A Unifying Framework to Characterize the Power of a Language to Express Relations*

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Abstract

In this extended abstract we provide a unifying framework that can be used to characterize and compare the expressive power of query languages for different data base models. The framework is based upon the new idea of valid partition, that is a partition of the elements of a given data base, where each class of the partition is composed by elements that cannot be separated (distinguished) according to some level of information contained in the data base. We describe two applications of this new framework, first by deriving a new syntactic characterization of the expressive power of relational algebra which is equivalent to the one given by Paredaens, and subsequently by studying the expressive power of a simple graph-based data model.

1 Introduction

The relational data base model, introduced by Codd in [7], has been particularly successful since it is a mathematically elegant model well suited to describe almost all “real world” situations. Since the query languages associated to such model (the relational algebra and the relational calculus) have a formal and simple definition, an interesting field of research is to study the expressive power of such language. Codd [8] has proved that the relational algebra is equivalent to the relational calculus, in the sense that both query languages can compute the same set of relations.

A breakthrough in this field [4, 12] has been a syntactic characterization of the set of relations that can be computed in a give data base. These results, also known as BP-completeness, are based on the principle of data independency from the physical representation: the information that can be extracted from the data base is completely determined at the logical level of such data base. This fact can be stated in a simple way: a relation \( R \) can be computed from a data base \( D \) if and only if all permutations over the elements of \( D \) which preserve \( D \)

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(that is, all permutations that produce a data base isomorphic to \( D \)), also preserve \( R \). An interesting interpretation of this property is that only the information given by the structure of the data can be used to differentiate data values; consequently, a query is expressible if and only if it does not add any additional differentiation to the one initially available \([1]\).

This idea can be rephrased by stating that the result of a query is invariant w.r.t. permutations of indistinguishable values; such a permutation was captured with the notion of automorphism in \([4, 12]\). While the \( BP \)-criterion is a natural requirement, it refers to properties of relations in a given data base instead of queries as a whole. We recall that a query is an expression of the query language that can be applied to different data bases leading to possibly different results. Thus it has been extended to a property of queries as partial functions from data bases to data bases, which is known nowadays as genericity \([6]\): it has been recognized as the capability of the calculus to preserve isomorphisms between data bases, rather than automorphisms. Genericity is a common requirement for query languages and it is traditionally related to the data independence principle that assumes that the data base is constructed over an abstract domain which is independent from the internal representation of data. Subsequent research has shown that this approach to the analysis of the expressiveness of a query language has certain shortcomings \([1, 10]\), mainly when new data models, such as the object-based model, are introduced. Other notions have been proposed to analyze properties of queries in some new models \([5, 13, 2]\) pointing out the importance of extending genericity to be used in more complex models. In \([5]\) languages are classified w.r.t. the degree of the use of the equality predicate, by analyzing the invariance property of queries under different mappings (not necessarily isomorphisms) over the data domain, which are compatible with the relational structure of the data base.

Subsequent advances in data base theory have led to different models that take into account the limitations of the relational model when it comes to describe complex situations. Most of such models have been introduced in the graph-based or object-oriented frameworks, but usually their mathematical foundations do not allow a complete study of the expressive power of the query languages introduced. In fact, to our knowledge, the only exception is the graph-based model GOOD \([3]\).

In this paper we introduce a different syntactic characterization of queries computable in a data base. Our characterization relies upon the notion of partitions of the domain, where each partition represents a level of undifferentiation among objects, values or vertices. Notice that an automorphism also can represent a certain level of undifferentiation. Initially we will exploit such notion to give two new characterizations of relations expressible in a relational data base. Subsequently, we will show how to apply the new framework to analyze a simple graph-based model, hence proving that our characterization can be useful in comparing the expressive power of different data languages.

Following the approach of \([12]\), the data models studied in this paper are domain-preserving, that is, it is not possible to create new vertices or values, but only to query an existing data base. In our framework, a binary relation over sets of data values is defined, denoted by \( \leftrightarrow \), which relates those sets of values that cannot be differentiated. From the relation \( \leftrightarrow \) we build some sets of partitions that respect \( \rightarrow \), that is, all classes in a partition are preserved by \( \leftrightarrow \). We prove that expressiveness of a query language can be stated as the conservation of some of those partitions, where the exact set of partitions that must be preserved depends on the data model. The expressibility results we obtain have the following form: Given a data base
2 Preliminaries

All sets considered in this paper are assumed to be finite and nonempty. Given a set $U$, a relation $R$ over $U$ is a subset of the cartesian product $U^a = U \times U \times \cdots \times U$ ($a$ times) for some fixed integer $a > 0$, that is a set of tuples of length $a$, where all components of a tuple are elements of $U$. The number $a \in \mathbb{N}$ is called the order or arity of the relation. Given a set $R = \{R_1, R_2, \ldots, R_p\}$ of relations over $U$, the pair $(U, R)$ is called a relational database; in this setting, $U$ is the domain of the database, and $R$ is the set of relations of the database.

Given a relation $R \in R$ of a database $(U, R)$, we denote with $D(R)$ the data domain of $R$, that is the subset of the elements of the database domain $U$ that are in at least one tuple of $R$. The notion of data domain is easily extended to the set $R$ of relations as the set union of relations’ data domains: $D(R) = \bigcup_{R \in R} D(R)$. Without loss of generality, we can assume that $D(R) = U$ for every considered database $(U, R)$. This seemingly trivial requirement is indeed very important, as it will become evident after Theorem 3.2, therefore we will omit the universe set unless it is necessary to avoid any ambiguities.

Just as in [12], when referring to a relational database, we use the relational algebra as a query language. In relational algebra two binary operators (union and product) and three unary operators (projection, equality restriction and inequality restriction) are given. In the following definition all relations are defined over the same database domain $U$.

**Definition 2.1 (Relational Algebra).** Let $R$ and $S$ be two relations with the same arity; the union of $R$ and $S$, denoted by $R \cup S$, is simply the set–theoretical union of the two sets of tuples.

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Given two relations $R$ and $S$ (not necessarily with the same arity), the (cartesian) product of $R$ and $S$, denoted by $R \times S$, is the set of all possible concatenations of a tuple of $R$ with a tuple of $S$: $\{r \cdot s | r \in R, s \in S\}$. The abbreviation $R^k$ is used to express the relation $R \times \cdots \times R$ ($k$ times).

Let $m$ be the arity of a relation $R$, $q \leq m$ a positive integer and $f : \{1, \ldots, q\} \to \{1, \ldots, m\}$ a function. The projection of $R$ over $(f(1), \ldots, f(q))$, denoted by $R_{\pi(f(1), \ldots, f(q))}$, is the relation: $\{(r_{f(1)}, \ldots, r_{f(q)}) : (r_1, \ldots, r_m) \in R\}$.

Now, let $j_1$ and $j_2$ be two integers such that $1 \leq j_1, j_2 \leq m$, where $m$ is the arity of a relation $R$. The equality restriction of $R$ on $j_1$ and $j_2$ is the relation, denoted by $R|_{j_1 = j_2}$, that is obtained by taking from $R$ all the tuples for which the $j_1$-th and the $j_2$-th components are equal: $\{(r_1, \ldots, r_m) \in R : r_{j_1} = r_{j_2}\}$. Analogously, the inequality restriction of $R$ on $j_1$ and $j_2$, denoted by $R|_{j_1 \neq j_2}$, is the relation obtained by taking from $R$ all the tuples for which the $j_1$-th and the $j_2$-th components are different: $\{(r_1, \ldots, r_m) \in R : r_{j_1} \neq r_{j_2}\}$.

The five operations just described are sufficient to generate the operations of intersection, difference, join and division, usually assumed as primitives in Codd’s relational algebra; a proof of this fact can be found, for example, in [8].

Given a relational data base $D = (U, R)$, we will denote by $M_D(D)$ the relation which is the result of applying the expression (of the relational algebra) $E$ to the data base $D$. 


Moreover a relation $S$ over $U$ is told to be expressible from $\mathcal{R}$ if there exists an expression $E$ whose operands are all relations in $\mathcal{R}$, and such that $M_E(D)$ is equal to $S$. Following [12], we denote with $\text{BI}(\mathcal{R})$ (basic information contained in the set of relations $\mathcal{R}$) the set of relations that can be expressed from $\mathcal{R}$.

As observed in [12], $\text{BI}(\mathcal{R})$ is the set of the answers to all possible queries that can be asked to a relational database that contains the relations $\mathcal{R}$. In [12], Paredaens gives a characterization of the class $\text{BI}(\mathcal{R})$ based upon appropriate automorphisms, that is permutations of the elements of the database domain.

Let $R$ be a relation of order $m$ over a set $U$. As in [12], an automorphism is a bijective function (that is, a permutation) on $U$. We say that the automorphism $\psi : U \to U$ respects the relation $R$ if, equivalently, that $\psi$ is $R$-compatible if, for each tuple $(a_1, a_2, \ldots, a_m) \in U^m$, $(\psi(a_1), \psi(a_2), \ldots, \psi(a_m)) \in R$.

The compatibility of an automorphism $\psi : U \to U$ with respect to a relation $R$ can be naturally extended to a set $\mathcal{R}$ of relations in the following way: $\psi$ respects the relations in $\mathcal{R}$ if, equivalently, $\psi$ is $\mathcal{R}$-compatible if $\psi$ is $R$-compatible for each relation $R$ in $\mathcal{R}$. Notice that the set of automorphisms $\mathcal{R}$-compatible, is a group where the operation is the composition of functions and the identity is the identity function (i.e. the function defined as $f(x) = x$).

As in [12], we denote with $\text{Aut}(\mathcal{R})$ the set of all the automorphisms $\psi : U \to U$ which are $\mathcal{R}$-compatible; with a small abuse of notation, if $\mathcal{R} = \{R\}$, we will usually write $\text{Aut}(R)$ instead of $\text{Aut}(\{R\})$. It will be very useful to consider the following representation of $\text{Aut}(\mathcal{R})$.

**Definition 2.2.** Let $\langle U, \mathcal{R} \rangle$ be a relational database, with $U = \{d_1, d_2, \ldots, d_n\}$, and let $\text{Aut}(\mathcal{R}) = \{\psi_1, \psi_2, \ldots, \psi_l\}$ be the set of $\mathcal{R}$-compatible automorphisms. The following relation of arity $n$:

$$
cgr(\mathcal{R}) = \\
\begin{align*}
\psi_1(d_1) & \cdots & \psi_1(d_n) \\
\vdots & & \vdots \\
\psi_l(d_1) & \cdots & \psi_l(d_n)
\end{align*}
$$

is called the cogroup–relation of $\langle U, \mathcal{R} \rangle$.

As we can see, each row (tuple) of the relation $\text{cgr}(\mathcal{R})$ represents one of the $\mathcal{R}$-compatible automorphisms. Since we do not associate any particular meaning to the elements of the domain $U$, if $|U| = n$ we can assume, without loss of generality, $U = \{1, 2, \ldots, n\}$. We can also assume that the first tuple of $\text{cgr}(\mathcal{R})$ represents the identity function on $U$ (which is always present in $\text{Aut}(\mathcal{R})$, since it is compatible with every nonempty set of relations); as a consequence, it can always be assumed that the first row of $\text{cgr}(\mathcal{R})$ is the tuple $(1, 2, \ldots, n)$.

**Example 2.1.** Let $\langle U, \mathcal{R} \rangle$ be a relational database, with:

- $U = \{1, 2, 3, 4\}$
- $\mathcal{R} = \{R_1, R_2, R_3\}$, with:

\[
\begin{align*}
R_1 &= \\
 1 & 2 & 3 & 4 \\
2 & 1 & 3 & 1 \\
3 & 4 & 2 & 3 \\
4 & 3 & 4 & 2 \\
\end{align*}
\]

\[
\begin{align*}
R_2 &= \\
 1 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
2 & 4 & 2 & 3 \\
4 & 1 & 3 & 2 \\
\end{align*}
\]

\[
\begin{align*}
R_3 &= \\
 1 & 4 & 1 & 3 \\
4 & 1 & 3 & 2 \\
3 & 2 & 1 & 4 \\
2 & 3 & 1 & 4 \\
\end{align*}
\]

\[\text{Aut}(\mathcal{R}) \] a group consists of a set $G$ of elements, a binary associative operation on $G$, and an identity element $1_G \in G$, such that the operation is closed and invertible in $G$.}

\[\]
It is easily verified that:

\[ \text{Aut}(\{R_1, R_2\}) = \text{Aut}(\{R_1, R_3\}) = \text{Aut}(\{R_2, R_3\}) = \text{Aut}(\{R_1, R_2, R_3\}) \]

\[ \text{cgr}(R) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{array} \]

If we look at the \( R \)-compatible automorphisms as permutations over \( U \), we can express \( \text{Aut}(R) \) as follows:

\[ \text{Aut}(R) = \begin{align*}
\text{Identity} \\
(1 & 2) (3 & 4) \\
(1 & 3) (2 & 4) \\
(1 & 4) (2 & 3) \\
\end{align*} \]

It is not difficult to see that, for a given database \( \langle U, R \rangle \), the set \( \text{Aut}(R) \) of \( R \)-compatible automorphisms is indeed a group with respect to function composition, with the identity function over \( U \) as unitary element. In fact, the identity over \( U \) is always in \( \text{Aut}(R) \), the inverse of an \( R \)-compatible automorphism is still an \( R \)-compatible automorphism, and the composition between two \( R \)-compatible automorphisms is again an \( R \)-compatible automorphism. Since we can always assume \( U = \{1, 2, \ldots, n\} \), we can think of \( \text{Aut}(R) \) as a finite permutation group over the set \( \{1, 2, \ldots, n\} \), that is a subgroup of the symmetric group \( S_n \).

In this paper we investigate the relation between expressive power and partitions of the database domain. More precisely, we investigate the possibility to characterize the expressive power of relational and graph-based databases via one or more theorems abiding to the following meta theorem.

**Theorem 2.1** (Meta theorem). Let \( \langle U, R \rangle \) be a relational database, and let \( S \) be a relation over \( U \). Then \( S \in \text{BI}(R) \iff P(R) = P(R \cup \{S\}) \), where \( P(R) \) and \( P(R \cup \{S\}) \) are sets of partitions over \( U \), built from the sets \( R \) and \( R \cup \{S\} \) of relations respectively.

### 3 Expressiveness in Relational Databases

The relevance of the main result in [12] is that it is the first syntactic characterization of the relations that can be obtained from a given database \( \langle U, R \rangle \) when the relational algebra is used as a query language. More precisely, in [12] the following theorem is proved.

**Theorem 3.1.** Let \( \langle U, R \rangle \) be a relational database, and let \( S \) be a relation over \( U \). Then \( S \in \text{BI}(R) \iff \text{Aut}(R) \subseteq \text{Aut}(S) \) and \( D(S) \subseteq D(R) \).

Basically, Paredaens has been able to point out the fundamental relation between expressiveness in a database and the set of automorphisms in the relational model. Such result has been successively extended in [6] to define in a formal way the notion of *genericity*, that is computable queries [6] have to be invariant with respect to the isomorphisms between databases. We can restate Theorem 3.1 in a form that will be more convenient for our purposes.
Theorem 3.2. Let \( \langle U, \mathcal{R} \rangle \) be a relational database, and let \( S \) be a relation over \( U \). Then \( S \in \text{BI}(\mathcal{R}) \iff \text{Aut}(\mathcal{R}) = \text{Aut}(\mathcal{R} \cup \{S\}) \).

Proof. First of all, we show that \( S \in \text{BI}(\mathcal{R}) \iff \text{BI}(\mathcal{R}) = \text{BI}(\mathcal{R} \cup \{S\}) \). Proving that \( S \in \text{BI}(\mathcal{R}) \implies \text{BI}(\mathcal{R}) = \text{BI}(\mathcal{R} \cup \{S\}) \) is trivial as \( \text{BI}(\mathcal{R}) \subseteq \text{BI}(\mathcal{R} \cup \{S\}) \). The latter stems from the fact that the relations which are expressible from \( \mathcal{R} \) are those obtained from \( \mathcal{R} \cup \{S\} \) simply ignoring the relation \( S \). Let now be \( S \in \text{BI}(\mathcal{R}) \) and \( T \in \text{BI}(\mathcal{R} \cup \{S\}) \). If the expression that gives \( T \) from \( \mathcal{R} \cup \{S\} \) does contain some occurrence of the relation \( S \), it is sufficient to replace such occurrence with the expression that gives \( S \) from \( \mathcal{R} \) to conclude that \( T \in \text{BI}(\mathcal{R}) \), and thus \( \text{BI}(\mathcal{R} \cup \{S\}) \subseteq \text{BI}(\mathcal{R}) \). It is immediate to notice that \( \text{BI}(\mathcal{R}) = \text{BI}(\mathcal{R} \cup \{S\}) \) implies \( S \in \text{BI}(\mathcal{R} \cup \{S\}) \).

Since we have established that \( S \in \text{BI}(\mathcal{R}) \iff \text{BI}(\mathcal{R}) = \text{BI}(\mathcal{R} \cup \{S\}) \), the two databases \( \langle U, \mathcal{R} \rangle \) and \( \langle U, \mathcal{R} \cup \{S\} \rangle \) are basic information equivalent – that is, every relation of the first database can be obtained from the relations of the second database and vice versa – if and only if \( S \) is expressible from \( \langle U, \mathcal{R} \rangle \). A direct consequence of Theorem 3.1 is that two databases \( \langle U, \mathcal{R}_1 \rangle \) and \( \langle U, \mathcal{R}_2 \rangle \) are basic information equivalent if and only if \( D(\mathcal{R}_1) = D(\mathcal{R}_2) \) (which are assumed to be both equal to \( U \)) and \( \text{Aut}(\mathcal{R}_1) = \text{Aut}(\mathcal{R}_2) \); thus, we can conclude that \( S \in \text{BI}(\mathcal{R}) \iff \text{Aut}(\mathcal{R}) = \text{Aut}(\mathcal{R} \cup \{S\}) \) as stated.

We observe that, given our assumption that \( U = D(\mathcal{R}) \), in Theorem 3.2 we can get rid of the inclusion between the domains, since it is implicit from the fact that \( S \) is a relation over \( U \). On the other hand, we cannot ignore the inclusion condition if we suppose that \( D(\mathcal{R}) \subset U \), since in such a situation it is not difficult to show two relations \( R \) and \( S \) such that \( \text{Aut}(R) = \text{Aut}(\{R, S\}) \) but \( S \notin \text{BI}(R) \).

A notion that seems tightly related to the expressiveness of relations in a database is that of indistinguishability between elements of the domain. Intuitively, the idea is that the elements of a subset of the domain of a given database are indistinguishable if and only if no query to the database is able to divide the set in two parts, one made of the elements that occur in the relation resulting from the query and the other made of the elements that do not occur in the relation. In such a situation, we say that the set of indistinguishable elements cannot be separated by any of the queries that can be presented to the database. Thus, a relation resulting from a query to the database can only contain all or none of the elements of a non-separable set.

Theorem 2.1 defines the general framework we propose to investigate the expressive power of query languages. In this framework different notions of expressible queries can be studied by considering different sets of partitions. For a given database \( \langle U, \mathcal{R} \rangle \), we say that a set \( \mathcal{P}(\mathcal{R}) \) of partitions of \( U \) is a set of valid partitions if and only if it satisfies Theorem 2.1. By the results in [12], it seems to us quite natural to define the following sets of valid partitions, namely the orbit partitions and the cycle partitions; indeed later we will be able to prove that, in the context of Theorem 2.1, they are equivalent to the characterization of relations obtainable in a relational data base of [12].

Definition 3.1. Let \( \langle U, \mathcal{R} \rangle \) be a relational database, and let \( \mathcal{P} = \{P_1, P_2, \ldots, P_k\} \) be a partition of \( U \). \( \mathcal{P} \) is an orbit partition of \( U \) with respect to \( \mathcal{R} \) if both the following conditions hold:

1. for each relation \( R \in \mathcal{R} \) and for each class \( P_i \in \mathcal{P} \), \( P_i \cap D(R) = \emptyset \) or \( P_i \subseteq D(R) \);
2. for each class $P_i \in \mathcal{P}$ and for each pair $a_1, a_2$ of elements of $P_i$ there exists an automorphism $\phi \in \text{Aut}(\mathcal{R})$ such that $\phi(a_1) = a_2$, and $\phi(P_j) = P_j$ for every class $P_j \in \mathcal{P}$.

We denote with $\text{OP}(\mathcal{R})$ the set of all orbit partitions of the given database $\langle U, \mathcal{R} \rangle$.

**Definition 3.2.** Let $\langle U, \mathcal{R} \rangle$ be a relational database, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a partition of $U$. $\mathcal{P}$ is a cycle partition of $U$ with respect to $\mathcal{R}$ if both the following conditions hold:

1. for each relation $R \in \mathcal{R}$ and for each class $P_i \in \mathcal{P}$, $P_i \cap D(R) = \emptyset$ or $P_i \subseteq D(R)$;
2. there exists an automorphism $\phi \in \text{Aut}(\mathcal{R})$ such that for each class $P_i \in \mathcal{P}$ and for each pair $a_1, a_2$ of elements of $P_i$ there exists an integer $n$ such that $\phi^n(a_1) = a_2$ and $\phi(P_j) = P_j$ for every class $P_j \in \mathcal{P}$.

We denote with $\text{CP}(\mathcal{R})$ the set of all cycle partitions of the given database $\langle U, \mathcal{R} \rangle$.

As already stated for $\text{Aut}(\mathcal{R})$, if $R$ is a relation we will write $\text{OP}(R)$ and $\text{CP}(R)$ instead of $\text{OP}(\{R\})$ and $\text{CP}(\{R\})$ respectively.

The following theorem is an alternative formulation of the main result of [12] (the equivalence of the two formulation follows from Theorem 3.2) which is more useful for our purposes.

**Theorem 3.3.** Let $\langle U, \mathcal{R} \rangle$ be a relational database, and let $S$ be a relation over $U$. Then $\text{Aut}(\mathcal{R}) = \text{Aut}(\mathcal{R} \cup \{S\}) \iff \text{P}(\mathcal{R}) = \text{P}(\mathcal{R} \cup \{S\})$.

Let $\langle U, \mathcal{R} \rangle$ be a relational database, and let $\text{Aut}(\mathcal{R})$ and $\text{cgr}(\mathcal{R})$ be respectively the group of $\mathcal{R}$-compatible automorphisms and the cogroup-relation of $\mathcal{R}$. A useful fact proved in [12] is that the cogroup-relation is expressible from $\mathcal{R}$, that is $\text{cgr}(\mathcal{R}) \in \text{BI}(\mathcal{R})$. Using this fact, we are able to prove the following theorem.

**Theorem 3.4.** $\text{Aut}(\mathcal{R}) = \text{Aut}(\text{cgr}(\mathcal{R}))$.

*Proof.* Since $\text{cgr}(\mathcal{R}) \in \text{BI}(\mathcal{R})$, by Theorem 3.1 we can conclude that $\text{Aut}(\mathcal{R}) \subseteq \text{Aut}(\text{cgr}(\mathcal{R}))$. Now, let $\phi \in \text{Aut}(\text{cgr}(\mathcal{R}))$; as we have already observed, $\phi$ is a permutation of the set $U = D(\mathcal{R})$, as well as of the tuples that compose the relation $\text{cgr}(\mathcal{R})$. Thus, for each tuple $t \in \text{cgr}(\mathcal{R})$, we have that $\phi(t) \in \text{cgr}(\mathcal{R})$. In particular, by letting $n$ be the cardinality of $U$, we have:

$$\phi((1, 2, \ldots, n)) = (\phi(1), \phi(2), \ldots, \phi(n)) \in \text{cgr}(\mathcal{R})$$

Thus, the elements of $U$ are mapped by $\phi$ in such a way that the result is a row of the cogroup-relation; so we can conclude that $\phi \in \text{Aut}(\mathcal{R})$. \hfill \Box

A direct consequence of Theorem 3.4 is that not only $\text{cgr}(\mathcal{R}) \in \text{BI}(\mathcal{R})$, as established by Paredaens, but also $R \in \text{BI}(\text{cgr}(\mathcal{R}))$ for every relation $R \in \mathcal{R}$, since $D(R) \subseteq D(\text{cgr}(\mathcal{R})) = U$ and $\text{Aut}(\text{cgr}(\mathcal{R})) = \text{Aut}(\mathcal{R}) \subseteq \text{Aut}(\mathcal{R})$. As a corollary of Theorem 3.4, if we are interested to study the expressive power of a given relational database $\langle U, \mathcal{R} \rangle$ then we can work as well on the database $\langle U, \{\text{cgr}(\mathcal{R})\} \rangle$, which has only one relation and, moreover, such relation is an explicit representation of the finite permutation group $\text{Aut}(\mathcal{R})$.

We now turn our attention to the structure of $\text{OP}(\mathcal{R})$ and $\text{CP}(\mathcal{R})$. First of all we observe that, thanks to Theorem 3.4 we can get rid of item 1 in Definitions 3.1 and 3.2 since, by
considering the database \( \langle U, \{cgr(R)\} \rangle \), there is only one relation and, for such relation, it holds \( P_i \subseteq D(cgr(R)) = U \) for each \( P_i \in \mathcal{P} \).

To characterize the sets of cycle and orbit partitions we need to recall some notions from basic abstract algebra.

**Definition 3.3.** Let \( X \) be a set and \( \langle G, \cdot, e \rangle \) a group. An action of \( G \) on \( X \) is a map \* : \( G \times X \rightarrow X \) such that

1. \( \forall x \in X, \ e \ast x = x \);
2. \( \forall g_1, g_2 \in G, \ \forall x \in X \ (g_1 \cdot g_2) \ast x = g_1 \ast (g_2 \ast x) \)

In group theory it is customary to omit the operators symbols from expressions when confusion does not arise; so, the expression in item 2 above is usually written as: \( (g_1g_2)x = g_1(g_2x) \).

**Definition 3.4.** Let \( G \) be a group acting on a set \( X \). For \( x_1, x_2 \in X \), let \( x_1 \sim x_2 \) if and only if there exists \( g \in G \) such that \( gx_1 = x_2 \). It is not difficult to see that \( \sim \) is an equivalence relation on \( X \), and thus it induces a partition \( \mathcal{P} \) on \( X \). The classes of \( \mathcal{P} \) are called the orbits in \( X \) under \( G \). If \( x \in X \), the class containing \( x \) — denoted by \( Gx \) — is called the orbit of \( x \) under \( G \). In other words, \( Gx = \{y \in X \mid y = gx \mathrm{~for~some~} g \in G\} \).

It is not difficult to see that the partition induced by the orbits of \( \mathrm{Aut}(\mathcal{R}) \) on \( U \) satisfies Definition 3.1. In fact, every automorphism \( \phi \in \mathrm{Aut}(\mathcal{R}) \) maps each orbit \( \mathrm{Aut}(\mathcal{R})x \) into itself and, given a pair \( a_1, a_2 \) of elements of \( U \), there exists an automorphism that maps \( a_1 \) to \( a_2 \) if and only if \( a_1 \) and \( a_2 \) are in the same orbit. Moreover, if \( H \) is a subgroup of a group \( G \) acting on the set \( X \), then every orbit \( Hx \) is a subset of the orbit \( Gx \); more precisely, it is not difficult to prove that the orbits induced by \( H \) are a refinement of the orbits induced by \( G \). Since each partition induced by the orbits of every subgroup of \( \mathrm{Aut}(\mathcal{R}) \) satisfies Definition 3.1, we have that \( \mathrm{OP}(\mathcal{R}) \) contains the set of those partitions.

Vice versa, let \( \mathcal{P} \in \mathrm{OP}(\mathcal{R}) \). It is not difficult to see that the set of automorphisms \( \phi \in \mathrm{Aut}(\mathcal{R}) \) that map each class of \( \mathcal{P} \) into itself and that map each element of a class to an element of the same class forms a subgroup of \( \mathrm{Aut}(\mathcal{R}) \); moreover, the orbit partition induced by such a subgroup is just \( \mathcal{P} \). As a consequence, \( \mathrm{OP}(\mathcal{R}) \) is a subset of the set of partitions induced by all the subgroups of \( \mathrm{Aut}(\mathcal{R}) \); since also the converse inclusion holds, the two sets indeed coincide.

**Definition 3.5.** Let \( G \) be a group acting on the set \( X \), and let \( g \in G \). For \( x_1, x_2 \in X \), let \( x_1 \sim x_2 \) if and only if there exists an integer \( n \) such that \( x_2 = g^n x_1 \), where \( g^n \) is the application of \( g \) for \( n \) times. It is not difficult to see that \( \sim \) is an equivalence relation on \( X \), and thus it induces a partition \( \mathcal{P} \) on \( X \). The classes of \( \mathcal{P} \) are called the cycles of \( g \) on \( X \).

Analogously to what said about orbits, it is not difficult to see that the partitions induced by the cycles of the automorphisms of \( \mathrm{Aut}(\mathcal{R}) \) satisfy Definition 3.2. We observe that, while an orbit partition is induced by a subgroup of \( \mathrm{Aut}(\mathcal{R}) \), a cycle partition is induced by an automorphism, that is by an element of \( \mathrm{Aut}(\mathcal{R}) \). The class \( \mathrm{CP}(\mathcal{R}) \) is thus the set of cycle partitions obtained by considering every element of \( \mathrm{Aut}(\mathcal{R}) \).
Definition 3.6. Let $G$ be a group acting on the set $X$ and let $g$ be a permutation in $G$. Then the orbits of the (cyclic) group $\langle g \rangle$ generated by $g$ are the cycles of $g$. Since $\langle g \rangle$ is a subgroup of $G$, we have immediately that every cycle partition of $G$ is also an orbit partition of $G$, that is, $\text{CP}(\mathcal{R}) \subseteq \text{OP}(\mathcal{R})$.

Example 2.1 can be used to show that the converse does not generally hold: not every orbit partition is also a cycle partition. In fact we have:

$$\text{CP}(\mathcal{R}) = \left\{ \{\{1\},\{2\},\{3\},\{4\}\}, \{\{1\},\{3\},\{2\},\{4\}\}, \{\{1\},\{4\},\{2\},\{3\}\} \right\}$$

$$\text{OP}(\mathcal{R}) = \text{CP}(\mathcal{R}) \cup \left\{ \{\{1\},\{2\},\{3\},\{4\}\} \right\}$$

As noted above, Theorem 3.4 allows us to deal only with cogroup-relations instead of sets of arbitrary relations. The same can be done when working with cycle and orbit partitions: since cycles and orbits that form the partitions in $\text{CP}(\mathcal{R})$ and $\text{OP}(\mathcal{R})$ are completely determined from the elements and the subgroups of $\text{Aut}(\mathcal{R})$ respectively, by Theorem 3.4 we can conclude that $\text{CP}(\mathcal{R}) = \text{CP}(\text{egr}(\mathcal{R}))$ and $\text{OP}(\mathcal{R}) = \text{OP}(\text{egr}(\mathcal{R}))$.

It is possible to show that both the set $\text{CP}(\mathcal{R})$ of cycle partitions and the set $\text{OP}(\mathcal{R})$ of orbit partitions of a given database $(U, \mathcal{R})$ constitute a partially ordered set (poset) with respect to the binary relation $\leq$, where $\mathcal{P}_1 \leq \mathcal{P}_2$ iff each class of $\mathcal{P}_1$ is contained in some class of $\mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions in $\mathcal{P}(\mathcal{R})$, $\mathcal{P}(\mathcal{R})$ is equal to $\text{CP}(\mathcal{R})$ or $\text{OP}(\mathcal{R})$. In fact, it is not difficult to see that $\leq$ is reflexive, antisymmetric and transitive: that is, $\leq$ is an order relation over both $\text{CP}(\mathcal{R})$ and $\text{OP}(\mathcal{R})$. One notably difference between the posets $\langle \text{OP}(\mathcal{R}), \leq \rangle$ and $\langle \text{CP}(\mathcal{R}), \leq \rangle$ is that the first has always a maximum element, corresponding to the orbits of the entire $\text{Aut}(\mathcal{R})$, while the second may not have a maximum element, as shown above referring to Example 2.1, where $\text{Aut}(\mathcal{R})$ is the so called Klein group. Instead, both the posets have a minimum element, corresponding to the cycles (equal to the orbits) induced by the identity element of $\text{Aut}(\mathcal{R})$: that is, the trivial partition, where each class is a singleton.

In order to prove our main results we need some definitions and some well known properties of finite groups. Here we just recall the notion of stabilizer; we address the reader to an introductory book on abstract algebra, such as [9], for the notion of coset and its properties.

Definition 3.7. Let $G$ be a group acting on a set $X$, and let $x \in X$. The subgroup $G_x$ of $G$ defined as $G_x = \{g \in G \mid gx = x\}$ is called the stabilizer of $x$ in $G$.

It is not difficult to see that if $G$ is a group which acts on the set $X$, and $x \in X$, then the stabilizer $G_x$ of $x$ can be considered as a group which acts on the set $X \setminus \{x\}$. The following are two well known results in group theory: Lagrange's theorem – which correlates the cardinality of a given group $G$ and the cardinality of a given subgroup $H$ of $G$ with the number of left cosets of $G$ with respect to $H$ – and a theorem which expresses the cardinality of the orbit of $G$ containing $x$ as the number of left cosets of $G$ with respect to the stabilizer $G_x$.

Theorem 3.5 (Lagrange’s Theorem). Let $G$ be a finite group, and let $H$ be a subgroup of $G$. Then $|G| = (G : H) \cdot |H|$, where $(G : H)$ is the number of left cosets of $G$ with respect to $H$, and is usually called the index of $H$ in $G$. 

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**Theorem 3.6.** Let $G$ be a finite group acting on a set $X$, and let $x \in X$. Then $|Gx| = (G : G_x)$, that is there exists a one-to-one correspondence between the elements of the orbit $Gx$ of $x$ under $G$ and the left cosets of the stabilizer $G_x$ in $G$.

We are now able to prove the following theorem.

**Theorem 3.7.** Let $G$ be a subgroup of the symmetric group $S_n$, and let $H$ be a subgroup of $G$. If the orbit partitions of $G$ and $H$ are the same, then $H = G$.

**Proof.** We prove the assertion by induction on $n$. For $n \leq 2$ the theorem can be proved by direct inspection of the subgroups of $S_n$.

Now, let us suppose that the theorem is true for $n - 1$, and let us show that it holds also for $n$. We first observe that since the orbit partitions of $G$ and $H$ are the same, then also the orbits $Gn$ and $Hn$ of the element $n$ with respect to $G$ and $H$ are the same. Now, if we take all the partitions having $\{n\}$ as a class, we get the orbit partitions induced by the stabilizers $G_n$ and $H_n$ of the element $n$ with respect to $G$ and $H$. These orbit partitions are equal and thus, by induction hypothesis, $G_n = H_n$. By Lagrange’s theorem, we can express the cardinalities of $G$ and $H$ with respect to the cardinalities of their stabilizers as $|G| = (G : G_n) \cdot |G_n|$ and $|H| = (H : H_n) \cdot |H_n|$, where $(G : G_n)$ and $(H : H_n)$ are the indices, respectively, of the stabilizer $G_n$ in $G$ and of the stabilizer $H_n$ in $H$. By Theorem 3.6 we can infer that $|G| = |G_n| \cdot |G_n|$ and $|H| = |H_n| \cdot |H_n|$. Since $|G_n| = |H_n|$ and $|G_n| = |H_n|$, we can conclude that $G$ and $H$ have the same order, and thus $G = H$. 

**Corollary 3.8.** Let $(U, R)$ be a relational database, and let $S$ be a relation over $U$. Then $\text{Aut}(R) = \text{Aut}(R \cup \{S\}) \iff \text{OP}(R) = \text{OP}(R \cup \{S\})$

**Proof.** If $\text{Aut}(R) = \text{Aut}(R \cup \{S\})$, since the orbit partitions are completely determined from the subgroups of $\text{Aut}(R)$, we obtain that $\text{OP}(R) = \text{OP}(R \cup \{S\})$.

For the converse, we observe that $\text{Aut}(R)$ is a subgroup of the symmetric group $S_n$, and $\text{Aut}(R \cup \{S\})$ is a subgroup of $\text{Aut}(R)$. By hypothesis, the orbit partitions of $\text{Aut}(R)$ and $\text{Aut}(R \cup \{S\})$ are equal and thus, by Theorem 3.7, $\text{Aut}(R) = \text{Aut}(R \cup \{S\})$.

A second characterization of expressible queries in relational databases can be obtained by considering cycle partitions instead of orbit partitions. We need the following lemma.

**Lemma 3.1.** Let $G$ be a subgroup of the symmetric group $S_n$, and let $H$ be a subgroup of $G$. If the cycle partitions of $G$ and $H$ are the same, then also the orbits of $G$ and $H$ are the same, that is $Hx = Gx$ for every $x \in \{1, 2, \ldots, n\}$.

**Proof.** Since the orbit $Gx$ is the set of elements of $\{1, 2, \ldots, n\}$ which are reachable from $x$ through some element $g$ of $G$, while a cycle containing $x$ is the set of elements which are reachable from $x$ through one element $g$ of $G$, one method to build $Gx$ from the cycle partitions of $G$ is given by Algorithm [1].
Algorithm 1: BuildOrbit

Data: an integer \( x \in \{1, \ldots, n\} \), a subgroup \( G \) of \( S_n \), a set \( CP \) of cycle partitions

Result: Result

1. Result ← \( \{x\} \);

2. repeat
   3. Modified ← false;
   4. foreach partition \( P \) in \( CP \) do
      5. Cycles ← the smallest union of cycles of \( P \) which covers Result;
      6. if \( Cycles \setminus \text{Result} \neq \emptyset \) then
         7. Result ← Result \( \cup \) Cycles;
         8. Modified ← true;
      end
   end

9. until Modified = false;

Algorithm 1 computes the least subset \( O \) of \( \{1, 2, \ldots, n\} \) which contains \( x \) and such that, for every cycle partition \( P \) of \( G \), \( O \) is the union of some cycles in \( P \); it is not difficult to see that \( O \) is, indeed, the orbit \( Gx \).

Since the cycle partitions of \( G \) and \( H \) are the same by hypothesis, the orbits computed by the algorithm above will be the same for \( G \) and \( H \), for every choice of \( x \in \{1, 2, \ldots, n\} \).

We are now ready to prove the following theorem.

**Theorem 3.9.** Let \( G \) be a subgroup of the symmetric group \( S_n \), and let \( H \) be a subgroup of \( G \). If the cycle partitions of \( G \) and \( H \) are the same, then \( H = G \).

**Proof.** By Lemma 3.1, the orbits of \( G \) and \( H \) are the same. Thus we can prove the theorem by the same argument used for Theorem 3.7.

A direct consequence of Theorem 3.9 is that the cycle partitions of a given database satisfy Theorem Schema II; thus, the following theorem provides a second characterization of expressible queries in relational databases alternative to the one originally given by Paredaens. The proof is analogous to the one given for Theorem 3.8.

**Theorem 3.10.** Let \( \langle U, \mathcal{R} \rangle \) be a relational database, and let \( S \) be a relation over \( U \). Then:

\[
\text{Aut}(\mathcal{R}) = \text{Aut}(\mathcal{R} \cup \{S\}) \iff \text{CP}(\mathcal{R}) = \text{CP}(\mathcal{R} \cup \{S\})
\]

A final observation is due about Theorems 3.8 and 3.10. Even though there is a strong resemblance between our meta Theorem 2.1 and Theorem 3.2, our results cannot be expressed neither in the form \( S \in \text{BI}(\mathcal{R}) \iff \text{OP}(\mathcal{R}) \subseteq \text{OP}(S) \) and \( D(S) \subseteq D(\mathcal{R}) \) nor in the form \( S \in \text{BI}(\mathcal{R}) \iff \text{CP}(\mathcal{R}) \subseteq \text{CP}(S) \) and \( D(S) \subseteq D(\mathcal{R}) \), as shown in the next example.

**Example 3.1.** Let \( \langle U, \{R\} \rangle \) and \( \langle U, \{S\} \rangle \) be two relational databases, with:
• $U = \{1, 2, 3, 4, 5\}$

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
R = & 3 & 4 & 5 & 1 & 2 \\
S = & 5 & 4 & 1 & 3 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4 \\
\end{array}
\]

Notice that $\text{Aut}(R)$ is the cyclic group generated by the permutation $(1 \ 2 \ 3 \ 4 \ 5)$, while $\text{Aut}(S)$ is the cyclic group generated by the permutation $(1 \ 2 \ 3 \ 5 \ 4)$:

\[
\begin{array}{cccc}
\text{Identity} & \text{Identity} \\
(1 \ 2 \ 3 \ 4 \ 5) & (1 \ 2 \ 3 \ 5 \ 4) \\
(1 \ 4 \ 2 \ 5 \ 3) & (1 \ 5 \ 2 \ 4 \ 3) \\
(1 \ 5 \ 4 \ 3 \ 2) & (1 \ 4 \ 5 \ 3 \ 2) \\
\end{array}
\]

From $\text{Aut}(R)$ and $\text{Aut}(S)$ we can easily obtain $\text{CP}(R) = \text{OP}(R) = \text{CP}(S) = \text{OP}(S) = \{(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}), \{(1, 2, 3, 4, 5)\}\}$. Clearly, $S$ is not expressible from $R$, since we have $D(R) = D(S)$ but $\text{Aut}(R) \not\subseteq \text{Aut}(S)$; on the other hand, $\text{OP}(R) \subseteq \text{OP}(S)$ and $D(S) \subseteq D(R)$, and $\text{CP}(R) \subseteq \text{CP}(S)$ and $D(S) \subseteq D(R)$. The fact that $S$ is not expressible from $R$ can be correctly determined through orbit partitions or through cycle partitions by observing that: $\text{OP}(\{R, S\}) = \{(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \neq \text{OP}(R) \text{ or } \text{CP}(\{R, S\}) = \{(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}\neq \text{CP}(R)$.

### 4 Expressiveness in graph-based data bases

In this section we study a simple graph-based model where two labeled graphs are used to model data bases. A data base consists of two distinct layers: a schema layer and a structure layer; the objects can be found in the latter, while the former describe the data organization. Each layer is a labeled weakly-connected directed graph, moreover there exists a function that maps a schema into a structure: such function will be called an extension. Both vertices and edges of the graphs are labeled, and we can assume that the sets of edge labels and vertex labels, as well as schema labels and structure labels, are disjoint. An example of data base is represented in Figures 1, 2, from which it is easy to note how the schema and the structure are closely related, the following definitions only formalize the intuitive idea.

**Definition 4.1 (Schema).** A schema graph, in short schema, is a triple $\Sigma = (G, \lambda_1, \lambda_2)$, where $G = (V, E)$ is an oriented, weakly-connected graph, and $\lambda_1, \lambda_2$ are respectively the injective functions that maps each node (resp. edge) to its label.

**Definition 4.2 (Structure).** A structure is a triple $S = (S, \lambda'_1, \lambda'_2)$, with $S$ a colored oriented graph $S = (V, E, \mu)$, where $V$ is the set of nodes of the structure, $E \subseteq V \times V$ is the set of edges, $\lambda'_1, \lambda'_2$ are respectively the injective functions that maps each node (resp. edge) to its label, and $\mu : E \rightarrow \Gamma$, is a labeling of the edges over the finite alphabet $\Gamma$, called coloring of the structure.
In the following, we will use the set $\Gamma = \{true, false\}$ of colors that allows to specify that a link between object instances in $S$ is actual or not. In the example of Fig. 2 only the links labeled true are represented, and the presence (or the absence) of links labeled false does not change the data stored in the data base. In Fig. 3 is represented a part of the structure, where false links are represented with dotted arrows.

The schema and the structure must be strongly correlated; in fact there must exist a function, called extension (denoted by $Ext$), mapping the schema into the structure. In order to have a sound definition of extension some restrictions must be enforced, as pointed out in the following definition, where $Pow(A)$ stands for the family of all nonempty subsets of $A$.

Informally $Ext$ maps each vertex of the schema into some vertices of the structure and each edge of the schema into some edges of the structure.

**Definition 4.3 (Extension).** Let $\Sigma = (G = (V,E), \lambda_1, \lambda_2)$ be a schema and $S = (S, X_1, X_2)$ a structure, where $S = (V', E', \mu)$. Then $S$ is an extensional structure of $\Sigma$ if there is a function (the extension) from $\Sigma$ to $S$, $Ext : V \rightarrow Pow(V')$, such that:

1. $\{Ext(v) : v \in V(G)\}$ is a partition $\{V_1, \ldots, V_n\}$ of the set $V'$,
2. for every $x \in V_i, y \in V_j$, the pair $(x, y) \in E'$ iff $(Ext^{-1}(x), Ext^{-1}(y)) \in E$;

Notice that the first point of the definition of extension implies that the function $Ext^{-1}$ is well defined. In the following, if $S$ is the extensional structure of $\Sigma$, then we write $S = Ext(\Sigma)$ and we will simply say that $S$ is a structure of $\Sigma$. Given two vertices $v_1$ and $v_2$ of the schema, connected with a link $(v_1, v_2)$ then in the structure there must exist all links $(w_1, w_2)$ for $w_1 \in Ext(v_1), w_2 \in Ext(v_2)$. Such requirement justifies the introduction of a labeling (and especially of a true-false labeling) in order to have a reasonable graph-based model.

**Definition 4.4 (Data base).** A data base $B$ is a pair $(\Sigma, S)$, where $\Sigma$ is a schema and $S$ is an extensional structure of $\Sigma$.

The schema describes the conceptual organization of the data, while the data content or instantiation of the data base is given by the extensional structure associated to the schema.
Figure 2: Example of structure

It is not hard to notice that, given a schema, there is a one-to-one correspondence between structures and extension functions, therefore we will sometimes use the pair $(\Sigma, Ext)$ as a data base.

Some preliminary definitions are required for introducing our query language. Given a partial function $f : A \mapsto B$ (i.e. a function where each element of $A$ can be associated to one or none of the elements of $B$), by $Dom(f)$ we denote the domain of $f$, that is the set of elements $x \in A$ such that $f(A)$ is defined. Let $f, g$ be two partial functions from the set $A$ to the set $Pow(B)$. Then $f$ is a restriction of $g$, denoted by $f \leq g$, if $Dom(f) \subseteq Dom(g)$ and for every $x \in Dom(f)$, $f(x) \subseteq g(x)$. Moreover by $Im(f)$ we denote the set obtained as union of all images of elements in $Dom(f)$: formally $Im(f) = \bigcup_{x \in Dom(f)} f(x)$.

**Definition 4.5** (Instance). Let $B = (\Sigma, Ext)$ be a data base. An instance of $B$ is a restriction $f$ of $Ext$ such that $Dom(f)$ induces a weakly-connected subgraph of $\Sigma$.

The following notations will be used in the rest of the paper. The set $I(B)$ is the set
of all instances of $B$. Let $\mathcal{I}$ be a subset of $\mathcal{I}(B)$, then by $\text{Im}(\mathcal{I})$, we mean the set of nodes of the structure of $B$ that is the union of all images of instances in $\mathcal{I}$, while $\text{Domain}(\mathcal{I})$ is the union of all domains of instances in $\mathcal{I}$. An element in $\text{Im}(\mathcal{I})$ is called a value, while an element in $\text{Domain}(\mathcal{I})$ is called a name. Then the image of a name $x \in \text{Domain}(\mathcal{I})$ is the subset $A$ of $\text{Im}(\mathcal{I})$ such that $A = \bigcup_{f \in \mathcal{I}} f(x)$. For a value $y \in \text{Im}(\mathcal{I})$, the inverse image of $y$, denoted by $\text{name}(y)$, is the name of $\text{Domain}(\mathcal{I})$ that is mapped by $\text{Ext}$, to a set containing the element $y$. Similarly, given a set $A$ of values, the inverse image of $A$ is the set $\text{names}(A)$ which is union of all inverse images of the values in $A$.

4.1 The graph algebra

Our graph data model is proposed as a domain-preserving data base, along the same lines as other papers where the expressiveness of the relational algebra is studied [12, 4], and it gives a formal embedding for languages used for the retrieval of graph-structured information [11]. The requirement that we are dealing with domain-preserving data bases reflects in the query language: in fact we have no operation for creating new elements or modifying the schema graph, and all operations must preserve the schema and the original structure.

The main consequence of the assumption that our model is domain preserving consists in the fact that we will deal with a schema which is mapped to an instance through an extensional mapping. Therefore there is a complete equivalence between subgraphs of the structure and restrictions of the extensional mapping. We are now able to introduce the operations of our graph algebra: according to our reasoning above we can describe the operation as over partial functions whenever it allows a simpler formulation.

Definition 4.6 (Addition). Let $B = (\Sigma, \mathcal{S})$ be a data base and let $f_1, f_2 \in \mathcal{I}(B)$. The Addition of $f_1$ and $f_2$, denoted as $f_1 \oplus f_2$, is the following function over domain $\text{Dom}(f_1)$.

The operation is defined only if $\text{Dom}(f_1) = \text{Dom}(f_2)$:

$$(f_1 \oplus f_2)(x) = f_1(x) \cup f_2(x)$$

Definition 4.7 (Product). Let $B = (\Sigma, \mathcal{S})$ be a data base, and let $f_1, f_2$ be two functions in $\mathcal{I}(B)$. The Product of $f_1$ and $f_2$, denoted as $f_1 \otimes f_2$ is the instance in $\mathcal{I}(B)$ defined as follows:

$$(f_1 \otimes f_2)(x) = \begin{cases} f_1(x) \cap f_2(x) & \text{if } x \in \text{Dom}(f_1) \cap \text{Dom}(f_2), f_1(x) \cap f_2(x) \neq \emptyset \\ f_2(x) & \text{if } x \in \text{Dom}(f_2) - \text{Dom}(f_1) \\ f_1(x) & \text{if } x \in \text{Dom}(f_1) - \text{Dom}(f_2) \\ \text{undefined} & \text{otherwise} \end{cases}$$

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The product is defined only if \( \text{Dom}(f_1 \otimes f_2) \) induces a weakly-connected subgraph of \( \Sigma \).

**Definition 4.8** (Projection). Let \( B = (\Sigma, S) \) be a data base. Let \( f \) be a function in \( \mathcal{I}(B) \) and let \( A \) be a subset of the domain of \( f \), such that \( A \) induces in \( \Sigma \) a weakly-connected subgraph. The projection of \( f \) on \( A \), denoted as \( \Pi_A(f) \), is the instance defined as follows:

\[
\Pi_A(f)(x) = \begin{cases} 
  f(x) & \text{if } x \in A \\
  \text{undefined} & \text{if } x \notin A
\end{cases}
\]

**Definition 4.9** (Difference). Let \( B = (\Sigma, S) \) be a data base. Let \( f_1, f_2 \) be two functions in \( \mathcal{I}(B) \) over the same domain \( A \). The difference of \( f_1 \) by \( f_2 \), denoted as \( f_1 \ominus f_2 \) is the following instance:

\[
f_1 \ominus f_2(x) = \begin{cases} 
  f_1(x) - f_2(x) & \text{if } x \in A, f_1(x) - f_2(x) \neq \emptyset \\
  \text{undefined} & \text{otherwise}
\end{cases}
\]

The difference is defined only if \( \text{Dom}(f_1 \ominus f_2) \) induces a weakly-connected subgraph of \( \Sigma \).

Since the coloring of the edges encodes the fact that a relation between two objects is actual or not, it is natural that the query language has some tools for exploiting such coloring. In our model we will need to extract instances where “similar” edges are the same color. The definition of selector is the first step in such direction.

**Definition 4.10** (Selector). Let \( \Sigma \) be the schema of a data base \( B \). Then a selector of \( \Sigma \) is a pair \((G, \sigma)\) consisting of a weakly-connected subgraph \( G \) of \( \Sigma \) and a coloring \( \sigma : E \to \Gamma \) of the edges of \( G \).

Querying for a selector in a data base returns all subgraphs of the structure that are isomorphic to the selector: each such subgraph is indeed called a simple instance. Moreover, it is natural to define an operation of selection that allows to obtain instances which are compatible with a coloring of the schema over the alphabet \( \Gamma \). This is the last operation of our algebra.

**Definition 4.11** (Simple instance). Let \( (\Sigma, \text{Ext}) \) be a data base, where \( S = (V', E', \mu) \). Let \((G_s, \sigma)\) be a selector of \( \Sigma \), where \( G_s = (V(G_s), E(G_s)) \). Then a simple instance induced by the selector \((G_s, \sigma)\) is a restriction \( f \) of \( \text{Ext} \) such that \( \text{Domain}(f) = V, |f(v)| = 1 \) for each \( v \in \text{Domain}(f) \) and \( \mu(f(x), f(y)) = \sigma(x, y) \) for each \( (x, y) \in E(G_s) \).

Notice that all simple instances have the same domain.

**Definition 4.12** (Selection). Let \( B = (\Sigma, S) \) be a data base. Let \( f \) be a function in \( \mathcal{I}(B) \) and \((G_s, \sigma)\) a selector. Let \( F \) be the set of all simple instances induced by \( G_s \) that are also subinstances of \( f \). The selection of \( f \) by \((G_s, \sigma)\), denoted as \( f|(G_s, \sigma) \), is \( \bigoplus_{g \in F} g \).

5 Stability

Given a set \( \mathcal{I} \) of instances, our first aim is to give a characterization of all instances that can be obtained with a query that uses only the information contained in the instances in \( \mathcal{I} \), or equivalently by an expression of the algebra that has only instances in \( \mathcal{I} \) as operands.
In such direction the main result of this section is that expressiveness in our graph algebra is equivalent to the conservation of a certain partition. It is natural to associate a notion of undistinguishability to a partition, where all elements in a set of the partition are deemed undistinguishable. We share the goals of [12], but we have introduced in this paper a new framework, that is we are looking for a notion of expressiveness that is coherent with our meta theorem. Just as the notion of automorphisms, introduced in [12, 4] for relations, gives a global description of the logical dependencies among data that must be preserved when querying the data base, in our model a partition (or an equivalence relation) will represent all such logical dependencies. The equivalence partition over elements of the structure that we will study is called stability and is denoted by \( \equiv \) (where \( I \) is an instance).

The simplest possible form of undifferentiation (called 0-stability) is based on the idea that we are able to distinguish images of different vertices of the instance and vertices of the extensional structure belonging to different functions of \( I \). Such notion basically consists of using expressions in our algebra that do not contain any selection.

**Definition 5.1.** Let \( A \) be a subset of the image of a set \( I \) of instances. Then \( A \) is split by \( I \) iff there is a function \( f_1 \) in \( I \) such that \( A \cap \text{Im}(f_1) \neq \emptyset \) and \( A - \text{Im}(f_1) \neq \emptyset \).

We are now able to introduce formally the definition of 0-stability, as follows:

**Definition 5.2 (0-stable).** Let \( I \) be a set of instances over a data base \((\Sigma, S)\), and let \( A \) be a subset of \( \text{Im}(I) \). Then \( A \) is 0-stable w.r.t. \( I \), if the two following conditions hold:

1. \( A \subseteq \text{Im}(f(x)) \), where \( f \in I \), and \( x \in \text{Domain}(I) \) is a name of the schema.
2. for each function \( f \in I \), then \( A \) and \( \text{Im}(f) \) are disjoint or one is contained into the other one.

A more refined notion of undifferentiation is called 1-stability; informally a set \( A \) is 1-stable w.r.t. \( B \) if \( A \) is 0-stable and \( B \) is not able to distinguish two vertices of \( A \) with edges outgoing from \( B \) and ingoing in \( A \) (or outgoing from \( A \) and ingoing in \( B \)). Notice that 1-stability is a binary relation over subsets of \( \text{Im}(I) \), while 0-stability is a unary relation. The formal definition is:

**Definition 5.3 (1-stable).** Let \( I \) be a set of instances over a data base \((\Sigma, S)\) and let \( A \) and \( B \) be two disjoint subsets \( \text{Im}(I) \). Then \( A \) is 1-stable w.r.t. \( B \) and \( I \), denoted as \( B \leftrightarrow_{1,I} A \) if the following conditions are verified:

1. \( A \) is 0-stable w.r.t. \( I \);
2. for each edge \((a_1, b_1)\) of \( S \), with \( a_1 \in A, b_1 \in B \) and for each \( a_2 \in A \) there exists \( b_2 \in B \) such that \( \mu(a_1, b_1) = \mu(a_2, b_2) \);
3. for each edge \((b_1, a_1)\) of \( S \), with \( a_1 \in A, b_1 \in B \) and for each \( a_2 \in A \) there exists \( b_2 \in B \) such that \( \mu(b_1, a_1) = \mu(b_2, a_2) \).

Informally 1-stability means that whenever there is an colored edge (say a red edge) from a vertex of \( A \) to a vertex of \( B \), then all vertices of \( A \) have a red edge ingoing in \( B \). In other words if we assume that \( B \) is undistinguishable, then also \( A \) is undistinguishable, by any single-edge path. The notion of 1-stability can be further generalized, but first we need a new definition.
Definition 5.4 (Path). Let $G = (V, E)$ be a labeled graph. Then a colored path in $G$ is a pair $(p, s)$ where $p = < v_1, e_1, v_2, \ldots, v_{l-1}, e_l, v_l >$, and for every $1 \leq i \leq l$, $v_i$ belongs to $V$ and $e_i$ is an edge of $G$ such that $e_i = (w_i, w_{i+1})$ or $e_i = (w_{i+1}, w_i)$. Moreover, $s$ is the sequence $< \mu(e_1), \ldots, \mu(e_{l-1}) >$ where $\mu(e_i)$ is the color of the edge $e_i$ in $G$.

Notice that the definition of path used in the paper is different from the one that can be usually found in a graph theory textbook, as arcs can also be in the reverse direction. The length of a path is the number of edges it contains. Let $I = (\Sigma, S)$ be a data base and let $I = \langle p, s \rangle$ be a colored path of $S$, with $p = < v_1, e_1, v_2, \ldots, e_n, v_{n+1} >$. Then the path schema of $(p, s)$ is the pair $(p', s)$, with $p' = < v'_1, e'_1, v'_2, \ldots, e'_n, v'_{n+1} >$, where for every $1 \leq i \leq n + 1$, $v'_i = Ext^{-1}(v_i)$ and $e'_i Ext^{-1}(e_i)$.

Definition 5.5. Let $x, y$ be two nodes of the $I$-structure, and let $Z$ be a subset of $Im(I)$, then the path dependencies from $x$ to $y$ in $Z$, denoted as $PD_{k,Z,I}(x, y)$, is the set of path schemata of all paths of the $I$-structure that are starting in $x$ and ending in $y$ and entirely contained in $Z$.

Informally given $x, y, Z$, their path dependencies is obtained by removing all vertices not in $Z$, then computing all possible paths from $x$ to $y$, and finally computing the respective path schemata. The next step is to generalize 1-stability to $k$-stability, that is taking into account length-$k$ paths instead of simple edges (that is length-1 paths).

Definition 5.6 (k-stable). Let $I$ be a set of instances over a data base $(\Sigma, S)$, let $k$ be an integer larger than one, and let $A$ and $B$ be two disjoint subsets of $Im(I)$. Then $A$ is $k$-stable w.r.t. $B$ and $I$, denoted as $B \hookrightarrow_{k,I} A$ if the following conditions are verified:

1. $A$ is 0-stable w.r.t. $I$;
2. $A$ is $(k - 1)$-stable w.r.t. $B$ and $I$;
3. for each $a_1, a_2 \in A$, $b_1 \in B$ there exists $b_2 \in B$ such that $PD_{k,A\cup B,I}(a_1, b_1) \subseteq PD_{k,A\cup B,I}(a_2, b_2)$.

The main idea is that when $B \hookrightarrow_{k,I} A$ then if $B$ is undistinguishable also $A$ is undistinguishable when only paths no longer than $k$ are taken into account. Our main definition follows:

Definition 5.7 (Stability). Let $I$ be a set of instances, and let $A, B \subseteq Im(I)$. Then $A$ is stable w.r.t. $B$ in $I$, denoted as $B \hookrightarrow_{I} A$, if $B \hookrightarrow_{k,I} A$ for all $k \in \mathbb{N}$.

By Def. 5.7, it is immediate to verify the following properties of stability:

Lemma 5.1. Let $I$ be a set of instances, and let $A, B, C \subseteq Im(I)$, then:

1. if $B \cup C \hookrightarrow_{I} A$ and $names(B) \cap names(C) = \emptyset$, then $B \hookrightarrow_{I} A$ and $C \hookrightarrow_{I} A$,
2. if $names(B) = names(C) = \{x\}$, $B \hookrightarrow_{I} A$ and $C \hookrightarrow_{I} A$, then $B \cup C \hookrightarrow_{I} A$,
3. if $names(B) = names(C) = \{x\}$, $B \cap C \neq \emptyset$, $A \hookrightarrow_{I} B$ and $A \hookrightarrow_{I} C$, then $A \hookrightarrow_{I} B \cup C$. 


Stability is a relation between disjoint subsets of the domain. The definition of expressiveness in the query language that we want to obtain is based on partitions, and now we are able to introduce the class of partitions we are interested into. A partition $P$ of nodes of the structure is called valid if and only if for each set $A$ of the partition and every set $B$ that is a union of sets of $P$, then $B$ cannot differentiate $A$.

**Definition 5.8.** Let $I$ be a set of instances. A partition $P = \{P_1, \ldots, P_k\}$ of $Im(I)$ is valid if for every $P_i \in P$, $L \subset \{1, \ldots, k\}$, $L \neq \emptyset$, $i \notin L$, then $\bigcup_{i \in L} P_i \not\rightarrow_I P_i$.

Given a set $I$ of instances, then there may be various valid partitions of $Im(I)$, and at least one valid partition always exists (the partition where each vertex of the extensional structure is a set). Some valid partitions are more representative of the actual undifferentiation, in fact we will assume as a measure of the undifferentiation induced by $I$ the coarsest valid partition, which we will call canonical partition and denote as $C_I$. We can show that the definition of canonical partition is well-formed.

**Theorem 5.1.** Every set $I$ of instances has a unique canonical partition $C_I$.

**Proof.** Clearly the partition of $Im(I)$ into singletons is a valid partition, so there exists at least one canonical partition. Now assume to the contrary that there exist two coarsest valid partitions $P_1$ and $P_2$. Let $R_1$ and $R_2$ be the equivalence relations induced by the partitions $P_1$ and $P_2$, respectively. Let $R^*$ be the transitive closure of the relation $R$ defined as follows: $xR_1y$ if and only if $x$ and $y$ are in the same set of $P_1$ or $P_2$. We can prove that the partition $P$ induced by $R^*$ is a valid one of index strictly less than $k$. By construction of $R^*$ each set of $P$ is a union of sets in $P_1$ and also a union of sets in $P_2$, moreover each set in $P$ is contained in the image of a single name (since each set must be 0-stable). Notice that $R^* \neq P_1$ iff $P_1 \neq P_2$.

Let $X_i$ be a set of $P$, and let $Z_k$ be a class of $P_1$ contained in $X_i$. Since $P_1$ is a valid partition, and $X_i$ is a union of disjoint sets of $P_1$, by Lemmata 5.7, 5.1 we have that $Z_i \not\rightarrow_I X_i$ and $X_i \not\rightarrow_I Z_i$. Let $X_i, X_j$ be two sets of $P$, with $X_i = Z_{i1} \cup \cdots Z_{ik}$ and $X_j = Z_{j1} \cup \cdots Z_{jl}$. We have already proved that $Z_j \not\rightarrow_I X_i$ and $X_i \not\rightarrow_I Z_j$, applying again Lemmata 5.7, 5.1 and noting that $X_j = Z_{j1} \cup \cdots Z_{jl}$ we obtain $X_j \not\rightarrow_I X_i$. By the generality of $X_i$ and $X_j$ the partition $P$ is valid. Moreover $P$ is coarser than $P_1$, which is a contradiction. □

6 Expressiveness

In this section we will prove our main result regarding the expressiveness of the graph-based query language by showing that a function can be computed if and only if adding such function does not change the canonical partition. In the following we will assume that the union of the images of all functions in $I$ is exactly the universe set; such assumption does not violate the generality since otherwise we would simply have some sets of the canonical partition whose union consists of exactly those elements of the universe set that are not in any function in $I$.

**Theorem 6.1.** Let $BI(I)$ be the set of functions that are a result of an expression of the graph algebra where only functions of $I$ are operand. Then $f \in BI(I)$ if and only if the canonical partition induced by $I$ is equal to the canonical partition induced by $I \cup \{f\}$, that is $C_I = C_{I \cup \{f\}}$. 

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The following two properties, that are consequences of Def. 5.8, will be useful to prove the main result of the paper.

**Proposition 6.2.** Let $\mathcal{I}$ be a set of instances and let $\mathcal{P}$ be a valid partition. Then the image of every instance $g \in BI(\mathcal{I})$ is the union of sets of $\mathcal{P}$.

**Proof.** We prove the lemma by induction on the number $n$ of operations of the expression for $g$. If $n = 0$, then $g$ is a function in $\mathcal{I}$. Since all sets of a valid partition are 0-stable, no set of a valid partition has both an element in $Im(g)$ and an element not in $Im(g)$, therefore the union of all sets of $\mathcal{P}$ that are contained in $Im(g)$ is contained in $Im(g)$. To prove that such containment is not strict (i.e. such union is equal to $Im(g)$) it is sufficient to note that all elements of $Im(g)$ belong to some set of $\mathcal{P}$.

Assume now that $n > 0$ and $g$ is obtained by the application of an operation to two expressions $f_1$ and $f_2$ in $BI(\mathcal{I})$, or one expression for a selection. Clearly, by inductive hypothesis the images of $f_1$ and $f_2$ are obtained as the union of some sets of $\mathcal{P}$. It is immediate to verify the lemma for the case that $g = f_1 \oplus f_2$, $g = f_1 \otimes f_2$, $g = f_1 \ominus f_2$ and $g = \Pi_A(f_1)$. Finally, assume that $g = f_1|G_s$, where $G_s$ is a selector. By definition of the selection, then $Im(g)$ is the union of the images of all simple instances induced by $G_s$. Assume to the contrary that there exists a set $A$ of the valid partition such that $A \not\subseteq Im(g)$ and $A \cap Im(g) \neq \emptyset$. Now, let $y \in A - Im(g)$ and $x \in A \cap Im(g)$. Hence, by construction of selection, $y$ cannot be in the image of any simple instance induced by $G_s$, while $x$ is contained in the image of a simple instance induced by $G_s$. By inductive hypothesis $Im(f_1)$ is union of sets of $\mathcal{P}$, moreover since $A$ is a set of $\mathcal{P}$, also $Im(f_1) - A$ is union of sets of the valid partition, implying that $Im(f_1) - A \leftrightarrow_\mathcal{I} A$. It follows that, for each $z \in Im(f_1) - A$, $PD_{Im(f_1)}(x,z) \subseteq PD_{Im(f_1)}(y,v)$ for some $v \in Im(f_1) - A$. This implies that there is a simple instance induced by $G_s$ that has in its image $y$, which is a contradiction with the above assumption. Consequently, the image of $g$ must be union of sets of $\mathcal{P}$. \qed

We will prove that an alternative characterization of canonical partition is as the partition induced by the equivalence relation $R_T$ between elements of $Im(\mathcal{I})$, where $xR_Ty$ if and only if for every instance $f \in BI(\mathcal{I})$, $x \in Im(f) \Leftrightarrow y \in Im(f)$. In the following of the paper let $P_T^{BI}$ denote the partition induced by the equivalence relation $R_T$. Successively we will prove that a function $f$ belongs to $BI(\mathcal{I})$ if and only if $Im(f)$ can be obtained as union of sets of $P_T^{BI}$, completing the proof of our main result, in two steps. First we will prove that a function $f$ belongs to $BI(\mathcal{I})$ if and only if $Im(f)$ is union of sets in $P_T^{BI}$, then we will prove that $P_T^{BI} = C_T$. The following proposition is an immediate consequence of the definitions of $xR_Ty$ and of projection.

**Proposition 6.3.** Let $x, y \in Im(\mathcal{I})$ such that $xR_Ty$. Then both $x$ and $y$ belong to the set $Ext(z)$ for some name $z$.

**Corollary 6.4.** Let $P_T^{BI}$ be the partition induced by a set $\mathcal{I}$ of instances. Then, the inverse image of every set of the partition consists of a single vertex.

**Lemma 6.1.** Let $\mathcal{I}$ be a set of instances over $B$, let $P_T^{BI}$ be the partition induced by $\mathcal{I}$, and let $A \in P_T^{BI}$. Then there exists an instance $f \in BI(\mathcal{I})$ such that $Im(f) = A$. 

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Proof. Let $\mathcal{F}$ be the set of functions $f \in BI(\mathcal{I})$ such that $Im(f) \subseteq A$. By Cor. 6.4 all functions in $\mathcal{F}$ have the same domain, therefore the expression $g = \bigoplus_{f \in \mathcal{F}} f$ is well-formed; by construction $Im(g) \subseteq A$. By definition of $P^B_{\mathcal{I}}$ all functions $f \in BI(\mathcal{I})$ are such that $Im(f) \subseteq A$ or $Im(g) \cap A = \emptyset$, therefore all functions $f \in BI(\mathcal{I})$ whose image intersect $A$ are such that $Im(f) \subseteq A$, which in turn implies that are also in $\mathcal{F}$. Hence $Im(g) = A$, for otherwise there would be an element of $A$ not belonging to the image of any function in $\mathcal{I}$. \qed

**Lemma 6.2.** Let $\mathcal{I}$ be a set of instances over $B$, let $P^B_{\mathcal{I}}$ be the partition induced by $\mathcal{I}$ and let $A$ be a union of sets in $P^B_{\mathcal{I}}$, such that the inverse image of $A$ induces a weakly-connected subgraph of the schema, then $A$ is the image of an instance $f \in BI(\mathcal{I})$.

Proof. Let $A_1, \cdots, A_n$ be the sets of $P^B_{\mathcal{I}}$ whose union is $A$, and notice that, by Lemma 6.1, it is possible to associate to each set $A_i$ the instance $f_i \in BI(\mathcal{I})$ whose image is $A_i$, moreover for each such $f_i$, $|Dom(f_i)| = 1$. For each vertex $x$ in the inverse image of $A$ we can construct the function $g_x$ as $\bigoplus_{Dom(f_i) = (x)} f_i$. Then let $g = \bigotimes g_x$; it is immediate to note that $Im(g) = A$. \qed

An immediate consequence of Lemmata 6.2 and 6.2 is the following:

**Corollary 6.5.** Let $\mathcal{I}$ be a set of instances over $B$ and let $f \in \mathcal{I}$. Then $f \in BI(\mathcal{I})$ if and only if $Im(f)$ is union of sets in $P^B_{\mathcal{I}}$ and the inverse image of $Im(f)$ induces a weakly-connected subgraph of the schema.

With Lemma 6.2 we have proved that all interesting unions of sets of the partition $P^B_{\mathcal{I}}$ can be obtained with an expression of the graph algebra where all operands are taken from $\mathcal{I}$, therefore $P^B_{\mathcal{I}}$ conveys all expressibility information. But $P^B_{\mathcal{I}}$ is defined on the set $BI(\mathcal{I})$, we still need to correlate the definition of canonical partition with that of $P^B_{\mathcal{I}}$.

**Lemma 6.3.** Let $\mathcal{I}$ be a set of instances. Then $P^B_{\mathcal{I}}$ is a valid partition of $\mathcal{I}$.

Proof. Let $A, B$ be two disjoint sets where $A \in P^B_{\mathcal{I}}$ and $B$ is union of sets in $P^B_{\mathcal{I}}$, we will prove that $B \rightarrow_{\mathcal{I}} A$. First of all we will show that $A$ is 0-stable. Remember that, by definition of $P^B_{\mathcal{I}}$, for each $f \in BI(\mathcal{I})$ either $Im(f) \supseteq A$ or $Im(f) \cap A = \emptyset$. Since $BI(\mathcal{I})$ contains $\mathcal{I}$, it is immediate to not that $A$ is 0-stable. In the following let $a$ be the single-vertex inverse image of $A$.

If the inverse image of $A \cup B$ does not induce a weakly-connected subgraph of the instance, then 0-stability of $A$ suffices to prove that $B \rightarrow_{\mathcal{I}} A$, therefore assume that the inverse image of $A \cup B$ induces a weakly-connected subgraph of the instance. Let us assume that $B \rightarrow_{\mathcal{I}} A$ does not hold, then we will get a contradiction. Without loss of generality we can assume that $B$ is a minimum set for which $B \rightarrow_{\mathcal{I}} A$ does not hold. The new assumption implies that $k$-stability does not hold for some $k$. It follows that there are two elements $x, y \in A$ and an element $z \in B$ such that $PD_{A \cup B, \mathcal{I}}(x, z)$ is not contained in $PD_{A \cup B, \mathcal{I}}(y, v)$, for every $v \in B$. Now, by Lemma 6.2 there is an instance $g \in BI(\mathcal{I})$ such that $Im(g) = A \cup B$. Let $name(x) = x_1$ and $name(z) = z_1$ and let $v \in B$ such that $name(v) = z_1$. Let us consider the function $h = \bigoplus_{G_\alpha \in PD_{A \cup B, \mathcal{I}}(x, z)} (\Pi_a(g(G_\alpha)))$. By construction $x$ belongs to $Im(h)$, but $y$ does not; since $h \in BI(\mathcal{I})$ we have found a function in $BI(\mathcal{I})$ containing $x$ but not $y$, contradicting the assumption that $xR_I y$. \qed
Lemma 6.4. Let $\mathcal{I}$ be a set of instances. Then $P^{BI}_I = \mathcal{C}_I$.

Proof. By Lemma 6.3, $P^{BI}_I$ must be a valid partition of $\mathcal{I}$. By Lemma 6.1 every set $A \in P^{BI}_I$ is obtained as the image of some instance $f \in BI(\mathcal{I})$. Clearly, since $f \in BI(\mathcal{I})$, by Lemma 6.2 the image $A$ of $f$ is the union of sets of the canonical partition of $\mathcal{I}$. Hence $P^{BI}_I = \mathcal{C}_I$.

Theorem 6.6. Let $\mathcal{I}$ be a set of instances. Then an instance $g$ belongs to $BI(\mathcal{I})$ if and only if $P^{BI}_I = P^{BI}_{I \cup \{g\}}$.

Proof. Clearly by Lemma 6.4 it suffices to show that $g \in BI(\mathcal{I})$ if and only if $P^{BI}_I = P^{BI}_{I \cup \{g\}}$, moreover it is immediate to note that, by construction of $P^{BI}_I$, if $g \in BI(\mathcal{I})$ then $P^{BI}_I = P^{BI}_{I \cup \{g\}}$. Assume now that $P^{BI}_I = P^{BI}_{I \cup \{g\}}$. By Lemma 6.3, $P^{BI}_{I \cup \{g\}}$ must be a valid partition. By Lemma 6.2 and exploiting the assumption that $P^{BI}_I = P^{BI}_{I \cup \{g\}}$, for each $x \in \text{Dom}(g)$, $g(x)$ must be the union of some sets in $P^{BI}_{I \cup \{g\}}$, that is $\text{Im}(g(x)) = \bigcup A_j$, for some sets $A_j \in P_I \cup \{g\}^{BI}$. But by Lemma 6.1 for each set $A_j$ there is an instance $f_j \in BI(\mathcal{I})$ such that $A_j$ is the image of $f_j$. Consequently, $g(x) = \bigoplus f_i$, and hence $g = \bigotimes_{x \in \text{Dom}(g)} g(x)$, which proves that $g \in BI(\mathcal{I})$ as required.

Theorem 6.6 and Lemma 6.4 lead to our main result.

Corollary 6.7. Let $\mathcal{I}$ be a set of instances. Then an instance $g$ belongs to $BI(\mathcal{I})$ if and only if $\mathcal{C}_I = \mathcal{C}_{I \cup \{g\}}$.

7 Conclusions

We have introduced the idea that partitions of the domain set can be used for characterizing the set of relations or graphs that can be extracted in a data base in the relational or in a graph-based model. By formally proving those expressiveness results we have effectively given a new framework for the analysis of data base query languages.

The graph-based model presented here is not rich enough to be considered of practical use, therefore it would be interesting to use our framework for analyzing a more sophisticated graph-based model.

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