RAPID DECAY AND THE METRIC APPROXIMATION PROPERTY

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A $C^*$-algebra $A$ is said to have the Metric Approximation Property (MAP) if the identity map $\text{id} : A \to A$ can be approximated in the point-norm topology by a net of finite rank contractions. The main purpose of this note is to prove the following theorem.

**Theorem.** Let $\Gamma$ be a discrete group satisfying the rapid decay property with respect to a length function $\ell$ which is conditionally negative. Then the reduced $C^*$-algebra $C^*_r(\Gamma)$ has the metric approximation property.

The central point of our proof is an observation that the proof of the same property for free groups due to Haagerup [2] transfers directly to this more general situation. We also note that under the same hypotheses, the Fourier algebra $A(\Gamma)$ has a bounded approximate identity, which implies that it too has the MAP.

A discrete group $\Gamma$ satisfies property (RD) (Rapid Decay) with respect to a length function $\ell$ on $\Gamma$ if the operator norm of any element of the group ring can be uniformly majorised by a Sobolev norm determined by $\ell$. In detail, this means the following.

The left action of a group $\Gamma$ on itself extends to the convolution action of the group ring $\mathbb{C} \Gamma$ on the Hilbert space $\ell^2(\Gamma)$. This is the left regular representation $\lambda$ of $\Gamma$ which embeds the group ring in the $C^*$-algebra $\mathcal{B}(\ell^2(\Gamma))$ of all bounded linear operators on $\ell^2(\Gamma)$. The reduced $C^*$-algebra $C^*_r(\Gamma)$ is the $C^*$-subalgebra of $\mathcal{B}(\ell^2(\Gamma))$ generated by $\lambda(\mathbb{C} \Gamma)$.

For any positive real number $s$ we define a Sobolev norm associated with the length function $\ell$ by

$$\|f\|_{\ell,s} = \sqrt[2s]{\sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s}}$$

for any $f \in \mathbb{C} \Gamma$.

Following Jolissaint [5] (see also [1]) we say that $\Gamma$ has property (RD) with respect to the length function $\ell$ if and only if it satisfies the following property: There exist a $C > 0$ and $s > 0$ such that for all $f \in \mathbb{C} \Gamma$

$$\|\lambda(f)\| \leq C\|f\|_{\ell,s}.$$
where the norm on the left hand side is the operator norm in $\mathfrak{B}(\ell^2(\Gamma))$.

As we shall see this is the key property that is required to make Haagerup’s method work in a greater generality.

Following Haagerup [2, Def. 1.6] we say that a function $\phi : \Gamma \to \mathbb{C}$ is a multiplier of $C_r^*(\Gamma)$ if and only if there exists a unique bounded operator $M_\phi : C_r^*(\Gamma) \to C_r^*(\Gamma)$ such that

$$M_\phi \lambda(\gamma) = \phi(\gamma) \lambda(\gamma).$$

for all $\gamma \in \Gamma$. This condition can be written equivalently as:

$$M_\phi \lambda(f) = \lambda(\phi \cdot f).$$

An important situation in which such operators arise is given by the following lemma, which is a generalisation of [2, Lemma 1.7].

**Lemma.** Let $\Gamma$ be a discrete group equipped with a length function $\ell$. Assume that $(\Gamma, \ell)$ satisfies the rapid decay inequality for given $C, s > 0$.

Let $\phi$ be any function on $\Gamma$ such that

$$K = \sup_{\gamma \in \Gamma} |\phi(\gamma)|(1 + \ell(\gamma))^s < \infty.$$

Then $\phi$ is a multiplier of $C_r^*(\Gamma)$ and $\|M_\phi\| \leq CK$.

In particular this holds for any element $f \in \mathbb{C}\Gamma$ and for any such element $M_f$ has finite rank.

**Proof.** This proof is essentially identical to the proof of Lemma 1.7 in [2]. For any discrete group $\Gamma$ the characteristic function $\delta_e$ of the identity element $e$ of $\Gamma$ is the identity of the group ring $\mathbb{C}\Gamma$. Since $\delta_e$ is a unit vector in $\ell^2(\Gamma)$ we have that for any $f \in \mathbb{C}\Gamma$, $\|\lambda(f)\| \geq \|\lambda(f)(\delta_e)\|_2 = \|f \ast \delta_e\|_2 = \|f\|_2$.

Then for any $f \in \mathbb{C}\Gamma$, the pointwise product $\phi \cdot f$ is also an element of $\mathbb{C}\Gamma$, so we can apply the rapid decay inequality to get:

$$\|\lambda(\phi \cdot f)\| \leq C \sqrt{\frac{\sum_{\gamma \in \Gamma} |\phi(\gamma)f(\gamma)|^2(1 + \ell(\gamma))^{2s}}{\sum_{\gamma \in \Gamma} \{\phi(\gamma)(1 + \ell(\gamma))^s\} \|f(\gamma)\|^2}}$$

$$= C K \|f\|_2$$

Putting together the two inequalities we have that

$$\|\lambda(\phi \cdot f)\| \leq CK \|f\|_2 \leq CK \|\lambda(f)\|.$$  

This shows that the map from $\mathbb{C}\Gamma$ to $C_r^*(\Gamma)$ which sends $\lambda(f)$ to $\lambda(\phi \cdot f)$ is continuous and so extends to a unique map $M_\phi : C_r^*(\Gamma) \to C_r^*(\Gamma)$ with the property that $M_\phi \lambda(f) = \lambda(\phi \cdot f)$. 

It is also clear that \( \| M_\phi \| \leq CK \). Finally it is clear that if \( \phi \) has finite support then \( M_\phi \) has finite rank.

\[ \square \]

We are now ready to prove the main technical statement of this note.

**Theorem.** Let \( \Gamma \) be a discrete group with a conditionally negative length function \( \ell \), which satisfies the property (RD) for \( C, s > 0 \). Then there exists a net \( \{ \phi_\alpha \} \) of functions on \( \Gamma \) with finite support such that

1. For each \( \alpha \), \( \| M_{\phi_\alpha} \| \leq 1 \);
2. \( \| M_{\phi_\alpha}(x) - x \| \to 0 \) for all \( x \in C^*_r(\Gamma) \).

**Proof.** Since the length function \( \ell \) is conditionally negative, it follows from Schoenberg’s lemma that for any \( r > 0 \) the function \( \phi_r(\gamma) = e^{-r\ell(\gamma)} \) is of positive type. Thus by [2, Lemma 1.1] (see also [4, Lemma 3.2 and 3.5]), for every \( r \) there exists a unique completely positive operator \( M_{\phi_r} : C_r^*(\Gamma) \to C_r^*(\Gamma) \) such that \( M_{\phi_r}(\lambda(\gamma)) = \phi_r(\gamma)\lambda(\gamma) \) for all \( \gamma \in \Gamma \) and \( \| M_{\phi_r} \| = \phi_r(e) = 1 \).

Let us now define a family \( \phi_{r,n} \) of finitely supported functions on \( \Gamma \) by truncating the functions \( \phi_r \) to balls of radius \( n \) with respect to the length function \( \ell \). For every \( \gamma \in \Gamma \) we put:

\[ \phi_{r,n}(\gamma) = \begin{cases} e^{-r\ell(\gamma)}, & \text{if } \ell(\gamma) \leq n \\ 0, & \text{otherwise} \end{cases} \]

Since \( e^{-x}(1+x)^s \to 0 \) for any positive \( s \) and \( x \to \infty \), sup \( \gamma \in \Gamma |\phi_r(\gamma)|(1 + \ell(\gamma))^s < \infty \). If we denote this finite number by \( K \), then clearly sup \( \gamma \in \Gamma |\phi_{r,n}(\gamma)|(1 + \ell(\gamma))^s \leq K \). Thus, for every \( r \) and \( n \), these functions are multipliers of \( C^*_r(\Gamma) \), and the corresponding operators \( M_{\phi_r} \) and \( M_{\phi_{r,n}} \) have norms bounded by \( CK \). Since the functions \( \phi_{r,n} \) have finite support, the corresponding operators \( M_{\phi_{r,n}} \) are of finite rank.

On the other hand, since

\[ (\phi_r - \phi_{r,n})(\gamma) = \begin{cases} 0, & \ell(\gamma) \leq n \\ e^{-r\ell(\gamma)}, & \ell(\gamma) > n \end{cases} \]

we have that

\[ \sup_{\gamma \in \Gamma} |(\phi_r - \phi_{r,n})(\gamma)|(1 + \ell(\gamma))^s \]

\[ = \sup_{\ell(\gamma) > n} |(\phi_r - \phi_{r,n})(\gamma)|(1 + \ell(\gamma))^s \]

\[ \leq K_n < \infty \]

where \( K_n \to 0 \) as \( n \to \infty \). Thus these functions are multipliers of \( C^*_r(\Gamma) \) and the corresponding operators \( M_{\phi_r - \phi_{r,n}} \) are such that \( \| M_{\phi_r - \phi_{r,n}} \| \leq CK_n \to 0 \), as \( n \to \infty \).
Since
\[ \| M_{\phi r} - M_{\phi r,n} \| = \| M_{\phi r} - \phi_{r,n} \| \]
we have \( \| M_{\phi r} - M_{\phi r,n} \| \to 0 \) as \( n \to \infty \). This implies that \( \| M_{\phi r,n} \| \to \| M_{\phi r} \| = \phi_r(e) = 1 \).

To get the correct bound on the norm of these operators we introduce scaled functions:
\[ \rho_{r,n} = \frac{1}{\| M_{\phi r,n} \|} \phi_{r,n}. \]
The algebraic identity satisfied by the multipliers, as stated in (2), guarantees that on \( \lambda(C\Gamma) \) we have the following identity
\[ (3) \quad M_{\rho_{r,n}} = \frac{1}{\| M_{\phi r,n} \|} M_{\phi r,n}. \]

We now want to show that each operator \( M_{\rho_{r,n}} \) is a finite rank contraction on \( C^*_r(\Gamma) \) and that the strong operator closure of the family \( \{ M_{\rho_{r,n}} \} \) contains the identity map \( id : C^*_r(\Gamma) \to C^*_r(\Gamma) \). This means that for every positive \( \epsilon \) there exists an operator \( M_{\rho_{r,n}} \) such that
\[ \| M_{\rho_{r,n}} x - x \| < \epsilon \]
for all \( x \in C^*_r(\Gamma) \).

First, a simple use of the triangle inequality leads to the following argument.
\[ \| M_{\rho_{r,n}} - M_{\phi r} \| \leq \| M_{\rho_{r,n}} - M_{\phi r,n} \| + \| M_{\phi r,n} - M_{\phi r} \| \]
\[ = \| (1 - 1/\| M_{\phi r,n} \|) M_{\phi r,n} \| + \| M_{\phi r,n} - M_{\phi r} \| \]
\[ \to 0 \quad \text{as} \quad n \to \infty \]

Let \( x \in C^*_r(\Gamma) \). Then \( x \) is a limit of a sequence of elements \( x_m \in \lambda(C\Gamma) \) so that \( \| M_{\rho_{r,n}}(x) \| = \lim_{m \to \infty} \| M_{\rho_{r,n}}(x_m) \| \), and equation (3) implies that \( \| M_{\rho_{r,n}}(x_m) \| = \| M_{\phi r,n}^{-1}(M_{\phi r,n}(x_m)) \| \).

This leads to the following estimate:
\[ \| M_{\rho_{r,n}}(x) \| = \lim_{m \to \infty} \frac{1}{\| M_{\phi r,n} \|} \| M_{\phi r,n}(x_m) \| \]
\[ \leq \lim_{m \to \infty} \frac{1}{\| M_{\phi r,n} \|} \| M_{\phi r,n} \| \| x_m \| = \lim_{m \to \infty} \| x_m \| = \| x \|. \]

It follows that \( \| M_{\rho_{r,n}} \| \leq 1 \).

Finally, it is clear that for any \( \gamma \in \Gamma \), \( e^{-r\ell(\gamma)} \to 1 \) as \( r \to 0 \). Thus for any \( x = \sum_{\gamma \in \Gamma} \mu_{\gamma} \lambda(\gamma) \in C\Gamma \) we have
\[ M_{\phi_r}(x) = \sum_{\gamma \in \Gamma} \mu_{\gamma} \phi_r(\gamma) \lambda(\gamma) \]
so that
\[
\lim_{r \to 0} M_{\phi_r}(x) = \lim_{r \to 0} \sum \mu_\gamma \phi_r(\gamma) \lambda(\gamma)
\]
\[= \sum \mu_\gamma (\lim_{r \to 0} \phi_r(\gamma)) \lambda(\gamma) = \sum \mu_\gamma \lambda(\gamma) = x
\]

Since any \(x \in C^*_r(\Gamma)\) can be approximated by a sequence \(x_m \in \lambda(\mathbb{C}\Gamma)\)
we have
\[
\|M_{\phi_r}(x) - x\| \leq \|M_{\phi_r}(x) - M_{\phi_r}(x_m)\| + \|M_{\phi_r}(x_m) - x_m\| + \|x_m - x\|
\]
\[\leq \|x - x_m\| < \epsilon/3 \text{ for all large enough } n \text{ and independently of } r.\]

For sufficiently small \(r > 0\), \(\|M_{\phi_r}(x) - M_{\phi_r}(x_m)\| \leq \epsilon/3\) and so the middle term will be smaller than \(\epsilon/3\) for all sufficiently small \(r > 0\).

As \(r \to 0\), \(\|M_{\phi_r}(x) - x\| \to 0\) for all \(x \in C^*_r(\Gamma)\).

Let \(\epsilon > 0\). Then it follows from (4) that for every \(r > 0\) and all sufficiently large \(n\), \(\|M_{\rho_{r,n}} - M_{\phi_r}\| < \epsilon/2\). Secondly, as we have just shown, for all sufficiently small \(r\), \(\|M_{\phi_r}(x) - x\| < \epsilon/2\). Given that
\[
\|M_{\rho_{r,n}}x - x\| \leq \|M_{\rho_{r,n}}x - M_{\phi_r}x\| + \|M_{\phi_r}(x) - x\|
\]
for every \(x \in C^*_r(\Gamma)\), the norm on the left hand side can be made smaller than \(\epsilon\) by taking a sufficiently large \(n\) and a sufficiently small \(r > 0\).

This means that the strong closure of the family \(\mathcal{M} = \{M_{\rho_{r,n}}\}\) of finite rank contractions contains the identity map on the algebra \(C^*_r(\Gamma)\). This implies that there exists a net of finitely supported functions \(\phi_\alpha\) with corresponding finite rank contractions \(M_{\phi_\alpha}\) \(\in\mathcal{M}\) such that \(\|M_{\phi_\alpha}x - x\| \to 0\). This concludes the proof.

As a corollary we obtain the main result of this note.

**Theorem.** Let \(\Gamma\) be a discrete group satisfying the rapid decay property with respect to a length function \(\ell\) which is conditionally negative. Then the reduced \(C^*\)-algebra \(C^*_r(\Gamma)\) has the metric approximation property.

Now according to Niblo and Reeves [6] given a group acting on a CAT(0) cube complex we obtain a conditionally negative kernel on the group which gives rise to a conditionally negative length function. By
results of Chatterji and Ruane \cite{ChatterjiRuane} the group will have the rapid decay property with respect to this this length function provided that the action is properly discontinuous, stabilisers are uniformly bounded and the cube complex has finite dimension. Hence we obtain:

**Corollary.** Groups acting properly discontinuously on a finite dimensional $\text{CAT}(0)$ cube complex with uniformly bounded stabilisers have the metric approximation property.

This class of examples includes free groups, finitely generated Coxeter groups \cite{Coxeter}, and finitely generated right angled Artin groups for which the Salvetti complex is a $\text{CAT}(0)$ cube complex. A rich class of interesting examples is furnished by Wise, \cite{Wise}, in which it is shown that many small cancellation groups act properly and co-compactly on $\text{CAT}(0)$ cube complexes. The examples include every finitely presented group satisfying the B(4)-T(4) small cancellation condition and all those word-hyperbolic groups satisfying the B(6) condition.

Another class of examples where the main theorem applies is furnished by groups acting co-compactly and properly discontinuously on real or complex hyperbolic space. According to a result of Faraut and Harzallah \cite{FarautHarzallah} the natural metrics on these hyperbolic spaces are conditionally negative and they give rise to conditionally negative length functions on the groups. See \cite{Jolissaint} for a discussion and generalisation of this fact. The fact that these metrics satisfy rapid decay for the group was established by Jolissaint in \cite{Jolissaint}.

Finally we remark that the net $\phi_\alpha$ of Theorem provides an approximate identity for the Fourier algebra $A(\Gamma)$ of the group $\Gamma$ which is bounded in the multiplier norm. The proof of \cite[Theorem 2.1]{Haagerup} carries over verbatim to the present situation. This implies, as in \cite[Corollary 2.2]{Haagerup}, that if a group $\Gamma$ satisfies the (RD) property with respect to a conditionally negative length function then its Fourier algebra $A(\Gamma)$ has the metric approximation property.

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