A RECURSIVE APPROACH TO THE MATRIX MOMENT PROBLEM

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Abstract. In this paper, we study the truncated matrix moment problem in one variable through recursive matrix extensions. We give necessary and sufficient conditions for a recursive matrix extension of finite data to be a matrix moment sequence in the classical cases of Hamburger, Stieltjes, and Hausdorff moment problems. We also discuss matricial subnormal completion and matricial $k$–hyponormal completion problems and provide an analog of Stampfli’s Theorem on flat propagation for 2–hyponormal matricial weighted shifts.

1. Introduction

The real and complex one-dimensional moment problems are well-known in classical analysis and have been widely studied by many mathematicians and engineers since the late 19th. century. Stieltjes studied the problem on the positive real line [26], Hamburger on the real line [12], and Hausdorff on a closed and bounded interval [13]. The key to resolving the full moment problem lies in the utilization of Riesz’s Theorem [23] on one hand, and an appropriate characterization of positive polynomials on the other. The solution to the one-dimensional moment problem is expressed in terms of the positivity of certain associated Hankel matrices.

In the truncated case, a more constructive approach is employed, relying on techniques from finite-dimensional linear algebra in conjunction with extension methods. For truncated moment problems, R. Curto and L. Fialkow led a comprehensive investigation centered around the concept of flat extensions; see [5, 6, 7] for further information. D. Kimsey and H. Werderman provided a solution based on commutativity conditions of certain matrices [15]. Several other mathematicians have also tackled the truncated

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moment problem using various approaches, such as K. Schmüdgen’s operator approach [24] and F.H. Vasilescu’s dimension stability approach [29], among others.

Matrix moment problems have become modern and actively studied topics in recent years, thanks to their applications, which attract the attention of a growing number of researchers. H. Dette and W. J. Studden, in [8], used matrix moment problem techniques to delve into a matrix version of the $q-d$ algorithm to be applied in numerical analysis. They demonstrated that this algorithm can be employed to derive the coefficients of recurrence relations for matrix orthogonal polynomials on intervals such as $[0,1]$ and $[0, +\infty)$, based on their moment generating functional. Their paper includes illustrative examples that extend the classical orthogonal polynomials defined on the real line.

Dette and Studden investigated in [9] the maximum value of the determinant of canonical Hankel matrices associated with Hausdorff matrix moment sequences. The obtained results were then used in optimal design problems within linear models. This highlights a statistical application that involves maximizing matrix Hankel determinants constructed from matrix moments.

Furthermore, J.B. Lasserre, in [19], combined global optimization with polynomials and addressed the moment problem in a broader context. He emphasized the intersection of global optimization techniques using sums of squares of polynomials and other related techniques.

In the present paper, we adopt the recursive relations approach widely studied by M. Rachidi et al. in the one-dimensional case, as shown in [1] and [2]. Our paper is organized as follows. In Section 2, we introduce the matrix moment problem and give the general preparatory results needed for its resolution. We exhibit the description of positive matrix-valued polynomials on the classical intervals $I = [0,1]$, $I = [0, +\infty)$ and $I = \mathbb{R}$. Section 3 is devoted to the study of recursive matrix sequences. We show that a recursive matrix sequence admits a representing matrix charge if and only if the associated minimal polynomial has all its roots distinct and real. We characterize in Section 4 recursive matrix moment sequences in terms of the positivity of some finite Hankel matrix. We give several computational results in the case of recursive matrix sequences of order 2 and 3. In Section 5, we apply our results to the subnormal and $k$-hyponormal completion.
problem for matricial weighted shifts. We also extend Stampfli's Theorem on the propagation phenomena to 2-hyponormal matricial weighted shifts.

2. Preliminaries

In this paper, we use the following notation and symbols. \( \mathbb{R} \) and \( \mathbb{N} \) denote the sets of real numbers and positive integers, respectively. The polynomial algebra \( \mathbb{R}[X] \) consists of all polynomials with real coefficients. For \( P \in \mathbb{R}[X] \), \( \mathcal{Z}(P) \) denotes the zeros set of \( P \). The algebra of \( p \times p \) matrices with real entries is denoted by \( \mathcal{M}_p(\mathbb{R}) \), where the unit \( I_p \) is the identity matrix and where \( 0_p \) is the null matrix in \( \mathcal{M}_p(\mathbb{R}) \). We use \( S_p(\mathbb{R}) \) to denote the subspace of symmetric matrices in \( \mathcal{M}_p(\mathbb{R}) \) and \( S_p^+(\mathbb{R}) \) to denote the set of nonnegative matrices in \( S_p(\mathbb{R}) \), equipped with the Loewner order, denoted by “\( \succeq \)”. We will write \( \mathcal{M}_p(\mathbb{R}[X]) \) for the algebra of \( p \times p \) matrices with entries in \( \mathbb{R}[X] \). Alternatively, \( \mathcal{M}_p(\mathbb{R}[X]) \) can be viewed as \( \mathcal{M}_p(\mathbb{R})[X] \), the algebra of polynomials with coefficients in \( \mathcal{M}_p(\mathbb{R}) \). In other words, every polynomial \( P \in \mathcal{M}_p(\mathbb{R}[X]) \) may take one of two different forms:

\[
P(X) = \sum_{k=0}^{n} A_k X^k = (P_{i,j})_{1 \leq i, j \leq p},
\]

where \( A_k = (a_{ij}^{(k)})_{1 \leq i, j \leq p} \in \mathcal{M}_p(\mathbb{R}) \) for \( 0 \leq k \leq n \), or

\[
P_{i,j}(X) = \sum_{k=0}^{n} a_{ij}^{(k)} X^k \in \mathbb{R}[X]
\]

for every \( 1 \leq i, j \leq p \). If \( A_n \neq 0_p \), we will say that \( P \) is of degree \( n \) and we write \( \deg(P) = n \).

Before introducing the matrix \( K \)-moment problem and the tracial \( K \)-moment problem as it is stated in [20], recall that a matrix-valued function \( \sigma : \mathcal{B}(\mathbb{R}) \to \mathcal{M}_p(\mathbb{R}) \) is called a matrix charge if

\[
\sigma(B) := (\sigma_{ij}(B))_{i,j=1,...,p} \in \mathcal{M}_p(\mathbb{R}), \text{ for every } B \in \mathcal{B}(\mathbb{R}),
\]

where \( \mathcal{B}(\mathbb{R}) \) is the Borel \( \sigma \)-algebra of Borel subsets in \( \mathbb{R} \) and \( \sigma_{ij} \) are scalar charges on \( \mathcal{B}(\mathbb{R}) \) for \( 1 \leq i, j \leq p \).

A matrix charge \( \sigma \) is called a matrix measure when for all \( B \in \mathcal{B}(\mathbb{R}) \) and for all \( v \in \mathbb{R}^p \), we have \( v \sigma(B)v^T \geq 0 \). Equivalently, when \( \sigma(B) \in S_p^+(\mathbb{R}) \).

The support of a matrix charge \( \sigma = (\sigma_{ij})_{i,j=1,...,p} \), denoted by \( \text{supp}(\sigma) \), is defined as the smallest closed subset \( F \subseteq \mathbb{R} \) such that \( \sigma(F) = 0 \) for every
Borel set $F \subseteq \mathbb{R} \setminus S$. In the case of a matrix measure, the support coincides with the smallest closed subset $S \subseteq \mathbb{R}$ such that $\sigma(\mathbb{R} \setminus F) = 0_p$.

For a given matrix charge $\sigma$, it is not difficult to prove the following general characterization of the support

$$\text{supp}(\sigma) = \bigcup_{i,j=1}^{p} \text{supp}(\sigma_{ij}).$$

Moreover, if $\sigma$ is a matrix measure, then for every $B \in \mathcal{B}(\mathbb{R})$ and every $i,j = 1, \ldots, p$, we have

$$\begin{pmatrix} \sigma_{ii}(B) & \sigma_{ij}(B) \\ \sigma_{ij}(B) & \sigma_{jj}(B) \end{pmatrix} \succeq 0.$$ 

It follows, in particular that, $\sigma_{ij}(B)^2 \leq \sigma_{ii}(B)\sigma_{jj}(B)$. This implies that

$$\text{supp}(\sigma_{ij}) \subseteq \text{supp}(\sigma_{ii}) \cap \text{supp}(\sigma_{jj}),$$

and then

$$\text{supp}(\sigma) = \bigcup_{i=1}^{p} \text{supp}(\sigma_{ii}).$$

An important class of matrix charges consists of those with finite support, called finitely atomic matrix charges. They are written as

$$\sigma = \sum_{i=1}^{k} T_i \delta_{\lambda_i},$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $T_1, \ldots, T_k \in \mathcal{M}_p(\mathbb{R})$.

Let $\sigma = (\sigma_{ij})_{1 \leq i,j \leq p}$ be a matrix charge. A function $f : \mathbb{R} \to \mathbb{R}$ is called $\sigma$-measurable if $f$ is $\sigma_{ij}$-measurable for every $i,j = 1, \ldots, p$. In this case, the matrix-valued integral of $f$ with respect to the matrix charge $\sigma$ is defined by

$$\int_{\mathbb{R}} f(x) d\sigma(x) := \left( \int_{\mathbb{R}} f(x) d\sigma_{ij}(x) \right)_{i,j=1,\ldots,p} \in \mathcal{M}_p(\mathbb{R}).$$

In particular, if $\sigma$ is a finitely atomic matrix charge of the form (2.1), then

$$\int_{\mathbb{R}} f(x) d\sigma(x) = \sum_{i=1}^{k} f(\lambda_i) T_i \in \mathcal{M}_p(\mathbb{R}).$$

Let $S = (S_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}_p(\mathbb{R})$. We consider the real-valued linear form $L_S$ on $\mathcal{M}_p(\mathbb{R}[X])$ defined by

$$L_S : \mathcal{M}_p(\mathbb{R}[X]) \mapsto \mathbb{R}, \quad P = \sum_{k=0}^{m} A_k X^k \mapsto L_S(P) := \sum_{k=0}^{m} \text{tr}(A_k S_k),$$

where $\text{tr}(A)$ is the trace of the matrix $A$. 
The linear form $L_S$ is known as the Riesz functional associated with the matrix sequence $S$. Given a closed set $K \subseteq \mathbb{R}$, the tracial $K$–moment problem is introduced as follows.

**Problem 2.1.** The tracial $K$–moment problem asks for necessary and sufficient conditions for the existence of a matrix measure $\sigma = (\sigma_{ij})_{1 \leq i,j \leq p}$ satisfying the following conditions

$$\text{supp}(\sigma) \subseteq K \quad \text{and} \quad L_S(P) = \int_K \text{tr}(P(x)d\sigma(x)) \text{ for every } P \in M_p(\mathbb{R}[X]).$$

The matrix $K$–moment problem is formulated as follows.

**Problem 2.2.** The matrix $K$–moment problem asks for necessary and sufficient conditions for the existence of a matrix measure $\sigma = (\sigma_{ij})_{1 \leq i,j \leq p}$ satisfying

$$\text{supp}(\sigma) \subseteq K \quad \text{and} \quad S_k = \int_K x^k d\sigma(x) \text{ for every } k \in \mathbb{N}.$$

We retain the usual definitions in the case of scalar moment problems. A matrix sequence (finite matrix sequence) for which the corresponding $K$–matrix moment problem (truncated matrix moment problem) has a solution is referred to as a $K$–matrix moment sequence (resp. truncated $K$–matrix moment sequence). Problems 2.1 and 2.2 are referred to as the Hamburger matrix moment problems if $K = \mathbb{R}$, the Stieltjes matrix moment problems if $K = [0, +\infty)$, and the Hausdorff matrix moment problems if $K = [0, 1]$.

Recall a useful result from [20].

**Theorem 2.3** ([20, Proposition 2.8 and Theorem 2.9]). The following statements are equivalent:

1. $S$ is a $K$–matrix moment sequence;
2. $L_S$ is a tracial $K$–moment functional;
3. $L_S(P) \geq 0$ for all $P \in \mathcal{P}(K)$,

where $\mathcal{P}(K) = \{P \in M_p(\mathbb{R}[X]) \mid P|_K \succeq 0\}$.

**Remark 2.4.** In the truncated case, Theorem 2.3 has been extended by C. Mädler and K. Schmüdgen to all measure spaces [21].

It follows from the previous discussion that a necessary condition for $S$ to be a $K$–matrix moment sequence can be expressed as follows:

$$L_S(PP^t) \geq 0, \text{ for every } P \in M_p(\mathbb{R}[X]).$$
Equivalently, if

\( \sum_{i,j=0}^{n} \text{tr}(A_i S_{i+j} A_j^t) = \text{tr} \left( \sum_{i,j=0}^{n} A_i S_{i+j} A_j^t \right) \geq 0 \)

for every \( A_0, \ldots, A_n \in \mathcal{M}_p(\mathbb{R}) \).

For every \( k, m \in \mathbb{N} \), we denote by \( H_m^{(k)} = (S_{i+j})_{0 \leq i,j \leq m} \) the moment block matrix, and we let \( H_m := H_m^{(0)} \).

**Remark 2.5.** Condition (2.2) is equivalent to \( H_m = (S_{i+j})_{0 \leq i,j \leq m} \succeq 0 \), for every \( m \in \mathbb{N} \). Thus, for an arbitrary set \( K \) of real numbers, a necessary solubility condition for the \( K \)-matrix moment problem is \( H_m \succeq 0 \) for every \( m \in \mathbb{N} \).

This last criterion turns out to be necessary in all classical moment problems, \( K = [0,1], K = [0, +\infty) \) and \( K = \mathbb{R} \). For more specific results, we have:

**Theorem 2.6.** ([3, Corollary 1]). Let \( S = (S_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{S}_p(\mathbb{R}) \). Then:

1. \( S \) is a Hamburger matrix moment sequence \( \iff \) \( H_m \succeq 0 \) for every \( m \in \mathbb{N} \).
2. \( S \) is a Stieltjes matrix moment sequence \( \iff \) \( H_m \succeq 0 \) and \( H_m^{(1)} \succeq 0 \) for every \( m \in \mathbb{N} \).
3. \( S \) is a Hausdorff matrix moment sequence \( \iff \) \( H_m \succeq 0 \), \( H_m^{(1)} \succeq 0 \), \( H_m - H_m^{(1)} \succeq 0 \), and \( H_m^{(1)} - H_m^{(2)} \succeq 0 \) for every \( m \in \mathbb{N} \).

Recall that \( P \in \mathcal{M}_p(\mathbb{R}[X]) \) is said to be a sum of squares matrix polynomial if there exists \( P_1, \ldots, P_k \in \mathcal{M}_p(\mathbb{R}[X]) \) such that \( P(X) = P_1(X)P_1^t(X) + \cdots + P_k(X)P_k^t(X) \), where \( P^t \) is the transpose of the matrix polynomial \( P \).

The most common approach to obtain a proof of Theorem 2.6 is to combine Theorem 2.4 with the following description of positive matrix polynomials.

**Proposition 2.7** ([3, Proposition 3]). For \( P \in \mathcal{M}_p(\mathbb{R}[X]) \), the following statements hold.

1. \( P \succeq 0 \) on \( \mathbb{R} \) \( \iff \) \( P \) is a sum of two square matrix polynomials.
2. \( P \succeq 0 \) on \( [0, +\infty[ \) \( \iff \) \( P = A + XB \), where \( A \) and \( B \) are sums of square matrix polynomials.
3. \( P \succeq 0 \) on \( [0, 1] \) \( \iff \) \( P = A + XB + (1 - X)C + X(1 - X)D \), where \( A,B,C, \) and \( D \) are sums of square matrix polynomials.
Another solution to the Hausdorff matrix moment problem is obtained by combining Theorem 2.6 above together with the description of positive matrix polynomials on \([0, 1]\) and by using Bernstein decomposition, as provided in [20, Corollary 3.7].

**Proposition 2.8.** A matrix sequence \((S_n)_{n \geq 0}\) is a Hausdorff matrix moment sequence if and only if \(\sum_{i=1}^{k} (-1)^i \binom{k}{i} \text{tr}(S_{i+l}G) \geq 0\), for all positive definite matrices \(G\) and \(k, l \in \mathbb{N}\).

Thanks to the expressions \(X = X^2 + X(1 - X)\) and \(1 - X = (1 - X)^2 + X(1 - X)\), we derive the following finer characterization of non-negative matrix polynomials, improving assertion 3 of Proposition 2.7.

**Proposition 2.9.** Let \(P \in \mathcal{M}_p(\mathbb{R}[X])\). Then

\[
P \succeq 0 \text{ on } [0, 1] \iff P(X) = A + X(1 - X)B,
\]

where \(A\) and \(B\) are sums of squares of matrix polynomials.

As a consequence of these results, we obtain the next solution of the Hausdorff matrix moment problem, with a simpler condition than in statement 3 of Theorem 2.6.

**Theorem 2.10.** The following statements are equivalent:

1. \(S\) is a Hausdorff matrix moment sequence;
2. \(H_m \succeq 0\) and \(H_m^{(1)} - H_m^{(2)} \succeq 0\) for every \(m \in \mathbb{N}\);
3. \(\sum_{i=1}^{k} (-1)^i \binom{k}{i} H_m^{(i)} \succeq 0\), for every \(m \in \mathbb{N}\).

3. **The truncated matrix moment problem and recursive matrix sequences**

In the truncated matrix moment problem, only finite data \(S = (S_0, \cdots, S_{r-1})\) of symmetric matrices is provided. The truncated tracial \(K\)–moment problem and the truncated matrix \(K\)–moment problem are introduced as follows.

**Problem 3.1.** The truncated tracial \(K\)–moment problem asks for necessary and sufficient conditions for the existence of a matrix measure \(\sigma\) with \(\text{supp}(\sigma) \subseteq K\) and such that

\[
L_S(P) = \int_K \text{tr}(P(x)d\sigma(x)), \text{ for all } P \in \mathcal{M}_p(\mathbb{R}[X]) \text{ with } \deg(P) \leq r - 1.
\]
Problem 3.2. The truncated matrix $K$–moment problem asks for necessary and sufficient conditions for the existence of a matrix measure $\sigma$ with $\text{supp}(\sigma) \subseteq K$ and such that
\[ S_k = \int_K x^k d\sigma(x) \quad \text{for} \quad 0 \leq k \leq r - 1. \]

The Richter–Tchakaloff Theorem (cf. [24, Thm. 1.24]) asserts that each scalar moment functional on a finite-dimensional vector space of functions admits a finitely atomic representing measure. The Richter–Tchakaloff Theorem has been extended recently to the matrix case by C. Mädl er and K. Schmüdgen in [21, Theorem 5.1]. It follows that the above-mentioned truncated matrix moment problems, the full matrix moment problem, and the matrix moment problems for recursive matrix sequences are all equivalent.

This last fact motivates our restriction, in the remainder of this paper, to sequences of matrices given by $r$ initial data items and satisfying a linear recurrence relation of order $r$.

It turns out that, under an additional condition on the recurrence relation, all the solubility conditions of the classical matrix moment problem can be reduced to a unique condition, expressed as the positivity of $H_{r-1}$.

To be more precise, let $a_0, a_1, \ldots, a_{r-1}$ be some fixed real numbers with $a_{r-1} \neq 0$ and consider the sequence of matrices $S = (S_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_p(\mathbb{R})$ defined by the following linear recurrence relation:
\[
S_{n+1} = a_0 S_n + a_1 S_{n-1} + \cdots + a_{r-1} S_{n-r+1}, \quad n \geq r - 1,
\]
where $r \geq 1$ is an integer and $S_0, S_1, \ldots, S_{r-1} \in \mathcal{M}_p(\mathbb{R})$ are the initial conditions. In this paper, we will refer to sequences satisfying (3.1) as recursive matrix sequences.

The scalar case, $p = 1$, has been widely studied in the literature. These sequences are generally called $r$-generated Fibonacci sequences and have been a key point in solving several known problems; see, for example, [11] [10].

For the matrix case, we write $S = (S_n)_n = ((s^{(n)}_{ij})_{1 \leq i,j \leq p})_n$. Then for $n \in \mathbb{N}$, it follows that (3.1) is equivalent to the next family of linear scalar relations:
\[
s^{(n+1)}_{ij} = a_0 s^{(n)}_{ij} + a_1 s^{(n-1)}_{ij} + \cdots + a_{r-1} s^{(n-r+1)}_{ij},
\]
for every $n \geq r - 1$, and $1 \leq i, j \leq p$.

The polynomial $Q_S(X) := X^r - a_0 X^{r-1} - \cdots - a_{r-2} X - a_{r-1}$ is called a characteristic polynomial associated with the recursive matrix sequence $S$. 
As shown in the following example, a recursive matrix sequence always has many characteristic polynomials.

**Example 3.3.** Let $M \in \mathcal{M}_p(\mathbb{R})$, and $S_n = (n + 1)M$. Then the sequence $S = (S_n)_{n \in \mathbb{N}}$ verifies the following two recursive matrix relations.

$S_0 = M, S_1 = 2M, S_2 = 3M$ and $S_{n+1} = S_n + S_{n-1} - S_{n-2}$, for $n \geq 2$,

$S_0 = M, S_1 = 2M$ and $S_{n+1} = 2S_n - S_{n-1}$, for $n \geq 1$.

It follows that $P(X) = X^3 - X^2 - X + 1$ and $Q(X) = X^2 - 2X + 1$ are two characteristic polynomials for the sequence $S = (S_n)_{n \in \mathbb{N}}$.

Let $\mathcal{P}_S$ be the vector space generated by all characteristic polynomials associated with the recursive matrix sequence $S = (S_n)_{n \in \mathbb{N}}$. We have the following result.

**Proposition 3.4.** For every recursive matrix sequence $S = (S_n)_{n \in \mathbb{N}}$, the set $\mathcal{P}_S$ is an ideal in $\mathbb{R}[X]$. In particular, there exists a unique monic characteristic polynomial $P_S \in \mathcal{P}_S$ with minimal degree $r$ such that $P_S = P_S \mathbb{R}[X]$.

**Proof.** Let $\mathcal{M}_p(\mathbb{R})^N$ be the vector space of all sequences $(M_n)_{n \in \mathbb{N}}$ of $\mathcal{M}_p(\mathbb{R})$ and $\mathcal{L}(\mathcal{M}_p(\mathbb{R})^N)$ be the algebra of linear maps on $\mathcal{M}_p(\mathbb{R})^N$. The shift operator $W$ on $\mathcal{M}_p(\mathbb{R})^N$ is defined by

$$ (3.3) \quad W((M_n)_{n \in \mathbb{N}}) = (M_{n+1})_{n \in \mathbb{N}}. $$

Consider also the algebra homomorphism

$$ \Phi : \mathbb{R}[X] \rightarrow \mathcal{L}(\mathcal{M}_p(\mathbb{R})^N), $$

$$ Q = \sum_{i=0}^{m} a_i X^i \mapsto Q(W) = \sum_{i=0}^{m} a_i W^i. $$

It follows from (3.1) that

$$ Q \in \mathcal{P}_S \iff Q(W)(S) = (0_p)_{n \in \mathbb{N}} \iff Q \in \ker(\Phi). $$

Clearly, for $Q_1, Q_2 \in \ker(\Phi)$ and $P \in \mathbb{R}[X]$, we have $Q_1 + Q_2 \in \ker(\Phi)$ and $PQ_1 \in \ker(\Phi)$. It follows that $\mathcal{P}_S = \ker(\Phi)$ is an ideal in the principal ideal domain $\mathbb{R}[X]$. Hence, there exists a unique monic polynomial $P_S \in \mathcal{P}_S$ such that

$$ \mathcal{P}_S = P_s \mathbb{R}[X]. $$

□
The monic polynomial $P_S$ will be called the minimal polynomial associated with $S = (S_n)_{n \in \mathbb{N}}$. We will say in the sequel that a recursive matrix sequence is of order $r$ if the associated minimal polynomial has degree $r$.

### 3.1. Minimal polynomial and finitely atomic representing matrix charge

We investigate next the relation between the minimal polynomial associated with recursive matrix sequences and the minimal polynomials associated with the scalar entries of the sequence. More precisely, we have:

**Theorem 3.5.** Let $S = (S_n)_{n \in \mathbb{N}} = ((s_{ij}^{(n)})_{1 \leq i,j \leq p})_{n \in \mathbb{N}}$ be a recursive sequence with minimal polynomial $P_S$. For $1 \leq i, j \leq p$, let $P_{ij}$ be the minimal polynomial associated with the recursive scalar sequence $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$. Then

$$P_S = \text{lcm}\{P_{ij} \mid 1 \leq i, j \leq p\},$$

where $\text{lcm}\{P_{ij} \mid 1 \leq i, j \leq p\}$ stands for the least common multiple of the family of polynomials $\{P_{ij} \mid 1 \leq i, j \leq p\}$.

**Proof.** Let $P_{\text{lmc}}$ be the least common multiple of the family of polynomials $\{P_{ij} \mid 1 \leq i, j \leq p\}$, and let us show that $P_S = P_{\text{lmc}}$. Since $P_S$ is a characteristic polynomial for $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$, it is a multiple of $P_{ij}$ for every $1 \leq i, j \leq p$, and then $P_S$ is a multiple of $\text{lcm}\{P_{ij} \mid 1 \leq i, j \leq p\} = P_{\text{lmc}}$. Conversely, $P_{\text{lmc}}$ is a multiple of $P_{ij}$ for every $1 \leq i, j \leq p$ and then is a characteristic polynomial of $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$ for every $1 \leq i, j \leq p$. Now, since addition and scalar multiplication act linearly on entries, it follows that $P_{\text{lmc}}$ is a characteristic polynomial for $S$. We deduce that $P_{\text{lmc}}$ is a multiple of $P_S$, which completes the proof. \[\square\]

**Remark 3.6.** For an alternative proof of the previous theorem, one can observe that for every $1 \leq i, j \leq p$, the polynomial $P_{ij}$ generates the ideal $P_{ij}\mathbb{R}[X]$, and that

$$P_S \mathbb{R}[X] = \bigcap_{i,j=1}^{p} P_{ij} \mathbb{R}[X] = \text{lcm}\{P_{ij} \mid 1 \leq i, j \leq p\}\mathbb{R}[X].$$

We derive some additional useful properties.

**Corollary 3.7.** Using the above notation, the following statements hold:

1. $\mathcal{Z}(P_S) = \bigcup_{i,j=1}^{p} \mathcal{Z}(P_{ij})$;
2. $P_S$ has distinct roots if and only if $P_{ij}$ has distinct roots for every $1 \leq i, j \leq p$. 
Now, we give a cornerstone result in the solution of the matrix moment problem associated with recursive matrix sequences. This result characterizes the existence of representing charges for recursive matrix sequences.

**Theorem 3.8.** Let $S = (S_n)_{n \in \mathbb{N}}$ be a recursive matrix sequence, with a minimal polynomial $P_S$. The following statement are equivalent.

1. $S$ admits a representing matrix charge $\sigma$;
2. $P_S$ has real distinct roots.

Moreover, when a representing charge $\sigma$ exists, we have $\text{supp}(\sigma) = \mathbb{Z}(P_S)$.

**Proof.** Let $S = (S_n)_{n \in \mathbb{N}}$ be a matrix sequence. It is clear that $S = (S_n)_{n \in \mathbb{N}}$ is recursive if and only if the scalar sequences $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$ are recursive for all $1 \leq i, j \leq p$.

Similarly, the matrix sequence $S = (S_n)_{n \in \mathbb{N}}$ admits a representing matrix charge $\sigma = (\sigma_{ij})_{1 \leq i, j \leq 1}$ if and only if the scalar sequence $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$ admits a representing scalar charge $\sigma_{ij}$ for every $1 \leq i, j \leq p$.

Hence if $S = (S_n)_{n \in \mathbb{N}}$ is recursive (and so are $s_{ij} = (s_{ij}^{(n)})_{n \in \mathbb{N}}$), according to [1, Proposition 2.4], this is equivalent to the fact that $P_{ij}$ has distinct roots for every $1 \leq i, j \leq p$. By Corollary 3.7, the latter is equivalent to $P_S$ having distinct roots.

Moreover, $\text{supp}(\sigma_{ij}) = \mathbb{Z}(P_{ij})$ for every $1 \leq i, j \leq p$, and then

$$\text{supp}(\sigma) = \bigcup_{i,j=1}^{p} \text{supp}(\sigma_{ij}) = \bigcup_{i,j=1}^{p} \mathbb{Z}(P_{ij}) = \mathbb{Z}(P_S).$$

\[ \square \]

**Example 3.9.**

1. For $S = (S_n)_{n \in \mathbb{N}}$ where $S_n = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$, the minimal polynomial is $P_S(X) = (X - 1)^2$, has not distinct roots. Thus, the sequence $S$ admits no representing matrix charge.

2. Let $S = (S_n)_{n \geq 0}$ be a matrix sequence such that $S_{n+2} = -S_n$. Then $P_S(X) = X^2 + 1$ and since $\mathbb{R} \cap \mathbb{Z}(P_S) = \emptyset$, we have $S_n$ admits no representing matrix charge.

3. Denote $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $S_n = A^n$ for $n \in \mathbb{N}$. The characteristic polynomial of $A$ is given by $P_A(X) := X^2 - 1 = (X + 1)(X - 1)$. Using the Cayley-Hamilton Theorem, we get $A^n(A^2 - I_2) = 0_2$ for every $n \geq 0$ and hence

$$S_n = S_{n-2}, \quad n \geq 2$$

with $S_0 = I_2$ and $S_1 = A$. 

In particular, $P_A$ is the minimal polynomial associated with $(S_n)_{n \in \mathbb{N}}$, and then $(S_n)_{n \in \mathbb{N}}$ admits a representing matrix charge. We obtain easily

$$S_n = \frac{1}{2} \begin{pmatrix} 1 - (-1)^n & 1 + (-1)^n \\ 1 + (-1)^n & 1 - (-1)^n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(-1)^n}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

We can express $S_n$ as

$$S_n = \int_{\text{supp}(\sigma)} t^n d\sigma(t),$$

where

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_1 + \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \delta_{-1},$$

and $\text{supp}(\sigma) = Z(P_A) = \{-1, 1\}$.

Here, $\sigma$ is not a matrix measure since $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is not positive.

We mention at this stage that the Cayley-Hamilton Theorem shows that $A^n$ is a recursive sequence for every matrix $A \in M_p(\mathbb{R})$ associated with the characteristic polynomial $P_A(X) = \det(A - XI_p)$. For a symmetric matrix $A$, its minimal polynomial has only real simple roots, and $A$ is similar to a diagonal matrix in $M_p(\mathbb{R})$. It follows that a representing matrix charge always exists for $(A^n)_{n \geq 0}$ in the case $A$ is symmetric.

### 3.2. Minimal polynomial and finitely atomic representing matrix measure

In the remainder of this paper, we will focus on recursive matrix sequences $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$ whose minimal polynomial $P_S$ has all its roots distinct and real. We will write $Z(P_S) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and we denote $\sigma = \sum_{q=1}^r T_q \delta_{\lambda_q}$, with $T_1, \ldots, T_r \in M_p(\mathbb{R})$ an associated finitely atomic matrix charge provided by Theorem 3.8.

Our next objective is to determine necessary and sufficient conditions on the initial data of $\mathcal{S} = (S_0, \ldots, S_{r-1})$ to guarantee that $\sigma$ is a finitely atomic matrix measure. That is, when are the coefficients $T_1, \ldots, T_r$ in $S^+_p(\mathbb{R})$. According to Remark 2.5, this is equivalent to

$$H_n = (S_{i+j})_{0 \leq i,j \leq n} \succeq 0,$$

for every $n \in \mathbb{N}$.

As in the scalar case [1, Proposition 3.2], the following proposition provides a simple equivalent condition to (3.4) in the case of recursive matrix sequences of order $r$. 
**Proposition 3.10.** Let $S$ be a recursive matrix sequence of order $r$, and suppose that its minimal polynomial $P_S$ has only real distinct roots. The following statements are equivalent:

1. $H_{r-1} \succeq 0$;
2. $H_n \succeq 0$ for every $n \in \mathbb{N}$.

**Proof.** We only need to show (1) $\implies$ (2). To this aim, denote by $S = (S_n)_{n \in \mathbb{N}}$ the recursive matrix sequence of order $r$. Let $Z(P_S) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be the set of distinct roots of $P_S$, and let $\sigma$ be an associated finitely atomic matrix charge as in Theorem 3.8. We have

$$\sigma = \sum_{q=1}^{r} T_q \delta_{\lambda_q}$$

for some $T_1, \ldots, T_r \in \mathcal{M}_p(\mathbb{R})$.

Assume that $H_{r-1} \succeq 0$ and let $\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_n \in \mathbb{R}^p$. We write

$$\sum_{i,j=0}^{n} \vec{c}_i S_{i+j} \vec{c}_j^T = \sum_{q=1}^{r} \vec{P}(\lambda_q) T_q \vec{P}(\lambda_q)^T,$$

where $\vec{P}(X) = \sum_{k=0}^{n} \vec{c}_k X^k \in \mathbb{R}_n^p[X]$ denotes a vector polynomial with coefficients in $\mathbb{R}^p$. By setting $\vec{c}_k = (c_{k,i})_{1 \leq i \leq p}$ for $0 \leq k \leq n$, we obtain $\vec{P}(X) = (P_i(X))_{1 \leq i \leq p}$ with $P_i(X) = \sum_{k=0}^{n} c_{k,i} X^k \in \mathbb{R}_n[X]$. Now, for fixed $1 \leq i \leq p$, there exist two polynomials $Q_i$ and $R_i$ such that

$$P_i = P_S Q_i + R_i, \text{ with } \deg(R_i) < \deg(P_S) = r.$$

Hence, for every vector-valued polynomial, we have

$$\vec{P} = P_S \vec{Q} + \vec{R},$$

with $\vec{Q}(X) = (Q_i(X))_{1 \leq i \leq p}$ and $\vec{R}(X) = (R_i(X))_{1 \leq i \leq p} = \sum_{k=0}^{r-1} \vec{c}_k X^k$. Here $\vec{c}_0, \ldots, \vec{c}_{r-1} \in \mathbb{R}^p$ are the coefficients of the vector-valued polynomial $\vec{R}$. Moreover, for every $1 \leq q \leq r$, we have $\vec{P}(\lambda_q) = \vec{R}(\lambda_q)$, since $P_S(\lambda_q) = 0$. Thus, according to the hypothesis and (3.5), we have

$$\sum_{i,j=0}^{n} \vec{c}_i S_{i+j} \vec{c}_j^T = \sum_{q=1}^{r} \vec{R}(\lambda_q) T_q \vec{R}(\lambda_q)^T = \sum_{i,j=0}^{r-1} \vec{c}_i S_{i+j} \vec{c}_i^T \succeq 0.$$  

It readily follows that $H_n \succeq 0$ for every $n \geq 0$. \qed
Remark 3.11. Note that condition (2) in the previous proposition is equivalent to the positivity of the following infinite block moment matrix

\[ H_\infty := (S_{i+j})_{i,j \geq 0} = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots \\ S_1 & S_2 & S_3 & \ldots \\ S_2 & S_3 & S_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

We identify the polynomial \( P(X) = \sum_{k=0}^n a_k X^k \) with the column matrix vector \( \hat{P} = (a_0 I_p, \ldots, a_n I_p, 0_p, 0_p, \ldots)^T \), and we define the canonical inner product on \( \mathbb{R}[X] \) by

\[ \langle \sum_{k=0}^n a_k X^k, \sum_{k=0}^n b_k X^k \rangle = \sum_{k=0}^n a_k b_k. \]

Setting \( H_\infty P := H_\infty \hat{P} \), we can consider \( H_\infty \) as a linear map on \( \mathbb{R}[X] \). With this notation, we derive the following trivial properties.

Lemma 3.12. Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive matrix sequence with associated minimal polynomial \( P_S \). Then,

(1) \( P \in P_S \) if and only if \( H_\infty P = 0 \). In particular, we have \( H_\infty P_S = 0 \).

(2) For every \( P, Q \in \mathbb{R}[X] \) we have \( \langle Q, H_\infty P \rangle = \langle 1, H_\infty P Q \rangle \).

Proposition 3.13. Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive matrix sequence of order \( r \) with associated minimal polynomial \( P_S \). If \( H_{r-1} = (S_{i+j})_{0 \leq i,j \leq r-1} \geq 0 \), then

(1) The following statements are equivalent:
   (a) \( P \in P_S \);
   (b) \( P^n \in P_S \) for every \( n \in \mathbb{N}^* \);
   (c) \( P^m \in P_S \) for some \( m \in \mathbb{N}^* \).

(2) The minimal polynomial \( P_S \) has distinct roots.

Proof. (1) Suppose that \( H_{r-1} \geq 0 \). By Proposition 3.10 \( H_n \geq 0 \) for every \( n \geq 1 \). Due to the fact that \( P_S \) is an ideal of \( \mathbb{R}[X] \), we have \( (a) \Rightarrow (b) \), and it is evident that \( (b) \Rightarrow (c) \) holds. Then, we only need to prove that \( (c) \Rightarrow (a) \). Assume that there exists a positive integer \( m \) such that \( P^m \in P_S \). According to Lemma 3.12 this is equivalent to \( H_\infty P^m = 0 \). If \( m = 1 \), then it is straightforward that \( P \in P_S \).

Now, for the case where \( m \geq 2 \), we use Lemma 3.12 to get

\[ \langle P^{m-1}, H_\infty P^{m-1} \rangle = \langle P^{m-2}, H_\infty P^m \rangle = 0, \]
and since $H_\infty \geq 0$ we obtain $H_\infty P^{m-1} = 0$. By arguing recursively on $m$, we deduce that $H_\infty P = 0$, and therefore, that $P \in P_S$.

(2) Let $P_S(X) = \prod_{i=1}^{k} (X - \lambda_i)^{m_i}$, and let $m \geq \max \{ m_i \mid i = 1, \ldots, k \}$. We have $(\prod_{i=1}^{k} (X - \lambda_i))^m \in P_S$, since $P_S$ is an ideal of $\mathbb{R}[X]$. Using (1), we obtain $P_S(X) = \prod_{i=1}^{k} (X - \lambda_i)$, since $P_S$ is the minimal polynomial. □

4. Recursive moment sequences

We discuss in this section the classical recursive matrix moment sequences on $\mathbb{R}$.

4.1. Hamburger recursive matrix moment sequences.

**Theorem 4.1.** Let $S = (S_n)_{n \in \mathbb{N}}$ be a recursive matrix sequence of order $r$ with $P_S$ the associated minimal polynomial has real distinct roots. The following statements are equivalent:

(1) $S$ is a Hamburger matrix moment sequence;

(2) $H_{r-1} \succeq 0$.

**Proof.** Let $S = (S_n)_{n \in \mathbb{N}}$ be a recursive matrix sequence of order $r$, and let $P_S$ be its minimal polynomial. The direct implication is clear. Now, we assume that $H_{r-1} \succeq 0$. Since the minimal polynomial $P_S$ has $r$ real distinct roots, $Z(P_S) = \{ \lambda_1, \ldots, \lambda_r \}$, from Proposition 3.10 we derive that $S = (S_n)_{n \in \mathbb{N}}$ admits a finitely atomic generating charge $\sigma$, such that $\sigma = \sum_{i=1}^{r} T_i \delta_{\lambda_i}$ for some $T_1, \ldots, T_r \in M_p(\mathbb{R})$. Thus, it remains to show that $T_i \in S_S^+(\mathbb{R})$ for $i = 1, \ldots, r$. To this aim, since $H_{r-1} \succeq 0$, we have:

(4.1) $\sum_{i,j=0}^{r-1} \bar{c}_i S_{i+j} \bar{c}_j^T = \sum_{i=0}^{r} \bar{P}(\lambda_i) T_i \bar{P}(\lambda_i)^T \succeq 0,$

for every vector polynomial $\bar{P}(X) = \sum_{k=0}^{r-1} \bar{c}_k X^k$, where $\bar{c}_k \in \mathbb{R}^p$. 

Now, we fix \( i = 1, \ldots, r \), and we consider the interpolating Lagrange polynomial
\[
L_i(X) = \prod_{j=1}^{r} \frac{(X - \lambda_j)}{(\lambda_i - \lambda_j)}.
\]

Also, for an arbitrary vector \( \vec{v} \) in \( \mathbb{R}^p \), we define \( \vec{Q}_i(X) := \vec{v} \cdot L_i(X) \). It is clear that \( \vec{Q}_i(\lambda_j) = \vec{v} \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. Applying (4.1) to the vector polynomial \( \vec{P} = \vec{Q}_i \), we obtain \( \vec{v}^T \vec{P} \vec{v}^T \geq 0 \), and then \( T_i \in S_p^+ (\mathbb{R}) \) for every \( 1 \leq i \leq r \).

**Remark 4.2.** We can recover the previous theorem as a consequence of the Theorem 2.6 and the Proposition 3.10.

As a corollary, we have the next completion result.

**Corollary 4.3.** Let \( r \geq 1 \) and \( S = (S_0, S_1, \ldots, S_{r-1}) \) be a finite family of symmetric matrices. The following statements are equivalent:

1. \( S \) is a Hamburger truncated moment sequence;
2. There exists \((S_r, \ldots, S_{2r-2})\) a family of symmetric matrices such that \((S_{i+j})_{0 \leq i, j \leq r-1} \succeq 0\).

### 4.2. Stieltjes and Hausdorff recursive matrix sequences.

Using similar proofs, we obtain the next result.

**Theorem 4.4.** Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive symmetric matrix sequence of order \( r \) with \( P_S \) the associated minimal polynomial has real distinct roots. Then

1. \( S \) is a Stieltjes matrix moment sequence if and only if \( H_{r-1} \succeq 0 \) and \( H_{r-1}^{(1)} \succeq 0 \).
2. \( S \) is a Hausdorff matrix moment sequence if and only if \( H_{r-1} \succeq 0 \) and \( H_{r-1}^{(1)} - H_{r-1}^{(2)} \succeq 0 \).

We further deduce the following corollary.

**Corollary 4.5.** Let \( k \geq 1 \) and \( S = (S_0, S_1, \ldots, S_{r-1}) \) be a finite family of symmetric matrices. The following statements are equivalent:

1. \( S \) is a Stieltjes truncated matrix moment sequence;
2. There exists \((S_r, \ldots, S_{2r-1})\) a family of symmetric matrices such that \((S_{i+j})_{0 \leq i, j \leq r-1} \succeq 0 \) and \((S_{i+j+1})_{0 \leq i, j \leq r-1} \succeq 0 \).

**Corollary 4.6.** Let \( k \geq 1 \) and \( S = (S_0, S_1, \ldots, S_{r-1}) \) be a finite family of symmetric matrices. The following statements are equivalent:
(1) \( S \) is a Hausdorff truncated matrix moment sequence;
(2) There exists \((S_r, \cdots, S_{2r})\) a family of symmetric matrices such that 
\((S_{i+j})_{0 \leq i+j \leq r-1} \succeq 0\) and 
\((S_{i+j+1} - S_{i+j+2})_{0 \leq i+j \leq r-1} \succeq 0\).

4.3. Some special cases. Next, we investigate two cases of recursive matrix sequences of special interest. Without any loss of generality, we will take \( S_0 = I_p \).

4.3.1. Case \( r=2 \). Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive matrix sequence, with initial data \((I_p, S_1)\), such that 
\( S_{n+1} = a_0 S_n + a_1 S_{n-1} \) for \( n \geq 1 \), where 
\( a_0^2 + 4a_1 > 0 \). We write the associated minimal polynomial as 
\( P_S(X) = (X - \lambda_0)(X - \lambda_1) \) with \( \lambda_0 < \lambda_1 \). According to Theorem 3.8, the sequence \( S \) admits a finitely atomic generating matrix charge \( \sigma = T_0 \delta_{\lambda_0} + T_1 \delta_{\lambda_1} \), for some 
\( T_0, T_1 \in M_p(\mathbb{R}) \). In particular, we have 
\[
\begin{cases}
T_0 + T_1 = I_p, \\
\lambda_0 T_0 + \lambda_1 T_1 = S_1,
\end{cases}
\]
and then,
\[
T_0 = \frac{1}{\lambda_1 - \lambda_0} (\lambda_1 I_p - S_1) \quad \text{and} \quad T_1 = \frac{1}{\lambda_1 - \lambda_0} (S_1 - \lambda_0 I_p).
\]
We now derive the next result.

Theorem 4.7. The following statements are equivalent:
(1) \( \sigma \) is a matrix measure;
(2) \( \lambda_0 I_2 \preceq S_1 \preceq \lambda_1 I_2 \).

The next corollary highlights the impact of the choice of \( \lambda_0 \) and \( \lambda_1 \).

Corollary 4.8. Given a matrix \( A \in S_p(\mathbb{R}) \), then
(1) \((I_p, A)\) is a truncated Hamburger moment problem;
(2) \((I_p, A)\) is a truncated Stieljes moment problem if and only if \( A \in S_p^+(\mathbb{R}) \);
(3) \((I_p, A)\) is a truncated Hausdorff moment problem if and only if \( A \) is a contraction in \( S_p^+(\mathbb{R}) \).

Proof. For (1), using Theorem 4.7, it suffices to choose \( \lambda_0 < \lambda_1 \) such that 
\( \lambda_0 I_p \preceq A \preceq \lambda_1 I_p \) and define \( S = (S_n)_{n \in \mathbb{N}} \) as follows:
\[
S_{n+1} = (\lambda_0 + \lambda_1) S_n - \lambda_0 \lambda_1 S_{n-1}, \quad n \geq 1,
\]
with initial conditions \( S_0 = I_p \) and \( S_1 = A \).
We obtain the second assertion by noticing that necessarily \( \lambda_0 \in \mathbb{R}^+ \), and the third one by observing that we should have \( \lambda_0, \lambda_1 \in [0, 1] \). \( \square \)
Remark 4.9. We notice here that for every \( A \in \mathcal{S}_p(\mathbb{R}) \), the sequence \( S = (A^n)_{n \in \mathbb{N}} \) is a recursive matrix sequence that admits a finitely atomic representing matrix charge \( \sigma_A \). However, \( \sigma_A \) is matrix measure if and only if \( A \in \mathcal{S}_p^+(\mathbb{R}) \). Thus, the previous corollary deals with a more general setting.

In the scalar case \( p = 1 \), we recover Proposition 4.1 of [1] in the following corollary.

Corollary 4.10. A recursive real sequence \( S = (s_n)_{n \in \mathbb{N}} \) such that \( s_0 = 1, s_1 \in \mathbb{R} \) is given, and \( s_{n+1} = a_0 s_n + a_1 s_{n-1} \). The following statements are equivalent:

1. \( S \) is a real moment sequence;
2. \( P_S(s_1) \leq 0 \).

Here, \( P_S(X) = X^2 + a_0 X - a_1 \).

4.3.2. Case \( r = 3 \). Let \( S = (I, S_1, S_2) \) be a given initial data of symmetric matrices. By using Corollary 4.3, we have \( S \) is a truncated moment sequence if and only if there exists \( S_3 \) and \( S_4 \) such that \( H_2 \geq 0 \). It is then necessary to have \( S_2 \geq S_1 \). Thus, as in the scalar case, there exist initial data \( (I, S_1, S_2) \) without any matrix moment sequence extension. See [1], for the scalar case. We exhibit below a constructive method to get the representing matrix measure when \( (I, S_1, S_2) \) is a truncated moment sequence.

Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive matrix sequence of order \( r = 3 \), with initial data \( S_0 = I_p \), and \( S_1, S_2 \in \mathcal{S}_p(\mathbb{R}) \). Let \( P_S \) be the associated minimal polynomial, such that \( P_S(X) = (X - \lambda_0)(X - \lambda_1)(X - \lambda_2) \), with \( \lambda_0 < \lambda_1 < \lambda_2 \). Denote \( \sigma = T_0 \delta_{\lambda_0} + T_1 \delta_{\lambda_1} + T_2 \delta_{\lambda_2} \) for the associated finitely atomic representing matrix charge obtained from Theorem 3.8. The coefficients \( T_1, T_2, T_3 \in \mathcal{M}_p(\mathbb{R}) \) will satisfy the following Vandermonde system:

\[
\begin{align*}
T_0 + T_1 + T_2 &= I_p, \\
\lambda_0 T_0 + \lambda_1 T_1 + \lambda_2 T_2 &= S_1, \\
\lambda_0^2 T_0 + \lambda_1^2 T_1 + \lambda_2^2 T_2 &= S_2.
\end{align*}
\]

Solving this linear matrix system, we obtain the expression of \( T_0, T_1, \) and \( T_2 \)

\[
\begin{align*}
T_0 &= (\lambda_3 - \lambda_2) \Delta (S_2 - S_1 (\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 I_p), \\
T_1 &= (\lambda_2 - \lambda_0) \Delta (-S_2 + S_1 (\lambda_0 + \lambda_2) - \lambda_0 \lambda_2 I_p), \\
T_2 &= (\lambda_1 - \lambda_0) \Delta (S_2 - S_1 (\lambda_0 + \lambda_1) + \lambda_0 \lambda_1 I_p),
\end{align*}
\]

where

\[
\Delta = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)} > 0.
\]

Hence, we derive the following result.
Proposition 4.11. The following statements are equivalent:

1. \( \sigma \) is matrix measure;
   
   \[
   \begin{align*}
   S_2 - (\lambda_1 + \lambda_2)S_1 + \lambda_1\lambda_2 I_p & \succeq 0, \\
   -S_2 + (\lambda_0 + \lambda_2)S_1 - \lambda_0\lambda_2 I_p & \succeq 0, \\
   S_2 - (\lambda_0 + \lambda_1)S_1 + \lambda_0\lambda_1 I_p & \succeq 0;
   \end{align*}
   \]

2. \( \lambda (S_1 - \lambda_2 I_p) \preceq S_2 - \lambda_2 S_1 \preceq \lambda_0 (S_1 - \lambda_2 I_p) \),
   \( \lambda (S_1 - \lambda_0 I_p) \preceq S_2 - \lambda_0 S_1 \preceq \lambda_2 (S_1 - \lambda_0 I_p) \).

Notice that some necessary conditions to determine if \( \sigma \) is a matrix measure are given by

\[
S_2 \succeq S_1^2, \quad \lambda_0 I_p \preceq S_1 \preceq \lambda_2 I_p \quad \text{and} \quad \lambda_0^2 I_p \preceq S_2 \preceq \lambda_2^2 I_p.
\]

In the extremal cases \( S_1 = \lambda_0 I_p \) or \( S_1 = \lambda_2 I_p \), \( (I_p, S_1, S_2) \) is a truncated matrix moment sequence if and only if \( S_2 = S_1^2 \). This corresponds to the one-atomic matrix measure \( S_n = A\lambda^n \), for some \( A \in S^+_R(\mathbb{R}) \) and \( \lambda \in \mathbb{R} \) as in the scalar case.

Example 4.12. Let \( S = (S_n)_{n \in \mathbb{N}} \) be a recursive sequence of \( S^+_R(\mathbb{R}) \) satisfying \( S_{n+1} = 6S_n - 10S_{n-1} + 4S_{n-2} \) for \( n \geq 2 \), and with initial data

\[
S_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad S_1 = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix}
5 & -4 & 1 \\
-4 & 6 & -4 \\
1 & -4 & 5
\end{pmatrix}.
\]

The associated minimal polynomial is given by

\[
P_S(X) = X^3 - 6X^2 + 10X - 4 = (X - \lambda_0)(X - \lambda_1)(X - \lambda_2).
\]

With \( \lambda_0 = 2 - \sqrt{2} \), \( \lambda_1 = 2 \) and \( \lambda_2 = 2 + \sqrt{2} \). We have

\[
\begin{align*}
S_2 - (\lambda_1 + \lambda_2)S_1 + \lambda_1\lambda_2 I_3 &= \begin{pmatrix}
1 & \sqrt{2} & 1 \\
\sqrt{2} & 2 & \sqrt{2} \\
1 & \sqrt{2} & 1
\end{pmatrix} \succeq 0, \\
-S_2 + (\lambda_0 + \lambda_2)S_1 - \lambda_0\lambda_2 I_3 &= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 1
\end{pmatrix} \succeq 0, \\
S_2 - (\lambda_0 + \lambda_1)S_1 + \lambda_0\lambda_1 I_3 &= \begin{pmatrix}
1 & -\sqrt{2} & 1 \\
-\sqrt{2} & 2 & -\sqrt{2} \\
1 & -\sqrt{2} & 1
\end{pmatrix} \succeq 0.
\end{align*}
\]

Therefore, according to Proposition 4.11, \( S = (S_n)_{n \in \mathbb{N}} \) admits finitely atomic representing matrix measure \( \sigma \). Moreover,

\[
\sigma = T_0 \delta_{\lambda_0} + T_1 \delta_{\lambda_1} + T_2 \delta_{\lambda_2},
\]
where
\[
T_0 = \begin{pmatrix}
\frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\
\frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{\sqrt{2}}{4} \\
\frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4}
\end{pmatrix},
T_1 = \begin{pmatrix}
\frac{1}{4} & 0 & -\frac{1}{2} \\
0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\text{ and } T_2 = \begin{pmatrix}
\frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \\
-\frac{\sqrt{2}}{4} & 2 & -\frac{\sqrt{2}}{4} \\
\frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4}
\end{pmatrix}.
\]

In particular, \(S_n = T_0\lambda_0^n + T_1\lambda_1^n + T_2\lambda_2^n\) for every \(n \in \mathbb{N}\).

**Remark 4.13.** If \(S = (S_n)_{n \in \mathbb{N}} = ((s_{i,j}^{(n)})_{1 \leq i,j \leq p})_{n \in \mathbb{N}}\) is a matrix moment sequence, then \((s_{ii}^{(n)})_{n \in \mathbb{N}}\) is also a scalar moment sequence for every \(i = 1, \ldots, p\). Moreover, Example 4.12 shows that this is no longer true for \(s_{ij}\) when \(i \neq j\). In the reverse direction, the next sequence
\[
S_n = \begin{pmatrix}
1 + 2^n & 1 + 3.2^n \\
1 + 3.2^n & 2^n
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} 1^n + \begin{pmatrix}
1 & 3 \\
3 & 1
\end{pmatrix} 2^n
\]
has as entries real moment sequences. Since the associated matrix charge
\[
\mu = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} \delta_1 + \begin{pmatrix}
1 & 3 \\
3 & 1
\end{pmatrix} \delta_2
\]
is not a matrix measure, \(S\) is not a matrix moment sequence.

5. **Propagation phenomena for \(k\)-hyponormal matricial weighted Shifts**

5.1. **\(k\)-hyponormal and subnormal matricial weighted Shifts.** An operator \(T\) on a Hilbert space \(H\) is normal if \([T^*, T] = T^*T - TT^* = 0_H\), is hyponormal if \([T^*, T] \succeq 0\), and is subnormal if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace.

The class of \(k\)-hyponormal operators introduced as a bridge between hyponormal operators and subnormal operators consists of operators such that the matrix \([(T^*)^i, T^j]_{i,j=0}^k\) is positive. For arbitrary \(k \geq 1\), the next implications hold

\text{subnormal} \Rightarrow \text{\(k\)-hyponormal} \Rightarrow \text{(\(k-1\))-hyponormal} \Rightarrow \ldots \Rightarrow \text{hyponormal}.

It is also well-known that the reverse implications are not always true (see [4]).

Let \(\mathbb{C}^p\) represent the \(p\)-dimensional complex Hilbert space. We consider the Hilbert space
\[
\ell_2^p = \left\{ (x_n)_{n \geq 0} : \forall n \geq 0, x_n \in \mathbb{C}^p \text{ and } \sum_{n=0}^{+\infty} \|x_n\|_{\mathbb{C}^p}^2 < \infty \right\}
\]
equipped with the inner product defined as

$$(x, y) = \sum_{n=0}^{+\infty} \langle x_n, y_n \rangle_{\ell^p},$$

where $x = (x_n)_{n \geq 0}$ and $y = (y_n)_{n \geq 0}$ belong to $\ell^p$.

Let $A = (A_n)_{n=0}^{+\infty}$ be a uniformly bounded sequence of $p \times p$ matrices (i.e., $\sup_{n \in \mathbb{N}} \|A_n\| < \infty$). The matricial weighted shift associated with the matricial weighted sequence $A$ is the bounded linear operator on $\ell^p$ given by

$$W_A(x_0, x_1, \ldots) = (0, A_0x_0, A_1x_1, \ldots),$$

with $\|W_A\| = \sup_{n \in \mathbb{N}} \|A_n\|$.

It is easy to check that $W_A$ is hyponormal if and only if $A_n A_n^* \preceq A_{n+1}^2 A_{n+1}$ for every $n \in \mathbb{N}$.

From [18], every matricial weighted shift with invertible weights is unitarily equivalent to a matricial weighted shift with positive definite weights. For this reason, we will focus, in the sequel, on shifts with positive definite weights. In this particular case, we have $W_A$ is hyponormal if and only if $A_n^2 \preceq A_{n+1}^2$ for every $n \in \mathbb{N}$.

Denote $B_0 = I_p$ and $B_n = A_{n-1} \ldots A_0$ for $n \geq 1$. The next characterization of subnormal matricial weighted shifts is given in [14, Theorem 1.1] as a consequence of [11] and [17].

**Theorem 5.1.** Let $A = (A_n)_{n \in \mathbb{N}}$ be a sequence of $p \times p$ complex positive definite matrices satisfying $\sup_{n \in \mathbb{N}} \|A_n\| < +\infty$. Then $W_A$ is subnormal if and only if there exists a matrix measure $\sigma$ supported in $[0, \|W_A\|^2]$, with $\|W_A\|^2 \in \text{supp}(\sigma)$, such that

$$B_n^* B_n = \int_0^{\|W_A\|^2} t^n d\sigma(t) \text{ for every } n \in \mathbb{N}.$$ 

In this case, $\sigma$ is unique and referred to as Berger’s matrix measure corresponding to the weight $A$.

For $k$–hyponormality of scalar weighted shifts, the next characterization is provided by S. McCullough and V. Paulsen in [22]. Let $T$ be a scalar weighted shift with a nonzero weight sequence. For $k \geq 1$, we have

$$T \text{ is } k \text{– hyponormal} \iff (T^{*i+j}T^{i+j})_{i,j=0}^{k} \succeq 0.$$ 

The same arguments as in [22, Theorem 2.2], extend (5.1) to the case of matricial weighted shifts with positive-definite weights. To be precise, if
$W_A$ is a matricial weighted shift with positive definite weights, then

$$W_A \text{ is } k-\text{hyponormal} \iff (W_A^{r^i+j} W_A^{r^i+j})_{i,j=0}^k \succeq 0.$$ 

. The latter is equivalent to the following extension of [4, Theorem 4] to matricial weighted shifts.

(5.2) $W_A \text{ is } k-\text{hyponormal} \iff (B_{m+k}^* B_{m+k})_{k=0}^{2k} \succeq 0$ for every $m \geq 0$.

A sequence $(B_n B_n)_{n \in \mathbb{N}}$ satisfying (5.2) will be called $k$–positive. Thus a sequence $(B_n^* B_n)_{n \in \mathbb{N}}$ is a moment sequence if and only if it is $k$–positive for every $k \geq 1$.

5.2. **Subnormal and $k$–hyponormal completion of matricial weighted shifts.** For $A_r = (A_0, \cdots, A_{r-1})$ a family of matrices, a matrix sequence $\tilde{A} = (\tilde{A}_i)_{i \geq 0}$ extends $A_k$ if $\tilde{A}_i = A_i$ for every $i \leq r - 1$. The $k$–hyponormal (subnormal) completion of matricial weighted shifts is stated as follows. Given $A_r = (A_0, \cdots, A_{r-1})$ positive matrices, Is there any extension $\tilde{A}$ of $A_r$ such that $W_{\tilde{A}}$ is $k$–hyponormal (subnormal). We will say that $A_r$ extends to a $k$–hyponormal weight shift or subnormal weight shift, respectively.

We highlight the link between the $k$–hyponormal and subnormal completions of matricial weights and truncated completion problems by using the previous section. We have the following

**Proposition 5.2.** Let $A_r = (A_0, \cdots, A_{r-1})$ be a finite family of positive definite matrices and define $B_n = A_{n-1} \cdots A_1 A_0$, for $1 \leq n \leq r$. The following are equivalent:

1. $A_r$ extends to a $k$–hyponormal weighted shift (resp. subnormal weighted shift);

2. $(I_p, B_1^* B_1, \cdots, B_r^* B_r)$ extends to a $k$–positive sequence (moment sequence).

As a consequence, we obtain the following results.

**Corollary 5.3.** Let $A_r = (A_0, \cdots, A_{r-1})$ be a finite family of positive definite matrices; Then

1. $A_r$ extends to a $k$–hyponormal weighted shift $\iff A_r$ extends to a subnormal weighted shift.

2. $A_2$ always extends to a subnormal weighted shift.
5.3. **Propagation phenomena for matricial weighted shifts.** In the scalar case, Stampfli’s Theorem establishes a propagation property for certain subnormal scalar weighted shifts, as follows.

**Theorem 5.4.** [27, Theorem 6] Let \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) be a sequence of non-zero positive real numbers. If \( W_\alpha \) is a subnormal weighted shift, and \( \alpha_k = \alpha_{k+1} \) for some \( k \geq 1 \), then \( \alpha_n = \alpha_0 \) for every \( n \geq 1 \).

Stampfli’s Theorem has been extended by the first author to 2–hyponormal weighted shifts as follows.

**Proposition 5.5.** [4, Corollary 6] As above, if \( W_\alpha \) is a 2–hyponormal weighted shift, and \( \alpha_k = \alpha_{k+1} \) for some \( k \geq 0 \), then \( \alpha_n = \alpha_1 \) for every \( n \geq 1 \).

It is natural to ask whether Stampfli’s Theorem (or, more generally, Curto’s extension) holds for matricial weighted shifts (\( p \geq 2 \)). Specifically, if \( W_A \) is subnormal (2–hyponormal) such that \( A_k = A_{k+1} \) for some \( k \geq 0 \), is it true that \( A_n = A_1 \) for all \( n \geq 1 \)?

In 1973, N. Ivanovski established the following result in his Ph.D. thesis.

**Theorem 5.6.** [16, Corollary 1.23] Let \( A = (A_n)_{n \in \mathbb{N}} \) be a sequence of \( p \times p \) complex positive definite matrices satisfying \( \sup_{n \in \mathbb{N}} \| A_n \| < +\infty \). If \( W_A \) is a subnormal matricial weighted shift, and \( A_k = A_{k+1} \) for some \( k \in \mathbb{N} \), then \( A_n = A_k \) for all \( n \geq k \).

In the previous theorem, no information is provided for \( n < k \). We give an alternative proof valid for the more general class of 2–hyponormal matricial weighted shifts. We also show the inner propagation phenomena to fill the case \( n < k \) in Ivanovski’s result. This extends both Stampfli’s and Curto’s results to the matricial case.

**Theorem 5.7.** Let \( A = (A_n)_{n \in \mathbb{N}} \) be a sequence of \( p \times p \) positive definite matrices satisfying \( \sup_{n \in \mathbb{N}} \| A_n \| < +\infty \). If \( W_A \) is a 2–hyponormal matricial weighted shift, and \( A_k = A_{k+1} \) for some \( k \in \mathbb{N} \), then \( A_n = A_k \) for all \( n \geq k \).

**Proof.** Suppose that \( W_A \) is a 2–hyponormal matricial weighted shift and \( A_k = A_{k+1} \) for some \( k \in \mathbb{N} \). The restriction of \( W_A \) to the \( W_A \)-invariant subspace \( \{ (x_n)_{n \geq 0} \in \ell_2^p : x_n = 0 \text{ for } n < k \} \) is a 2–hyponormal matricial weighted shift. Thus, we may suppose without any loss of generality that \( k = 0 \). Suppose that \( A_0 = A_1 (= A) \).
Notice that since $W_A$ is a 2–hyponormal, we have

$$
\begin{pmatrix}
I & B_1^1B_1 & B_2^1B_2 \\
B_1^1B_1 & B_2^2B_2 & B_3^2B_3 \\
B_2^2B_2 & B_3^3B_3 & B_4^3B_4
\end{pmatrix} \succeq 0.
$$

By expanding $B_k^kB_k$ for $k \leq 4$, we obtain

$$
\begin{pmatrix}
I & A^2 & A^4 \\
A^2 & A^4 & A^2A_2^2A^2 \\
A^4 & A^2A_2^2A^2 & A^2A_2A_2^2A^2
\end{pmatrix} \succeq 0.
$$

To prove that $A_2 = A$, we use the following extension theorem due to Y. Smul’jan.

**Lemma 5.8.** \cite{25} Let $X \in \mathcal{M}_k(\mathbb{C})$, $X \succeq 0$, $Y \in \mathcal{M}_{k,\ell}(\mathbb{C})$, and $Z \in \mathcal{M}_\ell(\mathbb{C})$. The following are equivalent:

1. $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \succeq 0$;

2. There exists $W \in \mathcal{M}_{k,\ell}(\mathbb{C})$ such that $XW = Y$ and $Z \succeq W^*XW$.

Using Lemma 5.8 with $X = \begin{pmatrix} I & A^2 \\ A^2 & A^4 \end{pmatrix}$ and $Y = \begin{pmatrix} A^4 \\ A^2A_2^2A^2 \end{pmatrix}$, there exists $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, such that

$$
\begin{pmatrix} A^4 \\ A^2A_2^2A^2 \end{pmatrix} = \begin{pmatrix} I & A^2 \\ A^2 & A^4 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} W_1 + A^2W_2 \\ A^2W_1 + A^4W_2 \end{pmatrix}.
$$

It follows that $A^6 = A^2A_2^2A^2$ and then $A_2 = A$. We now use induction to complete the proof of the outer propagation phenomena.

To show the inner propagation ( $A_n = A_k$ for $n < k$), we will show that $A_{k-1} = A_k$ ($k \geq 2$), using our results on recursive matrix sequences. Since the operator $S := A_k^2 \geq 0$, its minimal polynomial $P = \sum_{i=0}^{s} a_i X^i$ has single roots and $a_0 \neq 0$. From the identity $B_k^n P(S)B_n = 0_p$ and $A_n = A_k$ for $n \geq k$, we obtain

$$
\sum_{j=0}^{s} a_j B_{n+j}^* B_{n+j} = \sum_{j=0}^{s} a_j B_k^* S^{n+j-k} B_k^* = B_k^* S^{n-k} P(S)B_k = 0_p.
$$
It follows that \((B_n^*B_n)_{n \geq 0}\) satisfies the recursive matrix relation associated with the polynomial \(P\). Then, we see easily that \(X^k P(X)\) is a characteristic polynomial for the sequence \((B_n^*B_n)_{n \geq 0}\). Now, since \((B_n^*B_n)_{n \geq 0}\) is a matrix moment sequence, the polynomial \(XP(X)\) is also an associated characteristic polynomial. We deduce that for every \(n \geq 1\), we have
\[
a_s B_{n+s}^* B_{n+s} + a_{s-1} B_{n+s-1}^* B_{n+s-1} + \cdots + a_1 B_{n+1}^* B_{n+1} + a_0 B_n^* B_n = 0.
\]
For \(n = k - 1\), we find
\[
-a_0 B_{k-1}^* B_{k-1} = a_s B_{k+s-1}^* B_{k+s-1} + a_{s-1} B_{k+s-2}^* B_{k+s-2} + \cdots + a_1 B_k^* B_k
\]
\[
= B_k^* [a_s S^{s-1} + a_{s-1} S^{s-2} + \cdots + a_1 I_p] B_k
\]
\[
= B_k^* [Q(S)] B_k.
\]
Where \(Q(X) = \frac{P(X) - a_0}{X}\). Since \(P(S) = 0\), we have
\[
-a_0 I_p = P(S) - a_0 I_p = (P - a_0)(S) = SQ(S),
\]
and then \(Q(S) = -a_0 S^{-1}\). We obtain
\[
B_{k-1}^* B_{k-1} = B_k^* S^{-1} B_k = B_k^* A_k^{-2} B_k.
\]
Hence
\[
B_{k-1}^* B_{k-1} = B_k^* A_k^{-2} B_k \Rightarrow B_{k-1}^* B_{k-1} = B_{k-1}^* A_{k-1} A_k^{-2} A_{k-1} B_{k-1} \Rightarrow A_{k-1} A_k^{-2} A_{k-1} = I_p \Rightarrow A_k = A_{k-1}.
\]
We again use induction to complete the proof of the inner propagation phenomena.

\[\square\]

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References

[1] R. Ben Taher, M. Rachidi, and E.H. Zerouali, Recursive subnormal completion and the truncated moment problem, Bull. Amer. Math. Soc. 33 (2001), no. 4, 425–432.
[2] C.H. Chidume, M. Rachidi, and E.H. Zerouali, Solving the general truncated moment problem by the \( r \)-generalized Fibonacci sequences method, J. Math. Anal. Appl. 256 (2001), no. 2, 625–635.
[3] J. Cimprič and A. Zalar, Moment problems for operator polynomials, J. Math. Anal. Appl. 401 (2013), no. 1, 307–316.
[4] R.E. Curto, Quadratically hyponormal weighted shifts, Integral Equations Operator Theory 13 (1990), 49–66.
[5] R.E. Curto and L.A. Fialkow, Solution of the truncated complex moment problem for flat data, in Mem. Amer. Math. Soc., vol. 568, American Mathematical Society, Providence (1996).
[6] R.E. Curto and L.A. Fialkow, Flat extensions of positive moment matrices: Recursively generated relations, in Mem. Amer. Math. Soc., vol. 648, American Mathematical Society, Providence (1998).
[7] R.E. Curto and L.A. Fialkow, The truncated complex \( K \)-moment problem, Trans. Amer. Math. Soc. 352 (2000), no. 6, 2825–2855.
[8] H. Dette and W.J. Studden, A note on the matrix valued \( q-d \) algorithm and matrix orthogonal polynomials on \([0,1]\) and \([0,\infty]\), J. Comp. Appl. Math., 148 (2002), no. 2, 349–361.
[9] H. Dette and W.J. Studden, A note on the maximization of matrix valued Hankel determinants with applications, J. Comp. Appl. Math., 177 (2005), no. 1, 129–140.
[10] F. Dubeau, W. Motta, M. Rachidi, and O. Saeki, On weighted \( r \)-generalized Fibonacci sequences, Fibonacci Quarterly 35 (1997), no. 2, 102–110.
[11] P. Ghatage, Subnormal shifts with operator-valued weights, Proc. Amer. Math. Soc. 57 (1976), no. 1, 107–108.
[12] H. Hamburger, Über eine erweiterung des stieltjesschen momentenproblems, Math. Ann. 81 (1920), no. 2, 235–319.
[13] F. Hausdorff, Summationsmethoden und momentfolgen. I, Mathematische Zeitschrift 9 (1921), no. 1, 74–109.
[14] D. Kimsey, The matricial subnormal completion problem, Linear Algebra Appl. 645 (2022), 170–193.
[15] D. Kimsey and H. Woerdeman, The truncated matrix-valued \( K \)-moment problem on \( \mathbb{R}^d \), \( \mathbb{C}^d \) and \( \mathbb{T}^d \), Trans. Amer. Math. Soc. 365 (2013), no. 10, 5393–5430.
[16] N. Ivanovski, Subnormality of operator valued weighted shifts, Ph.D. dissertation, Indiana University, 1973.
[17] A. Lambert, Subnormality and weighted shifts, J. London Math. Soc. 14 (1976), 2(3), 476–480.
[18] A. Lambert, Unitary equivalence and reductibility or invertibly weighted shifts. Bull. Aust. Math. Soc. 5 (1971), 157–173.
[19] J.B. Lasserre, Global optimization with polynomials and the problem of moments, SIAM J. Optim. 11 (2001), no. 3, 796–817.
[20] C.T. Le, Tracial moment problems on hypercubes, Oper. Matrices 14 (2020), no. 4, 1015—1027.

[21] C. Mädl and K. Schmüdgen, On the Truncated Matricial Moment Problem. I, arXiv preprint arXiv:2310.00957 (2023)

[22] S. McCullough, and V. Paulsen, $k$-Hyponormality of Weighted Shifts, Proc. Amer. Math. Soc. 116 (1992) 165—69.

[23] M. Riesz, Sur le problème des moments, Arkiv för Mat., Astr. och Fysik 17 (1923), no. 12, 1–52.

[24] K. Schmüdgen, The Moment Problem, Vol. 9, Berlin: Springer, 2017.

[25] Y.L.V. Shmul’yan, An operator Hellinger integral, Mat. Sb. 91 (1959), no. 4, 381-430.

[26] T.J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse Math. 8 (1894), no. 4, J1–J122.

[27] J. Stampfli, Which weighted shifts are subnormal, Pacific J. Math. 17 (1966) no. 2, 367-379.

[28] J. Stochel, Solving the truncated moment problem solves the full moment problem, Glasgow Math. J. 43 (2001), no. 3, 335–341.

[29] F.-H. Vasilescu, Dimensional stability in truncated moment problems, J. Math Anal. Appl. 388 (2012), no. 1, 219–230.

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