TOWARDS SECOND ORDER LAX PAIRS TO DISCRETE
PAINLEVÉ EQUATIONS OF FIRST DEGREE

Robert Conte† and Micheline Musette‡

†Service de physique de l’état condensé, CEA Saclay,
F-91191 Gif-sur-Yvette Cedex, France
‡Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel,
B-1050 Brussel, Belgique

PACS : 02.30.Ks, 02.90.+p, 05.50.+q,

Keywords
Painlevé equations
discrete Painlevé equations
Lax pairs
discrete Lax pairs

Abstract – We investigate the question of finding discrete Lax pairs for the six
discrete Painlevé equations (Pn). The choice we make is to discretize the pairs of
Garnier, once converted to matricial form.

Integrability and chaos in discrete systems, Chaos, solitons and fractals (1998),
eds. I. Antoniou and F. Lambert. Brussels, 2–6 July 1997.
# Contents

1. **Introduction** ................................................................. 3

2. **The five (Pn) equations of Garnier and their coalescence** ............................ 4

3. **Various Lax pairs of (Pn) in the continuous case** .................................... 5
   - 3.1 Lax pairs by scalar isomonodromy (Garnier) ........................................ 5
   - 3.2 Lax pairs by matricial isomonodromy (Jimbo and Miwa) .......................... 6
   - 3.3 Lax pairs by reduction of a PDE ......................................................... 7
   - 3.4 Comparison of these three kinds of Lax pairs ....................................... 10

4. **The Garnier matricial Lax pairs** ............................................. 10

5. **Rules of discretization of matricial Lax pairs** ....................................... 12

6. **Application to the discretization of the (Pn) equations and of their Lax pair** ............................... 13

7. **Conclusion** ..................................................................... 14
1 Introduction

In their programme of defining functions by nonlinear ordinary differential equations (ODEs), Painlevé and Gambier built a list of fifty second order, first degree such equations, the general solution of which is uniformizable; among them, only six define new functions, called the Painlevé transcendents (Pn). There have been recently many achievements in obtaining some discrete analogues of these fifty equations. However, at present time, the nice features of the continuous case (degree one in $u''$ for the ODEs, order two for the Lax pairs of the six (Pn) equations, coalescence cascade from (P6) to (P1) for the ODEs as well as for their Lax pairs) are far from being all implemented in the discrete case. Table 1 outlines some of the gaps to be filled.

Table 1: Current state of the discrete (Pn) equations and their Lax pair. Column 1 contains the type (difference or $q$–difference) of the discrete (Pn), column 2 the degree of this discrete equation in $u$, column 3 the order of the Lax pair if one is known, column 4 the appropriate references. A blank denotes a missing information. Ideally, there should be at least one entry for each d–(Pn), only digits 1 in column 2 and digits 2 in column 3.

| d–(Pn) | degree | order of Lax pair | references |
|--------|--------|-------------------|------------|
| d–(P6) | 2      |                   | [25]       |
| q–(P6) | 2      | 2                 | [16]       |
| d–(P5) | 1      |                   | [24]       |
| q–(P5) | 1      |                   | [24]       |
| d–(P4) | 1      | 4                 | [24, 21]   |
| d–(P3) | 2      | 4                 | [12]       |
| q–(P3) | 1      | 4                 | [24, 21]   |
| d–(P2) | 1      | 2                 | [12]       |
| d–(P1) | 1      | 2                 | [7]        |
| d–℘    | 1      | 2                 | [23]       |

In this paper, we address the question of finding a second order Lax pair for each discrete (Pn) equation.

In section 2, we present the five (Pn) equations chosen by Garnier, which have some advantages over the six usual ones. In section 3, we review the different kinds of continuous Lax pairs and discuss them according to their relevance for discretization. In section 4, we convert the scalar “Lax” pairs given by Garnier in 1911 into an equivalent traceless matricial form, so as to remove the apparent singularity unavoidable in the scalar form. In section 5, we recall the discretization rules which we previously established. In section 6, we give some preliminary results.
2 The five (Pn) equations of Garnier and their coalescence

The (P6) equation for \( u(x) \) possesses in the plane of \( u \) four poles \((\infty, 0, 1, x)\) with the same residue equal to \(1/2\), and it depends on four parameters \( \alpha, \beta, \gamma, \delta \). The scheme of the successive coalescences of these four poles

\[(1/2, 3/2) \rightarrow (1/2, 1/2) \rightarrow (1/2, 1, 1/2) \rightarrow (1, 1)\]

defines, from (P6)(\( u, x, \alpha, \beta, \gamma, \delta \)), four other equations with four parameters, chosen as follows by Garnier [11]

\[
\begin{align*}
(P6) \quad u'' &= \frac{1}{2} \left[ \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right] u'^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right] u' \\
&\quad + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{u^2} + \gamma \frac{x-1}{(u-1)^2} + \delta \frac{x(x-1)}{(u-x)^2} \right], \\
(P5) \quad u'' &= \left[ \frac{1}{2u} + \frac{1}{u-1} \right] u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2} \left[ \alpha u + \beta \right] + \gamma \frac{u}{x} + \delta \frac{u(u+1)}{u-1}, \\
(P4') \quad u'' &= \frac{u'^2}{2u} + \gamma \left( \frac{3}{2} u^3 + 4xu^2 + 2x^2u \right) + 4\delta(u^2 + xu) - 2\alpha u + \frac{\beta}{u}, \\
(P3') \quad u'' &= \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \gamma u^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u}, \\
(P2') \quad u'' &= \delta(2u^3 + xu) + \gamma(6u^2 + x) + \beta u + \alpha, \\
(J') \quad u'' &= 2\delta u^3 + 6\gamma u^2 + \beta u + \alpha.
\end{align*}
\]

Under the group of homographic transformations on \( u \), the (P4') equation is equivalent to (P4) for \( \gamma \neq 0 \); the (P3') equation is that defined by Painlevé (Ref. [20] p. 1115) in the class of (P3), the (P2') belongs to the class of (P2) for \( \delta \neq 0 \) and to that of (P1) for \( \delta = 0, \gamma \neq 0 \). This list has been terminated by the elliptic equation which is the autonomous limit of (P2'), which we denote (J') (like “Jacobi”).

This is the most advantageous list of equations to be considered for the study of Lax pairs, for it allows to take full advantage of the four parameters \( \alpha, \beta, \gamma, \delta \) appearing in each equation.

The successive coalescences

\[(P4') \quad (P6) \rightarrow (P5) \rightarrow (P2') \rightarrow (J') \rightarrow (P3')\]

from an equation \( E(x, u, \alpha, \beta, \gamma, \delta) = 0 \) to another equation \( E(X, U, A, B, C, D) = 0 \) are described by the following Poincaré perturbations
$(x, u, \alpha, \beta, \gamma, \delta) \to (X, U, A, B, C, D, \varepsilon)$

$6 \to 5: \quad (x, u, \alpha, \beta, \gamma, \delta) = (1 + \varepsilon X, U, A, B, \varepsilon^{-1} C - \varepsilon^{-2} D, \varepsilon^{-2} D),$

$5 \to 4': \quad (x, u, \alpha, \beta, \gamma, \delta) = (1 + \varepsilon X, \varepsilon U / 2, 2 C \varepsilon^{-1} + 28 D \varepsilon^{-3}, B / 4, -4 C \varepsilon^{-4} - 60 D \varepsilon^{-3}, 2 A \varepsilon^{-2} - 2 C \varepsilon^{-4} - 32 D \varepsilon^{-3}),$

$5 \to 3': \quad (x, u, \alpha, \beta, \gamma, \delta) = (X, 1 + \varepsilon U, \varepsilon^{-1} A / 4 + \varepsilon^{-2} C / 8, -\varepsilon^{-2} C / 8, \varepsilon B / 4, \varepsilon^2 D / 8),$

$4' \to 2': \quad (x, u, \alpha, \beta, \gamma, \delta) = (\varepsilon^2 X / 4, 1 + \varepsilon U, -4 B \varepsilon^{-4} + 96 C \varepsilon^{-5} - 24 D \varepsilon^{-6}, 16 A \varepsilon^{-3} - 8 B \varepsilon^{-4} + 48 C \varepsilon^{-5} - 8 D \varepsilon^{-6}, -32 C \varepsilon^{-5} + 16 D \varepsilon^{-6}, 48 C \varepsilon^{-5} - 16 D \varepsilon^{-6}),$

$3' \to 2': \quad (x, u, \alpha, \beta, \gamma, \delta) = (1 + \varepsilon^2 X / 2, 1 + \varepsilon U, 16 B \varepsilon^{-4} - 64 C \varepsilon^{-5} - 32 D \varepsilon^{-6}, 32 D \varepsilon^{-6}, -8 B \varepsilon^{-4} + 48 C \varepsilon^{-5} + 16 D \varepsilon^{-6}, 16 A \varepsilon^{-3} - 8 B \varepsilon^{-4} + 16 C \varepsilon^{-5} - 16 D \varepsilon^{-6}),$

$2' \to J': \quad (x, u, \alpha, \beta, \gamma, \delta) = (\varepsilon X, \varepsilon^{-1} U, \varepsilon^{-3} A, \varepsilon^{-2} B, \varepsilon^{-1} C, D)$

in which $\varepsilon$ goes to zero. These six transformation laws on $x$ and $u$ are affine, contrary to those of the usual coalescence cascade.

3 Various Lax pairs of (Pn) in the continuous case

Lax pairs can be obtained either by the isomonodromic deformation of a given second order linear differential equation (defined either in scalar form or in matricial form), or by the reduction of the Lax pair for some partial differential equation (PDE) [5], which makes three possibilities.

3.1 Lax pairs by scalar isomonodromy (Garnier)

In order to find (P6), Richard Fuchs had to assume in the second order scalar equation

$$\partial_t^2 \psi + (S/2)\psi = 0,$$  \hspace{1cm} (3)

the presence of four Fuchsian singularities, put a priori at the points $t = \infty, 0, 1, x$. As shown by Poincaré (Ref. [22] pp. 217–20), the maximum number of apparent singularities (a singularity is apparent iff in its neighborhood the ratio of any two solutions $\psi$ is singlevalued) which can be added in this case (second order, four Fuchsian singularities) is one. If no apparent singularity is added, a case which corresponds to the Heun equation [23, 27], the monodromy is trivial. By adding one apparent singularity, at a location denoted $t = u$, R. Fuchs [24, 25] found that $u(x)$ had to satisfy the (P6) equation. This result has been completed by Garnier, who established a coalescence cascade starting from (P6) and yielding scalar pairs for (P5), (P4'), (P3'), (P2') and (J') while respecting a remarkable symmetry between $t$, $x$ and $u$.

These scalar Lax pairs of Garnier are linear in $\alpha, \beta, \gamma, \delta$ and characterized by the two quantities $(S, C)$, with the cross-derivative condition $Z = 0$

$$\partial_t^2 \psi + (S/2)\psi = 0, \quad (4)$$

$$\partial_x \psi + C \partial_t \psi - (1/2) C t \psi = 0, \quad (5)$$

$$Z \equiv S_x + C u u t + C S_t + 2 C_t S = 0. \quad (6)$$
The value of $S$ is given by

$$-\frac{S}{2} = \frac{3/4}{(t-u)^2} + \frac{\beta_1 u' + \beta_0}{(t-u)e_1} + \frac{[(\beta_1 u')^2 - \beta_0^2]e_0 + f_G(u)}{e_2} + f_G(t),$$

in which the function $f_G$ and the various scalars are defined in Table 2.

Table 2: Scalar Lax pairs of the Painlevé equations

| $\beta_1$   | $\beta_0$   | $f_G(z)$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $e_0$ | $e_1$ | $e_2$ | $-C$     |
|-------------|-------------|----------|----------|---------|----------|----------|-------|-------|-------|----------|
| $\frac{x(x-1)}{2(u-x)}$ | $-u+1/2$ | $\frac{a}{z^2} + \frac{b}{(z-1)^2}$ | $\frac{a}{(z-x)^2} + \frac{b}{z(z-1)}$ | $2(a+b+c+d+1)$ | $-2(a+1/4)$ | $2(b+1/4)$ | $-2c$ | $t(t-1)$ | $t(t-1)(t-x)$ | $\frac{t(t-1)}{(t-u)x(x-1)}$ |
| $\frac{x}{2(u-1)}$ | $-u+1/2$ | $\frac{a}{z^2} - \frac{b}{(z-1)^3}$ | $\frac{a}{(z-x)^2} + \frac{b}{z(z-1)^2}$ | $2(a+b+c+d)$ | $-2(a+1/4)$ | $2(b+1/4)$ | $-2c$ | $t(t-1)$ | $t(t-1)^2$ | $\frac{t(t-1)(u-1)}{(t-u)x(x-1)}$ |
| $\frac{1}{2}$ | $1/2$ | $\frac{a}{z^2} - \frac{b}{(z-1)^4}$ | $\frac{a}{(z-x)^2} + \frac{b}{z(z-1)^3}$ | $-4b$ | $-8a-2$ | $16a$ | $-16c$ | $\frac{2t}{t-u}$ | $\frac{tu}{(t-u)x}$ | $1/2$ |

3.2 Lax pairs by matricial isomonodromy (Jimbo and Miwa)

Equivalently, the second order deformation equation can be replaced by a first order, two-component, matricial system. The advantage is the possibility to get rid of the apparent singularity, unavoidable in the scalar case, by a change of basis. The four Fuchsian singularities are similarly put at the locations $t = \infty, 0, 1, x$. The resulting matricial Lax pairs obtained by Jimbo and Miwa [15], some of these parameters are restricted to numerical values, and the Lax pair of (P6) is not traceless. In the following formulae, the symbols $\sigma_k$ denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_j \sigma_k = \delta_{jk} + i \varepsilon_{jkl} \sigma_l, \quad (8)$$
\[ \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

while the symbol \( Z \) is the commutator of the Lax pair
\[ \partial_x \psi = L \psi, \quad \partial_t \psi = M \psi, \quad Z = [\partial_x - L, \partial_t - M] \tag{9} \]

and the symbol \( E \) denotes the \((P_n)\) equation.

The pair of \((P3)\) \cite{15} for \((\gamma, \delta) \neq (0, 0)\) is
\[
L = \frac{t}{x} \left[ (dx/4 + t^{-2}(cx/4 - z)) \sigma_3 + (t^{-2} - t^{-1}u)w\sigma^+ + (t^{-1}V - t^{-2}z(z - cx/2))w^{-1}\sigma^- \right] \tag{10}
\]
\[
M = (dx/4 - t^{-2}(cx/4 - z) - t^{-1}(\beta d^{-1} - 2)/4) \sigma_3 - (t^{-2} - t^{-1}u)w\sigma^+ + (t^{-1}V + t^{-2}z(z - cx/2))w^{-1}\sigma^- \tag{11}
\]
\[
u^2 Z = (1/8)t^{-2}(2x\sigma_3 + (x^2u^{-2}(u' - d) + (\beta d^{-1} - 1)xu^{-1})w^{-1}\sigma^-) - (1/8)t^{-1}(x^2u^{-2}(u' - d) + x)w^{-1}\sigma^- E \tag{12}
\]
\[
4z = xu'/u^2 + cx - dx/u^2 + (\beta d^{-1} - 1)/u, \tag{13}
\]
\[
V = -z^2u + z(cxu + \beta d^{-1} - 2)/2 + (\alpha + 2c - \beta cd^{-1})x/8, \tag{14}
\]
\[
w'/w = -u'/u + d/u + (1 - (1/2)\beta d^{-1})/x \tag{15}
\]
\[
c^2 = \gamma, \quad d^2 = -\delta. \tag{16}
\]

The pair of \((P5)\) \cite{15} for \(\delta \neq 0\) is
\[
L = \frac{dt\sigma_3/2}{x} + x^{-1}(z + \theta_0 - u(z + (\theta_0 - \theta_1 + \Theta_{\infty})/2))w\sigma^+ \tag{17}
\]
\[
x^{-1}(-z + u^{-1}(z + (\theta_0 - \theta_1 + \Theta_{\infty})/2))w^{-1}\sigma^- \tag{18}
\]
\[
M = (\delta x/2 + \frac{z + \theta_0/2}{t} - \frac{z + (\theta_0 + \Theta_{\infty})/2}{t - 1}) \sigma_3 + ((z + \theta_0)/t - u(z + (\theta_0 - \theta_1 + \Theta_{\infty})/2)/(t - 1))w\sigma^+ \tag{19}
\]
\[
+ (-z/t + u^{-1}(z + (\theta_0 + \theta_1 + \Theta_{\infty})/2)/(t - 1))w^{-1}\sigma^- \]
\[
Z = \left( \frac{-u}{(t + u - tu)/w} \right) \frac{x}{t(t - 1)u(u - 1)} E \tag{20}
\]
\[
z = -x \frac{u' - du}{2(u - 1)^2} + \frac{(3\theta_0 + \theta_1 + \Theta_{\infty}) - (\theta_0 - \theta_1 + \Theta_{\infty})u}{4(u - 1)} \tag{21}
\]
\[
w'/w = \left[ -2z - \theta_0 + u(z + (\theta_0 - \theta_1 + \Theta_{\infty})/2) + u^{-1}(z + (\theta_0 + \theta_1 + \Theta_{\infty})/2) \right]/x \tag{22}
\]
\[
d^2 = -2\delta, \quad (\theta_0 - \theta_1 + \Theta_{\infty})^2 = 8\alpha, \quad (\theta_0 - \theta_1 - \Theta_{\infty})^2 = -8\beta, \quad (1 - \theta_0 - \theta_1) = \gamma/d.
\]

The traceless pair of \((P6)\) for \((2\alpha - 1, 2\beta + 1, 2\gamma - 1, \delta) \neq (0, 0, 0, 0)\) can be found in Ref. \cite{18}.

### 3.3 Lax pairs by reduction of a PDE

One first selects a PDE which admits a second order Lax pair. The reductions create singularities at four points at most, which must then be moved by homography
to $t = \infty, 0, 1, x$. The resulting Lax pairs are generically algebraic in $\alpha, \beta, \gamma, \delta$, maybe with some restrictions on them.

One such system is the pumped Maxwell-Bloch system, defined either in the complex form

\[ e_T = \rho, \quad \rho_X = N e, \quad N_X + (\rho e + \rho \bar{e})/2 - 4s = 0, \quad (23) \]

with $s$ constant (the system is “pumped” when $s$ is nonzero), or in the real form

\[ e_T = \rho, \quad \rho_X = N e, \quad N_X + \rho e - 4s = 0. \quad (24) \]

One does not know a possible common parent to these two systems. These systems describe phenomena in nonlinear optics, stimulated Raman scattering and self-induced transparency. Their respective Lax pairs are, for the complex system

\[ L = \frac{1}{2} \left( \begin{array}{cc} 0 & e \\ -\bar{e} & 0 \end{array} \right) + f \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad M = \frac{1}{4f} \left( \begin{array}{cc} N & -\rho \\ -\rho & -N \end{array} \right), \quad (25) \]

for the real system

\[ L = \frac{1}{2} \left( \begin{array}{cc} 0 & e \\ -e & 0 \end{array} \right) + f \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad M = \frac{1}{4f} \left( \begin{array}{cc} N & -\rho + m_1/(2f) \\ -\rho - m_1/(2f) & -N \end{array} \right) \quad (26) \]

with $f^2 = 2sT + \lambda^2$ and $\lambda$ an arbitrary complex constant.

A Lax pair of (P3) is obtained by the following reduction of the pumped real Maxwell-Bloch system

\[ x = XT^{1/2}, \quad e = -iT^{1/2}[u'/u + cu + d/u], \quad m_1 = -i(\alpha d - \beta c), \quad s = -cd/2, \quad (27) \]
\[ t = T^{-1/2} f(T), \quad (28) \]
\[ L = (1/2)(u'/u + cu + d/u)\sigma_3 + t\sigma_1, \quad (29) \]
\[ M = \left[ (x/t)L + t^{-2}cdx\sigma_1/2 + t^{-3}(-\alpha d + \beta c)\sigma_3/4 \right] \frac{t^2}{t^2 + cd}, \quad (30) \]
\[ Z = \frac{x}{2tu}(\sigma_3 - t^{-1}(cu\sigma^+ + du^{-1}\sigma^-)) \frac{t^2}{t^2 + cd} E, \quad (31) \]
\[ c^2 = \gamma, d^2 = -\delta. \quad (32) \]

The transition matrix \( \left( \begin{array}{cc} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{array} \right) \) makes this pair invariant by parity on $t$. For $\gamma \delta = 0$, it has two nonFuchsian singularities $t^2 = \infty, 0$. For $\gamma \delta \neq 0$, it has three singularities in the variable $-t^2/(cd)$: two Fuchsian 0 and 1, one nonFuchsian at infinity, i.e. one more than expected for (P3), and there exists no coalescence of two of these three singularities which preserves (P3). Therefore this is in fact a pair of (P5) for $\delta = 0$ since (P5) for $\delta = 0, \gamma \neq 0$ is equivalent to the full (P3) under a birational transformation. 

Another Lax pair of (P3) is obtained by a reduction of the unpumped complex Maxwell-Bloch system [23], but this one is restricted to \((\gamma, \delta) \neq (0, 0)\)

\[
x = X T^\theta, \quad e = T^{(1-2\theta)} e, \quad \varphi = T^{(1+2\theta)} \varphi, \quad s = 0, \quad t = T^{-\theta} f(T),
\]

\[
L = -(1/2)(u'/u + d/u + 1/x)\sigma_1 + (\alpha/cx + cu)(\sigma_3 + \sigma_2/i)/2 - tux/c,
\]

\[
M = \left[ t^{-2}cdx^{-1}u^{-1}(u'/u + d/u - (\beta + d)/(d\varphi)) + t^{-1}\alpha/c \right] (\sigma_3 + \sigma_2/i)/4
\]

\[
+ \left[ -nu^2/c\sigma_3 + t^{-1}(\alpha/c)\sigma_2 - t^{-1}x(u'/u + d/u + 1/x)\sigma_1 - t^{-2}\nu^{-1}c^2d\sigma_1 \right]/4
\]

\[
Z = (1/4)u^{-2}[t^{-2}\nu^{-1}(\sigma_3 + \sigma_2/i) - t^{-1}xu\sigma_1]E,
\]

\[
\theta_k = -\alpha/(4c)
\]

\[
c^2 = \gamma, \quad d^2 = -\delta.
\]

in which \(\theta\) is arbitrary. The arbitrary parameter \(\nu\) just reflects the scaling law of (P3) and it can be set for instance to \(c, d\) or 1. This pair is much simpler than the one of Jimbo and Miwa (10)-(11), and it possesses two non-Fuchsian singularities \(t = \infty, 0\).

A Lax pair of the full (P5) is obtained by a reduction of the pumped complex system [23] [17] with the restriction \(\delta \neq 0\)

\[
x = XT^{1/2}, \quad e = T^{1/2-t_j} F, \quad \varphi = T^{1/2+t_j} G, \quad s = -\delta/16,
\]

\[
t = 2T^{-1/2} f(T)/d + 1/2
\]

\[
L = \begin{pmatrix}
(2t - 1)d & F/2 \\
-G/2 & -(2t - 1)4d
\end{pmatrix}
\]

\[
M = \frac{C_0}{t} + \frac{C_1}{t - 1} + (d/2)x\sigma_3
\]

\[
Z = \frac{d^2 x}{8(t - 1)(u - 1)^5}[\sigma_3 + u^{-1/2}((t + u - tu)v\sigma^+ + (t - 1 - tu)v^{-1}\sigma^-)]E,
\]

\[
F = u^{-1/2} \left[ \frac{u' + du}{u - 1} + 2(t_c(u - 1) - t_ju)/x \right] v
\]

\[
G = u^{-1/2} \left[ \frac{u' + du}{u - 1} - 2(t_c(u - 1) - t_ju)/x \right] v^{-1}
\]

\[
\nu'/\nu = (t_j(u^2 + 1) - t_c(u^2 - 1))/(xu)
\]

\[
C_0 + C_1 = -t_j\sigma_3 - (x/2) \begin{pmatrix} 0 & F \\ -G & 0 \end{pmatrix}
\]

\[
C_0 - C_1 = \left[ \frac{x(u' + du)}{(u - 1)^2} - \frac{(\gamma + d)(u + 1)}{2d(u - 1)} \right] \sigma_3
\]

\[
+ u^{-1/2}((p + q)v\sigma^+ + (p - q)v^{-1}\sigma^-)
\]

\[
p = -\frac{x(u + 1)(u' + du)}{2(u - 1)^2} u' + \frac{(\gamma + d)u}{d(u - 1)}, \quad q = t_c(u + 1) - t_ju
\]

\[
d^2 = -2\delta, \quad t_c^2 = -\beta/2, \quad (t_c - t_j)^2 = \alpha/2.
\]
3.4 Comparison of these three kinds of Lax pairs

Our goal is to discretize as easily as possible without introducing restrictions on $\alpha, \beta, \gamma, \delta$.

The three kinds of Lax pairs are compared in Table 3 according to several criteria.

Table 3: Advantages and inconveniences of the three kinds of Lax pairs. The ideal situation would be that all entries be “yes”.

|                          | scalar Garnier | matricial Jimbo and Miwa by reduction |
|--------------------------|----------------|----------------------------------------|
| no apparent singularity  | no             | yes                                    |
| linearity in $\alpha, \beta, \gamma, \delta$ | yes           | no                                     |
| easy confluence          | yes            | no                                     |
| arbitrary $\alpha, \beta, \gamma, \delta$ | yes           | no                                     |

We therefore choose the scalar pairs of Garnier and, before attempting to discretize, we remove the apparent singularity by converting them to matricial form.

Remark. If one first discretizes a PDE then performs a discrete reduction, the resulting discrete ODE is not explicit but is defined by the system made of the discrete PDE and the discrete nonlinear relation defining the reduction; for details, see section 2.4 in Ref. [13].

4 The Garnier matricial Lax pairs

From the scalar Garnier pairs (4)–(5), let us build an equivalent matricial pair which does not contain any more the apparent singularity $t = u$.

Let us choose the two matrices as traceless

$$\begin{pmatrix} \partial_x - \left( \begin{array}{cc} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{array} \right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} \partial_t - \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{array} \right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$ (50)

The elimination of one component, say $\psi_1$, defines the scalar system

$$\left( \partial_t^2 - \frac{M_{21,t}}{M_{21}} \partial_t + M_{11,t} - M_{11} \frac{M_{21,t}}{M_{21}} - M_{12}M_{21} \right) \psi_2 = 0,$$

$$\left( \partial_x - \frac{L_{21}}{M_{21}} \partial_t + L_{11} - M_{11} \frac{L_{21}}{M_{21}} \right) \psi_2 = 0.$$ (52)

The coefficients of this scalar Lax pair have simple poles at $M_{21} = 0$. Let us first convert the double pole $t = u$ of the coefficients of the pair of Garnier into a simple pole, by the transformation

$$\Psi = \sqrt{t - u} \psi,$$

$$\left( \partial_t^2 - \frac{1}{t - u} \partial_t + \left( \frac{S}{2} + \frac{3/4}{(t - u)^2} \right) \right) \Psi = 0,$$

$$\left( \partial_x + C \partial_t + \frac{u' - (C + (t - u)C_t)}{2(t - u)} \right) \Psi = 0.$$ (54)
with $S$ given by (7) and $C$ by Table 2. The six coefficients $(L_{jk}, M_{jk})$ of the Lax pair are obtained by identifying the singularities $M_{21} = 0$ and $t - u = 0$ in (52)–(53) and (55)–(56). This results in

\[
M_{11} = \frac{\beta_1 u' + \beta_0}{e_1}, \tag{57}
\]

\[
M_{12} = M_{11,t} - M_{11}^2 - \frac{M_{11}}{(t-u)^2} - \frac{1}{t-u} \left( S + \frac{3/4}{(t-u)^2} \right) \tag{58}
\]

\[
M_{21} = t - u, \tag{59}
\]

\[
L_{21} = -(t-u)C, \tag{60}
\]

\[
L_{11} = -CM_{11} + \frac{u' - (C + (t-u)C_t)}{2(t-u)} \tag{61}
\]

\[
L_{12} = -L_{11,t} + M_{11,x} + L_{21}M_{12} \tag{62}
\]

Once expressed in terms of $(t, x, u, u', \alpha, \beta, \gamma, \delta)$ only, this matricial Lax pair has the following features: regularity at $t = u$, concentration of the dependence on $\alpha, \beta, \gamma, \delta$ in the elements $(L_{12}, M_{12})$, linearity of these two elements in $\alpha, \beta, \gamma, \delta$, absence of any restriction on $\alpha, \beta, \gamma, \delta$. For instance, the element $L_{11}$ evaluates to

\[
L_{11} = \frac{(u-x)}{(x(x-1))}, \frac{u-1}{x}, 0, 0, 0 \tag{63}
\]

for the five successive $(P_n)$ of Garnier, and the element $L_{12}$ to

\[
L_{12} = \frac{(\beta_1 u')^2 - \beta_0^2}{t(t-1)(t-x)u(u-1)}, 0, 0, 0 \tag{64}
\]

for $a = b = c = d = 0$. The confluence operates on each element separately.

For instance, the matricial Lax pair of (P2') is

\[
L = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[ \beta + 4\gamma(t+2u) + \delta(t^2 + 2tu + 3u^2 + x) \right]/2, \tag{65}
\]

\[
M = \begin{pmatrix} -u' & 0 \\ t-u & u' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[ 2\alpha + \beta(t+u) + \gamma(4(t^2+tu+u^2)+2x) + \delta(t^3+t^2u+tu^2+u^3+x(t+u)) \right], \tag{66}
\]

and the one of (J') is

\[
L = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[ \beta + 4\gamma(\lambda+2u) + \delta(\lambda^2 + 2\lambda u + 3u^2) \right]/2, \tag{67}
\]

\[
M = \begin{pmatrix} -u' & 0 \\ \lambda-u & u' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[ 2\alpha + \beta(\lambda+u) + 4\gamma(\lambda^2 + \lambda u + u^2) + \delta(\lambda^3 + \lambda^2 u + \lambda u^2 + u^3) \right], \tag{68}
\]

in which $\lambda$ is an arbitrary constant.
5 Rules of discretization of matricial Lax pairs

In order to discretize a given ODE having the Painlevé property, some basic discretization rules have been proposed [3].

The definition of the (continuous) matricial Lax pair \((L,M)\) can be discretized as
\[
\psi(x + h/2) = A(x,z)\psi(x - h/2), \quad \partial_z\psi(x - h/2) = B(x - h/2, z)\psi(x - h/2),
\]
\[
\partial_z A(x,z) + A(x,z)B(x - h/2, z) - B(x + h/2, z)A(x,z) = 0.
\]

The continuum limit \(h \to 0\) is then
\[
\frac{A - 1}{h} \to L, \quad (dz/dt)B \to M,
\]
\[
(dz/dt)(\partial_z A + AB - BA)/h \to \partial_t L - \partial_x M + LM - ML,
\]
with some link \(F(t, z, h) = 0\) between the spectral parameters \(t\) and \(z\). For a second order equation \(E(\pi, u, \dot{u}, x, h) = 0\), the operators \(A\) and \(B\) must have the \(u\)–dependences \(A(\pi, u, \dot{u}), B(\pi, u, \dot{u})\).

Recently, we proposed [4] a direct method for finding the Lax pair of a discrete equation whose continuous pair is known. This consists in discretizing the continuous Lax pair by obeying the following common sense rules

1. conserve the matricial order. This is indeed the differential order of the scalar Lax pair, which must be conserved;
2. replace the continuous spectral parameter \(t\) by an unspecified function \(T(z, h)\);
3. discretize the operator \(L\) centered at the three points \(x - h, x, x + h\). If \(L\) is traceless, so is its discretization;
4. discretize the operator \(M\) centered at the two points \(x - h, x\). If \(M\) is traceless, so is its discretization;
5. replace each monomial \((du/dx)^k\) by its discretization obeying the general rules, multiplied by the \(k\)-th power of an unspecified function \(g(z, h)\). This function \(g\), whose continuum limit must be 1 for any \(z\), represents the ratio of the stepsize \(h\) to the differential element \(dx\);
6. take \(B\) as the product of the discretized \(M\) by an unspecified function \(J(z, h)\) (like Jacobian) representing a discretization of the derivative \(dT/dz\);
7. take \((A - 1)/h\) equal to the sum of the discretized operator \(L\) and a diagonal matrix of unspecified functions of \((z, h)\) only, \(\text{diag}(g_1, g_2)\); these functions, whose continuum limit must be zero, account for the dissymmetry of the formula defining \(A\).
6 Application to the discretization of the (Pn) equations and of their Lax pair

The (Pn) equations and the pairs of Garnier, in their scalar form as well as in their matricial one, are linear in the four parameters $\alpha, \beta, \gamma, \delta$. If one requires this property to be conserved in the discrete case, the problem of finding a discrete equation and its discrete Lax pair splits into five successive subproblems: all four parameters zero (or numerical constants), coefficient of $\alpha$ alone, of $\beta$ alone, etc.

When applied to the autonomous limit $(J')$ of $(P2')$ and its Lax pair $(77)$–$(85)$, the discretization rules give straightforwardly the desired result, namely

\[-\frac{\pi - 2u + u}{h^2} + \delta u^2(\pi + u) + 2\gamma u(\pi + u + u) + \beta \left(\lambda_1 u + \left(1 - \lambda_1\right)\frac{\pi + u}{2}\right) + \alpha = 0, \quad (72)\]

\[A/h = \begin{pmatrix} g/h & 0 \\ 1/2 & g/h \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[\beta/2 + 2\gamma(T + 2u) + \delta \frac{F(x + h/2) - F(x - h/2)}{u(x + h) - u(x - h)}\right], \quad (73)\]

\[B/J = \begin{pmatrix} -g(u - u)/h \\ T - (u + u)/2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[2\alpha + \beta(T + (u + u)/2) + \gamma(4T^2 + 2T(u + u) + 4u^2) + \delta F(x - h/2)\right], \quad (74)\]

\[F(x) = \frac{T^4 - (u(x + h/2)u(x - h/2))^2 - T^2(x + h/2)u(x + h/2) - u(x - h/2))^2}{2(T - (u + h/2) + u(x - h/2))/2}, \quad (75)\]

\[g(z)^2 - 1 = \beta \frac{2\lambda_1 - 1}{4} + \gamma T + \delta \frac{T^2}{2}, \quad (76)\]

\[T' = 0. \quad (77)\]

In these expressions, $\lambda_1$ is an arbitrary constant (which could be absorbed in the discrete second derivative), the functions $J(z), T(z)$ and $g(z)$ are constant, the spectral parameter is the constant value of $T$.

One notices an unexpected drawback: the apparent singularity $t = u$, absent from the continuous matricial pair, reappears in the coefficient of $\delta$, because the polynomial $T^3 + T^2u + Tu^2 + u^3$ is discretized as the quotient of a discretization of $T^4 - u^4$ by a discretization of $T - u$.

This is the reason why the Lax pair of d–$(P2')$ obtained by this method is, for the moment, restricted to $\delta = 0$ i.e. to d–$(P1)$

\[-\frac{\pi - 2u + u}{h^2} + \delta u^2(\pi + u) + xu + \gamma(2u(\pi + u + u) + x) + \beta u + \alpha = 0, \quad (78)\]

\[A/h = \begin{pmatrix} g/h & 0 \\ 1/2 & g/h \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[\beta/2 + 2\gamma(T + 2u)\right], \quad (79)\]

\[B/J = \begin{pmatrix} -g(u - u)/h \\ T - (u + u)/2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[2\alpha + \beta(T + (u + u)/2) + \gamma(4T^2 + 2T(u + u) + 4u^2)\right], \quad (80)\]

\[g(z)^2 - 1 = \beta/4 + \gamma T, \quad (81)\]
\[ \gamma T'(z) = \gamma J(z)g(z), \]  
\[ g'(z) = \frac{1}{2}\gamma h^2 J(z), \]  
\[ (82) \]
\[ (83) \]

with \( J(z) \) arbitrary. The contributions \( \delta G(x, z) \) and \( \delta F(x - h/2, z) \) in the elements \( A_{12} \) and \( B_{12} \) lead to an impossibility.

For (P3'), the result given in [4] deals with the case \( \alpha = \beta = \gamma = \delta = 0 \) where the degree of the d–(P3) given in Ref. [12] reduces from two to one (see Table [1]); in this case a second order discrete Lax pair can be obtained.

7 Conclusion

This direct approach, which serializes the difficulties, can be improved in the following directions. If one sticks to the matricial form, one should design more efficient discretization rules, so as to at least find the contribution of \( \delta \) in the Lax pair of d–(P2'). The other possibility is to directly discretize the Garnier pairs on their scalar form (55)–(56).

Acknowledgments

The financial support of the Tournesol grant T 95/004 is gratefully acknowledged. M. M. acknowledges the financial support extended within the framework of the IUAP contract P4/08 funded by the Belgian government.
References

[1] S. P. Burtsev, V. E. Zakharov and A. V. Mikhailov, Inverse scattering method with variable spectral parameter, Teoreticheskaya i Matematicheskaya Fizika 70 (1987) 323–341 [English: Theor. and Math. Phys. 70 (1987) 227–240].

[2] P. A. Clarkson, E. L. Mansfield and A. E. Milne, Symmetries and exact solutions of a 2 + 1–dimensional sine-Gordon system, Phil. Trans. Roy. Soc. London A 354 (1996) 1807–1835.

[3] R. Conte and M. Musette, A new method to test discrete Painlevé equations, Phys. Lett. A 223 (1996) 439–438.

[4] R. Conte and M. Musette, Rules of discretization for Painlevé equations, Theory of nonlinear special functions : the Painlevé transcendentts, 20 pages, eds. L. Vinet and P. Winternitz (Springer, Berlin, 1998).

[5] H. Flaschka and A. C. Newell, Monodromy and spectrum preserving deformations, I, Commun. Math. Phys. 76 (1980) 65–116.

[6] A. S. Fokas and M. J. Ablowitz, On a unified approach to transformations and elementary solutions of Painlevé equations, JMP 23 (1982) 2033–2042.

[7] A. S. Fokas, B. Grammaticos and A. Ramani, From continuous to discrete Painlevé equations, Journal of mathematical analysis and applications 180 (1993) 342–360.

[8] R. Fuchs, Sur quelques équations différentielles linéaires du second ordre, C. R. Acad. Sc. Paris 141 (1905) 555–558.

[9] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, Math. Annalen 63 (1907) 301–321.

[10] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes, Thèse, Paris (1909); Acta Math. 33 (1910) 1–55.

[11] R. Garnier, Sur des équations différentielles du troisième ordre dont l’intégrale générale est uniforme et sur une classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses points critiques fixes, Thèse, Paris (1911); Ann. Éc. Norm. 29 (1912) 1–126.

[12] B. Grammaticos, F. W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, Linearization and solutions of the discrete Painlevé III equation, Phys. Lett. A 185 (1994) 446–452.

[13] B. Grammaticos, F. W. Nijhoff and A. Ramani, Discrete Painlevé equations, The Painlevé property, one century later, 114 pages, ed. R. Conte, CRM series in mathematical physics (Springer, Berlin, 1998).

[14] V. I. Gromak, Theory of Painlevé’s equation, Differentsial’nye Uravneniya 11 (1975) 373–376 [English: Diff. equ. 11 (1975) 285–287].
[15] M. Jimbo and T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II, Physica D 2 (1981) 407–448.

[16] M. Jimbo and H. Sakai, A $q$–analog of the sixth Painlevé equation, Lett. Math. Phys. 38 (1996) 145–154.

[17] A. V. Kitaev, A. V. Rybin and J. Timonen, Similarity solutions of the deformed Maxwell-Bloch system, J. Phys. A 26 (1993) 3583–3595.

[18] G. Mahoux, Introduction to the theory of isomonodromic deformations of linear ordinary differential equations with rational coefficients, The Painlevé property, one century later, 44 pages, ed. R. Conte, CRM series in mathematical physics (Springer, Berlin, 1998).

[19] A. Milne, Ph. D. Thesis (University of Exeter, UK, 1995).

[20] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sc. Paris 143 (1906) 1111–1117.

[21] V. G. Papageorgiou, F. W. Nijhoff, B. Grammaticos and A. Ramani, Isomonodromic deformation problems for discrete analogs of Painlevé equations, Phys. Lett. A 164 (1992) 57–64.

[22] H. Poincaré, Sur les groupes des équations linéaires, Acta mathematica 4 (1883) 201–312.

[23] R. B. Potts (1987), Weierstrass elliptic difference equations, Bull. Austral. Math. Soc. 35 (1987) 43–48.

[24] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations, Phys. Rev. Lett. 67 (1991) 1829–1832.

[25] A. Ramani, B. Grammaticos and Y. Ohta, The Painlevé of discrete equations and other stories, Theory of nonlinear special functions : the Painlevé transcendants, eds. L. Vinet and P. Winternitz (Springer, Berlin, 1998).

[26] K. Heun, Zur Theorie der Riemann’schen Funktionen zweiter Ordnung mit vier Verzweigungspunkten, bf 33 (1889) 161–179.

[27] A. Seeger and W. Lay (eds.), Centennial workshop on Heun’s equation – Theory and applications (Max-Planck-Institut, Stuttgart, 1990).

[28] P. Winternitz, Physical applications of Painlevé type equations quadratic in the highest derivatives, Painlevé transcendants, their asymptotics and physical applications, 425–431, eds. D. Levi and P. Winternitz (Plenum, New York, 1992).