LOWER FUNCTIONS AND CHUNG’S LILS OF THE GENERALIZED FRACTIONAL BROWNIAN MOTION

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Abstract: Let $X := \{X(t)\}_{t \geq 0}$ be a generalized fractional Brownian motion (GFBM) introduced by Pang and Taqqu (2019):

$$\{X(t)\}_{t \geq 0} \overset{d}{=} \left\{ \int_{\mathbb{R}} \left( (t-u)_+^\gamma - (-u)_+^\gamma \right) |u|^{-\gamma} B(du) \right\}_{t \geq 0},$$

with parameters $\gamma \in (0, 1/2)$ and $\alpha \in \left( -\frac{1}{2} + \gamma, \frac{1}{2} + \gamma \right)$.

Continuing the studies of sample path properties of GFBM $X$ in Ichiba, Pang and Taqqu (2021) and Wang and Xiao (2021), we establish integral criteria for the lower functions of $X$ at $t = 0$ and at infinity by modifying the arguments of Talagrand (1996). As a consequence of the integral criteria, we derive the Chung-type laws of the iterated logarithm of $X$ at the $t = 0$ and at infinity, respectively. This solves a problem in Wang and Xiao (2021).

Keyword: Generalized fractional Brownian motion; lower function; Chung’s LIL; small ball probability.

MSC: 60G15, 60G17, 60G18, 60G22.

1. Introduction

The generalized fractional Brownian motion (GFBM, in short) $X := \{X(t)\}_{t \geq 0}$ is a centered Gaussian self-similar process introduced by Pang and Taqqu [21] as the scaling limit of a sequence of power-law shot noise processes. It has the following integral representation:

$$\{X(t)\}_{t \geq 0} \overset{d}{=} \left\{ \int_{\mathbb{R}} \left( (t-u)_+^\gamma - (-u)_+^\gamma \right) |u|^{-\gamma} B(du) \right\}_{t \geq 0},$$

where the parameters $\gamma$ and $\alpha$ satisfy

$$\gamma \in \left( 0, \frac{1}{2} \right), \quad \alpha \in \left( -\frac{1}{2} + \gamma, \frac{1}{2} + \gamma \right),$$

and $B$ is a two-sided Brownian motion on $\mathbb{R}$. It follows that the Gaussian process $X$ is self-similar with index $H$ given by

$$H = \alpha - \gamma + \frac{1}{2} \in (0, 1).$$

If $\gamma = 0$, then $X$ becomes an ordinary fractional Brownian motion (FBM, in short) $B^H$, which can be represented as:

$$\{B^H(t)\}_{t \geq 0} \overset{d}{=} \left\{ \int_{\mathbb{R}} \left( (t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) B(du) \right\}_{t \geq 0}.$$ 

As shown by Pang and Taqqu [21], GFBM $X$ preserves the self-similarity property while the factor $|u|^{-\gamma}$ introduces non-stationarity of the increments, which is useful for reflecting
the non-stationarity increments property in physical systems. Ichiba, Pang and Taqqu [13] established the H"{o}lder continuity, the functional and local laws of the iterated logarithm of GFBM and showed that these properties are determined by the self-similarity index $H = \alpha - \gamma + 1/2$. More recently, Ichiba, Pang and Taqqu [14] studied the semimartingale properties of GFBM $X$ and its mixtures and applied them to model the volatility processes in finance.

In [29], we studied some precise sample path properties of GFBM $X$, including the exact uniform modulus of continuity, small ball probabilities, and Chung’s LIL at any fixed point $t > 0$. In contrast to the theorems of Ichiba, Pang and Taqqu [14], our results show that the uniform modulus of continuity and Chung’s LIL at any fixed point $t > 0$ are determined mainly by the parameter $\alpha$, while $\gamma$ plays a less important role. Roughly speaking, for $\alpha < 1/2$, the results in [29] on uniform modulus of continuity and Chung’s LIL at $t > 0$ are analogous to the corresponding results for a fractional Brownian motion with index $\alpha + 1/2$.

For example, Theorem 1.5 in [29] shows the following Chung’s LILs for GFBM $X$ and its derivative $X'$ (which exists when $\alpha > 1/2$) at any fixed $t > 0$:

(a) If $\alpha \in (-1/2 + \gamma/2, 1/2)$, then there exists a constant $c_{1,1} \in (0, \infty)$ such that for every $t > 0$,

\[
\liminf_{r \to 0^+} \sup_{0 \leq r \leq r^0 + r^{1/2}/(\ln \ln r^{-1})^{\alpha + 1/2}} \frac{|X(t + h) - X(t)|}{c_{1,1} t^{-\gamma}, \ a.s.}
\]  

(b) If $\alpha \in (1/2, 1/2 + \gamma/2)$, then there exists a constant $c_{1,2} \in (0, \infty)$ such that for every $t > 0$,

\[
\liminf_{r \to 0^+} \sup_{0 \leq r \leq r^{1/2}/(\ln \ln r^{-1})^{\alpha - 1/2}} \frac{|X'(t + h) - X'(t)|}{c_{1,2} t^{-\gamma}, \ a.s.}
\]

As $t \to 0^+$, the terms on the right-hand sides of (1.5) and (1.6) tend to $+\infty$. This suggests that the scaling functions on the left-hand sides of (1.5) and (1.6) are not optimal in the neighborhood of the origin. The problem on Chung's LIL for GFBM $X$ at $t = 0$ was left open in [29].

The main objective of the present paper is to establish Chung’s LILs of GFBM $X$ at $t = 0$ and at infinity. In fact, we will prove more precise results, namely, integral criteria for the lower functions of $M := \{M(t)\}_{t \geq 0} := \{\sup_{0 \leq s \leq t} |X(s)|\}_{t \geq 0}$ at $t = 0$ and at infinity, which imply the following Chung’s LILs of GFBM $X$.

**Theorem 1.1.** Let $X = \{X(t)\}_{t \geq 0}$ be a GFBM with parameters $\alpha$ and $\gamma$. Suppose $\alpha \in (-1/2 + \gamma, 1/2)$.

(a) There exists a positive constant $\kappa_1 \in (0, \infty)$ such that

\[
\liminf_{t \to 0^+} \sup_{0 \leq s \leq t} \frac{|X(s)|}{t^{H}/(\ln \ln t)^{\alpha + 1/2}} = \kappa_1, \ a.s.
\]  

(b) There exists a positive constant $\kappa_2 \in (0, \infty)$ such that

\[
\liminf_{t \to \infty} \sup_{0 \leq s \leq t} \frac{|X(s)|}{t^{H}/(\ln \ln t)^{\alpha + 1/2}} = \kappa_2, \ a.s.
\]

Similarly to the theorems of Ichiba, Pang and Taqqu [14] mentioned above, the self-similarity index $H$ plays an essential role in (1.7) and (1.8). The results in [14, 29] and the present paper show that GFBM $X$ is an interesting example of self-similar Gaussian processes which has richer sample path properties than the ordinary FBM and its close
relatives such as the Riemann-Liouville FBM (cf. e.g., [5, 10]), bifractional Brownian motion (cf. [12, 17, 23, 28]), and the sub-fractional Brownian motion (cf. [2, 11, 26, 31]). In this sense, GFBM is a good object (see also [27] for other examples related to stochastic partial differential equations driven by a fractional-colored Gaussian noise) that can be studied for the purpose to develop a general theoretical framework for studying the fine properties of all (or at least a wide class of) self-similar Gaussian processes which, to the best of our knowledge, is still not complete yet.

In the literature, limit theorems of the forms (1.7) and (1.8) are also called “the other law of the iterated logarithm” and there has been a long history of studying them. Chung [6] proved that if $S_n = \eta_1 + \cdots + \eta_n$, where $\{\eta_k\}$ is a sequence of i.i.d. random variables with mean 0, variance 1 and finite third moment, then

$$\lim inf_{n \to \infty} \frac{\max_{1 \leq k \leq n} |S_k|}{\sqrt{n/\ln \ln n}} = \frac{\pi}{\sqrt{8}}, \text{ a.s.}$$

Chung [6] also gave the corresponding large time result for Brownian motion. The extra condition of finite third moment on $\eta_1$ in [6] was removed by Jain and Pruitt [15]. There have been many extensions of these results. For example, Csáki gave a converse of lower/upper class in [8], and he found an interesting connection between Chung’s “other law” for Brownian motion and Strassen’s LIL in [9]. Kuelbs et al [16] studied Chung’s functional LIL for Banach space-valued Gaussian random vectors. Their results are applicable to Brownian motion and provide interesting refinements to those in [9]. Monrad and Rootzén [20] proved Chung’s LIL for a large class of Gaussian processes that have the property of strong local nondeterminism. Li and Shao [18] and Xiao [30] extended the Chung’s LIL in [20] to Gaussian random fields with stationary increments. Chung’s LILs have also been studied for non-Gaussian processes, we refer to Buchmann and Maller [4] and the references therein for more information.

**Remark 1.1.** The following are some remarks about Chung’s LILs in Theorem 1.1.

(i). Notice that the cases of $\alpha = 1/2$ and $\alpha \in (1/2, 1/2 + \gamma)$ are excluded in Theorem 1.1. In the first case, the sample functions of $X$ are not differentiable, while in the second case the sample functions of $X$ are differentiable on $(0, \infty)$. In both cases, we have not be able to solve the problem whether (1.7) and (1.8) hold or not because the optimal small ball probability estimates for $\max_{t \in [0, 1]} |X(t)|$ have not been established for GFBM $X$ yet. See [29, Remark 5.1] for more information.

It is worth mentioning that, when $\alpha \in (1/2, 1/2 + \gamma)$, $X$ has a modification that is continuously differentiable and its derivative $X'$ is a self-similar process with index $H' = \alpha - \gamma - 1/2 < 0$. The exact uniform modulus of continuity on $[a, b]$ with $0 < a < b < \infty$ and Chung’s LIL at any fixed $t > 0$ have been proved for $X'$ in [29]. However, the asymptotic properties of $X'$ at $t = 0$ or at infinity have not been studied. Since $H' < 0$, it is expected that $X'(t) \to -\infty$ as $t \to 0^+$ and $X'(t) \to 0$ as $t \to \infty$. It would be interesting from the viewpoint of the aforementioned general theoretical framework for self-similar Gaussian processes to prove both ordinary LIL and Chung’s LIL of $X'$ at $t = 0$ and at $\infty$.

(ii). To prove Theorem 1.1, we modify the argument of Talagrand [25], which is concerned with the lower functions of FBM at $\infty$, to establish integral criteria for the lower functions of GFBM $X$ at $t = 0$ and at infinity. We remark that, in [10, 11], El-Nouty has extended Talagrand’s result to the Riemann-Liouville FBM and the sub-FBM to characterize their lower functions at $\infty$. 
For studying the lower functions of $M = \{\sup_{0 \leq s \leq t} |X(s)|\}_{t \geq 0}$ at $t = 0$ in this paper, the main difficulty comes from the singularity of the second moment of the increment of $X$ at $t = 0$. To elaborate, we consider a decomposition of GFBM:

$$X(t) = \int_{-\infty}^{0} ((t - u)^{\alpha} - (-u)^{\alpha})(-u)^{-\gamma}B(du) + \int_{0}^{t} (t - u)^{\alpha}u^{-\gamma}B(du)$$

$$=: Y(t) + Z(t).$$

The Gaussian processes $Y = \{Y(t)\}_{t \geq 0}$ and $Z = \{Z(t)\}_{t \geq 0}$ are independent. The process $Z$ in (1.9) is called a generalized Riemann-Liouville FBM, using the terminology of Ichiba, Pang and Taqqu [13]. By [29, Lemmas 2.1 and 3.1], there exist positive constants $c_{1,i}, i = 3, \cdots , 6$, such that for all $0 < s < t$,

$$c_{1,3} \frac{|t - s|^2}{t^{2-2\gamma}} \leq \mathbb{E} \left[ (Y(t) - Y(s))^2 \right] \leq c_{1,4} \frac{|t - s|^2}{s^{2-2\gamma}}. \quad (1.10)$$

and

$$c_{1,5} \frac{|t - s|^{2\alpha + 1}}{t^{2\gamma}} \leq \mathbb{E} \left[ (Z(t) - Z(s))^2 \right] \leq c_{1,6} \frac{|t - s|^{2\alpha + 1}}{s^{2\gamma}}. \quad (1.11)$$

These bounds are optimal when $s \leq t \leq c_{1,7}s$ for any constant $c_{1,7} > 1$. Therefore, the small values of $s$ have some effects on $Y$ and $Z$ when $H < 1$ and $\gamma > 0$, respectively. The singularity at $s = 0$ brings two technical difficulties when we modify the approach in Talagrand [25]: one is in estimating the metric entropy for proving Proposition 3.1; the other is in constructing of the sequences in Section 4.2.1 in order to use the (generalized) Borel-Cantelli lemma.

(iii). From the proofs of Theorems 1.1 and 2.1, we can verify that the conclusions of Theorem 1.1 also hold for the generalized Riemann-Liouville FBM $Z$ in (1.9). For example, when $\alpha \in (-1/2 + \gamma, 1/2)$,

$$\liminf_{t \to 0^+} \sup_{0 \leq s \leq t} \frac{|Z(s)|}{t^H / (\ln \ln t^{-1})^{\alpha + 1/2}} = c_{1,8} \in (0, \infty) \quad a.s. \quad (1.12)$$

For the process $Y$ defined in (1.9), when $\alpha \in (-1/2 + \gamma, 1/2)$ we obtain that by (1.7) and (1.12),

$$\liminf_{t \to 0^+} \sup_{0 \leq s \leq t} \frac{|Y(s)|}{t^H / (\ln \ln t^{-1})^{\alpha + 1/2}} = c_{1,9} \in [0, \infty) \quad a.s. \quad (1.13)$$

Since an optimal upper bound for the small ball probability estimates has not been established for $Y$ yet (see [29, Lemma 7.1]), we are not able to decide if $c_{1,9} > 0$ or $c_{1,9} = 0$.

The rest of this paper is organized as follows. In Section 2, we state Theorems 2.1 and 2.2 which provide integral criteria for the lower functions of $M$ at $t = 0$ and at infinity. From these integral criteria, we derive Chung’s LILs for $X$ in Theorem 1.1.

Section 3 contains some preliminary results. In Section 4 and Section 5, we prove Theorems 2.1 and 2.2, respectively.

2. Main results and Proof of Theorem 1.1

The following definition of the lower classes for the process $M = \{M(t)\}_{t \geq 0}$ is adapted from [22]. This book provides a systematic and extensive account on the studies of lower and upper classes for Brownian motion, random walks and their functionals.
Definition 2.1. (a) A function \( f(t), t > 0 \), belongs to the lower-lower class of the process \( M \) at \( \infty \) (resp. at 0), denoted by \( f \in LLC(X)(M) \) (resp. \( f \in LLC_0(M) \)), if for almost all \( \omega \in \Omega \) there exists \( t_0 = t_0(\omega) \) such that \( M(t) \geq f(t) \) for every \( t > t_0 \) (resp. \( t < t_0 \)).

(b) A function \( f(t), t > 0 \), belongs to the lower-upper class of the process \( M \) at \( \infty \) (resp. at 0), denoted by \( f \in LUC(X)(M) \) (resp. \( f \in LUC_0(M) \)), if for almost all \( \omega \in \Omega \) there exists a sequence \( 0 < t_1(\omega) < t_2(\omega) < \cdots < \) with \( t_n(\omega) \uparrow \infty \) (resp. \( t_1(\omega) > t_2(\omega) > \cdots > \) with \( t_n(\omega) \downarrow 0 \)), as \( n \to \infty \), such that \( M(t_n(\omega)) \leq f(t_n(\omega)) \), \( n \in \mathbb{N} \).

Since in the present paper, \( M(t) = \sup_{0 \leq s \leq t} |X(s)| \), we will also write the lower classes in Definition 2.1 as \( LLC_0(X) \) and \( LUC_0(X) \) and call a function \( f \in LLC_0(M) \) (resp. \( f \in LUC_0(M) \)) a lower-lower (resp. lower-upper) function of \( X \) at \( t = 0 \). It is known from Talagrand [25] that small ball probability estimates are essential for studying the lower classes of a stochastic process.

By the self-similarity of GFBM \( X \), we have
\[
\mathbb{P}(M(t) \leq \theta) = \mathbb{P}(M(1) \leq \theta) =: \varphi(\theta). 
\] (2.1)

Wang and Xiao [29] proved the following small ball probability estimates for GFBM: If \( \alpha \in (-1/2 + \gamma, 1/2) \), then there exist constants \( \kappa_3 > \kappa_4 > 0 \) such that for all \( t > 0 \) and \( 0 < \theta < 1 \),
\[
\exp\left(-\kappa_3\left(\frac{t^H}{\theta}\right)^{\frac{1}{\theta}}\right) \leq \mathbb{P}\left\{ \sup_{s \in [0,t]} |X(s)| \leq \theta \right\} \leq \exp\left(-\kappa_4\left(\frac{t^H}{\theta}\right)^{\frac{1}{\theta}}\right).
\]

Here and in the sequel, \( \beta = \alpha + 1/2 \). It follows that for any \( \theta \in (0,1) \),
\[
\exp\left(-\kappa_3\theta^{-\frac{1}{\theta}}\right) \leq \varphi(\theta) \leq \exp\left(-\kappa_4\theta^{-\frac{1}{\theta}}\right). 
\] (2.2)

Now we state our first main result, which gives an integral criterion for the lower functions of GFBM \( X \) at \( t = 0 \). It shows that, besides \( \beta = \alpha + 1/2 \), the self-similarity index \( H = \alpha - \gamma + \frac{1}{2} \) plays an essential role.

Theorem 2.1. Assume \( \alpha \in (-1/2 + \gamma, 1/2) \). Let \( \xi : (0, e^{-\varepsilon}] \to (0, \infty) \) be a nondecreasing continuous function.

(a) (Sufficiency). If
\[
\frac{\xi(t)}{t^H} \text{ is bounded and } I_0(\xi) := \int_0^{e^{-\varepsilon}} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t} < +\infty,
\] (2.3)

then \( \xi \in LLC_0(X) \).

(b) (Necessity). Conversely, if \( \frac{\xi(t)}{t^{(1+\varepsilon)H}} \) is non-increasing for some constant \( \varepsilon_0 > 0 \) and if \( \xi \in LLC_0(X) \), then (2.3) holds.

Theorem 2.2 is an analogous result for GFBM \( X \) at infinity.

Theorem 2.2. Assume \( \alpha \in (-1/2 + \gamma, 1/2) \). Let \( \xi : [e^\varepsilon, \infty) \to (0, \infty) \) be a nondecreasing continuous function. Then \( \xi \in LLC_0(X) \) if and only if
\[
\frac{\xi(t)}{t^H} \text{ is bounded and } I_\infty(\xi) := \int_{e^\varepsilon}^{\infty} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t} < +\infty.
\] (2.4)
The proofs of Theorems 2.1 and 2.2 will be given in Sections 4 and 5 below. First, let us apply them to prove Theorem 1.1.

We will make use of the following zero-one laws at $t = 0$ and $\infty$. Eq. (2.5) follows from [28, Proposition 3.3] which provides a zero-one law for the lower class of (not necessarily Gaussian) self-similar processes with ergodic scaling transformations. Recall from [24, 28] that for every $a > 0$, $a \neq 1$, the scaling transformation $S_a$ of $X$ is defined by $S_a(X) = \{a^{-H}X(at)\}_{t \geq 0}$. Notice that for any $H$-self-similar process $X$, the scaling transformation $S_a$ preserves the distribution of $X$. Hence the notions of ergodicity and mixing of $S_a$ can be defined in the usual way, cf. Cornfeld et al. [7]. Following Takashima [24], we say that an $H$-self-similar process $X = \{X(t)\}_{t \geq 0}$ is ergodic (resp. strong mixing) if for every $a > 0$, $a \neq 1$, the scaling transformation $S_a$ is ergodic (resp. strong mixing). This, in turn, is equivalent to saying that all the shift transformations for the corresponding stationary process obtained via Lamperti’s transformation $L(X) = \{e^{-H}X(e^t)\}_{t \in \mathbb{R}}$ are ergodic (resp. strong mixing).

For GFBM $X$ in (1.1), in order to verify the ergodicity of its scaling transformations, we use the representation (1.1) to show that the autocovariance function of $L(X)$ satisfies $e^{-Ht}E(X(e^t)X(1)) = O(e^{-\kappa t^\gamma})$ as $t \to \infty$, where $\kappa_5 = \min\{\frac{1}{2} - \gamma, \frac{1}{2} + \gamma - \alpha\} > 0$. By the Fourier inversion formula, $L(X)$ has a continuous spectral density function. It follows from [19, Theorem 8] that $L(X)$ is strong mixing and thus is ergodic. Hence, [28, Proposition 3.3] is applicable to GFBM $X$ and (2.5) follows.

The proof of (2.6) is similar to that of [28, Proposition 3.3] with minor modifications. See [24] for more zero-one laws for self-similar processes with ergodic scaling transformations.

**Lemma 3.1.** Assume $\alpha \in (-1/2 + \gamma, 1/2)$. There exist constants $c_{2,1}, c'_{2,1} \in [0, \infty)$ such that

$$\liminf_{t \to 0^+} \sup_{0 \leq s \leq t} \frac{|X(s)|}{t^{H/(\ln \ln t - 1)^\beta}} = c_{2,1}, \quad a.s. \quad (2.5)$$

and

$$\liminf_{t \to \infty} \sup_{0 \leq s \leq t} \frac{|X(s)|}{t^{H/(\ln \ln t)^\beta}} = c'_{2,1}, \quad a.s. \quad (2.6)$$

For a constant $\lambda > 0$, let $f_\lambda$ be the function defined by

$$f_\lambda(t) := \frac{\lambda t^H}{(\ln \ln t)^\beta}, \quad t > 0. \quad (2.7)$$

**Proof of Theorem 1.1.** When $\alpha \in (-1/2 + \gamma, 1/2)$, by combining Theorem 2.1 (resp. Theorem 2.2) with (2.2), we derive that if $\lambda < \kappa_4^\beta$, then $f_\lambda \in LLC_0(X)$ (resp. $f_\lambda \in LLC_\infty(X)$), else if $\lambda > \kappa_3^\beta$, then $f_\lambda \in LUC_0(X)$ (resp. $f_\lambda \in LUC_\infty(X)$). These, together with the zero-one law in Lemma 2.1, imply the desirable results in Theorem 1.1. \qed

3. SOME PRELIMINARY RESULTS

In this section, we provide some preliminary results that will be useful for proving Theorems 2.1 and 2.2. Their proofs are modifications of those in Talagrand [25].

**Lemma 3.1.** If $\alpha \in (-1/2 + \gamma, 1/2)$, then there exists a constant $c_{3,1} > 0$ such that for all $0 < t < u$, $\theta, \eta > 0$,

$$\mathbb{P}(M(t) \leq \theta t^H, M(u) \leq \eta) \leq 2\varphi(\theta) \exp \left(-\frac{u - t}{c_{3,1} u^{\frac{3}{2}} \eta^{\frac{1}{\beta}}} \right). \quad (3.1)$$
Proof. If \((u - t)/(u^2 \eta^2) \leq 2\), then it is obvious that (3.1) holds with \(c_{3.1} = 2/\ln 2\). Hence, we only need to prove (3.1) in the case of \((u - t)/(u^2 \eta^2) > 2\). The proof is divided into two steps.

**Step 1.** We define an increasing sequence \(\{t_n\}_{n \geq 0}\) as follows. Set \(t_0 = t\). For any \(n \geq 1\), if \(t_{n-1}\) has been defined, then we choose \(t_n > t_{n-1}\) such that
\[
t_n - t_{n-1} \approx u^2 \eta^2 = t_{n-1}.
\]
Consider the event
\[
A_k := \{M(t) \leq \theta t^H\} \cap \{M(t_k) \leq \eta\}.
\]
It suffices to prove that for any \(k \geq 1\), we have
\[
\mathbb{P}(A_k) \leq \varphi(\theta)\rho^k,
\]
where \(\rho \in (0, 1)\) is a constant that depends on \(\beta\) and \(\gamma\) only. Indeed, if \(k\) is the largest integer such that \(t_k \leq u\), then \(k+1 \geq (u - t)/(u^2 \eta^2)\), which implies \(k \geq (u - t)/(2u^2 \eta^2)\) and
\[
A_k \supset \{M(t) \leq \theta t^H\} \cap \{M(u) \leq \eta\}.
\]
Hence, (3.2) implies (3.1).

**Step 2.** We prove (3.2) by induction over \(k\). The result holds for \(k = 0\) by (2.1). For the induction step, we observe that
\[
A_{k+1} \subset A_k \cap \{|U| \leq 2\eta\},
\]
where \(U := X(t_{k+1}) - X(t_k)\). By using (1.1), \(U\) can be rewritten as follows \(U = U_1 + U_2\), where
\[
U_1 := \int_{t_k}^{t_{k+1}} (t_{k+1} - u)^{\alpha} u^{-\gamma} B(du),
\]
\[
U_2 := \int_{-\infty}^{t_k} \left[(t_{k+1} - u)^{\alpha} - (t_k - u)^{\alpha}\right] |u|^{-\gamma} B(du).
\]
Notice that \(U_1\) is a Gaussian random variable with
\[
\mathbb{E}[U_1] = 0 \quad \text{and} \quad \text{Var}(U_1) \geq \frac{1}{\tilde{t}_{k+1}^{\beta}} |t_{k+1} - t_k|^{2\beta} = \eta^2.
\]
Thus, we have
\[
\mathbb{P}(|U_1| \leq 2\eta) \leq \Phi(2) - \Phi(-2),
\]
where \(\Phi\) denotes the distribution function of a standard Gaussian random variable. Consequently, by Anderson’s inequality [1] and the independence of \(U_1\) and \(\sigma\{B(s); s \leq t_k\}\), we have
\[
\mathbb{P}(A_{k+1}) \leq \mathbb{E}\left[\mathbb{P}\left(A_k \cap \{|U_1 + U_2| \leq 2\eta\} \mid \sigma\{B(s); s \leq t_k\}\right)\right]
\]
\[
= \mathbb{E}\left[\mathbb{1}_{A_k} \cdot \mathbb{P}(|U_1 + U_2| \leq 2\eta \mid \sigma\{B(s); s \leq t_k\})\right]
\]
\[
\leq \mathbb{P}(A_k) \cdot \mathbb{P}(|U_1| \leq 2\eta)
\]
\[
\leq \mathbb{P}(A_k) \cdot (\Phi(2) - \Phi(-2)).
\]
Therefore, we have proved (3.2) with \(\rho = \Phi(2) - \Phi(-2)\). This finishes the proof of (3.1). \(\square\)
Set
\[ \psi(\theta) := -\log \varphi(\theta). \]  
(3.3)

Then \( \psi \) is positive and non-increasing. According to Borell [3], we know that \( \psi \) is convex which implies the existence of the right derivative \( \psi' \) of \( \psi \). Thus, \( \psi' \leq 0 \) and \( |\psi'| \) is non-increasing.

By the small ball probability estimates in (2.2), we see that there exists a constant \( K_1 \geq 1 \) such that for all \( \theta < 1 \),
\[ \frac{1}{K_1 \theta^{1/\beta}} \leq \psi(\theta) \leq \frac{K_1}{\theta^{1/\beta}}. \]  
(3.4)

The following lemmas give more properties of the functions \( \varphi \) and \( \psi \), which are similar to those in Talagrand [25, Section 2].

**Lemma 3.2.** There exists a constant \( K_2 \geq \max \{ 2^{1+1/\beta}, 2(2K_1^2)\alpha K_1 \} \) such that for all \( \theta \in (0, 1/K_2) \),
\[ -\frac{K_2}{\theta^{1+1/\beta}} \leq \psi'(\theta) \leq -\frac{1}{K_2 \theta^{1+1/\beta}}. \]  
(3.5)

**Lemma 3.3.** There exists a constant \( K_3 \geq K_2 2^{1+1/\beta} \) such that for all \( \theta \leq \varepsilon \leq 2\theta < 1 \),
\[ \exp \left( -K_3 \frac{\varepsilon - \theta}{\theta^{1+1/\beta}} \right) \leq \frac{\varphi(\varepsilon)}{\varphi(\theta)} \leq \exp \left( K_3 \frac{\varepsilon - \theta}{\theta^{1+1/\beta}} \right). \]  
(3.6)

**Lemma 3.4.** For all \( \theta < \theta_0 := (\beta/K_2)^2 \), the function \( \theta^{-1/\beta} \varphi(\theta) \) is increasing.

Since the proofs of the above three lemmas are similar to the analogous lemmas for FBM in Talagrand [25, Section 2], we only prove Lemma 3.4 as an example.

**Proof of Lemma 3.4.** The right derivative of the function \( \theta^{-1/\beta} \varphi(\theta) \) is
\[ \left( -\frac{1}{\beta \theta} - \psi'(\theta) \right) \theta^{-1/\beta} \varphi(\theta), \]
which is positive in the interval \( (0, \theta_0) \) for the positive constant \( \theta_0 = (\beta/K_2)^2 \) by (3.5). The proof is complete. \( \square \)

Given a Gaussian process \( \{X(t)\}_{t \in \mathbb{T}} \), let us denote by \( N(\mathbb{T}, d_X, \varepsilon) \) the smallest number of open balls of radius \( \varepsilon \) for the canonical distance \( d_X(s, t) = \|X(s) - X(t)\|_2 \) that are needed to cover \( \mathbb{T} \), where \( \| \cdot \|_2 \) is the \( L^2(\mathbb{P}) \)-norm. Recall the following lemma from Talagrand [25].

**Lemma 3.5.** [25, Lemma 2.1] There exists a universal constant \( K_4 \) such that for all \( t_0 \in \mathbb{T} \) and \( x > 0 \),
\[ \mathbb{P} \left( \sup_{t \in \mathbb{T}} |X(t) - X(t_0)| \geq K_4 x \int_0^\infty \sqrt{\log N(\mathbb{T}, d_X, \varepsilon)} d\varepsilon \right) \leq \exp \left( -x^2 \right). \]  
(3.7)

The following proposition will be important for proving the necessity in Theorems 2.1 and 2.2.
Proposition 3.1. Assume \( \tau = \min \{ \frac{1}{4}(1-H), \frac{H}{4}, \frac{1}{4} (\frac{1}{2} - \gamma) \} \). Then for any \( u > t > 0, \theta > 0, \eta > 0, \) we have
\[
\mathbb{P} \left( \left\{ M(t) \leq \theta t^H \right\} \cap \left\{ M(u) \leq \eta u^H \right\} \right) \\
\leq \exp \left[ -\frac{1}{K} \left( \frac{u}{t} \right)^\tau \right] + \varphi(\theta) \varphi(\eta) \exp \left[ K \left( \frac{u}{t} \right)^{-\tau} \left( \theta^{-1/\beta} + \eta^{-1/\beta} \right) \right].
\]

Proof. We set \( v := \sqrt{ut} \). The idea is that if \( t \ll u \), then \( t \ll v \ll u \). We recall the stochastic integral representation in (1.1), and define
\[
G(s, x) := \left( (s-x)^\alpha + (-x)^\alpha \right) |x|^{-\gamma}, \quad \text{for } s, x \in \mathbb{R}.
\]
Consider the following two processes:
\[
X_1(s) := \int_{|x| \leq v} G(s, x) B(dx), \quad X_2(s) := \int_{|x| > v} G(s, x) B(dx).
\]
Thus, \( X(s) = X_1(s) + X_2(s) \) and the processes \( X_1 \) and \( X_2 \) are independent.

For any \( \delta > 0 \), we have
\[
\mathbb{P} \left( \left\{ M(t) \leq \theta t^H \right\} \cap \left\{ M(u) \leq \eta u^H \right\} \right) \\
= \mathbb{P} \left( \sup_{0 \leq s \leq t} |X(s)| \leq \theta t^H, \sup_{0 \leq s \leq u} |X(s)| \leq \eta u^H \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq t} |X_1(s)| \leq (\theta + \delta) t^H, \sup_{0 \leq s \leq u} |X_2(s)| \leq (\eta + \delta) u^H \right) \\
+ \mathbb{P} \left( \sup_{0 \leq s \leq t} |X_2(s)| \geq \delta t^H \right) + \mathbb{P} \left( \sup_{0 \leq s \leq u} |X_1(s)| \geq \delta u^H \right).
\]

By the independence of \( X_1 \) and \( X_2 \), we have
\[
\mathbb{P} \left( \sup_{0 \leq s \leq t} |X_1(s)| \leq (\theta + \delta) t^H, \sup_{0 \leq s \leq u} |X_2(s)| \leq (\eta + \delta) u^H \right) \\
= \mathbb{P} \left( \sup_{0 \leq s \leq t} |X_1(s)| \leq (\theta + \delta) t^H \right) \cdot \mathbb{P} \left( \sup_{0 \leq s \leq u} |X_2(s)| \leq (\eta + \delta) u^H \right).
\]

Notice that
\[
\mathbb{P} \left( \sup_{0 \leq s \leq t} |X_1(s)| \leq (\theta + \delta) t^H \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq t} |X(s)| \leq (\theta + 2\delta) t^H \right) + \mathbb{P} \left( \sup_{0 \leq s \leq u} |X_2(s)| \geq \delta t^H \right)
\]
and
\[
\mathbb{P} \left( \sup_{0 \leq s \leq u} |X_2(s)| \leq (\theta + \delta) u^H \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq u} |X(s)| \leq (\theta + 2\delta) u^H \right) + \mathbb{P} \left( \sup_{0 \leq s \leq u} |X_1(s)| \geq \delta u^H \right).
\]
Plugging (3.11) to (3.13) into (3.10), we get
\[ \mathbb{P}\left(\{M(t) \leq \theta t^H\} \cap \{M(u) \leq \eta u^H\}\right) \]
\[ \leq \varphi(\theta + 2\delta)\varphi(\eta + 2\delta) + 2\mathbb{P}\left(\sup_{0 \leq s \leq t} |X_2(s)| \geq \delta t^H\right) + 2\mathbb{P}\left(\sup_{0 \leq s \leq u} |X_1(s)| \geq \delta u^H\right). \]  

(3.14)

By the convexity of \(\psi\) and (3.5), we have
\[ \psi(\theta + 2\delta) \geq \psi(\theta) + 2\delta \psi'(\theta) \geq \psi(\theta) - \frac{2\delta K_2}{\theta^{1+1/\beta}}. \]

Hence, we have
\[ \varphi(\theta + 2\delta) \leq \varphi(\theta) \exp\left(\frac{2\delta K_2}{\theta^{1+1/\beta}}\right). \]

This and a similar inequality for \(\varphi(\eta + 2\delta)\) show that it suffices to show that when \(\delta = (t/u)^\tau\) with \(\tau = \min\left\{\frac{1}{4}(1 - H), \frac{H}{4}, \frac{1}{4}(1 - \gamma)\right\}\), the last two terms of (3.14) are bounded by \(\exp\left(-\frac{1}{K}(u/t)^\tau\right)\). This will be done through the following two lemmas whose proofs are postponed.

**Lemma 3.6.** If \(\alpha \in (-1/2 + \gamma, 1/2)\), then there exist constants \(c_{3.2}, c_{3.3} > 0\) satisfying that

(a) For any \(0 \leq s \leq u\), we have
\[ \|X_1(s)\|_2 \leq c_{3.2} \begin{cases} v^H, & \text{if } \beta \in \left(\gamma, \frac{1}{2}\right); \\ v^{1/2 - \gamma}u^\beta - \frac{1}{2}, & \text{if } \beta \in \left[\frac{1}{2}, 1 + \gamma\right). \end{cases} \]

(3.15)

(b) For any \(0 \leq s \leq t\), we have
\[ \|X_2(s)\|_2 \leq c_{3.3}tv^{\beta - \gamma - 1}. \]

(3.16)

Recall \(X = Y + Z\) in (1.9). Observing the moment estimates of \(Y\) and \(Z\) in (1.10) and (1.11), it is easy to see that for any \(b > a > 0, \varepsilon > 0\), the covering numbers of \([a, b]\) in the canonical metrics of \(Y\) and \(Z\) satisfy
\[ N([a, b], d_Y, \varepsilon) \leq c_{1.4}^4(b - a)a^{H-1}\varepsilon_1^{-1}, \quad N([a, b], d_Z, \varepsilon) \leq c_{1.6}^4(b - a)a^{-\frac{\gamma}{2}}\varepsilon^{\frac{\gamma}{2}}. \]

The next lemma gives their estimates when \(a = 0\).

**Lemma 3.7.** There exist constants \(c_{3.4}, c_{3.5} > 0\) satisfying that for any \(b > 0, \varepsilon > 0\),
\[ N([0, b], d_Y, \varepsilon) \leq c_{3.4}b^H\varepsilon^{-1} \]  

and
\[ N([0, b], d_Z, \varepsilon) \leq c_{3.5}b^H\varepsilon^{-\frac{1}{\beta}}. \]  

(3.17)

(3.18)

Combining Lemma 3.6 and Lemma 3.7, we derive that there exist constants \(c_{3.6}, c_{3.7} > 0\) satisfying that
\[ \int_0^\infty \sqrt{\log N([0, u], d_{X, 1}, \varepsilon)} d\varepsilon \leq c_{3.6} \begin{cases} v^H \sqrt{\log \left(\frac{u}{v}\right)}, & \text{if } \beta \in \left(\gamma, \frac{1}{2}\right); \\ u^H \left(\frac{u}{v}\right)^{1/2 - \gamma} \sqrt{\log \left(\frac{u}{v}\right)}, & \text{if } \beta \in \left[\frac{1}{2}, 1 + \gamma\right), \end{cases} \]

(3.19)
and
\[\int_0^\infty \sqrt{\log N([0, t], d_{X, \varepsilon})} d\varepsilon \leq c_{3,7} \left( \frac{t}{v} \right) v^H \sqrt{\log \left( \frac{v}{t} \right)}. \tag{3.20}\]

Since \( t/v = v/u = \sqrt{t/U} \), the conclusion in Proposition 3.1 follows from Lemma 3.5, (3.19), (3.20) and the choice of \( \tau \).

To prove (3.19) and (3.20), we use the decomposition of \( X = Y + Z \) in (1.9). Since
\[N([0, u], d_{X, \varepsilon}) \leq N([0, u], d_{Y, \varepsilon/2}) + N([0, u], d_{Z, \varepsilon/2}), \quad i = 1, 2,\]
it suffices to prove (3.19) and (3.20) for \( Z \) and \( Y \) separately.

If \( \beta \in (\gamma, 1/2) \), then
\[
\int_0^\infty \sqrt{\log N([0, u], d_{Z, \varepsilon})} d\varepsilon \leq \int_0^\infty \sqrt{\log \left( c_{3,5} u^H \beta \varepsilon^{-1/\beta} \right)} d\varepsilon
\]
\[= \beta u^H \int_{c_{3,2}}^{\infty} \sqrt{\log \left( c_{3,5} x \right)} x^{-\beta - 1} dx\]
\[= 2\beta c_{3,5} u^H \int_{c_{3,2}}^{\infty} \sqrt{\log \left( c_{3,5} c_{3,2}^{\beta} (u/\varepsilon)^{H/\beta} \right)} t^2 e^{-\beta t^2} dt\]
\[\leq c_{3,8} v^H \sqrt{\log \left( \frac{1}{u} \right)},\]
for some \( c_{3,8} > 0 \), here in the second and third steps, the changes of variables \( \varepsilon = u^H x^{-\beta} \) and \( x = e^{t^2 / c_{3,5}} \) are used, respectively, and in the last step we have used the following element inequality: there exists a constant \( c_{3,9} > 0 \) satisfying that for all \( a \) large enough,
\[
\int_a^\infty t^2 e^{-\beta t^2} dt \leq c_{3,9} a e^{-\beta a^2}. \tag{3.21}\]

Similarly,
\[
\int_0^\infty \sqrt{\log N([0, u], d_{Y, \varepsilon})} d\varepsilon \leq \int_0^\infty \sqrt{\log \left( c_{3,4} u^H \varepsilon^{-1} \right)} d\varepsilon
\]
\[= u^H \int_{c_{3,2}}^{\infty} \sqrt{\log \left( c_{3,4} x \right)} x^{-2} dx\]
\[= 2c_{3,4} u^H \int_{c_{3,2}}^{\infty} \sqrt{\log \left( c_{3,4} c_{3,2}^{1/2} (u/\varepsilon)^{H/2} \right)} t^2 e^{-t^2} dt\]
\[\leq c_{3,10} v^H \sqrt{\log \left( \frac{1}{u} \right)},\]
for some constant \( c_{3,10} > 0 \), here the changes of variables \( \varepsilon = u^H x^{-1} \) and \( x = e^{t^2 / c_{3,4}} \) are used in the second and third steps, respectively, and (3.21) is used in the last step.

By using the same procedures, we can prove the remainder of (3.19) and (3.20). The details are omitted here. The proof of Proposition 3.1 is complete. \[\Box\]
Proof of Lemma 3.6. (a) If $\beta \in (\gamma, 1/2)$, then $\alpha \in (\gamma - 1/2, 0)$. By using the inequality $|(s + x)\alpha - x\alpha| \leq x\alpha$ for any $s, x > 0$, we have
\[
\|X_1(s)\|_2^2 = \int_{-\infty}^{0} |(s - x)^\alpha - (-x)^\alpha|^2 |x|^{-2\gamma} dx + \int_{0}^{\infty} (s - x)^{2\alpha} x^{-2\gamma} dx \\
= \int_{0}^{\infty} |(s + x)^\alpha - x^\alpha|^2 x^{-2\gamma} dx + \int_{0}^{\infty} (s - x)^{2\alpha} x^{-2\gamma} dx \\
\leq \int_{0}^{\infty} x^{2\alpha - 2\gamma} dx + \int_{0}^{\infty} (v \wedge s)^{2\alpha} x^{-2\gamma} dx \\
= \frac{1}{2\alpha - 2\gamma + 1} v^{2\alpha - 2\gamma + 1} + B(2\alpha + 1, 1 - 2\gamma)(v \wedge s)^{2\alpha - 2\gamma + 1} \\
\leq \left( \frac{1}{2\alpha - 2\gamma + 1} + B(2\alpha + 1, 1 - 2\gamma) \right) v^{2H}.
\]
If $\beta \in [1/2, 1 + \gamma)$, then $\alpha \geq 0$. In this case, by using the inequality $|(s - x)\alpha^+ - (-x)\alpha^+| \leq 2u\alpha$ for all $s \in [0, u]$ and $|x| \leq u$, we have
\[
\|X_1(s)\|_2^2 \leq 4u^{2\alpha} \int_{|x| \leq u} |x|^{-2\gamma} dx = \frac{4}{1 - 2\gamma} u^{1 - 2\gamma} u^{2\beta - 1}.
\]
(b). For any $0 \leq s \leq t$, by using the inequality $|(s + x)^\alpha - x^\alpha| \leq |\alpha|x^{\alpha - 1}s$ for any $\alpha \in (-1/2 + \gamma, 1/2), s, x \geq 0$, we have
\[
\|X_2(s)\|_2^2 = \int_{|x| \geq u} \left( (s - x)^\alpha_+ - (-x)^\alpha_+ \right)^2 |x|^{-2\gamma} dx \\
= \int_{x \geq u} \left( (s + x)^\alpha - x^\alpha \right)^2 x^{-2\gamma} dx \\
\leq |\alpha|^2 s^2 \int_{x \geq u} x^{2\alpha - 2\gamma - 2} dx \\
= \frac{|\alpha|^2}{1 + 2\gamma - 2\alpha} s^2 u^{2\beta - 2\gamma - 2}.
\]
This implies (3.16).

Proof of Lemma 3.7. We use the argument in the proof of Lemma 4.1 in [29]. Since the proofs of (3.17) and (3.18) are similar, we only prove (3.18) here.

It follows from (1.11) that there exists a constant $c_{3,11} > 0$ such that for any $t > s > 0$
\[
d_Z(0, t) = \|Z(t) - Z(0)\|_2 \leq c_{3,11} |t|^H, \tag{3.22}
\]
and
\[
d_Z(s, t) = \|Z(s) - Z(t)\|_2 \leq \frac{c_{3,11}}{s^\gamma} |t - s|^\beta. \tag{3.23}
\]

Let $t_0 = 0, t_1 = \varepsilon^{1/H}$. For any $n \geq 2$, if $t_{n-1}$ has been defined, we define
\[
t_n = t_{n-1} + t_{n-1}^{\gamma} \varepsilon^{\frac{1}{\beta}}. \tag{3.24}
\]
It follows from (3.23) that for all $n \geq 2$,
\[
\|Z(t_n) - Z(t_{n-1})\|_2 \leq \frac{c_{3,11}}{t_{n-1}^{\gamma}} |t_n - t_{n-1}|^\beta \leq c_{3,11} \varepsilon.
\]
Hence \( d_Z(t_n, t_{n-1}) \leq c_{3,11} \varepsilon \) for all \( n \geq 1 \).

Since \([0,u]\) can be covered by the intervals \([t_{n-1}, t_n]\) for \( n = 1, 2, \ldots, L_{\varepsilon} \), where \( L_{\varepsilon} \) is the largest integer \( n \) such that \( t_n \leq b \), we have \( N([0,b], d_Z, \varepsilon) \leq L_{\varepsilon} + 1 \leq 2L_{\varepsilon} \).

In order to estimate \( L_{\varepsilon} \), we write \( t_n = a_n \varepsilon^{1/H} \) for all \( n \geq 1 \). Then, by (3.24), we have \( a_1 = 1 \) and

\[
a_n = a_{n-1} + \frac{2}{a_{n-1}}, \quad \forall \, n \geq 2.
\]

Denote by \( \rho = \frac{H}{\alpha} \). By (4.9) in [29], we know that there exist positive and finite constants \( c_{3,12} \leq 2^{-\gamma/(H \rho)} \rho^{1/\rho} \) and \( c_{3,13} \geq 1 \) such that

\[
c_{3,12} \varepsilon^{1/\rho} \leq a_n \leq c_{3,13} \varepsilon^{1/\rho}, \quad \forall \, n \geq 1.
\]

By (3.25), we have

\[
L_{\varepsilon} = \max \left\{ n; a_n \varepsilon^{1/\rho} \leq b \right\} \leq c_{3,12} b^{\rho} \varepsilon^{-\frac{1}{\rho}}.
\]

This implies that for all \( \varepsilon \in (0,1) \),

\[
N([0,b], d_Z, \varepsilon) \leq 2c_{3,12} b^{\rho} \varepsilon^{-\frac{1}{\rho}}.
\]

The proof of Lemma 3.7 is complete. \( \square \)

We end this section with the following lemma from Talagrand [25], which will be used to prove the necessary parts in Theorems 2.1 and 2.2.

**Lemma 3.8.** [25, Corollary 2.3] Let \( J \subseteq \mathbb{N} \). If there exist some positive numbers \( K_5 \) and \( \varepsilon \) such that for all \( i \in J \),

\[
\sum_{j<i} \mathbb{P}(A_i \cap A_j) \leq \mathbb{P}(A_i) \left( K_5 + (1 + \varepsilon) \sum_{j<i} \mathbb{P}(A_j) \right),
\]

and assume that \( \sum_{i \in J} \mathbb{P}(A_i) \geq (1 + 2K_5)/\varepsilon \), then we have \( \mathbb{P}(\bigcup_{i \in J} A_i) \geq (1 + 2\varepsilon)^{-1} \).

4. **Proof of Theorem 2.1**

4.1. **Sufficiency of the integral condition.** In this part, we always assume that \( \xi(t) \) is a nondecreasing continuous function such that \( \xi(t)/t^H : (0, e^\theta] \to (0, \infty) \) is bounded and \( I_0(\xi) < +\infty \). We prove that \( \xi(t) \leq M(t) \) for all \( t \) small enough in probability one.

Using the argument in the proof of [25, Lemma 3.1], we first prove the following result.

**Lemma 4.1.** Suppose that \( \xi(t) \) is a nondecreasing continuous function such that \( \xi(t)/t^H \) is bounded and \( I_0(\xi) < +\infty \). Then

\[
\lim_{t \to 0} \frac{\xi(t)}{t^H} = 0.
\]

**Proof.** Recall the number \( \theta_0 \) in Lemma 3.4. By the boundedness and the continuity of \( \xi(t)/t^H \), we know

\[
c_{4,1} := \inf \left\{ \frac{\varphi(\theta)}{\theta^{1/\beta}}; \, \theta_0 \leq \theta \leq \sup_{t > 0} \frac{\xi(t)}{t^H} \right\} > 0.
\]

For any \( t \leq u \leq 2t \), we have

\[
\frac{\xi(u)}{u^H} \geq \frac{\xi(t)}{(2t)^H}.
\]
By Lemma 3.4, we know that the function $\varphi(\theta) / \theta^{1/\beta}$ is increasing in $(0, \theta_0)$. Thus for any $t \leq u \leq 2t$,

$$\left(\frac{\xi(u)}{u^H}\right)^{-1/\beta} \varphi \left(\frac{\xi(u)}{u^H}\right) \geq \min \left\{ \left(\frac{\xi(t)}{(2t)^H}\right)^{-1/\beta} \varphi \left(\frac{\xi(t)}{(2t)^H}\right), \ c_{4.1} \right\}.$$  

Consequently, we get

$$\int_t^{2t} \left(\frac{\xi(u)}{u^H}\right)^{-1/\beta} \varphi \left(\frac{\xi(u)}{u^H}\right) \, du \geq \log 2 \cdot \min \left\{ \left(\frac{\xi(t)}{(2t)^H}\right)^{-1/\beta} \varphi \left(\frac{\xi(t)}{(2t)^H}\right), \ c_{4.1} \right\}.$$  

This, together with the fact that $I_0(\xi) < +\infty$ and the monotonicity of $\varphi(\theta) / \theta^{1/\beta}$ in $(0, \theta_0)$, implies (4.1). The proof is complete.  

In order to prove the sufficiency in Theorem 2.1, we construct the sequences $\{t_n\}_{n \geq 1}$, $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ by recursion as follows. Let $L > 2H$ be a constant, whose value will be chosen later. We start with $t_1 = e^{-\varepsilon}$. Having constructed $t_n$, we set

$$u_{n+1} := \inf \left\{ u < t_n; \ t_n + \frac{2^n}{k} \xi(u) \frac{1}{\beta} \geq t_n \right\};$$  

$$v_{n+1} := \inf \left\{ u < t_n; \ t_n \xi(u) \left(1 + L \left(\frac{\xi(u)}{u^H}\right)^{1/\beta}\right) \geq \xi(t_n) \right\};$$  

$$t_{n+1} := \max\{u_{n+1}, v_{n+1}\}. \tag{4.4}$$  

By the continuity of $\xi$, we have

$$u_{n+1} = t_n - \frac{2^n}{k} \xi(u_{n+1}) \frac{1}{\beta}. \tag{4.5}$$  

**Lemma 4.2.** The sequence $\{t_n\}_{n \geq 1}$ is decreasing and

$$\lim_{n \to \infty} t_n = 0. \tag{4.6}$$  

**Proof.** By the construction of $\{t_n\}_{n \geq 1}$, it is obvious to see that $\{t_n\}_{n \geq 1}$ is decreasing and $t_\infty := \lim_{n \to \infty} t_n \geq 0$ exists. Next, we prove $t_\infty = 0$. Suppose, otherwise, that $t_\infty \geq t_\infty > 0$ for all $n \geq 1$. By the continuity of $\xi$, we have $\lim_{n \to \infty} \xi(t_n) = \xi(t_\infty) > 0$. By (4.5), we have $t_n = v_n$ for $n$ large enough but then. Since $\xi$ is continuous, we have that for all $n$ large enough,

$$\xi(t_n) = \xi(t_{n+1}) \left(1 + L \left(\frac{\xi(t_{n+1})}{t_{n+1}^H}\right)^{1/\beta}\right) \geq \xi(t_{n+1}) \left(1 + L \left(\frac{\xi(t_\infty)}{t_\infty^H}\right)^{1/\beta}\right),$$

which contradicts with the convergence of $\{\xi(t_n)\}_{n \geq 1}$. The proof is complete.  

**Lemma 4.3.** If $M(t_n) \geq \xi(t_n) \left(1 + L \left(\frac{\xi(t_n)}{t_n^H}\right)^{1/\beta}\right)$ for all $n \geq n_0$, then $M(t) \geq \xi(t)$ for all $t \in (0, t_{n_0})$.

**Proof.** Assume $n \geq n_0$ and $t_{n+1} < t \leq t_n$. By (4.3) and (4.4), we have

$$\xi(t) \leq \xi(t_n) \leq \xi(t_{n+1}) \left(1 + L \left(\frac{\xi(t_{n+1})}{t_{n+1}^H}\right)^{1/\beta}\right) \leq M(t_{n+1}) \leq M(t).$$

Thus, $M(t) \geq \xi(t)$ for all $t \in (0, t_{n_0})$, as desired.
By (2.1), we have
\[ P\left(M(t_n) < \xi(t_n) \left(1 + L \left(\frac{\xi(t_n)}{t_n^H}\right)^{1/\beta}\right)\right) = \varphi\left(\frac{\xi(t_n)}{t_n^H} \left(1 + L \left(\frac{\xi(t_n)}{t_n^H}\right)^{1/\beta}\right)\right). \] (4.7)

By using the Borell-Cantell lemma and Lemma 4.3, the proof of the sufficiency in Theorem 2.1 will be finished if we show that the terms in (4.7) form a convergent series. Applying (3.6) with \( \theta = \xi(t_n)/t_n^H, \ v = \theta + L\theta^{1+1/\beta} \) (since \( \lim_{n \to \infty} \xi(t_n)/t_n^H = 0 \)) to see that it suffices to show that
\[ \sum_{n=1}^{\infty} \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) < +\infty. \] (4.8)

This is done by using the following lemma.

**Lemma 4.4.**

(a) If \( n \) is large enough and \( t_n = u_n \), then we have
\[ \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) \leq c_{4.2} \int_{t_n}^{t_{n-1}} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t}, \] where \( c_{4.2} \) depends on \( \beta \) and \( H \) only.

(b) If \( n \) is large enough and \( t_n = v_n \), then there exists a constant \( L > 2H \) depending on \( \beta \) and \( H \) only such that
\[ \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) \leq 2^{-p} \varphi\left(\frac{\xi(t_{n-1})}{t_{n-1}^H}\right). \] (4.10)

We postpone the proof of Lemma 4.4. Let us first finish the proof of (4.8), hence the sufficiency in Theorem 2.1. Consider the set
\[ J := \{ n \in \mathbb{N}; t_{n_k} = u_{n_k} \geq v_{n_k} \}. \]

By (4.9), we have
\[ \sum_{n_k \in J} \varphi\left(\frac{\xi(t_{n_k})}{t_{n_k}^H}\right) \leq c_{4.2} \sum_{n_k \in J} \int_{t_{n_k}}^{t_{n_k-1}} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t} \]
\[ \leq c_{4.2} \int_0 e^{-e} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t} < \infty. \] (4.11)

Let \( n_{k-1} \) and \( n_k \) be two consecutive terms of \( J \). If there exists an integer \( n \) such that \( n_{k-1} < n < n_k \), then set \( p := n - n_{k-1} \). Since \( t_n = v_n \) for all \( n \in \mathbb{N} \setminus J \), Eq. (4.10) implies that
\[ \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) \leq 2^{-p} \varphi\left(\frac{\xi(t_{n_{k-1}})}{t_{n_{k-1}}^H}\right). \]

Thus, we obtain by setting \( n_0 = 1 \),
\[ \sum_{n \in J} \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) = \sum_{k=1}^{\infty} \sum_{n_{k-1} < n < n_k} \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) \leq \sum_{k=1}^{\infty} \varphi\left(\frac{\xi(t_{n_{k-1}})}{t_{n_{k-1}}^H}\right) \]
\[ \leq c_{4.2} \int_0 e^{-e} \left(\frac{\xi(t)}{t^H}\right)^{-1/\beta} \varphi\left(\frac{\xi(t)}{t^H}\right) \frac{dt}{t} < \infty. \]

This, together with (4.11), implies (4.8). Therefore, the sufficiency in Theorem 2.1 has been proved. □
Next, we prove Lemma 4.4.

Proof of Lemma 4.4. (a) Assume \( t_n = u_n \geq v_n \). By (4.3), we have

\[
\xi(t_{n-1}) \leq \xi(t_n) \left( 1 + L \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right). \tag{4.12}
\]

By (4.5) and the monotonities of \( \varphi \) and \( \xi \), we have

\[
\int_{t_n}^{t_{n-1}} \left( \frac{\xi(t)}{t^H} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^H} \right) \, dt \geq \int_{t_n}^{t_{n-1}} \left( \frac{\xi(t)}{t_{n-1}^H} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t_{n-1}^H} \right) \, dt = \left( \frac{\xi(t_n)}{\xi(t_{n-1})} \right)^{1/\beta} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) \geq \frac{1}{2} \varphi \left( \frac{\xi(t_n)}{t_{n-1}^H} \right),
\]

where in the last inequality, we have used the fact that \( \xi(t_n) \geq \xi(t_{n-1})/2^\beta \) for all \( n \) large enough by (4.12), Lemma 4.1 and Lemma 4.2.

Now, using (4.5) again, we know that for all \( n \) large enough,

\[
\frac{\xi(t_n)}{t_{n-1}^H} = \frac{\xi(t_n)}{t_n^H} \left( \frac{t_n}{t_{n-1}} \right)^H = \frac{\xi(t_n)}{t_n^H} \left( 1 - \left( \frac{\xi(t_n)}{t_{n-1}^H} \right)^{1/\beta} \right)^H \geq \frac{\xi(t_n)}{t_n^H} \left( 1 - c_{4,3} \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right).
\]

Here \( c_{4,3} > 0 \) which only depends on \( H \). Thus, (4.9) follows from (3.6).

(b) If \( t_n = v_n \geq u_n \), then by the continuity of \( \xi \) we have

\[
\xi(t_{n-1}) = \xi(t_n) \left( 1 + L \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right). \tag{4.13}
\]

Since \( t_n \geq u_n \), it follows from (4.5) and (4.13) that

\[
\frac{\xi(t_{n-1})}{t_{n-1}^H} = \frac{\xi(t_n)}{t_n^H} \left( \frac{t_n}{t_{n-1}} \right)^H \geq \frac{\xi(t_n)}{t_n^H} \left( 1 + L \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right) \left( 1 - \left( \frac{\xi(t_n)}{t_{n-1}^H} \right)^{1/\beta} \right)^H \geq \frac{\xi(t_n)}{t_n^H} \left( 1 + (L - c_{4,4}) \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right), \tag{4.14}
\]

for some positive constant \( c_{4,4} \) which only depends on \( H \) and \( \beta \).

Set \( \theta = \xi(t_n)/t_n^H \), \( \varepsilon = \theta + (L - c_{4,4})\theta^{1+1/\beta} \). It follows from Lemma 4.1 that \( \theta < \varepsilon < 2\theta < 1 \) for all \( n \) large enough. Hence, by (3.5) and the convexity of \( \psi \), we have

\[
\psi(\varepsilon) \leq \psi(\theta) + (\varepsilon - \theta) \psi'(2\theta) \leq \psi(\theta) - \frac{\varepsilon - \theta}{K_2(2\theta)^{1+1/\beta}} \leq \psi(\theta) - \frac{L - c_{4,4}}{2^{1+1/\beta}K_2},
\]
This, together with (4.14), implies that for all $L$ large enough,

$$\varphi\left(\frac{\xi(t_{n-1})}{t_{n-1}^H}\right) \geq \varphi(\varepsilon) = \exp\left(-\psi(\varepsilon)\right) \geq 2\exp\left(-\psi(\theta)\right) = 2\varphi(\theta).$$

The proof of Lemma 4.4. is complete. \qed

4.2. **Necessity of the integral condition.** Suppose that with positive probability, $\xi(t) \leq M(t)$ for all $t > 0$ small enough. We will prove that $\xi(t)/t^H$ is bounded and $I_0(\xi) < +\infty$. The first fact is a direct consequence of (2.1) and $\lim_{\theta \to \infty} \varphi(\theta) = 1$. Let us prove $I_0(\xi) < +\infty$ by using Lemma 3.8.

4.2.1. **Construction of $\{t_n\}_{n \geq 1}$.**

**Lemma 4.5.** If with positive probability, $\xi(t) \leq M(t)$ for all $t > 0$ small enough, then

$$\lim_{t \to 0} \frac{\xi(t)}{t^H} = 0. \quad (4.15)$$

**Proof.** Otherwise, we can find a sequence $\{t_n\}_{n \geq 1}$ such that $\delta := \inf_{n \geq 1} \frac{\xi(t_n)}{t_n^H} > 0$. Denote by $A_n := \{M(t_n) \leq \xi(t_n)\}$. By (2.1), we have

$$\inf_{n \geq 1} \mathbb{P}(A_n) = \inf_{n \geq 1} \mathbb{P}(M(t_n) \leq \xi(t_n)) \geq \mathbb{P}(M(1) \leq \delta) > 0. \quad (4.16)$$

In the following, we show that the events $A_n$ occur infinitely often almost surely. This contradicts the assumption of Lemma 4.5. Without loss of generality, we assume that $t_n/t_{n+1} \geq 2$. Denote by $\mathbb{P}(A_n) = a_n$ for all $n \geq 1$. Applying Proposition 3.1 with $t = t_n, u = t_m, \theta = \xi(t_n)/t_n^H, \eta = \xi(t_m)/t_m^H$ for $m < n$, we have

$$\mathbb{P}(A_n \cap A_m) \leq a_na_m \left\{ \exp\left(-\frac{1}{K} \left(\frac{t_m}{t_n}\right)^\tau\right) + \varphi\left(\frac{\xi(t_m)}{t_m^H}\right) + \varphi\left(\frac{\xi(t_n)}{t_n^H}\right) \right\}$$

$$\quad + \exp\left[K \left(\frac{t_m}{t_n}\right)^{-\tau} \left(\frac{\xi(t_m)}{t_m^H}\right)^{-1-1/\beta} + \left(\frac{\xi(t_n)}{t_n^H}\right)^{-1-1/\beta}\right]\right\}

$$\leq a_na_m \left\{ \exp\left(-\frac{1}{K} \left(\frac{t_m}{t_n}\right)^\tau + 2K_1\delta^{-1/\beta}\right) + \exp\left(2K \left(\frac{t_m}{t_n}\right)^{-\tau}\delta^{-1-1/\beta}\right) \right\},$$

where (3.4) is used in the last step. For any $l \geq 1, N \geq 1$, let $\mathbb{N}_{i,N} := \{n_k\}_{k \geq N} \subset \mathbb{N}$ such that for $t_{n_k}/t_{n_k+1} \geq l$ for any $n_k, n_{k+1} \in \mathbb{N}_{i,N}$. Thus, we have

$$\mathbb{P}(A_n \cap A_m) \leq a_na_m \left\{ \exp\left(-\frac{1}{K} l^\tau + 2K_1\delta^{-1/\beta}\right) + \exp\left(2K l^{-\tau}\delta^{-1-1/\beta}\right) \right\}$$

$$=: a_na_m (1 + o(l)),$$

where $o(l) \to 0$ as $l \to +\infty$. By (4.16), we have $\sum_{m \in \mathbb{N}_{i,N}} a_m = +\infty$. Consequently, by Lemma 3.8, we have

$$\mathbb{P}\left(\bigcup_{m \in \mathbb{N}_{i,N}} A_m\right) \geq \frac{1}{1 + o(l)}.$$

This implies that $\lim_{N \to \infty} \mathbb{P}\left(\bigcup_{n \geq N} A_n\right) = 1$. Hence, the events $A_n$ occur infinitely often almost surely. The proof is complete. \qed
Lemma 4.6. Assume $I_0(\xi) = +\infty$ and (4.15). Then there exists a sequence $\{t_n\}_{n \geq 1}$ with the following properties:

(i) 

$$\sum_{n=1}^{\infty} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) = +\infty; \quad (4.17)$$

(ii) 

$$t_{n+1} = t_n - \frac{\gamma/\beta}{t_{n+1}} \xi(t_{n+1})^{1/\beta}. \quad (4.18)$$

Proof. The construction is given by induction over $n$. We take $t_1 = e^{-e}$. Having constructed $t_n$, we define

$$t_{n+1} := \inf \left\{ u \leq t_n; u + u^{\gamma/\beta} \xi(u)^{1/\beta} \geq t_n \right\}.$$

It is obvious by the continuity of $\xi$ that (4.18) holds.

To prove (4.17), it is sufficient to prove that for all $n$ large enough,

$$I_n := \int_{t_{n+1}}^{t_n} \left( \frac{\xi(t)}{t^H} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^H} \right) \frac{dt}{t} \leq c_{4.5} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right), \quad (4.19)$$

where $c_{4.5}$ depends on $H$ and $\beta$ only. By the monotonicities of $\{t_n\}_{n \geq 1}$ and $\xi$, we derive

$$I_n \leq \frac{(t_n - t_{n+1})}{t_{n+1}^H \xi(t_{n+1})^{1/\beta}} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) = \varphi \left( \frac{\xi(t_n)}{t_n^H} \right). \quad (4.20)$$

It follows from (4.15) and (4.18) that $t_{n+1} \geq t_n/2$ for all $n$ large enough. Hence

$$\frac{\xi(t_n)}{t_n^H} \leq \frac{\xi(t_n)}{t_{n+1}^H} \left( 1 + \left( \frac{2 \xi(t_n)}{t_{n+1}^H} \right)^{1/\beta} \right)^H \leq \frac{\xi(t_n)}{t_n^H} \left( 1 + c_{4.6} \left( \frac{\xi(t_n)}{t_n^H} \right)^{1/\beta} \right),$$

for some constant $c_{4.6} > 0$ which only depends on $H$. The inequality (4.19) follows now from (3.6), (4.20) and (4.21). The proof is complete. \qed

For each $n \geq 1$, we define $k(n)$ by

$$2^{k(n)} \leq \left( \frac{t_n^H}{\xi(t_n)} \right)^{1/\beta} < 2^{k(n)+1}, \quad (4.22)$$

and set $I_k := \{ n \in \mathbb{N}; k(n) = k \}$ for any $k \geq 1$.

Recall the constant $K_1$ given in (3.4). Without loss of generality, we assume that $K_1 \geq c_{2.2}$ where $c_{2.2}$ is the constant in (3.1). For $k \geq 1$, we set

$$N_k := \exp \left( 2^{k-2}/K_1 \right).$$

The following is a refinement of Lemma 4.6.

Proposition 4.1. Assume $I_0(\xi) = +\infty$, (4.15), and $\xi(t)/t^{(1+\varepsilon_0)H}$ is nonincreasing for some $\varepsilon_0 > 0$. Then there exist a positive constant $c_{4.7}$ depending on $\beta$ and $H$ only and a set $J$ with the following properties:
(i)  
\[ \sum_{n \in I} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) = +\infty; \]  
(4.23)

(ii) Given \( n, m \in J \) with \( m < n \) such that  
\[ \text{card}(I_{k(m)} \cap [m, n]) > N_{k(m)}, \]  
we have  
\[ \frac{t_m}{t_n} \geq \exp \left( \exp \left( \frac{2^{\max\{k(m), k(n)\}}}{c_{4.7}} \right) \right). \]  
(4.25)

Proof. Set \( a_n := \varphi(\xi(t_n)/t_n^H) \). We recall from (3.4) that  
\[ \exp \left( -K_12^{k(n)+1} \right) \leq a_n \leq \exp \left( -2^{K(n)} / K_1 \right). \]

Given \( m, k \in \mathbb{N} \), define  
\[ U_{m,k} := \{ i > m; i \in I_k, \text{card}(I_k \cap [m, i]) \leq N_k \}. \]

Thus,  
\[ \sum_{i \in U_{m,k}} a_i \leq N_k \exp \left( -\frac{2^k}{K_1} \right) \leq \exp \left( -\frac{2^{k-1}}{K_1} \right). \]

Denote by \( k_0 \) the smallest integer such that \( 2^{k_0} \geq 2K_1 + 4K_1^2 \). Then there exists a constant \( c_{4,8} \in (0, 1) \) satisfying that  
\[ \sum_{k \geq k(m)+k_0} \sum_{i \in U_{m,k}} a_i \leq a_m \sum_{k \geq k(m)+k_0} \exp \left( K_12^{k(m)+1} - \frac{2^{k-1}}{K_1} \right) \]
\[ \leq a_m \sum_{l \geq 0} \exp \left( 2^{k(m)} \left( 2K_1 - \frac{2^{l-1}+k_0}{K_1} \right) \right) \]
\[ \leq a_m \sum_{l \geq 0} \exp \left( -2^l \right) \leq c_{4,8}a_m. \]

For each \( m \geq 1 \), set  
\[ V_m := \bigcup_{k \geq k(m)+k_0} U_{m,k}. \]

It follows from (4.26) that  
\[ \sum_{k \leq p \in I_k} \sum_{i \in V_m} a_i \leq c_{4,8} \sum_{k \leq p \in I_k} a_m. \]

(4.27)

Set  
\[ J_0 := \mathbb{N} \cap \left( \bigcup_{m \geq 1} V_m \right)^c. \]
By the definition of $V_m$, we know that $k(m) + k_0 \leq k(i)$ if $i \in V_m$. Thus,

$$
\left( \bigcup_{m \geq 1} V_m \right) \cap \left( \bigcup_{k \leq p} I_k \right) \subset \bigcup_{k(m) \leq p} V_m.
$$

This, together with (4.27), implies that

$$
\sum_{i \notin J_0, k(i) \leq p} a_i \leq c_{4.8} \sum_{k(i) \leq p} a_i,
$$

and then

$$
\sum_{i \in J_0, k(i) \leq p} a_i \geq (1 - c_{4.8}) \sum_{k(i) \leq p} a_i.
$$

Letting $p \to \infty$, we obtain that

$$
\sum_{n \in J_0} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) = +\infty. \tag{4.28}
$$

Let $J := J_0 \cap [w, \infty)$ for a constant $w \in \mathbb{N}$, whose value will be chosen later. Then (4.28) implies (4.23).

Next, we prove (ii). For any $n, m \in J$ with $n < m$. If $i \in I_{k(m)}$, then by (4.18) we have

$$
t_{i-1} \geq t_i \left( 1 + \left( \frac{\xi(t_i)}{t_i^H} \right)^{1/\beta} \right) \geq t_i \left( 1 + 2^{-k(m)-1} \right).
$$

Thus, when (4.24) holds and when $w$ (hence $k(m)$) is large enough, we have

$$
\frac{t_m}{t_n} \geq \left( 1 + 2^{-k(m)-1} \right)^{N_{k(m)}} \geq \exp \left( 2^{-k(m)-2} N_{k(m)} \right)

\geq \exp \left( \exp \left( 2^{k(m)-3}/c_{4.9} \right) \right), \tag{4.29}
$$

for some constant $c_{4.9} > 0$. This implies (4.25) whenever $k(m) \geq k(n) - k_0$. If $k(m) < k(n) - k_0$, then by definition of $J$ we must have $n \notin U_{m,k(n)}$. This means that

$$
\text{card}(I_{k(n)} \cap [m, n]) > N_{k(n)},
$$

and the argument leading to (4.29) shows that (4.29) holds for $k(n)$ rather than $k(m)$. Thus, we complete the proof of this proposition by choosing $w$ large enough. \hfill \square

4.2.2. Proof of the necessity in Theorem 2.1. Let $\{t_n\}_{n \in J}$ be as in Proposition 4.1. Set $A_n = \{M(t_n) < \xi(t_n)\}$, then $\mathbb{P}(A_n) = a_n = \varphi(\xi(t_n)/t_n^H)$. According to (4.23) and Lemma 3.8, it suffices to prove the following statement:

Given $\varepsilon > 0$, there exist positive constants $K$ and $q$ such that for any $n \in J$ with $n \geq q$,

$$
\sum_{m \in J, m < n} \mathbb{P}(A_n \cap A_m) \leq \mathbb{P}(A_n) \left( K + (1 + \varepsilon) \sum_{m \in J, m < n} \mathbb{P}(A_m) \right). \tag{4.30}
$$
Thus, we have that for any $n, k \in \mathbb{N}$, set

$$J' := \{m \in J; t_m < t_n \leq 2t_n\};$$

$$J_k := \{m \in J \cap I_k; t_m > 2t_n, \text{ card}(I_k \cap [m, n]) \leq N_k\};$$

$$J'' := J \setminus \bigcup_{k \geq 1} J_k.$$

Applying Eq. (3.1) with $t = t_n, u = t_m, \theta = \xi(t_n)/t_n^H$ and $\eta = \xi(t_m)$ for any $m < n$, we have

$$\mathbb{P}(A_n \cap A_m) \leq 2a_n \exp \left( -\frac{t_m - t_n}{c_2,2t_m^\gamma \xi(t_m)^{1/\beta}} \right).$$  \hfill (4.31)

(1). When $t_n < t_m \leq 2t_n$, the monotonicity of $\xi(t)/t^{(1+\varepsilon_0)H}$ implies that $\xi(t_m) \leq 2^{(1+\varepsilon_0)H}\xi(t_n)$. Hence, by (4.31), we have

$$\mathbb{P}(A_n \cap A_m) \leq 2a_n \exp \left( -\frac{t_m - t_n}{c_2,2^{\gamma/\beta}2^{(1+\varepsilon_0)H/\beta}t_m^\gamma \xi(t_m)^{1/\beta}} \right).$$ \hfill (4.32)

By (4.18) and the monotonicity of $\xi$, we have that for all $m \leq i < n$,

$$t_i = t_{i+1} + t_{i+1}^\gamma \xi(t_{i+1})^{1/\beta} \geq t_{i+1} + t_n^\gamma \xi(t_n)^{1/\beta}.$$  

This, together with (4.32), implies that

$$\mathbb{P}(A_n \cap A_m) \leq 2a_n \exp \left( -\frac{n - m}{c_2,2^{\gamma/\beta}2^{(1+\varepsilon_0)H/\beta}} \right).$$

Therefore, we have

$$\sum_{m \in J'} \mathbb{P}(A_n \cap A_m) \leq c_{4,10}a_n,$$ \hfill (4.33)

for some constant $c_{4,10} > 0$.

(2). When $t_m > 2t_n$, by (4.31) we know

$$\mathbb{P}(A_n \cap A_m) \leq 2a_n \exp \left( -\frac{t_m^H}{2c_2,2t_m^\gamma \xi(t_m)^{1/\beta}} \right).$$

Thus, we have that for any $m \in I_k$,

$$\mathbb{P}(A_n \cap A_m) \leq 2a_n \exp \left( -\frac{2^{k-1}}{c_2,2} \right).$$

Since $J_k \subset I_k$ and $\text{card}(J_k) \leq N_k$, using the definition of $N_k$, we have

$$\sum_{m \in J_k} \mathbb{P}(A_n \cap A_m) \leq 2N_ka_n \exp \left( -\frac{2^{k-1}}{c_2,2} \right) \leq 2a_n \exp \left( -\frac{2^{k-2}}{K_1} \right).$$

Thus,

$$\sum_{m \in \bigcup_{k \geq 1} J_k} \mathbb{P}(A_n \cap A_m) \leq c_{4,11}a_n,$$ \hfill (4.34)

for some constant $c_{4,11} > 0$.

(3). By using (4.33) and (4.34), in order to prove (4.30), it suffices to prove the following result:
Given $\varepsilon > 0$, there exists a constant $c_{4,12} > 0$ such that for any

$$n > \sup_{t \leq c_{4,12}} \sup \{k; k \in I_t\}$$

and $m \in J''$ with $m < n$,

we have

$$\mathbb{P}(A_n \cap A_m) \leq a_n a_m (1 + \varepsilon).$$

(4.35)

Assume $m \in J''$ such that $m < n$. Applying (3.8) with $t = t_m$, $u = t_m$, $\theta = \xi(t_m)/t_m^H$ and

$$\eta = \xi(t_m)/t_m^H,$

and using the facts of $\theta \geq 2^{-(k(n)+1)/\beta}$ and $\eta \geq 2^{-(k(m)+1)/\beta}$, we have

$$\mathbb{P}(A_n \cap A_m) \leq a_n a_m \left\{ \exp \left[ -1 \frac{t_m}{t_n} \right] + \psi(\theta) + \psi(\eta) \right\} + \exp \left[ K \left( \frac{t_m}{t_n} \right)^{-\tau} \left( 2^{(k(n)+1)(1+1/\beta)} + 2^{(k(m)+1)(1+1/\beta)} \right) \right].$$

(4.36)

Since $m \in J''$, $m \notin J_{k(m)}$. By the definition of $J_{k(m)}$, we have

$$\text{card}(J_{k(m)} \cap [m,n]) > N_{k(m)}.$$

Thus, (4.24) holds. By (4.25), we have

$$\frac{t_m}{t_n} \geq \exp \left( \exp \left( 2^\max\{k(m),k(n)\}/c_{4,7} \right) \right).$$

(4.37)

By (3.4) and (4.22), $\psi(\theta) \leq K_2 2^{k(n)+1}$ and $\psi(\eta) \leq K_2 2^{k(m)+1}$. Those, together with (4.37), imply that the coefficient of $a_n a_m$ in (4.36) gets close to 1 as $\max\{k(m),k(n)\}$ becomes large. Hence, we get (4.35). The proof of the necessity in Theorem 2.1 is complete.

5. Proof of Theorem 2.2

In this section, we prove an integral criterion for the lower classes of GFBM at infinity. The setting is the same as in Talagrand [25] and the arguments are similar to those in [25] or Section 4 of this paper. In order not to make the paper too lengthy, we only give a sketch of the proof.

5.1. Sufficiency. Suppose that $\xi(t)$ is a nondecreasing continuous function such that $\xi(t)/t^H$ is bounded and $I_{\infty}(\xi) < +\infty$. We prove that $\xi(t) \leq M(t)$ for $t$ large enough in probability one.

By using the argument in the proof of [25, Lemma 3.1], one can obtain the following analogue of Lemma 4.1. The details of the proof are omitted here.

**Lemma 5.1.** Suppose that $\xi(t)$ is a nondecreasing continuous function such that $\xi(t)/t^H$ is bounded and $I_{\infty}(\xi) < +\infty$. Then

$$\lim_{t \to \infty} \frac{\xi(t)}{t^H} = 0.$$  

(5.1)

In order to prove the sufficiency, we construct the sequences $\{t_n\}_{n \geq 1}$, $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ recursively as follows. Let $L > 2H$ be a constant. We start with $t_1 = e^L$. Having constructed $t_n$, we set

$$u_{n+1} := t_n + \frac{1}{t_n^{2H}} \xi(t_n)^{\frac{1}{2}},$$

(5.2)
\begin{align*}
v_{n+1} & := \inf \left\{ u > t_n; \xi(u) \geq \xi(t_n) \left(1 + L \left( \frac{\xi(t_n)}{t_n^{1/H}} \right)^{1/\beta} \right) \right\}, \quad (5.3) \\
t_{n+1} & := \min \{ u_{n+1}, v_{n+1} \}. \quad (5.4)
\end{align*}

Similar to the proofs of [25, Lemmas 3.2 and 3.3], we have the following lemmas.

**Lemma 5.2.** \( \{t_n\}_{n \geq 1} \) is increasing and \( \lim_{n \to \infty} t_n = +\infty \).

**Lemma 5.3.** If \( M(t_n) \geq \xi(t_n) \left(1 + L \left( \frac{\xi(t_n)}{t_n^{1/H}} \right)^{1/\beta} \right) \) for all \( n \geq n_0 \), then \( M(t) \geq \xi(t) \) for all \( t \geq t_{n_0} \).

By (2.1), we have
\[\text{P} \left( M(t_n) < \xi(t_n) \left(1 + L \left( \frac{\xi(t_n)}{t_n^{1/H}} \right)^{1/\beta} \right) \right) = \varphi \left( \frac{\xi(t_n)}{t_n^{1/H}} \left(1 + L \left( \frac{\xi(t_n)}{t_n^{1/H}} \right)^{1/\beta} \right) \right).\] \( (5.5) \)

One could show that this later series converges by using the same argument in [25, Section 3] or [10, Section 3]. Therefore, the proof of the sufficiency is finished by the Borel-Cantelli lemma, Lemma 5.2 and Lemma 5.3.

5.2. **Necessity.** In this part, we suppose that with positive probability, \( \xi(t) \leq M(t) \) for all \( t \) large enough. We prove that \( \xi(t) \) is bounded and \( I_x(\xi) < +\infty \). The first fact is a direct consequence of (2.1) and the fact that \( \lim_{\theta \to \infty} \varphi(\theta) = 1 \). To prove the second statement, we use Lemma 3.8 and the following lemmas in a way similar to the proof in Section 4.2.

**Lemma 5.4.** Suppose that with positive probability \( \xi(t) \leq M(t) \) for all \( t \) large enough. Then we have
\[\lim_{t \to \infty} \frac{\xi(t)}{t^{1/H}} = 0.\] \( (5.6) \)

**Proof.** Since its proof is similar to that of [25, Lemma 4.1], we omit the details here. \( \square \)

To prove the necessity, we will show that \( \xi \in LUC_x(M) \) when
\[\text{I}_x(\xi) = +\infty \quad \text{and} \quad \lim_{t \to \infty} \frac{\xi(t)}{t^{1/H}} = 0.\] \( (5.7) \)

The first step is to construct a suitable sequence.

**Lemma 5.5.** Under (5.7), there exists a sequence \( \{t_n\}_{n \geq 1} \) with the following three properties:

(i) \( t_{n+1} \geq t_n \left(1 + \left( \frac{\xi(t_n)}{t_n^{1/H}} \right)^{1/\beta} \right); \) \( (5.8) \)

(ii) \[\sum_{n=1}^{\infty} \varphi \left( \frac{\xi(t_n)}{t_n^{1/H}} \right) = +\infty; \] \( (5.9) \)

(iii) For all \( m \geq n \) large enough,
\[\frac{\xi(t_m)}{t_m^{2/H}} \leq 2 \frac{\xi(t_n)}{t_n^{2/H}}.\] \( (5.10) \)
Proof. The construction is given by induction over \( n \). We take \( t_1 = e^s \). Having constructed \( t_n \), we find \( s_n \geq t_n \) such that
\[
\sup \left\{ \frac{\xi(t)}{t^{2H}}, t \geq t_n \right\} = \frac{\xi(s_n)}{s_n^{2H}}.
\]
Using the continuity of \( \xi \) and (5.6), we know \( s_n \in (t_n, \infty) \). We then set
\[
t_{n+1} := s_n + s_n^{\gamma/\beta} \xi(s_n)^{1/\beta} = s_n \left( 1 + \left( \frac{\xi(s_n)}{s_n^{2H}} \right)^{1/\beta} \right).
\]
(5.11)
It is obvious that (5.8) holds. By the construction of \( s_{n-1} \) and by (5.6), we have that for \( m \geq n \),
\[
\frac{\xi(t_m)}{t_m^{2H}} \leq \frac{\xi(s_{n-1})}{s_{n-1}^{2H}}.
\]
By (5.6) and (5.11), we know for all \( n \) large enough, \( \xi(s_{n-1}) \leq \xi(t_n) \) and \( t_n^{2H} \leq 2s_n^{2H} \). Thus, (5.10) holds. To prove (5.9), we show that for all \( n \) large enough,
\[
I_n := \int_{t_n}^{t_{n+1}} \left( \frac{\xi(t)}{t^{2H}} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^{2H}} \right) \frac{dt}{t} \leq c_{5.1} \varphi \left( \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \right),
\]
where the constant \( c_{5.1} \) depends on \( H \) and \( \beta \) only.
We write
\[
I_n = \left[ \int_{t_n}^{s_n} + \int_{s_n}^{t_{n+1}} \right] \left( \frac{\xi(t)}{t^{2H}} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^{2H}} \right) \frac{dt}{t} =: I_n^1 + I_n^2.
\]
(5.13)
First, we have
\[
I_n^2 = \int_{s_n}^{t_{n+1}} \left( \frac{\xi(t)}{t^{2H}} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^{2H}} \right) \frac{dt}{t}
\]
\[
\leq (t_{n+1} - s_n) s_n^{-\gamma/\beta} \xi(s_n)^{-1/\beta} \varphi \left( \frac{\xi(t_{n+1})}{s_n^{2H}} \right)
\]
\[
= \varphi \left( \frac{\xi(t_{n+1})}{s_n^{2H}} \right).
\]
(5.16)
By (5.11), we know that \( t_{n+1} \leq 2s_n \) for all \( n \) large enough. Hence,
\[
\frac{\xi(t_{n+1})}{s_n^{2H}} = \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \left( \frac{t_{n+1}}{s_n} \right)^H = \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \left( 1 + \left( \frac{\xi(s_n)}{s_n^{2H}} \right)^{1/\beta} \right)^H
\]
\[
\leq \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \left( 1 + 2^{H/\beta} \left( \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \right)^{1/\beta} \right)^H
\]
\[
\leq \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \left( 1 + c_{5.2} \left( \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \right)^{1/\beta} \right),
\]
(5.17)
here \( c_{5.2} \) is a positive constant which only depends on \( H \) and \( \beta \).
By (3.6), (5.14) and (5.17), we know that there exists a constant \( c_{5.3} > 0 \) satisfying that
\[
I_n^2 \leq c_{5.3} \varphi \left( \frac{\xi(t_{n+1})}{t_{n+1}^{2H}} \right).
\]
Now, we turn to study $I^1_n$. By the construction of $s_n$, we have for any $t_n \leq t \leq s_n$,  
\[
\frac{\xi(t)}{t^H} \leq t^H \frac{\xi(s_n)}{s_n^H} \leq \frac{\xi(s_n)}{s_n^H}.
\]
By (5.6), we can assume $\xi(s_n)/s_n^H$ is arbitrarily small for all large $n$, so that Lemma 3.4 implies that for any $t_n \leq t \leq s_n$,  
\[
\left( \frac{\xi(t)}{t^H} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^H} \right) \leq \left( \frac{\xi(s_n)}{s_n^H} \right)^{-1/\beta} \varphi \left( \frac{t^H \xi(s_n)}{s_n^H} \right),
\]
and thus  
\[
I^1_n \leq \int_{t_n}^{s_n} \left( \frac{\xi(t)}{t^H} \right)^{-1/\beta} \varphi \left( \frac{\xi(t)}{t^H} \right) \frac{dt}{t} \leq \int_{t_n}^{s_n} \left( \frac{t^H \xi(s_n)}{s_n^H} \right)^{-1/\beta} \varphi \left( \frac{t^H \xi(s_n)}{s_n^H} \right) \frac{dt}{t}.
\]
Making the change of variable $t = u s_n$, we have  
\[
I^1_n \leq \int_0^1 \left( u^H a \right)^{-1/\beta} \varphi \left( u^H a \right) \frac{du}{u},
\]
where $a = \xi(s_n)/s_n^H$. It remains to prove that this later integral is at most $c_{5,4} \varphi(a)$ for some positive constant $c_{5,4}$. By (2.2), it suffices to prove that for any $c_{5,5} \in (0,1)$, there exists $c_{5,6} > 0$ such that  
\[
\int_{c_{5,5}}^1 \left( u^H a \right)^{-1/\beta} \varphi \left( u^H a \right) \frac{du}{u} \leq c_{5,6} \varphi(a).
\]
Setting $v = u^H$, it suffices to prove that  
\[
\int_0^1 \varphi(v a) dv = \int_0^1 \varphi((1-v)a) dv \leq c_{5,6} a^{1/\beta} \varphi(a). \tag{5.18}
\]
By the convexity of $\psi$ and (3.5), we have  
\[
\psi(a-a v) \geq \psi(a) + a v \psi'(a) \geq \psi(a) + \frac{a v}{K_2 a^{1/\beta}}.
\]
This implies that  
\[
\varphi((1-v)a) \leq \varphi(a) \exp \left( -\frac{v}{K_2 a^{1/\beta}} \right).
\]
Since  
\[
\int_0^1 \exp \left( -\frac{v}{K_2 a^{1/\beta}} \right) dv \leq K_2 a^{1/\beta},
\]
Eq. (5.18) is proved and the proof is complete.  

For each $n \geq 1$, we define $k(n)$ by  
\[
2^{k(n)} \leq \left( \frac{t_n^H}{\xi(t_n)} \right)^{1/\beta} \leq 2^{k(n)+1},
\]
and we set $I_k := \{ n \in \mathbb{N}; k(n) = k \}$ for any $k \geq 1$. By (5.6), we know that each $I_k$ is finite.  

We recall the constant $K_1$ given in (3.4). Without loss of generality, we can assume that $K_1 \geq c_{2.2}$, where $c_{2.2}$ is the constant in (3.1). For any $k \geq 1$, we set $N_k := \exp(2^{k-2}/K_1)$.  

By using the argument in [25, Proposition 4.2] or Proposition 4.1, we obtain the following result.
Lemma 5.6. Under (5.7), there exist a positive constant $c_{5,7}$ depending on $\beta$ and $H$ only and a set $J$ with the following properties:

(i) 
\[
\sum_{n \in J} \varphi \left( \frac{\xi(t_n)}{t_n^H} \right) = +\infty;
\] (5.19)

(ii) Given $n, m \in J$ with $n < m$ such that
\[
\operatorname{card}(I_{k(m)} \cap [n, m]) > N_{k(m)},
\] (5.20)
we have
\[
\frac{t_m}{t_n} \geq \exp \left( \exp \left( \frac{2 \max\{k(n), k(m)\}}{c_{5,7}} \right) \right).
\] (5.21)

By using Lemmas 5.5 and 5.6 and using the argument in Section 4.2.2 (also see [25, Section 5]), one can prove the necessity part of Theorem 2.2. The details are omitted.

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