Multiple ground-state instabilities in the anisotropic quantum Rabi model

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I. INTRODUCTION

The Quantum Rabi model (QRM) describes a two-level system coupled to a single electromagnetic mode (an oscillator) via the dipole term [1], the simplest form of light-matter interaction. As such it has many applications in numerous fields ranging from quantum optics and quantum information science to condensed matter physics. In conventional (weak coupling) applications to quantum optics, the rotating-wave (RW) terms are kept and the counter-rotating-wave (CRW) terms are neglected [2]. This so-called rotating-wave approximation (RWA) is equivalent to the Jaynes-Cummings model [3], which is solvable in closed form.

Over the past decade, developments in circuit QED [4] have allowed to reach the ultra-strong coupling regime where the coupling between the superconducting qubit and the resonator can reach 10% of the mode frequency \( \omega \). In this ultrastrong-coupling regime, evidence for the breakdown of the RWA and the importance of CRW terms has been provided by measurements of transmission spectra [5, 6]. More recently, even the deep strong coupling region has been realized experimentally, where the coupling strength is of the same order as the mode frequency [7]. In this regime, the RWA cannot even qualitatively describe the system [8]. Although the spectrum of the full QRM is very easy to obtain numerically by working in a truncated bosonic Hilbert space, the exact analytical solution and with it qualitative statements about the spectrum are difficult to obtain compared to the super-integrable Jaynes-Cummings model. Analytical approximations have been obtained at different levels, such as the limit of small energy splitting \( \Delta \) of the qubit (\( \Delta/\omega < 0.5 \)) and the deep strong coupling regime \( g/\omega \sim 1 \) [9, 10], weak and intermediate coupling (\( g/\omega < 0.4 \)) [11], and also in the whole parameter range [12, 13]. All these approximations, while numerically often satisfying, miss some qualitative features of the exact spectral graph like true level crossings and narrow avoided crossings.

An analytical solution based on the \( Z_2 \)-symmetry of the model and using the Bargmann representation of \( L^2(\mathbb{R}) \) introduced a transcendental function, called \( G \)-function in [10], whose zeros yield the exact spectrum of the QRM. Shortly afterwards, it was found that the \( G \)-function can be written in terms of Heun functions, known from the theory of linear differential equations in the complex domain [14]. The \( G \)-function has a characteristic pole structure, giving information about the form of the eigenstates and the distribution of the eigenvalues along the real axis [15, 16]. With its help, one may classify the eigenvalues as belonging either to the regular or to the exceptional spectrum, the former always non-degenerate, while the latter is comprised of a degenerate and a non-degenerate part [16, 20]. The \( G \)-function can be derived also in the more familiar Hilbert space \( L^2(\mathbb{R}) \), using the Bogoliubov operator approach, and thus in a physically more intuitive way [21].

The “anisotropic” generalization of the QRM where RW and CRW terms have different coupling strengths (AiQRM) has been studied for quite a long time, initially out of pure theoretical interest. It appeared first in the form of the anisotropic variant of the Dicke model [22]. Recently, Goldstone and Higgs modes have been experimentally demonstrated in optical systems with only a few (artificial) atoms, which can be described by the anisotropic Dicke model with a small number of qubits [22]. This experimental progress motivated theoretical studies of the anisotropic QRM (a single qubit) [23, 24]. The AiQRM can also model a two-dimensional electron gas with Rashba (\( \alpha_R \), corresponding to RW coupling) and Dresselhaus (\( \alpha_D \), corresponding...
ing to CRW coupling) spin–orbit interactions, subject to 
a perpendicular magnetic field \[26\]. The two types of 
couplings can be tuned by external electric and magnetic 
fields, allowing the exploration of the whole parameter 
space of the model. It can also be directly realized in 
both cavity QED \[27\] and circuit QED \[4\]. For example, 
Ref. \[28\] proposes a realization of the AiQRM based on 
resonant Raman transitions in an atom interacting with 
a high finesse optical cavity mode. Very recently, it has 
been proposed that the AiQRM can also be realized in 
the dispersive regime via momentum states instead of 
electronic states \[29\].

The exact solution of the AiQRM has been obtained 
using the Bargmann representation \[24\]. The G–
function was obtained by Xie et al. \[24\], and both 
regular and exceptional eigenvalues have been studied. 
The isolated exact solutions at the level crossings (i.e. a part 
of the exceptional spectrum) were found by Tomka et al. \[23\]. 
The surprising finding of Ref. \[24\] was that for 
certain parameter values the first excited state may form 
a degenerate doublet with the ground state and belongs 
to the exceptional spectrum \[30\]. The RW and CRW coupling 
operators \[24\] are the same, and

\[
H = \frac{1}{2} \Delta \sigma_z + \omega a^\dagger a + g_1 (a^\dagger \sigma_x + a \sigma_+ + g_2 (a^\dagger \sigma_x + a \sigma_-),
\]

where \( \Delta \) is qubit level splitting, \( a^\dagger \) (a) is the photonic creation (anihilation) operator of the single radiation mode with frequency \( \omega \) (set to \( 1 = \hbar = \omega \) in the following and the figures), \( g_1 \) and \( g_2 \) are the RW and CRW coupling constants respectively, and \( \sigma_k (k = x, y, z) \) are the Pauli matrices. Set \( r = g_2/g_1 \) as the anisotropic parameter and \( g = g_1 \) below.

The anisotropic QRM possesses the same \( \mathbb{Z}_2 \) symmetry as the isotropic one. The parity operator is 
defined as \( \Pi = \exp(i \pi \hat{N}) \), where \( \hat{N} = a^\dagger a + \sigma_+ \sigma_- \) with \( \sigma_\pm = (\sigma_x \pm i \sigma_y) / 2 \) is the operator of the total excitation number. Note that while \( \hat{N} \) is not conserved, the Hamiltonian \( H \) conserves its parity, \( (\hat{N} \mod 2) \) and therefore \( \Pi \). The parity operator \( \Pi \) has two eigenvalues \( \Pi = \pm 1 \), corresponding to even and odd parity of \( \hat{N} \).

We proceed to derive the G-function of \( H \). Employing the following transformation

\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{r} & 1 \\ -\sqrt{r} & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\frac{1}{\sqrt{r}} \\ 1 & \frac{1}{\sqrt{r}} \end{pmatrix},
\]

the Hamiltonian becomes

\[
H_1 = PHP^{-1} = \begin{pmatrix} a^\dagger a + \beta (a + a^\dagger) + \lambda_- / \beta & \frac{1}{2} \Delta - \frac{\lambda_+}{\beta} a^\dagger \\
-\frac{1}{2} \Delta + \frac{\lambda_+}{\beta} a & a^\dagger a - \beta (a + a^\dagger) - \left( \frac{\lambda_-}{\beta} - \beta \right) a^\dagger \end{pmatrix}
\]

where \( \lambda_\pm = g^2 (1 \pm r^2) / 2 \) and \( \beta = g \sqrt{r} \).

We introduce two displaced bosonic operators with opposite displacements

\[
A_+ = a^\dagger + \beta; \quad A_- = a^\dagger - \beta.
\]

The bosonic number state in terms of the new photonic
where \( D(\beta) = \exp(\beta a^\dagger - \beta a) \) is the unitary displacement operator, acting on the original vacuum state \( |0\rangle \) \((a|0\rangle = 0\). The states \( \{|n\rangle_{A_+}\}_n \), respectively \( \{|n\rangle_{A_-}\}_n \), form an orthonormal basis of the bosonic Hilbert space \( L^2(\mathbb{R}) \) and are called extended coherent states (ECS) \(^{[33]}\).

The Hamiltonian in terms of \( A_+^\dagger, A_+ \) reads

\[
H_1 = \left( A_+^\dagger A_+ + \left( \frac{\lambda_+}{\beta} - \beta \right) A_+^\dagger - \lambda_+ \right) - \left( -\frac{1}{2}\Delta + \lambda_- \right) + \frac{\lambda_-}{2\beta} A_+^\dagger A_+ - \left( \beta + \lambda_+ / \beta \right) A_+^\dagger A_+ - 2\beta A_+ + 2\beta^2 + \lambda_+ \right),
\]

Multiplying the time-independent Schrödinger equation with the bra-vector \( A_+ \langle m | \) from the left yields a recurrence relation for the coefficients

\[
\epsilon_m = \frac{(\beta - \frac{\lambda_+}{\beta}) \epsilon_{m-1} + (\frac{1}{2}\Delta - \lambda_-) f_m + \frac{\lambda_-}{2\beta} f_{m-1}}{m - x},
\]

\[
f_m = \frac{(-\frac{1}{2}\Delta - \lambda_-) \epsilon_{m-1} + \frac{\lambda_-}{2\beta} \epsilon_{m-2} + (m - 1 + 2\beta^2 + 2\lambda_+ - x) f_{m-1} - (\beta + \lambda_+ / \beta) f_{m-2}}{2\beta m}.
\]

Note that the hermitian conjugated operators \( A_+ = (A_+^\dagger)^\dagger \) annihilate the respective displaced vacua \( |0\rangle_{A_\pm} = D(\mp \beta)|0\rangle \).

Eq. \(^{[11]}\) can be written as

\[
G_+(x)G_-(x) = 0
\]
where we have defined the \( G \)-functions

\[
G_\pm(x) = \sum_{n=0}^{\infty} (f_n \pm e_n) \beta^n.
\]

\( E = x - \lambda_+ \) is an eigenvalue of \( H \) with positive (negative) parity if \( G_+(x) \) \((G_-(x)) \) vanishes at \( x \). These \( G \)-functions are equivalent to the \( G \)-functions obtained in Ref. \(^{[24]}\) using the Bargmann space in the sense that they have the same zeros as function of \( E \).

Let’s emphasize that all non-degenerate eigenvalues \( (\text{formal the regular spectrum of the AiQRM}) \) are given by the zeros of one of the \( G \)-functions in Eq. \(^{[13]}\), while for the (at most doubly) degenerate states Eq. \(^{[11]}\) is not valid because Eq. \(^{[10]}\) presumes non-degeneracy, i.e, the eigenstate must have a fixed parity. We will particularly pay attention to the latter case in the next section, discussing the exceptional spectrum. The energy spectra for \( \Delta = 0.5, 2 \) and \( r = 0.2, 2 \) obtained in this way are presented in Fig. \(^{[4]}\).

We will shown in the next section that the open circles correspond to degenerate exceptional solutions and the crosses to non-degenerate exceptional solutions. Both are accompanied by the lifting of a pole in both \( G_+(x) \)
has even parity according to the definition of $\hat{\Pi}$. so that $e$ either (A), $E$ with energy $E = n - \lambda_+$ are associated with the $n$-th pole line of $G_{\pm}(x)$. Energies of this form comprise the degenerate and non-degenerate exceptional spectrum.

A. Degenerate exceptional states

Equation (15) provides a constraint on the model parameters. This constraint is actually a necessary and sufficient condition for the occurrence of a doubly degenerate eigenvalue without specified parity, because the pole at $x = n$ is lifted in both $G_{\pm}(n)$ and $G_{\mp}(n)$ and neither function vanishes or diverges at the exceptional energy parameter $x = n$. The model parameters $\{g, \Delta, r\}$ satisfying Eq. (15) indicate therefore a level crossing in the spectral graph located on the pole line $E(g, r) = n - \lambda_x(g, r)$.

By eliminating the $\epsilon_i$, Eq. (15) can be written explicitly as

$$F_n(g) = \sum_{i=0}^{n} \Gamma_i(n) f_i = 0$$

where

$$\Gamma_i(n) = \frac{\left(\frac{\lambda_+ - \beta}{\lambda_+ - \beta^2}\right)^{(n-i)}}{(n-i)!} \left[\left(\frac{n-i}{\lambda_+ - \beta^2} - 1\right)\lambda_- + \frac{1}{2}\Delta\right]$$

Given the parameters $\Delta$ and $r$, the coupling strength $g_k^{(n)}$ where the doubly degenerate state on the $n$-the pole line occurs follows from Eq. (17). In general, there is more than one solution for given $n$, these solutions are labeled here with the index $k$. These states are marked with open circles in Fig. 1. Let us analyze these doubly degenerate states in detail below.

For $n = 0, x = 0$, the solution is unique and given by

$$g_1^{(0)} = \sqrt{\frac{\Delta}{1 - r^2}},$$

which is the same as Eq. (11) in Ref. [24] using the Bargmann space approach and Eq. (15) in Ref. [31]. It follows that the first excited state and the ground state can only intersect if the CRW coupling is different and weaker than the RW one. The parity in the lowest energy state switches passing through this point, so a first-order quantum phase transition occurs at $g_1^{(0)}$, in sharp contrast with the isotropic QRM. From Eq. (18), the first level crossing in the left panel of Fig. 1 occurs at $g_1^{(0)} = 0.7217$ and 1.4434 for $r = 0.2$, $\Delta = 0.5$ and 2, respectively, consistent with numerical calculations. As shown in the right panel of Fig. 1 the first two levels do not cross. No real solutions exist for $r > 1$. 

III. THE EXCEPTIONAL SPECTRUM

From Eq. (17), one notes that the coefficient $e_n$ diverges at $x = n$ due to its denominator if

$$E = n - \lambda_+.$$ (14)

Energies of this form are excluded from the regular spectrum, because they correspond to poles, not zeros of $G_{\pm}(x)$. But let’s assume there is nevertheless a state with energy $E = n - \lambda_+$. In this case the numerator of the right-hand-side of Eq. (7) should vanish at $x = n$ so that $e_n$ remains finite, which results in two cases, i.e. either (A),

$$e_{n-1} = 0, f_n = 0, f_{n-1} = 0.$$ (16)
changes at level crossings with the first excited state at clearly. One sees that for
in the right panels exhibit the double degenerate points only revealed on an enlarged scale (insets), the functions
ings are barely recognizable in the plots of the left panels, level crossings occur. This shows that although the cross-
whose zeros give the position of the couplings where the first excited state are almost degenerate anyway.
parity change occurs also for \( n = 1 \) but at level crossings level crossing or a numerical artefact? To this end, we re-
with anisotropic constant \( r > 0 \) and \( r > 1 \) (low panel) for \( \Delta = 0.5 \). Solutions to Eq. (17) are marked
\( n = 2 \) (square). The zeros of \( F_1(g) \) corresponding to open symbols in the left panels are indicated by the same symbols in the right panels. Note that some zeros associated with level crossings of states above those present in the left panels are not marked with symbols.

For the second pole line, \( n = 1, x = 1 \), Eq. (17) reduces to a cubic equation for \( g = g^2 \) as

\[
2y (1 + r^2) - 1 + \frac{\Delta^2 - y^2 (1 - r^2)^2}{4} + \frac{2}{\sqrt{1 - r^2}} - 1 = 0.
\]

Unlike Eq. (18), real solutions of Eq. (19) and Eq. (17) for all \( n > 1 \) exist for \( r > 1 \).

Surprisingly, as shown in the left panel of Fig. \[1\], the first two levels seem to intersect the pole line \( n = 1 \) after they have intersected the line \( n = 0 \) at \( g_1^{(0)} \). Is this a true level crossing or a numerical artefact? To this end, we re-
plot the spectral graph for \( \Delta = 0.5, r = 0.2 \) and 2, for the first four levels in Figs. \[2\] (a) and \[2\] (c), respectively. In Figs. \[2\] (b) and (d) we plot the functions \( F_1(g) \) and \( F_2(g) \) whose zeros give the position of the couplings where the level crossings occur. This shows that although the crossings are barely recognizable in the plots of the left panels, only revealed on an enlarged scale (insets), the functions in the right panels exhibit the double degenerate points clearly. On sees that for \( r < 1 \) the ground state parity changes at level crossings with the first excited state at the pole lines \( n = 0, 1, 2 \) for increasing \( g \), whereas the parity change occurs also for \( r > 1 \) but at level crossings belonging to \( n = 1, 2 \), as the line \( n = 0 \) does not support a degeneracy for \( r > 1 \). These quantum phase transitions for \( r > 1 \) happen only in the deep strong coupling limit (for \( r = 2, g/g_c \sim 6 \)), where the ground state and the first excited state are almost degenerate anyway.

To show the level crossings more clearly, we present the energy spectra \( E + \lambda_\pm \) as a function of the coupling \( g \) in Fig. \[3\]. The pole lines are now horizontal (red dashed lines). The most interesting feature of the AiQRM, which sets it apart from the isotropic QRM, is the fact that for increasing \( g \), the ground state rises in energy compared to the pole lines. In the isotropic case \( r = 1 \), the ground state never crosses the first pole line \( n = 0 \) (middle panels in Fig. \[3\]) and the asymptotic energies in the deep strong coupling regime are the pole energies (all energies are asymptotically doubly degenerate). In the AiQRM for \( r \neq 1 \), the GS energy crosses all pole lines for \( n > 1 \) if the coupling grows and this allows for a succession of first order phase transitions if these crossings belong to the de-
generate exceptional spectrum. This is not necessary the case, because they could also be non-degenerate exceptional solutions, discussed in Sec. 1111. However, we find that the largest zero \( g_{\text{max}} \) of \( F_n(g) \) always corresponds to a crossing of the ground state and the first excited state because all non-degenerate exceptional solutions \( g_{n,d}^{(n)} \) for given \( n \) satisfy \( g_{n,d}^{(n)} < g_{\text{max}}^{(n)} \).

Quasi-exactness of the doubly degenerate eigenvalues:- For a doubly degenerate eigenvalue \( E = m - \lambda_\pm \), the model parameters satisfy Eq. (15). This means that \( e_m \) is not determined by the recurrence relations and may take an arbitrary value before normalizing the wave function. If we fix it to

\[
e_m = \left[ \frac{\lambda_\pm e_{m-1} + (2\beta^2 + 2\lambda_\pm) f_m - (\beta + \lambda_\pm/\beta) f_{m-1}}{(2\beta^2 + 2\lambda_\pm) f_m - (\beta + \lambda_\pm/\beta) f_{m-1}} \right],
\]

we have \( f_{m+1} = e_{m+1} = 0 \). It can be easily shown that all coefficients \( f_k \) and \( e_k \) for \( k > m \) vanish whereas the coefficients \( e_n, f_n \) for \( n < m \) are defined by the recurrence

\[
E_{\text{pred}} = E_{\text{pred}} + \lambda_\pm \frac{g \beta}{c_g} \frac{1}{\sqrt{1 - r^2}} - 1 = 0.
\]
relations (7) and (8). This leads to the expression of one of the doubly degenerate eigenfunctions in terms of the first ECS,

$$|A_+\rangle_m = \left( \sum_{n=0}^{m} \sqrt{n!} e_n |n\rangle_{A+} \right) / \sum_{n=0}^{m} \sqrt{n!} f_n |n\rangle_{A+}.$$  \hspace{1cm} (20)

This eigenfunction has no fixed parity, but it can be written as a finite polynomial in the shifted oscillator states $|n\rangle_{A+}$. Applying now the parity operator $\Pi$ to this state, we obtain

$$|A_-\rangle_m = \left( \sum_{n=0}^{m} (-1)^n \sqrt{n!} f_n |n\rangle_{A-} \right) / \sum_{n=0}^{m} (-1)^n \sqrt{n!} e_n |n\rangle_{A-},$$  \hspace{1cm} (21)

which is obviously another state with energy $m - \lambda_+$, linearly independent from $|A_+\rangle_m$. Both states have a finite expansion in their respective ECS bases but the expansion in the original bosonic Fock states does not terminate, in contrast to the “dark states” occurring in the Dicke models (34).

In the isotropic QRM, the proof is even simpler. If $E = m - g^2$, which is the $m$-th pole energy, then the condition for double degeneracy reads

$$f_m(x) = 0,$$  \hspace{1cm} (22)

using

$$(m - g^2 - E) e_m = \Delta f_m,$$

Thus $e_m$ would be arbitrary. If we set

$$e_m = -4g \Delta f_{m-1},$$

then $f_{m+1} = 0$, further $e_{m+1} = 0$. Because both $f_m$ and $f_{m+1}$ are zero, $f_{m+2}$ and $e_{m+2}$ vanish as well. In this case the coefficients $f_k$ and $e_k$ for $k > m + 1$ vanish, thus one of the degenerate eigenfunctions is given as a finite polynomial in the ECS basis $\{|n\rangle_{A+}\}$ (see also (19)). These states are the quasi-exact solutions of the QRM found originally by Judd (35, 36).

### B. Non-degenerate exceptional states

The non-degenerate exceptional states correspond also to states with $E = m - \lambda_+$ but this energy is non-degenerate because the pole at $x = m$ is only lifted in one of the $G$-functions, $G_+(x)$ or $G_-(x)$, but not in both. The state is therefore a parity eigenstate. These states have been first analyzed for the QRM with the Bargmann space technique in (37) and later in (19) and (30).

We shall analyze them now with the ECS technique for the AiQRM. If condition (16) holds, $e_m$ can take an arbitrary value, while all coefficients $e_k, f_k$ for $k < m$ vanish. Setting $e_m = 1, f_m = 0$ all coefficients for $e_n$ and $f_n$ with $n \geq m + 1$ are fixed by the recurrence relations (7) and (8). Imposing now that the constructed state is a parity eigenstate, we find that one of the $G$-functions

$$G_m^\pm(g, \Delta, r) = \pm \beta^m + \sum_{n=m+1}^{\infty} (f_n \pm e_n) \beta^n$$  \hspace{1cm} (23)

must vanish. These $G$-functions are associated with the exceptional energy $m - \lambda_+$ and the $m$-th pole line. They are functions of the model parameters and their vanishing puts a constraint on these parameters, similarly to Eq. (17) for the degenerate eigenvalues. The non-degenerate exceptional eigenstates are marked by crosses in Fig. 1 and correspond to zeros of either $G_m^+(g, \Delta, r)$ or $G_m^-(g, \Delta, r)$. They are not quasi-exact states like the doubly-degenerate eigenstates, because the functions $G_m^\pm(g, \Delta, r)$ are not polynomials in $\beta$.

The $G$-functions $G_m^\pm(g, \Delta, r)$, $m = 0, \ldots, 4$, for several $\Delta$ at $r = 2$ are plotted in Fig. 4 (c.f. the lower panel in Fig. 5). We see that each of them has at most one zero at the coupling $g_{n.d.}$, which happens to be always smaller than $(m)_{g_{n.d.}}$. While it would be interesting to show this conjecture analytically, we confine ourselves here to a numerical check. It entails that the AiQRM exhibits an infinite series of phase transitions for increasing coupling, similar to the Jaynes-Cummings model, not only for $r < 1$, where the RW terms dominate but also in the dual case $r > 1$.
IV. GROUND STATE INSTABILITY

The intersections of energy levels in the spectral graph are directly related to the symmetries of the Hamiltonian. In the case of the isotropic QRM and the AiQRM, we have seen that all level crossings in these models have the same origin, namely the manifest $\mathbb{Z}_2$-symmetry. The crucial feature of these degeneracies is that they are located always on the lines where the $G$-functions have poles. If the ground state energy crosses one of these lines, as is the case for any non-vanishing anisotropy ($r \neq 1$), the concomitant degeneracy of the ground state indicates a quantum phase transition of first order, where the symmetry of the ground state is not defined. The otherwise well-defined parity of the ground state changes sign upon crossing these points. Because the ground state energy crosses eventually all pole lines in the AiQRM, the system undergoes infinitely many such phase transitions for increasing coupling. The ground state phase diagram in the $g/r$-plane is shown in Fig. 5 for three different values of $\Delta$. The parity of the ground state in the different phases is either $+1$ or $-1$. The parity is unique and positive in a region around the isotropy line $r = 1$, while we recover the infinitely many phase transitions of the Jaynes-Cummings model for $r = 0$. The phase diagram for $r < 1$ is consistent with that proposed recently by Ying using numerical exact diagonalization [32].

In principle, the crossing of an energy level with a pole line could be due to a non-degenerate exceptional eigenstate [as in Figs. 1(b) and 1(d)] and would not indicate a quantum phase transition. However, our numerical checks have shown so far that all maximal solutions of Eq. (17) belong indeed to a degeneracy of the ground state, although most of them occur in the deep strong coupling regime, where we have already almost perfect degeneracy of the two lowest energy eigenstates.

Finally, we would like to point out that the switch between even and odd parity of the ground state has been observed in the anisotropic spin-boson model by two of the present authors [38].

V. CONCLUSIONS

In this work, we derive the two parity $G$-functions for the anisotropic quantum Rabi model employing its manifest $\mathbb{Z}_2$-symmetry by the Bogoliubov operator (ECS) approach in the physical Hilbert space $L_2(\mathbb{R})$. Zeros of the $G$-functions yield the regular spectrum with well-defined parity. The exceptional solutions are located at the pole lines of these $G$-functions and comprise all doubly degenerate eigenvalues. The condition for their occurrence is derived in closed form. This allows us to identify an infinity of first-order quantum phase transitions in the AiQRM, whenever the model is not fully isotropic. The importance of the analytical treatment becomes clear as in many cases the numerical resolution of the spectra is very difficult, especially in the deep strong coupling regime (see Fig. 1).

At each crossing of the two lowest energy states the parity of the ground state switches between the discrete values $+1$ and $-1$ for increasing coupling strength. For the extreme case $r = 0$ (Jaynes-Cummings model), the value of the excitation number $\hat{N}$ rises by $+1$ at each phase transition point. While $\hat{N}$ is not conserved for $r \neq 0$, the infinite number of phase transitions remains in the anisotropic case at any value $r \neq 1$. The only model with no phase transition in the ground state for any coupling is the isotropic QRM. The rich phase diagram of the AiQRM is solely due to its manifest $\mathbb{Z}_2$-symmetry. In contrast, the level crossings of higher excited states occurring at special values of the symmetry-breaking parameter $\epsilon$ in the asymmetric QRM (where the $\mathbb{Z}_2$-symmetry is broken by the term $\epsilon \sigma_x$ in the Hamiltonian) is due to a non-manifest, hidden symmetry [39] [42].

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