Investigation of Phase Portraits Belonging to Polynomial Dynamic Systems in a Poincare Disk

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Abstract. The paper describes especially developed research methods and results for some original study of a broad dynamic systems’ family, which is characterized with the reciprocal polynomials in the right parts of differential equations which compose a system. The goal of this work is to obtain and analyze all the existing (and different in the topological meaning) phase portraits in a Poincare disk, as well as to indicate the criteria of each topological type of phase portraits appearance. Poincare methods of sequential transformations and displays have been successfully used along with a whole set of new research techniques developed for the purposes of this study. Over than 250 of different portraits were obtained and depicted eventually for different subsystems belong to the dynamic systems’ family under consideration. All portraits were described up to each the so-called invariant cell of the phase portrait, a boundary of it, and features of a phase flow in it, including it’s a source and a sink. Results of dynamic systems’ behavior research are useful for a wide spectrum of applications such as the mathematical modeling of physical processes, vital problems of civil engineering, for example, in the consideration of seismic stability of buildings etc., computing and producing systems, biological phenomena, and of sociological events.

1. Introduction

In the different branches of contemporary science and technology, in a considerably broad spectrum of them, i.e. in the mathematical modeling of physical processes (see, for example, [22, 23]), in the long series of complicated and pressing problems of civil engineering, for example, in the consideration of seismic stability of buildings (see [27 – 30]), as well as in many other problems of architectural and construction analysis; in the fundamental studies of computing and producing systems [20, 21], of biological phenomena, and of sociological events (such as an economical functioning of a state [17, 18], military and peaceful interactions between the two different states [16, 19]), dynamic systems play a very important and even a key role. Researcher uses a dynamic system as a mathematical apparatus in case of necessity to describe some phenomena and conditions, under which fluctuations or any statistical events are not important, hence they can be disregarded, and their influence omitted.

In a whole the main task of the theory of dynamic systems is to study curves, which differential equations define. During such a research, firstly we need to split a dynamic system’s phase space into trajectories. Secondly, it’s important to investigate a limit behavior of trajectories [10, 11, 24 – 26]. This stage of research is assumed to reveal equilibrium positions and make their thorough classification. Also, here we find and investigate sinks and sources of the system’s phase flow.

Necessary is to study:
1) the so-called stability of equilibrium states - this concept, or notion, means an ability of a taken dynamic system to remain near an equilibrium state in a case of some (although satisfactory small) changes to the initial data, or to remain on a taken manifold, for a (considerably long) time interval,

2) a question of a so-called roughness of a considered dynamic system. The notion of the system’s roughness means its ability to save system’s properties in the case of some (also tiny enough) changes in the very model described with the system. Rough systems hold their (qualitative) characteristics of motion stable in the case of (small arbitrary) changes made in their parameters.

The proposed article represents investigating methods and strict mathematical conclusions of the original study. This study was conducted in the field of the qualitative theory of ordinary differential equations and dynamic systems. Precise research methods, directly designed here for the aims of this investigation, being new and working effective, would be useful for further works over some applied dynamic systems, especially over the dynamic systems with polynomial right parts of different orders.

The foundations of the qualitative theory of ordinary differential equations and dynamic systems (as well as the foundations of the topology and the theory of relativity) were laid by one of the last encyclopedists in mathematics, physics, celestial mechanics and some other branches of science simultaneously, the ingenious French researcher Jules Henri Poincare (29.04.1854 – 17.07.1912). From the J.H. Poincare point of view, see the sources [12 - 14], any normal autonomous differential system of the second order with polynomial forms in the right parts of its equations on an extended arithmetical (or real) plane \( R_{x,y}^2 \), may be completely qualitatively investigated [1].

Mathematicians of the later times have investigated successfully the quadratic dynamic systems [2], along with the dynamic systems including linear nonzero terms. Also, there were studied dynamic systems in which the homogeneous nonlinear terms have odd degrees equal to 3, 5, and 7 [2, 3], having a singular point \( O (0, 0) \) of a center or a focus type [4], and several other concrete and special categories of dynamic systems.

In this manuscript authors consider the broad family of dynamic systems defined on a real phase plane \( x, y \)

\[
\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),
\]

where \( X(x, y) \), \( Y(x, y) \) are considered to be reciprocal polynomial forms, \( X \) be a cubic form, and \( Y \) be a square form, for which we propose the following conditions:

\[
X (0, 1) > 0, \quad Y (0, 1) > 0.
\]

The Poincare method of sequential mappings allows to achieve the established above main research goal of constructing in a Poincare disk of all existing for these differential systems topologically different phase portraits. This method involves the two following main steps:

1) Using the central display (from the center \( (0, 0, 1) \) of the Poincare sphere \( \Sigma \)) we map the (augmented with the line at infinity) phase plane \( R_{x,y}^2 \) of the considered system on the Poincare sphere \( \Sigma \):

\[
x^2 + y^2 + z^2 = 1
\]

where the diametrically opposite points we would consider identified, and

2) Via the orthogonal display the lower enclosed semi sphere of the Poincare sphere \( \Sigma \) we map on the (enclosed) Poincare disk \( \Omega \):

\[
x^2 + y^2 \leq 1
\]
where the diametrically opposite points belonging to the boundary (the circle) \( \Gamma \) of the Poincare disk \( \Omega \) we also agree to consider identified [1].

2. Some definitions of the key importance

\( \varphi(t, p), \ p = (x, y) \) - a fixed point \( := \) be a motion (a solution) of a system defined by equations (1) with the initial data \((0, p)\).

\( L_p^\varphi : \varphi(t, p), t \in I_{\text{max}} \) be a trajectory of an abovementioned motion \( \varphi(t, p) \).

\( L_p^\varphi(\pm) := \pm \) be a semi trajectory of an abovementioned trajectory \( L_p \).

\( O \)-curve of a dynamic system \( := \) be a semi trajectory \( L_p^s \) \((p \neq O, s \in \{+,-\})\) of a considered system, which is adjoining to a point \( O \) with the condition that \( s t \to +\infty \).

\( O^{\pm(-)} \) curve of a dynamic system \( := \) be the system’s \( O \)-curve \( L_p^{\pm(-)} \).

\( O^{\pm(-)}_s \)-curve of a dynamic system \( := \) be the system’s \( O \)-curve, which is adjoining to a point \( O \) from the domain \( x > 0 \) (or from the domain \( x < 0 \) correspondingly).

\( TO \)-curve of a system \( := \) be the dynamic system’s \( O \)-curve, such as, if it will be supplemented by a point \( O \), will touch some ray in this point.

A nodal bundle of \( NO \)-curves of a dynamic system \( := \) be an open continuous family of this dynamic system’s \( TO \)-curves \( L_p^s \), where \( s \in \{+,-\} \) be a fixed index, \( p \in \mathcal{A}, \mathcal{A} \) - be a simple open arc, such as \( L_p^s \cap \mathcal{A} = \{p\} \).

A saddle bundle of \( SO \)-curves of a dynamic system, or a separatrix of the point \( O := \) be a fixed \( TO \)-curve, such as this curve isn’t belonging to any bundle of \( NO \)-curves of a dynamic system.

\( H, P, E \) - be the \( O \)-sectors of a dynamic system: a hyperbolic sector, a parabolic sector, and an elliptical sector.

A T-type (a topological type) of a singular point \( O \) of a dynamic system \( := \) be a word \( A_0 \) constructed of the letters \( N, S \), which is describing an order of bundles \( N, S \) of this system’s \( O \)-curves when going around the point \( O \) counter clockwise, starting with some of them (that means in the \(<\to>)\)-direction).

Also we are going to indicate the topological type of a singular point \( O \) of a dynamic system using the closely connected to the abovementioned method word \( B_0 \) constructed of letters \( E, H, P \), which we use to describe a circular order of \( O \)-sectors \( E, H, P \) also when going around the finite singular point \( O \) in the \(<\to>)\)-direction (similarly starting with some of them).

We introduce the special polynomials:

\[
P(u) := X(1, u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,\]
\[
Q(u) := Y(1, u) \equiv a + bu + cu^2.\]  

(5)

For all dynamic systems belonging to the family under investigation, described with the equations (1):

1) The singular point’s \( O \) T-type, initially described in the words (using the forms) \( A_0 \) or \( B_0 \), can easily be “translated” or rewritten from one mentioned form into another.

2) All real roots of the introduced above special polynomial \( Q(u) \) actually play the role of angular coefficients of the isoclines of a zero.

3) All real roots of the special polynomial \( P(u) \) similarly play the role of angular coefficients of the isoclines of the infinity.

4) When we writing them out, we always number the real roots of polynomials \( P(u), Q(u) \) in the ascending order.

\( T \) – types of a singular point \( O \) \((0, 0)\) and infinitely remote singular points of dynamic systems belonging to this family were described in articles [5 - 9].
3. Cases of special subfamilies. Right parts decomposed into 3 and 2 multipliers

In order to study the whole wide Eq. (1) – systems family, it is vitally necessary to split it into subfamilies. As the detailed further investigation has shown, there exist several hierarchical levels of such subfamilies. The first level of their hierarchy is based on the amounts of different multipliers in the decompositions of the polynomials situated in the right parts of the system’s equations.

Firstly, we’ll describe the subfamily including those dynamic Eq. (1) – systems, for which the decompositions into the lower degree forms for their real polynomial right parts $X (x, y), Y (x, y)$ include three and two different multipliers correspondingly:

$$
X(x, y) = p_3(y - u_1 x)(y - u_2 x)(y - u_3 x), \quad Y(x, y) = c(y - q_1 x)(y - q_2 x).
$$

(6)

$p_3 > 0$, $c > 0$, $u_1 < u_2 < u_3$, $q_1 < q_2$, $u_i \neq q_j$ for each $i$ and $j$.

Their study must involve several key steps [8 - 10].

Let the so-called double change (DC)-transformation be a double exchange of variables in the considered system, with the rule: $(t, y) \rightarrow (-t, -y)$. Such an exchange we use in order to transform the studied system into another one, where the numbers (and also signs) of the roots of forms $P(u), Q(u)$, and directions of motion along the system’s trajectories (with the time increasing) are reversed. We’ll denote a pair of different dynamic systems as “the mutually inversed via the DC-transformation”, if that kind of transformation converts those systems into each other, and denote them independent of a DC-transformation oppositely.

It is clear, that namely 10 independent variants of $RSPQ$ (this abbreviation means the sequence of ascending roots of polynomial forms $P(u), Q(u)$) may exist for the taken system described by the Eq. (6):

$$
C_5^2 = \frac{5!}{2!3!} = 10.
$$

(7)

Six dynamic systems among them proved to be independent in pairs via the described DC-transformation, while each one of the rest four systems has the mutually inversed system among the first six systems.

On the basis of these facts let’s number by $r \in \{1, \ldots, 10\}$ different $RSPQ$’s belonging to of the Eq.- (6) subfamily, keeping in the mind: let $RSPQ$ having $r = \frac{1}{5}$ be independent in pairs, but $RSPQ$ numbered with $r = \frac{7}{10}$ be mutually inversed to the sequences numbered by $r = \frac{1}{4}$.

So, it’s necessary to note: we just have come to the second hierarchical level of a whole wide dynamic systems’ family division:

an $r$-family of Eq. (6) – systems $\vdash r$ be a totality of systems which $RSPQ$ has the number $r$.

Further investigation of (sub)subfamilies with the numbers $r = \frac{1}{5}$ follows the common program. For families having numbers $r = \frac{7}{10}$ are being received using the DC-transformation of families numbered by $r = \frac{1}{4}$.

The common program for each subfamily is:

1) It’s needed to enlist all singular points of systems belong to the investigated family in the enclosed Poincare disk $\mathbb{N}$. Those are the finite singular point $O (0, 0) \in \Omega$ and infinitely remote points $O^F_i (u_i, 0) \in \Gamma$, $i = 0, 3$, $u_0 = 0$. For them we use the notions of a nodal (N) and a saddle (S) bundles of semi trajectories (being adjacent to the point), of a separatrix and a topo-dynamical type of the taken singular point.

2) Then to split the given subfamily into (sub)subfamilies ($s = \frac{1}{7} \ldots$). For every of those we find the topo-dynamical types and their separatrices.

3) For each taken singular point of every given system belonging to the chosen (sub)subfamily $\forall s \in \{1, \ldots, 7\}$ investigate a separatrices’ behavior: here mainly interesting are uniqueness of continuation of a taken separatrix from a small neighborhood of a studied singular point to all this.
separatrix’s lengths, together with a study of all separatrices’ mutual arrangement in a Poincare disk \( \Omega \). We answer all mentioned questions for all families of studied systems.

4) Construct all phase portraits in the Poincare disk for a taken family and detail criteria of them [10]. The family with the number \( r=1 \) has 25 different topological types of phase portraits in the Poincare disk.

For the families numbered by 2 and 3 were revealed 9 types of portraits (per each family).

For the families having the numbers 4 and 5 there exist and totally described 7 types of phase portraits for every family.

The systems of the family number \( r=6 \) reveal 36 independent portraits’ types.

In the total 93 different types for systems characterized by 3 and 2 (different) multipliers in their polynomial right parts’ decompositions were revealed. Lots of types? But remember: every taken into consideration family, subfamily, (sub)subfamily includes an uncountable number of concrete differential systems [9 – 11, 15].

4. Systems with combinations of 2 different multipliers in both right parts

Let us discuss in the present section the special and common features of those dynamic systems, where decompositions of polynomial forms \( X(x, y), Y(x, y) \) into the real forms of lowest degrees will contain two multipliers:

\[
X(x, y) = p(y - u_1 x)^{k_1}(y - u_2 x)^{k_2}, \quad Y(x, y) = q(y - q_1 x)(y - q_2 x),
\]

where \( p, q, u_1, u_2, q_1, q_2 \in \mathbb{R}, \; p > 0, \; q > 0, \; u_1 < u_2, \; q_1 < q_2, \; u_i \neq q_j \) for each \( i, j \in \{1, 2\}, \; k_1, k_2 \in \mathbb{N}, \; k_1 + k_2 = 3 \).

There exist the two natural classes of such Eq. (8) -systems. The first (A) class includes dynamic systems with \( k_1 = 1, \; k_2 = 2 \), while the second (B) class includes systems with \( k_1 = 2, \; k_2 = 1 \).

The program of their investigation involves steps similar to the items described in the previous section of the paper.

For an arbitrary system under consideration we use the following notations.

Let \( P(u), Q(u) \) be a considered dynamic system’s polynomials \( P, Q \):

\[
P(u) := X(1, u) \equiv p(u - u_1)(u - u_2)^2, \quad Q(u) := Y(1, u) \equiv q(u - q_1)(u - q_2),
\]

for the A-class of systems, and

\[
P(u) := X(1, u) \equiv p(u - u_1)^2(u - u_2), \quad Q(u) := Y(1, u) \equiv q(u - q_1)(u - q_2),
\]

for the B-class of them.

While \( RSP \) (\( RSQ \)) be an ascending sequence of the real roots of the polynomial \( P(u) \) (\( Q(u) \)), and thus \( RSPQ \) – be an (ascending) sequence of all real roots of both polynomials \( P(u) \) and \( Q(u) \). There are possible six different combinations of \( RSPQ \) since \( C_4^2 = \frac{4!}{2!2!} = 6 \). We number them from 1 to 6 for each class of systems (A or B) independently.

The research plan points for every fixed subfamily with the corresponding number are the follows.

1) For all singular points of a taken system, belonging to the considered subfamily, we introduce - the important and similar to the mentioned in the previous section - notions of S (saddle) and N (nodal) bundles of the (adjacent to a chosen singular point) semi trajectories, as well as the notion of this singular point separatrix and the notion of its topo-dynamical type also.
2) The subfamily must be divided into the (sub)subfamilies \( s \in \{1, ..., 5\} \). Now determine the TD-types of singular points existing for the systems of (sub)subfamilies, and separatrices of singular points \( \gamma_s = \frac{1}{15} \).

3) For all the five subfamilies we study the behavior of their separatrices and answer a question of uniqueness of the continuation of every taken separatrix from some small neighborhood of a singular point to all its prolongation within the Poincare disk \( \Omega \), and also a question of their mutual arrangement in \( \Omega \).

The mutual arrangement of all separatrices in the Poincare disk appears to be invariable in the case if for a taken \( s \) the global continuation of each given separatrix of each singular point of the subfamily number \( s \) is unique. As a result, in this case all systems of a taken subfamily \( s \) will have within a Poincare disk one common phase portrait.

But oppositely, if for a fixed \( s \) the systems of such a subfamily have, for example, \( m \) separatrices with global continuations which aren’t unique, such a subfamily must be subjected to a further division into \( m \) additional (sub)subfamilies of the next level of hierarchy.

As we revealed from their thorough study, for every of such (sub)subfamilies the global continuation of each separatrix is unique. Thus, the mutual arrangement of those separatrices in the Poincare disk \( \Omega \) becomes already invariable.

Consequently, a topological type of a phase portrait of all systems belonging to this (sub)subfamily in the enclosed Poincare disk \( \overline{\Omega} \) is common and unique for the taken (sub)subfamily.

4) We construct and describe phase portraits in the enclosed Poincare disk \( \overline{\Omega} \) for all the studied systems using the two admissible forms - the table form and the graphic one, indicating per each portrait close to coefficient criteria of it appearance.

Another types of decompositions of right parts are the topic for coming further papers.

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Recommendations
Paper will be useful for students, postgraduates and researchers in the fields of the qualitative theory of differential equations, dynamic systems and applications of them in different directions of modern technology and science, such as the mathematical modeling of processes in several problems of civil engineering, for example, in the investigations and analysis of seismic stability of constructions and buildings [27 – 30] and others; studies of computing and producing systems [20, 21], of biological, economical and sociological phenomena [16 – 19, 22, 23].

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