Wave-packet Formalism of Full Counting Statistics

F. Hassler, M.V. Suslov, G.M. Graf, M.V. Lebedev, G.B. Lesovik, and G. Blatter

1Theoretische Physik, ETH Zurich, CH-8093 Zurich, Switzerland
2Institute of Solid State Physics RAS, 142432 Chernogolovka, Moscow Region, Russia
3L.D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia

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We make use of the first-quantized wave-packet formulation of the full counting statistics to describe charge transport of noninteracting electrons in a mesoscopic device. We derive various expressions for the characteristic function generating the full counting statistics, accounting for both energy and time dependence in the scattering process and including exchange effects due to finite overlap of the incoming wave packets. We apply our results to describe the general stochastic properties of a two-fermion scattering event and find, among other features, sub-binomial statistics for nonentangled incoming states (Slater rank 1), while entangled states (Slater rank 2) may generate super-binomial (and even super-Poissonian) noise, a feature that can be used as a spin singlet-triplet detector. Another application is concerned with the constant-voltage case, where we generalize the original result of Levitov-Lesovik to account for energy-dependent scattering and finite measurement time, including short time measurements, where Pauli blocking becomes important.

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I. INTRODUCTION

Charge transport across an obstacle in a wire is a statistical process, whose complete description is provided by the probability function $P(n, t)$, telling how many charge carriers $n$ are transmitted through the wire during the time $t$. The calculation of this full counting statistics usually aims at the generating function $\chi(\lambda, t) = \sum_n P(n, t) e^{i\lambda n}$ for this process, from which the probability distribution $P(n, t)$ follows through simple Fourier transformation $\mathcal{F}[\chi(\lambda, t)] = P(n, t)$. The proper physical definition of the generating function $\chi(\lambda, t)$ is a nontrivial problem and has been solved by Levitov and Lesovik back in 1993 see also Ref. [2] with numerous applications to follow[2]. The original definition includes a ‘charge counter’ in the form of a spin, coupled via the gauge potential to the moving charges, and has been cast in a second-quantized formalism of appreciable complexity. The recent observations[4] of the correspondence between the generating function $\chi_1(\lambda)$ of the full counting statistics for one particle and the notion of fidelity in a (one-particle, chaotic) quantum system[5] has lead to a much simpler first-quantized formulation of full counting statistics, including the generalization $\chi_N(\lambda)$ to $N$ particles. In fact, a first-quantized version of charge transport to calculate noise has been already introduced some years ago[6]. Furthermore, such a wave-packet formalism naturally describes the statistics of pulsed transport, where unit-flux voltage pulses generate single-particle excitations feeding the device of interest[2,7,8,9,10] (a source injecting individual electrons into a quantum wire has been realized in a recent experiment[11]). The simplicity of the first-quantized formalism then has allowed to obtain nontrivial results on the full counting statistics for an energy dependent scatterer, including its dependence on the exchange symmetry of the transported charge[12].

In this paper, we make intense use of this wave-packet formalism of charge transport and (re-)derive various expressions for the characteristic function $\chi_N(\lambda)$ in a much simplified manner. We start with an $N$-particle Slater determinant made from orthonormalized single-particle wave functions $\phi_m$ describing fermions incident from the left and derive the associated characteristic function describing the full counting statistics in determinant form,

$$\chi_N(\lambda) = \det \langle \phi_m | 1 - \mathcal{T} + \mathcal{T} e^{i\lambda} | \phi_n \rangle, \quad (1)$$

with the operator $\mathcal{T}$ describing the energy dependent transmission across the scatterer, $\mathcal{T} = \int (dk/2\pi) T_k |k\rangle \langle k|$ in momentum ($k$) representation (here, the particle number $N$ replaces the time variable $t$ in the original formula1). The determinant in Eq. (1) can be cast in a product form

$$\chi_N(\lambda) = \prod_{m=1}^{N} (1 - \tau_m + \tau_m e^{i\lambda}), \quad (2)$$

where $\tau_m$ are the eigenvalues of the Hermitian operator $\mathcal{T}$ in the space spanned by the basis states $|\phi_n\rangle$. We denote the distribution in (2) as ‘generalized binomial’.

In a real experiment, the unit-flux voltage pulses generating the incoming wave packets may overlap. For this situation, we rederive the simple and elegant expression (2) for the full counting statistics, but with the coefficients $\tau_m$ now replaced by the roots of a generalized eigenvalue problem incorporating all effects of fermionic statistics and the full energy dependence of the transmission. The results (1) and (2) apply to a nonentangled incident state in the form of a Slater determinant[12,13] an extension to include entangled states of Slater rank 2 is provided as well[14].

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where the determinant in (1) has to be taken over all the single-particle Hilbert space.

Subsequently, we analyze the situation with \( N \) fermions and derive the full counting statistics for a constant voltage (\( V \)) drive, thereby generalizing the original result of Levitov and Lesovik\(^{18}\) to describe transport with an energy dependent scattering transmission (cf. Ref. 19).

Our result,

\[
\log \chi_N(\lambda) = N \frac{2 \pi \hbar v_F}{e V} \int_0^{e V/h v_F} \frac{dk}{2 \pi} \log(1 - T_k + T_k e^{i \lambda}),
\]

admits the simple interpretation of the full counting statistics as deriving from the transmission of the unbalanced Fermi sea residing between energies \( E_F \) and \( E_F + e V \), with \( E_F \) denoting the Fermi energy and \( V \) the applied bias. Using an alternative derivation based on \(3\) and stationary scattering states, we determine the short-time limit of the counting statistics and rederive the binomial result \(5\) in the long-time situation, with the particle number \( N \) replaced by the measuring time \( t \), \( N \to t e V/2 \pi \hbar \). The use of our determinant formula combined with Szegő’s theorem\(^{20,21}\) will allow us to present a rigorous derivation of these results.

In the following, we give a short review of previous work on the subject and then derive the characteristic functions \(1\) and \(2\) of \( N \) incoming fermions. In Sec. II, we apply these results to discuss the statistical transport-properties of two fermions. Section IV is devoted to the calculation of the characteristic function for the constant voltage case starting from \( N \)-particle trains and letting the width of the individual wave packets go to infinity. In Sec. V we derive the results \(3\) and \(4\) describing the setup involving a time dependent scattering and counting incoherent superpositions of incoming particles. We rederive the constant voltage result as an application, including the short-time limit.

### II. FULL COUNTING STATISTICS

The first suggestion\(^{22}\) of a generating function for full counting statistics relied on the straightforward expression \( \chi(\lambda, t) = \langle \exp[i \lambda \int dt' T(t')] \rangle \), where \( T(t) \) denotes the current operator. It then was soon realized\(^{22}\) that this definition does not correspond to any known (even on the level of a ‘Gedanken Experiment’) measuring procedure; still, this first definition produced the correct results for all irreducible zero-frequency current-current correlators \( \langle \langle I_0(t_0) \rangle \rangle \) (see also the discussion in Ref. 23). The first ‘practical’ definition\(^{23}\) of a generating function \( \chi(\lambda, t) \), corresponding (at least in principle) to a realistic counting experiment, involved a spin-galvanometer as a measurement device (see also Ref. 2). Recently, it has been pointed out\(^{24}\) that this suggestion (corresponding rather to a ‘Gedanken Experiment’) could actually be realized with qubits serving as a measuring device, whereby the ‘environmental noise’ generated by the transmitted charge serves as the measurement signal for the full counting statistics. This contrasts with the usual interpretation of the ‘environmental noise’ as being responsible for the qubit’s dephasing\(^{25}\) expressed through the fidelity, and also relates to the competition between the gain of information and dephasing\(^{26}\) in quantum measurement theory.

The insight on the equivalence between the notions of fidelity and full counting statistics has motivated a first-quantized formalism of the counting problem in terms of wave packets. Fidelity \( |\chi_{\text{id}}| \), the modulus of the overlap \( \chi_{\text{id}} = \langle \Psi_2 | \Psi_1 \rangle \), was introduces by Peres\(^{27}\) in the context of chaotic systems. It measures the overlap between two wave functions \( |\Psi_{1/2} \rangle \) which describe an initial state \( |\Psi_0 \rangle \) which has evolved under the action of two slightly different Hamiltonians. In the context of full counting statistics of a single particle measured by a spin counter, the wave functions \( \Psi_1 \) and \( \Psi_2 \) are substituted by scattering states \( \Psi_{\text{out}}^+ \) and \( \Psi_{\text{out}}^- \) interacting with the spin counter in the states \( |\uparrow \rangle \) and \( |\downarrow \rangle \), resulting in an expression for the generating function in the form \( \chi_1 = \langle \Psi_{\text{out}}^- | \Psi_{\text{out}}^+ \rangle \). This new first-quantized formulation in terms of wave-packets provides a drastic simplification as compared to the original second-quantized formalism.\(^{28}\) While the use of a second-quantized formalism is mandatory for the description of particles describing bosonic excitations of fields (photons, phonons, etc.), here, we deal with non-
relativistic electrons where the particle number is fixed, thus allowing for an alternative first-quantized description. Moreover, our wave-packet formalism has technical merits (e.g., in the description of energy dependent scattering or in the classification of two-particle scattering events) and also provides a better physical understanding. We remark, however, that in dealing with finite temperatures we make use of the second-quantized formalism in Fock space.

An alternative method, to the procedure based on a spin counter, was pursued in several contributions27-29 where the full counting statistics and, in particular, its generating function $\chi(\lambda, t)$, was constructed using only basic quantum mechanical definitions; starting with an initial state in the form of an eigenstate of the particle number operator with a fixed particle number to the right of the scatterer (or the ‘counter’), a second projection (to eigenstates of the number operator) onto the final state is carried out after the observation time $t$. Both procedures, projection and spin-counting, lead to the same expressions for the generating function $\chi$, provided that the incoming state involves no superposition across the scatterer. In the latter situation, the explicit calculation using a spin-counter produces a fidelity describing the decoherence of the spin, while an interpretation in terms of a generating function can produce probabilities for non-integer charge transport and hence is unphysical. On the other hand, the projection method, destroying such a superposition in the course of the first measurement, always admits an interpretation in terms of probabilities.

A. One particle

In this paper, we make extensive use of the first-quantized formulation of the generating function: starting with a simple one-particle problem, we exploit the equivalence between the notion of fidelity and full counting statistics.\(^1\) Consider an incoming wave packet $\psi(x; t \rightarrow -\infty)$ from the left of the form

$$\psi(x; t) = \int \frac{dk}{2\pi} \phi_1(k) e^{i k x - i e(k) t}$$

with normalization $\int (dk/2\pi)|\phi_1(k)|^2 = 1$, cf. Fig. 1. In the following, we assume (for simplicity) a linear spectrum $\epsilon = v_F k$ with $v_F$ the Fermi velocity; at low temperatures and voltages the interesting physics usually takes place near the Fermi points. The momentum $hk$ and the energy $\hbar \epsilon$ are measured with respect to the Fermi momentum $k_F$ and the Fermi energy $E_F$. Here and below, the wave-packets include only momenta with $k > 0$ in order not to disturb the Fermi sea which is considered to be the vacuum in our analysis. The scatterer at $x = 0$ is characterized by momentum(energy)-dependent transmission (reflection) amplitudes $t_k$ ($r_k$), particle reflection takes us to the branch $\epsilon = -v_F k$, with $k$ measured relative to $-k_F$. The spin- (or qubit-) counter, placed to the right of the scatterer, contributes a phase-factor $e^{\pm i \lambda/2}$ to the wave function, where the sign depends on the state $|\uparrow\rangle$, $|\downarrow\rangle$ of the spin-counter. The outgoing ($t \rightarrow \infty$) wave function assumes the form (we place the counter right behind the scatterer at $x = 0$)

$$\psi_\text{out}^\pm(x; t) = \int \frac{dk}{2\pi} [r_k e^{-ik(x+vt)} \Theta(-x)$$

$$+ t_k e^{ik(x-vt)} e^{\pm i \lambda/2} \Theta(x)] \phi_1(k)$$

and consists of reflected ($x < 0$) and transmitted ($x > 0$) parts; $\Theta(x)$ is the unit step-function. The fidelity $\chi_1(\lambda)$ is given by the overlap of wave functions with slightly different perturbations in their evolution, here, with coupling to opposite spin-configurations $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$\chi_1(\lambda) = \int dx \psi^-\text{out}(x; t)^* \psi^+\text{out}(x; t)$$

$$\rightarrow \int \frac{dk}{2\pi} (1 - T_k + T_k e^{i\lambda}) |\phi_1(k)|^2$$

$$= \langle \phi_1 | 1 - \mathcal{T} + \mathcal{T} e^{i\lambda} | \phi_1 \rangle$$

and

$$\langle \mathcal{N} \rangle = \sum_m P_m e^{i\lambda m}$$

of the full counting statistics as defined in Ref. 24 where a spin-galvanometer has been used as a measuring device. The Fourier-coefficients $P_m$ are the probabilities for transmitting $m$ particles. For the simple example of one incoming particle only two possibilities are possible, particle reflection with probability $P_0 = 1 - \langle \mathcal{T} \rangle$ and particle transmission with $P_1 = \langle \mathcal{T} \rangle$, where $\langle \mathcal{T} \rangle = \langle \phi_1 | \mathcal{T} | \phi_1 \rangle$ denotes the average transmission probability. Knowing the characteristic function, the cumulants $\langle \mathcal{N}^3 \rangle$ can be obtained as the coefficients in the Taylor series of $\log \chi(\lambda)$,

$$\langle \mathcal{N}^3 \rangle = \left( \frac{d}{d\lambda} \right)^3 \log \chi(\lambda) \bigg|_{\lambda=0}$$

The ratio $F = \langle \mathcal{N}^2 \rangle / \langle \mathcal{N} \rangle$ between the second and the first cumulant, called Fano-factor, will be of special interest later.

B. $N$ particles

Next, we extend the above description to $N$ particles with an incoming wave function $\Psi(k)$ defined in momentum space; the vector $k = (k_1, \ldots, k_N)$ specifies the $N$
momenta of the particles. We consider independent particles without interaction which scatter independently. After scattering, the outgoing wave function assumes the asymptotic \((t \to \infty)\) form

\[
\psi_{\text{out}}(x; t) = \left\{ \sum_{m=1}^{N} \int \frac{dk_m}{2\pi} \left[ t_{km} e^{-ik_m(x_m + v_r t)} \Theta(-x_m)ight.ight. \\
+ t_{km} e^{ik_m(x_m - v_r t)} e^{\pm i\lambda/2} \Theta(x_m) \left. \right] \right\} \Psi(k), \tag{11}
\]

i.e., the evolution is the product of the single-particle evolutions in expression (7). The characteristic function of the full counting statistics \(\chi_N(\lambda) = \int dx \psi_{\text{out}}^*(x; t) \psi_{\text{out}}(x; t)\) then can be cast into the form

\[
\chi_N(\lambda) = \left\{ \sum_{m=1}^{N} \int \frac{dk_m}{2\pi} (1-T_{km} + T_{km} e^{i\lambda}) \right\} |\Psi(k)|^2. \tag{12}
\]

So far, we did not specify the specific type of incoming wave function. If we limit ourselves to Slater determinant states composed of orthonormalized single particle states \(\phi_m\),

\[
\Psi(k_1, \ldots, k_N) = \frac{1}{\sqrt{N!}} \det \phi_m(k_n), \tag{13}
\]

the expression Eq. (12) can be rewritten as a single determinant (see Eq. (10))

\[
\chi_N(\lambda) = \det \int \frac{dk}{2\pi} \phi_m^*(k)(1-T_k + T_k e^{i\lambda}) \phi_n(k) = \det \langle \phi_m | 1 - T + Te^{i\lambda} | \phi_n \rangle \tag{14}
\]

involving the single-particle matrix elements \(\langle \phi_m | O | \phi_n \rangle\) of the operator \(O = 1 - T + Te^{i\lambda}\).

C. Nonorthogonal basis

In a physical realization of such a scattering experiment, one usually does not populate orthogonal states as used in the above construction of the Slater determinant. E.g., in the setup of Fig. 1 the electrons typically occupy states \(f_1\) and \(f_2\) with a finite overlap, i.e., they are nonorthogonal. Of course, an \(N\)-particle Slater determinant can be constructed as well out of nonorthogonal states \(|f_m\rangle\), provided they are linearly independent, i.e., \(\det(f_m|f_n) \neq 0\). The properly antisymmetrized and normalized wave function (13) then acquires the form

\[
\Psi'(k_1, \ldots, k_N) = \frac{1}{\sqrt{N! \det(f_m|f_n)}} \det f_m(k_n). \tag{15}
\]

Inserting this expression into (12) and repeating the calculation that led to (14), we obtain the generating function in the form of a ratio of two determinants,

\[
\chi_N(\lambda) = \frac{\det(f_m|1 - T + Te^{i\lambda}|f_n)}{\det(f_m|f_n)} = \frac{\det(S' - T' + T'e^{i\lambda})}{\det S'} \tag{16}
\]

with the two \(N \times N\) matrices

\[
S'_{mn} = \langle f_m|f_n \rangle \quad \text{and} \quad T'_{mn} = \langle f_m|T|f_n \rangle. \tag{17}
\]

D. Invariance of Slater Determinants under Linear Transformations

It turns out that the expression (16) for the generating function can be drastically simplified and rewritten in a generalized binomial form. As a first step towards this goal, one has to realize that an \(N\)-dimensional Hilbert space \(H_N\), spanned by the single-particle wave-functions \(f_m(k)\), defines exactly one properly antisymmetrized wave function, or, equivalently, there exists (up to a phase factor) only one associated \(N\)-particle Slater determinant state. The antisymmetrized \(N\)-particle state is thus a property of the Hilbert space \(H_N\) and is independent on the basis chosen.\(^{20}\)

Consider, as a simple example, a two-particle Slater-determinant state (in second-quantized notation) \(|\Psi\rangle = a_1^\dagger a_2^\dagger |0\rangle\), with the vacuum-state \(|0\rangle\) and Fermionic operators \(a_{1,2}\). Defining the new operators \(a_{1,2} = (a_1 \pm a_2)/\sqrt{2}\), we easily see that the two-particle state

\[
a_1^\dagger a_2^\dagger |0\rangle = \frac{1}{2}(a_1^1 + a_2^2) (a_1^1 - a_2^2) |0\rangle = a_2^1 a_1^1 |0\rangle = |\Psi\rangle \tag{18}
\]

remains unchanged. Consider then a general \(N\)-particle Slater determinant state of the form Eq. (15). Transforming the basis states \(f_m(k)\) to new states \(g_m(k)\) via the complex linear transformation

\[
g_m(k) = \sum_n A_{nm} f_n(k), \quad \det A \neq 0, \tag{19}
\]
the antisymmetric combination
\[ \det g_m(k_n) = (\det A) \det f_m(k_n) \]  
(20)
remains invariant up to the factor \( \det A \); here, we have used the fact that the determinant of the product of two matrices is the product of the individual determinants. Furthermore, the normalized \( N \)-particle Slater-determinant states \( \Psi_f \) and \( \Psi_g \) obey the relation
\[ \Psi_g(k_1, \ldots, k_N) = \text{sgn}(\det A) \Psi_f(k_1, \ldots, k_N) \]  
(21)
with \( \text{sgn}(x) = x/|x| \). The only effect of adopting a new basis is the appearance of an overall phase factor \( \text{sgn}(\det A) \) which drops out of the characteristic function \( \chi \). Therefore, the full counting statistics calculated in the bases \( f \) and \( g \) give identical results.

**E. Diagonalization**

The above invariance can be used to simplify the calculation of the full counting statistics. Furthermore, even without specification of the (time-independent) scatterer, one can obtain valuable insights about the structure of possible outcomes in the counting statistics. In particular, it turns out that the most general full counting statistics for a Slater-determinant state is given by a generalized binomial expression of the form \( (12) \). Therefore, the full counting statistics calculated in the bases \( f \) and \( g \) give identical results.

Let us first investigate how the invariance under linear transformations, Eq. \( (19) \), manifests itself in the determinant formula Eq. \( (16) \). To this end, we note that any one-particle matrix \( B \) of the form \( (17) \) transforms under the linear transformation \( A \) of the basis functions according to
\[ B^g = A^f B^f A, \quad B = S, T. \]  
(22)
Since \( \det(AB) = \det A \det B \), we find that the characteristic function \( \chi_N \) (we define \( X^f = S^f - T^f + T^f e^{i\lambda} \)) is invariant under the change of basis,
\[ \chi_N = \frac{\det X^f}{\det S^f} = \frac{\det A}{|\det A|^2} \frac{\det X^f}{\det S^f} = \frac{\det X^g}{\det S^g}. \]  
(23)
This invariance can be exploited by going over to new orthogonal basis functions \( g_m(k) \) with an overlap matrix \( S^g_{mn} = \delta_{mn} \) and a transmission matrix assuming a diagonal form \( T^g_m = \tau_m g_m \). The possibility of simultaneous diagonalization of the matrices \( T^g_{mn} \) and \( S^g_{mn} \) is a consequence of the transformation law \( (22) \), characteristic of bilinear forms. The corresponding eigenbasis \( g_m \) and eigenvalues \( \tau_m \) of \( T^g_{mn} \) can be found by solving the generalized eigenvalue problem
\[ (T^f - \tau_m S^f)a_m = 0 \]  
(24)
with the normalization \( a_m^H S_f a_m = 1 \). The eigenvectors \( a_m \) constitute the column vectors of the transformation matrix \( A = (a_1, \ldots, a_N) \). The eigenvalues are given by the roots of the characteristic polynomial \( \det(T^f - \tau S^f) = 0 \). The full counting statistics, Eq. \( (16) \), written in the new basis \( g_m(k) \) assumes the generalized binomial form
\[ \chi_N(\lambda) = \prod_{m=1}^N (1 - \tau_m + \tau_m e^{i\lambda}), \]  
(25)
where the determinant has been evaluated explicitly and the result depends only on the eigenvalues \( \tau_m \). The generalized eigenvalue problem can be reduced to a normal one by rewriting the problem in a orthogonalized basis \( \phi_m(k) \), with \( S^g = 1_N \), which can be obtained by the Gram-Schmidt procedure or by setting \( \phi_m(k) = \sum_n [(S^f)^{-\frac{1}{2}]}_{nm} f_n(k) \).

From the above, we see that the concrete form of the eigenvalue problem \( (24) \) is basis dependent, whereas the eigenvalues and eigenvectors are simply a property of the transmission operator \( T \) operating in the Hilbert space \( H_N \) with the scalar product \( \langle f | g \rangle \). Indeed, it is possible to find the eigenvalues and eigenvectors in a basis independent way using the positive definite quadratic forms \( T(g) = \langle g | T | g \rangle \) and \( S(g) = \langle g | g \rangle \). Representing the bilinear form \( T(g) \) with fixed \( S(g) = 1 \) as a polar plot with \( T(g) \) the radius and \( g \) defining the direction in \( H_N \), we obtain an ellipsoid in \( N \)-dimensional space. The lengths of the main axes of this ellipsoid then constitute the eigenvalues and the associated directions the eigenvectors of the problem \( (24) \). The eigenvalues \( \tau_m \) are constrained to the interval \([0,1]\) as \( T(g) \geq 0 \) and \( T(g) \leq S(g) \) due to unitarity.

**F. Full Counting Statistics for Entangled States**

The above discussion has concentrated on incoming states described by a single Slater determinant, i.e., nonentangled states with Slater rank 1. It is instructive to generalize this discussion to entangled states involving a coherent superposition of Slater determinants. We start from an incoming state of \( N \) particles with Slater rank 2,
\[ \Psi(k) = \alpha \Psi^I(k) + \beta \Psi^H(k), \]  
(26)
where \( \Psi^I(k) \) and \( \Psi^H(k) \) are normalized \( N \)-particle Slater determinants describing particles incoming from the left and made from single particle states \( f^I_{m}(k) \) and \( f^H_{m}(k), \) \( m = 1, \ldots, N \); the complex numbers \( \alpha \) and \( \beta \) have been chosen such as to make \( \Psi(k) \) normalized. The characteristic function for the full counting statistics \( (12) \) assumes the form
\[
\chi_N(\lambda) = \prod_{m=1}^{N} \int \frac{dk_m}{2\pi} (1 - T_{km} + T_{km} e^{i\lambda}) \left[ |\alpha|^2 |\Psi^I(k)|^2 + |\beta|^2 |\Psi^II(k)|^2 + 2\Re\{\alpha\beta^* \Psi^I(k) \Psi^II(\kappa)\} \right],
\]

where Re denotes the real part. The first two terms reduce to generating functions for simple Slater determinant states and we can write

\[
\chi_N(\lambda) = |\alpha|^2 \chi^I_N(\lambda) + |\beta|^2 \chi^II_N(\lambda) + \alpha\beta^* \chi^\text{mix}_N(\lambda) + \alpha^*\beta \chi^\text{mix}_N(-\lambda)^* \tag{28}
\]

with

\[
\chi^I_N(\lambda) = \frac{\det(S^I - T^I + T^I e^{i\lambda})}{\det S^I},
\]
\[
\chi^II_N(\lambda) = \frac{\det(S^{II} - T^{II} + T^{II} e^{i\lambda})}{\det S^{II}},
\]
\[
\chi^\text{mix}_N(\lambda) = \frac{\det(S^{\text{mix}} - T^{\text{mix}} + T^{\text{mix}} e^{i\lambda})}{\sqrt{\det S^{\text{mix}}}}. \tag{29}
\]

The matrices with superscripts \( f^I \) and \( f^{II} \) have been defined in Eq. (17), while the new Hermitian matrices with a superscript ‘mix’ are given by the mixed matrix elements

\[
S^{\text{mix}}_{nm} = \langle f^I_m | f^I_n \rangle, \quad T^{\text{mix}}_{nm} = \langle f^I_m | T | f^I_n \rangle. \tag{30}
\]

The first two terms in (28) can be diagonalized as before, cf. (24),

\[
\chi^I_N(\lambda) = \prod_{m=1}^{N} (1 - \tau^I_m + \tau^I_m e^{i\lambda}), \tag{31}
\]
\[
\chi^II_N(\lambda) = \prod_{m=1}^{N} (1 - \tau^{II}_m + \tau^{II}_m e^{i\lambda}), \tag{32}
\]

with the eigenvalues \( \tau^I_m \) and \( \tau^{II}_m \) given by the roots of \( \det(T^I - T^I S^I) = 0 \) and \( \det(T^{II} - T^{II} S^{II}) = 0 \).

Let us then concentrate on the characteristic function \( \chi^{\text{mix}}(\lambda) \). Unfortunately, there is no generic procedure to follow in this case, as the matrices \( S^{\text{mix}} \) and \( T^{\text{mix}} \) are not Hermitian any more and hence the expression (27) cannot be further simplified in general. In particular, the characteristic function \( \chi^{\text{mix}}(\lambda) \) is not invariant under individual transformations of the bases \( f^I_m \) and \( f^{II}_m \) (such basis transformations leave the Slater-determinants invariant only up to a phase factor, which dropped out in the calculation of the characteristic function of a single Slater determinant state but does not when two Slater determinants are superimposed coherently). In order to proceed further, we restrict ourselves to specific situations where \( S^{\text{mix}} = 0 \) or \( \det S^{\text{mix}} \neq 0 \). The most trivial case is realized for mutually orthogonal sets of basis functions \( f^I_m \) and \( f^{II}_m \) where \( S^{\text{mix}} = 0 \); if, in addition, \( \det T^{\text{mix}} = 0 \), we have \( \chi^{\text{mix}}(\lambda) = 0 \) (see also Sec. III C below), else \( \chi^{\text{mix}}(\lambda) = \tau^{\text{mix}}(e^{i\lambda} - 1)^N \) with \( \tau^{\text{mix}} = \det T^{\text{mix}} / \sqrt{\det S^I S^{II}} \).

Second, let us assume that \( S^{\text{mix}} \) is invertible, \( \det S^{\text{mix}} \neq 0 \). Let \( \tau^{\text{mix}}_m \) be the roots of the polynomial

\[
\det[T^{\text{mix}} - \tau^{\text{mix}} S^{\text{mix}}] = 0. \tag{33}
\]

The matrix \( T^{\text{mix}}(S^{\text{mix}})^{-1} \) then can be brought into a Jordan canonical form with \( \tau^{\text{mix}}_m \) on the diagonal and the characteristic function assumes the simple form

\[
\chi_N^{\text{mix}}(\lambda) = \frac{\det S^{\text{mix}}}{\sqrt{\det S^I S^{II}}} \prod_{m=1}^{N} (1 - \tau^{\text{mix}}_m + \tau^{\text{mix}}_m e^{i\lambda}). \tag{34}
\]

The procedure outlined above is straightforwardly generalized to states with higher Slater rank.

### III. TWO PARTICLES

#### A. Full counting statistics

The above findings have interesting generic consequences for the charge transport of fermionic particles; in the following, we discuss the simplest case of two particles, see Fig. 1 where nontrivial exchange properties manifest themselves. For \( N = 2 \) particles the diagonalization (24) can be carried out explicitly for arbitrary matrices \( T^I \) and \( S^I \). The two eigenvalues \( \tau_{1,2} \) are given by

\[
\tau_{1,2} = \frac{\alpha \mp \sqrt{\alpha^2 - \det T^I \det S^I}}{\det S^I}, \tag{35}
\]

with the parameter \( 2\alpha = S^I \tau^{II}_{11} + S^I \tau^I_{11} - 2\Re(S^I \tau^I_{21}) \).

Alternatively, the eigenvalues \( 0 \leq \tau_m \leq 1 \) are given by a minimum/maximum property

\[
\tau_1 = \min_{g \in H_2 | S(g) = 1} T(g), \quad \tau_2 = \max_{g \in H_2 | S(g) = 1} T(g), \tag{36}
\]

with the eigenvectors \( g_{1,2}(k) \) given by those functions where the minimum/maximum values are attained, i.e., \( T(g_{1,2}) = \tau_{1,2} \). Once the eigenvalues \( \tau_m \) are known, the characteristic function \( \chi_2 \) assumes the simple generalized binomial form

\[
\chi_2(\lambda) = (1 - \tau_1 + \tau_1 e^{i\lambda})(1 - \tau_2 + \tau_2 e^{i\lambda}). \tag{37}
\]

As a result, we find that in the new basis \( g_m \), the two particles traverse the scatterer independent of one another, i.e., the characteristic function is a simple product of independent one-particle characteristic functions.
Even more, the characteristic function is determined by the Hilbert space spanned by the incoming states \( f_{1,2} \) and is independent of the choice of basis. Exchange effects manifest themselves when comparing the result \((47)\) for the Slater determinant \( \Psi^f \propto \det f_m(k_n) \) with the result \( \chi^\text{dist}_2(\lambda) = (1 - T_{11}^f + T_{11}^f e^{i\lambda})(1 - T_{22}^f + T_{22}^f e^{i\lambda}) \) for distinguishable particles, \( \Psi^\text{dist} \propto f_1(k_1) f_2(k_2) \): Exchange effects are absent if both matrix elements \( S_{21}^f = \langle f_2|f_1 \rangle = 0 \) and \( T_{22}^f = 0 \), i.e., for orthogonal initial and transmitted states. On the other hand, a finite overlap of at least one pair of these states generates finite exchange effects via the substitution of \( T_{mm}^f \) in \( \chi^\text{dist}_2 \) by the eigenvalues \( \tau_m \) in \( \chi^\text{dist}_2 \).

The minimum/maximum property described above entails a set of \textit{a priori} inequalities for the transmission probabilities \( P_n \) involving the transmission matrix elements \( T_{\text{min}} = \min\{T_{11}, T_{22}\} \) and \( T_{\text{max}} = \max\{T_{11}, T_{22}\} \); note that while the probabilities \( P_n \) do account for exchange effects, the single particle matrix elements \( T_{mm} \) obviously do not. With initial (nonorthogonal) wave packets \( f_m \) normalized to unity, \( S(f_m) = 1 \), the search for the extrema in Eq. \((56)\) includes these states as well. We then obtain the set of inequalities \( 0 \leq \tau_1 \leq T_{\text{min}} \leq T_{\text{max}} \leq \tau_2 \leq 1 \). Using them to estimate \( P_0 = (1 - \tau_1)(1 - \tau_2) \leq (1 - \tau_2), P_2 = \tau_1 \tau_2 \leq \tau_1, \) and \( P_1 = 1 - P_0 - P_2 \), we can derive the following bounds

\[
P_0 \leq 1 - T_{\text{max}}, \quad P_1 \geq T_{\text{max}} - T_{\text{min}}, \quad P_2 \leq T_{\text{min}}.
\]

for the transmission probabilities for two particles. The above bounds set an upper limit on bunching \( (P_2 \) and \( P_0 \) \) and a lower limit on anti-bunching \( (P_1 \) \). Note though, that the bound on \( P_2 \) does not exclude an increase (due to exchange) of the transmission probability beyond the ‘classical’ value \( P^\text{dist}_2 = T_{11}^f T_{22}^f \) for distinguishable particles, see \( \chi^\text{dist}_2 \) above. Indeed, since \( T_{11}^f T_{22}^f \leq T_{\text{min}} \), a value \( P_2 \gg T_{11}^f T_{22}^f \) remains possible. Such a result has been recently observed\(^{37}\) the probability of two-electron events in the electron emission from a Cs\(_3\)Sb photocathode in a photomultiplier tube has been found to be much larger than the square of the probability for single-electron emission. This was observed both in the case of thermal emission without photocathode illumination and photoemission under weak photocathode illumination. Furthermore, as detailed calculation shows, a large \( P_2 \) can also be obtained for wave packets with amplitudes \( f_2(k) = f_1(k + \delta k) \) shifted in \( k \)-space and a large overlap integral \( S_{21}^f \), combined with a transmission amplitude suppressing \( k \)-values in the overlap region.

**B. Restrictions due to binomial statistics**

An arbitrary two-particle scattering process is fully characterized by the three parameters \( P_0, P_1, P_2 \), from which only two are independent; here, we assume that we can transmit only integer charges (no charge fractionalization). In Figs. \( 2(a) \) and \( (b) \), we find the regions with different statistical properties that can be generated in a two-fermion scattering process, both in \( P_0-P_2 \) parameter space as well as in the noise \( \langle n^2 \rangle \) versus average number \( \langle n \rangle \) diagram. We start with the definition of the physically accessible regime in these diagrams: requiring that \( P_1 = 1 - P_0 - P_2 \geq 0 \) (Fig. \( 2(a) \)) and \( P_0, P_1, P_2 \geq 0 \) (Fig. \( 2(b) \)), we find that the black regions are forbidden.

Traditionally, starting from Poissonian statistics \( (F = 1) \) relevant for the coherent light emitted from a laser or for the transport of a classical electron gas in a vacuum tube, much emphasis has been put on the distinction between sub- and super-Poissonian statistics, with reduced and enhanced noise intensity as quantified by Fano factors \( F < 1 \) (sub-Poissonian noise) and \( F > 1 \) (super-Poissonian processes). It appears to us that in the context of degenerate fermions, the generic starting point is the binomial statistics, instead, and more relevant qualifications are given by the regimes of sub-binomial and super-binomial processes introduced below.

Nevertheless, let us start our analysis with the traditional classification comparing a process with Poissonian statistics, which is realized on the dotted line in Fig. \( 2(a) \) defined through the relation

\[
F = P_0(1 - P_0) + P_2(1 - P_2) + 2P_0P_2 = 1,
\]

i.e.,

\[
P_2 = P_0 - \sqrt{2P_0 - 1} \quad \text{with} \quad P_0 \geq 1/2.
\]

Within the dark-gray region noise is super-Poissonian, which is usually associated with the bunching of particles and therefore with bosonic statistics. Note that Fano-factors larger than the Poissonian value \( 1 \) require a large reflection probability \( P_0 > 1/2 \); only when most of the particles are reflected one can observe the ‘bunching’ of the remaining transmitted objects.

A much more natural classification for our fermion system is in terms of (deviations from) binomial statistics. The characteristic function \( \chi_2 \) for two fermions in a Slater determinant state can be cast into the generalized binomial form Eq. \((67)\), which depends on two parameters \( \tau_1, \tau_2 \). As a consequence, the probabilities satisfy the additional inequality

\[
\sqrt{P_0} + \sqrt{P_2} \leq 1.
\]

This condition follows from expressing the parameters \( \tau_1, \tau_2 \) through the probabilities \( P_0, P_2 \) using the relations \( P_0 = (1 - \tau_1)(1 - \tau_2) \) and \( P_2 = \tau_1 \tau_2 \); requiring a positive discriminant of the resulting (quadratic) equation implies the constraint \((11)\) which defines the light gray region in Fig. \( 2(a) \), naturally termed the ‘sub-binomial’ regime. The (thick) black line bounding the general binomial (or sub-binomial) region is the line of usual binomial statistics, which is realized for the case of degenerate transmission coefficients \( \tau_1 = \tau_2 \) as they appear if the scattering does not depend on energy.
The region with super-Poissonian noise (dark gray) and the sub-binomial region (light gray) are distinct, with the statistics of fermions incoming in a Slater-determinant state always residing in the sub-binomial domain. Note that the counting statistics of an arbitrary two-particle process (without specification of exchange properties) also depends on two out of the three parameters \( P_0, P_1, P_2 \) (as the constraint \( P_0 + P_1 + P_2 = 1 \) needs to be fulfilled) but cannot be cast into the form Eq. (37) in general, hence these processes are devoid of such an additional restriction.

The \( P_2-P_0 \) diagram can be transcribed to the (experimentally more relevant) \( \langle n^2 \rangle - \langle n \rangle \) diagram, cf. Fig. 2(b). The physical constraints \( 0 \leq P_0, P_1, P_2 \) lead to the set of inequalities,

\[
\begin{align*}
\langle n^2 \rangle &\geq \langle n \rangle (1 - \langle n \rangle), \\
\langle n^2 \rangle &\leq \langle n \rangle (2 - \langle n \rangle), \\
\langle n^2 \rangle &\geq (\langle n \rangle - 1)(2 - \langle n \rangle),
\end{align*}
\]
(42)

which can be cast into the more compact form \((m + 1 - \langle n \rangle)(\langle n \rangle - m) \leq \langle n^2 \rangle \leq \langle n \rangle (2 - \langle n \rangle)\), with \( m = 0, 1 \). The single large- and two small parabolas bounding the unphysical (black) regions are given by the second and the two (for \( m = 0, 1 \)) first inequalities. For the generalized (or sub-) binomial statistics, the additional constraint assumes the form

\[
F = \frac{\langle n^2 \rangle}{\langle n \rangle} \leq 1 - \langle n \rangle/2,
\]
(43)

with the equality applying to the binomial case with \( \tau_1 = \tau_2 \). Within the gray region of the diagram the noise is sub-binomial \( F \leq 1 - \langle n \rangle/2 \) and hence trivially sub-Poissonian, \( F \leq 1 \). Note that noiseless transmission of charge requires that an integer average charge is transmitted.

The generalization of the above analysis to \( N \) incoming particles in a Slater determinant state is straightforward. The generalized binomial characteristic function is given by Eq. (2). The positivity of the probabilities \( P_m \geq 0 \), \( m = 0, \ldots, N \) imposes the \( N + 1 \) restrictions on the first two momenta \( \langle n \rangle \) and \( \langle n^2 \rangle \), \( (m + 1 - \langle n \rangle)(\langle n \rangle - m) \leq \langle n^2 \rangle \leq \langle n \rangle (N - \langle n \rangle) \), with \( m = 0, \ldots, N - 1 \), defining a simple generalization of Fig. 2(b) with one large and \( N \) small parabolas. In the generalized binomial case, the additional constraint

\[
F = \frac{\langle n^2 \rangle}{\langle n \rangle} \leq 1 - \langle n \rangle/N \leq 1
\]
(44)
tells that the incoming Slater determinant states produce a sub-binomial noise statistics. A similar result was found recently in the context of adiabatic pumping. The authors considered a time-dependent scattering matrix in the instant scattering approximation (i.e., an energy independent scatterer) and obtained a generating function in a product form describing a generalized binomial statistics with parameters \( u_m \leq 0 \); the \( u_m \) relate to our \( \tau_m \) via \( \tau_m = (1 - u_m)^{-1} \).

C. Entangled states

The above discussion for two particles lets us conclude that incoming Slater determinant states generate Fano factors \( F \leq 1 - \langle n \rangle/2 \leq 1 \); such states are nonentangled. On the other hand, an entangled two-particle state can be generated with a sum of two Slater determinants; such an entangled state (with \( 0 < \alpha < 1 \), i.e., a state with
Slater rank 2, see Sec. II

\[ \Psi(k_1, k_2) = \sqrt{\alpha} \Psi^I(k_1, k_2) + \sqrt{1 - \alpha} \Psi^{II}(k_1, k_2), \]

is sufficient to generate all possible types of two-particle statistics: we choose the Slater-determinant wave functions \( \Psi^I \) and \( \Psi^{II} \) (incoming from the left) such that they occupy different parts of momentum space, e.g., \( \Psi^I \) has only components below \( k_c \) and \( \Psi^{II} \) above. Furthermore, let the transmission be \( T_1 = T_{k < k_c} \) below \( k_c \) and \( T_2 = T_{k > k_c} \) above. For such a setup, all the overlap integrals vanish, e.g., \( \int dk_1 dk_2 / (2\pi)^2 \Psi^{(*)*(k_1, k_2)} \Psi^I(k_1, k_2) = 0 \), and we obtain (cf. Eq. (28))

\[ \chi_2(\lambda) = \alpha(1 - T_1 + T_1 e^{i\lambda}) + (1 - \alpha)(1 - T_2 + T_2 e^{i\lambda}), \]

that is, the generating function is simply the weighted sum of the two individual generating functions for the Slater-determinant states. The statistics of such entangled wave functions is described by points in the \( P_2-P_0 \) diagram of Fig. 2(a) which lie on a straight line between the point \( p^I \) for \( \Psi^I \) and the point \( p^{II} \) for \( \Psi^{II} \) with \( \alpha \) parameterizing the line. Both \( p^I \) and \( p^{II} \) are situated on the binomial line, while the line connecting them may enter the super-binomial or even the super-Poissonian region: for example setting \( T_1 = 0 \) and \( T_2 = 1 \), the characteristic function is given by \( \chi_2 = \alpha + (1 - \alpha) e^{i\lambda} \) and \( F = 2\alpha \), which assumes values between zero and two (note that in the limit \( \alpha \rightarrow 1 \), the wave function \( |\Psi\rangle \) is of Slater rank 1, but nevertheless, the Fano factor approaches \( F = 2 \). As the Fano factor for \( P_1 = 1 \) assumes the form \( 0/0 \) its value depends on the direction from which \( P_0 = 1 \) is approached). As simple Slater determinants produce only Fano factors up to \( 1 - \langle n \rangle / 2 \), a larger value serves as a test for the entanglement of the two particles. For \( N \) incoming particles in an entangled state of rank 2, the analogous construction (cf. [44,45]) produces a Fano factor \( F = N\alpha \) with \( 0 < \alpha < 1 \), i.e., super-Poissonian statistics can be admitted for sufficiently large \( \alpha \).

### D. Two spin 1/2 particles

Next, we consider the situation in the setup of Fig. 1 with incoming particles in normalized states \( f_1(k) \) and \( f_2(k) \) with overlap \( S = S_{21}^f = \langle f_2 | f_1 \rangle \) and carrying a spin 1/2 degree of freedom. We consider the case of spin-independent interaction, hence the coefficients in \( T^I \) depend exclusively on \( f_1(k) \) and \( f_2(k) \). The four properly symmetrized states available to the two incoming particles are denoted by \( \Psi_{s,m_s}(k) \), with \( s = 0 \) the singlet \((m_s = 0)\) state and \( s = 1 \) the three \((m_s = -1, 0, +1)\) triplet states. The degrees of freedom \( k \) involve the momenta \( k_m \) and spins \( s_m \) of the particles, \( k = (k_1, s_1; k_2, s_2) \). The triplet states with \( m_s = \pm 1 \) are simple Slater determinant states

\[ \Psi_{1,\pm 1}(k) = \frac{1}{\sqrt{2(1 - |S|^2)}} \left[ f_1(k_1) \chi_{1/1}(s_1) f_2(k_2) \chi_{1/1}(s_2) \right. \]

\[ - \left. [(k_1, s_1) \leftrightarrow (k_2, s_2)] \right]. \]

The characteristic function \( \chi_2 \) for the full counting statistics then is of the generalized binomial form with \( \tau_{1/2} \) given by Eq. (35),

\[ \chi_{1,\pm 1}(\lambda) = \left( 1 - \tau_1 + \tau_1 e^{i\lambda} \right) \left( 1 - \tau_2 + \tau_2 e^{i\lambda} \right) \]

\[ = \frac{1 - \left( T_{11} + T_{11} e^{i\lambda} \right) \left( 1 - T_{22} + T_{22} e^{i\lambda} \right)}{1 - |S|^2} \]

\[ - \frac{(S - T_{21}^f + T_{21}^f e^{i\lambda})(S^* - T_{12}^f + T_{12}^f e^{i\lambda})}{1 - |S|^2}. \]

The states with \( m_s = 0 \) are more interesting as they are of Slater rank 2. Defining

\[ f_{1/2}^I(k, s) = f_1(k) \chi_1(s), \quad f_{1/2}^I(k, s) = f_2(k) \chi_1(s), \]

\[ f_{1/2}^{II}(k, s) = f_1(k) \chi_1(s), \quad f_{1/2}^{II}(k, s) = f_2(k) \chi_1(s), \]

we have

\[ \Psi_{0/1,0}(k) = \frac{1}{\sqrt{2(1 - |S|^2)}} [\Psi^I(k) \mp \Psi^{II}(k)] \]

with \( \Psi^{II}(k) \) the normalized two-particle Slater determinants made from the states \( f_{1/2}^{II} \). The calculation of the characteristic function follows the procedure outlined above: As the matrices \( T_{mn}^{II} = T_{mn}^f \delta_{mn} \) and \( S_{mn}^{II} = \delta_{mn} \) diagonal (the particles 1 and 2 are distinguishable), we immediately have

\[ \chi_1^{II}(\lambda) = \left( 1 - T_{11}^f + T_{11}^f e^{i\lambda} \right) \left( 1 - T_{22}^f + T_{22}^f e^{i\lambda} \right). \]

For the calculation of \( \chi_{\text{mix}}^{II}(\lambda) \), the matrices \( T^{\text{mix}} \) and \( S^{\text{mix}} \) need to be evaluated. In the present case, they are purely off-diagonal with the off-diagonal matrix element given by \( T_{12}^{\text{mix}} = T_{21}^f \) and \( S_{12}^{\text{mix}} = S \). Calculating the determinants in Eq. (49), we obtain the mixed component in the form

\[ \chi_{0/1,0}^{\text{mix}}(\lambda) = \frac{1 - \left( T_{11}^f + T_{11}^f e^{i\lambda} \right) \left( 1 - T_{22}^f + T_{22}^f e^{i\lambda} \right)}{1 - |S|^2} \]

\[ - \frac{(S - T_{21}^f + T_{21}^f e^{i\lambda})(S^* - T_{12}^f + T_{12}^f e^{i\lambda})}{1 - |S|^2}. \]

The result (53) agrees with the results in Ref. [12]. The characteristic functions for the two spin triplet states \( s = 1 \) with maximal magnetization \( m_s = \pm 1 \) and the characteristic function for the triplet \( s = 1, m_s = 0 \) with zero magnetization coincide with the one for a Slater determinant of spinless fermions, Eq. (37). This is because all three states involve identical orbital wave functions and the scattering process does not depend on the spin part of the wave function. The corresponding average number of particles \( \langle n \rangle_{1,m_s} \) and noise \( \langle \langle n^2 \rangle \rangle_{1,m_s} \) reside
within the region of generalized binomial statistics, cf. Fig. 2.

\[ F_{1,m_x} \leq 1 - \langle n \rangle_{1,m_x}/2. \] (54)

The entangled singlet state (with \( s = 0 \)) does not necessarily fulfill this condition. Rather opposite, for the case where the individual transmission probabilities of the two particles are equal, \( T_{11}^f = T_{22}^f \), the moments and the Fano factor always reside outside the region allowed by the generalized binomial statistics,

\[ F_{0,0} \geq 1 - \langle n \rangle_{0,0}/2, \] (55)

as a lengthy but straightforward calculation shows. Hence, this rather trivial setup can be used to discriminate singlet from triplet states and also serves as an indicator of entanglement (as long as the inequalities are fulfilled and the inequalities are strict which is the case as long as \( S \neq 0 \) and \( T_{21}^f \neq \ast T_{11}^f \)).

A similar experiment was proposed by Burkard et al. Ref. 15, which had two particles with equal energy come in from different arms in a symmetric beam splitter, see Ref. [14] for a calculation of the full counting statistics for this setup. Our setup involves one single lead only, at the expense of requiring an energy dependent transmission probability (otherwise we end up on the binomial line which is devoid of any separation power). Furthermore, the discrimination between singlet and triplet states is determined by the presence or absence of generalized binomial statistics and hence involves the binomial bound \( 1 - \langle n \rangle_{0,0}/2 \) on \( F \).

### IV. N-PARTICLE TRAINS

We consider the case of \( N \) incoming particles, all with the same shape of the wave function \( f(k) \) aligned regularly in real space with separation \( a \); the wave function of the \( m \)-th particle then is given by \( f_m(k) = f(k)e^{-imk} \). The overlap and transmission matrices [17] are given by the Fourier transforms

\[ S_{mn}^f = \int \frac{dk}{2\pi} |f(k)|^2 e^{i(m-n)ka}, \]
\[ T_{mn}^f = \int \frac{dk}{2\pi} |f(k)|^2 T_k e^{i(m-n)ka}, \] (56)

these are Toeplitz matrices as their elements depend only on the difference \( m - n \) between indices. In the limit \( N \to \infty \), the determinants of the Toeplitz matrices \( S^f \) and \( S^f - T^f + T^f e^{i\lambda} \) can be evaluated by reducing the integral over \( k \)-space to an integral over the first Brillouin zone \([0, 2\pi/a]\) and using Szegő’s theorem, see Ref. 20 and App. A.

The logarithm of these determinants scales linearly with \( N \), a result that has to be expected as correlations between particles vanish at large separation. Combining the results (57) and replacing the integration over the angle \( \theta \) by an integration over the first Brillouin zone \( k \in [0, 2\pi/a] \), we find the generating function in the form

\[ \log \det S^f - T^f + T^f e^{i\lambda} \sim N \int_0^{2\pi/a} \frac{d\theta}{2\pi} \log \left\{ \frac{1}{a} \sum_{m \in \mathbb{Z}} |f([\theta + 2\pi m]/a)|^2 \right\}, \] (57)

\[ \log \det S^f \sim N \int_0^{2\pi/a} \frac{d\theta}{2\pi} \log \left\{ \frac{1}{a} \sum_{m \in \mathbb{Z}} |f([\theta + 2\pi m]/a)|^2 [1 - T([\theta + 2\pi m]/a + T([\theta + 2\pi m]/a)e^{i\lambda}] \right\}. \]

The effective scattering probabilities

\[ \tau_k = \frac{\sum_{m \in \mathbb{Z}} |f(k + 2\pi m/a)|^2 T_{k+2\pi m/a}}{\sum_{m \in \mathbb{Z}} |f(k + 2\pi m/a)|^2}, \] (59)

which denote transmission probabilities (with \( 0 \leq \tau_k \leq 1 \)) averaged over higher harmonics \( 2\pi m/a \) with weight \( |f(k + 2\pi m/a)|^2 \).

Let us apply this result to wave packets generated by Lorentzian voltage pulses. As shown in Ref. 10, a unit-flux (i.e., \( c \int dt V(t) = hc/e = \Phi_0 \)) Lorentzian voltage-pulse \( eV_{\Phi_0}(t) = 2h\gamma/(t - t_0)^2 + \gamma^2 \) parametrized by its width \( \gamma \) and time of appearance \( t_0 \), excites a single particle with wave function \( f_{x_0}(k) = \sqrt{4\pi\xi} e^{-\xi k - i\gamma k\Theta(k)} \) moving through the quantum wire (\( \Theta(k) \) denotes the unit-step function; we remind that \( k \) is measured with respect to the Fermi momentum \( k_F \)). Here \( e \) is the charge of the particle, \( x_0 = v_F t_0 \) parametrizes the position, and \( \xi = v_F \gamma \) the real-space width of the wave packet. A periodic sequence of unit-flux voltage-pulses \( V(t) = \sum_{m \in \mathbb{Z}} V_{ma/v_F}(t) \) applied to an interval to the left of the scatterer and driving one particle per time interval \( a/v_F \) generates the transmission probabilities

\[ \tau_k = (1 - e^{-4\pi\xi/a}) \sum_{m \geq 0} e^{-4\pi m\xi/a} T_{k+2\pi m/a}. \] (60)

For nonoverlapping wave packets \( \xi \ll a \), cf. Fig. 3(a), exchange effects are absent. The sum in Eq. (60) be-
the generating function assumes the form
\[ \Phi = \int \frac{dk}{2\pi} |f(k)|^2 T_k \]

The particles probe the transmission within this energy of magnitude \(eV\). The integral over one voltage pulse generates one flux unit. Each particle is transmitted independent of the others with its weight \( |f(k)|^2 \). In the opposite limit, \( \xi \gg a \), the transmission probabilities for momenta up to \( 1/\xi \) are \( \approx \). If the particles were transmitted independent of each other (no exchange effects), they would probe transmission probabilities with its weight \( |f(k)|^2 \). However, exchange effects force the transmission probabilities for momenta in the interval \([0, 2\pi/a]\) (as easily obtained from (59) by replacing the sums with \( N \)). The correction for the noise term is given by
\[
\Delta \langle n^2 \rangle = \left( \frac{T_{2\pi/a} - T_0}{2\pi} \right) \log N \]

and similar corrections are obtained for the third- and higher-order cumulants.

V. GENERALIZATIONS

A. Unitary evolution and time-dependent counting

We want to generalize the generating function \( \chi_N \) as given by Eq. (13) to account both for the specific time-evolution of the scattering state and for different counting procedures. Throughout this discussion, it is convenient to apply the Dirac notation and we rewrite the Slater determinant (13) in the form
\[
|\Psi \rangle = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \text{sgn}(\pi) |\phi_{\pi(1)} \rangle \otimes \cdots \otimes |\phi_{\pi(N)} \rangle; \quad (64)
\]

Eq. (64) describes the initial \( N \)-particle wave function at time \( t = 0 \) composed of orthonormalized one-particle states \( |\phi_m \rangle \) (here, \( \pi \) denotes an element of the permutation group \( S_N \)). The choice of orthonormalized wave packets is only for convenience: as seen in Sec. (11) a Slater determinant is invariant under general linear combination of states it is composed of; in particular, an orthonormalized basis can be chosen. Let
\[
U = \exp \left[ -\frac{i}{\hbar} \int_0^t dt' \mathcal{H}(t') \right] \]

be the unitary evolution operator generated by the single particle Hamiltonian \( \mathcal{H}(t) \). In the absence of interaction, the evolution of the total system is governed by the product operator \( \Gamma_N(U) \), where, given a one-particle operator \( \mathcal{O} \), we define the \( N \)-particle operator
\[
\Gamma_N(\mathcal{O}) = \mathcal{O} \otimes \cdots \otimes \mathcal{O}_{\text{N-times}} \quad (66)
\]
acting simultaneously on all \(N\)-particles. While we restrict ourselves to noninteracting systems, we still allow for a time-dependent scattering potential which can generate inelastic processes. The final state at time \(t\) is given by \(\langle \Psi_{\text{out}} | = \Gamma_{N}(U)|\Psi\rangle\). Including the counting field \(e^{i\lambda Q/2}\), the wave-function assumes the form

\[
|\Psi_{\text{out}}\rangle = \Gamma_{N}(e^{i\lambda Q/2})|\Psi_{\text{out}}\rangle = \Gamma_{N}(e^{i\lambda Q/2}U)|\Psi\rangle,
\]

where \(Q\) is a projector (\(Q^2 = Q\) and \(Q^\dagger = Q\)) on that part of the wave-function that has been counted. E.g., in the original setup of Ref. 4 with a spin at position \(x_0\) and particles incoming from the left, the operator \(Q_\ell = \int_I dx |x\rangle\langle x|\) projects onto the causal interval \(I = [x_0, x_0 + v_\ell t]\) (no such operator \(Q\) mimicking a spin-counter can be defined for particles incident from both sides); hence that part of the wave function which passed the counter during the time \(t\) picks up an additional phase \(e^{i\lambda/2}\). Note that it is always the full phase \(\lambda\) which is picked up, as the particle is either measured (eigenvalue 1 of \(Q\)) or not (eigenvalue 0 of \(Q\)). The characteristic function of the full counting statistics is given by the overlap (fidelity)

\[
\chi_N(\lambda) = \langle \Phi^-_{\text{out}} | \Phi^+_{\text{out}} \rangle = \langle \Phi | \Gamma_{N}(\mathcal{U}^t e^{i\lambda Q} \mathcal{U}) | \Phi \rangle
\]

of the forward- and back-propagating wave-functions measured with opposite spin states.

Next, we exploit that the expectation value of a product operator \(\Gamma_{N}(O)\) in a Slater-determinant state can be written as a determinant of one-particle matrix elements \(\langle \phi_m|O|\phi_n\rangle\) in the Hilbert space \(H_N\) spanned by the states \(|\phi_m\rangle\),

\[
\langle \Psi | \Gamma_{N}(O) | \Psi \rangle = \frac{1}{N!} \sum_{\pi,\pi' \in S_N} \text{sgn}(\pi \circ \pi') \prod_{m=1}^{N} \langle \phi_{\pi(m)} | O | \phi_{\pi'(m)} \rangle
\]

\[
= \frac{1}{N!} \sum_{\pi,\pi' \in S_N} \text{sgn}(\pi'') \prod_{m=1}^{N} \langle \phi_{\pi(m)} | O | \phi_{\pi''(m)} \rangle
\]

\[
= \text{det}(\langle \phi_m | O | \phi_n \rangle);
\]

this formula is at the origin of (most) results which cast the characteristic function of the full counting statistics into a determinant form. Making use of Eq. (65), we can rewrite the characteristic function Eq. (68) as the determinant

\[
\chi_N(\lambda) = \text{det}(\langle \phi_m | e^{i\lambda t} \mathcal{Q} \mathcal{U} | \phi_n \rangle)
\]

\[
= \text{det}(\langle \phi_m | 1 - T_Q + T_Q e^{i\lambda} | \phi_n \rangle)
\]

with

\[
T_Q = \mathcal{U}^t \mathcal{Q} \mathcal{U};
\]

in going from the first to the second line in Eq. (70), the exponential has been expanded and use has been made of the fact that \(T_Q\) is a projector. With \(T_Q\) a projector in the one particle Hilbert space \(H\), its eigenvalues in the subspace \(H_N\) lie between 0 and 1 and Eq. (70) leads to a generalized binomial statistics.

In order to familiarize us with this new formula, we reproduce the results of the above section. We then are interested in the situation where the initial state \(|\Psi\rangle\) is localized to the left of the scattering region and the final state describes the \(t \to \infty\) asymptotic behavior where all particles have completed the scattering process. Within the basis of states left/right of the scatterer with momentum \(k\), the asymptotic form of the propagator is given by the unitary (scattering) matrix

\[
U_k^\infty = \left(\begin{array}{cc} r_k & t_k' \\ t_k & r_k' \end{array} \right),
\]

where the coefficients \(r_k\) (\(r_k'\)) and \(t_k\) (\(t_k'\)) are the reflection and transmission amplitudes of a particle coming from the left (right). The total propagator assumes the form \(U_k^\infty = \int (dk/2\pi) |k\rangle_{\text{out}} U_k^\infty |k\rangle_{\text{in(out)}}\) where we have introduced the asymptotic states \(|k\rangle_{\text{in(out)}} = (|k\rangle_{L,\text{in(out)}}, |k\rangle_{R,\text{in(out)}})\) which are in-(out-)going plane waves in the left/right lead: in a formal derivation, we have to consider the \(t \to \infty\) limit of the evolution in (65) within an interaction picture with a trivial reference dynamics \(U_0 = \int (dk/2\pi) e^{-i\lambda/k}(|k\rangle_{\text{in}} \langle k| + |k\rangle_{\text{out}} \langle k|)\). The counting operator \(Q\) is given by the projection on the right outgoing lead, \(Q_R = (0,1)^{t}(0,1)\), and we obtain

\[
T_{Q_R}^\infty = \int (dk/2\pi) |k\rangle_{\text{in}} \langle k|_t t_k' \rangle_\pi (t_k, r_k') \rangle_\text{in}\langle |k|.
\]

Since the initial single-particle wave functions \(\langle k| \phi_m \rangle = (\langle k| \phi_m \rangle, 0)\) are located to the left of the scatterer, the characteristic function assumes the form

\[
\chi(\lambda) = \text{det}(\langle \phi_m | 1 - T + T e^{i\lambda} | \phi_n \rangle),
\]

with \(T = \int (dk/2\pi) T_k |k\rangle \langle k|\), in agreement with (14); here, we have shortened the notation \(|k\rangle = |k\rangle_{\text{in(out)}}\) in agreement with previous sections. The generalization of the result Eq. (14) to many channels is straightforward: the propagator Eq. (72) exhibits a block structure with matrices \(t_k\) and \(r_k\) describing the transmission and reflection in the channel basis, the transmission probabilities \(T_k = t_k^t t_k\) assume a matrix form and the state vector \(|\phi_m\rangle\) adopts an additional channel index. Assuming an implicit summation over channel indices, the form of Eq. (74) remains unchanged. The same comment holds for the spin index.

### B. Density matrix – finite temperatures

The determinant in Eq. (74) is restricted to the subspace spanned by the initial states \(|\phi_m\rangle\). Introducing the projection operator \(P = \sum_{m=1}^{N} |\phi_m\rangle \langle \phi_m|\) onto the subspace spanned by the initial states \(|\phi_m\rangle\), the determinant can be elevated to cover the whole Hilbert space. We split the total Hilbert space into the sector defined by the projector \(P\) and its complement projected onto
\(P_\perp = 1 - P\). The operator \(1 - PT_Q + PT_Qe^{i\lambda}\) can be expressed in block form

\[
[1 + PT_Q(e^{i\lambda} - 1)] = \begin{bmatrix} 1 + T_Q(e^{i\lambda} - 1) & 0 \\ 0 & 1 \end{bmatrix},
\]

(75)

with the blocks operating in the \(PH\) and \(P_\perp H\) subspaces. The determinant of the upper block-diagonal matrix Eq. (75) is given as the product of the determinant \(\rho\) in the \(P\)-block and the determinant of 1 in the \(P_\perp\)-block and thus the generating function assumes the form

\[
\chi_N(\lambda) = \det(1 - PT_Q + PT_Qe^{i\lambda}),
\]

(76)

determined in the entire one-particle Hilbert space \(H\).

Interestingly, this formula can be generalized to the case when the initial state is not a single Slater determinant, but an incoherent superposition of many Slater determinants with a density matrix of the form \(\Gamma(\rho)/Z\) in Fock space \(F = \bigoplus N H_N, \Gamma(\rho) = \bigoplus N \Gamma_N(\rho), Z = \text{Tr}_F \Gamma(\rho)\), and \(\rho\) is the one-particle density matrix, e.g., \(\rho = e^{-\beta(H - \mu)}\) for a thermal ensemble with temperature \(\beta^{-1}\), chemical potential \(\mu\) and time-independent single-particle Hamiltonian \(\mathcal{H}\); here \(F_\rho\) denotes the antisymmetric sector of the Fock space. Using the trace formula, \(\text{Tr}_F \Gamma(\rho) = \det(1 + \rho)\), where the determinant is over the one-particle Hilbert space, the characteristic function \(\chi(\lambda) = \text{Tr}_{F_\rho} [\Gamma(\rho) \Gamma(U^\dagger e^{i\lambda} U)]/Z\), cf. Eq. (68), assumes the form

\[
\chi(\lambda) = \det(1 + \rho e^{i\lambda} \Omega(U))/\det(1 + \rho) = \det(1 - \eta + \eta e^{i\lambda} T_Q)
\]

(77)

with the one-particle occupation-number operator \(\eta = \rho/(1 + \rho)\) (and arbitrary one-particle density matrix \(\rho\); note again that the spectrum of \(\eta T_Q\) resides between 0 and 1 so that \(\chi(\lambda)\) denotes a generalized binomial statistics.

As an example, consider the situation of two particles incident from the left, with wave functions \(\phi_1(k)\) and \(\phi_2(k)\), where the process of particle generation is not deterministic but involves some success probability: let \(p_1\) \((p_2)\) be the probability that the first (second) particle is successfully created. In order to keep the discussion simple, we assume \(\phi_1(k)\) and \(\phi_2(k)\) to be orthonormalized \(\langle \phi_m | \phi_n \rangle = \delta_{mn}\). The initial state can be written as a density matrix \(\Gamma(\rho)/Z\) with the one-particle density matrix

\[
\rho = \frac{p_1}{1 - p_1} \langle \phi_1 | \phi_1 \rangle + \frac{p_2}{1 - p_2} \langle \phi_2 | \phi_2 \rangle.
\]

(78)

The weights in \(\chi(\lambda)\) are chosen to make the single particle occupation operator \(\eta = \rho/(1 + \rho)\) have the form

\[
\eta = p_1 \langle \phi_1 | \phi_1 \rangle + p_2 \langle \phi_2 | \phi_2 \rangle,
\]

(79)

i.e., \(p_1\) \((p_2)\) are the probabilities to occupy the state 1 \((2)\). The normalized density matrix [we use the fact that \(Z = \text{Tr}_F \Gamma(\rho) = \det(1 + \rho) = 1/(1 - p_1)(1 - p_2)\)]

\[
\Gamma(\rho)/Z = (1 - p_1)(1 - p_2) \oplus p_1 \langle \phi_1 | \phi_1 \rangle \oplus p_2 \langle \phi_2 | \phi_2 \rangle
\]

(80)

consists of three terms: The first term describes the zero particle sector which occurs with probability \((1 - p_1)(1 - p_2)\). The second term involves one particle states: \(p_1 \langle \phi_1 | \phi_1 \rangle \oplus p_2 \langle \phi_2 | \phi_2 \rangle\) is the probability that only the first \([\text{second}]\) particle is created. The third term shows that the probability to observe a Slater determinant of both states \(\phi_1\) and \(\phi_2\) is \(p_1 p_2\); we have omitted terms which involve tensor products of more than one projector on the same state as they have no weight on the antisymmetric part of the Hilbert space. The generating function of full counting statistics is given by \(\chi(\lambda)\):

\[
\chi(\lambda) = \det(1 - \eta T_Q^\infty + \eta T_Q e^{i\lambda})
\]

(81)

for the simplest case of asymptotic scattering with an energy-independent transmission probability, \(T_k = T\).

VI. CONSTANT VOLTAGE

Many results in the literature so far have been obtained in the stationary regime where a constant voltage \(V\) is applied across the wire for long measuring times \(teV/\hbar \gg 1\). Here, we discuss a wave-packet analog of the constant-voltage case. Contrary to the discussion in Sec. IV involving a nonstationary finite train of \(N\) particles with the spin counter measuring all the time \(t \to \infty\), here, we consider a stationary situation in the thermodynamic limit \((N, L \to \infty)\) with fixed density \(n = N/L, L\) the system size] with two reservoirs disbalanced by the applied voltage \(V\) and the counting extending over a finite time \(t\).

We start with \(N\) particles residing in \(N\) scattering states

\[
\varphi_k(x) = (e^{i k x} + r_k e^{-i k x}) \Theta(-x) + t_k e^{i k x} \Theta(x),
\]

(82)

with energies \(\hbar \varepsilon = \hbar v_x k\) between \(E_v\) and \(E_v + eV\). The scatterer is positioned at the origin. In order to regularize the problem, we go over to wave packets \(\phi_m(x)\): We split the momentum interval \([0, eV/v_x]\) into compartments of width \(h \varepsilon = eV/v_x N\) and define the weights

\[
f_m(k) = \begin{cases} \sqrt{2\pi/h\varepsilon} e^{i(m - 1)k} & 0 \leq k \leq \varepsilon m, \\ 0, & \text{elsewhere}, \end{cases}
\]

(83)

with \(m \in \{1, \ldots, N\}\). With the real weights \(f_m(k)\), the (normalized) wave packets

\[
\phi_m(x) = \frac{1}{2\pi} \int dk f_m(k) \varphi_k(x)
\]

(84)
define states centered around the origin. Note that adding arbitrary global phases to the wave packets $\phi_m(x)$ does not change their Slater determinant (up to a trivial global phase of the many-body wave function, see \cite{34}). Keeping $V$ constant and letting $\varepsilon \to 0$, the wave packets spread out in real space, the particle number $N$ goes to infinity, the homogeneous particle density assumes the finite value $eV/2\pi\hbar v_F$, and the resulting current $\langle I \rangle = (e/\hbar)V$ is constant in time, cf. \cite{111}. This procedure then properly emulates the constant voltage setup, as it generates the identical zero temperature density matrix as the one obtained in a second quantization formulation by filling scattering states within the interval of width $eV$.

In making use of the expression \cite{70}, we need the time evolution of the wave packets,

$$\phi_m(x; t) = \int \frac{dk}{2\pi} e^{-i\varepsilon v_F kt} f_m(k) \phi_k(x), \quad (85)$$

as well as the counting operator $Q_t = \int_I dx \langle x \rangle x$ projecting particles on the space interval $I = [x_0, x_0 + v_F t]$, where we assume the counter to be placed to the right of the origin, $x_0 > 0$.

### A. Generalized binomial statistics

The characteristic function $\chi_t(\lambda)$ is the determinant of the matrix, cf. Eq. \cite{70},

$$\langle \phi_m | e^{i\varepsilon t Q_t} | \phi_n \rangle = \langle \phi_m(t) | e^{i\lambda Q_t} | \phi_n(t) \rangle = \int dx \phi_m(x; t)^* (x) e^{i\lambda Q_t} | x \rangle \phi_n(x; t) = \delta_{mn} + (e^{i\lambda} - 1) Q_{mn}, \quad (86)$$

$$\chi_t(\lambda) = \det[\delta_{mn} + (e^{i\lambda} - 1) Q_{mn}] \quad (87)$$

with

$$Q_{mn} = \int \frac{dk' dk}{4\pi^2} \int_k f_n(k) K_t(k-k') t_{k'}^* f_m(k') \quad (88)$$

and the kernel

$$K_t(q) = \int x_0 v_F t dx \ e^{iq(x-v_F t)} = \frac{e^{iqx_0} (1 - e^{-iqv_F t})}{iq} \quad (89)$$

$$= 2e^{iq(x_0-v_F t/2)} \sin(qv_F t/2) \frac{q}{q}. \quad (89)$$

The matrix $Q_{mn}$ is a Hermitian matrix with real eigenvalues $0 \leq \tau_{mn}(t) \leq 1$, hence the associated full counting statistics is generalized binomial for all times. The same result can be retrieved from Ref. \cite{34} with an appropriate choice for the time-dependent scatterer. In the following, we discuss various limits for the generating function $\chi_t(\lambda)$.

### B. Short measuring time

Assuming that $N$ is large enough so that $t_k$ does not change appreciably over the interval $\varepsilon$, i.e., $\varepsilon \partial_t t_k \ll 1$, the amplitude $t_k$ can be taken out of the integral in Eq. \cite{73}. Assuming furthermore that the measurement time $t$ is short, $|q| v_F t \leq \varepsilon V/\hbar \ll 1$, we can expand $K_t(q)$ and obtain (to lowest order in $\varepsilon V/\hbar$)

$$\chi_t^{\varepsilon}(\lambda) = \det[\delta_{mn} + (e^{i\varepsilon} - 1) t_{k_m} t_{k_n} \varepsilon v_F t \times e^{iqx_0 (n-m)} \frac{2\sin^2(\varepsilon x_0/2)}{\pi x^2} \frac{N}{m=1} T_{mn}]. \quad (90)$$

The second term involves a matrix product $(v_1, v_2, \ldots, v_N)^\dagger (v_1, v_2, \ldots, v_N)$ of a vector and its dual, where $v_m = t_{k_m} e^{iqx_{0m}}$, and hence can be written as a projector, in Dirac notation, $\mu | v \rangle \langle v | v \rangle$ with $\mu = 2(e^{i\varepsilon} - 1) \varepsilon v_F t \sin^2(\varepsilon x_0/2)/\pi x^2$. The determinant $\det(1 + \mu | v \rangle \langle v | v \rangle)$ then is given by the product of eigenvalues $1 + \mu | v \rangle \langle v | v \rangle$ (in the direction of $| v \rangle$ and 1 (in the complement), $\det(1 + \mu | v \rangle \langle v | v \rangle) = 1 + \mu | v \rangle \langle v | v \rangle$, and we obtain

$$\chi_t^{\varepsilon}(\lambda) = 1 + (e^{i\varepsilon} - 1) \varepsilon v_F t \frac{2\sin^2(\varepsilon x_0/2)}{\pi x^2} \frac{N}{m=1} T_{mn} \frac{(\varepsilon=0)}{1 + \alpha(e^{i\varepsilon} - 1)}, \quad (91)$$

with (note that $N \varepsilon = eV/\hbar v_F$)

$$\alpha = tv_F \int_0^{eV/\hbar v_F} \frac{dk}{2\pi} T_k \leq \varepsilon V/2\pi \hbar \ll 1 \quad (92)$$

(note that the dependence on the counter position $x_0$ has disappeared from the parameter $\alpha \propto t$ in the constant voltage limit $\varepsilon \to 0$). Pushing the calculation to higher order in $\varepsilon V/\hbar$, we find that in the expansion of $\chi_t^{\varepsilon}(\lambda)$ both, second- and third-order terms, vanish and the next correction appears only in fourth order,

$$\Delta \chi_t^{\varepsilon}(\lambda) = \frac{(e^{i\varepsilon} - 1)^2}{24} \int_0^{eV/\hbar v_F} \frac{dkdk'}{(2\pi)^2} T_k T_{k'} (k-k')^2 \leq \frac{(e^{i\varepsilon} - 1)^2}{24\pi} (\varepsilon V/\hbar)^4 \quad (93)$$

hence, for short measuring times the majority of counts involve either no or a single particle, while the observation of two-particle events $P_2 \leq (\varepsilon V/\hbar)^4/(24\pi)^2$ is strongly suppressed, a consequence of the Pauli exclusion principle. Note that in the short measuring time limit, the specific nature of the counting device matters. Above, we have assumed that all intrinsic timescales of the counter are much shorter than the measuring time. Furthermore, we have neglected the effect of the Fermi sea which will produce an additional contribution. Nevertheless, even modeling the counter more realistically, the Pauli principle with its reduction of two- and more-particle events is expected to reveal itself. Furthermore,
it is possible to realize an experiment where the effect of the additional Fermi sea is absent: by applying a voltage to a quantum wire which is larger than the Fermi energy, particles incident from the right are blocked by the band bottom and only left going states within an energy interval $E_F$ (replacing the bias $eV$) contribute to the particle current.$^{12}$

Note that the generalized binomial statistics Eq. $\text{(3)}$ reduces to the simple Poissonian result

$$\log \chi(\lambda) = \sum_m \tau_m (e^{\lambda} - 1)$$  \hspace{1cm} (94)

in the limit of small generalized transmission probabilities $\tau_m \ll 1$; the result then only depends on one parameter $\sum_m \tau_m = \text{tr} T_Q$. In the limit of short measuring times, the smallness of the transmission eigenvalues is imposed by the small space interval in the projection $Q$, and $\sum_m \tau_m = \alpha$, see Eq. $\text{(92)}$. The same result is obtained in the long time limit, see $\text{(98)}$, provided the transmission probabilities $T_k$ themselves are small.

C. Large measuring times

In the asymptotic limit of $t \to \infty$, the kernel $K_t(q)$ ensures energy/momentum conservation, rendering the problem diagonal in the momentum basis.$^{11,12}$ However, adopting the $t \to \infty$ asymptotic limit is incompatible with a regular derivation of a finite result. Here, we consider instead the case of large but finite measuring time $t$, while adopting the limit of infinite particle number $N \to \infty$ when letting the width $h\alpha = eV/v_N$ go to zero at constant voltage $V$.

In the limit $N \to \infty$, the characteristic function $\chi_t(\lambda)$, which is the determinant of the matrix in Eq. $\text{(88)}$, cf. Eq. $\text{(76)}$, is given by

$$\chi_t(\lambda) = \det(1 - PT Q_t + PT Q_t e^{i\lambda})$$  \hspace{1cm} (95)

where $P = \int_{\pi} e^{iV/\hbar v_F} dk/(2\pi)$ is the projector on the subspace of occupied states; the form $\text{(95)}$ can be obtained from $\text{(88)}$ introducing the projector $P$ to extend the determinant over the whole Hilbert space, cf. Eqs. $\text{(75)}$ and $\text{(76)}$, and using the determinant identity $\text{det}(1 + AB) = \text{det}(1 + B)A$ to shuffle $t_k^*$ to the left of $K_t$, which itself is the momentum representation of the projector $Q_t$. The expression Eq. $\text{(95)}$ then corresponds to Eq. $\text{(76)}$ with the substitution $T_Q = T Q_t$. As $Q_t$ is a projector, $Q_t^2 = Q_t$, we can rewrite Eq. $\text{(95)}$ as $\text{det}[1 + (e^{i\lambda} - 1)Q_t PT Q_t]$. This determinant only needs to be calculated in the subspace $Q_t H$ as the matrix is unity in the complement. In the subspace $Q_t H$, we use the orthonormal real-space (rather than $k$-space, see Eq. $\text{(88)}$) basis

$$g_l(x) = \begin{cases} 1/\sqrt{T}, & l \leq x - x_0 \leq l, \\ 0, & \text{elsewhere} \end{cases} \hspace{1cm} (96)$$

with $\epsilon = tv_F/L$ the width of a real space segment and $l \in \{1, \ldots, L\}$. The matrix elements of $PT$ assume the form

$$\langle g_l | PT | g_m \rangle = \int_0^{\pi} \frac{dv}{2\pi} T_k \langle g_l | k \rangle \langle k | g_m \rangle$$  \hspace{1cm} (97)

$$= \int_0^{\pi} \frac{dv}{2\pi} T_k \frac{4 \sin^2(ek/2)}{ek^2} e^{i(l-m)ke},$$

i.e., they form a Toeplitz matrix. Applying Szegő’s theorem and taking the limit of large $t$ and $L$ with $\epsilon = v_N t/L$ fixed but small $e \ll hv_F/eV$, hence $4 \sin^2(ek/2)/ek^2 \approx \epsilon$) we obtain the generating function

$$\log \chi_t(\lambda) = tv_F \int_0^{\pi} \frac{dv}{2\pi} \log(1 - T_k + T_k e^{i\lambda}),$$  \hspace{1cm} (98)

cf., App. A.

Using the generalization of Szegő’s theorem (Fisher-Hartwig conjecture, see $\text{A7}$ and $\text{A10}$),$^{13}$ it is possible to calculate the next order term. As the argument of the logarithm (cf. $\text{A10}$), note that $x(\theta)$ is to be replaced by $\sum_{n \in \mathbb{Z}} P(\theta + 2\pi n)/e[1 + (e^{i\lambda} - 1)T(\theta + 2\pi n)/e]$ with $P_k = 1, k \in [0, eV/hv_F]$ and $P_k = 0$ otherwise) exhibits discontinuities at $k = 0$ and $k = eV/hv_F$, the correction to the leading term is given by the two contributions originating from the jumps at $k = 0, eV/hv_F$,

$$\Delta \log \chi_t(\lambda) = \frac{\log(t/t_0)}{4\pi^2} \sum_{\kappa = 0, eV/hv_F} \log^2[1 + (e^{i\lambda} - 1)T_k],$$  \hspace{1cm} (99)

with $t_0$ some small time cutoff; this result leads to logarithmic corrections for the second-order and all higher cumulants. For the noise, the correction is given by$^{19,27}$

$$\Delta \langle \langle n^2 \rangle \rangle = \frac{T_0^2 + T_e^2 e^{2iV/hv_F}}{2\pi} \log(t/t_0).$$  \hspace{1cm} (100)

Here, the logarithmic corrections in Eqs. $\text{(99)}$ and $\text{(100)}$ are due to fluctuations in the number of particles in a finite interval of length $v_N t$. Therefore, fluctuations do not disappear for $T = 1$ as in Eqs. $\text{(92)}$ and $\text{(93)}$ where the number of particles is fixed and noise stems only from partitioning. In addition to the noise originating from the voltage bias, there is an equilibrium contribution due to the Fermi sea at any finite measuring time, cf. Sec. VI B. For asymptotically large times, the first contribution grows logarithmically in time.$^2$}

The reason why Szegő’s theorem is applicable to the matrix Eq. $\text{(97)}$ is the presence of time translation invariance: matrix elements between states localized at two different places/times depend only on their space/time separation and hence they form a Toeplitz matrix. The same reasoning does not apply to the momentum basis and that is why we could not apply Szegő’s theorem directly to the matrix in Eq. $\text{(88)}$. The result Eq. $\text{(98)}$ (and its generalization to finite temperatures, cf. Eq. $\text{(113)}$) has been found by Schönhämmers using a double projection in his counting procedure: instead of relying on Szegő’s theorem, use has been made
of the relation \( \log(1 + M) = \log(1 + M) \), followed by an expansion of the logarithm. Evaluating the trace of each term, the phase factors (see Eq. (99)) appearing in the cyclic product of the kernel \( K_t \) cancel mutually. In the long-time limit, the kernels become diagonal (ensuring energy-conservation) with one of them contributing a factor of \( t \), thus rendering the cumulant generating function \( \log \chi_t(\lambda) \) linear in \( t \). In alternative approaches, use has been made of a mapping onto the Riemann-Hilbert problem (this procedure enables the calculation of the leading term as well as the logarithmic correction Eq. (99)), or of time periodicity introduced in order to render \( \log \chi_t \) extensive in \( t \) (this way, only the term linear in \( t \) is obtained).

D. Fano factor for intermediate regime

In order to understand the crossover between the short and long time behavior of the carrier distribution, we calculate the Fano factor \( F \) and present the result as a function of \( \nu_t = tv/2\pi \hbar \) (the incident particle number during time \( t \)) in Fig. 4(a) for several values of the transmission coefficient \( T \) (for a scatterer with energy independent transmission). For small times, the distribution is Poissonian and hence \( F(\nu_t \to 0) \to 1 \). The binomial distribution valid at large times provides the asymptotics \( F(\nu_t \to \infty) \to (1 - T) \). In order to find the crossover in between, we determine the matrix \( Q \) (see Eq. (100)),

\[
Q_{mn} \to t^* \tau \frac{e^{i(n-m)\tau(x_0-v_{F}t)}}{\pi(n-m)},
\]

in terms of which the characteristic function assumes the simple form (87) and hence \( \log \chi_t(\lambda) = \tau \log(\delta_{mn} + e^{i\lambda} - 1)Q_{mn} \) (again, we consider the limit \( \tau \to 0 \) at fixed voltage \( V \)). The average transmitted charge \( \langle n \rangle = -i\partial \lambda \log \chi_t|_{\lambda = 0} \),

\[
\langle n \rangle = \tau Q = tv \int_0^{eV/hv\nu} \frac{dk}{2\pi} T_k,
\]

grows linearly with the measuring time \( t \); the above result coincides with those obtained from the short and long time expressions (91) and (98). The noise \( \langle n^2 \rangle \) = \(-\partial^2 \log \chi_t|_{\lambda = 0} \) assumes the form

\[
\langle n^2 \rangle = \tau Q - \tau Q^2
\]

\[
= \langle n \rangle - \int_0^{eV/hv\nu} \frac{dk}{2\pi} T_k \frac{\sin^2[(k' - k)v_{F}t/2]}{(k' - k)^2}.
\]

(in the limit \( \tau \to 0 \) considered here, both momenta do not depend on the position \( x_0 \) of the counter, as the wave packets are infinitely spread). In order to keep the analysis simple, we assume an energy independent transmission probability, \( T_k = T \), over the interval \( [0,eV/hv\nu] \). The average charge then is given by

\[
\langle n \rangle = TteV/2\pi \hbar = tvt.
\]

The Fano factor \( F = \langle n^2 \rangle/\langle n \rangle \) can be cast into the form

\[
F = 1 - Tf(\nu_t)
\]

with

\[
f(\nu_t) = \frac{1}{\pi^2} \sin^2(\pi v_{F}t).
\]

For small times \( \nu_t \ll 1 \),

\[
f(\nu_t) = \nu_t - \frac{\pi^2}{18} \nu_t^3 + O(\nu_t^5),
\]

while \( f \) approaches unity in the long time limit \( \nu_t \gg 1 \) (\( \gamma \approx 0.5772 \) is Euler’s constant),

\[
f(\nu_t) = 1 - \frac{\log(2\pi \nu_t) + 1 + \gamma}{\pi^2 \nu_t} + O(\nu_t^{-3}).
\]

The corrections to the simple binomial result produce a logarithmic in time increase of the noise \( \langle n^2 \rangle \); the result (107) coincides with Eq. (100) for the case of energy independent scattering probabilities \( T_k = T \). This logarithmic dependence in the noise is due to the fluctuations in the number of electrons in a finite segment of the wire.

Analogously, the third cumulant \( \langle n^3 \rangle \) can be calculated; the (numerical) results, shown in Fig. 4(b), interpolate between the Poissonian value \( \langle n^3 \rangle/\langle n \rangle = 1 \) for short times and the binomial result \( \langle n^3 \rangle/\langle n \rangle = T(1 - T)(1 - 2T) \) for long measuring times.

E. Finite temperature

We consider the case where particles are emitted from a lead at finite temperature into vacuum, i.e., we assume a single Fermi reservoir of particles (incident from the left) which are scattered with energy-dependent transmission probabilities (to the right). At finite temperature, scattering states are occupied according to the Fermi-Dirac occupation as described by the one-particle density operator \( \eta = [e^{i\nu - \mu} + 1]^{-1} \). The characteristic function \( \chi_t(\lambda) \) is given by Eq. (104), with \( T_\nu = T_Q t \) and the interval \( I = [x_0, x_0 + v_{F}t] \) defining the projector \( Q_I \), cf. Sec. VI C. Following essentially the calculation in Sec. VI C, i.e., calculating the determinant in the basis (99) and applying Szegő’s theorem, the result

\[
\log \chi_t(\lambda) = tvt \int \frac{dk}{2\pi} \log \left[ 1 + T_k n_k(\lambda) \right].
\]

with \( n_k(\lambda) = |\langle k | \lambda \rangle| \approx [e^{i\nu - \mu} + 1]^{-1} \) can be obtained for fixed but long measurement times \( t \). For high temperatures at constant particle density \( \beta \to 0 \), \( n_k(\lambda) \approx e^{-\beta_H k} \), all transmission eigenvalues \( \tau_k = T_k / e^{\beta_H k} \) approach zero. The logarithm in Eq. (108) can be expanded and the emission statistics for electrons leaving a Fermi reservoir in
the high temperature regime is given by a Poissonian statistics:\cite{44}

$$\log \chi_l(\lambda) = (e^{i\lambda} - 1) t V \int \frac{dk}{2\pi} T_k e^{-\beta(h v_F k - \mu_L)} \tag{109}$$

The Fano factor assumes the value $F = 1$, independent of $T_k$.

To complete the analysis, we discuss the extension of the constant-voltage result with two reservoirs to finite temperatures. We model the setup by two Fermi reservoirs with occupation numbers $n_{L/R}(k) = |\exp[\beta(h v_F k - \mu_{L/R})]|^{-1}$ for particles incoming from the left (L) or right (R), respectively. The voltage enters via the bias of the chemical potentials $eV = \mu_L - \mu_R$.

Unfortunately, it is not possible to define a projection operator $Q$ which acts after the evolution and which can emulate the action of the spin counter, cf. the discussion below Eq. (17). The reason is that there is no way to tell for a particle outgoing to the left at time $t \to \infty$ whether it was coming in from the left and was reflected at the scatterer (hence no counting is done with the counter to the right of the scatterer) or whether it was coming in from the right and had been transmitted through the scatterer (hence passing the spin counter once). One solution to this problem is to perform a first projective measurement at the initial time\cite{27,28} (this corresponds to replacing Eq. (17) by the expression

$$\chi_l(\lambda) = \frac{\det(1 + \rho \mathcal{U} e^{i\lambda Q_L} e^{-i\lambda Q_R} \mathcal{U}^\dagger e^{i\lambda Q_L} \mathcal{U}^\dagger)}{\det(1 + \rho)} \tag{110}$$

with the occupation-operator $\eta = \rho/(1 + \rho) = \int (dk/2\pi)|k|_m \text{diag}[n_L(k), n_R(k)]_m\langle k |$, the single-particle evolution $\mathcal{U}$ involving the scatterer but not the spin counter, cf. Eq. (72), and $Q_L$ a projector emulating the counting measurement of transmitted particles via projection of the wave functions onto the lead to the right of the scatterer. The additional factor $\exp(-i\lambda Q_L)$, as compared to (77), corresponds to the additional measurement before the evolution.

In this article, we want to stick with the spin counter as a measurement apparatus. Contrary to the situation in Sec. V the action of the spin counter cannot be modeled by a projection onto the outgoing states, i.e., the operators $\mathcal{U}_k$ do not separate any more into factors describing the scatterer and the counter. Therefore, we have to make use of the full evolution operators $\mathcal{U}_k$, $|\Psi_{out}\rangle = \Gamma(\mathcal{U}_k)|\Psi\rangle$, in the presence of both the scatterer and the spin counter, where the index $\pm$ refers to the two spin states of the counter. The overall evolution (cf. Fig. 5) then can be written as

$$\mathcal{U}_k = e^{\pm i\lambda Q_L/2} \mathcal{U} e^{i\lambda Q_L/2}, \tag{111}$$

where $\mathcal{U}$, cf. Eq. (72), is the evolution without accounting for the presence of the spin-counter. The generating function for the full counting statistics assumes the form

$$\chi_l(\lambda) = \frac{\det(1 + \rho e^{-i\lambda Q_L/2} \mathcal{U} \mathcal{U}^\dagger e^{i\lambda Q_L} \mathcal{U}^\dagger e^{-i\lambda Q_L/2})}{\det(1 + \rho)} \tag{112}$$

with the occupation-operator $\eta = \rho/(1 + \rho) = \int (dk/2\pi)|k|_m \text{diag}[n_L(k), n_R(k)]_m\langle k |$, the single-particle evolution $\mathcal{U}$ involving the scatterer but not the spin counter, cf. Eq. (72), and $Q_L$ a projector emulating the counting measurement of transmitted particles via projection of the wave functions onto the lead to the right of the scatterer. The additional factor $\exp(-i\lambda Q_L)$, as compared to (77), corresponds to the additional measurement before the evolution.

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$$\chi_l(\lambda) = \frac{\det(1 + \rho e^{-i\lambda Q_L/2} \mathcal{U} \mathcal{U}^\dagger e^{i\lambda Q_L} \mathcal{U}^\dagger e^{-i\lambda Q_L/2})}{\det(1 + \rho)} \tag{112}$$

The two counting procedures Eqs. (110) and (112) agree if the particles are only incident from the left, as the additional counting factors, compared to Eq. (77), contribute unity. For particles incoming from both left and right the two counting procedures do not necessarily coincide; only if $Q_L$ commutes with $\rho$, we can shift the factor $e^{-i\lambda Q_L/2}$ to the left of $\rho$ and then cyclically permute the factors in the second term of the determinant to assert the equivalence of (110) and (112).
We have used the first-quantized wave packet formalism to calculate the generating function \( \chi^k(\lambda) \) of full counting statistics of fermionic particles in various physical situations, such as \( N \) particles incident in Slater determinant states of rank 1 (nontangled), rank 2 (entangled), or incoherent superpositions of Slater determinants in Fock space with undetermined particle number. Our formalism captures various features such as energy dependent scattering probabilities as well as time-dependent scattering and time-dependent counting.

We have presented our results in determinantal form, with further simplifications explicitly unveiling a generalized binomial statistics in various cases. Applications of our results include a classification of possible statistical behavior of two-particle scattering events and a particularly simple singlet-triplet and entanglement detector. In the context of coherent transport of noninteracting (degenerate) fermions, the natural reference point in the discussion of statistical properties is the binomial distribution; energy dependent scattering naturally shifts the noise into the sub-binomial (or generalized binomial) regime, whereas additional correlations through entanglement can generate super-binomial noise statistics.

Our results, calculated at zero temperature, remain valid for \( \beta^{-1} \ll \hbar v_F/\xi \), i.e., sufficiently narrow wave packets with a small width \( \xi \) in real space. Furthermore, we have calculated the generating function for the constant voltage case in the long-time limit for any temperature. For short measuring times our results are valid in the temperature regime \( \beta^{-1} \ll eV \) and we have found a strong suppression of \( P_{n \geq 2} \) due to Pauli blocking.

The central element underlying the appearance of a (sub-)binomial statistics in fermionic systems is the absence of interparticle interactions and entanglement. This result remains valid for a time-dependent scattering potential and finite temperature. We have analyzed the modification introduced by entanglement and have found that super-binomial statistics may be generated. The inclusion of interaction, particularly within the scatterer where interacting particles become entangled, remains an interesting open problem.

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APPENDIX A: (STRONG) SZEGÖ THEOREM

The (strong) Szegő theorem applies to Toeplitz matrices and reduces the calculation of the asymptotic be-
behavior of their determinants to a simple integration (plus summation) problem. We define a Toeplitz matrix starting from a complex-valued periodic function $a(\theta)$ with $a(\theta + 2\pi) = a(\theta)$. In addition, we require that its winding number with respect to the origin is equal to zero. We define the Fourier coefficients

$$a_m = \int_0^{2\pi} \frac{d\theta}{2\pi} a(\theta) e^{-im\theta} \quad (A1)$$

and the associated $N \times N$ Toeplitz matrix with elements

$$[A_N(a)]_{m,n} = a_{m-n} \quad (A2)$$

depending only on the difference between the indices $m$ and $n$ (banded matrix), $m, n = 1, \ldots, N$. The strong form of the Szegő theorem\textsuperscript{31,45} states that

$$\log \det A_N(a) \sim N \log |a|_0 + \sum_{n=1}^{\infty} n |\log a|_n |\log a|_{-n} \quad (A3)$$

asymptotically for $N \to \infty$, with

$$|\log a|_n = \int_0^{2\pi} \frac{d\theta}{2\pi} |\log(a(\theta))| e^{-im\theta} \quad (A4)$$

the Fourier coefficients of $\log |a(\theta)|$. The first term in (A3) scaling with $N^1$ is the result of Szegő’s theorem, while its strong form applies once the sum in the second term converges — this correction then scales with $N^0$.

Given the Toeplitz matrix $X^f = S^f + (e^{i\lambda} - 1)T^f$, cf., Eq. (A9), we show how to find its determinant Eq. (57) in the asymptotic limit of large $N$. Specifying the matrix elements

$$x_{m-n} = \int \frac{dk}{2\pi} |f(k)|^2 (1 - T_k + T_k e^{i\lambda}) e^{i(m-n)ka} \quad (A5)$$

we find the original periodic function $x(\theta)$ by calculating the Fourier series

$$x(\theta) = \sum_{m \in \mathbb{Z}} x_m e^{im\theta}$$

$$= \frac{1}{a} \sum_{m \in \mathbb{Z}} |f((\theta + 2\pi m)/a)|^2$$

$$\times [1 - T(\theta + 2\pi m)/a + T(\theta + 2\pi m)/ae^{i\lambda}] \quad (A6)$$

Note that, while the original function $x(k) = |f(k)|^2 (1 - T_k + T_k e^{i\lambda})$ was defined on the real axis, the new expression $\kappa(k) = x(\theta = ak)$ is restricted to the first Brillouin zone $k \in [0, 2\pi/a]$. Fourier transforming the logarithm of $x(\theta)$ according to Eq. (A4), we obtain the asymptotic expression for the determinant

$$\log \det X_N^f = N \int_0^{2\pi} \frac{d\theta}{2\pi} \log |x(\theta)|$$

$$+ \sum_{n=1}^{\infty} n \log |x|_n \log |x|_{-n} + o(1), \quad (A7)$$

consisting of a main term $\propto N$, a first correction staying constant as $N \to \infty$, and a remaining correction $o(1)$ vanishing as $N \to \infty$. The (logarithm of the) determinant $S^f$ in Eq. (57) is derived by setting $T \equiv 0$ in (A6). Finally, we obtain the (log of the) characteristic function by simple subtraction (we replace the angle $\theta$ on the unit circle $[0, 2\pi]$ by $k = \theta/a$ in the first Brillouin zone $[0, 2\pi/a]$), to leading order in $N$

$$\log \chi_N(\lambda) = N a \int_0^{2\pi/a} \frac{dk}{2\pi} \log(1 - \tau_k + \tau_k e^{i\lambda}) \quad (A8)$$

with the effective scattering probabilities

$$\tau_k = \frac{\sum_{m \in \mathbb{Z}} |f(k + 2\pi m/a)|^2 T_{k+2\pi m/a}}{\sum_{m \in \mathbb{Z}} |f(k + 2\pi m/a)|^2} \quad (A9)$$

For a function $x(\theta)$ which is continuous on the unit circle, i.e., $x(2\pi) = x(0)$, the sum in (A2) converges and the corrections to (A8) are constant when $N \to \infty$ (and similar for $s(\theta) = \sum_{m \in \mathbb{Z}} |f[(\theta + 2\pi m)/a]|^2/2|a|$ in the calculation of log det $S_N^f$). A more subtle situation appears in the situation where $x(\theta)$ and/or $s(\theta)$ are discontinuous across the Brillouin zone (Fisher-Hartwig conjecture).\textsuperscript{43}

This situation is the usual case as the wave function $f(k)$ is discontinuous at the Fermi level $k = 0$. Thus, the sum in (A7) is divergent and the next term in the expansion of (A8) scales with $\log N$ (cf. also Eq. (99)),

$$\Delta \log \chi_N(\lambda) = \frac{\log^2[x(0^+)/x(0^-)] - \log^2[s(0^+)/s(0^-)]}{4\pi^2} \log N, \quad (A10)$$

and is followed by a constant term. Here, the number of particles $N$ is fixed and noise is due to partitioning; hence, the logarithmic corrections (A10) have to be attributed to partitioning (and not to fluctuations in the number of particles as for the constant voltage result Eq. (99)). This is also consistent with the vanishing of the correction (A10) for $T = 1$ where $x(\theta) = e^{i\lambda} s(\theta)$ and $x(0^+)/x(0^-) = s(0^+)/s(0^-)$.

1. L.S. Levitov and G.B. Lesovik, JETP Lett. 58, 230 (1993).
2. L.S. Levitov, H.W. Lee, and G.B. Lesovik, J. Math. Phys. 37, 4845 (1996).
3. Yu.V. Nazarov, ed., Quantum Noise in Mesoscopic Physics (Kluwer Academic Publishers, 2003).
4. G.B. Lesovik, F. Hassler, and G. Blatter, Phys. Rev. Lett.
20

96, 106801 (2006).
5 A. Peres, Phys. Rev. A 30, 1610 (1984).
6 Th. Martin and R. Landauer, Phys. Rev. B 45, 1742 (1992).
7 H. Lee and L.S. Levitov, cond-mat/9507011 (1995).
8 D.A. Ivanov, H. Lee, and L.S. Levitov, Phys. Rev. B 56, 6839 (1997).
9 A.V. Lebedev, G.B. Lesovik, and G. Blatter, Phys. Rev. B 72, 245314 (2005).
10 J. Keeling, I. Klich, and L.S. Levitov, Phys. Rev. Lett. 97, 116403 (2006).
11 G. Fève, A. Mahé, J.-M. Berroir, T. Kontos, B. Plaçais, D.C. Glattli, A. Cavanna, B. Etienne, and Y. Jin, Science 316, 1169 (2007).
12 F. Hassler, G.B. Lesovik, and G. Blatter, Phys. Rev. Lett. 99, 076804 (2007).
13 We use the term ‘entangled-state’ for indistinguishable particles in the sense of J. Schliemann, J.I. Cirac, M. Kuś, M. Lewenstein, and D. Loss, Physical Review A 64, 022303 (2001).
14 F. Taddei and R. Fazio, Phys. Rev. B 65, 075317 (2002).
15 K.E. Cahill and R.J. Glauber, Phys. Rev. A 59, 1538 (1999). Cahill et al. denote their generating function (generating counting probabilities upon taking derivatives) by $Q(\lambda)$ where their $\lambda$ is equivalent to $1 - e^{i\lambda}$ in our article.
16 R.J. Glauber, in Quantum Optics and Electronics, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965).
17 S. Braungardt, A. Sen(De), U. Sen, R.J. Glauber, and M. Lewenstein, arXiv:0802.4276 (2008).
18 G. Burkard, D. Loss, and E.V. Sukhorukov, Phys. Rev. B 61, R16303 (2000).
19 K. Schönhammer, Phys. Rev. B 75, 205329 (2007).
20 G. Szegő, Math. Ann. 76, 490 (1915).
21 G. Szegő, Comm. Sém. Math. Univ. Lund pp. 228–238 (1952).
22 L.S. Levitov and G.B. Lesovik, JETP Lett. 55, 555 (1992).
23 G.B. Lesovik and N.M. Chctchelkatchev, JETP Lett. 77, 393 (2003).
24 L.S. Levitov and G.B. Lesovik, cond-mat/9401004 (1994).
25 I. Neder and F. Marquardt, New J. Phys. 9, 112 (2007).
26 D.V. Averin and E.V. Sukhorukov, Phys. Rev. Lett. 95, 126803 (2005).
27 B.A. Muzykantskii and Y. Adamov, Phys. Rev. B 68, 155304 (2003).
28 A. Shelankov and J. Rammer, Europhys. Lett. 63, 485 (2003).
29 J.E. Avron, S. Bachmann, G.M. Graf, and I. Klich, Commun. Math. Phys. 280, 807 (2008).
30 C.C.J. Roothaan, Rev. Mod. Phys. 23, 69 (1951).
31 In condensed matter theory this statement is made use of in the Bogoliubov-Valatin transformation where the quadratic Hamiltonian is diagonalized while keeping the commutation relations invariant.
32 R. Courant and D. Hilbert, Methoden der mathematischen Physik, 4th edition (Springer-Verlag, Berlin, Heidelberg, New York, 1993).
33 M.V. Lebedev, A.A. Shchekin, and O.V. Mishchik, Quantum Electron. 38, 710 (2007).
34 A.G. Abanov and D.A. Ivanov, Phys. Rev. Lett. 100, 086602 (2008).
35 M.J.M. de Jong and C.W.J. Beenakker, Phys. Rev. B 49, 16070 (1994).
36 F. Bodoky, W. Belzig, and C. Bruder, Phys. Rev. B 77, 035302 (2008).
37 We assume that the Hamilton operators $\mathcal{H}(t)$ and $\mathcal{H}(t')$ for different times $t \neq t'$ commute. Otherwise a time-ordering operator has to be introduced.
38 I. Klich, in Ref. 3.
39 B.A. Muzykantski and D.E. Khmelnitskii, Phys. Rev. B 50, 3982 (1994).
40 Yu.V. Nazarov and D.A. Bagrets, Phys. Rev. Lett. 88, 196801 (2002).
41 S. Pilgram and M. Büttiker, Phys. Rev. B 67, 235308 (2003).
42 R.J. Brown, M.J. Kelly, M. Pepper, H. Ahmed, D.G. Hasko, D.C. Peacock, J.E.F. Frost, D.A. Ritchie, and G.A.C. Jone, J. Phys.: Condens. Matter 1, 6285 (1989).
43 E.L. Basor, Indiana Univ. Math. J. 28, 975 (1979).
44 W. Schottky, Ann. Phys. (Leipzig) 57, 541 (1918).
45 B. Simon, in Geometry, Spectral Theory, Groups, and Dynamics, vol. 387 of Contemporary Mathematics, pp. 253–275 (American Mathematical Society, Providence, 2005).