THE BRAUER GROUP OF 1-MOTIVES

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Abstract. Let $S$ be a site. Our first main result is that if $F$ is an abelian sheaf on $S$, the Picard 2-stack $\mathcal{G}erbe(F)$ of $F$-gerbes is equivalent (as Picard 2-stack) to the Picard 2-stack associated to the complex $\tau_{\leq 0} R\Gamma(S, F[2])$.

Let $X$ be a stack over $S$ to which we associate the site $S(X)$. Using sheaves theory over stacks, we introduce the Picard 2-stack $\mathcal{G}erbe(F)$ of $F$-gerbes on $X$, with $F$ an abelian sheaf on $X$. As a corollary of our first main theorem we get that $F$-equivalence classes of $F$-gerbes on $X$ are parametrized by cohomological classes of $H^2(X, F)$.

Let $S$ be an arbitrary scheme and let $S_{et}$ be the étale site on $S$. Let $X = (X, O_X)$ be a locally ringed $S$-stack and denote by $S_{et}(X)$ its associated étale site. Always using sheaves theory over stacks, we define the Brauer group $\text{Br}(X)$ of the locally ringed $S$-stack $X$ as the group of equivalence classes of Azumaya algebras over $X$. Using our first main result, we construct an injective homomorphism $\delta : \text{Br}(X) \to H^2_{et}(X, \mathbb{G}_m, X)$ which extends to stacks Grothendieck’s injective homomorphism for schemes.

Let $\mathcal{G}erbe_S(\mathbb{G}_m, M)$ be the Picard 2-stack of $\mathbb{G}_m, M$-gerbes on the Picard stack $\mathcal{M}$ associated to a 1-motive $M$ defined over a scheme $S$. Using the corollary of our first main result, which asserts that $\mathbb{G}_m, M$-equivalence classes of $\mathbb{G}_m, M$-gerbes on $M$ are parametrized by cohomological classes of $H^2(M, \mathbb{G}_m, M)$, we can prove our second main theorem: 1-motives, which are defined over a connected, reduced, geometrically unibranch and noetherian scheme $S$, satisfy the generalized Theorem of the Cube for any prime $\ell$ distinct from the residue characteristics of $S$.

Let $M = [u : X \to G]$ be a 1-motive defined over a noetherian scheme $S$. We define the Brauer group $\text{Br}(M)$ of $M$ as the Brauer group of the Picard $S$-stack $M$ associated to $M$. Our third main result is the following: if the base scheme $S$ is normal and noetherian, and if the extension $G$ underlying $M$ satisfies the generalized Theorem of the Cube for a prime $\ell$ distinct from the residue characteristics of $S$, then the $\ell$-primary component of $\ker[H^2_{et}(M, \mathbb{G}_m, M) \to H^2_{et}(S, \mathbb{G}_m, S)]$ is contained in $\text{Br}(M)$.

Contents

Introduction  2
Acknowledgment  4
Notation  4
1. Recall on sheaves, gerbes and Picard stacks on a stack  6
2. Gerbes with abelian band on a stack  7
3. The Brauer group of a locally ringed stack  11
4. Gerbes and Azumaya algebras over 1-motives  13
5. The generalized Theorem of the Cube for 1-motives and its consequences  16
6. Proof of Theorem 0.3  21
References  25

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Introduction

In class field theory, the Brauer group of a field $k$ classifies central simple algebra over $k$. This definition was generalized to schemes (and even to locally ringed toposes) by Grothendieck who has defined the Brauer group $\text{Br}(X)$ of a scheme $X$ as the group of similarity classes of Azumaya algebras over $X$. In [13, I, §1] Grothendieck constructed an injective group homomorphism

$$\delta : \text{Br}(X) \rightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)$$

from the Brauer group of $X$ to the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ which classifies the $\mathbb{G}_m$-gerbes over $X$. This homomorphism is not in general bijective, as pointed out by Grothendieck in [18, II, §2], where he found a scheme $X$ whose Brauer group is a torsion group but whose étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is not torsion. However, since the hypothesis of quasi-compactness on $X$ implies that the elements of $\delta(\text{Br}(X))$ are torsion elements of $H^2_{\text{ét}}(X, \mathbb{G}_m)$, Grothendieck asked in loc. cit. the following question:

QUESTION: For a quasi-compact scheme $X$, is the image of $\text{Br}(X)$ via the homomorphism $\delta$ the torsion subgroup $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{Tors}}$ of $H^2_{\text{ét}}(X, \mathbb{G}_m)$?

Grothendieck showed that if $X$ is regular, the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is a torsion group, and so under this hypothesis the question becomes:

QUESTION': For a regular scheme $X$, is $\delta(\text{Br}(X)) = H^2_{\text{ét}}(X, \mathbb{G}_m)$?

The following well-known results are related to this question: Auslander and Goldman proved that if $X$ is a regular scheme of dimension $\leq 2$, then the Brauer group of $X$ is all of $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Moreover, if $X$ is an smooth variety over a field, then $\delta(\text{Br}(X)) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Gabber showed that the Brauer group of a quasi-compact scheme $X$ endowed with an ample invertible sheaf is isomorphic to $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{Tors}}$ (see [11]). If $A$ is an abelian scheme, which is defined over a noetherian scheme $S$ and which satisfies the generalized Theorem of the Cube for a prime number $\ell$ distinct from the residue characteristics of $S$, Hoobler proved in [20, Thm 3.3] that the $\ell$-primary component of $\ker[H^2_{\text{ét}}(\epsilon) : H^2_{\text{ét}}(A, \mathbb{G}_m) \rightarrow H^2_{\text{ét}}(S, \mathbb{G}_m)]$, where $\epsilon : S \rightarrow A$ is the unit section of $A$, is contained in the Brauer group of $A$.

The aim of this paper is to investigate Grothendieck’s QUESTION in the case of 1-motives defined over a scheme $S$. We proceed in the following way:

Let $S$ be a site. Let $X$ be a stack in groupoids over $S$. In Section 1 we associate to $X$ the site $S(X)$, which allows us to introduce the notion of sheaf and gerbe on a stack. In [14, Chp IV] Giraud defined and studied $F$-gerbes with $F$ an abelian sheaf on the site $S$. Endowed with the contracted product, $F$-gerbes build a Picard 2-stack that we denote $\text{Gerbe}_S(F)$. We start Section 2 associating to the Picard 2-stack $\text{Gerbe}_S(F)$ its classifying groups $\text{Gerbe}^i_S(F)$ for $i = 2, 1, 0$, which are abelian groups. In particular $\text{Gerbe}^2_S(F)$ is the abelian group of $F$-equivalence classes of $F$-gerbes. Our first main result furnishes the homological interpretation of $F$-gerbes.

Theorem 0.1. Let $F$ be an abelian sheaf on a site $S$. Then the Picard 2-stack $\text{Gerbe}_S(F)$ of $F$-gerbes is equivalent (as Picard 2-stack) to the Picard 2-stack associated to the complex $\tau_{\leq 0}\Gamma(S, F[2])$, where $F[2] = [F \rightarrow 0 \rightarrow 0]$ with $F$ in degree $-2$ and $\tau_{\leq 0}$ is the good truncation in degree 0:

$$\text{Gerbe}_S(F) \cong \text{2st}(\tau_{\leq 0}\Gamma(S, F[2])).$$

In particular, for $i = 2, 1, 0$, we have an isomorphism of abelian groups between the $i$-th classifying group $\text{Gerbe}^i_S(F)$ and the cohomological group $H^i(S, F)$.

The link between Picard 2-stacks and length 3 complexes of abelian sheaves is done in [31, Cor 6.5]. This theorem contains the following classical result: $F$-equivalence classes of $F$-gerbes are parametrized by elements of the cohomological group $H^2(S, F)$. Always in Section
By Theorem 3.5 we have an injective group homomorphism \( \delta \): \( \text{Br}(\mathcal{X}) \rightarrow H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m, \mathcal{M}). \)

which extends Grothendieck’s group homomorphism \( \text{Br}(\mathcal{M}) \rightarrow H^2_{\text{ét}}(\mathcal{M}, \mathbb{G}_m, \mathcal{M}) \). By Theorem 3.5 we have an injective group homomorphism \( \delta : \text{Br}(\mathcal{M}) \rightarrow H^2_{\text{ét}}(\mathcal{M}, \mathbb{G}_m, \mathcal{M}). \)

Let \( M = [u : X \rightarrow G] \) be a 1-motive defined over a scheme \( S \), with \( X \) an \( S \)-scheme which is, locally for the étale topology, a constant group scheme defined by a finitely generated \( \mathbb{Z} \)-module, \( G \) an extension of an abelian \( S \)-scheme by an \( S \)-torus, and finally \( u : X \rightarrow G \) a morphism of \( S \)-group schemes. Since in [14, Exposé XVIII, §1.4] Deligne associates to any extension \( \mathcal{M} \) of abelian sheaves on the site \( X \) an Azumaya algebra over \( X \), we get that \( \text{Br}(\mathcal{M}) \) is injective (see Definition 5.1). In Section 5 we study the consequences of the generalized Theorem of the Cube for a prime number \( \ell \) distinct from the residue characteristics of \( S \).

Theorem 0.2. 1-motives, which are defined over a connected, reduced, geometrically unibranch and noetherian scheme \( S \), satisfy the generalized Theorem of the Cube for any prime \( \ell \) distinct from the residue characteristics of \( S \).

In Section 6 we investigate Grothendieck’s QUESTION for 1-motives and our answer is contained in our third main result, which is

Theorem 0.3. Let \( M = [u : X \rightarrow G] \) be a 1-motive defined over a normal and noetherian scheme \( S \). Assume that the extension \( G \) underlying \( M \) satisfies the generalized Theorem of the Cube for a prime number \( \ell \) distinct from the residue characteristics of \( S \). Then the \( \ell \)-primary
component of the kernel of the homomorphism $H^2_{\text{ét}}(e) : H^2_{\text{ét}}(M, \mathbb{G}_m,M) \to H^2_{\text{ét}}(S, \mathbb{G}_m,S)$ induced by the unit section $e : S \to M$ of $M$, is contained in the Brauer group of $M$:

$$\ker [H^2_{\text{ét}}(e) : H^2_{\text{ét}}(M, \mathbb{G}_m,M) \to H^2_{\text{ét}}(S, \mathbb{G}_m,S)](\ell) \subseteq \text{Br}(M).$$

We prove this result as follows: first we show this theorem for an extension of an abelian scheme by a torus recovering Hoobler’s Theorem [20, Thm 3.3] (see Proposition 6.1). Then, using the effectiveness of the descent of Azumaya algebras and of $\mathbb{G}_m$-gerbes with respect to the quotient map $\iota : G \to [G/X] \cong M$ (see Lemmas 4.2 and 4.3), we prove the required statement for $M$.

Since $H^2_{\text{ét}}(S, \mathbb{G}_m,S) = 0$ if $S$ is the spectrum of an algebraically closed field, Theorem 0.3 has the following immediate consequence

**Corollary 0.4.** If $M = [u : X \to G]$ is a 1-motive defined over an algebraically closed field $k$, then $\text{Br}(M) \cong H^2_{\text{ét}}(M, \mathbb{G}_m,M)$.

Remark that in the above corollary we don’t need the hypothesis on the extension $G$ because of (6.4). We finish Section 6 recalling some results about the Brauer groups of the pure motives underlying a 1-motive $M$. In particular, in Proposition 6.2 we prove geometrically that for the stack of $X$-torsors, with $X$ an $S$-group scheme which is, locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, the injective group homomorphism (0.2) is in fact a bijection. This last result is a positive answer to Grothendieck’s QUESTION in the case of $X$-torsors over an arbitrary noetherian scheme $S$.

The Picard stack $M$ associated to a 1-motive $M$ is not an algebraic stack! In the last years, several authors have worked with the Brauer group of stacks (see for example [1], [12], [25]) but they all focus for their applications to algebraic stacks. So far as we know, our paper is the first one which studies the Brauer group of a stack $M$ which is not algebraic. Moreover, the non algebraicity of the Picard stack $M$ has implied that we had to start from zero (notion of site associated to a stack, sheaf theory on a stack, ...) since we found no references in the literature.

An important role in this paper is played by the descent theory of gerbes. Also here, so far as we now, we are the first one involved in descent theory of stacks, and we hope that this work will shed some light on the notion of “descent” for higher categories.

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**Notation**

**Geometrical objects involved in this paper**

Let $S$ be an arbitrary scheme. The geometrical objects involved in this paper are abelian $S$-schemes, $S$-tori, $S$-group schemes which are, locally for the étale topology, constant group schemes defined by finitely generated free $\mathbb{Z}$-modules, and 1-motives.

**Topologies**

The main results of this paper are stated in terms of the étale or $fppf$ site on the base scheme $S$: the étale site $S_{\text{ét}}$ is the category of étale $S$-schemes endowed with the étale topology, and the $fppf$ site $S_{\text{fppf}}$ is the category of locally of finite presentation $S$-schemes endowed with the $fppf$ (faithfully flat and of finite presentation) topology. We have a morphism of sites $\sigma : S_{\text{fppf}} \to S_{\text{ét}}$. Grothendieck has shown that if $F$ is the sheaf $\mathbb{G}_m$ of units or the sheaf $\mu_n$ of $n$-roots of unity for $n$ relatively prime to all the residue characteristics of $S$, then $H^n(S_{\text{ét}}, \sigma^*F) \cong H^n(S_{\text{fppf}}, F)$ for $n > 0$. 

Stack language
Let $S$ be a site. A stack over $S$ is a fibered category $\mathcal{X}$ over $S$ such that
- (Gluing condition on objects) descent is effective for objects in $\mathcal{X}$, and
- (Gluing condition on arrows) for any object $U$ of $S$ and for every pair of objects $X, Y$ of the category $\mathcal{X}(U)$, the presheaf of arrows $\text{Arr}_{\mathcal{X}(U)}(X, Y)$ of $\mathcal{X}(U)$ is a sheaf over $U$.

For the notions of morphisms of stacks (i.e. cartesian functors), modifications of cartesian functors and equivalences of stacks, we refer to [14, Chp II 1.2]. An isomorphism of stacks $F : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks which is an isomorphism of fibered categories over $S$, that is $F(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is an isomorphism of categories for any object $U$ of $S$. A stack in groupoids over $S$ is a stack $\mathcal{X}$ over $S$ such that for any object $U$ of $S$, the category $\mathcal{X}(U)$ is a groupoid, i.e. a category in which all arrows are invertible. From now on, all stacks will be stacks in groupoids.

A gerbe over the site $S$ is a stack $\mathcal{G}$ over $S$ such that
- $\mathcal{G}$ is locally non-empty: for any object $U$ of $S$, there exists a covering $\{\phi_i : U_i \to U\}_{i \in I}$ for which the set of objects of the category $\mathcal{G}(U_i)$ is not empty for all $i \in I$;
- $\mathcal{G}$ is locally connected: for any object $U$ of $S$ and for each pair of objects $g_1$ and $g_2$ of $\mathcal{G}(U)$, there exists a covering $\{\phi_i : U_i \to U\}_{i \in I}$ of $U$ such that the set of arrows from $g_1|_{U_i}$ to $g_2|_{U_i}$ in $\mathcal{G}(U_i)$ is not empty for all $i \in I$.

A morphism (resp. isomorphism) of gerbes is just a morphism (resp. isomorphism) of stacks whose source and target are gerbes, and a morphism of morphisms of gerbes is a morphism of cartesian functors. An equivalence of gerbes is an equivalence of the underlying stacks.

A Picard stack over the site $S$ is a stack $\mathcal{P}$ over $S$ endowed with a morphism of stacks $\otimes : \mathcal{P} \times_S \mathcal{P} \to \mathcal{P}$, called the group law of $\mathcal{P}$, and two natural isomorphisms $\alpha$ and $\gamma$, expressing the associativity and the commutativity constraints of the group law of $\mathcal{P}$, such that $\mathcal{P}(U)$ is a strictly commutative Picard category for any object $U$ of $S$ (i.e. it is possible to make the sum of two objects of $\mathcal{P}(U)$, this sum is associative and commutative, and any object of $\mathcal{P}(U)$ has an inverse with respect to this sum). An additive functor $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$ between two Picard stacks is a morphism of stacks $F : \mathcal{P}_1 \to \mathcal{P}_2$ endowed with a natural isomorphism $\sum : F(a \otimes_{\mathcal{P}_1} b) \cong F(a) \otimes_{\mathcal{P}_2} F(b)$ (for all $a, b \in \mathcal{P}_1$) which is compatible with the natural isomorphisms $\alpha$ and $\gamma$ underlying $\mathcal{P}_1$ and $\mathcal{P}_2$.

A 2-stack over the site $S$ is a fibered 2-category $\mathcal{X}$ over $S$ such that
- 2-descent is effective for objects in $\mathcal{X}$, and
- for any object $U$ of $S$ and for every pair of objects $X, Y$ of the 2-category $\mathcal{X}(U)$, the fibered category of arrows $\text{Arr}_{\mathcal{X}(U)}(X, Y)$ of $\mathcal{X}(U)$ is a stack over $S_U$.

For the notions of morphisms of 2-stacks (i.e. cartesian 2-functors), modifications of cartesian 2-functors, modifications of 2-stacks and equivalences of 2-stacks, we refer to [19, Chp I]. A 2-stack in 2-groupoids over $S$ is a 2-stack $\mathcal{X}$ over $S$ such that for any object $U$ of $S$ the 2-category $\mathcal{X}(U)$ is a 2-groupoid, i.e. a 2-category in which 1-arrows are invertible up to a 2-arrow and 2-arrows are strictly invertible. From now on, all 2-stacks will be 2-stacks in 2-groupoids.

Let $S$ be an arbitrary scheme and denote by $S$ the site of $S$ for a Grothendieck topology that we will fix later. We will call a stack, a Picard stack, a 2-stack over $S$ respectively an $S$-stack, a Picard $S$-stack, an $S$-2-stack.

Notation
Let $\ell$ be a prime number. If $H$ is an abelian group or an abelian sheaf on $S$, we denote by $\ell H, H(\ell), \ell H, H_{\ell}$, the $\ell$-torsion elements of $H$ (i.e. the kernel of the multiplication by $\ell$ on $H$),
the \(\ell\)-primary component of \(H\), the image of \(H\) under the multiplication by \(\ell\) and the cokernel of the multiplication by \(\ell\) on \(H\) respectively.

1. Recall on sheaves, gerbes and Picard stacks on a stack

Let \(S\) be a site. Let \(X\) be a stack over \(S\). The site \(S(X)\) associated to \(X\) is the site defined in the following way:

- the category underlying \(S(X)\) consists of the objects \((U, u)\) with \(U\) an object of \(S\) and \(u\) an object of \(X(U)\), and of the arrows \((\phi, \Phi) : (U, u) \to (V, v)\) with \(\phi : U \to V\) a morphism of \(S\) and \(\Phi : \phi^*v \to u\) an isomorphism in \(X(U)\). We call the pair \((U, u)\) an open of \(X\) with respect to the chosen topology.
- the topology on \(S(X)\) is the one generated by the pre-topology for which a covering of \((U, u)\) is a family \(\{(\phi_i, \Phi_i) : (U, u) \to (U, u)\}_i\) such that the morphism of \(S\) \(\coprod \phi_i : \coprod U_i \to U\) is surjective.

Using the above notion, we can define as in the classical case the notion of sheaf of sets on the site \(S(X)\). Following [16, Exp II, Prop. 6.7] and [15, Thm 1.10.1], we have the following equivalent definition of sheaf on \(X\), which is more useful for our aim:

**Definition 1.1. A sheaf (of sets) \(F\) on \(X\) is a system \((F_{U,u}, \theta_{\phi,\Phi})\), where for any object \((U, u)\) of \(S(X)\), \(F_{U,u}\) is a sheaf on \(S_U\), and for any arrow \((\phi, \Phi) : (U, u) \to (V, v)\) of \(S(X)\), \(\theta_{\phi,\Phi} : F_{U,u} \to \phi_*F_{V,v}\) is a morphism of sheaves on \(S_V\), such that

(i) if \((\phi, \Phi) : (U, u) \to (V, v)\) and \((\gamma, \Gamma) : (V, v) \to (W, w)\) are two arrows of \(S(X)\), then

\[ \gamma_*\theta_{\phi,\Phi} \circ \theta_{\gamma,\Gamma} = \theta_{\gamma_*\phi,\Phi \circ \Gamma} \]

(ii) if \((\phi, \Phi) : (U, u) \to (V, v)\) is an arrow of \(S(X)\), the morphism of sheaves \(\phi^{-1}F_{V,v} \to F_{U,u}\), obtained by adjunction from \(\theta_{\phi,\Phi}\), is an isomorphism.

We recall that if \(F\) is a sheaf on \(X\), then \(F_{U,u}\) is just the restriction of \(F\) to the open \((U, u)\) of \(X\). Reciprocally, given the system \((F_{U,u}, \theta_{\phi,\Phi})\), for any open \((U, u)\) of \(X\) we set \(F(U, u) = F_{U,u}(U)\), and for any arrow \((\phi, \Phi) : (U, u) \to (V, v)\) of \(S(X)\) we set \(\theta_{\phi,\Phi}(V) = \text{res}_{\phi} : F(V, v) \to F(U, u)\) for the restriction map. To simplify notations, we denote just \((F_{U,u}, \theta_{\phi,\Phi})\) the sheaf.

The set of **global sections** of a sheaf \(F\) on \(X\), that we denote by \(\Gamma(X, F)\), is the set of families \((s_{U,u})\) of sections of \(F\) on the objects \((U, u)\) of \(S(X)\) such that for any arrow \((\phi, \Phi) : (U, u) \to (V, v)\) of \(S(X)\), \(\text{res}_{\phi} s_{V,v} = s_{U,u}\).

A **sheaf of groups** (resp. an **abelian sheaf**) \(F\) on \(X\) is a system \((F_{U,u}, \theta_{\phi,\Phi})\) verifying the conditions (i) and (ii) of Definition 1.1 where the \(F_{U,u}\) are sheaves of groups (resp. abelian sheaves) on \(S_U\). We denote by \(\text{Gr}(X)\) (resp. \(\text{Ab}(X)\)) the category of sheaves of groups (resp. the category of abelian sheaves) on \(X\). According to [16, Exp II, Prop. 6.7] and [15, Thm 1.10.1], the category \(\text{Ab}(X)\) is an abelian category with enough injectives. Let \(R\Gamma(X, -)\) be the right derived functor of the functor \(\Gamma(X, -) : \text{Ab}(X) \to \text{Ab}\) of global sections (here \(\text{Ab}\) is the category of abelian groups). The \(i\)-th cohomology group \(H^i(R\Gamma(X, -))\) of \(R\Gamma(X, -)\) is denoted by \(H^i(X, -)\).

A **stack** on \(X\) is a stack over the site \(S(X)\). It is a stack \(Y\) over \(S\) endowed with a morphism of stacks \(P : Y \to X\) (called the structural morphism) such that for any object \((U, x)\) of \(S(X)\) the fibered product \(U \times_{x, X, P} Y\) is a stack over \(S_U\).

A **gerbe** on \(X\) is a gerbe over the site \(S(X)\). It is a stack \(G\) over \(S\) endowed with a morphism of stacks \(P : G \to X\) (called the structural morphism) such that for any object \((U, x)\) of \(S(X)\) the fibered product \(U \times_{x, X, P} G\) is a gerbe over \(S_U\). A **morphism** (resp. an **isomorphism**) of **gerbes** on \(X\) is a morphism (resp. an isomorphism) of gerbes which is compatible with the underlying structural morphisms.
A Picard stack on $X$ is a Picard stack over the site $S(X)$. It is a stack $P$ over $S$ endowed with a morphism of stacks $P : \mathcal{P} \to X$ (called the structural morphism), with a morphism of stacks $\otimes : \mathcal{P} \times_{P,X,P} \mathcal{P} \to \mathcal{P}$ (called the group law of $\mathcal{P}$), and with two natural 2-transformations $a$ and $c$, expressing the associativity and the commutativity constraints of the group law of $\mathcal{P}$, such that $U \times_{X,P} \mathcal{P}$ is a Picard stack over $S_U$ for any object $(U,x)$ of $S(X)$.

A Picard 2-stack on $X$ is a Picard 2-stack over the site $S(X)$. It is a 2-stack $P$ over $S$ endowed with a morphism of 2-stacks $P : \mathcal{P} \to X$ (called the structural morphism - here we see $X$ as a 2-stack), with a morphism of 2-stacks $\otimes : \mathcal{P} \times_{P,X,P} \mathcal{P} \to \mathcal{P}$ (called the group law of $\mathcal{P}$), and with two natural 2-transformations $a$ and $c$, expressing the associativity and the commutativity constraints of the group law of $\mathcal{P}$, such that $U \times_{X,P} \mathcal{P}$ is a Picard 2-stack over $S_U$ for any object $(U,x)$ of $S(X)$ (for more details see [5,1]). Remark that the theory of Picard stacks is included in the theory of Picard 2-stacks. An additive 2-functor $(F,\lambda_F) : P_1 \to P_2$ between two Picard 2-stacks on $X$ is given by a morphism of 2-stacks $F : P_1 \to P_2$ and a natural 2-2 transformation $\lambda_F : \otimes_{P_2} \circ F^2 \to F \circ \otimes_{P_1}$, which are compatible with the structural morphisms of 2-stacks $P_i : P_1 \to X$ and $P_2 : P_2 \to X$ and with the natural 2-transformations $a$ and $c$ underlying $P_1$ and $P_2$. An equivalence of Picard 2-stacks on $X$ is an additive 2-functor whose underlying morphism of 2-stacks is an equivalence of 2-stacks.

Denote by $2\text{Picard}(X,S)$ the category whose objects are Picard 2-stacks on $X$ and whose arrows are isomorphism classes of additive 2-functors. Applying [31, Cor 6.5] to the site $S(X)$, we have the following equivalence of categories

\begin{equation}
2\text{st} : \mathcal{D}^{[-2,0]}(S(X)) \to 2\text{Picard}(X,S).
\end{equation}

where $\mathcal{D}^{[-2,0]}(S(X))$ is the derived category of length three complexes of abelian sheaves on $X$. Via this equivalence Picard stacks on $X$ correspond to length two complexes of abelian sheaves on $X$. We denote by $[\ ]$ the inverse equivalence of $2\text{st}$.

If $P$ is a Picard stack over a site $S$ we define its classifying groups $P^i$ for $i = 1, 0$ in the following way: $P^1$ is the group of isomorphism classes of objects of $P$ and $P^0$ is the group of automorphisms of the neutral object $e$ of $P$. We define the classifying groups $P^i$ for $i = 2, 1, 0$ of a Picard 2-stack $P$ over a site $S$ recursively: $P^2$ is the group of equivalence classes of objects of $P$, $P^1 = \text{Aut}^1(e)$ and $P^0 = \text{Aut}^0(e)$ where $\text{Aut}(e)$ is the Picard stack of automorphisms of the neutral object $e$ of $P$.

If two Picard 2-stacks $P$ and $P'$ are equivalent as Picard 2-stacks, then their classifying groups are isomorphic: $P^i \cong P'^i$ for $i = 2, 1, 0$. The inverse affirmation is not true as explained in [2, Rem 1.3].

Let $S$ be an arbitrary scheme and denote by $S$ the site of $S$ for a Grothendieck topology. Let $X$ be an $S$-stack. A stack (resp. a Picard 2-stack) on $X$ will be called an $S$-stack (resp. a Picard $S$-2-stack) on $X$.

2. Gerbes with abelian band on a stack

Let $F$ be an abelian sheaf on a site $S$. The contracted product of $F$-gerbes (see [14, Chp IV 2.4.3]) endows the 2-stack of $F$-gerbes with a structure of Picard 2-stack, that we denote $\text{Gerbe}_S(F)$. The classifying groups $\text{Gerbe}_S^i(F)$ for $i = 2, 1, 0$ of the Picard 2-stack $\text{Gerbe}_S(F)$ are

- $\text{Gerbe}_S^2(F)$, the abelian group of $F$-equivalence classes of $F$-gerbes;
- $\text{Gerbe}_S^1(F)$, the abelian group of isomorphism classes of morphisms of $F$-gerbes from a $F$-gerbe to itself.
- $\text{Gerbe}_S^0(F)$, the abelian group of automorphisms of a morphism of $F$-gerbes from a $F$-gerbe to itself.
Proof of Theorem 0.1. It is a classical result that via the equivalence of categories stated in [16, Exposé XVIII, Prop 1.4.15], the complex \( \tau_{\leq 0} R \Gamma(S, F[1]) \) corresponds to the Picard stack \( \text{tors}(F) \) of \( F \)-torsors: \( \text{tors}(F) = 2st(\tau_{\leq 0} R \Gamma(S, F[1])) \). A higher dimensional analogue of the notion of torsor under an abelian sheaf is the notion of torsor under a Picard stack, which was introduced by Breen in [8, Def 3.1.8] and studied by the first author in [3, Remark that in fact in [3] the first author introduces the notion of torsor under a Picard 2-stack]. Hence we have the notion of \( \text{tors}(\text{tors}(F)) \)-torsors. The contracted product of torsors under a Picard 2-stack, introduced in [3, Def 2.11], endows the 2-stack \( \text{tors}(\text{tors}(F)) \) of \( \text{tors}(F) \)-torsors with a Picard structure, and by [3, Thm 0.1] this Picard 2-stack \( \text{tors}(\text{tors}(F)) \) corresponds, via the equivalence of categories (1.1), to the complex \( \tau_{\leq 0} R \Gamma(S, \text{tors}(F)[1]) \):

\[
\text{tors}(\text{tors}(F)) = 2st(\tau_{\leq 0} R \Gamma(S, F[2])).
\]

In [10, Prop 2.14] Breen constructs a canonical equivalence of Picard 2-stacks between the Picard 2-stack \( \text{gerbe}_S(F) \) of \( F \)-gerbes and the Picard 2-stack \( \text{tors}(\text{tors}(F)) \) of \( \text{tors}(F) \)-torsors:

\[
\text{gerbe}_S(F) \cong \text{tors}(\text{tors}(F)).
\]

This equivalence and the equality (2.1) furnish the expected equivalence \( \text{gerbe}_S(F) \cong 2st(\tau_{\leq 0} R \Gamma(S, F[2])) \). If we denote by \( \text{tors}^i(\text{tors}(F)) \) for \( i = 1, 0, -1 \) the classifying groups of the Picard 2-stack \( \text{tors}(\text{tors}(F)) \), by [3, Thm 0.1] we have that for \( i = 2, 1, 0 \)

\[
\text{gerbe}^i_S(F) = \text{tors}^{i-1}(\text{tors}(F)) \cong H^{i-1}(S, F[1]) = H^i(S, F).
\]

\[\square\]

Remark 2.1. Via the cohomological interpretation (2.1) of torsors under the Picard stack of \( F \)-torsors, the equivalence of Picard 2-stacks (2.2) is the geometrical counterpart of the canonical isomorphism in cohomology \( H^2(S, F) \cong H^1(S, F[1]) \).

Now let \( \mathcal{X} \) be a stack over a site \( S \) and denote by \( S(\mathcal{X}) \) the site associated to \( \mathcal{X} \). Applying [14, Chp IV] to the site \( S(\mathcal{X}) \) we get the notion of \( \mathcal{F} \)-gerbes on the stack \( \mathcal{X} \), with \( \mathcal{F} \) an abelian sheaf on \( \mathcal{X} \). We recall briefly this notion.

The stack of bands on \( \mathcal{X} \), denoted by \( \text{band}_S(\mathcal{X}) \), is the stack on \( \mathcal{X} \) associated to the fibered category whose objects are sheaves of groups on \( \mathcal{X} \) and whose arrows are morphisms of sheaves of groups modulo inner automorphisms. By construction we have a morphism of stacks \( \text{band}_S^\mathcal{X} : \text{Gr}(\mathcal{X}) \to \text{band}_S(\mathcal{X}) \). A band over \( S(\mathcal{X}) \) is a cartesian section \( \mathcal{L} : S(\mathcal{X}) \to \text{band}_S(\mathcal{X}) \) of the stack \( \text{band}_S(\mathcal{X}) \) on \( \mathcal{X} \). A representable band is a band \( \mathcal{L} \) for which it exists an isomorphism \( \mathcal{L} \cong \text{band}_S^\mathcal{X}(\mathcal{F}) \) with \( \mathcal{F} \) a sheaf of groups on \( \mathcal{X} \).

Let \( \mathcal{G} \) be a gerbe on the stack \( \mathcal{X} \). Let \( P : \mathcal{G} \to \mathcal{X} \) be the structural morphism underlying \( \mathcal{G} \). For any object \( (U, x) \) of \( S(\mathcal{X}) \), for any object \( g \) of \( \mathcal{G}(U) \) such that \( P(g) = x \), denote by \( \text{Aut}(g)_{U,x} \) the sheaf of automorphisms of \( g \) on \( S_U \). The system \( (\text{Aut}(g)_{U,x}) \) verifies the conditions (i) and (ii) of Definition 1.1 and therefore it defines a sheaf of groups on \( \mathcal{X} \), denoted by \( \text{Aut}(g) \). We can therefore define the morphism of stacks \( \text{band}_S^\mathcal{X}(\mathcal{F}) : \mathcal{G} \to \text{band}_S(\mathcal{X}), g \mapsto \text{band}_S^\mathcal{X}(\text{Aut}(g)) \).

An \( (\mathcal{L}, a) \)-gerbe on \( \mathcal{X} \), or simply an \( \mathcal{L} \)-gerbe, is a gerbe \( \mathcal{G} \) on \( \mathcal{X} \) endowed with a pair \( (\mathcal{L}, a) \) where \( \mathcal{L} \) is a band and

\[
a : \mathcal{L} \circ f \Rightarrow \text{band}_S^\mathcal{X}
\]

is an isomorphism of cartesian functor with \( f : \mathcal{G} \to S \) the structural morphism of \( \mathcal{G} \). The notion of \( \mathcal{L} \)-gerbe becomes more explicit if the band \( \mathcal{L} \) is an abelian band, i.e. it is representable by an abelian sheaf \( \mathcal{F} \) on \( \mathcal{X} \) (see [14, Chp IV Prop 1.2.3]): in fact in this case, a \( \text{band}_S^\mathcal{X}(\mathcal{F}) \)-gerbe, called just an \( \mathcal{F} \)-gerbe, is a gerbe \( \mathcal{G} \) such that for any object \( (U, x) \) of \( S(\mathcal{X}) \)
and for any object \( g \) of \( \mathcal{G}(U) \) such that \( P(g) = x \), there is an isomorphism \( \mathcal{F}_{U,x} \to \text{Aut}(g)_{U,x} \) of sheaves of groups on \( S_U \).

Consider now an \((\mathcal{L}, a)\)-gerbe \( \mathcal{G} \) and an \((\mathcal{L}', a')\)-gerbe \( \mathcal{G}' \) on \( X \). Let \( u : \mathcal{L} \to \mathcal{L}' \) a morphism of bands. A morphism of gerbes \( m : \mathcal{G} \to \mathcal{G}' \) is an \( u \)-\textbf{morphism} if

\[
\text{band}_{\mathcal{G}}(m) \circ a = (a' * m)(u * f)
\]

with \( \text{band}_{\mathcal{G}}(m) : \text{band}_{\mathcal{G}} \Rightarrow \text{band}_{\mathcal{G}'} \circ m \) and \( f : \mathcal{G} \to \mathcal{S} \) the structural morphism of \( \mathcal{G} \). An \( u \)-\textbf{isomorphism} is an \( u \)-morphism \( m : \mathcal{G} \to \mathcal{G}' \) which is an isomorphism of gerbes. As in \cite[Chp IV Prop 2.2.6]{[14]} an \( u \)-morphism \( m : \mathcal{G} \to \mathcal{G}' \) is fully faithful if and only if \( u : \mathcal{L} \to \mathcal{L}' \) is an isomorphism, in which case \( m \) is an equivalence of gerbes. Let \( \mathcal{G} \) and \( \mathcal{G}' \) be two \( \mathcal{F} \)-gerbes on \( X \), with \( \mathcal{F} \) abelian sheaf on \( X \). Instead of \( \text{id}_{\text{band}_{\mathcal{G}}(\mathcal{F})} \)-morphism \( \mathcal{G} \to \mathcal{G}' \) we use the terminology \( \mathcal{F} \)-\textbf{equivalence} \( \mathcal{G} \to \mathcal{G}' \) of \( \mathcal{F} \)-\textbf{gerbes} on \( X \).

Generalizing \cite[Chp IV §1.6]{[14]} it is possible to define the contracted product of two bands. In particular by \cite[Chp IV 1.6.1.3]{[14]} the contracted product of bands represented by abelian sheaves on \( X \) is just the band represented by the fibered product of the involved abelian sheaves on \( X \). Moreover as in \cite[Chp IV 2.4.3]{[14]} we define the \textbf{contracted product} of two \( \mathcal{F} \)-gerbes \( \mathcal{G} \) and \( \mathcal{G}' \) as the \( \mathcal{F} \)-gerbe \( \mathcal{G} \wedge_{\text{band}_{\mathcal{G}}(\mathcal{F})} \mathcal{G}' \) obtained in such a way that \( \text{band}_{\mathcal{G}}(\mathcal{F}) \) acts on \( \mathcal{G} \times \mathcal{G}' \) via the morphism of band represented by \( (\text{id}_{\mathcal{F}}, \text{id}_{\mathcal{F}}) : \mathcal{F} \to \mathcal{F} \times \mathcal{F} \).

\( \mathcal{F} \)-gerbes on \( X \) build a Picard 2-stack on \( X \), denoted by \( \text{Gerbes}_{\mathcal{F}}(\mathcal{F}) \), whose group law is given by the contracted product of \( \mathcal{F} \)-gerbes over \( X \). The neutral element of this group law is the stack \( \text{Fors}(\mathcal{F}) \) of \( \mathcal{F} \)-torsors on \( X \), which is called the \textbf{neutral} \( \mathcal{F} \)-\textbf{gerbe}. Applying Theorem \([4,11]\) to the abelian sheaf \( \mathcal{F} \) on \( X \) we get

**Corollary 2.2.** We have the following equivalence of Picard 2-stacks

\[
\text{Gerbes}_{\mathcal{F}}(\mathcal{F}) \cong \text{2st}(\tau_{\leq 0} \text{R}(\mathcal{X}, \mathcal{F}[2])).
\]

In particular, \( \text{Gerbes}^i_{\mathcal{F}}(\mathcal{F}) \cong H^i(\mathcal{X}, \mathcal{F}) \) for \( i = 2, 1, 0 \).

Hence, \( \mathcal{F} \)-equivalence classes of \( \mathcal{F} \)-gerbes on \( X \), which are the elements of the 0th-homotopy group \( \text{Gerbes}_{\mathcal{F}}(\mathcal{F}) \), are parametrized by cohomological classes of \( H^2(\mathcal{X}, \mathcal{F}) \).

Let \( S \) be an arbitrary scheme and denote by \( \mathcal{S} \) the site of \( S \) for a Grothendieck topology that we will fix later. Let

\[
0 \to \mathcal{X} \xrightarrow{1} \mathcal{Y} \xrightarrow{P} \mathcal{Z} \to 0
\]

be a short exact sequence of Picard \( S \)-stacks (see \([2, \text{Def } 3.2]\)). We finish this section associating to this short exact sequence a long exact sequence involving the classifying groups of the Picard \( S \)-2-stacks of \( \mathcal{G}_{m,S} \)-gerbes on \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{Z} \).

**Definition 2.3.** Let \( F : \mathcal{X} \to \mathcal{Y} \) be a morphism of \( S \)-stacks and let \( \mathcal{G} \) be a \( \mathcal{G}_{m,\mathcal{Y}} \)-gerbe on \( \mathcal{Y} \).

The **pull-back of the \( \mathcal{G}_{m,\mathcal{Y}} \)-gerbe \( \mathcal{G} \) via \( F \)** is the fibered product

\[
F^* \mathcal{G} := \mathcal{X} \times_{F, \mathcal{Y}, P} \mathcal{G}
\]

of \( \mathcal{X} \) and \( \mathcal{G} \) via the morphism \( F : \mathcal{X} \to \mathcal{Y} \) and the structural morphism \( P : \mathcal{G} \to \mathcal{Y} \) underlying \( \mathcal{G} \) (see \([3, \text{Def } 2.14]\) for the definition of fibered product of \( S \)-2-stacks).

Since the fibered product and the contracted product commute, the pull-back of \( \mathcal{G}_{m,S} \)-gerbes induces an additive 2-functor between the Picard \( S \)-2-stacks \( \text{Gerbes}_{S}(\mathcal{G}_{m,\mathcal{Y}}) \) and \( \text{Gerbes}_{S}(\mathcal{G}_{m,\mathcal{X}}) \)

\[
F^* : \text{Gerbes}_{S}(\mathcal{G}_{m,\mathcal{Y}}) \to \text{Gerbes}_{S}(\mathcal{G}_{m,\mathcal{X}})
\]

(2.3)

which associates to each \( \mathcal{G}_{m,\mathcal{Y}} \)-gerbe \( \mathcal{G} \) on \( \mathcal{Y} \) the \( \mathcal{G}_{m,\mathcal{X}} \)-gerbe \( F^* \mathcal{G} \) on \( \mathcal{X} \).
In our setting, we can associate to the short exact sequence \( 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0 \) of Picard \( S \)-stacks two additive 2-functors

\[
\text{Gerbes}_S(G_m, Z) \xrightarrow{\Pi'} \text{Gerbes}_S(G_m, y) \xrightarrow{\Pi} \text{Gerbes}_S(G_m, x)
\]

such that we have a morphism of additive 2-functors \( \Pi' \circ \Pi^* \Rightarrow 0 \) induced by the isomorphism of additive functors \( \Pi \circ I \cong 0 \). Denote by \( \text{Aut}(e_{\text{Gerbes}(G_m,x)}) \) the Picard \( S \)-stack of automorphisms of the neutral object of \( \text{Gerbes}_S(G_m,x) \). By the equivalence of Picard 2-stacks \( \text{Gerbes}_S(G_m,x) \cong \text{Tors}(\text{Tors}(G_m,x)) \) recalled in (2.2) and by [5, Lem 3.1], we have that \( \text{Aut}(e_{\text{Gerbes}(G_m,x)}) \) is equivalent (as Picard stack) to the Picard \( S \)-stack \( \text{Tors}(G_m,x) \) of \( G_m,x \)-torsors on \( X \). Now \( \text{Tors}(G_m,x) \) is equivalent (as Picard stack) to the Picard \( S \)-stack \( \mathfrak{H} \text{om}(X, \text{st}(G_m,S[1])) \), whose objects over \( U \in \text{Ob}(S) \) are additive functors from \( U \times_S X \) to \( \text{st}(G_m,U[1]) \) and whose arrows are morphisms of additive functors: in fact, to have an additive functor \( F : X \to \text{st}(G_m,S[1]) \) is equivalent to have, for any \( U \in \text{Ob}(S) \) and for any \( x \in \text{X}(U) \), a \( G_m \)-torsor \( F(U)(x) := L_{U,x} \) on \( U \), that is to have a \( G_m,x \)-torsor \( (L_{U,x}(U,x))_{(U,x) \in \text{S}(X)} \) on \( X \). The composite of these two equivalences of Picard stacks furnishes that \( \text{Aut}(e_{\text{Gerbes}(G_m,x)}) \) is equivalent (as Picard stack) to \( \mathfrak{H} \text{om}(X, \text{st}(G_m,S[1])) \). Moreover, observe that the pull-back of \( G_m \)-torsors via the structural morphism \( P : Z \to S \) of the Picard \( S \)-stack \( Z \) induces an additive functor \( P^* : \text{st}(G_m,S[1]) \to \text{Tors}(G_m,z) \).

Now if \( F : X \to \text{st}(G_m,S[1]) \) is an additive functor, the push-down

\[(P^* \circ F)_* Y\]

of \( Y \) via \( P^* \circ F : X \to \text{Tors}(G_m,z) \) (called also the fibered sum of \( \text{Tors}(G_m,z) \) and \( Y \) under \( X \) via \( P^* \circ F \) and \( I : X \to Y \), see [5, Def 2.11]) is an extension of \( Z \) by \( \text{Tors}(G_m,z) \), that is in particular a \( \text{Tors}(G_m,z) \)-torsor over \( Z \). Via the equivalence of Picard 2-stacks recalled in (2.2), \( (P^* \circ F)_* Y \) defines a \( G_m,z \)-gerbe on \( Z \). With these notation we have

**Proposition 2.4.** Let \( 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0 \) be a short exact sequence of Picard \( S \)-stacks. The additive 2-functor

\[
C : \text{Aut}(e_{\text{Gerbes}(G_m,x)}) \cong \mathfrak{H} \text{om}(X, \text{st}(G_m,S[1])) \longrightarrow \text{Gerbes}_S(G_m,z)
\]

\[
(F : X \to \text{st}(G_m,S[1])) \quad \longmapsto \quad (P^* \circ F)_* Y
\]

furnishes two connecting homomorphisms

\[
C^1 : \text{Gerbe}^1_S(G_m,z) = \text{Aut}^1(e_{\text{Gerbes}(G_m,x)}) \longrightarrow \text{Gerbe}^2_S(G_m,z),
\]

\[
C^0 : \text{Gerbe}^0_S(G_m,z) = \text{Aut}^0(e_{\text{Gerbes}(G_m,x)}) \longrightarrow \text{Gerbe}^1_S(G_m,z).
\]

such that the sequence of abelian groups

\[
0 \to \text{Gerbe}^0_S(G_m,z) \xrightarrow{\Pi^*} \text{Gerbe}^0_S(G_m,y) \xrightarrow{\Pi} \text{Gerbe}^0_S(G_m,x) \xrightarrow{C^0} \text{Gerbe}^1_S(G_m,z)
\]

\[
\xrightarrow{\Pi^*} \text{Gerbe}^1_S(G_m,y) \xrightarrow{\Pi} \text{Gerbe}^1_S(G_m,x) \xrightarrow{C^1} \text{Gerbe}^2_S(G_m,z)
\]

\[
\xrightarrow{\Pi^*} \text{Gerbe}^2_S(G_m,y) \xrightarrow{\Pi} \text{Gerbe}^2_S(G_m,x) \to 0
\]

is a long exact sequence.

We left the proof of this statement to the reader. In Proposition 5.5, we will check the exactness of the higher terms of the long exact sequence associated to an extension \( G \) of an abelian \( S \)-scheme by an \( S \)-torus: \( 0 \to T \overset{i}{\to} G \overset{\pi}{\to} A \to 0 \).
3. The Brauer group of a locally ringed stack

We start recalling the notion of the Brauer group of a scheme according to [18] and [21].

Let \( X \) be a scheme with structural sheaf \( \mathcal{O}_X \). An Azumaya algebra \( A \) over \( X \) is an \( \mathcal{O}_X \)-algebra of finite presentation as \( \mathcal{O}_X \)-module such that there exists an étale covering \( \{ U_i \rightarrow X \}_i \) on \( X \) for which \( A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong \mathcal{M}_r(\mathcal{O}_{U_i}) \) for any \( i \). The Brauer group \( \text{Br}(X) \) of \( X \), denoted by \( \text{Br}(X) \), is the group of the equivalence classes (with respect to similarity) of Azumaya algebras on \( X \). In [18, I, Prop 1.4] Grothendieck constructed a canonical map \( \delta : \text{Br}(X) \rightarrow H^2_{\text{ét}}(X, \mathbb{G}_m) \) which is an injective group homomorphism. We have the following well-known results concerning the image \( \delta(\text{Br}(X)) \) of \( \text{Br}(X) \) via this injective homomorphism:

**Theorem 3.1.** (i) If \( X = \text{Spec}(k) \) with \( k \) an algebraically closed field, then \( \text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m) \). (\[2\], Chp X, §5)

(ii) If \( X \) is a regular scheme, then \( H^q_{\text{ét}}(X, \mathbb{G}_m) \) is a torsion group for \( q \geq 2 \). (\[18\], II, Prop 1.4)

(iii) If \( X \) has dimension \( \leq 1 \) or if \( X \) is regular and of dimension \( \leq 2 \), then \( \text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m) \). (\[18\], II, Cor 2.2)

(iv) If \( X \) is an algebraic curve over an algebraically closed field, then \( \text{Br}(X) = 0 \). (\[18\], III, Cor 1.2)

(v) If \( X \) is a quasi-compact and separated scheme endowed with an ample invertible sheaf (in particular if \( X \) is an affine scheme), then \( \text{Br}(X) \cong H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{Tors}} \). (\[18\] and \[13\]).

(vi) If \( X \) is a smooth variety over a field, then \( \text{Br}(X) \cong H^2_{\text{ét}}(X, \mathbb{G}_m) \). (\[22\], IV, Prop 2.15)

Let \( \mathcal{X} \) be a stack over a site \( S \) and let \( S(\mathcal{X}) \) be its associated site. A sheaf of rings \( \mathcal{A} \) on \( \mathcal{X} \) is a system \((\mathcal{A}_{U,u})\) verifying the conditions (i) and (ii) of Definition [14] where the \( \mathcal{A}_{U,u} \) are sheaves of rings on \( S_U \). Consider the sheaf of rings \( \mathcal{O}_\mathcal{X} \) on \( \mathcal{X} \) given by the system \((\mathcal{O}_X_{U,u})\) with \( \mathcal{O}_X_{U,u} \) the structural sheaf of \( U \). The sheaf of rings \( \mathcal{O}_\mathcal{X} \) is the **sheaf of the stack** \( \mathcal{X} \) and the pair \((\mathcal{X}, \mathcal{O}_\mathcal{X})\) is a **ringed stack**. An \( \mathcal{O}_\mathcal{X} \)-module \( \mathcal{M} \) is a system \((\mathcal{M}_{U,u})\) verifying the conditions (i) and (ii) of Definition [14] where the \( \mathcal{M}_{U,u} \) are sheaves of \( \mathcal{O}_U \)-modules on \( S_U \). An \( \mathcal{O}_\mathcal{X} \)-algebra \( \mathcal{A} \) is a system \((\mathcal{A}_{U,u})\) verifying the conditions (i) and (ii) of Definition [14] where the \( \mathcal{A}_{U,u} \) are sheaves of \( \mathcal{O}_U \)-algebras on \( S_U \). An \( \mathcal{O}_\mathcal{X} \)-module \( \mathcal{M} \) is of **finite presentation** if the \( \mathcal{M}_{U,u} \) are sheaves of \( \mathcal{O}_U \)-modules of finite presentation.

Now let \( S \) be an arbitrary scheme and let \( S_{\text{ét}} \) be the étale site on \( S \). Let \( \mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \) be a locally ringed \( S \)-stack, i.e. for any object \((U, u)\) of the associated étale site \( S_{\text{ét}}(\mathcal{X}) \), and for any section \( f \in \mathcal{O}_{\mathcal{X}_{U,u}}(U) \), we have \( U_f \cup U_{1-f} = U \) with \( U_f \) the biggest subobject of \( U \) over which the restriction of \( f \) is invertible. An **Azumaya algebra** over \( \mathcal{X} \) is an \( \mathcal{O}_\mathcal{X} \)-algebra \( \mathcal{A} = (\mathcal{A}_{U,u}) \) of finite presentation as \( \mathcal{O}_\mathcal{X} \)-module which is, locally for the topology of \( S_{\text{ét}}(\mathcal{X}) \), isomorphic to a matrix algebra, i.e. for any open \((U, u)\) of \( \mathcal{X} \) there exists a covering \( \{(\phi_i, \Phi_i) : (U_i, u_i) \rightarrow (U, u)\}_i \) in \( S_{\text{ét}}(\mathcal{X}) \) such that \( \mathcal{A}_{U_i,u_i} \otimes_{\mathcal{O}_{U_i,u_i}} \mathcal{O}_{U_i} \cong \mathcal{M}_r(\mathcal{O}_{U_i,u_i}) \) for any \( i \). Azumaya algebras over \( \mathcal{X} \) build an \( S \)-stack on \( \mathcal{X} \), that we denote by \( \text{Az}(\mathcal{X}) \). Two Azumaya algebras \( \mathcal{A} \) and \( \mathcal{A}' \) over \( \mathcal{X} \) are **similar** if there exist two locally free \( \mathcal{O}_\mathcal{X} \)-modules \( \mathcal{E} \) and \( \mathcal{E}' \) of finite rank such that

\[
\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_\mathcal{X}} \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}').
\]

The above isomorphism defines an equivalence relation because of the isomorphism of \( \mathcal{O}_\mathcal{X} \)-algebras \( \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) \otimes_{\mathcal{O}_\mathcal{X}} \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}') \cong \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}') \). We denote by \( [\mathcal{A}] \) the equivalence class of an Azumaya algebra \( \mathcal{A} \) over \( \mathcal{X} \). The set of equivalence classes of Azumaya algebra is a group under the group law given by \([\mathcal{A}] [\mathcal{A}'] = [\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{A}'] \). A **trivialization** of an Azumaya algebra \( \mathcal{A} \) over \( \mathcal{X} \) is a couple \((\mathcal{L}, a)\) with \( \mathcal{L} \) a locally free \( \mathcal{O}_\mathcal{X} \)-module and \( a : \text{End}_{\mathcal{O}_\mathcal{X}}(\mathcal{L}) \rightarrow \mathcal{A} \) an isomorphism of sheaves of \( \mathcal{O}_\mathcal{X} \)-algebras. An Azumaya algebra \( \mathcal{A} \) is **trivial** if it exists a
trivialization of $A$. The class of any trivial Azumaya algebra is the neutral element of the above group law. The inverse of a class $[A]$ is the class $[A^0]$ with $A^0$ the opposite $\mathcal{O}_X$-algebra of $A$.

**Definition 3.2.** Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a locally ringed $S$-stack. The **Brauer group** of $\mathcal{X}$, denoted by $\text{Br}(\mathcal{X})$, is the group of equivalence classes of Azumaya algebras over $\mathcal{X}$.

$\text{Br}(-)$ is a functor from the category of locally ringed $S$-stacks (objects are locally ringed $S$-stacks and arrows are isomorphism classes of morphisms of locally ringed $S$-stacks) to the category $\text{Ab}$ of abelian groups. Remark that the above definition generalizes to stacks the classical notion of Brauer group of a scheme: in fact if $\mathcal{X}$ is a locally ringed $S$-stack associated to an $S$-scheme $X$, then $\text{Br}(\mathcal{X}) = \text{Br}(X)$.

Consider the following sheaves of groups on $X$: the multiplicative group $G_{m,X}$, the linear general group $GL(n, X)$, and the sheaf of groups $\text{PGL}(n, X)$ on $X$ defined by the system $(\text{PGL}(n, X)_{U,u})$ where $\text{PGL}(n, X)_{U,u} = \text{Aut}(M_n(\mathcal{O}_X U,u))$ (automorphisms of $M_n(\mathcal{O}_X U,u)$ as a sheaf of $\mathcal{O}_X U,u$-algebras). We have the following

**Lemma 3.3.** Assume $n \geq 0$. The sequence of sheaves of groups on $X$

$$1 \rightarrow G_{m,X} \rightarrow GL(n, X) \rightarrow \text{PGL}(n, X) \rightarrow 1$$

is exact.

**Proof.** It is enough to show that for any étale open $(U, u)$ of $X$, the restriction to the Zariski site of $U$ of the sequence $1 \rightarrow G_{m,U,u} \rightarrow GL(n,U,u) \rightarrow \text{PGL}(n,U,u) \rightarrow 1$ is exact and this follows by [27, IV, Prop. 2.3. and Cor 2.4.].

Let $\text{Lf}(\mathcal{X})$ be the $S$-stack on $X$ of locally free $\mathcal{O}_X$-modules. Let $A$ be an Azumaya algebra over $X$. Consider the morphism of $S$-stacks on $X$

$$\text{End} : \text{Lf}(\mathcal{X}) \rightarrow \text{Az}(\mathcal{X}), \quad \mathcal{L} \mapsto \text{End}_{\mathcal{O}_X}(\mathcal{L})$$

Following [14, Chp IV 2.5], let $\delta(A)$ be the fibered category over $S_{\text{ét}}$ of trivializations of $A$ defined in the following way:

- for any $U \in \text{Ob}(S_{\text{ét}})$, the objects of $\delta(A)(U)$ are trivializations of $A|_U$, i.e. pairs $(\mathcal{L}, a)$ with $\mathcal{L} \in \text{Ob}(\text{Lf}(\mathcal{X})(U))$ and $a \in \text{Isom}_{U}(\text{End}_{\mathcal{O}_X}(\mathcal{L}), A|_U)$,
- for any arrow $f : V \rightarrow U$ of $S_{\text{ét}}$, the arrows of $\delta(A)$ over $f$ with source $(\mathcal{L}', a')$ and target $(\mathcal{L}, a)$ are arrows $\varphi : \mathcal{L}' \rightarrow \mathcal{L}$ of $\text{Lf}(\mathcal{X})$ over $f$ such that $\text{Az}(\mathcal{X})(f) \circ a' = a \circ \text{End}(\varphi)$, with $\text{Az}(\mathcal{X})(f) : A|_V \rightarrow A|_U$.

Since $\text{Lf}(\mathcal{X})$ and $\text{Az}(\mathcal{X})$ are $S$-stacks on $X$, $\delta(A)$ is also an $S$-stack on $X$ (see [14, Chp IV Prop 2.5.4 (i)]). Observe that the morphism of $S$-stacks $\text{End} : \text{Lf}(\mathcal{X}) \rightarrow \text{Az}(\mathcal{X})$ is locally surjective on objects by definition of Azumaya algebra. Moreover, it is locally surjective on arrows by exactness of the sequence (3.1). Therefore as in [14, Chp IV Prop 2.5.4 (ii)], $\delta(A)$ is a gerbe over $X$, called the **gerbe of trivializations of $A$**. For any object $(U, u)$ of $S_{\text{ét}}(X)$ the morphism of sheaves of groups on $U$

$$(G_{m,X})_{U,u} = (\mathcal{O}_X)_{U,u} \rightarrow (\text{Aut}(\mathcal{L}, a))_{U,u},$$

that sends a section $g$ of $(\mathcal{O}_X)_{U,u}$ to the multiplication $g : (\mathcal{L}, a)_{U,u} \rightarrow (\mathcal{L}, a)_{U,u}$ by this section, is an isomorphism. This means that the gerbe $\delta(A)$ is in fact a $G_{m,X}$-gerbe. By Corollary 2.2 we can then associate to any Azumaya algebra $A$ over $X$ a cohomological class in $H^2_{\text{ét}}(X, G_{m,X})$, denoted by $\overline{\delta(A)}$, which is given by the $G_{m,X}$-equivalence class of $\delta(A)$ in $\text{Gerbe}^2_{S}(G_{m,X})$.

**Proposition 3.4.** An Azumaya algebra $A$ over $X$ is trivial if and only if its cohomological class $\overline{\delta(A)}$ in $H^2_{\text{ét}}(X, G_{m,X})$ is zero.
Proof. The Azumaya algebra $A$ is trivial if and only if the gerbe $\delta(A)$ admits a global section if and only if its corresponding class $\overline{\delta(A)}$ is zero in $H^2_{\text{et}}(\mathcal{X}, \mathbb{G}_m)$. \hfill $\square$

Theorem 3.5. The morphism

$$\delta : \text{Br}(\mathcal{X}) \longrightarrow H^2_{\text{et}}(\mathcal{X}, \mathbb{G}_m)$$

$$[A] \longmapsto \overline{\delta(A)}$$

is an injective group homomorphism.

Proof. Let $A, B$ be two Azumaya algebras over $\mathcal{X}$. For any $U \in \text{Ob}(S_{\text{et}})$, the morphism of gerbes

$$\delta(A)(U) \times \delta(B)(U) \longrightarrow \delta(A \otimes_B B)(U)$$

is a ++-morphism, where $+: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is the group law underlying the sheaf $\mathbb{G}_m$. Therefore

$$(\mathcal{L}, a), (\mathcal{M}, b) \longmapsto (\mathcal{L} \otimes \mathcal{O}_X \mathcal{M}, a \otimes \mathcal{O}_X b)$$

is an injective group homomorphism.

Proof. Let $A, B$ be two Azumaya algebras over $\mathcal{X}$. For any $U \in \text{Ob}(S_{\text{et}})$, the morphism of gerbes

$$\delta(A)(U) \times \delta(B)(U) \longrightarrow \delta(A \otimes_B B)(U)$$

is well-defined and injective. Finally always from the equality (3.3) we get that $\delta$ is a group homomorphism. \hfill $\square$

4. Gerbes and Azumaya algebras over 1-motives

Let $M = [X \to G]$ be a 1-motive defined over a noetherian scheme $S$ and denote by $M$ its associated Picard $S$-stack (see [10, Exposé XVIII, §1.4]).

Definition 4.1. The Brauer group of the 1-motive $M$ is the Brauer group of its associated Picard $S$-stack $M$:

$$\text{Br}(M) := \text{Br}(M).$$

Moreover the Picard $S$-2-stack of $\mathbb{G}_m$-gerbes on $M$ is the Picard $S$-2-stack of $\mathbb{G}_m$-gerbes on $M$:

$$\text{Gerbes}(\mathbb{G}_m) := \text{Gerbes}(\mathbb{G}_m).$$

By [24, (3.4.3)] the associated Picard $S$-stack $M$ is isomorphic to the quotient stack $[G/X]$ (where $X$ acts on $G$ via the given morphism $\mu : X \to G$). Note that in general it is not algebraic in the sense of [24] because it is not quasi-separated. However the quotient map

$$\iota : G \to [G/X] \cong M$$

is representable, étale and surjective. The fiber product $G \times_{[G/X]} G$ is isomorphic to $X \times_S G$. Via this identification, the projections $\pi_i : G \times_{[G/X]} G \to G$ (for $i = 1, 2$) correspond respectively to the second projection $p_2 : X \times_S G \to G$ and to the map $\mu : X \times_S G \to G$ given by the action $(x, g) \mapsto u(x)g$. We can further identify the fiber product $G \times_{[G/X]} G \times_{[G/X]} G$ with $X \times_S X \times_S G$ and the partial projections $q_{13}, q_{23}, q_{12} : G \times_{[G/X]} G \times_{[G/X]} G \to G \times_{[G/X]} G$ respectively with the map $m_X \times \text{id}_G : X \times_S X \times_S G \to X \times_S G$ where $m_X$ denotes the group law of $X$, the map $\text{id}_X \times \mu : X \times_S X \times_S G \to X \times_S G$, and the partial projection $p_{23} : X \times_S X \times_S G \to X \times_S G$. The descent of Azumaya algebras with respect to the quotient map $\iota : G \to [G/X]$ is proved in the following Lemma (see [29, (9.3.4)] for the definition of pull-back of $\mathcal{O}_M$-algebras):
Lemma 4.2. The pull-back \( \iota^* : \text{Az}(M) \to \text{Az}(G) \) is an equivalence of \( S \)-stacks between the \( S \)-stack of Azumaya algebras on \( M \) and the \( S \)-stack of \( X \)-equivariant Azumaya algebras on \( G \). More precisely, to have an Azumaya algebra \( A \) on \( M \) is equivalent to have a pair

\[(A, \varphi)\]

where \( A \) is an Azumaya algebra on \( G \) and \( \varphi : p^*_2 A \to \mu^* A \) is an isomorphism of Azumaya algebras on \( X \times S \) \( G \) that satisfies (up to canonical isomorphisms) the cocycle condition

\[(m_X \times \text{id}_G)^* \varphi = \left( (\text{id}_X \times \mu)^* \varphi \right) \circ \left( (p^*_{23})^* \varphi \right).\]

Proof. For any object \( U \) of \( \mathbf{S}_U \) and any object \( x \) of \( \mathcal{M}(U) \), the descent of quasi-coherent modules is known for the morphism \( \iota_U : G \times _{\mathcal{M}, x} U \to U \) obtained by base change (see [24, Thm (13.5.5)]). The additional algebra structure descends by [23, II Thm 3.4]. Finally the Azumaya algebra structure descends by [22, III, Prop 2.8]. Since an Azumaya algebra on \( M \) is by definition a collection of Azumaya algebras on the various schemes \( U \), the general case follows.

As observed in [23, 134], the pull-back of \( \mathbb{G}_m \)-gerbes via the quotient map \( \iota : G \to M \) induces an additive 2-functor

\[\iota^* : \text{Gerbes}_S(\mathbb{G}_m, M) \to \text{Gerbes}_S(\mathbb{G}_m, G)\]

which associates to each \( \mathbb{G}_m, M \)-gerbe \( \mathcal{G} \) on \( M \) the \( \mathbb{G}_m, G \)-gerbe \( \iota^* \mathcal{G} \) on \( G \). Using the same notation as in Lemma 4.2, we can now state our result concerning the descent of \( \mathbb{G}_m \)-gerbes via the quotient map.

Lemma 4.3. The Picard \( S \)-2-stack \( \text{Gerbes}_S(\mathbb{G}_m, M) \) of \( \mathbb{G}_m, M \)-gerbes on \( M \) is equivalent (as Picard 2-stack) to the Picard \( S \)-2-stack of triplets

\[(\mathcal{G}', \varphi, \gamma)\]

where \( \mathcal{G}' \) is an object of \( \text{Gerbes}_S(\mathbb{G}_m, G) \) and \((\varphi, \gamma)\) is a descent datum on \( \mathcal{G}' \) with respect to \( \iota : G \to [G/X] \). More precisely, to have a \( \mathbb{G}_m, M \)-gerbe \( \mathcal{G} \) on \( M \) is equivalent to have a triplet \((\mathcal{G}', \varphi, \gamma)\) where

- \( \mathcal{G}' \) is a \( \mathbb{G}_m, G \)-gerbe on \( G \),
- \( \varphi : p^*_2 \mathcal{G}' \to \mu^* \mathcal{G}' \) is an \( \text{id}_{\mathbb{G}_m} \)-isomorphism of gerbes on \( X \times S \) \( G \) (hence in particular an isomorphism of gerbes), which restricts to the identity when pulled back via the diagonal morphism \( \Delta : G \to G \times _{[G/X]} G \cong X \times S \) \( G \), and
- \( \gamma : ((\text{id}_X \times \mu)^* \varphi) \circ ((p^*_{23})^* \varphi) \) is an isomorphism of cartesian \( S \)-functors between morphisms of \( S \)-stacks on \( X \times S \) \( X \times S \) \( G \cong G \times _{[G/X]} G \times _{[G/X]} G \), which satisfies the compatibility condition

\[p^*_{134} \gamma \circ [p^*_{34} \varphi \ast p^*_{123} \gamma] = p^*_{124} \gamma \circ [p^*_{234} \gamma \ast p^*_{12} \varphi]\]

when pulled back to \( X \times S \) \( X \times S \) \( G \cong G \times _{[G/X]} G \times _{[G/X]} G \times _{[G/X]} G \) := \( G^4 \) (here \( \text{pr}_{i,j,k} : G^4 \to G \times _{[G/X]} G \times _{[G/X]} G \) and \( \text{pr}_{i,j} : G^4 \to G \times _{[G/X]} G \) are the partial projections).

Under this equivalence, the pull-back \( \iota^* : \text{Gerbes}_S(\mathbb{G}_m, M) \to \text{Gerbes}_S(\mathbb{G}_m, G) \) is the additive 2-functor which forgets the descent datum: \( \iota^*(\mathcal{G}', \varphi, \gamma) = \mathcal{G}' \).

Proof. A \( \mathbb{G}_m, M \)-gerbe on \( M \) is by definition a collection of \( \mathbb{G}_m, U \)-gerbes over the various objects \( U \) of \( \mathbf{S} \). Hence it is enough to prove that for any object \( U \) of \( \mathbf{S} \) and any object \( x \) of \( \mathcal{M}(U) \), the descent of \( \mathbb{G}_m \)-gerbes with respect to the morphism \( \iota_U : G \times _{\mathcal{M}, x} U \to U \) obtained by base change is effective. This will be done in the following Proposition.
In order to prove the effectiveness of the descent of $\mathbb{G}_m$-gerbes with respect to the morphism $\iota_U : G \times_{\text{Spec} \mathbb{Z}} U \to U$, we need the **semi-local description of gerbes** done by Breen in [9, §2.3.], that we recall only in the case of $\mathbb{G}_m$-gerbes. According to Breen, to have a $\mathbb{G}_m$-gerbe $\mathcal{G}$ over a site $\mathbf{S}$ is equivalent to have the data
\begin{equation}
(\mathcal{T}(\mathbb{G}_m, U), \Psi_x), (\psi_x, \xi_x)_{x \in \mathcal{G}(U), U \in \mathbf{S}}
\end{equation}
indexed by the objects $x$ of the $\mathbb{G}_m$-gerbe $\mathcal{G}$ (recall that $\mathcal{G}$ is locally non-empty), where

- $\Psi_x : \mathcal{G}_U \to \mathcal{T}(\mathbb{G}_m, U)$ is an equivalence of $U$-stacks between the restriction $\mathcal{G}_U$ to $U$ of the $\mathbb{G}_m$-gerbe $\mathcal{G}$ and the neutral gerbe $\mathcal{T}(\mathbb{G}_m, U)$. This equivalence is determined by the object $x$ in $\mathcal{G}(U)$,
- $\psi_x = pr^*_1 \Psi_x \circ (pr^*_2 \Psi_x)^{-1} : \mathcal{T}(pr^*_2 \mathbb{G}_m, U) \to \mathcal{T}(pr^*_1 \mathbb{G}_m, U)$ is an equivalence of stacks over $U \times_S U$ (here $pr^*_i : U \times_S U \to U$ are the projections), which restricts to the identity when pulled back via the diagonal morphism $\Delta : U \to U \times_S U$, and
- $\xi_x : pr^*_{23} \psi_x \circ pr^*_1 \psi_x = pr^*_{12} \xi_x \circ [pr^*_3 \xi_x \circ pr^*_{23} \psi_x]$ when pulled back to $U \times_S U \times_S U := U^4$ (here $pr^*_{ijk} : U^4 \to U \times_S U \times_S U$ and $pr^*_i : U^4 \to U \times_S U$ are the partial projections), which satisfies the compatibility condition
\begin{equation}
pr^*_i \xi_x \circ [pr^*_3 \psi_x \circ pr^*_2 \xi_x] = pr^*_i \xi_x \circ [pr^*_3 \xi_x \circ pr^*_2 \psi_x]
\end{equation}

The $\mathbb{G}_m$-gerbe $\mathcal{G}$ may be reconstructed from the local data $(\mathcal{T}(\mathbb{G}_m, U), \Psi_x)_x$ using the transition data $(\psi_x, \xi_x)$.

**Remark 4.4.** In this paper, Breen’s semi-local description of gerbes allows us to reduce of one the degree of the cohomology groups involved: instead of working with gerbes, which are cohomology classes of $H^2(S, \mathbb{G}_m)$, we can work with torsors, which are cohomology classes of $H^1(S, \mathbb{G}_m)$.

**Proposition 4.5.** Let $p : S' \to S$ be a faithfully flat morphism of schemes which is quasi-compact or locally of finite presentation. The Picard $S$-2-stack $\text{Gerbe}_S(\mathbb{G}_m, S)$ of $\mathbb{G}_m$-gerbes over $S$ is equivalent (as Picard 2-stack) to the Picard $S$-2-stack of triplets $(\mathcal{G}', \varphi, \gamma)$ where

- $\mathcal{G}'$ is an object of $\text{Gerbe}_S(\mathbb{G}_m, S')$ and $(\varphi, \gamma)$ is a descent datum on $\mathcal{G}'$ with respect to $p : S' \to S$. More precisely, to have a $\mathbb{G}_m$-gerbe $\mathcal{G}$ over $S$ is equivalent to have a triplet $(\mathcal{G}', \varphi, \gamma)$ where

- $\mathcal{G}'$ is a $\mathbb{G}_m$-gerbe over $S'$,
- $\varphi : p^*_1 \mathcal{G}' \to p^*_2 \mathcal{G}'$ is an id$_{\mathbb{G}_m}$-isomorphism of gerbes on $S' \times_S S'$ (hence in particular an isomorphism of gerbes - here $p_i : S' \times_S S' \to S'$ are the projections), which restricts to the identity when pulled back via the diagonal morphism $\Delta : S' \to S' \times_S S'$, and
- $\gamma : p^*_{12} \varphi \circ p^*_{12} \varphi = p^*_{12} \varphi$ is an isomorphism of cartesian $S$-functors between morphisms of $S$-stacks on $S' \times_S S' \times_S S'$ (here $p_{ij} : S' \times_S S' \times_S S' \to S' \times_S S'$ are the partial projections), which satisfies the compatibility condition

\begin{equation}
p^*_{12} \gamma \circ [p^*_{13} \gamma \circ p^*_{12} \gamma] = p^*_{12} \gamma \circ [p^*_{13} \gamma \circ p^*_{12} \gamma]
\end{equation}

when pulled back to $S' \times_S S' \times_S S' := (S')^4$ (here $p_{ijk} : (S')^4 \to S' \times_S S' \times_S S'$ and $p_{ij} : (S')^4 \to S' \times_S S'$ are the partial projections).

Under this equivalence, the pull-back $p^* : \text{Gerbe}_S(\mathbb{G}_m, S) \to \text{Gerbe}_S(\mathbb{G}_m, S')$ is the additive 2-functor which forgets the descent datum: $p^*(\mathcal{G}', \varphi, \gamma) = \mathcal{G}'$. 

Proof. Let $U$ be an object of $S$. Let $(G', \varphi, \gamma)$ be a triplet as in the statement. By the semi-local description of gerbes done by Breen, to have the $G_{m,S'}$-gerbe $\mathcal{G}'$ over $S'$ is equivalent to have the data

$$\left( (\text{Tors}(G_m, U \times S U), \Psi_x), (\psi_x, \xi_x) \right)_{x \in \mathcal{G}'(U \times S U), U' \in S'}. $$

For any $x \in \mathcal{G}'(U' \times S U)$ with $U' \in S'$, the isomorphism of gerbes $\varphi: p_1^* \mathcal{G}' \to p_2^* \mathcal{G}'$ over $S' \times S S'$ defines an isomorphism of $(U' \times S U) \times S' (U' \times S U)\times S U$-stacks

$$\varphi(U' \times S U) : p_1^*U' \times S U \text{Tors}(G_m, U \times S U) \to p_2^*U' \times S U \text{Tors}(G_m, U \times S U),$$

where $p_{i(U' \times S U)} : (U' \times S U) \times S U \to (U' \times S U) \times S U$ are the projections. This isomorphism $\varphi_{U' \times S U}$ and the isomorphism of cartesian $S'$-functors $\gamma|U' \times S U$ satisfying (1.2) endow each object of the category $\text{Tors}(G_m, U' \times S U)(V')$ (here $V'$ an object of the site of $U' \times S U$) with a descent datum with respect to the morphism of schemes $p_{U' \times S U} : U' \times S U \to U$. By effectiveness of this descent, the $U' \times S U$-stack $\text{Tors}(G_m, U' \times S U)$ with the descent datum $(\varphi_{U' \times S U}, \gamma|U' \times S U)$ is equivalent to the $U$-stack $\text{Tors}(G_m, U)$. Since $\varphi : p_1^* \mathcal{G}' \to p_2^* \mathcal{G}'$ is also an $\text{id}_{G_m}$-morphism of gerbes on $S' \times S S'$, the equivalence of stacks $\psi_x : \text{Tors}(pr_2^* G_m, U \times S U) \to \text{Tors}(pr_1^* G_m, U \times S U)$ over $(U' \times S U) \times S'(U' \times S U)$ and the morphism of cartesian $S'$-functors $\xi_x : pr_1^* \psi_x \circ pr_2^* \psi_x \Rightarrow pr_1^* \psi_x$ satisfying (1.2) induce an equivalence of stacks $\psi_{x,U} : \text{Tors}(pr_2^* G_m, U \times S U) \to \text{Tors}(pr_1^* G_m, U \times S U)$ over $U \times S U$ and a morphism of cartesian $S'$-functors $\xi_{x,U} : pr_2^* \psi_{x,U} \circ pr_1^* \psi_{x,U} \Rightarrow pr_1^* \psi_{x,U}$ satisfying (1.2) (here $pr_i : (U' \times S U) \times S'(U' \times S U) \to (U' \times S U)$, $pr_{ij} : (U' \times S U)^2 \to (U' \times S U)^2$ are the projections involved in the semi-local description of the $G_{m,S'}$-gerbe $\mathcal{G}'$ over $S'$, and $pr_{i,U} : U \times S U \to U$, $pr_{ij,U} : U^3 \to U^2$ are the projections involved in the semi-local description of a $G_{m,S}$-gerbe over $S$).

Using the transition data $(\psi_{x,U}, \xi_{x,U})$, the local data $(\text{Tors}(G_m, U))_U$ glue together and furnish a $G_m$-gerbe over $S$.

\[\square\]

5. The generalized Theorem of the Cube for 1-motives and its consequences

We use the same notation of the previous Section. We denote by $M^3 = M \times S M \times S M$ (resp. $M^2 = M \times S M$) the fibered product of 3 (resp. 2) copies of $M$. Since any Picard stack admits a global neutral object, it exists a unit section denoted by $\epsilon : S \to M$. Consider the map

$$s_{ij} := M \times S M \to M \times S M \times S M$$

which inserts the unit section $\epsilon : S \to M$ into the k-th factor for $k \in \{1, 2, 3\} - \{i, j\}$.

Definition 5.1. Let $M$ be a 1-motive defined over a scheme $S$. Let $\ell$ be a prime number distinct from the residue characteristics of $S$. The 1-motive $M$ satisfies the generalized Theorem of the Cube for the prime $\ell$ if the natural homomorphism

$$\prod s_{ij}^* : H^2_{\acute{e}t}(M^3, G_{m,M^3})(\ell) \to H^2_{\acute{e}t}(M^2, G_{m,M^2})(\ell)^3 \to (s_{12}^*(x), s_{13}^*(x), s_{23}^*(x))$$

is injective.

Proposition 5.2. Let $M$ be a 1-motive satisfying the generalized Theorem of the Cube for a prime $\ell$ distinct from the residue characteristics of $S$. Let $N : M \to M$ be the multiplication by $N$ on the Picard $S$-stack $M$. Then for any $y \in H^2_{\acute{e}t}(M, G_{m,M})(\ell)$ we have that

$$N^* (y) = N^2 y + \left( \frac{N^2 - N}{2} \right) (-id_M)^*(y) - y.$$
Proof. First we prove that given three contravariant functors \( F, G, H : \mathcal{P} \to \mathcal{M} \), we have the following equality for any \( y \) in \( H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}})(\ell) \)

\[
(5.2) \quad (F+G+H)^*(y)-(F+G)^*(y)-(F+H)^*(y)-(G+H)^*(y)+F^*(y)+G^*(y)+H^*(y) = 0.
\]

Let \( pr_i : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) the projection onto the \( i \)-th factor. Put \( m_{i,j} = pr_i \otimes pr_j : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) and \( m = pr_1 \otimes pr_2 \otimes pr_3 : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \), where \( \otimes \) is the law group of the Picard \( S \)-stack \( \mathcal{M} \). The element

\[
z = m^*(y) - m^*_{1,2}(y) - m^*_{1,3}(y) - m^*_{2,3}(y) + pr_1^*(y) + pr_2^*(y) + pr_3^*(y)
\]

of \( H^2_{\text{et}}(\mathcal{M}^3, \mathcal{G}_{m,\mathcal{M}^3})(\ell) \) is zero when restricted to \( \mathcal{M} \times \mathcal{M} \times \mathcal{M} \) and \( \mathcal{M} \times \mathcal{M} \times S \) and \( \mathcal{M} \times \mathcal{M} \times \mathcal{M} \). Thus it is zero in \( H^2_{\text{et}}(\mathcal{M}^3, \mathcal{G}_{m,\mathcal{M}^3})(\ell) \) by the generalized Theorem of the Cube for \( \ell \). Finally, pulling back \( z \) by \( (F, G, H) : \mathcal{P} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M} \) we get (5.2).

Now, setting \( F = N, G = id_M, h = (-id_M) \) we get

\[
N^*(y) = (N + id_M)^*(y) + (N - id_M)^*(y) + 0^*(y) - N^*(y) - (id_M)^*(y) - (-id_M)^*(y).
\]

We rewrite this as

\[
(N + id_M)^*(y) - N^*(y) = N^*(y) - (N - id_M)^*(y) + (id_M)^*(y) + (-id_M)^*(y).
\]

If we denote \( z_1 = y \) and \( z_N = N^*(y) - (N - id_M)^*(y) \), we obtain \( z_{N+1} = z_N + y + (-id_M)^*(y) \).

By recursion, we get \( z_N = y + (N - id_M)(y + (-id_M)^*(y)) \). From the equality \( N^*(y) = z_N + (N - id_M)^*(y) \) we have

\[
N^*(y) = z_N + z_{N-1} + \cdots + z_1.
\]

and therefore we are done. \( \square \)

Corollary 5.3. Let \( M \) be a 1-motive satisfying the generalized Theorem of the Cube for a prime \( \ell \). Then, if \( \ell \neq 2 \),

\[
el^n H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \subseteq \ker \left[ (\ell^e_M)^* : H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \longrightarrow H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \right].
\]

and if \( \ell = 2 \),

\[
2^n H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \subseteq \ker \left[ (2^{n+1}_M)^* : H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \longrightarrow H^2_{\text{et}}(\mathcal{M}, \mathcal{G}_{m,\mathcal{M}}) \right].
\]

We finish this section searching the hypothesis we should put on the base scheme \( S \) in order to have that the 1-motive \( M = [X \xrightarrow{u} G] \) satisfies the generalized Theorem of the Cube. From now on we will switch freely between the two equivalent notion of invertible sheaf \( \mathcal{L} \) on the extension \( G \) and \( \mathcal{G}_m \)-torsor \( \text{Isom}(0_G, \mathcal{L}) \) on \( G \). First a result on tori:

Lemma 5.4. If \( T \) is a torus defined over a normal scheme \( S \), then \( \text{Gerbe}^2_\mathcal{G}(\mathcal{G}_m, T) = 0 \).

Proof. Since the question is local on \( S \), we may assume that \( T = \mathcal{G}_m^r \). The hypothesis of normality on the base scheme \( S \) implies that \( H^1(S, \mathbb{Z}/\ell) = 0 \) in \( S_{\text{fppf}} \) (see [28, Rem 7.2.4]), and so the group of isomorphism classes of \( \mathcal{G}_m, T \)-torsors on \( T \) is trivial, that is

\[
(5.3) \quad \text{Tors}^1(\mathcal{G}_m, T) = 0.
\]

Let \( \mathcal{F} \) be an element of \( \text{Gerbe}^2_\mathcal{G}(\mathcal{G}_m, T) \), that is the \( \mathcal{G}_m, T \)-equivalence class of a \( \mathcal{G}_m, T \)-gerbe \( \mathcal{F} \) on \( T \). By Breen’s semi-local description of gerbes ([13], 5.3), to have \( \mathcal{F} \) is equivalent to have the local data \( (\text{Tors}(\mathcal{G}_m, T, U), \Psi_x)_{x \in \mathcal{G}(U), U \in S_{\text{fppf}}} \), endowed with the transition data \( (\psi_x, \xi_x) \), where \( \Psi_x : \mathcal{G}(U) \to \text{Tors}(\mathcal{G}_m, T, U) \) is an equivalence of \( U \)-stacks. By equality (5.3), \( \text{Tors}^1(\mathcal{G}_m, T, U) = 0 \) for any \( U \in \text{Ob}(S_{\text{fppf}}) \) and therefore, modulo \( \mathcal{G}_m \)-equivalences, \( \mathcal{F} \) is globally equivalent to \( \text{Tors}(\mathcal{G}_m, T) \), that is \( \mathcal{F} = 0 \). \( \square \)
Now we investigate the case of an extension $G$ of an abelian $S$-scheme $A$ by an $S$-torus $T$. From the short exact sequence

$$0 \to T \xrightarrow{i} G \xrightarrow{\pi} A \to 0,$$

by Proposition 2.3 we get a long exact sequence involving the homotopy groups of the Picard $S$-2-stacks $\mathcal{G}erbes(S(G,A), \mathcal{G}erbes(S(G,T))$ and $\mathcal{G}erbes(S(G,T))$. We will compute explicitly the higher terms of this long exact sequence. Let $p^* : st(G_m,S[1]) \to T_{\mathcal{O}}(S(G,A))$ the additive functor defined by the pull-back of torsors via the structural morphism $p : A \to S$ of the abelian scheme $A$.

**Proposition 5.5.** Let $S$ be a normal scheme. There exists an additive 2-functor

$$C : \mathcal{A}ut(\mathcal{G}erbes(S(G,T))) \cong \mathcal{H}om(T, st(G_m,S[1])) \to \mathcal{G}erbes(S(G_m,A))$$

such that the sequence of abelian sheaves

$$0 \to \mathcal{G}erbe^0(S(G_m,A)) \xrightarrow{\pi^*} \mathcal{G}erbe^0(S(G_m,G)) \xrightarrow{i^*} \mathcal{G}erbe^0(S(G_m,T)) \xrightarrow{\mathcal{C}^0} \mathcal{G}erbe^1(S(G_m,A)) \xrightarrow{\pi^*}$$

$$\xrightarrow{\pi^*} \mathcal{G}erbe^1(S(G_m,G)) \xrightarrow{i^*} \mathcal{G}erbe^1(S(G_m,T)) \xrightarrow{\mathcal{C}^1} \mathcal{G}erbe^2(S(G_m,A)) \xrightarrow{\pi^*} \mathcal{G}erbe^2(S(G_m,G)) \to 0$$

is a long exact sequence.

**Proof.** Denote by $\mathcal{T}_{\mathcal{O}}r_{\mathcal{O}}(S(G,G_m))$ the Picard $S$-stack of $S_m$-torsors on $G$ with rigidification along the unit section $\epsilon_G : S \to G$. Because of this unit section $\epsilon_G$, the group of isomorphism classes of $S_m$-torsors with rigidification is canonically isomorphic to the group of isomorphism classes of $S_m$-torsors:

$$\mathcal{T}_{\mathcal{O}}r_{\mathcal{O}}^1(S(G,G_m)) \cong \mathcal{T}_{\mathcal{O}}r_{\mathcal{O}}^1(S(G_m,G)).$$

Denote by $\mathcal{C}ub(G,G_m)$ the Picard $S$-stack of $S_m$-torsors on $G$ with cubical structure and by $\mathcal{C}ub^i(G,G_m)$ for $i = 1, 0$ its classifying groups. In [2], Prop 2.4, Breen proves the theorem of the Cube for extensions of abelian schemes by tori which are defined over a normal scheme, that is the forgetful additive functor $\mathcal{C}ub(G,G_m) \to \mathcal{T}_{\mathcal{O}}r_{\mathcal{O}}(S(G,G_m))$ is an equivalence of Picard $S$-stacks. In particular

$$\mathcal{C}ub^1(G,G_m) \cong \mathcal{T}_{\mathcal{O}}r_{\mathcal{O}}^1(S(G,G_m)).$$

By [28, Chp I, Prop 7.2.2], the Picard $S$-stack $\mathcal{C}ub(A,G_m)$ is equivalent to the Picard $S$-stack of pairs $(L,s)$, where $L$ is a $S_m$-torsor on $G$ with cubical structure, and $s$ is a trivialization of $i^*L$ in the category $\mathcal{C}ub(T,G_m)$. Under this identification, the pullback functor $\pi^* : \mathcal{C}ub(A,G_m) \to \mathcal{C}ub(G,G_m)$ is the forgetful functor that maps a pair $(L,s)$ to $L$. Generalizing the results of [2], Prop 2] from categories to stacks, to the short exact sequence $0 \to T \xrightarrow{i} G \xrightarrow{\pi} A \to 0$ we associate the following long exact sequence involving the classifying groups of the Picard $S$-stacks $\mathcal{C}ub(G,G_m), \mathcal{C}ub(A,G_m)$ and $\mathcal{C}ub(T,G_m)$

$$0 \to \mathcal{H}om(G,G_m) \xrightarrow{i^*} \mathcal{H}om(T,G_m) \xrightarrow{\xi} \mathcal{C}ub^1(A,G_m) \xrightarrow{\pi^*} \mathcal{C}ub^1(G,G_m) \xrightarrow{i^*} \mathcal{C}ub^1(T,G_m),$$

where $\mathcal{H}om(G,G_m) = \mathcal{C}ub^0(G,G_m), \mathcal{H}om(T,G_m) = \mathcal{C}ub^0(T,G_m), \mathcal{C}ub^0(A,G_m) = 0$, and $\xi$ is the morphism of sheaves which sends a morphism of group schemes $\alpha : T \to G_m$ to the class $[\alpha_*G]$ of the push-down of $G$ via $\alpha$ (recall that extensions of $A$ by $G_m$ are in particular $G_m$-torsors on $A$). Always generalizing from categories to stacks [28, Prop 7.2.1], we have that the Picard $S$-stack $\mathcal{C}ub(T,G_m)$ of $S_m$-torsors on $T$ with cubical structure is equivalent (as Picard stack) to the Picard $S$-stack $\mathcal{E}xt(T,G_m)$ of extensions of $T$ by $G_m$. Now since the question is local over $S$, we may assume that the torus $T$ underlying the extension $G$ is $G_m^r$. The hypothesis of normality on the base scheme $S$ implies that $\mathcal{E}xt^1(S,G_m^r) = H^1(S,\mathcal{O}_S) = 0$. The
in $\mathbf{S}_{fppf}$, and therefore $\text{Cub}^1(T, \mathbb{G}_m) \cong \text{Ext}^1(T, \mathbb{G}_m) \cong (\text{Ext}^1(\mathbb{G}_m, \mathbb{G}_m))^\tau = 0$. Using the isomorphisms (5.5) and (5.6), from the long exact sequence (5.7) we get the long exact sequence of abelian sheaves

\[(5.8) \quad 0 \to \text{Hom}(G, \mathbb{G}_m) \overset{i^*}{\to} \text{Hom}(T, \mathbb{G}_m) \overset{\xi}{\to} \text{Tors}^1(\mathbb{G}_m, A) \overset{\pi^*}{\to} \text{Tors}^1(\mathbb{G}_m, G) \to 0.\]

In particular, the group $\text{Tors}^1(\mathbb{G}_m, A)$ is a Hom$(T, \mathbb{G}_m)$-torsor over $\text{Tors}^1(\mathbb{G}_m, G)$. Breen’s semi-local description of gerbes (5.4) asserts that to give a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on $G$ is equivalent to have the local data $(\text{Tors}(\mathbb{G}_m, G, U), \Psi_x)_{x \in S(U), U \in \mathbf{S}_{fppf}}$ endowed with the transition data $(\psi_x, \xi_x)$. Since the equivalences of $U$-stacks $\Psi_x : \mathcal{G}|_U \to \text{Tors}(\mathbb{G}_m, G, U)$ and the transition data $(\psi_x, \xi_x)$ are compatible with the pull-back, the above long exact sequence of abelian sheaves (5.8) is the local version of the higher terms of the long exact sequence expected.

More precisely, let $y = \pi(P(x)) \in A(U)$, where $P : G \to G$ is the structural morphism of $\mathcal{G}$. The equivalence of $U \times \mathbb{G}_m$-stacks $\psi_x : \text{Tors}(\mathbb{G}_m, G, U) \to \text{Tors}(\mathbb{G}_m, G, U)$ induces an isomorphism between the classifying groups $\psi_x^1 : \text{Tors}^1(\mathbb{G}_m, G, U) \to \text{Tors}^1(\mathbb{G}_m, G, U)$. Since our question is local over $S$ and since $\text{Tors}^1(\mathbb{G}_m, A)$ is a Hom$(T, \mathbb{G}_m)$-torsor over $\text{Tors}^1(\mathbb{G}_m, G)$, $\psi_x^1$ induces an isomorphism $\overline{\psi}_y : \text{Tors}^1(\mathbb{G}_m, A, U) \to \text{Tors}^1(\mathbb{G}_m, A, U)$.

Consider the morphism of cartesian $S$-functors $\overline{\xi}_y : \psi_{23}^1 \overline{\psi}_y \circ \psi_{12}^1 \overline{\psi}_y = \psi_{13}^1 \overline{\psi}_y$. By construction, $\overline{\xi}_y$ satisfies the equality (4.4). The local data $(\text{Tors}^1(\mathbb{G}_m, A, U))_{y = \pi(P(x)), x \in S(U)}$ endowed with the transition data $(\overline{\psi}_y, \overline{\xi}_y)$ define a $\mathbb{G}_m$-equivalence class $\mathcal{G}'$ of a $\mathbb{G}_m$-gerbe on $A$ such that $\pi^*(\mathcal{G}') = \mathcal{G}$.

\[\text{Corollary 5.6. Let } S \text{ be a normal scheme. The group } \text{Gerbe}^2_S(\mathbb{G}_m, A) \text{ is a Gerbe}^1_S(\mathbb{G}_m, T)-\text{torsor over } \text{Gerbe}^2_S(\mathbb{G}_m, G).\]

In particular, the morphism of abelian groups $\pi^* : \text{Gerbe}^2_S(\mathbb{G}_m, A) \to \text{Gerbe}^2_S(\mathbb{G}_m, G)$ induced by the pull-back of gerbes is surjective, that is any $\mathbb{G}_m$-gerbe on $G$ comes from a $\mathbb{G}_m$-gerbe on $A$ modulo $\mathbb{G}_m$-equivalences.

\[\text{Corollary 5.7. Let } S \text{ be a normal scheme. The homomorphism of abelian group } \pi^*_1 : \text{Gerbe}^2_S(\mathbb{G}_m, A)(\ell) \to \text{Gerbe}^2_S(\mathbb{G}_m, G)(\ell), \text{ given by the restriction of } \pi^* \text{ to } \ell\text{-primary components, is a bijection.}\]

**Proof.** First observe that $\pi^*_1$ is well defined: Since the fibered product and the contracted product commute, the elements of $\text{Gerbe}^2_S(\mathbb{G}_m, A)$ which are killed by a power of $\ell$ are sent via $\pi^*_1$ to elements of $\text{Gerbe}^2_S(\mathbb{G}_m, G)(\ell)$.

Now we prove that $\pi^*_1$ is injective: Let $\mathcal{G}'$ be the $\mathbb{G}_m$-equivalence class of a $\mathbb{G}_m$-gerbe $\mathcal{G}'$ on $A$ such that $\Lambda^n \mathcal{G}' = 0$ for some integer $n > 0$, where $\Lambda$ is the contracted product of $\mathbb{G}_m$-gerbes, and such that $\pi^*_1(\mathcal{G}') = 0$, that is, modulo $\mathbb{G}_m$-equivalences, $\pi^*_1(\mathcal{G}')$ is equivalent to $\text{Tors}(\mathbb{G}_m, G)$. Assume that $\mathcal{G}' \neq 0$. By Breen’s semi-local description of gerbes (5.5), to have $\mathcal{G}'$ is equivalent to have the local data $(\text{Tors}(\mathbb{G}_m, A, U), \Psi_x)_{x \in S(U), U \in \mathbf{S}_{fppf}}$ endowed with the transition data $(\psi_x, \xi_x)$, where $\Psi_x : \mathcal{G}'|_U \to \text{Tors}(\mathbb{G}_m, A, U)$ is an equivalence of $U$-stacks. Since $\mathcal{G}' \neq 0$, there exists two objects $t_{U,x}$ and $t_{V,y}$ of $\text{Tors}(\mathbb{G}_m, A, U)$ and $\text{Tors}(\mathbb{G}_m, A, V)$ respectively such that the isomorphism class $t_{U,x|U \times V} - t_{V,y|U \times V}$ is not trivial in $\text{Tors}^1(\mathbb{G}_n, A|U \times U \times V)$. In this local setting, the condition $\pi^*_1(\mathcal{G}') = 0$ means that $\pi^*(t_{U,x|U \times V} - t_{V,y|U \times V}) = 0$ in $\text{Tors}^1(\mathbb{G}_m, G|U \times U \times V)$, and therefore using the long exact sequence (5.8), we get that

\[(5.9) \quad t_{U,x|U \times V} - t_{V,y|U \times V} \in \xi(\text{Hom}(T, \mathbb{G}_m)).\]
Moreover, always in this local setting, the condition $\Lambda^{\ell^n}G = 0$ becomes
\[(5.10)\]
\[\ell^n(t_{U,x}(U \times V) - t_{V,y}(U \times V)) = 0\]
in $\mathfrak{Tors}(G_{m,A,\mathcal{X} \times_{\mathcal{S},G} V})$. Since $\text{Hom}(T, G_m)$ is torsion free and since $G$ is not the trivial extension of $A$ by $T$, the image $\xi(\text{Hom}(T, G_m))$ is torsion free and so \([5,9]\) and \([5,10]\) furnish a contradiction. Therefore, modulo $G_m$-equivalences, $G'$ is globally equivalent to $\mathfrak{Tors}(G_{m,A})$, that is $\mathfrak{G} = 0$.

Finally we check that $\pi'$ is surjective: Let $\mathfrak{G}$ be the $G_{m,G}$-equivalence class of a $G_{m,G}$-gerbe $\mathfrak{G}$ on $G$ such that $\Lambda^{\ell^n}G = 0$ for some integer $n > 0$, where $\Lambda$ is the contracted product of $G_{m,G}$-gerbes. Assume that $\mathfrak{G} \neq 0$. By Proposition 5.5 it exists a $G_{m,A'}$-equivalence class $\mathfrak{G}$ of a $G_{m,A'}$-gerbe $G'$ on $A$ such that $\mathfrak{G} = \pi'(\mathfrak{G})$. Since the fibered product and the contracted product commute, modulo $G_{m,G}$-equivalences we have that
\[(5.11)\]
\[\Lambda^{\ell^n}G \cong \pi^*(\Lambda^{\ell^n}G') \cong G \times_{\pi,A,P'} \Lambda^{\ell^n}G'\]
where $P' : \Lambda^{\ell^n}G' \to A$ is the structural morphism underlying $\Lambda^{\ell^n}G'$. According to [1062 (2.1.1)] the equality $\Lambda^{\ell^n}G = 0$ means that it exists an $S$-point $g : S \to G$ such that $S \times_{g,G,P} \Lambda^{\ell^n}G \neq 0$, where $P : \Lambda^{\ell^n}G \to G$ is the structural morphism underlying $\Lambda^{\ell^n}G$. Let $x$ be an object of $S \times_{g,G,P} \Lambda^{\ell^n}G$ over $g \in G(S)$. Via the $G_{m,G}$-equivalence \([5,11]\), the object $x$ corresponds to an object $y$ of $S \times_{g,G,P'} \Lambda^{\ell^n}G' \cong S \times_{g,G,P'} \Lambda^{\ell^n}G'$ over $\pi(g) \in A(S)$ (here $P' : G \times_{\pi,A,P'} \Lambda^{\ell^n}G' \to G$ is the projection to the first factor).

If the composite $\pi \circ g : S \to A$ is the zero map, it exists an $S$-point $t : S \to T$ such that $i \circ t = g$, and we can consider $\Lambda^{\ell^n}G$ as a $G_{m,T}$-gerbe $\mathfrak{G}$ on $T$. Since tori are divisible in $S_{fpf}$, it exists $t'$ such that $\ell^n t' = t$. But then, by Lemma [7,3] $i \circ t' : S \to G$ is an $S$-point of $G$ such that $S \times_{i \circ t',G,P} \mathfrak{G} \neq 0$, and the neutralization of the gerbes $\Lambda^{\ell^n}G$ and $\mathfrak{G}$ over $t$ and $t'$ are compatible. This is a contradiction since $\mathfrak{G} \neq 0$. Thus, we have an object $y$ of $S \times_{g,G,P} \Lambda^{\ell^n}G'$ over a non trivial point $\pi(g) \in A(S)$ and by [1062 (2.1.1)], this means that $\mathfrak{G}$ is in fact an element of $\text{Gerbe}^G_2(G_{m,A}((\ell))$.}

Let $G_i$ be an extension of an abelian $S$-scheme $A_i$ by an $S$-torus for $i = 1, 2, 3$, and denote by $\epsilon^G_i : S \to G_i$ its unit section. Let $s^G_{ij} := G_i \times_S G_j \to G_1 \times_S G_2 \times_S G_3$ be the map obtained from the unit section $\epsilon^G_i : S \to G_i$ after the base change $G_i \times_S G_j \to S$ (i.e. the map which inserts $\epsilon^G_i : S \to G_i$ into the $k$-th factor for $k \in \{1, 2, 3\} - \{i, j\}$). We denote by $\pi_i : G_i \to A_i$ the surjective morphism of $S$-schemes underlying the extension $G_i$. If $\epsilon^A_i : S \to A_i$ is the unit section of the abelian scheme $A_i$, we have that $\epsilon^A_i = \pi_i \circ \epsilon^G_i$. Let $s^A_{ij} := A_i \times_S A_j \to A_1 \times_S A_2 \times_S A_3$ be the map obtained from the unit section $\epsilon^A_i : S \to A_i$ after the base change $A_i \times_S A_j \to S$. Since $\epsilon^A_i = \pi_i \circ \epsilon^G_i$, observe that
\[(\pi_1 \times \pi_2 \times \pi_3) \circ s^G_{ij} = s^A_{ij} \circ (\pi_i \times \pi_j).
\]
As an immediate consequence of Corollaries [5.7, 22] and of [20, Cor 2.6], we can then conclude that

**Corollary 5.8.** Let $S$ be a connected, reduced, geometrically unibranch and noetherian scheme and let $G_i$ be an extension of an abelian $S$-scheme $A_i$ by an $S$-torus $T_i$ for $i = 1, 2, 3$. Let $\ell$ be a prime distinct from the residue characteristics of $S$. Then, with the above notations, the natural homomorphism
\[(5.12)\]
\[\Pi_{x} \mathcal{H}^2_{\ell}(G_1 \times_S G_2 \times S G_3, G_m)(\ell) \quad \leftrightarrow \quad \Pi_{(i,j) \in \{1,2,3\}} \mathcal{H}^2_{\ell}(G_i \times_S G_j, G_m)(\ell)
\]
\[s^G_{ij}(x), s^G_{ij}(x), s^G_{ij}(x))\]
is injective.

In particular, if the base scheme is connected, reduced, geometrically unibranch and noetherian, an extension of an abelian \( S \)-scheme by an \( S \)-torus satisfies the generalized Theorem of the Cube for any prime \( \ell \) distinct from the residue characteristics of \( S \).

Using the effectiveness of the descent data for \( \mathbb{G}_m \)-gerbes via the quotient map \( \iota : G \to M \) (Lemma 13), from the above corollary we get the generalized Theorem of the Cube for the 1-motive \( M = [X \to G] \) (Theorem 12).

6. Proof of Theorem 1.3

Let \( S \) be a scheme. We will need the finite site on \( S \): first recall that a morphism of schemes \( f : X \to S \) is said to be \textit{finite locally free} if it is finite and \( f_*\big(\mathcal{O}_X\big) \) is a locally free \( \mathcal{O}_S \)-module. In particular, by [17, Prop (18.2.3)] finite étale morphisms are finite locally free. The finite site on \( S \), denoted \( S_f \), is the category of finite locally free schemes over \( S \), endowed with the topology generated from the pretopology for which the set of coverings of a finite locally free scheme \( T \) over \( S \) is the set of single morphisms \( u : T' \to T \) such that \( u \) is finite locally free and \( T = u(T') \) (set theoretically). There is a morphism of site \( \tau : S_{fppf} \to S_f \). If \( F \) is sheaf for the étale topology, then

\[
F(T)_f = \{ y \in F(T) \mid \text{there is a covering } u : T' \to T \text{ in } S_f \text{ with } F(u)(y) = 0 \}
\]
i.e. \( F(T)_f \) are the elements of \( F(T) \) which can be split by a finite locally free covering.

**Proposition 6.1.** Let \( G \) be an extension of an abelian scheme by a torus, which is defined over a normal and noetherian scheme \( S \), and which satisfies the generalized Theorem of the Cube for a prime number \( \ell \) distinct from the characteristics of \( S \). Then the \( \ell \)-primary component of the kernel of the homomorphism \( H^2_{\text{ét}}(\epsilon) : H^2_{\text{ét}}(G, \mathbb{G}_m, G) \to H^2_{\text{ét}}(S, \mathbb{G}_m, S) \) induced by the unit section \( \epsilon : S \to G \) of \( G \) is contained in the Brauer group of \( G \):

\[
\ker \left[ H^2_{\text{ét}}(\epsilon) : H^2_{\text{ét}}(G, \mathbb{G}_m, G) \to H^2_{\text{ét}}(S, \mathbb{G}_m, S) \right] (\ell) \subseteq \text{Br}(G).
\]

**Proof.** In order to simplify notations we denote by \( \ker(H^2_{\text{ét}}(\epsilon)) \) the kernel of the homomorphism \( H^2_{\text{ét}}(\epsilon) : H^2_{\text{ét}}(G, \mathbb{G}_m, G) \to H^2_{\text{ét}}(S, \mathbb{G}_m, S) \).

1. First we show that \( H^2_{\text{ét}}(G_f, \tau_\ell \mu_\ell^n) \) is isomorphic to \( H^2_{\text{ét}}(G, \mu_\ell^n)_f \). By definition, \( R^1\tau_\ell \mu_\ell^n \) is the sheaf on \( G_f \) associated to the presheaf \( U \to H^1(U_{fppf}, \mu_\ell^n) \). This latter group classifies torsors in \( U_{fppf} \) under the finite locally free group scheme \( \mu_\ell^n \), but since any invertible sheaf over a semi-local ring is free, we have that \( R^1\tau_\ell \mu_\ell^n = (0) \). The Leray spectral sequence for the morphism of sites \( \tau : S_{fppf} \to S_f \) (see [27, page 309]) gives then the isomorphism

\[
H^2_{\text{ét}}(G_f, \tau_\ell \mu_\ell^n) \cong \ker \left[ H^2_{\text{ét}}(G, \mu_\ell^n) \to H^0_f(G_f, R^2\tau_\ell \mu_\ell^n) \right] = H^2_{\text{ét}}(G, \mu_\ell^n)_f
\]

where the map \( \pi \) is the edge morphism which can be interpreted as the canonical morphism from the presheaf \( U \to H^2_{\text{ét}}(U_{fppf}, \mu_\ell^n) \) to the associate sheaf \( R^2\tau_\ell \mu_\ell^n \).

2. Now we prove that \( H^2_{\text{ét}}(G, \mu_\ell) \) maps onto \( \ker(H^2_{\text{ét}}(\epsilon))(\ell) \). Let \( x \) be an element of \( \ker(H^2_{\text{ét}}(\epsilon)) \) with \( \ell^n x = 0 \) for some \( n \). The filtration on the Leray spectral sequence for
Moreover, we have to show that if \( \ell^n x = 0 \), then

\[
\ker(H^2_{et}(G, \mu_{\ell^n})) = (\ell^n) \subseteq \ker(H^2_{et}(G, \mu_{\ell^n})).
\]

(3) Here we show that ker\((H^2_{et}(\ell))(\ell) \subseteq \tau^*H^2_f(G_f, \tau_\ell, \mu_{\ell^n})\). By the first two steps, \( H^2_f(G_f, \tau_\ell, \mu_{\ell^n}) \) maps onto ker\((H^2_{et}(\ell))(\ell) \subseteq H^2_f(G_f, \tau_\ell, \mu_{\ell^n})\), we can then conclude that ker\((H^2_{et}(\ell))(\ell) \subseteq \tau^*H^2_f(G_f, \tau_\ell, \mu_{\ell^n})\).

(4) Let \( x \) be an element of ker\((H^2_{et}(\ell))(\ell) \subseteq \tau^*H^2_f(G_f, \tau_\ell, \mu_{\ell^n})\). By [20, Lem 3.2.], the group \( \tau^*H^2_f(G_f, \tau_\ell, \mu_{\ell^n}) \) is contained in the Čech cohomology group \( \check{H}^2_{et}(G_f, \tau_\ell, \mu_{\ell^n}) \) and so, from step (3), \( x \) is an element of \( \check{H}^2_{et}(G_f, \tau_\ell, \mu_{\ell^n}) \). By Corollary 5.3, the isogeny \( \ell^n : G \to G \) is a finite locally free covering which splits \( x \), that is \( x \in \check{H}^2_{et}(G_f, \tau_\ell, \mu_{\ell^n}) \).

Finally, we can prove our second main result.

**Proof of Theorem 0.3.** We have to show that if \( G \) satisfies the generalized Theorem of the Cube for a prime \( \ell \), then

\[
\ker\left[H^2_{et}(\ell) : H^2_{et}(M, \mathbb{G}_m, M) \to H^2_{et}(S, \mathbb{G}_m, S)\right](\ell) \subseteq \text{Br}(M).
\]

In order to simplify notations we denote by ker\((H^2_{et}(\ell))\) the kernel of the homomorphism \( H^2_{et}(\ell) : H^2_{et}(M, \mathbb{G}_m, M) \to H^2_{et}(S, \mathbb{G}_m, S) \). Let \( x \) be an element of ker\((H^2_{et}(\ell))\) such that \( \ell^n x = 0 \) for some \( n \). Let \( y = \ell^n x \) be the image of \( x \) via the homomorphism \( \ell^n : H^2_{et}(M, \mathbb{G}_m, M) \to H^2_{et}(S, \mathbb{G}_m, S) \) induced by the quotient map \( \ell : G \to [G/X] \). Because of the commutativity of the following diagram

(6.2)

\[
\begin{array}{ccc}
H^2_{et}(M, \mathbb{G}_m, M) & \xrightarrow{x} & H^2_{et}(G, \mathbb{G}_m, G) \\
\downarrow \quad x & \quad & \downarrow \quad \\
H^2_{et}(S, \mathbb{G}_m, S) & \to & H^2_{et}(S, \mathbb{G}_m, S)
\end{array}
\]

(since \( x : S \to M \) is the unit section of \( M \) and \( x_G : S \to G \) is the unit section of \( G \), \( x \circ x_G = x \)), \( y \) is in fact an element of ker\((H^2_{et}(\ell))(\ell)\). By Proposition 6.1, we know that
By the effectiveness of the descent of $L$, $\ker(H^2_\delta(M, G_m, G))$ obtained in Corollary 2.2 the element $x$ corresponds to the $G_m$-equivariant class $\overline{K}$ of a $G_m$-gerbe $\mathcal{K}$ on $M$, and the element $y$ corresponds to the $G$-equivariant class $\iota^*\mathcal{K}$ of the $G$-gerbe $\iota^*\mathcal{K}$ on $G$, which is the pull-back of $\mathcal{K}$ via the quotient map $\iota: G \to [G/X]$. By the effectiveness of the descent of $G_m$-gerbes with respect to $\iota$ proved in Lemma 4.3, we can identify the $G_m$-gerbe $\mathcal{K}$ on $M$ with the triplet $(\iota^*\mathcal{K}, \varphi, \gamma)$, where $(\varphi, \gamma)$ is a descent datum on the $G_m$-gerbe $\iota^*\mathcal{K}$ with respect to $\iota$. More precisely, $\varphi: p_2^*\iota^*\mathcal{K} \to \mu^*\iota^*\mathcal{K}$ is an id$G_m$-isomorphism of gerbes on $X \times_S G$ and $\gamma: (\iota^*\mathcal{K})^\varphi \circ ((p_{23})^*\varphi) \Rightarrow (m_X \times id_G)^\varphi$ is an isomorphism of cartesian $S$-functors which satisfies the compatibility condition (4.2). Moreover via the inclusions $\ker(H^2_\delta(M, G)) \hookrightarrow Br(G) \xrightarrow{\delta} H^2_\delta(G, G_m, G)$, in $\text{Gerbe}_G^2(M, G_m, G)$ the class $\iota^*\mathcal{K}$ coincides with the class $\delta([A])$ of the gerbe of trivializations of the Azumaya algebra $A$.

Now we will show that the descent datum $(\varphi, \gamma)$ with respect to $\iota$ on the $G_m$-gerbe $\iota^*\mathcal{K}$ induces a descent datum $\varphi^A: p_2^*A \to \mu^*A$ with respect to $\iota$ on the Azumaya algebra $A$, which satisfies the cocycle condition (6.1). Since the statement of our main theorem involves classes of Azumaya algebras and $G_m$-equivariance classes of $G_m$-gerbes, we may assume that $\iota^*\mathcal{K} = \delta(A)$, and so the pair $(\varphi, \gamma)$ is a canonical descent datum on $\delta(A)$. The isomorphism of gerbes $\varphi: p_2^*\delta(A) \to \mu^*\delta(A)$ on $X \times_S G$ implies an isomorphism of categories $\varphi(U): p_2^*\delta(A)(U) \to \mu^*\delta(A)(U)$ for any object $U$ of $S_{\text{et}}$ and hence we have the following diagram

\[
\begin{array}{ccc}
\text{End}_{\mathcal{O}_{X \times S/G}}(p_2^*\mathcal{L}_1) & \xrightarrow{a_1} & p_2^*A|_U \\
\downarrow \text{End}_\varphi(U) & & \\
\text{End}_{\mathcal{O}_{X \times S/G}}(\mu^*\mathcal{L}_2) & \xrightarrow{a_2} & \mu^*A|_U
\end{array}
\]


with $\mathcal{L}_1$ and $\mathcal{L}_2$ objects of $\text{L}(G)(U)$. For any object $U$ of $S_{\text{et}}$, we define $\varphi^A_U := a_2 \circ \text{End}_\varphi(U) \circ a_1^{-1}: p_2^*A|_U \to \mu^*A|_U$. It is an isomorphism of Azumaya algebras over $U$. The collection $(\varphi^A_U)_U$ of all these isomorphisms furnishes the expected isomorphism of Azumaya algebras $\varphi^A: p_2^*A \to \mu^*A$ on $X \times_S G$. The descent datum $\gamma$, which satisfies the compatibility condition (4.2), implies that $\varphi^A$ satisfies the cocycle condition (6.1).

By Lemma 4.2 the descent of Azumaya algebras with respect to $\iota$ is effective, and so the pair $(A, \varphi^A)$ corresponds to an Azumaya algebra $A$ on $M$, whose equivalence class $[A]$ is an element of Br($M$).

Now we recall some results about the Brauer groups of the pure motives underlying a 1-motive $M = [u: X \to G]$ defined over an arbitrary field $k$:

- the Brauer group of the cocharacter group of the torus $G_m^n$ is zero, i.e. Br($\mathbb{Z}^r$) = 0.
- by [21, Cor 1] if $T$ is a torus defined over an arbitrary field $k$, then Br($T$) = $H^2_\delta(T, G_m, T)$. In [26] Magid computes explicitly the Brauer group of a $d$-dimensional torus $T$ defined over an algebraic closed field of characteristic zero: Br($T$) = ($\mathbb{Q}/\mathbb{Z}$)$^n$ where $n = d(d-1)/2$.
- by Theorem 3.1 (vi) if $A$ is an abelian variety defined over an arbitrary field $k$, then $\text{Br}(A) \cong H^2_\delta(A, G_m, A)$.
- the group variety $G$ underlying the 1-motive $M$ is an extension of an abelian $k$-variety $A$ by a $k$-torus. Since $T$ is smooth and since smoothness is stable under base extensions, the extension $G$ is smooth over $A$. But the abelian variety is smooth and
so $G$ is smooth. Hence by Theorem 5.1 (vi),

$$\text{Br}(G) \cong H^2_{\text{ét}}(G, \mathbb{G}_{m,G}).$$

(6.4)

Because of the weight filtration $W_*$ of the 1-motive $M$, we have the exact sequence $0 \to G \to M \to X[1] \to 0$, where $X[1] = [X \to 0]$ is the complex with $X$ in degree -1. Therefore it is interesting to study the Brauer group of $X[1]$. We will do it over an arbitrary noetherian scheme $S$. By Deligne in [10], Exposé XVIII, §1.4 the Picard $S$-stack $\text{st}(X[1])$ associated to the complex $X[1]$ is just the $S$-stack of $X$-torsors. The Brauer group of $X[1]$ is then the Brauer group of the Picard $S$-stack of $X$-torsors:

$$\text{Br}(X[1]) \coloneqq \text{Br}(\text{st}(X[1])).$$

By [24, (3.4.3)] the associated Picard $S$-stack $\text{st}(X[1])$ is isomorphic to the quotient stack $[S/X]$. The structural morphism $\tau : [S/X] \to S$ admits a section $\epsilon : S \to [S/X]$, and so

- the pull-back $\epsilon^* : \text{Az}(X[1]) \to \text{Az}(S)$ is an equivalence of $S$-stacks between the $S$-stack of Azumaya algebras on $X[1]$ and the $S$-stack of Azumaya algebras on $S$ with descent data with respect to $\epsilon$, and
- the pull-back $\tau^* : \text{Br}(S) \to \text{Br}(X[1])$ is an injective homomorphism.

**Proposition 6.2.** Let $X$ be a group scheme, which is defined over a noetherian scheme $S$, and which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module. Then the injective group homomorphism

$$\delta : \text{Br}(X[1]) \to H^2_{\text{ét}}(\text{st}(X[1]), \mathbb{G}_{m,\text{st}(X[1])}),$$

constructed in Theorem 3.2, is in fact a bijective group homomorphism.

**Proof.** In this proof, in order to simplify notation, we write $X[1]$ instead of $\text{st}(X[1])$. We will construct a group homomorphism $\lambda : H^2_{\text{ét}}(X[1], \mathbb{G}_{m,X[1]}) \to \text{Br}(X[1])$ such that $\delta \circ \lambda = \text{id}$. By Corollary 2.2, the elements of $H^2_{\text{ét}}(X[1], \mathbb{G}_{m,X[1]})$ can be seen as $\mathbb{G}_{m,X[1]}$-equivalence classes of $\mathbb{G}_{m,X[1]}$-gerbes on $X[1]$. Therefore it is enough to associate to any $\mathbb{G}_{m,X[1]}$-gerbe $\mathcal{G}$ on $X[1]$ an Azumaya algebra $\mathcal{A}$ on $X[1]$ such that $\delta(\mathcal{A}) = \mathcal{G}$, in other words $\lambda(\mathcal{G}) = [\mathcal{A}]$.

Denote by $P : \mathcal{G} \to X[1]$ the structural morphism underlying $\mathcal{G}$. By Breen’s semi-local description of gerbes (recalled in Section 4), for any object $U$ of $\mathbf{S}_{\text{ét}}$, for any $X$-torsor $t : U \to X[1]|U)$ over $U$, and for any object $U'$ of $\mathbf{S}_{\text{ét}}$ such that $U \times_{X[1],P} \mathcal{G}(U') \neq \emptyset$, the $\mathbb{G}_{m,X[1]}$-gerbe $\mathcal{G}_{U'}$ is equivalent as $U'$-stack to the stack $\text{tors}(\mathbb{G}_{m,X[1]|U'}) = \text{st}(\mathbb{G}_{m,X[1]|U'|1})$. Therefore, locally over $\mathbf{S}_{\text{ét}}$, the structural morphism $P : \mathcal{G} \to X[1]$ is given by morphisms of complexes $\mathbb{G}_{m,X[1]|U'|1} \to X[U'|1]$ modulo quasi-isomorphisms, that is by morphisms of group $U'$-schemes

$$p_{U'} : \mathbb{G}_{m|U'} \to X[U'] .$$

Denote by $q_{U'} : X[U'] \to \mathbb{G}_{m|U'}$ the morphism of group $U'$-schemes such that $p_{U'} \circ q_{U'} = \text{id}_{X[U']}$ ($q_{U'}$ is a character of $X[U']$ and $p_{U'}$ its co-character). By hypothesis on $X$, restricting $U'$ if necessary, we can suppose that $X[U'] = \mathbb{Z}'$. Since $\text{Hom}(\mathbb{Z}', \mathbb{G}_m) \cong \text{Hom}(\mathbb{Z}, \mathbb{G}_m)'$, we have that $q_{U'} = \prod_{i=1}^{r'} q_{U'i}$, with $q_{U'i} : \mathbb{Z} \to \mathbb{G}_{m|U'}$, a morphism of group $U'$-schemes. To have the $\mathbb{Z}'$-torsor $t_{U'|1}$ over $U'$ is equivalent to have $\mathbb{Z}$-torsors $t_{U'i}$ over $U'$ for $i = 1, \ldots, r'$. Denote by $q_{U'i}(t_{U'i})$ the $\mathbb{G}_m$-torsor over $U'$ obtained from the $\mathbb{Z}$-torsor $t_{U'i}$ by extension of the structural group via the character $q_{U'i} : \mathbb{Z} \to \mathbb{G}_m$. We set

$$\mathcal{A}_{U',U'} := \text{End}(\mathcal{L}_{U'}$$
with $\mathcal{L}_{U'}$ the locally free $\mathcal{O}_{U'}$-module of finite rank $\oplus_{i=1}^{r'} q_{U'i}(t_{U'i})$ which is the direct sum of the invertible sheaves corresponding to the $\mathbb{G}_m$-torsors $q_{U'i}(t_{U'i})$ over $U'$. By construction $A = (A_{U,t})$ is an Azumaya algebra over $X[1]$ such that $\delta(A) = \mathfrak{g}$. □

**Remark 6.3.** Since we can consider the Picard stack $X[1]$ as a stack on $X[1]$ via the structural morphism $\text{id} : X[1] \to X[1]$, the local morphisms of group $U'$-schemes $q|_{U'} : X|_{U'} \to \mathbb{G}_m$ induce a morphism of gerbes on $X[1]$ from $X[1]$ to $\mathfrak{g}$.

**Remark 6.4.** Because of the weight filtration $W_*$ of the 1-motive $M$, we have the exact sequence $0 \to G \xrightarrow{\beta} M \xrightarrow{\beta} X[1] \to 0$. The Brauer group is a contravariant functor, and so we get the following diagram:

$$
\begin{array}{ccc}
\text{Br}(X[1]) & \xrightarrow{\beta^*} & \text{Br}(M) & \xrightarrow{\epsilon^*} & \text{Br}(G) \\
\cong & & \cong & & \cong \\
H^2_{\text{ét}}(X[1], \mathbb{G}_m, X[1]) & \xrightarrow{\beta^*} & H^2_{\text{ét}}(M, \mathbb{G}_m, M) & \xrightarrow{\epsilon^*} & H^2_{\text{ét}}(G, \mathbb{G}_m, G).
\end{array}
$$

If $M$ is defined over an algebraically closed field, the vertical arrows of this diagram are isomorphisms by Lemma 6.2, Corollary 0.4 and (6.4). The sequence $\text{Br}(X[1]) \to \text{Br}(M) \to \text{Br}(G)$ is not exact in general.

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