ALMOST GLOBAL EXISTENCE FOR NONLINEAR WAVE EQUATIONS IN AN EXTERIOR DOMAIN IN TWO SPACE DIMENSIONS

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Abstract. In this paper we deal with the exterior problem for a system of nonlinear wave equations in two space dimensions, assuming that the initial data is small and smooth. We establish the same type of lower bound of the lifespan for the problem as that for the Cauchy problem, despite of the weak decay property of the solution in two space dimensions.

Keywords. Exterior problem, Nonlinear wave equation, Lifespan

Dedicated to Professor Yoshihiro Shibata on the occasion of his 60th birthday

1. Introduction and statement of main results

Let $\Omega$ be an unbounded domain in $\mathbb{R}^n$ ($n \geq 2$) with compact and smooth boundary $\partial \Omega$. We put $\mathcal{O} := \mathbb{R}^n \setminus \Omega$, which is called an obstacle and is assumed to be non-empty. We consider the mixed problem for a system of nonlinear wave equations:

\begin{align}
(\partial^2_t - \Delta) u_i &= F_i(\partial u, \nabla_x \partial u), \quad (t, x) \in (0, \infty) \times \Omega; \\
u(t, x) &= 0, \quad (t, x) \in (0, \infty) \times \partial \Omega; \\
u(0, x) &= \varepsilon \phi(x), \quad \partial_t \nu(0, x) = \varepsilon \psi(x), \quad x \in \Omega
\end{align}

for $i = 1, \ldots, N$, where $u = (u_1, u_2, \ldots, u_N)$ is an unknown function, $\Delta = \sum_{j=1}^n \partial^2_j$, $\partial_t = \partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ ($j = 1, \ldots, n$), and $\varepsilon > 0$. We assume $\phi, \psi \in C_0^\infty(\Omega; \mathbb{R}^N)$, namely, they are smooth functions on $\Omega$ vanishing outside some ball. We also assume that $F_i(\partial u, \nabla_x \partial u)$ is a smooth function satisfying

\begin{equation}
F_i(\partial u, \nabla_x \partial u) = O(|\partial u|^q + |\nabla_x \partial u|^q), \quad 1 \leq i \leq N
\end{equation}

around $(\partial u, \nabla_x \partial u) = 0$ for some integer $q \geq 2$, together with the energy symmetric condition.

We suppose, in addition, that $(\phi, \psi, F)$ satisfies the compatibility condition to infinite order for the mixed problem (1.1)-(1.3), that is, $(\partial^j_t u)(0, x)$, formally determined by (1.1) and (1.3), vanishes on $\partial \Omega$ for any non-negative integer $j$ (notice that the values

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(\partial_t^2 u)(0, x) are determined by \((\phi, \psi, F)\) successively; for example we have \(\partial_t^2 u(0) = \Delta_x \phi + F(\psi, \nabla_x \phi)\), and so on.

It was firstly shown by Shibata and Tsutsumi [21] that the mixed problem for \((1.1)-(1.3)\) admits a unique global solution for sufficiently small initial data, when either \(n \geq 6\) and \(q \geq 2\) or \(3 \leq n \leq 5\) and \(q \geq 3\), provided \(\Omega\) is non-trapping. Although the dispersive property is getting weaker as the spatial dimension is lower, there are already many contributions for the case where \(3 \leq n \leq 5\) and \(q = 2\) (see [4], [5], [6], [9], [10], [11], [15], [17], [18], [19] and the references cited therein).

However, up to the author’s knowledge, there is no literature about the exterior problem \((1.1)-(1.3)\) for the case \(n = 2\). The aim of this paper is to treat the problem in that case, by assuming that \(q = 3\) in \((1.4)\) and \(\Omega\) is star-shaped. We remark that when \(n = 2\), the cubic nonlinearity is on the critical level concerning the global existence theorem for small initial data. Indeed, if \(N = 1\) and \(F_1 = (\partial_t u)^3\), then one can show a blow-up result from a corresponding result for the Cauchy problem (see e.g. [14]), because of the domain of dependance.

Let us denote the lifespan by \(T_\epsilon\), i.e., the supremum of all \(T > 0\) such that a classical solution to the problem \((1.1)-(1.3)\) exists in \([0, T) \times \Omega\). Then we find that \(T_\epsilon \leq \exp(A \epsilon^{-2})\) holds for some positive constant \(A\), in view of the argument in [14]. Therefore, it is natural to ask whether the above upper bound of the lifespan is optimal with respect to \(\epsilon\) or not. In this paper we shall establish an affirmative answer to this question as follows.

**Theorem 1.1.** Let \(n = 2\) and let \(\phi, \psi \in C^\infty(\Omega; \mathbb{R}^N)\) vanish outside certain ball. Assume that \((\phi, \psi, F)\) satisfies the compatibility condition to infinite order for the problem \((1.1)-(1.3)\), \(\Omega\) is star-shaped, and \(F\) satisfies \((1.4)\) with \(q = 3\). Then there exist positive constants \(\epsilon_0, C\) such that for all \(\epsilon \in (0, \epsilon_0]\), we have \(T_\epsilon \geq \exp(C \epsilon^{-2})\).

Our proof of the theorem is based on the cut-off method used in [21]. Because the decaying rate of the local energy is actually weak when \(n = 2\) (see \((3.3)\) below), we need a careful treatment for getting weighted pointwise estimates given in Theorem 4.2 below from those for the corresponding Cauchy problem due to Kubota [16], Di Flaviano [2] and Hoshiga and Kubo [8]. Unfortunately, the resulting pointwise estimate for derivatives of the solution is not good enough, unlike the case of \(n = 3\), for handling the boundary term arising from the integration-by-parts argument. The main idea to overcome the difficulty is to make use of the stronger decay property for the time derivative of the solution than that for the space derivatives of the solution. This stronger decay property for the time derivative is deduced from the fact that the boundary condition is preserved under the differentiation in time.

As in the work of [8], we shall use a part of the vector fields of the Lorentz invariance: \(\partial_1, \partial_2\), and \(O_{12} = x_1 \partial_2 - x_2 \partial_1\), because the boundary condition makes difficult to use \(t \partial_1 + x_1 \partial_1\) \((j = 1, 2)\) and \(t \partial_2 + x \cdot \nabla_x\). We also remark that the geometric assumption on the obstacle will be used for assuring the decay of the local energy.

This paper is organized as follows. In the next section we collect several notation. In the section 3 we give some preliminaries needed later on. The section 4 is devoted to
establish the weighted pointwise estimates \( [4.8] \) and \( [4.9] \) below. Making use of these estimates, we give a proof of the almost global existence theorem in the section 5.

2. Notation

For \( \Xi = (v_0, v_1, f) \in H^1(\Omega) \times L^2(\Omega) \times L^\infty((0, T); L^2(\Omega)) \), we denote by \( S[\Xi](t, x) \) the solution of the mixed problem:

\[
\begin{align*}
(2.1) & \quad (\partial_t^2 - \Delta_x) v = f, & (t, x) & \in (0, T) \times \Omega, \\
(2.2) & \quad v(t, x) = 0, & (t, x) & \in (0, T) \times \partial \Omega, \\
(2.3) & \quad v(0, x) = v_0(x), \quad (\partial_t v)(0, x) = v_1(x), & x & \in \Omega.
\end{align*}
\]

We sometimes write \( \vec{v}_0 = (v_0, v_1) \) in what follows. We denote by \( X(T) \) the set of all \( \Xi = (v_0, v_1, f) \in C^\infty_C(\overline{\Omega}; \mathbb{R}^2) \times C^\infty_C([0, T] \times \overline{\Omega}; \mathbb{R}) \) satisfying the compatibility condition to infinite order for \( [2.1]-[2.3] \), i.e., \( (\partial_t^j v)(0, x) \), determined formally from \( [2.1] \) and \( [2.3] \) by

\[
(2.4) \quad v_j(x) = \Delta v_{j-2}(x) + (\partial_t^{j-2} f)(0, x) \quad (j \geq 2)
\]

vanishes on \( \partial \Omega \) for any non-negative integer \( j \). Here \( f \in C^\infty_C([0, T] \times \overline{\Omega}; \mathbb{R}) \) means that \( f \in C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}) \) and \( f(t, \cdot) \in C^\infty_C(\overline{\Omega}) \) for any fixed \( t \in [0, T) \). In addition, for \( a > 1 \), \( X_a(T) \) denotes the set of all \( \Xi = (v_0, v_1, f) \in X(T) \) satisfying

\[
v_0(x) = v_1(x) = f(t, x) \equiv 0 \text{ for } |x| \geq a \text{ and } t \in [0, T).
\]

Besides we set \( K[\vec{v}_0] = S[[\vec{v}_0, 0]] \) and \( L[f] = S[(0, 0, f)] \).

Similarly, for \( (w_0, w_1, g) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^\infty((0, T); L^2(\mathbb{R}^2)) \), we denote by \( S_0[(w_0, w_1, g)](t, x) \) the solution of the following Cauchy problem:

\[
\begin{align*}
(2.5) & \quad (\partial_t^2 - \Delta_x) w = g, & (t, x) & \in (0, T) \times \mathbb{R}^2, \\
(2.6) & \quad w(0, x) = w_0(x), \quad (\partial_t w)(0, x) = w_1(x), & x & \in \mathbb{R}^2.
\end{align*}
\]

Besides we put \( K_0[\vec{w}_0] = S_0[(\vec{w}_0, 0)] \) and \( L_0[g] = S_0[(0, 0, g)] \), where \( \vec{w}_0 = (w_0, w_1) \).

We denote

\[
Z_0 = \partial_0 = \partial_t, \quad Z_j = \partial_j (j = 1, 2), \quad Z_3 = O_{12} = x_1 \partial_2 - x_2 \partial_1.
\]

Then we have

\[
(2.7) \quad Z_j(\partial_t^2 - \Delta) = (\partial_t^2 - \Delta) Z_j \quad \text{for } j = 0, 1, 2, 3.
\]

Denoting \( Z^\alpha = Z_0^{\alpha_0} \cdots Z_3^{\alpha_3} \) with a multi-index \( \alpha = (\alpha_0, \ldots, \alpha_3) \), we set

\[
(2.8) \quad |\varphi(t, x)|_m = \sum_{|\alpha| \leq m} |Z^\alpha \varphi(t, x)|, \quad ||\varphi(t)||_m = ||\varphi(t, \cdot)||_m : L^2(\Omega)
\]

for a real or \( \mathbb{R}^N \)-valued smooth function \( \varphi(t, x) \) and a non-negative integer \( m \).
For $\nu, \kappa \in \mathbb{R}$, we define

$$
\Phi_{\nu}(t, x) = \begin{cases} 
\langle t + |x| \rangle^{\nu} & \text{if } \nu < 0, \\
\log^{-1} \left( 2 + \langle t + |x| \rangle \right) & \text{if } \nu = 0, \\
\langle t - |x| \rangle^{\frac{1}{2}} \langle t - |x| \rangle^{-[\frac{1}{2} - \nu]} & \text{if } \nu > 0,
\end{cases}
$$

(2.9)

$$
\Psi_{\kappa}(t) = \begin{cases} 
\log(2 + t) & \text{if } \kappa = 1, \\
1 & \text{if } \kappa \neq 1,
\end{cases}
$$

(2.10)

where we have denoted

$$
A^{[a]} = A^a \text{ if } a > 0; \quad A^{[a]} = 1 \text{ if } a < 0; \quad A^{[0]} = 1 + \log A
$$

for $A \geq 1$, and $\langle s \rangle = \sqrt{1 + |s|^2}$ for $s \in \mathbb{R}^n$. Besides, for $\rho, \kappa \in \mathbb{R}$ and $c \geq 0$, we put

$$
z_{\rho, \kappa; c}(t, x) = \langle t + |x| \rangle^{\rho} (ct - |x|)^{\kappa},
$$

(2.11)

$$
W_{\rho, \kappa}(t, x) = \langle t + |x| \rangle^{\rho} \left( \min \{ \langle |x| \rangle, \langle t - |x| \rangle \} \right)^{\kappa},
$$

(2.12)

$$
w_{\rho}(t, x) = \langle x \rangle^{-1/2} (t - |x|)^{-\rho} + \langle t + |x| \rangle^{-1/2} (t - |x|)^{-1/2}.
$$

(2.13)

Note that for $1/2 \leq \rho \leq 1$, we have

$$
w_{\rho}(t, x) \leq C \left( W_{1/2, 1/2}(t, x) \right)^{-1},
$$

(2.14)

$$
w_{\rho}(t, x) \leq C \langle t \rangle^{-\rho} \quad \text{if } |x| \leq 2.
$$

(2.15)

We define

$$
\| f(t) : N_k(W) \| = \sup_{(s,x) \in [0,t] \times \Omega} \langle x \rangle^{1/2} W(s, x) |f(s, x)|_k
$$

(2.16)

for $t \in [0, T)$, a non-negative integer $k$ and any non-negative function $W(s, x)$. Similarly we put

$$
\| g(t) : M_k(W) \| = \sup_{(s,x) \in [0,t] \times \mathbb{R}^2} \langle x \rangle^{1/2} W(s, x) |g(s, x)|_k.
$$

(2.17)

Let $\rho \geq 0$, and $k$ be a non-negative integer. We define

$$
A_{\rho, k}[v_0, v_1] = \sup_{x \in \Omega} \langle x \rangle^{\rho} \left( |v_0(x)|_k + |\nabla_x v_0(x)|_k + |v_1(x)|_k \right)
$$

(2.18)

for a smooth function $(v_0, v_1)$ on $\Omega$, and

$$
B_{\rho, k}[w_0, w_1] = \sup_{x \in \mathbb{R}^2} \langle x \rangle^{\rho} \left( |w_0(x)|_k + |\nabla_x w_0(x)|_k + |w_1(x)|_k \right)
$$

(2.19)

for a smooth function $(w_0, w_1)$ on $\mathbb{R}^2$.

For $a \geq 1$, let $\psi_a$ be a smooth radially symmetric function on $\mathbb{R}^2$ satisfying

$$
\psi_a(x) = 0 \quad (|x| \leq a), \quad \psi_a(x) = 1 \quad (|x| \geq a + 1).
$$

(2.20)

We put $B_R = \{ x \in \mathbb{R}^2; |x| < R \}$ for $R > 0$. We may assume, without loss of generality, that $\Omega \subset B_1$ by the translation and scaling. Hence we always assume it in the following. For $R \geq 1$, we set $\Omega_R = \Omega \cap B_R$. 


3. Preliminaries

First we introduce an elliptic estimate, whose proof will be given in Appendix A for the sake of completeness.

Lemma 3.1. Assume that $\mathcal{O}$ is star-shaped. For $\varphi \in H^m(\Omega) \cap H_0^1(\Omega)$ with an integer $m (\geq 2)$, we have

\begin{equation}
\sum_{|\alpha|=m} \| \partial_\alpha^\varphi \|_{L^2(\Omega)} \leq C (\| \Delta \varphi \|_{H^{-2}(\Omega)} + \| \nabla \varphi \|_{L^2(\Omega)}).
\end{equation}

Next we derive an estimate for the local energy of solutions to \((2.1)-(2.3)\). We put $H^m(\Omega) = H^{m+1}(\Omega) \times H^m(\Omega)$.

Lemma 3.2. Assume that $\mathcal{O}$ is star-shaped. Let $a, b > 1$, $\gamma \in (0, 1]$, and $m$ be a non-negative integer. Then for $(\vec{v}_0, f) \in X_a(T)$, there exists a positive constant $C = C(\gamma, a, b, m)$ such that for $t \in [0, T)$, we have

\begin{equation}
\sum_{|\alpha|\leq m} \langle t \rangle^\gamma \| \partial_\alpha S((\vec{v}_0, f)) \|_{L^2(\Omega)} \leq C \left( \| \vec{v}_0 \|_{H^{m-1}(\Omega)} + \sum_{|\alpha|\leq m-1} \sup_{0 \leq s \leq t} \langle s \rangle^\gamma \| \partial_\alpha f(s) \|_{L^2(\Omega)} \right).
\end{equation}

Proof. It is known that there exists a positive constant $C$ depending on $a, b$ such that

\begin{equation}
\sum_{|\alpha|\leq 1} \| \partial_\alpha K[\vec{\phi}_0](t) \|_{L^2(\Omega)} \leq C \langle t \rangle^{-1} (\log(2 + t))^{-2} \| \vec{\phi}_0 \|_{H^0(\Omega)}
\end{equation}

for any $\vec{\phi}_0 = (\phi_0, \phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying $\phi_0(x) = \phi_1(x) \equiv 0$ for $|x| \geq a$ (see for instance Morawetz [20], Vainberg [22]).

Now let $(\vec{v}_0, f) = (v_0, v_1, f) \in X_a(T)$. Then, by Duhamel’s principle, it follows that

\begin{equation}
\partial_\alpha S((\vec{v}_0, f))(t, x) = K[(v_j, v_{j+1}))(t, x) + \int_0^t K[(0, (\partial^j_t f)(s)))(t - s, x)ds
\end{equation}

for any non-negative integer $j$ and any $(t, x) \in [0, T) \times \Omega$, where $v_j$ are given by \((2.4)\). Apparently we have $(\partial^j_t f)(s, \cdot) \in L^2(\Omega)$ for $0 \leq s \leq t$. Thanks to the compatibility condition, we also find $v_j \in H^1_0(\Omega)$ for any $j \geq 0$. Therefore, by \((3.3)\), for $|\alpha| \leq 1$ and $\gamma \in (0, 1]$, we have

\begin{equation}
\| \partial_\alpha K[\vec{v}_j](t) \|_{L^2(\Omega)} \leq \langle t \rangle^{-1} (\log(2 + t))^{-2} \| \vec{v}_j \|_{H^0(\Omega)}
\end{equation}

\begin{equation}
\leq C \langle t \rangle^{-\gamma} \left( \| \vec{v}_0 \|_{H^0(\Omega)} + \sum_{|\alpha|\leq j-1} \| \partial_\alpha (\partial^j_t f)(0) \|_{L^2(\Omega)} \right)
\end{equation}
with \( \vec{v}_j = (v_j, v_{j+1}) \), and

\[
\int_0^t \| \partial^\alpha K[(0, (\partial^2 f)(s))](t-s) : L^2(\Omega_b) \| ds \\
\leq C \int_0^t (t-s)^{-1} (\log(2 + t-s))^{-1/2} \| (\partial^2 f)(s) : L^2(\Omega) \| ds \\
\leq C \| t \|^{-\gamma} \sup_{0 \leq s \leq t} \langle s \rangle^2 \| (\partial^2 f)(s) : L^2(\Omega) \|.
\]

Hence for \(|\alpha| \leq 1\) and \(j \geq 0\), we get from (3.4)

\[
\| \partial^\alpha \partial^j_x v \|_{L^2(\Omega_b)} \\
\leq C \| t \|^{-\gamma} \left( \| \vec{v}_0 : \mathcal{H}^j(\Omega) \| + \sum_{|\alpha| \leq j} \sup_{0 \leq s \leq t} \langle s \rangle^2 \| \partial^\alpha f(s) : L^2(\Omega) \| \right).
\]

In order to evaluate \( \partial^2_x \partial^\alpha v \) with \(|\alpha| \geq 2\) and \(j + |\alpha| \leq m\), we make use of the following variant of (3.1):

\[
\| \gamma : H^m(\Omega_b) \| \leq C(\| \Delta_x \gamma : H^{m-2}(\Omega_{\nu'}) \| + \| \gamma : H^1(\Omega_{\nu'}) \|),
\]

where \(1 < b < b'\) and \(\varphi \in H^m(\Omega) \cap H^1_0(\Omega)\) with \(m \geq 2\), together with the fact that \(\Delta_x S[(\vec{v}_0, f)] = \partial^2_x S[(\vec{v}_0, f)] + f\). In this way, we obtain (3.2). This completes the proof.

Next we prepare three lemmas concerning the Cauchy problem. The first one is the decay estimate for solutions of the homogeneous wave equation, due to [16, Proposition 2.1] (observe that the general case can be reduced to the case \(m = 0\), thanks to (2.7)). We recall that \(\Phi_\nu(t, x)\) and \(\Psi_\kappa(t)\) were defined by (2.9) and (2.10), respectively.

**Lemma 3.3.** For \(\vec{v}_0 \in (C_0^\infty(\mathbb{R}^2))^2\), \(\nu > 0\) and a non-negative integer \(m\), there is a positive constant \(C = C(\nu, m)\) such that

\[
\langle t + |x| \rangle^{1/2} \Phi_{\nu-1}(t, x) |K_0[\vec{v}_0](t, x)|_m \leq C\mathcal{B}_{\nu+(1/2), m}[\vec{v}_0]
\]

for \((t, x) \in [0, T) \times \mathbb{R}^2\).

The second one is the decay estimates for the inhomogeneous wave equation.

**Lemma 3.4.** If \(\nu > 0\) and \(\kappa \geq 1\) are non-negative integers, then there exists a positive constant \(C = C(\nu, \kappa, m)\) such that

\[
\langle t + |x| \rangle^{1/2} \Phi_{\nu-1}(t, x) |L_0[g](t, x)|_m \\
\leq C\Psi_\kappa(t + |x|) \| g(t) : M_m(W_{\nu, \kappa}) \|
\]

for \((t, x) \in [0, T) \times \mathbb{R}^2\).

**Proof.** Let \(|\alpha| \leq m\). It follows from (2.7) that

\[
Z^\alpha L_0[g] = L_0[Z^\alpha g] + K_0(\phi_{\alpha}, \psi_{\alpha}),
\]
From the equation (2.5) we get
\[
\phi_\alpha(x) = (Z^\alpha L_0[g])(0, x), \quad \psi_\alpha(x) = (\partial_1 Z^\alpha L_0[g])(0, x).
\]
From the equation (2.5) we get
\[
\phi_\alpha(x) = \sum_{|\beta| \leq |\alpha|-2} C_\beta (Z^\beta g)(0, x), \quad \psi_\alpha(x) = \sum_{|\beta| \leq |\alpha|-1} C_\beta' (Z^\beta g)(0, x)
\]
with suitable constants \(C_\beta, C_\beta'.\) Therefore, by \((3.9),\) we get
\[
\langle t + |x| \rangle^{1/2} \Phi_{\nu-1}(t, x)|K_0[(\phi_\alpha, \psi_\alpha)]| \leq C \sum_{|\alpha| \leq m-1} \| \langle \cdot \rangle^{\nu+1/2} (Z^\alpha g)(0) : L^\infty(\mathbb{R}^2) \|
\]
\[
\leq C \|g(0) : W_{m-1}(\nu, \kappa)\|.
\]
Hence, in view of \((3.11),\) it is enough to show \((3.10)\) for \(m = 0.\)
If we set \(z_{\nu,\kappa}(s, y) = \langle s + |y| \rangle^{\nu} \langle |y| - cs \rangle^\kappa,\) then we have
\[
(W_{\nu,\kappa}(s, y))^{-1} \leq (z_{\nu,\kappa,0}(s, y))^{-1} + (z_{\nu,\kappa,1}(s, y))^{-1},
\]
so that
\[
|L_0[g](t, x)| \leq \|g(t) : M_0(W_{\nu,\kappa})\| |L_0[W_{\nu,\kappa}^{-1}](t, x)|
\]
\[
\leq \|g(t) : M_0(W_{\nu,\kappa})\| (|L_0[z_{\nu,\kappa,0}^{-1}](t, x)| + |L_0[z_{\nu,\kappa,1}^{-1}](t, x)|)
\]
for \((t, x) \in [0, T) \times \mathbb{R}^2.\) Since it was shown by \([2, \text{Proposition 3.1}]\) that
\[
\langle t + |x| \rangle^{1/2} \Phi_{\nu-1}(t, x)|L_0[z_{\nu,\kappa,1}^{-1}](t, x)| \leq C\Psi_\kappa(t)
\]
holds, we have only to show
\[
\langle t + |x| \rangle^{1/2} \Phi_{\nu-1}(t, x)|L_0[z_{\nu,\kappa,0}^{-1}](t, x)| \leq C\Psi_\kappa(t).
\]
Following the proof of \([2, \text{Proposition 3.1}],\) we obtain
\[
|L_0[z_{\nu,\kappa,0}^{-1}](t, x)| \leq C \left\{ \int_{|t-r|}^{t+r} \langle \alpha \rangle^{-\nu+(1/2)} (\alpha + r - t)^{-1/2} V(\alpha) d\alpha \right. \\
\left. + \phi(r, t) \int_0^{t-r} \langle \alpha \rangle^{-\nu+(1/2)} (t - r - \alpha)^{-1/2} V(\alpha) d\alpha \right\},
\]
where we put \(r = |x|,\)
\[
V(\alpha) = \int_{-\alpha}^{\alpha} \langle \alpha + \beta \rangle^{-\kappa+(1/2)} (\beta + r + t)^{-1/2} d\beta,
\]
\[
\phi(r, t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq r, \\
1 & \text{if } t > r
\end{cases}
\]
(Notice that \(\alpha + r - t > 0\) if \(\alpha > |t - r|).\) Therefore, once we find
\[
V(\alpha) \leq C(\alpha)^{1/2} \langle \alpha \rangle^{1 - \alpha} (t + r)^{-1/2}
\]
for \(0 < \alpha < t + r,\) then we get \((3.14)\) by Lemmas 3.3 and 3.4 in \([2,\).]
We are going to show (3.15). When \(0 < t + r < 1\) or \((t + r)/2 < \alpha < t + r\), it suffices to show that \(V(\alpha) \leq C(\alpha)^{|1 - \nu|}\). Splitting the integral at \(\beta = -\alpha + 1\), we get

\[
V(\alpha) \leq \int_{-\alpha}^{-\alpha + 1} (\beta + \alpha)^{-1/2} d\beta + C \int_{-\alpha + 1}^{\alpha} \left(\frac{\alpha + \beta}{2}\right)^{-\kappa} d\beta,
\]

since \(\beta + r + t > \beta + \alpha > (\beta + \alpha + 1)/2\) if \(\alpha < t + r\) and \(\beta > -\alpha + 1\). This estimate yields (3.15). On the other hand, when \(t + r > 1\) and \(0 < \alpha < (t + r)/2\), we have \(\beta + r + t > (t + r)/2\) if \(\beta > -\alpha\), hence

\[
V(\alpha) \leq C(t + r)^{-1/2} \int_{-\alpha}^{\alpha} \left(\frac{\alpha + \beta}{2}\right)^{-\kappa + (1/2)} d\beta.
\]

The last integral is bounded by \(C(\alpha)^{-(\kappa + (3/2))} \leq C(\alpha)^{1/2}\) when \(\kappa \geq 1\). Thus we obtain (3.15). This completes the proof. \(\square\)

The third one is the decay estimates for derivatives of solutions of the inhomogeneous wave equation.

**Lemma 3.5.** If \(0 < \nu < 3/2\), \(\mu \geq 0\), \(\kappa \geq 1\), \(0 < \eta < 1\), and \(m\) is a non-negative integer, then there exists a positive constant \(C = C(\nu, \kappa, \mu, m)\) such that we have

\[
(3.16) \quad (w_{\nu}(t, x))^{-1} |\partial_{t,x} L_0[g](t, x)|_m \
\leq C \Psi_{1+\mu}(t + |x|)\Psi_\kappa(t + |x|)\|g(t) : M_{m+1}(z_{\nu+\mu, \kappa, 0})\| ,
\]

\[
(3.17) \quad (w_{1-\eta}(t, x))^{-1} |\partial_{t,x} L_0[g](t, x)|_m \
\leq C \log(2 + t + |x|)\|g(t) : M_{m+1}(W_{1,1})\|
\]

for \((t, x) \in [0, T) \times \mathbb{R}^2\).

**Proof.** Let \(|\alpha| \leq m\). Then we have

\[
(3.18) \quad \partial_{t,x} Z^\alpha L_0[g] = \partial_{t,x} L_0[Z^\alpha g] + \partial_{t,x} K_0[\langle \phi_\alpha, \psi_\alpha \rangle]
\]

from (3.11), and

\[
(3.19) \quad \partial_t L_0[Z^\alpha g] = L_0[\partial_t Z^\alpha g] \quad (\ell = 1, 2),
\]

\[
(3.20) \quad \partial_t L_0[Z^\alpha g] = L_0[\partial_t Z^\alpha g] + K_0[(0, (Z^\alpha g)(0))].
\]

First we prove (3.16). By (3.6) it suffices to show

\[
(3.21) \quad (w_{\nu}(t, x))^{-1} |\partial_{t,x} K_0[\langle \phi_\alpha, \psi_\alpha \rangle](t, x)| \
\leq C \Psi_{1+\mu}(t + |x|)\|g(0) : M_{m+1}(z_{\nu+\mu, 1, 0})\|,
\]

\[
(3.22) \quad (w_{\nu}(t, x))^{-1} |K_0[(0, (Z^\alpha g)(0))](t, x)| \
\leq C \Psi_{1+\mu}(t + |x|)\|g(0) : M_{m+1}(z_{\nu+\mu, 1, 0})\|.
\]

We shall show only (3.20), because the proof of the other is similar.
When $0 < \nu < 1/2$, by (3.9) with $\nu$ replaced by $1 + \nu$, we get
\[
\langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{1/2} |\partial_{t,x} K_0[\phi_\alpha, \psi_\alpha](t, x)| \leq C \sum_{|\alpha| \leq m} \| \langle \cdot \rangle^{(3/2) + \nu}(Z^\alpha g)(0) : L^\infty(\mathbb{R}^2) \| \\
\leq C \| g(0) : M_m(z_{\nu,1,0}) \|.
\]
On the other hand, when $\nu \geq 1/2$, by (3.9) with $\nu = (3/2) + \mu$, we get
\[
\langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{1/2} |\partial_{t,x} K_0[\phi_\alpha, \psi_\alpha](t, x)| \leq C \Psi_{1+\mu}(t - |x|) \| g(0) : M_m(z_{1/2+\mu,1,1}) \|.
\]
Thus we obtain (3.20).
Next we prove (3.17). It follows from (B.6) and (B.7) with $\nu = 1 - \eta$, $\mu = \eta$, and $\kappa = 1$ that
\[
(w_{1-\eta}(t, x))^{-1} |L_0[\partial_{t,x} g](t, x)|_m \leq C \log(2 + t + |x|) \| g(t) : M_{m+1}(W_{1,1}) \|.
\]
Since it is easy to see that (3.22) is still valid if we replace $z_{(1/2)+\mu,1,0}$ by $z_{(1/2)+\mu,1,1}$, we find (3.17). This completes the proof.

Finally, we introduce the following Sobolev type inequality, whose counterpart for the Cauchy problem is due to Klainerman [12]. Since the proof is similar to the case of $n = 3$ (see e.g. [9]), we omit it.

**Lemma 3.6.** Let $\varphi \in C_0^2(\Omega)$. Then we have
\[
(3.23) \sup_{x \in \Omega} \langle x \rangle^{1/2} |\varphi(x)| \leq C \sum_{|\alpha| + k \leq 2} \| \partial^\alpha_x O_{12}^k \varphi : L^2(\Omega) \|.
\]

4. Basic estimates

First of all, we prepare the following lemma which will be used to prove Theorem 4.2 below. Recall that we have assumed $\mathcal{O} \subset B_1$.

**Lemma 4.1.** Let $\mathcal{O}$ be star-shaped. Let $a, b > 1$, $0 < \rho \leq 1$, $\mu \geq 0$, $\kappa \geq 1$, $\eta > 0$, and $m$ is a non-negative integer.

(i) Suppose that $\chi$ is a smooth function on $\mathbb{R}^2$ satisfying $\operatorname{supp} \chi \subset B_a$. If $\Xi = (\vec{v}_0, f) \in X_a(T)$, then there exists a positive constant $C = C(\rho, a, b, m)$ such that
\[
(4.1) \langle t \rangle^\rho |\chi S[\Xi](t, x)|_m \leq C A_{\rho+1, m+1}[\vec{v}_0] + C \sum_{|\beta| \leq m+1} \sup_{(s, x) \in [0, \epsilon] \times \Omega_a} \langle s \rangle^\rho |\partial^\beta f(s, x)|
\]
for $(t, x) \in [0, T) \times \overline{\Omega}$.

(ii) Let $\vec{w}$ and $g$ are smooth functions on $\mathbb{R}^2$ and on $[0, T) \times \mathbb{R}^2$, respectively. If $\operatorname{supp} g(t, \cdot) \subset B_a \setminus B_1$ for any $t \in [0, T)$, then there exists a positive constant $C = $
We have seen that the Sobolev inequality and (3.2) imply that
\[ |L_0[g](t, x)|_m \leq C \sum_{|\beta| \leq m} \sup_{(s, x) \in [0,t] \times \Omega} (s)^{1/2} |\partial^\beta g(s, x)| \]
and
\[ (w_\rho(t, x))^{-1} |\partial L_0[g](t, x)|_m \leq C \Psi_{1+\mu}(t + |x|) \sum_{|\beta| \leq m+1} \sup_{(s, x) \in [0,t] \times \Omega} (s)^{\rho+\mu} |\partial^\beta g(s, x)| \]
for \((t, x) \in [0, T] \times \overline{\Omega}\).

On the other hand, if \(\tilde{w}_0(x) = g(t, x) = 0\) for any \((t, x) \in [0, T] \times B_1\), then there exists a positive constant \(C = C(\kappa, b, m)\) such that
\[ \langle t \rangle^{1/2} |S_0[(\tilde{w}_0, g)](t, x)|_m \leq CA_{3/2, m}[\tilde{w}_0] + C \Psi_\kappa(t + |x|) \|g(t) : N_{m}(W_{1,\kappa})\| \]
and
\[ \langle t \rangle^{1-\eta} |\partial S_0[(\tilde{w}_0, g)](t, x)|_m \leq CA_{2, m+1}[\tilde{w}_0] + C \log(2 + t + |x|) \|g(t) : N_{m+1}(W_{1,1})\| \]
for \((t, x) \in [0, T] \times \Omega_b\).

Proof. First we note that
\[ |h(t, x)|_m \leq C \sum_{|\beta| \leq m} |\partial^\beta h(t, x)| \]
holds for any smooth function \(h\) on \([0, T] \times \overline{\Omega}\) (or on \([0, T] \times \mathbb{R}^2\)) with \(\text{supp } h(t, \cdot) \subset B_R\) for some \(R (> 1)\).

We start with the proof of (4.1). Let \(\Xi = (\tilde{v}_0, f) \in X_a(T)\) and \(0 < \rho \leq 1\). For \((t, x) \in [0, T] \times \Omega\), by (4.6), the Sobolev inequality and (3.2), we get
\[
\langle t \rangle^\rho \chi_S(\Xi)(t, x)|_m \leq C \langle t \rangle^\rho \sum_{|\beta| \leq m+2} \|\partial^\beta S(\Xi)(t) : L^2(\Omega_b)\|
\leq C\|\tilde{v}_0 : \mathcal{H}^{m+1}(\Omega)\| + C \sup_{s \in [0, t]} \langle s \rangle^\rho \sum_{|\beta| \leq m+1} \|\partial^\beta f(s) : L^2(\Omega)\|.
\]
Since \(\text{supp } f(t, \cdot) \subset \overline{\Omega}_a\) implies \(\|\partial^\beta f(s) : L^2(\Omega)\| \leq C\|\partial^\beta f(s) : L^\infty(\Omega_a)\|\), we obtain (4.1).

Next we prove (4.2). By (3.10) with \(\nu = 1/2\) and \(\kappa > 1\), we find that the left-hand side on (4.2) is estimated by
\[ C\|g(t) : M_m(W_{1/2, \kappa})\| \leq C\|g(t) : N_m(W_{1/2, \kappa})\| \]
because \(\text{supp } g(t, \cdot) \subset B_a \setminus B_1\). Since \(|x| \leq a\) on \(\text{supp } g(t, x)\), we get (4.2). Similarly, if we use (3.16) with \(\nu = \rho\) and \(\kappa > 1\), instead of (3.10), then we get (4.3).
Next we prove (4.4). From (3.9) and (3.17) we have
\[
\langle t + |x| \rangle^{1/2} \Phi_0(t,x)|S_0((\vec{w}_0,g))(t,x)|_m \leq CB_{3/2,m}[\vec{w}_0] + C\Psi_\kappa(t + |x|) \|g(t):M_m(W_{1,1})\| 
\]
for \((t,x) \in [0,T) \times \mathbb{R}^2\). Since \(\Phi_0(t,x)\) is equivalent to a constant when \(x \in \overline{\Omega}_b\), we get (4.4), because of the assumption on \(\vec{w}_0\) and \(g\).

Finally, we prove (4.5). From (3.9) and (3.17) we have
\[
(w_{1-\eta}(t,x))^{-1}\partial S_0((\vec{w}_0,g))(t,x)|_m \leq CB_{2,m+1}[\vec{w}_0] + C\log(2 + t + r) \|g(t):M_{m+1}(W_{1,1})\| 
\]
for \((t,x) \in [0,T) \times \mathbb{R}^2\). Recalling (2.15), we get (4.4). This completes the proof. \(\square\)

Now we are in a position to state our basic estimates for solutions to the linear mixed problem.

**Theorem 4.2.** Let \(\mathcal{O}\) be star-shaped and \(\Xi = (\vec{v}_0,f) \in X(T)\). Let \(k\) be a nonnegative integer and \(\kappa \geq 1\).

(i) It holds that
\[
|S[\Xi](t,x)|_k \leq CA_{3/2,k+3}[\vec{v}_0] + \Psi_\kappa(t + |x|)\|f(t):N_{k+3}(W_{1,1})\| 
\]
for \((t,x) \in [0,T) \times \overline{\Omega}\).

(ii) For \(0 < \rho \leq 1/2\) and \(\delta > 0\), we have
\[
|\partial S[\Xi](t,x)|_k \leq CW_{1/2}(t,x)\left(A_{2+\delta,k+4}[\vec{v}_0] + \log(2 + t + |x|) \|f(t):N_{k+3}(W_{1,1})\| + CW_{\rho}(t,x)\Psi(3/2-\rho)(t + |x|) \Psi_\kappa(t + |x|)\|f(t):N_{k+4}(W_{1,1})\| \right) 
\]
for \((t,x) \in [0,T) \times \overline{\Omega}\).

Moreover, for \(0 < \eta < 1\) and \(\delta > 0\), we have
\[
|\partial t S[\Xi](t,x)|_k \leq CW_{1-\eta}(t,x)\left(A_{2+\delta,k+5}[\vec{v}_0] + (\log(2 + t + |x|))^2 \|f(t):N_{k+5}(W_{1,1})\| \right) 
\]
for \((t,x) \in [0,T) \times \overline{\Omega}\).

**Proof.** First we prove (4.7). We use the following representation formula based on the cut-off method:
\[
S[\Xi](t,x) = \psi_1(x)S_0[\psi_2\Xi](t,x) + \sum_{i=1}^4 S_i[\Xi](t,x) 
\]
for \((t, x) \in [0, T) \times \overline{\Omega}\), where \(\psi_a\) is defined by (2.20) and we have set
\begin{align}
(4.11) & \quad S_1[\Xi](t, x) = (1 - \psi_2(x)) L \left[ \psi_1, -\Delta x \right] S_0[\psi_2 \Xi](t, x), \\
(4.12) & \quad S_2[\Xi](t, x) = -L_0 \left[ \psi_2, -\Delta x \right] L \left[ \psi_1, -\Delta x \right] S_0[\psi_2 \Xi](t, x), \\
(4.13) & \quad S_3[\Xi](t, x) = (1 - \psi_3(x)) S [(1 - \psi_2) \Xi](t, x), \\
(4.14) & \quad S_4[\Xi](t, x) = -L_0 \left[ \psi_3, -\Delta x \right] S [(1 - \psi_2) \Xi](t, x).
\end{align}

By (3.9) and (3.10) we have
\begin{align*}
\langle t + |x| \rangle^{1/2} \Phi_0(t, x) \left| \psi_1(x) S_0[\psi_2 \Xi](t, x) \right|_k \\
& \leq C A_{3/2, k}[\tilde{v}_0] + C \Psi_n(t + |x|) \| f(t) : N_{k+1}(W_{1, \alpha}) \|.
\end{align*}

Next we shall estimate \(S_1[\Xi]\) and \(S_3[\Xi]\). It is easy to check that
\begin{align*}
[\psi_a, -\Delta x] h(t, x) = h(t, x) \Delta x \psi_a(x) + 2 \nabla_x h(t, x) \cdot \nabla_x \psi_a(x)
\end{align*}
for \((t, x) \in [0, T) \times \overline{\Omega}\), \(a \geq 1\) and any smooth function \(h\). Note that this identity implies \((0, 0, [\psi_a, -\Delta x] h) \in X_{a+1}(T)\) for any smooth function \(h\) and \(a \geq 1\), because \(\sup \nabla_x \psi_a \cup \sup \Delta x \psi_a \subset B_{a+1} \setminus B_{a}\). Therefore, by (4.1) with \(\rho = 1/2\) and (4.4), we obtain
\begin{align}
\langle t + |x| \rangle^{1/2} |S_1[\Xi](t, x)|_k \\
& \leq C \sum_{|\beta| \leq k+2} \sup_{(s, x) \in [0, t] \times \Omega_2} \langle s \rangle^{1/2} \partial^\beta S_0[\psi_2 \Xi](s, x) \\
& \leq C A_{3/2, k+2}[\tilde{v}_0] + C \Psi_n(t + |x|) \| f(t) : N_{k+2}(W_{1, \alpha}) \|.
\end{align}
for \((t, x) \in [0, T) \times \overline{\Omega}\). Similarly, since we have \((1 - \psi_2) \Xi \in X_3(T)\) for any \(\Xi \in X(T)\), (4.1) with \(\rho = 1/2\) leads to
\begin{align}
\langle t \rangle^{1/2} \langle \Xi \rangle(t, x) \leq C A_{3/2, k+1}[\tilde{v}_0] \\
+ C \sum_{|\beta| \leq k+1} \sup_{(s, x) \in [0, t] \times \Omega_3} \langle s \rangle^{1/2} \partial^\beta f(s, x)
\end{align}
for \((t, x) \in [0, T) \times \overline{\Omega}\). Since \(\sup S_i[\Xi] \subset B_4 (i = 1, 3)\), (4.15) and (4.16) imply
\begin{align*}
\langle t + |x| \rangle^{1/2} |S_i[\Xi](t, x)|_k \leq C A_{3/2, k+2}[\tilde{v}_0] \\
+ C \Psi_n(t + |x|) \| f(t) : N_{k+2}(W_{1, \alpha}) \| \quad (i = 1, 3)
\end{align*}
for \((t, x) \in [0, T) \times \overline{\Omega}\).

Set \(g_i[\Xi] = (\partial^2_t - \Delta x) S_i[\Xi]\) for \(i = 2, 4\). Observing that \(g_2\) and \(g_4\) have the almost same structures as \(S_1\) and \(S_3\), respectively, we find
\begin{align*}
\sum_{|\beta| \leq m} \sup_{(s, x) \in [0, t] \times \Omega_4} \langle s \rangle^{1/2} |\partial^\beta g_i[\Xi](s, x)| \leq C A_{3/2, m+2}[\tilde{v}_0] \\
+ C \Psi_n(t + |x|) \| f(t) : N_{m+2}(W_{1, \alpha}) \|
\end{align*}
for \( i = 2, 4 \) and \((t, x) \in [0, T) \times \overline{\Omega} \), in a similar fashion. Note that \( g_2 \) and \( g_4 \) are supported on \( \overline{B_3 \setminus B_2} \). By (4.12), (4.14), and (4.2), we obtain
\[
|S_i[\Xi](t, x)|_k \leq CA_{3/2,k+3}[\vec{v}_0] + C\Psi_\kappa(t + |x|)\|f(t) : N_{k+3}(W_{1,\kappa})\|
\]
for \( i = 2, 4 \) and \((t, x) \in [0, T) \times \overline{\Omega} \), and hence (4.7).

Next we prove (4.8) by using (4.10). Let \( 0 < \rho \leq 1/2 \) and \( \delta > 0 \). Writing \( \zeta_0 = S_0[\psi_2 \Xi] \), we get
\[
|\partial_a(\psi_1(x)\zeta_0(t, x))|_k \leq C|\partial_a\zeta_0(t, x)|_k + C|\partial_a\psi_1(x)|_k|\zeta_0(t, x)|_k.
\]
By (3.9) and (3.17) with \( \eta = 1/2 \), we see that the first term on the right-hand side is estimated by
\[
C \langle t + |x| \rangle^{-1/2} \langle t - |x| \rangle^{-1/2}A_{2+\delta,k+1}[\vec{v}_0] + Cw_{1/2}(t, x) \log(2 + t + |x|)\|f(t) : N_{k+1}(W_{1,1})\|.
\]
Since \( \langle t \rangle^{-1/2} \leq C \langle x \rangle^{-1/2} \langle t - |x| \rangle^{-1/2} \) for \( |x| \leq 2 \), (4.4) shows that the second term on the right-hand side is estimated by
\[
C \langle x \rangle^{-1/2} \langle t - |x| \rangle^{-1/2} (A_{3/2,k}[\vec{v}_0] + \log(2 + t + |x|)\|f(t) : N_k(W_{1,1})\|).
\]
Therefore we have
\[
\text{(4.17)} \quad |\partial(\psi_1(x)S_0[\psi_2 \Xi](t, x))|_k \leq Cw_{1/2}(t, x)(A_{2+\delta,k+1}[\vec{v}_0] + \log(2 + t + |x|)\|f(t) : N_{k+1}(W_{1,1})\|)
\]
for \((t, x) \in [0, T) \times \overline{\Omega} \).

From (4.15) with \( \kappa = 1 \) and (4.16) we have
\[
\text{(4.18)} \quad \langle x \rangle^{1/2} \langle t - |x| \rangle^{1/2}|\partial S_i[\Xi](t, x)|_k \leq CA_{3/2,k+3}[\vec{v}_0] + C\log(2 + t + |x|)\|f(t) : N_{k+3}(W_{1,1})\| \quad (i = 1, 3)
\]
for \((t, x) \in [0, T) \times \overline{\Omega} \).

As for \( g_4[\Xi] = (\partial_t^2 - \Delta_x)S_4[\Xi] \), it follows from (4.11) with \( \rho = 1 \) that
\[
\sum_{|\beta| \leq m} \sup_{(s, x) \in [0, t) \times \Omega_4} \langle s \rangle^{1/2} \partial^\beta g_4[\Xi](s, x) \leq CA_{2,m+2}[\bar{v}_0] + C\|f(t) : N_{m+2}(W_{1,1})\|.
\]
Therefore, by (4.14) and (4.3) with \( \rho = \mu = 1/2 \), we get
\[
\text{(4.19)} \quad (w_{1/2}(t, x))^{-1}|\partial S_4[\Xi](t, x)|_k \leq CA_{2,k+3}[\bar{v}_0] + C\|f(t) : N_{k+3}(W_{1,1})\|
\]
for \((t, x) \in [0, T) \times \overline{\Omega} \).

To estimate \( g_2[\Xi] = -[\psi_2, -\Delta_x]L[ [\psi_1, -\Delta_x]S_0[\psi_2 \Xi] ] \), we define \( g_{2,0}[\bar{v}_0] \) and \( g_{2,1}[f] \) by replacing \( S_0[\psi_2 \Xi] \) with \( K_0[\psi_2 \bar{v}_0] \) and \( L_0[\psi_2 f] \), respectively. Then we have \( g_2[\Xi] = \ldots \)
\[ g_{2,0}[\tilde{v}_0] + g_{2,1}[f]. \] By (4.14) with \( \rho = (1/2) + \mu \) and (3.9) with \( \nu = 1 + \mu \) (0 < \mu < 1/2), we get

\[
\langle t \rangle^{(1/2)+\mu} |g_{2,0}[\tilde{v}_0](t, x)|_m \\
\leq C \sum_{|\beta| \leq m+3} \sup_{(s,x) \in [0,t] \times \Omega_2} \langle s \rangle^{(1/2)+\mu} |\partial^\beta K_0[\psi_2 \tilde{v}_0](s, x)| \\
\leq C A_{(3/2)+\mu,m+3}[\tilde{v}_0]
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\). Applying (4.13) with \( \rho = 1/2 \) and \( \mu > 0 \), we find

\[
(w_{1/2}(t, x))^{-1} |\partial L_0[g_{2,0}[\tilde{v}_0]](t, x)|_k \leq C A_{(3/2)+\mu,k+4}[\tilde{v}_0]
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\).

On the other hand, similarly to the proof of (4.15), we get

\[
\langle t \rangle^{1/2} |g_{2,1}[f](t, x)|_m \leq C \Psi_\kappa(t + |x|) \|f(t) : N_{m+3}(W_{1,\kappa})\|
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\). By (4.13) with \( \mu = (1/2) - \rho \), we obtain

\[
(w_\mu(t, x))^{-1} |\partial L_0[g_{2,1}[f]](t, x)|_k \\
\leq C \Psi_{(3/2)-\rho}(t + |x|) \Psi_\kappa(t + |x|) \|f(t) : N_{k+4}(W_{1,\kappa})\|
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\). Thus we get

\[
(4.20) \quad |\partial S_2[\Xi](t, x)|_k \leq C w_{1/2}(t, x) A_{(3/2)+\mu,k+4}[\tilde{v}_0] \]
\[
+ C w_\mu(t, x) \Psi_{(3/2)-\rho}(t + |x|) \Psi_\kappa(t + |x|) \|f(t) : N_{k+4}(W_{1,\kappa})\|
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\). Now (4.8) follows from (4.17), (4.18), (4.19), and (4.20).

Finally we prove (4.9). Let \( 0 < \eta < 1 \) and \( \delta > 0 \). Since \( \partial_t S[\Xi] \) satisfies the boundary condition, we have

\[
(4.21) \quad \partial_t S[\Xi](t, x) = \psi_1(x) S_0[\psi_2 \tilde{v}](t, x) + \sum_{i=1}^4 S_i[\tilde{v}](t, x)
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\), where we have set \( \tilde{\Xi} = (\tilde{v}_1, \psi_2, \partial_t f) \) with \( \psi_2 = \Delta v_0 + f \). Since \( S_0[\psi_2 \tilde{v}] = \partial_t S_0[\psi_2 \Xi] \), we find from (3.9) and (3.17) that

\[
(4.22) \quad |\partial(\psi_1(x) S_0[\psi_2 \tilde{v}](t, x))|_k \\
\leq C \langle t + |x| \rangle^{-1/2} \langle t - |x| \rangle^{-1/2} A_{2+\delta,k+2}[\tilde{v}_0] \]
\[
+ C w_{1/2}(t, x) \log(2 + t + |x|) \|f(t) : N_{k+2}(W_{1,1})\|
\]

for \((t, x) \in [0, T) \times \overline{\Omega}\).
By (4.11) and (4.5), we obtain
\begin{equation}
(4.23) \quad (t)^{1-\eta} |\partial S_1[\tilde{\Xi}](t, x)|_k \\
\leq C \sum_{|\beta| \leq k+3} \sup_{(s, x) \in [0, t] \times \Omega} \langle s \rangle^{1-\eta} |\partial^\beta S_0[\psi_2 \tilde{\Xi}](s, x)| \\
\leq CA_{2,k+4}[\tilde{v}_0] + C \log(2 + t + |x|) \|f(t) : N_{k+4}(W_{1,1})\|
\end{equation}
for \((t, x) \in [0, T) \times \Omega\).

Since \(S[(1 - \psi_2)\Xi] = \partial_t S[(1 - \psi_2)\Xi]\), by (4.11) we have
\begin{equation}
(4.24) \quad (t)|\partial S_3[\tilde{\Xi}](t, x)|_k \leq CA_{2,k+3}[\tilde{v}_0] \\
+ C \sum_{|\beta| \leq k+3} \sup_{(s, x) \in [0, t] \times \Omega} \langle s \rangle |\partial^\beta f(s, x)|
\end{equation}
for \((t, x) \in [0, T) \times \Omega\), if we use (4.3) with \(\rho = 1 - \eta, \mu = \eta\).

Next we evaluate \(S_4[\tilde{\Xi}]\). Since \(S_4[\tilde{\Xi}] = L_0[\partial_t g_4[\Xi]]\), analogously to the proof of (4.19), we get
\begin{equation}
(4.25) \quad (w_{1-\eta}(t, x))^{-1}|\partial S_4[\tilde{\Xi}](t, x)|_k \\
\leq CA_{2,k+4}[\tilde{v}_0] + C \|f(t) : N_{k+4}(W_{1,1})\|
\end{equation}
for \((t, x) \in [0, T) \times \Omega\), if we use (4.3) with \(\rho = 1 - \eta, \mu = \eta\).

Next we evaluate \(S_2[\tilde{\Xi}]\). We define \(\tilde{g}_{2,0}[\tilde{v}_0]\) and \(\tilde{g}_{2,1}[f]\) by replacing \(\partial_t S_0[\psi_2 \Xi]\) with \(\partial_t K_0[\psi_2 \tilde{v}_0]\) and \(\partial_t L_0[\psi_2 f]\) in
\[
g_2[\tilde{\Xi}] = -[\psi_2, -\Delta_x]L[\psi_1, -\Delta_x] \partial_t S_0[\psi_2 \Xi],
\]
respectively, so that \(g_2[\tilde{\Xi}] = \tilde{g}_{2,0}[\tilde{v}_0] + \tilde{g}_{2,1}[f]\). By (4.11) and (3.9), we get
\begin{equation}
(4.26) \quad (t)^{1-\eta} |\tilde{g}_{2,0}[\tilde{v}_0](t, x)|_m \leq C \sum_{|\beta| \leq m+3} \sup_{(s, x) \in [0, t] \times \Omega} \langle s \rangle^{1-\eta} |\partial^\beta \partial_t K_0[\psi_2 \tilde{v}_0](s, x)| \\
\leq CA_{2,m+4}[\tilde{v}_0]
\end{equation}
for \((t, x) \in [0, T) \times \Omega\). Applying (4.3) with \(\rho = 1 - \eta, \mu = \eta\), we find
\begin{equation}
(4.27) \quad (w_{1-\eta}(t, x))^{-1}|\partial L_0[\tilde{g}_{2,0}[\tilde{v}_0]](t, x)|_k \leq CA_{2,k+5}[\tilde{v}_0]
\end{equation}
for \((t, x) \in [0, T) \times \Omega\). On the other hand, by (4.11) and (4.3), we obtain
\begin{equation}
(4.28) \quad (t)^{1-\eta} |\tilde{g}_{2,1}[f](t, x)|_m \leq C \sum_{|\beta| \leq m+3} \sup_{(s, x) \in [0, t] \times \Omega} \langle s \rangle^{1-\eta} |\partial^\beta \partial_t L_0[\psi_2 f](s, x)| \\
\leq C \log(2 + t + |x|) \|f(t) : N_{k+4}(W_{1,1})\|
\end{equation}
for \((t, x) \in [0, T) \times \Omega\). By (4.3) with \(\rho = 1 - \eta \text{ and } \mu = 0\), we obtain
\begin{equation}
(4.29) \quad (w_{1-\eta}(t, x))^{-1}|\partial L_0[\tilde{g}_{2,1}[f]](t, x)|_k \\
\leq C(\log(2 + t + |x|))^2 \|f(t) : N_{k+5}(W_{1,1})\|
\end{equation}
for \((t, x) \in [0, T) \times \overline{\Omega}\). Thus we get
\[
(w_{1-\eta}(t, x))^{-1}|\partial S_2[\tilde{\Xi}](t, x)| \leq CA_{2,k+5}[\tilde{v}_0]
+C(\log(2 + t + |x|))^2 \|f(t)\|_{N_{k+5}(W_{1,1})}
\]
for \((t, x) \in [0, T) \times \overline{\Omega}\). Now (4.9) follows from (4.22), (4.23), (4.24), (4.25), and (4.26). This completes the proof. 

5. Proof of Theorem 1.4

In this section we prove Theorem 1.1. Let all the assumptions of Theorem 1.1 be fulfilled and let us assume \(O \subset B_1\). Though there is no essential difficulty in treating the quasilinear case, we concentrate on the semilinear case to keep our exposition simple. Namely, we assume that the nonlinearity is of the form
\[
F_i(\partial u) = \sum_{a=0}^N \sum_{j,k,l=1}^N g^a_{i,j,k,l}(\partial^a u_j)(\partial^b u_k)(\partial^c u_l),
\]
where \(g^a_{i,j,k,l} (a, b, c = 0, 1, 2)\) are real constants.

Since the local existence for the mixed problem (1.1)-(1.3) can be proved by a standard argument, we have only to deduce an \(apriori\) estimate. Let \(u\) be a smooth solution to (1.1)-(1.3) on \([0, T) \times \overline{\Omega}\). For a nonnegative integer \(k\), we set
\[
e_k[u](t, x) = (w_{1/2}(t, x))^{-1}|\partial u(t, x)|.
\]
Since \(\phi, \psi \in C_0^\infty(\overline{\Omega}; \mathbb{R}^N)\), we have \(\|e_k[u](0) : L^\infty(\Omega)\| \leq C\varepsilon\). Let \(k \geq 29\), and assume that
\[
\sup_{0 \leq t < T} \|e_k[u](t) : L^\infty(\Omega)\| \leq M\varepsilon
\]
holds for some large \(M(>1)\) and small \(\varepsilon(>0)\), satisfying \(M\varepsilon \leq 1\).

Because the decay property is weak when \(n = 2\), we need to refine a treatment of the boundary term arising from the integration-by-parts argument, compared with the case \(n = 3\). Namely, we shall make use of rather stronger decay of the time derivative based on (4.9).

5.1. Estimates of the energy. In this subsection we shall prove
\[
\sum_{|\alpha| \leq 2k} \|\partial^\alpha \partial u(t) : L^2(\Omega)\| \leq CM\varepsilon(1 + t)^{C_0 M^2 \varepsilon^2}, \quad t \in [0, T)
\]
where \(C_0\) is a universal constant which is independent of \(M, \varepsilon\) and \(T\).

For \(0 \leq m \leq 2k\), we define \(z_m(t) = \sum_{p=0}^{2k-m} \|\partial^p \partial u(t) : H^m(\Omega)\|\). To prove (5.3), it suffices to show
\[
z_m(t) \leq CM\varepsilon(1 + t)^{C_0 M^2 \varepsilon^2} \quad \text{for } 0 \leq m \leq 2k.
\]
First we evaluate $z_0(t)$. For $0 \leq p \leq 2k$, from (5.2) we get

$$|\partial_t^p F(\partial u)(t, x)| \leq CM^2 \varepsilon^2 (1 + t)^{-1} \sum_{q=0}^{2k} |\partial_t^q \partial u(t, x)|,$$

so that

$$\|\partial_t^p F(\partial u)(t) : L^2(\Omega)\| \leq C_0 M^2 \varepsilon^2 (1 + t)^{-1} z_0(t).$$

Therefore, noting that the boundary condition (1.2) implies $\partial_\nu^p u(t, x) = 0$ for $(t, x) \in [0, T) \times \partial \Omega$ and $0 \leq p \leq 2k + 1$, we see from the energy inequality for the wave equation that

$$\frac{dz_0(t)}{dt} \leq C_0 M^2 \varepsilon^2 (1 + t)^{-1} z_0(t),$$

which yields

(5.5) \hspace{1cm} z_0(t) \leq C M \varepsilon (1 + t)^{C_0 M^2 \varepsilon^2}.

Next suppose $m \geq 1$. Then, from the definition of $z_m$, we have

$$z_m(t) \leq C \sum_{p=0}^{2k-m} \left( \|\partial_t^p \partial u(t) : L^2(\Omega)\| + \sum_{1 \leq |\alpha| \leq m} \|\partial_t^p \partial_{\alpha} \partial u(t) : L^2(\Omega)\| \right. $$

$$\left. + \sum_{1 \leq |\alpha| \leq m} \|\partial_t^p \partial_{\alpha}^2 \nabla_x u(t) : L^2(\Omega)\| \right)$$

$$\leq C \left( z_0(t) + z_{m-1}(t) + \sum_{p=0}^{2k-m} \sum_{2 \leq |\alpha| \leq m+1} \|\partial_t^p \partial_{\alpha}^2 u(t) : L^2(\Omega)\| \right),$$

where we have used

$$\sum_{1 \leq |\alpha| \leq m} \|\partial_t^p \partial_{\alpha}^2 \partial u(t) : L^2(\Omega)\| \leq C \sum_{|\alpha'| \leq m-1} \|\partial_t^p \partial_{\alpha'}^2 \nabla_x u(t) : L^2(\Omega)\|.$$ 

For $2 \leq |\alpha| \leq m + 1$, (3.1) yields

$$\|\partial_t^p \partial_{\alpha}^2 u(t) : L^2(\Omega)\| \leq C(\|\Delta_x \partial_t^p u(t) : H^{m-1}(\Omega)\| + \|\nabla_x \partial_t^p u(t) : L^2(\Omega)\|).$$

For $0 \leq p \leq 2k - m$, we see that the second term on the right-hand side in the above is bounded by $z_0(t)$. On the other hand, by using (1.1), the first term is estimated by

$$C(\|\partial_t^{p+2} u(t) : H^{m-1}(\Omega)\| + \|\partial_t^p F(\partial u)(t) : H^{m-1}(\Omega)\|)$$

$$\leq C(z_{m-1}(t) + M \varepsilon^2 (1 + t)^{-1} z_{m-1}(t))$$

for $0 \leq p \leq 2k - m$. Since $M \varepsilon \leq 1$, we obtain

(5.6) \hspace{1cm} z_m(t) \leq C(z_{m-1}(t) + z_0(t))$$

for $m \geq 1$. Using (5.5), we find (5.3).
5.2. Estimates of the generalized energy, part 1. In this subsection we evaluate the generalized derivatives $\partial Z^\alpha u$ in $L^2(\Omega)$ for $1 \leq |\alpha| \leq 2k - 1$. It follows from (2.7) that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |\partial_1 Z^\alpha u_i|^2 + |\nabla_x Z^\alpha u_i|^2 \right) dx
$$

(5.7)

$$
= \int_{\Omega} Z^\alpha F_i(\partial u) \partial_1 Z^\alpha u_i dx + \int_{\partial \Omega} (\nu \cdot \nabla_x Z^\alpha u_i)(\partial_1 Z^\alpha u_i) dS,
$$

where $\nu = \nu(x)$ is the unit outer normal vector at $x \in \partial \Omega$, and $dS$ is the surface measure on $\partial \Omega$. Since $\partial \Omega \subset B_1$ implies $|\partial Z^\alpha u(t,x)| \leq C \sum_{1 \leq |\beta| \leq |\alpha|} |\partial^\beta \partial u(t,x)|$ for $(t, x) \in [0, T) \times \partial \Omega$, we see from the trace theorem that the second term on the right-hand side of (5.7) is evaluated by

$$
C \sum_{1 \leq |\beta| \leq |\alpha| + 1} \|\partial^\beta \partial u(t)\|_{L^2(\Omega)}^2 \leq CM^2 \varepsilon^2 (1 + t)^2 C M^2 \varepsilon^2,
$$

in view of (5.3). On the other hand, it follows that

$$
\|Z^\alpha F(\partial u)(t)\|_{L^2(\Omega)} \leq C_1 M^2 \varepsilon^2 (1 + t)^{-1} \|\partial u(t)\|_{|\alpha|}
$$

(5.8)

for $|\alpha| \leq 2k - 1$, where $C_1$ is a constant independent of $\alpha$, $M$, $\varepsilon$, and $T$.

Let $\delta > 0$ be a sufficiently small number that is fixed later on, and take $\varepsilon_0 > 0$ in such a way that $C_1 M^2 \varepsilon_0^2 + C_0 M^2 \varepsilon_0^2 \leq \delta$. Then for $0 < \varepsilon \leq \varepsilon_0$, we get

$$
\frac{d}{dt} \|\partial u(t)\|_{2k-1}^2 \leq \frac{C_1 M^2 \varepsilon^2 (1 + t)^{-1} \|\partial u(t)\|_{2k-1}^2}{2} + CM^2 \varepsilon^2 (1 + t)^{2\delta},
$$

which leads to

$$
\|\partial u(t)\|_{2k-1} \leq CM \varepsilon (1 + t)^{\delta + (1/2)}, \quad t \in [0, T).
$$

(5.9)

5.3. Estimates of the generalized energy, part 2. By (3.23) and (5.9) we have

$$
\langle x \rangle^{1/2} \|\partial u(t, x)\|_{2k-3} \leq CM \varepsilon (1 + t)^{\delta + (1/2)},
$$

(5.10)

which yields

$$
\|F(\partial u)(t) : N_{2k-3}(W_{1,1})\| \leq C M^3 \varepsilon^3 (1 + t)^{\delta + (1/2)},
$$

by (2.14) with $\nu = 1/2$. For a sufficiently small number $\eta$, we set $\nu = 1 - \eta$, in the following. Applying (1.9), we get

$$
|\partial \partial_1 u(t, x)|_{2k-8} \leq C w \nu \varepsilon (x) (\varepsilon + M^3 \varepsilon^3 (\log(2 + t))^{2} (1 + t)^{\delta + (1/2)}),
$$

because of the finite speed of propagation. When $(t, x) \in [0, T) \times \overline{\Omega_1}$, by (2.15) we have

$$
|\partial \partial_1 u(t, x)|_{2k-8} \leq CM \varepsilon (1 + t)^{-\nu + 2\delta + (1/2)},
$$

because $M > 1$ and $M \varepsilon \leq 1$. Using this inequality, (5.3), and (5.8), we arrive at

$$
\frac{d}{dt} \|\partial u(t)\|_{2k-7}^2 \leq C_1 M^2 \varepsilon^2 (1 + t)^{-1} \|\partial u(t)\|_{2k-7}^2 + CM^2 \varepsilon^2 (1 + t)^{-\nu + 3\delta + (1/2)},
$$
which leads to
\begin{equation}
\|\partial u(t)\|_{2k-7} \leq CM\varepsilon(1 + t)^{-\nu/2 + (3\delta/2) + (3/4)}, \quad t \in [0, T),
\end{equation}
provided $0 < \varepsilon \leq \varepsilon_0$.

### 5.4. Estimates of the generalized energy, part 3.

Repeating the argument in the previous step, we get
\begin{equation}
\|F(\partial u)(t) : N_{2k-9}(W_{1,1})\| \leq CM^3\varepsilon^3(1 + t)^{-\nu/2 + (3\delta/2) + (3/4)}.
\end{equation}
Using (4.8) with $\rho = 1/2, \kappa = 1$, and (4.9) with $\nu = 1 - \eta$, we obtain
\begin{align*}
|\partial \nabla u(t, x)|_{2k-14} &\leq Cw_{1/2}(t, x)(\varepsilon + M^3\varepsilon^3(\log(2 + t))^2)(1 + t)^{-\nu/2 + (3\delta/2) + (3/4)} \\
|\partial \partial u(t, x)|_{2k-14} &\leq Cw_{\nu}(t, x)(\varepsilon + M^3\varepsilon^3(\log(2 + t))^2)(1 + t)^{-\nu/2 + (3\delta/2) + (3/4)} \\
&\leq CM\varepsilon(1 + t)^{-3\nu/2 + 3\delta + (3/4)}
\end{align*}
for $(t, x) \in [0, T) \times \overline{\Omega}_1$. Therefore, we get
\begin{equation}
\frac{d}{dt}\|\partial u(t)\|_{2k-13}^2 \leq C_1M^2\varepsilon^2(1 + t)^{-1}\|\partial u(t)\|_{2k-13}^2 + CM^2\varepsilon^2(1 + t)^{-2\nu + 4\delta + 1},
\end{equation}
which yields
\begin{equation}
\|\partial u(t)\|_{2k-13} \leq CM\varepsilon(1 + t)^{-\nu + 2\delta + 1}, \quad t \in [0, T),
\end{equation}
provided $0 < \varepsilon \leq \varepsilon_0$.

### 5.5. Estimates of the generalized energy, part 4.

As before, we have
\begin{equation}
\|F(\partial u)(t) : N_{2k-15}(W_{1,1})\| \leq CM^3\varepsilon^3(1 + t)^{-\nu + 2\delta + 1},
\end{equation}
so that
\begin{align*}
|\partial \nabla u(t, x)|_{2k-20} &\leq CM\varepsilon(1 + t)^{-\nu + 3\delta + (1/2)} \\
|\partial \partial u(t, x)|_{2k-20} &\leq CM\varepsilon(1 + t)^{-2\nu + 3\delta + 1}
\end{align*}
for $(t, x) \in [0, T) \times \overline{\Omega}_1$. Hence we get
\begin{equation}
\frac{d}{dt}\|\partial u(t)\|_{2k-19}^2 \leq C_2M^2\varepsilon^2(1 + t)^{-1}\|\partial u(t)\|_{2k-19}^2 + CM^2\varepsilon^2(1 + t)^{-3\nu + 6\delta + (3/2)}.
\end{equation}
If we choose $\delta$ so small that $3\nu - 6\delta - (3/2) > 1$, then we get $\|\partial u(t)\|_{2k-19} \leq CM\varepsilon(1 + t)^{C_1M^2\varepsilon^2}$ for $0 < \varepsilon \leq \varepsilon_0$. Taking $T$ in such a way that
\begin{equation}
(2 + T)^{M^2\varepsilon^2} \leq e,
\end{equation}
we have
\begin{equation}
\|\partial u(t)\|_{2k-19} \leq CM\varepsilon, \quad t \in [0, T).
\end{equation}
5.6. Pointwise estimates, part 2. By virtue of (5.15), we get
\[ \|F(\partial u)(t) : N_{2k-21}(W_{1,1})\| \leq CM^3 \varepsilon^3. \]
Let \( 0 < \rho < 1/2 \). Then it follows from (4.8) that
\[ |\partial u(t, x)|_{2k-25} \leq Cw_\rho(t, x)(\varepsilon + M^3 \varepsilon^3 \log(2 + t)) \]
for \((t, x) \in [0, T) \times \Omega\). Therefore, assuming (5.14), we get
\[ |\partial u(t, x)|_{2k-25} \leq CM_\varepsilon w_\rho(t, x) \]
for \((t, x) \in [0, T) \times \Omega\), provided \( 0 < \varepsilon \leq \varepsilon_0 \).

5.7. Pointwise estimates, final part. For \( \kappa = 1 + \rho \), we get
\[ \|F(\partial u)(t) : N_{2k-25}(W_{1,\kappa})\| \leq CM^3 \varepsilon^3, \]
by (5.16). Using (4.8) with \( \rho = 1/2 \) and \( \kappa > 1 \), we have
\[ |\partial u(t, x)|_{2k-29} \leq C_2 w_{1/2}(t, x)(\varepsilon + M^3 \varepsilon^3 \log(2 + t)) \]
for \((t, x) \in [0, T) \times \Omega\), provided \( 0 < \varepsilon \leq \varepsilon_0 \). Here \( C_2 \) is a constant independent of \( M, \varepsilon \) and \( T \). From (5.17) we find that (5.2) with \( M \) replaced by \( M/2 \) is true for \( M \geq 4C_2 \) and \( C_2 M^2 \varepsilon^2 \log(2 + T) \leq 1/4 \). Then, for \( \varepsilon \in (0, \varepsilon_0] \), the standard continuity argument implies that \( e_k[u](t) \) stays bounded as long as the solution \( u \) exists (observe that \( \|e_k[u](t) : L^{\infty}(\Omega)\| \) is continuous with respect to \( t \), because \( u \) is smooth and \( \text{supp } u(t, \cdot) \subset B_{t+R} \) for \( t \in [0, T) \) with some \( R > 0 \)). Theorem 1.1 follows immediately from this \textit{a priori} bound and a restriction on \( T \). This completes the proof. \( \square \)

Appendix A: Proof of Lemma 3.1

Suppose \( m \geq 2 \) and \( \varphi \in H^m(\Omega) \cap H_0^1(\Omega) \). Let \( \chi \) be a \( C_0^\infty(\mathbb{R}^2) \) function such that \( \chi \equiv 1 \) in a neighborhood of \( O \). Let \( \text{supp } \chi \subset B_R \) for some \( R > 1 \). We set \( \varphi_1 = \chi \varphi \) and \( \varphi_2 = (1 - \chi) \varphi \), so that \( \varphi = \varphi_1 + \varphi_2 \).

First we estimate \( \varphi_1 \). The following elliptic estimate (see Chapter 9 in [3] for instance)
\[ \|v : H^{k+2}(\Omega_R)\| \leq C(\|\Delta v : H^k(\Omega_R)\| + \|v : L^2(\Omega_R)\|) \]
holds for \( v \in H^{k+2}(\Omega_R) \cap H_0^1(\Omega_R) \) with a non-negative integer \( k \). On the other hand, we have
\[ \|v : L^2(\Omega_R)\| \leq C\|\nabla v : L^2(\Omega)\| \]
for \( v \in H_0^1(\Omega) \). Indeed, one can show (A.2) as follows. We define a positive number \( r(\omega) \) for each \( \omega \in S^1 \) so that \( r(\omega) \omega \in \partial \Omega \), and put \( r_0 = \text{dist } (0, \partial \Omega) \). Then, for \( v \in C_0^\infty(\Omega) \)
we have

\[
|v(r\omega)|^2 = \left| \int_{r(\omega)}^r (\omega \cdot \nabla v)(s\omega) ds \right|^2 \\
\leq \left( \int_{r(\omega)}^r \frac{ds}{s} \right) \left( \int_{r(\omega)}^r |\nabla v(s\omega)|^2 s ds \right) \\
\leq \frac{r}{r_0} \int_{r(\omega)}^R |\nabla v(s\omega)|^2 s ds
\]

for \( r(\omega) < r < R \), because \( r(\omega) \geq r_0 \). Multiplying it by \( r \) and integrating the resulting inequality over \( \Omega_R \), we find (A.2).

Since \( \varphi \in H^1_0(\Omega) \) and \( \text{supp } \chi \subset B_R \), we have \( \varphi_1 \in H^1_0(\Omega_R) \). Therefore, the application of (A.1) in combination with (A.2) gives

(A.3) \[
\|\varphi_1 : H^m(\Omega)\| \leq C(\|\Delta_x \varphi : H^{m-2}(\Omega)\| + \|\nabla_x \varphi : L^2(\Omega)\|).
\]

Now our task is to show

(A.4) \[
\sum_{|\alpha|=m} \|\partial^\alpha_x \varphi_2 : L^2(\Omega)\| \leq C(\|\Delta_x \varphi : H^{m-2}(\Omega)\| + \|\nabla_x \varphi : L^2(\Omega)\|),
\]

because it implies (3.1) in view of (A.3).

Since \( \|\partial^\alpha w : L^2(\mathbb{R}^2)\| \leq C\|\Delta_x w : L^2(\mathbb{R}^2)\| \) for \( |\alpha| = 2 \) and \( w \in H^2(\mathbb{R}^2) \), the left-hand side of (A.4) with \( m = 2 \) is estimated by

\[
C\|\Delta_x \varphi_2 : L^2(\Omega)\| \leq C(\|\Delta_x \varphi : L^2(\Omega)\| + \|\nabla_x \varphi : L^2(\Omega)\| + \|\varphi : L^2(\Omega_R)\|).
\]

Hence, using (A.2), we obtain (A.4) for \( m = 2 \).

For \( k \geq 3 \), similar argument to the above gives

\[
\sum_{|\alpha|=k} \|\partial^\alpha_x \varphi_2 : L^2(\Omega)\| \leq C(\|\Delta_x \varphi : H^{k-2}(\Omega)\| + \|\nabla_x \varphi : H^{k-2}(\Omega)\|),
\]

and the second term on the right-hand side is bounded by \( C(\|\Delta_x \varphi : H^{k-3}(\Omega)\| + \|\nabla_x \varphi : L^2(\Omega_R)\|) \), if we know (3.1) for \( m = k - 1 \). Hence we inductively obtain (A.4) for \( m \geq 2 \).

□

APPENDIX B: BASIC ESTIMATES FOR THE CAUCHY PROBLEM

Here we prove the basic estimates used in the proof of Lemma 3.5. In [13], [1], and [8], some weighted \( L^\infty \) estimates for derivatives of solutions to the Cauchy problem are well examined. However, we need their variants under different assumptions on the right-hand member. Although the proof of (B.6) below can be done in a similar way as in the previous works, we give its proof, because the case where \( 0 < \nu < 1 \) has not been considered at all.
First of all, we introduce a couple of functions:

\[ K_1(\lambda, \psi; r, t) = (2\pi)^{-\frac{1}{2}} \left\{ t^2 - r^2 - \lambda^2 + 2r\lambda \cos \psi \right\}^{-\frac{1}{2}}, \]

\[ K_2(\lambda, \tau; r, t) = (2\pi)^{-\frac{1}{2}} \left\{ 2r\lambda \tau(1 - \tau)(2 - (1 - \cos \varphi)\tau) \right\}^{-\frac{1}{2}}, \]

\[ \varphi(\lambda; r, t) = \arccos \left[ \frac{r^2 + \lambda^2 - t^2}{2r\lambda} \right], \]

\[ \Psi(\lambda, \tau; r, t) = \arccos[1 - (1 - \cos \varphi(\lambda; r, t))\tau], \]

\[ K_3^{(\ell)}(\lambda, \psi; r, t) = -\frac{(x_\ell - \lambda \xi_\ell)}{2\pi(t^2 - r^2 - \lambda^2 + 2r\lambda \cos \psi)^{\frac{3}{2}}} \quad (\ell = 1, 2). \]

As for these functions, we shall use the following estimates. For the proof of (B.1), (B.2), and (B.4), see for instance Proposition 5.3 in [1]. Concerning (B.3) and (B.5), see the proof of (4.14) and (4.34) in [8], respectively.

**Lemma B.2.** We set \( \lambda_- = |t - s - r| \) and \( \lambda_+ = t - s + r \). If \( 0 < s < t \) and \( \lambda_- < \lambda < \lambda_+ \), then we have

\[ \int_{-\varphi}^{\varphi} K_1(\lambda, \psi; r, t - s)d\psi = 2 \int_0^1 K_2(\lambda, \tau; r, t - s)d\tau \leq \frac{C}{(r\lambda)^{\frac{3}{2}}} \log \left[ 2 + \frac{r\lambda}{(\lambda - \lambda_+)(\lambda + \lambda)}H(t - s - r) \right], \]

(B.1)

\[ \int_0^1 |\partial_\lambda K_2(\lambda, \tau; r, t - s)|d\tau \leq \frac{C}{(r\lambda)^{\frac{3}{2}}(\lambda + s + r - t)}, \]

(B.2)

\[ \int_0^1 |\partial_\lambda \Psi \cdot K_2(\lambda, \tau; r, t - s)|d\tau \leq \frac{C}{(r\lambda)^{\frac{3}{2}}} \left( \frac{1}{\sqrt{(\lambda + \lambda_+)(\lambda - \lambda_-)}} + \frac{1}{\sqrt{\lambda^2 - \lambda_-^2}} \right), \]

(B.3)

where \( H(s) = 1 \) for \( s > 0 \) and \( H(s) = 0 \) otherwise.

On the other hand, if \( 0 < s < t - r \) and \( 0 < \lambda < \lambda_- \), then we have

\[ \int_0^\pi K_1(\lambda, \psi; r, t - s)d\psi \leq \frac{C}{\sqrt{(\lambda + \lambda_-)(\lambda_+ - \lambda)}} \log \left[ 2 + \frac{r\lambda}{(\lambda_+ - \lambda)(\lambda_+ + \lambda)} \right], \]

(B.4)

\[ \int_{-\pi}^{\pi} |K_3^{(\ell)}(\lambda, \psi; r, t - s)|d\psi \leq \frac{C}{(\lambda_+ - \lambda)(\lambda_+ + \lambda)} \sqrt{(\lambda_+ - \lambda)^2 - (\lambda + \lambda_+)(\lambda - \lambda_-)^2}. \]

(B.5)

Now we are in a position to state our basic estimates for solutions to the Cauchy problem.
Proposition B.3. Let $0 < \nu < 3/2$, $\mu \geq 0$, $\kappa \geq 1$, and $\eta > 0$. Then we have

\begin{align}
(B.6) & \quad |L_0[\partial_t g](x,t)|(1 + |x|)^{1/2}(1 + |t - |x||)^\nu \\
& \leq C \Psi_{1+\mu}(t + |x|) \Psi_\kappa(|x|) \|g(t) : M_1(z_{\nu,\mu,\kappa,0})\|,
\end{align}

\begin{align}
(B.7) & \quad |L_0[\partial_x g](x,t)|(1 + |x|)^{1/2}(1 + |t - |x||)^{1-\eta} \\
& \leq C \log(2 + t + |x|) \|g(t) : M_1(z_{1,1,\kappa,0})\|
\end{align}

for $(x,t) \in \mathbb{R}^2 \times [0,T)$, where $C$ depends on $\nu$, $\mu$, $\kappa$, and $\eta$.

Proof. We prove only (B.6), because the other can be treated analogously. In addition, we evaluate only the spatial derivatives, since the time derivative can be handled by using Proposition 5.3 in [1]. Besides, since the case where $\mu > 0$ is treated by modifying a little the argument for handling the case $\mu = 0$, we let $\mu = 0$ in the following.

We set

\begin{align}
E_1 &= \{(y,s) \in \mathbb{R}^2 \times [0,t) : |y| + s > t - r, \ |x - y| < t - s\}, \\
E_2 &= \{(y,s) \in \mathbb{R}^2 \times [0,t) : |y| + s < t - r\},
\end{align}

so that $\overline{E_1 \cup E_2} = \{(y,s) \in \mathbb{R}^2 \times [0,t) : |x - y| < t - s\}$. According to this decomposition, we define

\begin{equation}
P_j[g](x,t) = \frac{1}{2\pi} \int_{E_j} \frac{g(y,s)}{\sqrt{(t-s)^2 - |x-y|^2}} \, dyds \quad (j = 1,2). \tag{B.8}
\end{equation}

Then we have $L_0[\partial_t g](x,t) = P_1[\partial_t g](x,t) + P_2[\partial_t g](x,t)$ with $\ell = 1,2$.

Firstly we deal with $P_1[\partial_t g](x,t)$. Following the computation made in the section 4 of [7], we find that

\begin{equation}
|P_1[\partial_t g](x,t)| \leq \|g(t) : M_1(z_{\nu,\kappa,0})\| \sum_{k=0}^5 I_k, \tag{B.9}
\end{equation}

where we have set

\begin{align}
I_1 &= \int_{D_1} \frac{\lambda \psi}{z_{\nu,\kappa,0}(\lambda,s)} \, d\lambda ds \int_{-\varphi}^{\varphi} K_1(\lambda,\psi;r,t-s) \, d\psi, \\
I_2 &= \int_{D_2} \frac{\lambda \rho}{z_{\nu,\kappa,0}(\lambda,s)} \, d\lambda ds \int_0^1 K_2(\lambda,\varphi;r,t-s) \, d\varphi, \\
I_3 &= \int_{D_2} \frac{1}{z_{\nu,\kappa,0}(\lambda,s)} \, d\lambda ds \int_0^1 K_2(\lambda,\varphi;r,t-s) \, d\varphi, \\
I_4 &= \int_{D_2} \frac{\lambda \varphi}{z_{\nu,\kappa,0}(\lambda,s)} \, d\lambda ds \int_0^1 |\partial_\lambda K_2(\lambda,\varphi;r,t-s)| \, d\varphi, \\
I_5 &= \int_{D_2} \frac{\lambda \varphi}{z_{\nu,\kappa,0}(\lambda,s)} \, d\lambda ds \int_0^1 |(\partial_\lambda \Psi \cdot K_2)(\lambda,\varphi;r,t-s)| \, d\varphi
\end{align}
and
\[ D_1 = \{(\lambda, s) \in (0, \infty) \times (0, t) : \lambda_- < \lambda \leq \lambda_- + \delta \text{ or } \lambda_+ - \delta \leq \lambda < \lambda_+\}, \]
\[ D_2 = \{(\lambda, s) \in (0, \infty) \times (0, t) : \lambda_- + \delta \leq \lambda \leq \lambda_+ - \delta\}, \]
\[ D'_2 = \{(\lambda, s) \in (0, \infty) \times (0, t) : \lambda = \lambda_- + \delta \text{ or } \lambda = \lambda_+ - \delta\} \]
with \( \delta = \min\{r, 1/2\} \) and \( \lambda_- = |t - s - r|, \lambda_+ = t - s + r \).

Now we are going to show that
\[ I_k \leq C(1 + r)^{-\frac{1}{2}}(1 + |t - r|)^{-\nu} \log(2 + t + r) \Psi_\kappa(t + r) \]
holds for \( k = 1, \ldots, 5 \). First we evaluate \( I_1 \). Notice that when \( 0 < s < t - r \) and \( \lambda > \lambda_+ - \delta \), we have \( \lambda - \lambda_- > r \), so that
\[ \log \left[ 2 + \frac{r\lambda}{(\lambda - \lambda_-)(\lambda_+ + \lambda)} \right] \leq \log 3. \]

For \( 0 < s < t - r \) and \( \lambda > \lambda_- \), we get
\[ \log \left[ 2 + \frac{r\lambda}{(\lambda - \lambda_-)(\lambda_+ + \lambda)} \right] \leq \log \left[ 2 + \frac{\lambda}{\lambda - \lambda_-} \right]. \]
Moreover, we note that \( z_{\nu, \kappa, 0}(\lambda, s) \) is equivalent to \( z_{\nu, \kappa, 0}(\lambda_+, s) \) (resp. \( z_{\nu, \kappa, 0}(\lambda_-, s) \)) for \( \lambda_+ - \delta < \lambda < \lambda_+ \) (resp. \( \lambda_- < \lambda < \lambda_- + \delta \)). Hence by (B.11), we get
\[ I_1 \leq Cr^{-\frac{1}{2}}[A_{1,0} + A_{2,0} + A_{3,0}], \]
where we have set
\[ A_{1,0} = \int_0^t \int_{\lambda_- + \delta}^{\lambda_+} \frac{1}{z_{\nu, \kappa, 0}(\lambda_+, s)} d\lambda ds, \]
\[ A_{2,0} = \int_0^{(t-r)_+} \int_{\lambda_- + \delta}^{\lambda_-} \frac{1}{z_{\nu, \kappa, 0}(\lambda_-, s)} \log \left[ 2 + \frac{\lambda}{\lambda - \lambda_-} \right] d\lambda ds, \]
\[ A_{3,0} = \int_{(t-r)_+}^t \int_{\lambda_- + \delta}^{\lambda_-} \frac{1}{z_{\nu, \kappa, 0}(\lambda_-, s)} d\lambda ds. \]

It is easy to see that
\[ A_{1,0} \leq \frac{\delta}{(1 + t + r)^\nu} \int_0^t \frac{1}{1 + \lambda_+} ds \leq C\delta(1 + t + r)^{-\nu} \Psi_\kappa(t + r). \]

To evaluate \( A_{2,0} \), observe that
\[ \int_{\lambda_-}^{\lambda_- + \delta} \log \left[ 2 + \frac{\lambda}{\lambda - \lambda_-} \right] d\lambda \leq C\delta^{1/2} \log(3 + \lambda_-). \]
Indeed, the left-hand side is equal to
\[ \delta \log \left[ 3 + \frac{\lambda_-}{\delta} \right] + \int_{\lambda_-}^{\lambda_- + \delta} \frac{\lambda_-}{3(\lambda - \lambda_-) + \lambda_-} d\lambda, \]
which is bounded by the right-hand side, if we use $0 \leq \delta \leq 1/2$, and an inequality $|x^{1/2}\log x| \leq 2e^{-1}$ for $0 < x < 1$.

Since $s + \lambda_- \geq |t - r|$, we get

$$A_{2,0} \leq C \delta^{1/2} \frac{(1 + |t - r|)\nu}{(1 + \lambda_-)^\kappa} \int_0^{(t-r)+} \log(3 + \lambda_-) ds$$

$$\leq C \delta^{1/2}(1 + |t - r|)^{-\nu}(\Psi_\kappa(t - r))^2.$$

We easily have

$$A_{3,0} \leq C\delta(1 + |t - r|)^{-\nu}\Psi_\kappa(t + r).$$

Summing up (B.13), (B.15) and (B.16), we see from (B.12) that (B.10) holds for $k = 1$.

In the following, we assume $r \geq 1/2$ so that $\delta = 1/2$, because $D_2$ is the empty set when $0 < r < 1/2$. Since $\lambda = \lambda_- + (1/2)$ or $\lambda = \lambda_+ - (1/2)$ for $(\lambda, s) \in D_2$, we get (B.10) for $k = 2$ similarly to the previous argument.

Next we evaluate $I_3$. Note that $\lambda \geq 1/2$ if $(\lambda, s) \in D_2$ and that

$$\log \left(2 + \frac{r\lambda}{(\lambda - \lambda_+)(\lambda_+ + \lambda)}\right) \leq C\log(2 + \lambda)$$

for $0 < s < t$ and $\lambda \geq \lambda_- + (1/2)$. Therefore we get from (B.1)

$$r^{1/2}I_3 \leq C \int_{D_2} \frac{\log(2 + \lambda) d\lambda ds}{(1 + \lambda)\nu(\lambda, s)} = C \int_{D_2} \frac{\log(2 + \lambda) d\lambda ds}{(1 + s + \lambda)^\nu(1 + \lambda)^{1+\kappa}}.$$

Since $s + \lambda \geq |t - r|$ for $(\lambda, s) \in D_2$, the right-hand side is bounded by

$$\frac{C}{(1 + |t - r|)^\nu} \int_{D_2} \frac{\log(2 + \lambda) d\lambda ds}{(1 + \lambda)^{1+\kappa}} \leq \frac{C(\Psi_\kappa(t, r))^2}{(1 + |t - r|)^\nu}.$$

Therefore, (B.10) holds for $k = 3$.

Next we evaluate $I_4$. Since $\lambda + s + r - t \geq 1/2$ for $\lambda \geq \lambda_- + (1/2)$, we get from (B.2)

$$r^{1/2}I_4 \leq C \int_{D_2} \frac{d\lambda ds}{z_{\nu, r, 0}(\lambda, s)(\lambda + s + r - t + 1)}$$

$$\leq C \int_{|t-r|}^{t+r} \frac{d\alpha}{(\alpha - t + r + 1)(1 + \alpha)^\nu} \int_{-\alpha}^\alpha \frac{d\beta}{(1 + \alpha + \beta)^\kappa}$$

$$\leq C \int_{|t-r|}^{t+r} \frac{\Psi_\kappa(\alpha)}{(1 + |t - r|)^\nu} d\alpha$$

$$\leq C(1 + |t - r|)^{-\nu}\Psi_\kappa(t + r)\log(2 + r),$$

where we have changed the variables by

$$\alpha = \lambda + s, \quad \beta = \lambda - s.$$

Next we evaluate $I_5$. It follows from (B.3) that

$$r^{1/2}I_5 \leq C(A_{5,0} + B_{5,0} + C_{5,0}),$$
where we have set

\[
A_{5,0} = \int_{D_2} \frac{d\lambda ds}{Z_{\nu,\kappa;0}(\lambda, s) \sqrt{t - s + r - \lambda + 1} \sqrt{\lambda - t + s + r + 1}}, \\
B_{5,0} = \int_{D_2} \frac{d\lambda ds}{Z_{\nu,\kappa;0}(\lambda, s) \sqrt{t - s - \lambda + 1} \sqrt{\lambda + t - s - r + 1}}, \\
C_{5,0} = \int_{D_2} \frac{d\lambda ds}{Z_{\nu,\kappa;0}(\lambda, s) \sqrt{\lambda - t + s + r + 1} \sqrt{\lambda + t - s - r + 1}}.
\]

Changing the variables by (B.17), we have

\[
A_{5,0} \leq C \int_{|t - r|}^{t+r} \frac{d\alpha}{(1 + \alpha)^\nu \sqrt{t + r - \alpha} \sqrt{\alpha - t + r}} \int_{-\alpha}^{\alpha} \frac{d\beta}{(1 + \alpha + \beta)^\kappa} \\
\leq C (1 + |t - r|)^{1/2 - \nu} \int_{|t - r|}^{t+r} \frac{d\alpha}{\Psi_{\kappa}(\alpha)} \int_{t-r}^{\alpha} \frac{d\beta}{\sqrt{t + r - \alpha} \sqrt{\alpha - t + r}} \\
= C \pi (1 + |t - r|)^{-\nu} \Psi_{\kappa}(t + r).
\]

Moreover, we see that $B_{5,0}$ is bounded by

\[
C \int_{|t - r|}^{t+r} \frac{d\alpha}{(1 + \alpha)^\nu \sqrt{t + r - \alpha + 1}} \int_{r-t}^{\alpha} \frac{d\beta}{(1 + \alpha + \beta)^\kappa \sqrt{t - r + \beta + 1}}.
\]

By integrating by parts in the $\beta$-integral, it is estimated by

\[
2(1 + 2\alpha)^{(1/2) - \kappa} + 2\kappa \int_{r-t}^{\alpha} (1 + \alpha + \beta)^{-(1/2) - \kappa} d\beta \leq C (1 + \alpha + r - t)^{(1/2) - \kappa},
\]

if $\alpha \geq |r - t|$ and $\kappa > 1/2$. Therefore we get

\[
(1 + |t - r|)^{\nu} B_{5,0} \leq C \int_{|t - r|}^{t+r} \frac{d\alpha}{\sqrt{t + r - \alpha + 1} (1 + \alpha + r - t)^{\kappa - (1/2)}} \leq C,
\]

for $\kappa \geq 1$. Similarly, one can show

\[
(1 + |t - r|)^{\nu} C_{5,0} \leq C \Phi_{\kappa}(t + r).
\]

Thus we obtain (B.10) for all $k = 1, \ldots, 5$ in conclusion.

Secondly we deal with $P_2[\partial_t g](x, t)$. First, suppose $0 \leq t - r \leq 2$. Switching to the polar coordinates as

(B.18) \[ x = (r \cos \theta, r \sin \theta), \quad y = \lambda \xi = (\lambda \cos(\theta + \psi), \lambda \sin(\theta + \psi)) \]

in (B.8) with $j = 2$, we get

\[
P_2[\partial_t g](x, t) = \int_0^{t-r} \int_0^{t-s-r} \int_{-\pi}^{\pi} \lambda \partial_t g(\lambda \xi, s) K_1(\lambda, \psi; r, t - s) d\psi d\lambda ds.
\]
For $0 < s < t - r$, $0 < \lambda < \lambda_-$, we have $0 < \lambda_- - \lambda \leq 2$, so that (B.4) yields

\[(B.19) \quad \int_{-\pi}^{\pi} K_1(\lambda, \psi; r, t - s) d\psi \leq \frac{C}{\sqrt{\lambda\sqrt{\lambda_- - \lambda}}} \log \left[ 2 + \frac{\lambda}{\lambda_- - \lambda} \right] \leq \frac{C}{\sqrt{\lambda\sqrt{r + 1}} \sqrt{\lambda_- - \lambda}} \log \left[ 2 + \frac{\lambda}{\lambda_- - \lambda} \right].\]

Since $\lambda < \lambda_- \leq 2$, we get

\[
\sqrt{r + 1} |P_2[\partial_t g](x, t)| \\
\leq C\|g(t) : M_1(z_{\nu, \kappa, 0})\| \int_0^{t-r} \int_0^{\lambda_-} \frac{1}{\sqrt{\lambda_- - \lambda}} \log \left[ 2 + \frac{2}{\lambda_- - \lambda} \right] d\lambda ds.
\]

The last integral is bounded, because $0 \leq t - r \leq 2$. Hence we obtain

\[(B.20) \quad (1 + r)^{1/2}(1 + |t - r|)^\nu |P_2[\partial_t g](x, t)| \leq C\|g(t) : M_1(z_{\nu, \kappa, 0})\|.
\]

In the following, suppose $t - r \geq 2$, so that $t - r - 1 \geq (t - r)/2$. We decompose $P_2[\partial_t g](x, t)$ as

\[
P_2[\partial_t g](x, t) = \frac{1}{2\pi} \int_{E_2} \int H(|y| > t-s-r-1) \frac{\partial_t g(y, s)}{\sqrt{(t-s)^2 - |x-y|^2}} dyds \\
+ \frac{1}{2\pi} \int_{E_2} \int H(|y| < t-s-r-1) \frac{\partial_t g(y, s)}{\sqrt{(t-s)^2 - |x-y|^2}} dyds \\
\equiv Q_1(x, t) + Q_2(x, t).
\]

Since $0 < \lambda_- - \lambda < 1$ for $t-s-r-1 < \lambda < t-s-r$, one can proceed as in the previous case and get

\[
\sqrt{r + 1} |Q_1(x, t)| \leq C\|g(t) : M_1(z_{\nu, \kappa, 0})\| \\
\times \int_0^{t-r} \int_{(t-s-r-1)_+} \log(2 + \lambda) - \log(\lambda_- - \lambda) d\lambda ds \\
\]

Changing the variables by (B.17), the last integral is estimated by

\[
C \int_{t-r-1}^{t-r} \frac{d\alpha}{(1 + \alpha)^\nu(t-r-\alpha)} \int_{-\alpha}^{\alpha} \frac{\log(4 + \alpha + \beta) - \log(t-r-\alpha) - \log(4 + \alpha + \beta)}{(1 + \alpha + \beta)^\nu} d\beta \\
\leq C(1 + |t - r|)^{-\nu}(\Psi_\kappa(t - r))^2.
\]

Thus we get

\[(B.21) \quad \sqrt{r + 1} (1 + |t - r|)^\nu |Q_1(x, t)| \\
\leq C\|g(t) : M_1(z_{\nu, \kappa, 0})\|(\Psi_\kappa(t - r))^2.
\]
Finally, we deal with $Q_2(x,t)$. Making the integration by parts in $y$ and switching to the polar coordinates as (B.18), we get

$$Q_2(x,t) = \int_0^{t-r-1} \int_0^{t-s-r-1} \frac{1}{\lambda_+ - \lambda} \log \left[ 2 + \frac{\lambda}{\lambda_+ - \lambda} \right] ds$$

We see from (B.19) that the second term in the right-hand side of (B.22) is estimated by $C\|g(t)\|_r M_0(z_{\nu,\kappa,0})/\sqrt{r-1}$ times

$$\int_0^{t-r-1} \int_0^{t-s-r-1} \frac{1}{\lambda_+ - \lambda} \log \left[ 2 + \frac{\lambda}{\lambda_+ - \lambda} \right] \frac{d\lambda ds}{\sqrt{r-1}}$$

Noting $\lambda_+ - \lambda \geq 2r + 1$ for $\lambda < t - s - r - 1$, we see from (B.22) that the first term in the right-hand side of (B.22) is estimated by $C\|g(t)\|_r M_0(z_{\nu,\kappa,0})/\sqrt{r-1}$ times

$$\int_0^{t-r-1} \int_0^{t-s-r-1} \frac{1}{\lambda_+ - \lambda} \log \left[ 2 + \frac{\lambda}{\lambda_+ - \lambda} \right] \frac{d\lambda ds}{\sqrt{r-1}}$$

because we have assumed $\nu < 3/2$. Thus we get

$$\sqrt{r-1} (1 + |t - r|)^\nu |Q_2(x,t)| \leq C\|g(t)\|_r M_0(z_{\nu,\kappa,0}) (2 + t - r) \Psi_\kappa(t - r).$$

Now, (B.6) follows from (B.9), (B.10), (B.20), (B.21) and (B.23). This completes the proof of Proposition B.2.

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