Upwind-SAV approach for constructing bound-preserving and energy-stable schemes of the Cahn–Hilliard equation with degenerate mobility

Qiong-Ao Huang\textsuperscript{a,b}, Wei Jiang\textsuperscript{c,d}, Jerry Zhijian Yang\textsuperscript{c,d}, Cheng Yuan\textsuperscript{c,∗}

\textsuperscript{a}School of Mathematics and Statistics, Henan University, Kaifeng 475004, China
\textsuperscript{b}Center for Applied Mathematics of Henan Province, Henan University, Zhengzhou 450046, China
\textsuperscript{c}School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
\textsuperscript{d}Hubei Key Laboratory of Computational Sciences, Wuhan University, Wuhan 430072, China

Abstract

This paper establishes an unconditionally bound-preserving and energy-stable scheme for the Cahn–Hilliard equation with degenerate mobility. More specifically, by applying a finite volume method (FVM) with upwind numerical fluxes to the degenerate Cahn–Hilliard equation rewritten by the scalar auxiliary variable (SAV) approach, we obtain an unconditionally bound-preserving, energy-stable and fully-discrete scheme, which, for the first time, addresses the boundedness of the classical SAV approach under $H^{-1}$-gradient flow. Furthermore, the dimensional-splitting technique is introduced in high-dimensional case, which greatly reduces the computational complexity while preserves original structural properties. Several numerical experiments are presented to verify the bound-preserving and energy-stable properties of the proposed scheme. Moreover, by applying the scheme to the moving interface problem, we have numerically demonstrated that surface diffusion can be modeled by the Cahn–Hilliard equation with degenerate mobility and Flory–Huggins potential at low temperature, which was only shown theoretically before by formal matched asymptotics.

Keywords: Cahn–Hilliard equation; Degenerate mobility; Bound-preserving; Upwind-Scalar auxiliary variable; Flory–Huggins potential.

1. Introduction

The famous Cahn–Hilliard equation was originally established to model phase separation and coarsening processes in binary alloys [1], and is now widely used in many scientific and engineering fields such as image inpainting [2,3], polymer blends [4–7], solid-state dewetting [8–10], multi-phase flow [11–13] and tumor growth [14–16], all of which are built on the following total free energy with respect to the conserved order parameter (i.e., phase variable) $\phi(x, t)$,

\[
E[\phi(x, t)] = \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\nabla \phi|^2 + F(\phi) \right) dx, \quad (x, t) \in \Omega \times [0, T],
\]

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain with the boundary $\partial \Omega$, $\varepsilon > 0$ denotes the thickness of the interface between the two phases and $F(\phi)$ is the Helmholtz free energy density of the system.

A typical thermodynamically relevant form of $F(\phi)$ is the so-called logarithmic Flory–Huggins potential as follows [1,17]

\[
F_{\log}(\phi) = \frac{\theta}{2} \left[ (1 + \phi) \ln(1 + \phi) + (1 - \phi) \ln(1 - \phi) \right] + \frac{\theta_c}{2}(1 - \phi^2), \quad 0 < \theta < \theta_c,
\]

\*Corresponding author.

Email addresses: huangqiongao@henu.edu.cn (Qiong-Ao Huang), jiangwei1007@whu.edu.cn (Wei Jiang), zjyang.math@whu.edu.cn (Jerry Zhijian Yang), yuancheng@whu.edu.cn (Cheng Yuan)
where $\theta$ and $\theta_c$ are the absolute and critical temperatures, respectively. Furthermore, it is easy to check that the above logarithmic potential $\log(\chi)$ has a double-well structure with two minima at $\pm \beta_{s,c} \in (-1, 1)$, where $\beta_{s,c}$ is the positive root of $F_{\log}(\chi) = 0$ and approaches 1 as $\theta/\theta_c \to 0$. However, due to the singularity of the logarithmic function in (1.2), a simplified polynomial version is used more frequently both in theory and in practice, namely,

$$F_{pol}(\phi) = \frac{1}{4}(1 - \phi^2)^2,$$

which is obtained by using the Taylor-series for (1.2) near the origin and omitting the high-order infinitesimal terms. Similarly, (1.3) is also a double-well potential with two minima at $\pm 1$.

By applying the conserved $H^{-1}$-gradient flow to the energy functional (1.1), we can obtain the following desired Cahn–Hilliard equation

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot J, \quad J = -M(\phi)\nabla \mu, \quad \mu = -\varepsilon^2 \Delta \phi + F'(\phi), \quad in \; \Omega \times (0, T],$$

equipped with the Neumann and no-flux boundary conditions:

$$\nabla \phi \cdot n = 0, \quad J \cdot n = 0, \quad on \; \partial \Omega \times (0, T],$$

where $J$ is the mass flux, $M(\phi) \geq 0$ is the diffusion mobility, $\mu$ is the chemical potential and $n$ is the outwards pointing normal unit vector onto $\partial \Omega$.

Equipping different types of diffusion mobility $M(\phi)$ into the Cahn–Hilliard equation (1.4)-(1.5), although not changing the energy landscape, can have a significant effect on the kinetic process of the solution. For constant mobility $M(\phi) \equiv 1$, Pego [15] showed by formally matched asymptotic analysis that the sharp-interface limit of the Cahn–Hilliard equation is the Mullins–Sekerka problem [19, 20], which is rigorously presented later by Alikakos et al. in [21]. One of the main features of the Mullins–Sekerka problem is that there is not only surface diffusion along the interface but also bulk diffusion between the two phases, which results in smaller parts being swallowed up by larger parts between the same phases that are not adjacent. On the other hand, the mobility function being degenerate at or near the minima of the Helmholtz free energy density is also commonly used, which is typically phase-dependent and takes the form of

$$M(\phi) = (1 - \phi^2)^k, \quad k = 1, 2, 3, \cdots$$

Due to the degeneracy of the above mobility, bulk diffusion will be suppressed and the kinetics are dominated by surface diffusion along the interface of the two phases [23, 24]. In particular, Lee et al. [25] showed by matched asymptotic analysis that the sharp-interface limit of the Cahn–Hilliard equation with polynomial potential (1.6) at $k = 2$ is surface diffusion, at least to leading order. However, numerical results show that there exists some oscillations in the profile of solution when the pure state is reached, which causes the solution to be outside its physical range $[-1, 1]$, and that smaller parts may still be absorbed by larger ones in the same phase [24, 25]. Pesce and Muench [26] pointed out that the above undesirable results may be a numerical artifact, and suggested that a high-degeneracy mobility should be considered in numerical simulations, at least $k = 4$ in [16]. Moreover, by combining the logarithmic potential (1.2) at $\theta = 1$ and the degenerate mobility (1.6) at $k = 1$, Cahn et al. [29] proved that the sharp-interface limit of the Cahn–Hilliard equation is surface diffusion when the temperature $\theta = O(\varepsilon^\alpha)$, $\alpha > 0$. Unfortunately, due to the singularity of the logarithmic function, constructing a structure-preserving scheme for this combination still remains an open challenge.

From a continuous point of view, the Cahn–Hilliard equation with degenerate mobility possesses several important properties. Firstly, although the maximum bound principle (MBP) similar to the Allen–Cahn equation (i.e., the $L^2$-gradient flow of energy functional (1.1)) has not been established for the degenerate Cahn–Hilliard equation [22-25, 30, 31], it is still necessary to impose boundedness (bounded by 1) on the phase variable $\phi$, especially for the logarithmic potential. Otherwise, the logarithmic arithmetic will lead to complex value solutions and the mobility function may become negative. Secondly, as a direct consequence of the conserved $H^{-1}$-gradient flow, we may derive the law of mass conservation as follows,

$$\frac{d m(t)}{d t} = \frac{d}{d t} \int_\Omega \phi \; dx = \int_\Omega \phi_t \; dx = -\int_\Omega \nabla \cdot J \; dx = -\int_{\partial \Omega} J \cdot n \; ds = 0. \quad (1.7)$$
Finally, the evolution process described by the gradient flow naturally satisfies the law of energy dissipation, namely,

\[ \frac{dE(t)}{dt} = \int_{\Omega} (\varepsilon^2 \nabla \phi \cdot \nabla \phi_t + F'(\phi) \phi_t) \, dx - \int_{\Omega} (\varepsilon^2 \Delta \phi + F'(\phi)) \phi_t \, dx + \int_{\partial \Omega} \varepsilon^2 (\nabla \phi \cdot n) \phi_t \, ds \]

\[ = \int_{\Omega} \mu \nabla \cdot (M(\phi) \nabla \mu) \, dx - \int_{\Omega} M(\phi)|\nabla \mu|^2 \, dx \leq 0. \tag{1.8} \]

As we can see, all the above properties except the first one are determined by the inherent nature of the physical problem, while the boundedness of \( \phi \) is an additional requirement imposed by the established model. On the other hand, in order to avoid non-physical effects in the simulations over a long period of time, it is highly desirable to design a structure-preserving scheme at the fully-discrete level. Fortunately, while the mass conservation can be achieved for many numerical methods, several widely used approaches aimed at the unconditional energy stability have also been developed in recent years, such as the convex splitting approach [32–34], the exponential time differencing (ETD) approach [35–37], the invariant energy quadratization (IEQ) approach [38–40], the scalar auxiliary variable (SAV) approach [41–43] and its variants [44–48] (collectively referred to as classical SAV approach), and the new SAV approach [49, 50]. As for the most challenging issue on the bound/positivity preserving, some explorations, but not limited to, are summarized as follows:

- Finite element approach [51]: by using a practical finite element approximation scheme with an intentionally designed variational inequality, the author embedded the boundedness property naturally into the solution for Cahn–Hilliard equation with both polynomial potential and the logarithmic Flory–Huggins potential. This approximation is proved to be well-posed and stable, while the law of energy dissipation for the scheme still needs to be clarified.

- Function transform approach [52, 53]: this approach achieves bound/positivity preserving via a suitable function transform (e.g., \( \phi = \tanh(\psi) \) for bound-preserving and \( \phi = \exp(\psi) \) for positivity-preserving). This approach usually results in transformed equation to be more complicated than the original one, and likewise fails to address energy stability and mass conservation.

- Cut-off approach [54–56]: this approach is to artificially cut off values outside the desired range. In addition to being easy to implement, the main advantage of this approach is that it can achieve arbitrarily high-order accuracy in time for some situations (e.g., for Allen–Cahn equation [55]), while the energy stability and mass conservation are usually difficult to guarantee.

- Implicit-explicit approach [57–59]: this method usually adopts implicit-explicit discretization and central difference in time and space respectively, which leads to the negative diagonally dominant property of the discrete matrix of the Laplace operator under appropriate boundary conditions. Especially, this property is crucial for establishing the bound-preserving scheme for Allen–Cahn type equations.

- ETD approach [30, 60, 61]: this approach comes from the Duhamel principle with the nonlinear terms approximated by polynomial interpolations in time, followed by the exact temporal integration. Nowadays, an unconditionally bound-preserving scheme has been achieved for Allen–Cahn type equations, which also benefits from the good properties of the Laplace operator in the discrete sense obtained by central difference or lumped-mass finite element methods.

- Convex splitting approach [62, 64]: the key idea of this approach is to treat the contractive and expansive parts of the potential as implicit and explicit, respectively. The main advantage of this approach is that it can simultaneously ensure energy stability, mass conservation and boundedness, but it is usually useless for situations such as degenerate mobility and potential where convex-concave decomposition is difficult.

- Upwind approach [63, 67]: this method applies upwind scheme to deal with the flux term, leading to a scheme suitable for degenerate PDEs and can guarantee mass conservation, but usually has first-order
accuracy in time. Recently, this method was further combined with the convex splitting strategy to achieve the unconditional energy stability [68, 69].

- **Lagrange multiplier approach** [71]: this approach introduces a space-time Lagrange multiplier and takes the famous Karush–Kuhn–Tucker (KKT) conditions as a constraint to achieve bound/positivity preserving, and the previous cut-off approach can be regarded as a special case. However, this approach does not guarantee the energy stability, although the mass conservation can be ensured.

In order to obtain a numerical method that simultaneously or partially captures the three properties mentioned before (boundedness, mass conservation and energy dissipation), several works on the combination of the SAV method and above explorations have been studied, including

- **New SAV approach with function transform strategy** [72, 73]: this combination solve the difficulty that the function transform approach cannot achieve energy stability and mass conservation, and can achieve arbitrarily high-order accuracy in time, but currently it can only be applied to homogeneous boundary conditions (e.g., Neumann and periodic boundary conditions), and the achieved energy stability is a somewhat weakened form.

- **Original SAV approach with cut-off strategy** [74]: this joint strategy constructs unconditionally bound-preserving, energy dissipation, and arbitrarily high-order (in time) scheme for the Allen–Cahn equation. However, since the discrete energy in the law of energy dissipation is modified, there may exist some undesired increase in the original energy, which is unphysical.

- **Exponential SAV approach with stabilized implicit-explicit strategy** [75]: with the effect of such stabilization, the proposed first-order scheme unconditionally maintains boundedness and energy dissipation for the Allen–Cahn type equations, while the boundedness of the second-order one is constrained by the time step size.

- **Generalized SAV approach with stabilized ETD strategy** [76]: with the appropriate stabilization terms, the proposed first- and second-order SAV-ETD schemes are unconditionally bound-preserving and energy dissipation for the Allen–Cahn type equations. Still, how to choose the appropriate type of scalar auxiliary variable in practice is a problem that needs to be further solved.

Nevertheless, it should be pointed out that most of above bound-preserving schemes established for Allen–Cahn type equations are difficult to extend to Cahn–Hilliard type equations, due to that negative biharmonic operator (i.e., $-\Delta^2$) involved in Cahn–Hilliard equation does not possess the key property of being negative diagonally dominant of Laplace operator in Allen–Cahn equation under the same spatial discretization [30, 55, 59]. In other words, among the above approaches, only the upwind-convex splitting method given in [68] can supply us with an original structure-preserving scheme for the Cahn–Hilliard equation under the same spatial discretization with degenerate mobility [1.0] and logarithmic potential [1.2] or polynomial potential [1.3]. However, despite the many advantages of the convex splitting approach, several obvious shortcomings including the difficulty in applying to anisotropic potential and constructing high-order scheme in the time direction, requires us to further develop an effective and simple to be generalized structure-preserving scheme.

In this paper, inspired by the fact that the SAV-like approaches are easy to deal with various forms of potential, and one of the Lagrange multiplier-type SAV approach can guarantee the original energy stability at the fully-discrete level, we choose to replace the convex splitting with the Lagrange multiplier-type SAV in the upwind-convex splitting approach, resulting in a structure-preserving scheme for the Cahn–Hilliard equation with degenerate mobility, and further numerically verify whether it can capture the main features of surface diffusion. In practice, it is also important to reduce the computational effort in high dimensional case while maintaining the structural properties. To this end, unlike various spatial adaptive strategies [77–79] that are widely used, the dimensional-splitting technique [67, 68] is first introduced into the spatial discretization under the SAV approach, which magically decouples the discrete equation in $d$-dimensional into solving a series of discrete equations in one-dimensional, thereby greatly improving the computational efficiency. To sum up, our main contributions of this article are as follows
In the framework of classical SAV approach, the boundedness of Cahn–Hilliard equation is additionally realized for the first time.

The dimensional-splitting technique is applied to SAV approach for the first time, which greatly reduces the computational effort while maintaining the original structural properties.

Based on our approach, we numerically verify that the surface diffusion flow can be captured by the combination of degenerate Cahn–Hilliard equation with Flory–Huggins potential at low temperature, which was only proved theoretically.

The rest of this paper is organized as follows. In Section 2, the upwind strategy is applied to the Lagrange multiplier-type SAV approach, leading to an unconditionally bound-preserving and energy-stable scheme for the Cahn–Hilliard equation with degenerate mobility. In Section 3, the dimensional-splitting technique is applied to the situation of high-dimensional space to reduce the computational effort caused by space discretization while ensuring the original structural properties. In Section 4, ample examples will be provided to show the boundedness, mass conservation and energy stability of the proposed scheme, and the claim that the sharp-interface limit of the Cahn–Hilliard equation with degenerate mobility and logarithmic potential at low temperature is surface diffusion will be verified numerically. Finally, some conclusions will be given in Section 5.

2. Upwind-SAV approach

In this section, the upwind approach and the SAV approach are combined for the first time to construct a structure-preserving scheme for the Cahn–Hilliard equation with degenerate mobility.

To begin with, applying the key idea of the Lagrange multiplier-type SAV approach \[47, 80, 81\] to the degenerate Cahn–Hilliard equation (1.4)-(1.5) yields

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= -\nabla \cdot J, \\
J &= -M(\phi)\nabla \mu, \\
\mu &= -\varepsilon^2 \Delta \phi + \xi(t) F'(\phi), \\
\frac{d}{dt} \int_{\Omega} F(\phi) d\mathbf{x} &= \xi(t) \int_{\Omega} F'(\phi) \frac{\partial \phi}{\partial t} d\mathbf{x}, \\
\text{in } \Omega \times (0, T],
\end{align*}
\]

subject to the following Neumann and no-flux boundary conditions

\[\nabla \phi \cdot \mathbf{n} = 0, \quad J \cdot \mathbf{n} = 0, \quad \text{on } \partial \Omega \times (0, T],\]

where \(\xi(t)\) is the newly introduced scalar auxiliary variable whose role is the usual Lagrange multiplier and \(M(\phi) \geq 0\) is the diffusion mobility defined in \(1.6\). If the initial condition of \(\xi(t)\) is taken as \(\xi(0) = 1\), it is easy to check that the above rewritten system is equivalent to the original one, i.e., \(\xi(t) \equiv 1\). Therefore, the above equivalent system \(2.1-2.2\) also obeys the properties of mass conservation and energy dissipation at the PDE level, while the boundedness of \(\phi\) should also be an essential requirement to be satisfied.

Secondly, to facilitate subsequent numerical discretization, we introduce the following two notations

\[
\chi^+ = \max\{\chi, 0\}, \quad \chi^- = \min\{\chi, 0\},
\]

and express the diffusion mobility given in \(1.6\) as

\[
M(\chi_1, \chi_2) = [(1 + \chi_1)^+(1 - \chi_2)^+]^k, \quad k = 1, 2, 3, \ldots
\]

In fact, due to the boundedness of phase variable, the above expression style is not substantially different from the original one.

Lastly, the finite volume method (FVM) with upwind numerical fluxes is considered for the discretization of \(2.1-2.2\). Starting with the one dimensional case, we divide the computational domain \(\Omega = \Omega \cup \partial \Omega\)
into $N_x$ cells $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 1, 2, \ldots, N_x$, all with uniform size $\Delta x$, so that the centre of each cell is $x_i = (i - \frac{1}{2})\Delta x + x_{\frac{1}{2}}$. By defining the cell average $\phi_i$ on $C_i$ as

$$\phi_i(t) = \frac{1}{\Delta x} \int_{C_i} \phi(x,t)dx$$

and using the backward Euler formula with FVM in temporal and spatial discretization respectively, the continuous system (2.1) can be approximated as

$$\phi_i^{n+1} - \phi_i^n = -\frac{\Delta t}{\Delta x} \left( J_i^{n+1} - J_i^n \right), \tag{2.6}$$

$$J_i^{n+1} = \left( V_i^{n+1} \right)^\top M(\phi_i^{n+1}, \phi_{i+1}^{n+1}) + \left( V_i^{n+1} \right)^\top M(\phi_i^{n+1}, \phi_{i+1}^{n+1}), \tag{2.7}$$

$$V_i^{n+1} = -\frac{1}{\Delta x} (\mu_i^{n+1} - \mu_i^n), \tag{2.8}$$

$$\mu_i^{n+1} = -\varepsilon^2 (\Delta \phi)_i^{n+1} + \xi^{n+1} F'(\phi_i^{n+1}), \tag{2.9}$$

$$\sum_{i=1}^{N_x} (F(\phi_i^{n+1}) - F(\phi_i^n)) = \xi^{n+1} \sum_{i=1}^{N_x} F'(\phi_i^{n+1})(\phi_i^{n+1} - \phi_i^n), \tag{2.10}$$

where in (2.8) the key idea of the upwind approach [65, 68] has been used. Here $\Delta t > 0$ denotes the time step size, $\chi^n_i$ is the numerical approximation of $\chi_i(t)$ ($\chi$ refers to the variables appeared before, including $\phi, J, V, \mu, \xi$) at time $t = t^n = \Delta t \cdot n$ for $n = 0, 1, \ldots, N$ with $T = N \Delta t$, and the Laplacian term $(\Delta \phi)_i^{n+1}$ is discretized by the following central difference formula

$$(\Delta \phi)_i^{n+1} = \frac{(\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n+1}_{i-\frac{1}{2}}}{\Delta x} = \frac{1}{\Delta x} \left( \phi_{i+1}^{n+1} - \phi_i^{n+1} \right) - \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\Delta x} = \phi_{i+1}^{n+1} - \frac{\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}. \tag{2.11}$$

Moreover, the Neumann and no-flux boundary conditions are implemented as

$$(\nabla \phi)^{n+1}_{N_x+\frac{1}{2}} = 0, \quad (\nabla \phi)^{n+1}_{N_x+\frac{1}{2}} = 0, \quad J^{n+1}_{N_x} = 0, \quad J^{n+1}_{N_x} = 0, \tag{2.12}$$

resulting in that the Laplacian term $(\Delta \phi)_i^{n+1}$ at the boundaries can be approximated as

$$(\Delta \phi)_i^{n+1} = \frac{\phi_1^{n+1} - \phi_k^{n+1}}{\Delta x^2}, \quad (\Delta \phi)_i^{n+1} = \frac{-\phi_{N_x}^{n+1} + \phi_{N_x}^{n+1}}{\Delta x^2}. \tag{2.13}$$

Now we have achieved a fully-discrete scheme (2.6)-(2.13) satisfying the properties of bound-preserving, mass conservation and energy dissipation, which can be rigorously proved as follows.

**Theorem 2.1.** (Boundedness) The fully-discrete scheme (2.6)-(2.13) can ensure the boundedness of the phase average $\phi_i$. That is, for all $i$, if $|\phi_i^n| < 1$, then $|\phi_i^{n+1}| < 1$.

**Proof.** First, for all $i$, $|\phi_i^n| < 1$ leads to $\phi_i^{n+1} < 1$. Otherwise, suppose there is a group of contiguous cells $\{\phi_0^{n+1}, \phi_1^{n+1}, \ldots, \phi_k^{n+1}\}$ such that $\phi_i^{n+1} \geq 1$, then sum the two ends of (2.6) over the cells, resulting in

$$0 < \frac{\Delta x}{\Delta t} \sum_{i=j}^{k} (\phi_i^{n+1} - \phi_i^n) = - \sum_{i=j}^{k} \left( J_{i+\frac{1}{2}}^{n+1} - J_{i-\frac{1}{2}}^n \right) = J_{j+\frac{1}{2}}^{n+1} - J_{j-\frac{1}{2}}^n$$

$$= \left( V_{j+\frac{1}{2}}^{n+1} \right)^\top M(\phi_{j-1}^{n+1}, \phi_j^{n+1}) + \left( V_{j-\frac{1}{2}}^{n+1} \right)^\top M(\phi_j^{n+1}, \phi_{j-1}^{n+1})$$

$$+ \left( V_{k+\frac{1}{2}}^{n+1} \right)^\top M(\phi_{k+1}^{n+1}, \phi_k^{n+1}) + \left( V_{k-\frac{1}{2}}^{n+1} \right)^\top M(\phi_k^{n+1}, \phi_{k-1}^{n+1})$$

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\[ -\left(V_{k+\frac{1}{2}}^{n+1}\right)^{-} M(\phi_{k}^{n+1}, \phi_{k+1}^{n+1}) - \left(V_{k+\frac{1}{2}}^{n+1}\right)^{+} M(\phi_{k+1}^{n+1}, \phi_{k}^{n+1}). \]  

(2.14)

Since

\[ M(\phi_{j-1}^{n+1}, \phi_{j}^{n+1}) = 0, \quad M(\phi_{j}^{n+1}, \phi_{j+1}^{n+1}) > 0, \quad M(\phi_{k}^{n+1}, \phi_{k+1}^{n+1}) > 0, \quad M(\phi_{k+1}^{n+1}, \phi_{k}^{n+1}) = 0, \]

which leads to the right end of (2.13) to be non-positive. Therefore, there must be \( \phi_{i}^{n+1} < 1 \). Similarly, we can prove \( \phi_{i}^{n+1} > -1 \) by following the same procedure.

\[ \text{Remark 2.1.} \quad \text{If the phase-dependent mobility function degenerates at the extreme point of the logarithmic Flory–Huggins potential (1.2), i.e., } M(\phi) = (\beta s_{n} - \phi^{2})^{k}, \text{ it can be similarly obtained from the above Theorem 2.2 that for any } \|\phi_{i}^{n}\|_{\infty} \leq \beta_{s_{n}}, \text{ the fully-discrete scheme (2.10)–(2.13) guarantees } \|\phi_{i}^{n+1}\|_{\infty} \leq \beta_{s_{n}}. \]

\[ \text{Theorem 2.2. (Mass conservation)} \quad \text{The fully-discrete scheme (2.10)–(2.13) ensures that the total mass is conserved during the evolution, i.e.} \]

\[ \sum_{i=1}^{N_{x}} \phi_{i}^{n+1} = \sum_{i=1}^{N_{x}} \phi_{i}^{n} = \cdots = \sum_{i=1}^{N_{x}} \phi_{i}^{0}. \]  

(2.16)

\[ \text{Proof.} \quad \text{Sum the two ends of (2.6) over all cells } C_{i}, \text{ it obtains} \]

\[ \sum_{i=1}^{N_{x}} (\phi_{i}^{n+1} - \phi_{i}^{n}) = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{N_{x}} \left( E_{i+\frac{1}{2}}^{n+1} - E_{i+\frac{1}{2}}^{n} \right) = -\frac{\Delta t}{\Delta x} \left( J_{N_{x}}^{\frac{1}{2}} - J_{0}^{\frac{1}{2}} \right) = 0, \]

(2.17)

where the last equality is derived from the non-flux boundary conditions (2.12).

\[ \text{Theorem 2.3. (Energy dissipation)} \quad \text{The fully-discrete scheme (2.10)–(2.13) is unconditionally energy stable, and satisfies the following discrete energy dissipation law:} \]

\[ E^{n+1} - E^{n} \leq -\Delta t \Delta x \sum_{i=1}^{N_{x}-1} \min \left\{ M(\phi_{i}^{n+1}, \phi_{i+1}^{n+1}), M(\phi_{i+1}^{n+1}, \phi_{i}^{n+1}) \right\} \left| V_{i+\frac{1}{2}}^{n+1} \right|^{2} \leq 0, \]  

(2.18)

where

\[ E^{n} = \Delta x \sum_{i=1}^{N_{x}} \frac{\varepsilon}{2} \left( |(\nabla \phi)^{n}_{i+\frac{1}{2}}|^{2} + \Delta x \sum_{i=1}^{N_{x}} F(\phi_{i}^{n}) \right). \]

\[ \text{Proof.} \quad \text{By subtract the discrete free energy in (2.19) at subsequent times, it obtains} \]

\[ \frac{E^{n+1} - E^{n}}{\Delta x} = \frac{\varepsilon}{2} \sum_{i=1}^{N_{x}-1} \left( |(\nabla \phi)^{n+1}_{i+\frac{1}{2}}|^{2} - |(\nabla \phi)^{n}_{i+\frac{1}{2}}|^{2} \right) + \sum_{i=1}^{N_{x}} \left( F(\phi_{i}^{n+1}) - F(\phi_{i}^{n}) \right) \]

\[ = \frac{\varepsilon}{2} \sum_{i=1}^{N_{x}-1} \left( |(\nabla \phi)^{n+1}_{i+\frac{1}{2}}|^2 - |(\nabla \phi)^{n}_{i+\frac{1}{2}}|^2 \right) - \frac{\varepsilon}{2} \sum_{i=1}^{N_{x}-1} |(\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}}|^2 \]

\[ + \xi^{n+1} \sum_{i=1}^{N_{x}} F(\phi_{i}^{n+1})(\phi_{i}^{n+1} - \phi_{i}^{n}) \]

\[ = \frac{\varepsilon}{2} \sum_{i=1}^{N_{x}-1} \frac{\phi_{i}^{n+1} - \phi_{i}^{n+1}}{\Delta x} \left( \frac{\phi_{i}^{n+1} - \phi_{i+1}^{n+1}}{\Delta x} - \frac{\phi_{i+1}^{n+1} - \phi_{i+1}^{n}}{\Delta x} \right) \]

\[ - \frac{\varepsilon}{2} \sum_{i=1}^{N_{x}-1} |(\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}}|^2 + \xi^{n+1} \sum_{i=1}^{N_{x}} F(\phi_{i}^{n+1})(\phi_{i}^{n+1} - \phi_{i}^{n}) \]
\begin{align*}
&= \varepsilon^2 \sum_{i=1}^{N_x-1} \phi_i^{n+1} - \phi_i^{n+1} \Delta x^2 (\phi_{i+1}^{n+1} - \phi_i^{n+1}) - \varepsilon^2 \sum_{i=1}^{N_x-1} \phi_{i+1}^{n+1} - \phi_{i+1}^{n} \Delta x^2 (\phi_{i+1}^{n+1} - \phi_{i+1}^{n}) \\
&- \varepsilon^2 \sum_{i=1}^{N_x-1} \phi_{i+1}^{n+1} - 2 \phi_{i+1}^{n+1} + \phi_{i}^{n+1} (\phi_{i+1}^{n+1} - \phi_i^{n+1}) \\
&+ \varepsilon^2 \phi_{i+1}^{n+1} - \phi_{i+1}^{n+1} (\phi_{i+1}^{n+1} - \phi_{i+1}^{n}) - \varepsilon^2 \phi_{i+1}^{n+1} - \phi_{i}^{n+1} (\phi_{i+1}^{n+1} - \phi_i^{n+1}) \\
&- \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 + \xi^n \sum_{i=1}^{N_x} F'(\phi_i^{n+1})(\phi_i^{n+1} - \phi^n_i) \\
&= - \sum_{i=1}^{N_x} \varepsilon^2 \xi^n \epsilon_i^{n+1} (\phi_i^{n+1} - \phi^n_i) - \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&+ \xi^n \sum_{i=1}^{N_x} F'(\phi_i^{n+1})(\phi_i^{n+1} - \phi^n_i) \\
&= - \sum_{i=1}^{N_x} \frac{\Delta t}{\Delta x} \left( J_{i+\frac{1}{2}}^{n+1} - J_{i-\frac{1}{2}}^{n+1} \right) \mu_i^{n+1} - \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&= - \sum_{i=1}^{N_x} \frac{\Delta t}{\Delta x} \left( \mu_i^{n+1} - \mu_i^{n+1} \right) - \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&= - \Delta t \sum_{i=1}^{N_x-1} J_{i+\frac{1}{2}}^{n+1} V_{i+\frac{1}{2}}^{n+1} - \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x-1} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&= - \Delta t \sum_{i=1}^{N_x-1} \left[ \left( V_{i+\frac{1}{2}}^{n+1} \right) M(\phi_i^{n+1}, \phi_{i+1}^{n+1}) \right. \\
&\left. \left( V_{i+\frac{1}{2}}^{n+1} \right) - M(\phi_i^{n+1}, \phi_{i+1}^{n+1}) \right] V_{i+\frac{1}{2}}^{n+1} \\
&- \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x-1} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&\leq - \Delta t \sum_{i=1}^{N_x-1} \min \left\{ M(\phi_i^{n+1}, \phi_{i+1}^{n+1}), M(\phi_i^{n+1}, \phi_{i+1}^{n+1}) \right\} \left| V_{i+\frac{1}{2}}^{n+1} \right|^2 - \frac{\varepsilon^2}{2} \sum_{i=1}^{N_x-1} \left| (\nabla \phi)^{n+1}_{i+\frac{1}{2}} - (\nabla \phi)^{n}_{i+\frac{1}{2}} \right|^2 \\
&\leq 0.
\end{align*}

Therefore, the theorem has been proved. \hfill \Box

Theorem 2.1 shows that the key to establishing the boundedness of numerical solution is to use the upwind idea to deal with the mass flux numerically, and its bound is only determined by the zero point of the degenerate mobility function. In addition, SAV approach is still used to ensure the stability of energy, while the conservation of mass, consistent with most methods, comes from the implicit discretization of non-flux boundary condition.

Furthermore, the above scheme (2.6)–(2.13) in one-dimensional is easy to be extended to the case of high-dimensional space, while still satisfying the properties of boundedness, mass conservation and energy dissipation. For the sake of saving computational effort, however, we will utilize the dimensional-splitting technique in dealing with high-dimensional problems in the next section considering the excellent performance of this technique.
3. Dimensional-splitting technique

In this section, the upwind-SAV approach established above will be combined with the dimensional-splitting technique \cite{67, 68} and applied to the discretization of high-dimensional space. Specifically, we begin with dividing the computational domain \( \Omega \) uniformly into \( N_x \times N_y \) cells \( C_{i,j} = [x_i-\frac{1}{2}, x_i+\frac{1}{2}] \times [y_j-\frac{1}{2}, y_j+\frac{1}{2}] \), while the spatial steps are denoted as \( \Delta x \) and \( \Delta y \) respectively. In each cell \( C_{i,j} \), the corresponding cell average \( \phi_{i,j} \) is defined as
\[
\phi_{i,j}(t) = \frac{1}{\Delta x \Delta y} \int_{C_{i,j}} \phi(x, y, t) \, dx \, dy.
\] (3.1)

Next, according to the dimensional-splitting technique, at each step \( n \) in the outer loop, we first update \( \phi_{i,j} \) along \( x \)-direction for each fixed \( y_j \) one by one, and then solve the systems along \( y \)-direction for every fixed \( x_i \). More precisely, we can use \( \tilde{\phi}_{i,j}^{n,q} \) to stands for the solution at the \( q \)-th update in the first inner loop while \( q (= 1, 2, \cdots, N_y) \) denotes the index of the fixed \( y_j \) in this loop. Similarly, the solution at the \( p \)-th update in the second inner loop can be written as \( \tilde{\phi}_{i,j}^{n,p} \) while \( p = 1, 2, \cdots, N_x \). Based on these notations, the initial condition of the first inner loop can be expressed as \( \tilde{\phi}_{i,j}^{n,q} |_{q=0} = \phi_{i,j}^0 \) and the dimensional-splitting technique can be formulated as:

**Step 1. for \( q = 1, 2, \cdots, N_y \) do:**
\[
\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1} = \begin{cases} 
-\frac{\Delta t}{\Delta x} \left( \tilde{J}_{i+\frac{1}{2},j}^{n,q} - \tilde{J}_{i-\frac{1}{2},j}^{n,q} \right), & \text{if } j = q; \\
0, & \text{otherwise},
\end{cases}
\] (3.2)
\[
\tilde{J}_{i+\frac{1}{2},j}^{n,q} = \left( \tilde{\varphi}_{i+\frac{1}{2},j}^{n,q} \right)^+ M(\tilde{\varphi}_{i,j}^{n,q}, \tilde{\varphi}_{i+1,j}^{n,q}) + \left( \tilde{\varphi}_{i+\frac{1}{2},j}^{n,q} \right)^- M(\tilde{\varphi}_{i+1,j}^{n,q}, \tilde{\varphi}_{i,j}^{n,q}),
\] (3.3)
\[
\tilde{\varphi}_{i+\frac{1}{2},j}^{n,q} = -\frac{1}{\Delta x} (\tilde{\mu}_{i+1,j}^{n,q} - \tilde{\mu}_{i,j}^{n,q}),
\] (3.4)
\[
\tilde{\mu}_{i,j}^{n,q} = -\varepsilon^2 (\Delta \tilde{\phi}_{i,j}^{n,q}) + \tilde{\epsilon}_{i,j}^{n,q} F'(\tilde{\phi}_{i,j}^{n,q}),
\] (3.5)
\[
\sum_{j=1}^{N_y} \sum_{i=1}^{N_x} \left( F'(\tilde{\phi}_{i,j}^{n,q}) - F'(\tilde{\phi}_{i,j}^{n,q-1}) \right) \right) = \xi_{i,j}^{n,q} \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} F'(\tilde{\phi}_{i,j}^{n,q})(\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1}),
\] (3.6)

where the Laplacian term \((\Delta \tilde{\phi}_{i,j}^{n,q})\) is discretized by the following central difference formula in the domain
\[
(\Delta \tilde{\phi}_{i,j}^{n,q}) = \frac{\tilde{\phi}_{i+1,j}^{n,q} - 2\tilde{\phi}_{i,j}^{n,q} + \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j+1}^{n,q} - 2\tilde{\phi}_{i,j}^{n,q} + \tilde{\phi}_{i,j-1}^{n,q}}{\Delta y^2},
\] (3.7)
and at the boundaries are discretized as
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{1,1} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\]
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{N_x,1} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\]
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{1,N_y} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\]
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{N_x,N_y} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\]
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{2,j} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\] (3.8)
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{j,2} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\] for \( j = 2, 3, \cdots, N_y - 1, \)
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{N_x-1,j} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\] for \( j = 2, 3, \cdots, N_y - 1, \)
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{i,2} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\] for \( i = 2, 3, \cdots, N_x - 1, \)
\[
(\Delta \tilde{\phi}_{i,j}^{n,q})_{i,N_y-1} = \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x^2} + \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y^2},
\] for \( i = 2, 3, \cdots, N_x - 1, \)
Moreover, the no-flux boundary conditions for mass flux are implemented as
\[
\tilde{J}^{n,q}_{i,j} = 0, \quad \tilde{J}^{n,q}_{N_+ \pm \frac{1}{2}, j} = 0, \quad j = 1, 2, \cdots, N_y.
\] (3.9)

Once the above first inner loop is completed, the cell average for each of the \(y\)-direction is continued. The initial condition for second inner loop is taken as \(\tilde{\phi}^{n,p}_{i,j}|_{p=0} = \tilde{\phi}^{n,q}_{i,j}|_{q=N_y}\) and the scheme for each \(y\)-direction satisfies:

**Step 2. for \(p = 1, 2, \cdots, N_\chi\) do**:
\[
\begin{align*}
\tilde{\phi}^{n,p}_{i,j} - \tilde{\phi}^{n,p-1}_{i,j} &= \begin{cases} 
-\frac{\Delta y}{\Delta x} \left( \tilde{J}^{n,p}_{i,j+\frac{1}{2}} - \tilde{J}^{n,p}_{i,j-\frac{1}{2}} \right), & \text{if } i = p; \\
0, & \text{otherwise},
\end{cases} \\
\tilde{\phi}^{n,p}_{i,j+\frac{1}{2}} &= \left( \tilde{\phi}^{n,p}_{i,j+1} + \tilde{\phi}^{n,p}_{i,j} \right) + \left( \tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i,j-1} \right), \quad \text{if } i = 1, \quad (3.10) \\
\tilde{\phi}^{n,p}_{i,j} &= \frac{1}{\Delta y} \left( \tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j} \right), \quad \text{if } i = N_\chi, \quad (3.11) \\
\tilde{\phi}^{n,p}_{i,j} &= -\varepsilon^2 (\Delta \tilde{\phi})^{n,p}_{i,j} + \tilde{\phi}^{n,p} F'(\tilde{\phi}^{n,p}_{i,j}), \quad \text{if } i = 1, \quad (3.12) \\
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( F'(\tilde{\phi}^{n,p}_{i,j}) - F'(\tilde{\phi}^{n,p-1}_{i,j}) \right) &= \tilde{\phi}^{n,p}_{i,j} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} F'(\tilde{\phi}^{n,p}_{i,j})(\tilde{\phi}^{n,p}_{i,j} - \tilde{\phi}^{n,p-1}_{i,j}), \quad (3.13)
\end{align*}
\]

where the Laplacian term \((\Delta \tilde{\phi})^{n,p}_{i,j}\) is discretized by the following central difference formula in the domain
\[
(\Delta \tilde{\phi})^{n,p}_{i,j} = \frac{\tilde{\phi}^{n,p}_{i+1,j} - 2\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i-1,j}}{\Delta x^2} + \frac{\tilde{\phi}^{n,p}_{i,j+1} - 2\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i,j-1}}{\Delta y^2}, \quad (3.15)
\]

and at the boundaries are discretized as
\[
\begin{align*}
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{\tilde{\phi}^{n,p}_{i+1,j} - \tilde{\phi}^{n,p}_{i,j}}{\Delta x^2} + \frac{\tilde{\phi}^{n,p}_{i,j} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta y^2}, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{-\tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta x^2} + \frac{\tilde{\phi}^{n,p}_{i,j} - \tilde{\phi}^{n,p}_{i-1,j}}{\Delta y^2}, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{\tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta x^2} + \frac{-\tilde{\phi}^{n,p}_{i,j} - \tilde{\phi}^{n,p}_{i+1,j}}{\Delta y^2}, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{-\tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta x^2} + \frac{-\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i+1,j}}{\Delta y^2}, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{-\tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta x^2} + \frac{-\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i-1,j}}{\Delta y^2}, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{-\tilde{\phi}^{n,p}_{i,j+1} - \tilde{\phi}^{n,p}_{i,j-1}}{\Delta x^2} + \frac{-\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i+1,j}}{\Delta y^2}, \quad j = 2, 3, \cdots, N_y - 1, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{\tilde{\phi}^{n,p}_{i+1,j+1} - 2\tilde{\phi}^{n,p}_{i+1,j} + \tilde{\phi}^{n,p}_{i+1,j-1}}{\Delta x^2}, \quad j = 2, 3, \cdots, N_y - 1, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{\tilde{\phi}^{n,p}_{i,j+1} - 2\tilde{\phi}^{n,p}_{i,j} + \tilde{\phi}^{n,p}_{i,j-1}}{\Delta y^2}, \quad i = 2, 3, \cdots, N_x - 1, \\
(\Delta \tilde{\phi})^{n,p}_{i,j} &= \frac{\tilde{\phi}^{n,p}_{i+1,j+1} - 2\tilde{\phi}^{n,p}_{i+1,j} + \tilde{\phi}^{n,p}_{i+1,j-1}}{\Delta y^2}, \quad i = 2, 3, \cdots, N_x - 1.
\end{align*}
\] (3.16)

Moreover, the no-flux boundary conditions for mass flux are also implemented as
\[
\tilde{J}^{n,p}_{i,j} = 0, \quad \tilde{J}^{n,p}_{i,N_y \pm \frac{1}{2}} = 0, \quad i = 1, 2, \cdots, N_x.
\] (3.17)

Finally, when the above second inner loops are completed, the cell average and the Lagrange multiplier at the \((n + 1)\)-th step can be obtained as follows
\[
\begin{align*}
\tilde{\phi}^{n+1,p}_{i,j} &= \tilde{\phi}^{n,p}_{i,j}|_{p=N_\chi} \quad \text{and} \quad \xi^{n+1} = \frac{1}{N_y + N_\chi} \left( \sum_{q=1}^{N_y} \tilde{\phi}^{n,q}_{i,j} + \sum_{p=1}^{N_\chi} \tilde{\phi}^{n,p}_{i,j} \right).
\end{align*}
\] (3.18)
In fact, for the dimensional-splitting technique, when updating from the \(n\)-th step to the \((n+1)\)-th step, the Lagrange multiplier \(\xi^{n+1}\) has no explicit value, and the average is chosen here in order to utilize all the data obtained from one iteration in time.

Excitingly, the above dimensional-splitting scheme in 2D still satisfies the properties of bound-preserving, mass conservation and energy dissipation, and the rigorous proof are given below.

**Theorem 3.1.** (Boundedness) The dimensional-splitting scheme (3.2)\textsuperscript{1} - (3.18) can ensure the boundedness of the phase average \(\phi_{i,j}\). That is, for \(\forall i,j\), if \(|\phi_{i,j}| < 1\), then \(|\phi_{i,j}^{n+1}| < 1\).

**Proof.** Since \(|\tilde{\phi}_{i,j}^{n,q}|_{q=0} = |\phi_{i,j}^{n}| < 1\), it is easy to prove that \(|\tilde{\phi}_{i,j}^{n,q}| < 1\) for \(\forall q\) by following the contradiction strategy employed for one-dimension case in Theorem 2.1. Similarly, \(|\tilde{\phi}_{i,j}^{n,p}| < 1\) for \(\forall p\) can be obtained from \(|\tilde{\phi}_{i,j}^{n,p}|_{p=0} = |\phi_{i,j}^{n}| < 1\), and thus \(|\phi_{i,j}^{n+1}| = |\tilde{\phi}_{i,j}^{n,p}|_{p=N_y} < 1\).

\(\square\)

**Theorem 3.2.** (Mass conservation) The dimensional-splitting scheme (3.2)\textsuperscript{1} - (3.18) ensures that the total mass is conserved during the evolution, i.e.

\[
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{n+1} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{n} = \cdots = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{0}. \tag{3.19}
\]

**Proof.** Sum the two ends of (3.2) and (3.10) over all cells \(C_{i,j}\), respectively, it obtains

\[
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1}) = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{N_x} \left( \tilde{J}_{i+\frac{1}{2},j}^{n,q} - \tilde{J}_{i-\frac{1}{2},j}^{n,q} \right) = -\frac{\Delta t}{\Delta y} \left( \tilde{J}_{i,j+\frac{1}{2},q}^{n,q} - \tilde{J}_{i,j-\frac{1}{2},q}^{n,q} \right) = 0, \quad \forall q, \tag{3.20}
\]

\[
\sum_{j=1}^{N_y} \sum_{i=1}^{N_x} (\tilde{\phi}_{i,j}^{n,p} - \tilde{\phi}_{i,j}^{n,p-1}) = -\frac{\Delta t}{\Delta y} \sum_{j=1}^{N_y} \left( \tilde{J}_{i,j+\frac{1}{2},p}^{n,p} - \tilde{J}_{i,j-\frac{1}{2},p}^{n,p} \right) = -\frac{\Delta t}{\Delta y} \left( \tilde{J}_{i,N_y+\frac{1}{2},q}^{n,p} - \tilde{J}_{i,p}^{n,p} \right) = 0, \quad \forall p. \tag{3.21}
\]

Therefore,

\[
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{n+1} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{n} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^{0}, \tag{3.22}
\]

where the initial conditions \(\tilde{\phi}_{i,j}^{n,p}_{|p=0} = \tilde{\phi}_{i,j}^{n,q}_{|q=N_y}\) and \(\tilde{\phi}_{i,j}^{n,p}_{|q=0} = \phi_{i,j}^{n}\) are applied. \(\square\)

**Theorem 3.3.** (Energy dissipation) The dimensional-splitting scheme (3.2)\textsuperscript{1} - (3.18) is unconditionally energy stable, and satisfies the following discrete energy dissipation law:

\[
\mathcal{E}^{n+1} - \mathcal{E}^{n} \leq -\Delta t \Delta x \Delta y \sum_{p=1}^{N_p-1} \sum_{j=1}^{N_y} \min \left\{ M(\tilde{\phi}_{p,j+1}^{n,q}, \tilde{\phi}_{p,j}^{n,p}), M(\tilde{\phi}_{p,j}^{n,p}, \tilde{\phi}_{p,j-1}^{n,p}) \right\} \left| \tilde{V}_{p,j+\frac{1}{2}}^{n,q} \right|^2
\]

\[
-\Delta t \Delta x \Delta y \sum_{q=1}^{N_q-1} \sum_{i=1}^{N_x} \min \left\{ M(\tilde{\phi}_{i+1,q}^{n,q}, \tilde{\phi}_{i,q+1}^{n,q}), M(\tilde{\phi}_{i,q+1}^{n,q}, \tilde{\phi}_{i,q-1}^{n,q}) \right\} \left| \tilde{V}_{i+\frac{1}{2},q}^{n,q} \right|^2 \leq 0, \tag{3.23}
\]
where

\[ \mathcal{E}^n = \Delta x \Delta y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varepsilon^2 \left( \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \right)^2 + \Delta x \Delta y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varepsilon^2 \left( \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} \right)^2 + \Delta x \Delta y \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} F(\phi_{i,j}^n). \] (3.24)

**Proof.** First prove that the energy dissipation in each \( x \)-direction iteration, namely,

\[ \tilde{\mathcal{E}}_{n,q} - \tilde{\mathcal{E}}_{n,q-1} \leq -\Delta t \Delta x \Delta y \sum_{i=1}^{N_x-1} \min \left\{ M(\tilde{\phi}_{i,q}^{n,q}, \tilde{\phi}_{i+1,q}^{n,q}), M(\tilde{\phi}_{i+1,q}^{n,q}, \tilde{\phi}_{i,q}^{n,q}) \right\} \left| \tilde{V}_{i+\frac{1}{2},q}^{n,q} \right| ^2 \leq 0, \] (3.25)

where

\[ \tilde{\mathcal{E}}_{n,q} = \Delta x \Delta y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varepsilon^2 \left( \frac{\tilde{\phi}_{i+1,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x} \right)^2 + \Delta x \Delta y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varepsilon^2 \left( \frac{\tilde{\phi}_{i,j+1}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y} \right)^2 + \Delta x \Delta y \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \tilde{F}(\tilde{\phi}_{i,j}^{n,q}). \] (3.26)

Directly subtract the first term on the right side of the discrete energy [3.20] at subsequent times, it obtains

\[
\begin{align*}
\varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} & \left[ \left( \frac{\tilde{\phi}_{i+1,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta x} \right)^2 - \left( \frac{\tilde{\phi}_{i+1,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta x} \right)^2 \right] \\
= & \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left[ \left( \frac{\tilde{\phi}_{i+1,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta x} \right) - \left( \frac{\tilde{\phi}_{i+1,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta x} \right) \right] \\
& - \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left[ \left( \frac{\tilde{\phi}_{i+1,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta x} \right) - \left( \frac{\tilde{\phi}_{i+1,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta x} \right) \right] \\
= & \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1} \right) - \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right) \\
& - \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right) \\
= & -\varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1} \right) - \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right) \\
& - \varepsilon^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta x^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right),
\end{align*}
\]

(3.27)

Similarly, it can be obtained

\[
\begin{align*}
\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} & \left[ \left( \frac{\tilde{\phi}_{i,j+1}^{n,q} - \tilde{\phi}_{i,j}^{n,q}}{\Delta y} \right)^2 - \left( \frac{\tilde{\phi}_{i,j+1}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1}}{\Delta y} \right)^2 \right] \\
= & -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta y^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i,j}^{n,q-1} \right) \\
& - \varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta y^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right) \\
& - \varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\tilde{\phi}_{i,j}^{n,q} - \tilde{\phi}_{i-1,j}^{n,q}}{\Delta y^2} \cdot \left( \tilde{\phi}_{i,j}^{n,q-1} - \tilde{\phi}_{i,j}^{n,q-1} \right),
\end{align*}
\]

(3.28)
\[
\begin{align*}
&+ \varepsilon^2 \sum_{i=1}^{N_x} \left( \frac{\hat{\phi}^{n,q}_{i,N_y} - \hat{\phi}^{n-1,q}_{i,N_y}}{\Delta y^2} \right) \cdot \left( \hat{\phi}^{n,q}_{i,N_y} - \hat{\phi}^{n-1,q}_{i,N_y} \right) \cdot \left( \hat{\phi}^{n,q+1}_{i+1,N_y} - \hat{\phi}^{n,q}_{i+1,N_y} \right) \cdot \left( \hat{\phi}^{n,q+1}_{i+1,N_y} - \hat{\phi}^{n,q-1}_{i+1,N_y} \right) \\
&- \varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-1} \left[ \frac{\hat{\phi}^{n,q}_{i,j+1} - \hat{\phi}^{n,q}_{i,j}}{\Delta y} \right] \cdot \left( \hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q}_{i,j+1} \right) \cdot \left( \hat{\phi}^{n,q}_{i,j+1} - \hat{\phi}^{n,q-1}_{i,j+1} \right) \cdot \left( \hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q-1}_{i,j+1} \right) \cdot \left( \hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q-1}_{i,j+1} \right).
\end{align*}
\]

Therefore,
\[
\frac{\varepsilon^2}{\Delta x \Delta y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ \left( \frac{\hat{\phi}^{n,q}_{i,j+1} - \hat{\phi}^{n,q}_{i,j}}{\Delta y} \right)^2 - \left( \frac{\hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q}_{i,j+1}}{\Delta y} \right)^2 \right] \\
+ \frac{\varepsilon^2}{\Delta x \Delta y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ \left( \frac{\hat{\phi}^{n,q}_{i,j+1} - \hat{\phi}^{n,q}_{i,j}}{\Delta y} \right)^2 - \left( \frac{\hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q}_{i,j+1}}{\Delta y} \right)^2 \right] \\
+ \frac{\varepsilon^2}{\Delta x \Delta y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ \left( \frac{\hat{\phi}^{n,q}_{i,j+1} - \hat{\phi}^{n,q}_{i,j}}{\Delta y} \right)^2 - \left( \frac{\hat{\phi}^{n,q+1}_{i,j+1} - \hat{\phi}^{n,q}_{i,j+1}}{\Delta y} \right)^2 \right] \\
+ \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ F(\hat{\phi}^{n,q}_{i,j}) - F(\hat{\phi}^{n,q+1}_{i,j}) \right] \\
\leq -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} = -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} \\
= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} = -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} \\
= -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} = -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} \\
= -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} = -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} \\
\leq -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} = -\varepsilon^2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( \Delta \hat{\phi}^{n,q}_{i,j} - \hat{\phi}^{n,q+1}_{i,j} \right) \hat{\phi}^{n,q}_{i,j} \\
\leq 0.
\end{align*}
\]
\[ \begin{align*}
&\leq -\Delta t \Delta x \Delta y \sum_{p=1}^{N_{x}} \sum_{j=1}^{N_{y}-1} \min \left\{ M(\hat{\phi}_{p,j}^{n,p}, \tilde{\phi}_{p,j+1}^{n,p}), M(\hat{\phi}_{p,j+1}^{n,p}, \tilde{\phi}_{p,j}^{n,p}) \right\} \left| \tilde{V}_{p,j+\frac{1}{2}}^{n,p} \right|^{2} \\
&- \Delta t \Delta x \Delta y \sum_{q=1}^{N_{x}} \sum_{i=1}^{N_{y}-1} \min \left\{ M(\tilde{\phi}_{i,q}^{n,q}, \hat{\phi}_{i+1,q}^{n,q}), M(\hat{\phi}_{i+1,q}^{n,q}, \tilde{\phi}_{i,q}^{n,q}) \right\} \left| \tilde{V}_{i+\frac{1}{2},q}^{n,q} \right|^{2} \\
&\leq 0,
\end{align*} \] 

(3.32)

Therefore, the proof is completed. \qed

The above theorems shows that the dimensional-splitting technique can effectively decouple the multi-dimensional discrete problem into a series of one-dimensional discrete problems while preserving original structural properties. Since the upwind-SAV approach leads to an implicit scheme at each time step, and the main computational effort of the usual nonlinear equation solvers (such as the commonly used Newton–Raphson method and trust region method) is to find the inverse of the Jacobian matrix, a quantitative estimate of the computational complexity saved by the dimensional-splitting technique can be obtained from the aspect of matrix inversion. Consider an \( N \times N \) full matrix, the computational complexity of finding its inverse is \( O(N^3) \) for \( 2 < \beta \leq 3 \) \cite{24, 53}. Therefore, by decoupling a \( d \)-dimensional problem (the size of the Jacobian matrix is \( N^d \times N^d \)) into \( dN^{d-1} \) one-dimensional problems via dimensional-splitting technique, the computational complexity will decrease from \( O(N^{d^2}) \) to \( O(dN^{\beta+d-1}) \), which saves a huge computational cost.

It is also straightforward to develop the scheme in higher dimensions through the dimensional-splitting technique, which are omitted here for brevity.

4. Numerical results

In this section, several numerical examples will be presented to verify the structure-preserving property of the proposed scheme, and to check that the dynamic process of degenerate Cahn–Hilliard equation with Flory–Huggins potential at low temperature is surface diffusion.

Unless otherwise specified, the time step \( \Delta t \), the spatial step \( \Delta x (= \Delta y) \) and the interface parameter \( \varepsilon \) are set as \( 10^{-4} \), 0.004 and 0.02 respectively. The initial value of the phase variable is chosen be

\[ \phi(x, t)|_{t=0} = \lambda \tanh \left( \frac{\text{dist}(x, \Gamma)}{\sqrt{2} \varepsilon} \right), \quad x \in \Omega, \]  

(4.1)

where \( \Gamma \) represents some curve/surface in the domain \( \Omega \), \( \text{dist}(x, \Gamma) \) denotes the signed distance from point \( x \) to \( \Gamma \) and \( \lambda \) is set as \( 1 - 10^{-4} \) to ensure that the initial value of the phase variable does not exceed 1 (\( ||\phi||_{\infty} < 1 \)). The critical temperature \( \theta_{c} \) in the logarithmic Flory–Huggins potential \( V_{F} \) is chosen to be 1 and we select \( M(\phi) = 1 - \phi^2 \) as the phase-dependent mobility function in numerical simulation. For the nonlinear equations appeared in our scheme, we shall perform a Newton-type iteration to solve them at each time step.

4.1. Boundedness and energy stability

Following the example given in \cite{62}, the first initial condition we consider is the random form defined in \( \Omega = [0, 1]^2 \) as

\[ \phi(x, y, t)|_{t=0} = 0.2 + 0.05 \times \text{Rand}(x, y) \]  

(4.2)

where \( \text{Rand}(x, y) \) represents the uniform random distribution in \([-1, 1]\).

Fig. 1 shows several snapshots of the solution under different absolute temperatures \( \theta \), from which it can be clearly observed that a smaller \( \theta \) leads to a faster initial growth rate. Furthermore, the coarsening processes and results are also significantly different. The boundedness of the proposed scheme is verified in Fig. 2 in which its maximum (or minimum) approaches 1 (or \(-1\)) as \( \theta \to 0 \). Next, Fig. 3 confirms the principle of mass conservation and energy dissipation. Finally, we can see from Fig. 4 that the value
Figure 1: Several contour plots of numerical solutions with three different absolute temperatures $\theta = 0.25$ (first column), $\theta = 0.30$ (second column) and $\theta = 0.50$ (third column) under initial condition (4.2), respectively.
Figure 2: The evolution of $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$ with different $\theta$ under initial condition (4.2), where (a) $\theta = 0.25$, (b) $\theta = 0.30$ and (c) $\theta = 0.50$.

Figure 3: The evolution of the normalized energy and mass with different $\theta$ under initial condition (4.2), where (a) $\theta = 0.25$, (b) $\theta = 0.30$ and (c) $\theta = 0.50$.

Figure 4: The evolution of the Lagrange multiplier $\xi(t)$ with different $\theta$ under initial condition (4.2), where (a) $\theta = 0.25$, (b) $\theta = 0.30$ and (c) $\theta = 0.50$. 
of Lagrange multiplier $\xi(t)$ is always near 1 during the evolution, which is consistent with the theoretical expectation.

To further examine the effect of time step $\Delta t$ on the evolution morphology, boundedness and energy stability of numerical solutions, we select following four-leaved closed curve \[84, 85\] at the center of $\Omega = [0, 1]^2$ as initial condition,

$$\rho = \frac{2 + \cos(4\alpha)}{8}, \quad \text{where} \quad \rho = \sqrt{x^2 + y^2} \quad \text{and} \quad \alpha = \arctan \frac{y}{x},$$  

(4.3)

and fix $\theta = 0.3$ in logarithmic Flory–Huggins potential.

Figure 5: Several snapshots of simulating a four-leaved closed curve (4.3) with different time step $\Delta t$. 

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The interface evolution of the initial condition with different time step $\Delta t$ is shown in Fig. 5. We can see that the evolution would become slower with a larger time step, while it always converges to a perfect circle as equilibrium state. Furthermore, the corresponding evolution of the maximum value of phase variable (the minimum value has a similar result), normalized energy and mass, and Lagrange multiplier are given in Fig. 6, which numerically verifies that the boundedness, mass conservation and energy stability of our scheme are unconditionally satisfied. However, when time step is large, the evolution curve is obviously different from the smaller time step case (e.g., $\Delta t \leq 10^{-4}$), which indicates that in practical a smaller time step can avoid a slow evolution of the solution. Finally, we give the area change rate $\delta S = \frac{S(t) - S(0)}{S(0)}$ at $t = 1$ in Table 1, from which we can find that smaller time step results in a smaller enclosed area loss.

### Table 1: The area change rate $\delta S$ at $t = 1$ under initial condition

| $\Delta t$     | $\delta S$       |
|----------------|------------------|
| $10^{-3}$      | $-0.2996\%$      |
| $5 \times 10^{-4}$ | $-0.2395\%$      |
| $10^{-4}$      | $-0.1625\%$      |
| $5 \times 10^{-5}$ | $-0.1388\%$      |
| $10^{-5}$      | $-0.1158\%$      |

#### 4.2. Surface diffusion

The focus of this subsection is to numerically verify that the sharp-interface limit of the Cahn–Hilliard equation with degenerate mobility and logarithmic Flory–Huggins potential at low temperature (i.e., the absolute temperature $\theta$ is small enough) is surface diffusion. To begin with, we first consider the evolution of two isolated circles with different sizes under different potential, one of which has center $(x_1, y_1) = (0.4, 0.4)$ with radius $r_1 = 0.2$, and the other one has center $(x_2, y_2) = (0.75, 0.75)$ with radius $r_2 = 0.1$. Fig. 7 represents several snapshots of solutions under logarithmic potential $F_{\log}(\phi)$ with different absolute temperatures $\theta$ and solutions under polynomial potential $F_{pol}(\phi)$. It can be clearly observed that the process caused by logarithmic potential at large absolute temperatures $\theta$ and polynomial potential violates the main characteristics of surface diffusion, which is consistent with the theoretical results [26, 27, 29]. On the other hand, when the absolute temperature $\theta$ is small enough (e.g., $\theta \leq 0.30$ in this simulation), the kinetic process accords well with the surface diffusion. Furthermore, the total area change rate $\delta S$ at $t = 2$ is also examined. As shown in Table 2, the area changes very little when $\theta \leq 0.30$, which again verifies that a sufficiently low absolute temperature can capture the main feature of surface diffusion. Finally, the evolution of maximum/minimum value of phase variable, normalized energy and mass, and Lagrange multiplier under several potential are plotted in Figs. 8-10, respectively, which are consistent with theoretical expectation.

As the second example, we simply replace the larger circle in the previous example by an ellipse with the major semi axis $r_a = \sqrt{2}$ and the minor semi axis $r_b = \sqrt{10}$ (that is, keeping their initial areas equal), leaving...
Figure 7: Several snapshots of simulating isolated large and small circles with different potential energy densities.

Table 2: The total area change rate $\delta S$ at $t = 2$ under initial two different size circles.

|                 | $F_{\text{log}}(\phi; \theta = 0.15)$ | $F_{\text{log}}(\phi; \theta = 0.30)$ | $F_{\text{log}}(\phi; \theta = 0.45)$ | $F_{\text{log}}(\phi; \theta = 0.60)$ | $F_{\text{pol}}(\phi)$ |
|----------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|--------------------------|
| $\delta S$     | 0.1397%                                 | -0.2829%                               | -11.1564%                              | -27.2918%                              | -8.2058%                 |

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Figure 8: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under logarithmic potential $F_{\text{log}}(\phi)$ at $\theta = 0.15$ in Fig. 7.

Figure 9: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under logarithmic potential $F_{\text{log}}(\phi)$ at $\theta = 0.45$ in Fig. 7.

Figure 10: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under polynomial potential $F_{\text{pol}}(\phi)$ in Fig. 7.
the other conditions unchanged, to test the influence of the ellipse on small circle during the evolution process. The results of Fig. 11 illustrate that when $\theta \leq 0.30$, the ellipse gradually evolves into a circle and the small circle remains stable, while in other case the small circle is completely absorbed. Furthermore, in Table 3, we quantitatively examine the total area change rate $\delta S$ at $t = 3$. It can be seen that the change of total area is rather small when $\theta \leq 0.30$, which again verifies that the kinetic process of logarithmic potential at low temperature conforms to the geometric characteristics of surface diffusion. Finally, Figs. 12-14 present the evolution of related parameters (e.g., maximum/minimum value of phase variable, normalized energy and mass, Lagrange multiplier) under different potentials, which again numerically illustrate the bound-preserving, mass-conservation and energy-stable properties of the proposed scheme.

Figure 11: Several snapshots of simulating isolated ellipse and small circle with different potential energy densities.
Table 3: The total area change rate $\delta S$ at $t = 3$ under initial an ellipse and a small circle.

| $F_{\log}(\phi; \theta)$ | $F_{\log}(\phi; \theta = 0.3)$ | $F_{\log}(\phi; \theta = 0.45)$ | $F_{\log}(\phi; \theta = 0.6)$ | $F_{\text{pol}}(\phi)$ |
|--------------------------|-------------------------------|-------------------------------|-------------------------------|--------------------------|
| $\delta S$               | $0.7401\%$                   | $-0.9826\%$                  | $-7.2635\%$                  | $-28.0316\%$             | $-6.6129\%$             |

Figure 12: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under logarithmic potential $F_{\log}(\phi)$ at $\theta = 0.15$ in Fig. 11.

Figure 13: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under logarithmic potential $F_{\log}(\phi)$ at $\theta = 0.45$ in Fig. 11.

Figure 14: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under polynomial potential $F_{\text{pol}}(\phi)$ in Fig. 11.
At last, since the pinch-off dynamics is also an important phenomenon in the interface evolution problem, we choose to consider a long rectangle with aspect ratio being 20 as the initial shape. Fig. 15 depicts the evolution under the logarithmic potential $F_{\text{log}}(\phi)$ at $\theta = 0.2$ and polynomial potential $F_{\text{pol}}(\phi)$. For polynomial potential $F_{\text{pol}}(\phi)$, Fig. 15 shows that the pinch-off occurs at $t = 1.066$ and the long rectangle splits into three independent closed curves. After that, the middle smaller closed curve is gradually absorbed until it eventually disappears at $t = 4.791$. The total area change rate $\delta S$ equals to $-8.3825\%$ at $t = 6$. Obviously, this process is inconsistent with the property of surface diffusion. As for the case under logarithmic potential $F_{\text{log}}(\phi)$ at $\theta = 0.2$, the long rectangle splits into two isolated closed curves at $t = 2.852$, and finally evolves into two perfect circles, while the total area change rate $\delta S$ at $t = 6$ is only $0.7804\%$. These results demonstrate that the logarithmic potential at low temperatures with a degenerate mobility can well simulate the pinch-off dynamics of surface diffusion. Moreover, for polynomial potential, it is interesting that when the middle small closed curve disappears completely, the Lagrange multiplier will have a relatively bigger deviation from 1, which may be caused by the rapid decrease of energy when phase transition occurs, so the Lagrange multiplier makes a larger correction to the numerical results of fully-implicit scheme (i.e., the Lagrange multiplier term is removed from the proposed scheme) to ensure the stability of energy.

Figure 15: Several snapshots of simulating a long rectangle with logarithmic potential $F_{\text{log}}(\phi)$ at $\theta = 0.2$ (left column) and polynomial potential $F_{\text{pol}}(\phi)$ (right column).
Figure 16: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under logarithmic potential $F_{\log}(\phi)$ at $\theta = 0.2$ in Fig. 15.

Figure 17: The evolution of (a) $\max_{i,j} \phi_{i,j}$ and $\min_{i,j} \phi_{i,j}$, (b) normalized energy and mass and (c) Lagrange multiplier $\xi(t)$ under polynomial potential $F_{pol}(\phi)$ in Fig. 15.

5. Conclusions

In this paper, the upwind-scheme and SAV approach are successfully combined to construct an unconditionally bound-preserving and energy-stable scheme for the Cahn–Hilliard equation with degenerate mobility. In particular, for a high-dimensional problem, the dimensional-splitting technique is introduced in the SAV framework for the first time to decouple it into a series of one-dimensional problems while maintaining the original structural properties, thereby reducing the computational complexity from $O(N^{d\beta})$ to $O(dN^{\beta+d-1})$, which greatly saves computational costs. Numerical results confirm the boundedness and energy stability of the proposed scheme for solving degenerate Cahn–Hilliard equations, and successfully capture the main features of surface diffusion numerically when the absolute temperature in the logarithmic Flory–Huggins potential is sufficiently low. In future work, we plan to generalize the upwind-SAV approach to other gradient flows with degenerate term, and the well-posedness and error estimates of this scheme will also be studied.

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