On extracting plasma compression signatures from white light coronal images

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Abstract. We offer the intriguing possibility that the divergence of the plasma velocity (\(\nabla \cdot V\)) in outward-moving density features can be extracted directly from coronal and heliospheric white-light images of coronal mass ejections (CMEs). The relevant empirical formula is quite simple to apply. It follows directly from the conservation of electrons. We derive an approximate relation between the measured white-light intensities in the brightest image pixels and the dominant electron densities along the pixel line of sight by invoking the generalized mean value theorem for integrals. If the formula holds with acceptable accuracy, we then may have a measure of not only (\(\nabla \cdot V\)) itself, but also the related compressive acceleration rate of solar energetic particles within the CME.

1. Introduction

The quantity (\(-\nabla \cdot V\)) measures the compression of the plasma within the feature. This is a fundamental measure of plasma dynamics in a CME. In particular, the nature of the compression in the sheath (or driver) region of the CME should vary strongly with the altitude of the driver. For instance, the velocities of Type II shock-associated CMEs can exhibit a variety of acceleration signatures (~\(\pm 10 \text{ m/s}^2\)) out to several solar radii (\(R_S\)), e.g., [1]. Those CMEs with constant velocities can be characterized by approximately steady expansion in which \(\nabla \cdot V > 0\). However, very low in the corona during their launch and acceleration phase, the driver region should contain a volume of strong over-pressure with the consequent compression of the mass ahead of it, i.e., \(\nabla \cdot V < 0\).

We are proposing an observational method for extracting the quantity \(\nabla \cdot V\) directly from white-light images of the solar corona. The method should be valid in either the compression (\(\nabla \cdot V < 0\)) or expansion (\(\nabla \cdot V > 0\)) regimes, but, for the reasons given above, we expect that compression (particularly strong compression) is most likely to be observed at the lowest altitudes in the corona, i.e., close to the innermost edges of coronograph images. This poses an observational challenge, but we believe it will be well worth the effort. There is an additional and closely related scientific motivation for being able to detect compression regions in the low corona. It should not be too surprising that the plasma compression can be related to the acceleration of solar energetic particles (SEPs). A simple and direct quantitative relation between the characteristic e-folding time for SEP acceleration (\(\tau\)) and the plasma “transverse” compression \((\nabla \cdot V)\) was derived in the companion paper [2]. The inverse (1/\(\tau\)) of the particle...
acceleration time is defined as the fractional time-rate-of-change of the particle’s momentum (p) along its guiding center trajectory, and the relationship to plasma compression is

\[ \frac{1}{\tau} = \frac{d \ln p}{dt} \approx \frac{1}{3} (-\nabla \cdot \mathbf{V}_\perp) \]  

(0)

The relationship holds even if the plasma flow and the related electric and magnetic fields are time-dependent. It also holds for both non-relativistic and relativistic charged particles, and (remarkably) for all charged particles (since neither mass, energy, nor charge appears on the RHS).

2. Observations of CMEs associated with SEP events

We first present some observations that suggest to us that there are compressional regions within outgoing CMEs that will accelerate SEPs. Kahler and Vourlidas [3] made a comparison of the properties of 116 CMEs observed by LASCO and the peak intensities of 20 MeV protons measured by Wind/EPACT during the associated western hemisphere SEP events (1998-2002). The proton peak intensities generally increased with increasing CME velocity, but had a wide spread (factor>10\(^3\)) about this trend. Calling most intense 15 events matching the trend “SEP-Rich” and the least intense 16 events “SEP-Poor”, the authors concluded that the SEP-Rich events were “clearly larger events in the low corona”.

![Figure 1](https://via.placeholder.com/150)

*Figure 1. Upper row: Examples of ‘SEP-Rich’ CMEs as defined in Kahler and Vourlidas [3]. They are characterized by a bright front (‘edge’) behind the fainter shock front. In some cases, there’s a clear separation between the shock and driver edge (marked by arrows) while in other cases they appear to overlap. Lower Row: Examples of ‘SEP-Poor’ CMEs. Dark disk is inner edge of the SOHO/LASCO coronograph at r=2R\(_S\). The SEP-Rich CMEs are clearly larger events that are brighter lower in the corona.*

Eight CME images drawn from this joint data set are displayed in Figure 1. The dark central disc is the occultation by the SOHO/LASCO coronagraph for r<2R\(_S\). The SOHO/LASCO images have been calibrated (excess mass) and scaled identically; for more details of the image preparation see [4] and [5]. The upper row displays CME images associated with “SEP-rich” events. The shocks (at ~5R\(_S\)) are easily identifiable and are well separated by ~1R\(_S\) from the bright leading edges of the CMEs (at ~4R\(_S\)). The lower row displays images associated with “SEP-poor” events. The lack of sharp and bright fronts suggests that a shock is unlikely to have developed, but the CME did in fact produce an SEP event (albeit a relatively small one). There would still be regions of plasma compression within
the CME (too weak to drive a strong shock) that nonetheless could have accelerated the observed SEPs. Although the strongest brightening may occur when the leading edge of the CME is closer to the Sun than \( r=2R_S \) (the inner limit of the images), we believe that a careful measurement of such images will yield a useful estimate of the compression \(-\nabla \cdot \mathbf{V}\) for \( r>2R_S \), sufficient to discriminate between positive values (implying compression) and negative values (implying expansion).

3. Basic relation between electron density and plasma compression

The extraction procedure for obtaining estimates of the plasma compression \(-\nabla \cdot \mathbf{V}\) from white-light images of the corona follows from a version of the familiar conservation equation for electron density \(N\). As explained in the companion paper [2], expansion and re-arrangement of the terms yields

\[
\frac{\partial N}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0 \quad \text{becomes} \quad D\ln N/Dt = -\nabla \cdot \mathbf{V}
\]

where \(D/Dt = \partial/\partial t + \mathbf{V} \cdot \nabla\) is the Lagrange convective derivative for a function co-moving (but evolving) with the plasma velocity \(\mathbf{V}\).

Figure 2. Schematic of the outward motion of the density contours of a bright feature in a white-light “movie” using the evolution of its brightness ratio (enhancement/pre-event=\(I/I_q\)). Note that the plane-of-sky of the camera at 1 AU is \textit{lies in the plane} the Figure, so the blue density contours of the feature are in the ecliptic plane and “N” marks the north (ecliptic) pole of the Sun. Subject to assumptions concerning the electron density distribution (\(N\)), the plasma compression \(-\nabla \cdot \mathbf{V}\) may be estimated from the information in the image \((I/I_q)\) using the basic relationship among them (written in the box).
Then we can formally integrate the convective derivative along the characteristic plasma path \(dr=V(r,t)dt\) over the time range \(t_1<t<t_2\), thus obtaining the local path-averaged plasma compression \(<-\nabla \cdot V>\), which has the units of inverse time.

\[
\int_{t_1}^{t_2} dt \cdot [-\nabla \cdot V(r,t)] = \Delta t \cdot \nabla \cdot V(r,t)= \ln\left[\frac{N(r_2,t_2)}{N(r_1,t_1)}\right] \quad (2)
\]

Here \(N(r_1,t_1)\) and \(N(r_2,t_2)\), are the electron densities at heights \(r_1, r_2\) at times \(t_1, t_2\), respectively and \(\Delta t= t_2- t_1\). The geometry is sketched in Figure 2.

4. Extraction of compression from coronal white light images

Now suppose an outward moving brightness feature within the CME itself, not necessarily at the shock, can be identified in successive coronal or heliospheric images. See Figure 2. Note that the brightest pixel of the feature need not necessarily be propagating radially. Since the corona is optically thin in white light, the (calibrated) image brightness \(I(k,t)\) in the pixel direction specified by the unit vector \((k)\) is proportional to the line-of-sight (LOS) integral (over \(dr=kd\)) of the electron density \(N(r,t)\)

\[
I(k,t) = \int dl \cdot N(kl,t) \cdot \int d\omega \cdot \sigma(\omega \cdot k) = I_0 \cdot \int dl \cdot N(kl,t) \cdot \Delta \omega(kl) \cdot \sigma_{ave}(kl) \quad (3)
\]

where \(I_0\) is the photospheric brightness in white light and \(\sigma_{ave}(kl)\) is the value of the Thomson scattering cross-section of a photospheric photon (unit direction \(\omega\)) averaged over \(\Delta \omega(kl)\), the solid angle subtended by the Sun at LOS position \(r=kl\). Consistent with the usual method of analyzing coronagraph images of moving structures, we compare the ratio \(\Gamma(k;t,t_0)\) of the LOS intensity \(I(k,t)\) in the pixel direction \((k)\) within the CME to the pre-event intensity of the “quiet” corona, \(I(q(k,t_0)\) in the same pixel \((k)\) at an earlier time \(t_0<t\).

\[
\Gamma(k;t,t_0) = I(k,t)/I(q(k,t_0)) \quad (4)
\]

It is this pixel brightness ratio \((\Gamma)\) that is actually depicted in coronograph movies of outgoing CMEs. It is therefore a directly observable quantity, and it is particularly useful because many of the coronograph calibration factors required for estimating the absolute intensities \((I\) and \(I^0)\) cancel out of the ratio.

4.1. Simplification of the LOS integral of the intensity

A more useful representation of the LOS position \(r=kl\) involves the angle \((\psi)\) from the center of the Sun between the point of closest approach \((a)\) and the point \((r)\) on the LOS. Please refer to Figure 3 and note that the plane of sky is out of the page. The coordinate transformation is \(r=a+aktan\psi\) so that \(r=asec\psi\) and \(dl=asec^2\psi\) with \(-\pi/2<\psi<\pi/2\). Then a simple calculation will reveal that the solid angle subtended by the photospheric limb from position \((r)\) is

\[
\Delta \omega = 2\pi(1-cos\alpha) = 2\pi[1-(1-cos^2\psi \cdot R_s^2/a^3)\]^{1/2} \quad (5)
\]

because since \(R_s/\alpha = (R_s/a)cos\psi\). We ignore limb darkening (so the illumination is uniform across the apparent solar disk), because it occupies a negligible fraction disk as viewed along the LOS. When the LOS is tangent to the photosphere \((a=R_s)\) we have \(\Delta \omega=2\pi(1-sin\psi)\), i.e., the solid angle encompasses the whole Sunward hemisphere at closest approach itself, when \(\psi=0\). For \(a>R_s\) we can apply the binomial expansion to \(\Delta \omega\) with \(x=(a/R_s)cos\psi\) that converges for all \(x^2<1\).

\[
\Delta \omega = \pi x^2(1+x^2/4+x^4/8+5x^6/64+...) \quad (5')
\]
For LOSs with points of closest approach $a > 2R_S$, we need therefore retain only the leading term in Eq. (5'),
\[ \Delta \omega \approx \pi \left( \frac{R_S}{a} \right)^2 \cos^2 \psi, \]
because the next order correction term ($x^2/4$) is never more than $\sim 6\%$ above the true value. On the other hand, for LOSs that pass within a few $R_S$ of the Sun, the incident-direction-integrated Thomson (polarization) scattering term ($\sigma_{ave}$) will depend only weakly on the photon scattering direction ($\omega$) through $\cos \psi = k \cdot \omega$. From here on, we will both approximate $\Delta \omega$ using only its leading term in Eq. (5') and also pull $\sigma_{ave}$ out from under the LOS integral.

Figure 3. The generalized mean value theorem for integrals delimits the electron density profile (blue) as a function of time by the angular “shoulders” ($\psi^*, \psi^*$) of the density function along pixel LOS. The plane of sky is out of the page while the LOS pixel directions ($k_1$ and $k_2$) are in the page, so that we are looking down on the north polar region (N) of the Sun.

Thus the equation we will use for the LOS intensity in pixel ($k$) becomes
\[
I(k, t) = \pi a I_0 \int_0^\pi d\psi \sec^2 \psi \left[ 1 - (1 - \cos^2 \psi \left( \frac{R_S}{a} \right)^2) \right] \sigma_{ave}(a + a_k \tan \psi) N(a + a_k \tan \psi, t)
\]
\[ \approx \left( \pi R_S^2 I_0 \sigma_{ave}/a \right) \int_0^\pi d\psi N(a + a_k \tan \psi, t) \] (6)

For the reasons that will soon become apparent, we re-express the electron density $N(r, t)$ at time $t$ in terms of its ratio ($\gamma$) to the density $N_0(r, t_0)$ at the same position prior to the launch of the CME at time $t_0 < t$. 
\[ \gamma(a + a_k \tan \psi; t, t_0) = N(a + a_k \tan \psi; t)/N_q(a + a_k \tan \psi, t_0) \]  

(7)

We then re-group these factors under the integral in Eq. (6) to obtain our basic imaging relation

\[ I(k, t) = (\pi R_s^2 \sigma_{ave}/a) \int_{\psi} \gamma(a + a_k \tan \psi; t, t_0) N(a + a_k \tan \psi, t) \, \text{d} \psi \]  

(8)

4.2. Application of the generalized mean-value theorem for integrals (GMVTI)

In order to estimate the plasma compression using Eq. (2), we need an estimate of the actual electron density ratio \( N(r_2, t_2)/N(r_1, t_1) \). We obviously need to make further assumptions and approximations in order to use Eq. (8) to estimate the distribution of the electron density along different LOSs, e.g., see [3] for a detailed discussion of extracting mass densities (or electron densities) from coronagraph images of CMEs. We are assisted in this by the generalized mean value theorem for integrals (GMVTI), a very useful form of which is

\[ \int \text{d}x f(x)g(x) = f(x^*) \int \text{d}x g(x). \]

The theorem states that if \( g(x) > 0 \) is a continuous function and \( \int \text{d}x g(x) \) is bounded by some value \( \Delta y < \infty \) (even if the range of \( x \) is infinite), then there must be at least one value of \( x^* \) (and there may be more) at which the theorem is satisfied. The theorem follows trivially from the ordinary theory of the mean after a change of variable of integration to \( \text{d}y = \text{d}x g(x) \) so that \( \int \text{d}y = \Delta y \), because the equation transforms to

\[ \int \text{d}y f[x(y)] = f[x(y^*)] \Delta y, \]

revealing that \( f(x^*) = f[x(y^*)] \) is the mean value of \( f[x(y)] \) over the interval \( \Delta y \). For brevity, we can write \( f^* = f(x^*) \) where we also write \( x^* = x(y^*) \).

We apply the GMVTI directly to Eq. (8).

\[ I(k, t) = (\pi R_s^2 \sigma_{ave}/a) \gamma^* \int_{\psi} \gamma(a + a_k \tan \psi; t, t_0) N(a + a_k \tan \psi, t_0) \]  

(9)

The quantity \( \gamma^* \) is the mean value of the density enhancement ratio \( (\gamma = N/N^0) \) for the LOS integral weighted by the pre-event density \( (N^0) \), in a manner that we will describe in the next Section. By the GMVTI, \( \gamma^* \) must take on its mean value \( \gamma^* \) at some point(s) along the LOS, which we designate by \( r^* = a + a_k \tan \psi^* \). This is how we know that there is at least one angle \( \psi^* \) such that \( \gamma^* = \gamma(a + a_k \tan \psi^*; t, t_0) \). We have also recognized that the remaining integral in Eq. (9) is just the pre-event LOS intensity \( I_q(k, t_0) \), as can be seen immediately from Eq. (6). Then Eq. (9) reduces to an equality between the intensity ratio of Eq. (4) and the density ratio of Eq. (7), evaluated at the angle(s) \( \psi^* \) defined by the GMVTI.

\[ \Gamma(k, t, t_0) = \gamma(a + a_k \tan \psi^*; t, t_0) \]  

(10)

We have therefore obtained a relation between the ratio \( \Gamma = I/I^0 \) of the enhanced intensity at time \( t \) to the intensity in the same pixel at the pre-event time \( t_0 < t \) and the ratio \( \gamma = N/N^0 \) of the corresponding electron densities at the same (unknown) angle \( \psi^* \) along the LOS for pixel \( (k) \) which has a distance of closest approach \( (a) \). This equation between intensity and density ratios is our fundamental analytical tool for extracting the electron densities, as can be seen by re-writing it explicitly in terms of the densities.

\[ N(k, a, \psi^*, t) = N^0(k, a, \psi^*, t_0) \Gamma(k, t, t_0) \]  

(11)

The utility of Eq. (11) depends upon two things: 1) our knowledge of the pre-event electron density distribution \( N^0(k, a, \psi^*, t_0) \), and 2) the properties of the angle(s) \( \psi^* \) that come out of our application of the generalized mean theorem.
Figure 4. Functional relationships in the application of the generalized mean value theorem for integral (GMVTI) to the LOS density ratio profiles $\gamma(\psi)$ given by the black curves. (A) The mapping of $\gamma(\psi)$ in black to $\gamma(\psi(\Psi))$ in blue in terms of the pre-event-density-weighted angle ($\Psi$). The GMVTI implies the existence of the two “shoulders” ($\Psi^*, \Psi^*'$) where the density ratio equals its average value $\gamma^*=\gamma(\psi(\Psi^*))$ over the LOS integral. The inverse mapping ($\Psi \rightarrow \psi$) then gives the “shoulders” in terms of the actual LOS angles ($\psi^*, \psi^*'$). (B) If the shape of the CME is self-similar in two successive images ($t_1, t_2$) of the white-light “movie”, then the pairs of values of corresponding the “shoulder” angles ($\psi_1^*, \psi_1^*$) and ($\psi_2^*, \psi_2^*$) should be approximately equal in the corresponding brightest pixels ($k_1, k_2$).

4.3. Interpreting the GMVTI

Eq. (11) cannot be interpreted until we can understand the behavior of the values of the angle ($\psi^*$) implied by the GMVTI. We can do this, because we actually possess a considerable amount of information, once we assume that our LOS pixel direction ($k$) of the (relatively) brightest feature tracks the maximum of the (relative) electron density with time across the plane of sky. Many CMEs appear to have a single dominant density maximum along the LOS, and this is the way we drew our sketch in Fig. 3. This means that there will be two angles ($\psi^*, \psi^*$) that will bracket that dominant density maximum. The mean GMVTI is rather robust in this sense, because secondary density maxima (as long as they are significantly smaller than the dominant peak), will have little effect on the values of ($\psi^*, \psi^*$). We will say that this pair of angles delimits the “shoulders” of the dominant density feature along the LOS. In other words, these “shoulders” provide a rather robust means of
localizing the density peak somewhere between \((\psi^*)\) and \((\psi'^*)\), without having to specify the exact mathematical shape of the density peak itself.

Another simplifying assumption – one commonly made in white-light imaging – is that the pre-event electron distribution 1) depends locally predominantly on the radius \((r)\) and 2) can be characterized by a local power-law dependence \(N^0(r)=N^0(a)(a/r)^{\gamma_0}\), where \((a)\) is some normalizing radius. The formulas we will need are derived in the Appendix. In particular, when we set \((a)\) equal to the LOS radius of closest approach and use the geometric reaction \(r = \sec(\psi)\), the power-law relation becomes

\[
N^0(k,a,\psi)=N^0(k,a)\cos^{\gamma_0}\psi \quad \text{where} \quad N^0(k,a) = N^0(k,a,\psi=0) \tag{12}
\]

We can now go back to the basic LOS integral in Eq. (9) and insert this locally spherically symmetric pre-event density distribution.

\[
\ldots \int_{-\pi/2}^{\pi/2} d\psi \, \gamma(k,a,\psi;t,t_0) \, N^0(k,a,\psi) = \ldots \int_{-\pi/2}^{\pi/2} d\psi \, \gamma(k,a,\psi;t,t_0) \, \cos^{\gamma_0}\psi
\]

where \(\gamma(k,a,\psi;t,t_0)=N(a+ak\tan(\psi);t)/N(a+ak\tan(\psi),t_0)\) is the event-to-pre-event density ratio. This allows us to introduce the new variable of integration \((\Psi)\) and transform the integral over \((\psi)\) to one over \((\Psi)\).

\[
d\Psi = \cos^{\gamma_0}\psi \, d\psi \tag{13}
\]

This angular function \(\Psi(\psi)\) and its implied inverse \(\psi(\Psi)\) are essential to the GMVTI. For mathematical convenience in our illustrative example, we will take \(b=2\) exactly, because we saw that there was a relatively small variation in the power-law index \(2.0<\beta<2.5\) for \(2R_S<r<8R_S\). Then, with the condition \(\Psi(0)=0\), the new variable of integration is \(\Psi(\psi)=(\psi/2)+(1/4)\sin^2\psi\). The original integration range \(-\pi/2<\psi<\pi/2\) maps into \(\pi/4<\Psi<\pi/4\) in the transformed integral. The inverse function \(\psi=\psi(\Psi)\) is transcendental, but it always exists. The transformed integral in Eq. (9) now simply equals \((\pi/2)\gamma^*\).

\[
\ldots \int_{-\pi/2}^{\pi/2} d\psi \, \gamma(k,a,\psi;t,t_0) \, \cos^{\gamma_0}\psi = \ldots \int_{-\pi/2}^{\pi/2} d\Psi \, \gamma(k,a,\psi(\Psi);t,t_0) = (\pi/2) \, \gamma(k,a,\psi(\Psi^*);t,t_0) \tag{14}
\]

The effect is to shrink the angular range by a factor of two, but with little distortion \(\Psi=\psi+O(\psi^3)\) near zero values for both variables. These relations are sketched in Figure 4A. Note that the mean value \((\gamma^*)\) of the density ratio is not affected by the mapping of the independent variable: \(\gamma^* = \gamma[\psi(\Psi^*)]=\gamma(\Psi^*)\).

4.4. Tracking the brightest pixels in the outgoing CME

Although we cannot know the location \((r=a+ak\tan(\psi^*)\) precisely (only because we do not precisely know the mean-value angle \((\psi^*\) precisely), we are justified in assuming that it is somewhere along the LOS of the pixel within the bright feature, because that is how we selected the pixel \(k\) at time \(t=t_0\). The feature cannot be bright unless it contains a localized enhancement of electron density. This simple inference from our assumption, that the density has a single (dominant) maximum along the LOS, allows us to extract significant quantitative information on the plasma compression without detailed knowledge of the precise geometry of the enhancement.

We do this by tracking the bright feature in two successive images. From Eq. (11) we have the ratio

\[
N(k_2,a_2,\psi^*,t_2)/N(k_1,a_1,\psi_1^*,t_1) = N^0(k_2,a_2,\psi_2^*,t_2)/N^0(k_1,a_1,\psi_1^*,t_1) \, \Gamma(k_2;t_2,t_0)/\Gamma(k_1;t_1,t_0) \tag{15}
\]

It is worth noting at this stage of the argument that if we are following pixels from a spacecraft at 1AU of an outgoing feature \((k_1,k_2,...)\), then the successive LOS vectors are very nearly parallel. If the change \(a(k_1)-a(k_2)<1R_S\), then they are parallel to within \(<1/214.3\) rad = 0.3°.
The actual plasma “track” of shoulder of bright feature that originates at \( \mathbf{r}_1 = a_1 + a_1 \mathbf{k}_1 \tan \psi_1 \) moves to a position \( \mathbf{r}_2 = a_2 + a_2 \mathbf{k}_2 \tan \psi_2 \) during time interval \( t_1 < t < t_2 \) that is determined by the actual plasma velocity \( \mathbf{V}[\mathbf{r}(t), t] \) so that

\[
\mathbf{r}_2 = \mathbf{r}_1 + \int_{t_1}^{t_2} \mathbf{V}[\mathbf{r}(t), t] \, dt
\]

(16)

Because the pixel directions \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) actually “track” the brightest region, we know the LOS radii of closest approach \( a_1 \) and \( a_2 \) in the plane of sky. We have just shown that we can interpret the angle pairs \( (\psi_1, \psi_1^*) \) and \( (\psi_2, \psi_2^*) \) as specifying the angular width (the “shoulders”) of the density enhancement along the LOSs. Many observational studies over the past decades have indicated a nearly-constant angular width with radius for individual CMEs (the “ice-cream-cone” model), so we may set \( \psi_1 = \psi_2 \) and \( \psi_1^* = \psi_2^* \). That will be our basic “tracking” assumption for deriving the compression of the plasma.

Therefore, we can take the shoulder track defined by Eq. (16) as going approximately from \( \mathbf{r}_1^* = a_1 + a_1 \mathbf{k}_1 \tan \psi_1^* \) to \( \mathbf{r}_2^* = a_2 + a_2 \mathbf{k}_2 \tan \psi_2^* \), or from \( \mathbf{r}_1^* = a_1 + a_2 \mathbf{k}_2 \tan \psi_1^* \) to \( \mathbf{r}_2^* = a_2 + a_1 \mathbf{k}_1 \tan \psi_2^* \), on the other shoulder. In other words, the density enhancement profile along the two LOSs is taken to be self-similar between to LOSs not too far separated in distances of closest approach (\( a_1 \) and \( a_2 \)). See Fig. 4B. By the same reasoning, the tracks of the pair of shoulders must bracket the actual track of the maximum of the density enhancement. Since we are assuming that all LOS density profiles are self-similar between the shoulders, then the plasma compression (which only depends on density ratios) should be comparable along all three tracks. This assumption will indeed be justified if the compression is mainly caused by the radial to the divergence

\[
\nabla \cdot \mathbf{V} = (1/r^2) \partial (r^2 V_r)/\partial r + (1/\sin \theta) \partial (\sin \theta V_\theta)/\partial \theta + (1/\sin \theta) \partial V / \partial \phi = (1/r^2) \partial (r^2 V_r)/\partial r
\]

(17)

with a relatively weaker contribution transverse to the radial axis of the CME. Another way of saying this is that the self-similarity of the shapes of expanding CMEs implies that the transverse expansion is weak. Thus, even though we don’t know the locations of the “shoulder” tracks at angles \( (\psi_1^*, \psi_2^*) \) implied by the MVTI, we have some justification that the self-similarity implies that we can assign the average value of the compression \( \langle -\nabla \cdot \mathbf{V} \rangle \) to that within the density peak itself.

5. Observational formula for the plasma compression

The relations from the previous Section allow us to “extract” the time-averaged plasma compression \( \langle -\nabla \cdot \mathbf{V} \rangle \) from successive disturbed images using the exact relation of Eq. (2) from the conservation of electrons and our derived relations in Eq. (9) and Eq. (11) from the GMVTI from either shoulder \( (\psi_1^* \) or \( \psi_2^* \)).

\[
(1/\Delta t) \int_0^\infty \langle -\nabla \cdot \mathbf{V}(\mathbf{r}, t) \rangle \, dt = \ln[N(\mathbf{r}_2, t_2), N(\mathbf{r}_1, t_1)] = \ln[N(\mathbf{k}_2, \mathbf{a}_2, \psi_2^*, t_2)/N(\mathbf{k}_1, \mathbf{a}_1, \psi_1^*, t_1)]
\]

\[
= \ln[N(\mathbf{k}_2, \mathbf{a}_2, \psi_2^*, t_0)/N(\mathbf{k}_1, \mathbf{a}_1, \psi_1^*, t_0)] \Gamma(\mathbf{k}_2, t_2, t_0) \Gamma(\mathbf{k}_1, t_1, t_0)
\]

(18)

We can immediately apply the result from our discussion of the pre-event local spherical density model (see the Appendix) to the estimate of the pre-event ratio

\[
N(\mathbf{k}_2, \mathbf{a}_2, \psi_2^*, t_0)/N(\mathbf{k}_1, \mathbf{a}_1, \psi_1^*, t_0) \approx (r_1^*/r_2^*)^{b_{12}} = (a_1 \cos \psi_1^*/a_2 \cos \psi_2^*)^{b_{12}} \approx (a_1/a_2)^{b_{12}}
\]

(19)

because self-similarity implies that \( \psi_1^* = \psi_2^* \) which in turn implies (to even better approximation) that \( \cos \psi_1^* = \cos \psi_2^* \).

This gives us our final formula for plasma compression, in the approximation of a spherically symmetric model for the pre-event coronal electron density distribution.
\[ (\Delta t) \cdot \langle -\nabla \cdot V(\mathbf{r},t) \rangle = -b_{12} \ln(a_2/a_1) + \ln\left( \frac{\Gamma(k_2; t_2, t_0)}{\Gamma(k_1; t_1, t_0)} \right) \]  

(20)

The first logarithm in the square brackets stems from the radial dependence of the pre-event density. It will always be negative, because \( a_1 < a_2 \) for an outgoing CME. Consequently, if there is to be plasma compression, the density feature will have to brighten sufficiently as it moves outward so that the second logarithmic term \( \ln(\frac{\Gamma_2}{\Gamma_1}) \) can overcome the pre-event density ratio term containing \( (a_2/a_1) \), if there is to be time-averaged plasma compression \( \langle \nabla \cdot V \rangle < 0 \). This condition can be stated as an inequality:

\[ \frac{\Gamma(k_2; t_2, t_0)}{\Gamma(k_1; t_1, t_0)} > (a_2/a_1)^{b_{12}} \]  

(21)

implies that \( \nabla \cdot V < 0 \)

The only un-observed quantity is the position-averaged power-law index \( b_{12} \) of the pre-event density. It may be taken from published studies like [6] or [7], but perhaps it may be possible in some events to derive a pre-event value \( b_{12} \) from “quiet” images over the several days prior to the SEP event. In any case, the remaining logarithm involves only the ratio of the brightness ratios at the two times \( \Gamma_2/\Gamma_1 \). These are precisely the measured pre-event-normalized pixel brightnesses that are used to make movies of CMEs. Note that any linear instrument calibration and response factors will cancel out of the brightness ratio, and we have argued that any Thomson scattering corrections will be negligible in the low corona.

6. Summary

Under the assumptions made here, the normalized intensity ratio \( \Gamma \), corresponds directly to the density compression ratio \( \Gamma_{CR} \) introduced in [4] and [5] to estimate the strength of CME shocks in white light images. Also note that we do not have to estimate the plasma velocity \( \mathbf{V} \) itself, nor the time derivative \( \partial N/\partial t \) nor the spatial gradient \( \nabla N \) of the electron density! All we have to do is track the feature’s normalized white-light brightness: if it is increasing sufficiently to satisfy the inequality in Eq. (21), then we have a measure of the compression of the plasma within the region occupied by that feature, and conversely, an insufficient increase implies plasma expansion.

We suggest that the strongest brightening is much more likely to occur when the leading edge is closer to the Sun, perhaps inside of the radius of the occultation disk (\( \sim 1.5-2 R_S \)) on contemporary spacecraft coronagraphs. Nonetheless, even with this observational limitation, we believe that a careful measurement of such images will yield a useful estimate of the quantity \( \langle -\nabla \cdot V \rangle \), according to Eq. (4), regardless of whether its value is positive (implying compression) or negative (implying expansion). When there is compression, Eq. (0) -- taken from the related paper [2] -- may be used to estimate the rate of compressive acceleration of solar energetic particles occurring within the observed CME, thus providing observational “closure” for the proposed mechanism for SEP acceleration.

7. References

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Appendix A. Simple model for a spherically symmetric electron density distribution

We will treat a simple example in this paper, that of a spherically symmetric “quiet” corona $N_q(r)$ within which the electron densities obey a “local” power-law dependence on radius. The statistical analysis of electron densities using dekameter solar radio Type III bursts observed by ISEE-3 from ~2R_s to 1 AU by Leblanc et al. [6] found an average density $N \propto 1/r^2$ from 1 AU into $r=8R_s$. The power law gradually steepened to $N \propto 1/r^{2.5}$ from $8R_s$ to $2R_s$, in good agreement with the oft-quoted estimates of Saito et al. [7]. We can model this behavior as a “local” power law in radius with a radius-dependent index $b(r)$.

$$\frac{d \ln N_q}{d \ln r} = -b(r) \quad \ln \left[ \frac{N_q(r_1)}{N_q(r_2)} \right] = \int_{r_2}^{r_1} \frac{d \ln N_q}{d \ln r} b(r)$$

If we expand $\gamma(r)$ in a Taylor series about either the radius $b(r)=b_2+\ln(r/r_2)db/d\ln r_2+\ldots$ or the outer radius $b(r)=b_2+\ln(r/r_2)db/d\ln r_2+\ldots$, integrate the two expressions as above, and then average the results, we obtain

$$\ln \left[ \frac{N(r_2)}{N(r_1)} \right] = b_{12} \ln \left( \frac{r_2}{r_1} \right) + (1/4) \ln^2 \left( \frac{r_2}{r_1} \right) \left( \frac{db/d\ln r_2}{db/d\ln r_1} - \frac{db/d\ln r_1}{db/d\ln r_2} \right) +\ldots$$

where $b_{12}=(1/2)(b_1+b_2)$. The coefficient of second-order term in the already small quantity $\ln(r_2/r_1)$ involves the difference of the slope derivatives, so it can clearly be neglected for two radii relatively close together such that $\ln(r_2/r_1)<1$ for $r_2>r_1$. In our LOS coordinates (with the pixel direction $k$ held constant), we have $r^*={\text{asec}}\psi^*$, so our estimate of the pre-event ratio finally becomes

$$\frac{N_{q2}}{N_{q1}} \approx (r_2/r_1)^{b_{12}} = (a_2/a_1)^{b_{12}} (\cos \psi_2/\cos \psi_1)^{b_{12}}$$