The Maxwell-BF theory in the presence of magnetic defects via the Julia-Toulouse approach

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We use the Julia-Toulouse approach for condensation of defects to show that the topological Abelian BF term can be induced due to a condensation of electric charges and discuss how to include external magnetic defects into the Maxwell-BF theory in a way consistent with the Elitzur’s theorem. We also discuss a new way of obtaining the Dirac’s veto, which is based on the so-called (Dirac) brane symmetry.

Keywords: Topological defects, monopoles, vortices, confinement, charge quantization, Dirac’s veto.

I. INTRODUCTION

Recently, the pure topological Abelian BF theory with adequate couplings with external fields has been attracting attention as an effective description of topological insulators in (2+1) and (3+1)-dimensions. This pure topological theory can be seen as a long distance limit of the Maxwell-BF (MBF) theory. The MBF theory, in turn, generalizes for an arbitrary number of spacetime dimensions the topological mass generation of the Maxwell-Chern-Simons (MCS) theory. In this Letter we make use of a generalization of the Julia-Toulouse Approach (JTA) for condensation of defects, as put forward in [12], and show that the topological Abelian BF term can be induced through a condensation of electric charges. We then discuss the subtle issue of how to include external probe monopoles into the MBF theory without violating the Elitzur’s theorem [2]. We also discuss the equivalence of the MBF theory with the hydrodynamic limit of the Abelian Higgs model given by Proca’s theory and the confinement of the external magnetic defects in the electric condensed regime.

The main idea involved in the JTA is to allow one to study gauge theories in the presence of defects that eventually condense. If we are not interested in the details of the condensation process we can ask ourselves if, having the knowledge of the model that describes the system before the condensation (i.e., in the diluted regime), we are able to determine the effective model describing the system in the condensed regime. The condensate of topological defects establishes a new medium in which

the defects constitute a continuous distribution in space. The low energy excitations of such a medium represent the new degrees of freedom of the condensed regime. Julia and Toulouse specified a prescription to identify these new degrees of freedom, knowing the model that describes the regime with diluted defects. This prescription does not deal with the dynamical reasons responsible for the condensation process: this is considered a separate issue, beyond the scope of the prescription that is concerned only with the properties of the new degrees of freedom once the condensation of topological defects has taken place. However, the work of Julia and Toulouse has taken place in the context of ordered systems in condensed matter and due to the possible non-linearity of the topological currents, the absence of relativistic symmetry and the need for the introduction of dissipative terms in this scenario, the construction of effective actions is a very complicated issue. Latter, the JTA was extended by Quevedo and Trugenberger who showed that in theories involving p-forms, which are very common in effective descriptions of string theories, these difficulties do not show up. They showed that in this context the prescription can be defined into a more precise form, which leads to the determination of the effective action describing the system in the condensed regime.

Based mainly on the ideas developed in [3, 4], we developed a general procedure to address the condensation of these defects, whose main features are a careful treatment of a local symmetry called as (Dirac) brane symmetry (which is independent of the usual gauge symmetry, as discussed by Kleinert [9]), which consists in the freedom of deforming the unphysical Dirac strings without any observable consequences, and the development of the JTA to be completely compatible with the Elitzur’s theorem.

As some of us have shown in [10, 12], it is possible to induce the topological Abelian Chern-Simons (CS) term in (2+1)-dimensions through a condensation of electric
charges via the JTA. The electric condensate described by the MCS theory (actually, a condensate of spins) gives a topological mass to the photons. The inclusion of magnetic instantons in this system is a subtle issue, since a naive substitution of the electromagnetic curvature tensor by a non-minimal coupling with the magnetic charges violates the Elitzur’s theorem through the breaking of the magnetic brane symmetry, spoiling the charge quantization in the electric condensed regime as a consequence; furthermore, the violation of the Elitzur’s theorem in this case also spoils the charge conservation when we include external electric charges in the system [11]. With the treatment given in [11] we were able to include external probe magnetic defects into the system [11].

Similar issues also arise in the MBF system in the presence of magnetic defects and in this Letter we proceed to generalize the earlier programs that we applied to the MCS system [10, 12] and show that the topological Abelian BF term can also be induced by an electric condensation. Hence, it is expected that if one includes external probe magnetic defects into the system, they should become confined due to the Meissner effect generated by the electric condensate. In fact, using the JTA we show how to include monopoles into the MBF theory without violating the Elitzur’s theorem, obtaining in the electric condensed regime the magnetic confinement, the charge quantization and the vortex density conservation in a consistent way. We also discuss a new way of obtaining the Dirac’s veto, which is based on the brane symmetry.

II. THE ABELIAN BF TERM AS AN ELECTRIC CONDENSATE

In this Letter we shall work in Minkowski spacetime $\mathcal{M}_{d+1}$ with metric determinant $g = -1$ and units of $c = \hbar = 1$. We are also going to use the differential forms notation (see appendix A).

Let us begin with the definition of the partition function of the Maxwell theory in the regime with diluted electric charges and external electric monopoles:

$$Z[A_1, J_{d-2}] = \int_{G.F.} D A_1 \exp \left\{ i \int_{\mathcal{M}_{d+1}} \left[ -\frac{1}{2} (d A_1 + \right.\left. -g \ast \chi_{d-1}) \wedge (d A_1 - g \ast \chi_{d-1}) - e A_1 \wedge \ast J_1) \right] \right\},$$

where $J_1 = \delta \Sigma_2$ is the topological electric current which localizes the world-line of the electric charge $e$, the physical boundary of the world-surface of the electric Dirac string (electric brane) localized by the Chern-Kernel $\chi_{d-1}$ and $J_{d-2} = \delta \chi_{d-1}$ is the external topological magnetic probe current which localizes the world-hypersurface of the magnetic charge $g$, the physical boundary of the world-hypersurface of the magnetic Dirac hyperstring (magnetic brane) localized by the Chern-Kernel $\chi_{d-1}$.

The acronym “G.F.” in the functional integral stands for some arbitrary “gauge fixing” procedure which must be implemented at some stage of the calculations.

The non-minimal coupling $(d A_1 - g \ast \chi_{d-1})$ describes the physical electromagnetic field in the presence of magnetic defects [1, 9, 17, 18]: the gauge field $A_1$ as well as its exterior derivative $d A_1$ are singular over the magnetic Dirac branes and the singularity of $d A_1$ is canceled out by $\ast \chi_{d-1}$, thus the non-minimal coupling is regular everywhere in spacetime. The minimal coupling of the gauge field $A_1$ with the prescribed electric current $\ast J_1$ vanishes almost everywhere, being non-trivial only along the world-line of the electric charge. Notice then, that if the magnetic Dirac branes touch the electric world-line at any point, at this point the minimal coupling becomes singular, since $A_1$ is singular over the magnetic Dirac branes. Hence, for the minimal coupling to be regular along the whole trajectory of the electric charge, the magnetic Dirac branes must not touch the electric world-line; this is the famous Dirac’s veto [1, 17].

As we show now, the Dirac’s veto as well as the charge quantization actually come together from the (Dirac) brane symmetry, which corresponds to the unobservability of the unphysical Dirac branes. Under an arbitrary local deformation of the magnetic hyperstrings attached to the monopoles:

$$\ast \chi_{d-1} \mapsto \ast \chi_{d-1} + d \ast \tau_d,$$

$$A_1 \mapsto A_1 + g \ast \tau_d,$$

where $\ast \tau_d$ is a delta-distribution that localizes the hypersurface spanned by the deformation of the magnetic Dirac brane keeping fixed its physical boundary corresponding to the monopole current, the non-minimal coupling $(d A_1 - g \ast \chi_{d-1})$ is trivially invariant, while the minimal coupling $e A_1 \wedge \ast J_1$ changes the Boltzmann factor in (1) by the following factor:

$$\exp \left\{ -ie g \int_{\mathcal{M}_{d+1}} \ast \tau_d \wedge \ast J_1 \right\} = \exp \left\{ -ie g N \right\},$$

where the integer $N$ corresponds to the intersection number between (the branes Poincare-dual to) $\ast \tau_d$ and $\ast J_1$. Hence, in order to the theory to be invariant under deformations of the unphysical Dirac strings, one must impose the following consistency condition [9]:

$$eg = 2\pi n, n \in \mathbb{Z},$$

which is the famous Dirac charge quantization [1, 17]. However, this is not the whole story. Suppose that initially the magnetic branes do not touch the electric world-lines. Then, as explained before, the minimal coupling is regular along the whole electric world-lines. If the deformed magnetic branes cross the electric world-lines,
then at the points where these crossings happen the minimal coupling is singular and hence the operation of deforming the Dirac branes is not a symmetry of the system, since we begin with a regular minimal coupling and end up with a singular one. Hence, for the brane transformation [2] to be a symmetry of [1], we must require that the magnetic branes never cross the electric world-lines at any point and this is the Dirac’s veto. Thus, the Dirac’s veto as well as the charge quantization follow as consequences of the brane symmetry [2].

Notice also that this is a new way of deriving the Dirac’s veto as well as the charge quantization (form (2) to be a symmetry of (1), we must require that the magnetic (Dirac) branes (in a space of (d + 1)-dimensions) and the electric world-line. However, in 3-dimensions it is easy to see that a closed curve by crossing this curve (at an even number of crossings) can cross the surface bordered by another closed curve (the electric world-line) can cross a d-volume without crossing its boundary (which is the union of the original magnetic Dirac brane and the deformed one). Hence, the intersection number in (3) can be non-trivial in general and still respect the Dirac’s veto, since what is involved in (3) is only the intersection number between the d-volume spanned by the deformation of the Dirac branes (in a space of (d + 1)-dimensions) and the electric world-line. However, the Dirac’s veto automatically excludes all kinds of deformations of the magnetic Dirac branes that implies in a crossing of the deformed magnetic Dirac branes with the electric world-lines. In this way, the magnetic (Dirac) brane symmetry corresponds actually to the transformation [3] subjected to the constraint of the Dirac’s veto, or in other words, the brane symmetry corresponds to the freedom of moving the unphysical magnetic Dirac strings through the geometric place not occupied by the electric charges.

In order to allow the electric charges to proliferate until they establish a continuum (the electric condensate), we add to the action present in the Boltzmann factor in the partition function [1] an activation term for the electric loops (which gives dynamics to the electric Dirac branes) such that it preserves the relevant symmetries of the system (P, T, Lorentz and the local gauge and brane symmetries) and gives the dominant contribution for the dynamics of the condensate in the infrared regime [8, 12]:

\[ S_{activation}[J_1] = \int_{M_{d+1}} \frac{(-1)^d}{2\Lambda^{d-1}} J_1 \wedge *J_1 \]

where \( \Lambda \) is a phenomenological mass scale associated to the electric condensation energy (see the discussion about mass dimensions at the end of this section).

Introducing also a sum over (the branes Poincare-dual to \( *\Sigma_2 \)) we define the partition function for the electric condensed regime as:

\[ Z_c[j_{d-2}] := \sum_{\{*\Sigma_2\}} \int_{G.F.} DA_1 \exp \left\{ i \int_{M_{d+1}} \left[ -\frac{1}{2}(dA_1 + - g * \chi_{d-1}) \wedge * (dA_1 - g * \chi_{d-1}) - eA_1 \wedge d * \Sigma_2 + \frac{(-1)^{d-1}}{2\Lambda^{d-1}} d * \Sigma_2 \wedge *d * \Sigma_2 \right] \right\}. \]  

(6)

We now introduce the identity \( 1 = \int D * P_2 \delta[*P_2 - *\Sigma_2] \) into (6) and rewrite the partition function for the condensed regime as:

\[ Z_c[j_{d-2}] = \int_{G.F.} DA_1 D * P_2 \left( \sum_{\{*\Sigma_2\}} \delta[*P_2 - *\Sigma_2] \right) \]

\[ \exp \left\{ i \int_{M_{d+1}} \left[ -\frac{1}{2}(dA_1 - g * \chi_{d-1}) \wedge * (dA_1 - g * \chi_{d-1}) + - eA_1 \wedge d * P_2 + \frac{(-1)^{d-1}}{2\Lambda^{d-1}} d * P_2 \wedge *d * P_2 \right] \right\}. \]  

(7)

We are now going to use the following version of the generalized Poisson’s identity (GPI) [8, 10]:

\[ \sum_{\{*\Sigma_2\}} \delta[*P_2 - *\Sigma_2] = \sum_{\{*\Omega_{d-1}\}} \exp \left\{ 2\pi i \int_{M_{d+1}} *\Omega_{d-1} \wedge *P_2 \right\}, \]  

(8)

where \( *\Omega_{d-1} \) is the brane Poisson-dual to \( *\Sigma_2 \). The GPI works as a geometric analogue of the Fourier transform: when the brane configurations on the right hand side of (8) proliferate (condense), the brane configurations on the right hand side become diluted and vice-versa (see appendix A of [10] for a detailed discussion and derivation of the GPI in the general case). Hence, the proliferation of the electric Dirac branes \( *\Sigma_2 \) (which is directly associated to the proliferation of the electric currents \( J_1 \) that live on their boundaries) is accompanied by the dilution of the branes of complementary dimension \( *\Omega_{d-1} \) and vice-versa, what tells us that the branes \( *\Omega_{d-1} \) must be interpreted as magnetic vortices over the electric condensate.

Using (8) and redefining \( *P_2 = -\Lambda^{(d-1)/2} B_{d-1} \), we rewrite (7) as:

\[ Z_c[j_{d-2}] = \sum_{\{*\Omega_{d-1}\}} \int_{G.F.} DA_1 DB_{d-1} \exp \left\{ i \int_{M_{d+1}} \left[ -\frac{1}{2}(dA_1 + - g * \chi_{d-1}) \wedge * (dA_1 - g * \chi_{d-1}) + m A_1 \wedge dB_{d-1} + \frac{(-1)^{d-1}}{2} dB_{d-1} \wedge *d B_{d-1} - 2\pi \Lambda^{(d-1)/2} B_{d-1} \wedge *\Omega_{d-1} \right] \right\}, \]  

(9)
where \( m := e A^{(d-1)/2} \) is the topological mass: the BF term is topological and its coefficient determines the physical massive poles of the propagators of the fields \( A_1 \) and \( B_{d-1} \) in the MBF theory (just like it happens in the MCS theory, where the coefficient of the CS term corresponds to the topological mass of the gauge boson).

Under application of the exterior derivative the equation of motion for the field describing the electric condensate, \( B_{d-1} \), implies the constraint \( d * \Omega_{d-1} = 0 \), which can be solved locally by the introduction of a current \(* \lambda_d\) according to \(* \Omega_{d-1} = d * \lambda_d\). This means that the magnetic vortices are closed and hence the magnetic vortex density is conserved: \( d \phi_{d-1} = 0 \).

Defining the shift (which contributes with a trivial Jacobian in the functional integral):

\[
A_1 \mapsto A_1 + \frac{2\pi}{e} * \lambda_d, \tag{10}
\]

we rewrite the partition function \( \mathcal{Z}_c \) in the following form:

\[
\mathcal{Z}_c[J_{d-2}] = \sum_{\{ \lambda_d \}} \int_{G.F.} \mathcal{D}A_1 \mathcal{D}B_{d-1} \exp \left\{ i \int_{\mathcal{M}_{d+1}} \left[ \frac{1}{2} (dA_1 + g * L_{d-1}) \right] \right\}, \tag{11}
\]

where we used the Dirac quantization condition \( \{ \} \) to define the magnetic brane invariant:

\[
* L_{d-1} := * \chi_{d-1} - d * \lambda_d. \tag{12}
\]

Notice that discarding the monopoles and the magnetic vortices and looking at the system in the very long distance regime (where we can also discard the Maxwell terms, which are of second order in the derivatives of the fields), we have from \( \{ \} \) the pure topological Abelian BF term, which arises here as a result of a condensation of electric charges.

We now return to the issue of the brane symmetry, but in the context of the electric condensed regime described by \( \{ \} \). As we stated before, the brane symmetry corresponds to the freedom of moving the unphysical Dirac strings through the geometric place not occupied by the electric charges. Notice that the “size” of this geometric place is not defined \textit{a priori}: it depends on the electric charge content of the system. In the electric condensed regime, the electric world-lines proliferated in such a way that they established a continuum, which is described here by the field of the electric condensate, \( B_{d-1} \). Hence, due to the Dirac’s veto, in the electric condensed regime the only place allowed for the magnetic Dirac strings is the interior of the magnetic vortices connected to the monopoles. In such a setup, which can be read off from \( \{ \} \) with non-trivial \(* \lambda_{d-1}\) (in regions where \(* \chi_{d-1} = 0 \) and \(* \Omega_{d-1} = d * \lambda_d \neq 0 \) we have from \( \{ \} \) the closed magnetic vortices disconnected from the monopoles), the flux inside the magnetic Dirac strings cancels out part of the flux inside the closed vortices, leaving only the magnetic open vortices with a pair of monopole-antimonopole in their ends. These open vortices correspond to the confining magnetic flux tubes.

Using the nilpotency of the exterior derivative, we easily rewrite the monopole currents in terms of the brane invariants as:

\[
\delta J_{d-2} = \delta \chi_{d-1} = \delta L_{d-1}. \tag{13}
\]

The partition function \( \{ \} \) (or, equivalently, the partition function \( \{ \} \)) defines the MBF theory in the presence of external monopoles in a way consistent with the Elitzur’s theorem. A naive try of introducing monopoles into the MBF system would consist in making the substitution \( dA_1 \mapsto (dA_1 - g * \chi_{d-1}) \) in the curvature tensor for the electric field \( A_1 \) in the MBF theory (which appears in the Maxwell term, \( dA_1 + dA_1 \), and also in the BF term, \( B_{d-1} + dA_1 \)), what would not take into account the ensemble of internal defects \(* \lambda_d\). In such a case, the brane symmetry \( \{ \} \) would be violated and the charge quantization would be spoiled in the condensed regime (basically, without taking the ensemble of internal vortices \(* \lambda_d\) into account, the unphysical Dirac strings “would become real, constituting the confining flux tubes”). Hence, it is clear that in the presence of external magnetic sources it is impossible to have a complete electric condensation, what would imply in the complete dilution of the ensemble of magnetic vortices \(* \lambda_d\), destroying the brane invariants and spoiling the brane symmetry. Such a restriction over the electric condensation when there are external magnetic sources embedded in the system is easily understandable in physical terms: the (topological) Meissner effect (generated by the topological mass \( m \)) expels the magnetic fields generated by the external monopoles of almost the whole space constituted by the electric condensate, however, these fields cannot simply vanish - they become confined in regions of the space with minimal volume corresponding to the magnetic confining flux tubes described by the brane invariants \(* L_{d-1}\).

The naive prescription referred above for the inclusion of monopoles into the MBF system also violates the magnetic vortex density conservation, what can be seen from the equations of motion obtained from the variation of the field \( B_{d-1} \) in such a naive effective action if we add a minimal coupling of the field \( B_{d-1} \) with an external vortex current. This problem does not happen in our approach, as discussed after equation \( \{ \} \).

It is instructive to close this section by pointing out the mass dimensions involved: \( [A_1] = [B_{d-1}] = (d-1)/2, [e] = (3-d)/2, [g] = (d-3)/2, [J_1] = d \) (point-deltas), \( \left[ \Sigma_2 \right] = d - 1 ((d - 1) \) point-deltas), \( [J_{d-2}] = 3 \) (3 point-deltas), \( [\chi_{d-1}] = [L_{d-1}] = 2 \) (2 point-deltas) and \( [\lambda_d] = 1 \) (1 point-delta).
III. THE EQUIVALENCE WITH THE HYDRODYNAMIC LIMIT OF THE ABELIAN HIGGS MODEL AND MONOPOLE CONFINEMENT

Let us specialize in this section to the case of (3+1)-dimensional Minkowski spacetime.

We have seen in the last section that it is not possible to completely dilute the magnetic vortices over the electric condensate in the presence of external monopoles. The best the Meißner effect can do is to dilute the closed vortices disconnected from the Dirac strings. These are exactly Dirac strings, giving origin to open vortices with a pair of monopole-antimonopole in their ends. These are exactly the brane invariants described in equation (12). Hence, after the complete dilution of the vortices disconnected from the Dirac strings, the partition function reads:

$$Z_c[j_1] = \sum_{\{L_2\}} \mathcal{D}A_1 \mathcal{D}B_2 \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ -\frac{1}{2} (dA_1 - g \ast L_2) \right. \right.$$

$$\left. \ast \left( dA_1 - g \ast L_2 \right) + mA_1 \wedge dB_2 + \frac{1}{2} dB_2 \wedge \ast dB_2 \right\},$$

(14)

where the sum over branes is now taken over all the possible configurations of the magnetic flux tubes.

Integrating out the Kalb-Ramond field $B_2$ in the partition function (14) using the Lorentz gauge condition, $\delta A_1 = 0$, we get:

$$Z_c[j_1] = \sum_{\{L_2\}} \mathcal{D}A_1 \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ -\frac{1}{2} (dA_1 - g \ast L_2) \right. \right.$$

$$\left. \ast \left( dA_1 - g \ast L_2 \right) + \frac{m}{2} \left( m \ast dA_1 \right) \wedge dA_1 \right\} \right.$$

$$= \sum_{\{L_2\}} \mathcal{D}A_1 \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ -\frac{1}{2} (dA_1 - g \ast L_2) \right. \right.$$

$$\left. \ast \left( dA_1 - g \ast L_2 \right) + \frac{m^2}{2} A_1 \wedge \ast A_1 \right\},$$

(15)

where we have fixed the Lorentz gauge condition in the mass term, $\delta A_1 = 0$.

Equation (15) defines the Proca’s theory in the presence of external monopoles in a way compatible with the Elitzur’s theorem. Again, this a non-trivial result, since the naive prescription of substituting the electromagnetic curvature tensor by a non-minimal coupling with the magnetic sources in the Proca’s theory violates the Elitzur’s theorem, rendering some inconsistent results. Notice also that if one makes the identification $\Lambda = \langle \phi_{Higgs} \rangle$ and discards the monopoles, then equation (15) corresponds exactly to the hydrodynamic limit of the Abelian Higgs model, what establishes its equivalence with the MBF theory. This is expected, since both theories describe the infrared regime of an electric condensate with the very same symmetries. Hence, the hydrodynamic limit of a relativistic superconductor can also be described by the MBF theory, as pointed out in [7,8].

Integrating out the gauge field $A_1$ in (15) using the Lorentz gauge condition, $\delta A_1 = 0$, we get:

$$Z_c[j_1] = \sum_{\{L_2\}} \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ -\frac{g^2}{2} \left( -\frac{g}{\Delta - m^2} \ast dL_2 \right) \right. \right.$$

$$\left. \ast \left( \frac{g^2}{2} L_2 \wedge \ast L_2 \right) \right\} \right.$$

$$= \sum_{\{L_2\}} \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ -\frac{g^2}{2} L_2 \wedge \ast \left( \frac{\delta d}{\Delta - m^2} - 1 \right) L_2 \right] \right\} \right.$$

$$= \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ \frac{g^2}{2} j_1 \wedge \frac{1}{\Delta - m^2} \ast j_1 \right] \right\} \right.$$

$$\sum_{\{L_2\}} \exp \left\{ i \int_{\mathcal{M}_{3+1}} \left[ \frac{m^2 g^2}{2} L_2 \wedge \frac{1}{\Delta - m^2} \ast L_2 \right] \right\},$$

(16)

where we used (13).

Using the metric diag(+1,−1,−1,−1), we have from (A15) that:

$$\Delta = -\partial_i^2 + \nabla^2 \Rightarrow -\Delta + m^2 = \partial_i^2 - \nabla^2 + m^2.$$ (17)

Moreover, switching to the index notation, we have from (A11) that:

$$j_1 \wedge * j_1 = j_\mu j^\mu d^4 x, \quad L_2 \wedge * L_2 = \frac{1}{2} L_{\mu \nu} L^{\mu \nu} d^4 x = -\frac{1}{2} \tilde{L}_{\mu \nu} \tilde{L}^{\mu \nu} d^4 x,$$ (18)

(19)

where we defined the dual tensor $\tilde{L}_{\mu \nu} := \frac{i}{2} \omega_{\mu \nu \alpha \beta} L^{\alpha \beta}$. Hence, in the index notation we write (16) as:

$$Z_c[j_\mu] = \exp \left\{ i \int_{\mathcal{M}_{3+1}} d^4 x \left[ -\frac{g^2}{2} j_\mu \partial_i^2 \frac{1}{\Delta - \nabla^2 + m^2} j^\mu \right] \right.$$

$$\sum_{\{L_{\mu \nu}\}} \exp \left\{ i \int_{\mathcal{M}_{3+1}} d^4 x \left[ -\frac{m^2 g^2}{4} \tilde{L}_{\mu \nu} \partial_i^2 \frac{1}{\Delta - \nabla^2 + m^2} \tilde{L}^{\mu \nu} \right] \right\}. \quad (20)$$

Considering a stationary monopole-antimonopole configuration and the asymptotic time regime $T \to \infty$, the dominant contribution for the energy of the system in the sum over configurations of the magnetic flux tubes in (20) is given by a linear magnetic flux tube (corresponding to the minimal distance between the monopole and the antimonopole) [12]. In this limit, the second term in (20) gives a magnetic confining potential that is linear in the monopole-antimonopole separation, while the first term gives a short-range Yukawa interaction [18].
IV. CONCLUDING REMARKS

In this Letter we used the Julia-Toulouse approach for condensation of defects to show that it is possible to induce the topological Abelian BF term through an electric condensation process. We then discussed how to include external magnetic defects into the Maxwell-BF theory without violating the Elitzur’s theorem, obtaining in the electric condensed regime the magnetic confinement, the charge quantization and the vortex density conservation in a consistent way. These are our mains results.

The (Dirac) brane symmetry is seen to imply both, the charge quantization and the Dirac’s veto, the latter being obtained here through a new argument based on the behavior of a minimal coupling under deformations of the Dirac strings. With the use of the generalized Poisson’s identity it is clear that the currents minimally coupled do the field of the electric condensate in the Maxwell-BF theory are naturally interpreted as magnetic vortices describing regions of the space where the electric condensate has not been established. Also, the Maxwell-BF theory is seen to be equivalent to the hydrodynamic limit of the Abelian Higgs model when a proper identification of parameters is made.

In the electric condensed regime the monopoles are confined and their presence prohibits the realization of a complete electric condensation in the system, since the magnetic open vortices with a pair of monopole-antimonopole in their ends cannot be undone by the Meissner effect, which can at most completely dilute the antimonopole in their ends cannot be undone by the Meissner effect, which can at most completely dilute the antimonopole in their ends cannot be undone by the Meissner effect, which can at most completely dilute the antimonopole in their ends cannot be undone by the Meissner effect.

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Appendix A: Differential forms notation

In this Letter we make use of the differential forms notation. So it is useful to point out some basic definitions and properties in this appendix.

A p-(differential)form is an element of the completely antisymmetric subspace of the cotangent space of a \( (D = d + 1) \)-dimensional differentiable manifold \( \mathcal{M}_{d+1} \) at a given point given by:

\[
A_p := \frac{1}{p!} A_{\mu_1 \cdots \mu_p} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p},
\]

where \( \{ dx^i, i = 1 \cdots D \} \) is a local basis for the cotangent space and \( dx^1 \wedge dx^2 := dx^1 \otimes dx^2 - dx^2 \otimes dx^1 \) is the wedge or exterior product of the base elements \( dx^1 \) and \( dx^2 \). It can be shown that:

\[
A_p \wedge B_{D-p} = (-1)^{p(D-p)} B_{D-p} \wedge A_p.
\]  

Let \( g := \det(g_{\mu\nu}) \) denote the determinant of the metric tensor \( g_{\mu\nu} \, dx^\mu \otimes dx^\nu \) of a (pseudo)Riemannian manifold \( \mathcal{M}_{d+1} \), with \( (g_{\mu\nu})^{-1} =: g^{\mu\nu} \). Notice that \( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} \) is the completely antisymmetric product of the base elements of the cotangent space with the highest rank that can be defined on a \( D \)-dimensional manifold. Hence, one identifies \( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} \) as a completely antisymmetric symbol with the same properties of \( \epsilon^{\mu_1 \cdots \mu_D} \), except that it has even indices. Then, we can define the totally antisymmetric version of \( \epsilon^{\mu_1 \cdots \mu_D} \) by \( \epsilon^{\mu_1 \cdots \mu_D} := g^{\mu_1 \nu_1} \cdots g^{\mu_D \nu_D} \epsilon_{\nu_1 \cdots \nu_D} = g^{-1} \epsilon_{\mu_1 \cdots \mu_D} \), and also the covariant version of \( \epsilon^{\mu_1 \cdots \mu_D} \) by \( \epsilon^{\mu_1 \cdots \mu_D} := g_{\mu_1 \nu_1} \cdots g_{\mu_D \nu_D} \epsilon^{\nu_1 \cdots \nu_D} = g \epsilon_{\mu_1 \cdots \mu_D} \). Thus, while \( \epsilon^{\mu_1 \cdots \mu_D} \) and \( \epsilon_{\mu_1 \cdots \mu_D} \) do not depend on the metric, \( \epsilon^{\mu_1 \cdots \mu_D} \) do depend on it. The contraction rules for these symbols are then given by:

\[
\epsilon^{\mu_1 \cdots \mu_{p+1} \cdots \mu_D} \epsilon_{\mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_D} = g^{\mu_p \nu_{p+1}} \cdots g^{\mu_D \nu_D} \delta_{\nu_{p+1} \cdots \nu_D}^{\mu_{p+1} \cdots \mu_D},
\]

\[
\epsilon^{\mu_1 \cdots \mu_{p+1} \cdots \mu_D} \epsilon_{\mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_D} = -g^{-1} \delta_{\nu_{p+1} \cdots \nu_D}^{\mu_{p+1} \cdots \mu_D}.
\]

where the indices between brackets are completely anti-symmetrized (for example, \( \partial_\mu A_\nu := \partial_\mu A_\nu - \partial_\nu A_\mu \)).

We can construct a \( D \)-form in terms of the symbol \( \epsilon_{\mu_1 \cdots \mu_D} \), which is known as the Levi-Civita form:

\[
\omega_D := \frac{1}{D!} \omega_{\mu_1 \cdots \mu_D} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} := \frac{\sqrt{|g|}}{D!} \epsilon_{\mu_1 \cdots \mu_D} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D},
\]

from which we immediately see that the Levi-Civita form corresponds to the volume form on a orientable (pseudo)Riemannian manifold \( \mathcal{M}_D \): \( \omega_D = \sqrt{|g|} \, dx^D \). We also see also see from [A1] that the contraction rule for the Levi-Civita tensor \( \omega_{\mu_1 \cdots \mu_D} := \sqrt{|g|} \epsilon_{\mu_1 \cdots \mu_D} \) is given by:

\[
\omega^{\mu_1 \cdots \mu_{p+1} \cdots \mu_D} \omega_{\mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_D} = \text{sgn}(g) \sqrt{|g|} \delta^{\mu_{p+1} \cdots \mu_D}_{\nu_{p+1} \cdots \nu_D},
\]

where \( \text{sgn}(g) := |g|/g \) is the sign function.

The Hodge dual operator, \( * \), maps a \( p \)-form into a \( (D-p) \)-form according to:

\[
* (dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) := \frac{1}{(D-p)!} \omega_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D} \, dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}.
\]
The following properties are easily demonstrated:

\[ *A_p = \frac{\sqrt{|g|}}{(D - p)!p!} A_{\mu_1 \cdots \mu_p} \epsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D} \]  
\[ dx^{q_{p+1}} \wedge \cdots \wedge dx^{q_D}, \]  
\[ A_p \wedge B_p = (D - P - p) ! \text{sgn}(g) A_p, \]  
\[ A_p \wedge B_p = (D - p - p)! A_p \wedge *B_p \]  
\[ A_p \wedge B_p = 1 / p! A_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p} \sqrt{|g|} d^D \! x. \]  

From (A8) we see that the Hodge dual of a 0-form (an ordinary function) is the volume form, hence, one usually defines the Hodge dual of the identity to be the volume form, \( * 1 \equiv \omega_D \).

The exterior derivative, \( d := \partial_{\mu} dx^\mu \), is a nilpotent operator which maps a \( p \)-form into a \((p - 1)\)-form:

\[ dA_p = \frac{1}{p!} \partial_{\mu} A_{\mu_1 \cdots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \]  

and satisfies the Leibniz rule:

\[ d(A_p \wedge B_q) = dA_p \wedge B_q + (1 - p)A_p \wedge dB_q. \]  

The adjoint of the exterior derivative is the exterior coderivative or codifferential, \( \delta \), a nilpotent operator that maps a \( p \)-form into a \((p - 1)\)-form according to:

\[ \delta A_p := (-1)^{p(D + p) + 1} \text{sgn}(g) * d * A_p. \]  

The Laplace-de Rham operator is defined by \( \Delta := (\delta d + d \delta) \). In an Euclidean or Minkowski spacetime with metric \( g_{\mu \nu} \) written in Cartesian coordinates, it can be shown that this operator reduces to minus the Laplacian or to minus the D’Alembertian, respectively:

\[ \Delta = -\eta_{\mu \nu} \partial^\mu \partial^\nu. \]  

The generalized Stokes theorem is given by:

\[ \int_{\partial M_p} dA_{p-1} = \int_{M_p} A_{p-1}, \]  

where \( \partial M_p \) is the boundary of \( M_p \).

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