JACOB’S LADDERS, THEIR ITERATIONS AND THE NEW CLASS OF INTEGRALS CONNECTED WITH PARTS OF THE HARDY-LITTLEWOOD INTEGRAL OF THE FUNCTION $|\zeta\left(\frac{1}{2} + it\right)|^2$

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Abstract. In this paper we introduce the iterations of the Jacob’s ladder and the new type of integral containing certain product of the factors $|\zeta|^2$ corresponding to the components of some disconnected set of the critical line. Next, we obtain an asymptotic formula for this integral, its factorization and, for example, the essential generalization of two Selberg’s formulae (1946).

Dedicated to the 500th anniversary of rabbi Löw.

1. Introduction

1.1. Let

$$Z(t) = e^{it\vartheta(t)}\left(\frac{1}{2} + it\right), \quad \vartheta(t) = -\frac{t}{2}\ln\pi + \text{Im} \ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right)$$

be the signal generated by the Riemann zeta-function on the critical line. Hardy and Littlewood started to study the following integral in 1918

$$\int_0^T |\zeta\left(\frac{1}{2} + it\right)|^2 \, dt = \int_0^T Z^2(t) \, dt,$$

and they have derived the following formula (see [1], pp. 151-156)

$$\int_0^T Z^2(t) \, dt \sim T \ln T, \quad T \to \infty.$$  

In this direction, the Titchmarsh-Kober-Atkinson (TKA) formula

$$\int_0^\infty Z^2(t)e^{-2\delta t} \, dt = \frac{c - \ln(4\pi \delta)}{2 \sin \delta} + \sum_{n=0}^N c_n \delta^n + O(\delta^{N+\epsilon})$$

(see [6], p. 141) where $c$ is the Euler constant, remained as an isolated result for the period of 56 years. We have obtained in our paper [3] the nonlinear integral equation

$$\int_0^{\mu[y(T)]} Z^2(t)e^{-\frac{2}{2\pi T^2}t} \, dt = \int_0^T Z^2(t) \, dt,$$

where

$$\mu(y) \geq 7y \ln y.$$
Each function $\mu(y)$ generates a solution

$$y = \varphi_{\mu}(T) = \varphi(T).$$

Namely, we have shown in [3] that the following infinite set of the almost exact expressions of the Hardy-Littlewood integral (1.1) takes place

$$\int_0^T Z^2(t) dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi ) \varphi_1(T) + c_0 + O \left( \frac{\ln T}{T} \right),$$

$$\varphi_1(T) = \frac{1}{2} \varphi(T),$$

where $\varphi(T)$ is the Jacob’s ladder, i.e. the solution of the integral equation (1.5).

**Remark 1.** Hence, we have proved that except the asymptotic formula (1.3) possessing an unbounded error term (it is clear that the corresponding formulae of Ingham, Titchmarsh and Balasubramanian also possess the unbounded errors, comp. [3], Remark 1) there is an infinite set of almost exact representations (1.6) of the Hardy-Littlewood integral (1.2).

**Remark 2.** The result (1.6) can be formulated as follows: the Jacob’s ladders $\varphi_1(t)$ (comp. (1.6) and the extension in [3], p. 415; $G[\varphi(T)]$) are the asymptotic solutions of the following transcendental equation

$$\int_0^T Z^2(t) dt = V(T) \ln V(T) + (c - \ln 2\pi ) V(T) + c_0.$$

1.2. Next, we have proved (see [4], (2.5)) the asymptotic formula

$$\int_T^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim U \tan[\alpha(T,U)] \ln T,$$

$$U \in \left( 0, \frac{T}{\ln T} \right), \quad T \to \infty,$$

where $\alpha(T,U)$ is the angle of the chord of the curve

$$y = \varphi_1(T)$$

that binds the points $[T, \varphi_1(T)]$ and $[T + U, \varphi_1(T + U)]$ and, in addition to this, the asymptotic formula (see [4], (8.3))

$$\int_T^{T+U_1} \left| \zeta \left( \frac{1}{2} + it \varphi_1(t) \right) \right|^4 \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{1}{2\pi^2} U_1 \ln^5 T,$$

$$U_1 = T^{7/8+2\epsilon}, \quad T \to \infty,$$

(small improvements to of the exponent $\frac{7}{8}$ and, similarly, of the analogous exponents $\frac{1}{3}$, $\frac{4}{9}$ are irrelevant for our purpose. On the contrary, the case $U \in (0,1)$ is relevant).

**Remark 3.** The formula (1.8) is the first integral asymptotic formula of the sixth order in the theory of the function $\zeta \left( \frac{1}{2} + it \right)$.

Let us remind that the proof of the formula (1.8) is simultaneously the proof of the following theorem (see [4], Theorem 3, p. 219): every Jacob’s ladder

$$\varphi_1(t) = \frac{1}{2} \varphi(t)$$
where $\varphi(t)$ is an exact solution of the nonlinear integral equation (1.5) is the asymptotic solution of the following nonlinear integral equation

\[
\int_T^{T+U} \left| \zeta \left( \frac{1}{2} + i x(t) \right) \right|^4 \left| \zeta \left( \frac{1}{2} + i t \right) \right|^2 dt = \frac{1}{2\pi^2} U_1 \ln^5 T.
\]

1.3. Certain motivation for the next step is the well-known multiplicative formula

\[
M \left( \prod_{k=1}^n X_k \right) = \prod_{k=1}^n M(X_k)
\]

from the theory of probability. In the formula (1.10) $X_k$, $k = 1, \ldots, n$ stand for independent random variables and $M$ stands for the population mean.

Let

\[
y = \frac{1}{2} \varphi(t) = \varphi_1(t), \quad \varphi_1^0(t) = t, \quad \varphi_1^1(t) = \varphi_1(t),
\]

\[
\varphi_1^2(t) = \varphi_1[\varphi_1(t)], \ldots, \varphi_1^k(t) = \varphi_1[\varphi_1^{k-1}(t)], \ldots, t \in [T, T + U],
\]

where $\varphi_1^k(t)$ denotes the $k$-th iteration of the Jacob’s ladder $\varphi_1(t)$, $t \geq T_0[\varphi_1]$. Let us remind that the functions $\varphi_1^k(t)$, $k = 2, \ldots$ are increasing since the function $\varphi_1(t)$ is increasing.

In this paper we obtain an asymptotic formula for a new kind of the transcendental integrals

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 dt, \quad U \in \left( 0, \frac{T}{\ln T} \right]
\]

for every fixed $n \in \mathbb{N}_0$ and for every Lebesgue-integrable function

\[
F(t), \quad t \in [\varphi_1^{n+1}(T), \varphi_1^{n+1}(T + U)], \quad F(t) \geq 0 (\leq 0).
\]

**Remark 4.** The integral (1.12) is a natural multiplicative generalization of the part of the Hardy-Littlewood integral (1.2):

\[
\int_T^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \to
\]

\[
\int_T^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left| F \left( \varphi_1^{n+1}(t) \right) \right|^2 \left| \prod_{k=1}^n \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 dt.
\]

The following is connected with the formula (1.12):

(A) if $T \to \infty$ then the disconnected set

\[
\bigcup_{k=0}^{n+1} [\varphi_1^k(T), \varphi_1^k(T + U)]
\]

looks like a one dimensional Friedmann-Hubble expanding universe,

(B) new class of integro-iterative equations,

(C) the set of $p(n+1) - 1$ factorization formulae for the integral (1.12) generated by all proper Euler’s partitions of the number $n + 1$ where, for example,

\[
p(200) - 1 = 3,972,999,029,387.
\]
and in the general case

\[ p(n + 1) \sim \frac{1}{4(n + 1)^{4/3}} e^{K \sqrt{n+1}}, \quad K = \pi \sqrt{\frac{2}{3}}, \quad n \to \infty \]

by the well-known Hardy-Ramanujan-Rademacher formula (see [2], pp. 164, 166),

(D) an essential generalization of two Selberg’s formulae.

**Remark 5.** The integral (1.12) expresses the energy of the complicated signal

\[ \sqrt{|F[\varphi_{n+1}^T(t)]| \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_{1}^k(t) \right) \right|^2} \quad t \in [T, T + U], \]

and formulae corresponding to (A) – (D) give some properties of this energy.

**Remark 6.** The integral (1.12) is not accessible neither for the classical methods of Hardy-Littlewood and Selberg, nor for the current methods in the theory of the Riemann zeta-function.

2. **THEOREM AND A NEW CLASS OF INTEGRO-ITERATIVE EQUATIONS**

2.1. The following Theorem holds true.

**Theorem.** For every fixed \( n \in \mathbb{N}_0 \) and for every Lebesgue-integrable function

\[ F(t), \quad t \in [\varphi_{1}^{n+1}(T), \varphi_{1}^{n+1}(T + U)], \quad F(t) \geq 0 \ (\leq 0), \]

\[ \int_{\varphi_{1}^{n+1}(T)}^{\varphi_{1}^{n+1}(T+U)} F(t)dt \neq 0 \]

we have

\[ \int_{T}^{T+U} \sqrt{|F[\varphi_{1}^{n+1}(t)]|} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_{1}^k(t) \right) \right|^2 dt \sim \]

\[ \sim \left\{ \int_{\varphi_{1}^{n+1}(T)}^{\varphi_{1}^{n+1}(T+U)} F(t)dt \right\} \ln^{n+1}T, \quad T \in \left( 0, \frac{T}{\ln^2 T} \right], \quad T \to \infty, \]

and the set

\[ \bigcup_{k=0}^{n+1} [\varphi_{1}^k(T), \varphi_{1}^k(T + U)] \]

has the following properties:

\[ t \sim \varphi_{1}^1(t), \quad \varphi_{1}^k(T) \geq (1-\epsilon)T, \quad k = 0, 1, \ldots, n+1, \]

\[ \varphi_{1}^k(T + U) - \varphi_{1}^k(T) < \frac{1}{2n+5} \frac{T}{\ln T}, \quad k = 1, \ldots, n+1, \]

\[ \varphi_{1}^k(T) - \varphi_{1}^{k+1}(T + U) > 0.18 \times \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n. \]

Next, in the macroscopic domain, i. e. for

\[ U \in \left[ T^{1/3+\epsilon}, \frac{T}{\ln^2 T} \right], \]

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we have

\[ \int_{T}^{T+U} \sqrt{|F[\varphi_{1}^{n+1}(t)]|} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_{1}^k(t) \right) \right|^2 dt \sim \]

\[ \sim \left\{ \int_{\varphi_{1}^{n+1}(T)}^{\varphi_{1}^{n+1}(T+U)} F(t)dt \right\} \ln^{n+1}T, \quad U \in \left( 0, \frac{T}{\ln^2 T} \right], \quad T \to \infty, \]

and the set

\[ \bigcup_{k=0}^{n+1} [\varphi_{1}^k(T), \varphi_{1}^k(T + U)] \]

has the following properties:

\[ t \sim \varphi_{1}^1(t), \quad \varphi_{1}^k(T) \geq (1-\epsilon)T, \quad k = 0, 1, \ldots, n+1, \]

\[ \varphi_{1}^k(T + U) - \varphi_{1}^k(T) < \frac{1}{2n+5} \frac{T}{\ln T}, \quad k = 1, \ldots, n+1, \]

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Next, in the macroscopic domain, i. e. for

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By the well-known Hardy-Ramanujan-Rademacher formula (see [2], pp. 164, 166),

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we have more detailed information about the set (2.2):

\[(2.7) \quad \left[ \phi^k_1(T), \phi^{k+1}_1(T+U) \right] \sim \phi^k_1(T + U) - \phi^k_1(T), \quad k = 1, \ldots, n + 1,\]

\[(2.8) \quad \phi^k_1(T) - \phi^{k+1}_1(T + U) \sim (1 - c) \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n,\]

\[(2.9) \quad \rho \left\{ \left[ \phi^{k-1}_1(T), \phi^{k-1}_1(T+U) \right]; \left[ \phi^k_1(T), \phi^k_1(T+U) \right] \right\} \sim \rho \left( 1 - c \right) \frac{T}{\ln T}, \quad k = 1, \ldots, n + 1\]

where \(\rho\) denotes the distance of corresponding segments.

**Remark 7.** First of all we have that the set (2.2) is disconnected (see (2.5) for every admissible \(U\) (see (2.1)). The components of connectedness of the set (2.2) are distributed from the right to the left (see (2.5)).

**Remark 8.** Below listed properties of the set (2.2) hold true in the macroscopic domain (2.6):

(a) the components of connectedness of the set (2.2) have asymptotically equal measures (see (2.7), i.e., the transformations

\[\phi^k_1 : [T, T + U] \to \left[ \phi^k_1(T), \phi^k_1(T + U) \right]\]

asymptotically preserve the measure of the segment [T, T + U],

(b) the adjacent intervals of this set have asymptotically equal measures (see (2.8))

\[\left| \left( \phi^{k+1}_1(T + U), \phi^k_1(T) \right) \right| \sim (1 - c) \frac{T}{\ln T}\]

Hence, by (a) and (b) the components of connectedness of the disconnected set (2.2) are distributed with remarkably asymptotic regularity.

**Remark 9.** The asymptotic behavior of the disconnected set (2.2) is as follows: if \(T \to \infty\) then the components of connectedness of this set receding unboundedly each from other (see (2.6), (2.8)) and all together are receding to infinity. Hence, if \(T \to \infty\) then the set (2.2) behaves as a one-dimensional Friedmann-Hubble expanding universe (comp. Introduction, (A)).

2.2. A new class of nonlinear equations is connected with our Theorem. Namely, we obtain from our Theorem the following corollary.

**Corollary 1.** Every Jacob’s ladder

\[\varphi_1(t) = \frac{1}{2} \varphi(t)\]

where \(\varphi(t)\) is an exact solution of the nonlinear integral equation (1.5) is an asymptotic solution of the following integro-iterative equation (comp. (1.9))

\[(2.10) \quad \frac{1}{U} \int_T^{T + U} F[x^{n+1}(t)] \prod_{k=0}^{n} \left| \left( \frac{1}{2} + ix^k(t) \right) \right|^2 dt = \left\{ \int_{x^{n+1}(T + U)}^{x^{n+1}(T)} F(t) dt \right\} \ln^{n+1} T\]
in the sense that
\[
\frac{1}{\pi} \int_T^{T+U} F[\varphi^{n+1}_1(t)] \prod_{k=0}^n \left| \zeta \left( \frac{1}{2} + i \varphi^k_1(t) \right) \right|^2 dt \\
\int_{\varphi^{n+1}_1(T)}^{\varphi^{n+1}_1(T+U)} F(t) dt \\n\sim \ln^{n+1} T, \; T \to \infty,
\]
where
\[
x^0(t) = t, \; x^1(t) = x(t), \; x^2(t) = x(x(t)), \ldots,
\]
i.e. the function \( x^k(t) \) is the \( k \)-th iteration of the function \( x(t) \).

**Remark 10.** There are fixed-point methods and other methods of the functional analysis used to study nonlinear equations. What can be obtained by using these methods in the case of the nonlinear integro-iterative equation (2.10) (at least in the case \( F(t) = 1 \))?

3. **On the set of factorizations of the integral (1.12) generated by the set of all proper partitions of \( n+1 \)**

3.1. Since (see (2.1), \( F(t) = 1 \))

\[
\int_T^{T+U} \prod_{k=0}^n \left| \zeta \left( \frac{1}{2} + i \varphi^k_1(t) \right) \right|^2 dt \sim \{ \varphi^{n+1}_1(T+U) - \varphi^{n+1}_1(T) \} \ln^{n+1} T
\]

then for every proper partition \( (n+1 = n+1 \text{ is excluded}) \)

\[
n + 1 = a_{j_1} + a_{j_2} + \cdots + a_{j_s}, \; a_{j_l} \in [1, n], \; l = 1, \ldots, s
\]

we have

\[
\ln^{a_{j_l}} T \sim \frac{1}{\varphi^{a_{j_l}}_1(T+U) - \varphi^{a_{j_l}}_1(T)} \int_T^{T+U} \prod_{k=0}^{a_{j_l}-1} \left| \zeta \left( \frac{1}{2} + i \varphi^k_1(t) \right) \right|^2 dt.
\]

Next, we obtain from Theorem by (3.2), (3.3) the following formula

\[
\int_T^{T+U} F[\varphi^{n+1}_1(t)] \prod_{k=0}^n \left| \zeta \left( \frac{1}{2} + i \varphi^k_1(t) \right) \right|^2 dt \sim \int_{\varphi^{n+1}_1(T)}^{\varphi^{n+1}_1(T+U)} F(t) dt \\
\times \prod_{l=1}^{s} \frac{1}{\varphi^{a_{j_l}}_1(T+U) - \varphi^{a_{j_l}}_1(T)} \int_T^{T+U} \prod_{k=0}^{a_{j_l}-1} \left| \zeta \left( \frac{1}{2} + i \varphi^k_1(t) \right) \right|^2 dt.
\]

Thus, if we use the weight factors

\[
g_l = \frac{U}{\varphi^{a_{j_l}}_1(T+U) - \varphi^{a_{j_l}}_1(T)}, \; l = 1, \ldots, s,
\]

\[
g_{n+1} = \frac{U}{\varphi^{n+1}_1(T+U) - \varphi^{n+1}_1(T)},
\]

then we obtain from Theorem by (3.2), (3.5) the following corollary.
Corollary 2. The set of all proper partitions \((\ref{3.2})\) generates for every fixed \(n \in \mathbb{N}_0\) the set of \(p(n + 1) - 1\) factorization formulae for the weighted mean-value of the integral \((\ref{1.12})\)

\[
g_{n+1} \frac{1}{U} \int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt \sim \frac{1}{\varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T)} \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt \times \\
\prod_{l=1}^{s} g_l \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{\alpha_{n+1}-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt, \ U \in \left( 0, \frac{T}{\ln^2 T} \right], \ T \to \infty,
\]

and, consequently, for \(F(t) = 1\) we have

\[
g_{n+1} \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt \sim \prod_{l=1}^{s} g_l \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{\alpha_{n+1}-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt, \ T \to \infty.
\]

3.2. Next, the set of all \(p(n + 1) - 1\) asymptotic equalities \((\ref{3.6})\) generates the set of all

\[
\frac{1}{2} \{p(n + 1) - 1\} \{p(n + 1) - 2\}, \ n \geq 2
\]

asymptotic equalities between the right-hand sides of \((\ref{3.6})\).

Example. Let us consider two partitions of the number 6:

\[
6 = 2 + 2 + 2, \ a_{j_1} = 2; \ 6 = 3 + 3, \ a_{j_1} = 3.
\]

Then

\[
\left\{ g_2 \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt \right\}^3 \sim \\
\sim \left\{ g_3 \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{2} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt \right\}^2.
\]

Since

\[
\int_T^{T+U} \prod_{k=0}^{2} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt = \\
= \left| \zeta \left( \frac{1}{2} + i \varphi_1^{1}(t_1) \right) \right|^4 \int_T^{T+U} \prod_{k=0}^{1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt, \ t_1 \in (T, T + U)
\]

then it follows from \((\ref{3.8})\) that

\[
\left| \zeta \left( \frac{1}{2} + i \varphi_1^{1}(t_1) \right) \right|^4 \sim \frac{(g_2)^3}{(g_3)^2} \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^2 dt = \\
= \frac{(g_2)^3}{(g_3)^2} \left| \zeta \left( \frac{1}{2} + i \rho \right) \right|^2 \left| \zeta \left( \frac{1}{2} + i \varphi_1^{1}(\rho) \right) \right|^2, \ \rho \in (T, T + U).
\]
Hence, denoting
\[ \rho = \tau_0(T), \ \varphi_1(\rho) = \tau_1(T), \ \varphi_2^2(t_1) = \tau_2(T) \]
we obtain from (3.9) the following corollary.

**Corollary 3.** There are values (a continuum set of these if \( T \to \infty \))
\[ \tau_k(T) \in (\varphi_1^k(T), \varphi_1^k(T+U)), \ k = 0, 1, 2 \]
such that the asymptotic equality
\[ (3.10) \quad \left| \zeta \left( \frac{1}{2} + i\tau_2 \right) \right|^2 \sim \left( \frac{g_3}{g_3} \right)^{3/2} \left| \zeta \left( \frac{1}{2} + i\tau_1 \right) \right| \left| \zeta \left( \frac{1}{2} + i\tau_0 \right) \right|, \ T \to \infty \]
holds true.

**Remark 11.** The formula (3.10) gives us a new kind of result about the distribution of the values
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right| \]
with respect to the disconnected set
\[ \bigcup_{k=0}^{2} [\varphi_1^k(T), \varphi_1^k(T+U)]. \]

**Remark 12.** We hope that this example gives the sufficient information about the construction of the analogue of (3.10) for the elements of the second set.

3.3.

**Remark 13.** The formula (3.7) gives us, in the direction from the left to the right, the factorization of the energy and, from the left to the right, we have the multiplicative synthesis of the elementary energies in the following sense. For every fixed natural number \( n + 1 \) we have the reservoir
\[ R = \bigcup_{L=1}^{n} \{ J_L, J_L, \ldots, J_L \} \]
where
\[ J_L = g_L \int_T^{T+U} \prod_{k=0}^{L-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt, \ n_L = \left[ \frac{n+1}{L} \right], \]
of the weighted elementary energies (of the same integral form). Now, the proper partition (3.2) chooses (an analogue of the *shem ha-meforash of a golem*) corresponding elementary energies from the reservoir \( R \), and by multiplication of these, we obtain the resulting energy (of the same integral form).

4. **Other factorizations of the integral (1.12)**
4.1. We obtain by the substitution
\[ T \to \varphi_1^k(T), \quad T+U \to \varphi_1^k(T+U) \]
in the formula (1.7) (see (2.3), (2.4))

\[
\frac{1}{\varphi_1^k(T + U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T + U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim
\]

(4.1)

\[
\frac{\varphi_1^{k+1}(T + U) - \varphi_1^{k+1}(T)}{\varphi_1^k(T + U) - \varphi_1^k(T)} \ln T, \quad k = 0, 1, \ldots, n
\]

\[
\ln \varphi_1^k(T) \sim \ln T,
\]

and the multiple of these factors gives us the following formula

\[
\prod_{k=0}^{n} \frac{1}{\varphi_1^k(T + U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T + U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim
\]

(4.2)

\[
\sim \frac{1}{U} \{ \varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T) \} \ln^{n+1} T.
\]

Next, we have from (2.1)

\[
\frac{1}{U} \int_{T}^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt \sim
\]

(4.3)

\[
\sim \frac{1}{\varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T)} \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T + U)} F(t) dt \times
\]

\[
\times \frac{1}{U} \{ \varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T) \} \ln^{n+1} T.
\]

Hence, we obtain by (4.2), (4.3) the following corollary.

**Corollary 4.**

\[
\frac{1}{U} \int_{T}^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt \sim
\]

(4.4)

\[
\sim \frac{1}{\varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T)} \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T + U)} F(t) dt \times
\]

\[
\times \prod_{k=0}^{n} \frac{1}{\varphi_1^{k+1}(T + U) - \varphi_1^{k+1}(T)} \int_{\varphi_1^{k+1}(T)}^{\varphi_1^{k+1}(T + U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt
\]

(comp. with the formula (1.10)).

4.2. In addition to (4.3) there are also other (degenerate) factorizations. Namely, it follows from (4.1) that

\[
\frac{1}{\varphi_1^{l+1}(T + U) - \varphi_1^{l+1}(T)} \int_{\varphi_1^l(T)}^{\varphi_1^l(T + U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \ln T,
\]

(4.5)

\[
l = 0, 1, \ldots, n; \quad T \to \infty.
\]

Now, we obtain from (2.1) by (4.5) the following corollary.
Corollary 5.

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^n \left| \zeta \left( \frac{1}{2} + \imath \varphi_1^k(t) \right) \right|^2 dt \sim \\
\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt \times \\
\left\{ \frac{1}{\varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)} \int_{\varphi_1(T)}^{\varphi_1(T+U)} \left| \zeta \left( \frac{1}{2} + \imath t \right) \right|^2 dt \right\}^{n+1}, \\
l = 0, 1, \ldots, n, \ \ T \to \infty.
\]

For example, we obtain in the case (4.6) with \( l = 0 \) the following formula

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^n \left| \zeta \left( \frac{1}{2} + \imath \varphi_1^k(t) \right) \right|^2 dt \sim \\
\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt \\
\sim \left\{ \frac{1}{\varphi_1(T+U) - \varphi_1(T)} \int_T^{T+U} \left| \zeta \left( \frac{1}{2} + \imath t \right) \right|^2 dt \right\}^{n+1}, \ T \to \infty.
\]

Remark 14. The formula (4.7) shows a curious effect: its left-hand side depends on all iterations \( \varphi_1^0, \ldots, \varphi_1^{n+1} \) while its right-hand side depends only the two first iterations \( \varphi_1^0 \) and \( \varphi_1^1 \), where

\[ \varphi_1^0(t) = t, \ \varphi_1^1(t) = \varphi_1(t). \]

5. Generalization of some Selberg’s formulae and the nonlocal and nonlinear interactions of the functions \( |\zeta \left( \frac{1}{2} + \imath t \right) | \) and \( \arg \zeta \left( \frac{1}{2} + \imath t \right) \)

5.1. First, let us remind the Selberg’s formula (see [5], p. 126, Theorem 6)

\[
\int_T^{T+U} (S(t))^2 dt \sim \frac{(2l)!}{l!(2\pi)^{2l}} U(\ln T)^l, \\
\ U \in [T^{1/2+\delta}, T], \ T \to \infty,
\]

where \( l \) is arbitrary fixed natural number, and

\[
S(t) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + \imath t \right)
\]

with the arg function defined as usually (comp. [6], p. 179). By making use of the formulae (see [2.3], [2.7])

\[
\varphi_1^{n+1}(T) \sim T, \ \varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T) \sim U
\]

we obtain from (5.1) that

\[
\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} (S(t))^2 dt \sim \\
\sim \frac{(2l)!}{l!(2\pi)^{2l}} \left\{ \varphi_1^{n+1}(T + U) - \varphi_1^{n+1}(T) \right\} \left\{ \ln \varphi_1^{n+1}(T) \right\}^l \\
\sim \frac{(2l)!}{l!(4\pi)^{2l}} U(\ln T)^l, \ T \to \infty,
\]
and consequently, we obtain from our Theorem in the case

\[ F(t) = S(t) \]

(see (5.2)) the following corollary.

**Corollary 6.**

\[ \frac{1}{U} \int_{T}^{T+U} \left\{ \arg \zeta \left( \frac{1}{2} + i \varphi_1^{n+1}(t) \right) \right\}^{2l} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^{2} \, dt \sim \]

\[ \sim \frac{(2l)!}{4l!} (\ln \ln T)^{l} \ln^{n+1} T, \quad U \in \left[ T^{1/2+\delta}, \frac{T}{\ln^{2} T} \right], \quad T \to \infty, \]

for every fixed \( l \in \mathbb{N} \), \( n \in \mathbb{N}_0 \).

**Remark 15.** The formula (5.5) in the form

\[ \frac{1}{U} \int_{T}^{T+U} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2l} \left\{ \arg \zeta \left( \frac{1}{2} + i \varphi_1^{n+1}(t) \right) \right\}^{2} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^{2} \, dt \sim \]

\[ \sim \frac{(2l)!}{4l!} (\ln \ln T)^{l} \ln^{n+1} T \]

is simultaneously the generalization of the corresponding part of the Hardy-Littlewood integral (1.2), (comp. (1.13)).

5.2. Next, let us remind another Selberg’s formula (see [5], p. 130, Theorem 7)

\[ \int_{T}^{T+U} \{ S_1(t) \}^{2l} \, dt \sim d_{l} U, \quad U \in \left[ T^{1/2+\delta}, T \right], \quad T \to \infty \]

where

\[ S_1(T) = \int_{0}^{T} S(t) \, dt. \]

We obtain, similarly to (5.4), that

\[ \int_{\varphi_1^{n+1}(T+U)}^{\varphi_1^{n+1}(T)} \{ S_1(t) \}^{2l} \, dt \sim d_{l} \{ \varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T) \} \, d_{l} U, \]

and consequently we obtain from the Theorem \((F(t) = S_1(t))\) the following corollary.

**Corollary 7.**

\[ \frac{1}{U} \int_{T}^{T+U} \{ S_1[\varphi_1^{n+1}(t)] \}^{2l} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k}(t) \right) \right|^{2} \, dt \sim \]

\[ \sim d_{l} \ln^{n+1} T, \quad U \in \left[ T^{1/2+\delta}, \frac{T}{\ln^{2} T} \right], \quad T \to \infty \]

for every fixed \( l \in \mathbb{N} \), \( n \in \mathbb{N}_0 \).
Remark 16. We obtain from (5.5), (5.7) in the case \( l = 1, n = 0 \) the following minimal formulae

\[
\frac{1}{U} \int_{T}^{T+U} \left\{ \arg \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right\}^2 \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{1}{2} \ln \ln T \ln T, \\
\frac{1}{U} \int_{T}^{T+U} \{ \varphi_1(t) \}^2 \zeta \left( \frac{1}{2} + it \right)^2 dt \sim d_1 \ln T, \ T \to \infty.
\]

(5.8)

The simplest form of the nonlocal and nonlinear interactions of the pairs of signals

\[
\left\{ \arg \zeta \left( \frac{1}{2} + it \right), \left| \zeta \left( \frac{1}{2} + it \right) \right| \right\}, \left\{ \varphi_1(t), \zeta \left( \frac{1}{2} + it \right) \right\}
\]
on the disconnected set

\[
[\varphi_1(T), \varphi_1(T + U)] \bigcup [T, T + U].
\]
is expressed by the formulae (5.8).

6. Proof of the Theorem

We use our formula (see [3], (6.2))

\[
t - \varphi_1(t) \sim (1 - c) \pi(t); \ \pi(t) \sim \frac{t}{\ln t},
\]
where \( \pi(t) \) is the prime-counting function.

Remark 17. The fundamental geometric property of the set of Jacob’s ladders is expressed by the formula (6.1): the difference of the abscissa and the ordinate of the point

\[
[t, \varphi_1(t)]
\]
of every curve

\[
y = \varphi_1(t)
\]
is asymptotically equal to \((1 - c) \pi(t)\).

6.1. We have (see (6.1), \( t \to \varphi_k^1(t) \))

\[
\varphi_k^1(t) - \varphi_{k+1}^1(t) \sim (1 - c) \frac{\varphi_k^1(t)}{\ln \varphi_1^1(t)}, \ k = 0, 1, \ldots, n + 1,
\]

(6.2)

\[
t \in [T, T + U], \ U \in \left( 0, \frac{T}{\ln^2 T} \right]: \varphi_0^1(t) = t
\]
(comp. (1.11)) for arbitrary fixed \( n \in \mathbb{N}_0 \), and (see (6.1), (6.2))

\[
t \sim \varphi_1^1(t) \sim \varphi_2^1(t) \sim \ldots \sim \varphi_{n+1}^1(t), \ T \to \infty,
\]

(6.3)

\[
t > \varphi_1^1(t) > \varphi_2^1(t) > \ldots > \varphi_{n+1}^1(t).
\]
Next, we have (see (6.2), (6.3))

\[
\varphi_k^1(t) - \varphi_{k+1}^1(t) \sim \frac{t}{\ln t}, \ k = 0, 1, \ldots, n,
\]

(6.4)
and consequently we obtain by addition of (6.4)

\[ t - \varphi_1^{n+1}(t) \sim (1 - c)(n + 1)\frac{t}{\ln t}, \]

(6.5)

\[ \varphi_1^{n+1}(t) \sim \left\{1 - \frac{(1 - c)(n + 1)}{\ln t}\right\} t, \]

\[ 0 < 1 - c < 1, \]

\[ \varphi_1^{n+1}(t) > \left(1 - \frac{\epsilon}{2}\right) \left\{1 - \frac{(1 - c)(n + 1)}{\ln t}\right\} t > \]

\[ (1 - \epsilon)t \geq (1 - \epsilon)T, \quad t \in [T, T + U], \]

i.e. from (6.3), (6.5) the properties (2.3) follow. Especially, the following holds true

(6.6)

\[ (1 - \epsilon)T < \varphi_1^{n+1}(T) < T. \]

6.2. First of all we have (see (6.1), (6.2) – condition for U)

\[ T - \varphi_1(T) = \{1 + o(1)\}(1 - c)\frac{T}{\ln T}, \]

\[ T + U - \varphi_1(T + U) = \{1 + o(1)\}(1 - c)\frac{T}{\ln T}, \]

then

\[ \left(1 - \frac{1}{(2n + 5)^2}\right)(1 - c)\frac{T}{\ln T} < T + U - \varphi_1(T + U) < \]

\[ < \left(1 + \frac{1}{(2n + 5)^2}\right)(1 - c)\frac{T}{\ln T}, \]

\[ \left(1 - \frac{1}{(2n + 5)^2}\right)(1 - c)\frac{T}{\ln T} < T - \varphi_1(T) < \]

\[ < \left(1 + \frac{1}{(2n + 5)^2}\right)(1 - c)\frac{T}{\ln T}, \quad T \rightarrow \infty. \]

Next, we have (0 < 1 - c < 1)

\[ |T + U - \varphi_1(T + U) - \{T - \varphi_1(T)\}| < \]

\[ < \frac{2}{(2n + 5)^2}(1 - c)\frac{T}{\ln T} < \frac{2}{(2n + 5)^2}\frac{T}{\ln T}. \]

i.e.

(6.7)

\[ \varphi_1(T + U) - \varphi_1(T) - U < \frac{2}{(2n + 5)^2}\frac{T}{\ln T}, \]

and (see (6.2) – the condition for U)

\[ 0 < \varphi_1(T + U) - \varphi_1(T) < 2\frac{T}{(2n + 5)^2}\frac{T}{\ln T} + U < \frac{3}{(2n + 5)^2}\frac{T}{\ln T}. \]

Hence (see (2.1))

(6.8)

\[ U \leq \frac{T}{\ln^2 T} \Rightarrow \varphi_1(T + U) - \varphi_1(T) < \frac{3}{(2n + 5)^2}\frac{T}{\ln T}. \]

Similarly, from the formula (see (5.2))

\[ \varphi_1(t) - \varphi_1(t) \sim (1 - c)\frac{t}{\ln t}, \quad t \rightarrow \infty, \]
we obtain (comp. (6.7), (6.8))

$$
\varphi_1^2(T + U) - \varphi_1^2(T) < \frac{2}{(2n+5)^2} \frac{T}{\ln T} + \varphi_1^1(T + U) - \varphi_1^1(T) < \frac{5}{(2n+5)^2} \frac{T}{\ln T}.
$$

Next, if the estimate (the function $\varphi_k^k(t)$ is increasing)

$$
\varphi_k^k(T + U) - \varphi_k^k(T) < \frac{2k + 1}{(2n+5)^2} \frac{T}{\ln T}
$$

holds true then we obtain by a similar way that

$$
\varphi_{k+1}^k(T + U) - \varphi_{k+1}^k(T) < \frac{2(k+1) + 1}{(2n+5)^2} \frac{T}{\ln T}.
$$

Hence, the following estimates hold true

$$
(6.9) \quad \varphi_{k}^k(T + U) - \varphi_{k+1}^k(T) < \frac{2k + 1}{(2n+5)^2} \frac{T}{\ln T} < \frac{1}{2n+5} \frac{T}{\ln T}, \quad k = 1, \ldots, n+1,
$$

i. e. we have the inequalities (2.4).

6.3. Next we have (see (6.4))

$$
\varphi_{k}^k(T) - \varphi_{k+1}^k(T) > \left(1 - \frac{1}{(2n+5)^2}\right) (1 - c) \frac{T}{\ln T},
$$

i. e. (see (6.9))

$$
\varphi_{k}^k(T) - \varphi_{k+1}^k(T) = \varphi_{k}^k(T) - \varphi_{k+1}^k(T + U) + \varphi_{k+1}^k(T + U) - \varphi_{k+1}^k(T) > \left(1 - \frac{1}{(2n+5)^2}\right) (1 - c) \frac{T}{\ln T} - \left\{ \varphi_{k}^k(T + U) - \varphi_{k+1}^k(T) \right\} > \left(1 - c - \frac{1}{T \ln T} - \frac{1}{2n+5} \frac{T}{\ln T} \right) \geq (1 - c - 0.24) \frac{T}{\ln T} > 0.18 \times \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n, \quad n \geq 0
$$

($c < 0.58 \Rightarrow 1 - c > 0.42$), i. e. (2.5) holds true.

6.4. We use the Hardy-Littlewood-Ingham formula

$$
(6.10) \quad \int_T^{T+U} Z^2(t) dt \sim U \ln T, \quad U \in \left[ \frac{T^{1/3+c}}{\ln^2 T}, \frac{T}{\ln^2 T} \right]
$$
(here the exponent $\frac{1}{3}$ is called the Balasubramanian exponent) in what follows. Next, we use also our formula (1.7)

$$\int_T^{T+U} Z^2(t) dt \sim \{\varphi_1(T+U) - \varphi_1(T)\} \ln T.$$  

We compare the formulae (6.10) and (6.11) in order to obtain

$$\varphi_1^k(T+U) - \varphi_1^k(T) \sim U.$$  

Similarly, by such a comparison in the cases

$$T \rightarrow \varphi_1^1(T), \; T + U \rightarrow \varphi_1^1(T+U), \ldots$$

where (see (2.3))

$$\ln \varphi_1^k(T) \sim \ln T,$$

we obtain

$$\varphi_1^k(T+U) - \varphi_1^k(T) \sim U, \; k = 1, \ldots, n+1,$$

i. e. the formula (2.7) holds true.

6.5. We have by (6.4)

$$\varphi_1^k(T) - \varphi_1^{k+1}(T) \sim (1-c) \frac{T}{\ln T},$$

and consequently (see (6.2) – the condition for $U$, (6.12))

$$\varphi_1^k(T) - \varphi_1^{k+1}(T+U) \sim (1-c) \frac{T + \mathcal{O}\left(\frac{T}{\ln^2 T}\right)}{\ln T} \sim (1-c) \frac{T}{\ln T},$$

i. e. the formula (2.8) holds true. The formula (2.9) follows from (2.8).

6.6. Let us remind (see [4], (9.1), (9.2)) that

$$\tilde{Z}^2(t) = \frac{d}{dt} \varphi_1(t), \; \varphi_1(t) = \frac{1}{2} \varphi(t), \; t \geq T_0[\varphi_1]$$

where

$$\tilde{Z}^2(t) = \frac{Z^2(t)}{2 \Phi'[\varphi(t)]} = \frac{\zeta\left(\frac{1}{2} + it\right)^2}{\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\} \ln t}, \; t \in [T, T+U], \; U \in \left(0, \frac{T}{\ln T}\right).$$

If we use the formula (6.13) for the iterations (1.11) we obtain

$$\prod_{k=0}^{n} \tilde{Z}^2[\varphi_1^k(t)] = \frac{d}{dt} \frac{d}{d\varphi_1} \cdots \frac{d}{d\varphi_1^n} = \frac{d}{dt} \varphi_1^{n+1}$$

by the rule of differentiation of the composite function. Consequently we have (see (1.12), (6.15))

$$\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_1^k(t)] dt =$$

$$= \int_T^{T+U} F[\varphi_1^{n+1}(t)] d\varphi_1^{n+1}(t) = \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt,$$
i. e.

\begin{equation}
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_1^k(t)] \, dt = \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) \, dt.
\end{equation}

Because (comp. (6.6))

\((1 - \epsilon)T < \varphi_1^{n+1}(T) < T + U\)

we have the following

\begin{equation}
T' \in (\varphi_1^{n+1}(T), T + U) \Rightarrow \ln T' = \ln T + O(1).
\end{equation}

Next, if we use the mean-value theorem on the left-hand side of the formula (6.16), we obtain (see (2.3) – the first equality, (6.3), (6.14), (6.17))

\begin{equation}
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_1^k(t)] \, dt \sim \\
\sim \frac{1}{\ln^{n+1} T} \int_T^{T+U} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 \, dt.
\end{equation}

Hence, from (6.16) and (6.18) the asymptotic formula (2.1) follows.

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