Two Qubit Quantum Computing in a Projected Subspace

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Abstract

A formulation for performing quantum computing in a projected subspace is presented, based on the subdynamical kinetic equation (SKE) for an open quantum system. The eigenvectors of the kinetic equation are shown to remain invariant before and after interaction with the environment. However, the eigenvalues in the projected subspace exhibit a type of phase shift to the evolutionary states. This phase shift does not destroy the decoherence-free (DF) property of the subspace because the associated fidelity is 1. This permits a universal formalism to be presented - the eigenprojectors of the free part of the Hamiltonian for the system and bath may be used to construct a DF projected subspace based on the SKE. To eliminate possible phase or unitary errors induced by the change in the eigenvalues, a cancellation technique is proposed, using the adjustment of the coupling time, and applied to a two qubit computing system. A general criteria for constructing a DF projected subspace from the SKE is discussed. Finally, a proposal for using triangulation to realize a decoherence-free subsystem based on SKE is presented. The concrete formulation for a two-qubit model is given exactly. Our approach is novel and general, and appears applicable to any type of decoherence.
I. INTRODUCTION

Solid-state proposals for quantum computers have progressed markedly recently: these include superconducting junctions, quantum dots, electron spin resonance, nuclear spins of impurity atoms, and nuclear spins in a crystal lattice\cite{1-5}. The working conditions for most proposals require low operating temperatures (typically a few $K$). Ideally, if such a system can be created as isolated from the environment, system evolution for a given quantum computing process, may be described by a unitary operator with time reversal symmetry. Often this is not the case for real quantum computing systems because the interaction between such systems and their environment introduce decoherence, which destroys the superpositions of qubits that enable the quantum logical operations to be validated\cite{6,7}. Such decoherence is a major obstacle in developing a practical quantum computer. Recent publications have formulated a theory for a decoherence-free (DF) subspace in which quantum computing is performed. The first formulation of DF conditions was performed by Paolo Zanardi and Mario Rasetti\cite{8,9}. This original formulation is quite general, which is not in the form of a master equation and does not invoke the Born-Markov approximation. Further work by Viola and Zanardi deal with DF subspaces by quantum control theory in a non-Markovian setting\cite{10,11,12}. Then an important direction of the DF subspace theory was developed from a Master equation (such as the Lindblad equation) for the open system within the powerful semigroup approach\cite{13-17}. A quantum jump description for the state of the atoms has also derived a consistent result with Master equation for dipole interaction atoms, but may give more insight into the time evolution of a single system\cite{18}. Experimentally, DF subspace
have recently been observed, which shows that such DF subspaces do indeed exist, allowing logical qubits to be encoded without decoherence \cite{19,20}. These provide the motivation for building appropriate open quantum computing systems suitable for (a) canceling the effects of intrinsic decoherence, and thus behaving as ideal quantum computing systems in the subspaces, or (b) construct DF subspace without invoking any approximations (such as the Born-Markov approximation) or restrictions on the type of decoherence (e.g., symmetric and collective decoherence). In this open system, self-adjoint operators and unitary evolution groups are not intrinsically required for quantum computation. Quantum computation can then be performed in a more general functional space including triplet structures: Rigged Hilbert Space (RHS), $\Phi \subset \mathcal{H} \subset \Phi^\times$, or Rigged Liouville Space (RLS), $\Phi \otimes \Phi \subset \mathcal{H} \otimes \mathcal{H} \subset \times \Phi \otimes \Phi^\times$, where $\mathcal{H}$ is Hilbert space, $\Phi$ is dense subspace of $\mathcal{H}$ and $\Phi^\times$ is dual space of $\Phi$, rather than just Hilbert or Liouville Space \cite{21−23}.

In this work, we present a formulation for performing quantum computation in a DF projected subspace based on the subdynamical equation (SKE). In section II, we briefly introduce a subdynamical formulation in the Hamiltonian representation. In section III, we propose a type of DF projected subspace. In sections IV and V, these concepts are applied to the example of a two qubit spin computing system plus Bosonic bath; a procedure to cancel the phase error induced by the change of eigenvalues is discussed. In section VI, a general approach for obtaining the necessary and sufficient condition for DF behavior is discussed. In section VII, a triangulating method is introduced for determining the DF projected subspace exactly.

**II. SUBDYNAMICS FORMULATION**

Consider a quantum system which is composed of $N$ quantum registers and interacts with a large thermal reservoir. We denote $H_S(t)$, $H_B$ and $H_{int}$ as the Hamiltonian of the system $S$, the Hamiltonian of the thermal reservoir $B$, and the interaction between $S$ and $B$, respectively. Then the total Hamiltonian of the system plus the reservoir can be expressed
as $H_S(t) \otimes I_B + I_S \otimes H_B + \lambda H_{\text{int}}$, with the corresponding Schrödinger equation as $i \frac{\partial}{\partial t} \psi(t) \equiv H(t) \psi(t)$. If one chooses the time-independent eigenprojectors of $H_S(t) \otimes I_B + I_S \otimes H_B$ as $P_\nu \equiv |\varphi_\nu\rangle \langle \varphi_\nu|$ and $Q_\nu$ with $P_\nu + Q_\nu = 1$ and the eigenprojectors of the total Hamiltonian as $\Pi_\nu(t)$ to satisfy the Heisenberg equation, then, from the definition of $\Pi_\nu(t)$ one can induce the definition of the creation correlation operator $C_\nu(t) (D_\nu(t))$ as $Q_\nu \Pi_\nu(t) = C_\nu(t) \Pi_\nu(t)$ ($\Pi_\nu(t) Q_\nu = \Pi_\nu(t) D_\nu(t)$), respectively. Note that the operator $C_\nu(t)$ creates the $Q_\nu$-part of $\Pi_\nu(t)$ from $P_\nu$ and the operator $D_\nu(t)$ destroys the $Q_\nu$-part of $\Pi_\nu(t)$ from $P_\nu$, since $C_\nu(t) = Q_\nu C_\nu(t) P_\nu$ and $D_\nu(t) = P_\nu D_\nu(t) Q_\nu$. This enables a projected kinetic equation to be constructed in the projected subspace by

$$ i \frac{\partial}{\partial t} \sum_\nu (P_\nu \Pi_\nu(t) \psi(t)) = \Theta(t) \sum_\nu (P_\nu \Pi_\nu(t) \psi(t)), \quad (1) $$

with an intermediate operator defined as

$$ \Theta(t) \equiv \left( H_S^0(t) + H_B^0 + H_{\text{int}}^0 \right) + \left( H_S^1(t) + H_B^1 + H_{\text{int}}^1 \right) C(t) \quad (2) $$

$$ = \Theta_0(t) + H_{\text{int}}^0 + \left( H_S^1(t) + H_B^1 + H_{\text{int}}^1 \right) C(t), $$

here $H_S^0, B, \text{int}(t)$ and $H_S^1, B, \text{int}(t)$ are the diagonal and off-diagonal parts of the corresponding total Hamiltonian, respectively. $\Pi_\nu(t)$ can be found from the subdynamics formulation as $(P_\nu + C_\nu(t)) (P_\nu + D_\nu(t) C_\nu(t))^{-1} (P_\nu + D_\nu(t))^{-1}$. The creation (destruction) operator can be obtained from the basic operator equations in the subdynamics formulation. Indeed, from the definition of the eigenprojectors $\Pi_\nu(t)$ we have

$$ (E_\nu(t) - Q_\nu H(t) Q_\nu) Q_\nu \Pi_\nu(t) = Q_\nu H_{\text{int}} P_\nu \Pi_\nu(t), \quad (3) $$

and hence yielding the crucial relationship

$$ Q_\nu \Pi_\nu(t) = Q_\nu C(t) P_\nu \Pi_\nu(t), \quad (4) $$

with definitions

$$ C(t) \equiv \sum_\mu \frac{1}{E_\mu(t) - Q_\nu H(t) Q_\nu H(t) P_\mu}, \quad (5) $$
and

\[ C_\nu(t) \equiv Q_\nu C(t) P_\nu \]

\[ = \frac{1}{E_\nu(t) - Q_\nu H(t) Q_\nu} Q_\nu H_{\text{int}} P_\nu, \] (6)

where \( E_\nu(t) \) is \( \nu \)th eigenvalue of the total Hamiltonian \( H(t) \). In the same way,

\[ D_\nu(t) = P_\nu H_{\text{int}} Q_\nu \frac{1}{E_\nu(t) - Q_\nu H(t) Q_\nu}. \] (7)

If one defines a projected wavefunction, \( \psi^{\text{proj}}(t) \), by

\[ \psi(t) \equiv \sum_\nu (P_\nu + C_\nu(t)) \psi^{\text{proj}}_\nu(t), \]

then

\[ \sum_\nu P_\nu \Pi_\nu(t) \psi(t) \equiv \sum_\nu \psi^{\text{proj}}_\nu(t) \equiv \psi^{\text{proj}}(t), \]

giving the formal evolution equation for a projected wavefunction \( \psi^{\text{proj}}(t) \) as

\[ \psi^{\text{proj}}(t) = \hat{T} e^{-i \int_0^t d\tau \Theta(\tau)} \psi^{\text{proj}}(t_0) \]

\[ = \hat{T} e^{-i \int_0^t d\tau \Theta(\tau)} \sum_\nu (P_\nu + D_\nu(t_0) C_\nu(t_0))^{-1} (P_\nu + D_\nu(t_0)) \psi(t_0). \] (8)

### III. DF PROJECT SUBSPACE

Using the above subdynamics formulation, it is apparent that before the interaction occurs, the spectral decomposition for \((H_S(t) + H_B)\) is the same as that for the total intermediate operator \( \Theta(t) \), i.e.,

\[ H_S(t) + H_B = \Theta(t) = \sum_\nu E^0_\nu(t) P_\nu, \] (9)

where \( E^0_\nu(t) \) is the \( \nu \)th eigenvalue of \((H_S(t) + H_B)\). But once the interaction with the environment has occurred, the eigenvalues change from \( E^0_\nu(t) \) to \( E^0_\nu(t) + \Delta E_\nu(t) \), where the definition

\[ \Delta E_\nu(t) = \langle \phi_\nu | H_{\text{int}}^1 C(t) | \phi_\nu \rangle, \] (10)

is made. The corresponding eigenprojectors \( P_\nu \) of \( \Theta(t) \) still remain invariant. The spectral decomposition of the intermediate operator \( \Theta(t) \) thus is different from \((H_S(t) + H_B)\),

\[ \Theta(t) = \sum_\nu \left( E^0_\nu(t) + \Delta E_\nu(t) \right) P_\nu. \] (11)
The corresponding mixed-state fidelity in the projected subspace can be calculated from

\[ F(t) = \text{Tr}\sqrt{\rho^{\text{proj}}(t_0) \rho^{\text{proj}}(t)} \]

\[ = \text{Tr}\sqrt{\rho^{\text{proj}}(t_0) \hat{T} e^{-i \int \Theta(t') dt'} \rho^{\text{proj}}(t_0) \hat{T} e^{i \int \Theta(t') dt'}} \]

\[ = \text{Tr}\left( \sum_{\nu,\mu} \left(\rho^{\text{proj}}(t_0)\right)_{\nu\mu} P_{\nu} e^{-i \int_{t_0}^{t} [E_\nu(t') + \Delta E_\mu(t')] dt'} \sum_{\nu,\mu} \left(\rho^{\text{proj}}(t_0)\right)_{\nu\mu} P_{\nu} e^{i \int_{t_0}^{t} [E_\nu(t') + \Delta E_\mu(t')] dt'} \right) \]

\[ = \sum_{\nu} \sqrt{\langle \phi_\nu | \sum_{\nu,\mu} \left(\rho^{\text{proj}}(t_0)\right)_{\nu\mu} | \phi_\nu \rangle} e^{-i \int_{t_0}^{t} [E_\nu(t') + \Delta E_\mu(t')] dt'} \sum_{\nu,\mu} \left(\rho^{\text{proj}}(t_0)\right)_{\nu\mu} P_{\nu} e^{i \int_{t_0}^{t} [E_\nu(t') + \Delta E_\mu(t')] dt'} \langle \psi^{\text{proj}}_\nu | \phi_\nu \rangle = 1. \]

where \(\left(\rho^{\text{proj}}(t_0)\right)_{\nu\mu}\) are the matrix elements of the density operator \(\rho^{\text{proj}}(t_0)\). This exposes an exciting result: there is no decoherence in the projected subspace for states of the system since the eigenvectors remain invariant. However, a change in the eigenvalues introduces a phase shift in the evolution of the states,

\[ e^{-i \int E_\nu^0(t) dt} | \psi^{\text{proj}}_\nu (t_0) \rangle \quad \text{After the interaction} \quad e^{-i \int (E_\nu^0(t) + \Delta E_\nu(t)) dt} | \psi^{\text{proj}}_\nu (t_0) \rangle. \]

From these findings there are two conclusions:

1. In general, for any system \(S + B\), we can use the eigenprojectors of \(H^S\) to construct a DF projected subspace in which the eigenprojectors remain invariant before and after the interaction between \(S\) and \(B\), while the eigenvalues induce a phase shift in the eigenstates. The encoded states in the projected subspace are the projected states which are related to the original states by \(P_\nu \Pi_\nu \psi_\nu\). In particular, in this projected subsystem the states useful for performing quantum computing are the reduced projected states \(\rho^S_{\text{proj}}(t)\). These can be obtained by using a projection operator \(Tr_R P_\nu \Pi_\nu\) in Liouville space to act on the density operator \(\rho\) of the total system. Although \(\rho^S_{\text{proj}}(t)\) is not a reduced density operator \(\rho_S(t)\), one can consider \(\rho^S_{\text{proj}}(t)\) to be a generalization of \(\rho_S(t)\) (i.e., when \(\Pi_\nu \rightarrow P_\nu\), \(\sum_\nu P_\nu \Pi_\nu \rightarrow 1\), \(\rho^S_{\text{proj}}(t) = Tr_B \sum_\nu P_\nu \Pi_\nu \rho(t) = Tr_B \rho(t) = \rho_S(t)\)). We argue that the information encoded in this projected subspace can be measured because the projected bases are orthogonal.
and distinguishable in Liouville space - namely \( \left\langle \rho_{\nu}^{proj,S} \right| \rho_{\mu}^{proj,S} \right\rangle = 0 \), for \( \nu \neq \mu \), because of \( P_{\mu}P_{\nu} = 0 \) in Liouville space too.

(2) The phase shift induced by the eigenvalues may cost a unitary type error for quantum computation. Although this sort of error may be eliminated by developing standard quantum error correction schemes (such as multiqubit code\[24\]), the error recovery is not easy since phase error induced by the systematic phase shift may occur, at the same time, to many different clusters (inducing a phase or bit-flip errors in the encoded data). The increase in the number of phase errors for different qubits will cause a fast increase in the number of clusters, which likely leads to impractical implementations. On the other hand, the property of invariant eigenvectors in subspace provides the possibility of eliminating the unitary error by adjusting the appropriate time scale for the evolution operator to remain invariant under certain conditions; e.g., by choosing the time delay \( \Delta t \) to allow
\[
\sum_{\nu} e^{-i \int_0^t E_0(\tau) d\tau} P_\nu = \sum_{\nu} e^{-i \int_0^{t+\Delta t} (E_0(\tau) + \Delta E_\nu(\tau)) d\tau} P_\nu.
\]
Below, a concrete application of these concepts is presented.

IV. COMPUTING SYSTEM

We consider a two qubit quantum computing system \( S \), consisting of spins \( S_1 \) and \( S_2 \), such as those corresponding to two electrons around two \( ^{31}P \) confined in a germanium/silicon heterostructures in an electron spin-resonance transistor\[4\], or for two electrons confined in two quantum dots\[2\,3\]. Ignoring the influence of the environment, the Hamiltonian may be written using the Heisenberg model as
\[
H_S(t) = J(t) S_1 \cdot S_2,
\]
where \( J(t) \) is the time-dependent exchange coupling parameter determined by the specific model considerations. In the case of spins of the two electrons (e.g., confined in two vertically (laterally) coupled quantum dots), \( J \) is the difference in the energies of two-electrons ground state, a spin singlet at zero magnetic field, and the lowest spin-triplet state; \( J \) is
also a function of the electric and magnetic field and the interdot distance\cite{25,26}. Using the relationship between $S_1 \cdot S_2$ and the square of the sum of $S_1$ and $S_2$, the eigenvalues and eigenvectors of $S_1 \cdot S_2$ can be found from $S_1 \cdot S_2 = \frac{1}{2} \left(S^2 - \frac{3}{2}\right)$ by

$$E_S^1 = \frac{1}{4} \left\{ \begin{array}{l}
|\phi_1\rangle = |11\rangle,
|\phi_2\rangle = |00\rangle,
|\phi_3\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle),
\end{array} \right.$$

$$E_S^2 = \frac{3}{4} |\phi_4\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle).$$

A quantum XOR gate is given by the sequence of operations\cite{2}, $U_{\text{XOR}} = e^{i(\pi/2)S_z^1}e^{-i(\pi/2)S_z^2}U_{\text{sw}}^{1/2}e^{i\pi S_z^1}U_{\text{sw}}^{1/2}$, where $U_{\text{sw}}$ is an (ideal) swap operator and determined generally by an evolution operator $U_s (J (0) \tau)$ by adjusting the coupling time between the two spins in the evolution of the system. For the particular spin-spin coupling duration, $\tau_s$ where $\int_0^{\tau_s} J (t) \, dt = \pi \pmod{2\pi}$, $U_{\text{sw}} = U_s (\pi)$, the swap operator, is given by

$$U_{\text{sw}} = e^{-i \int_0^{\tau_s} H_s (\tau) \, d\tau} = \sum_{j=1}^3 e^{-i \frac{\pi}{4}} |\phi_j\rangle \langle \phi_j| + e^{i \frac{\pi}{4}} |\phi_4\rangle \langle \phi_4|$$

and can exchange the quantum states of qubit 1 and 2.

If we consider the influence of the environment, the non-ideal action of the swap operation must be considered because the influence of environment introduces decoherence. To treat this decoherence, one needs to understand the behavior of the evolution of the system. Here it is assumed that the environment consists of a set of harmonic-oscillators whose Hamiltonian is given by $H_B = \sum_k \omega_k b_k^+ b_k$, and the Hamiltonian coupling to the two qubit spin system is $H_{\text{int}} = \lambda \sum_k (\sigma_z^1 + \sigma_z^2) \left( g_k b_k^+ + g_k^* b_k \right)$, where $b_k^+$, $b_k$ are bosonic operators for the $k$th field mode, characterized by a generally complex coupling parameter $g_k$ which characterizes the case as being one of either independent or collective decoherence\cite{27}. The Hamiltonian operator for the total system is given by $H (t) = H_S (t) + H_B + \lambda H_{\text{int}}$, and the corresponding Schrödinger equation is $i \frac{\partial}{\partial t} \psi (t) = (H_S (t) + H_B + H_{\text{int}}) \psi (t)$. Choosing the time-independent eigenprojectors of $H_S (t) + H_B$ as $P_\nu$ and $Q_\nu$ with $Q_\nu + P_\nu = 1$, with

$$P_\nu \equiv |\varphi_\nu\rangle \langle \varphi_\nu| = |n_1 \cdots n_k \cdots \rangle \langle \phi_j| \langle \phi_j| \langle \cdot \cdots n_k \cdots n_1|$$

and
for $\nu = (j, n_1 \cdots n_k \cdots)$ and $j = 1, \cdots, 4$; $n_k = 1, \cdots, 4$, respectively, then the eigenprojectors $\Pi_\nu (t)$ for the total Hamiltonian $H (t)$ can be written in terms of the Heisenberg equation and satisfy the usual properties of projection operators.

V. CODE CORRECTION IN SUBSPACE

Using above subdynamics formulation, it is apparent that before the interaction occurs, the spectral decomposition for $(H_S (t) + H_B)$ is the same as that for the total intermediate operator $\Theta (t)$, i.e.,

\begin{equation}
H_S (t) + H_B = \Theta (t)
\end{equation}

\begin{equation}
= \sum_{j=1}^{4} \sum_{n_1 \cdots n_k \cdots} E_{j,n_1 \cdots n_k \cdots}^0 (t) P_{j,n_1 \cdots n_k \cdots},
\end{equation}

where $E_{j,n_1 \cdots n_k \cdots}^0 (t) = \left( \frac{1}{4} - \delta_{j,4} \right) J (t) + \sum_k \omega_k n_k$. But once interaction with the environment occurs, the eigenvalues change from $E_{j,n_1 \cdots n_k \cdots}^0 (t)$ to $E_{j,n_1 \cdots n_k \cdots}^0 (t) + \Delta E_{j,n_1 \cdots n_k \cdots} (t)$, where we have made the following definitions:

\begin{equation}
\Delta E_{j,n_1 \cdots n_k \cdots} (t) = \langle n_1 \cdots n_k \cdots | \langle \phi_j | H_{int} C_{j,n_1 \cdots n_k \cdots} (t) | \phi_j \rangle | n_1 \cdots n_k \cdots \rangle,
\end{equation}

while the corresponding the eigenvectors of $\Theta (t)$, $|\phi_j \rangle | n_1 \cdots n_k \cdots \rangle$ remain invariant. The spectral decomposition of the intermediate operator $\Theta (t)$ is thus different from $H_0 (t)$,

\begin{equation}
\Theta (t) = \sum_{j=1}^{4} \sum_{n_1 \cdots n_k \cdots} \left( E_{j,n_1 \cdots n_k \cdots}^0 (t) + \Delta E_{j,n_1 \cdots n_k \cdots} (t) \right) P_{j,n_1 \cdots n_k \cdots}. \tag{19}
\end{equation}

This shows that there is no decoherence in the projected subspace for the stationary states of the system since the eigenvectors remain invariant, but for the evolutionary states, the change of the eigenvalues can introduce a type of unitary like error in the system evolution,

\begin{equation}
|\psi_{\text{proj}}^\text{proj} (t) \rangle
= \sum_{j=1}^{4} \sum_{n_1 \cdots n_k \cdots} e^{-i \int_0^t \left( E_{j,n_1 \cdots n_k \cdots}^0 (\tau) + \Delta E_{j,n_1 \cdots n_k \cdots} (\tau) \right) d\tau} |\psi_{\text{proj}}^\text{proj}_{j,n_1 \cdots n_k \cdots} (0) \rangle. \tag{20}
\end{equation}
As mentioned previously, the quantum XOR operator can be constructed from a sequence of operations related to the ideal swap operator, adjusted by controlling the coupling time for the interaction between the two qubits without any influence from the environment. But if the effects of the environment are now included, the ideal swap operator changes to the non-ideal swap operator, owing to the unitary error. To cancel this type of decoherence, in terms of the subdynamics formulation, it is proposed that one allow the quantum logical operators to work on the projected subspaces. For example, if the quantum XOR operator previously introduced is considered, the ideal swap operator should be adjusted by controlling the coupling time between the two spins without considering interactions with the environment; then the ideal swap operator is given in a projected subspace as

\[
U_{\text{sw}}(\tau_s) = e^{-i(\pi S_1 \cdot S_2 \otimes I_B + I_S \otimes H_B) \tau_s},
\]

where a specific coupling duration \( \tau_s \) is given by

\[
\int_0^{\tau_s} J(\tau) \, d\tau = \pi \pmod{2\pi}.
\]

The unitary error is related to the non-ideal action of the swap operator which can be adjusted by the evolution operator in a projected subspace as

\[
U'_{\text{sw}}(\tau_s + \Delta t) = e^{-i \int_0^{\tau_s+\Delta t} d\tau [J(\tau) S_1 \cdot S_2 \otimes I_B + I_S \otimes H_B + H_{\text{int}} C_{j,n_1\cdots n_k\cdots}]}.
\]

The spectral decomposition of \( U'_{\text{sw}}(\tau_s + \Delta t) \) can be expressed by adjusting the interaction time \( \tau_s \) to \( \tau_s + \Delta t \), allowing the non-ideal swap operator to be equal to the ideal swap operator,

\[
U'_{\text{sw}}(\tau_s + \Delta t) = U_{\text{sw}}
\]

\[
= \sum_{n_1\cdots n_k\cdots} \left\{ 3 \sum_{j=1}^3 e^{-i \int_0^{\tau_s+\Delta t} d\tau \left[ \frac{3}{4} J(\tau) + \sum_k \omega_k n_k + \Delta E_{j,n_1\cdots n_k\cdots}(\tau) \right]} P_{j,n_1\cdots n_k\cdots} + e^{i \pi/4 - i \tau_s} \sum_{j=1}^3 e^{-i \int_0^{\tau_s+\Delta t} d\tau \left[ \frac{3}{4} J(\tau) + \sum_k \omega_k n_k + \Delta E_{j,n_1\cdots n_k\cdots}(\tau) \right]} P_{4,n_1\cdots n_k\cdots} \right\}
\]

This induces the integral equation for determining \( \Delta t \) which depends on \( j, n_1 \cdots n_k \cdots \),

\[
\int_0^{\tau_s+\Delta t} d\tau \left( \frac{J(\tau)}{4} - \delta j \frac{3}{4} J(\tau) + \Delta E_{j,n_1\cdots n_k\cdots}(\tau) \right) = \begin{cases} \frac{\pi}{4}, & \text{for } j = 1, 2, 3, \\ -\frac{3\pi}{4}, & \text{for } j = 4. \end{cases}
\]

For instance, if the exchange interaction coupling \( J(B,E,d) \) is a time-independent function of the external magnetic field \( B \), electric field \( E \) and the interdot distance \( d \) as in the case
of vertically or laterally tunnel-coupled quantum dots, then $\Delta t$ can be solved from Eq. (22) as:

$$\Delta t = - \frac{\tau_s}{4\Delta E_{j,n_1 \cdots n_k \cdots}} + 1 = - \frac{\tau_s}{4\Delta E_{j,n_1 \cdots n_k \cdots}} + 1,$$

(23)

where the concrete formula for $J(B, E, d)$, in the case of vertically or laterally tunnel-coupled quantum dots, can be found in refs. [25, 26]. Assuming, in this case, that the energy of the total system is uniformly distributed, i.e.,

$$J(B, E, d) \frac{4\Delta E_{j,n_1 \cdots n_k \cdots}}{4\Delta E_{j,n_1 \cdots n_k \cdots}} = \text{const}, \text{ for } j = 1, 2, 3; n_1, \cdots, n_k = 1, 2, \cdots.$$

(24)

The shift energy can be obtained from Eq. (23),

$$\Delta E_{j,n_1 \cdots n_k \cdots} = \sum_{j=1}^{4} \sum_{n_1' \cdots n_k' \cdots} (H_{\text{int}})_{j,n_1 \cdots n_k \cdots \cdots j', n_1' \cdots n_k' \cdots} C_{j', n_1' \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots},$$

(25)

and the matrix of the creation operator is given by

$$C_{j', n_1' \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots} = \langle \varphi_{j', n_1' \cdots n_k' \cdots} | C_{j,n_1 \cdots n_k \cdots} | \varphi_{j,n_1 \cdots n_k \cdots} \rangle,$$

(26)

which can be calculated from formula (23); for example, the first order $C_{j,n_1 \cdots n_k \cdots}^{[1]}$ is divided by degenerate part and non-degenerate part:

$$C_{j,n_1 \cdots n_k \cdots}^{[1]} = \sum_{n_1' \cdots n_k' \cdots} \left[ \frac{3}{\Delta E_{j,n_1 \cdots n_k \cdots}} \frac{1}{\Delta E_{j,n_1 \cdots n_k \cdots}} (H_{\text{int}})_{j,n_1 \cdots n_k \cdots \cdots j', n_1' \cdots n_k' \cdots} \right] P_{j,n_1' \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots},$$

(27)

$$+ \delta_{j', 4} \frac{1}{\Delta E_{j,n_1 \cdots n_k \cdots}} \frac{1}{\Delta E_{j,n_1 \cdots n_k \cdots}} (H_{\text{int}})_{j,n_1 \cdots n_k \cdots \cdots j', n_1' \cdots n_k' \cdots} P_{j,n_1' \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots},$$

where

$$\Delta E_{j,n_1 \cdots n_k \cdots} = \sum_{n_1' \cdots n_k' \cdots} (H_{\text{int}})_{j,n_1 \cdots n_k \cdots \cdots j', n_1' \cdots n_k' \cdots} P_{j,n_1' \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots}, \text{ for } j = 1, 2,$$

$$0, \text{ for } j = 3, 4,$$

which results in the second order $\Delta E_{j,n_1 \cdots n_k \cdots}^{[2]}$ by Eq. (25)

$$\Delta E_{j,n_1 \cdots n_k \cdots}^{[2]} = \sqrt{\sum_{j=1}^{2} \sum_{n_1' \cdots n_k' \cdots} (H_{\text{int}})_{j,n_1 \cdots n_k' \cdots \cdots j,n_1 \cdots n_k \cdots}}^2,$$

(28)

$$0, \text{ for } j = 3, 4,$$
with the matrix elements given by

$$(H_{\text{int}})_{j,n_1\ldots n_k\ldots j',n'_1\ldots n'_k} = \langle n_1 \cdots n_k \cdots | \phi_i | H_{\text{int}} | \phi_{j'} \rangle | n'_1 \cdots n'_k \cdots \rangle$$

$$= \left\{ \begin{aligned}
\lambda (\delta_{1,j'} - \delta_{2,j'}) & \sum_k \left( \delta_{n_k,n'_k} + g_k \sqrt{n'_k + 1} + \delta_{n_k,n'_k - 1} g_k \sqrt{n'_k} \right), \\
0, & \text{for } j = 3, 4,
\end{aligned} \right.$$  

Eq. (23) shows that although the interaction introduces the sort of phase shift in the swap operator, this sort of phase shift can be cancelled by adjusting the coupling time between the two spins under the assumptions of homogeneous distribution of energy (i.e., owing to the invariance of the eigenvectors in the projected subspace). In the same way, the second-order projected states in the projected subspace are given by

$$\psi_{j,n_1\ldots n_k\ldots}^{\text{proj}[2]} (t) = (P_{j,n_1\ldots n_k\ldots} + D_{j,n_1\ldots n_k\ldots} - D_{j,n_1\ldots n_k\ldots} C_{j,n_1\ldots n_k\ldots}) \psi (t)$$

$$= \left\{ \begin{aligned}
P_{j,n_1\ldots n_k\ldots} - \frac{(\delta_{1j} + \delta_{2j})}{\Delta E_{j,n_1\ldots n_k\ldots}^{|2|}} \sum_{n'_1\ldots n'_k\ldots} (H_{\text{int}})_{j,n_1\ldots n_k\ldots;j,n'_1\ldots n'_k\ldots} P_{j,n_1\ldots n_k\ldots;j,n'_1\ldots n'_k\ldots} \\
- \frac{(\delta_{1j} + \delta_{2j})}{(\Delta E_{j,n_1\ldots n_k\ldots}^{|2|})^2} \sum_{n'_1\ldots n'_k\ldots} \sum_{n''_1\ldots n''_k\ldots} (H_{\text{int}})_{j,n_1\ldots n_k\ldots;j,n''_1\ldots n''_k\ldots} (H_{\text{int}})_{j,n''_1\ldots n''_k\ldots;j,n_1\ldots n_k\ldots} P_{j,n_1\ldots n_k\ldots} \right\} \psi (t),
\end{aligned} \right.$$  

where the first order destruction operator is

$$D_{j,n_1\ldots n_k\ldots}^{[1]} = \left\{ \begin{aligned}
\sum_{n'_1\ldots n'_k\ldots} \frac{1}{\Delta E_{j,n_1\ldots n_k\ldots}^{|2|}} (H_{\text{int}})_{j,n_1\ldots n_k\ldots;j,n'_1\ldots n'_k\ldots} P_{j,n'_1\ldots n'_k\ldots;j,n_1\ldots n_k\ldots}, & \text{for } j = 1, 2, \\
0, & \text{for } j = 3, 4,
\end{aligned} \right.$$  

By above cancelling procedure, the evolution formula for second-order reduced projected density operator, before or after the interaction, remains the same and is given by

$$\rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S} (t)$$

$$= Tr_B e^{-i(\Theta^{[2]}_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}) t} \rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S} (0) \langle \varphi_{j,n_1\ldots n_k\ldots}^{\text{proj}[2]} (0) | \varphi_{j',n'_1\ldots n'_k\ldots}^{\text{proj}[2]} (0) \rangle e^{i\Theta^{[2]}_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots} t}$$

$$= e^{-i(E^0_j - E^0_{j'}) \rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S}} (0),$$

where $\rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S}$ represent probability of existence for the state $| \varphi_{j,n_1\ldots n_k\ldots}^{\text{proj}[2]} (0) \rangle \langle \varphi_{j',n'_1\ldots n'_k\ldots}^{\text{proj}[2]} (0) |$, noting that $\rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S} (0)$ is given by initial decoupling condition, $\rho_{j,n_1\ldots n_k\ldots;j',n'_1\ldots n'_k\ldots}^{\text{proj}[2],S} (0) = Tr_B \left( P_{j,n_1\ldots n_k\ldots} \rho_S (0) \otimes \rho_B (0) P_{j',n'_1\ldots n'_k\ldots} \right)$. The
fidelity, which measures the decoherence between the initial state $\psi^{\text{proj}[2]}(t)$ and the evolution reduced project density operator $\rho^{\text{proj}[2],S}(t)$ in the subspaces, is given by

$$
F \left( \left| \psi^{\text{proj}[2]}(t) \right\rangle, \rho^{\text{proj}[2],S}(t) \right) = \sqrt{\left\langle \psi^{\text{proj}[2]}(t) \right| \sum_{j,j',n_1 \ldots n_k} \rho^{\text{proj}[2],S}_{j,n_1 \ldots n_k} \left\langle j',n_1 \ldots n_k \right| \psi^{\text{proj}[2]}(t) \rangle}
= \sqrt{\sum_{j,n_1 \ldots n_k} \varphi^{\text{proj}[2],S}_{j,n_1 \ldots n_k} (0) \sum_{j',n_1 \ldots n_k} \rho^{\text{proj}[2],S}_{j',n_1 \ldots n_k} (0) \sum_{j,n_1 \ldots n_k} \varphi^{\text{proj}[2],S}_{j',n_1 \ldots n_k} (0)}
= \sum_{j,n_1 \ldots n_k} p_{j,n_1 \ldots n_k} \sum_{j',n_1 \ldots n_k} p_{j',n_1 \ldots n_k}
= 1,
$$
$H_{\text{int}}$ (i.e., $H_{\text{int}} P_\mu = \varepsilon \alpha P_\mu$, $\forall$ degenerate $\alpha \in$ set of complex number $c$) then it means having $P_\nu H_{\text{int}} P_\mu C_\nu (t) P_\nu = 0$, for any $\nu$. But in the general case, without restrictions on the type of decoherence (e.g., a non-Markovian process and non-symmetric and non-collective decoherence) how can one find the conditions for constructing the DF subspace? For generality, we propose a procedure to find the condition for the DF subspace starting directly from Eq. (34) by considering weak coupling between the system and the environment, without making any of the above assumptions. In fact, from Eq. (34) we have:

$$P_{j,n_1 \cdots n_k \cdots} \left( H_{\text{int}} C_{j,n_1 \cdots n_k \cdots} (t) \right)$$

$$= \sum_{j'=1}^4 \sum_{n'_k \cdots} \frac{1}{E_{j,n_1 \cdots n_k \cdots} (t) - E_{j',n'_1 \cdots n'_k \cdots} (t) + \Delta E_{j,n_1 \cdots n_k \cdots} (t)} P_{j,n_1 \cdots n_k \cdots} H_{\text{int}} P_{j',n'_1 \cdots n'_k \cdots} H_{\text{int}} P_{j,n_1 \cdots n_k \cdots}$$

$$= \lambda^2 \left( \frac{E_{1,n_1 \cdots n_k \cdots} (t) - E_{2,n_1 \cdots n_k \cdots} (t) + \Delta E_{2,n_1 \cdots n_k \cdots} (t)}{E_{1,n_1 \cdots n_k \cdots} (t) - E_{1,n_1 \cdots n_k \cdots} (t) + \Delta E_{1,n_1 \cdots n_k \cdots} (t)} \right) \sum_k \left( g_k g_{k+1}^* (n_k + 1) + g_{k-1} g_k^* n_k \right) \delta_{j1} P_{1,n_1 \cdots n_k \cdots}$$

$$+ \lambda^2 \left( \frac{E_{0,n_1 \cdots n_k \cdots} (t) - E_{0,n_1 \cdots n_k \cdots} (t) + \Delta E_{0,n_1 \cdots n_k \cdots} (t)}{E_{0,n_1 \cdots n_k \cdots} (t) - E_{0,n_1 \cdots n_k \cdots} (t) + \Delta E_{0,n_1 \cdots n_k \cdots} (t)} \right) \sum_k \left( g_k g_{k+1}^* (n_k + 1) + g_{k-1} g_k^* n_k \right) \delta_{j2} P_{2,n_1 \cdots n_k \cdots}$$

$$= 0.$$

This gives a DF condition which is a restriction on the bath operator by

$$\sum_k \left( g_k g_{k+1}^* (n_k + 1) + g_{k-1} g_k^* n_k \right) = 0. \quad (36)$$

In one special case it is

$$\frac{n_k + 1}{n_k} = - \frac{g_{k-1} g_k^*}{g_k g_{k+1}}. \quad (37)$$

Under the DF conditions, e.g. with Eqs. (36) or (37) holding, Eq. (1) reduces to

$$i \frac{\partial |\varphi^{\text{proj}} (t)\rangle}{\partial t} = H_0 (t) \mid \varphi^{\text{proj}} (t) \rangle \quad (38)$$

with evolution of the projected state described by

$$\mid \varphi^{\text{proj}} (t) \rangle = \hat{T} e^{-itH_0 (t)} \mid \varphi^{\text{proj}} (0) \rangle$$

$$= \hat{T} e^{-itH_0 (t)} \sum_\nu \left( P_\nu + D_\nu (0) \right)^{-1} \left( P_\nu + D_\nu (0) \right) \mid \varphi (0) \rangle,$$

which enables the constructed projected subspaces to become a DF subspace spanned by the set of $\{ P_{j,n_1 \cdots n_k \cdots} \}$, although the total space is subject to decoherence. The projected
subspace is closed with respect to the intermediate operator, \( \Theta_\nu (t) \Phi^\text{proj} \subset \Phi^\text{proj} \), for any projected state in the projected subspace \( \Phi^\text{proj} \), \( (P_\nu \Pi_\nu (t) \psi (t)) \). Furthermore, if one generalizes the initial test projector \( P_\mu \) to be up (or down) triangular from the partial eigenprojector of \( H^\text{int} \), i.e. \( P_\nu H^\text{int} P_\mu \neq 0 \) for \( \nu < \mu \), otherwise it is zero, then Eq.(2) gives \( C_\nu (t) P_\nu = 0 \) which enables Eq.(34) to generally hold in the projected subspace without any approximation. In this sense, if one can construct a triangular basis for \( H^\text{int} \) to span a projected subspace, then this space is DF although the total space is decoherent.

**VII. REALIZATION OF A DF SUBSPACE USING TRIANGULATION**

As a starting point we consider a necessary and sufficient condition for DF behavior in the projected subspace. This can be determined from the fact that the interaction part of the intermediate operator \( \Theta (t) \) is zero or that the evolution of the projected (reduced) state is independent of the interaction part of the Liouvillian. That is

\[
L^\text{int}_0 + \left( L^S_1 (t) + L^B_1 + L^\text{int}_1 \right) C (t) = 0.
\]

(40)

If the constructed projectors \( P_\nu \) and \( Q_\nu \) are triangular with respect to \( L^S_1 (t) + L^B_1 + L^\text{int}_1 \) (i.e., \( P_\nu L^\text{int}_1 P_\mu \neq 0 \), \( P_\nu \left( L^S_1 (t) + L^B_1 \right) P_\mu \neq 0 \) for \( \nu < \mu \) (up-triangular) or \( \nu > \mu \) (down-triangular), then \( L^\text{int}_0 + \left( L^S_1 (t) + L^B_1 + L^\text{int}_1 \right) C (t) = L^\text{int}_0 + \sum_\nu P_\nu \left( L^S_1 (t) + L^B_1 + L^\text{int}_1 \right) Q_\nu \frac{1}{E_\nu (t) - Q_\nu L(t) Q_\nu} Q_\nu \left( L^S_1 (t) + L^B_1 + L^\text{int}_1 \right) P_\nu = 0 \). This enables the evolution of the reduced projected density operator to be independent of the interaction part of the total Liouvillian in the projected Liouville subspace,

\[
\rho^\text{proj},S (t) = \sum_\nu Tr_B \left( \left[ (P_\nu + D_\nu (t)) \rho (t) \right] \right) = \sum_\nu Tr_B \left( e^{-i \int_{t_0}^{t} dt' \Theta_0 (t') P_\nu \rho (t_0)} \right),
\]

(41)

where \( \Theta_0 (t) \) is a \( \nu \)th eigenvalue of \( \Theta_0 (t) \) in Liouvillian representation. The corresponding mixed-state fidelity is equal to 1 (i.e., \( F (t) = Tr_S \left\{ \rho^\text{proj},S (0) \sum_\nu Tr_B \left( e^{-i \int_{t_0}^{t} dt' \Theta_0 (t')} P_\nu \rho (t_0) \right) \right\} = 1 \)). This shows that there is no decoherence introduced by the environment in the projected subspace, although the total system is subject to decoherence introduced by the environment. The formulation for
this DF projected subspace is exact (i.e., there are no approximations including the Born-Markov approximation). The method is general. Thus for any combined system, one may construct a triangular basis for the total Hamiltonian to span a projected subspace in which the evolution of the projected density operator is independent of the interaction part of the total Hamiltonian, with a fidelity of 1. Furthermore, one can construct a partial triangular basis for the computing system or bath, by choosing a partial diagonal basis for the left part of total system to form a DF projected subspace, based on the subdynamical Liouville equation. Thus triangulation procedure places no restriction on the type of coupling between the system and the environment (i.e., Markovian or collective decoherence).

**VIII. AN EXAMPLE**

To illustrate the above method, we revisit the example of the two qubit quantum computing system $S$, consisting of spins $S_1$ and $S_2$. For example, this includes the two electrons around two $^{31}P$ confined in a germanium/silicon heterostructures in an electron spin-resonance transistor$^1$, or two electrons confined in two quantum dots$^2$. Ignoring the influence of environment, the Hamiltonian can be written using the Heisenberg model as $H_S(t) = J(t) S_1 \cdot S_2$, where $J(t)$ is the time-dependent exchange coupling parameter determined by the specific model considerations. The coupling to the environment is assumed to be described by a Caldeira-Leggett-type model, consisting of a set of harmonic oscillators coupled linearly to $S$ by $H_{int} = \lambda \sum_{k=1}^{2} \sigma_k \cdot b_k$, where $b_k^\dagger = \sum_{\alpha} g_{\alpha,kj}^{\dagger} \left( a_{\alpha,kj} + a_{\alpha,kj}^\dagger \right)$ is a fluctuation quantum field. The Hamiltonian of bath is given by $H_B = \sum_{\alpha} \omega_{\alpha}^{kj} a_{\alpha,kj}^\dagger a_{\alpha,kj}$, where $a_{\alpha,kj}^\dagger (a_{\alpha,kj})$ are bosonic creation (annihilation) operators with $j = x, y, z$ and $\omega_{\alpha}^{kj}$ are the corresponding frequencies with spectral distribution function $J_{kj}(\omega) = \pi \sum_{\alpha} \left( g_{\alpha}^{kj} \right)^2 \delta (\omega - \omega_{\alpha})$ $^{28}$.

We show how to triangulate the partial basis of the bath since, in principle, triangulating the partial basis of the system follows the same approach. The key point is that, in general, the triangular property of $L$ results in $P_{\nu} L Q_{\nu} C_{\nu} P_{\nu}$ being in the SKE.
The matrix for $H_B + H_{int}$, with respect to a $\alpha$th element of the basis of $B$, in a repeat subspace, $(|n_{\alpha, k}^j\rangle, |n_{\alpha, k}^j + 1\rangle, \langle n_{\alpha, k}^j|, \langle n_{\alpha, k}^j + 1|)$, can be written as

$$M = \begin{pmatrix} \omega_{\alpha}^{kj} n_{\alpha, k}^j a_k j \sqrt{n_{\alpha, k}^j + 1} \\ g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} \omega_{\alpha}^{kj} n_{\alpha, k}^j + 1 \end{pmatrix}. \quad (42)$$

Using a similarity transformation to up-triangulate $M$,

$$M_{tri}^{tr} = \frac{1}{ad - bc} \begin{pmatrix} d - b & 0 & 0 & 0 \\ -c & a & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \omega_{\alpha}^{kj} n_{\alpha, k}^j a_k j \sqrt{n_{\alpha, k}^j + 1} \\ g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} \omega_{\alpha}^{kj} n_{\alpha, k}^j + 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (43)$$

we have

$$-bc\omega_{\alpha}^{kj}(n_{\alpha, k}^j + 1) + (dc - ab) g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} = 0,$$

$$(-c^2 + a^2) g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} + ac\omega_{\alpha}^{kj} = 0,$$

which gives

$$a = \pm \sqrt{\frac{\omega_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1}}{2g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} + 1}} c = \gamma_{a, kj} c,$$

$$d = \omega_{\alpha}^{kj} n_{\alpha, k}^j + 1 + \gamma_{a, kj} g_{\alpha}^{kj} b = \zeta_{a, kj} b. \quad (44)$$

This determines the triangular basis for $H_B + H_{int}$ in the repeat subspace as $(\phi_n \otimes \{n_{\alpha, k}^j\}), \langle n_{\alpha, k}^j| \otimes \phi_n\rangle$. For example, by choosing $c = 1$ and $b = -1$ we have

$$M_{tri}^{tr} = \begin{pmatrix} -\gamma_{a, kj} \zeta_{a, kj} \omega_{\alpha}^{kj} n_{\alpha, k}^j & \left( \zeta_{a, kj}^2 - 1 \right) g_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} - \zeta_{a, kj} \omega_{\alpha}^{kj} \\ 0 & \omega_{\alpha}^{kj} j \sqrt{n_{\alpha, k}^j + 1} - \gamma_{a, kj} \zeta_{a, kj} \omega_{\alpha}^{kj} (n_{\alpha, k}^j + 1) \end{pmatrix}, \quad (45)$$

defining,

$$\begin{pmatrix} \gamma_{a, kj} n_{\alpha, k}^j - n_{\alpha, k}^j + 1 \\ n_{\alpha, k}^j - \zeta_{a, kj} n_{\alpha, k}^j + 1 \end{pmatrix}, \quad (46)$$
and

\[
\begin{pmatrix}
\langle n_{a,k}^{j,+} \rangle \\
\langle n_{a,k}^{j,-} \rangle
\end{pmatrix} = \frac{1}{-\gamma_{a,k}\zeta_{a,k} + 1} \begin{pmatrix}
-\zeta_{a,k} \langle n_{a,k}^{j,+} \rangle - \langle n_{a,k}^{j,-} + 1 \rangle \\
\langle n_{a,k}^{j,+} \rangle + \gamma_{a,k} \langle n_{a,k}^{j,-} + 1 \rangle
\end{pmatrix}
\]

(47)

with \(\{n_{a,k}^{j,+} = (n_{1,k}^{j,+} \cdots n_{a,k}^{j,+} \cdots)\). Then we can construct a triangular basis for the total Liouvilian \(L(t)\) by \(\{\phi_n \otimes \{n_{a,k}^{j,+}\} \langle \{n_{a,k}^{j,+}\} \otimes \phi_m\rangle, (\phi_m \otimes \{n_{a,k}^{j,+}\} \langle \{n_{a,k}^{j,+}\} \otimes \phi_n\rangle\}\). This gives triangulation of matrix of \(L\), define the projectors as \(P_\nu = \langle \phi_n \otimes \{n_{a,k}^{j,+}\} \rangle \langle \{n_{a,k}^{j,+}\} \otimes \phi_n\rangle\) and \(Q_\nu = 1 - P_\nu\). Taking into account the definitions of the creation and destruction operators, one has \(P_\nu (\hat{B}_1 + \hat{B}_1) P_\nu (\hat{B}_1 + \hat{B}_1) P_\nu \rightarrow 0\) for \(\mu > \nu\) (up-triangular property). Therefore one finds the DF condition: \(L_0^\mu + (L_1^B + L_1^B) C(t) = 0\) with \(\Theta(t) = 0\) for \(L_0^\mu + (L_1^B + L_1^B)\). Under this DF condition, the SKE reduces to \(\frac{\partial \rho_{\text{proj}}(t)}{\partial t} = \Theta_0(t) \rho_{\text{proj}}(t)\). The evolution of the reduced projected state, in the initial decoupling condition, becomes to Eq.(41). This shows that \(\rho_{S}^{\text{proj}}(t)\) is independent on \(L_{\text{int}}\). Therefore there is no decoherence introduced by \(L_{\text{int}}\) in the projected subsystem although the total system is subject to the decoherence introduced by \(L_{\text{int}}\). In this projected subsystem, the quantum Control-Not logic operation is still given by a sequence of operations and the swap operator \(U_{sw}\) remains invariant before and after the interaction, and is given by

\[
\text{Tr}_B \hat{T} e^{-i \int_0^t \Theta(r) dr} \rho_{\text{proj}}(0) = \text{Tr}_B \hat{T} e^{-i \int_0^t L_0^\mu(r) dr} \rho_{\text{proj}}(0)
\]

(48)

\[
= \text{Tr}_B \left( \hat{T} e^{-i \int_0^t H_0^\mu(r) dr} \rho_{\text{proj}}(0) \hat{T} e^{i \int_0^t H_0^\mu(r) dr} \right)
\]

\[
= \text{Tr}_B \left( U_{sw} \rho_{\text{proj}}(0) U_{sw}^{-1} \right),
\]

where

\[
U_{sw} = \sum_{\alpha} \left( \sum_{n=1}^3 e^{-i \frac{\pi}{2} \phi_n \otimes 1_\alpha} \langle 1_\alpha \otimes \phi_n \rangle + e^{i \frac{\pi}{2} \phi_4 \otimes 1_\alpha} \langle 1_\alpha \otimes \phi_4 \rangle \right).
\]

The following table clarifies the difference between the total space and the projected subspace constructed by the triangulation:
If one assumes that in the Schrödinger picture, an initial state of $S$ is $\phi_S(0) = \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - \frac{1}{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, then by choosing a projector $P_\nu$ as $\frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes \left\{ \left\{ n_{a,k}^{j,\pm} \right\} \otimes \frac{1}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right\}$, after taking into account the obtained triangular basis, an initial state in the DF $P_\nu$-projected subspace is given by $\text{Tr}_B P_\nu \phi_S(0) = \frac{1}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ under the initial decoupling condition. This correspondence is realized by the projection: $\phi_S(0) \rightarrow P_\nu^S \phi_S(0)$. The advantage of using $P_\nu^S \phi_S(0)$ compared with $\phi_S(0)$, is that the evolution of $P_\nu^S \phi_S(0)$ is independent of the interaction between $S$ and $B$ in the projected subspace. Therefore, one can encode information onto $P_\nu^S \phi_S(0)$ to perform quantum computing (or quantum communication), protecting against decoherence introduced by the interaction between $S$ and $B$.

Finally, it should be noted that it is not necessary that the evolution operator $U_\Theta$ in the projected subspace is unitary, because a quantum computing system projected in a DF subspace may be an open quantum system, obeying the semigroup evolution rules\cite{17}. In this open system, self-adjoint operators and unitary evolution groups are not intrinsically necessary to govern quantum computation. Quantum computation can then be performed in a more general functional space, such as RHS, rather than just Hilbert space. The projected state $\psi^{\text{proj}}$ may exist in the test space $\Phi^{\text{proj}}$, which is a dense subspace of the Hilbert space $\mathcal{H}^{\text{proj}}$ constructed by $\sum_\nu P_\nu \Pi_\nu \mathcal{H}$, representing the physical states which can be prepared in an actual experiment. Its adjoint $\tilde{\psi}^{\text{proj}}$ lies in the dual space $(\Phi^{\text{proj}})^\times$, representing a procedure that associates with each state a number, while preserving the linear structure which results from the superposition principle, i.e., the triplet structure $\Phi^{\text{proj}} \subset \mathcal{H}^{\text{proj}} \subset (\Phi^{\text{proj}})^\times$ \cite{23}. This is a RHS structure which facilitates describing irreversible processes like decoherence and dissipation due to interaction with the environment. In this space the evolution of the states are permitted to be time asymmetric, providing a framework for describing the irreversibility.
of practical open systems. This irreversibility does not change quantum reversible logical operations to quantum irreversible logical operation in the quantum universal Controlled-Not logical gate. To appreciate this one must distinguish between irreversibility of a quantum logical operation, introduced by the structure of logical gate, and irreversibility of the process induced by interactions with the environment. Reversible computation means reversible logical operations on the structure of the logical gate. In this sense, quantum computing in RHS is compatible with reversible quantum logical operations and permits computing any reversible function, although irreversible processes do in fact exist.

IX. CONCLUSION

In conclusion, a subdynamics based formulation in the Schrödinger picture was presented for an open quantum system. Based on the subdynamical kinetic equation for an open quantum system, a proposal for quantum computing in the projected subspaces is developed. The eigenvectors of the intermediate operator in this subspace were shown to remain invariant before and after interaction with the environment, while the eigenvalues of the intermediate operator in this subspace change. The fidelity of mixed states in the projected subspace is 1 which means that the constructed projected subspace is definitely DF. This reveals a universal property for any system plus reservoir: one can construct a DF projected subspace by using eigenprojectors of the free part of the Hamiltonian in which the encoded states are projected states, which are themselves determined by relevant formulae of subdynamics. On the other hand, changes of the eigenvalues after interaction may introduce a type of unitary error in the ideal swap operator. This sort of error can be cancelled by adjusting the interaction coupling time between two spins in the subspace, since the eigenvectors remain invariant, although decoherence exists in the total space for the total system. Finally, the general case for completely DF behavior in the projected subspaces was discussed, and it was shown that using a general DF condition (i.e., the second term of the subdynamics kinetic equation is zero), one can find a condition to allow the constructed projected sub-
space to be DF. This reveals that this condition is a necessary and sufficient condition for constructing a DF projected subspace. We wish to emphasize two points here: (1) the constructed projected subspace is spanned by a set of \( \{ \rho^{\text{proj}}_{\nu} (t) \} \), which is closed with respect to the intermediate operator. Indeed, \( \Theta (t) \rho^{\text{proj}} (t) = \sum_{\nu} P_{\nu} L_{\nu} (t) (P_{\nu} + C_{\nu} (t)) \Pi_{\nu} (t) \rho (t) = \sum_{\nu} l_{\nu}^{\Theta} (t) \rho^{\text{proj}}_{\nu} (t) \), where \( l_{\nu}^{\Theta} (t) \) is a \( \nu \)th eigenvalue of \( \Theta (t) \) in Liouvillian representation. (2) The Born-Markov assumption and various other types of restrictions for DF behavior does not need to be made, indeed as we show a general approach can be used.

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