EQUIVALENCE AMONG VARIOUS VARIABLE EXPONENT HARDY OR BERGMAN SPACES

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ABSTRACT. We study the question of when two weighted variable exponent Bergman spaces or Hardy spaces are equivalent. As an application, we show that variable exponent Hardy spaces have a close relation to classical Hardy spaces when the exponent is log-Hölder continuous and has bounded harmonic conjugate (when extended from its boundary values to be harmonic in the disc). We use this to characterize Carleson measures for these variable exponent Hardy spaces. We also prove under certain conditions an analogue of Littlewood subordination and a result on the boundedness of composition operators.

1. Introduction

Let \( p(z) \) be a measurable function defined on \( \mathbb{D} \), the unit disc. Let \( p_\leq = \text{ess inf } p(z) \) and \( p_\geq = \text{ess sup } p(z) \). If \( 0 < p_\leq \leq p_\geq < \infty \) we say that \( p(\cdot) \) is a variable exponent on \( \mathbb{D} \). Let \( dA \) denote normalized Lebesgue area measure on the unit disc and let 
\[
dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \quad \text{for } -1 < \alpha < \infty.
\]
Note that \( dA_\alpha(\mathbb{D}) = 1 \). We say that \( f \in L^{p(\cdot)}_\alpha \) if
\[
\rho_{p(\cdot)}(f) := \int_\mathbb{D} |f(z)|^{p(z)} dA_\alpha(z) < \infty.
\]
We can define the norm of a function \( f \in L^{p(\cdot)}_\alpha \) by
\[
\inf \left\{ \lambda > 0 : \int_\mathbb{D} |f(z)/\lambda|^{p(z)} dA(z) < 1 \right\}.
\]
This norm is called the Luxemborg-Nakano norm and makes \( L^{p(\cdot)}_\alpha \) into a Banach space if \( p_\leq \geq 1 \) (see [5]).

Recall the inequality \((a + b)^p \leq a^p + b^p\) that holds for \( a, b \geq 0 \) and \( 0 < p \leq 1 \). Thus if \( 0 < p(\cdot) \leq 1 \) then as in the constant exponent case the distance \( d(f, g) = \rho_{p(\cdot)}(f - g) \) defines a translation invariant metric (in the sense of addition of vectors, i.e. functions) under which \( L^{p(\cdot)} \) is complete. Also, since \( p_\leq > 0 \), the Luxemborg-Nakano norm

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makes $L^p(\cdot)$ into a quasi-Banach space (which satisfies the axioms for a Banach space except that the triangle inequality only needs to hold up to a constant multiple). If $0 < p_- < 1 < p_+ < \infty$ then the distance $d(f, g) = \rho_p(\cdot)[(f-g)\chi_{0<p(\cdot)<1}] + \| (f-g)\chi_{p(\cdot)\geq 1}\|_{L^p(\cdot)}$ defines a translation invariant metric on $L^p(\cdot)$ under which $L^p(\cdot)$ is complete. With this fact, it is not hard to see that $L^p(\cdot)$ is an $F$-space, and thus the closed-graph theorem applies to it [6, II.2, Theorem 4]. Also, the Luxemborg-Nakano norm still makes $L^p(\cdot)$ into a quasi-Banach space.

We let $A^p(\cdot)$ be the subspace of $L^p(\cdot)$ consisting of all analytic functions in $L^p(\cdot)$. Since $\| \cdot \|_{L^p(\cdot)} \leq \| \cdot \|_{L^p(\cdot)}$ and point evaluation is a bounded linear functional on $A^p(\cdot)$ with uniform bound on compact subsets, the Bergman spaces $A^p(\cdot)$ are closed.

More generally, we may define the space $A^\mu(\cdot)$ where $\mu$ is a (not identically zero) finite measure on the open unit interval by letting the norm of a function in $L^\mu(\cdot)$ be

$$\inf \left\{ \lambda > 0 : \int_0^1 \int_0^{2\pi} |f(z)/\lambda|^{p(\cdot)} d\theta d\mu(r) < 1 \right\}.$$ 

We will not study the $A^\mu(\cdot)$ spaces in detail, so we leave aside the question of when $A^\mu(\cdot)$ is a closed subspace of $L^\mu(\cdot)$.

We also define the integral mean $M_p(\cdot)(r, f)$ of an function analytic in the unit disc to be the Luxemborg-Nakano norm of $f$ restricted to the circle of radius $r$ with normalized Lebesgue measure. In other words,

$$M_p(\cdot)(r, f) = \inf \left\{ \lambda > 0 : \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})/\lambda|^{p(\cdot)} d\theta < 1 \right\}.$$ 

We also define

$$M^p(\cdot)(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p(\cdot)} d\theta.$$ 

(Note that the superscript $p(\cdot)$ is not really an exponent, but is suggestive of the notation $M_p(\cdot)(r, f)$ used with classical integral means.) We let $H^p(\cdot)$ be the class of all functions analytic in $\mathbb{D}$ such that

$$\| f \|_{H,p(\cdot),\sup} = \sup_{0 \leq r < 1} M_p(\cdot)(r, f) < \infty.$$ 

For a log-Hölder continuous exponent $p$, we define

$$\| f \|_{H,p(\cdot),\lim} = \lim_{r \to 1} M_p(\cdot)(r, f).$$

It is a consequence of [5, Theorem 5.11] that the limit in question always exists and

$$\| f \|_{H,p(\cdot),\lim} \leq \| f \|_{H,p(\cdot),\sup} \leq C \| f \|_{H,p(\cdot),\lim}$$
for some constant $C$ depending only on $p(\cdot)$ but not on $f$. (These two norms are equal in the classical case because the integral means are increasing). From now on, we will let $\|f\|_{H,p(\cdot)}$ denote $\|f\|_{H,p(\cdot),\lim}$ for Hardy spaces with log-Hölder continuous exponent.

We will also let $A_{-1}^{p(\cdot)}$ be an alternate notation for $H^{p(\cdot)}$. Note that if $f \in H^{p(\cdot)}$, then $f \in H^{p_{-}}$, so that $f$ has nontangential limit almost everywhere. Furthermore, by Fatou’s lemma, if we let $p(e^{i\theta}) = \lim \inf_{r \to 1} -p(re^{i\theta})$, then $f(e^{i\theta}) \in L^{p(e^{i\theta})}$. As with Bergman spaces, these Hardy spaces can be made into $F$-spaces or quasi-Banach spaces, and they are Banach spaces if $p_{-} \geq 1$.

We study several questions in this paper. First, we prove that the $A_{-1}^{p(\cdot)}$ and $H^{p(\cdot)}$ spaces depend only on the values of $p$ on the unit circle if $p$ satisfies a condition that is slightly weaker than log-Hölder continuity on the closed unit circle. Log-Hölder continuity is well known to be a natural condition when working with variable exponent Lebesgue spaces (see [5]).

We then discuss the case of variable exponent Hardy spaces in more detail. In the classical case, it is important that if $f \in H^{p}$ then $f = Bg$ for the Blaschke product $B$ formed from the zeros of $f$. Also, $\|f\|_{H,p} = \|g\|_{H,p}$ and $g^{p/2} \in H^{2}$ and $\|g^{p/2}\|_{H,2} = \|g\|_{H,p}^{p/2}$. We prove an analogous result for variable exponent Hardy spaces, under the condition that the exponent is log-Hölder continuous and that when restricted to the unit circle and then extended harmonically into the disc, the exponent has bounded harmonic conjugate. This allows us to pass immediately from certain classical results about Hardy spaces to corresponding results for these variable exponent Hardy spaces. Two examples are the characterization of Carleson measures and an analogue of the Littlewood subordination theorem. This allows us to prove that composition operators are bounded between certain variable exponent Hardy spaces.

In the last section, we discuss a case for the unweighted Bergman space where the exponent $p$ is assumed to be radial and $q \leq p(r) \leq \text{ess sup} p(r) < \infty$ for some $q > 0$. (We do not need to assume any continuity on $p$.) We find a necessary and sufficient condition for $A_{-1}^{p(\cdot)}$ to equal $A^{q}$. This leads to some interesting and surprising corollaries, such as Corollary 4.5.

2. **Uniformly Radially Log-Hölder Continuous Exponents**

In this section, we discuss uniformly radially log-Hölder continuous exponents and prove that the Bergman and Hardy spaces they define depend only on the values of the exponents on the unit circle. The
proof is very general and uses only the fact that functions in these spaces satisfy a bound of the form $|f(re^{i\theta})| = O((1 - r)^{-a})$ for some $a > 0$. We then discuss some applications of the result.

**Definition 2.1.** Let $p$ be an exponent on $\overline{D}$ with $0 < p_- \leq p_+ < \infty$ and let $0 \leq R < 1$ be fixed. Suppose that there is an absolute constant $C$ such that for each fixed $\theta$, one has

$$|p(re^{i\theta}) - p(e^{i\theta})| \leq C$$

for $R \leq r < 1$. We then say that $p$ is uniformly radially log-Hölder continuous.

**Definition 2.2.** Let $p$ be an exponent on $\overline{D}$ with $0 < p_- \leq p_+ < \infty$. We say that $p$ is log-Hölder continuous if there is a constant $C'$ such that for all $z, w \in \overline{D}$ with $|z - w| < 1$ we have

$$|p(z) - p(w)| \leq \frac{C'}{\log \frac{a}{|z-w|}}.$$

A slight problem with the above definition is that the right hand side of the inequality becomes infinite as $|z - w| \to 1$. However, we still get a uniform bound on $|z - w|$ by applying the inequality first to $z$ and $(2z + w)/3$ and then to $(2z + w)/3$ and $(z + 2w)/3$, and finally to $(z + 2w)/3$ and $w$. An alternate route is to note that we can change the inequality in the definition to

$$|p(z) - p(w)| \leq \frac{C''}{\log \frac{a}{|z-w|}}.$$

for any $a > 2$ and any $z, w \in \overline{D}$ without affecting the class of log-Hölder continuous functions. Furthermore, $C''$ can be bounded by a number depending on $C$ and $a$, and $C$ can be bounded by a number depending on $C''$ and $a$.

**Theorem 2.1.** Let $-1 \leq \alpha < \infty$. Suppose that $p(z)$ is a uniformly radially log-Hölder continuous exponent and let $\widehat{p}(re^{i\theta}) = p(e^{i\theta})$. Then $A_{\alpha}^{p(-)} = A_{\alpha}^{\widehat{p}(-)}$.

**Proof.** First let $-1 < \alpha < \infty$. Suppose that $f \in A_{\alpha}^{p(-)}$. We may assume without loss of generality that it has unit norm. Then $f \in A_{\alpha}^{p_-}$ so from [16],

$$|f(re^{i\theta})| \leq \frac{1}{(1 - r)^{(2+\alpha)/p_-}}.$$

Note that

$$(1 - r)^{-|p(re^{i\theta}) - p(e^{i\theta})|} \leq (1 - r)^{C/\log(1-r)} = e^C$$
Thus if $|f(z)| \geq 1$ we have that
\[
|f(re^{i\theta})|^{p(e^{i\alpha}) - p(e^{i\alpha})} \leq e^{(2+\alpha)C/p_-}.
\]
Thus
\[
\int_{\{|f(z)| \geq 1\}} |f(z)|^\hat{p}(z) dA_\alpha(z) \leq e^{(2+\alpha)C/p_-} \int_{\{|f(z)| \geq 1\}} |f(z)|^{p(z)} dA_\alpha(z).
\]
Since $\int_{\{|f(z)| < 1\}} |f(z)|^\hat{p}(z) dA_\alpha(z)$ is bounded by 1, we have that
\[
\rho_\hat{p}(f) \leq e^{2C/p_-} \rho_p(f) + 1 = e^{2C/p_-} + 1.
\]
Thus for any $f \in A^p(\cdot)$, one has that $\|f\|_{A,\hat{p}(\cdot)} \leq (e^{(2+\alpha)C/p_-} + 1) \|f\|_{A,p(\cdot)}$.
A similar argument shows that $\|f\|_{A,p(\cdot)} \leq (e^{(2+\alpha)C/\hat{p}_-} + 1) \|f\|_{A,\hat{p}(\cdot)}$.
For the case $\alpha = -1$, performing similar calculations shows that $M_{\hat{p}(\cdot)}(r,f) \leq (e^{C/p_-} + 1)M_{\hat{p}(\cdot)}(r,f)$ and that $M_{\hat{p}(\cdot)}(r,f) \leq (e^{C/\hat{p}_-} + 1)M_{\hat{p}(\cdot)}(r,f)$, which gives the result.

We need the following result found in [10] Lemma 1.

**Lemma 2.2.** Let $p > 1$ and $0 < r < 1$. Then
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^p} d\theta \leq \frac{\Gamma(p-1)}{\Gamma(p/2)^2} (1-r^2)^{1-p}.
\]
Furthermore, the bound is sharp, in the sense that the integral in question, divided by $(1-r^2)^{1-p}$, is always less than or equal to $\frac{\Gamma(p-1)}{\Gamma(p/2)^2}$, but the quotient approaches $\frac{\Gamma(p-1)}{\Gamma(p/2)^2}$ as $r \to 1$.

**Theorem 2.3.** Suppose that $p$ and $q$ are two exponents in $\mathbb{D}$ with $0 < p_-, q_-$ and $p_+, q_+ < \infty$ and that $\text{ess inf}_{z \in D} p(z) > \text{ess sup}_{z \in D} q(z)$ where $D$ is a disc centered on the boundary of the unit disc. Let $-1 \leq \alpha < \infty$. Then $A^p_\alpha \neq A^q_\alpha$.

**Proof.** Assume without loss of generality that $D$ is centered at 1. Let $s$ be such that $p_- D = \text{ess inf}_{z \in D} p(z) > s > \text{ess sup}_{z \in D} q(z) = q_+ D$. It follows from the lemma that the function $f(z) = (1-z)^{-(2/\alpha) s}$ is in $A^{q_+ D}$ but not in $A^{p_- D}$. Thus $f(z) \in A^q(\cdot)$ but $f \notin A^p(\cdot)$. \hfill $\Box$

**Corollary 2.4.** If $p$ and $q$ are two exponents continuous in $\overline{\mathbb{D}}$ with $0 < p_-, q_-$ and $p_+, q_+ < \infty$ then if $p(e^{\theta}) \neq q(e^{\theta})$ for some $\theta$, then $A^p_\alpha \neq A^q_\alpha$.

**Theorem 2.5.** Suppose that $p$ and $q$ are two exponents in $\overline{\mathbb{D}}$ with $0 < p_- \leq p_+ < \infty$ and $0 < q_- \leq q_+ < \infty$ and assume that $p$ and $q$ are log-Hölder continuous. Then $A^p = A^q$ if and only if $p(e^{i\theta}) = q(e^{i\theta})$ for all $\theta$. \hfill $\Box$
3. Complex Exponents and Hardy Spaces

In this section we find a condition under which we can use complex variable exponentials to easily derive many results for certain variable exponent Hardy spaces. The following proposition lets us simply our definitions somewhat.

**Proposition 3.1.** If a continuous harmonic function has log-Hölder continuous boundary values on the unit circle, then it is log-Hölder continuous in the closed unit disc.

*Proof.* Let $f$ be the harmonic function. It is enough to show that $|f(re^{i\theta}) - f(r)| \leq C/\log|2\pi/\theta|$ for any $r$ and that $|f(1) - f(r)| \leq -C/\log|1-r|$. The first inequality follows easily from the fact that

$$f(re^{i\theta}) - f(r) = \frac{1}{2\pi} \int_0^{2\pi} P_r(e^{i\psi})[f(e^{i(\theta - \psi)}) - f(e^{-i\psi})] \, d\psi$$

so that

$$|f(re^{i\theta}) - f(r)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{i\psi}) \frac{C}{\log(2\pi/|\theta|)} \, d\psi \leq \frac{C}{\log(2\pi/|\theta|)},$$

for $|\theta| < 1$.

To prove the second inequality, first note that it is enough to show that

$$|f(1) - f(r)| \leq \frac{C}{\log \frac{1}{1-r}}$$

for $r$ close enough to 1. But

$$f(1) - f(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} P_r(e^{it})(f(1) - f(e^{it})) \, dt$$

Note that

$$P_r(t) \leq \frac{C(1-r)}{(1-r)^2 + t^2}$$

for some constant $C$ (see [7, p. 74]). Thus we want to show that

$$\int_{-\pi}^{\pi} \frac{1-r}{(1-r)^2 + t^2} \frac{|\log(1-r)|}{\log |\pi e^2/t|} \, dt \leq C$$

Now let $u = t/(1-r)$ to rewrite this integral as

$$\int_{-\pi/(1-r)}^{\pi/(1-r)} \frac{du}{1 + u^2} \frac{|\log(1-r)|}{|\log((1-r)|u|/(\pi e^2)|)} \, du$$

Note that the integrand is even and that

$$\int_{0}^{1} \frac{du}{1 + u^2} \frac{|\log(1-r)|}{|\log((1-r)|u|/(\pi e^2)|} \, du$$
is bounded independently of $r$ for $1 - r$ small. Let $a = 1 - r$ and consider the integral

\begin{equation}
\int_1^{\pi/a} \frac{1}{u^2} \frac{|\log a|}{|\log(au/(\pi e^2))|} \, du.
\end{equation}

We may assume without loss of generality that $a < \pi/e$. Let $n$ be the first integer such that $\pi/a \leq e^n$, so that in particular $n \geq 1$. Let $j + 1 \leq n$. Notice that

\begin{equation}
\int_{e^j}^{e^{j+1}} \frac{1}{u^2} \frac{|\log a|}{|\log(au/(\pi e^2))|} \, du \leq e^{-j} \frac{|\log a|}{-\log a + 1 + \log \pi - j}.
\end{equation}

Then we have that the above expression is at most

\[ e^{-j} \frac{n}{n - j} \leq (j + 1)e^{-j}. \]

But since $\sum_{j=0}^{\infty} (j + 1)e^{-j}$ converges, the integral in (3.1) is bounded by a number independent of $a$, which finishes the proof. \hfill \square

**Definition 3.1.** Suppose that $p(\cdot)$ is log-Hölder continuous. By its harmonic conjugate we mean the harmonic conjugate of the harmonic function with the same boundary values as $p(\cdot)$. We say that an exponent $p$ is log-Hölder continuous with bounded conjugate if $p$ is log-Hölder continuous in the closed unit disc and its harmonic conjugate is bounded.

Note that if $p$ is log-Hölder continuous in the closed unit disc then so is the harmonic function with the same boundary values as $p$, by the above Proposition. The main advantage of working with exponents satisfying this definition is that for such exponents, many results about classical Hardy spaces immediately imply corresponding results for variable Hardy spaces.

Definition 3.1 is only slightly more restrictive than log-Hölder continuity. For example, if $p$ is Hölder continuous then $p$ is log-Hölder continuous with bounded conjugate. This also happens if $p$ satisfies a condition like log-Hölder continuity but with an exponent greater than 1 on the logarithm.

We now discuss some basic facts about variable exponent Hardy spaces. For more information see [4, 13–15]. If $p(\cdot)$ is log-Hölder continuous and $p_- \geq 1$, then $\|f(e^{i\theta})\|_{L^p(e^{i\theta})} = \|f\|_{H^{p(\cdot)}}$. Furthermore, we have mean convergence to boundary values. Both of these facts follow from [5, Theorem 5.11].
If $0 < p_- < 1$, we may write $f = Bg$ where $g$ is the Blaschke product of the zeros of $f$. Then $g^{p_-} \in H^{p/p_-}$, so
\[ M_{p(\cdot)}(r, g)^{p_-} = M_{p(\cdot)/p_-}(r, g^{p_-}) \rightarrow \|g(e^{i\cdot})^{p_-}\|_{L^{p(\cdot)/p_-}} = \|g(e^{i\cdot})\|_{L^{p(\cdot)}}^{p_-} \]
as $r \to 1$. But also $g(r e^{i\theta}) \to g(e^{i\theta})$ pointwise almost everywhere, so by an application of Egorov’s theorem, we have that $g$ converges to its boundary values in the mean (see [7, Lemma 1 after Theorem 2.6]). The fact that $f$ converges to its boundary values in the mean follows from Fatou’s lemma and a similar application of Egorov’s theorem. (see [7, Proof of Theorem 2.6]). Furthermore, $\|f\|_{H^{p(\cdot)}} = \|g\|_{H^{p(\cdot)}}$.

We will not need the following Theorem, but we include it because it is simpler to prove than Theorem 3.3. It may generally be used in place of that theorem when $p_- > 1$. For the remainder of this section, we will let $\tilde{p}$ be the harmonic conjugate of the extension of $p$ as a harmonic function in the unit disc, and let $\tilde{p} = p + i\tilde{p}$.

**Theorem 3.2.** Suppose that $f \in H^p$ and that $p$ is log-Hölder continuous with bounded conjugate. Assume that $p$ is harmonic in $\mathbb{D}$. Also assume that $1 < p_- \leq p_+ < \infty$. Then we may write
\[ f = f_1 - f_2 \]
where each $f_j$ satisfies either the inequality $-\frac{\pi}{2} < \arg f_j(z) < \frac{\pi}{2}$ for all $z \in \mathbb{D}$, or satisfies $f_j(z) = 0$ for all $z \in \mathbb{D}$. Furthermore, we have $\|f_j\|_{H^{p(\cdot)}} \leq C \|f\|_{H^{p(\cdot)}}$ for some constant $C$ depending only on $p_-, p_+$, and the log-Hölder constant of $p$. Also,
\[ \exp(-\pi \|\tilde{p}\|_{\infty}/2) |f_j|^{p(z)} \leq |f_j(z)|^{\tilde{p}(z)/2} \leq \exp(\pi \|\tilde{p}\|_{\infty}/2) |f_j|^{p(z)} \]

If $\|f\|_{H^{p(\cdot)}} \leq 1/C$ then
\[ \|f_j^{\tilde{p}(\cdot)/2}\|_{H^2} \leq \exp(\pi \|\tilde{p}\|_{\infty}/4) \]
where $C$ is the same constant as above. If $\|f_j\|_{H^{p(\cdot)}} \leq C'$ for some constant $C'$ then
\[ \|f_j^{\tilde{p}(\cdot)/2}\|_{H^2} \leq \max(C'^{p_-}/2, C'^{p_+}/2) \exp(\pi \|\tilde{p}\|_{\infty}/2). \]

A converse also holds: if $\|f_j^{\tilde{p}(\cdot)/2}\|_{H^2} \leq C'$ then
\[ \|f_j\|_{H^{p(\cdot)}} \leq \max(C'^{2/p_-}, C'^{2/p_+}) \exp(\pi \|\tilde{p}\|_{\infty}/2). \]

**Proof.** Let $f_1$ be the analytic function whose real part equals $\max(f, 0)$ on the unit circle, and let $f_2$ be the analytic function whose real part equals $-\min(f, 0)$ on the unit circle. Then $f = f_1 - f_2$. Also Re $f_j \geq 0$ and
\[ \|f_j\|_{H^{p(\cdot)}} \leq C \|f\|_{H^{p(\cdot)}} \]
since \( p \) is log-Hölder continuous and thus the harmonic conjugation operator is bounded on \( H^p(\cdot) \) [5, Theorem 5.39].

Note that
\[
|f_j(z)^{\tilde{p}(z)/2}| = |f_j(z)|^{p(z)/2}e^{-\tilde{p}(z)\arg f_j(z)/2}
\]
Thus equation (3.2) holds. If \( \|f\|_{H^p(\cdot)} \leq 1/C \) then \( \|f_j\|_{H^p(\cdot)} \leq 1 \), so
\[
\lim_{r \to 1} \sup 1 \int_0^{2\pi} |f_j(re^{i\theta})|^{p(re^{i\theta})} d\theta \leq 1.
\]
But this implies that
\[
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f_j(re^{i\theta})^{\tilde{p}(re^{i\theta})}|^2 d\theta \leq \exp(\pi \|\tilde{p}\|_{\infty}/2).
\]
Note that the limit above exists because the \( M_2 \) integral means are increasing. The rest of the proof follows from the definition of the Luxemborg-Nakano norm and equation (3.2).

The following theorem is similar to the previous one, but also applies in the case that \( p_- < 1 \).

**Theorem 3.3.** Suppose that \( f \in H^p \) and that \( p \) is log-Hölder continuous with bounded conjugate. Also assume that \( p \) is harmonic. Also suppose that \( 0 < p_- \leq p_+ < \infty \). Let \( n \) be the least integer such that \( np_+ \geq 2 \). Then we may write
\[
f = B \sum_{j=1}^{n+1} f_j
\]
where \( B \) is the Blaschke product of the zeros of \( f \) and each \( f_j \) satisfies either the inequality \( -\frac{\pi n}{2} < \arg f_j(z) < \frac{\pi n}{2} \) for all \( z \in \mathbb{D} \), or satisfies \( f_j(z) = 0 \) for all \( z \in \mathbb{D} \). Furthermore we have \( \|f_j\|_{H^p(\cdot)} \leq C\|f\|_{H^p(\cdot)} \) for some constant \( C \) depending only on \( n, p_+ \), and on the log-Hölder constant of \( p \). Also,
\[
\exp(-\pi n \|\tilde{p}\|_{\infty}/2)|f_j|^{p(z)} \leq |f_j(z)^{\tilde{p}(z)/2}|^2 \leq \exp(\pi n \|\tilde{p}\|_{\infty}/2)|f_j|^{p(z)}.
\]
If \( \|f\|_{H^p(\cdot)} \leq 1/C \) then
\[
\|f^{\tilde{p}(\cdot)/2}\|_{H^2} \leq \exp(\pi n \|\tilde{p}\|_{\infty}/4)
\]
where \( C \) is the same constant as above. If \( \|f_j\|_{H^p(\cdot)} \leq C' \) for some constant \( C' \) then
\[
\|f_j^{\tilde{p}(\cdot)/2}\|_{H^2} \leq \max(C'^{p_-/2}, C'^{p_+/2}) \exp(\pi n \|\tilde{p}\|_{\infty}/2).
\]
A converse also holds: if \( \| f_j^{\tilde{\mu}^{(j)}/2} \|_{H,2} \leq C' \) then
\[
\| f_j \|_{H,p(\mu)} \leq \max(C'^{2/p_-,} C'^{2/p_+}) \exp(\pi n \| \tilde{\mu} \|_\infty /2).
\]

Proof. Let \( f = B g \) where \( B \) is the Blaschke product formed with the zeros of \( f \). Then \( g^{1/n} \in H^{np(\mu)} \). Thus we may write \( g^{1/n} = g_1 - g_2 \), where \( \text{Re} g_j \geq 0 \) and
\[
\| \tilde{g}_j \|_{H^{np(\mu)}} \leq C \| g^{1/n} \|_{H^{np(\mu)}} = C \| f \|_{H,p(\mu)}^n
\]
since \( p \) is log-Hölder continuous and thus the harmonic conjugation operator is bounded on \( H^{p(\cdot)} \). We will assume that no \( g_j \) is identically zero (the proof where one is identically zero is nearly identical but easier). Since \( g_j(z) > 0 \) for all \( z \in \mathbb{D} \), we may define \( \arg g_j(z) \) so that \(-\pi/2 < \arg g_j(z) < \pi/2 \). Thus \( g(z) = (g_1 - g_2)^n \), and so \( g(z) \) can be written as a sum of \( n+1 \) functions, say \( f_1, \ldots, f_{n+1} \), such that each one satisfies \(-\pi/n < \arg f_j < \pi/n \). Also, each \( f_j \) satisfies \( \| f_j \|_{H^{p(\cdot)}} \leq C \| f \|_{H^{p(\cdot)}} \) by Hölder’s inequality for variable Lebesgue spaces [5, Corollary 2.28]. Then
\[
|f_j(z)^{\tilde{\mu}(z)/2}| = |f_j(z)|^{p(z)/2} e^{-\tilde{\mu}(z) \arg f_j(z)/2}
\]
Thus equation (3.3) holds. The rest of the proof follows as in Theorem 3.2.

We now discuss several results that follow quickly from the corresponding results for classical Hardy spaces. The first deals with Carleson measures. We say that a measure \( \mu \) on the unit disc is a Carleson measure for \( H^p \), where \( 0 < p < \infty \), if \( \| f \|_{L^p(\mu)} \leq C \| f \|_{H^p} \) for all \( f \in H^p \). It is well known that Carleson showed that a necessary and sufficient condition for a measure to be a Carleson measure for \( H^p \) is that \( \mu(S) \leq C'h \) for every Carleson square
\[
S_{h,\theta_0} = \{ z : 1 - h \leq |z| < 1 \text{ and } \theta_0 - h/2 < \arg z < \theta_0 + h/2 \}
\]
(see [12]).

Duren in [9] generalized this. His method is based on Hörmander’s proof in [12] of the Carleson measure theorem. Let \( 0 < p \leq q < \infty \) and let \( \mu \) and \( S \) be as before. Then a necessary and sufficient condition for \( \| f \|_{L^p(\mu)} \leq C \| f \|_{H^p} \) to hold for every \( H^p \) function \( f \) is that \( \mu(S) \leq C'h^{a/p} \) for every Carleson square \( S \). We generalize this result to variable exponent Hardy spaces.

**Theorem 3.4.** Suppose that that \( 0 < p_- \leq p(\cdot) \leq p_+ < \infty \) and let \( 1 \leq a < \infty \) be a constant. Let \( p \) be log-Hölder continuous with bounded conjugate. Then a necessary and sufficient condition that there exist a constant \( C \) such that \( \| f \|_{L^{p(\cdot)}(\mu)} \leq C \| f \|_{H^{p(\cdot)}} \) for every function \( f \) in \( H^{p(\cdot)} \) is that \( \mu(S) < C'h^a \) for every Carleson square \( S \).
Proof. The proof of the necessity uses a testing function similar to the one used in [2] but some difficulties arise because of the variable exponents. The calculation involved is similar to one in [18]. Let \( |z_0| = \rho \) and let \( h = 1 - \rho \). It is enough to show the result holds when \( h \) is near 0. Let \( q(z) = ap(z) \). We may assume that the support of \( \mu \) is in \( S \). Consider \( f(z) = (1 - \overline{z_0}z)^{-2/p(z)} \). Note that \( 1 - z_0z \) has bounded argument and so \( f(z) \) is well defined and

\[
|f(z)| \asymp \frac{1}{|1 - \overline{z_0}z|^{2/p(z)}}.
\]

Thus

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(z)|^{p(z)} d\theta \asymp \frac{1}{2\pi} \int_0^{2\pi} |1 - \overline{z_0}z|^{2} d\theta = (1 - \rho^2)^{-1},
\]

where \( A \asymp B \) means the quantities \( A \) and \( B \) are equivalent up to a multiplicative constant.

Now let

\[
g(z) = (1 - \rho^2)^{1/p(\theta_0)} f(z).
\]

For \( \rho \) near 1, we have that \( |1 - \overline{z_0}z| \asymp d \), where \( d \) is the distance from \( z \) to \( 1/z_0 \). It is also clear that \( d \asymp |\theta - \theta_0| + h \), where \( \theta = \arg(z) \) and \( \theta_0 = \arg(z_0) \). Thus \( g(z) \asymp h^{1/p_0} d^{-2/p(z)} \), where \( p_0 = p(e^{i\theta_0}) \).

It follows that \( |g(z)|^{p(z)} \leq C \) for \( |\theta - \theta_0| \geq Ch^{2/p_0} \). Recall that in a disc \( D \), if \( p(\cdot) \) is log-Hölder continuous then \( |D|^{(p_+D) - (p_-D)} \leq C \) for some constant \( C \), where \( |D| \) is the area of \( D \), and \( p_+,D \) and \( p_-,D \) are the maximum and minimum, respectively, of \( p(\cdot) \) restricted to \( D \). For \( |\theta - \theta_0| < Ch^{2/p_0} \) we thus have that

\[
h^{p(z)/p_0} \leq Ch^a
\]

by the log-Hölder continuity of \( p \). Thus \( \|g\|_{H,p(\cdot)} \leq C \). The Carleson measure condition applied to \( g \) implies that

\[
\int_S (1 - \rho^2)^{ap(z)/p_0} |(1 - \overline{z_0}z)|^{-2a} d\mu(z) \leq C
\]

But this implies that

\[
\mu(S)h^{-2a} \leq \int_S |(1 - \overline{z_0}z)|^{-2a} d\mu(z) \leq C(1 - \rho^2)^{-ap_+/s + p_-s}.
\]

But by log-Hölder continuity, \( (1 - \rho^2)^{-ap_+/s + p_-s} \) is bounded by \( Ch^{-a} \). Now use the fact that in \( S \) we have that \( (1 - \overline{z_0}z)^{-1} \geq Ch^{-1} \) (see [7, Proof of Theorem 9.3]) to see that \( \mu(S) \leq Ch^a \).

For the sufficiency let the \( f_j \) be defined as in Theorem 3.3 and assume without loss of generality that \( \|f\|_{H,p(\cdot)} \leq 1 \). Then \( \|f_j^{p(z)/2}\|_{H^2} \leq C \),
which implies that $f_j^{(\cdot)/2} \in L^{2\alpha}(\mu)$ and $\|f_j^{(\cdot)/2}\|_{L^{2\alpha}(\mu)} \leq C$. And since $|f_j(z)|^{p(z)/2} \leq C|f_j(z)\tilde{p}(z)/2|$, this shows that $\|f_j(z)\|_{L^{2\alpha}(\cdot)} \leq C$, so that $\|f(z)\|_{L^{2\alpha}(\cdot)} \leq C$. □

The following two corollaries follow immediately. The sharp form of the classical case of the first can be found in [17].

**Corollary 3.5.** Suppose that $p$ is an exponent with $0 < p_- \leq p_+ < \infty$. Also suppose that $p$ is log-Hölder continuous with bounded conjugate. If $f \in H^{p(\cdot)}$ then $f \in A^{2p(\cdot)}$ and $\|f(z)\|_{A^{2p(\cdot)}} \leq C\|f(z)\|_{H^{p(\cdot)}}$.

The classical Fejer-Riesz inequality (see [7, Theorem 3.13]) says that if $f \in H^p$ then
\[
\int_{-1}^1 |f(x)|^p \, dx \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.
\]
We have the following analogy.

**Corollary 3.6.** Let $f \in H^{p(\cdot)}$ where $p$ is log-Hölder continuous with bounded conjugate. Then
\[
\|f\|_{L^p((-1,1))} \leq C\|f\|_{H^{p(\cdot)}}
\]
for some constant $C$. Here we may define $p(-x) = p(-1)$ and $p(x) = p(1)$ for $x > 0$.

We now give a variable exponent form of another classical result, which has applications to composition operators. Let $f$ and $F$ be two analytic functions in the unit disc. Recall that $f$ is subordinate to $F$, written $f \prec F$, if $f(z) = F(\omega(z))$ where $\omega : \mathbb{D} \to \mathbb{D}$ and $|\omega(z)| \leq |z|$. If we wish to specify $\omega$, we will write $f \prec_\omega F$.

In this theorem, we take $p(\cdot)$ to be harmonic. The reason is that $p(\omega(\cdot))$ may not be log-Hölder continuous, so we need to specify its values inside the unit disc.

**Theorem 3.7** (Littlewood Subordination). Suppose that $f \prec_\omega F$ and that $F \in H^{p(\cdot)}$, where $p$ is harmonic and log-Hölder continuous with bounded conjugate. Then $f \in H^{p(\omega(\cdot))}$ and $\|f\|_{H^{p(\omega(\cdot))}} \leq C\|F\|_{H^{p(\cdot)}}$. Furthermore,
\[
M_{p(\omega(\cdot))}(r,f) \leq CM_{p(\cdot)}(r,F).
\]

**Proof.** Assume without loss of generality that $\|F\|_{H^{p(\cdot)}} = 1$. Decompose $F$ as in Theorem 3.3 so that
\[
F = B \sum_{j=1}^{n+1} F_j.
\]
Then

\[ f(z) = B(\omega(z)) \sum_{j=1}^{n+1} F_j(\omega(z)). \]

Recall that \(|B(\omega(z))| < 1\). Also \(\|F_j(\omega(\cdot))\|_{H,2}^{(\omega)} \leq \|F_j(\cdot)\|_{H,2}^{(\omega)}\) by the Littlewood subordination theorem for \(H^2\). Since \(F_j(\omega(z))\) has bounded argument, for \(C' = \exp(\|\omega\|_{\infty})\|\arg F_j\|_{\infty}/2\) we have that

\[ (1/C')|F_j(\omega(z))|^{p(\omega(z))} \leq |F_j(\omega(z))|^{p(\omega(z))} \leq C'|F_j(\omega(z))|^{p(\omega(z))}. \]

This implies that \(\|F_j(\omega(\cdot))\|_{H,p(\omega(\cdot))} \leq C\) for some constant \(C\). And thus

\[ \|f\|_{H,p(\omega(\cdot))} \leq \sum_{j=1}^{n+1} \|F_j(\omega(\cdot))\|_{H,p(\omega(\cdot))} \leq C \]

for a different constant \(C\). This implies in general that \(\|f\|_{H,p(\omega(\cdot))} \leq C\|F\|_{H,p(\omega(\cdot))}\), even if we drop the assumption that \(\|F\|_{H,p(\cdot)} = 1\). This proves the first part of the theorem. The second part follows from applying the first part to the functions \(f(rz)\) and \(F(rz)\).

\[ \square \]

**Corollary 3.8** (see [3 Theorem 3.1]). Let \(\mu\) be a finite positive measure on the closed unit interval and let \(p(\cdot)\) and \(\omega\) be as above. Then the composition operator \(C_\omega\) is bounded from \(A^p_{\mu(\cdot)}\) to \(A^p_{\mu(\omega(\cdot))}\), where \(C_\omega(f)(z) = f(\omega(z))\).

**Proof.** Suppose that \(f\) has unit \(A^p_{\mu(\cdot)}\) norm. For each \(0 \leq r < 1\) we have by the above theorem that

\[ \int_0^{2\pi} |f(\omega(re^{i\theta}))|^{p(\omega(re^{i\theta}))} d\theta \leq C. \]

Integration with respect to \(d\mu(r)\) now gives the result. \[ \square \]

**Corollary 3.9.** Let \(\phi : \mathbb{D} \to \mathbb{D}\), and let \(C_\phi\) be the composition operator defined by \((C_\phi f)(z) = f(\phi(z))\). Let \(p\) be a harmonic log-Hölder continuous exponent with bounded conjugate such that \(0 < p_- \leq p_+ < \infty\). Then \(C_\phi\) is bounded from \(A^p_{\alpha(\cdot)}\) to \(A^p_{\alpha(\phi(\cdot))}\) for \(-1 \leq \alpha < \infty\).

**Proof.** Suppose that \(\phi(0) = \lambda\). We may write \(\phi = \phi_\lambda \circ \omega\), where \(\omega : \mathbb{D} \to \mathbb{D}\) and \(\omega(0) = 0\), and

\[ \phi_\lambda(z) = \frac{\lambda - z}{1 - \lambda z}. \]

Note that \(\phi_\lambda\) is its own inverse. The previous corollary implies that \(C_\omega\) is bounded from \(H^{p(\phi_\lambda(\cdot))}\) to \(H^{p(\omega(\phi_\lambda(\cdot)))} = H^{p(\phi(\cdot)))\}. We will show that
$C_{\phi\lambda}$ is bounded from $H^p(\cdot)$ to $H^{p(\phi\lambda(\cdot))}$. Note that (see \[8, p. 38\])

$$1 - |\phi\lambda(z)|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{(1 - \overline{\lambda}z)^2}$$

and $|\phi'\lambda(z)|$ is bounded above and below in the unit disc. Thus

$$\int_{\mathbb{D}} |f(\phi\lambda(z))|^{p(\phi\lambda(z))}(1 - |z|^2)^\alpha dA(z) = \int_{\mathbb{D}} |f(\zeta)|^{p(\zeta)}(1 - |\phi\lambda(\zeta)|^2)^\alpha |\phi'\lambda(\zeta)|^2 dA(\zeta) \leq C \int_{\mathbb{D}} |f(\zeta)|^{p(\zeta)}(1 - |\zeta|^2)^\alpha dA(\zeta),$$

where we have used the substitution $\zeta = \phi\lambda(z)$. Here $C$ may depend on $\lambda$. Thus $C_{\phi\lambda}$ is bounded from $H^p(\cdot)$ to $H^{p(\phi\lambda(\cdot))}$. Since $C_{\phi} = C\omega C_{\phi\lambda}$, the result follows.

\[\Box\]

4. **Radial Exponents and $A^q$**

For the results of this section, we will assume that $p$ is a radial exponent, and that $0 < q \leq p(z) < \infty$ for some constant exponent $q$. We will find a necessary and sufficient condition that $A^q = A^{p(\cdot)}$ (as sets). Note that if $A^q = A^{p(\cdot)}$, then the two spaces have equivalent norms by the closed graph theorem. By Hölder’s inequality for variable exponent spaces we see that $A^{p(\cdot)} \subset A^q$, so the only question is whether $A^q \subset A^{p(\cdot)}$.

We will need the following theorem, which may be of interest in itself.

**Theorem 4.1.** Suppose that $f(x)$ is a nonnegative measurable function on $[0, 1]$. The following are equivalent:

1. For any nonnegative increasing $g$ in $L^1$, we have $\int_0^1 f(x)g(x) \, dx \leq C \int_0^1 g(x) \, dx$ for some constant $C$.

2. For all $0 \leq x < 1$ we have $\frac{1}{x} \int_{1-x}^1 f(t) \, dt \leq C$ for some constant $C$.

3. For all $0 \leq x < 1$ we have $\frac{2}{x} \int_{1-x}^{1-(x/2)} f(t) \, dt \leq C$ for some constant $C$.

Note that condition (2) may be thought of as saying that the maximal function of $f$ is finite at 1.
Proof. It is clear that (1) implies (2) by taking $g(x) = (1/x)\chi_{[1-x,1]}(x)$. It is also clear that (2) implies (3).

Now suppose that (3) holds. Let $g$ be a function as in (1). Then
\[
\int_{1-2^{-n}}^{1-2^{-n+1}} f(x)g(x) \, dx \leq g(1-2^{-n}) \int_{1-2^{-n+1}}^{1-2^{-n}} f(x) \, dx \leq C 2^{-n} g(1-2^{-n}).
\]
So
\[
\int_0^1 f(x)g(x) \, dx \leq C \sum_{n=1}^{\infty} 2^{-n} g(1-2^{-n}) \leq 2C \sum_{n=1}^{\infty} 2^{-n-1} g(1-2^{-n})
\]
\[
\leq 2C \int_{1/2}^1 g(x) \, dx
\]
which shows that (1) holds.\[\Box\]

For a radial exponent $p(\cdot)$ let
\[
M_{p(\cdot)}^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p(r)} \, d\theta.
\]
Then we have
\[
\int_D |f(z)|^{p(z)} \, dA(z) = \int_0^1 M_{p(r)}^p(r, f) \, r \, dr.
\]

We need the following lemma on the growth of integral means. Here $p$ can be thought of as a constant exponent, since the radius is fixed.

Lemma 4.2. Suppose that $\|f\|_{A^q} \leq 1$ and $q \leq p < \infty$. Then
\[
M^q_p(r, f) \leq M^q_q(r, f) \frac{1}{(1-r)^{2(p-q)/q}}.
\]

Proof. The hypothesis on $f$ implies that $M^q(1-r) = (1-r)^{-2/q}$ (see [16]). Thus
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leq \left( \sup_{0 \leq \theta < 2\pi} |f(re^{i\theta})|^{p-q} \right) \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta
\]
\[
\leq \frac{1}{(1-r)^{2(p-q)/q}} M^q_q(r, f).
\]
\[\Box\]

We now come to the main result.

Theorem 4.3. Let $q > 0$ be given and let $p(z)$ be a variable exponent on the disc with $q \leq p(z) \leq \text{ess sup } p < \infty$. Also assume that $p(z) = p(|z|)$ for all $z$. Then the following are equivalent.

(1) $A^q = A^{p(\cdot)}$. 

(2) For all \( f \) in \( A^q \) with \( \| f \|_{A^q} = 1 \), there is a constant \( K \) such that
\[
\int_0^1 M_{p(r)}^q(r, f) r \, dr \leq K \int_0^1 M_q^q(r, f) r \, dr
\]

(3) For all \( f \) in \( A^q \) there is a constant \( K \) such that
\[
\int_0^1 \frac{1}{(1 - r)^{2(p(r) - q)/q}} M_q^q(r, f) r \, dr \leq K \int_0^1 M_q^q(r, f) r \, dr.
\]

(4) There is a constant \( K \) such that for any increasing left continuous nonnegative function \( g \) we have
\[
\int_0^1 \frac{1}{(1 - r)^{2(p(r) - q)/q}} g(r) \, dr \leq K \int_0^1 g(r) \, dr.
\]

(5) There is a constant \( K \) such that for all \( x \) such that \( 0 < x \leq 1 \), it is the case that \( \frac{1}{x} \int_{1-x}^1 \frac{1}{(1 - r)^{2(p(r) - q)/q}} \leq K \).

(6) There is a constant \( K \) such that for all \( x \) such that \( 0 < x \leq 1 \), it is the case that \( \frac{2}{x} \int_{1-x}^{1-(x/2)} \frac{1}{(1 - r)^{2(p(r) - q)/q}} \leq K \).

(7) For some \( a \) such that \( aq > 2 \) there is a constant \( C' \) independent of \( \lambda \) so that
\[
\int_D |K_{\lambda,a,q}(z)|^{p(z)} \, dz \leq C',
\]
where
\[
K_{\lambda,a,q}(z) = \frac{(1 - |\lambda|^2)^a}{(1 - \lambda z)^a}.
\]

(8) The previous statement holds for all \( a \) with \( aq > 2 \) (where \( C' \) may depend on \( a \) and \( q \)).

Note that the following estimate holds:
\[
I_{\alpha,\beta}(z) := \int_D \frac{(1 - |w|^2)^\alpha}{|1 - z\overline{w}|^{2+\alpha+\beta}} \, dA(w) < C(1 - |z|^2)^{-\beta}
\]
for \(-1 < \alpha < \infty \) and \( \beta > 0 \) [11, Theorem 1.7]. This implies that the \( A^q \) norm of \( K_{\lambda,a,q} \) is bounded by a constant depending on \( q \) and \( a \) but not on \( \lambda \).

\textbf{Proof.} The fact that (2) implies (11) is by definition and the fact that \( A^{p(r)} \subset A^q \). The fact that (11) implies (2) is by the closed graph theorem. The fact that (3) implies (2) is from Lemma 4.2. It is clear that (4) implies (3) since integral means are increasing. The equivalence of (4), (5) and (6) follows from Theorem 4.1. The fact that (2) implies (8) is clear, and (8) trivially implies (7).
We will be done if we can show that (7) implies (6). Let $0 < r < 1$ and let $K_r = K_{r,a,q}$ for some $a$ such that $aq > 2$. Note that

$$M^p_p(\rho, K_r) \approx \frac{(1 - r^2)^{(aq-2)p/q}}{(1 - r^\rho)^{ap-1}}$$

by a well known estimate [7, p. 84, Lemma 3]. In fact, by Lemma 2.2, for $s > 1$ one has

$$(1 - r^2)^{1-s} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^s} d\theta \leq \frac{\Gamma(s-1)}{\Gamma(s/2)^2} (1 - r^2)^{1-s}.$$ 

Because $ap$ and $aq$ are bounded away from 1 and $\infty$, taking $s = ap$ or $s = aq$ in the above expression shows that the implied constants in (4.1) are independent of $r$ and $\rho$. So

$$M^p_p(\rho, K_r) \approx M^q_q(\rho, K_r) \frac{(1 - r^2)^{(aq-2)}[(p/q)-1]}{(1 - r^\rho)^{ap-aq}}$$

Let $r' = (r + 1)/2$. For $r \leq \rho \leq r'$ we have that

$$(1 - r^2)^{(aq-2)}[(p/q)-1] \leq (1 - r^2)^{(aq-2)}[(p/q)-1].$$

Thus we have that $1 - r^2 \approx 1 - r^\rho \approx 1 - \rho^2 \approx 1 - \rho$. Therefore

$$(1 - r^2)^{(aq-2)}[(p/q)-1] \approx \frac{1}{(1 - r^\rho)^{2(p-q)/q}}$$

where the implied constant is independent of $\rho$. We also have for $\rho \geq r$ that $M^q_q(\rho, K_r) \geq C(1 - r)^{-1}$ for some constant $C$. Thus

$$\frac{2}{1 - r} \int_r^{r'} \frac{1}{(1 - \rho)^{2[p(\rho)-q]/q}} d\rho \leq C \frac{2}{1 - r} \int_r^{r'} \frac{(1 - r^2)^{(aq-2)[p(\rho)/q]-1}}{(1 - r^\rho)^{ap(\rho)-aq}} d\rho.$$ 

But this is at most

$$C \frac{2}{1 - r} \int_r^{r'} \frac{(1 - r^2)^{(aq-2)[p(\rho)/q]-1}}{(1 - r^\rho)^{ap(\rho)-aq}} M^q_q(\rho, K_r)(1 - r) d\rho$$

$$\leq C \int_r^{r'} M^{p(\rho)}_{p(\rho)}(\rho, K_r) d\rho$$

$$\leq C \int_r^{r'} M^{p(\rho)}_{p(\rho)}(\rho, K_r) \rho d\rho \leq CK.$$ 

This finishes the proof. □

The following corollary is clear.
Corollary 4.4. Let \( p(\cdot) \) be as in Theorem 4.3. If \((1-r)^{q-p(r)} \) is bounded then \( A^\prime = A^{p(\cdot)} \).

This corollary implies that Theorem 2.1 holds in the special case of log-Hölder continuous radial exponents, but it also applies in cases where that theorem does not.

The following surprising corollary also follows.

Corollary 4.5. Let \( P > q \) be given. There is a radial function \( p(r) \geq q \) such that \( \limsup_{0 \leq r < 1} p(r) = P \) and \( A^{p(\cdot)} = A^q \).

Proof. Define \( p(r) \) so that for \( n \geq 1 \) we have

\[
p(r) = \begin{cases} 
P & \text{if } 1 - 2^{-n+1} \leq r < R_n \\
q & \text{if } R_n \leq r < 1 - 2^{-n}
\end{cases}
\]

where \( R_n \) is chosen close enough to \( 1 - 2^{-n+1} \) so that

\[
\int_{1-2^{-n+1}}^{R_n} (1-r)^{2(q-P)/q} \, dr < 1/2^n.
\]

Then

\[
\int_{1-2^{-n+1}}^1 (1-r)^{2(q-p(r))/q} \, dr < 2^{-n+1},
\]

so for any \( 0 < x \leq 1 \) we have

\[
\frac{1}{x} \int_{1-x}^1 \frac{1}{(1-r)^{2(p(r)-q)/q}} \, dr < 2.
\]

\( \square \)

The following corollary also follows.

Corollary 4.6. For \( 0 < q < \infty \) there is a radial exponent \( p(\cdot) \) such that \( p(r) \to q \) as \( r \to 1 \) but that \( A^{p(\cdot)} \neq A^q \).

Proof. Let \( p(r) = q + (q/2)[-\log(1-r)]^{-1/2} \). Then

\[
\frac{1}{(1-r)^{2(p(r)-q)/q}} = \exp \left[ \left( \log \frac{1}{1-r} \right)^{1/2} \right]
\]

which is an increasing function that is unbounded as \( r \to 1 \). Thus its average value on intervals of the form \([r, 1)\) is unbounded. \( \square \)
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