The cone conjecture for abelian varieties

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The purpose of this paper is to write down a complete proof of the Morrison–Kawamata cone conjecture for abelian varieties. The conjecture predicts, roughly speaking, that for a large class of varieties (including all smooth varieties with numerically trivial canonical bundle) the automorphism group acts on the nef cone with rational polyhedral fundamental domain. (See Section 1 for a precise statement.) The conjecture has been proved in dimension 2 by Sterk–Looijenga, Namikawa, Kawamata, and Totaro [Ste85, Nam85, Kaw97, Tot10], but in higher dimensions little is known in general.

Abelian varieties provide one setting in which the conjecture is tractable, because in this case the nef cone and the automorphism group can both be viewed as living inside a larger object, namely the real endomorphism algebra. In this paper we combine this fact with known results for arithmetic group actions on convex cones to produce a proof of the conjecture for abelian varieties. It should be stressed that our proof is a more or less straightforward deduction from well-known results, and it is written out in full here to serve as a reference.

Here is the main result.

**Theorem 0.1** Let $X$ be an abelian variety and $\overline{A(X)}$ its effective nef cone. Then there is a rational polyhedral fundamental domain for the action of the automorphism group $\text{Aut}(X)$ on $\overline{A(X)}$.

The conclusion of the theorem was already known in some cases. It was proved for abelian surfaces by Kawamata [Kaw97], adapting the proof of Sterk–Looijenga for $K3$ surfaces. In the same paper, Kawamata also proved the conjecture for all self-products of an elliptic curve without complex multiplication. Finally, Bauer [Bau98] showed that for an abelian variety, the nef cone is rational polyhedral if and only if the variety is isogenous to a product of mutually non-isogenous abelian varieties of Picard number 1, which in particular implies the theorem for abelian varieties of this special type.

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1 The cone conjecture

We work throughout the paper over an arbitrary algebraically closed field.

Morrison [Mor93] gave the original statement of the cone conjecture for Calabi–Yau threefolds, motivated by considerations from mirror symmetry. The statement was generalised by Kawamata [Kaw97] to families of varieties with numerically trivial canonical bundle, and from there to so-called klt Calabi–Yau pairs [Tot10]. As mentioned in the introduction, the conjecture has been verified in dimension 2, but in general it remains wide open. See [Tot10, Section 1] for history and a summary of the current status.
Here we state the conjecture in a rather simple form applicable to abelian varieties. The symbol \( \equiv \) denotes numerical equivalence of divisors, and for a projective variety \( N^1(X) \) denotes the real vector space \((\text{Div}(X))/\equiv) \otimes \mathbb{R}\) where \text{Div}(X) is the free abelian group spanned by Cartier divisors. The cones \( \overline{A(X)} \) and \( \overline{M(X)} \) are the closed cones in \( N^1(X) \) spanned by the classes of nef or movable divisors. The cone \text{Eff}(X) is the cone spanned by the classes of all effective divisors, and \( A(X)^e \) and \( M(X)^e \) denote the intersections \( \overline{A}(X) \cap \text{Eff}(X) \) and \( \overline{M}(X) \cap \text{Eff}(X) \). Finally, a pseudo-automorphism of \( X \) is a birational map \( X \to X \) which is an isomorphism outside a subset of codimension 2. Note that a pseudo-automorphism maps a movable or effective divisor to another movable or effective divisor, therefore preserves the cone \( \overline{M(X)^e} \).

**Conjecture 1.1 (Morrison–Kawamata)** Let \( X \) be a smooth projective variety with \( K_X \equiv 0 \). Then:

1. There exists a rational polyhedral cone \( \Pi \) which is a fundamental domain for the action of \( \text{Aut}(X) \) on \( \overline{A(X)^e} \) in the sense that
   - (a) \( \overline{A(X)^e} = \text{Aut}(X) \cdot \Pi \),
   - (b) For all \( g \in \text{Aut}(X) \) we have \( (\text{Int} \Pi) \cap g^*(\text{Int} \Pi) = \emptyset \) for \( g^* \neq 1 \) in \( GL(N^1(X)) \).

2. There exists a rational polyhedral cone \( \Pi' \) which is a fundamental domain (in the sense above) for the action of the pseudo-automorphism group \( \text{PsAut}(X) \) on \( \overline{M(X)^e} \).

Part (1) of the conjecture would imply in particular that any variety with numerically trivial canonical bundle has finitely many contractions up to automorphisms. This is because any contraction of a projective variety is determined by a semi-ample line bundle, and two semi-ample line bundles give the same contraction if they belong to the interior of the same face in the nef effective cone.

For abelian varieties part (2) of the conjecture is implied by part (1), because the effective nef cone \( \overline{A(X)^e} \) and effective movable cone \( \overline{M(X)^e} \) are the same for abelian varieties. Indeed, on an abelian variety any effective divisor is semi-ample by the ‘theorem of the square’ [Mum70, §6, Corollary 4], so \( \overline{A(X)^e} = \overline{M(X)^e} = \text{Eff}(X) \). As a consequence Theorem 0.1 implies the full cone conjecture for abelian varieties.

The equality of cones in the last paragraph can be strengthened to give the following result; a reference is [Bau98, Proposition 1.1].

**Proposition 1.2** Let \( D \) be a Cartier divisor on an abelian variety \( X \) and let \( [D] \) denote the class of \( D \) in \( N^1(X) \). Then \( [D] \in \text{Eff}(X) \) if and only if \( [D] \in \overline{A(X)} \).

This implies that \( \overline{A(X)^e} \) is equal to the rational hull \( A(X)_+ \) of the ample cone \( A(X) \), defined as the convex hull of the rational points in the closure \( \overline{A(X)} \). This will be important later, because from the point of view of reduction theory for arithmetic groups \( A(X)_+ \) is a more natural object to consider than \( \overline{A(X)} \).

Finally it should be emphasised that for abelian varieties, it makes no difference to the conjecture whether \( \text{Aut} \) denotes the group of automorphisms in the category \( \text{Var}/k \) of varieties over the ground field \( k \) or in the category \( \text{GrVar}/k \) of group varieties over \( k \). This is because any \( (\text{Var}/k) \)-automorphism of an abelian variety can be composed with a translation automorphism to give a \( (\text{GrVar}/k) \)-automorphism, and translations act trivially on \( N^1(X) \).

For the rest of the paper we will use \( \text{Aut} \) to denote the group of \( (\text{GrVar}/k) \)-automorphisms.

**Examples.** The following examples illustrate how the nef effective cone and automorphism group of an abelian variety can vary, compatibly with the cone conjecture. We will
consider two abelian surfaces of Picard number 2. By standard results on quadratic forms and the Hodge Index Theorem, for any surface $X$ we can choose a rational basis $\{v_1, v_2\}$ for $N^1(X)$ in which the intersection form has matrix $\text{diag}(a, -b)$, with $a$ and $b$ positive. When $X$ is an abelian surface, Proposition 1.2 says that a Cartier divisor $D$ on $X$ is ample if and only if $D^2 > 0$ and $D \cdot H > 0$ for some fixed ample divisor $H$. So if we choose a basis $\{v_1, v_2\}$ for $N^1(X)$ as above, the ample cone of $X$ is described as

$$A(X) = \{x_1v_1 + x_2v_2 \in N^1(X) \mid ax_1^2 - bx_2^2 > 0, x_1 > 0\}.$$ 

The two extremal rays of $A(X)$ are spanned by the vectors $v_1 \pm (\sqrt{a/b})v_2$, so $A(X)^e = A(X)_+$ is a rational polyhedral cone if and only if $a/b$ is a square in $Q$.

Before giving our examples we mention the following useful fact. To verify the cone conjecture for a variety $X$, it suffices to find a rational polyhedral cone $\Pi \subset A(X)^e$ whose translates by $\text{Aut}(X)$ cover the whole nef effective cone. This fact will follow from Theorem 2.5 in the next section, but we mention it now to clarify our examples.

For the first example we take $X$ to be a product $E_1 \times E_2$ of non-isogenous elliptic curves. Then $\text{Aut}(X) = \text{Aut}(E_1) \times \text{Aut}(E_2)$, and $\text{Aut}(E_i)$ is a cyclic group of order 2, 4, or 6. Therefore in this case $\text{Aut}(X)$ is a finite group acting on $A(X)^e$. On the other hand, by taking suitable rational linear combinations of the divisors $E_1 \times \{0\}$ and $\{0\} \times E_2$, the intersection form on $N^1(X)$ can be transformed to have matrix $\text{diag}(1, -1)$. By our description of the extremal rays a few paragraphs ago, $A(X)^e$ is therefore a rational polyhedral cone, so by the fact in the previous paragraph the cone conjecture is true for $X$, taking $\Pi = A(X)^e$.

For the second example we take $X$ to be an abelian surface with real multiplication, by which we mean that the endomorphism algebra $\text{End}^0(X) := \text{End}(X) \otimes Q$ is isomorphic to the number field $Q(\sqrt{d})$ for some square-free integer $d > 0$. (The simplest examples of such surfaces are Jacobians of certain genus 2 curves; explicit models can be found in [Wil00].) By Dirichlet’s unit theorem, the automorphism group $\text{Aut}(X)$ has rank equal to $r_1 + r_2 - 1$, where $r_1$ is the number of embeddings $Q(\sqrt{d}) \hookrightarrow R$ and $r_2$ is the number of conjugate pairs of embeddings $Q(\sqrt{d}) \hookrightarrow C$ whose image is not contained in $R$. One checks easily that $r_1 = 2$ and $r_2 = 0$, so $\text{Aut}(X)$ has rank 1. What about the cone $A(X)^e$? In this case the matrix of the intersection form diagonalises to $\text{diag}(1, -d)$, so the boundary rays are irrational. To find the required rational polyhedral fundamental domain, we proceed as follows. Choose an arbitrary rational ray $R \subset A(X)^e$ and an element $g$ of infinite order in $\text{Aut}(X)$. A little thought shows $\{g^i(R) \mid i = 1, 2, 3 \ldots\}$ is a sequence of rational rays, which either converges to one extremal ray, or decomposes into two subsequences, one converging to each extremal ray. Composing $g$ with a torsion element of $\text{Aut}(X)$, we can assume that the first case occurs. Then the cone $\Pi$ spanned by the rays $R$ and $g(R)$ is a rational polyhedral cone whose translates by $\text{Aut}(X)$ cover the whole nef effective cone (Figure 1). Again by the fact above, this proves the cone conjecture for $X$.

## 2 Homogeneous self-dual cones and reduction theory

In this section we give the results we need about reduction theory for arithmetic group actions on homogeneous self-dual convex cones. It should be mentioned that this theory has a rich history that we touch on here only briefly; see [AMRT1] for historical discussion and references.

From now on $V$ will always denote a finite-dimensional real vector space. By a cone in $V$ we always mean a convex cone $C \subset V$ which is non-degenerate (meaning that its closure $\bar{C}$
contains no nonzero subspaces of $V$). The dual cone $C^* \subset V^*$ is defined to be the interior of the cone $C^\vee$ consisting of linear forms on $V$ which are nonnegative on $C$.

Now suppose $C$ is an open cone in a vector space $V$. We define the automorphism group $G(C)$ of $C$ to be the subgroup of $GL(V)$ consisting of linear transformations which preserve $C$. The cone $C$ is said to be homogeneous if $G(C)$ acts transitively on $C$. Suppose further that $V$ carries an inner product, giving an identification of $V$ with $V^*$. We say $C$ is self-dual if this identification takes $C$ to its dual $C^*$. (This condition depends on the choice of inner product, but the dependence will not matter in what follows.) The basic theorem about the automorphism group of a homogeneous self-dual cone is the following, due to Vinberg [Vin65].

**Theorem 2.1 (Vinberg)** Let $C \subset V$ be a homogeneous self-dual convex cone. Then the automorphism group $G(C)$ is the group of real points of a reductive algebraic group $G(C)$.

It is an amazing fact that homogeneous self-dual convex cones can be completely classified into a small number of cases. More precisely, define the direct sum of cones $C_1$ and $C_2$ in vector spaces $V_1$ and $V_2$ to be the cone $C_1 \oplus C_2 := \{ v_1 + v_2 \in V_1 \oplus V_2 | v_i \in C_i \}$ and call a cone indecomposable if it is not the direct sum of 2 nontrivial cones. The classification theorem, due to Koecher and Vinberg, is the following [Vin63 I, §1, Proposition 2], [Vin65 p. 71].

**Theorem 2.2 (Koecher—Vinberg)** Any convex cone $C$ can be written as a direct sum $\bigoplus_i C_i$ of indecomposable cones. The product $\prod G(C_i)$ is a finite-index subgroup of $G(C)$. The cones $C_i$ are homogeneous and self-dual if and only if $C$ is. Any indecomposable homogeneous self-dual cone is isomorphic to one of the following:

1. the cone $P_r(\mathbb{R})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbb{R})$ of $r \times r$ real symmetric matrices;
2. the cone $P_r(\mathbb{C})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbb{C})$ of $r \times r$ complex Hermitian matrices;

![Figure 1: Nef cone of an abelian surface with real multiplication](image-url)
3. the cone $P_r(H)$ of positive-definite matrices in the space $\mathcal{H}_r(H)$ of $r \times r$ quaternionic Hermitian matrices;

4. the spherical cone $\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 > \sqrt{x_1^2 + \cdots x_n^2} \right\}$;

5. the 27-dimensional cone of positive-definite $3 \times 3$ octonionic Hermitian matrices.

The inner product for which the cone is self-dual is $\langle x, y \rangle = \text{Tr}(xy^*)$ in all cases except 4, and the usual inner product on $\mathbb{R}^{n+1}$ in case 4.

The proof uses the surprising correspondence between self-dual homogeneous convex cones and formally real Jordan algebras; see [AMRT, Chapter II, §2] for details.

Vinberg [Vin65] computed the automorphism groups of all the cones in the list of Theorem 2.2. Here we state the part of his result which will be relevant to abelian varieties.

**Theorem 2.3 (Vinberg)** Let $C$ be one of the cones $P_r(k)$ in the previous theorem: that is, the cone of positive-definite matrices in the vector space $\mathcal{H}_r(k)$ of $r \times r$ symmetric or Hermitian matrices over $k$, where $k = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then the identity component $G(\mathbb{C})^0$ of the automorphism group of $C$ consists of all $\mathbb{R}$-linear transformations of $\mathcal{H}_r(k)$ of the form $D \mapsto M^*DM$ for some $M \in \text{GL}(r,k)$.

Now we come to reduction theory. For a $\mathbb{Q}$-algebraic group $G$, a subgroup $\Gamma \subset G(\mathbb{Q})$ is called an arithmetic subgroup of $G$ if $\Gamma$ and the group $G(\mathbb{Z})$ of integer points of $G$ are commensurable (meaning their intersection inside $G(\mathbb{Q})$ has finite index in each). The basic problem of reduction theory for convex cones is the following: given a convex cone $C$ and an arithmetic subgroup $\Gamma$ of the automorphism group $G(\mathbb{C})$, can we find a rational polyhedral fundamental domain for the action of $\Gamma$ on $C$? The first results of this kind go back to Hermite and Minkowski, who found rational polyhedral fundamental domains for the adjoint action of $\text{SL}(r,\mathbb{Z})$ on the cone $P_r(\mathbb{R})$ of positive-definite real symmetric matrices. That is, there is a finite set of integral linear inequalities such that any quadratic form can be reduced by an integral change of basis to a form whose coefficients satisfy those inequalities. This explains the name ‘reduction theory’.

The main result of the theory we will use is the following, due to Ash [AMRT, Chapter II].

**Theorem 2.4 (Ash)** Let $C$ be a self-dual homogeneous convex cone in a real vector space $V$ with $\mathbb{Q}$-structure. Let $G(\mathbb{C})$ be the automorphism group of $C$ and $G(\mathbb{C})$ the associated reductive algebraic group (as in Theorem 2.1). Assume $G(\mathbb{C})^0$ is defined over $\mathbb{Q}$. Then for any arithmetic subgroup $\Gamma$ of $G$, there exists a rational polyhedral cone $\Pi \subset C_+$ such that $(\Gamma \cdot \Pi) \cap C = C$.

Here as before $C_+$ is the rational hull of $C$, meaning the convex hull of the rational points in $C$.

Applied in the case of abelian varieties, this theorem will provide us with a rational polyhedral cone whose translates by automorphisms cover the ample cone. The cone conjecture asks for more, however: we need to find a precise fundamental domain for the action of automorphisms on the effective nef cone. This is supplied by the following result of Looijenga [Loo09, Proposition 4.1, Application 4.15].
Theorem 2.5 (Looijenga) Let $C$ be a convex cone in a vector space $V$ with $\mathbb{Q}$-structure, and $\Gamma$ a subgroup of $GL(V)$ which preserves $C$ and some rational lattice in $V$. If there exists a rational polyhedral cone $\Pi \subset C_+$ such that $(\Gamma \cdot \Pi) \cap C = C$, then in fact $\Gamma \cdot \Pi = C_+$, and in fact there exists a rational polyhedral fundamental domain for the action of $\Gamma$ on $C_+$.

We observed in Section 1 that if $X$ is an abelian variety then $A(X)_+ = A(X)^\oplus$, so together Theorems 2.4 and 2.5 will imply the cone conjecture for abelian varieties if we can show that for any abelian variety, the ample cone is a self-dual homogeneous cone and the automorphism group of the variety acts as an arithmetic subgroup of the automorphism group of the cone. This is what we do in the next two sections.

3 The endomorphism algebra of an abelian variety

In this section we describe the endomorphism algebra of an abelian variety as a product of certain matrix algebras. In particular this gives a description of the automorphism group as an arithmetic group, connecting the cone conjecture for abelian varieties to the reduction theory of the previous section. For a full exposition of the structure theory of the endomorphism algebra, read Chapter 4 of Mumford’s beautiful book [Mum70].

The first result we need is a form of the Poincaré complete reducibility theorem [Mum70, §19, Theorem 1, Corollary 1].

**Theorem 3.1** Let $X$ be an abelian variety. Then $X$ is isogenous to a product $X_1^{n_1} \times \cdots \times X_k^{n_k}$ where the $X_i$ are simple abelian varieties, not isogeneous for $i \neq j$. The isogeny type of the $X_i$ and the natural numbers $n_i$ are uniquely determined by $X$.

Here a *simple* abelian variety is one that has no proper abelian subvarieties. An *isogeny* $X \to Y$ of abelian varieties is a surjective homomorphism with finite kernel, and $X$ and $Y$ are *isogenous* if there exists an isogeny $X \to Y$. In fact given an isogeny $f : X \to Y$ there exists an isogeny $g : Y \to X$ such that $gf = n_X \in \text{End}(X)$ for some natural number $n$; in particular, the relation ‘is isogenous to’ is indeed an equivalence relation.

For an abelian variety $X$ we write $\text{End}^0(X)$ to denote the tensor product $\text{End}(X) \otimes \mathbb{Q}$. Note that if $X$ and $Y$ are isogenous, then $\text{End}^0(X) \cong \text{End}^0(Y)$ via pullback by the isogenies in either direction. Therefore as a corollary of Theorem 3.1 we get the following [Mum70 §19, Corollary 2].

**Corollary 3.2** Let $X$ be an abelian variety. If $X$ is simple, then $\text{End}^0(X)$ is a division algebra over $\mathbb{Q}$. For any abelian variety, if $X$ is isogenous to $X_1^{n_1} \times \cdots \times X_k^{n_k}$ a product of mutually non-isogenous simple abelian varieties, then

$$\text{End}^0(X) = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

where $D_i$ is the division algebra $\text{End}^0(X_i)$ and $M_{n}(D_i)$ is the ring of $n \times n$ matrices over $D_i$.

So reduction to the case of simple abelian varieties is straightforward. It is a much deeper problem to determine the $\mathbb{Q}$-division algebras which can appear as $\text{End}^0(X)$ for $X$ a simple abelian varieties. The key point is that the endomorphism algebra has finite rank and is equipped with a positive involution [Mum70 §19 Corollary 3, §21 Theorem 1]:
Proposition 3.3 Let \( X \) be a simple abelian variety. Then \( D = \text{End}^0(X) \) is a finite-rank \( \mathbb{Q} \)-division algebra. Moreover \( D \) carries an involution \( x \mapsto x' \) called the Rosati involution. This involution is positive-definite in the sense that if \( x \in D \) is any nonzero element, then \( \text{Tr}_\mathbb{Q}(xx') > 0 \), where \( \text{Tr}_\mathbb{Q} \) is the reduced trace over \( \mathbb{Q} \) of the division algebra \( D \).

Now there is a classification (due to Albert) of all finite-rank \( \mathbb{Q} \)-division algebras with a positive involution, and together with some extra geometric restrictions this gives a complete list of possibilities for \( \text{End}^0(X) \) when \( X \) is a simple abelian variety. Chapter 4 of Mumford’s book gives an exposition of the classification; here we only state a weak form of the result.

Theorem 3.4 Let \( X \) be a simple abelian variety and \( D = \text{End}^0(X) \) its endomorphism algebra. Then \( D \otimes \mathbb{R} \) is isomorphic as an \( \mathbb{R} \)-algebra with involution to one of the following algebras:

- \( \mathbb{R} \times \cdots \times \mathbb{R} \), with \( x \mapsto x' \) the trivial involution;
- \( \mathbb{H} \times \cdots \times \mathbb{H} \) where \( \mathbb{H} \) is the algebra of quaternions, and \( x' = \bar{x} \), the usual conjugate;
- \( M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R}) \) where \( M_2(\mathbb{R}) \) is the algebra of \( 2 \times 2 \) real matrices, and \( x' = x^t \), the transpose;
- \( M_2(\mathbb{C}) \times \cdots \times M_2(\mathbb{C}) \), where \( M_2(\mathbb{C}) \) is the algebra of \( 2 \times 2 \) complex matrices, and \( x' = x^* \), the conjugate transpose.

Combining this result with Corollary 3.2 one can deduce the following description of \( \text{End}^0(X) \otimes \mathbb{R} \) for an arbitrary abelian variety \( X \).

Corollary 3.5 Let \( X \) be an abelian variety. Then \( \text{End}^0(X) \otimes \mathbb{R} \) is isomorphic as an algebra with involution to a product

\[
\prod_i M_{r_i}(\mathbb{R}) \times \prod_j M_{s_j}(\mathbb{C}) \times \prod_k M_{t_k}(\mathbb{H})
\]

with involution given by conjugate transpose on each factor. The bilinear pairing \( \langle x, y \rangle = \text{Tr}(xy^*) \) defines an inner product on \( \text{End}^0(X) \otimes \mathbb{R} \).

Finally we must explain how the automorphism group \( \text{Aut}(X) \) sits inside the algebra \( \text{End}^0(X) \otimes \mathbb{R} \). If \( X \) is an abelian variety, \( \text{Aut}(X) \) is the group of units \( \text{End}(X)^\times \) in the endomorphism ring \( \text{End}(X) \). Furthermore \( \text{End}(X) \) is a lattice in the vector space \( \text{End}^0(X) \otimes \mathbb{R} \), therefore induces a \( \mathbb{Q} \)-structure on \( \text{End}^0(X) \otimes \mathbb{R} \), and this \( \mathbb{Q} \)-structure determines \( \text{End}(X) \) as a subring of the \( \mathbb{R} \)-algebra \( \text{End}^0(X) \otimes \mathbb{R} \) up to finite index. So from the previous corollary we have the following:

Corollary 3.6 Let \( X \) be an abelian variety. Then \( (\text{End}^0(X) \otimes \mathbb{R})^\times \) is an algebraic group defined over \( \mathbb{Q} \), and \( \text{Aut}(X) \) is an arithmetic subgroup.
4 The Néron–Severi space of an abelian variety

In this final section, we explain how the Néron–Severi space of an abelian variety can be identified with a subspace of the space $\text{End}^0(X) \otimes \mathbb{R}$. This identification allows us to describe the action of the automorphism group on the ample cone in terms of matrices, and applying the results of Section 2 we get a proof of Theorem 0.1.

First we define the Néron–Severi space of an abelian variety $X$ to be the finite-dimensional real vector space $N^1(X) := \left[\text{Pic}(X)/\text{Pic}^0(X)\right] \otimes \mathbb{R}$. Our first task is to identify $N^1(X)$ with a subspace of $\text{End}^0X \otimes \mathbb{R}$. By linearity, to make such an identification it suffices to identify a Cartier divisor with an element of $\text{End}^0(X)$, which we do in the following way:

$$\text{Pic}(X) \xrightarrow{\phi} \text{Hom}(X, \hat{X}) \otimes \mathbb{Q} \xrightarrow{\psi} \text{End}^0(X)$$

The notation of the diagram is as follows. The variety $\hat{X}$ is the dual abelian variety of $X$, which can be identified with $\text{Pic}^0(X)$. The homomorphism $\phi_D : X \to \hat{X}$ is defined by $\phi_D(x) = T^*_x(D) \otimes D^{-1}$, where $T_x$ is translation by the point $x \in X$. Finally, $L$ is any (fixed) ample line bundle on $X$; ampleness implies that $\phi_L$ is an isogeny $X \to \hat{X}$, and therefore has an inverse $\phi_L^{-1} \in \text{Hom}(\hat{X}, X) \otimes \mathbb{Q}$. One checks that the kernel of $\phi$ is exactly the subgroup $\text{Pic}^0(X)$ of numerically trivial line bundles on $X$, and that $\psi$ is an isomorphism. Therefore tensoring with $\mathbb{R}$ gives the claimed embedding $N^1(X) \hookrightarrow \text{End}^0X \otimes \mathbb{R}$. (See [Mum70] for proofs of all the assertions here.)

The automorphism group $\text{Aut}(X)$ acts naturally on $N^1(X)$ via pullback of divisors: an automorphism $f$ maps a divisor $D$ to $f^*D$. Using the diagram above, we can extend this to an action of $\text{Aut}(X)$ on the whole algebra $\text{End}^0(X) \otimes \mathbb{R}$. To compute the action we need the following formula [Mum70] §15, proof of Theorem 1:

**Lemma 4.1** Let $f : X \to Y$ be an isogeny of abelian varieties, with dual isogeny $\hat{f} : \hat{Y} \to \hat{X}$. Let $D$ be a line bundle on $Y$. Then $\phi_{f^*D} = \hat{f} \circ \phi_D \circ f$.

Via the diagram we can then work out the extension to an action of $\text{Aut}(X)$ on $\text{End}^0(X) \otimes \mathbb{R}$: one computes that $f \in \text{Aut}(X)$ acts by the formula

$$f \cdot x = \phi_L^{-1} \circ \hat{f} \circ \phi_L \circ x \circ f$$

for any $x \in \text{End}^0(X) \otimes \mathbb{R}$. Moreover the same formula defines an action of the whole group of units $(\text{End}^0(X) \otimes \mathbb{R})^\times$ on $\text{End}^0(X) \otimes \mathbb{R}$, and this action fixes the subspace $N^1(X)$.

To complete the picture, we observe [Mum70] p.189 that the map $x \mapsto \phi_L^{-1} \circ \hat{e} \circ \phi_L$ is by definition exactly the Rosati involution on $\text{End}^0(X) \otimes \mathbb{R}$ mentioned in Proposition 3.3. (The involution therefore depends on the choice of an ample line bundle on $X$, but the dependence does not matter for our purposes.) For an element $e \in \text{End}^0(X) \otimes \mathbb{R}$ let us denote $\phi_L^{-1} \circ \hat{e} \circ \phi_L$ by $e'$, so that the action of $(\text{End}^0(X) \otimes \mathbb{R})^\times$ on $\text{End}^0(X) \otimes \mathbb{R}$ is now given by the formula $x \mapsto f' \circ x \circ f$. By Theorem 3.4 there is an isomorphism of $\text{End}(X)^0 \otimes \mathbb{R}$ with a product of matrix algebras which takes the Rosati involution $e \mapsto e'$ to the conjugate transpose involution $x \mapsto x^*$. Using this isomorphism to translate the action above into matrix terms, we get the following theorem.
Theorem 4.2 Let $X$ be an abelian variety. Then the $\mathbb{Q}$-algebraic group $(\text{End}^0(X) \otimes \mathbb{R})^\times$ acts on $N^1(X)$ by the formula $F : D \to F^*DF$, and this extends the action of $\text{Aut}(X)$ on $N^1(X)$ by pullback of line bundles.

At this point we have an explicit description in terms of matrices of the action of the automorphism group on the Néron–Severi space. To complete the proof of the cone conjecture, we need to identify the ample cone in the same terms (i.e. as a cone in a space of matrices), and then apply the results of reduction theory from Section 2.

The first step is to identify the image of our embedding $N^1(X) \hookrightarrow \text{End}^0(X) \otimes \mathbb{R}$, viewing the target as a product of matrix algebras as in Corollary 3.5. Again the key here is the Rosati involution: as we have just seen, in matrix terms this is simply conjugate-transposition $x \mapsto x^\ast$.

Theorem 4.3 Let $X$ be an abelian variety. Then $N^1(X) \subset \text{End}^0(X) \otimes \mathbb{R}$ is exactly the fixed subspace of the Rosati involution. If $\text{End}^0(X) \otimes \mathbb{R}$ is isomorphic to a product of matrix algebras

\[ \prod_i M_{r_i}(\mathbb{R}) \times \prod_j M_{s_j}(\mathbb{C}) \times \prod_k M_{t_k}(\mathbb{H}) \]

then $N^1(X)$ is isomorphic to the subspace

\[ \bigoplus_i \mathcal{H}_{r_i}(\mathbb{R}) \oplus \bigoplus_j \mathcal{H}_{s_j}(\mathbb{C}) \oplus \bigoplus_k \mathcal{H}_{t_k}(\mathbb{H}) \]

where $\mathcal{H}_r$ denotes the space of $r \times r$ symmetric or Hermitian matrices. Moreover, the ample cone $A(X)$ is the direct sum of the positive-definite cones $P_r(k)$ in each of the direct summands $\mathcal{H}_r(k)$ of $N^1(X)$.

We use additive notation for $N^1(X)$ to emphasise the point that it need not be a subalgebra of $\text{End}^0(X) \otimes \mathbb{R}$: for divisors $D_1$ and $D_2$ fixed by the Rosati involution, their product $D_1D_2 \in \text{End}^0(X) \otimes \mathbb{R}$ need not be fixed, as one can check using suitable matrices.

Theorem 4.3 shows in particular that the ample cone of an abelian variety is a self-dual homogeneous cone, since it is a direct sum of cones on the list of Theorem 2.2. Indeed since it is a sum of cones of the form $P_r(k)$, Theorem 2.3 tells us that $G(A(X))^0$ acts transitively on $A(X)$. Moreover from Theorem 4.3 together with the description of the automorphism group in Theorem 2.3 we can also deduce the following:

Corollary 4.4 Let $X$ be an abelian variety. Then the homomorphism

\[ (\text{End}^0(X) \otimes \mathbb{R})^\times \to G(\text{Amp}(X))^0 \]

\[ M \mapsto (D \mapsto M^*DM) \]

given by the action of $(\text{End}^0(X) \otimes \mathbb{R})^\times$ on $N^1(X)$ is surjective.

Proof: Suppose for simplicity that $N^1(X)$ has a single direct summand, say $N^1(X) = \mathcal{H}_r(\mathbb{R})$. By Theorem 2.3 the identity component $G(A(X))^0$ of the automorphism group of the ample cone is exactly the group of linear transformations of $N^1(X)$ of the form $D \mapsto M^*DM$ with $M \in GL(r, \mathbb{R})$. Such a linear transformation is the image under the homomorphism
of $M \in GL(r, \mathbb{R}) = (\text{End}^0(X) \otimes \mathbb{R})^\times$, which proves surjectivity. The proof in the case of more than one direct summand works in the same way, since by Theorem 2.2 the identity component $G(A(X))^0$ is the direct product of the identity components of the automorphism groups of the direct summand cones. QED Corollary 4.4

We can now complete the proof of Theorem 0.1. The inclusion $N^1(X) \subset \text{End}^0(X) \otimes \mathbb{R}$ endows $N^1(X)$ with a $\mathbb{Q}$-structure, and by Corollary 4.4 the connected component $G(A(X))^0$ is a $\mathbb{Q}$-algebraic subgroup of $GL(N^1(X))$. By Corollary 3.6 the automorphism group $\text{Aut}(X)$ is an arithmetic subgroup of $(\text{End}^0(X) \otimes \mathbb{R})^\times$, so the image $\Gamma$ of $\text{Aut}(X)$ in $GL(N^1(X))$ is an arithmetic subgroup of $G(A(X))^0$. Therefore by Theorem 2.4 and 2.5 there is a rational polyhedral fundamental domain for the action of $\Gamma$ on $A(X)_+ = A(X)$ as required. QED

Theorem 0.1

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