Cosmology with Mimetic Matter

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Abstract. We consider minimal extensions of the recently proposed Mimetic Dark Matter and show that by introducing a potential for the mimetic non-dynamical scalar field we can mimic nearly any gravitational properties of the normal matter. In particular, the mimetic matter can provide us with inflaton, quintessence and even can lead to a bouncing nonsingular universe. We also investigate the behaviour of cosmological perturbations due to a mimetic matter. We demonstrate that simple mimetic inflation can produce red-tilted scalar perturbations which are largely enhanced over gravity waves.

Keywords: modified gravity, inflation, cosmological perturbation theory, dark energy theory

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1 Introduction

Recent work [1] proposed a modification of general relativity where the metric $g_{\mu\nu}$ is defined in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$ as

$$ g_{\mu\nu} = \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi . $$

(1.1)

The conformal mode of gravity is thus encoded into the scalar field $\phi$. The theory is manifestly invariant with respect to the Weyl transformations of the auxiliary metric. The resulting equations of motion are equivalent to the Einstein equations except for the appearance of an extra longitudinal mode of the gravitational field which is dynamical even in the absence of normal matter. It was shown that this extra mode of the gravitational field can serve as a source of cold Dark Matter. In this paper we generalize this model in a minimal way and show that Mimetic Matter can imitate practically any gravitational properties of the normal matter. In particular, this matter can provide us with quintessence, inflation and even bouncing universe.

The equations of motion derived from the action formulated in terms of the auxiliary metric $\tilde{g}_{\mu\nu}$ are equivalent to those ones obtained by variation of the Einstein action with respect to the physical metric $g_{\mu\nu}$ if we impose an extra constraint on the scalar field [2, 3]. In fact, the theory with action

$$ S = \int d^4x \sqrt{-g (\tilde{g}_{\mu\nu}, \phi)} \left[ -\frac{1}{2} R (g_{\mu\nu} (\tilde{g}_{\mu\nu}, \phi)) + \mathcal{L}_m \right] , $n

(1.2)

yields traceless Einstein equations of motion

$$ (G^{\mu\nu} - T^{\mu\nu}) - (G - T) g^{\mu\nu} \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi = 0 , $n

(1.3)
because
\[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1 , \] (1.4)
as it follows from definition (1.1). This suggests to use identify (1.4) as a constraint by employing a Lagrange multiplier
\[ S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R (g_{\mu\nu}) + \mathcal{L}_m (g_{\mu\nu}, \ldots) + \lambda (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1) + \bar{\lambda} (\nabla_\mu V^\mu - 1) \right] , \] (1.5)
so that both Dark Matter and Dark Energy arise as constants of integration in the resulting equations. The first constraint here is responsible for the appearance of mimetic Dark Matter \[ 1 \], while the second one gives the cosmological constant \[ 4 \]. This can be easily seen by examining the equations of motion of the action (1.5) which in addition to imposing the constraints (1.4) and \[ \nabla_\mu V^\mu = 1 \] also give
\[ G_{\mu\nu} - T_{\mu\nu} + 2\lambda \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \bar{\lambda} = 0 , \] (1.6)
\[ \partial_\mu \bar{\lambda} = 0 . \] (1.7)
Thus the cosmological constant \[ \bar{\lambda} = \Lambda \] arises as a a constant of integration, and the Lagrange multiplier \[ \lambda \] is then determined from the trace of the Einstein equations (1.6)
\[ \lambda = -\frac{1}{2} (G - T + 4\Lambda) . \] (1.8)
The metric \[ g_{\mu\nu} \] and the scalar field \[ \phi \] are then determined by equations
\[ (G_{\mu\nu} - T_{\mu\nu}) - (G - T) \partial_\mu \phi \partial_\nu \phi + (g_{\mu\nu} - 4\partial_\mu \phi \partial_\nu \phi) \Lambda = 0 , \] (1.9)
together with (1.4). Therefore both Dark Matter and Dark Energy can arise from the minimal modification of General Relativity (GR) by adding non-dynamical scalar and vector fields.

The purpose of the present work is to generalize the model above by introducing an arbitrary potential \[ V (\phi) \] and study the cosmological solutions in this theory. In particular, we shall show how the appropriate choice for the potential \[ V (\phi) \] can lead to various cosmological solutions for bouncing universe, inflation and quintessence.

2 Potential for Mimetic Matter
Consider the theory with the action (compare to \[ 5 \])
\[ S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R (g_{\mu\nu}) + \lambda (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1) - V (\phi) + \mathcal{L}_m (g_{\mu\nu}, \ldots) \right] , \] (2.1)
where we have skipped the cosmological constant constraint because its only role for the classical solutions is to shift the potential \[ V (\phi) \] by a constant \[ \Lambda \]. Variation with respect to \[ \lambda \] gives equation (1.4) while varying with respect to \[ g^{\mu\nu} \] we obtain
\[ G_{\mu\nu} - 2\lambda \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} V (\phi) = T_{\mu\nu} , \] (2.2)
where \[ G_{\mu\nu} \] and \[ T_{\mu\nu} \] are the Einstein tensor and the energy-momentum tensor for the normal matter. Taking trace of these equations we can express the Lagrange multiplier as
\[ \lambda = \frac{1}{2} (G - T - 4V) , \] (2.3)
and hence equations (2.2) become

\[ G_{\mu\nu} = (G - T - 4V) \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} V(\phi) + T_{\mu\nu}, \] (2.4)

which taken together with

\[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1, \] (2.5)

replace the Einstein equations. They are equivalent to the Einstein equations with an extra longitudinal degree of freedom. In distinction from General Relativity, if we take the trace of (2.4) the resulting equation will be satisfied identically. The missing equation in our case is replaced by the constraint (2.5).

We would like to stress that the extra longitudinal degree of freedom of the gravitational field cannot be entirely attributed to the scalar field \( \phi \). In fact this field satisfies the first order Hamilton-Jacobi type differential equation and therefore it is not dynamical by itself.

Taking the covariant derivative \( \nabla^\nu \) of equation (2.4) and using the Bianchi identity \( \nabla^\nu G_{\mu\nu} = 0 \) together with conservation of the energy-momentum tensor \( \nabla^\nu T_{\mu\nu} = 0 \), we obtain

\[ \nabla^\nu [(G - T - 4V) \partial_\nu \phi] = 0. \] (2.6)

This equation can be simplified further if we take into account that

\[ \nabla^\rho (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = 2g^{\mu\nu} (\nabla^\rho \partial_\mu \phi) \partial_\nu \phi = 0, \] (2.7)

and \( \partial_\mu \phi \neq 0 \) at least for one index \( \mu \). The result is

\[ \nabla^\nu ((G - T - 4V) \partial_\nu \phi) = -V'(\phi), \] (2.8)

where \( V'(\phi) = \partial V/\partial \phi \). Equation (2.8) also follows directly from the action (2.1) when we vary it with respect to \( \phi \) and take into account (2.3).

It is easy to see that equations (2.4) are equivalent to the Einstein equations with an extra ideal fluid with pressure

\[ \tilde{p} = -V, \] (2.9)

and energy density

\[ \tilde{\varepsilon} = G - T - 3V. \] (2.10)

The scalar field \( \phi \) plays the role of the velocity potential and the constraint (2.5) is simply the normalization condition for the 4-velocities.

Let us now investigate the solutions of equations (2.4), (2.5) in a flat universe with the metric

\[ ds^2 = dt^2 - a^2(t) \delta_{ik} dx^i dx^k, \] (2.11)

assuming that ordinary matter is absent, that is, \( T_{\mu\nu} = 0 \). A general solution of (2.5) is in this case

\[ \phi = \pm t + A, \] (2.12)

where \( A \) is a constant of integration. Without lose of generality we shall identify the field \( \phi \) with time

\[ \phi = t. \] (2.13)

Taking into account that the pressure and the energy density, defined in (2.9) and (2.10), depend only on time, equation (2.8) becomes

\[ \frac{1}{a^3} \frac{d}{dt} \left( a^3 (\tilde{\varepsilon} - V) \right) = -\dot{V}, \] (2.14)
where we took into account that $V' = \dot{V}$, where dot denotes the derivative with respect to time $t$. This equation can be easily integrated and we obtain the following expression for the energy density in terms of the potential $V$,

$$
\tilde{\varepsilon} = V - \frac{1}{a^3} \int a^3 \dot{V} dt = \frac{3}{a^3} \int a^2 V da ,
$$

(2.15)

while

$$
\tilde{p} = -V .
$$

(2.16)

One can easily check that the energy density and the pressure given by these expressions satisfy the conservation law

$$
\dot{\tilde{\varepsilon}} = -3H (\tilde{\varepsilon} + \tilde{p}) ,
$$

(2.17)

where

$$
H \equiv \frac{\dot{a}}{a} ,
$$

is the Hubble constant. A constant of integration in (2.15) determines the amount of mimetic dark matter, which decays as $a^{-3}$. On the other hand for a nonvanishing $V$ there is an extra contribution to mimetic matter in the amount entirely fixed by the potential $V$. In this sense an extra mimetic matter is similar to a cosmological constant added to the Lagrangian with a fixed value. Therefore, the number of degrees of freedom in the system does not increase compared to the case of mimetic dust. Because the field $\phi$ is not a dynamical field, and in Friedmann universe it is a linear function of time for any $V$, nothing can prevent us to consider negative potentials along with positive ones.

The Friedmann equation, obtained from the time-time component of equation (2.4), takes the form

$$
H^2 = \frac{1}{3} \tilde{\varepsilon} = \frac{1}{a^3} \int a^2 V da ,
$$

(2.18)

and for a given $V(\phi) = V(t)$ could be solved for $a(t)$. However, instead of solving this integral equation, it is more convenient to differentiate it first and reduce it to an ordinary differential equation. Multiplying equation (2.18) by $a^3$ and differentiating it with respect to time we obtain

$$
2\ddot{H} + 3H^2 = V(t) .
$$

(2.19)

Note that equation (2.19) could be also obtained from the space-space component of equation (2.4). It can be simplified further if instead of $a$ we introduce the new variable

$$
y = a^{\frac{3}{2}} ,
$$

(2.20)

then

$$
H = \frac{2}{3} \frac{\dot{y}}{y} , \quad \dot{H} = \frac{2}{3} \left( \frac{\ddot{y}}{y} - \left( \frac{\dot{y}}{y} \right)^2 \right) ,
$$

(2.21)

and equation (2.19) becomes a linear differential equation

$$
\ddot{y} - \frac{3}{4} V(t) y = 0 .
$$

(2.22)

A point which should not be underestimated is that the identification of the scalar field $\phi$ with time greatly simplifies the problem because for a given potential the pressure becomes known function of time and $y = a^{3/2}$ satisfies linear differential equation. This allows us easily to find cosmological solutions.
3 Cosmological solutions

First we consider the potential

\[ V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2}, \quad (3.1) \]

where \( \alpha \) is a constant. The general solution of the equation

\[ \ddot{y} - \frac{3\alpha}{4t^2} y = 0, \quad (3.2) \]

is

\[ y = \begin{cases} 
C_1 t^{\frac{1}{2}} \cos \left( \frac{1}{2} \sqrt{1+3\alpha} \ln t + C_2 \right), & \text{for } \alpha < -1/3, \\
C_1 t^{\frac{1}{2}} (1+\sqrt{1+3\alpha}) + C_2 t^{\frac{1}{2}} (1-\sqrt{1+3\alpha}), & \text{for } \alpha \geq -1/3,
\end{cases} \quad (3.3) \]

where \( C_1 \) and \( C_2 \) are constants of integration. It is interesting to note that for large negative \( \alpha \) the solution describes an oscillating flat universe with singularities and the amplitude of oscillation grows with time. This, however, is not surprising because this case corresponds to a large positive pressure. In flat universe the scale factor is defined up to an overall normalization factor and therefore, assuming \( C_1 \neq 0 \), we can write the general solution for the scale factor in case \( \alpha \geq -1/3 \) as

\[ a(t) = t^{\frac{1}{3}} \left( 1 + \frac{A}{t^{-\sqrt{1+3\alpha}}} \right)^{2/3}, \quad (3.4) \]

where \( A = C_2/C_1 \) is a constant of integration.

One can substitute this solution in (2.18) to find the energy density

\[ \dot{\varepsilon} = 3H^2 = \frac{1}{3t^2} \left( 1 + \sqrt{1+3\alpha} \frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}} \right)^2. \quad (3.5) \]

The appearance of the constant of integration \( A \) is related to the freedom of having an extra constant of integration in (2.18). Taking into account that

\[ \dot{p} = -\frac{\alpha}{t^2}, \]

we will find the equation of state for the mimetic matter

\[ w = \frac{\dot{p}}{\dot{\varepsilon}} = -3\alpha \left( 1 + \sqrt{1+3\alpha} \frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}} \right)^{-2}, \quad (3.7) \]

In general this equation of state depends on time but in the limit of small and large \( t \) approaches a constant. For \( \alpha = -1/3 \) we have ultra-hard equation of state \( \dot{p} = \dot{\varepsilon} \) and \( a \propto t^{1/3} \). The case \( \alpha = -1/4 \) corresponds to ultra-relativistic fluid with \( \dot{p} = \frac{1}{3} \dot{\varepsilon} \) at large time and \( \dot{p} = 3 \dot{\varepsilon} \) when \( t \to 0 \) if \( A \neq 0 \). When \( \alpha \) is very small then we have mimetic dark matter with negligible pressure, Finally for positive \( \alpha \) the pressure is negative and if \( \alpha \gg 1 \) the equation of state approaches the cosmological constant, \( \dot{p} = -\dot{\varepsilon} \).

In case of an arbitrary power law potential

\[ V(\phi) = \alpha \phi^n = \alpha t^n, \quad (3.8) \]
$n \neq -2$, the general solution of the differential equation
\[ \ddot{y} - \frac{3\alpha}{4} \dot{t}^n y = 0, \]
is given in terms of Bessel functions
\[
y = t^{1/2} \frac{Z_{1/2}}{n+2} \left( \frac{\sqrt{-3\alpha}}{n+2} t^{n+2} \right). \tag{3.9}
\]
If $n < -2$, that is, the potential decays faster than $1/\phi^2$, the asymptotic at large $t$ is $y \propto t$ and, correspondingly, the scale factor in the leading order behaves as in dust dominated universe, $a \propto t^{2/3}$. For $n > -2$
\[
y \propto t^{-n/4} \exp \left( \pm i \frac{\sqrt{-3\alpha}}{n+2} t^{n+2} \right), \tag{3.10}
\]
as $t \to \infty$. Here the behavior of the scale factor drastically depends on the sign of $\alpha$. For negative $\alpha$, corresponding to positive pressure, the mimetic matter leads to an oscillating universe with singularities. The case of positive $\alpha$ or negative pressure corresponds to an accelerated, inflationary universe. In particular, for $n = 0$ we have an exponential expansion corresponding to the cosmological constant, while $n = 2$ leads to inflationary expansion with
\[
a \propto t^{-1/3} \exp \left( \sqrt{\frac{\alpha}{12}} t^2 \right), \tag{3.11}
\]
similar to chaotic inflation with quadratic potential.

4 Mimetic matter as quintessence

Let us consider the behavior of mimetic matter in the case when the universe is dominated by some other matter with constant equation of state $p = w\varepsilon$ and where the potential is given by
\[
V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2}, \tag{4.1}
\]
In this case the scale factor is
\[
a \propto t^{\frac{2}{3(1+w)}}, \tag{4.2}
\]
and if $\phi = t$ then the energy density of mimetic matter given by (2.15) decays as
\[
\tilde{\varepsilon} = -\frac{\alpha}{wt^2}, \tag{4.3}
\]
if we set the constant of integration in (2.15) to zero. Because $\tilde{p} = -\alpha/t^2$ the mimetic matter imitates the equation of state of the dominant matter. However, since the total energy density is equal to
\[
\varepsilon = 3H^2 = \frac{4}{3(1+w)^2 t^2}, \tag{4.4}
\]
this mimetic matter can be subdominant only if $\alpha/w \ll 1$. The more general solution for subdominant mimetic matter, $\phi = t + t_0$, first corresponds to a cosmological constant for $t < t_0$ and only at $t > t_0$ starts to behave similar to a dominant matter.
5 Mimetic matter as an inflaton

One can easily construct the inflationary solutions using the mimetic matter. In fact, one can take any scale factor \( a(t) = y^{2/3} \) and using (3.2) find the potential

\[
V(\phi) = V(t) = \frac{4}{3} \frac{\ddot{y}}{y},
\]

for the theory where this scale factor will be a solution of the corresponding equations. For example, the potential

\[
V(\phi) = \frac{\alpha \phi^2}{\exp(\phi) + 1},
\]

with positive \( \alpha \) describes inflation with graceful exit to matter dominating universe. In fact, the scale factor grows as

\[
a \propto \exp \left( -\sqrt{\frac{\alpha}{12}} t^2 \right),
\]

at large negative \( \phi = t \) and it is proportional to \( t^{2/3} \) for positive \( t \). Playing with potentials one can easily get any “wishful” behavior for the scale factor during inflation and after it. Thus we see that the mimetic matter can easily provide us with an inflaton. The question is then how one can generate the radiation and baryons we observe. This can be done either via gravitational particle production at the end of inflation [7–9], or via direct coupling of the other fields to \( \phi \). We will leave it to the reader to elaborate this question, but there are no obvious obstacles which would prevent us from making the models of inflation using the mimetic matter.

6 Mimetic matter and bouncing universe

Using the same strategy as for building inflationary solutions we can easily find a theory which provide us with non-singular bounce in a contracting flat universe. For example, let us consider the potential

\[
V(\phi) = \frac{4}{3} \frac{1}{(1 + \phi^2)^2} = \frac{4}{3} \frac{1}{(1 + t^2)^2}.
\]

The general exact solution of the equation

\[
\frac{d^2y}{dt^2} - \frac{1}{(t^2 + 1)^2} y = 0,
\]

is

\[
y = \sqrt{t^2 + 1} (C_1 + C_2 \arctan t),
\]

and correspondingly the scale factor is

\[
a = \left( \sqrt{t^2 + 1} (1 + A \arctan t) \right)^{2/3},
\]

where we have assumed that \( C_1 \neq 0 \) and used the freedom for normalization of the scale factor in a flat universe. Let us consider for simplicity \( A = 0 \). In this case

\[
a = (t^2 + 1)^{1/3}.
\]
The energy density and pressure are

\[ \tilde{\varepsilon} = 3H^2 = \frac{4}{3} \frac{t^2}{(1 + t^2)^2}, \quad \tilde{p} = -\frac{4}{3} \frac{1}{(1 + t^2)^2}, \]

respectively. For large negative \( t \) the universe is dominated by dust with negligible pressure and it contracts. The energy density first grows as \( a^{-3} \) and then during a very short time interval around \( |t| \sim 1 \) it drastically drops to zero, the universe stops contraction and begins to expand. After the beginning of expansion the energy density first increases to the Planckian value within short interval corresponding to the Planckian time and then the expansion proceeds as in dust dominated universe. The interesting property of the model is the change of equation of state from the normal one, satisfying condition \( \tilde{\varepsilon} + \tilde{p} > 0 \) to the phantom one \( \tilde{\varepsilon} + \tilde{p} < 0 \).

In fact, \( \tilde{\varepsilon} + \tilde{p} = \frac{4}{3} \frac{t^2 - 1}{(1 + t^2)^2} \),

and for \( |t| < 1 \) the mimetic matter is a phantom and therefore we can have a non-singular bounce. In the general case for \( A \neq 0 \), the bounce is non-singular if \( |A| < 2/\pi \), otherwise the contracting universe ends up in the singularity. The value of \( A \) is determined by the relative balance of phantom and dark matter “components” in mimetic matter when we are close to the point of bounce. Therefore it is not surprising that the universe becomes singular when dark matter dominates. Using this phantom mimetic matter one can also avoid a singularity in contracting universe in the presence of other matter.

In the model we have considered the bounce happens at Planckian scales when the theory is not under control because of quantum gravitational effects. However slightly modifying the potential \( V \) we can easily lower the bounce scale and make the bounce duration to be longer than the Planckian time. In fact let us consider the potential

\[ V(\phi) = \frac{4}{3} \frac{\alpha}{(\phi_0^2 + \phi^2)^2} = \frac{4}{3} \frac{\alpha}{(t_0^2 + t^2)^2}. \]

For this potential the differential equation (2.22) becomes

\[ \frac{d^2 y}{d\tilde{t}^2} - \frac{\alpha t_0^2}{(\tilde{t}^2 + 1)^2} y = 0, \]

where \( \tilde{t} = t/t_0 \), and its general solution is

\[ a(t) = y^2 = \left( \sqrt{\left(\frac{t}{t_0}\right)^2 + 1} \left( \cos \left( \beta \arctan \left( \frac{t}{t_0}\right) \right) + A \sin \left( \beta \arctan \left( \frac{t}{t_0}\right) \right) \right) \right)^\frac{2}{3}, \]

where \( \beta = \sqrt{1 - \alpha t_0^{-2}}. \) In this case the bounce happens at scales about \( \alpha t_0^{-2} \) during the time interval \( t_0 \).

7 Cosmological perturbations

Finally we will consider the behavior of small longitudinal metric perturbations in the universe dominated by mimetic matter. Because the off-diagonal components of the energy-momentum tensor for mimetic matter vanish in the linear order, the metric of perturbed
universe in the Newtonian gauge can be written as [6]

\[ ds^2 = (1 + 2\Phi) dt^2 - (1 - 2\Phi) a^2 \delta_{ik} dx^i dx^k, \]  

(7.1)

where \( \Phi \) is the Newtonian gravitational potential. Considering perturbations of the scalar field,

\[ \phi = t + \delta \phi, \]  

(7.2)

from equation (2.5) we find

\[ \Phi = \delta \dot{\phi}. \]  

(7.3)

The equation for perturbations which follows from the linearized 0–i components of Einstein equations is (see equations (7.39) and (7.45) in [6])

\[ \left( \dot{\Phi} + H \Phi \right)_i = \frac{1}{2} \left( \dot{\epsilon} + \dot{p} \right) \delta \phi_i. \]  

(7.4)

Taking into account that \( \dot{\epsilon} + \dot{p} = -2H \) and substituting here \( \Phi \) from (7.3) we obtain the following equation for \( \delta \phi \)

\[ \ddot{\delta \phi} + H \dot{\delta \phi} + H\delta \phi = 0. \]  

(7.5)

As one can easily verify by direct substitution the general solution of this equation is

\[ \delta \phi = A \frac{1}{a} \int adt, \]  

(7.6)

where \( A \) is a constant of integration which depends only on the spatial coordinates (the other constant of integration corresponding to decaying mode can be always included in the integral). The corresponding gravitational potential is

\[ \Phi = \delta \dot{\phi} = A \frac{d}{dt} \left( \frac{1}{a} \int adt \right) = A \left( 1 - \frac{H}{a} \int adt \right). \]  

(7.7)

This is exactly the general solution we normally have for the long wavelength cosmological perturbations when one can neglect the spatial derivative terms which are multiplied by the speed of sound for normal hydrodynamical fluid [6]. However, in our case the solution above is universally valid for all perturbations irrespective of their wavelength. In this sense the perturbations behave as a dust with vanishing speed of sound even for mimetic matter with nonvanishing pressure (in agreement with [5]). As a result one cannot define the quantum fluctuations of mimetic matter in the usual way. If so the mimetic inflation considered above would fail in one of its major tasks, namely, in explaining of the large scale structure as originated from quantum fluctuations. To make the mimetic inflation “viable” one either has to use one more scalar field playing the role of curvaton [11] or slightly modify the model for mimetic matter. The use of extra scalar field makes the theory not very plausible because such a theory can “explain” nearly everything and predict nothing. For this reason we prefer to modify the model above.

8 Modified mimetic matter action and cosmological perturbations

Let us add to the Lagrangian (2.1) an extra term

\[ + \frac{1}{2} \gamma (\square \phi)^2, \]  

(8.1)
where $\gamma$ is a constant and $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$. Note that because the scalar field satisfies the constraint
\begin{equation}
    g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi = 1, \tag{8.2}
\end{equation}
by adding the terms with many derivatives of $\phi$ we do not change the total number of degrees of freedom. Varying the action
\begin{equation}
    S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R(g_{\mu\nu}) + \lambda (g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - 1) - V(\phi) + \frac{1}{2} \gamma (\Box \phi)^2 \right], \tag{8.3}
\end{equation}
with respect to the metric we obtain
\begin{equation}
    G^\mu_\nu = \tilde{T}^\mu_\nu, \tag{8.4}
\end{equation}
where
\begin{equation}
    \tilde{T}^\mu_\nu = \left( V + \gamma \left( \phi,\alpha \chi^\alpha + \frac{1}{2} \chi^2 \right) \right) \delta^\mu_\nu + 2\lambda \phi\phi \phi^{\mu} - \gamma (\phi,\nu \chi^{\mu} + \chi,\nu \phi^{\mu}), \tag{8.5}
\end{equation}
and
\begin{equation}
    \chi = \Box \phi. \tag{8.6}
\end{equation}
Taken together with constraint (8.2) the equations (8.4) completely determine the metric, scalar field and Lagrange multiplier $\lambda$. Instead of making analogy with perfect fluid it is more convenient to solve equations (8.4) directly. The general solution of equation (8.2) in the Friedmann universe is
\begin{equation}
    \phi = t + A, \tag{8.7}
\end{equation}
and hence
\begin{equation}
    \chi = \Box \phi = \dot{\phi} + 3H \phi = 3H. \tag{8.8}
\end{equation}
Taking this into account, the $0-0$ Einstein equation then reduces to
\begin{equation}
    H^2 = \frac{1}{3} V + \gamma \left( \frac{3}{2} H^2 - \dot{H} \right) + \frac{2}{3} \lambda, \tag{8.9}
\end{equation}
and the $i-j$ equations give
\begin{equation}
    2\dot{H} + 3H^2 = V(t) + \frac{3\gamma}{2} \left( 2\dot{H} + 3H^2 \right). \tag{8.10}
\end{equation}
Thus, instead of equation (2.19) we obtain
\begin{equation}
    2\dot{H} + 3H^2 = \frac{2}{2 - 3\gamma} V, \tag{8.11}
\end{equation}
which is different from (2.19) only by overall normalization of potential $V$. Therefore, in the presence of the extra $(\Box \phi)^2$ term the cosmological solutions derived above for homogeneous universe will remain unchanged up to the numerical factor of order unity. However, this term changes the behavior of the short wave cosmological perturbations drastically. The linear perturbation of $0-i$ component of the energy momentum tensor is equal to
\begin{equation}
    \delta T^0_i = 2\lambda \delta \phi,i - 3\gamma \dot{H} \delta \phi,i - \gamma \delta \chi,i. \tag{8.12}
\end{equation}
Taking into account that
\[ \delta \chi = \delta (\Box \phi) = -4 \dot{\Phi} - 6H \Phi + \delta \ddot{\phi} + 3H \dot{\delta} \phi - \frac{\Delta}{a^2} \delta \phi = \]
\[ = -3 \delta \ddot{\phi} - 3H \dot{\delta} \phi - \frac{\Delta}{a^2} \delta \phi , \] (8.13)
and
\[ \lambda = (3\gamma - 1) H , \] (8.14)
as it follows from equations (8.9) and (8.10), we find that the perturbed 0\text{--}i Einstein equation reduces to
\[ \delta \ddot{\phi} + H \delta \dot{\phi} - \frac{c_s^2}{a^2} \Delta \delta \phi + \dot{H} \delta \phi = 0 , \] (8.15)
where
\[ c_s^2 = \frac{\gamma^2}{2 - 3\gamma} . \] (8.16)
This equation is different from (7.5) only by the presence of the gradient terms multiplied by the speed of sound \( c_s \). For a plane wave perturbation \( \propto \exp (ikx) \) equation (8.15) becomes
\[ \delta \phi_{kk}'' + \left( c_s^2 k^2 + \frac{a''}{a} - 2 \left( \frac{a'}{a} \right)^2 \right) \delta \phi_k = 0 . \] (8.17)
where prime denotes the derivative with respect to conformal time \( \eta = \int dt/a \). Considering the short wavelength perturbations with \( c_s k \eta \gg 1 (\lambda_{ph} = a/k \ll c_s H^{-1}) \) we can neglect the time derivative terms inside the bracket and the corresponding solution is
\[ \delta \phi_k \propto e^{\pm ic_s k \eta} . \] (8.18)
On the other hand for the long wavelength perturbations with \( c_s k \eta \ll 1 (\lambda_{ph} = a/k \gg c_s H^{-1}) \) the \( c_s^2 k^2 \)–term can be neglected and we obtain the same solution as in (7.6), that is,
\[ \delta \phi = A \frac{1}{a} \int a^2 d\eta . \] (8.19)
To fix the amplitude of quantum fluctuations we have to identify the canonical quantization variable. Expanding action to the second order in perturbations gives
\[ S = -\frac{1}{2} \int d\eta d^3x \left( \frac{\gamma}{c_s^2} \delta \phi' \Delta \delta \phi' + \ldots \right) , \] (8.20)
and hence for the short wavelength quantum initial perturbations the canonically normalized variable is
\[ v_k \sim \sqrt{\frac{\gamma}{c_s}} \ k \ \delta \phi_k , \] (8.21)
whose fluctuation in vacuum is
\[ \delta v_k \sim \frac{1}{\sqrt{\omega_k}} \sim \frac{1}{\sqrt{c_s k}} , \] (8.22)
so that consequently
\[ \delta \phi_k \sim \sqrt{\frac{c_s}{\gamma}} k^{-3/2} . \]
This implies a flat power-spectrum for $\delta \phi_\lambda$ and $\Phi_\lambda \sim \lambda^{-1}$ for short length-scales $\lambda = 1/k$. Let us consider an inflationary stage when

$$\frac{1}{a} \int a dt \simeq H^{-1}. \quad (8.23)$$

Taking this into account and matching the solution (8.18) and (8.19) at $c_s k \eta \sim 1$ we find that

$$A_k \sim \sqrt{\frac{c_s}{\gamma}} \frac{H_{c_s,k=H\alpha}}{k^{3/2}}. \quad (8.24)$$

Therefore the typical amplitude of the gravitational potential ($\Phi_k \sim A_k$) in comoving scales $\lambda \sim 1/k$ after inflation is

$$\Phi_\lambda \sim A_k \sim A_k k^{3/2} \sim \sqrt{\frac{c_s}{\gamma}} H_{c_s,k=H\alpha}. \quad (8.25)$$

Similarly to $k$-inflation [12, 13] one obtains that

$$\text{for } c_s \ll 1 \quad : \quad \Phi_\lambda \sim c_s^{-1/2} H_{c_s,k=H\alpha}, \quad (8.26)$$

which is $c_s^{-1/2}$ enhanced with respect to the amplitude of the gravity waves $h_\lambda \sim H_{k=H\alpha}$. On the other hand,

$$\text{for } c_s \gg 1 \quad : \quad \Phi_\lambda \sim c_s^{1/2} H_{c_s,k=H\alpha}, \quad (8.27)$$

which is again $c_s^{1/2}$ enhanced\(^1\) with respect to the amplitude of the gravity waves. This is a completely opposite effect to what happens in the $k$-inflationary case where for $c_s \gg 1$ the scalar fluctuations are suppressed similarly to (8.26) and effectively the gravity waves are enhanced, see e.g. [14]. In fact in this mimetic model the scalar perturbations are always larger than the tensor perturbations.

Here it is important to mention that this suppression of the gravity waves occurs purely due to the quadratic term with the d’Alembertian. Thus, one can expect that, contrary to $k$-inflation, this suppression does not induce any non-Gaussianity.

We would like to stress that similar to usual inflation the spectral index for the adiabatic fluctuations is also red-tilted for the mimetic inflation.

### 9 Conclusions

In this paper we have extended the Mimetic Dark Matter to mimic any background cosmology. This can be achieved by adding an appropriate potential $V(\phi)$ to the original metric. As simple examples we have discussed quintessence, inflation and bouncing universe with vanishing speed of sound for perturbations.

Further, we have found another interesting novel extension which allows for the non-trivial speed of sound. This can be achieved by adding higher-order-derivative terms to the action without increasing the number of degrees of freedom in the system. This allows one to quantize the inflationary scalar perturbations using standard techniques. It is demonstrated that these perturbations can have novel observational features absent in the case of $k$-inflation models. In particular it is possible to strongly suppress the gravitational waves from inflation,

\(^1\)Note that this superluminality does not imply any causal paradoxes or other inconsistencies on the level of effective field theory, see e.g. [10].
seemingly without any non-Gaussianity. It would be very interesting to analyze whether one can observationally distinguish Mimetic Inflation from other models.

Finally, such a modification opens up a new interesting playground for modeling Dark Matter, where the speed of sound can be very small but not exactly vanishing and the behavior of the mimetic matter can deviate from the usual perfect-fluid-like dust.

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