ALGEBRAIC EMBEDDINGS OF $\mathbb{C}$ INTO $\text{SL}_n(\mathbb{C})$

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Abstract. We prove that any two algebraic embeddings $\mathbb{C} \to \text{SL}_n(\mathbb{C})$ are the same up to an algebraic automorphism of $\text{SL}_n(\mathbb{C})$, provided that $n$ is at least 3. Moreover, we prove that two algebraic embeddings $\mathbb{C} \to \text{SL}_2(\mathbb{C})$ are the same up to a holomorphic automorphism of $\text{SL}_2(\mathbb{C})$.

1. Introduction

There are many results concerning algebraic embeddings of some variety into the affine space $\mathbb{C}^n$. Let me recall two of them. Any two algebraic embeddings of a smooth affine variety $X$ into $\mathbb{C}^n$ are the same up to an algebraic automorphism of $\mathbb{C}^n$, provided that $n > 2 \dim X + 1$. This result is due to Nori, Srinivas [Sri91] and Kaliman [Kal91]. If one relaxes the condition that the automorphism of $\mathbb{C}^n$ must be algebraic, Kaliman [Kal13] and independently, Feller and the author [FS14] proved the following improvement: Any two algebraic embeddings of a smooth affine variety $X$ into $\mathbb{C}^n$ are the same up to a holomorphic automorphism of $\mathbb{C}^n$, provided that $n > 2 \dim X$.

As a further development of these results, we study algebraic embeddings of $\mathbb{C}$ into $\text{SL}_n$. This article can be seen as a first example to understand algebraic embeddings of a curve into an arbitrary affine algebraic variety with a large automorphism group.

In dimension zero, Arzhantsev, Flenner, Kaliman, Kutzschebauch and Zaidenberg proved that two embeddings of a finite set into any irreducible smooth affine flexible variety $Z$ are the same up to an algebraic automorphism of $Z$, provided that $\dim Z > 1$ [AFK+13]. Our main result is based on this work.

Main Theorem (see Theorem 4 and Theorem 7). Let $f, g: \mathbb{C} \to \text{SL}_n$ be algebraic embeddings. If $n \geq 3$, then $f$ and $g$ are the same up to an algebraic automorphism of $\text{SL}_n$ and if $n = 2$, then $f$ and $g$ are the same up to a holomorphic automorphism of $\text{SL}_n$.

To the author’s knowledge it is not known, whether all algebraic embeddings $\mathbb{C} \to \text{SL}_2$ are the same up to an algebraic automorphism of $\text{SL}_2$. Also for algebraic embeddings $\mathbb{C} \to \mathbb{C}^3$ it is an open problem, whether all these embeddings are the same up to an algebraic automorphism of $\mathbb{C}^3$, see [Sha92] for potential examples that are not equivalent to linear embeddings.

In fact, in a certain sense the class of algebraic embeddings $\mathbb{C} \to \text{SL}_2$ is as big as the class of algebraic embeddings $\mathbb{C} \to \mathbb{C}^3$. More precisely,

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the following holds. If \( g: \mathbb{C} \rightarrow \mathbb{C}^3, t \mapsto (g_1(t), g_2(t), g_3(t)) \) is an algebraic embedding, then one can apply a (tame) algebraic automorphism of \( \mathbb{C}^3 \) such that afterwards the polynomial \( g_2 \) divides \( g_1g_3 - 1 \) and thus the following map is an algebraic embedding

\[
\mathbb{C} \longrightarrow \text{SL}_2, \quad t \mapsto \begin{pmatrix} g_1(t) & (g_1(t)g_3(t) - 1)/g_2(t) \\ g_2(t) & g_3(t) \end{pmatrix}.
\]

The construction of the claimed (tame) algebraic automorphism of \( \mathbb{C}^3 \) can be seen as follows. First one can apply a map of the form \( (x, y, z) \mapsto (x, y + \lambda, z) \) such that afterwards the polynomial \( g_2 \) has only finitely many simple roots, say \( t_1, \ldots, t_n \). Now, it is enough to apply some (tame) algebraic automorphism of the form \( (x, y, z) \mapsto (\varphi_1(x, z), y, \varphi_3(x, z)) \), which sends the points \( g(t_1), \ldots, g(t_n) \) to the curve \( \{xz = 1, y = 0\} \subseteq \mathbb{C}^3 \), see [KZ99, Lemma 5.5].

The proof of the main theorem gives a method to construct the claimed automorphism. However, the proof does not produce a computer algorithm that would give such an automorphism. This is because the construction in the proof depends on certain zero sets of polynomials.

2. Algebraic automorphisms of \( \text{SL}_n \)

Let us introduce first some notation. For \( i, j \) in \( \{1, \ldots, n\} \), we denote the \( ij \)-th entry of a matrix \( X \in \text{SL}_n \) by \( X_{ij} \). The projection \( \text{SL}_n \rightarrow \mathbb{C}, X \mapsto X_{ij} \) we denote by \( x_{ij} \).

In the first lemma, we list algebraic automorphisms of \( \text{SL}_n \) that we use constantly. The proof is straight forward.

**Lemma 1.** Let \( n \geq 2 \) and let \( i \neq j \) be integers in \( \{1, \ldots, n\} \). Then, for every polynomial \( p \) in the functions \( x_{kl}, k \neq i \), the map

\[
\text{SL}_n \longrightarrow \text{SL}_n, \quad X \mapsto E_{ij}(p(X)) \cdot X
\]

is an automorphism, where \( E_{ij}(a) \) denotes the elementary matrix with \( ij \)-th entry equal to \( a \). Similarly, for every polynomial \( q \) in the functions \( x_{kl}, l \neq j \), the map

\[
\text{SL}_n \longrightarrow \text{SL}_n, \quad X \mapsto X \cdot E_{ij}(q(X))
\]

is an automorphism.

Recall that the group of tame automorphisms of \( \mathbb{C}^n \) is the subgroup of the automorphisms of \( \mathbb{C}^n \) generated by the affine linear maps and the elementary automorphisms, i.e. the automorphisms of the form

\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i + h_i(x_1, \ldots, \hat{x}_i, \ldots, x_n), \ldots, x_n),
\]

where \( h_i \) is a polynomial not depending on \( x_i \). In the next result we list automorphisms of \( \mathbb{C}^n \) that can be lifted to automorphisms of \( \text{SL}_n \) via the projection to the first column \( \pi_1: \text{SL}_n \rightarrow \mathbb{C}^n \), i.e. automorphisms \( \psi \) of \( \mathbb{C}^n \) such that there exists an automorphism \( \Psi \) of \( \text{SL}_n \) (depending on \( \psi \)) that
makes the following diagram commutative:

\[
\begin{array}{ccc}
\text{SL}_n & \xrightarrow{\psi} & \text{SL}_n \\
\pi_1 \downarrow & & \downarrow \pi_1 \\
\mathbb{C}^n & \xrightarrow{\psi} & \mathbb{C}^n.
\end{array}
\]

**Lemma 2.** Let \( n \geq 2 \). Every tame automorphism of \( \mathbb{C}^n \) that preserves the origin can be lifted to some automorphism of \( \text{SL}_n \) via \( \pi_1 : \text{SL}_n \to \mathbb{C}^n \).

**Proof.** First, remark that the group of tame automorphisms of \( \mathbb{C}^n \) that preserve the origin is generated by the linear group \( \text{GL}_n \) and by the elementary automorphisms that preserve the origin. For every \( A \in \text{GL}_n \), the linear map \( x \mapsto A \cdot x \) of \( \mathbb{C}^n \) can be lifted to the automorphism \( \text{SL}_n \xrightarrow{\pi_1} \text{SL}_n \), \( X \mapsto \pi_1(A \cdot X) \cdot \text{diag}(\lambda_1, \ldots, \lambda_n) \) where \( \text{diag}(\lambda_1, \ldots, \lambda_n) \) denotes the \( n \times n \)-diagonal matrix with entries \( \lambda_1, \ldots, \lambda_n \).

Let \( \psi \) be an elementary automorphism of \( \mathbb{C}^n \) that preserves the origin, i.e. there exist \( i \in \{1, \ldots, n\} \) and polynomials \( p_1, \ldots, p_i, \ldots, p_n \) in the variables \( x_1, \ldots, \hat{x}_i, \ldots, x_n \) such that \( \psi(x_1, \ldots, x_n) = (x_1, \ldots, x_i + \sum_{j \neq i} x_j p_j(x_1, \ldots, \hat{x}_i, \ldots, x_n), \ldots, x_n) \).

The automorphism \( \psi \) can be lifted to some automorphism of \( \text{SL}_n \), e.g. to the automorphism

\[
\text{SL}_n \xrightarrow{\pi_1} \text{SL}_n, \quad X \mapsto \left( \prod_{j \neq i} E_{ij}(p_j(X_{11}, \ldots, \hat{X}_{i1}, \ldots, X_{n1})) \right) \cdot X,
\]

cf. also Lemma 1. This finishes the proof. \( \square \)

### 3. A generic projection result

Let \( V \) be an algebraic variety. We say that a statement is true for **generic** \( v \in V \) if there exists a Zariski dense open subset \( U \subseteq V \) such that the statement is true for all \( v \in U \).

**Lemma 3.** Let \( n \geq 3 \). If \( f : \mathbb{C} \to \text{SL}_n \) is an algebraic embedding such that the matrices \( f(0) - f(1) \) and \( f'(0) \) have maximal rank, then, for generic \( A \in M_{n,n-1} \) the map

\[
\mathbb{C} \xrightarrow{f} \text{SL}_n \xrightarrow{\pi_A} M_{n,n-1}
\]

is an algebraic embedding, where \( M_{n,n-1} \) denotes the space of \( n \times (n-1) \)-matrices and \( \pi_A \) is given by \( X \mapsto X \cdot A \).

**Proof.** Let \( \Delta \subseteq \mathbb{C}^2 \) be the diagonal. Consider the following (Zariski) locally closed subsets of \( \mathbb{C}^2 \setminus \Delta \):

\[
C_i = \{ (t, r) \in \mathbb{C}^2 \setminus \Delta \mid \text{rank}(f(t) - f(r)) = i \}.
\]

Consider for every \( A \in M_{n,n-1} \) the composition

\[
C_i \xrightarrow{(t, r) \mapsto f(t) - f(r)} M_{n,n} \xrightarrow{\pi_A} M_{n,n-1}.
\]

This map is never zero for generic \( A \in M_{n,n-1} \); indeed:
• If $1 < i \leq n$, then $\pi$ is never zero provided that $A \in M_{n,n-1}$ has maximal rank.

• If $i = 1$, then $\dim C_1 \leq 1$, since $\dim C_n = 2$ (note that $f(0) - f(1)$ has maximal rank). For $(t,r) \in C_1$, let $Z_{(t,r)} = \ker(f(t) - f(r))$.

Since $\dim C_1 \leq 1 < n - 1$, a generic $(n-1)$-dimensional subspace of $\mathbb{C}^n$ is different from $Z_{(t,r)}$ for all $(t,r) \in C_1$. Thus, for generic $A$ the composition $\pi$ is never zero.

Clearly, $C_0 = \emptyset$. Hence, we proved that the composition $\pi_A \circ f$ is injective for generic $A \in M_{n,n-1}$. Clearly, $\pi_A \circ f$ is proper for generic $A \in M_{n,n-1}$.

For the immersivity, we have to show for generic $A \in M_{n,n-1}$ that

$$f'(t) \cdot A \neq 0$$

for all $t \in \mathbb{C}$. Since $\operatorname{rank} f'(0) = n$, the set $U = \{ t \in \mathbb{C} \mid \operatorname{rank} f'(t) = n \}$ is Zariski dense and open in $\mathbb{C}$. Thus $\pi$ is satisfied for all $A \neq 0$ and for all $t \in U$. Since $f$ is immersive, we have $f'(t) \neq 0$ for all $t \in \mathbb{C}$. This implies that for generic $A$ we have $f'(t) \cdot A \neq 0$ for all $t \in \mathbb{C}$. □

4. Algebraic embeddings of $\mathbb{C}$ into $\text{SL}_n$ for $n \geq 3$

Theorem 4. For $n \geq 3$, any two algebraic embeddings of $\mathbb{C}$ into $\text{SL}_n$ are the same up to an algebraic automorphism of $\text{SL}_n$.

Lemma 5. Let $n \geq 2$. Assume that $f : \mathbb{C} \to \text{SL}_n$ is an algebraic embedding such that

$$\mathbb{C} \xrightarrow{f} \text{SL}_n \xrightarrow{\pi_{n-1}} M_{n,n-1}$$

is an algebraic embedding, where $\pi_{n-1}$ denotes the projection to the first $n-1$ columns. Then there exists an algebraic automorphism $\varphi$ of $\text{SL}_n$ such that

$$\mathbb{C} \xrightarrow{f} \text{SL}_n \xrightarrow{\varphi} \text{SL}_n \xrightarrow{\pi_1} \mathbb{C}^n$$

is given by $t \mapsto (1,0,\ldots,0,t)^T$.

Proof of Lemma 4. Assume that $n = 2$. Since two algebraic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ are the same up to an algebraic automorphism of $\mathbb{C}^2$ (Abhyankar-Moh-Suzuki Theorem, see [AM75, Suz74]), one can see that there exists an algebraic automorphism of $\mathbb{C}^2$ that preserves the origin and changes the embedding $\pi_1 \circ f : \mathbb{C} \to \mathbb{C}^2$ to the embedding $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (1,t)$. Using the fact that every algebraic automorphism of $\mathbb{C}^2$ is tame (Jung’s Theorem, see [Jung42]), it follows from Lemma 2 that there exists an algebraic automorphism $\varphi$ of $\text{SL}_2$ such that $\pi_1 \circ \varphi \circ f(t) = (1,t)$.

Assume that $n \geq 3$. Let $A(t) = \pi_{n-1} \circ f(t)$. Since the kernel of $A(t)^T$ is one-dimensional for all $t$, the following affine variety

$$E = \{ (v,t) \in \mathbb{C}^n \times \mathbb{C} \mid A(t)^T \cdot v = 0 \}$$

defines the total space of a line bundle over $\mathbb{C}$ with projection map $(v,t) \mapsto t$.

Since $n \geq 3 > \dim E$, this implies that there exists a vector $v \in \mathbb{C}^n$ such that $v^T \cdot A(t)$ is non-zero for all $t \in \mathbb{C}$. Now, complete $v^T$ to a matrix $B \in \text{SL}_n$ with last row equal to $v^T$. Since $n \geq 3$, there exists a permutation matrix $P \in \text{SL}_n$, with first column equal to $(0,\ldots,0,1)^T$. After applying the automorphism $X \mapsto B \cdot X \cdot P$ of $\text{SL}_n$, we can assume that
i) the map \( \mathbb{C} \to M_{n,n-1} \) given by \( t \mapsto (f_{ij}(t))_{1 \leq i \leq n, 2 \leq j \leq n} \) is an algebraic embedding and

ii) the vector \( (f_{n2}(t), f_{n3}(t), \ldots, f_{nn}(t)) \) is non-zero for all \( t \in \mathbb{C} \), where \( f_{ij}(t) \) denotes the \( ij \)-th entry of the matrix \( f(t) \). By ii), there exist polynomials \( \tilde{p}_k \in \mathbb{C}[t] \), \( 2 \leq k \leq n \) such that

\[
\sum_{k=2}^{n} f_{nk}(t) \tilde{p}_k(t) = t - f_{n1}(t).
\]

By i), there exist polynomials \( p_k \) in the functions \( x_{ij} \) with \( 1 \leq i \leq n \), \( 2 \leq j \leq n \) such that \( \tilde{p}_k(t) = p_k(\ldots, f_j(t), \ldots) \). Let \( \varphi : SL_n \to SL_n \) be the automorphism

\[
X \mapsto X \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p_2(X) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n(X) & 0 & \cdots & 1 \end{pmatrix}.
\]

Clearly, the left down corner of the matrix \( \varphi \circ f(t) \) is equal to \( t \). Now, one can construct with the aid of Lemma \( 2 \) an automorphism \( \psi \) of \( SL_n \) such that the first column of \( \psi \circ \varphi \circ f(t) \) is equal to \((1,0,\ldots,0,t)^T\). This proves the lemma.

**Lemma 6.** Let \( n \geq 2 \) and let \( f : \mathbb{C} \to SL_n \) be an algebraic embedding such that the first column of \( f(t) \) is equal to \((1,0,\ldots,0,t)^T\). Then \( f \) is the same as

\[
\mathbb{C} \to SL_n, \quad t \mapsto E_{n1}(t)
\]

up to an algebraic automorphism of \( SL_n \), where \( E_{n1}(t) \) denotes the elementary matrix with left down corner equal to \( t \).

**Proof of Lemma 6.** Let \( \psi \) be the automorphism of \( SL_n \) defined by

\[
X \mapsto X \cdot f(X_{n1})^{-1} \cdot E_{n1}(X_{n1})
\]

where \( X_{ij} \) denotes the \( ij \)-th entry of the matrix \( X \). Now, one can easily check that \( \psi \circ f \) is the embedding \( t \mapsto E_{n1}(t) \).

**Proof of Theorem 4.** Start with an algebraic embedding \( f : \mathbb{C} \to SL_n \). As \( SL_n \) is flexible, for any finite set \( Z \) in \( SL_n \) there exists an automorphism of \( SL_n \) which fixes \( Z \) and has prescribed volume preserving differentials in the points of \( Z \), see \[AFK-13\] Theorem 4.14 and Remark 4.16]. Using the fact that \( \text{Aut}(SL_n) \) acts 2-transitively on \( SL_n \), see e.g. \[AFK-13\] Theorem 0.1], we can assume that

\[
\det(f(0) - f(1)) \neq 0 \quad \text{and} \quad \det f'(0) \neq 0.
\]

Since \( n \geq 3 \), by Lemma 3 there exists a matrix \( A \in M_{n,n-1} \) of maximal rank, such that \( t \mapsto f(t) \cdot A \) defines an algebraic embedding of \( \mathbb{C} \) into \( M_{n,n-1} \). Extend \( A \) with an additional column \( v \in \mathbb{C}^n \) to a \( n \times n \)-matrix \( (A|v) \) of determinant one. After applying the algebraic automorphism \( X \mapsto X \cdot (A|v) \) of \( SL_n \), we can assume that the composition

\[
\mathbb{C} \xrightarrow{f} SL_n \xrightarrow{(A|v)} M_{n,n-1}
\]

is an algebraic embedding. After an algebraic coordinate change of \( SL_n \), we can assume that the first column of \( f(t) \) is equal to \((1,0,\ldots,0,t)^T\) by
Lemma 5. Thus, up to an algebraic automorphism of $\text{SL}_n$, $f$ is the same as $t \mapsto E_{n1}(t)$ by Lemma 6. This finishes the proof. □

5. ALGEBRAIC EMBEDDINGS OF $\mathbb{C}$ INTO $\text{SL}_2$

Theorem 7. Any two algebraic embeddings $\mathbb{C} \to \text{SL}_2$ are the same up to a holomorphic automorphism of $\text{SL}_2$.

Remark 8. Since for all $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ the embeddings

\[ t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t \mapsto \begin{pmatrix} 1 & at + b \\ 0 & 1 \end{pmatrix} \]

are the same up to an algebraic automorphism of $\text{SL}_2$, it is enough to prove Theorem 7 up to an algebraic reparametrization of the embeddings $\mathbb{C} \to \text{SL}_2$.

For the proof of Theorem 7 we need the following rather technical result, which enables us to bring an arbitrary algebraic embedding $\mathbb{C} \to \text{SL}_2$ in a “nice” position.

Proposition 9. Let $f : \mathbb{C} \to \text{SL}_2$ be an algebraic embedding. Then there exists a holomorphic automorphism $\varphi$ of $\text{SL}_2$ and a constant $a \in \mathbb{C}$ such that the embedding

\[ \mathbb{C} \longrightarrow \text{SL}_2, \quad t \mapsto \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} := (\varphi \circ f)(t + a) \]

satisfies:

1. for all $t \in g_{11}^{-1}(0)$ we have $g_{12}(t) = t$;
2. the map $t \mapsto (g_{11}(t), g_{21}(t))$ is a proper, bimeromorphic immersion such that the image $\Gamma$ has only simple normal crossing singularities;
3. the singularities of $\Gamma$ are distinguished by the first coordinate of $\mathbb{C}^2$;
4. the line $\{0\} \times \mathbb{C}$ intersects $\Gamma$ transversally; in particular, $\Gamma$ is smooth in every point of $\Gamma \cap \{0\} \times \mathbb{C}$;
5. the map $t \mapsto g_{11}(t)$ is polynomial.

The proof of this proposition uses the following easy result which is a direct application of the Baire category theorem:

Lemma 10. Let $\mathcal{H}(\mathbb{C}^n)$ be the Fréchet space of holomorphic functions on $\mathbb{C}^n$ with the compact-open topology. If $S$ is the countable union of closed proper subspaces of $\mathcal{H}(\mathbb{C}^n)$, then $\mathcal{H}(\mathbb{C}^n) \setminus S$ is dense in $\mathcal{H}(\mathbb{C}^n)$.

Let $p \in \mathbb{C}^n$ and let $i \in \{1, \ldots, n\}$. In our proof of Proposition 9 we use the fact that the linear functionals on $\mathcal{H}(\mathbb{C}^n)$

\[ h \mapsto h(p) \quad \text{and} \quad h \mapsto D_{x_i}h(p) \]

are continuous and thus their kernels are proper closed subspaces of $\mathcal{H}(\mathbb{C}^n)$.

Additionally, we use for the proof of Proposition 9 the following, again rather technical result:

Lemma 11. Let $f : \mathbb{C} \to \text{SL}_2$ be an algebraic embedding. Then there exists an algebraic automorphism $\varphi$ of $\text{SL}_2$ such that the embedding

\[ \mathbb{C} \longrightarrow \text{SL}_2, \quad t \mapsto (\varphi \circ f)(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \]

satisfies:
a) the maps $t \mapsto x(t)$ and $t \mapsto w(t)$ are non-constant polynomials;
b) the maps $t \mapsto (x(t), z(t))$ and $t \mapsto (x(t), w(t))$ are bimeromorphic and immersive;
c) the singularities of the image of $t \mapsto (x(t), z(t))$ lie inside $(\mathbb{C}^*)^2$;
d) the image of $t \mapsto (x(t), z(t))$ intersects $\{0\} \times \mathbb{C}$ transversally.

Proof of Lemma 11. Clearly, we can assume that $f(0) = E_2$ is the identity matrix $E_2 \in \text{SL}_2$. By [AFK'13, Theorem 4.14 and Remark 4.16], there exists an algebraic automorphism of $\text{SL}_2$ which fixes $E_2$ and maps the tangent vector $f'(0) \in T_{E_2} \text{SL}_2 = \text{Lie SL}_2$ to the matrix

$$F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{Lie SL}_2.$$ 

Thus we can assume that $f(0) = E_2$ and $f'(0) = F_2$. In particular, property a) is satisfied. Since $f'(t)$ is never zero and since $f'(t)$ is invertible for generic $t$ (note that $f'(0) = F_2$ is invertible) it follows that $f'(t) \cdot v$ is non-zero for generic $v \in \mathbb{C}^2 \setminus \{(0, 0)\}$. For generic $\mu \in \mathbb{C}$, this implies that the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

satisfies still property a) and the projection to the first column gives an immersive map. Let us fix such a $\mu$. For generic $\lambda \in \mathbb{C}$ the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

still satisfies property a) and the projection to the first column and the projection to the diagonal give immersive maps. Since any immersive morphism of $\mathbb{C}$ to an irreducible affine curve is birational, we can assume that $f$ satisfies properties a) and b). Now, for generic $a \in \mathbb{C}$ the embedding

$$t \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t)$$

satisfies properties a) and b) and the singularities of the image of the projection to the first column lie inside $\mathbb{C} \times \mathbb{C}^*$. Let us fix such an $a$. For generic $b \in \mathbb{C}$ the embedding

$$t \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t)$$

satisfies now the properties a) to c). Let $(p(t), q(t))^T$ be the first column of the embedding $\text{(***)}$. Then the top left entry of the embedding $\text{(***)}$ is given by $p(t) + bq(t)$. Now, if $\text{(***)}$ satisfies properties a) to c), then $\text{(***)}$ satisfies property d) if and only if $p(t) + bq(t)$ has only simple roots. However, this last condition is satisfied for generic $b$, since $p(t) + bq(t)$ has only simple roots if and only if for all $t$ the vector $(1, b)^T$ lies not in the kernel of the matrix

$$\begin{pmatrix} p(t) & q(t) \\ p'(t) & q'(t) \end{pmatrix}$$

and since this last matrix is invertible for generic $t$ and never vanishes. This finishes the proof. \[\square\]
Proof of Proposition 8. Using Lemma 11 we can assume that \( f \) satisfies the properties a) to d) of Lemma 11. As a consequence of b) and c) we get that the map \( t \mapsto (x(t), z(t), w(t)) \) is a proper holomorphic embedding.

Let \( t_1, \ldots, t_n \), be the roots of \( x(t) = 0 \) (which are simple according to property d)). After a reparametrization of \( f \) of the form \( t \mapsto t + a \) one can assume that \( w(t_i) \neq w(t_j) \) for all \( i \neq j \) and \( t_i \neq 0 \) for all \( i \). Let \( a_i \in \mathbb{C} \) such that \( e^{-a_i} = -t_i z(t_i) \) and let \( b: \mathbb{C} \to \mathbb{C} \) be a polynomial map such that \( b(w(t_i)) = a_i \) and \( b'(w(t)) = 0 \) for all \( t \) with \( x'(t) = 0 \). After applying the holomorphic automorphism

\[
\text{SL}_2 \to \text{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & e^{-b(w)}y \\ z e^{b(w)} & w \end{pmatrix}
\]

we can assume that the embedding \( f \) satisfies \( y(t_i) = t_i \) for all \( i \) and \( f \) still satisfies the properties a) to d).

Let \( \rho \) be the embedding \( t \mapsto (x(t), y(t), z(t)) \). Fix \( x_0 \neq 0 \) such that

i) \( z(s) \neq 0 \) and \( x'(s) \neq 0 \) for all \( s \in x^{-1}(x_0) \) and

ii) the maps \( t \mapsto z(t) \) and \( t \mapsto w(t) \) are injective on \( x^{-1}(x_0) \).

Let \( \{s_1, \ldots, s_n\} = x^{-1}(x_0) \). With the aid of Lemma 10 one can see that there exists a holomorphic function \( c: \mathbb{C}^2 \to \mathbb{C} \) that satisfies the following:

i) for all \( (x, z, w) \neq (x, z, w') \in \rho(\mathbb{C}) \) we have \( c(x, w) \neq c(x, w') \);

ii) for all \( t \) with \( x'(t) = 0 \), the partial derivative \( D_w c \) vanishes in \( (x(t), w(t)) \); and

iii) for all \( i = 1, \ldots, n \) we have \( c(0, w(t_i)) = 0 \).

iv) for all integers \( k, q \) and for all 2-element sets \( \{i, j\} \neq \{l, p\} \) we have

\[
[\log z(s_i) - \log z(s_p) + 2 \pi i q] \cdot [c(x_0, w(s_j)) - c(x_0, w(s_i))] = \left[ c(x_0, w(s)) - c(x_0, w(s)) \right];
\]

v) for all integers \( k \) and for all \( i \neq j \) we have

\[
[\log z(s_i) - \log z(s_j) + 2 \pi i k] \cdot [x'(s_i)c(x, w)'(s_j) - x'(s_j)c(x, w)'(s_i)] = \left[ c(x_0, w(s_j)) - c(x_0, w(s_j)) \right].
\]

Let \( V \subseteq \mathbb{C}^* \) be the largest subset such that for all \( x_0 \in V \) the properties I) and II) are satisfied. By property a), the complement \( \mathbb{C} \setminus V \) is a closed discrete (countable) subset of \( \mathbb{C} \). The inequalities in iv) and v) are locally holomorphic in \( x_0 \in V \) after a local choice of sections \( s_1, \ldots, s_n \) of the covering \( x^{-1}(V) \to V \) and a local choice of the branches of the logarithms.

Since \( V \) is path-connected, one can now deduce that there exists a subset \( U \subseteq V \) such that \( \mathbb{C} \setminus U \) is countable and for all \( x_0 \in U \) the properties iv) and v) are satisfied.

According to i) and c) there exists \( \lambda \in \mathbb{C}^* \) such that for all \( x_1 \in \mathbb{C} \setminus U \) we have the following: If \( (x_1, z, w) \neq (x_1, z', w') \in \rho(\mathbb{C}) \), then \( e^{\lambda c(x_1, w)z} \neq e^{\lambda c(x_1', w')z} \). Now, let \( \varphi \) be the following holomorphic automorphism

\[
\text{SL}_2 \to \text{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & e^{-\lambda c(x, w)}y \\ e^{\lambda c(x, w)}z & w \end{pmatrix}
\]

and let \( g = \varphi \circ f \). According to iii), \( g \) satisfies property (1) of the proposition. Property ii) implies that \( t \mapsto (g_1(t), g_2(t)) \) is immersive. Clearly, \( t \mapsto (g_{11}(t), g_{21}(t)) \) is proper and \( g \) satisfies property (5) of the proposition. By iii), it follows that \( g \) satisfies property (4) of the proposition and thus
$t \mapsto (g_{11}(t), g_{21}(t))$ is bimeromorphic. By the choice of $\lambda$, it follows for $x_1 \notin U$ that $g_{21}(t) \neq g_{21}(t')$ for all $t \neq t' \in x^{-1}(x_1)$. Since all $x_0 \in U$ satisfy iv) and v) the image of $t \mapsto (g_{11}(t), g_{21}(t))$ has only simple normal crossings, which have distinct first coordinates in $\mathbb{C}^2$. This implies properties (2) and (3) of the proposition. □

Proof of Theorem 7 Let $f: \mathbb{C} \to \text{SL}_2$ be an algebraic embedding. We will prove that up to a holomorphic automorphism of $\text{SL}_2$ and up to an algebraic reparametrization, $f$ is the same as the standard embedding $t \mapsto E_{12}(t)$.

After applying a holomorphic automorphism of $\text{SL}_2$ and performing an algebraic reparametrization we can assume that $f$ satisfies properties (1) to (5) of Proposition 9. We denote

$$f(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}.$$  

As usual, $\pi_1: \text{SL}_2 \to \mathbb{C}^2$ denotes the projection onto the first column. Let $S$ be the (countable) closed discrete set of points $s \in \mathbb{C}^2 \setminus \{0\}$ such that $(\pi_1 \circ f)^{-1}(s) = \{s_1, s_2\}$ with $s_1 \neq s_2$, see property (2). For every $s$ in $S$, it holds that $y(s_1) \neq y(s_2)$, since $f$ is an embedding and since all simple normal crossings of the image of $\pi_1 \circ f$ lie inside $\mathbb{C}^* \times \mathbb{C}$ due to property (4). Thus, we can choose $a_s \in \mathbb{C}$ such that

$$s_1 - e^{a_s} y(s_1) = s_2 - e^{a_s} y(s_2).$$

Let $\psi_1: \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $\psi_1(0) = 0$ such that for all $s \in S$ we have $\psi_1(x(s_1)) = a_s$. This function exists, since $x(s_1) = x(s_2) \neq 0$ for all $s \in S$ (by property (4)), since $x((\pi_1 \circ f)^{-1}(S))$ is a closed analytic subset of $\mathbb{C}$ (by property (5)) and since $x(s_1) \neq x(s'_1)$ for distinct $s, s'$ of $S$ (by property (3)). Let $\alpha_1$ be the holomorphic automorphism of $\text{SL}_2$ defined by

$$\alpha_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & e^{\psi_1(x)} y \\ e^{-\psi_1(x)} z & w \end{pmatrix}.$$  

By composing $f$ with $\alpha_1$, we can assume that $s_1 - y(s_1) = s_2 - y(s_2)$ for all $s \in S$. The embedding $f$ still satisfies the properties (1) to (5).

Let $\Gamma \subset \mathbb{C}^2$ be the image of $\pi_1 \circ f: \mathbb{C} \to \mathbb{C}^2$. By Remmert’s proper mapping theorem [Rem57, Satz 23], $\Gamma$ is a closed analytic subvariety of $\mathbb{C}^2$. Now, using that $\pi_1 \circ f$ is immersive and $\Gamma$ has only simple normal crossings, we get a holomorphic factorization

$$\mathbb{C} \xrightarrow{\pi_1 \circ f} \Gamma \xrightarrow{e} \mathbb{C}.$$  

Using properties (1) and (4), it follows that the map

$$\tilde{e}: \Gamma \to \mathbb{C}, \quad (x, z) \mapsto e(x, z)$$

is holomorphic. Using Cartan’s Theorem B [Car53 Théorème B], we can extend $\tilde{e}$ to a holomorphic map $\psi_2: \mathbb{C}^2 \to \mathbb{C}$. Let $\alpha_2$ be the holomorphic
automorphism of $\text{SL}_2$ defined by

$$\alpha_2 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x + \psi_2(x, z) & y + \psi_2(x, z) \\ z + \psi_2(y, z) & w + \psi_2(y, z) \end{pmatrix}.$$  

After applying the automorphism $\alpha_2$ to $f$ we can assume that $y(t) = t$. This implies that $x(0)t = x(0) - x(0)$. Let $p, q$ be the holomorphic functions such that $p(t) = x(t) - x(0)$ and $q(t) = w(t) - w(0)$. After applying the holomorphic automorphism

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} w(0) & 0 \\ 0 & x(0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q(y) & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p(y) & 1 \end{pmatrix}$$

we can additionally assume that $w(t) = x(t) = 1$, which implies $z(t) = 0$. The statement follows now from Remark 8. □

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