NONCOMMUTATIVE WEIGHTED INDIVIDUAL ERGODIC THEOREMS WITH CONTINUOUS TIME

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Abstract. We show that ergodic flows in noncommutative fully symmetric spaces (associated with a semifinite von Neumann algebra) generated by continuous semigroups of positive Dunford-Schwartz operators and modulated by bounded Besicovitch almost periodic functions converge almost uniformly. The corresponding local ergodic theorem is also discussed.

1. Introduction

In the classical ergodic theory, Besicovitch-weighted individual ergodic theorems have been studied quite extensively (see, for example, [33, 2, 3, 25, 11]).

In the noncommutative setting, first individual ergodic theorem with bounded Besicovitch weights in a von Neumann algebra was obtained in [19]. Later, in [7], a similar result concerning the so-called bilaterally almost uniform convergence (in Egorov’s sense) was established in the $L^1$-space associated with a semifinite von Neumann algebra. In [29], utilizing the approach of [20], a multi-parameter version of [7, Theorem 4.6] was proved for every noncommutative $L^p$-space with $1 < p < \infty$. Recently, almost uniform convergence (in Egorov’s sense) of Besicovitch-weighted ergodic averages in fully symmetric spaces of measurable operators was established in [8, Theorem 4.7 and Sec. 6].

Note that all of the above were concerned with bounded Besicovitch sequences - generated by the well-studied Besicovitch almost periodic functions [11] - leaving open the problem what happens in the case of actions of continuous semigroups, when one has to turn to Bisicovitch almost periodic functions. To this end, a noncommutative local ergodic Besicovitch-weighted theorem was first considered in [30]; see remarks following Theorem 3.2.

Let $\{T_t\}_{t \geq 0}$ be a (continuous) semigroup of positive Dunford-Schwartz operators in a noncommutative $(L^1 + L^\infty)$-space. Our goal is to show that the corresponding ergodic averages modulated by a bounded Besicovitch (zero-Besicovitch) almost periodic function $\beta(t)$, $t \geq 0$, in a noncommutative fully symmetric space converge almost uniformly as $t \to \infty$ (Theorem 4.2) (respectively, as $t \to 0$ (Theorem 4.3)). Since the results appear to be new for the commutative setting, a relevant discussion is given in the last section of the article.

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1
2. Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of projections in $\mathcal{M}$. If $1$ is the identity of $\mathcal{M}$ and $e \in \mathcal{P}(\mathcal{M})$, we write $e^\perp = 1 - e$. Also, if $\mathcal{M}$ acts in a Hilbert space $\mathcal{H}$, then, given $\{ e_\alpha \}_{\alpha \in \Lambda} \subset \mathcal{P}(\mathcal{M})$, denote $\bigwedge e_\alpha$ the projection on the subspace $\bigcap_{\alpha \in \Lambda} e_\alpha \mathcal{H}$. Note that $\bigwedge e_\alpha \in \mathcal{P}(\mathcal{M})$ and

$$\tau \left( \bigwedge_{n=1}^\infty e_n \right) \leq \sum_{n=1}^\infty \tau (e_n^\perp)$$

for any sequence $\{ e_n \} \subset \mathcal{P}(\mathcal{M})$.

Let $L^0 = L^0(\mathcal{M}, \tau)$ be the *-algebra of $\tau$-measurable operators affiliated with $\mathcal{M}$, and let $\| \cdot \|_\infty$ be the uniform norm in $\mathcal{M}$. Equipped with the measure topology given by the system

$$\mathcal{N}(\epsilon, \delta) = \{ x \in L^0 : \|xe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon \},$$

$\epsilon > 0, \delta > 0$, of (closed) neighborhoods of zero, $L^0$ is a complete metrizable topological *-algebra 31.

Let $L^p = L^p(\mathcal{M}, \tau), 1 \leq p \leq \infty, (L^\infty = \mathcal{M})$ be the noncommutative $L^p$-space associated with $(\mathcal{M}, \tau)$, and let $\| \cdot \|_p$ be the standard norm in the space $L^p, 1 \leq p < \infty$. For detailed accounts on the noncommutative $L^p$-spaces, $p \in \{ 0 \} \cup [1, \infty)$, see 33, 31, 30, 32.

A net $\{ x_\alpha \} \subset L^0$ is said to converge almost uniformly (a.u.) (bilaterally almost uniformly (b.a.u.)) to $x \in L^0$ if for any given $\epsilon > 0$ there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e^\perp) \leq \epsilon$ and $\| (x - x_\alpha)e \|_\infty \to 0$ (respectively, $\| e(x - x_\alpha)e \|_\infty \to 0$).

It is well-known that if a sequence in $L^0$ converges in measure, then it has a subsequence converging a.u. Besides, a sequence in $L^0$ converging in $L^p$ for some $1 \leq p \leq \infty$ also converges in measure.

A linear operator $T : L^1 + \mathcal{M} \to L^1 + \mathcal{M}$ is called a Dunford-Schwartz operator (writing $T \in DS$) if

$$\| T(x) \|_1 \leq \| x \|_1 \quad \forall x \in L^1 \quad \text{and} \quad \| T(x) \|_\infty \leq \| x \|_\infty \quad \forall x \in L^\infty.$$ 

If a Dunford-Schwartz operator $T$ is positive, that is, $T(x) \geq 0$ whenever $x \geq 0$, we will write $T \in DS^+$. Note that positive absolute contractions in $L^1$, considered in 37 and then in 7, 29, 30, can be uniquely extended to positive Dunford-Schwartz operators - see 4.

Given $x \in L^1 + \mathcal{M}$ and $T \in DS$, denote

$$A_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x).$$

The following fundamental result is due to Yeadon 37, Theorem 1).

**Theorem 2.1.** Let $T \in DS^+$. Then for every $x \in L^1_+$ and $\lambda > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e^\perp) \leq \frac{\| x \|_1}{\lambda} \quad \text{and} \quad \sup_n \| e A_n(x) e \|_\infty \leq \lambda.$$
Let a semigroup \( \{T_t\}_{t \geq 0} \subset DS \) be strongly continuous in \( L^1 \), that is,\[
\|T_t(x) - T_{t_0}(x)\|_1 \to 0 \quad \text{whenever} \quad t \to t_0 \quad \text{for all} \quad x \in L^1.
\]
Then, given \( g \) it is possible to find \( \lambda > 0 \) implying, that \( \lambda > 0 \) by Theorem 2.1, given \( A \) Lebesgue measurable function \( C \) Let \( e \)

\[\lambda > 0 \]

\[\text{Here is a continuous extension of Theorem 2.1 (cf. [20, Remark 4.7]):}\]

**Theorem 2.2.** If \( \{T_t\}_{t \geq 0} \subset DS^+ \) is strongly continuous in \( L^1 \), then, given \( x \in L^1_\lambda \) and \( \lambda > 0 \), there exists \( e \in \mathcal{P}(\mathcal{M}) \) such that \[
\tau(e^+) \leq \frac{2\|x\|_1}{\lambda} \quad \text{and} \quad \sup_{t > 0} \|eA_t(x)e\|_\infty \leq \lambda.
\]

**Proof.** Let \( \mathbb{N} \) (\( \mathbb{Q} \)) be the set of all natural (respectively, rational) numbers, and let \( \frac{n}{m} \in \mathbb{Q} \), where \( n, m \in \mathbb{N} \). Denote \( y = \int_0^1 T_{s/m}(x)ds \). We have
\[
0 \leq A_{\frac{n}{m}}(x) = \frac{m}{n} \int_0^\infty T_s(x)ds = \frac{1}{n} \int_0^n T_{s/m}(x)ds
\]
\[
= \frac{1}{n} \left( \int_0^1 T_{s/m}(x)ds + \cdots + \int_{n-1}^n T_{s/m}(x)ds \right) = \frac{1}{n} \sum_{k=0}^{n-1} T_{1/m}(y).
\]

By Theorem 2.1 given \( \lambda > 0 \), there is \( f \in \mathcal{P}(\mathcal{M}) \) such that \[
\tau(f^+) \leq \frac{\|y\|_1}{\lambda} \leq \frac{\|x\|_1}{\lambda} \quad \text{and} \quad \sup_n \left\| f \frac{1}{n} \sum_{k=0}^{n-1} T_{1/m}(y) f \right\|_\infty \leq \lambda,
\]

implying, that \[
\sup_{0 < r \in \mathbb{Q}} \|fA_r(x)f\|_\infty \leq \lambda.
\]

If \( t > 0 \) and \( 0 < r_n \to t \), \( r_n \in \mathbb{Q} \), then we have \( A_{r_n}(x) \to A_t(x) \) in \( L^1 \), hence in measure. Therefore \( A_{r_{n_k}}(x) \to A_t(x) \) a.u. for a subsequence \( \{r_{n_k}\} \subset \{r_n\} \). Thus, it is possible to find \( g \in \mathcal{P}(\mathcal{M}) \) such that \[
\tau(g^+) \leq \frac{\|x\|_1}{\lambda} \quad \text{and} \quad \|gA_{r_{n_k}}(x)g\|_\infty \to \|gA_t(x)g\|_\infty \quad \text{as} \quad k \to \infty.
\]

Letting \( e = f \land g \), we obtain the desired inequalities. \( \square \)

3. **Convergence in the space \( L^1(\mathcal{M}, \tau) \)**

Let \( \mathbb{C} \) be the field of complex numbers, and let \( \mathbb{C}_1 = \{ z \in \mathbb{C} : |z| = 1 \} \). A function \( p : \mathbb{R}_+ \to \mathbb{C} \) is called a *trigonometric polynomial* if \( p(t) = \sum_{j=1}^n w_j \lambda_j^t \), where \( n \in \mathbb{N} \), \( \{w_j\}_1^n \subset \mathbb{C} \), and \( \{\lambda_j\}_1^n \subset \mathbb{C}_1 \).

A Lebesgue measurable function \( \beta : \mathbb{R}_+ \to \mathbb{C} \) will be called *bounded Besicovitch (zero-Besicovitch) function* if \( \sup_{t \geq 0} |\beta(t)| < C < \infty \), and for every \( \epsilon > 0 \) there is a trigonometric polynomial \( p_\epsilon \) such that
\[
\lim_{t \to \infty} \sup_{t \geq 0} \frac{1}{t} \int_0^t |\beta(s) - p_\epsilon(s)|ds < \epsilon
\]
Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and assume that \( \beta : \mathbb{R}_+ \to \mathbb{C} \) be a Lebesgue measurable function with \( \|\beta\|_\infty < \infty \). Fix \( x \in L^1 \). Then for any given \( y \in M \) the function \( \varphi_{x,y}(t) = \tau(T_t(x)y) \) is continuous on \( \mathbb{R}_+ \). Therefore, if \( \mu \) is the Lebesgue measure on \( \mathbb{R}_+ \), then the map \( U_x : \mathbb{R}_+ \to L^1 \) defined as \( U_x(t) = T_t(x) \) is weakly \( \mu \)-measurable [38, Ch.V, §4]. Since, in addition, \( U_x(\mathbb{R}) \) is a separable subset in \( L^1 \), Pettis theorem [38, Ch.V, §4] entails that the map \( U_x \) is strongly \( \mu \)-measurable and the real function \( \|U_x(t)\|_1 = \|T_t(x)\|_1 \) is \( \mu \)-measurable on \( \mathbb{R}_+ \). Since \( \|T_t(x)\|_1 \leq \|x\|_1 \), it follows that \( \|T_t(x)\|_1 \) is an integrable function on \([0,t]\) for any \( t > 0 \). Consequently, \( \|\beta(s)T_s(x)\|_1 = |\beta(s)| \cdot \|T_s(x)\|_1 \) is also integrable on \([0,t]\) for any \( t > 0 \). By [38, Ch.V, §5, Theorem 1], the function \( \beta(s)T_s(x) \) is Bochner \( \mu \)-integrable on \([0,t]\), \( t > 0 \). Therefore, for any \( x \in L^1 \) and \( t > 0 \) there exists

\[
B_t(x) = \frac{1}{t} \int_0^t \beta(s)T_s(x) \, ds \in L^1.
\]

**Lemma 3.1.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and let \( \beta : \mathbb{R}_+ \to \mathbb{C} \) be a Lebesgue measurable function such that \( \sup_{t \geq 0} |\beta(t)| \leq C < \infty \).

If \( x \in L^1 \) and \( \epsilon > 0 \), then there is a projection \( e \in \mathcal{P}(M) \) satisfying inequalities

\[
\tau(e^+) \leq \frac{4\|x\|_1}{\epsilon} \quad \text{and} \quad \sup_{t > 0} \|eB_t(x)e\|_\infty \leq 48C\epsilon.
\]

**Proof.** We have \( x = (x_1 - x_2) + i(x_3 - x_4) \), where \( x_j \in L^1 \) and \( \|x_j\|_1 \leq \|x\|_1 \) for each \( j = 1, 2, 3, 4 \). By Theorem 2.2, given \( j \), there exists \( e_j \in \mathcal{P}(M) \) such that

\[
\tau(e_j^+) \leq \frac{\|x_j\|_1}{\epsilon} \quad \text{and} \quad \sup_{t > 0} \left\|e_j \frac{1}{t} \int_0^t T_s(x_j) ds e_j \right\|_\infty \leq 2\epsilon.
\]

Next, we have \( 0 \leq \Re \beta(s) + C \leq 2C \) and \( 0 \leq \Im \beta(s) + C \leq 2C, \ s \geq 0 \), implying that

\[
0 \leq |\Re \beta(s) + C|T_s(x_j) \leq 2CT_s(x_j) \quad \text{and} \quad 0 \leq |\Im \beta(s) + C|T_s(x_j) \leq 2CT_s(x_j)
\]

for all \( s \geq 0 \), hence

\[
0 \leq \frac{1}{t} \int_0^t |\Re \beta(s) + C|T_s(x_j) \, ds \leq 2C \frac{1}{t} \int_0^t T_s(x_j) \, ds
\]

and

\[
0 \leq \frac{1}{t} \int_0^t |\Im \beta(s) + C|T_s(x_j) \, ds \leq 2C \frac{1}{t} \int_0^t T_s(x_j) \, ds.
\]

for each \( j \). This, together with the decomposition

\[
B_t(x_j) = \frac{1}{t} \int_0^t |\Re \beta(s) + C|T_s(x_j) \, ds + i \frac{1}{t} \int_0^t |\Im \beta(s) + C|T_s(x_j) \, ds
\]

\[
- C(1 + i) \frac{1}{t} \int_0^t T_s(x_j) \, ds,
\]

implies that

\[
\sup_{t > 0} \left\|e_j B_t(x_j) e_j \right\|_\infty \leq 12C\epsilon, \quad j = 1, 2, 3, 4.
\]
Now, letting $e = \sum_{j=1}^{d} e_j$, we obtain the desired inequalities. \hfill \square

The following is a noncommutative counterpart of the classical notion of continuity at zero of the maximal operator (see [26, Proposition 1.1 and ensuing remarks]).

**Definition 3.1.** Let $(X, \| \cdot \|)$ be a Banach space. A sequence of linear maps $M_n : X \to L^0$ is called \textit{bilaterally uniformly equicontinuous in measure (b.u.e.m.)} at zero if, given $\epsilon > 0$ and $\delta > 0$, there exists $\gamma > 0$ such that for every $\| x \| < \gamma$ there exists a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying conditions

$$\tau(e^+) \leq \epsilon \quad \text{and} \quad \sup_n \| eM_n(x)e \|_{\infty} \leq \delta.$$ 

In order to establish a.u. convergence - which is generally stronger than b.a.u. convergence - of the averages $B_i(x)$, we will need the following lemma a proof of which can be found in [27, Lemma 3.2]; see also [28].

**Lemma 3.2.** Let $M_n : L^1 \to L^1$ be sequence of linear maps that is b.u.e.m. at zero. If $\{x_m\} \subset L^1$ is such that $\| x_m \|_1 \to 0$, then for every $\epsilon > 0$ and $\delta > 0$ there are $e \in \mathcal{P}(\mathcal{M})$ and $x_{m_0} \in \{x_m\}$ satisfying conditions

$$\tau(e^+) \leq \epsilon \quad \text{and} \quad \sup_n \| eM_n(x_{m_0})e \|_{\infty} \leq \delta.$$ 

In what follows we shall assume that $\mathcal{M}$ has separable predual. Denote by $\nu$ the normalized Lebesgue measure on $\mathbb{C}_1$. Let $\tilde{\mathcal{M}}$ be the von Neumann algebra of essentially bounded ultraweakly measurable functions $\tilde{f} : (\mathbb{C}_1, \nu) \to \mathcal{M}$ equipped with the trace

$$\tilde{\tau}(\tilde{f}) = \int_{\mathbb{C}_1} \tau(\tilde{f}(z))d\nu(z), \quad \tilde{f} \geq 0.$$ 

Let $\tilde{L}^1$ be the Banach space of Bochner $\nu$-integrable functions $\tilde{f} : (\mathbb{C}_1, \nu) \to L^1(\tilde{\mathcal{M}}, \tilde{\tau})$. As the predual of $\tilde{\mathcal{M}}$ [34, Theorem 1.22.13], the space $\tilde{L}^1$ is isomorphic to $L^1(\tilde{\mathcal{M}}, \tilde{\tau})$.

Repeating the argument in [34, Lemma 2] (see also [4, Lemma 4.1]), we obtain the following.

**Lemma 3.3.** If $\tilde{L}^1 \ni \tilde{f}_t \to \tilde{f} \in \tilde{L}^1$ a.u. as $t \to \infty$, then $\tilde{f}_t(z) \to \tilde{f}(z)$ a.u. as $t \to \infty$ for $\nu$-almost all $z \in \mathbb{C}_1$.

Let $\{T_t\}_{t \geq 0} \subset DS^+$ be a strongly continuous semigroup in $L^1$. Pick $\lambda \in \mathbb{C}_1$ and define

$$T_t^{(\lambda)}(\tilde{f})(z) = T_t(\tilde{f}(\lambda^t z)) = T_t(\tilde{f}(\lambda^s z)) = T_t(T_s(\tilde{f}(\lambda^s z))) = T_t(T_s^{(\lambda)}(\tilde{f})(\lambda^s z)) = (T_t^{(\lambda)}T_s^{(\lambda)})(\tilde{f})(z),$$

that is, $\{T_t^{(\lambda)}\}_{t \geq 0}$ is a semigroup.

**Proposition 3.1.** The semigroup $\{T_t^{(\lambda)}\}_{t \geq 0}$ is strongly continuous in $L^1(\tilde{\mathcal{M}}, \tilde{\tau})$. 

Proof. Let us show that if $0 \leq s_n \to 0$ and $\bar{f} \in L^1(\widetilde{\mathcal{M}}, \tilde{\tau})$, then

$$\|T_{s_n}(\bar{f}) - \bar{f}\|_{L^1(\widetilde{\mathcal{M}}, \tilde{\tau})} = \int_{C_1} \|T_{s_n}(\bar{f}(\lambda^{s_n} z)) - \bar{f}(z)\|_1 \, d\nu(z) \to 0 \text{ as } n \to \infty.$$ 

We have

$$\|T_{s_n}(\bar{f}(\lambda^{s_n} z)) - \bar{f}(z)\|_1 \leq \|T_{s_n}(\bar{f}(\lambda^{s_n} z)) - T_{s_n}(\bar{f}(z))\|_1 + \|T_{s_n}(\bar{f}(z)) - \bar{f}(z)\|_1$$

$$\leq \|\bar{f}(\lambda^{s_n} z) - \bar{f}(z)\|_1 + \|T_{s_n}(\bar{f}(z)) - \bar{f}(z)\|_1.$$ 

Since $\{T_t\}_{t \geq 0}$ is strongly continuous in $L^1$, it follows that $\|T_{s_n}(\bar{f}(z)) - \bar{f}(z)\|_1 \to 0$ for $\nu$-almost all $z \in C_1$. Besides,

$$\|T_{s_n}(\bar{f}(z)) - \bar{f}(z)\|_1 \leq 2\|\bar{f}(z)\|_1 \quad \nu \text{-a.e.,}$$

where $\|\bar{f}(z)\|_1 \in L^1(C_1, \nu)$, which allows us to conclude that

$$\int_{C_1} \|T_{s_n}(\bar{f}(z)) - \bar{f}(z)\|_1 \, d\nu(z) \to 0 \text{ as } n \to \infty.$$ 

Thus, it remains to verify that

$$\int_{C_1} \|\bar{f}(\lambda^{s_n} z) - \bar{f}(z)\|_1 \, d\nu(z) \to 0 \text{ as } n \to \infty.$$ 

Let

$$\bar{h}(z) = \sum_{i=1}^{m} x_i \chi_{(\lambda_i, \lambda_{i+1})}(z), \quad z \in C_1, \quad x_i \in L^1, \quad i = 1, \ldots, m$$

be a simple Bochner measurable function, where $\{\lambda_i\}_{i=1}^{m}$ is a partition of the circle $C_1$. Then, given $s \in \mathbb{R}$, we have

$$\bar{h}(\lambda^s z) = \sum_{i=1}^{m} x_i \chi_{(\lambda^s \lambda_i, \lambda^s \lambda_{i+1})}(z),$$

where $\{\lambda^s \lambda_i\}_{i=1}^{m}$ is another partition of $C_1$. Since $s_n \to 0$, it follows that $\lambda^s \lambda_i \to \lambda_i$ as $n \to \infty$ for all $i = 1, \ldots, m$, implying that

$$\int_{C_1} \|\bar{h}(\lambda^s z) - \bar{h}(z)\|_1 \, d\nu(z) \to 0 \text{ as } n \to \infty.$$ 

Let $\mathcal{A}$ be the subalgebra of the $\sigma$-algebra of Lebesgue measurable sets of $(C_1, \nu)$ generated by the arcs of the circle $C_1$. Since any $A \in \mathcal{A}$ is a finite union of pairwise disjoint arcs of $C_1$, a simple Bochner measurable function

$$\bar{h}(z) = \sum_{i=1}^{m} x_i \chi_{A_i}(z), \quad A_i \in \mathcal{A}, \quad i = 1, \ldots, m$$

has the form $\text{[A]}$ for some partition $\{\lambda_i\}_{i=1}^{m}$ of $C_1$. Therefore, $\text{[A]}$ holds for any simple Bochner measurable function $\bar{h}$ of the form $\text{[A]}$.

Since the $\sigma$-algebra generated by the subalgebra $\mathcal{A}$ coincides $\nu$-almost everywhere with the $\sigma$-algebra $\Sigma$ of Lebesgue measurable sets in $(C_1, \nu)$, given an arbitrary simple Bochner measurable function $\bar{g}(z) = \sum_{i=1}^{m} x_i \chi_{A_i}(z), A_i \in \Sigma, i = 1, \ldots, m$, there exists a sequence $\{\bar{h}_k(z)\}_{k=1}^{\infty}$ of simple Bochner measurable functions of the form $\text{[B]}$ such that

$$\int_{C_1} \|\bar{g}(z) - \bar{h}_k(z)\|_1 \, d\nu(z) \to 0 \text{ as } k \to \infty.$$
In view of Proposition 3.1,

\[
\int_{C_1} \|\tilde{g}(\lambda^nz) - \bar{g}(z)\|_1 \, d\nu(z) \leq \int_{C_1} \|\tilde{g}(\lambda^nz) - \tilde{h}_k(\lambda^nz)\|_1 \, d\nu(z)
\]
\[
+ \int_{C_1} \|\tilde{h}_k(\lambda^nz) - \tilde{h}(z)\|_1 \, d\nu(z) + \int_{C_1} \|\tilde{h}(z) - \bar{g}(z)\|_1 \, d\nu(z)
\]
\[
= 2 \int_{C_1} \|\tilde{h}_k(z) - \bar{g}(z)\|_1 \, d\nu(z) + \int_{C_1} \|\tilde{h}(z) - \tilde{h}_k(z)\|_1 \, d\nu(z)
\]

implies that (5) holds for any simple Bochner measurable function \( \tilde{g} \).

As \( \tilde{f} \in L^1(\mathcal{M}, \tau) \), there exists a sequence \( \{\tilde{g}_k(z)\}_{k=1}^\infty \) of simple Bochner measurable functions for which

\[
\int_{C_1} \|\tilde{f}(z) - \tilde{g}_k(z)\|_1 \, d\nu(z) \to 0 \quad \text{as} \quad k \to \infty.
\]

Then, repeating the previous argument, we conclude that the convergence in (5) holds for \( \tilde{f} \). Therefore, it now follows that

\[
\|T_n^{(\lambda)}(\tilde{f}) - \tilde{f}\|_{L^1(\mathcal{M}, \tau)} \to 0 \quad \text{as} \quad n \to \infty.
\]

Finally, let \( t \geq 0, t_n > 0, t_n \downarrow t \) and denote \( s_n = t_n - t \). Then we have

\[
\|T_{t_n}^{(\lambda)}(\tilde{f}) - T_t^{(\lambda)}(\tilde{f})\|_{L^1(\mathcal{M}, \tau)} = \|T_{t+s_n}^{(\lambda)}(\tilde{f}) - T_t^{(\lambda)}(\tilde{f})\|_{L^1(\mathcal{M}, \tau)}
\]
\[
\leq \|T_{s_n}^{(\lambda)}(\tilde{f}) - \tilde{f}\|_{L^1(\mathcal{M}, \tau)} \to 0
\]
as \( n \to \infty \). The case \( t_n \uparrow t > 0 \) is similar. \( \square \)

**Lemma 3.4.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and let \( p(t) = \sum_{j=1}^n w_j \lambda_j^t \) be a trigonometric polynomial. If \( x \in L^1 \) and

\[
P_t(x) = \frac{1}{t} \int_0^t p(s)T_s(x) \, ds,
\]
then

(i) the averages \( P_t(x) \) converge a.u. as \( t \to \infty \);
(ii) the averages \( P_t(x) \) converge a.u. to \( p(0)x \) as \( t \to 0 \).

**Proof.** (i) Fix \( \lambda \in C_1 \) and let

\[
T_t^{(\lambda)}(\tilde{f})(z) = T_t(\tilde{f}(\lambda^t z)), \quad t \geq 0, \quad \tilde{f} \in L^1(\mathcal{M}, \tau) + \tilde{\mathcal{M}}, \quad z \in C_1.
\]

In view of Proposition 3.1, \( \{T_t^{(\lambda)}\}_{t \geq 0} \subset DS^+(\mathcal{M}, \tau) \) is a strongly continuous semigroup on \( L^1(\mathcal{M}, \tau) \). Then, by [8, Corollary 5.2], given \( \tilde{f} \in L^1(\mathcal{M}, \tau) \), the averages

\[
(7) \quad \frac{1}{t} \int_0^t T_s^{(\lambda)}(\tilde{f}) \, ds
\]
converge a.u. as \( t \to \infty \). Therefore, by Lemma 3.4, the averages

\[
\frac{1}{t} \int_0^t T_s^{(\lambda)}(\tilde{f})(z) \, ds = \frac{1}{t} \int_0^t T_s(\tilde{f}(\lambda^sz)) \, ds
\]
converge a.u. as \( t \to \infty \) for \( \nu \)-almost all \( z \in \mathbb{C}_1 \). In particular, letting \( \tilde{f}(z) = z \), we conclude that the averages
\[
\frac{1}{t} \int_0^t \lambda^s T_s(z) ds
\]
converge a.u. as \( t \to \infty \) for some \( 0 \neq z \in \mathbb{C}_1 \), implying that the averages
\[
\frac{1}{t} \int_0^t \lambda^s T_s(z) ds
\]
converge a.u. as \( t \to \infty \). Therefore, by linearity, the averages \( P_t(x) \) converge a.u. as \( t \to \infty \).

(ii) Now, by [9, Theorem 5.1], if \( \tilde{f} \in L^1(\mathcal{M}, \tau) \), it follows that the averages (7) converge a.u. to \( \tilde{f} \) as \( t \to 0 \). Then, letting \( \tilde{f}(z) = z \), we see as above that
\[
\frac{1}{t} \int_0^t \lambda^s T_s(z) ds \to x \quad \text{a.u.}
\]
as \( t \to 0 \), and the result follows by linearity. \( \square \)

Here is a Besicovitch-weighted noncommutative individual ergodic theorem for flows in \( L^1 \) generated by \( L^1 \)-strongly continuous semigroups of positive Dunford-Schwartz operators:

**Theorem 3.1.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and let \( \beta(t) \) be a bounded Besicovitch function with \( \|\beta\|_\infty < C < \infty \). Then, given \( x \in L^1 \), the averages (3) converge a.u. to some \( \tilde{x} \in L^1 \) as \( t \to \infty \).

**Proof.** Assume first that \( x \in L^1 \cap \mathcal{M} \). Fix \( \epsilon > 0 \) and choose a trigonometric polynomial \( p = p_\epsilon \) to satisfy condition (1). Let \( \{P_t(x)\}_{t \geq 0} \) be the corresponding averages from Lemma 3.4. Then we have
\[
\|B_t(x) - P_t(x)\|_\infty \leq \frac{1}{t} \int_0^t |\beta(s) - p(s)| ds \|x\|_\infty < \epsilon \|x\|_\infty
\]
for all big enough values of \( t \). Since, by Lemma 3.4, the averages \( P_t(x) \) converge a.u., it follows that the net \( \{B_t(x)\}_{t \geq 0} \) is a.u. Cauchy as \( t \to \infty \).

Now, let \( x \in L^1 \). Without loss of generality, assume that \( x \in L^1_+ \), and let \( \{e_\lambda\} \) be the spectral family of \( x \). Given \( m \in \mathbb{N} \), if we define \( y_m = \int_0^m \lambda d\epsilon_\lambda \) and \( x_m = x - y_m \), then \( \{y_m\} \subset L^1_+ \cap \mathcal{M} \), \( \{x_m\} \subset L^1_+ \) and \( \|x_m\|_1 \to 0 \).

Fix \( \epsilon > 0 \) and \( \delta > 0 \). If \( \{t_n\} \) is a sequence of positive rational numbers which is dense in \( (0, \infty) \), then, by Lemma 3.1, the sequence \( \{B_{t_n}\} \) is b.u.e.m. at zero on \( L^1_+ \), hence on \( L^1 \) (see [20, Lemma 4.1]). Applying Lemma 3.2 we find a projection \( e \in \mathcal{P}(\mathcal{M}) \) and \( x_{m_0} \in \{x_m\} \) such that
\[
\tau(e^+) \leq \frac{\epsilon}{2} \quad \text{and} \quad \sup_n \|B_{t_n}(x_{m_0})e\|_\infty \leq \frac{\delta}{3}.
\]

If \( t > 0 \), then \( t_{nk} \to t \) for a subsequence \( \{t_{nk}\} \), and it easily verified that
\[
\|B_t(x_{m_0}) - B_{t_{nk}}(x_{m_0})\|_1 \to 0.
\]
Therefore \( B_{t_{nk}}(x_{m_0}) \to B_t(x_{m_0}) \) in measure, which implies that there is a subsequence \( \{t_{nk}\} \) such that \( B_{t_{nk}}(x_{m_0}) \to B_t(x_{m_0}) \) a.u.
Since \( \|B_{t_{n_k}}(x_{m_0})e\|_\infty < \delta/3 \) for each \( l \), it follows from [8] Lemma 5.1 that

\[
(8) \quad \sup_{t>0} \|B_t(x_{m_0})e\|_\infty \leq \frac{\delta}{3}.
\]

Because \( y_{m_0} \in L^1 \cap \mathcal{M} \), the net \( B_t(y_{m_0}) \) is a.u. Cauchy as \( t \to \infty \). Therefore, there exist \( g \in \mathcal{P}(\mathcal{M}) \) and \( t_0 > 0 \) such that

\[
(9) \quad \tau(g^+) \leq \frac{\epsilon}{2} \quad \text{and} \quad \|(B_t(y_{m_0}) - B_{t'}(y_{m_0}))g\|_\infty \leq \frac{\delta}{3}.
\]

for all \( t, t' \geq t_0 \).

If \( h = e \wedge g \), then \( \tau(h^+) \leq \epsilon \) and, in view of (8) and (9), we have

\[
\|(B_t(x) - B_{t'}(x))h\|_\infty \leq \|(B_t(y_{m_0}) - B_{t'}(y_{m_0}))h\|_\infty + \|B_t(x_{m_0})h\|_\infty +\|B_{t'}(x_{m_0})h\|_\infty \leq \delta
\]

for all \( t, t' \geq t_0 \). Thus, the net \( \{B_t(x)\} \) is a.u. Cauchy as \( t \to \infty \). Since \( L^0 \) is complete with respect to a.u. convergence (see proof of [7, Theorem 2.3]), there is \( \tilde{x} \in L^0 \) such that \( B_t(x) \to \tilde{x} \) a.u. In particular, \( B_t(x) \to \tilde{x} \) in measure. Since \( |\beta(t)| < C < \infty \) for all \( t \geq 0 \), each map \( C^{-1}B_t \) is a contraction in \( L^1 \), which, because the unit ball of \( L^1 \) is closed in measure topology, implies that \( \tilde{x} \in L^1 \).

The following theorem is a local ergodic theorem in \( L^1 \) for \( L^1 \)-continuous semigroups of positive Dunford-Schwartz operators modulated by bounded zero-Besicovitch functions.

**Theorem 3.2.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and let \( \beta(t) \) be a zero-bounded Besicovitch function with \( \|\beta\|_{\infty} < C < \infty \). Then, given \( x \in L^1 \), the averages (3) converge a.u. to \( \alpha(x) \) as \( t \to 0 \) for some \( \alpha(x) \in \mathbb{C} \).

**Proof.** Assume first that \( x \in L^1 \cap \mathcal{M} \). Fix \( \epsilon > 0 \) and choose a trigonometric polynomial \( p = p_\epsilon \) to satisfy condition (2). If \( \{P_t(x)\}_{t \geq 0} \) are the averages from Lemma 3.4, then

\[
\|B_t(x) - P_t(x)\|_\infty \leq \frac{1}{t} \int_0^t |\beta(s) - p(s)|ds \|x\|_\infty < \epsilon \|x\|_\infty
\]

for all small enough values of \( t \). Since, by Lemma 3.4, the averages \( P_t(x) \) converge a.u. as \( t \to 0 \), it follows that the net \( \{B_t(x)\}_{t \geq 0} \) is a.u. Cauchy as \( t \to 0 \). Then, repeating the proof of the Theorem 3.1, we obtain that for any \( x \in L^1 \) the averages \( \{B_t(x)\}_{t \geq 0} \) converge a.u. to some \( \tilde{x} \in L^1 \) as \( t \to 0 \).

Let \( p_n \) be a trigonometric polynomial to satisfy condition (2) with \( \epsilon = 1/n \). If \( \{P^{(n)}_t(x)\}_{t \geq 0} \) are the corresponding averages from Lemma 3.4, then there is \( t_n > 0 \) such that

\[
(10) \quad \|B_t(x) - P^{(n)}_t(x)\|_1 \leq \frac{1}{t} \int_0^t |\beta(s) - p(s)|ds \|x\|_1 < \frac{\|x\|_1}{n}
\]

for all \( 0 < t < t_n \).

By Lemma 3.4, \( P^{(n)}_t(x) \to p_n(0) \) a.u., hence \( B_t(x) - P^{(n)}_t(x) \to \tilde{x} - p_n(0) \) a.u., implying that

\[
B_t(x) - P^{(n)}_t(x) \to \tilde{x} - p_n(0) \quad \text{in measure as} \quad t \to 0.
\]
Since the unit ball of $L^1$ is closed in measure topology, (10) entails that
\[
\|\widehat{x} - p_n(0) x\|_1 \leq \frac{\|x\|_1}{n},
\]
hence $\widehat{x} = \|\cdot\|_1 - \lim_{n \to \infty} p_n(0) x$, and we conclude that $\widehat{x} = \alpha(x) x$ for some $\alpha(x) \in \mathbb{C}$.

Now, let $0 \leq x \in L^1$, $x \neq 0$, $e_n = \{x \leq n\}$, $n \in \mathbb{N}$, and let $x_n = xe_n$. It is clear that $\{x_n\} \subset L^1 \cap M$ and $\|x - x_n\|_1 \to 0$ as $n \to \infty$. As shown above, $B_t(x) \to \widehat{x}$ a.u. as $t \to 0$ for some $\widehat{x} \in L^1$ and $B_t(x_n) \to \alpha(x_n) x_n$ as $t \to 0$ for for every $n$ and some $\alpha(x_n) \in \mathbb{C}$. Consequently, $B_t(x) - B_t(x_n) \to \widehat{x} - \alpha(x_n) x_n$ in measure.

Besides,
\[
\|B_t(x) - B_t(x_n)\|_1 \leq \frac{1}{t} \int_0^t \|\beta(s)\| ds \|x - x_n\|_1 \leq C \|x - x_n\|_1.
\]
Since the unit ball of $L^1$ is closed in measure topology, it follows that (11)
\[
\|\widehat{x} - \alpha(x_n) x_n\|_1 \leq C \|x - x_n\|_1 \to 0 \quad \text{as} \quad n \to \infty.
\]
Choose $k \in \mathbb{N}$ such that $x_k = xe_k \neq 0$. If $n > k$, then we have
\[
\|\widehat{x} e_k - \alpha(x_n) x_k\|_1 = \|\widehat{x} e_k - \alpha(x_n) x_n e_k\|_1 \leq \|\widehat{x} - \alpha(x_n) x_n\|_1,
\]
so, (11) implies that
\[
\widehat{x} e_k = \|\cdot\|_1 - \lim_{n \to \infty} \alpha(x_n) x_n.
\]
Therefore, there exists $\lim_{n \to \infty} \alpha(x_n) = \alpha(x)$, implying that $\|\alpha(x_n) x_n - \alpha(x) x\|_1 \to 0$ as $n \to \infty$, hence $\widehat{x} = \alpha(x) x$, in view of (11).

If $x \in L^1$, we employ the decomposition $x = x_1 - x_2 + i(x_3 - x_4), 0 \leq x_i \in L^1, i = 1, 2, 3, 4$, and apply the above argument to each $x_i$. \qed

A weaker version of Theorem 3.2 - for b.a.u. convergence and with no identification of the limit - was announced in [30]. However, some steps leading to the main Theorem 3.1 of the paper, such as the fact given above in Proposition 3.1, were left unjustified.

4. Extension to Noncommutative Fully Symmetric Spaces

Now we will extend the results of Section 3 to the noncommutative fully symmetric spaces $E \subset L^1 + L^\infty$ with $1 \notin E$, in particular, to the spaces $L^p, 1 < p < \infty$.

Let $x \in L^p$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of $x$. If $t > 0$, then the $t$-th generalized singular number of $x$ (the non-increasing rearrangement of $x$) is defined as
\[
\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^+) \leq t\}
\]
(see [18]).

Let $\mu$ be the Lebesgue measure on $(0, \infty)$. It is well known that
\[
L^p = L^p(M, \tau) = \left\{ x \in L^0 : \int_0^\infty \mu_t(x)^p d\mu(t) < \infty \right\},
\]
and $\|x\|_p = \|\mu_t(x)\|_p, x \in L^p, 1 \leq p < \infty$ (see, for example, [18]).

Let $(E, \|\cdot\|_E)$ be a symmetric function space on $((0, \infty), \mu)$ (see, for example, [22, Ch.II, §4]). Define
\[
E(M) = E(M, \tau) = \{ x \in L^0 : \mu_t(x) \in E \}
and set
\[ \|x\|_{E(M)} = \| \mu_t(x) \|_E, \quad x \in E(M). \]

It is shown in [21] that \((E(M), \| \cdot \|_{E(M)})\) is a Banach space and that conditions
\[ x \in E(M), \quad y \in L^0, \quad \mu_t(y) \leq \mu_t(x) \quad \text{for all} \quad t > 0 \]

imply that \(y \in E\) and \(\|y\|_E \leq \|x\|_E\), in which case \((E(M), \| \cdot \|_{E(M)})\) is said to be a noncommutative symmetric space.

A noncommutative symmetric space \((E(M), \| \cdot \|_{E(M)})\) is called fully symmetric if conditions
\[ x \in E(M), \quad y \in L^0, \quad \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \quad \forall \ s > 0 \quad (\text{writing} \quad y \prec x) \]

imply that \(y \in E\) and \(\|y\|_E \leq \|x\|_E\). For example, \(L^p(M) = L^p, 1 \leq p \leq \infty,\) and the Banach spaces
\[ (L^1 \cap L^\infty)(M) = L^1 \cap M \quad \text{with} \quad \|x\|_{L^1 \cap M} = \max\{\|x\|_1, \|x\|_\infty\} \quad \text{and} \]
\[ (L^1 + L^\infty)(M) = L^1 + M \quad \text{with} \]
\[ \|x\|_{L^1 + M} = \inf \{\|y\|_1 + \|z\|_\infty : x = y + z, \quad y \in L^1, \quad z \in M\} = \int_0^1 \mu_t(x) dt \]

are noncommutative fully symmetric spaces (see [14]).

Since, given a symmetric function space \(E = E(0, \infty),\)
\[ L^1(0, \infty) \cap L^\infty(0, \infty) \subset E \subset L^1(0, \infty) + L^\infty(0, \infty), \]
with continuous embedding [22, Ch. II, §4, Theorem 4.1], it follows that
\[ L^1(M) \cap M \subset E(M) \subset L^1(M) + M, \]

with continuous embedding.

Define
\[ R_\tau = \{x \in L^1 + M : \mu_t(x) \to 0 \quad \text{as} \quad t \to \infty\}. \]

It is known that \(R_\tau\) is the closure of \(L^1 \cap M\) in \(L^1 + M\) [14, Proposition 2.7], in particular, \((R_\tau, \| \cdot \|_{L^1 + M})\) is a noncommutative fully symmetric space [8]. In addition, if \(\tau(1) = \infty,\) then a symmetric space \(E(M, \tau)\) is contained in \(R_\tau\) if and only if \(1 \notin E(M, \tau)\) [8, Proposition 2.2].

Every noncommutative fully symmetric space \(E = E(M)\) is an exact interpolation space for the Banach couple \((L^1(M), M)\) [13]. Therefore \(T(E) \in E\) and \(\|T\|_{E \to E} \leq 1\) for \(T \in DS;\) in particular,
\[ T(R_\tau) \subset R_\tau \quad \text{and} \quad \|T\|_{L^1 + M \to L^1 + M} \leq 1. \]

Let \(\{T_t\}_{t \geq 0} \subset DS^+\) be a strongly continuous semigroup in \(L^1, \beta : \mathbb{R}_+ \to \mathbb{C}\) a Lebesgue measurable function with \(\|\beta\|_\infty < C < \infty,\) and let \(B_t(x), \ x \in L^1,\)
\[ t > 0, \quad \text{be given by} \ \beta. \quad \text{We have} \ \beta(t) = \beta_1(t) - \beta_2(t) + i(\beta_3(t) - \beta_4(t)),\]

where \(\beta_j : \mathbb{R}_+ \to \mathbb{R}_+\) is Lebesgue measurable function such that \(\|\beta_j\|_\infty < C < \infty\) for each \(j = 1, \ldots, 4.\) Denote
\[ B_t^{(j)}(x) = \frac{1}{t} \int_0^t \beta_j(s) T_s(x) ds, \quad x \in L^1, \quad t > 0. \]

Then for each \(j,\)
\[ \|B_t^{(j)}(x)\|_1 \leq C \|x\|_1 \quad \forall \ x \in L^1 \quad \text{and} \quad \|B_t^{(j)}(x)\|_\infty \leq C \|x\|_\infty \quad \forall \ x \in L^1 \cap L^\infty. \]
Therefore, \( C^{-1}B_t^{(j)} \) is a positive absolute contraction in \( L^1 \), which, by Proposition 1.1, admits a unique extension to a positive Dunford-Schwartz operator \( D_t^{(j)} \). Therefore, \( CD_t^{(j)}, t > 0 \), is the unique extension of \( B_t^{(j)} \) to the Banach space \( L^1 + L^\infty \). By linearity, \( B_t \) admits a unique extension to \( L^1 + L^\infty \), which we will also denote by \( B_t \).

Since a noncommutative fully symmetric space \( E = E(\mathcal{M}) \) is an exact interpolation space for the Banach couple \( (L^1, \mathcal{M}) \), it now follows that \( B_t(E) \subset E \) and \( \|B_t\|_{E \to E} \leq C \) for every \( t > 0 \).

**Theorem 4.1.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), and let \( \beta(t) \) be a bounded Besicovitch function such that \( \|\beta(t)\|_\infty < C \). Then, given \( x \in \mathcal{R}_r \), the averages \( e_m \) converge a.u. to some \( \hat{x} \in \mathcal{R}_r \) as \( t \to \infty \).

**Proof.** Without loss of generality assume that \( x \geq 0 \), and let \( \{e_\lambda\}_{\lambda \geq 0} \) be the spectral family of \( x \). Given \( m \in \mathbb{N} \), denote \( x_m = \int_{1/m}^\infty \lambda de_\lambda \) and \( y_m = \int_0^{1/m} \lambda de_\lambda \). Then we have \( 0 \leq y_m \leq m^{-1}1) \), \( x_m \in L^1 \), and \( x = x_m + y_m \) for all \( m \).

Fix \( \epsilon > 0 \). By Theorem 3.1, \( B_t(x_m) \to \hat{x}_m \in L^1 \) a.u. as \( t \to \infty \) for each \( m \). Therefore, there exists a sequence \( \{e_m\} \subset \mathcal{P}(\mathcal{M}) \) such that

\[
\tau(e_m) \leq \frac{\epsilon}{2m} \quad \text{and} \quad \|(B_t(x_m) - \hat{x}_m)e_m\|_\infty \to 0 \quad \text{as} \quad t \to \infty.
\]

Then it follows that, for for some \( t(m) > 0 \),

\[
\|(B_t(x_m) - B_{t'}(x_m))e_m\|_\infty < \frac{1}{m} \quad \forall \ t, t' \geq t(m).
\]

Since \( \|y_m\|_\infty \leq m^{-1} \), we have

\[
\|(B_t(x) - B_{t'}(x))e_m\|_\infty \leq \|(B_t(x_m) - B_{t'}(x_m))e_m\|_\infty + \|(B_t(y_m) - B_{t'}(y_m))e_m\|_\infty
\]

\[
< \frac{1}{m} + \|B_t(y_m)e_m\|_\infty + \|B_{t'}(y_m)e_m\|_\infty \leq \frac{1 + 2C}{m}
\]

for each \( m \) and all \( t, t' \geq t(m) \).

If \( e = \bigwedge_{m \in \mathbb{N}} e_m \), then

\[
\tau(e) \leq \epsilon \quad \text{and} \quad \|(B_t(x) - B_{t'}(x))e\|_\infty < \frac{1 + 2C}{m} \quad \forall \ t, t' \geq t(m).
\]

This means that \( \{B_t(x)\}_{t > 0} \) is a Cauchy net with respect to a.u. convergence. Since \( L^0 \) is complete with respect to a.u. convergence [Remark 2.4], we conclude that the net \( \{B_t(x)\}_{t > 0} \) converges a.u. to some \( \hat{x} \in L^0 \).

As \( B_t(x) \to \hat{x} \) a.u., it is clear that \( B_t(x) \to \hat{x} \) in measure. Since \( L^1 + \mathcal{M} \) satisfies the Fatou property [§4], its unit ball is closed in measure topology [Theorem 4.1], and the inequality \( \|B_t(x)\|_{L^1 + \mathcal{M}} \leq C \|x\|_{L^1 + \mathcal{M}} \) implies that \( \hat{x} \in L^1 + \mathcal{M} \). In addition, \( B_t(x) \to \hat{x} \) in measure implies that \( \mu_s(B_t(x)) \to \mu_s(\hat{x}) \) almost everywhere on \((0, \infty), \mu\); for each \( s > 0 \) (this can be shown as in the commutative case; see, for example, [Ch.II, &sect;2, Property 11c]).

Since \( C^{-1}B_t \in DS^+ \), we have \( C^{-1}\mu_s(B_t(x)) = \mu_s(C^{-1}B_t(x)) \ll \mu_t(x) \). This means that

\[
C^{-1}\int_0^u \mu_s(B_t(x))ds = \int_0^u \mu_s(C^{-1}B_t(x))ds \leq \int_0^u \mu_s(x)ds \quad \forall \ u > 0, \ t > 0.
\]
Now, Fatou property for \( L^1((0, u), \mu) \) and \( \mu_s(B_t(x)) \rightarrow \mu_s(\hat{x}) \) in measure on \((0, u), \mu)\) imply that
\[
\int_0^u \mu_s(\hat{x}) ds \leq C \int_0^u \mu_s(x) ds = \int_0^u \mu_s(Cx) ds \quad \forall \ u > 0,
\]
that is, \( \mu_s(\hat{x}) \prec \prec \mu_s(Cx) \). Therefore, since \( \mathcal{R}_\tau \) is a fully symmetric space and \( Cx \in \mathcal{R}_\tau \), it follows that \( \hat{x} \in \mathcal{R}_\tau \). \( \square \)

The following is an application of Theorem 4.1 to fully symmetric spaces.

**Theorem 4.2.** Let \( E = E(M, \tau) \) be a noncommutative fully symmetric space such that \( 1 \notin E \), and let \( \{T_t\}_{t \geq 0} \) be as in Theorem 4.1. Then, given \( x \in E \), the averages \( \{\hat{T}_t(x)\}_{t \geq 0} \) converge a.u. to some \( \hat{x} \in E \) as \( t \to \infty \).

**Proof.** Since \( 1 \notin E \), it follows that \( E \subset \mathcal{R}_\tau \). Thus, by Theorem 4.1, the averages \( \{B_t(x)\}_{t \geq 0} \) converge a.u. to some \( \hat{x} \in \mathcal{R}_\tau \) as \( t \to \infty \). Since \( \mu_s(\hat{x}) \prec \prec \mu_s(Cx) \) (see the proof of Theorem 4.1) and \( E \) is a fully symmetric space, we conclude that \( \hat{x} \in E \). \( \square \)

Repeating the proofs of Theorems 4.1 and 4.2, we obtain the following extension of Theorem 4.2.

**Theorem 4.3.** Let \( \{T_t\}_{t \geq 0} \subset DS^+ \) be a strongly continuous semigroup in \( L^1 \), \( \beta(t) \) be a bounded zero-Besicovitch function, and let \( E = E(M, \tau) \) be a noncommutative fully symmetric space such that \( 1 \notin E \). Then, given \( x \in E \), the averages \( \{\beta(s)T_t(x)ds\} \) converge a.u. to \( \alpha(x)x \) as \( t \to 0 \) for some \( \alpha(x) \in \mathbb{C} \).

**Remark 4.1.** Employing the approach in the proof of [8, Theorem 5.7] (see also [9]), one can see that the assertion of Theorem 4.2 (Theorem 4.3) remains valid when the semigroup \( \mathbb{R}_+ \) is replaced by \( \mathbb{R}^d \), \( d \in \mathbb{N} \), so that the averages
\[
\mathbf{B}_t(x) = \frac{1}{t^d} \int_{[0, t^d]} \beta(s)T_s(x)ds, \quad x \in E,
\]
converge a.u. to some \( \hat{x} \in E \) as \( t \to \infty \) (respectively, to \( \alpha(x)x \) as \( t \to 0 \)).

5. **Applications to noncommutative Orlicz, Lorentz and Marcinkiewicz spaces**

Below we give applications of Theorems 4.2 and 4.3 to noncommutative Orlicz, Lorentz and Marcinkiewicz spaces.

1. Let \( \Phi \) be an Orlicz function, that is, \( \Phi : [0, \infty) \rightarrow [0, \infty) \) is left-continuous, convex, increasing and such that \( \Phi(0) = 0 \) and \( \Phi(u) > 0 \) for some \( u \neq 0 \) (see, for example [17, Ch.2, §2.1]). If \( 0 \leq x \in L^0 \) and \( x = \int_0^\infty \lambda d\lambda \) its spectral decomposition, one can define \( \Phi(x) = \int_0^\infty \Phi(\lambda)d\lambda \in L^0 \).

The noncommutative Orlicz space associated with \( (M, \tau) \) is the set
\[
L^\Phi = L^\Phi(M, \tau) = \left\{ x \in L^0 : \Phi(a^{-1}|x|) \in L^1 \text{ for some } a > 0 \right\}.
\]

The Luxembourg norm of an operator \( x \in L^\Phi \) is defined as
\[
\|x\|_\Phi = \inf \left\{ a > 0 : \|\Phi(a^{-1}|x|)\|_1 \leq 1 \right\}.
\]

It is known that \( (L^\Phi, \|\cdot\|_\Phi) \) is a noncommutative fully symmetric space (see, for example, [9 Corollary 2.2]). In addition,
\[
L^\Phi = \left\{ x \in L^0 : \mu_t(x) \in L^\Phi(0, \infty) \right\} \quad \text{and} \quad \|x\|_\Phi = \|\mu_t(x)\|_\Phi \quad \forall \ x \in L^\Phi.
\]
The pair \((L^\Phi, \| \cdot \|_{\Phi})\) is called the noncommutative Orlicz space (see, for example [22]).

If \(\tau(1) < \infty\), then \(L^\Phi \subset L^1\). If \(\tau(1) = \infty\) and \(\Phi(u) > 0\) for all \(u \neq 0\), then \(\Phi(a^{-1} 1) \notin L^1\) for each \(a > 0\), hence \(1 \notin L^\Phi\). Therefore, Theorems 4.2 and 4.3 imply the following.

**Theorem 5.1.** Let \(\Phi\) be an Orlicz function such that \(\Phi(u) > 0\) for all \(u > 0\). Let \(\{T_t\}_{t \geq 0} \subset DS^+\) be a strongly continuous semigroup in \(L^1\), and let \(\beta(t)\) be a bounded Besicovitch (zero-Besicovitch) function. Then, given \(x \in L^\Phi\), the averages (3) converge a.u. to some \(\hat{x} \in L^\Phi\) as \(t \to \infty\) (respectively, to \(\alpha(x) x\) as \(t \to 0\) for some \(\alpha(x) \in \mathbb{C}\)).

2. Let \(\mu\) be the Lebesgue measure on the interval \((0, \infty)\). Let \(L^0(0, \infty)\) be the algebra of (equivalence classes of) almost everywhere finite real-valued measurable functions \(f\) on the measure space \(((0, \infty), \mu)\) with \(\mu\{|f| > \lambda\} < \infty\) for some \(\lambda > 0\). The non-increasing rearrangement of a function \(f \in L^0(0, \infty)\) is the function \(f^*\) on \((0, \infty)\) defined by

\[
f^*(t) = \inf\{\lambda > 0 : \mu\{|f| > \lambda\} \leq t\}
\]

(see, for example, [22, Ch.II, §2]). Let \(\psi\) be a concave function on \([0, \infty)\) with \(\psi(0) = 0\) and \(\psi(t) > 0\) for all \(t > 0\), and let

\[
\Lambda_\psi(0, \infty) = \left\{ f \in L^0(0, \infty) : \| f \|_{\Lambda_\psi} = \int_0^\infty f^*(t) d\psi(t) < \infty \right\}
\]

be the corresponding Lorentz space.

It is well known that \((\Lambda_\psi(0, \infty), \| \cdot \|_{\Lambda_\psi})\) is a fully symmetric function space; in addition, if \(\psi(\infty) = \lim_{t \to \infty} \psi(t) = \infty\), then \(1 \notin \Lambda_\psi(0, \infty)\) [22, Ch.II, §5].

Define the noncommutative Lorentz space (see, for example, [3]) as

\[
\Lambda_\psi = \Lambda_\psi(M, \tau) = \{ x \in L^0 : \mu_t(x) \in \Lambda_\psi(0, \infty) \}
\]

and set

\[
\| x \|_{\Lambda_\psi} = \| \mu_t(x) \|_{\Lambda_\psi}, \ x \in \Lambda_\psi.
\]

It is known [21] that \((\Lambda_\psi, \| \cdot \|_{\Lambda_\psi})\) is a noncommutative fully symmetric space. If \(\tau(1) < \infty\), then \(\Lambda_\psi \subset L^1\). If \(\tau(1) = \infty\) and \(\psi(\infty) = \infty\), then it follows that \(1 \notin \Lambda_\psi\). Therefore, Theorems 4.2 and 4.3 imply the following.

**Theorem 5.2.** Let \(\psi\) be a concave function on \([0, \infty)\) with \(\psi(0) = 0\) and \(\psi(t) > 0\) for all \(t > 0\), and let \(\psi(\infty) = \infty\). Let \(\{T_t\}_{t \geq 0}\) and \(\beta(t)\) be as in Theorem 5.1. Then, given \(x \in \Lambda_\psi\), the averages (3) converge a.u. to some \(\hat{x} \in \Lambda_\psi(M, \tau)\) as \(t \to \infty\) (respectively, to \(\alpha(x) x\) as \(t \to 0\) for some \(\alpha(x) \in \mathbb{C}\)).

3. Let \(\psi\) be as above, and let

\[
M_\psi(0, \infty) = \left\{ f \in L^0(0, \infty) : \| f \|_{M_\psi} = \sup_{0 < s < \infty} \frac{1}{\psi(s)} \int_0^s f^*(t) dt < \infty \right\}
\]

be the corresponding Marcinkiewicz space. It is known that \((M_\psi(0, \infty), \| \cdot \|_{M_\psi})\) is a fully symmetric function space. Besides, \(1 \notin M_\psi(0, \infty)\) if and only if \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\) [22, Ch.II, §5]. Also, if \(\psi(+0) > 0\) and \(\psi(\infty) < \infty\), then \(M_\psi(0, \infty) = L^1(0, \infty)\) as sets.

Define the noncommutative Marcinkiewicz space as

\[
M_\psi = M_\psi(M, \tau) = \{ x \in L^0 : \mu_t(x) \in M_\psi(0, \infty) \} 
\]
and set
\[ \|x\|_{M_\psi} = \|\mu_t(x)\|_{M_\psi}, \quad x \in M_\psi(\mathcal{M}, \tau) \]
(see, for example, [4]). It is known [21] that \((M_\psi, \|\cdot\|_{M_\psi})\) is a noncommutative fully symmetric function space. If \(\tau(1) < \infty\) or \(\psi(+0) > 0\) and \(\psi(\infty) < \infty\), then \(M_\psi \subset L^1\). If \(\tau(1) = \infty\) and \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\), then \(1 \notin M_\psi\).
Thus, the corresponding version of Theorem 5.2 holds for Marcinkiewicz spaces \(M_\psi(\mathcal{M}, \tau)\) if we replace condition \(\psi(\infty) = \infty\) by \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\).

6. THE COMMUTATIVE CASE

In this section we present applications of Theorems 4.2 and 4.3 to fully symmetric function spaces.

Let \((\Omega, \nu) = (\Omega, \mathcal{A}, \nu)\) be a Maharam measure space (see, for example, [21] Ch.1, §1.2, Sections 1.1.7, 1.1.8), that is,

(i) \(\nu\) is a countably additive function defined on a \(\sigma\)-algebra \(\mathcal{A}\) of subsets of a set \(\Omega\) with the values in the extended half-line \([0, \infty]\);
(ii) if \(A \subset \Omega\) and \(A \cap F \in \mathcal{A}\) for all \(F \in \mathcal{A}\) with \(\nu(F) < \infty\), then \(A \in \mathcal{A}\);
(iii) if \(A \subset F \in \mathcal{A}\) and \(\nu(F) = 0\), then \(A \in \mathcal{A}\);
(iv) for any \(A \in \mathcal{A}\) there exists \(F \in \mathcal{A}\) such that \(F \subset A\) and \(\nu(F) < \infty\);
(v) the Boolean algebra \(\mathcal{A}\) of classes of \(\nu\)-almost everywhere equal sets in \(\mathcal{A}\) is order complete.

Clearly, every complete \(\sigma\)-finite measure space is Maharam measure space.

It is well known that the \(\ast\)-algebra \(L^\infty(\Omega, \nu)\) of (equivalence classes of) essentially bounded measurable complex-valued functions defined on a Maharam measure space \((\Omega, \nu)\) is a noncommutative von Neumann algebra with a semifinite normal faithful trace \(\nu(f) = \int_\Omega f \, d\nu, \quad 0 \leq f \in L^\infty(\Omega, \nu)\). The converse is also true: any commutative von Neumann algebra \(\mathcal{M}\) with a semi-finite normal faithful trace \(\tau\) is \(\ast\)-isomorphic to the \(\ast\)-algebra \(L^\infty(\Omega, \nu)\) for some Maharam measure space \((\Omega, \nu)\) such that \(\tau(f) = \int_\Omega f \, d\nu, \quad 0 \leq f \in L^\infty(\Omega, \nu)\) (see, for example, [22] Ch.7, §7.3).

If \((\Omega, \nu)\) is a Maharam measure space and \(\mathcal{M} = L^\infty(\Omega, \nu)\) with \(\tau(f) = \int_\Omega f \, d\nu, \quad 0 \leq f \in L^\infty(\Omega, \nu)\), then \(L^0 = L^0(\mathcal{M}, \tau) = L^0(\Omega, \nu)\) is the algebra of (equivalence classes of) almost everywhere finite measurable complex-valued functions on \((\Omega, \nu)\) with \(\nu(|f| > \lambda) < \infty\) for some \(\lambda > 0\). The non-increasing rearrangement of a function \(f \in L^0(\Omega, \nu)\) is the function \(f^*\) on \((0, \infty)\) defined by [12].

Let \((E(0, \infty), \|\cdot\|_E)\) be a symmetric function space on \(((0, \infty), \mu)\). Let
\[ E(\Omega, \nu) = E(\mathcal{M}, \tau) = \{ f \in L^0(\Omega, \nu) : f^*(t) \in E(0, \infty) \}\]
and
\[ \|f\|_{E(\Omega)} = \|f\|_{E(\mathcal{M})} = \|f^*(t)\|_E, \quad f \in E(\Omega, \nu). \]

Then \((E(\Omega, \nu), \|\cdot\|_{E(\Omega)})\) is a symmetric function space on \((\Omega, \nu)\). Recall that this space is a fully symmetric function space if conditions
\[ f \in E(\Omega, \nu), \quad g \in L^0(\Omega, \nu), \quad \int_0^s g^*(t) \, dt \leq \int_0^s f^*(t) \, dt, \quad \forall \, s > 0 \]
imply that \(g \in E(\Omega, \nu)\) and \(\|g\|_{E(\Omega)} \leq \|f\|_{E(\Omega)}\). Note that if \(\nu(\Omega) < \infty\), then \(E(\Omega, \nu) \subset L^1(\Omega, \nu)\).
A net \( \{ f_\alpha \} \subset L^0(\Omega, \nu) \) converges almost uniformly (a.u.) to \( f \in L^0(\Omega, \nu) \) if for any \( \epsilon > 0 \) there is a set \( A \in \mathcal{A} \) such that \( \tau(\Omega \setminus A) \leq \epsilon \) and \( \| f - f_\alpha \chi_A \|_\infty \to 0 \), where \( \chi_A \) is the characteristic function of \( A \).

Now we can state the following corollary of Theorems 4.2 and 4.3.

**Theorem 6.1.** Let \((\Omega, \nu)\) be a Maharam measure space. Let \( E = E(\Omega, \nu) \) be a fully symmetric function space on \((\Omega, \nu)\) such that \( 1/\infty \in E \) if \( \nu(\Omega) = \infty \). If \( \{ T_t \}_{t \geq 0} \subset DS^+ \) is a strongly continuous semigroup in \( L^1(\Omega, \nu) \) and \( \beta(t) \) a bounded Besicovitch (zero-Besicovitch) function, then, given \( f \in E \), the averages (3) converge a.u. to some \( \hat{f} \in E \) as \( t \to \infty \) (respectively, to \( \alpha(f) f \) as \( t \to 0 \) for some \( \alpha(f) \in \mathbb{C} \)).

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