NEW DEFORMATIONS ON SPHERICAL CURVES AND ÖSTLUND CONJECTURE

MEGUMI HASHIZUME AND NOBORU ITO

Abstract. In [11], a deformation of spherical curves called deformation type α was introduced. Then, it was showed that if two spherical curves $P$ and $P'$ are equivalent under the relation consisting of deformations of type RI and type RIII up to ambient isotopy, and satisfy certain conditions, then $P'$ is obtained from $P$ by a finite sequence of deformations of type α. In this paper, we introduce a new type of deformations of spherical curves, called deformation of type β. The main result of this paper is: Two spherical curves $P$ and $P'$ are equivalent under (possibly empty) deformations of type RI and a single deformation of type RIII up to ambient isotopy if and only if reduced($P$) and reduced($P'$) are transformed each other by exactly one deformation which is of type RIII, type α, or type β up to ambient isotopy, where reduced($Q$) is the spherical curve which does not contain a 1-gon obtained from a spherical curve $Q$ by applying deformations of type RI up to ambient isotopy.

1. Introduction

A spherical curve is the image of a generic immersion of a circle into a 2-sphere. Every spherical curve is transformed into the simple closed curve by a finite sequence of deformations, each of which is either one of type RI, type RII, or type RIII that is a replacement of a part of the spherical curve contained in a disk as shown in Figure 1 and ambient isotopies. The deformations of type RI, type RII, and type RIII are obtained from Reidemeister moves of type $\Omega_1$, type $\Omega_2$, and type $\Omega_3$ on knot diagrams by ignoring over/under informations near the crossing points. We note that each of the deformations means a replacement of a part of a immersed circle fixing the ambient 2-sphere.

In 2001, from viewpoints of singularity theory, Östlund [11] raised a problem that deformations of type RI and type RIII would be sufficient to describe a homotopy from every generic immersion $S^1 \to \mathbb{R}^2$ to the simple closed curve. In the same paper [11], from viewpoints of mathematical physics and topology, Östlund proved...
that if any Vassiliev-type function is invariant under $\Omega_1$ and $\Omega_3$, then it is invariant under $\Omega_2$. It was an evidence to support the above problem.

In 2014, Hagge and Yazinski [2] found a counterexample to this problem with 16 double points. Recently, in [6], Takimura and the second author of this paper obtained a counterexample with 15 double points and infinitely many counterexamples, and they further showed that there exist infinitely many equivalence classes of spherical curves under the relation consisting of deformations of type RI and type RIII up to ambient isotopy. For more details, see [7]. However, their arguments work for a restricted class of spherical curves. It is still difficult to detect whether a given pair of spherical curves are equivalent under the relation consisting of deformations of type RI and type RIII up to ambient isotopy. In [1], a deformation of spherical curve, called deformation of type $\alpha$ (Figure 3) which is a combination of deformations of type RI and type RII, was introduced and it was shown that if two spherical curves $P$ and $P'$ are equivalent under the relation consisting of deformations of type RI and type RIII up to ambient isotopy, and satisfy certain technical conditions, then $P'$ is obtained from $P$ by a finite sequence of deformations of type $\alpha$. In Section 2 of this paper, we introduce another type of deformations of spherical curves, called deformation of type $\beta$ (Figure 4). The main result of this paper (Theorem 1) shows that the three deformations (deformations of type RIII, type $\alpha$ and type $\beta$) are used to describe the equivalence class. For the statement of Theorem 1, we introduce one terminology.

A spherical curve $Q$ is called RI-minimal if $Q$ does not contain a 1-gon. For a deformation of type RI in Figure 1, we say that the type of the deformation from the left (the right, resp.) to the right (the left, resp.) is of type RI$^+$ (RI$^-$, resp.). For a spherical curve $Q$, let reduced($Q$) be an RI-minimal spherical curve obtained from $Q$ by successively applying deformations of type RI$^-$.  

**Theorem 1.** Two spherical curves $P$ and $P'$ are equivalent under (possibly empty) deformations of type RI and a single deformation of type RIII up to ambient isotopy if and only if reduced($P$) and reduced($P'$) are transformed each other by exactly one deformation which is of type RIII, type $\alpha$, or type $\beta$ up to ambient isotopy.

**Corollary 1.** Let $P$ and $P'$ be two spherical curves. Then, $P$ and $P'$ are equivalent under deformations of type RI and type RIII up to ambient isotopy if and only if there is a sequence of RI-minimal spherical curves $P_0(=\text{reduced}(P))$, $P_1$, $\ldots$, $P_n(=\text{reduced}(P'))$ such that $P_{i+1}$ is obtained from $P_i$ ($i = 0, 1, \ldots, n-1$) by a deformation of type RIII, type $\alpha$, or type $\beta$ up to ambient isotopy.

We note that Kobayashi-Kobayashi [10] introduced a new invariant called a stable double point number of pairs of spherical curves and proved that the invariant is non-trivial in a sense by using Theorem 1.

### 2. Preliminaries

In this section, we introduce some terminologies, notations and facts related to this work. We start with the following fact.

**Fact 1** ([4], [9]). For any spherical curve $P$, any pair of RI-minimal spherical curves obtained from $P$ by applying deformations of type RI$^-$ are mutually ambient isotopic.
A spherical curve is called *trivial* if it is a simple closed curve on $S^2$. Let $P$ be a non-trivial spherical curve and $m_1, m_2, \ldots, m_k$ the double points of $P$. Each component of $P \setminus \bigcup_{i=1}^{k} m_i$ is called an *arc*.

**Definition 1** (spherical curve with dots). Let $Q$ be a non-trivial spherical curve with (possibly empty) specified points, $\{p_i\}_{i=1}^{r}$ called *dot(s)* such that: each $p_i$ is not a double point; and each arc contains at most one dot. Then, $Q$ is called a spherical curve with dots $\{p_i\}_{i=1}^{r}$.

We note that a spherical curve sometimes denotes that a spherical curve with dots.

**Definition 2** (RI-minimal spherical curve with dots). Let $Q$ be a spherical curve with dots $\{p_i\}_{i=1}^{r}$. We say that the spherical curve with dots $\{p_i\}_{i=1}^{r}$ is RI-minimal if the boundary of each 1-gon of $Q$ contains (exactly one) dot.

It is easy to see that for each spherical curve $Q$ with dots $\{p_i\}_{i=1}^{r}$, we can obtain an RI-minimal spherical curve with dots by applying deformations of type RI$^-$ for 1-gons without dots. Further it is easy to see that (cf. Fact[1]) such spherical curve with dots are mutually ambient isotopic as spherical curve with dots.

**Definition 3** (connected sum). Let $P$ ($P'$, resp.) be a spherical curve with dot(s) $\{p_i\}$ ($\{p'_j\}$, resp.), where $\{p_i\}$ means $\{p_i\}_{i=1}^{r}$ ($\{p_j\}$ means $\{p'_j\}_{j=1}^{s}$, resp.). Let $d$ ($d'$, resp.) be a sufficiently small disk with the center $p_1$ ($p'_1$, resp.) where $d \cap P$ ($d' \cap P'$, resp.) consists of an arc properly embedded in $d$ ($d'$, resp.). Let $\hat{d} = cl(S^2 \setminus d)$ ($\hat{d}' = cl(S^2 \setminus d')$, resp.) and $\hat{P} = P \cap \hat{d}$ ($\hat{P}' = P' \cap \hat{d}'$, resp.); let $h: \partial \hat{d} \to \partial \hat{d}'$ be a homeomorphism such that $h(\partial P) = \partial \hat{P}'$ where $h$ is an orientation reversing or preserving homeomorphism (i.e., we consider the possibilities orientations with respect to both $S^1$ and $S^2$). Then, we have a spherical curve $\hat{P} \cup_h \hat{P}'$ with dots $\{(p_i) \setminus \{p_1\} \cup \{p'_j\} \setminus \{p'_1\}\}$ (if $r > 1$ or $r' > 1$), or a spherical curve $\hat{P} \cup_h \hat{P}'$ without a dot (that is a canonical spherical curve) (if $r = 1$ and $r' = 1$) in the oriented 2-sphere $\hat{d} \cup_h \hat{d}'$. The spherical curve $\hat{P} \cup_h \hat{P}'$ with dots $\{(p_i) \setminus \{p_1\} \cup \{p'_j\} \setminus \{p'_1\}\}$ or without a dot in the oriented 2-sphere is denoted by $P_{\#(p_1, p'_1), h} P'$ (or $P_{\#(p_1, p'_1), h} P'$ and is called a connected sum of the spherical curves $P$ (with dot(s) $\{p_i\}$) and $P'$ (with dot(s) $\{p'_j\}$) at the pair of dots $p_1$ and $p'_1$ (see Figure 2). The circle corresponding to $\partial \hat{P} (= \partial \hat{P}')$ is called the *decomposing circle* of the connected sum.

**Figure 2.** An example of a connected sum $P_{\#(p_1, p'_1), h} P'$ of two spherical curves $P$ with dots $\{p_1, p_2\}$ and $P'$ with dots $\{p'_1, p'_2\}$. In the right figure, the dotted circle denotes the decomposing circle which means $\partial \hat{P}_1 (= \partial P_2)$. 

[1]
A spherical curve \( P \) is prime if \( P \) is nontrivial and is not a connected sum of two nontrivial spherical curves. It is elementary to show that any spherical curve admits a prime connected sum decomposition, and hence a set of mutually disjoint decomposing circles corresponding to a prime connected sum decomposition.

**Definition 4** (deformations of type \( \alpha \), deformation of type \( \beta \)). For spherical curves \( P \) and \( P' \), we say that \( P' \) is obtained from \( P \) by a deformation of type \( \alpha \), if \( P' \) is obtained by replacing the part of \( P \) contained in a disk as in Figure 3. We say that \( P' \) is obtained from \( P \) by a deformation of type \( \alpha + \) (type \( \alpha - \), resp.), if the number of double points of \( P' \) is greater (less, resp.) than that of \( P \). See Figure 3.

\[ \xymatrix{ \alpha^+ \ar@{<->}[r] & \alpha^- } \]

**Figure 3.** deformation of type \( \alpha \)

Let \( 3_1 \) be a prime spherical curve with exactly three double points with a dot \( \{p\} \). Then, we say that \( P' \) is obtained from \( P \) by a deformation of type \( \beta^+ \) or \( \beta^+(m) \), where \( m \) is a non-negative integer, if \( P' \) is a connected sum \( P \# (q, p_2) \) \((\infty^m \# (p_1, p), h_3)\), where \( \{q\} \) is a dot in \( P \), \( \infty^m \) denotes an RI-minimal spherical curve with two dots \( \{p_1, p_2\} \) obtained from a trivial spherical curve by applying \( m \) deformations of type RI\(^+\). Then a connected sum \( \infty^m \# (p_1, p), h_3 \) is RI-minimal spherical curve with a dot \( \{p_2\} \). We say that \( P \) is obtained from the connected sum \( P' = P \# (\infty^m \# 3_1) \) by the deformation of type \( \beta^- \) or \( \beta^-(m) \).

By Definition 4, we may regard a deformation of type \( \beta^\pm(m) \) as a deformation fixing an ambient 2-sphere.

**Remark.** We note that \( \alpha^+ \) is expressed as a combination of a deformation of type RI\(^+\) and a deformation of type RI\(^-\), and that \( \beta^+(m) \) is expressed as a combination of \( m + 3 \) deformations of type RI\(^+\) and a deformation of type RI\(^-\).

**Example.** Let \( 4_1 \) be the spherical curve with a dot \( \{p\} \) as in Figure 4 (a). Then Figure 4 (b) is the list of spherical curves obtained from \( 4_1 \) (with a dot \( \{p\} \)) by applying \( \beta^+(2) \) up to ambient isotopy.

**Definition 5.** Let \( P \) be a spherical curve. A double point \( d \) is said to be nugatory if there exists a circle on \( S^2 \) which transversely intersects \( P \) only in \( d \). (Figure 5).

**Notation 1** \((f_c)\). For a spherical curve \( P \), let \( f_c(P) \) be the number of prime factors of \( P \).

Let \( Q \) and \( Q' \) be non-trivial spherical curves. Suppose that \( Q' \) is obtained from \( Q \) by a single deformation of type \( \alpha^+ \). Let \( d_1 \), and \( d_2 \) be the double points of \( Q \) relevant to the deformation of type \( \alpha^+ \). Then we have Lemmas 1 and 2 below.

**Lemma 1.** Suppose that \( Q \) is prime. Then \( Q' \) is prime.
Proof. For a contradiction, suppose that $Q'$ is not prime. Let $C$ be a set of circles that divides prime factors of $Q'$. Let $D$ be a small (open) disk ($\subset S^2$) where the deformation of type $\alpha^+$ is applied. We note that $D$ is a subsurface of $S^2$. Hence, since $Q$ is applied the deformation of type $\alpha^+$, there are $d_1$ and $d_2$ in $D$ as in the left figure of Figure 3 first. After $Q'$ was obtained from $Q$ by the deformation of type $\alpha^+$ within $D$, there are three double points of $Q'$ in $D$ as in the right figure of Figure 3. Then we claim that $D \cap C = \emptyset$. For a contradiction, suppose that there exists $C \in C$ such that $C \cap D \neq \emptyset$. Then $C \cap (D \cap Q')$ looks within $D$ as in Figure 4 since $C \cap (D \cap Q')$ consists of at most two points. However if $C \cap (D \cap Q')$ is as in Case a, then there is a trivial factor for the prime decomposition of $Q'$, a contradiction. If $C \cap (D \cap Q')$ is as in Case b, then it is easy to see that $Q'$ is a union of more than one spherical curves, a contradiction. Hence $D \cap C = \emptyset$. Let $d'_1, d'_2, d'_3$ be double points of $Q'$ relevant to the deformation of type $\alpha^+$. Since $D \cap C = \emptyset$, we see that $d'_1, d'_2, d'_3$ are contained in a prime factor. Hence, we may regard $C$ as a system of decomposing circles for $Q$. Obviously, $C$ gives a non-trivial connected sum decomposition of $Q$, a contradiction.
Lemma 2. Suppose that $d_1$ and $d_2$ are not nugatory. Then, $f_c(Q') = f_c(Q)$.

Proof. Let $C$ be a set of circles that divides prime factors of $Q$. Let $D$ be a small disk where the deformation of type $\alpha^+$ is applied. First we show that $D \cap C = \emptyset$. For a contradiction, suppose that $D \cap C \neq \emptyset$. Let $C$ be a component of $C$ such that $D \cap C \neq \emptyset$. Since $Q \cap C$ consists of two points, $C \cap (D \cap Q)$ consists of at most two points. Let $a_1, a_2$ be the edges of the 2-gon (hence $\partial a_i = d_1 \cup d_2$). Here, we note that $C \cap a_1 \neq \emptyset$, and $C \cap a_2 \neq \emptyset$, because if $C \cap a_i = \emptyset$, then we see that a prime factor of $P$ decomposed by $C$ is trivial (see Figure 7). Hence $C \cap D$ will look as in Figure 8. By using an isotopy along the triangle in Figure 8 we can obtain a circle $C'$ from $C$ such that $C' \cap D$ consists of a single transverse point $d_i$ ($i = 1, 2$), hence $d_1$ and $d_2$ are nugatory, a contradiction.

Finally, we show that $f_c(Q') = f_c(Q)$. Let $Q_0$ be a prime factor including $d_1$ and $d_2$. Then $Q$ admits the connected sum decomposition $Q = Q_0 \sharp Q_1 \sharp \cdots \sharp Q_s$ (possibly $Q_1 \sharp \cdots \sharp Q_s$ is empty) where $Q_1, \ldots, Q_s$ are prime factors. Since $D \cap C = \emptyset$, $Q'$ admits a connected sum decomposition $Q' = Q'_0 \sharp Q_1 \sharp \cdots \sharp Q_s$, where $Q'_0$ is obtained from $Q_0$ by a deformation of type $\alpha^+$ performed within $D$. These facts together with Lemma 4 shows that $Q'_0$ is prime. Therefore $f_c(Q') = f_c(Q) (= s + 1)$.

\[ \square \]
Let $P$ and $P'$ be spherical curves. Suppose that $P'$ is obtained from $P$ a single deformation of type $R III$. Let $c_1$, $c_2$ and $c_3$ be the three double points of $P$ relevant to the deformation of type $R III$. Then we have Lemma 3 below.

**Lemma 3.** If some $c_i$ is nugatory, then $f_c(P') < f_c(P)$ (in particular, $f_c(P') = f_c(P) + 1$ or $f_c(P') = f_c(P) + 2$).

**Proof.** Note that we have essentially the following two cases.

**Case 1:** Exactly one of $c_1$, $c_2$, $c_3$, say $c_1$ is nugatory.
Since $c_1$ is nugatory, $P$ admits a connected sum decomposition $P_1 \# \infty \# P_2$ where $P_1$ and $P_2$ are obtained from $P$ by smoothing at $c_1$ and $\infty$ means $\infty^3$ in Definition 4.

Hence $f_c(P) = f_c(P_1) + 1 + f_c(P_2)$.

On the other hand, by Figure 9, it is directly observed that $P'$ admits a connected sum decomposition $P_1' \# P_2'$ where $P_2'$ is obtained from $P_2$ by a deformation of type $\alpha^+$. Hence $f_c(P') = f_c(P_1) + f_c(P_2)$.

Since $c_2$ and $c_3$ are not nugatory, by Lemma 2 we see that $f_c(P_2') = f_c(P_2)$. Hence $f_c(P) = f_c(P_1) + 1 + f_c(P_2) = f_c(P_1) + 1 + f_c(P_2) = f_c(P') + 1$. Then we see $f_c(P') < f_c(P)$.

![Figure 9](image)

**Figure 9.** $P$ admits a connected sum decomposition $P_1 \# \infty \# P_2$, whereas $P'$ admits a connected sum decomposition $P_1' \# P_2'$.

**Case 2:** At least two of $c_1$, $c_2$, $c_3$, say $c_1$, $c_2$, are nugatory.
Since $c_1$, $c_2$ are nugatory, it is easy to see that $c_3$ is nugatory also. (Recall the relationship between spherical curves with dots and connected sums as in Definition 3.) Let $\infty^3$ be a spherical curve with dots as Figure 10. Then $P$ admits a connected sum decomposition $P_1' \# P_2' \# P_3' \# \infty^3$. Then $f_c(P) = f_c(P_1) + f_c(P_2) + f_c(P_3) + 3$ since $f_c(\infty^3) = 3$.

On the other hand, since $3_1$ is obtained from $\infty^3$ by a deformation of type $R III$, $P'$ admits a connected sum decomposition $P_1' \# P_2' \# P_3' \# 3_1$, where $3_1$ is as in Definition 4. Then $f_c(P') = f_c(P_1) + f_c(P_2) + f_c(P_3) + f_c(3_1) = f_c(P_1) + f_c(P_2) + f_c(P_3) + 1$. Hence $f_c(P) = f_c(P') + 2$. Then we see $f_c(P') < f_c(P)$.

![Figure 10](image)
Notation 2. Let $P$ and $P'$ be two spherical curves that are related by a finite sequence of deformations of type RI and type RIII, i.e., there exists a finite sequence of spherical curves $P = P_0, P_1, \ldots, P_r = P'$, where $P_i$ is obtained from $P_{i-1}$ by a single deformation of type RI or type RIII. Then, $O_{P_i}$ denotes the deformation from $P_{i-1}$ to $P_i$, and these settings are expressed by

$$P = P_0 \xrightarrow{O_{P_1}} P_1 \xrightarrow{O_{P_2}} \cdots \xrightarrow{O_{P_r}} P_r = P'.$$

In the reminder of this section, for the proof of Theorem 1, we prepare Claim 1. Let $P$ be a spherical curve and $f_c(Q) \geq f_c(Q')$. Suppose that $Q'$ is obtained from $Q$ by a single deformation of type RIII and $f_c(Q') \geq f_c(Q)$. Let $D$ be a small disk where the deformation of type RIII is applied. Suppose that $c_1, c_2$ and $c_3$ are the double points in $Q \cap D$. Since $f_c(Q') \geq f_c(Q)$, every $c_\lambda$ is not a nugatory double point ($\lambda = 1, 2, 3$) by Lemma 3.

On the other hand, by Fact 1, there exists a finite sequence of deformations of type RI from $Q$ to reduced($Q$). Since $c_1, c_2$ and $c_3$ are not nugatory, this sequence does not affect to $Q \cap D$. Then, let $R$ be the spherical curve obtained from reduced($Q$) by the single deformation of type RIII within $D$.

Moreover, since the sequence consisting of deformations of type RI does not affect to $Q \cap D$, applying the sequence of deformations of type RI does not affect $D$, these facts imply $R \cap D = Q' \cap D$.

Thus, $R$ and $Q'$ are ambient isotopic.

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1.

Proof of if part of the statement of Theorem 1

By Remark of Definition 4, a deformation of type $\alpha (\beta \pm (m), \text{resp.})$ consists of a single deformation of type RIII and a single deformation of type RI ($3 + m$ deformations of type RI, resp.). This together with the definition of reduced(·) implies if part of the statement of Theorem 1.

Proof of only if part of the statement of Theorem 1

By the assumption, there exists a finite sequence denoted by:

$$P = P_0 \xrightarrow{O_{P_1}} P_1 \xrightarrow{O_{P_2}} \cdots \xrightarrow{O_{P_r}} P_r = P'$$

where $\{O_{P_i}\}_{i=1}^r$ consists deformations of type RI and a single deformation of type RIII (Notation 2). Let $j$ be the integer such that $O_{P_j}$ is the deformation of type RIII.
By exchanging $P$ and $P'$, if necessary, we may suppose that $f_\ast(P_j) \geq f_\ast(P_{j-1})$. Hence it is clear that the next claim gives the proof of the only if part of the statement of Theorem I. Hence in the remainder of this section, we give a proof of Claim 2.

**Claim 2.** If $f_\ast(P_j) \geq f_\ast(P_{j-1})$, then reduced($P$) is obtained from reduced($P'$) by a single deformation of type $\mathbb{RI}_3$, type $\alpha^-$ or type $\beta^-$. 

Let $D$ be a small disk to which $Op_j$ is applied. Let $f$ be the generic immersion satisfying that $f(S^1) = P_{j-1}$. By definition, since $D$ is a disk containing a 3-gon corresponding to a deformation of type $\mathbb{RI}_3$, $f^{-1}(D)$ consists of three components $J_1, J_2$ and $J_3$. Then the number of components of $S^1 \setminus f^{-1}(D)$ is also three. The three components of $S^1 \setminus f^{-1}(D)$ are denoted by $J_1, J_2,$ and $J_3$. Let $c_1$ ($c_2,$ $c_3,$ resp.) be the double point that is the intersection of $f(J_2)$ and $f(J_3)$ ($f(J_3)$ and $f(J_1)$, $f(J_1)$ and $f(J_2)$, resp.). Let $\delta$ be the number of the components of $P_{j-1} \setminus D = P_{j-1} \cap D'$, where $D' = cl(S^2 \setminus D)$.

Let $g$ be a generic immersion satisfying that $g(S^1) = \text{reduced}(P_{j-1})$ such that $S^1 \setminus g^{-1}(D)$ consists of the tree components $I_1, I_2$ and $I_3$.

**Case 1:** $\delta = 1$ (i.e., $f(I_\lambda) \cap f(I_\mu) \neq \emptyset$ and $f(I_\mu) \cap f(I_\nu) \neq \emptyset$, where $\lambda, \mu$ and $\nu$ are mutually distinct numbers in $\{1, 2, 3\}$).

Since $\delta = 1$, we see that each $c_i$ is not nugatory. This together with Fact 1 shows that the sequence of the deformations of type $RI^-$ from $P_{j-1}$ to reduced($P_{j-1}$) dose not affect $D$. Hence we may apply $Op_j$ to reduced($P_{j-1}$), and $S_\mathbb{RI}_3$ denotes the obtained spherical curve.

**Claim 3.** $S_\mathbb{RI}_3$ is RI-minimal.

**Proof.** By the definition of a deformation of type $\mathbb{RI}_3$, there is no 1-gon completely included in $S_\mathbb{RI}_3 \cap D$. Since the sequence from $P_{j-1}$ to reduced($P_{j-1}$) resolves 1-gons in $P_{j-1} \cap D'$, there is no 1-gon completely included in $S_\mathbb{RI}_3 \cap D'$. If there exists a 1-gon on the 2-sphere, then the 1-gon intersects $S_\mathbb{RI}_3 \cap D$. Here, for example, the 1-gon will look as in Figure 11. However, this contradicts the condition of Case 1.

![Figure 11](image-url)
the proof of Claim 1 with regarding $Q = P_{j-1}$, $Q' = P_j$, $R = S_\mathbb{W}$ and $\tilde{P}_j = \tilde{Q}'$, it is clear that $S_\mathbb{W}$ is ambient isotopic to $\tilde{P}_j$. This fact together with Claim 3 we see that $\tilde{P}_j$ is ambient isotopic to reduced($P_j$). On the other hand, the sequence $\{Op_i\}_{i=j+1}^{j-1}$ from $P_j$ to $P_j = P'$ consist of deformations of type RI, by Fact 1 it is clear that reduced($P'$) is ambient isotopy. Hence reduced($P'$) = $\tilde{P}_j$ up to ambient isotopy.

On the other hand, since the sequence $\{Op_i\}_{i=2}^{j-1}$ from $P = P_0$ to $P_{j-1}$ consists of deformations of type RI, by Fact 1 it is clear that reduced($P$) = reduced($P_{j-1}$) up to ambient isotopy.

Recall that $S_\mathbb{W}$ is obtained from reduced($P_{j-1}$) by applying $Op_j$. This fact together with the above shows that reduced($P'$) is obtained from reduced($P$) by a single deformation of type RI.$\mathbb{W}$

Case 2: $\delta = 2$ (i.e., $f(I_3) \cap f(I_\mu) \neq \emptyset$, $f(I_\nu) \cap f(I_\sigma) = \emptyset$ and $f(I_\nu) \cap f(I_\lambda) = \emptyset$, where $\lambda$, $\mu$ and $\nu$ are mutually distinct numbers in $\{1, 2, 3\}$). Without loss of generality, we may suppose that $f(I_1) \cap f(I_2) \neq \emptyset$, $f(I_2) \cap f(I_3) = \emptyset$, and $f(I_3) \cap f(I_1) = \emptyset$.

Case 2-1: $g(I_3)$ is a simple arc.

By Fact 1 there exists a sequence of deformations of type RI$^{-}$ from $P_{j-1}$ to reduced($P_{j-1}$). By the assumption: $f_c(P_j) \geq f_c(P_{j-1})$, we see that $c_1, c_2, c_3$ are not nauty (Lemma 3). Hence $P_{j-1} \cap G$ is not affected by the sequence. Then, we apply the sequence of deformations of type RI$^{-}$ to $P_j (\subset S^2)$. Let $\tilde{P}_j$ be the spherical curve obtained from $P_j$ by applying the sequence. Moreover let $S_\mathbb{W}$ be a spherical curve obtained from reduced($P_{j-1}$) by applying a deformation of type RI.$\mathbb{W}$ to reduced($P_{j-1}$) $\cap D$. Since $f_c(P_j) \geq f_c(P_{j-1})$, we can apply Claim 1 with regarding $Q = P_{j-1}$, $Q' = P_j$, $R = S_\mathbb{W}$ and $\tilde{P}_j = Q'$, we have:

Claim 4. $\tilde{P}_j$ and $S_\mathbb{W}$ are ambient isotopic.

By the assumption of Case 2-1, $g(I_3)$ is a simple arc. Let $E$ be a sufficiently small disk completely including $g(I_3)$ and the 3-gon of reduced($P_{j-1}$) $\cap D$. We note that the configuration $(E, P_{j-1} \cap E)$ is the same as that of the right figure of Figure 3 since a simple arc connects a 3-gon in $E$. Hence we can apply $\alpha^{-}$ to reduced($P_{j-1}$) in $E$. Then, we denote the resulting spherical curve by $S_\alpha$. Then, by the definition of $\alpha^{-}$, we have:

Claim 5. $S_\alpha$ is obtained from $S_\mathbb{W}$ by a deformation of type RI$^{-}$ performed within $E$.

Furthermore, we see Claim 4 below.

Claim 6. $S_\alpha$ is RI-minimal.

Proof. Assume, for a contradiction, that $S_\alpha$ admits a 1-gon, denoted by $G$. By the construction of $S_\alpha$, we see that $G$ is not completely included in $\mathbb{T}$. Hence $G \cap D \neq \emptyset$, and this shows that $G$ is as the right side in Figure 3 However, it is easy to see that the configuration implies $g(I_1) \cap g(I_2) = \emptyset$, a contradiction. \[\square\]

By Fact 1 reduced($P$) = reduced($P_{j-1}$) up to ambient isotopy. Then $S_\alpha$ is regarded as a spherical curve obtained from reduced($P$) by applying a deformation of type $\alpha^{-}$ in $E$. Furthermore, by Claim 3 $S_\alpha = \tilde{P}_j = Q'$, and the 3-gon of reduced($P_{j-1}$) $\cap D$. Without loss of generality, we may suppose that $g(I_3)$ is not affected by the sequence. Then, we apply the sequence of deformations of type RI$^{-}$ to $P_j (\subset S^2)$. Let $\tilde{P}_j$ be the spherical curve obtained from $P_j$ by applying the sequence. Moreover let $S_\mathbb{W}$ be a spherical curve obtained from reduced($P_{j-1}$) by applying a deformation of type RI.$\mathbb{W}$ to reduced($P_{j-1}$) $\cap D$. Since $f_c(P_j) \geq f_c(P_{j-1})$, we can apply Claim 1 with regarding $Q = P_{j-1}$, $Q' = P_j$, $R = S_\mathbb{W}$ and $\tilde{P}_j = Q'$, we have:

Claim 4. $\tilde{P}_j$ and $S_\mathbb{W}$ are ambient isotopic.

By the assumption of Case 2-1, $g(I_3)$ is a simple arc. Let $E$ be a sufficiently small disk completely including $g(I_3)$ and the 3-gon of reduced($P_{j-1}$) $\cap D$. We note that the configuration $(E, P_{j-1} \cap E)$ is the same as that of the right figure of Figure 3 since a simple arc connects a 3-gon in $E$. Hence we can apply $\alpha^{-}$ to reduced($P_{j-1}$) in $E$. Then, we denote the resulting spherical curve by $S_\alpha$. Then, by the definition of $\alpha^{-}$, we have:

Claim 5. $S_\alpha$ is obtained from $S_\mathbb{W}$ by a deformation of type RI$^{-}$ performed within $E$.

Furthermore, we see Claim 4 below.

Claim 6. $S_\alpha$ is RI-minimal.

Proof. Assume, for a contradiction, that $S_\alpha$ admits a 1-gon, denoted by $G$. By the construction of $S_\alpha$, we see that $G$ is not completely included in $\mathbb{T}$. Hence $G \cap D \neq \emptyset$, and this shows that $G$ is as the right side in Figure 3 However, it is easy to see that the configuration implies $g(I_1) \cap g(I_2) = \emptyset$, a contradiction. \[\square\]

By Fact 1 reduced($P$) = reduced($P_{j-1}$) up to ambient isotopy. Then $S_\alpha$ is regarded as a spherical curve obtained from reduced($P$) by applying a deformation of type $\alpha^{-}$ in $E$. Furthermore, by Claim 3 $S_\alpha = \tilde{P}_j = Q'$. Moreover, by Claim 5 $S_\alpha = \tilde{P}_j = Q'$, we have:

Claim 4. $\tilde{P}_j$ and $S_\mathbb{W}$ are ambient isotopic.
Case 2-2: of type α

The arguments in Case 1 work to show that reduced(P) sequence from λ ambient isotopy.

Case 3: by a deformation of type RI I I. Details of the arguments are left to the reader.

Case 3-1: For some λ, µ (λ, µ ∈ {1, 2, 3}, λ ≠ µ), each of g(Iλ) and g(Iµ) is a simple arc. Without loss of generality, we may suppose that (λ, µ) = (1, 2).

By Fact 11 there exists a finite sequence of deformations of type RI− from Pj−1 to reduced(Pj−1). By the assumption: fc(Pj) ≥ fc(Pj−1), we see that c1, c2, c3 are not nugatory (Lemma 11). Hence Pj−1 ∩ D is not affected by the sequence. Then, we apply the sequence of deformations of type RI− to Pj (⊂ S2). Let Pj be the spherical curve obtained from Pj by applying the sequence. Moreover let S[I] be a spherical curve obtained from reduced(Pj−1) by applying a deformation of type RI to reduced(Pj−1) ∩ D. By the assumption: fc(Pj) ≥ fc(Pj−1), we apply Claim 11 with regarding Q = Pj−1, Q′ = Pj, R = S[I] and Pj = Q′, we have;

Claim 7. Pj and S[I] are ambient isotopic.

By the assumption of Case 3-1, g(Iλ) (λ = 1, 2) is a simple arc. Let F be a sufficiently small disk completely including g(I1), g(I2) and the 3-gon of reduced(Pj−1) ∩ D. We note that the configuration (F, reduced(Pj−1) ∩ F) is as in Figure 11. Then this figure shows that reduced(Pj−1) admits a connected sum decomposition reduced(Pj−1) ⨿ 31, where ⨿ denotes a certain decomposition. Here, we note that reduced(Pj−1) ⨿ may not be RI-minimal. For example, we obtained reduced(Pj−1) as in the left figure of Figure 11. Then the spherical curve admits a connected sum decomposition as in the right figure of Figure 11. However, since reduced(Pj−1) is RI-minimal, we see that reduced(Pj−1) ⨿ 31 may be further decompose into reduced(Pj−1) ⨿ ⨿ ⨿ 31, where reduced(Pj−1) ⨿ is RI-minimal. These show that reduced(Pj−1) is obtained from reduced(Pj−1) ⨿ by deformation of type β+.
On the other hand, since $S$ is obtained reduced($P_{j-1}$) by deformation of type $R\exists$, it is directly obtained from Figure 13 that reduced($S$) and reduced($P_{j-1}$)** are ambient isotopic.

These facts together with Claim 7 shows that reduced($P_j$) is obtained from reduced($P_{j-1}$) by a deformation of type $\beta^-$. Hence reduced($P'$) is obtained from reduced($P$) by a deformation of type $\beta^-$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13}
\caption{Figure 13.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14}
\caption{Figure 14.}
\end{figure}

- **Case 3-2:** $g(I_\lambda)$ is a simple arc each of the others is not a simple arc. Without loss of generality, we may suppose that $g(I_1)$, $g(I_2)$ are not simple arcs by and $g(I_3)$ is a simple arc.

  In this case, it is easy to see that the arguments in Case 2-1 works to show that reduced($P'$) is obtained from reduced($P$) by a single deformation of type $\alpha^-$. Details are left to the reader.

- **Case 3-3:** Every $g(I_\lambda)$ ($\lambda = 1, 2, 3$) is not a simple arc.

  The treatment of this case is essentially the same as the arguments in Case 1. The difference is;

  In Case 1, we appealed the connectedness of $P_{j-1} \setminus D$ (i.e., $\delta = 1$), for contradictions. Here we should use the condition of Case 3-3 for the connectedness of $P_{j-1} \setminus D$ to derive contradictions.

  Then we can show that reduced($P'$) is obtained from reduced($P$) by deformation of type $R\exists$.

  For every case, the statement is proved, which completes the proof.

\[\square\]
Acknowledgements

The authors would like to thank Professor Tsuyoshi Kobayashi for giving many advice. The authors also thank Professor Kouki Taniyama for comments. The authors also thank Mr. Yusuke Takimura for sharing his table and comments. M. H. is a researcher supported by Meiji University Organization for the Strategic Coordination of Research and Intellectual Properties.

References

[1] Y. Funakoshi, M. Hashizume, N. Ito, T. Kobayashi, and H. Murai, A distance on the equivalence classes of spherical curves generated by deformations of type RI, J. Knot Theory Ramifications, 27, No.12 (2018), 1850066, 22pp.
[2] T. Hagge and J. Yazinski, On the necessity of Reidemeister move 2 for simplifying immersed planar curves, Banach Center Publ. 103 (2014), 101–110.
[3] N. Ito, Knot projections, CRC Press, Boca Raton, FL, 2016.
[4] N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, J. Knot Theory Ramifications 22 (2013), 135–85, 14pp.
[5] N. Ito and Y. Takimura, Sub-chord diagrams of knot projections, Houston J. Math. 41 (2015), 701–725.
[6] N. Ito and Y. Takimura, On a nontrivial knot projection under (1, 3) homotopy, Topology Appl. 210 (2016), 22–28.
[7] N. Ito and Y. Takimura, RI number of knot projections, Kobe J. Math., accepted.
[8] N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, Osaka J. Math. 52 (2015), 617–646.
[9] M. Khovanov, Doodle groups, Trans. Amer. Math. Soc., 349, (1997), 2297–2315.
[10] T. Kobayashi and S. Kobayashi, Stable Double point numbers of pairs of spherical curves, JP J. of Geometry and Topology, 22, No. 2 (2019), 129–163.
[11] Olof-Peter Östlund, Invariants of knot diagrams and diagrammatic knot invariants. Thesis (Ph.D.)–Uppsala Universitet (Sweden). ProQuest LLC, Ann Arbor, MI, 2001. 67 pp.

4-21-1 NAKANO, NAKANO-KU, TOKYO, JAPAN 164-8525, MEIJI UNIVERSITY ORGANIZATION FOR THE STRATEGIC COORDINATION OF RESEARCH AND INTELLECTUAL PROPERTIES

Current address: TAKABATAKECHYO, NARA-SHI, NARA-KEN, JAPAN, 8528, NARA UNIVERSITY OF EDUCATION CENTER FOR EDUCATIONAL RESEARCH OF SCIENCE AND MATHEMATICS
E-mail address: hashizume.megumi.y9@nara-edu.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1, KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN
E-mail address: noboru@ms.u-tokyo.ac.jp