On the probability interpretation of wave functions in the Dirac theory

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It is suggested symmetric relative to particles and antiparticles formulation of the Dirac theory, in which the states with negative energy are excluded. The fields of particles and antiparticles are associated with the wave functions, for which there is valid the Born interpretation of being the probability amplitudes. In doing so, one eliminates different kinds of “paradoxes” in the theory, which existence is caused by an incorrect account of the states with negative energy.

Key words: Dirac equation, electron, positron, particle, antiparticle, wave function, probability amplitude, charge conjugation

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I. INTRODUCTION

The relativistic equation for an electron was obtained by Dirac in his classical works [1]. A presentation of the Dirac theory is available in a large number of textbooks and monographs, some of which became classical as well [2][15]. Like the nonrelativistic Schrödinger equation describing the spatial and temporal evolution of the complex function, the Dirac equation describes the complex function incorporating the four components. The solutions of the nonrelativistic Schrödinger equation for a free particle with positive energy constitute a full set of states by which an arbitrary solution can be decomposed. In the Dirac theory, along with the solutions with positive energy, there are the solutions with the opposite sign of energy. In contrast to the nonrelativistic theory, the solutions of the Dirac equation corresponding to the states with positive energy do not constitute a full set of states, and in order to obtain the general solution one have to take into account also the solutions with negative energy. In this connection, after it became clear that it is impossible to eliminate negative energies from the theory, the problem has arisen regarding the physical interpretation of such solutions [1]. With this aim Dirac assumed that in the nature almost all states with negative energy are occupied, and the unoccupied states (holes) behave themselves as the positively charged particles. Since the electrons occupying the states with negative energy are charged, they should have created the electric field with an infinite density of energy. To overcome this difficulty Dirac made a rather exotic assumption that the electric field is produced only by the electrons situated over “the electron sea” or by the holes. Initially Dirac identified the positive states with the protons, but from the symmetry consideration it followed that the mass of electrons and holes should be equal. The result of awareness of this circumstance was the hypothesis that the holes correspond to the new particles unknown to the science at that time and with the mass equal the electron’s mass but the opposite charge. The Dirac theory describes both the positively and negatively charged particles in a completely equal way, whereas the hypotheses about “the electron sea” breaks the symmetry between them. The Dirac interpretation which in itself does not follow from the mathematical formalism was rather coldly accepted by a lot of leading physicists. Only the discovery of the positron changed the attitude towards it. However, the experimental confirmation of some predictions of a theory by no means makes a theory in itself logically perfect.

Although eventually it became clear that the Dirac theory does not need additional constructions for its interpretation, the treatment at the level “particle-hole” is reproduced in many books on the relativistic quantum mechanics and field theory [6]. At present, there exists quite consistent interpretation of the Dirac theory not attracting additional, internally unnatural to it, qualitative considerations [2], although in our opinion this right point of view is far from always being consistently advanced. In the interpretation of the theory one deals with a central question: “What is the physical meaning of the complex multicomponent field, which is described by the Dirac equation?” This question is note merely a speculative one, but it has an important practical meaning, since depending on the answer to it there should be established the rules for the calculation of the observable quantities. While according to the Born interpretation the complex Schrödinger field is the probability amplitude and its phase-invariant combinations correspond to the probability density and the probability flow density, the general solution of the Dirac equation contains the contributions of the states with both positive and negative energies and does not allow such an interpretation. Since such general solution of the Dirac equation does not have the meaning of the probability amplitude, it cannot be used

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for the calculation of the average values of the observable quantities and other probability characteristics. The usual probability interpretation is allowed only for the solutions with positive energies, through which one can also express the general solution of the Dirac equation. But this general solution is not a new allowable state in accordance with the superposition principle since it includes the operation of the charge conjugation. The neglecting of this fact leads to the appearance of various “paradoxes”, such as the Klein paradox [2, 0], and also to such phenomena in the theory as the “jittering” of electrons [2, 13].

In this work we propose a variant of the Dirac theory, in which the equations for the particles with the opposite charges and equal masses are considered simultaneously and symmetrically and where the independent variables describing the physically realizable states are the solutions for the particles with the opposite signs of charges and positive energies. The solutions of the equations to which correspond the negative energies are not independent but they are expressed through the solutions with positive energies by means of the operation of the charge conjugation. In this way the problem of negative energies in the theory is resolved. The complex wave functions describing the physically realizable states are the solutions for the particles with the opposite signs of charges and equal masses are considered simultaneously and symmetrically and where the independent variables represent the charged particles enter into the theory completely symmetrically, then for the charge-symmetric consideration it is necessary to consider separately.

II. THE CHARGE-SYMMETRIC FORM OF THE DIRAC EQUATIONS

The Dirac equation describes the evolution of the four-component complex function $\psi(x) \equiv \psi_\sigma(x, t)$, where the index takes the values $\sigma = 1, 2, 3, 4$. In the matrix notation the Dirac equation for the function $\psi(x)$ and the conjugate function $\bar{\psi}(x) = \psi^+(x)\gamma_4$ has the form

$$h c \gamma_\mu \frac{\partial \psi(x)}{\partial x_\mu} + mc^2 \psi(x) = 0, \quad h c \frac{\partial \bar{\psi}(x)}{\partial x_\mu} \gamma_\mu - mc^2 \bar{\psi}(x) = 0,$$

where $x \equiv x_\mu \equiv (x_0, x_\mu) = (x, ic\tau)$, $m$ – the mass of a particle, $c$ – the speed of light, $\hbar$ – the Planck constant, $\gamma_\mu$ – the hermitian 4 × 4 Dirac matrices. The sign $+$ in $\psi^+(x)$ denotes the hermitian conjugation. A summation is everywhere implied over the repeated indices. We will mainly follow the notations of the book [2]. The scalar product of the two 4-vectors $a = (a_0, a_\mu) = (a, ia_0)$ and $b = (b_0, b_\mu) = (b, ib_0)$ is written in the form $ab = a_0b_0 + a_\mu b_\mu = ab - a_0b_0$. The account for the interaction with the electromagnetic field is performed by means of the famous replacement

$$\frac{\partial}{\partial x_\mu} \rightarrow \frac{\partial}{\partial x_\mu} - \frac{ie}{hc} A_\mu(x),$$

where $A_\mu(x)$ is the 4-vector potential, and the charge $e = \mp |e|$, or $e = 0$ for electrically neutral particles. Further we deal with the charged particles. Usually when writing down the Dirac equation one chooses the electron charge $e = -|e|$ (electron) and a particle with positive charge $e = |e|$ (positron) – an “antiparticle”. We do not discuss here the asymmetry between particles and antiparticles observed in the nature which is, quite probably, not related to the breaking of the charge symmetry in the fundamental equations. Note also, that the state of a particle can be additionally characterized by the “inner” (lepton) quantum number $\Lambda$, which is assumed to be positive for a particle whereas an antiparticle is characterized by the lepton quantum number of the opposite sign. The existence of the “inner” quantum number allows to distinguish the fields of particles and antiparticles also in the absence of the electromagnetic field and in the case of neutral particles. Thus, the Dirac equations for the fields of a charged particle and antiparticle in the electromagnetic field $A_\mu = (A_0, iA_\mu)$ take the form

$$a) \quad hc \gamma_\mu \left( \frac{\partial \psi_\Lambda}{\partial x_\mu} - i \frac{e}{hc} A_\mu \psi_\Lambda \right) + mc^2 \psi_\Lambda = 0, \quad b) \quad hc \gamma_\mu \left( \frac{\partial \eta_{-\Lambda}}{\partial x_\mu} + i \frac{e}{hc} A_\mu \eta_{-\Lambda} \right) + mc^2 \eta_{-\Lambda} = 0,$$

Let us also give the equations for the conjugate functions $\bar{\psi}_\Lambda \equiv \psi^+_\Lambda \gamma_4$ and $\bar{\eta}_{-\Lambda} \equiv \eta^{+}_{-\Lambda} \gamma_4$:

$$a) \quad hc \left( \frac{\partial \bar{\psi}_\Lambda}{\partial x_\mu} + i \frac{e}{hc} A_\mu \bar{\psi}_\Lambda \right) \gamma_\mu - mc^2 \bar{\psi}_\Lambda = 0, \quad b) \quad hc \left( \frac{\partial \bar{\eta}_{-\Lambda}}{\partial x_\mu} - i \frac{e}{hc} A_\mu \bar{\eta}_{-\Lambda} \right) \gamma_\mu - mc^2 \bar{\eta}_{-\Lambda} = 0.$$
In the following the index of the inner quantum number $\Lambda$ will be everywhere omitted for brevity. The functions $\psi$ and $\eta$ are not independent. Their relationship can be established by means of the unitary matrix of the charge conjugation, which satisfies the conditions \(2\):

\[
C^+ C = CC^+ = 1, \quad C\gamma_\mu C^+ = -\gamma_\mu, \quad C = -\bar{C}.
\]  

(5)

The symbol $\sim$ denotes the transposition. This matrix can be chosen in the form $C = \gamma_2\gamma_4$. Then from equations \(3\) and \(4\) there follow the relations which connect the solutions of the Dirac equations with the opposite charges:

\[
a) \; \psi = C^+ \bar{\eta}, \quad b) \; \eta = C^* \bar{\psi}.
\]

(6)

Let us decompose the solution of the Dirac equation into the Fourier integral:

\[
\psi(x) = \int_{-\infty}^{\infty} c(\omega) \psi(x, \omega) e^{-i\omega t} d\omega = \psi_+(x) + \psi_-(x),
\]

(7)

where the positive-frequency and negative-frequency functions are defined by the formulas:

\[
\psi_{\pm}(x) = \int_{0}^{\infty} c(\pm \omega) \psi(x, \pm \omega) e^{\mp i\omega t} d\omega.
\]

(8)

It is assumed that there exists the integral

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x) e^{i\omega t} dt < \infty.
\]

(9)

Note that the integration in (8) is performed over only positive frequencies. In a similar way one can represent the solution of the equation with the opposite sign of the charge:

\[
\eta(x) = \int_{-\infty}^{\infty} b(\omega) \eta(x, \omega) e^{-i\omega t} d\omega = \eta_+(x) + \eta_-(x),
\]

(10)

where

\[
\eta_{\pm}(x) = \int_{0}^{\infty} b(\pm \omega) \eta(x, \pm \omega) e^{\mp i\omega t} d\omega.
\]

(11)

Then from equations \(11\) there follow the relations expressing the negative-frequency functions through the positive-frequency functions:

\[
\psi_-(x) = C^+ \bar{\eta}_+(x), \quad \bar{\psi}_-(x) = \bar{\eta}_+(x)C,
\]

\[
\eta_-(x) = C^* \bar{\psi}_+(x), \quad \bar{\eta}_-(x) = \bar{\psi}_+(x)C.
\]

(12)

Thus, the general solutions of the Dirac equation with the charge of an arbitrary sign can be expressed through only the positive-frequency particular solutions $\psi_+(x)$ and $\eta_+(x)$ of the Dirac equations with the opposite signs of the charge, which are those to be considered as the wave functions of a particle and an antiparticle allowing the Born probability interpretation:

\[
\psi(x) = \psi_+(x) + C^+ \bar{\eta}_+(x), \quad \eta(x) = \eta_+(x) + C^* \bar{\psi}_+(x).
\]

(13)

Since the positive-frequency particular solutions are interpreted as the wave functions of a particle and an antiparticle having the meaning of the probability amplitudes, they should be normalized by the conditions:

\[
\int |\psi_+(x)|^2 dx = 1, \quad \int |\eta_+(x)|^2 dx = 1.
\]

(14)

The general solutions of the Dirac equations \(13\) are expressed through both the wave function of a particle and the wave function of an antiparticle, but they are not the linear superposition of these functions since contain the antilinear transformation of the complex conjugation, and so they do not have the meaning of the probability amplitudes. Thus, the functions \(13\) containing the contribution of the states with negative energies are not the physically realizable states having the meaning of the probability amplitudes and, therefore, they cannot be used in calculating of the transition probabilities and average values of the operators of the observables. It is precisely the improper use in calculations of such functions including the states with negative energies that leads to the appearance in the theory of various “paradoxes”. 
III. THE DIRAC EQUATIONS FOR THE PROBABILITY AMPLITUDES

Now let us substitute the functions (13) into the equations (3) and (4). As a result we obtain the equations containing only the functions with positive frequencies:

\[ \hbar c \gamma_\mu \frac{\partial}{\partial x_\mu} \left( \psi_+ + C^* \bar{\eta}_+ \right) - ie A_\mu \gamma_\mu \left( \psi_+ + C^* \bar{\eta}_+ \right) + mc^2 \left( \psi_+ + C^* \bar{\eta}_+ \right) = 0, \]  
\[ \hbar c \frac{\partial}{\partial x_\mu} \left( \bar{\psi}_+ + \bar{\eta}_+ C \right) \gamma_\mu + ie A_\mu \left( \bar{\psi}_+ + \bar{\eta}_+ C \right) \gamma_\mu - mc^2 \left( \bar{\psi}_+ + \bar{\eta}_+ C \right) = 0, \]  
\[ \hbar c \gamma_\mu \frac{\partial}{\partial x_\mu} \left( \eta_+ + C^* \bar{\psi}_+ \right) + ie A_\mu \gamma_\mu \left( \eta_+ + C^* \bar{\psi}_+ \right) + mc^2 \left( \eta_+ + C^* \bar{\psi}_+ \right) = 0, \]  
\[ \hbar c \frac{\partial}{\partial x_\mu} \left( \bar{\eta}_+ + \bar{\psi}_+ C \right) \gamma_\mu - ie A_\mu \left( \bar{\eta}_+ + \bar{\psi}_+ C \right) \gamma_\mu - mc^2 \left( \bar{\eta}_+ + \bar{\psi}_+ C \right) = 0. \]

It is convenient to introduce the notations

\[ Q(x) \equiv \hbar c \gamma_\mu \frac{\partial}{\partial x_\mu} \psi_+ - ie A_\mu \gamma_\mu \psi_+ + mc^2 \psi_+, \quad \Pi(x) \equiv \hbar c \gamma_\mu \frac{\partial}{\partial x_\mu} \eta_+ + ie A_\mu \gamma_\mu \eta_+ + mc^2 \eta_. \]

Then in these notations the equations (15) – (18) take the form

\[ Q(x) + C^* \bar{\Pi}(x) = 0, \quad \bar{Q}(x) + \bar{\Pi}(x) C = 0, \]
\[ \Pi(x) + C^* \bar{Q}(x) = 0, \quad \bar{\Pi}(x) + \bar{Q}(x) C = 0, \]

where \( \bar{Q}(x) \equiv Q^+(x) \gamma_4, \quad \bar{\Pi}(x) \equiv \Pi^+(x) \gamma_4. \) Each of the three equations (20) are the consequence of the fourth equation, so that the relations (21) give the various forms of presentation of the single equation. First we mainly consider the case of the stationary electromagnetic field, setting \( A_\mu(x) = A_\mu(\mathbf{x}). \) Using the decompositions (8) and (11) for the functions \( \psi_+(x) \) and \( \eta_+(x), \) we obtain

\[ Q(x) = \int_0^\infty c(\omega) Q(x, \omega) e^{-i\omega t} d\omega, \quad \Pi(x) = \int_0^\infty b(\omega) \Pi(x, \omega) e^{-i\omega t} d\omega, \]

where

\[ Q(x, \omega) = \left[ \hbar c \left( \gamma \nabla - \frac{\omega}{c} \gamma_4 \right) + mc^2 - ie A_\mu(\mathbf{x}) \gamma_\mu \right] \psi(x, \omega), \]
\[ \Pi(x, \omega) = \left[ \hbar c \left( \gamma \nabla - \frac{\omega}{c} \gamma_4 \right) + mc^2 + ie A_\mu(\mathbf{x}) \gamma_\mu \right] \eta(x, \omega). \]

In these notations the equations (20) are equivalent to the equation

\[ \int_0^\infty \left[ c(\omega) Q(x, \omega) e^{-i\omega t} + b^*(\omega) C^* \bar{\Pi}(x, \omega) e^{i\omega t} \right] d\omega = 0. \]

Multiplying (23) first by \( e^{i\omega' t}, \) where \( \omega' > 0, \) and integrating over time, and then multiplying by \( e^{-i\omega' t} \) and also performing integration over time, we obtain that there must be \( Q(x, \omega) = 0 \) and \( \bar{\Pi}(x, \omega) = 0. \) Thus, we arrive at the two independent equations for the wave functions of a particle and an antiparticle, which with account of the notations (24) has the form

\[ \left[ \hbar c \gamma \left( \nabla - i \frac{e}{\hbar c} A(\mathbf{x}) \right) - (\hbar \omega - eA_0(\mathbf{x}) ) \gamma_4 + mc^2 \right] \psi(x, \omega) = 0, \]
\[ \left[ \hbar c \gamma \left( \nabla + i \frac{e}{\hbar c} A(\mathbf{x}) \right) - (\hbar \omega + eA_0(\mathbf{x}) ) \gamma_4 + mc^2 \right] \eta(x, \omega) = 0. \]
Here the functions $\psi(x, \omega)$ and $\eta(x, \omega)$, according to [8] and [11], depend only on positive frequency. From equations [24], [25] there follow the conditions of orthonormality for the electron and positron wave functions:

$$\int dx \psi^+(x, \omega)\psi(x, \omega') = \delta(\omega - \omega'), \quad \int dx \eta^+(x, \omega)\eta(x, \omega') = \delta(\omega - \omega').$$

The conditions of orthogonality of the electron and positron wave functions are also satisfied:

$$\int dx \psi^+(x, \omega')\gamma_4 C^*\eta^+(x, \omega) = 0, \quad \int dx \eta^+(x, \omega)\gamma_4 \psi(x, \omega') = 0.$$  \hspace{1cm} (27)

For the wave functions depending on time, the equations for particles and antiparticles have the form of the Dirac equations which differ only by the sign of the charge:

$$\hbar c \gamma_\mu \left( \frac{\partial \psi_+}{\partial x_\mu} - i \frac{e}{\hbar c} A_\mu(x)\psi_+ \right) + mc^2\psi_+ = 0, \quad \hbar c \gamma_\mu \left( \frac{\partial \eta_+}{\partial x_\mu} + i \frac{e}{\hbar c} A_\mu(x)\eta_+ \right) + mc^2\eta_+ = 0,$$ \hspace{1cm} (28)

Thus, in the stationary fields the equations for the wave functions of particles and antiparticles with positive energies are unconnected and, therefore, the states of particles and antiparticles should be considered independently.

**IV. LAGRANGIAN FORMALISM**

Let us formulate the developed approach for the description of particles and antiparticles in terms of the probability amplitudes on the basis of the Lagrangian formalism, which will allow to obtain the energy-momentum tensor and the conservation laws. The Dirac equations (28) can be obtained if the Lagrangian function density is represented as a sum of the Lagrangians of a particle $\Lambda_\psi(e)$ and an antiparticle $\Lambda_\eta(-e)$:

$$\Lambda_\psi(e) = -\frac{\hbar c}{2} \left( \bar{\psi}_+ \gamma_\mu \frac{\partial \psi_+}{\partial x_\mu} - \frac{\partial \bar{\psi}_+}{\partial x_\mu} \gamma_\mu \psi_+ \right) + ieA_\mu(x)\bar{\psi}_+ \gamma_\mu \psi_+ - mc^2\bar{\psi}_+ \psi_+, \quad \Lambda_\eta(-e) = -\frac{\hbar c}{2} \left( \bar{\eta}_+ \gamma_\mu \frac{\partial \eta_+}{\partial x_\mu} - \frac{\partial \bar{\eta}_+}{\partial x_\mu} \gamma_\mu \eta_+ \right) - ieA_\mu(x)\bar{\eta}_+ \gamma_\mu \eta_+ + mc^2\bar{\eta}_+ \eta_+.$$ \hspace{1cm} (29)

As the independent dynamical variables we consider the functions $\psi_+, \bar{\psi}_+$ and $\eta_+, \bar{\eta}_+$. Since, as was shown, in the stationary field particles and antiparticles are described independently, then it is sufficient to consider the case of particles and the similar relations for antiparticles are obtained if there are performed the replacements $e \to -e$ and $\psi_+ \to \eta_+$. From the Euler-Lagrange equations, with account of the form of the Lagrangians [29], [30], there follow the equations (28), which were obtained above directly from the Dirac equation. From the condition of the invariance of the particle Lagrangian [29] relative to the phase transformations:

$$\psi_+(x) \to \psi'_+(x) = \psi_+(x)e^{i\alpha}, \quad \bar{\psi}_+(x) \to \bar{\psi}'_+(x) = \bar{\psi}_+(x)e^{-i\alpha},$$ \hspace{1cm} (31)

where $\alpha$ is a real parameter, there follows the continuity equation for the particle’s probability density

$$\frac{\partial j_{\psi\mu}}{\partial x_\mu} = 0,$$ \hspace{1cm} (32)

where the 4-vector of the probability flow density has the form

$$j_{\psi\mu} = i\bar{\psi}_+ \gamma_\mu \psi_+.$$ \hspace{1cm} (33)

From [32] there follows the law of conservation of the total probability for a particle $\int dx j_{\psi0}(x) = \text{const}$. These relations are similar to those taking place in the nonrelativistic quantum theory [10].

The Lagrangian function densities [29], [31] depend on the wave functions of a particle and an antiparticle and on the electromagnetic field which is considered as an external field, but they do not depend explicitly on $x$. Consequently
the form of the Lagrangians should not change under the translation of the whole system, including an external field, by an arbitrary 4-vector \( a \). From these considerations we find the equation for the energy-momentum tensor

\[
\frac{\partial T_{\psi \mu \nu}}{\partial x_\mu} = -\frac{\partial \Lambda_\psi}{\partial A_\nu} \frac{\partial A_\nu(x)}{\partial x_\mu},
\]

(34)

where the energy-momentum tensor pertaining to a particle is defined by a known relation

\[
T_{\psi \mu \nu} = \frac{\partial \Lambda_\psi}{\partial \psi} \frac{\partial \psi}{\partial x_\mu} + \frac{\partial \bar{\psi}_+}{\partial x_\nu} \frac{\partial \Lambda_\psi}{\partial \bar{\psi}_+} - \Lambda_\psi \delta_{\mu \nu}.
\]

(35)

It is customary to introduce the 4-vector of the energy-momentum

\[
P_{\psi \mu} = \frac{i}{c} \int T_{\psi \mu 4} dx,
\]

(36)

where \( P_{\psi \mu} \equiv \left( P_\psi, \frac{i}{c} W_\psi \right) \). Thus the total momentum \( P_\psi \) and the total energy \( W_\psi \) are defined by the formulas

\[
P_\psi = \frac{i}{c} \int T_{\psi 4 4} dx, \quad W_\psi = \int T_{\psi 4 4} dx.
\]

(37)

With account of the form of the Lagrangian (29) and also the fact that this Lagrangian vanishes for the functions \( \psi_+, \bar{\psi}_+ \) satisfying the Dirac equation, we find the energy-momentum tensor expressed through the wave functions of a particle with positive energy:

\[
T_{\psi \mu \nu} = -\frac{\hbar c}{2} \psi_+ \gamma_\nu \frac{\partial \psi_+}{\partial x_\mu} + \frac{\hbar c}{2} \bar{\psi}_+ \gamma_\nu \psi_+.
\]

(38)

Taking into account the law of conservation of probability (32), the 4-vector of the energy-momentum can be represented in the form

\[
P_{\psi \mu} = -i\hbar \int \bar{\psi}_+ \gamma_\mu \frac{\partial \psi_+}{\partial x_\mu} dx,
\]

(39)

so that the total momentum \( P_\psi \) and energy \( W_\psi = -i\hbar P_{\psi 4} \) of a particle are given by the relations

\[
P_\psi = -i\hbar \int \bar{\psi}_+ \nabla \psi_+ dx, \quad W_\psi = i\hbar \int \bar{\psi}_+ \frac{\partial \psi_+}{\partial t} dx.
\]

(40)

These formulas are similar to those for the calculation of the average momentum and the average energy in the nonrelativistic quantum mechanics. The peculiarity of the field relations obtained here consists in that the fields therein have the meaning of the complex probability amplitudes for particles and antiparticles, and the temporal dependence of the fields is determined by the Fourier decompositions (8), (11) over only positive frequencies.

V. FREE PARTICLES AND ANTIPARTICLES

Let us apply the proposed interpretation of the Dirac theory to the description of the free particles and antiparticles. In the absence of an external field the equations for a particle and an antiparticle are the same:

\[
\begin{align*}
\left[ \hbar c \gamma_\nu - E \gamma_4 + mc^2 \right] \psi_+ (x, E) &= 0, \\
\left[ \hbar c \gamma_\nu - E \gamma_4 + mc^2 \right] \eta_+ (x, E) &= 0,
\end{align*}
\]

(41)

where the notation \( E \equiv \hbar \omega \) is introduced for positive energy. We look for the solutions of these equations in the form of plane waves

\[
\psi_+ (x, E) = \frac{1}{\sqrt{V}} \psi(k)e^{ikx}, \quad \eta_+ (x, E) = \frac{1}{\sqrt{V}} \eta(k)e^{ikx}.
\]

(42)
In this case:

\[
[i\hbar c\gamma - E\gamma_4 + mc^2] \psi(k) = 0, \quad [i\hbar c\gamma - E\gamma_4 + mc^2] \eta(k) = 0.
\]

(43)

The bispinors can be written in the form of the columns of the spinors

\[
\psi(k) = \begin{bmatrix} \varphi(k) \\ \chi(k) \end{bmatrix}, \quad \eta(k) = \begin{bmatrix} \zeta(k) \\ \psi(k) \end{bmatrix}.
\]

(44)

In the Dirac-Pauli representation

\[
\gamma = \begin{bmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

(45)

the equation for a particle takes the form

\[
\begin{bmatrix} -E + mc^2 & \hbar c\sigma k \\ -\hbar c\sigma k & E + mc^2 \end{bmatrix} \begin{bmatrix} \varphi(k) \\ \chi(k) \end{bmatrix} = 0.
\]

(46)

From here it follows the expression for positive energy of a particle:

\[
E = \sqrt{(\hbar c k)^2 + (mc^2)^2}.
\]

(47)

Since there are considered the functions (8), (11) for which the Fourier decomposition is carried out over only positive frequencies, then the root with the negative sign for energy should not be taken into account. Similar relations are valid for an antiparticle. Thus, the solutions of the equations (43) can be written in the form

\[
\psi(k) = \begin{bmatrix} \varphi(k) \\ \hbar c\sigma k \varphi(k) \\ E + mc^2 \end{bmatrix}, \quad \eta(k) = \begin{bmatrix} \zeta(k) \\ \hbar c\sigma k \zeta(k) \\ E + mc^2 \end{bmatrix}.
\]

(48)

For the fulfillment of the normalization for the bispinors \(\psi^+(k)\psi(k) = 1\) and \(\eta^+(k)\eta(k) = 1\) the following normalization of the spinors is necessary:

\[
\varphi^+(k)\varphi(k) = \zeta^+(k)\zeta(k) = \frac{1}{2} \left( 1 + \frac{mc^2}{E} \right).
\]

(49)

Thus, the general solution of the Dirac equation for a free particle with the momentum \(\delta k\) can be represented in one of the two forms:

\[
\psi(x, t) = \frac{1}{\sqrt{V}} \left[ \psi(k)e^{i(kx - \omega t)} + C^*\eta(k)e^{-i(kx - \omega t)} \right],
\]

\[
\eta(x, t) = \frac{1}{\sqrt{V}} \left[ \eta(k)e^{i(kx - \omega t)} + C^*\psi(k)e^{-i(kx - \omega t)} \right],
\]

(50)

where \(\omega = E/h = \sqrt{(\hbar c k)^2 + (mc^2)^2}/h\) is the positive frequency. However, as was outlined, these functions do not have the meaning of the probability amplitudes and cannot be used for the calculation of the probability characteristics. As in the nonrelativistic theory, the wave functions (42) having the meaning of the probability amplitudes describe the delocalized particle and antiparticle with the definite momentum and positive energy.

In the general case the wave functions of particles and antiparticles decomposed over the plane waves have the form

\[
\psi_+(x, t) = \frac{1}{\sqrt{V}} \sum_{k, r} c_r(k)\psi(k, r)e^{i(kx - \omega t)}, \quad \eta_+(x, t) = \frac{1}{\sqrt{V}} \sum_{k, r} b_r(k)\eta(k, r)e^{i(kx - \omega t)}
\]

(51)

Here the index \(r = \pm 1\) enumerates the spinors with a different in the rest frame projection of the spin to the axis \(z\). For the bispinors in (51) the orthonormality conditions are fulfilled

\[
\psi^+(k, r)\psi(k, r') = \delta_{rr'}, \quad \eta^+(k, r)\eta(k, r') = \delta_{rr'}.
\]

(52)
For the coefficients of the decomposition (51) the normalization conditions are also fulfilled

$$\sum_{k,r} |c_r(k)|^2 = \sum_{k,r} |b_r(k)|^2 = 1.$$  \hfill (53)

The completeness conditions for the system of the wave functions of a particle and an antiparticle with positive energies can be written in one of the equivalent forms:

$$\sum_{r=\pm 1} \left[ \psi(k, r)\overline{\psi}(k, r) + C^*\overline{\eta}(-k, r)\eta(-k, r)C \right] = 1,$$

$$\sum_{r=\pm 1} \left[ \eta(k, r)\overline{\eta}(k, r) + C^*\overline{\psi}(-k, r)\psi(-k, r)C \right] = 1.$$  \hfill (54)

According to (40) the energy and momentum of an electron and a positron are determined by the formulas

$$W = \sum_{k,r} E(k) \left[ c^*_r(k)c_r(k) + b^*_r(k)b_r(k) \right],$$  \hfill (55)

$$P = \sum_{k,r} \hbar k \left[ c^*_r(k)c_r(k) + b^*_r(k)b_r(k) \right],$$  \hfill (56)

where $E(k) = \sqrt{\left(\hbar ck\right)^2 + \left(mc^2\right)^2}$. Naturally, in the proposed interpretation the contribution of the states with negative energy into the total energy is absent. One should use the formulas (51) – (56) in proceeding towards the quantum-field description of electrons and positrons.

The Dirac equations describing electrons and positrons in stationary fields have the same form as in the standard approach [2]. The difference consists in that there is no necessity to take into account the states with negative energies $E < -mc^2$ which do not exist. Therefore, naturally, there is absent the tunneling probability into such states and there are absent the “paradoxes” conditioned by taking into account of such states (different variants of “the Klein paradox” [2, 6]).

VI. THE ELECTRON AND POSITRON STATES IN THE NONSTATIONARY ELECTROMAGNETIC FIELD

It was shown above that the states of electrons and positrons with positive energies in the stationary electromagnetic field are independent and they are described by the same unconnected among themselves equations which differ only by the sign of the charge. Let us consider the vectors in the presence of a nonstationary field. The vector potential can be represented as a sum of a stationary and a nonstationary in time terms:

$$A_\mu(x) = A_\mu^s(x) + A_\mu^n(x).$$

In this case the equation (20) takes the form

$$Q(x) + C^*\Pi(x) = ieA_\mu^s(x)\gamma_\mu \left( \psi_+(x) + C^*\overline{\eta}_+(x) \right),$$  \hfill (57)

where $Q(x)$ and $\Pi(x)$ are defined by the formulas (19). The equation (57) also can be decomposed into the equation for particles when $\eta_+(x) = 0$ and the equation for antiparticles when $\psi_+(x) = 0$:

$$\hbar c\gamma_\mu \frac{\partial \psi_+}{\partial x_\mu} - ieA_\mu(x)\gamma_\mu \psi_+ + mc^2\psi_+ = ieA_\mu(x)\gamma_\mu \psi_+(x),$$  \hfill (58)

$$\hbar c\gamma_\mu \frac{\partial \eta_+}{\partial x_\mu} + ieA_\mu(x)\gamma_\mu \eta_+ + mc^2\eta_+ = -ieA_\mu(x)\gamma_\mu \eta_+(x).$$  \hfill (59)

In the alternating external field the energy does not conserve.
VII. CONCLUSION

Usually, when difficulties of the Dirac theory are discussed the attention is paid to the fact that this theory is “one-particle”, that is describes one relativistic electron or positron being free or in the external electromagnetic field. Hence, it is believed that the emerging difficulties of the theory should be overcome when proceeding towards the quantum-field description on the basis of the secondary quantization method, that is, undoubtedly, true. However, as considered above, many difficulties fall away already within the frames of the one-particle theory if one consistently follows the probability interpretation of the functions describing the particle state. Obviously, not any complex function of coordinates and time can be treated as the probability amplitude. This in particular applies to the multicomponent complex field described by the Dirac equation. The general solution of the Dirac equation contains both the positive-frequency part describing the states with positive energies and the negative-frequency part to which correspond the states with energies of the opposite sign. Clearly, only the states with positive energies have the physical meaning, but the negative-frequency part cannot be directly interpreted as the function describing the real states. Thus, the general solution of the Dirac equation cannot be also interpreted as the wave function of the physical state. However, one cannot simply discard the states with negative energies since the functions of the states with positive energies do not constitute a full set. In order to overcome this contradiction, one should take into account that, as many years investigations have shown, the Dirac theory describes the two sorts of particles with the equal masses but opposite charges. Therefore it is natural, along with the Dirac equation containing the charge of a certain sign, to consider as well the Dirac equation with the charge of the opposite sign. The positive-frequency solutions of these equations prove to be independent and their unobservable negative-frequency solutions are expressed through these positive-frequency solutions by means of the matrix of the charge conjugation. As a result, to get in the Dirac theory a full set of the physical states having the meaning of the probability amplitudes, we should use the positive-frequency solutions of the Dirac equations with the equal mass and opposite signs of the charge.

In connection with the performed consideration, the question may arise why under the fully charge-symmetric treatment only the positive-frequency solutions correspond to the physical states. Where from and at what stage does the asymmetry arise regarding the functions with the different sign of energy? As is evident, this is conditioned by the assumption that in the nature there occur the states which correspond to the minimum of energy. In the opposite case, as the observables we should choose the functions describing the states with negative energies.

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