A characterization of the edge connectivity of direct products of graphs

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Abstract

The direct product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph, denoted as $G \times H$, with vertex set $V(G \times H) = V(G) \times V(H)$, where vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent in $G \times H$ if $x_1 x_2 \in E(G)$ and $y_1 y_2 \in E(H)$. The edge connectivity of a graph $G$, denoted as $\lambda(G)$, is the size of a minimum edge-cut in $G$. We introduce a function $\psi$ and prove the following formula

$$\lambda(G \times H) = \min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H), \psi(G,H), \psi(H,G)\}.$$ 

We also describe the structure of every minimum edge-cut in $G \times H$.

Key words: Direct product, edge connectivity

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1 Introduction

Weichsel observed in [17] that the direct product of graphs $G$ and $H$ is connected if and only if both graphs are connected and at least one of them is nonbipartite. Several decades after this observation a detailed study of connectivity of graph products was initiated by different authors. The aim of this study is to determine the connectivity of the product and express it in terms of connectivities of the factors. The second aim is to describe the structure and other properties of a minimum vertex-cut or a minimum edge-cut in the product. The size and the structure of a minimum edge-cut in the Cartesian product of graphs was determined in [9] where the following result is proved

$$\lambda(G \Box H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta(G \Box H)\}.$$ 

The authors also prove that every minimum edge-cut in the Cartesian product of graphs is the preimage (under projection) of a minimum edge-cut of a factor, or the set of

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edges incident to a vertex of minimum degree. Therefore the connected components of a minimum edge-cut in the Cartesian product coincide with one of the first three cases of Fig. 1. For the strong product of graphs a similar formula for the edge connectivity and an analogous result about the structure of a minimum edge-cut was proved, see [1]. A partial answer to these questions was obtained for the vertex connectivity of the Cartesian and the strong product of graphs, see [12, 13].

The problem of determining the connectivity of direct products of graphs appears to be more challenging (compared to other graph products). This comes from the fact that the direct product of two bipartite graphs is not connected although both graphs may have high edge and vertex connectivities. Therefore it is not possible to express the connectivity of the direct product of graphs exclusively in terms of connectivities (and minimum degree, size and order) of the factors. There is an extensive list of articles on connectivity of direct products of graphs where the authors consider special examples and determine their connectivity, or they obtain upper and lower bounds for the connectivity, or they study related concepts such as super connectivity, see [2, 3, 5, 6, 10, 11, 14, 15, 16, 7]. In this article we settle the question of edge connectivity of direct products as well as the question about the structure of a minimum edge-cut in the direct product of graphs.

![Figure 1: Connected components of a minimum edge cut in $G \times H$.](image)

Before we state our main result we give the definitions and the notation we use. Let $G$ and $H$ be graphs and $G \times H$ their direct product. For $x \in V(G)$ and $y \in V(H)$, the $H$-layer $H_x$, and the $G$-layer $G_y$, are defined as

$$H_x = \{(x, h) \mid h \in V(H)\} \quad \text{and} \quad G_y = \{(g, y) \mid g \in V(G)\}.$$ 

If $S$ is a subset of $E(G)$ then we call an edge of $S$ an $S$-edge. The graph obtained from $G$ by deleting all $S$-edges is denoted by $G - S$. A set $S \subseteq E(G)$ is an edge-cut in
$G$ if $G - S$ is not connected, and the edge connectivity of $G$ is the size of a minimum edge-cut in $G$. If $S$ is a minimum edge-cut in $G \times H$ then we denote by $B$ and $W$ the connected components of $(G \times H) - S$. We refer to $B$ as black and $W$ as white. For example, when we say that a vertex $(x, y)$ is white, we mean that $(x, y) \in W$. We say that an edge cut $S$ in $G \times H$ is of type 1 if $B = B' \times V(H)$ and $W = W' \times V(H)$ where $B' \cup W' = V(G)$. The definitions of all other types of edge-cuts are evident from Fig. 1. For example $S$ is of type 3 if $|B| = 1$ or $|W| = 1$. The bipartite edge frustration of a graph $G$, denoted as $\varphi(G)$, is the minimum number of edges whose deletion makes $G$ a bipartite graph. The concept was introduced in [8] and studied in [4, 8, 19]. In section 2 we formally define the function $\psi$ which is needed to determine the edge connectivity of direct products, moreover the bipartite edge frustration of factors is crucial in this definition. It follows from the definition that for a bipartite graph $H$, $\psi(G, H)$ is equal to the size of a minimum edge cut of type 4 or 5, and if $G$ is bipartite, then $\psi(H, G)$ is equal to the size of a minimum edge cut of type 4 or 6. We prove the following theorem in Section 2.

**Theorem 1.1** For any graphs $G$ and $H$

$$\lambda(G \times H) = \min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H), \psi(G, H), \psi(H, G)\}.$$ 

The structure theorem is proved in Section 3.

**Theorem 1.2** If $G$ and $H$ are not equal to a path on three vertices or a 4-cycle then every minimum edge-cut in $G \times H$ is of type 1, \ldots, 7 or 8.

It also follows from our results, that for every graph product $G \times H$ there is minimum edge-cut in $G \times H$ of type 1, 2, 3, 4, 5, or 6. In particular, if there is a minimum edge-cut in $G \times H$ of type 8 (resp. 7), then there is also a minimum edge-cut of type 6 or 3 (resp. 5 or 3). In Fig. 2 we give an example of a graph product that has a minimum edge-cut (of size 2) with the structure that cannot be categorized as one of the eight types.

![Figure 2: An example of an edge-cut with no structure.](image-url)
2 The size of a minimum edge cut

For each vertex \((x, y) \in V(G \times H)\) we denote by \(K(x, y)\) the subgraph of \(G \times H\) induced by
\[
(N_G(x) \times \{y\}) \cup (\{x\} \times N_H(y)).
\]
Note that this is a complete bipartite subgraph of \(G \times H\).

**Observation 2.1** The edge \(e = (x', y)(x, y')\) is an edge of \(K(u, v)\) if and only if \((u, v) = (x, y)\) or \((u, v) = (x', y')\).

Observation 2.1 is depicted in Fig. 3. The following observation is due to Weichsel [17].

![Fig. 3](image)

**Observation 2.2** Let \(G\) be a connected graph. If \(G \times K_2\) is not connected then \(G\) is bipartite and connected components of \(G \times K_2\) are
\[
C_1 = (A \times \{x\}) \cup (B \times \{y\}) \quad \text{and} \quad C_2 = (A \times \{y\}) \cup (B \times \{x\}),
\]
where \(V(G) = A \cup B\) is the bipartition of graph \(G\), and \(V(K_2) = \{x, y\}\).

**Lemma 2.3** Let \(G\) and \(H\) be graphs such that \(\delta(G) \leq \delta(H)\). Let \(S\) be a minimum edge cut in \(G \times H\) and \(B, W\) be connected components of \((G \times H) - S\). If \(|S| < \delta(G \times H)\) then for every \((x, y) \in V(G \times H)\) either \(N(x) \times \{y\} \subseteq B\) or \(N(x) \times \{y\} \subseteq W\).

**Proof.** Let \(S\) be a minimum edge cut in \(G \times H\) and let \(B, W\) be connected components of \((G \times H) - S\). We refer to \(B\) as black and \(W\) as white. Let \((x, y)\) be any vertex of \(G \times H\) and let \(B_G(x, y)\) and \(W_G(x, y)\) be subsets of \(N_G(x)\) such that \(B_G(x, y) \times \{y\}\) is black and \(W_G(x, y) \times \{y\}\) is white. Additionally let \(B_H(x, y)\) and \(W_H(x, y)\) be subsets of \(N_H(y)\) such that \(\{x\} \times B_H(x, y)\) is black and \(\{x\} \times W_H(x, y)\) is white (see Fig. 4).
We first prove the lemma under assumption that $\delta(H) \geq 3$. Assume (contrary to the claim of the lemma) that $|B_G(x,y)|, |W_G(x,y)| \geq 1$. To prove the lemma we will show that $|S| \geq \delta(G \times H)$. To prove this we count the number of $S$-edges in each subgraph $K(u,v)$, where

$$(u,v) \in (B_G(x,y) \times W_H(x,y)) \cup (W_G(x,y) \times B_H(x,y)) = X.$$  

For each $(u,v) \in X$ let us denote by $\alpha'(u,v)$ the number of $S$-edges of $K(u,v)$ that have both endvertices in $B_G(x,y) \times W_H(x,y)$, or both endvertices in $W_G(x,y) \times B_H(x,y)$, or one endvertex in $W_G(x,y) \times W_H(x,y)$ and the other in $B_G(x,y) \times B_H(x,y)$. Note that $\alpha'(u,v)$ is the number of $S$-edges of $K(u,v)$ that are counted twice when counting $S$-edges of $K(u,v)$'s and $(u,v)$ goes through the set $X$ (see Observation 2.1). Let $\alpha''(u,v)$ be the number of all other $S$-edges of $K(u,v)$ and set

$$\alpha(u,v) = \frac{1}{2} \alpha'(u,v) + \alpha''(u,v).$$

Clearly

$$|S| \geq \sum_{(u,v) \in X} \alpha(u,v).$$

We claim that $\alpha(u,v) \geq \delta(G)$ for all $(u,v) \in X$. Denote by $B_G, B_H, W_G$ and $W_H$ the size of black and white parts of $N_G(u) \times \{v\}$ and $\{u\} \times N_H(v)$ (see Fig. 5). Since $(x,v)$ is black and $(u,y)$ is white (if $(u,v) \in W_G(x,y) \times B_H(x,y)$, otherwise the converse
is true) we find that $B_G \geq 1$ and $W_H \geq 1$. Note also that $S$-edges of $K(u,v)$ incident to $(x,v)$ or $(u,y)$ are not counted twice, and hence contribute to $\alpha''(u,v)$. Therefore

$$\alpha(u,v) \geq B_G + W_H - 1 + \frac{1}{2}((B_G - 1)(W_H - 1) + B_HW_G).$$

If $B_H = 0$ or $W_G = 0$, then $W_H \geq \delta(H)$ or $B_G \geq \delta(G)$, and hence $\alpha(u,v) \geq \delta(H)$ or $\alpha(u,v) \geq \delta(G)$. So assume $B_H > 0$ and $W_G > 0$. Since $B_G \geq \delta(G) - W_G$ and $W_H \geq \delta(H) - B_H$ we find that

$$\alpha(u,v) \geq \frac{1}{2}(B_G + W_H) - 1 + \frac{1}{2}((B_G - 1)(W_H - 1) + B_HW_G - B_H - W_G).$$

Since $B_H > 0$ and $W_G > 0$ we see that $B_HW_G - B_H - W_G \geq -1$. Therefore the only possibility for $\alpha(u,v) < \delta(G)$ is when

$$B_HW_G - B_H - W_G = -1, B_G + W_H = 2 \text{ and } \delta(G) = \delta(H).$$

If the above equalities are true we have $W_H = 1$ and $B_G = 1$. Since $W_H = 1$ we find that $B_H \geq 2$ (for otherwise $\delta(H) \leq 2$) and therefore $B_HW_G - B_H - W_G = -1$ only if $B_H = 2$ and $W_G = 1$. But then $\delta(G) = 2$ and hence $\delta(G) \neq \delta(H)$. This proves that $\alpha(u,v) \geq \delta(G)$ for all $(u,v) \in X$, and therefore

$$|S| \geq \sum_{(u,v) \in X} \alpha(u,v) \geq \sum_{(u,v) \in X} \delta(G) \geq \delta(G)\delta(H) \quad (1)$$

and the equality holds only if $|X| = \delta(H)$ and $\alpha(u,v) = \delta(G)$ for all $(u,v) \in X$.

Now assume that $2 \geq \delta(H) \geq \delta(G)$. If $\delta(H) = \delta(G) = 2$ then $\alpha(u,v) \geq 3/2$ for every $(u,v) \in X$. So if $|X| \geq 3$ we have

$$|S| \geq \sum_{(u,v) \in X} \alpha(u,v) \geq 9/2 > \delta(G)\delta(H).$$
If \(|X| = 2\) we have one of the three cases presented in Fig. 6. Let \(N(x) = \{a, b\}\) and \(N(y) = \{c, d\}\).

For the second and third case we have \(\alpha(u, v) \geq 2\) for all \((u, v) \in X\) (because the set \(X\) is a subset of a single \(H\)-layer and therefore \(\alpha'(u, v) = 0\)), and thus \(|S| \geq \sum_{(u, v) \in X} \alpha(u, v) \geq 4\). For the first case we argue that \(\alpha(u, v) = 2\) unless \((a, c)\) is adjacent to \((b, d)\) and one of these two vertices is black and the other white. Since \(\alpha(b, c) + \alpha(a, d) \geq 3\) and there is one additional \(S\)-edge (either \((x, y)(a, c)\) or \((x, y)(b, d)\)) we find that \(|S| \geq 4\). Finally if \(\delta(G) = 1\) we have \(\alpha(u, v) \geq \delta(G)\) and since \(|X| \geq \delta(H)\) we conclude \(|S| \geq \delta(G)\delta(H)\).

□

Several details of the above proof are needed in the following section where we describe the structure of a minimum edge-cut in a direct product of graphs. We use the notation of the above proof and write these details in remarks that follow. Additionally we define

\[ T = S \cap \{e \in E(K(u, v)) \mid (u, v) \in X\} \]

and use this notation in the rest of the paper.

**Remark 2.4** Let \(G\) and \(H\) be graphs with \(\delta(G) \leq \delta(H)\) and \(\delta(H) \geq 3\). Suppose that \((u, v) \in X, \alpha'(u, v) = 0\) and \(\alpha(u, v) = \delta(G)\).

(i) If \((u, v) \in W_G(x, y) \times B_H(x, y)\) then \(W_H = 1\) and \(W_G = 0\), or \(B_G = 1\) and \(B_H = 0\). Moreover if \(\delta(G) < \delta(H)\) then \(W_H = 1\) and \(W_G = 0\).

(ii) If \((u, v) \in B_G(x, y) \times W_H(x, y)\) then \(B_H = 1\) and \(B_G = 0\), or \(W_G = 1\) and \(W_H = 0\). Moreover if \(\delta(G) < \delta(H)\) then \(B_H = 1\) and \(B_G = 0\).

**Proof.** Since \(\alpha'(u, v) = 0\) we find that \(\alpha(u, v) = B_G W_H + W_G B_H\). The result follows from \(\delta(G) \leq \delta(H)\) and \(\delta(H) \geq 3\).

□

The following remark follows directly from inequality (1) and the fact that \(|X| \geq \delta(H)\) and \(\alpha(u, v) \geq \delta(G)\).

**Remark 2.5** Let \(G\) and \(H\) be graphs with \(\delta(G) \leq \delta(H)\) and \(\delta(H) \geq 3\), and let \(S\) be a minimum edge-cut in \(G \times H\). If there is a vertex \((x, y) \in V(G \times H)\) such that
$W_G(x, y) \neq \emptyset$ and $B_G(x, y) \neq \emptyset$ then $|X| = \delta(H)$ and $\alpha(u, v) = \delta(G)$ for every $(u, v) \in X$.

In proof of Lemma 2.3 we have showed that $|T| \geq \delta(G)\delta(H)$ except in the case when $\delta(G) = \delta(H) = 2$ where we needed an additional $S$-edge to prove that $|S| \geq \delta(G)\delta(H)$. We write this in the following remark.

**Remark 2.6** Let $G$ and $H$ be graphs such that $\delta(G) \leq \delta(H)$ and $\delta(H) \geq 3$. If there is a vertex $(x, y) \in V(G \times H)$ such that $B_G(x, y) \neq \emptyset$ and $W_G(x, y) \neq \emptyset$ then $|T| \geq \delta(G)\delta(H)$.

For a graph $G$ and $A, B \subseteq V(G)$, we denote by $[A, B]$ the set of all vertices with one endvertex in $A$ and the other in $B$. Additionally, we denote by $[A]$ the subgraph of $G$ induced by $A$.

Let $G$ and $H$ be graphs. If $S$ is a minimum edge cut of $G$ and $X, Y$ are connected components of $G - S$ then

$$[X \times V(H), Y \times V(H)]$$

is an edge cut of $G \times H$ whose connected components are $X \times V(H)$ and $Y \times V(H)$ as shown in case 1 of Fig. 1. Clearly

$$|[X \times V(H), Y \times V(H)]| = 2\lambda(G)|E(H)|,$$

and hence $\lambda(G \times H) \leq 2\lambda(G)|E(H)|$. Similarly we prove $\lambda(G \times H) \leq 2\lambda(H)|E(G)|$, and obviously $\lambda(G \times H) \leq \delta(G \times H)$.

Recall that the bipartite edge frustration of a graph $G$, denoted as $\varphi(G)$, is the minimum number of edges whose delition makes $G$ a bipartite graph. Denote by $\mathcal{P}(V(G))$ the set of all partitions of $V(G)$ on two subsets (one of them may be empty), and let

$$\rho(G) = \min\{2\varphi([A]) + |[A, B]| : \{A, B\} \in \mathcal{P}(V(G)), A \neq \emptyset\}.$$  

For graphs $G$ and $H$ we define

$$\psi(G, H) = \rho(G)|E(H)| + 2\varphi(H)|E(G)|.$$  

We claim that $\lambda(G \times H) \leq \psi(G, H)$. To see this let $\{A, B\} \in \mathcal{P}(V(G))$ be a partition of $V(G)$ for which $2\varphi([A]) + |[A, B]|$ is minimum, and let $C \cup D$ be a partition of $V(H)$ such that $|[C, D]| = |E(H)| - \varphi(H)$. Assume also that $A = A_1 \cup A_2$ is a partition of $A$ with $|[A_1, A_2]| = |E([A])| - \varphi([A])$ (so there are $\varphi([A])$ edges with both endvertices in $A_1$ or both in $A_2$). Let $(A_1 \times C) \cup (A_2 \times D)$ be black, and $(A_2 \times C) \cup (A_1 \times D) \cup (B \times V(H))$ white, see Fig. 7. We will show that the number of edges between black and white part of $V(G \times H)$ is at most $\rho(G)|E(H)| + 2\varphi(H)|E(G)|$.

The number of edges with both endvertices in $V(G) \times C$ or both in $V(G) \times D$ is at most $2\varphi(H)|E(G)|$. So we only need to count the edges with one endvertex in $V(G) \times C$ and the other in $V(G) \times D$. For each edge in $[A, B]$ and each edge in $[C, D]$ there is exactly one edge in $G \times H$ with one endvertex black and the other white, so the total number of such edges is at most $|E(H)||[A, B]|$. For each edge with both endvertices
in $A_1$ (or both in $A_2$), and each edge in $[C, D]$, there are two edges in $G \times H$ with one endvertex black and the other white. Therefore the total number of such edges is at most $2\phi(|A|)|E(H)|$. This proves that $\lambda(G \times H) \leq \psi(G, H)$. If $H$ is bipartite then $[C, D] = V(H)$ so the number of $S$-edges with one endvertex in $V(G) \times C$ and the other in $V(G) \times D$ is equal to $\rho(G)|E(H)|$. This explains the following remark.

**Remark 2.7** If $H$ is bipartite with $E(H) = [C, D]$, then $\psi(G, H)$ is equal to the minimum size of an edge-cut of type 4 or 5 such that $(A_1 \times C) \cup (A_2 \times D)$ is black (and the rest white) for some $A_1, A_2 \subseteq V(G)$. Analogously, if $G$ is bipartite with $E(G) = [A, B]$, then $\psi(H, G)$ is equal to the minimum size of an edge-cut of type 4 or 6 such that $(A \times C_1) \cup (B \times C_2)$ is black (and the rest white) for some $C_1, C_2 \subseteq V(H)$.

Combining all upper bounds into one we get

$$
\lambda(G \times H) \leq \min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H), \psi(G, H), \psi(H, G)\}.
$$

We prove next that the above inequality is in fact an equality.

**Proof of Theorem 1.1.** If $G$ or $H$ is not connected then both sides are 0. So assume that $G$ and $H$ are connected. Let $S$ be a minimum edge-cut in $G \times H$ and $B, W$ be connected components of $(G \times H) - S$. We need to prove that

$$
|S| \geq \min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H), \psi(G, H), \psi(H, G)\}.
$$

If $|S| \geq \delta(G \times H)$ then we are done. So assume that $|S| < \delta(G \times H)$. Without loss of generality assume also $\delta(G) \leq \delta(H)$. By Lemma 2.3 for every $(x, y) \in V(G \times H)$ either $N(x) \times \{y\} \subseteq B$ or $N(x) \times \{y\} \subseteq W$. Let $y \in V(H)$ be any vertex. If there are two adjacent vertices $x_1, x_2 \in V(G)$ such that $(x_1, y)$ and $(x_2, y)$ are both black (or both white) then we find that $G_y \subseteq B$ (or $G_y \subseteq W$). If there are no such vertices, then we have a proper 2-coloring of $G$, and hence $G$ is bipartite. Moreover if $X \cup Y$ is the unique (recall that $G$ is connected) bipartition of $V(G)$ then either $X \times \{y\} \subseteq W$ and $Y \times \{y\} \subseteq B$, or $X \times \{y\} \subseteq B$ and $Y \times \{y\} \subseteq W$. Since this is true for every $y \in V(H)$ we find that the edge-cut $S$ is of type 2, 4, 6, or 8 (see Fig. 1). Moreover types 4, 6 and
8 can occur only if $G$ is bipartite. If $S$ is of type 2 we see that $|S| \geq 2\lambda(H)|E(G)|$. Otherwise $G$ is bipartite and therefore $\varphi(G) = 0$. It follows that $\psi(H, G) = \rho(H)|E(G)|$. In a partition of type 8 we may change the color of all black $G$-layers to become white. In this way we get a partition of type 6, moreover such partition requires deletion of fewer or equal number of edges (because $G$ is bipartite). If $S$ is of type 4 or 6 we have $|S| \geq \rho(H)|E(G)| = \psi(H, G)$ (see Remark 2.7), and therefore the same applies for type 8 (because it requires deletion of at least as much edges as type 6). 

The proof of the above theorem gives a more precise information regarding the structure of connected components. We write this in the following corollary.

**Corollary 2.8** Let $G$ and $H$ be graphs with $\delta(G) \leq \delta(H)$. If $S$ is a minimum edge-cut in $G \times H$ and for every $(x, y) \in V(G \times H)$ either $N(x) \times \{y\} \subseteq B$ or $N(x) \times \{y\} \subseteq W$ then $S$ is of type 2, 4, 6 or 8.

## 3 The structure of a minimum edge cut

In this section we use the notation and definitions of the previous section.

**Proof of Theorem 1.2** Let $S$ be a minimum edge-cut in $G \times H$ and $B, W$ be connected components of $(G \times H) - S$. Assume that $\delta(G) \leq \delta(H)$ and $\delta(H) \geq 3$.

Corollary 2.8 settles the structure of a minimum edge-cut if $N(x) \times \{y\} \subseteq B$ or $N(x) \times \{y\} \subseteq W$ for every $(x, y) \in V(G \times H)$. Assume now that there is $(x, y) \in V(G \times H)$ such that

$$B_G(x, y) = (N(x) \times \{y\}) \cap B \neq \emptyset \quad \text{and} \quad W_G(x, y) = (N(x) \times \{y\}) \cap W \neq \emptyset.$$ 

![Figure 8: The four cases when $B_G(x, y) \neq \emptyset$ and $W_G(x, y) \neq \emptyset$.](image)

By Remark 2.3 we know that $|X| = \delta(H)$ and $\alpha(u, v) = \delta(G)$ for every $(u, v) \in X$. It follows from Remark 2.6 that $|T| \geq \delta(G)\delta(H)$ and therefore $S = T$. This is particularly
noteworthy since most of the arguments in the proof below reduce to the fact that there are no $S$-edges other than the edges in $T$.

Since $|X| = \delta(H)$ we see that one of the four cases shown in Fig. 8 occurs. Note that in all four cases the color of vertex $(x, y)$ is the same as the color of vertices in $N_{G \times H}(x, y)$ because no edge incident to $(x, y)$ is in $T$ (and therefore also not in $S$). Therefore for every $(u, v) \in X$ we have $\alpha'(u, v) = 0$. Since also $\alpha(u, v) = \delta(G)$ for every $(u, v) \in X$ we may use Remark 2.4 in the sequel. Observe also that

\[
\alpha(u, v) \geq \min\{\deg_G(u), \deg_H(v)\}. \tag{2}
\]

Case (i) and (ii). The color of vertex $(x, y)$ is irrelevant, so we prove both cases simultaneously. We claim that $G$ is $P_3$ or $C_4$. Let $N(x) = \{a, b\}$. Consider open neighborhoods of vertices $a$ and $b$ as shown in Fig. 9 and note that $x \in N(a) \cap N(b)$.

We claim that $(N(a) \cup N(b)) \times B_H(x, y)$ is black and $(N(a) \cup N(b)) \times W_H(x, y)$ is white. Since every vertex in $N(a) \times B_H(x, y)$ is adjacent to $(a, y)$, and the corresponding edges are not in $T$, we find that $N(a) \times B_H(x, y)$ is black. Since $\delta(G) \leq 2$ we find that $\delta(H) > \delta(G)$ and therefore $N(b) \times B_H(x, y)$ is black by Remark 2.4 (because $W_H = 1$ and $W_G = 0$ for every $(u, v) \in W_G(x, y) \times B_H(x, y)$). Similar arguments prove that $(N(a) \cup N(b)) \times W_H(x, y)$ is white.

![Figure 9: Cases (i) and (ii).](image-url)

Next we claim that $N(N(a) \cup N(b)) \subseteq \{a, b\}$. If not, there is a $t \neq a, b$ adjacent to a vertex $t' \in N(a) \cup N(b)$. Hence $(t, y)$ is adjacent to black and white vertices in $\{t'\} \times B_H(x, y)$ and $\{t'\} \times W_H(x, y)$, respectively. Thus there is at least one $S$-edge not in $T$, a contradiction. If $N(a) = N(b) = \{x\}$ then $G = P_3$. If $N(a) \neq \{x\}$ or $N(b) \neq \{x\}$ then $N(a) = N(b)$, for otherwise $\delta(G) = 1$ and hence $\alpha(u, v) > \delta(G)$ for a $(u, v) \in X$ (see (2)). Since $\delta(G) \leq 2$ we find that $|N(a)| \leq 2$ (for otherwise inequality (2) implies $\alpha(u, v) \geq 3 > \delta(G)$ for all $(u, v) \in X$). Since $a$ and $b$ are nonadjacent and the vertices in $N(a)$ are nonadjacent (otherwise we have $S$-edges that are not in $T$) we find that $G$ is either $P_3$ or $C_4$.

Case (iii). Let $u \in B_G(x, y)$ and observe that $H_x$ and $H_u$ are adjacent and in both layers there are black and white vertices. Since $S = T$ and hence $[H_x, H_u] \cap S = \emptyset$ we
find that the subgraph of $G \times H$ induced by $H_x \cup H_u$ is not connected, and therefore $H$ is bipartite by Observation 2.2. Let $W_G(x, y) = \{b\}$ and $E(H) = [C, D]$. Consider the graph $G - b$ (vertex deleted subgraph of $G$), and observe that no $S$-edge has both endvertices in $V(G - b) \times V(H)$ (because $S = T$ and each $T$-edge has one endvertex in $H_b$). Suppose that $G - b$ has no isolated vertices and consider two adjacent $H$-layers $H_s$ and $H_t$ where $s, t \neq b$. By Observation 2.2 one of the following occurs:

1. $H_s$ and $H_t$ are both white
2. $H_s$ and $H_t$ are both black
3. $(\{s\} \times C) \cup (\{t\} \times D)$ is white and $(\{s\} \times D) \cup (\{t\} \times C)$ is black
4. $(\{s\} \times C) \cup (\{t\} \times D)$ is black and $(\{s\} \times D) \cup (\{t\} \times C)$ is white

The layers $H_u$ and $H_x$ fall into the last two cases. Therefore each $H$-layer, except $H_b$, is either black, or white, or partitioned (into black and white parts) like $H_x$, or like $H_u$. Now the $H_b$ layer may only be adjacent to layers that are white or partitioned like $H_x$ (because if $b'$ is adjacent to $b$, then $(b', y)$ is white unless there is an $S$-edge not in $T$). By Remark 2.4 either all layers adjacent to $H_b$ are white or all are partitioned as $H_x$. Moreover there are no black $H$-layers. In the first case $H_b$ is white and $S$ is of type 5, and in the second case $G$ is bipartite (since also $H$ is bipartite this is a contradiction). Finally if $G - b$ has an isolated vertex then $\delta(G) = 1$ and therefore $N(b) = \{x\}$ by (2). In this case $G$ is bipartite, a contradiction.

**Case (iv).** Here we have two cases (see Fig. 10). In any case, the vertex $(x, y)$ is black, and so are also parts $E$ and $F$. Since $A$ is adjacent to $G$, and $G$ is black, we find that also $A$ is black. Parts $C, D$ and $I$ are black because they are adjacent to $F$. The only difference of these two cases is in parts $B$ and $H$. Either both are black or both white (see Remark 2.4). If both are black, then there is only one white vertex and hence $|W| = 1$, otherwise $S$ is of type 5. This proves the theorem if $\delta(H) \geq 3$.

![Figure 10: Possibilities in case (iv).](image-url)
Now we consider the case $\delta(G) \leq \delta(H) \leq 2$. We prove it by a case analysis. Assume that $\delta(G) = \delta(H) = 2$. If $|X| \geq 3$ then

$$|S| \geq \sum_{(u,v) \in X} \alpha(u,v) \geq 3 \cdot \frac{3}{2} > 4 = \delta(G)\delta(H),$$

because $\alpha(u,v) \geq 3/2$ for all $(u,v) \in X$. So assume that $|X| = 2$. One of the cases shown in Fig. 6 occurs. Let $N(x) = \{a, b\}$ and $N(y) = \{c, d\}$. Since $\delta(G) = \delta(H) = 2$ there is at least one neighbor $a' \neq x$ of $a$, $b' \neq x$ of $b$, $c' \neq y$ of $c$, and $d' \neq y$ of $d$.

Consider the first case where $(a, y)$ and $(x, c)$ are black, and $(b, y)$ and $(x, d)$ are white. Since $(a', c)$ is adjacent to $(a, y)$ we find that $(a', c)$ is black. If $(a', d)$ is white then $N(a') \subseteq \{a, b\}$, and since $\delta(G) = 2$ we find that $N(a') = \{a, b\}$. If $a' \neq b'$ then $\{x, a', b'\} \subseteq N(b)$ and hence $\alpha(b, c) \geq 3 > \delta(G)$, a contradiction. Otherwise, if $a' = b'$, then $G = C_4$. Assume now that both $(a', c)$ and $(a', d)$ are black (note that in this case $a' \neq b'$ because of the edge $(b', d)(b, y)$). If $N(a') \subseteq \{a, b\}$ then we argue the same as before. If not, there is a vertex $a'' \neq a, b$ adjacent to $a'$. By Observation 2.2 $H$ is bipartite (because $a'$-layer has black and white vertices, and no $T$-edge has one endvertex in $a'$-layer and the other in $a''$-layer). Similar arguments prove that either $\alpha(a, d) \geq 3$ (if $c'$ is adjacent to $d$) or $G$ is biparite. In both cases we have a contradiction (either both graphs are biparite or $\alpha(u, v) > \delta(G)$ for a $(u, v) \in X$).

Consider now the second case of Fig. 6. We claim that $S$ is of type 3 or $H$ is $C_4$. Since $\delta(G) = 2$ we find that $a'$ has a neighbor $a'' \neq a$. If $(a', c)$ is black and $(a', d)$ is white, or vice versa, we have an $S$-edge incident to $(a'', y)$, a contradiction. Therefore either $(a', c)$ and $(a', d)$ are both black or both white. If they are both black, then $H$ is biparite. Since $G$ is not biparite we find that layers $G_{c'}$ and $G_{d'}$ are both white unless $c' = d'$. If $c' = d'$ then $H = C_4$, otherwise $(a', c')$ and $(a', d')$ are both white, which is in a contradiction to Observation 2.2. If $(a', c)$ and $(a', d)$ are both white then $S$ is of type 3.

In third case of Fig. 6 the graph $H$ is biparite. If $H \neq C_4$ we find that $c'$ has a neighbor $c'' \neq c$. Since $G$ is not biparite we find that layers $G_{c'}$ and $G_{c''}$ are both white and hence $(x, c')$ and $(x, c'')$ are white, a contradiction.

Assume that $\delta(G) = 1$ and $\delta(H) = 2$. If $(x, c)$ is black and $(x, d)$ white we find that $G = P_3$ (because $N(x) = \{a, b\}$). If $(x, c)$ and $(x, d)$ are both black then $H$ is biparite. If $c' = d'$ then $H = C_4$, otherwise also $G$ is bipartite, a contradiction.

Finally we look at the case $\delta(G) = \delta(H) = 1$. Assume that $|V(G)|, |V(H)| \geq 3$. If all but one $G$-layer (or $H$-layer) are white we find that $S$ is of type 3. Otherwise both graphs are bipartite. If $H = K_2$ we find that $S$ is of type 3 or 5.

\[\square\]

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