PERSISTENT MARKOV PARTITIONS FOR RATIONAL MAPS

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Abstract. A construction is given of Markov partitions for some rational maps, which persist over regions of parameter space, not confined to single hyperbolic components. The set on which the Markov partition exists, and its boundary, are analysed.

The first result of this paper is a construction of Markov partitions for some rational maps, including non-hyperbolic rational maps (Theorem 1.1). Of course results of this type have been around for many decades. We comment on this below. There is considerable freedom in the construction. In particular, the construction can be made so that the partition varies isotopically to a partition for all maps in a sufficiently small neighbourhood of the original one (Lemma 2.1). So the partition is not specific, like the Yoccoz puzzle, and also less specific than other partitions which have been developed to exploit the ideas on analysis of dynamical planes and parameter space which were pioneered using the Yoccoz puzzle. We then investigate the boundary of the set of rational maps for which the partition exists in section 2, in particular in Theorem 2.2. We also explore the set in which the partition does exist, in section 3, in particular in Theorem 3.2. We show how parameter space is partitioned, using a partition which is related to the Markov partitions of dynamical planes – in much the usual manner – and show that all the sets in the partition are nonempty. We are able to apply some of the results of [12] in our setting, in particular in the analysis of dynamical planes. The main tool used in the results about the partitioning the subset of parameter space admitting a fixed Markov partition is the \( \lambda \)-lemma [9].

It is natural to start our study with hyperbolic rational maps. For some integer \( N \) which depends on \( f \), the iterate \( f^N \) of a hyperbolic map \( f \) is expanding on the Julia set \( J = J(f) \) with respect to the spherical metric. The full expanding property does not hold for a parabolic rational map on its Julia set, but a minor adjustment of it does. Given any closed subset of the Julia set disjoint from the prabolic orbits map \( f^N \) is still expanding with respect to the spherical metric, for a suitable \( N \).
We shall use the following definition of Markov partition for a rational map \( f : \mathbb{C} \to \mathbb{C} \).

**Definition.** A Markov partition for \( f \) is a set \( \mathcal{P} = \{P_1, \cdots, P_n\} \) such that:

- \( \text{int}(P_i) = P_i \);
- \( P_i \) and \( P_j \) have disjoint interiors if \( i \neq j \);
- \( \bigcup_{i=1}^{n} P_i = \mathbb{C} \);
- each \( P_i \) is a union of connected components of \( f^{-1}(P_j) \) for varying \( j \).

## 1. Construction of partitions

Our first theorem applies to a familiar “easy” class of rational maps. In particular, we assume that every critical orbit is attracted to an attractive or parabolic periodic orbit. The most important property of the Markov partitions yielded by this theorem, however, is that the set of rational maps for which they exist is open.

**Theorem 1.1.** Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational map such that every critical point is in the Fatou set, and such that the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Let \( F_0 \) be the union of the periodic Fatou components. Let \( Z \) be a finite forward invariant set which includes all parabolic points. Let \( G_0 \subset \mathbb{C} \) be a connected piecewise \( C^1 \) graph such that the following hold.

- All components of \( \mathbb{C} \setminus G_0 \) are topological discs, as are the closures of these components.
- \( G_0 \cap (F_0 \cup Z) = \emptyset \), any component of \( \mathbb{C} \setminus G_0 \) contains at most component of \( F_0 \cup Z \), and \( G_0 \) has at most one component of intersection with any Fatou component.
- \( G_0 \) is trivalent, that is, exactly three edges meet at each vertex.
- The closures of any two components of \( \mathbb{C} \setminus G_0 \) intersect in at most a single component, which, if it exists, must be either an edge together with the endpoints of this edge, or a single vertex, by the previous conditions.

Then there exists \( G' \subset \mathbb{C} \setminus (F_0 \cup Z) \) isotopic to \( G_0 \) in \( \mathbb{C} \setminus (F_0 \cup Z) \) and such that \( G' \subset f^{-N}(G') \) for some \( N \). Given any neighbourhood \( U \) of \( G_0 \), \( G' \) can be chosen with \( G' \subset U \), for sufficiently large \( N \).

Moreover, \( G \bigcup_{i=0}^{N-1} f^{-i}(G') \) is a graph with finitely many vertices and edges, with \( G \subset f^{-1}(G) \), and hence \( \mathcal{P} = \{ \overline{U} : U \text{ is a component of } \mathbb{C} \setminus G \} \) is a Markov partition for \( f \). The boundary of the closure of any component of \( \mathbb{C} \setminus G \) is a quasi-circle.
Remarks 1. Although it is a folklore result that expanding maps of compact metric spaces admit Markov partitions, there does not seem to be any well-known reference from the period when this result might have been expected to be proved for the first time, the early 70’s or late 60’s. Expanding maps of compact metric spaces are, of course, always non-invertible. The corresponding result for Axiom A diffeomorphisms was proved by Rufus Bowen [2], who developed the whole theory of describing invertible hyperbolic systems in terms of their symbolic dynamics in a remarkable series of papers. Although related results in the expanding case were certainly obtained, [13], for example, I have been unable to find a published proof of the result from that time. The existence of Markov partitions appears as Theorem 4.5.2 in the recent book by Przytycki and Urbanski [10]. But there is no statement, there, about topological properties of the sets in the partition. So I am not aware of another proof that the sets in the partition are closed topological discs, disjoint from the given finite set \(Z\) and further, that the boundaries are quasi-circles, under suitable conditions. I am not sure that this property holds, even for general expanding maps on repellers in surfaces. There are many pathologies concerning Markov partitions. [3] gives just one (mild, though interesting).

2. If \(P\) is a Markov partition for \(f^N\), then \(P' = \bigvee_{i=0}^{N-1} f^{-i}(P)\) is a Markov partition for \(f\). If \(Z\) is a forward invariant set and \(Z \cap (\bigcup_{P \in P} \partial P) = \emptyset\), then \(Z \cap (\bigcup_{P \in P'} \partial P) = \emptyset\) also. However, it is not clear that the interiors of the sets in \(P'\) are connected, nor, even, that they have finitely many components.

3. For \(f\) as stated in the theorem, it can be shown that any component of \(\bigvee_{k=0}^{\infty} f^{-k}(P)\) is either a point or the closure of a single Fatou component.

As an immediate corollary of Theorem 1.1 we have the following.

**Corollary 1.2.** Let \(f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\) be a rational map with connected Julia set \(J\), such that the forward orbit of each critical point is attracted to an attractive parabolic periodic orbit, and such that the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Then there exists a graph \(G \subset \overline{\mathbb{C}}\) such that the following hold.

1. \(G \subset f^{-1}G\).
2. \(G\) does not intersect the closure of any Fatou component.
3. All components of \(\overline{\mathbb{C}} \setminus G\) are topological discs.
4. Any component of \(\overline{\mathbb{C}} \setminus G\) contains at most one periodic Fatou component of \(f\).
5. The boundary of any component of \(\overline{\mathbb{C}} \setminus G\) is a quasi-circle.

In particular, the set of closures of components of \(\overline{\mathbb{C}} \setminus G\) is a Markov partition for \(f\).
Proof. We can choose the graph $G_0$ of Theorem 1.1 to satisfy the conditions of 1.1 and also property 2 above. Then by Theorem 1.1 we can find $G'$ in an arbitrarily small neighbourhood of $G_0$ and hence satisfying all the properties above, including property 2. The graph $G = \bigcup_{i=0}^{N-1} f^{-i}(G')$ then also satisfies property 2.

The first step in the proof of 1.1 is a lemma about the existence of subgraphs.

Lemma 1.3. Let $f$, $F_0$, $Z$ and $G_0$ be as in 1.1. Let $F(G_0)$ denote the union of $G_0$ and all sets $\overline{F}$ such that $F$ is a Fatou component intersected by $G_0$. Then the following holds for $\delta$ sufficiently small given $\delta_1$. Let $\Gamma$ be another graph which also has these properties, and such that every component of $\overline{\mathbb{C}\setminus\Gamma}$ within $2\delta_1$ of $F(G_0)$ is either within $\delta$ of a Fatou component intersected by $G_0$, or has diameter $<\delta$. Then there is a subgraph $G_1$ of $\Gamma$ which is in the $\delta_1$-neighbourhood of $F(G_0)$, such that $G_1$ can be isotoped to $G_0$ in this neighbourhood.

Remark Many of the vertices of $G_1$ are likely to be bivalent rather than trivalent, but these are the only types which occur.

Proof. Perturbing all intersections of $G_0$ with Fatou components to the boundaries of those components, we can assume that $G_0$ is contained in the Julia set of $f$. The hypotheses on $\Gamma$ then ensure that there is a point of $\Gamma$ within $\delta$ of each point of $G_0$. Write $\delta_0 = \delta_1/3$ and suppose $\delta < \delta_1/18$. So now we aim to find $G_1 \subset \Gamma$ within a $3\delta_0$-neighbourhood of $G_0$ itself, which can be isotoped to $G_0$ in this neighbourhood. We identify a vertex $v_1 = v_1(v)$ of $\Gamma$ within $\delta$ of each vertex $v$ of $G_0$. These are to be the vertices of $G_1$. Let $G_v$ be the connected component of $G_0 \cap B_{\delta_0}(v)$ which contains $v$. We shall find three arcs in $\Gamma \cap B_{\delta_0}(v_1)$ starting from $v_1$, disjoint apart from the starting point at $v_1$, with exactly one ending within $\delta$ of each of the endpoints of $G_v$. Suppose that we can find such sets for each vertex $v$ of $\Gamma$. We denote by $G_{v,1}$ this union of $v_1(v)$ and the three attached arcs in $\Gamma$. Then $G_{v,1}$ can be isotoped to $G_v$ within a $3\delta_0$-neighbourhood of $G_v$ (because $B_{\delta_0}(v_1)$ has diameter $2\delta_0$, and $B_{\delta_0}(v_1) \subset B_{\delta_0+\delta}(v)$). We can assume without loss of generality that the $2\delta_1$-neighbourhoods of vertices of $G_0$ are disjoint, and let $\delta_2$ be such that the $2\delta_2$-neighbourhoods of the components of $G_0 \setminus \bigcup_{v} G_v$ are disjoint. Now we assume that $\delta < \delta_2$. Then for each edge of $G_0$ between a pair of vertices $v$ and $v_2$, there is a unique pair of endpoints from $G_{v,1}$ and $G_{v_2,1}$ within $\delta$ of this edge, and we can find an arc between them in $\Gamma$, staying within $\delta$ of the edge in $G_0$. The resulting path between $v_1(v)$ and $v_1(v_2)$ might not be an arc, but if it is not an arc, then any self-intersections only occur within $2\delta$ of $\partial B_{\delta_0}(v) \cup \partial B_{\delta_0}(v_2)$, and
the path can then be replaced by an arc in \( \Gamma \) within \( \delta \) of this path. The construction ensures that all the arcs between distinct pairs of vertices are disjoint, apart from intersections at the vertices. The union of these arcs, joined at the vertices \( v_1(v) \), is then the required graph \( G_1 \).

So it remains to construct the sets \( G_{v,1} \). For the moment, all edges and vertices referred to are edges and vertices of \( G \). By a 2-cell (of \( G \)) we mean the closure of a component of \( \overline{\mathbb{C}} \setminus \Gamma \). We fix a vertex \( v_1 = v_1(v) \) of \( \Gamma \), given a vertex \( v \) of \( G_0 \). We denote by \( S_r(v_1) \) the collection of points which can be reached by crossing at most \( r \) 2-cells from \( v_1 \). Then each \( S_r(v_1) \) is a connected topological surface with boundary. \( S_1(v_1) \) is a closed topological disc, but in general \( S_r(v_1) \) might have several boundary components. However, so long as \( S_r(v_1) \subset B_{\delta_0}(v_1) \), there is a particular boundary component \( \partial_1 S_r(v_1) \) which separates \( S_r(v_1) \) from \( \partial B_{\delta_0}(v_1) \). Now we claim we can draw three arcs from \( v_1 \) in \( \Gamma \cap B_{\delta_0}(v_1) \) which successively cross \( \partial_1 S_r(v_1) \) for increasing \( r \), so long as \( S_r(v_1) \subset B_{\delta_0}(v_1) \).

We prove this by induction on \( r \). It is true for \( r = 1 \) because there are three edges from \( v_1 \) to \( \partial S_1(v_1) \). Suppose inductively it is true for \( r - 1 \), and suppose that \( S_r(v_1) \subset B_{\delta_0}(v_1) \). By definition, \( \partial_1 S_r(v_1) \) and \( \partial_1 S_{r-1}(v_1) \) are disjoint. All we need are three disjoint paths through edges joining these two closed curves. To do this, we consider the 2-cells in \( S_r(v_1) \setminus \text{int}(S_{r-1}(v_1)) \) which have boundary intersecting \( \partial_1 S_r(v_1) \). Any such 2-cell must also have boundary intersecting \( \partial_1 S_{r-1}(v_1) \). There are at least three vertices on \( \partial S_r(v_1) \), because otherwise we have two 2-cells with disconnected intersection between boundaries. There must also be at least three vertices in the boundaries of these 2-cells on \( \partial_1 S_{r-1}(v_1) \), for the same reason. If there are not three disjoint paths through these 2-cell boundaries from \( \partial_1 S_r(v_1) \) to \( \partial_1 S_{r-1}(v_1) \), then there must be two 2-cells which intersect in at least one vertex on each of \( \partial_1 S_r(v_1) \) and \( \partial_1 S_{r-1}(v_1) \), and do not intersect along the edges in between. But then once again, there are two 2-cells with disconnected intersection.

We can continue this process for \( S_r(v_1) \) so long as \( S_r(v_1) \subset B_{\delta_0}(v_1) \). So now consider the largest \( r \) such that this holds. The diameter of \( \partial_1 S_r(v_1) \) is \( \geq \delta_0 - \delta > \delta_0/2 \). Since we are assuming that \( \delta < \delta_0/6 = \delta_1/18 \), the same is true for \( \partial_1 S_{r-1}(v_1) \) and \( \partial_1 S_{r-2}(v_1) \). We can then modify the three arcs to \( \partial_1 S_r(v_1) \), by cutting one off at \( \partial_1 S_{r-2}(v_1) \), one at \( \partial_1 S_{r-1}(v_1) \), and then extending the three arcs round \( \partial_1 S_i(v_1) \) for each of \( r - 2 \leq i \leq r \), to the nearest points on each of these boundaries to the three endpoints of \( G_v \). For \( \delta \) sufficiently small given \( \delta_2 \), the endpoints of these arcs are distance \( > 4\delta \) apart. The arcs can then be extended by disjoint arcs outside \( S_i(v_1) \) (for \( r - 2 \leq i \leq r \) in the respective cases) in \( \Gamma \), to within \( \delta \) of each of the endpoints of \( G_v \), as required.

\( \square \)
We will prove Theorem 1.1 using Lemma 1.4. Lemma 1.4 probably works without holomorphicity of \( f \), provided that \( f \) is \( C^2 \).

**Lemma 1.4.** Let \( f, Z, G_0 \) be as in 1.1 to 1.3. As in 1.3, let \( F(G_0) \) be the union of \( G_0 \) and the closures of any components of the Fatou set of \( f \) which are intersected by \( G_0 \). Let \( U \) be a neighbourhood of \( F(G_0) \) with \( C^1 \) boundary such that the distance from any point of \( F(G_0) \) to \( \partial U \) is at least a bounded proportion of the internal diameter at that point of \( F(G_0) \). Let \( \epsilon > 0 \). Then for all sufficiently large \( N \), depending on \( G_0 \) and \( \epsilon \), there are a graph \( G_1 \) and a piecewise \( C^1 \) homeomorphism \( h \) of \( \overline{C} \) such that:

- \( G_1 \subset f^{-N}(G_0) \) and \( G_1 \) is contained in the \( \epsilon \) - neighbourhood of \( F(G_0) \);
- \( h \) is isotopic to the identity, is the identity outside \( U \), and \( h(G_0) = G_1 \);
- \( g = f^N \circ h \) is expanding on \( U_1 \), with expansion constant \( > 2 \), where \( U_1 \) is the component of \( g^{-1}(U) \) containing \( G_0 \), and \( \overline{U_1} \subset U \).

**Proof.** Throughout this proof, by “derivative”, we mean “spherical derivative”.

If \( N \) is sufficiently large given \( \delta \) then every component of \( f^{-N}(\overline{C} \setminus G_0) \) either has diameter \( < \delta \), or is within the \( \delta \)-neighbourhood of some Fatou component. This is simply because, if \( B_1 \) is any closed set, and \( S \) is any univalent local inverse of \( f^n \) defined on an open set \( B_2 \) containing \( B_1 \) then the diameter of \( SB_1 \) tends to zero uniformly with \( n \), independent of \( S \). This is true whenever \( f \) has no Siegel discs or Herman rings, so holds under our assumptions. In fact, our assumptions ensure that we can take \( B_2 \) to be any open set which is disjoint from the closures of the critical forward orbits. In particular, we can take \( B_2 \) to be a sufficiently small neighbourhood of the closure \( B_3 \) of any component of \( \overline{C} \setminus G_0 \) which does not contain a periodic Fatou component. We can also take \( B_1 \) to be any closed simply-connected set in the closure of complement in a component of \( \overline{C} \setminus G_0 \) of a periodic Fatou component, and in the component of a neighbourhood of the set of parabolic points. It follows from the fact that \( G_0 \) satisfies the properties of 1.3, that \( f^{-N}(G_0) \) satisfies the properties of \( \Gamma \) of 1.3 if \( \delta \) is sufficiently small given \( \delta_1 \). So, for \( \delta_1 < \epsilon/2 \), we choose

\[
G_1 \subset f^{-N}(G_0) \cap B_{\delta_1}(G_0)
\]

as in 1.3. In particular, \( G_1 \) is isotopic to \( G_0 \), and the isotopy can be performed within a \( \delta_1 \)-neighbourhood of \( F(G_0) \). We assume that \( \delta_1 \) is sufficiently small that this neighbourhood is tubular and contained in \( U \).

It is clear that, if we choose \( G_0 \) to be piecewise \( C^1 \), then we can find a piecewise \( C^1 \) \( h \) isotopic to the identity mapping \( G_0 \) to \( G_1 \). It remains to
show that we can ensure the required expanding properties of \( f^N \circ h \). For this, it suffices to bound the derivative of \( h \) from 0, independently of \( N \), because the minimum of the derivative of \( f^N \) on \( U_{2,N} \) tends to \( \infty \) with \( N \), where \( U_{2,N} \) is the component of \( f^{-N}(U) \) which contains \( G_1 \). Of course we have to map vertices of \( G_0 \) to the nearby vertices of \( G_1 \), and edges of \( G_0 \) to the corresponding edges of \( G_1 \), and then we need to extend the map to 2-cells with bounded derivative. We will do this by choosing an appropriate partition of \( G_1 \) and of the 2-cells for \( G_1 \), and will also choose appropriate corresponding partitions of \( G_0 \) and the 2-cells for \( G_0 \), and map these across by \( h \). We also need to choose \( h \) to be the identity outside \( U \). We will define \( U_1 = h^{-1}(U_{2,N}) \).

Let \( W \) be a closed set which is disjoint from the closure of the forward orbits of critical points, and covers all non-periodic Fatou components except in a chosen small neighbourhood of parabolic periodic points. Cover \( W \) by finitely many closed balls, again disjoint from the closures of forward orbits of critical points. We will call this cover \( \mathcal{Y}_0 \). Choose a subset of these balls which covers \( G_0 \). All local inverses of \( f^n \), for all \( n \geq 1 \), are defined and univalent on all of these balls covering \( W \). So we have a covering of \( G_1 \) by sets \( SB \), where \( S \) is a local inverse of \( f^n \), and \( B \) is any of the balls in the cover of \( W \). For each \( n \geq 1 \), we can also cover \( G_1 \) by sets \( SB \) where \( B \in \mathcal{Y}_0 \), and \( S \) is a local inverse of \( f^n \). There is a constant \( C_1 > 0 \) which depends only on \( f \), not on \( N \), or \( S \), such that

\[
C_1^{-1} \leq \left| \frac{S'(w_1)}{S'(w_2)} \right| \leq C_1
\]

for all \( w_1, w_2 \) in the domain of \( S \).

Now we use these sets \( SB \), for varying \( n \), to make partitions of \( G_1 \), and of neighbourhoods of \( G_1 \). Since \( G_1 \subset f^{-N}(G_0) \), it is piecewise smooth. We partition \( G_1 \) by partitioning each edge into arcs which contain \( SB \cap G_1 \) for one set \( B \) in the cover of \( G_0 \), and for one local inverse \( S \) of \( f^N \), and intersect at most \( r_1 \) such sets, for a suitable integer \( r_1 \), depending only on the index of \( \mathcal{Y}_0 \). We call this partition \( \mathcal{X}_N \). By the properties of the local inverses, the length of each arc \( I \subset G_1 \), for \( I \in \mathcal{X}_N \), is boundedly proportional to the diameter of \( SB \), for any set \( SB \) intersecting \( I \), for \( S \) a local inverse of \( f^N \). We also have a cover \( \mathcal{Y}_N \) of a neighbourhood \( A_N \) of \( G_1 \) by sets \( SB \), for balls \( B \) in the cover of \( W \) and local inverses \( S \) of \( f^N \). We choose this neighbourhood so that the intersection of \( A_N \) with each 2-cell for \( G_1 \) is an annulus, any set \( SB \in \mathcal{Y}_N \) which intersects \( \partial A_N \) does not intersect \( G_1 \), and there is a path from any point of \( \partial A_N \) to \( G_1 \) which crosses \( \leq r_1 \) of the sets \( SB \in \mathcal{Y}_N \), assuming that \( r_1 \) is large enough, again depending only on the index of \( \mathcal{Y}_0 \). We can form similar covers \( \mathcal{Y}_n \), for each \( n > 0 \), using local
inverses of $f^n$, for neighbourhoods $A_n$ of $G_1$ with similar properties. For $r_2$
depending only on $r_1$ and the index of the cover $\mathcal{Y}_0$, we have $A_n \subset A_{n-r_2}$
for all $n \geq 0$, and any path from $\partial A_n$ to $A_{n-r_2}$ has to pass through at
least $r_1$ sets of $\mathcal{Y}_n$. For $r_3$ depending only on $r_1$, $r_2$ and the index of $\mathcal{Y}_n$,
there is a path from any point on $\partial A_{n-r_2}$ to $\partial A_n$ which intersects at most
$r_3$ sets in $\mathcal{Y}_n$. Now we make a partition of $A_N$ using paths in from the
endpoints of the sets in $\mathcal{X}_N$ to $\partial A_N$, where these paths pass through $\leq r_1$
sets $SB \in \mathcal{Y}_N$, and are images under the local inverses $S$ of spherical geodesic
segments (for example) in the sets $B$. Thus, these paths are boundedly $C_1$.
This gives a partition $\mathcal{Z}_N$ of $A_N$. The sets in the partition are topological
rectangles, with vertices given by the endpoints of the paths between $G_1$ and
$\partial A_N$. Now, inductively, we make a partition $\mathcal{Z}_{N-kr_2}$ of $A_{N-(k+1)r_2} \setminus A_{N-kr_2}$
for each $k \geq 0$ with $(k+1)r_2 < N$, with rectangles of moduli bounded
from 0. So suppose that $\mathcal{Z}_{n-(k-1)r_2}$ has been constructed for some $k \geq 1$.
We include in the inductive hypothesis that the intersection of each set
of $\mathcal{Z}_{N-(k-1)r_2}$ with $\partial A_{N-kr_2}$ intersects at least $r_1$, and at most $r_2$, of the
sets in $\mathcal{Y}_{N-kr_2}$. This gives a partition of $\partial A_{N-kr_2}$. Each component of
$A_{N-(k+1)r_2} \setminus A_{N-kr_2}$ is an annulus. We then partition $A_{N-(k+1)r_2} \setminus A_{N-kr_2}$
by $C^1$ arcs between the boundary components of each annulus, using some,
but not necessarily all, of the vertices of partition elements of $\mathcal{Z}_{n-(k-1)r_2}$
on $\partial A_{N-kr_2}$. We choose vertices on $\partial A_{N-(k-1)r_2}$ so that any path between
adjacent vertices on $\partial A_{N-(k-1)r_2}$ must cross at least $r_1$, and at most $r_2$, of the
sets in $\mathcal{Y}_{N-(k-1)r_2}$. We choose the same number of vertices on $\partial A_{N-kr_2}$,
with the same number of vertices on corresponding boundary components
and join them by paths with bounded derivative, crossing at most $r_3$ of the
sets in $\mathcal{Y}_{N-(k-1)r_2}$.
Now we construct similar neighbourhoods of $G_0$, and partition in a similar
way. We start by partitioning $G_0$ itself into arcs. We choose this partition
$\mathcal{X}_N'$ corresponding to $\mathcal{X}_N$ so that each edge is partitioned into the same
number of sets as the corresponding edge of $G_1$ by $\mathcal{X}_N$. It is also convenient
to choose each set of $\mathcal{X}_N'$ within $2\delta_1$ of either the corresponding set in $\mathcal{X}_N$,
or within $2\delta_1$ of some Fatou component intersecting it. Here, $\delta_1$ is as in
and this can be done if $\delta$ is sufficiently small given $\delta_1$, that is, $N$ is
sufficiently large given $\delta_1$. We also choose the partition so that, for a suitable
constant $C_2$, the length of a set in $\mathcal{X}_N'$ is $\leq C_2$ times the length of the
corresponding set in $\mathcal{X}_N$. This is possible because all paths in $G_0$ are paths
with uniformly bounded derivative. Then we successively construct annulus
neighbourhoods $A_{N-kr_2}'$ of $G_0$, so that $A_{N-(k+1)r_2} \setminus A_{N-kr_2}'$ is split into
rectangles in a partition $\mathcal{Z}_{N-kr_2}'$, of bounded moduli, and the lengths of
edges of these rectangles are $\leq C_2$ times the lengths of the edges of the
corresponding rectangles in $A_{N-(k+1)r_2} \setminus A_{N-kr_2}$. In order to do this, we
claim that we can choose the $\partial A'_n$ to be uniformly bounded in length, for any $n = N - k r_2$. This is because, if we take any rectangle in $Z_n$ and extend the arcs successively, along partition set edges in $Z_{N-\ell r_2}$ in $A_{N-\ell r_2} \setminus A_{N-(\ell -1)r_2}$ ($1 \leq \ell \leq k$) and in $Z_N$ in $A_N \setminus G_1$ between vertices, to meet $G_1$ in the corresponding vertices from $X_N$, the modulus of this larger rectangle with one edge on $G_1$ is also bounded, because the diameters of rectangles in $Z_n$ decreases geometrically as $n$ increases. So provided we choose $A'_n$ and $Z'_n$ so that lengths along $A'_n$ between vertices are boundedly proportional to corresponding lengths on $G_0$, the upper bound on the ratio of lengths along $G_0$ to lengths along $G_1$ translates to an upper bound on the ratio of lengths along $A'_n$ to lengths along $A_n$. It follows that we can choose $h$ with bounded derivative to map the sets of $Z'_n$ to the sets of $Z_n$ for all $n = N - k r_2$ for $0 \leq k$ and $k r_2 \leq N$, and in addition if $k_0$ is such that $G_0 \subset A_{N-kr_2}$ we can choose $h$ to be the identity outside $A_{N-kr_2}$ for all $k > k_0$ with $kr_2 \leq N$.

Proof of Theorem 1.1 for some $N$. Let $G_0$ and $G_1$ be the graphs as in Lemma 1.4 and let $h$ be as in Lemma 1.4, so that $f^N$ and $f^N \circ h = g$ are expanding on $U_1$. Define $h = h_1$, and, inductively, define $h_n : U \to U$ to be $h_{n-1}$ on $U \setminus g^{-(n - 1)}(U)$, and $f^N \circ h_n = h_{n-1} \circ f^N$ on $g^{-n}(U)$. Also define $G_n = h_n(G_{n-1})$ and $\varphi_n = h_n \circ \cdots \circ h_1$ on $g^{-n}(U)$ for all $n \geq 1$. Then inductively we see that

$$f^N \circ \varphi_n = \varphi_{n-1} \circ g$$

on $U_1$, where we define $\varphi_0$ to be the identity and $\varphi_n = \varphi_{n-1}$ on $U \setminus U_1$. We also have

$$g \circ \varphi_n^{-1} = \varphi_n^{-1} \circ f^N \circ h(U_1).$$

If follows that, if $\lambda^{-1} > 1$ is th expansion constant of $f^N$ on $h(U_1)$,

$$d(\varphi_{n+1}(w), \varphi_n(w)) \leq \lambda^N d(w, \varphi_1(w))$$

for all $w \in U$ and, if $\lambda^{-1}_1 > 1$ is the expansion constant of $g$ on $U_1$,

$$d(\varphi_{n+1}^{-1}(w), \varphi_n^{-1}(w)) \leq \lambda_1^N d(w, \varphi_1^{-1}(w))$$

for all $w \in U$. It follows that $\varphi_n$ converges uniformly on $U$ to a homeomorphism $\varphi : U \to U$ and $\varphi_n^{-1}$ converges uniformly on $U$ to $\varphi^{-1}$. The graph $G' = \varphi(G_0)$ is then the required graph with $G' \subset f^{-N}(G')$.

Proof of Theorem 1.1

We shall construct from this the graph $G$ required for Theorem 1.1, which turns out to be the graph $G = \bigcup_{i=0}^{N-1} f^{-i}(G')$. It is not immediately clear that this works, because it is not clear that the number of intersections between $G_0$ and $f^{-i}(G_0)$ is finite. So we proceed indirectly. For each $i$, there is a graph $G_{1,0}$ which is isotopic to $f^{-i}(G')$, via an isotopy preserving $Z \cup F_0$,
and such that $G_{i,0}$ has only essential intersections with $G'$, that is, every component of $\overline{G \setminus (G' \cup G_{i,0})}$ either intersects $Z \cup F_0$, or the boundary contains at least two components of intersection with each of $G'$ and $G'_i$, or the boundary contains a vertex of $G'$ or $G_{i,0}$. Then we can find a sequence $G_{i,n}$ of graphs with $G_{i,n+1} \subset f^{-N}(G_{i,n})$, such that $G_{i,n}$ also has only essential intersections with $G'$, there is an isotopy $h$ between $G_{i,0}$ and $G_{i,1}$ which lifts under $f^{nN}$ to give an isotopy $h_n$ between $G_{i,n}$ and $G_{i,n+1}$. As before, we can ensure that $f^N \circ h$ is expanding in a neighbourhood of $G' \cup G_{i,0}$, and hence can prove that $G'_i = \lim_{n \to \infty} G_{i,n}$ exists with $G'_i \subset f^{-N}(G'_i)$ is isotopic to $f^{-i}(G_0)$ and has only transverse intersections with $G'$. In fact, $G'_i = f^{-i}(G')$. We see this as follows. Suppose they are not equal. Then the homotopy between $G'_i$ and $f^{-i}(G')$ lifts to a homotopy between $f^{-N}(G'_i)$ and $f^{-i-N}(G')$, for which the distance is strictly less that the distance between $G'_i$ and $f^{-i}(G')$. But $G'_i$ and $f^{-i}(G')$ are equal to subsets of $f^{-N}(G'_i)$ and $f^{-i-N}(G')$, which is a contradiction. So we must have $G'_i = f^{-i}(G')$.

So $G = \bigcup_{i=0}^{N-1} f^{-i}(G')$ is our required graph. Then $G$ is a graph with finitely many edges and vertices, disjoint from $Z \cup F_0$, and $G \subset f^{-1}(G)$. It remains to show that every simple closed loop $\gamma$ in $G$ is a quasi-circle. We use the characterisation of quasi-circles using the bounded turning property of [1]. The technique is similar to the above, but now that we have a Markov partition for $G$, we can use that. For a suitable $n$, let $U_n$ be the neighbourhood of $G$ consisting of the closures of all components of $f^{-n}(U \setminus G)$ which either intersect $G$ or are separated from $G$ by at most two other components. For a suitable $k_0$, the ratios of the minimum diameters of paths between two components of $f^{-n}(U \setminus G)$ which are not adjacent, through at most $k_0$ components, is bounded, where the bound depends only on $f$ and $k_0$ and the minimum distance between vertices of $G$. This ratio is independent of $n$, and $G \subset f^{-n}(G)$ for all $n$. So for any two points $w_1$ and $w_2 \in G$ we can choose $n$ so that they are separated by between $k_1$ and $k_0$ sets in $f^{-n}(U \setminus G)$, for a suitable $n$, and the bounded turning property follows.

2. Boundary of existence of Markov partition

The main motivation for constructing Markov partitions as in Section [1] is that Markov partitions with such properties exist on an open subset of a suitable parameter space. One can then use such partitions to analyse dynamical planes of maps in a subset of parameter space, and this subset of parameter space itself, and try to follow at least part of the programme introduced by Yoccoz for quadratic polynomials, and generalised by others, including Roesch [12] to other families of rational maps.

We have the following lemma.
Lemma 2.1. Let $f$ be a rational map. Let $G \subset \overline{\mathbb{C}}$ be a graph, and $U$ a connected closed neighbourhood of $G$ such that the following hold.

- $G \subset f^{-1}(G)$.
- $U$ is disjoint from the set of critical values of $f$.
- $U$ contains the component of $f^{-1}(U)$ containing $G$, and, for some $N > 0$, $\text{int}(U)$ contains the component of $f^{-N}(U)$ containing $G$.

Then for all $g$ sufficiently close to $f$ in the uniform topology, the properties above hold with $g$ replacing $f$ and a graph $G(g)$ isotopic to the graph $G = G(f)$ above, and varying continuously with $g$.

In particular, these properties hold for nearby $g$, if $f$ is a rational map such that the forward orbit of every critical point is attracted to an attractive or parabolic periodic orbit, the closures of any two periodic Fatou components are disjoint, and $G$ is a graph with the properties above, and which is also disjoint from the closure of any periodic Fatou component.

Proof. First we note that the hypotheses do hold for $f$ and $G$ as in the final sentence. For if we take any sufficiently large $n$ given $\varepsilon$, every component of $\overline{\mathbb{C}} \setminus f^{-n}(G)$ is either within $\varepsilon$ of a single Fatou component or has diameter $< \varepsilon$. Under the given hypotheses on $f$, only finitely many Fatou components of $f$ have diameter $< \varepsilon$. So for sufficiently large $n$, if we take $U$ to be the union of the closures of all components of $\overline{\mathbb{C}} \setminus f^{-n}(G)$ which intersect $G$, then $U$ contains the component of $f^{-1}(U)$ containing $G$. Moreover, if we take $N$ sufficiently large given $f$ and $U$, the maximum diameter of any component of $f^{-N}(W)$, for any component $W$ of $\overline{\mathbb{C}} \setminus f^{-n}(G)$, is strictly less than the minimum distance of $G$ from $\partial U$. For this $N$, the component $U_1$ of $f^{-N}(U)$ which contains $G$ is contained in $\text{int}(U)$. We can also assume, by taking $N$ sufficiently large, that $f^N : U_1 \rightarrow U$ is expanding in the spherical metric, with expansion constant suitably large for what follows. (Proof of expansion is a standard argument, but the proof of a slightly more precise statement is given in Lemma 2.3 below.)

For future purpose, we write $G_0$ for $G$. Then for $g$ sufficiently close to $f$ there is a component $U_1(g)$ of $g^{-N}(U)$ which varies isotopically for $g$ near $f$, with $U_1(f) = U_1, \overline{U_1(g)} \subset U$ and $g^N : U_1(g) \rightarrow U$ is sufficiently strongly expanding for the methods of Section 1 to work. It follows that there is a graph $G_1(g) \subset g^{-N}(G_1(g)) \cap U_1(g)$ isotopic to $G_0$ with $G_1(f) = G_0$. Then as in Section 1 we construct $G_{n,g}$ inductively varying isotopically for $g$ near $f$, with:

- $G_n(f) = G_0$ for all $n$;
- $G_{n+1}(g) \subset g^{-N}(G_n(g))$ for all $n$;
- $G_0 = G_0(g)$ for all $g$ near $f$. 

For $g$ sufficiently near $f$, the expansion constant of $g^N$ on $U_1(g)$ is sufficiently strong that \( \square \) holds, and hence we obtain $G(g) = \lim_{n \to \infty} G_n(g)$ isotopic to $G_0$ with $G(g) \subset \text{int}(U) \cap g^{-N}(G(g))$. We claim that we also have $G(g) \subset g^{-1}(G(g))$. For suppose not so. Then we have an isotopy of $G(g)$ into $g^{-1}(G(g))$, which extends continuously from the inclusion of $G_0 = G_0(f)$ in $f^{-1}(G_0)$. This isotopy lifts to an isotopy of $g^{-N}(G(g))$ into $g^{-N-1}(G(g))$, for which the maximum distance is strictly less. But this gives a contradiction because this lifted isotopy includes the original one. So $G(g) \subset g^{-1}(G(g))$, as required.

So we see that there are natural conditions under which an isotopically varying graph $G(f)$ exists, with $G(f) \subset f^{-1}(G(f))$, for an open connected set of $f$ which are not all hyperbolic. In fact these open connected sets will intersect infinitely many hyperbolic components. We also have an isotopically varying Markov partition $\mathcal{P}(f)$ given by

$$\mathcal{P}(f) = \{U : U \text{ is a component of } \overline{C} \setminus G(f)\}.$$  

We now proceed to investigate the boundary of the set of $f$ in which $G(f)$ and $\mathcal{P}(f)$ exist.

**Theorem 2.2.** Let $V$ be a connected component of an affine variety over $\mathbb{C}$ of rational maps $V$ in which the set $Y(f)$ of critical values varies isotopically. Let $V_1$ be a maximal connected subset of $V$ such that, for $f \in V_1$, there exist a finite connected graph $G(f)$, a closed neighbourhood $U(f)$ of $G(f)$, and an integer $n(f) > 0$ with the following properties.

- $G(f)$ varies isotopically with $f$ for $f \in V_1$.
- $G(f) \subset f^{-1}(G(f))$.
- $\partial U(f) \subset f^{-n(U)}(G(f)) \setminus G(f)$.
- $U(f)$ contains the component of $f^{-1}(U(f))$ which contains $G(f)$.
- $Y(f) \cap U(f) = \emptyset$.
- For any component $U$ of $\overline{C} \setminus G(f)$, all components of $f^{-1}(U)$ are discs.

Then if $V_2 \subset V_1$ is a set such that $\overline{V_2} \setminus V_1 \neq \emptyset$, where the closure denotes closure in $V$, the integer $n(g)$ is unbounded for $g \in V_2$.

**Definition** We shall say that $Y(g)$ is combinatorially bounded from $G(g)$ for $g \in V_2$ if $n(g)$ as above is bounded for $g \in V_2$, that is, for some $N$, there is a closed neighbourhood $U(g)$ of $G(g)$ with boundary in $g^{-N}(G(g)) \setminus G(g)$ which is disjoint from $Y(g)$, for all $g \in V_2$, and such that $U(g)$ contains the component of $g^{-1}(U(g))$ which contains $G(g)$.

**Remarks** 1. Because the critical value set $Y(f)$ varies isotopically for $f \in V_1$, the set of critical points also varies isotopically.
2. For $f \in V_1$, the hypotheses of [2.1] are satisfied by the set $U = U(f)$ as in Theorem [2.2] with $N = n(f)$ because $f^{-n(f)}(\partial U(f)) \subset f^{-2n(f)}(G(f)) \setminus f^{-n(f)}(G(f))$ is disjoint from $\partial U(f)$.

We now establish some basic properties of the dynamics of $g$ in a neighbourhood of $G(g)$, for $g \in V_1$. 

**Lemma 2.3.** Let $g \in V_1$, for $V_1$ as in [2.2] and let $U(g)$ be as in [2.2]. Then for sufficiently large $N$, $g^N : G(g) \rightarrow G(g)$ is expanding with respect to the spherical metric. If $U_1(g)$ denotes the component of $g^{-N}((U(g))$ which contains $G(g)$, and the modulus of any component of int$(U(g)) \setminus U_1(g)$ adjacent to $\partial U(g)$ is bounded below, then the expansion constant of $g^N$ is bounded from 1.

**Proof.** Since there are no critical values of $g$ in $U$, $g^N : U \rightarrow U(g)$ is a local isometry with respect to the Poincaré metrics on $U_1$ and $U$. But the Poincaré metric $d_1$ on $U_1$ is strictly larger than the restriction $d$ to $U_1$ of the Poincaré metric $d_1$ on $U$. If $\text{modulus}(A) \geq c > 0$ for any component $A$ of $\text{int}(U) \setminus U_1$ adjacent to $\partial U(g)$, then $d_1 \geq \mu(c)d$ for $\mu(c) > 1$. So the derivative of $g^N$ on $U_1$ with respect to the Poincaré metric on $U$ is strictly $> \mu(c)$ in modulus. \hfill \Box

2.4. **Real-analytic coordinates on** $G(g)$. A key idea in the proof of [2.2] is to use real-analytic coordinates on the graph $G(g)$ for $g \in V_1$, provided by the normalisations of the sets in the complement of the graph. Let $P_i(g)$ be the closures of the components of $\mathbb{C} \setminus G(g)$ for $1 \leq i \leq k$, so that 

$$G(g) = \bigcup_{i=1}^k \partial P_i(g).$$

We have uniformising maps $\varphi_{i,g} : P_i(g) \rightarrow \{z | z| \leq 1\}$ for each $1 \leq i \leq k$, which are holomorphic between interiors, and unique up to post-composition with Möbius transformations. Then we have a collection of maps $\varphi_{i,g} \circ g \circ \varphi_{i,g}^{-1}$, defined on subsets of the closed unit disc, and mapping onto the closed unit disc. Each of these maps is holomorphic on the intersection of its domain with the open unit disc, and extends by the Schwarz reflection principle to a holomorphic map on the reflection $z \mapsto \overline{z}$ of this domain in the unit circle. In particular, each such map is real analytic on the intersection of its domain with the unit circle.

Now $g : g^{-1}(P_i(g)) \rightarrow P_i(g)$ is a branched covering, and, by assumption, each component of $g^{-1}(P_i(g))$ is conformally a disc, and the closure of each component is a closed topological disc, and let $I(i)$ denote the (finite)set of components of $g^{-1}(P_i(g))$. Let $\psi_{i,g} : g^{-1}(P_i(g)) \rightarrow \{z : |z| \leq 1\} \times I(i)$ be a uniformising map, once again, holomorphic on the interior and unique up to post-composition with a Möbius transformation on each component. Then $\varphi_{i,g} \circ g \circ \psi_{i,g}^{-1}$ is a disc-preserving Blaschke product on each of a finite union
of discs, mapping each one to the same disc whose degree is the degree of \( g|P_i(g) \) — with no other restriction, unless we normalise the maps \( \varphi_{i,g} \) and \( \psi_{i,g} \) in some way, which we might want to do. Each map \( \varphi_{i,g} \circ g \circ \varphi_{j,g}^{-1} \) where defined, is of the form \( \varphi_{i,g} \circ g \circ \varphi_{j,g}^{-1} \circ \psi_{j,g} \circ \varphi_{j,g}^{-1} \). Now we establish an expansion property of these maps.

**Lemma 2.5.** Let \( X(g) \) denote the vertex set of \( G(g) \). Suppose that \( N \) is such that for any \( i \) and \( j \) and component \( Q \) of \( g^{-N}(P_j(g)) \) with \( Q \subset P_i(g) \), at least one component of \( \partial P_i(g) \setminus \partial Q \) contains at least two vertices of \( G(g) \), and the moduli of

\[
\left( \bigcup_{i \in I} P_i(g), g^{-N}(X(g)) \right) \cap \partial \left( \bigcup_{i \in I} P_i(g) \right)
\]

are bounded for any finite set \( I \) such that \( \bigcup_{i \in I} P_i(g) \) is a topological disc. Then the expansion constants of the maps \( \varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1} \) with respect to the Euclidean metric, where defined, are bounded from 1 for some bounded \( \ell \geq 1 \).

**Remark** If \( D \) denotes the closed unit disc and \( A \subset D \) is a finite set, then we say that the moduli of \( (D, A) \) are bounded \( A \) contains less than four points, or if the cross-ratio of any subset of \( A \) consisting of four points is bounded above and below. If \( Q \) is a closed topological disc and \( B \subset \partial Q \) is finite, then we say that the moduli of \( (Q, B) \) are bounded if the moduli of \( (\varphi(Q), \varphi(B)) \) are bounded, where \( \varphi : Q \to D \) is a homeomorphism which is holomorphic on the interior of \( Q \).

**Proof.** For the maps \( \varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1} \), it suffices to bound below the derivative of \( \varphi_{i,g} \circ g^{N} \circ \varphi_{j,g}^{-1} \), with respect to a suitable metric \( d_p \) which we can show to be boundedly Lipschitz equivalent to the Euclidean metric \( d_e \). Then the derivative of \( \varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1} \) with respect to \( d_p \) is \( \geq \mu^k \), and if \( d_p/d_e \) is bounded between \( C_{\pm 1} \) for some \( C \geq 1 \), we see that the derivative with respect to \( d_e \) is \( \geq C^{-1}\mu^{Nk} \), giving expansion for all \( k \) such that \( C^{-1}\mu^{Nk} > 1 \). So it remains to define \( d_p \) so that these properties are satisfied. This is the restriction of a Poincaré metric on a suitable surface, one for each component \( e \) of \( \partial Q \cap \partial P_i(g) \), or union of two such components round a vertex of \( g^{-N}(G(g)) \) in \( \partial Q \), where \( Q \) is the closure of a component of \( \overline{C} \setminus g^{-N}(G(g)) \) with \( Q \subset P_i(g) \) and \( e \subset \partial Q \). For each such component, we consider a union \( Q' \) of closures of components of \( \overline{C} \setminus g^{-N}(G(g)) \) contained in \( P_i(g) \), such that \( Q' \) is a topological disc and such that the connected component \( e' \) of \( \partial Q' \cap \partial P_i(g) \) which contains \( e \) has \( e \) in its interior. We can assume without loss of generality, replacing \( G(g) \) by \( g^{-M}(G(g)) \) if necessary, that the image of \( Q' \) under \( g^{N} \) is also a closed topological disc — obviously of the form \( \bigcup_{j \in I} P_j(g) \) — and that \( g^{N} \) is a homeomorphism on \( e' \). So there is a map of \( Q' \) to \( \{ z : |z| \leq 1, \text{Im}(z) \geq 0 \} \) which maps \( e' \) to the interval
[−1, 1], and which is conformal on the interior. We then take the restriction of the Poincaré metric on the unit disc to (−1, 1). This is the metric $d_p$ on \( \text{int}(e') \supseteq e \). The image of $e$ under $g^N$ is an edge of $G(g)$ in $\partial P_j(g)$, or a union of two edges round a vertex in $\partial P_j(g)$, for some $j \in J$. We take the corresponding metric $d_p$ on each edge of $g^{-N}(G(g))$ in $\partial P_j(g)$. Take any edge $e_1$ of $g^{-N}(G(g))$ or union of two edges of $g^{-N}(G(g))$ which are subsets of edges of $G(g)$, adjacent to a vertex of $G(g)$ in $P_j(g)$, with $e_1 \subset e$. Let $Q_1$ be the component of $\mathbb{C} \setminus g^{-N}(G(g))$, and $e_1 \subset \partial Q_1$ and $Q_1 \subset g^N(Q)$. Let $Q'_1$ union of closures of components of $\mathbb{C} \setminus g^{-N}(G)$ with $Q_1 \subset Q'_1$ which is used to define the metric $d_p$ on $e_1$. Then $Q'_1 \subset g^N(Q')$, and by the hypotheses, if we double $g^N(Q')$ across $g^N(e')$ by Schwarz reflection, and then normalise, the image of the double of $Q'_1$ within this is contained in \( \{z : |z| \leq r\} \), for some $r$ bounded by 1. It follows that $g^N$ is expanding with respect to the metric $d_p$. 

Now each edge of $G(g)$ is in the image of two maps $\varphi_{i_1,g}$ and $\varphi_{i_2,g}$, where the edge is a connected component of $\partial P_{i_1}(g) \cap \partial P_{i_2}(g)$. Since $G(g) \subset g^{-1}(G(g))$, it is also the case that each edge is contained in a union of components of sets $g^{-1}(P_{i_1}(g) \cap P_{i_2}(g))$, where these sets are disjoint apart from some common endpoints. It follows that from $g$, we obtain two real-analytic maps $h_{1,g}$ and $h_{2,g}$, each mapping a finite union of intervals to itself, mapping endpoints to endpoints, except for being two-valued at finitely many interior points in the intervals, but at these points, the right and left-derivatives exist and coincide, so that the derivative is single valued at such points, and extends continuously in the neighbourhood of any such point. These two maps are quasi-symmetrically conjugate, because the maps $\varphi_{i,g}$ are quasi-conformal. The quasi-symmetry is unique, and the pair $(\mathbb{C}, g^{-1}(G))$ can be reconstructed from it, up to Möbius transformation of $\mathbb{C}$. Now we can make this idea more precise. Lemma 2.5 shows that the hypotheses are satisfied.

**Lemma 2.6.** Let $I_{i,r}$ be finite intervals for $1 \leq i \leq k$ and $r = 1, 2$. Let $h_1 : \bigcup_{i=1}^k I_{i,1} \to \bigcup_{i=1}^k I_{i,1}$ and $h_2 : \bigcup_{i=1}^k I_{i,2} \to \bigcup_{i=1}^k I_{i,2}$ be two $C^2$ maps which are multivalued just at points which are mapped to endpoints of intervals, but with well-defined continuous derivatives at such points, such that $h_r(I_{i,r})$ is a union of intervals $I_{j,r}$ for each of $r = 1, 2$, and $I_{i,1} \subset h_1(I_{i,1})$ if and only if $I_{j,2} \subset h_2(I_{i,2})$, and $I_{i,r} \cap h_{-1}^1(I_{j,r})$ has at most one component, for both $r = 1, 2$. Suppose also that there is $N$ such that $h_1^N$ and $h_2^N$ are expanding with respect to the Euclidean metric for all $n \geq N$. Then $h_1$ and $h_2$ are quasi-symmetrically conjugate, with the norm of the quasi-symmetric conjugacy bounded in terms of $N$ and of the bound of the expansion constants of $h_1^N$ and $h_2^N$ from 1.
Proof. This is standard. We simply choose \( \varphi_0 : \bigcup_{i=1}^{k} I_{i,1} \to \bigcup_{i=1}^{k} I_{i,2} \) to be an affine transformation (for example) restricted to \( I_{i,1} \), mapping \( I_{i,1} \) to \( I_{i,2} \), for each \( 1 \leq i \leq k \). Then \( \varphi_n \) is defined inductively by the properties \( h_2 \circ \varphi_{n+1} = \varphi_n \circ h_1 \) and \( \varphi_{n+1}(I_{i,1}) = I_{i,2} \) for each \( 1 \leq i \leq k \). Then \( \varphi_0 \circ h_1^n = h_2^n \circ \varphi_n \) for all \( n \), and we deduce from this that \( |\varphi_n(x) - \varphi_{n+1}(x)| \leq C_2 \lambda^n \) for all \( x \) and \( n \), for some constant \( C_2 \) depending on \( C_1 \), where \( |h_2^n(x) - h_2^n(y)| \geq C_1 \lambda^{-n} \) for all \( n \) and all \( x \) and \( y \) such that \( h^m(x) \) and \( h^m(y) \) are in the same set \( I_{i,m} \), for all \( 0 \leq m \leq n \). Then \( \varphi_n \) converges uniformly to \( \varphi \), with \( \varphi \circ h_1 = h_2 \circ \varphi \). Similarly, using the expanding properties of \( h_1 \), we deduce that \( \varphi_{n}^{-1} \) converges uniformly to \( \varphi^{-1} \).

To prove quasi-symmetry of \( \varphi \), we use the standard result that \( (h^n_p)' \) varies by a bounded proportion on any interval \( J \) such that \( h^n_p(J) \) is a union of at most two subintervals of \( \bigcup_{i=1}^{k} I_{i,n} \). This uses continuity of the derivative across the finitely many discontinuities of \( h_r \). So then given any \( x \neq y \in \bigcup_{i=1}^{k} I_{i,1} \) such that \( |x - y| \) is sufficiently small, we choose the greatest \( n \) such that \( |h^n_1(x) - h^n_1(y)| \leq c \), for a suitable constant \( c > 0 \) such that any interval of \( \bigcup_{i=1}^{k} I_{i,1} \) which has length \( \leq c \) is mapped to a union of at most two intervals of \( \bigcup_{i=1}^{k} I_{i,1} \). Then \( |h^{n+p}_1(x) - h^{n+p}_1(y)| \) is bounded above and below for any bounded \( p \), and \( (h^{n+p}_1)' \) varies by a bounded proportion on the interval \([x, y]\). So does the derivative \( S' \), on the smallest interval containing \( h^{n+p}_1(x), h^{n+p}_1(y) \), where \( S \) is the branch of \( h^{-(n+p)}_2 \) such that \( \varphi_{n+p} = S \circ \varphi_0 \circ h^{n+p}_1 \).

We can choose \( p \) so that each of the points \( h^n_1(x), h^n_1(y), h^n_1((x + y)/2) \) is separated by at least two points from \( \bigcup_{i=1}^{k} h^{-(n+p)}_1(\partial I_{i,1}) \) — but only boundedly many, by the bound on \( p \). Now \( \varphi_m = \varphi_{n+p} \) on \( \bigcup_{i=1}^{k} h^{-(n+p)}_1(\partial I_{i,1}) \) for all \( m \geq n + p \), and hence \( \varphi = \varphi_{n+p} \) on \( \bigcup_{i=1}^{k} h^{-(n+p)}_1(\partial I_{i,1}) \). If \( z_1, z_2 \) and \( z_3 \) are any three distinct points of \( \bigcup_{i=1}^{k} h^{-(n+p)}_1(\partial I_{i,1}) \) which are either between \( x \) and \( y \), or the nearest point on one side, then \( |\varphi_{n+p}(z_1) - \varphi_{n+p}(z_2)/| |\varphi_{n+p}(z_1) - \varphi_{n+p}(z_3)| \) is bounded and bounded from 0, that is, \( |\varphi(z_1) - \varphi(z_2)/| |\varphi(z_1) - \varphi(z_3)| \) is bounded and bounded from 0. But then since \( |\varphi(x) - \varphi((x + y)/2)| \) is bounded between some such \( |\varphi(z_1) - \varphi(z_2)| \) and \( |\varphi(z_3) - \varphi(z_4)| \), and similarly for \( |\varphi(y) - \varphi((x + y)/2)| \), we have upper and lower bounds on \( |\varphi(x) - \varphi((x + y)/2)|/|\varphi(y) - \varphi((x + y)/2)| \), and quasi-symmetry follows. \( \Box \)

We deduce the following.

**Lemma 2.7.** Let \( V_1 \) be as in Theorem 2.3. For \( f \in V_1 \), let \( P_i(f) \), \( \varphi_{i,f} \) and \( \psi_{i,f} \) be as previously defined. Let \( \{g_n : n \geq 0\} \) be any sequence in \( V_1 \) such that \( Y(g_n) \) is combinatorially bounded from \( G(g_n) \) for \( n \geq 0 \), and let \( g_n \to g \). Let \( X(g_n) \) denote the vertex set of \( G(g_n) \). Then \( g \in V_1 \) if the moduli of

\[
\left( \bigcup_{i=1}^{k} P_i(g_n), g^{-\ell}(X(g_n)) \cap \partial(\bigcup_{i=1}^{k} P_i(g_n)) \right)
\]
are bounded as \( n \to \infty \) for any fixed \( \ell \), and any finite set \( I \) such that \( \cup_{i \in I} P_i(g_n) \) is a topological disc, and, using this to normalise the maps \( \varphi_{i,g_n} \) and \( \psi_{i,g_n} \), the disc-preserving Blaschke products \( \varphi_{i,g_n} \circ g_n \circ \psi_{i,g_n}^{-1} \) are also bounded.

**Proof.** The bounds on moduli and Blaschke products ensure that the real analytic maps \( h_{1,g_n} \) and \( h_{2,g_n} \) have derivatives which are bounded above and below. Also, they extend to Blaschke products on neighbourhoods of intervals of the unit circle. By the hypotheses, there is a closed neighbourhood \( U(g_n) \) of the graph \( G(g_n) \), disjoint from \( Y(g_n) \), such that \( U(g_n) \) has boundary in \( g_n^{-r}(G(g_n)) \) for some \( r \) independent of \( n \). Moreover, \( U(g_n) \) contains the component of \( g_n^{-1}(U(g_n)) \) containing \( G(g_n) \), and there is \( N \) independent of \( n \), such that \( \text{int}(U(g_n)) \cap g_n^{-N}(U(g_n)) \) containing \( G(g_n) \). We have seen from [2.5] and [2.6] that the maps \( h_{1,g_n} \) and \( h_{2,g_n} \) are boundedly quasi-symmetrically conjugate, that is, there is a quasi-symmetric homeomorphism \( \varphi_n \) whose domain is the domain and image of \( h_{1,g_n} \) and whose image is the domain and image of \( h_{2,g_n} \), that is, a finite union of intervals in each case, such that \( \varphi_n \circ h_{1,g_n} = h_{2,g_n} \circ \varphi_n \).

Then \( \varphi_n \) can be used to define a Beltrami differential \( \mu_n \) on \( \overline{\mathbb{C}} \), which is uniformly bounded independently of \( n \), as follows. This sphere is, topologically, a finite union of discs, with the boundary of each disc written as a finite union of arcs, and with each arc identified with one other, from a different disc, by \( \varphi_n \) in one direction and \( \varphi_n^{-1} \) in the other. It is convenient to identify this sphere with the Riemann sphere \( \overline{\mathbb{C}} \), in such a way that each of the discs has piecewise smooth boundary, and the maps identifying the copies of the closed unit disc with the image discs in \( \overline{\mathbb{C}} \) are piecewise smooth. The union of the images of copies of the unit circle form a graph \( \Gamma \subset \overline{\mathbb{C}} \). We then define a quasi-conformal homeomorphism \( \psi_n \) from the union of copies of the closed unit disc to \( \mathbb{C} \) such that, whenever \( I_1 \) and \( I_2 \) are arcs on the boundaries of discs \( D_1 \) and \( D_2 \), identified by \( \varphi_n : I_1 \to I_2 \), we have \( \psi_n \) on \( I_2 \) is defined by \( \psi_n \circ \varphi_n^{-1} \), using \( \varphi_n^{-1} : I_2 \to I_1 \) and \( \psi_n : I_1 \to \overline{\mathbb{C}} \). The q-c norm of \( \psi_n \) can clearly be bounded in terms of the q-s norm of \( \varphi_n \), and the identification we choose of the copies of the closed unit disc with their images in \( \overline{\mathbb{C}} \). This means that the q-c norm of \( \varphi_n \) can be bounded independently of \( n \). We then define \( \mu_n = (\varphi_n)_*0 \) on the image of each copy of the open unit disc, where 0 simply denotes the Beltrami differential which is 0 everywhere on the open unit disc. Then \( \mu_n \) is defined a.e. on \( \overline{\mathbb{C}} \), and is uniformly bounded, in \( n \), in the \( L_\infty \) norm.

So there is a quasi-conformal map \( \chi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \), with q-c norm which is uniformly bounded in \( n \), such that \( \mu_n = \chi_n^*0 \), where, here, 0 denotes the Beltrami differential which is 0 everywhere on \( \overline{\mathbb{C}} \). By construction, there is a
conformal map of $\mathbb{C}$ which maps $\chi_n(\Gamma)$ to $G(g_n)$. So we can assume without loss of generality that $\chi_n(\Gamma) = G(g_n)$. By taking limits, we can assume that $\chi_n$ has a limit $\chi$ in the uniform topology, which is a quasi-conformal homeomorphism. So $\chi_n(\Gamma)$ has a limit $\chi(\Gamma)$, which is also a graph, and since $G(g_n) \subset g_n^{-1}(G(g_n))$, we have $\chi(\Gamma) \subset g^{-1}(\chi(\Gamma))$. The sequence of sets $U(g_n)$ also has a limit $U(g)$ with boundary in $g^{-\ell}(\chi(\Gamma))$, such that $U(g)$ is a closed neighbourhood of $g(\chi(\Gamma))$, contains the component of $g^{-1}(U(g))$ which contains $\chi(\Gamma)$, and such that $\text{int}(U(g))$ contains the component of $g^{-N}(U(g))$ containing $\chi(\Gamma)$. So we have $g \in V_1$, with $\chi(\Gamma) = G(g)$, as required. 

Since $G(f)$ varies isotopically in $V_1$, the set $X(f)$ of vertices of $G(f)$ also varies isotopically in $V_1$. But $X(f)$ is a finite forward invariant set for all $f \in V_1$. Hence $X(g)$ varies locally isotopically for $g$ in the dense open subset $V_0$ of $V$ such that the multiplier of any periodic points in $X(g)$ is not 1, and there are no critical points in $X(g)$. We have $V_1 \subset V_0$.

**Definition** A path $\alpha$ with endpoints in $X(g)$ is has homotopy length $\leq M$ if it can be isotoped, by an isotopy which is the identity on $X(g)$, to be arbitrarily uniformly close to a path in $G(g)$ which crosses $\leq M$ edges of $G(g)$.

**Lemma 2.8.** Let $V$ and $V_1$ be as in 2.2. Let $V_0$ be as above. Fix $g_0 \in V_1$. Let $W_0$ be a path-connected compact subset of $V$ containing $g_0$, and let $M_0 > 0$ be given. There is $M_1 = M_1(M_0, W_0)$ with the following property. Let $g \in V_1$ be joined in $V$ to $g_0$ by a path in $V_0$. If $e$ is an edge of $G(g)$ and $e' \subset e$ is a connected set which shares its first endpoint with $e$, and $\alpha$ is any extension of $e'$ by spherical length $\leq M_0$ to a path with both endpoints in $X(g)$, then $\alpha$ has homotopy length $\leq M_1$.

**Proof.** Let $g_t$ be a path between $g_0$ and $g = g_1$. Since $V \setminus V_0$ has codimension two, we can assume without loss of generality, enlarging $W_0$ if necessary, that $g_t \in V_0 \cap W_0$ for all $t$, so that $X(g_t)$ varies isotopically. We can choose the path $g_t$ so that its length is bounded in terms of $W_0$, using any suitable Riemannian metric on $V$, for example, that coming from the embedding of $V$ in $\mathbb{C}^n$ (since $V$ is an affine variety).

Now given $N > 1$, there is $k$ such that $g^k(e'')$ is a union of at least $N$ edges for each edge $e''$ of $G(g)$. This is true for all $g \in V_1$, because the dynamics of the map $g : G(g) \to G(g)$ is independent of $g$. We take $N = 2$. For this $k$ (or, indeed, any strictly positive integer), $\cup_{t \geq 0} g^{-\ell k}(X(g))$ is dense in $G(g)$, because, for any edge $e$ of $G(g)$, the maximum diameter of any component of $g^{-n}(e)$ tends to 0 as $n \to \infty$. So it suffices to prove the lemma for $e' \subset e$ sharing first endpoint with $e$ and with the second endpoint in $g^{-\ell k}(X(g))$ for some $\ell \geq 0$, but we cannot obtain any bound on $\ell$. So fix such an $e'$. For
each \( i \leq \ell \), let \( e_{ik} = e_{ik}(g) \subseteq e \) such that \( g^i(e_{ik}) \) is an edge of \( G(g) \), hence with endpoints in \( X(g) \), such that the second endpoint of \( e' \) is in \( e_{ik} \), and is not the first endpoint of \( e_{ik} \).

Any point of \( \mathbb{C} \) is spherical distance \( \leq \pi \) from a point of \( X(g) \) (assuming the sphere has radius 1). Any path of bounded (spherical) distance between points of \( X(g) \) is homotopically bounded, because of the bounded distance between \( X(g_0) \) and \( X(g) \). We suppose for contradiction that, for any path \( \alpha_0 \) of length \( \leq M_0 \) from the second endpoint of \( e' \) to a point of \( X(g) \), the path \( e' \ast \alpha_0 \) has homotopy length \( \geq l_1 \). Then \( g^i(e' \cup \alpha_0) \) has homotopy length \( \geq 2M_1 \). Now let \( \alpha_k \) be a path of length \( \leq M_0 \) connecting the second endpoint of \( g^k(e') \) to \( X(g) \). Now we have a bound on the homotopy length of \( g^k(e' \setminus e_k) \) depending only on \( k \), because this is a union of a number of edges of \( G(g) \), where the number is bounded in terms of \( k \). We also have a bound in terms of \( k \) and \( M_0 \) (and on \( g_0 \), but \( g_0 \) is fixed throughout) on the spherical length of \( \overline{\alpha_k} \ast g^k(\alpha_0) \), where \( \overline{\alpha_k} \) denotes the reverse of \( \alpha_k \). This is because the bound on the path between \( g_0 \) and \( g \) gives a bound on the spherical derivative of \( g_k \) in terms of \( M_0 \) and \( k \). If \( \varphi \) is the homeomorphism of \( \mathbb{C} \) given by the isotopy from the identity mapping \( X(g) \) to \( X(g_0) \), then \( \varphi \) is bounded in terms of \( M_0 \). So we have a bound on the spherical length of \( \varphi(\overline{\alpha_k} \ast g^k(\alpha_0)) \). This is a path between points of \( X(g_0) \). So we have a bound on the homotopy length of this path in terms of \( M_0 \) and \( k \) (and \( g_0 \), but this is fixed throughout). But the homotopy length is the same as the homotopy length of \( \overline{\alpha_k} \ast g^k(\alpha_0) \). So both \( g^k(e' \setminus e_k) \) and \( \overline{\alpha_k} \ast g^k(\alpha_0) \) have homotopy length \( \leq M'_0 \) where \( M'_0 \) is bounded in terms of \( M_0 \) and \( k \). So then \( g^k(e' \cap e_k) \ast \alpha_k \) has homotopy length \( \geq 2M_1 - 2M'_0 > M_1 \) assuming that \( M_1 \) is sufficiently large given \( M'_0 \) and \( k \), that is, sufficiently large given \( M_0 \). Similarly, for each \( i \), \( g^k((e' \cap e_{(i-1)k}) \setminus e_{ik}) \) and \( \overline{\alpha_{ik}} \ast g^k(\alpha_{(i-1)k}) \) have homotopy length \( \leq M'_0 \), and hence we prove by induction that \( g^i(e_{ik} \cap e') \) has homotopy length \( > M_1 \) for all \( i \geq 0 \). For \( i = \ell \) we obtain the required contradiction, because \( g^\ell(e' \cap e_{\ell k}) \) is a single edge. \( \square \)

**Corollary 2.9.** Let \( V, V_1, g_0, M_0, W_0 \) and \( g \) be as in 2.8. There is \( M_2 > 0 \), depending on \( M_0, W_0 \) and \( g_0 \) with the following property. If \( e' \) is any path in an edge of \( G(g) \) then \( e' \) is homotopic, via a homotopy fixing endpoints and \( X(g) \), to a path of (spherical) length \( \leq M_2 \).

**Proof.** It suffices to prove this for paths with one endpoint at \( X(g) \), because \( e' = e'_{ik} \ast e'_{\ell} \) for two such paths in the same edge as \( e' \). So now assume that \( e' \) shares an endpoint with \( e \). Then by 2.8, we can extend \( e' \) by spherical length \( \leq M_0 \) to a path \( \alpha \) with both endpoints in \( X(g) \) so that \( \alpha \) is homotopic, via a homotopy fixing \( X(g) \), to an arbitrarily small neighbourhood of a path crossing \( \leq M_1 \) edges of \( G(g) \). Because the movement of \( X(g_0) \) to \( X(g) \)
is bounded, this means that \( \alpha \) is homotopic, via a homotopy fixing \( X(g) \),
to a path of spherical length \( \leq M'_2 \). Then since \( e' \) can be obtained from \( \alpha \) by adding length \( M_0 \), we obtain the required bound on \( \gamma \) with \( M_2 = M'_2 + M_0 \).

\[ \square \]

**Lemma 2.10.** Let \( V, V_1, g_0, M_0, W_0 \) and \( g \) be as in \[2.8\]. There is \( \varepsilon > 0 \) depending on \( M_0 \) and \( g_0 \) such that for each \( i \), there is some point in \( P_i(g) \) which is distance \( \geq \varepsilon \) from \( \partial P_i(g) \).

**Proof.** It suffices, for some \( x \in P_i(g) \) and for some fixed \( n \), to find a lower bound on the length of \( g^n \alpha \), where \( \alpha \) is any path from \( x \) to \( \partial P_i(g) \). By \[2.9\] we can extend \( g^n \alpha \) by a path \( \gamma \) in some \( \partial P_i(g) \cap g^n(\partial P_i(g)) \) to a point of \( X(g) \), such that \( \gamma \) is homotopic, via a homotopy fixing endpoints and \( X(g) \), to a path of length \( \leq M_2 \), and such that any extension of \( \gamma \) at the other end by a path of length \( \leq M_0 \) to a point of \( X(g) \) has homotopy length \( \leq M_1 \), by \[2.8\]. Both \( M_1 \) and \( M_2 \) are independent of \( n \). But we can choose \( x \in g^{-n}(X(g)) \), for some \( n \), so that if \( \alpha' \) is any path from \( x \) to \( \partial P_i(g) \cap X(g) \) then the homotopy length of \( g^n \alpha' \) is \( > M_3 \), where \( M_3 \) is sufficiently long to force spherical length \( > 2M_2 \). We do this using the bound on the isotopy distance between \( X(g) \) and \( X(g_0) \), and the dynamics of \( g_0 \) on the graph \( G(g_0) \). Then the spherical length of \( g^n \alpha \) is \( > M_2 \), which gives us a strictly positive lower bound on the spherical length of \( \alpha \): in terms if \( n \), which means, ultimately, in terms of \( M_0 \). \[ \square \]

In a similar way, we can prove the following.

**Lemma 2.11.** Let \( V, V_1, g_0, M_0, W_0 \) and \( g \) be as in \[2.8\]. Let \( A \) be any embedded annulus which is a union of \( N_1 \geq 1 \) components of sets \( g^{-1}(P_i(g)) \) (for varying \( i \)) surrounding a union of \( N_2 \geq 1 \) components of sets \( g^{-1}(P_j(g)) \) (for varying \( j \)). Then the modulus of \( A \) is bounded and bounded from 0, where the bounds depend on \( N_1, N_2, M_0, g_0 \) and \( r \).

**Proof.** It suffices to prove this with \( r = 0 \), since the result remains true under branched covers, just depending on \( r \) and the degree of \( g_0 \). The upper bound is clear, from the bound on the diameter of the sets \( P_i(g) \) from \[2.8\] and on the lower bound on the interior of sets \( P_j(g) \) in \[2.10\]. Actually a lower bound on the diameter of the sets \( P_j(g) \) is enough, and this is easily obtained. So now we need to bound the modulus below. For this, we need to bound below the length (in the spherical metric) of any path \( \gamma \) between the two boundary components of \( A \). As in \[2.10\] it suffices to bound below the length of \( g^n(\gamma) \), for some fixed \( n \), and it suffices to show that this length tends to \( \infty \) with \( n \). As in \[2.10\] it suffices to prove this for paths with endpoints in \( X(g) \), in distinct components of \( \partial A \), and this length tends to
∞ because of the bounded homotopy distance of points in \(X(g)\) from \(X(g_0)\), and the homotopy length tends to ∞.

Then using this, we can prove the following.

**Lemma 2.12.** Let \(V, V_1, g_0, M_0, W_0\) and \(g\) be as in 2.8. The moduli of
\[ (\cup_{i \in I} P_i(g), g^{-N}(X(g)) \cap \partial(\cup_{i \in I} P_i(g))) \]
are bounded whenever \(\cup_{i \in I} P_i(g)\) is a topological disc.

**Proof.** Write \(Q = \cup_{i \in I} P_i(g)\), for any fixed \(I\) such that \(Q\) is a topological disc. If \((x_1, x_2, x_3, x_4)\) is an ordered quadruple of four points of \(\partial Q \cap g^{-N}(X(g))\), with \(x_1\) and \(x_2\) not separated in \(\partial Q\) by the set \(\{x_3, x_4\}\), then we define the modulus of \((x_1, x_2, x_3, x_4)\) to be the modulus of the rectangle \(\varphi(Q)\) where \(\varphi\) is conformal on the interior and the vertices are the points \(\varphi(x_i)\). In turn, we define modulus to be the modulus of the annulus formed by identifying the edge of the rectangle joining \(\varphi(x_1)\) and \(\varphi(x_2)\) to the edge joining \(\varphi(x_3)\) and \(\varphi(x_4)\). So it suffices to bound below the modulus of each such quadruple \((x_1, x_2, x_3, x_4)\). But then it suffices to do it in the case when \(x_1\) and \(x_2\) come from adjacent points of \(q^{-N}(X(g))\) on \(\partial Q\), and similarly for \(x_3\) and \(x_4\), because modulus \((A_1) \leq \text{modulus}(A_2)\) if \(A_1 \subset A_2\) and the inclusion is injective on \(\pi_1\). But if we have two disjoint edges on \(\partial Q\), we can make an annulus which includes \(Q\) and encloses a union of partition elements \(P_j(g)\). The partition elements \(P_j(g)\) are those with edges on one path in \(\partial Q\) between the edges associated with \((x_1, x_2)\) and \((x_3, x_4)\). So the lower bound on the modulus of \((x_1, x_2, x_3, x_4)\) comes from the lower bound of this annulus, which was obtained in 2.11.

2.13. **Proof of Theorem 2.2.** We recall that we are making the assumption that \(Y(g_n)\) is combinatorially bounded from \(G(g_n)\). We need to check that the assumptions of Lemma 2.7 are satisfied, since Theorem 2.2 will then immediately follow. Lemma 2.12 gives the bounds on the moduli of
\[ (\cup_{i \in I} P_i(g_n), g_n^{-N}(X(g_n)) \cap \partial(\cup_{i \in I} P_i(g_n))) \]
By 2.11, the set \(Y(g_n)\) is bounded from \(G(g_n)\) by a union of annuli of moduli bounded from 0. Together with the bound on the moduli of \((P_i(g_n), g_n^{-N}(X(g_n)) \cap \partial P_i(g_n)))\), which is just used for normalisation, this gives the required bound on the Blaschke products \(\varphi_{i,g_n} \circ g_n \circ \psi_{i,g_n}^{-1}\) of 2.7 and the proof is completed.

3. **Parametrisation of existence set of Markov partition**

In Section 2 the parameter space \(V\) was a connected component of an affine variety over \(\mathbb{C}\). In this section, we put more restrictions on \(V\). In particular, the restrictions include that \(V\) is of complex dimension one. This means that we are looking at a familiar scenario, in which it is reasonable to suppose that parameter space can be described by movement of a single
critical value. It is certainly possible that the ideas generalise to higher
dimensions. But there are still new features to consider, even for $V$ of
complex dimension one.

We consider the case when $V$ is a parameter space of quadratic rational
maps $g$ with numbered critical points for which one critical point $c_1(g)$ is
periodic of some fixed period and the other, $c_2(g)$, is free to vary. The family
of such maps, quotiented by Möbius conjugation, is of complex dimension
one, and is well known to have no finite singular points. (See, for example,
Theorem 2.5 of [6]). So $V$, or a natural quotient of it, is a Riemann surface,
with some punctures at $\infty$, where the degree of the map degenerates. So we
assume from now on that $V$ is a Riemann surface. We write $v_1(g) = g(c_1(g))$
and $v_2(g) = g(c_2(g))$ for the critical values. Fix a critically finite $g_0 \in V$
for which a graph $G(g_0)$ exists with $G(g_0) \subset g^{-1}(G(g_0))$ and $v_2(g_0) \notin G(g_0)$.

There are simple conditions on $G(g_0)$ under which the results of Section 2
hold. It is enough to assume that $G(g_0)$ does not intersect the boundary
of any periodic Fatou component, separates critical values and separates
periodic Fatou components. In particular, this ensures that the diameters
of the components of $\overline{\mathbb{C}} \setminus f^{-n}(G_0)$, with closures intersecting $G(g_0)$, tend to
0 as $n \to \infty$. Write

$$\mathcal{P} = \mathcal{P}(g_0) = \{ U : U \text{ is a component of } \overline{\mathbb{C}} \setminus G(g_0) \}.$$ 

We write $V(G(g_0), g_0)$ for the largest connected set of $g \in V$ containing $g_0$
for which there exists a graph $G(g)$ varying isotopically from $G(g_0)$ with:

- $G(g) \subset g^{-1}(G(g))$;
- a neighbourhood $U(g)$ of $G(g)$ with boundary in $g^{-r}(G(g))$ for some
  $r$ and not containing $v_2(g)$;
- $U(g)$ contains the component of $g^{-1}(G(g))$ which contains $G(g)$;
- $G(g)$ separates the critical values $v_1(g)$ and $v_2(g)$.

Thus, $V(G(g_0), g_0)$ is the set $V_1$ defined directly after [2.11] if we replace $f$
and $G(f)$ by $g_0$ and $G(g_0)$, and assume suitable conditions, as above, on
$G(g_0)$. We write $V(G(g_0))$ for the union of sets $V(G(g_0), g_1)$ for which there
is a homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\varphi(G(g_0)) = G(g_1), \quad \varphi(g_0(v_1(g))) = g_1^i(v_1(g)) \quad \text{for } i \geq 0, \quad \varphi(v_2(g_0)) = v_2(g_1),$$

and

$$\varphi \circ g_0 = g_1 \circ \varphi \text{ on } G(g_0).$$

Thus, $G(g)$ exists for all $g \in V(G(g_0))$, and varies isotopically on each
component of $V(G(g_0))$, and $V(G(g), g)$ is one of the sets above, so that
there is a homeomorphism from $G(g_0)$ to $G(g)$ with properties as above.
This is slightly ambiguous notation, because the definition of $V(G(g_0))$ uses
the isomorphism class of the dynamical system $(G(g_0), g_0)$, not just the
homeomorphism class of the graph $G(g_0)$, but this seems the best option available.

For $g \in V(G(g_0))$, we write

$$P(g) = \{P : P \text{ is a component of } \overline{G} \setminus G(g)\}.$$  

Where it is convenient to do so, we shall write $G_0(g)$ for $G(g)$. In Section 2 we found a partial characterisation of the boundary of this set. Now we want to try and obtain a parametrisation of the set $V(G(g_0), g_0)$. For any $g \in V(G(g_0))$, and integer $n \geq 0$, we define

$$G_n(g) = g^{-n}(G(g)),$$  

$$P_n(g) = \bigcup_{i=0}^{n} g^{-i}(P(g)) = \{U : U \text{ is a component of } \overline{G} \setminus G_n(g)\}.$$  

3.1. The possible graphs. Let $g_0 \in V$ and $G(g_0)$ be as above. Following a common strategy, we want to use the dynamical plane of $g_0$ to investigate the variation of dynamics in $V(G(g_0), g_0)$. Let $G(g)$ be the graph which varies isotopically from $G(g_0)$ for $g \in V(G(g_0), g_0)$. Then $G_1(g) = g^{-1}(G(g))$ also varies isotopically with $g$. This is not true for $n > 1$. But nevertheless, it is possible to determine inductively all the possible graphs $G_n(g)$ up to isotopy, for $g \in V(G(g_0))$. The different possibilities for $G_n(g)$, up to isotopy, are determined from the different possibilities for $G_{n-1}(g)$ up to isotopy, together with the position, up to homeomorphism fixing $G_{n-1}(g)$, of $v_2(g)$ in $G_{n-1}(g)$ or its complement. Inductively, this means that the different possibilities for $G_n(g)$ (and $P_n(g)$), up to isotopy, are determined by $(Q_i(g) : 0 \leq i \leq n - 1)$, where:

- $Q_0 = Q_0(g)$ is the set in $P(g)$ with $v_2(g) \in \text{int}(Q_0)$;
- $Q_{i+1}(g) \subset Q_i(g)$ for $0 \leq i \leq n - 1$;
- $Q_i(g) \in P_i(g)$ or $Q_i(g)$ is an edge of $G_i(g)$ or a vertex of $G_i(g)$;
- $v_2(g) \in Q_i(g)$ for $i \leq n - 1$ and $v_2(g) \in \text{int}(Q_i(g))$ if $Q_i \in P_i(g)$, and $v_2(g)$ is not an endpoint of $Q_i(g)$ if $Q_i(g)$ is an edge of $G_i(g)$.

Inductively, this means that the different possibilities for $Q_n(g)$ are determined by $Q_i(g)$, for $0 \leq i \leq n - 1$, and hence so is the graph $G_n(g)$, up to homeomorphism of $\overline{G}$, and the dynamical system $(G_n(g), g)$, up to isomorphism. So the different possibilities for any sequence $(Q_i : 0 \leq i \leq n - 1)$ as above, or even any infinite sequence $(Q_i : i \geq 0)$ with these properties, are determined by $g_0 : G_1(g_0) \to G(g_0)$, up to homeomorphism of $\overline{G}$ which is the identity on $\partial Q_0$. We will write $Q$ for the set of sequences, either finite or infinite, up to equivalence, where two sequences $(Q_i : i \geq 0)$ and $(Q'_i : i \geq 0)$ are regarded as equivalent if there is a homeomorphism $\varphi$ of $\overline{G}$ which maps $Q_i$ to $Q'_i$ for all $i \geq 0$. We will write $Q_\infty$ for the set of infinite
sequences in $Q$, and $Q_n$ for the set of finite sequences $(Q_0, \cdots Q_n)$ in $Q$. For $Q = (Q_0, \cdots Q_{n-1}) \in Q$, we write $V(Q)$ for the set of $g \in V(G(g_0))$ such that $(Q_i(g) : 0 \leq i \leq n - 1)$ is equivalent to $(Q_0, \cdots Q_{n-1})$. We write $G(Q_0, \cdots Q_{n-1})$ and $P(Q_0, \cdots Q_{n-1})$ for the graph $G_n(g)$ and $P_n(g)$, up to isotopy, for any $g$ such that $(Q_0(g), \cdots Q_{n-1}(g))$ is equivalent to $Q$. This means that all the dynamical systems $(G_n(g), g)$, for $g \in V(Q)$, are isomorphic. If $g_1 \in V(Q)$, we write $V(Q; g_1)$ for the component of $V(Q)$ containing $g_1$. In particular, all the graphs $G_n(g)$ for $g \in V(Q)$ are homeomorphic. For $g_1 \in V(Q)$ and $g \in V(Q; g_1)$, the graph $G(g)$ varies isotopically. This isotopy is, of course, an ambient isotopy, because any isotopy of a graph in a two-dimensional manifold is an ambient isotopy.

This isotopy is actually a bit more general, which will be important later. Let $(Q_0, \cdots Q_n) \in Q$, so that $Q_i \in P(Q_0, \cdots Q_{i-1})$ for $1 \leq i \leq n$. Let $Q'_{n-1} \subset Q_{n-1}$ be an edge or point of $G(Q_0, \cdots Q_{n-1})$. Then $g^{-1}(G(Q_0, \cdots Q_{n-1}) \setminus Q'_{n-1})$ varies isotopically for $g \in V(Q_0, \cdots Q_{n-1}; g_0) \cup V(Q_0, \cdots Q_{n-2}, Q'_{n-1}; g_0)$. This means that if $g \in V(Q_0, \cdots Q_{n-1}; g_0)$ and $h \in V(Q_0, \cdots Q_{n-2}, Q'_{n-1}; g_0)$, then

$$G_n(g) \setminus g^{-1}(Q'_{n-1}(g)), \ G_n(h) \setminus h^{-1}(Q'_{n-1}(h))$$

are isotopic, where $Q'_{n-1}(g)$ and $Q'_{n-1}(h)$ are the images of $Q'_{n-1}$ under the isotopic homeomorphisms of $G(Q_1, \cdots Q_{n-1})$ to $G_{n-1}(g)$ and $G_{n-1}(h)$.

For $g \in V(G(g_0))$, we also define

$$P_\infty(g) = \{ \cap_{n=0}^\infty Q_n : Q_n \subset Q_{n-1}, Q_n \in P_n(g) \text{ for all } n \geq 0 \}.$$  

Then $P_\infty(g)$ is a collection of closed sets whose union is the whole sphere. If $v_2(g)$ is not persistently recurrent then all the sets in $P_\infty(g)$ are either points or Fatou components for $g$. This follows from [2].

For any $Q = (Q_i : i \geq 0) \in Q_\infty$, we also define

$$V(Q) = \cap_{n=1}^\infty (V(Q_0, \cdots Q_n) \cup V(Q_0, \cdots Q_{n-1}, \partial Q_n)),$$

where $V(Q_0, \cdots Q_{n-1}, \partial Q_n)$ is the union of all those $V(Q_0, \cdots Q_{n-1}, Q'; g_0)$ such that $Q' \subset Q$ and $Q'$ is an edge of $G(Q_0, \cdots Q_{n-1}) \setminus G(g_0)$. For each $n$, we have

$$V(G(g_0)) = \cup_{Q \in Q_n} V(Q)$$

and

$$V(G(g_0)) = \cup_{Q \in Q_\infty} V(Q; g_0).$$

We now have the notation in place to state the main theorem of this section. A branched overing $f$ of $\overline{\mathbb{C}}$ is said to be critically finite if the postcritical set $Z(f) = \{ f^n(c) : c \text{ critical }, n > 0 \}$ is finite.
Theorem 3.2. Let $V$ be the Riemann surface consisting of a connected component of the set of quadratic rational maps $f$ with numbered critical values $v_1(f)$ and $v_2(f)$, such that $v_1(f)$ is of some fixed period, quotiented by Möbius conjugation (all as previously stated). Let $g_0 \in V$ be such that there exists a finite connected graph $G(g_0) \subset \mathbb{C}$ with the following properties.

- $G(g_0) \subset g^{-1}(G(g_0))$
- $G(g_0)$ separates the critical values.
- $G(g_0)$ does not intersect the boundary of any periodic Fatou component intersecting the forward orbit of $v_1(g_0)$.
- Any component of $\mathbb{C} \setminus G(g_0)$ contains at most one Fatou component intersecting the forward orbit of $v_1(g_0)$.
- $v_2(g_0) \in g_0^{-r}(G(g_0)) \setminus G(g_0)$ for some $r \geq 1$, and is eventually periodic.

Let $Q$ be defined using $G(g_0)$.

Let $Q \in Q$. Let $n$ be any integer $\geq 1$, and let $Q = (Q_0, \ldots Q_n) \in Q_n$. Then the following hold.

- If $g_1 \in V$ is also critically finite, and there is a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ which preserves critical points and periodic Fatou components, maps $G(g_0)$ to $G(g_1)$ and $\varphi \circ g_0 = g_1 \circ \varphi$ on the union of $G(g_0)$ and periodic critical orbits of $g_0$, then $V(G(g_0), g_0) = V(G(g_1), g_1)$.
- $V(Q)$ is nonempty, connected and its complement is connected.
- If there is some $n$ such that $Q_i \subset G(Q_0, \ldots Q_{n-1})$ for all $i \geq n$, or if there is $n$ such that
  
  $$Q(g) \subset \text{int}(G_n(g)) \text{ for all } g \in V(Q_0, \ldots Q_n),$$
  
  $$g^m(Q(g)) \cap \text{int}(Q_n(g)) = \emptyset \text{ for all } m > 0,$$
  
  then $V(Q; g_0)$ is a single point.
- If $Q = (Q_0, \ldots Q_n) \in Q_n$ and if $Q_i \in \mathcal{P}(Q_0, \ldots Q_{i-1})$ for each $1 \leq i \leq n$, then $V(Q)$ is open, and
  
  $$V(Q) \subset V(Q) \cup V(Q_0, \ldots Q_{n-1}, \partial Q_n),$$
  
  where the closure is taken in $V(G(g_0))$.

Remark As already explained at the start of this section, the properties specified for $g_0$ and $G(g_0)$ ensure that $V(G(g_0), g_0)$ satisfies the conditions for a set $V_1$ as in section [2] in particular in [2.1].

For the rest of this section, we keep the hypotheses of Theorem 3.2 and we use the notation that we have established. The following proposition
shows that the possibilities for $Q$ can be analysed by simply looking at those $Q = (Q_i) \in \mathcal{Q}$ for which all the $Q_i$ are topological discs.

**Proposition 3.3.** For any $(Q_0, \cdots, Q_n) \in \mathcal{Q}_n$, there is $(Q_0, Q_1', \cdots, Q_n') \in \mathcal{Q}_n$ such that $Q_i'$ is a topological disc for all $0 \leq i \leq n$, and $Q_i \subset Q_i'$ for $0 < i \leq n$, and there are isotopic subgraphs $G'(Q_0, \cdots, Q_{n-1})$ and $G'(Q_0, Q_1', \cdots, Q_{n-1}')$ of $G(Q_0, \cdots, Q_{n-1})$ and $G(Q_0, Q_1', \cdots, Q_{n-1}')$ such that $Q_i' \subset G'(Q_0, Q_1', \cdots, Q_{n-1})$ for all $1 \leq i \leq n - 1$ with $Q_i' \neq Q_i$, and the isotopy between $G(Q_0, Q_1, \cdots, Q_i)$ and $G(Q_0, Q_1', \cdots, Q_i')$ extends to the isotopy between $G(Q_0, Q_1, \cdots, Q_{n-1})$ and $G(Q_0, Q_1', \cdots, Q_{n-1}')$ for all $0 \leq i < n - 1$.

This is not difficult. The main step is the following.

**Lemma 3.4.** If $e$ is any edge of $G_n(g) \setminus G(g)$, for any $g \in V(G(g_0))$ and any integer $n \geq 1$, then $e \cap g^{-m}(e) = \emptyset$ for any $m \geq 1$.

**Proof.** It suffices to prove this for $n = 1$, because any edge $e$ of $G_1(g) \setminus G(g)$ is a contained in $g^{1-n}(e')$ for some edge $e'$ of $G_1(g) \setminus G(g)$. So now we assume that $e$ is an edge of $G_1(g) \setminus G(g)$. Now $G_1(g) = g^{-1}(G(g))$. So

$$g^{-m}(G_1(g) \setminus G(g)) = g^{-(m+1)}(G(g)) \setminus g^{-m}(G(g)).$$

So

$$g^{-m}(G_1(g) \setminus G(g)) \cap g^{-m}(G(g)) = \emptyset$$

for all $m \geq 0$. But $G(g) \subset g^{-1}(G(g)) = G_1(g)$, and hence $G(g) \subset g^{-m}(G(g))$ for all $m \geq 0$ and $G_1(g) \subset g^{-m}(G(g))$ for all $m \geq 1$. So

$$g^{-m}(G_1(g) \setminus G(g)) \cap G_1(g) = \emptyset$$

for all $m \geq 1$, as required.

**Proof of the proposition.** We prove this by induction on $n$. If $n = 1$ then there is nothing to prove, because $G(g)$ is isotopic to $G(g_0)$. So we assume it is true for $n - 1 \geq 1$, and we need to prove that it is also true for $n$. If $Q_n$ is a topological disc, there is nothing to prove. Otherwise there is a least $1 \leq i \leq n$ such that $Q_i$ is not a topological disc. Then $Q_i$ is an edge or point of $G(Q_0, \cdots, Q_{i-1})$. Let $Q_i(g)$ be the corresponding isotopically varying edge or point of $G(Q_0, \cdots, Q_{i-1})$ for $g \in V(Q_0, \cdots, Q_{i-1})$. Fix such a $g$. Write $e = Q_i(g)$ if $Q_i(g)$ is an edge of $G_i(g)$. Otherwise, let $e$ be an edge of $G_i(g)$ in $\partial Q_{i-1}(g)$ which contains the point $Q_i(g)$. Let $Q_i'$ be any closed topological disc such that $(Q_0, \cdots, Q_{i-1}, Q_i') \in \mathcal{Q}_i$ with $Q_i \subset Q_i'$. It has already been noted in [3.1] that if $g \in V(Q_0, \cdots, Q_{i-1}, Q_i)$ and $h \in V(Q_0, \cdots, Q_{i-1}, Q_i')$ then $g^{-1}(Q_i(g) \setminus Q_i(h))$ and $h^{-1}(Q_i(g) \setminus Q_i(h))$ are isotopic. Then by [3.3] $e \cap g^{-m}(e) = \emptyset$ for all $g \in V(Q_0, \cdots, Q_i) \cup V(Q_0, \cdots, Q_{i-1}, Q_i')$ and all $m > 0$. So $Q_i \cap g^{-\ell}(e) = \emptyset$ for all $i < \ell \leq n$ and for all such $g$. For $i \leq \ell \leq n$ we choose a
topological disc $Q'_\ell$ so that $(Q_0, \cdots, Q_{i-1}, Q'_i \cdots Q'_\ell) \in Q_\ell$ and $Q_\ell \subset Q'_\ell$. Once $Q'_i$ has been chosen, the choice of $Q'_\ell$ for $\ell > i$ is unique. So then by induction on $\ell$, we have that if $g \in V(Q_0, \cdots Q_\ell)$ and $h \in V(Q_0, \cdots, Q_{i-1}, Q'_i \cdots Q'_\ell)$, then $G_{\ell+1}(g) \setminus g^{-1}(Q_\ell(g))$ and $G_{\ell+1}(h) \setminus h^{-1}(Q_\ell(h))$ are isotopic. This gives the required result if we define $G'(Q_0, \cdots Q_{\ell+1})$ to be the subgraph of $G(Q_0, \cdots Q_{\ell+1})$ which is isotopic to $G_{\ell+1}(g) \setminus g^{-1}(Q_\ell(g))$, and similarly for $G'(Q_0, \cdots, Q_i-1, Q'_i \cdots Q'_{\ell+1})$. The claimed extension properties hold, by construction.

The following lemma uses Thurston’s theorem for critically finite branched coverings, and the set-up for this. See [11] or [5] for more details. Two critically finite branched coverings $f_0$ and $f_1$ are said to be Thurston equivalent if there is a homotopy $f_t$ ($t \in [0, 1]$) through critically finite branched coverings, such that the postcritical set $Z(f_t)$ varies isotopically for $t \in [0, 1]$. Thurston’s theorem gives a necessary and sufficient condition for a critically finite branched covering $f$ of $\mathbb{C}$ to be Thurston equivalent to a critically finite rational map. The rational map is then unique up to conjugation by a Möbius transformation. The condition is in terms of non-existence of loop sets in $\mathbb{C} \setminus Z(f)$ with certain properties. In the case of degree two branched coverings, the criterion reduces to the non-existence of a \textit{Levy cycle}, as is explained in the proof below.

**Lemma 3.5.** Let $(Q_0, \cdots, Q_n) \in Q_n$ where $Q_n$ is a closed topological disc (that is, the closure of a component of $\mathbb{C} \setminus G(Q_0, \cdots Q_{n-1})$ if $n \geq 1$, or $Q_0 = Q_0(g_0)$ if $n = 0$) and that $Q = (Q_i : 0 \leq i < N) \in Q$ for $N > n + 1$, possibly $N = \infty$, with $Q_i \subset Q_n \cap G(Q_0, \cdots, Q_n) \cap \text{int}(Q_{n-1})$ for $i > n$ and such that $\bigcap_{i \geq 0} Q_i$ represents an eventually periodic point. Suppose that $V(Q_0, \cdots, Q_n) \neq \emptyset$. Then $V(Q) = \{g_1\}$ for some $g_1 \in V$.

**Remark.** Note that there is no statement, as yet, that $g_1 \in V(G(g_0), g_0)$. That will come later.

**Proof.** Let $g \in V(Q_1, \cdots Q_n)$. Then $G(Q_0, \cdots, Q_n)$ is isotopic to $G_n+1(g)$, and the isotopy carries $\bigcap_{i \geq 0} Q_i$ to a point $z_0$ in $G_n+1(g)$, which, like $v_2(g)$, is in $\text{int}(Q_{n-1}(g)) \cap Q_n(g)$. We can construct a path $\beta : [0, 1] \to Q_n(g) \cap \text{int}(Q_{n-1}(g))$ with $\beta(0) = v_2(g)$ and $\beta(1) = \bigcap_{0 \leq i < N} Q_i(1) = z_0$. We can also choose $\beta$ so that $\beta([0, 1]) \subset \text{int}(Q_n(g))$. The hypotheses ensure that either $z_0 \in G_{n+1}(g) \setminus G_n(g)$ or $z_0 \in G_n(g) \setminus G_{n-1}(g)$. Either way, the endpoint-fixing homotopy class of $\beta$ is uniquely determined in $\mathbb{C} \setminus \{g^i(z_0) : i > 0\}$. This means that the Thurston-equivalence class of the post-critically finite branched covering $\sigma_\beta \circ g$ is well defined, where $\sigma_\beta$ is a homeomorphism which is the identity outside an arbitrarily small neighbourhood of $\beta$ and maps $\beta(0)$ to $\beta(1) = z_0$. 


Then we claim that $\sigma_\beta \circ g$ is Thurston equivalent to a rational map. Since this is a branched covering of degree two, it suffices to prove the non-existence of a Levy cycle. By definition, a Levy cycle is an isotopy class of a collection of distinct and disjoint simple closed loops, where the isotopy is in the complement of the postcritical set. In the present case, it is convenient to consider isotopy in the complement of a potentially larger forward invariant set $X$ consisting of the union of the forward orbits of $z_0$, $c_1(g)$ and the vertices of $G_0(g)$. Thurston’s Theorem adapts naturally to this setting. A Levy cycle for $\sigma_\beta \circ g$ is then the isotopy class in $\mathbb{C} \setminus X$ of a finite set $\{\gamma_i \setminus 1 \leq i \leq r\}$ of distinct and disjoint simple closed loops, such that there is a component $\gamma'_i$ of $(\sigma_\beta \circ g)^{-1}(\gamma_{i+1})$ (writing $\gamma_1 = \gamma_{r+1}$, so that this also makes sense if $i = r$), such that $\gamma_i$ and $\gamma'_i$ are isotopic in $\mathbb{C} \setminus X$, for $1 \leq i \leq r$. We consider the case when $z_0 \in \partial Q_n(g) \cap \text{int}(Q_{n-1}(g) \subseteq G_n(g) \setminus G_{n-1}(g))$. The other case, when $z_0 \in G_{n+1}(g) \cap \text{int}(Q_n(g) \subseteq G_{n+1}(g) \setminus G_n(g)$ can be dealt with similarly. The $\gamma_i$ can also be chosen to have only transversal intersections with $G_{n-1}(g)$. We have $z_0 \notin G_{n-1}(g)$. So $(\sigma_\beta \circ g)^{-1}(G_{n-1}(g)) = g^{-1}(G_{n-1}(g)) = G_n(g)$. Now $(\sigma_\beta \circ g)^{-1}(\gamma_{i+1})$ has two components $\gamma'_i$ and $\gamma''_i$, each of them mapped homeomorphically to $\gamma_{i+1}$ by $\sigma_\beta \circ g$. Each transverse intersection between $\gamma_i$ and $G_{n-1}(g)$ in $\mathbb{C} \setminus X$ lifts to two transverse intersections between $\gamma'_i \cup \gamma''_i$ and $G_n(g) \supset G_{n-1}(g)$ in $\mathbb{C} \setminus (\sigma_\beta \circ g)^{-1}(X)$, one of these intersections with $\gamma'_i$ and one with $\gamma''_i$. Because of the isotopy between $\gamma_i$ and $\gamma'_i$, the intersection on $\gamma'_i$ must be in $G_{n-1}(g)$ and must be essential in $\mathbb{C} \setminus X$. So this means that each arc on $\gamma_{i+1}$ between essential intersections in $G_{n-1}(g)$ lifts to an arc on $\gamma'_i$ between essential intersections in $G_{n-1}(g)$, and this arc can be isotoped in the complement of $X$ to an arc on $\gamma_i$ between essential intersections in $G_{n-1}(g)$. Since $g^{-1}(G_{n-j}(g) \setminus G_{n-j-1}(g)) = G_{n-j+1}(g) \setminus G_{n-j}(g)$, it follows by induction on $j \geq 1$ that all intersections between $\gamma_i$ and $G_{n-1}(g)$ are in $G_0(g)$. So every arc of intersection of $\gamma_i$ with $G_{n-1}(g)$ must be with $G_0(g)$, and in a single set of $\mathcal{P}_{n-1}(g)$ adjacent to a vertex of $G_0(g) = G(g)$. If $n$ is large enough, this is clearly impossible, because successive arcs are too far apart. But we can assume $n$ is large enough to make this impossible, by replacing $\gamma_i$ by $\gamma'^m_i$ if necessary, where $\gamma'^0_i = \gamma_i$ and $\gamma'_i = \gamma_i$ and $\gamma'^m_{i+1}$ is isotopic to $\gamma'^{m}_i$, obtained by lifting, under $\sigma_\beta \circ g$, the isotopy between $\gamma'^m_{i+1}$ and $\gamma'^m_{i+1}$, writing $\gamma'^m_i = \gamma'^m_{i+1}$. It follows that all intersections between $\gamma'^m_i$ and $G_0(g)$ are in a single set of $\mathcal{P}_{n+m-1}(g)$, adjacent to a vertex of $G_0(g)$. If $m$ is large enough, this is, once again, impossible.

So Thurston’s Theorem for critically finite branched coverings implies that $\sigma_\beta \circ g$ is Thurston equivalent to a unique rational map $g_1$. From the definitions, we have $g_1 \in V(Q)$. By the uniqueness statement in Thurston’s Theorem, we have $V(Q) = \{g_1\}$. For if $g_2 \in V(Q)$ and $v_1(g_1) \in G_{m+1}(g) \setminus \cdots$
$G_m(g_1)$ for $m = n$ or $n - 1$ then there is a homeomorphism $\varphi$ of $\mathbb{C}$ which maps $G_m(g_1)$ to $G_m(g_2)$ which conjugates dynamics of $g_1$ and $g_2$ on these graphs, and maps $v_2(g_1)$ to $v_2(g_2)$ and $g_1'(v_1(g_1))$ to $g_2'(v_1(g_2))$ for all $i \geq 0$. So $\varphi \circ g_1 \circ \varphi^{-1}$ and $g_2$ are homotopic through branched coverings which are constant on $G_m(g_2)$, and on the postcritical sets.

The following lemma, like the preceding one, gives a condition under which $V(Q)$ is nonempty. It has some overlap with the preceding one, but is of a rather different type. It uses the $\lambda$ lemma of Mane, Sullivan and Sad [9] rather than Thurston’s Theorem, and is a result about connected sets of maps rather than critically finite maps. [3,6] has no uniqueness statement. The two lemmas complement each other in the proof of [3,2].

**Lemma 3.6.** Let $g_1 \in V(G(g_0))$. Let $Q_{n-1} \in \mathcal{P}_{n-1}(G_1)$ and let $v_2(g_1) \in \text{int}(Q_{n-1}) \cap G_n(g_1)$ for some $n \geq 1$. Then $V(Q, g_1) \neq \emptyset$ for all $(Q) = (Q_i')$ with $Q_i' = Q_i$ for $i \leq n - 1$ such that $\cap_i Q_i$ is in the same component of $G_n \cap \text{int}(Q_0) \cap Q_n - 1 = v_2(g_1)$.

**Proof.** From the hypotheses on $g_1$, the graph $G_n(g)$ varies isotopically for $g \in V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1)$, and the dynamics of maps in $V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1)$ are conjugate in the following sense. There is a homeomorphism $\varphi_{g, h} : G_n(h) \to G_n(g)$, $(g, h) \in (V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1) \cap V)^2$, such that the map $(g, h) \mapsto \varphi_{g, h}$ is continuous, using the uniform topology on the image and $\varphi_{g, h} \circ h = g_1 \circ \varphi_{g, h}$ on $G_i(h)$, and $\varphi_{h, h}$ is the identity. Each preperiodic point in $G_n(g)$ varies holomorphically for $g \in V(Q_0, \cdots Q_{n-1}; g_1)$, that is, $\varphi_{g, h_i}(z)$ varies holomorphically with $g$ for each preperiodic point $z \in G_n(g_1)$. But preperiodic points are dense in $G_n(g_1)$. (For example, the backward orbits of vertices of $G_n(g_1)$ are dense in $G_n(g_1)$, by the expansion properties of $g_1$ on $G_n(g_1)$ established in [24]) It follows by the $\lambda$-Lemma [9] that $(z, g) \mapsto \varphi_{g, h}(z)$ is continuous in $(z, g)$, and holomorphic in $g \in V(Q_0, \cdots Q_{n-1}; g_1)$ for each $z \in G_n(g_1)$. (In fact it is also possible to prove this by standard hyperbolicity arguments.) Now we assume without loss of generality, conjugating by a Möbius transformation if necessary, that $Q_{n-1}(g) \subset \mathbb{C}$ for $g \in V(Q_0, \cdots Q_{n-1}; g_1)$, in particular, $\{v_2(g)\} \cup (G_n(g) \cap Q_n - 1(g)) \subset \mathbb{C}$. We consider the maps

$$\psi(z, g) = \varphi_{g, g_1}(z) - v_2(g)$$

for $z \in G_n(g_1) \cap Q_{n-1}(g_1)$. The map $(z, g) \mapsto \psi(z, g)$ is, once again, continuous in $(z, g)$ and holomorphic in $g \in V(Q_0, \cdots Q_{n-1}; g_1)$. Now write $z_0 = v_2(g_1)$, so that $z_0 \in G_n(g_1) \setminus G_n(g_1)$. The map $g \mapsto \psi(z_0, g)$ is holomorphic in $g$ and the inverse image of a disc round 0 is a topological disc containing $z_0$ in its interior. By continuity, the same is true for $z$ sufficiently
near \( z_0 \). Hence for all \( z \) sufficiently near \( z_0 \), the map \( g \mapsto \psi(z, g) \) has a zero. This argument shows that the set of \( z \in Q_{n-1}(g) \cap \text{int}(Q_0) \cap G_n(g_1) \) for which \( g \mapsto \psi(z, g) : V(Q_0, \ldots, Q_{n-1}) \) has a zero in \( V(Q_0, \ldots, Q_{n-1}) \) is open, because \( z_0 \) can be replaced by any other point \( z \) in \( Q_{n-1}(g_0) \cap \text{int}(Q_0) \cap G_n(g_0) \). But the set is also closed in \( \text{int}(Q_0(g_1)) \cap Q_{n-1}(g_1) \cap G_n(g_1) \). For suppose \( \psi(z_k, g_k) = 0 \) and \( z_k \to z \). Then either some subsequence of \( g_k \) has a limit \( g \), in which case \( \psi(z, g) = 0 \) for any such \( g \), and the proof is finished, or \( g_k \to \infty \) in \( V \).

We now have to deal with the situation that \( g_k \to \infty \) in \( V \). In this case, we can assume that all \( z_k \) are in a single edge of \( G_n(g_1) \). We will now show that this implies the existence of a Levy cycle for the unique map \( h_1 \in G(Q_0, \ldots, Q_{n-1}, Q'_n) \), where \( Q'_n \) is a vertex of \( G_n(g_1) \setminus G_0(g_1) \). This contradicts the result of [3,3] and hence \( g_k \to \infty \) is impossible. We use certain facts about the ends of \( V \). These appear in Stimson’s thesis [13] and in various other papers, for example [6]. Choosing suitable representatives of \( g_k \) up to Möbius conjugation, chosen, in particular, so that \( c_1(g_k) = 1 \) for all \( k \), \( g_k \) converges to a periodic Möbius transformation \( g(z) = e^{2\pi i r/q}z \) for some integer \( q \geq 2 \) and some \( r \geq 1 \) which is coprime to \( q \), and the set \( \{g^i_k(v_1(g_k)) : i \geq 0\} \cup \{v_2(g_k)\} = Z_1(g_k) \) converges \( Z_1(g) = \{e^{2\pi ij/q} : 0 \leq j \leq q - 1\} \). Let \( \overline{V} \) be the compactification of \( V \) obtained by adding the Möbius transformations at infinity and consider a fixed \( g \in \overline{V} \setminus V \). The parametrisation can be chosen so that the other critical point \( c_2(g_k) = 1 + \rho_k \) where \( \lim_{k \to \infty} \rho_k = 0 \). Passing to a subsequence if necessary, we may assume that \( g_k \) is in a single branch of \( V \) near \( g \). Then \( (g_k^q(1 + z\rho_k) - 1)/\rho_k \) has a limit as \( k \to \infty \) for \( z \) bounded and bounded from \( 1/2 \), which is the quadratic map

\[
h : z \mapsto qa + z + \frac{1}{4(z - \frac{1}{2})}
\]

for a constant \( a \neq 0 \).

Because of the nature of \( h \), it follows that all the eventually periodic points of \( g_k \) whose forward orbits have size \( \leq N \) lie in the \( C_{\rho_k} \)-neighbourhood of \( Z(g) \), if \( k \) is sufficiently large given \( N \), for a suitable constant \( C \). We will call this neighbourhood \( U_1 \). So if \( N \) is a bound on the number of vertices of \( G_n(g_k) \) — which is, of course, the same for all \( k \) — then all vertices of \( G_n(g_k) \) lie in \( U_1 \), for all sufficiently large \( k \). If the edge \( e \) of \( G_n(g_k) \) between one vertex and \( v_2(g_k) \) is contained in a single component of \( U_1 \), then the boundary of \( U_1 \) provides a Levy cycle for \( h_1 \), where \( Q'_n \) is taken to be this vertex, and this gives the required contradiction. Now \( e \subset G_n(g_k) \setminus G(g_k) \), and we claim that \( e \subset U_1 \), up to isotopy preserving the set \( X \) which is the union of the vertex set of \( G_n(g_k) \) and the set \( \{g^i_k(v_1(g_k)) : i \geq 0\} \). We consider only essential intersections between \( G_n(g_k) \setminus G(g_k) \) and \( \partial U_1 \) under
isotopies preserving $X$. If $\gamma$ is an arc of essential intersection then it must be in the inverse image under $g_k$ of an arc which contains one or more arcs of essential intersection. Since the number of such arcs is finite, each arc must be in the inverse image of exactly one other, and the inverse image of each arc contains exactly one other. But then each edge must be contained in a periodic edge of $G_n(g_k) \setminus G_0(g_k)$. But there are none. So there are no essential intersections with $\partial U_1$. In particular, $e \subset U_1$ up to isotopy preserving $X$, as required.

\begin{corollary}
For all $(Q_0, \cdots, Q_n) \in Q_n$, if $V(Q_0, \cdots, Q_n) \neq \emptyset$, then it is connected.
\end{corollary}

\begin{proof}
By \textsection 3.6 for any nonempty component $V(Q_0, \cdots, Q_n; g_1)$ of $V(Q_0, \cdots, Q_n)$,
\[ V(Q_0, \cdots, Q_n; g_1) \cap V(Q) \neq \emptyset \]
for any $Q \in Q$ such that $Q$ extends $(Q_0, \cdots, Q_n)$.

In particular, if $V(Q_0, \cdots, Q_n, g_2)$ is another component of $V(Q_0, \cdots, Q_n)$, then there is $Q$ such that $\cap_{i \geq 0} Q_i$ representing an eventually periodic point such that $V(Q)$ which intersects both components. But this is impossible, because $V(Q)$ contains a single critically finite map. So $V(Q_0, \cdots, Q_n)$ is connected.
\end{proof}

\begin{lemma}
$V(Q) \neq \emptyset$ for all $Q \in Q$. Moreover, $V(Q) \subset V(G(g_0), g_0)$ for all $Q \in Q$.
\end{lemma}

\begin{proof}
By \textsection 3.6 $V(Q) \neq \emptyset$ for all $Q$ with $\cap_{i \geq 0} Q_i \subset \partial Q_n$ for any $(Q_0, \cdots, Q_n) \in Q_n$ and such that $V(Q_0, \cdots, Q_{n-1}, \partial Q_n) \neq \emptyset$ with $Q_n \subset \text{int}(Q_0)$, because then $\partial Q_n \cap \text{int}(Q_0)$ is connected. This means that if $V(Q) \neq \emptyset$, then we have $V(Q') \neq \emptyset$ for any $Q'$ which can be connected to $Q$ by sets $\partial Q_{n_i}$, for varying $n_i$ and $Q_i = (Q_0, \cdots, Q_{n_i})$ with $Q_{n_i} \subset \text{int}(Q_0)$. But any $Q$ and $Q'$ can be connected in this way.

This also shows that any component of $V(G(g_0))$ which is intersected by $V(Q)$ for some $Q \in Q$ is intersected by $V(Q')$ for all $Q' \in Q$. Since $V(G(g_0), g_0)$ is intersected by $V(Q)$ for at least one $Q$ with $Q_i \subset G_{r}(g_0)$ for all $i \geq r$, simply by choosing $Q$ with $V(Q) = \{g_0\}$, we see that $V(Q') \cap V(G(g_0), g_0) \neq \emptyset$ for all $Q' \in Q$, as required.
\end{proof}

\begin{lemma}
$V(Q)$ is singleton if there is $n \geq 1$ such that $\cap_{i=0}^{\infty} Q_i(g) = Q(g) \subset \text{int}(Q_n(g))$ and such that $g^k(Q(g)) \cap \text{int}(Q_n(g)) = \emptyset$ for all $k > 0$, and for at least one $g \in V(Q)$.
\end{lemma}
Proof. The set \( Q(g) = \cap_{i=0}^{\infty} Q_i(g) \) is well-defined for all \( g \in V(Q_0, \cdots, Q_n) \). It is a point, which follows from the result of [12] about non-persistently-recurrent points, but in any case the construction of a nested sequence of annuli of moduli bounded from 0 is straightforward. Moreover \( z(g) = Q(g) \) is the limit of a sequence \( z_n(g) \) of eventually periodic points with the same property of being defined for all \( g \in V(Q_0, \cdots, Q_n) \). Fix \( g_0 \in V(Q_0, \cdots, Q_n) \) and write \( z(g_0) = Q(Q_0) \) and \( z_n(g_0) \) for the sequence of eventually preperiodic points under \( g_0 \) with \( \lim_{n \to \infty} z_n(g_0) = z(g_0) \). Then since \( g \mapsto \psi(z_n(g_0), g) \) is holomorphic in \( g \) and has a single zero, the same is true for the limiting holomorphic function \( g \mapsto \psi(z(g_0), g) \). The single zero is the unique point in \( V(Q) \).

Now the following lemma completes the proof of Theorem 3.2.

Lemma 3.10. The complement of \( \overline{V(Q; g_0)} \) has exactly one component in \( V(G(g_0)) \) for all \( Q \in Q \), for \( Q \neq Q_0 \).

Proof. If \( Q = (Q_i: i \geq 0) \in Q_\infty \) and the complement of \( V(Q) \) has more than one component in \( V(G(g_0)) \), then the same is true for the complement of \( \overline{V(Q_0, \cdots, Q_n; g_0)} \), for some \( n \). So it suffices to show that the complement of \( \overline{V(Q_0, \cdots, Q_n)} \) has at most one component in for each \( (Q_0, \cdots, Q_n) \in Q_n \). So suppose this is not true. Then \( \partial V(Q_0, \cdots, Q_n; g_0) \cap V(G(g_0)) \) is disconnected. But

\[
\partial V(Q_0, \cdots, Q_n; g_0) \cap V(G(g_0), g_0) \subset V(Q_0, \cdots, Q_{n-1}, \partial Q_n \setminus \partial Q_0).
\]

Moreover, if we fix \( h \in V(Q_0, \cdots, Q_n) \) there is a continuous surjective map

\[
\Phi : V(Q_0, \cdots, Q_{n-1} \setminus \partial Q_0) \to \partial Q_n \setminus \partial Q_0(h),
\]

defined by

\[
\Phi(g) = \varphi_{g,h}^{-1}(v_2(g)),
\]

where \( \varphi_{g,h} \) is as in the proof of 3.6. By 3.6 to 3.8, \( \Phi^{-1}(\Phi(g)) \) is connected for each \( g \). In fact if \( G_2(g) \) is critically finite, then this already follows from 3.6. Also, \( \Phi(\partial V(Q_0, \cdots, Q_n)) \supset \partial Q_n(g) \cap \text{int}(Q_0(g)) \) by the proof of 3.6. So if \( \partial V(Q_0, \cdots, Q_n) \cap V(G(g_0)) \) can be written as a disjoint union of two nonempty closed sets \( X_1 \) and \( X_2 \) in \( V(G(g_0)) \), we have \( X_j = \Phi^{-1}(\Phi(X_j)) \). Since \( X_j \) is closed and bounded (and hence compact), we see that \( \Phi(X_j) \) is also closed (and bounded and compact) and the sets \( \Phi(X_1) \) and \( \Phi(X_2) \) are also disjoint. So then \( \partial Q_n(g) \cap \text{int}(Q_0(g)) \) is disconnected, giving a contradiction.

\[\square\]
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