Poisson brackets with divergence terms in field theories: two examples

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Abstract

In field theories one often works with the functionals which are integrals of some densities. These densities are defined up to divergence terms (boundary terms). A Poisson bracket of two functionals is also a functional, i.e., an integral of a density. Suppose the divergence term in the density of the Poisson bracket be fixed so that it becomes a bilinear form of densities of two functionals. Then the left-hand side of the Jacobi identity written in terms of densities is not necessarily zero but a divergence of a trilinear form. The question is: what can be said about this trilinear form, what kind of a higher Jacobi identity (involving four fields) it enjoys? Two examples whose origin is the theory of integrable systems are given.

In field theories one often works with the functionals which are integrals of some densities. These densities are defined up to divergence terms (boundary terms). A Poisson bracket of two functionals is also a functional, i.e., an integral of a density. Suppose the divergence term in the density of the Poisson bracket be fixed so that it becomes a bilinear form of densities of two functionals. Then the left-hand side of the Jacobi identity written in terms of densities is not necessarily zero but a divergence of a trilinear form. The question is: what can be said about this trilinear form, what kind of a higher Jacobi identity (involving four fields) it enjoys? Our examples relate to the simplest 1-dimensional case. Their origin is in the theory of integrable systems.

My attention to this topic was called by J. Stasheff, to whom I am very thankful. In a series of articles by V. O. Soloviev [1] a close problem was posed: is it possible, using the freedom of choice of the divergence term in a local Poisson bracket, to make the Jacobi identity exact?

1. Scalar example.

We have the following structures.

1) Differential algebra $A$ consisting of differential polynomials of $u$ with the derivation $\partial = d/dx$.

2) A space $B = A/\partial A$.

3) Derivations $\partial_a = \sum_{i=0}^{\infty} a^{(i)} \partial/\partial u^{(i)}$ where $a \in A$. They commute with $\partial$ and, therefore, can be transferred to $B$. They form a Lie algebra with respect of the commutator $[\partial_a, \partial_b] = \partial_a \partial_b - \partial_b \partial_a = \partial_{\partial_a b - \partial_b a}$ called the Lie algebra of vector fields.

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4) Vector fields $\partial_{X'}$, where $X \in \mathcal{A}$, form a subalgebra,

$$[\partial_{X'}, \partial_{Y'}] = \partial(\partial_{X'}Y' - \partial_{Y'}X').$$

(1.1)

5) We define a skew symmetric bilinear form on the above subalgebra:

$$\omega(\partial_{X'}, \partial_{Y'}) = (X'Y - XY')/2.$$  

(1.2)

**Lemma 1.1.** The form $\omega$ is closed with respect to the derivation $\delta$, acting as

$$(\delta \omega)(\partial_{X'}, \partial_{Y'}, \partial_{Z'}) = \partial_{X'} \omega(\partial_{Y'}, \partial_{Z'}) - \omega([\partial_{X'}, \partial_{Y'}], \partial_{Z'}) + \text{(cyclic)}.$$  

Proof.

$$\partial_{X'} 2 \omega(\partial_{Y'}, \partial_{Z'}) - 2 \omega([\partial_{X'}, \partial_{Y'}], \partial_{Z'}) + \text{(cyclic)}$$
$$= \partial_{X'} (Y'Z - YZ') - (\partial_{X'}Y' - \partial_{Y'}X')Z + (\partial_{X'}Y - \partial_{Y'}X)Z' + \text{(cyclic)}$$
$$= Y'\partial_{X'}Z - Y\partial_{X'}Z' + Z\partial_{Y'}X' - Z'\partial_{Y'}X + \text{(cyclic)} = 0. \Box$$

Notice that this is an exact equality, i.e., in $\mathcal{A}$.

6) Now, we define a quasi-Poisson bracket in $\mathcal{A}$:

$$\{f, g\} = \omega(\partial_{X'}, \partial_{Y'}) = (X'Y - XY')/2,$$  

where $X = \delta f/\delta u$, $Y = \delta g/\delta u$.

The word quasi refers to the fact that the Jacobi identity holds, as we will see, only up to an exact derivative, i.e., in $\mathcal{B}$. This bracket is also well defined in $\mathcal{B}$ since a variational derivative of an exact derivative is always zero. Thus, considered in $\mathcal{B}$, this bracket is a genuine Poisson bracket.

Introduce a notation $\xi_f = \partial_{X'}$ where $X = \delta f/\delta u$. The Poisson bracket is a pull-back of the “symplectic” form $\omega$ under the mapping $f \mapsto \xi_f$.

7) Integration by parts yields a formula

$$\partial_{X'}g = X'Y + \partial \Omega_g(\partial_{X'})$$

(1.3)

where $Y = \delta g/\delta u$ and $\Omega_g$ is a 1-form linearly depending on $g$ (the Poincaré invariant for the Lagrangian $g$; in terms of canonical variables $p dq$); $X$ is not necessarily a variational derivative. If it is, then we write $\partial_{X'} = \xi_f$.

**Lemma 1.2.**

$$\xi_f g - \xi_g f = 2\{f, g\} + \partial \Phi(f, g)$$  

where $\Phi(f, g) = \Omega_g(\xi_f) - \Omega_f(\xi_g)$.

This is obvious.

$\Phi$ is a skew symmetric bilinear form in $\mathcal{A}$ which cannot be transferred to $\mathcal{B}$ since $\Phi(f, g') \neq 0$.

**Lemma 1.3.**

$$\xi_f g = \Phi(f, g').$$

2The Leibniz property of a Poisson bracket is not required and even does not make any sense since there is no multiplication in $\mathcal{B}$. 

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Proof. Take a derivative of Eq. (1.3): \( \xi g' = \partial (XY + \Omega g(\xi f)) \). On the other hand, this is \( \partial \Omega g(\xi f) \) (from the same equation taking into account that the variational derivative of \( g' \) vanishes). Thus, \( \Omega g(\xi f) = X'Y + \partial \Omega g(\xi f) = \xi fg \). Further, \( \xi g' = 0 \), hence \( \Phi(f, g') = \Omega g(\xi f) = \xi fg \).

**Proposition 1.1.**

\[ \xi_{(f, g)} = [\xi f, \xi g]. \]

**Proof.** For any vector field \( \xi = \partial_{Z'} \) we have

\[
0 = (\delta \omega)(\xi f, \xi g, \xi) = \xi f\omega(\xi g, \xi) - \xi g\omega(\xi f, \xi) + \omega(\xi f, \xi g)
\]

\[
-\omega([\xi f, \xi g], \xi) + \omega([\xi f, \xi], \xi g) - \omega([\xi g, \xi], \xi f).
\]

Using (1.3), we have

\[
\omega(\xi f, \xi) = (X'Z - XZ')/2 = -XZ' + (XZ)'/2 = -\xi f + \partial (\Omega g(\xi f) + (XZ)/2),
\]

therefore,

\[
0 = -\xi f\xi g + \partial (\xi f\Omega g(\xi f) + \xi f(YZ)/2) + \xi g\xi f - \partial (\xi g\Omega f(\xi f) + \xi g(XZ)/2)
\]

\[
+ \xi\{f, g\} - \omega([\xi f, \xi g], \xi) + [\xi f, \xi g] - \partial (\Omega g([\xi f, \xi]) + Y(\partial X'Z - \partial Z'X)/2)
\]

\[
- [\xi g, \xi f] + \partial (\Omega f([\xi g, \xi]) + X(\partial Y'Z - \partial Z'Y)/2)
\]

\[
= -\xi (\xi fg - \xi g f) + \xi\{f, g\} - \omega([\xi f, \xi g], \xi) + \partial (\xi f\Omega g(\xi f) - \xi g\Omega f(\xi f) - \Omega g([\xi f, \xi]) + \Omega f([\xi g, \xi]))
\]

\[
+ \partial (\partial X'YZ - \partial Y'XZ - \partial X'Z + \partial Z'X + X\partial Y'Z - X\partial Z'Y)/2.
\]

Lemma 1.2 implies:

\[
-\xi (\xi fg - \xi g f) + \xi\{f, g\} = -\xi\{f, g\} + \partial \xi (\xi f\Omega g(\xi f) + \Omega f(\xi g)).
\]

Replacing \( f \) by \( \{f, g\} \) in (1.4), we have

\[
-\xi\{f, g\} = \omega(\xi_{(f, g)}, \xi) - \partial (\Omega_{(f, g)}(\xi) + TZ/2)
\]

(1.5)

where \( \xi_{(f, g)} = \partial_{Y'} \).

First of all, we notice now that \( \omega(\xi_{(f, g)}, \xi) - \omega([\xi f, \xi g], \xi) \) is an exact derivative. The vector \( \xi_{(f, g)} - [\xi f, \xi g] \) has a form \( \partial_{Y'} \). An expression of the type \( UZ' - U'Z \) is an exact derivative for an arbitrary \( Z \) iff \( U = \text{const} \) and \( \partial_{U'} = 0 \). We arrive at the statement of the proposition. □

We actually did not use yet the exact form of terms with \( \partial \). They will be important in what follows. Therefore, we must collect all remaining terms. Notice that Proposition 1.1 implies that \( T = \partial X'Y - \partial Y'X \). These terms cancel with two other terms. The rest of terms are

\[
\partial (\xi g\Omega f(\xi f) + \xi f\Omega g(\xi f) - \xi f\Omega g(\xi f) - \xi g\Omega f(\xi f) - \Omega g([\xi f, \xi]) + \Omega f([\xi g, \xi])
\]

\[
+ (Y\partial Z'X - X\partial Z'Y)/2 = 0.
\]

(1.6)

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Proposition 1.2. The Poisson bracket \( \{f, g\} \) satisfies the Jacobi identity up to an exact derivative term:

\[
\{\{f, g\}, h\} + \text{(cyclic)} = \partial \Psi(f, g, h)
\]

where \( \Psi = \delta \Phi \) is a trilinear form, \( \Phi \) was defined earlier, and \( \delta \):

\[
(\delta \Phi)(f, g, h) = \xi_f \Phi(g, h) - \Phi(\{f, g\}, h) + \text{(cyclic)}.
\]

Proof. Let \( X = \delta f/\delta u \), \( Y = \delta g/\delta u \), \( Z = \delta h/\delta u \). We have (using Proposition 1.1)

\[
0 = (d\omega)(\xi_f, \xi_g, \xi_h) = \xi_f \{g, h\} - \omega(\{f, g\}, \xi_h) + \text{(cyclic)}.
\]

Transform this with the help of (1.5):

\[
0 = -2\omega(\xi_{(g,h)}, \xi_f) + \partial (\Omega_{(g,h)}(\xi_f) + (\partial Y Z - \partial Z Y) X) + \text{(cyclic)}.
\]

In the right-hand side one can perform any cyclic permutation of \( X, Y \) and \( Z \) in every term. The equation can be rewritten as

\[
2\{\{f, g\}, h\} + \text{(cyclic)} = \partial (\Omega_{(f,g)}(\xi_h) + (Y \partial Z - X \partial Z Y)/2) + \text{(cyclic)}.
\]

The last term can be eliminated with the help of (1.6):

\[
2\{\{f, g\}, h\} + \text{(cyclic)} = \partial (\Omega_{(f,g)}(\xi_h) + \xi_h \Omega_g(\xi_f) - \xi_g \Omega_f(\xi_h) + \Omega_{(f,g)}(\xi_h) - \xi_f \Omega_g(\xi_h) + \xi_g \Omega_f(\xi_h) + \Omega_g(\Omega_{(f,g)}(\xi_h) - \Omega_f(\Omega_{(f,g)}(\xi_h))) + \text{(cyclic)})
\]

\[
= 2\partial (\Omega_{(f,g)}(\xi_f) - \xi_f \Omega_g(\xi_h) + \Omega_{(f,g)}(\xi_h) - \Omega_g(\Omega_{(f,g)}(\xi_h))) + \text{(cyclic)}
\]

\[
= 2\partial (\xi_h \Phi(f, g) - \Phi(\{f, g\}, h) + \text{(cyclic)}) = 2\partial \delta \Phi(f, g, h). \quad \square
\]

Proposition 1.3. The 3-form \( \Psi \) satisfies the identity (a “higher Jacobi”)

\[
\sum_{i<j} (-1)^{i+j} \Psi(\{f_i, f_j\}, ..., \hat{f_i}, ..., \hat{f_j}, ...) = 0
\]

for arbitrary 4 functions \( f_1, f_2, f_3 \) and \( f_4 \).

Proof. The derivation \( \delta \) is a differential (i.e., \( \delta^2 = 0 \)) only in \( B \), since in \( A \) the Jacobi identity holds not precisely but up to a term. Therefore, the fact that \( \Psi = \delta \Phi \) does not imply \( \delta \Psi = 0 \). (The proclaimed statement involves a part of the expression for \( \delta \Psi \).) Nevertheless, we can keep \( \delta^2 \) under control.

It is not very difficult to compute that

\[
(\delta^2 \Phi)(f_1, f_2, f_3, f_4) = \sum_{i<j<k} (-1)^l \Phi(\{f_i, f_j\}, f_k) + \text{(cyclic)}, \quad \text{where } l \neq i, j, k,
\]

the only properties of the Poisson bracket used here are bilinearity, skew symmetry and Proposition 1.1. The first argument of \( \Phi \) is not zero, it is \( \partial \Psi(f_i, f_j, f_k) \). Thus,

\[
\delta \Psi(f_1, f_2, f_3, f_4) = \sum_{i<j<k} (-1)^l \Phi(\partial \Psi(f_i, f_j, f_k), f_l) = \sum_{i<j<k} (-1)^{l-1} \Phi(f_i, \partial \Psi(f_i, f_j, f_k)).
\]
Using lemma 1.3, one gets

\[ \delta \Psi(f_1, f_2, f_3, f_4) = \sum_{i<j<k} (-1)^{i-1} \xi_{f_l} \Psi(f_i, f_j, f_k) = \sum_l (-1)^{l-1} \xi_{f_l} \Psi(..., \hat{f}_l, ...). \]

On the other hand, by definition of the derivation $\delta$,

\[ \delta \Psi(f_1, f_2, f_3, f_4) = \sum_l (-1)^{l-1} \xi_{f_l} \Psi(..., \hat{f}_l, ...) + \sum_{i<j} (-1)^{i+j} \Psi(\{f_i, f_j\}, ..., \hat{f}_i, ..., \hat{f}_j, ...) \]

Equating two expressions, we obtain the required statement. \(\square\)

2. Matrix example.

1) Elements $u_{jk}$ of an $n \times n$ matrix $U$ are taken as generators of a differential algebra $\mathcal{A}$.

2) Let $\mathcal{B} = \mathcal{A}/\partial \mathcal{A}$.

3) If $a$ is a matrix with entries belonging to $\mathcal{A}$ then a derivation in $\mathcal{A}$ (a “vector field”) can be defined:

\[ \partial_a = \sum_{k,ij} a_{ij}^{(k)} \frac{\partial}{\partial u_{ij}^{(k)}} = \text{tr} a^{(k)} \frac{\partial}{\partial U^{(k)}}, \quad \text{where} \quad \left( \frac{\partial}{\partial U^{(k)}} \right)_{ij} = \frac{\partial}{\partial u_{ij}^{(k)}}. \]

4) To any matrix $X$ (hereafter we always assume that all scalars, elements of matrix, etc always belong to $\mathcal{A}$) another matrix is assigned, $H(X) = X' + [U, X]$ and also a vector field $\partial_{H(X)}$.

   It is easy to check that

\[ [\partial_{H(X)}, \partial_{H(Y)}] = \partial_{H([X,Y]+\partial_{H(X)}Y-\partial_{H(Y)}X)}. \]

This, in particular, means that the vector fields of a special type $\partial_{H(X)}$ make a subalgebra.

5) Define a skew symmetric bilinear form on the above subalgebra:

\[ \omega(\partial_{H(X)}, \partial_{H(Y)}) = \text{tr} (H(X)Y - H(Y)X)/2. \]

**Lemma 2.1.** The relation

\[ \delta \omega(\partial_{H(X)}, \partial_{H(Y)}, \partial_{H(Z)}) = 0 \]

holds, i.e., the form is closed exactly, in $\mathcal{A}$.

The lemma can be verified by a direct calculation.

6) The Poisson bracket is

\[ \{f, g\} = \omega(\xi_f, \xi_g), \quad \text{where} \quad \xi_f = \partial_{H(X)}, \xi_g = \partial_{H(Y)}, \quad \text{and} \quad X = \delta f/\delta U, \quad Y = \delta g/\delta U. \]

Here, $\delta f/\delta U$ is a matrix with the entries: $(\delta f/\delta U)_{ij} = \delta f_j/\delta u_{ji}$.

7) The analogue to the formula (1.3) is now

\[ \partial_{H(X)}g = \text{tr} H(X)Y + \partial \Omega_g(\partial_{H(X)}) \]

where $\Omega_g$ is a 1-form, $X$ is an arbitrary matrix, $Y = \delta g/\delta U$. An obvious corollary is:
Lemma 2.2. The formula
\[ \xi_f g - \xi_g f = 2\{f, g\} + \partial \Phi(f, g) \] where \( \Phi(f, g) = \Omega_g(\xi_f) - \Omega_f(\xi_g) \)
holds.

Lemma 2.3.
\[ \xi_f g = \Phi(f, g') \]
The lemma has absolutely the same proof as Lemma 1.3 for the first example.

Proposition 2.1.
\[ \xi_{\{f, g\}} = [\xi_f, \xi_g] \]

Proof. For any vector field \( \xi = \partial H(Z) \) we have
\[ 0 = (\delta \omega)(\xi_f, \xi_g, \xi) = \xi_f \omega(\xi_g, \xi) - \xi_g \omega(\xi_f, \xi) + \xi \omega(\xi_f, \xi_g) \]
\[ -\omega([\xi_f, \xi_g], \xi) + \omega([\xi_f, \xi], \xi_g) - \omega([\xi_g, \xi], \xi_f), \]
We have
\[ \omega(\xi_f, \xi) = \text{tr} \left( H(X)Z - XH(Z) \right)/2 = \text{tr} \left( -H(Z)X + (ZX)' \right)/2 \]
\[ = -\xi f + \partial(\Omega_f(\xi) + \text{tr}(ZX)/2), \]
\( \omega(\xi_f, \xi) = -\xi g + \partial(\Omega_g(\xi) + \text{tr}(ZY)/2), \)
\[ \omega([\xi_f, \xi], \xi_g) = -\omega(\xi_g, [\xi_f, \xi]) = [\xi_f, \xi]g - \partial(\Omega_g([\xi_f, \xi]) + \text{tr} ([X, Z] + \xi_f Z - \xi X)Y/2), \]
\[ \omega([\xi_g, \xi], \xi_f) = [\xi_g, \xi]f - \partial(\Omega_f([\xi_g, \xi]) + \text{tr} ([Y, Z] + \xi g Z - \xi Y)X/2). \]
Now, we have
\[ 0 = -\xi(\xi_f g - \xi_g f) + \xi \omega(\xi_f, \xi_g) - \omega([\xi_f, \xi_g], \xi) \]
\[ + \partial(\xi_f \Omega_g(\xi) - \xi_g \Omega_f(\xi) - \Omega_g([\xi_f, \xi]) + \Omega_f([\xi_g, \xi])) \]
\[ + \partial \text{tr}(\xi_f(\xi_f Z) - \xi_g(\xi_f Z) - Y \xi_f Z + X \xi_g Z + Y \xi X - X \xi Y \]
\[ - Y [X, Z] + X [Y, Z])/2. \]
Using lemma 2.2, we have
\[ 0 = -\xi(2\{f, g\} + \partial \Phi(f, g)) + \xi\{f, g\} - \omega([\xi_f, \xi_g], \xi) \]
\[ + \partial (\xi_f \Omega_g(\xi) - \xi_g \Omega_f(\xi) - \Omega_g([\xi_f, \xi]) + \Omega_f([\xi_g, \xi])) \]
\[ + \partial \text{tr}(Z(\xi_f Y - \xi_g X) + Y \xi X - X \xi Y + 2Z[X, Y])/2. \]
Replacing \( f \) by \( \{f, g\} \) in (2.4), we get
\[ -\xi \{f, g\} = \omega(\xi_{\{f, g\}}, \xi) - \partial(\Omega_{\{f, g\}}(\xi) + \text{tr}(ZT)/2) \]
(2.5)
where \( \xi_{\{f, g\}} = \partial H(T) \). Now,
\[ \omega(\xi_{\{f, g\}} - [\xi_f, \xi_g], \xi) = \partial (-\xi \Phi(f, g) - \Omega_{\{f, g\}}(\xi) - \text{tr} ZT/2) \]
\[ + \partial (\xi_f \Omega_g(\xi) - \xi_g \Omega_f(\xi) - \Omega_g([\xi_f, \xi]) + \Omega_f([\xi_g, \xi])) \]
\[ + \partial \text{tr} \left( Z (\xi_f Y - \xi_g X) + Y \xi X - X \xi Y + 2Z[X,Y] \right)/2. \]

the fact that \( \omega(\xi_{\{f,g\}}, \xi) \) is an exact derivative for all \( \xi \)'s implies that \( \xi_{\{f,g\}} - [\xi_f, \xi_g] = 0 \), as required. In particular, \( T = [X, Y] + \xi_f Y - \xi_g X \).

At the same time, we obtained an identity

\[ 0 = \partial(-\xi \Phi(f, g) - \Omega_{f,g} \xi + \xi_f \Omega_g \xi - \xi_g \Omega_f \xi - \Omega_g(\xi_f, \xi) + \Omega_f(\xi_g, \xi)) + (Y \xi X - X \xi Y + Z[X,Y])/2. \]  

(2.6)

**Proposition 2.2.** The Poisson bracket \( \{f, g\} \) satisfies the Jacobi identity up to an exact derivative term:

\[ \{\{f, g\}, h\} + (\text{cyclic}) = \partial \Psi(f, g, h) \]

where \( \Psi = \delta \Phi \) is a trilinear form, with earlier defined \( \Phi \) and

\[ (\delta \Phi)(f, g, h) = \xi_f \Phi(g, h) - \Phi(\{f, g\}, h) + (\text{cyclic}). \]

**Proof.** Let \( X = \delta f/\delta U, Y = \delta g/\delta U, Z = \delta h/\delta U \). We have

\[ 0 = (d\omega)(\xi_f, \xi_g, \xi_h) = \xi_f \{g, h\} - \omega(\xi_{\{f,g\}}, \xi_h) + (\text{cyclic}). \]

With (2.5), this yields

\[ 0 = -2\omega(\xi_{\{g,h\}}, \xi_f) + \partial (\Omega_{\{g,h\}}(\xi_f) + ([Y, Z] + \xi_g Z - \xi_h Y) X) + (\text{cyclic}). \]

Then,

\[ 2\{\{f, g\}, h\} + (\text{cyclic}) = \partial (\Omega_{\{f,g\}}(\xi_h) + ([X, Y] + Y \xi_h X - X \xi_h Y)/2) + (\text{cyclic}). \]

Eliminating the last term with the help of (2.6), we have the same equation as for the first example; the end of the proof coincides with that of Proposition 1.2.

**Proposition 1.3** holds in the matrix case, too, along with its proof:

**Proposition 2.3.** The 3-form \( \Psi \) satisfies the identity (a “higher Jacobi”)

\[ \sum_{i<j}(-1)^{i+j} \Psi(\{f_i, f_j\}, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots) = 0 \]

for arbitrary 4 functions \( f_1, f_2, f_3 \) and \( f_4 \).

**References.**

[1] Soloviev, V.O., Boundary values as Hamilton variables, I, hep-th 9305133, 1993; II, q-alg 9501017, 1995

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