An LMI Approach to Stability Analysis of Coupled Parabolic Systems

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Abstract—We analyze the exponential stability of a class of distributed parameter systems. The system we consider is described by a coupled parabolic partial differential equation with spatially varying coefficients. We approximate the coefficients by splitting space domains but take into account approximation errors during stability analysis. Using a quadratic Lyapunov function, we obtain sufficient conditions for exponential stability in terms of linear matrix inequalities.

Index Terms—Exponential stability, linear matrix inequalities (LMIs), Lyapunov functional, partial differential equations (PDE).

I. INTRODUCTION

Consider the following parabolic partial differential equation (PDE) on a bounded open set \( \Omega \subset \mathbb{R}^n \):

\[
\begin{align*}
\partial_t z &= A \Delta z + B(x)z \quad \text{in } \Omega \times (0, T) \\
z &= 0 \quad \text{on } \partial \Omega \times (0, T) \\
z(0, \cdot) &= z^0 \quad \text{in } \Omega
\end{align*}
\]  

(1)

where \( A \in \mathbb{R}^{n \times n}, B \in L^\infty(\Omega)^{n \times n} \), \( \Delta \) is the Laplacian acting component-wise, \( z = [z_1, \ldots, z_n]^{\top} : \Omega \times [0, T) \rightarrow \mathbb{R}^n \) is the state, and \( z^0 : \Omega \rightarrow \mathbb{R}^n \) is a given initial data in \( L^2(\Omega)^n \). This PDE is a subclass of abstract parabolic equations (see, e.g., [1, Sec. 11.1]), and we call the PDE in (1) a coupled parabolic system. In this paper, we study the stability analysis of this coupled parabolic system (1) by using linear matrix inequalities (LMIs) and a quadratic Lyapunov function

\[
V(z) := \int_{\Omega} z(\cdot)^{\top} P(x) z(x) dx \quad \forall z \in L^2(\Omega)^n.
\]  

(2)

Lyapunov-based stability analysis without approximation has recently been developed for distributed parameter systems. The authors of [2]–[5] have proposed semidefinite programming approaches for the stability of one-dimensional (1-D) and 2-D PDEs with polynomial data. LMI-based exponential stability conditions have been obtained for various classes of distributed parameter systems, for example, 1-D heat/wave equations with time-varying delays in [6], 1-D semilinear parabolic systems in [7], coupled \( n \)-D semilinear diffusion equations with time-varying delays (and spatially constant coefficients) in [8], and \( n \)-D wave equations in [9]. Moreover, in [10], a sufficient dissipative boundary condition has been derived to guarantee the exponential stability of coupled 1-D hyperbolic systems. In terms of the coupled parabolic system (1), its controllability has been extensively investigated, e.g., in [11] and [12]. However, relatively little work has been done on the stability analysis of this class of distributed parameter systems. The difficulties in the stability analysis of the parabolic system (1) are the following three points.

1) The state \( z \) is vector-valued.
2) The coefficient function \( B \) may not be constant or even polynomial.
3) The set \( \Omega \) is multidimensional.

To the parabolic PDE (1), we apply the gridding methods that have been proposed for establishing the stability of networked control systems with aperiodic sampling and time-varying delays, e.g., in [13]–[17]. First, we consider a general bounded open set \( \Omega \) and transform the coefficient function \( B(x) \) to a piecewise constant function plus an approximation error by splitting the set \( \Omega \). This approximation error is taken into account during the stability analysis. We obtain an LMI-based sufficient condition for exponential stability, using a Lyapunov function in (2) where \( P(x) \) is a constant function. Second, we focus on the case where the set \( \Omega \) is a polytope. In this case, we approximate \( B(x) \) by a piecewise linear function and use a Lyapunov function in (2) where \( P(x) \) is piecewise linear on \( \Omega \). This means that we use a wider class of Lyapunov functions to analyze the stability of the coupled parabolic system (1). As a result, we can obtain a less conservative sufficient LMI condition for exponential stability in the case of a polytope \( \Omega \).

This paper is organized as follows. In Section II, we recall preliminary results on Sobolev spaces and the concept of weak solutions of the parabolic PDE (1). In Section III, we analyze the exponential stability of the coupled PDE (1) with general set \( \Omega \), using Lyapunov functions with constant \( P \). In Section IV, stability analysis by Lyapunov functions with piecewise linear \( P \) is presented in the case where \( \Omega \) is a polytope. We illustrate numerical examples in Section V.

Notation: For a set \( \Omega \subset \mathbb{R}^n \), its closure, interior, and boundary are denoted by \( \overline{\Omega}, \Omega^\circ \), and \( \partial \Omega \), respectively. Let us denote the Euclidean norm of a vector \( \xi \in \mathbb{R}^m \) by \( \| \xi \| \). For a matrix \( M \in \mathbb{R}^{m \times p} \), we denote by \( M^\top \) and \( \| M \| \) its transpose and Euclidean-induced norm, respectively. Let us denote by \( \{ e_i \}_{i=1}^m \) the standard basis in \( \mathbb{R}^m \), namely, \( e_1 = [1 \ 0 \ \cdots \ 0]^{\top}, \ldots, e_m = [0 \ \cdots \ 0 \ 1]^{\top} \). For a square matrix \( P \in \mathbb{R}^{m \times m} \), the notation \( P \succ 0 \) means that \( P \) is symmetric and positive definite. The Kronecker product of two real matrices \( A \) and \( B \) is denoted by \( A \otimes B \). For simplicity, we write a partitioned real symmetric matrix

\[
\begin{bmatrix}
A & B \\
B^\top & C
\end{bmatrix}
\]

as

\[
\begin{bmatrix}
A & B \\
B^\top & C
\end{bmatrix}.
\]

Let \( \Omega \subset \mathbb{R}^n \) be an open set. We denote by \( L^2(\Omega)^n \) the space of all measurable functions \( f : \Omega \rightarrow \mathbb{R}^n \) satisfying \( \| f(x) \|^2 dx < \infty \). The norm and inner product of \( L^2(\Omega)^n \) are defined by

\[
\| f \|_{L^2(\Omega)^n} := \left( \int_{\Omega} \| f(x) \|^2 dx \right)^{1/2}, \quad (f,g)_{L^2(\Omega)} := \int_{\Omega} f(x)^\top g(x) dx
\]  

respectively. The space \( L^\infty(\Omega)^{n \times p} \) consists of all measurable functions \( F : \Omega \rightarrow \mathbb{R}^{n \times p} \) satisfying \( \text{ess sup}_{x \in \Omega} \| F(x) \| < \infty \). The norm of \( L^\infty(\Omega)^{n \times p} \) is defined by

\[
\| F \|_{L^\infty(\Omega)^{n \times p}} := \text{ess sup}_{x \in \Omega} \| F(x) \|.
\]
We write $L^\infty(\Omega)^n$ for $L^\infty(\Omega)^{n\times 1}$, and if $n = 1$, then we will drop the superscript $n$. Let us denote by $H^1(\Omega)^n$ the space of all functions $f = [f_1, f_2, \ldots, f_n] \in L^2(\Omega)^n$ such that the first-order partial derivatives of $f_1, \ldots, f_n$ exist in the weak sense and belong to $L^2(\Omega)$. We denote the gradient of a scalar-valued function $f \in H^1(\Omega)$ by $\nabla f$ and define in the vector-valued case

$$\nabla f := \begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{bmatrix} \quad \text{for} \quad f := \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in H^1(\Omega)^n.$$  

The norm of $H^1(\Omega)^n$ is defined by

$$\|f\|_{H^1(\Omega)^n} := \sqrt{\|f\|_{L^2(\Omega)^n}^2 + \|\nabla f\|_{L^2(\Omega)^{n\times 1}}^2}.$$  

The space $C_0^\infty(\Omega)^n$ comprises all infinitely many times differentiable functions $f : \Omega \to \mathbb{R}^n$ such that $\text{supp} f := \{x \in \Omega : f(x) \neq 0\}$ is compactly contained in $\Omega$. The space $H^1_0(\Omega)^n$ is the closure of $C_0^\infty(\Omega)^n$ in $H^1(\Omega)^n$. We denote by $H^{-1}(\Omega)^n$ the dual space of $H^1_0(\Omega)^n$, that is, the space of bounded linear maps $g : H^1_0(\Omega)^n \to \mathbb{R}$. Elements of $H^{-1}(\Omega)^n$ can be regarded as n-dimensional vectors whose entries belong to $H^{-1}(\Omega)$. The duality pairing between $H^1_0(\Omega)^n$ and its dual $H^{-1}(\Omega)^n$ is denoted by $(g, f) : H^{-1}(\Omega)^n \times H^1_0(\Omega)^n \to \mathbb{R}$. For simplicity of notation, we will drop the dimension $n$ and the set $\Omega$ in the norms and the inner product, e.g., write $\|f\|_{H^1(\Omega)}$ for $\|f\|_{H^1(\Omega)^n}$.

Let $X$ be a Banach space with norm $\| \cdot \|_X$. We denote by $L^2(0, T; X)$ the space of all strongly measurable functions $f : (0, T) \to X$ such that $\int_0^T \|f(t)\|_X^2 \, dt < \infty$. The space $C([0, T]; X)$ comprises all continuous functions $f : [0, T] \to X$.

II. PRELIMINARIES

In what follows, we write $z(t) = z(\cdot, t)$ from the vector-valued viewpoint. The following theorem will be useful to study the coupled parabolic PDE (1):

**Theorem 2.1 (Sec. 5.9.2 in [18]):** Let

$$z \in L^2(0, T; H^1_0(\Omega)^n) \quad \text{and} \quad \frac{dz}{dt} \in L^2(0, T; H^{-1}(\Omega)^n).$$

Then,

1. $z \in C([0, T]; L^2(\Omega)^n)$.
2. The mapping $t \mapsto \|z(t)\|_{L^2}$ is absolutely continuous with

$$\frac{d}{dt} \|z(t)\|_{L^2} = 2 \left( \frac{dz}{dt}(t), z(t) \right) \quad \text{a.e.} \ t \in [0, T].$$

Although only the case $n = 1$ is considered in [18, Sec. 5.9.2], one can obtain Theorem 2.1, the case $n \geq 1$, by applying the result of the case $n = 1$ to each element of $z$.

We define a weak solution of the coupled parabolic PDE (1).

**Definition 2.2 (Weak solution):** A function $z \in L^2(0, T; H^1_0(\Omega)^n)$ with $\frac{dz}{dt} \in L^2(0, T; H^{-1}(\Omega)^n)$ is a weak solution of the coupled parabolic PDE (1) with the initial data $z_0^i \in L^2(\Omega)^n$ if the following two conditions hold.

1. For every $v \in H^1_0(\Omega)^n$ and for a.e. $t \in [0, T]$,

$$\left\langle \frac{dz}{dt}(t), v \right\rangle = -(A \otimes I_m) \nabla z(t), \nabla v \right\rangle_{L^2} + (Bz(t), v)_{L^2}.$$  

2. $z(0) = z_0^i$.

Since $z \in L^2(0, T; H^1_0(\Omega)^n)$ and $\frac{dz}{dt} \in L^2(0, T; H^{-1}(\Omega)^n)$ in Definition 2.2, it follows that 1) of Theorem 2.1 yields $z \in C([0, T]; L^2(\Omega)^n)$. Hence, the initial condition 2) makes sense.

We place the following coercivity condition on the coefficient matrix $A$ in the PDE (1), which is used to guarantee the existence and uniqueness of weak solutions.

**Assumption 2.3:** There exists $\alpha > 0$ such that $A \in R^{n \times n}$ in the PDE (1) satisfies $\langle A \zeta, \zeta \rangle \geq \alpha \|\zeta\|^2$ for every $\zeta \in \mathbb{R}^n$.

Applying Galerkin’s method, we see that if Assumption 2.3 is satisfied, then for every initial data $z_0^i \in L^2(\Omega)^n$, there exists a unique weak solution of the coupled parabolic PDE (1); see, e.g., the appendix of the online version of this paper [19], [1, Sec. 11.1], and [18, Sec. 7.1]. We define the exponential stability of the coupled parabolic system (1).

**Definition 2.4 (Exponential stability):** The coupled parabolic system (1) is exponentially stable if there exist $M \geq 1$ and $\gamma > 0$ such that for each $T > 0$, the weak solution $z$ of the PDE (1) satisfies

$$\|z(t)\|_{L^2} \leq Me^{-\gamma t} \|z_0\|_{L^2}, \quad \forall t \in [0, T].$$

To analyze the exponential stability of the coupled PDE, we employ Poincaré–Friedrichs’ inequality.

**Theorem 2.5 (Poincaré–Friedrichs’ inequality):** For every bounded open set $\Omega \subset \mathbb{R}^n$, there exists a constant $c = c(\Omega) > 0$ such that

$$\|z\|_{L^2} \leq c\|\nabla z\|_{L^2}, \quad \forall z \in H^1_0(\Omega)^n.$$

If $\Omega$ is contained between a pair of parallel hyperplanes situated at a distance $\delta > 0$, then the constant $c$ of Poincaré–Friedrichs’ inequality is given by $\delta$; see, e.g., [20, Proposition 13.4.10]. If $\Omega = (a, b) \subset \mathbb{R}$, then $c = (b - a)/\pi$, which cannot be improved; see, e.g., [21, Sec. 1.7.2]. Applying Poincaré–Friedrichs’ inequality to each element of $z \in H^1_0(\Omega)^n$, we obtain the following result.

**Corollary 2.6:** Let $\Omega \subset \mathbb{R}^m$ be bounded and open. For every $z \in H^1_0(\Omega)^n$ and every positive definite diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$, we obtain

$$\int_{\Omega} \left[ z(x) \right]^T \begin{bmatrix} -\Lambda & 0 \\ 0 & c^2 \Lambda \otimes I_m \end{bmatrix} \left[ \nabla z(x) \right] dx \geq 0$$

where $c > 0$ is a constant of Poincaré–Friedrichs’ inequality (3).

III. STABILITY ANALYSIS BY LYAPUNOV FUNCTIONS WITH CONSTANT $P$

First, we study the stability of the coupled parabolic PDE (1), by using Lyapunov function with constant $P$. We place the following assumptions on the bounded open set $\Omega$.

**Assumption 3.1:** For a bounded open set $\Omega \subset \mathbb{R}^n$, let Lebesgue measurable sets $\Omega_k \subset \mathbb{R}^n$ ($k = 1, \ldots, N$) satisfy the following conditions:

1. $\Omega = \bigcup_{k=1}^N \Omega_k$;
2. $\Omega_k \cap \Omega_\ell = \emptyset \quad \forall k, \ell = 1, \ldots, N$ with $k \neq \ell$.

**Assumption 3.2:** For every $k = 1, \ldots, N$, the matrix $B_k \in \mathbb{R}^{n \times n}$ and the scalar $\rho_k > 0$ satisfy

$$\|B_k - B_k\| \leq \rho_k \quad \text{a.e.} \ x \in \Omega_k.$$  

For example, we can choose $B_k = B(\omega_k)$, where $\omega_k \subset \mathbb{R}^m$ is the “center” of $\Omega_k$. The scalar $\rho_k$ is the approximation error of $B_k$. The disjoint subsets $\Omega_1, \ldots, \Omega_N$ are tuning parameters in our stability analysis.

We then have the following sufficient LMI condition for stability.

**Theorem 3.3:** Let Assumptions 2.3, 3.1, and 3.2 hold. The coupled parabolic system (1) is exponentially stable if there exist a positive definite matrix $P \in \mathbb{R}^{n \times n}$, a positive definite diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$, and a positive scalar $\delta_k$ such that the following LMIs are feasible.
for all \( k = 1, \ldots, N \):
\[
\begin{bmatrix}
\Lambda - \sigma_k I_n - B_k^T P - PB_k & \rho_k P \\
\times & \sigma_k I_n
\end{bmatrix} \succ 0 \quad \text{(6a)}
\]
\[
A^T P + PA^T - c^2 A \succeq 0
\quad \text{(6b)}
\]
where \( c > 0 \) is a constant of Poincaré–Friedrichs’ inequality (3).

\textbf{Proof:} 1) Using the positive definite matrix \( P \in \mathbb{R}^{n \times n} \), we define
\[
V(z) := (z, Pz)_{L^2} \quad \forall z \in L^2(\Omega)^n.
\]
We use the notation \( V(t) := V(z(t)) \) for simplicity, where \( z \) is the weak solution of the PDE (1). One can see that \( V(t) \) is absolutely continuous on \([0, T]\) and
\[
d\frac{dV}{dt}(t) = 2 \left( \frac{d}{dt}(t), Pz(t) \right) \quad \text{a.e. } t \in [0, T]
\]
from the same argument as in the proof of 2) of Theorem 2.1. Since \( Pz(t) \in H_0^1(\Omega)^n \) for a.e. \( t \in [0, T] \), it follows from the condition 1) in Definition 2.2 that
\[
\frac{dV}{dt}(t) = \int_\Omega \begin{bmatrix} z(x, t) \nabla z(x, t) \end{bmatrix}^T M(x) \begin{bmatrix} z(x, t) \\ \nabla z(x, t) \end{bmatrix} dx \quad \text{a.e. } t \in [0, T],
\]
where
\[
M(x) := \begin{bmatrix} B(x)^T P + PB(x) & 0 \\
0 & -(A^T P + PA) \otimes I_m \end{bmatrix}.
\]
On the other hand, since \( z(t) \in H_0^1(\Omega)^n \) for a.e. \( t \in [0, T] \), Corollary 2.6 shows that for a.e. \( t \in [0, T] \)
\[
\int_\Omega \begin{bmatrix} z(x, t) \\ \nabla z(x, t) \end{bmatrix}^T \begin{bmatrix} -\Lambda & 0 \\
0 & c^2 A \otimes I_m \end{bmatrix} \begin{bmatrix} z(x, t) \\ \nabla z(x, t) \end{bmatrix} dx \succeq 0.
\quad (8)
\]
Therefore, if there exists \( \epsilon > 0 \) such that for a.e. \( x \in \Omega \)
\[
\Lambda - B(x)^T P - PB(x) \succeq \epsilon I_n \quad \text{(9a)}
\]
\[
(A^T P + PA - c^2 A) \otimes I_m \succeq 0 \quad \text{(9b)}
\]
then it follows from (7) and (8) that
\[
\frac{dV}{dt}(t) \leq -\epsilon \|z(t)\|_{L^2}^2 \quad \text{a.e. } t \in [0, T].
\quad (10)
\]

2) We next show that if the LMIs (6) are feasible, then there exists a constant \( \epsilon > 0 \) such that the inequalities (9) hold for a.e. \( x \in \Omega \). By the LMI (6a), there exists \( \epsilon > 0 \) such that for all \( k = 1, \ldots, N \)
\[
G_k := \begin{bmatrix} \Lambda - (\sigma_k + \epsilon) I_n - B_k^T P - PB_k & \rho_k P \\
\times & \sigma_k I_n \end{bmatrix} \succ 0.
\quad (11)
\]
By (5), for every \( k = 1, \ldots, N \), there exists a measurable function \( \Phi_k : \Omega_k \to \mathbb{R}^{n \times n} \) such that for a.e. \( x \in \Omega_k \)
\[
B(x) - B_k = \rho_k \Phi_k(x) \quad \|\Phi_k(x)\| \leq 1.
\quad (12a)
\]
Since \( I_n - \Phi_k^T(x)\Phi_k(x) \succeq 0 \) for a.e. \( x \in \Omega_k \) by (12b), it follows from (12a) that
\[
\begin{bmatrix} I_n \\ -\Phi_k(x) \end{bmatrix}^T G_k \begin{bmatrix} I_n \\ -\Phi_k(x) \end{bmatrix} = \Lambda - (B_k + \rho_k \Phi_k(x))^T P - P(B_k + \rho_k \Phi_k(x)) - \epsilon I_n
\]
\[
- \sigma_k (I_n - \Phi_k(x))^T \Phi_k(x) \succeq 0 \quad \text{a.e. } x \in \Omega_k.
\]
Thus, the inequality (11) yields (9a). Moreover, since \( (A^T P + PA - c^2 A) \otimes I_m \) and \( A^T P + PA - c^2 A \) have the same eigenvalues, it follows that (6b) implies (9b).

3) Finally, we show that the inequality (10) leads to the exponential stability of the coupled parabolic system (1). Let \( \delta_{\min} \) and \( \delta_{\max} \) be the minimum and maximum eigenvalues of \( P \), respectively. From the inequality (10), we find that
\[
\frac{dV}{dt}(t) \leq -2\gamma V(t) \quad \text{a.e. } t \in [0, T]
\quad (13)
\]
where \( \gamma := \epsilon/(2\delta_{\max}) \). Since \( V(t) \) is absolutely continuous on \([0, T] \), Gronwall’s inequality (see, e.g., [18, Appendix B.2]) yields
\[
V(t) \leq V(0)e^{-\gamma t} \quad \forall t \in [0, T].
\]
Thus, for each \( T > 0 \) and each initial state \( z^0 \in L^2(\Omega)^n \), the solution \( z(t) \) of the parabolic PDE (1) satisfies
\[
\|z(t)\|_{L^2} \leq \sqrt{\max_{\delta_{\min}} \delta_{\max}} e^{-\gamma t} \|z^0\|_{L^2} \quad \forall t \in [0, T].
\]
This completes the proof.

\textbf{Remark 3.4 (Complexity of LMIs in Theorem 3.3):} Let us study the numbers of variables in the LMIs of Theorem 3.3. In these LMIs, the matrices \( P \) and \( \Lambda \) have \( O(n^2) \) and \( O(n) \) variables, respectively. On the other hand, the number of the scalar variables \( \sigma_1, \ldots, \sigma_N \) is \( O(N) \). In total, the LMIs of Theorem 3.3 contain \( O(n^2 + N) \) variables. Suppose that the number \( N \) of the disjoint subsets \( \Omega_1, \ldots, \Omega_k \) is given by \( N = 2^m \), which makes sense due to the curse of dimensionality. Then, the worst-case number is given by \( O(n^2 + 2^m) \).

\section{IV. Stabilility Analysis by Lyapunov Functions With Piecewise Linear \( P \)}

In this section, we analyze the stability of the coupled parabolic system (1), by using Lyapunov functions that depend on the space variable in a piecewise linear fashion. We impose the following assumption on the bounded open set \( \Omega \).

\textbf{Assumption 4.1:} For a bounded open set \( \Omega \subseteq \mathbb{R}^m \), let \( m \)-simplices \( \Omega_1, \ldots, \Omega_N \subseteq \mathbb{R}^m \) satisfy the following conditions:

1) \( \Omega_k = \bigcup_{i = 1}^{m} \Omega_k^i \); 
2) \( \Omega_1 \cap \Omega_k \neq \emptyset \Rightarrow \Omega_1 \cap \Omega_k \) is a face of \( \Omega_k \) and \( \Omega_k \).

For \( k = 1, \ldots, N \), let \( \xi_{k1}^1, \ldots, \xi_{km}^k \) be the vertices of the \( m \)-simplex \( \Omega_k \) of \( \mathbb{R}^m \). We reorder the set \( \{\xi_{k1}^1, \ldots, \xi_{km}^k : k = 1, \ldots, N\} \) into \( \{\xi_1, \ldots, \xi_N\} \) without duplication. Namely, \( \{\xi_{k1}^1, \ldots, \xi_{km}^k : k = 1, \ldots, N\} = \{\xi_1, \ldots, \xi_N\} \) and \( \xi_i \neq \xi_i \) for every \( k, \ell = 1, \ldots, N \) with \( k \neq \ell \). Let \( \xi_{p(k, \ell)} \) be the vertices of \( \Omega_k \) for every \( k, \ell = 1, \ldots, N \). Define a matrix \( B_k := B(\xi_{p(k, \ell)}) \) for each \( p = 1, \ldots, N_0 \), and let \( x \in \Omega_k \) be represented as
\[
x = \sum_{\ell = 0}^{n} \alpha_{p(k, \ell)}(x) \xi_{p(k, \ell)}
\quad (14)
where the coefficients $\alpha_{p(k,0)}(x), \ldots, \alpha_{p(k,m)}(x)$ are nonnegative and satisfy $\sum_{p=0}^{m} \alpha_{p(k,0)}(x) = 1$.

We formulate the remaining assumption.

**Assumption 4.2:** For every $k = 1, \ldots, N$, a scalar $\rho_k > 0$ satisfies

$$
\left\| B(x) - \sum_{l=0}^{m} \alpha_{p(k,l)}(x)B_{p(k,l)} \right\| \leq \rho_k \quad \text{a.e. } x \in \Omega_k.
$$

(15)

As in Assumption 3.1, the tuning of the disjoint sets $\Omega_1, \ldots, \Omega_N$ is needed in our stability analysis to obtain a less conservative result.

For the second main result, we use the following lemma on LMIs, inspired by the stability analysis of systems with polytopic uncertainty developed, e.g., in [22].

**Lemma 4.3:** For every symmetric matrix $M$ and for every matrices $B$ and $P$, the inequality

$$
M - B^T P - P^T B \succeq 0
$$

is satisfied if and only if there exist (not necessarily symmetric) matrices $\Xi$ and $\Xi'$ such that

$$
\begin{bmatrix}
M - B^T \Xi - \Xi' B & \Xi^T - P^T - B^T \Xi' \\
* & \Xi' + \Xi^T
\end{bmatrix} \succeq 0.
$$

(17)

Proof: Since

$$
\begin{bmatrix}
I & B \\
B^T & \begin{bmatrix}
M - B^T \Xi - \Xi' B & \Xi^T - P^T - B^T \Xi' \\
* & \Xi' + \Xi^T
\end{bmatrix}
\end{bmatrix}
$$

it follows that (17) leads to (16). On the other hand, (17) with $\Xi = P$ and $\Xi' = 0$ is equivalent to (16). This completes the proof.

The following lemma provides a sufficient condition for the product of Sobolev functions to belong to $H_0^1$.

**Lemma 4.4:** For a bounded open set $\Omega \subset \mathbb{R}^m$, if $f \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\nabla f \in L^\infty(\Omega)^m$ and if $g \in H^1_0(\Omega)$, then $fg \in H^1_0(\Omega)$.

Proof: First, we show $fg \in H^1_0(\Omega)$. Since $f \in H^1(\Omega)$ and $g \in H^1_0(\Omega)$, it follows that $fg$ possesses weak derivatives and $\nabla (fg) = (\nabla f)g + f(\nabla g)$. Recall that for every $v, w \in \mathbb{R}^m$

$$
\left\| v + w \right\|^2 \leq \left( \left\| v \right\| + \left\| w \right\| \right)^2 \leq 2 \left( \left\| v \right\|^2 + \left\| w \right\|^2 \right).
$$

(18)

Since $f \in L^\infty(\Omega)$ and $\nabla f \in L^\infty(\Omega)^m$ yield

$$
\int_{\Omega} \left| f(x)g(x) \right|^2 \, dx \leq \left\| f \right\|_{L^\infty(\Omega)}^2 \left\| g \right\|_{L^1(\Omega)}^2,
$$

$$
\frac{1}{2} \int_{\Omega} \left\| \nabla (fg)(x) \right\|^2 \, dx \leq \frac{1}{2} \int_{\Omega} \left\| \nabla f(x)g(x) \right\|^2 \, dx + \int_{\Omega} \left\| f(x)\nabla g(x) \right\|^2 \, dx
$$

$$
\leq \left( \left\| f \right\|_{L^\infty(\Omega)}^2 + \left\| \nabla f \right\|_{L^1(\Omega)}^2 \right) \left\| g \right\|_{L^1(\Omega)}^2,
$$

(19)

it follows that $fg \in H^1_0(\Omega)$.

To show $fg \in H^1_0(\Omega) = C_0^\infty(\Omega)$, it is enough to prove that for every $\epsilon > 0$, there exists $h \in C_0^\infty(\Omega)$ such that

$$
\left\| h - fg \right\|_{H^1(\Omega)} < \epsilon.
$$

Fix $\epsilon > 0$ arbitrarily. Since $g \in H^1_0(\Omega)$, there exists $g_0 \in C_0^\infty(\Omega)$ such that

$$
\left\| g - g_0 \right\|_{H^1(\Omega)} \leq \frac{\epsilon^2}{12 \left( \left\| f \right\|_{L^\infty(\Omega)}^2 + \left\| \nabla f \right\|_{L^1(\Omega)}^2 \right)}.
$$

Choose an open set $U$ such that $\text{supp} \ g_0 \subset U \subset \overline{U} \subset \Omega$. Using a mollifier (see, e.g., [23, Sec. 1.1.5]), we obtain $f \in C^\infty(\Omega)$ satisfying

$$
\left\| f - f \right\|_{H^1(\Omega)} \leq \frac{\epsilon^2}{12 \left( \left\| f \right\|_{L^\infty(\Omega)}^2 + \left\| \nabla f \right\|_{L^1(\Omega)}^2 \right)}.
$$

We define

$$
h(x) := \begin{cases} f(x)g_0(x), & \text{if } x \in U \\ 0, & \text{if } x \in \Omega \setminus U. \end{cases}
$$

Similarly, we obtain

$$
\frac{1}{2} \int_{\Omega} \left\| \nabla h(x) - \nabla (fg)(x) \right\|^2 \, dx
$$

$$
\leq \frac{1}{2} \int_{\Omega} \left\| \nabla (f(x)g_0)(x) \right\|^2 \, dx + \frac{1}{2} \int_{\Omega} \left\| \nabla (f(x)(g_0(x) - g(x))) \right\|^2 \, dx
$$

$$
\leq \left\| f \right\|_{H^1(U)}^2 \left\| g \right\|_{L^\infty(\Omega)}^2 + \left\| f \right\|_{H^1(U)}^2 \left\| \nabla g \right\|_{L^\infty(\Omega)}^2
$$

$$
< \epsilon^2 / 12.
$$

Therefore,

$$
\frac{1}{2} \int_{\Omega} \left\| \nabla h(x) - \nabla (fg)(x) \right\|^2 \, dx
$$

$$
\leq \frac{1}{2} \int_{\Omega} \left\| \nabla h(x) - \nabla (f(x)g)(x) \right\|^2 \, dx + \int_{\Omega} \left\| \nabla (f(x)g)(x) - \nabla (fg)(x) \right\|^2 \, dx
$$

$$
< \epsilon^2 / 3.
$$

Finally, since $\text{supp} \ g_0 \subset U$, it follows that $h \in C_0^\infty(\Omega)$.

The next result provides the partial derivatives of the coefficients of simplices.

**Lemma 4.5:** For an $m$-simplex $\Omega \subset \mathbb{R}^m$, let $\xi_0, \ldots, \xi_m \in \mathbb{R}^m$ be its vertices and $x \in \Omega$ be represented as

$$
\begin{align*}
x &= \sum_{\ell=0}^{m} \alpha_{\ell}(x)\xi_{\ell} \\
\text{where the coefficients } \alpha_{0}(x), \ldots, \alpha_{m}(x) \text{ are nonnegative and satisfy:}
\end{align*}
$$

$$
\sum_{\ell=0}^{m} \alpha_{\ell}(x) = 1.
$$

(20)
Then, the coefficients \( \alpha_0(x), \ldots, \alpha_m(x) \) are continuous on \( \Omega \). Furthermore, for every \( x \in \Omega' \)
\[
\nabla \alpha_0(x) = -\sum_{\ell=1}^{m} v_{\ell} \quad \forall \ell = 1, \ldots, m \tag{22a}
\]
\[
\nabla \alpha_0(x) = \sum_{\ell=1}^{m} v_{\ell} \quad \forall \ell = 1, \ldots, m \tag{22b}
\]
where
\[
v_{\ell} := \begin{bmatrix} \xi_{1,\ell} - \xi_{0,\ell} \\ \vdots \\ \xi_{m,\ell} - \xi_{0,\ell} \end{bmatrix}^{-1} \varepsilon_{\ell} \quad \forall \ell = 1, \ldots, m. \tag{22c}
\]

**Proof:** By (20) and (21)
\[
x = \left( 1 - \sum_{\ell=1}^{m} \alpha_{\ell}(x) \right) \xi_0 + \alpha_1(x) \xi_1 + \cdots + \alpha_m(x) \xi_m.
\]
Therefore,
\[
x - \xi_0 = D \begin{bmatrix} \alpha_1(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix}
\]
where the matrix \( D \in \mathbb{R}^{m \times m} \) is defined by
\[
D := \begin{bmatrix} \xi_1 - \xi_0 \\ \vdots \\ \xi_m - \xi_0 \end{bmatrix}.
\]

Since \( \Omega \) is an \( m \)-simplex, the matrix \( D \) is invertible and
\[
\begin{bmatrix} \alpha_1(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix} = D^{-1} (x - \xi_0). \tag{23}
\]

Hence \( \alpha_1(x), \ldots, \alpha_m(x) \) are continuous on \( \Omega \). By (21), \( \alpha_0(x) = 1 - \sum_{\ell=1}^{m} \alpha_{\ell}(x) \) is also continuous on \( \Omega \).

Next, we investigate the gradients \( \nabla \alpha_0, \ldots, \nabla \alpha_m \). Choose \( \ell = 1, \ldots, m \) arbitrarily. Since \( \alpha_{\ell}(x) = e_{\ell}^T D^{-1} (x - \xi_0) \) by (23), it follows that
\[
\nabla \alpha_{\ell}(x) = \begin{bmatrix} e_{\ell}^T D^{-1} e_1 \\ \vdots \\ e_{\ell}^T D^{-1} e_m \end{bmatrix} \quad \forall x \in \Omega'. \tag{24}
\]

The vector of the right-hand side of (24) is equal to the transpose of the \( \ell \)th row vector of \( D^{-1} \), namely, the vector \( v_{\ell} \) defined by (22c). Thus, we obtain (22a). Moreover, since (21) leads to
\[
\nabla \alpha_0(x) = -\sum_{\ell=1}^{m} \nabla \alpha_{\ell}(x) \quad \forall x \in \Omega',
\]
it follows that (22b) holds.

For every \( k = 1, \ldots, N \) and every \( \ell = 1, \ldots, m \), define
\[
v_{p(k,\ell)}(t) := \begin{bmatrix} e_{p(k,\ell)} - e_{p(k,0)} \\ \vdots \\ e_{p(k,m)} - e_{p(k,0)} \end{bmatrix}^{-1} e_{\ell}, \quad v_{p(k,0)} := -\sum_{\ell=1}^{m} v_{p(k,\ell)}.
\]
We are in a position to state the second main result.

**Theorem 4.6:** Let Assumptions 2.3, 4.1, and 4.2 hold. The coupled parabolic system (1) is exponentially stable if there exist positive definite matrices \( P_1, \ldots, P_N \in \mathbb{R}^{n \times n} \), a positive definite diagonal matrix \( \Lambda \in \mathbb{R}^{n \times n} \), and positive scalars \( \sigma_{k,\ell} \) \( (k = 1, \ldots, N, \ell = 0, \ldots, m) \) such that the LMIs in (A) at the bottom of this page, where \( c > 0 \) is a constant of Poincaré–Friedrichs’ inequality (3), are feasible for all \( k = 1, \ldots, N \) and \( \ell = 0, \ldots, m \).

**Proof:** 1) Using \( P_{p(k,\ell)}, B_{p(k,\ell)} \) and \( \sigma_{k,\ell} \) in the LMIs in (A), we define the functions \( P_k, B_k \), and \( \sigma_k \) on \( \Omega_k \) by
\[
P_k(x) := \sum_{\ell=0}^{m} \alpha_{p(k,\ell)}(x) P_{p(k,\ell)}
\]
\[
B_k(x) := \sum_{\ell=0}^{m} \alpha_{p(k,\ell)}(x) B_{p(k,\ell)}
\]
\[
\sigma_k(x) := \sum_{\ell=0}^{m} \alpha_{p(k,\ell)}(x) \sigma_{k,\ell}
\]
for every \( k = 1, \ldots, N \), where the coefficients \( \alpha_{p(k,0)}, \ldots, \alpha_{p(k,m)} \) are given as in (14). First, we show that the weak solution \( z(t) \) of the parabolic PDE (1) satisfies \( P z(t) \in H_0^1(\Omega)^n \) for a.e. \( t \in [0, T] \), where \( P : \Omega \to \mathbb{R}^{n \times n} \) is defined by
\[
P(x) := P_k(x) \quad \forall x \in \Omega_k, \quad \forall k = 1, \ldots, N. \tag{25}
\]
To this end, we need to see that the values of \( P_k \) and \( P_\omega \) with \( \Omega_1 \cap \Omega_\omega \neq \emptyset \) are not different on the intersection \( \Omega_1 \cap \Omega_\omega \). For every \( j, k = 1, \ldots, N \) with \( \Omega_j \cap \Omega_\omega \neq \emptyset \), let
\[
\xi_0(0), \ldots, \xi_{m(\omega)}(0) \in \{ \xi_{0,j} \}_0^{m(j)} \cap \{ \xi_{0,k} \}_0^{m(k)}
\]
be the vertices of the face \( \Omega_j \cap \Omega_\omega \), which is guaranteed by 2) of Assumption 4.1. Then, for every \( x \in \Omega_j \cap \Omega_\omega \), there exist
\[
\alpha_{q(0)}(x), \ldots, \alpha_{q(m(\omega))}(x) \geq 0 \quad \text{with} \quad \sum_{\ell=0}^{m(\omega)} \alpha_{q(\ell)}(x) = 1
\]
such that
\[
\sum_{\ell=0}^{m(\omega)} \alpha_{q(j,\ell)}(x) P_{q(j,\ell)} = \sum_{\ell=0}^{m(\omega)} \alpha_{q(k,\ell)}(x) P_{q(k,\ell)}
\]
\[
= \sum_{\ell=0}^{m(\omega)} \alpha_{p(k,\ell)}(x) P_{p(k,\ell)}.
\]
Hence, the values of \( P \) are consistent on the boundaries. By Lemma 4.5, the coefficients \( \alpha_{p(k,0)}(x), \ldots, \alpha_{p(k,m)}(x) \) are continuous in \( \Omega_k \), which implies that \( P_k \) is continuous in \( \Omega_k \) for every \( k = 1, \ldots, N \).
Thus, $P$ is continuous in $\overline{\Omega}$. Furthermore, Lemma 4.5 shows that $\nabla \alpha_{p(k,\ell)}$ is constant for every $k = 1, \ldots, N$ and every $\ell = 0, \ldots, m$. The restriction of each element of $P$ to every line parallel to the coordinate directions is continuous piecewise linear and hence absolutely continuous. Thus, every element of $P$ belongs to $H^1(\Omega)$ [23, Sec. 1.1.3, Th. 2], which is called the absolutely continuous on lines characterization of Sobolev functions. Since $z(t) \in H^1(\Omega)$ for a.e. $t \in [0, T]$, it follows from Lemma 4.4 that $Pz(t) \in H^1(\Omega)$ for a.e. $t \in [0, T]$.

For the function $P(x)$ defined by (25) with positive definite matrices $P_1, \ldots, P_N \in \mathbb{R}^{n \times n}$, we set
\[ V(z) := (z, Pz) \] for all $z \in H^1(\Omega)$. Since the Lebesgue measure of the boundary $\partial\Omega_k$ is zero for every $k = 1, \ldots, N$, Lemma 4.5 and (26) yield
\begin{align*}
2 \left( (A \otimes I_m) \nabla z(t), (\nabla P(z(t)) \right)_L^2 \\
= 2 \sum_{k=1}^N \int_{\Omega_k} ((A \otimes I_m) \nabla z(x, t))^T \nabla (P(z)x(t)) \, dx \\
= \sum_{k=1}^N \int_{\Omega_k} \left[ \begin{array}{c} z(x, t) \\ \nabla z(x, t) \end{array} \right]^T \left[ \begin{array}{c} 0 \\ (P + P_\Sigma) \nabla z(x, t) \end{array} \right] \, dx.
\end{align*}

Moreover, (15) guarantees that for every $k = 1, \ldots, N$, there exists a measurable function $\Phi_k : \Omega_k \rightarrow \mathbb{R}^{n \times n}$ such that for a.e. $x \in \Omega_k$
\begin{align*}
B(x) - B_k(x) &= \rho_k \Phi_k(x) \\ \|\Phi_k(x)\| &\leq 1.
\end{align*}

It follows from the condition 1) in Definition 2.2 that for a.e. $t \in [0, T]$, the Lyapunov function $V(t)$ satisfies
\begin{equation}
\frac{dV}{dt}(t) = \sum_{k=1}^N \int_{\Omega_k} \left[ \begin{array}{c} z(x, t) \\ \nabla z(x, t) \end{array} \right]^T M_k(x) \left[ \begin{array}{c} z(x, t) \\ \nabla z(x, t) \end{array} \right] \, dx.
\end{equation}

Here, we defined $M_k : \Omega_k \rightarrow \mathbb{R}^{(m+1)n \times (m+1)n}$ by
\[ M_k(x) := \begin{bmatrix} M_k^{(1)}(x) & M_k^{(2)}(x) \\ * & M_k^{(3)}(x) \end{bmatrix} \]
with
\begin{align*}
M_k^{(1)}(x) &:= \left( B_k(x) + \rho_k \Phi_k(x) \right)^T P_k(x) \\
&\quad + P_k(x) \left( B_k(x) + \rho_k \Phi_k(x) \right) \\
M_k^{(2)}(x) &:= -\sum_{r=0}^m (P_{p(k,r)} A) \otimes v^r_{p(k,r)} \\
M_k^{(3)}(x) &:= - (A^T P_k(x) + P_k(x) A) \otimes I_m.
\end{align*}

3) Let $\epsilon > 0$. For every $k = 1, \ldots, N$, define $G_k : \Omega_k \rightarrow \mathbb{R}^{(m+2)n \times (m+2)n}$ by
\[ G_k(x) := \begin{bmatrix} G_k^{(1)}(x) & G_k^{(2)}(x) \\ * & G_k^{(3)}(x) \end{bmatrix} \]
where
\begin{align*}
G_k^{(1)}(x) &= \Lambda - (\sigma_k(x) + \epsilon) I_n \\
&\quad - B_k(x)^T P_k(x) - P_k(x) B_k(x) \\
G_k^{(2)}(x) &= \left[ \rho_k P_k(x) \sum_{r=0}^m (P_{p(k,r)} A) \otimes v^r_{p(k,r)} \right] \\
G_k^{(3)}(x) &= \left[ \sigma_k(x) I_n \right. \\
&\quad \left. * \left( (A^T P_k(x) + P_k(x) A - c^2 \Lambda) \otimes I_m \right) \right].
\end{align*}

We now show that if the LMIs (A) are feasible for all $k = 1, \ldots, N$ and for all $\ell = 0, \ldots, m$, then there exists $\epsilon > 0$ such that $G_k \geq 0$ for every $x \in \Omega_k$ and every $k = 1, \ldots, N$. Define
\begin{align*}
\Theta_{k,\ell}^{(1)}(x) &= \Lambda - (\sigma_k(x) + \epsilon) I_n - B_{p(k,\ell)} \Psi_k - \Psi_k^T B_{p(k,\ell)} \\
&\quad - \left[ \rho_k P_{p(k,\ell)} \sum_{r=0}^m (P_{p(k,r)} A) \otimes v^r_{p(k,r)} \right] \\
\Theta_{k,\ell}^{(2)}(x) &= \left[ \sigma_k(x) I_n \right. \\
&\quad \left. * \left( (A^T P_{p(k,\ell)} + P_{p(k,\ell)} A - c^2 \Lambda) \otimes I_m \right) \right].
\end{align*}

For every $x \in \Omega_k$ and every $k = 1, \ldots, N$, we obtain
\begin{align*}
\Theta_{k,\ell}^{(1)}(x) &= \Lambda - (\sigma_k(x) + \epsilon) I_n - B_k(x)^T \Psi_k - \Psi_k^T B_k(x) \\
\Theta_{k,\ell}^{(2)}(x) &= \left[ \sigma_k(x) I_n \right. \\
&\quad \left. * \left( (A^T P_k(x) + P_k(x) A - c^2 \Lambda) \otimes I_m \right) \right].
\end{align*}

If the LMIs (A) are feasible for all $k = 1, \ldots, N$ and for all $\ell = 0, \ldots, m$, then there exists $\epsilon > 0$ such that
\[ \left[ \Theta_{k,\ell}^{(1)}(x) \quad \Theta_{k,\ell}^{(2)}(x) \right] \geq 0 \quad \forall k = 1, \ldots, N, \quad \forall \ell = 0, \ldots, m. \]
for all \( x \in \Omega_k \) and for all \( k = 1, \ldots, N \). Since \( \Theta_i^{(3)}(x) > 0 \) for all \( x \in \Omega_k \), provided that the inequalities (A) hold, the Schur complement formula shows that

\[
\begin{bmatrix}
\Theta_k^{(1)}(x) & \Theta_k^{(2)}(x) \\
\ast & \Theta_k^{(3)}(x)
\end{bmatrix} \geq 0
\]

for all \( x \in \Omega_k \) and for all \( k = 1, \ldots, N \), where \( R_k(x) \) is defined by

\[
R_k(x) := Q_k(x) - B_k(x) \top \eta_k - \eta_k \top B_k(x)
\]

with

\[
Q_k(x) := \Lambda - (\sigma_k(x) + \epsilon) I_n - G_k^{(2)}(x) G_k^{(1)}(x)^{-1} G_k^{(2)}(x)^\top.
\]

Applying Lemma 4.3 to the inequality (29), we obtain

\[
Q_k(x) - B_k(x) \top P_k(x) - P_k(x) B_k(x) \succeq 0
\]

for all \( x \in \Omega_k \) and for all \( k = 1, \ldots, N \). Using the Schur complement formula again, we derive \( G_i(x) \succeq 0 \) for every \( x \in \Omega_k \) and every \( k = 1, \ldots, N \).

4) By (27b), \( I - \Phi_k(x)^\top \Phi_k(x) \succeq 0 \) for a.e. \( x \in \Omega_k \) and every \( k = 1, \ldots, N \). Since \( G_k(x) \succeq 0 \), it follows that

\[
0 \preceq \begin{bmatrix}
I_n & 0 \\
0 & I_n
\end{bmatrix} \succeq \begin{bmatrix}
\Phi_k(x)^\top \\
-\Phi_k(x)
\end{bmatrix}
\]

for a.e. \( x \in \Omega_k \) and for all \( k = 1, \ldots, N \). Applying Corollary 2.6, we obtain \( \frac{d}{dt} \rho_k \leq -\| \mathbf{z}(t) \|_{L^2}^2 \) by (28). Thus, the coupled parabolic system is exponentially stable from the same argument in 3) of the proof of Theorem 3.3.

**Remark 4.7 (Complexity of LMI s in Theorem 4.6):** In Theorem 4.6, the total number \( N_0 \) of the vertices satisfies \( N_0 \leq (m + 1)N \). Therefore, there are \( O(n^2N_0) = O(n^2mN) \) variables in the matrices \( P_1, \ldots, P_{N_0} \). The number of variables in the diagonal matrix \( \Lambda \) is \( O(n) \), and the number of scalar variables \( \sigma_{k,\ell} \) (\( k = 1, \ldots, N \), \( \ell = 0, \ldots, m \)) is \( O(mN) \). Hence, the LMIs of Theorem 4.6 contain \( O(n^2mN) \) variables in total. If \( N = 2^n \), then the number of variables satisfies \( O(n^22^{mn}) \). Let us next consider the case where all the intersections of the \( m \)-simplices are their facets, i.e., \((m-1)\)-simplices. Then, \( N_0 \leq m + N \), and hence the LMIs of Theorem 4.6 has \( O(n^2mN) \) variables. If \( N = 2^n \), then the worst-case number is given by \( O((n^2 + m)2^m) \).

### V. Examples

#### A. 1-D Case

Let \( b \geq 0 \) and consider the coupled 1-D parabolic system (1) with \( \Omega = (0, 1) \) and

\[
A = \begin{bmatrix}
1 & 0.1 \\
0.5 & 1
\end{bmatrix}, \quad B(x) = \begin{bmatrix}
\sin(2\pi x) & \tan(x) \\
\cos(\pi x) & 2x
\end{bmatrix} + bI_2.
\]

Since the coefficient matrix \( B(x) \) is not polynomial, the techniques developed in the previous studies [2]–[5], [8] cannot be applied to this system. To use the obtained results, we divide \( \Omega \) into \( N = 100 \) intervals

\[
\Omega_k := \left( \frac{k-1}{N}, \frac{k}{N} \right], \quad \text{if } k = 1, \ldots, N - 1
\]

and

\[
\left( \frac{N-1}{N}, 1 \right), \quad \text{if } k = N.
\]

The constant \( B_{\ell}(x) \) in (5) for Theorem 3.3 is chosen as

\[
B_{\ell}(x) = B \left( \frac{2k - 1}{2N} \right) \quad \forall k = 1, \ldots, N
\]

which is the value of \( B \) at the center of the interval \( \Omega_k \). Since the vertices of \( \Omega_k \) are \( \xi_{p(k,0)} = (k-1)/N \) and \( \xi_{p(k,1)} = k/N \), the constant \( B_{p(k,\ell)}(x) \) in (15) is given by

\[
B_{p(k,\ell)}(x) = B \left( \frac{k-1}{N} \right), \quad B_{p(k,\ell)}(x) = B \left( \frac{k}{N} \right) \quad \forall k = 1, \ldots, N.
\]

We numerically compute the bound \( \rho_k \) in (5) for Theorem 3.3 based on the following approximation:

\[
\rho_k = \max_{x \in \Omega_k} \| B(x) - B_k \| \approx \max_{x \in \Omega_k} \| B(x) - B_k \| \quad \forall k = 1, \ldots, N
\]

where

\[
\Omega_k^a := \left( \frac{k-1}{N}, \frac{k-1}{N} + \frac{1}{20N}, \ldots, \frac{k-1}{N} + \frac{2}{20N}, \ldots, \frac{k}{N} \right)
\]

The bound \( \rho_{k} \) in (15) for Theorem 4.6 is computed in the same brute force way. The constants \( c \) in Poincaré–Friedrichs’ inequality (3) and \( v_{p(k,\ell)}(x) \) in the LMI (A) are given by \( c = 1/\pi \), \( v_{p(k,0)} = -N \), and \( v_{p(k,1)} = N(k = 1, \ldots, N) \), respectively. Using finite differences with 1000 uniformly distributed spatial points, we find that the approximated parabolic PDE is stable if \( b \leq 8.35 \). The LMIs in Theorems 3.3 and 4.6 are feasible for \( b \leq 6.66 \) and \( b \leq 6.84 \), respectively. From this example, we observe the effectiveness of Lyapunov functions that depend on the space variable in a piecewise linear fashion.

#### B. 3-D Case

Next, we illustrate the advantage of Lyapunov functions with constant \( P \), which allow us to analyze the stability of parabolic PDEs on a general set \( \Omega \). We consider the coupled 3-D parabolic system (1) with the unit ball \( \Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < 1\} \) and the coefficient matrices \( A \) in (30) and

\[
B(x_1, x_2, x_3) = 2 \begin{bmatrix}
\sin(2\pi (x_1 + x_2)) & \tan(x_3) \\
\cos(\pi (x_2 + x_3)) & 2x_1
\end{bmatrix} + bI_2.
\]

where \( b \geq 0 \). The previous studies [2], [8], [9] for multidimensional PDEs focus on the case where \( \Omega \) is a box. Although balls are also basic sets, relatively little work has been done on stability analysis for parabolic PDEs on balls. Using the fact on the Rayleigh quotient for the Laplace operator (see, e.g., [18, Sec. 6.5.1, Th. 2]), we choose the constant \( c \) in Poincaré–Friedrichs’ inequality (3) as \( c = 1/\pi \). We divide \( \Omega = \{(r, \theta, \phi) : r \in [0, 1], \theta \in [0, \pi], \phi \in [0, 2\pi]\} \)}
by uniformly splitting the intervals $[0, 1)$, $[0, \pi]$, and $[0, 2\pi)$ into $N \in \{5, 10, 15, 20, 25, 30\}$ segments, respectively. As in the 1-D case mentioned above, the constant $D_k$ in (5) is set to the value of $B$ at the center of each segment, and the bound $\rho_k$ in (5) is numerically computed with a sufficiently fine grid. Table I describes the maximum $b \geq 0$ for which the LMIs in Theorem 3.3 are feasible. This table shows that a large $N$ is required to obtain less conservative results.

### VI. Conclusion

We have studied the stability analysis of coupled parabolic systems with spatially varying coefficients. Employing the gridding method developed for systems with aperiodic sampling and time-varying delays, we have obtained LMI-based sufficient conditions for exponential stability. Future work will focus on extending this gridding method to various classes of distributed parameter systems. Another interesting direction for future research would be to make stability analysis more accurate by using integration operators with kernels for Lyapunov functions as in [4] and [5]. If $B$ is a polynomial and $\Omega$ is a convex polytope, then sum-of-squares-based analysis through Pólya’s theorem and Handelman representations is expected to be a less conservative approach.

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