The role of media coverage on the dynamical behavior of smoking model with and without spatial diffusion

Ahmed Ali Mohsen¹ and Raid Kamel Naji²

¹Department of Mathematics, Ibn-Al-Haitham College of Education, Baghdad University. E-mail: aamuhsseen@gmail.com;
²Department of mathematics, College of science, University of Baghdad. rknaji@scbaghdad.edu.iq

Abstract: The spread of epidemic diseases still a major threat to the life of communities. Therefore, with the great development of the technology, the spread of diseases can be reduced by using media coverage awareness. In this paper a smoking model incorporating media coverage for warranting the population is proposed and studied. The dynamics of the model is investigated in two different cases: nonexistence and existence of diffusion. The existence, positivity and bounded-ness of solutions are investigated. The local and global stability by the help of Lyapunov function of all possible equilibrium points are investigated. Moreover, numerical simulations are carried out to validate the analytical results and specify the effect of varying the parameters.

Keywords: Smoking model, media, diffusion, stability.

1. Introduction

The smoke from the Cigarette is a very complex chemical mixture that is dangerous to human health and all the elements of the environment. It contains more than 3,800 toxic chemicals, the most important of which is the carbon monoxide (Co), which is one of the poisonous and dangerous gases on human life, ammonia (NH₃), Hydrogen sulfide (H₂S), formaldehyde (HCHO), Acetaldehyde (CH₂CHO), hydrogen cyanide (HCN), in addition to a large number of acids including: Carbonic acid (H₂CO₃), nitric acid (HNO₃), acetic acid (CH₃COOH) and formic acid (HCOOH), see [1].

Cigarette smoke also carries a huge range of organic compounds, which have proved dangerous, classified globally as highly dangerous. These substances include benzopyrene, which works to destroy the mucous membranes of the respiratory tract of smokers, and also destroys the airways of smokers. In one of the statistics from 2013, the number of premature deaths due to smoking to 5950 deaths, as well as 200,000 cases of hospitalization. And there are many diseases caused by smoking such as 44% Cancer, 30% Circulatory diseases, 25% Respiratory diseases and other [2-3]. All these reasons have invited many authors to understand and study the smoking epidemic for example: In [4], Castillo-Garsow et al. suggested the tobacco model with recovery. Lahrouz, et al [5] proposed and studied mathematical model of smoking. Al-Shareef and Batarfi studied the effect of chain, mild and passive smoke see [6]. In [7], Sharomi and Gumel provided a rigorous mathematical study for assessing the dynamics of smoking and their impact on public health in a community. Zaman, studied the smoking dynamics with control strategy, he discussed qualitative behavior of tobacco model [8, 9]. Erturk and Momani [10] proposed analytic method for approximating a giving up smoking model. Zainab et al [11] studied global dynamics of a mathematical model on Smoking. Moreover many researchers proposed and studied models showed how the media effect of the spread of the diseases for example: Misra et al [12] studied the effects of awareness programs by media on the spread of infectious diseases. Smith et al [13] investigated the impact of media coverage on the influenza disease. Cui and Zhu [14] studied the impact of media on control of infectious disease. On the other hand, it is well known that location play a critical role in disease dynamics see for example [15-19]. In this work, we proposed and studied a mathematical model describing the effect of awareness through media program on the spread of smoking. Further, the effect of location on outbreak the smoking in
the population is also considered through studying the model with reaction diffusion. Finally, local as well as global stability analysis of the proposed model are also investigated.

2. Construction of the model

The mathematical model offers us more understanding about spreading the infection disease, we know that the disease is transmitted by direct contact between healthy individuals with infected individuals. In fact, outbreak the smoking is very similar to the spread of epidemic and hence some populations start smoking due to contact with smokers. Consider a population of size \( N \) at time \( t \). It is assumed that, the population divided into four classes: the 1st class consisting of individuals who do not smoke tobacco and maybe become smokers in future (potential smokers) and the size of individuals at time \( t \) for this class denoted by \( P(t) \); 2nd class involving the smoker individuals and denoted their size at time \( t \) by \( S(t) \); \( Q(t) \) represents the size of individuals at time \( t \) in the 3rd class that contains individuals who temporarily quit smoking; \( R(t) \) stands for the size of individuals at time \( t \) in the 4th class, which contains the recovery from smoking. On the other hand, the efficiency of awareness by media coverage to reducing the number of smokers (or smoking prevention) at time \( t \) will be denoted by \( M(t) \). Accordingly, the dynamics of smoking model with the effect of awareness by media coverage to outbreak the smoking can be describe by the following system of nonlinear ODEs.

\[
\begin{align*}
\dot{P} &= \psi - \beta PS - \mu P - \gamma PM \\
\dot{S} &= \beta PS + \sigma \gamma PM - \mu S - \gamma SM + \epsilon \delta Q \\
\dot{Q} &= \epsilon \gamma SM - \mu Q - \delta Q \\
\dot{R} &= \gamma(1-\sigma)PM + \gamma(1-\epsilon)SM - \mu R + \delta(1-\epsilon)Q \\
\dot{M} &= \alpha(S + P) - \theta M
\end{align*}
\]

As the fourth equation is a linear differential equation with respect to variable \( R(t) \), which is not appear in the other equations of system (1), hence system (1) can be reduced to the following system:

\[
\begin{align*}
\dot{P} &= \psi - \beta PS - \mu P - \gamma PM \\
\dot{S} &= \beta PS + \sigma \gamma PM - \mu S - \gamma SM + \epsilon \delta Q \\
\dot{Q} &= \epsilon \gamma SM - \mu Q - \delta Q \\
\dot{M} &= \alpha(S + P) - \theta M
\end{align*}
\]

with initial condition \( P(0) > 0, S(0) \geq 0, Q(0) \geq 0 \) and \( M(1) > 0 \). Therefore, by solving system (2) and substituting the solution, say \( (P^*, S^*, Q^*, M^*) \), it in the fourth equation of system (1) and solving the obtained linear differential equation we get for \( t \to \infty \) that:

\[
R = \frac{\gamma((1-\sigma)P^* + (1-\epsilon)S^*)M^* + \delta(1-\epsilon)Q^*}{\mu}
\]

Moreover, all the parameters are assumed to be nonnegative with, \( \psi > 0 \) represents the recruitment of potential smokers population, \( \mu > 0 \) represents the natural death rate of the human populations. The parameter \( \beta > 0 \) is the contact rate between potential smokers and smokers. On other hand, the awareness level through media coverage that reached to the individuals is denoted by \( \gamma > 0 \), however portion of individuals who received awareness transfers to smoker class and temporarily quit smoking class with rates \( (0 \leq \sigma \leq 1) \) and \( (0 \leq \epsilon \leq 1) \) respectively. The parameter \( \delta > 0 \) represents the rate of losing the temporary quitters smoking individuals, in fact fraction of them with rate \( (0 \leq \epsilon \leq 1) \) transfers to smoker’s class while the rest of individuals will transfer to recovery from smoking class. The parameter \( \alpha > 0 \) represents media campaigns rate performed by both smokers and nonsmokers, however the rate of disappearance of media coverage represented by \( \theta > 0 \).
Keeping the above description of variables and parameters, it is easy to prove that system (1), and hence system (2), is defined on the following positively invariant set:

$$\Gamma = \{(P, Q, S, R, M) \in \mathbb{R}^5_+: 0 \leq N \leq \frac{\psi}{\mu}, 0 \leq M \leq \frac{\alpha \gamma \phi}{\theta \mu}\}$$

where $N = P + S + Q + R$.

3. The existence of equilibrium points of system (2)

In this section, the existence conditions of all possible equilibrium points are determine. It is easy to shows that system (2) has three equilibrium points. The points and their existence conditions can be described as following:

- In the absence of smokers, that is $S = 0$. Then, system (2) has a unique positive equilibrium point in the interior of positive quadrant of $PM$-plane, namely smoking free equilibrium point (SFEP), which denoted by $E_0 = (P_0, 0, 0, M_0)$ where

$$F_0 = \frac{\psi_0}{2\gamma \theta_0}$$

$$M_0 = \frac{\psi_0 + \theta_0 - \theta_0 \psi_0 + \theta_0 \psi_0}{2\gamma \theta_0}$$

provided that the following condition holds

$$\sigma = 0$$

- In the absence of temporarily quit smokers ($Q = 0$). Hence, system (2) has an equilibrium point is the interior of positive octant of $PSM$-space, namely free temporarily quit smoking equilibrium point (FTQSEP), which denoted by $E_1 = (P_1, S_1, 0, M_1)$ where

$$M_1 = \frac{\alpha \sigma (S_1 + P_1)}{\sigma}$$

while $(P_1, S_1)$ is a positive root to the following two isoclines:

$$f(P, S) = \theta \psi - \theta \psi - \theta \psi + \alpha \gamma \psi S \psi - \theta \psi - \alpha \gamma \psi = 0$$

$$g(P, S) = (\theta \psi + \alpha \gamma (\sigma - 1))\psi S + \alpha \gamma \psi^2 - \theta \psi - \alpha \gamma S = 0$$

Clearly, as $S \to 0$, the two isoclines reduced to:

$$\theta \psi - \theta \psi - \alpha \gamma \psi^2 = 0$$

$$\alpha \gamma \psi^2 = 0$$

- The coexistence equilibrium point or endemic equilibrium point (EEP), which denoted by $E_2 = (P_2, S_2, Q_2, M_2)$ where

$$M_2 = \frac{\alpha \sigma (S_2 + P_2)}{\sigma}$$

while $(P_2, S_2)$ represents a positive intersection point of the two isoclines $f(S, P) = 0$, which is given by Eq. (5b), while the other isocline is given by

$$g_1(S, P) = [(\theta \psi + \alpha \gamma (\sigma - 1))\psi S + \alpha \gamma \psi^2 - \theta \psi - \alpha \gamma S^2] (\mu + \delta) + \alpha \gamma \psi \delta S (S + P) = 0$$

Clearly, as $S \to 0$, the last two isocline reduced to the same polynomial equation given in Eq. (5d) and (5e). Hence they have the same nonnegative roots fall on the $P$-axis. Accordingly, $(P_2, S_2)$ exists uniquely in the interior of positive quadrant $PS$-plane provided that
Hence the EEP exists uniquely in the interior of positive octant of $PSM$ space under the condition (7b).

4. Stability analysis of system (2)

In this section, the stability analysis of all equilibrium points of system (2) is studied. The Jacobian matrix of system (2) at $E_0$ can be written in the following form.

$$J(E_0) = \begin{pmatrix}
0 & -\gamma P_0 & 0 & 0 \\
0 & -\beta P_0 & 0 & 0 \\
0 & 0 & -\mu & -\delta \\
0 & 0 & -\mu & 0
\end{pmatrix}$$

where

- $a_{11} = -\beta S - \mu - \gamma M, a_{12} = -\beta P, a_{13} = 0, a_{14} = -\gamma P$
- $a_{21} = \beta S + \gamma M, a_{22} = \beta P - \mu - \gamma M, a_{23} = 0, a_{24} = \gamma P - \gamma S$
- $a_{31} = 0, a_{32} = \gamma M, a_{33} = -\mu - \delta, a_{34} = \gamma S$
- $a_{41} = 0, a_{42} = 0, a_{43} = 0, a_{44} = 0$

Consequently, the local stability of SFEP is investigated in the following theorem.

Theorem 1: The SFEP of system (2) is locally asymptotically stable (LAS) if the following sufficient conditions hold

$$0 < \mu + \gamma M_0$$

$$0 < (\mu + \delta)P_0 + \gamma \mu \delta M_0$$

Proof: The Jacobian matrix of system (2) at $E_0$ can be written as:

$$J(E_0) = \begin{pmatrix}
0 & -\beta P_0 & 0 & 0 \\
0 & -\gamma P_0 & 0 & 0 \\
0 & 0 & -\mu & -\delta \\
0 & 0 & -\mu & 0
\end{pmatrix}$$

Hence, the characteristic equation can be written as

$$\lambda^4 + B_1 \lambda^3 + B_2 \lambda^2 + B_3 \lambda + B_4 = 0$$

Such that

- $B_1 = -[B_{11} + b_{22} + b_{33} + b_{44}]$
- $B_2 = b_{11} (b_{22} + b_{33}) + b_{12} (b_{23} + b_{32}) + b_{13} b_{23} + b_{44} (b_{22} + b_{33})$
- $B_3 = -(b_{11} + b_{44}) (b_{23} + b_{32}) + (b_{22} + b_{33})(b_{11} b_{44} - b_{13} b_{24})$
- $B_4 = (b_{22} b_{33} - b_{23} b_{32}) (b_{11} b_{44} - b_{13} b_{24})$

while by using some algebraic computation we obtain that

$$B_1 = b_{11} (b_{22} + b_{33}) + b_{12} (b_{23} + b_{32}) + b_{13} b_{23} + b_{44} (b_{22} + b_{33})$$

and

$$B_2 = -[b_{11} b_{44} (b_{22} + b_{33}) (b_{23} + b_{32}) - (b_{11} + b_{44}) (b_{23} + b_{32}) (b_{22} + b_{33})]$$

$$B_3 = -(b_{11} b_{44}) (b_{23} + b_{32}) (b_{22} + b_{33}) - (b_{22} + b_{33}) (b_{11} b_{44} - b_{13} b_{24})$$

$$B_4 = (b_{22} b_{33} - b_{23} b_{32}) (b_{11} b_{44} - b_{13} b_{24})$$

However $\Delta = B_1 (B_2 - B_3) - B_2 (B_1 - B_4)$ can be written as:

$$\Delta = -B_1 [B_{11} + b_{44}] (b_{23} + b_{32}) (b_{22} + b_{33}) - B_1 b_{11} b_{44} (b_{22} + b_{33})$$

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of $J(E_0)$ have negative real parts and then the SFEP of system (2) is locally asymptotically stable provided that $B_1 > 0$ for $i = 1, 2, 3, 4; B_1 B_2 - B_3 > 0$ and $\Delta > 0$. It is easy to verify that condition (11a) guarantees that the term $b_{22} b_{33} - b_{23} b_{32} > 0$ Hence due to the sign of matrix elements and the sufficient conditions (11a) and (11b) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.
**Theorem 2:** The FTQSEP of system (2) is LAS if the following sufficient conditions hold
\begin{align}
\beta P_1 < \mu + \gamma M_2 \\
\sigma P_1 < \xi \\
\alpha \gamma S_1 < \alpha \gamma P_1 + \theta \delta S_1 + \theta \gamma M_1 \\
\alpha \gamma P_1 [\beta P_1 + \gamma M_1] < 2 \theta \delta S_1 (\mu + \gamma M_1) - \beta F_1
\end{align}

**Proof:** The Jacobian matrix of system (2) at $E_1$ can be written:
\begin{equation}
J(E_1) = (c_{ij})_{4 \times 4} =
\begin{pmatrix}
\beta P_1 & 0 & -\gamma F_1 \\
0 & \beta P_1 - (\mu + \gamma M_1) & \alpha \\
\alpha & \alpha & 0 \\
0 & 0 & -\theta
\end{pmatrix}
\end{equation}

Hence, the characteristic equation can be written as
\begin{equation}
(c_{32} - \lambda)(\lambda^3 + C_1 \lambda^2 + C_2 \lambda + C_3) = 0
\end{equation}

where the eigenvalue in the $G$ direction is given by $\lambda_G = - (\mu + \delta) < 0$, while $C_1 = -\{c_{11} + c_{22} + c_{44}\}$, $C_2 = \{c_{11}c_{22} - c_{12}c_{21} + c_{14}c_{44} - c_{14}c_{41} + c_{22}c_{44} - c_{24}c_{42}\}$, $C_3 = -\{c_{11}c_{22}c_{44} - c_{12}c_{21}c_{44} + c_{14}c_{24}c_{42} + c_{16}c_{24}c_{42} - c_{24}c_{42}\}$

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of $J(E_1)$ have negative real parts and then the FTQSEP of system (2) is locally asymptotically stable provided that $C_i > 0$ for $i = 1, 3$ and $C_1C_2 - C_3 > 0$.

It is easy to verify that condition (14a) guarantees that the element $c_{22}$ is negative and condition (14b) guarantees that the element $c_{24}$ is negative, while condition (14c) guarantees that the term $c_{24}c_{42} - c_{22}c_{44} > 0$. On the other hand condition (14d) ensure that $-2c_{11}c_{22}c_{44} + c_{12}c_{24}c_{42} + c_{14}c_{24}c_{42} > 0$. Hence due to the sign of matrix elements and the sufficient conditions (14a)-(14d) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.

**Theorem 3:** The EEP of system (2) is LAS if the following sufficient conditions hold
\begin{align}
\beta P_1 < \mu + \gamma M_2 \\
\sigma P_1 < \xi \\
\beta (\mu + \delta) P_2 + \epsilon \delta c M_2 < (\mu + \gamma M_2) (\mu + \delta) \\
P_2 (\gamma S_2 + \sigma \gamma M_2) + \epsilon \delta S_2 < \gamma P_2 (\beta S_1 + \mu + \gamma M_2 + \theta)
\end{align}

**Proof:** The Jacobian matrix of system (2) at $E_2$ is written as
\begin{equation}
J(E_2) = (z_{ij})_{4 \times 4}
\end{equation}

where $z_{ij} = a_{ij} (P_j, \gamma S_2, Q_0, M_2)$. For $i, j = 1, 2, 3, 4$. Hence, the characteristic equation can be written as
\begin{equation}
\lambda^4 + Z_1 \lambda^3 + Z_2 \lambda^2 + Z_3 \lambda + Z_4 = 0
\end{equation}

where
\begin{align}
Z_1 &= -[Z_{11} + Z_{22} + Z_{33} + Z_{44}] \\
Z_2 &= Z_{11}Z_{22} - Z_{12}Z_{21} + Z_{13}Z_{33} + Z_{14}Z_{44} - Z_{14}Z_{41} + Z_{22}Z_{33} - Z_{23}Z_{32} - Z_{23}Z_{32} \\
Z_3 &= -[(Z_{11} + Z_{44})(Z_{22}Z_{33} - Z_{23}Z_{32}) + (Z_{12} + Z_{33})(Z_{14}Z_{44} - Z_{14}Z_{41})] \\
Z_4 &= (Z_{14}Z_{44} - Z_{14}Z_{41})(Z_{22}Z_{33} - Z_{23}Z_{32}) + (Z_{11}Z_{44} - Z_{12}Z_{21})(Z_{22}Z_{33} - Z_{23}Z_{32})
\end{align}

Moreover, we have that
\[ Z_1 Z_2 - Z_3 = -(z_{11} + z_{22})(z_{11} z_{22} - z_{14} z_{21}) - z_{11} z_{33}(z_{11} + z_{22} + z_{13}) \\
- z_{14} z_{14}(z_{11} + z_{22} + z_{13} + z_{44})(z_{11} + z_{22} + z_{13} + z_{44}) + z_{14} z_{24} z_{41} \\
- (z_{12} + z_{13})(z_{12} z_{33} - z_{23} z_{12}) + z_{14} z_{14}(z_{11} + z_{44}) \\
+ z_{14} z_{41}(z_{12} + z_{44}) + z_{14} z_{14} z_{21} - z_{23} z_{34} \]

and

\[ \Delta = Z_3 (Z_1 Z_2 - Z_3) - Z_1 Z_4 \] can be written as:

\[ \Delta = X_1 (X_2 + X_1) - X_3 \]

where

\[ X_1 = (z_{11} + z_{44})(z_{22} z_{33} - z_{23} z_{12}) + (z_{22} + z_{33})(z_{11} z_{44} - z_{14} z_{41}) \\
- z_{12} z_{21}(z_{33} + z_{44}) + z_{22} z_{12}(z_{11} + z_{33}) + z_{22} z_{12}(z_{44}) - z_{12} z_{21} - z_{14} z_{41} \\
+ z_{14} z_{24}(z_{22} + z_{33}) - z_{14} z_{13} z_{33} + z_{23} z_{32} \]

\[ X_2 = (z_{21} + z_{22} + z_{44})(z_{11} z_{33} + z_{44} + z_{14} z_{21}) - z_{12} z_{21} - z_{14} z_{41} \]

\[ X_3 = z_{11} + z_{32} + z_{33} - z_{22} \]

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of \( J(F_2) \) have negative real parts and then the EEP of system (2) is locally asymptotically stable provided that \( z_i > 0 \) for \( i = 1, 2, 3; Z_1 - Z_3 > 0 \) and \( \Delta > 0 \).

It is easy to verify that conditions (17a) and (17b) guarantees that the elements \( z_{22} \) and \( z_{24} \) are negative respectively and condition (17c) guarantees that the term \( z_{22} z_{33} - z_{23} z_{32} > 0 \) while the term \( z_{14} z_{24} (z_{11} + z_{44}) - z_{14} z_{13} z_{33} > 0 \) if the conditions (17c) holds. Hence due to the sign of matrix elements and the sufficient conditions (17a) and (17d) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.

It is well known that, for each equilibrium point there is a specific basin of attraction and the point will be a globally asymptotically stable if and only if their basin of attraction is the total domain. Therefore, in the following theorems, the basin of attraction or the global stability conditions of each point is determined.

**Theorem 4:** Assume that the SFEP is LAS. Then it has a basin of attraction that satisfies the following conditions

\[ \left( \frac{\alpha P - \beta P_0}{PM} \right)^2 \leq 4 \left( \frac{\mu + \frac{M}{P} \gamma}{P} \right) \left( \frac{\theta}{M} \right) \]  

(20a)

\[ (\alpha + \beta P_0) < \mu \]  

(20b)

**Proof:** Consider the following positive definite Lyapunov function, which is defined for all \( P > 0 \) and \( M > 0 \) in the domain of system (2),

\[ V_1 = \left( P - P_0 - P_0 \ln \frac{P}{P_0} \right) + S + Q \left( M - M_0 - M_0 \ln \frac{M}{M_0} \right) \]

Clearly, by differentiating \( V_1 \) with respect to \( t \) along the solution curve of system (2), it’s obtaining that:

\[ V_1' = -\left( \frac{\alpha P - \beta P_0}{P} \right)^2 (P - P_0)^2 + \left( \frac{\alpha P - \beta P_0}{PM} \right) (P - P_0)(M - M_0) \]

\[ - \left( \mu + (1 - \epsilon) \delta \right) Q - \gamma (1 - \epsilon) y S M \]

\[ - \left( \mu + (1 - \epsilon) \delta \right) S \]

\[ - \left( \mu + (1 - \epsilon) \delta \right) S - \frac{M_0}{M} S \]

Therefore by using the above conditions, it’s observed that

\[ V_1' < \left[ \frac{\alpha P - \beta P_0}{P} - \left( \frac{\mu + (1 - \epsilon) \delta}{\sqrt{M - M_0}} \right)^2 \right] (P - P_0)^2 - \left( \mu + (1 - \epsilon) \delta \right) Q - \gamma (1 - \epsilon) y S M \]

\[ - \left( \mu + (1 - \epsilon) \delta \right) \frac{M_0}{M} S \]

Obviously, \( V_1' = 0 \) at \( F_0 = (P_0, 0, 0, M_0) \), moreover \( V_1' < 0 \) otherwise. Hence \( V_1' \) is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to SFEP. Hence the proof is complete.
Theorem 5: Assume that the \( \text{FTQSEP} \) is LAS. Then it has a basin of attraction that satisfies the following conditions

\[
\begin{align*}
\beta P_1 &< \gamma M_1 + \mu \quad \text{(21a)} \\
\left( \frac{\alpha \gamma M}{S} \right)^2 &< \left( \frac{\beta^2 + \beta P_1 + M}{\mu} \right) (\gamma M + \mu - \beta P_1) \\
\left( \frac{\alpha}{M} \right)^2 &< \left( \frac{\beta \gamma M}{S} + \mu \right) (\gamma M + \mu - \beta P_1) \\
\left( \frac{\alpha}{M} \right)^2 &< \left( \frac{\gamma M + \mu - \beta P_1}{S} \right) (\gamma M + \mu - \beta P_1) \\
(21d)
\end{align*}
\]

Proof: Consider the following positive definite Lyapunov function, which is defined for all \( P > 0, S > 0 \) and \( M > 0 \) in the domain of system (2)

\[
V_2 = \left( P - P_1 - P_1 \ln \frac{P}{P_1} + (S - S_1 - S_1 \ln \frac{S}{S_1}) + Q + (M - M_1 - M_1 \ln \frac{M}{M_1}) \right)
\]

Clearly, by differentiating with respect to \( t \) along the solution curve of system (2), it’s obtaining that:

\[
\begin{align*}
V_2' &= \left( \frac{\beta S + \beta P + \gamma M}{S} \right) (P - P_1)^2 - \left( \frac{\gamma M + \mu - \beta P_1}{S} \right) (P - P_1) (S - S_1) \\
&\quad - \left( \frac{\beta S + \beta P + \gamma M}{S} \right) (P - P_1) (S - S_1)^2 - \left( \frac{\gamma M + \mu - \beta P_1}{S} \right) (P - P_1) (M - M_1) \\
&\quad + \left( \frac{\alpha}{M} \right) (P - P_1) (M - M_1) - \frac{6}{2M} (M - M_1)^2 \\
&\quad - \left( \frac{\gamma M + \mu - \beta P_1}{S} \right) (S - S_1)^2 + \left( \frac{\alpha}{M} \right) (P - P_1) (M - M_1) \\
&\quad - \frac{\alpha}{2M} (M - M_1)^2 - (\mu + (1 - \delta) \varphi) Q - \frac{\varepsilon \delta S^2}{S}
\end{align*}
\]

Therefore by using the above conditions, it’s observed that

\[
V_2' < -\left[ \frac{x_{11}}{\varepsilon} \right] \left( P - P_1 \right)^2 - \left( \frac{x_{22}}{\varepsilon} \right) (S - S_1)^2 - \left( \frac{x_{44}}{\varepsilon} \right) (M - M_1)^2 - (\mu + (1 - \delta) \varphi) Q - \frac{\varepsilon \delta S^2}{S}
\]

where \( x_{11} = \left( \frac{\beta S + \beta P + \gamma M}{S} \right) \); \( x_{22} = \left( \frac{\gamma M + \mu - \beta P_1}{S} \right) \); \( x_{44} = \frac{\alpha}{2M} \).

Obviously, \( V_2 = 0 \) at \( E_1 = (P_1, S_1, 0, M_1) \), moreover \( V_2' < 0 \) otherwise. Hence \( V_2' \) is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to \( \text{FTQSEP} \). Hence the proof is complete.

Furthermore, in the following theorem the conditions that specify the basin of attraction of \( \text{EEP} \) are established.

Theorem 6: Assume that the \( \text{EEP} \) is LAS. Then it has a basin of attraction that satisfies the following conditions

\[
\begin{align*}
\beta P_2 &< \mu + \gamma M \quad \text{(22a)} \\
(\sigma M + \beta S - \beta P_2)^2 &< \frac{2}{3} (\beta S + \beta P_2 + \mu) (\gamma M + \mu - \beta P_2) \quad \text{(22b)} \\
(\alpha - \gamma P_2)^2 &< \frac{2}{3} \theta (\beta S + \beta P_2 + \mu) \quad \text{(22c)} \\
(\varepsilon \delta + \sigma M)^2 &< \frac{2}{3} (\mu + \delta) (\gamma M + \mu - \beta P_2) \quad \text{(22d)} \\
(\alpha + \sigma M)^2 &< \frac{4}{9} \theta (\gamma M + \mu - \beta P_2) \quad \text{(22e)} \\
(\varepsilon \delta)^2 &< \frac{2}{3} \theta (\mu + \delta) \quad \text{(22f)}
\end{align*}
\]

Proof: Consider the following positive definite Lyapunov function

\[
V_3 = \left( \frac{P - P_2}{2} \right)^2 + \left( \frac{S - S_2}{2} \right)^2 + \left( \frac{Q - Q_2}{2} \right)^2 + \left( \frac{M - M_2}{2} \right)^2
\]
Hence, by differentiating $V_3$ with respect to $t$ along the solution curve of system (2), we get that
\[
V_3' = -\frac{g_1 + g_3}{g_2} (P - P_2)^2 - \frac{g_1 + g_2}{g_3} (S - S_2)^2 - \frac{g_1 + g_3}{g_2} (P - P_2)(S - S_2)
\]
\[
+ (\alpha + \gamma S_2)(S - S_2)^2 + (\delta + \delta M + M S - S_2)(S - S_2)
\]
\[
= -\frac{g_1 + g_3}{g_2} (Q - Q_2)^2 - \frac{g_1 + g_2}{g_3} (S - S_2)^2
\]
\[
+ (\alpha + \gamma S_2)(S - S_2)^2 - \frac{g_1 + g_3}{g_2} (Q - Q_2)^2 + \gamma S_2 (M - M_2)^2 - \frac{g_1 + g_3}{g_2} (M - M_2)^2
\]
Therefore by using the above conditions, it's observed that
\[
V_3' = -\left[\frac{g_1 + g_3}{g_2} (P - P_2)^2 - \frac{g_1 + g_2}{g_3} (S - S_2)^2\right] - \left[\frac{g_1 + g_3}{g_2} (P - P_2)^2 - \frac{g_1 + g_2}{g_3} (M - M_2)^2\right]
\]
\[
= \frac{g_1 + g_3}{g_2} (Q - Q_2)^2 - \frac{g_1 + g_2}{g_3} (S - S_2)^2 - \frac{g_1 + g_3}{g_2} (M - M_2)^2
\]
\[
\frac{g_1 + g_3}{g_2} (Q - Q_2)^2 + \gamma S_2 (M - M_2)^2 - \frac{g_1 + g_3}{g_2} (M - M_2)^2
\]
\[
\text{Here } \quad \frac{g_1 + g_3}{g_2} = \beta S + \beta P_2 + \mu; \quad \frac{g_2}{g_3} = \gamma M + \mu - \beta P_2; \quad \frac{g_1 + g_3}{g_2} = \mu + \delta; \quad \frac{g_1 + g_3}{g_2} = \theta.
\]
Obviously, $V_3' = 0$ at $P_2 = (P_2, S_2, Q_2, M_2)$, moreover $V_3' < 0$ otherwise. Hence $V_3'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to EEP. Hence the proof is complete.

5. Smoking model with diffusion

Obviously, system (1) does not consider the structure of smokers spreading and hence it is not suitable to understand the transmission of smoking in case of moving the individuals. Therefore, it is important to consider the diffusion terms in the model structure in order to investigate whether and how spatial heterogeneity can affect the smoking transmission dynamics. Consequently, the smoking model with diffusion is considered in this section, which is extended to the smoking model given in Eq. (1). Let $\Omega$ is a bounded domain in $\mathbb{R}^n_+$ with smooth boundary $\partial \Omega$ and $\eta$ is the outward unit normal vector on the boundary, then the smoking model with diffusion can be written as:
\[
\frac{\partial P}{\partial t} = \psi - \beta PS - \mu P - \gamma PM + D_1 \Delta P
\]
\[
\frac{\partial S}{\partial t} = \beta PS + \sigma \gamma PM - \mu S - \gamma SM + e \delta Q + D_2 \Delta S
\]
\[
\frac{\partial Q}{\partial t} = e \gamma SM - \mu Q - \delta Q + D_3 \Delta Q
\]
\[
\frac{\partial R}{\partial t} = \gamma (1 - \sigma) PM + \gamma (1 - \epsilon) SM - \mu R + \delta (1 - \epsilon) Q + D_4 \Delta R
\]
\[
\frac{\partial M}{\partial t} = \alpha (S + P) - \theta M + D_5 \Delta M
\]
with homogeneous Neumann boundary condition
\[
\frac{\partial P}{\partial \eta} = \frac{\partial S}{\partial \eta} = \frac{\partial Q}{\partial \eta} = \frac{\partial R}{\partial \eta} = \frac{\partial M}{\partial \eta} = 0, \quad x \in \partial \Omega, \quad t > 0
\]
and initial conditions
\[
\frac{\partial P}{\partial t} = \frac{\partial S}{\partial t} = \frac{\partial Q}{\partial t} = \frac{\partial R}{\partial t} = \frac{\partial M}{\partial t} = 0, \quad x \in \partial \Omega, \quad t = 0
\]
\begin{align}
\frac{\partial P}{\partial t} &= \psi - \beta P - \mu P + \gamma M + D_1 \Delta P \\
\frac{\partial S}{\partial t} &= \beta PS - \mu S - \phi S + \sigma P + D_2 \Delta S \\
\frac{\partial Q}{\partial t} &= e \gamma SM - \mu Q - \delta Q + D_3 \Delta Q \\
\frac{\partial M}{\partial t} &= \alpha (S + P) - \theta M + D_5 \Delta M
\end{align}

So that \( R \) can be determined from
\[
R(x,t) = N - \left[ P(x,t) + S(x,t) + Q(x,t) \right], \quad x \in \Omega, \ t > 0
\]

As the initial values are positive and the growth functions in the interaction functions of system (26) are assumed to be sufficiently smooth in \( \mathbb{R}^+ \), then standard partial differential equations theory shows that the solution of (26) is unique and continuous for all the positive time in \( \Omega \). Furthermore, we recall the positivity lemma in order to using it to proof the positivity and the uniformly bounded of the solution of (26).

**Lemma 7 [17]:** Suppose \( K \in C(\overline{\Omega} \times [0,\tau]) \cap C^{2,1}(\Omega \times (0,\tau]) \) and satisfies
\[
K_t - D \Delta K \geq c(z,t) K, \quad z \in \Omega, \quad 0 < t \leq \tau,
\]
\[
\frac{\partial K}{\partial \eta} \geq 0, \quad z \in \partial \Omega, \quad 0 < t \leq \tau,
\]
\[
K(z,0) \geq 0, \quad z \in \overline{\Omega}
\]

where \( c(z,t) \in C(\overline{\Omega} \times [0,\tau]) \). Then \( K(z,t) \geq 0 \) on \( \overline{\Omega} \times [0,\tau] \). Moreover, \( K(z,t) > 0 \) or \( K \equiv 0 \) in \( \Omega \times [0,\tau] \).

Hence, according to lemma (7), we have the following theorem.

**Theorem 8:** Any solution of system (26) with a positive initial condition is positive.

**Proof:** Assume that \((P, S, Q, M)\) be a solution of system (26) in \( \Omega \times [0,T_{\text{max}}) \). Then for any \( \tau \) with \( 0 < \tau < T_{\text{max}} \), we get from 1st equation of system (26) that:
\[
P_t - D_1 \Delta P \geq - (\beta S + \mu + \gamma M) P, \quad x \in \Omega, \ 0 < t \leq \tau
\]

Since \(- (\beta S + \mu + \gamma M)\) is bounded due to the boundedness of the population in \( \Omega \times [0,\tau] \), then by using the Lemma (7) we obtain \( P > 0 \) in \( \Omega \times (0,\tau] \). By the same way we have \( S > 0 \) in \( \Omega \times [0,\tau] \) since that
\[
S_t - D_2 \Delta S \geq - (\mu + \gamma M) S, \quad x \in \Omega, \ 0 < t \leq \tau
\]

Similarly, we have \( Q > 0 \), due to the following
\[
Q_t - D_3 \Delta Q \geq - (\mu + \delta) Q, \quad x \in \Omega, \ 0 < t \leq \tau
\]

Again we applied the same lemma on last equation of system (26) we obtain that
\[
M_t - D_5 \Delta M \geq \theta M, \quad x \in \Omega, \ 0 < t \leq \tau
\]
Hence, \(M > 0\). Now, since \(\tau\) is arbitrary in \((0, T_{\text{max}})\), we obtain that \(P > 0\), \(S > 0\), \(Q > 0\) and \(M > 0\) in \(\Omega \times [0, T_{\text{max}})\).

Now, we show the boundedness of solution of system (26) and investigate that in following theorem

**Theorem 8:** Let \((P, S, Q) \in [C([0, T_{\text{max}}]]) \cap C^2 \Omega \times [0, T_{\text{max}})]^3\) be the solution of system (26) with non-negative non-trivial initial values. Then \(T = \infty\) and \(P(x, t) + S(x, t) + Q(x, t) \leq \max \{N, \|P_0(x) + S_0(x) + Q_0(x)\|_\infty\}\), where \(N = \frac{\psi}{\mu}\).

**Proof:** We show that \(P(x, t), S(x, t)\) and \(Q(x, t)\) are bounded by \(\Omega \times [0, T_{\text{max}}]\). Since \(0 < P(x, 0) + S(x, 0) + Q(x, 0) \leq \|P_0(x) + S_0(x) + Q_0(x)\|_\infty\) and
\[
(P + S + Q)_t - D\Delta (P + S + Q) \leq \psi - \mu (P + S + Q)
\]
with \(D = \max \{D_1, D_2, D_3\}\), then for \(t \in [0, \infty)\), we have that
\[
P(x, t) + S(x, t) + Q(x, t) \leq \left[\frac{\psi}{\mu} + \left(\|P_0(x) + S_0(x) + Q_0(x)\|_\infty - \frac{\psi}{\mu}\right)e^{-\mu t}\right]
\]
is the solution of the inequalities
\[
\frac{dL(t)}{dt} = \psi - \mu L(t); \quad L(0) = \|P_0(x) + S_0(x) + Q_0(x)\|_\infty
\]
Such that, \(L = (P + S + Q)\), hence, we have
\[
0 < L(t) \leq \max \left\{\frac{\psi}{\mu} \left\|P_0(x) + S_0(x) + Q_0(x)\right\|_\infty\right\}, \text{ for } t \in [0, \infty) \text{ and thus,}
\]
\[
P(x, t) + S(x, t) + Q(x, t) \leq L(x, t) \leq \max \left\{\frac{\psi}{\mu} \left\|P_0(x) + S_0(x) + Q_0(x)\right\|_\infty\right\}
\]
As well, by the same way we have shown that the media equation is bounded by \(\Omega \times [0, T_{\text{max}}]\). Such that, \(M(x, 0) \leq \left\|M_0(X)\right\|_\infty\), then
\[
M_t - D\Delta M \leq \alpha (P + S) - \theta M
\]
We have
\[
M(x, t) \leq \frac{\alpha (P + S)}{\theta} + \left(\left\|M_0(X)\right\|_\infty - \frac{\alpha (P + S)}{\theta}\right)e^{-\mu t}
\]
If \(t \to \infty\), we get
\[
M(x, t) \leq \max \left\{\frac{\alpha \psi}{\theta \mu} \left\|M_0(X)\right\|_\infty\right\}
\]
Thus the proof is complete.

6. **Stability analysis of system (26)**

In this section, the local and global stabilities of the equilibrium points of diffusion system (26) are discussed. It is easy to verify that the equilibrium points of diffusion system (26) and those of system (2) are the same. Then the stability analysis for each of them can be studied as in the following theorems

**Theorem 9:** The SFEp of diffusion system (26) is LAS if the following sufficient conditions hold
\[
\begin{align}
\beta R_0 &< \mu + \gamma M - kD_2 \\
(\mu + \delta + kD_3)\delta P_0 + \epsilon \gamma &\leq (\mu + \delta + kD_3)(\mu + \gamma M_0 + kD_2)
\end{align}
\]

**Proof:** The Jacobian matrix of system (26) at the SFEp is given by
where $b_{ij}; i,j = 1,2,3,4$ are given by Eq. (12). Then the characteristic equation can be written as
\[ \lambda^4 + \tilde{b}_1 \lambda^3 + \tilde{b}_2 \lambda^2 + \tilde{b}_3 \lambda + \tilde{b}_4 = 0 \]  
(30b)\)

Such that
\[ \tilde{b}_1 = -(\tilde{b}_{11} + \tilde{b}_{22} + \tilde{b}_{33} + \tilde{b}_{44}) \]
\[ \tilde{b}_2 = \tilde{b}_{11}(\tilde{b}_{22} + \tilde{b}_{33}) + \tilde{b}_{11} \tilde{b}_{44} - \tilde{b}_{12} \tilde{b}_{33} - \tilde{b}_{13} \tilde{b}_{22} + \tilde{b}_{14}(\tilde{b}_{22} + \tilde{b}_{33}) \]
\[ \tilde{b}_3 = -(\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{22} \tilde{b}_{33} - \tilde{b}_{23} \tilde{b}_{32}) + (\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{11} \tilde{b}_{44} - \tilde{b}_{14} \tilde{b}_{41}) \]
\[ \tilde{b}_4 = (\tilde{b}_{12} \tilde{b}_{33} - \tilde{b}_{23} \tilde{b}_{32})(\tilde{b}_{11} \tilde{b}_{44} - \tilde{b}_{14} \tilde{b}_{41}) \]

As well
\[ \tilde{b}_1 \tilde{b}_2 - \tilde{b}_3 = -(\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{11} + \tilde{b}_{22}) \]
\[ -(\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{12} + \tilde{b}_{44}) \]
\[ -(\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{14} \tilde{b}_{44} - \tilde{b}_{14} \tilde{b}_{41}) \]
\[ -(\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{22} \tilde{b}_{33} - \tilde{b}_{23} \tilde{b}_{32}) \]

while $\Delta = \tilde{b}_3(\tilde{b}_1 \tilde{b}_2 - \tilde{b}_3) - \tilde{b}_1^2 \tilde{b}_4$ can be written as
\[ \Delta = -\tilde{b}_1(\tilde{b}_{11} + \tilde{b}_{44})^2 (\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{22} \tilde{b}_{33} - \tilde{b}_{23} \tilde{b}_{32}) \]
\[ -\tilde{b}_1(\tilde{b}_{11} + \tilde{b}_{44})^2 (\tilde{b}_{12} + \tilde{b}_{44})^2 (\tilde{b}_{11} \tilde{b}_{44} - \tilde{b}_{14} \tilde{b}_{41}) \]
\[ + (\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{22} + \tilde{b}_{33})[(\tilde{b}_{22} \tilde{b}_{12} - \tilde{b}_{23} \tilde{b}_{32}) - (\tilde{b}_{11} \tilde{b}_{44} - \tilde{b}_{14} \tilde{b}_{41})]^2 \]

where
\[ \tilde{b}_{11} = -(\mu + \gamma M_4 + kD_1) ; \quad \tilde{b}_{22} = (\beta P_0 - \mu - \gamma M_0 - kD_2) \]
\[ \tilde{b}_{33} = -(\mu + \delta + kD_3) ; \quad \tilde{b}_{44} = -(\theta + kD_5) \]

Note that, all the Routh-Hurwitz conditions that guarantee the LAS of the SFEP of system (26) are satisfied provided that the conditions (29a)-(29b) hold.

**Theorem 10:** The FTQSEP of diffusion system (26) is LAS if in addition to condition (14b) the following sufficient conditions hold
\[ \beta P_1 < (\mu + \gamma M_1 + kD_2) \]  
(31a)\)
\[ \alpha_2 S_1 < \alpha \gamma P_1 + (\theta + kD_1) \gamma S_1 + (\delta + kD_2) \gamma M_1 \]  
(31b)\)
\[ \alpha_2 S_1 [\beta P_1 + \gamma M_1] < 2(\theta + kD_1)(\gamma S_1 + \mu + \gamma M_1 + kD_1) \]  
(31c)\)

**Proof:** The Jacobian matrix of system (26) at FTQSEP can be written:
\[ J(E_1) = \begin{pmatrix} c_{11} - kD_1 & c_{12} & 0 & c_{14} \\ c_{21} & c_{22} - kD_2 & c_{23} & c_{24} \\ 0 & 0 & c_{33} - kD_3 & 0 \\ c_{41} & c_{42} & 0 & c_{44} - kD_5 \end{pmatrix} \]  
(32a)\)

where $c_{ij}; i,j = 1,2,3,4$ are given in Eq. (15). Hence, the characteristic equation can be written as
\[ (\tilde{c}_{33} - i)(\lambda^3 + \tilde{c}_1 \lambda^2 + \tilde{c}_2 \lambda + \tilde{c}_3) = 0 \]  
(32b)\)

here the eigenvalue in the $Q$ direction is given by $\lambda_0 = -(\mu + \delta + kD_3) < 0$, while the other three eigenvalues are the roots of the third degree polynomial, where
\[ \tilde{c}_1 = -[c_{11} + c_{22} + c_{44}] \]
\[ \tilde{c}_2 = c_{11} c_{22} - c_{12} c_{21} + c_{11} c_{44} - c_{14} c_{41} + c_{22} c_{44} - c_{24} c_{42} \]
\[ \tilde{c}_3 = -[c_{11} c_{22} c_{44} - c_{24} c_{42}] + c_{12} (c_{24} c_{41} - c_{21} c_{44}) + c_{14} (c_{21} c_{42} - c_{22} c_{41}) \]
\[ \dot{c}_1 \dot{c}_2 - \dot{c}_3 = -(c_{11} + c_{22})(c_{11}c_{22} - c_{12}c_{11}) - (c_{11} + c_{44})(c_{31}c_{44} - c_{14}c_{31}) - (c_{12} + c_{44})(c_{22}c_{44} - c_{24}c_{42}) - 2(c_{11}c_{22}c_{44} + c_{12}c_{24}c_{41} + c_{14}c_{21}c_{42}) \]

Here
\[ \dot{c}_{11} = -(\beta S_1 + \mu + \gamma M_1 + k D_1) \]
\[ \dot{c}_{22} = (\beta P_1 - \mu - \gamma M_1 - k D_2) \]
\[ \dot{c}_{33} = -(\mu + \delta + k D_3) \]
\[ \dot{c}_{44} = -(\theta + k D_5) \]

Note that, it is easy to verify that all the Routh-Hurwitz conditions that guarantee the LAS of the \textit{FTQSEP} of system (26) are satisfied provided that the conditions (31a)-(31c) and (14b) hold.

**Theorem 11:** The \textit{EEP} of diffusion system (26) is LAS if in addition to condition (17b) the following sufficient conditions hold:

\[ \beta P_2 < \mu + \gamma M_2 + k D_2 \]  \quad (33a)
\[ \beta (\mu + \delta + k D_1) P_2 + e \delta M_1 < (\mu + \gamma M_2 + k D_2)(\mu + \delta + k D_5) \]  \quad (33b)
\[ P_2(S_{21} + \gamma M_2) + e \delta S_2 < \beta P_2(\beta S_2 + \mu + \gamma M_2 + k D_1 + \theta + k D_5) \]  \quad (33c)

**Proof:** The Jacobian matrix of system (26) at \textit{EEP} can be written:

\[ J(\dot{E}_2) = \begin{bmatrix}
    z_{21} & z_{22} - kD_2 & z_{23} & z_{24} \\
    z_{31} & z_{32} - kD_3 & z_{33} & z_{34} \\
    z_{41} & z_{42} & z_{44} - kD_5 & 0
\end{bmatrix} \]  \quad (34a)

where \(z_{ij}, i, j = 1, 2, 3, 4\) are given in Eq. (18). So the characteristic equation can be written as:

\[ \hat{X}^4 + \hat{Y}_1 \hat{X}^3 + \hat{Y}_2 \hat{X}^2 + \hat{Y}_3 \hat{X} + \hat{Y}_4 = 0 \]  \quad (34b)

where
\[ \hat{Y}_1 = -[[z_{11} + z_{22} + z_{33} + z_{44}]] \]
\[ \hat{Y}_2 = \left[ z_{11}z_{22} - z_{12}z_{21} + z_{13}z_{31} + z_{14}z_{41} - z_{21}z_{32} - z_{22}z_{42} + z_{23}z_{34} + z_{24}z_{34} \right] \]
\[ \hat{Y}_3 = 
\left( \left( z_{11} + z_{44} \right) z_{22}z_{33} - z_{22}z_{44} + \left( z_{11}z_{23} - z_{12}z_{31} \right) \right) 
\left( z_{11}z_{22} + z_{12}z_{21} - z_{11}z_{23} + z_{12}z_{24} \right) 
\left( z_{11}z_{33} - z_{13}z_{31} \right) 
\left( z_{11}z_{44} - z_{14}z_{41} \right) 
\left( z_{12}z_{34} - z_{13}z_{41} \right) 
\left( z_{14}z_{23} - z_{12}z_{34} \right) - z_{33}z_{44} \]
\[ \hat{Y}_4 = \hat{X}_1 \hat{X}_2 \hat{X}_3 \hat{X}_4 \]

Moreover, we have that
\[ \hat{Z}_1 \hat{Z}_2 - \hat{Z}_3 = 
\left( \left( \left( z_{11} + z_{22} \right) \left( z_{11} + z_{22} - z_{12}z_{21} \right) \right) \right) 
\left( z_{11}z_{22} - z_{12}z_{21} + z_{13}z_{31} + z_{14}z_{41} - z_{21}z_{32} - z_{22}z_{42} + z_{23}z_{34} + z_{24}z_{34} \right) 
\left( \left( z_{11} + z_{33} \right) z_{22}z_{33} - z_{22}z_{33} + \left( z_{11}z_{23} - z_{13}z_{21} \right) \right) 
\left( z_{11}z_{33} - z_{13}z_{31} \right) \]
\[ \hat{Z}_4 \]
\[ \text{and } \Delta = \hat{Z}_1 (\hat{Z}_2 - \hat{Z}_3) - \hat{Z}_4 \text{ can be written as:} \]
\[ \Delta = \hat{X}_1 \hat{X}_2 \hat{X}_3 \hat{X}_4 \]

Such that
\[ \hat{Z}_{11} = -(\beta S_2 + \mu + \gamma M_2 + k D_1) \]
\[ \hat{Z}_{22} = (\beta P_2 - \mu - \gamma M_2 - k D_2) \]
\[ \hat{Z}_{33} = -(\mu + \delta + k D_3) \]
\[ \hat{Z}_{44} = -\theta + k D_5) \]

Note that, it is easy to verify that all the Routh-Hurwitz conditions that guarantee the LAS of the \textit{FTQSEP} of system (26) are satisfied provided that the conditions (31a)-(31c) and (14b) hold.
Again by using Routh-Hurwitz criterion, we get that the EEP is LAS if the sufficient conditions (33a)-(33c) with (17b) hold.

Note that, according to the above theorems it’s clear that, the equilibrium points of diffusion system (26) are always LAS if they are stable in system (2), that is mean without diffusion, but the converse is not necessarily true.

Next, in following theorems the globally asymptotically stability (GAS) of diffusion system (26) at SFEP, FTQSEP and EEP is carried out using the method described in [15].

**Theorem 12:** Assume that the SFEP of the diffusion system (26) is LAS, then it is GAS if the conditions (20a)-(20b) hold.

**Proof:** Consider the following candidate Lyapunov function with \( u(x,t) \) is a solution of diffusion system (26)

\[
W_1 = \int_{\Omega} V(u(x,t)) \, dx
\]

where \( V(u) \) is a continuously differentiable function defined on some \( \mathbb{R}_+^4 \). Then the time derivative of \( W_1 \) along the positive solution of system (26) is written as

\[
\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u) \cdot (f(u) + D\Delta u) \, dx
\]

where \( f(u) \) is the vector field that given in right hand side of system (26) without diffusion, while \( D\Delta u \) is the diffusion term with \( D = (D_1, D_2, D_3, D_5) \) and \( D_i \geq 0 \). Therefore, we obtain that

\[
\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u) \cdot f(u) \, dx + \int_{\Omega} \nabla V(u) \cdot D\Delta u \, dx
\]

which gives

\[
\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u) \cdot f(u) \, dx + \sum_{i=1}^{4} D_i \int_{\Omega} \frac{\partial V}{\partial u_i} \Delta u_i \, dx
\]

Assume that, the integrand of the first term in Eq. (36) is already calculated as that for the system (2) given by theorem (4). However, the second term is simplified by using Green’s formula, and we obtain

\[
\int_{\Omega} \frac{\partial V}{\partial u_i} \Delta u_i \, dx = - \int_{\Omega} \nabla u_i \cdot \nabla \left( \frac{\partial V}{\partial u_i} \right) \, dx
\]

Therefore, Eq. (37) becomes

\[
\int_{\Omega} \frac{\partial V}{\partial u_i} \Delta u_i \, dx = - \int_{\Omega} \nabla u_i \cdot \nabla \left( \frac{\partial V}{\partial u_i} \right) \, dx
\]

Accordingly, by using Eq. (38) in Eq. (36), it’s obtain that

\[
\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u) \cdot f(u) \, dx - \sum_{i=1}^{4} D_i \int_{\Omega} \nabla u_i \cdot \nabla \left( \frac{\partial V}{\partial u_i} \right) \, dx
\]

Therefore, in order to construct the function \( V \) we should have

\[
D_i \int_{\Omega} \nabla u_i \cdot \nabla \left( \frac{\partial V}{\partial u_i} \right) \, dx \geq 0, \text{ for all } i = 1, 2, 3, 4
\]

Now by using the function \( V = V_1 \), that given in theorem (4)

\[
V = (P - P_0 - P_0 \ln \frac{P}{P_0} + S + Q + (M - M_0 - M_0 \ln \frac{M}{M_0})
\]

Hence, in this case we have that...
Consequently, we obtain that
\[
\frac{dW_1}{dt} = -\left(\frac{\mu+\gamma M}{p} (F - P_0)^2 + \frac{(\alpha P - \beta P_0)}{pM} (P - P_0)(M - M_0) \right)
- \frac{\theta}{M} (M - M_0)^2 - \left[\mu + (1 - \varepsilon)\delta\right](Q - \gamma(1 - \varepsilon)\delta)SM
- \left[\mu - (\alpha + \beta P_0)S - \frac{\alpha M_0}{M}\right]S - D \int_\Omega \left[ \frac{\left[|\nabla p|^2\right]}{p^2} + M_0 \frac{|\nabla M|^2}{M^2} \right] dx
\]
where \(D = \min\{\rho_1, D_3\}\). Therefore by using the conditions (20a)-(20b), it's observed that
\[
\frac{dW_3}{dt} < -\left[\frac{\mu+\gamma M}{p} (P - P_0)^2 - \frac{\left[\mu+\gamma M\right]}{p} \left( M - M_0 \right) \right]^2 - \left[\mu + (1 - \varepsilon)\delta\right]Q
- \left[\mu - (\alpha + \beta P_0)S - \frac{\alpha M_0}{M}\right]S - D \int_\Omega \left[ \frac{\left[|\nabla p|^2\right]}{p^2} + M_0 \frac{|\nabla M|^2}{M^2} \right] dx
\]
Obviously, \(W_1' = 0\) at \(E_0 = (P_0, 0, 0, M_0)\), moreover \(W_1' < 0\) otherwise. Hence \(W_1'\) is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to SFEP. Hence the proof is complete. \(\square\)

**Theorem 13:** Assume that the FTQSEP of the diffusion system (26) is LAS, then it is GAS if the conditions (21a)-(21d) hold

**Proof:** Similarly as in proof of theorem (12), we consider the following candidate Lyapunov function with \(u(x, t)\) be a solution of diffusion system (26).

\[
W_2 = \int_\Omega \frac{1}{2} (u(x, t))^2 dx
\]
with the function \(V_2\) that given in theorem (5). Therefore, direct computation gives that
\[
W_2' = -\left(\frac{\left[|\nabla p|^2\right]}{p^2} + M_0 \frac{|\nabla M|^2}{M^2} \right) (P - P_0)^2 - \frac{\left[\mu+\gamma M\right]}{p} \left( M - M_0 \right)
- \frac{\alpha}{M} \left( P - P_1 \right) (M - M_1) - \frac{\theta}{2M} \left( S - S_1 \right)^2
- \frac{\gamma M_0}{p} \left( S - S_1 \right) - \frac{\alpha}{M} \left( P - P_1 \right) (M - M_1) - \frac{\theta}{2M} \left( S - S_1 \right)^2 + \frac{\alpha}{M} \left( P - P_1 \right) (M - M_1)
- D \int_\Omega \left[ \frac{\left[|\nabla p|^2\right]}{p^2} + M_0 \frac{|\nabla M|^2}{M^2} \right] dx
\]
where \(D = \min\{D_1, D_2, D_3\}\). Therefore by using the conditions (21a)-(21d), it's observed that
\[
W_2' < -\left[\frac{x_{11}}{2} (P - P_1) - \frac{x_{22}}{2} (S - S_1) \right]^2 - \left[\frac{x_{44}}{2} (M - M_1) \right]^2
- \left[\frac{x_{22}}{2} (S - S_1) - \frac{x_{44}}{2} (M - M_1) \right]^2 - \left(\mu + (1 - \varepsilon)\delta\right)Q - \frac{\left[\nabla S, Q\right]_Q}{\delta}
- D \int_\Omega \left[ \frac{\left[|\nabla p|^2\right]}{p^2} + M_0 \frac{|\nabla M|^2}{M^2} \right] dx
\]
where \(x_{11}, x_{22}, x_{44}\) and \(x_{44}\) are given theorem (5). Obviously, \(W_2' = 0\) at \(E_1 = (P_1, S_1, 0, M_1)\), moreover \(W_2' < 0\) otherwise. Hence \(W_2'\) is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to FTQSEP. Hence the proof is complete. \(\square\)
Theorem 14: Assume that the EEP of the diffusion system (26) is LAS, then it is GAS if the conditions (22a)-(22f) hold

Proof: Consider the following candidate Lyapunov function with \( u(x,t) \) be a solution of diffusion system (26).

\[
W_3 = \int_{I_2} V_3(u(x,t))dx
\]

with the function \( V_3 \) that given in theorem (6). Therefore, direct computation gives that

\[
W_3' = -\frac{(\beta S + \beta P_2 + \mu)}{2}(P - P_2)^2 + (\sigma \gamma M + \beta S - \beta P_2)(P - P_2)(S - S_2)
\]

\[
+ (\alpha - \gamma P_2)(P - P_2)(M - M_2) - \frac{\theta}{3}(M - M_2)^2
\]

\[
- \frac{(\gamma M + \mu - \beta P_2)}{3}(S - S_2)^2 + (\epsilon \delta + \epsilon \gamma M)(S - S_2)(Q - Q_2)
\]

\[
- \frac{\mu - \delta}{2}(Q - Q_2)^2 - \frac{(\gamma M + \mu - \beta P_2)}{3}(S - S_2)^2
\]

\[
+ (\alpha + \sigma \gamma P_2 - \gamma S_2)(S - S_2)(M - M_2) - \frac{\theta}{3}(M - M_2)^2
\]

\[
- \frac{\mu - \delta}{2}(Q - Q_2)^2 + \epsilon \gamma S_1(Q - Q_2)(M - M_2) - \frac{\theta}{3}(M - M_2)^2
\]

\[
- \frac{\mu - \delta}{2}(Q - Q_2)^2 + \epsilon \gamma S_1(Q - Q_2)(M - M_2) - \frac{\theta}{3}(M - M_2)^2
\]

\[
- D \left[ \frac{|\nabla u|^2}{p^2} + \frac{|\nabla S|^2}{2} + \frac{|\nabla Q|^2}{Q^2} + \frac{\|\nabla M\|^2}{M^2} \right]dx
\]

where \( D = \min(\Omega_1, \Omega_2, D_3, E_5) \). Therefore by using the conditions (22a)-(22f), it’s observed that

\[
W_3' = -\left[ \frac{q_{11}}{2}(P - P_2) - \sqrt{\frac{q_{22}}{S}}(S - S_2) \right]^2 - \left[ \frac{q_{11}}{2}(P - P_2) - \sqrt{\frac{q_{44}}{3}}(M - M_2) \right]^2
\]

\[
- \left[ \frac{q_{22}}{S}(S - S_2) - \sqrt{\frac{q_{44}}{3}}(M - M_2) \right]^2
\]

\[
- \left[ \frac{q_{44}}{3}(S - S_2) - \sqrt{\frac{q_{44}}{3}}(M - M_2) \right]^2
\]

\[
- D \left[ \frac{|\nabla u|^2}{p^2} + \frac{|\nabla S|^2}{2} + \frac{|\nabla Q|^2}{Q^2} + \frac{\|\nabla M\|^2}{M^2} \right]dx
\]

here \( q_{11}, q_{22}, q_{33}, \) and \( q_{44} \) are given in theorem (6). Obviously, \( W_3' = 0 \) at \( E_2 = (P_2, S_2, Q_2, M_2) \), moreover \( W_3' < 0 \) otherwise. Hence \( W_3' \) is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to EEP. Hence the proof is complete.

8. Numerical simulation of systems (1)

In a bid to check our computation, some numerical simulations are carried out. The objective is to understand the global dynamics if system (1) and then study the effects of varying the parameters values. For the following set of hypothetical values of the parameters with different initial conditions the dynamical behavior of system (1) is investigated using the following sets of initial conditions \((0.7, 0.9, 0.6, 0.5, 0.5), (1, 2, 3, 1, 4), \) and \((3, 0.5, 5, 3, 1)\) respectively. The obtained trajectories are drawn in Fig. (1) below.

\[
\psi = 3, \beta = 0.03, \mu = 0.1, \gamma = 0.1, \sigma = 0, \epsilon = 0.03, \epsilon = 0.1
\]

\[
\delta = 0.1, \alpha = 0.05, \theta = 0.02
\]

(43)
Fig. 1: The trajectory of system (1) approaches asymptotically to a globally stable SFEP given by $E_0 = (3.2, 0.026, 7.8, 1)$. (a) Trajectory of $P(t)$, (b) Trajectory of $S(t)$, (c) Trajectory of $Q(t)$, (d) Trajectory of $R(t)$, (e) Trajectory of $M(t)$.

Clearly, as shown in Fig. (1), system (1) has a globally asymptotically stable SFEP for the data (43). Now, for the following set of hypothetical parameters values with the same initial sets of values used in Fig. (1), the trajectories of system (1) are drawn in Fig. (2) below.

$$\psi = 3, \beta = 0.3, \mu = 0.1, \gamma = 0.1, \sigma = 0.1, \epsilon = 0.03, e = 0.1$$

$$\delta = 0.1, \alpha = 0.05, \theta = 0.02$$

(44)
Fig. 2: The trajectory of system (1) approaches asymptotically to a globally stable EEP given by $E_2 = (2.4, 0.95, 0.6, 25.9, 8.4)$. (a) Trajectory of $P(t)$, (b) Trajectory of $S(t)$, (c) Trajectory of $Q(t)$, (d) Trajectory of $R(t)$, (e) Trajectory of $M(t)$.

Now, we used the same set of hypothetical parameters values in Eq. (44) with $\epsilon = 0$, and the same initial sets of values used in Fig. (1), then system (1) has a globally asymptotically stable FTQSEP, hence the trajectories of system (1) are drawn in Fig. (3) below.
Fig. 3: The trajectory of system (1) approaches asymptotically to a globally stable FTQSEP given by $E_1 = (2.4,0.95,0.26,6.8,4)$. (a) Trajectory of $P(t)$, (b) Trajectory of $S(t)$, (c) Trajectory of $Q(t)$, (d) Trajectory of $R(t)$, (e) Trajectory of $M(t)$.

Now, for the data set (44) with different values of contact rate $\beta$ given by the parameters values $0.0001, 0.3, 0.5, 0.0001$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (4) below.

Fig. (4): Time series of the trajectory of system (1) for the data (44) with different values of contact rate. (a) Trajectory of system (1) for $\beta = 0.3$, (b) Trajectory of system (1) for $\beta = 0.5$, (c) Trajectory of system (1) for $\beta = 0.0001$.

According to Fig. (4), as the contact rate between the potential smoker individuals and smoker individuals increases, then the trajectory of system (1) approaches asymptotically to the $(EEP)$ point as shown in the typical figure given by Fig. (4). In fact as $\beta$ increases, it is observed that the populations of smoker, quit smoker and media coverage increase while the populations of potential smokers and recovered decrease. On the other hand, as the contact rate $\beta$ decreases then the trajectory of system (1) still approaches asymptotically to the $(EEP)$ but with opposite size of populations.
Now, for the data (44) with awareness level given by $\gamma = 0.2$ and different values of response to media coverage from the potential smoker individuals such that $1 - \sigma = 0.99999, 0.8, 0$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (5) below.

![Graphs showing trajectories](image)

**Fig. (5):** Time series of the trajectory of system (1) for the data (44) with $\gamma = 0.2$ and different values of response rate to the media coverage. (a) Trajectory of system (1) for $1 - \sigma = 0.99999$, (b) Trajectory of system (1) for $1 - \sigma = 0.8$, (c) Trajectory of system (1) for $1 - \sigma = 0$.

Clearly, as shown in Fig. (5), increase the efficiency rate of the media coverage makes the trajectory of system (1) approaches asymptotically to the (SFEP) gradually and vice versa.

Similarly, for the data (44) with awareness level given by $\gamma = 0.2$ and different values of response to media coverage from the smoker individuals such that $1 - \epsilon = 1, 0.5, 0$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (6) below.

![Graphs showing trajectories](image)
Fig. (6): Time series of the trajectory of system (1) for the data (44) with $\gamma = 0.2$ and different values of response rate of smoker individuals to the media coverage. (a) Trajectory of system (1) for $1 - e = 1$, (b) Trajectory of system (1) for $1 - e = 0.5$, (c) Trajectory of system (1) for $1 - e = 0$.

Clearly, as shown in Fig. (6), increase the efficiency rate of the media coverage on the smoker individuals makes the trajectory of system (1) approaches asymptotically to the (FTQSEP) gradually and vice versa.

9. Discussion

In this paper, a mathematical model has been studied and analyzed to study the effect of a warning by media on the dynamical behavior of smoking epidemic model. The existence and the stability analysis of all possible equilibrium points are studied analytically as well as numerically. Finally according to the numerically simulation the following results are obtained:

1. As the contact rate between the individuals of potential smokers and smokers increase the trajectory of system (1) approaches asymptotically to the (EEP).
2. As the response to the media coverage from the potential smokers increases then the trajectory of system (1) approaches asymptotically to the (SFEP). Otherwise the trajectory still approaches asymptotically to (EEP).
3. As the response to the media coverage from the smokers increases then the trajectory of system (1) approaches asymptotically to the (SFEP). Otherwise the trajectory still approaches asymptotically to (EEP).
4. The stability of the smoking system in presence of diffusion follows if the smoking system without diffusion is stable, but the converse is not necessarily true.

References

[1] J. E. Harris, 1996, Smoking and Tobacco Control Monograph, Chapter 5, pp. 59-75.
[2] M. M. Bassiony, 2009, “Smoking in Saudi Arabia,” Saudi Medical Journal, vol. 30, no. 7, pp. 876–881.
[3] P. Tonnesen, L. Carrozzi, C. Jimenez et al., 2007, Smoking cessation in patients with respiratory diseases a high priority integral component of therapy, Eur. Respir J., 29: 390-417. https://DOI:10.1183/0903 1936.00060806.
[4] C. Castillo-Garsow, G. Jordan-Salivia, and A. R. Herrera, 1997, “Mathematical models for the dynamics of tobacco use, recovery, and relapse,” Technical Report Series BU-1505-M, Cornell University, Ithaca, NY, USA.
[5] Lahrouz, L. Omari, D. Kiouach, A. Belmaati, 2011, Deterministic and stochastic stability of a mathematical model of smoking. J. Statistics and Probability Letters.

[6] A. A. Al-shareef and H. A. Batarfi, 2020, Stability Analysis of chain, mild and passive smoking model, American J. of Comp. Math., 10, 31-42. https://doi.org/10.4236/ajcm.2020.101003.

[7] O. Sharomi, A.B. Gumel, 2008, Curtailing smoking dynamics a mathematical modeling approach, J. Applied Math., and Comp., 195, 475-499.

[8] G. Zaman, 2011, Optimal Campaign in the Smoking Dynamics, J., Hindawi Publishing Corp., Comp., and Math., Meth., in Med., ID 163834, pp 9.

[9] G. Zaman, 2011, “Qualitative behavior of giving up smoking models,” Bulletin of the Malaysian Mathematical Sciences Society, vol. 34, no. 2, pp. 403–415.

[10] V. S. Ert’urk, G. Zaman, and S. Momani, 2012, “A numeric-analytic method for approximating a giving up smoking model containing fractional derivatives,” Computers & Mathematics with Applications, vol. 64, no. 10, pp. 3065–3074.

[11] Zainab Alkhudhari, Sarah Al-Sheikh, and Salma Al-Tuwairqi, 2014, Global Dynamics of a Mathematical Model on Smoking, Hindawi journal, , Article ID 847075. http://dx.doi.org/10.1155/2014/847075.

[12] K. Misra, A. Sharma, J. B. Shukla, 2011, Modeling and analysis of effects of awareness programs by media on the spread of infectious diseases, Math. Comput. Model. 53, 1221–1228.

[13] R. J. Smith, J. M. Tchueneche, N. Dube, C. P. Bhunu, C. T. Bauch, 2011, The impact of media coverage on the transmission dynamics of human influenza, BMC Public Health 11.

[14] J. Cui, H. Zhu, 2008, The impact of media on the spreading and control of infectious disease, J. Dyn. Differ. Equations, 20(1):31-53.

[15] M. Lotfi, M. Maziane, K. Hattaf and N. yousf, 2014, Partial differential equations of an epidemic model with diffusion, Hindawi journal, , Article ID 186437. http://dx.doi.org/10.1155/2014/186437.

[16] K. Hattaf, A. A. Lashari, Y. Louartassi, and N. Youssi, 2013, “A delayed SIR epidemic model with general incidence rate,” Electronic Journal of Qualitative Theory of Differential Equations, vol. 3, pp. 1–9.

[17] S. Chinviriyasit and W. Chinviriyasit, 2010, Numerical modeling of an SIR epidemic model with diffusion, journal of applied mathematics and computation, 216, 395-409.

[18] R. Peng and F. Yi, 2013, Asymptotic profile of the positive steady state for an SIS epidemic model reaction-diffusion model effects of epidemic risk and population movement, Physica D journal, 259, pp. 8-25.

[19] K. Hattaf and N. Youssi, 2013, “Global stability for reaction-diffusion equations in biology,” Computers and Mathematics with Applications, vol. 66, pp. 1488–1497.