Dunkl-Gamma Type Operators Including Appell Polynomials

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Received: 5 January 2019 / Accepted: 3 July 2019 / Published online: 16 July 2019
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Abstract
The aim of the present paper is to introduce Dunkl-gamma type operators in terms of Appell polynomials and to investigate approximation properties of these operators.

Keywords Dunkl exponential · Appell polynomial · Gamma function · Modulus of continuity · Peetre’s K-functional · Lipschitz class

Mathematics Subject Classification Primary 41A25 · 41A36; Secondary 33C45

1 Introduction

Recently, linear positive operators constructed by means of generating functions and their several extensions are intensively considered by many research authors, for example, we refer to [13–17,19,21,25–27]. In [14], Jakimovski et al. introduced linear...
positive operators in terms of Appell polynomials as follows
\[
(T_n f) (x) = \frac{e^{-nx}}{g (1)} \sum_{k=0}^{\infty} p_k (nx) \left( \frac{k}{n} \right)
\]
under the assumption
\[
\frac{a_k}{g (1)} \geq 0, \quad k = 0, 1, 2, \ldots
\]
where \( g (1) \neq 0 \). For Appell polynomials \( p_k (x) \) the generating function as follows
\[
g (t) e^{xt} = \sum_{k=0}^{\infty} p_k (x) t^k,
\]
where \( g (t) \) is an analytic function in the disc \( |t| < R \) \((R > 1)\)
\[
g (t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0
\]
(see [5]). In [6], Ciupa defined the following Durrmeyer type integral modification of the operators (1.1)
\[
(P_n f) (x) = \frac{e^{-nx}}{g (1)} \sum_{k=0}^{\infty} p_k (nx) \frac{n^{\lambda+k+1}}{\Gamma (\lambda + k + 1)} \int_{0}^{\infty} e^{-nt} t^{\lambda+k} f (t) \, dt
\]
under the assumption given by (1.2) where \( \lambda \geq 0 \). Sucu [24] introduced Dunkl analogue of the Szász operators by
\[
S_n^\mu (f; x) = \frac{1}{e_\mu (nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu (k)} f \left( \frac{k + 2\mu \theta_k}{n} \right), \quad n \in \mathbb{N}
\]
for any \( x \in [0, \infty), \, n \in \mathbb{N}, \, \mu \geq 0 \) and \( f \in C [0, \infty) \) by using Dunkl generalization of the exponential function \( e_\mu (x) \) defined by [23]
\[
e_\mu (x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_\mu (k)},
\]
where the coefficients \( \gamma_\mu \) are in the form
\[
\gamma_\mu (2k) = \frac{2^{2k} k!}{\Gamma (k + \mu + \frac{1}{2})} \quad \text{and} \quad \gamma_\mu (2k + 1) = \frac{2^{2k+1} k! \Gamma (k + \mu + \frac{3}{2})}{\Gamma (\mu + \frac{1}{2})}
\]
for \( k \in \mathbb{N}_0, \mu > -\frac{1}{2} \). Moreover, the next recursion formula is satisfied

\[
\gamma_\mu (k + 1) = (k + 1 + 2\mu \theta_{k+1}) \gamma_\mu (k), \quad k \in \mathbb{N}_0,
\]

where \( \theta_k \) is

\[
\theta_k = \begin{cases} 
0, & \text{if } k = 2p \\
1, & \text{if } k = 2p + 1
\end{cases}.
\]

Now, let us recall the Dunkl derivative operator \([9,10]\).

Let \( \mu \) be a real number which satisfies \( \mu > -\frac{1}{2} \). The Dunkl operator \( T_\mu \) is given

\[
T_\mu \phi (x) = \phi'(x) + \mu \frac{\phi(x) - \phi(-x)}{x},
\]

where \( \phi(x) \) is an entire function. For \( \mu = 0 \), the operator \( T_\mu \) gives the derivative operator. It is clear that

\[
T_\mu e_\mu (xt) = te_\mu (xt),
\]

\[
T_\mu x^n = \frac{\gamma_\mu (n)}{\gamma_\mu (n-1)} x^{n-1}.
\]

Moreover, the Dunkl generalization of the product of two function is given by

\[
T_\mu (fg)(x) = f(x) T_\mu g(x) + g(-x) T_\mu f(x) + f'(x) [g(x) - g(-x)],
\]

which gives the next result if the function \( g \) is an even function

\[
T_\mu (fg)(x) = f(x) T_\mu g(x) + g(x) T_\mu f(x).
\]

By the motivation of the paper \([24]\), many authors studied Dunkl analogue of the several approximation operators for example, we refer to \([1,4,7,11,12,18,20,22]\).

Wafi and Rao \([28]\) constructed Dunkl analogue of Szász Durrmeyer operators as

\[
D_n (f; x) = \frac{1}{e_\mu (nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu (k)} \frac{n^{k+2\mu \theta_k + \lambda + 1}}{\Gamma (k + 2\mu \theta_k + \lambda + 1)} \int_0^\infty e^{-nt} t^{k+2\mu \theta_k + \lambda} f(t) \, dt
\]

for any \( \lambda \geq 0, x \in [0, \infty), n \in \mathbb{N}, \mu \geq 0 \) and \( f \in C [0, \infty) \). The authors gave local approximation results and order of the approximation for functions with derivative of the bounded variation. They also studied weighted approximation results by means of ordinary convergence and \( A \)-statistical convergence.
In [3], Ben Cheikh studied some properties of Dunkl–Appell $d$-orthogonal polynomials. In that work, Dunkl–Appell polynomials $p_k(x)$ defined by

$$p_k(x) = \sum_{n=0}^{k} \binom{k}{n} \mu a_{k-n}x^n, \quad (a_k)_{k \geq 0}$$

are generated by

$$A(t)e_\mu(x) = \sum_{k=0}^{\infty} \frac{p_k(x)}{\gamma_\mu(k)}t^k, \quad (1.10)$$

where $A(t)$ is an analytic function in the disc $|t| < R$ ($R > 1$)

$$A(t) = \sum_{r=0}^{\infty} \frac{a_r}{\gamma_\mu(r)}t^r, \quad a_0 \neq 0 \quad (1.11)$$

and Dunkl-binomial coefficient is as follows

$$\binom{k}{n}_{\mu} = \frac{\gamma_\mu(k)}{\gamma_\mu(n)\gamma_\mu(k-n)}.$$

Note that $\gamma_0(k) = k!$ and $\binom{k}{n}_0 = \binom{k}{n}$.

Inspired by the above works, for any $x \in [0, \infty), \ f \in C[0, \infty)$, we define Dunkl analogue of the Appell Szász Durrmeyer operators as

$$D^*_n(f; x) = \frac{1}{e_\mu(nx)A(1)}\sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^{\infty} e^{-nt} t^{k+2\mu\theta_k+\lambda} f(t) \, dt, \quad (1.12)$$

where $\mu, \lambda \geq 0$, $A(1) \neq 0$, $\frac{a_{k-n}}{A(1)} \geq 0$, $(0 \leq n \leq k), \ k = 0, 1, 2, \ldots$, and $\gamma_\mu$ is defined by (1.4) and $A(t)$ is given as in (1.11).

Note that in the case of $\mu = 0$ the operator (1.12) gives the operator (1.3), and for $A(t) = 1$ the operator (1.12) reduces to the operator (1.9).

We organize the paper as follows. In Sect. 2, some lemmas are given and the convergence of the operators (1.12) is presented by universal Korovkin-type theorem. In Sect. 3, the rates of convergence of the operators $D^*_n(f)$ to $f$ are given by the usual and second modulus of continuity and Lipschitz class functions.
2 Approximation Properties of the Operators $D_n^*$

In what follows, we first give some lemmas and then prove convergence of the operators (1.12) using universal Korovkin-type theorem.

Lemma 2.1 \textit{From the generating function (1.10), the following equalities are satisfied}

\begin{equation}
\sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_\mu(k)} = A(1)e_\mu(nx),
\end{equation}

\begin{equation}
\sum_{k=0}^{\infty} \frac{p_{k+1}(nx)}{\gamma_\mu(k)} = (nx)A(1)e_\mu(nx)
+ \mu e_\mu(-nx)[A(1) - A(-1)] + A'(1)e_\mu(nx)
\end{equation}

and

\begin{equation}
\sum_{k=0}^{\infty} \frac{p_{k+2}(nx)}{\gamma_\mu(k)} = n^2x^2 A(1)e_\mu(nx) + 2nx e_\mu(nx)A'(1)
+ 2\mu e_\mu(-nx)\left[A'(1) - \frac{A(1) - A(-1)}{2}\right]
+ A''(1)e_\mu(nx).
\end{equation}

\textbf{Proof} Taking $t \to 1, x \to nx$ in (1.10), we get the first one. When we apply the Dunkl operator $T_\mu$ to both of sides of the equality (1.10), by using the relations (1.6), (1.7) and (1.8) we obtain the second and third relations. \qed

Lemma 2.2 \textit{For the operators $D_n^*$, one can have}

\begin{align*}
D_n^*(1; x) &= 1 \\
D_n^*(t; x) &= x + \frac{\mu}{n}\left\{\frac{e_\mu(-nx)}{e_\mu(nx)} \left[\frac{A(1) - A(-1)}{A(1)}\right]\right\} + \frac{1}{n}\left[\frac{A'(1) + \lambda + 1}{A(1)}\right] \\
D_n^*(t^2; x) &= x^2 + \frac{x}{n}\left\{2\mu\frac{e_\mu(-nx)}{e_\mu(nx)} \frac{A(-1)}{A(1)} + 2A'(1) + 2\lambda + 4\right\}
+ \frac{2\mu e_\mu(-nx)A'(1) + A'(-1)}{n^2}\left[A(1) - A(-1)\right]
+ \mu \frac{e_\mu(-nx)}{e_\mu(nx)}\left[A(1) - A(-1)\right] \\
&\quad + \frac{1}{n^2}\left[A''(1) + (2\lambda + 4)\frac{A'(1)}{A(1)} + (\lambda + 1)(\lambda + 2)\right].
\end{align*}
Proof For \( f(t) = 1 \) in the operator (1.12), we have

\[
D_n^*(1; x) = \frac{1}{e^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_{\mu}(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^{\infty} e^{-nt} t^{k+2\mu\theta_k+\lambda} f(t) \, dt
\]

= \frac{1}{e^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_{\mu}(k)},

from (2.1), it follows \( D_n^*(1; x) = 1 \). For \( f(t) = t \), the operator (1.12) reduces to

\[
D_n^*(t; x) = \frac{1}{e^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_{\mu}(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^{\infty} e^{-nt} t^{k+2\mu\theta_k+\lambda+1} \, dt
\]

= \frac{1}{e^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_{\mu}(k)} (k + 2\mu\theta_k + \lambda + 1)

= \frac{1}{ne^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_{k+1}(nx)}{\gamma_{\mu}(k)} + \frac{\lambda + 1}{ne^{nx} A(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{\gamma_{\mu}(k)}.

By considering the equalities (2.1) and (2.2), we obtain

\[
D_n^*(t; x) = \frac{(nx) A(1) e^{nx} + \mu e^{nx} (-nx) [A(1) - A(-1)] + A'(1) e^{nx} (nx) + \lambda + 1}{ne^{nx} A(1)}
\]

= \left[ x + \frac{\mu}{n} \left\{ \frac{e^{nx} (-nx)}{e^{nx}} \left[ \frac{A(1) - A(-1)}{A(1)} \right] \right\} + \frac{1}{n} \left\{ A'(1) A(1) + \lambda + 1 \right\} \right].

Similarly, for \( f(t) = t^2 \), by means of the equalities (2.1)–(2.3), it is seen that the equality (2.4) holds. \( \square \)

Lemma 2.3 For each \( x \in [0, \infty) \), it follows from the results in Lemma 2.2

\[
\Omega_n^1(x) := D_n^* ((t - x); x)
\]

= \left[ \frac{\mu}{n} \left\{ \frac{e^{nx} (-nx)}{e^{nx}} \left[ \frac{A(1) - A(-1)}{A(1)} \right] \right\} + \frac{1}{n} \left\{ A'(1) A(1) + \lambda + 1 \right\} \right]

\[
\Omega_n^2(x) := D_n^* ((t - x)^2; x)
\]

= \left[ \frac{2x}{n} \left\{ 1 + \frac{\mu e^{nx} (-nx)}{e^{nx}} \left[ \frac{2A(-1) - A(1)}{A(1)} \right] \right\} \right]

\left[ + \frac{1}{n^2} \frac{e^{nx} (-nx)}{e^{nx}} \left\{ 2\mu A'(1) + A'(-1) \frac{A(1) - A(-1)}{A(1)} + \mu (2\lambda + 3) \frac{A(1) - A(-1)}{A(1)} \right\} \right]
\[ + \frac{1}{n^2} \left\{ \frac{A''(1)}{A(1)} + 2(\lambda + 2) \frac{A'(1)}{A(1)} + (\lambda + 1)(\lambda + 2) \right\} + \frac{2\mu^2}{n^2} \left( \frac{A(1) - A(-1)}{A(1)} \right). \] 

(2.5)

**Theorem 2.4** Let \( f \in C [0, \infty) \cap E \) and \( D^*_n \) be the operators defined by (1.12). If the following convergence is verified

\[ \lim_{n \to \infty} D^*_n (f; x) = f(x), \]

uniformly on each compact subset of \([0, \infty)\), where

\[ E := \left\{ f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty \right\}. \]

**Proof** Using Lemma 2.2, we have that

\[ \lim_{n \to \infty} D^*_n \left( t^i; x \right) = x^i, \ i = 0, 1, 2, \]

uniformly in each compact subset of \([0, \infty)\). Taking into account of the property (vi) in Theorem 4.1.4 in [2], the proof is completed. \( \square \)

### 3 Rates of Convergence

In this part, we calculate the order of approximation by means of the usual and second modulus of continuity and Lipschitz class functions. First of all, we recall some definitions as follows.

Let \( f \in \tilde{C}[0, \infty) \) and \( \delta > 0 \). The modulus of continuity of \( f \) denoted by \( \omega (f; \delta) \) is defined by

\[ \omega (f; \delta) := \sup_{x, y \in [0, \infty) : |x - y| \leq \delta} |f(x) - f(y)| \]

where \( \tilde{C}[0, \infty) \) is the space of uniformly continuous functions on \([0, \infty)\). Then, for any \( \delta > 0 \) and each \( x \in [0, \infty) \), we have the following inequality

\[ |f(x) - f(y)| \leq \omega (f; \delta) \left( \frac{|x - y|}{\delta} + 1 \right). \] (3.1)

Let us denote the space

\( C_B [0, \infty) = \{ f : [0, \infty) \to \mathbb{R}, \ f \text{ is bounded and uniformly continuous} \} \)

it is given with the norm \( \| f \|_{C_B} = \sup_{x \in [0, \infty)} |f(x)| \).
The second modulus of continuity is given by

$$\omega_2(f; \delta) := \sup_{0 < t \leq \delta} \| f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot) \|_{C_B},$$

for $f \in C_B[0, \infty)$.

Now, we recall the following definitions.

**Definition 3.1** A continuous function $f : [0, \infty) \to \mathbb{R}$ is said to be Lipschitz continuous of order $\gamma$ on $[0, \infty)$ if the following inequality is satisfied

$$|f(x) - f(y)| \leq M |x - y|^{\gamma}$$

for $x, y \in [0, \infty)$ with $M > 0$ and $0 < \gamma \leq 1$. We denote the set of Lipschitz continuous functions with Lip$_M(\gamma)$.

**Definition 3.2** [8] Peetre’s $K$-functional of the function $f \in C_B[0, \infty)$ is defined by

$$K(f; \delta) := \inf_{h \in C_B^2[0, \infty)} \left\{ \| f - h \|_{C_B} + \delta \| h \|_{C_B^2} \right\}$$

(3.2)

where

$$C_B^2[0, \infty) := \{ h \in C_B[0, \infty) : h', h'' \in C_B[0, \infty) \}$$

and the norm

$$\| h \|_{C_B^2} := \| h \|_{C_B} + \| h' \|_{C_B} + \| h'' \|_{C_B}.$$

It is clear that the following inequality

$$K(f; \delta) \leq M \left\{ \omega_2(f; \sqrt{\delta}) + \min(1, \delta) \| f \|_{C_B} \right\}$$

(3.3)

holds for all $\delta > 0$. The constant $M$ is independent of $f$ and $\delta$.

**Theorem 3.3** For $f \in \tilde{C}[0, \infty) \cap E$, we have

$$\left| D_n^* (f; x) - f(x) \right| \leq 2\omega(f; \sqrt{\Omega_n^2(x)}),$$

where $\Omega_n^2$ is given as in Lemma 2.3.

**Proof** From linearity and positivity of the operators $D_n^*$, by applying (3.1), we get

$$\left| D_n^* (f; x) - f(x) \right| \leq \frac{1}{e^{\mu(nx)} A(1)} \sum_{k=0}^{\infty} p_k(nx) \gamma_k^{\mu/k} \mu_k^{n+2\mu k+1}$$

$$\frac{\Gamma(k+2\mu k+\lambda+1)}{\Gamma(k+\lambda+1)}$$
\[
\times \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} |f(t) - f(x)| dt
\]

\[
\leq \frac{1}{e_\mu (nx) A(1)} \sum_{k=0}^\infty \frac{p_k(nx)}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k + \lambda + 1}}{\Gamma(k + 2\mu\theta_k + \lambda + 1)}
\times \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} \frac{|t-x|}{\delta} \omega(f; \delta) dt
\]

\[
\leq \left\{ 1 + \frac{1}{\delta} \frac{1}{e_\mu (nx) A(1)} \sum_{k=0}^\infty \frac{p_k(nx)}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k + \lambda + 1}}{\Gamma(k + 2\mu\theta_k + \lambda + 1)}
\times \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} |t-x| dt \right\} \omega(f; \delta).
\] (3.4)

From the Cauchy–Schwarz inequality for integration, one may write

\[
\int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} |t-x| dt
\]

\[
\leq \left( \frac{\Gamma(k + 2\mu\theta_k + \lambda + 1)}{n^{k+2\mu\theta_k + \lambda + 1}} \right)^{1/2} \left( \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} (t-x)^2 dt \right)^{1/2},
\]

by using this inequality, it follows that

\[
\sum_{k=0}^\infty \frac{p_k(nx)}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k + \lambda + 1}}{\Gamma(k + 2\mu\theta_k + \lambda + 1)} \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} |t-x| dt
\]

\[
\leq \sum_{k=0}^\infty \frac{p_k(nx)}{\gamma_\mu(k)} \left( \frac{n^{k+2\mu\theta_k + \lambda + 1}}{\Gamma(k + 2\mu\theta_k + \lambda + 1)} \right)^{1/2} \left( \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} (t-x)^2 dt \right)^{1/2}.
\] (3.5)

If we now apply the Cauchy–Schwarz inequality for sum on the right hand side of (3.5), we get

\[
\sum_{k=0}^\infty \frac{p_k(nx)}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k + \lambda + 1}}{\Gamma(k + 2\mu\theta_k + \lambda + 1)} \int_0^\infty e^{-nt} t^{k+2\mu\theta_k + \lambda} |t-x| dt
\]

\[
\leq \sqrt{e_\mu(nx) A(1) \left( e_\mu(nx) A(1) D_n^* \left( (t-x)^2; x \right) \right)^{1/2}}.
\]
\[ e_\mu (nx) A (1) \left( D_n^* \left( (t - x)^2 ; x \right) \right)^{1/2} = e_\mu (nx) A (1) \left( \Omega_n^2 (x) \right)^{1/2} , \]  

(3.6)

where \( \Omega_n^2 (x) \) is as in the equality (2.5). When we consider (3.6) in (3.4), we obtain

\[ |D_n^* (f; x) - f (x)| \leq \left\{ 1 + \frac{1}{\delta} \sqrt{\Omega_n^2 (x)} \right\} \omega (f; \delta). \]

If we choose \( \delta = \sqrt{\Omega_n^2 (x)} \), we arrive at

\[ |D_n^* (f; x) - f (x)| \leq 2 \omega \left( f; \sqrt{\Omega_n^2 (x)} \right). \]

We note that \( \Omega_n^2 (x) \) goes to zero when \( n \to \infty \). \( \square \)

**Theorem 3.4** For \( f \in \text{Lip}_M (\alpha) \), such that \( 0 < \alpha \leq 1, M \in \mathbb{R}^+ \) we have

\[ |D_n^* (f; x) - f (x)| \leq M \left( \Omega_n^2 (x) \right)^{\frac{\alpha}{2}}, \]

where \( \Omega_n^2 \) is given in Lemma 2.3.

**Proof** Since \( f \in \text{Lip}_M (\alpha) \), we can write from linearity

\[ |D_n^* (f; x) - f (x)| \leq D_n^* (|f (t) - f (x)| ; x) \leq M D_n^* (|t - x|^\alpha ; x). \]

By taking into account Lemma 2.3 and Hölder inequality, we get

\[ |D_n^* (f; x) - f (x)| \leq M \left( \Omega_n^2 (x) \right)^{\frac{\alpha}{2}}, \]

which ends the proof. \( \square \)

Now, we give rate of convergence of the operators \( D_n^* \) via Peetre’s K-functional.

**Lemma 3.5** For any \( h \in C_B^2 [0, \infty) \), we have

\[ |D_n^* (h; x) - h (x)| \leq \lambda_n (x) \|h\|_{C_B^2 [0, \infty)} \]

where

\[ \lambda_n (x) = \Omega_n^1 (x) + \frac{\Omega_n^2 (x)}{2}. \]

(3.7)
\textbf{Proof} From the Taylor’s series of the function \( h \in C^2_B[0, \infty) \), we can write

\[ h(t) = h(x) + h'(x)(t-x) + (t-x)^2 \frac{h''(\xi)}{2!}, \quad \xi \in (x,t). \]

By applying \( D_n^* \) to the above equality and then using the linearity of the operator, we get

\[ D_n^*(h;x) - h(x) = h'(x) D_n^*((t-x);x) + \frac{h''(\xi)}{2} D_n^*((t-x)^2;x). \]

By considering Lemma 2.3, one can have

\[ \left| D_n^*(h;x) - h(x) \right| \leq \left| h'(x) D_n^*((t-x);x) \right| + \left| \frac{h''(\xi)}{2} D_n^*((t-x)^2;x) \right| \]

\[ \leq \left\| h' \right\|_{C_B[0,\infty)} \Omega_1^1(x) + \frac{1}{2} \left\| h'' \right\|_{C_B[0,\infty)} \Omega_2^2(x) \]

\[ \leq \left( \Omega_1^1(x) + \frac{\Omega_2^2(x)}{2} \right) \| h \|_{C_B^2[0,\infty)}. \]

So the proof is completed. \( \square \)

\textbf{Theorem 3.6} For any \( f \in C_B[0, \infty) \), we have

\[ \left| D_n^*(f;x) - f(x) \right| \leq 2M \left\{ \min\left(1, \frac{\lambda_n(x)}{2}\right) \| f \|_{C_B[0,\infty)} + \omega_2\left(f;\sqrt{\frac{\lambda_n(x)}{2}}\right) \right\}, \]

where \( M \) is a positive constant which is independent of \( n \) and \( \lambda_n(x) \) is given by (3.7).

\textbf{Proof} Let \( h \in C^2_B[0, \infty) \). In view of Lemma 3.5, one can have

\[ \left| D_n^*(f;x) - f(x) \right| \leq \left| D_n^*(f-h;x) \right| + \left| D_n^*(h;x) - h(x) \right| + \left| f(x) - h(x) \right| \]

\[ \leq 2 \| f - h \|_{C_B[0,\infty)} + \lambda_n(x) \| h \|_{C_B^2[0,\infty)} \]

\[ = 2 \left\{ \| f - h \|_{C_B[0,\infty)} + \frac{\lambda_n(x)}{2} \| h \|_{C_B^2[0,\infty)} \right\}, \]

from (3.2) it follows

\[ \left| D_n^*(f;x) - f(x) \right| \leq 2K\left(f;\frac{\lambda_n(x)}{2}\right). \]
By (3.3), we obtain
\[
\left| D_n^* (f; x) - f (x) \right| \leq 2M \left\{ \min \left( 1, \frac{\lambda_n (x)}{2} \right) \| f \|_{C_B [0, \infty)} + \omega_2 \left( f; \sqrt{\frac{\lambda_n (x)}{2}} \right) \right\},
\]
which completes the proof. \(\Box\)

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