Constructing a bivariate distribution function with given marginals and correlation: application to the galaxy luminosity function

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ABSTRACT

We provide an analytic method to construct a bivariate distribution function (DF) with given marginal distributions and correlation coefficient. We introduce a convenient mathematical tool, called a copula, to connect two DFs with any prescribed dependence structure. If the correlation of two variables is weak (Pearson’s correlation coefficient $|\rho| < 1/3$), the Farlie–Gumbel–Morgenstern (FGM) copula provides an intuitive and natural way to construct such a bivariate DF. When the linear correlation is stronger, the FGM copula cannot work anymore. In this case, we propose using a Gaussian copula, which connects two given marginals and is directly related to the linear correlation coefficient between two variables. Using the copulas, we construct the bivariate luminosity function (BLF) and discuss its statistical properties.

We focus especially on the far-infrared–far-ultraviolet (FUV–FIR) BLF, since these two wavelength regions are related to star-formation (SF) activity. Though both the FUV and FIR are related to SF activity, the univariate LFs have a very different functional form: the former is well described by the Schechter function whilst the latter has a much more extended power-law-like luminous end. We construct the FUV–FIR BLFs using the FGM and Gaussian copulas with different strengths of correlation, and examine their statistical properties. We then discuss some further possible applications of the BLF: the problem of a multiband flux-limited sample selection, the construction of the star-formation rate (SFR) function, and the construction of the stellar mass of galaxies ($M_\ast$)–specific SFR ($\text{SFR}/M_\ast$) relation. The copulas turn out to be a very useful tool to investigate all these issues, especially for including complicated selection effects.

Key words: methods: statistical – dust, extinction – galaxies: evolution – galaxies: luminosity function, mass function – infrared: galaxies – ultraviolet: galaxies.

1 INTRODUCTION

A luminosity function (LF) of galaxies is one of the fundamental tools used to describe and explore the distribution of luminous matter in the Universe (see for example Binggeli, Sandage & Tammann 1988; Lin et al. 1996; Takeuchi 2000; Takeuchi, Yoshikawa & Ishii 2000; Blanton et al. 2001; de Lapparent et al. 2003; Takeuchi, Buat & Burgarella 2005; Willmer et al. 2006). Up to now, studies of the LFs have been restricted to univariate ones, i.e. LFs based on a single selection wavelength band. However, such a situation is drastically changing in the era of large and/or deep Legacy surveys. Indeed, a vast number of recent studies are multiband-oriented: they require data from various wavelengths, from the ultraviolet (UV) to the infrared (IR) and radio bands. A bivariate luminosity function (BLF) would be a very convenient tool in such studies. However, to date, it has often been defined and used in a confused manner, without careful consideration of complicated selection effects in both bands. This confusion might arise partially because of the intrinsically complicated nature of multiband surveys, but also because of the lack of proper recipes to describe a BLF. Thus, the situation would be remedied if we had a proper analytic BLF model.

However, it is not a trivial task to determine the corresponding bivariate function from its marginal distributions, if the distribution is not multivariate Gaussian. In fact, there exist infinitely many distributions with the same marginals because the correlation structure is not specified. In general astronomical applications (not only BLFs), for instance, a bivariate distribution is often obtained in either an ad hoc or a heuristic manner (e.g. Choloniewski 1985; Chapman et al. 2003; Schafer 2007), though the methods used are quite well designed for their purposes. Further, analytic bivariate distribution models are often required to interpret the distributions obtained by non-parametric methods (e.g. Cross & Driver 2002; Ball et al. 2006; Driver et al. 2006). For such purposes, a general method by which to
construct a bivariate distribution function with pre-defined marginal distributions and correlation coefficient is desired.

In econometrics and mathematical finance, such a function has been commonly used to analyse two covariate random variables. This is called a ‘copula’. Especially in a bivariate context, copulas are useful to define non-parametric measures of dependence for pairs of random variables (e.g. Trivedi & Zimmer 2005). In astrophysics, however, it is only recently that copulas have attracted researchers’ attention and they are not very widely known yet; there are still only a handful of astrophysical applications (Benabed et al. 2009; Jiang et al. 2009; Koen 2009; Scherrer et al. 2010). Hence the usefulness and limitations of copulas are still not well understood in the astrophysical community.

In this paper, we first introduce a relatively rigorous definition of a copula. We then choose two specific copulas, the Farlie–Gumbel–Morgenstern (FGM) and the Gaussian, to adopt for the construction of a model BLF. Both of them have the ideal property that they are explicitly related to the linear correlation coefficient. Although, as we show in the following, the linear correlation coefficient is not a perfect measure of the dependence of two quantities, this is the most familiar and thus fundamental statistical tool for physical scientists. We focus on the far-infrared–far-ultraviolet (FIR–FUV) BLF as a concrete example, and discuss its properties and some applications.

This paper is organized as follows: in Section 2 we define a copula and present its dependence measures. We also introduce two concrete functional forms, the FGM copula and the Gaussian copula. In Section 3, we make use of these copulas to construct a BLF of galaxies. We emphasize in particular the FIR–FUV BLF. We discuss some implications and further applications in Section 4. Section 5 is devoted to a summary and conclusions. In Appendix A, we show an iterated extension of the FGM copula. We present statistical estimators of the dependence measures of two variables in Appendix B to complete the discussion.

Throughout this paper, we adopt a cosmological model ($h$, $\Omega_{m0}$, $\Omega_{\Lambda0}$) = (0.7, 0.3, 0.7) ($h \equiv H_0/100$ km s$^{-1}$) unless otherwise stated.

2 FORMULATION

2.1 The copula

As we discussed in the Introduction, there is an infinite number of degrees of freedom to choose the dependence structure of two variables with a given marginal distribution. However, very often we need a systematic procedure to construct a bivariate distribution function (DF) of two variables. Copulas have a very desirable property from this point of view. In short, copulas are functions that join multivariate DFs to their one-dimensional marginal DFs. However, this statement does not serve as a definition. We first introduce its abstract framework, and move on to a more concrete form which is suitable for the aim of this work (and many other physical studies).

Before defining the copula, we prepare some mathematical concepts in the following.

Definition 1. Let $S_1$ and $S_2$ be non-empty subsets of $\mathbb{R}$ [a union of real number $\mathbb{R}$ and $(-\infty, \infty]$]. Let $H$ be a real function with two arguments (referred to as bivariate or 2-place) such that $\text{Dom } H = S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle, all vertices of which are in $\text{Dom } H$. Then, the $H$-volume of $B$ is defined by

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).$$

Definition 2. A bivariate real function $H$ is 2-increasing if $V_H \geq 0$ for all rectangles $B$ the vertices of which lie in $\text{Dom } H$.

Definition 3. Suppose $S_1$ has a least element $a_1$ and $S_2$ has a least element $a_2$. Then a function $H : S_1 \times S_2 \to \mathbb{R}$ is grounded if $H(x, a_2) = 0$ and $H(a_1, y) = 0$ for all $(x, y)$ in $S_1 \times S_2$.

With these concepts, we now define a two-dimensional copula.

Definition 4. A two-dimensional copula (or in brief 2-copula) is a function $C$ with the following properties:

(i) $\text{Dom } C = [0, 1] \times [0, 1]$;

(ii) $C$ is grounded and 2-increasing;

(iii) For every $u$ and $v$ in $[0,1]$.

$$C(u, 1) = u \text{ and } C(1, v) = v.$$ (2)

It may be useful to show that there are upper and lower limits for the ranges of copulas, which are given by the following theorem.

Theorem 1. Let $C$ be a copula. Then, for every $(u, v)$ in $\text{Dom } C$,

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v).$$ (3)

Often the notations $W(u, v) \equiv \max(u + v - 1, 0)$ and $M(u, v) \equiv \min(u, v)$ are used. The former and the latter are referred to as the Fréchet–Hoeffding lower bound and Fréchet–Hoeffding upper bound, respectively.

The Fréchet–Hoeffding lower and upper bounds are illustrated in Fig. 1. Any copula has its value between $W(u, v)$ and $M(u, v)$.

Any bivariate function that satisfies the above conditions can be a copula. Thus, there is an infinite number of degrees of freedom for a set of copulas. Using a copula $C$, we can construct a bivariate DF $G$ with two margins $F_1$ and $F_2$ as

$$G(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$ (4)

However, a natural question arises: can any bivariate DF be written in the above form? This is guaranteed by Sklar’s theorem (Sklar 1959).

Theorem 2. (Sklar’s theorem) Let $G$ be a joint distribution function with margins $F_1$ and $F_2$. Then, there exists a copula $C$ such that for all $x_1, x_2$ in $\mathbb{R}$

$$G(x_1, x_2) = C[F_1(x_1), F_2(x_2)].$$ (5)

If $F_1$ and $F_2$ are continuous, then $C$ is unique: otherwise, $C$ is uniquely determined on $\text{Range } F_1 \times \text{Range } F_2$.

A comprehensive proof of Sklar’s theorem is found in e.g. Nelsen (2006). This theorem gives the basis that any bivariate DF with given margins can be expressed with a form of equation (5). Thus, finally, the somewhat abstract definition of a copula turns out to be really useful for our aim, i.e. to construct a bivariate DF when its marginals are known in some way.

Up to now, we have discussed only bivariate DFs and their copulas. It is straightforward to introduce multivariate DFs as a natural extension of the formulation presented here.

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2.2 Copulas and dependence measures between two variables

The most important statistical aspect of bivariate DFs is their dependence properties between variables. Since the dependence can never be given by the marginals of a DF, this is the most non-trivial information that a bivariate DF provides. Since any bivariate DFs are described by equation (5), all the information regarding the dependence is carried by their copulas.

The most familiar measure of dependence among physical scientists (and others) may be correlation coefficients, especially Pearson’s product-moment correlation coefficient $\rho$. The bivariate PDF of $x_1$ and $x_2$, $g(x_1, x_2)$, is written as

$$g(x_1, x_2) = \frac{\partial^2 C[F_1(x_1), F_2(x_2)]}{\partial x_1 \partial x_2} f_1(x_1) f_2(x_2)$$

$$= c[F_1(x_1), F_2(x_2)] f_1(x_1) f_2(x_2),$$

where $f_1(x_1)$ and $f_2(x_2)$ are PDFs of $f_1(x_1)$ and $f_2(x_2)$, respectively. Then the correlation coefficient is expressed as

$$\rho = \sqrt{\frac{\int (x_1 - \bar{x}_1)^2 f_1(x_1) \, dx_1 \int (x_2 - \bar{x}_2)^2 f_2(x_2) \, dx_2}{\left[\int (x_1 - \bar{x}_1)^2 f_1(x_1) \, dx_1 \int (x_2 - \bar{x}_2)^2 f_2(x_2) \, dx_2\right]^{\frac{1}{2}}}}.$$

(7)

We should note that $\rho$ can measure only a linear dependence of two variables. However, in general the dependence of two variables would be not linear at all, and it cannot be a sufficient measure of dependence. Further, and more fundamentally, equation (7) depends not only on the dependence of two variables (copula part) but also the marginals $f_1(x_1)$, $f_2(x_2)$, i.e. the linear correlation coefficient $\rho$ does not measure only the dependence. In such a situation, a more flexible and genuine measure of dependence, e.g. Spearman’s $\rho_S$ or Kendall’s $\tau$, would be more appropriate. Spearman’s rank correlation is a non-parametric version of Pearson’s correlation using ranked data. The population version of Spearman’s $\rho_S$ is expressed by copula as

$$\rho_S = 12 \int_0^1 \int_0^1 u_1 u_2 \, dC(u_1, u_2) - 3$$

$$= 12 \int_0^1 \int_0^1 C(u_1, u_2) \, du_1 \, du_2 - 3.$$  (8)

The definition of Kendall’s $\tau$ is more complicated. We define the concept of concordance as follows: when we have pairs of data $(x_{1i}, x_{2i})$ and $(x_{1j}, x_{2j})$, they are said to be concordant if $x_{1i} > x_{1j}$ and $x_{2i} > x_{2j}$ or $x_{1i} < x_{1j}$ and $x_{2i} < x_{2j}$ (i.e. $(x_{1i} - x_{1j})(x_{2i} - x_{2j}) > 0$), and otherwise discordant. Let $n_c$ denote the number of concordant pairs, and $n_d$ the number of discordant ones. Then, Kendall’s $\tau$ for the sample, $t$, is defined as

$$t = \frac{n_c - n_d}{n_c + n_d}$$  (9)

The population version of $\tau$ is also expressed in a simple form in terms of copula as

$$\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) \, dC(u_1, u_2) - 1$$

$$= 4 \int_0^1 \int_0^1 C(u_1, u_2) c(u_1, u_2) \, du_1 \, du_2 - 1.$$  (10)

Note that both equations (8) and (10) are independent of the distributions $F_1$, $F_2$ and $G$, but they only show the dependence structure described by the copula, unlike the linear correlation coefficient. This is a direct consequence of the non-parametric (i.e. distribution-free) nature of these estimators. This is the reason why both dependence measures are almost always used in the context of copulas in the literature. Estimators of $\rho_S$ and $\tau$ from a sample are found in Appendix B.
2.3 Farlie–Gumbel–Morgenstern (FGM) copula

As seen in the discussion above, the usefulness of the linear correlation coefficient is quite limited, and distribution-free measures of dependence are more appropriate for general joint DFs with non-Gaussian marginals. However, even if this is true, physicists may cling to the most familiar linear correlation coefficient \( \rho \) and hence a copula that has an explicit dependence on \( \rho \) would be convenient. We introduce two special types of copula with this ideal property.

For cases where the correlation between two variables is weak, a systematic method has been proposed by Morgenstern (1956) and Gumbel (1960) for specific functional forms, and later generalized to arbitrary functions by Farlie (1960). These forms are known as FGM distributions after the inventors’ names. Although the study of FGM distributions does not seem tightly connected to that of copulas, we will see later that they can be expressed in terms of the copulas that have an explicit dependence on \( \rho \).

The correlation structure of FGM distributions was studied by Schucany, Parr & Boyer (1978). Let \( F_1(x_1) \) and \( F_2(x_2) \) be the cumulative distribution functions (CDFs) of stochastic variables \( x \) and \( y \), respectively, and let \( F_1(x_1) \) and \( F_2(x_2) \) be their probability density functions (PDFs). The bivariate FGM system of distributions \( G(x_1, x_2) \) is written as

\[
G(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \kappa (1 - F_1(x_1))(1 - F_2(x_2))],
\]

where \( G(x_1, x_2) \) is the joint DF of \( x_1 \) and \( x_2 \). Here \( \kappa \) is a parameter related to the correlation (see below), and to make \( G(x_1, x_2) \) have an appropriate property as a bivariate DF [\( \kappa \leq 1 \) is required (for a proof, see Cambanis 1977)]. Its PDF can be obtained by a direct differentiation of \( G(x_1, x_2) \) as

\[
g(x_1, x_2) = \frac{\partial^2 G}{\partial x_1 \partial x_2} \bigg|_{x_1, x_2} = \frac{\partial^2}{\partial x_1 \partial x_2} \left[ F_1(x_1)F_2(x_2)[1 + \kappa (1 - F_1(x_1))(1 - F_2(x_2))] \right]_{x_1, x_2} = f_1(x_1)f_2(x_2)[1 + \kappa (2F_1(x_1) - 1)(2F_2(x_2) - 1)].
\]

From equation (12), it is straightforward to obtain its covariance function \( \text{Cov}(x_1, x_2) \) as

\[
\text{Cov}(x_1, x_2) = \kappa \int x_1 [2F_1(x_1) - 1] f_1(x_1) \, dx_1 \times \int x_2 [2F_2(x_2) - 1] f_2(x_2) \, dx_2.
\]

Then we have a correlation function of two stochastic variables \( x_1 \) and \( x_2 \), \( \rho(x_1, x_2) \) (equation 7) as follows:

\[
\rho(x_1, x_2) = \frac{\text{Cov}(x_1, x_2)}{\sigma_1 \sigma_2} = \frac{\kappa}{\sigma_1 \sigma_2} \int x_1 [2F_1(x_1) - 1] f_1(x_1) \, dx_1 \int x_2 [2F_2(x_2) - 1] f_2(x_2) \, dx_2,
\]

where \( \sigma_1 \) and \( \sigma_2 \) are the standard deviations of \( x_1 \) and \( x_2 \) with respect to \( f_1(x_1) \) and \( f_2(x_2) \). It is straightforwardly confirmed that \( g(x_1, x_2) \) really has marginals \( f_1(x_1) \) and \( f_2(x_2) \), by a direct integration with respect to \( x_1 \) or \( x_2 \). It is also clear that if we want a bivariate PDF with a prescribed correlation coefficient \( \rho \) we can determine the parameter \( \kappa \) from equations (7) and (13).

Here we consider the case in which both \( f_1(x_1) \) and \( f_2(x_2) \) are Gamma distributions, i.e.

\[
f_j(x_j) = \frac{x_j^{a_j-1}e^{-x_j/b_j}}{b_j^a_j \Gamma(a_j)},
\]

where \( \Gamma(x) \) is the gamma function \( (j = 1, 2) \). In this case, after some algebra, \( \rho \) can be written analytically as

\[
\rho(x_1, x_2) = \frac{\kappa}{\sqrt{ab}} \left[ \frac{2 - 2(1-\rho) - 2\kappa}{B(a, a)} \right] \left[ \frac{2 - 2a(1-\rho) - 2\kappa}{B(b, b)} \right],
\]

where

\[
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\]

This result was obtained by D’Este (1981). In the case in which \( b = 1 \) this corresponds to a bivariate extension of the Schechter function (Schechter 1976), and we expect some astrophysical applications.

As we mentioned, the correlation of the FGM distributions is restricted to be weak: indeed, the correlation coefficient cannot exceed 1/3. Here we prove this. For all (absolutely continuous) \( F(x) \),

\[
\int \left( \int \frac{[2F(x) - 1]}{f(x)} \, dx \right)^2 \leq \int \left( \int \frac{[2F(x) - 1]}{f(x)} \, dx \right)^2 \int (x - \bar{x})^2 f(x) \, dx = \frac{\sigma^2}{3},
\]

where \( \bar{x} \) is the average of \( x \). The second line follows from the Cauchy–Schwarz inequality. From equations (13) and (7), and the condition \( |\kappa| \leq 1 \), we obtain \( |\rho| \leq 1/3 \).

The copula of the FGM family of distributions is expressed as

\[
C_{\text{FGM}}(u_1, u_2; \kappa) = u_1 u_2 + \kappa u_1 u_2 (1 - u_1)(1 - u_2),
\]

with \(-1 < \kappa < 1\). The differential form of the FGM copula follows from equations (6) and (19):

\[
c_{\text{FGM}}(u_1, u_2; \kappa) = 1 + \kappa (1 - 2u_1)(1 - 2u_2).
\]

2.4 Gaussian copula

As seen in Section 2.3, although the FGM distribution is one of the most ‘natural’ examples of a bivariate DF with an explicit \( \rho \)-dependence, the limitation of the correlation coefficient of the FGM family hampers a flexible application of this DF, though there have been many attempts to extend its range of application (see Appendix A). The second natural candidate may therefore be a copula related to a bivariate Gaussian DF. The Gaussian copula also has an explicit dependence on a linear correlation coefficient because of its construction.

Let

\[
\psi_1(x) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2}\right),
\]

\[
\Psi_1 = \int_{-\infty}^x \psi_1(x') \, dx',
\]

\[
\psi_2(x_1, x_2; \rho) = \frac{1}{\sqrt{(2\pi)^2(1 - \rho^2)}} \exp \left[-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1 - \rho^2)}\right]
\]

and

\[
\Psi_2(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \psi_2(x_1', x_2') \, dx_1' \, dx_2'.
\]
By using the covariance matrix \( \Sigma \),

\[
\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},
\]

(25)
equation (23) is simplified as

\[
\psi_2(x_1, x_2; \rho) = \frac{1}{\sqrt{2\pi} \det \Sigma} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right),
\]

(26)
where \( x = (x_1, x_2)^T \) and superscript ‘T’ stands for the transpose of a matrix or vector.

We then define a Gaussian copula \( C^G(u_1, u_2; \rho) \) as

\[
C^G(u_1, u_2; \rho) = \psi_2 \left( \phi^{-1}(u_1), \phi^{-1}(u_2); \rho \right).
\]

(27)
The density of \( C^G, c^G \), is obtained as

\[
c^G(u_1, u_2; \rho) = \frac{\partial^2 C^G(u_1, u_2; \rho)}{\partial u_1 \partial u_2} = \frac{\partial^2 \psi_2 \left[ \phi^{-1}(u_1), \phi^{-1}(u_2); \rho \right]}{\partial u_1 \partial u_2}
\]

\[
= \frac{\psi_2(x_1, x_2; \rho)}{\psi_1(x_1) \psi_1(x_2)} \\
\quad \times \frac{1}{\sqrt{2\pi} \det \Sigma} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right)
\]

\[
= \frac{1}{\sqrt{\det \Sigma}} \exp \left[ -\frac{1}{2} \left[ \Sigma^{-1} - I \right] x \right]
\]

\[
= \frac{1}{\sqrt{\det \Sigma}} \exp \left[ -\frac{1}{2} \left[ \psi^{-1}(u_1), \psi^{-1}(u_2); \rho \right] \right],
\]

(28)
where \( \psi^{-1} = \left[ \psi^{-1}(u_1), \psi^{-1}(u_2) \right]^T \) and \( I \) stands for the identity matrix. The second line follows from equation (6).

3 APPLICATION TO CONSTRUCT THE BIVARIATE LUMINOSITY FUNCTION (BLF) OF GALAXIES

3.1 Construction of the BLF

We define the luminosity for a certain wavelength band as \( L \equiv L_\nu \) (\( \nu \) is the corresponding frequency). Then the luminosity function is defined as the number density of galaxies with luminosity in the logarithmic interval \([\log L, \log L + \Delta \log L] \):

\[
\phi^{(1)}(L) = \frac{d n}{d \log L},
\]

(29)
where we denote \( \log x \equiv \log_{10} x \) and \( \ln x \equiv e^x \). For mathematical simplicity, we define the LF as being normalized, i.e.

\[
\int \phi^{(1)}(L) d \log L = 1.
\]

(30)
Hence this corresponds to a probability density function (PDF), a commonly used terminology in the field of mathematical statistics. We also define the cumulative LF as

\[
\Phi^{(1)}(L) = \int_{\log L_{\min}}^{\log L} \phi^{(1)}(L') \, d \log L',
\]

(31)
where \( L_{\min} \) is the minimum luminosity of galaxies considered. This corresponds to the DF.

If we denote the univariate LFs as \( \phi^{(1)}_1(L_1) \) and \( \phi^{(1)}_2(L_2) \), then the bivariate PDF \( \phi^{(2)}(L_1, L_2) \) is described by a differential copula \( c(u_1, u_2) \) as

\[
\phi^{(2)}(L_1, L_2) \equiv c \left[ \phi^{(1)}_1(L_1), \phi^{(1)}_2(L_2) \right].
\]

(32)
For the FGM copula, the BLF follows from equation (20):

\[
\phi^{(2)}(L_1, L_2; \kappa) \equiv \{ 1 + \kappa \left[ 2 \Phi^{(1)}_1(L_1) - 1 \right] \left[ 2 \Phi^{(1)}_2(L_2) - 1 \right] \}
\]

\[
\times \phi^{(1)}_1(L_1) \phi^{(1)}_2(L_2),
\]

(33)
The parameter \( \kappa \) is proportional to the correlation coefficient \( \rho \) between \( \log L_1 \) and \( \log L_2 \). For the Gaussian copula, the BLF is obtained as

\[
\phi^{(2)}(L_1, L_2; \rho) = \frac{1}{\sqrt{\det \Sigma}} \exp \left[ -\frac{1}{2} \left[ \psi^{-1} \left( \Phi^{(1)}_1(L_1) \right), \psi^{-1} \left( \Phi^{(1)}_2(L_2) \right) \right]^T \right]
\]

\[
\times \phi^{(1)}_1(L_1) \phi^{(1)}_2(L_2),
\]

(34)
and \( \Sigma \) is again defined by equation (25).

3.2 The FIR–FUV BLF

Here, to make our model BLF astrophysically realistic, we construct the FUV–FIR BLF by the copula method. For the IR, we use the parameters presented by Wyder et al. (2005) for the \( \text{IRAS} \) \( PSC_z \) galaxies (Saunders et al. 2000). For the UV, we adopt the Schechter function (Schechter 1976),

\[
\phi^{(1)}_2(L) = \left( \ln 10 \right) \phi_2 \left( \frac{L}{L_{\eff}} \right)^{1-\alpha_2} \exp \left[ -\frac{1}{2 \sigma_2^2} \left( \log \left( 1 + \frac{L}{L_{\eff}} \right) \right)^2 \right].
\]

(36)
We adopt the parameters estimated by Takeuchi, Yoshikawa & Ishii (2003), which are obtained from the \( \text{IRAS} PSC_z \) galaxies (Saunders et al. 2000). For the UV, we adopt the Schechter function (Schechter 1976),

\[
\phi^{(1)}_2(L) = \left( \ln 10 \right) \phi_2 \left( \frac{L}{L_{\eff}} \right)^{1-\alpha_2} \exp \left[ -\left( \frac{L}{L_{\eff}} \right) \right].
\]

(37)
We use the parameters presented by Wyder et al. (2005) for the \textit{GALEX} FUV \( \lambda_{\text{eff}} = 1530 \, \text{Å} \); \( \alpha_2, L_{\eff}, \phi_2 = (1.21, 1.81 \times 10^9 \, h^{-2} \, \text{L}_\odot, 1.35 \times 10^{-2} \, h^3 \, \text{Mpc}^{-3}) \). For simplicity, we neglect the K-correction. We use the renormalized version of equations (36) and (37) so that they can be regarded as PDFs, as mentioned above.

3.3 Results

We show the BLFs constructed from the FGM and Gaussian copulas in Figs 2 and 3, respectively. The FGM-based BLF cannot have a linear correlation coefficient larger than \( \simeq 0.3 \) as explained above, while the Gaussian-based BLF may have a much higher linear correlation. We note that both copulas allow negative correlations, which are not discussed in this paper.

First, even if the linear correlation coefficients are the same, the detailed structure of the BLFs with FGM and Gaussian copulas is different (see the case of \( \rho = 0.0–0.3 \)). For the Gaussian-based BLFs we see a decline at the faint end, while we do not encounter such a structure in the FGM-based BLFs (see the closed contours in Fig. 3). This structure is introduced by the Gaussian copula, and...
Figure 2. The BLFs constructed with the Farlie–Gumbel–Morgenstern copula, with model LFs of UV- and IR-selected galaxies. The BLFs are normalized so that integrating over the whole ranges of $L_1$ and $L_2$ gives 1. From top left to bottom right, the linear correlation coefficient is $\rho = 0.0, 0.1, 0.2$ and 0.3, corresponding to $\kappa = 0.0, 0.33, 0.67$ and 1.0, respectively. The contours are logarithmic with an interval $\Delta \log \phi^{(2)} = 0.5$ drawn from the peak probability.

from a physical point of view it might not be strongly desired. The FGM-based BLF has a more ideal shape.

Secondly, since the univariate LF shapes are different at FIR and FUV wavelengths, the ridge of the BLF is not a straight line but clearly non-linear. This feature is more clearly visible in higher correlation cases in Fig. 3, but always exists for the whole range of $\rho$. This trend is indeed found in the $L_{\text{FIR}}-L_{\text{FUV}}$ diagram (Martin et al. 2005). The underlying physics is that galaxies with high star-formation rates are extinguished more by dust (e.g. Buat et al. 2007a,b).

Observational applications relevant to this topic will be presented elsewhere (Takeuchi et al., in preparation).

4 DISCUSSION

4.1 Flux selection effect in multiband surveys

Since we have an explicit form of BLF, we can discuss the flux selection effect formally. For simplicity, we consider the bivariate case (i.e. a sample selected in two bands) but it will be straightforward to extend the formulation to a multiband case (or more generally, one selected using any physical properties). The flux selection is described in terms of luminosity as putting a lower bound $L_{\text{lim}}$ on a luminosity–luminosity ($L_1-L_2$) plane. The lower-bound luminosity $L_{\text{lim}}$ is defined by the flux (density) detection limit $S_{\text{lim}}$ as a function of redshift. In most surveys, a certain wavelength band is chosen as the primary selection band, e.g. $B$ band, $K_s$ band, 60-$\mu$m-selected. The schematic description of a survey is presented in Fig. 4.

If we select a sample of objects (in our case galaxies) in band 1, the objects with $L_1 < L_{\text{lim}}^1(z)$ would not be included in the sample at a certain redshift $z$. Thus the detected sources should have $L_1 > L_{\text{lim}}^1(z)$ and $L_2 > L_{\text{lim}}^2(z)$. Hence, on the $L_1-L_2$ plane, the two-dimensional distribution of the detected objects is expressed as

$$\Sigma^{\text{det}}(L_1, L_2) \equiv \int_0^z \frac{d^2V}{dz' dz} \phi^{(2)}$$

$$\times \left( L_1, L_2 \right) \Theta \left( L_{\text{lim}}^1(z') \right) \Theta \left( L_{\text{lim}}^2(z') \right) \; dz' ,$$

(38)
Figure 3. The analytical BLF constructed with the Gaussian copula with model LFs of UV- and IR-selected galaxies. The BLFs are again normalized so that integrating over the whole ranges of \( L_1 \) and \( L_2 \) gives 1. The linear correlation coefficient \( \rho \) varies from 0.0–0.9 from top left to bottom right. As in Fig. 2, the contours are logarithmic with an interval \( \Delta \log \varphi^{(2)} = 0.5 \) drawn from the peak probability.
4.2 Other possible applications

4.2.1 The star-formation rate function

The star-formation rate (SFR) is one of the most fundamental quantities used to investigate the formation and evolution of galaxies. The SFR is often estimated from the FUV flux of galaxies (or other related observables like Hα etc.) after ‘correcting’ for dust extinction. However, some problems have been pointed out for this method. For instance, the relation between the UV slope β (or equivalently, FUV–NUV colour) and the FIR–FUV flux ratio $L_{\text{FIR}}/L_{\text{FUV}}$ (often referred to as the IRX–β relation) is frequently used to correct the extinction, but this relation is not always the same for various categories of star-forming galaxies (e.g. Buat et al. 2005; Boissier et al. 2007; Boquien et al. 2009; Takeuchi et al. 2010a).

Instead, the total SFR obtained from the FUV and FIR luminosities would be a more reliable measure of the SFR, since both are directly observable values (e.g. Iglesias-Páramo et al. 2004; Buat et al. 2005; Iglesias-Páramo et al. 2006; Buat et al. 2007a,b, 2009; Takeuchi et al. 2010a). Assuming a constant SFR over 10$^7$ yr, and a Salpeter initial mass function (IMF) (Salpeter 1955, mass range 0.1–100 M$\odot$), we have the relation between the SFR and $L_{\text{FUV}}$

$$\log \text{SFR}_{\text{FUV}} = \log L_{\text{FUV}} - 9.51.$$  (42)

For the FIR, to transform dust emission to the SFR, we assume that all the stellar light is absorbed by dust. We then obtain the following formula under the same assumptions for both SFR history and IMF as for the FUV,

$$\log \text{SFR}_{\text{dust}} = \log L_{\text{TIR}} - 9.75 - \log(1 - \eta).$$  (43)

Here, η is the fraction of the dust emission from old stars that is not related to the current SFR (Hirashita, Buat & Inoue 2003) and $L_{\text{TIR}}$ is the FIR luminosity integrated over $\lambda = 8$–1000 μm. Thus, the total SFR is simply

$$\text{SFR}_{\text{tot}} = \text{SFR}_{\text{FUV}} + \text{SFR}_{\text{dust}}.$$  (44)

(Iglesias-Páramo et al. 2006). Since the total SFR is basically estimated from the luminosities in the FUV and FIR (note that SFR$_{\text{FUV}} \propto L_{\text{FUV}}$ and SFR$_{\text{dust}} \propto L_{\text{TIR}}$), the estimation of the PDF of the total

where Ω is a solid angle and Θ is the Heaviside step function, defined as

$$\Theta(a) = \begin{cases} 
0, & L < a, \\
1, & L \geq a.
\end{cases}$$  (39)

The quantity $\Sigma_{\text{UL}}$ is proportional to the surface number density of objects detected in both bands on the $L_1$–$L_2$ plane. If we start from a primary selection in band 1, we would then have objects detected in band 1 but not detected in band 2. In such a case we have only upper limits for these objects. The two-dimensional distribution of the upper limits in band 2 is similarly formulated as

$$\Sigma_{\text{UL}}(L_1, L_2) = \int_0^L \frac{d^2V}{dz^2 d\Omega} \phi^{\text{(2)}}(z') \times (L_1, L_2) \Theta \left( L_{\text{lim}}^{\text{UL}}(z') \right) \left[ 1 - \Theta \left( L_{\text{lim}}^{\text{UL}}(z') \right) \right] \, dz'.$$  (40)

The superscript ‘UL’ stands for ‘upper limit in band 2’. In statistical terminology the upper-limit case, i.e. one in which we know there is an object but only have the upper (or lower) limits of a certain quantity, is referred to as ‘censored’. Although we can define the distribution $\Sigma_{\text{UL}}(L_1, L_2)$ from equation (40), since the sample objects belonging to this category appear only as upper limits on the plot, a special statistical treatment, referred to as survival analysis, is required to estimate $\Sigma_{\text{UL}}(L_1, L_2)$ from the data. Since we select objects in band 1, we do not have upper limits in band 1, because we do not know whether there might be an object below the limit. This case is called ‘truncated’ in statistics.

If we select objects in band 2, we can formulate the two-dimensional distribution of detected objects and upper limits in exactly the same way as the band-1-selected sample. For the objects detected in both bands, the two-dimensional distribution is expressed by equation (38). The objects detected in band 2 but not detected in band 1 are expressed as

$$\Sigma_{\text{UL}}(L_1, L_2) = \int_0^L \frac{d^2V}{dz^2 d\Omega} \phi^{\text{(2)}}(z') \times (L_1, L_2) \left[ 1 - \Theta \left( L_{\text{lim}}^{\text{UL}}(z') \right) \right] \Theta \left( L_{\text{lim}}^{\text{UL}}(z') \right) \, dz'.$$  (41)

If we can model $L_{\text{lim}}^{\text{UL}}(z')$ and $L_{\text{lim}}^{\text{UL}}(z')$ precisely including K-correction, evolutionary effects, etc., we can use the observed bivariate luminosity distribution to estimate the correlation coefficient, or more generally the dependence structure of two luminosities through equations (38)–(40). We can deal with these cases in a unified manner with techniques developed in survival analysis. We discuss this issue in a subsequent work (Takeuchi et al., in preparation).
SFR reduces to the estimation of the FIR–FUV BLF (e.g. Takeuchi et al. 2010b). However, since the total SFR is the sum of two dependent variables, it is not straightforward to formulate the function, unlike the cases we have seen above. This analysis will be discussed with a more specific methodology in our future work.

4.2.2 The distribution of the specific star-formation rate

Another direct application is the distribution of the specific SFR (SSFR), SFR/M∗, where M∗ is the total stellar mass of a galaxy. The SSFR has gained much attention in the last decade, since the relation between M∗ and the SSFR of galaxies turns out to be a very important clue by which to understand the SF history of galaxies: more massive galaxies have ceased their SF activity earlier in cosmic time than less massive galaxies (downsizing in redshift: e.g. Cowie et al. 1996; Boselli et al. 2001; Heavens et al. 2004; Feulner et al. 2005; Noeske et al. 2007a,b; Panter et al. 2007; Damen et al. 2009a,b, among others). For a comprehensive summary of the downsizing, readers are encouraged to read the introduction of Fontana et al. (2009). Despite its importance, the treatment of multiwavelength data for this analysis is inevitably complicated and does not seem to be well understood, because we must deal with data related to SFR and M∗ estimation. This might be, at least partially, the reason why the quantitative values of the M∗–SSFR relation are different for different studies.

As may be easily guessed from the above discussions, the M∗–SSFR relation can be reduced to the relation between the luminosity at a certain mass-related band (often near-IR) and a SF-related one (FUV, FIR, etc.) We can then model, for example, a L∗-L∗TIR bivariate luminosity function (where L∗ is the K-band luminosity)2 to examine the observed relation including all selection effects. This is particularly useful for this topic, since Takeuchi et al. (2010b) found that the SFR function cannot be described by the Schechter function, in contrast to the assumptions adopted in previous studies, but one much more similar to the Saunders IR LF (equation 36). The selection effect would be more complicated than in the case of the same Schechter marginals, but can be treated in the same way as discussed above. Thus, this may also be an interesting application of the copula-based BLF in the epoch of future large surveys.

5 SUMMARY AND CONCLUSIONS

In this work, we introduced an analytic method to construct a bivariate distribution function (DF) with given marginal distributions and correlation coefficient by making use of a convenient mathematical tool called a copula. Using this mathematical tool, we presented an application to construct the bivariate LF of galaxies (BLF). Specifically, we focused on the FUV–FIR BLF, since these two luminosities are related to the star-formation (SF) activity. Though both the FUV and FIR are related to the SF activity, the marginal univariate LFs have a very different functional form: the former is well described by a Schechter function whilst the latter has a much more extended power-law-like luminous end. We constructed the FUV–FIR BLFs using FGM and Gaussian copulas with different strengths of correlation, and examined their statistical properties. We then discussed some further possible applications of the BLF: the problem of a multiband flux-limited sample selection, the construction of the SF rate (SFR) function, and the construction of the stellar mass of galaxies (M∗)–specific SFR (SFR/M∗) relation.

2 More precisely, a L∗-L∗−L∗FUV trivariate function might be appropriate for this issue.

We summarize our conclusions as follows.

(i) If the correlation of two variables is weak (Pearson’s correlation coefficient |ρ| < 1/3), the Farlie–Gumbel–Morgenstern (FGM) copula provides an intuitive and natural way to construct such a bivariate DF.

(ii) When the linear correlation is stronger, the FGM copula becomes inadequate, in which case a Gaussian copula should be preferred. The latter connects two marginals and is directly related to the linear correlation coefficient between two variables.

(iii) Even if the linear correlation coefficient is the same, the structure of a BLF is different depending on the choice of copula. Hence, a proper copula should be chosen for each case.

(iv) The model FIR–FUV BLF was constructed. Since the functional shape of the LF at each wavelength is very different, the BLF obtained has a clearly non-linear structure. This feature was indeed found in actual observational data (e.g. Martin et al. 2005).

(v) We formulated the problem of multiwavelength selection effects by using the BLF. This enables us to deal with data sets derived from surveys presenting complex selection functions.

(vi) We discussed the estimation of the SFR function of galaxies. The copula-based BLF will be a convenient tool with which to extract detailed information from the observationally estimated SFR function because of its bivariate nature.

(vii) The stellar mass–specific SFR relation was also discussed. This relation can be reduced to a BLF of luminosities for a mass-related band and a SF-related band. An analytic BLF model constructed by a copula will provide us with a powerful tool to analyze the downsizing phenomenon while addressing complicated selection effects.

As the copula becomes better known to the astrophysical community and statisticians develop copula functions, we envision many more interesting applications in the future. In a series of forthcoming papers, we will present more observationally oriented applications of copulas.

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REFERENCES

Ball N. M., Loveday J., Brunner R. J., Baldry I. K., Brinkmann J., 2006, MNRAS, 373, 845
Benabed K., Cardoso J.-F., Prunet S., Hivon E., 2009, MNRAS, 400, 219
Binggeli B., Sandage A., Tammann G. A., 1988, ARA&A, 26, 509
Blanton M. R. et al., 2001, AJ, 121, 2358
Boissier S. et al., 2007, ApJS, 173, 524
Bivariate distribution function for the LF

Boquien M. et al., 2009, ApJ, 706, 553
Boselli A., Gavazzi G., Donas J., Scodellino M., 2001, AJ, 121, 753
Buat V. et al., 2005, ApJ, 619, L51
Buat V. et al., 2007a, ApJS, 173, 404
Buat V., Marcillac D., Burgarella D., Le Floc’h E., Takeuchi T. T., Iglesias-Páramo J., Xu C. K., 2007b, A&A, 469, 19
Buat V., Takeuchi T. T., Burgarella D., Giovannelli E., Murata K. L., 2009, A&A, 507, 693
Cambanis S., 1977, J. Multivariate Analysis, 7, 551
Chapman S. C., Helou G., Lewis G. F., Dale D. A., 2003, ApJ, 588, 186
Choloniewski J., 1985, MNRAS, 214, 197
Cowie L. L., Songaila A., Hu E. M., Cohen J. G., 1996, AJ, 112, 839
Cross N., Driver S. P., 2002, MNRAS, 329, 579
Damen M., Labbé I., Franx M., van Dokkum P. G., Taylor E. N., Gawiser E. J., 2009a, ApJ, 690, 937
Damen M., Förster Schreiber N. M., Franx M., Labbé I., Toft S., van Dokkum P. G., Wy IMPs, 2009b, ApJ, 705, 617
de Lapparent V., Galaz G., Bardelli S., Arnouts S., 2003, A&A, 404, 831
D’Este G. M., 1981, Biometrika, 68, 339
Driver S. P. et al., 2006, MNRAS, 368, 414
Farlie D. J. G., 1960, Biometrika, 47, 307
Feulner G., Gabasch A., Salvato M., Droy N., Hopp U., Bender R., 2005, ApJ, 633, L9
Fontanot F., De Lucia G., Monaco P., Somerville R. S., Santini P., 2009, MNRAS, 397, 1776
Gumbel E. J., 1960, J. Amer. Statist. Assoc., 55, 698
Heavens A., Panter B., Jimenez R., Dunlop J., 2004, Nat, 428, 625
Hettmansperger T. P., 1984, Statistical Inference Based on Ranks. John Wiley & Sons, New York
Hirashita H., Buat V., Inoue A. K., 2003, A&A, 410, 83
Hollander M., Wolfe D. A., 1999, Nonparametric Statistical Methods, 2nd edn. John Wiley & Sons, New York
Huang J. S., Kotz S., 1984, Biometrika, 71, 633
Iglesias-Páramo J., Buat V., Donas J., Boselli A., Milliard B., 2004, A&A, 419, 109
Iglesias-Páramo J. et al., 2006, ApJS, 164, 38
Jiang I.-G., Yeh L.-C., Chang Y.-C., Hung W.-L., 2009, AJ, 137, 329
Johnson N. L., Kotz S., 1977, Comm. Stat. Ser. A (Theory and Methods), 6, 485
Koen C., 2009, MNRAS, 393, 1370
Kotz S., Balakrishnan N., Johnson N. L., 2000, Continuous Multivariate Distributions, Volume 1: Models and Applications, 2nd edn. John Wiley & Sons, New York, pp 51–62
Lin H., Kirchner R. P., Shectman S. A., Landy S. D., Oemler A., Tucker D. L., Schechter P. L., 1996, ApJ, 464, 60
Martin D. C. et al., 2005, ApJ, 619, L59
Morgenstern D., 1956, Mitt. Math. Stat., 8, 234
Nelsen R. B., 2006, An Introduction to Copulas, 2nd edn. Springer, New York, Section 2
Noeske K. G. et al., 2007a, ApJ, 660, L43
Noeske K. G. et al., 2007b, ApJ, 660, L47
Panter B., Jimenez R., Heavens A. F., Charlton S., 2007, MNRAS, 378, 1550
Salpeter E. E., 1955, ApJ, 121, 161
Saunders W., Rowan-Robinson M., Lawrence A., Efstathiou G., Kaiser N., Ellis R. S., Frenk C. S., 1990, MNRAS, 242, 318
Saunders W. et al., 2000, MNRAS, 317, 55
Schechter P. L., 1976, ApJ, 203, 297
Scherrer R. J., Berfield A. A., Mao Q., McBride C. K., 2010, ApJ, 708, L9
Schucany W. R., Parr W. C., Boyer J. E., 1978, Biometrika, 65, 650
Sklar A., 1959, Publ. Inst. Stat. Univ. Paris, 8, 229
Stuart A., Ord K., 1994, Kendall’s Advanced Theory of Statistics, 6th edn, Vol. 1, Distribution Theory, Arnold, London, pp. 275–276
Takeuchi T. T., 2000, Ap&SS, 271, 213
Takeuchi T. T., Yoshikawa K., Ishii T. T., 2000, ApJS, 129, 1
Takeuchi T. T., Yoshikawa K., Ishii T. T., 2003, ApJ, 587, L89
Takeuchi T. T., Buat V., Burgarella D., 2005, A&A, 440, L17

Takeuchi T. T., Buat V., Heinis S., Giovannelli E., Yuan F.-T., Iglesias-Páramo J., Murata K. L., Burgarella D., 2010a, A&A, 514, A4
Takeuchi T. T., Buat V., Burgarella E., Giovannelli E., Murata K. L., Iglesias-Páramo J., Hernández-Fernández J., 2010b, in Debattista V. P., Popescu C. C., eds, AIP Conf. Proc., Hunting for the Dark: The Hidden Side of Galaxy Formation. Am. Inst. Phys., New York, in press
Trivedi P. R., Zimmer D. M., 2005, Foundations and Trends in Econometrics, 1, 1
Valz P. D., McLeod A. L., 1990, Am. Stat., 44, 39
Willmer C. N. A. et al., 2006, ApJ, 647, 853
Wyder T. K. et al., 2005, ApJ, 619, L15

APPENDIX A: AN EXTENSION OF THE FGM SYSTEM BY JOHNSON & KOTZ

Although the FGM system of distributions provides us with a convenient tool with which to construct a statistical model, its usefulness is restricted by the limitation of the correlation strength described above. To overcome this drawback, many attempts have been made to extend the FGM distributions (see e.g. Stuart & Ord 1994; Kotz, Balakrishnan & Johnson 2000). Among them, Johnson & Kotz (1977) introduced the following iterated generalization of equation (11):

\[ G(x_1, x_2) = \sum_{j=0}^{k} \kappa_j [F_1(x_1)F_2(x_2)]^{j/2+1} \times \left[1 - F_1(x_1)\right]\left[1 - F_2(x_2)\right]^{(j+1)/2}, \]  

(A1)

where the symbol in the exponent \(j/2\) means the maximum number which does not exceed \(j/2\). We set \(\kappa_0 = 1\). Huang & Kotz (1984) examined the dependence structure of equation (A1) for the particular case of \(k = 2\), and showed that the correlation can be stronger than that of the original FGM distribution by these extension formulae. In the case of the one-iteration family \(k = 2\), we have the DF as

\[ G(x_1, x_2) = F_1(x_1)F_2(x_2) [1 + \kappa_1 \left[1 - F_1(x_1)\right] [1 - F_2(x_2)]] + \kappa_2 F_1(x_1)F_2(x_2) [1 - F_1(x_1)][1 - F_2(x_2)]. \]  

(A2)

The corresponding PDF is

\[ g(x_1, x_2) = f_1(x_1)f_2(x_2) [1 + \kappa_1 \left[2F_1(x_1) - 1\right][2F_2(x_2) - 1]] + \kappa_2 f_1(x_1)f_2(x_2) [3F_1(x_1) - 2][3F_2(x_2) - 2]. \]  

(A3)

Then, as in the case of the original FGM distribution \(k = 1\), we obtain the covariance

\[ \text{Cov}(x_1, x_2) = \iint (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)f_1(x_1)f_2(x_2) \times [1 + \kappa_1 \left[2F_1(x_1) - 1\right][2F_2(x_2) - 1]] + \kappa_2 F_1(x_1)F_2(x_2) [1 - F_1(x_1)][1 - F_2(x_2)]. \]  

(A4)
The FIR–FUV BLF given by equation (A9) is shown in Fig. A1. Clearly the dependence between the two luminosities is stronger than for the original FGM-based BLF. However, it is now neither intuitive nor straightforward to relate the two parameters of dependence $\kappa_1$ and $\kappa_2$ to the linear correlation coefficient.

**APPENDIX B: ESTIMATORS OF THE NON-PARAMETRIC DEPENDENCE MEASURES $\rho_S$ AND $\tau$**

Here we present the estimators of the non-parametric measure of dependence introduced in Section 2.2. More detailed derivation and properties of these non-parametric measures of dependence are found in e.g. Hettmansperger (1984) and Hollander & Wolfe (1999).

Let \( \{R_i\}_{i=1,...,n} \) and \( \{S_i\}_{i=1,...,n} \) be the ranks of \( \{x_{1i}\}_{i=1,...,n} \) and \( \{x_{2i}\}_{i=1,...,n} \), respectively. If we denote the estimator of $\rho_S$ for a sample as $r_S$, then

$$r_S = \frac{\sum_{i=1}^n (R_i - \frac{n+1}{2}) (S_i - \frac{n+1}{2})}{\sqrt{\sum_{i=1}^n (R_i - \frac{n+1}{2})^2 \sum_{i=1}^n (S_i - \frac{n+1}{2})^2}}.$$  \hspace{1cm} (B1)

This is also expressed as

$$r_S = \frac{12 \sum_{i=1}^n (R_i - \frac{n+1}{2}) (S_i - \frac{n+1}{2})}{n(n^2-1)}.$$  \hspace{1cm} (B2)

This form is an exact sample counterpart of equation (8). If we define $d_i \equiv S_i - R_i$, equation (B2) reduces to the following simpler form:

$$r_S = 1 - 6 \frac{\sum_{i=1}^n d_i^2}{n(n^2-1)}.$$  \hspace{1cm} (B3)

The variance of $r_S$ in the large-sample limit is given by

$$\text{Var}[r_S] = \frac{1}{n-1}.$$  \hspace{1cm} (B4)

The most basic form of Kendall’s $\tau$ for a sample has already been shown in Section 2.2 (equation 9). It is also expressed as

$$t = \frac{2(n_a - n_d)}{n(n-1)} = 1 - \frac{4}{n(n-1)} n_d,$$  \hspace{1cm} (B5)

since $n_a + n_d = n(n-1)/2$. The variance of $t$ is given by

$$\text{Var}[t] = \frac{2(2n+5)}{9n(n-1)}.$$  \hspace{1cm} (B6)

(see Valz & Mcleod 1990 for a very concise derivation).