Irregular Assignment of Series Parallel Networks

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Irregular Assignment of Series Parallel Networks

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Abstract. In this paper, we determined that irregularity strength and the total vertex irregularity strength of the parallel networks, $sp(m, r, 2)$ for the natural number $m, r \geq 3$, is $mr + 1$ for $m \geq 4, r \geq 3$ and $\left\lceil \frac{2mr+2}{3} \right\rceil$, for $m, r \geq 3$, respectively.

1. Introduction
In 1736, a Swiss mathematician, Leonhard Euler was introduced graph theory. Research on graph theory continues to grow along with the development of information technology. One of the most developed is a graphs labelling. The study object of graph labelling are vertex set, edge set, and number (generally positive integers). Graph labelling introduced by Sadlack, then Stewart, [1], Kotzig, and Rosa [2].

Formally, graph labelling is a function that pairing graph elements to a set of numbers (usually integer number). If the domain of this labelling is vertex set, then this labelling called vertex labelling. If the domain of this labelling is edge set, then this labelling called edge labelling. If the domain of this labelling is a joint of vertex and edge sets, then called total labelling [3].

An edge labelling of a graph can be done in many ways. One of them is irregular labelling. The irregular labelling on a graph was introduced by Chartrand. A edge labelling $f: E \rightarrow \{1, 2, \ldots, s\}$ is called a irregular s-labelling of $G$ if every two distinct vertices $x$ and $y$ in $V$ satisfy $wt(x) \neq wt(y)$, where $wt(x) = \sum_{xz \in E(G)} f(xz)$. The minimum $s$ for which a graph $G$ has an irregular $s$-labelling, denoted by $(G)$, is called the irregularity labelling of $G$ [4].

In 2007, Bača [5] introduced a vertex and edge irregular total labellings. A total labelling $f: V \cup E \rightarrow \{1, 2, \ldots, k\}$ is called a vertex irregular total $k$-labelling of $G$ if every two distinct vertices $x$ and $y$ in $G$ satisfy $wt(x) \neq wt(y)$, where $wt(x) = f(x) + \sum_{xz \in E(G)} f(xz)$. The minimum $k$ for which a graph $G$ has a vertex irregular total $k$-labelling, denoted by $tvs(G)$, is called the total vertex irregularity strength of $G$ [5].

Some experts have determined the value of irregularity labelling of some graphs. Ahmad et al, have determined irregular labelling of helm and sun graphs [6]. Besides that, Ahmad et.al have determined the total vertex and the total edge irregularity strength of Halin graph [7]. In 2012, Anolcher and Palmer determined the irregular labelling of circulant graphs [8]. Chartrand et al. [4], has determined irregular labelling of regular-$d$ graph and posed the following.

Theorem 1. Let $G$ is a regular-$d$ graph, then the lower bound for irregularity strength of $G$, for $d \geq 2$ is $s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil$. 
The total vertex irregularity strength of a graph has been done by some experts. Rajasingh have determined the total vertex irregularity strength of triangular related graph [9]. Nurdin et al. determined the total vertex irregularity strength of tree graphs and posed the following [10]. In [11], Nurdin et al. have determined the total irregularity strength of network constructed by cycle. Recently, Nurdin have determined the total vertex irregularity strength of butterfly network [12].

**Theorem 2.** Let $G$ be a connected graph having $n_i$ vertices of degree $i$ ($i = \delta, \delta + 1, \delta + 2, ..., \Delta$) where $\delta$ and $\Delta$ are minimum and maximum degree of $G$, respectively. Then

$$tv(G) \geq \max\left\{ \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1} \right\rceil \right\}.$$ 

In 2015, Rajasingh have determined the total edge irregularity strength of series parallel graphs [9]. It has not yet determined the irregular labelling of series parallel graph and the total vertex irregularity strength of series parallel graph. This study aims to determine the greatest lower bound and the smallest upper bound to obtain the irregular labelling of series parallel graphs and the total vertex irregularity strength of series parallel graphs are exact.

2. Series Parallel Graph

Rajasingh [9, 13] gave the definition of series parallel graph as follows.

**Definition 1.** The series parallel graph of $G$ is a chain graph in which each block is generalized theta graph. The series parallel is denoted by $sp(m, r, l)$ for $m, r \geq 3$ and $l \geq 2$.

![Figure 1. Series Parallel Graph sp(m, r, 2).](image)

Figure 1 show the series parallel graph for $l = 2$. Defined vertex set of series parallel graph $V = \{x_{i,j}, y_{i,j} | j = 1, 2, ..., r\} \cup \{z_1, z_2, z_3\}$ and edge set of is $E = \{z_3, x_{i,j}, z_1, y_{i,j}, z_1, y_{i,j+1}, z_2, y_{i,j}, z_1, y_{i,j+1} | j = 1, 2, ..., r - 1\}$, for $i = 1, 2, ..., m$.

3. Irregular Labelling of Series Parallel Graph

In determining of the irregular labelling of series parallel graph, begin with determining the lower and upper bound of irregularity strength of the graph. The lower bound of irregularity strength of a series parallel graph $sp(m, r, 2)$ for $m = 4, 5, 6$ and $r = 3$ is shown in table 1 which is analyzed by using series parallel properties and based on Theorem 1.
The upper bound is analyzed by edge labelling on the series parallel \( sp(m, r, 2) \) for \( m = 4, 5, 6 \) and \( r = 3 \) by maintaining pattern labelling as follow.

![Figure 2. Irregular Labelling-13 of sp(4,3,2).](image)

![Figure 3. Irregular Labelling-16 of sp(5,3,2).](image)

Based on table 1, figure 2 and figure 3, in general for any \( m \geq 4, r \geq 3 \), irregular labelling of series parallel \( (sp(m, r, 2)) \) can be estimated as

| Table 2. Irregular Labelling \( sp(m, r, 2) \). |
|---|---|---|---|---|
| \( m \) | \( r \) | 3 | 4 | 5 | \( \cdots \) | \( r \) |
| 4 | 13 | 17 | 21 | \( \cdots \) | \( (4, r) + 1 \) |
| 5 | 16 | 21 | 26 | \( \cdots \) | \( (5, r) + 1 \) |
| 6 | 19 | 25 | 31 | \( \cdots \) | \( (6, r) + 1 \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( M \) | \( (m, 3) + 1 \) | \( (m, 4) + 1 \) | \( (m, 5) + 1 \) | \( \cdots \) | \( mr + 1 \) |

Table 1. Lower Bound of Irregularity Strength \( sp(m, 3, 2) \).

| \( m \) | \( \delta \) | \( n_\delta \) | \( \left\lfloor \frac{n_\delta + \delta - 1}{\delta} \right\rfloor \leq s(sp(m, 3, 2)) \) |
|---|---|---|---|
| 4 | 2 | 24 | 13 |
| 5 | 2 | 30 | 16 |
| 6 | 2 | 36 | 19 |

**Table 1. Lower Bound of Irregularity Strength \( sp(m, 3, 2) \).**
Based on table 2 it is assumed that irregular labelling of series parallel graph is \((sp(m, r, 2)) = mr + 1\). These results are written on the theorem as follows.

**Theorem 3.** For \(m \geq 4\) and \(r \geq 3\), then irregular labelling of series parallel \(sp(m, r, 2)\) is

\[
s(sp(m, r, 2)) = mr + 1.
\]

**Proof**

To prove that \(s(sp(m, r, 2)) \geq mr + 1\), use Theorem 1. Since the minimum degree of \(sp(m, r, 2)\) is \(\delta = 2\), the number of vertex of degree 2 is \(n_2 = 2mr\). Then based on Theorem 1 obtained

\[
s(sp(m, r, 2)) \geq \left\lfloor \frac{n_2 + \delta - 1}{\delta} \right\rfloor = \left\lfloor \frac{2mr + 2 - 1}{2} \right\rfloor = mr + 1.
\]

Next, we will determine the upper bound of \(s(sp(m, r, 2))\) is \(mr + 1\). To prove it, we will construction an irregular labelling on the series parallel \(sp(m, r, 2)\) as follows. Let \(s = mr + 1\).

**Case I. For \(r\) is odd**

For \(i = 1, 2, ..., m\), defined \(\lambda\) as follows:

\[
\lambda(z_{3x1}) = s - \left(\frac{m}{2} (r + 1) - i + 1\right)
\]

\[
\lambda(x_{ij}x_{i,j+1}) = \begin{cases} 
  s - \left(\frac{m}{2} (r + j)\right), & j = 1, 3, ..., r - 2 \\
  s - \left(\frac{m}{2} (r + j + 1) - i + 1\right), & j = 2, 4, ..., r - 1
\end{cases}
\]

\[
\lambda(z_{x_1r}) = 1
\]

\[
\lambda(z_{1y1}) = s - \left(\frac{m}{2} (r + 1) - i + 1\right)
\]

\[
\lambda(y_{ij}y_{i,j+1}) = \begin{cases} 
  s - \left(\frac{m}{2} (r - j)\right), & j = 1, 3, ..., r - 2 \\
  s - \left(\frac{m}{2} (r - j + 1) - i + 1\right), & j = 2, 4, ..., r - 1
\end{cases}
\]

\[
\lambda(z_{2y1}) = s.
\]

**Case II. For \(r\) is even**

For \(i = 1, 2, ..., m\), defined \(\lambda\) as follows:

\[
\lambda(z_{3x1}) = s - \left(\frac{mr}{2}\right)
\]

\[
\lambda(x_{ij}x_{i,j+1}) = \begin{cases} 
  s - \left(\frac{m}{2} (r + j + 1) - i + 1\right), & j = 1, 3, ..., r - 1 \\
  s - \left(\frac{m}{2} (r + j)\right), & j = 2, 4, ..., r - 2
\end{cases}
\]

\[
\lambda(z_{x_1r}) = 1
\]

\[
\lambda(z_{1y1}) = s - \left(\frac{mr}{2}\right)
\]

\[
\lambda(y_{ij}y_{i,j+1}) = \begin{cases} 
  s - \left(\frac{m}{2} (r - j + 1) - i + 1\right), & j = 1, 3, ..., r - 1 \\
  s - \left(\frac{m}{2} (r - j)\right), & j = 2, 4, ..., r - 2
\end{cases}
\]

\[
\lambda(z_{2y1}) = s.
\]

Based on the labelling function \(\lambda\), we obtained the weight of all vertices of graph \(sp(m, r, 2)\) for \(i = 1, 2, ..., m\) and \(j = 1, 2, ..., r - 1\) as follows.

**Case I. For \(r\) is odd**

\[
wt(x_{i,r}) = \lambda(x_{i+1,r}) + \lambda(z_{x_1r}) = s - mr + i.
\]

\[
wt(x_{ij}) = \lambda(x_{i,j+1}) + \lambda(z_{x_1r}) = 2s - 2m(r + j) - \left(\frac{m}{2}\right) + i - 1.
\]

\[
wt(x_{1i}) = \lambda(x_{2x1}) + \lambda(z_{x_1r}) = 2s - mr - m + i - 1.
\]

\[
wt(y_{i,1}) = \lambda(z_{x_1}) + \lambda(y_{1,2}) = 2s - mr + i - 1.
\]
\[ wt(y_{ij}) = \lambda(y_{ij-1}y_{ij}) + \lambda(y_{ij}y_{ij+1}) = 2s - m(r - j) - \frac{m}{2} + i - 1. \]
\[ wt(y_{1r}) = \lambda(y_{1r-1}y_{1r}) + \lambda(z_2y_{1r}) = 2s - m + i - 1. \]
\[ wt(z_3) = \sum_{i=1}^{m} \lambda(z_3x_{i,1}) = \sum_{i=1}^{m} \left(s - \left(\frac{m}{2}(r + 1) - i + 1\right)\right). \]
\[ wt(z_1) = \sum_{i=1}^{m} \lambda(z_1x_{1r}) = \sum_{i=1}^{m} \lambda(z_1y_{1i}) = m + \sum_{i=1}^{m} \left(s - \left(\frac{m}{2}(r + 1) - i + 1\right)\right). \]
\[ wt(z_2) = \sum_{i=1}^{m} \lambda(z_2y_{1i}) = ms. \]

**Case II. For \( r \) is even**
\[ wt(x_{ij}) = \lambda(x_{ij-1}x_{ij}) + \lambda(z_1x_{ij}) = s - mr + i. \]
\[ wt(x_{i,1}) = \lambda(z_2x_{i,1}) + \lambda(x_{i,1}x_{i,2}) = 2s - mr + i - 1. \]
\[ wt(y_{i,1}) = \lambda(y_{i,1-1}y_{i,1}) + \lambda(y_{i,1}y_{i,2}) = 2s - mr + i - 1. \]
\[ wt(y_{1r}) = \lambda(y_{1r-1}y_{1r}) + \lambda(z_2y_{1r}) = 2s - m + i - 1. \]
\[ wt(z_3) = \sum_{i=1}^{m} \lambda(z_3x_{i,1}) = \sum_{i=1}^{m} \left(s - \frac{mr}{2}\right). \]
\[ wt(z_1) = \sum_{i=1}^{m} \lambda(z_1x_{1r}) = \sum_{i=1}^{m} \lambda(z_1y_{1i}) = m + \sum_{i=1}^{m} \left(s - \frac{mr}{2}\right). \]
\[ wt(z_2) = \sum_{i=1}^{m} \lambda(z_2y_{1i}) = ms. \]

Based on the definition of the weight of the vertices, obtained
\[ wt(x_{1,r}) < wt(x_{2,r}) < \cdots < wt(x_{m,r}) < wt(x_{1,r-1}) < \cdots < wt(x_{m,r-1}) \]
\[ < wt(x_{1,r-2}) < \cdots < wt(x_{1,1}) < wt(x_{2,1}) < \cdots < wt(x_{m,1}) < wt(y_{1,1}) < wt(y_{2,1}) < \cdots \]
\[ < wt(y_{m,1}) < wt(y_{1,2}) < wt(y_{2,2}) < \cdots < wt(y_{m,2}) < wt(y_{1,3}) < \cdots < wt(y_{1,r}) \]
\[ < wt(y_{2,r}) < \cdots < wt(y_{m,r}) < wt(z_3) < wt(z_1) < wt(z_2). \]

So it can be concluded that the weight of each vertex on \( sp(m, r, 2) \) is different. Then the \( \lambda \) constructed is an irregular labelling on \( sp(m, r, 2) \). Thus \( \lambda \) is an irregular labelling with \( s = mr + 1 \) for \( m \geq 4, r \geq 3 \). That is \( s(sp(m, r, 2)) \leq mr + 1 \).

### 4. Total Vertex Irregularity Strength of Series Parallel

In determining the total vertex irregularity strength of series parallel graph, begin from determine the lower and upper bound of the total vertex irregularity strength. The lower bound of series parallel for \( m = 3,4,5 \) and \( r = 3 \) is shown in Table 2 which is analyzed by using series parallel properties and based on Theorem 2.

| Table 3. Lower Bound of Total Labelling \( sp(m, 3, 2) \) |
|---|---|---|---|
| \( m \) | \( \delta \) | \( n_\delta \) | \( \left[\frac{\delta + n_\delta}{\delta + 1}\right] \leq tvs(sp(m, 3, 2)) \) |
| 3 | 2 | 18 | 7 |
| 4 | 2 | 24 | 9 |
| 5 | 2 | 30 | 11 |
The upper bound is analyzed by total labelling on the series parallel \( \text{sp}(m, r, 2) \) for \( m = 3, 4, 5 \) and \( r = 3 \) by maintaining pattern labelling as follows.

**Figure 4.** Irregular total vertex 7 – labelling \( \text{sp}(3, 3, 2) \).

**Figure 5.** Irregular Total Vertex 9 – labelling \( \text{sp}(4, 3, 2) \).

**Figure 6.** Irregular Total Vertex 11 – labelling \( \text{sp}(5, 3, 2) \).

Based on table 3, figure 3 and figure 4, in general for any \( m, r \geq 3 \), the total vertex irregularity strength of series parallel \( \text{sp}(m, r, 2) \) can be estimated like in table 4.

Based on table 3 it is assumed that total vertex irregularity strength of series parallel graph is \( \text{tv}_v\text{sp}(m, r, 2) = \left\lceil \frac{2mr+2}{3} \right\rceil \). These results are written on the theorem as follows.

**Theorem 4.** For \( m, r \geq 3 \), then total vertex irregularity strength of series parallel \( \text{sp}(m, r, 2) \) is

\[
\text{tv}_v\text{sp}(m, r, 2) = \left\lceil \frac{2mr+2}{3} \right\rceil.
\]

**Proof.**

To prove that \( tv_v\text{sp}(m, r, 2) \geq \left\lceil \frac{2mr+2}{3} \right\rceil \), use Theorem 2. The minimum degree of \( \text{sp}(m, r, 2) \) is \( \delta = 2 \), the number of vertex of degree 2 is \( n_2 = 2mr \). Then based on Theorem 2 we obtained...
\[
tvs(sp(m, r, 2)) \geq \text{maks} \left\{ \left\lfloor \frac{2mr + 2}{3} \right\rfloor, \left\lfloor \frac{2mr + m + 4}{m} \right\rfloor, \left\lfloor \frac{2mr + 3m + 3}{2m + 1} \right\rfloor \right\} = \left\lfloor \frac{2mr + 2}{3} \right\rfloor.
\]

**Table 4.** Total Vertex Irregularity Strength \(sp(m, r, 2)\).

| \(m\) | \((2.3.m) + 2\) | \((2.4.m) + 2\) | \((2.5.m) + 2\) | \(...\) | \((2.m.r) + 2\) |
|-------|----------------|----------------|----------------|------|----------------|
| 3     | 7              | 9              | 11             | \(...\) | \(\left\lfloor \frac{2mr + 2}{3} \right\rfloor\) |
| 4     | 9              | 12             | 14             | \(...\) | \(\left\lfloor \frac{2mr + 2}{3} \right\rfloor\) |
| 5     | 11             | 14             | 18             | \(...\) | \(\left\lfloor \frac{2mr + 2}{3} \right\rfloor\) |

Next determine the upper bound of \(tvs(sp(m, r, 2)) \leq \left\lfloor \frac{2mr+2}{3} \right\rfloor\). To prove it we will construct a vertex irregular total labelling on the series parallel \(sp(m, r, 2)\). Let \(t = \left\lfloor \frac{2mr+2}{3} \right\rfloor\).

**Case I. For** \(m = 0 \pmod{3}\)

\[
f(y_{ij}) = t - \left(\frac{m}{3}(r - j + 4) - i - 1\right)
\]

\[
f(y_{ir}) = t - (m - i)
\]

\[
f(z_i) = i, \quad i = 1, 2, 3
\]

\[
f(z_3x_{i1}) = t - (m - 2)
\]

\[
f(x_{i1}x_{i+1}) = t - \left(\frac{m}{3}(r + j + 1)\right)
\]

\[
f(z_1x_{i1}) = 1
\]

\[
f(z_1y_{i1}) = t - (m - i)
\]

\[
f(y_{ij}y_{i,j+1}) = t - \left(\frac{m}{3}(r - j + 1) + 1\right)
\]

\[
f(y_{i1}) = t
\]

\[
f(x_{i1}) = t - \left(\frac{m}{3}(2r - 2) - i + 3\right)
\]

\[
f(x_{ij}) = t - \left(\frac{m}{3}(r + j - 1) - i + 1\right)
\]

\[
f(x_{ir}) = t - \left(\frac{2mr}{3} - i + 1\right)
\]

\[
f(y_{i1}) = t - \left(\frac{m}{3}(2r - 1)\right)
\]
Case II. For $m = 1 \text{ (mod 3)}$

$$f(x_{i,i}) = \begin{cases} 
    t - \left( \frac{m}{3} (2r - 2) - \frac{r}{3} - i + \frac{14}{3} \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{m}{3} (2r - 2) - \frac{r}{3} - i + \frac{16}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{m}{3} (2r - 2) - \frac{r}{3} - i + \frac{12}{3} \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(x_{i,j}) = \begin{cases} 
    t - \left( \frac{m}{3} (r + j - 1) - \frac{2(j-r)}{3} - i + \frac{4}{3} \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r + j - 1) - \frac{2(j-r)}{3} - i + \frac{5}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r + j - 1) - \frac{2(j-r)}{3} - i + \frac{3}{3} \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(x_{i,r}) = \begin{cases} 
    t - \left( \frac{2mr}{3} - i + 1 \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{2mr}{3} - i + \frac{4}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{2mr}{3} - i + \frac{2}{3} \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(y_{i,i}) = \begin{cases} 
    t - \left( \frac{m}{3} (2r - 1) + \frac{1}{3} \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{m}{3} (2r - 1) + \frac{2}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{m}{3} (2r - 1) + 1 \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(y_{i,j}) = \begin{cases} 
    t - \left( \frac{m}{3} (r - j + 4) - \frac{2(r-j)}{3} - i + \frac{4}{3} \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r - j + 4) - \frac{2(r-j)}{3} - i + \frac{1}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r - j + 4) - \frac{2(r-j)}{3} - i + \frac{1}{3} \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(y_{i,r}) = \begin{cases} 
    t - (m - i), & r = 0 \text{ mod 3} \\
    t - (m - i + 1), & r = 1 \text{ mod 3} \\
    t - (m - i), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(z_i) = i, \quad i = 1, 2, 3$$

$$f(z_3 x_{i,i}) = t - (m - 3)$$

$$f(x_{i,j} x_{i,j+1}) = \begin{cases} 
    t - \left( \frac{m}{3} (r + j + 1) + \frac{r-j-1}{3} \right), & r = 0 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r + j + 1) + \frac{r-j}{3} \right), & r = 1 \text{ mod 3} \\
    t - \left( \frac{m}{3} (r + j + 1) + \frac{r-j-2}{3} \right), & r = 2 \text{ mod 3} 
\end{cases}$$

$$f(z_1 x_{i,r}) = 1$$
\[ f(z, y_{i, i+1}) = \begin{cases} 
- (m + \frac{r - 3}{3} - i), & r = 0 \mod 3 \\
- (m + \frac{r - 1}{3} - i), & r = 1 \mod 3 \\
- (m + \frac{r - 5}{3} - i), & r = 2 \mod 3 
\end{cases} \]

\[ f(y, y_{i, i+1}) = \begin{cases} 
- \left( \frac{m}{3} (r - j - 1) + \frac{j - r + 4}{3} \right), & r = 0 \mod 3 \\
- \left( \frac{m}{3} (r - j - 1) + \frac{j - r + 4}{3} \right), & r = 1 \mod 3 \\
- \left( \frac{m}{3} (r - j - 1) + \frac{j - r + 3}{3} \right), & r = 2 \mod 3 
\end{cases} \]

**Case III. For** \( m = 2 \mod 3 \)

\[ f(x, y) = \begin{cases} 
- \left( \frac{m}{3} (2r - 2) - \frac{2r}{3} - i + \frac{19}{3} \right), & r = 0 \mod 3 \\
- \left( \frac{m}{3} (2r - 2) - \frac{2r}{3} - i + \frac{17}{3} \right), & r = 1 \mod 3 \\
- \left( \frac{m}{3} (2r - 2) - \frac{2r}{3} - i + \frac{21}{3} \right), & r = 2 \mod 3 
\end{cases} \]

\[ f(x, y) = \begin{cases} 
- \left( \frac{2mr}{3} - i + \frac{1}{3} \right), & r = 0 \mod 3 \\
- \left( \frac{2mr}{3} - i + \frac{2}{3} \right), & r = 1 \mod 3 \\
- \left( \frac{2mr}{3} - i + \frac{4}{3} \right), & r = 2 \mod 3 
\end{cases} \]

\[ f(x, y) = \begin{cases} 
- \left( \frac{m}{3} (2r - 1) + \frac{2}{3} \right), & r = 0 \mod 3 \\
- \left( \frac{m}{3} (2r - 1) + \frac{4}{3} \right), & r = 1 \mod 3 \\
- \left( \frac{m}{3} (2r - 1) + \frac{1}{3} \right), & r = 2 \mod 3 
\end{cases} \]

\[ f(y, y_{i, i+1}) = \begin{cases} 
- \left( \frac{m}{3} (r - j + 4) - \frac{4(r - j)}{3} - i + \frac{5}{3} \right), & r = 0 \mod 3 \\
- \left( \frac{m}{3} (r - j + 4) - \frac{4(r - j)}{3} - i + \frac{2}{3} \right), & r = 1 \mod 3 \\
- \left( \frac{m}{3} (r - j + 4) - \frac{4(r - j)}{3} - i - \frac{2}{3} \right), & r = 2 \mod 3 
\end{cases} \]

\[ f(y, y_{i, i+1}) = \begin{cases} 
- (m - i), & r = 0 \mod 3 \\
- (m - i), & r = 1 \mod 3 \\
- (m - i + 1), & r = 2 \mod 3 
\end{cases} \]

\[ f(z, i) = i, \quad i = 1, 2, 3 \]
\[ f(z_3x_{i,1}) = t - (m - 4) \]

\[ f(x_{i,j}x_{i,j+1}) = \begin{cases} 
  t - \left(\frac{m}{3} (r + j + 1) + \frac{2(r - j)}{3} - \frac{2}{3}\right), & r = 0 \mod 3 \\
  t - \left(\frac{m}{3} (r + j + 1) + \frac{2(r - j)}{3} - \frac{3}{3}\right), & r = 1 \mod 3 \\
  t - \left(\frac{m}{3} (r + j + 1) + \frac{2(r - j)}{3} - \frac{1}{3}\right), & r = 2 \mod 3 
\end{cases} \]

\[ f(z_1x_{i,r}) = \begin{cases} 
  t - \left(\frac{m + 2(r-3)}{3} - i\right), & r = 0 \mod 3 \\
  t - \left(\frac{m + 2(r-4)}{3} - i\right), & r = 1 \mod 3 \\
  t - \left(\frac{m + 2(r-2)}{3} - i\right), & r = 2 \mod 3 
\end{cases} \]

\[ f(z_1y_{i,1}) = \begin{cases} 
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{5}{3}\right), & r = 0 \mod 3 \\
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{2}{3}\right), & r = 1 \mod 3 \\
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{5}{3}\right), & r = 2 \mod 3 
\end{cases} \]

\[ f(y_{i,j}y_{i,j+1}) = \begin{cases} 
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{5}{3}\right), & r = 0 \mod 3 \\
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{2}{3}\right), & r = 1 \mod 3 \\
  t - \left(\frac{m}{3} (r - j + 1) + \frac{2(j - r)}{3} + \frac{5}{3}\right), & r = 2 \mod 3 
\end{cases} \]

\[ f(z_2y_{i,r}) = t \]

Based on the labelling function \( f \), then we obtained that the weight of all vertices of a graph \( sp(m,r,2) \) for \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,r - 1 \) as follows.

**Case I For \( m = 0 \ (\mod 3) \)**

\[ wt(x_{i,r}) = f(x_{i,r}) + f(z_1x_{i,r}) + f(x_{i,r-1}x_{i,r}) = 2t - \frac{4mr}{3} + i \]

\[ wt(x_{i,j}) = f(x_{i,j}) + f(x_{i,j-1}x_{i,j}) + f(x_{i,j}x_{i,j+1}) = 3t - m(r + j) + i - 1 \]

\[ wt(x_{i,1}) = f(x_{i,1}) + f(z_3x_{i,1}) + f(x_{i,1}x_{i,2}) = 3t - mr - m + i - 1 \]

\[ wt(y_{i,1}) = f(y_{i,1}) + f(z_1y_{i,1}) + f(y_{i,1}y_{i,2}) = 3t - mr + i - 1 \]

\[ wt(y_{i,j}) = f(y_{i,j}) + f(y_{i,j-1}y_{i,j}) + f(y_{i,j}y_{i,j+1}) = 3t - m(r - j + 1) + i - 1 \]

\[ wt(y_{i,r}) = f(y_{i,r}) + f(z_2y_{i,r}) + f(y_{i,r-1}y_{i,r}) = 3t - m + i - 1 \]

\[ wt(z_3) = \sum_{i=1}^{m} f(z_3x_{i,1}) + f(z_3) = m(t - (m - 2)) + 3 \]

\[ wt(z_1) = \sum_{i=1}^{m} f(z_1x_{i,r}) + f(z_1) + \sum_{i=1}^{m} f(z_1y_{i,1}) = m + 1 + \sum_{i=1}^{m} t - (m - i) \]

\[ wt(z_2) = \left(\sum_{i=1}^{m} f(z_2y_{i,r})\right) + f(z_2) = mt + 2 \]

**Case II For \( m = 1 \ (\mod 3) \).** There are three subcases as follows.  

**Subcase I.** For \( r = 0 \ (\mod 3) \)

\[ wt(x_{i,r}) = f(x_{i,r}) + f(z_1x_{i,r}) + f(x_{i,r-1}x_{i,r}) = 2t - \frac{4mr}{3} + i \]

\[ wt(x_{i,j}) = f(x_{i,j}) + f(x_{i,j-1}x_{i,j}) + f(x_{i,j}x_{i,j+1}) = 3t - m(r + j) + i - 1 \]

\[ wt(x_{i,1}) = f(x_{i,1}) + f(z_3x_{i,1}) + f(x_{i,1}x_{i,2}) = 3t - mr - m + i - 1 \]
\[ wt(y_{i,1}) = f(y_{i,1}) + f(z_1 y_{i,1}) + f(y_{i,1} y_{i,2}) = 3t - mr + i - 1 \]
\[ wt(y_{i,j}) = f(y_{i,j}) + f(y_{i,j-1} y_{i,j}) + f(y_{i,j} y_{i,j+1}) = 3t - m(r - j + 1) + i - 1 \]
\[ wt(y_{i,r}) = f(y_{i,r}) + f(z_2 y_{i,r}) + f(y_{i,r-1} y_{i,r}) = 3t - m + i - 1 \]
\[ wt(z_3) = \sum_{i=1}^{m} f(z_3 x_{i,1}) + f(z_3) = m(t - (m - 3)) + 3 \]
\[ wt(z_1) = \sum_{i=1}^{m} f(z_1 x_{i,r}) + f(z_1) + \sum_{i=1}^{m} f(z_1 y_{i,1}) = m + 1 + \sum_{i=1}^{m} t - (m + \frac{r-3}{3} - i) \]
\[ wt(z_2) = (\sum_{i=1}^{m} f(z_2 y_{i,r})) + f(z_2) = mt + 2 \]

Subcase 2. For \( r = 1 \) (mod 3)
\[ wt(x_{i,r}) = f(x_{i,r}) + f(z_1 x_{i,r}) + f(x_{i,r-1} x_{i,r}) = 2t - \frac{4mr}{3} + i - \frac{2}{3} \]
\[ wt(x_{i,j}) = f(x_{i,j}) + f(x_{i,j-1} x_{i,j}) + f(x_{i,j} x_{i,j+1}) = 3t - m(r + j) + i - 2 \]
\[ wt(x_{i,1}) = f(x_{i,1}) + f(z_3 x_{i,1}) + f(x_{i,1} x_{i,2}) = 3t - mr - m + i - 2 \]
\[ wt(y_{i,1}) = f(y_{i,1}) + f(z_1 y_{i,1}) + f(y_{i,1} y_{i,2}) = 3t - mr + i - 2 \]
\[ wt(y_{i,j}) = f(y_{i,j}) + f(y_{i,j-1} y_{i,j}) + f(y_{i,j} y_{i,j+1}) = 3t - m(r - j + 1) + i - 2 \]
\[ wt(y_{i,r}) = f(y_{i,r}) + f(z_2 y_{i,r}) + f(y_{i,r-1} y_{i,r}) = 3t - m + i - 2 \]
\[ wt(z_3) = \sum_{i=1}^{m} f(z_3 x_{i,1}) + f(z_3) = m(t - (m - 3)) + 3 \]
\[ wt(z_1) = \sum_{i=1}^{m} f(z_1 x_{i,r}) + f(z_1) + \sum_{i=1}^{m} f(z_1 y_{i,1}) = m + 1 + \sum_{i=1}^{m} t - (m + \frac{r-5}{3} - i) \]
\[ wt(z_2) = (\sum_{i=1}^{m} f(z_2 y_{i,r})) + f(z_2) = mt + 2 \]

Subcase 3. For \( r = 2 \) (mod 3)
\[ wt(x_{i,r}) = f(x_{i,r}) + f(z_1 x_{i,r}) + f(x_{i,r-1} x_{i,r}) = 2t - \frac{4mr}{3} + i + \frac{2}{3} \]
\[ wt(x_{i,j}) = f(x_{i,j}) + f(x_{i,j-1} x_{i,j}) + f(x_{i,j} x_{i,j+1}) = 3t - m(r + j) + i \]
\[ wt(x_{i,1}) = f(x_{i,1}) + f(z_3 x_{i,1}) + f(x_{i,1} x_{i,2}) = 3t - mr - m + i \]
\[ wt(y_{i,1}) = f(y_{i,1}) + f(z_1 y_{i,1}) + f(y_{i,1} y_{i,2}) = 3t - mr + i \]
\[ wt(y_{i,j}) = f(y_{i,j}) + f(y_{i,j-1} y_{i,j}) + f(y_{i,j} y_{i,j+1}) = 3t - m(r - j + 1) + i \]
\[ wt(y_{i,r}) = f(y_{i,r}) + f(z_2 y_{i,r}) + f(y_{i,r-1} y_{i,r}) = 3t - m + i \]
\[ wt(z_3) = \sum_{i=1}^{m} f(z_3 x_{i,1}) + f(z_3) = m(t - (m - 3)) + 3 \]
\[ wt(z_1) = \sum_{i=1}^{m} f(z_1 x_{i,r}) + f(z_1) + \sum_{i=1}^{m} f(z_1 y_{i,1}) = m + 1 + \sum_{i=1}^{m} t - (m + \frac{r-5}{3} - i) \]
\[ wt(z_2) = (\sum_{i=1}^{m} f(z_2 y_{i,r})) + f(z_2) = mt + 2 \]

Case III For \( m = 2 \) (mod 3). In this case, there are three cases as follows.
Subcase 1. For \( r = 0 \) (mod 3)
\[ wt(x_{i,r}) = f(x_{i,r}) + f(z_1 x_{i,r}) + f(x_{i,r-1} x_{i,r}) = 2t - \frac{4mr}{3} + i \]
\[ wt(x_{i,j}) = f(x_{i,j}) + f(x_{i,j-1} x_{i,j}) + f(x_{i,j} x_{i,j+1}) = 3t - m(r + j) + i - 1 \]
So it can be concluded that the weight of each vertex on \( \text{sp}(m, r, 2) \) is different. Thus \( f \) is a total irregular \( t \)-labelling with \( t = \left\lceil \frac{2mr + 2}{3} \right\rceil \) for \( m, r \geq 3 \). That is, \( \text{tvs}(\text{sp}(m, r, 2)) \leq \left\lceil \frac{2mr + 2}{3} \right\rceil \).
5. Conclusion
There are two theorems in this paper as conclusions, i.e. the irregularity strength of the parallel networks is $mr + 1$ for the natural numbers $m, r \geq 3$, and the total vertex irregularity strength of the parallel networks is $\left\lfloor \frac{2mr+2}{3} \right\rfloor$, for the natural numbers $m, r \geq 3$, and the total vertex irregularity strength of the parallel networks, $sp(m, r, 2)$ for the natural number $m, r \geq 3$, is $mr + 1$ for $m \geq 4, r \geq 3$, respectively.

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