Slow dynamics in the 3–D gonihedric model

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Abstract

We study dynamical aspects of three-dimensional gonihedric spins by using Monte-Carlo methods. The interest of this family of models (parametrized by one self-avoidance parameter $\kappa$) lies in their capability to show remarkably slow dynamics and seemingly glassy behaviour below a certain temperature $T_g$ without the need of introducing disorder of any kind. We consider first a hamiltonian that takes into account only a four-spin term ($\kappa = 0$), where a first order phase transition is well established. By studying the relaxation properties at low temperatures we confirm that the model exhibits two distinct regimes. For $T_g < T < T_c$, with long lived metastability and a supercooled phase, the approach to equilibrium is well described by a stretched exponential. For $T < T_g$ the dynamics appears to be logarithmic. We provide an accurate determination of $T_g$. We also determine the evolution of particularly long lived configurations. Next, we consider the case $\kappa = 1$, where the plaquette term is absent and the gonihedric action consists in a ferromagnetic Ising with fine-tuned next-to-nearest neighbour interactions. This model exhibits a second order phase transition. The consideration of the relaxation time for configurations in the cold phase reveals the presence of slow dynamics and glassy behaviour for any $T < T_c$. Type II aging features are exhibited by this model.

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1 Introduction

Glassy systems are very common in nature, yet not quite well understood. Lattice models may serve as good candidates to describe some properties of these systems. In recent years some interest has been raised by remarkably simple Ising spin systems that originated from the consideration of a model of random surfaces in the context of string theory [1], [2]. The version of this model in a discretized space defines the so-called gonihedric spin model which consists, in general, of an Ising model with finely tuned nearest, next-to-nearest neighbour and plaquette interactions. The relation among the couplings of the hamiltonian depends also on the dimensionality of the system. The geometric origins of the model show up in a remarkable simple way of writing the energy of a given configuration: the surfaces corresponding to the interfaces between up and down spins are weighed by

\[ E = n_2 + \kappa n_4, \]

where \( n_2 \) is the number of edges of such an interface and \( n_4 \) is the number of four plaquettes that share a common link. The parameter \( \kappa \) can thus be interpreted as an indicator of the self-avoidance of the model. Notice that there is no microscopic surface tension.

Up to now there has been a considerable amount of numerical work on the three–dimensional case, to which we shall refer in the following and also some preliminary results in four–dimensional case [3]. In two dimensions, the model with \( \kappa = 0 \) is actually trivial (no phase transition [4]) but the solution with \( \kappa \neq 0 \) is unknown.

For the three dimensional case which is of our concern in this paper the Hamiltonian of the model takes the form

\[
H(\sigma) = -2\kappa \sum_{\vec{r},\vec{a}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{a}} + \frac{\kappa}{2} \sum_{\vec{r},\vec{a},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{a}+\vec{\beta}} - \frac{1}{2} \kappa \sum_{\vec{r},\vec{a},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{a}}\sigma_{\vec{r}+\vec{a}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}},
\]

(1)

where \( \vec{a} \) and \( \vec{\beta} \) are lattice unit vectors. The model is defined on an cubic lattice. The system exhibits a very high degree of symmetry due to the particular ratio of the couplings. This symmetry implies that there is no cost by the flipping of any plane of spins. This results in a highly degenerate ground state. Note in passing that this last feature is common in every glassy system.

Some numerical evidence regarding dynamical and equilibrium properties of the hamiltonian [1] has been accumulated in the past, particularly for \( \kappa = 0 \), but also for \( \kappa \neq 0 \). These studies revealed many interesting features and provided evidence that a glassy phase is present in the phase diagram [1]. First of all, it is known that the model has a first order phase transition at a temperature \( T_c \) where the solid to liquid transition is present [1]. There is also good evidence that a dynamical transition exists at \( T_g < T_c \) which seems to mark the onset of the glassy behaviour [1]. The existence of \( T_g \) and the study of the dynamical properties of the system above and below that value have been considered in [1]-[13]. Preliminary results for \( \kappa \neq 0 \) were given in [14] in what respects the equilibrium properties, and in [15] in what respects the dynamical properties of the system.

In section 2 we provide a qualitative view of the slow dynamics behaviour of the cold phase by considering the relaxation properties of the model and we estimate with good accuracy \( T_g \) by measuring the spin-spin autocorrelation functions. We show that there is a dramatic change in the behaviour of this function above and below \( T_g \). In section 3 we study a different case of the spin gonihedric action by taking \( \kappa = 1 \) in [1], apparently a much simpler system (the

\footnote{For the interesting case of the four–spin model where randomly distributed couplings are considered, see [16].}
plaquette term in the hamiltonian is absent for this value). For that value of the parameter $\kappa$ the system is described by nearest and next-to-nearest interactions. We confirm, by studying the energy susceptibility, that the system has a second order phase transition \footnote{Note that from some preliminary results referred in \cite{6} the first order transition present for $\kappa = 0$ gets weaker and possibly becomes second order at $\kappa \sim 0.5$.}. Furthermore we find slow dynamics behaviour anywhere in the cold phase (below the critical temperature). By studying spin–spin autocorrelation function and the overlap spin distribution function, we provide evidence that the system exhibits type II aging \cite{17}, which is a feature of the glassy systems.

The physical interest of this model is twofold. On the formal side, the model is of such simplicity that a theoretical understanding of the mechanisms underlying slow dynamics and glassy behaviour appears possible. On a more practical side, it would be extremely interesting to be able to understand and produce magnetic materials and coatings with such finely tuned (or approximately so) couplings. The extremely long relaxation times would make them very robust against thermal noise and fluctuations, yet encoding information there would be as simple as in a normal magnetic material. This possibility has been suggested in \cite{18}. While the plaquette term seems hard to imitate in real materials, the fact that many of the interesting features persist for $\kappa = 1$ makes perhaps such possibility less remote.

2 Four–spin interaction ($\kappa = 0$)

In this section we will study the case $\kappa = 0$, i.e, a spin model with only four–spin (plaquette) interaction. In this case the hamiltonian \footnote{\textsuperscript{**}This simulation refers to the system expressed in dual variables \cite{20} and it has been carried out using a cluster algorithm.} takes the form:

$$H = \frac{1}{2} \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} \sigma_{\vec{r}+\vec{\beta}}$$

(2)

This form of the interaction leads to a highly degenerated ground state. Flipping every spin in any plane of the cubic lattice implies invariant ground state energy. Taking this symmetry into account, the degeneracy of the ground state is equal to $2^{3L}$ due to the $3L$ different planes in a cubic lattice. This degeneracy survives even at $T \neq 0$.

It is well known that this model exhibits a first order phase transition at $T_c = 1.95$ along with a dynamical transition at $T_g \sim 1.7$, which is the temperature where the glassy behaviour shows up \cite{9}, \cite{10}, \cite{11}. Even though our main interest is the study of the glassy characteristics by looking at the relaxation as well as the autocorrelation of the order parameters (to be defined below) in the glassy phase, the region $T_g < T < T_c$ is interesting as well. In this region, numerical simulations clearly indicate the presence of metastability. This is exemplified by the result presented in Fig. 1a where the time evolution of an $8^3$ volume at a temperature value just below $T_c$ is shown \footnote{This simulation refers to the system expressed in dual variables \cite{20} and it has been carried out using a cluster algorithm.}. This figure also gives clear evidence that there is a first order phase transition. Fig. 1b consists of two different curves. One of them corresponds to a heating process starting from an initial ordered configuration. The other one describes the result of quenching a random initial configuration for each one of the temperature values shown. In both cases the simulation has been carried out using the Metropolis algorithm on a $24^3$ volume by performing $10^4$ measurements at each temperature value. As we will see in a while, the approach to equilibrium
in the region $T_g < T < T_c$ is non-standard and it is well described by a stretched exponential, instead of a simple exponential. For $T < T_g$ one immediately sees that the results from the quench of the random configuration after $10^4$ MC steps differ from those obtained starting from an ordered configuration (any of the $2^{3L}$ vacua). The difference appears to be constant all over this region (for a fixed number of thermalization steps). This clearly hints to the coexistence of two different dynamics. Initially fast dynamics quickly brings an initial configuration which is badly out of equilibrium to some sort of approximate equilibrium. At that point slow dynamics takes over and the evolution of the system is considerably freezed.

Before getting into the more quantitative aspects of these results, it is perhaps interesting to turn to one of our motivations, namely to test whether the appearance of slow dynamics makes the transition between two approximate ground states so slow as to make a given configuration virtually indestructible by thermal fluctuations, thus providing a convenient way of storing information.

To this end, we simulate the system on a cubic lattice and we use a Glauber algorithm, which is assumed to provide a good approximation to the thermal mechanism of fluctuations. Metropolis or heat bath give very similar results. We look at the decay of an artificial initial configuration consisted of an inner volume with a chessboard-like arrangement of the spins, which is one of the ground states, while the outer volume spins are fixed at $+1$ i.e, they form a ferromagnetic ground state. We use fixed boundary conditions in order to make the system decay to a ferromagnetic ground state. Our measured quantity is the number of the minus spins, $N_-(t)$, in terms of the computing time divided by the number of spins at time $t = 0$, $N_-(0)$. The quantity $N_-(t)$ is clearly related with the magnetization, $M$, since: $M \approx N_+ + N_- = N_{\text{tot}} - 2N_-$. Two examples of our results are given in Fig. 2a and Fig. 2b which have been produced from simulations over two lattice volumes namely $16^3$ and $30^3$ which at time $t = 0$ have enclosed $8^3$ and $20^3$ lattice volumes respectively with a chessboard–like arrangement for the spins.

The two curves in each subfigure have been generated after averaging over a sample of 50–100 copies where each one evolves starting from the same initial configuration, in order to reduce the noise. Two cases are shown. One is for temperature $T = 1.83$, in the supercooled phase, while the other one, $T = 1.0$, lies in the glassy phase . The difference on the relaxation time
between the two temperatures is obvious. For the $T = 1.83$ case the system reaches a stable value relatively fast for both lattice volumes, but this is not the case for $T = 1.0$. Note that the slope of the curve keeps taking non-zero value even for remarkably large times. This is perhaps clearer in Fig. 3 where long runs are depicted for $T = 1.0$, confirming that we are in presence of slow dynamics. This behaviour can be seen in a more apparent way in Fig. 4 where the results of Fig. 3 are presented in a logarithmic time scale. In Fig. 4 the logarithmic decay is present clearly enough.

The very distinct dynamics between the glassy phase and the evolution of the supercooled phase in the metastability region $T_g < T < T_c$ can be seen in a more definite way in Fig. 5. In this plot the results for four temperatures are presented for the bigger volume used i.e. $30^3$. 

Figure 2: The relaxation for two different temperatures, $T = 1.0$ and $T = 1.83$: (a) $16^3$ lattice volume with an initial chessboard–like configuration in a $8^3$ volume, (b) $30^3$ lattice volume with an initial chessboard–like configuration in a $20^3$ volume.

Figure 3: The long time evolution at $T=1.0$ for inner chessboard–like configuration in a $8^3$ (a) and $20^3$ volume (b).
All of them correspond to a random starting configuration. For $T = 1.83$, which lies in the metastability region the evolution seems very fast all the way to the equilibrium value. On the contrary for the other three temperatures, a fast evolution is initially observed followed by a very slow one which persists up to very long times.

These results are very suggestive and indeed show that the system finds very difficult to overcome dynamical energy barriers that are created along the evolution and this is undoubtedly the reason for the slow dynamics. Recall that the energy of the model is concentrated on the edges; the system has vanishing microscopic surface tension. To reduce the volume of the excitation with local moves, the total edge length must temporarily increase by a substantial amount. That makes excitations such as the one we have been analysing virtually stable.

In spite of the unambiguity of the previous results, in order to determine some properties associated to the very slow dynamics observed at low temperatures we shall proceed to analysing several dynamical correlators.

We shall first study the spin–spin autocorrelation function for temperatures lying in the metastability region after a random start, thus forcing the system to be in a supercooled phase. Its definition is given by

$$C_{\text{spin}}(t, t_w) = \frac{1}{N} \langle \sum_{\vec{r}} \sigma_{\vec{r}}(t_w) \sigma_{\vec{r}}(t + t_w) \rangle$$

The brackets mean that we take the average value over copies starting from a random configuration (200-400 in our case). The waiting time $t_w$ is the time for which the system is being thermalized before taking the measurements at subsequent times denoted by $t$. The waiting time $t_w$ is taken to be about 300 by noticing that from that value on the resultant values for $C_{\text{spin}}$ are quite identical as long as the temperature lies in the metastability region. In Fig. 6 we present some of our results for $C_{\text{spin}}$ in a $40^3$ lattice volume and for four temperature values. The fittings are stretched exponentials of the form $ae^{-(t/\tau)^b}$. We denote by $\tau$ the relaxation time. In all cases we found $0.60 < b < 0.80$ with an error smaller than 0.004. Since the fits seem quite good we go on and make a plot of the resulting values of $\tau$ versus the temperature $T$. The result is...
Figure 5: For $T = 1.83$ the system is in the supercooled region and exhibits fast relaxation. For the three other temperatures values which stand in the cold phase the slow decay is obvious when it is presented in logarithmic time scale.

shown in Fig. 4 where the corresponding fit to a function having the form, $\text{const} / (T - T_g)^c$, is quite good and leads to the prediction $T_g = 1.698(1)$ with $c = 0.41$. At $T_g$ the autocorrelation time is expected to diverge because of the onset of the slow dynamics which turns the stretched exponential behaviour into a power law (with a small exponent) or a logarithm.

Figure 6: Spin–spin autocorrelation function versus time. The fittings to the points are of stretched exponential type.

Although that result is in good agreement with previous simulations [9], [11] the method relying on the stretched exponential fit may prove to be too risky for an exact prediction of $T_g$ due to ambiguities in the fitting process and perhaps is not trustable to that accuracy. As
Figure 7: $\tau$ versus $T$. The fitting procedure leads to divergent $\tau$ for $T_g = 1.698$.

an alternative and a cross check, for two different temperature values, namely $T = 1.720$ and $T = 1.695$ we show the behaviour of the $C_{spin}$ for two very different values of the waiting time, $t_w = 300, 2000$. One sees that for the higher temperature the behaviour of the $C_{spin}$ is identical but for the smaller one there is a strong dependence on $t_w$ showing that for larger waiting times the system exhibits different, and in particular, slow dynamics behaviour which is a strong signal that the system has passed to the glassy phase. Hence we are able to estimate the value of $T_g$ to lie in between the interval $1.695 < T_g < 1.720$ which indeed agrees also with the previously obtained value.

Figure 8: $C_{spin}$ versus time for two values for the waiting time $t_w$. For the system being in the glassy phase we see the dependence on $t_w$. 
3 Next and next-to-nearest neighbour interaction ($\kappa = 1$)

As we have indicated the gonihedric spin model is actually a family of models which are parametrized by a real quantity $\kappa$. All the members of this family share the common feature of having their spin interfaces weighed with the total edge length. The value of $\kappa$ simply indicates the degree of self-avoidance of the surfaces. The plaquette model whose dynamical properties we just discussed corresponds to $\kappa = 0$. An interesting member of this family is given by $\kappa = 1$. The degeneracy of the ground state in this case is $3 \cdot 2^L$.

In this case the Hamiltonian is given by

$$H = -2 \sum_{\vec{r}, \vec{\alpha}} \sigma_{\vec{r}} \sigma_{\vec{r}+\vec{\alpha}} + \frac{1}{2} \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}$$

(4)

As we see, the spin plaquette term has disappeared. From a practical point of view this model may be particularly interesting as the plaquette term is obviously hard to get in real materials. From the standpoint of a spin system this is just a model with nearest and next-to-nearest neighbour interactions (though a finely tuned one).

We shall proceed to studying the dynamical properties of the system by using Monte–Carlo methods. In performing the simulations we used the Metropolis algorithm for several lattice volumes, namely $10^3$, $16^3$, $20^3$, $24^3$, $30^3$, $40^3$ and $46^3$. We imposed either fixed or periodic boundary conditions depending on the kind of the measurement that was performed. During the presentation of our results it will be explicitly mentioned when we used one or the other.

We begin by giving the behaviour for the susceptibility of the energy with the lattice volume which is given by

$$S(E) = V(\langle E^2 \rangle - \langle E \rangle^2)$$

We denote by $V$ the lattice volume and by the symbol $\langle \rangle$, the average value over sweeps. In Fig. 9 the susceptibility for the system energy as a function of the temperature for four lattice volumes is depicted. For every point in the figure we performed $10^5$ thermalization sweeps followed by more than $10^5$ measurements, using periodic boundary conditions and starting from the ordered configuration. The peak for each volume clearly increases with it although with an exponent less than one, which is a signal for a second order phase transition. Also, the positions of the peaks for the bigger volumes are seen to concentrate around the value $T_c = 2.329$ which is the pseudocritical value for the up to the volumes we used. It should be mentioned that while the second order character of the phase transition was already mentioned in [4], [16] however our prediction for $T_c$ gives a somewhat smaller value.

Having estimated the critical temperature value for the system we want now to study the dynamic behaviour at low temperatures, i.e. below $T_c$. To this end we use first the same method as for the $\kappa = 0$ case, described in the previous section, which consists of choosing an initial chessboard–like configuration defined in a cubic subvolume of the system. We impose fixed boundary conditions and we study the behaviour of the dynamical quantity

$$\frac{E(t) - E_{eq}}{E(0) - E_{eq}}$$

(5)

with time for various temperatures in the cold phase and for different sizes of the inner chessboard–like volume. In Eq.(5) $E(t)$ and $E(0)$ denote the energy of the system at time $t$ and just after the first sweep performed, respectively and $E_{eq}$ is the system energy when an ordered configuration
is taken as the initial one. In Fig. 10a we give an example for lattice volume $16^3$. We fix the temperature at the value $T = 1.1$ and we are interested in the behaviour as the dimension of the inner chessboard–like volume increases from $L = 2$ to $L = 8$. Each curve has been produced after averaging several hundreds of repetitions starting from the same initial configuration in order for the noise to be reduced. The results show a dramatic increase of the relaxation time with increasing system size. For the maximum value of $L$ used, the observable (5) shows very slow variation with the time and it follows a power law behaviour with estimated exponent equal to 0.18, so a logarithmic behaviour cannot be really excluded.

We can reach similar conclusions by studying the above relaxation quantity for decreasing values of temperature at fixed $L = 3$. In Fig. 10b the corresponding results are depicted for a $16^3$ total volume. In particular for the value $T = 0.2$ the dynamics is so slow that it seems really hard for the system to get to the ground state for accessible computing times.

Next we shall move to a more quantitative study following the same lines of the previous section. We look for aging features by considering the autocorrelation spin–spin function defined by (3). Brackets indicate that we take the average value over 100-200 trials starting from a random configuration. $t_w$ is the time for which the system is being thermalized before taking the measurements at subsequent times denoted by $t$. We follow the same lines of analysis as in [11] which concerns the $\kappa = 0$ case. In Fig. 11 we show an example of the behaviour of the autocorrelation function (3) for the temperature value $T=2.2$ and for a $24^3$ lattice volume. Six different curves for the $C_{\text{spin}}$ are depicted each corresponding to a different value of $t_w$. The very slow evolution of the autocorrelation function combined with the clear dependence on the time $t_w$ give strong evidence in favour of aging. In order to determine the aging (whether I or II) we consider the overlap function [17]:

$$Q(t + t_w, t + t_w) = \frac{1}{N} \langle \sum_{\vec{r}} \sigma^{(1)}_\vec{r}(t + t_w) \sigma^{(2)}_\vec{r}(t + t_w) \rangle$$

The measuring process for the above consists of the relaxation of the system for time $t_w$ starting from the random configuration. At that moment we take two copies $\sigma^{(1)}$ and $\sigma^{(2)}$ each
Figure 10: The relaxation behaviour of the initial configuration as seen by the study of quantity defined in (5): a) for various inner volume sizes at fixed temperature (from top to bottom, $L = 2, 3, 4, 6, 8$) and b) for fixed inner volume size with decreasing temperature (from top to bottom $T = 0.2, 0.5, 0.9, 1.1$).

Figure 11: The spin–spin autocorrelation function as depicted for different waiting times $t_w$ (from top to bottom $t_w = 1000, 500, 200, 100, 50, 10$).

evolving independently for time $t$. In Fig. 12 the behaviour of $Q(t + t_w, t + t_w)$ in terms of $C_{\text{spin}}(t, t_w)$ for the same values for the volume and temperature as in Fig. 11 and for four values of the waiting time $t_w$ is shown. It clearly follows that as $C_{\text{spin}}(t, t_w) \to 0$, $Q(t + t_w, t + t_w) \to 0$. The fact that the two system copies are moving independently since they show zero overlapping indicates strong evidence for type II aging.
Figure 12: The $Q(t + t_w, t + t_w) \to 0$ behaviour as $C_{spin}(t, t_w) \to 0$ is clearly seen for the four values of $t_w$.

4 Conclusions

At this point we would like to summarize our main conclusions.

We have carried out a rather complete analysis of the dynamical properties of the gonihedric spin model. The model naturally exhibits large potential barriers that are dynamically generated along the evolution process when trying to adjust to the environment. This is due to the fact that the area term of the interfaces is completely subleading and the dynamics is driven by the total edge length of the excitations. This creates basins of stability from where it is virtually impossible to kick out the system by local thermal fluctuations.

Several algorithms have been used: Glauber dynamics, heat bath and standard Metropolis. Modulo an overall rescaling of Monte–Carlo time, the results are fully equivalent.

In this paper we have studied numerically two cases of the gonihedric spin model. For the $\kappa = 0$ case we give evidence of a glassy transition and we estimate the temperature value $T_g$ at which the glassy phase arises being in the interval $[1.695, 1.720]$. We show that the dynamics is most likely logarithmic for $T < T_g$. In the mextatability region, above $T_g$ but below the true thermodynamical transition $T_c$ (where the system has a first order transition) the approach to equilibrium is well described by a stretched exponential, whose characteristic time scale diverges as $T \to T_g^+$.

We confirm that for the case $\kappa = 1$ the system exhibits a second order phase transition. The critical properties of this transition have not been elucidated yet; a tentative determination of the exponents using finite size scaling carried out in [14] gives a rather non-standard set of values. Work on this is in progress, but the problem is agrivated by the presence of slow dynamics. Indeed we have detected in the cold phase of this system very slow dynamical behaviour, quite similar to what takes place with $\kappa = 0$ for $T < T_g$. This dynamical behaviour has all the features of type II aging and is again compatible with logarithmic evolution.

Possible applications of these simple, but interesting, spin models are discussed.
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