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Aspects of Six Dimensional Supersymmetric Theories†

S. Randjbar–Daemi††
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT

In this contribution some aspects of supergravity and super Yang-Mills systems in $D = 6$ are briefly reviewed and, in some cases, are contrasted with the analogous features in $D = 4$. Particular emphasis is laid on the stringy solutions of the $D = 6$ super Yang-Mills systems.

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†† e-mail address: daemi@ictp.trieste.it
1. Introduction.

I was indeed very privileged to be for many years a close associate of Abdus Salam. I have learned many things from him. Salam combined the vigorous western thought in a coherent manner with his oriental culture. He believed deeply that the social life of an individual has a sense and purpose only in relationship with those of others. It becomes richer and more purposeful if its guiding principles are compassion and tolerance. He himself was a proverbially generous person.

Salam deeply appreciated the relevance of Science to the enrichment of human life. He spent a major part of his active life to disseminate scientific knowledge among the less privileged nations. Being a singularity as he was, he also contributed substantially to the advancement of the fundamental science. In fact the very best existing theory of Nature, the Standard Model of Particle Physics, bears his name.

My scientific collaboration with Abdus Salam started with a study of theories of Kaluza - Klein type in a space time of six dimensions [1]. I have therefore chosen to review in this memorial contribution some of the recent developments in 6-dimensional theories. The presentation will be mostly, but not always, non technical and elementary.

1. Particles and Strings in D=4 and D=6

Physical theories in a six dimensional manifold of Lorenzian signature differ in many respects from the four dimensional theories. In $D = 6$ in addition to spinors, scalars, vectors, and second rank symmetric tensors, which are the basic objects of 4-dimensional field theories, we also have second rank antisymmetric tensor potentials. Also the fact that the fundamental spinor representation of $SO(1,5)$ is pseudo real, as opposed to the complex Weyl spinors of $SO(1,3)$, has some significance in constructing anomaly free models in D=6. We shall start with a summary of differential forms and the extended objects to which they couple [2].

In $D = 4$ the only interesting forms are the 1-forms and their exterior derivatives which correspond to, respectively, a Maxwell (or Yang-Mills) potential $A$ and its field strength $F$. Being a 2 form, $F$ admits a dual, $^*F$, which is also a 2 form. Maxwell’s equations are
essentially symmetric under the exchange of $F$ and $*F$. This is called the electromagnetic duality under which the electric and magnetic charges interchange their roles [3]. This type of duality is the prototype of a larger class of duality symmetries which can occur in space times of higher than 4 dimensions. Note that in $D$ dimensions the dual of a $p$ form is a $D - p$ form. It is thus only for $D = 4$ that the dual of the electromagnetic 2 form $F$ is again a 2 form. To appreciate the physical significance of this simple fact let us recall that the electromagnetic potential $A$ couples to particles through the term $\int_C A$ where $C$ denotes the world line of the charged particle. If $*F$ is derived from a dual potential $\tilde{A}$ then there will be a dual particle which could couple to $\tilde{A}$ through $\int_{\tilde{C}} \tilde{A}$, where $\tilde{C}$ is the world line of the dual particle. In $D$ dimensions a $p + 1$ form potential couples naturally to a $p$ dimensional extended object, called a $p$-brane. This coupling is a direct generalization of the electromagnetic coupling, namely, $\int_{\Sigma_{p+1}} A$, where $\Sigma_{p+1}$ denotes the $p + 1$ dimensional world volume of the $p$ dimensional extended object. Note that a $p$-brane will occupy a $p$ dimensional subspace of the $D$ dimensional space time. For example, we can think of a $p$ dimensional hyperplane extended along $p$ of the $D - 1$ space coordinates. Therefore, in the remaining $D - 1 - p$ space dimensions our object will look like a point, which we call the position of the $p$-brane. A large sphere around this position will have $D - 2 - p$ dimensions, which equals to the rank of the dual of the $p + 1$ form $F = dA$. We can thus integrate $*F$ over this large sphere and call it the electric charge of the $p$-brane. On the other hand the integral of the $p + 2$ form $F$ over a $p + 2$ dimensional sphere is called the magnetic charge of the considered $p$-brane. These are direct extensions of the well known definitions in $D = 4$. Thus in $D$ space time dimensions the dual of a $p$-brane is an extended object with $D - p - 4$ space dimensions, i.e. a $D - p - 4$ brane.

With the above definitions, in $D = 6$ the dual of a particle is a 2 brane, while the dual of a 1 brane is again a 1 brane. This indicates that in $D = 6$ one dimensionally extended objects, namely, strings, and the two form potentials to which they couple, play a role analogous to the role of particles and vector potentials in $D = 4$.

To mention other interesting differences let us impose the condition $F = *F$. This
condition is meaningful only for a 2 form $F$ in $D = 4$ and a 3 form $F$ in $D = 6$. Thus in $D = 4$, if the self-duality equation had a real solution, there could exist self dual particles. However, it turns out that the self duality equation in $D = 4$ has interesting solutions only if the 4 dimensional manifold is Euclidean and the gauge group is non Abelian. These solutions are called instantons. They are localised finite action solutions of the 4 dimensional Euclidean Yang-Mills equations. The value of their Euclidean action, appropriately normalised, equals to their topological charge. These solutions do not have particle type interpretation in 4 dimensional Minkowski space time. In contrast with $D = 4$, the self duality equation $F = \ast F$ has a solution for an Abelian 3 form $F$ in $D = 6$ Minkowski space-time. These are self-dual strings [4]. Such strings will carry both electric and magnetic charges and their magnitude will be equal. The non Abelian version of higher rank forms are not yet known.

One may wonder what type of interpretation the ordinary Yang-Mills instantons can have in $D = 6$. To answer this question we need to consider a non Abelian gauge theory in a 6-dimensional space time. One can consider a 4-dimensional Euclidean subspace and an instanton configuration localised in this subspace. From the point of view of the 6-dimensional space time this object will look like a string with a thickness given by the size of the instanton. As the size goes to zero it will look more and more like a fundamental string[5]. We shall make use of this interpretation in section three.

2. Supersymmetry and Chiral Anomalies

The possibility of having self dual or anti-self dual gauge fields in $D = 6$ make the six dimensional supersymmetric theories more akin to D=10 theories rather than their D=4 counterparts. The conditions for the cancellation of chiral anomalies are also more stringent in $D = 6$. For example, there are no pure gravitational anomalies in $D = 4$ theories. Such anomalies can exist, however, in $D = 6$. The requirement that the pure gravitational anomalies do cancel imposes restrictions on the supermultiplet structure of the six dimensional supergravity theories.

A model with a minimum number of supersymmetries in $D = 6$ has four independent
complex supercharges which can be assembled into a Weyl spinor of $SO(1,5)$. It is customary to double the number of components and impose a symplectic Majorana condition. For this reason such models are sometimes called $(2,0)$. We shall refer to them as $(1,0)$. These models like their extended versions with 16 real supercharges which we shall denote as $(2,0)$ are chiral, while their $(1,1)$ and $(2,2)$ theories, which have respectively 16 and 32 supercharges, are non chiral.

In section 5 we shall give a little more detail about the models with more than 8 real supercharges and explain briefly how they can be obtained from the $D = 10$ superstring theories upon compactification on a four manifold. In the rest of this section we shall exclusively discuss theories with $(1,0)$ supersymmetry.

The $(1,0)$ models admit the following type of super multiplets:

1) gravity: $E^m_a, \Psi_\mu, B_{ab}$
2) hypermatter : $\psi^r, \phi^\alpha$
3) Yang-Mills: $\lambda, A_a, Y$
4) Tensor : $\chi, B^+_a, \sigma$

The spinors in the gravity and the Yang-Mills multiplets are left handed, while those in the tensor and hypermultiplets are right handed with respect to $SO(1,5)$. Furthermore, the spinors in the gravity, Yang-Mills and the tensor multiplets are doublets of an automorphism $Sp(1) = SU(2)$, and they are Majorana symplectic in the sense that they satisfy a constraint of the type $\psi = \Omega \psi_c$ where $\Omega$ is the $Sp(1)$ invariant metric and $\psi_c$ is the charge conjugate of $\psi$. The superscripts $\pm$ on the antisymmetric tensor potentials indicate that their field strengths are self dual (+) or anti-self dual (-). The scalars in the hypermultiplets are the $2n$ complex coordinates of a quaternionic manifold. Thus $\alpha = 1, \ldots, 2n$ and the index $r$ in the hypermatter fermions, $r = 1, \ldots, n$, where, $n$ counts the number of the hypermatter multiplets. $Y$ in the Yang-Mills multiplet is an auxiliary field. It is a triplet of the automorphism $Sp(1)$.

In any $D = 6$ coupled Yang-Mills supergravity theory, the condition for the cancellation of pure gravitational anomalies imposes a restriction on the number of multiplets
listed above. This condition is

\[ n = m + 273 - 29k \]

where \( m, n \) and \( k \) are respectively the number of Yang-Mills, hyper and tensor multiplets.

Anomaly free supergravity models with \((1,0)\) supersymmetry in \( D = 6 \) can be obtained from the compactifications of the ten dimensional heterotic strings on a particular class of complex manifolds called \( K_3 \) [7].

Having cancelled the pure gravitational anomalies the remaining anomalies can be cancelled with the help of a mechanism discovered by Green and Schwarz in the context of ten dimensional superstring models [8] and extended to the six dimensional models in [6, 7].

Let us consider models with \( k = 1 \). For such models one can construct invariant supergravity actions. The anomaly condition simplifies to \( n = m + 244 \). The Green-Schwarz anomaly cancellation mechanism starts from an 8 form \( P_8 = X_4 \tilde{X}_4 \), where ”.” indicates a wedge product. The 4 forms \( X_4 \) and \( \tilde{X}_4 \) have the following general structures

\[
X_4 = tr R^2 - \Sigma v_\alpha tr F^\alpha_2
\]

\[
\tilde{X}_4 = tr R^2 - \Sigma \tilde{v}_\alpha tr F^\alpha_2
\]

where \( v_\alpha \) and \( \tilde{v}_\alpha \) are numerical constants and \( F_\alpha \) is the field strength 2 form associated with the gauge group \( G_\alpha \). Note that both \( X_4 \) and \( \tilde{X}_4 \) are closed. Locally we can write \( X_4 = d\Omega_3 \), where \( \Omega_3 \) is a Chern-Simons three form. It is not gauge invariant. Since \( X_4 \) is gauge invariant, under a gauge transformation, we need to have \( \Omega_3 \rightarrow \Omega_3 + do_\alpha \). The 2 -form \( \alpha \) can be constructed explicitly [9]. Up to total derivatives, the gauge and gravitational chiral anomaly is proportional to the integral of the 6 form \( \alpha . \tilde{X}_4 \) over the Euclideanised six-dimensional space time.

The GS mechanism requires the addition of a local counter term of the form \( B . \tilde{X}_4 \) to the effective Lagrangian. If we demand that the two form potential \( B \) undergoes a gauge transformation of the form \( B \rightarrow B - \alpha \) then the one loop effective action will be gauge
invariant. A gauge invariant field strength associated with the 2 form potential $B$ should thus be defined by $H = dB + \Omega_3$.

All this is very similar to the application of the Green-Schwarz mechanism to the anomaly problem in $D = 10$ heterotic string theory. In that case the anomaly polynomial is a 12 form which factorises as $P_{12} = X_4 \cdot X_8$, where $X_4$ and $X_8$ are closed forms and furthermore $X_4 = \frac{1}{30} Tr F^2 - tr R^2$. Here $Tr$ and $tr$ refer, respectively, to the adjoint and the fundamental representations.

The modified definition of $H$ has very interesting consequences for $K_3$ compactifications. First note that

$$dH = tr R^2 - \Sigma \nu_\alpha tr F^2_\alpha$$

Integrate this expression on $K_3$. If $dH$ has no $\delta$-function type singularities on $K_3$ its integral will vanish. We then obtain $24 = \int \Sigma \nu_\alpha tr F^2_\alpha$, where 24 is the Euler number of $K_3$. The integral on the right-hand side of this expression is the Chern number (instanton number) associated with the background gauge field configuration. For the $E_8 \times E_8$ heterotic string it will equal to the sum of instanton numbers embedded in each $E_8$ factor. We thus obtain $24 = n_1 + n_2$.

If $dH$ has singularities they can be interpreted as 5 branes and the result of integration becomes $24 = n_1 + n_2 + n_5$ for the $E_8 \times E_8$ and $24 = n + n_5$ for the $SO(32)$ heterotic strings, where $n_5$ indicates the number of 5 branes.

In the above discussion we considered only models with one tensor multiplet. A more general framework has been developed in [10] for models with $k > 1$

In addition to anomalies in local symmetries there is also the possibility of global anomalies in the supergravity and super Yang-Mills theories in $D = 6$. Such anomalies exist in $D = 4$ theories if the fourth homotopy group, $\Pi_4$, of the gauge group is non trivial [11]. If $\Pi_6$ of the gauge group is non trivial, a super Yang-Mills theory in $D = 6$ can be inconsistent due to the presence of global gauge anomalies. This happens for the groups $SU(2)$, $SU(3)$ and $G_2$. The requirement of the absence of such anomalies imposes further restrictions on the structure of the consistent six dimensional models. We shall return to
this issue in section 4.

Very interesting physics can be extracted by studying the moduli space of scalars. We saw above that there are two types of scalars in the spectrum of \((1,0)\) models, namely, those in the hypermultiplets and those in the tensor multiplets. Seiberg and Witten parametrise the Coulomb branch of the \((1,0)\) theories by the expectation values of the scalars in the tensor multiplets [12]. The reason for naming the tensor moduli space as the Coulomb branch is that upon compactification to lower dimensions the tensor multiplets become vector multiplets of the lower dimensional theories.

The metric in the hypermultiplet moduli space is independent of the scalars in the tensor multiplets. Likewise the metric in the Coulomb branch and the kinetic energy terms for the vector and the tensor fields are independent of the scalars in the hypermultiplets. In the infrared region, where the physics is described by classical field theory, there is no supersymmetric coupling which could lead to masses for massless particles by wondering in the moduli space of scalars in the tensor multiplets. Conversely, it is impossible for massive particles to become massless at a particular point on the Coulomb branch, by any mechanism that can be described at low energies by a free field theory. Seiberg and Witten then conclude that the singularities in the Coulomb branch necessarily involve non infrared free physics which is associated with non-critical tensionless strings, where non-critical means that gravity is not a mode of the string. Thus the non trivial infrared physics associated with the Coulomb branch singularities is occurring in flat six dimensional Minkowski space time. In the next section we shall study a flat space model which exhibits some of these features.

3. Super Yang-Mills coupled to tensors in \(D=6\) and non critical Strings

As it was argued in the preceding section the singularities in the Coulomb branch of \((1,0)\) models are conjectured to be related to non critical strings which do not couple to gravity and therefore they can be studied in a flat Minkowski space time. Supersymmetric models in flat \(D = 6\) which involve only the Yang-Mills and the tensor multiplets have been constructed in [13]. In addition to the argument given at the end of the last section, the
introduction of tensor multiplets is called for also by the requirement of anomaly cancellation, because, it is the presence of such tensor fields in six dimensions which enable us to make use of the Green-Schwarz anomaly cancellation mechanism. The tensor multiplet by itself does not have nonsingular solitonic solutions. This is one more reason for considering the tensor multiplet coupled to Yang-Mills fields. *

The supersymmetry transformations of the fields in our system are as follows [13].

\[
\begin{align*}
\delta A_a &= -\bar{\epsilon}_i \Gamma_a \lambda^i \\
\delta \lambda^i &= \frac{1}{8} \Gamma^{ab} F_{ab} \epsilon^i - \frac{1}{2} Y^{ij} \epsilon_j \\
\delta Y^{ij} &= -\bar{\epsilon} (i \Gamma^a D_a \lambda^j)
\end{align*}
\]  

(1)

where the index \(i\) is a doublet index of the automorphism \(Sp(1)\). The corresponding rules for the tensor multiplet coupled to Yang–Mills are given by

\[
\begin{align*}
\delta \sigma &= \bar{\epsilon} \chi \\
\delta \chi^i &= \frac{1}{48} \Gamma^{abc} H_{abc}^+ \epsilon^i + \frac{1}{4} \Gamma^a \partial_a \sigma \epsilon^i - \frac{\alpha'}{4} \text{Tr} \Gamma^a \lambda^i \bar{\epsilon} \Gamma_a \lambda \\
\delta B_{ab} &= -\bar{\epsilon} \Gamma_{ab} \chi - \alpha' \text{Tr} A_{[a} \bar{\epsilon} \Gamma_{b]} \lambda
\end{align*}
\]  

(2)

where

\[
\begin{align*}
H_{abc} &= 3 \partial_{[a} B_{bc]} + 3 \alpha' \text{Tr} (A_{[a} \partial_b A_{c]} + \frac{2}{3} A_a A_b A_c) \\
H_{abc}^\pm &= \frac{1}{2} \left( H_{abc} \pm \bar{H}_{abc} \right).
\end{align*}
\]  

(3)

The closure of the supersymmetry algebra leads to the following field equations for various fields,

\[
\begin{align*}
H_{abc} &= -\frac{\alpha'}{2} \text{Tr} (\bar{\lambda} \Gamma_{abc} \lambda) \\
\Gamma^a \partial_a \chi^i &= \alpha' \text{Tr} \left( \frac{1}{4} \Gamma^{ab} F_{ab} \lambda^i + Y^{ij} \lambda_j \right) \\
\partial^2 \sigma &= \alpha' \text{Tr} \left( -\frac{1}{4} F^{ab} F_{ab} - 2 \bar{\lambda} \Gamma^a D_a \lambda + Y^{ij} Y_{ij} \right).
\end{align*}
\]  

(4a)  (4b)  (4c)

* This section and section 4 follow closely ref.[14]. Similar results have been obtained from the flat space limit of a \(D = 6\) supergravity model in [15].
Further, by virtue of its definition, $H_{abc}$ satisfies the identity

$$\partial_{[a}H_{bcd]}^+ = \alpha' \text{tr} \left( \frac{3}{4} F_{[ab}F_{cd]} - \bar{\lambda} \Gamma_{[abc}D_{d]} \lambda \right)$$  \hspace{1cm} (5)$$

We shall look for a bosonic background configuration in which all the fermions as well as the auxiliary field $Y_{ij}$ will vanish. The six-dimensional coordinates will be chosen as $x^+$, $x^-$ and $x^\mu$ where $\mu = 1, \ldots, 4$. We shall consider a multi instanton-type configuration in the $\mathbb{R}^4$ spanned by $x^\mu$. It will be shown that that the moduli of this instanton can depend on $x^+$. This will require that the $A_+\text{-component}$ of the vector potential is different from zero. In this sense the solution looks like a static monopole configuration in the six-dimensional spacetime in which $x^-$ is taken to be the time coordinate. This configuration will preserve half the six-dimensional $N = 1$ supersymmetry.

It follows from (4a) that if $\lambda = 0$, then $H$ is self dual.* Now setting $\delta \lambda$ and $\delta \chi$ equal to zero we obtain

$$\Gamma^{ab}F_{ab}\epsilon = 0, \quad \left( \Gamma^a \partial_a \sigma + \frac{1}{12} \Gamma^{abc}H_{abc} \right) \epsilon = 0$$  \hspace{1cm} (6)$$

To satisfy these equations, we can choose $\epsilon = \left( \begin{smallmatrix} \epsilon \\ 0 \end{smallmatrix} \right)$, where, $\gamma_5 \epsilon = \pm \epsilon$, and $\gamma_5$ gives the four-dimensional chirality. We shall first discuss the case of positive chirality; the case of negative chirality can be obtained by essentially trivial change of some self duality conditions. With this choice the fields must obey the equations

$$H_{05\mu} = -\partial_\mu \sigma$$  \hspace{1cm} (7a)$$
$$H_{0\mu\nu} = \tilde{H}_{0\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta}H_{0\alpha\beta}$$  \hspace{1cm} (7b)$$
$$F_{\mu\nu} = \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$  \hspace{1cm} (7c)$$

together with $F_{+\mu} = 0$, $F_{-\mu} = 0$, $\partial_- \sigma = 0$. Choosing the gauge $A_- = 0$, these reduce to $\partial_- A_\mu = \partial_- A_+ = \partial_- \sigma = 0$.

* We shall henceforth drop the superscript $+$ from $H$. 

10
The constraint (5) for $H_{abc}$, expressing its coupling to the Yang-Mills fields via the Chern-Simons 3-form, now gives the following conditions,

$$\partial_- H_{+\mu\nu} = 0$$

$$\partial_\lambda H_{+\lambda\alpha} = \partial_+ \partial_\alpha \sigma - 2c \text{Tr}(F_{\lambda\alpha} F_{+\lambda})$$

$$\partial_\mu \partial_\mu \sigma = -\frac{c}{2} \text{Tr}(F_{\mu\nu} F_{\mu\nu})$$

where $c = 3\alpha'/4$. Further, $H_{-\mu\nu} = 0$ and $H_{+\mu\nu} = \tilde{H}_{+\mu\nu}$. Setting the auxiliary field $Y^{ij}$ to zero implies $D_a F^{ab} = 0$ [13]. The only nontrivial surviving component of this equation is

$$D_\lambda (D_\lambda A_+ - \partial_+ A_\lambda) = 0$$

where $D_\lambda A_+ = \partial_\lambda A_+ + [A_\lambda, A_+]$.

The strategy for solving these equations is as follows. We first choose $F_{\mu\nu}$ to be a multi-instanton configuration in $\mathbb{R}^4$. Then equation (8c) gives $\sigma$, and (7a) gives $H_{05\mu}$.

Since $D_\lambda D_\lambda$ is invertible in the instanton background, (9) can be uniquely solved for $A_+$. Finally, equation (8b) can be solved, consistently with its self-duality, to get $H_{+\mu\nu}$. As a consequence of $\partial_- A_\mu = \partial_- A_+ = 0$, the instanton parameters, collectively denoted by $\xi$, obey the condition $\partial_- \xi = 0$, but they can, of course, depend on $x^+$. (They are thus left-moving modes in the $(x^0, x^5)$-subspace.)

Using the self-duality of $H_{+\mu\nu}$, we can rewrite (8b) as

$$\partial_\lambda \partial_\lambda H_{+\mu\nu} = (\partial_\mu J_\nu - \partial_\nu J_\mu) + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial_\alpha J_\beta - \partial_\beta J_\alpha)$$

where $J_\alpha = \partial_+ (\partial_\alpha \sigma) - 2c \text{Tr}(F_{\lambda\alpha} F_{+\lambda})$. It is easy to see that $\partial_+ J_\alpha = 0$, as required by the consistency of the equations. Since the four-dimensional Laplacian is invertible, the above equation can easily be solved, once we have $J_\alpha$. For gauge group $SU(2)$, $A_+$ is given by

$$A^a_+ = \int d^4y \Delta^{ab}(x, y) \epsilon^{bkl}(A^k_\lambda \partial_+ A^l_\lambda)(y)$$

where the Green’s function $\Delta^{ab}(x, y)$ for $D_\lambda D_\lambda$ in the instanton background is given in reference [16]. To make the above solutions explicit, we can, for example, take the ’t Hooft...
ansatz for instantons, viz., \( A_\mu^a = \bar{\eta}^a_\mu \partial_\nu (\log \phi) \) where \( \phi = 1 + \sum_{i=1}^N \rho_i^2 / (x - a_i)^2 \) and insert it in various equations above. In this case, \( \sigma \), for example, becomes \( 2c \partial_\mu \phi \partial_\mu \phi / \phi^2 \).

4. String Interpretation.

To see the stringy interpretation of our solution, we need to analyze its moduli or zero mode structure. From the above equations, we see that, given the gauge field \( F_{\mu \nu} \), all the fields are uniquely determined up to the addition of the freely propagating six-dimensional waves for the tensor multiplet. * Therefore the only zero modes correspond to the moduli of the instantons.

In order for our models to be mathematically meaningful they should be free from local and global gauge anomalies. In the absence of hypermatter, the gauge groups \( SU(2), SU(3), G_2, F_4, E_6, E_7 \) and \( E_8 \) can be made perturbatively anomaly-free with the help of the Green-Schwarz prescription. However, since the homotopy group \( \Pi_6 \) of the first three groups in this list are nontrivial, these theories will harbour global gauge anomalies. To make them consistent we need to introduce hypermatter for these theories[17]. The allowed matter contents for the cancellation of the global [17] as well as the local [18] anomalies in the presence of one tensor multiplet are \( n_2 = 4, 10 \) for \( SU(2) \), \( n_3 = 0, 6, 12 \) for \( SU(3) \) and \( n_7 = 1, 4, 7 \) for \( G_2 \), where \( n_2, n_3 \) and \( n_7 \) represent the number of the doublets for \( SU(2) \), triplets for \( SU(3) \) and 7-dimensional representation of \( G_7 \), respectively. All other gauge groups are free from global anomalies and they can be made free from perturbative anomalies (using the Green-Schwarz prescription) if an appropriate amount of hypermultiplets are taken together with the gauge and the tensor multiplets [17, 18].

For the gauge group \( SU(2) \), for the four-dimensional space being \( \mathbb{R}^4 \) and for instanton number \( k \), we have \( 8k \) bosonic moduli corresponding to the instanton positions, scale sizes and group orientations. (The equations of motion, despite the appearance of the dimensional parameter \( c \), have scale invariance and give the scale size parameter in the solutions.) These moduli appear in the solution for the fields \( B_{ab} \) as well.

The surviving supersymmetry has \( \gamma_5 \varepsilon = \varepsilon \), i.e., left-chirality in the four-dimensional

* Note the soliton does not modify their propagation.
sense corresponding to a \((4,0)\) world-sheet supersymmetry for the solitonic string. There must necessarily be fermionic zero modes. For the gauginos, we have \(4k\) zero modes for the gauge group \(SU(2)\), which are of right-chirality in the four-dimensional sense and are in the right-moving sector. The Dirac equation for the gauginos along with the half-supersymmetry condition shows that the gaugino zero mode parameters are constants; the bosonic parameters are constant as well, by supersymmetry. The fermionic zero mode parameters are complex, i.e., we have \(8k\) real Grassman parameters which balance the \(8k\) bosonic parameters. Some of the fermionic zero modes correspond to the supersymmetries which are broken by the background and can be obtained by such supersymmetry variations. With hypermatter, there are also hyperino zero modes, which are in the left-moving sector. There is no supersymmetry for these modes and generically there are no hyperscalar zero modes.

For higher gauge groups, there will be more moduli. Thus, for example, for \(SU(3)\), with the standard embedding of the instanton and \(n_3 = 0\), we have \(12k\) bosonic parameters and \(6k\) fermionic parameters. It is easy to see that the number of moduli for all of the anomaly-free gauge groups listed above is always a multiple of 4. We may thus interpret these solutions as six-dimensional strings with 4 transverse coordinates corresponding to the zero modes for the broken translational symmetries. The remaining zero modes can be regarded as additional world-sheet degrees of freedom. In this way for instanton number \(k\), we have \(k\) strings with \((4,0)\) world-sheet supersymmetry.

As an example, consider an \(SU(2)\) theory with 10 hypermatter doublets [19]. In this case, for instanton number equal to one, we have eight instanton moduli, eight gaugino zero modes for the right-moving sector and 20 hypermatter zero modes for the left-moving sector. The \(SU(2)\) symmetry can be spontaneously broken by vacuum expectation values of the scalars originating from the moduli corresponding to the global \(SU(2)\) rotations and the scale size of the instanton. By supersymmetry this should remove four of the gaugino zero modes from the right moving sector by giving them a non zero mass, which will also eat up four hyperino zero modes in the left moving sector. One is left with four moduli
for the instanton, four gaugino modes in the right-moving sector and 16 hyperino zero modes in the left-moving sector. These 16 hyperino zero modes presumably generate a left moving $E_8$ current algebra. This looks like the spectrum of the non critical string which lives in the boundary of a membrane joining a 5-brane to a 9-brane in $M$-theory and which becomes tensionless as the 5-brane approaches the 9-brane \[20\]. It has been argued in \[21\] that the same model corresponds to one of the phases of the $F$-theory.

There are also independent solutions with the opposite chirality. The choice $\gamma_5 \epsilon = -\epsilon$ leads to anti-self dual $H_{\mu\nu}$, $F_{\mu\nu}$ with $A_+ = 0$ and $\partial_+ \xi = 0$.

The solution we have obtained is a static one. The choice of four-dimensional chirality as $\gamma_5 \epsilon = \pm \epsilon$ leads to static solitons. By Lorentz boosts, it is possible to obtain a solution whose center of mass is moving at a constant velocity. For a moving soliton, the condition $\gamma_5 \epsilon = \pm \epsilon$ must be modified. Consider, for example, the one-soliton (one-instanton) solution. We choose the supersymmetry parameters $\epsilon$ as $S\epsilon(0)$ where $S = \exp(-\frac{1}{2} \omega^\mu \gamma_\mu) \approx 1 - \frac{1}{2} \omega^\mu \gamma_\mu$ and $\epsilon(0)$ obeys $\gamma_5 \epsilon(0) = \epsilon(0)$. (For small velocities, the parameter $\omega^\mu \approx v^\mu$, the velocity.) The vanishing of the gaugino variation, viz., $\Gamma^{ab} F_{ab} \epsilon = 0$, now gives, to first order in $v^\mu$,

\[
\begin{align*}
F_{\mu\nu} - \tilde{F}_{\mu\nu} &= 0 \\
F_{-\mu} + \frac{1}{\sqrt{2}} F_{\mu\nu} v^\nu &= 0 \\
F_{+-} - \frac{1}{\sqrt{2}} F_{+\nu} v^\nu &= 0
\end{align*}
\]

To this order, $F_{\mu\nu}$ is still self dual. The other two equations are seen to be satisfied if we take the instanton position $a^\alpha$ to move with velocity $v^\alpha$, i.e., $\partial_0 a^\alpha = v^\alpha$. (We can make a gauge transformation $A_- \to A_- - (1/\sqrt{2}) A_\mu v^\mu$ to restore the $A_- = 0$ gauge.) There is a similar set of statements for the vanishing of the tensorino variation. What we have shown is that a soliton whose center of mass is moving at a constant velocity $v^\alpha$ is also a supersymmetric solution with supersymmetry parameters being $S\epsilon(0)$, $\epsilon(0)$ having definite four-dimensional chirality.
5. $D = 6$ Models with Extended Supersymmetries.

In the foregoing sections we discussed only the $D = 6$ models with a minimum number of supersymmetries. Apart from the $(1, 0)$ type supersymmetry in $D = 6$ there are also models with $(1, 1)$, $(2, 0)$ and $(2, 2)$ type supersymmetries. The number of real components of the supercharges are respectively 16, 16 and 32. Out of these three types only the $(2, 0)$ models are chiral and therefore can be anomalous. Like the four dimensional theories, if we do not want to have a physical field of spin larger than 2, then the total number of real supercharges should not exceed 32. This is the number of supersymmetries of the $D = 11$ supergravity which is conjectured to be the low energy limit of a unifying theory of all known $D = 10$ string theories and is called the M theory. When we obtain a lower dimensional theory from the $D = 10$ or $D = 11$ some of the supersymmetries can be broken. For example the $K_3$ compcatification which takes us from $D = 10$ to $D = 6$ breaks $1/2$ of supersymmetries. Thus starting from the type IIB theory, which has 32 real chiral supersymmetries in $D = 10$, and compactifying on a $K_3$ we obtain a $D = 6$ theory with 16 chiral supersymmetries. This is an example of a $(2, 0)$ model with 21 tensor multiplets [22]. It is exactly 21 tensor multiplets which is required by the anomaly cancellation in $D = 6$. Although an invariant Lagrangian has not yet been constructed for these models, the field equations with an arbitrary number of tensor multiplets have been known for some time [23].

The $(2, 0)$ models involve self dual and anti-self dual tensor fields. One can then contemplate self dual or anti-self dual string like solutions of the type discussed in the previous section. Presumably these strings are also tensionless. An intuitive way of understanding this is to remember that in $D = 10$ the IIB theory has a four form potential whose field strength is self dual. There are also self dual 3-brane solutions [24]. One can imagine that a self dual brane wraps around a 2 cycle of the $K_3$ to produce an object which will look like a string from the $D = 6$ point of view. In general this string will have some thickness. But as the area of the cycle shrinks to zero the thickness also will decrease. Furthermore, the string will be self dual by construction and its tension will be proportional to the area.
of the 2 cycle and hence will vanish as the area goes to zero.

The type IIA theory has the same number of supersymmetries in $D = 10$ as the type IIB, however it is a non chiral theory. For that reason upon compactification from $D = 10$ to $D = 6$ on a $K_3$ one obtains a non chiral theory in $D = 6$ with 16 real supercharges which generate a $(1, 1)$ supersymmetry. The same type of model can be obtained from the compactification of the heterotic strings on a $T^4$. There exist many compelling evidence that the theories obtained from the type IIA compactifications on a $K_3$ and the $E_8 \times E_8$ compactifications on $T^4$ are dual in the sense that the strong coupling limit of one can be set in correspondence with the weak coupling limit of the other [25]. At a first glance this looks puzzling, because, although at a generic point on the moduli space of the heterotic compactification the six dimensional gauge group is $U(1)^{24}$ it is known from Narain’s work that at some special points the gauge symmetry can be enlarged to a non Abelian group [26]. For the duality to work one needs to find mechanisms for the generation of non Abelian gauge symmetries on the type IIA side. The possibility which has been suggested [27] is that the 2-branes of the type II can wrap around 2 cycles of $K_3$ and produce, in the limit that the area of the cycles shrink to zero, particle like objects in $D = 6$. The masses of these particles will be proportional to the area of the 2 cycles and will vanish in the limit of the vanishing cycles. These massless particles should then match with the massless particles generated at the special points in the heterotic moduli space at which the gauge symmetries are enhanced.

Finally the compactifications of type IIA or the type IIB theories on a four dimensional torus will produce non chiral $D = 6$ supergravity models with $(2, 2)$ supersymmetries.

An important role in the study of these models is played by the moduli space of vacua, i.e. the expectation values of the massless scalars. It has been conjectured that all known compactifications with $(1, 1)$, $(0, 2)$ and $(2, 2)$ supersymmetries belong to the same moduli space of vacua.

For example, we mentioned above that the heterotic string on $T^4$ is dual to the type IIA compactification on $K_3$. It is also known that if we compactify type IIA on a $S^1$ of
radius $R$ from $D = 10$ to $D = 9$ then it gives the same theory as the one obtained from the compactification of type IIB from $D = 10$ to $D = 9$ on a circle of radius $1/R$. It then follows that the compactification of the heterotic string on a $T^5$ should produce a theory dual to the compactification of the type IIB on $S^1 \times K_3$. This duality is a strong weak duality. On the heterotic side the string coupling is given by $e^{\phi}$, where $\phi$ denotes the vev of the dilaton field. On the type IIB side this modulus corresponds to the radius of $S^1$. This implies that the strong coupling limit of the heterotic string in $D = 5$ is dual to the large radius limit of the type IIB string in $D = 5$. The large radius limit is, of course the same as the decompactification limit which takes us back to $D = 6$ space time again. Thus by moving around in the moduli space of type IIB or heterotic in $D = 5$ we can end up with the $(2,0)$ type IIB in $D = 6$.

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