Expansion for quantum perturbations in random spin systems

C. Itoi\textsuperscript{1}, K. Horie\textsuperscript{1}, H. Shimajiri\textsuperscript{1} and Y. Sakamoto\textsuperscript{2},
\textsuperscript{1}Department of Physics, GS & CST, Nihon University, Kandasurugadai, Chiyoda, Tokyo 101-8308, Japan
\textsuperscript{2}Laboratory of Physics, CST, Nihon University, Narashinodai, Funabashi-city, Chiba 274-8501, Japan

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Abstract

Energy eigenstates in the random transverse field Edwards-Anderson (EA) model and the random bond quantum Heisenberg XYZ model in a \(d\)-dimensional finite cubic lattice are obtained for sufficiently weak interactions. The Datta-Kennedy-Kirkwood-Thomas convergent perturbative expansion using the contraction mapping theorem is developed for quantum spin systems with site- and bond-dependent interactions. This expansion enables us to obtain energy eigenstates in the random transverse field free spin model perturbed by sufficiently weak longitudinal exchange interactions. This expansion is useful also for the EA model perturbed by sufficiently weak transverse fields and bond-dependent XY exchange interactions. In these models, their perturbations split the two fold degenerate energy eigenvalues because of the \(\mathbb{Z}_2\) symmetry in the unperturbed EA model. It is shown that the energy gap between split energy eigenvalues is exponentially small in the system size. We provide a sufficient condition on the perturbation for absence of level crossing between arbitrary energy eigenstates.

1 Introduction

Quantum spin systems with random interactions have been studied extensively. Many physicists, mathematicians and computer scientists have studied these systems including the transverse field Ising model with random interactions, since D-Wave Systems actually devised and produced a quantum annealer based on these models \[5\]. Quantum annealer is an optimization hardware, which is theoretically proposed by Finnila-Gomez-Sebenik-Stenson-Dol \[1, 2\] and Kadowaki-Nishimori \[3, 4\] on the basis of the adiabatic theorem in quantum mechanics. The adiabatic theorem claims that the time developed state from the ground state is preserved as the corresponding instantaneous ground state by an infinitely slow time dependent perturbation, if the perturbation gives no level crossing between the ground state and an excited state. Absence of level crossing in the ground state is necessary for the precise solution obtained by the quantum annealer for the optimization problem of the Hamiltonian in the Ising model with random interactions. Apart from problems in quantum annealing, generally it is great concern whether or not, quantum perturbations give a level crossing of the energy eigenvalues in spin modes with random interactions. In the present paper, \(S = 1/2\) quantum spin systems with site- and bond-dependent interactions are studied in the Kirkwood-Thomas convergent perturbative expansion \[7\] developed by Datta and Kennedy \[11, 12\]. Quantum spin systems are regarded as classical Ising systems with quantum perturbations. The perturbative expansion is performed around the classical Ising systems with diagonalized Hamiltonians. Some sufficient conditions on quantum interactions for the absence of level crossing is obtained by a convergent perturbative expansion for sufficiently weak quantum interactions. This expansion enables us to obtain energy eigenstates in the model for weak quantum perturbations. Datta and Kennedy develop the Kirkwood-Thomas expansion method and study uniform transverse field Ising model \[1, 11\] and the Heisenberg XXZ model \[2\]. The expansion method given by Kirkwood and Thomas can be applied only to a restricted class of systems whose Hamiltonians satisfy the Perron-Frobenius condition. Datta and Kennedy have improved the Kirkwood-Thomas method by removing this condition. Banach’s fixed point theorem for an arbitrary contraction mapping is used for functions which define quantum spin states.

Consider \(d\)-dimensional hyper cubic lattice \(\Lambda_L = \mathbb{Z}^d \cap (-L/2, L/2]^d\) with an even integer \(L > 0\). Note that the lattice \(\Lambda_L\) contains \(L^d\) sites. \(S_\Lambda := 2^{\Lambda_L}\) denotes the collection of sub-lattices of \(\Lambda_L\). Each lattice
site \( i \in \Lambda_L \) has an operator-valued spin vector \((\sigma_i^x, \sigma_i^y, \sigma_i^z)\) defined by the Pauli matrices. For a sub-lattice \( X \in \mathcal{S}_L \), denote
\[
\sigma_X^w := \prod_{i \in X} \sigma_i^w
\]
for \( w = x, y, z \). Define a set of nearest neighbor bonds by
\[
B_\Lambda = \{ \{ i, j \}| i, j \in \Lambda_L, |i - j| = 1 \}.
\]
Note \(|B_\Lambda| = |\Lambda_L|d. A bond spin \( \sigma_b \) denotes \( \sigma_b^a = \sigma_b^x\sigma_b^y\sigma_b^z \) for a bond \( b = \{ i, j \} \in B_\Lambda \) and \( w = x, y, z \). Let \( h := (h_i)_{i \in \Lambda_L}, J := (J_b)_{b \in B_\Lambda}, \) and \( \epsilon := (\epsilon_b^x, \epsilon_b^y, \epsilon_b^z)_{b \in B_\Lambda} \) be sequences of arbitrary real numbers. Although these numbers can be random variables, here we do not assume their specific distributions. Define the following functions of a sequence of spin operators \( \sigma := (\sigma_i^w)_{i \in \Lambda_L, w=x,y,z} \) and sequences of coupling constants \( h, J, \epsilon \)
\[
\begin{align*}
H^R_\Lambda(\sigma, h) &:= -\sum_{i \in \Lambda_L} h_i \sigma_i^z, \\
H^Z_\Lambda(\sigma, J) &:= -\sum_{b \in B_\Lambda} J_b \sigma_b^z \\
H^{XY}_\Lambda(\sigma, \epsilon) &:= -\sum_{b \in B_\Lambda} (\epsilon_b^x \sigma_b^x + \epsilon_b^y \sigma_b^y).
\end{align*}
\]
In the present paper, we study the random transverse field Edwards-Anderson model
\[
H^{ZL}_\Lambda(\sigma, h, J) := H^R_\Lambda(\sigma, h) + H^Z_\Lambda(\sigma, J),
\]
and the random bond Heisenberg XYZ model
\[
H^{XYZ}_\Lambda(\sigma, J, \epsilon) := H^R_\Lambda(\sigma, J) + H^{XY}_\Lambda(\sigma, \epsilon),
\]
in a convergent expansion.

First in the present paper, the random transverse field Edwards-Anderson (EA) model defined by the Hamiltonian \( H^{ZL}_\Lambda(\sigma, h, J) \) is studied in the expansion around unperturbed model \( J = 0 \). If longitudinal exchange interactions are switched off, then this model defined by the Hamiltonian \( H^{ZL}_\Lambda(\sigma, h, 0) \) becomes a free spin model under random transverse fields. All eigenstates and energy eigenvalues are obtained trivially in the free spin model, and it can be shown that the energy eigenstates in this free spin model are not degenerate for almost all transverse fields \( h \). The weak longitudinal exchange interactions can be treated by a simple perturbative expansion.

Next, the random transverse field EA model defined by the Hamiltonian \( H^{R}_\Lambda(\sigma, h) \) is studied in the expansion around \( h = 0 \). Although the unperturbed Hamiltonian \( H^{R}_\Lambda(\sigma, 0, J) \) is also diagonalized, all energy eigenvalues are two fold degenerate because of the \( \mathbb{Z}_2 \) symmetry of the global spin flip \( 3 \). For sufficiently weak transverse fields, a perturbative expansion around \( h = 0 \) can be done for almost all longitudinal exchange interactions \( J \). The degenerate unperturbed energy eigenvalues are split by the perturbation of transverse fields. It is proven that the energy gap between these split energy eigenvalues is exponentially small in the system size. It is proven also that the sufficiently weak perturbation cannot produce the level crossing under a condition to break the \( \mathbb{Z}_2 \) symmetry.

Finally, the random bond Heisenberg XYZ model defined by the Hamiltonian \( H^{XYZ}_\Lambda(\sigma, J, \epsilon) \) is studied in the perturbative expansion around \( \epsilon = 0 \). This model is regarded as the EA model \( H^{XY}_\Lambda(\sigma, J, 0) \) perturbed by the XY-exchange interactions. We obtain all energy eigenstates in random bond quantum Heisenberg model, if its XY exchange interactions are sufficiently weak. Similar properties to those in weak transverse field EA model are obtained also in this model because of the \( \mathbb{Z}_2 \) symmetry. As in the weak transverse field EA model, the weak XY interactions split the unperturbed degenerate energy eigenvalues, and the energy gap between these is also exponentially small in the system size. It is proven also that the weak XY interactions cannot produce the level crossing under a condition to break the \( \mathbb{Z}_2 \) symmetry.

The present paper is organized as follows. In section two, an arbitrary energy eigenstate in the transverse field EA model defined by the Hamiltonian \( H^{R}_\Lambda(\sigma, h) \) is obtained in the Datta-Kenedy-Kirkwood-Thomas expansion for the perturbation \( \epsilon \) with sufficiently weak longitudinal exchange interactions. In section three, this model is studied in another expansion for perturbation \( \epsilon \) with weak transverse fields. In section four, the random bond Heisenberg XYZ model defined by \( H^{XYZ}_\Lambda(\sigma, J, \epsilon) \) is studied in the expansion for perturbation \( \epsilon \) with weak XY-exchange interactions.
2 Transverse field EA model around the free spin model

In this section, we study the model defined by the Hamiltonian \(|H|\) around \(J = 0\) in the Datta-Kennedy-Kirkwood-Thomas expansion.

2.1 Unperturbed model

The following lemma guarantees the non-degenerate energy eigenstates in the random transverse field free spin model defined by the Hamiltonian \(|H|\).

Lemma 2.1 Consider the unperturbed model defined by the Hamiltonian \(|H|\) at \(J = 0\) in \(d\)-dimensional hyper cubic lattice \(\Lambda_L\). The Hamiltonian takes different values

\[
H^x_\Lambda(\sigma, h) \neq H^x_\Lambda(\sigma', h),
\]

for any two different spin configurations \(\sigma, \sigma' \in \{1, -1\}^{\Lambda_L}\) for almost all \(h \in \mathbb{R}^\Lambda\).

Proof. Let \(i_1, i_2, i_3, \cdots \in \Lambda_L\) be a sequence of different sites. Define a sub-lattice \(X_N(\in \mathcal{S}_\Lambda)\) by

\[
X_N := \bigcup_{n=1}^N \{i_n\}.
\]

The following mathematical inductivity with respect to \(N\) enables us to prove this lemma.

For \(N = 1\), \(X_1 = \{i_1\}\) is a single bond.

\[
H^x_{X_1}(\sigma, h) = -h_{i_1}\sigma_{i_1}.
\]

Since the Hamiltonian takes different values for \(\sigma_{i_1} = \pm 1\), this lemma is valid for \(N = 1\) and \(h_{i_1} \neq 0\).

For an arbitrary positive integer \(N\), assume the validity of this lemma. Then,

\[
H^x_{X_N}(\sigma, h) \neq H^x_{X_N}(\sigma', h),
\]

is valid for any two different configurations \(\sigma, \sigma' \in \{1, -1\}^{X_N}\) for almost all \(h\).

For \(N + 1\), let \(\sigma, \sigma' \in \{1, -1\}^{X_{N+1}}\) be two different configurations. Consider the equation for \(h_{i_{N+1}}\)

\[
H^x_{X_{N+1}}(\sigma, h) = H^x_{X_{N+1}}(\sigma', h),
\]

which has the following representation in terms of \(H^x_{X_N}\) and \(h_{i_{N+1}}\)

\[
H^x_{X_N}(\sigma|_{X_N}, h) - h_{i_{N+1}}\sigma_{i_{N+1}} = H^x_{X_N}(\sigma'|_{X_N}, h) - h_{i_{N+1}}\sigma'_{i_{N+1}}.
\]

Since \(|X_{N+1} \setminus X_N| = 1\), \(\sigma_{i_{N+1}} = \sigma'_{i_{N+1}}\) implies the assumption \((6)\). Then, the equation \((7)\) has no solution for \(\sigma_{i_{N+1}} = \sigma'_{i_{N+1}}\). For \(\sigma_{i_{N+1}} - \sigma'_{i_{N+1}} = \pm 2\), the corresponding solutions of the equation \((7)\) are given by

\[
h_{i_{N+1}} = \pm \frac{1}{2} |H^x_{X_N}(\sigma'|_{X_N}, h) - H^x_{X_N}(\sigma|_{X_N}, h)|.
\]

Therefore,

\[
H^x_{X_{N+1}}(\sigma, h) \neq H^x_{X_{N+1}}(\sigma', h),
\]

is valid also for \(N + 1\) for almost all \(h_{i_{N+1}} \in \mathbb{R}\) except the solutions \((8)\). Then, this lemma is valid for an arbitrary positive integer \(N\). This completes the proof. □

2.2 Reference state

Define

\[
\sigma_X := \prod_{i \in X} \sigma_i,
\]
and for \( X = \emptyset, \sigma_\emptyset := 1 \). \( \Sigma_\Lambda := \{ 1, -1 \}^{\Lambda_L} \) denotes the collection of sequences of eigenvalues of \( \sigma_i^z \). Note the following identity for \( X,Y \subset \Lambda_L \)

\[
2^{-|\Lambda_L|} \sum_{\sigma \in \Sigma_\Lambda} \sigma_X \sigma_Y = I(X = Y) =: \delta_{X,Y},
\]

where an indicator \( I(e) \) for an arbitrary event \( e \) is defined by \( I(\text{true}) = 1 \) and \( I(\text{false}) = 0 \). Here, we discuss a convergent expansion around \( J = \mathbf{0} \). In the case for \( J = \mathbf{0} \), the model defined by the Hamiltonian \( \mathcal{H} \) becomes the free spin model, and Lemma 2.1 guarantees absence of degeneracy in energy levels. For an arbitrary sub-lattice \( D \in S_\Lambda \), define a sequence \( s^D \in \{ 1, -1 \}^{\Lambda_L} \) by

\[
s^D_i = \begin{cases} -1 & (i \in D) \\ 1 & (i \notin D) \end{cases}
\]

The following state satisfies

\[
- \sum_{i \in \Lambda_L} h_i \sigma_i^T \sum_{\sigma \in \Sigma_\Lambda} \sigma_D|\sigma\rangle = - \sum_{i \in \Lambda_L} h_i s_i^D \sum_{\sigma \in \Sigma_\Lambda} \sigma_D|\sigma\rangle.
\]

This state gives each state corresponding to \( D \in S_\Lambda \) for \( J = \mathbf{0} \).

Let \( \psi(\sigma) \) be a function \( \psi : \{-1, 1\}^{\Lambda_L} \to \mathbb{R} \) of a sequence, and express the energy eigenstate of the Hamiltonian \( \mathcal{H} \) for weak exchange interactions

\[
|D\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_\Lambda} \sigma_D \psi(\sigma)|\sigma\rangle.
\]

The normalization \( \langle D|D \rangle = 1 \) requires

\[
\sum_{\sigma \in \Sigma_\Lambda} \psi(\sigma)^2 = 2^{|\Lambda_L|}.
\]

Note that \( \psi(\sigma) = 1 \) for \( J_b = 0 \) is given by the state corresponding to \( D \in S_\Lambda \). This state \( |D\rangle \) defined by \( D := \{ i \in \Lambda_L | h_i < 0 \} \) becomes the ground state for \( J_b = 0 \), and \( s_i^D = h_i / |h_i| \). The eigenvalue equation defined by

\[
\mathcal{H}_\Lambda^Z(\sigma, h, J)|D\rangle = E_D|D\rangle
\]

is written in

\[
- (\sum_{b \in B_\Lambda} J_b \sigma_b^z + \sum_{i \in \Lambda_L} h_i \sigma_i^T)|D\rangle = E_D|D\rangle.
\]

Using \( \sigma_i^T|\sigma\rangle = |\sigma^{(i)}\rangle \) and \( \sigma_i^z|\sigma\rangle = \sigma_b|\sigma\rangle \), the eigenvalue equation can be represented in terms of \( \psi(\sigma) \).

\[
\sum_{b \in B_\Lambda} J_b \sigma_b \sigma_D \psi(\sigma) + \sum_{i \in \Lambda_L} h_i s_i^D \psi(\sigma^{(i)}) = -E_D \sigma_D \psi(\sigma),
\]

where \( \sigma^{(i)} \) denotes a spin configuration replaced by \( \sigma_i \rightarrow -\sigma_i \). Therefore

\[
\sum_{b \in B_\Lambda} J_b \sigma_b + \sum_{i \in \Lambda_L} h_i s_i^D \psi(\sigma^{(i)}) = -E_D.
\]

To obtain the Kirkwood-Thomas equation for the state, represent the function \( \psi(\sigma) \) in terms of a real valued function \( g(X) \) of an arbitrary sub-lattice \( X \in S_\Lambda \),

\[
\psi(\sigma) = \exp \left[ - \frac{1}{2} \sum_{X \in S_\Lambda} g(X) \sigma_X \right].
\]

Note the following relations

\[
\psi(\sigma^{(i)}) = \exp \left[ - \frac{1}{2} \sum_{X \in S_\Lambda} g(X) \sigma_X + \sum_{X \in S_\Lambda} I(i \in X) g(X) \sigma_X \right],
\]

then

\[
\frac{\psi(\sigma^{(i)})}{\psi(\sigma)} = \exp \left[ \sum_{X \in S_\Lambda} I(i \in X) g(X) \sigma_X \right].
\]
Note also
\[ \frac{\sigma_D^{(i)}}{\sigma_D} = s_i^D. \]

These and the eigenvalue equation (13) give
\[ \sum_{b \in B_\Lambda} J_b \sigma_b + \sum_{i \in \Lambda_L} h_i s_i^D \exp \left[ \sum_{X \in S_\Lambda} I(i \in X)g(X)\sigma_X \right] = -E_D. \] (17)

We expand the exponential function in power series.
\[ \sum_{b \in B_\Lambda} J_b \sigma_b + E_D + \sum_{i \in \Lambda_L} h_i s_i^D + \sum_{X \in S_\Lambda} \sum_{i \in \Lambda_L} h_i s_i^D I(i \in X)g(X)\sigma_X \]
\[ + \sum_{i \in \Lambda_L} h_i s_i^D \exp(2) \left[ \sum_{X \in S_\Lambda} I(i \in X)g(X)\sigma_X \right] = 0, \] (18)

The orthonormalization property (9) gives
\[ \frac{\sigma_D^{(i)}}{\sigma_D} = s_i^D. \]

We expand the exponential function in power series.
\[ \sum_{b \in B_\Lambda} J_b \sigma_b + E_D + \sum_{i \in \Lambda_L} h_i s_i^D + \sum_{X \in S_\Lambda} \sum_{i \in \Lambda_L} h_i s_i^D I(i \in X)g(X)\sigma_X \]
\[ + \sum_{i \in \Lambda_L} h_i s_i^D \exp(2) \left[ \sum_{X \in S_\Lambda} I(i \in X)g(X)\sigma_X \right] = 0, \] (18)

The orthonormalization property (9) gives
\[ E_D = -\sum_{i \in \Lambda_L} h_i s_i^D - \sum_{i \in \Lambda_L} h_i s_i^D \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_\Lambda} \delta_{X_1, \ldots, X_k} \prod_{l=1}^{k} g(X_l)I(i \in X_l), \] (19)

and for \( X \neq \phi \)
\[ g(X) = \frac{-1}{\sum_{i \in X} h_i s_i^D} \left[ \sum_{j \in \Lambda_L} h_j s_j^D \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_\Lambda} \delta_{X_1, \ldots, X_k, X} \prod_{l=1}^{k} g(X_l)I(j \in X_l) \right] =: F(g)(X), \] (20)

where \( X \triangle Y := (X \cup Y) \setminus (X \cap Y) \) for arbitrary sets \( X, Y, \) and we have used \( \sigma_X \sigma_Y = \sigma_{X \triangle Y}. \) The normalization (12) fixes \( g(\phi). \) The first term in the energy eigenvalue is identical to that of the eigenvalue configuration for \( J = 0, \) and the excited energy of a spin configuration \( \sigma \) for \( J = 0 \) is \( 2 \sum_{i \in \Lambda_L} h_i s_i^D, \) where \( X := \{ i \in \Lambda_L | \sigma_i \neq s_i^D \}. \) To prove uniqueness of the function \( g \) for the energy eigenstate \( |D \rangle \) in the transverse field EA model with a given \( h \) for sufficiently weak \( J, \) define a norm for the function \( g(X) \) by
\[ \| g \| := \sup_{i \in \Lambda_L} \sum_{X \in S_\Lambda} I(i \in X) \left| \sum_{j \in X} h_j s_j^D \right| |g(X)|, \] (21)

Then, the following theorem can be proven.

**Theorem 2.2** Consider the transverse field EA model defined by the Hamiltonian (4). For sufficiently weak exchange coupling constants \( J \) and for almost all \( h \in \mathbb{R}^{\Lambda_L}, \) there is a unique energy eigenstate, which corresponds to the unperturbed energy eigenstate \( \sum_{\sigma \in \Sigma_\Lambda} \sigma_D|\sigma\rangle. \)

The following lemma and the contraction mapping theorem enable us to prove this theorem.

**Lemma 2.3** Define a set \( \partial X \) by \( \partial X := \{ i, j \} \in B_\Lambda | i \in X, j \notin X \}. \) There exists a small constant \( \delta > 0 \) such that if \( \sup_{j \in \Lambda_L} \sum_{b \in \partial(j)} |J_b| < \frac{\delta}{2}, \)
\[ \| F(g) - F(g') \| \leq \| g - g' \| / 2, \quad \| F(g) \| \leq \delta, \quad \text{for} \quad \| g \|, \| g' \| \leq \delta, \] (22)

for almost all \( h \in \mathbb{R}^{\Lambda_L}. \)
Proof. For lighter notations, define \( \Delta_k := X_1 \Delta \cdots \Delta X_k \). The norm \( \| F(g) - F(g') \| \) is represented in

\[
\| F(g) - F(g') \| = \sup_{j \in A_L} \sum_{X \in S_A} I(j \in X) \left| \sum_{i \in \Lambda_L} h_i s_i \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{\Delta_k, X} \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \prod_{l=1}^{k} I(i \in X_l) \right|
\]

\[
\leq \sup_{j \in A_L} \sum_{X \in S_A} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} I(j \in \Delta_k) \delta_{\Delta_k, X} \sum_{i \in A_L} h_i s_i \prod_{l=1}^{k} I(i \in X_l) \left| \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \right|
\]

\[
\leq \sup_{j \in A_L} \sum_{X \in S_A} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} I(j \in \Delta_k) \delta_{\Delta_k, X} \sum_{i \in \Lambda_L} h_i s_i \prod_{l=1}^{k} I(i \in X_l) \left| \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \right|
\]

\[
\leq \sup_{j \in A_L} \sum_{X \in S_A} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} I(j \in \Delta_k) \delta_{\Delta_k, X} \sum_{i \in \Lambda_L} h_i s_i \prod_{l=1}^{k} I(i \in X_l) \left| \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \right|
\]

where \( I(j \in \Delta_k) \leq \sum_{l=1}^{k} I(j \in X_l) \) and permutation symmetry in the summation over \( X_1, \ldots, X_k \) have been used. The inequality

\[
\left| \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \right| \leq \sum_{l=1}^{k-1} \prod_{j=1}^{l} I(g(X_j) - g'(X_j)) \prod_{j=l+1}^{k} |g'(X_j)|
\]

enables us to evaluate the norm as follows:

\[
\| F(g) - F(g') \| \leq \sup_{m \in A_L} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \ldots, X_k \in S_A} I(m \in X_1) \left| \sum_{i \in \Lambda_L} h_i s_i \prod_{l=1}^{k} I(i \in X_l) \right| \left| \sum_{i \in X_1} g(X_1) - g'(X_1) \prod_{l=1}^{k} I(i_l \in X_l) \right|
\]

\[
\times \left| \sum_{i=1}^{k-1} \prod_{j=1}^{l-1} |g(X_j) - g'(X_j)| \prod_{j=l+1}^{k} |g'(X_j)| \right|
\]

\[
\leq \sup_{m \in A_L} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left[ \sum_{X_1 \in S_A} I(m \in X_1) \left| \sum_{i \in X_1} g(X_1) - g'(X_1) \prod_{j=2}^{k} I(i_j \in X_j) \right| \prod_{j=1}^{k} \sup_{i_j \in \Lambda_L} \sum_{X_j \in S_A} |g'(X_j)| I(i_j \in X_j) \right]
\]

\[
+ \sum_{l=2}^{k} \sum_{X_1 \in S_A} I(m \in X_1) \left| \sum_{i \in X_1} g(X_1) - g'(X_1) \prod_{j=2}^{l-1} I(i_j \in X_j) \right| \prod_{j=1}^{l-1} \sup_{i_j \in \Lambda_L} \sum_{X_j \in S_A} |g'(X_j)| I(i_j \in X_j)
\]

\[
\times \sum_{i=1}^{k} \prod_{j=1}^{l-1} |g(X_j) - g'(X_j)| \prod_{j=l+1}^{k} |g'(X_j)|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left[ \left( \prod_{j=2}^{k} \| g' \| / \Delta \right) + \sum_{l=2}^{k} \| g \| \left( \prod_{j=2}^{l-1} \| g \| / \Delta \right) \| g - g' \| / \Delta \right]
\]

\[
= \left( \prod_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^{k} \left( \prod_{j=2}^{l-1} \| g \| / \Delta \right)^{l-1} \left( \prod_{j=2}^{l} \| g \| / \Delta \right)^{l-k} \right) \leq \| g - g' \| \sum_{k=2}^{\infty} \frac{k(\delta / \Delta)^{k-1}}{(k-1)!} = [e^{\delta / \Delta}(1 + \delta / \Delta) - 1] \| g - g' \|,
\]
where the energy gap $\Delta > 0$ in the unperturbed model is defined by

$$\Delta := \inf_{X \in \mathcal{S}_{\Lambda}} \left| \sum_{i \in X} h_i s_i^D \right|,$$

and the following inequality for any $g$

$$\sup_{j \in \Lambda_L} \sum_{X \in \mathcal{S}_{\Lambda}} I(j \in X) |g(X)| \leq \frac{\|g\|}{\Delta} \leq \frac{\delta}{\Delta},$$

has been used. The condition $\frac{1}{2} = e^{\delta/\Delta}(1 + \delta/\Delta) - 1$ fixes $\alpha = \delta/\Delta$. To obtain the bound on $\|F(0)\|$, let us evaluate $\|F(0)\|$ first. Since

$$F(0)(X) = -\sum_{b \in B_{\Lambda}} J_b b_{\Lambda}^1,$$

the norm is given by

$$\|F(0)\| = \sup_{j \in \Lambda_L} \sum_{X \in \mathcal{S}_{\Lambda}} I(j \in X) \sum_{b \in B_{\Lambda}} J_b b_{\Lambda}^1 = \sup_{j \in \Lambda_L} \sum_{b \in \partial(j)} |J_b|. \quad (27)$$

For $\sup_{j \in \Lambda_L} \sum_{b \in \partial(j)} |J_b| \leq \frac{\delta}{2} := \frac{\alpha \Delta}{2}$ and $\|g\| \leq \delta$,

$$\|F(g)\| = \|F(g) - F(0) + F(0)\| \leq \|F(g) - F(0)\| + \|F(0)\| \leq \frac{\|g\|}{2} + \frac{\delta}{2} \leq \delta.$$

This completes the proof of Lemma 2.3 $\square$

**Proof of Theorem 2.2** Let $\alpha > 0$ be the solution of the equation $e^{\alpha(1 + \alpha)} - 1 = \frac{1}{2}$, and define a positive constant $\delta := \alpha \inf_{i \in \Lambda_L} |h_i|$. Lemma 2.3 and the contraction mapping theorem enable us to prove that the fixed point equation $F(g) = g$ has a unique solution $g$, which corresponds to the unperturbed energy eigenstate $|s^D\rangle$ and satisfies $\|g\| \leq \delta$, if $\sup_{j \in \Lambda_L} \sum_{b \in \partial(j)} |J_b| \leq \frac{\delta}{2}$. $\square$

### 2.3 Expansion for energy gap

Here, we discuss an energy gap $E_C - E_D$ between two arbitrary energy eigenstates $|C\rangle$ and $|D\rangle$. The eigenvalue equation

$$H_{\Lambda \Sigma}^{L}(\sigma, h, J)|C\rangle = E_C|C\rangle$$

is written in

$$-(\sum_{b \in B_{\Lambda}} J_b \sigma_b^z + \sum_{i \in \Lambda_L} h_i \sigma_i^+)|C\rangle = E_C|C\rangle.$$  

There exists a real valued function $\phi : \{1, -1\}^{\Lambda_L} \rightarrow \mathbb{R}$, and the state $|C\rangle$ is represented in

$$|C\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_D \psi(\sigma) \phi(\sigma) |\sigma\rangle,$$

where the function $\psi$ defines the reference state

$$|D\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_D \psi(\sigma) |\sigma\rangle.$$  

Assume that in the unperturbed model $J = 0$, there exists a sub-lattice $C \in \mathcal{S}_{\Lambda}$, such that

$$|C\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_C |\sigma\rangle.$$  

The eigenvalue equation can be represented in terms of $\phi(\sigma)$.

$$-\sum_{b \in B_{\Lambda}} J_b \sigma_b \sigma_D \psi(\sigma) \phi(\sigma) - \sum_{i \in \Lambda_L} h_i \sigma_i^{(i)} \psi(\sigma^{(i)}) \phi(\sigma^{(i)}) = E_C \sigma_D \psi(\sigma) \phi(\sigma). \quad (28)$$
Therefore
\[ - \sum_{b \in B_A} J_b \phi_b(\sigma) \sum_{i \in \Lambda_L} h_i \frac{\sigma^{(i)}_D}{\sigma^{(i)}_D} \psi(\sigma^{(i)}) \phi(\sigma^{(i)}) = E_C \phi(\sigma). \] (29)

The following relation
\[ \frac{\sigma^{(i)}_D}{\sigma^{(i)}_D} \psi(\sigma^{(i)}) = s_i^D \exp \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] \]
and the eigenvalue equation (29) give
\[ \sum_{b \in B_A} J_b \phi_b(\sigma) + \sum_{i \in \Lambda_L} h_i s_i^D \phi(\sigma^{(i)}) \exp \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] = -E_C \phi(\sigma). \] (30)

The eigenvalue equation with the energy eigenvalue \( E_D \) times \( \phi(\sigma) \) is
\[ \sum_{b \in B_A} J_b \phi_b(\sigma) + \sum_{i \in \Lambda_L} h_i s_i^D \phi(\sigma) \exp \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] = -E_D \phi(\sigma). \] (31)

The difference between above two equations gives
\[ - \sum_{i \in \Lambda_L} h_i s_i^D [\phi(\sigma^{(i)}) - \phi(\sigma)] \exp \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] = (E_C - E_D) \phi(\sigma). \] (32)

To obtain the Kirkwood-Thomas equation for the first excited state, represent the function \( \phi(\sigma) \) in terms of a real valued function \( f(X) \) of an arbitrary subset \( X \in S_A \),
\[ \phi(\sigma) = \sum_{X \in S_A} f(X)\sigma_X. \] (33)

This gives
\[ \phi(\sigma^{(i)}) - \phi(\sigma) = -2 \sum_{X \in S_A} f(X)\sigma_X I(i \in X) \]
Then we have
\[ 2 \sum_{i \in \Lambda_L} h_i s_i^D \sum_{Y \in S_A} I(i \in Y)f(Y)\sigma_Y \exp \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] = (E_C - E_D) \sum_{X \in S_A} f(X)\sigma_X. \]

Define a function \( \exp^{(1)} \) by
\[ \exp^{(1)}(x) := e^x - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k!}, \]
then we have
\[ \sum_{Y \in S_A} \Delta_Y f(Y)\sigma_Y + 2 \sum_{i \in \Lambda_L} h_i s_i^D \sum_{Y \in S_A} I(i \in Y)\sigma_Y \exp^{(1)} \left[ \sum_{X \in S_A} I(i \in X)g(X)\sigma_X \right] \]
\[ = (E_C - E_D) \sum_{X \in S_A} f(X)\sigma_X. \] (34)

where an energy gap \( \Delta_Y \) for \( Y \in \Lambda_L \) is defined by
\[ \Delta_Y := 2 \sum_{i \in Y} h_i s_i^D. \]

The orthonormalization property (33) gives
\[ 2 \sum_{i \in \Lambda_L} h_i s_i^D \sum_{Y \in S_A} I(i \in Y)f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{X_1, \ldots, X_k, Y} \Delta Z \prod_{l=1}^{k} g(X_l) I(i \in X_l) \]
\[ = (E_C - E_D - \Delta Z) f(Z). \] (35)
Define a function $e(X)$ of a sub-lattice $X \in S_A$, by $e(C) := E_C - E_D - \Delta_C$ for $X = C$ and $e(X) := f(X)/f(C)$ for $X \neq C$.

For $Z = C$,

$$e(C) = 2 \sum_{i \in \Lambda_L} h_i s^D_i \sum_{Y \in S_A} I(i \in Y) e(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{X_1, \Delta \cdots \Delta X_k, Y \Delta M} \prod_{l=1}^{k} g(X_l) I(i \in X_l) =: F(e)(C),$$

For $Z \neq C$,

$$e(Z) = \frac{1}{\Delta_Z - \Delta_C} \left[ e(C) e(Z) - 2 \sum_{i \in \Lambda_L} h_i s^D_i \sum_{Y \in S_A} I(i \in Y) e(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{X_1, \Delta \cdots \Delta X_k, Y \Delta Z} \prod_{l=1}^{k} g(X_l) I(i \in X_l) \right] =: F(e)(Z).$$

These two equations define a fixed point equation $F(e) = e$, whose solution $e$ gives the state $|C\rangle$ except its normalization. To prove the uniqueness of the solution, define a norm of the function $e$ by

$$\|e\| := |e(C)| + \sum_{X \in S_A} |\Delta_X - \Delta_C||e(X)|.$$  \hfill (37)

The following theorem implies that there is no level crossing against a sufficiently small perturbation $J$.

**Theorem 2.4** Consider the transverse field EA model defined by the Hamiltonian $H^{\Lambda_Z}$. For two different sub-lattices $C, D \in S_A$, let $s^C, s^D \in \{1, -1\}^{\Lambda_L}$ be their corresponding sequences defined by $H^{\Lambda_Z}$. If the sequence of exchange interactions $J$ is sufficiently weak, then there exists a sufficiently small constant $\delta > 0$ depending on the sequence of coupling constants $(J, h)$, such that the energy gap $E_C - E_D$ in the perturbed model satisfies

$$H^{\Lambda_Z}(s^C, 0, h) - H^{\Lambda_Z}(s^D, 0, h) - \delta < E_C - E_D < H^{\Lambda_Z}(s^C, 0, h) - H^{\Lambda_Z}(s^D, 0, h) + \delta,$$

for almost all $h \in \mathbb{R}^{B_1}$.

Then, the following lemma can be proven.

**Lemma 2.5** Consider the model under the conditions in Lemma 2.3, and assume that the lower bound on energy gap is sufficiently large. There exist a sufficiently small constant $\delta > 0$ and $0 < K < 1$, such that

$$\|F(e) - F(e')\| \leq K\|e - e'\|, \quad \|F(e)\| \leq \delta, \quad \text{for} \quad \|e\|, \|e'\| \leq \delta,$$

for almost all $h \in \mathbb{R}^{B_1}$.

**Proof.** The difference between two evaluations of energy gap

$$|F(e)(C) - F(e')(C)| = \left| 2 \sum_{i \in \Lambda_L} h_i s^D_i \sum_{Y \in S_A} I(i \in Y) [e(Y) - e'(Y)] \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{X_1, \Delta \cdots \Delta X_k, Y \Delta C} \prod_{l=1}^{k} g(X_l) I(i \in X_l) \right|

\leq \sum_{Y \in S_A} |\Delta_Y| |e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} \sup_{h \in \Lambda_L} \sum_{X_l \in S_A} |g(X_l)| I(i_l \in X_l)

\leq \sum_{Y \in S_A} |\Delta_Y - \Delta_C + \Delta_C||e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{\delta}{\Delta} \right)^k

\leq \|e - e'\|(1 + |\Delta_C|/\Delta')|e^{\delta/\Delta} - 1|,$$

\hfill (40)
where $\Delta' := \inf_{Y} |\Delta_{Y} - \Delta_{C}|$.

$$
\|F(e) - F(e')\| = \|F(e)(C) - F(e')(C)\| + \sum_{Z \in \mathcal{S}_{A}} |e(C)e(Z) - e'(C)e'(Z) |
$$

$$
-2 \sum_{i \in \Lambda} h_{i} s_{i}^{D} \sum_{Y \in \mathcal{S}_{A}} I(i \in Y) |e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_{1}, \ldots, X_{k} \in \mathcal{S}_{A}} \sum_{\delta_{X_{1}, \Delta \ldots \Delta X_{k}, Y \Delta Z} g(X_{i}) I(i \in X_{i})}
$$

$$
\leq |F(e)(C) - F(e')(C)| + \sum_{Z \in \mathcal{S}_{A}} |e(C)e(Z) - e'(C)e'(Z) |
$$

$$
+ \sum_{Y \in \mathcal{S}_{A}} \sum_{i \in Y} |2 \sum_{h_{i} s_{i}^{D}}|e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_{1}, \ldots, X_{k} \in \mathcal{S}_{A}} \sum_{\delta_{X_{1}, \Delta \ldots \Delta X_{k}, Y \Delta Z} g(X_{i}) I(i \in X_{i})}
$$

$$
\leq |F(e)(C) - F(e')(C)| + |e(C) - e'(C)| \sum_{Z \in \mathcal{S}_{A}} |e(Z) - e'(Z) |
$$

$$
+ \sum_{Y \in \mathcal{S}_{A}} |\Delta_{Y}||e(Y) - e'(Y)|| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i \in \Lambda} \sum_{X_{i} \in \mathcal{S}_{A}} |g(X_{i}) I(i \in X_{i})|
$$

$$
\leq \|e - e'||(1 + |\Delta_{C}|/\Delta') (e^{\delta/\Delta} - 1) + \delta/\Delta'\|e - e'|| + (1 + |\Delta_{C}|/\Delta') (e^{\delta/\Delta} - 1)\|e - e'||
$$

$$
= 2(1 + |\Delta_{C}|/\Delta') (e^{\delta/\Delta} - 1) + \delta/\Delta'\|e - e'|| = K\|e - e||. \tag{41}
$$

If $K < 1$, $F$ is a contraction mapping. This completes the proof of Lemma 2.8. \ \square

**Proof of Theorem 2.4** For two different sub-lattices $C \neq D$, Lemma 2.8 and the contraction mapping theorem imply that unique function $e : 2^{\Lambda} \rightarrow \mathbb{R}$ exists and satisfies $|E_{C} - E_{D} - \Delta_{C}| =: |e(C)| \leq \delta$ for sufficiently small $\delta > 0$. Therefore, the energy gap $E_{C} - E_{D}$ in the perturbed model satisfies

$$
H_{\Lambda}^{Z}(s^{C}, 0, h) - H_{\Lambda}^{Z}(s^{D}, 0, h) - \delta < E_{C} - E_{D} < H_{\Lambda}^{Z}(s^{C}, 0, h) - H_{\Lambda}^{Z}(s^{D}, 0, h) + \delta,
$$

for almost all $J \in \mathbb{R}^{\Lambda}$. This completes the proof of Theorem 2.4. \ \square

### 3 Transverse field EA model around the EA model

Define discrete transformation

$$
P_{w} := \sigma_{c_{w}}^{w} = \prod_{i \in \Lambda} \sigma_{i}^{w}, \tag{42}
$$

for $w = x, y, z$. The Hamiltonian is invariant

$$
P_{x} H_{\Lambda}^{Z}(\sigma, h, J) P_{x} = H_{\Lambda}^{Z}(\sigma, h, J),
$$

for $w = x$. The unitary operator $P_{x}$ transforms each spin to $\sigma_{z} \rightarrow P_{x} \sigma_{z} P_{x} = -\sigma_{z}$ and $\sigma_{x} \rightarrow P_{x} \sigma_{x} P_{x} = \sigma_{x}$, $\sigma_{y} \rightarrow P_{x} \sigma_{y} P_{x} = -\sigma_{y}$. This symmetry corresponds to $Z_{2}$ symmetry in the EA model for $h = 0$. Define a state $|\sigma\rangle$ with a sequence of eigenvalues $\sigma \in \{1, -1\}^{\Lambda}$ of spin operators $(\sigma_{i})_{i \in \Lambda}$ by

$$
\sigma_{i}|\sigma\rangle = \sigma_{i}|\sigma\rangle.
$$

To remove the trivial two-fold degeneracy due to the global $Z_{2}$ symmetry, assume a symmetry breaking condition at an arbitrarily fixed site $i_{0} \in \Lambda$, such that

$$
\sigma_{i_{0}} = 1. \tag{43}
$$

Define sub-lattice $\Lambda'_{L} := \Lambda_{L} \setminus \{i_{0}\}$. Any $Z_{2}$ symmetry broken state $|\sigma\rangle$ is given by $\sigma \in \{1, -1\}^{\Lambda'}$. 10
3.1 Unperturbed system

The following lemma guarantees the non-degenerate property of energy eigenstates under the condition (43) in the classical EA model defined by the Hamiltonian (3). In the following, we study eigenstates of the Hamiltonian (4) under the condition (43) in a convergent perturbative expansion.

Lemma 3.1 Consider the unperturbed model defined by the Hamiltonian (3) in d-dimensional hyper cubic lattice $\Lambda_L$. For any two different eigenvalue sequences $\sigma, \sigma' \in \{1, -1\}^{X_L}$ of the operator sequence $(\sigma_i^\dagger)_{i \in \Lambda_L}$ satisfying the fixed spin condition (43), the Hamiltonian takes different values

$$H^Z_{\Lambda}(\sigma, J) \neq H^Z_{\Lambda}(\sigma', J),$$

for almost all $J \in \mathbb{R}^{B_L}$.

Proof. Let $b_1, b_2, b_3, \ldots \in B_L$ be a sequence of bonds, such that $i_0 \in b_1$ and each collection $\{b_n|n \leq N\}$ is connected for an arbitrary positive integer $N$. Define a connected sub-lattice $X_N(\subset S_L)$ for this bond sequence $(b_n)_{n \leq N}$ by

$$X_N := \bigcup_{n=1}^{N} b_n.$$ 

The following mathematical inductivity with respect to $N$ enables us to prove this lemma. For $N = 1$, $X_1 = b_1 = \{i_0, i_1\}$ is a single bond. The condition $\sigma_{i_0} = 1$ implies

$$H^Z_{X_1}(\sigma, J) = -Jb_1\sigma_{i_1}.$$ 

Since the Hamiltonian takes different values for $\sigma_{i_1} = \pm 1$, this lemma is valid for $N = 1$. For an arbitrary positive integer $N$, assume the validity of this lemma. Then,

$$H^Z_{X_N}(\sigma, J) \neq H^Z_{X_N}(\sigma', J),$$

is valid for any two different configurations $\sigma, \sigma' \in \{1, -1\}^{X_N}$ satisfying (43) for almost all $J$.

For $N + 1$, let $\sigma, \sigma' \in \{1, -1\}^{X_{N+1}}$ be two different configurations satisfying (43). Consider the equation for $J_{b_{N+1}}$

$$H^Z_{X_{N+1}}(\sigma, J) = H^Z_{X_{N+1}}(\sigma', J),$$

which has the following representation in terms of $H_{X_N}$ and $J_{b_{N+1}}$

$$H^Z_{X_N}(\sigma|X_N, J) - J_{b_{N+1}}\sigma_{b_{N+1}} = H^Z_{X_N}(\sigma'|X_N, J) - J_{b_{N+1}}\sigma'_{b_{N+1}}.$$ 

Since $|X_{N+1} \setminus X_N| \leq 1$, $\sigma_{b_{N+1}} = \sigma'_{b_{N+1}}$ implies the assumption (44). Then, the equation (45) has no solution for $\sigma_{b_{N+1}} = \sigma'_{b_{N+1}}$. For $\sigma_{b_{N+1}} - \sigma'_{b_{N+1}} = \pm 2$, the corresponding solutions of the equation (45) are given by

$$J_{b_{N+1}} = \pm \frac{1}{2}[H^Z_{X_N}(\sigma'|X_N, J) - H^Z_{X_N}(\sigma|X_N, J)].$$

Therefore,

$$H^Z_{X_{N+1}}(\sigma, J) \neq H^Z_{X_{N+1}}(\sigma', J),$$

is valid also for $N + 1$ for almost all $J_{b_{N+1}} \in \mathbb{R}$ except the solutions (46). Then, this lemma is valid for an arbitrary positive integer $N$. This completes the proof. □

3.2 Expansion method

To obtain an arbitrary eigenstate in the transverse EA model, consider the following unitary transformed Hamiltonian

$$H^z_H(\sigma, h, J) := U H^z_{\Lambda} U^\dagger = -\sum_{b \in B_L} J_b \sigma_b^z - \sum_{i \in \Lambda_L} h_i \sigma_i^z,$$

where $U \sigma_i^z U^\dagger = \sigma_i^z$ and $U \sigma_i^x U^\dagger = -\sigma_i^x$. The discrete unitary transformation defined by (42) is transformed into

$$P_z = \sigma^z_{\Lambda_L} = U^\dagger P_z U.$$
In the unperturbed model for \( h = 0 \), Lemma 1 guarantees non-degeneracy of energy eigenstates satisfying the condition (43). For an arbitrary sub-lattice \( D \) (\( \in S_A \)), the corresponding eigenstate in the unperturbed model is represented in

\[
2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_A} \sigma_D|\sigma\rangle,
\]

and another degenerate eigenstate is

\[
2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_A} \sigma_D^+|\sigma\rangle.
\]

Note that the unitary transformed energy eigenstate in the unperturbed model can be represented in

\[
U|s^D\rangle = \sum_{\sigma \in \{1,-1\}^{\Lambda_L}} \sigma_D|\sigma\rangle,
\]

where a sequence \( (s_i^D)_{i \in \Lambda_L} \) is defined by (10). For the perturbed model, let \( \psi_\pm(\sigma) \) be a function \( \psi_\pm : \{-1,1\}^{\Lambda_L} \to \mathbb{R} \) of spin configuration, and express the energy eigenstate of the Hamiltonian with respect to the unperturbed state

\[
|\pm\rangle = 2^{-|\Lambda_L|+1/2} \sum_{\sigma \in \Lambda_L} \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma)|\sigma\rangle.
\]

These are eigenstates of \( P_z \) satisfying \( P_z|\pm\rangle = \pm|\pm\rangle \). Note that \( \psi_\pm(\sigma) = 1 \) for \( h = 0 \). The normalization condition \( \langle \pm | \pm \rangle = 1 \) gives

\[
\sum_{\sigma} \psi_\pm(\sigma)^2 = 2^{2|\Lambda_L|}.
\]

(48)

The eigenvalue equation defined by

\[
\tilde{H}_A(\sigma, h, J)|\pm\rangle = E_\pm|\pm\rangle
\]

is written in

\[
-(\sum_{b \in B_A} J_b \sigma_b^\mp + \sum_{i \in \Lambda_L} h_i \sigma_i^\mp)|\pm\rangle = E_\pm|\pm\rangle.
\]

If \( \sigma_i^\mp|\sigma\rangle = |\tau_i\rangle \), \( \tau_i = -\sigma_i \) and \( \tau_j = \sigma_j \) for \( j \neq i \).

\[
\sigma_b^\mp|\sigma\rangle = \sigma_i^\mp \sigma_j^\mp|\sigma\rangle = |\sigma^{(i,j)}\rangle,
\]

where \( \sigma^{(i,j)} \) denotes a spin configuration replaced by \( (\sigma_i, \sigma_j) \to (-\sigma_i, -\sigma_j) \). This eigenvalue equation can be represented in terms of \( \psi(\sigma) \).

\[
\sum_{b \in B_A} J_b \sigma_b^{(b)}(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma^{(b)}) + \sum_{i \in \Lambda_L} h_i \sigma_i \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma) = -E_\pm \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma).
\]

(49)

Therefore

\[
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_A} J_b \sigma_b^{(b)} \psi_\pm(\sigma^{(b)}) \frac{\sigma_D \psi_\pm(\sigma)}{\sigma_D \psi_\pm(\sigma)} + \sum_{i \in \Lambda_L} h_i \sigma_i + E_\pm \right] = 0.
\]

(50)

This equation determines \( \psi_\pm(\sigma) \) as a function of \( \sigma \) satisfying \( \sigma_{\Lambda_L} = \pm 1 \). To obtain the Kirkwood-Thomas equation for the state, represent the functions \( \psi_\pm(\sigma) \) in terms of a real valued function \( g(X) \) of an arbitrary non-empty sub-lattice \( X \in S_A \),

\[
\psi_\pm(\sigma) = \frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_A} g(X)\sigma_X \right].
\]

(51)

Note the following relations

\[
\psi_\pm(\sigma^{(b)}) = \frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_A} g(X)\sigma_X + \sum_{X \in S_A} I(b \in \partial X)g(X)\sigma_X \right].
\]

(52)
Define
\[ s_b^D := \frac{\sigma_D^{(b)}}{\sigma_D}. \]

These and the eigenvalue equation (50) give
\[ \frac{1}{2} (1 + \sigma_{\Lambda_L}) \left( \sum_{b \in B_\Lambda} J_b s_b^D \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X + \sum_{i \in \Lambda_L} h_i \sigma_i + E_{\pm} \right) = 0. \] (53)

We expand the exponential function in power series. The first order term in the exponential function is given by
\[ \sum_{b \in B_\Lambda} J_b s_b^D \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X = \sum_{X \in S_\Lambda} \sum_{b \in \partial X} J_b s_b^D g(X) \sigma_X, \] (54)

then we have
\[ \frac{1}{2} (1 + \sigma_{\Lambda_L}) \left( \sum_{b \in B_\Lambda} J_b s_b^D + E_{\pm} + \sum_{X \in S_\Lambda} \sum_{b \in \partial X} J_b s_b^D g(X) \sigma_X \right. \]
\[ \left. + \sum_{b \in B_\Lambda} J_b s_b^D \exp(2) \left[ \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right] \sum_{i \in \Lambda_L} h_i \sigma_i \right) = 0, \] (55)

where
\[ \exp(2) x := e^x - 1 - x = \sum_{k=2}^{\infty} \frac{x^k}{k!}. \]

The orthonormalization property (9) gives
\[ E_{\pm} = - \sum_{b \in B_\Lambda} J_b s_b^D - \sum_{c \in B_\Lambda} J_{c \in S_\Lambda} \sum_{i=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k} \delta_{X_1, \Delta \ldots \Delta X_k} \phi \prod_{i=1}^{k} g(X_i) I(c \in \partial X_i) \]
\[ \mp \sum_{c \in B_\Lambda} J_{c \in S_\Lambda} \sum_{i=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k} \delta_{X_1, \Delta \ldots \Delta X_k, \Lambda_L} \prod_{i=1}^{k} g(X_i) I(c \in \partial X_i), \] (56)

and \( g \) should satisfy
\[ g(X) = \sum_{b \in \partial X} J_b s_b^D \left[ \sum_{c \in B_\Lambda} J_{c \in S_\Lambda} \sum_{i=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k} \delta_{X_1, \Delta \ldots \Delta X_k} \phi \prod_{i=1}^{k} g(X_i) I(c \in \partial X_i) \right. \]
\[ \left. + \sum_{i \in \Lambda_L} h_i \delta_{X_i, \{i\}} \right] =: F(g)(X), \] (57)

for \( X \neq \phi, \Lambda_L \). The normalization (18) fixes \( g(\phi) \pm g(\Lambda_L) \). The first term in the energy eigenvalue is identical to the energy of the spin configuration \( s^D \) for \( h_i = 0 \). To obtain the energy eigenstate in the transverse field EA model with a given \( J \) for a sufficiently small \( h := \sup_{b \in B_\Lambda} \sum_{c \in c} |h_i| \), define a norm of the function \( g(X) \) with a constant \( M > 0 \) by
\[ \| g \| := \sup_{c \in B_\Lambda} \sum_{X \in S_\Lambda} I(c \in \partial X) \sum_{b \in \partial X} J_b s_b^D \left| g(X) \right| (hM)^{-w(X)}, \] (58)

where \( w(X) \) is defined by the cardinality of the smallest connected sub-lattice that contains \( X \in S_\Lambda \). We say that a sub-lattice \( X \) is connected, if for any \( i, j \in X \), there exists a sequence \( i_1, i_2, \ldots, i_n \in X \), such that \( i_1 = i \), \( i_n = j \) and \( \{ i_k, i_{k+1} \} \in B_\Lambda \) for \( k = 1, \ldots, n - 1 \). Then, the following theorem can be proven.

**Theorem 3.2** Consider the transverse field EA model defined by the Hamiltonian (7) with sufficiently weak coupling constants \( h \). Let \( D(\in S_\Lambda) \) be an arbitrary sub-lattice. There are two energy eigenstates with energy eigenvalues \( E_{\pm} \) corresponding to the two-fold degenerate energy eigenstates \( |s^D \rangle \pm |s^D \rangle \) in the unperturbed model, such that the energy gap \( |E_{\pm} - E_{\pm} \| \) is exponentially small in the system size \( |\Lambda_L| \) for almost all \( J \in \mathbb{R}^{B_\Lambda} \).

Theorem 3.2 is proven by the following lemma and the contraction mapping theorem.
Lemma 3.3 There exist a constant \( \delta > 0 \) and define \( M := \frac{2}{\delta} \), such that if \( hM < 1 \),

\[
\| F(g) - F(g') \| \leq \| g - g' \|/2, \quad \| F(g) \| \leq \delta, \quad \text{for } \| g \|, \| g' \| \leq \delta,
\]

for almost all \( J \in \mathbb{R}^{B_2} \). Then, there exists a constant \( A > 0 \) depending on \( \delta \), such that the energy gap \( |E_+ - E_-| \) satisfies the following exponentially small bound

\[
|E_+ - E_-| \leq A \sum_{b \in B_\lambda} J_b s_b^D \big( hM \big)^{\Lambda_L}.
\]

Proof. The norm \( \| F(g) - F(g') \| \) is represented in

\[
\| F(g) - F(g') \| = \sup_{c \in B_A} \sum_{X \in S_A} I(c \in \partial X) \sum_{b \in B_\lambda} J_b s_b^D \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \delta_{\Delta_k, X}
\]

\[
\times \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \prod_{l=1}^{k} I(b \in \partial X_l) (hM)^{-w(X_l)}
\]

\[
\leq \sup_{c \in B_A} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \ldots, X_k \in S_A} I(c \in \partial \Delta_k) \sum_{b \in B_\lambda} J_b s_b^D \prod_{l=1}^{k} I(b \in \partial X_l) \left| g(X_l) - g'(X_l) \right| (hM)^{-w(\Delta_k)}
\]

\[
\leq \sup_{c \in B_A} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \ldots, X_k \in S_A} I(c \in \partial X_1) \sum_{b \in \partial X_1} J_b s_b^D \prod_{l=2}^{k} I(b \in \partial X_l)
\]

\[
\times \left| \sum_{l=1}^{k-1} \prod_{j=1}^{l-1} \left| g(X_j) \right| \prod_{j=l+1}^{k} \left| g'(X_j) \right| \right| (hM)^{-\sum_{j=1}^{k} w(X_j)}
\]

\[
\leq \sup_{b_1, \ldots, b_k \in \Lambda_L} \sum_{X_1, \ldots, X_k \in S_A} \sum_{b \in \partial X_1} J_b s_b^D
\]

\[
\times \left| \sum_{l=1}^{k-1} \prod_{j=1}^{l-1} \left| g(X_j) \right| I(b_j \in \partial X_j) (hM)^{-w(X_j)} \right| \left| g(X_l) - g'(X_l) \right| I(b_l \in \partial X_l) (hM)^{-w(X_l)}
\]

\[
\times \prod_{j=l+1}^{k} \left| g'(X_j) \right| I(b_j \in \partial X_j) (hM)^{-w(X_j)}
\].

(60)
The norm is bounded in terms of norms of $g, g'$

$$\|F(g) - F(g')\|$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left[ \|g - g'\| \prod_{j=2}^{k} \|g'||/\Delta + \sum_{l=2}^{k} \|g\| \prod_{j=2}^{l-1} \|g\|/\Delta \|g - g'||/\Delta \prod_{j=l+1}^{k} \|g'||/\Delta \right]$$

$$= \|g - g'\| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^{k} \left( (\|g\|/\Delta)^{l-1} (\|g'||/\Delta)^{k-l} \right)$$

$$\leq \|g - g'\| \sum_{k=2}^{\infty} \frac{k(\delta/\Delta)^{k-1}}{(k-1)!} = [e^{\delta/\Delta} (1 + \delta/\Delta) - 1] \|g - g'\|,$$

where the energy gap $\Delta > 0$ from the state in the unperturbed model is defined by

$$\Delta := \inf_{X \in S} \left| \sum_{b \in \partial X} J_{b} s_{b}^{D} \right|,$$  \hspace{1cm} (61)

and the following inequality for $\|g\| \leq \delta$,

$$\sup_{c \in B} \sum_{X \in S} I(c \in \partial X) |g(X) (hM)^{-w(X)}| \leq \left\| g \right\| \Delta \leq \frac{\delta}{\Delta},$$

has been used. The condition $\frac{1}{\Delta} = K = e^{\delta/\Delta} (1 + \delta/\Delta) - 1$ fixes $\alpha = \delta/\Delta$. To obtain the bound on $\|F(g)\|$, let us evaluate $\|F(0)\|$ first. Since

$$F(0)(X) = -\sum_{i \in \Lambda} h_{i} \delta_{X_{i}}^{X},$$

the norm is given by

$$\|F(0)\| = \sup_{c \in B} \sum_{X \in S} I(c \in \partial X) \left| \sum_{i \in \Lambda} h_{i} \delta_{X_{i}}^{X} \right| (hM)^{-w(X)}$$

$$= \sup_{c \in B} \sum_{X \in S} \sum_{i \in \Lambda} I(c \in \partial X) h_{i} \delta_{X_{i}}^{X} (hM)^{-w(X)}$$

$$= \sup_{c \in B} \sum_{i \in \Lambda} I(c \in \partial X) h_{i} (hM)^{-1} = \sup_{c \in B} \sum_{i \in \Lambda} h_{i} (hM)^{-1} = M^{-1}. \hspace{1cm} (62)$$

Define $M := \frac{2}{\Delta} := \frac{\delta}{\Delta}$, and then

$$\|F(g)\| = \|F(g) - F(0) + F(0)\| \leq \|F(g) - F(0)\| + \|F(0)\| \leq \frac{\|g\|}{2} + \frac{\delta}{2} \leq \delta.$$  \hspace{1cm} (63)

The energy gap

$$E^{+} - E^{-} = 2 \sum_{c \in B} J_{c} s_{c}^{D} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{k} \in S} \delta_{X_{1}, \ldots, X_{k}}^{X} \prod_{l=1}^{k} g(X_{l}) I(c \in \partial X_{l}),$$

can be evaluated in terms of the norm $\|g\|$ using an inequality $\sum_{l=1}^{k} w(X_{k}) \geq |\Lambda|_{l}$ for $\Delta_{k} = \Lambda_{L}$

$$|E^{+} - E^{-}| = \left| 2 \sum_{c \in B} J_{c} s_{c}^{D} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{k} \in S} \delta_{X_{1}, \ldots, X_{k}}^{X} \prod_{l=1}^{k} g(X_{l}) I(c \in \partial X_{l}) \right|$$

$$= \left| 2 \sum_{c \in B} J_{c} s_{c}^{D} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{k} \in S} \delta_{X_{1}, \ldots, X_{k}}^{X} (hM)^{w(X)} \prod_{l=1}^{k} \sup_{c_{l} \in B} \sum_{X_{l} \in S} g(X_{l}) I(c_{l} \in \partial X_{l}) (hM)^{-w(X_{l})} \right|$$

$$\leq \left| 2 \sum_{c \in B} J_{c} s_{c}^{D} \left| (hM)^{w(X)} \prod_{l=1}^{k} \sup_{c_{l} \in B} \sum_{X_{l} \in S} g(X_{l}) I(c_{l} \in \partial X_{l}) (hM)^{-w(X_{l})} \right| \right|_{L}$$

$$\leq \left| 2 \sum_{c \in B} J_{c} s_{c}^{D} \left| (hM)^{|\Lambda|} \prod_{k=2}^{\infty} \frac{1}{k!} \left( \left\| g \right\| \right)^{k} \right|_{L} \right| = \left| A \right| \sum_{c \in B} J_{c} s_{c}^{D} \left| (hM)^{|\Lambda|} \right|,$$  \hspace{1cm} (64)
where $A$ is defined by $A := \exp(2) \alpha$. This completes the proof of Lemma. □

**Proof of Theorem 3.2.** Let $\alpha > 0$ be the solution of an equation $e^\alpha(1+\alpha) - 1 = \frac{1}{2}$, and define a constant $\delta > 0$ by

$$
\delta := \alpha \inf_{X \in S_A} \left| \sum_{b \in \partial X} J_b s_b^D \right|. 
$$

(65)

Lemma 3.2 and the contraction mapping theorem enable us to prove that the fixed point equation $F(g) = g$ has unique solution $g$ satisfying $\|g\| \leq \delta$, if $h := \sup_{C \in B_A} \sum_{i \in \Lambda} |h_i| < \frac{1}{2}$. This solution gives the energy eigenstates

$$
|\pm\rangle = 2^{-(|\Lambda_L|+1)/2} \sum_{\sigma \in \{1,-1\}^{\Lambda_L}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_A} g(X) \sigma_X \right] |\sigma\rangle, 
$$

(66)

corresponding to $|D\rangle \pm |D^c\rangle$ in the unperturbed model. The energy gap $|E_+ - E_-|$ satisfies the following exponentially small bound

$$
|E_+ - E_-| \leq \exp(2) \alpha \left| \sum_{b \in B_A} J_b s_b^D \right| (hM)^{|\Lambda_L|} = \exp(2) \alpha \left| \sum_{b \in B_A} J_b s_b^D \right| (2h/\delta)^{|\Lambda_L|}. 
$$

for $M := \frac{2}{\delta}$. This completes the proof of Theorem 3.2. □

### 3.3 Expansion for energy gap

Here, we discuss an energy eigenstate $|\pm\rangle'$ different from the state $|\pm\rangle$ obtained in Lemma 3.2. The eigenvalue equation

$$
\tilde{H}_\Lambda(\sigma, h, J)|\pm\rangle' = E_\pm'|\pm\rangle',
$$

is written in

$$
-(\sum_{b \in B_A} J_b \sigma_b^x + \sum_{i \in \Lambda_L} h_i \sigma_i^x)|\pm\rangle' = E_\pm'|\pm\rangle'.
$$

There exists a real valued function $\phi_\pm : \{1, -1\}^{\Lambda_L} \to \mathbb{R}$, and the state $|\pm\rangle'$ is represented in

$$
|\pm\rangle' = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma) \phi_\pm(\sigma)|\sigma\rangle,
$$

where the function $\psi$ defines the reference state

$$
|\pm\rangle = 2^{-(|\Lambda_L|+1)/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_D(1 \pm \sigma_{\Lambda_L}) |\pm\rangle|\sigma\rangle.
$$

Assume that for the unperturbed model $h = 0$, there exists a sub-lattice $C \in S_A$, such that

$$
|\pm\rangle' = 2^{-(|\Lambda_L|+1)/2} \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_C(1 \pm \sigma_{\Lambda_L})|\sigma\rangle.
$$

The eigenvalue equation can be represented in terms of $\phi(\sigma)$.

$$
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_A} J_b \psi_\pm(b) \phi_\pm(b) \right] + \sum_{i \in \Lambda_L} h_i \sigma_i \psi_\pm(\sigma) \phi_\pm(\sigma) + E_\pm \sigma_D \psi_\pm(\sigma) \phi_\pm(\sigma) = 0. 
$$

(67)

Therefore

$$
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_A} J_b \sigma_D^{(b)} \psi_\pm(\sigma) \phi_\pm(\sigma) + \sum_{i \in \Lambda_L} h_i \sigma_i \phi_\pm(\sigma) + E_\pm \phi_\pm(\sigma) \right] = 0. 
$$

(68)

The following relation

$$
\sigma_D^{(b)} \psi_\pm(\sigma) \phi_\pm(\sigma) = s_b^D \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right]
$$

and the eigenvalue equation (68) give

$$
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_A} J_b s_b^D \phi_\pm(b) \right] \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] + \sum_{i \in \Lambda_L} h_i \sigma_i \phi_\pm(\sigma) + E_\pm \phi_\pm(\sigma) \right] = 0. 
$$

(69)
The energy eigenvalue equation with $E_{\pm}$ times $\phi(\sigma)$ is

$$(1 \pm \sigma_{L}) \left[ \sum_{b \in B_{\Lambda}} J_{bs_{b}} \phi_{\pm}(\sigma) \exp \left[ \sum_{X \in S_{\Lambda}} I(b \in \partial X) g(X) \sigma_{X} \right] + \sum_{i \in \Lambda_{L}} h_{i} \sigma_{i} \phi_{\pm}(\sigma) + E_{\pm} \phi_{\pm}(\sigma) \right] = 0. \quad (70)$$

The difference between above two equations gives

$$(1 \pm \sigma_{L}) \left[ \sum_{b \in B_{\Lambda}} J_{bs_{b}} \phi_{\pm}(\sigma^{(b)}) - \phi_{\pm}(\sigma) \right] \exp \left[ \sum_{X \in S_{\Lambda}} I(b \in \partial X) g(X) \sigma_{X} \right] = 0. \quad (71)$$

To obtain the Kirkwood-Thomas equation for the first excited state, represent the function $\phi(\sigma)$ in terms of a real valued function $f(X)$ of an arbitrary subset $X \in S_{\Lambda}$,

$$\phi_{\pm}(\sigma) = \frac{1}{2} (1 \pm \sigma_{L}) \sum_{X \in S_{\Lambda}} f(X) \sigma_{X}. \quad (72)$$

This gives

$$\phi_{\pm}(\sigma^{(b)}) - \phi_{\pm}(\sigma) = -(1 \pm \sigma_{L}) \sum_{X \in S_{\Lambda}} f(X) \sigma_{X} I(b \in \partial X)$$

Then we have

$$\left[ 2 \sum_{b \in B_{\Lambda}} J_{bs_{b}} \sum_{Y \in S_{\Lambda}} I(b \in \partial Y) f(Y) \sigma_{Y} \exp \left[ \sum_{X \in S_{\Lambda}} I(b \in \partial X) g(X) \sigma_{X} \right] - (E_{\pm} - E_{\pm}) \sum_{X \in S_{\Lambda}} f(X) \sigma_{X} \right] (1 \pm \sigma_{L}) = 0. \quad (73)$$

Define a function $\exp^{(1)}$ by

$$\exp^{(1)} x := e^{x} - 1 = \sum_{k=1}^{\infty} \frac{x^{k}}{k!},$$

then we have

$$\left[ \sum_{Y \in S_{\Lambda}} \Delta_{Y} f(Y) \sigma_{Y} + 2 \sum_{b \in B_{\Lambda}} J_{bs_{b}} \sum_{Y \in S_{\Lambda}} I(b \in \partial Y) f(Y) \sigma_{Y} \exp^{(1)} \left[ \sum_{X \in S_{\Lambda}} I(b \in \partial X) g(X) \sigma_{X} \right] \right] - (E'_{\pm} - E_{\pm}) \sum_{X \in S_{\Lambda}} f(X) \sigma_{X} (1 \pm \sigma_{L}) = 0. \quad (74)$$

where an energy gap $\Delta_{Y}$ for $Y \in \Lambda_{L}$ is defined by

$$\Delta_{Y} := 2 \sum_{b \in \partial Y} J_{bs_{b}}.$$

The orthonormalization property (9) gives

$$2 \sum_{b \in B_{\Lambda}} J_{bs_{b}} \sum_{Y \in S_{\Lambda}} I(b \in \partial Y) f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_{1}, \ldots, X_{k} \in S_{\Lambda}} \delta_{X_{1} \Delta \cdots \Delta X_{k} \Delta Y \Delta Z, \phi} \pm \delta_{X_{1} \Delta \cdots \Delta X_{k} \Delta Y \Delta Z, L} \prod_{l=1}^{k} g(X_{l}) I(b \in \partial X_{l}) - (E'_{\pm} - E_{\pm} - \Delta_{Z}) [f(Z) \pm f(Z')] = 0. \quad (75)$$

Therefore,

$$2 \sum_{b \in B_{\Lambda}} J_{bs_{b}} \sum_{Y \in S_{\Lambda}} I(b \in \partial Y) f(Y) \pm f(Y') \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_{1}, \ldots, X_{k} \in S_{\Lambda}} \delta_{X_{1} \Delta \cdots \Delta X_{k} \Delta Y \Delta Z, \phi} \times \prod_{l=1}^{k} g(X_{l}) I(b \in \partial X_{l}) - (E'_{\pm} - E_{\pm} - \Delta_{Z}) [f(Z) \pm f(Z')] = 0. \quad (76)$$
Define a function $e_{\pm}(X)$ of non-empty sub-lattice $X \in S_\Lambda$ by
\[
e_{\pm}(X) := \begin{cases} 

\frac{E_{\pm} - E_{\pm} - \Delta_{C}}{f(\xi) \pm f(\xi)} & (X = C) \\
\frac{E_{\pm} - E_{\pm} - \Delta_{C}}{f(\xi) \pm f(\xi)} & (X \neq C) 
\end{cases} \quad (77)
\]

For $Z = C$,
\[
e_{\pm}(C) = \sum_{Y \in S_\Lambda} \frac{1}{2} \sum_{b \in \partial Y} J_b s_b^D e_{\pm}(Y) \sum_{k=1}^{\infty} \sum_{x_1, \ldots, x_k \in S_\Lambda} \frac{1}{k!} \delta_{x_1 \Delta \cdots \Delta x_k \Delta Y,C} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) =: F(e_{\pm})(C),
\]

For $Z \neq C$,
\[
e_{\pm}(Z) = \frac{1}{\Delta_Z - \Delta_C} \left[ e_{\pm}(C)e_{\pm}(Z) - \sum_{Y \in S_\Lambda} \frac{1}{2} \sum_{b \in \partial Y} J_b s_b^D e_{\pm}(Y) \sum_{k=1}^{\infty} \sum_{x_1, \ldots, x_k \in S_\Lambda} \frac{1}{k!} \delta_{x_1 \Delta \cdots \Delta x_k \Delta Y,Z} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right] =: F(e_{\pm})(Z).
\]

These equations define a fixed point equation $F(e) = e$, whose solution $e$ gives the state $|\pm\rangle$ except its normalization. To prove the uniqueness of the solution, define a norm for the function $e_{\pm}$ by
\[
||e_{\pm}|| := |e_{\pm}(C)| + \sum_{X \in S_\Lambda} |\Delta_{X} - \Delta_{C}| |e_{\pm}(X)|.
\]

The following theorem implies that there is no level crossing against a sufficiently small perturbation $h$.

**Theorem 3.4** Consider the random transverse field EA model defined by the Hamiltonian \[.\] . For two different sub-lattices $C, D \in S_\Lambda$, let $s^C, s^D \in \{1, -1\}^{\Lambda_l}$ be their corresponding sequences defined by \[.\]. If the sequence of transverse fields $h$ is sufficiently weak, there exists a sufficiently small constant $\delta > 0$ depending on the sequence of coupling constants $(J, e)$, such that the energy gap $E'_{\pm} - E_{\pm}$ in the perturbed model satisfies
\[
H_\Lambda^{s^C}(s^C, J, 0) - H_\Lambda^{s^D}(s^D, J, 0) - \delta < E'_{\pm} - E_{\pm} < H_\Lambda^{s^C}(s^C, J, 0) - H_\Lambda^{s^D}(s^D, J, 0) + \delta,
\]
for almost all $J \in \mathbb{R}^{B_\Lambda}$.

Theorem 3.4 is proven by the following lemma.

**Lemma 3.5** Consider the model under the conditions in Lemma 3.2 and assume that lower bound on energy gap $\inf_{X,Y \in S_\Lambda} |\Delta_{X} - \Delta_{Y}|$ in the unperturbed model is sufficiently large. There exist constants $\delta > 0$ and $0 < K < 1$, such that
\[
||F(e_{\pm}) - F(e'_{\pm})|| \leq K ||e_{\pm} - e'_{\pm}||, \quad ||F(e_{\pm})|| \leq \delta, \quad \text{for} \quad ||e_{\pm}||, ||e'_{\pm}|| \leq \delta.
\]

for almost all $J \in \mathbb{R}^{B_\Lambda}$.

**Proof.** For lighter notation, we remove indices $\pm$ from $e_{\pm}$. The difference between two evaluations of energy
where $\Delta' := \inf_{Y} |\Delta_Y - \Delta_C|$. 

$$
\|F(e) - F(e')\| = |F(e(C)) - F(e'(C))| + \sum_{Z \in S_A} \left| e(C)e(Z) - e'(C)e'(Z) \right| \\
-2 \sum_{b \in B_{\Lambda}} J_{b}^{P} \sum_{Y \in S_{A}} |I(b)\|e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_{A}} \delta \sum_{X_1, \ldots, X_k \in Y} \prod_{l=1}^{k} g(X_l) |I(b)\|e'(Z)| \\
\leq |F(e(C)) - F(e'(C))| + \sum_{Z \in S_{A}} |e(C)e(Z) - e'(C)e'(Z)| \\
+ \sum_{Y \in S_{A}} 2 \sum_{b \in B_{\Lambda}} J_{b}^{P} |e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_{A}} \delta \sum_{X_1, \ldots, X_k \in Y} \prod_{l=1}^{k} g(X_l) |I(b)\|e'(Z)| \\
\leq |F(e(C)) - F(e'(C))| + |e(C) - e'(C)| \sum_{Z \in S_{A}} |e(Z) + |e(C)| \sum_{Z \in S_{A}} |e(Z) - e'(Z)| \\
+ \sum_{Y \in S_{A}} 2 \sum_{b \in B_{\Lambda}} J_{b}^{P} |e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_{A}} \delta \sum_{X_1, \ldots, X_k \in Y} \prod_{l=1}^{k} g(X_l) |I(b)\|e'(Z)| \\
\leq |F(e(C)) - F(e'(C))| + |e(C) - e'(C)| \sum_{Z \in S_{A}} |e(Z) - e'(Z)| \\
+ \sum_{Y \in S_{A}} |\Delta_Y||e(Y) - e'(Y)| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_{A}} \delta \sum_{X_1, \ldots, X_k \in Y} \prod_{l=1}^{k} g(X_l) |I(c_l)\|e'(Z)| \\
\leq |e - e'||(1 + |\Delta_C|/\Delta') (e^{\delta/\Delta} - 1) + \delta/\Delta' |e - e'|| + (1 + |\Delta_C|/\Delta') (e^{\delta/\Delta} - 1) |e - e'|| = K|e - e'||, 
$$
where $K := 2(1 + |\Delta_C|/\Delta') (e^{\delta/\Delta} - 1) + \delta/\Delta'$. If $K < 1$, $F$ is a contraction mapping. This completes the proof of Lemma 3.3. \hfill \Box 

**Proof of Theorem 3.4.** Lemma 3.3 implies that 

$$
|E'_{\pm} - E_{\pm} - \Delta_C| = |e(C)| \leq \delta.
$$

This implies 

$$
H^{Z}_{A}(s^{C}, J, 0) - H^{Z}_{A}(s^{D}, J, 0) - \delta < E'_{\pm} - E_{\pm} < H^{Z}_{A}(s^{C}, J, 0) - H^{Z}_{A}(s^{D}, J, 0) + \delta,
$$
for almost all $J \in \mathbb{R}^{B_{\Lambda}}$. This completes the proof of Theorem 3.4. \hfill \Box 

### 3.4 Remarks on results for the random transverse field EA model

1. For the random transverse field EA model, the Perron-Frobenius theorem enables us to prove the uniqueness of the ground state. For an arbitrary sequence $(h_{i})_{i \in A_{L}^{+}}$, define a sequence $\theta := (\theta_{i})_{i \in A_{L}^{+}}$ by
\[ \theta_i = 0 \text{ for } h_i > 0 \text{ and } \theta_i = \pi/2 \text{ for } h_i < 0. \]

The following unitary transformation

\[ U_\theta := \exp \left( i \sum_{i \in \Lambda_L} \theta_i \sigma_i^z \right) \]

transforms the perturbation Hamiltonian

\[ U_\theta \sum_{i \in \Lambda_L} h_i \sigma_i^x U_\theta^\dagger = \sum_{i \in \Lambda_L} |h_i| \sigma_i^x. \]

For the Hamiltonian defined by (4), the transformed Hamiltonian \( H_{\theta} := U_\theta H_{\Lambda}^Z U_\theta^\dagger \) satisfies (i) non-positivity of all off-diagonal matrix elements \( \langle \sigma | H_{\theta} | \tau \rangle \leq 0 \) for \( \sigma \neq \tau \) and (ii) connectivity condition \( \langle \sigma | H_{\theta} | \tau \rangle \neq 0 \) for any \( \sigma \neq \tau \) for some positive integer \( n \). These conditions allow the application of the Perron-Frobenius theorem to the transverse field EA model. This theorem implies that the unique ground state is given by \( |GS\rangle = \sum_{\sigma \in \Sigma} \psi(\sigma)|\sigma\rangle \) with positive coefficients \( \psi(\sigma) \). This result is consistent to Theorem 2.2 and 3.2 in the present paper. This Perron-Frobenius argument is valid also for the EA model under an arbitrary sequence of vector-valued fields \((\tilde{h}_i)_{i \in \Lambda_L} := (h_i^x, h_i^y, h_i^z)_{i \in \Lambda_L}\).

As far as the ground state in the random transverse field EA model is concerned, our expansion method cannot yield new results other than that obtained in the Perron-Frobenius argument. For excited states, however, our method concludes that there is no level crossing between any energy eigenstates against sufficiently weak transverse exchange interactions in the following.

2. Consider a problem to obtain the ground state in the EA model by the quantum annealing with the transverse field EA model. Since Theorem 3.4 cannot guarantee the absence of level crossing between energy eigenvalues \( E_+ \) and \( E_- \), a condition to break \( \mathbb{Z}_2 \) in the EA model is necessary to obtain precise solution of the ground state. Theorem 3.4 guarantees that the EA model has no level crossing for sufficiently weak transverse field, if there is no \( \mathbb{Z}_2 \) symmetry.

3. Consider the transverse field EA model with weak transverse fields. Theorem 3.2 claims that an arbitrary energy gap \( |E_+ - E_-| \) between split energy eigenvalues is exponentially small in the system size \( |\Lambda_L| \). This fact suggests a spontaneous symmetry breaking of the \( \mathbb{Z}_2 \) symmetry in this model. In our expansion method, however, any results cannot be concluded in the infinite-volume limit.

4. The random bond Heisenberg XYZ model

4.1 Energy eigenstate

Let us consider the random bond Heisenberg XYZ model defined by the Hamiltonian (5), which is invariant

\[ P_w H_{\Lambda}^{XYZ}(\sigma, h, J) P_w^\dagger = H_{\Lambda}^{XYZ}(\sigma, h, J), \]

for any \( w = x, y, z \). \( P_\Lambda \) corresponds to the \( \mathbb{Z}_2 \) invariance in the EA model. To construct a perturbative expansion method, we define several notations. For an arbitrary sub-lattice \( D \in S_\Lambda \), define an eigenvalue sequence \( s^D \in \{1, -1\}^{\Lambda_L} \) by (10). In the unperturbed model defined by the Hamiltonian \( H_{\Lambda}^{Z}(\sigma, J) \), there is a one-to-one correspondence between a sub-lattice \( D \in S_\Lambda \) and an arbitrary energy eigenstate \( |s^D\rangle \), which breaks \( \mathbb{Z}_2 \) symmetry. To obtain all energy eigenstates in the random bond Heisenberg XYZ model, consider the following unitary transformed Hamiltonian

\[ \tilde{H}_{\Lambda}(\sigma, J, \epsilon) := U H_{\Lambda}^{XYZ} U^\dagger = - \sum_{\sigma \in \mathbb{Z}_2} (J_0 \sigma_0^z + \epsilon_0^x \sigma_0^x + \epsilon_0^y \sigma_0^y), \quad (83) \]

where \( U \sigma_i^x U^\dagger = \sigma_i^x \) and \( U \sigma_i^y U^\dagger = -\sigma_i^x \). Let \( \psi_{\pm}(\sigma) \) be functions \( \psi_{\pm} : \{-1, 1\}^{\Lambda_L} \rightarrow \mathbb{R} \) of spin eigenvalues, and express an eigenstate of the Hamiltonian \( \tilde{H}_{\Lambda} \) corresponding to \( s^D \)

\[ |\pm\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \{-1, 1\}^{\Lambda_L}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_{\pm}(\sigma) |\sigma\rangle. \]

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Note the eigenvalue equation $P_z|\pm\rangle = \pm|\pm\rangle$. The normalization of the state $|\pm\rangle$ requires
\[
\sum_{\sigma \in \{1, -1\}^\Lambda_L} \psi_\pm(\sigma)^2 = 2|\pm\rangle.
\] (84)

Note that $\psi_\pm(\sigma) = 1$ for $\epsilon = 0$ is given by a spin configuration $s^D \in \{1, -1\}^\Lambda_L$ regarded as an energy eigenstate in the EA model for $\epsilon = 0$. The eigenvalue equation defined by
\[
\tilde{H}_\Lambda(\sigma, J, \epsilon)|\pm\rangle = E_\pm|\pm\rangle
\]
is written in
\[
- \sum_{b \in B_\Lambda} (J_b \sigma_b^\tau + \epsilon_b^x \sigma_b^x - \epsilon_b^y \sigma_b^y)|\pm\rangle = E_\pm|\pm\rangle,
\]
where $\sigma_y^j = -i\sigma_j^x\sigma_j^y$ has been used. If $\sigma^x_\tau|\sigma\rangle = |\tau\rangle$, $\tau_i = -\sigma_i$ and $\tau_j = \sigma_j$ for $j \neq i$.
\[
\sigma_b^x|\sigma\rangle = \sigma_i^x\sigma_j^x|\sigma\rangle = |\sigma^{(i,j)}\rangle,
\]
where $\sigma^{(i,j)}$ denotes a spin configuration replaced by $(\sigma_i, \sigma_j) \rightarrow (-\sigma_i, -\sigma_j)$. This eigenvalue equation can be represented in terms of $\psi(\sigma)$.
\[
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_\Lambda} [J_b \sigma^D_b \psi_\pm(\sigma^b) + \epsilon_b^x \sigma_b \psi_\pm(\sigma) - \epsilon_b^y \sigma_b \psi_\pm(\sigma^b)] + E_\pm \psi_\pm(\sigma) \right] = 0.
\] (85)

Therefore
\[
(1 \pm \sigma_{\Lambda_L}) \left[ \sum_{b \in B_\Lambda} [(J_b - \epsilon_b^y \sigma_b) \frac{\sigma^D_b \psi_\pm(\sigma^b)}{\sigma_D \psi_\pm(\sigma)} + \epsilon_b^x \sigma_b] + E_\pm \right] = 0.
\] (86)

To obtain the Kirkwood-Thomas equation for energy eigenstates, represent the function $\psi(\sigma)$ in terms of a real valued function $g(X)$ of an arbitrary sub-lattice $X \in S_\Lambda$,
\[
\psi_\pm(\sigma) = \frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_\Lambda} g(X) \sigma_X \right].
\] (87)

Note the following relations
\[
\psi_\pm(\sigma^b) = \frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_\Lambda} g(X) \sigma_X + \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right].
\] (88)

\[
\frac{\psi_\pm(\sigma^b)}{\psi_\pm(\sigma)} = \frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \exp \left[ \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right].
\]

Note also,
\[
\frac{s^D_b}{\sigma_D} = s^D_b.
\]

These and the eigenvalue equation (86) give
\[
\frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \left( \sum_{b \in B_\Lambda} [(J_b - \epsilon_b^y \sigma_b) s^D_b] \exp \left[ \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right] + \epsilon_b^x \sigma_b] + E_\pm \right) = 0.
\] (89)

We expand the exponential function in power series. The first order term in the exponential function gives
\[
\sum_{b \in B_\Lambda} J_b s^D_b \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X = \sum_{X \in S_\Lambda} \sum_{b \in \partial X} J_b s^D_b g(X) \sigma_X,
\] (90)

then we have
\[
\frac{1}{2}(1 \pm \sigma_{\Lambda_L}) \left( \sum_{b \in B_\Lambda} [J_b s^D_b + (\epsilon_b^x - \epsilon_b^y s^D_b) s_b] + E_\pm + \sum_{X \in S_\Lambda} \sum_{b \in \partial X} J_b s^D_b g(X) \sigma_X \right.
\]
\[
+ \sum_{b \in B_\Lambda} J_b s^D_b \exp^{(2)} \left[ \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right] - \sum_{b \in B_\Lambda} \epsilon_b^x s^D_b \exp^{(1)} \left[ \sum_{X \in S_\Lambda} I(b \in \partial X) g(X) \sigma_X \right] \right) = 0,
\] (91)
where for a positive integer $n$, the function is defined by
\[
\exp(n) x := \sum_{k=n}^{\infty} \frac{x^k}{k!}.
\]

The equation (91) summed over all $\sigma \in \{1, -1\}^{A_L}$ and the orthonormalization property (9) give
\[
E_{\pm} = -\frac{1}{2} \sum_{b \in B_A} J_b s^D_b - \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} J_c \delta_{\Delta_k, \phi} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l)
\]
\[
+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} \epsilon c \delta_{\Delta_k, \phi} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l),
\]
\[
+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} J_c \delta_{\Delta_k, \Lambda L} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l)
\]
\[
+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} \epsilon c \delta_{\Delta_k, \Lambda L} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l).
\] (92)

where The first term in the right hand side of (92) for the energy eigenvalue $E_{\pm}$ is identical to that of the spin configuration $s^D$ for $\epsilon = 0$. For $X \neq \phi, \Lambda L$, the summation of the equation (91) multiplied by $\sigma_Y$ over $\sigma \in \{1, -1\}^{A_L}$ and the orthonormalization property (9) yield
\[
g(X) = \frac{-1}{\sum_{b \in \partial X} J_b s^D_b} \left[ \sum_{c \in B_A} (\epsilon c - \epsilon c s^D_c) \delta_{c, X} + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} J_c \delta_{\Delta_k, X} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l) \right]
\]
\[- \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A, \, c \in B_A} \epsilon c \delta_{\Delta_k, X} s^D_c \prod_{l=1}^{k} g(X_l) I(c \in \partial X_l) \right] =: F(g)(X),
\] (93)

which is a fixed point equation for $g$. The normalization (54) fixes $g(\phi) = \pm g(\Lambda L)$. For $\epsilon = 0$, $2 \sum_{b \in \partial X} J_b s^D_b$ represents the energy gap of a spin configuration given by $X := \{i \in \Lambda_L : \sigma_i \neq s^D_i\}$. Lemma 3.1 guarantees $\sum_{b \in \partial X} J_b s^D_b \neq 0$ for $X \neq D$.

We provide the following theorem for the energy eigenstates in the random bond Heisenberg XYZ model on the basis of convergent perturbative expansions around $\epsilon = 0$.

To prove the convergence of expansion for the energy eigenstate corresponding to the unperturbed state $|s^D\rangle$ in the random bond XYZ model with a given $J$ for sufficiently small $\epsilon$, define a norm for the function $g(X)$ with positive constants $\epsilon := \sup_{c \in B_A, \sum_{b \in \partial c} |\epsilon c - \epsilon c s^D_c|}$ and $M > 0$ by
\[
\|g\| := \sup_{c \in B_A, X \in S_A} \sum_{X \in S_A} I(c \in \partial X) \left| \sum_{b \in \partial X} J_b s^D_b \right| g(X) |(\epsilon M)^{-w(X)}|,
\] (94)

where $w(X)$ is the cardinality of the smallest set of connected bonds whose union contains $X$. We say that two bonds $\{i_1, i_2\}, \{j_1, j_2\} \in B_A$ are connected, if $\inf_{1 \leq k, l \leq 2} |i_k - j_l| \leq 1$.

**Theorem 4.1** Consider the Heisenberg XYZ model defined by the Hamiltonian (39). For an arbitrary sub-lattice $D \subset S_A$, the energy eigenstate $|\pm\rangle$ with energy eigenvalues $E_{\pm}$ corresponding to the unperturbed energy eigenstate $|s^D\rangle$ exist and satisfy $\sigma_{\Lambda L}^\pm |\pm\rangle = |\pm\rangle$ for almost all $J \in \mathbb{R}^{B_A}$, if the XY-exchange coupling constants $\epsilon$ are sufficiently weak. The energy gap $|E_+ - E_-|$ is exponentially small in the system size $|\Lambda L|$.

The following lemma and the contraction mapping theorem enable us to prove Theorem 4.1.
Lemma 4.2 Consider the Heisenberg XYZ model defined by the Hamiltonian \( \mathcal{H}_0 \). For an arbitrary sublattice \( D \subseteq S_L \), let \( s^D \in \{1,-1\}^{\Lambda^D} \) be its corresponding sequence defined by \( \mathcal{H}_0 \). Define \( \Delta, \epsilon, \delta > 0 \) by

\[
\Delta := \inf_{X \in S_L} \left| 2 \sum_{b \in \partial X} J_b s_b^D \right|, \quad \epsilon_g := \Delta \sup_{c \in B_X, X \in S_L} I(c \in \partial X) \sum_{b \in \partial X} |\epsilon_b^c| / \left| \sum_{b \in \partial X} J_b s_b^D \right|, \quad \epsilon := \sup_{c \in B_X, b \in \partial c} |\epsilon_b^c s_b^D - \epsilon_b^c|.
\]

For a constant \( \delta > 0 \), define \( M := 2/\delta \). If \( \epsilon \) and \( \delta \) satisfy

\[
e^{2\epsilon/\Delta} [2\delta/\Delta + 1 + (\epsilon M)^{-1} \epsilon_g/\Delta (2\delta/\Delta + 4d - 1)] \leq \frac{3}{2}, \quad \epsilon < \delta/2,
\]

then the following norms are bounded by

\[
\|F(g) - F(g')\| \leq \|g - g'\|/2, \quad \|F(g)\| \leq \delta, \quad \text{for } \|g\|, \|g'\| \leq \delta,
\]

for almost all \( J \in \mathbb{R}^{B_X} \). This unique function \( g \) defines the energy eigenstates \( |\pm\rangle \) with energy eigenvalues \( E_{\pm} \) corresponding to the unperturbed energy eigenstate \( |s^D\rangle \pm |s'^D\rangle \) for almost all \( J \). There exists a constant \( A > 0 \) depending on \( \delta > 0 \) such that the energy gap \( |E_+ - E_-| \) has an exponentially small bound

\[
|E_+ - E_-| \leq A \left( \sum_{c \in B_X} J_c s_c^D + \sum_{c \in B_X} |\epsilon_b^c| \right) (\epsilon M)^{\Lambda^L},
\]

Proof. The norm \( \|F(g) - F(g')\| \) is represented in

\[
\|F(g) - F(g')\| = \sup_{c \in B_X} \sum_{X \in S_L} I(c \in \partial X) \left| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_L} s_b^D \sum_{b \in B_X} J_b \delta_{\Delta_k, X} \sum_{l=1}^{k} I(b \in \partial X_l) \right| (\epsilon M)^{-w(X)},
\]

where each term in the last line is defined by

\[
(I) := \sup_{c \in B_X} \sum_{X \in S_L} I(c \in \partial X) \left| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_L} s_b^D \sum_{b \in B_X} J_b \delta_{\Delta_k, X} \sum_{l=1}^{k} I(b \in \partial X_l) \right| (\epsilon M)^{-w(X)},
\]

\[
(II) := \sup_{c \in B_X} \sum_{X \in S_L} I(c \in \partial X) \left| \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_L} s_b^D \sum_{b \in B_X} J_b \delta_{\Delta_k, X} \sum_{l=1}^{k} I(b \in \partial X_l) \right| (\epsilon M)^{-w(X)}.
\]

Let us evaluate each term. An upper bound on (I) is

\[
(I) \leq \sup_{c \in B_X} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \ldots, X_k \in S_L} I(c \in \partial \Delta_k) \left| \sum_{b \in B_X} J_b s_b^D \sum_{l=1}^{k} I(b \in \partial X_l) \right| (\epsilon M)^{-w(\Delta_k)}.
\]

where \( w(\Delta_k) = \sum_{l=1}^{k} w(X_l), I(c \in \partial \Delta_k) \leq \sum_{l=1}^{k} I(c \in \partial X_l) \) and permutation symmetry in the summation.
over \(X_1, \ldots, X_k\) have been used. An inequality (24) enables us to evaluate (I) as follows:

\[
\text{(I)} \leq \sup_{c \in B_A} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^{k-2} \prod_{j=1}^{l} |g(X_j)|(|\epsilon M|^{-w(X_j)} - |g(X_j)|) \prod_{j=l+1}^{k} |g'(X_j)|(|\epsilon M|^{-w(X_j)}) \sum_{b \in B_A} I(b \in \partial X_l)
\]

\[
= \sup_{c \in B_A} \sum_{X_1, \ldots, X_k \in S_A} \sum_{b \in \partial X_k} \prod_{l=1}^{k} |g(X_j)|(|\epsilon M|^{-w(X_j)}) \prod_{j=1}^{k} \left[ \|g(X_j)|(|\epsilon M|^{-w(X_j)}) \sum_{b \in \partial X_l} I(b \in \partial X_l) \right]
\]

\[
= \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sup_{c_1, \ldots, c_k \in B_A} \sum_{X_1, \ldots, X_k \in S_A} \left[ \sum_{b \in \partial X_1} \prod_{l=1}^{k} |g(X_j)|(|\epsilon M|^{-w(X_j)}) \prod_{j=1}^{k} \left[ \|g(X_j)|(|\epsilon M|^{-w(X_j)}) \sum_{b \in \partial X_l} I(b \in \partial X_l) \right] \right]
\]

\[
\times |g(X_j)| - |g'(X_j)|(|\epsilon M|^{-w(X_j)}) I(c_j \in \partial X_j)(|\epsilon M|^{-w(X_j)}) \prod_{j=l+1}^{k} |g'(X_j)| I(c_j \in \partial X_j)(|\epsilon M|^{-w(X_j)}) \right)
\]

\[
= (II) := \sup_{c \in B_A} \sum_{X_1, \ldots, X_k \in S_A} I(c \in \partial X) \sum_{l=1}^{k-1} \prod_{j=1}^{l} |g(X_j)|(|\epsilon M|^{-w(X_j)}) \prod_{j=l+1}^{k} |g'(X_j)| I(b \in \partial X_l)(|\epsilon M|^{-w(X_j)})
\]

\[
\leq \sup_{c \in B_A} \sum_{X_1, \ldots, X_k \in S_A} I(c \in \partial X) \sum_{l=1}^{k-1} \prod_{j=1}^{l} |g(X_j)|(|\epsilon M|^{-w(X_j)}) \prod_{j=l+1}^{k} |g'(X_j)| I(b \in \partial X_l)(|\epsilon M|^{-w(X_j)})
\]

\[
\sum_{c_1, \ldots, c_k \in B_A} \sum_{X_1, \ldots, X_k \in S_A} \sum_{b \in \partial X_k} \prod_{l=1}^{k} |g(X_j)|(|\epsilon M|^{-w(X_j)}) \prod_{j=1}^{k} \left[ \|g(X_j)|(|\epsilon M|^{-w(X_j)}) \sum_{b \in \partial X_l} I(b \in \partial X_l) \right]
\]

\[
\times |g(X_j)| - |g'(X_j)|(|\epsilon M|^{-w(X_j)}) I(c_j \in \partial X_j)(|\epsilon M|^{-w(X_j)}) \prod_{j=l+1}^{k} |g'(X_j)| I(c_j \in \partial X_j)(|\epsilon M|^{-w(X_j)}) \right)
\]

\[
= (II_A) + (II_B).
\]

An explicit decomposition

\[
p_{\partial c} := \bigcup_{a=1}^{2(2d-1)} \{ c_a \in B_A \},
\]

\[
(101)
\]
enables us to evaluate the first term \((\Pi_A)\) in the last line,

\[
\Pi_A := \sup_{c \in B_A} \sum_{k=1}^\infty \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \sum_{b \in B_A} |\epsilon_b^k| I(c(b \in \partial c)) \prod_{l=1}^k g(X_l) \prod_{l=1}^k g'(X_l) \prod_{l=1}^k I(b \in \partial X_l)(\epsilon M)^{-\sum_{j=1}^k w(X_j) - 1}
\]

\[
\leq \sup_{c \in B_A} \sum_{k=1}^\infty \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} |\epsilon_b^k| \prod_{l=1}^k I(b \in \partial X_l) \sum_{j=1}^{k-1} \prod_{l=1}^k |g(X_j)||g(X_l) - g'(X_l)| \prod_{j=l+1}^k |g'(X_j)|(\epsilon M)^{-\sum_{j=1}^k w(X_j) - 1}
\]

\[
\leq \sum_{k=1}^\infty \frac{1}{k!} \sum_{l=1}^{k-1} \sup_{b_j \in B_A} \sum_{X_j \in S_A} |g(X_j)||I(b_j \in \partial X_j)(\epsilon M)^{-w(X_j)}
\]

\[
\times \sup_{c \in B_A} \sum_{X_j \in S_A} \sum_{b \in \partial c \cap \partial X} |\epsilon_b^j| |g(X_j) - g'(X_j)|(\epsilon M)^{-w(X_j) - 1} \prod_{j=l+1}^k \sup_{b_j \in B_A} \sum_{X_j \in S_A} |g'(X_j)||I(b_j \in \partial X_j)(\epsilon M)^{-w(X_j)}
\]

\[
\leq \sum_{k=1}^\infty \frac{1}{k!} \sum_{l=1}^{k-1} \sup_{b_j \in B_A} \sum_{X_j \in S_A} |g(X_j)||I(b_j \in \partial X_j)(\epsilon M)^{-w(X_j)}
\]

\[
\times \sup_{c \in B_A} \sum_{X_j \in S_A} \sum_{b \in \partial c \cap \partial X} |\epsilon_b^j| |g'(c_a \in \partial X_j)||g(X_j) - g'(X_j)|(\epsilon M)^{-w(X_j) - 1} \prod_{j=l+1}^k \sup_{b_j \in B_A} \sum_{X_j \in S_A} |g'(X_j)||I(b_j \in \partial X_j)(\epsilon M)^{-w(X_j)}
\]

\[
(\Pi_A) \text{ is bounded in terms of norms of } g, g'.
\]

\[
(\Pi_A) \leq (4d - 2)(\epsilon M)^{-1} \epsilon_y / \Delta ||g - g'|| \sum_{k=1}^\infty \frac{1}{k!} \sum_{l=1}^k \left( \frac{2||g||}{\Delta} \right)^{l-1} \left( \frac{2||g'||}{\Delta} \right)^{k-l}
\]

\[
\leq (4d - 2)(\epsilon M)^{-1} \epsilon_y / \Delta ||g - g'|| \sum_{k=1}^\infty \frac{1}{(k-1)!} \left( \frac{2\delta}{\Delta} \right)^{k-1} = (4d - 2)(\epsilon M)^{-1} \epsilon_y / \Delta e^{2\delta / \Delta} ||g - g'||, (102)
\]

where \(\epsilon_y\) is defined by (55). The second term is evaluated as follows:

\[
(\Pi_B) := \sup_{c \in B_A} \sum_{k=1}^\infty \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} |\epsilon_b^k| I(c \in \partial X_1)(\epsilon M)^{-\sum_{j=1}^k w(X_j) - 1} \prod_{l=1}^k g(X_l) - \prod_{l=1}^k g'(X_l) \prod_{l=1}^k I(b \in \partial X_l)
\]

\[
\leq \sum_{k=1}^\infty \frac{1}{(k-1)!} \sup_{c \in B_A} \sum_{X_1, \ldots, X_k \in S_A} |\epsilon_b^k| I(c \in \partial X_1)(\epsilon M)^{-\sum_{j=1}^k w(X_j) - 1} \prod_{l=1}^k g(X_l) - \prod_{l=1}^k g'(X_l) \prod_{l=1}^k I(c \in \partial X_l)
\]

\[
\leq (\epsilon M)^{-1} \epsilon_y / \Delta ||g - g'|| \sum_{k=1}^\infty \frac{1}{(k-1)!} \sum_{l=1}^k \left( \frac{2||g||}{\Delta} \right)^{l-1} \left( \frac{2||g'||}{\Delta} \right)^{k-l} \leq (\epsilon M)^{-1} \epsilon_y / \Delta ||g - g'|| \sum_{k=1}^\infty \frac{k}{(k-1)!} \left( \frac{2\delta}{\Delta} \right)^{k-1}
\]

\[
= (\epsilon M)^{-1} \epsilon_y / \Delta e^{2\delta / \Delta} (2\delta / \Delta + 1) ||g - g'||,
\]

where the inequality (55) has been used. These imply that an upper bound on \((\Pi)\) is given by

\[
\Pi = (\Pi_A) + (\Pi_B) \leq (\epsilon M)^{-1} \epsilon_y / \Delta e^{2\delta / \Delta} (2\delta / \Delta + 4d - 1) ||g - g'||. (103)
\]

Therefore, evaluations of (I) and (II) give the following upper bound on the norm

\[
\|F(g) - F(g')\| \leq (I) + (II) \leq [e^{2\delta / \Delta}(2\delta / \Delta + 1) - 1 + (\epsilon M)^{-1} \epsilon_y / \Delta e^{2\delta / \Delta}(2\delta / \Delta + 4d - 1)] ||g - g'||. (104)
\]

Assume

\[
e^{2\delta / \Delta}(2\delta / \Delta + 1) - 1 + (\epsilon M)^{-1} \epsilon_y / \Delta e^{2\delta / \Delta}(2\delta / \Delta + 4d - 1) \leq \frac{1}{2}. (105)
\]

To obtain the bound on \(\|F(g)\|\), let us evaluate \(\|F(0)\|\) first. Since

\[
F(0)(X) = \frac{\sum_{b, \in B_A} (\epsilon_b^k s_b^D - \epsilon_b^k) \delta_{X,b}}{\sum_{b \in \partial X} d_b s_b^D},
\]
and the definition $\epsilon := \sup_{c \in B_A} \sum_{b \in \partial c} |\epsilon^b_i s^D_c - \epsilon^b_i|$, the norm of $F(0)$ is given by

$$\|F(0)\| = \sup_{c \in B_A} \sum_{X \in S_A} I(c \in \partial X) \left| \sum_{b \in B_A} (\epsilon^b_i s^D_c - \epsilon^b_i) \delta_{X,b} \right| (\epsilon M)^{-w(X)}$$

$$= \sup_{c \in B_A} \sum_{b \in \partial B_A} I(c \in \partial b) |\epsilon^b_i s^D_c - \epsilon^b_i| (\epsilon M)^{-1}$$

$$= \sup_{c \in B_A} \sum_{b \in \partial c} |\epsilon^b_i s^D_c - \epsilon^b_i| (\epsilon M)^{-1} = M^{-1}. \quad (106)$$

If $M^{-1} \leq \delta/2$, then $\|F(g)\| \leq \delta$ is proven as follows:

$$\|F(g)\| = \|F(g) - F(0) + F(0)\| \leq \|F(g) - F(0)\| + \|F(0)\| \leq \frac{\|g\|}{2} + \frac{\delta}{2} \leq \delta.$$

The energy gap between $E_+$ is given by

$$E_+ - E_- = -\sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \sum_{c \in B_A} J_c \delta_{\Delta_k, \Lambda_L} s^D_c \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \sum_{c \in B_A} \epsilon^y_c \delta_{\Delta_k, \Lambda_L} s^D_c \prod_{l=1}^k g(X_l) I(c \in \partial X_l), \quad (107)$$

An upper bound on $|E_+ - E_-|$ is evaluated as

$$|E_+ - E_-| \leq \left| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \sum_{c \in B_A} J_c \delta_{\Delta_k, \Lambda_L} s^D_c \prod_{l=1}^k g(X_l) I(c \in \partial X_l) - \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \sum_{c \in B_A} \epsilon^y_c \delta_{\Delta_k, \Lambda_L} s^D_c \prod_{l=1}^k g(X_l) I(c \in \partial X_l) \right|$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} J_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} \epsilon^y_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} J_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} (\epsilon M)^{\sum_{l=1}^k w(X_l)} \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} \epsilon^y_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} (\epsilon M)^{\sum_{l=1}^k w(X_l)} \prod_{l=1}^k g(X_l) I(c \in \partial X_l).$$

The inequality $\sum_{l=1}^k w(X_l) \geq |\Lambda_L|$ for $\Delta_k = \Lambda_L$ gives

$$|E_+ - E_-| \leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} J_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} (\epsilon M)^{|\Lambda_L|} \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{X_1, \ldots, X_k \in S_A} \left| \sum_{c \in B_A} \epsilon^y_c s^D_c \right| \delta_{\Delta_k, \Lambda_L} (\epsilon M)^{|\Lambda_L|} \prod_{l=1}^k g(X_l) I(c \in \partial X_l)$$

$$\leq \left( \sum_{c \in B_A} J_c s^D_c \right) + \sum_{c \in B_A} \epsilon^y_c \prod_{k=2}^{\infty} \frac{1}{k!} \sum_{l=1}^k \sup_{c \in B_A} \sum_{X_1 \in S_A} g(X_l) (\epsilon M)^{-w(X_l)} I(c \in \partial X_l)$$

$$\leq \left( \sum_{c \in B_A} J_c s^D_c \right) + \sum_{c \in B_A} \epsilon^y_c \prod_{k=2}^{\infty} \frac{1}{k!} \left( \frac{\|g\|}{\Delta} \right)^k$$

$$\leq \left( \sum_{c \in B_A} J_c s^D_c \right) + \sum_{c \in B_A} \epsilon^y_c \prod_{k=2}^{\infty} \left( \frac{\|g\|}{\Delta} \right)^k \exp(2) \left( \frac{\delta}{\Delta} \right). \quad (108)$$
Therefore,
\[ |E_+ - E_-| \leq A \left( \left| \sum_{c \in B_A} J_c s_c^D \right| + \sum_{c \in B_A} |\epsilon_c^D| \right) (\epsilon M)^{|\Lambda_L|}, \]
where \( A := \exp(\alpha) \) This completes the proof. \( \square \)

**Proof of Theorem 4.1** Lemma 4.2 and the contraction mapping theorem enable us to prove that the fixed point equation \( F(g) = g \) has the unique solution \( g \) for sufficiently weak perturbation of the XY interactions, under the condition in Lemma 4.2. This solution gives energy eigenstates
\[ |\pm\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \{1,-1\}^{\Lambda_L}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \exp \left[ -\frac{1}{2} \sum_{X \in S_A} g(X) \sigma_X \right] |\sigma\rangle, \]
corresponding to \(|D\rangle \pm |D'\rangle\) in the unperturbed model. The condition in Lemma 4.2 gives the bound on the energy gap
\[ |E_+ - E_-| \leq A \left( \left| \sum_{c \in B_A} J_c s_c^D \right| + \sum_{c \in B_A} |\epsilon_c^D| \right) (2\epsilon/\delta)^{|\Lambda_L|}. \]
If \( \epsilon < \delta/2 \), this energy gap becomes exponentially small in the system size \(|\Lambda_L|\). This completes the proof. \( \square \)

### 4.2 Expansion for energy gap

Let \( C, D \in S_A \) be two different sub-lattices which define sequences of eigenvalues \( s^C, s^D \in \{1, -1\}^{\Lambda_L} \) in the unperturbed model \( \epsilon = 0 \). Lemma 4.1 guarantees that there is no degeneracy \( H_A(s^C, J, 0) \neq H_A(s^D, J, 0) \) in the unperturbed model. Here we evaluate energy gap between corresponding two energy eigenstates \(|\pm\rangle'\) and \(|\pm\rangle\) in perturbed model. These obey the following eigenvalue equations
\[ -\sum_{b \in B_A} (J_b \sigma_b^\pm + \epsilon_b^\pm \sigma_b^\pm + \epsilon_b^\pm \sigma_b^\pm) |\pm\rangle = E_\pm |\pm\rangle, \]
\[ -\sum_{b \in B_A} (J_b \sigma_b^\pm + \epsilon_b^\pm \sigma_b^\pm + \epsilon_b^\pm \sigma_b^\pm) |\pm\rangle' = E_\pm' |\pm\rangle'. \]

Lemma 4.2 guarantees that there exists a function \( \psi_\pm : \{1, -1\}^{\Lambda_L} \to \mathbb{R} \), such that
\[ |\pm\rangle = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \{1,-1\}^{\Lambda_L}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma) |\sigma\rangle. \]

Here, we show that a real valued function \( \phi_\pm : \{1, -1\}^{\Lambda_L} \to \mathbb{R} \) exists uniquely and the state \(|\pm\rangle'\) can be represented in
\[ |\pm\rangle' = 2^{-|\Lambda_L|/2} \sum_{\sigma \in \{1,-1\}^{\Lambda_L}} \sigma_D(1 \pm \sigma_{\Lambda_L}) \psi_\pm(\sigma) \phi_\pm(\sigma) |\sigma\rangle. \]

The following relation
\[ \frac{\sigma_D^{(b)} \psi_\pm^{(b)}}{\psi_\pm^{(b)}} = s_D^D(1 \pm \sigma_{\Lambda_L}) \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] \]
and the eigenvalue equation \( (110) \) give
\[ (1 \pm \sigma_{\Lambda_L}) \left( \sum_{b \in B_A} [J_b - \epsilon_b^{\pm} \sigma_b] \sigma_D^{(b)} \phi_\pm^{(b)} \right) \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X + \epsilon_b^{\pm} \sigma_b \phi_\pm^{(b)} \right] + E_\pm' \phi_\pm(\sigma) = 0. \]
The eigenvalue equation for the reference state times $\phi(\sigma)$ is

$$
(1 \pm \sigma_{AL}) \left( \sum_{b \in B_A} \left[ (J_b - \epsilon_b^b \sigma_b) s_b^D \phi(\sigma) \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] + \epsilon_b^b \sigma_b \phi(\sigma) \right] + E_{\pm} \phi_{\pm}(\sigma) \right) = 0. \quad (112)
$$

The difference between above two equations gives

$$
(1 \pm \sigma_{AL}) \left( \sum_{b \in B_A} \left[ (J_b - \epsilon_b^b \sigma_b) s_b^D \left[ \phi_{\pm}(\sigma^{(b)}) - \phi_{\pm}(\sigma) \right] \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] + (E'_{\pm} - E_{\pm}) \phi_{\pm}(\sigma) \right) = 0. \quad (113)
$$

To obtain the Kirkwood-Thomas equation for the state $|\pm\rangle$, represent the function $\phi(\sigma)$ in terms of a real valued function $f(X)$ of an arbitrary subset $X \in S_A$,

$$
\phi_{\pm}(\sigma) = \frac{1}{2} (1 \pm \sigma_{AL}) \sum_{X \in S_A} f(X) \sigma_X. \quad (114)
$$

This gives

$$
\phi_{\pm}(\sigma^{(b)}) - \phi_{\pm}(\sigma) = -2 (1 \pm \sigma_{AL}) \sum_{X \in S_A} f(X) \sigma_X I(b \in \partial X)
$$

Then we have

$$
(1 \pm \sigma_{AL}) \left[ 2 \sum_{b \in B_A} (J_b - \epsilon_b^b \sigma_b) s_b^D \sum_{Y \in S_A} I(b \in \partial Y) f(X) \sigma_Y \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] - (E'_{\pm} - E_{\pm}) \sum_{X \in S_A} f(X) \sigma_X \right] = 0.
$$

Therefore

$$
(1 \pm \sigma_{AL}) \left[ \sum_{Y \in S_A} \Delta_Y f(Y) \sigma_Y + 2 \sum_{b \in B_A} J_b s_b^D \sum_{Y \in S_A} I(b \in \partial Y) f(Y) \sigma_Y \exp^{(1)} \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] \right.
$$

$$
- 2 \sum_{b \in B_A} \epsilon_b^b \sigma_b s_b^D \sum_{Y \in S_A} I(b \in \partial Y) f(Y) \sigma_Y \exp \left[ \sum_{X \in S_A} I(b \in \partial X) g(X) \sigma_X \right] - (E'_{\pm} - E_{\pm}) \sum_{X \in S_A} f(X) \sigma_X = 0,
$$

where an energy gap $\Delta_Y$ for $Y \in S_A$ is defined by

$$
\Delta_Y := 2 \sum_{b \in \partial Y} J_b s_b^D.
$$

The orthonormalization property (10) gives

$$
\sum_{Y \in S_A} 2 \sum_{b \in \partial Y} f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l)
$$

$$
- \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l)
$$

$$
\pm \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l)
$$

$$
= \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} f(Y) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) = (E'_{\pm} - E_{\pm} - \Delta_Z)[f(Z) \pm f(Z')].
$$

$$
\sum_{Y \in S_A} 2 \sum_{b \in \partial Y} \left[ f(Y) \pm f(Y') \right] \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l)
$$

$$
- \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} \left[ f(Y) \pm f(Y') \right] \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l)
$$

$$
= (E'_{\pm} - E_{\pm} - \Delta_Z)[f(Z) \pm f(Z')].
$$
Define \( e_\pm(C) := E_\pm' - E_\pm - \Delta C \) and \( e_\pm(X) := [f(X) \pm f(X^c)]/[f(C) \pm f(C^c)] \). for \( X \neq C \).

For \( Z = C \),

\[
e_\pm(C) = \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} J_b s_b^p e_\pm(Y) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{x_1, \ldots, x_k \in S_A} \delta_{\Delta, Y} \Delta D \sum_{l=1}^{k} g(X_l) I(b \in \partial X_l)
\]

\[
- \sum_{Y \in S_A} 2 \sum_{b \in \partial Y} e_\pm(Y) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{x_1, \ldots, x_k \in S_A} \delta_{\Delta, Y} \Delta D \sum_{l=1}^{k} g(X_l) I(b \in \partial X_l) =: F(e)(C),
\]

Then these two equations defines a fixed point equation \( F(e) = e \), whose solution \( e \) gives the state \( |\pm\rangle \) except its normalization. To prove the uniqueness of the solution, define a norm for the function \( e \) by

\[
\|e\| := |e(C)| + \sum_{X \in S_A} |\Delta X - \Delta C| |e(X)|.
\]

The following theorem implies that there is no level crossing between two arbitrary states \( |\pm\rangle \), \( |\pm\rangle \) against a sufficiently weak perturbation of the XY-exchange interactions, if we impose a condition to break the \( \mathbb{Z}_2 \) symmetry.

**Theorem 4.3** Consider the random bond Heisenberg XYZ model defined by the Hamiltonian \( H \). For two different sub-lattices \( C, D \in S_A \), let \( s^C, s^D \in \{1, -1\}^A \) be their corresponding sequences defined by \( I() \). If the sequence of \( X \)-exchange \( e \) is sufficiently small, there exists a sufficiently small constant \( \delta > 0 \) depending on the sequence of coupling constants \( (J, \epsilon) \), such that the energy gap \( E_\pm' - E_\pm \) in the perturbed model satisfies

\[
H_A^{\text{XYZ}}(s^C, J, 0) - H_A^{\text{XYZ}}(s^D, J, 0) - \delta < E_\pm' - E_\pm < H_A^{\text{XYZ}}(s^C, J, 0) - H_A^{\text{XYZ}}(s^D, J, 0) + \delta,
\]

for almost all \( J \in \mathbb{R}^B \). Theorem 4.3 implies that the sign of \( E_\pm' - E_\pm \) is identical to that of \( H_A(s^C, J, 0) - H_A(s^D, J, 0) \) for \( \delta < | H_A(s^C, J, 0) - H_A(s^D, J, 0) | \).

The following lemma and the contraction mapping theorem are helpful to prove Theorem 4.3.

**Lemma 4.4** Consider the random bond Heisenberg XYZ model under the condition in Lemma 4.3, namely

\[
\frac{3}{2} \geq e^{25/\Delta}(2\delta/\Delta + 1) + \epsilon_y/\Delta e^{25/\Delta}(2\delta/\Delta + 4d - 1).
\]

If \( K := 2(1 + |\Delta C|/|\Delta'|)(e^{25/\Delta} - 1) + 2\epsilon_y'/\Delta' e^{25/\Delta} + \delta/\Delta' < 1 \) for

\[
\Delta' := \inf_{Y \in S_A} |\Delta Y - \Delta C|, \quad \epsilon_y' := \Delta' \sup_{c \in \partial A, Y \in S_A} I(c \in \partial Y) \frac{2}{|\Delta Y - \Delta C|}
\]

then for almost all \( J \in \mathbb{R}^B \), the following norms are bounded by

\[
\|F(e_\pm) - F(e_\pm')\| \leq K \|e_\pm - e_\pm'\|, \quad \|F(e_\pm)\| \leq \delta, \quad \text{for } \|e_\pm\|, \|e_\pm'\| \leq \delta.
\]
Proof. For lighter notation, we remove indices ± from $e_{±}$. The difference between two evaluations of energy gap is

$$\begin{align*}
|F(e)(C) - F(e')(C)| &= \left| \sum_{Y \in S_A} \left( \sum_{b \in \partial Y} J_{b,b} D_e [e(Y) - e'(Y)] \right) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right| \\
&- \sum_{Y \in S_A} \sum_{b \in \partial Y} \epsilon_{b,b} D_e [e(Y) - e'(Y)] \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l),
\end{align*}$$

(123)

The triangle inequality gives

$$\begin{align*}
|F(e)(C) - F(e')(C)| &\leq \sum_{Y \in S_A} \left| \sum_{b \in \partial Y} \left( J_{b,b} D_e ||e(Y) - e'(Y)|| \right) \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right| \\
&+ \sum_{Y \in S_A} \sum_{b \in \partial Y} \epsilon_{b,b} D_e ||e(Y) - e'(Y)|| \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \\
&\leq \sum_{Y \in S_A} \left| \Delta Y ||e(Y) - e'(Y)|| \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right| \\
&+ \sum_{Y \in S_A} \sum_{b \in \partial Y} \epsilon_{b,b} D_e ||e(Y) - e'(Y)|| \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \\
&\leq \sum_{Y \in S_A} \left| \Delta Y ||e(Y) - e'(Y)|| \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right| \\
&\leq ||e - e'|| \left[ (1 + |\Delta C|/\Delta') (e^{2\delta/\Delta} - 1) + \epsilon'/\Delta' e^{2\delta/\Delta} \right],
\end{align*}$$

(124)

where $\Delta'$ and $\epsilon'$ are defined by [121]. The norm between $F(g)$ and $F(g')$ is

$$\begin{align*}
||F(e) - F(e')|| &= ||F(e)(C) - F(e')(C)|| + \sum_{Z \in S_A} |e(C) e(Z) - e'(C) e'(Z)| \\
&- \sum_{Y \in S_A} \sum_{b \in \partial Y} J_{b,b} D_e [e(Y) - e'(Y)] \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \\
&+ \sum_{Y \in S_A} \sum_{b \in \partial Y} \epsilon_{b,b} D_e [e(Y) - e'(Y)] \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \\
&\leq ||F(e)(C) - F(e')(C)|| + (I) + (II) + (III),
\end{align*}$$

(125)

where each term in the last line is defined by

(I) := \sum_{Z \in S_A} |e(C) e(Z) - e'(C) e'(Z)|

(II) := \sum_{Y,Z \in S_A} \left| \sum_{b \in \partial Y} J_{b,b} D_e [e(Y) - e'(Y)] \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right|

(III) := \sum_{Y,Z \in S_A} \left| \sum_{b \in \partial Y} \epsilon_{b,b} D_e [e(Y) - e'(Y)] \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{l=1}^{k} g(X_l) I(b \in \partial X_l) \right|.

Each can be bounded as follows:

(I) \leq |e(C) - e'(C)| \sum_{Z \in S_A} |e(Z)| + |e(C)| \sum_{Z \in S_A} |e(Z) - e'(Z)|

= |e(C) - e'(C)| \sum_{Z \in S_A} |\Delta Z - \Delta C||e(Z)|/\Delta' + |e(C)|/\Delta' \sum_{Z \in S_A} |\Delta Z - \Delta C||e(Z) - e'(Z)|

\leq \|e\|/\Delta'|\|e(C) - e'(C)| + \sum_{Z \in S_A} |\Delta Z - \Delta C||e(Z) - e'(Z)| \leq \delta/\Delta' \|e - e'||,

(126)
EA model with weak transverse fields. A spontaneous symmetry breaking of the

4.3 Remarks on results for the random bond Heisenberg XYZ model

Theorem 4.1 claims that an arbitrary energy gap small in the system size

This completes the proof of Theorem 4.3.

Proof of Theorem 4.3 Lemma 4.4 and the contraction mapping theorem guarantee the unique solution $e_\pm(C)$ of the fixed point equation under the condition on the perturbation of the sequence of coupling constants in Lemma 4.4. Then, there exists $\delta > 0$, such that

$|E'_\pm - E_\pm - \Delta_C| = |e_\pm(C)| \leq \delta$.

This completes the proof of Theorem 4.3.

4.3 Remarks on results for the random bond Heisenberg XYZ model

Theorem 4.1 claims that an arbitrary energy gap $|E_+ - E_-|$ between split energy eigenvalues is exponentially small in the system size $|\Lambda_L|$, also in the random bond Heisenberg XYZ model, as in the transverse field EA model with weak transverse fields. A spontaneous symmetry breaking of the $\mathbb{Z}_2$ symmetry is expected also in this model. In our expansion method, however, we cannot give any results in the infinite-volume limit.

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