Eigenvalue and eigenspace anholonomies in hierarchical systems

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Abstract – An adiabatic cycle of parameters in a quantum system can yield the quantum anholonomies, non-trivial evolution not just in the phase of states, but also in eigenvalues and eigenstates. Such exotic anholonomies imply that an adiabatic cycle rearranges eigenstates even without spectral degeneracy. We show that an arbitrarily large quantum circuit generated by recursive extension can also exhibit the eigenvalue and eigenspace anholonomies.

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A quantum system described by a parametric Hamiltonian goes through an adiabatic change connecting instantaneous eigenstates when it is subjected to a slow variation of parameters. Berry pointed out that this adiabatic change can yield non-trivial anholonomy in quantum phase when the trajectory of the parameter variation is closed to form a loop, for which he coined the term geometric phase [1].

Anholonomy under cyclic adiabatic parameter variation need not be limited to the phase of an eigenstate, but can involve the interchange of eigenvalues and eigenstates, since what is required after the return to the original parameter value is the identity of the whole set of eigensystems. Curiously, however, this possibility has been overlooked until recently, when examples of a quantum system exhibiting eigenvalue and eigenspace anholonomy have been found, first in the Hamiltonian spectra of a one-dimensional system with generalized point interaction [2], and then in the Floquet spectra of time-periodic kicked spin [3] (see also, fig. 1). In hindsight, the quantum anholonomy of Wilczek and Zee, in which eigenstates belonging to a single degenerate eigenvalue undergo mutual exchange and mixing [4], can be thought of as a precursor of this new type. Since then, further examples of a novel type of quantum anholonomy have been found both in Hamiltonian [5,6] and Floquet systems [7,8]. In one example, even the requirement of adiabaticity has been lifted, and the new type of anholonomy is shown to persist in a system with non-adiabatic cyclic parameter variation [6].

Lately, it has been shown that all quantum holonomies, namely, the Berry, the Wilczek-Zee and the novel-type ones showing eigenvalue exchange can be described by a unified formulation which is built upon the generalized Mead-Truhlar-Berry gauge connection [8]. The structure of the gauge connection in the anholonomy of the new type has been examined in terms of the theory of Abelian gerbes [9]. It is also clarified that the exceptional point, i.e. the singularity of gauge connection in complex plane, plays a crucial role in the new type of quantum anholonomy [10]. Applications to adiabatic manipulations of quantum states, including quantum computation [11], are promising [3,12], since the eigenvalue and eigenspace anholonomies are stable against perturbations that retain the periodicity of the parameter space [3,7].

Until now, there has been no known composite system that exhibits the new type of quantum anholonomy. In other words, all the conventional examples are “one-body”–type systems. This raises the question on whether there is any way to realize the eigenvalue and eigenspace anholonomies in quantum composite systems. This question is not only fundamental for the anholonomies, but also important for the application to quantum computation [12]: It is essential to deal with more than one qubit and to find a systematic way to generate multi-qubit systems starting from a single qubit.

In this letter, we propose a systematic way to construct quantum circuits that exhibit the eigenvalue and eigenspace anholonomies based upon multiple qubits. The obtained multi-qubit systems clearly show that a hierarchical structure exists in the resultant quantum circuits. We also examine the influence of the hierarchical
structure on the non-Abelian gauge connection associated with the eigenspace anholonomy.

Preliminaries — a one-body example. — We first explain the constituent building block of our many-body examples. This is a quantum circuit on a qubit. The eigenvalue and eigenspace anholonomies in the constituent are also explained using a gauge theoretical approach for the anholonomies [8].

We introduce our simplest example (fig. 1):

$$\hat{u}(\lambda) = e^{i\lambda \langle y|y^\prime \rangle Z},$$

(1)

where \( \hat{X} \equiv |0\rangle \langle 1| + h.c., \) \( \hat{Y} \equiv i|1\rangle \langle 0| + h.c., \) and \( \hat{Z} \equiv |0\rangle \langle 0| - |1\rangle \langle 1| \). \( |y\rangle \equiv ((0) - i|1\rangle)/\sqrt{2} \) satisfies \( \hat{Y}|y\rangle = -i|y\rangle \). The first factor in eq. (1) is the phase shift gate where the y-axis of the control qubit is chosen. We solve the eigenvalue problem of \( \hat{u}(\lambda) \). Let \( z(n; \lambda) \) denote the \( n \)-th eigenvalue of the unitary operator \( \hat{u}(\lambda) \) (\( n = 0, 1 \)). Since \( z(n; \lambda) \) is a unimodular complex number, we introduce a real number \( \theta(n; \lambda) \) that satisfies \( z(n; \lambda) = \exp\{i\theta(n; \lambda)\} \):

$$\theta(n; \lambda) = n\pi + \frac{\lambda}{2},$$

(2)

which is called an eigenvangle. The corresponding eigenvectors are

$$|0; \lambda\rangle \equiv \cos \frac{\lambda}{4} |0\rangle + \sin \frac{\lambda}{4} |1\rangle,$$

$$|1; \lambda\rangle \equiv \cos \frac{\lambda}{4} |1\rangle - \sin \frac{\lambda}{4} |0\rangle,$$

(3)

where the phases of the eigenvectors are chosen so as to simplify the following analysis.

\( \hat{u}(\lambda) \) is periodic with \( \lambda \), namely \( \hat{u}(\lambda + 2\pi) = \hat{u}(\lambda) \). Let \( C \) denote the closed path of \( \lambda \) from \( \lambda = 0 \) to \( 2\pi \). Both the spectral set \( \{e^{i\theta(n;\lambda)}\}_{n=0,1} \) and the set of projectors \( \{|n; \lambda\rangle \langle n; \lambda|\}_{n=0,1} \) obey the same periodicity of \( \hat{u}(\lambda) \). However, each eigenvalue \( e^{i\theta(n;\lambda)} \) and each projector \( |n; \lambda\rangle \langle n; \lambda| \) have a longer period. Namely, \( \hat{u}(\lambda) \) exhibits eigenvalue and eigenspace anholonomies under adiabatic parametric change along the closed path \( C \), where eigenvalues and eigenvectors are, respectively, exchanged.

We outline a theoretical framework for the eigenvalue and eigenspace anholonomies using the one-body example \( \hat{u}(\lambda) \). First, we examine the anholonomy in eigenangles. Using integers \( s(n) \) and \( r(n) \), the parametric dependence of an eigenangle is arranged as

$$\theta(n; \lambda + 2\pi) = \theta(s(n); \lambda) + 2\pi r(n).$$

(4)

Namely, the \( n \)-th eigenangle arrives at the \( s(n) \)-th eigenangle after a cycle \( C \). On the other hand, \( r(n) \) describes the transition between “Brillouin zones” of eigenvalues [9] (it is shown that \( \sum_{r} r(n) \) offers a topological character for the “Floquet operator” \( \hat{u}(\lambda) \) [13]). We introduce a \( 2 \times 2 \) matrix \( S(C) \) whose elements are defined as

$$\{S(C)\}_{n',n} \equiv \delta_{n',s(n)}. \quad (5)$$

From eq. (2), we have \( s(n) = \bar{n} \) and \( r(n) = n \), where \( \bar{n} = 1 \) and \( \bar{I} = 0 \). Accordingly, the permutation matrix \( \{S(C)\}_{n',n} \) describes a cycle whose length is 2.

Second, we examine the geometric phase factors associated with the adiabatic time evolution along the cycle \( C \). We introduce a holonomy matrix \( M(C) \) [8], whose \( (n',n) \)-th element is the overlapping integral \( \langle n'; \lambda|n; \lambda + 2\pi \rangle \), where \( |n; \lambda\rangle \) is supposed to satisfy the parallel transport condition [14] along \( C \) for the non-degenerate eigenspace, i.e., \( (n; \lambda)|\partial_{\lambda}|n; \lambda\rangle = 0 \). Due to the periodicity of \( \hat{u}(\lambda), \langle n'; \lambda + 2\pi |n; \lambda\rangle \) is either parallel or perpendicular to \( |n; \lambda\rangle \). In the former case, the diagonal elements of \( M(C) \) are Berry’s geometric phase factors [1]. The latter case implies the presence of the eigenvalue and eigenspace anholonomies and the off-diagonal elements of \( M(C) \) provide Manin-P Kristen’s gauge invariants [15]. It is worth remarking that the nodal-free geometric phase factors, which are the eigenvalues of \( M(C) \), offer the geometric phases for both cases [16]. An extension of the Fujikawa formalism for the geometric phase [17] offers a gauge covariant expression of \( M(C) \) [8]:

$$M(C) = \exp\left(-i \int_{C} A(\lambda) d\lambda\right) \exp\left(i \int_{C} A^{D}(\lambda) d\lambda\right), \quad (6)$$

where \( \exp_{e_{c}} \) and \( \exp_{p_{c}} \) are path-ordered and anti-path-ordered exponentials, respectively. Gauge connections \( \{A(\lambda)\}_{n',n} \equiv i(n'; \lambda)|\partial_{\lambda}|n; \lambda\rangle \) and \( A^{D}(\lambda) \equiv \delta_{n',n} \{A(\lambda)\}_{n',n} \) are also introduced. The second factor in the right-hand side of eq. (6) describes the time evolution in terms of adiabatic basis vectors \( |n; \lambda(t)\rangle \) [18] (see also, eq. (3) in ref. [8]). On the other hand, the first factor in eq. (6) is introduced so as to incorporate the multiple-valuedness of \( |n; \lambda\rangle \), which is nothing but the eigenspace anholonomy [8].

The gauge connection in our model (1), under the gauge specified by eq. (3), is

$$A(\lambda) = \frac{1}{4} Y, \quad \text{where} \quad [Y_{00}, Y_{01}, Y_{10}, Y_{11}] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (7)$$

which implies \( A^{D}(\lambda) = 0 \). Accordingly, we obtain \( M(C) = -iY \), or, equivalently

$$\{M(C)\}_{n',n} = \{S(C)\}_{n',n}(-1)^{n}. \quad (8)$$

Namely, \( M(C) \) is composed by two parts, the permutation matrix \( S(C) \) and the phase factors \((-1)^{n}\).
Recursive construction of an $N$-qubit circuit. –

A crucial ingredient of our $N$-body extension of $\hat{u}(\lambda)$ is the following “super-operator” $\hat{D}$:

$$\hat{D}[\hat{U}] \equiv \hat{C}^{y}[\hat{U}] (\hat{Z} \otimes 1)$$ \hspace{1cm} (9)

where

$$\hat{C}^{y}[\hat{U}] \equiv (\hat{1} - |y\rangle\langle y|) \otimes 1 + |y\rangle\langle y| \otimes \hat{U}$$ \hspace{1cm} (10)

is a controlled unitary gate, where the “axis” of the control bit is chosen to be in the “$y$-direction”.

In order to expose the one-body quantum circuit $\hat{u}(\lambda)$ hidden in $\hat{D}[\cdot]$, we examine $\hat{D}[\cdot]$ with the global phase gate $e^{i\lambda}1_{A}$ for an ancilla, where $1_{A}$ is the identity operator of the ancilla: $\hat{D}[e^{i\lambda}1_{A}] = \hat{u}(\lambda) \otimes 1_{A}$. Namely, we may say that $\hat{u}(\lambda)$ is an extension of the global phase gate on an ancilla with $\hat{D}[\cdot]$. This interpretation suggests the following $N$-body extension.

A family of quantum circuits $\hat{U}^{(N)}(\lambda)$ on $N$-qubits is recursively defined in the following. For $N = 1$, we set $\hat{U}^{(1)}(\lambda) = \hat{u}(\lambda)$, which is examined above. For $N > 1$, we compose $\hat{U}^{(N)}(\lambda)$ from a $(N - 1)$-qubit circuit $\hat{U}^{(N-1)}(\lambda)$, adding a qubit (fig. 2):

$$\hat{U}^{(N)}(\lambda) \equiv \hat{D}[\hat{U}^{(N-1)}(\lambda)].$$ \hspace{1cm} (11)

We depict the quantum circuit with $N = 3$ in fig. 3. It is worth pointing out the scalability of $\hat{U}^{(N)}(\lambda)$, i.e., only poly($N$) quantum gates are required to construct $\hat{U}^{(N)}(\lambda)$.

We will solve the eigenvalue problem of $\hat{U}^{(N)}(\lambda)$. A complete set of quantum numbers for $\hat{U}^{(N)}(\lambda)$ is \((n_{N}, n_{N-1}, \ldots, n_{1})\), where \(n_{j} \in \{0, 1\}\). Let $|n_{N}, n_{N-1}, \ldots, n_{1}; \lambda\rangle$ and $\theta^{(N)}(n_{N}, n_{N-1}, \ldots, n_{1}; \lambda)$ denote an eigenvector and the corresponding eigenangle of $\hat{U}^{(N)}(\lambda)$, respectively. We will obtain recursion relations for eigenvalues and eigenangles. Suppose that we have an eigenvector $|\Psi\rangle$ and the corresponding eigenangle $\Theta$ of the smaller quantum circuit $\hat{U}^{(N-1)}(\lambda)$. From eq. (11), $|n_{N}; \Theta\rangle \otimes |\Psi\rangle$ is an eigenvector of $\hat{U}^{(N)}(\lambda)$, where $n_{N} \in \{0, 1\}$. The corresponding eigenangle is $\theta^{(1)}(n_{N}; \Theta)$. Accordingly the recursion relations are

$$\theta^{(N)}(n_{N}, n_{N-1}, \ldots, n_{1}; \lambda) =$$

$$\theta^{(1)}(n_{N}; \theta^{(N-1)}(n_{N-1}, \ldots, n_{1}; \lambda))$$ \hspace{1cm} (12)

and

$$|n_{N}, n_{N-1}, \ldots, n_{1}; \lambda\rangle =$$

$$|n_{N}; \theta^{(N-1)}(n_{N-1}, \ldots, n_{1}; \lambda)\rangle \otimes |n_{N-1}, \ldots, n_{1}; \lambda\rangle. \hspace{1cm} (13)$$

Hence, we obtain the eigenangle

$$\theta^{(N)}(n_{N}, n_{N-1}, \ldots, n_{1}; \lambda) =$$

$$\frac{2\pi}{2N} \left\{ m_{N}(n_{N}, n_{N-1}, \ldots, n_{1}) + \frac{\lambda}{2\pi} \right\}$$, \hspace{1cm} (14)

where

$$m_{N}(n_{N}, n_{N-1}, \ldots, n_{1}) \equiv \sum_{j=1}^{N} 2^{j-1} n_{j}. \hspace{1cm} (15)$$

This is called a principal quantum number. Note that $\hat{U}^{(N)}(\lambda)$ has no spectral degeneracy. Equation (15) shows that $n_{N}, \ldots, n_{1}$ are the coefficients of the binary expansion of $m_{N}$. Because of the simplicity of the correspondence between $m_{N}$ and $(n_{N}, \ldots, n_{1})$, we will identify them in the following. An eigenvector of $\hat{U}^{(N)}(\lambda)$ is

$$|n_{N}, n_{N-1}, \ldots, n_{1}; \lambda\rangle =$$

$$|n_{N}; \theta^{(N-1)}(m_{N-1}; \lambda)\rangle \otimes \ldots \otimes |n_{2}; \theta^{(1)}(m_{1}; \lambda)\rangle \otimes |n_{1}; \lambda\rangle.$$ \hspace{1cm} (16)

Analysis of exotic anholonomies. – We examine the parametric dependences of eigenangles and eigenprojectors of $\hat{U}^{(N)}(\lambda)$, along the cycle $C$, i.e., $\lambda \rightarrow \lambda + 2\pi$. Let $(n_{N}, \ldots, n_{1})$, or equivalently, $m_{N}$, be the set of quantum numbers of the initial eigenstate. After the completion of the parametric variation along the cycle $C$, the quantum numbers are rearranged, which shows that the anholonomies take place. Let $\tilde{s}^{(N)}(m_{N}; C) \in \{0, 1\}$ denote the value of the $j$-th quantum number ($1 \leq j \leq N$).
We introduce a $2^N \times 2^N$ permutation matrix $S^{(N)}(C)$ whose elements are determined by $s^{(N)}_{ij}(m_N; C)$:

$$\{S^{(N)}(C)\}_{m'_N, m_N} = \prod_{j=1}^{N} \delta_{m'_j, m_j}(m_N; C),$$

which represents an anholonomy in quantum numbers.

Now we show that $S^{(N)}(C)$ can be obtained from the parametric dependence of eigenvalues:

$$\theta^{(N)}(m_N; \lambda + 2\pi) = \theta^{(N)}(s^{(N)}(m_N; C); \lambda) + 2\pi r^{(N)}(m_N; C),$$

where we abbreviate the collection of the quantum numbers $s^{(N)}(m_N; C), \ldots, s^{(N)}_{j}(m_N; C)$ as $s^{(N)}(m_N; C)$. An integer $r^{(N)}(m_N; C)$ is a “winding number” of $\theta^{(N)}(m_N; \lambda)$ in the periodic space of eigenangle.

In our model (eq. (11)), $s^{(N)}_N$ and $r^{(N)}$ can be obtained through recursion relations. Here we show only the solutions:

$$s^{(N)}_N(n_N, \ldots, n_1) = \begin{cases} \pi n_N, & \text{for } n_{N-1} \cdots n_1 = 1, \\ n_N, & \text{otherwise} \end{cases}$$

and

$$r^{(N)}(n_N, \ldots, n_1) = n_N \cdots n_1,$$

for $N > 1$, and $s^{(N)}_1(n_1) = \pi n_1$ and $r^{(N)}(n_1) = n_1$. Hence the permutation matrix $S^{(N)}$ contains only a cycle whose length is $2^N$. The itinerary of $n_N, \ldots, n_1$ with $N = 3$, for example, is the following:

$$000 \rightarrow 001 \rightarrow 010 \rightarrow 011 \rightarrow 100 \rightarrow 101 \rightarrow 110 \rightarrow 111 \rightarrow 000.$$

In terms of the principal quantum number $m_N$, this itinerary can be described in a simple way: When $m_N < 2^N - 1$, at every parametric variation of $\lambda$ by $2\pi$, $m_N$ increases by unity. Otherwise, $m_N$ becomes zero. Hence, after encircling the path $C$ by $2^N$ times, $m_N$ reverts to the initial point.

**Adiabatic geometric phase.** – We examine the holonomy matrix

$$\{M^{(N)}(C)\}_{m'_N, m_N} = \langle m'_N; \lambda|\psi^{(C)}(m_N; \lambda)\rangle,$$

where $|\psi^{(C)}(m_N; \lambda)\rangle$ is obtained by the parallel transport of $|m_N; \lambda\rangle$ along the path $C$. $M^{(N)}(C)$ incorporates two aspects of the adiabatic cycle along $C$. One is the change in the quantum numbers, which is described by $S^{(N)}(C)$ (eq. (17)). The other involves the geometric phase, in a generalized sense [15,19]. Let $\sigma^{(N)}(m_N)$ be the phase factor associated with the eigenspace initially labeled by $m_N$. These two factors comprise the holonomy matrix $\{M^{(N)}(C)\}_{m'_N, m_N} = \{S^{(N)}(C)\}_{m'_N, m_N} \sigma^{(N)}(m_N; C)$. We will obtain the phase factor $\sigma^{(N)}(m_N; C)$ using a gauge covariant expression of $M^{(N)}(C)$ (eq. (6)). Here the non-Abelian gauge connection is

$$\{A^{(N)}(\lambda)\}_{m'_N, m_N} \equiv \langle m'_N; \lambda|\partial_\lambda m_N; \lambda\rangle.$$  

Because of the absence of the spectral degeneracy in $U^{(N)}(\lambda)$, the diagonal part of $A^{(N)}(\lambda)$ is

$$\{A^{D(N)}(\lambda)\}_{m'_N, m_N} \equiv \delta_{m'_N, m_N} \{A^{(N)}(\lambda)\}_{m_N, m_N}.$$

We will obtain $A^{(N)}(\lambda)$ through recursion relations. Note that the results for the case $N = 1$ are already obtained above. For $N > 1$, the Leibniz rule in the derivative of eq. (13) suggests the decomposition of the gauge connection

$$A^{(N)}(\lambda) = A^{(N)}_H(\lambda) + A^{(N)}_L(\lambda),$$

where

$$\{A^{(N)}_H(\lambda)\}_{m'_N, m_N} \equiv \{A^{(N)}(\lambda)\}_{m'_N, m_N},$$

and

$$\{A^{(N)}_L(\lambda)\}_{m'_N, m_N} \equiv \langle m'_N; \lambda|\theta^{(N)}(m_N - 1; \lambda)|m_N; \lambda\rangle \prod_{j=1}^{N-1} \delta_{m'_j, m_j}.$$  

We also have a similar recursion relation for “diagonal” gauge connection $A^{D(N)}_L(\lambda)$. As we have already chosen the gauge that satisfies $A^{D(1)}(\lambda) = 0$ in the one-body problem, we obtain $A^{D(N)}_L(\lambda) = 0$ for all $N$ from the recursion relations. It is straightforward to obtain

$$A^{(N)}_L(\lambda) = e^{i\pi Y \otimes J^{(N)}_D} \times e^{-i\pi Y \otimes J^{(N)}_D},$$

and

$$A^{(N)}_H(\lambda) = \frac{1}{2^{N+1}} Y \otimes 1^{(N-1)},$$

where $\{J^{(N)}_D\}_{m'_N, m_N} \equiv m_N \delta_{m'_N, m_N}$. Because $A^{(N)}_H(\lambda)$ is independent of $\lambda$, $A^{(N)}_L(\lambda)$ as well as $A^{(N)}(\lambda)$ are also independent of $\lambda$. Hence, in our model, $A^{(N)}(\lambda)$ is independent of $\lambda$. Furthermore, $A^{(N)}_H(\lambda)$ commutes with $A^{(N)}_L(\lambda)$. Now it is straightforward to obtain

$$\{M^{(N)}(C)\}_{m'_N, m_N} = \{S^{(N)}(C)\}_{m'_N, m_N} (-1)^{r^{(N)}(m_N)} \times \sigma^{(N)}(m_N; C),$$

which implies a recursion relation for $\sigma^{(N)}$:

$$\sigma^{(N)}(m_N; C) = (-1)^{r^{(N)}(m_N)} \sigma^{(N-1)}(m_{N-1}; C).$$
Manini-Pistolesi gauge invariant. – The holonomy matrix $M^{(N)}(C)$ is a gauge covariant quantity, from which we can construct a gauge invariant Manini-Pistolesi phase factor [15]

$$\gamma^{(N)}_{\text{MP}}(C) = \prod_{n_N=0}^{1} \cdots \prod_{n_1=0}^{1} \sigma^{(N)}(n_N, \ldots, n_1; C),$$ (30)

which turns out to be a sole non-trivial invariant phase factor of the system. We explain how eq. (30) is obtained through the $2^N$ repetitions of the adiabatic cycle $C$. Assume we start from an eigenstate specified by a set of quantum numbers $(n_N, \ldots, n_1)$. Let $(n_N(j), \ldots, n_1(j))$ denote the value of the quantum numbers after the completion of the $j$-th cycle. At $j = 2^N$, $(n_N(j), \ldots, n_1(j))$ returns to the initial point. From $M^{(N)}(C)$, $\gamma^{(N)}_{\text{MP}}(C)$ is defined as

$$\gamma^{(N)}_{\text{MP}}(C) = 2^{N-1} \prod_{j=1}^{2^N} \{M(C)\}_{(n_N(j+1), \ldots, n_1(j+1));(n_N(j), \ldots, n_1(j))},$$ (31)

Because $(n_N(j), \ldots, n_1(j))$ experiences all the combinations of $\{0, 1\}^N$, we have

$$\gamma^{(N)}_{\text{MP}}(C) = \prod_{n_N=0}^{1} \cdots \prod_{n_1=0}^{1} \{M(C)\}_{s(n_N, \ldots, n_1), (n_N, \ldots, n_1)},$$

which implies eq. (30). The meaning of $\gamma^{(N)}_{\text{MP}}(C)$ is straightforward; it is the Berry phase obtained after $2^N$ repetitions of the loop $C$ in the parameter space. In our model, we have

$$\gamma^{(N)}_{\text{MP}}(C) = -1$$

for arbitrary $N > 0$, which suggests that the anholonomy found in the model is a “halfway evolution” to Longuet-Higgins anholonomy [20].

Summary and discussion. – We have introduced a family of multi-qubit systems that display the eigenvalue and the eigenspace anholonomies. The systems considered here can be regarded as a quantum map under a rank-one perturbation [7,21]. As this family is composed in a recursive way, their eigenvalues and eigenvectors exhibit a hierarchical structure. Furthermore, since these examples have explicit analytic expressions of eigenvalues and eigenvectors, we have examined the resultant eigenspace anholonomy using the extended Fujikawa formalism. The structure of the non-Abelian gauge connection also reflects the recursive construction. The path-ordered exponential of the gauge connection has an explicit analytical expression, which allows us to examine the holonomy matrix throughout.

Our quantum circuit $\hat{U}^{(N)}(\lambda)$ can be utilized as a reference to construct another quantum circuit that retains the eigenvalue and the eigenspace anholonomies. More precisely, while we vary $\hat{U}^{(N)}(\lambda)$ keeping both the periodicity in $\lambda$ and the unitarity, the anholonomies survive until we encounter a spectral degeneracy [3,7]. Because $\hat{U}^{(N)}(\lambda)$ has no spectral degeneracy as shown in eq. (14), there are a lot of quantum circuits that exhibit the anholonomies around $\hat{U}^{(N)}(\lambda)$.

On the other hand, the eigenspace and eigenvalue anholonomies are generally fragile against the increment of the degrees of freedom. For example, when two qubits, each of which exhibits the anholonomies, are composed without any interaction, i.e., $\hat{u}(\lambda) \otimes \hat{1} + \hat{1} \otimes \hat{u}(\lambda)$, the anholonomies do not survive in the resultant composite system. Hence the realization of eigenspace and eigenvalue anholonomies in a quantum composite system is not straightforward. This also explains why the eigenspace and eigenvalue anholonomies are rather uncommon.

Nevertheless, our recursive construction offers a way to realize the anholonomies in quantum composite systems including systems with large degree of freedom. It is also possible to extend our method to construct systems which have different topological features (i.e., $S^{(N)}(C)$ in eq. (17)) from the present one. This will be reported in a future publication [22]. Also, such a recursive construction might be useful to construct many-body systems with phase anholonomy. Because quantum circuits allow such a recursive construction in a straightforward manner, this suggests that quantum circuits offer an interesting playground for many-body quantum anholonomies.

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