Synthetic turbulence prediction in non-Kolmogorov turbulence

F C G A Nicolleau¹, A F Nowakowski¹ and T M Michelitsch²

1 Sheffield Fluid Mechanics Group - Mechanical Engineering, University of Sheffield, United Kingdom,
2 Institut Jean le Rond d’Alembert, University of Paris, France
E-mail: F.Nicolleau@Sheffield.ac.uk

Abstract. We use Kinematic Simulation to assess the role of sweeping in the theory of Richardson for particle pair dispersion. We use different spectral laws to show the particular case of Kolmogorov spectrum. KS shows good consistency with Richardson’s locality assumption for a large range of spectral laws (1.15 ≤ p ≤ 1.96).

1. Synthetic turbulence and Kinematic Simulation models

Synthetic turbulence models can be regarded as any techniques where a Eulerian velocity field is postulated (or synthetised). Historically such approach was limited to Lagrangian prediction though at present Eulerian applications such as initial condition field or subgrid model can be found [1]. Kinematic Simulation (KS) is probably the most popular of such models. In this case the predetermined Eulerian field \( u(x, t) \) based on [2] for incompressible isotropic turbulence is reduced to a truncated Fourier series, sum of \( N_k \) random Fourier modes:

\[
    u(x, t) = \sum_{n=0}^{N_k} a_n \cos(k_n x + \omega_n t) + b_n \sin(k_n x + \omega_n t)
\]

where \( N_k \) is the total number of modes included, \( a_n \) and \( b_n \) are decomposition coefficients corresponding to the wave vector \( k_n \), and \( \omega_n \) is the unsteadiness frequency. The coefficients \( a_n \) and \( b_n \) are fixed by the turbulent energy spectrum which in KS is given as an input. It is then very easy to vary the form of the spectrum. Furthermore, as there is no decay of the spectrum in KS (it is given as a formula and independent of time) there is no need for complicated mode-forcing. In the present work the spectrum is chosen as a power law with an exponent, \( p \), varying from 1.15 to 1.96:

\[
    E(k_n) \sim u_{\text{rms}}^2 L (k_n L)^{-p} \quad \text{for} \quad k_1 \leq k_n \leq k_N
\]

Though there has been some attempts at constructing spectral laws from particular kinds of interactions [3], it is not clear how to generalise such approaches to turbulent flows. However, spectral laws are found in nature which depart from the classical Kolmogorov 5/3 and from a mathematical point of view it is important to understand what is so peculiar about 5/3 spectra.
2. Kinematic Simulation and non-Kolmogorov spectra
The quantity of interest is the separation $\Delta$ between two fluid particles $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\Delta = |\mathbf{x}_1 - \mathbf{x}_2|$$

The evolution of the particle pair diffusivity was predicted by [4] based on a locality-in-scale assumption. Richardson derived the $4/3$ law for the diffusivity for $5/3$ spectral laws:

$$\frac{d}{dt} \langle \Delta^2(t) \rangle \sim \langle \Delta^2(t) \rangle^{2/3}$$

The locality assumption can be summarised as “only eddies comparable in size with the separation are effective in further statistical increase of the mean square separation”. The law (4) is probably better known by its extension as an integrated form: the famous $t^3$ law for diffusion in isotropic turbulence

$$\langle \Delta^2(t) \rangle = G_{\Delta} \epsilon t^3$$

where $G_{\Delta}$ is the Richardson universal dimensionless constant, $\epsilon$ the turbulent energy dissipation rate and $t$ time, which was deduced for Kolmogorov type spectra

$$E(k) \sim u_{\text{rms}}^2 L(kL)^{-5/3}$$

where $u_{\text{rms}}$ is the turbulence rms velocity fluctuation and $L$ the turbulence integral scale. Richardson’s locality assumption can be extended to different spectra [5] which is very easy to implement in KS.

3. Present study
In our study we varied the spectral power law $p$:

$$E(k) \sim u_{\text{rms}}^2 L(kL)^{-p}$$

from 1.15 to 1.96. Morel and Larchevêque [5] Generalised [6]’s locality assumption to any spectral power law. It is assumed that the diffusivity depends only the spectrum $E(k)$ and a wavenumber $k_\Delta$ which is of the order of $\sqrt{\langle \Delta^2 \rangle}$ so that:

$$\frac{d}{dt} \langle \Delta^2 \rangle = f \{ E(k_\Delta), k_\Delta \}$$

with $k_\Delta \sim \sqrt{\langle \Delta^2 \rangle}$. Using dimensional arguments, the diffusivity must be of the form:

$$\frac{d}{dt} \langle \Delta^2(t) \rangle \sim \langle \Delta^2 \rangle^{1/4} \sqrt{E(\sqrt{\langle \Delta^2 \rangle})}$$

Eq. (4) can then be written in a general form for a turbulence energy spectrum (2) as follows:

$$\langle \Delta^2(t) \rangle \sim L^2 \left( \frac{u_{\text{rms}}}{L} \right)^{4-p}$$

This formula relies on the locality assumption. Thomson and Devenish [7] proposed another formula for the diffusivity in KS for a $5/3$ spectrum they tried to account for the absence of sweeping of the small scales by the large scales in KS. We generalised [7]’s formula to spectral powers $1 < p < 2$:

$$\frac{d}{dt} \langle \Delta^2(t) \rangle \sim u_{\text{rms}} L \left( \frac{\langle \Delta^2(t) \rangle}{L^2} \right)^{4-p-2 \frac{p-2}{3p+1}}$$
Figure 1. \( \frac{d\langle \Delta^2 \rangle}{dt}/\langle \Delta^2 \rangle^b \) as a function of \( \langle \Delta^2 \rangle/L^2 \).

The full derivation of (9) can be found in [8].

Figure (1) shows \( \frac{d\langle \Delta^2 \rangle}{dt}/\langle \Delta^2 \rangle^b \) as a function of \( \langle \Delta^2 \rangle/L^2 \) where \( b \) is given by Eq. (8) for different spectral power laws. For easier interpretation we plot the cases \( p < 5/3 \) in Fig. 1a and the cases \( p > 5/3 \) in Fig. 1b. All the curves show a remarkable consistency of KS with Richardson’s locality-in-scale hypothesis and [5, 9]’s prediction (8) for the small range \( \sqrt{\langle \Delta^2 \rangle}/L \) given in by the two arrows in the figures. For smaller \( \sqrt{\langle \Delta^2 \rangle}/L \) the diffusivit departs from [5, 9]’s generalisation of Richardson power law.

4. Assessment of the departure from the locality assumption

Figure 2. Power \( c \) from Eq. 10: (a) solid line, theoretical value (9), points, results from KS, dash-line value as predicted from (10); (b) relative error in % between the power \( c \) measured from KS and the values.

At small scales where KS starts to depart from [5]’s prediction, KS predictions are compared to [5]’s prediction and to the formula (9) obtained by generalising [7]’s argument to \( p \neq -5/3 \). In Figure 2 we plot the power \( c \) defined as

\[
\frac{d}{dt} \langle \Delta^2(t) \rangle \sim u_{rms} L \left( \frac{\langle \Delta^2(t) \rangle}{L^2} \right)^c
\]  

(10)
obtained from KS.

As can be seen from Fig. 2a, the departure from [5, 9]'s theory increases up to $p = 5/3$ and then levels off around a value $c = 0.77$ very close to 0.75 the limit value for $p = 2$.

Figure 2b shows the relative error of the two theories when compared to the KS values. Interestingly, the maximum discrepancy between KS and [5, 9]'s is observed for the case $p = 5/3$ where [7]'s theory gives a better prediction of the KS result. Apart from that range around $p = 5/3$, KS results are very close to [5, 9]'s predictions.

It is worth noting that the generalisation of [7] converges to [5, 9]'s predictions for $p = 1$ and $p = 3$. $p = 1$ corresponds to the lower limit of integrability for the energy spectrum. Such spectra as illustrated in Fig. 3 would have a much more even distribution of energy than the classical 5/3.

$\begin{align*}
E(k) &\sim k^{-1} \\
E(k) &\sim k^{5/3} \\
E(k) &\sim k^3
\end{align*}$

**Figure 3.** Different power spectra.

$p = 2$ would correspond to a turbulence where the dissipation rate distribution is constant, $p > 2$ to turbulent flows were the dissipation rate distribution would be an increasing function of scales. (Though the dissipation itself is not.) Turbulent flows with a range of scales obeying such spectral laws may be encountered in nature but they would involve a completely different physics and it seems not worth generalising Richardson’s and Kolmogorov’s approaches to a flat dissipation spectrum even less to a spectrum where dissipation distribution would be the highest on large scales that is why we limited our study to $1 < p < 2$. Furthermore, it is worth noting that it is difficult to get numerically converged results for $p > 2$ because the law for the pair diffusion is diverging for $p \to 3$.

The largest discrepancy between the two theories occurs for $p = 1.775$. This value could be thought of as the point where the absence of sweeping is the most harmful to KS, however, as noted before KS departs the most significantly from [5, 9] earlier at $p = 5/3$ where it gets closer to [7]'s generalisation.

It is reasonable to believe that, approaching the two limiting cases $p = 1$ and $p = 3$, the lack of sweeping of the small eddies becomes less relevant, for two opposite reasons.

- As $p$ gets closer to 1, the energy spectrum becomes flatter, the characteristic velocity is more or less the same over the range of scales modelled by KS so that a sweeping of small eddies by large eddies looses its relevance as indicated by the convergence of both theories to the same prediction $c = 0.5$
Similarly for $p \to 3$ both theories converge to $c = 1$. In this case the energy spectrum tends to a very sharp distribution on the large scales. The particle advection is completely dominated by the large scales in the flow. The contribution of the small eddies where KS struggles with Richardson’s locality assumption becomes less important. Therefore the accurate modelling of their sweeping by large scales is not so important anymore. This is supported by our observation that KS follow remarkably Richardson’s locality assumption at large scales.

5. Conclusion

- The discrepancy between kinematic simulations and Richardson’s locality-in-scale hypothesis only appears for inertial ranges $L/\eta$ larger than $10^4$, where $L$ is the integral length scale and $\eta$ the Kolmogorov length scale, which corresponds to significant Reynolds numbers. In other words, we can argue that the discrepancies between the different authors on the ability of kinematic simulation to predict Richardson power law may be linked to the inertial subrange they have used. For small inertial subrange, KS is efficient and the significance of the sweeping can be ignored, as a result we limit the KS agreement with the Richardson scaling law $t^3$ for inertial subranges $L/\eta \leq 10000$.
- We show that Large scales are not affected by the absence of sweeping in KS.
- The sweeping effect is less important than predicted by [7].
- The sweeping effect is most important for spectral laws $E(k) \approx k^{-5/3}$.

References

[1] Nicolleau F, Cambon C, Redondo J M, Vassilicos J, Reeks M and Nowakowski A (eds) 2011 New approaches in modelling multiphase flows and dispersion in turbulence, fractal methods and synthetic turbulence (Ercoftac Series vol (In Press)) (Springer Science)
[2] Fung J, Hunt J, Malik N and Perkins R 1992 J. Fluid Mech. 236 281–317
[3] Michelitsch T, Maugin G, Nicolleau F, Nowakowski A and Derogar S 2009 Phys. Rev. E 80 011135
[4] Richardson L F 1926 Beitr. Phys. Frei. Atmos. 15 24–29
[5] Morel P and Larchevêque M 1974 J. Atm. Sc. 31 2189
[6] Richardson L F 1926 Proc. Roy. Soc. A 110 24–29
[7] Thomson D J and Devenish B J 2005 J. Fluid Mech. 526 277–302
[8] Nicolleau F and Nowakowski A 2011 Phys. Rev. E 83 056317
[9] Fung J and Vassilicos J 1998 Phys. Rev. E 57 1677–1690