ON THE ENERGY DECAY RATE OF THE FRACTIONAL WAVE EQUATION ON $\mathbb{R}$ WITH RELATIVELY DENSE DAMPING

WALTON GREEN

Abstract. We establish upper bounds for the decay rate of the energy of the damped fractional wave equation when the averages of the damping coefficient on all intervals of a fixed length are bounded below. If the power of the fractional Laplacian, $s$, is between 0 and 2, the decay is polynomial. For $s \geq 2$, the decay is exponential.

Consider the following damped fractional wave equation on $\mathbb{R}$ for $s > 0$ and $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$:

$$w_{tt}(x, t) + \gamma(x)w_t(x, t) + (-\partial_{xx})^{s/2}w(x, t) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$  \hfill (1)

The damping force is represented by $\gamma w_t$. Herein, we study the decay rate of the energy of $w$, defined by

$$E(t) = \|(w(t), w_t(t))\|_{H^{s/2} \times L^2}^2 := \int_{\mathbb{R}} |(-\partial_{xx})^{s/4}w(x, t)|^2 + |w_t(x, t)|^2 \, dx.$$  \hfill (2)

Of course, if $\gamma = 0$, then the energy is conserved, i.e. there is no decay. On the other hand, for constant damping $\gamma = c > 0$, it can be easily shown that $E(t)$ decays exponentially.

For the case $s = 2$, the problem was initially studied on bounded domains under the so called Geometric Control Condition (GCC) [1, 15] which requires $\gamma$ to be positive (in some sense) on certain geodesic curves determined by the geometry of the domain. On $\mathbb{R}$, or more generally $\mathbb{R}^d$, the GCC simplifies to the following: there exist $R$ and $c > 0$ such that for all line segments $\ell \in \mathbb{R}^d$ of length $R$,

$$\int_{\ell} \gamma(x) \, dx > c.$$  \hfill (2)

In [11], when $d = 1$ and $\gamma \in L^\infty(\mathbb{R})$, it is shown that condition (2) is equivalent to exponential decay of the energy of solutions to (1). For $d \geq 1$ and for more general Laplacians, (2) is shown to be sufficient for exponential decay of $E(t)$ in [2], with the requirement that $\gamma$ be uniformly continuous in addition to being bounded. In certain cases where GCC fails, the energy may still decay polynomially [16, 7] or logarithmically [3].

The damped fractional wave equation was recently introduced in [10]. It was noticed that for $\gamma$ such that $\{x \in \mathbb{R} : \gamma(x) \geq \varepsilon\}$ contains a periodic set, the rate of decay depends on the power $s$ of the Laplacian in [11]. For $s < 2$, the rate is polynomial, and for $s \geq 2$ it is exponential. Herein, we relax this periodic condition on $\{\gamma \geq \varepsilon\}$ to simply require
that \( \{ \gamma \geq \varepsilon \} \) be relatively dense \([5]\), which means there exists \( R > 0 \) such that
\[
\inf_{a \in \mathbb{R}} m(\{ \gamma \geq \varepsilon \} \cap [a - R, a + R]) > 0
\]
where \( m \) is the Lebesgue measure. We will prove the following result.

**Theorem 1.** Let \( 0 \leq \gamma \in L^\infty(\mathbb{R}) \) for which there exists \( R > 0 \) such that
\[
\inf_{a \in \mathbb{R}} \int_{a-R}^{a+R} \gamma(x) \, dx > 0.
\]
Then, there exists \( C, \omega > 0 \) such that
\[
E(t) \leq \begin{cases} 
C(1 + t)^{-\frac{2s}{1-2s}} \|w(0), w_t(0)\|^2_{H^s \times H^{s/2}} & \text{if } 0 < s < 2 \\
Ce^{-\omega t} E(0) & \text{if } s \geq 2
\end{cases}
\]
for all \( t > 0 \) whenever the right-hand side is finite.

Notice that for \( \gamma \) bounded (in fact even for \( \gamma \in L^p(\mathbb{R}) \) with \( p > 1 \)) the condition \((4)\) is equivalent to \( \{ x \in \mathbb{R} : \gamma(x) \geq \varepsilon \} \) being a relatively dense set for \( \varepsilon \) small enough.

The main ingredient in our proof is a resolvent estimate for the fractional Laplacian (Proposition 1 below). In proving this, we will rely on the study of uncertainty principles for the Fourier transform \([5]\) which is defined by
\[
\mathcal{F}(f)(\xi) := \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx
\]
for \( \xi \in \mathbb{R} \), \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). \( \mathcal{F} \) then uniquely extends to a unitary operator on \( L^2(\mathbb{R}) \). The uncertainty principle we will use is a generalization due to O. Kovrijkine of the classical Panceah-Logvinenko-Sereda Theorem \([12, 9]\).

**Theorem 2** (Thm 2 from \([8]\)). Let \( \{J_k\}_{k=1}^n \) be intervals in \( \mathbb{R} \) with \( |J_k| = b \). Let \( E \subset \mathbb{R} \) which is relatively dense. Then, there exists \( c > 0 \) such that
\[
\|f\|_{L^p(E)} \geq c \|f\|_{L^p(\mathbb{R})}
\]
for all \( f \in L^p, p \in [1, \infty] \) with \( \text{supp} \hat{f} \subset \bigcup_{k=1}^n J_k \). Moreover, \( c \) depends only on the number and size of the intervals, not on how they are placed.

In the proof of the proposition, we will only need the case when \( p = 2 \) and there are two intervals \( J_1, J_2 \).

In order to conclude the polynomial or exponential decay in Theorem \([1]\), we will use the following two results on semigroups which connect resolvent bounds for the generator to the decay of the semigroup. For exponential decay, there is the following characterization from \([6]\) Theorem 3 (See also \([4, 14]\)).

**Theorem 3** (Gearhart-Pruss Test). Let \( e^{tA} \) be a \( C_0 \)-semigroup in a Hilbert space \( \mathcal{H} \) and assume there exists \( M > 0 \) such that \( \|e^{tA}\| \leq M \) for all \( t \geq 0 \). Then, there exists \( C, \omega > 0 \) such that
\[
\|e^{tA}\| \leq Ce^{-\omega t}
\]
if and only if \( i\mathbb{R} \subset \rho(A) \) and \( \sup_{\lambda \in \mathbb{R}} \|(A - i\lambda)^{-1}\| < \infty \).
For the polynomial decay, we use the following result from [2, Theorem 2.4]:

**Theorem 4** (Borichev-Tomilov). Let $e^{tA}$ be a $C_0$-semigroup on a Hilbert space $H$. Assume there exists $M > 0$ such that $\|e^{tA}\| \leq M$ for all $t \geq 0$ and $i\mathbb{R} \subset \rho(A)$. Then, for a fixed $\alpha > 0$,

$$\|e^{tA}A^{-1}\| = O(t^{-1/\alpha}) \text{ as } t \to \infty$$

if $\|(A-i\lambda)^{-1}\| = O(\lambda^{\alpha})$ as $\lambda \to \infty$.

**Resolvent Estimates**

**Proposition 1.** Let $\Omega \subset \mathbb{R}$ be relatively dense, $s > 0$. There exists $c > 0$ (depending on $s$) such that for all $f \in L^2(\mathbb{R})$,

$$c\|f\|_{L^2(\mathbb{R})}^2 \leq (1 + \lambda)^{\frac{s}{2} - 2}\|(-\partial_{xx})^{s/2} - \lambda\|f\|_{L^2(\mathbb{R})}^2 + \|f\|_{L^2(\Omega)}^2$$

for all $\lambda \geq 0$.

Throughout, we denote by $\| \cdot \|$ the norm $\| \cdot \|_{L^2(\mathbb{R})}$. We begin with the following algebraic lemma.

**Lemma 1.** Let $s > 0$. There exists $c_s > 0$ such that

$$\|\xi^s - \lambda\| \geq c_s(1 + \lambda)^{1-1/s}$$

for all $(\xi, \lambda) \in \mathbb{R} \times \mathbb{R}_+$ in the region $\|\xi\| - \lambda^{1/s} > 1$.

**Proof.** First, for any $s > 0$, there exists $d_s > 0$ such that

$$d_s \max(x,y)^{s-1}|x-y| \leq |x^s - y^s|$$

for all $x, y \in \mathbb{R}_+$. Now, consider two cases.

(i) If $|\xi| \geq \lambda^{1/s} + 1$, then

$$\|\xi^s - \lambda\| \geq d_s \max(|\xi|, \lambda^{1/s})^{s-1}|\xi| - \lambda^{1/s}| = |\xi^{s-1}||\xi| - \lambda^{1/s}|.$$

The function $x \mapsto x^{s-1}(x - \mu)$ is positive and increasing for $x > \mu + 1$, so we can bound the final term from below by its value at $|\xi| = \lambda^{1/s} + 1$ which yields

$$\|\xi^s - \lambda\| \geq d_s(\lambda^{1/s} + 1)^{s-1}.$$

(ii) If $|\xi| \leq \lambda^{1/s} - 1$, then this implies $\lambda \geq 1$. Therefore,

$$\|\xi^s - \lambda\| \geq d_s \max(|\xi|, \lambda^{1/s})^{s-1} \cdot 1 = d_s(\lambda^{1/s})^{s-1}.$$

If $s < 1$, then $s - 1 < 0$ so $(\lambda^{1/s})^{s-1} \geq (\lambda^{1/s} + 1)^{s-1}$. If $s \geq 1$, then

$$(\lambda^{1/s})^{s-1} = \frac{(\lambda^{1/s} + \lambda^{1/s})^{s-1}}{2^{s-1}} \geq \frac{(\lambda^{1/s} + 1)^{s-1}}{2^{s-1}}.$$

Therefore, there exists $c_s$ such that

$$\|\xi^s - \lambda\| \geq c_s(\lambda^{1/s} + 1)^{s-1} \geq c_s(\lambda + 1)^{s-1} - \frac{1}{s}$$

where in the final step, we have used the fact that for $p \leq q$, $(x^q + y^q)^{1/q} \leq (x^p + y^p)^{1/p}$.

$\square$
Proof of Proposition 1. Let $\lambda \geq 0$, $s > 0$ and $g \in L^2(\mathbb{R})$ such that $\text{supp } \hat{g} \subset [-\lambda^{1/s} - 1, -\lambda^{1/s} + 1] \cup [\lambda^{1/s} - 1, \lambda^{1/s} + 1] =: A_\lambda$. Notice that $A_\lambda = \{\xi \in \mathbb{R} : |\xi| - \lambda^{1/s} \leq 1\}$ and that $A_\lambda$ is the union of two intervals of length 2. Therefore, by Theorem 2 there exists $C > 0$ (independent of $\lambda$ and $g$) such that

$$\|g\|_{L^2(\mathbb{R})} \leq C\|g\|_{L^2(\Omega)}$$

Denote by $P_\lambda$ the projection $P_\lambda f = \mathcal{F}^{-1}(\mathbf{1}_{A_\lambda} \mathcal{F}(f))$. Then, for $f \in L^2(\mathbb{R})$,

$$\|f\|^2 = \|P_\lambda f\|^2 + \|(I - P_\lambda)f\|^2 \leq C\|P_\lambda f\|_{L^2(\Omega)}^2 + \|(I - P_\lambda)f\|^2 \leq C\|f - (I - P_\lambda)f\|_{L^2(\Omega)}^2 + \|(I - P_\lambda)f\|^2 \leq 2C\|f\|_{L^2(\Omega)}^2 + 2C\|(I - P_\lambda)f\|_{L^2(\Omega)}^2 + \|(I - P_\lambda)f\|^2 \leq 2C\|f\|_{L^2(\Omega)}^2 + (2C + 1)\|(I - P_\lambda)f\|^2.$$

It only remains to estimate the final term. By Lemma 1

$$\|((\partial_{xx})^{s/2} - \lambda)f\|_{L^2(\mathbb{R})}^2 = \int (|\xi|^s - \lambda)^2 |\hat{f}(\xi)|^2 \, d\xi \geq \int_{A_\lambda^\circ} (|\xi|^s - \lambda)^2 |\hat{f}(\xi)|^2 \, d\xi \geq c_s(\lambda + 1)^{2-2_s} \int_{A_\lambda^\circ} |\hat{f}(\xi)|^2 \, d\xi = c_s(\lambda + 1)^{2-2_s} \|(I - P_\lambda)f\|^2.$$ 

To apply (5) to the wave equation (1), we first represent the wave equation as a semigroup: Setting $W(t) = (w(t), w_t(t))$, we see that (1) is equivalent to

$$\frac{d}{dt}W(t) = A_\gamma W(t)$$

where $A_\gamma : H^{s/2} \times L^2 \to H^{s/2} \times L^2$ is densely defined on $H^s \times H^{s/2}$ by $A_\gamma(u_1, u_2) = (u_2, -(-\partial_{xx})^{s/2}u_1 - \gamma u_2)$. It is standard that $A_0$ is a closed skew-adjoint operator therefore $e^{tA_0}$ is a semigroup of unitary operators. Then, since $\gamma \geq 0$, for $U = (u_1, u_2) \in H^{s/2} \times L^2$,

$$\text{Re}\langle A_\gamma^* U, U \rangle = \text{Re}\langle A_\gamma U, U \rangle = \text{Re}\langle A_0 U, U \rangle_{H^{s/2} \times L^2} - \langle \gamma u_2, u_2 \rangle_{L^2} = -\langle \gamma u_2, u_2 \rangle_{L^2} \leq 0.$$ 

Moreover, since $\gamma \in L^\infty(\mathbb{R})$, the domain of $A_\gamma$ is the same as $A_0$. So, by classical semigroup theory [13] $e^{tA_\gamma}$ is a $C_0$-semigroup of contractions. We now apply Proposition 1 to $A_0$ and $A_\gamma$. The first step is an observability inequality for the undamped wave equation (1).

Proposition 2. Let $\Omega \subset \mathbb{R}$ be relatively dense, $s > 0$. Then, there exists $c > 0$ such that

$$c\|U\|_{H^{s/2} \times L^2}^2 \leq (|\lambda| + 1)^{4_s} \|(A_0 - i\lambda)U\|_{H^{s/2} \times L^2}^2 + \|u_2\|_{L^2(\Omega)}^2$$

for all $U = (u_1, u_2) \in H^{s/2} \times L^2$ and $\lambda \in \mathbb{R}$. 

Proof. For $U = (u_1, u_2) \in H^{s/2}(\mathbb{R}) \times L^2(\mathbb{R})$, set $w_1 = (-\partial_x)^{s/4}u_1 - iu_2$ and $w_2 = (-\partial_x)^{s/4}u_1 + iu_2$. First, by the parallelogram identity,

$$\|w_1\|^2_{L^2(\mathbb{R})} + \|w_2\|^2_{L^2(\mathbb{R})} = 2\|(-\partial_x)^{s/4}u_1\|^2 + 2\|w_2\|^2 = 2\|U\|^2_{H^{s/2} \times L^2}.$$ 

Secondly,

$$\|(A_0 - \lambda I)U\|^2_{H^{s/2} \times L^2} = \|(-\partial_x)^{s/4}(-\lambda u_1 + u_2)\|^2 + \|(-\partial_x)^{s/2}u_1 - \lambda u_2\|^2$$

$$\begin{aligned}
&= \| - \lambda \frac{w_1 + w_2}{2} + i(-\partial_x)^{s/4}\frac{w_1 - w_2}{2}\| + \|(-\partial_x)^{s/4}\frac{w_1 + w_2}{2} - i\lambda \frac{w_1 - w_2}{2}\|
&= \| - i\lambda w_1 - (-\partial_x)^{s/4}w_1\| + \| - i\lambda w_2 + (-\partial_x)^{s/4}w_2\|^2.
\end{aligned}$$

So, applying Proposition 1 to $w_1$ with $s$ replaced by $s/2$, we have, for $\lambda \geq 0$,

$$2c\|U\|^2_{H^{s/2} \times L^2} = c(\|w_1\|^2 + \|w_2\|^2)$$

$$\leq \left(\|\lambda\| + 1\right)\frac{4}{5} - 2\|(\lambda - \partial_x)^{s/4}\|w_1\| + \|w_1\|_{L^2(\Omega)}^2 + c\|w_2\|^2$$

$$\leq \left(\|\lambda\| + 1\right)\frac{4}{5} - 2\|(\lambda - \partial_x)^{s/4}\|w_1\| + 2\|w_1 - w_2\|_{L^2(\Omega)}^2 + (c + 2)\|w_2\|^2$$

$$\leq \left(\|\lambda\| + 1\right)\frac{4}{5} - 2\|(\lambda - \partial_x)^{s/4}\|w_1\| + 2\|w_2\|_{L^2(\Omega)}^2 + \frac{c + 2}{\left(\|\lambda\| + 1\right)^2}\|w_2\|^2$$

$$\leq \left(\|\lambda\| + 1\right)\frac{4}{5} - 2\|(A_0 - i\lambda I)U\|_{H^{s/2} \times L^2}^2 + 8\|w_2\|_{L^2(\Omega)}^2.$$ 

We get the case $\lambda < 0$ by exchanging the roles of $w_1$ and $w_2$. \qed

Finally we extend this to $A_\gamma - i\lambda I$ and prove Theorem 1.

Proof of Theorem 1

First notice that for any $\varepsilon > 0$,

$$\int_{a-R}^{a+R} \gamma(x) \, dx \leq \|\gamma\|_\infty m(\{\gamma \geq \varepsilon\} \cap [a - R, a + R]) + 2R \varepsilon.$$ 

So, (4) implies that $\{\gamma \geq \varepsilon\}$ is relatively dense for $\varepsilon$ small enough. Therefore, taking $\Omega = \{\gamma \geq \varepsilon\}$ and applying Proposition 2,

$$c\|U\|^2_{H^{s/2} \times L^2} \leq \left(\|\lambda\| + 1\right)^{\frac{4}{5} - 2}\|(A - i\lambda I)U\|_{H^{s/2} \times L^2}^2 + \|w_2\|_{L^2(\Omega)}^2$$

$$\leq \left(\|\lambda\| + 1\right)^{\frac{4}{5} - 2}\|(A_\gamma - i\lambda I)U\|_{H^{s/2} \times L^2}^2 + \left[2(\|\lambda\| + 1)^{\frac{4}{5} - 2} + \varepsilon^{-2}\right]\|u_2\|_{L^2(\Omega)}^2.$$ 

We estimate the final term. Since $A_0$ is skew-adjoint,

$$\text{Re} \langle (A_\gamma - i\lambda I)U, U \rangle = \text{Re} \langle (A_0 - i\lambda I)U, U \rangle - \langle u_2, u_2 \rangle = -\|\sqrt{\gamma}u_2\|^2$$ 

which implies

$$D\|\gamma u_2\|^2 \leq D\|\gamma\|_\infty \|\sqrt{\gamma}u_2\|^2 \leq \frac{D^2\|\gamma\|^2}{\delta^2}\|(A_\gamma - i\lambda I)U\|^2 + \delta\|U\|^2$$ 

for any $D, \delta > 0$. Choosing $D = 2(\|\lambda\| + 1)^{\frac{4}{5} - 2} + \varepsilon^{-2}$ and $\delta = c/2$, from (6) we obtain

$$c\|U\|^2_{H^{s/2} \times L^2} \leq C \left[\left(\|\lambda\| + 1\right)^{\frac{4}{5} - 2} + \left(\|\lambda\| + 1\right)^{\frac{8}{5} - 4} + 1\right]\|(A_\gamma - i\lambda I)U\|_{H^{s/2} \times L^2}^2 + \frac{c}{2}\|U\|^2_{H^{s/2} \times L^2}.$$
Thus, we have proved the following estimate for \((A_\gamma - i\lambda I)^{-1}\):

\[
\| (A_\gamma - i\lambda I)^{-1} \|_{H^{s/2} \times L^2 \to H^{s/2} \times L^2} \leq \begin{cases} 
C(|\lambda| + 1)^{\frac{4}{s} - 2} & 0 < s < 2 \\
C & s \geq 2.
\end{cases}
\]

Applying the Theorems\ref{thm:1} and \ref{thm:2} allows us to conclude the main Theorem\ref{thm:main} from (7).

ACKNOWLEDGEMENTS

The author is thankful to Milena Stanislavova for introducing him to this problem.

REFERENCES

[1] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim., 30(5):1024–1065, 1992.
[2] Alexander Borichev and Yuri Tomilov. Optimal polynomial decay of functions and operator semigroups. Mathematische Annalen, 347(2):455–478, 2010.
[3] Nicolas Burq and Romain Joly. Exponential decay for the damped wave equation in unbounded domains. Communications in Contemporary Mathematics, 18(06):1650012, 2016.
[4] Larry Gearhart. Spectral theory for contraction semigroups on hilbert space. Transactions of the American Mathematical Society, 236:385–394, 1978.
[5] Victor Havin and Burglind Jörickie. The uncertainty principle in harmonic analysis, volume 28. Springer Science & Business Media, 2012.
[6] Falun Huang. Characteristic conditions for exponential stability of linear dynamical systems in hilbert spaces. Ann. of Diff. Eqs., 1:43–56, 1985.
[7] Romain Joly and Julien Royer. Energy decay and diffusion phenomenon for the asymptotically periodic damped wave equation. Journal of the Mathematical Society of Japan, 70(4):1375–1418, 2018.
[8] Oleg Kovrijkine. Some results related to the logvinenko-sereda theorem. Proceedings of the American Mathematical Society, 129(10):3037–3047, 2001.
[9] VN Logvinenko and Ju F Sereda. Equivalent norms in spaces of entire functions of exponential type. Teor. FunkciıFunkcional. Anal. i Prilozen. Vyp, 20:102–111, 1974.
[10] Satbir Malhi and Milena Stanislavova. On the energy decay rates for the 1d damped fractional klein-gordon equation. arXiv preprint arXiv:1809.09531, 2018.
[11] Satbir Malhi and Milena Stanislavova. When is the energy of the 1d damped klein-gordon equation decaying? Mathematische Annalen, 372(3-4):1459–1479, 2018.
[12] Boris Petrovich Paneah. Some theorems of paley–wiener type. In Doklady Akademii Nauk, volume 138, pages 47–50. Russian Academy of Sciences, 1961.
[13] Amnon Pazy. Semigroups of linear operators and applications to partial differential equations, volume 44. Springer-Verlag, 1983.
[14] Jan Prüss. On the spectrum of c0-semigroups. Transactions of the American Mathematical Society, 284(2):847–857, 1984.
[15] Jeffrey Rauch, Michael Taylor, and Ralph Phillips. Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana university Mathematics journal, 24(1):79–86, 1974.
[16] Jared Wunsch. Periodic damping gives polynomial energy decay. Mathematical Research Letters, 24(2):571–580, 2017.