A note of pointwise estimates on Shishkin meshes

Jin Zhang∗†

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Abstract

We propose the estimates of the discrete Green function for the streamline diffusion finite element method (SDFEM) on Shishkin meshes.

1 Problem

We consider the singularly perturbed boundary value problem

\begin{align}
Lu &:= -\varepsilon \Delta u + b \cdot \nabla u + u = f \quad \text{in} \quad \Omega = (0,1)^2, \\
u & = 0 \quad \text{on} \quad \partial \Omega,
\end{align}

where $\varepsilon \ll 1$ is a small positive parameter and $b = (b_1, b_2)^T > (0,0)^T$ is constant. It is also assumed that $f$ is sufficiently smooth.

2 The SDFEM on Shishkin meshes

2.1 Shishkin meshes

Let $N > 4$ be a positive even integer. We use a piecewise uniform mesh — a so-called Shishkin mesh — with $N$ mesh intervals in both $x$– and $y$–direction which condenses in the layer regions. For this purpose we define the two mesh transition parameters

$$\lambda_x := \min \left\{ \frac{1}{2}, \frac{2 \varepsilon}{\beta_1} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{2}, \frac{2 \varepsilon}{\beta_2} \ln N \right\}.$$

Assumption 1. We assume in our analysis that $\varepsilon \leq N^{-1}$, as is generally the case in practice. Furthermore we assume that $\lambda_x = 2 \varepsilon \beta_1^{-1} \ln N$ and $\lambda_y = 2 \varepsilon \beta_2^{-1} \ln N$ as otherwise $N^{-1}$ is exponentially small compared with $\varepsilon$.

∗Email: JinZhangalex@hotmail.com
†Address: School of Science, Xi’an Jiaotong University, Xi’an, 710049, China
Figure 1: Dissection of $\Omega$ and triangulation $\Omega^N$.

The domain $\Omega$ is dissected into four parts as $\Omega = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$ (see FIG. 1), where

$$
\begin{align*}
\Omega_s &:= [0, 1 - \lambda_x] \times [0, 1 - \lambda_y], & \Omega_x &:= [1 - \lambda_x, 1] \times [0, 1 - \lambda_y], \\
\Omega_y &:= [0, 1 - \lambda_x] \times [1 - \lambda_y, 1], & \Omega_{xy} &:= [1 - \lambda_x, 1] \times [1 - \lambda_y, 1].
\end{align*}
$$

We introduce the set of mesh points $\{(x_i, y_j) \in \Omega : i = 0, \cdots, N\}$ defined by

$$
\begin{align*}
x_i &= \begin{cases} 
2i(1 - \lambda_x)/N, & \text{for } i = 0, \cdots, N/2, \\
1 - 2(N - i)\lambda_x/N, & \text{for } i = N/2 + 1, \cdots, N
\end{cases}, \\
y_j &= \begin{cases} 
2j(1 - \lambda_y)/N, & \text{for } j = 0, \cdots, N/2, \\
1 - 2(N - j)\lambda_y/N, & \text{for } j = N/2 + 1, \cdots, N.
\end{cases}
\end{align*}
$$

By drawing lines through these mesh points parallel to the x-axis and y-axis the domain $\Omega$ is partitioned into rectangles. This triangulation is denoted by $\Omega^N$ (see FIG. 1). If $D$ is a mesh subdomain of $\Omega$, we write $D^N$ for the triangulation of $D$. The mesh sizes $h_{x,\tau} = x_i - x_{i-1}$ and $h_{y,\tau} = y_j - y_{j-1}$ satisfy

$$
\begin{align*}
h_{x,\tau} &= \begin{cases} 
H_x := \frac{1 - \lambda_x}{N/2}, & \text{for } i = 1, \cdots, N/2, \\
h_x := \frac{\lambda_x}{N/2}, & \text{for } i = N/2 + 1, \cdots, N
\end{cases}.
\end{align*}
$$
and

\[ h_{y,\tau} = \begin{cases} 
H_y := 1 - \lambda_y \frac{N}{2}, & \text{for } j = 1, \cdots, N/2, \\
\lambda_y \frac{N}{2}, & \text{for } j = N/2 + 1, \cdots, N.
\end{cases} \]

The mesh sizes \( h_{x,\tau} \) and \( h_{y,\tau} \) satisfy

\[ N^{-1} \leq H_x, H_y \leq 2N^{-1} \quad \text{and} \quad C_1 \varepsilon N^{-1} \ln N \leq h_x, h_y \leq C_2 \varepsilon N^{-1} \ln N, \]

where \( C_1 \) and \( C_2 \) are positive constants and independent of \( \varepsilon \) and of the mesh parameter \( N \). The above properties are essential when inverse inequalities are applied in our later analysis.

For the mesh elements we shall use two notations: \( \tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \) for a specific element, and \( \tau \) for a generic mesh rectangle.

### 2.2 The streamline diffusion finite element method

Let \( V := H^1_0(\Omega) \). On the above Shishkin mesh we define a finite element space

\[ V^N := \{ v^N \in C(\overline{\Omega}) : v^N|_{\partial \Omega} = 0 \text{ and } v^N|_{\tau} \text{ is bilinear}, \forall \tau \in \Omega^N \}. \]

In this case, the SDFEM reads as

\[ (2.1) \quad \begin{cases} 
\text{Find } U \in V^N \text{ such that for all } v^N \in V^N \\
\varepsilon(\nabla U, \nabla v^N) + (b \cdot \nabla U + U, v^N + \delta b \cdot \nabla v^N) = (f, v^N + \delta b \cdot \nabla v^N),
\end{cases} \]

where \( \delta = \delta(x) \) is a user-chosen parameter (see \cite{3}).

We set

\[ b := \sqrt{b_1^2 + b_2^2}, \quad \beta := \left( \frac{b_1}{b_2} \right)/b, \quad \eta := \left( -\frac{b_2}{b_1} \right)/b \quad \text{and} \quad \nu \zeta := \zeta^T \nabla v \]

for any vector \( \zeta \) of unit length. By an easy calculation one shows that

\[ (\nabla w, \nabla v) = (w_\beta, v_\beta) + (w_\eta, v_\eta). \]

We rewrite (2.1) as

\[ \varepsilon(U_\beta, v^N_\beta) + \varepsilon(U_\eta, v^N_\eta) + (bU_\beta + U, v^N_\eta + \delta bv^N_\eta) = (f, v^N + \delta bv^N_\eta) \]

and, following usual practice, we set

\[ \delta(x) := \begin{cases} 
N^{-1}, & \text{if } x \in \Omega_s, \\
0, & \text{otherwise}.
\end{cases} \]

For technical reasons in the later analysis, we increase the crosswind diffusion (see \cite{4}) by replacing \( \varepsilon(U_\eta, v^N_\eta) \) by \( \tilde{\varepsilon}(U_\eta, v^N_\eta) \) where

\[ \tilde{\varepsilon} := \max(\varepsilon, N^{-3/2}) \]
and
\[
\hat{\varepsilon}(x) := \begin{cases} 
\tilde{\varepsilon}, & x \in \Omega_s, \\
\varepsilon, & x \in \Omega \setminus \Omega_s.
\end{cases}
\]

We now state our streamline diffusion method with artificial crosswind:

\begin{equation}
(2.2) \quad \begin{cases} 
\text{Find } U \in V^N \text{ such that for all } v^N \in V^N \\
B(U, v^N) = (f, v^N + \delta b v^N),
\end{cases}
\end{equation}

with

\begin{equation}
(2.3) \quad B(U, v^N) := (\varepsilon + b^2 \delta)(U, v^N) + \hat{\varepsilon}(U, v^N) - b(1 - \delta)(U, v^N) + (U, v^N).
\end{equation}

### 3 The discrete Green function

Let \( x^* \) be a mesh node in \( \Omega \). The discrete Green’s function \( G \in V^N \) associated with \( x^* \) is defined by

\[
B(v^N, G) = v^N(x^*), \forall v^N \in V^N.
\]

The weighted function \( \omega \):

\[
\omega(x) := g \left( \frac{(x - x^*) \cdot \beta}{\sigma_\beta} \right) g \left( \frac{(x - x^*) \cdot \eta}{\sigma_\eta} \right) g \left( \frac{-(x - x^*) \cdot \eta}{\sigma_\eta} \right)
\]

where

\[
g(r) = \frac{2}{1 + e^r} \quad \text{for } r \in (-\infty, \infty).
\]

and \( \sigma_\beta = k N^{-1} \ln N \) and \( \sigma_\eta = k \tilde{\varepsilon}^{1/2} \ln N \).

\[
\| G \|_\omega^2 := (\varepsilon + b^2 \delta)\| \omega^{-1/2}G_\beta \|^2 + \hat{\varepsilon}\| \omega^{-1/2}G_\eta \|^2 + \frac{b}{2} \| (\omega^{-1})^{1/2}G \|^2 + \| \omega^{-1/2}G \|^2
\]

and

\begin{equation}
(3.1) \quad \| G \|_\omega^2 = B(\omega^{-1}G, G) - (\varepsilon + b^2 \delta)((\omega^{-1})_\beta G, G_\beta) - \hat{\varepsilon}((\omega^{-1})_\eta G, G_\eta) - b(1 - \delta)(\omega^{-1}G, G_\beta).
\end{equation}

Thus, we obtain

\[
B(\omega^{-1}G, G) = B(E, G) + B((\omega^{-1}G)^T, G)
\]

\[
= B(E, G) + (\omega^{-1}G)(x^*)
\]

where \( E := \omega^{-1}G - (\omega^{-1}G)^T \).

**Lemma 1.** If \( \sigma_\beta = k N^{-1} \ln N \) and \( \sigma_\eta = k \ln N \tilde{\varepsilon}^{1/2} \), then for \( k > 1 \) sufficiently large and independent of \( N \) and \( \varepsilon \), we have

\[
B(\omega^{-1}G, G) \geq \frac{1}{4} \| G \|_\omega^2.
\]
Proof. From (3.1), we estimate the following terms.

\[
(\varepsilon + \delta) \left| (\omega^{-1})_\beta G, G_\beta \right| \leq C(\varepsilon + \delta)^{1/2} \sigma_\beta^{-1/2} \cdot \left\| (\omega^{-1})_\beta G \right\| \cdot (\varepsilon + \delta)^{1/2} \| \omega^{-1/2} G_\beta \|
\]

and

\[
\hat{\varepsilon} \left| (\omega^{-1})_\eta G, G_\eta \right| \leq C\hat{\varepsilon}^{1/2} \sigma_\eta^{-1} \cdot \left\| \omega^{-1/2} G \right\| \cdot \hat{\varepsilon}^{1/2} \| \omega^{-1/2} G_\eta \|
\]

\[
\leq C\hat{\varepsilon}^{1/2} \sigma_\eta^{-1} \left\| G \right\|_0^2
\]

For \( b\delta(\omega^{-1} G, G_\beta) \), we make use of integration by parts.

From the definition of \( \sigma_\beta \) and \( \sigma_\eta \) and \( \varepsilon \leq N^{-1} \), we take \( k \) sufficiently large and we are done. \( \square \)

Lemma 2. If \( \sigma_\beta = kN^{-1} \ln N \), with \( k > 0 \) sufficiently large and independent of \( N \) and \( \varepsilon \). Then for each mesh point \( \mathbf{x}^* \in \Omega \setminus \Omega_{xy} \), we have

\[
\left| (\omega^{-1} G)(\mathbf{x}^*) \right| \leq \frac{1}{16} \left\| G \right\|_0^2 + CN \ln N.
\]

where \( C \) is independent of \( N \), \( \varepsilon \) and \( \mathbf{x}^* \).

Proof. First let \( \mathbf{x}^* \in \Omega_s \). Let \( \tau^* \) be the unique triangle that has \( \mathbf{x}^* \) as its north-east corner. Then

\[
\left| (\omega^{-1} G)(\mathbf{x}^*) \right| \leq CN \left\| G \right\|_{\tau^*}
\]

\[
\leq CN \max_{\tau^*} \left| (\omega^{-1})_\beta^{-1/2} \right| \cdot \left\| (\omega^{-1})_\beta^{-1/2} G \right\|_{\tau^*}
\]

Calculating \( (\omega^{-1})_\beta^{-1}(\mathbf{x}) \) explicitly, we see that

\[
(\omega^{-1})_\beta^{-1}(\mathbf{x}) \leq C\sigma_\beta = CkN^{-1} \ln N \quad \forall \mathbf{x} \in \tau^*
\]

Thus

\[
\left| (\omega^{-1} G)(\mathbf{x}^*) \right| \leq CN \ln N + \frac{1}{16} \left\| G \right\|_0^2
\]

by means of the arithmetic-geometric mean inequality.

Next, let \( \mathbf{x}^* \in \Omega_x \). (The case \( \mathbf{x}^* \in \Omega_y \) is similar.) Write \( \mathbf{x}^* = (x_i, y_j) \). Then

\[
\left| \omega^{-1} G(\mathbf{x}^*) \right| = \left| G(\mathbf{x}^*) \right|
\]

\[
= \left| \int_{x_i}^{1} G_x(t, y_j) dt \right|
\]

\[
\leq CH_\gamma^{-1} \int_{x_i}^{1} \int_{y_j}^{y_{j+1}} |G_x(t, y)| dy dt
\]

\[
\leq CN \left( \varepsilon \ln N \cdot N^{-1/2} \right) \| G \|_{\Omega_x}
\]

\[
\leq CN^{1/2} \ln^{1/2} N \| G \|
\]

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where \( G_x(t, y_j) = \frac{G_k - G_{k+1}}{h_x} \) for \((t, y_j) \in \tau_{kj}\).

Analysis: for the relation of boundary integral and domain integral, we analyze

\[
\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |G_x(t, y)|
\]

where \( f \parallel \) for \( \Delta_1 \)

\[
\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| f(y) \frac{y_{j+1} - y}{H_y} + f(y_{j+1}) \frac{y - y_j}{H_y} \right| dy
\]

where \( f(y) = \frac{G_{k+1} - G_k}{h_x} \) and \( f(y_{j+1}) = \frac{G_{k+1} - G_{k+1}}{h_x} \).

For \( \Delta_1 := \int_{y_j}^{y_{j+1}} \left| f(y) \frac{y_{j+1} - y}{H_y} + f(y_{j+1}) \frac{y - y_j}{H_y} \right| dy \), we have

\[
\Delta_1 = \left\{ \begin{array}{ll}
\frac{1}{2} \max \{|f(y)|, |f(y_{j+1})|\} H_y & \text{if } f(y) f(y_{j+1}) \geq 0 \\
\frac{1}{2} \max \{|f(y)|, |f(y_{j+1})|\} H_y & \text{if } f(y) f(y_{j+1}) < 0
\end{array} \right.
\]

For \( \forall v \in C^2(\tau) \), we have

\[
|v_x| \leq C(|v_\beta| + |v_\eta|)
\]

\[
|v_{xx}| \leq C(|v_\beta| + |v_\eta|)
\]

Similarly, we have

\[
|v_\beta| \leq C(|v_x| + |v_y|)
\]

\[
|\beta| \leq C(|v_x| + |v_y|)
\]

**Lemma 3.** Let \( \tau \in \Omega^N \). Then

\[
\|\omega^{1/2} D^\alpha E\|_{\Omega_\tau} \leq C k^{1/2} N^{1/2} ||G|| \omega
\]

\[
\|\omega^{1/2} D^\alpha E\|_{\Omega^N \setminus \Omega_\tau} \leq C k^{-1/2} \ln^{-1} N ||G|| \omega
\]

where \( H_{\tau} = \max\{h_{x,\tau}, h_{y,\tau}\} \) and \( |\alpha| = 1 \), \( D^\alpha G \) are \( G_\beta \) and \( G_\eta \).

**Proof.** Assume \( p \in [1, \infty] \) and \( g \in C^3(\tau) \). Then (see [2] Theorem 4)

\[
\|g - g^l_x\|_{L^p(\tau)} \leq C ( \frac{h^2}{h} \|g_{xxx}\|_{L^p(\tau)} + h_x \|h_{x,\tau} g_{xxy}\|_{L^p(\tau)} + h_y \|g_{xyy}\|_{L^p(\tau)})
\]

\[
+ Ch_{x,\tau} \|g_{xx}\|_{L^p(\tau)},
\]

\[
\|g - g^l_y\|_{L^p(\tau)} \leq C ( \frac{h^2}{h} \|g_{xyy}\|_{L^p(\tau)} + h_x \|h_{x,\tau} g_{xxy}\|_{L^p(\tau)} + h_y \|g_{xyy}\|_{L^p(\tau)})
\]

\[
+ Ch_{y,\tau} \|g_{yy}\|_{L^p(\tau)},
\]

or (see [1] Comment 2.15)

\[
\|g - g^l_x\|_{L^2(\tau)} \leq C h_{x,\tau}^2 \|g_{xyy}\|_{L^2(\tau)} + Ch_{x,\tau} \|g_{xx}\|_{L^2(\tau)}
\]

\[
\|g - g^l_y\|_{L^2(\tau)} \leq C h_{x,\tau}^2 \|g_{xyy}\|_{L^2(\tau)} + Ch_{x,\tau} \|g_{yy}\|_{L^2(\tau)}.
\]
In the following analysis, $D^n v$ denotes the directional derivative of $v$ along $\beta$ or $\eta$ for different orders. The following analysis makes use of the former estimates (The latter will make the analysis more shorter).

For $\tau \in \Omega^N$, we have

\[
\|\omega^{1/2} E_{\beta}\|_\tau \leq C \max_{\tau} \omega^{1/2}(\|E_x\|_\tau + \|E_y\|_\tau)
\]

\[
\leq C \max_{\tau} \omega^{1/2} \left\{ h_{\tau,\tau}^2 \| (\omega^{-1})_{xx} \|_\tau + h_{\tau,\tau} \| (\omega^{-1})_{xx} \|_\tau + h_{\tau,\tau} H_{\tau} \| (\omega^{-1})_{xy} \|_\tau \right\} + H_{\tau} h_{\tau,\tau} \| (\omega^{-1})_{xy} \|_\tau + h_{\tau,\tau} \| (\omega^{-1})_{yy} \|_\tau + h_{\tau,\tau} \| (\omega^{-1})_{ys} \|_\tau \}
\]

\[
\leq C h_{\tau,\tau}^2 \| (\omega^{-1})_{xx} \|_\tau + \| (\omega^{-1})_{xx} \|_\tau + \| (\omega^{-1})_{xy} \|_\tau + C h_{\tau,\tau} \| (\omega^{-1})_{xy} \|_\tau + \| (\omega^{-1})_{yy} \|_\tau + \| (\omega^{-1})_{ys} \|_\tau + \| (\omega^{-1})_{ys} \|_\tau + \| (\omega^{-1})_{ys} \|_\tau
\]

\[
+ H_{\tau} h_{\tau,\tau} \| (\omega^{-1})_{xy} \|_\tau + \| (\omega^{-1})_{xx} \|_\tau + \| (\omega^{-1})_{xy} \|_\tau + \| (\omega^{-1})_{yy} \|_\tau + \| (\omega^{-1})_{ys} \|_\tau + \| (\omega^{-1})_{ys} \|_\tau
\]

\[
\leq C h_{\tau,\tau}^2 \sum_{k=2}^3 \sum_{|\alpha| + |\gamma| = 3} \| D^{\alpha} (\omega^{-1}) D^{\gamma} g \|_\tau + C H_{\tau} \sum_{k=1}^2 \sum_{|\alpha| + |\gamma| = 2} \| D^{\alpha} (\omega^{-1}) D^{\gamma} g \|_\tau
\]

where we have used the following analysis for $\sum_{|\alpha| = 1, |\gamma| = 2} D^{\alpha} (\omega^{-1}) D^{\gamma} g$ :

\[
\| (\omega^{-1})_{xx} \|_\tau \leq C \| \sum_{|\alpha| = 1} |D^{\alpha} (\omega^{-1})| \cdot G_{xx} \|_\tau
\]

\[
\leq C \max_{\tau} \sum_{|\alpha| = 1} |D^{\alpha} (\omega^{-1})| \cdot \| G_{xx} \|_\tau
\]

\[
\leq C h_{\tau,\tau}^{-1} \max_{\tau} \sum_{|\alpha| = 1} |D^{\alpha} (\omega^{-1})| \cdot \| G_{x} \|_\tau
\]

\[
\leq C h_{\tau,\tau}^{-1} \sum_{|\alpha| = 1} |D^{\alpha} (\omega^{-1})| \cdot \| G_{x} \|_\tau
\]

The same analysis can be applied to $\|\omega^{1/2} E_{\eta}\|_\tau$.

For $\tau \in \Omega_\star$, we have

\[
\|\omega^{1/2} E_{\eta}\|_\tau \leq C k^{-5/2} N^{-2} \left[ (\sigma_{\beta}^{-5/2} + \sigma_{\beta}^{-3/2} \sigma_{\eta}^{-1} + \sigma_{\beta}^{-1/2} \sigma_{\eta}^{-2} \max_{\tau} (\omega^{-1})_{\beta}^{1/2} + \sigma_{\eta}^{-3} \max_{\tau} (\omega^{-1})_{\beta}^{-1/2}) \right] \|G\|_\tau
\]

\[
+ C k^{-3/2} H_{\tau} \left[ (\sigma_{\beta}^{-3/2} + \sigma_{\beta}^{-1/2} \sigma_{\eta}^{-1}) \max_{\tau} (\omega^{-1})_{\beta}^{1/2} + \sigma_{\eta}^{-2} \max_{\tau} (\omega^{-1})_{\beta}^{-1/2} \right] \cdot N \|G\|_\tau
\]

\[
+ C k^{-3/2} H_{\tau} \left[ (\sigma_{\beta}^{-3/2} + \sigma_{\beta}^{-1/2} \sigma_{\eta}^{-1}) \max_{\tau} (\omega^{-1})_{\beta}^{1/2} + \sigma_{\eta}^{-2} \max_{\tau} (\omega^{-1})_{\beta}^{-1/2} \right] \|G\|_\tau
\]

\[
+ C \max_{\tau} \omega^{1/2} H_{\tau} \sum_{|\alpha| = 1} \|D^{\alpha} (\omega^{-1})\|_{L^\infty(\tau)} \cdot \sum_{|\gamma| = 1} \|D^{\gamma} g\|_\tau
\]

\[
\leq C k^{-1/2} N^{1/2} \|\|G\|_\omega
\]

where we have used the estimates of $\omega^{-1}$, standard inverse estimates and Hölder.
inequalities. Similarly, we have
\[ \| \omega^{1/2} E_\eta \|_{\Omega_4} \leq C k^{-1/2} N^{1/2} \|G\|_\omega. \]

For \( \tau \in \Omega_4 \setminus \Omega_4^N \), we have
\[
\| \omega^{1/2} E_\beta \|_\tau \leq C k^{-5/2} H_\tau^2 \left[ (\sigma_\beta^{1/2} + \sigma_\beta^{3/2} \sigma_q^{-1} + \sigma_\beta^{-1/2} \sigma_q^{-2} ) \max(\omega^{-1})_\beta^{1/2} + \sigma_q^{-3/2} \max(\omega^{-1})_\omega^{1/2} \right] \| G \|_\tau
\]
\[
+ C k^{-2} \| \varepsilon^{-1/2} H_\tau^2 [ (\sigma_\beta^{1/2} + \sigma_\beta^{-1/2} \sigma_q^{-1} ) \max(\omega^{-1})_\beta^{1/2} + \sigma_q^{-2} \max(\omega^{-1})_\omega^{1/2} ] \| G \|_\tau
\]
\[
+ C k^{-3/2} H_\tau [ (\sigma_\beta^{1/2} + \sigma_\beta^{-1/2} \sigma_q^{-1} ) \max(\omega^{-1})_\beta^{1/2} + \sigma_q^{-2} \max(\omega^{-1})_\omega^{1/2} ] \| G \|_\tau
\]
\[
\leq C k^{-1} \| \varepsilon^{-1/2} N \| \| G \|_\omega
\]

Similarly, we have
\[ \| \omega^{1/2} E_\eta \|_{\Omega_4 \setminus \Omega_4} \leq C k^{-1} \| \varepsilon^{-1/2} N \| \| G \|_\omega. \]

\[ \square \]

**Lemma 4.** Let \( \tau \in \Omega_4^N \). Let \( E = (\omega^{-1} G - (\omega^{-1} G)^1 \) where \( (\omega^{-1} G)^1 \) denote the bilinear function that interpolates to \(\omega^{-1} G \) at the vertices of \( \tau \). Then
\[
\| \omega^{1/2} E \|_{\Omega_4} \leq C k^{-1} N^{-1/2} \|G\|_\omega
\]
\[
\| \omega^{1/2} E \|_{\Omega_4 \setminus \Omega_4} \leq C k^{-1} \| \varepsilon^{-1/2} N \| \| G \|_\omega
\]

where \( H_\tau = \max \{ h_{x,\tau}, h_{y,\tau} \} \).

**Proof.** We make use of the following standard interpolation error bounds
\[
\| u - u^I \|_{L^p(\tau)} \leq h_{x,\tau}^2 \| u_{xx} \|_{L^p(\tau)} + h_{y,\tau}^2 \| u_{yy} \|_{L^p(\tau)}
\]
where \( p \in [1, \infty] \) and \( u \in C(\tau) \cap W^{2,p}(\tau) \).

Then, we have
\[
\| E \|_{\tau} \leq h_{x,\tau}^2 \| (\omega^{-1} G)_{xx} \|_{\tau} + h_{y,\tau}^2 \| (\omega^{-1} G)_{yy} \|_{\tau}
\]
\[
\leq CH_\tau^2 \sum_{|\alpha|=2} \| D^\alpha (\omega^{-1}) \cdot G \|_{\tau} + CH_\tau^2 \int_{\tau} \frac{1}{(\omega^{-1})_\beta} \| (\omega^{-1})_\beta \|_{\tau} + \int_{\tau} \frac{1}{(\omega^{-1})_\eta} \| (\omega^{-1})_\eta \|_{\tau}
\]
\[
\leq CH_\tau^2 \left\{ \| (\omega^{-1})_\beta G_\beta \|_{\tau} + \| (\omega^{-1})_\beta G_\eta \|_{\tau} + \| (\omega^{-1})_\eta G_\beta \|_{\tau} + \| (\omega^{-1})_\eta G_\eta \|_{\tau} \right\}
\]
\[
+ CH_\tau^2 \sum_{|\alpha|=2} | D^\alpha (\omega^{-1}) \cdot G |_{\tau}.
\]

From the above inequality, we have
\[ \| \omega^{1/2} E \|_{\Omega_4} \leq C k^{-1} N^{-1/2} \|G\|_\omega. \]
and
\[ \|\omega^{1/2}E\|_{\Omega \setminus \Omega_x} \leq C k^{-1} \varepsilon^{-1/2} N^{-1} ||G||_\omega. \]

Following the techniques of (see [9, Lemma 4.4]), we have
\[ (\omega^{1/2}E)(x) = \int_x^{\Gamma(x)} (\omega^{1/2}E)_\eta ds \]
where \( x \in \Omega \setminus \Omega_x \), \( \Gamma(x) \in \Gamma \) satisfies \( (x - \Gamma(x)) \cdot \beta = 0 \) and the following condition:
For \( \forall y \in \Gamma, (x - y) \cdot \beta = 0 \),
\[ |x - \Gamma(x)| = \min_y |x - y|. \]

From the above representation of \( \omega^{1/2}E \), we have
\[
\begin{aligned}
\|\omega^{1/2}E\|_{L^2}^2 &= \int_{\lambda_0}^{1-\lambda_y} \left[ \int_{\Gamma(x)} (\omega^{1/2}E)_\eta ds \right]^2 d\Omega \\
&+ \int_{1-\lambda_y}^{1} \left[ \int_{\Gamma(x)} (\omega^{1/2}E)_\eta ds \right]^2 d\Omega + \int_{0}^{\lambda_y} \left[ \int_{\Gamma(x)} (\omega^{1/2}E)_\eta ds \right]^2 d\Omega \\
&\leq C \lambda_x^2 \left\{ ||(\omega^{1/2})_\eta E||_{L^2}^2 + ||\omega^{1/2}E||_{L^2}^2 \right\} \\
&\leq C \varepsilon^2 \ln^2 N \left\{ \sigma_x^{-2} ||\omega^{1/2}E||_{L^2}^2 + ||\omega^{1/2}E||_{L^2}^2 \right\} \\
&\leq C k^{-2} \varepsilon^2 \ln^2 N \left\{ N^{3/2} \ln^{-2} N \cdot \varepsilon^{-1} N^{-2} + \varepsilon^{-1} \ln^{-2} N \right\} ||G||_\omega^2 \\
&\leq C k^{-2} \varepsilon ||G||_\omega^2
\end{aligned}
\]
where \( \lambda_0 = \frac{b_1}{b_2} \lambda_x \) and
- \( x_{lu} \in \{(1 - \lambda_x, y) : \lambda_0 \leq y \leq 1 - \lambda_y\}; \)
- \( x_{ld} \in \{(1 - \lambda_x, y) : 0 \leq y \leq \lambda_0\}; \)
- \( x_u \in \{(x, 1 - \lambda_y) : 1 - \lambda_x \leq x \leq 1\}. \)

\[ \square \]

**Lemma 5.** If \( \sigma_\beta = kN^{-1} \ln N \) and \( \sigma_\eta = k\varepsilon^{1/2} \ln N \), where \( k > 1 \) sufficiently large and independent of \( N \) and \( \varepsilon \). Then
\[ B((\omega^{-1}G)^t - \omega^{-1}G, G) \leq \frac{1}{16} ||G||_\omega^2. \]

**Proof.** Cauchy-Schwarzs inequality gives
\[
\begin{aligned}
|B(E, G)| &\leq (\varepsilon + b^2 \delta)^{1/2} \|\omega^{1/2}E_\beta\| \cdot (\varepsilon + b^2 \delta)^{1/2} \|\omega^{-1/2}G_\beta\| + \varepsilon^{1/2} \|\omega^{1/2}E_\eta\| \cdot \varepsilon^{1/2} \|\omega^{-1/2}G_\eta\| \\
&+ C \|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G_\beta\| + \|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G\|.
\end{aligned}
\]

From Lemma 3 and Lemma 4 we are done. \[ \square \]
Theorem 3.1. Assume that $\sigma_\beta = kN^{-1}\ln N$ and $\sigma_\eta = k\varepsilon^{1/2}\ln N$, where $k > 0$ is sufficiently large and independent of $\varepsilon$ and $N$. Let $x^* \in \Omega\setminus\Omega_{xy}$. Then for each nonnegative integer $v$, there exists a positive constant $C = C(v)$ and $K = K(v)$ such that

$$\|G\|_{W^{1,\infty}(\Omega_s\setminus\Omega'_0)} \leq CN^{-v},$$

$$\varepsilon|G|_{W^{1,\infty}((\Omega_s\cup\Omega_y)\setminus\Omega'_0)} + \|G\|_{L^\infty((\Omega_s\cup\Omega_y)\setminus\Omega'_0)} \leq C\varepsilon^{-1/2}N^{-v},$$

and

$$\varepsilon|G|_{W^{1,\infty}(\Omega_{xy}\setminus\Omega'_0)} + \|G\|_{L^\infty(\Omega_{xy}\setminus\Omega'_0)} \leq C\varepsilon^{-1/2}N^{-v}.$$ 

Proof. On $\Omega_s$, we apply an inverse estimate. On $\Omega\setminus\Omega_s$ the application of an inverse estimate does not yield a satisfactory result, so we use a different technique.

Let $x \in \Omega_s \setminus \Omega'_0$ be arbitrary. Starting from $x$ we choose a polygonal curve $\Gamma \subset (\Omega \setminus \Omega_{xy}) \setminus \Omega'_0$ that joints $x$ with some point on outflow boundaries. If $(x - x^*) \cdot \eta < 0$, we can choose $\Gamma$ as a line parallel to $\beta$. If $(x - x^*) \cdot \eta > 0$, the situation is a little complicated. We can choose $\Gamma$ as follows:

In $\Omega_s \setminus \Omega'_0$, we choose the direction of $\Gamma$ along $\eta$ or the negative direction of $x$-axis so that $\Gamma \cap \Omega_{xy} = \Phi$. In $(\Omega_s \cup \Omega_y) \setminus \Omega'_0$, we choose the direction of $\Gamma$ along $\eta$ or the positive direction of $y$–axis.

Let $T^N$ be the set of mesh rectangle $\tau$ in $(\Omega \setminus \Omega_{xy}) \setminus \Omega'_0$ that $\Gamma$ intersects. Note that the length of the segment of $\Gamma$ that lies in each $\tau$ is at most $C\varepsilon N^{-1}\ln N$ if $\tau \in \Omega_s$ or $\tau \in \Omega_y$.

Then, by the fundamental theorem of calculus and inverse estimates in different domain, we can obtain the results.

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