Robust Adaptive Output Feedback Control for a Class of Underactuated Aerial Vehicles with Input and Output Constraints

Trong-Toan Tran, Tan-No Nguyen, Duc-Duy Nguyen, Viet-Long Nguyen, and Nguyen-Vu Truong

1Graduate University of Science and Technology, Hanoi, Vietnam
2Institute of Applied Mechanics and Informatics, Vietnam Academy of Science and Technology, Hanoi, Vietnam
3Becamex IDC Corp, Thu Dau Mot, Binh Duong, Vietnam

Correspondence should be addressed to Trong-Toan Tran; toan1003@gmail.com

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1.Introduction

In recent years, control of underactuated systems has attracted much attention. Unlike the fully actuated system in which the number of independent control inputs is equal to degrees of freedom (DOFs), the underactuated system has fewer independent control inputs than DOFs [1, 2]. Because of the underactuated property, several undesired properties such as higher relative degree and non-minimum phase behavior are manifested. Therefore, powerful techniques developed for fully actuated systems may not be directly applied. For these reasons, control system design for underactuated systems is not an easy task. In fact, based on recent surveys [3–7], control of general underactuated systems is currently a major open problem.

In the control design for underactuated systems, all DOFs cannot be controlled; instead, a portion of DOFs is controlled to have desired behaviours while remainders are required to be bounded. This requirement leads the system in the question to a non-minimum phase system [8]. Finding the general solution of non-minimum phase systems is still a quite challenging research topic [9]. In some special cases, underactuated systems with collocated partial feedback linearization, i.e., actuated configuration variables of the underactuated system with noninteracting inputs, can be globally linearized by the use of an invertible change of control as shown in [10]. An alternative approach is the reduction of the order of the system; i.e., the underactuated system is separated into fully actuated subsystems by adding virtual controls, and these subsystems are constructed as a cascaded structure for the control design [11]. However, this approach is only applied in some classes of systems because control of cascaded systems is not solvable in general [12, 13].

Constraints appear in most mechanical control systems due to different reasons such as physical restrictions and/or control tasks. There are different kinds of constraints, for example, input constraint, output constraint, and mixed constraints. Optimal control is one of the powerful tools in dealing with constraints [14, 15]. However, the computation
is expensive. With the rapid development of control techniques, model predictive control (MPC) has effectiveness to handle constraints [16]. The computation is cheaper in comparison with optimal control because of the receding horizon control. In particular, when constraints have the form of saturations, a combination of the saturation function, barrier function, and direct Lyapunov method is a good choice to seek an analytic solution. The excellent work dealing with bounded control by using the nested saturation function was proposed in [17]. Based on this work, several bounded controllers have been developed [18, 19]. Concerning with output constraints, different approaches have been proposed [20–22]. Among these approaches, the barrier function, and direct Lyapunov method is a good form of saturations, a combination of the saturation function with input and output constraints. In particular, we focus on the class of underactuated aerial vehicles with a single thrust direction and full torque actuation [2]. In this case, several robust adaptive output feedback control schemata have been proposed to solve the problem [26, 27].

In this paper, we are interested in the input constraints as well as the underactuated systems and constraints are taken into account. In this situation, the control system design becomes much more challenging because it is necessary to simultaneously deal with underactuated property, uncertain dynamics, disturbance rejection, state estimation, and constraint satisfaction.

Motivated by the above consideration, in this paper, we are interested in designing the robust adaptive output feedback control for a class of underactuated aerial vehicles with input and output constraints. In particular, we focus on the class of underactuated aerial vehicles with a single thrust direction and full torque actuation. To solve the problem, the modular design strategy is proposed for the control system design. Namely, by constructing the underactuated aerial vehicle as a cascaded structure consisting of fully actuated subsystems, the robust adaptive output feedback control can be separately designed for each subsystem. In this setting, the observers are employed to reconstruct system states for the feedback and the radial basis function NNs (RBF NNs) are used to approximate the system uncertainties. To deal with the input and output constraints, a combination of saturation and barrier functions is integrated into the Lyapunov control design. Our design methodology yields the ultimate boundedness of all the states in the closed-loop system, while the tracking error converges to the small neighbourhood of the origin which can be made arbitrarily small.

The main contributions in this paper include the following aspects: (i) The modular design strategy is proposed to design the robust adaptive output feedback control for the class of underactuated aerial vehicles with taking into account multiple constraints. (ii) The assumption on limitations of orientation angles in the existing studies is removed by dealing with the constraints on orientation angles. It follows that the problem of the gimbal lock is solved, and the vehicle will be never overruled. (iii) The proposed methodology yields the new result for the control cascaded system design. Namely, we show that an ISS subsystem driven by a subsystem enjoying UB property results in the UB property of the overall cascaded subsystem.

The rest of the paper is organized as follows. In Section 2, the control problem is formulated. Section 3 presents the control design while the stability of the overall system is analysed in Section 4. In Section 5, the simulations are provided to illustrate the effectiveness of the proposed control. The conclusion of this paper is presented in Section 6.

1.1. Notation. Throughout this paper, \( \mathbb{R}^n \) denotes the Euclidean space with \( n \)-dimension. \( \| \cdot \| \) is the Euclidean norm of vector \( \cdot \). \( \stackrel{\rightarrow}{\cdot} \rightarrow 0 \) denotes that \( \stackrel{\rightarrow}{\cdot} \) converges to 0. For integer indices \( i \) and \( j \), the symbol \( \sigma_i \in \mathbb{R}^n \) denotes the 1, \( \cdots , n \). \( A \in \mathbb{R}^{m \times n} \) denotes the \( n \times m \)-matrix, and \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote the minimum and maximum eigenvalues of matrix \( A \), respectively. Given two vector \( \tilde{a}, \tilde{b} \in \mathbb{R}^n \), then \( \tilde{a} \leq \tilde{b} \) denotes \( a_i \leq b_i, i = 1, \cdots , n \).

2. Problem Formulation and Preliminaries

2.1. Problem Formulation. The system considered in this paper is the class of underactuated aerial vehicles with a single thrust direction and full torque actuation [2]. In particular, the dynamics of the considered system can be described by the following Euler–Lagrange equations:

\[
\begin{align*}
\dot{q} &= R_3u - mge_3 + d_q, \\
J(\eta)\ddot{\eta} &= -C(\eta, \dot{\eta})\dot{\eta} + \tau + d_\eta,
\end{align*}
\]

where \( q = [x, y, z]^T \in \mathbb{R}^3 \) denotes the position of the vehicle in the reference frame; \( \eta = [\phi, \theta, \psi]^T \in \mathbb{R}^3 \) denotes the orientation angles with respect to the roll, pitch, and yaw; \( u \in \mathbb{R} \) is the total thrust; \( \tau = [\tau_\phi, \tau_\theta, \tau_\psi]^T \in \mathbb{R}^3 \) is the torque; \( m \) is the mass of the rigid body; \( g \) is the acceleration due to gravity; \( e_3 = [0, 0, 1]^T \) is the unit vector; \( J(\eta) \in \mathbb{R}^{3 \times 3} \) is the effective inertial matrix; \( C(\eta, \dot{\eta}) \in \mathbb{R}^{3 \times 3} \) is the Coriolis and centrifugal force matrix; \( R \in SO(3) \) is the rotation matrix having the form

\[
R = \begin{bmatrix}
C_\psi C_\theta & C_\psi S_\theta S_\psi - S_\psi C_\phi & S_\psi S_\phi S_\psi + S_\phi C_\psi \\
S_\psi C_\theta & S_\psi S_\theta S_\psi + C_\psi C_\phi & -S_\phi S_\psi \\
-S_\theta & C_\theta S_\phi & C_\phi C_\theta 
\end{bmatrix};
\]

and \( d_q \) and \( d_\eta \in \mathbb{R}^3 \) are unknown disturbances with the known bounds

\[
\|d_q\| \leq d_q^*; \\
\|d_\eta\| \leq d_\eta^*,
\]

where \( d_q^* \) and \( d_\eta^* \) are the known positive constants.

In this paper, we are interested in the input constraints as follows:
0 < u ≤ π, \quad \text{(5)}
\|τ\| ≤ τ, \quad \text{(6)}

and we are also interested in the following output constraints:

\|φ\| ≤ π/2; \quad |θ| < π/2; \quad |ψ| ≤ π,
\text{(7)}

with \(ω \in \mathbb{R}\) and \(τ \in \mathbb{R}^3\) being the positive constants.

**Remark 1.** Constraints (5) and (6) ensure the final values of the control inputs. Constraint (7) ensures that the vehicle will be never overturned [28–30].

We notice that the considered system has four control inputs \(\{u, τ^T\} \in \mathbb{R}^3\) and six DOFs \(\{q^T, \psi^T\} \in \mathbb{R}^6\). Accordingly, it is impossible for such systems to track an arbitrary reference trajectory. Instead, we select four outputs \(\{q^T, \psi^T\}\) for the tracking purpose as usual in [2, 29]. Therefore, the control problem considered in this paper can be stated as follows.

**Problem 1.** Consider the class of underactuated vehicles described by (1) and (2). Let \(\{q_d(t)^T, ψ_d(t)^T\} \in \mathbb{R}^4\) be a given smooth time-varying desired trajectory with its time derivatives bounded up to the second order. Under conditions that the velocities \(q\) and \(ψ\) are not available for the feedback control, the terms \(J(η), C(η, η), \) and \(m\) are uncertain and the disturbances \(d_q\) and \(d_ψ\) are bounded; design a robust adaptive output feedback control \(\hat{U} = [u, τ^T]^T\) for system (1) and (2) such that the following criteria are met:

(i) Constraints (5)–(7) are satisfied

(ii) All the signals in the closed-loop system are bounded

(iii) The output \(\{q^T, ψ^T\}\) follow an adjustable neighborhood of its desired trajectory \(\{q_d^T, ψ_d^T\}\).

As the problem is formulated, the velocities \(q\) and \(ψ\) are unavailable for the feedback while the uncertain terms \(J(η), C(η, η), \) and \(m\), constraints (5)–(7), and disturbances (4) are taken into account. Therefore, direct design of the controls \(u\) and \(τ\) is not an easy task. To overcome this difficulty, the control strategy based on the modular design is proposed as follows. First, we decouple and construct systems (1) and (2) as the cascaded structure in which position subsystem (1) is driven by orientation subsystem (2). Second, the robust output feedback control is designed for position subsystem (1) such that subsystem (1) is ISS while constraint (5) is satisfied. Third, the robust adaptive output feedback control is designed to achieve the stability of orientation subsystem (2) and simultaneously tackle constraints (6) and (7).

**Remark 2.** We notice that by dealing with constraint (7), the following two issues are ensured. First, the assumption on limitations of orientation angles \(\{|φ|, |θ| < π/2\}\) in the existing studies [1, 31] is rejected, and the so-called problem gimbal lock is solved. Second, the vehicle will be never overturned.

2.2. **Preliminaries.** This section briefly recalls the notions of RBF NN, high-gain observer, and saturation function which are used in the control design in the next section.

2.2.1. **RBF Neural Networks.** The RBF neural networks consist of two layers, in which the hidden layer completes a fixed nonlinear transformation. Then, the output layer linearly combines the outputs of the hidden layer. Therefore, the networks can be simply represented by [32]

\[
f_{m}(X) = W^T G(X),
\]

with the input vector \(X = [x_1, x_2, \ldots, x_m]^T \in \mathbb{R}_x \subset \mathbb{R}^m\), weight vector \(W \in \mathbb{R}^\ell\), node number \(\ell \geq 1\), and basis function \(G(X) = [g_1(X), g_2(X), \ldots, g_\ell(X)]^T \in \mathbb{R}^\ell\). Here, \(g_j(X)\) is the Gaussian function as follows:

\[
g_j(X) = \exp \left[-\frac{\|X - k_j\|^2}{h_j^2}\right], \quad j = 1, 2, \ldots, \ell,
\]

where \(k_j = [k_{j1}, k_{j2}, \ldots, k_{j\ell}]^T\) and \(h_j\) are the center and the width, respectively.

It is shown that by universe approximation results that any continuous function \(f(X)\) defined on a compact set \(\Omega_X \subset \mathbb{R}^m\) can be approximated by \(W^T G(X)\) if \(\ell\) is chosen sufficiently large. This is described as

\[
f(X) = W^*^T G(X) + \epsilon(X), \quad \forall X \in \Omega_X \subset \mathbb{R}^m,
\]

where \(\epsilon(X)\) is the function approximation error and \(W^*\) is the ideal weigh vector. The following assumption for the function approximation is made.

Assumption 1. (see [32]) Over the compact set \(\Omega_X \subset \mathbb{R}^m\), the neural network ideal weight vector \(W^*\), and the function approximation error \(\epsilon(X)\) of neural networks is bounded by

\[
\|W^*\| \leq W_m,
\]
\[
|\epsilon(X)| \leq \epsilon_m, \quad \forall X \in \Omega_X,
\]

where \(W_m\) and \(\epsilon_m\) are the positive constants.

**Lemma 1.** (see [32]) For Gaussian RBF (9), if \(\tilde{X} = X - β\theta\) with constant \(β > 0\) and \(θ\) being a bounded vector, then

\[
G(\tilde{X}) = G(X) - βG_θ,
\]

where \(G_θ\) is a bounded function vector.

2.2.2. **High-Gain Observer.** Since only the positions and orientation angles are measurable and the linear and angular velocities are not available, we need to estimate the velocities \(q\) and \(ψ\) for the output feedback control. The following high-gain observer is used later.
Lemma 2. (see [33]) Assume that the output $y(t)$ of a system and its first $n-1$ derivatives are bounded, that is, $|y^{(k)}|<Y_k$, $k=1,\cdots,n-1$, with positive constants $Y_k$. Consider the following linear system:

$$\begin{align*}
\dot{\xi}_i &= \xi_{i+1}, \quad i = 1,\cdots,n-1, \\
\dot{\xi}_n &= \nu_1\xi_n - \nu_2\xi_{n-1} - \cdots - \nu_{n-1}\xi_2 - \xi_1 + y(t),
\end{align*}$$

with $\epsilon$ being any small positive constant, where $\xi_i, i = 1,\cdots,n$, are the state variables of the observer. Choosing the parameters $\nu_1,\cdots,\nu_n$ such that the polynomial $s^n + \nu_1 s^{n-1} + \cdots + \nu_{n-1} s + 1$ is Hurwitz, then we have the following properties:

(i) \( (\xi_{k+1})/\epsilon^k) - y(k) = -\epsilon\hat{\theta}(k+1), k = 0,1,\cdots,n-1, \) where $\hat{\theta} = \xi_n + \nu_1\xi_{n-1} + \cdots + \nu_{n-1}\xi_1$ with $\hat{\theta}$ denoting the $k$ derivatives of $\theta$

(ii) There exist positive constants $t^*$ and $b_k$ only depending on $Y_{k-1}$ ($k=1,2,\cdots,n$),$\epsilon$, and $\nu_1, (i = 1,\cdots,n-1)$, such that we have $|\hat{\theta}(k)| \leq b_k$ for all $t > t^*$.

2.2.3. Saturation Function. To deal with input constraint (5), the following saturation function is used.

Definition 1 (see [17]). Given a positive constant $\sigma$, a function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ is said to be a linear saturation with the bound $\sigma$ if it is a continuous, nondecreasing function satisfying (i) $\sigma(0) = 0, \forall \epsilon \in \mathbb{R}$; (ii) $\sigma(c) = \frac{c}{\epsilon}, \forall |c| \leq \epsilon$; (iii) $\sigma(c) \leq \sigma, \forall \epsilon \in \mathbb{R}$; and (iv) $\partial \sigma/\partial c \leq 1, \forall \epsilon \in \mathbb{R}$.

3. Control Design

This section presents the control design for systems (1) and (2) according to the proposed control strategy in the previous section.

3.1. Position Control by Bounded Robust Output Feedback.

The goal of this section is to decouple subsystem (1) from orientation subsystem (2) and design the robust output feedback control for (1) such that the position $q$ tracks its desired trajectory $q_d$, constraint (5) is satisfied, and subsystem (1) is ISS. To this end, we assume that the uncertain mass $m$ in (1) can be written as

$$m = \bar{m} + \Delta_m,$$

where $\bar{m}$ is the known nominal term and $\Delta_m$ is an uncertain term with the known bound as

$$0 \leq |\Delta_m| \leq \Delta_m^* < \bar{m}.$$  \hspace{1cm} (14)

Denoting the states as

$$\begin{align*}
q_1 &= q, \\
q_2 &= \dot{q},
\end{align*}$$

and using the relationship $1/m = 1/\bar{m} - \Delta_m/m\bar{m}$ from (14), system (1) can be rewritten as

$$\begin{align*}
\dot{q}_1 &= q_2, \\
\dot{q}_2 &= \frac{u}{m} Re_3 - g e_3 + d_p,
\end{align*}$$

where $d_p = d_q - uRe_3/m\bar{m}$ is bounded by

$$\|d_p\| \leq d^*_p,$$  \hspace{1cm} (18)

with the known bound

$$d^*_p = \frac{d^* + \Delta^*_m}{(m - \Delta^*_m)m}.$$  \hspace{1cm} (19)

Remark 3. The upper bound $d^*_p$ is determined by (19) only when the control $u$ is bounded by $p$, i.e., constraint (5) must be satisfied. The satisfaction of this constraint will be guaranteed in the control design later.

3.1.1. Position Observer Design. Because the velocity $q$ is not available, we will estimate $\dot{q}$ for the output feedback control by using the following observer:

$$\begin{align*}
\dot{q}_1 &= \ddot{q}_2 + L_1(q_1 - \ddot{q}_1), \\
\dot{q}_2 &= \frac{u}{m} Re_3 - g e_3 + L_2(q_1 - \ddot{q}_1),
\end{align*}$$

where $\ddot{q}_1$ and $\ddot{q}_2$ are the estimations of $q_1$ and $q_2$, respectively, and $L_1$ and $L_2 \in \mathbb{R}^{3\times 3}$ are the positive constant observer gain matrices. By defining the observer error

$$\ddot{q} = [\ddot{q}_1^T, \ddot{q}_2^T]^T = [(q_1 - \ddot{q}_1)^T, (q_2 - \ddot{q}_2)^T]^T$$

and subtracting (20) from (17), the dynamic equation of the observer error is given by

$$\ddot{q} = L\ddot{q} + d^p_q,$$

where

$$L = \begin{bmatrix}
-L_1 & I_{3\times 3} \\
-L_2 & 0
\end{bmatrix};$$

$$d^p_q = \begin{bmatrix}
0_{3\times 1} \\
d_p
\end{bmatrix}.$$  \hspace{1cm} (23)

Lemma 3. For system (22), if $L_1$ and $L_2$ are chosen such that $L$ is Hurwitz, then there exists a time instant $T_o$ such that the observer error satisfies

$$\begin{align*}
\|\ddot{q}(t)\| &\leq k_o\|\ddot{q}(0)\|e^{-\rho_o t}, \quad 0 < t < T_o, \\
\|\ddot{q}(t)\| &\leq \delta_o, \quad \forall t \geq T_o,
\end{align*}$$

for some positive constants $k_o, \rho_o$ and $\delta_o$ where $\delta_o$ can be made small by adjusting $L_1$ and $L_2$.

Proof. See Appendix A. \hfill \Box
Remark 4. It is shown from (24) that the observer error $\tilde{q}$ exponentially decays with the convergent rate $\rho_o$. Moreover, the observer error is uniformly ultimately bounded with the ultimate bound \( \delta_o \) which can be adjusted by the observer design matrices \( L_1 \) and \( L_2 \).

### 3.1.2. Bounded Robust Output Feedback Control Design.

Having position observer (20), we proceed to decouple (1) from (2) and design the control \( u \) for position subsystem (1) to solve the tracking problem. To this end, defining the tracking errors

\[
\begin{align*}
\bar{p}_1 &= \bar{q}_1 - \bar{q}_d; \\
\bar{p}_2 &= \bar{q}_2 - \bar{q}_d,
\end{align*}
\]

and using the relationship \( \bar{q}_2 = \bar{q}_2 + \bar{q}_d \), the position tracking error system is obtained as

\[
\begin{align*}
\dot{\bar{p}}_1 &= \frac{\bar{u}}{m} R e_3 - g e_3 - \bar{q}_d + L_2 \bar{q}_1, \\
\dot{\bar{q}}_2 &= L\bar{q} + \bar{d}_\bar{q}.
\end{align*}
\]

We now decouple (26) from (2) by adding and subtracting the right-hand side of \( \bar{p}_2 \)-equation in (26) by the virtual control \( F \in \mathbb{R}^3 \) of the following form:

\[
F = \frac{\bar{u}}{m} R e_3,
\]

where \( R_d \) has the same form with \( R \) except that the angles \( \phi, \theta, \) and \( \psi \) are replaced with their desired \( \phi_d, \theta_d, \) and \( \psi_d \), respectively. Accordingly, \( \bar{p}_2 \)-equation in (26) can be further rewritten as

\[
\begin{align*}
\dot{\bar{p}}_2 &= F + \bar{\omega} - g e_3 - \bar{q}_d + L_2 \bar{q}_1, \\
\bar{\omega} &= \frac{\bar{u}}{m} (R - R_d) e_3.
\end{align*}
\]

Having (28), we are able to freely design the control \( F \) for (26). Furthermore, solving (27) with respect to \( u \) \( [29] \), we obtain the real control \( u \) and the desired orientation angles as

\[
\begin{align*}
u &= \bar{m} \sqrt{F_1^2 + F_2^2 + F_3^2}, \\
\phi_d &= \arctan \left( \frac{S_{\psi} F_1 - C_{\psi} F_2}{\sqrt{(C_{\psi} F_1 + S_{\psi} F_2)^2 + F_3^2}} \right), \\
\theta_d &= \arctan \left( \frac{C_{\psi} F_1 + S_{\psi} F_2}{F_3} \right),
\end{align*}
\]

where we use the denotations \( S_{\psi} = \sin \psi_d \) and \( C_{\psi} = \cos \psi_d \).

In view of (30), \( u \) satisfies constraint (5) if \( F \) is bounded with the known bound. For this reason, the control \( F \) is designed as

\[
F = -\sigma_1 (p_1 + \bar{p}_2) - \sigma_2 (\bar{p}_2) + \bar{q}_d + g e_3,
\]

where \( \sigma_1 \) and \( \sigma_2 \in \mathbb{R}^3 \) are the saturation function vectors in Definition 1.

**Assumption 2.** The desired trajectory \( q_d \) is bounded by

\[
|\dot{q}_d| \leq \rho,
\]

where \( \rho = [\rho_x, \rho_y, \rho_z]^T \in \mathbb{R}^3 \) is a given positive vector.

Under Assumption 2, constraint (5) is ensured if the control \( F \) satisfies

\[
\begin{align*}
&(\sigma_{11} + \sigma_{21} + \rho_x)^2 + (\sigma_{12} + \sigma_{22} + \rho_y)^2 + (\sigma_{13} + \sigma_{23} + \rho_z + g)^2 \\ &\leq \frac{\sigma^2}{m^2},
\end{align*}
\]

where \( \sigma_{ij} \) is the bound of \( \sigma_{ij} \) for \( i = 1, 2 \) and \( j = 1, 2, 3 \).

From (28), (26), and (31) we obtain the closed loop position tracking error system as

\[
\begin{align*}
\dot{\bar{p}}_1 &= \hat{\bar{p}}_2 + \bar{q}_d, \\
\dot{\bar{p}}_2 &= -\sigma_1 (p_1 + \bar{p}_2) - \sigma_2 (\bar{p}_2) + L_2 \bar{q}_1 + \sigma, \\
\dot{\bar{q}} &= L\bar{q} + \bar{d}_\bar{q}.
\end{align*}
\]

**Theorem 1.** Consider system (34) under Assumption 2 and Lemma 3. If condition (33) is satisfied, then for any initial condition \( p(0) \), the following statements hold:

(i) Constraint (5) is ensured.

(ii) If \( (\bar{\omega}, \bar{d}_\bar{q}) \) is bounded by

\[
\begin{align*}
\|\bar{\omega}\| &\leq \sigma^\ast, \\
\|\bar{d}_\bar{q}\| &\leq \bar{d}^\ast_\bar{q},
\end{align*}
\]

where \( \bar{d}^\ast_\bar{q} \) is defined in (19) and \( \sigma^\ast \) is a positive constant satisfying

\[
0 < 8\sigma^\ast \leq \bar{\sigma}_1, \\
0 < 8\sigma^\ast \leq \bar{\sigma}_2,
\]

then all the system states are ultimately bounded, i.e., there exist positive definite functions \( \bar{V}_1 (p_1, \bar{p}_2, \bar{q}) \) and \( \bar{V}_2 (p_1, \bar{p}_2, \bar{q}) \), a positive number \( C_1 \), and a time instant \( T_1 > 0 \) such that

\[
\begin{align*}
\dot{\bar{V}}_1 &< -\bar{W}_1 + C_1, \quad \forall t > T_1.
\end{align*}
\]

(iii) System (34) is ISS with the state \( (p_1, \bar{p}_2, \bar{q}) \) and the input \( (\bar{\omega}, \bar{d}_\bar{q}) \) under restriction (35).

**Proof.** See Appendix B.  

Remark 5. It is shown from (34) that by using observer (20) and \( \bar{p}_2 \)-equation for the control design, the disturbance \( d_p \) is moved from \( \bar{p}_2 \)-equation to \( \bar{q} \)-equation. This allows us to
deal with the effect of $d_p$ on the tracking performance by adjusting the observer gain $L$ as shown in Lemma 3.

Remark 6. By introducing the virtual control $F$ for (26), on the one hand, we decouple (26) from (2). On the other hand, we deal with the underactuated property of the considered system. Indeed, in view of (26), the one-dimensional control $u \in \mathbb{R}^1$ cannot be used to control the three-dimensional output $\tilde{p}_2 \in \mathbb{R}^3$.

In view of (34), the tracking errors ($p_1, \tilde{p}_2$) are driven by $\bar{q}$ and $\bar{\omega}$. The observer error $\bar{q}$ can be made small by adjusting the observer gain matrix $L$ as shown in Lemma 3. The remainder is how to make $\bar{\omega}$ small. In view of (29), $R \rightarrow R_d$ implies $\bar{\omega} \rightarrow 0$, or we need to have $\eta \rightarrow \eta_d$, where $\eta_d$ is the desired orientation angles. This goal will be achieved in the next section.

3.2. Orientation Control by Bounded Robust Adaptive Output Feedback Control. The goal of this section is the design of the control $\tau$ for orientation subsystem (2) to solve the orientation tracking problem, i.e., $\eta$ tracks its desired $\eta_d$ and constraints (6) and (7) are simultaneously tackled. To this end, we denote the states as

$$
\begin{align*}
\eta_1 &= \eta, \\
\eta_2 &= \dot{\eta},
\end{align*}
$$

and rewrite system (2) in the form

$$
\dot{\eta}_1 = \eta_2,
$$

$$
J(\eta_1)\dot{\eta}_2 = -C(\eta_1, \eta_2)\eta_2 + \tau + d_r.
$$

3.2.1. High-Gain Observer for Orientation Subsystems. Because the angular velocity $\eta_1$ in (39) is not available, we will estimate $\eta_1$ by using the following high-gain observer:

$$
\begin{align*}
\dot{\bar{\eta}}_1 &= X_2, \\
\dot{\bar{\eta}}_2 &= -\nu_1 X_2 - X_1 + \eta_1(t),
\end{align*}
$$

where $X_1$ and $X_2 \in \mathbb{R}^3$ are the state vectors of observer (40) and $\nu_1$ is chosen such that $s^2 + \nu_1 s + 1$ is Hurwitz. Then, the estimations $\bar{\eta}_1$ and $\bar{\eta}_2$ of $\eta_1$ and $\eta_2$, respectively, are reconstructed as

$$
\begin{align*}
\bar{\eta}_1 &= \dot{X}_1, \\
\bar{\eta}_2 &= \frac{1}{\epsilon} \dot{X}_2.
\end{align*}
$$

Lemma 4 (see [33]). For orientation subsystem (2), high-gain observer (40) provides the estimations $(\bar{\eta}_1, \bar{\eta}_2)$ of $(\eta_1, \eta_2)$, and the estimation errors are bounded by

$$
\begin{align*}
\|\eta_1 - \bar{\eta}_1\| &\leq \epsilon \theta_1, \\
\|\eta_2 - \bar{\eta}_2\| &\leq \epsilon \theta_2,
\end{align*}
$$

for some positive constants $\theta_1$ and $\theta_2$.

3.2.2. Bounded Robust Adaptive Output Feedback Control Design. Having high-gain observer (40), we proceed to design the control $\tau$ for orientation subsystem (39). In the control design, constraints (6) and (7) are simultaneously dealt with by the use of the saturation and barrier functions. The RBF NNs are used to approximate the unknown terms $J(\eta_1)$ and $C(\eta_1, \eta_2)$ in (39). The control design is based on the backstepping method. For more detail, the design schema consists of following two steps:

Step 1 Define the tracking errors as

$$
\begin{align*}
\zeta_1 &= \eta_1 - \eta_d, \\
\zeta_2 &= \eta_2 - \alpha, \\
\zeta_2 &= \tilde{\eta}_2 - \alpha,
\end{align*}
$$

where $\alpha$ is the virtual control. Let $\bar{\zeta}_2$ be the error between $\zeta_2$ and $\bar{\zeta}_2$, involving (42); $\zeta_2$ is bounded by

$$
\bar{\zeta}_2 = \zeta_2 - \bar{\zeta}_2 = \dot{\eta} - \dot{\bar{\eta}} \leq \epsilon \theta_2,
$$

and dynamic equation of $\zeta_1$ is

$$
\zeta_1 = \dot{\eta}_1 - \dot{\eta}_d = \zeta_2 + \alpha - \dot{\eta}_d.
$$

To deal with output constraint (7), let us consider the barrier Lyapunov function candidate [24] as follows:

$$
V_2 = \sum_{i=1}^{3} \frac{1}{2} \log \left( \frac{b_{2i}^2}{b_{2i}^2 - \zeta_{1i}^2} \right),
$$

where $b_i$ for $i = 1, 2, 3$ is defined as

$$
\begin{align*}
b_1 &= \frac{\pi}{2} - \bar{\psi}_d, \\
b_2 &= \frac{\pi}{2} - \bar{\psi}_d, \\
b_3 &= \pi - \bar{\psi}_d,
\end{align*}
$$

with $\bar{\psi}_d$ being the bound of $\psi_d$. The time derivative of (46) along (45) is given by

$$
\dot{V}_2 = \sum_{i=1}^{3} \frac{\zeta_{1i} \dot{\zeta}_{1i}}{b_{2i}^2 - \zeta_{1i}^2} = \zeta_{1i}^T B_{\zeta_1} \dot{\zeta}_1,
$$

where

$$
B_{\zeta_1} = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{b_{21}^2 - \zeta_{11}^2} & 0 \\
0 & 0 & \frac{1}{b_{22}^2 - \zeta_{12}^2}
\end{bmatrix}.
$$

Substituting (45) into (48) yields...
\[
\dot{V}_2 = \zeta_i^T B_{i1} (\alpha - \eta_{i\alpha}) + \zeta_i^T B_{i2} \zeta_2.
\] (50)

Choosing
\[
\alpha = -\left( A_0 + B_{i1} A_1 \right) \zeta_1 + \hat{\eta}_{i\alpha},
\] (51)

where \(A_0\) and \(A_1\) ∈ \(\mathbb{R}^{3 \times 3}\) are positive matrices, and substituting (51) into (50), yield
\[
\dot{V}_2 = -\zeta_i^T B_{i1} A_0 \zeta_1 - \zeta_i^T A_1 \zeta_1 + \zeta_i^T B_{i2} \zeta_2.
\] (52)

Using Lemma 2 in [34], in the set \(|\zeta_i| < b_i, i = 1, 2, 3\), we have
\[
\log \frac{b_i^2}{b_i^2 - \zeta_i^2} \leq \frac{c_i}{b_i^2 - \zeta_i^2},
\] (53)

and the first term in (52) satisfies
\[
-\zeta_i^T B_{i1} A_0 \zeta_1 < -2 \lambda_{\min} (A_0) \sum_{i=1}^{3} \frac{1}{2} \log \left( \frac{b_i^2}{b_i^2 - \zeta_i^2} \right)
\] (54)

\[
< -2 \lambda_{\min} (A_0) V_2.
\]

Substituting (54) into (52) yields
\[
\dot{V}_2 < -2 \lambda_{\min} (A_0) V_2 - \zeta_i^T A_1 \zeta_1 + \zeta_i^T B_{i2} \zeta_2.
\] (55)

The first two terms in (55) are negative; the last term will be cancelled in the next step.

Step 2 Taking the time derivative \(\dot{\xi}_2\) in (43) along (2) yields
\[
\dot{J}(\eta_1) \dot{\xi}_2 = J(\eta_1) \dot{\xi}_2 - J(\eta_1) \dot{u} = -C(\eta_1, \eta_2) \xi_2 - C(\eta_1, \eta_2) \alpha + \tau + \dot{d}_\tau - J(\eta_1) \dot{u}.
\] (56)

Consider the Lyapunov function candidate as
\[
V_3 = V_2 + \frac{1}{2} \zeta_i^T J(\eta_1) \zeta_2,
\] (57)

and its time derivative along (56) is given by
\[
\dot{V}_3 = \dot{V}_2 + \frac{1}{2} \zeta_i^T J(\eta_1) \dot{\xi}_2 + \zeta_i^T \dot{J}(\eta_1) \zeta_2
\] (58)

\[
= \dot{V}_2 + \frac{1}{2} \zeta_i^T \left( J(\eta_1) - 2C(\eta_1, \eta_2) \right) \zeta_2 + \zeta_i^T \left( -C(\eta_1, \eta_2) \alpha + \tau + \dot{d}_\tau - J(\eta_1) \dot{u} \right).
\]

Thus, the real control \(\tau\) appears in (58). We need to design the control \(\tau\) to make \(\dot{V}_3 \leq 0\). To this end, we notice that the term \(\zeta_i^T \left( J(\eta_1) - 2C(\eta_1, \eta_2) \right) \zeta_2 = 0\) according to the skew symmetric property. The unknown terms \(C(\eta_1, \eta_2) \alpha\) and \(J(\eta_1) \dot{u}\) are estimated by using the RBF NNs approximation function from (10) as
\[
C(\eta_1, \eta_2) \alpha + J(\eta_1) \dot{u} = W^* T \Phi(Z) + \epsilon(Z),
\] (59)

where \(Z = [\eta_1^T, \eta_2^T]^T\) is the input vector of the NNs and \(W^*\) is the ideal unknown constant weight matrix, which has the following form:
\[
W^* = \begin{bmatrix}
0_{1 \times \ell} & 0_{1 \times \ell} & 0_{1 \times \ell} \\
0_{1 \times \ell} & W^*_2 & 0_{1 \times \ell} \\
0_{1 \times \ell} & 0_{1 \times \ell} & W^*_3
\end{bmatrix} \in \mathbb{R}^{3 \times \ell},
\] (60)

with \(w_i^* \in \mathbb{R}^\ell, i = 1, 2, 3\), and \(W^*\) is bounded by
\[
\|W^*\| \leq W_m,
\] (61)

where \(W_m\) is a positive constant; \(\ell\) is the node number of the corresponding NN; and \(\Phi(Z) \in \mathbb{R}^\ell\) is the basis function vector of the following form:
\[
\Phi(Z) = [\Phi_1(Z), \Phi_2(Z), \Phi_3(Z)]^T \in \mathbb{R}^\ell,
\] (62)

with \(\Phi_i(Z) \in \mathbb{R}^\ell; \epsilon(Z)\) is the error vector with the known bound as
\[
\|\epsilon(Z)\| \leq \epsilon^*,
\] (63)

where \(\epsilon^*\) is a positive constant.

It was mentioned that the optimal weight \(W^*\) is unavailable for the approximation purpose. Furthermore, the state \(\eta_3\) is also unavailable for the feedback and hence cannot be the input of the NNs. Therefore, approximation (59) cannot be implemented. To deal with this situation, consider the relation between \(\eta_3\) and \(\hat{\eta}_3\) in Lemma 4; we use the estimated state \(\hat{\eta}_3\) as the input of the NNs for the approximation of the left-hand side of (59). The possibility of the use of the estimated state \(\hat{\eta}_3\) is shown in Appendix C. Accordingly, the input of the NNs is \(\hat{Z} = [\eta_1^T, \eta_2^T]^T\), and hence, the basis function vector is \(\Phi(\hat{Z})\).

Next, we need to deal with input constraint (6). To do this, the control \(\tau\) is designed in the following form:
\[
\tau_j = \begin{cases} 
\tilde{\tau}_p, & \text{if } \tilde{\tau}_p > \tilde{\tau}_p, \\
\tau_{\alpha j}, & \text{if } |\tau_{\alpha j}| \leq \tilde{\tau}_j, \\
-\tilde{\tau}_p, & \text{if } \tilde{\tau}_p < -\tilde{\tau}_j,
\end{cases}
\] (64)

for \(i = \phi, \theta, \psi, \) with \(\tau_{\alpha j} = [\tau_{\alpha \phi}, \tau_{\alpha \theta}, \tau_{\alpha \psi}]^T\) being the free control to be designed. Denoting the control input error as
\[
\Delta \tau = \tau - \tau_0
\] (65)

and substituting (55), (59), and (65) into (58), we obtain
\[
\dot{V}_3 = -2 \lambda_{\min} (A_0) V_2 - \zeta_i^T A_1 \zeta_1 + \zeta_i^T B_{i2} \zeta_2 + \zeta_i^T \left( \frac{W^* T \Phi(Z)}{\tau} + \epsilon(Z) + \tau_0 + \Delta \tau + d_\tau \right),
\] (66)

In view of (66), \(\Delta \tau \neq 0\) when the saturation appears. We need to reject \(\Delta \tau\). This can be accomplished by using the following auxiliary system [35]:
\[
\dot{\upsilon} = \begin{cases} 
-K_v \upsilon - \frac{\eta_i^2}{\zeta_i^2} \Delta \tau + \|\Delta \tau\| \|\upsilon\| k_0 + \Delta \tau, & \|\upsilon\| \geq q, \\
0, & \|\upsilon\| < q,
\end{cases}
\] (67)
where \( v \in \mathbb{R}^3 \) is the auxiliary vector; \( K_v \) is a positive definite matrix; \( k_o \) is a positive number; and \( q \) is a positive constant that should be chosen as an appropriate value in accordance with the requirement of the tracking performance.

Thus, we have transformed the control \( \tau \) under input constraint (6) to the free control \( \tau_o \) by adding the auxiliary variable \( v \). Consequently, in order to obtain the control \( \tau_o \) based on the Lyapunov method, we consider the following Lyapunov function candidate:

\[
V_4 = V_3 + \frac{1}{2} \sum_{i=1}^{3} \tilde{w}_i^T \Gamma_i^{-1} \tilde{w}_i + \frac{1}{2} b^T v, \tag{68}
\]

where \( \tilde{w}_i = w_i^* - \tilde{w}_i \), for \( i = 1, 2, 3 \), is the estimation error and \( \tilde{w}_i \) is the estimation of \( w_i^* \). The goal now is to select the control \( \tau_o \) to make the time derivative \( \dot{V}_4 \) negative. The process of selecting \( \tau_o \) is detailed in Appendix C. At this point, in order to state the result of this section, we propose the control \( \tau_o \) and the update law \( \tilde{w}_i \) in advance as follows:

\[
\tau_o = -B_1 \zeta_3 - A_2 \tilde{q}_2 + \tilde{W}^T \Phi (\tilde{Z}) + v, \tag{69}
\]

\[
\dot{\tilde{w}}_i = -\Gamma_i (\Phi_i (\tilde{Z}) \tilde{x}_2 + \mu_i \tilde{w}_i), \quad i = 1, 2, 3, \tag{70}
\]

where \( A_2 \) is a positive matrix and \( \mu_i \) is a positive constant.

In control (69), the first term is proposed to cancel the second term in (55). The second term is used for the stabilization of \( \zeta_2 \) and dealing with \( d_i \) and \( e_i \). The third term is the estimation of (59). The last term is used to tackle the appearance of the input saturation.

**Theorem 2.** For orientation subsystem (2), under high-gain observers (40) and (41), controllers (64), (69), and (67), and update law (70), if the initial conditions are such that \( |\zeta_1(0)| < b \), where \( b \) is defined in (47), then

(i) All the system states are ultimately bounded, i.e., there exist positive numbers \( \kappa \) and \( C_4 \) such that

\[
\dot{V}_4 \leq -\kappa V_4 + C_4, \tag{71}
\]

(ii) Constraints (6) and (7) are satisfied; in particular, the tracking error \( \zeta_1 \) is bounded by

\[
\|\zeta_1(t)\| \leq \|b\sqrt{1 - e^{2V_1(t)}}\}, \quad \forall t > 0. \tag{72}
\]

(iii) The tracking error \( \zeta(t) \) converges to a small neighbourhood of the origin which can be made arbitrarily small.

**Proof.** See Appendix C. \( \square \)

**Remark 7.** It is noticed that the second term of \( V_3 \) in (57) is the function of the state \( \zeta_2 \) instead of \( \zeta_3 \). This allows us to know the behavior of the state \( \zeta_3 \) even when \( \zeta_2 \) is unavailable for the feedback.

**Remark 8.** In order to deal with constraint (7), we use the Euler angles in the representation of the attitude. This makes the control design become much easier because the use of the unit quaternion may lead to inconsistent behavior, i.e., the unwinding phenomenon [2]. Although the use of Euler angles has drawback which is the so-called gimbal lock [36], i.e., the rotation matrix is singular when \( \phi = \pi/2 \) or \( \theta = \pi/2 \), this drawback is tackled by the satisfaction of the tracking performance.

### 4. Stability Analysis

This section investigates the stability of the overall closed-loop system under the controllers designed in the previous section. The overall error system consists of the closed-loop position error subsystem from (34) as follows:

\[
\dot{\bar{p}}_1 = \bar{p}_2 + \bar{q}_1, \tag{73}
\]

\[
\dot{\bar{p}}_2 = -\sigma_1 (\bar{p}_1 + \bar{p}_2) - \sigma_2 (\bar{p}_2) + L_2 \bar{q}_1 + \omega, \tag{73}
\]

\[
\dot{\bar{q}} = L\bar{q} + d_q.
\]

Also, it has the closed-loop orientation error subsystem from (40), (41), (45), (56), (64), (69), and (70) as follows:

\[
\dot{\bar{\zeta}}_1 = \zeta_2 + \alpha - B_{\zeta_1} \zeta_1, \tag{73}
\]

\[
J (\eta_1) \bar{\zeta}_2 = -C (\eta_1, \eta_2) \bar{\zeta}_2 - W^T \Phi (Z) + \hat{W}_3^T \Phi (\tilde{Z})
\]

\[-B_{\zeta_2} \zeta_2 + v + \Delta r + e(Z) + d_r, \tag{74}\]

\[
\dot{v} = \begin{cases} 
-K_v v - \frac{\|\zeta_2\|}{\|v\|^2} |\Delta r| k_0 \|v\|, & \|v\| \geq \theta, \\
0, & \|v\| < \theta. 
\end{cases} \tag{74}
\]

Systems (73) and (74) have the cascaded structure in which (73) is driven by (74). The result proposed in [11, 37] cannot be directly applied because of constraint (7) and disturbances. In the following, we show that the controllers designed in the previous section achieve the ultimate boundedness property of the overall systems (73) and (74).

**Theorem 3.** Consider the closed-loop overall tracking error system described by (73) and (74) under Assumption 2. If the initial condition is such that \( |\bar{\zeta}_1(0)| < b \), then the considered system enjoys the ultimate boundedness property.

**Proof.** The proof consists of two stages: (i) the solution of systems (73) and (74) does not have finite escape time, and (ii) all the system states are ultimately bounded. That is,

(i) Completeness of the solution: since the solution of system (74) is ultimately bounded from Theorem 2, the remainder is to show that the solution of system (73) is bounded for all the bounded time, or in other words, will not approach infinity in the finite time. To do this, considering the time derivative \( V_1 \) from (B.10) by noticing
(ii) Ultimate boundedness property of the overall system: we need to prove that the time derivative of the total Lyapunov function \( V_1 + V_4 \) is negative outside some compact set. It is noticed that \( V_4 \) satisfies (71) from Theorem 2. The remainder is \( V_1 \). We shall show that \( V_1 \) satisfies (37) in Theorem 1, or in other words, we need to verify condition (36) of Theorem 1. To this end, by direct computation from (29), we have

\[
\|q\| \leq k_0 \|\xi_1\|, \tag{81}
\]

where \( k_0 = 3 \sqrt{2/(\bar{n}/m)} > 0 \). From (72) of Theorem 2, the solution \( \xi_1(t) \) is bounded by

\[
\|\xi_1(t)\| \leq \|b\| \sqrt{1 - e^{-2V_4(t)}}. \tag{82}
\]

To evaluate the behavior of \( V_4 \), solving (71) with respect to \( V_4 \), we obtain

\[
V_4(t) \leq V_4(0)e^{-\xi t} + \frac{C_4}{k}(1 - e^{-\xi t}). \tag{83}
\]

Inequality (83) shows that \( V_4(t) \) exponentially decays and the size of the final value is \( C_4/k \) which can be made arbitrarily small. Accordingly, from (82) and (83), there exists a time instant \( T_\omega \) such that

\[
\|\xi_1(t)\| \leq \|b\| \sqrt{1 - e^{-2V_4(t)}} \leq \frac{\omega^*}{k_0}, \quad \forall t > T_\omega, \tag{84}
\]

with \( \omega^* \) being the constant in (36). Involving (81), we have

\[
\|q\| \leq k_0 \|\xi_1(t)\| \leq \omega^*, \quad \forall t > T_\omega. \tag{85}
\]

Condition (85) implies the satisfaction of condition (36) in Theorem 1. According to Theorem 1, there exists a time instant \( T_2 = \max\{T_1, T_\omega\} \) such that inequality (37) in Theorem 1 is achieved or we have

\[
V_1 \leq -\mathcal{W}_1 + C_1, \quad \forall t > T_2. \tag{86}
\]

Now let us consider the Lyapunov function candidate

\[
V_c = V_1 + V_4, \tag{87}
\]

from (86) and (71); the time derivative \( \dot{V}_c \) along closed-loop system (73) and (74) is given by

\[
\dot{V}_c = \dot{V}_1 + \dot{V}_2 \leq -\mathcal{W}_1 + C_1 - kV_4 + C_4, \quad \forall t > T_2, \tag{88}
\]

or

\[
\dot{V}_c \leq -\mathcal{W}_c + C_c, \quad \forall t > T_2, \tag{89}
\]

where \( \mathcal{W}_c = \mathcal{W}_1 + kV_4 \) and \( C_c = C_1 + C_4 \).

The foregoing inequality (89) shows that the solution of the overall closed loop system is ultimately bounded with the ultimate bound \( C_c \) which can be made arbitrarily small by making \( C_1 \) and \( C_4 \) small. The proof is completed. \( \square \)

Remark 9. It is noticed from Theorem 3 that cascaded systems (73) and (74), in which ISS system (73) with the bounded input is driven by system (74) with ultimate boundedness property, enjoy ultimate boundedness property. This fact is a novelty and gives us a flexible tool in the control design for the cascaded system. In comparison with the existing studies \([11, 37]\), this fact allows us not only to deal with constraints, disturbances, and uncertainties of the system but also take observers for state estimation into account.

Based on Theorem 3, our final result is stated in the following theorem.
Theorem 4. Consider the class of underactuated vehicles described by (1) and (2) with constraints (5)–(7), the velocities estimated by observers (20) and (40), respectively, the controls \( u \) and \( \tau \) obtained by (30), (31), (64), (69), and (67), and update laws described by (70). Let design parameters be chosen according to Theorems 1 and 2. Then, for every initial condition such that \( |\zeta_1(0)| < b \), Problem 1 is solved, i.e., constraints (5)–(7) are ensured, all the signals in the closed loop system are ultimately bounded, and the tracking error converges to adjustable neighbourhoods of the origin.

Proof. Constraint (5) is satisfied from Theorem 1, and constraints (6) and (7) are satisfied from Theorem 2. All the signals in the closed loop are ultimately bounded, and the tracking error converges to the adjustable neighbourhoods of the origin from Theorem 3.

5. Simulation Results

This section presents the numerical simulation results of the proposed controller. The system in the simulation is taken from the model of a quadrotor [1, 38]. Matrices \( J(\eta) \) and \( C(\eta, \dot{\eta}) \) are given in Appendix D. The nominal parameters are \( m = 1 \) kg; \( \bar{J} = \text{diag}[0.125/4, 0.125/4, 0.125/4] \) kgm\(^2\); and \( g = 9.81 \) m/s\(^2\). We assume that \( \Delta m = 0.05 \) m; \( J = \bar{J} + 0.1 \bar{J} \); and \( d_q \) and \( d_{\eta} \) have random values in \([-0.02; 0.02]\) and \([-0.01; 0.01]\), respectively. The upper bounds of constraints (5) and (6) are \( \bar{\tau} = 14 \) N and \( \bar{\tau} = [0.05; 0.05; 0.05]^T \) Nm. The desired trajectory is \( x_d(t) = 0.5 \cos(0.5t) \) m, \( y_d(t) = 0.5 \sin(0.5t) \) m, \( z_d(t) = -0.5t \) m, and \( \psi_d(t) = 0.2e^{-t} \) rad. The initial conditions are \( q(0) = [0, 0, 4]^T \) m; \( q(0) = [0.5, 0.5, 0]^T \) m/s; and \( \eta(0) = [0.1, 0.1, 0.1]^T \) rad. The computation of the controller is as follows.
For the position subsystem, the control $u$ is computed from (30) and (31). We set $\sigma_{11} = \sigma_{21} = 1$, $\sigma_{12} = \sigma_{22} = 1$, and $\sigma_{13} = \sigma_{23} = 1.8$ for (31) and $L_1 = L_2 = \text{diag}(20, 20, 20)$ for the position observer (20). Thus, condition (33) is satisfied.

For the orientation subsystem, we set $\gamma_1 = 2$ and $\epsilon = 0.005$ for observer (40); $A_3 = \text{diag}(3, 3, 3)$ for (69); $k_0 = 1, K_0 = \text{diag}(1, 1, 1)$, and $\rho = 0.05$ for (67). The update laws are designed based on (70). The neural network has $2^7$ nodes, the centers $\kappa_i$ of the basis function vector are evenly spaced in the range of $[-2, 2] \times [-2, 2] \times [-1, 1]$, the widths are $h_i = 1$ for all setting, and $\Gamma_i = \text{diag}(1, \cdots, 1) \in \mathbb{R}^{2^7 \times 2^7}$ and $\mu_i = 0.01, i = 1, \cdots, 3$ for (70).

For the comparison purpose, we carry out the simulation with the full-state feedback control proposed in [1]. The position and linear velocities are shown in Figures 1 and 2. The orientation angles and angular velocities are shown in Figures 3 and 4. It can be seen in Figures 2 and 4 that the estimated translational and angular velocities of the proposed control converge to the desired velocities. Furthermore, our proposed control can reject the velocity errors in
comparison with the controller proposed in [1]. The control inputs are shown in Figures 5 and 6. It is seen that the constraints on input are protected. Finally, the trajectory is shown in Figure 7.

6. Conclusion

This paper studies the robust adaptive output feedback control for the class of underactuated aerial vehicles with multiple constraints. By constructing the system as the cascaded structure, the modular design strategy is proposed to separately design the controller for each subsystem. By integrating observers, saturations, and barrier functions and NN approximation into the control system design, the controller has ability to simultaneously deal with system uncertainties, unavailable velocities, and constraints. The overall system enjoys the ultimate boundedness property. The simulation on the quadrotor helicopter shows that the control objectives are completed.

Appendix

A. Proof of Lemma 3

L is Hurwitz; then, there exists a symmetric positive definite matrix $P_0 \in \mathbb{R}^{3 \times 3}$ satisfying the Lyapunov equation [39, Theorem 4.6]:

$$ L^T P_0 + P_0 L = -Q_o, $$  \hspace{1cm} (A.1)

where $Q_o \in \mathbb{R}^{3 \times 3}$ is a positive definite matrix.

Consider the Lyapunov function candidate as follows:

$$ V_o(q) = \frac{1}{2} q^T P_o q, $$  \hspace{1cm} (A.2)

and its time derivative along (22) is given by

$$ V_o = \dot{q}^T P_o \dot{q} + \ddot{q}^T P_o \ddot{q} \leq -\lambda_{\min}(Q_o) \|q\|^2 + 2d_2^2 \lambda_{\max}(P_o) \|q\|. $$  \hspace{1cm} (A.3)

To use the term $-\lambda_{\min}(Q_o) \|q\|^2$ to dominate $+2d_2^2 \lambda_{\max}(P_o) \|q\|$, let $0 < c_o < 1$ and rewrite the above inequality as

$$ V_o \leq -(1 - c_o) \lambda_{\min}(Q_o) \|q\|^2 - \|q\| \left(c_o \lambda_{\min}(Q_o) \|q\| - 2d_2^2 \lambda_{\max}(P_o) \right) \leq -2\rho_o \|q\|^2 \forall \|q\| \geq \delta_o, $$  \hspace{1cm} (A.4)

where

$$ \rho_o = \frac{(1 - c_o) \lambda_{\min}(Q_o)}{2 \lambda_{\max}(P_o)}, $$  \hspace{1cm} (A.5)

$$ \delta_o = \frac{2 \lambda_{\max}(P_o)}{c_o \lambda_{\min}(Q_o)}. $$

Using the fact that $\lambda_{\min}(P_o) \|q\|^2 \leq V_o(q) \leq \lambda_{\max}(P_o) \|q\|^2$, the observer error satisfies

$$ \|\hat{q}(t)\| \leq k_o \|q(0)\| e^{-\rho_o t}, \forall \|q\| > \delta_o, $$  \hspace{1cm} (A.6)

with $k_o = \sqrt{\lambda_{\max}(P_o) / \lambda_{\min}(P_o)}$. Inequality (A.6) shows that there exists the time instant $T_o$, the observer error starting outside of the ball $\Omega_o = \{\|q\| < \delta_o\}$ enters $\Omega_o$ after the time $T_o$ and stays therein all time, and hence, (24) is achieved. Furthermore, the ultimate bound $\delta_o$ can be made arbitrarily small by adjusting the matrices $Q_o$ and $P_o$.

B. Proof of Theorem 1

B.1. Satisfaction of Constraint (5). From (30), we have

$$ u^2 = \overline{m}^2 \sum_{i=1}^{3} F_i^2 = \overline{m}^2 \sum_{i=1}^{3} \left(\sigma_{i1} + \sigma_{i2} + \delta_{di} + g e_{3i}\right). $$  \hspace{1cm} (B.1)

Using condition (33), we conclude that constraint (5) is ensured.

B.2. Ultimate Boundedness of State. Consider the Lyapunov function candidate $V_1(p_1, p_2, \dot{q})$ of the following form:

$$ V_1 = \sum_{i=1}^{3} \int_{0}^{\cdot} (p_{i1} \overline{p}_{i2}) \sigma_i(\dot{q}) \, dt \leq \frac{1}{\sigma_i} \int_{0}^{\cdot} (p_{i1})^2 \overline{p}_{i2}^2 + \frac{1}{\sigma_i^2} \overline{p}_{i2}^2 \overline{p}_{i1}^2 + \frac{1}{\sigma_i^2} \overline{p}_{i1}^2. $$  \hspace{1cm} (B.2)

where $p_{i1}, \overline{p}_{i2}, \sigma_i$, and $\dot{q}_i$ for $i = 1, \ldots, 3$ are the elements of $p_1, \overline{p}_2, \sigma$, and $\dot{q}$, respectively. We need to show that the time derivative $V_1$ along the trajectory of system (34) is negative outside a compact set. To do this, taking the time derivative of $V_1$ along (34) yields

$$ \dot{V}_1 = (\dot{p}_1 + \overline{p}_2) \sigma_1 (p_1 + \overline{p}_2) + \overline{p}_2 \overline{p}_1 \overline{q}^T P_o \overline{q} + \overline{q}^T P_o \overline{q}. $$  \hspace{1cm} (B.3)

For convenience, we use short denotations as follows:
\[
\sigma_1 := \sigma_1(p_1 + \hat{p}_2), \\
\sigma_2 := \sigma_2(p_2),
\]
and the first term in (B.3) satisfies
\[
(\hat{p}_1 + \hat{p}_2)\sigma_1 = \hat{p}_2\sigma_1 + \hat{q}_2\sigma_1 - \sigma_1^2 - \sigma_1^T L_2\sigma_1 + \omega^T \sigma_1
\leq -\|\sigma_1\|^2 + \|\sigma_1\|\|\sigma_2\| + \|\hat{q}_2\| + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|
+ \|\omega\|\|\sigma_1\| + \tilde{p}_2^T \sigma_1
\leq -\|\sigma_1\|^2 + \|\sigma_1\|\|\sigma_2\| + \sqrt{1 + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|}
+ \|\omega\|\|\sigma_1\| + \tilde{p}_2^T \sigma_1.
\]
(B.5)

Denoting \(c_1 = \sqrt{1 + \lambda_{\text{max}}(L_2)}\) and using the following inequalities
\[
\|\sigma_1\|\|\sigma_2\| \leq \frac{\|\sigma_1\|^2}{2} + \frac{\|\sigma_2\|^2}{2},
\]
\[
c_2\|\hat{q}_2\|\|\sigma_1\| \leq \frac{\|\sigma_1\|^2}{4} + c_1^2\|\hat{q}_2\|^2,
\]
inequality (B.5) can be further written as
\[
(\hat{p}_1 + \hat{p}_2)\sigma_1 \leq -\frac{1}{4}\|\sigma_1\|^2 + \frac{1}{2}\|\sigma_2\|^2 + c_1\|\hat{q}_2\|^2 + \|\omega\|\|\sigma_1\| + \tilde{p}_2^T \sigma_1.
\]
(B.7)

The second term in (B.3) satisfies
\[
\tilde{p}_2^T \sigma_2 = -\tilde{p}_2^T \sigma_2 - \frac{1}{2}\|\sigma_2\|^2 + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|
\leq -\tilde{p}_2^T \sigma_2 - \frac{1}{2}\|\sigma_2\|^2 - \frac{1}{2}\|\sigma_2\|^2 + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|
\leq -\frac{1}{2}\|\sigma_2\|^2 - \frac{1}{2}\|\sigma_2\|^2 + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|
\]
(B.8)

The second term in (B.3) satisfies
\[
\hat{p}_2^T \sigma_2 - \frac{1}{4}\|\sigma_2\|^2 - \frac{1}{2}\|\sigma_2\|^2 - \lambda_{\text{min}}(Q_o)\|\hat{q}\|^2 + 2\lambda_{\text{max}}(P_o)d_{q}^2\hat{q},
\]
where we use \(\hat{p}_2^T \sigma_2(\hat{p}_2) \geq \tilde{p}_2^T \sigma_2(\hat{p}_2)/2 + \|\sigma_2(\hat{p}_2)\|^2/2\).

The last two terms in (B.3) satisfy
\[
\hat{q}^T P_o \hat{q} + \hat{p}^T P_o \hat{q} \leq -\lambda_{\text{min}}(Q_o)\|\hat{q}\|^2 + 2\lambda_{\text{max}}(P_o)d_{q}^2\hat{q},
\]
where we use the result (A.3) in the proof of Lemma 3.

Substituting (B.6)–(B.8) into (B.3), we obtain
\[
V_1 \leq -\frac{1}{4}\|\sigma_1\|^2 - \frac{1}{2}\|\sigma_2\|^2 - \lambda_{\text{min}}(Q_o)\|\hat{q}\|^2 + \|\omega\|\|\sigma_1\|
+ \tilde{p}_2^T \omega + \lambda_{\text{max}}(L_2)\|\hat{q}_2\|\|\sigma_1\|
+ 2\lambda_{\text{max}}(P_o)d_{q}^2\hat{q},
\]
(B.10)

In view of (B.10), the term \(\tilde{p}_2^T \sigma_2/2\) will become \(\tilde{p}_2^T \tilde{p}_2/2\) if \(\hat{q}_2\) is in the linear region of saturation function \(\sigma_2\). In the following, we show that if the state \(\hat{p}_2\) reaches the saturation, then \(\hat{p}_2\) will enter the linear region of saturation function \(\sigma_2\) after a final time and stays therein. To do this, we consider the following two cases.

Case 1. The state \(\hat{p}_2\) reaches the saturation. In this case, we have \(\tilde{p}_2^T \sigma_2(\hat{p}_2) = \tilde{p}_2^T (\sigma_2^1 + \sigma_2^2) = |\tilde{p}_2|^2 \sigma_2^2\) and (B.10) can be rewritten as
\[
V_1 \leq -\frac{1}{8}\|\sigma_1\|^2 - \frac{1}{4}\tilde{p}_2^T \sigma_2 - \frac{1}{2}\|\sigma_1\|^2 - \frac{1}{4}\|\sigma_2\|^2 - \lambda_{\text{min}}(Q_o)\|\hat{q}\|^2 + \|\omega\|\|\sigma_1\|
+ \frac{1}{8}\|\sigma_1\|^2 - \frac{1}{4}\|\sigma_2\|^2 - \lambda_{\text{min}}(Q_o)\|\hat{q}\|^2
\]
(B.11)

where \(\|\tilde{p}_2\| = \min_{a=1,2,3}\|\tilde{p}_2\| > 0\). Choosing matrices \(P_o, Q_o,\) and \(L_2\) such that
\[
\lambda_{\text{min}}(Q_o) - c_1^2 > 0;
\]
\[
\frac{4\lambda_{\text{max}}(P_o)d_{q}^2}{\lambda_{\text{min}}(Q_o) - c_1^2} < \frac{\|\sigma_1\|}{8\lambda_{\text{max}}(L_2)}.
\]
(B.12)

and using result (24) from Lemma 3, then there exists a time instant \(T_0 > 0\) such that
\[
\|\hat{q}\| > \frac{\|\sigma_1\|}{8\lambda_{\text{max}}(L_2)} \quad \forall t > T_0,
\]
(B.13)

and the last term in (B.11) is negative.

Next, defining the set
\[
\Omega_1 = \{(p_1, \hat{p}_2, \hat{q}) \in \mathbb{R}^{12} \mid \|\sigma_1(p_1 + \hat{p}_2)\| \geq 8\sigma^*, \|\sigma_2(\hat{p}_2)\| \geq \sigma^*, \|\hat{q}\| \geq \frac{4\lambda_{\text{max}}(P_o)d_{q}^2}{\lambda_{\text{min}}(Q_o) - c_1^2}\}
\]
(B.14)

and using condition (36), if \(\forall (p_1, \hat{p}_2, \hat{q}) \notin \Omega_1\) and \(t > T_0\), we have
\[
\|\sigma_1\| > \|\hat{q}\| = \frac{\|\sigma_1\|}{8\lambda_{\text{max}}(L_2)} \|\hat{q}\| > \|\hat{q}\| \geq \frac{4\lambda_{\text{max}}(P_o)d_{q}^2}{\lambda_{\text{min}}(Q_o) - c_1^2},
\]
\[
\|\sigma_2\| > \|\hat{q}\| = \|\sigma_1\| > \|\hat{q}\| > \|\hat{q}\| \geq \frac{4\lambda_{\text{max}}(P_o)d_{q}^2}{\lambda_{\text{min}}(Q_o) - c_1^2},
\]
(B.15)

and the last four terms in (B.11) are negative, and (B.11) becomes
\[
V_1 \leq -\frac{1}{8}\|\sigma_1\|^2 - \frac{1}{4}\tilde{p}_2^T \sigma_2 - \frac{1}{2}\|\sigma_1\|^2 - \frac{1}{4}\|\sigma_2\|^2 - \lambda_{\text{min}}(Q_o)\|\hat{q}\|^2
\]
(B.16)

Inequality (B.16) shows that there exists a time instant \(T_1 > T_0 > 0\) such that the solution \((p_1, \hat{p}_2, \hat{q})\) starting outside \(\Omega_1\) enters \(\Omega_1\) after the finite time \(T_1\) and stays therein for all \(t > T_1\). In the other words, the state \(\hat{p}_2\) enters the linear region of the saturation function \(\sigma_2\).
Case 2. The state $\tilde{p}_2$ is in the linear region of $\sigma_2$. In this case, we have $\tilde{p}_2^T \sigma_2 (\tilde{p}_2) = \tilde{p}_2^T \tilde{p}_2$ and (B.10) is rewritten as

$$
V_1 \leq -\frac{1}{8}\|\sigma_1\|^2 - \frac{1}{2}\|\tilde{p}_2\|^2 - (\lambda_{\min}(Q_o) - c^2_\psi)\|q\|^2 + \|\omega\|\|\sigma_1\| + \tilde{p}_2^T \omega + \lambda_{\max}(L_2)\tilde{p}_2^T q_1 + 2\lambda_{\max}(P_o)d^*_q.
$$

(B.17)

Since $\|\omega\| \leq \omega^*$ and $\|d^*_q\| \leq d^*_q$, and

$$
\|\omega\|\|\sigma_1\| \leq \frac{\|\sigma_1\|^2}{8} + 2\|\omega\|^2;
$$

$$
\|\tilde{p}_2\|\|\omega\| \leq \frac{\|\tilde{p}_2\|^2}{8} + 2\|\omega\|^2;
$$

$$
\lambda_{\max}(L_2)\|\tilde{p}_2\|\|q_1\| \leq \frac{\|\tilde{p}_2\|^2}{8} + 2\lambda_{\max}(L_2)\|q\|^2;
$$

$$
2\lambda_{\max}(P_o)d^*_q \|q\|^2 \leq \lambda_{\max}(P_o)\|q\|^2 + \lambda_{\max}(P_o)\|d^*_q\|,
$$

(B.18)

inequality (B.17) can be further rewritten as

$$
V_1 \leq -\frac{1}{8}\|\sigma_1\|^2 - \frac{1}{4}\|\tilde{p}_2\|^2 - (\lambda_{\min}(Q_o) - 1 - 2\lambda_{\max}(L_2))\|q\|^2 + C_1.
$$

(B.19)

Defining

$$\mathcal{W}_1 = \frac{1}{8}\|\sigma_1\|^2 + \frac{1}{4}\|\tilde{p}_2\|^2 + (\lambda_{\min}(Q_o) - 1 - 2\lambda_{\max}(L_2))\|q\|^2 - \lambda_{\max}(P_o)d^*_q,$$

$$C_1 = 4\omega^* + \lambda_{\max}(P_o)d^*_q,$$

and choosing $Q_o$, $P_o$, and $L_2$ such that $\lambda_{\min}(Q_o) > 1 + 2\lambda_{\max}(L_2) + \lambda_{\max}(P_o)$, then $\mathcal{W}_1$ is the positive definite function, and we obtain (37) from (B.18), or

$$V_1 \leq -\mathcal{W}_1 (p_1, \tilde{p}_2, \tilde{q}) + C_1.
$$

(B.20)

In view of (B.20), $V < 0$, outside a compact set as follows:

$$\Omega_{C_1} = \{ (p_1, \tilde{p}_2, \tilde{q}) \mid \mathcal{W}_1 (p_1, \tilde{p}_2, \tilde{q}) \leq C_1 \}.
$$

(B.22)

According to an extension of Lyapunov theorem [32], we conclude that all the system states are ultimately bounded.

### B.3. ISS with Restrictions on the Inputs

Using Lemma 2 in [40], we need to prove that (1) system (34) enjoys the UB property with suitable restrictions; (2) it is globally asymptotically stable with zero inputs (0–GAS). The UBND property is ensured from the item (2). The remainder is 0–GAS of system (34). For this case, we have $(\omega, d^*_q) = 0$, implying from (B.19) that $C_1 = 0$, and (B.20) becomes

$$V_1 \leq -\mathcal{W}_1 (p_1, \tilde{p}_2, \tilde{q}) < 0, \quad \forall (p_1, \tilde{p}_2, \tilde{q}) \neq 0,
$$

and $V_1 (0) = 0$. It follows the Lyapunov theorem [39, Theorem 4.1] that system (34) is GAS. We conclude that system (34) is ISS with the state $(p_1, \tilde{p}_2, \tilde{q})$ and the input $(\omega, d^*_q)$ with restriction (36). The proof is completed.

### C. Proof of Theorem 2

#### C.1. Ultimate Boundedness Property of the Solution

The time derivative of $V_4$ in (68) along (56), (67), and (70) with the use of $\hat{w}_i = -\hat{\hat{w}}_i$ is given by

$$V_4 = -\sum_{i=1}^{3} \hat{w}_i^T \hat{\Gamma}_1^{-1} \hat{\hat{w}}_i + V_3 + u^T \hat{\hat{u}}$$

$$= -\sum_{i=1}^{3} \hat{w}_i^T \hat{\Gamma}_1^{-1} \hat{\hat{w}}_i - \zeta_1^T A_1 \zeta_1 + \zeta_2^T B_1 \zeta_2 + \zeta_3^T ( -W^* \Phi (Z)$$

$$- \epsilon (Z) + \tau_o + \Delta \tau + d_o)$$

$$+ u^T \left( -K_{\hat{\tau}} \hat{\tau} + \frac{\zeta_2 \Delta \tau}{\|u\|^2/k_0} u + \Delta \tau \right).$$

(C.1)

Substituting $\tau_o$ from (69) into (C.1) yields

$$V_4 = -\zeta_1^T A_1 \zeta_1 + V_2 + V_3 + V_4,$$

(C.2)

where

$$V_2 = -\zeta_2^T A_2 \zeta_2;$$

$$V_3 = -\sum_{i=1}^{3} \hat{w}_i^T \hat{\Gamma}_1^{-1} \hat{\hat{w}}_i - \zeta_1^T \left( W^* \Phi (Z) - \tilde{W}^* \Phi (\tilde{Z}) \right);$$

$$V_4 = \zeta_2^T \nu - \nu^T K \nu + \zeta_3^T ( d_o + \epsilon + \zeta_2 \Delta \tau + \frac{\Delta \tau^2}{k_0} u + \Delta \tau).$$

(C.3)

First, we calculate $V_2$. Using the relationship $	ilde{\zeta}_2 = \tilde{\zeta}_2 - \tilde{\zeta}_2$, $V_2$ becomes

$$V_2 = -\zeta_1^T A_2 \zeta_2 + \zeta_2 \Delta \tau \leq -\lambda_{\min}(A_2) \|\zeta_2\|^2 + \lambda_{\max}(A_2) \|\zeta_2\|^2$$

$$\leq -2k_1 \lambda_{\min}(A_2) \|\zeta_2\|^2 + k_2 \lambda_{\max}(A_2) \|\zeta_2\|^2$$

(C.4)

where $k_1$ is a positive designed parameter.

Second, we calculate $V_3$. Considering the second term in $V_3$, using the relationship $\|Z - \tilde{Z}\| \leq \epsilon \Phi$ from Lemma 4, we have $W^* \Phi (Z) - \Phi (\tilde{Z}) = \epsilon W^* \Phi \Phi$, [32, Lemma 3.1], where $\Phi$ is the bounded function vector, and
\[ -\zeta_2^T \left( W^T \Phi (Z) - \tilde{W}^T \Phi (\tilde{Z}) \right) = -\zeta_2^T \left( W^T \Phi (Z) - W^T \Phi (\tilde{Z}) + W^T \Phi (\tilde{Z}) - \tilde{W}^T \Phi (\tilde{Z}) \right) \]
\[ = -\zeta_2^T W^T \Phi (Z) - \zeta_2^T \tilde{W}^T \Phi (\tilde{Z}) \leq \epsilon \| \zeta_2 \| W^T \Phi_i^2 - \zeta_2^T \tilde{W}^T \Phi (\tilde{Z}), \] (C.5)

where \( k_2 \) is a positive number. Substituting (C.5) into \( \mathcal{V}_3 \) with the use of \( \zeta_2^T \tilde{W}^T \Phi (\tilde{Z}) = \sum_{i=1}^{3} \tilde{w}_i^T \Phi_i (\tilde{Z}) \zeta_{2i} \) yields
\[ \mathcal{V}_3 \leq \frac{\epsilon}{2k_3} \| \zeta_2 \|^2 + \frac{\epsilon k_3}{2} \| W^T \Phi_i \|^2 + \mathcal{V}_5, \] (C.6)

where
\[ \mathcal{V}_5 = -3 \sum_{i=1}^{3} \tilde{w}_i^T \left( \Phi_i (\tilde{Z}) \zeta_{2i} + \Phi_i (\tilde{Z}) \zeta_{2i} - \Phi_i (\tilde{Z}) \zeta_{2i} - \mu_i \tilde{w}_i \right). \] (C.7)

Substituting update law (70) into (C.7) with the use of \( \zeta_2 = \zeta_2 + \zeta_2 \), \( \mathcal{V}_5 \) becomes
\[ \mathcal{V}_5 = -3 \sum_{i=1}^{3} \tilde{w}_i^T \Phi_i (\tilde{Z}) \tilde{w}_i - 3 \sum_{i=1}^{3} \tilde{w}_i^T \Phi_i (\tilde{Z}) \tilde{w}_i. \] (C.8)

The first term in (C.8) satisfies
\[ \sum_{i=1}^{3} \mu_i \tilde{w}_i^T \tilde{w}_i = \sum_{i=1}^{3} \mu_i \left( \frac{1}{2} \| \tilde{w}_i \|^2 + \frac{1}{2} \| w_i^* - \tilde{w}_i \|^2 \right) \leq \frac{3}{2} \| \mu \| w_i^* \|^2, \] (C.9)

The second term in (C.8) satisfies
\[ -3 \sum_{i=1}^{3} \tilde{w}_i^T \Phi_i (\tilde{Z}) \tilde{w}_i \leq \sum_{i=1}^{3} \| \tilde{w}_i \| \| \Phi_i (\tilde{Z}) \| \| \zeta_{2i} \| \leq \frac{3}{2k_3} \sum_{i=1}^{3} \| \tilde{w}_i \|^2 + \frac{k_3}{2} \sum_{i=1}^{3} \| \Phi_i (\tilde{Z}) \|^2 \| \zeta_{2i} \|^2, \] (C.10)

where \( k_3 \) is a positive constant. Because every element of \( \Phi_i \) is not larger than one, we know that \( \| \Phi_i \|^2 \leq \ell \). Therefore, (C.10) satisfies
\[ \mathcal{V}_3 \leq \frac{\epsilon}{2k_3} \| \zeta_2 \|^2 + \frac{\epsilon k_3}{2} \| W^T \Phi_i \|^2 + \frac{k_3}{2} \sum_{i=1}^{3} \| \Phi_i (\tilde{Z}) \|^2 \| \zeta_{2i} \|^2, \] (C.11)

Substituting (C.8), (C.9), and (C.11) into (C.6), we obtain \( \mathcal{V}_3 \) as
\[ \mathcal{V}_3 \leq \frac{\epsilon}{2k_3} \| \zeta_2 \|^2 + \frac{\epsilon k_3}{2} \| W^T \Phi_i \|^2 - a_{31} \sum_{i=1}^{3} \| \tilde{w}_i \|^2 \]
\[ + \frac{1}{2} \sum_{i=1}^{3} \| w_i^* \|^2 + \frac{\epsilon k_3}{2} \| \zeta_2 \|^2, \] (C.12)

where
\[ a_{31} = \frac{1}{2} \left( \mu - \frac{1}{k_3} \right); \] (C.13)

Third, we calculate \( \mathcal{V}_4 \). Using the following inequalities
\[ \zeta_2^T v \leq \frac{\epsilon}{2k_4} \| \zeta_2 \|^2 + \frac{k_4}{2} \| v \|^2; \]
\[ \zeta_2^T (d_i - \epsilon) \leq \frac{\epsilon}{2k_3} \| \zeta_2 \|^2 + k_3 \left( \| d_i \|^2 + \| e_i \|^2 \right); \] (C.14)
\[ \tilde{v}^T \Delta e \leq k_0 \| \tilde{v} \|^2 + \frac{k_0}{2} \| \Delta e \|^2; \]
\[ v^T \Delta e \leq k_0 \| v \|^2 + \frac{k_0}{2} \| \Delta e \|^2, \]
equation \( \mathcal{V}_4 \) becomes
\[ \mathcal{V}_4 \leq \frac{1}{2} \left( \frac{1}{k_4} + \frac{1}{k_5} \right) \| \zeta_2 \|^2 - \frac{1}{2} \left( 2k_3 \mu_i (K_i) - k_4 - k_5 \right) \| v \|^2 \]
\[ + \frac{k_3}{2} \| \zeta_2 \|^2 + k_3 \left( \ell^2 + \| d_i \|^2 \right). \] (C.15)
Now, substituting (C.4), (C.12), and (C.15) into (C.2) yields
\[
\dot{\zeta}_i \leq -\zeta_i^T A_{\zeta_i} \zeta_i - \frac{2 k_1 \lambda_{\min}(A_2) - \lambda_{\max}(A_2)}{2k_1} \|\zeta_i\|^2 \\
+ \frac{k_1 \lambda_{\max}(A_2)}{2} \|\zeta_i\|^2 + \frac{\varepsilon}{2k_2} \|\zeta_i\|^2 + \frac{\varepsilon k_2}{2} \|W^T \Phi_i\|^2 \\
- a_{3i} \sum_{i=1}^{3} \|\bar{w}_i\|^2 + \frac{3}{2} \sum_{i=1}^{3} \|u_i\|^2 + \frac{\varepsilon k_2}{2} \|\zeta_i\|^2 \\
+ \frac{1}{2} \left( \frac{1}{k_4} + \frac{1}{k_5} \right) \|\zeta_i\|^2 \\
- \frac{1}{2} \left( 2 \lambda_{\min}(K_v) - k_4 - k_5 \right) \|v\|^2 + \frac{k_4}{2} \|\zeta_i\|^2 + k_2 \left( \varepsilon^2 + d^2\right).
\]
(C.16)

Denoting
\[
k_1 = \lambda_{\min}(A_1);
\]
k_2 = \frac{2 k_1 \lambda_{\min}(A_2) - \lambda_{\max}(A_2)}{2k_1} - \frac{\varepsilon}{2k_2} - \frac{1}{2k_4} - \frac{1}{2k_5},
\]
k_3 = a_{3i};
\]
k_4 = \frac{1}{2} (2 \lambda_{\min}(K_v) - k_4 - k_5),
\]
\[
C_4 = \frac{k_1 \lambda_{\max}(A_2)}{2} \|\zeta_i\|^2 + \frac{\varepsilon k_2}{2} \|\zeta_i\|^2 + \frac{k_4}{2} \|\zeta_i\|^2 + \frac{\varepsilon k_2}{2} \|W^T \Phi_i\|^2 \\
+ \frac{1}{2} \sum_{i=1}^{3} \|u_i\|^2 + k_2 \left( \varepsilon^2 + d^2\right),
\]
(C.17)

inequality (C.16) can be rewritten as
\[
\dot{\zeta}_i \leq -k_1 \|\zeta_i\|^2 - k_2 \|\zeta_i\|^2 - k_3 \sum_{i=1}^{3} \|\bar{w}_i\|^2 - k_4 \|v\|^2 + C_4.
\]
(C.18)

Now define
\[
\Omega_{C_i} = \left\{ \left( \zeta, \bar{w}, v \right) \mid k_1 \|\zeta_i\|^2 + k_2 \|\zeta_i\|^2 + k_3 \sum_{i=1}^{3} \|\bar{w}_i\|^2 + k_4 \|v\|^2 \leq C_4 \right\}.
\]
(C.19)

\[\kappa_i\] for \( i = 1, \ldots, 4 \) are positive and can be adjusted by choosing the design parameters \( A_1, A_2, \mu_i, K_v, \) and \( k_0 \). Inequality (C.18) shows that \( V_4 \leq 0 \) once the errors are outside the compact set \( \Omega_{C_i} \) in (C.19). According to an extension of Lyapunov theorem [32, page 78], we conclude that all the system signals are ultimately bounded. Furthermore, in order to obtain (71), let us define
\[
\kappa = \min_{i=1,\ldots,4} \left\{ 2k_i \right\},
\]
(C.20)
and inequality (71) is obtained directly from (C.18).

C.2. Satisfaction of Constraints (6) and (7): Considering (64), constraint (6) is satisfied automatically. Next, we show that constraint (7) is also satisfied. To this end, solving (71) with respect to \( V_4 \), we obtain
\[
V_4(t) \leq V_4(0) e^{-\kappa t} + \frac{C_4}{\kappa} (1 - e^{-\kappa t}).
\]
(C.21)

Considering \( V_2 \) from (46), we have
\[
\frac{1}{2} \log \left( \frac{b_i^2}{b_i^2 - \kappa \rho_i} \right) \leq V_4(t), \quad i = 1, 2, 3.
\]
(C.22)

Since \( V_4(t) > 0 \) for all \( t > 0 \), solving (C.22) with respect to \( \zeta_{ii} \) yields
\[
\|\zeta_{ii}\| \leq |b_i| \sqrt{1 - e^{-2V_4(t)}} \leq |b_i|, \quad i = 1, 2, 3.
\]
(C.23)

Taking the square of (C.23) and summing over \( i = 1, 2, 3 \), we have
\[
\sum_{i=1}^{3} \|\zeta_{ii}\|^2 \leq (1 - e^{-2V_4(t)}) \sum_{i=1}^{3} |b_i|^2 \leq \sum_{i=1}^{3} |b_i|^2.
\]
(C.24)

From (C.24), inequality (72) is achieved. Consequently, considering (43), for the case \( \phi = \zeta_1 + \phi_\theta \), we have \( |\zeta_{ii}| \leq \theta_1 \) and \( |\phi_\theta| \leq \Phi_\theta \). This implies from (47) that \( |\phi| \leq b_i + \Phi_\theta = 0.5 \pi \). Similarly, for the cases \( \theta \) and \( \psi \), we have \( |\theta| \leq 0.5 \pi \) and \( |\psi| \leq \pi \). Thus, constraint (7) is ensured.

C.3. Convergence of the Tracking Error to a Small Neighborhood of the Origin: Inequality (C.21) shows that the value of \( V_4(t) \) exponentially decays with the converging rate \( \kappa \) and the size of the final value is \( C_4/\kappa \). Furthermore, the tracking error \( \zeta_1 \) is bounded by (72) as
\[
\|\zeta_1\| \leq |b| \sqrt{1 - e^{-2V_4(t)}},
\]
(C.25)
and the size of the final tracking error is \( \|b\| \sqrt{1 - e^{-2V_4(t)}} \), which can be freely adjusted by the design parameters \( A_1, A_2, \mu_i, K_v, \) and \( k_0 \) in (C.18).

Finally, we show the tracking error \( \zeta_2 \) also converges to a small neighborhood of the origin. We notice from (C.18) and (C.19) that \( V_4 \leq 0 \) once \( \zeta_2 \geq C_4/\kappa \), i.e., \( \zeta_2 \) enters \( \Omega_{C_i} \) after a final time and stays therein. We conclude that the tracking error \( \zeta_2 \) will converge to a small neighborhood of the origin, whose size is adjusted by \( C_4 \) and \( \kappa \). The proof is completed.

D. Matrices \( J(\eta) \) and \( C(\eta, \dot{\eta}) \)

Details of matrices \( J(\eta) \) and \( C(\eta, \dot{\eta}) \) are as follows [38]:
\[
J(\eta) = \begin{bmatrix}
I_{xx} & 0 & -I_{xx} S_0 \\
0 & I_{yy} C_0^2 + I_{xx} S_0^2 & J_{32} \\
-I_{xx} S_0 (I_{yy} - I_{xx}) C_0 S_0 C_0 & J_{33}
\end{bmatrix},
\]
(D.1)
with \( j_{32} = (I_{yy} - I_{zz})C_\psi S_\theta C_\theta; j_{33} = I_{xx}S_\theta^2 + I_{yy}C_\theta^2 + I_{zz}C_\phi^2 C_\theta^2 \)

and

\[
C(\eta, \dot{\eta}) = \left[ \begin{array}{c} \dot{c}_{i,j} \end{array} \right], \quad i, j = 1, 2, 3, \quad (D.2)
\]

with \( c_{11} = 0; c_{12} = -I_{xx}\psi C_\psi/2 + (I_{yy} - I_{zz})\theta S_\psi C_\theta - (I_{yy} - I_{zz})\theta S_\psi C_\theta; c_{13} = -I_{xx}\psi C_\psi/2 - (I_{yy} - I_{zz})\theta (C_\phi^2 - S_\phi^2) C_\psi/2; c_{21} = I_{xx} C_\psi C_\theta/2 - (I_{yy} - I_{zz})\theta S_\psi C_\phi + (I_{yy} - I_{zz})\psi (C_\phi^2 - S_\phi^2) C_\psi/2; c_{23} = I_{xx} C_\psi C_\theta/2 - (I_{yy} - I_{zz})\theta S_\psi C_\phi + (I_{yy} - I_{zz})\psi (C_\phi^2 - S_\phi^2) C_\psi/2; c_{31} = -I_{xx}\psi C_\psi/2 + (I_{yy} - I_{zz})\theta \psi S_\phi/2 + (I_{yy} - I_{zz})\phi (C_\phi^2 - S_\phi^2) C_\psi/2; c_{32} = -I_{xx}\psi C_\psi/2 + (I_{yy} - I_{zz})\phi (C_\phi^2 - S_\phi^2) C_\psi/2; c_{33} = (I_{yy} - I_{zz})\phi S_\phi/2 + (I_{yy} - I_{zz})\theta S_\phi C_\phi/2 + (I_{yy} - I_{zz})\psi C_\phi/2. \]

\( \phi, \theta, \psi; \theta; \phi; S_\phi/2; C_\phi^2; \theta; \phi; S_\phi/2; C_\phi^2 \) are the moments of inertia with respect to axes \( O_x, O_y, \) and \( O_z \), respectively, of a fixed inertial frame \( Oxyz \) of the vehicle.

**Data Availability**

Data used in this paper are available upon request by email to the corresponding author.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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