Towards a Singularity-Free Inflationary Universe?

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Abstract
We consider the problem of constructing a non-singular inflationary universe in stringy gravity via branch changing, from a previously superexponentially expanding phase to an FRW-like phase. Our approach is based on the phase space analysis of the dynamics, and we obtain a no-go theorem which rules out the efficient scenario of branch changing catalyzed by dilaton potential and stringy fluid sources. We furthermore consider the effects of string-loop corrections to the gravitational action in the form recently suggested by Damour and Polyakov. These corrections also fail to produce the desired branch change. However, focusing on the possibility that these corrections may decouple the dilaton, we deduce that they may lead to an inflationary expansion in the presence of a cosmological constant, which asymptotically approaches Einstein-deSitter solution.

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1 Introduction

The standard cosmological model, based on Einstein’s General Relativity and the Cosmological Principle, represents a coherent, elegant and consistent picture of much of our current understanding and observations of the Universe. In fact, it is in excellent agreement with the data going as far back as the era of primordial nucleosynthesis. However, there remain well known gaps in this account, believed to originate mainly from the times prior to nucleosynthesis, and manifest in such problems as the size of the Universe, its smoothness and flatness on very large scales and the lack of it at smaller scales, the absence of topological relics, such as domain walls, cosmic strings and magnetic monopoles, the whereabouts of the missing matter, the absence (or near absence in Planck units) of a cosmological constant, and perhaps the greatest mystery of all, the initial singularity itself. The majority of the former problems can be elegantly dealt with, at least in principle, by the inflationary paradigm \[1\], postulating an era of accelerated expansion before the era of nucleosynthesis. Nevertheless, there yet remains the task to construct a concrete, plausible dynamical scenario which predicts such evolution, in agreement with observations and free of internal inconsistencies. To render the matters worse, even if a viable scenario is eventually brought to the light of the day, it will still fail to address the problem of singularity without resorting to major modifications of the theory of gravity, as demonstrated by the Hawking-Penrose singularity theorems \[2\].

As a consequence, there has been much labor in recent years in the attempt to find alternative theories of gravity, reflecting the silent consensus that certain alterations of Einstein’s theory are imminent. In most cases, the route taken was minimalistic in philosophy, consisting of inventing and incorporating changes to account for some of the phenomenologically dictated mechanisms. This approach has been only partially satisfactory, failing to produce a model capable of dealing comprehensively with the shortcomings of Einstein’s theory, which proved deserving of respect even in its demise.

On the other hand, the ongoing quest for the unified theory of interactions in Nature has finally produced a promising contender, which has withstood theoretical scrutiny up to date. The advent of string theory \[3\], based on a fundamentally different assumption about the nature of matter, has been supported mainly by the description of gravity in a manner equivalent to other forces, and perhaps even more importantly, by the absence of some of the
obstacles encountered in the failed attempts to quantize General Relativity. Thrusting from
the initial success, many studies have sprung up investigating gravitational aspects of string
theory, and in particular the early universe cosmology \[4\] - \[20\]. The justification for this
interest can be naturally found in the fact that string theory, while claiming to unify gravity
with the other forces of nature, must give us the means to investigate the regions where the
standard model has failed to give satisfactory answers. The aforementioned cosmological
problems, especially the problem of the initial singularity, fall precisely in this category. In
addition, results of these investigations should provide us with ways to test the compatibility
of string theory with the Nature.

While the early investigations of string cosmology have indicated the presence of some
of the coveted mechanisms to tackle the encountered problems, they have also burdened us
with a host of new difficulties. In a typical cosmological setting, most of the advantages and
the difficulties can be attributed to the presence of a new field, the scalar dilaton, which
comes in response to the requirement of conformal invariance of string world-sheet theories.
The dilaton has been recognized as a natural candidate for the inflaton \[4, 5\], a weakly
coupled scalar which is a necessary ingredient of many a generic inflationary scenario. How-
ever, in addition to preordaining its existence, conformal invariance also dictates the form of
the dilaton’s couplings to gravity and other matter fields. Specifically, all particle coupling
constants and masses (believed to come about via some symmetry-breaking mechanism) ac-
quire dependence on the dilaton expectation value. This yields to obstacles in implementing
most of the conventional inflationary scenarios in string theory \[5\], compelling us to resort
to more obscure and often contrived arguments, thus diminishing the overall appeal of the
theory. These obstacles are best summarized by saying that the dilaton tends to roll a bit
too eagerly, influencing other fields in the model in such ways, for example, as to preclude
some well-known solutions such as de-Sitter \[6, 10\], affect nucleosynthesis rates by inducing
time-dependence of particle masses and coupling constants \[18, 22\], and give rise to scalar
components of gravity, similar to the fifth force \[23\]. It is therefore important to see if there
are ways to keep the dilaton in check, and prevent it from meddling in the dynamics at late
times. Usually this is done by endowing the dilaton with a large mass, coming from a dilaton
self-interaction with a highly curved potential well.

Another approach has been proposed recently by Damour and Polyakov \[18\]. Their model
represents a further generalization of the effective string gravity action, motivated mainly by
the previously described need to decouple the dilaton. The model is based on the observation that scalar-tensor tensor theories with the scalar couplings to other fields given by functions with a minimum at a non-zero value naturally evolve towards Einstein’s theory, and the inference that such couplings can arise in string theory when the higher genus corrections are accounted for. The resulting effect is that the dilaton can be stopped even in the absence of conventional mass term-generating self-interactions. Obviously, the model could become even more appealing if it contains additional features compatible with our expectations from cosmology.

A very different approach towards string-driven inflation, dubbed the pre-Big-Bang inflation, has been suggested [15, 16]. It strives to induce inflation deriving from genuinely stringy mechanisms, relying on the dilaton, scale factor duality symmetry of string theory (not present in any other model of gravity), and somewhat less well understood possibility of existence of string-induced topology-changing solutions [24]. In its simplest form, this proposal rests on the extension of a simple power-law expanding, scalar field-dominated universe, to negative times by string scale factor duality and time inversion. The scenario is defined in the string world-sheet frame, where the two branches are characterized by super-exponential inflation for $t < 0$ and a milder, power-law expansion for $t > 0$. If the jump (branch change) at $t = 0$ can be made smoothly from the superinflationary phase to the power-law one, the dreaded singularity may be avoided [20].

In this paper we will examine the possibilities of avoiding the cosmological singularity à la Brustein and Veneziano [20]. We will first revisit the case of dilaton self-interaction and stringy fluid sources, originally investigated by these authors, and improve their conjecture that these terms cannot provide for a successful graceful exit, by promoting it to an exact no-go theorem. We will then address the possibility for branch changing using the Damour-Polyakov model, as well as investigate further inflationary capabilities of this model. We shall, however, generalize the action given by Damour and Polyakov one step further, to allow for a non-trivial dilaton potential associated either with supersymmetry breaking or a stringy cosmological term coming from the target-space central charge deficit. Such extensions have also been considered recently in [25]. We will again arrive at the concrete proof of the no-go theorem for a branch change induced by the higher genus terms in the gravitational action. We will also note that our extension of the Damour-Polyakov model brings de-Sitter-like solutions back into the game, as asymptotic attractors reached when the dilaton decouples,
as was also discussed in [25].

# 2 The Gravitational Action and The Higher-Genus Contributions

There is a considerable amount of literature concerning the tree level gravitational action in string theory and its expansion in the string tension ($\alpha'$) [26]. To $O(\alpha')$ the effective Lagrangian describing the dynamics of the modes of interest to us may be written as

$$ S = \int d^4x \sqrt{g} \left\{ \frac{R}{2} - \partial_{\mu}\phi \partial^{\mu}\phi + \frac{\alpha'}{16} e^{-2\phi} \left( \hat{R}^2 - F_{\mu\nu}F^{\mu\nu} \right) - e^{2\phi} \Lambda(\phi) \right\} $$  (1)

in the Einstein conformal frame. In (1), $\phi$ is the dilaton and $\Lambda(\phi)$ is the dilaton potential (we shall be most interested in the case where $\Lambda(\phi)$ is a constant, but we retain the $\phi$ dependence to discuss the results of [20]). We are using units such that $8\pi G_N = 1$. We have here ignored the contribution of the axion since it enters the equations of motion in the form $c/a(t)^6$ (where $a(t)$ is the FRW scale factor and $c$ a constant of integration), so during expansion an axion dominated universe can be expected to quickly evolve to one dominated by the dilaton [8, 12, 19]. Indeed, exact results [21] show that the axion is significant only in the past of any of the cosmological branches, and that it can neither facilitate inflation nor provide for the solution of the graceful exit problem. Moreover, in what follows we will ignore the corrections of the derivative expansion of order $O(\alpha')$ and higher. Though these terms would be expected to be important near the singularity, we neglect them now solely for reasons of simplicity.

We will, however, be interested in higher genus corrections. For this reason, we re-express our action, truncated to contain only the metric and dilaton-dependent terms up to two derivatives, in the string world-sheet frame, using the conformal transformation $g_{\mu\nu} \to \exp(-2\phi)g_{\mu\nu}$. The action then becomes

$$ S = \int d^4x \sqrt{g} e^{-2\phi} \left\{ \frac{R}{2} + 2\partial_{\mu}\phi \partial^{\mu}\phi - \Lambda(\phi) \right\} $$  (2)

Then, to parametrize the contributions from the higher genus terms to the action, we make the Damour-Polyakov ansatz [18]

$$ S = \int d^4x \sqrt{g} \left( B_g(\phi)R/2 + 2 B_\phi(\phi) \left( \Box \phi - (\nabla \phi)^2 \right) - B_\Lambda(\phi)\Lambda(\phi) \right) $$  (3)
The couplings $B_i(\phi)$ are the functions accounting for the loop corrections and should admit a weak coupling expansion of the form:

$$B_i(\phi) = e^{-2\phi} + c_0^{(i)} + c_1^{(i)} e^{2\phi} + c_2^{(i)} e^{4\phi} + \ldots$$  \hspace{1cm} (4)

realized in the limit where $\phi \to -\infty$.

These corrections were examined in [18] where the authors point out that at the tree level the dilaton couples universally to all terms in the action ($c_j^{(i)} = 0$ for all $i$ and $j$). Extending this presumption of universality to the loop expansion (taking $B_i(\phi) = B(\phi)$ for all $i$) and conformally rescaling to the Einstein frame results in mass dependencies of the fields of the form $m_A(\phi) = m_A(B(\phi))$. Since the equations of motion of the dilaton allow it to be attracted to extrema of the mass functions as the universe passes through mass thresholds, and all mass functions will generically have extrema coincident with those of $B(\phi)$, the authors conclude that the dilaton can be naturally decoupled in one such extremum of $B(\phi)$.

In light of this, we adopt the following action

$$S = \int d^4x \sqrt{g} B(\phi) \left( R/2 + 2 \left( \Box \phi - (\nabla \phi)^2 \right) - \Lambda(\phi) \right)$$ \hspace{1cm} (5)

The decoupling of the dilaton is necessary for a realistic cosmology. A rolling dilaton will result in the time variation in particle masses and gauge coupling constants. Their variations can be sharply constrained by the requirement that they must not disturb the delicate agreement between observed light element abundances and their calculated values from the era of primordial nucleosynthesis [22]. It is known that the dilaton does in fact decouple in a radiation dominated universe [12, 13]. On the other hand, in a matter dominated universe, at the tree level in the gravitational action without a dilaton potential, the dilaton rolls [12]. As mentioned above, in [18], it was shown that higher genus corrections of the form given in eq. (3) can decouple the dilaton without a dilaton self-interaction potential. It is also known that the dilaton must decouple in order for the universe to achieve a deSitter like expansion [2, 14], i.e. simply the presence of a cosmological constant would not bring about exponential expansion as (at the tree level) in the Einstein frame the cosmological term would carry a factor of $e^{2\phi}$. While a potential for the dilaton which is expected to be generated by supersymmetry breaking can trap the dilaton leading to exponential expansion [10, 12], below we will consider the possibility that the higher genus terms in the action play a similar role, thus allowing for inflation without a dilaton potential.
We choose to extract the equations of motion directly in the string frame, since here we will be able to express them in terms of a first-order system of non-linear differential equations and apply dynamical methods to analyze the evolution. We also specialize to the case of a Friedmann-Robertson-Walker cosmology and take a metric of the form:

\[ ds^2 = -n(t)^2 dt^2 + a(t)^2 d\tilde{x}^2 \]  

(6)

where \( d\tilde{x}^2 \) is the three dimensional volume element for a space of constant curvature \( k \). We will concentrate on the case of \( k = 0 \). Expressing the contents of the action in terms of the functions \( a(t) \) and \( n(t) \) we can factor out the spatial integration and perform the variation with respect to the functions \( a(t), n(t) \) and \( \phi(t) \), and finally set \( n(t) = 1 \). The equations will contain \( B'(\phi) \) and \( B''(\phi) \). In view of the expansion (4) it will be convenient to define \( \beta(\phi) = B'(\phi)/B(\phi) + 2 \) and therefore \( B''(\phi)/B(\phi) = \beta'(\phi) + (\beta(\phi) - 2)^2 \) so that the weakly coupled regime \( \phi \to -\infty \) corresponds to \( \beta \to 0 \). The equations of motion for the tree level in the string loop expansion (\( c_0, c_1, c_2 \ldots = 0 \)) thus correspond to \( \beta = 0 \).

The resulting equations of motion are:

\[ 0 = 3 h^2 - \Lambda(\phi) + 2 \dot{h} + 2 (\beta(\phi) - 2) h \dot{\phi} + (2 - 2 \beta(\phi) + \beta(\phi)^2 + \beta'(\phi)) \ddot{\phi}^2 + (\beta(\phi) - 2) \ddot{\phi} \]  \hspace{1cm} (7)

\[ 0 = 6 (\beta(\phi) - 2) h^2 + (2 - \beta(\phi)) \Lambda(\phi) + 3 (\beta(\phi) - 2) \dot{h} - \Lambda'(\phi) + 12 (1 - \beta(\phi)) h \dot{\phi} + 2 (-2 + 3 \beta(\phi) - \beta(\phi)^2 - \beta'(\phi)) \ddot{\phi}^2 + 4 (1 - \beta(\phi)) \ddot{\phi} \]  \hspace{1cm} (8)

\[ 0 = 3 h^2 - \Lambda(\phi) + 3 (\beta(\phi) - 2) h \dot{\phi} + 2 (1 - \beta(\phi)) \dot{\phi}^2 \]  \hspace{1cm} (9)

(4) results from the variation of the scale factor \( a(t) \), (8) from the variation of dilaton \( \phi(t) \) and (9) from the variation of the lapse factor \( n(t) \).

3 Branch Changing and the Graceful Exit

Before we begin our analysis, we should review the main ingredients of solution to the graceful exit problem proposed by Brustein and Veneziano [20]. This possibility was originally described in the genus-zero \( O(\alpha'^0) \) approximation, with some dilaton potential \( \Lambda(\phi) \), where
the dynamics is defined by the action (3). Generalization to the case when stringy fluid is present is straightforward, and we will reflect on it later. To recover the corresponding equations of motion we set $\beta(\phi) = 0$ in (7-9). It was noted [20] that the resulting equations of motion can be solved explicitly for $\dot{\phi}$ and $\dot{h}$. The canonical first order system takes the relatively simple form:

$$\dot{\phi} = \frac{3h \pm \sqrt{3h^2 + 2\Lambda(\phi)}}{2}$$  \hspace{1cm} (10)

$$\dot{h} = \pm h \sqrt{3h^2 + 2\Lambda(\phi) - \Lambda'(\phi)/2}$$ \hspace{1cm} (11)

with the $\pm$ sign chosen for both equations simultaneously. These equations are easily solved in the case $\Lambda = 0$ resulting in four different solutions, two for each of the two branches corresponding to the choice of $(+)$ and $(-)$ sign. The $(+)$ branch is defined in the domain $t < 0$ and the $(-)$ branch in $t > 0$. The solutions are [20]

$$a = a_0 |t|^\frac{1}{\sqrt{3}}, \quad h = \mp \frac{1}{\sqrt{3}t}, \quad \phi = \phi_0 + \mp \sqrt{3} - 1 \ln |t| \quad \text{for} \quad t < 0 \quad (+ \text{ branch})$$

$$a = a_0 t^{\pm 1/\sqrt{3}}, \quad h = \pm \frac{1}{\sqrt{3}t}, \quad \phi = \phi_0 + \pm \sqrt{3} - 1 \ln t \quad \text{for} \quad t > 0 \quad (- \text{ branch})$$ \hspace{1cm} (12)

Note that the expanding solution for $t < 0$ begins in the weak coupling regime ($\phi$ large and negative) and evolves toward the strong coupling region ($\phi$ positive), compatible with the form of the action which represents a weak-coupling truncation of the full effective action of string theory. In the following, we will want to choose the expanding solution for $t > 0$ as we are motivated by a desire to match this solution with a Robertson-Walker cosmology having decelerated expansion. In contrast, the contracting solution for $t < 0$, begins in the strong coupling region and evolves towards the weakly coupled region of field space. In some sense, unless string loops are taken into account, this truncation is somewhat ad hoc for the contracting solutions. Now, if we look at the time evolution of the scale factor, we can have either expansion or contraction at $t < 0$, yet we are interested in expansion only at $t > 0$. For the expanding/expanding (contracting/expanding) solutions we note that for $t < 0$ we have a pole-driven expansion (contraction), reaching the singularity at $t = 0$, and for $t > 0$ we get a power-law expanding universe emerging from the singularity.

If these two temporal branches could be viewed as a single solution, ignoring the presence of the curvature singularity at $t = 0$ for the moment, the compound configuration could
perhaps possess quite remarkable properties, carrying out most of the commandments of the inflationary doctrine. Namely, prior to the instant $t = 0$ we’d see a superexponential, pole-driven inflation. The inflation would be driven solely by the dilaton kinetic energy, thus doing away with the need for more complicated sources. Moreover, if the singularity at the pole $t = 0$ can be surmounted and the two temporal branches joined smoothly, the resulting solution would represent a completely nonsingular cosmology. As the curvature approaches the Planck scale, from a prior expanding or contracting solution, to eventually exit this region and metamorphose to a cooling, expanding universe which can be joined onto our own as $t \to \infty$. The bump around $t = 0$ would then resemble the Big Bang and therefore, in addition to possibly solving the problems usually assigned to inflation, it would also give an elegant resolution to the question of initial singularity.

In general, we will see that the asymptotic properties of the (+) and (−) branches will require that we switch from one to the other to stay within the limits of our theory and to get a desirable late time cosmology. These properties can be summarized in the observation that (+) branch solutions evolve towards singularities in their future while (−) branch evolve away from singularities in their past. It is obvious from the equations of motion that the two branches can never connect smoothly in the regions where the potential is positive (cf. eqs. (10) and (11)). Namely, if the two branches are to be continuously attached to each other, at the location of contact the values of derivatives must be the same. This requires that the potential be negative in a certain region. If we represent the dynamics by the phase space portrait in the phase plane $(\phi, h)$, the regions where branch changes can occur are closed curves symmetric around the $\phi$-axis, given by $3h^2 + 2\Lambda = 0$. They were conveniently named the “eggs” because of their concave shape in the regions containing a single negative minimum of $\Lambda$.

Before considering whether such successful branch changes can be catalyzed by eggs in general, we will present here the special case when $\Lambda(\phi) = const$. We look at this case for two reasons. The case $\Lambda < 0$ gives us the simplest example of a potential with an egg - in fact, with nothing else but the egg, because the potential is negative everywhere, and thus the egg is just two lines parallel with the $\phi$-axis. The other case, $\Lambda > 0$, has no eggs, but it gives us a clear description of the generic properties of solutions in the regions of fairly flat potentials, and allows us to identify the associated attractors and repellers as the linear dilaton vacua, which is a well understood conformal field theory construction. This shows that we can think
of the linear dilaton vacua as seeds for (+) branch superexponential inflation. Furthermore, in these two cases the equations of motion and can be integrated exactly, and we can use the solutions to develop our grasp of the qualitative properties of solutions in more general cases.

We present only the solutions where \( h > 0 \) since the \( h < 0 \) solutions may be obtained from these by time-reversal \( t \rightarrow -t \) (changing \( h \rightarrow -h \)). In the case \( \Lambda < 0 \) we obtain:

\[
\begin{align*}
    h &= \sqrt{2|\Lambda|/(\sqrt{3}\sin(\sqrt{2|\Lambda|}t))) \\
    \phi &= (1/2)(\sqrt{3}\ln(\tanh(\sqrt{2|\Lambda|}t/2)) - \ln(\sinh(\sqrt{2|\Lambda|}t))) + \phi_0
\end{align*}
\]  

(13)  

(14)

where \( 0 < t < \pi/\sqrt{2|\Lambda|} \). These solutions feature a branch change (from (−) to (+) at \( t = \pi/(2\sqrt{2|\Lambda|}) \) in analytic form. To see this, notice the sign change in \( \dot{h} \) and recall the presence of the upper boundary of the egg (the upper of the lines \( |h| = \sqrt{2|\Lambda|}/\sqrt{3} \)). Notice that the single trajectory is this case is singular at both endpoints. (The simple identification of the (+) branch being associated with negative times naturally followed by a (−) branch at positive times is no longer convenient when a potential is present. As always, the (+) branch evolves towards a singularity and the (−) branch away from one.) Qualitatively, we see that the picture is as follows: the universe begins on a (−) branch near the Big-Bang singularity, which in the phase space approach corresponds to the limit \( \phi \rightarrow -\infty, h \rightarrow \infty \). The universe then evolves along the (−) branch towards the strong coupling regime, with its expansion being decelerated. Eventually, the evolution brings it down near the upper egg line, which it touches tangentially, and moves away on the (+) branch. This branch goes steadily upwards, never returning to the vicinity of the egg line, and thus we end up with a branch change in the direction reverse to the one we are looking for. An example is shown in Fig. (1).

In the case \( \Lambda > 0 \) there is no egg, trajectories from different branches never link up. One can see that now there exist two classes of solutions. The phase trajectories with variable Hubble parameter are directly analogous to the two \( \Lambda = 0 \) branches. The (−) branch is given by:

\[
\begin{align*}
    h &= \sqrt{2\Lambda}/(\sqrt{3}\sinh(\sqrt{2\Lambda}t)) \\
    \phi &= (1/2)(\sqrt{3}\ln(\tanh(\sqrt{2\Lambda}t/2)) - \ln(\sinh(\sqrt{2\Lambda}t))) + \phi_0
\end{align*}
\]  

(15)  

(16)

where \( t > 0 \). Similarly, the (+) branch solution is given by:
\[ h = \frac{\sqrt{2\Lambda}}{\sqrt{3}\sinh(-\sqrt{2\Lambda}t)} \quad (17) \]

\[ \phi = \frac{1}{2}(-\sqrt{3}\ln(tanh(-\sqrt{2\Lambda}t/2)) - \ln(sinh(-\sqrt{2\Lambda}t))) + \phi_0 \quad (18) \]

where \( t < 0 \). In addition there are the linear dilaton vacua themselves, given by [7, 8]

\[ h = 0 \quad \phi = \phi_0 \pm \sqrt{\Lambda/2} \quad t \quad (19) \]

They generalize the trivial solutions \( h = 0, \phi = \text{const} \) present in the \( \Lambda = 0 \) case. The linear dilaton vacuum solutions do not appear when \( \Lambda < 0 \), as \( \dot{\phi} \) would become imaginary. From comparing these solutions we see that the \((-)\) branch solutions emerge from the singularity at \( t = 0^+ \) and approach the linear dilaton vacuum with the \(-\) sign in the above equation as \( t \to \infty \). For the \((+)\) branch case, the solutions begin at the linear dilaton vacuum with the positive sign chosen in the equation above as \( t \to -\infty \), and evolve towards the singularity as \( t \to 0^- \). This identifies for us the asymptotic conditions for the solutions with negligible potential gradients. Generically, the \((+)\) branch solutions evolve away from linear dilaton vacua (even in the case without the cosmological constant, which we can think of as \( \Lambda = \epsilon^+ \to 0 \)), and \((-)\) evolve towards the linear dilaton vacua. In particular, this singles out a specific initial condition for the universe which starts on a \((+)\) branch, and thus does away with the initial condition problem, as we have indicated above. These solutions are graphically represented in Figs. (2) and (3).

The family of solutions for constant \( \Lambda \) (the non-zero central charge deficit) may also be found implicitly in [19]. There the effects of the axion term were included too. The above solutions with \( \Lambda = \text{const} \) may be seen at the boundary of their figures in the \((\dot{\phi}, h)\) plane where the axion goes to zero.

Armed with these examples and intuitive arguments, we can delineate the properties of a non-singular cosmology and simultaneously with it the properties of a potential that would guide its evolution. To avoid all singularities, we must have a branch change, so we require a potential to become negative, producing one or more eggs. Given such a favorable potential, in investigating the possibility of a branch change from \((+)\) to \((-)\), Brustein and Veneziano arrived at the “graceful exit” problem. Based on numerical integration of the equations of motion augmented with some qualitative arguments, they concluded that while in all the
cases they have analyzed it was possible to induce a change $++ \rightarrow --$, it was always followed by another change $-- \rightarrow ++$, and the problem persisted. Due to the approximate nature of these arguments, they referred to their results as a “very-hard-to-go-theorem”. In fact, an exact result can be obtained. We will present this no-go theorem in the next section.

With the failure of the dilaton potential alone to produce the required branch changing, the authors of [20] attempted to improve the situation by including stringy fluid sources, higher dimensional embeddings, and combinations of all of them without success. After presenting the proof of the exact no-go theorem for dilaton potential, we will generalize it by outlining the proof when the fluid sources are present.

It has been advocated that the singular behavior of the cosmological solutions may be resolved with the help of higher derivative terms, important in the regions of large curvature, which have been shown to lead to interchanges of duality-related branches in asymptotically weakly coupled, flat regions [24]. This is further supported by the existence of completely non-singular, non-perturbative cosmologies in string-like models with the dilaton coupled to the Gauss-Bonnet higher derivative curvature combination [17]. We underline here that all the nonsingular solutions presented in [17] are nonperturbative in the strength of the Gauss-Bonnet coupling, which one should expect on the grounds that there are no nonsingular FRW cosmologies in the absence of this term. This points to the fact that these solutions cannot be immediately regarded as string cosmologies, because other higher order corrections may be important. Concrete information is still lacking, however, due to the absence of a general procedure to treat the higher order corrections to all orders in a systematic way, distinguishing between the physically relevant contributions and counterterms arising due to the redefinition ambiguity. In the absence of this, we feel that it is of interest to look at other options and attempt to clarify the essentials of this graceful exit problem.

4 The Proof of The No-Go Theorem

In this section we will consider the problem of potential-catalyzed branch changing and prove that it cannot occur in the scenario envisioned in the previous section. This therefore rules out potentials as possible solutions of the graceful exit problem in stringy cosmology. We will also show that it is straightforward to generalize this result to the case when stringy fluids are present.
The main thrust of this argument will concern the behavior of solutions which bounce off the egg, which we will show can’t lead to favorable branch changes. But before we embark on this, we remark that numerical experiments show that it is extremely difficult to get a ‘good’ (+) initial trajectory (in the sense of arising from no past singularity) to hit an egg at all for a simple looking potential. The reason for this is the presence of a saddle point in the flow ahead of the egg which divides the ‘good’ trajectories into two streams which flow around the egg. In fact, we can show analytically that no such trajectories can hit the egg generated by a positive curvature quadratic potential. But since we can’t rule out the possibility of a first touch on the egg of a sufficiently bizarre shape, we must consider the possibility of a bounce. In this case, we stress that the proof applies to any (+) trajectory originating outside of the egg region.

To begin the inquiry into the global properties of the solutions for a general $\Lambda(\phi)$ we shall find the loci of points where the direction of flow of trajectories in the phase plane changes. These are given by the curves where $\dot{h} = 0$ and $\dot{\phi} = 0$. The detailed properties of these curves are not important to our conclusions, and we will state merely those which we require as we quote the results. If we solve (11) with the condition $\dot{h} = 0$, we find:

$$h^2 = \frac{1}{6}(-\Lambda(\phi) + \sqrt{\Lambda(\phi)^2 + (3/4)\Lambda'(\phi)^2})$$

$$h\Lambda' \geq 0 \text{ for (+) branch} \quad h\Lambda' \leq 0 \text{ for (−) branch} \quad (20)$$

By directly examining (11) for large positive and negative values of $h$, we notice that for the (+) branch, the $h$-flow is away from the curve ($\dot{h} > 0$ above it and $\dot{h} < 0$ below), whereas for the (−) it is towards it ($\dot{h} < 0$ above the curve and $\dot{h} > 0$ below it). This behavior justifies our characterization of (+) trajectories as singular in the future (positive feedback) and (−) branch as singular in the past (negative feedback).

If we solve (11) with $\dot{\phi} = 0$, we obtain two curves where the flow of $\phi$ changes sign:

$$h^2 = \Lambda(\phi)/3$$

$$h \leq 0 \text{ for (+) branch} \quad h \geq 0 \text{ for (−) branch} \quad (21)$$

Notice that these curves touch the vertical strip containing the egg only at the very ends (where $\Lambda(\phi) = 0$) and extend away from the egg region. Thus, the $\phi$ flow vertically above the egg is from left to right (from the weak coupling towards the strong coupling) and reversed.
below them, for both branches. Putting these facts together we see that trajectories tend to flow clockwise around the egg.

At the intersection of these curves lie fixed points. The conditions for these are most easily read off by setting the dotted quantities to zero in the second order equations

\[ 3h^2 = \Lambda(\phi) = -\Lambda'(\phi)/2 \geq 0 \]
\[ h \leq 0 \text{ for (+) branch} \quad h \geq 0 \text{ for (−) branch} \quad (22) \]

These fixed points come in pairs, above the \( \phi \)-axis for the (−) branch and below it for (+), except when they coincide for the cases when \( h = 0 \). We can analyze the nature of these fixed points directly in the string frame and determine that if the quantity \( \Lambda'(\phi) + \Lambda''(\phi)/2 \) is negative, the points are hyperbolic (saddle points) and where it is non-negative the (−) branch \( h \geq 0 \) is an attractor and the (+) branch \( h \leq 0 \) point is a repeller. None of this should come as a surprise, since thinking of these fixed points in terms of the potential in the Einstein frame \( e^{2\phi} \Lambda(\phi) \), we see that the saddle points correspond to positive maxima and the attractor/repeller pairs to positive minima of this potential. This correspondence is exact since near the fixed points \( \dot{\phi} \) is moving very slowly, and the conformal rescaling between the frames is nearly constant, so that the notion of the character of a fixed point does not depend on the frame.

Following [20] we define the egg function:

\[ e = \sqrt{3h^2 + 2\Lambda(\phi)} \quad (23) \]

Then it is easy to show that

\[ \dot{e} = \pm(1/2)(6h^2 + \Lambda'(\phi)) = \pm(2h\dot{\phi} - \dot{h}) \quad (24) \]

The condition \( \dot{e} = 0 \) defines curves separating regions where the egg attracts or repels different branches and extends the infinitesimal condition in [20] governing approach to the egg. Of more interest than the curves themselves is the fact that dividing both sides of (24) by \( \dot{\phi} \) and integrating the result over \( \phi \) along a trajectory, we obtain:

\[ \pm (e(t_1) - e(t_0)) + h(t_1) - h(t_0) = 2 \int_{\phi(t_0)}^{\phi(t_1)} h d\phi \quad (25) \]

The integral here is to be understood as a line integral along the path of the system between \( \phi(t_0) \) and \( \phi(t_1) \). As we will now show, the sign of this integral represents an exceptionally
strong constraint on the behavior of trajectories, and provides the needed tool to obtain the no-go theorem.

Let us now outline our proof. We should recall that under a successful branch change we mean a trajectory which enters the egg region as a (+), since they can have non-singular pasts, and leaves on a (−) branch, since they have have non-singular futures and indeed may be captured by a (−) attractor if we wish to decouple the dilaton (recall there are no (+) branch attractors). We will show that such a successful branch change is impossible, and will base the proof on three important details. First, we reemphasize that all trajectories flow from left to right in the region vertically above the egg and right to left below. Thus any trajectory hitting the top of the egg must come from the left and any trajectory hitting below the egg must come from the right, if they are to have any extension outside of the egg region. This, of course, allows that a trajectory can flow around the egg without hitting it for “half” a cycle, e.g. coming from the right in the far past, flowing below the egg and reemerging above the \( \phi \) axis to the left of an egg, flowing towards it. Second, we will prove that any (−) branch trajectory, originating from anywhere on the upper side of the egg cannot escape over the right end of the egg but must hit it again. The third ingredient of our proof is a time-reversed corollary of the second, that any (+) trajectory coming from the right and flowing below the egg cannot hit the egg below, or on, the \( \phi \)-axis. These latter two impossible trajectories are illustrated in fig. (4).

Combining these together we see that any (+) branch entering an egg region from the left must go over the top of the egg, possibly experiencing several branch changes, and must exit the region of the egg to the right still being on the (+) branch. Any (+) branch entering an egg region from the right is prohibited from hitting below, and so it must remain (+) while flowing under the egg. Thus any (+) trajectory entering the egg region cannot leave on a (−) branch, and there is no graceful exit. The egg can only convert (−) to (+). Clearly, multiple eggs cannot change this conclusion.

Now it remains to establish the second and third of our claims. First we show that a (−) branch bounce, originating from anywhere on the upper side of the egg cannot escape over the right end of the egg but must fall down on it again. To see this recall the integral formula (25) for a (−) trajectory:

\[
-e(t_1) + e(t_0) + h(t_1) - h(t_0) = 2 \int_{\phi(t_0)}^{\phi(t_1)} h d\phi
\]  

(26)
Let $t_0$ be the time of the origin of the $(-)$ bounce, and $t_1$ the later time when the trajectory leaves the end of the egg. Then, at the egg $e(t_0) = 0$, and $h(t_0) \geq 0$. At the end of the egg $h(t_1) \geq 0$ and $e(t_1) = \sqrt{3}h(t_1)$, since the end of the egg is defined by the condition $\Lambda(\phi(t_1)) = 0$. Finally, in this region the flow is to the right and $h \geq 0$. Therefore, the integral is equal to the area between the segment and the $\phi$-axis and hence strictly positive (we will remark below on the degenerate cases where this area may be zero):

$$\int_{\phi(t_0)}^{\phi(t_1)} h d\phi = A > 0 \quad (27)$$

Substituting these in (26) we arrive at the sought contradiction:

$$0 < 2A = -(\sqrt{3} - 1)h(t_1) - h(t_0) \leq 0 \quad (28)$$

Therefore, the $(-)$ bounce emerging from the upper side of the egg must terminate back on it, as we claimed.

For the third claim consider a $(+)$ branch entering the region below the egg and passing the right end of the egg. Recall the integral formula (25) for a $(+)$ trajectory:

$$e(t_1) - e(t_0) + h(t_1) - h(t_0) = 2 \int_{\phi(t_0)}^{\phi(t_1)} h d\phi \quad (29)$$

Let $t_0$ be the time of passing the end of the egg, defined by the condition that $h(t_0) \leq 0$ and $\Lambda(\phi(t_0)) = 0$, so that $e(t_0) = \sqrt{3}|h(t_0)|$. Let $t_1$ the later time when the trajectory hits below the egg, so that $e(t_1) = 0$ and $h(t_1) \leq 0$. In this region the flow is to the left and $h \leq 0$ so the integral is again equal to the area between the segment and the $\phi$-axis and strictly positive:

$$\int_{\phi(t_0)}^{\phi(t_1)} h d\phi = A > 0 \quad (30)$$

Substituting these in (29) we arrive at the sought contradiction:

$$0 < 2A = -(\sqrt{3} - 1)|h(t_0)| - |h(t_1)| \leq 0 \quad (31)$$

This contradiction shows the $(+)$ trajectory cannot hit below the egg.

This concludes the main line of argument for the “no-go” theorem. However, several exceptional cases remain which we will deal with briefly. While these cases require an infinite fine tuning of initial conditions, we will show they can be dismissed. One may worry
about the case of tangential hits on the very ends of the egg where it meets the $\phi$ axis, or possibly passes through “pinches” where the egg narrows to a single point. If this happens at the left end of the egg, where generically $\Lambda'(\phi) < 0$, we can substitute $h = \Lambda(\phi) = 0$ into the equations of motion (7), (8) and (9) to conclude the $\dot{\phi} = 0$ and $\dot{h} - \ddot{\phi} = -\Lambda'(\phi)/2 > 0$. Hence this point on the curve is a minimum of $\phi(t)$, and the trajectory is curving from the region vertically below the egg (+) into the region vertically above the egg (−), and we can easily see this cannot lead to any exceptional behavior. A hit on the right point is a change from (−) above to (+) below and again does not lead to exceptional cases.

Hits at inflection points where $h = \Lambda(\phi) = \Lambda'(\phi) = 0$ (similar to a “pinch”, but occurring at the ends of the egg, or even more generally corresponding to a region where $\Lambda(\phi) = 0$ for an interval on the $\phi$ axis) may seem more troublesome, since it will be difficult to extract information about the past and future of these trajectories. But here we can refer to a general property of the second-order equations of motion. We note that they can be written in the form of a normal system [27], i.e. that the second derivatives $\ddot{\phi}$ and $\ddot{h}$ can be written as functions of the first derivatives and values of $\phi$ and $h$. If we require that these functions are Lipschitz in their arguments in a neighborhood of the point of interest, we may conclude that the trajectories are unique for given initial conditions there. (The Lipschitz condition is a weaker form of a bounded derivative condition). Since this is a natural local condition for $\Lambda(\phi)$ and $\Lambda'(\phi)$ we conclude that no two trajectories of the same branch can intersect.

Now to return to the inflection points, we notice that they are also well-behaved fixed points, since there is a trivial solution ($\phi$ and $h$ constant) sitting in them, and thus no other solution can cross through them, but only approach them asymptotically. Therefore the inflexions cannot be used for branch changing, as no bounces can originate from them.

In addition, one might wonder whether solutions can circle and change branches on the egg ad infinitum, leading to a curious quasi-cyclic cosmology. Briefly, the answer is no. The equation (25) can be used to show that each hit on the egg must be higher than the previous, and in fact must be at least twice the area of the egg higher for each rotation about it. This result can be sharpened to show that no incoming “good” (+) can circle the egg and rehit it on the top. A “zero area” egg is no solution either, since this is a (+) branch repeller. To see this we note that the motion around such a point is clockwise and the accumulation of area in the integral formula will push (+) branch solutions away.

Finally, simply for completeness, we note the presence of fine tuned (+) branch solutions
whose evolution asymptotically slows down to a halt at a saddle point at $h < 0$, approaching a contracting deSitter phase. With the exception of these, and the constant $h$ solutions sitting at fixed points to which these solution tend, our arguments show that all other evolutions must begin or end in singularities, or both.

As we have mentioned before, this no-go theorem can be generalized to the case when stringy fluid sources are present. We will now outline the proof for this case. We extend the phase space of the model to three dimensions, the third coordinate being the energy density of the fluid $\rho$. The associated equations of motion are given by the following generalization of (10-11) [20]:

$$
\dot{\phi} = \frac{(3h \pm \sqrt{3h^2 + 2\Lambda(\phi) + \rho \exp(2\phi)})}{2}
$$
$$
\dot{h} = \pm h \sqrt{3h^2 + 2\Lambda(\phi) + \rho \exp(2\phi) - \Lambda'(\phi)/2 + \gamma^2 \rho \exp(2\phi)}
$$
$$
\dot{\rho} = -3(1 + \gamma) h \rho
$$

Here $\gamma = p/\rho$, is a constant representing the fluid equation of state, for which we will only require $\gamma > -1/3$. This includes a wide range of fluids, both stringy with $\gamma \in (-1/3, 1/3)$ [20], and relativistic, corrected by the dilaton coupling as discussed in [12, 18] and references therein. We note that the physical restriction $\rho \geq 0$ is consistent with the equations of motion, as the $\rho$ flow terminates at the $\rho = 0$ plane, which is like a potential barrier. Moreover, we note that the trajectories completely confined in this plane are governed by our previous theorem, so there is no graceful exit for them. Now we look at the fully three-dimensional trajectories. The egg function is given by

$$
e = \sqrt{3h^2 + 2\Lambda(\phi) + \rho \exp(2\phi)}
$$

Taking a time derivative of (33), dividing the resulting equation by $\dot{\phi}$ and integrating over $\phi$ along a trajectory, we obtain the modified integral formula, analogous to (25) when sources are present:

$$
\pm (e(t_1) - e(t_0)) + h(t_1) - h(t_0) = 2 \int_{\phi(t_0)}^{\phi(t_1)} h d\phi + \frac{1 + \gamma}{2} \int_{t_0}^{t_1} \rho e^{2\phi} dt
$$

This equation differs from (25) only in the presence of the last term, which is a nonnegative quantity for all trajectories, as the energy density is restricted by $\rho \geq 0$. Now, in this case
the other relevant characteristics of the phase space, given by equations (20-22) are easily generalized to three dimensions. We will not present the details here. It is sufficient to see that qualitatively the picture remains the same: the egg is now a two-dimensional compact surface cut by the plane $\rho = 0$, and the only fixed points on it may again be “pinches” or inflexions in the $\rho = 0$ plane. The flow of trajectories around the egg is generically along helical paths, which if projected onto the $\rho = 0$ plane turn clockwise. We then need to consider trajectories crossing the cylindrical surface enclosing the egg, obtained by translating the curve representing the boundary of the egg in the $h = 0$ plane vertically upwards. Using the formula (34), we see that the first integral on the LHS, representing the area enclosed by the projection of the trajectory onto the $\rho = 0$ plane, remains positive for all such trajectories, due to the clockwise flow of projections. As we mentioned above, the second integral is always nonnegative, since the integrand is. Hence we can show that all the arguments we derived for the sourceless case extend to this case, again preventing favorable branch changes from occurring. As no new pathologies appear, we conclude that the no-go theorem must hold for this case too.

5 The Higher Genus Corrections

Here we shall consider the possibility of branch changing induced by the string-loop corrections when a stringy cosmological constant is present. This corresponds to the action given by equation (5). Equivalently, we will allow a general $\beta(\phi)$ and constant $\Lambda$ in the equations of motion (7-9). We will also briefly reflect on the case when $\Lambda$ is not constant, as the situation then becomes a combination of the two previous cases.

Our analysis is analogous to the genus-one case. We will again find ($+$) and ($-$) branches and egg regions where branch changing may take place. However, in this case we will find that all fixed points are located on the egg boundaries, with attractors for both ($+$) and ($-$) branches located on the upper surface of the egg. By the weak coupling expansion (4) we know that $B(\phi)$ is positive in a large region of $\phi$, starting from $\phi \to -\infty$. If we restrict ourselves to eggs in this region, we will show that the ($+$) branches originating outside of the egg region cannot reach these fixed points, nor can they use the eggs to change branches. Allowing the conformal factor, $B(\phi)$, to become negative will produce completely different behavior. In this case, we will see that we can construct completely nonsingular cosmologies.
in the string frame, ending in a deSitter phase. However, translating to the Einstein frame we find such phase trajectories consist of two singular Einstein branches separated by the point where \( B(\phi) = 0 \), where the conformal transformation to the Einstein frame is singular.

Solving the constraint equation for \( \dot{\phi} \) we find that we should define the analog of the egg function (23) by:

\[
e = \sqrt{3(4 - 4\beta + 3\beta^2)h^2 + 8(1 - \beta)\Lambda}
\]  

(35)

Also, the equation for \( \dot{\phi} \) is given by:

\[
\dot{\phi} = (\pm e + 3h(2 - \beta))/(4(1 - \beta))
\]  

(36)

In the region where \( B(\phi) > 0 \), the upper sign refers to the \((+\)) and the lower to the \((-\)) branch. This ansatz is in accord with the definition of the two branches in the previous section, as can be verified by setting \( \beta = 0 \). In the region where \( B(\phi) < 0 \) we will reverse this sign convention, the upper sign will refer to \((-\)) branch and the lower to \((+\)). This curious reversal of our conventions is needed to keep \( \dot{\phi} \) continuous across a sign change in \( B(\phi) \) where \( \beta \) becomes singular. We will find continuous evolution through this line (where \( B = 0 \)). To further justify this convention, consider the equations of motion in terms of the non-singular quantities \( B(\phi), B'(\phi) \) and \( B''(\phi) \). We see that the reduction to the form (36) requires extracting the quantity \( B(\phi)^2 \) from the radical, creating a sensitivity to the sign of \( B(\phi) \). The equation for \( \dot{h} \) is complicated and will not be needed here.

Next we examine the locations and character of fixed points. Inserting \( \dot{\phi} = \ddot{\phi} = \dot{h} = 0 \) into the equations of motion we obtain the two conditions:

\[
0 = 3h^2 - \Lambda
\]  

(37)

\[
0 = (6h^2 - \Lambda)(\beta - 2) = 0
\]  

(38)

We notice that we need \( \Lambda > 0 \) to get any fixed points at all. Furthermore, in this case a solution of these equations is given by \( 3h^2 = \Lambda \) and \( \beta = 2 \). Putting this solution into (35) we find \( e = 0 \), and therefore all the fixed points are on the egg, in contrast to the genus-zero case with dilaton self-interactions.

The condition \( \beta = 2 \) corresponds to \( B'(\phi) = 0 \), so we see that the fixed points are extrema of \( B(\phi) \). As we noted before, this is to be expected from the corresponding problem
in the Einstein frame, which is given as a dilaton evolution in a potential of the general form $V(\phi) = \Lambda/B(\phi)$. We will use this fact to classify the fixed points, as it will be difficult to study their properties directly in the string frame. Thus we see that positive minima of $B(\phi)$ (equivalent to $\beta' > 0$) are maxima of $V(\phi)$, and we conclude these are hyperbolic saddle points. Positive maxima of $B(\phi)$ (equivalent to $\beta' < 0$) are positive minima of $V(\phi)$, and thus these will be attractors for $h > 0$ and repellers for $h < 0$. We can easily extend these results to regions where $B(\phi) < 0$ by noting that the sign of $B(\phi)$ does not enter into the equations of motion, so we can simply invert $B(\phi)$.

At this point, we need to investigate the shape and location of the egg. Solving $e = 0$ for $h^2$ we find:

$$h^2 = 8(\beta - 1)\Lambda/(3(4 - 4\beta + 3\beta^2)) \quad (39)$$

Since the quadratic in the denominator is positive definite, we find an egg boundary where $1 \leq \beta \leq \infty$. Thus the egg begins at $h = 0$ at a value of $\beta = 1$, increases to a peak of $h^2 = \Lambda/3$ at $\beta = 2$, and decreases again to $h = 0$ as $\beta \to \infty$. Comparing this with our results about the locations of the fixed points, we conclude that the fixed points are precisely at the peaks above and below the egg (the topological saddle points of the egg boundary).

Finally, we examine the nature of curves along which $\dot{\phi}$ changes sign. From (38) we see that in the case where $B(\phi) > 0$, these are simply the horizontal segments defined by $3h^2 - \Lambda = 0$ with $h(2 - \beta) < 0$ for the $(+$) branch and $h(2 - \beta) > 0$ for the $(−)$ branch (these begin or end on one of the aforementioned fixed points) and the vertical segments (where $\dot{\phi}$ is singular) defined by $\beta = 1$ with $h > 0$ for the $(+$) branch and $h < 0$ for the $(−)$ branch (which begin at the ends of the egg). In the region where $B(\phi) < 0$, the same results hold with a reversal of the branch designations. At the boundaries between different signs of $B(\phi)$, we find singular lines of $\dot{\phi}$ sign change. At a line where $B(\phi) = 0$, $B'(\phi) < 0$ we find this discontinuity for $h < 0$ for $(+$) branch and $h > 0$ for $(−)$ branch, and where $B'(\phi) > 0$ we just reverse the branch labels.

Specializing to the case $B(\phi) > 0$, marking each of these regions with the sign of $\dot{\phi}$ flow gives us the complete picture of $\phi$ flow in the phase plane around a $B(\phi) > 0$ egg, see Figs. (5) and (6). Consideration of the direction of flow alone leads us to the no-go result in this case. We can easily see that $(+)$ branch solutions approaching the egg region from the right or the left cannot get into the region vertically above the egg where they could...
find an attractor. Thus if they are to hit the egg they must do it below and convert to \((-\)). But now we notice that a \((-\)) branch solution below the egg cannot emerge from this region without converting to a \((+\)). Combining these two facts together, we see that the egg is again unable to change a \((+\)) branch to a \((-\)) branch. The case of a trajectory passing through an endpoint of the egg is easily dealt with. Putting the conditions characterizing the egg endpoints \((\beta = 1 \text{ and } h = 0)\) into (9), we conclude \(\Lambda = 0\). Since we are interested in the case \(\Lambda > 0\), the trajectories do not flow through the endpoints.

If we begin with a \(B(\phi)\) with a negative minimum, we reach quite different conclusions since the barriers surrounding attractors for \((+\)) branch flow have fallen. Although one might suspect that all solutions beginning in the region where \(B(\phi) > 0\) will not be able to cross over a point where \(B(\phi) = 0\), this is not true. In the string frame there exist solutions which cross this region without hesitation, as numerical integration shows (see Figs. (7) and (8)). In Fig. (7), we show a solution which is always undergoing expansion, while in Fig. (8), we show a solution which starts out in a contracting phase, undergoes multiple branch changes and finishes in an expanding, asymptotically deSitter state (as does the former solution). Both of these solutions are non-singular in past as well as in the future. These solutions are quite peculiar in nature. It is actually quite easy to connect our linear dilaton solutions in the asymptotic past to deSitter solutions in the asymptotic future, using these solutions. The resulting configurations have several attractive features: i) the evolution very naturally flows to an attractor, ii) examining (4), we see that it is much easier and more natural for the loop corrections to have a negative minimum than a positive maximum without introducing large unperturbative coefficients. However, analyzed in the Einstein frame, the continuous string frame evolution splits into two Einstein evolutions, the first contracting to a singularity and the second expanding out of a singularity. This is symptomatic of the fact that the conformal transformation to the Einstein frame is ill-defined. Even in the string frame, the apparent change of signature is quite curious. Though our future history is described by a metric with a \((-,-,+,+)\) signature, it begins with a \((+,-,-,+)\) signature, and because all of the terms in the Lagrangian also change sign, the equations of motion are unaffected. The curiosity occurs just at the point when \(B=0\), where there is no metric, and we are dealing with a topological field theory. We hope to address this issue in more detail elsewhere.

In retrospect, we can now see that even a non-constant \(\Lambda\) is extremely unlikely to solve the graceful exit problem. Namely, the situation would correspond to a combination of the two
cases we developed the no-go theorems for, i.e. the dilaton self interaction superimposed with the higher-genus corrections. We would end up with eggs of both types considered, which separately cannot facilitate a favorable branch change. A novelty would be that the integral formula (25) could not be given a simple geometric interpretation for all cases. However, for (+) branch solutions extendible to linear dilaton vacua in the past, these aberrations would typically be small, and should not induce any qualitatively new behavior. Thus it appears that the only way to salvage the Pre-Big-Bang scenario is to resort to the higher order terms in the \( \alpha' \) expansion, with the difficulties which this approach brings, as explained in section 4.

We close this section with the note that there still exists a possibility to incorporate inflation in string theory using the model including the higher genus corrections as described by the action (5). Namely, it is not difficult to see that we can trap a (−) branch originating away from the egg in a fixed point with \( h > 0 \), resulting in asymptotically deSitter inflation. This scenario was recently analyzed in detail by Damour and Vilenkin [25], in Einstein conformal frame. We should only mention here that this scenario is very similar to standard inflationary models, in the sense that the resulting universe starts from a Big-Bang cosmological singularity.

6 Conclusion

We have derived an exact no-go theorem for string cosmology in very general circumstances, ruling out the possibility of resolving the graceful exit problem by branch changing catalyzed by dilaton selfinteraction, fluid sources and higher genus corrections. Our analysis was based on the investigation of phase space properties of the model, resulting in precise and strong answers concerning the evolution of the universe governed by the graviton-dilaton sector. We can still incorporate inflation in the model using a combination of a nonzero stringy cosmological constant with the higher-genus corrections, with the resulting cosmology looking like a universe starting from a Big Bang singularity and asymptotically evolving towards a deSitter phase, with a decoupled dilaton.

In addition, we have found a class of nonsingular solutions in the string frame, for the case when the conformal coupling \( B(\phi) \) becomes negative for some values of the dilaton. These solutions evolve out of the linear dilaton vacua in the past and asymptotically approach de-
Sitter expansion in the future, passing through the value $B = 0$ without hesitation. During this evolution, however, the relative sign of the action changes, which may require a topological description of the Universe at the point where $B=0$. We hope to address these solutions in more detail in the future.

Finally we observe that the only option still open for incorporating the original Pre-Big-Bang scenario is to resort to higher derivative terms in the $\alpha'$ expansion. In this approach we must consider systematically all the terms in the $\alpha'$ expansion, and this can only be implemented via the exact conformal field theory construction. At this moment, it appears that this goal is still beyond our means.

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Figure Captions

Figure 1: Evolution in the case with a negative cosmological constant ($\Lambda = -2$), with egg boundaries along the line $h = \pm \sqrt{4/3}$. Since the branch change is in the wrong direction ($(-) \rightarrow (+)$), this solutions has singularities in both the past and the future.

Figure 2: A $(-)$ branch solution with the positive cosmological constant ($\Lambda = 2$). The universe evolves from a past singularity into a linear dilaton vacuum in the future.

Figure 3: A $(+)$ branch solution with the positive cosmological constant ($\Lambda = 2$). The universe evolves from a linear dilaton vacuum into a future singularity.

Figure 4: Examples of trajectory segments ruled out by the second and third arguments of the no-go theorem. The $(-)$ branch cannot leave the egg and exit to the right, nor can the $(+)$ branch enter from the right and hit the egg. Here the vertical lines only demark the ends of the egg, and are not the boundaries of different directions of $\phi$ flow.

Figure 5: The boundaries of the regions of uniform direction of $\phi$ flow of the $(+)$ trajectories. Notice that a $(+)$ trajectory outside of the region above the egg cannot enter the region where the deSitter attractors are located. To get the corresponding picture for the $(-)$ branch trajectories, we need to invert this picture and reverse the arrows.

Figure 6: As in Figure 5 for the $(-)$ branch.

Figure 7: A nonsingular $(+)$ branch trajectory ending in a deSitter phase attractor on a $B < 0$ egg, $B(\phi) = e^{-2\phi} - 2 + 0.5e^{2\phi}$ and $\Lambda = 1$.

Figure 8: A nonsingular $(+)$ branch trajectory ends in a deSitter phase attractor on a $B > 0$ egg. This solution evades the terms of our no-go theorem by using a $B < 0$ egg in the strong coupling region to perform the $((+) \rightarrow (-))$ before the capture on the $B > 0$ egg. $B(\phi) = e^{-2\phi} + 0.5e^{2\phi} - 0.03e^{4\phi}$ and $\Lambda = 1$. 
Figure 2
Figure 3
Figure 4
Figure 5

\[ \text{Egg } B > 0 \]
Figure 6

Egg $B > 0$
Figure 7

Egg $B<0$

Egg $B>0$
Figure 8

Egg $B > 0$

Egg $B < 0$