Control of topological structure of solitons in a laser with saturable absorption

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Abstract. We present a theoretical and numerical investigation of effect of a weak coherent holding radiation on transverse field structures in a wide-aperture laser with fast saturable absorption. In the absence of the coherent driving, the laser is able to support various types of “free” bright solitons. We show that even weak driving can control their topology making the scheme promising for topologically protected information processing.

1. Introduction; the governing equation
Spatial dissipative solitons in a wide-aperture laser with saturable absorption were first demonstrated using numerical simulation in [1]. An analysis of this laser scheme followed the study of solitons in wide-aperture nonlinear interferometers with coherent holding radiation, found numerically in [2]. Compared with these “driven” solitons, laser “free” solitons are characterized by a higher contrast, and due to the absence of fixation of the radiation phase, their family is much more diverse and includes topological solitons of various types. The results of numerous subsequent theoretical and experimental studies of dissipative optical solitons are summarized in monographs and reviews [3-10]. We note here also the original studies of the wide-aperture laser scheme (without saturable absorption) with coherent support radiation [11-16], akin to the nonlinear interferometer scheme mentioned above.

At the same time, the arbitrariness of the phase of “free” laser solitons causes a drift of the phase and an increase in the level of fluctuations under the influence of noise, including that of quantum nature. Therefore, it is worth to consider a scheme in which a laser soliton would be synchronized by holding radiation with an amplitude much lower than the maximum amplitude of the field of the free soliton. A number of such studies have recently been performed in [17-20]. The objective of this work is the comparison of features of free and driven laser solitons, as well as a systematic analysis of the topological structure of driven laser solitons and of the possibility of controlling this structure using holding radiation.

In the approximation of slowly varying field envelope, fast gain and loss, and mean-field approximation, the dimensionless governing equation for the electric field envelope $E$ is

$$\frac{\partial E}{\partial t} = (i + d)\nabla^2 E + i\theta E + f(|E|^2)E + E_{in}.$$ (1)

Here $t$ is time, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian, $x$ and $y$ are the transverse Cartesian coordinates, $d$ is the positive effective diffusion coefficient, $\theta$ is detuning between the frequency of longitudinal mode and the carrier frequency of holding radiation, $f$ is the nonlinear function of
radiation intensity \( I = |E|^2 \) representing the balance of fast gain and loss, as well as nonlinear refraction, and \( E_{in} \) is the holding radiation envelope. We will suppose that the holding radiation is weak, its amplitude is much less than the maximum amplitude of the soliton, \( I_{in} = |E_{in}|^2 \ll I_{max} \). Next, when neglecting frequency detuning between coinciding the central frequencies of the gain and loss spectra and the field essential frequencies, the function \( f \) is real and can be taken in the form [1]:

\[
f(I) = \frac{g_0}{1+I/b} - \frac{a_0}{1+I} - 1. \tag{2}
\]

Here \( g_0 \) is the small-signal gain, \( b \) is the ratio of saturation intensities for gain and absorption, \( a_0 \) is the small-signal absorption, and the term \(-1\) represents the nonresonant loss (with the time normalization used).

Because localization and stability of the field structures in this dissipative scheme result due to the balance of energy gain and loss, their important characteristic is the energy flow distribution. In the paraxial approximation, the transverse energy flow, or the Poynting vector \( \mathbf{S}_\perp = \mathbf{S} \), is

\[
\mathbf{S}_\perp = \frac{\text{Im} \mathbf{E} \nabla \mathbf{E}^\perp}{\mathbf{E}} = \nabla \mathbf{\Phi}
\]

where \( \mathbf{\Phi} = \text{arg} \mathbf{E} \) is the radiation phase. At a fixed time moment, the energy flow lines, i.e. lines \( \mathbf{r} = (x,y) \) tangent to which at each their point is parallel to the vector \( \mathbf{S}_\perp \), is determined by the ordinary equations:

\[
\frac{\text{d}x}{\text{d}l} = \frac{\text{d}y}{\text{d}l} = \frac{\mathbf{S}_x}{\mathbf{S}_y} = \mathbf{S}_x, \quad \frac{\text{d}y}{\text{d}l} = \mathbf{S}_y.
\]

Here \( l \) is the current length of the line. The vector field \( \mathbf{S}_\perp \) has singularities at \( \mathbf{S}_\perp = 0 \), where its direction is not defined.

2. “Free” laser

In the absence of coherent driving, \( E_{in} = 0 \), the trivial solution of Eq. (1) \( E = 0 \) corresponds to the non-lasing mode. It is stable against small perturbations if \( f_0 = f(0) = g_0 - a_0 - 1 < 0 \). There are also modes of monochromatic lasing with homogeneous intensity distribution \( I_k = |A_k|^2 \) and the wave front inclination corresponding to the envelope form \( E = A_k \exp(\mathbf{i} \mathbf{K} \cdot \mathbf{r} - i \Omega t) \) with \( \mathbf{K}_\perp = (K_x, K_y) \) and \( \mathbf{r}_\perp = (x, y) \). Substitution of this form into (1) gives \( \Omega = K^2 \), \( f(I_k) = dK^2 \). The last equation allows one to determine the dependence of the intensity \( I_k = |A_k|^2 \) on the inclination \( \mathbf{K}_\perp \) caused by the angular selectivity of loss. The inclination cannot be large because it follows from this equation that \( K^2 < \max f(I_k) / d \). For the form of the nonlinear function (2), this equation is just the quadratic one and, correspondingly, has two solutions. Within the certain range of parameters, one of these solutions is unstable (intermediate branch, \( df / dI > 0 \) ) and the other is stable (upper branch, \( df / dI < 0 \) ) coexisting with the stable nonlasing mode \( I_k = |A_k|^2 \). Therefore for any fixed transverse wave vector below \( |\mathbf{K}_\perp| \), bistability takes place with nonlasing and lasing (upper branch) modes. Evidently, the same is the intensity of various plane waves with different directions but the same \( K_\perp \) (a cone of wave vectors).

Further, we will be interested only in structures with an inhomogeneous, localized intensity distribution. Of special interest are solid-like solitons whose intensity and energy flow distributions do not change with time in the system coordinates connected with the solitons (they can move and rotate). The stability of the nonlasing mode, \( f_0 < 0 \), is necessary for the localization, otherwise fluctuations on the structure’s tails will grow. Bright solitons can clearly be interpreted as the generation island at the laser aperture (the regime with a uniform intensity distribution) against the background of the non-generation regime. The simplest two-dimensional (2D) laser solitons have an axially symmetric intensity distribution. Their envelope has the form \( E(x, y, t) = a(r) \exp[i\Phi(r) - ivt] \). Here \( r \)
and $\varphi$ are the polar coordinates in the coordinate system centered at the singularity of phase and the Poynting vector, $m$ is an integer topological charge, $\Phi_r (r)$ is the regular function, and $\nu$ is the nonlinear frequency shift. The asymptotic behavior of the field is $a(r) = r^m e^{i\nu}$, $\Phi(r) = \Phi_0 - \Phi_2 r^2$ at $r \to 0$ and $a(r) = r^{-3/2} \exp(-\gamma r)$, $\Phi(r) = - \Im \nu r$ at $r \to \infty$. For fixed laser parameters and fixed $m$, there is the discrete spectrum of such stable solitons found from the nonlinear eigen problem with the eigenvector $a(r)$ and eigenvalue $\nu$. More general is characterization valid also for pulsating structures when we introduce the integral values over the laser aperture:

$$E_s (t) = \int E(x, y, t) dx dy, \Phi_s (t) = \arg E_s, \nu_s (t) = - \frac{d\Phi}{dt}, \mathbf{r}_s (t) = \int \mathbf{r}_s(t) ds / \int \left| \mathbf{r}_s(t) \right| ds.$$  \hspace{1cm} (3)

For monochromatic solitons, their frequency coincides with the integral frequency $\nu_s$. For symmetric structures, the structure center $\mathbf{r}_s$ coincides with the symmetry center.

We illustrate some of these symmetrical solitons with figure 1a,b. The fundamental soliton, figure 1a, has a bell-like intensity distribution. The lines of energy flow are purely radial for this soliton, the symmetry center is a singular point ($S_m = 0$), a stable node whose attraction domain is given by a circle – a collection of singular points where $S_m = 0$ and the direction of the radial lines of energy flow reverses. Correspondingly, in the phase plane of the lines of energy flow for fundamental laser soliton, there are two cells with qualitatively different types of flow separated by a closed line (the circle in the case). In figure 1b we show a multiple ($m = 3$) vortex in the center that is a stable focus. There are three closed lines, 1, 2, and 3 representing unstable (1, 3) and stable (2) limit cycle. In the case, one can see 4 cells with different type of energy flow. In figure 1c one can see a triangular

**Figure 1.** (a)-(c): Transverse intensity distributions (upper row) and energy flow (bottom row) for free axisymmetric fundamental soliton ($m = 0$, $g_0 = 2.14$) (a), axisymmetric vortex soliton (b), and triangular soliton (c), $m = 3$, $g_0 = 2.09$ (b, c). $F$, $N$ and $S$ mark focuses, nodes and saddles of the energy flow. (d), (e): Dependence of the integral frequency shift $\nu_s$ of solitons with charge $m$ (the numbers on the curves). In (e) $S$ marks axially symmetric and $T$ triangular solitons. The boundaries of dashed domains show the range of the instantaneous frequency modulation in the beating mode. $E_{in} = 0$. 

soliton with three separated dislocations (stable focuses $F$) surrounded by three closed lines: 1 – unstable limit cycle, 2 and 3 – stable and unstable lines composed from saddles and nodes, both stable and unstable, and connecting them separatrices. Additional saddles and nodes are present in the soliton central part. All these solitons are solid-like. Due to the symmetry rules [8], their symmetry center is motionless, and the triangular soliton rotates with a constant angular velocity (it has the symmetry to rotation through angle $2\pi/3$).

The solitons are stable within some range of the scheme parameters. In figure 1(d,e) we demonstrate the soliton bifurcations with transition to periodically oscillating one when the coefficient of small-signal amplification increases (the Andronov-Hopf bifurcations). One can see coexisting of different types of solitons for the same scheme parameters. Correspondingly, possible are various hysteretic phenomena discussed below for driven solitons.

3. Driven laser

Now let us consider what changes makes the holding radiation, a weak plane monochromatic wave, $E_\text{in} = \text{const} \neq 0$, $I_\text{in} = |E_\text{in}|^2 < < 1$, to the field structures. Instead of the non-lasing mode $E = 0$ we have now a background with a small-amplitude $E \approx -E_\text{in}/(f_0 + i\theta)$. Modes with the homogeneous lasing intensity $I_\text{in} > 0$ become weakly modulated in space and time (modulation depth is proportional to $E_\text{in}$). The only exception is the mode with $K_\perp = 0, \Omega = 0$, for which the intensity is still stationary and the same over the entire laser cross section. Of course, the boundaries of the stability region of these regimes are also shifted weakly due to weak driving.

A new important phenomenon is the possibility of soliton synchronization by the holding radiation. Because the frequency of this radiation (accepted as the carrier frequency here) differs, in the general case, of the free soliton frequency shifted from the carrier frequency by $\nu$, soliton synchronization is not possible for very small amplitudes $E_\text{in}$ below some threshold value $E_\text{in,thr}$ (this value depends also on the frequency detuning $\theta$). There we have beats between the soliton and the synchronized background, with the resulting beat frequency close to $\nu$. Synchronization of fundamental solitons above the threshold was studied in [18,19].

For vortex solitons, driving leads to some additional consequences (see also [18,20]). Then the full synchronization is impossible even with significant driving amplitudes. Next, new wave front dislocations come from the periphery and approach to the center with increase of $E_\text{in}$ up to merging with the central dislocations with their mutual annihilation and destroying the soliton. In figure 2(a,b) we illustrate this statement for the case of free soliton with the unit topological charge $m = 1$. The driven soliton is solid-like, but due to the appearance of the peripheral dislocation with charge $m = -1$, it has no any symmetry. Therefore, according to the symmetry rule [8], the driven soliton center revolves along a circle (circle 3 in figure 2a with the center in point C) and simultaneously the structure rotates around the center with the same period that is close to $2\pi/\nu$. The trajectories of the two dislocations also rotate with the same period along circles 1 and 2 in figure 2a. The central part of the energy flow plane for the driven soliton, figure 2b, is deformed as compared with the free soliton, and an additional cell appears in the form of a loop around the peripheral dislocation.

For free solitons with a multiple dislocation, $|m| > 1$, driving induces this central dislocation splitting into $|m|$ dislocations with unit topological charge. Simultaneously, the same number of peripheral dislocations appear with the opposite charge. Below the integral synchronization threshold, we have beating mode with rotation of vortices. Domains of the full (for fundamental solitons) and integral (for vortex solitons) synchronization and beating mode are presented in figure 2(c,d). Outside the domain of beating mode, vortex solitons lose their topological charge and then, due to the absence of stable fundamental solitons in this region of parameters, disappear with time. The triangular soliton...
(figure 1c) when driven, retains symmetry to rotation through an angle $\frac{2\pi}{3}$ and, accordingly, remains solid-state in a certain region of the laser parameters.

**Figure 2.** (a), (b): Characteristics of a soliton generated by driving of the axisymmetric soliton with topological charge $m_0 = 1$. Signs $<><\pm>$ indicate the sign of dislocations' topological charge. (a): Concentric circles 1-3 are trajectories of the central (1) and peripheral (2) dislocations and of the soliton center (3). Points mark instantaneous position of the dislocation, and $C$ is the center of the circles. Arrows show the direction of rotation. (b): Instantaneous distribution of energy flows for the driven soliton. Closed curve 1 is unstable limit cycle and curves 2 and 3 are stable and unstable limit cycles, respectively. Additional cell in the energy flow pattern is bounded by a loop around the peripheral dislocation including a saddle $S$. Arrows show the direction of energy flow. Parameters: $g_\theta = 2.1, \ E_\text{in} = 0.01, \ \theta = 0.08$. (c): Domains of the homogeneous lasing hysteresis (I) and full synchronization (phase locking) of 1D and 2D symmetric fundamental ($m = 0$) solitons on the plane of parameters $(\theta, I_\text{in})$, $g_\theta = 2.08$ (1D), $2.11$ (2D). (d): Regions of integral synchronization (solid lines, solitons are rotating solid-like there) and periodic beats (dashed lines) of solitons with the initial topological charge of the generating axisymmetric solitons $m_0 = 1$ and 3 ($m_0$ is indicated inside the regions). Minimum intensity of holding radiation necessary for the synchronization $I_{\text{in,thr}} \approx 4 \cdot 10^{-6}$; $g_\theta = 2.09$. Vertical arrows show the value of frequency nonlinear shift for free solitons. The inset to figure 2d demonstrates hysteretic variation of the integral frequency when, starting with the free axisymmetric soliton with charge $m_0 = 3$ we get the triangular soliton after increase and subsequent decrease of the holding beam amplitude $E_\text{in}$ ($\theta = 0.04$).

Overlapping the stability domains of vortex solitons evident from figure 2d, leads to various hysteresis phenomena, illustrated by the inset in the figure. The solitons are characterized here by their integral frequency $\nu_s$. Initially, without holding radiation, we have a free axisymmetric soliton with topological charge $m_0 = 3$. As was indicated above, the driving induces beatings with modulation depth increasing with increase of $E_\text{in}$. For some value of $E_\text{in}$ bifurcation occurs with transition to the triangular soliton, also in the beating mode. If the holding beam amplitude $E_\text{in}$ decreases then, the triangular soliton retains its structure, becoming stationary at $E_\text{in} = 0$. This example show that the introduction of driving allows controlling the topological structure of laser solitons.

**4. Conclusion**

In summary, we have presented interplay of “free” solitons in a laser with saturable absorption and weak coherent holding radiation driving the solitons. As was shown above, even small-amplitude driving can significantly change the basic properties of the initial “free” soliton, including its topology. Changes are manifested in the appearance of new wavefront dislocations, the internal structure of solitons reflected by energy flow distribution, symmetry and motion of “driven” solitons. Taking into account the maintaining of solitons high contrast – the ratio of the maximum soliton intensity to the background intensity and significant suppression of random drift of the radiation phase [19,20], this
scheme provides real prospects for topologically protected information processing. Some results of this research were published in [21].

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