Space-time noncommutativity and (1+1) Higgs Model

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Abstract

We compare the classical scattering of kinks in (1+1) Higgs model with its analogous noncommutative counterpart. While at a classical level we are able to solve the scattering at all orders finding a smooth solution, at a noncommutative level we present only perturbative results, suggesting the existence of a smooth solution also in this case.
1 Introduction

The concept of time in physics has been subject of endless discussions. For Einstein time is equivalent to a spatial coordinate, its role reduced to a mere parameter and the distinction between past, present and future only an illusion of the macroscopic level.

More recently time has emerged as a central theme in the problem of irreversibility in both living and inanimate systems, as in Prigogine’s approach to nonequilibrium thermodynamics. For Prigogine introducing in physics the concept of ”arrow of time” [1]-[4], irreversibility can have a constructive, positive role creating order out of chaos. In this framework the present-day laws of physics emerged from an initial chaotic universe, where there was no law of physics at all, and the order that we measure now is just an evolution under nonequilibrium conditions due to the irreversibility of time.

However the physical community has always avoided discussing the possibility of a microscopic irreversibility in quantum field theory, conserving the spirit of Einstein. All the major physical theories, i.e. quantum mechanics and general relativity, are unitary by construction, in the sense that the quantum S-matrix relating the ”in” and ”out” states must be unitary and conserve in the scattering process the basis of the Hilbert space. Till now no internal contradiction to this scheme has been found in ordinary gauge theories and the experimental tests are coherent with it.

Problems arise when we try to unify quantum mechanics with general relativity, i.e. when we perform ideal tests on the structure of space-time. Hawking and Bekenstein in 1975 predicted that black hole physics, taking into account quantum mechanics, is strictly connected with thermodynamics, and that black holes can slowly radiate. This prediction created an information loss problem, in the sense that some information of the quantum system is lost during the scattering process with a black hole. Starting with a quantum system in a pure state, the black hole is able to transform it into a mixed state, described quantum-mechanically by a density matrix rather than a wave function. The transformation of pure states into mixed states can be taken as a paradigm of microscopic irreversibility, being a non-unitary process.

Quite recently an indetermination principle for space-time coordinates has been suggested by a gedanken experiment [5]-[7], based on classical black holes and quantum mechanics, pointing out to the study of quantum field theory on noncommutative spacetimes [8]-[16]. An isomorphism allows translating the noncommutativity of the coordinates to a noncommutative star product of the fields on a commutative space-time. This new type of nonlocality, especially in time, can be very dangerous to the maintenance of the basic principles of standard quantum field theory. In fact quantum field theories with space-time
noncommutativity have no straightforward Hamiltonian quantization, which is usually the warranty for unitarity and causality. Such theories are defined only through the Lagrangian and via perturbation theory. To check unitarity in a Feynman diagram it is enough to verify the cutting rules that relate the amplitude of the Feynman diagram to its imaginary part. It has been found that space-time noncommutativity indeed breaks the standard cutting rules and therefore unitarity is lost \cite{17}-\cite{18}. This fundamental result has been confirmed by an analysis based on string theory propagating in an electric background field. It is possible to restore unitarity in the quantum field theory nonlocal in time by embedding it into string theory, which is unitary, but the additional states, required for recovering unitarity, cannot be decoupled in the field theory limit. So it seems that there is no hope to maintain the structure of standard quantum field theory in the case of space-time noncommutativity \cite{17}-\cite{20}. We not able to judge other approaches, based on axiomatic field theory, which promise that restoring unitarity is possible modifying the definition of quantum field theory on noncommutative space-times \cite{7}.

We instead want to push the idea that there is no mystery behind these results, since the main physical motivation for introducing noncommutative coordinates comes from black holes and the Heisenberg indetermination principle, and black hole physics is one of the best examples in physics where unitarity is lost and the scattering process modifies the nature of quantum states, from pure states to mixed ones. Therefore our interest in space-time noncommutativity is motivated by interpreting it as a beautiful model of microscopic irreversibility, which at least can be controlled with a Lagrangian, while the black hole information paradox is hard to study.

As a preliminary step, we have studied the classical scattering of wave packets in a simple system with space-time noncommutativity where perturbative resummations are possible, since we believe that only nonperturbative results can be physically meaningful. The system we have chosen is the Higgs model in $(1 + 1)$ dimensions, where classical solitons are the so-called kink solutions. Since noncommutativity switches on only for an interacting solution and the one-kink solution is trivial, we are forced to study the scattering of two kinks, one left-mover according to the equation $u = x - t = 0$, and the other right-mover ($v = x + t = 0$).

We have simplified our ansatz of solution by taking two sharp waves (completely localized in space) which can interact only after $t > 0$, when an extra interacting solution is needed to solve the equations of motion. The simplicity of our ansatz allows us to reduce the 2D equations of motion to a single nonlinear differential equation depending on a single variable, which we are able to solve exactly.

In the second part of the paper we attempt to generalize our interacting classical solution
to the case of noncommutative kinks. Although we haven’t been able to find the complete solution we report our partial results postponing the exact solution to a future research. In particular, the first-order correction in the coupling constant is interesting because it modifies the profiles of the two shock waves, which lose their sharpness and get a dimension of order $\sqrt{\theta}$. Moreover the scattering doesn’t develop anymore at $t > 0$, but it is anticipated due to the width of the wave packets. Noncommutativity is able to modify the asymptotic states permanently, also when the two wave packets are far apart. This result suggests that the rules of quantum mechanics must be modified in presence of an infinite range force (as noncommutativity appears to be). However it would be more interesting to interpret a complete solution, which we leave to a future research.

2 Higgs model for real scalar field

Our aim was to find an exactly solvable scattering problem to compare with the noncommutative case, possibly at a nonperturbative level. Our choice has been to consider the classical (1 + 1) Higgs model with a real scalar field defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4$$  \hspace{1cm} (2.1)

The Higgs mechanics is based on making perturbation theory around the nontrivial minimum of the potential. To find it we need to introduce the corresponding Hamiltonian:

$$\pi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} = \partial_0 \phi$$

$$\mathcal{H} = \frac{1}{2} (\pi^2 + |\partial_i \phi|^2) + V(\phi)$$

$$V(\phi) = \frac{1}{4} \lambda \phi^4 - \frac{1}{2} m^2 \phi^2$$  \hspace{1cm} (2.2)

The minimum of the potential (2.2) is reached when

$$\frac{\partial V(\phi)}{\partial \phi} = 0 \rightarrow \phi^2 = \frac{m^2}{\lambda}$$  \hspace{1cm} (2.3)

Let us choose for simplicity the coupling constants such that $m = \sqrt{\lambda}$ and $\phi = \pm 1$ are the two nontrivial minima.

The corresponding equations of motion
\[ \Box \phi = \lambda \phi (1 - \phi^2) \]  \hspace{1cm} (2.4)

can be simplified by defining the light-cone variables:

\[ u = \frac{x - t}{2} \quad v = \frac{x + t}{2} \]
\[ \Box = -\partial_u \partial_v \]
\[ \partial_u \partial_v \phi = -\lambda \phi (1 - \phi^2) \]  \hspace{1cm} (2.5)

Analyzing such equation one notice that \( \phi = 0, \pm 1 \) are possible solutions. Let us generalize them by introducing as an ansatz a sum of step functions which interpolate between the three values 0, ±1:

\[ \phi^{(0)} = -1 + p(\Theta(u) - \Theta(v)) \quad p = 1, 2 \]  \hspace{1cm} (2.6)

Due to the dependence from single light-cone variables, \( \phi^{(0)} \) is certainly a zero of the first member of the equation \( (2.5) \), however the second member is null only if \( t < 0 \). For \( t > 0 \) the ansatz \( \phi^{(0)} \) is no longer valid and must be generalized to

\[ \phi = -1 + p(\Theta(u) - \Theta(v)) + f(uv)\Theta(v)\Theta(-u) \]  \hspace{1cm} (2.7)

where the unknown function \( f \) is dependent on the single variable \( uv \). This interacting solution must evolve such that asymptotically, at infinite time and finite space, i.e. for \( uv \to -\infty \), the function reaches a constant value. The particular value \( f(-\infty) \) is determined by solving the equations of motion \( (2.5) \).

The solution with \( p = 1 \) has as ”in state” \( \phi_{in} = 0 \) (for \( t \to -\infty \)) , while the ”out state” (for \( t \to +\infty \)) has to be determined by solving the equations of motion \( (2.5) \), and it is parameterized as \( f_{p=1}(-\infty) - 2 \).

The solution with \( p = 2 \) has as ”in state” \( \phi_{in} = 1 \) (for \( t \to -\infty \)) and the ”out state” \( f_{p=2}(-\infty) - 3 \).

In synthesis

\[ p = 1 \quad \phi_{in} = 0 \quad \to \quad \phi_{out} = f_{p=1}(-\infty) - 2 \]

*To avoid confusion we define the step functions as \( \Theta(u) = 1 \) if \( u > 0 \), or 0 otherwise, \( \Theta(v) = 1 \) if \( v > 0 \), or 0 otherwise, and \( \Theta(-u) = 1 - \Theta(u) \)
\[ p = 2 \quad \phi_{in} = 1 \quad \rightarrow \quad \phi_{out} = f_{p=2}(-\infty) - 3 \quad (2.8) \]

By introducing the complete ansatz \((2.7)\) into the equations of motion \((2.5)\) one is able to show that the ansatz closes if all the following conditions are met:

\[ f(0) = 0 \quad f'(uv) \quad \text{regular around} \quad uv \sim 0 \]
\[ (uv)f''(uv) + f'(uv) = \lambda(f(uv) - p)(f(uv) - p - 1)(f(uv) - p - 2) \]
\[ p(p - 1)(p - 2) = 0 \quad (2.9) \]

The ansatz closes for the values \(p = 0, 1, 2; \) since the value \(p = 0\) is trivial, in the following we will discuss only the solution to the differential equation \((2.9)\) for the values \(p = 1, 2.\)

By introducing the variable

\[ x = -6\lambda uv \quad (2.10) \]

we are led to discuss the solution to the following equation:

\[ xf''(x) + f'(x) + (f(x) - p)(f(x) - p - 1)(f(x) - p - 2)/6 = 0 \quad (2.11) \]

in the range \(0 < x < +\infty.\)

We will show that the nonlinear equation \((2.11)\) is consistent with the following boundary values:

\[ f_{p=1}(x = +\infty) = 1 \]
\[ f_{p=2}(x = +\infty) = 4 \quad (2.12) \]

leading to classify the possible scenarios:

\[ p = 1 \quad \phi_{in} = 0 \quad \rightarrow \quad \phi_{out} = -1 \]
\[ p = 2 \quad \phi_{in} = 1 \quad \rightarrow \quad \phi_{out} = 1 \quad (2.13) \]

In the first case, the scattering of kinks allows to describe the decay from the unstable state \(\phi_{in} = 0\) to the stable minimum of the potential \(\phi_{out} = -1;\) in the second case, the scattering of kinks doesn’t alter the stability of the minimum \(\phi_{in} = \phi_{out} = 1.\)
Let us start to solve (2.11) with $p = 1$. In this case we define

\[ f(x) = 1 - g(x) \quad g(0) = 1 \]
\[ xg''(x) + g'(x) + g(x)(g(x) + 1)(g(x) + 2)/6 = 0 \] (2.14)

Being a second order differential equation, it seems that the only boundary value $g(0) = 1$ is not enough to determine completely the solution, but it turns out that another physical requirement is necessary to obtain a smooth solution, i.e. the absence of logarithmic terms around $x = 0$. The request is sufficient to determine the asymptotic value $g(+\infty) = 0$, which can be achieved or by a direct numerical computation with Mathematica, or with a careful inspection of the differential equation. We have done both checks and they completely agree.

Firstly, let us suppose that asymptotically $g(x) \to 0$ for $x \to +\infty$, then the nonlinearity can be avoided and the nonlinear problem (2.11) can be linearized to

\[ xg''(x) + g'(x) + g(x)/3 = 0 \]
\[ g(0) = 1 \quad g(x) \quad \text{regular around } x = 0 \] (2.15)

which can be solved by the Bessel function:

\[ g(x) = J_0\left(2\sqrt{\frac{x}{3}}\right) \] (2.16)

Since it is well known the asymptotic behaviour of the Bessel function

\[ \lim_{x \to +\infty} J_0(x) = \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4}) \] (2.17)

we conclude that the asymptotic value $g(+\infty) = 0$ is consistent with the hypothesis, and we have learned that the solution of (2.11) has asymptotic damped oscillations behaving as $x^{-\frac{3}{2}}$.

The difference between linear and nonlinear solutions is concentrated around $x = 0$; however supposing that the value of $g(x)$ is always greater than $-1$, at the minima where $g'(x) = 0$, the value of the second derivative $g''(x)$ is always opposite to the value of the function $g(x)$, confirming the oscillating character of the solution around the value $g = 0$, also at the nonlinear level. At this point, we are ready to compare these considerations with a direct numerical computation with Mathematica.
We chose the following method. Firstly we developed in power series the solution around zero and put the recurrence relations for the coefficients of the series into Mathematica. The power series truncated at, let’s say, 100 steps has a certain convergence radius, inside it we can trust the approximated values of the function $g(x)$. Then we used these approximated values to build the recurrence relations around another fixed point inside the convergence radius of the first power series, and we iterated the procedure. In this way we have found the locus of the first 6 minima (these results should be taken as indicative values due to the imprecision of the extrapolated values)

\[
\begin{align*}
x &= 11 & g(x) &= -0.62 \\
x &= 90 & g(x) &= -0.49 \\
x &= 244 & g(x) &= -0.36 \\
x &= 462 & g(x) &= -0.32 \\
x &= 745 & g(x) &= -0.28 \\
x &= 1089 & g(x) &= -0.25
\end{align*}
\]

and of the first 5 local maxima:

\[
\begin{align*}
x &= 41 & g(x) &= 0.37 \\
x &= 161 & g(x) &= 0.29 \\
x &= 346 & g(x) &= 0.25 \\
x &= 598 & g(x) &= 0.23 \\
x &= 911 & g(x) &= 0.21
\end{align*}
\]

In Fig.1 we have plotted with Mathematica two curves representing the approximated value of the function $g(x)$ calculated from the power series truncated at an even and odd number respectively (for example 100 and 99 steps):

By analyzing these results we find agreement with all the preliminary discussion, since the minimum value of $g(x)$ ($-0.62$) is greater than $-1$, from which damped oscillations follow until reaching the linear behaviour (2.17). Thus by combining numerical and analytic methods we have full control of the nonlinear equation (2.14).

Let us now discuss the solution to the equations of motion for $p = 2$, in which case the final state of the Higgs field $\phi_{out} = f_{p=2}(\rightarrow -\infty) - 3$. By defining $f(x) = 2(1 - g(x))$ and rescaling $x \rightarrow 2x$, we obtain the following differential equation for $g(x)$:
At first sight this equation looks very similar to the one discussed before, but in reality its solution is quite different. Firstly we notice that it is not clear what is the final point of oscillation. There are at least two possible choices:

i) \( g(x) \) oscillates around the value \( g = 0 \), then to be self-consistent, at the stationary points, the value of the second derivative must be opposite to the value of the function and this happens if the function is confined over the minimum value \(-1/2\).

ii) \( g(x) \) oscillates around the value \( g = -1 \); this is possible if the local maxima and minima are confined under the maximum value \(-1/2\).

Only with a numerical computation we have been able to discern the right value. By using the same method illustrated before we have found the locus of the first 4 minima

\[
x = 41 \quad g(x) = -1.14
\]
\[
x = 212 \quad g(x) = -1.10
\]
\[
x = 506 \quad g(x) = -1.08
\]
\[
x = 922 \quad g(x) = -1.07
\]

and of the first 4 local maxima:

\[
x = 108 \quad g(x) = -0.85
\]
\[
x = 342 \quad g(x) = -0.89
\]
\[
x = 697 \quad g(x) = -0.91
\]
We conclude that the possibility ii) is realized, and therefore \( g(x) \) has as asymptotic value -1, with oscillations that are damped by a factor \( x^{-1/4} \), the typical factor of the Bessel function (2.17). This completes the demonstration of the boundary values depicted in (2.12).

### 3 Noncommutative case

We are going to deform the Higgs model with a noncommutative relation between the coordinates, for example

\[
[u, v] = i\theta \leftrightarrow [x, t] = 2i\theta
\]

This can be accomplished by deforming the ordinary product of fields into an associative star product as follows:

\[
\phi_1(u, v) \ast \phi_2(u, v) = \lim_{u_1 \to u_2} \lim_{v_1 \to v_2} e^{i\theta(\partial_{u_1} \partial_{v_2} - \partial_{u_2} \partial_{v_1})} \phi_1(u_1, v_1) \phi_2(u_2, v_2)
\]

The lagrangian of the Higgs field with a noncommutative star product is defined as:

\[
\mathcal{L} = \int d^2x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi \ast \phi \ast \phi \ast \phi \right)
\]

and the equations of motion (with \( m^2/\lambda = 1 \)) are nonlocal in time:

\[
\partial_\mu \partial^\mu \phi = \lambda \phi \ast (\phi \ast \phi - 1)
\]
It is not clear at this point how to complete the ansatz \( (2.0) \)

\[
\phi^{(0)} = -1 + p(\Theta(u) - \Theta(v)) \quad (3.5)
\]

in order to reduce the equations of motion \( (3.4) \) into a self-consistent set of equations. We prefer to setup a perturbative method. In this framework the natural perturbative parameter is the coupling constant \( \lambda \), and we develop the solution \( \phi \) as a sum

\[
\phi = \phi^{(0)} + \lambda \phi^{(1)} + O(\lambda^2) \quad (3.6)
\]

Starting from the ansatz \( \phi^{(0)} \) \( (3.5) \), we calculate the source \( (3.4) \) at the first perturbative order in \( \lambda \). In the case of the ordinary Higgs model we would obtain

\[
\partial_u \partial_v \phi^{(1)} = \lambda \phi^{(0)}(\phi^{(0)2} - 1) = \lambda p(p - 1)(p - 2)(\Theta(u) - \Theta(v)) - 6\lambda p^2 \Theta(-u) \Theta(v) \quad (3.7)
\]

For the special cases \( p = 1, 2 \) the first term in the second member cancels out and the main contribution comes from the term \( \Theta(-u) \Theta(v) \) which is different from zero only for \( t > 0 \), and in the interval \( -t < x < t \).

The interacting field \( \phi^{(1)} \) is then proportional to \( x^2 - t^2 \)

\[
\phi^{(1)} = -6\lambda p^2 uv \Theta(-u) \Theta(v) \quad (3.8)
\]

In the noncommutative case we limit ourself to a calculation of \( \phi^{(1)} \) always starting from the ansatz \( \phi^{(0)} \) but replacing the ordinary product with the star product

\[
\partial_u \partial_v \phi^{(1)}_{NC} = \lambda \phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1) \quad (3.9)
\]

However we encounter the first difficulty, i.e. the star product involving step functions seems to be ill-defined, being a sum of infinite distributions. We will do the following trick, i.e. solving the star product using the Fourier transform.

As an exercise, let us calculate the ordinary product \( \phi^{(0)2} \) using the Fourier transform of \( \phi^{(0)} \). It is more convenient to perform the Fourier transform of \( \phi^{(0)}(u, v) \) with respect to the single variable \( u \):
\[ \tilde{\phi}(k, v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} du e^{-iku} \phi^{(0)}(u, v) = \sqrt{2\pi} \left[ -(1 + p\Theta(v))\delta(k) + \lim_{\epsilon \to 0} \frac{p}{2\pi i(k - i\epsilon)} \right] \] (3.10)

The ordinary product \( \phi^{(0)} \) is mapped, under the Fourier transform, to a convolution product:

\[ \phi^{(0)}(u, v) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq \tilde{\phi}(q, v) \tilde{\phi}(k - q, v) = \sqrt{2\pi} \left[ (1 + p(p + 2)\Theta(v))\delta(k) + \lim_{\epsilon \to 0} \frac{p(p - 2) - 2p^2\Theta(v)}{2\pi i(k - i\epsilon)} \right] \] (3.11)

which is exactly the Fourier transform of

\[ \phi^{(0)}(u, v) = 1 + p(p + 2)\Theta(v) + (p(p - 2) - 2p^2\Theta(v))\Theta(u) \] (3.12)

The tool that we need during the calculation of the noncommutative case is how the star product of two functions of \( u \) and \( v \) is translated into a convolution product of their Fourier transforms. It is not difficult to show that \( \phi_1 \ast \phi_2 \), defined the Fourier transforms

\[ \begin{align*}
\phi_1(u_1, v_1) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq_1 e^{iq_1u_1} \tilde{\phi}_1(q_1, v_1) \\
\phi_2(u_2, v_2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq_2 e^{iq_2u_2} \tilde{\phi}_2(q_2, v_2)
\end{align*} \] (3.13)

is mapped to the following NC convolution product

\[ (\tilde{\phi} \ast \tilde{\phi})(k, v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq \tilde{\phi}_1 \left( q, v + \frac{\theta}{2}(k - q) \right) \tilde{\phi}_2 \left( k - q, v - \frac{\theta}{2}q \right) \] (3.14)

We notice from this definition that if the two functions are equal ( \( \phi_1 = \phi_2 \) ), then, due to the symmetry of the convolution product \( q \to k - q \), the dependence on the \( \theta \) variable becomes even, as it happens for the star product of the same function.

Let us calculate the star product \( \phi^{(0)} \ast \phi^{(0)} \) using the rule (3.14)
\[
(\phi^{(0)}*\phi^{(0)})(k, v) = \sqrt{2\pi} \left[ (1 + p(p + 2)\Theta(v))\delta(k) + \lim_{\epsilon \to 0} \frac{p(p - 2) - p^2(\Theta(v + \frac{p}{2}k) + \Theta(v - \frac{p}{2}k))}{2\pi i(k - i\epsilon)} \right] \tag{3.15}
\]

Therefore the pure noncommutative contribution is

\[
(\phi^{(0)}*\phi^{(0)})_{NC}(k, v) = -\frac{p^2}{\sqrt{2\pi i}} \lim_{\epsilon \to 0} \left( \frac{\Theta(v + \frac{\theta}{2}k) + \Theta(v - \frac{\theta}{2}k) - 2\Theta(v)}{k - i\epsilon} \right) \tag{3.16}
\]

In the following we will assume for simplicity that \( uv > 0 \) and \( \theta > 0 \), otherwise some signs function should to be added, with the result of making the notations heavier.

Let us perform the anti-Fourier transform of (3.15) to have a better idea of the noncommutative source:

\[
(\phi^{(0)}*\phi^{(0)})_{NC} = \int_{0}^{\theta} d\theta \frac{\partial}{\partial \theta} (\phi^{(0)}*\phi^{(0)})_{NC} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\theta} d\theta \int_{-\infty}^{+\infty} dk \ e^{iku} \frac{\partial}{\partial \theta} (\phi^{(0)}*\phi^{(0)})_{NC} =
\]

\[
= -\frac{p^2}{\pi} \int_{+\infty}^{2uv} dx \ \frac{\sin x}{x} = -\frac{p^2}{\pi} \left[ Si \left( \frac{2uv}{\theta} \right) - Si(+\infty) \right] \tag{3.17}
\]

This function is a special function, known as the Sine Integral, defined as

\[
Si(z) = \int_{0}^{z} dx \ \frac{\sin x}{x} \quad Si(+\infty) = \frac{\pi}{2} \tag{3.18}
\]

Therefore we conclude that the noncommutative part of the source has support only in the small region \( 0 < uv \lesssim \theta \) (otherwise the contribution is negligible) around the sharp wave packets, and its role is to give a size of order \( \sqrt{\theta} \) to the noncommutative kinks.

The next step is to perform the complete calculation of the source (3.9). We will compare again the commutative case with the noncommutative one, to be able to extract the pure noncommutative part. In the commutative case we need to compute in Fourier transform the product \( \phi^{(0)}(\phi^{(0)} \phi^{(0)} - 1) \) where

\[
\tilde{\phi}^{(0)} = \sqrt{2\pi} \left[ -(1 + p(\Theta(v))\delta(k) + \lim_{\epsilon \to 0} \frac{p}{2\pi i(k - i\epsilon)} \right] \quad (\phi^{(0)} \phi^{(0)} - 1) = \sqrt{2\pi} \left[ p(p + 2)\Theta(v)\delta(k) + \lim_{\epsilon \to 0} \frac{p(p - 2) - 2p^2\Theta(v)}{2\pi i(k - i\epsilon)} \right] \tag{3.19}
\]
We obtain

\[
\phi^{(0)} (\widetilde{\phi^{(0)}} \phi^{(0)} - 1)(k, v) = \sqrt{2\pi} \left[ -p(p + 1)(p + 2)\Theta(v)\delta(k) + \lim_{\epsilon \to 0} \frac{p(p - 1)(p - 2) + 6p^2\Theta(v)}{2\pi i(k - i\epsilon)} \right]
\]

(3.20)

It is easy to verify the correctness of this result, remembering that

\[
\phi^{(0)}(\phi^{(0)} - 1) = -p(p + 1)(p + 2)\Theta(v) + p(p - 1)(p - 2)\Theta(u) + 6p^2\Theta(u)\Theta(v)
\]

(3.21)

Let us compute the noncommutative case:

\[
\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1)
\]

(3.22)

where

\[
(\phi^{(0)} * \phi^{(0)} - 1) = \sqrt{2\pi} \left[ p(p + 2)\Theta(v)\delta(k) + \frac{1}{2\pi i} \lim_{\epsilon \to 0} \frac{p(p - 2)}{k - i\epsilon} \right]
\]

\[
- \lim_{\epsilon \to 0} \frac{p^2(\Theta(v + \frac{\theta}{2}k) + \Theta(v - \frac{\theta}{2}k))}{2\pi i(k - i\epsilon)}
\]

(3.23)

After simple but tedious calculations we arrive at the following result

\[
\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1)(k, v) = \sqrt{2\pi} \left[ -p(p + 1)(p + 2)\Theta(v)\delta(k) + \lim_{\epsilon \to 0} \frac{p(p - 1)(p - 2)}{2\pi i(k - i\epsilon)} \right]
\]

\[
+ \lim_{\epsilon \to 0} \frac{p^3(\Theta(v + \frac{\theta}{2}k)\Theta(v - \frac{\theta}{2}k) - \Theta(v))}{2\pi i(k - i\epsilon)}
\]

\[
+ \lim_{\epsilon \to 0} \frac{3p^2(\Theta(v + \frac{\theta}{2}k) + \Theta(v - \frac{\theta}{2}k))}{2\pi i(k - i\epsilon)}
\]

\[
- \frac{p^3}{2(2\pi i)^2} \lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} dq \frac{\Theta(v + \theta q) + \Theta(v - \theta q) - 2\Theta(v)}{(q + k - i\epsilon)(\frac{k}{2} - q - i\epsilon)}
\]

(3.24)

Let us extract the pure noncommutative part which is made of three parts:

\[
\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1)_{NC}(k, v, \theta) = I_1(k, v, \theta) + I_2(k, v, \theta) + I_3(k, v, \theta)
\]

(3.25)
where

\[
I_1(k, v, \theta) = \lim_{\epsilon \to 0} \frac{3p^2(\Theta(v + \frac{\theta}{2}k) + \Theta(v - \frac{\theta}{2}k) - 2\Theta(v))}{\sqrt{2\pi i(k - i\epsilon)}}
\]

\[
I_2(k, v, \theta) = \lim_{\epsilon \to 0} \frac{p^3(\Theta(v + \frac{\theta}{2}k)\Theta(v - \frac{\theta}{2}k) - \Theta(v))}{\sqrt{2\pi i(k - i\epsilon)}}
\]

\[
I_3(k, v, \theta) = -\frac{p^3\sqrt{2\pi}}{2(2\pi i)^2} \lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} dq \frac{\Theta(v + \theta q) + \Theta(v - \theta q) - 2\Theta(v)}{(q + \frac{k}{2} - i\epsilon)(\frac{k}{2} - q - i\epsilon)}
\]

(3.26)

Let us compute the anti-Fourier transforms of \( I_i \):

\[
I_i(u, v) = \frac{1}{\sqrt{2\pi}} \int_0^\theta d\theta \int_{-\infty}^{\infty} dk e^{iku} \frac{\partial}{\partial \theta} I_i(k, v, \theta)
\]

(3.27)

With the trick of integrating and deriving with respect to \( \theta \), all the integrals can be done, obtaining the following results:

\[
\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1)_{NC}(u, v, \theta) = \frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{\pi} \left[ S_i \left( \frac{2uv}{\theta} \right) - S_i(+\infty) \right]
\]

(3.28)

Let’s start integrating this source by taking the position

\[
\phi_i^{(1)}(u, v) = \frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{2\pi} \int_0^\theta d\theta \phi_i^{(1)} \left( \frac{2uv}{\theta} \right)
\]

(3.29)

where

\[
\partial_u \partial_v \phi_i^{(1)} \left( \frac{2uv}{\theta} \right) = -\frac{2\lambda}{\theta} \sin \left( \frac{2uv}{\theta} \right)
\]

(3.30)

Let us define \( x = 2uv/\theta \), then equation (3.30) is equivalent to

\[
\partial_x(\partial_x \phi_i^{(1)}(x)) = -\lambda \sin x \quad \rightarrow \quad \phi_i^{(1)}(x) = \lambda Ci \left( \frac{2uv}{\theta} \right)
\]

(3.31)

where we have introduced the Cosine Integral function defined as:

\[
Ci(z) = \int_{+\infty}^{z} dx \frac{\cos x}{x}
\]

(3.32)

However the steps functions give rise to additional contributions that we need to subtract
\[ \phi_{ii}^{(1)}(u, v) = -\lambda \left( \frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{2\pi} \right) \log \left( \frac{2uv}{\theta} \right) \]  

(3.33)

We have arrived at the final formula:

\[
\phi^{(1)} = \lambda \left( \frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{2\pi} \right) \left[ \theta \left( \cos \left( \frac{2uv}{\theta} \right) + Ci \left( \frac{2uv}{\theta} \right) - \log \left( \frac{2uv}{\theta} \right) \right) + 
+ 2uv \left( Si \left( \frac{2uv}{\theta} \right) - Si(+\infty) \right) \right]
\]  

(3.34)

Fortunately the divergent terms around the wave packets \( uv \sim 0 \) cancels out. Again this field has support primarily only in a small region around the wave packets \( 0 < uv < \theta \), apart from the oscillating cosine term and a logarithmic term which is less divergent than the corresponding classical term for \( uv \) large, as in eq. (3.3). The asymptotic states are modified permanently by noncommutativity also when the two wave packets are far apart. It is probably this characteristic which complicates the picture at the quantum level, since the \( S \)-matrix approach is useful only in those cases where the interaction switches off at large times.

## 4 Conclusions

In this preliminary investigation we have found an example of scattering which hopefully can be solved at a nonperturbative level. We have compared the classical scattering of kinks in \((1 + 1)\) Higgs model with the noncommutative case. At a classical level we have found a smooth solution without divergencies. This solution is based on introducing an ansatz, which reduces the equations of motions to a single nonlinear differential equation. We have been able to have full control of it by combining analytic and numerical methods. The solution of this equation is similar to a Bessel function of order zero, which contains damped oscillations towards a constant asymptotic value.

In the noncommutative case, the solution we have found is only perturbative; at this level there appear logarithmic terms which are divergent both near the wavefronts and at infinity. However the whole combination of terms cooperates to eliminate the divergencies near the wavefronts. Instead the divergence at infinity cannot be eliminated at a fixed order of perturbation theory, but only resuming all orders of perturbation theory.

The peculiar characteristic of noncommutativity is to dress the sharp wavefronts of the kinks giving them a size of order \( \sqrt{\theta} \) permanently, also when the wavefronts are far apart.
Noncommutativity modifies the asymptotic states in such a way that the asymptotic states for two-body cannot be factorized into a product of one-body states. These properties complicate the quantization of such theories, since the $S$-matrix approach is useful only for short range interactions switching off at large times.

This work leaves open many questions; firstly it would be nice to solve this model exactly at all orders and prove that there is a smooth solution describing the scattering of noncommutative kinks. Then investigating deeply the characteristics of the noncommutative scattering we can look for the right axioms on which to base the quantization of such theories. For example we remember that there are many efforts to introduce an arrow of time in quantum mechanics, by extending the ordinary Hilbert space into a Rigged Hilbert space $[21]-[24]$. We personally believe that only field theory, rather than one-particle quantum mechanics, is able to produce microscopic irreversibility, because it contains infinite degrees of freedom and thermodynamical behaviours are possible only for systems with large number of degrees of freedom.

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