := \mathcal{F}(0^0). \quad (14)$$

Now, take a vertex, say for example $C_m = \mathcal{F}(a)$, where $a$ is one of the admitted words of size $q$, and determine the follower sets of the words $a1$ and $a0$. The follower sets $\mathcal{F}(a1)$ and $\mathcal{F}(a0)$ are empty or belong necessarily to the family $\{ C_i : 0 \leq i \leq q \}$. Assume for the moment that such follower sets are not empty. Then there are $j,k$ such that $C_j = \mathcal{F}(a0)$ and $C_k := \mathcal{F}(a1)$. Then draw an edge from $C_m$ to $C_j$ labeled with ‘0’ and draw an edge from $C_m$ to $C_k$ labeled with ‘1’. On the other hand if either $\mathcal{F}(a1)$ or $\mathcal{F}(a0)$ is empty, then no edge corresponding to such label is drawn. Repeat this procedure for all the vertices. With this procedure we obtain the follower set graph (an edge labeled graph) $\mathcal{G}_q$ that turns out to be a right-resolving presentation [20] for sofic subshift $Y^{(q)}$. This

Figure 1. Right-resolving presentation for the sofic subshift $Y^{(q)}$, which corresponds to the ground-state for the potential $\phi_c$. 

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graph is shown in figure 1. The follower set graph allows us to write down the adjacency matrix $A_q$ corresponding to this labelled graph. The adjacency matrix for $G_q$ is given by

$$A_q = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 2 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}.$$  

(15)

It is not difficult to obtain the eigenvalues of $A_q$. After some calculations we can observe that the characteristic polynomial determining the eigenvalues can be written as,

$$(1 - \lambda)(2 - \lambda^q) = 0.$$  

The largest eigenvalue $\lambda_0$ of $A_q$ gives the topological entropy as follows (see [20]),

$$h_{top} = \log(\lambda_0) = \log(2^{1/q}) = \frac{\log(2)}{q},$$

from which it follows immediately the result stated in corollary 3.1.

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