The plane strain analysis for one-dimensional hexagonal piezoelectric quasicrystals strip in aperiodical plan

Huaimin Guo¹, Ming Gao¹*, Guozhong Zhao¹, Lijuan Jiang²
¹ Mathematics Science, Bao Tou Teacher’s college, Bao Tou, 014030, China
² Education Science, Bao Tou Teacher’s college, Bao Tou, 014030, China
*Corresponding author’s e-mail: gming99@126.com

Abstract. A new stress potential function is introduced, the non periodic plane problem in one-dimensional hexagonal piezoelectric quasicrystals is discussed and the physical equation of the stress-strain relationship in the non periodic plane is constructed. The exact solution of the straight crack in the periodic direction of the one-dimensional hexagonal piezoelectric quasicrystal is obtained. As an application, the problem of straight crack perpendicular to the direction of quasi-periodical in one-dimensional hexagonal piezoelectric quasicrystal with long and narrow body is solved. When the width of the long body becomes infinitely large, the Griffith crack solution is obtained. The results show that the stress at the crack tip remains singularity, which is basically consistent with the crack problem that penetrates along the quasi periodic direction. When the phonon field and the phase field get to zero, the above analytical solution degenerates into the fracture problem of isotropic piezoelectric materials, the results are in agreement with the existing results.

1. Introduction

A quasiperiodic crystal, or quasicrystal, is a structure that is ordered but not periodic. A quasicrystalline pattern can continuously fill all available space, but it lacks translational symmetry. While crystals, according to the classical crystallographic restriction theorem, can possess only two, three, four, and six-fold rotational symmetries, the Bragg diffraction pattern of quasicrystals shows sharp peaks with other symmetry orders, for instance five-fold.

Shechtman et al. [2,1] firstly discovered the fivefold symmetry in the diffraction pattern of Al-Mn alloys and claimed that there is a new structure of solid state in nature. Levine and Steinhardt named the new structure order as quasicrystals (QCs) and he was awarded the Nobel Prize in chemistry in 2011. As a new structure of solid matter, QCs have many desirable properties, such as high hardness, low friction coefficients, low surface energy, low heat-transfer, low adhesion, corrosion resistance and high wear resistance [3,4]. Recently, scientists have been considering to replace the traditional materials used to be employed in the aerospace industry with the quasicrystal materials, such as coating surface of spacecraft’s wings and fuselage, as well as the thermal barrier coating.

Because of the particular structure of QCs, which is sensitive to electrical, thermal, magnetic and other physical and chemical properties, these properties are essentially different from ordinary crystals and have been investigated intensively [5]. In 2012, Altary and Domeci [6], firstly gave the fundamental equations of piezoelectricity of QCs, which establish the theoretical foundation for the study of fracture mechanics of piezoelectricity of QCs. Li et al. [7] obtained the 3D fundamental
solution for 1D hexagonal QCs with piezoelectric effect, and the propagation of cracks may lead to premature failure of these materials produced during their manufacturing process when QCs are subjected to mechanical and electrical loadings in service. Yang\(^{[19]}\) reviewed the anti-plane shear problem of two symmetric cracks originating from an elliptical hole in 1D hexagonal piezoelectric QCs. Yu and Guo\(^{[20,21]}\) proposed the general solutions of plane problem and addressed complex variable method in 1D hexagonal piezoelectric QCs. Jiang et al.\(^{[22]}\) developed the interaction between a screw dislocation and a wedge-shaped crack in 1D hexagonal QCs with piezoelectric effect. However, to date, there has been relatively little research on the fracture problems of 1D hexagonal piezoelectric QCs in aperiodical plane.

2. Basic equations

We establish Cartesian coordinate system, let the coordinate axis \(x_3\) is the quasi periodic directions of 1D hexagonal piezoelectric QCs with point group 6 mm and the plane perpendicular to the quasi periodic direction is the coordinate plane \(x_1x_2\), after Ref.\([9]\), the generalized Hooke’s law of elasticity problem in 1D hexagonal piezoelectric QC are given by

\[
\begin{align*}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{12} + C_{13}\varepsilon_{13} + R_i\omega_i - \varepsilon_{11}^3 E_3 \\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_i\omega_i - \varepsilon_{11}^3 E_3 \\
\sigma_{33} &= C_{13}\varepsilon_{11} + C_{33}\varepsilon_{33} + C_{33}\varepsilon_{33} + R_i\omega_i - \varepsilon_{11}^3 E_3 \\
\sigma_{31} &= 2C_{44}\varepsilon_{31} + R_i\omega_i - \varepsilon_{11}^3 E_2 \\
\sigma_{13} &= 2C_{44}\varepsilon_{31} + R_i\omega_i - \varepsilon_{11}^3 E_1 \\
\sigma_{12} &= 2C_{66}\varepsilon_{12} \\
H_{31} &= 2R_i\varepsilon_{31} + \kappa_5 \omega_i - \varepsilon_{11}^3 E_2 \\
H_{32} &= 2R_i\varepsilon_{32} + \kappa_5 \omega_i - \varepsilon_{11}^3 E_2 \\
H_{33} &= R_i(\varepsilon_{11} + \varepsilon_{22}) + R_i\varepsilon_{33} + \kappa_5 \omega_i - \varepsilon_{11}^3 E_3 \\
D_1 &= 2\varepsilon_{11}^j\varepsilon_{31} + \varepsilon_{11}^j\varepsilon_{31} + \varepsilon_{11}^j\omega_i + \varepsilon_{11} E_1 \\
D_2 &= 2\varepsilon_{11}^j\varepsilon_{32} + \varepsilon_{11}^j\varepsilon_{32} + \varepsilon_{11}^j\omega_i + \varepsilon_{11} E_2 \\
D_3 &= \varepsilon_{31}(\varepsilon_{11} + \varepsilon_{22}) + \varepsilon_{31}\varepsilon_{33} + \varepsilon_{31}^j\varepsilon_{33} + \varepsilon_{31}^j\omega_i + \varepsilon_{33} E_3 \\
\end{align*}
\]

(1)

where \(\varepsilon_{ij}\), \(\sigma_{ij}\), \(\kappa_i\), and \(H_{ij}\) \((i,j = 1,2,3)\) are the phonon strains, phason strain, the phonon stress and phason stress, respectively; \(E_j\), \(D_j\) stand for the electric field and the electric displacement, respectively; \(C_{ij}, k_j, R_i\) stand for the phonon elastic, phason elastic and phonon-phason coupling modulus, \(e_{ij}^1\) and \(e_{ij}^3\) denote piezoelectric constants of the phonon and phason fields, respectively; \(\varepsilon_{11}\) and \(\varepsilon_{33}\) denote the dielectric permittivity. Besides, the geometry equations of 1D hexagonal piezoelectric QCs are given by

\[
\begin{align*}
\varepsilon_{ij} &= \frac{1}{2}(\partial_i u_j + \partial_j u_i), \omega_j = \partial_j \phi \\
E_j &= -\partial_j \phi \quad (i,j = 1,2,3)
\end{align*}
\]

(2)

where \(u_i\), \(\omega\), \(\phi\) denote the displacement of phonon field and phason field and the electric potential, respectively. Equilibrium equations in the absence of body forces are given by
3. Plane elasticity in aperiodical plane

When defects such as cracks and holes, etc parallel to the periodic axis of 1D hexagonal piezoelectric QCs, the geometric properties of the material will not change in the quasi-periodic direction. If we take quasi-periodic axis of 1D hexagonal piezoelectric QCs for \( x_i \) axis, then

\[
\partial_i u_i = 0, \quad \partial_i v = 0, \quad \partial_i \sigma_y = 0, \quad \partial_i D_y = 0, \quad \partial_i D_j = 0 \quad (i, j = 1, 2, 3) \tag{4}
\]

Substituting Eqs.(4) into Eqs.(1)-(3), we get the physical equation in \( x_2 - x_3 \) plane as follows:

\[
\begin{align*}
\varepsilon_{22} &= a_1 \sigma_{22} + a_2 \sigma_{33} + b_1 H_{33} + c_1 D_3 \\
\varepsilon_{33} &= a_3 \sigma_{22} + a_4 \sigma_{33} + b_2 H_{33} + c_2 D_3 \\
\omega_z &= a_2 \sigma_{22} + a_4 \sigma_{33} + b_3 H_{33} + c_3 D_3 \\
E_2 &= a_3 \sigma_{22} + d_2 \sigma_{33} + e_2 D_3 \\
E_3 &= a_4 \sigma_{22} + a_6 \sigma_{33} + b_3 H_{33} + c_4 D_3
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{\kappa_2 c_{33} \varepsilon_{33} + c_{33} e_{31}^2 \varepsilon_{33}^2 - R_2^2 \varepsilon_{33} - R_2 R_3 e_{31}^2 \varepsilon_{33} - R_2 e_{33}^2 \varepsilon_{33}^2 + \kappa_1 (e_{33}^2)^2}{\Delta_1} \\
a_2 &= \frac{R_2 R_3 \varepsilon_{33} + R_1 e_{31}^2 \varepsilon_{33}^2 + R_1 e_{33}^2 \varepsilon_{33}^2 - \kappa_1 (e_{33}^2)^2 - \kappa_3 \varepsilon_{33}}{\Delta_1} \\
a_3 &= \frac{R_3 R_2 \varepsilon_{33} + R_1 e_{31}^2 \varepsilon_{33}^2 + R_1 e_{33}^2 \varepsilon_{33}^2 - \kappa_1 (e_{33}^2)^2 - \kappa_3 \varepsilon_{33} - c_{33} e_{31}^2 \varepsilon_{33}^2}{\Delta_1} \\
a_4 &= \frac{\kappa_2 c_{22} \varepsilon_{22} + c_{33} e_{31}^2 \varepsilon_{33}^2 - R_1^2 \varepsilon_{33} - R_1 e_{31}^2 \varepsilon_{33}^2 - R_1 e_{31}^2 \varepsilon_{33}^2 + \kappa_1 (e_{33}^2)^2}{\Delta_1} \\
a_5 &= \frac{c_{33} R_3 R_2 \varepsilon_{33} + c_{33} e_{31}^2 \varepsilon_{33}^2 - c_{33} R_1 \varepsilon_{33} - c_{33} e_{31}^2 \varepsilon_{33}^2 - R_1 e_{31}^2 \varepsilon_{33}^2 + R_2 e_{31}^2 \varepsilon_{33}^2}{\Delta_1} \\
a_6 &= \frac{R_3 c_{15} e_{33}^2 - c_{13} R_3 e_{33}^2 + c_{33} R_1 e_{33}^2 - c_{33} \kappa_1 e_{31}}{\Delta_1} \\
a_7 &= \frac{c_{33} R_3 e_{33}^2 - c_{13} R_3 e_{31}^2 + c_{33} R_3 e_{33}^2 - R_1 R_2 e_{33}^2 + R_2 e_{33}^2}{\Delta_1} \\
a_8 &= \frac{c_{22} R_2 e_{33}^2 - c_{22} \kappa_1 e_{33}^2 - c_{13} R_2 e_{33}^2 + c_{33} \kappa_1 e_{31} + R_1 e_{33}^2 - R_1 R_2 e_{33}^2 + R_2 e_{33}^2}{\Delta_1} \\
a_9 &= \kappa_2 \varepsilon_{11} + (e_{15}^2)^2, \quad d_i = -(R_1 e_{11} + e_{15}^2), \quad e_1 = -R_3 e_{15}^2 + \kappa_2 e_{15}^2
\end{align*}
\]
\[ a_{10} = 2(R_2 \varepsilon_{11} + e_{15}^1 e_{15}^2 ) , \quad d_2 = 2(c_{44} \varepsilon_{11} + (e_{13}^1)^2 ) , \quad e_2 = 2(c_{44} e_{15}^1 - e_{15}^1 R_3 ) \]

\[ a_{11} = 2(R_2 e_{13}^2 - \kappa e_{23}^1 ) , \quad d_3 = -2(c_{44} e_{15}^1 - R e_{15}^1 ) , \quad e_3 = 2(c_{44} - R^2 ) \]

\[ b_2 = \frac{R c_{13} \varepsilon_{33} + R e_{13}^1 e_{13}^2 + c_{13} e_{15}^2 e_{33} - R e_{13} e_{33}^1 - R c_{22} e_{33}^1 - c_{22} e_{33}^2 e_{33} }{\Delta_1} \]

\[ b_3 = \frac{c_{22} c_{33} \varepsilon_{33} + c_{22} (e_{33}^1)^2 - (e_{33}^1)^2 \varepsilon_{33} - 2c_{13} e_{13}^1 e_{33} + c_{33} (e_{33}^1)^2 }{\Delta_1} \]

\[ b_4 = \frac{R c_{22} c_{33}^1 + (c_{22} e_{33}^1)^2 + R c_{13} e_{13}^2 - c_{22} c_{33}^1 e_{33} - R c_{22} e_{33} e_{33} - R c_{13} e_{13}^1 }{\Delta_1} \]

\[ c_{1} = \frac{\kappa_1 c_{22} c_{33}^1 + c_{13} R_2 e_{33}^1 + R c_{22} e_{33}^1 - R c_{13} c_{33}^1 e_{33} + c_{33} \kappa_1 e_{33}^1 c_{33}^1 - \kappa_1 c_{33} e_{33}^2 }{\Delta_1} \]

\[ c_{2} = \frac{c_{22} c_{33}^2 e_{33} - R c_{22} e_{33}^1 - (c_{13}^2 e_{33}^2 e_{33} + R c_{13} e_{33}^2 + R c_{13} e_{33}^1 - R c_{22} e_{33}^1 }{\Delta_1} \]

\[ c_{3} = c_{22} c_{33}^2 e_{33} - R c_{22} e_{33}^1 - (c_{13}^2 e_{33}^2 e_{33} + R c_{13} e_{33}^2 + R c_{13} e_{33}^1 - R c_{22} e_{33}^1 }{\Delta_1} \]

\[ c_{4} = c_{22} c_{33}^2 e_{33} - R c_{22} e_{33}^1 - (c_{13}^2 e_{33}^2 e_{33} + R c_{13} e_{33}^2 + R c_{13} e_{33}^1 - R c_{22} e_{33}^1 }{\Delta_1} \]

\[ d_{1} = \frac{-R_3 \varepsilon_{11} + (e_{15}^1)^2 }{\Delta_2} , \quad e_{2} = \frac{\kappa_2 e_{15}^1 - R e_{15}^2 }{\Delta_2} , \quad e_{1} = -R_3 e_{15}^1 + \kappa_2 e_{15}^1 \]

\[ \Delta_1 = c_{22} c_{33}^1 \kappa_1 e_{33} + c_{22} e_{33}^1 e_{33} + c_{22} R_2^2 e_{33}^1 - c_{22} R_2 e_{33}^1 e_{33} - c_{22} R_2 e_{33}^1 e_{33} - c_{22} \kappa_1 (e_{33}^1)^2 \]

\[ -c_{13} \kappa e_{33} + c_{22}^2 e_{33}^2 e_{33} + c_{13} R e_{33}^2 + c_{13} R e_{33}^1 e_{33} - c_{13} \kappa e_{33} e_{33}^1 \]

\[ + c_{13} R e_{33} - c_{22} c_{33}^1 c_{33}^1 - c_{33} R e_{33}^1 e_{33} - c_{33} R e_{33}^1 e_{33} - R^2 (e_{33}^1)^2 + R_3 R e_{33} e_{33} \]

\[ + c_{13} R e_{33} e_{33} - c_{33} \kappa_1 e_{33} e_{33} + c_{33} \kappa_1 e_{33} e_{33} + c_{33} \kappa_1 e_{33} e_{33} - R^2 (e_{33}^1)^2 \]

\[ \Delta_2 = 2[c_{44} \kappa_2 e_{33} + c_{44} (e_{33}^1)^2 - (R_2)^2 e_{33} - R_3 e_{15}^1 e_{15} - 2 R_3 (e_{33}^1)^2 + \kappa_2 (e_{33}^1)^2 ] \]

The corresponding equilibrium equations in plane \( x_2 - x_3 \) are

\[ \frac{\partial^2 \sigma_{22}}{\partial x_2^2} + \frac{\partial^2 \sigma_{23}}{\partial x_2 \partial x_3} + \frac{\partial^3 \sigma_{33}}{\partial x_2^3} = 0 , \quad \frac{\partial^2 \sigma_{32}}{\partial x_2^2} + \frac{\partial^2 \sigma_{33}}{\partial x_2 \partial x_3} + \frac{\partial^3 \sigma_{33}}{\partial x_2^3} = 0 , \quad \frac{\partial^2 H_{32}}{\partial x_2^2} + \frac{\partial^2 H_{33}}{\partial x_2 \partial x_3} = 0 , \quad \frac{\partial^2 D_2}{\partial x_2^2} + \frac{\partial^2 D_3}{\partial x_2 \partial x_3} = 0 \]

\[ (12) \]

The distortion equation of compatibility are

\[ \frac{\partial^2 e_{22}}{\partial x_2^2} + \frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} - 2 \partial^2_2 \partial x_2 e_{23} = 0 , \quad \frac{\partial^2 e_{32}}{\partial x_2^2} - \frac{\partial^2 e_{33}}{\partial x_2 \partial x_3} = 0 , \quad \frac{\partial^2 \omega_3}{\partial x_2^2} - \frac{\partial^2 \omega_3}{\partial x_2 \partial x_3} = 0 \]

\[ (13) \]

Now three new stress potential functions are introduced as followings

\[ \sigma_{22} = \sigma_{22}^0 U , \quad \sigma_{33} = \sigma_{33}^0 U , \quad \sigma_{23} = -\partial_2 \partial_3 U , \quad H_{32} = \partial_3 V , \quad H_{33} = -\partial_3 V , \quad D_{22} = \partial_2 W , \quad D_{33} = -\partial_3 W \]

\[ (14) \]

Where \( U(x_2, x_3) , \quad V(x_2, x_3) , \quad W(x_2, x_3) \) are three new stress potential functions introduced.

Equation (14) satisfies the equation (12). Substituting Eq. (14) into Eq. (5)-(9), then substituting the result into Eq. (13), by simple calculation, we have

\[ L_1 U - L_2 V - L_3 W = 0 , \quad L_4 U - L_5 V - L_6 W = 0 , \quad L_7 U - L_8 V - L_9 W = 0 \]

\[ (15) \]

where differential operator \( L_i (i = 1, 2, 3, 4, 5, 6, 7, 8, 9) \) such that

\[ L_1 = a_1 \partial_2^4 + (a_2 + a_3 + 2a_4) \partial_2^2 \partial_3^2 + a_4 \partial_2^2 , \quad L_2 = (b_1 + 2d_1) \partial_2 \partial_3^2 + b_2 \partial_3^2 , \]

\[ L_4 = (a_5 + a_{10}) \partial_2 \partial_3^2 + a_6 \partial_2^2 , \quad L_5 = (c_1 + 2e_1) \partial_2 \partial_3^2 + c_2 \partial_3^2 , \]
By Eq. (15), we eliminate $V$ and $W$, then we get a partial differential equation as follows:

\[
L_c - 2L_{bd} - L_{aa} = 0
\]

(17)

Which is a partial differential equations of order eight. By literature\[15\], using four generalized analytic functions $\Phi_k(z_k) (k = 1, 2, 3, 4)$, the solution of Eq. (17) can be expressed as

\[
U(x_2, x_3) = 2 \text{Re} \sum_{k=1}^{4} \Phi_k(z_k), \quad z_k = x_2 + \mu_k x_3
\]

(18)

where $\text{Re}$ stand for the real part of the corresponding complex function, $\mu_k = \alpha_k + i \beta_k \quad (k = 1, 2, 3, 4, i = \sqrt{-1})$ are the characteristic roots of differential equation(17), $\alpha_k$, $\beta_k$ are real constants, which depends on the piezoelectric quasicrystal elasticity only. If the eigenvalue is repeated root. From Eq.(15) we can obtain

\[
V(x_2, x_3) = 2 \text{Re} \sum_{k=1}^{4} \eta_k \Phi_k'(z_k), \quad W(x_2, x_3) = 2 \text{Re} \sum_{k=1}^{4} \zeta_k \Phi_k'(z_k)
\]

(19)

where

\[
\eta_k = \frac{[a_k \mu_k^4 + (a_2 + a_3 + 2a_9) \mu_k^3] (c_1 + e_2 \mu_k^2) - [(a_3 + a_{10}) \mu_k^2 + a_9] [(c_1 + 2e_1) \mu_k^2 + c_1]}{\Pi(\mu_k)}
\]

\[
\zeta_k = \frac{[(b_1 - 2d_1) \mu_k^2 + b_2] [(a_1 + a_{10}) \mu_k^2 + a_6] - (b_3 + d_3 \mu_k^2) [a_4 \mu_k^4 + (a_2 + a_3 + 2a_9) \mu_k^2]}{\Pi(\mu_k)}
\]

(20)

Substituting Eq. (18)and(19)into Eq.(14) yields

\[
\sigma_{22} = 2 \text{Re} \sum_{k=1}^{4} \mu_k^2 \Phi_k'(z_k), \quad \sigma_{33} = 2 \text{Re} \sum_{k=1}^{4} \phi_k'(z_k), \quad \sigma_{23} = -2 \text{Re} \sum_{k=1}^{4} \mu_k \phi_k'(z_k)
\]

\[
H_{32} = 2 \text{Re} \sum_{k=1}^{4} \eta_k \mu_k \phi_k'(z_k), \quad H_{33} = -2 \text{Re} \sum_{k=1}^{4} \phi_k'(z_k), \quad D_{32} = 2 \text{Re} \sum_{k=1}^{4} \zeta_k \mu_k \phi_k'(z_k), \quad D_{33} = -2 \text{Re} \sum_{k=1}^{4} \zeta_k \phi_k'(z_k)
\]

(21)

where $\phi_k(z_k) = \partial z_k \Phi_k(z_k) = \Phi_k'(z_k)$

By Eq.(14), the complex representation of boundary conditions can be represented by the following formula

\[
\partial_z U = 2 \text{Re} \sum_{k=1}^{4} \phi_k(z_k) = -\int T_3 ds
\]

(22)

\[
\partial_z U = 2 \text{Re} \sum_{k=1}^{4} \mu_k \phi_k(z_k) = \int T_3 ds
\]

(23)

\[
V = 2 \text{Re} \sum_{k=1}^{4} \eta_k \phi_k(z_k) = -\int T_3 ds
\]

(24)

\[
W = 2 \text{Re} \sum_{k=1}^{4} \zeta_k \phi_k(z_k) = \int T_3 ds
\]

(25)
Where $T_2$ and $T_3$ are plane stress acting on the boundary, $T_a$ and $T_e$ stand for generalized stress acting on the phason space and the plane stress acting on electric field separately.

4. The problem of straight cracks in a strip of 1D hexagonal piezoelectric quasicrystal body in the direction perpendicular to direction of quasi-period

We now discuss a penetrating straight crack, along the periodic direction $x_1$, in a strip of 1D hexagonal piezoelectric QCs with point 6 mm. Then the geometric properties of the materials will not change in the periodic direction $x_1$, the problem in the plane perpendicular to periodic direction is also plane elasticity problem.

As shown in Fig.1, we have the following boundary conditions:

$$\left\{ \begin{array}{l}
\sigma_{33} = P, \sigma_{32} = \tau_x, D_2 = T_2, D_3 = T_2, -a < x_1 < a, x_1 = 0 \\
H_{33} = \tau_y, H_{32} = \tau_y, T_1 = T_e = 0 \quad \{ x_1 < a, x_1 = 0 \}
\end{array} \right. \tag{26}$$

In order to obtain the complex potential function in the region in $\Omega$, suppose that the Laurent expansion of function $\psi(z)$ is:

$$\psi(z) = C_0 + \sum_{j=1}^{\infty} C_j z^j + \phi_1^k (z) \tag{28}$$

where

$$\phi_1^k (z) = a_k + \sum_{j=1}^{\infty} a_j z^j \tag{29}$$

$C_0$, $C_j$ and $a_j$ is the undetermined complex constant.

Substituting function $\psi(z)$ into Eq.(21), by condition (27) we have

$$C_k^{(j)} = 0, \quad k=1,2,3,4; j=2,3,\ldots \tag{30}$$

$$2 \text{Re} \sum_{k=1}^{4} \mu_k \chi_k^0 C_k = 0, \quad 2 \text{Re} \sum_{k=1}^{4} \chi_k^0 C_k = p, \quad 2 \text{Re} \sum_{k=1}^{4} \eta_k C_k = 0 \tag{31}$$

There are 8 real constants in the above equations $\text{Re} \chi_k$, $\text{Im} \chi_k (k=1,2,3)$, but there are 7 independent equations only, therefore a constant can be chosen freely. Take $\text{Re} C_3 = 0$

Substituting function $\psi(z)$ into Eq.(22)-(25) again, by condition (27) we get

$$\sum_{k=1}^{4} [\phi_k^0 (z_k) + \phi_k^0 (z_k)] = - \sum_{k=1}^{4} [\mu_k \chi_k^0 + \mu_k \chi_k^0] \tag{32}$$
\[
\sum_{k=1}^{4} [H_k \phi_k^0(z_k) + \mu_k \phi_k^0(\bar{z}_k)] = -\sum_{k=1}^{4} [c_k \mu_k z_k + \overline{c_k \mu_k z_k}] \tag{33}
\]
\[
\sum_{k=1}^{4} [\eta_k \phi_k^0(z_k) + \overline{\eta_k \phi_k^0(z_k)}] = -\sum_{k=1}^{4} [c_k \eta_k z_k + \overline{c_k \eta_k z_k}] \tag{34}
\]
\[
\sum_{k=1}^{4} [\zeta_k \phi_k^0(z_k) + \overline{\zeta_k \phi_k^0(\bar{z}_k)}] = -\sum_{k=1}^{4} [c_k \zeta_k z_k + \overline{c_k \zeta_k z_k}] \tag{35}
\]

where we take the boundary value on the crack surface for \(z_k\), and \((\ )^*\) stand for complex conjugate.

We introduce a generalized conformal mapping function \([16]\)
\[
z_k = \omega_k(\zeta_k) = \frac{\beta_k}{\pi} \log\left[\frac{\alpha}{2(\zeta_k + \frac{1}{\zeta_k})} + \beta\right] \tag{36}
\]
where
\[
\beta_k = \frac{\mu_k - \mu_k}{2i}, \quad \alpha = e^{\pi \alpha/\beta} - e^{-\pi \alpha/\beta}, \quad \beta = \frac{e^{\pi \alpha/\beta} + e^{-\pi \alpha/\beta}}{2}, \quad k = 1, 2, 3
\]

Which maps the interior area \(\Omega_k\) of \(z_k\)-plane into the exterior of a unit circle in the \(\zeta_k\)-plane, \(\zeta_k = \sigma = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi\), then there are three points on \(\Omega_k\) can be transformed into the same point \(\sigma\) on the unit circle, and \(\Phi_k^{(0)}(z_k)\) represented a function which is mapped by function \(\phi_k^0(z_k)\), then the condition (32)-(35) can be rewritten as:

\[
\sum_{i=1}^{4} [\Phi_i^{(0)}(\sigma) + \Phi_i^{(0)}(\overline{\sigma})] = l_{1} \sigma + l_{\overline{\sigma}} \tag{37}
\]
\[
\sum_{i=1}^{4} [\mu_i \Phi_i^{(0)}(\sigma) + \mu_i \Phi_i^{(0)}(\overline{\sigma})] = l_{1} \sigma + l_{\overline{\sigma}} \tag{38}
\]
\[
\sum_{i=1}^{4} [\eta_i \Phi_i^{(0)}(\sigma) + \eta_i \Phi_i^{(0)}(\overline{\sigma})] = l_{1} \sigma + l_{\overline{\sigma}} \tag{39}
\]
\[
\sum_{i=1}^{4} [\zeta_i \Phi_i^{(0)}(\sigma) + \zeta_i \Phi_i^{(0)}(\overline{\sigma})] = l_{1} \sigma + l_{\overline{\sigma}} \tag{40}
\]

where
\[
l_{1} = \sum_{k=1}^{4} a(C_k + \overline{C_k}) + ib(C_k \mu_k + \overline{C_k \mu_k})
\]
\[
l_{2} = \sum_{k=1}^{4} a(C_k \mu_k + \overline{C_k \mu_k}) + ib(C_k \mu_k^2 + \overline{C_k \mu_k^2})
\]
\[
l_{3} = \sum_{k=1}^{4} a(C_k \eta_k + \overline{C_k \eta_k}) + ib(C_k \mu_k \eta_k + \overline{C_k \mu_k \eta_k})
\]
\[
l_{4} = \sum_{k=1}^{4} a(c_k \zeta_k + \overline{c_k \zeta_k}) + ib(c_k \mu_k \zeta_k + \overline{c_k \mu_k \zeta_k})
\]

Multiplying equations (37)-(40) by \(\frac{d\sigma}{\sigma - \zeta}\), where \(\zeta\) is point outside of the unit circle, according to Cauchy’s integral formula for infinite region, we get
\[
\int_{\gamma} \Phi_k^{(0)}(\sigma) d\sigma = -2\pi i \Phi_k^{(0)}(\zeta_k), \quad \int_{\gamma} \Phi_i^{(0)}(\overline{\sigma}) d\overline{\sigma} = \int_{\gamma} \sigma d\sigma = 0, \quad \int_{\gamma} \frac{\sigma d\sigma}{\sigma - \zeta} = 0, \quad \int_{\gamma} \frac{d\sigma}{\sigma(\sigma - \zeta)} = -2\pi i \frac{1}{\zeta}
\]

Which yields

\[
\sum_{k=1}^{4} [H_k \phi_k^0(z_k) + \mu_k \phi_k^0(\bar{z}_k)] = -\sum_{k=1}^{4} [c_k \mu_k z_k + \overline{c_k \mu_k z_k}] \tag{33}
\]
\[
\sum_{k=1}^{4} [\eta_k \phi_k^0(z_k) + \overline{\eta_k \phi_k^0(z_k)}] = -\sum_{k=1}^{4} [c_k \eta_k z_k + \overline{c_k \eta_k z_k}] \tag{34}
\]
\[
\sum_{k=1}^{4} [\zeta_k \phi_k^0(z_k) + \overline{\zeta_k \phi_k^0(\bar{z}_k)}] = -\sum_{k=1}^{4} [c_k \zeta_k z_k + \overline{c_k \zeta_k z_k}] \tag{35}
\]
$$\sum_{i=1}^{4} \Phi_i = \frac{l_i}{\zeta}, \quad \sum_{i=1}^{4} \mu_i \Phi_i = \frac{l_i}{\zeta}, \quad \sum_{i=1}^{4} \eta_i \Phi_i = \frac{\mu_i}{\zeta}, \quad \sum_{i=1}^{4} \zeta_i \Phi_i = \frac{l_i}{\zeta} \quad (42)$$

When k takes 1, 2, 3, 4, $\zeta_i$ correspond to $\zeta_k$. The above solutions can be further represented as

$$\Phi_i = \frac{l_i}{\zeta_k} \sum_{j=1}^{4} A_{ij} l_j, k = 1, 2, 3, 4 \quad (43)$$

where $$|A_{ij}| = \frac{1}{\Delta} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$\Delta = \mu_1 \eta_1 (\zeta_1 - \zeta_1) + \mu_2 \eta_1 (\zeta_2 - \zeta_1) + \mu_3 \eta_1 (\zeta_3 - \zeta_1) + \mu_4 \eta_1 (\zeta_4 - \zeta_1) + \mu_1 \eta_2 (\zeta_1 - \zeta_2) + \mu_2 \eta_2 (\zeta_2 - \zeta_2) + \mu_3 \eta_2 (\zeta_3 - \zeta_2) + \mu_4 \eta_2 (\zeta_4 - \zeta_2) + \mu_1 \eta_3 (\zeta_1 - \zeta_3) + \mu_2 \eta_3 (\zeta_2 - \zeta_3) + \mu_3 \eta_3 (\zeta_3 - \zeta_3) + \mu_4 \eta_3 (\zeta_4 - \zeta_3) + \mu_1 \eta_4 (\zeta_1 - \zeta_4) + \mu_2 \eta_4 (\zeta_2 - \zeta_4) + \mu_3 \eta_4 (\zeta_3 - \zeta_4) + \mu_4 \eta_4 (\zeta_4 - \zeta_4) + \mu_1 \eta_5 (\zeta_1 - \zeta_5) + \mu_2 \eta_5 (\zeta_2 - \zeta_5) + \mu_3 \eta_5 (\zeta_3 - \zeta_5) + \mu_4 \eta_5 (\zeta_4 - \zeta_5) + \mu_1 \eta_6 (\zeta_1 - \zeta_6) + \mu_2 \eta_6 (\zeta_2 - \zeta_6) + \mu_3 \eta_6 (\zeta_3 - \zeta_6) + \mu_4 \eta_6 (\zeta_4 - \zeta_6) + \mu_1 \eta_7 (\zeta_1 - \zeta_7) + \mu_2 \eta_7 (\zeta_2 - \zeta_7) + \mu_3 \eta_7 (\zeta_3 - \zeta_7) + \mu_4 \eta_7 (\zeta_4 - \zeta_7) + \mu_1 \eta_8 (\zeta_1 - \zeta_8) + \mu_2 \eta_8 (\zeta_2 - \zeta_8) + \mu_3 \eta_8 (\zeta_3 - \zeta_8) + \mu_4 \eta_8 (\zeta_4 - \zeta_8)$$

From Eqs. (28) and (36), we have

$$\varphi_k(z_k) = c_k z_k - \frac{\exp(\pi \beta_k H z_k) - \beta}{\alpha} \sqrt{1 - 2 \beta \exp(\pi \beta_k H z_k) + \exp(\frac{2 \pi}{\beta_k H} z_k)} \sum_{j=1}^{4} A_{ij} l_j, \quad (44)$$

$$\varphi_k(z_k) = c_k - \frac{\exp(\frac{\pi}{\beta_k H} z_k) - \exp(\frac{2 \pi}{\beta_k H} z_k) + \alpha}{\sqrt{1 - 2 \beta \exp(\frac{\pi}{\beta_k H} z_k) + \exp(\frac{2 \pi}{\beta_k H} z_k)}} \sum_{j=1}^{4} A_{ij} l_j, \quad (45)$$

Substituting Eq.(45) into the Eqs. (21), all stress components of the elastic field of piezoelectric quasicrystals are obtained as follows:

$$\sigma_{22} = 2 \text{Re} \sum_{i=1}^{4} \mu_i c_i - \frac{\exp(\frac{\pi}{\beta_k H} z_k)}{\beta_k H} \left[ \sqrt{1 - 2 \beta \exp(\frac{\pi}{\beta_k H} z_k) + \exp(\frac{2 \pi}{\beta_k H} z_k)} \right] \sum_{j=1}^{4} A_{ij} l_j \right]$$

$$\sigma_{33} = 2 \text{Re} \sum_{i=1}^{4} \mu_i c_i - \frac{\exp(\frac{\pi}{\beta_k H} z_k)}{\beta_k H} \left[ \sqrt{1 - 2 \beta \exp(\frac{\pi}{\beta_k H} z_k) + \exp(\frac{2 \pi}{\beta_k H} z_k)} \right] \sum_{j=1}^{4} A_{ij} l_j \right]$$

$$\sigma_{23} = -2 \text{Re} \sum_{i=1}^{4} \mu_i c_i - \frac{\exp(\frac{2 \pi}{\beta_k H} z_k)}{\beta_k H} \left[ \sqrt{1 - 2 \beta \exp(\frac{\pi}{\beta_k H} z_k) + \exp(\frac{2 \pi}{\beta_k H} z_k)} \right] \sum_{j=1}^{4} A_{ij} l_j \right]$$
It is not difficult to find that there is singularity of $-\frac{1}{2}$ order at the crack tip ($z = \pm a$). By document[14], the stress intensity factors of mode III crack of phonon field near the crack tip $z = a$ can be defined as follow

$$K = \begin{bmatrix} K_{u2} & iK_{u3} \\ K_{u3} & -iK_{u2} \end{bmatrix} = \lim_{x \to a} \sqrt{2\pi(x-a)} \begin{bmatrix} \sigma_{32} - i\sigma_{33} \\ H_{33} - iH_{32} \end{bmatrix} \quad (47)$$

Substituting Eq. (46) into the Eqs. (47), we can get

$$K_{u2} = \sqrt{H} \Re \sum_{i=1}^{4} \eta_i \mu \frac{\alpha + \beta - 1}{\sqrt{\alpha(\alpha + \beta)}} \sum_{j=1}^{4} A_{ij} l_{ij}, \quad K_{u3} = \sqrt{H} \Re \sum_{i=1}^{4} \mu_i \sqrt{\beta_i} \frac{\alpha + \beta - 1}{\sqrt{\alpha(\alpha + \beta)}} \sum_{j=1}^{4} A_{ij} l_{ij} \quad (48)$$

When the phonon field stress, phase field stress and their coefficients get to zero, Eq. (48) can be rewritten as

$$K_{u2} = \sqrt{H} \Re \sum_{i=1}^{4} \eta_i \mu \frac{\alpha + \beta - 1}{\sqrt{\alpha(\alpha + \beta)}} \sum_{j=1}^{4} A_{ij} l_{ij}$$

where $l_i = 0$, this is the plane elastic problem of a straight crack in an isotropic piezoelectric narrow body, which is in accordance with the results [23]. The stress intensity factor are given by

$$K_l = \sqrt{H} \frac{\alpha + \beta - 1}{\sqrt{\alpha(\alpha + \beta)}} \quad (49)$$
Fig. 2 reveals the law of $K_i$ with crack length. It is shown that the magnitude of stress intensity factor always increases with the increase of crack length and decreases with the increase of strip width. Fig. 3 indicates $\beta_i/\alpha_i$ has a strong influence on $K_{\eta_i}/K_i$.

As $h \to \infty$, Eqs. (45) can be reduced to

$$\varphi_k(z_k) = c_k - \frac{1}{a} \left[ 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right] \sum_{j=1}^{4} A_j l_j$$

(50)

Substituting Eq. (50) into (21), the solution of the aperiodic plane problem for piezoelectric quasicrystals is obtained as follows

$$\sigma_{22} = 2 \text{Re} \sum_{k=1}^{4} \mu_k^2 \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right],$$

$$\sigma_{33} = 2 \text{Re} \sum_{k=1}^{4} \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right],$$

$$\sigma_{23} = -2 \text{Re} \sum_{k=1}^{4} \mu_k \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right],$$

$$H_{32} = -2 \text{Re} \sum_{k=1}^{4} \eta_k \mu_k \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right],$$

$$H_{33} = -2 \text{Re} \sum_{k=1}^{4} \eta_k \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right],$$

$$D_j = 2 \text{Re} \sum_{k=1}^{4} \xi_k \eta_k \left[ c_k - \frac{1}{a} \left( 1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right) \sum_{j=1}^{4} A_j l_j \right]$$

The analytical solution of the stress intensity factor at the tip of a Griffith crack in an infinite one-dimensional hexagonal piezoelectric quasicrystal can be obtained by equation (47)

$$\begin{cases}
K_{\eta_{11}} = p \sqrt{\pi a}, & K_{\eta_{12}} = \tau_{2} \mu_{1} \sqrt{\pi a} \\
K_{\eta_{13}} = \tau_{2} \eta_{1} \sqrt{\pi a}, & K_{\eta_{33}} = \tau_{1} \mu_{1} \sqrt{\pi a} \\
K_{\eta_{31}} = \tau_{3} \mu_{1} \sqrt{\pi a}, & K_{\eta_{33}} = \tau_{2} \eta_{1} \sqrt{\pi a} \\
K_{\eta_{33}} = \tau_{3} \xi_{1} \sqrt{\pi a}, & K_{\eta_{33}} = \zeta_{1} \mu_{1} \sqrt{\pi a}
\end{cases}$$

(52)
When the piezoelectric constants, permittivity and quasicrystal elastic constants change into zero, the Eqs. (52) can be rewritten as

\[ K(x) = p\sqrt{x}a \]

This is an analytical solution of the stress intensity factor at the tip of a Griffith crack in an infinitely isotropic material, which is consistent with the classical results.

5. Conclusion and Discussion

The theory of defects in the aperiodic plane of one dimensional six dimensional piezoelectric quasicrystals is established, and the governing equations and fundamental solutions of the elastic problems are given. As an application, the Griffith crack in an infinitely long and narrow body is studied by means of generalized conformal transformation in complex functions, the analytical solution of the elastic field is given. In the limit state, the solution of the crack problem is given. The result shows that the stress still has \( -\frac{1}{2} \) order singularity at crack tip \( z = \pm a \), this is basically the same as cracks passing through periodic plane in one-dimensional hexagonal quasicrystal. The stress field is related to the elastic constants of the phason field, which is different from that of quasi crystal.

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