Measurement theory in local quantum physics:
Based on local state formalism in AQFT

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Abstract

We aim at establishing measurement theory in local quantum physics. For this purpose, we first develop a general theory of quantum measurement. We discuss a representation theory of completely positive (CP) instruments in section 3: the necessary preliminaries are given in section 2. We introduce a condition called the normal extension property (NEP) for CP instruments to represent physical processes of measurement. In section 4, we shall characterize the set of CP instruments by cohomology theory of von Neumann algebras. The existence problem of a family of a posteriori states is discussed in section 5. It is shown that NEP is equivalent to the existence of a strongly $F$-measurable family of a posteriori states for every normal state. In the last section, local measurements in DHR-DR theory are developed. For any representation satisfying the DHR selection criterion, every CP instrument, which is defined on a local algebra and described by a measuring process, can be extended to a local CP instrument.

1 Introduction

We aim at establishing measurement theory in local quantum physics. As a first step, we study it on the basis of local state formalism of algebraic quantum field theory (AQFT). AQFT based on the concept of local state is given in the study [34] of one of the authors, K. Okamura (K.O. for short), with his collaborators.

Quantum measurement theory is one area of quantum theory. Quantum information technology and this theory are now developing and having a mutual, good influence on each other. In particular, mathematical theory of quantum measurement makes a great contribution to understanding of measurement. There are two examples: one is the resolution of the dispute on a quantum mechanical analysis on the performance of gravitational wave detectors [39, 40, 31], and the other is experimental demonstrations of error-disturbance uncertainty relations [23, 49, 6, 24, 27, 47]. Therefore, mathematical theory of quantum measurement is much worth studying. From the viewpoint of mathematics, we believe that it is very interesting and should be deepened.

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Mathematical study of quantum measurement begins with the famous book [33] written by von Neumann. After the studies by Lüders [30], Nakamura and Umegaki [32], and Arveson [3], Davies and Lewis [15] introduced the concept of instruments [15, 14] in order to analyze general measurements which do not always satisfy the repeatability hypothesis. Taking this opportunity, the theory greatly developed. At almost the same time, Kraus [28, 29] treated completely positive maps in the context of quantum measurement but his interest is only the yes-no measurements. Following these studies, one of the author, M. Ozawa (M.O. for short), established the basis of present quantum measurement theory in [36]. In this paper, completely positive (CP) instruments are introduced in the general setting, and are shown to correspond to statistical equivalence classes of measuring processes in quantum mechanical situations [36, Theorem 5.1]. This result finished the characterization of general measurements in quantum mechanics (see also [41, 42] for an axiomatic characterization of generalized quantum measurement). Introductions of families of a posteriori states and of disintegrations and the resolution of Davies-Lewis conjecture in [37, 38] also played the crucial role of development of quantum measurement theory.

Here, we develop a representation theory of CP instruments defined on general (σ-finite) von Neumann algebras and apply to quantum systems of infinite degrees of freedom, specially, to algebraic quantum field theory (AQFT). We expect that measurement theory in infinite quantum systems is soon needed since quantum information technology is developing very rapidly. We believe that this course of action of the study is very natural.

We discuss a representation theory of CP instruments in section 3: the necessary preliminaries are given in section 2. We introduce a condition called the normal extension property (NEP) for CP instruments to represent physical processes of measurement. Let \( \mathcal{M} \) be a (σ-finite) von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \( (S, \mathcal{F}) \) a measurable space. In fact, this condition for a CP instrument \( I \) for \( (\mathcal{M}, S) \) is equivalent to the existence of a (faithful) measuring process \( \mathcal{M} = (K, \sigma, E, U) \) such that

\[
I(\Delta)M = (id \otimes \sigma)[U^*(M \otimes E(\Delta))U] \tag{1}
\]

for all \( \Delta \in \mathcal{F} \) and \( M \in \mathcal{M} \). This equivalence is given in the main theorem of this section. The one-to-one correspondence between CP instruments for \( (\mathcal{M}, S) \) and statistical equivalence classes of measuring processes for \( (\mathcal{M}, S) \) follows from the result. This fact is a generalization of [36, Theorem 5.1] to general (σ-finite) von Neumann algebras. The class of CP instruments described by measuring processes is completely characterized in this way.

In section 4, we shall characterize the set \( \text{CPInst}(\mathcal{M}, S) \) of CP instruments for \( (\mathcal{M}, S) \) by cohomology theory of von Neumann algebras. We begin with a basic result that every CP instrument for \( (\mathcal{M}, S) \) has NEP if there exists a normal conditional expectation from \( B(\mathcal{H}) \) into \( \mathcal{M} \) A CP instrument is said to have the approximately normal extension property (ANEP) if it is approximated by CP instruments with NEP. It is proved that the set \( \text{CPInst}_{\text{AN}}(\mathcal{M}, S) \) of CP instruments with ANEP is equal to \( \text{CPInst}(\mathcal{M}, S) \) if \( \mathcal{M} \) is injective. This beautiful consequence is a natural extension of the basic result mentioned above.

The existence problem of a family of a posteriori states introduced by [37] is discussed in section 5. The main theorem of the section shows that NEP is equivalent to the existence of a strongly \( \mathcal{F} \)-measurable family of a posteriori states for every normal state. Also, it is proved that a weakly repeatable CP instrument has NEP if and only if it is discrete. Following this result, an important example of CP instruments without NEP is given.
It demonstrates the conceptual superiority of CP instruments in quantum measurement theory.

In the last section, local measurements in DHR-DR theory are developed. We can prove the following theorem under some condition including Haag duality:

Theorem 6.1. Suppose that a local net \( \{ \pi_0(\mathcal{A}(\mathcal{O})) \}_{\mathcal{O} \in \mathcal{K}} \) satisfies the split property. Let \((\Gamma, \mathcal{B}(\Gamma)) \) be a standard Borel space, \( \mathcal{O} \) a double cone, \( \pi \) a representation of \( \mathcal{A} \) on \( \mathcal{H}_0 \) such that \( \pi_{|\mathcal{A}(\mathcal{O})'} \cong \pi_0_{|\mathcal{A}(\mathcal{O})} \). For every CP instrument \( \mathcal{I} \) for \((\pi(\mathcal{A})(\mathcal{O}))'', \Gamma \) with NEP and \( \Lambda \in \mathcal{K}_{DC}^\infty \) such that \( \mathcal{O}_1^\Lambda = \mathcal{O} \), there exists a local CP instrument \( \tilde{\mathcal{I}} \) for \((\pi(\mathcal{A})'', \Gamma, \Lambda) \) such that \( \tilde{\mathcal{I}}(\Delta)A = \mathcal{I}(\Delta)A \) for all \( A \in \pi(\mathcal{A})(\mathcal{O})'' \). Then the minimal dilation \((\mathcal{K}, E, V)\) of a CP instrument for \((\mathcal{B}(\mathcal{H}_0), \Gamma)\) extending \( \mathcal{I} \) satisfies the following intertwining relation:

\[
VA = (A \otimes 1)V
\]

for all \( A \in \pi(\mathcal{A}((\mathcal{O}_2')'))'' \).

This shows that, for any representation \( \pi \) of \( \mathcal{A} \) satisfying the DHR selection criterion, every CP instrument for \((\pi(\mathcal{A})(\mathcal{O}))'', \Gamma \) described by a measuring process can be extended to a local CP instrument for \((\pi(\mathcal{A})'', \Gamma, \Lambda) \) for every \( \Lambda \in \mathcal{K}_{DC}^\infty \) such that \( \mathcal{O}_1^\Lambda = \mathcal{O} \), and that the intertwining relation (2) appears as a mathematical expression of a causal structure of space-time. We are now ready to use both local states discussed in [34] and local CP instruments in order to describe quantum phenomena occurring in local space-time regions.

2 Preliminaries

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be \( C^* \)-algebras and \( \mathcal{H} \) be a Hilbert space. For any category \( E \), we denote by \( \text{Ob}(E) \) the set of objects of \( E \). We denote by \( \text{Rep}(\mathcal{X}) \) by the category of representations of \( \mathcal{X} \), \( \text{Rep}(\mathcal{X}; \mathcal{H}) \) by the category of representations of \( \mathcal{X} \) on \( \mathcal{H} \) and by \( \text{Hilb} \) the category of Hilbert spaces. We define two norms \( \| \cdot \|_{\text{min}} \) and \( \| \cdot \|_{\text{max}} \) on the algebraic tensor product \( \mathcal{X} \otimes_{\text{alg}} \mathcal{Y} \) of \( \mathcal{X} \) and \( \mathcal{Y} \) by

\[
\|A\|_{\text{min}} = \sup_{(\pi_1, \pi_2) \in \text{Ob}(\text{Rep}(\mathcal{X}) \times \text{Rep}(\mathcal{Y}))} \| \sum_{j=1}^n \pi_1(X_j) \otimes \pi_2(Y_j) \|, \tag{3}
\]

\[
\|A\|_{\text{max}} = \sup_{(\pi_1, \pi_2) \in I_{\text{max}}(\mathcal{X}, \mathcal{Y})} \| \sum_{j=1}^n \pi_1(X_j)\pi_2(Y_j) \|, \tag{4}
\]

respectively, for every \( A = \sum_{j=1}^n X_j \otimes_{\text{alg}} Y_j \in \mathcal{X} \otimes_{\text{alg}} \mathcal{Y} \), where

\[
I_{\text{max}}(\mathcal{X}, \mathcal{Y}) = \bigcup_{\mathcal{H} \in \text{Ob}(\text{Hilb})} I_{\text{max}}(\mathcal{X}, \mathcal{Y}; \mathcal{H}), \tag{5}
\]

\[
I_{\text{max}}(\mathcal{X}, \mathcal{Y}; \mathcal{H}) = \{ (\pi_1, \pi_2) \in \text{Ob}(\text{Rep}(\mathcal{X}; \mathcal{H}) \times \text{Rep}(\mathcal{Y}; \mathcal{H})) \mid [\pi_1(X), \pi_2(Y)] = 0, X \in \mathcal{X}, Y \in \mathcal{Y} \}. \tag{6}
\]

We call the completion \( \mathcal{X} \otimes_{\text{min}} \mathcal{Y} \) (\( \mathcal{X} \otimes_{\text{max}} \mathcal{Y} \), resp.) of \( \mathcal{X} \otimes_{\text{alg}} \mathcal{Y} \) with respect to the norm \( \| \cdot \|_{\text{min}} \) (\( \| \cdot \|_{\text{max}}, \) resp.) the minimal (maximal, resp.) tensor product of \( \mathcal{X} \) and \( \mathcal{Y} \). The maximal tensor product \( \mathcal{X} \otimes_{\text{max}} \mathcal{Y} \) has the following property:
\textbf{Proposition 2.1} \cite{Chapter IV, Proposition 4.7}. Let $\mathcal{X}$ and $\mathcal{Y}$ be $C^*$-algebras, and $\mathcal{H}$ be a Hilbert space. For every $(\pi_1, \pi_2) \in \mathcal{I}_{\text{max}}(\mathcal{X}, \mathcal{Y}; \mathcal{H})$, there exists a representation $\pi$ of $\mathcal{X} \otimes_{\text{max}} \mathcal{Y}$ on $\mathcal{H}$ such that

$$\pi(X \otimes Y) = \pi_1(X)\pi_2(Y)$$

for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{Y}$ be a $C^*$-algebra. We denote by $\text{Rep}_n(\mathcal{M}; \mathcal{H})$ the category of normal representations of $\mathcal{M}$ on $\mathcal{H}$. We call the completion $\mathcal{M} \otimes_{\text{nor}} \mathcal{Y}$ of $\mathcal{M} \otimes_{\text{alg}} \mathcal{Y}$ with respect to the norm $\| \cdot \|_{\text{nor}}$ defined below the normal tensor product of $\mathcal{M}$ and $\mathcal{Y}$:

$$\|X\|_{\text{nor}} = \sup_{(\pi_1, \pi_2) \in \mathcal{I}_{\text{nor}}(\mathcal{M}, \mathcal{Y})} \left\| \sum_{j=1}^{n} \pi_1(M_j)\pi_2(Y_j) \right\|$$

for every $X = \sum_{j=1}^{n} M_j \otimes_{\text{alg}} Y_j \in \mathcal{M} \otimes_{\text{alg}} \mathcal{Y}$, where

$$\mathcal{I}_{\text{nor}}(\mathcal{M}, \mathcal{Y}) = \bigcup_{\mathcal{H} \in \text{Ob}(\mathcal{H})} \mathcal{I}_{\text{nor}}(\mathcal{M}, \mathcal{Y}; \mathcal{H}),$$

where

$$\mathcal{I}_{\text{nor}}(\mathcal{M}, \mathcal{Y}; \mathcal{H}) = \{ (\pi_1, \pi_2) \in \text{Ob}(\text{Rep}_n(\mathcal{M}; \mathcal{H}) \times \text{Rep}(\mathcal{Y}; \mathcal{H})) \mid [\pi_1(M), \pi_2(Y)] = 0, M \in \mathcal{M}, Y \in \mathcal{Y} \}.\ (10)$$

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. We call the completion $\mathcal{M} \otimes_{\text{bin}} \mathcal{N}$ of $\mathcal{M} \otimes_{\text{alg}} \mathcal{N}$ with respect to the norm $\| \cdot \|_{\text{bin}}$ defined below the binormal tensor product of $\mathcal{M}$ and $\mathcal{N}$:

$$\|X\|_{\text{bin}} = \sup_{(\pi_1, \pi_2) \in \mathcal{I}_{\text{bin}}(\mathcal{M}, \mathcal{N})} \left\| \sum_{j=1}^{n} \pi_1(M_j)\pi_2(N_j) \right\|$$

for every $X = \sum_{j=1}^{n} M_j \otimes_{\text{alg}} N_j \in \mathcal{M} \otimes_{\text{alg}} \mathcal{N}$, where

$$\mathcal{I}_{\text{bin}}(\mathcal{M}, \mathcal{N}) = \bigcup_{\mathcal{H} \in \text{Ob}(\mathcal{H})} \mathcal{I}_{\text{bin}}(\mathcal{M}, \mathcal{N}; \mathcal{H}),$$

where

$$\mathcal{I}_{\text{bin}}(\mathcal{M}, \mathcal{N}; \mathcal{H}) = \{ (\pi_1, \pi_2) \in \text{Ob}(\text{Rep}_n(\mathcal{M}; \mathcal{H}) \times \text{Rep}_n(\mathcal{N}; \mathcal{H})) \mid [\pi_1(M), \pi_2(N)] = 0, M \in \mathcal{M}, N \in \mathcal{N} \}.\ (13)$$

A $C^*$-algebra $\mathcal{X}$ is nuclear if, for every $C^*$-algebra $\mathcal{Y}$,

$$\mathcal{X} \otimes_{\text{min}} \mathcal{Y} = \mathcal{X} \otimes_{\text{max}} \mathcal{Y}.\ (14)$$

It is known that $C^*$-tensor products with nuclear $C^*$-algebras are unique. A $C^*$-algebra $\mathcal{X}$ on a Hilbert space $\mathcal{H}$ is injective if there exists a norm one projection from $\mathcal{B}(\mathcal{H})$ into $\mathcal{X}$. It is proven in \cite{22} that a von Neumann algebras $\mathcal{M}$ is injective if and only if, for every $C^*$-algebra $\mathcal{Y}$,

$$\mathcal{M} \otimes_{\text{min}} \mathcal{Y} = \mathcal{M} \otimes_{\text{nor}} \mathcal{Y}.\ (15)$$

Abelian $C^*$-algebras are both nuclear and injective. A characterization of von Neumann algebras which are nuclear as $C^*$-algebras is given in \cite{7} Proposition 2.4.9.

\textbf{Theorem 2.1} \cite{43 Theorem 1.3.1, 15 Theorem 12.7}. Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $\mathcal{B}$ be a unital $C^*$-subalgebra of $\mathcal{B}(\mathcal{K})$, and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{K} = \text{span}(\mathcal{B}V\mathcal{H})$. For every $A \in (V^*\mathcal{B}V)'$, there exists a unique $A_1 \in \mathcal{B}'$ such that $VA = A_1V$. Furthermore, the map $\pi' : A \in (V^*\mathcal{B}V)' \ni A \mapsto A_1 \in \mathcal{B}' \cap \{VV^*\}'$ is a $^*$-homomorphism.
Theorem 2.2 (Arveson [41 Theorem 1.2.3], [45 Theorem 7.5]). Let $\mathcal{X}$ and $\mathcal{Y}$ be $C^*$-algebras such that $\mathcal{X} \subset \mathcal{Y}$, and $\mathcal{H}$ be a Hilbert space. For every $T \in \text{CP}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$, there exists $\tilde{T} \in \text{CP}(\mathcal{Y}, \mathcal{B}(\mathcal{H}))$ such that $\tilde{T}(X) = T(X)$, $X \in \mathcal{X}$.

This theorem is known as Arveson extension theorem.

Definition 2.1 (CP-measure space [34, Definition 5.1]). A triplet $(S, \mathcal{F}, \mu)$ is called a CP-measure space if it satisfies the following conditions:
(1) $(S, \mathcal{F})$ is a measurable space;
(2) $\mu$ is a CP($\mathcal{X}, \mathcal{M}$)-valued measure on $(S, \mathcal{F})$ satisfying
$$\rho(\mu(\cup_i \Delta_i, X)) = \sum_i \rho(\mu(\Delta_i, X)), \quad (16)$$
for all mutually disjoint subset $\{\Delta_i\}_{i \in \mathbb{N}}$ of $\mathcal{F}$ and $\rho \in \mathcal{M}_*$, and $\mu(S, \cdot) \in \text{CP}(\mathcal{X}, \mathcal{M})$.

A CP-measure space $(S, \mathcal{F}, \mu)$ is called a CP-measure space with barycenter $T \in \text{CP}(\mathcal{X}, \mathcal{M})$ or a CP-measure space of $T$ if $T = \mu(S, \cdot)$.

For a normal positive linear functional $\rho$ on $\mathcal{M}$, a positive measure $\rho \circ \mu$ on $S$ is defined by $(\rho \circ \mu)(\Delta) = \rho(\mu(\Delta, 1))$ for all $\Delta \in \mathcal{F}$. For every measure space $(S, \mathcal{F}, \nu)$ and $1 \leq p \leq \infty$, we write $L^p(S, \nu)$ or $L^p(\nu)$ as $L^p(S, \mathcal{F}, \nu)$ for short in the paper.

Lemma 2.1 ([34, Lemma 5.3]). If $(S, \mathcal{F}, \mu)$ is a CP-measure space of $T \in \text{CP}(\mathcal{X}, \mathcal{M})$, then there exists a unique linear map $V^\infty(S, \nu) \ni f \mapsto \kappa_\mu(f) \in \pi_T(\mathcal{X})'$ defined by
$$V_T^\mu \kappa_\mu(f) \pi_T(X)V_T = \int f(s) \, d\mu(s, X), \quad f \in L^\infty(S, \nu), \quad X \in \mathcal{X}, \quad (17)$$
i.e., for every $\rho \in \mathcal{M}_*$,
$$\rho(V_T^\mu \kappa_\mu(f) \pi_T(X)V_T) = \int f(s) \, d(\rho(\mu(s, X))), \quad f \in L^\infty(S, \nu), \quad X \in \mathcal{X}, \quad (18)$$
which is positive and contractive, where $\nu$ is a (scalar-valued) positive measure which is equivalent to $\varphi \circ \mu$ for some normal faithful state $\varphi$ on $\mathcal{M}$. Furthermore, if $f \in L^\infty(\nu)_+$ satisfies $\kappa_\mu(f) = 0$, then $f = 0$.
If $L^\infty(S, \nu)$ is equipped with the $\sigma(L^\infty(S, \nu), L^1(S, \nu))$-topology and $\pi_T(\mathcal{X})'$ with the weak topology, then the map $\kappa_\mu$ is continuous.

Theorem 2.3 ([46, Part I, Chapter 4, Theorem 3], [51, Chapter IV, Theorem 5.5]). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If $\pi$ is a normal homomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$, there exist a Hilbert space $\mathcal{K}$, a projection $E$ of $\mathcal{M}_1' \otimes \mathcal{B}(\mathcal{K})$, and an isometry $U$ of $E(\mathcal{H}_1 \otimes \mathcal{K})$ onto $\mathcal{K}_2$ such that
$$\pi(M) = U j_E(M \otimes 1_\mathcal{K}) U^*, \quad M \in \mathcal{M}_1, \quad (19)$$
where $j_E$ is a CP map of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K})$ onto $E \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K})E$ defined by $j_E(X) = EXE$, $X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K})$.

We use the following form of Theorem 2.3

Corollary 2.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. If $\pi$ is a normal representation of $\mathcal{B}(\mathcal{H}_1)$ on $\mathcal{H}_2$, there exist a Hilbert space $\mathcal{K}$ and a unitary operator $U$ of $\mathcal{H}_1 \otimes \mathcal{K}$ onto $\mathcal{K}_2$ such that
$$\pi(X) = U(X \otimes 1_\mathcal{K}) U^*, \quad X \in \mathcal{B}(\mathcal{H}_1). \quad (20)$$
3 CP Instrument and Quantum Measuring Process

See [12, 43] for details of quantum measurement theory around CP instruments and measuring processes. Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, F)\) be a measurable space. In the paper, we assume that von Neumann algebras are \( \sigma \)-finite. We denote by \( P_n(\mathcal{M}) \) the set of normal positive linear maps from \( \mathcal{M} \) into itself.

**Definition 3.1.** An instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is a \( P_n(\mathcal{M}) \)-valued measure on \((S, F)\) satisfying the following three conditions:

1. \( \mathcal{I}(S)1 = 1 \);
2. For each countable mutually disjoint sequence \( \{ \Delta_j \} \subset F \),
   \[
   \rho(\mathcal{I}(\bigcup_j \Delta_j)M) = \sum_j \rho(\mathcal{I}(\Delta_j)M) \tag{21}
   \]
   for all \( \rho \in \mathcal{M}_* \), \( M \in \mathcal{M} \).

We also use the notation \( \mathcal{I}(\cdot, \cdot) \) for an instrument \( \mathcal{I} \) in such a way \( \mathcal{I}(\Delta, M) = \mathcal{I}(\Delta)M \) for all \( \Delta \in F \) and \( M \in \mathcal{M} \). An instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is called a CP instrument for \((\mathcal{M}, S)\) if, for every \( \Delta \in F \), \( \mathcal{I}(\Delta) \) is completely positive. We denote by \( \text{CPInst}(\mathcal{M}, S) \) the set of CP instruments for \((\mathcal{M}, S)\).

**Lemma 3.1** (Ozawa [36, Proposition 4.2]). For a CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\), there are a Hilbert space \( \mathcal{K} \), a spectral measure \( E : F \rightarrow \mathcal{B} (\mathcal{K}) \), a nondegenerate normal representation \( \pi : \mathcal{M} \rightarrow \mathcal{B} (\mathcal{K}) \) and an isometry \( V \in \mathcal{B} (\mathcal{H}, \mathcal{K}) \) satisfying

\[
\mathcal{I}(\Delta)M = V^* E(\Delta) \pi(M)V, \tag{22}
\]
\[
E(\Delta) \pi(M) = \pi(M) E(\Delta) \tag{23}
\]
for any \( \Delta \in F \) and \( M \in \mathcal{M} \).

**Proof.** Let \( \nu \) be a finite positive measure on \( S \) such that \( \nu \sim \rho \circ \mathcal{I} \) for some normal faithful state \( \rho \) on \( \mathcal{M} \). By Lemma 2.1 there exists a POVM \( \kappa_\mathcal{I} \) in \( \pi_T(\mathcal{M})t \) such that

\[
\mathcal{I}(f, M) = V^*_T \kappa_\mathcal{I}(f) \pi_T(M)V_T \tag{24}
\]
for all \( M \in \mathcal{M} \) and \( f \in L^\infty(S, \nu) \). Let \((E, \mathcal{K}, W)\) be the minimal Stinespring representation of \( \kappa_\mathcal{I} \). By Theorem 2.1 there exists a nondegenerate normal representation \( \pi \) of \( \mathcal{M} \) on \( \mathcal{K} \) such that

\[
\pi(M)W = W \pi_T(M), \tag{25}
\]
\[
E(f) \pi(M) = \pi(M) E(f)
\]
for all \( M \in \mathcal{M} \) and \( f \in L^\infty(S, \nu) \). We denote \( WV_T \) by \( V \), which is seen to be an isometry.

\[
\mathcal{I}(f, M) = V^*_T \kappa_\mathcal{I}(f) \pi_T(M)V_T = V^*_T W^* E(f)W \pi_T(M)V_T = (WV_T)^* E(f) \pi(M)WV_T = V^* E(f) \pi(M)V. \tag{26}
\]

\( \square \)

Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras. For every \( \sigma \in \mathcal{N}_* \), the map \( id \otimes \sigma : \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \) is defined by \( (\rho \otimes \sigma)(X) = \rho((id \otimes \sigma)(X)) \) for all \( X \in \mathcal{M} \otimes \mathcal{N} \) and \( \rho \in \mathcal{M}_* \).
Definition 3.2 (Measuring process). A measuring process $\mathbb{M}$ for $(\mathcal{M}, S)$ is a 4-tuple $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ consisting of a Hilbert space $\mathcal{K}$, a state $\sigma$ on $B(\mathcal{K})$, a spectral measure $E : F \to B(\mathcal{K})$, and a unitary operator $U$ on $\mathcal{H} \otimes \mathcal{K}$ satisfying $\{\mathcal{I}_M(\Delta)M \mid M \in \mathcal{M}, \Delta \in F\} \subset \mathcal{M}$, where $\mathcal{I}_M$ is a CP instrument for $(B(\mathcal{H}), S)$ defined by $\mathcal{I}_M(\Delta)X = (id \otimes \sigma)[U^*(X \otimes E(\Delta))U]$ for all $X \in B(\mathcal{H})$ and $\Delta \in F$.

Definition 3.3 (Faithfulness of measuring process). A measuring process $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ for $(\mathcal{M}, S)$ is faithful if there exists a normal faithful representation $\tilde{E} : L^\infty(S, \rho \circ \mathcal{I}_M) \to B(\mathcal{K})$ such that $E(\Delta) = \tilde{E}(\chi_\Delta)$ for all $\Delta \in F$.

For a faithful measuring process $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ for $(\mathcal{M}, S)$, a unital normal CP map $\Psi_\mathbb{M} : \mathcal{M} \otimes L^\infty(S, \rho \circ \mathcal{I}_M) \to \mathcal{M}$ is defined by

$$
\Psi_\mathbb{M}(X) = (id \otimes \sigma)[U^*(((id_\mathcal{M} \otimes \tilde{E}))(X))U]
$$

for all $X \in \mathcal{M} \otimes L^\infty(S, \rho \circ \mathcal{I}_M)$, where $\tilde{E}$ is a normal faithful representation of $L^\infty(S, \rho \circ \mathcal{I}_M)$ on $\mathcal{K}$ such that $E(\Delta) = \tilde{E}(\chi_\Delta)$ for all $\Delta \in F$.

Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ and $\nu$ a finite positive measure on $S$ such that $\nu \sim \rho \circ \mathcal{I}$ for some normal faithful state $\rho$ on $\mathcal{M}$. By Lemma 3.1, the nuclearity of $L^\infty(S, \nu)$ and Proposition 2.1, there exists a binormal representation $\tilde{\pi} : \mathcal{M} \otimes_{bin} L^\infty(S, \nu) \to B(\mathcal{K})$ such that $\tilde{\pi}(M \otimes f) = \pi(M)E(f)$, for every $M \in \mathcal{M}$, $f \in L^\infty(S, \nu)$. We can define a CP map $\Psi_\mathcal{I} : \mathcal{M} \otimes_{bin} L^\infty(S, \nu) \to B(\mathcal{H})$ by

$$
\Psi_\mathcal{I}(X) = V^*\tilde{\pi}(X)V, \quad X \in \mathcal{M} \otimes_{bin} L^\infty(S, \nu).
$$

Since $L^\infty(S, \nu)$ is a nuclear $C^*$-algebra, it holds that

$$
\mathcal{M} \otimes_{bin} L^\infty(S, \nu) = \mathcal{M} \otimes_{min} L^\infty(S, \nu) = \mathcal{M} \otimes_{max} L^\infty(S, \nu),
$$

and $\mathcal{M} \otimes_{min} L^\infty(S, \nu)$ is a dense $C^*$-subalgebra of a von Neumann algebra $\mathcal{M} \otimes L^\infty(S, \nu)$. By Theorem 2.2, there then exists a CP map $\Psi_\mathcal{I} : \mathcal{M} \otimes L^\infty(S, \nu) \to B(\mathcal{H})$ such that $\Psi_\mathcal{I}|_{\mathcal{M} \otimes_{bin} L^\infty(S, \nu)} = \Psi_\mathcal{I}$, which is usually nonnormal.

Definition 3.4 (Normal extension property). Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ and $\nu$ a finite positive measure on $S$ such that $\nu \sim \rho \circ \mathcal{I}$ for some normal faithful state $\rho$ on $\mathcal{M}$.

1. $\mathcal{I}$ has normal extension property (NEP) if there exists a unital normal CP map $\widetilde{\Psi}_\mathcal{I} : \mathcal{M} \otimes L^\infty(S, \nu) \to B(\mathcal{H})$ such that $\widetilde{\Psi}_\mathcal{I}|_{\mathcal{M} \otimes_{bin} L^\infty(S, \nu)} = \Psi_\mathcal{I}$.

2. $\mathcal{I}$ has unique normal extension property (UNEP) if there exists a unique unital normal CP map $\Psi_\mathcal{I} : \mathcal{M} \otimes L^\infty(S, \nu) \to B(\mathcal{H})$ such that $\Psi_\mathcal{I}|_{\mathcal{M} \otimes_{bin} L^\infty(S, \nu)} = \Psi_\mathcal{I}$.

We denote by CPInst$_{NE}(\mathcal{M}, S)$ the set of CP instruments for $(\mathcal{M}, S)$ with NEP.

We gave the name “normal extension property” in the light of unique extension property [5] used in operator system theory.

Lemma 3.2. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, $\mathcal{M}_0$ be a dense $^*$-subalgebra of $\mathcal{M}$ such that $\overline{\mathcal{M}_0}^{\omega} = \mathcal{M}$, and $T$ be a normal CP map from $\mathcal{M}$ into $B(\mathcal{K})$ such that $T(\mathcal{M}_0) \subset \mathcal{N}$. Then $T(\mathcal{M}) \subset \mathcal{N}$.

Proof. For any $M \in \mathcal{M}$, there exists a net $\{M_\alpha\}$ of $\mathcal{M}_0$ converging to $M$ in the ultraweak topology. Then $\{T(M_\alpha)\}$ of $\mathcal{N}$ is a net in $\mathcal{N}$ converging to $T(M) \in B(\mathcal{K})$ in the ultraweak topology. By von Neumann density theorem, $T(M) \in \mathcal{N}$. This proves the lemma. □
Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ with NEP and $\tilde{\Psi}_I : \mathcal{M} \otimes L^\infty(S, \nu) \to \mathcal{B}(\mathcal{H})$ be a unital normal CP map such that $\tilde{\Psi}_I|_{\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \nu)} = \Psi_I$. Since $\mathcal{M} \otimes L^\infty(S, \nu) = \mathcal{M} \otimes_{\text{alg}} L^\infty(S, \nu)$ and $\tilde{\Psi}_I|_{\mathcal{M} \otimes_{\text{alg}} L^\infty(S, \nu)} = \Psi_I$, it follows that $\tilde{\Psi}_I(\mathcal{M} \otimes L^\infty(S, \nu)) \subset \mathcal{M}$ from Lemma 3.2.

**Theorem 3.1.** For a CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, the following conditions are equivalent:

(i) $\mathcal{I}$ has NEP.

(ii) $\mathcal{I}$ has UNEP.

(iii) There exists a CP instrument $\tilde{\mathcal{I}}$ for $(\mathcal{B}(\mathcal{H}), S)$ such that $\tilde{\mathcal{I}}(\Delta)M = \mathcal{I}(\Delta)M$ for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$.

(iv) There exists a faithful measuring process $M = (\mathcal{K}, \sigma, E, U)$ for $(\mathcal{M}, S)$ such that

$$\mathcal{I}(\Delta)M = (\text{id} \otimes \sigma)[U^*(M \otimes E(\Delta))U]$$

(30)

for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$.

**Proof.** Let $\nu$ be a finite positive measure on $S$ such that $\nu \sim \rho \circ \mathcal{I}$ for some normal faithful state $\rho$ on $\mathcal{M}$.

(ii) $\Rightarrow$ (i) Trivial.

(i) $\Rightarrow$ (ii) Let $T_1, T_2 : \mathcal{M} \otimes L^\infty(S, \nu) \to \mathcal{B}(\mathcal{H})$ be normal CP maps such that

$$T_j|_{\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \nu)} = \Psi_I$$

(31)

for $j = 1, 2$. By assumption, for every $M \in \mathcal{M} \otimes_{\text{alg}} L^\infty(S, \nu)$, we have $T_1(M) = T_2(M)$. Since $\mathcal{M} \otimes_{\text{alg}} L^\infty(S, \nu)$ is dense in $\mathcal{M} \otimes L^\infty(S, \nu)$, and $T_1$ and $T_2$ are normal on $\mathcal{M} \otimes L^\infty(S, \nu)$, it is seen that $T_1$ is equal to $T_2$ on $\mathcal{M} \otimes L^\infty(S, \nu)$.

(iv) $\Rightarrow$ (i) Let $M = (\mathcal{K}, \sigma, E, U)$ be a faithful measuring process for $(\mathcal{M}, S)$ such that

$$\mathcal{I}(\Delta)M = (\text{id} \otimes \sigma)[U^*(M \otimes E(\Delta))U] = \Psi_M(M \otimes \chi_\Delta),$$

(32)

for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. Then $\Psi_M$ satisfies $\Psi_M|_{\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \nu)} = \Psi_I$.

(i) $\Rightarrow$ (iii) By assumption, there exists a unital normal CP map $\tilde{\Psi}_I : \mathcal{M} \otimes L^\infty(S, \nu) \to \mathcal{M}$ such that $\tilde{\Psi}_I|_{\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \nu)} = \Psi_I$. There then exists a minimal Stinespring representation $(\pi, \mathcal{L}_1, W_1)$ of $\tilde{\Psi}_I$, i.e.,

$$\Psi_I(X) = W_1^*\pi(X)W_1, \quad X \in \mathcal{M} \otimes L^\infty(S, \nu).$$

(33)

Furthermore, by Theorem 2.3, there exist a Hilbert space $\mathcal{L}_2$, a projection $E$ of $(\mathcal{M} \otimes L^\infty(S, \nu))^\prime \otimes \mathcal{B}(\mathcal{L}_2)$ and an isometry $W_2 : E(\mathcal{H} \otimes L^2(S, \nu) \otimes \mathcal{L}_2) \to \mathcal{L}_1$ such that

$$\pi(X) = W_2j_{E}(X \otimes 1_{\mathcal{L}_2})W_2^*, \quad X \in \mathcal{M} \otimes L^\infty(S, \nu),$$

(34)

where a normal CP map $j_E : \mathcal{B}(\mathcal{H} \otimes L^2(S, \nu) \otimes \mathcal{L}_2) \to E(\mathcal{B}(\mathcal{H} \otimes L^2(S, \nu) \otimes \mathcal{L}_2))E$ defined by $j_E(X) = EXE, \ X \in \mathcal{B}(\mathcal{H} \otimes L^2(S, \nu) \otimes \mathcal{L}_2)$. We then define a normal CP map $\tilde{\pi} : \mathcal{B}(\mathcal{H}) \otimes L^\infty(S, \nu) \to \mathcal{B}(\mathcal{L}_1)$ by

$$\tilde{\pi}(X) = W_2j_{E}(X \otimes 1_{\mathcal{L}_2})W_2^*, \quad X \in \mathcal{B}(\mathcal{H}) \otimes L^\infty(S, \nu).$$

(35)

A CP instrument $\tilde{\mathcal{I}}$ for $(\mathcal{B}(\mathcal{H}), S)$ is defined by

$$\tilde{\mathcal{I}}(\Delta)X = W_1^*\tilde{\pi}(X \otimes \chi_\Delta)W_1$$

(36)
for every $X \in \mathcal{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$. For every $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$, it is seen that

$$\tilde{\mathcal{I}}(\Delta)M = W_1^*\tilde{\pi}(M \otimes \chi_\Delta)W_1$$

$$= W_1^*\pi(M \otimes \chi_\Delta)W_1 = \tilde{\Psi}_\mathcal{T}(M \otimes \chi_\Delta) = \mathcal{I}(\Delta)M$$

for every $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$. 

(iii) $\Rightarrow$ (iv) Let $\tilde{\mathcal{I}}$ be a CP instrument for $(\mathcal{B}(\mathcal{H}), \mathcal{S})$ such that $\tilde{\mathcal{I}}(\Delta)M = \mathcal{I}(\Delta)M$ for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. We denote $\tilde{\mathcal{I}}(S)$ by $\tilde{\mathcal{I}}$. By Corollary 2.1, a normal representation of $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to the representation $id \otimes 1_\mathcal{L}$, where $\mathcal{L}$ is a Hilbert space. Therefore, there exist a Hilbert space $\mathcal{L}_1$ and a unitary operator $W_1 : \mathcal{K}_\mathcal{T} \to \mathcal{H} \otimes \mathcal{L}_1$ such that

$$\tilde{\mathcal{I}}(X) = V_\mathcal{T}^*W_1^*(X \otimes 1)W_1V_\mathcal{T}$$

for all $X \in \mathcal{B}(\mathcal{H})$. By Lemma 2.1, there exists a positive contractive $\kappa_\mathcal{T} : L^\infty(S, \nu) \to \mathcal{B}(\mathcal{L}_1)$ such that

$$\tilde{\mathcal{I}}(f, X) = V_\mathcal{T}^*W_1^*(X \otimes \kappa_\mathcal{T}(f))W_1V_\mathcal{T}$$

for all $X \in \mathcal{B}(\mathcal{H})$ and $f \in L^\infty(S, \nu)$, and that, if $f \in L^\infty(S, \nu)_+$ satisfies $\kappa_\mu(f) = 0$ then $f = 0$. Let $(E_0, \mathcal{L}_2, W_2)$ be the minimal Stinespring representation of $\kappa_\mathcal{T}$. Then $E_0$ is a normal faithful representation of $L^\infty(S, \nu)$ on $\mathcal{L}_2$. Denote $W_2W_1V_\mathcal{T}$ by $V$. It holds that

$$\tilde{\mathcal{I}}(f, X) = V^*(X \otimes E_0(f))V$$

for all $X \in \mathcal{B}(\mathcal{H})$ and $f \in L^\infty(S, \nu)$.

Let $\mathcal{L}_3$ be an infinite-dimensional Hilbert space, $\eta_3$ be a unit vector of $\mathcal{L}_3$ and $\eta_2$ be a unit vector of $\mathcal{L}_2$. We define an isometry $U_0$ from $\mathcal{H} \otimes \mathbb{C}\eta_2 \otimes \mathbb{C}\eta_3$ to $\mathcal{H} \otimes \mathcal{L}_2 \otimes \mathcal{L}_3$ by

$$U_0(\xi \otimes \eta_2 \otimes \eta_3) = V\xi \otimes \eta_3,$$

for all $\xi \in \mathcal{H}$. Since it holds that

$$\dim(\mathcal{H} \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 - \mathcal{H} \otimes \mathbb{C}\eta_2 \otimes \mathbb{C}\eta_3) = \dim(\mathcal{H} \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 - U_0(\mathcal{H} \otimes \mathbb{C}\eta_2 \otimes \mathbb{C}\eta_3)),$$

there is a unitary operator $U$ on $\mathcal{H} \otimes \mathcal{L}_2 \otimes \mathcal{L}_3$, which is an extension of $U_0$. We then define a Hilbert space $\mathcal{K}$ by $\mathcal{K} = \mathcal{L}_2 \otimes \mathcal{L}_3$, a state $\sigma$ on $\mathcal{B}(\mathcal{K})$ by

$$\sigma(Y) = \langle \eta_2 \otimes \eta_3| Y(\eta_2 \otimes \eta_3) \rangle$$

for all $Y \in \mathcal{B}(\mathcal{K})$, and a spectral measure $E : \mathcal{F} \to \mathcal{B}(\mathcal{K})$ by

$$E(\Delta) = E_0(\chi_\Delta) \otimes 1_{\mathcal{L}_3}$$

for all $\Delta \in \mathcal{F}$. It is checked that

$$\langle \xi|\tilde{\mathcal{I}}(\Delta)X\xi \rangle = \langle \xi|V^*(X \otimes E_0(\chi_\Delta))V\xi \rangle$$

$$= \langle V\xi \otimes \eta_3|(X \otimes E_0(\chi_\Delta))(V\xi \otimes \eta_3) \rangle$$

$$= \langle U(\xi \otimes \eta_2 \otimes \eta_3)|(X \otimes E_0(\chi_\Delta) \otimes 1_{\mathcal{L}_3})U(\xi \otimes \eta_2 \otimes \eta_3) \rangle$$

$$= \langle \xi \otimes \eta_2 \otimes \eta_3|U^*(X \otimes E(\Delta))U(\xi \otimes \eta_2 \otimes \eta_3) \rangle$$

$$= \langle \xi|\{(id \otimes \sigma)[U^*(X \otimes E(\Delta))U]\} \xi \rangle$$

(45)
for all \( X \in \mathcal{B}(\mathcal{H}) \) and \( \Delta \in \mathcal{F} \), and, by the definition of \( E \), there exists a normal faithful representation \( \tilde{E} \) of \( L^\infty(S, \nu) \) on \( \mathcal{K} \) such that \( \tilde{E}(\chi_\Delta) = E(\Delta) \) for all \( \Delta \in \mathcal{F} \). Thus there exists a faithful measuring process \( \tilde{M} = (\mathcal{K}, \sigma, E, U) \) for \((\mathcal{M}, S)\) such that

\[
\tilde{I}(\Delta)X = (id \otimes \sigma)[U^*(X \otimes E(\Delta))U] \tag{46}
\]

for all \( X \in \mathcal{B}(\mathcal{H}) \) and \( \Delta \in \mathcal{F} \). Therefore, for every \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \), it holds that

\[
I(\Delta)M = (id \otimes \sigma)[U^*(M \otimes E(\Delta))U]. \tag{47}
\]

Let \( \mathcal{M} \) be a von Neumann algebra on a separable Hilbert space \( \mathcal{H} \) and \((\Gamma, \mathcal{B}(\Gamma))\) be a standard Borel space, i.e., a Borel space associated to a Polish space, For a CP instrument \( I \) for \((\mathcal{M}, \Gamma)\), there exists a measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for \((\mathcal{M}, \Gamma)\) such that \( \mathcal{K} \) is separable [36, Corollary 5.3].

**Corollary 3.1.** For an instrument \( I \) for \((\mathcal{M}, S)\), the following conditions are equivalent:
(i) For every \( \Delta \in \mathcal{F} \), \( I(\Delta) \) is completely positive and \( I \) has NEP;
(ii) There exists a measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for \((\mathcal{M}, S)\) such that

\[
I(\Delta)M = (id \otimes \sigma)[U^*(M \otimes E(\Delta))U] \tag{48}
\]

for all \( \Delta \in \mathcal{F} \) and \( M \in \mathcal{M} \).

**Definition 3.5** (Statistical equivalence class of measuring processes [36]). Two measuring processes \( \mathcal{M}_1 = (\mathcal{K}_1, \sigma_1, E_1, U_1) \) and \( \mathcal{M}_2 = (\mathcal{K}_2, \sigma_2, E_2, U_2) \) for \((\mathcal{M}, S)\) are said to be statistically equivalent if, for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \),

\[
(id \otimes \sigma_1)[U_1^*(M \otimes E_1(\Delta))U_1] = (id \otimes \sigma_2)[U_2^*(M \otimes E_2(\Delta))U_2]. \tag{49}
\]

We summarize the result of this section as follows (compare with [36, Theorem 5.1]):

**Theorem 3.2.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, \mathcal{F})\) be a measurable space. Then there is a one-to-one correspondence between statistical equivalence classes of measuring processes \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for \((\mathcal{M}, S)\) and CP instruments \( I \) for \((\mathcal{M}, S)\) with NEP, which is given by the relation

\[
I(\Delta)M = (id \otimes \sigma)[U^*(M \otimes E(\Delta))U] \tag{50}
\]

for all \( \Delta \in \mathcal{F} \) and \( M \in \mathcal{M} \).

## 4 Approximations by CP Instruments with NEP

**Proposition 4.1.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) such that there exists a normal conditional expectation \( \mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M} \), and \((S, \mathcal{F})\) be a measurable space. Every CP instrument \( I \) for \((\mathcal{M}, S)\) has NEP.

**Proof.** Let \( \mathcal{I} \) be a CP instrument for \((\mathcal{M}, S)\). We define a CP instrument \( \tilde{\mathcal{I}} \) for \((\mathcal{B}(\mathcal{H}), S)\) by

\[
\tilde{\mathcal{I}}(\Delta)X = \mathcal{I}(\Delta, \mathcal{E}(X)) \tag{51}
\]

for \( \Delta \in \mathcal{F} \) and \( X \in \mathcal{B}(\mathcal{H}) \). For every \( \Delta \in \mathcal{F} \) and \( M \in \mathcal{M} \),

\[
\tilde{\mathcal{I}}(\Delta)M = \mathcal{I}(\Delta, \mathcal{E}(M)) = \mathcal{I}(\Delta, M) = \mathcal{I}(\Delta)M. \tag{52}
\]

By Theorem 3.2 \( \mathcal{I} \) has NEP. \( \square \)
Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ such that there exists a normal conditional expectation $\mathcal{E}: B(\mathcal{H}) \to \mathcal{M}$, and $(S, \mathcal{F})$ be a measurable space. Every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, we call the CP instrument $\tilde{\mathcal{I}}$ for $(B(\mathcal{H}), S)$ defined by Eq. (51) the $\mathcal{E}$-canonical extension of $\mathcal{I}$.

Denote by $M_\alpha \to^{uw} M$ when a net $\{M_\alpha\}_\alpha$ of $\mathcal{M}$ ultraweakly converges to an element $M$ of $\mathcal{M}$.

**Proposition 4.2** (Anantharaman-Delaroche [1]). Let $\mathcal{M} \subset \mathcal{N}$ be a pair of von Neumann algebras and assume that there exists a normal faithful semifinite weight $\varphi$ on $\mathcal{M}$ such that, for all $t \in \mathbb{R}$, the modular automorphism $\sigma^\varphi_t$ is induced by a unitary operator of $\mathcal{N}$. Then the following conditions are equivalent:

1. There exists a norm one projection from $\mathcal{N}$ onto $\mathcal{M}$.
2. There exists a net $\{T_\alpha\}$ of normal CP maps from $\mathcal{N}$ to $\mathcal{M}$, such that $T_\alpha(1) \leq 1$ for all $\alpha$, and $T_\alpha(M) \to^{uw} M$ for all $M \in \mathcal{M}$.
3. There exists a net $\{T_\alpha\}$ of unital normal CP maps from $\mathcal{N}$ to $\mathcal{M}$, such that $T_\alpha(M) \to^{uw} M$ for all $M \in \mathcal{M}$.

**Proof.** The proof of the equivalence between (1) and (2) is given in [1] Corollary 3.9.

(2) $\Rightarrow$ (3) Let $\{T_\alpha\}$ be a net of normal CP maps from $\mathcal{N}$ to $\mathcal{M}$, such that $T_\alpha(1) \leq 1$ for all $\alpha$, and $T_\alpha(M) \to^{uw} M$ for all $M \in \mathcal{M}$. Choose a normal state $\omega$ on $\mathcal{M}$. We define a net $\{T'_\alpha\}$ of unital normal CP maps from $\mathcal{N}$ to $\mathcal{M}$ by

$$T'_\alpha(M) = T_\alpha(M) + (1 - T_\alpha(1))\omega(M), \quad M \in \mathcal{M}.$$  \hspace{1cm} (53)

For every $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $\sum_n \|x_n\|^2 < \infty, \sum_n \|y_n\|^2 < \infty$, and $M \in \mathcal{M}$,

$$\left| \sum_n \langle x_n | (T'_\alpha(M) - M)y_n \rangle \right|$$

$$\leq \sum_n \langle x_n | (T'_\alpha(M) - T_\alpha(M))y_n \rangle + \left| \sum_n \langle x_n | (T_\alpha(M) - M)y_n \rangle \right|$$

$$= \left| \sum_n \langle x_n | (1 - T_\alpha(1))y_n \rangle \cdot |\omega(M)| \right| + \left| \sum_n \langle x_n | (T_\alpha(M) - M)y_n \rangle \right|$$

$$\to 0. \hspace{1cm} (54)$$

This shows that $\{T'_\alpha\}$ has the desired property.

(3) $\Rightarrow$ (2) Trivial. \hspace{10cm} \square

Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ and $\{\mathcal{I}_\alpha\}_\alpha$ be a net of CP instruments for $(\mathcal{M}, S)$. Denote by $\mathcal{I}_\alpha \to^{uw} \mathcal{I}$ when $\mathcal{I}_\alpha(\Delta)M \to^{uw} \mathcal{I}(\Delta)M$ for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$.

**Theorem 4.1.** Let $\mathcal{M}$ be an injective von Neumann algebra on a Hilbert space $\mathcal{H}$, and $(S, \mathcal{F})$ be a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, there exists a net $\{\mathcal{I}_\alpha\}$ of CP instruments with NEP such that $\mathcal{I}_\alpha \to^{uw} \mathcal{I}$.

**Proof.** Suppose that $\mathcal{M}$ is in a standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$. Since $\mathcal{M}$ is injective, there exists a norm one projection $\mathcal{P}$ from $B(\mathcal{H})$ to $\mathcal{M}$. Also, for every normal faithful semifinite weight $\varphi$ on $\mathcal{M}$ and for all $t \in \mathbb{R}$, the modular automorphism $\sigma^\varphi_t$ is induced by a unitary operator of $B(\mathcal{H})$. Thus, by Proposition 4.2 there exists a net $\{T_\alpha\}$ of unital normal CP maps from $B(\mathcal{H})$ to $\mathcal{M}$, such that $T_\alpha(M) \to^{uw} M$ for all $M \in \mathcal{M}$.

Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$. For every $\alpha$, CP instrument $\tilde{\mathcal{I}}_\alpha$ for $(B(\mathcal{H}), S)$ is defined by

$$\tilde{\mathcal{I}}_\alpha(\Delta)X = \mathcal{I}(\Delta, T_\alpha(X)) \hspace{1cm} (55)$$
for $\Delta \in \mathcal{F}$ and $X \in \mathcal{B}(\mathcal{H})$. In addition, we define a net $\{\mathcal{I}_\alpha\}$ of CP instruments for $(\mathcal{M}, S)$ by
\[
\mathcal{I}_\alpha(\Delta) M = \mathcal{I}(\Delta, T_\alpha(M))
\]
for $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. For every $\alpha$, $\mathcal{I}_\alpha$ satisfies that
\[
\tilde{\mathcal{I}}_\alpha(\Delta) M = \mathcal{I}_\alpha(\Delta) M
\]
for $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. By Theorem 3.1, it is seen that $\{\mathcal{I}_\alpha\}$ is a net of CP instruments with NEP.

For every $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $\sum_n \|x_n\|^2 < \infty$, $\sum_n \|y_n\|^2 < \infty$, and for every $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$,
\[
|\sum_n \langle x_n | (\mathcal{I}_\alpha(\Delta) M - \mathcal{I}(\Delta) M) y_n \rangle| = |\omega_{\{x_n\}, \{y_n\}}(\mathcal{I}_\alpha(\Delta) M - \mathcal{I}(\Delta) M)| = |\sum_n \langle z_n | (T_\alpha(M) - M) w_n \rangle| \to 0,
\]
where $\omega_{\{x_n\}, \{y_n\}} = \sum_n \langle x_n | (\cdot) y_n \rangle$ is a normal state on $\mathcal{M}$ and $\{z_n\}_{n \in \mathbb{N}}, \{w_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $\sum_n \|z_n\|^2 < \infty$, $\sum_n \|w_n\|^2 < \infty$, $\mathcal{I}(\Delta) \omega_{\{x_n\}, \{y_n\}} = \sum_n \langle z_n | (\cdot) w_n \rangle$. \qed

**Definition 4.1** (Approximately normal extension property). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and $(S, \mathcal{F})$ be a measurable space. A CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ has approximately normal extension property (ANEP) if there exists a net $\{\mathcal{I}_\alpha\}$ of CP instruments with NEP such that $\mathcal{I}_\alpha \to^\text{uw} \mathcal{I}$. We denote by $\text{CPInst}_{\text{AN}}(\mathcal{M}, S)$ the set of CP instruments for $(\mathcal{M}, S)$ with ANEP.

**Theorem 4.2.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and $(S, \mathcal{F})$ be a measurable space. The following statements holds:
(1) $\text{CPInst}_{\text{NE}}(\mathcal{M}, S) = \text{CPInst}_{\text{AN}}(\mathcal{M}, S) = \text{CPInst}(\mathcal{M}, S)$ if there exists a normal conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$;
(2) $\text{CPInst}_{\text{AN}}(\mathcal{M}, S) = \text{CPInst}(\mathcal{M}, S)$ if $\mathcal{M}$ is injective.

Equivalently, if there exists a CP instrument without ANEP defined on a von Neumann algebra $\mathcal{M}$, then $\mathcal{M}$ is not injective, We believe that the converse is true under some condition.

A von Neumann algebra $\mathcal{M}$ is said to be approximately finite dimensional (AFD) if there exists a increasing net $\{\mathcal{M}_\alpha\}_\alpha$ of finite-dimensional von Neumann subalgebras of $\mathcal{M}$ such that
\[
\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha^{\text{uw}}
\]
It is known that von Neumann algebras describing observable algebras of physical systems are AFD and separable. In addition, a separable von Neumann algebra is injective if and only if it is AFD. This famous result is established by Connes, Wassermann, Haagerup, Popa and other researchers [12, 51]. Therefore, the second statement of Theorem 4.2 always holds for physically realizable von Neumann algebras. We may understand the physical necessity of CP instruments without NEP by the discussion in the next section.
5 Existence of A Family of A Posteriori States and Its Consequences

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space.

Definition 5.1. Let $\mu$ be a positive finite measure on $S$.

1. A family $\{\rho_s^\ast\}_{s \in S}$ of (not necessarily normal) positive linear functionals on $\mathcal{M}$ is said to be $\mathcal{F}$-measurable if, for all $M \in \mathcal{M}$, the function $s \mapsto \rho_s(M)$ is $\mathcal{F}$-measurable.

2. A family $\{\rho_s^\ast\}_{s \in S}$ of (not necessarily normal) positive linear functionals on $\mathcal{M}$ is said to be $\mu$-measurable if, for all $M \in \mathcal{M}$, there exists a $\mathcal{F}$-measurable function $F$ on $S$ such that $\rho_s(M) = F(s)$, $\mu$-a.e..

Definition 5.2 (Disintegration and family of a posteriori states). Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, \Gamma)$.

1. A family $\{\rho_s^\ast\}_{s \in S}$ of (not necessarily normal) states on $\mathcal{M}$ is called a disintegration with respect to $(\mathcal{I}, \rho)$ if it satisfies the following two conditions:
   (i) The function $s \mapsto \rho_s$ is $\rho \circ \mathcal{I}$-measurable;
   (ii) For all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$,
   \[
   (\mathcal{I}(\Delta)^*\rho)(M) = \int_{\Delta} \rho_s(M) \, d(\rho \circ \mathcal{I})(s). \tag{60}
   \]

2. A disintegration $\{\rho_s^\ast\}_{s \in S}$ with respect to $(\mathcal{I}, \rho)$ is called a family of a posteriori states with respect to $(\mathcal{I}, \rho)$ if, for all $s \in S$, $\rho_s \in \mathcal{M}_{s,1}$.

A family $\{\rho_s^\ast\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$ is unique in the following sense: If $\{\rho_s^{\ast'}\}_{s \in S}$ is another disintegration with respect to $(\mathcal{I}, \rho)$, then $\rho_s^\ast(M) = \rho_s^{\ast'}(M)$, $\rho \circ \mathcal{I}$-a.e. for all $M \in \mathcal{M}$.

Definition 5.3.

1. A family $\{\rho_s^\ast\}_{s \in S}$ of normal positive linear functionals on $\mathcal{M}$ is said to be strongly $\mathcal{F}$-measurable if there exists a sequence $\{F_n\}$ of $\mathcal{M}_{s,1}$-valued simple functions on $S$ such that $\lim_n \|\rho_s - F_n(s)\| = 0$ for all $s \in S$.

2. A disintegration $\{\rho_s^\ast\}_{s \in S}$ with respect to $(\mathcal{I}, \rho)$ is said to be proper if, for any positive element $M$ of $\mathcal{M}$ such that $(\mathcal{I}(\Gamma)^*\rho)(M) = 0$, $\rho_s(M) = 0$ for all $s \in S$.

3. A family $\{\rho_s^\ast\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$ is said to be good if it is strongly $\mathcal{F}$-measurable and proper.

For two strongly $\mathcal{F}$-measurable families $\{\rho_s^\ast\}_{s \in S}$, $\{\rho_s^{\ast'}\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$, it holds that $\rho_s^\ast = \rho_s^{\ast'}$, $\rho \circ \mathcal{I}$-a.e.. The following result is known:

Theorem 5.1 (Ozawa [38, Theorem 4.3]). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. Proper disintegrations $\{\rho_s^\ast\}_{s \in S}$ with respect to $(\mathcal{I}, \rho)$ always exist. If $\{\rho_s^{\ast'}\}_{s \in S}$ is another disintegration with respect to $(\mathcal{I}, \rho)$, then $\rho_s^\ast(M) = \rho_s^{\ast'}(M)$, $\rho \circ \mathcal{I}$-a.e. for all $M \in \mathcal{M}$.

Our interest in this section is the existence of a posteriori states.

Proposition 5.1. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ with NEP and normal state $\rho \in \mathcal{M}_{s,1}$, there exists a proper family $\{\rho_s^\ast\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$.
Proof. Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ with NEP. There then exists a CP instrument $\tilde{\mathcal{I}}$ for $(\mathcal{B}(\mathcal{H}), S)$ such that $\tilde{\mathcal{I}}(\Delta)M = \mathcal{I}(\Delta)M$, for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. Let $\rho$ be a normal state on $\mathcal{M}$. There exists a normal state $\tilde{\rho}$ on $\mathcal{B}(\mathcal{H})$ such that $\tilde{\rho}(M) = \rho(M)$ for all $M \in \mathcal{M}$. Then, for all $\Delta \in \mathcal{F}$,

$$\tilde{(\rho \circ \tilde{\mathcal{I}})}(\Delta) = \tilde{\rho}(\tilde{\mathcal{I}}(\Delta)1) = \tilde{\rho}(\mathcal{I}(\Delta)1) = \rho(\mathcal{I}(\Delta)1) = (\rho \circ \mathcal{I})(\Delta)$$ \hspace{1cm} (61)

By [37] Theorem 4.4, there exists a proper family $\{\tilde{\rho}_s^\mathcal{I}\}_{s \in S}$ of a posteriori states with respect to $(\tilde{\mathcal{I}}, \tilde{\rho})$. It is then obvious that the function $s \mapsto \tilde{\rho}_s^\mathcal{I}|_{\mathcal{M}}$ is $\rho \circ \mathcal{I}$-measurable, and that, for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$,

$$(\mathcal{I}(\Delta)^* \rho)(M) = (\tilde{\mathcal{I}}(\Delta)^* \tilde{\rho})(M) = \int_{\Delta} \tilde{\rho}_s^\mathcal{I}(M) \, d(\tilde{\rho} \circ \tilde{\mathcal{I}})(s) = \int_{\Delta} \tilde{\rho}_s^\mathcal{I}(M) \, d(\rho \circ \mathcal{I})(s). \hspace{1cm} (62)$$

Thus $\{\tilde{\rho}_s^\mathcal{I}|_{\mathcal{M}}\}_{s \in S}$ is a family of a posteriori states with respect to $(\mathcal{I}, \rho)$. The properness of $\{\tilde{\rho}_s^\mathcal{I}|_{\mathcal{M}}\}_{s \in S}$ follows from that of $\{\tilde{\rho}_s^\mathcal{I}\}_{s \in S}$.

The following two corollary are shown to hold by observations in [37] p.291, ll.19-20 and in the previous section:

**Corollary 5.1.** Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ with NEP and normal state $\rho \in \mathcal{M}_{*,1}$, there exists a good family $\{\rho_s^\mathcal{I}\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$.

**Corollary 5.2.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ such that there exists a normal conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$, and $(S, \mathcal{F})$ a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ and every normal state $\rho \in \mathcal{M}_{*,1}$, there exists a proper family $\{\rho_s^\mathcal{I}\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$.

The next is an example that not all CP instruments defined on injective von Neumann algebras have NEP, and is strongly related to Theorems 5.2 and 5.3.

**Example 5.1** (Ozawa [37] pp.292-293]). Let $m$ be a Lebesgue measure on $[0,1]$ and $\mathcal{I}$ a CP instrument for $(L^\infty([0,1], m), [0,1])$ defined by $\mathcal{I}(\Delta)f = \chi_{\Delta}f$ for all $\Delta \in \mathcal{B}([0,1])$ and $f \in L^\infty([0,1], m)$. Let $\rho = m \in L^\infty([0,1], m)_{*,1}$. There exists no family $\{\rho_x\}_{x \in [0,1]}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$. $L^\infty([0,1], m)$ is an injective (maximal abelian) von Neumann subalgebra of the von Neumann algebra $\mathcal{B}(L^2([0,1], m))$ of bounded operators on a separable Hilbert space $L^2([0,1], m)$. It is well-known that there is no normal conditional expectation from $\mathcal{B}(L^2([0,1], m))$ into $L^\infty([0,1], m)$.

**Proposition 5.2.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ with NEP and normal state $\rho \in \mathcal{M}_{*,1}$, there exists a strongly $\mathcal{F}$-measurable family $\{\rho_s^\mathcal{I}\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$.

**Proof.** The proof of this proposition is very similar to that of Proposition 5.1.

Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ with NEP. There then exists a CP instrument $\tilde{\mathcal{I}}$ for $(\mathcal{B}(\mathcal{H}), S)$ such that $\tilde{\mathcal{I}}(\Delta)M = \mathcal{I}(\Delta)M$, for all $\Delta \in \mathcal{B}(\mathcal{I})$ and $M \in \mathcal{M}$. Let $\rho$ be a normal state on $\mathcal{M}$. There exists a normal state $\tilde{\rho}$ on $\mathcal{B}(\mathcal{H})$ such that $\tilde{\rho}(M) = \rho(M)$ for all $M \in \mathcal{M}$. Then it holds that $(\tilde{\rho} \circ \tilde{\mathcal{I}})(\Delta) = (\rho \circ \mathcal{I})(\Delta)$, for all $\Delta \in \mathcal{F}$.
By \cite{37} Theorem 4.5, there exists a strongly $\mathcal{F}$-measurable family $\{\tilde{\rho}_s^\alpha\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$. It is then obvious that the function $s \mapsto \tilde{\rho}_s^\alpha|_{\mathcal{M}}$ is $\rho \circ \mathcal{I}$-measurable, and that, for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$,

\[
(\mathcal{I}(\Delta)^* \rho)(M) = \int_{\Delta} \tilde{\rho}_s^\alpha(M) \ d(\rho \circ \mathcal{I})(s).
\]

(63)

Thus $\{\rho_s^\alpha := \tilde{\rho}_s^\alpha|_{\mathcal{M}}\}_{s \in S}$ is a family of a posteriori states with respect to $(\mathcal{I}, \rho)$. The strong $\mathcal{F}$-measurability of $\{\rho_s^\alpha\}_{s \in S}$ follows from that of $\{\tilde{\rho}_s^\alpha\}_{s \in S}$.

\[\square\]

**Theorem 5.2.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $(\mathcal{S}, \mathcal{F})$ a measurable space, and $\mathcal{I}$ a CP instrument for $(\mathcal{M}, \mathcal{S})$. The following conditions are equivalent:

1. $(\mathcal{I}, \rho)$ has NEP.
2. For every normal state $\rho \in \mathcal{M}_{\ast, 1}$, there exists a strongly $\mathcal{F}$-measurable family $\{\rho_s^\alpha\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$.

**Proof.**

(1) $\Rightarrow$ (2) The proof is already given in Proposition 5.2.

(2) $\Rightarrow$ (1) Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, \mathcal{S})$. Suppose (2) for $\mathcal{I}$. Let $\rho \in \mathcal{M}_{\ast, 1}$. Then there exists a strongly $\mathcal{F}$-measurable family $\{\rho_s^\alpha\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$. By the definition of a family of a posteriori states, $\rho_s \in L^1_{\mathcal{M}_s}(\mathcal{S}, \rho \circ \mathcal{I})$. In addition, it holds that $\rho \circ \mathcal{I} \ll \varphi \circ \mathcal{I}$. There exists a $\mathcal{F}$-measurable function $\lambda$ on $\mathcal{S}$ such that $\lambda = d(\rho \circ \mathcal{I})/d(\varphi \circ \mathcal{I})$, $\varphi \circ \mathcal{I}$-a.e.. A family $\{\lambda(s) \cdot \rho_s\}_{s \in S}$ of normal positive linear functionals on $\mathcal{M}$ is strongly $\mathcal{F}$-measurable, and

\[
\int_{\mathcal{S}} \|\lambda(s) \cdot \rho_s\|_{\mathcal{M}_s} d(\varphi \circ \mathcal{I})(s) = \int_{\mathcal{S}} \lambda(s) \cdot \|\rho_s\|_{\mathcal{M}_s} d(\varphi \circ \mathcal{I})(s) = \int_{\mathcal{S}} \frac{d(\rho \circ \mathcal{I})}{d(\varphi \circ \mathcal{I})}(s) d(\varphi \circ \mathcal{I})(s) = 1.
\]

Thus $\lambda(\cdot) \cdot \rho(\cdot)$ is an element of $L^1_{\mathcal{M}_s}(\mathcal{S}, \varphi \circ \mathcal{I})$. By \cite{51} Chapter IV, Theorem 7.17, there exists a unique normal state $\tilde{\rho}$ on $\mathcal{M} \otimes L^\infty(\mathcal{S}, \varphi \circ \mathcal{I})$ satisfying

\[
\tilde{\rho}(M \otimes f) = \int_{\mathcal{S}} f(s) \rho(\varphi \circ \mathcal{I})(s) \ d(\varphi \circ \mathcal{I})(s).
\]

(64)

We show that the map $\mathcal{M}_{\ast, 1} \ni \rho \mapsto \tilde{\rho} \in (\mathcal{M} \otimes L^\infty(\mathcal{S}, \varphi \circ \mathcal{I}))_{\ast, 1}$ is affine. Let $\rho_1, \rho_2 \in \mathcal{M}_{\ast, 1}$, $0 \leq \alpha \leq 1$, and denote $\alpha \rho_1 + (1 - \alpha)\rho_2$ by $\rho$. For every $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$,

\[
\tilde{\rho}(M \otimes \chi_\Delta) = \int_{\mathcal{S}} \chi_\Delta(s) \rho(\varphi \circ \mathcal{I})(s) \ d(\varphi \circ \mathcal{I})(s) = \int_{\mathcal{S}} \chi_\Delta(s) \rho(\varphi \circ \mathcal{I})(s) \ d(\rho \circ \mathcal{I})(s)
\]

\[\begin{align*}
&= |\mathcal{I}(\Delta)^*(\alpha \rho_1 + (1 - \alpha)\rho_2)|(M) \\
&= \alpha (\mathcal{I}(\Delta)^* \rho_1)(M) + (1 - \alpha)(\mathcal{I}(\Delta)^* \rho_2)(M) \\
&= \alpha \tilde{\rho}_1(M \otimes \chi_\Delta) + (1 - \alpha)\tilde{\rho}_2(M \otimes \chi_\Delta).
\end{align*}\]

(65)

Since $L^\infty(\mathcal{S}, \varphi \circ \mathcal{I}) = \text{span} \{\chi_\Delta \mid \Delta \in \mathcal{F}\}$ and $\mathcal{M} \otimes L^\infty(\mathcal{S}, \varphi \circ \mathcal{I}) = \mathcal{M} \otimes_{\text{alg}} L^\infty(\mathcal{S}, \varphi \circ \mathcal{I})^{uw}$, and since $\tilde{\rho}$, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are normal, it holds that

\[
(\alpha \rho_1 + (1 - \alpha)\rho_2) = \alpha \tilde{\rho}_1 + (1 - \alpha)\tilde{\rho}_2.
\]

(66)
For any \( \rho \in \mathcal{M}_{s, +} \), we define \( \tilde{\rho} \in (\mathcal{M} \otimes L^\infty(S, \varphi \circ I))_{s, +} \) by

\[
\tilde{\rho} = \begin{cases} 
\rho(1) \left( \frac{\rho}{\rho(1)} \right) & (\rho(1) \neq 0) \\
0 & (\rho(1) = 0).
\end{cases}
\]  

It is easily checked that the map \( \mathcal{M}_{s, +} \ni \rho \mapsto \tilde{\rho} \in (\mathcal{M} \otimes L^\infty(S, \varphi \circ I))_{s, +} \) is also affine. Furthermore, we define the map \( \mathcal{M}_s \ni \rho \mapsto \tilde{\rho} \in (\mathcal{M} \otimes L^\infty(S, \varphi \circ I))_s \) by, for all \( \rho \in \mathcal{M}_s \), by

\[
\tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2 + i(\tilde{\rho}_3 - \tilde{\rho}_4),
\]
where \( \rho_1, \rho_2, \rho_3, \rho_4 \in \mathcal{M}_{s, +} \) such that \( \rho = \rho_1 - \rho_2 + i(\rho_3 - \rho_4) \). By a similar discussion in [11, 42], \( \tilde{\rho} \) does not depend on the choice of \( \rho_1, \rho_2, \rho_3, \rho_4 \in \mathcal{M}_{s, +} \) such that \( \rho = \rho_1 - \rho_2 + i(\rho_3 - \rho_4) \). We now define a unital ultraweakly continuous linear map \( \Psi_I : \mathcal{M} \otimes L^\infty(S, \varphi \circ I) \to \mathcal{M} \) by

\[
\rho(\tilde{\Psi}_I(X)) = \tilde{\rho}(X)
\]
for all \( M \in \mathcal{M} \) and \( \rho \in \mathcal{M}_s \). By the definition of \( \tilde{\Psi}_I \), \( \tilde{\Psi}_I(X) = \Psi_I(X) \), for all \( X \in \mathcal{M} \otimes_{\text{bin}} L^\infty(S, \varphi \circ I) = \mathcal{M} \otimes_{\text{min}} L^\infty(S, \varphi \circ I) \). Therefore, \( \tilde{\Psi}_I \) is completely positive on \( \mathcal{M} \otimes_{\text{bin}} L^\infty(S, \varphi \circ I) \).

For every \( n \in \mathbb{N} \) and \( C^* \)-algebra \( X \), we denote by \( M_n(X) \) that of \( n \times n \) matrices with entries from \( X \). For every \( n \in \mathbb{N} \), since \( M_n(\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \varphi \circ I)) = M_n(\mathcal{M} \otimes_{\text{min}} L^\infty(S, \varphi \circ I)) \) is a dense \( C^* \)-subalgebra of \( M_n(\mathcal{M} \otimes L^\infty(S, \varphi \circ I)) \), for all \( X \in M_n(\mathcal{M} \otimes L^\infty(S, \varphi \circ I))_+ \), there exists a net \( \{X_\alpha\} \) of \( M_n(\mathcal{M} \otimes_{\text{bin}} L^\infty(I, \varphi \circ I))_+ \) such that \( X_\alpha \to X \). By a similar discussion appeared in Lemma 3.2, it holds that \( \tilde{\Psi}_I^{(n)}(X_\alpha) = \tilde{\Psi}_I^{(n)}(X) \). Therefore, \( \tilde{\Psi}_I \) is a unital normal CP map from \( \mathcal{M} \otimes L^\infty(S, \varphi \circ I) \) to \( \mathcal{M} \) such that \( \tilde{\Psi}_I|_{\mathcal{M} \otimes_{\text{bin}} L^\infty(S, \varphi \circ I)} = \Psi_I \).

**Corollary 5.3.** Let \( \mathcal{M} \) be a von Neumann algebra on a separable Hilbert space \( H(\Gamma, \mathcal{B}(\Gamma)) \) a standard Borel space, and \( I \) be a CP instrument for \( (\mathcal{M}, \Gamma) \). The following conditions are equivalent:

1. \( I \) has NEP.
2. For every normal state \( \rho \in \mathcal{M}_{s, 1} \), there exists a good family \( \{\rho_\gamma^I\}_{\gamma \in \Gamma} \) of a posteriori states with respect to \( (\mathcal{I}, \rho) \).

**Corollary 5.4.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( H, (S, \mathcal{F}) \) a measurable space, and \( I \) an instrument for \( (\mathcal{M}, S) \). The following conditions are equivalent:

1. There exists a unital normal positive map \( \Phi_I : \mathcal{M} \otimes L^\infty(S, \varphi \circ I) \to \mathcal{M} \) such that

\[
\mathcal{I}(\Delta)M = \Phi_I(M \otimes \chi_\Delta)
\]

for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).
2. For every normal state \( \rho \in \mathcal{M}_{s, 1} \), there exists a strongly \( \mathcal{F} \)-measurable family \( \{\rho_s^I\}_{s \in S} \) of a posteriori states with respect to \( (\mathcal{I}, \rho) \).

**Definition 5.4 (Discreteness and weak repeatability).**

1. A CP instrument \( \mathcal{I} \) for \( (\mathcal{M}, S) \) is said to be discrete if there exists a countable subset \( S_0 \) of \( S \) such that \( \mathcal{I}(S \setminus S_0) = 0 \).
2. A CP instrument \( \mathcal{I} \) for \( (\mathcal{M}, S) \) is said to be weakly repeatable if it satisfies \( \mathcal{I}(\Delta_1) \circ \mathcal{I}(\Delta_2) = \mathcal{I}(\Delta_1 \cap \Delta_2) \) for all \( \Delta_1, \Delta_2 \in \mathcal{F} \).
Davies-Lewis conjecture was resolved in [36, Theorem 6.6], and completely in [37, Theorem 5.1]. We can strengthen the former result by using the latter one and the method in this section:

**Theorem 5.3.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\Gamma, \mathcal{B}(\Gamma))$ a standard Borel space. A weakly repeatable CP instrument $\mathcal{I}$ for $(\mathcal{M}, \Gamma)$ has NEP if and only if it is discrete.

**Proposition 5.3.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. Every discrete CP instrument $\mathcal{I}$ for $(\mathcal{M}, \mathcal{S})$ has NEP.

**Proof.** For every normal state $\rho \in \mathcal{M}_{\ast,1}$, since $\mathcal{I}$ is discrete, a strongly $\mathcal{F}$-measurable proper family $\{\rho_s^\mathcal{I}\}_{s \in S}$ of a posteriori states with respect to $(\mathcal{I}, \rho)$ is given by

$$
\rho_s^\mathcal{I}(M) = \begin{cases} 
\frac{(\mathcal{I}(\{s\})^\ast \rho)(M)}{(\mathcal{I}(\{s\})^\ast \rho)(1)} & (s \in \text{supp}(\rho \circ \mathcal{I})) \\
\frac{(\mathcal{I}(S)^\ast \rho)(M)}{(\mathcal{I}(S)^\ast \rho)(1)} & (s \not\in \text{supp}(\rho \circ \mathcal{I})) 
\end{cases}
$$

(71)

for all $s \in S$. By Theorem 5.2, $\mathcal{I}$ has NEP. □

**Proof of Theorem 5.3.** By Proposition 5.3, discreteness implies NEP. The converse follows from Theorem 5.2 and [37, Theorem 5.1]. □

We shall show another example of a CP instrument without NEP in addition to Example 5.1. Let $\mathcal{N}$ be a AFD von Neumann algebra of type II$_1$ on a separable Hilbert space $\mathcal{H}$, $A$ a self-adjoint element of $\mathcal{N}$ with continuous spectrum, and $\mathcal{E}$ a conditional expectation from $\mathcal{N}$ into $\{A\}^\prime \cap \mathcal{N}$. We define a CP instrument $\mathcal{I}_A$ for $(\mathcal{N}, \mathbb{R})$ by

$$
\mathcal{I}_A(\Delta)N = \mathcal{E}(N)E_A(\Delta)
$$

(72)

for all $N \in \mathcal{N}$ and $\Delta \in \mathcal{B}(\mathbb{R})$, where $E_A$ is the spectral measure of $A$. By the property of conditional expectation, $\mathcal{I}_A$ precisely measures the spectrum of $A$ and is weakly repeatable. Hence it does not have NEP but has ANEP. This is the reason why we cannot ignore CP instruments without NEP. We conclude that all CP instruments defined on von Neumann algebras describing physical systems are physically realizable and are approximated by measuring processes.

6 DHR-DR Theory and Local Measurement

First, we give assumptions of algebraic quantum field theory (AQFT). We refer readers to [2, 25] for standard reference on AQFT.

1 (Local net). Let $\{\mathcal{A}(O)\}_{O \in \mathcal{K}}$ be a family of $W^*$-algebras over a causal poset $\mathcal{K}$ of bounded subregions of the four dimensional Minkowski space $(\mathbb{R}^4, \eta)$, where $\eta$ is the Minkowski metric on $\mathbb{R}^4$, satisfying the following four conditions: (i) $O_1 \subset O_2 \in \mathcal{K} \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$; (ii) if $O_1$ and $O_2$ are causally separated each other, then $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ mutually commute; (iii) $\bigcup_{O \in \mathcal{K}} \mathcal{A}(O)$ is a dense $^*$-subalgebra of a $C^*$-algebra $\mathcal{A}$; (iv) there is a strongly continuous automorphic action $\alpha$ on $\mathcal{A}$ of the Poincare group $\mathcal{P}_+^\Gamma$ such that, for any $g = (a, L) \in \mathcal{P}_+^\Gamma = \mathbb{R}^4 \rtimes \mathcal{L}_+^\Gamma$ and $O \in \mathcal{K}$, $\alpha_{(a,L)}(\mathcal{A}(O)) = \mathcal{A}(LO + a) = \mathcal{A}(k_{(a,L)}O)$, where $\mathcal{L}_+^\Gamma$ is the Lorentz group and, for every $g = (a, L) \in \mathcal{P}_+^\Gamma$, $k_g : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $k_{(a,L)}x = Lx + a$ for all $x \in \mathbb{R}^4$. We call a family $\{\mathcal{A}(O)\}_{O \in \mathcal{K}}$ of $W^*$-algebras satisfying the above conditions a $(W^*)$-local net of observables.
In the setting of AQFT, it is assumed that all physically realizable states on \( \mathcal{A} \) and representations of \( \mathcal{A} \) are locally normal, i.e., normal on \( \mathcal{A}(\mathcal{O}) \) for all \( \mathcal{O} \in \mathcal{K} \).

2 (Vacuum state and representation). A vacuum state \( \omega_0 \) is a \( \mathcal{D}_+^\mathcal{K} \)-invariant locally normal pure state on \( \mathcal{A} \). We denote by \((\pi_0, \mathcal{H}_0, U, \Omega)\) the GNS representation of \((\mathcal{A}, \mathcal{P}_+^\mathcal{K}, \alpha, \omega_0)\). In addition, it is assumed that the spectrum of the generator \( P = (P_\mu) \) of the translation part of \( U \) is contained in the closed future lightcone \( \overline{V_+} \).

For every \( \mathcal{O} \in \mathcal{K} \), we denote by \( \overline{\mathcal{O}} \) the closure of \( \mathcal{O} \) and define the causal complement \( \mathcal{O}' \) of \( \mathcal{O} \) by \( \mathcal{O}' = \{ x \in \mathbb{R}^4 \mid \eta(x-y, x-y) = (x-y)^2 < 0, y \in \mathcal{O} \} \). For \( \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K} \), we denote by \( \mathcal{O}_1 \subset \mathcal{O}_2 \) whenever \( \overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2} \). We denote by \( \mathcal{K}_{DC} \) the subset \( \{ (a + V_+) \cap (b - V_+) \in \mathcal{K} \mid a, b \in \mathbb{R}^4 \} \) of \( \mathcal{K} \) consisting of double cones. Furthermore, we adopt the following notations:

\[
\mathcal{K}_\oplus = \{ \Lambda = (\mathcal{O}_1^\Lambda, \mathcal{O}_2^\Lambda) \in \mathcal{K} \times \mathcal{K} \mid \mathcal{O}_1 \subset \mathcal{O}_2 \},
\]

\[
\mathcal{K}_{\oplus}^{DC} = \{ \Lambda = (\mathcal{O}_1^\Lambda, \mathcal{O}_2^\Lambda) \in \mathcal{K}_{\oplus} \mid \mathcal{O}_1^\Lambda \text{ and } \mathcal{O}_2^\Lambda \text{ are double cones} \}.
\]

For a local net \( \{ \mathcal{A}(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{K}} \) and a vacuum state \( \omega_0 \) on \( \mathcal{A} \), we assume the following three conditions:

A (Property B). \( \{ \mathcal{A}(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{K}} \) has property B: for every pair \( (\mathcal{O}_1^\Lambda, \mathcal{O}_2^\Lambda) \in \mathcal{K}_{\oplus} \) of regions and projection operator \( E \in \mathcal{A}(\mathcal{O}_1^\Lambda) \), there is an isometry operator \( W \in \mathcal{A}(\mathcal{O}_2^\Lambda) \) such that \( WW^* = E \) and \( W^*W = 1 \).

B (Haag duality). We define a dual net \( \{ \mathcal{A}^d(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{K}_{DC}} \) of \( \{ \mathcal{A}(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{K}} \) with respect to the vacuum representation \( \pi_0 \) by \( \mathcal{A}^d(\mathcal{O}) := \pi_0(\mathcal{A}(\mathcal{O}'))' \) for all \( \mathcal{O} \in \mathcal{K}_{DC} \), where \( \mathcal{A}(\mathcal{O}') = \bigcup_{\mathcal{O}_1 \in \mathcal{K}_{\oplus}^{DC}, \mathcal{O}_1 \subset \mathcal{O}} \mathcal{A}(\mathcal{O}_1)^{\mathcal{O}_1} \). \( \{ \mathcal{A}(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{K}} \) satisfies Haag duality in \( \pi_0 \): \( \mathcal{A}^d(\mathcal{O}) = \pi_0(\mathcal{A}(\mathcal{O}))'' \) for all \( \mathcal{O} \in \mathcal{K}_{DC} \).

C (Separability). \( \mathcal{H}_0 \) is separable.

A representation \( \pi \) of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \) is said to satisfy the DHR selection criterion in support with a double cone \( \mathcal{O} \) if the restriction \( \pi|_{\mathcal{A}(\mathcal{O}')} \) of \( \pi \) to \( \mathcal{A}(\mathcal{O}') \) is unitarily equivalent to \( \pi_0|_{\mathcal{A}(\mathcal{O}')} \):

\[
\pi|_{\mathcal{A}(\mathcal{O}')} \cong \pi_0|_{\mathcal{A}(\mathcal{O}')}.
\]

For a representation \( \pi \) of \( \mathcal{A} \) on \( \mathcal{H}_0 \), this criterion is reduced to the form

\[
\pi|_{\mathcal{A}(\mathcal{O}')} = \pi_0|_{\mathcal{A}(\mathcal{O}')}.
\]

Next, we give the definition of strictly local CP instrument.

Definition 6.1. Let \( (\Gamma, \mathcal{B}(\Gamma)) \) be a standard Borel space and \( \pi \) a representation of \( \mathcal{A} \) on \( \mathcal{H}_0 \) satisfying the DHR selection criterion in support with a double cone \( \mathcal{O} \). A strictly local CP instrument \( \mathcal{I} \) for \( (\pi(\mathcal{A})'', \Gamma, \mathcal{O}) \) is a CP instrument for \( (\pi(\mathcal{A})'', \Gamma) \) satisfying

\[
\mathcal{I}(\Delta, AB) = \mathcal{I}(\Delta, A)B
\]

for all \( \Delta \in \mathcal{B}(\Gamma) \), \( A \in \pi(\mathcal{A})'' \) and \( B \in \pi(\mathcal{A}(\mathcal{O}'))'' \) \( (\cong \pi_0(\mathcal{A}(\mathcal{O}'))) \), and

\[
\mathcal{I}(\Delta, A) \in \pi(\mathcal{A}(\mathcal{O}'))''
\]

for all \( \Delta \in \mathcal{B}(\Gamma) \) and \( A \in \pi(\mathcal{A}(\mathcal{O}'))'' \).
This definition is a generalization of that of Halvorson [26] to general representations satisfying the DHR selection criterion.

In DHR-DR theory, it is usually assumed that all factor representations satisfying the DHR selection criterion are quasi-equivalent to irreducible ones. By this assumption and the categorical analysis by Doplicher and Roberts [20, 21], all representations satisfying the DHR selection criterion generate type I von Neumann algebras with separable discrete center. There then exists a normal conditional expectation \( E : B(H_0) \to \pi(A)'' \) for any representation \( \pi \) on \( H_0 \) satisfying the DHR selection criterion.

**Definition 6.2** (Minimal dilation [44]). For a CP instrument for \( (B(H), S) \), the triplet \( (K, E, V) \) is called a minimal dilation of \( \mathcal{I} \) if \( K \) is a Hilbert space, \( E : F \to B(K) \) is a spectral measure and \( V \) is an isometry from \( H \) into \( H \otimes K \) such that

\[
\mathcal{I}(\Delta, X) = V^*(X \otimes E(\Delta))V
\]

for all \( X \in B(H) \), and

\[
H \otimes K = \text{span}\{(X \otimes E(\Delta))V\xi \mid \xi \in H, X \in B(H), \Delta \in F\}.
\]

The following proposition holds:

**Proposition 6.1.** Let \( (\Gamma, B(\Gamma)) \) be a standard Borel space, \( O \) a double cone, and \( \pi \) a representation of \( A \) on \( H_0 \) such that \( \pi|_{A(O')} \cong \pi_0|_{A(O')} \). Then every strictly local CP instrument \( I \) for \( (\pi(A)'', \Gamma, O) \) has NEP, and the minimal dilation \( (K, E, V) \) of the \( E_\pi \)-canonical extension \( \tilde{I} \) of \( I \) satisfies the following intertwining relation:

\[
VA = (A \otimes 1)V
\]

for all \( A \in \pi(A(O'))'' \).

A typical example of strictly local CP instruments is a von Neumann model of an observable affiliated to \( \pi(A(O'))'' \). Even if a CP instrument \( I \) for \( (\pi(A(O'))'', \Gamma) \) has NEP, there does not always exist a strictly local CP instrument \( \tilde{I} \) for \( (\pi(A)'', \Gamma, O) \) such that \( \tilde{I}(\Delta)A = I(\Delta)A \) for all \( A \in \pi(A(O'))'' \). In contrast to this fact, we can show that every CP instrument \( I \) for \( (\pi(A(O'))'', \Gamma) \) with NEP is extended into a local CP instrument \( \tilde{I} \) for \( (\pi(A)'', \Gamma, O) \) defined as follows, where \( \Lambda \in K^{DC}_{\pi} \) such that \( O_1^\Lambda = O \):

**Definition 6.3.** Let \( (\Gamma, B(\Gamma)) \) be a standard Borel space, \( O \) a double cone, \( \Lambda \in K^{DC}_{\pi} \) such that \( O_1^\Lambda = O \), \( \pi \) a representation of \( A \) on \( H_0 \) such that \( \pi|_{A(O')} \cong \pi_0|_{A(O')} \). A local CP instrument \( I \) for \( (\pi(A)'', \Gamma, \Lambda) \) is a CP instrument for \( (\pi(A)'', \Gamma, \Lambda) \) satisfying

\[
\mathcal{I}(\Delta, AB) = \mathcal{I}(\Delta, A)B
\]

for all \( \Delta \in B(\Gamma) \), \( A \in \pi_0(A)'' \) and \( B \in \pi(A((O_2^\Lambda)''))'' = \pi_0(A(O_2^\Lambda))' \), and

\[
\mathcal{I}(\Delta, A) \in \pi(A(O))''
\]

for all \( \Delta \in B(\Gamma) \) and \( A \in \pi(A(O'))'' \).

**Definition 6.4.** Let \( \{(A(O))_{O \in K}\} \) be a local net of observables. \( \{(A(O))_{O \in K}\} \) satisfies split property if, for all \( \Lambda \in K_{\pi} \), there exists a type I factor \( \mathcal{N} \) such that \( \mathcal{A}(O_1^\Lambda) \subset \mathcal{N} \subset \mathcal{A}(O_2^\Lambda) \).
Theorem 6.1. Suppose that a local net \( \{ \pi_0(\mathcal{A}(\mathcal{O})) \}_{\mathcal{O} \in \mathcal{K}} \) satisfies the split property. Let \((\Gamma, \mathcal{B}(\Gamma))\) be a standard Borel space, \(\mathcal{O}\) a double cone, \(\pi\) a representation of \(\mathcal{A}\) on \(\mathcal{H}_0\) such that \(\pi|_{\mathcal{A}(\mathcal{O}_1^A)} \equiv \pi_0|_{\mathcal{A}(\mathcal{O}_1^A)}\). For every CP instrument \(\mathcal{I}\) for \((\pi(\mathcal{A}(\mathcal{O})), \Gamma)\) with NEP and \(\Lambda \in \mathcal{K}^{DC}\) such that \(\mathcal{O}_1^A = \mathcal{O}\), there exists a local CP instrument \(\widetilde{\mathcal{I}}\) for \((\pi(\mathcal{A})), \Gamma, \Lambda)\) such that \(\widetilde{\mathcal{I}}(\Delta)A = \mathcal{I}(\Delta)A\) for all \(A \in \pi(\mathcal{A}(\mathcal{O}))\). Then the minimal dilation \((\mathcal{K}, E, V)\) of a CP instrument for \((\mathcal{B}(\mathcal{H}_0), \Gamma)\) extending \(\widetilde{\mathcal{I}}\) satisfies the following intertwining relation:

\[
VA = (A \otimes 1)V
\]

for all \(A \in \pi(\mathcal{A}(\mathcal{O}_2^A)))\).

Proof. Let \(\mathcal{I}\) be a CP instrument for \((\pi(\mathcal{A}(\mathcal{O})))^\prime, \Gamma\) with NEP and \(\Lambda \in \mathcal{K}^{DC}\) such that \(\mathcal{O}_1^A = \mathcal{O}\). Since there exists a type I factor \(\mathcal{N}\) such that \(\pi(\mathcal{A}(\mathcal{O}_1^A)) \subset \pi_0(\mathcal{A}(\mathcal{O}_1^A)) \subset \mathcal{N} \subset \pi_0(\mathcal{A}(\mathcal{O}_2^A))\), it holds by [8] that

\[
\pi(\mathcal{A}(\mathcal{O}_1^A))' \supset \pi(\mathcal{A}(\mathcal{O}_1^A))' \cong \pi(\mathcal{A}(\mathcal{O}_1^A))'' \otimes \pi_0(\mathcal{A}(\mathcal{O}_2^A)).
\]

There then exists a CP instrument \(\widetilde{\mathcal{I}}_0\) for \((\pi(\mathcal{A}(\mathcal{O})))'' \otimes \pi(\mathcal{A}(\mathcal{O}_2^A)))', \Gamma\) with NEP such that

\[
\widetilde{\mathcal{I}}_0(\Delta, X \otimes Y) = \mathcal{I}(\Delta, X) \otimes Y
\]

for all \(\Delta \in \mathcal{B}(\Gamma), X \in \pi(\mathcal{A}(\mathcal{O}_1^A))''\) and \(Y \in \pi(\mathcal{A}(\mathcal{O}_2^A))'\). We identify the CP instrument \(\widetilde{\mathcal{I}}_0\) for \((\pi(\mathcal{A}(\mathcal{O})))'' \otimes \pi(\mathcal{A}(\mathcal{O}_2^A)))', \Gamma\) with that for \((\pi(\mathcal{A}(\mathcal{O})))' \otimes \mathcal{I}(\mathcal{A}(\mathcal{O}_2^A))', \Gamma\) with NEP. Let a CP instrument \(\widetilde{\mathcal{I}}_1\) for \((\mathcal{B}(\mathcal{H}_0), \Gamma)\) such that \(\widetilde{\mathcal{I}}_1(\Delta, X) = \widetilde{\mathcal{I}}_0(\Delta, X)\) for all \(\Delta \in \mathcal{B}(\Gamma)\) and \(X \in \pi(\mathcal{A}(\mathcal{O})))'' \otimes \pi_0(\mathcal{A}(\mathcal{O}_2^A))'\). We define a CP instrument \(\widetilde{\mathcal{I}}_2\) for \((\mathcal{B}(\mathcal{H}_0), \Gamma)\) by

\[
\widetilde{\mathcal{I}}_2(\Delta, X) = \mathcal{E}_\pi(\mathcal{I}_1(\Delta, X))
\]

for all \(\Delta \in \mathcal{B}(\Gamma)\) and \(X \in \mathcal{B}(\mathcal{H}_0)\), and a CP instrument \(\widetilde{\mathcal{I}}\) for \((\pi(\mathcal{A})), \Gamma\) by

\[
\widetilde{\mathcal{I}}(\Delta, X) = \widetilde{\mathcal{I}}_2(\Delta, X)
\]

for all \(\Delta \in \mathcal{B}(\Gamma)\) and \(X \in \pi(\mathcal{A})'\). It is easily seen that \(\widetilde{\mathcal{I}}\) is a local CP instrument for \((\pi(\mathcal{A})), \Gamma, \Lambda\) such that \(\mathcal{I}(\Delta)A = \widetilde{\mathcal{I}}(\Delta)A\) for all \(\Delta \in \mathcal{B}(\Gamma)\) and \(A \in \pi(\mathcal{A}(\mathcal{O})))'\), and that \(\mathcal{I}(\Delta)X = \mathcal{I}_2(\Delta)X\) for all \(\Delta \in \mathcal{B}(\Gamma)\) and \(X \in \pi(\mathcal{A})'\). In addition, the minimal dilation \((\mathcal{K}, E, V)\) of \(\widetilde{\mathcal{I}}_2\) satisfies the desired relation. \(\square\)

In this section, we formulated local measurement on the basis of AQFT. Our attempt is very natural because actual measurements are genuinely local. On the other hand, there exist observables such as charges and as particle numbers, both of which are affiliated to grobal algebras but not to local algebras [10] [13]. This fact follows from origins of them. It is, however, known that we can actually measure them in local regions. A typical example is photon counting measurement in quantum theoretical light detections, which should be taken into account even when we treat light as quantum electro-magnetic field. It is proved by [13] that there exists a net of self-adjoint operators affiliated to \(W^\ast\)-algebras of local observables which converges to a global charge under the split property. In addition, in terms of nonstandard analysis [11] [48], we can describe “infinitely large” local regions. Therefore, it is expected that we can mathematically justify local measurements of global charges and of particle numbers. This may be related to the reason why particle numbers should be treated as non-conserved quantities in wide situations in contrast to low-energy situations their ammounts are conserved. Temperature and chemical potential are of the same kind in thermal situations. We hope big innovations in AQFT in order to formulate local measurement in more actual sense.
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