Integral solutions of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space

M. Mallika Arjunan

Abstract
The main objective of this article is to examine the existence and uniqueness of integral solutions for a class of fractional order mixed type integro-differential equations with non-instantaneous impulses and non-densely defined linear operators in Banach spaces. Based on the Banach contraction principle, we develop the main results.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K37, 37L05, 47J35, 26A33.

1 Department of Mathematics, Vel Tech High Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Avadi-600062, Tamil Nadu, India.
*Corresponding author: arjunphd07@yahoo.co.in
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1. Introduction
Various certifiable issues include abrupt shifts in their states; these abrupt changes are known as impulsive impacts on issues. There are two types of impulsive processes in the present hypothesis: the instantaneous impulsive system, and the non-instantaneous impulsive system. In the instantaneous impulsive system, the duration of these abrupt shifts has nothing to do with the period of the whole phase of progression. Such abrupt impulses arise in cardiac pulses, vibrations, and natural events, while in non-instantaneous impulses the duration of these rapid changes proceeds over a short time interval. Hernandez & O’Regan [4] developed a new class of differential equations with non-instantaneous impulses and established the existence of mild and classical solutions. For more details, see for instance [1–3, 6].

Motivated by [1–5], in this paper, we consider a class of fractional mixed type integro-differential systems with non-instantaneous impulses of the form

\[ {}^cD^\alpha x(t) = Ax(t) + f(t,x(t),E_1x(t),E_2x(t)), \]
\[ t \in (s_i,s_{i+1}], \quad i = 0, 1, 2, \ldots, m \]
\[ x(t) = g_i(t,x(t)), \quad t \in (t_i,s_i], \quad i = 1, 2, \ldots, m \]
\[ x(0) = x_0, \]

where \( {}^cD^\alpha \) is the Caputo fractional derivative of order \( \alpha \in (0,1) \), \( A : D(A) \subset X \to X \) not necessarily a densely defined closed operator on the Banach space \( (X,\|\cdot\|) \), \( 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T \) are fixed numbers, \( g_i \in C \left( [t_i,s_i] \times X ; D(A) \right) \), \( f : [0,T] \times X^3 \to X \) is a non-linear function and the functions \( E_1 \) and \( E_2 \) are defined by

\[ E_1 x(t) = \int_0^t k(t,s,x(s))ds \quad \text{and} \quad E_2 x(t) = \int_0^T k(t,s,x(s))ds, \]

where \( k, \tilde{k} : \Delta \times X \to X \), \( \Delta = \{ (x,s) : 0 \leq s \leq x \leq \tau \} \) are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence and uniqueness results of problem (1.1).
Let us set \( J = [0, T] \), \( J_0 = [0, t_1] \), \( J_1 = (t_1, t_2] \), \( \ldots \), \( J_{m-1} = (t_{m-1}, t_m] \) and introduce the space \( PC(J, X) := \{ u : J \to X \mid u \in C(J, X), k = 0, 1, 2, \ldots, m \), and there exist \( x(t_k^+), x(t_k^-), k = 1, 2, \ldots, m \), with \( x(t_k^+) = x(t_k^-) \}. \) It is clear that \( PC(J, X) \) is a Banach space with the norm \( \|x\|_{PC} = \sup \{\|x(t)\| : t \in J \} \).

Let \( X_0 = D(A) \) and \( A_0 \) be the part of \( A \) in \( D(A) \) defined by

\[
D(A_0) = \{ x_1 \in D(A) : Ax_1 \in D(A) \}, A_0(x_1) = A(x_1)
\]

Throughout our analysis, the following hypotheses will be considered:

\[ \text{(H1)} \quad A : D(A) \subset X \to X \text{ satisfies the Hille-Yosida condition,} \]

that is, there exist two constants \( \omega \in \mathbb{R} \) and \( A_0 \geq 0 \) such that \( (\omega, \infty) \subset \rho(A) \) and

\[
\| (LI - A)^{-1} \|_{L(X, X)} \leq \frac{A_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1
\]

\[ \text{(H2)} \quad \text{The part } A_0 \text{ of } A \text{ generates a compact } C_0 \text{-semigroup } \{ Q(t) \}_{t \geq 0} \text{ in } X_0, \text{ which is uniformly bounded, that is, there exists } A \geq 1 \text{ such that } \sup_{t \in [0, \infty)} \|Q(t)\| < \Lambda.
\]

Let \( (U(t))_{t \geq 0} \) be the integrated semigroup generated by \( A \). It is to be noted that \( (U(t))_{t \geq 0} \) is a \( C_0 \)-semigroup on \( D(A) \) generated by \( A_0 \) and \( \|U'(t)\| \leq \Lambda e^{\omega t}, t \geq 0 \) and \( \Lambda, \omega \) are the constants used in the Hille-Yosida condition.

Let \( B_\lambda = \lambda R(\lambda, A) := (\lambda(I - A))^{-1} \). Then for all \( x_1 \in X_0, B_\lambda x_1 \to x_1 \) as \( \lambda \to \infty. \) Also from Hille-Yosida condition, it is clear that \( \lim_{\lambda \to \infty} \|B_\lambda\| \leq A_0. \)

Based on the above discussion along with \( \text{[2]} \), we define the integral solution of the given system \( \text{(1.1)} \).

**Definition 2.1.** \( \text{[2]} \) A function \( x \in PC(J, X) \) is said to be an integral solution of the Cauchy problem \( \text{(1.1)} \) if it satisfies \( x(0) = x_0 \in X_0, \quad x(t) = g_i(t, x(t)) \) for all \( t \in (t_i, s_i], i = 1, 2, \ldots, m : \)

\[
x(t) = U_a(t)x_0 + \lim_{\lambda \to \infty} \int_0^t V_a(t-s)B_\lambda f(s, x(s), E_1x(s), E_2x(s))ds, \quad t \in [0, t_1] \quad \text{and}
\]

\[
x(t) = U_a(t - s_i)g_i(s_i, x(s_i)) + \lim_{\lambda \to \infty} \int_{s_i}^t V_a(t-s)B_\lambda f(s, x(s), E_1x(s), E_2x(s))ds, \quad t \in (s_i, s_{i+1}],
\]

where

\[
U_a(t) = t_{0+}^{-\alpha}V_a(t), V_a(t) = t^{\alpha-1}W_a(t),
\]

\[
W_a(t) = \int_0^t \alpha \omega \Lambda a(\omega) Q(t^\omega d\omega.
\]

**Remark 2.2.** For any fixed \( t \geq 0 \), \( V_a(t) \) and \( U_a(t) \) are linear operators, and for any \( x_1 \in X_0, \)

\[
\|V_a(t)x_1\| \leq A_0^{-1} \|x_1\| \text{ and } \|U_a(t)x_1\| \leq \Lambda \|x_1\|.
\]
Let \( x, y \in PC(J, X) \). For \( t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m, \) we obtain
\[
\|Yx(t) - Yy(t)\| \\
\leq \|U\alpha(t-s_i)g_\alpha(s_i, x(s_i)) - U\alpha(t-s_i)g_\alpha(s_i, y(s_i))\| \\
+ \lim_{\lambda \to \infty} \int_s^t \|V\alpha(t-s)B_\alpha f(s, x(s), E_1 x(s), E_2 x(s)) \]
\[
- V\alpha(t-s)B_\alpha f(s, y(s), E_1 y(s), E_2 y(s))\| \, ds \\
\leq \Lambda L g_{\alpha} \|x-y\|_{PC} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \int_s^t \|A_{\alpha}-1\|L_{\alpha}(s)[1 + L_4 + L_2]ds \|x-y\|_{PC} \\
\leq \Lambda L g_{\alpha} \|x-y\|_{PC} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \left[ \int_s^t (t-s)^{\alpha-1} \right]^{1-\alpha} ds \\
\leq \Lambda L g_{\alpha} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \left[ (1+a)^{1-\alpha} \right] \left[ \|L_{\alpha}\|L_{\alpha}^{\alpha-1}(s, t_{i+1}, \mathbb{R}^+) \right] \|x-y\|_{PC}.
\]
\[
\|x-y\|_{PC}.
\]

For \( t \in [0, t_1] \)
\[
\|Yx(t) - Yy(t)\| \leq \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \left[ (1+a)^{1-\alpha} \right] \left[ \|L_{\alpha}\|L_{\alpha}^{\alpha-1}(s, t_{i+1}, \mathbb{R}^+) \right] \|x-y\|_{PC}.
\]

For \( t \in (t_i, t_{i+1}], i = 1, 2, \ldots, m, \) we have
\[
\|Yx(t) - Yy(t)\| \leq L g_{\alpha} \|x-y\|_{PC} \leq \Lambda L g_{\alpha} \|x-y\|_{PC}.
\]

From above, we observe that
\[
\|Y(x) - Y(y)\|_{PC} \leq C \|x-y\|_{PC}
\]
which implies that \( Y(\cdot) \) is a contraction and there exists a unique integral solution of the system (1.1).

\[\Box\]

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