Abstract

We find the best possible constant $C$ in the inequality

$$\|\varphi\|_{L^r} \leq C \|\varphi\|_{L^p}^{\frac{p}{r}} \|\varphi\|_{\text{BMO}}^{1-\frac{p}{r}}$$

for all possible values of parameters $p$ and $r$ such that $1 \leq p < r < +\infty$. We employ the Bellman function technique to solve this problem. The Bellman function of three variables corresponding to this problem has a rather complicated structure, however, we managed to provide the explicit formulas for this function. First, we solve the problem on an interval and then transfer our results to the circle and the line. We also obtain explicit estimates in multi-dimensional cases.

2010 MSC subject classification: 42B35, 60G45.

Keywords: bounded mean oscillation, Bellman function, interpolation.

1 Introduction

This is a continuation of the paper [9], where a partial case of the following problem was considered: for fixed parameters $p$ and $r$, $1 \leq p < r < +\infty$, find the sharp constant $C = C(p, r)$ such that the inequality

$$\|\varphi\|_{L^r} \leq C \|\varphi\|_{L^p}^{\frac{p}{r}} \|\varphi\|_{\text{BMO}}^{1-\frac{p}{r}}$$

(1.1)

is true for all functions $\varphi$ from BMO. It was proved in [9] that in the case $r \geq 2$ the best possible constant is

$$C(p, r) = \left( \frac{\Gamma(r + 1)}{\Gamma(p + 1)} \right)^{\frac{1}{r-1}}.$$  

(1.2)

We also mention that the same estimate was obtained in [6] for the partial case $1 \leq p \leq 2 \leq r < +\infty$. Here we find the best constant $C(p, r)$ for the remaining case $1 \leq p < r < 2$. The expression for this constant is implicit and too difficult to be presented at the very beginning of the paper. All the details can be found in Section 9. So we complete the work announced in Remark 1.4 of [9].

Lebesgue spaces and BMO here can be considered either on the line $\mathbb{R}$, or on the unit circle $\mathbb{T}$, or on an interval $I$. In order to speak about sharp constant, we need to specify the BMO-norm. We will consider $L^2$-based BMO-norm, namely,

$$\|\varphi\|_{\text{BMO}}^2 \overset{\text{def}}{=} \sup_J (|\varphi - \langle \varphi \rangle_J|^2)_J,$$

(1.3)

where the supremum is taken over all subintervals $J$. Here and in what follows we use the notation $\langle \psi \rangle_E$ to denote the average of a function $\psi$ over a set $E$ of positive finite measure, that is

$$\langle \psi \rangle_E \overset{\text{def}}{=} \frac{1}{|E|} \int_E \psi.$$
Since the relation (1.3) defines a seminorm and for constant function it is zero, we need to impose the additional restriction $\langle \varphi \rangle = 0$ for the cases of a circle and an interval to obtain (1.1).

We will not repeat here the motivation and the references concerning possible applications. All of that may be found in [9]. We stress that we are not so much interested in the inequality itself but rather in the corresponding Bellman function due to its importance for the future development of the Bellman function method.

We state now formally the main results of the paper.

**Theorem 1.1.** Let $I, I \subset \mathbb{R}$, be an interval. The inequality

$$
\| \varphi \|_{L^p(I)} \leq C(p, r) \| \varphi \|_{L^p(I)}^{\frac{r}{p}} \| \varphi \|_{\text{BMO}(I)}^{1-\frac{r}{p}}, \quad \varphi \in \text{BMO}, \quad \langle \varphi \rangle = 0,
$$

holds true for $1 \leq p \leq r < 2$ with the sharp constant $C(p, r)$ described in Section 9.

**Remark 1.2.** For $p = 1$ we have explicit expression for the constant

$$
C^r(1, r) = \begin{cases} 
2^{r-1}, & 1 < r \leq 2; \\
\Gamma(r + 1), & 2 \leq r.
\end{cases}
$$

Theorem 1.1 implies the corresponding inequality for the circle and for the line.

**Theorem 1.3.** For $1 \leq p \leq r < 2$ the inequality

$$
\| \varphi \|_{L^p(\mathbb{T})} \leq C(p, r) \| \varphi \|_{L^p(\mathbb{T})}^{\frac{r}{p}} \| \varphi \|_{\text{BMO}(\mathbb{T})}^{1-\frac{r}{p}}, \quad \varphi \in \text{BMO}(\mathbb{T}), \quad \int_{\mathbb{T}} \varphi = 0,
$$

holds true with the same sharp constant $C(p, r)$.

**Theorem 1.4.** For $1 \leq p \leq r < 2$ the inequality

$$
\| \varphi \|_{L^p(\mathbb{R})} \leq C(p, r) \| \varphi \|_{L^p(\mathbb{R})}^{\frac{r}{p}} \| \varphi \|_{\text{BMO}(\mathbb{R})}^{1-\frac{r}{p}}, \quad \varphi \in L^p(\mathbb{R}),
$$

holds true with the same sharp constant $C(p, r)$.

We deduce Theorems 1.3 and 1.4 from Theorem 1.1 in Section 10.

We also prove several statements for higher dimensions. Let $\mathcal{C}(n) = 4(1 + 2\sqrt{n} - 1)$ for $n \in \mathbb{N}$.

**Theorem 1.5.** If $1 \leq p \leq r < \infty$, then the inequality

$$
\| \varphi \|_{L^p(Q)} \leq C(p, r) \| \varphi \|_{L^p(Q)}^{\frac{r}{p}} \left( \mathcal{C}(n) \| \varphi \|_{\text{BMO}(Q)} \right)^{1-\frac{r}{p}}, \quad \varphi \in \text{BMO}(Q), \quad \langle \varphi \rangle_Q = 0,
$$

holds, where either $Q = I^n$ is a cube in $\mathbb{R}^n$ or $Q = \mathbb{T}^n$ is an $n$-dimensional torus. The BMO-norm in (1.7) is defined by (1.3), where supremum is taken over subcubes $J$.

This theorem follows immediately from Theorem 1.1 (and Theorem 1.1 in [9]) and the fact that the monotone rearrangement operator acts from $\text{BMO}(I^n)$ and $\text{BMO}(\mathbb{T}^n)$ to $\text{BMO}(I)$ with the norm bounded by $\mathcal{C}(n)$. For this estimate see [11], where the estimate for the monotone rearrangement operator is obtained for $L^1$-based BMO-norm. The corresponding estimate for $L^2$-based norm may be deduced by a straightforward modification.

One may use the limiting arguments (see the details in paper [9], Section 6.1) to obtain the following result.

**Corollary 1.6.** If $1 \leq p \leq r < \infty$, then we have the following inequality:

$$
\| \varphi \|_{L^p(\mathbb{R}^n)} \leq C(p, r) \| \varphi \|_{L^p(\mathbb{R}^n)}^{\frac{r}{p}} \left( \mathcal{C}(n) \| \varphi \|_{\text{BMO}(\mathbb{R}^n)} \right)^{1-\frac{r}{p}}, \quad \varphi \in L^p(\mathbb{R}^n).
$$
Another natural norm on $\text{BMO}(\mathbb{R}^n)$ is the norm defined by (1.3), where supremum is taken over balls $J$. We will call it the ball-based norm. In order to obtain estimates like we have in Corollary 1.6 related to such ball-based norm, we apply another approach inspired by [7]. Following [7], we prove dimension-free estimate using the Garcia-type norm on $\text{BMO}(\mathbb{R}^n)$. For $y \in \mathbb{R}^n$ and $t > 0$ consider the Poisson kernel $P_t$ and the heat kernel $H_t$:

$$P_t(y) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |y|^2)^{\frac{n+1}{2}}} , \quad H_t(y) = \frac{1}{(4\pi t)^\frac{n}{2}} e^{-\frac{|y|^2}{4t}} .$$

We will write $K_t$ for both $P_t$ and $H_t$.

For a function $\varphi$ on $\mathbb{R}^n$ consider its $K$-extension onto $\mathbb{R}^n \times \mathbb{R}_+$ given by convolution:

$$\varphi_K(y, t) = (K_t \ast \varphi)(y), \quad y \in \mathbb{R}^n, \ t > 0 .$$

For $y \in \mathbb{R}^n$, $t > 0$, this value may be considered as the average of $\varphi$ with the weight $K_t$ centered at $y$ instead of the average over a ball with radius $t$ centered at $y$. It is well-known that

$$\|\varphi\|_K := \sup_{(y,t) \in \mathbb{R}^n \times \mathbb{R}_+} \left( (\varphi^2)_K(y, t) - (\varphi_K(y, t))^2 \right)^{\frac{1}{2}}$$

is an equivalent norm on $\text{BMO}(\mathbb{R}^n)$. For $K = P$ this norm is called the Garcia norm. For the case $n = 1$ you can refer to [2], Chapter VI, Theorem 1.2.

In [7] it is shown how to prove dimension-free estimates on $\text{BMO}$ and $A_p$ equipped with Garcia-type norms having the corresponding estimates for one-dimensional case. We use this approach to prove the following theorem.

**Theorem 1.7.** Let $n \in \mathbb{N}$. If $1 \leq p < r < \infty$, then the inequality

$$\left\| \varphi \right\|_{L^r(\mathbb{R}^n)} \leq C(p, r) \left\| \varphi \right\|_{L^p(\mathbb{R}^n)} \left\| \varphi \right\|_{K}^{\frac{r-p}{r}} , \quad \varphi \in L^p(\mathbb{R}^n),$$

holds, where the kernel $K$ is either the Poisson kernel or the heat kernel, see (1.8).

The following estimate is proved in [7]:

$$\left\| \varphi \right\|_{K} \leq \tilde{C} n^{\frac{n}{2}} \left\| \varphi \right\|_{\text{BMO}(\mathbb{R}^n)} ,$$

(1.9)

where $\tilde{C}$ is an absolute constant, and the BMO-norm is the ball-based one. This estimate together with Theorem 1.7 implies the following corollary.

**Corollary 1.8.** Let $n \in \mathbb{N}$. If $1 \leq p < r < \infty$, then the following inequality holds:

$$\left\| \varphi \right\|_{L^r(\mathbb{R}^n)} \leq C(p, r) \tilde{C} n^{\frac{n}{2}} \left\| \varphi \right\|_{L^p(\mathbb{R}^n)} \left\| \varphi \right\|_{\text{BMO}(\mathbb{R}^n)}^{\frac{r-p}{r}} , \quad \varphi \in L^p(\mathbb{R}^n),$$

(1.10)

where BMO-norm is the ball-based one.

In the next section we repeat the definition of the main Bellman function as well as the definition and the properties of some auxiliary Bellman functions after [9] for convenience of the reader.

The Bellman function method allows to obtain various estimates (sometimes sharp) in analysis and probability reducing the infinite dimensional extremal problems to finite dimensional ones by using some auxiliary function, which now is usually called the Bellman function. For more information regarding this method and its application we refer the reader to the monographs [4] and [11].

### 2 Optimization problem

We introduce the main characters. These are the following Bellman functions:
\[ B^+_{p,r;\varepsilon}(x_1, x_2, x_3) = \sup \left\{ (|\varphi|^p)_i : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2, \langle |\varphi|^p \rangle_i = x_3 \right\}. \tag{2.1} \]

and

\[ B^-_{p,r;\varepsilon}(x_1, x_2, x_3) = \inf \left\{ (|\varphi|^p)_i : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2, \langle |\varphi|^p \rangle_i = x_3 \right\}. \tag{2.2} \]

We say that \( \varphi \) is a test function for the point \( x \in \mathbb{R}^3 \) if

\[ \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2, \langle |\varphi|^p \rangle_i = x_3. \tag{2.3} \]

The main purpose of this paper is to find explicit formulas for \( B^\pm_{p,r;\varepsilon} \). We state this result by referring to the formulas appearing in the forthcoming sections. After rather long preliminaries we define our Bellman candidates: \( B_1(x; p, r, \varepsilon) \) in Subsection 4.4.1 and \( B_2(x; p, r, \varepsilon) \) in Subsection 4.4.2. After that we will prove the following theorem.

**Theorem 2.1.** In the case \((r - 2)(r - p) > 0\) the function \( B^+_{p,r;\varepsilon} \) coincides with \( B_1 \) and the function \( B^-_{p,r;\varepsilon} \) coincides with \( B_2 \). In the case \((r - 2)(r - p) < 0\) the function \( B^+_{p,r;\varepsilon} \) coincides with \( B_2 \) and the function \( B^-_{p,r;\varepsilon} \) coincides with \( B_1 \).

The function \( B_1 \) was found in [9], where the statement of Theorem 2.1 was proved in the part concerning \( B_1 \). Our aim here is to find much more complicated function \( B_2 \) and to prove the rest of Theorem 2.1.

To describe the function \( B_1 \), we will need two auxiliary Bellman functions \( B^\pm_{p;\varepsilon} : \mathbb{R}^2 \to \mathbb{R} \cup \{ \pm \infty \} \) defined as follows:

\[ B^+_{p;\varepsilon}(x_1, x_2) = \sup \left\{ (|\varphi|^p)_i : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2 \right\}, \tag{2.4} \n\]
\[ B^-_{p;\varepsilon}(x_1, x_2) = \inf \left\{ (|\varphi|^p)_i : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2 \right\}. \tag{2.5} \]

The latter two functions were studied in detail in [3]. We survey these results, since they will play an important role in our study.

### 2.1 Description of \( B^\pm_{p;\varepsilon} \)

The domain of both functions \( B^\pm_{p;\varepsilon} \) is

\[ \Omega^2_\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2 \right\}. \]

By the domain of a Bellman function we mean the set of \( x \) such that the set of test functions \( \varphi \) (i.e., the functions over which we optimize in formulas (2.4) and (2.5)) is non-empty for this \( x \) (compare with (2.3)). Both functions also satisfy the boundary condition \( B^\pm_{p;\varepsilon}(t, t^2) = |t|^p, t \in \mathbb{R} \). From now on we omit the index \( \varepsilon \) in the notation of domains and functions.

To describe \( B^\pm \), we need some auxiliary functions. For \( s \geq 1 \) define

\[ m_s(u) = \frac{s}{\varepsilon} \int_u^{+\infty} e^{(u-t)/\varepsilon} t^{s-1} dt, \quad u \geq 0; \tag{2.6} \]
\[ k_s(u) = \frac{s}{\varepsilon} \int_u^{-\infty} e^{(t-u)/\varepsilon} t^{s-1} dt, \quad u \geq \varepsilon. \tag{2.7} \]

For any \( u \in \mathbb{R} \) we denote the segment connecting the points \((u, u^2)\) with \((u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2)\) by \( S_+(u) \) and the segment connecting \((u, u^2)\) with \((u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2)\) by \( S_-(u) \). Note that these segments touch
Figure 1: Left and right tangents passing through $x$

upon the upper boundary of $\Omega^2$, that is the parabola $x_2 = x_1^2 + \varepsilon^2$. For any $(x_1, x_2) \in \Omega^2$ there exist unique $u_\pm = u_\pm(x_1, x_2) \in \mathbb{R}$ such that $(x_1, x_2) \in S_\pm(u_\pm)$, $u_+ \leq x_1 \leq u_-$, see Figure 1. Namely,

$$u_\pm = x_1 \mp \left( \varepsilon - \sqrt{\varepsilon^2 + x_1^2 - x_2^2} \right).$$

Define the function $A_{m_p}$ on $\Omega^2$ in the following way. We put

$$A_{m_p}(x) = u^p + m_p(u)(x_1 - u), \quad x \in S_+(u), \quad u \geq 0,$$

$$A_{m_p}(x) = |u|^p - m_p(|u|)(x_1 - u), \quad x \in S_-(u), \quad u \leq 0. \quad (2.8)$$

In the curvilinear triangle between the tangents $S_-(0)$ and $S_+(0)$, we set

$$A_{m_p}(x) = \frac{m_p(0)}{2\varepsilon} x_2, \quad |x_1| \leq \varepsilon, \quad 2\varepsilon|x_1| \leq x_2 \leq x_1^2 + \varepsilon^2. \quad (2.9)$$

Formulas (2.8) and (2.9) define the function $A_{m_p}$ on the entire domain $\Omega^2$. Note that $A_{m_p}$ is even with respect to $x_1$ and $C^1$-smooth for $p > 1$.

Define the function $A_{k_p}$ on $\Omega^2$ as follows. We put

$$A_{k_p}(x) = u^p + k_p(u)(x_1 - u), \quad x \in S_+(u), \quad u \geq \varepsilon,$$

$$A_{k_p}(x) = |u|^p - k_p(|u|)(x_1 - u), \quad x \in S_-(u), \quad u \leq -\varepsilon. \quad (2.10)$$

In the domain $x_2 \leq \varepsilon^2$, we set

$$A_{k_p}(x_1, x_2) = x_2^{p/2}, \quad x_1^2 \leq x_2 \leq \varepsilon^2. \quad (2.11)$$

Formulas (2.10) and (2.11) define the function $A_{k_p}$ on the entire domain $\Omega^2$. This function is also even with respect to $x_1$ and $C^1$-smooth for $p \geq 1$.

Now we are ready to describe the functions $B^\pm$:

$$B^+_p = \begin{cases} A_{m_p}, & \text{if } 2 \leq p < \infty, \\ A_{k_p}, & \text{if } 1 \leq p \leq 2, \end{cases} \quad B^-_p = \begin{cases} A_{k_p}, & \text{if } 2 \leq p < \infty, \\ A_{m_p}, & \text{if } 1 \leq p \leq 2. \end{cases} \quad (2.12)$$

Here we collect some useful relations for derivatives of the functions $m_s$ and $k_s$:

$$m''_s(u) = \frac{s(s-1)(s-2)}{\varepsilon} \int_u^{+\infty} e^{(u-t)/\varepsilon} t^{s-3} dt, \quad (2.13)$$
Proposition 2.2. The set

\[ \mathcal{B} \]

The domain of the functions \( B \) follows from condition (iii) and the definition of the Bellman function \( G \).

We note that for \( x \notin \Omega^3 \) there are no test functions and we definitely have \( B_{\pm}(x) = \mp\infty \) in this case. On the other hand, for any \( x \in \Omega^3 \) there is a test function \( \varphi \). The proof of Proposition 2.2 can be found in [9] (Proposition 2.2 there).

In what follows by the skeleton of \( \Omega^3 \) we mean the set of points \( \{(t, t^2, |t|^\ell) : t \in \mathbb{R}\} \).

A function \( G : \omega \to \mathbb{R} \cup \{\pm\infty\} \), where \( \omega \subset \mathbb{R}^d \) is an arbitrary set, is called locally concave/convex if for any segment \( \ell \subset \omega \), the restricted function \( G|_{\ell} \) is concave/convex.

We collect standard facts concerning Bellman functions of such kind.

Proposition 2.3. 1) The functions \( B_{p,r}^\pm \) satisfy the boundary conditions on the skeleton of \( \Omega^3 \):

\[ B_{p,r}^\pm(t, t^2, |t|^\ell) = |t|^\ell, \quad t \in \mathbb{R}. \] (2.18)

2) The function \( B_{p,r}^+ \) is locally concave and the function \( B_{p,r}^- \) is locally convex on \( \Omega^3 \).

3) The function \( B_{p,r}^+ \) is the pointwise minimal among all locally concave on \( \Omega^3 \) functions \( B \) that satisfy the boundary condition (2.18). The function \( B_{p,r}^- \) is the pointwise maximal among all locally convex on \( \Omega^3 \) functions \( B \) that satisfy the boundary condition (2.18).

Proposition 2.3 is proved in [9] for the case of concave functions (see Proposition 2.3 there). The proof of the other case is literally the same.

In view of Proposition 2.3, to find the Bellman function \( B_{p,r}^\pm \), it suffices to construct a \( C^1 \)-smooth function \( B : \Omega^3 \to \mathbb{R} \) such that

(i) the function \( B \) is locally concave/convex on \( \Omega^3 \);

(ii) the function \( B \) fulfills the boundary conditions (2.18);

(iii) for any point \( x \in \Omega^3 \) there is a function \( \varphi_x \in \text{BMO}(I) \) such that

\[ \|\varphi_x\|_{\text{BMO}(I)} \leq \varepsilon, \quad \langle \varphi_x \rangle_1 = x_1, \quad \langle \varphi_x^2 \rangle_{1,2} = x_2, \quad \langle |\varphi_x|^p \rangle_1 = x_3, \quad \langle |\varphi_x|^\ell \rangle_1 = B(x); \] (2.19)

If all of the above requirements hold, then \( B = B_{p,r}^+ \) for locally concave function \( B \) and \( B = B_{p,r}^- \) for locally convex function \( B \). Indeed, the inequality \( B(x) \geq B_{p,r}^+(x), x \in \Omega^3 \), follows from conditions (i) and (ii), and the third statement of Proposition 2.3. The reverse inequality \( B(x) \leq B_{p,r}^+(x) \) for \( x \in \Omega^3 \) follows from condition (iii) and the definition of the Bellman function \( B_{p,r}^+ \). We will provide more details in Section 7 and Section 8.

A function \( \varphi_x \) satisfying (2.19) is called an optimizer for \( B \) at \( x \).
3 Foliation. Definitions

Our aim is to construct the function $B$ on $\Omega_3^3$ described at the end of the previous section. First, we build a foliation of the domain, i.e., we split the whole domain into a union of one-dimensional and two-dimensional sets, where our function is linear and its gradient is constant. These subsets of linearity we call leaves of the foliation. Having a foliation we can reconstruct the Bellman function using the boundary values. Recall the construction performed in [9] for the Bellman function $B_1$. We had there only two-dimension leaves of foliation. We repeat here description of this foliation.

Domain $\Omega_3^3$ was split into three subdomains $\Xi_+, \Xi_0, \Xi_-$ with foliation of different types:

$$\Xi_0 = \{x \in \Omega_3^3: |x_1| \leq 2\varepsilon, x_2 \geq 4\varepsilon|x_1| - 3\varepsilon^2; (p - 2)(x_3 - \varepsilon^p - \frac{x_2 - \varepsilon^2}{4\varepsilon}m_p(\varepsilon)) \geq 0\},$$

$$\Xi_+ = \{x \in \Omega_3^3 \setminus \Xi_0: x_1 > 0\},$$

$$\Xi_- = \{x \in \Omega_3^3 \setminus \Xi_0: x_1 < 0\}. \tag{3.1}$$

3.1 Foliation far from the origin

Since our problem is symmetric with respect to the change of sign of the first coordinate, we will assume that $x_1 > 0$. We consider the subdomain $\Xi_+$. Let us denote by $U(v)$ the point of the skeleton with the first coordinate $v$: $U(v) = (v, v^2, v^3)$. Together with the point $U(v)$ we consider two points $W_\pm(v)$ that belong to the upper or lower boundary of $\Omega_3^3$ and are the second endpoints of tangents from the point $U(v)$ to the parabolic part of the boundary, namely

$$W_+(v) = \left(v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2, A_{m_p}(v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2)\right)$$

$$= \left(v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2, v^p + \varepsilon m_p(v)\right),$$

$$W_-(v) = \left(v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2, A_{k_p}(v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2)\right)$$

$$= \left(v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2, v^p - \varepsilon k_p(v)\right). \tag{3.2}$$

Note that the point $W_+(v)$ is on the upper boundary if $p > 2$ and on the lower boundary if $p < 2$. The converse position is taken by $W_-(v)$. Note that the projection of the segments $[U(v), W_\pm(v)]$ onto the $x_1x_2$-plane are the segments $S_\pm(v)$. Here and in what follows, by $[A, B]$ we denote the straight line segment with the endpoints $A$ and $B$.

For $v \geq \varepsilon$ we consider the two-dimensional plane that passes through $U(v)$ and the points $W_\pm(v)$. Its equation is

$$x_3 = v^p + \frac{m_p - k_p}{4\varepsilon} \cdot (x_2 - x_1^2 + (x_1 - v)^2) + \frac{m_p + k_p}{2} \cdot (x_1 - v). \tag{3.3}$$

Here and in what follows, we omit the argument of $m_p$ and $k_p$ if this does not lead to ambiguity.

Let $T(v)$ be the intersection of $\Omega_3^3$ and the triangle with the vertices $U(v)$, $W_-(v)$, $W_+(v)$. So, $T(v)$ is a curvilinear triangle with linear sides on $\partial \Omega_3^3$: $[U(v), W_+(v)]$ is the graph of $A_{m_p}$ restricted on $S_+(v)$ and $[U(v), W_-(v)]$ is the graph of $A_{k_p}$ restricted on $S_-(v)$. That is for $p > 2$ the segment $[U, W_+]$ lies on the upper boundary of $\Omega_3^3$, for $p < 2$ it lies on the lower boundary. For the segment $[U, W_-]$ we have the opposite situation. This difference of the cases $p < 2$ and $p > 2$ explains rather cumbersome definition of the domain $\Xi_+$: depending on $p$ we have to consider the points either above or below the triangle $T(v)$. The domain $\Xi_-$ is foliated by such curvilinear triangles $T(v)$, $v \geq \varepsilon$.

This was the description of the foliation of $\Xi_+$ for $B_1$ found in [9]. We start with some heuristic arguments concerning possible foliation for $B_2$. We need to find another foliation with the same boundary condition: on the boundary we assume the same extremal lines $[U(v), W_+(v)]$ and $[U(v), W_-(v)]$. The unique possibility to have a two-dimensional leaf of linearity is already used in the described foliation, therefore, we have to look for one dimensional extremals. All extremals have to start from the skeleton $\{x = (v, v^2, |v|^p)\}$ and cannot go transversal to the free parabolic boundary $\{x = (x_1, x_1^2 + \varepsilon^2, x_3)\}$. Therefore, there are only two possibilities: either a fan $F_1(u)$ of left tangents to parabolic boundary
from a point $U(u - \varepsilon)$, or a fan $F_R(u)$ of right tangents to parabolic boundary from a point $U(u + \varepsilon)$. The tangency points of each such fan lie on the line $(u, u^2 + \varepsilon^2, \cdot)$. We would like to pay attention of the reader that symbols $L$ and $R$ mean left and right tangents, what means that these lines lies on the left and on the right of the tangency point, correspondingly. But if we look from their common point $U(u \pm \varepsilon)$ the situation is opposite: the fan $F_L(u)$ lies on the right from $U(u - \varepsilon)$ (its projection on $(x_1, x_2)$-plane is $S_+(u - \varepsilon)$) and the fan $F_R(u)$ lies on the left of $U(u + \varepsilon)$ (its projection on $(x_1, x_2)$-plane is $S_-(u + \varepsilon)$). We illustrate all of this by Figure 2. To visualize our construction, it is more natural to consider what happens in a tangent plane to the parabolic boundary. Let us intersect our domain $\Omega^3_\varepsilon$ by the two-dimensional plane $2ux_1 - x_2 - u^2 + \varepsilon^2 = 0$, we call this intersection $P(u)$. It touches the parabolic boundary along the vertical line $x_1 = u$, $x_2 = u^2 + \varepsilon^2$. The picture of $P(u)$ on Figure 2 illustrates the situation for the case $1 < p < 2$ and $u > 3\varepsilon$. The chord $[U(u - \varepsilon), U(u + \varepsilon)]$ has a very large slope for large values of $u$, but we place it horizontally, i.e., this picture presents an affine transform of the true graph.

We see here that $P(u)$ has two linear parts of the boundary. These are two straight line segments: the extremal $[U(u - \varepsilon), W_+(u - \varepsilon)]$ on the upper boundary and the extremal $[U(u + \varepsilon), W_-(u + \varepsilon)]$ on the lower boundary. The fans $F_R(u)$ and $F_L(u)$ start from these two extremals. The “long chord” $[U(u - \varepsilon), U(u + \varepsilon)]$ is a natural second border for both fans. The curvilinear line on the Figure 2 connecting $U(u - \varepsilon)$ and $U(u + \varepsilon)$ is the trace of the other long chords on this plane. That is the set of points of intersection of the chords $[U(u - \varepsilon + t), U(u + \varepsilon + t)]$, $-2\varepsilon < t < 2\varepsilon$, with the plane $P(u)$. In each plane $P(u + t)$ there are own fans $F_R(u + t)$ and $F_L(u + t)$ that intersect our plane $P(u)$ along some vertical segments between the boundary point and the corresponding long chord. So, we have described the extremals for all points of $P(u)$ except of ones between the long chord $[U(u - \varepsilon), U(u + \varepsilon)]$ and the curvilinear line connecting its endpoints. It appears that the extremals for the points between these two lines are short chords of the form $[U(a), U(b)]$, $0 < b - a < 2\varepsilon$, passing through them, i.e., the chords that do not touch the parabolic boundary. In Lemma 5.5 below it will be proved that for each such point there exists exactly one chord passing through it. Observe that

Figure 2: Traces of extremals on the plane $P(u)$ far from the origin for $1 < p < 2$. 
for $x \in [U(u - \varepsilon), U(u + \varepsilon)]$ we have the following relation:

$$
x_3 = \begin{cases} 
\Delta_+(x_1 - \Delta_+)^p + \Delta_+(x_1 + \Delta_-)^p, & \text{if } u \leq x_1 \leq u + \varepsilon, \\
\Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p, & \text{if } u - \varepsilon \leq x_1 \leq u,
\end{cases}
$$

where $\Delta_{\pm} \overset{\text{def}}{=} \varepsilon \pm d$ and $d \overset{\text{def}}{=} \sqrt{x_1^2 + \varepsilon^2} - x_2$.

Now, instead of one domain $\Xi_+$ generated by the triangles $T(u)$, we have three domains: $\Xi_{l+}$ is the domain foliated by fans $F_L(u)$ of left tangents, $\Xi_{r+}$ is the domain foliated by fans $F_R(u)$ of right tangents, and $\Xi_{ch+}$ is the domain between two preceding ones foliated by chords. So, approximately, we can describe these domains as follows:

$$
\Xi_{l+} = \left\{ x \in \Omega_\varepsilon^3 : x_1 > 0 \text{ and sufficiently far from } 0, \right. \\
\left. (p - 2) \frac{\Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p}{2\varepsilon} \leq (p - 2)x_3 \leq (p - 2)A_{m_p}(x_1, x_2) \right\};
$$

$$
\Xi_{r+} = \left\{ x \in \Omega_\varepsilon^3 : x_1 > 0 \text{ and sufficiently far from } 0, \\
(p - 2)A_{k_p}(x_1, x_2) \leq (p - 2)x_3 \leq (p - 2) \left( \frac{\Delta_-(x_1 - \Delta_+)^p + \Delta_+(x_1 + \Delta_-)^p}{2\varepsilon} \right) \right\};
$$

$$
\Xi_{ch+} = \left\{ x \in \Omega_\varepsilon^3 : x_1 > 0 \text{ and sufficiently far from } 0, \\
(p - 2) \left( \frac{\Delta_-(x_1 - \Delta_+)^p + \Delta_+(x_1 + \Delta_-)^p}{2\varepsilon} \right) \leq (p - 2)x_3 \leq (p - 2) \left( \frac{\Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p}{2\varepsilon} \right) \right\}.
$$

It is clear that we have domains $\Xi_{r-}$, $\Xi_{l-}$, and $\Xi_{ch-}$ symmetrical to $\Xi_{l+}$, $\Xi_{r+}$, and $\Xi_{ch+}$ respectively. And now the problem arises how to connect these domains in a neighbourhood of the origin and specify what does it mean “sufficiently far” in these formulas. Furthermore, near the origin, the description of all these three domains should be slightly changed due to the influence of two new specific domains that appear here. The traces of all these domains near the origin can be seen on Figures 3, 4, and 5.

Figure 3: Traces of extremals on the plane $P(u)$ for $2\varepsilon < u < 3\varepsilon$ and $1 < p < 2$. 

9
3.2 Foliation near the origin

In [9] the domain $\Xi_0$ for $B_1$ (see (3.1)) was obtained in a rather natural way: when $u$ decreases, the triangle $T(u)$ and the symmetrical triangle $T(-u)$ get closer and closer, and at the moment $u = \varepsilon$ they touch each other at the point $(0, \varepsilon^2, \varepsilon^p)$. At this moment they lie in the same plane and two of their sides
form a single chord $[U(-\varepsilon), U(\varepsilon)]$. For each $u, u < \varepsilon$, we have a two-dimensional domain of linearity in the form of curvilinear trapezoid $T(u)$ with three straight line sides $[U(-u), U(u)]$, $[U(u), W_+(u)]$, and $[U(-u), W_-(-u)]$.

Now, for $B_2$, the situation is much less clear, the question is how to gather nearly the origin three domains from the right and three domains from the left. It appears that the domains foliated by the chords $\Xi_{ch+}$ and $\Xi_{ch-}$ are separated automatically, they have the only common point — the origin. It is rather easy to build an interlacing domain between $\Xi_{L+}$ and $\Xi_{R-}$. Each of these domains has a border fan $F_{L}(\varepsilon)$ and $F_{R}(\varepsilon)$. They have the common point $U(0)$. If we take an extremal from $F_{L}(\varepsilon)$ and the symmetric extremal from $F_{R}(\varepsilon)$ it is natural to consider a curvilinear triangle between these two extremals as a domain of linearity. In such a way we obtain at the origin a fan of two-dimensional extremals. We denote this fan simply by $F(0)$ and it is just the interlacing domain between $\Xi_{L+}$ and $\Xi_{R-}$.

The construction of the interlacing domain between $\Xi_{L-}$ and $\Xi_{R+}$ is more difficult. It appears that the connecting foliation consists of two-dimensional linearity domains $R(v)$, $0 < v \leq 2\varepsilon$, describing as follows: the leaf $R(v)$ is the induced (in $\Omega^3_{2\varepsilon}$) convex hull of the points $U(0)$ and $U(\pm v)$. Two types of linearity domains $R(v)$ are presented on Fig. 6:

- if $0 < v \leq \varepsilon$, then $R(v)$ is simply the convex hull of these points, i.e., it is the triangle with these vertices;
- if $\varepsilon \leq v \leq 2\varepsilon$, then $R(v)$ is the induced convex hull of these points, i.e., it is the plane curvilinear triangle with two sides being the chords $[U(0), U(\pm v)]$, and the third side consist of two symmetrical line straight segments starting from $U(\pm v)$ and tangent to the parabolic boundary at the points with the first coordinates $\pm(v - \varepsilon)$ and a curve on the parabolic boundary connecting these two tangency points.

Now we give a formal description of all subdomains. The fan of two-dimensional extremals foliates the domain

$$F(0) = \left\{ x \in \Omega^3_{2\varepsilon}: |x_1| \leq \varepsilon, \ 2\varepsilon|x_1| \leq x_2 \leq x_1^2 + \varepsilon^2, \ (2 - p)x_3 \leq (2 - p)(2\varepsilon)^{p-2}x_2 \right\}.$$

We denote by $R$ the domain foliated by two-dimensional leaves $R(v)$. To describe this domain ana-

---

1 For the reader familiar with \[3\] we give the following analogy. We know that in two-dimensional cases, when two angles simultaneously touch a cup, they form a birdie. Here in the three-dimensional case, we get a family of “birdies”.

2 We refer to a set $X, X \subset \Omega$, as the induced convex set in $\Omega$ if for any pair $x, y \in X$ such that the straight line segment $[x,y]$ lies in $\Omega$, it also lies in $X$. We say that $X$ is an induced convex hull of $X_1$ in $\Omega$ if it is the minimal by inclusion induced convex subset of $\Omega$ containing $X_1$. 

Figure 6: Two examples of two-dimensional extremals $R(v)$. 

11
lytically, we first split the underlying domain $\Omega^2_\varepsilon$ into the following subregions:

\[
\begin{align*}
\omega_0 &= \left\{ x \in \Omega^2_\varepsilon : 2\varepsilon|x_1| \leq x_2 \leq \varepsilon^2 \right\}; \\
\omega_1 &= \left\{ x \in \Omega^2_\varepsilon : x_2 \geq \varepsilon^2, \ x_2 \geq 2\varepsilon x_1, \ 0 \leq x_1 \leq \varepsilon \right\}; \\
\omega_2 &= \left\{ x \in \Omega^2_\varepsilon : x_2 \leq \varepsilon^2, \ x_2 \leq 2\varepsilon x_1 \right\}; \\
\omega_3 &= \left\{ x \in \Omega^2_\varepsilon : \varepsilon^2 \leq x_2 \leq 2\varepsilon x_1 \right\}; \\
\omega_4 &= \left\{ x \in \Omega^2_\varepsilon : x_1 \geq \varepsilon, \ x_2 \geq 2\varepsilon x_1 \right\}; \\
\end{align*}
\]  
(3.6)

and $\omega_i = \left\{ x = (x_1, x_2) : (-x_1, x_2) \in \omega_i \right\}$.

![Figure 7: Splitting of $\Omega^2_\varepsilon$ into subdomains $\omega_i$.](image)

The projection of the domain $R$ to the first two coordinates covers all subregions $\omega_i$ except $\omega_{\pm 4}$.

\[
R = \bigcup_{i=3}^{i=1} R_i,  
\]
(3.7)

where

\[
\begin{align*}
R_0 &= \left\{ x : (x_1, x_2) \in \omega_0, \ (2-p)(2\varepsilon)^{p-2}x_2 \leq (2-p)x_3 \leq (2-p)x_2^\frac{\varepsilon}{\varepsilon} \right\}; \\
R_{\pm 1} &= \left\{ x : (x_1, x_2) \in \omega_{\pm 1}, \ (2-p)(2\varepsilon)^{p-2}x_2 \leq (2-p)x_3 \leq (2-p)(|x_1| + \Delta_-)^{p-2}x_2 \right\}; \\
R_{\pm 2} &= \left\{ x : (x_1, x_2) \in \omega_{\pm 2}, \ (2-p)x_2^{\frac{\varepsilon}{p}}|x_1|^{2-p} \leq (2-p)x_3 \leq (2-p)x_2^\frac{\varepsilon}{\varepsilon} \right\}; \\
R_{\pm 3} &= \left\{ x : (x_1, x_2) \in \omega_{\pm 3}, \ (2-p)x_2^{\frac{\varepsilon}{p}}|x_1|^{2-p} \leq (2-p)x_3 \leq (2-p)(|x_1| + \Delta_-)^{p-2}x_2 \right\}. \\
\end{align*}
\]  
(3.8, 3.9, 3.10, 3.11)

As before, we use the notation $\Delta_\pm = \varepsilon \pm d$ and $d = \sqrt{x_1^2 + \varepsilon^2 - x_2^2}$. The different parts of $R$ have different neighbours in vertical direction: $R_0$ has common boundary with $F(0)$, $R_1$ lies between $F(0)$ and $\Xi_{\pm}$, $R_2$ has common boundary with $\Xi_{\pm}$, $R_3$ lies between $\Xi_{\pm}$ and $\Xi_{\pm}$. We will check in Lemma 5.2 that indeed the domain $R$ is foliated by the leaves $R(v), \ 0 \leq v \leq 2\varepsilon$.

Now we can complete the definition of the domains (3.4):

\[
\Xi_{\pm} = \left\{ x \in \Omega^2_\varepsilon : x_1 \geq \Delta_-, \ (2-p)A_{mp}(x_1, x_2) \leq (2-p)x_3 \leq (2-p)(\Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p) \right\}. 
\]  
(3.12)
In Lemma 5.3 we will check that the domain $\Xi_{L_r}$ is foliated by the fans $F_v(u)$, $\varepsilon < u < \infty$.

Near the origin the domain foliated by the right fans should be changed a bit:

\[
\Xi_{r+} = \left\{ x \in \Omega_x^3; x_1 \geq 0, \ x_2 \geq \varepsilon^2, \ (p-2)A_{h_p}(x_1, x_2) \leq (p-2)x_3 \right\}
\]

\[
\Xi_{r+} = \left\{ x \in \Omega_x^3; x_1 \geq \Delta_+, \ (p-2)\frac{(x_1 + \Delta_-)^p - x_2}{2\varepsilon} \leq (2-p)x_3 \right\}
\]

In Lemma 5.4 we will check that the domain $\Xi_{r+}$ is foliated by the fans $F_v(u)$, $0 \leq u < \infty$.

The remaining subdomain

\[
\Xi_{U+} = \left\{ x \in \Omega_x^3; x_1 \geq \Delta_+, \ (2-p)\frac{(x_1 + \Delta_-)^p + \Delta_+(x_1 - \Delta_-)^p}{2\varepsilon} \leq (2-p)x_3 \right\}
\]

is foliated by the chords connecting two points of the skeleton. In Lemma 5.5 we will check that for every point $x$ from the domain $\Xi_{U+}$ there exists a unique pair of non-negative numbers $a = a(x)$ and $b = b(x)$ such that $0 \leq b - a \leq 2\varepsilon$ and the chord $[U(a), U(b)]$ passes through $x$.

In the latter two cases we have two different analytic expressions describing the domains, because the different parts of the domains in (3.13) and (3.14) have different neighbours.

We collect the proofs of all the results concerning the described foliation in Section 5.

### 4 Construction of a Bellman candidate

We start with the simplest domain $\Xi_{L_+}$ foliated by chords of the form $[U(a), U(b)]$. Since the Bellman function is assumed to be linear on these chords, we can define a Bellman candidate $B$ by the formula

\[
B(x) = \frac{x_1 - a}{b - a} b^r + \frac{b - x_1}{b - a} a^r.
\]

(4.1)

It is also easy to find a Bellman candidate on the domain $R$ foliated by two-dimensional leaves $R(v)$, that are induced convex hulls of the points $U(\pm v)$ and the origin, because the linear function there is also determined completely by the values at these three points:

\[
B(x) = v^{-2}x_2 = x_3^{\frac{r-2}{2}}x_2^{\frac{p-r}{2}}, \quad x \in R(v).
\]

(4.2)

Formally, it is easy to define a candidate on the fan of triangles $F(0)$: the function there depends only on $x_2$ and $x_3$ and is completely determined by the values on the sides of each triangle. However, the values on the boundary fans $F_\varepsilon(\varepsilon)$ and $F_{-\varepsilon}(-\varepsilon)$ should be obtained from the domains $\Xi_{L_+}$ and $\Xi_{R_+}$, where it is not so simple to find an appropriate Bellman candidate. Due to the symmetry, it is sufficient to consider only one of these domains, and we start with $\Xi_{L_+}$.

#### 4.1 Bellman function in the domain $\Xi_{L_+}$

Recall that the domain $\Xi_{L_+}$ is foliated by the fans $F_v(u)$, $u \geq \varepsilon$ (see Fig. 2, 3, and 4). Recall that all the extremals of the fan $F_v(u)$ lie in the plane $P(u)$ and are tangent to the parabolic part of the boundary at the points $(u, u^2 + \varepsilon^2, \cdot)$. Recall that the plane $P(u)$ is determined by the equation

\[
x_2 = 2ux_1 + \varepsilon^2 - u^2.
\]

(4.3)
Since we consider the left tangents, \( u = u(x) \geq x_1 \), i.e.,

\[
u = x_1 + \sqrt{x_1^2 + \varepsilon^2 - x_2}.
\] (4.4)

The tangent lines forming the fan \( F(u) \) have the common endpoint \( U(v) \), \( v = u - \varepsilon = x_1 - \Delta_- \). The second endpoint of the extremal passing through \( x \) will be denoted by \( H(u,h) \), where \( h = h(x) \) is determined by the third coordinate of this endpoint by the formula

\[
H(u,h) = (u,u^2 + \varepsilon^2,v^p + \varepsilon m_p(v) + h).
\] (4.5)

This second endpoint runs over a segment \([W_+(v),H_L(u)]\) on the parabolic part of the boundary, where

\[
H_L(u) = [W_+(u-\varepsilon),W_-(u+\varepsilon)] \cap [U(u-\varepsilon),U(u+\varepsilon)] = \left(u,u^2 + \varepsilon^2,\frac{1}{2}((u+\varepsilon)^p + (u-\varepsilon)^p),\right)
\] (4.6)

and \( W_\pm \) were defined in (3.2). This means that \( h \) runs from 0 till \( \frac{1}{2}((u+\varepsilon)^p - (u-\varepsilon)^p) - \varepsilon m_p(u-\varepsilon) \):

\[
H(u,0) = W_+(u-\varepsilon), \quad H\left(u,\frac{1}{2}((u+\varepsilon)^p - (u-\varepsilon)^p) - \varepsilon m_p(u-\varepsilon)\right) = H_L(u).
\]

We see that \( \text{sign} h = \text{sign}(2-p) \), because the point \( W_+(v) \) is on the lower boundary of the domain if \( p < 2 \) and on the upper boundary if \( p > 2 \).

Since \( B \) has to be linear along the extremal \([U(v),H(u,h)]\), we write

\[
B(x) = v^r + K_L(u,h)(x_1 - v), \quad x \in [U(v),H(u,h)],
\] (4.7)

where \( K_L = K_L(u,h) \) is the unknown slope of this linear function. Now, we find the slope \( K_L \) using the property of \( \text{grad} B \) to be constant along the extremal. First, we have to calculate the partial derivatives of the variables \( u \) and \( h \).

The function \( u = u(x) \) is defined by (4.3), \( v = u - \varepsilon = x_1 - \Delta_- \), whence

\[
\frac{\partial v}{\partial x_2} = \frac{\partial u}{\partial x_2} = \frac{1}{2(x_1 - u)}; \quad \frac{\partial v}{\partial x_3} = \frac{\partial u}{\partial x_3} = 0.
\] (4.8)

The function \( h = h(x) \) is determined by the fact that the points \( H(u,h), x, \) and \( U(v) \) lie on the same straight line, i.e.,

\[
x_1 - v = \frac{x_3 - v^p}{\varepsilon m_p(v) + h}.
\] (4.9)

This yields

\[
h = \varepsilon \left(\frac{x_3 - v^p}{x_1 - v} - m_p(v)\right),
\] (4.10)

and

\[
x_3 = v^p + \left(\frac{h}{\varepsilon} + m_p(v)\right)(x_1 - v) = (u-\varepsilon)^p + \left(\frac{h}{\varepsilon} + m_p(u-\varepsilon)\right)\Delta_-.
\] (4.11)

Since \( v \) depends only on \( x_1 \) and \( x_2 \), we have

\[
\frac{\partial h}{\partial x_3} = \frac{\varepsilon}{x_1 - v}; \quad \frac{\partial h}{\partial x_2} = \frac{h + \varepsilon m_p'(v)(u-x_1)}{x_1 - v} \cdot \frac{\partial v}{\partial x_2}.
\] (4.12)

Differentiating the function \( B \) with respect to \( x_3 \), we obtain

\[
\frac{\partial B}{\partial x_3} = \frac{\partial K_L}{\partial h} \cdot (x_1 - v) \cdot \frac{\partial h}{\partial x_3} = \varepsilon \frac{\partial K_L}{\partial h}.
\] (4.13)

The expression for the derivative of \( B \) with respect to \( x_2 \) is more complicated:

\[
\frac{\partial B}{\partial x_2} = \frac{1}{2(x_1 - u)} \cdot \left[rv^{r-1} + \frac{\partial K_L}{\partial u} \cdot (x_1 - v) - K_L + \frac{\partial K_L}{\partial h} \cdot (h + \varepsilon m_p'(v)(u-x_1))\right]
\]

\[
= \frac{1}{2(x_1 - u)} \cdot \left[rv^{r-1} + \varepsilon \frac{\partial K_L}{\partial u} - K_L + h \frac{\partial K_L}{\partial h} \right] + \frac{1}{2} \left(\frac{\partial K_L}{\partial u} - \varepsilon m_p'(v) \frac{\partial K_L}{\partial h}\right).
\] (4.14)
Since the derivative of the function \( B \) with respect to \( x_2 \) has to be constant on the extremal line with the fixed parameters \((u, h)\), we conclude that

\[
rv^{r-1} + \varepsilon \frac{\partial K_u}{\partial u} - K_u + \frac{h}{\partial h} = 0 \tag{4.15}
\]

and

\[
\frac{\partial B}{\partial x_2} = \frac{1}{2} \left( \frac{\partial K_u}{\partial u} - \varepsilon m_p'(v) \frac{\partial K_h}{\partial h} \right). \tag{4.16}
\]

From the formula for the function \( B \) at the point \( W(v) \), we get the boundary value \( K_u(u, 0) = m_r(v) \). The general solution of the differential equation \( (4.15) \) with this boundary condition has the following form:

\[
K_u(u, h) = e^{-\frac{v}{\varepsilon}} \Psi_u(e^{-\frac{v}{\varepsilon}} h) + m_r(v), \tag{4.17}
\]

where \( \Psi_u \) is an arbitrary sufficiently smooth function with \( \Psi_u(0) = 0 \). We remind the reader that in this subsection we set \( v = u - \varepsilon \).

We find \( \Psi_u \) from the boundary value at the point \( H_1(u) \). Recall that \( H_1(u) \) coincides with \( H(u, h) \) for

\[
h = \frac{1}{2} \left( (u + \varepsilon)^p - (u - \varepsilon)^p \right) - \varepsilon m_p(u - \varepsilon).
\]

Since we assume the segment \([U(u - \varepsilon), U(u + \varepsilon)]\) to be an extremal line, \( B \) has to be linear on it, i.e.,

\[
B = \frac{1}{2} \left( (u - \varepsilon)^r + (u + \varepsilon)^r \right)
\]

at the midpoint of this segment. As a result, we come to the equation

\[
B(H_1(u)) = (u - \varepsilon)^r + K_u\left(u, \frac{1}{2} \left( (u + \varepsilon)^p - (u - \varepsilon)^p \right) - \varepsilon m_p(v)\right) \cdot \varepsilon = \frac{1}{2} \left( (u + \varepsilon)^r + (u - \varepsilon)^r \right),
\]

whence

\[
K_u\left(u, \frac{1}{2} \left( (u + \varepsilon)^p - (u - \varepsilon)^p \right) - \varepsilon m_p(v)\right) = \frac{(u + \varepsilon)^r - (u - \varepsilon)^r}{2\varepsilon}, \tag{4.18}
\]

or

\[
\Psi_u\left(e^{-\frac{v}{\varepsilon}} \left[ \frac{1}{2} \left( (u + \varepsilon)^p - (u - \varepsilon)^p \right) - \varepsilon m_p(u - \varepsilon) \right]\right) = \frac{1}{\varepsilon} e^{-\frac{v}{\varepsilon}} \left[ \frac{1}{2} \left( (u + \varepsilon)^r - (u - \varepsilon)^r \right) - \varepsilon m_r(u - \varepsilon) \right]. \tag{4.19}
\]

Let us introduce a function \( \xi \mapsto w_L(\xi; s, \varepsilon) \) by the formula

\[
w_L(\xi; s, \varepsilon) = e^{-\frac{\varepsilon}{2} \left[ \frac{1}{2} \left( (\xi + \varepsilon)^s - (\xi - \varepsilon)^s \right) - \varepsilon m_s(\xi - \varepsilon) \right]}, \quad \xi \geq s. \tag{4.20}
\]

The symbols \( s \) and \( \varepsilon \) are considered here as fixed parameters, we will sometimes omit them when it does not lead to misunderstanding. Then we can rewrite relation \( (4.19) \) in terms of \( w_L \):

\[
\Psi_u\left(w_L(u; p, \varepsilon)\right) = \frac{1}{\varepsilon} w_L(u; r, \varepsilon). \tag{4.21}
\]

In Lemma \( 4.4 \) below we will prove that the function \( w_L \) is monotone, and thereby, the inverse function is correctly defined.

This representation suggests the following change of parametrization for the extremals of the fan \( F_1(u) \). Till now, they were parametrized by \( h \). Let us introduce a new parameter \( \xi \):

\[
\xi = w_L^{-1}(e^{-\frac{v}{\varepsilon}} h; p, \varepsilon), \tag{4.22}
\]

i.e.,

\[
h = e^{-\frac{v}{\varepsilon}} w_L(\xi; p, \varepsilon). \tag{4.23}
\]

When the variable \( \xi \) is running from \( u \) till \( \infty \), then \( h \) is running from \( \frac{1}{2} \left( (u + \varepsilon)^p - (u - \varepsilon)^p \right) - \varepsilon m_p(u - \varepsilon) \) till zero. Therefore, the range of \( w_L \) covers the domain of \( \Psi_u \), and we can write down the following expression for \( \Psi_u \):

\[
\Psi_u(\xi) = \frac{1}{\varepsilon} w_L\left(\xi^{-1}(\cdot; p, \varepsilon): r, \varepsilon\right). \tag{4.24}
\]
Rewriting (4.11) in terms of $\xi$ we get:

$$x_3 = \nu^p + \left(\frac{1}{\varepsilon} e^{\frac{-x}{\varepsilon}} w_\nu(\xi; p, \varepsilon) + m_p(v)\right)(x_1 - v).$$

(4.25)

Taking into account (4.21), we rewrite (4.7) as follows:

$$B(x) = \nu^r + \left(\frac{1}{\varepsilon} e^{\frac{-x}{\varepsilon}} w_\nu(\xi; r, \varepsilon) + m_r(v)\right)(x_1 - v).$$

(4.26)

At the end of this subsection, we collect some formulas for the derivatives of the function $B$. We will use them in various proofs. For brevity, we omit the argument $e^{-\frac{x}{\varepsilon}} h$ of the function $\Psi_L$ and its derivatives, as well as the argument $v$ of the functions $m_s$ and their derivatives. From (4.17) we see that

$$\frac{\partial K_L}{\partial h} = \Psi'_L; \quad \frac{\partial K_L}{\partial u} = \frac{1}{\varepsilon} e^{\frac{-x}{\varepsilon}} \Psi_L - \frac{h}{\varepsilon} \Psi'_L + m'_r.$$  

(4.27)

Recall that $\frac{\partial B}{\partial x_3}$ was computed in (4.13). We have

$$\frac{\partial B}{\partial x_3} = \varepsilon \Psi'_L, \quad \frac{\partial^2 B}{\partial x_3^2} = \frac{\varepsilon^2}{x_1 - v} e^{-\frac{x}{\varepsilon}} \Psi''_L.$$  

(4.28)

Now, using relations (4.27), we rewrite formula (4.16) as follows

$$\frac{\partial B}{\partial x_2} = \frac{1}{2} \left(\frac{1}{\varepsilon} e^{\frac{-x}{\varepsilon}} \Psi'_L - \frac{h}{\varepsilon} \Psi'_L + m'_r - \varepsilon m'_p \Psi''_L\right).$$

(4.29)

Differentiating (4.29), we obtain

$$\frac{\partial^2 B}{\partial x_2^2} = -\frac{1}{2\varepsilon} \frac{\partial h}{\partial x_2} e^{-\frac{x}{\varepsilon}} \left(h + \varepsilon^2 m_p\right)\Psi''_L + \frac{1}{2\varepsilon^2} \frac{\partial h}{\partial x_2} \left(e^{\frac{-x}{\varepsilon}} \Psi_L - \left(h + \varepsilon^3 m''_p\right)\Psi'_L + \varepsilon e^{-\frac{x}{\varepsilon}} \left(h + \varepsilon^2 m_p\right)\Psi''_L + \varepsilon^2 m''_p\right)$$

(4.30)

Collecting (4.28), (4.30), and (4.31), we finally obtain

$$\det \begin{pmatrix} B_{x_2 x_2} & B_{x_2 x_3} \\ B_{x_3 x_2} & B_{x_3 x_3} \end{pmatrix} = \frac{e^{-\frac{x}{\varepsilon}} \Psi''_L}{4(x_1 - u)(x_1 - v)} \left(e^{\frac{-x}{\varepsilon}} \Psi_L - \left(h + \varepsilon^3 m''_p\right)\Psi'_L + \varepsilon^2 m''_p\right).$$

(4.32)

4.2 Bellman function in the domain $\Xi_{R+}$

The construction of a Bellman candidate in $\Xi_{R+}$ is quite similar to that for $\Xi_{L-}$. The additional difficulty appears here due to the fact that we have to distinguish the cases $0 \leq u \leq \varepsilon$ and $u \geq \varepsilon$, compare Fig. 2 with Fig. 3. If $u > \varepsilon$ then the situation is completely the same as in $\Xi_{L-}$. However, if $u < \varepsilon$ the corresponding long chord cannot be an extremal, because our function $B^{\pm}_{p}$ is symmetric with respect to the plane $x_1 = 0$, but this chord intersects (not orthogonally) the plane of symmetry. For $u < \varepsilon$ the fan $F_{h}(u)$ is continued till the domain $R$. This gives us another boundary condition for the corresponding function $\Psi_{R}$. Since we consider right tangents, now we have $u = u(x) \leq x_1$, and we choose the smaller root of (4.3), i.e.,

$$u = x_1 - \sqrt{x_1^2 + \varepsilon^2 - x_2}$$

(4.33)
or in coordinates
\[ H_n(u) = \begin{cases} 
[u(u - \varepsilon), (u + \varepsilon)] \cap [W_-(u + \varepsilon), W_+(u - \varepsilon)], & \text{for } u \geq \varepsilon, \\
\{ x: x_1 = u, x_2 = u^2 + \varepsilon^2 \} \cap R(v), & \text{for } 0 \leq u \leq \varepsilon,
\end{cases} \] (4.35)
or in coordinates
\[ H_n(u) = \begin{cases} 
(u, u^2 + \varepsilon^2, \frac{1}{2}((u + \varepsilon)^p + (u - \varepsilon)^p)), & \text{for } u \geq \varepsilon, \\
(u, u^2 + \varepsilon^2, v^p - (u^2 + \varepsilon^2)), & \text{for } 0 \leq u \leq \varepsilon.
\end{cases} \] (4.36)

This means that in the fan \( F_n(u) \) the variable \( h \) runs from 0 till \( \varepsilon k_p(v) - \frac{1}{2}((u + \varepsilon)^p - (u - \varepsilon)^p) \) for \( u \geq \varepsilon \) and till \( \varepsilon k_p(v) - 2\varepsilon uv^{p-2} \) for \( 0 \leq u \leq \varepsilon \).

We see that \( \text{sign } h = \text{sign } (p - 2) \), because the point \( W_-(u) \) is on the upper boundary of the domain if \( p < 2 \) and on the lower boundary if \( p > 2 \).

Since \( B \) has to be linear along the extremal \([U(v), H(u, h)]\), we write
\[ B(x) = v^r + K_n(u, h)(x_1 - v), \quad x \in [U(v), H(u, h)], \] (4.37)

In the same way as we got (4.9), we obtain:
\[ \frac{x_1 - v}{-\varepsilon} = \frac{x_3 - v^p}{-\varepsilon k_p(v) + h}. \] (4.38)
This yields
\[ h = \varepsilon \left(k_p(v) - \frac{x_3 - v^p}{x_1 - v}\right) \] (4.39)
or
\[ x_3 = v^p - \left(\frac{h}{\varepsilon} - k_p(v)\right)(x_1 - v) = (u + \varepsilon)^p + \left(\frac{h}{\varepsilon} - k_p(u + \varepsilon)\right)\Delta_. \] (4.40)

We collect the expressions for the derivatives of \( u \) and \( h \):
\[ \frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_2} = \frac{1}{2(x_1 - u)}; \quad \frac{\partial u}{\partial x_3} = \frac{\partial v}{\partial x_3} = 0, \] (4.41)
\[ \frac{\partial h}{\partial x_2} = \frac{h + \varepsilon(x_1 - u)k_p'(v)}{x_1 - v} \quad \frac{\partial v}{\partial x_2} = \frac{\partial h}{\partial x_3} = -\frac{\varepsilon}{x_1 - v}. \] (4.42)

Similarly to (4.13) and (4.14), we have
\[ \frac{\partial B}{\partial x_3} = -\varepsilon \frac{\partial K_n}{\partial h}, \] (4.43)
\[ \frac{\partial B}{\partial x_2} = \frac{1}{2(x_1 - u)} \cdot \left[ rv^{r-1} + \frac{\partial K_n}{\partial u} \cdot (x_1 - v) - K_n + h \frac{\partial K_n}{\partial h} \cdot (h - \varepsilon k_p'(v)(u - x_1)) \right] \] (4.44)
\[ = \frac{1}{2(x_1 - u)} \cdot \left[ rv^{r-1} - \varepsilon \frac{\partial K_n}{\partial u} - K_n + h \frac{\partial K_n}{\partial h} \right] + \frac{1}{2} \left( \frac{\partial K_n}{\partial u} + \varepsilon k_p'(v) \frac{\partial K_n}{\partial h} \right). \]

The requirement for \( \frac{\partial B}{\partial x_2} \) to be constant along the extremal line yields the following differential equation (compare with (4.15)):
\[ rv^{r-1} - \varepsilon \frac{\partial K_n}{\partial u} - K_n + h \frac{\partial K_n}{\partial h} = 0. \] (4.45)
As before, we know the value $B(W_-(v)) = A_{k_r}(W_-(v)) = v^r - \varepsilon k_r(v)$, and it implies the boundary condition $K_r(u, 0) = k_r(v)$. The general solution of (4.44) with this boundary condition has the form

$$K_r(u, h) = e^{-\frac{r}{\varepsilon}} \Psi_r(e^{\frac{r}{\varepsilon}} h) + k_r(v),$$  \hspace{1cm} (4.46)$$

where $\Psi_r$ is arbitrarily sufficiently smooth function with $\Psi_r(0) = 0$ (compare with (4.17)). Identities (4.44) and (4.45) imply

$$\frac{\partial B}{\partial x_2} = \frac{1}{2} \left( \frac{\partial K_r}{\partial u} + \varepsilon k'_r(v) \frac{\partial K_r}{\partial h} \right)$$  \hspace{1cm} (4.47)$$

Now, we consider the cases $0 \leq u \leq \varepsilon$ and $u \geq \varepsilon$ separately. Let $0 \leq u \leq \varepsilon$. From formula (4.42) we have $B(H_r) = v^r - 2\varepsilon uv^r - 2$, whence using (4.37) we obtain

$$K_r(u, k_r(v) - 2\varepsilon uv^p - 2) = 2uv^r - 2,$$

or

$$\Psi_r(e^{\frac{r}{\varepsilon}} [k_r(v) - 2\varepsilon uv^p - 2]) = e^{\frac{r}{\varepsilon}} [2uv^r - 2 - k_r(v)].$$  \hspace{1cm} (4.48)$$

Now, let $u \geq \varepsilon$. Similarly to the case of the function $\Psi_{11}$, we have the endpoint $H_r$ at the middle of the extremal segment $[U(u - \varepsilon), U(u + \varepsilon)]$, where $B$ is linear, i.e., $B(H_r) = \frac{1}{2} ((u + \varepsilon)^r + (u - \varepsilon)^r)$. As a result, we come to the equation

$$B(H_r(u)) = v^r + K_r \left( u, \frac{1}{2} ((u - \varepsilon)^p - (u + \varepsilon)^p) + \varepsilon k_r(p) \right) \cdot (-\varepsilon) = \frac{1}{2} ((u + \varepsilon)^r + (u - \varepsilon)^r),$$

or

$$\Psi_r \left( e^{\frac{r}{\varepsilon}} \left[ \frac{1}{2} ((u - \varepsilon)^p - (u + \varepsilon)^p) + \varepsilon k_r(p) \right] \right) = \frac{1}{\varepsilon} e^{\frac{r}{\varepsilon}} \left[ \frac{1}{2} ((u + \varepsilon)^r - (u - \varepsilon)^r) - \varepsilon k_r(v) \right].$$  \hspace{1cm} (4.49)$$

Let us introduce a function $\xi \mapsto w_\eta(\xi; s, \varepsilon)$ by the formula

$$w_\eta(\xi; s, \varepsilon) = \begin{cases} e^{s} \left[ k_\eta(\xi + \varepsilon) - 2(\xi + \varepsilon)^{s-2} \right], & \text{if } 0 \leq \xi \leq \varepsilon, \\ e^{s} \left[ \frac{1}{2} ((\xi - \varepsilon)^s - (\xi + \varepsilon)^s) + \varepsilon k_\eta(\xi + \varepsilon) \right], & \text{if } \xi \geq \varepsilon. \end{cases}$$  \hspace{1cm} (4.50)$$

The symbols $s$ and $\varepsilon$ are considered here as fixed parameters, we will sometimes omit them if it does not lead to ambiguity. Then we can rewrite relations (4.48) and (4.49) in terms of $w_\eta$:

$$\Psi_r \left( w_\eta u(p, \varepsilon) \right) = -\frac{1}{\varepsilon} w_\eta(u; r, \varepsilon).$$  \hspace{1cm} (4.51)$$

In Lemma 4.2 we prove that the function $w_\eta$ is monotone, and thereby, the inverse function is correctly defined.

This representation suggests the following change of parametrization for the extremals of the fan $F_r(u)$. Till now, they were parametrized by $h$. Let us introduce a new parameter $\xi$.

$$\xi = w_\eta^{-1} \left( e^{\frac{r}{\varepsilon}} h; p, \varepsilon \right),$$  \hspace{1cm} (4.52)$$

i.e.,

$$h = e^{-\frac{r}{\varepsilon}} w_\eta(\xi; p, \varepsilon).$$  \hspace{1cm} (4.53)$$

When the variable $\xi$ is running from zero till $u$, then $h$ is running from zero till $\varepsilon \left[ k_\eta(v) - 2uv^p - 2 \right]$ if $u \leq \varepsilon$ and till $\frac{1}{2} ((u - \varepsilon)^p - (u + \varepsilon)^p) + \varepsilon k_\eta(p)$ if $u \geq \varepsilon$. Since the range of $w_\eta$ cover the domain of $\Psi_r$, we can write down the following expression for $\Psi_r$:

$$\Psi_r(\cdot) = -\frac{1}{\varepsilon} w_\eta(\cdot; p, \varepsilon; r, \varepsilon).$$  \hspace{1cm} (4.54)$$
Rewriting (4.40) in terms of $\xi$, we get:

$$x_3 = v^p - \left(\frac{1}{\varepsilon} e^{-\frac{\xi}{\varepsilon}} w_L(\xi;p,\varepsilon) - k_p(v)\right)(x_1 - v).$$

(4.55)

Taking into account (4.46) and (4.51), we rewrite (3.7) as follows:

$$B(x) = v^p - \left(\frac{1}{\varepsilon} e^{-\frac{\xi}{\varepsilon}} w_L(\xi;r,\varepsilon) - k_r(v)\right)(x_1 - v).$$

(4.56)

As at the end of the preceding subsection, we collect here some formulas for the function $B$ and calculate $\det \{B_{x_i x_j}\}_{2 \leq i,j \leq 3}$. As before, we omit the argument $\varepsilon h$ of the function $\Psi_n$ and its derivatives, as well as the argument $v$ of the functions $k_s$ and their derivatives.

Differentiating (4.46), we get

$$\frac{\partial K_R}{\partial h} = \Psi_R', \quad \frac{\partial K_R}{\partial v} = -\frac{1}{\varepsilon} e^{-\frac{\xi}{\varepsilon}} \Psi_R + \frac{h}{\varepsilon} \Psi_R' + k_r'.$$

(4.57)

We use these relations to rewrite (4.43) and (4.47):

$$\frac{\partial B}{\partial x_3} = -\varepsilon \Psi_R'; \quad \frac{\partial B}{\partial x_2} = \frac{1}{2} \left(k_r' - \frac{1}{\varepsilon} e^{-\frac{\xi}{\varepsilon}} \Psi_R + \left(\frac{h}{\varepsilon} + \varepsilon k_p'\right) \Psi_R'\right).$$

(4.58)

Using these formulas together with (4.41) and (4.42), we can check the following relations

$$\frac{\partial^2 B}{\partial x_3^2} = \frac{\varepsilon^2}{x_1 - v} e^{\frac{\xi}{\varepsilon}} \Psi_R'';$$

(4.59)

$$\frac{\partial^2 B}{\partial x_2 \partial x_3} = \frac{\varepsilon^2 (h + \varepsilon k_p')}{2(v - x_1)} \Psi_R''$$

(4.60)

(compare with (4.31));

$$\frac{\partial^2 B}{\partial x_2^2} = \frac{1}{2\varepsilon} \frac{\partial h}{\partial x_2} e^{\frac{\xi}{\varepsilon}} \left( h + \varepsilon k_p' \right) \Psi_R'' + \frac{1}{2\varepsilon} \frac{\partial u}{\partial x_2} \left( e^{-\frac{\xi}{\varepsilon}} \Psi_R - (h - \varepsilon k_p' \Psi_R' + \varepsilon^2 (h + \varepsilon k_p') \Psi_R' + \varepsilon^2 k_r'\right)$$

$$= \frac{1}{2\varepsilon} \frac{\partial u}{\partial x_2} \left( e^{-\frac{\xi}{\varepsilon}} \Psi_R - (h - \varepsilon k_p' \Psi_R' + \varepsilon^2 k_r' \Psi_R' + \varepsilon^2 (h + \varepsilon k_p') \Psi_R' + \varepsilon^2 k_r') \Psi_R' \cdot (h + \varepsilon \frac{h + \varepsilon (x_1 - u)k_r'}{x_1 - v}) \right)$$

$$= \frac{1}{4\varepsilon^2 (x_1 - u)} \left( e^{-\frac{\xi}{\varepsilon}} \Psi_R - (h - \varepsilon k_p' \Psi_R' + \varepsilon^2 k_r' \Psi_R' + \varepsilon^2 (h + \varepsilon k_p') \Psi_R' \cdot \frac{x_1 - u}{x_1 - v} \right)$$

(4.61)

(compare with (4.30)).

Finally, we obtain

$$\det \begin{pmatrix} B_{x_2 x_2} & B_{x_2 x_3} \\ B_{x_2 x_3} & B_{x_3 x_3} \end{pmatrix} = \frac{\varepsilon^2 \Psi_R''}{4(x_1 - u)(x_1 - v)} \left( e^{-\frac{\xi}{\varepsilon}} \Psi_R - (h - \varepsilon k_p' \Psi_R' + \varepsilon^2 k_r') \right)$$

(4.62)

(compare with (4.32)).

Let us note that instead of calculating all these derivatives we could replace the subindex L by R, m by k, and $\varepsilon$ by $-\varepsilon$ in all formulas we deduced for the case of $\Psi_L$.

### 4.3 Correctness of definitions

First, to ensure that the function $\Psi_L$ is defined correctly we will show that $w_L$ is strictly monotone function.

**Lemma 4.1.** The equality $\text{sign} w_L' (\xi; s, \varepsilon) = \text{sign}(s - 2)$ holds for $\xi \geq \varepsilon$ and $s \in (1, +\infty)$.
Proof. Direct differentiation of \( w_L \) from (4.20) leads to
\[
w'_L = e^{-\frac{1}{2}x} \left( -\frac{1}{2x}((\xi + \varepsilon)^s - (\xi - \varepsilon)^s) + m_s(\xi - \varepsilon) + \frac{1}{2s}((\xi + \varepsilon)^s - (\xi - \varepsilon)^s) - \varepsilon m'_s(\xi - \varepsilon) \right).
\]
Applying relation (2.15), we get
\[
w'_L = \frac{1}{2}e^{-\frac{1}{2}x} \left( - (\xi + \varepsilon)^s + (\xi - \varepsilon)^s + s\varepsilon(\xi + \varepsilon)^s - 1 + s\varepsilon(\xi - \varepsilon)^s - 1 \right). \tag{4.63}
\]
We introduce the following function
\[
A(\alpha, \beta, s) = 2((\alpha - \beta)^s - (\alpha + \beta)^s + s\beta(\alpha + \beta)^s - 1 + s\beta(\alpha - \beta)^s - 1), \tag{4.64}
\]
which admits the integral representation
\[
A(\alpha, \beta, s) = s(s - 1)(s - 2) \int_{-\beta}^{\beta}(\beta^2 - \lambda^2)(\lambda + \alpha)^s - 3 d\lambda. \tag{4.65}
\]
The sign of \( w'_L \) is clear from the relation
\[
w'_L(\xi; s, \varepsilon) = \frac{1}{2}e^{-\frac{1}{2}x} A(\xi, \varepsilon, s). \tag{4.66}
\]

Lemma 4.2. The function \( w_R \) is \( C^1 \)-smooth and the equality \( \text{sign } w'_R(\xi; s, \varepsilon) = \text{sign } (s - 2) \) holds for \( \xi \geq 0 \) and \( s \in (1, +\infty) \).

Proof. By the direct differentiation of
\[
w_R = \begin{cases} \varepsilon e^{\frac{s}{2}x}(k_s(\xi + \varepsilon) - 2\xi(\xi + \varepsilon)^s - 2), & \text{if } 0 \leq \xi \leq \varepsilon, \\ e^{\frac{s}{2}x}(\frac{1}{2}((\xi - \varepsilon)^s - (\xi + \varepsilon)^s) + \varepsilon k_s(\xi + \varepsilon)), & \text{if } \xi \geq \varepsilon \end{cases}
\]
and the usage of (2.15), we get
\[
w'_R = \begin{cases} e^{\frac{s}{2}(s - 2)(\xi + \varepsilon)^s - 2\xi(\xi + \varepsilon)^s - 2), & \text{if } 0 \leq \xi \leq \varepsilon, \\ \frac{1}{2}e^{\frac{s}{2}x}(s\varepsilon(\xi - \varepsilon)^s - 1 + s\varepsilon(\xi + \varepsilon)^s - 1 + (\xi - \varepsilon)^s - (\xi + \varepsilon)^s), & \text{if } \xi \geq \varepsilon. \end{cases} \tag{4.67}
\]
Continuity of \( w'_R \) immediately follows from this formula, and \( w'_R(\xi; s, \varepsilon) = e(s - 2)2^{s-2}\varepsilon^{s-1} \). The statement \( \text{sign } w'_R = \text{sign } (s - 2) \) is clear directly from the first line of (4.67) if \( 0 \leq \xi \leq \varepsilon \). If \( \xi \geq \varepsilon \), then comparing (4.67) with (4.63) we note that
\[
w'_R(\xi) = e^{\frac{2s}{2}}w'_L(\xi), \quad \xi \geq \varepsilon. \tag{4.68}
\]
Applying Lemma 4.1 we finish the proof. \( \Box \)

4.4 Definitions. Summary
In this subsection we collect all formulas defining our Bellman function (we call it \( B_2 \)) together with the formulas for the Bellman function (we call it \( B_1 \)) constructed in [9].

First we recall notation used in the description of foliations.

- \( U(v) \) is the point at the skeleton with the first coordinate \( v \), i.e., \( U(v) = (v, v^2, |v|^p) \).
• $W_\pm(v)$ are the following points

$$W_+(v) = (v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2, v^p + \varepsilon m_p(v)), \quad v \geq 0,$$

$$W_-(v) = (v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2, v^p - \varepsilon k_p(v)), \quad v \geq \varepsilon,$$

and

$$W_+(v) = (v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2, |v|^p - \varepsilon k_p(|v|)), \quad v \leq -\varepsilon,$$

$$W_-(v) = (v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2, |v|^p + \varepsilon m_p(|v|)), \quad v \leq 0.$$

• The two-dimensional linearity domain $T(v)$ is the induced (in $\Omega^3_2$) convex hull of $U(v)$, $W_\pm(v)$, i.e., it is the curvilinear triangle being the intersection of the triangle whose vertices are $U(v)$, $W_\pm(v)$ with the domain $\Omega^3_2$.

• The two-dimensional linearity domain $\tilde{T}(v)$, $0 < v < \varepsilon$, is the induced (in $\Omega^3_2$) convex hull of the points $U(\pm v)$, $W_\pm(\pm v)$, i.e., it is the curvilinear trapezoid being the intersection of the usual trapezoid whose vertices are $U(\pm v)$, $W_\pm(\pm v)$ with the domain $\Omega^3_2$.

• The two-dimensional linearity domain $R(v)$ is the induced (in $\Omega^3_2$) convex hull of the points $U(0)$ and $U(\pm v)$.

• $H_\varepsilon(u) = (u, u^2 + \varepsilon^2, \frac{1}{2}((u + \varepsilon)^p + (u - \varepsilon)^p))$, for $u \geq \varepsilon$ (see (4.6)).

• $H_\varepsilon(u) = \begin{cases} (u, u^2 + \varepsilon^2, \frac{1}{2}((u + \varepsilon)^p + (u - \varepsilon)^p)) & \text{for } u \geq \varepsilon, \\ (u, u^2 + \varepsilon^2, (u + \varepsilon)^{p-2}(u^2 + \varepsilon^2)) & \text{for } 0 \leq u \leq \varepsilon \end{cases}$ (see (4.36)).

• The two-dimensional domain $F_\varepsilon(u)$ ($u \geq \varepsilon$) is the triangle with the vertices $U(v)$, $W_+(v)$, and $H_\varepsilon(u)$ foliated by a fan of extremals from the point $U(v)$, $v = u - \varepsilon$.

• The two-dimensional domain $F_\varepsilon(u)$ ($u \geq 0$) is the triangle with the vertices $U(v)$, $W_-(v)$, and $H_\varepsilon(u)$ foliated by a fan of extremals from the point $U(v)$, $v = u + \varepsilon$.

• $\Delta_\pm = \varepsilon \pm d, \quad d = \sqrt{x_1^2 + \varepsilon^2 - x_2^2}$.

### 4.4.1 Definition of $B_1$

**Domain $\Xi_o$**.

The domain

$$\Xi_o = \left\{ x \in \Omega^3_2 : |x_1| \leq 2\varepsilon, \quad x_2 \geq 4\varepsilon|x_1| - 3\varepsilon^2, \quad (p - 2)\left(x_3 - \varepsilon^p - \frac{x_2 - \varepsilon^2}{4\varepsilon}m_p(\varepsilon)\right) \geq 0 \right\},$$

is foliated by the two-dimensional linearity domains $\tilde{T}(v)$, $0 \leq v \leq \varepsilon$. The curvilinear trapezoid $\tilde{T}(v)$ with three straight line sides $[U(-v), U(v)]$, $[U(v), W_+(v)]$, and $[U(-v), W_-(v)]$ belongs to the plane

$$x_3 = v^p + \frac{m_p(v)}{2(v + \varepsilon)}(x_2 - v^2).$$

The function $B_1$ on $\tilde{T}(v)$ is defined by the formula

$$B_1(x) = v^p + \frac{m_p(v)}{2(v + \varepsilon)}(x_2 - v^2).$$
Domain $\Xi_+$. The domain

$$\Xi_+ = \left\{ x \in \Omega^2 \setminus \Xi_0 : x_1 \geq 0 \right\},$$

(4.73)
is foliated by the two-dimensional extremals $T(v)$, $v \geq \varepsilon$. The curvilinear triangle $T(v)$ is the intersection of the domain $\Omega^2$ with the plane

$$x_3 = v^p + \frac{m_p(v) - k_p(v)}{4\varepsilon}(x_2 - 2v x_1 + v^2) + \frac{m_p(v) + k_p(v)}{2}(x_1 - v).$$

(4.74)
The function $B_1$ on $T(v)$ is defined by the formula

$$B_1(x) = v^r + \frac{m_r(v) - k_r(v)}{4\varepsilon}(x_2 - 2v x_1 + v^2) + \frac{m_r(v) + k_r(v)}{2}(x_1 - v).$$

(4.75)

Domain $\Xi_-$. The domain

$$\Xi_- = \left\{ x \in \Omega^2 \setminus \Xi_0 : x_1 \leq 0 \right\},$$

(4.76)
is foliated by the two-dimensional extremals $T(v)$, $v \leq -\varepsilon$. The function $B_1$ on $\Xi_-$ is defined by the formula

$$B_1(x_1, x_2, x_3) = B_1(-x_1, x_2, x_3).$$

(4.77)

It is possible to join all the formulas above in the following way. Let us introduce a function $\xi \mapsto w_1(\xi; s, \varepsilon, x_1, x_2)$ by the formula

$$w_1(\xi; s, \varepsilon, x_1, x_2) = \begin{cases} 
\xi^s + \frac{m_s(\xi)}{2(\xi + \varepsilon)}(x_2 - \xi^2), & \text{if } 0 \leq \xi \leq \varepsilon, \\
\xi^s + \frac{m_s(\xi) - k_s(\xi)}{4\varepsilon}(x_2 - 2\xi x_1 + \xi^2) + \frac{m_s(\xi) + k_s(\xi)}{2}(x_1 - \xi), & \text{if } \xi \geq \varepsilon.
\end{cases}$$

(4.78)
The symbols $s, \varepsilon, x_1$, and $x_2$ are considered here as fixed parameters. Then we can rewrite relations (4.72), (4.75), and (4.77) in terms of $w_1$:

$$B_1(x_1, x_2, w_1(v; p, \varepsilon, |x_1|, x_2)) = w_1(v; r, \varepsilon, |x_1|, x_2),$$

(4.79)
in other words

$$B_1(x_1, x_2, x_3) = w_1(\xi_1^{-1}(x_3; p, \varepsilon, |x_1|, x_2); r, \varepsilon, |x_1|, x_2).$$

(4.80)

4.4.2 Definition of $B_2$

Domain $\Xi_{l+}$. The domain

$$\Xi_{l+} = \left\{ x \in \Omega_{l+}^2 : x_1 \geq \Delta_-, \right. \\
(2 - p)A_{mp}(x_1, x_2) \leq (2 - p)x_3 \leq (2 - p)\frac{\Delta_-(x_1 + \Delta_+)p + \Delta_+(x_1 - \Delta_-)p}{2\varepsilon}, \left. \right\},$$

(4.81)
is foliated by the fans $F_1(u)$, $\varepsilon \leq u < \infty$ (see Lemma 5.3 below). The function $B_2$ on $F_1(u)$ is defined by (4.26):

$$B_2(x) = v^r + \left[ m_r(v) + \frac{1}{\varepsilon}e^\frac{2\varepsilon}{\xi} w_1(\xi; r, \varepsilon) \right] \Delta_-, \quad (4.82)$$

where, $v = x_1 - \Delta_-$, $u = v + \varepsilon$, the function $\xi \mapsto w_1(\xi; s, \varepsilon)$ was defined by (4.20):

$$w_1(\xi; s, \varepsilon) = e^{-\xi}\left[ \frac{1}{2}((\xi + \varepsilon)^s - (\xi - \varepsilon)^s) - \varepsilon m_s(\xi - \varepsilon) \right], \quad \xi \in [\varepsilon, +\infty].$$
We would like to note that $\xi = +\infty$ is included in the domain of $w_L$ and $w_L(+\infty) = 0$. The value of the variable $\xi = \xi(x; p, \varepsilon)$ in (4.82) is obtained as the solution of the equation

$$x_3 = v^p + \left[m_p(v) + \frac{1}{\varepsilon} e^{\frac{v}{\varepsilon}} w_L(\xi; p, \varepsilon)\right] \Delta_-$$

running from $u = x_1 + d$ to $+\infty$.

We are ready to write down the function $B_2$ in the form (4.80), as it was made for the function $B_1$. From (4.82) and (4.83) we see that we should introduce the following function $w_2$:

$$w_2(\xi; s, \varepsilon, x_1, x_2) = v^s + \left[m_s(v) + \frac{1}{\varepsilon} e^{\frac{v}{\varepsilon}} w_L(\xi; s, \varepsilon)\right] \Delta_-, \quad u \leq \xi \leq \infty, \quad u = v + \varepsilon = x_1 + d.$$  (4.84)

Then, for $x \in \Xi_{\varepsilon+}$ we have

$$B_2(x_1, x_2, x_3) = w_2^{-1}(x_3; p, \varepsilon, x_1, x_2; r, \varepsilon, x_1, x_2).$$  (4.85)

**Domain $\Xi_{\varepsilon+}$**. The domain

$$\Xi_{\varepsilon+} = \left\{ x \in \Omega_{\varepsilon}^2 : x_1 \geq 0, \; x_2 \geq \varepsilon^2, \; (p - 2)A_{k_p}(x_1, x_2) \leq (p - 2)x_3 \right\}$$

is foliated by the fans $F_{\varepsilon}(u)$, $0 \leq u < \infty$ (see Lemma 5.3 below).

The function $B_2$ on $F_{\varepsilon}(u)$ is defined by the formula

$$B_2(x) = v^r - \left[k_r(v) - \frac{1}{\varepsilon} e^{-\frac{v}{\varepsilon}} w^r(\xi; r, \varepsilon)\right] \Delta_-,$$  (4.87)

where $v = x_1 + \Delta_-$, $u = v - \varepsilon$, the function $\xi \mapsto w^r(\xi; s, \varepsilon)$ was defined by (4.50):

$$w^r(\xi; s, \varepsilon) = \left\{ \begin{array}{ll} \varepsilon e^{\frac{v}{\varepsilon}} \left[k_s(\xi + \varepsilon) - 2s(\xi + \varepsilon)^{s-2}\right], & \text{if } 0 \leq \xi \leq \varepsilon, \\
\varepsilon \left[\left(\xi - \xi\right)^s - s(\xi + \varepsilon)^{s-2}\right] + \varepsilon k_s(\xi + \varepsilon), & \text{if } \xi \geq \varepsilon. \end{array} \right.$$  (4.88)

The value of the variable $\xi = \xi(x; p, \varepsilon)$ in (4.87) is the solution of the equation

$$x_3 = v^p - \left[k_p(v) - \frac{1}{\varepsilon} e^{-\frac{v}{\varepsilon}} w^r(\xi; p, \varepsilon)\right] \Delta_-$$  (4.89)

running from 0 till $u = x_1 - d$.

Comparing (4.87) and (4.89) we see that we should introduce the following function $w_2$:

$$w_2(\xi; s, \varepsilon, x_1, x_2) = v^s - \left[k_s(v) - \frac{1}{\varepsilon} e^{-\frac{v}{\varepsilon}} w^r(\xi; s, \varepsilon)\right] \Delta_-, \quad 0 \leq \xi \leq u, \quad u = v - \varepsilon = x_1 - d.$$  (4.90)

Recall that here $v = x_1 + \Delta_+$, not as in (4.84), where $v = x_1 - \Delta_-$. Again, for $x \in \Xi_{\varepsilon+}$ we have (4.85).

**Domain $F(0)$**. The domain

$$F(0) = \left\{ x \in \Omega_{\varepsilon}^3 : |x_1| \leq \varepsilon, \; \varepsilon |x_1| \leq x_2 \leq x_1^2 + \varepsilon^2, \; (2 - p)x_3 \leq (2 - p)(2\varepsilon)^{p-2} x_2 \right\}$$  (4.91)

is foliated by the two-dimensional domains of linearity. The function $B_2$ does not depend on $x_1$:

$$B_2(x_1, x_2, x_3) = B_2\left(\frac{x_2}{2\varepsilon}, x_2, x_3\right),$$  (4.92)
and the former point lies in $F_L(\varepsilon)$. We can use (4.17), (4.10), and (4.7) for $v = 0$ to rewrite the right-hand side of (4.92) to obtain
\begin{equation}
B_2(x) = \left[ e\Psi_L\left(\frac{2s^2x_3}{e\varepsilon_2} - \frac{e^p}{e}\Gamma(p + 1)\right) + \varepsilon^{-1}\Gamma(r + 1)\right]\frac{x_2}{2\varepsilon}, \quad x \in F(0).
\end{equation}

However, we would like to write down this formula in another form. We prefer to have a description on each leaf of the foliation of $F(0)$ separately. Now we have appropriate tools to describe the foliation of $F(0)$. The right boundary of $F(0)$ (that is the boundary between $F(0)$ and $\Xi_{L^+}$) consists of the fan $F_L(\varepsilon)$, and the symmetrical boundary (that is the boundary between $F(0)$ and $\Xi_{R^-}$) consists of the symmetrical fan $F_L(-\varepsilon)$. The extremals of a fan $F_L(u)$ are segments $[U(u - \varepsilon), H(u, h)]$ that are parametrized either by the parameter $h$, or by the corresponding parameter $\xi$ (see (4.22)). The parameter $h$ is running from zero (when the endpoint $H(u, h)$ is on the boundary) till $\frac{1}{2}((u + \varepsilon)^p - (u - \varepsilon)^p) - \varepsilon m_p(u - \varepsilon)$ (when $H(u, h)$ is at the middle of the chord $[U(u - \varepsilon), U(u + \varepsilon)]$). Therefore, the extremals of the fan $F_L(\varepsilon)$ are parametrized by the parameter $h$ running from zero till $\varepsilon^p(2p - 1 - \Gamma(p + 1))$. Each such extremal is a boundary of the two-dimensional domain of linearity $F_0(h)$ being the induced convex hull of the points $(0, 0, 0)$ and $(\pm \varepsilon, 2e^2, \varepsilon m_p(0) + h)$. The whole domain $F(0)$ is foliated by $F_0(h)$. The triangle $F_0(0)$ is on the boundary of $\Omega_2^3$ and the triangle $F_0((2p - 1 - \Gamma(p + 1))\varepsilon^p)$ separates $F(0)$ from $R$.

Let us compare now formula (4.10) with (4.83) taking into account that $\Delta_\perp = x_1 = \frac{2s}{2e}$ for $F_L(\varepsilon)$ (see (4.3)). We see that $h = e \cdot w_L(\xi; p, \varepsilon)$, and (4.82) supplies us with the formula for $B_2$ on $F_0(h)$:
\begin{equation}
B_2(x) = \left[ m_r(0) + \frac{e}{\varepsilon}w_L(\varepsilon, \xi; p, \varepsilon)\right]\frac{x_2}{2\varepsilon}.
\end{equation}

This formula coincides with (4.93), since $m_s(0) = \varepsilon^{-1}\Gamma(s + 1)$.

Instead of parameter $h$ we can parametrize the leaves of $F(0)$ by the parameter $\xi$ from (4.22) running from $\varepsilon$ to $+\infty$. Then
\begin{equation}
x_3 = \left[ m_p(0) + \frac{e}{\varepsilon}w_L(\xi; p, \varepsilon)\right]\frac{x_2}{2\varepsilon}
\end{equation}
and
\begin{equation}
B_2(x) = \left[ m_r(0) + \frac{e}{\varepsilon}w_L(\xi; r, \varepsilon)\right]\frac{x_2}{2\varepsilon}.
\end{equation}

Therefore, for this domain it is natural to introduce the function $w_2$ by the formula
\begin{equation}
w_2(\xi; s, \varepsilon, x_1, x_2) = \left[ \varepsilon^s\Gamma(s + 1) + e w_L(\xi; s, \varepsilon)\right]\frac{x_2}{2\varepsilon^2}, \quad \varepsilon \leqslant \xi \leqslant +\infty,
\end{equation}
and then, as before,
\begin{equation}
B_2(x_1, x_2, x_3) = w_2(w_2^{-1}(x_3; p, \varepsilon, |x_1|, x_2); r, \varepsilon, |x_1|, x_2).
\end{equation}

\textbf{Domain $R$}. The function $B_2$ has a very simple description in the domain $R$ (see (4.2)):
\begin{equation}
B_2(x_1, x_2, x_3) = \frac{x_3^{\frac{r-2}{2}}}{x_2^{\frac{p-2}{2}}}. \quad x \in R(v), \quad 0 \leqslant v \leqslant 2\varepsilon,
\end{equation}
However, if we would like to present $B_2$ in a form such as (4.97), we should use another representation from (4.2):
\begin{equation}
B_2(x_1, x_2, x_3) = v^{r-2}x_2, \quad x \in R(v), \quad 0 \leqslant v \leqslant 2\varepsilon,
\end{equation}
together with formula for the third coordinate
\begin{equation}
x_3 = v^{r-2}x_2, \quad x \in R(v), \quad 0 \leqslant v \leqslant 2\varepsilon.
\end{equation}
This gives us the same formula (4.97), if we take
\begin{equation}
w_2(\xi; s, \varepsilon, x_1, x_2) = v(\xi)^{s-2}x_2,
\end{equation}
with some monotone parametrization $v(\xi)$. In what follows we will use not the definition of the form (4.97), but the direct definition (4.98). It is the reason why we will not specify the function $v(\xi)$, this may be done in many different ways.
Domain $\Xi_{\alpha \beta}$. Finally, on the domain $\Xi_{\alpha \beta}$ we will use formula (4.1) for our Bellman candidate $B_2$ on a chord $[U(a), U(b)]$:

$$B_2(x) = \frac{x_1 - a}{b - a} b^r + \frac{b - x_1}{b - a} a^r, \quad (4.101)$$

where the parameters $a$ and $b$ are considered as functions of $x$ determined by the pair of equations

$$x_2 = (a + b)x_1 - ab \quad (4.102)$$

and

$$x_3 = \frac{x_1 - a}{b - a} b^p + \frac{b - x_1}{b - a} a^p. \quad (4.103)$$

We have used the fact that $x \in [U(a), U(b)]$.

Comparing (4.101) with (4.103), it is easy to represent the function $B_2$ in the same form (4.97). To do this it suffices to parametrize all chords passing through a point $x$ with fixed $x_1$ and $x_2$ by some parameter $\xi$ and consider some functions $\xi \rightarrow a(\xi; x_1, x_2)$ and $\xi \rightarrow b(\xi; x_1, x_2)$ rather than functions $a$ and $b$ dependent on $x$. After such a choice we can put

$$w_2(\xi; \varepsilon, x_1, x_2) = \frac{x_1 - a}{b - a} b^s + \frac{b - x_1}{b - a} a^s \quad (5.1)$$

to obtain the representation (4.97). This also may be done in many different ways. We will not use such a representation, since we have not found the choice of the parameter $\xi$ that essentially simplifies all calculations.

5 Foliation. Proofs

In this section we formally prove that the foliation corresponding to the candidate $B_2$ is precisely the foliation announced in Section [3]

Lemma 5.1. Fix a point $(x_1, x_2) \in \omega_2 \cup \omega_3 \cup \omega_4$ (see (3.6)). Consider all the pairs $a, b > 0$ such that the chord $[U(a), U(b)]$ contains the point of the form $(x_1, x_2, \cdot)$. Then, the third coordinate of this point, considered as a function of $a$, is strictly increasing for $p > 2$ and strictly decreasing for $p < 2$.

Proof. First we differentiate relation (4.102) with respect to $x_3$. As a result, we get

$$b_{x_3}(x_1 - a) = a_{x_3}(b - x_1). \quad (5.1)$$

Therefore, both endpoints of the chord $[U(a), U(b)]$ move in one and the same direction. Without lost of generality, we may assume $a < x_1 < b$.

Now, we calculate the derivative of the function

$$w(a, b; s) = \frac{(b - x_1)a^s + (x_1 - a)b^s}{b - a}. \quad (5.2)$$

We have

$$w_{x_3} = \frac{a_x b_{x_3} + (b - x_1) s a^{s-1} a_{x_3} - b^s a_{x_3} + (x_1 - a) s b^{s-1} b_{x_3}}{(b - a)^2}$$

$$= \frac{a_{x_3}(b - x_1) [s a^{s-1} (b - a) + a^s - b^s] + b_{x_3}(x_1 - a) [s b^{s-1} (b - a) + a^s - b^s]}{(b - a)^2}$$

$$\frac{a_{x_3}(b - x_1) [s(b - a)(a^{s-1} + b^{s-1}) + 2a^s - 2b^s]}{(b - a)^2} \quad \frac{a_{x_3}(b - x_1)}{(b - a)^2} A\left(\frac{1}{2}(b + a), \frac{1}{2}(b - a), s\right). \quad (5.3)$$

where the function $A$ was defined in (4.64). We see that sign $A(\alpha, \beta, s) = \text{sign}(s - 2)$ from the integral representation (4.65). Since $x_3 = w(a, b; p)$ (see (4.103)) and $x_1 < b$, we immediately conclude that sign $a_{x_3} = \text{sign}(p - 2)$. This gives us the conclusion of the lemma.

Lemma 5.2. The domain $R$ is foliated by leaves $R(v)$, $0 \leq v \leq 2\varepsilon$. If we keep the first two coordinates of a point $x \in R(v)$ fixed, we get a function $v \mapsto x_3$. This function $x_3(v)$ is strictly increasing for $p > 2$ and strictly decreasing for $p < 2$. 

25
Recall that the extremal lines in this fan are parametrized by \( x_3 = v^{p-2} x_2 \). From this formula the second statement of the Lemma is clear: the function \( x_3(v) \) is strictly increasing for \( p > 2 \) and strictly decreasing for \( p < 2 \).

To prove the first statement we need to consider four subdomains \( R_i \), separately (see (3.7)–(3.11)). The projection on the \( u \)-plane of the points from \( R_0 \) (see (3.8)) lie in \( \omega_0 \) (see (3.6)), see Fig. 7. Now, we will look at the domains \( R(v) \) when \( v \) increases from zero to \( 2\varepsilon \). For a fixed point \( (x_1, x_2) \in \omega_0 \) the first moment when the projection of \( R(v) \) to the first two coordinates contains the point \( (x_1, x_2) \) is \( v = \sqrt{2} \), and the point \( (x_1, x_2, x_3(v)) \) lies in \( R(v) \) till \( v = 2\varepsilon \). That means that \( x_3(v) \) continuously varies from \( x_2^p/2 \) till \( (2\varepsilon)^p/2 - x_2 \). This is just what is written in formula (3.8), where the factor \( (2 - p) \) reflects the fact that the function \( x_3(v) \) is increasing for \( p > 2 \) and decreasing for \( p < 2 \).

Now, we consider in a similar way the case when \( (x_1, x_2) \in \omega_1 \). The first moment when the projection of \( R(v) \) contains this point \( v = x_1 + \Delta_- \), i.e., when the point \( (x_1, x_2) \) is on the tangent line passing through \( (v, v^2) \). The point \( (x_1, x_2, x_3(v)) \) lies in \( R(v) \) till \( v = 2\varepsilon \). That means that \( x_3(v) \) continuously varies from \( x_1 + \Delta_- \) till \( (2\varepsilon)^p/2 \), as it is written in formula (3.9).

For the case when \( (x_1, x_2) \in \omega_2 \), the first moment when the projection of \( R(v) \) contains this point is the same as for \( \omega_0 \), i.e., \( v = \sqrt{2} \). But now, the point \((x_1, x_2, x_3(v)) \) lies in \( R(v) \) only till the moment when \( v = x_2/x_1 \). That means that \( x_3(v) \) continuously varies from \( x_2^p/2 \) till \( x_1^p x_2^p \), as it is written in formula (3.11).

Finally, for the case when \( (x_1, x_2) \in \omega_3 \), the first moment when projection of \( R(v) \) contains this point is the same as for \( \omega_1 \), i.e., \( v = x_1 + \Delta_- \). As in the preceding case, the point \( (x_1, x_2, x_3(v)) \) lies in \( R(v) \) till the moment when \( v = x_2/x_1 \). That means that \( x_3(v) \) continuously varies from \( x_1 + \Delta_- \) till \( x_1^p x_2^p/2 \), as it is written in formula (3.11).

**Lemma 5.3.** The domain \( \Xi_{L+} \) is foliated by the fans \( F_i(u) \), \( \varepsilon \leq u < \infty \).

**Proof.** First, we note that the fans \( F_i(u) \) do not intersec. Indeed, the projection of \( F_i(u) \) onto the \( x_1 x_2 \)-plane is the tangent \( S_+ \) \( u - \varepsilon \) and these lines do not intersect (see the beginning of Subsection 2.1 and Fig. 1 there).

The fact that \( F_i(u) \) foliate the whole domain \( \Xi_{L+} \) is almost evident. Indeed, for every point \( x \in \Xi_{L+} \) (see (3.12)) we have \( x_1 \geq \Delta_- \). This is equivalent to the assertion that a left tangent line \( S_+ \) passes through \( (x_1, x_2) \) with \( v = u + (x_1, x_2) \geq 0 \), see Fig. 1. Thus, we have to consider the fan \( F_i(u) \) with \( u = v + \varepsilon \). Recall that the extremal lines in this fan are parametrized by \( h \) (see (4.5)), which runs from zero till \( \frac{1}{2}((u + \varepsilon)^p - (u - \varepsilon)^p) - \varepsilon m_p(u + \varepsilon) \). This means that \( x_3 \) (see (4.11)) runs from \( A_{m_p}(x_1, x_2) \) till

\[
(u - \varepsilon)^p + \frac{(u + \varepsilon)^p - (u - \varepsilon)^p}{2\varepsilon} \Delta_- = \frac{(x_1 + \Delta_+)^p \Delta_- + (x_1 - \Delta_-)^p \Delta_+}{2\varepsilon},
\]

because \( u + \varepsilon = x_1 + \Delta_+ \) and \( u - \varepsilon = x_1 - \Delta_- \). These are the boundary values for \( x_3 \) described in (3.12).

**Lemma 5.4.** The domain \( \Xi_{R+} \) is foliated by the fans \( F_i(u) \), \( 0 \leq u < \infty \).

**Proof.** The reasoning here is the same as in the preceding lemma. The fans \( F_i(u) \) do not intersec because their projections onto the \( x_1 x_2 \)-plane are disjoint tangent lines \( S_-(u + \varepsilon) \). Thus, we have to check that for any \( x, x \in \Xi_{R+} \) (see (3.13)), there exists \( u \geq 0 \) such that \( x \in F_i(u) \).

First, we note that \( x_2 \geq \varepsilon^2 \) for \( x \in \Xi_{R+} \). This is equivalent to the assertion that some right tangent line \( S_-(v) \) passes through \( (x_1, x_2) \) with \( v \geq \varepsilon \). Thus, we have to consider the fan \( F_i(u) \) with \( u = v - \varepsilon \). Recall that the extremal lines in this fan are parametrized by \( h \) (see (4.34)) running from zero till \( \frac{1}{2}((u - \varepsilon)^p - (u + \varepsilon)^p) + \varepsilon k_p(u + \varepsilon) \) if \( u \geq \varepsilon \), and till \( \varepsilon k_p(u + \varepsilon) - 2\varepsilon u(u + \varepsilon)^p \) if \( 0 \leq u < \varepsilon \). This means that in the case \( u \geq \varepsilon \) (i.e., \( x_1 \geq \Delta_+) \) \( x_3 \) runs (see (4.40)) from \( A_p(x_1, x_2) \) till

\[
(u + \varepsilon)^p + \frac{(u - \varepsilon)^p - (u + \varepsilon)^p}{2\varepsilon} \Delta_- = \frac{(x_1 - \Delta_+)^p \Delta_- + (x_1 + \Delta_-)^p \Delta_+}{2\varepsilon},
\]

(5.4)
because \( u + \varepsilon = x_1 + \Delta_- \) and \( u - \varepsilon = x_1 - \Delta_- \). In the case \( 0 \leq u \leq \varepsilon \) the coordinate \( x_3 \) runs from \( A_k(x_1, x_2) \) till 
\[
(x_1 + \Delta_-)^{p-2}x_2,
\]
(5.5)
because \( v = x_1 + \Delta_- = u + \varepsilon \) and \( v^2 - 2u\Delta_- = x_2 \). The boundary values for \( x_3 \) described in (5.4) and (5.5) are given in (3.13).

**Lemma 5.5.** The domain \( \Xi_{ch}^+ \) is foliated by the chords \([U(a), U(b)], 0 \leq a < \infty, a \leq b \leq a + 2\varepsilon \).

**Proof.** The fact that the chords \([U(a), U(b)]\) are disjoint was proved in Lemma 5.1. Thus, we only have to verify that for any \( x, x \in \Xi_{ch}^+ \) (see (3.14)), there exist \( a \geq 0 \) and \( b \geq a \) such that \( x \in [U(a), U(b)] \).

If \((x_1, x_2) \in \omega_4\) (i.e., if \( x_1 \geq \Delta_+ \)), then \( x \in [U(a), U(b)] \) when \( a \) runs from \( x_1 - \Delta_+ \) till \( x_1 - \Delta_- \), and therefore, \( b \) runs from \( x_1 + \Delta_- \) till \( x_1 + \Delta_+ \). Due to formula (4.103), we see that \( x_3 \) runs from \( \Delta_-(x_1 - \Delta_+)^p + \Delta_+(x_1 + \Delta_-)^p) / (2\varepsilon) \) till \( \Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p) / (2\varepsilon) \). These are exactly the bounds for \( x_3 \) as stated in (3.14).

If \((x_1, x_2) \in \omega_2 \cup \omega_3\) (i.e., if \( \Delta_- \leq x_1 \leq \Delta_+ \)), then \( x \in [U(a), U(b)] \) when \( a \) runs from zero till \( x_1 - \Delta_- \), and therefore, \( b \) runs from \( x_2 / x_1 \) till \( x_1 + \Delta_+ \). Again formula (4.103) yields that \( x_3 \) runs from \( x_2^{-p-1} - x_1^{-p} \) till \( \Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p) / (2\varepsilon) \). These are the bounds for \( x_3 \) given by another part of (3.14).

Completing this section, we collect some information concerning the order of subdomains with different types of foliation over an arbitrary point \((x_1, x_2) \in \Omega^2\) and the formulas for the boundaries between different layers. This will be especially important when we glue the solutions in different subdomains and verify that the obtained candidate is \( C^1 \)-smooth.

For a fixed point \((x_1, x_2) \in \omega_j\) we list the domains and their boundaries over this point separated by \( | \) symbol. These objects are listed from up to down in the case \( p < 2 \) and in the reverse order in the case \( p > 2 \).

\[
(x_1, x_2) \in \omega_0: \quad x_3 = A_k(x_1, x_2) \quad | \quad R \quad | \quad x_3 = (2\varepsilon)^{p-2}x_2 \quad | \quad F(0) \quad | \quad x_3 = A_m(x_1, x_2); \quad (5.6)
\]

\[
(x_1, x_2) \in \omega_1: \quad x_3 = A_k(x_1, x_2) \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = (x_1 + \Delta_-)^{p-2}x_2 \quad | \quad R \quad | \quad x_3 = (2\varepsilon)^{p-2}x_2 \quad | \quad F(0) \quad | \quad x_3 = A_m(x_1, x_2); \quad (5.7)
\]

\[
(x_1, x_2) \in \omega_2: \quad x_3 = A_k(x_1, x_2) \quad | \quad R \quad | \quad x_3 = x_2^{-p-1}x_1^{-p} \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = A_m(x_1, x_2); \quad (5.8)
\]

\[
(x_1, x_2) \in \omega_3: \quad x_3 = A_k(x_1, x_2) \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = (x_1 + \Delta_-)^{p-2}x_2 \quad | \quad R \quad | \quad x_3 = x_2^{-p-1}x_1^{-p} \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = A_m(x_1, x_2); \quad (5.9)
\]

\[
(x_1, x_2) \in \omega_4: \quad x_3 = A_k(x_1, x_2) \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = \frac{\Delta_-(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p}{2\varepsilon} \quad | \quad \Xi_{ch}^+ \quad | \quad x_3 = A_m(x_1, x_2); \quad (5.10)
\]
6 Properties of the Bellman candidate $B_2$

In this section we formulate local concavity/convexity property of our candidate $B_2$ defined in Subsection 4.4.2. Our aim is to prove the following result.

**Theorem 6.1.** Let $B_2$ be the function defined in Subsection 4.4.2. It is $C^1$-smooth and locally concave if $(r - 2)(r - p) < 0$ and locally convex if $(r - 2)(r - p) > 0$.

The required calculations are rather long. By this reason we decided to move the proof of the theorem to Appendix. The $C^1$-smoothness of $B_2$ is proved in Appendix A and the concavity/convexity of $B_2$ is proved in Appendix B.

7 Optimizers

In this section we provide optimizers for $B_2$. Namely, for any point $x \in \Omega_3$ we find a function $\varphi_x$ (called optimizer for $B_2$ at $x$) that satisfies (2.19). The construction of optimizers depends on the foliation, therefore, we consequently investigate different subdomains of $\Omega_3$.

7.1 Optimizers for points in $\Xi_{L^+}$

We start with integration by parts and use the change of variable to prove the following identity:

\[ \varepsilon m_s(v) = \int_0^{+\infty} e^{\frac{\tau - \varepsilon t}{\varepsilon}} t^{v - 1} dt = -v^s + e^\varepsilon \int_0^\varepsilon (-\varepsilon \ln \tau)^s d\tau, \]

and use this identity to rewrite the function $w_L$ from (4.20):

\[ w_L(\xi; s, \varepsilon) = e^{-\frac{\xi + \varepsilon}{2}} \cdot \frac{(\xi + \varepsilon)^s + (\xi - \varepsilon)^s}{2} - \frac{e^{-\frac{\xi - \varepsilon}{\varepsilon}}}{\varepsilon} \int_0^\varepsilon (-\varepsilon \ln \tau)^s d\tau, \quad \xi \geq \varepsilon. \]

Now, using $u = v + \varepsilon$, the function $w_2$ defined in (4.84) can be rewritten in the following form:

\[ w_2(\xi; s, \varepsilon, x_1, x_2) = v^s + \Delta_- \left[ \varepsilon m_s(v) + e^\varepsilon w_L(\xi; s, \varepsilon) \right] \]

\[ = \frac{\varepsilon - \Delta_-}{\varepsilon} v^s + \Delta_- \left[ e^{\frac{\xi - \varepsilon}{\varepsilon}} \cdot \frac{(\xi + \varepsilon)^s + (\xi - \varepsilon)^s}{2} + e^\varepsilon \int_0^\varepsilon (-\varepsilon \ln \tau)^s d\tau \right] \]

\[ = \frac{1}{l} \int_0^l \varphi_x(\tau)^s d\tau, \]

where $l = \frac{\varepsilon - \Delta_-}{\Delta_-} e^\varepsilon$ and

\[ \varphi_x(\tau) = \begin{cases} 
\xi + \varepsilon, & 0 \leq \tau < \tau_1 \equiv \frac{1}{2} e^{-\frac{\xi - \varepsilon}{\varepsilon}}, \\
\xi - \varepsilon, & \tau_1 \leq \tau < \tau_2 \equiv e^{-\frac{\xi - \varepsilon}{\varepsilon}}, \\
-\varepsilon \ln \tau, & \tau_2 \leq \tau < \tau_3 \equiv e^{-\frac{\xi - \varepsilon}{\varepsilon}}, \\
v, & \tau_3 \leq \tau \leq \frac{\varepsilon}{\Delta_-} e^{-\frac{\xi - \varepsilon}{\varepsilon}} = l.
\end{cases} \]

Take $s = 1$. Then, we trivially have $m_1(v) = 1$, $w_L(\xi; 1, \varepsilon) = 0$, and

\[ \langle \varphi_x \rangle_{[0, l]} = w_2(\xi; 1, \varepsilon, x_1, x_2) = v + \Delta_- = x_1. \]
Take \( s = 2 \). We have \( m_2(v) = 2(v + \varepsilon) \), \( w_2(\xi; 2, \varepsilon) = 0 \), and
\[
\langle \varphi^2_x \rangle_{[0, t]} = w_2(\xi; 2, \varepsilon, x_1) = v^2 + 2(v + \varepsilon)\Delta \cdot x_2.
\]
According to \((4.83)\) and \((4.82)\) we have
\[
\langle \varphi^0_x \rangle_{[0, t]} = x_3, \quad \langle \varphi^1_x \rangle_{[0, t]} = B_2(x_1, x_2, x_3).
\]
Let us verify that the BMO-norm of \( \varphi_x \) does not exceed \( \varepsilon \). To do that we use the techniques of the so-called delivery curves, see Chapter 5 in \([3]\). Consider the two-dimensional curve
\[
\gamma(\tau) = \left(\langle \varphi_2 \rangle_{[0, \tau]}, \langle \varphi^1_x \rangle_{[0, \tau]}\right), \quad 0 < \tau < l.
\]
This curve is called the delivery curve generated by \( \varphi_x \). We see that this curve starts at the point \((\xi + \varepsilon, (\xi + \varepsilon)^2)\) on the lower boundary of \( \Omega^2_\varepsilon \) (when \( \tau \in (0, \tau_1) \)), then goes along the tangent line \( S_-(\xi + \varepsilon) \) and arrives at \((\xi, \xi^2 + \varepsilon^2)\) when \( \tau = \tau_2 \) (see Fig. 1 for the notation \( S_\pm \)). The point \( \gamma(\tau) \) goes along the upper boundary of \( \Omega^2_\varepsilon \) when \( \tau \in (\tau_2, \tau_3) \) and arrives at \((v + \varepsilon, (v + \varepsilon)^2 + \varepsilon^2)\) when \( \tau = \tau_3 \). Then it goes along the tangent line \( S_+(v) \) till the point \((x_1, x_2)\). We see that this curve is a graph of a convex function and that any tangent line to this curve does not cross the upper boundary of \( \Omega^2_\varepsilon \), only touches it. Therefore, by Corollary 5.1.6 from \([3]\) we have
\[
\|\varphi_x\|_{BMO} \leq \varepsilon.
\]
We summarize: for any \( x = (x_1, x_2, x_3) \in \Xi_{\varepsilon,l} \) we have constructed the desired optimizer \( \varphi_x \) for \( B_2 \) at \( x \) (see \((2.19)\)).

### 7.2 Optimizers for points in \( \Xi_{\varepsilon,l} \)

We again integrate by parts and use the change of variable to prove the following identity:
\[
\varepsilon k_s(v) = \int_{\varepsilon}^{u} e^{(t-u)/\varepsilon s - 1} dt = v^s - e^{\varepsilon e^{s} - e^{-\varepsilon s}} \left( e^{\varepsilon s} \int_{\varepsilon}^{u} (\varepsilon \ln \tau)^s d\tau \right),
\]
and use this relation to rewrite the function \( w_r \) from \((4.50)\). Consider first the case \( \xi \geq \varepsilon \). We obtain
\[
\begin{align*}
\int_{\varepsilon}^{u} e^{(t-u)/\varepsilon s - 1} dt &= \int_{\varepsilon}^{u} e^{(t-u)/\varepsilon s - 1} dt \\
&= \int_{\varepsilon}^{u} e^{(t-u)/\varepsilon s - 1} dt \\
&= \int_{\varepsilon}^{u} e^{-\varepsilon s - \varepsilon^s} d\tau \\
&= \int_{\varepsilon}^{u} (\varepsilon \ln \tau)^s d\tau.
\end{align*}
\]

By the two preceding formulas, we may represent the function \( w_2 \) defined in \((4.90)\) in the following form:
\[
\begin{align*}
w_2(\xi; s, \varepsilon, x_1, x_2) &= v^s - \Delta \varepsilon \left[ k_s(v) - e^{-\varepsilon s} w_r(\xi; s, \varepsilon) \right] \\
&= \varepsilon - \Delta \varepsilon v^s + \Delta \varepsilon \int_{\varepsilon}^{u} (\varepsilon \ln \tau)^s d\tau \\
&= \frac{\varepsilon - \Delta \varepsilon v^s + \Delta \varepsilon}{\varepsilon} \int_{\varepsilon}^{u} (\varepsilon \ln \tau)^s d\tau \\
&= \frac{1}{l} \int_{0}^{l} \varphi_x(\tau)^s d\tau,
\end{align*}
\]
where \( l = \frac{\varepsilon}{2} e^{\frac{\varepsilon}{2}} \) and

\[
\varphi_x (\tau) = \begin{cases} 
\xi - \varepsilon, & 0 \leq \tau < \tau_1 = \frac{1}{2} e^{\frac{\varepsilon}{2}}, \\
\xi + \varepsilon, & \tau_1 \leq \tau < \tau_2 = e^{\frac{\varepsilon}{2}}, \\
\varepsilon \ln \tau, & \tau_2 \leq \tau < \tau_3 = e^\varepsilon, \\
v, & \tau_3 \leq \tau \leq l \left( e^{\frac{\varepsilon}{2}} - 1 \right) = l.
\end{cases}
\]

Let us verify that the function \( \varphi_x \) is an optimizer for \( B_2 \) at \( x \). For \( s = 1 \) we have

\[ k_1 (v) = 1 - e^{1 - \frac{\varepsilon}{2}}; \quad w_1 (\xi; 1, \varepsilon) = e^\varepsilon (-\varepsilon + \varepsilon (1 - e^{1 - \frac{\varepsilon}{2}})) = -\varepsilon. \]

Therefore,

\[ \langle \varphi_x \rangle_{[0,1]} = w_2 (\xi; 1, \varepsilon, x_1, x_2) = v - \Delta_+ (1 - e^{1 - \frac{\varepsilon}{2}} + e^{\frac{\varepsilon}{2}}) = v - \Delta_+ = x_1. \]

For \( s = 2 \) we have

\[ k_2 (v) = 2 (v - \varepsilon); \quad w_2 (\xi; 2, \varepsilon) = e^\varepsilon (2 \varepsilon + e^\varepsilon) = 0, \]

therefore

\[ \langle \varphi_x \rangle_{[0,1]} = w_2 (\xi; 2, \varepsilon, x_1, x_2) = v^2 - 2 (v - \varepsilon) \Delta_+ = x_2. \]

Applying (4.89) and (4.87), we obtain

\[ \langle \varphi_x \rangle_{[0,1]} = w_2 (\xi; p, \varepsilon, x_1, x_2) = x_3, \quad \langle \varphi_x \rangle_{[0,1]} = w_2 (\xi; r, \varepsilon, x_1, x_2) = B_2 (x_1, x_2, x_3). \]

Thus, to prove that \( \varphi_x \) is an optimizer for \( B_2 \) at \( x \) it suffices to verify \( \| \varphi_x \|_{\text{BMO}} \leq \varepsilon \). We will again argue using delivery curves. Consider the two-dimensional curve \( \gamma \) defined by (7.1). In this case it starts at the point \( (\xi - \varepsilon, (\xi - \varepsilon)^2) \) on the lower boundary of \( \Omega_\varepsilon \) when \( \tau \in (0, \tau_1) \), then goes along the tangent line \( S_+ (\xi - \varepsilon) \) and arrives at \( (\xi, \varepsilon^2 + \varepsilon^2) \) when \( \tau = \tau_2 \). The point \( \gamma (\tau) \) goes along the upper boundary of \( \Omega_\varepsilon \) when \( \tau \in (\tau_2, \tau_3) \) and arrives at \( (v - \varepsilon, (v - \varepsilon)^2 + \varepsilon^2) \) when \( \tau = \tau_3 \). Then it follows the tangent line \( S_- (v) \) till the point \( (x_1, x_2) \). We see that this curve is a graph of a convex function and that any tangent line to this curve does not cross the upper boundary of \( \Omega_\varepsilon \) transversally. Therefore, by Corollary 5.1.6 from [3] we have

\[ \| \varphi_x \|_{\text{BMO}} \leq \varepsilon. \]

We now turn to the case \( 0 \leq \xi \leq \varepsilon \). By (7.2) we have

\[ w_2 (\xi; s, \varepsilon) \overset{(4.90)}{=} e^\varepsilon (\varepsilon k_s (\xi + \varepsilon) - 2 \xi \varepsilon (\xi + \varepsilon)^{s-2}) \]

\[ = e^\varepsilon ((\xi + \varepsilon)^s - 2 \xi \varepsilon (\xi + \varepsilon)^{s-2}) - e^s - e^{-1} \int_{\varepsilon}^{\xi} (\varepsilon \ln \tau)^s d\tau. \]

Thus, the function \( w_2 \) defined in (4.90) has the following representation (recall that here \( u = v - \varepsilon \)):

\[ w_2 (\xi; s, \varepsilon, x_1, x_2) = u^s - \frac{\Delta_-}{\varepsilon} \left[ \varepsilon k_s (u) - e^{-\varepsilon} w_1 (\xi; s, \varepsilon) \right] \]

\[ \overset{(7.2)}{=} \frac{\varepsilon - \Delta_-}{\varepsilon} u^s + \frac{\Delta_-}{\varepsilon} \left[ e^{\frac{\varepsilon}{2}} ((\xi + \varepsilon)^s - 2 \xi \varepsilon (\xi + \varepsilon)^{s-2}) + e^{-\frac{\varepsilon}{2}} \int_{\varepsilon}^{\xi} (\varepsilon \ln \tau)^s d\tau \right] \]

\[ = \frac{1}{l} \int_0^l |\varphi_x (\tau)|^s d\tau, \]
where \( l = \frac{-r}{\lambda} e^\frac{r}{\lambda} \) and

\[
\varphi_x(\tau) = \begin{cases} 
- (\xi + \varepsilon), & 0 \leq \tau < \tau_1 \overset{\text{def}}{=} \alpha_- e^\frac{r}{\lambda}, \\
0, & \tau_1 \leq \tau < \tau_2 \overset{\text{def}}{=} (1 - \alpha_+) e^\frac{r}{\lambda}, \\
\xi + \varepsilon, & \tau_2 \leq \tau < \tau_3 \overset{\text{def}}{=} e^\frac{r}{\lambda}, \\
\varepsilon \ln \tau, & \tau_3 \leq \tau < \tau_4 \overset{\text{def}}{=} e^\frac{r}{\lambda}, \\
v, & \tau_4 \leq \tau \leq \frac{\varepsilon}{\Delta} e^\frac{r}{\lambda} = l,
\end{cases}
\]

(7.4)

here \( \alpha_- \) and \( \alpha_+ \) are any non-negative numbers that satisfy \( \alpha_- + \alpha_+ = \frac{\varepsilon^2 + \varepsilon^2}{(\xi + \varepsilon)\lambda} \). As before, one can easily verify that for \( s = 2 \) we have \( w_2(\xi; 2, \varepsilon) = 0 \) and

\[
\langle \varphi^2_x \rangle_{[0, l]} = w_2(\xi; 2, \varepsilon, x_1, x_2) = v^2 - 2(v - \varepsilon)\Delta_+ = x_2.
\]

We also have (7.3). The function \( \varphi_x \) is no longer non-negative, therefore, \( \langle \varphi^2_x \rangle_{[0, l]} \neq \langle \langle \varphi^2_x \rangle \rangle_{[0, l]} \), thus we cannot argue as before to calculate the average \( \langle \varphi_x \rangle_{[0, l]} \). On the other hand, we can choose \( \alpha_- \) and \( \alpha_+ \) in such a way that \( \langle \varphi^2_x \rangle_{[0, l]} = x_1 \). Let us verify that we may take

\[
\alpha_- = \frac{\varepsilon^2 - \varepsilon \xi}{2(\xi + \varepsilon)^2}, \quad \alpha_+ = \frac{\varepsilon^2 + \varepsilon \xi + 2 \xi^2}{2(\xi + \varepsilon)^2}.
\]

Indeed,

\[
\int_0^l \varphi_\tau d\tau = (\alpha_- - \alpha_+) (\xi + \varepsilon) e^\frac{r}{\lambda} + \varepsilon \int_0^l \ln \tau d\tau + \varepsilon \frac{v e^\frac{r}{\lambda}}{\Delta_-} (\frac{\varepsilon}{\Delta_-} - 1)
\]

\[
= \xi e^\frac{r}{\lambda} + \varepsilon \frac{v e^\frac{r}{\lambda}}{\Delta_-} - \xi + \varepsilon \frac{\xi e^\frac{r}{\lambda} + e^\frac{r}{\lambda}}{\Delta_-} (v - \varepsilon) - \xi \frac{e^\frac{r}{\lambda}}{\Delta_-} (v - \varepsilon) - \xi e^\frac{r}{\lambda} \\
= e^\frac{r}{\lambda} (v - \varepsilon + \frac{v e}{\Delta_-} - v) = e^\frac{r}{\lambda} \frac{\varepsilon}{\Delta_-} (v - \Delta_-) = e^\frac{r}{\lambda} \frac{\varepsilon}{\Delta_-} x_1 = l x_1.
\]

It remains to prove that \( \| \varphi_x \|_{\text{BMO}} \leq \varepsilon \). Consider the two-dimensional curve \( \gamma \) defined by (7.1). In this case it starts at the point \( (-\xi + \varepsilon, (\xi + \varepsilon)^2) \) on the lower boundary of \( \Omega^2_\varepsilon \) (when \( \tau \in (0, \tau_1) \)), then goes in the direction of origin and arrives at \( \gamma(\tau_2) = \frac{\alpha_-}{1 - \alpha_+} (\xi + \varepsilon, (\xi + \varepsilon)^2) \) when \( \tau = \tau_3 \) and arrives at \( \gamma(\tau_3) = (\xi, \xi^2 + \varepsilon^2) \) when \( \tau = \tau_4 \). Finally, for \( \tau \in [\tau_4, l] \) the point \( \gamma(\tau) \) goes along the tangent line \( S_\tau(v) \) till the point \( (x_1, x_2) \). Figure 8 illustrates the curve \( \gamma \). This curve is a graph of a convex function and any tangent line to this curve does not cross the upper boundary of \( \Omega^2_\varepsilon \) transversally. Therefore, by Corollary 5.1.6 from [3] we obtain

\[
\| \varphi_x \|_{\text{BMO}} \leq \varepsilon.
\]

### 7.3 Description of optimizers for points in other domains

In this subsection we propose the optimizers at the points of the domains \( R, F(0), \) and \( \Xi_{a, b} \). We omit some details of the proof because we do not need these optimizers to prove that \( B_2 \) coincides with either \( B^+_{a, r, \xi} \) or \( B^-_{a, r, \xi} \) (depending on \( p \) and \( r \)).

For any \( x \in \Xi_{a, b} \) there is a chord of the form \([U(a), U(b)]\), \( 0 \leq a \leq b \leq a + 2\varepsilon \), that passes through \( x \) such that the function \( B_2 \) is linear on this chord. The function

\[
\varphi_x(\tau) = \begin{cases} 
b, & a \leq \tau < x_1, \\
a, & x_1 \leq \tau \leq b,
\end{cases}
\]

31
defined on an interval $[a, b]$ is an optimizer for $B_2$ at $x$. Moreover, any function with the same distribution function will also be an optimizer at $x$.

For any $x \in R$ there is $v \in [0, 2\varepsilon]$ such that $x \in R(v)$, which is the induced convex hull of three points on the skeleton: $U(v), U(-v)$, and $U(0) = 0$, see Subsection 3.2. The function $B_2$ is linear on $R(v)$, therefore, it is natural to find an optimizer that attains only three values: $-v, 0, v$. As $x \in R(v)$ lies inside the triangle with the vertices $U(v), U(-v)$, and $U(0) = 0$, there are coefficients $\alpha_-, \alpha_0, \alpha_+ \in [0, 1]$ such that

$$x = \alpha_- U(-v) + \alpha_0 U(0) + \alpha_+ U(v), \quad \alpha_- + \alpha_0 + \alpha_+ = 1,$$

namely, they are

$$\alpha_- = \frac{x_2 - vx_1}{2v^2}, \quad \alpha_+ = \frac{x_2 + vx_1}{2v^2}, \quad \alpha_0 = \frac{v^2 - x_2}{v^2}.$$

Define the function

$$\varphi_x(\tau) = \begin{cases} -v, & 0 \leq \tau < \alpha_-, \\ 0, & \alpha_- \leq \tau < \alpha_- + \alpha_0 = 1 - \alpha_+, \\ v, & 1 - \alpha_+ \leq \tau \leq 1. \end{cases}$$

It is obvious that $\langle \varphi_x \rangle_{[0,1]} = x_1$, $\langle \varphi_x^2 \rangle_{[0,1]} = x_2$, and $\langle |\varphi_x|^p \rangle_{[0,1]} = x_3$. The identity $\langle |\varphi_x|^r \rangle_{[0,1]} = B_2(x)$ follows from the linearity of $B_2$ on $R(v)$. It is easy to verify that $\varphi_x$ has BMO-norm not greater than $\varepsilon$, therefore, the function $\varphi_x$ constructed above is an optimizer for $B_2$ at $x$.

The domain $F(0)$ is foliated by the two-dimensional domains of linearity $F_0(h)$, each such domain is the curvilinear triangle with the vertices $U(0) = (0, 0, 0)$ and $F_{\pm}(h) = (\pm \varepsilon, 2\varepsilon^2, \varepsilon m_p(0) + h)$. The last two vertices, $F_{\pm}(h)$, lie on the boundary of the domains $\Xi_{+}^L$ and $\Xi_{-}^{R}$, we already know the optimizers for $B_2$ at these points. The optimizer at $x$ can be constructed by gluing those two optimizers together with the zero function (optimizer at the origin) in appropriate proportions. Namely, let $\alpha_0, \alpha_{\pm} \in [0,1]$ be coefficients such that

$$x = \alpha_- F_-(h) + \alpha_0 U(0) + \alpha_+ F_+(h), \quad \alpha_- + \alpha_0 + \alpha_+ = 1.$$
Let $\varphi_{\pm}$ be optimizers for $B_2$ at the points $F_\pm(h)$ such that $\varphi_+$ is non-decreasing and non-negative, $\varphi_-$ is non-decreasing and non-positive, both defined on $[0, 1]$. Then the function

$$\varphi_x(\tau) = \begin{cases} 
\varphi_-(\frac{\tau}{\alpha_-}), & 0 \leq \tau < \alpha_-; \\
0, & \alpha_- \leq \tau < \alpha_- + \alpha_0; \\
\varphi_+(\frac{\tau - \alpha_- - \alpha_0}{\alpha_0}), & \alpha_- + \alpha_0 \leq \tau \leq 1
\end{cases} \quad (7.5)$$

is an optimizer for $B_2$ at $x$. The proof of the fact that BMO-norm of $\varphi_x$ does not exceed $\varepsilon$ is more cumbersome in this case. The arguments repeat the proof of Proposition 5.2.9 in [3].

8 Proof of Theorem 2.1

To prove Theorem 2.1 we only need to deal with the function $B_2$, because the same theorem concerning $B_1$ was proved in [9].

Consider the case $(r - 2)(r - p) < 0$. By Proposition 2.3 the function $B^+_{p,r;\varepsilon}$ is the pointwise minimal locally concave function on $\Omega^3_\varepsilon$ satisfying boundary condition (2.18) on the skeleton, whereas, by Theorem 6.1 $B_2$ is locally concave there, thus

$$B_2 \geq B^+_{p,r;\varepsilon} \quad \text{on } \Omega^3_\varepsilon.$$

Let us now prove the reverse inequality. The whole domain $\Omega^3_\varepsilon$ is the union of $\Xi_{\varepsilon L}, \Xi_{\varepsilon L}, \Xi_{\varepsilon h}, F(0)$, and $R$. In Section 7 we have constructed the optimizer $\varphi_x$ for each point $x \in \Xi_{\varepsilon L} \cup \Xi_{\varepsilon L} \cup \Xi_{\varepsilon h}$, therefore, by the definition of $B^+_{p,r;\varepsilon}$, we have

$$B^+_{p,r;\varepsilon}(x) \geq \langle \varphi_x \rangle = B_2(x).$$

The domain $R$ is foliated by $R(v), v \in (0, 2\varepsilon)$. Each leaf $R(v)$ is a curvilinear triangle, its vertices lie on the skeleton, therefore, $B_2$ and $B^+_{p,r;\varepsilon}$ coincide at the vertices. The function $B_2$ is linear on $R(v)$ whereas $B^+_{p,r;\varepsilon}$ is locally concave there, thus $B^+_{p,r;\varepsilon} \geq B_2$ on $R(v)$.

The same argument works for the domain $F(0)$. It is foliated by $F_0(h)$. Each $F_0(h)$ is a curvilinear triangle. Two of its vertices lie in $\Xi_{\varepsilon L} \cup \Xi_{\varepsilon L}$, and the third one is the origin. Therefore, $B_2$ and $B^+_{p,r;\varepsilon}$ coincide at the vertices. The function $B_2$ is linear on $F_0(h)$ whereas $B^+_{p,r;\varepsilon}$ is locally concave there, thus $B^+_{p,r;\varepsilon} \geq B_2$ on $F_0(h)$.

The case $(r - 2)(r - p) > 0$ is completely symmetric, all the inequalities change to the opposite, and concavity changes to convexity. Theorem 2.1 is proved.

9 Computation of the constant

Recall that we investigate the optimal constant $C = C(p, r)$ in the inequality

$$\|\varphi\|_{L^p(I)} \leq C(p, r)\|\varphi\|_{L^p(I)}^\frac{p}{2}\|\varphi\|_{BMO(I)}^{1 - \frac{p}{2}}, \quad \varphi \in BMO, \quad \langle \varphi \rangle_I = 0. \quad (9.1)$$

In this section, we do not provide the explicit formula for the constant $C(r, p)$. However, we show that for given values $r$ and $p$ it can be computed as a unique maximum point of a known function of a single variable.

Without loss of generality we may assume that $\|\varphi\|_{BMO} = 1$, i.e., set $\varepsilon = 1$ throughout this section, and rewrite the inequality above in the form

$$\int I |\varphi|^r \leq C^r(p, r) \int I |\varphi|^p. \quad (9.2)$$

From the definition of the Bellman function (see (2.1)), we get

$$C^r(p, r) = \sup_{x_2, x_3} \frac{B^+_{p,r;1}(0, x_2, x_3)}{x_3}. \quad (9.3)$$
In this section we assume that \( p > 1 \). We postpone the investigation of the case \( p = 1 \) till Section 11. Till the end of this section we skip indexes \( p, r, 1 \) and symbol + in the notation \( \mathcal{B}_{p,r,1}^+ \), i.e., we set \( \mathcal{B} = \mathcal{B}_{p,r,1}^+ \). We have to consider the points \( x \) from the domains \( \mathcal{F}(0) \) and \( \mathcal{R}(v) \). We will show that supremum in (9.3) is attained at the points \( (0, x_3) \in \mathcal{F}(0) \).

First, we consider \( x \in \mathcal{R} \) and show that the maximum of \( \mathcal{B}/x_3 \) is attained at the boundary with \( \mathcal{F}(0) \), i.e., at \( x_3 = 2^{r-2}x_2 \), see (3.8). Indeed, considering \( x_3 \geq 2^{r-2}x_2 \), from (4.98) we have that

\[
\frac{B}{x_3} = x_3^{\frac{r-2}{x_2^{r-2}}} = \left( \frac{x_2}{x_3} \right)^{\frac{r-2}{x_2^{r-2}}} \leq 2^{r-p}.
\]

Second, we consider \( x \in \mathcal{F}(0) \). From (4.94) and (4.95) we see that the ratio \( \mathcal{B}/x_3 \) does not depend on the variable \( x_2 \) and therefore, we may set \( x_2 = 1 \). Recall that the function \( \mathcal{B}(0, 1, \cdot) \) is concave on the segment

\[
J = \left[ 2^{r-2}, \Lambda, (0, 1) \right] \quad (9.5)
\]

Consider the value

\[
x_3^2 \left( \frac{B}{x_3} \right)_{x_3} = x_3 B_{x_3} - B. \quad (9.4)
\]

Clearly, this function is decreasing, since its derivative is equal to \( x_3 B_{x_3} < 0 \). Now, we show that the function from (9.4) attains the values of different signs at the endpoints of \( J \), and thereby, it has the unique zero value inside \( J \).

It is convenient to use the parameter \( \xi \) instead of parameter \( x_3 \). Recall that \( \xi \) varies from 1 to \(+\infty\), see (4.94) and (4.95). In the domain \( \mathcal{F}(0) \), using (B.1) from Appendix B and (4.65), we get

\[
B_{x_3} = \frac{r(r-1)(r-2)}{p(p-1)(p-2)} \int_{-1}^{1} (1 - \lambda^2)(\lambda + \xi)^{r-3}d\lambda.
\]

Therefore, since \( 1 < p < r < 2 \), for \( \xi = +\infty \), we deduce that \( B_{x_3} = +\infty \). From (4.94) and (4.95), using \( u_\xi(+\infty) = 0 \), for \( \xi = +\infty \) we obtain

\[
\frac{B}{x_3} = \frac{\Gamma(r+1)}{\Gamma(p+1)}, \quad (9.6)
\]

whence, we obtain \( B/x_3 < B_{x_3} \) at the endpoint of \( J \) corresponding to \( \xi = +\infty \).

At the other endpoint \( \xi = 1 \), the required quantities are computed in Appendix A

\[
\frac{B_{x_3}}{x_3} \quad \frac{B}{x_3} \quad 2^{r-p} 2^{r-p} 2^{r-p}.
\]

Therefore, we have the opposite inequality \( B_{x_3} < \frac{B}{x_3} \).

Hence, we finished the proof that supremum of \( B/x_3 \) is attained at the unique interior point of the interval \( J \), where the derivative of \( B/x_3 \) with respect to \( x_3 \) vanishes.

Thus, the sharp constant \( C(p, r) \) can be calculated as follows: \( C(p, r)' = \frac{B_{x_3}(0, 1, x_3)}{x_3} \), where

\[
x_3 \quad 4.93 \quad \frac{1}{2} \left[ \Gamma(p+1) + Cw_\xi(\xi; p, 1) \right]
\]

and \( \xi \) is the solution of the equation \( x_3 B_{x_3} = B \), which can be rewritten as

\[
\frac{r(r-1)(r-2)}{p(p-1)(p-2)} \int_{-1}^{1} (1 - \lambda^2)(\lambda + \xi)^{r-3}d\lambda = \frac{\Gamma(r+1) + Cw_\xi(\xi; r, 1)}{\Gamma(p+1) + Cw_\xi(\xi; p, 1)}. \quad (9.7)
\]
10 On the circle and on the line: proofs of Theorems 1.3 and 1.4

It is easy to prove inequalities (1.5) and (1.6) having Theorem 1.1 at hand, see Subsection 6.1 of [9] for details. It is much harder to prove the sharpness of these inequalities. Fortunately, we can directly apply lemmas from Subsection 6.2 of [9] to our problem. Namely, let \( \phi_0 \) be an optimizer for \( B_2 \) at the point \((0, 1, x_3)\), where supremum in (1.3) is attained (this function \( \phi_0 \) can be found explicitly, see (7.5)). We use Lemma 6.3 of [9] and for any \( \delta > 0 \) find a 1-periodic function \( \psi_0 \) on \( \mathbb{R} \) (i.e., a function on \( \mathbb{T} \)) such that

\[
\langle \psi_0 \rangle_{\{0,1\}} = 0, \quad \langle \psi_0^2 \rangle_{\{0,1\}} = 1,
\]

\[
\langle |\psi_0|^p \rangle_{\{0,1\}} = \langle |\phi_0|^p \rangle_{\{0,1\}} + O(\delta), \quad \delta \to 0+,
\]

\[
\langle |\psi|^p \rangle_{\{0,1\}} = \langle |\phi|^p \rangle_{\{0,1\}} + O(\delta), \quad \delta \to 0+,
\]

\[
\|\psi_0\|_{\text{BMO}(\mathbb{T})} \leq 1 + \delta.
\]

This means that the constant in (1.5) is attained on a sequence of functions \( \psi_0 \) when \( \delta \) tends to zero. This proves Theorem 1.3.

We use Lemma 6.4 of [9] to prove the sharpness of (1.6) (and thereby Theorem 1.4). For any \( \delta > 0 \) we find a function \( \psi \) on \( \mathbb{R} \) such that

\[
\psi = 0 \quad \text{on} \quad \mathbb{R} \setminus [0, 1],
\]

\[
\langle |\psi|^p \rangle_{\{0,1\}} = \langle |\psi_0|^p \rangle_{\{0,1\}} = \langle |\phi_0|^p \rangle_{\{0,1\}} + O(\delta), \quad \delta \to 0+,
\]

\[
\langle |\psi|^r \rangle_{\{0,1\}} = \langle |\psi_0|^r \rangle_{\{0,1\}} = \langle |\phi_0|^r \rangle_{\{0,1\}} + O(\delta), \quad \delta \to 0+,
\]

\[
\|\psi_0\|_{\text{BMO}(\mathbb{R})} \leq 1 + \delta.
\]

We conclude that the constant in (1.6) is attained on a sequence of functions \( \psi \) when \( \delta \) goes to zero. This proves Theorem 1.4.

11 Special case: \( p = 1 \)

We will prove that the Bellman function \( B_{1,r;\varepsilon}^+ \) is the limit of the functions \( B_{p,r;\varepsilon}^+ \) when \( p \to 1+ \). The limit function coincides with \( B_2 \) described in Subsection 4.4.2.

First, we note that the domain \( \Omega_\varepsilon^4 = \Omega_\varepsilon^{3,p} \) for \( p = 1 \) is the limit of the corresponding domains in a natural sense: for any point \((x_1, x_2) \in \Omega_\varepsilon^4 \) the values \( B_{1,\varepsilon}^+(x_1, x_2) \) that define the upper and the lower boundaries of \( \Omega_{\varepsilon;1}^4 \) are the limits of \( B_{p,\varepsilon}^+(x_1, x_2) \), when \( p \to 1+ \), see Subsection 2.1. For \( p = 1 \) we have \( m_1(u) = 1 \) for \( u \geq 0 \), therefore, the function \( A_{m_1}(x_1, x_2) \) has a very simple description:

\[
A_{m_1}(x_1, x_2) = \begin{cases} [x_1], & (x_1, x_2) \in \Omega_\varepsilon^2 \setminus (\omega_0 \cup \omega_{\pm 1}) \\
\{x_2\}, & (x_1, x_2) \in \omega_0 \cup \omega_{\pm 1}. \end{cases} \tag{11.1}
\]

Second, the domains \( \Xi_{1,\varepsilon}, \Xi_{\varepsilon;\varepsilon} \), and \( F(0) \) do not exist for \( p = 1 \) (see (3.12), (3.14), and (3.5)). This happens because all the chords of the form \([U(a), U(b)]\), \( 0 \leq a \leq b \leq a + 2\varepsilon \) lie on the lower boundary of \( \Omega_{\varepsilon;1}^4 \). We conclude that the whole domain \( \Omega_{\varepsilon;1}^5 \) is the union of \( \Xi_{1,\varepsilon} \), its symmetric domain \( \Xi_{1,\varepsilon} \), and the domain \( R \) foliated by the two-dimensional leaves \( R(v), v \in [0, 2\varepsilon] \). We state that the function \( B_2 \) defined on these domains in Subsection 4.4.2 for \( p = 1 \) coincides with \( B_{1,\varepsilon}^+ \).

Note that \( w_2(\cdot, p, \varepsilon, x_1, x_2) \) defined in (4.9) converges to \( w_2(\cdot, 1, \varepsilon, x_1, x_2) \) uniformly in \( \xi \in [0, u] \) as \( p \to 1+ \) when \( \varepsilon \) and \((x_1, x_2)\) are fixed. The limit function \( w_2(\cdot, 1, \varepsilon, x_1, x_2) \) is continuous and strictly increasing. Therefore, for every \( x_3 \in (w_2(0, 1, \varepsilon, x_1, x_2), w_2(0, 1, \varepsilon, x_1, x_2)) \) we have the convergence

\[
w_2(w_2^{-1}(x_3; p, \varepsilon, x_1, x_2); r; \varepsilon, x_1, x_2) \to w_2(w_2^{-1}(x_3; 1, \varepsilon, x_1, x_2); r; \varepsilon, x_1, x_2), \quad p \to 1+.
\]

Thus, in the interior of \( \Xi_{1,\varepsilon} \) we have the pointwise convergence of the functions \( B_2 \) to the limit function when \( p \to 1+ \). Convergence on \( R \) is obvious. As a result, we get convergence on the interior of \( \Omega_{\varepsilon;1}^5 \). All
these functions $B_2$ are locally concave for $p > 1$, therefore, the limit function is also locally concave on
the interior of $\Omega^3_{\xi;1}$. Since it is also continuous on $\Omega^3_{\xi;1}$, it is locally concave on the entire $\Omega^3_{\xi;1}$, therefore,
due to item 3) of Proposition 2.3

$$B_2(x) \geq B^+_{1,r;\xi}(x), \quad x \in \Omega^3_{\xi;1}.$$  

For each point $x \in \Xi_{\mathbb{R}^+}$ one can construct an optimizer in the same way as it was done in (7.2). The
only difference is that for $p = 1$ the variable $\xi$ runs from 0 to $\min(u, \varepsilon)$, and by this reason we only use
optimizers given by (7.4). Therefore,

$$B_2(x) \leq B^+_{1,r;\xi}(x), \quad x \in \Xi_{\mathbb{R}^+}.$$  

In the domain $R$ we use the same reasoning as before: the function $B_2$ is linear on $R(v)$ and coincides
with the locally concave function $B^+_{1,r;\xi}$ at the vertices of $R(v)$, therefore, $B_2 \leq B^+_{1,r;\xi}$ on $R(v)$.

Thus, we have proved that $B_2 = B^+_{1,r;\xi}$ everywhere on $\Omega^3_{\xi;1}$. As a consequence, we have that
the statements of Theorems [1.1] [1.3] [1.4] [1.5] and [1.7] and Corollaries [1.6] [1.8] hold for $p = 1$.

Finally, we compute the sharp constant in the inequality (1.1) for $p = 1$. Repeating the argument
from Section 9 we arrive at

$$C'(1, r) = \sup_{x_2, x_3} \frac{B^+_{1,r;\xi}(0, x_2, x_3)}{x_3}.  \tag{11.2}$$  

Since the domain $F(0)$ does not exist for the case $p = 1$, it suffices to find the maximum of the function
$B^+_{1,r;\xi}$ in the domain $R$. Recall that in this domain our function is defined by a simple formula (4.98),
whence

$$\frac{B^+_{1,r;\xi}(0, x_2, x_3)}{x_3} = \left(\frac{x_2}{x_3}\right)^{-1}.  \tag{11.3}$$  

From (3.8) we have $x_3 > \frac{1}{2}x_2$ and the maximum value is attained on the lower boundary, i.e., $x_3 = \frac{1}{2}x_2$,
which implies

$$C(1, r) = 2^{1-\frac{1}{r}}.$$  

Thus, we have proved the formula stated in Remark 1.2 for the case $1 < r \leq 2$. For $r \geq 2$ it was proved
in [8], see (1.2).

12 Comments on multidimensional case

The proof of Theorem 1.7 follows the reasoning from [7]. Following the arguments of Lemma 5.2 there
it is possible to prove the following assertion. If $G$ is a non-negative locally concave function on $\Omega^3_p$ and
$\phi \in \text{BMO}(\mathbb{R}^d)$, $\|\phi\|_\kappa < \varepsilon$, then

$$G(\phi_\kappa(y, t), (\phi^2)_\kappa(y, t), (|\phi|^p)_\kappa(y, t)) \geq G(\phi, \phi^2, |\phi|^p)_\kappa(y, t), \quad y \in \mathbb{R}^d, \quad t > 0.$$  

Fix a point $(y, t) \in \mathbb{R}^d \times \mathbb{R}^+$ and use this inequality for the function $G = B^+_{p,r;\xi}$ and for $\phi = \varphi - \varphi_\kappa(y, t)$.
Then we obtain

$$B^+_{p,r;\xi}(0, ((\varphi - \varphi_\kappa(y, t))^2)_\kappa(y, t), (|\varphi - \varphi_\kappa(y, t)|^p)_\kappa(y, t)) \geq (|\varphi - \varphi_\kappa(y, t)|^r)_\kappa(y, t).  \tag{12.1}$$

Now, we use the definitions of the Bellman function $B^+_{p,r;\xi}$ and the constant $C(p, r)$ in the corresponding
inequality. We estimate the left-hand side of (12.1):

$$B^+_{p,r;\xi}(0, ((\varphi - \varphi_\kappa(y, t))^2)_\kappa(y, t), (|\varphi - \varphi_\kappa(y, t)|^p)_\kappa(y, t)) \leq C(p, r)^r \varepsilon^{-p} (|\varphi - \varphi_\kappa(y, t)|^r)_\kappa(y, t).$$  

Therefore, we obtain

$$\varepsilon^{-p}C(p, r)^r (|\varphi - \varphi_\kappa(y, t)|^r)_\kappa(y, t) \geq (|\varphi - \varphi_\kappa(y, t)|^r)_\kappa(y, t).$$  

36
Rewrite the last inequality in the form
\[ \varepsilon^{r-p} C(p, r)^r \int_{\mathbb{R}^d} |\varphi(\tilde{y}) - \varphi_K(y, t)|^p K_t(y - \tilde{y})d\tilde{y} \geq \int_{\mathbb{R}^d} |\varphi(\tilde{y}) - \varphi_K(y, t)|^r K_t(y - \tilde{y})d\tilde{y}. \]

At this moment we recall that \( K_t \) coincides either with \( P \) or \( H \). For each of these kernels there exists the function
\[ Q(t) = \begin{cases} \frac{x^{d+1}}{\Gamma\left( \frac{d+2}{2} \right)} t^d, & K = P, \\ (4\pi t)^{\frac{d}{2}}, & K = H, \end{cases} \]
such that \( Q(t)K_t(y) \to 1 \) for any \( y \in \mathbb{R}^d \) when \( t \to +\infty \). Moreover, this convergence is monotone and uniform on compact subsets of \( \mathbb{R}^d \). We also note that since the function \( \varphi \) lies in \( L^p(\mathbb{R}^d) \), we have \( \varphi_K(y, t) \to 0 \) for any \( y \in \mathbb{R}^d \) when \( t \to +\infty \). Multiplying both sides of (12.2) by \( Q(t) \) and passing \( t \to +\infty \), we finally get
\[ \varepsilon^{r-p} C(p, r)^r \int_{\mathbb{R}^d} |\varphi(\tilde{y})|^p d\tilde{y} \geq \int_{\mathbb{R}^d} |\varphi(\tilde{y})|^r d\tilde{y}. \]

Theorem 1.7 is proved.

### Appendix A  Smoothness of the Bellman function \( B_2 \)

In this appendix we prove \( C^1 \)-smoothness of \( B_2 \). Since we will deal only with this candidate we omit index 2. Then it will be convenient to denote the partial derivatives of \( B \) placing the corresponding variable in index: \( B_{x_1} = \frac{\partial B}{\partial x_1} \).

**Lemma A.1.** The function \( B \) is \( C^1 \)-smooth.

**Proof.** By the definition, it is clear that \( B \) is \( C^2 \)-smooth on each subdomain. Therefore, we need to check that this function is \( C^1 \)-smooth on the borders between subdomains. Since all borders consists of extremals, \( B \) is linear in corresponding direction. Therefore, it suffices to verify continuity of the derivatives in any two transversal directions. We will check continuity of \( B_{x_1} \) and \( B_{x_2} \). The only exception is the boundary between \( \Xi_{x_1} \) and \( F(0) \) that goes along \( x_2 = 2\varepsilon x_1 \). In this case we check continuity of \( B_{x_1} \) instead of \( B_{x_2} \).

Let us start with the domain \( \Xi_{x_1} \). On the boundary \( x_2 = 2\varepsilon x_1 \) (or more exactly \( x_1 = \Delta_+ \)) the function \( B \) is continuous by definition: we have defined \( B \) on \( F(0) \) using this continuity.

**Boundary \( \Xi_{x_1} \mid F(0) \).** To check \( C^1 \)-smoothness, we calculate the derivative \( B_{x_1} \) and show that it is zero on the boundary \( x_1 = \Delta_+ \). It is what we need because the function \( B \) does not depend on \( x_1 \) on the domain \( F(0) \). We use the representation of \( B \) in it initial form (1.7):
\[ B_{x_1} = rv^{r-1} \frac{\partial v}{\partial x_1} + \left( \frac{\partial K_L}{\partial u} \frac{\partial u}{\partial x_1} + \frac{\partial K_L}{\partial h} \frac{\partial h}{\partial x_1} \right) (x_1 - v) + K_L \left( 1 - \frac{\partial v}{\partial x_1} \right), \]
where \( v = u - \varepsilon = x_1 - \Delta_+ \), therefore,
\[ \frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} = \frac{d + x_1}{d}, \]
and \( v = 0 \) on the boundary. To find the derivative \( \frac{\partial h}{\partial x_1} \) we use formula (4.10) and the relation \( \varepsilon m_p'(v) = m_p(v) - pv^{p-1} \) (see (2.15)). As a result, we get
\[ \frac{\partial h}{\partial x_1} \bigg|_{x_1 = \Delta_-} = \frac{h}{d}. \]

From (4.15) we see that
\[ h \frac{\partial K_L}{\partial h} \bigg|_{x_1 = \Delta_-} = K_L - \varepsilon \frac{\partial K_L}{\partial u}. \]
Gathering all these formulas, we obtain

\[ B_{x_1} \big|_{x_1=\Delta_-} = \frac{\partial K_L}{\partial u} \cdot \frac{d+x_1}{d} \cdot x_1 + \left( K_L - \varepsilon \frac{\partial K_L}{\partial u} \right) \cdot \frac{x_1}{d} - \frac{x_1}{d} K_L = \frac{\partial K_L}{\partial u} \cdot \frac{x_1}{d} \cdot (x_1 + d - \varepsilon) = 0, \quad (A.2) \]

since \( \varepsilon - d = \Delta_- = x_1 \) on this boundary.

To prove continuity of \( B_{x_2} \), note that from (4.92) for the points in the intersection \( F(0) \cap \Xi_{L+} = F_L(\varepsilon) \) we have

\[ B_{x_2} \big|_{x_1=0} = \frac{1}{2\varepsilon} B_{x_1} \big|_{\xi_{L+}} + B_{x_2} \big|_{\xi_{L+}}, \]

and the first summand at the right-hand vanishes due to (A.2).

**Boundary \( F(0) \cap R \).** Now, we check \( C^1 \)-smoothness of \( B \) on the boundary \( x_3 = (2\varepsilon)^{p-2}x_2 \) between \( F(0) \) and \( R \) (see (5.6)). According to (4.93) the value of \( B \) from \( R \) is \( (2\varepsilon)^{r-2}x_2 \). The value from \( F(0) \) looks much more difficult (see (4.93)):

\[ \left[ e^{\Psi_L} \left( \frac{(2\varepsilon)^p}{2\varepsilon} - \varepsilon \frac{p}{e} \Gamma(p+1) \right) + \varepsilon^{r-1} \Gamma(r+1) \right] \frac{x_2}{2\varepsilon}. \]

However, the fact that these two expressions are equal is known: this is just formula (4.19) for \( u = \varepsilon \).

The calculations will be simpler if we use representation (4.94) for the function on \( F(0) \). Then the boundary \( x_3 = (2\varepsilon)^{p-2}x_2 \) corresponds to \( \xi = \varepsilon \) and we have

\[ w_L(\varepsilon; s, \varepsilon) = \frac{(2\varepsilon)^s}{2\varepsilon} - \varepsilon \frac{e}{e} m_s(0) \quad \text{and} \quad B(x) = (2\varepsilon)^{r-2}x_2. \quad (A.3) \]

Since this boundary is a piece of two-dimensional plane where \( B \) is linear, for continuity of the gradient it is sufficient to check continuity of \( B_{x_3} \) only. On \( R \) we have (see (4.98)):

\[ B_{x_3} = \frac{r-2}{p-2} \left( \frac{x_3}{x_2} \right)^{r/p} \quad \text{and} \quad B_{x_3} \big|_{x_3=(2\varepsilon)^{p-2}x_2} = \frac{r-2}{p-2} (2\varepsilon)^{r-p}. \]

On \( F(0) \) we use the same representation (4.94) and get

\[ B_{x_3} = \frac{w_L'(\xi; r, \varepsilon)}{w_L(\xi; p, \varepsilon)}. \quad (A.4) \]

Formula (4.63) yields \( w_L'(\varepsilon; s, \varepsilon) = \frac{1}{2\varepsilon} (s-2)(2\varepsilon)^{s-1} \), whence

\[ B_{x_3} \big|_{\xi=\varepsilon} = \frac{r-2}{p-2} (2\varepsilon)^{r-p}. \quad (A.5) \]

We have considered all boundaries of \( F(0) \) and now we consider boundaries of \( \Xi_{L+} \). It has two neighbours: already considered \( F(0) \) and \( \Xi_{B+} \), which we have to check.

**Boundary \( \Xi_{L+} \cap \Xi_{B+} \).** This is the boundary

\[ x_3 = \frac{\Delta_+(x_1 + \Delta_+)^p + \Delta_+(x_1 - \Delta_-)^p}{2\varepsilon} \]

(see (5.8)–(5.10)). This boundary is foliated by the left halves of the chords \( [U(u-\varepsilon), U(u+\varepsilon)] \), where \( B \) is linear with the prescribed values at the ends. Therefore, it takes the same values not depending on what side of the boundary we consider our function. However, we can verify this formally plugging the values \( \xi = u = v + \varepsilon = x_1 + d \) into (4.82):

\[ B(x) = (u-\varepsilon)^r + \left[ m_r(u-\varepsilon) + \frac{1}{2\varepsilon} (u+\varepsilon)^r - (u-\varepsilon)^r \right] \Delta_+ = \frac{\Delta_-(x_1 + \Delta_+)^r + \Delta_+(x_1 - \Delta_-)^r}{2\varepsilon}. \]

This expression coincides with (4.101), since \( a = u - \varepsilon = x_1 - \Delta_- \) and \( b = u + \varepsilon = x_1 + \Delta_+ \).
Now, we have to check continuity of $B_{x_3}$ and $B_{x_2}$ on this boundary. In $\Xi_{b+}$ we have:

$$B_{x_3} \frac{4.29}{\varepsilon \Psi_{b}^{\prime}(\xi)} \frac{4.24}{w_{l}^\prime(\xi; r, \varepsilon)} \frac{4.66}{w_{l}^\prime(\xi; p, \varepsilon)} \frac{4.66}{A(\xi; \varepsilon, r)} \frac{4.66}{A(\xi; \varepsilon, p)}.$$  \hspace{1cm} (A.6)

On $\Xi_{b+}$ formula $5.3$ yields

$$1 - w_{x_3}(a, b; p) = \frac{a_{x_3}(b - x_1)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), p), \hspace{1cm} (A.7)$$

$$B_{x_3} \frac{4.101}{w_{x_3}(a, b; r) = \frac{a_{x_3}(b - x_1)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), r), \hspace{1cm} (A.8)$$

whence

$$B_{x_3} = \frac{A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), r)}{A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), p). \hspace{1cm} (A.9)$$

Therefore, on the line $\xi = u$, $b = u + \varepsilon$, $a = u - \varepsilon$, the expressions from $(A.6)$ and $(A.9)$ are equal to

$$A(u, \varepsilon, r) \frac{4.66}{A(u, \varepsilon, p)}.$$  

On this boundary, it remains to check continuity of $B_{x_2}$. In $\Xi_{b+}$ we use formula $4.29$ rewritten in terms of $w_{l}$ with the help of $4.23$ and $4.24$:

$$B_{x_2}(x) = \frac{1}{2\varepsilon} \left[ \frac{1}{2\varepsilon} e^{\frac{w_{l}(\xi; r, \varepsilon)}{w_{l}(\xi; p, \varepsilon)}} \left( \frac{w_{l}^\prime(\xi; r, \varepsilon)}{w_{l}^\prime(\xi; p, \varepsilon)} + \varepsilon m_{r}^\prime(v) - \varepsilon m_{p}^\prime(v) \right) \right].$$

On the boundary $\xi = u = v + \varepsilon = x_1 + d$ we use relations $4.20$, $4.66$, $2.15$ and rewrite this formula as follows:

$$B_{x_2}(x)|_{\xi=u} = \frac{1}{2\varepsilon} \left[ \left( \frac{(u + \varepsilon)^{r} - (u - \varepsilon)^{r}}{2\varepsilon} - r v^{r-1} \right) - \left( \frac{(u + \varepsilon)^{p} - (u - \varepsilon)^{p}}{2\varepsilon} - p v^{p-1} \right) \right] \frac{A(u, \varepsilon, r)}{A(u, \varepsilon, p)}. \hspace{1cm} (A.10)$$

In $\Xi_{b+}$ we differentiate formula $5.2$ with respect to $x_2$ and as in $(5.3)$ we get

$$w_{x_2} = \frac{a_{x_2}(b - x_1) \left[ s a^{-1}(b - a) + a^{-b} \right] + b_{x_2}(x_1 - a) \left[ s b^{-1}(b - a) + a^{-b} \right]}{(b - a)^2}. \hspace{1cm} (A.11)$$

To obtain relation between $a_{x_2}$ and $b_{x_2}$ we differentiate $(4.102)$:

$$1 = b_{x_2}(x_1 - a) - a_{x_2}(b - x_1). \hspace{1cm} (A.12)$$

Now we replace the expression $a_{x_2}(b - x_1)$ in $(A.11)$ using $(A.12)$ and get

$$w_{x_2} = \frac{b_{x_2}(x_1 - a) \left[ s(b(a(t^{-1} + b^{-1}) + 2a^{t} - 2b^{t}) - [s a^{-1}(b - a) + a^{-b} \right]}{(b - a)^2}$$

$$= \frac{b_{x_2}(x_1 - a)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), s) - \frac{s a^{-1}(b - a) + a^{-b}}{(b - a)^2}.$$  

Whence,

$$B_{x_2} \frac{4.101}{w_{x_2}(a, b; r) = \frac{b_{x_2}(x_1 - a)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), r) - \frac{r a^{t-1}(b - a) + a^{t} - b^{t}}{(b - a)^2}, \hspace{1cm} (A.13)$$

$$0 \frac{4.103}{w_{x_2}(a, b; p) = \frac{b_{x_2}(x_1 - a)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), p) - \frac{pa^{p-1}(b - a) + a^{p} - b^{p}}{(b - a)^2}. \hspace{1cm} (A.14)$$
Therefore,
\[ B_{x_2} = \frac{b^r - a^r - r a^{r-1} (b - a)}{(b - a)^2} - \frac{b^p - a^p - p a^{p-1} (b - a)}{(b - a)^2} \cdot A\left(\frac{1}{2} (b + a), \frac{1}{2} (b - a), r\right). \] (A.15)

Since on the boundary we have \( b = u + \varepsilon, \ a = u - \varepsilon = v, \) we see that the expressions in (A.10) and in (A.15) coincide.

Now we consider two remaining boundaries of \( \Xi_{ch_+}. \) This is the boundary \( \chi_3 = x^p_2 - x^2_1 - p \) with \( R \) (if \( (x_1, x_2) \in \omega_2 \cup \omega_3, \) see (5.8) and (5.9)) and the boundary \( \chi_3 = \frac{\Delta (x_1 - \Delta \varepsilon) + \Delta (x_1 + \Delta \varepsilon)}{2\varepsilon} \) with \( \Xi_{r+} \) (if \( (x_1, x_2) \in \omega_4, \) see (5.10)).

**Boundary \( \Xi_{ch_+} \) \( R. \)** This is the boundary \( \chi_3 = x^p_2 - x^2_1 - p. \) It consists of the chords with \( \left[ U(0), U(b) \right], \ b \in [0, 2\varepsilon]: \ x_2 = b x_1, \ x_3 = b^{p-1} x_1. \) The value of our function on this chord is \( \chi = b^{r-1} x_1 \) in both domains.

The derivatives on \( R \) (see (4.98)) are:
\[ B_{x_3} = \frac{r - 2}{p - 2} \left( x_3 \right) \frac{\tilde{s}_2^2}{x_2^2} = \frac{r - 2}{p - 2} \cdot b^{r-2}, \] (A.16)
\[ B_{x_2} = \frac{p - r}{p - 2} \left( x_3 \right) \frac{\tilde{s}_2^2}{x_2^2} = \frac{p - r}{p - 2} \cdot b^{r-2}. \] (A.17)

Since \( A\left(\frac{1}{2} b, \frac{1}{2} b, s\right) \) \( (s - 2) b^s, \) on this boundary the value of (A.9) coincides with (A.16) and the value of (A.15) coincides with (A.17).

**Boundary \( \Xi_{ch_+} \) \( \Xi_{r+}. \)** This boundary is foliated by the right half of the chords \( \left[ U(u - \varepsilon), U(u + \varepsilon) \right] \) \( (u \geq \varepsilon), \) and \( \chi \) is linear on such chord with the prescribed values at the ends. Therefore, the boundary values from both sides of the boundary are the same.

Now, we check continuity of \( B_{x_3}. \) On \( \Xi_{ch_+} \) these derivatives are already calculated, we need only to plug the boundary values \( b = u + \varepsilon \) and \( a = u - \varepsilon \) into (A.9):
\[ B_{x_3} = \frac{A(u, \varepsilon, r)}{A(u, \varepsilon, p)}. \] (A.18)

On \( \Xi_{r+} \) we get the same expression if we substitute \( \xi = u \) to the following formula:
\[ B_{x_3} \bigg|_{\Xi_{r+}} = \frac{1}{2\varepsilon} \left[ \epsilon \Psi_n'(u) + \frac{1}{\varepsilon} e^{-\frac{\pi}{\varepsilon} w_n(\xi; p, \varepsilon)} - \left( \epsilon \Psi_n'(u) + \frac{1}{\varepsilon} e^{-\frac{\pi}{\varepsilon} w_n(\xi; p, \varepsilon)} \right) w_n' \left( \xi; r, \varepsilon \right) \right]. \] (A.19)

Verifying continuity of \( B_{x_2}, \) we use (4.53) and (4.54) to rewrite the second formula of (4.58) in terms of \( w_n: \)
\[ B_{x_2} (x) = \frac{1}{2\varepsilon} \left[ \left( u - \varepsilon \right)^r - \frac{u^{r+1} - \left( u + \varepsilon \right)^r}{2\varepsilon} + ru^{r-1} \right] - \left( \frac{(u - \varepsilon)^p - (u + \varepsilon)^p}{2\varepsilon} + pu^{p-1} \right) A(u, \varepsilon, r). \] (A.20)

On the boundary \( \xi = u = v - \varepsilon = x_1 - d \) we apply already used relations (4.50), (4.68), (4.66), (2.15), and rewrite this formula as follows:
\[ B_{x_2} (x) \bigg|_{\xi = u} = \frac{1}{2\varepsilon} \left[ \left( u - \varepsilon \right)^r - \frac{u^{r+1} - \left( u + \varepsilon \right)^r}{2\varepsilon} + ru^{r-1} \right] - \left( \frac{(u - \varepsilon)^p - (u + \varepsilon)^p}{2\varepsilon} + pu^{p-1} \right) A(u, \varepsilon, r). \] (A.21)

On the face of it, this expression differs from (A.15) with \( b = u + \varepsilon = v, \ a = u - \varepsilon. \) However, in fact they are equal. It can be verified by the direct calculation using the definition of the function \( A \) (see (4.64)). But we check this in other way, deducing an alternative formula for \( B_{x_2} \) on \( \Xi_{ch_+}. \) Namely, in (A.11) we remove not \( a_{x_2} \) but \( b_{x_2} \) using the same relation (A.12):
\[ w_{x_2} = \frac{a_{x_2}(b - x_1)}{(b - a)^2} \frac{(b - a)(a^{r-1} + b^{r-1} + 2a^r - 2b^r) + \left[s(b - a)(a^{r-1} + b^{r-1}) + 2a^r - 2b^r \right]}{(b - a)^2} \]
Then instead of (A.13) and (A.14) we get
\[ B_{x_2} \text{[4.101]} w_{x_2}(a, b; r) = \frac{a x_2(b - x_1)}{(b - a)^2} A(\frac{1}{2}(b + a), \frac{1}{2}(b - a), r) + \frac{r b^r - 1(b - a) + a^r - b^r}{(b - a)^2}, \] (A.22)
and this expression evidently coincides with (A.21) on the boundary \( b = u + \varepsilon = v, \ a = u - \varepsilon. \)

It remains to check one boundary, namely, between \( \Xi_{u\varepsilon} \) and \( R. \)

**Boundary \( \Xi_{u\varepsilon} \) of \( R. \)** This is the boundary \( x_3 = (x_1 + \Delta_+)p^2 x_2 \) or \( \xi = u = v - \varepsilon = x_1 - d \) and it is foliated by the extremal lines \( x_3 = v p^{-2} x_2, \ x_2 = 2(v - \varepsilon)x_1 - v^2 + 2v\varepsilon, \ \varepsilon \leq v \leq 2\varepsilon. \) On this boundary we have to use the first line of definition of \( w_{\nu} \text{[4.50]} \) for \( \xi \leq \varepsilon. \) Therefore, (see (4.87)), on any such extremal line we have
\[ B = v^r - (k_r(v) - k_r(v) + 2uv^{r-2})\Delta_+ = v^{r-2}(v^2 - 2uv\Delta_+) = v^{r-2}x_2, \]
that coincides with the value on \( R \) (see (4.99)).

Using as before (4.54) and the first formula in (4.58) for \( B_{x_3} \) on \( \Xi_{u\varepsilon} \), we get
\[ B_{x_3} = \frac{w'_{\nu}(\xi; r, \varepsilon) \text{[4.67]} \ (r - 2)}{\varepsilon (p - 2)(\xi + \varepsilon)^{r-p}}, \]
and for \( \xi = u = v - \varepsilon \) we get the same expression as in (A.16) for \( x_3 = v p^{-2} x_2. \) The second formula in (4.58) for \( B_{x_2} \) yields (A.20), where we have use another expression for \( w_{\nu} \) (see the first line of (4.50)). As a result, we get not (A.21), but
\[ B_{x_2} \big|_{\xi = u} = \frac{1}{2\varepsilon} \left[ (2v^{r-1} - 2uv^{r-2}) - (p v^{p-1} - 2uv^{p-2}) \frac{(r - 2)v^{r-3}}{(p - 2)v^{p-3}} \right] \\
= \frac{1}{2\varepsilon} \left[ ((r - 2)v^{r-1} + 2\varepsilon v^{r-2}) - ((p - 2)v^{p-1} + 2\varepsilon v^{p-2}) \frac{r - 2}{p - 2} v^{r-2} \right] = \frac{p - r}{p - 2} v^{r-2}, \]
therefore, we get the same expression as in (A.17) for \( x_3 = v p^{-2} x_2. \)

The only boundary intersecting the plane \( x_1 = 0 \) is the boundary between \( F(0) \) and \( R \) considered above. \( C^1 \)-smoothness of \( B \) in the domain \( x_1 < 0 \) follows from the symmetry. This completes the proof of \( C^1 \)-smoothness of the function \( B. \) \( \Box \)

**Appendix B  Convexity/concavity of the Bellman function \( B_2 \)**

In this appendix we prove convexity/concavity of \( B_2. \) Since we will deal only with this candidate we omit index 2, as it was done in Appendix A

**Lemma B.1.** *The equality sign \( B_{x_3 x_3} = \text{sign}(r - 2)(r - p) \) holds on subdomains \( F(0), \ R, \ \Xi_{\pm}, \ \Xi_{\pm \pm}, \ \Xi_{\pm \mp}. \)*

**Proof.** We start with \( F(0), \) where by (A.4) we have
\[ B_{x_3} = \frac{w'_{\nu}(\xi; r, \varepsilon) \text{[4.66]} \ A(\xi, \varepsilon, r)}{\varepsilon A(\xi, \varepsilon, p)}, \] (B.1)
The same expression we have in $\Xi_{L^+}$, as we have seen in \[\text{(A.6)}\]. In $\Xi_{R^+}$, we have the same formula for $\xi \geq \varepsilon$ due to \[\text{(A.19)}\]. Therefore, in all three cases we have

$$B_{x_3x_3} = \left(\frac{A(\xi, \varepsilon, r)}{A(\xi, \varepsilon, p)}\right)' \cdot \frac{\partial \xi}{\partial x_3}.$$ 

Recall that we use symbol $'$ to denote differentiation with respect to $\xi$. In the cases under consideration we have

\[
\begin{align*}
F(0): & \quad x_3' = \frac{4\beta}{4.84} \frac{e^x_2}{2\pi^2} w'_L(\xi; p, \varepsilon); \\
\Xi_{R^+}: & \quad x_3' = \frac{4.68}{\varepsilon} \frac{\Delta}{\varepsilon} e^{-\frac{\varepsilon}{2}} w'_L(\xi; p, \varepsilon) = \frac{4.68}{\varepsilon} e^{-\frac{2\varepsilon}{\varepsilon}} w'_L(\xi; p, \varepsilon), \quad \xi \geq \varepsilon; \\
\Xi_{L^+}: & \quad x_3' = \frac{4.68}{\varepsilon} \frac{\Delta}{\varepsilon} e^{-\frac{\varepsilon}{2}} w'_L(\xi; p, \varepsilon).
\end{align*}
\]

Therefore, in all cases Lemma \[\text{4.1}\] implies

$$\text{sign} \frac{\partial \xi}{\partial x_3} = \text{sign} x_3' = \text{sign} w'_L(\xi; p, \varepsilon) = \text{sign}(p - 2),$$

and we need to check that

$$\text{sign} \left(\frac{A(\xi, \varepsilon, r)}{A(\xi, \varepsilon, p)}\right)' = \text{sign} \left(\frac{A'(\xi, \varepsilon, r)A(\xi, \varepsilon, p) - A'(\xi, \varepsilon, p)A(\xi, \varepsilon, r)}{A(\xi, \varepsilon, p)}\right) = \text{sign}(p - 2)(r - 2)(r - p). \quad (B.2)$$

Differentiating \[\text{(4.65)}\] with respect to $\alpha$ and then integrating by parts, we get

$$\frac{\partial}{\partial \alpha} A(\alpha, \beta, s) = s(s - 1)(s - 2) \int_{-\beta}^{\beta} 2\lambda(\lambda + \alpha)^{s-3}d\lambda. \quad (B.3)$$

After dividing the expression in \[\text{(B.2)}\] over $r(r - 1)(r - 2)p(p - 1)(p - 2)$, we need to verify that $\text{sign}(r - p)$ is the sign of the following expression:

\[
\begin{align*}
&\int_{-\beta}^{\beta} 2\lambda(\lambda + \alpha)^{r-3}d\lambda \int_{-\beta}^{\beta} (\lambda^2 - \lambda^2)(\lambda + \alpha)^{p-3}d\lambda - \int_{-\beta}^{\beta} 2\lambda(\lambda + \alpha)^{p-3}d\lambda \int_{-\beta}^{\beta} (\lambda^2 - \lambda^2)(\lambda + \alpha)^{r-3}d\lambda \\
&= 2 \int_{-\beta}^{\beta} \int_{-\beta}^{\beta} \left[\lambda(\beta^2 - \lambda^2) - \mu(\beta^2 - \lambda^2)\right](\mu + \alpha)^{p-3}(\lambda + \alpha)^{r-3}d\lambda d\mu \\
&= 2 \int_{-\beta}^{\beta} \int_{-\beta}^{\beta} (\lambda - \mu)(\beta^2 + \lambda \mu)(\mu + \alpha)^{p-3}(\lambda + \alpha)^{r-3}d\lambda d\mu.
\end{align*}
\]

After symmetrization (interchanging $\mu$ and $\lambda$) we get:

$$\int_{-\beta}^{\beta} \int_{-\beta}^{\beta} (\lambda^2 + \lambda \mu)(\mu + \alpha)^{r-3}(\lambda + \alpha)^{p-3}\left[(\lambda - \mu)((\lambda + \alpha)^{r-3} - (\mu + \alpha)^{r-3})\right]d\lambda d\mu.$$ 

Since the function $t \mapsto t^s$ is increasing for $s > 0$ and decreasing for $s < 0$, the sign of expression in the square brackets coincides with $\text{sign}(r - p)$, and this is just what we need, because all other terms are positive.
whence therefor, (B.5) can be rewritten as follows:

\[ A(\alpha, \beta, r) = \alpha, \beta, p \]

For brevity, we omit arguments \( \alpha \) and \( \beta \) if this does not lead to misunderstanding. The principal step will be the same as before, but we need some auxiliary calculations here. First of all, we will use another integral form of the function \( A \). Changing the variable of integration in (4.65), we get the following formula

\[ A(s) = s(s - 1)(s - 2) \int_a^b (b - \lambda)(\lambda - a)\lambda^{s-3}d\lambda, \quad (B.4) \]

whence

\[ A_x(s) = s(s - 1)(s - 2) \int_a^b [bx_x(\lambda - a) - (b - \lambda)a_x]\lambda^{s-3}d\lambda. \quad (B.5) \]

From (A.7) and (5.1) we have

\[ (x_1 - a)bx_x = (b - x_1)a_x = \frac{(b - a)^2}{A(p),} \quad (B.6) \]

therefore, (B.5) can be rewritten as follows:

\[ A_x(s) = \frac{s(s - 1)(s - 2)(b - a)^2}{(x_1 - a)(b - x_1)A(p)} \int_a^b [(b - x_1)(\lambda - a) - (b - \lambda)(x_1 - a)]\lambda^{s-3}d\lambda \]

\[ = \frac{s(s - 1)(s - 2)(b - a)^2}{(x_1 - a)(b - x_1)A(p)} \int_a^b (\lambda - x_1)\lambda^{s-3}d\lambda, \quad (B.7) \]

whence

\[ B_{x_3} = \frac{r(r - 1)(r - 2)p(p - 1)(p - 2)(b - a)^3}{(x_1 - a)(b - x_1)A(p)^3} \times \]

\[ \times \left\{ \int_a^b (\lambda - x_1)\lambda^{r-3}d\lambda \int_a^b (b - \mu)(\mu - a)\mu^{r-3}d\mu - \int_a^b (b - \lambda)(\lambda - a)\mu^{r-3}d\lambda \int_a^b (\mu - x_1)\mu^{r-3}d\mu \right\} \]

\[ = \frac{r(r - 1)(r - 2)p(p - 1)(p - 2)(b - a)^3}{(x_1 - a)(b - x_1)A(p)^3} \int_a^b \int_a^b [\lambda\mu - ab + (a + b - \lambda - \mu)x_1] (\lambda - \mu)\lambda^{r-3}\mu^{r-3}d\lambda d\mu \]

\[ = \frac{r(r - 1)(r - 2)p(p - 1)(p - 2)(b - a)^3}{2(x_1 - a)(b - x_1)A(p)^3} \times \]

\[ \times \int_a^b \int_a^b [\lambda\mu - ab + (a + b - \lambda - \mu)x_1] (\lambda - \mu)(\lambda^{r-3} - \mu^{r-3})\lambda^{r-3}d\lambda d\mu. \quad (B.8) \]
In the latter equality, we use symmetrization with respect $\lambda$ and $\mu$, as it has been made before. Now it is easy to determine the sign of this expression. Since $a < x_1 < b$ and $\text{sign} A(p) = \text{sign}(p - 2)$, the sign of the factor in front of the integral is $\text{sign}(r - 2)$. The expression in the square brackets is positive. To check this it suffices to note that this is a linear function with respect to $x_1$ and it is positive at both endpoints of the interval $[a, b]$:

- at $x_1 = a$ we have $(\lambda - a)(\mu - a)$;
- at $x_1 = b$ we have $(b - \lambda)(b - \mu)$.

Clearly,

$$\text{sign}(\lambda r - \mu r - p)(\lambda - \mu) = \text{sign}(r - p).$$

Therefore, we get what is needed: $\text{sign} B_{x_2, x_3} = \text{sign}(r - 2)(r - p)$.

It remains to calculate $\text{sign} B_{x_2, x_3}$ on $R$. Since the function $B$ has a simple explicit expression there (see (4.98))

$$B = x_2^{\frac{p-2}{2}} x_3^{\frac{p-2}{2}},$$

it is easy to calculate $B_{x_2, x_3}$ directly:

$$\text{sign} B_{x_2, x_3} = \text{sign} \frac{r - 2}{p - 2} (r - 2) x_2^{\frac{p-r}{2}} x_3^{\frac{p-r}{2}} = \text{sign}(r - 2)(r - p).$$

The consideration of this last case completes the proof of the lemma. □

Since in the domains $F(0)$ and $R$ the function $B$ is linear on two-dimensional planes transversal to the direction of $x_3$, just proved Lemma B.1 ensures the required in Theorem 6.1 concavity/convexity of $B$ inside these domains. The other domains are foliated by one-dimensional extremal lines transversal to $x_2 x_3$-plane, therefore, to prove concavity/convexity there we need to check positivity of the minor $B_{x_2, x_2} B_{x_3, x_3} - B_{x_2, x_3}^2$.

We start with domain $\Xi_{l+}$.

**Lemma B.2.** $\det\{B_{x_i, x_j}\}_{2 \leq i, j \leq 3} = B_{x_2, x_2} B_{x_3, x_3} - B_{x_2, x_3}^2 \geq 0$ on $\Xi_{l+}$.

**Proof.** To calculate the sign of this determinant we use formula (4.32):

$$B_{x_2, x_2} B_{x_3, x_3} - B_{x_2, x_3}^2 = \frac{e^{-\frac{\Psi''}{2}}(e^{-\frac{\Psi'}{2} h})}{4(x_1 - u)(x_1 - v)} \left( e^{\frac{\Psi'}{2} h} (e^{\frac{\Psi}{2} h} - (h + \epsilon^3 m''_p(v))\Psi' (e^{-\frac{\Psi}{2} h}) + \epsilon^2 m''_p(v)) \right).$$

Since $v < x_1 < u = u + \epsilon$ and $\text{sign} \Psi'' = \text{sign}(r - 2)(r - p)$ (see (4.28) and Lemma B.1) we deduce that $\text{sign} \det\{B_{x_i, x_j}\}_{2 \leq i, j \leq 3}$ coincides with

$$-\text{sign}(r - 2)(r - p) \times \text{sign} \left( e^{\frac{\Psi'}{2} h} (e^{\frac{\Psi}{2} h} - (h + \epsilon^3 m''_p(v))\Psi' (e^{-\frac{\Psi}{2} h}) + \epsilon^2 m''_p(v)) \right).$$

So, we need to check that

$$\text{sign} \left( e^{\frac{\Psi'}{2} h} (e^{\frac{\Psi}{2} h} - (h + \epsilon^3 m''_p(v))\Psi' (e^{-\frac{\Psi}{2} h}) + \epsilon^2 m''_p(v)) \right)^{\frac{7}{2}} = \text{sign}(p - r)(r - 2). \quad (B.9)$$

We return to variable $\xi$ (see (4.22)) and use the definition of $\Psi_l$ in terms of $w_l$ (see (4.24)) and rewrite (B.9) as follows:

$$\text{sign} \left( \frac{\epsilon^{\frac{\Psi}{2}}}{\xi} w_l(\xi; r, \epsilon) - \frac{\epsilon^{\frac{\Psi'}{2}}}{\xi} w_l(\xi; p, \epsilon) + \epsilon^3 m''_p(v) \right)^{\frac{7}{2}} = \text{sign}(p - r)(r - 2),$$

which is equivalent to

$$\text{sign} \left( \frac{w_l(\xi; p, \epsilon) + \epsilon^{\frac{\Psi}{2}} m''_p(v)}{w_l(\xi; r, \epsilon)} \right) = \text{sign}(p - r), \quad (B.10)$$

44
because sign $w'_L(\xi; r, \varepsilon) = \text{sign}(r - 2)$ by Lemma 1.1. In other words, we need to prove that the function
\[
s \mapsto \frac{w_L(\xi; s, \varepsilon) + e^{-\frac{t}{2}\varepsilon^3 m''_s(v)}}{w'_L(\xi; s, \varepsilon)}
\]
decreases for all $\xi \geq u$.

We split this function into the sum of two
\[
\frac{w_L(\xi; s, \varepsilon) + e^{-\frac{t}{2}\varepsilon^3 m''_s(v)}}{w'_L(\xi; s, \varepsilon)} = \frac{w_L(\xi; s, \varepsilon) + e^{-\frac{t}{2}\varepsilon^3 m''_s(\xi - \varepsilon)}}{w'_L(\xi; s, \varepsilon)} + \frac{e^{-\frac{t}{2}\varepsilon^3 m''_s(v)} - e^{-\frac{t}{2}\varepsilon^3 m''_s(\xi - \varepsilon)}}{w'_L(\xi; s, \varepsilon)}
\] (B.11)
and prove that both of them are decreasing.

We begin with a simpler second function rewriting the expression in the numerator. By (2.13) we have
\[
e^{-\frac{t}{2}\varepsilon^3 m''_s(v)} - e^{-\frac{t}{2}\varepsilon^3 m''_s(\xi - \varepsilon)} = \varepsilon^2 s(s - 1)(s - 2) \int_u^\xi e^{-\frac{t}{2}(t - \varepsilon)s^{-3}} dt,
\]
and due to (4.66) and (4.65) the question is reduced to verification that the function
\[
s \mapsto \frac{\int_u^\xi e^{-\frac{t}{2}(t - \varepsilon)s^{-3}} dt}{\int_{-\varepsilon}^\xi (\varepsilon^2 - \lambda^2)(\lambda + \xi)s^{-3} d\lambda}
\] (B.12)
is decreasing. Indeed, for $\varepsilon < t < \xi$ and $\lambda > -\varepsilon$ we have $\frac{\lambda + \xi}{t - \varepsilon} > 1$, and therefore, the function
\[
s \mapsto \left(\frac{\lambda + \xi}{t - \varepsilon}\right)^{s^{-3}}
\]
is increasing. After integration of this family of increasing functions with positive weight we get the following increasing function:
\[
s \mapsto \int_{-\varepsilon}^\xi (\varepsilon^2 - \lambda^2) \left(\frac{\lambda + \xi}{t - \varepsilon}\right)^{s^{-3}} d\lambda,
\]
or the following decreasing function:
\[
s \mapsto \frac{1}{\int_{-\varepsilon}^\xi (\varepsilon^2 - \lambda^2)(\lambda + \xi)s^{-3} d\lambda} \int_{-\varepsilon}^\xi (\varepsilon^2 - \lambda^2)(\lambda + \xi)s^{-3} d\lambda.
\] (B.13)

We integrate once more family of functions (B.13) with the positive weight $e^{-\frac{t}{2}}$ and finally obtain the decreasing function (B.12).

Now, we consider the first summand in (B.11). Again, we start with simplifying the expression in the numerator. Using twice (2.15), we get
\[
\varepsilon^2 m''_s(v) = \varepsilon(m_s(v) - sv^{s-1})' = m_s(v) - sv^{s-1} - \varepsilon s(s - 1)v^{s-2}.
\]
Then, we use the definition (4.20) of $w_L$:
\[
w_L(\xi; s, \varepsilon) + e^{-\frac{t}{2}\varepsilon^3 m''_s(\xi - \varepsilon)}
\]
\[
e^{-\frac{t}{2}\left(\frac{\xi + \varepsilon}{s} - \frac{\xi - \varepsilon}{s}\right)} - e^{-\frac{t}{2}\varepsilon m_s(\xi - \varepsilon)} + e^{-\frac{t}{2}\varepsilon m_s(\xi - \varepsilon)} = e^{-\frac{t}{2}\left(\frac{\xi + \varepsilon}{s} - \frac{\xi - \varepsilon}{s}\right)} - e^{-\frac{t}{2}\varepsilon m_s(\xi - \varepsilon)}
\]
\[
= e^{-\frac{t}{2}\left(\frac{\xi + \varepsilon}{s} - \frac{\xi - \varepsilon}{s}\right)} - e^{-\frac{t}{2}\varepsilon m_s(\xi - \varepsilon)} = e^{-\frac{t}{2}\varepsilon m_s(\xi - \varepsilon)}
\]
\[
= \frac{1}{45}(s - 1)(s - 2)e^{-\frac{t}{2}} \int_{-\varepsilon}^\xi (\varepsilon - \mu)^2(\mu + \xi)^{s^{-3}} d\mu.
\]
Therefore, we need to check that the function

\[
s \mapsto \frac{\int_{-\epsilon}^{\epsilon} (x - \mu)^2(\mu + \xi)^{s-3}d\mu}{\int_{-\epsilon}^{\epsilon} (x^2 - \lambda^2)(\lambda + \xi)^{s-3}d\lambda}
\]

is decreasing. After differentiating this function, we get in numerator the following expression:

\[
\int_{-\epsilon}^{\epsilon} (x - \mu)^2(\mu + \xi)^{s-3}\log(\mu + \xi) d\mu \int_{-\epsilon}^{\epsilon} (x^2 - \lambda^2)(\lambda + \xi)^{s-3}d\lambda
\]

\[
- \int_{-\epsilon}^{\epsilon} (x - \mu)^2(\mu + \xi)^{s-3}d\mu \int_{-\epsilon}^{\epsilon} (x^2 - \lambda^2)(\lambda + \xi)^{s-3}\log(\lambda + \xi) d\lambda
\]

\[
\int_{-\epsilon}^{\epsilon} (x - \mu)^2(\mu + \xi)^{s-3}(\lambda + \xi)^{s-3}(\log(\mu + \xi) - \log(\lambda + \xi)) d\mu d\lambda.
\]

Now, we symmetrize this expression (interchanging \( \mu \) and \( \lambda \)):

\[
\frac{1}{2} \int_{-\epsilon}^{\epsilon} \left[ (x - \mu)^2(x^2 - \lambda^2) - (x - \lambda)^2(x^2 - \mu^2) \right] (\mu + \xi)^{s-3}(\lambda + \xi)^{s-3}(\log(\mu + \xi) - \log(\lambda + \xi)) d\mu d\lambda
\]

\[
= \epsilon \int_{-\epsilon}^{\epsilon} (\lambda - \mu)(x - \mu)(x - \lambda)(\mu + \xi)^{s-3}(\lambda + \xi)^{s-3}(\log(\mu + \xi) - \log(\lambda + \xi)) d\mu d\lambda.
\]

Since \( \log \) is an increasing function, we have

\[
(\lambda - \mu)(\log(\mu + \xi) - \log(\lambda + \xi)) < 0.
\]

This means that the derivative we have calculated is negative and the function \( \text{(B.14)} \) decreases. This completes the proof of the lemma.

**Lemma B.3.** \( B_{x_2x_2}B_{x_3x_3} - B^2_{x_2x_3} \geq 0 \) on \( \Xi_{R+} \).

**Proof.** To calculate the sign of this determinant, we use formula \( \text{(4.62)} \)

\[
B_{x_2x_2}B_{x_3x_3} - B^2_{x_2x_3} = \frac{e^{-\frac{3}{2}} \Psi'_R(e^{\frac{r}{2}}h)(e^{\frac{r}{2}}h)}{4(x_1 - u)(x_1 - v)} \left( e^{-\frac{r}{2}} \Psi'_R(e^{\frac{r}{2}}h) - (h - \varepsilon \kappa'_p(v)) \Psi'_R(e^{\frac{r}{2}}h) + \varepsilon^2 \kappa''_p(v) \right).
\]

Since \( u < x_1 < v = u + \varepsilon \) and \( \text{sign } \Psi'_R = -\text{sign}(r - 2)(r - p) \) (see \( \text{(4.59)} \) and Lemma \( \text{[B.1]} \)), we deduce that \( \text{sign } \det \{B_{x_i,x_j}\} \) coincides with

\[
\text{sign}(r - 2)(r - p) \times \text{sign} \left( e^{-\frac{r}{2}} \Psi'_R(e^{\frac{r}{2}}h) - (h - \varepsilon \kappa'_p(v)) \Psi'_R(e^{\frac{r}{2}}h) + \varepsilon^2 \kappa''_p(v) \right).
\]

So, we need to check that

\[
\text{sign} \left( e^{-\frac{r}{2}} \Psi'_R(e^{\frac{r}{2}}h) - (h - \varepsilon \kappa'_p(v)) \Psi'_R(e^{\frac{r}{2}}h) + \varepsilon^2 \kappa''_p(v) \right) \equiv \text{sign}(r - p)(r - 2) \quad \text{(B.15)}
\]

As in the proof of the preceding lemma, we return to variable \( \xi \) (see \( \text{(4.52)} \)) and use the definition of \( \Psi_R \) in terms of \( w_R \) (see \( \text{(4.54)} \)) to rewrite \( \text{(B.15)} \) as follows:

\[
\text{sign} \left( -\varepsilon \frac{w'_R(\xi; r, \varepsilon)}{\varepsilon} + (e^{-\frac{r}{2}} w_R(\xi; p, \varepsilon) - \varepsilon^3 \kappa''_p(v)) \frac{1}{\varepsilon} \frac{w'_R(\xi; r, \varepsilon)}{w_R(\xi; p, \varepsilon)} + \varepsilon^2 \kappa''_p(v) \right) \equiv \text{sign}(r - p)(r - 2),
\]
which is equivalent to
\[
\text{sign} \left( \frac{w_n(\xi; r, \varepsilon) - e^{\frac{\varepsilon}{r}} e^{3k''(v)}}{w_n'(\xi; r, \varepsilon)} \right) \equiv \text{sign}(p - r), \quad (B.16)
\]
because \( \text{sign} w'(\xi; r, \varepsilon) = \text{sign}(r - 2) \) by Lemma 4.2. In other words, we need to prove that the function
\[
s \mapsto \frac{w_n(\xi; s, \varepsilon) - e^{\frac{s}{\varepsilon}} e^{3k''(v)}}{w_n'(\xi; s, \varepsilon)}
\]
decreases for all \( \xi \leq u \).

We split this function into the sum of two
\[
\frac{w_n(\xi; s, \varepsilon) - e^{\frac{s}{\varepsilon}} e^{3k''(v)}}{w_n'(\xi; s, \varepsilon)} = \frac{w_n(\xi; s, \varepsilon) - e^{\frac{\xi}{s}} e^{3k''(\xi + \varepsilon)}}{w_n'(\xi; s, \varepsilon)} + \frac{e^{\frac{s}{\varepsilon}} e^{3k''(\xi + \varepsilon)} - e^{\frac{\xi}{s}} e^{3k''(v)}}{w_n'(\xi; s, \varepsilon)} \quad (B.17)
\]
and prove that both of them are decreasing.

Again, we begin with a simpler second function rewriting the expression in the numerator. From (2.14) we have
\[
e^{\frac{s}{\varepsilon}} e^{3k''(\xi + \varepsilon)} - e^{\frac{\xi}{s}} e^{3k''(v)} = -e^{2k''(s - 1)(s - 2) \int_0^\varepsilon (t + \varepsilon)^{s-3} dt},
\]
and, therefore, since \( v = u + \varepsilon \), we have
\[
e^{\frac{s}{\varepsilon}} e^{3k''(\xi + \varepsilon)} - e^{\frac{\xi}{s}} e^{3k''(v)} = -e^{2k''(s - 1)(s - 2) \int_\xi^u (t + \varepsilon)^{s-3} dt}.
\]
If \( \xi \geq \varepsilon \) then due to (4.68), (4.66), and (4.65) the question is reduced to verification that the function
\[
s \mapsto \int_\xi^u e^{\frac{3}{2} (t + \varepsilon)^{s-3} dt}
\]
is increasing. If \( \xi \leq \varepsilon \) we refer to formula (4.61) and check that the function
\[
s \mapsto \int_{-\varepsilon}^\xi e^{\frac{3}{2} (t + \varepsilon)^{s-3} dt} \quad (B.18)
\]
increases. Since \( s \mapsto s(s - 1)a^s \) increases on \((1, \infty)\) for \( a > 1 \), the function \( (B.19) \) is increasing as well, because \( t > \xi \). To prove that \( (B.18) \) increases, we repeat the chain of arguments we already used for proving that \( (B.12) \) is decreasing. Since \( t > \xi \) and \( \lambda < \varepsilon \), we have \( \frac{\lambda + \xi}{t + \varepsilon} < 1 \), and therefore, the function
\[
s \mapsto \left( \frac{\lambda + \xi}{t + \varepsilon} \right)^{s-3}
\]
is decreasing. After integrating this family of decreasing functions with a positive weight, we get the following decreasing function:
\[
s \mapsto \int_{-\varepsilon}^\varepsilon (\varepsilon^2 - \lambda^2) \left( \frac{\lambda + \xi}{t + \varepsilon} \right)^{s-3} d\lambda,
\]
or the following increasing function:

\[ s \mapsto \frac{(t + \varepsilon)^{s-3}}{\int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - \lambda^2)(\lambda + \varepsilon)^{s-3}d\lambda}. \] (B.20)

We integrate once more family of functions (B.20) with the positive weight \( e^{\frac{\lambda}{2}} \) and finally obtain the increasing function (B.18).

Now, we consider the first summand in (B.17). First, we simplify the expression in the numerator. Using twice (B.21), we get

\[ e^2 k''(v) = e(s^s - k_s(v))' = e s(s - 1) v^{s-2} - s v^{s-1} + k_s(v). \]

Then we use definition of \( w_n \) (see (4.50)) first for \( \xi \leq \varepsilon \):

\[
\begin{align*}
 w_n(\xi; s, \varepsilon) - e^\frac{\xi}{2} e^3 k''(\xi + \varepsilon) & = e^\frac{\xi}{2} \left( k_s(\xi + \varepsilon) - 2\xi(\xi + \varepsilon)^{s-2} \right) - e^\frac{\xi}{2} \left( e s(s - 1)(\xi + \varepsilon)^{s-2} - s(\xi + \varepsilon)^{s-1} + k_s(\xi + \varepsilon) \right) \\
 & = e^\frac{\xi}{2} (s - 2)(\xi - \varepsilon s)(\xi + \varepsilon)^{s-2},
\end{align*}
\]

and the function

\[
\frac{w_n(\xi; s, \varepsilon) - e^\frac{\xi}{2} e^3 k''(\xi + \varepsilon)}{w_n(\xi; s, \varepsilon)} \xrightarrow{\xi \geq \varepsilon} \frac{\varepsilon(\xi - \varepsilon s)(\xi + \varepsilon)}{\xi^2 + \varepsilon^2}
\]

decreases in \( s \).

For \( \xi \geq \varepsilon \) we have:

\[
\begin{align*}
 w_n(\xi; s, \varepsilon) - e^\frac{\xi}{2} e^3 k''(\xi + \varepsilon) & = e^\frac{\xi}{2} \left( (\xi - \varepsilon s)^{s-2} - (\xi + \varepsilon)^{s-2} \right) + e k_s(\xi + \varepsilon) - e^\frac{\xi}{2} \left( e s(s - 1)(\xi + \varepsilon)^{s-2} - s(\xi + \varepsilon)^{s-1} + k_s(\xi + \varepsilon) \right) \\
 & = e^\frac{\xi}{2} \left( (\xi - \varepsilon s)^{s-2} - (\xi + \varepsilon)^{s-2} \right) + e s(\xi + \varepsilon)^{s-1} - e^2 s(s - 1)(\xi + \varepsilon)^{s-2} \\
 & = -\frac{1}{4} e^\frac{\xi}{2} s(s - 1)(s - 2) \int_{-\varepsilon}^{\varepsilon} (\varepsilon + \mu)^2(\mu + \xi)^{s-3}d\mu.
\end{align*}
\]

Therefore, we need to check that the function

\[
 s \mapsto \frac{\int_{-\varepsilon}^{\varepsilon} (\varepsilon + \mu)^2(\mu + \xi)^{s-3}d\mu}{\int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - \lambda^2)(\lambda + \xi)^{s-3}d\lambda}
\]

is increasing. After differentiating this function, we get the following expression in the numerator:

\[
\begin{align*}
 & \int_{-\varepsilon}^{\varepsilon} (\varepsilon + \mu)^2(\mu + \xi)^{s-3}\log(\mu + \xi) d\mu \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - \lambda^2)(\lambda + \xi)^{s-3}d\lambda \\
 & - \int_{-\varepsilon}^{\varepsilon} (\varepsilon + \mu)^2(\mu + \xi)^{s-3}d\mu \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - \lambda^2)(\lambda + \xi)^{s-3}\log(\lambda + \xi) d\lambda \\
 & = \int_{-\varepsilon}^{\varepsilon} (\varepsilon + \mu)^2(\varepsilon^2 - \lambda^2)(\mu + \xi)^{s-3}(\lambda + \xi)^{s-3}\left( \log(\mu + \xi) - \log(\lambda + \xi) \right) d\mu d\lambda.
\end{align*}
\]
Now, we symmetrize this expression (interchanging $\mu$ and $\lambda$):

$$
\frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \left[ (\varepsilon + \mu)^2 (\varepsilon^2 - \lambda^2) - (\varepsilon + \lambda)^2 (\varepsilon^2 - \mu^2) \right] (\mu + \xi)^{s-3} (\lambda + \xi)^{s-3} \left( \log(\mu + \xi) - \log(\lambda + \xi) \right) d\mu d\lambda
$$

$$
= \varepsilon \int_{-\varepsilon}^{\varepsilon} (\mu - \lambda)(\varepsilon + \mu)(\mu + \xi)^{s-3} (\lambda + \xi)^{s-3} \left( \log(\mu + \xi) - \log(\lambda + \xi) \right) d\mu d\lambda.
$$

Since log is an increasing function, we have

$$(\mu - \lambda)(\log(\mu + \xi) - \log(\lambda + \xi)) > 0.$$ 

This means that the derivative we have calculated is positive and the function \([B.21]\) increases. This completes the proof of the lemma. □

**Lemma B.4.** $B_{x_2x_2}B_{x_3x_3} - B_{x_2x_3}^2 \geq 0$ on $\mathbb{Z}_{ch^+}$.

**Proof.** Using \([A.9]\), we take a half sum of expressions \([A.13]\) and \([A.24]\) to obtain:

$$B_{x_2} = D(r) - D(p)B_{x_3}, \quad \text{(B.22)}$$

where

$$D(s) = D(s; a, b) = \frac{s}{2} \frac{b^{s-1} - a^{s-1}}{b - a}.$$ 

Using this representation, we express the derivatives of $B$ with respect to $x_2$ via derivatives with respect to $x_3$:

$$B_{x_2x_3} = D_{x_3}(r) - D_{x_3}(p)B_{x_3} - D(p)B_{x_3x_3};$$

$$B_{x_2x_2} = D_{x_2}(r) - D_{x_2}(p)B_{x_2} - D(p)B_{x_2x_2};$$

$$= [D_{x_2}(r) - D(p)D_{x_3}(r)] - [D_{x_2}(p) - D(p)D_{x_3}(p)]B_{x_3} + D(p)^2 B_{x_3x_3}.$$ 

Using these formulas, after some simplifications, we rewrite the minor under consideration in the following form:

$$B_{x_2x_3}B_{x_3x_3} - B_{x_2}^2 = \left[ E(r) - E(p)B_{x_3} \right] B_{x_3x_3} - \left[ D_{x_3}(r) - D_{x_3}(p)B_{x_3} \right]^2, \quad \text{(B.23)}$$

where $E(s) = D_{x_2}(s) + D(p)D_{x_3}(s)$.

Now, we plan to find some integral representations for each factor in \((B.23)\), from where it will be clear that this expression is positive. We already have an integral representation for $B_{x_3x_3}$, this is formula \((B.8)\). Let us show that the factor $E(r) - E(p)B_{x_3}$ has almost the same representation, the difference consists in the factor in front of the integral.

We start with an integral representation for $D$:

$$D(s) = \frac{s(s-1)}{2(b-a)} \int_{a}^{b} \lambda^{s-2} d\lambda. \quad \text{(B.24)}$$

Whence

$$D_a(s) = \frac{s(s-1)(s-2)}{2(b-a)^2} \int_{a}^{b} (b - \lambda) \lambda^{s-3} d\lambda;$$

$$D_b(s) = \frac{s(s-1)(s-2)}{2(b-a)^2} \int_{a}^{b} (\lambda - a) \lambda^{s-3} d\lambda,$$
and
\[ D_{x_i}(s) = \frac{s(s-1)(s-2)}{2(b-a)^2} \int_a^b \left[ (b-\lambda)a_{x_i} + (\lambda-a)b_{x_i} \right] \lambda^{s-3}d\lambda. \] (B.25)

We use this expression to get a representation for \( E \):
\[ E(s) = D_{x_2}(s) + D(p)D_{x_3}(s) \]
\[ = \frac{s(s-1)(s-2)}{2(b-a)^2} \int_a^b \left[ (b-\lambda)(a_{x_2} + D(p)a_{x_3}) + (\lambda-a)(b_{x_2} + D(p)b_{x_3}) \right] \lambda^{s-2}d\lambda. \] (B.26)

We have formula (B.6) for \( a_{x_3} \) and \( b_{x_3} \):
\[ (b-x_1)a_{x_3} = (x_1-a)b_{x_3} = \frac{(b-a)^2}{A(p)}, \] (B.27)
and formulas (A.23) and (A.14) for \( a_{x_2} \) and \( b_{x_2} \):
\[ (b-x_1)a_{x_2} = \frac{b^p-a^p-pb^{p-1}(b-a)}{A(p)} \quad \text{and} \quad (x_1-a)b_{x_2} = \frac{b^p-a^p-pa^{p-1}(b-a)}{A(p)}. \]

If we plug these expressions into the parts of (B.26), we get
\[ a_{x_2} + D(p)a_{x_3} = \frac{1}{b-x_1} \left[ \frac{b^p-a^p-pb^{p-1}(b-a)}{A(p)} + \frac{p}{2} \cdot \frac{b^{p-1} - a^{p-1}}{b-a} \cdot \frac{(b-a)^2}{A(p)} \right] \]
\[ = \frac{1}{2A(p)(b-x_1)} \left[ 2b^p - 2a^p - p(b^{p-1} + a^{p-1})(b-a) \right] = -\frac{1}{2(b-x_1)}, \]
and similarly
\[ b_{x_2} + D(p)b_{x_3} = \frac{1}{2(x_1-a)}. \]

As a result, we can rewrite (B.26) as follows
\[ E(s) = \frac{s(s-1)(s-2)}{4(b-a)^2} \int_a^b \frac{\lambda-a}{x_1-a} - \frac{b-\lambda}{b-x_1} \lambda^{s-3}d\lambda \]
\[ = \frac{s(s-1)(s-2)}{4(b-a)(x_1-a)(b-x_1)} \int_a^b (\lambda-x_1)\lambda^{s-3}d\lambda = \frac{A(p)}{4(b-a)^3} A_{x_3}(s). \] (B.28)

In the last equality we use representation (B.7).

Finally, we calculate the first factor in (B.23):
\[ E(r) - E(p)B_{x_3} \]
\[ = \frac{A(r)A(p) - E(p)A(r)}{A(p)} = \frac{1}{4(b-a)^4} (A_{x_3}(r)A(p) - A_{x_3}(p)A(r)) \]
\[ = \frac{A(p)^2}{4(b-a)^4} \frac{\partial}{\partial x_3} \left( \frac{A(r)}{A(p)} \right) \]
\[ = \frac{A(p)^2}{4(b-a)^4} B_{x_3x_3}. \]

Therefore,
\[ \left[ E(r) - E(p)B_{x_3} \right] B_{x_3x_3} = \left( \frac{A(p)}{2(b-a)^2} B_{x_3x_3} \right)^2 \]
\[ = \left( \frac{r(r-1)(r-2)p(p-1)(p-2)(b-a)}{4(x_1-a)(b-x_1)A(p)^2} \times \right. \]
\[ \left. \times \int_a^b \int_a^b \left[ (\lambda-a) + (a-b-\lambda)x_1 \right] (\lambda-a)(\lambda^r-p - \mu^r-p)p\mu^{p-3}d\lambda d\mu \right)^2. \] (B.29)
Now, we find an integral representation for another term in (B.23):

\[ D_x^3(r) - D_x^3(p)B_x^3 = \frac{D_x^3(r)A(p) - D_x^3(p)A(r)}{A(p)} \times \]

\[ \int_a^b \int_a^b \left[ (\lambda - a)(\mu - a)b_{x_1} - (b - \lambda)(b - \mu)a_{x_1} \right] (\lambda - \mu)\lambda^{p-3}\mu^{p-3}d\lambda d\mu \]

(B.30)

where we, as usual, used symmetrization to get the last formula in the chain.

It remains to compare the different parts of expression in (B.29) and (B.30). We see that

\[ (b - a) \left[ \lambda \mu - ab + (a + b - \lambda - \mu)x_1 \right] \geq \left| (\lambda - a)(\mu - a)(b - x_1) - (b - \lambda)(b - \mu)(x_1 - a) \right| \]

for \( \lambda, \mu, x_1 \in [a, b] \). Indeed, on the left-hand side we have a linear function in \( x_1 \) and we have a convex piecewise linear function on the right-hand side. The values of both functions coincide at the endpoints of the interval \( a \leq x_1 \leq b \). Therefore, inside the interval the left function is strictly larger than the right one. Since the remaining terms are the same and they do not change its sign inside the interval, we have the required inequalities for the squares of integrals. \( \square \)

References

[1] A. Burchard, G. Dafni, R. Gibara, Mean oscillation bounds on rearrangements, https://arxiv.org/pdf/2011.09111.pdf.
[2] J. B. Garnett, Bounded analytic functions. Graduate Texts in Mathematics, 236. Springer, New York, 2007. 459 pp. ISBN: 978-0-387-33621-3; 0-387-33621-4.
[3] P. Ivanisvili, D. M. Stolyarov, V. I. Vasyunin, P. B. Zatitskiy, Bellman function for extremal problems in BMO II: evolution, Memoirs of the AMS, 255:1220 (2018); https://doi.org/10.1090/memo/1220, preprint: http://arxiv.org/abs/1510.01010.
[4] A. Osękowski, Sharp Martingale and Semimartingale Inequalities, Springer Basel, (2012), 464pp, ISBN: 978-3-0348-0369-4.
[5] L. Slavin, V. Vasyunin, Sharp results in the integral form John–Nirenberg inequality, Trans. Amer. Math. Soc. 363: 8 (2011), 4135–4169.
[6] L. Slavin and V. Vasyunin, Sharp \( L^p \) estimates on BMO, Indiana Univ. Math. J. 61: 3 (2012), 1051–1110.
[7] L. Slavin, P. B. Zatitskiy, Dimension-free estimates for semigroup BMO and \( A_p \), to appear in Indiana University Mathematics Journal, preprint: https://arxiv.org/abs/1908.02602.
[8] Dmitriy M. Stolyarov, Vasily I. Vasyunin, Pavel B. Zatitskiy, Monotonic rearrangements of functions with small mean oscillation, Studia Mathematica 231 (2015), 257–267. https://doi.org/10.4064/sm38326-2-2016, preprint: https://arxiv.org/pdf/1506.00502.pdf.
[9] Dmitriy Stolyarov, Vasily Vasyunin, Pavel Zatitskiy, *Sharp multiplicative inequalities with BMO I*, Journal of Mathematical Analysis and Applications, Volume 492, Issue 2, 15 December 2020, 124479, https://doi.org/10.1016/j.jmaa.2020.124479, preprint: https://arxiv.org/pdf/2001.09454.pdf.

[10] D. M. Stolyarov, P. B. Zatitskiy, *Theory of locally concave functions and its applications to sharp estimates of integral functionals*, Adv. Math. 291 (2016), 228–273.

[11] V. Vasyunin, A. Volberg, *The Bellman Function Technique in Harmonic Analysis*, Cambridge University Press, (2020), 460pp, ISBN:978-1-108-48689-7.

Vasily Vasyunin  
Department of Mathematics and Computer Science,  
St. Petersburg State University, 14-th Line Vasilyevsky Island, 29,  
199178, St. Petersburg, Russia  
vasyunin@pdmi.ras.ru

Pavel Zatitskiy  
Department of Mathematics and Computer Science,  
St. Petersburg State University, 14-th Line Vasilyevsky Island, 29,  
199178, St. Petersburg, Russia  
pavelz@pdmi.ras.ru

Ilya Zlotnikov  
Department of Mathematics and Computer Science,  
St. Petersburg State University, 14-th Line Vasilyevsky Island, 29,  
199178, St. Petersburg, Russia  
i.zlotnikov@spbu.ru