Jordan-Hölder theorems for derived categories of derived discrete algebras

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Abstract
For any positive integer $n$, $n$-derived-simple derived discrete algebras are classified up to derived equivalence. Furthermore, the Jordan-Hölder theorems for all kinds of derived categories of derived discrete algebras are obtained.

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1 Introduction

Throughout this paper, $k$ is an algebraically closed field and all algebras are finite dimensional associative $k$-algebras with identity. Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [6], and play an important role in algebraic geometry and representation theory. Recollements, more general $n$-recollements [17], of derived categories of algebras provide a reduction technique for some homological properties such as the finiteness of global dimension [31, 26, 3], the finiteness of finitistic dimension [22, 15], the finiteness of Hochschild dimension [10] and homological smoothness [17], and some homological invariants such as $K$-theory [32, 36, 29, 31, 14, 3], Hochschild homology and cyclic homology [25] and Hochschild cohomology [16]. Moreover, some homological conjectures such as the finitistic dimension conjecture [22, 15], the Hochschild homology dimension conjecture [17], the Cartan determinant conjecture [17], the Gorenstein
symmetry conjecture [17] and the (dual) vanishing conjecture [35], can be reduced to \( n \)-derived-simple algebras by \( n \)-recollements for appropriate \( n \geq 1 \). A recollement of derived categories of algebras is a short exact sequence of derived categories of algebras, and thus leads to the concepts of composition series and composition factors and Jordan-Hölder theorem analogous to those in group theory and module theory. Indeed, the composition factors in this context are the derived categories of derived simple algebras which were studied in [34, 21, 27, 28, 3]. Meanwhile, the Jordan-Hölder theorem of derived category was established for hereditary algebras [1] and later for piecewise hereditary algebras [2]. Nonetheless, this theorem was disproved for certain infinite dimensional algebras [13, 14], and later for a finite dimensional algebra [3]. Hence, the Jordan-Hölder theorem of the derived category of an algebra is a little bit subtle.

Derived discrete algebras were introduced by Vossieck [33], which are crucial in the Brauer-Thrall type theorems for derived category [18] and were explored in [8, 7, 11, 12, 19, 9]. In particular, Vossieck proved that a basic connected derived discrete algebra is isomorphic to either a piecewise hereditary algebra of Dynkin type or some one-cycle gentle algebra which does not satisfying the clock-condition [33]. The latter was further classified up to derived equivalence by Bobiński, Geiß and Skowroński [5]. Synthesizing the relevant results of Vossieck [33], Happel [20] and Bekkert-Merklen [5], all indecomposable objects in the bounded derived category of a derived discrete algebra are fairly clear.

In this paper, we will classify all \( n \)-derived-simple derived discrete algebras up to derived equivalence, and prove the Jordan-Hölder theorems for all kinds of derived categories of derived discrete algebras. More precisely, we will show that, for any positive integer \( n \), a derived discrete algebra is \( n \)-derived-simple if and only if it is derived equivalent to the underlying field \( k \) or a 2-truncated cycle algebra. In order to obtain simultaneously the Jordan-Hölder theorems for unbounded, upper bounded, lower bounded, bounded derived categories of derived discrete algebras, we will introduce \( n \)-composition-series and \( n \)-composition-factors of the unbounded derived category of an algebra. Furthermore, we will prove that the unbounded derived category of any derived discrete algebra admits an \( n \)-composition-series whose \( n \)-composition-factors are independent on the choice of \( n \)-composition-series up to reordering and derived equivalence. In particular, the Jordan-Hölder theorem holds for the unbounded (resp. upper bounded, lower bounded, bounded) derived category of any derived discrete algebra.
This paper is organized as follows: In section 2, we will study the \( n \)-recollements of derived categories of derived discrete algebras. In Section 3, we will classify all \( n \)-derived-simple derived discrete algebras up to derived equivalence. In section 4, we will define the \( n \)-composition-series and \( n \)-composition-factors of the unbounded derived category of an algebra, and prove the Jordan-Hölder theorems for derived categories of derived discrete algebras.

2 \( n \)-recollements on derived discrete algebras

In this section, we will study the \( n \)-recollements of derived categories of derived discrete algebras.

Let \( A \) be an algebra. Denote by \( \text{Mod} A \) the category of right \( A \)-modules, and by \( \text{mod} A \) (resp. \( \text{proj} A \)) its full subcategory consisting of all finitely generated modules (resp. finitely generated projective modules). Denote by \( D(\text{Mod} A) \) or just \( DA \) the unbounded derived category of \( A \). Denote by \( Db(\text{mod} A) \) or just \( Db(A) \) the bounded derived category of \( A \). Denote by \( Kb(\text{proj} A) \) the homotopy category of cochain complexes of objects in \( \text{proj} A \). An object \( X \) in \( DA \) is said to be compact if \( \text{Hom}_{DA}(X, -) \) commutes with direct sums. It is well-known that up to isomorphism the objects in \( Kb(\text{proj} A) \) are precisely all the compact objects in \( Da \) (Ref. [10]).

An algebra \( A \) is said to be derived discrete provided for every vector \( d = (d_p)_{p \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})} \) there are only finitely many isomorphism classes of indecomposable objects \( X \) in \( Db(A) \) of cohomology dimension vector \( (\dim_k H^p(X))_{p \in \mathbb{Z}} = d \) (cf. [2]). Some characterizations of derived discrete algebras in terms of (global) cohomological range and (global) cohomological length are provided in [18].

Vossieck proved in [33] that a basic connected derived discrete algebra is isomorphic to either a piecewise hereditary algebra of Dynkin type or a one-cycle gentle algebra not satisfying the clock-condition. The latter was shown by Bobiński-Geiβ-Skowroński in [3] to be derived equivalent to a gentle algebra \( \Lambda(r, s, t) \) with \( 1 \leq r \leq s \) and \( t \geq 0 \), which is the bound quiver algebra.

\[ \text{3} \]
generalized strings are defined up to inverse, i.e., the equivalence relation to the vertex \( p \) is composable \( \Lambda(s, s, t) \).

For \( 0 \leq p \leq s-1 \), let \( X_p \) be the simple \( \Lambda(s, s, t) \)-module corresponding to the vertex \( p \) of the quiver \( Q(s, t) \). For \( -t \leq q \leq -1 \), let \( Y_q \) be the indecomposable \( \Lambda(s, s, t) \)-module \( P_q/S_1 \) if \( s = 2 \) and \( P_q/S_0 \) if \( s = 1 \), where \( P_q \) is the indecomposable projective \( \Lambda(s, s, t) \)-module corresponding to the vertex \( q \) of the quiver \( Q(s, t) \). Note that \( Y_q \) has a minimal projective resolution

\[
\cdots \cdots \xrightarrow{\alpha_0} P_0 \xrightarrow{\alpha_{s-1}} P_{s-1} \xrightarrow{\alpha_{s-2}} \cdots \xrightarrow{\alpha_2} P_2 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_{-1} \cdots \alpha_0} P_q \xrightarrow{0}
\]

where all differentials are given by the left multiplications of the paths.

**Lemma 1.** Let \( A = \Lambda(s, s, t) \) with \( s \geq 1 \) and \( t \geq 0 \). Then, up to isomorphism and shift, \( X_0, X_1, \ldots, X_{s-1} \) and \( Y_{-t}, Y_{-t-1}, \ldots, Y_{-1} \) are all noncompact indecomposable objects in \( \mathcal{D}^b(A) \). Moreover, for all noncompact indecomposable objects \( X \) and \( Y \) in \( \mathcal{D}^b(A) \), the following statements hold:

1. \( \text{Hom}_{\mathcal{D}A}(X, X[h]) \cong k \) if \( h = 0, s, 2s, \ldots \), and \( = 0 \) otherwise.
2. \( \text{RHom}_{\mathcal{D}A}(X, Y) \cong 0 \).

**Proof.** For the gentle algebra \( A = \Lambda(s, s, t) \), it is not difficult to see \( \text{GST}_{c} = \{\alpha_p, (\alpha_q \cdots \alpha_1 \alpha_0) \mid 0 \leq p \leq s-1, -t \leq q \leq -1\} \) and \( \text{GST}_{e} = \{\alpha_p^{-1}, (\alpha_0^{-1} \cdots \alpha_{-1} \alpha_q^{-1}) \mid 0 \leq p \leq s-1, -t \leq q \leq -1\} \) (Ref. [4] 4.3). Since generalized strings are defined up to inverse, i.e., the equivalence relation \( \cong_s \) (Ref. [5] 4.1), we have \( \text{GST}_{c} = \text{GST}_{e} = \{\alpha_p, (\alpha_q \cdots \alpha_1 \alpha_0) \mid 0 \leq p \leq s-1, -t \leq q \leq -1\} \).
s − 1, −t ≤ q ≤ −1]. Due to [3, Definition 2, Lemma 6, and Theorem 3], up to isomorphism and shift, \(X_0, X_1, \ldots, X_{s-1}\) and \(Y_t, Y_{t+1}, \ldots, Y_{t-1}\) are all noncompact indecomposable objects in \(\mathcal{D}^b(A)\).

Let \(X\) and \(Y\) be two noncompact indecomposable objects in \(\mathcal{D}^b(A)\).

1. There are altogether two cases:
   1.1) \(X = X_p, \ 0 \leq p \leq s - 1\). Since \(X\) is the simple \(A\)-module \(S_p\) corresponding to the vertex \(p\) of the quiver \(Q(s, t)\), it is easy to calculate \(\text{Hom}_{\mathcal{D}A}(X, X[h]) \cong k\) if \(h = 0, s, 2s, \ldots\), and \(= 0\) otherwise.

   1.2) \(X = Y_q, -t \leq q \leq -1\). In the following, we only consider the case \(s \geq 2\). For the case \(s = 1\), it is enough to replace \(S_1\) with \(S_0\) in appropriate places. We have a short exact sequence \(S_1 \rightarrow P_q \rightarrow Y_q\), which induces a triangle \(P_q \rightarrow Y_q \rightarrow S_1[1] \rightarrow \mathcal{D}A\). Applying the derived functor \(\text{RHom}_A(Y_q, -)\) to this triangle, we obtain a triangle \(\text{RHom}_A(Y_p, P_q) \rightarrow \text{RHom}_A(Y_q, Y_q) \rightarrow \text{RHom}_A(Y_q, S_1[1]) \rightarrow \mathcal{D}k\). It is not difficult to prove \(\text{RHom}_A(Y_q, P_q) \cong 0\). Thus we have \(\text{RHom}_A(Y_q, Y_q) \cong \text{RHom}_A(Y_q, S_1[1])\). Now (1.2) follows from the obvious fact \(H^0(\text{RHom}_A(Y_q, S_1[1])) \cong k\) if \(h = 0, s, 2s, \ldots\), and \(= 0\) otherwise.

2. There are altogether four cases:
   2.1) \((X = X_p) \wedge (Y = X_{p'})\), \(0 \leq p, p' \leq s - 1\). Obviously, there exist infinitely many \(h \in \mathbb{N}\) such that \(\text{Ext}^h_A(S_p, S_{p'}) \neq 0\). Thus \(\text{RHom}_A(X_p, X_{p'}) \neq 0\).

   2.2) \((X = X_p) \wedge (Y = Y_q), 0 \leq p \leq s - 1, -t \leq q \leq -1\). Applying the derived functor \(\text{RHom}_A(X_p, -)\) to the triangle \(P_q \rightarrow Y_q \rightarrow S_1[1] \rightarrow \mathcal{D}A\), we obtain a triangle \(\text{RHom}_A(X_p, P_q) \rightarrow \text{RHom}_A(X_p, Y_q) \rightarrow \text{RHom}_A(X_p, S_1[1]) \rightarrow \mathcal{D}k\) which gives a long exact sequence. We can show \(\text{Hom}_{\mathcal{D}A}(X_p, P_q[h]) = 0\) for all \(h \neq 0\). Thus \(\text{Hom}_{\mathcal{D}A}(X_p, Y_q[h]) \cong \text{Hom}_{\mathcal{D}A}(X_p, S_1[h + 1])\) for all \(h \geq 1\). Since \(\text{Ext}^h_A(S_p, S_1) \neq 0\) for infinitely many \(h > 0\), there exist infinitely many \(h \in \mathbb{N}\) such that \(\text{Hom}_{\mathcal{D}A}(X_p, Y_q[h]) \neq 0\). Hence \(\text{RHom}_A(X_p, Y_q) \neq 0\).

   2.3) \((X = Y_q) \wedge (Y = X_p), -t \leq q \leq -1\), \(0 \leq p \leq s - 1\). Applying the derived functor \(\text{RHom}_A(-, S_p)\) to the triangle \(P_q \rightarrow Y_q \rightarrow S_1[1] \rightarrow \mathcal{D}A\), we obtain a triangle \(\text{RHom}_A(S_1[1], S_p) \rightarrow \text{RHom}_A(Y_q, S_p) \rightarrow \text{RHom}_A(P_q, S_p) \rightarrow \mathcal{D}k\). Obviously, \(\text{RHom}_A(P_q, S_p) \cong 0\). Therefore, we have \(\text{RHom}_A(Y_q, S_p) \cong \text{RHom}_A(S_1[1], S_p) \neq 0\) by (2.1), i.e., \(\text{RHom}_A(Y_q, X_p) \neq 0\).

   2.4) \((X = Y_q) \wedge (Y = Y_{q'}), -t \leq q, q' \leq -1\). Applying the derived functor \(\text{RHom}_A(Y_q, -)\) to the triangle \(P_{q'} \rightarrow Y_{q'} \rightarrow S_1[1] \rightarrow \mathcal{D}A\), we obtain a triangle \(\text{RHom}_A(Y_q, P_{q'}) \rightarrow \text{RHom}_A(Y_q, Y_{q'}) \rightarrow \text{RHom}_A(Y_q, S_1[1]) \rightarrow \mathcal{D}k\). It is not difficult to show \(\text{Hom}_{\mathcal{D}A}(Y_q, P_{q'}[h]) = 0\) for all \(h \neq 1\). Thus \(\text{Hom}_{\mathcal{D}A}(Y_q, Y_{q'}[h]) \cong \text{Hom}_{\mathcal{D}A}(Y_q, S_1[h + 1])\) for all \(h \geq 2\). Note that
\[ \text{Hom}_{\mathcal{D}_A}(Y_q, S_1[h + 1]) \neq 0 \] for infinitely many \( h \in \mathbb{N} \). Hence, there exist infinitely many \( h \in \mathbb{N} \) such that \( \text{Hom}_{\mathcal{D}_A}(Y_q, Y_q'[h]) \neq 0 \). Therefore, \( \text{RHom}_{A}(Y_q, Y_q') \not\cong 0 \).

Before giving some results on the \( n \)-recollements of derived categories of derived discrete algebras, we recall the definition of \( n \)-recollement of triangulated categories.

Let \( T_1, T \) and \( T_2 \) be triangulated categories. A recollement of \( T \) relative to \( T_1 \) and \( T_2 \) is given by

\[
\begin{array}{ccc}
T_1 & \xrightarrow{i^*} & \xleftarrow{i_!} T \\
\xrightarrow{j^*} & & \xleftarrow{j_!} T_2
\end{array}
\]

such that

- (R1) \((i^*, i_*), (i_!, i^!)\), \((j^*, j_!)\) and \((j^!, j_*)!\) are adjoint pairs of triangle functors;
- (R2) \( i_*, j_! \) and \( j_* \) are full embeddings;
- (R3) \( j^!i_* = 0 \) (and thus also \( i_!j_* = 0 \) and \( i^*j_! = 0 \));
- (R4) for each \( X \in T \), there are triangles

\[
j^!j_*X \rightarrow X \rightarrow i_*i^*X
\]

\[
i^!i_*X \rightarrow X \rightarrow j_*j^*X
\]

where the arrows to and from \( X \) are the counits and the units of the adjoint pairs respectively [6].

Let \( T_1, T \) and \( T_2 \) be triangulated categories, and \( n \) a positive integer. An \( n \)-recollement of \( T \) relative to \( T_1 \) and \( T_2 \) is given by \( n + 2 \) layers of triangle functors

\[
\begin{array}{ccc}
T_1 & \xrightarrow{i^*} & \xleftarrow{i_!} T \\
\xrightarrow{j^*} & & \xleftarrow{j_!} T_2
\end{array}
\]

such that every consecutive three layers form a recollement [17].

From now on, we only focus on the \( n \)-recollements of derived categories of algebras, i.e., all three triangulated categories in an \( n \)-recollement are the derived categories of algebras.

**Proposition 1.** Let \( A \) be a derived discrete algebra. Then every recollement of \( \mathcal{D}_A \) relative to \( \mathcal{D}_B \) and \( \mathcal{D}_C \) can be extended to an \( n \)-recollement for all \( n \geq 1 \).
Proof. If gl.dim$A < \infty$ then this is clear by [3, Proposition 3.3]. If gl.dim$A = \infty$, all the connected directed summands of $A$ that are of infinite global dimension are derived equivalent to $\Lambda(s, s, t)$ for some $s \geq 1$ and $t \geq 0$. Assume that there is a recollement

$$
\begin{array}{c}
DB \overset{i^*}{\longrightarrow} DA \overset{j^!}{\longrightarrow} DC,
\end{array}
$$

Then $i_*B \in D^b(A)$ by [17, Lemma 2]. If $i_*B$ is not compact then there exists an indecomposable projective $B$-module $P$ such that $i_*P$ is not compact. Therefore, $i_*P$ is a noncompact indecomposable object in $D^b(\Lambda(s, s, t))$ for some $s \geq 1$ and $t \geq 0$. By Lemma [11], $i_*P$ is isomorphic to a shift of $X_p$ or $Y_q$, and $\text{Hom}_{DA}(i_*P, i_*P[h]) \neq 0$ for infinitely many $h$. It is a contradiction, since $i_*B$, and thus $i_*P$, is exceptional. Hence, $i_*B$ is compact and the recollement can be extended to a 2-recollement by [17, Lemma 3]. Furthermore, the recollement can be extended to an $n$-recollement for all $n \geq 1$ by induction.

The following result is a generalization of [33, Proposition], which states that derived discrete algebras are invariant under derived equivalence.

**Proposition 2.** Let $A$, $B$ and $C$ be algebras, and $DA$ admit a recollement relative to $DB$ and $DC$. If $A$ is derived discrete then so are $B$ and $C$.

**Proof.** According to Proposition [11 and 17 Lemma 2], we have a recollement

$$
\begin{array}{c}
DB \overset{i^*}{\longrightarrow} DA \overset{j^!}{\longrightarrow} DC,
\end{array}
$$

where all six triangle functors can be restricted to $D^b(mod)$. Since $i_*$ is fully faithful, it induces a map from the set of isomorphism classes of indecomposable objects in $D^b(B)$ to the set of isomorphism classes of indecomposable objects in $D^b(A)$. Due to $i^*i_* \cong 1$, this map is injective. Moreover, by the same estimate on the cohomological dimension vector as [33, Section 1.1], we can show that the cohomological dimension vectors of the images of indecomposable objects in $D^b(B)$ of a fixed cohomological dimension vector under the triangle functor $i_*$ are “bounded”. Thus $B$ is derived discrete. Similarly, we can prove $C$ is derived discrete by considering the triangle functor $j_*$ instead of $i_*$. 

\[\square\]
Remark 1. The converse of Proposition 2 is not true in general. Kronecker algebra $A$, which is nothing but the path algebra $kQ$ of the quiver $Q$ given by two vertices 1 and 2 and two arrows from 1 to 2, provides a counterexample. Indeed, though $DA$ admits a recollement relative to $Dk$ and $Dk$, $A$ is not derived discrete since all derived discrete algebras are of finite representation type.

3 $n$-derived-simple derived discrete algebras

In this section, we will classify all $n$-derived-simple derived discrete algebras up to derived equivalence.

Let $n$ be a positive integer. An algebra $A$ is said to be $n$-derived-simple if its derived category $DA$ admits no nontrivial $n$-recollements of derived categories of algebras [17]. Note that 1- (resp. 2-, 3-) derived-simple algebras here are just the $D$(Mod)- (resp. $D^-(\text{Mod})$, $D^+(\text{Mod})$, $D^b(\text{mod})$, $D^b(\text{Mod})$) derived-simple algebras in [3].

Now we focus on the algebra $\Lambda(s, s, t)$, where $s \geq 1$ and $t \geq 0$. The following lemma shows that $\Lambda(s, s, t)$ is rigid when deconstructing its derived category by recollements.

Lemma 2. Let $A = \Lambda(s, s, t)$ with $s \geq 1$ and $t \geq 0$, and $DA$ admit a recollement relative to $DB$ and $DC$. Then either $\text{gl.dim} C < \infty$ and $B$ is derived equivalent to $\Lambda(s, s, t') \oplus B'$ with $0 \leq t' \leq t$ and $\text{gl.dim} B' < \infty$, or $\text{gl.dim} B < \infty$ and $C$ is derived equivalent to $\Lambda(s, s, t') \oplus C'$ with $0 \leq t' \leq t$ and $\text{gl.dim} C' < \infty$.

Proof. If $DA$ admits a recollement relative to $DB$ and $DC$, then both $B$ and $C$ are derived discrete by Proposition 2. Since $\text{gl.dim} A = \infty$, at least one of $B$ and $C$ is of infinite global dimension [3, Proposition 2.14]. Without loss of generality, we assume $\text{gl.dim} B = \infty$. Thus $B$ is derived equivalent to $\tilde{B} = \Lambda(s', s', t') \oplus B'$ for some integers $s' \geq 1$ and $t' \geq 0$ and some algebra $B'$. Therefore, there is a recollement

$$
\begin{array}{cccccc}
\mathcal{D}\tilde{B} & i^* & & & & j_* \\
& i & & & & \\
& & D\Lambda & & & \\
& & & j & & \\
& & & & & D\mathcal{C}.
\end{array}
$$

The proposition will be proved by the following three steps.

Step 1. We claim $s' = s$ and $t' \leq t$. 

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For this, let $S'_0$ be the simple $\Lambda(s',s',t')$-module corresponding to the vertex 0 of the quiver $Q(s',t')$. Then $S'_0$ is naturally a $\tilde{B}$-module as well and \( \text{Hom}_{\mathcal{D}B}(S'_0,S'_0[h]) \cong k \) if $h = 0, s', 2s', \ldots$, and 0 otherwise. Since $S'_0$ is a noncompact indecomposable object in $\mathcal{D}^b(\tilde{B})$ and $i^*$ preserves compactness, we have $i_*S'_0$ is a noncompact indecomposable object in $\mathcal{D}^b(A)$. By Lemma 2 we have \( \text{Hom}_{\mathcal{D}A}(i_*S'_0, i_*S'_0[h]) \cong k \) if $h = 0, s, 2s, \ldots$, and 0 otherwise. Therefore, $s' = s$, since the triangle functor $i_*$ is fully faithful. It follows from Proposition 6.5 that the rank $r$ of the Grothendieck group $K_0(\tilde{B})$ of $\tilde{B}$ is not larger than the rank $r(\Lambda(1))$ of the Grothendieck group $K_0(\Lambda(1))$ of $\Lambda(1)$. Hence, $t' \leq t$.

**Step 2.** We claim $\text{gl.dim}B' < \infty$.

Assume on the contrary $\text{gl.dim}B' = \infty$. Then $B'$ is derived equivalent to $\Lambda(s'',s'',t'') \oplus B''$ for some integers $s'' \geq 1$ and $t'' \geq 0$ and some algebra $B''$. Therefore, $i_*$ induces a triangle functor

\[
i'_* : \mathcal{D}(\Lambda(s',s',t')) \oplus \mathcal{D}(\Lambda(s'',s'',t'')) \oplus \mathcal{D}B'' \cong \mathcal{D}\tilde{B} \to \mathcal{D}A.
\]

Let $S''_0$ be the simple $\Lambda(s'',s'',t'')$-module corresponding to the vertex 0 of the quiver $Q(s'',t'')$. Then $\text{RHom}_A(i'_*S'_0, i'_*S''_0) \cong 0$. However, $i'_*S'_0$ and $i'_*S''_0$ are pairwise non-isomorphic noncompact indecomposable objects in $\mathcal{D}^b(A)$, and thus we have $\text{RHom}_A(i'_*S'_0, i'_*S''_0) \neq 0$ by Lemma 2. It is a contradiction.

**Step 3.** We claim $\text{gl.dim}C \leq \infty$.

Assume on the contrary $\text{gl.dim}C = \infty$. Then $C$ is derived equivalent to $\tilde{C} = \Lambda(s'''',s'''',t'''') \oplus C'$ for some integers $s'''' \geq 1$ and $t'''' \geq 0$ and some algebra $C'$. Therefore, there is a recollement

\[
\begin{array}{ccc}
\mathcal{D}\tilde{B} & \xrightarrow{i_*} & \mathcal{D}A \\
pullback & & \downarrow j^* \\
\mathcal{D}B & \xrightarrow{i^*} & \mathcal{D}C,
\end{array}
\]

where both $i_*$ and $j_*$ can be restricted to $\mathcal{D}^b(\text{mod})$ by Proposition 1 and Lemma 2. Let $S'''_0$ be the simple $\Lambda(s''',s''',t''')$-module corresponding to the vertex 0 of the quiver $Q(s''',t''')$. Then $\text{RHom}_A(i_*S'_0, j_*S'''_0) \cong 0$. However, $i_*S'_0$ and $j_*S'''_0$ are noncompact indecomposable objects in $\mathcal{D}^b(A)$, and thus $\text{RHom}_A(i_*S'_0, j_*S'''_0) \neq 0$ by Lemma 2(2). It is a contradiction. \[\square\]
As a consequence of Lemma 2, we have the following proposition which was obtained independently by Angeleri H"ugel, K"onig, Liu and Yang in different way in the earlier version of [3].

**Proposition 3.** Every 2-truncated cycle algebra is $n$-derived-simple for all $n \geq 1$.

*Proof.* Assume $A = \Lambda(s, s, 0)$ and $DA$ admits an $n$-recollement relative to $DB$ and $DC$. According to Lemma 2, we may assume that $\text{gl.dim} C < \infty$ and $B$ is derived equivalent to $\Lambda(s, s, 0) \oplus B'$ for some algebra $B'$. By [3, Proposition 6.5], we have $r(A) = r(B) + r(C) = r(A) + r(B') + r(C)$. Thus $r(B') = 0 = r(C)$. Hence $B' = 0 = C$, i.e., the $n$-recollement is trivial. \hfill $\square$

The next proposition implies that dropping an idecomposable projective module with projective radical in [24] induces a recollement of derived categories of algebras.

**Proposition 4.** Let $A$ be a basic algebra, $e$ a primitive idempotent, $\bar{e} := 1 - e$, and rade $A$ a projective $A$-module. Then $DA$ admits a recollement relative to $D(\bar{e}A\bar{e})$ and $Dk$.

*Proof.* Let $X$ be the projective $A$-module $\bar{e}A$, and $Y$ the simple $A$-module $eA/\text{rad} eA$. Since rade $A$ is projective, we have $\text{pd}_A Y \leq 1$. It is easy to see that both $X$ and $Y$ are compact exceptional objects in $DA$ with $\text{End}_A(X) \cong \bar{e}A\bar{e}$ and $\text{End}_A(Y) \cong k$, and $\text{RHom}_A(X, Y) \cong 0$. Moreover, for all $Z \in DA$ satisfying $\text{RHom}_A(X, Z) \cong 0 \cong \text{RHom}_A(Y, Z)$, we have $\text{RHom}_A(X \oplus Y, Z) \cong 0$. Clearly, rade $A \in \text{thick} X$, the smallest thick subcategory of $DA$ containing $X$, i.e., the smallest full triangulated subcategory of $DA$ containing $X$ and closed under direct summands. Thus $eA$, and further $A \cong eA \oplus X$, belongs to $\text{thick}(X \oplus Y)$. Hence $Z \cong \text{RHom}_A(A, Z) \cong 0$. Therefore, $X$ and $Y$ determine a recollement of $DA$ relative to $Dk$ and $D(\bar{e}A\bar{e})$ by [16, Proposition 1], and further a 2-recollement of $DA$ relative to $Dk$ and $D(\bar{e}A\bar{e})$ by [17, Proposition 1]. Hence, $DA$ admits a recollement relative to $D(\bar{e}A\bar{e})$ and $Dk$. \hfill $\square$

Let $B$ be a bound quiver algebra, $M$ a right $B$-module and $N$ a left $B$-module. Then $\begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$ is called a one-point extension of $B$, and $\begin{bmatrix} k & 0 \\ N & B \end{bmatrix}$ is called a one-point coextension of $B$ (Ref. [30]). The following lemma implies that one-point (co)extension induces a recollement, which will be used frequently later.
Lemma 3. Let $A$ be a one-point (co)extension of a bound quiver algebra $B$. Then $\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}k$.

Proof. Both one-point extensions and one-point coextensions are special triangular matrix algebras. So this lemma follows from [17, Example 1 (2)]. In fact, the one-point coextension case also can be deduced from Proposition 4 by considering the idempotent element corresponding to the coextension vertex which is a sink in the quiver of $A$ and corresponds to a simple projective $A$-module. □

The following result is our main theorem in this section.

Theorem 1. For any positive integer $n$, a derived discrete algebra is $n$-derived-simple if and only if it is derived equivalent to either $k$ or a 2-truncated cycle algebra.

Proof. Sufficiency. It follows from Proposition 3.

Necessity. Let $A$ be a basic $n$-derived-simple derived discrete algebra. Then $A$ must be connected. If $A$ is a piecewise hereditary algebra of Dynkin type then it is derived equivalent to $k$ by [2, Theorem 5.7]. Otherwise, $A$ is derived equivalent to $\Lambda(r, s, t)$ for some $s \geq r \geq 1$ and $t \geq 0$. If $t \geq 1$ then $A$ is a one-point extension of $\Lambda(r, s, t - 1)$. It follows from Lemma 3 and Proposition 4 that $\mathcal{D}A$ admits a non-trivial $n$-recollement relative to $\mathcal{D}\Lambda(r, s, t - 1)$ and $\mathcal{D}k$, which contradicts to the assumption. Hence, we have $t = 0$. If $s > r$ then applying Proposition 4 to the algebra $\Lambda(r, s, 0)$ and the idempotent $e$ corresponding to the vertex 0 of the quiver $Q(s, 0)$ and also Proposition 4 we can obtain a non-trivial $n$-recollement of $\mathcal{D}\Lambda(r, s, 0)$, and thus $\mathcal{D}A$. It contradicts to the assumption, which implies $s = r$ and $A$ must be derived equivalent to a 2-truncated cycle algebra $\Lambda(s, s, 0)$.

4 Jordan-Hölder theorems

In this section, we will show that the Jordan-Hölder theorems hold for all kinds of derived categories of derived discrete algebras.

Definition 1. Let $n$ be a positive integer. An $n$-composition-series of the derived category $\mathcal{D}A$ of an algebra $A$ is a chain of derived categories of algebras linked by fully faithful triangle functors

$$0 = \mathcal{D}B_l \xrightarrow{i_l} \mathcal{D}B_{l-1} \xrightarrow{i_{l-1}} \cdots \xrightarrow{i_2} \mathcal{D}B_1 \xrightarrow{i_1} \mathcal{D}B_0 = \mathcal{D}A$$
such that, for all \( p = 1, 2, \cdots, l \), the triangle functor \( DB_p \overset{i_p}{\rightarrow} DB_{p-1} \) can be completed to an \( n \)-recollement

\[
\begin{array}{ccc}
DB_p & \overset{i_p}{\longrightarrow} & DB_{p-1} \\
\downarrow & & \downarrow \\
& & \vdots \\
\end{array}
\]

for some \( n \)-derived-simple algebra \( C_p \). In this case, \( DC_1, DC_2, \cdots, DC_l \) are called the \( n \)-composition-factors of \( D A \). Moreover, \( DC_p \) is said to be of multiplicity \( m_p \) if it appears exactly \( m_p \) times in the sequence \( DC_1, DC_2, \cdots, DC_l \) up to derived equivalence.

**Remark 2.**

(1) By [3, Proposition 6.5], we know that the length \( l \) of an \( n \)-composition-series of \( D A \) is not larger than the rank \( r(A) \) of the Grothendieck group \( K_0(A) \) of \( A \).

(2) If \( n \neq n' \) then it is possible that the length of an \( n \)-composition-series of \( D A \) is not equal to the length of an \( n' \)-composition-series of \( D A \). For example, let \( A \) be the two-point algebra given in [3, Example 5.8] which is 2-derived simple but not 1-derived-simple. Then the length of any 1-composition-series of \( D A \) is 2. However, the length of any 2-composition-series of \( D A \) is 1.

(3) It is possible that the length of one \( n \)-composition-series of \( D A \) is equal to the length of the other \( n \)-composition-series of \( D A \) but two \( n \)-composition-series have different \( n \)-composition-factors, so the Jordan-Hölder theorems do not hold for derived categories of algebras in general. For example, let \( A \) be the two-point algebra given in [3, Example 7.6]. Then \( D A \) admits two 1-composition-series with completely different 1-composition-factors.

(4) According to [2, Theorem 5.7 and Corollary 5.9], for any piecewise hereditary algebra \( A \) and \( n = 1, 2, 3 \), \( D A \) admits an \( n \)-composition-series with only \( n \)-composition-factors \( Dk \) of multiplicity \( r(A) \) which are independent of the choice of \( n \)-composition-series up to derived equivalence. Thus the Jordan-Hölder theorem holds for the unbounded (resp. upper bounded, lower bounded, bounded) derived category of any piecewise hereditary algebra.

**Lemma 4.** Let \( A_1, \cdots, A_m \) be algebras, \( A = \bigoplus_{p=1}^m A_p \) and \( n \) a positive integer. If \( D A_p \) admits an \( n \)-composition-series with \( n \)-composition-factors \( DC_{p,1}, \cdots, DC_{p,l_p} \) for all \( 1 \leq p \leq m \) then \( D A \) admits an \( n \)-composition-series with \( n \)-composition-factors \( DC_{1,1}, \cdots, DC_{1,l_1}, \cdots, DC_{m,1}, \cdots, DC_{m,l_m} \).
Proof. It is clear that any \( n \)-recollement
\[
\begin{array}{ccc}
\mathcal{D}B & \overset{\oplus}{\longrightarrow} & \mathcal{D}A \\
\vdots & & \vdots \\
\end{array}
\]
can always induce an \( n \)-recollement
\[
\begin{array}{ccc}
\mathcal{D}(B \oplus E) & \overset{\oplus}{\longrightarrow} & \mathcal{D}(A \oplus E) \\
\vdots & & \vdots \\
\end{array}
\]
for all algebras \( A, B, C \) and \( E \). Let
\[
0 = \mathcal{D}B_{p,1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}B_{p,0} \hookrightarrow \mathcal{D}B_{p,0} = \mathcal{D}A_p
\]
be an \( n \)-composition-series of \( \mathcal{D}A_p \) with \( n \)-composition-factors \( \mathcal{D}C_{p,1}, \cdots, \mathcal{D}C_{p,l_p} \). Then they induce an \( n \)-composition-series
\[
0 = \mathcal{D}B_{m,1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}B_{m,0} = \mathcal{D}A_m = \mathcal{D}A_m \oplus \mathcal{D}B_{m-1,l_{m-1}} \hookrightarrow \mathcal{D}A_m \oplus \mathcal{D}B_{m-1,1,1} \hookrightarrow \mathcal{D}A_m \oplus \mathcal{D}B_{m-1,1,1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}A \text{ of } \mathcal{D}A \text{ with } \mathcal{D}C_{1,1}, \cdots, \mathcal{D}C_{1,1}, \ldots, \mathcal{D}C_{m,1}, \ldots, \mathcal{D}C_{m,l_m}.
\]

The following result is our main theorem in this section.

**Theorem 2.** Let \( n \) be a positive integer and \( A \) a derived discrete algebra, say derived equivalent to \( \oplus_{p=1}^m A_p \) where \( m \geq 1 \) and \( A_p = \Lambda(s_p, s_p, t_p) \) for \( s_p \geq 1, t_p \geq 0, 1 \leq p \leq u \) and \( 0 \leq u \leq m; A_p = \Lambda(r_p, s_p, t_p) \) for \( s_p > r_p \geq 1, t_p \geq 0, u + 1 \leq p \leq v \) and \( u \leq v \leq m; A_p \) is a basic connected piecewise hereditary algebra of Dynkin type for \( v + 1 \leq p \leq m \). Then \( \mathcal{D}A \) admits an \( n \)-composition-series with \( n \)-composition-factors \( \mathcal{D}(\Lambda(s_1, s_1, 0)), \cdots, \mathcal{D}(\Lambda(s_u, s_u, 0)), \) and \( \mathcal{D}k \) of multiplicity \( r(A) - \sum_{p=1}^m s_p \). Moreover, any \( n \)-composition-series of \( \mathcal{D}A \) has precisely these \( n \)-composition-factors up to reordering and derived equivalence.

**Proof.** For \( v + 1 \leq p \leq m \), \( A_p \) is a basic connected piecewise hereditary algebra of Dynkin type. By [23, Theorem 1.1 (i)], \( A_p \) is triangular, i.e., its quiver has no oriented cycles. Thus \( A_p \) can be constructed from \( k \) by \( r(A_p) - 1 \) times of one-point extensions. It follows from Lemma [3, Proposition 2] and Proposition [1] that \( \mathcal{D}A_p \) admits an \( n \)-composition-series with \( n \)-composition-factors \( \mathcal{D}k \) of multiplicity \( r(A_p) \).

For \( u + 1 \leq p \leq v \), \( A_p = \Lambda(r_p, s_p, t_p) \) with \( s_p > r_p \geq 1 \) and \( t_p \geq 0 \). The algebra \( \Lambda(r_p, s_p, t_p) \) can be constructed from \( \Lambda(r_p, s_p, 0) \) by \( t_p \) times one-point extensions. For \( \Lambda(r_p, s_p, 0), \) since \( s_p > r_p \), applying Proposition [3] to the
idempotent element \( e \) corresponding to the vertex 0 of the quiver \( Q(s_p, 0) \) and also Proposition [\ref{prop:1}] we get an \( n \)-recollement of \( \mathcal{D}\Lambda(r_p, s_p, 0) \) relative to \( \mathcal{D}E_p \) and \( \mathcal{D}k \) where \( E_p \) is a Nakayama algebra whose quiver is a line quiver. The algebra \( E_p \) can be constructed from \( k \) by \( r(E_p) - 1 = s_p - 2 \) times of one-point extensions. Therefore, \( \mathcal{D}\Lambda(r_p, s_p, t_p) \) admits an \( n \)-composition-series with \( n \)-composition-factors \( \mathcal{D}k \) of multiplicity \( r(\Lambda(r_p, s_p, t_p)) = t_p + s_p \).

For \( 1 \leq p \leq u \), \( A_p = \Lambda(s_p, s_p, t_p) \) with \( s_p \geq 1 \) and \( t_p \geq 0 \). The algebra \( \Lambda(s_p, s_p, t_p) \) can be constructed from \( \Lambda(s_p, s_p, 0) \) by \( t_p \) times one-point extensions. Thus \( \mathcal{D}\Lambda(s_p, s_p, t_p) \) admits an \( n \)-composition-series with \( n \)-composition-factors \( \mathcal{D}(\Lambda(s_p, s_p, 0)) \) and \( \mathcal{D}k \) of multiplicity \( t_p \).

By the above analyzes and Lemma [\ref{lem:1}], we know \( \mathcal{D}A \) admits an \( n \)-composition-series with \( n \)-composition-factors \( \mathcal{D}(\Lambda(s_1, s_1, 0)), \cdots, \mathcal{D}(\Lambda(s_u, s_u, 0)) \), and \( \mathcal{D}k \) of multiplicity \( r(A) - \sum_{p=1}^{u} s_p \).

Next we prove these \( n \)-composition-factors of \( \mathcal{D}A \) are independent of the choice of \( n \)-composition-series by induction on \( r(A) \). If \( r(A) = 1 \) then \( A \) is local. Thus \( A \) is \( n \)-derived-simple. By Theorem [\ref{thm:1}] \( A \) is derived equivalent to either \( k \) or \( \Lambda(1,1,0) \). In either case, we have nothing to say. Assume \( r(A) \geq 2 \) and the statement holds for all algebras \( B \) with \( r(B) < r(A) \). For any \( n \)-composition-series

\[
(*) \quad 0 = \mathcal{D}B_0 \hookrightarrow \mathcal{D}B_{l-1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}B_1 \hookrightarrow \mathcal{D}B_0 = \mathcal{D}A
\]

of \( \mathcal{D}A \), we assume that the triangle functor \( \mathcal{D}B_1 \hookrightarrow \mathcal{D}A \) is completed to an \( n \)-recollement

\[
\begin{array}{c}
\mathcal{D}B_1 \\
\vdots
\end{array}
\begin{array}{c}
\mathcal{D}A \\
\vdots
\end{array}
\begin{array}{c}
\mathcal{D}C_1
\end{array}
\]

where \( C_1 \) is \( n \)-derived-simple. Then \( C_1 \) is derived discrete by Proposition [\ref{prop:2}]. Thus up to derived equivalence \( C_1 = k \) or \( \Lambda(s,s,0) \) for some integer \( s \geq 1 \) by Theorem [\ref{thm:1}]

According to the proof of [\ref{ref:27}, Corollary 3.4], we get \( B_1 = \oplus_{p=1}^{m} B_{1,p} \), \( C_1 = \oplus_{p=1}^{m} C_{1,p} \) and the above \( n \)-recollement can be decomposed as the direct sum of \( n \)-recollements

\[
\begin{array}{c}
\mathcal{D}B_{1,p} \\
\vdots
\end{array}
\begin{array}{c}
\mathcal{D}A_p \\
\vdots
\end{array}
\begin{array}{c}
\mathcal{D}C_{1,p}
\end{array}
\]

The \( n \)-derived-simplicity of \( C_1 \) implies that \( C_1 \) is connected. Therefore, there exists some \( q \in \{1,2, \cdots, m\} \) such that \( C_1 = C_{1,q} \) and \( \mathcal{D}B_{1,p} \simeq \mathcal{D}A_p \) for all \( p \neq q \). Up to derived equivalence, we may assume \( B_1 = (\oplus_{p \neq q} A_p) \oplus B_{1,q} \).
If $1 \leq q \leq u$ then $A_q = \Lambda(s_q, s_q, t_q)$. By Lemma [2] we have either $C_1 = k$ and $B_{1,q} = \Lambda(s_q, s_q, t_q') \oplus B_{1,q}'$ for some integer $0 \leq t_q' \leq t_q$ and some algebra $B_{1,q}'$ with $\text{gl.dim} B_{1,q}' < \infty$, or $C_1 = \Lambda(s_q, s_q, 0)$ and $\text{gl.dim} B_{1,q} < \infty$. If $u + 1 \leq q \leq m$ then $\text{gl.dim} A_q < \infty$. Thus $C_1 = k$ and $\text{gl.dim} B_{1,q} < \infty$. Let’s discuss these three cases in detail.

Case 1. $A_q = \Lambda(s_q, s_q, t_q)$, $C_1 = k$ and $B_{1,q} = \Lambda(s_q, s_q, t_q') \oplus B_{1,q}'$ with $\text{gl.dim} B_{1,q}' < \infty$.

In this case $B_1 = (\oplus_{p \neq q} A_p) \oplus B_{1,q} = (\oplus_{p \neq q} A_p) \oplus \Lambda(s_q, s_q, t_q') \oplus B_{1,q}'$, and $r(B_1) = r(A) - 1$. By induction assumption, the $n$-composition-series

$$0 = DB_1 \leftrightarrow DB_{1-1} \leftrightarrow \cdots \leftrightarrow DB_2 \leftrightarrow DB_1$$

of $DB_1$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $Dk$ of multiplicity $r(B_1) - \sum_{p=1}^u s_p$ up to reordering and derived equivalence. Thus the $n$-composition-series (*) of $DA$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $Dk$ of multiplicity $r(B_1) - \sum_{p=1}^u s_p + 1 = r(A) - \sum_{p=1}^u s_p$ up to reordering and derived equivalence.

Case 2. $A_q = \Lambda(s_q, s_q, t_q)$, $C_1 = \Lambda(s_q, s_q, 0)$ and $\text{gl.dim} B_{1,q} < \infty$.

In this case $B_1 = (\oplus_{p \neq q} A_p) \oplus B_{1,q}$ with $\text{gl.dim} B_{1,q} < \infty$, and $r(B_1) = r(A) - s_q$. By induction assumption, the $n$-composition-series

$$0 = DB_1 \leftrightarrow DB_{1-1} \leftrightarrow \cdots \leftrightarrow DB_2 \leftrightarrow DB_1$$

of $DB_1$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $p \neq q$, and $Dk$ of multiplicity $r(B_1) - \sum_{p \neq q} s_p$, up to reordering and derived equivalence. Thus the $n$-composition-series (*) of $DA$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $Dk$ of multiplicity $r(B_1) - \sum_{p \neq q} s_p = r(A) - \sum_{p=1}^u s_p$ up to reordering and derived equivalence.

Case 3. $\text{gl.dim} A_q < \infty$, $C_1 = k$ and $\text{gl.dim} B_{1,q} < \infty$.

In this case $B_1 = (\oplus_{p \neq q} A_p) \oplus B_{1,q}$ with $\text{gl.dim} B_{1,q} < \infty$ and $r(B_1) = r(A) - 1$. By induction assumption, the $n$-composition-series

$$0 = DB_1 \leftrightarrow DB_{1-1} \leftrightarrow \cdots \leftrightarrow DB_2 \leftrightarrow DB_1$$

of $DB_1$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $Dk$ of multiplicity $r(B_1) - \sum_{p=1}^u s_p$ up to reordering and derived equivalence. Thus the $n$-composition-series (*) of $DA$ has exactly $n$-composition-factors $D(\Lambda(s_p, s_p, 0))$ with $1 \leq p \leq u$ and $Dk$ of multiplicity $r(B_1) - \sum_{p=1}^u s_p + 1 = r(A) - \sum_{p=1}^u s_p$ up to reordering and derived equivalence.

Now we finish the proof of the theorem. \[ \square \]
Remark 3. For \( n = 1 \) (resp. 2, 3), Theorem [2] implies that the Jordan-Hölder theorem holds for the unbounded (resp. upper bounded or lower bounded, bounded) derived category of any derived discrete algebra.

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