Perspective from Micro-Macro Duality*
–Towards non-perturbative
renormalization scheme–

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Abstract
The problem of renormalization procedure is re-examined from the viewpoint of Micro-Macro duality.

1 Micro-Macro duality

“Micro-Macro duality” is one of the basic features found between the invisible microscopic nature and its visible macroscopic manifestations, which can be understood in parallel with the Fourier duality between an abstract group and the concrete representations of the former. This viewpoint has played crucial roles in our analysis of the mutual relations between virtual dynamical levels and specific geometric ones in various contexts (see, \cite{1}). Using this general notion, we can provide the heuristic idea of “Quantum-Classical Correspondence” with precise mathematical formulations in which Micro and Macro are mutually and closely related with each other; the latter, Macro, emerges from the former, Micro, through the processes of condensation of infinitely many quanta and the essential features of the former can be determined and reconstructed to certain extent from the data structure at the levels of Macro, in close analogy with the above-mentioned duality in the context of groups and representations. From this viewpoint of Micro-Macro duality, we try here to sketch the essential ingredients for a natural reformulation of the traditional theory of renormalization procedures commonly adopted in the physical applications of quantum field theory (QFT for short). For this purpose, the most relevant notions in what follows are the group of scale transformations and the associated aspects of broken symmetry which is not unitarily implementable within a sector (defined by a quasi-equivalence class of factor representations of the algebra of observables) but which generates a family of (mutually disjoint) sectors along an orbit of symmetry transformations.

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2 Broken scale invariance: imaginary-time vs. real-time

Here we briefly summarize some consequences of broken scale invariance in relativistic QFT.

1) Imaginary-time version = “temperature as order parameter of broken scale invariance”:

**Theorem 1 (IO04 [2])** In the standard setting up of algebraic QFT, the inverse temperature \( \beta := (\beta^\mu \beta^\mu)^{1/2} \) is a macroscopic order parameter for parametrizing mutually disjoint sectors in the thermal situation arising from the broken scale invariance under the renormalization-group transformations, where \( \beta^\mu \) is an inverse temperature 4-vector of a relativistic KMS state \( \omega_{\beta^\mu} \) describing a thermal equilibrium in its rest frame.

This result is based on the notion of a scaling algebra due to Buchholz-Verch [3] in combination with Takesaki’s theorem [4] on the disjointness of KMS states at different temperatures valid for a system with physical observables constituting a von Neumann algebra of type III.

2) What should be the corresponding “real-time” version to the above?: Renormalization Theory (at \( T = 0 \)K).

2.1 How to formulate broken scale invariance

**Theorem 2 (Takesaki’70 [4])** For a quantum C*-dynamical system with type III representations in its KMS states, any pair of KMS states for different (inverse) temperatures \( \beta_1 \neq \beta_2 \) are mutually disjoint \( \omega_{\beta_1} \circ \omega_{\beta_2} \).

The claim of the first theorem due to myself is that the above disjointness allows us to interpret the inverse temperature \( \beta \) as an order parameter of broken scale invariance. In the usual situation, this kind of symmetry breakdown arises as a spontaneous breakdown of a symmetry described by a group acting on the algebra of physical quantities by automorphisms. In contrast, the present case of broken scale invariance usually involves explicit breaking terms such as mass, which seem to prevent scale transformations from being treated as automorphisms. However, the results on scaling algebra in algebraic QFT due to [3, 2] shows that the above negative anticipation can be avoided.

Their results can be summarized as follows. Let the following requirements be imposed on all the possible renormalization-group transformations \( R_\lambda \):

(i) \( R_\lambda \) should map the given net \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \) of local observables at spacetime scale 1 onto the corresponding net \( \mathcal{O} \rightarrow \mathcal{A}_\lambda(\mathcal{O}) \equiv \mathcal{A}(\lambda \mathcal{O}) \) at scale \( \lambda \), i.e.,

\[
R_\lambda : \mathcal{A}(\mathcal{O}) \rightarrow \mathcal{A}_\lambda(\mathcal{O})
\]

for every region \( \mathcal{O} \subset \mathbb{R}^4 \), through which the light velocity \( c \) is kept unchanged: \( (\lambda x)^0 / (\lambda x)^0 = x^0 / x^0 \).

(ii) In the Fourier-transformed picture, the subspace \( \tilde{\mathcal{A}}(\tilde{\mathcal{O}}) \) of all (quasi-local) observables carrying energy-momentum in the set \( \mathcal{O} \subset \mathbb{R}^4 \) is transformed as

\[
R_\lambda : \tilde{\mathcal{A}}(\tilde{\mathcal{O}}) \rightarrow \tilde{\mathcal{A}}_\lambda(\tilde{\mathcal{O}})
\]
for \( \forall \tilde{O} \), where \( \tilde{A}_\lambda(\tilde{O}) := \tilde{A}(\lambda^{-1}\tilde{O}) \), through which the Planck constant \( \hbar \) is unchanged: \((\lambda^{-1})p_\mu(\lambda x)^\mu /h = p_\mu x^\mu /h \).

(iii) For scale invariant theories \( R_\lambda \) may not be isomorphisms but are maps continuous and bounded uniformly in \( \lambda \).

Then, scaling net \( \mathcal{O} \rightarrow \tilde{A}(\mathcal{O}) \) corresponding to the original local net \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \) of observables is defined as the local net consisting of scale-changed observables under the action of all the possible choice of \( R_\lambda \) satisfying (i)-(iii). Mathematically the algebra \( \tilde{A}(\mathcal{O}) \) can be understood as the algebra \( \Gamma(\mathcal{R}^+ \times \mathcal{A}(\mathcal{O})) \) of sections \( \mathcal{R}^+ \ni \lambda \mapsto \tilde{A}(\lambda) \in \mathcal{A}(\mathcal{O}) \) of algebra bundle \( \Pi_{\lambda \in \mathcal{R}^+} \tilde{A}_\lambda(\mathcal{O}) \rightarrow \mathcal{R}^+ \) over the multiplicative group \( \mathcal{R}^+ \) of scale changes (= \( \mathcal{A}(\mathcal{O}) \times_\alpha \mathcal{R}^+ \): augmented algebra). Then, the scaling algebra \( \tilde{A} \) is defined by the \( \mathcal{C}^* \)-inductive limit of all local algebras \( \mathcal{A}(\mathcal{O}) \). Algebraic structures making \( \tilde{A}(\mathcal{O}) \) a unital \( \mathcal{C}^* \)-algebra are defined in a pointwise manner, for instance, by \( (\tilde{A} \cdot \tilde{B})(\lambda) := \tilde{A}(\lambda)\tilde{B}(\lambda) \), \((\tilde{A}^*)(\lambda) := \tilde{A}(\lambda)^* \), etc., and \( ||\tilde{A}|| := \sup_{\lambda \in \mathcal{R}^+} ||\tilde{A}(\lambda)|| \).

From the scaled actions \( \mathcal{A}_\lambda \otimes \mathcal{P}^+_1 \) of Poincaré group on \( \mathcal{A}_\lambda \) with \( \alpha^{(\lambda)}_{x,\Lambda} = \alpha_{x,\lambda,\Lambda} \), an action of \( \mathcal{P}^+_1 \) is induced on \( \tilde{A} \) by
\[
(\tilde{\alpha}_{x,\lambda}(\tilde{A}))(\lambda) := \alpha_{x,\lambda}(\tilde{A}(\lambda)).
\]

Then the conditions (ii), (iii) are expressed simply as the continuity of Poincaré-group action: \( ||\tilde{\alpha}_{x,\lambda}(\tilde{A}) - \tilde{A}||_{(x,\Lambda) \rightarrow (0,1)} \rightarrow 0 \). Then, the scaling net \( \mathcal{O} \rightarrow \tilde{A}(\mathcal{O}) \) is shown to satisfy all the properties to characterize a relativistic local net of observables if the original one \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \) does.

Now scale transformations can be defined by an automorphic action \( \tilde{\sigma}_{\mathcal{R}^+} \) of the \( \mathcal{R}^+ \) on the scaling algebra \( \tilde{A} \), given for \( \forall \mu \in \mathcal{R}^+ \) by
\[
(\tilde{\sigma}_\mu(\tilde{A}))(\lambda) := \tilde{A}(\mu\lambda), \quad \lambda > 0,
\]
satisfying
\[
\tilde{\sigma}_\mu(\tilde{A}(\mathcal{O})) = \tilde{A}(\mu\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4,
\]
\[
\tilde{\sigma}_\mu \circ \tilde{\alpha}_{x,\Lambda} = \tilde{\alpha}_{x,\mu\Lambda} \circ \tilde{\sigma}_\mu, \quad (x,\Lambda) \in \mathcal{P}^+_1.
\]

**Remark 3** Scaling transformations \( \tilde{\sigma}_{\mathcal{R}^+} \) play the role of renormalization group transformations to relate observables at different scales.

**Remark 4** Since a broken symmetry can always be restored by taking all breaking parameters as variables undergoing the broken symmetry transformations, there is no miracle in the results due to Buchholz and Verch through their complicated analysis: it can naturally be accommodated as a special case into the general definition of a augmented algebra \( \hat{\mathcal{F}} := \Gamma(\times \mathcal{G} \times \mathcal{F}) \) with the choice of \( \mathcal{H} := \mathcal{P}^+_1, \mathcal{G} = \mathcal{H} \times \mathcal{R}^+ \) (semidirect product) and together with slight modifications due to spacetime dependence \( \mathcal{F} \Rightarrow \mathcal{A}(\mathcal{O}) \) (which is affected by the action of \( \mathcal{R}^+ \) ) (and the intervention of the centre due to SSB: \( SO(3) \backslash \mathcal{L}^+_1 \cong \mathbb{R}^3 \) at \( T \neq 0 \) K) \( \mathcal{L}^+_1 \).

Scaled actions \( \alpha^{(\lambda)}_{x,\Lambda} = \alpha_{x,\lambda,\Lambda} \) of Poincaré group on \( \mathcal{A}_\lambda \) can also be naturally understood as the conjugacy change of the stability group \( H \rightarrow gHg^{-1} \) from the point \( He \) to \( Hg^{-1} \) on the base space \( H \backslash G = \mathcal{R}^+ \): \( s_\mu(x,\Lambda)s_\mu^{-1} = (\mu x, \Lambda) \).
2.2 Scale changes on states

Corresponding to each probability measure \( \mu \) on the centre \( \mathcal{F}(\hat{A}) = \mathcal{F}(\hat{A}(\emptyset)) = C(\mathbb{R}^+) \) due to the broken scale invariance, we have a conditional expectation \( \hat{\mu} \) from \( \hat{A} \) to \( A \):

\[
\hat{\mu} : \hat{A} \ni \hat{\omega} \mapsto \int_{\mathbb{R}^+} d\mu(\lambda) \hat{\omega}(\lambda) \in A.
\]

Instead of \( d\mu(\lambda) \), it is also possible to take the Haar measure \( d\lambda / \lambda \) of \( \mathbb{R}^+ \). As it is a positive unbounded measure but not a probability one with the total mass one, however, the corresponding map \( \hat{\mu} \) becomes an \textit{operator-valued weight} whose images are not guaranteed to be finite. Any state \( \hat{\omega} \in \hat{E}_A \) can be lifted onto \( \hat{A} \) through \( \hat{\mu} \) by

\[
E_A \ni \omega \mapsto \hat{\mu}^*(\omega) = \omega \circ \hat{\mu} = \omega \otimes \mu \in E_{\hat{A}},
\]

where we have used \( \hat{\mathcal{A}} \subset C(\mathbb{R}^+, A) \simeq A \otimes C(\mathbb{R}^+) \).

In \([3]\) the case \( \mu = \delta_{\lambda=1} \) (Dirac measure at the identity of \( \mathbb{R}^+ \)) is called a \textit{canonical lift} \( \hat{\omega} := \omega \circ \delta_1 \). The scale transformed state defined by

\[
\hat{\omega}_\lambda := \hat{\omega} \circ \hat{\sigma}_\lambda = \omega \circ \delta_\lambda
\]

describes the situation at scale \( \lambda \) due to the renormalization-group transformation of scale change \( \lambda \).

Conversely, starting from a state \( \hat{\omega} \) of \( \hat{A} \), we can obtain its central decomposition as follows: first, we call two natural embedding maps \( \iota : A \hookrightarrow \hat{A} \) \( \{i(A)\} = A \) and \( \kappa : C(\mathbb{R}^+) \simeq \mathcal{F}(\hat{A}) \to \hat{A} \). Pulling back \( \hat{\omega} \) by \( \kappa^* : E_{\hat{A}} \to E_{C(\mathbb{R}^+)} \), we can define a probability measure \( \rho_{\hat{\omega}} := \kappa^*(\hat{\omega}) = \hat{\omega} \circ \kappa = \hat{\omega} |_{C(\mathbb{R}^+)} \) on \( \mathbb{R}^+ \), namely, \( \hat{\omega} |_{C(\mathbb{R}^+)} (f) = \int_{\mathbb{R}^+} d\rho_{\hat{\omega}}(\lambda)f(\lambda) \) for \( \forall f \in C(\mathbb{R}^+) \).

For any positive operator \( \hat{A} = \int a dE_{\hat{A}}(a) \in \hat{A} \), we can consider the central supports \( c(E_{\hat{A}}(\Delta)) \in \text{Proj}(\mathcal{F}(\hat{A})) \) of \( E_{\hat{A}}(\Delta) \in \text{Proj}(\mathcal{F}(\hat{A})) \) with a Borel set \( \Delta \) in \( Sp(\hat{A}) \subset [0, +\infty) \) satisfying \( c(E_{\hat{A}}(\Delta)) E_{\hat{A}}(\Delta) = E_{\hat{A}}(\Delta) \). From this we see that \( \rho_{\hat{\omega}}^\mu(c(E_{\hat{A}}(\Delta)) = 0 \) implies \( \hat{\omega}''(E_{\hat{A}}(\Delta)) = 0 \), where \( \hat{\omega}'' \) and \( \rho_{\hat{\omega}}^\mu \) are the extensions of \( \hat{\omega} \) and \( \rho_{\hat{\omega}} \) to \( \hat{\pi}_\omega(\hat{A})'' \) and \( L^\infty(\mathbb{R}^+, dp_{\omega}) \), respectively. Thus, we can define the Radon-Nikodym derivative \( \omega_{\lambda} := \frac{d\hat{\omega}_\lambda}{d\omega_{\lambda}}(\hat{\omega}) \) w.r.t. \( \rho_{\hat{\omega}} \) as a state on \( \hat{\pi}_\omega(\hat{A})'' \) so that

\[
\hat{\omega}(\hat{A}) = \int d\rho_{\hat{\omega}}(\lambda) \omega_{\lambda}(\hat{A}(\lambda)) = \int d\rho_{\hat{\omega}}(\lambda) \omega_{\lambda}(\hat{\delta}_{\lambda}(\hat{A})) = \int d\rho_{\hat{\omega}}(\lambda) \left[ \omega_{\lambda} \otimes \hat{\delta}_{\lambda} \right](\hat{A}).
\]

Then, the pull-back \( \iota^*(\hat{\omega}) = \hat{\omega} \circ \iota \in E_A \) of \( \hat{\omega} \in E_{\hat{A}} \) by \( \iota^* : E_{\hat{A}} \to E_A \) is given by

\[
\iota^*(\hat{\omega}) = \int d\rho_{\hat{\omega}}(\lambda) \omega_{\lambda},
\]

owing to the relation

\[
\iota^*(\hat{\omega})(A) = \hat{\omega}(\iota(A)) = \int d\rho_{\hat{\omega}}(\lambda) \omega_{\lambda}(A) = \left[ \int d\rho_{\hat{\omega}}(\lambda) \omega_{\lambda} \right](A).
\]

Using this relation to the scaled canonical lift, \( \hat{\omega}_\lambda := \hat{\omega} \circ \hat{\sigma}_\lambda = (\omega \circ \delta_1) \circ \hat{\sigma}_\lambda = \omega \circ \delta_\lambda \), of a state \( \omega \in E_A \), we can easily see \( \iota^*(\omega \circ \delta_\lambda) = \iota^*(\hat{\omega}_\lambda) = \omega_{\lambda} = \frac{d\hat{\omega}_\lambda}{d\omega_{\lambda}}(\lambda) = \phi_{\lambda}(\omega) \),
where \( \phi_\lambda \) is the isomorphism introduced in [3] between \( \omega \) and the canonical lift \( \hat{\omega}_\lambda \in E_{\hat{A}} \) projected onto \( \hat{A}/\ker(\hat{\pi}_\lambda \circ \hat{\delta}_\lambda) \).

Thus we can lift any state \( \omega \in E_A \) canonically from \( A \) to \( \hat{\omega}_\lambda \in E_{\hat{A}} \), and, after the scale shift \( \hat{\delta}_\lambda \) on \( \hat{A} \), return \( \hat{\omega} \circ \hat{\delta}_\lambda \) back onto \( A \): \( \phi_\lambda(\omega) = \omega_\lambda = \iota^*(\omega \circ \hat{\delta}_\lambda) \), as result of which we obtain the scaled-shifted state \( \omega_\lambda \in E_A \) from \( \omega \in E_A \) in spite of the absence of scale invariance on \( A \).

Now applying this procedure to \( \omega = \omega_\beta \) (any state belonging to the family of relativistic KMS states with the same \( (\beta^2)^{1/2} \)), we have a genuine KMS state by going to their rest frames. Then we have \( \hat{\omega}_\lambda = (\hat{\omega}_\beta)_\lambda = \omega_\beta \circ \hat{\delta}_\lambda \) which is shown to be a KMS state at \( \beta/\lambda \):

\[
(\omega_\beta \circ \hat{\delta}_\lambda)(\hat{A}\hat{\alpha}_{\lambda}(\hat{B})) = \omega_\beta(\hat{A}(\lambda)\alpha_{\Lambda}(\hat{B}(\lambda)))
= \omega_\beta(\alpha_{\Lambda^{-1}\beta}(\hat{B}(\lambda))\hat{A}(\lambda))
= (\omega_\beta \circ \hat{\delta}_\lambda)(\alpha_{\Lambda^{-1}\beta/\lambda}(\hat{B})\hat{A}),
\]

and hence, \( (\hat{\omega}_\beta)_\lambda \in K_{\beta/\lambda} \), \( \phi_\lambda(\omega_\beta) \in K_{\beta/\lambda} \).

As already remarked, the above discussion is seen to apply equally to the spontaneous as well as explicitly broken scale invariance with explicit breaking parameters such as mass terms. The actions of scale transformations on such variables as \( x^\mu, \beta^\mu \) and also conserved charges are just straightforward, which is justified by such facts that the first and the second ones are of kinematical nature and that the second and the third ones exhibit themselves in the state labels for specifying the relevant sectors in the context of the superselection structures [6,5]. This gives an alternative verification to the so-called non-renormalization theorem of conserved charges. In sharp contrast, other such variables as coupling constants (to be read off from the data of correlation functions or Green’s functions) are affected by the scaled dynamics, and hence, may show non-trivial scaling behaviours with deviations from the canonical (or kinematical) dimensions, in such forms as the running couplings or anomalous dimensions. Thus, the transformations \( \hat{\delta}_\lambda \) (as “exact” symmetry on the augmented algebra \( \hat{A} \)) are understood to play the roles of the renormalization-group transformations (as broken symmetry on the original algebra \( A \)). As a result, we see that classical macroscopic observable \( \beta \) naturally emerging from a microscopic quantum system is verified to be an order parameter of broken scale invariance involved in the renormalization group.

In the present context of the scale transformations in real version, we can use these scale changes of states to compare different theories renormalized by renormalization conditions imposed at different scale points.

### 3 Nuclearity Condition & Renormalizability

For the purpose of controlling the phase space properties in algebraic QFT, the nuclearity condition is formulated as follows: the map \( \Phi_{\Omega, E} : \mathcal{A}(\Omega)_1 \ni A \rightarrow P_E A \Omega \in \mathfrak{f}_1 \) with \( P_E \) the spectral projection on state vectors having energy below
\( E \) is required to be \textit{nuclear}, admitting such a decomposition as

\[
\Phi_{\mathcal{O}, E}(A) = \sum_{i=1}^{\infty} \varphi_i(A) \xi_i \quad \text{for } \forall A \in \mathcal{A}(\mathcal{O})_1
\]

with \( \varphi_i \in \mathcal{A}(\mathcal{O})^* \) and \( \xi_i \in \mathcal{H} \) s.t. \( \sum_{i=1}^{\infty} ||\varphi_i|| ||\xi_i|| < \infty \),

on the unit ball \( \mathcal{A}(\mathcal{O})_1 := \{ A \in \mathcal{A}(\mathcal{O}); ||A|| \leq 1 \} \) of observables. This condition excludes such “unphysical” fields as \textit{generalized free fields} without discrete mass spectrum admitting no particle picture to be detected in scattering experiments. The nuclearity condition and the assumption of the approximate scale invariance are known [8] to imply that the local subalgebra \( \mathcal{A}(\mathcal{O}) \) is a factor von Neumann algebra of \textit{type III} with no minimal projections, i.e., any projection operator \( E \in \mathcal{A}(\mathcal{O}) \backslash \{0\} \) is equivalent to the identity operator \( I = id_{\mathcal{H}}; \exists v \in \mathcal{A}(\mathcal{O}) \) s.t. \( v^* v = I, v v^* = E \).

### 3.1 Point-like fields as idealized local observables

On the basis of the nuclearity condition [7] and the energy-bound, the notion of point-like field operators [8][10] has been established satisfying the operator-product expansion (OPE) in a \textit{non-perturbative} way in algebraic QFT by Bostelmann [12]. The \textit{energy bound} means the requirement that observed values of quantum fields \( \hat{\phi}(f) \) can become large only with large energy: for any \( l > 0 \), there is a sufficiently large \( m > 0 \) that the inequality

\[
|||(1 + H)^{-m} \hat{\phi}(f) (1 + H)^{-m}|| \leq c \int dx |(1 - \Delta)^{-l} f(x)|,
\]

holds with a (positive) Hamiltonian \( H \), operator norm \( || \cdot || \) in the vacuum sector \( \mathcal{H} \) and \( \Delta \): Laplacian on \( \mathbb{R}^4 \). When this holds, there exist a sequence of test functions tending to \( f_i \to \delta_x \): Dirac measure at \( x \) and a sufficiently large integer \( m > 0 \) such that

\[
\lim_{i \to \infty} (1 + H)^{-m} \hat{\phi}(f_i) (1 + H)^{-m} =: (1 + H)^{-m} \hat{\phi}(x) (1 + H)^{-m}.
\]

Then a \textit{field} \( \hat{\phi}(x) \) \textit{at a point} \( x \) is well-defined as a linear form on such states \( \omega \) in the vacuum sector that \( \omega(1 + H)^{2m} < \infty \). Hermitian elements in the sets \( Q_{m,x} := \{ \hat{\phi}(x); ||(1 + H)^{-m} \hat{\phi}(x)(1 + H)^{-m}|| < \infty \} \) of point-like fields are idealized observables at spacetime points \( x \) meaningful for such states \( \omega \) that \( \omega(1 + H)^{2m} < \infty \). The set \( Q_{m,x} \) of such point-like fields are, in general, finite-dimensional linear spaces satisfying \( Q_{m,x} \subset Q_{m',x} \) for \( m \leq m' \) and are invariant under the stability group of \( x \) in \( P_{\mathcal{H}}^\mathcal{V} \). The meaninglessness notion of \textit{product of fields at a point} \( x \) is replaced in \( Q_{m,x} \) by \textit{normal products} defined by the following OPE: for instance, ill-defined square \( \hat{\phi}(x)^2 \) is replaced by the subspaces \( \mathcal{N}(\hat{\phi}^2) \subset Q_{m,x} \) generated by normal products \( \Phi_j(x), j = 1, \cdots, J(q), \) appearing in OPE of \( \hat{\phi}(x + \frac{\xi}{2}) \hat{\phi}(x - \frac{\xi}{2}) \):

\[
|||(1 + H)^{-n} \left[ \hat{\phi}(x + \frac{\xi}{2}) \hat{\phi}(x - \frac{\xi}{2}) - \sum_{j=1}^{J(q)} c_j(\xi) \Phi_j(x) \right] (1 + H)^{-n}|| \leq c |\xi|^q,
\]
which is satisfied for any \( \hat{\phi} \in \mathcal{Q}_{m,x} \) for spacelike \( \xi \in \mathbb{R}^4 \to 0 \) with arbitrary \( q > 0 \), by choosing a finite number of fields \( \Phi_j(x) \in \mathcal{Q}_{n,x} \) and sufficiently large \( n \), and some analytic functions \( \xi \mapsto c_j(\xi) \), \( j = 1, \ldots, J(q) \). Using this definition, the spaces \( \mathcal{N}(\hat{\phi}^p)_{q,x}(\subset \mathcal{Q}_{n,x}) \) of normal products of higher powers \( p \) can similarly be defined. While the linear spaces \( \mathcal{Q}_{m,x} \) of pointlike fields lack the multiplication structure, the validity of OPE allows us to provide them with a structure generalizing a product system of Hilbert modules \( \mathcal{Q}_{n,x} \). It is also possible for the partial derivatives \( \partial_\xi \) of spacetime coordinates \( \xi \) to act on these spaces through the “balanced derivatives” \( \partial_\xi \hat{\phi}(x + \frac{\xi}{2})\hat{\phi}(x - \frac{\xi}{2}) \) \cite{13} which are contained in \( \mathcal{N}(\hat{\phi}^2)_{q,x} \) (for large \( q \)) as shown by the relation,

\[
\|(1 + H)^{-n}[\partial_\xi \hat{\phi}(x + \frac{\xi}{2})\hat{\phi}(x - \frac{\xi}{2}) - \sum_{j=1}^{J(q)} \partial_\xi c_j(\xi) \hat{\phi}_j(x)](1 + H)^{-n}\| \leq c|\xi|^r ,
\]

valid for \( \forall r > 0 \ \exists q \) and \( \exists n \) sufficiently large.

### 3.2 Comparison between OPE & Wigner-Eckhart theorem

To take advantage of the above OPE structure, we compare it with the basic feature of the Wigner-Eckhart theorem for an irreducible family of tensor operators \( \{F_{m_1}^{(\gamma)}; m_1 = -\gamma_1, -\gamma_1 + 1, \ldots, \gamma_1 - 1, \gamma_1 \} \) under the action of a (compact) group \( G \) (such as \( SU(2) \), typically):

\[
\langle \gamma m | F_{m_1}^{(\gamma)} | \gamma_2 m_2 \rangle = \langle \gamma | F^{(\gamma)} | \gamma_2 \rangle \langle \gamma m | (\gamma_1 m_1), (\gamma_2 m_2) \rangle ,
\]

where \( \langle \gamma m | (\gamma_1 m_1), (\gamma_2 m_2) \rangle \) are the Clebsch-Gordan coefficients describing a branching rule from the Kronecker tensor product \( [\gamma_1 \otimes \gamma_2] (g) = \gamma_1 (g) \otimes \gamma_2 (g) \) of representations \( [\gamma_i, V_{\gamma_i}] \) \( (i = 1, 2) \) into irreducible ones \( \{ (\gamma, V_\gamma) \} \in \text{Rep}(G) \) \( (\gamma, m) \in V_\gamma \) of \( G \). Note that the Kronecker tensor product \( \gamma_1 \otimes \gamma_2 \) of \( G \) is the restriction of the tensor product representation \( \gamma \otimes \gamma_2 (g_1, g_2) = \gamma (g_1) \otimes \gamma_2 (g_2) \) of \( G \times G \) onto a subgroup \( G \) embedded via the diagonal map \( \delta_\mathcal{C} : G \ni g \mapsto \delta_\mathcal{C} (g) = (g, g) \in G \times G \):

\[
[\gamma_1 \otimes \gamma_2] (g) = [(\gamma_1 \otimes \gamma_2) \circ \delta_\mathcal{C} ] (g) = \gamma_1 (g) \otimes \gamma_2 (g) .
\]

According to this formula, the matrix elements of the tensor operator \( \{F_{m_1}^{(\gamma)}; m_1 = -\gamma_1, -\gamma_1 + 1, \ldots, \gamma_1 - 1, \gamma_1 \} \) are decomposed into two factors, \( G \)-invariant dynamical one \( \langle \gamma | F^{(\gamma)} | \gamma_2 \rangle \) & purely kinematical one \( \langle \gamma m | (\gamma_1 m_1), (\gamma_2 m_2) \rangle \) determined completely by \( G \)-transformation property of \( F^{(\gamma)} \).

In the case of OPE,

\[
\varphi_1(x + \frac{\xi}{2})\varphi_2(x - \frac{\xi}{2}) \approx \sum \mathcal{N}(\varphi_1\varphi_2)_j(x)C_j(\xi) + \cdots ,
\]

the dual map \( \delta^* \) of \( \delta \) given by \( \delta^* (\varphi_1 \otimes \varphi_2)(x) = (\varphi_1 \otimes \varphi_2)(\delta(x)) = \varphi_1(x) \otimes \varphi_2(x) \) is ill-defined for operator-valued distributions \( \varphi_i \). In this context, therefore, the diagonal map \( \delta(x) = (x, x) \) should be understood in the limit: \( (x + \frac{\xi}{2}, x - \frac{\xi}{2}) \approx \delta(x) = (x, x) \) after “removing” such divergent terms as \( C_j(\xi) \). Except
for this difference, the essence of OPE formula is just in parallel with the above Wigner-Eckhart case: \textit{factorization} of the product $\varphi_1(x + \frac{\xi}{2})\varphi_2(x - \frac{\xi}{2})$ into two components, \textit{dynamical non-singular} factors $N(\varphi_1, \varphi_2)$ depending only on the “centre of mass” $[x + \frac{\xi}{2}] + [x - \frac{\xi}{2}]/2 = x$ and \textit{c-number kinematical singular functions} $C_1(\xi)$ of the relative coordinates $(x + \frac{\xi}{2}) - (x - \frac{\xi}{2}) = \xi$. Note that the \textit{singularity} of the product $\varphi_1(x + \frac{\xi}{2})\varphi_2(x - \frac{\xi}{2})$ in the limit of $\xi \to 0$ is isolated into these kinematical \textit{c-number factors} $C_1(\xi) = N_1(\lambda)C_1^{\text{reg}}(\xi)$, where $\lambda := |\xi|^{-1}$ represents the \textit{cutoff momentum} to regularize the UV divergences in a \textit{non-perturbative} way and $N_1(\lambda)$ can be taken as \textit{counter terms} to define \textit{renormalized field operators} (formally) by

$$\varphi_{\text{ren}}(x) := N_1(\lambda)^{-1/2}\varphi(x).$$

It may be instructive to find the analogy of the present structure with the time-localization scale $\Delta t$ of Hida derivatives $a_t, a_t^*$ in the White-Noise Analysis \cite{15}.

Since the limit $\xi \to 0$ means

$$||(1 + H)^{-\lambda} \left[ \hat{\varphi}(x + \frac{\xi}{2})\hat{\varphi}(x - \frac{\xi}{2}) - \sum_{j=1}^{J(q)} c_j(\xi) \hat{\Phi}_j(x) \right] (1 + H)^{-\lambda}|| \leq c|\xi|^q,$$

the convergence $\hat{\varphi}(x + \frac{\xi}{2})\hat{\varphi}(x - \frac{\xi}{2}) \to \sum_{j=1}^{J(q)} c_j(\xi) \hat{\Phi}_j(x)$ is \textit{state-dependent} so that

$$\omega\left( \left[ \hat{\varphi}(x + \frac{\xi}{2})\hat{\varphi}(x - \frac{\xi}{2}) - \sum_{j=1}^{J(q)} c_j(\xi) \hat{\Phi}_j(x) \right] \right) \to 0$$

holds only for those states $\omega$ which satisfy

$$\omega((1 + H)^{2n}) < \text{constant}.$$

Thus, states $\omega$ for which OPE is valid \textit{cannot be localized}, and hence, to such an extent, the spacetime point $x$ in $\varphi(x)$ is actually extended!

In the above situations the common essence can be found in the relevance of some selective filters depending on the choice of states $\omega$. We note here that the (approximate) \textit{diagonal maps} $\delta(x) = (x, x)$,

$$(x + \frac{\xi}{2}, x - \frac{\xi}{2}) \sim (x, x) = \delta(x);$$

$$[(\gamma_1 \otimes \gamma_2) \circ \delta](g) = [\gamma_1 \hat{\otimes} \gamma_2](g) = \gamma_1(g) \otimes \gamma_2(g),$$

play essential roles in the definition of \textit{Hopf algebra} structures with the harmonic-analytic dualities controlled by Kac-Takesaki operator (of the so-called duality transformations), which should play crucial roles in extending the above relations for two-point functions to arbitrary $n$-point functions.

By the above condition $\omega((1 + H)^{2n}) < \text{constant}$, the selective filter on the initial state $\omega$ is related with the nucularity condition $\Phi_{\varsigma, E}(A) = P_{E\Lambda} \Omega = \sum_{n=1}^{\infty} \varphi_1(A)\xi_n$ whose energy scale $E$ can be related to the above cutoff $\lambda$. In spite of the sharp contrast between \textit{almost finite-dimensionality} as nucularity and \textit{\infty-dimensionality} inherent to \textit{type III}, both properties are closely related.
with the **nuclearity condition** and are crucial for **renormalizability** and for shifts of the renormalization points by renormalization-group transformations:

1) **renormalizability** = finiteness of the number of graph types of divergent “1-particle irreducible (1PI)” diagrams is expected to follow from the very nuclearity condition (= *intra-sectorial* structure);

2) the **absence of minimal projection in type III** von Neumann factors (due to approximate scale invariance) allows the **shifts of renormalization points** by scale transformations = renormalization-group transformations. This gives the inter-sectorial relations among “sectors parametrized by renormalization conditions” at different renormalization points (on the centre \( \mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}(\mathcal{O})) = C(\mathbb{R}^+) \) of the scaling algebra).

In this sense, the **nuclearity condition** can be regarded as the mathematical version of the renormalizability condition and broken scale invariance inherent to local subalgebras \( \mathcal{A}(\mathcal{O}) \) of type III with no minimal projection requires the renormalization condition to be specified at some renormalization point which can, however, be **chosen arbitrarily**.

Here we present some new perspectives for understanding the conceptual and mathematical meaning of renormalization scheme in relation with such key notions as the nuclearity condition, broken scale invariance and the type III nature of local subalgebras of quantum fields in close relation with algebraic QFT. What remains to be clarified is the following:

1. **Counter terms** \( N_i(\lambda) \) are expected to be factors of automorphy associated to the fractional linear transformations of (approximate) conformal symmetry \( SO(2,4)(\simeq SU(2,2)) \) associated with (approximate) scale invariance. Along this line, the Callan-Symanzik type equation for \( N_i(\lambda) \) involving running coupling constants and anomalous dimensions should be established.

2. **In the opposite direction** to the conventional renormalization scheme based on perturbative expansion method starting from a “Lagrangian” (along such a flow chart as “Lagrangian” \( \rightarrow \) perturbative expansion \( \rightarrow \) renormalization + OPE), the perturbation expansion itself should be derived and justified as a kind of asymptotic analysis within the non-perturbative formulation of renormalization based on OPE: namely, we advocate such a flow chart as starting from OPE \( \rightarrow \) renormalization \( \rightarrow \) perturbative method as asymptotic expansion \( \rightarrow \) “Lagrangian” determined by \( \Gamma_{1PI} \) & renormalizability (= finite generation property).

3. More detailed mathematical connections should be clarified among nuclearity condition, renormalizability, renormalization conditions, renormalization group to shift renormalization point and broken scale invariance inherent to local subalgebras \( \mathcal{A}(\mathcal{O}) \) of type III from the viewpoint of non-standard analysis.

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