The Programs of the Extended Relativity in C-spaces: Towards the Physical Foundations of String Theory

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Abstract

An outline is presented of the Extended Scale Relativity (ESR) in C-spaces (Clifford manifolds), where the speed of light and the minimum Planck scale are the two universal invariants. This represents in a sense an extension of the theory developed by L.Nottale long ago. It is shown how all the dimensions of a C-space can be treated on equal footing by implementing the holographic principle associated with a nested family of p-loops of various dimensionalities. This is achieved by using poly-vector valued coordinates in C-spaces that encode in one stroke points, lines, areas, volumes,... In addition, we review the derivation of the minimal-length string uncertainty relations; the logarithmic corrections (valid in any dimension) to the black hole area-entropy relation. We also show how the higher derivative gravity with torsion and the recent results of kappa-deformed Poincare theories of gravity follow naturally from the geometry of C-spaces. In conclusion some comments are made on the cosmological implications of this theory with respect to the cosmological constant problem, the two modes of time, the expansion of the universe, number four as the average dimension of our world and a variable fine structure constant.

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1 Introduction

Since the inception of string theory there have been an incessant strive to find the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein’s general relativity. This principle might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. In this connection it should be said that a deeper understanding of geometry and its relation to algebra has always turned out be very useful for the advancement of physical theories. Without analytical geometry Newton mechanics, and later special relativity, could not have acquired its full power in the description of physical phenomena. Without development of the geometries of curved spaces, general relativity could not have emerged. The role of geometry is nowadays being investigated also within the context of string theory, and especially in the searches for $M$-theory. The need for suitable generalizations, such as non commutative geometries is being increasingly recognized.

It was recognized long time ago [6] that Clifford algebra provided a very useful tool for a description of geometry and physics, containing a lot of room for important generalizations of the current physical theories. Hence it was suggested in [3,5] that every physical quantity is in fact a poly-vector, that is, a Clifford number or a Clifford aggregate. It has turned out that spinors are the members of left or right minimal ideals of Clifford algebra, the fact that provided a framework for a description and a deeper understanding of supersymmetries, i.e., the transformations relating bosons and fermions.

Moreover, it was shown that the well known Fock-Stueckelberg theory of relativistic particle [6] can be embedded in the Clifford algebra of spacetime [3]. Many other fascinating aspects of Clifford algebra are described in a recent book [3] and [5]. A recent overview of Clifford algebras and their applications is to be found in the book [6].

In addition, the fruitfulness of Clifford algebra showed itself in the following. A significant body of work was devoted to the collective dynamics of $p$-branes in terms of area variables [7]. It has been observed [2] that this has connection to $C$-space, and also to the branes with variable tension [22] and wiggly branes [3]. Moreover, using these methods the bosonic $p$-brane propagator [7] and the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short $C$-space) was obtained [12].

In previous papers [2,3,11,12] we proposed a new physical theory where the arena for physics is no longer the ordinary spacetime, but a more general
manifold of Clifford algebra valued objects, polyvectors. Such a manifold has been called a pan-dimensional continuum [5] or C-space [2]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes,..., called 0-loops, 2-loops, 3-loops, etc.. It is a sort of a dimension category, where the role of functorial maps is played by C-space transformations. The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes.

The ordinary space-time then is just a subspace of C-space. A “point” of C-space can correspond to any p-loop in ordinary space-time. Rotations in C-space transform one point of C-space into another point of C-space, and this appears in the ordinary space as a transformation from a p-loop into another p′-loop of different dimensionality p′. Technically those transformations are generalizations of Lorentz transformations to C-space. In that sense, the C-space is roughly speaking a sort of generalized Penrose-Twistor space.

Furthermore, instead of a flat C-space we may consider a curved C-space. Since moving from flat Minkowski space-time to a curved spacetime had provided us with a very deep insight into the nature of one of the fundamental interactions, namely gravity, we may expect that introduction of a curved C-space will increase further our understanding of the other fundamental interactions and their unification with gravity.

Motivated by these important developments and prospects, we outline next the programs of the Extended Scale Relativity (ESR) in C-spaces.

**Extending Relativity from Minkowski spacetime to C-space**

We embark onto a trip into the extended relativity theory in C-spaces by a natural generalization of the notion of a space-time interval in Minkowski space to C-space:

\[ dX^2 = d\Omega^2 + dx_\mu dx^\mu + dx_\mu dx_\nu + ... \]  

(1)

The Clifford valued poly-vector:

\[ X = \Omega I + x^\mu \gamma_\mu + x_\mu x_\nu \gamma_\mu \wedge \gamma_\nu + ... \]

(2)

denotes the position in a manifolds, called Clifford space or C-space. If we take differential dX of X and compute the scalar product dX * dX we obtain:

\[ d\Sigma^2 = (d\Omega)^2 + \Lambda^{2D-2} dx_\mu dx^\mu + \Lambda^{2D-4} dx_\mu dx_\nu + .. \]

(3)
Here we have introduced the Planck scale \( \Lambda \) since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops, ..., \( p \)-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval:

\[
ds^2 = c^2 dt^2 - dx_i dx^i.
\] (4)

A similar unification is needed here to "unite" objects of different dimensions, such as \( x^\mu \), \( x^{\mu\nu} \), etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [2].

To continue along the same path, we consider The analog of Lorentz transformations in C-spaces transform a poly-vector \( X \) into another poly-vector \( X' \) given by \( X' = RXR^{-1} \) with

\[
R = \exp \left[ i(\theta I + \theta^\mu \gamma_{\mu} + \theta^{\mu\nu} \gamma_{\mu\nu}) \right].
\] (5)

and

\[
R^{-1} = \exp \left[-i(\theta I + \theta^\nu \gamma_{\nu} + \theta^{\nu\nu} \gamma_{\nu\nu}) \right].
\] (6)

where the theta parameters:

\[
\theta; \theta^\mu; \theta^{\mu\nu}; \ldots
\] (7)

are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: \( \|X\|^2 = \text{Trace } X^2 \) Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

\[
\text{Trace } X'^2 = \text{Trace } [RX^2R^{-1}] = \text{Trace } [RR^{-1}X^2] = \text{Trace } X^2.
\] (8)

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and \( RR^{-1} = 1 \).

**Planck scale as the minimum invariant in Extended Scale Relativity**

Long time ago L.Nottale proposed to view the Planck scale as the absolute minimum invariant (observer independent) scale in Nature in his formulation.
of scale relativity [1]. We can apply this idea to C-spaces by choosing the correct analog of the Minkowski signature:

\[ \|dX\|^2 = d\Sigma^2 = (d\Omega)^2[1 - \Lambda^{2D-2}\frac{(dx_\mu)^2}{(d\Omega)^2} - \Lambda^{2D-4}\frac{(dx_{\mu\nu})^2}{(d\Omega)^2} - \Lambda^{2D-6}\frac{(dx_{\mu\nu\rho})^2}{(d\Omega)^2} - \ldots] \]

\[ \|dx\|^2 = d\Sigma^2 = (d\Omega)^2[1 - \left(\frac{\Lambda}{\lambda_1}\right)^{2D-2} - \left(\frac{\Lambda}{\lambda_2}\right)^{2D-4} - \left(\frac{\Lambda}{\lambda_3}\right)^{2D-6} - \ldots]. \quad (9) \]

where the sequence of variable scales \(\lambda_1, \lambda_2, \lambda_3, \ldots\) are related to the generalized (holographic) velocities defined as follows:

\[ \frac{(dx_\mu)^2}{(d\Omega)^2} \equiv (V_1)^2 = \left(\frac{1}{\lambda_1}\right)^{2D-2}. \]

\[ \frac{(dx_{\mu\nu})^2}{(d\Omega)^2} \equiv (V_2)^2 = \left(\frac{1}{\lambda_2}\right)^{2D-4}. \]

\[ \frac{(dx_{\mu\nu\rho})^2}{(d\Omega)^2} \equiv (V_3)^2 = \left(\frac{1}{\lambda_3}\right)^{2D-6}. \quad (10) \]

It is clear now that if \(\|dX\|^2 \geq 0\) then the sequence of variable lengths \(\lambda_n\) cannot be smaller than the Planck scale \(\Lambda\). This is analogous to a situation with the Minkowski interval:

\[ ds^2 = c^2 dt^2[1 - \frac{v^2}{c^2}]. \quad (11) \]

when it is \(\geq 0\) if, and only if, the velocity \(v\) does not exceed the speed of light. If any of the \(\lambda_n\) were smaller than the Planck scale the C-space interval will become tachyonic-like \(d\Sigma^2 < 0\). Photons in C-space are tensionless branes/loops. Quite analogously one can interpret the Planck scale as the postulated minimum universal distance in nature, not unlike the postulate about the speed of light as the upper limit on the speed of signal propagation.

What seems remarkable in this scheme of things is the nature of the signatures and the emergence of two times. One of the latter is the local mode, a clock, represented by \(t\) and the other mode is a “global” one represented by the volume of the space-time filling brane \(\Omega\). For more details related to this
Stuckelberg-type parameter and the two modes of time in other branches of science see [13].

Another immediate application of this is that one may consider “strings” and “branes” in C-spaces as a unifying description of all branes of different dimensionality. As we have already indicated, since spinors are left/right ideals of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world volume and target space supersymmetry simultaneously [20].

We hope that the C-space “strings” and “branes” may lead us towards discovering the physical foundations of string and M-theory.

In this talk we shall explore several important topics currently under intensive investigation.

2 Planck-scale Relativity, $\kappa$-deformed Poincare from C-spaces

We will briefly summarize the main results of [19] that allowed us, among other things, to derive the Casimirs (masses) of kappa-deformed Poincare algebras [9,10]. The relativity in C-spaces is very closely connected to Planck-scale Relativity [9,10]. Below we will review how the minimal length string uncertainty relations can be obtained from C-spaces [2]. The norm of a momentum poly-vector was defined:

$$P^2 = \pi^2 + p_{\mu} p^\mu + p_{\mu \nu} p^{\mu \nu} + p_{\mu \nu \rho} p^{\mu \nu \rho} + \ldots = M^2$$  \hspace{1cm} (12)

Nottale has given convincing arguments why the notion of dimension is resolution dependent, and at the Planck scale, the minimum attainable distance, the dimension becomes singular, that is blows-up. If we take the dimension at the Planck scale to be infinity, then the norm $P^2$ will involve an infinite number of terms since the degree of a Clifford algebra in $D$-dim is $2^D$. It is precisely this infinite series expansion which will reproduce all the different forms of the Casimir invariant masses appearing in kappa-deformed Poincare algebras [9,10].

It was discussed recently why there is an infinity of possible values of the Casimirs invariant $M^2$ due to an infinite choice of possible bases. The parameter $\kappa$ is taken to be equal to the inverse of the Planck scale. The
classical Poincare algebra is retrieved when $\Lambda = 0$. The kappa-deformed Poincare algebra does not act in classical Minkowski spacetime. It acts in a quantum-deformed spacetime. We conjecture that the natural deformation of Minkowski spacetime is given by C-space.

The way to generate all the different forms of the Casimirs $M^2$ is by “projecting down” from the $2^D$-dim Clifford algebra to $D$-dim. One simply “slices” the $2^D$-dim mass-shell hyper-surface in C-space by a $D$-dim one. This is achieved by imposing the following constraints on the holographic components of the polyvector-momentum. In doing so one is explicitly breaking the poly-dimensional covariance and for this reason one can obtain an infinity of possible choices for the Casimirs $M^2$.

To demonstrate this, we impose the following constraints:

$$p_{\mu\nu}p^{\mu\nu} = a_2(p_\mu p^\mu)^2 = a_2p^4. \quad p_{\mu\nu\rho}p^{\mu\nu\rho} = a_3(p_\mu p^\mu)^3 = a_3p^6. \quad \ldots \quad (13)$$

Upon doing so the norm of the poly-momentum becomes:

$$P^2 = \sum_n a_n p^{2n} = M^2(1, a_2, a_3, \ldots, a_n, \ldots) \quad (14)$$

Therefore, by a judicious choice of the coefficients $a_n$, and by reinserting the suitable powers of the Planck scale, which have to be there in order to combine objects of different dimensions, one can reproduce all the possible Casimirs in the form:

$$M^2 = m^2[f(\Lambda m)]^2. \quad m^2 \equiv p_\mu p^\mu = p^2. \quad (15)$$

where the functions $f(\Lambda m)$ are the scaling functions with the property that when $\Lambda = 0$ then $f \to 1$.

**The Generalized String Uncertainty Relations**

To illustrate the relevance of poly-vectors, we will summarize our derivation of the minimal length string uncertainty relations [2]. The canonical quantization in C-space will be given in a future work. Because of the holographic variables one cannot naively impose $[x, p] = i\hbar$ due to the effects of the other components. The units of $[x_\mu, p^{\mu\nu}]$ are of $\hbar^2$ and of higher powers of $\hbar$ for the other commutators. To achieve covariance in C-space which reshuffles objects of different dimensionality, the effective Planck constant in C-space should be given by a sum of powers of $\hbar$. 
This is not surprising. Classical C-space contains the Planck scale, which itself depends on $\hbar$. This implies that already at the classical level, C-space contains the seeds of the quantum space. At the next level of quantization, we have an effective $\hbar$ that comprises all the powers of $\hbar$ induced by the commutators involving all the holographic variables. In general one must write down the commutation relations in terms of polyvector-valued quantities. In particular, the Planck constant will now be a Clifford number, a polyvector with multiple components.

The simplest way to infer the effects of the holographic coordinates of C-space on the commutation relations is by working with the effective $\hbar$ emerging from the “shadows” of C-space. For the relevance of these “shadows” of Planck scale physics to string theory see [18]. The mass-shell condition in C-space, after imposing the constraints among the holographic components, yields an effective mass $M = mf(\Lambda m)$. The generalized De Broglie relations, which are no longer linear, are [2]:

$$|P_{\text{effective}}| = |p|f(\Lambda m/\hbar) = h_{\text{effective}}|k|. \quad h_{\text{effective}} = h f(\Lambda m/\hbar) = \hbar \sum n a_n \frac{\Lambda m}{\hbar}^{2n}. \quad m^2 = p^2 = (\hbar k)^2.$$  (16)

Using the effective $h_{eff}$, the well known relation based on the Schwartz inequality and the fact that $|z| \geq |Im z|$ we obtain:

$$\delta x i \delta p j \geq \frac{1}{2} |< x i, p j > | = \frac{h_{\text{effective}}}{2} \delta_{ij}.$$  (17)

Using the relations

$$< p^2 > \geq (\delta p)^2. \quad < p^4 > \geq (\delta p)^4 \ldots.$$  (18)

and the series expansion of the effective $h_{eff}$, we get for each component (we omit indices for simplicity):

$$\delta x \delta p \geq \frac{1}{2} h + \frac{a \Lambda^2}{2 \hbar} (\delta p)^2 + \ldots$$  (19)

This yields the minimal length string uncertainty relations:

$$\delta x \geq \frac{h}{2 \delta p} + \frac{a \Lambda^2}{2 \hbar} \delta p \ldots.$$  (20)
One could include all the terms in the series expansion and derive a generalized string/brane uncertainty relation which still retains the minimal length condition, of the order of the Planck scale [2].

The physical interpretation of these uncertainty relations follow from the extended relativity principle. As we boost the string to higher energies part of the energy will always be invested into the string’s potential energy, increasing its length in bits of Planck scale sizes. This reminds one of ordinary relativity, where boosting a massive particle to higher energy increasing its speed and a part of the energy is also invested into increasing its mass. In this process the speed of light remains the maximum attainable speed (it takes an infinite energy to do so) and in our scheme the Planck scale is never surpassed. The effects of a minimal length can be clearly seen in Finsler geometries having both a maximum four acceleration \( c^2/\Lambda \) (maximum tidal forces) and a maximum speed [21]. The Riemannian limit is reached when the maximum four acceleration goes to infinity; i.e. the \( \Lambda = 0 \) and Finsler geometry collapses to Riemannian one.

**Effective Lorentz Boosts from C-space Lorentz Transformations**

We can also show that the effective Lorentz boosts transformations can be derived from the C-space Lorentz transformations by a judicious choice of the theta parameters. The effective boosts along the \( x_1 \) direction were obtained in [10] using the kappa-deformed Poincare algebra:

\[
t' = t \cosh[z(\xi)] + x_1 \sinh[z(\xi)] , \quad \dot{x}_1 = t \sinh[z(\xi)] + x_1 \cosh[z(\xi)]. \tag{21}
\]

where \( z(\xi) \) is the effective boost parameter that collapses to \( \xi \) when \( \Lambda = 0 \). The effective boost \( z(\xi) \) ensures that the minimum Planck scale is not surpassed after the (effective) Lorentz contraction. When one has an infinite amount of energy, the \( \beta = v/c = 1 \) and the ordinary boosts are: \( \xi = \text{artanh}(\beta) = \text{artanh}(1) = \infty \). But the effective boost \( z(\xi) = z(\infty) \) is finite meaning that boosts saturate at the Planck scale [10] and the (effective) Lorentz contraction factor doesn’t blow up which would otherwise have shrunk all lengths to zero.

The C-space Lorentz transformations of the \( X \) poly-vector can be written in the most general compact form:

\[
X' = X^N E_N = e^{i\theta_A E_A} (X^M E_M) e^{-i\theta_B E_B} . \tag{22}
\]
where

$$\theta_A = \{\theta, \theta_{\mu}, \theta_{\mu\nu}, \theta_{\mu\nu\rho}, \ldots\}.$$  \hfill (23)

are the C-space boosts parameters. And $E^A$ the C-space basis elements.

Performing a Taylor series expansion and taking the scalar product of both sides of eq. (22), by $*E^N$, it can be written in the form $X'^N \sim \mathcal{L}_M^N X^M$. It contains the following types of terms:

\[
(\theta_A \theta^A)^n (\theta_B \theta^B)^m (\theta_C \theta^C) (E_M * E^N) X^M.
\]

\[
(\theta_A \theta^A)^n (\theta_B \theta^B)^m \{\theta_C \theta_D E^C [E_M, E^D]\} * E^N X^M.
\]

\[
(\theta_A \theta^A)^n (\theta_B \theta^B)^m (E_M * E^N) X^M.
\]

\[
i(\theta_A \theta^A)^n (\theta_B \theta^B)^m \{\theta_C E^C E_M\} * E^N X^M. \hfill (24)
\]

Notice that the odd powers of the $\theta$'s are the ones which contain the imaginary unit.

One can perform a "dimensional reduction" from the C-space Lorentz to an effective Lorentz:

$$\theta_A E^A \rightarrow \theta_{01}^\text{eff} \gamma^0 \wedge \gamma^1$$ \hfill (25)

by imposing the conditions:

$$\theta_A \theta^A = \sum_k \alpha_k \xi^k.$$ \hfill (26)

the unknown coefficients $\alpha_k$ to be be determined later.

The terms in (24) become now:

\[
(\sum_{m,n} \sum_k \alpha_k \xi^{m+n+1}) (E_M * E^N) X^M = F_2(\xi) X^N. \hfill (27)
\]

\[
(\sum_{m,n} \sum_k \alpha_k \xi^{m+n}) \{\theta_C \theta_D E^C [E_M, E^D]\} * E^N X^M = F_1(\xi) \mathcal{L}_M^{(2), N} X^M. \hfill (28)
\]

\[
(\sum_{m,n} \sum_k \alpha_k \xi^{m+n}) (E_M * E^N) X^M = F_1(\xi) X^N. \hfill (29)
\]
\[ i(\sum_{m,n} (\sum_k \alpha_k \xi^k)^{m+n}) \{\theta C E^C E_M\} \ast E^N X^M = iF_1(\xi)\mathcal{L}_M^{(1)N} X^M. \]  

(30)

where we have defined:

\[ \sum_{m,n} (\sum_k \alpha_k \xi^k)^{m+n} \equiv F_1(\xi). \quad \sum_{m,n} (\sum_k \alpha_k \xi^k)^{m+n+1} \equiv F_2(\xi). \]  

(31)

Notice that both functions of \( \xi \) contain even and odd powers of \( \xi \).

If one sets all the components (except those of the vector \( x^\mu \) ) of the poly-vector \( X \) to zero, one would arrive at an effective Lorentz transformation of the form:

\[ x'_{\rho} = [e^{i\theta_A E_A} (x^\mu\gamma_\mu)e^{-i\theta_B E_B}] \ast \gamma^\rho \equiv L^\rho_{\nu}[\theta, \theta_\mu, \theta_{\mu\nu}, \theta_{\mu\nu\rho}; \ldots] x^\nu. \]  

(32)

Since ordinary boosts along the \( x_1 \) direction (with boost parameter \( \xi \) ) are rotations with an imaginary angle \( i\theta_{01} = i\xi \) (along the 01 directions) we can see from the identities:

\[ \cos(i\theta_{01}) = \cosh(\theta_{01}) = \cosh\xi. \quad \sin(i\theta_{01}) = isinh(\theta_{01}) = isinh\xi. \quad \theta_{01} \equiv \xi. \]  

(33)

that one will retrieve the standard Lorentz Transformations if, and only if, \( \theta_{01} = \xi \) and if one were to constrain all of the other thetas to zero.

In a more general case, this is not so, and one must include the contributions of all the other \( \theta_A \). Therefore, when one uses the constraints imposed on all the C-space theta parameters given in terms of \( \xi \) by eqs(26), and after defining the relations for the two functions \( F_1(\xi), F_2(\xi) \), one will have the desired effective Lorentz transformations in terms of the effective \( z(\xi) \) induced from C-space. The reason why one has an effective \( z(\xi) \) is due to the contributions of all the C-space theta parameters. These contributions are encoded (compounded) in the two functions \( F_1(\xi), F_2(\xi) \), compatible with the fact that we have two functions \( \cosh[z(\xi)]; \sinh[z(\xi)] \).

The sought-after equations that define the unknown coefficients \( \alpha_k \) (which determine the constraints among the thetas and \( \xi \) ) are obtained after using the double Taylor series expansion in the following equations:

\[ L_0^0 = L_1^1 = \cosh[z(\xi)] = \sum_{m,n} a_n(b_m \xi^m)^n. \]  

(34)
\[ L_1^0 = L_0^1 = \sinh[z(\xi)] = \sum_{m,n} c_n (b_m \xi^m)^n. \] (35)

These are the defining equations that determine the coefficients \( \alpha_k \) in terms of the known coefficients appearing in the double Taylor series expansion.

In order for the remaining matrix elements \( L_{\mu}^\nu[\theta, \theta_{\mu}, \theta_{\mu\nu}, ...] \) to be zero one will have to choose judiciously the thetas to satisfy these vanishing conditions; i.e. one will have to choose most of the thetas to be zero except those whose components contain the 01 directions. In particular, to ensure that no half-integer powers of \( \xi \) occur we should choose:

\[ \theta_{01} \sim \xi, \quad \theta_{012} = 0, \quad \theta_{0123} \sim \xi^2, \quad \theta_{01234} = 0, \quad \theta_{012345} \sim \xi^3, \ldots \] (36)

Otherwise one would have encountered half-integer valued powers \( \xi^{3/2}, \xi^{5/2}, \ldots, \xi^{n/2}, \ldots \). For more details see [19].

To conclude: the effective Lorentz boosts are obtained through a “dimensional reduction” procedure of the more general C-space Lorentz transformations

\[ \theta_A E^A \rightarrow \theta_{01}^{effective} \gamma^0 \wedge \gamma^1. \] (37)

For further details about this derivation and the derivation of the nonlinear addition law of energy-momenta in particle collisions in kappa-deformed Minkowski space directly from C-spaces we refer to [19]. The crux of the arguments lies in the fact that Planck scale relativity in C-spaces requires taking the \( D = \infty \) limit. Because the conformal group is contained within the Clifford algebra of space-time [11], the physics of C-space should in principle also yield the Casimirs for the deformed Weyl conformal algebra of spacetime [9]. Notice that C-space automatically incorporates non-commuting objects since poly-vectors, Clifford-valued matrices, do not commute.

3 On the geometry of curved C-space

In this section we will summarize the basic results of [11] that show why the C-space curvature can be written as a sum of products of the ordinary curvature with torsion. Thus, the analog of the Einstein-Hilbert action in C-space is given by a higher derivative gravity with torsion. This result is
reminiscent of the string effective action in curved backgrounds. Since the expansion is given in powers of the Planck scale.

Let us now consider a curved $C$-space. A basis in $C$-space is given by

$$E_A = \{\gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \ldots\}.$$  

where in an $r$-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < \ldots < \mu_r$. An element of $C$-space is a Clifford number, or Polyvector written in the form

$$X = X^A E_A = s I + x^\mu \gamma_\mu + x^{\mu \nu} \gamma_\mu \wedge \gamma_\nu + \ldots$$  \hspace{1cm} (38)

A $C$-space is parameterized not only by 1-vector coordinates $x^\mu$ but also by the 2-vector coordinates $x^{\mu \nu}$, 3-vector coordinates $x^{\mu \nu \rho}$, etc., called also holographic coordinates, since they describe the holographic projections of 1-lines, 2-loops, 3-loops, etc., onto the coordinate planes. By $p$-loop we mean a closed $p$-brane; in particular, a 1-loop is closed string, a 2-loop is a closed membrane, etc....

In order to avoid using the powers of the Planck scale length parameter $\Lambda$ in the expansion of the poly-vector $X$ we use the dilatationally invariant units in which $\Lambda$ is set to 1 [3]. The Planck scale in general is given in terms of the Newton constant: $\Lambda = (G_D)^{1/(D-2)}$, in units of $\hbar = c = 1$. If we imagine performing an analytical continuation of the dimension from $D = 2$ all the way to $D = \infty$, it is clear that that in order to have the Planck scale as a universal invariant, with the same value in all dimensions, the value $\Lambda = 1$ is compatible with the choice of $G_D = 1$ (for all dimensions). Hence the extended relativity principle in $C$-space admits the natural system of units $\hbar = c = G = \Lambda = 1$ which will set the Planck temperature and Boltzmann constant to be also $T_P = K_B = 1$. This unifying picture of setting all the fundamental constants of Nature to 1 was advocated by Wheeler long time ago by suggesting that information theory may lie at the core of things.

In a flat $C$-space the basis vectors $E^A$ are constants. In a curved $C$-space this is no longer true. Each $E_A$ is a function of the $C$-space coordinates $X^A = \{s, x^\mu, \sigma^{\mu \nu}, \ldots\}$ which include scalar, vector, bivector,..., r-vector,..., coordinates.

Now we define the connection $\Gamma^C_{AB}$ in $C$-space according to $\partial_A E_B = \Gamma^C_{AB} E_C$. The $C$-space curvature is thus defined in the geometric calculus notation:
The ‘star’ means the scalar product between two poly-vectors $A$ and $B$, defined as $A \ast B = \langle A | B \rangle_s$ where $s$ means ‘the scalar part’ of the geometric Clifford product $AB$.

Next we shall provide the relation for curvature and see how it is related to the curvature of the ordinary space. We will present the basic formulae and refer all the details of the derivation to the references [11]. We found, in particular that:

\[ \frac{\partial \gamma_{\mu}}{\partial x^{\nu}} = \Gamma_{\nu\alpha}^{\mu} \gamma_{\alpha} \quad \frac{\partial \gamma_{\mu}}{\partial x^{\alpha\beta}} = R_{\alpha\beta\mu}^{\rho} \gamma_{\rho}. \]  

For an arbitrary poly-vector

\[ V = V^{A}E_{A} = v + v^{\mu}\gamma_{\mu} + v^{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu} + ... \]  

we have the covariant derivatives:

\[ \frac{D}{DX^{B}} \frac{DV^{A}}{DX^{B}} = \frac{\partial V^{A}}{\partial X^{B}} + \Gamma_{BC}^{A}V^{C}. \quad \frac{DE^{A}}{DX^{B}} = 0. \]  

\[ \frac{Dv}{DX^{\mu\nu}} = [D_{\mu}, D_{\nu}]v = \frac{\partial v}{\partial X^{\mu\nu}} = K_{\mu\nu}^{\rho}\partial_{\rho}v. \]  

\[ \frac{Dv^{\alpha}}{DX^{\mu\nu}} = [D_{\mu}, D_{\nu}]v^{\alpha} = R_{\mu\nu\rho}^{\alpha}v^{\rho} + K_{\mu\nu}^{\delta}D_{\rho}v^{\alpha}. \]  

using these relations in the basic formula for the curvature:

\[ \mathcal{R}_{ABC}^{D} = ([\partial_{A}, \partial_{B}]E_{C}) \ast E^{D}. \]  

we have for a particular example the poly-vector valued multi-indices:

\[ A = [\mu\nu]. \quad B = [\alpha\beta]. \quad C = \tau. \quad D = \delta \]  

\[ ([\frac{\partial}{\partial \sigma^{\mu\nu}}, \frac{\partial}{\partial \sigma^{\alpha\beta}}]_{\gamma_{\tau}})_{\gamma^{\delta}} = \mathcal{R}_{\mu\nu}[\alpha\beta][\tau] = \]  

\[ R_{\mu\nu\tau}^{\rho}R_{\alpha\beta\rho}^{\delta} - R_{\alpha\beta\tau}^{\rho}R_{\mu\nu\rho}^{\delta}. \]
We can see now how one gets the product of two usual curvature tensors. We can proceed in analogous way to calculate the other components of the C-space curvature $R^D_{ABC}$ and find that these contain higher powers of the curvature in an ordinary space-time. After performing the appropriate contractions, we can see that the scalar curvature in C-space will contain the sums of products of the ordinary curvature tensors. In general one has also contributions from a non-vanishing torsion. This resembles the results based on non-linear quantum sigma models and used to evaluate the string effective action in curved backgrounds as an expansion in higher derivative terms (higher power of the curvature) [11].

One of the most important consequences of this result is that a flat C-space can be curved from the ordinary space-time point of view. If one takes the symmetric spaces, like de Sitter and Anti de Sitter, where the curvature tensors are suitable multiples of the scalar curvature, we see that the C-space scalar curvature is given as a sum of powers of the space-time scalar:

$$\mathcal{R} = \sum a_n R^n.$$

Clearly, if $R = 0$ then $\mathcal{R} = 0$. Inversely, $\mathcal{R} = 0$ may yield besides the trivial solution $R = 0$ other nonvanishing values for $R$, such as the solutions of a polynomial equation. Tele-parallellism theories of gravity have vanishing curvature but non-vanishing torsion.

Based on this fact, the logarithmic corrections in any dimension to black hole entropy were found with the help of a quantum p-loop harmonic oscillator in flat C-space [12]. The logarithm of the degeneracy of quantum states yielded the standard Bekenstein-Hawking area-entropy relation plus the logarithmic corrections expressed in terms of Clifford-bits or true quanta of spacetime: the holographic areas, holographic volumes, ...were all quantized in Planck scale units.

Furthermore, the expression for the Schwrazschild radius and Hawking temperature, in any dimension, were also found using these methods [12]. The open problem was to derive the Schwrazschild metric from the long distance limit of the condensation of the holographic quanta of area bits, volume bits, ... The most important result in [12] was that C-space methods appear to indicate naturally the origins of the thermodynamic properties of black holes and why string theory contains gravity.
4 Cosmological Implications and Further topics

To end this talk, I sketch the cosmological and other physical implications of the extended scale relativity in C-spaces.

- Cosmological constant. Flat C-space versus non-flat Riemann.

The cosmological constant is just one component of the momentum polyvector, the Fourier conjugate to the position polyvector $X$. The first component was the volume $\Omega$ of the spacetime filling p-loop/brane, $p+1 = D$. Its Fourier dual is the cosmological constant, the vacuum energy density. Clearly, since it is just a component of a polyvector, the cosmological constant is not a constant in C-spaces. Secondly, we have discussed above why a flat C-space may not imply a flat ordinary spacetime. Hence, we hope that C-space relativity may bring about the key to solve this problem. Perhaps, the problem was ill-posed due to the fact that the cosmological “constant” may not be a true constant. It is clearly not a constant in C-space!

- Two times. Universal time arrow. Why universe may expand forever.

We have also discussed why the volume $\Omega$ of the spacetime filling brane plays the role of the global time mode, the Stuckelberg parameter [3] and its relation to the two modes of time given in [13]. It is possible that the fundamental constants, like the fine structure constant, themselves depend on this global mode time parameter $\Omega$. As the universe expands, it does according to this universal time arrow. Since this arrow points forward then one should expect that $\Omega$ will increase and not surprisingly the universe will expand forever.

On the variable fine structure, we know from the above results that C-space demands an effective $\bar{\hbar}$ given by a sum of powers of $\Lambda$. If one uses $\alpha = e^2/\bar{\hbar}\text{eff}c$ then one should expect corrections to the ordinary value. However, this not new since results of the renormalization group program require running coupling constants. What is new is the dependence on all constants on the global mode of time $\Omega$. This idea is more compatible with Diracs picture of the changing values with time of the fundamental constants in Nature.

- Yang-Mills interactions. Some preliminary results on how to incorporate Wilson loops in C-spaces appeared in [15]. The construction of fiber bundles over p-loop spaces seems to be a challenging problem.
• Duality of small/large scales. Nottale had envisioned this duality long ago and wrote down the analog of scale relativistic transformations involving an upper impassible scale, another universal invariant, the dual picture of the minimum Planck scale. This was his proposal for the resolution of the cosmological constant problem: it is meaningless to compare the vacuum energy at two such separate scales, the Planck versus the Hubble regime, without including the scale relativistic corrections. This accounted perfectly for the $10^{60}$ factors.

• Quantum Clifford algebras. One can deform C-spaces by using q-Clifford algebras, like braided Hopf quantum Clifford algebras [16]. For an extensive report on q-spins see [17]. For other relevant work on deformed Poincare algebras, quantum groups, in the construction of q-deformed Lagrangians for gravity see [14].

• On four as the average dimension of the world.

Since Planck scale relativity involves an infinite number of dimensions one will immediately wonder why we perceive four dimensions. In [23] it was shown that the average dimension of a family of spheres of arbitrary dimensions was close to $4 + \phi^3 = 4.236$, where $\phi$ is the Golden Mean 0.618...

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References

1- L. Nottale: La Relativite dans tous ses Etats Hachette Lit. Paris 1999.
2-C. Castro: Chaos, Solitons and Fractals 11 (2000) 1663.
3-Foundations of Physics 30 (2000) 1301.
4-Chaos, Solitons and Fractals 11 (2000) 1721.
5-Chaos, Solitons and Fractals 12 (2001) 1585.
"The search for the origins of M theory ...." 

3-M. Pavsic: The Landscape of Theoretical Physics: A Global View. Kluwer, Dordrecht 1993.
Clifford algebra based polydimensional Relativity and Relativistic Dynamics
Talk presented at the IARD Conference in Tel Aviv, June 2000.
Foundations of Physics 31 (2001) 1185. hep-th/0011216.
Phys. Let A 242 (1998) 187.
Nuovo Cimento A 110 (1997) 369.
4-J. Fanchi: Parametrized Relativistic Quantum Theory. Kluwer, Dordrecht 1993.
5-W. Pezzaglia: “Physical Applications of a Generalized Geometric Calculus” gr-qc/9710027.
6-D. Hestenes: Spacetime Algebra. Gordon and Breach, New York, 1996.
D. Hestens and G. Sobcyk: Clifford Algebra to Geometric Calculus. D. Reidel Publishing Company, Dordrecht, 1984.
Clifford Algebras and their applications in Mathematical Physics Vol 1: Algebras and Physics. eds by R. Ablamowicz, B. Fauser.
Vol 2: Clifford analysis. eds by J. Ryan, W. Sprosig. Birkhauser, Boston 2000.
P. Lounesto: Clifford Algebras and Spinors. Cambridge University Press. 1997.
7-S. Ansoldi, A. Aurilia, E. Spallucci: Chaos, Solitons and Fractals 10 (2-3) (1999).
8-S. Ansoldi, A. Aurilia, C. Castro, E. Spallucci: Phys. Rev. D 64 026003 (2001) hep-th/0105027.
9-J. Lukierski, A. Nowicki: Double Special Relativity versus kappa-deformed Relativistic dynamics hep-th/0203063.
J. Lukierski, V. Lyakhovsky, M. Mozrzymas: kappa-deformations of D = 4 Weyl and conformal symmetries hep-th/0203182.
S.Majid, H. Ruegg: Phys. Lett B 334 (1994) 348.
J. Lukierski, H. Ruegg, W. Zakrzewski: Ann. Phys. 243 (1995) 90.
J. Lukierski, A. Nowicki, H, Ruegg, V. Tolstoy: Phys. Lett B 264 (1991) 331.
10-J. Kowalski-Glikman, S. Nowak: Doubly special Relativity theories as different bases of kappa-Poincare algebras. hep-th/0203040.
N. Bruno, G. Amelino-Camelia, J. Kowalski: Deformed boosts transformations that saturate at the Planck scale hep-th/0107039
G. Amelino-Camelia: Int. J. Mod. Phys D 11 (2002) 35. gr-qc/0012051
G. Amelino-Camelia: Phys. Lett B 510 (2001) 255.
A. Granik: A comment on the work of Bruno-Amelino-Camelia and Kowalski physics/0108050.
11-C. Castro, M. Pavsic: Higher Derivative Gravity and Torsion from the Geometry of C-spaces hep-th/0110079
C. Castro, M. Pavsic: The Clifford algebra of Spacetime and the Conformal Group
12-C. Castro, A. Granik: Extended Scale Relativity, p-loop harmonic oscillator and logarithmic corrections to the black hole entropy physics/0009088
C. Castro: Jour. of Entropy 3 (2001) 12-26.
13-D. Chakalov: Two modes of Time: Bicausality Talk to be presented at the NATO advance workshop on the Nature of Time: Geometry, Physics and Perception. Slovakia, May 2002.
14-L. Castellani: The Lagrangian of q-Poincare Gravity hep-th/940233
Differential Calculus on ISO_q(N), Quantum Poincare Algebra and q-Gravity hep-th/9312179.
15-C. Castro: On Wilson Loops and Confinement without Supersymmetry from Composite Antisymmetric Tensor Field Theories: hep-th/0204182
16-Z. Osiewicz: Clifford Hopf Algebra and bi-universal Hopf algebra q-alg/9709016
17-C. Blochmann Spin representations of the Poincare Algebra Ph. D Thesis math.QA/0110029.
18- K. Dienes, A. Mafi: Phys. Rev. Let 88 (11) (2002) 111602
19- C. Castro, A. Granik: Planck scale Relativity and variable fine structure from C-space To appear.
20- A. Aurilia, C. Castro, M. Pavsic, E. Spallucci: To appear.
21-H. Brandt: Chaos, Solitons and Fractals 10 (2-3) (1999) 267.
22-E. Guendelman: Class. Quant. Grav 17 (2000) 3673. hep-th/0005041
23-C. Castro, A. Granik: Chaos, Solitons and Fractals 12 (10) (2001) 1793.