LOCALIZATION OF \( u \)-MODULES. II.
CONFIGURATION SPACES AND QUANTUM GROUPS

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1. Introduction

1.1. This paper is a sequel to \([\text{FS}]\). We are starting here the geometric study of the tensor category \( \mathcal{C} \) associated with a quantum group (corresponding to a Cartan matrix of finite type) at a root of unity (see \([\text{AJS}]\), 1.3 and the present paper, 11.3 for the precise definitions).

The main results of this paper are Theorems 8.18, 8.23, 12.7 and 12.8 which
— establish isomorphisms between homogeneous components of irreducible objects in \( \mathcal{C} \) and spaces of vanishing cycles at the origin of certain Goresky-MacPherson sheaves on configuration spaces;
— establish isomorphisms of the stalks at the origin of the above GM sheaves with certain Hochschild complexes (which compute the Hochschild homology of a certain "triangular" subalgebra of our quantum group with coefficients in the corresponding irreducible representation);
— establish the analogous results for tensor products of irreducibles. In geometry, the tensor product of representations corresponds to a "fusion" of sheaves on configuration spaces — operation defined using the functor of nearby cycles, see Section 10.

We must mention that the assumption that we are dealing with a Cartan matrix of finite type and a root of unity appears only at the very end (see Chapter 4). We need these assumptions in order to compare our representations with the conventional definition of the category \( \mathcal{C} \). All previous results are valid in more general assumptions. In particular a Cartan matrix could be arbitrary and a deformation parameter \( \zeta \) not necessarily a root of unity.

1.2. A part of the results of this paper constitutes the description of the cohomology of certain "standard" local systems over configuration spaces in terms of quantum groups. These results, due to Varchenko and one of us, were announced several years ago in \([\text{SV2}]\).
The proofs may be found in \cite{N}. Our proof of these results uses completely different approach. Some results close to this paper were discussed in \cite{S}.

Certain results of a similar geometric spirit are discussed in \cite{FW}.

1.3. We are grateful to A.Shen who made our communication during the writing of this paper possible.

1.4. Notations. We will use all the notations from \cite{FS}. References to loc. cit. will look like I.1.1. If $a, b$ are two integers, we will denote by $[a, b]$ the set of all integers $c$ such that $a \leq c \leq b$; $[1, a]$ will be denoted by $[a]$. $\mathbb{N}$ will denote the set of non-negative integers. For $r \in \mathbb{N}$, $\Sigma_r$ will denote the group of all bijections $[r] \overset{\sim}{\longrightarrow} [r]$.

We suppose that our ground field $B$ has characteristic 0, and fix an element $\zeta \in B$, $\zeta \neq 0$. For $a \in \mathbb{Z}$ we will use the notation

\[ [a]_\zeta = 1 - \zeta^{-2a} \quad (1) \]

The word ”$t$-exact” will always mean $t$-exactness with respect to the middle perversity.
CHAPTER 1. Algebraic discussion.

2. FREE ALGEBRAS AND BILINEAR FORMS

Most definitions of this section follow [1] and [SV2] (with slight modifications). We also add some new definitions and computations important for the sequel. Cf. also [V], Section 4.

2.1. Untill the end of this paper, let us fix a finite set $I$ and a symmetric $\mathbb{Z}$-valued bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ (cf. [1], 1.1). We will denote by $X$ the dual abelian group $\text{Hom}(\mathbb{Z}[I], \mathbb{Z})$. Its elements will be called weights. Given $\nu \in \mathbb{Z}[I]$, we will denote by $\lambda_\nu \in X$ the functional $i \mapsto i \cdot \nu$. Thus we have

$$\langle \lambda_\nu, \mu \rangle = \nu \cdot \mu$$

for all $\nu, \mu \in \mathbb{N}[I]$.

2.2. Let $F$ denote a free associative $B$-algebra with 1 with generators $\theta_i$, $i \in I$. Let $\mathbb{N}[I]$ be a submonoid of $\mathbb{Z}[I]$ consisting of all linear combinations of elements of $I$ with coefficients in $\mathbb{N}$. For $\nu = \sum \nu_i i \in \mathbb{N}[I]$ we denote by $F_\nu$ the $B$-subspace of $F$ spanned by all monomials $\theta_i_1 \theta_i_2 \cdots \theta_i_p$ such that for any $i \in I$, the number of occurrences of $i$ in the sequence $i_1, \ldots, i_p$ is equal to $\nu_i$.

We have a direct sum decomposition $F = \bigoplus_{\nu \in \mathbb{N}[I]} F_\nu$, all spaces $F_\nu$ are finite dimensional, and we have $F_0 = B \cdot 1$, $F_\nu \cdot F_{\nu'} \subset F_{\nu + \nu'}$.

Let $\epsilon : F \longrightarrow B$ denote the augmentation — a unique $B$-algebra map such that $\epsilon(1) = 1$ and $\epsilon(\theta_i) = 0$ for all $i$. Set $F^+ := \text{Ker}(\epsilon)$. We have $F^+ = \bigoplus_{\nu \neq 0} F_\nu$.

An element $x \in F$ is called homogeneous if it belongs to $F_\nu$ for some $\nu$. We then set $|x| = \nu$. We will use the notation $\text{depth}(x)$ for the number $\sum_i \nu_i$ if $\nu = \sum_i \nu_i$; it will be called the depth of $x$.

2.3. Given a sequence $\vec{K} = (i_1, \ldots, i_N)$, $i_j \in I$, let us denote by $\theta_{\vec{K}}$ the monomial $\theta_{i_1} \cdots \theta_{i_N}$. For an empty sequence we set $\theta_{\emptyset} = 1$.

For $\tau \in \Sigma_N$ let us introduce the number

$$\zeta(\vec{K}; \tau) = \prod a^{i_a - i_b},$$

the product over all $a, b$ such that $1 \leq a < b \leq N$ and $\tau(a) > \tau(b)$.

We will call this number the twisting number of the sequence $\vec{K}$ with respect to the permutation $\tau$. 
We will use the notation
\[ \tau(\vec{K}) = (i_{\tau(1)}, i_{\tau(2)}, \ldots, i_{\tau(N)}) \] (4)

2.4. Let us regard the tensor product \( \mathfrak{F} \otimes \mathfrak{F} \) (in the sequel \( \otimes \) will mean \( \otimes_B \) unless specified otherwise) as a \( B \)-algebra with multiplication
\[ (x_1 \otimes x_2) \cdot (x'_1 \otimes x'_2) = \zeta^{(|x_2|,|x'_2|)} x_1 x'_1 \otimes x_2 x'_2 \] (5)
for homogeneous \( x_2, x'_1 \). Let us define a map
\[ \Delta : \mathfrak{F} \longrightarrow \mathfrak{F} \otimes \mathfrak{F} \] (6)
as a unique algebra homomorphism carrying \( \theta_i \) to \( \theta_i \otimes 1 + 1 \otimes \theta_i \).

2.5. Let us define a coalgebra structure on \( \mathfrak{F} \otimes \mathfrak{F} \) as follows. Let us introduce the braiding isomorphism
\[ r : \mathfrak{F} \otimes \mathfrak{F} \isom \mathfrak{F} \otimes \mathfrak{F} \] (7)
by the rule
\[ r(x \otimes y) = \zeta^{(|x|,|y|)} y \otimes x \] (8)
for homogeneous \( x, y \). By definition,
\[ \Delta_{\mathfrak{F} \otimes \mathfrak{F}} : \mathfrak{F} \otimes \mathfrak{F} \longrightarrow (\mathfrak{F} \otimes \mathfrak{F}) \otimes (\mathfrak{F} \otimes \mathfrak{F}) \] (9)
coincides with the composition \( (1_{\mathfrak{F}} \otimes r \otimes 1_{\mathfrak{F}}) \circ (\Delta_{\mathfrak{F}} \otimes \Delta_{\mathfrak{F}}) \).
The multiplication
\[ \mathfrak{F} \otimes \mathfrak{F} \longrightarrow \mathfrak{F} \] (10)
is a coalgebra morphism.

2.6. Let us describe \( \Delta \) more explicitely. Suppose a sequence \( \vec{K} = (i_1, \ldots, i_N), \) \( i_j \in I, \) is given. For a subset \( A = \{j_1, \ldots, j_a\} \subset [N], \) \( j_1 < \ldots < j_a, \) let \( A' = [N] - A = \{k_1, \ldots, k_{N-a}\}, \) \( k_1 < \ldots < k_{N-a}. \) Define a permutation \( \tau_A \) by the formula
\[ (\tau(1), \ldots, \tau(N)) = (j_1, j_2, \ldots, j_a, k_1, k_2, \ldots, k_{N-a}) \] (11)
Set \( \vec{K}_A := (i_{j_1}, i_{j_2}, \ldots, i_{j_a}), \) \( \vec{K}_{A'} := (i_{k_1}, i_{k_2}, \ldots, i_{k_{N-a}}). \)

2.6.1. **Lemma.**
\[ \Delta(\theta_{\vec{K}}) = \sum_{A \subset [N]} \zeta(\vec{K}; \tau_A) \theta_{\vec{K}_A} \otimes \theta_{\vec{K}_{A'}}, \]
the summation ranging over all subsets \( A \subset [N]. \)

**Proof** follows immediately from the definitions. \( \square \)
2.7. Let us denote by
\[ \Delta^{(N)} : \mathcal{F} \to \mathcal{F}^{\otimes N} \] (12)
iterated coproducts; by the coassociativity they are well defined.

Let us define a structure of an algebra on \( \mathcal{F}^{\otimes N} \) as follows:
\[ (x_1 \otimes \ldots \otimes x_N) \cdot (y_1 \otimes \ldots \otimes y_N) = \zeta \sum_{j < i} |x_i| |y_j| x_1 y_1 \otimes \ldots \otimes x_N y_N \] (13)
for homogeneous \( x_1, \ldots, x_N; y_1, \ldots, y_N \). The map \( \Delta^{(N)} \) is an algebra morphism.

2.8. Suppose we have a sequence \( \vec{K} = (i_1, \ldots, i_N) \). Let us consider an element \( \Delta^{(N)}(\theta_{\vec{K}}) \); let \( \Delta^{(N)}(\theta_{\vec{K}})^+ \) denote its projection to the subspace \( \mathcal{F}^{+\otimes N} \).

2.8.1. **Lemma.**
\[ \Delta^{(N)}(\theta_{\vec{K}})^+ = \sum_{\tau \in \Sigma_N} \zeta(\vec{K}; \tau) \theta_{i_\tau(1)} \otimes \ldots \otimes \theta_{i_\tau(N)} \]

**Proof** follows from 2.6.1 by induction on \( N \). \( \square \)

2.9. For each component \( \mathcal{F}_\nu \) consider the dual \( B \)-space \( \mathcal{F}_\nu^* \), and set \( \mathcal{F}^* := \bigoplus \mathcal{F}_\nu^* \). Graded components \( \Delta_{\nu,\nu'} : \mathcal{F}_{\nu+\nu'} \to \mathcal{F}_\nu \otimes \mathcal{F}_{\nu'} \) define dual maps \( \mathcal{F}_\nu^* \otimes \mathcal{F}_{\nu'}^* \to \mathcal{F}_{\nu+\nu'}^* \) which give rise to a multiplication
\[ \mathcal{F}^* \otimes \mathcal{F}^* \to \mathcal{F}^* \] (14)
making \( \mathcal{F}^* \) a graded associative algebra with 1 (dual to the augmentation of \( \mathcal{F} \)). This follows from the coassociativity of \( \Delta \), cf. [L], 1.2.2.

Here and in the sequel, we will use identifications \( (V \otimes W)^* = V^* \otimes W^* \) (for finite dimensional spaces \( V, W \)) by the rule \( \langle \phi \otimes \psi, x \otimes y \rangle = \langle \phi, x \rangle \cdot \langle \psi, y \rangle \).

The dual to (14) defines a comultiplication
\[ \delta : \mathcal{F}^* \to \mathcal{F}^* \otimes \mathcal{F}^* \] (15)
It makes \( \mathcal{F}^* \) a graded coassociative coalgebra with a counit.

The constructions dual to 2.4 and 2.5 equip \( \mathcal{F}^* \otimes \mathcal{F}^* \) with a structure of a coalgebra and an algebra. It follows from loc. cit that (14) is a coalgebra morphism, and \( \delta \) is an algebra morphism.

By iterating \( \delta \) we get maps
\[ \delta^{(N)} : \mathcal{F}^* \to \mathcal{F}^{*\otimes N} \] (16)
If we regard \( \mathcal{F}^{*\otimes N} \) as an algebra by the same construction as in (13), \( \delta^{(N)} \) is an algebra morphism.
2.10. **Lemma.** There exists a unique bilinear form
\[ S(\ ,
\ ) : \mathfrak{F} \otimes \mathfrak{F} \rightarrow \mathfrak{B} \]
such that
(a) \( S(1, 1) = 1 \) and \( (\theta_i, \theta_j) = \delta_{i,j} \) for all \( i, j \in I \);
(b) \( S(x, y'y'') = S(\Delta(x), y' \otimes y'') \) for all \( x, y', y'' \in \mathfrak{F} \);
(c) \( S(xx', y'') = S(x \otimes x', \Delta(y'')) \) for all \( x, x', y'' \in \mathfrak{F} \).

(The bilinear form \( (\mathfrak{F} \otimes \mathfrak{F}) \otimes (\mathfrak{F} \otimes \mathfrak{F}) \rightarrow \mathfrak{B} \) given by
\[ (x_1 \otimes x_2) \otimes (y_1 \otimes y_2) \mapsto S(x_1, y_1)S(x_2, y_2) \]
is denoted again by \( S(\ ,
\ )\).

The bilinear form \( S(\ ,
\ ) \) on \( \mathfrak{F} \) is symmetric. The different homogeneous components \( \mathfrak{F}_\nu \) are mutually orthogonal.

**Proof.** See [L], 1.2.3. Cf. also [SV2], (1.8)-(1.11). \( \square \)

2.11. Following [L], 1.2.13 and [SV2], (1.10)-(1.11), let us introduce operators \( \delta_i : \mathfrak{F} \rightarrow \mathfrak{F}, i \in I \), as unique linear mappings satisfying
\[ \delta_i(1) = 0; \delta_i(\theta_j) = \delta_{i,j}, j \in I; \delta_i(xy) = \delta_i(x)y + \zeta^{|x|}x\delta_i(y) \] (17)
for homogeneous \( x \).

It follows from 2.10 (c) that
\[ S(\theta_i x, y) = S(x, \delta_i(y)) \] (18)
for all \( i \in I, x, y \in \mathfrak{F} \), and obviously \( S \) is determined uniquely by this property, together with the requirement \( S(1, 1) = 1 \).

2.12. **Lemma.** For any two sequences \( \bar{K}, \bar{K}' \) of \( N \) elements from \( I \) we have
\[ S(\theta_{\bar{K}}, \theta_{\bar{K}'}) = \sum_{\tau \in \Sigma_N: \tau(\bar{K}) = \bar{K}'} \zeta(\bar{K}; \tau). \]

**Proof** follows from [IR] by induction on \( N \), or else from 2.8.1. \( \square \)

2.13. Let us define elements \( \theta_i^* \in \mathfrak{F}_i^* \) by the rule \( \langle \theta_i^*, \theta_i \rangle = 1 \). The form \( S \) defines a homomorphism of graded algebras
\[ S : \mathfrak{F} \rightarrow \mathfrak{F}^* \] (19)
carrying \( \theta_i \) to \( \theta_i^* \). \( S \) is determined uniquely by this property.
2.14. **Lemma.** The map $S$ is a morphism of coalgebras.

**Proof.** This follows from the symmetry of $S$. $\square$

**VERMA MODULES**

2.15. Let us pick a weight $\Lambda$. Our aim now will be to define certain $X$-graded vector space $V(\Lambda)$ equipped with the following structures.

(i) A structure of left $\mathfrak{g}$-module $\mathfrak{g} \otimes \mathfrak{b}(\Lambda) \rightarrow \mathfrak{b}(\Lambda)$;
(ii) a structure of left $\mathfrak{g}$-comodule $V(\Lambda) \rightarrow \mathfrak{g} \otimes \mathfrak{b}(\Lambda)$;
(iii) a symmetric bilinear form $S_\Lambda$ on $V(\Lambda)$.

As a vector space, we set $V(\Lambda) = \mathfrak{g}$. We will define on $V(\Lambda)$ two gradings. The first one, $\mathbb{N}[I]$-grading coincides with the grading on $\mathfrak{g}$. If $x \in V(\Lambda)$ is a homogeneous element, we will denote by depth$(x)$ its depth as an element of $\mathfrak{g}$.

The second grading — $X$-grading — is defined as follows. By definition, we set

$$V(\Lambda)_\lambda = \bigoplus_{\nu \in \mathbb{N}[I]} |\lambda - \lambda_\nu| \delta_\nu$$

for $\lambda \in X$. In particular, $V(\Lambda)_\Lambda = \mathfrak{g}_\lambda = \mathfrak{b} \cdot 1$. We will denote the element 1 in $V(\Lambda)$ by $v_\Lambda$.

By definition, multiplication

$$\mathfrak{g} \otimes \mathfrak{b}(\Lambda) \rightarrow \mathfrak{b}(\Lambda)$$

(20)

coincides with the multiplication in $\mathfrak{g}$.

Let us define an $X$-grading in $\mathfrak{g}$ by setting

$$\mathfrak{g}_\lambda = \bigoplus_{\nu \in \mathbb{N}[I]} \nu = \lambda \delta_\nu$$

for $\lambda \in X$. The map (20) is compatible with both $\mathbb{N}[I]$ and $X$-gradings (we define gradings on the tensor product as usually as a sum of gradings of factors).

2.16. **The form $S_\Lambda$.** Let us define linear operators $\epsilon_i : V(\Lambda) \rightarrow V(\Lambda), \ i \in I$, as unique operators such that $\epsilon_i(v_\Lambda) = 0$ and

$$\epsilon_i(\theta_j x) = [\langle \beta, i \rangle]_\zeta \delta_{i,j} x + \zeta^{i,j} \theta_j \epsilon_i(x)$$

(21)

for $j \in I, \ x \in V(\Lambda)_\beta$.

We define $S_\Lambda : V(\Lambda) \otimes V(\Lambda) \rightarrow B$ as a unique linear map such that $S_\Lambda(v_\Lambda, v_\Lambda) = 1$, and

$$S_\Lambda(\theta_i x, y) = S_\Lambda(x, \epsilon_i(y))$$

(22)

for all $x, y \in V(\Lambda), \ i \in I$. Let us list elementary properties of $S_\Lambda$. 
2.16.1. Different graded components $V(\Lambda)_\nu$, $\nu \in \mathbb{N}[\mathbb{I}]$, are orthogonal with respect to $S_\Lambda$. This follows directly from the definition.

2.16.2. The form $S_\Lambda$ is symmetric.
This is an immediate corollary of the formula

$$S_\Lambda(\epsilon_i(y), x) = S_\Lambda(y, \theta_i x)$$

which in turn is proved by an easy induction on depth($x$).

2.16.3. "Quasiclassical" limit. Let us consider restriction of our form to the homogeneous component $V(\Lambda)_\lambda$ of depth $N$. If we divide our form by $(\zeta - 1)^N$ and formally pass to the limit $\zeta \to 1$, we get the "Shapovalov" contravariant form as defined in [SV1], 6.4.1. The next lemma is similar to 2.12.

2.17. Lemma. For any $\vec{K}, \vec{K}'$ as in 2.12 we have

$$S_\Lambda(\theta_{\vec{K}} v_\Lambda, \theta_{\vec{K}'} v_\Lambda) = \sum_{\tau \in \Sigma_N: \tau(\vec{K}) = \vec{K}'} \zeta(\vec{K}; \tau) A(\vec{K}, \Lambda; \tau)$$

where

$$A(\vec{K}, \Lambda; \tau) = \prod_{a=1}^N \left[ (\Lambda - \sum_{b: b < a, \tau(b) < \tau(a)} \lambda_{b_i}, i_a) \right]_\zeta.$$

Proof. Induction on $N$, using definition of $S_\Lambda$. □

2.18. Coaction. Let us define a linear map

$$\Delta_\Lambda : V(\Lambda) \to \mathfrak{f} \otimes \mathfrak{v}(\Lambda)$$

as follows. Let us introduce linear operators $t_i : \mathfrak{f}^+ \otimes \mathfrak{v}(\Lambda) \to \mathfrak{f}^+ \otimes \mathfrak{v}(\Lambda)$, $i \in \mathfrak{I}$, by the formula

$$t_i(x \otimes y) = \theta_i x \otimes y - \zeta^{i,\nu-2(\lambda, i)} \cdot x \theta_i \otimes y + \zeta^{i,\nu} x \otimes \theta_i y$$

for $x \in \mathfrak{f}_\nu$ and $y \in V(\Lambda)_\lambda$.

By definition,

$$\Delta_\Lambda(\theta_{i_N} \cdot \ldots \cdot \theta_{i_1} v_\Lambda) = 1 \otimes \theta_{i_N} \cdot \ldots \cdot \theta_{i_1} v_\Lambda$$

$$+ \left[ (\Lambda - \lambda_{i_1} - \ldots - \lambda_{i_{N-1}}, i_N) \right]_\zeta \cdot \theta_{i_N} \otimes \theta_{i_{N-1}} \cdot \ldots \cdot \theta_{i_1} v_\Lambda$$

$$+ \sum_{j=1}^{N-1} \left[ (\Lambda - \lambda_{i_1} - \ldots - \lambda_{i_{j-1}}, i_j) \right]_\zeta \cdot t_{i_N} \circ t_{i_{N-1}} \circ \ldots \circ t_{i_{j+1}} (\theta_{i_j} \otimes \theta_{i_{j-1}} \cdot \ldots \cdot \theta_{i_1} v_\Lambda)$$
Let us define linear operators
\[ \text{ad}_{\theta,\lambda} : \mathfrak{g} \rightarrow \mathfrak{g}, \ i \in I, \ \lambda \in X \] (27)
by the formula
\[ \text{ad}_{\theta,\lambda}(x) = \theta_i x - \varsigma^{i\nu-2(\lambda,\nu)} \cdot x\theta_i \] (28)
for \( x \in \mathfrak{g}_\nu \).

Let us note the following relation
\[ (\delta_i \circ \text{ad}_{\theta,j,\lambda} - \varsigma^{ij} \cdot \text{ad}_{\theta,j,\lambda} \circ \delta_i)(x) = [\langle \lambda - \lambda_{\nu}, i \rangle] \varsigma \delta_{ij} x \] (29)
for \( x \in \mathfrak{g}_\nu \), where \( \delta_i \) are operators defined in 2.11, and \( \delta_{ij} \) the Kronecker symbol.

### 2.20. Formula for coaction.
Let us pick a sequence \( \vec{I} = (i_N, i_{N-1}, \ldots, i_1) \). To shorten the notations, we set
\[ \text{ad}_{j,\lambda} := \text{ad}_{\theta_{ij,\lambda}}, \ j = 1, \ldots, N \] (30)

### 2.20.1. Quantum commutators.
For any non-empty subset \( Q \subset [N] \), set \( \theta_{\vec{I},Q} := \theta_{\vec{I}_Q} \) where \( \vec{I}_Q \) denotes the sequence obtained from \( \vec{I} \) by omitting all entries \( i_j, \ j \in Q \). We will denote \( \mathfrak{g}_Q = \mathfrak{g}_{\nu_Q} \) where \( \nu_Q := \sum_{j \in Q} i_j \).

Let us define an element \( [\theta_{\vec{I},Q,\Lambda}] \in \mathfrak{g}_Q \) as follows. Set
\[ [\theta_{\vec{I},(j),\Lambda}] = \varsigma^{ij}(\sum_{k > j} \varsigma)_{ij} \theta_{ij} \] (31)
for all \( j \in [N] \).

Suppose now that \( \text{card}(Q) = l + 1 \geq 2 \). Let \( Q = \{ j_0, j_1, \ldots, j_l \}, \ j_0 < j_1 < \ldots < j_l \).
Define the weights
\[ \lambda_a = \Lambda - \lambda \sum_{k} i_k, \ a = 1, \ldots, l, \]
where the summation is over \( k \) from 1 to \( j_a - 1, k \neq j_1, j_2, \ldots, j_{a-1} \).

Let us define sequences \( \vec{N} := (N, N - 1, \ldots, 1), \vec{Q} = (j_l, j_{l-1}, \ldots, j_0) \) and \( \vec{N}_Q \) obtained from \( \vec{N} \) by omitting all entries \( j \in Q \). Define the permutation \( \tau_Q \in \Sigma_N \) by the requirement
\[ \tau_Q(\vec{N}) = \vec{Q} || \vec{N}_Q \]
where \( || \) denotes concatenation.

Set by definition
\[ [\theta_{\vec{I},Q,\Lambda}] = \varsigma^{i(\vec{I},\tau_Q)} \cdot \text{ad}_{j_l,\lambda_l} \circ \text{ad}_{j_{l-1},\lambda_{l-1}} \circ \ldots \circ \text{ad}_{j_1,\lambda_1}(\theta_{i_0}) \] (32)
2.20.2. Lemma. We have
\[
\Delta_\Lambda(\theta_I v_\Lambda) = 1 \otimes \theta_I v_\Lambda + \sum_Q[(\Lambda - \lambda_{i_1} - \lambda_{i_2} - \ldots - \lambda_{j(Q)-1}, i_{j(Q)})] \cdot \theta_{I,Q,\Lambda} \otimes \theta_{I,Q} v_\Lambda,
\]
the summation over all non-empty subsets \( Q \subset [N] \), \( j(Q) \) denotes the minimal element of \( Q \).

Proof. The statement of the lemma follows at once from the inspection of definition (20), after rearranging the summands. \(\square\)

Several remarks are in order.

2.20.3. Formula (33) as similar to [S], 2.5.4.

2.20.4. If all elements \( i_{j} \) are distinct then the part of the sum in the rhs of (33) corresponding to one-element subsets \( Q \) is equal to
\[
\sum_j^{N} \theta_{i_j} \otimes \epsilon_{i_j}(\theta_I v_\Lambda).
\]

2.20.5. "Quasiclassical" limit. It follows from the definition of quantum commutators that if we divide the rhs of (33) by \((\zeta - 1)^N\) and formally pass to the limit \( \zeta \to 1 \), we get the expression for the coaction obtained in [SV1], 6.15.3.2.

2.21. Let us define the space \( V(\Lambda)^* \) as the direct sum \( \oplus_\nu V(\Lambda)^*_\nu \). We define an \( N[\mathbb{I}] \)-grading on it as \( (V(\Lambda)^*)_\nu = V(\Lambda)^*_\nu \), and an \( X \)-grading as \( V(\Lambda)^*_\lambda = \oplus_{\nu : \Lambda - \lambda = \nu} V(\Lambda)^*_\nu \).

The form \( S_\Lambda \) induces the map \( S_\Lambda : V(\Lambda) \to V(\Lambda)^* \) (34)
compatible with both gradings.

2.22. Tensor products. Suppose we are given \( n \) weights \( \Lambda_0, \ldots, \Lambda_{n-1} \).

2.22.1. For every \( m \in \mathbb{N} \) we introduce a bilinear form \( S = S_{m;\Lambda_0,\ldots,\Lambda_{n-1}} \) on the tensor product \( \mathfrak{F}^{\otimes m} \otimes \mathfrak{V}(\Lambda_0) \otimes \ldots \otimes \mathfrak{V}(\Lambda_{n-1}) \) by the formula
\[
S(x_1 \otimes \ldots \otimes x_m \otimes y_0 \otimes \ldots \otimes y_{n-1}, x'_1 \otimes \ldots \otimes x'_m \otimes y'_0 \otimes \ldots \otimes y'_{n-1}) = \prod_{i=1}^{m} S(x_i, x'_i) \prod_{j=0}^{n-1} S_{\Lambda_j}(y_j, y'_j)
\]
(in the evident notations). This form defines mappings
\[
S : \mathfrak{F}^{\otimes m} \otimes \mathfrak{V}(\Lambda_0) \otimes \ldots \otimes \mathfrak{V}(\Lambda_{n-1}) \to \mathfrak{F}^{\otimes m} \otimes \mathfrak{V}(\Lambda_0)^* \otimes \ldots \otimes \mathfrak{V}(\Lambda_{n-1})^* \quad (35)
\]
We will regard $V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})$ as an $\mathfrak{g}^\otimes_n$-module with an action
\[
(u_0 \otimes \ldots \otimes u_{n-1}) \cdot (x_1 \otimes \ldots \otimes x_{n-1}) = \zeta^{\sum_{j<i} (\lambda_j, \nu_i)} u_0 x_0 \otimes \ldots \otimes u_{n-1} x_{n-1}
\] (36)
for $u_i \in \mathfrak{g}_{\nu_i}$, $x_i \in \mathfrak{u}(\Lambda_i)_{\lambda_i}$, cf. (13). Here we regard $\mathfrak{g}^\otimes_n$ as an algebra according to the rule of loc. cit.; one checks easily using (2) that we really get a module structure.

Using the iterated comultiplication $\Delta^{(n)}$, we get a structure of an $\mathfrak{g}$-module on $V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})$.

2.23. **Theorem.** We have an identity
\[
S_\Lambda(xy, z) = S_{1,\Lambda}(x \otimes y, \Delta_\Lambda(z))
\] (37)
for any $x \in \mathfrak{g}$, $y, z \in \mathfrak{u}(\Lambda)$ and any weight $\Lambda$.

2.24. **Proof.** We may suppose that $x, y$ and $z$ are monomials. Let $z = \theta_I v_\Lambda$ where $I = (i_N, \ldots, i_1)$.

(a) Let us suppose first that all indices $i_j$ are distinct. We will use the notations and computations from 2.20. The sides of (37) are non-zero only if $y$ is equal to $z_Q := \theta_I, Q v_\Lambda$ for some subset $Q \subset [N]$.

Therefore, it follows from Lemma 2.20.2 that it is enough to prove

2.24.1. **Lemma.** For every non-empty $Q \subset [N]$ and $x \in \mathfrak{g}_Q$ we have
\[
S_\Lambda(x z_Q, z) = [(A, i_Q(j)) - \mu_Q \cdot i_Q(j)] \cdot S(x, [\theta_I, Q, \Lambda]) \cdot S_\Lambda(z_Q, z_Q)
\] (38)
where $j(Q)$ denotes the minimal element of $Q$, and
\[
\mu_Q := \sum_{a=1}^{j(Q)-1} i_a.
\]

**Proof.** If $\text{card}(Q) = 1$ the statement follows from the definition (31). The proof will proceed by the simultaneous induction by $l$ and $N$. Suppose that $x = \theta_{I_p} \cdot x'$, so $x' \in \mathfrak{g}_{Q'}$, where $Q' = Q - \{i_p\}$, $p = j_a$ for some $a \in [0, l]$. Let us set $I' = I - \{i_p\}$, $z' = z_{i_p}$, so that $z_Q = z_{Q'}$.

We have
\[
S_\Lambda(\theta_{I_p} x' \cdot z_Q, z) = S_\Lambda(x' \cdot z_Q, \epsilon_{i_p}(z)) =
\]
\[
[(A, i_p) - \sum_{k<p} i_k'] \cdot \zeta^{(\sum_{k<p} i_k') i_p} \cdot S(x' \cdot z_Q, z') =
\]
\[
[(A, i_p) - \sum_{k<p} i_k'] \cdot [(A, i_Q(j')) - \mu_{Q'} \cdot i_Q(j')] \cdot \zeta^{(\sum_{k<p} i_k') i_p} \cdot S(x, [\theta_I, Q, \Lambda]) \cdot S_\Lambda(z'_{Q'}, z'_{Q'})
\]

2.24.2. We may suppose first that all indices $i_j$ are distinct. We will use the notations and computations from 2.20. The sides of (37) are non-zero only if $y$ is equal to $z_Q := \theta_I, Q v_\Lambda$ for some subset $Q \subset [N]$.

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We have
\[
S_\Lambda(\theta_{I_p} x' \cdot z_Q, z) = S_\Lambda(x' \cdot z_Q, \epsilon_{i_p}(z)) =
\]
\[
[(A, i_p) - \sum_{k<p} i_k'] \cdot \zeta^{(\sum_{k<p} i_k') i_p} \cdot S(x' \cdot z_Q, z') =
\]
\[
[(A, i_p) - \sum_{k<p} i_k'] \cdot [(A, i_Q(j')) - \mu_{Q'} \cdot i_Q(j')] \cdot \zeta^{(\sum_{k<p} i_k') i_p} \cdot S(x, [\theta_I, Q, \Lambda]) \cdot S_\Lambda(z'_{Q'}, z'_{Q'})
\]
by induction hypothesis. On the other hand,

\[ S(\theta_{\bar{f}} \cdot x', [\theta_{\bar{f}, Q, \Lambda}]) = S(x', \delta_{\bar{f}}([\theta_{\bar{f}, Q, \Lambda}])). \]

Therefore, to complete the induction step it is enough to prove that

\[ \langle \Lambda, i_j(Q) \rangle - \mu Q \cdot i_j(Q) \cdot [\delta_{\bar{f}}([\theta_{\bar{f}, Q, \Lambda}])] = \]

\[ = ([\Lambda, i_p] - \sum_{k<p} i_k \cdot i_p) \cdot [\Lambda, i_j(Q') - \mu Q' \cdot i_j(Q')] \cdot \zeta \cdot \delta_{\bar{f}}([\theta_{\bar{f}, Q', \Lambda}]) \]

This formula follows directly from the definition of quantum commutators (32) and formula (29). One has to treat separately two cases: \( a > 0 \), in which case \( j(Q') = j(Q) = 0 \) and \( a = 0 \), in which case \( j(Q) = j_0, j(Q') = j_1 \). Lemma is proven. \( \Box \)

This completes the proof of case (a).

(b) There are repeating indices in the sequence \( \bar{I} \). Suppose that \( \theta_{\bar{f}} \in \mathfrak{F}_{\nu} \). At this point we will use symmetrization constructions (and simple facts) from Section 4 below. The reader will readily see that there is no vicious circle. So, this part of the proof must be read after loc. cit.

There exists a finite set \( J \) and a map \( \pi : J \rightarrow I \) such that \( \nu = \nu_{\pi} \). Using compatibility of the coaction and the forms \( S \) with symmetrization — cf. Lemmata 4.5 and 4.8 below — our claim is immediately reduced to the analogous claim for the algebra \( \pi \mathfrak{F} \), the module \( V(\pi \Lambda) \) and homogeneous weight \( \chi_J \) which does not contain multiple indices and therefore follows from (a) above.

This completes the proof of the theorem. \( \Box \)

2.25. Let us pick a weight \( \Lambda \). We can consider numbers \( q_{ij} := \zeta_{i\bar{j}} \) and \( r_i := \langle \Lambda, i \rangle \), \( i, j \in I \) as parameters of our bilinear forms.

More precisely, for a given \( \nu \in \mathbb{N}[\mathfrak{I}] \) the matrix elements of the form \( S \) (resp., \( S_{\Lambda} \)) on \( \mathfrak{F}_{\nu} \) (resp., on \( V(\Lambda)_{\Lambda-\lambda} \)) in the standard bases of these spaces are certain universal polynomials of \( q_{ij} \) (resp., \( q_{ij} \) and \( r_i \)). Let us denote their determinants by \( \det(S_{\nu})(q) \) and \( \det(S_{\Lambda, \nu})(q; r) \) respectively. These determinants are polynomials of corresponding variables with integer coefficients.

2.25.1. **Lemma.** Polynomials \( \det(S_{\nu})(q) \) and \( \det(S_{\Lambda, \nu})(q; r) \) are not identically zero.

In other words, bilinear forms \( S \) and \( S_{\Lambda} \) are non-degenerate for generic values of parameters — ”Cartan matrix” \( (q_{ij}) \) and ”weight” \( (r_i) \).

**Proof.** Let us consider the form \( S_{\Lambda} \) first. The specialization of the matrix of \( S_{\Lambda, \nu} \) at \( \zeta = 1 \) is the identity matrix. It follows easily that \( \det(S_{\Lambda, \nu})(q; r) \neq 0 \).

Similarly, the matrix of \( S_{\nu} \) becomes identity at \( \zeta = 0 \), which implies the generic non-degeneracy. \( \Box \)
2.26. **Theorem.** _Coaction_ \( \Delta_A \) _is coassociative, i.e._

\[
(1 \otimes \Delta_A) \circ \Delta_A = (\Delta \otimes 1_{V(\Lambda)}) \circ \Delta_A.
\]

**Proof.** The equality (41) is a polynomial identity depending on parameters \( q_{ij} \) and \( r_i \) of the preceding subsection. For generic values of these parameters it is true due to associativity of the action of \( \mathfrak{F} \) an \( V(\Lambda) \), Theorem 2.23 and Lemma 2.25. Therefore it is true for all values of parameters. \( \square \)

2.27. The results Chapter 2 below provide a different, geometric proof of Theorems 2.23 and 2.26. Namely, the results of Section 8 summarized in Theorem 8.21 provide an isomorphism of our algebraic picture with a geometric one, and in the geometrical language the above theorems are obvious: they are nothing but the naturality of the canonical morphism between the extension by zero and the extension by star, and the claim that a Cousin complex is a complex. Lemma 2.25.1 also follows from geometric considerations: the extensions by zero and by star coincide for generic values of monodromy.

2.28. By Theorem 2.26 the dual maps

\[
\Delta_A^* : \mathfrak{F} \otimes V(\Lambda)^* \to V(\Lambda)^*
\]

give rise to a structure of a \( \mathfrak{F}^* \)-module on \( V(\Lambda)^* \).

More generally, suppose we are given \( n \) modules \( V(\Lambda_0), \ldots, V(\Lambda_{n-1}) \). We regard the tensor product \( V(\Lambda_0)^* \otimes \cdots \otimes V(\Lambda_{n-1})^* \) as a \( \mathfrak{F}^* \otimes^n \)-module according to the ”sign” rule (36). Using iterated comultiplication (16) we get a structure of a \( \mathfrak{F}^* \)-module on \( V(\Lambda_0)^* \otimes \cdots \otimes V(\Lambda_{n-1})^* \).

2.28.1. The square

\[
\begin{array}{ccc}
\mathfrak{F} \otimes V(\Lambda_0) \otimes \cdots \otimes V(\Lambda_{n-1}) & \to & V(\Lambda_0) \otimes \cdots \otimes V(\Lambda_{n-1}) \\
S \downarrow & & \downarrow S \\
\mathfrak{F}^* \otimes V(\Lambda_0)^* \otimes \cdots \otimes V(\Lambda_{n-1})^* & \to & V(\Lambda_0)^* \otimes \cdots \otimes V(\Lambda_{n-1})^*
\end{array}
\]

commutes.

This follows from 2.23 and 2.14.

3. **Hochschild complexes**

3.1. If \( A \) is an augmented \( B \)-algebra, \( A^+ \) — the kernel of the augmentation, \( M \) an \( A \)-module, let \( C_A^*(M) \) denote the following complex. By definition, \( C_A^*(M) \) is concentrated in non-positive degrees. For \( r \geq 0 \)

\[
C_A^{-r}(M) = A^+ \otimes^r \otimes M.
\]

We will use a notation \( a_r | \ldots | a_1 | m \) for \( a_r \otimes \cdots \otimes a_1 \otimes m \).
The differential \( d : C^{-r}_A(M) \rightarrow C^{-r+1}_A(M) \) acts as
\[
d(a_r | \ldots | a_1 | m) = \sum_{p=1}^{r-1} (-1)^p a_r | \ldots | a_{p+1} a_p | \ldots a_1 | m + a_r | \ldots a_2 | a_1 | m.
\]

We have canonically \( H^{-r}(C^\bullet_A(M)) \cong \text{Tor}^A_r(B, M) \) where \( B \) is considered as an \( A \)-module by means of the augmentation, cf. \([M]\), Ch. X, §2.

We will be interested in the algebras \( \mathfrak{F} \) and \( \mathfrak{F}^* \). We define the augmentation \( \mathfrak{F} \rightarrow \mathfrak{B} \) as being zero on all \( \mathfrak{F}_\nu, \nu \in \mathbb{N}[\mathbb{I}], \nu \not\equiv r \), and identity on \( \mathfrak{F}_0 \); in the same way it is defined on \( \mathfrak{F}^* \).

3.2. Let \( M \) be a \( \mathbb{N}[\mathbb{I}] \)-graded \( \mathfrak{F} \)-module. Each term \( C^{-r}_\mathfrak{F}(M) \) is \( \mathbb{N}[\mathbb{I}] \)-graded by the sum of gradings of tensor factors. We will denote \( \nu C^{-r}_\mathfrak{F}(M) \) the weight \( \nu \) component.

For \( \bar{\nu} = (\nu_0, \ldots, \nu_r) \in \mathbb{N}[\mathbb{I}] \setminus \{+\mathbb{I}\} \) we set
\[
\bar{\nu} C^{-r}_\mathfrak{F}(M) = \nu_r \ldots \nu_0 C^{-r}_\mathfrak{F}(M) = \mathfrak{F}_{\nu_0} \otimes \ldots \otimes \mathfrak{F}_{\nu_r} \otimes \mathcal{M}_{\nu_0}.
\]

Thus,
\[
\nu C^{-r}_\mathfrak{F}(M) = \oplus_{\nu_0 + \ldots + \nu_r = \nu} \nu_r \ldots \nu_0 C^{-r}_\mathfrak{F}(M).
\]

Note that all \( \nu_p \) must be \( > 0 \) for \( p > 0 \) since tensor factors lie in \( \mathfrak{F}^* \).

The differential \( d \) clearly respects the \( \mathbb{N}[\mathbb{I}] \)-grading; thus the whole complex is \( \mathbb{N}[\mathbb{I}] \)-graded:
\[
C^\bullet_\mathfrak{F}(M) = \oplus_{\nu \in \mathbb{N}[\mathbb{I}]} \nu C^\bullet_\mathfrak{F}(M).
\]

The same discussion applies to \( \mathbb{N}[\mathbb{I}] \)-graded \( \mathfrak{F}^* \)-modules.

3.3. Let us fix weights \( \Lambda_0, \ldots, \Lambda_{n-1}, n \geq 1 \). We will consider the Hochschild complex \( C^\bullet_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \) where the structure of an \( \mathfrak{F} \)-module on \( V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1}) \) has been introduced in 2.2.22.

3.3.1. In the sequel we will use the following notation. If \( K \subset I \) is a subset, we will denote by \( \chi_K := \sum_{i \in K} i \in \mathbb{N}[\mathbb{I}] \).

3.3.2. Suppose we have a map
\[
\varrho : I \rightarrow [-n + 1, r]
\]
where \( r \) is some non-negative integer. Let us introduce the elements
\[
\nu_a(\varrho) = \chi_{\varrho^{-1}(a)}, \tag{43}
\]

\( a \in [-n + 1, r] \). Let us denote by \( \mathcal{P}_\varrho(I; \setminus) \) the set of all maps \((43)\) such that \( \varrho^{-1}(a) \neq \emptyset \) for all \( a \in [r] \). It is easy to see that this set is not empty iff \( 0 \leq r \leq N \).

Let us assign to such a \( \varrho \) the space
\[
e C^{-r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) := \mathfrak{F}_{\nu_0(\varrho)} \otimes \ldots \otimes \mathfrak{F}_{\nu_r(\varrho)} \otimes \mathcal{V}(\Lambda_0)_{\nu_0(\varrho)} \otimes \ldots \otimes \mathcal{V}(\Lambda_{n-1})_{\nu_{r-1}(\varrho)} \tag{45}
\]
For each \( g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset) \) this space is non-zero, and we have
\[
\chi_l C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) = \bigoplus_{g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset)} \epsilon C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1}))
\]  
(46)

3.4. Bases. Let us consider the set \( \mathcal{P}_\mathcal{X}(\mathcal{I}; \emptyset) \). Obviously, if \( g \in \mathcal{P}_\mathcal{X}(\mathcal{I}; \emptyset) \) then \( g(I) = [N] \), and the induced map \( I \rightarrow [N] \) is a bijection; this way we get an isomorphism between \( \mathcal{P}_\mathcal{X}(\mathcal{I}; \emptyset) \) and the set of all bijections \( I \sim \rightarrow [N] \) or, to put it differently, with the set of all total orders on \( I \).

For an arbitrary \( r \), let \( g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset) \) and \( \tau \in \mathcal{P}_\mathcal{X}(\mathcal{I}; \emptyset) \). Let us say that \( \tau \) is a refinement of \( g \), and write \( g \leq \tau \), if \( g(i) < \tau(j) \) implies \( \tau(i) < \tau(j) \) for each \( i, j \in I \). The map \( \tau \) induces total orders on all subsets \( g^{-1}(a) \). We will denote by \( \mathcal{O}_\mathcal{V}[(g)] \) the set of all refinements of a given \( g \).

Given \( g \leq \tau \) as above, and \( a \in [-n+1, r] \), suppose that \( g^{-1}(a) = \{i_1, \ldots, i_p\} \) and \( \tau(i_1) < \tau(i_2) < \ldots < \tau(i_p) \). Let us define a monomial
\[
\theta_{g \leq \tau|a} = \theta_{i_p} \theta_{i_{p-1}} \cdot \ldots \cdot \theta_{i_1} \in \mathfrak{F}_{v_\alpha(g)}
\]
If \( g^{-1}(a) = \emptyset \), we set \( \theta_{g \leq \tau|a} = 1 \). This defines a monomial
\[
\theta_{g \leq \tau} = \theta_{g \leq \tau|0} \otimes \ldots \otimes \theta_{g \leq \tau|1} \otimes \theta_{v_\alpha(0)} \otimes \ldots \otimes \theta_{v_{\tau;-n+1}v_{\Lambda_{n-1}}} \in \epsilon C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1}))
\]
(47)

3.4.1. Lemma. The set \( \{\theta_{g \leq \tau|\tau} \in \mathcal{O}_\mathcal{V}[(g)]\} \) forms a basis of the space \( \epsilon C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \).

Proof is obvious. \( \square \)

3.4.2. Corollary. The set \( \{\theta_{g \leq \tau|\tau} | g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset), \tau \in \mathcal{O}_\mathcal{V}[(g)]\} \) forms a basis of the space \( \chi_l C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \). \( \square \)

3.5. We will also consider dual Hochschild complexes \( C^\bullet_\mathfrak{F}(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*) \) where \( V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^* \) is regarded as an \( \mathfrak{F}^* \)-module as in 2.28.1.

We have obvious isomorphisms
\[
C^{\tau - r}_\mathfrak{F}(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*) \cong C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1}))^*
\]

We define graded components
\[
\epsilon C^{\tau - r}_\mathfrak{F}(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*), \ g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset),
\]
as duals to \( \epsilon C^{\tau - r}_\mathfrak{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \).

We will denote by \( \{\theta_{g \leq \tau|\tau}^* | \ g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset), \tau \in \mathcal{O}_\mathcal{V}[(g)]\} \) the basis of \( \chi_l C^{\tau - r}_\mathfrak{F}(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*) \) dual to the basis \( \{\theta_{g \leq \tau|\tau} | \ g \in \mathcal{P}_\mathcal{V}(\mathcal{I}; \emptyset), \tau \in \mathcal{O}_\mathcal{V}[(g)]\}, \ 3.4.2 \).
3.6. The maps \( S_{r; \Lambda_0, \ldots, \Lambda_{n-1}} \), cf. (35), for different \( r \) are compatible with differentials in Hochschild complexes, and therefore induce morphism of complexes

\[
S : C^*_f(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \longrightarrow C^*_f(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*)
\]  

(48)

This follows from 2.28.1 and 2.10 (b).

4. Symmetrization

4.1. Let us fix a finite set \( J \) and a map \( \pi : J \longrightarrow I \). We set \( \nu_{\pi} := \sum_i N_{i} \in \mathbb{N}[I] \) where \( N_i := \text{card}(\pi^{-1}(i)) \). The map \( \pi \) induces a map \( \mathbb{N}[J] \longrightarrow \mathbb{N}[I] \) also to be denoted by \( \pi \). We will use the notation \( \chi_K := \sum_{j \in K} j \in \mathbb{N}[J] \) where \( K \subset J \). Thus, \( \pi(\chi_J) = \nu_{\pi} \).

We will denote also by \( \mu, \mu' \mapsto \mu \cdot \mu' := \pi(\mu) \cdot \pi(\mu') \) the bilinear form on \( \mathbb{N}[J] \) induced by the form on \( \mathbb{N}[I] \).

We will denote by \( \Sigma_\pi \) the group of all bijections \( \sigma : J \longrightarrow J \) preserving fibers of \( \pi \).

Let \( \pi \mathfrak{F} \) be a free associative \( B \)-algebra with 1 with generators \( \tilde{\theta}_j \), \( j \in J \). It is evidently \( \mathbb{N}[J] \)-graded. For \( \nu \in \mathbb{N}[J] \) the corresponding homogeneous component will be denoted \( \pi \mathfrak{F}_\nu \). The degree of a homogeneous element \( x \in \pi \mathfrak{F} \) will be denoted by \( |x| \in \mathbb{N}[J] \). The group \( \Sigma_\pi \) acts on algebras \( \pi \mathfrak{F}, \pi \mathfrak{F}^* \) by permutation of generators.

4.2. In the sequel, if \( G \) is a group and \( M \) is a \( G \)-module, \( M^G \) will denote the subset of \( G \)-invariants in \( M \).

Let us define a \( B \)-linear ”averaging” mapping

\[
\pi a : \pi \mathfrak{F}_\nu \longrightarrow (\pi \mathfrak{F}_{\chi_J})^{\Sigma_\pi}
\]  

(49)

by the rule

\[
\pi a(\theta_{i_1} \cdot \ldots \cdot \theta_{i_N}) = \sum_{j_1, \ldots, j_N} \tilde{\theta}_{j_1} \cdot \ldots \cdot \tilde{\theta}_{j_N},
\]  

(50)

the sum being taken over the set of all sequences \( (j_1, \ldots, j_N) \) such that \( \pi(j_p) = i_p \) for any \( p \). Note that this set is naturally a \( \Sigma_\pi \)-torsor. Alternatively, \( \pi a \) may be defined as follows. Pick some sequence \( (j_1, \ldots, j_N) \) as above, and consider an element

\[
\sum_{\sigma \in \Sigma_\pi} \sigma(\tilde{\theta}_{j_1} \cdot \ldots \cdot \tilde{\theta}_{j_N});
\]

this element obviously lies in \( (\pi \mathfrak{F}_{\chi_J})^{\Sigma_\pi} \) and is equal to \( \pi a(\theta_{i_1} \cdot \ldots \cdot \theta_{i_N}) \).

The map \( \pi \) induces the map between homogeneous components

\[
\pi : \pi \mathfrak{F}_{\chi_J} \longrightarrow \mathfrak{F}_{\nu_{\pi}}.
\]  

(51)

It is clear that the composition \( \pi \circ \pi a \) is equal to the multiplication by \( \text{card}(\Sigma_\pi) \), and \( \pi a \circ \pi \) — to the action of operator \( \sum_{\sigma \in \Sigma_\pi} \sigma \). As a consequence, we get

4.2.1. Lemma, [SV1], 5.11. The map \( \pi a \) is an isomorphism. □
4.3. Let us consider the dual to the map (51): $F^* \nu \pi^{-} \rightarrow \pi F^* \chi J$; it is obvious that it lands in the subspace of $\Sigma$-invariant functionals. Let us consider the induced map

\[ \pi a^* : F^* \nu \pi^{-} \sim \rightarrow (\pi F^* \chi J)_{\Sigma} \]  

(52)

It follows from the above discussion that $\pi a^*$ is an isomorphism.

4.4. Given a weight $\Lambda \in X = \text{Hom}(Z[J], Z)$, we will denote by $\pi \Lambda$ the composition $Z[J] \xrightarrow{\pi} Z[I] \xrightarrow{\Lambda} Z$, and by $V(\pi \Lambda)$ the corresponding Verma module over $\pi \mathcal{F}$.

Suppose we are given $n$ weights $\Lambda_0, \ldots, \Lambda_{n-1}$. Let us consider the Hochschild complex $C^*_\mathcal{F}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1}))$. By definition, its $(-r)$-th term coincides with the tensor power $\pi \mathcal{F}^{\otimes n+r}$. Therefore we can identify the homogeneous component $\chi J C^*_{\mathcal{F}}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1}))$ with $(\pi \mathcal{F}^{\otimes n+r})_{\chi J}$, which in turn is isomorphic to $\pi \mathcal{F}_{\chi J}$, by means of the multiplication map $\pi \mathcal{F}^{\otimes n+r} \rightarrow \pi \mathcal{F}$. This defines a map

\[ \chi J C^*_{\mathcal{F}}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1})) \rightarrow \pi \mathcal{F}_{\chi J} \]  

(53)

which is an embedding when restricted to polygraded components. The $\Sigma$-action on $\mathcal{F}$ induces the $\Sigma$-action on $\chi J C^*_{\mathcal{F}}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1}))$.

In the same manner we define a map

\[ \nu \pi \mathcal{F}^{\otimes n+r}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \rightarrow \nu \mathcal{F} \]  

(54)

Let us define an averaging map

\[ \pi a : \nu \pi \mathcal{F}^{\otimes n+r}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \rightarrow \chi J C^*_{\mathcal{F}}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1}))^{\Sigma} \]  

(55)

as the map induced by (49). It follows at once that this map is an isomorphism.

These maps for different $r$ are by definition compatible with differentials in Hochschild complexes. Therefore we get

4.4.1. **Lemma.** The maps (55) induce isomorphism of complexes

\[ \pi a : \nu \pi \mathcal{F}^{\otimes n+r}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \xrightarrow{\sim} \chi J C^*_{\mathcal{F}}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{n-1}))^{\Sigma} . \]  

(56)

4.5. **Lemma.** The averaging is compatible with coaction. In other words, for any $\Lambda \in X$ the square

\[
\begin{array}{ccc}
V(\Lambda) & \xrightarrow{\Delta} & \mathcal{F} \otimes \mathcal{O}(\Lambda) \\
\pi a \downarrow & & \downarrow \pi a \\
V(\pi \Lambda) & \xrightarrow{\Delta} & \pi \mathcal{F} \otimes \mathcal{O}(\pi \Lambda)
\end{array}
\]

commutes.

**Proof** follows at once by inspection of the definition (26). \(\square\)
4.6. Consider the dual Hochschild complexes. We have an obvious isomorphism

\[ \chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0})^{*} \otimes \cdots \otimes V(\pi \Lambda_{n-1})^{*}) \cong \chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0}) \otimes \cdots \otimes V(\pi \Lambda_{n-1}))^{*}; \]

using it, we define the isomorphism

\[ \chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0})^{*} \otimes \cdots \otimes V(\pi \Lambda_{n-1})^{*}) \xrightarrow{\sim} \pi^{\mathfrak{g}} \]

as the dual to \((53)\). The \(\Sigma_{\pi}\)-action on the target induces the action on \(\chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0})^{*} \otimes \cdots \otimes V(\pi \Lambda_{n-1}))^{*}\). Similarly, the isomorphism

\[ \nu \pi C^{r}_{\mathfrak{g}}(V(\Lambda_{0})^{*} \otimes \cdots \otimes V(\Lambda_{n-1})^{*}) \xrightarrow{\sim} \pi^{\mathfrak{g}} \]

is defined. We define the averaging map

\[ \pi a^{*} : \nu \pi C^{r}_{\mathfrak{g}}(V(\Lambda_{0})^{*} \otimes \cdots \otimes V(\Lambda_{n-1})^{*}) \longrightarrow \chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0})^{*} \otimes \cdots \otimes V(\pi \Lambda_{n-1}))^{\Sigma_{\pi}} \]

as the map which coincides with \((54)\) modulo the above identifications. Again, this map is an isomorphism.

Due to Lemma 4.5 these maps for different \(r\) are compatible with the differentials in Hochschild complexes. Therefore we get

4.6.1. **Lemma.** The maps \((53)\) induce isomorphism of complexes

\[ \pi a^{*} : \nu \pi C^{r}_{\mathfrak{g}}(V(\Lambda_{0})^{*} \otimes \cdots \otimes V(\Lambda_{n-1})^{*}) \longrightarrow \chi_{J} C^{r}_{\mathfrak{g}}(V(\pi \Lambda_{0})^{*} \otimes \cdots \otimes V(\pi \Lambda_{n-1}))^{\Sigma_{\pi}}. \]

\[ (57) \]

**BILINEAR FORMS**

4.7. Using the bilinear form on \(Z[\mathfrak{g}]\) introduced above, we define the symmetric bilinear form \(S( , )\) on \(\pi^{\mathfrak{g}}\) exactly in the same way as the form \(S\) on \(\mathfrak{g}\). Similarly, given \(\Lambda \in X\), we define the bilinear form \(S_{\pi \Lambda}\) on \(V(\pi \Lambda)\) as in \((2.16)\), with \(I\) replaced by \(J\).

4.7.1. **Lemma.** (i) The square

\[ \begin{array}{ccc}
\pi^{\mathfrak{g}} & \xrightarrow{S} & \pi^{\mathfrak{g}} \\
\pi a & \downarrow & \pi a^{*} \\
\pi^{\mathfrak{g}}_{\chi_{3}} & \xrightarrow{S} & \pi^{\mathfrak{g}}_{\chi_{3}}
\end{array} \]

commutes.
(ii) For any \( \Lambda \in X \) the square

\[
\begin{array}{ccc}
V(\Lambda)_{\nu_e} & \xrightarrow{S_\Lambda} & V(\Lambda)^*_e \\
\pi a \downarrow & & \downarrow \pi a^* \\
V(\pi \Lambda)_{X_J} & \xrightarrow{S_{\pi \Lambda} \chi_J} & V(\pi \Lambda)^*_X
\end{array}
\]

commutes.

**Proof.** (i) Let us consider an element \( \theta_I = \theta_{i_1} \cdots \theta_{i_N} \in \mathcal{F}_{\nu_e} \) (we assume that \( N = \text{card}(J) \)). The functional \( \pi a^* \circ S(\theta_I) \) carries a monomial \( \bar{\theta}_{j_1} \cdots \bar{\theta}_{j_N} \) to

\[ S(\theta_{i_1} \cdots \theta_{i_N}, \theta_{\pi(j_1)} \cdots \theta_{\pi(j_N)}) \]

On the other hand,

\[ S \circ \pi a(\bar{\theta}_{j_1} \cdots \bar{\theta}_{j_N}) = \sum S(\bar{\theta}_{k_1} \cdots \bar{\theta}_{k_N}, \bar{\theta}_{j_1} \cdots \bar{\theta}_{j_N}), \]

the summation ranging over all sequences \( \bar{K} = (k_1, \ldots, k_N) \) such that \( \pi(\bar{K}) = \bar{I} \). It follows from Lemma 2.12 that both expressions are equal.

(ii) The same argument as in (i), using Lemma 2.17 instead of 2.12. \( \square \)

More generally, we have

4.8. **Lemma.** For every \( m \geq 0 \) and weights \( \Lambda_0, \ldots, \Lambda_{n-1} \in X \) the square

\[
\begin{array}{ccc}
(\mathcal{F}^m \otimes \mathcal{V}(\Lambda_0) \otimes \cdots \otimes \mathcal{V}(\Lambda_{n-1}))_{\nu_e} & \xrightarrow{S_{m;\Lambda_0,\ldots,\Lambda_{n-1}} \chi_J} & (\mathcal{F}^m \otimes \mathcal{V}(\Lambda_0)^* \otimes \cdots \otimes \mathcal{V}(\Lambda_{n-1})^*)_{\nu_e} \\
\pi a \downarrow & & \downarrow \pi a^* \\
(\pi \mathcal{F}^m \otimes \mathcal{V}(\pi \Lambda_0) \otimes \cdots \otimes \mathcal{V}(\pi \Lambda_{n-1}))_{X_J} & \xrightarrow{S_{m;\pi \Lambda_0,\ldots,\pi \Lambda_{n-1}} \chi_J} & (\pi \mathcal{F}^m \otimes \mathcal{V}(\pi \Lambda_0)^* \otimes \cdots \otimes \mathcal{V}(\pi \Lambda_{n-1})^*)_{X_J}
\end{array}
\]

commutes.

**Proof** is quite similar to the proof of the previous lemma. We leave it to the reader. \( \square \)

**5. Quotient Algebras**

5.1. Let us consider the map \( [13] S : \mathcal{F} \longrightarrow \mathcal{F}^* \). Let us consider its kernel \( \text{Ker}(S) \). It follows at once from \( [18] \) that \( \text{Ker}(S) \) is a left ideal in \( \mathcal{F} \). In the same manner, it is easy to see that it is also a right ideal, cf. [1], 1.2.4.

We will denote by \( \mathfrak{f} \) the quotient algebra \( \mathcal{F}/\text{Ker}(\mathcal{G}) \). It inherits the \( \mathbb{N}[\Pi] \)-grading and the coalgebra structure from \( \mathcal{F} \), cf. loc.cit. 1.2.5, 1.2.6.

5.2. In the same manner, given a weight \( \Lambda \), consider the kernel of \( S_{\Lambda} : V(\Lambda) \longrightarrow V(\Lambda)^* \). Let us denote by \( L(\Lambda) \) the quotient space \( V(\Lambda)/\text{Ker}(S_{\Lambda}) \). It inherits \( \mathbb{N}[\Pi] \)- and \( X \)-gradings from \( V(\Lambda) \). Due to Theorem 2.23 the structure of \( \mathcal{F} \)-module on \( V(\Lambda) \) induces the structure of \( \mathfrak{f} \)-module on \( L(\Lambda) \).

More generally, due to the structure of a coalgebra on \( \mathfrak{f} \), all tensor products \( L(\Lambda_0) \otimes \cdots \otimes L(\Lambda_{n-1}) \) become \( \mathfrak{f} \)-modules (one should take into account the "sign rule" [36]).
5.3. We can consider Hochschild complexes $C_f^\bullet(L(\Lambda_0) \otimes \ldots \otimes L(\Lambda_{n-1})).$

5.3.1. **Lemma.** The map

$$S : C_f^\bullet(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{n-1})) \rightarrow C_f^\bullet(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{n-1})^*)$$

factors through the isomorphism

$$\text{Im}(S) \sim C_f^\bullet(L(\Lambda_0) \otimes \ldots \otimes L(\Lambda_{n-1})) \quad (59)$$

**Proof.** This follows at once from the definitions. \(\Box\)
CHAPTER 2. Geometric discussion.

6. DIAGONAL STRATIFICATION AND RELATED ALGEBRAS

6.1. Let us adopt notations of 4.1. We set $N := \text{card}(J)$. Let $\pi A_\mathbb{R}$ denote a real affine space with coordinates $t_j$, $j \in J$, and $\pi A$ its complexification. Let us consider an arrangement $\mathcal{H}_\emptyset$ consisting of all diagonals $\Delta_{ij}$, $i,j \in J$. Let us denote by $\mathcal{S}_\emptyset$ the corresponding stratification; $\mathcal{S}_{\emptyset,\mathbb{R}}$ will denote the corresponding real stratification of $A_\mathbb{R}$.

The stratification $\mathcal{S}_\emptyset$ has a unique minimal stratum

$$\Delta = \bigcap \Delta_{ij}$$  \hspace{1cm} (60)

— main diagonal; it is one-dimensional. We will denote by $\pi A_\emptyset$ (resp., $\pi A_{\emptyset,\mathbb{R}}$) the open stratum of $\mathcal{S}_\emptyset$ (resp., of $\mathcal{S}_{\emptyset,\mathbb{R}}$).

6.2. Let us describe the chambers of $\mathcal{S}_{\emptyset,\mathbb{R}}$. If $C$ is a chamber and $x = (x_j) \in C$, i.e. the embedding $J \hookrightarrow \mathbb{R}$, $j \mapsto \varpi_j$, it induces an obvious total order on $J$, i.e. a bijection

$$\tau_C : J \sim \rightarrow [N]$$  \hspace{1cm} (61)

Namely, $\tau_C$ is determined uniquely by the requirement $\tau_C(i) < \tau_C(j)$ iff $x_i < x_j$; it does not depend on the choice of $x$. This way we get a one-to-one correspondence between the set of chambers of $\mathcal{S}_\emptyset$ and the set of all bijections (61). We will denote by $C_\tau$ the chamber corresponding to $\tau$.

Given $C$ and $x$ as above, suppose that we have $i,j \in J$ such that $x_i < x_j$ and there is no $k \in J$ such that $x_i < x_k < x_j$. We will say that $i,j$ are neighbours in $C$, more precisely that $i$ is a left neighbour of $j$.

Let $x' = (x'_j)$ be a point with $x'_p = x_p$ for all $p \neq j$, and $x'_j$ equal to some number smaller than $x_i$ but greater than any $x_k$ such that $x_k < x_i$. Let $\gamma C$ denote the chamber containing $x'$. Let us introduce a homotopy class of paths $C \gamma_{ij}$ connecting $x$ and $x'$ as shown on Fig. 1 below.
We can apply the discussion I.4.1 and consider the groupoid $\pi_1(\mathcal{A}_0, \mathcal{A}_0 \times \mathbb{R})$. It has as the set of objects the set of all chambers. The set of morphisms is generated by all morphisms $C_{\gamma_{ij}}$ subject to certain evident braiding relations. We will need only the following particular case.

To define a one-dimensional local system $\mathcal{L}$ over $\pi_1(\mathcal{A}_0)$ is the same as to give a set of one-dimensional vector spaces $\mathcal{L}_C$, $C \in \pi_1(\mathcal{A}_0 \times \mathbb{R})$, together with arbitrary invertible linear operators

$$^CT_{ij} : \mathcal{L}_C \to \mathcal{L}_{C'}$$

("half-monodromies") defined for chambers having $i$ as a left neighbour of $j$.

6.3. We define a one-dimensional local system $\pi \mathcal{I}$ over $\pi \mathcal{A}_0$ as follows. Its fibers $\pi \mathcal{I}_C$ are one-dimensional linear spaces with fixed basis vectors; they will be identified with $B$.

Half-monodromies are defined as

$$^CT_{ij} = \zeta_{i,j}, \quad i, j \in J$$

6.4. Let $j : \pi \mathcal{A}_0 \to \pi \mathcal{A}$ denote an open embedding. We will study the following objects of $\mathcal{M}(\pi \mathcal{A}; S_0)$:

$$\pi \mathcal{I}_? = \left| \pi \mathcal{I}[\mathcal{N}] \right|,$$

where $? = !, \ast$. We have a canonical map

$$m : \pi \mathcal{I}_! \to \pi \mathcal{I}_\ast$$

and by definition $\pi \mathcal{I}_{!a}$ is its image, cf. I.4.5.
6.5. For an integer \( r \) let us denote by \( \mathcal{P}_\mathcal{V}(\mathcal{J}) \) the set of all surjective mappings \( J \rightarrow [r] \). It is evident that \( \mathcal{P}_\mathcal{V}(\mathcal{J}) \neq \emptyset \) if and only if \( 1 \leq r \leq N \). To each \( \rho \in \mathcal{P}_\mathcal{V}(\mathcal{J}) \) let us assign a point \( w_\rho = (\rho(j)) \in \mathcal{A}_\mathbb{R} \). Let \( F_\rho \) denote the facet containing \( w_\rho \). This way we get a bijection between \( \mathcal{P}_\mathcal{V}(\mathcal{J}) \) and the set of \( r \)-dimensional facets. For \( r = N \) we get the bijection from 6.2.

At the same time we have defined a marking of \( \mathcal{H}_\emptyset \): by definition, \( F_\rho w = w_\rho \). This defines cells \( D_F, S_F \).

6.6. The main diagonal \( \Delta \) is a unique 1-facet; it corresponds to the unique element \( \rho_0 \in \mathcal{P}_\infty(\mathcal{J}) \).

We will denote by \( \text{Ch} \) the set of all chambers; it is the same as \( \text{Ch}(\Delta) \) in notations of Part I. Let \( C_\tau \) be a chamber. The order \( \tau \) identifies \( C \) with an open cone in the standard coordinate space \( \mathbb{R}^N \); we provide \( C \) with the orientation induced from \( \mathbb{R}^N \).

6.7. Basis in \( \Phi_\Delta(\mathcal{I}_\ast) \). The construction I.4.7 gives us the basis \( \{c_{\Delta<C}\} \) in \( \Phi_\Delta(\mathcal{I}_\ast) \) indexed by \( C \in \text{Ch} \). We will use notation \( c_{\tau,!} := c_{\Delta<C_\tau} \).

A chain \( c_{\tau,!} \) looks as follows.

\[
\begin{array}{cccc}
\tau(j_1) & \tau(j_2) & \ldots & \tau(j_N) \\
\end{array}
\]

Fig. 2. A chain \( c_{\tau,!} \).

Here \( \tau(j_i) = i \). We will denote by \( \{b_{\tau,!}\} \) the dual basis in \( \Phi_\Delta(\mathcal{I}_\ast) \).

6.8. Basis in \( \Phi_\Delta(\mathcal{I}_\ast) \). Similarly, the definition I.4.9 gives us the basis \( \{c_{\Delta<C}\} \), \( C \in \text{Ch} \) in \( \Phi_\Delta(\mathcal{I}_\ast) \). We will use the notations \( c_{\tau,*} := c_{\Delta<C_\tau} \).

If we specify the definition I.4.9 and its explanation I.4.12 to our arrangement, we get the following picture for a dual chain \( c_{\tau,*} \).
This chain is represented by the section of a local system $\mathcal{I}^{-\infty}$ over the cell in $\hat{\mathcal{A}}_0$ shown above, which takes value 1 at the point corresponding to the end of the travel (direction of travel is shown by arrows).

To understand what is going on, it is instructive to treat the case $N = 2$ first, which essentially coincides with the Example I.4.10.

We will denote by $\{b_{\tau,*}\}$ the dual basis in $\Phi_\Delta(I_*)$.

6.9. Obviously, all maps $\tau: J \to [N]$ from $P_N(\mathcal{J})$ are bijections. Given two such maps $\tau_1, \tau_2$, define the sign $\text{sgn}(\tau_1, \tau_2) = \pm 1$ as the sign of the permutation $\tau_1 \tau_2^{-1} \in \Sigma_N$.

For any $\tau \in P_N(\mathcal{J})$ let us denote by $\vec{J}_\tau$ the sequence $(\tau^{-1}(N), \tau^{-1}(N-1), \ldots, \tau^{-1}(1))$.

6.10. Let us pick $\eta \in P_N(\mathcal{J})$. Let us define the following maps:

$$\pi \phi^{(\eta)}_\Delta: \Phi_\Delta(\pi \mathcal{I}_t) \to \pi \mathcal{F}_{\chi_3}$$

which carries $b_{\tau,!}$ to $\text{sgn}(\tau, \eta) \cdot \theta_{\vec{J}_\tau}$, and

$$\pi \phi^{(\eta)}_*: \Phi_\Delta(\pi \mathcal{I}_t) \to \pi \mathcal{F}_{\chi_3}$$

which carries $b_{\tau,*}$ to $\text{sgn}(\tau, \eta) \cdot \theta_{\vec{J}_\tau}^*$.

6.11. **Theorem.** (i) The maps $\pi \phi^{(\eta)}_\Delta,$ and $\pi \phi^{(\eta)}_*$ are isomorphisms. The square

$$
\begin{array}{ccc}
\Phi_\Delta(\pi \mathcal{I}_t) & \xrightarrow{\pi \phi^{(\eta)}_\Delta,} & \pi \mathcal{F}_{\chi_3} \\
m \downarrow & & \downarrow S \\
\Phi_\Delta(\pi \mathcal{I}_t) & \xrightarrow{\pi \phi^{(\eta)}_*} & \pi \mathcal{F}_{\chi_3}^*
\end{array}
$$

commutes.

(ii) The map $\pi \phi^{(\eta)}_\Delta,$ induces an isomorphism

$$\pi \phi^{(\eta)}_\Delta: \Phi_\Delta(\pi \mathcal{I}_t) \to \pi \mathcal{F}_{\chi_3}$$

(66)
Proof. This theorem is particular case of I.14.16, I.4.17. The claim about isomorphisms in (i) is clear. To prove the commutativity of the square, we have to compute the action of the canonical map \( m \) on our standard chains. The claim follows at once from their geometric description given above. Note that here the sign in the definition of morphisms \( \phi \) is essential, due to orientations of our chains. (ii) is a direct corollary of (i) \( \square \)

**SYMMETRIZED CONFIGURATIONAL SPACES**

6.12. **Colored configuration spaces.** Let us fix \( \nu = \sum \nu_i i \in \mathbb{N}[\mathbb{I}], \sum_i \nu_i = N \). There exists a finite set \( I \) and a morphism \( \pi : J \to I \) such that \( \text{card}(\pi^{-1}(i)) = \nu_i \) for all \( i \in I \). Let us call such \( \pi \) an unfolding of \( \nu \). It is unique up to a non-unique isomorphism; the automorphism group of \( \pi \) is precisely \( \Sigma_\pi \), and \( \nu = \nu_\pi \) in our previous notations.

Let us pick an unfolding \( \pi \). As in the above discussion, we define \( \overset{\pi}{A} \) as a complex affine space with coordinates \( t_j, j \in J \). Thus, \( \dim \overset{\pi}{A} = N \). The group \( \Sigma_\pi \) acts on the space \( \overset{\pi}{A} \) by permutations of coordinates.

Let us denote by \( \overset{\pi}{A}_\nu \) the quotient manifold \( \overset{\pi}{A}/\Sigma_\pi \). As an algebraic manifold, \( \overset{\pi}{A}_\nu \) is also a complex \( N \)-dimensional affine space. We have a canonical projection

\[
\pi : \overset{\pi}{A} \to \overset{\pi}{A}_\nu
\]

The space \( \overset{\pi}{A}_\nu \) does not depend on the choice of an unfolding \( \pi \). It will be called the configuration space of \( \nu \)-colored points on the affine line \( \overset{\pi}{A} \).

We will consider the stratification on \( \overset{\pi}{A}_\nu \) whose strata are \( \pi(S), S \in S_\emptyset \); we will denote this stratification also by \( S_\emptyset \); this definition does not depend on the choice of \( \pi \). We will study the category \( \mathcal{M}(\overset{\pi}{A}_\nu; S_\emptyset) \).

We will denote by \( \overset{\pi}{A}_\nu^{\emptyset} \) the open stratum. It is clear that \( \pi^{-1}(\overset{\pi}{A}_\nu^{\emptyset}) = \overset{\pi}{A}_\emptyset \). The morphism \( \pi \) is unramified over \( \overset{\pi}{A}_\nu^{\emptyset} \).

The action of \( \Sigma_\pi \) on \( \overset{\pi}{A} \) may be extended in the evident way to the local system \( \overset{\pi}{I} \), hence all our spaces of geometric origin — like \( \Phi_\Delta(\overset{\pi}{I}) \), etc. — get an action of \( \Sigma_\pi \).

6.12.1. If \( M \) is an object with a \( \Sigma_\pi \)-action (for example a vector space or a sheaf), we will denote by \( M^{\Sigma_\pi} \) the subobject \( \{ x \in M \mid \text{for every } \sigma \in \Sigma_\pi \sigma x = \text{sgn}(\sigma)x \} \) where \( \text{sgn}(\sigma) = \pm 1 \) is the sign of a permutation.

A morphism \( f : M \to N \) between two objects with \( \Sigma_\pi \)-action will be called skew \( (\Sigma_\pi) \)-equivariant if for any \( x \in M, \sigma \in \Sigma_\pi, f(\sigma x) = \text{sgn}(\sigma)\sigma f(x) \).
Let us define a local system over \( \mathcal{A}_\nu \)
\[
\mathcal{I}_\nu = (\pi_* \mathcal{I})^{\Sigma_{*,-}}
\] (68)

6.13. Let \( j : \pi_\Delta A_0 \hookrightarrow \pi_\Delta \mathcal{A}, \mathcal{J}_\Delta : A_{\nu,0} \hookrightarrow A_\nu \) be the open embeddings. Let us define the following objects of \( \mathcal{M}(\mathcal{A}_\nu; S_0) \):
\[
\mathcal{I}_\nu ? := |A?| \mathcal{I}_\nu [N]
\] (69)
where \( ? =!, * \) or \( !* \). We have by definition
\[
\mathcal{I}_\nu ! = (\pi_* \mathcal{I})^{\Sigma_{*,-}}
\] (70)
The morphism \( \pi \) is finite; consequently \( \pi_* \) is \( t \)-exact (see [BBD], 4.1.3) and commutes with the Verdier duality. Therefore,
\[
\mathcal{I}_{\nu !} = (\pi_* \mathcal{I})^{\Sigma_{*,-}}; \mathcal{I}_{\nu!*} = (\pi_* \mathcal{I})^{\Sigma_{!,*,-}}
\] (71)

6.14. Let us define vector spaces
\[
\Phi_\Delta(\mathcal{I}_{\nu ?}) := (\Phi_\Delta(\pi \mathcal{I}))^{\Sigma_{*,-}}
\] (72)
where \( ? =!, * \) or \( !* \).

Let us pick a \( \Sigma_{\pi} \)-equivariant marking of \( H_0 \), for example the one from 6.5; consider the corresponding cells \( D_{\Delta}, S_{\Delta} \). It follows from (70) and (71) that
\[
\Phi_\Delta(\mathcal{I}_{\nu ?}) = R\Gamma(\pi(\mathcal{D}_{\Delta}), \pi(S_{\Delta}); \mathcal{I}_{\nu ?})[-\infty]
\] (73)
where \( ? =!, * \) or \( !* \), cf. I.3.3.

6.15. The group \( \Sigma_{\pi} \) is acting on \( \pi \mathfrak{F} \). Let us pick \( \eta \in \mathcal{P}_N(\mathcal{J}) \). It follows from the definitions that the isomorphisms \( \pi \phi^{(\eta)}_{\Delta,1} : \pi \phi^{(\eta)}_{\Delta,*} \) are skew \( \Sigma_{\pi} \)-equivariant. Therefore, passing to invariants in Theorem 6.11 we get

6.16. Theorem. The maps \( \pi \phi^{(\eta)}_{\Delta,1}, \pi \phi^{(\eta)}_{\Delta,*} \) induce isomorphisms included into a commutative square
\[
\begin{array}{ccc}
\Phi_\Delta(\mathcal{I}_{\nu !}) & \xrightarrow{\phi^{(\eta)}_{\nu !}} & \mathfrak{F}_{\nu !} \\
m \downarrow & & \downarrow S \\
\Phi_\Delta(\mathcal{I}_{\nu !*}) & \xrightarrow{\phi^{(\eta)}_{\nu !*}} & \mathfrak{F}_{\nu ! *}
\end{array}
\]
and
\[
\phi^{(\eta)}_{\nu !*} : \Phi_\Delta(\mathcal{I}_{\nu !*}) \xrightarrow{\sim} \mathfrak{f}_{\nu ! *}
\] (74)
7. Principal stratification

The contents of this section is parallel to I, Section 3. However, we present here certain modification of general constructions from loc. cit.

7.1. Let us fix a finite set $J$ of cardinality $N$. In this section we will denote by $A_\mathbb{R}$ a real affine space with fixed coordinates $t_j : A_\mathbb{R} \rightarrow \mathbb{R}, \ j \in J$, and by $A$ its complexification. For $z \in \mathbb{C}$, $\mathbb{J}, \mathbb{J} \in \mathbb{J}$ denote by $H_j(z) \subset A$ a hyperplane $t_j = z$, and by $\Delta_{ij}$ a hyperplane $t_i = t_j$.

Let us consider an arrangement $H$ in $A$ consisting of hyperplanes $H_i(0)$ and $\Delta_{ij}, i, j \in J, i \neq j$. It is a complexification of an evident real arrangement $H_\mathbb{R}$ in $A_\mathbb{R}$. As usual, the subscript $\mathbb{R}$ will denote real points.

Denote by $S$ the corresponding stratification of $A$. To distinguish this stratification from the diagonal stratification of the previous section, we will call it the principal stratification. To shorten the notation, we will denote in this paper by $D(A, S)$ a category which would be denoted $D[A; S]$ in I. In this section we will study the category $\mathcal{M}(A; S)$.

7.1.1. Let us consider a positive cone

$$A_\mathbb{R}^+ = \{(\approx_j) | \text{ all } \approx_j \geq \mathcal{J} \} \subset A_\mathbb{R}$$

A facet will be called positive if it lies inside $A_\mathbb{R}^+$. A flag $F$ is called positive if all its facets are positive.

7.2. Let us fix a marking $w = \{ Fw \}$ of $H_\mathbb{R}$ (cf. I.3.2). For a positive facet $F$ define

$$D_F^+ = D_F \cap A_\mathbb{R}^+; \ S_F^+ = S_F \cap A_\mathbb{R}^+; \ D_F^+ = D_F^+ - S_F^+.$$  

Note that $D_F^+$ coincides with the union of $F \Delta$ over all positive flags beginning at $F$, and $S_F^+$ coincides with the union of $F \Delta$ as above with $\dim F \Delta < \text{codim } F$. It follows that only marking points $Fw$ for positive facets $F$ take part in the definition of cells $D_F^+$, $S_F^+$.

7.3. Let $K$ be an object of $D(A; S)$, $F$ a positive facet of dimension $p$. Let us introduce a notation

$$\Phi_F^+(K) = \Gamma(D_F^+, S_F^+; K)[-\sqrt{.}]$$

This way we get a functor

$$\Phi_F^+ : D(A; S) \rightarrow D(\sqrt{-\bot})$$ (75)
7.4. **Theorem.** Functors \( \Phi_F^+ \) commute with Verdier duality. More precisely, we have canonical natural isomorphisms

\[
D \Phi_F^+(\mathcal{K}) \sim \Phi_F^+(DK).
\]

**Proof** goes along the same lines as the proof of Theorem I.3.5.

7.5. First let us consider the case \( N = 1 \), cf. I.3.6. We will adopt notations from there and from I, Fig. 1. Our arrangement has one positive 1-dimensional facet \( E = \mathbb{R}_{>\nu} \), let \( w \in E \) be a marking.

We have an isomorphism

\[
\Phi_F^+(\mathcal{K}) \sim R\Gamma(A, \{\leq \} ; \mathcal{K})
\]

(77)

Denote \( j := j_{h-(\leq)} \). We have by Poincaré duality

\[
D \Phi_F^+(\mathcal{K}) \sim R\Gamma_j(A; \ll, \ll' \mathcal{D}\mathcal{K}) \sim R\Gamma(A, A_{\geq \nu}; \ll, \ll' \mathcal{D}\mathcal{K})
\]

(78)

Let us denote \( D_F^{+opp} := \mathbb{R}_{\geq \leq} \) and \( Y = \epsilon_i \cdot D_F^{+opp} \). We have

\[
R\Gamma(A, A_{\geq \nu}; \ll, \ll' \mathcal{D}\mathcal{K}) \sim R\Gamma(A, Y \cup A_{\geq \nu}; \ll, \ll' \mathcal{D}\mathcal{K})
\]

(79)

by homotopy. Consider the restriction map

\[
res : R\Gamma(A, Y \cup A_{\geq \nu}; \ll, \ll' \mathcal{D}\mathcal{K}) \rightarrow R\Gamma(A_{\leq \nu}, Y \cap A_{\leq \nu}; \mathcal{D}\mathcal{K})
\]

(80)

7.5.1. **Claim.** \( res \) is an isomorphism.

In fact, \( \text{Cone}(res) \) is isomorphic to

\[
R\Gamma(A, A_{\leq \nu} \cup A_{\geq \nu} \cup Y; \ll, \ll' \mathcal{D}\mathcal{K}) = R\Gamma_j(A_{\leq \nu}, A_{\leq \nu} \cup Y; \ll, \ll' \mathcal{D}\mathcal{K}) \cong D R\Gamma(A_{\leq \nu}, (A_{\leq \nu} \cup Y); \ll, \ll' \mathcal{K})
\]
But
\[ R\Gamma(\mathbb{A}_{\leq r''} - (\mathbb{A}_{\leq r'} \cup Y); \mathbb{J}\mathbb{J}'^\ast \mathcal{K}) = \mathcal{R}\Gamma(\mathbb{A}_{\leq r''} - (\mathbb{A}_{\leq r'} \cup Y), \{ \leq \}; \mathcal{K}) = \mathcal{T} \] (81)
evidently. This proves the claim. \( \square \)
A clockwise rotation by \( \pi/2 \) induces an isomorphism
\[ R\Gamma(\mathbb{A}_{\leq r'}, Y \cap \mathbb{A}_{\leq r'}; \mathbb{D}\mathcal{K}) \cong \mathcal{R}\Gamma(\mathbb{A}_{\leq r'}, \epsilon \cdot \mathbb{D}\mathcal{K}_{\infty}; \mathbb{D}\mathcal{K}), \]
and the last complex is isomorphic to \( \Phi^+_F(D\mathcal{K}) \). This proves the theorem for \( N = 1 \).

7.6. Now let us return to an arbitrary \( J \). Let us prove the theorem for \( F \) equal to the unique 0-dimensional facet.

Let us introduce the following subspaces of \( \mathbb{A}_R \) (as usually, a circle on the top will denote the interior).
\[ D^\text{opp}_F := \mathbb{A}_R - \mathbb{D}^+_F; \]
\[ D^+_F := \mathbb{A}^+_R - \mathbb{D}^+_F; \]
for each cell \( D_{F,C} \), \( C \in \text{Ch}(F) \) define \( D^\text{opp}_{F,C} := C - \mathcal{D}_{F,C} \).

It is easy to see that the restriction induces isomorphism
\[ \Phi^+_F(\mathcal{K}) \rightarrow \mathcal{R}\Gamma(\mathbb{A}, \mathbb{D}^+_F \times \mathcal{K}). \]
We use the notations of 1.3.8. Let us choose positive numbers \( r' < r'' \), \( \epsilon \), such that
\[ \epsilon D_{F} \subset \mathbb{A}_{\leq r'} \subset \mathbb{D}^+_F \subset \mathbb{D}_F \subset \mathbb{A}_{\leq r''} \]
Define the subspace
\[ Y^+ = \epsilon \cdot D^\text{opp}_{F}; \]
denote \( j := j_{\mathbb{A}^+_R \mathbb{D}^+_F} \). We have isomorphisms
\[ D\Phi^+_F(\mathcal{K}) \cong D\mathcal{R}\Gamma(\mathbb{A}, \mathbb{S}^+_F; \mathcal{K}) \cong \mathcal{R}\Gamma_j(\mathbb{A}, \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) \cong \mathcal{R}\Gamma(\mathbb{A}, \mathbb{Y}^+ \cup \mathbb{A}_{\geq r'}; \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) \]
(82)
Consider the restriction map
\[ \text{res} : \mathcal{R}\Gamma(\mathbb{A}, \mathbb{Y}^+ \cup \mathbb{A}_{\geq r'}; \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) \rightarrow \mathcal{R}\Gamma(\mathbb{A}_{\leq r'}, \mathbb{Y}^+ \cap \mathbb{A}_{\leq r'}; \mathbb{D}\mathcal{K}) \] (83)
Cone(\text{res}) is isomorphic to
\[ \mathcal{R}\Gamma(\mathbb{A}_{\leq r'}, \mathbb{A}_{\leq r'} \cup \mathbb{Y}^+; \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) = \mathcal{R}\Gamma_j(\mathbb{A}_{\leq r'}, \mathbb{A}_{\leq r'} \cup \mathbb{Y}^+; \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) \cong \]
\[ \cong D\mathcal{R}\Gamma(\mathbb{A}_{\leq r'}, \mathbb{A}_{\leq r'} \cup \mathbb{Y}^+; \mathbb{J}'^\ast \mathcal{D}\mathcal{K}) = \mathcal{R}\Gamma(\mathbb{A}_{\leq r'}, \mathbb{Y}^+; \mathbb{S}^+_F; \mathcal{K}) \]
7.6.1. Lemma. (Cf. I.3.8.1.) $R\Gamma(\mathbb{A}(\subseteq,\subseteq) - \mathbb{Y}^+, \mathbb{S}^+_R; \mathcal{K}) = t$.

Proof. Let us define the following subspaces of $\mathbb{A}$.

$A := \{(t_j) \mid \forall j \mid t_j < 1; \text{ there exists } j : t_j \neq 0\}; A^+_R := A \cap A^+_R$. Note that $A^+_R \cap i : A^+_R = \emptyset$. Due to monodromicity, it is easy to see that

$$R\Gamma(\mathbb{A}(\subseteq,\subseteq) - \mathbb{Y}^+, \mathbb{S}^+_R; \mathcal{K}) \cong R\Gamma(A - ) \cdot A^+_R, A^+_R; \mathcal{K})$$

Therefore, it is enough to prove the following

7.6.2. Claim. The restriction map

$$R\Gamma(A - i : A^+_R; \mathcal{K}) \longrightarrow R\Gamma(A^+_R; \mathcal{K})$$

is an isomorphism.

Proof of the Claim. Let us introduce for each $k \in J$ open subspaces

$$A_k = \{(t_j) \in A \mid t_j \not\in i : R_{\geq} \} \subset A - \bigcap A^+_R$$

and

$$A'_k = \{(t_j) \in A^+_R | t_j > 0 \} \subset A^+_R$$

Obviously $A'_k = A_k \cap A^+_R$. For each subset $M \subset J$ set $A_M := \cap_{k \in M} A_k; A'_M := \cap_{k \in M} A'_k$.

For each non-empty $M$ define the spaces $B_M := \{(t_j)_{j \in M} | \forall j \mid t_j < 1, t_j \not\in i : R_{\geq} \}$ and $B'_M := \{(t_j)_{j \in M} | \forall j \mid t_j \in \mathbb{R}, \forall < \approx < \| \}$. We have obvious projections

$f_M : A_M \longrightarrow B_M, f'_M : A'_M \longrightarrow B'_M$.

Let us look at fibers of $f_M$ and $f'_M$. Given $b = (t_j)_{j \in M} \in B_M$, the fiber $f_M^{-1}(b)$ is by definition $\{(t_k)_{k \in J - M} | t_k < 1\}$, the possible singularities of our sheaf $\mathcal{K}$ are at the hyperplanes $t_k = t_j$ and $t_k = 0$. It is easy to see that $f_M$ is "lisse" with respect to $\mathcal{K}$ which means in particular that a stalk $(f_M^* \mathcal{K})|_b$ is equal to $R\Gamma(f_M^{-1}(b); \mathcal{K})$. The same considerations apply to $f'_M$. Moreover, it follows from I.2.12 that the restriction maps

$$R\Gamma(f_M^{-1}(b); \mathcal{K}) \longrightarrow R\Gamma((f'_M)^{-\infty}(\cdot); \mathcal{K})$$

are isomorphisms for every $b \in B'_M$. This implies that $f'_M^* \mathcal{K}$ is equal to the restriction of $f_M^* \mathcal{K}$ to $B'_M$.

The sheaf $f_M^* \mathcal{K}$ is smooth along the diagonal stratification. For a small $\delta > 0$ let $U_\delta \subset B_M$ denote an open subset $\{(t_j) \in B_M | \| \arg(t_j)\| < \delta \text{ for all } j\}$. The restriction maps

$$R\Gamma(B_M; f_M^* \mathcal{K}) \longrightarrow R\Gamma(U_\delta; \{\mathcal{M}_* \mathcal{K}\})$$

are isomorphisms. We have $B'_M = \cap_{\delta} U_\delta$, therefore by I.2.12 the restriction

$$R\Gamma(B_M; f_M^* \mathcal{K}) \longrightarrow R\Gamma(B'_M; \{\mathcal{M}_* \mathcal{K}\})$$

is an isomorphism. This implies, by Leray, that restriction maps

$$R\Gamma(A_M; \mathcal{K}) \longrightarrow R\Gamma(A'_M; \mathcal{K})$$

are isomorphisms for every non-empty $M$.

Obviously $A - i : A^+_R = \cup_{k \in J} A_k$ and $A^+_R = \cup_{k \in J} A'_k$. Therefore, by Mayer-Vietoris the map $(84)$ is an isomorphism. This proves the claim, together with the lemma. $\Box$
A clockwise rotation by $\pi/2$ induces an isomorphism

$$R\Gamma(A_{\leq \langle}, Y^+ \cap A_{\leq \langle}'; DK) \cong R\Gamma(A_{\leq \langle}, \epsilon \cdot D_F^{+\times n}; DK) \cong R\Gamma(A_{\leq \langle}, \epsilon \cdot \mathcal{S}_F^+; DK)$$

and the last complex is isomorphic to $\Phi_F^+(DK)$ in view of [7.6]. This proves the theorem for the case of the 0-facet $F$.

7.7. Suppose that $F$ is an arbitrary positive facet. From the description of positive facets (see infra, 8.4) one sees that the cell $D_F^+$ is homeomorphic to a cartesian product of the form

$$D_{F_0}^+ \times D_{F_1}^+ \times \ldots \times D_{F_a}^+$$

where $F_0$ is a 0-facet of the principal stratification in some affine space of smaller dimension, and $D_{F_i}^+$ are the cells of the diagonal stratification discussed in the previous section.

Using this remark, we apply a combination of the arguments of the previous subsection (to the first factor) and of I.3.8 (to the remaining factors) to get the required isomorphism. We leave details to the reader.

The theorem is proved. □

7.8. **Theorem.** All functors $\Phi_F^+$ are $t$-exact. In other words, for all positive facets $F$,

$$\Phi_F^+(\mathcal{M}(A; S)) \subset V^\uparrow\sqcup \subset D^1(V\downarrow\sqcup).$$

**Proof.** The same as that of I.3.9. □.

7.9. Thus we get exact functors

$$\Phi^+: \mathcal{M}(A; S) \rightarrow V^\uparrow\sqcup$$

commuting with Verdier duality.

We will also denote vector spaces $\Phi_F^+(\mathcal{M})$ by $\mathcal{M}_F^+$.

7.10. **Canonical and variation maps.** Suppose we have a positive facet $E$. Let us denote by $+\mathcal{F}^{-}\uparrow\downarrow(\mathcal{E})$ the set of all positive facets $F$ such that $E < F$, $\dim F = \dim E + 1$. We have

$$S_E^+ = \bigcup_{F \in +\mathcal{F}^{-}\uparrow\downarrow(\mathcal{E})} D_F^+$$

(86)

Suppose we have $\mathcal{K} \in \mathcal{D}(A, S)$. 

[524x750]31
7.10.1. Lemma. We have a natural isomorphism
\[ R \Gamma (S_E^+, \bigcup_{F \in -I(E)} S_F^+; \mathcal{K}) \cong \bigoplus_{F \in +I(E)} R \Gamma (D_F^+, S_F^+; \mathcal{K}) \]

**Proof.** Note that \( S_E^+ - \bigcup_{F \in +I(E)} S_F^+ = \bigcup_{F \in -I(E)} D_F^+ \) (disjoint union). The claim follows now from the Poincaré duality. \( \square \)

Therefore, for any \( F \in +I(E) \) we get a natural inclusion map
\[ i^F_E : R \Gamma (D_F^+, S_F^+; \mathcal{K}) \hookrightarrow R \Gamma (S_F^+, \bigcup_{F \in -I(E)} S_F^+; \mathcal{K}) \] (87)

Let us define a map
\[ u^F_E(\mathcal{K}) : \Phi_F^+(\mathcal{K}) \longrightarrow \Phi_E^+(\mathcal{K}) \] as a composition
\[ R \Gamma (D_F^+, S_F^+; \mathcal{K})[-\sqrt{\cdot}] \xrightarrow{i^F_E} R \Gamma (S_F^+, \bigcup_{F \in -I(E)} S_F^+; \mathcal{K})[-\sqrt{\cdot}] \longrightarrow R \Gamma (S_E^+, S_F^+; \mathcal{K})[-\sqrt{\cdot}] \] (88)

where the last arrow is the coboundary map for the couple \((S_E^+, D_E^+)\), and the second one is evident.

This way we get a natural transformation
\[ ^+u^F_E : \Phi_F^+ \longrightarrow \Phi_E^+ \] (89)
which will be called a canonical map.

We define a variation morphism
\[ ^+u^F_E : \Phi_F^+ \longrightarrow \Phi_F^+ \] (90)
as follows. By definition, \(^+u^F_E(\mathcal{K})\) is the map dual to the composition
\[ D \Phi_F^+(\mathcal{K}) \sim \longrightarrow \Phi_F^+(\mathcal{D}\mathcal{K}) \longrightarrow \Phi_E^+(\mathcal{D}\mathcal{K}) \sim \longrightarrow D \Phi_E^+(\mathcal{K}). \]

7.11. Cochain complexes. For each \( r \in [0, N] \) and \( \mathcal{M} \in \mathcal{M}(\mathbb{A}, \mathcal{S}) \) introduce vector spaces
\[ ^+C^{-r}(\mathbb{A}; \mathcal{M}) = \bigoplus_{F, F \text{ positive}, \dim F = \nabla} \mathcal{M}^+_F \] (91)

For \( r > 0 \) or \( r < -N \) set \(^+C^r(\mathbb{A}; \mathcal{M}) = 0\).

Define operators
\[ d : ^+C^{-r}(\mathbb{A}; \mathcal{M}) \longrightarrow ^+C^{-r+\infty}(\mathbb{A}; \mathcal{M}) \]
having components \(^+u^F_E\).
7.11.1. **Lemma.** \( d^2 = 0. \)

**Proof.** The same as that of I.3.13.1. \( \square \)

This way we get a complex \( ^+C^\bullet (\mathbb{A}; \mathcal{M}) \) lying in degrees from \(-N\) to 0. It will be called the *complex of positive cochains* of the sheaf \( \mathcal{M} \).

7.12. **Theorem.** (i) A functor 
\[
\mathcal{M} \mapsto ^+C^\bullet (\mathbb{A}; \mathcal{M})
\]
is an exact functor from \( \mathcal{M}(\mathbb{A}; \mathcal{S}) \) to the category of complexes of vector spaces.

(ii) We have a canonical natural isomorphism in \( D(\{ \sqrt{\Box} \}) \)
\[
^+C^\bullet (\mathbb{A}; \mathcal{M}) \xrightarrow{\sim} R\Gamma(\mathbb{A}; \mathcal{M})
\]

**Proof.** One sees easily that restriction maps

\[
R\Gamma(\mathbb{A}; \mathcal{M}) \longrightarrow R\Gamma(\mathbb{A}_\mathbb{R}^+; \mathcal{M})
\]

are isomorphisms. The rest of the argument is the same as in I.3.14. \( \square \)

7.13. Let us consider a function \( \sum_j t_j : \mathbb{A} \longrightarrow \mathbb{A}^k \), and the corresponding vanishing cycles functor

\[
\Phi_{\Sigma t_j} : D^1(\mathbb{A}) \longrightarrow D^1(\mathbb{A}^k)
\]

where \( \mathbb{A}^k = \{ (\approx_2) | \sum_2 \approx_2 = k \} \), cf. [KS], 8.6.2.

If \( \mathcal{K} \in D(\mathbb{A}; \mathcal{S}) \), it is easy to see that the complex \( \Phi_{\Sigma t_j}(\mathcal{K}) \) has the support at the origin. Let us denote by the same letter its stalk at the origin — it is a complex of vector spaces.

7.13.1. **Lemma.** We have a natural isomorphism

\[
\Phi_{\Sigma t_j}(\mathcal{K}) \xrightarrow{\sim} \Phi_{\Sigma t_j}^+(\mathcal{K})
\]

**Proof** is left to the reader. \( \square \)

7.14. Let us consider the setup of 6.12. Let us denote by the same letter \( \mathcal{S} \) the stratification of \( \mathcal{A}_\nu \) whose strata are subspaces \( \pi(S) \), \( S \) being a stratum of the stratification \( \mathcal{S} \) on \( \pi\mathbb{A} \). This stratification will be called *the principal stratification* of \( \mathcal{A}_\nu \).

The function \( \Sigma t_j \) is obviously \( \Sigma_\pi \)-equivariant, therefore it induces the function for which we will use the same notation,

\[
\Sigma t_j : \mathcal{A}_\nu \longrightarrow \mathbb{A}^k
\]

Again, it is easy to see that for \( \mathcal{K} \in D^1(\mathcal{A}_\nu; \mathcal{S}) \) the complex \( \Phi_{\Sigma t_j}(\mathcal{K}) \) has the support at the origin. Let us denote by \( \Phi_{\nu}(\mathcal{K}) \) its stalk at the origin.
It is known that the functor of vanishing cycles is $t$-exact with respect to the middle perversity; whence we get an exact functor
\[ \Phi_\nu : \mathcal{M}(\mathcal{A}_\nu; \mathcal{S}) \to \mathcal{V} || \square \] (95)

7.14.1. **Lemma.** Suppose that $\mathcal{N}$ is a $\Sigma_\pi$-equivariant complex of sheaves over $\pi \mathbb{A}$ which belongs to to $\mathcal{D}^l(\pi \mathbb{A}; \mathcal{S})$ (after forgetting $\Sigma_\pi$-action). We have a natural isomorphism
\[ \Phi_\nu((\pi_* \mathcal{N})^{\Sigma_\pi,-}) \cong (\Phi_{\Sigma \sqcup \nu}(\mathcal{N}))^{\Sigma_\pi,-} \] (96)

**Proof** follows from the proper base change for vanishing cycles (see [D], 2.1.7.1) and the exactness of the functor $(\cdot)^{\Sigma_\pi,-}$. We leave details to the reader. $\square$

7.14.2. **Corollary.** For a $\Sigma_\pi$-equivariant sheaf $\mathcal{N} \in \mathcal{M}(\pi \mathbb{A}_\nu; \mathcal{S})$ we have a natural isomorphism of vector spaces
\[ \Phi_\nu(\mathcal{M}) \cong (\Phi_{\nu}(\mathcal{N}))^{\Sigma_\pi,-} \] (97)
where $\mathcal{M} = (\pi_* \mathcal{N})^{\Sigma_\pi,-}$. $\square$

8. **Standard sheaves**

Let us keep assumptions and notations of 4.1.

8.1. Let us denote by $\pi \mathbb{A}$ the complex affine space with coordinates $t_j, j \in J$. We will consider its principal stratification as in 7.1.

Suppose we are given a positive chamber $C$ and a point $x = (x_j)_{j \in J} \in C$. There exists a unique bijection
\[ \sigma_C : J \sim [N] \] (98)
such that for any $i, j \in J$, $\sigma_C(i) < \sigma_C(j)$ iff $x_i < x_j$. This bijection does not depend upon the choice of $x$. This way we can identify the set of all positive chambers with the set of all isomorphisms (98), or, in other words, with the set of all total orderings of $J$.

Given $C$ and $x$ as above, suppose that we have $i, j \in J$ such that $x_i < x_j$ and there is no $k \in J$ such that $x_i < x_k < x_j$. We will say that $i, j$ are *neighbours* in $C$, more precisely that $i$ is a left neighbour of $j$.

Let $x' = (x'_j)$ be a point with $x'_p = x_p$ for all $p \neq j$, and $x'_j$ equal to some number smaller than $x_i$ but greater than any $x_k$ such that $x_k < x_i$. Let $C'$ denote the chamber containing $x'$. Let us introduce a homotopy class of paths $C_{\gamma_{ij}}$ connecting $x$ and $x'$ as shown on Fig. 5(a) below.

On the other hand, suppose that $i$ and 0 are neighbours in $C$, there is no $x_j$ between 0 and $x_i$. Then we introduce the homotopy class of paths from $x$ to itself as shown on Fig. 5 (b).
Fig. 5

All chambers are contractible. Let us denote by $\pi^+_{\mathbb{R}}$ the union of all positive chambers.

We can apply the discussion I.4.1 and consider the groupoid $\pi_1(\pi^+_{\mathbb{R}})$. It has as the set of objects the set of all positive chambers. The set of morphisms is generated by all morphisms $C_{ij}$ and $C_{i0}$ subject to certain evident braiding relations. We will need only the following particular case.

To define a one-dimensional local system $\mathcal{L}$ over $\pi^+_{\mathbb{A}}$ is the same as to give a set of one-dimensional vector spaces $\mathcal{L}_C$, $C \in \pi^+_{\mathbb{R}}$, together with arbitrary invertible linear operators

$$C_{Tij} : \mathcal{L}_C \rightarrow \mathcal{L}_C$$

(“half-monodromies”) defined for chambers where $i$ is a left neighbour of $j$ and

$$C_{Tio} : \mathcal{L}_C \rightarrow \mathcal{L}_C$$

defined for chambers with neighbouring $i$ and 0.

8.2. Let us fix a weight $\Lambda \in X$. We define a one-dimensional local system $\mathcal{I}(\pi \Lambda)$ over $\pi^+_{\mathbb{A}}$ as follows. Its fibers $\mathcal{I}(\pi \Lambda)_C$ are one-dimensional linear spaces with fixed basis vectors; they will be identified with $B$.

Monodromies are defined as

$$C_{Tij} = \zeta^{ij}, \quad C_{Tio} = \zeta^{-2(\Lambda, \pi(i))},$$

for $i, j \in J$.

8.3. Let $j : \pi^+_{\mathbb{A}} \rightarrow \pi^+_{\mathbb{A}}$ denote an open embedding. We will study the following objects of $\mathcal{M}(\pi^+_{\mathbb{A}}; S)$:

$$\mathcal{I}(\pi \Lambda|_\gamma = |_\gamma \mathcal{I}(\pi \Lambda)[\mathcal{N}],$$
where $? = !, *$. We have a canonical map

$$m : \mathcal{I}(\pi \Lambda)_! \longrightarrow \mathcal{I}(\pi \Lambda)_*$$

(101)

and by definition $\mathcal{I}(\pi \Lambda)_*$ is its image, cf. I.4.5.

**COMPUTATIONS FOR $\mathcal{I}(\pi \Lambda)_!$**

8.4. We will use the notations of 3.3.2 with $I = J$ and $n = 1$. For each $r \in [0, N]$, let us assign to a map $\varrho \in \mathcal{P}_N(J; \infty)$ a point $w_\varrho = (\varrho(j))_{j \in J} \in \pi \Lambda_R$. It is easy to see that there exists a unique positive facet $F_\varrho$ containing $w_\varrho$, and the rule

$$\varrho \mapsto F_\varrho$$

(102)

establishes an isomorphism between $\mathcal{P}_N(J; \infty)$ and the set of all positive facets of $S_R$. Note that $\mathcal{P}_r(J; \infty)$ consists of one element — the unique map $J \longrightarrow [0]$; our stratification has one zero-dimensional facet.

At the same time we have picked a point $F_\varrho w := w_\varrho$ on each positive facet $F_\varrho$; this defines cells $D^+_F$, $S^+_F$ (cf. the last remark in 7.2).

8.5. Given $\varrho \in \mathcal{P}_N(J; \infty)$ and $\tau \in \mathcal{P}_N(J; \infty)$, it is easy to see that the chamber $C = F_\tau$ belongs to $\text{Ch}(F_\varrho)$ iff $\tau$ is a refinement of $\varrho$ in the sense of 3.4. This defines a bijection between the set of all refinements of $\varrho$ and the set of all positive chambers containing $F_\varrho$.

We will denote the last set by $\text{Ch}^+(F_\varrho)$.

8.5.1. **Orientations.** Let $F = F_\varrho$ be a positive facet and $C = F_\tau \in \text{Ch}^+(F)$. The map $\tau$ defines an isomorphism denoted by the same letter

$$\tau : J \sim \longrightarrow [N]$$

(103)

Using $\tau$, the natural order on $[N]$ induces a total order on $J$. For each $a \in [r]$, let $m_a$ denote the minimal element of $\varrho^{-1}(a)$, and set

$$J' = J_{\varrho \leq \tau} := J - \{m_1, \ldots, m_r\} \subset J$$

Let us consider the map

$$(x_j) \in D_{F < C} \mapsto \{x_j - m_{\varrho(j)} | j \in J'\} \in \mathbb{R}^r.'$$

It is easy to see that this mapping establishes an isomorphism of the germ of the cell $D_{F < C}$ near the point $F w$ onto a germ of the cone

$$\{0 \leq u_1 \leq \ldots \leq u_{N-r}\}$$

in $\mathbb{R}^r$ where we have denoted for a moment by $(u_i)$ the coordinates in $\mathbb{R}^r$ ordered by the order induced from $J$.  

This isomorphism together with the above order defines an orientation on $D_{F<C}$.

8.6. **Basis in $\Phi_F^+(\mathcal{I}(\pi \Lambda)_I)^*$**. We follow the pattern of I.4.7. Let $F$ be a positive facet of dimension $r$. By definition,

$$\Phi_F^+(\mathcal{I}(\pi \Lambda)_I) = \mathcal{H}^{-\nabla}(\mathcal{D}_F^+, S_F^+; |\mathcal{I}(\pi \Lambda)) \cong \mathcal{H}^{N-\nabla}(\mathcal{D}_F^+, S_F^+; |\mathcal{I}(\pi \Lambda)).$$

Note that we have

$$D_F^+ = (S_F^+ \cup \mathcal{H}_{\mathbb{R}}) = \bigcup_{C \in \text{Ch}^+(F)} \mathcal{D}_{F<C}$$

(disjoint union), therefore by additivity

$$H^{N-r}(D_F^+, S_F^+; j; \mathcal{I}(\pi \Lambda)) \cong \oplus_{C \in \text{Ch}^+(F)} \mathcal{H}^{N-\nabla}(\mathcal{D}_{F<C}, S_{F<C}; |\mathcal{I}(\pi \Lambda)).$$

By Poincaré duality,

$$H^{N-r}(D_F^+, S_F^+; j; \mathcal{I}(\pi \Lambda))^* \cong \mathcal{H}'(\mathcal{D}_{F<C}; \mathcal{I}(\pi \Lambda)^{-\infty})$$

— here we have used the orientations of cells $D_{F<C}$ introduced above. By definition of the local system $\mathcal{I}$, the last space is canonically identified with $B$.

If $F = F_\varrho$, $C = F_\tau$, we will denote by $c_{\varrho \leq \tau} \in \Phi_F^+(\mathcal{I}(\pi \Lambda)_I)^*$ the image of $1 \in H^0(\mathcal{D}_{F<C}; \mathcal{I}(\pi \Lambda)^{-\infty})$. Thus the chains $c_{\varrho \leq \tau}$, $\tau \in \mathcal{O}_\nabla[\varrho]$, form a basis of the space $\Phi_F^+(\mathcal{I}(\pi \Lambda)_I)^*$.

8.7. **Diagrams**. It is convenient to use the following diagram notations for chains $c_{\varrho \leq \tau}$.

Let us denote elements of $J$ by letters $a, b, c, \ldots$. An $r$-dimensional chain $c_{\varrho \leq \tau}$ where $\varrho : J \to [0, r]$, is represented by a picture:

```
0 \quad 1 \quad \ldots \quad r
```

Fig. 6. Chain $c_{\varrho \leq \tau}$.

A picture consists of $r + 1$ fragments:
where \( i = 0, \ldots, r \), the \( i \)-th fragment being a blank circle with a number of small vectors going from it. These vectors are in one-to-one correspondence with the set \( \varrho^{-1}(i) \); their ends are labeled by elements of this set. Their order (from left to right) is determined by the order on \( \varrho^{-1}(i) \) induced by \( \tau \). The point 0 may have no vectors (when \( \varrho^{-1}(0) = \emptyset \)); all other points have at least one vector.

8.8. Suppose we have \( \varrho \in \mathcal{P}_{\varnothing}(J; \infty) \), \( \varrho' \in \mathcal{P}_{\varnothing+\infty}(J; \infty) \). It is easy to see that \( F_{\varrho} < F_{\varrho'} \) if and only if there exists \( i \in [0, r] \) such that \( \varrho = \delta_i \circ \varrho' \) where \( \delta_i : [0, r+1] \to [0, r] \) carries \( a \) to \( a \) if \( a \leq i \) and to \( a - 1 \) if \( a \geq i + 1 \). We will write in this case that \( \varrho < \varrho' \).

Let us compute the dual to the canonical map

\[ +u^* : \Phi_F^{\varrho}(\mathcal{I}(\pi \Lambda))^* \to \Phi_F^{\varrho'}(\mathcal{I}(\pi \Lambda))^*. \tag{104} \]

Suppose we have \( \tau \in \mathcal{O}\mathcal{N}[\varrho] \); set \( C = F_\tau \), thus \( F_\varrho < F_{\varrho'} < C \). Let us define the sign

\[ \text{sgn}(\varrho < \varrho' \leq \tau) = (-1)^{\sum_{j=i+1}^{r+1} \text{card}((\varrho')^{-1}(j))} \tag{105} \]

This sign has the following geometrical meaning. The cell \( D_{F_{\varrho'} < C} \) lies in the boundary of \( D_{F_\varrho < C} \). We have oriented these cells above. Let us define the compatibility of these orientations as follows. Complete an orienting basis of the smaller cell by a vector directed outside the larger cell — if we get the orientation of the larger cell, we say that the orientations are compatible, cf. I.4.6.1.

It is easy to see from the definitions that the sign \( \text{(105)} \) is equal +1 iff the orientations of \( D_{F_{\varrho'} < C} \) and \( D_{F_\varrho < C} \) are compatible. As a consequence, we get

8.8.1. **Lemma.** The map \( \text{(104)} \) has the following matrix:

\[ +u^*(c_{\varrho \leq \tau}) = \sum \text{sgn}(\varrho < \varrho' \leq \tau) c_{\varrho' \leq \tau}, \]

the summation over all \( \varrho' \) such that \( F_\varrho < F_{\varrho'} < F_\tau \) and \( \dim F_{\varrho'} = \dim F_\varrho + 1 \). \( \square \)
8.9. Isomorphisms \( \pi \phi \). We will use notations of \( \mathcal{B} \) with \( I \) replaced by \( J \), \( \mathcal{F} \) by \( \pi \mathcal{F} \), with \( n = 1 \) and \( \Lambda_0 = \pi \Lambda \).

Thus, for any \( r \in [0, N] \) the set \( \{ b_{\varrho \leq r} | \varrho \in \mathcal{P}_{\mathcal{V}}(J; \infty) , \tau \in \mathcal{O}_{\mathcal{V}}[(\varrho)] \} \) is a basis of \( +C^{r}(\pi A; \mathcal{I}(\pi \Lambda)) \).

Let us pick \( \eta \in \mathcal{P}_{\mathcal{V}}(J) \). Any \( \tau \in \mathcal{P}_{\mathcal{V}}(J; \infty) \) induces the bijection \( \tau' : J \sim \to [N] \). We will denote by \( \text{sgn}(\tau, \eta) = \pm 1 \) the sign of the permutation \( \tau' \eta^{-1} \).

Let us denote by \( \{ b_{\varrho \leq r} | \tau \in \mathcal{O}_{\mathcal{V}}[(\varrho)] \} \) the basis in \( \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)) \) dual to \( \{ e_{\varrho \leq r} \} \).

Let us define isomorphisms

\[
\pi \phi^{(\eta)}_{\pi \mathcal{F}} : \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)) \sim \to \mathcal{C}_{\pi \mathcal{F}}^{-}(\mathcal{V}(\pi \Lambda)) \tag{106}
\]

by the formula

\[
\pi \phi^{(\eta)}_{\pi \mathcal{F}}(b_{\varrho \leq r}) = \text{sgn}(\tau, \eta) \text{sgn}(\varrho) \theta_{\varrho \leq r} \tag{107}
\]

(see \( \mathcal{I} \)) where

\[
\text{sgn}(\varrho) = (-1)^{\sum_{i=1}^{r} (r-i+1) \cdot \text{card}(e^{-1}(i)) - 1} \tag{108}
\]

for \( \varrho \in \mathcal{P}_{\mathcal{V}}(J; \infty) \). Taking the direct sum of \( \pi \phi^{(\eta)}_{\pi \mathcal{F}} \), \( \varrho \in \mathcal{P}_{\mathcal{V}}(J; \infty) \), we get isomorphisms

\[
\pi \phi^{(\eta)}_{\pi \mathcal{F}} : +C^{r}(\pi A; \mathcal{I}(\pi \Lambda)) \sim \to \chi_{\mathcal{J}} \mathcal{C}_{\pi \mathcal{F}}^{-}(\mathcal{V}(\pi \Lambda)) \tag{109}
\]

A direct computation using \( \mathcal{I} \) shows that the maps \( \pi \phi^{(\eta)}_{\pi \mathcal{F}} \) are compatible with differentials. Therefore, we arrive at

8.10. Theorem. The maps \( \pi \phi^{(\eta)}_{\pi \mathcal{F}} \) induce an isomorphism of complexes

\[
\pi \phi^{(\eta)}_{\pi \mathcal{F}} : +C^{\bullet}(\pi A; \mathcal{I}(\pi \Lambda)) \sim \to \chi_{\mathcal{J}} \mathcal{C}_{\pi \mathcal{F}}^{\bullet}(\mathcal{V}(\pi \Lambda)) \tag{110}
\]

Computations for \( \mathcal{I}(\pi \Lambda)_{*} \)

8.11. Bases. The Verdier dual to \( \mathcal{I}(\pi \Lambda)_{*} \) is canonically isomorphic to \( \mathcal{I}(\pi \Lambda)_{!}^{-\infty} \). Therefore, by Theorem \( \mathcal{I} \) for each positive facet \( F \) we have natural isomorphisms

\[
\Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)_{*})^{*} \cong \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)_{!}^{-\infty}) \tag{111}
\]

Let \( \{ \bar{e}_{\varrho \leq r} | \tau \in \mathcal{O}_{\mathcal{V}}[(\varrho)] \} \) be the basis in \( \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)_{!}^{-\infty})^{*} \) defined in \( \mathcal{I} \), with \( \mathcal{I}(\pi \Lambda) \) replaced by \( \mathcal{I}(\pi \Lambda)^{-\infty} \). We will denote by \( \{ c_{\varrho \leq r, \tau} | \tau \in \mathcal{O}_{\mathcal{V}}[(\varrho)] \} \) the dual basis in \( \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)_{*})^{*} \).

Finally, we will denote by \( \{ b_{\varrho \leq r, \tau} | \tau \in \mathcal{O}_{\mathcal{V}}[(\varrho)] \} \) the basis in \( \Phi_{\pi \mathcal{F}}^{+}(\mathcal{I}(\pi \Lambda)_{*})^{*} \) dual to the previous one.
Our aim in the next subsections will be the description of canonical morphisms \( m : \Phi^+_F(\mathcal{I}(\pi A)_i) \to \Phi^+_F(\mathcal{I}(\pi A)_*) \) and of the cochain complex \( +C^\bullet(\pi A; \mathcal{I}(\pi A)_*) \) in terms of our bases.

8.12. Example. Let us pick an element, \( i \in I \) and let \( \pi : J := \{i\} \to I \). Then we are in a one-dimensional situation, cf. 7.5. The space \( \pi A \) has one coordinate \( t_i \). By definition, the local system \( \pi A \) over \( \pi \circ A = \pi A - \{\not\in\} \) with the base point \( w : t_i = 1 \) has the fiber \( \pi \mathcal{I} = B \) and monodromy equal to \( \zeta^{-2(\Lambda, i)} \) along a counterclockwise loop.

The stratification \( S_R \) has a unique 0-dimensional facet \( F = F_{\tau} \) corresponding to the unique element \( \tau \in \mathcal{P} \), and a unique 1-dimensional positive facet \( E = F_{\tau} \) corresponding to the unique element \( \tau \in \mathcal{P} \).

Let us construct the dual chain \( c^*_i := c_{\tau \leq \tau, i} \). We adopt the notations of 7.5, in particular of Fig. 1. The chain \( \tilde{c}_1 := c_{\tau \leq \tau, i} \in H^1(\pi A - \{\not\in\}, \pi \mathcal{I}(\pi A)^{-}\mathcal{I}(\pi A))^* \) is shown on Fig. 8(a) below.

Next, we make a clockwise rotation of \( \tau \) on \( \pi/2 \), and make a homotopy inside the disk \( \pi A \leq \not\in \), to a chain \( c^*_0 := c_{\tau \leq \tau, i} \in H^1(\pi A - \{\not\in\}, \epsilon \cdot \mathcal{I}(\pi A))^* \) beginning and ending at \( \epsilon w \). This chain is shown on Fig. 8(b) (in a bigger scale). Modulo "homothety" identification

\[ H^1(\pi A - \{\not\in\}, \epsilon \cdot \mathcal{I}(\pi A))^* \cong H^\infty(\pi A - \{\not\in\}, \mathcal{I}(\pi A))^* = \Phi_F(\mathcal{I}(\pi A))^* \]

this chain coincides with \( c_{\tau \leq \tau, i} \).

The canonical map \( m^* : \Phi^+_F(\mathcal{I}(\pi A)_i)^* \to \Phi^+_F(\mathcal{I}(\pi A)_*)^* \) carries \( c_{\tau \leq \tau, i} \) to \( [(\Lambda, i)] \cdot c_{\tau \leq \tau} \). The boundary map

\[ d^* : \Phi^+_F(\mathcal{I}(\pi A)_i)^* \to \Phi^+_F(\mathcal{I}(\pi A)_*)^* \]
8.13. Vanishing cycles at the origin. Let us return to the general situation. First let us consider an important case of the unique 0-dimensional facet — the origin. Let 0 denote the unique element of \( \mathcal{P}(\mathcal{J}; \infty) \). To shorten the notation, let us denote \( \Phi^+_{\mathcal{F}_0} \) by \( \Phi_0^+ \).

The bases in \( \Phi_0^+(\mathcal{I}(\pi \Lambda)_i) \), etc. are numbered by all bijections \( \tau : J \rightarrow [N] \). Let us pick such a bijection. Let \( \tau^{-1}(i) = \{j_i\}, i = 1, \ldots, N \). The chain \( c_{0 \leq \tau} \in \Phi_0^+(\mathcal{I}(\pi \Lambda)_i)^* \) is depicted as follows:

![Fig. 9. \( c_{0 \leq \tau} \)](image)

Using considerations completely analogous to the one-dimensional case above, we see that the dual chain \( c_{0 \leq \tau, *} \in \Phi^+_0(\mathcal{I}(\pi \Lambda)_i)^* \) is portrayed as follows:

![Fig. 10. \( c_{0 \leq \tau, *} \)](image)

The points \( t_{j_i} \) are travelling independently along the corresponding loops, in the indicated directions. The section of \( \mathcal{I}(\pi \Lambda)^{-\infty} \) over this cell is determined by the requirement to be equal to 1 when all points are equal to marking points, at the end of their travel (coming from below).

8.14. Isomorphisms \( \phi_{\mathcal{F}_0}^{(n)} \). We will use notations of 3.4 and 3.5 with \( I \) replaced by \( J, \mathcal{F} \) — by \( \pi \mathcal{F} \), with \( n = 1 \) and \( \Lambda_0 = \pi \Lambda \). By definition, \( C^{(0)}_{\pi \mathcal{F}}(V(\pi \Lambda)) = V(\pi \Lambda) \). The space \( V(\pi \Lambda)_{\chi_J} \) admits as a basis (of cardinality \( N! \)) the set consisting of all monomials

\[
\theta_{0 \leq \tau} = \theta_{\tau^{-1}(N)} \theta_{\tau^{-1}(N-1)} \cdots \theta_{\tau^{-1}(1)} v_{\pi \Lambda},
\]
where $\tau$ ranges through the set of all bijections $J \sim [N]$. By definition, $\{\theta^*_0\}_{\theta \leq \tau}$ is the dual basis of $V(\pi^*\Lambda)^*$.

Let us pick $\eta \in P_N(\mathcal{J})$. Let us define an isomorphism

$$\pi_0(\eta)_0: \Phi_0^+(\mathcal{I}(\pi^*\Lambda)_+) \sim \mathcal{V}^{\pi^*\Lambda}_{\chi_J}$$

by the formula

$$\pi_0(\eta)(b_{0 \leq \tau}) = \text{sgn}(\tau, \eta)\theta^*_0 \leq \tau$$

8.15. **Theorem.** The square

$$\Phi_0^+(\mathcal{I}(\pi^*\Lambda)_+) \quad \Phi_0^+(\mathcal{I}(\pi^*\Lambda)_+)$$

$$\downarrow_{m} \quad \downarrow_S \quad \downarrow_S$$

commutes.

**Proof.** This follows directly from the discussion of 8.13 and the definition of the form $S_{\pi^*\Lambda}$. $\square$

8.16. Let us pass to the setup of 6.12 and 7.14. Let $j_\nu: A_\nu \hookrightarrow A_\nu$ denote the embedding of the open stratum of the principal stratification.

Set

$$\mathcal{I}_\nu(\Lambda) = (\pi_*\mathcal{I}(\pi^*\Lambda))^{\Sigma_{-}}$$

It is a local system over $A_\nu$.

Let us define objects

$$\mathcal{I}_\nu(\Lambda)_{?} := [\nu]\mathcal{I}_\nu(\Lambda)[N] \in \mathcal{M}(A_\nu; S)$$

where $? =!, *$ or !*. These objects will be called standard sheaves over $A_\nu$.

The same reasoning as in 6.13 proves

8.16.1. **Lemma.** We have natural isomorphisms

$$\mathcal{I}_\nu(\Lambda)_{?} \cong (\pi_*\mathcal{I}(\pi^*\Lambda))^{\Sigma_{-}}$$

for $? =!, *$ or !*. $\square$

8.17. For a given $\eta \in P_N(\mathcal{J})$ the isomorphisms $\pi_0(\eta)_0$ and $\pi_0(\eta)_!$ are skew $\Sigma_{\pi^*}$ equivariant. Therefore, after passing to invariants in Theorem 8.15 we get
8.18. **Theorem.** The maps $\pi \phi^{(n)}_{0,*}$ and $\pi \phi^{(n)}_{0,!}$ induce isomorphisms included into a commutative square

\[
\begin{array}{ccc}
\Phi_{\nu}(\mathcal{I}_{\nu}(\Lambda)_t) & \xrightarrow{\phi^{(n)}_{\nu,\Lambda,t}} & V(\Lambda)_\nu \\
\downarrow m & & \downarrow S_{\Lambda} \\
\Phi_{\nu}(\mathcal{I}_{\nu}(\Lambda)_*) & \xrightarrow{\phi^{(n)}_{\nu,\Lambda,*}} & V(\Lambda)_\nu
\end{array}
\]

and

\[
\phi^{(n)}_{\nu,\Lambda,!*} : \Phi_{\nu}(\mathcal{I}_{\nu}(\Lambda)_{!*}) \sim \to \mathcal{L}(\Lambda)_\nu
\]

**Proof** follows from the previous theorem and Lemma [4.7.1(ii)]. □

8.19. Now suppose we are given an arbitrary $r, \varrho \in \mathcal{P}_\nu(\mathcal{J}; \infty)$ and $\tau \in \mathcal{O}_\nu[(\varrho)]$. The picture of the dual chain $c_{\varrho \leq \tau,*}$ is a combination of Figures 10 and 3. For example, the chain dual to the one on Fig. 6 is portrayed as follows:

![Diagram of chain](image)

**Fig. 11.** Chain $c_{\varrho \leq \tau,*}$.

8.20. **Isomorphisms** $\pi \Phi_*$. Let us pick $\eta \in \mathcal{P}_\nu(\mathcal{J})$. Let us define isomorphisms

\[
\pi \phi^{(n)}_{\varrho,*} : \Phi_{\varrho}^+(\mathcal{I}(\pi \Lambda)_*) \sim \to \varrho C^\pi(\mathcal{V}(\pi \Lambda))^*
\]

by the formula

\[
\pi \phi^{(n)}_{\varrho,*}(b_{\varrho \leq \tau,*}) = \sgn(\tau, \eta) \sgn(\varrho) \theta^*_{\varrho \leq \tau}
\]

(119)

where $\sgn(\varrho)$ is defined in (108).

Taking the direct sum of $\pi \phi^{(n)}_{\varrho,*}$, $\varrho \in \mathcal{P}_\nu(\mathcal{J}; \infty)$, we get isomorphisms

\[
\pi \phi^{(n)}_{r,*} : +C^r(\pi A; \mathcal{I}(\pi \Lambda)_*) \sim \to \chi_{\mathcal{J}} C^\pi(\mathcal{V}(\pi \Lambda))^*
\]

(120)
8.21. **Theorem.** The maps $\pi \phi^{(n)}_{*}$ induce an isomorphism of complexes

$$\pi \phi^{(n)}_{*} : +C^\bullet(\pi A; \mathcal{I}(\pi \Lambda)_*) \xrightarrow{\sim} \chi_{\mathfrak{g}} C^\bullet_{\pi \mathfrak{g}^*}(V(\pi \Lambda)^*)$$

which makes the square

$$
\begin{array}{ccc}
+C^\bullet(\pi A; \mathcal{I}(\pi \Lambda)_*) & & \chi_{\mathfrak{g}} C^\bullet_{\pi \mathfrak{g}^*}(V(\pi \Lambda)) \\
m \downarrow & & \downarrow S \\
+C^\bullet(\pi A; \mathcal{I}(\pi \Lambda)_*) & & \chi_{\mathfrak{g}} C^\bullet_{\pi \mathfrak{g}^*}(V(\pi \Lambda)^*)
\end{array}
$$

commutative.

**Proof.** Compatibility with differentials is verified directly and commutativity of the square are checked directly from the geometric description of our chains (actually, it is sufficient to check one of these claims — the other one follows formally).

Note the geometric meaning of operators $t_i$ from (25) — they correspond to the deletion of the $i$-th loop on Fig. 10. □

8.22. Now let us pass to the situation 8.16. It follows from Theorem 7.12 (after passing to skew $\Sigma_{\pi^{-}}$ invariants) that the complexes $+C^\bullet(\pi A; \mathcal{I}_{\nu}^{\pi^{-}}, \Sigma_{\pi^{-}})$ compute the stalk of $\mathcal{I}_{\nu}^{\pi^{-}}$ at the origin. Let us denote this stalk by $\mathcal{I}_{\nu}^{\pi^{-}}$. Therefore, passing to $\Sigma_{\pi^{-}}$-invariants in the previous theorem, we get

8.23. **Theorem.** The isomorphisms $\pi \phi^{(n)}_{?}$ where $? = !, \ast$ or $!\ast$, induce isomorphisms in $\mathcal{D}(\sqrt{\bigvee})$ included into a commutative square

$$
\begin{array}{ccc}
\mathcal{I}_{\nu}(\Lambda)_{t,0} & & \nu C^\bullet_{\mathfrak{g}^*}(V(\Lambda)) \\
m \downarrow & & \downarrow S \\
\mathcal{I}_{\nu}(\Lambda)_{*,0} & & \nu C^\bullet_{\mathfrak{g}^*}(V(\Lambda)^*)
\end{array}
$$

and

$$
\lambda \phi^{(n)}_{\nu,!*} : \mathcal{I}_{\nu}(\Lambda)_{!*} \xrightarrow{\sim} \nu C^\bullet_{\mathfrak{f}}(\mathcal{L}(\Lambda)) □
$$

(122)
9. ADDITIVITY THEOREM

9.1. Let us start with the setup of 7.1. For a non-negative integer \( n \) let us denote by \( (n) \) the set \([-n, 0] \). Let us introduce the following spaces. \( \mathbb{A}^{(\kappa)} \) - a complex affine space with a fixed system of coordinates \((t_i), i \in (n)\). Let \( ^nJ \) denote the disjoint union \((n) \cup J; \ ^nA \) — a complex affine space with coordinates \( t_j, j \in ^nJ \).

In general, for an affine space with a distinguished coordinate system of, we will denote by \( S_\Delta \) its diagonal stratification as in 6.1.

Let \( \ ^nA \subset \ ^nA, \ ^nA(\kappa) \subset \mathbb{A}^{(\kappa)} \) be the open strata of \( S_\Delta \).

Let \( \ ^n\rho : \ ^nA \rightarrow \mathbb{A}^{(\kappa)} \) be the evident projection; \( \ ^nB = \ ^n\kappa_\iota^{-1}(\mathbb{A}^{(\kappa)}) \). Given a point \( z = (z_i) \in \mathbb{A}^{(\kappa)} \), let us denote by \( Z\mathbb{A} \) the fiber \( \ ^n\rho^{-1}(z) \) and by \( Z\mathbb{S} \) the stratification induced by \( S_\Delta \). We will consider \( t_j, j \in J \), as coordinates on \( Z\mathbb{A} \).

The subscript \( (.)_\mathbb{R} \) will mean as usually ”real points”.

9.2. Let us fix a point \( z = (z_1, z_0) \in \mathbb{A}_\mathbb{R}^{(\kappa)} \) such that \( z_1 < z_0 \). Let us concentrate on the fiber \( Z\mathbb{A} \). As an abstract variety it is canonically isomorphic to \( \mathbb{A} \) — a complex affine space with coordinates \( t_j, j \in J \); so we will suppress \( z \) from its notation, keeping it in the notation for the stratification \( Z\mathbb{S} \) where the dependence on \( z \) really takes place.

Let us fix a real \( w > z_0 \). Let us pick two open non-intersecting disks \( D_i \subset \mathbb{C} \) with centra at \( z_i \) and not containing \( w \). Let us pick two real numbers \( w_i > z_i \) such that \( w_i \in D_i \), and paths \( P_i \) connecting \( w \) with \( w_i \) as shown on Fig. 12 below.

![Fig. 12.](image-url)

Let us denote by \( Q_\infty(J) \) the set of all maps \( \rho : J \rightarrow [-1,0] \). Given such a map, let us denote by \( A_\rho \subset \mathbb{A} \) an open subvariety consisting of points \( (t_j)_{j \in J} \) such that \( t_j \in D_{\rho(j)} \) for all \( j \). We will denote by the same letter \( Z\mathbb{S} \) the stratification of this space induced by \( Z\mathbb{S} \).
Set $H_w := \bigcup_j H_j(w); P = P_{-1} \cup P_0; \tilde{P} = \{(t_j) \in A| \text{there exists } j \text{ such that } \approx_1 \in \mathbb{P}\}$.

Given $\mathcal{K} \in \mathcal{D}(A; \mathbb{Z}_S)$, the restriction map

$$R\Gamma(A, H \lessdot; \mathcal{K}) \rightarrow R\Gamma(A, \tilde{P}; \mathcal{K})$$

is an isomorphism by homotopy. On the other hand, we have restriction maps

$$R\Gamma(A, \tilde{P}; \mathcal{K}) \rightarrow R\Gamma(A_{\rho}, \tilde{P}_{\rho}; \mathcal{K})$$

where $\tilde{P}_{\rho} := \tilde{P} \cap A_{\rho}$. Therefore we have canonical maps

$$r_\rho : R\Gamma(A, H \lessdot; \mathcal{K}) \rightarrow R\Gamma(A_{\rho}, \tilde{P}_{\rho}; \mathcal{K}) \quad (123)$$

9.3. **Theorem.** For every $\mathcal{K} \in \mathcal{D}(A; \mathbb{Z}_S)$ the canonical map

$$r = \sum r_\rho : R\Gamma(A, H \lessdot; \mathcal{K}) \rightarrow \bigoplus_{\rho \in Q_\varepsilon(J)} R\Gamma(A_{\rho}, \tilde{P}_{\rho}; \mathcal{K}) \quad (124)$$

is an isomorphism.

9.4. **Proof.** Let us pick two open subsets $U_-, U_0 \subset \mathbb{C}$ as shown on Fig. 12. Set $U = U_- \cup U_0, A_U = \{(\approx_2) \in A| \approx_2 \in U \text{ for all } \mathfrak{J}\}$. It is clear that the restriction morphism

$$R\Gamma(A, \tilde{P}; \mathcal{K}) \rightarrow R\Gamma(A_U, \tilde{P}_U; \mathcal{K})$$

where $\tilde{P}_U := \tilde{P} \cap A_U$, is an isomorphism.

For each $\rho \in Q_\varepsilon(J)$ set

$$A_{U, \rho} := \{(\approx_2) \in A_U| \approx_2 \in U_{\rho(\mathfrak{J})} \text{ for all } \mathfrak{J}\}; \tilde{P}_{U, \rho} := \tilde{P} \cap A_{U, \rho} \quad (125)$$

We have $A_U = \bigcup_{\rho} A_{U, \rho}$.

9.4.1. **Lemma.** For every $\mathcal{K} \in \mathcal{D}(A; \mathbb{Z}_S)$ the sum of restriction maps

$$q : R\Gamma(A_U, \tilde{P}_U; \mathcal{K}) \rightarrow \sum_{\rho \in Q_\varepsilon(J)} R\Gamma(A_{U, \rho}, \tilde{P}_{U, \rho}; \mathcal{K}) \quad (126)$$

is an isomorphism.

**Proof.** Suppose we have distinct $\rho_1, \ldots, \rho_m$ such that $A_{U; \rho_1, \ldots, \rho_m} := A_{U, \rho_1} \cap \ldots \cap A_{U, \rho_m} \neq \emptyset$; set $\tilde{P}_{U; \rho_1, \ldots, \rho_m} := \tilde{P} \cap A_{U; \rho_1, \ldots, \rho_m}$.

Our lemma follows at once by Mayer-Vietoris argument from the following
9.4.2. **Claim.** For every \( m \geq 2 \)

\[
R\Gamma(\mathbb{A}_{U_\rho}; \mathbb{P}_{U_\rho}; \mathcal{K}) = 1.
\]

**Proof of the claim.** It is convenient to use the following notations. If \( J = A \cup B \) is a disjoint union, we will denote by \( \rho_{A,B} \) the map \( J \to [-1, 0] \) such that \( \rho^{-1}(1) = A \), \( \rho^{-1}(0) = B \), and by \( U_{A,B} \) the subspace \( \mathbb{A}_{U_\rho} \).

Let us prove the claim for the case \( N = 2 \). Let \( J = \{ i, j \} \). In this case it is easy to see that the only non-trivial intersections are \( U^{(1)} = U_{ij,\emptyset} \cap U_{ji,\emptyset} \) and \( U^{(2)} = U_{ij,\emptyset} \cap U_{\emptyset,ij} \).

To prove our claim for \( U^{(1)} \) we will use a *shrinking neighbourhood argument* based on Lemma I.2.12. Let \( \mathcal{K}' \) denote the sheaf on \( U^{(1)} \) obtained by extension by zero of \( \mathcal{K}|_{U^{(\infty)} - \tilde{P}} \).

For each \( \epsilon > 0 \) let us denote by \( U_{-1,\epsilon} \subset \mathbb{C} \) an open domain consisting of points having distance < \( \epsilon \) from \( D_{-1} \cup P \). Set

\[
U^{(1)}_\epsilon = \{(t_i, t_j) | t_i \in U_{-1,\epsilon}, t_j \in U_0 \cap U_{1,\epsilon}\}.
\]

It is clear that restriction maps \( R\Gamma(U^{(1)}_\epsilon; \mathcal{K}') \to R\Gamma(U^{(\infty)}_\epsilon; \mathcal{K}') \) are isomorphisms. On the other hand, it follows from I.2.12 that

\[
\lim_{\epsilon \to 0} R\Gamma(U^{(1)}_\epsilon; \mathcal{K}') \cong R\Gamma(\bigcap_\epsilon U^{(\infty)}_\epsilon; \mathcal{K}'),
\]

and the last complex is zero by the definition of \( \mathcal{K}' \) (the point \( t_j \) is confined to \( P \) in \( \bigcap_\epsilon U^{(1)}_\epsilon \)).

The subspace \( U^{(2)} \) consists of \( (t_i, t_j) \) such that both \( t_i \) and \( t_j \) lie in \( U_{-1} \cap U_0 \). This case is even simpler. The picture is homeomorphic to an affine plane with a sheaf smooth along the diagonal stratification; and we are interested in its cohomology modulo the coordinate cross. This is clearly equal to zero, i.e. \( R\Gamma(U^{(2)}; \tilde{P} \cap U^{(2)}; \mathcal{K}) = 1 \).

This proves the claim for \( N = 2 \). The case of an arbitrary \( N \) is treated in a similar manner, and we leave it to the reader. This completes the proof of the claim and of the lemma. \( \square \)

9.4.3. **Lemma.** For every \( \rho \in \mathcal{Q}_\epsilon(\mathcal{J}) \) the restriction map

\[
R\Gamma(\mathbb{A}_{U,\rho}; \mathbb{P}_{U,\rho}; \mathcal{K}) \to R\Gamma(\mathbb{A}_{\rho}, \mathbb{P}_\rho; \mathcal{K})
\]

is an isomorphism.

**Proof.** Again let us consider the case \( J = \{ i, j \} \). If \( \rho = \rho_{ij,\emptyset} \) or \( \rho_{\emptyset,ij} \) the statement is obvious. Suppose \( \rho = \rho_{ij} \). Let us denote by \( \mathbb{A}'_{U,\rho} \subset \mathbb{A}_{U,\rho} \) the subspace \( \{(t_i, t_j) | t_i \in D_{-1}, t_j \in U_0\} \). It is clear that the restriction map

\[
R\Gamma(\mathbb{A}'_{U,\rho}; \mathbb{P}'_{U,\rho}; \mathcal{K}) \to R\Gamma(\mathbb{A}_{\rho}, \mathbb{P}_\rho; \mathcal{K})
\]

where \( \mathbb{P}'_{U,\rho} := \tilde{P} \cap \mathbb{A}'_{U,\rho} \), is an isomorphism. Let us consider the restriction

\[
R\Gamma(\mathbb{A}_{U,\rho}; \mathbb{P}_{U,\rho}; \mathcal{K}) \to R\Gamma(\mathbb{A}'_{U,\rho}; \mathbb{P}'_{U,\rho}; \mathcal{K})
\]  

(127)
The cone of this map is isomorphic to $R\Gamma(A_{U,\rho}, \mathbb{P}_{U,\rho}; \mathcal{M})$ where the sheaf $\mathcal{M}$ has the same singularities as $\mathcal{K}$ and in addition is 0 over the closure $\bar{D}_{-1}$. Now, consider a system of shrinking neighbourhoods of $P_{-1} \cup \bar{D}_{-1}$ as in the proof af the claim above, we see that $R\Gamma(A_{U,\rho}, \mathbb{P}_{U,\rho}; \mathcal{M}) = t$, i.e. (127) is an isomorphism. This implies the lemma for this case.

The case of arbitrary $J$ is treated exactly in the same way. □

Our theorem is an obvious consequence of two previous lemmas. □

10. Fusion and tensor products

10.1. Fusion functors. The constructions below were inspired by [Dr].

For each integer $n \geq 1$ and $i \in [n]$, let us define functors

$$^n\psi_i : D^{(\bigtriangleup; S_{\Delta})} \to D^{(-\infty, A; S_{\Delta})}$$

(128)

as follows. We have the $t$-exact nearby cycles functors (see [L] or [KS], 8.6, but note the shift by $[-1]$!)

$$\Psi_{t_{-i} - t_{-i+1}, [-1]} : D^{(\bigtriangleup; S_{\Delta})} \to D^{(A'; S_{\Delta})}$$

where $A'$ denotes (for a moment) an affine space with coordinates $t_j$, $j \in ((n) \cup J) - \{-i\}$. We can identify the last space with $^{n-1}A$ simply by renaming coordinates $t_j$ to $t_{j+1}$ for $-n \leq j \leq -i - 1$. By definition, $^n\psi_i$ is equal to $\Psi_{t_{-i} - t_{-i+1}, [-1]}$ followed by this identification.

10.2. Lemma. (i) For each $n \geq 2$, $i \in [n]$ have canonical isomorphisms

$$^n\alpha_i : ^{n-1}\psi_i \circ ^n\psi_i \sim ^{n-1}\psi_i \circ ^n\psi_{i+1}$$

(129)

and equalities

$$^{n-1}\psi_j \circ ^n\psi_i = ^{n-1}\psi_i \circ ^n\psi_{j+1}$$

for $j > i$, such that

(ii) ("Stasheff pentagon" identity) the diagram below commutes:
10.3. Let us define a \( t \)-exact functor
\[
\psi : \mathcal{D}(\mathbb{A}; S_{\Delta}) \longrightarrow \mathcal{D}(\mathbb{A}; S)
\]
as a composition
\[
i^*_0[-1] \circ n\psi_1 \circ n^{-1}\psi_1 \circ \ldots \circ 1\psi_1,
\]
where
\[
i^*_0 : \mathcal{D}(\mathbb{A}; S_{\Delta}) \longrightarrow \mathcal{D}(\mathbb{A}; S)
\]
denotes the restriction to the subspace \( t_0 = 0 \). Note that \( i^*_0[-1] \) is a \( t \)-exact equivalence. (Recall that \( \mathbb{A} \) and \( S \) denote the same as in \( \mathbb{[1]} \)).

\section*{STANDARD SHEAVES}

The constructions and computations below generalize Section \( \mathbb{[3]} \).

10.4. Let us make the following assumptions. Let us denote by \( \mathbb{A} \) the \( I \times I \)-matrix \( (i \cdot j) \). Let us suppose that \( \det A \neq 0 \). There exists a unique \( \mathbb{Z}_{[\det A]} \)-valued symmetric bilinear form on \( X \) (to be denoted by \( \lambda, \mu \mapsto \lambda \cdot \mu \)) such that the map \( \mathbb{Z}[\mathbb{I}] \longrightarrow \mathbb{X}, \nu \mapsto \lambda_\nu \) respects scalar products.

Let us suppose that our field \( B \) contains an element \( \zeta' \) such that \( (\zeta')^{\det A} = \zeta \), and fix such \( \zeta' \). For \( a = \frac{c}{\det A}, c \in \mathbb{Z}, \) we set by definition \( \zeta^a := (\zeta')^c. \)
10.5. Let us fix \( \nu = \sum \nu_i i \in \mathbb{N}[I] \) and its unfolding \( \pi : J \to I \) as in 6.12, and an integer \( n \geq 1 \). We will use the preceding notations with this \( J \).

Let us fix \( n+1 \) weights \( \Lambda_0, \Lambda_{-1}, \ldots, \Lambda_{-n} \in X \). We define a one-dimensional local system \( I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu) \) over \( ^n \mathbb{A} \) exactly in the same manner as in 6.3, with half-monodromies equal to \( \zeta^{\pi(i), \pi(j)} \) if \( i, j \in J \), to \( \zeta^{\Lambda_i, \pi(j)} \) if \( i \in (n), j \in J \) and to \( \zeta^{\Lambda_i \Lambda_j} \) if \( i, j \in (n) \).

Let \( j : \ ^n \mathbb{A} \to ^n \mathbb{A} \) be the embedding. Let us introduce the sheaves

\[
I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu) \cong |I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu)|[-N - \infty] \in \mathcal{M}(\mathbb{A}; S_A)
\]

where ? = !, * or !*. Applying the functor \( \psi \), we get the sheaves \( \psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu) \in \mathcal{M}(\mathbb{A}; S) \).

All these objects are naturally \( \Sigma_\pi \)-equivariant. We define the following sheaves on \( A_\nu \):

\[
\psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu) \cong (\pi_* \psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu))^{\Sigma_\pi, -}
\]

(cf. 8.16).

The following theorem generalizes Theorem 8.18.

10.6. **Theorem.** Given a bijection \( \eta : J \to [N] \), we have natural isomorphisms included into a commutative square

\[
\Phi_\nu(\psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu)) \quad \Phi_\nu(\psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu))
\]

\[
\begin{array}{c}
\phi^{(n)}_m \downarrow \quad \phi^{(n)}_m \\
(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{-n})) \nu \quad (V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{-n})^*) \nu
\end{array}
\]

and

\[
\phi^{(n)} : \Phi_\nu(\psi I(\Lambda_r, \ldots, \Lambda_{-\infty}; \nu)) \cong (\mathcal{L}(\Lambda_r) \otimes \ldots \otimes \mathcal{L}(\Lambda_{-\infty})) \nu
\]

A change of \( \eta \) multiplies these isomorphisms by the sign of the corresponding permutation of \( [N] \).

**Proof.** We may suppose that \( \pi \) is injective, i.e. all \( \nu_i = 0 \) or 1. The general case immediately follows from this one after passing to \( \Sigma_\pi \)-skew invariants.

Let us consider the case \( n = 1 \). In this case one sees easily from the definitions that we have a canonical isomorphism

\[
\Phi_\nu(\psi I(\Lambda_r, \Lambda_{-\infty}; \nu)) \cong R\Gamma(A, \mathbb{H} \mathbb{L}_\nu; I(\Lambda_r, \Lambda_{-\infty}; \nu))
\]

in the notations of Additivity Theorem 9.3. On the other hand, the set \( Q_\infty(J) \) is in one-to-one correspondence with the set of all decompositions \( \nu = \nu_0 + \nu_{-1}, \nu_i \in \mathbb{N}[I], \) and
if \( \rho \) corresponds to such a decomposition, we have a natural isomorphism

\[
R\Gamma(\mathbb{A}_\rho; \bar{\mathcal{P}} \cap \mathbb{A}_\rho; \mathcal{I}(\Lambda; \Lambda_{-\infty})) \cong \Phi(\bar{\mathcal{I}}(\Lambda_\iota; \nu)) \otimes \Phi(\bar{\mathcal{I}}(\Lambda_{-\infty}; \nu_{-\infty}))
\]

(134)

by the K"unneth formula. Therefore, Additivity Theorem implies isomorphisms

\[
\Phi(\bar{\mathcal{I}}(\Lambda_\iota; \nu)) \cong \bigoplus_{\nu + \nu' = \nu} \Phi(\bar{\mathcal{I}}(\Lambda_\iota; \nu')) \otimes \Phi(\bar{\mathcal{I}}(\Lambda_{-\infty}; \nu_{-\infty}))
\]

which are the claim of our theorem.

The case \( n > 2 \) is obtained similarly by the iterated use of the Additivity Theorem. \( \Box \)

10.7. Next we will consider the stalks \((\mathcal{I})_0\) of our sheaves at 0, or what is the same (since they are \( \mathbb{R}^{++}\)-homogeneous), the complexes \( R\Gamma(\mathbb{A}_\nu; \mathcal{I}) \).

The next theorem generalizes Theorem 8.23.

10.8. \textbf{Theorem.} Given a bijection \( \eta : J \rightarrow [N] \), we have natural isomorphisms included into a commutative square

\[
\begin{array}{ccc}
\psi \mathcal{I}_\nu(\Lambda_1, \ldots, \Lambda_{-\iota})_y & \xrightarrow{\phi_{\nu,0}^{(\eta)}} & \nu C_\mathbb{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{-n})) \\
m \downarrow & & \downarrow S_\Lambda \\
\psi \mathcal{I}_\nu(\Lambda_1, \ldots, \Lambda_{-\iota})_s & \xrightarrow{\phi_{\nu,0}^{(\eta)}} & \nu C_\mathbb{F}^*(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{-n})^*)
\end{array}
\]

and

\[
\phi_{\nu,0}^{(\eta)} : \psi \mathcal{I}_\nu(\Lambda_1, \ldots, \Lambda_{-\iota})_s \xrightarrow{\sim} \nu C_\mathbb{F}^*(\mathcal{L}(\Lambda_0) \otimes \ldots \otimes \mathcal{L}(\Lambda_{-\iota}))
\]

(135)

A change of \( \eta \) multiplies these isomorphisms by the sign of the corresponding permutation of \([N]\).

\textbf{Proof.} It is not hard to deduce from the previous theorem that we have natural isomorphisms of complexes included into a commutative square

\[
\begin{array}{ccc}
C(A; \psi \mathcal{I}(\Lambda_1, \ldots, \Lambda_{-\iota}; \nu))_y & \xrightarrow{\chi_J C_\mathbb{F}(V(\pi \Lambda_0) \otimes \ldots \otimes V(\pi \Lambda_{-n}))} & \nu C_\mathbb{F}(V(\Lambda_0) \otimes \ldots \otimes V(\Lambda_{-n})) \\
m \downarrow & & \downarrow S_\Lambda \\
C(A; \psi \mathcal{I}(\Lambda_1, \ldots, \Lambda_{-\iota}; \nu))_s & \xrightarrow{\chi_J C_\mathbb{F}^*(V(\pi \Lambda_0)^* \otimes \ldots \otimes V(\pi \Lambda_{-n})^*)} & \nu C_\mathbb{F}^*(V(\Lambda_0)^* \otimes \ldots \otimes V(\Lambda_{-n})^*)
\end{array}
\]

This implies our claim after passing to \( \Sigma_\pi\)-(skew) invariants. \( \Box \)
CHAPTER 4. Category \( \mathcal{C} \).

11. Simply laced case

11.1. From now on until the end of the paper we will assume, in addition to the assumptions of [L], that \( \zeta \) is a primitive \( l \)-th root of unity, where \( l \) is a fixed integer \( l > 3 \) prime to 2 and 3.

11.2. We will use notations of [L], Chapters 1, 2, which we briefly recall.

11.2.1. Let \((I, \cdot)\) be a simply laced Cartan datum of finite type (cf. loc.cit., 1.1.1, 2.1.3), that is, a finite set \( I \) and a nondegenerate symmetric bilinear form \( \alpha, \beta \mapsto \alpha \cdot \beta \) on the free abelian group \( \mathbb{Z}[I] \). This form satisfies conditions

(a) \( i \cdot i = 2 \) for any \( i \in I \);

(b) \( i \cdot j \in \{0, -1\} \) for any \( i \neq j \) in \( I \).

11.2.2. We will consider the simply connected root datum of type \((I, \cdot)\), that is (see loc.cit., 2.2.2), two free abelian groups \( Y = \mathbb{Z}[I] \) and \( X = \text{Hom}_\mathbb{Z}(Y, \mathbb{Z}) \) together with

(a) the canonical bilinear pairing \( \langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z} \);

(b) an obvious embedding \( I \hookrightarrow Y \) \((i \mapsto i)\) and an embedding \( I \hookrightarrow X \) \((i \mapsto i')\), such that \( \langle i, j' \rangle = i \cdot j \) for any \( i, j \in I \).

We will call \( X \) the lattice of weights, and \( Y \) the lattice of coroots. An element of \( X \) will be typically denoted by \( \lambda, \mu, \nu, \ldots \); and an element of \( Y \) will be typically denoted by \( \alpha, \beta, \gamma, \ldots \).

11.3. We consider the finite dimensional algebra \( U \) over the field \( B \) defined as in the section 1.3 of [AJS]. We also consider the category \( \mathcal{C} \) of finite dimensional \( X \)-graded \( U \)-modules defined as in the section 2.3 of [AJS].

11.3.1. The algebra \( U \) is given by generators \( E_i, F_i, K_i^{\pm 1}, i \in I \), subject to relations

(a) \( K_i \cdot K_i^{-1} = 1; \ K_iK_j = K_jK_i; \)

(b) \( K_jE_i = \zeta^{i+1}E_jK_j; \)

(c) \( K_jF_i = \zeta^{-j+1}F_iK_j; \)

(d) \( E_iE_i = F_iF_i = 0; \)

(e) \( E_iE_j - E_jE_i = 0 \text{ if } i \cdot j = 0; \ E_i^2E_j - (\zeta + \zeta^{-1})E_iE_jE_i + E_iE_j^2 = 0 \text{ if } i \cdot j = -1; \)

(f) \( F_iF_j - F_jF_i = 0 \text{ if } i \cdot j = 0; \ F_i^2F_j - (\zeta + \zeta^{-1})F_iF_jF_i + F_iF_j^2 = 0 \text{ if } i \cdot j = -1. \)
The algebra $U$ has a unique $B$-algebra $X$-grading $U = \oplus U_\mu$ for which $|E_i| = i'$, $|F_i| = -i'$, $|K_i| = 0$.

We define a comultiplication

$$\Delta : U \longrightarrow U \otimes U$$

as a unique $B$-algebra mapping such that

$$\Delta(K_{i}^{\pm 1}) = K_{i}^{\pm 1} \otimes K_{i}^{\pm 1};$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i;$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

This makes $U$ a Hopf algebra (with obvious unit and counit).

11.3.2. The category $\mathcal{C}$ is by definition a category of finite dimensional $X$-graded $B$-vector spaces $V = \oplus_{\mu \in X} V_\mu$, equipped with a left action of $U$ such that the $U$-action is compatible with the $X$-grading, and

$$K_i x = \zeta^{(i, \mu)} x$$

for $x \in V_\mu$, $i \in I$.

Since $U$ is a Hopf algebra, $\mathcal{C}$ has a canonical structure of a tensor category.

11.4. We define an algebra $u$ having generators $\theta_i, \epsilon_i, K_i^{\pm 1}$, $i \in I$, subject to relations

(z) $K_i \cdot K_i^{-1} = 1$; $K_i K_j = K_j K_i$;
(a) $K_j \epsilon_i = \zeta^{j i} \epsilon_i K_j$;
(b) $K_j \theta_i = \zeta^{-j i} \theta_i K_j$;
(c) $\epsilon_i \theta_j - \zeta^{i j} \theta_j \epsilon_i = \delta_{ij} (1 - K_i^{-2})$
(d) if $f \in \text{Ker}(S) \subset \mathfrak{g}$ (see (19)) then $f = 0$;
(e) the same as (d) for the free algebra $\mathcal{E}$ on the generators $\epsilon_i$.

11.4.1. Let us define the comultiplication

$$\Delta : u \longrightarrow u \otimes u$$

by the formulas

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1};$$
$$\Delta(\theta_i) = \theta_i \otimes 1 + K_i^{-1} \otimes \theta_i;$$
$$\Delta(\epsilon_i) = \epsilon_i \otimes 1 + K_i^{-1} \otimes \epsilon_i$$

and the condition that $\Delta$ is a morphism of $B$-algebras.

This makes $u$ a Hopf algebra (with obvious unit and counit).

$u$ is an $X$-graded $B$-algebra, with an $X$-grading defined uniquely by the conditions $|K_i^{\pm 1}| = 0$; $|\theta_i| = -i'$; $|\epsilon_i| = i'$. 

11.5. We define $\tilde{C}$ as a category of finite dimensional $X$-graded vector spaces $V = \oplus V_\lambda$, equipped with a structure of a left $u$-module compatible with $X$-gradings and such that

$$K_i x = \zeta^{(i,\lambda)} x$$

for $x \in V_\lambda, i \in I$.

Since $u$ is a Hopf algebra, $\tilde{C}$ is a tensor category.

11.6. Recall that for any $\Lambda \in X$ we have defined in 5.2 the $X$-graded $\mathfrak{g}$-module $L(\Lambda)$. It is a quotient-module of the Verma module $V(\Lambda)$, and it inherits its $X$-grading from the one of $V(\Lambda)$ (see 2.15). Thus $L(\Lambda) = \oplus L(\Lambda)_\lambda$, and we define the action of generators $K_i$ on $L(\Lambda)_\lambda$ as multiplication by $\zeta^{(i,\lambda)}$. Finally, we define the action of generators $\epsilon_i$ on $V(\Lambda)$ as in 2.16. These operators on $V(\Lambda)$ satisfy the relations (a) — (c) above.

We check immediately that this action descends to the quotient $L(\Lambda)$. Moreover, it follows from Theorem 2.23 that these operators acting on $L(\Lambda)$ satisfy the relations (a) — (e) above. So we have constructed the action of $u$ on $L(\Lambda)$, therefore we can regard it as an object of $\tilde{C}$.

11.6.1. Lemma. $L(\Lambda)$ is an irreducible object in $\tilde{C}$.

Proof. Let $I(\Lambda)$ be the maximal proper homogeneous (with respect to $X$-grading) submodule of $V(\Lambda)$ (the sum of all homogeneous submodules not containing $v_\Lambda$). Then $V(\Lambda)/I(\Lambda)$ is irreducible, so it suffices to prove that $I(\Lambda) = \ker(S_\Lambda)$. The inclusion $\ker(S_\Lambda) \subset I(\Lambda)$ is obvious. Let us prove the opposite inclusion. Let $y \in I(\Lambda)$. It is enough to check that $S_\Lambda(y, x) = 0$ for any $x \in V(\Lambda)$ of the form $\theta_{i_1} \ldots \theta_{i_n} v_\Lambda$. By (22) we have $S_\Lambda(y, \theta_{i_1} \ldots \theta_{i_n} y, v_\Lambda) = S_\Lambda(\epsilon_{i_n} \ldots \epsilon_{i_1} y, v_\Lambda) = 0$ since $\epsilon_{i_n} \ldots \epsilon_{i_1} y \in I(\Lambda)$. □

11.7. Let us consider elements $E_i, F_i \in u$ given by the following formulas:

$$E_i = \frac{\zeta^2}{\zeta - \zeta^{-1}} \epsilon_i K_i; \quad F_i = \theta_i$$

(138)

It is immediate to check that these elements satisfy the relations 11.3.1 (a) — (c).

Moreover, one checks without difficulty that

$$\theta_i \theta_j - \theta_j \theta_i \in \ker(S) \text{ if } i \cdot j = 0,$$

and

$$\theta_i^2 \theta_j - (\zeta + \zeta^{-1}) \theta_j \theta_i + \theta_j \theta_i^2 \in \ker(S) \text{ if } i \cdot j = -1$$

(cf. [SV2], 1.16). Also, it is immediate that

$$S(\theta_i^a, \theta_i^b) = \prod_{p=1}^{a} \frac{1 - \zeta^{2p}}{1 - \zeta^2}.$$

It follows that $\theta_i^k \in \ker(S)$ for all $i$. 

It follows easily that the formulas (138) define a surjective morphism of algebras

\[ R : U \rightarrow u \]  

(139)

Moreover, one checks at once that \( R \) is a map of Hopf algebras.

Therefore, \( R \) induces a tensor functor

\[ Q : \tilde{C} \rightarrow C \]  

(140)

which is an embedding of a full subcategory.

11.8. **Theorem.** \( Q \) is an equivalence.

**Proof.** It is enough to check that \( \tilde{C} \) contains enough projectives for \( C \) (see e.g. Lemma A.15. of [KL] IV).

First of all, we know from [AJS], section 4.1, that the simple \( u \)-modules \( L(\Lambda), \Lambda \in X \) exhaust the list of simple objects of \( C \). Second, we know from [APK2], Theorem 4.6 and Remark 4.7, that the module \( L(-\rho) \) is projective where \( -\rho \in X \) is characterized by the property \( \langle i, -\rho \rangle = -1 \) for any \( i \in I \). Finally we know, say, from [APK2], Remark 4.7, and [APK1], Lemma 9.11, that the set of modules \( \{ L(\Lambda) \otimes L(-\rho), \Lambda \in X \} \) is an ample system of projectives for \( C \). □

11.9. Denote by \( u^0 \) (resp., \( U^0 \)) the subalgebra of \( u \) (resp., of \( U \)) generated by \( K_i^{\pm 1}, i \in I \). Obviously, both algebras are isomorphic to the ring of Laurent polynomials in \( K_i \), and the map \( R \) induces an identity isomorphism between them.

Denote by \( u^{\leq 0} \) (resp., by \( u^- \)) the subalgebra of \( u \) generated by \( \theta_i, K_i^{\pm 1} \) (resp., by \( \theta_i \)), \( i \in I \). The last algebra may be identified with \( \mathfrak{f} \). As a vector space, \( u^{\leq 0} \) is isomorphic to \( \mathfrak{f} \otimes u^0 \).

Denote by \( U^- \subset U \) the subalgebra generated by \( F_i, i \in I \), and by \( U^{\leq 0} \subset U \) the subalgebra generated by \( F_i, K_i^{\pm 1}, i \in I \). As a vector space it is isomorphic to \( U^- \otimes U^0 \).

11.10. **Theorem.** (a) \( R \) is an isomorphism;

(b) \( R \) induces an isomorphism \( U^- \sim \mathfrak{f} \).

**Proof.** Evidently it is enough to prove b). We know that \( R \) is surjective, and that \( U^- \) is finite dimensional. So it suffices to prove that \( \dim U^- \leq \dim \mathfrak{f} \). We know by [L] 36.1.5. that \( \dim U^- = \dim L(-\rho) \). On the other hand, the map \( \mathfrak{f} \rightarrow L(-\rho), \mathfrak{f} \mapsto \mathfrak{f}(v_{-\rho}) \) is surjective by construction. □

12. **Non-simply laced case**

In the non-simply laced case all main results of this paper hold true as well. However, the definitions need some minor modifications. In this section we will describe them.
12.1. We will use terminology and notations from [L], especially from Chapters 1-3. Let us fix a Cartan datum \((I, \cdot)\) of finite type, not necessarily simply laced, cf. loc. cit., 2.1.3. Let \((Y = \mathbb{Z}[I], X = \text{Hom}(Y, Z), \langle, \rangle, I \leftrightarrow X, I \leftrightarrow Y)\) be the simply connected root datum associated with \((I, \cdot)\), loc.cit., 2.2.2.

We set \(d_i := \frac{i \cdot i}{2}, i \in I\); these numbers are positive integers. We set \(\zeta_i := \zeta^{d_i}\).

The embedding \(I \leftrightarrow X\) sends \(i \in I\) to \(i'\) such that \(\langle d_j, j, i' \rangle = j \cdot i\) for all \(i, j \in I\).

12.2. The category \(\mathcal{C}\) is defined in the same way as in the simply laced case, where the definition of the Hopf algebra \(U\) should be modified as follows (cf. [AJS], 1.3).

By definition, \(U\) has generators \(E_i, F_i, K_i^{\pm 1}, i \in I\), subject to relations

\((z)\) \(K_i \cdot K_i^{-1} = 1; K_i K_j = K_j K_i\);

\((a)\) \(K_j E_i = \zeta^{\langle j, i' \rangle} E_i K_j\);

\((b)\) \(K_j F_i = \zeta^{-\langle j, i' \rangle} F_i K_j\);

\((c)\) \(E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{\zeta_i - \zeta_i^{-1}}\);

\((d)\) \(E_i^l = F_j^l = 0\);

\((e)\) \(\sum_{p=0}^{1-i-j} (-1)^p E_i^{(p)} E_j E_i^{(1-i-j-p)} = 0\) for \(i \neq j\);

\((f)\) \(\sum_{p=0}^{1-i-j} (-1)^p F_i^{(p)} F_j F_i^{(1-i-j-p)} = 0\) for \(i \neq j\),

where we have used the notations: \(\tilde{K}_i := K_i^{d_i}; G_i^{(p)} := G_i/[p]_i^1, G = E\) or \(F\),

\([p]_i^1 := \prod_{a=1}^{p} \frac{\zeta_i^a - \zeta_i^{-a}}{\zeta_i - \zeta_i^{-1}}\).

The \(X\)-grading on \(U\) is defined in the same way as in the simply laced case. The comultiplication is defined as

\[\Delta(K_i) = K_i \otimes K_i;\]

\[\Delta(E_i) = E_i \otimes \tilde{K}_i + 1 \otimes E_i;\]

\[\Delta(F_i) = F_i \otimes 1 + \tilde{K}_i^{-1} \otimes F_i.\]

12.2.1. Remark. This algebra is very close to (and presumably coincides with) the algebra \(U\) from [L], 3.1, specialized to \(v = \zeta\). We use the opposite comultiplication, though.

12.3. The definition of the Hopf algebra \(u\) should be modified as follows. It has generators \(\theta_i, \epsilon_i, K_i^{\pm 1}, i \in I\), subject to relations

\((z)\) \(K_i \cdot K_i^{-1} = 1; K_i K_j = K_j K_i\);

\((a)\) \(K_j \epsilon_i = \zeta^{\langle j, i' \rangle} \epsilon_i K_j\)
(b) \( K_j \theta_i = \zeta^{-ij} \theta_i K_j \);

(c) \( \epsilon_i \theta_j - \zeta_{ij} \theta_j \epsilon_i = \delta_{ij} (1 - \bar{K}_i^{-2}) \)

(d) if \( f \in \text{Ker}(S) \subset \mathfrak{F} \) (see [13]) then \( f = 0 \);

(e) the same as (d) for the free algebra \( \mathfrak{E} \) on the generators \( \epsilon_i \).

The comultiplication is defined as

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \\
\Delta(\theta_i) = \theta_i \otimes 1 + \bar{K}_i^{-1} \otimes \theta_i, \\
\Delta(\epsilon_i) = \epsilon_i \otimes 1 + \bar{K}_i^{-1} \otimes \epsilon_i
\]

The category \( \tilde{C} \) is defined in the same way as in the simply laced case.

12.4. We define an \( X \)-grading on the free \( \mathfrak{F} \)-module \( V(\Lambda) \) with generator \( v_\Lambda \) by setting \( V(\Lambda) = B \cdot v_\Lambda \) and assuming that operators \( \theta_i \) decrease the grading by \( i' \).

The definition of the form \( S_\Lambda \) on \( V(\Lambda) \) should be modified as follows. It is a unique bilinear from such that \( S_\Lambda(v_\Lambda, v_\Lambda) = 1 \) and \( S(\theta_i x, y) = S(x, \epsilon_i y) \) where the operators \( \epsilon_i : V(\Lambda) \to V(\Lambda) \) are defined by the requirements \( \epsilon_i(v_\Lambda) = 0, \)

\[
\epsilon_i(\theta_j x) = \zeta_{ij} \theta_j \epsilon_i(x) + \delta_{ij} [\langle i, \lambda \rangle]_{\zeta_i} x
\]

for \( x \in V(\Lambda) \).

We define \( L(\Lambda) \) as a quotient \( V(\Lambda) / \text{Ker}(S) \). As in the simply laced case, \( L(\Lambda) \) is naturally an object of \( \mathcal{C} \), and the same argument proves that it is irreducible.

12.5. We define the morphism

\[ R : U \to u \]

by the formulas

\[
R(E_i) = \frac{\zeta_i^2}{\zeta_i - \zeta_i^{-1}} \epsilon_i \bar{K}_i; \quad R(F_i) = \theta_i; \quad R(K_i) = K_i
\]

Using [4], 1.4.3, one sees immediately that it is correctly defined morphism of algebras. It follows at once from the definitions that \( R \) is a morphism of Hopf algebras. Hence, we get a tensor functor

\[ Q : \tilde{C} \to \mathcal{C} \]

and the same proof as in [11.8] shows that \( Q \) is an equivalence of categories.

It is a result of primary importance for us. It implies in particular that all irreducibles in \( \mathcal{C} \) (as well as their tensor products), come from \( \tilde{C} \).
12.6. Suppose we are given \( \Lambda_0, \ldots, \Lambda_{-n} \in X \) and \( \nu \in \mathbb{N}[I] \). Let \( \pi : J \to I, \pi : \pi A \to A \nu \) denote the same as in 6.12. We will use the notations for spaces and functors from Section 10.

The definition of the local system \( I(\Lambda'_0, \ldots, \Lambda'_{-n}; \nu) \) from loc. cit. should be modified: it should have half-monodromies \( \zeta^{-\langle \pi(j), \Lambda_i \rangle}_j \) for \( i \in (n), j \in J \), the other formulas stay without change.

After that, the standard sheaves are defined as in loc. cit.

Now we are arriving at the main results of this paper. The proof is the same as the proof of theorems 10.6 and 10.8, taking into account the previous algebraic remarks.

12.7. Theorem. Let \( L(\Lambda_0), \ldots, L(\Lambda_{-n}) \) be irreducibles of \( C, \lambda \in X, \lambda = \sum_{m=0}^{n} \Lambda_{-m} - \sum_{i=1}^{n} \nu_i i' \) for some \( \nu_i \in \mathbb{N} \). Set \( \nu = \sum \nu_i i \in \mathbb{N}[I] \).

Given a bijection \( \eta : J \to [N] \), we have natural isomorphisms

\[
\phi_{\eta}^{(q)} : \Phi_{\nu}(I(\Lambda_0, \ldots, \Lambda_{-n}^\eta)) \to (\mathcal{L}(\Lambda_0) \otimes \cdots \otimes \mathcal{L}(\Lambda_{-n}))_{\lambda} \tag{144}
\]

A change of \( \eta \) multiplies these isomorphisms by the sign of the corresponding permutation of \([N]\). \( \square \)

12.8. Theorem. In the notations of the previous theorem we have natural isomorphisms

\[
\phi_{\eta,0}^{(q)} : \psi I(\Lambda_0, \ldots, \Lambda_{-n}^\eta)_0 \to C^*(\mathcal{L}(\Lambda_0) \otimes \cdots \otimes \mathcal{L}(\Lambda_{-n}))_{\lambda} \tag{145}
\]

where we used the notation \( C^*(\ldots)_{\lambda} \) for \( \rho C^*(\ldots) \). A change of \( \eta \) multiplies these isomorphisms by the sign of the corresponding permutation of \([N]\). \( \square \)

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