OPTIMAL TRANSPORT AND TESSELLATION

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Abstract. Optimal transport from the volume measure to a convex combination of Dirac measures yields a tessellation of a Riemannian manifold into pieces of arbitrary relative size. This tessellation is studied for the cost functions $c_p(z, y) = \frac{1}{p}d^p(z, y)$ and $1 \leq p < \infty$. Geometric descriptions of the tessellations for all $p$ is obtained for compact subsets of the Euclidean space and the sphere. For $p = 1$ this approach yields Laguerre tessellations for all compact Riemannian manifolds.

1. Introduction

Given two probability measures $\mu$ and $\nu$ on some Riemannian manifold $M$, a transportation map from $\mu$ to $\nu$ is a map $\tilde{T} : M \to M$ pushing $\mu$ forward $\nu$, i.e. $\tilde{T} \# \mu = \nu$. In the theory of optimal transportation one is interested in a special transportation map, namely the one minimizing

$$\int_M c(x, \tilde{T}(x))d\mu(x)$$

the transportation cost among all possible transportation maps, for a given cost function $c : M \times M \to \mathbb{R}$. This minimizer, if it exists, is called optimal transportation map.

A Voronoi tessellation of a Riemannian manifold $M$ with respect to a discrete set of points $S$ is a decomposition of $M$ into different cells with center $s \in S$. Each cell consists of those points which are closer to its center than to any other center. A rather natural extension are the Laguerre tessellations where different centers can have different weights, see [LZ08].

The theory of optimal transport is a very natural setting for studying such tessellations. Just take $\mu$ as the normalized volume measure, $\nu$ as a convex combination of Dirac-measures and as the cost function $c(x, y) = d(x, y)$, the geodesic distance. In this framework the generalization of Laguerre tessellations to other cost functions like $c(x, y) = \frac{1}{p}d^p(x, y)$ is straightforward (at least to write the tessellation down).

Some of the phenomenons studied in this paper already appeared in the work of Ma, Trudinger and Wang, Loeper, Kim and McCann (see [MTW05, Loe09, KM07]), where they studied the regularity of the optimal transport map. In the case above for $p = 2$ and the manifold $M$ being the hyperbolic disc, the resulting "cells" do not have to be connected any more. If one now smears the Dirac points slightly

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one ends up with a measure which is absolutely continuous to the volume measure (the density can even be very nice) but with a discontinuous transportation map! Therefore, even the transport between two measures being absolutely continuous to the volume measure can be rather irregular.

1.1. General setting. In [McC01] McCann showed that on a compact connected complete Riemannian manifold \((M, g)\) without boundary there is always an optimal transport map \(T\) pushing some measure \(\mu\), absolutely continuous to the volume measure \(\text{vol}\), forward to an arbitrary Borel measure \(\nu\) on \(M\) under the condition that the cost function \(c\) is superdifferentiable. The map \(T\) is the exponential of the gradient of a \(c\)-convex function \(\Phi\). A function is \(c\)-convex iff 
\[
\Phi c(x) = \Phi, \\
\Phi c(x) = -\inf_{y \in M} [c(x, y) - \Phi(y)].
\]

We consider the special case of \(\mu(\cdot) = 1/\text{vol}(M, \cdot)\), the normalized volume measure, and \(\nu(\cdot) = \sum_{i \geq 0} \lambda_i \delta_{x_i}(\cdot)\), with \(\lambda_i \geq 0\), \(\sum \lambda_i = 1\) and \(x_i \in M\) for all \(i\), a (possibly infinite) convex combination of Dirac measures, and the cost function \(c_p(x, y) = \frac{1}{p} d_p(x, y)\) for \(1 \leq p < \infty\), where \(d(x, y)\) denotes the geodesic distance between \(x\) and \(y\). Then, the optimal transport map can be written down explicitly, i.e. the \(c\)-convex function \(\Phi\) can be written down explicitly (the apparent \(p\)-dependence is suppressed in the notation for the optimal transport map):

\[
\Phi(x) = \max_{1 \leq i \leq n} \Phi_i(z), \quad \text{with} \quad \Phi_i(z) = -\frac{d_p}{p}(x_i, z) + b_i, \quad b_i \in \mathbb{R} \quad \forall i.
\]

In particular, this shows that there is always a solution to the (even very much generalized) tessellation problem.

We are interested in the geometry of the mass being allocated to a certain Dirac point. More specific, how does the set \(T^{-1}(x_i)\), the cell allocated to \(x_i\), look like? Does it have to be connected? A nice way to look at these sets is to interpret them as intersections of “halfspaces”, i.e.:

\[
T^{-1}(x_i) = \{q \in M : \Phi_i(q) > \Phi_j(q) \forall j \neq i\} \\
= \bigcap_{j \neq i} \{q \in M : \Phi_i(q) > \Phi_j(q)\} \\
= \bigcap_{j \neq i} H^i_j.
\]

Thus, a good way to study the geometry of the cells \(T^{-1}(x_i)\) is to look for properties invariant under sections of these “halfspaces”. In this way we can reduce the problem from a many point problem to a two point problem.

This also directly settles the question of the regularity of the boundary of the cells \(T^{-1}(x_i)\) in the case of finitely many Dirac points. To study this regularity, it suffices to study the regularity of the boundary of the halfspaces \(\partial H^i_j = \{p \in M : \Phi_i(q) = \Phi_j(q)\}\). This question is answered by an implicit function theorem for subdifferentiable functions, e.g. like in the second appendix to chapter 10 in [Vil09], and yields

**Lemma 1.** \(\partial H^i_j\) is (locally) a \((n-1)\)-dimensional Lipschitz graph. For all \(q \in \partial H^i_j\) with \(q \notin \{\text{cutlocus}(x_i)\} \cup \{\text{cutlocus}(x_j)\}\) there is a neighbourhood \(U\) of \(q\) such that

\[
\partial H^i_j \text{ is (locally) a } (n-1)\text{-dimensional Lipschitz graph. For all } q \in \partial H^i_j \text{ with } q \notin \{\text{cutlocus}(x_i)\} \cup \{\text{cutlocus}(x_j)\} \text{ there is a neighbourhood } U\text{ of } q \text{ such that}
\]
Figure 1. $p = \frac{3}{2}, \alpha = 5$

$U \cap \partial H_j^i$ is the $(n-1)$ dimensional graph of a function which is as smooth as the cost function.

Thus, the boundary of the set $T^{-1}(x_i)$ is, as an intersection of sets with locally Lipschitz boundary, itself a set with locally Lipschitz boundary up to the points of intersection with the different “halfspaces”. In particular, $\partial H_j^i$ is a $\mu$-null set. Considering just finitely many Dirac points, we can thus define the cells $T^{-1}(x_i)$ to be open. In the case of infinitely many Dirac points, things become less regular. For example, take a dense countable subset of the unit disc as Dirac points. As the different cells are starlike, see section 4 they become stars consisting of many lines.

We now turn to the question of the geometry of the cells, and especially the connectedness. First, we will study the case of compact subsets of $\mathbb{R}^n$. Here we distinguish between convex and non convex subsets because they behave drastically different. Then, in section 3 we consider the problem on a curved space, the sphere. It turns out that this case is the nicest. We can describe the geometry of the cells well and moreover, opposed to the Euclidean setting, the cells are connected. In section 4 we deal with the case $p = 1$ which can be treated for all Riemannian manifolds yielding Laguerre tessellations.

2. Euclidean case

2.1. Convex subsets of $\mathbb{R}^n$. In this section, we will look at compact simply connected subsets $\Omega$ of $\mathbb{R}^n$. Let us assume until explicitly said otherwise that $\Omega$ is convex. In the case $p = 2$ and finitely many Dirac points it was shown in section 4 of [Stu09] and also in [AHA92] that the cells $T^{-1}(x_i)$ are convex polytopes and therefore simply connected. This can be easily seen from the characterization of this set as an intersection of halfspaces. Indeed, $\partial H_j^i$ is just a straight line. Thus, both $H_j^i$ and $H_i^j$ are convex polytopes. As intersections of convex sets are convex sets, $\bigcap_{j \neq i} H_j^i$ is still a convex polytope and we are done. However, this is the only case where we actually have connected sets. For all other cases of $p$, the sets $T^{-1}(x_i)$ do not have to be connected! Indeed, we have the following
Theorem 1. Let \( n \geq 2 \), \( \Omega \subset \mathbb{R}^n \) be convex compact and simply connected. Consider the optimal transport problem for \( \mu \) and \( \nu = \lambda \delta_{x_1} + (1 - \lambda)\delta_{x_2} \) and cost function \( c_p(x, y) \) with \( p \in (1, \infty) \setminus \{2\} \). Then, \( H^1_2 \) is in general not connected.

Proof. As the measure \( \nu \) is a convex combination of just two Dirac-measures, we can safely work in \( \mathbb{R}^2 \). We will take \( x_1 = (1, 0) \) and \( x_2 = (-1, 0) \). In order to characterize the set \( H^1_2 \) one needs to study its boundary \( \partial H^1_2 \). However, the boundary is just some level set \( \alpha \) of the function

\[
m(x, y, p) = \left| x_1 - (x, y) \right|^p - \left| x_2 - (x, y) \right|^p = \left( (x + 1)^2 - y^2 \right)^{p/2} - \left( (x - 1)^2 + y^2 \right)^{p/2},
\]

where \((x, y)\) denotes some point in \( \mathbb{R}^2 \). Figure 1 and figure 2 show how these level sets look like for certain values of \( p \) and \( \alpha \). The figures show the typical behaviour of \( m \) for \( p < 2 \) and for \( p > 2 \). They also show why theorem 1 holds.

In the case of \( p < 2 \) (figure 1) we can draw a straight line from a point in \( H^2_1 \), that is on the left hand side of the level set, to a point in \( H^1_2 \) on the right hand side of the level set which intersects the level sets three times. Completing this line to a convex polygon \( \Omega \) such that \( x_1 \) and \( x_2 \) are inside the polygon, we find a convex set \( \Omega \) such that \( H^1_2 \) is not connected. To show that this is indeed possible it suffices to show that the graph of the level set has a minimum in \( y = 0 \) and a change in sign in its second derivative for some \( y > 0 \).

The case \( p > 2 \) is slightly easier. The graph of the level set has a maximum at \( y = 0 \). For \( \alpha \) large enough, \( x_1 \in H^2_1 \). Thus, we can take a segment of the hyperplane \( x = c \) for some \( c > 1 \) such that \((c, 0) \in H^2_1 \), which intersects with the graph of the level set two times, and complete this segment to a polygon analogously to the case above. For this, it suffices to show that the graph of the level set has a maximum in \( y = 0 \).

By the implicit function theorem, if \( \frac{\partial m}{\partial y}(x_0, y_0) \neq 0 \) then there is a function \( \eta(y) \) such that in a neighbourhood of \((x_0, y_0)\) we have \( m(\eta(y), y, p) = \alpha \). This condition holds for all \( x > 1 \), the case we are interested in, and \( 1 < p < \infty \). Moreover, we...
have that

$$\frac{\partial \eta}{\partial y} = -\frac{\partial m/\partial y}{\partial m/\partial x}(\eta(y), y)$$

and for the second derivative of \( \eta \) we get

$$\frac{\partial^2 \eta}{\partial y^2} = \frac{1}{\partial m/\partial x} \left[ -\frac{\partial^2 m}{\partial x^2} \left( \frac{\partial \eta}{\partial y} \right)^2 + 2 \frac{\partial^2 m}{\partial y \partial x} \frac{\partial \eta}{\partial y} - \frac{\partial^2 m}{\partial y^2} \right]$$

(2.1)

This means, that even though we do not know how \( \eta \) explicitly looks like, we do know what its derivative at any point in \( \mathbb{R}^2 \) with \( x > 1 \) is. And this is enough for our purpose. We can check, if \( \eta \) has an extremum at \( y = 0 \) and if we know that for any fixed \( x > 1 \) the second derivative of \( \eta \) changes its sign, we are done. Let us first check the extremal points of \( \eta \).

Calculating the first derivative of \( \eta \) yields:

$$py \left( \frac{((1 + x)^2 + y^2)^{-\frac{1}{p}}}{-p(-1 + x) ((-1 + x)^2 + y^2)^{-\frac{1}{p}} + p(1 + x) ((1 + x)^2 + y^2)^{-\frac{1}{p}}} \right)$$

which is zero at \( y = 0 \). The second derivative at the point \((x, 0)\) is:

$$\frac{((1 + x)^2)^{-\frac{1}{p}} - ((1 + x)^2)^{-\frac{1}{p}}}{(-1 + x) ((-1 + x)^2)^{-\frac{1}{p}} + (1 + x) ((1 + x)^2)^{-\frac{1}{p}}}$$

As we consider the case \( x > 1, p > 1 \) the denominator is strictly positive. As the numerator is negative for \( p > 2 \) and positive for \( p < 2 \), \( \eta \) has at the point \((x, 0)\) a maximum for \( p > 2 \) and a minimum for \( p < 2 \).

This settles the case for \( p > 2 \). For \( p < 2 \) it remains to show that the second derivative of \( \eta \) has a change in sign for fixed \( x \). The second derivative at the point \((x, y)\) is
\[
\frac{p}{-p(-1 + x)((-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} + p(1 + x)((1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} \times \\
\times \left[\left(-2 + p\right)y^2 \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
- \left(-2 + p\right)y^2 \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} - \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
\frac{1}{p^2 y^2} \left(p(-1 + x)((-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} - p(1 + x)((1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)}\right)^2 \times \\
\times \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
\times \left((-2 + p)(-1 + x)^2 \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} - \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} \times \\
\frac{2(-2 + p)p y^2}{\left(p(-1 + x)((-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} + p(1 + x)((1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)}\right)^2} \times \\
\times \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
\times \left((-(-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)}\right)\right]\]

For \(x > 1\) the first fraction is positive. Thus, we can concentrate on the sum of the three terms inside the square brackets. The common denominator of these terms is a square. As we are interested in the sign, it suffices to look at the numerator after the expansion to the common denominator. This yields three terms, say \(u, v\) and \(w\). As we have already seen that the second derivative of \(\eta\) is positive at \((x, 0)\) (for \(1 < p < 2\)), it suffices to show that the numerator converges to zero from above as \(y\) tends to infinity to be able to conclude with the mean value theorem that there is a change in sign (a null with change of sign).

The first term reads

\[
u = \left[\left(-2 + p\right)y^2 \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} - \left(-2 + p\right)y^2 \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
\times \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-4 + p)} - \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} \times \\
\times \left(p(-1 + x)((-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} - p(1 + x)((1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)}\right)^2 \times \\
\times \left((-(-1 + x)^2 + y^2)^{\frac{1}{2}(-2 + p)} + \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)}\right)\right]\]

By using functions like \(g(z) = (x + 1)^{\frac{1}{2p}}\), we find the asymptotic \(u \sim 8p^2(p - 2)(3 - p)xy^{3p - 8}\). The second term reads

\[
v = \left[p^2 y^2 \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} - \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)}\right]^2 \times \\
\times \left[\left(p - 2\right)(x - 1)^2 \left((x - 1)^2 + y^2\right)^{\frac{1}{2}(p - 4)} - \left(p - 2\right)(1 + x)^2 \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(p - 4)} + \left((-1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)} - \left((1 + x)^2 + y^2\right)^{\frac{1}{2}(-2 + p)}\right],\]
which behaves like \( v \sim p^2(p - 2)^3 x^3 24y^{3p-10} \). The last term is
\[
w = 2(-2 + p)py^2 \left( -((1 + x)^2 + y^2)^{\frac{3}{2}(2+p)} + ((1 + x)^2 + y^2)^{\frac{3}{2}(2+p)} \right) \times \\
\times \left( (1 - x)((-1 + x)^2 + y^2)^{\frac{3}{2}(4+p)} + (1 + x)((1 + x)^2 + y^2)^{\frac{3}{2}(4+p)} \right) \times \\
\times \left( -p(-1 + x)((-1 + x)^2 + y^2)^{\frac{3}{2}(2+p)} + p(1 + x)((1 + x)^2 + y^2)^{\frac{3}{2}(2+p)} \right)
\]
This behaves asymptotically like \( w \sim 16p^2(p - 2)^2 x y^{3p-8} \). In total, we have an asymptotic for \( y \) tending to infinity like
\[
u + v + w \sim 8(p - 1)(p - 2)p^2 x y^{3p-8} \sim -y^{3p-8},
\]
as we assumed \( 1 < p < 2 \). Thus, there is a change in sign of the second derivative of \( \eta \) for any fixed \( x > 1 \) proving the theorem.

2.2. Non-convex subsets of \( \mathbb{R}^n \). If we now remove the restriction of \( \Omega \) being convex we are even worse off. Then, it can happen that \( T^{-1}(x_1) \) has even infinitely many components.

**Example 1.** Take two points in \( \mathbb{R}^2 \), say \( x_1 \) and \( x_2 \) from above, and for simplicity \( c(x, y) = \frac{\kappa}{2}(x, y) \) (this also works for other \( p > 1 \)). Arrange the constants \( b_i \) such that \( T(x_1) = x_2 \). By induction, we now build a set \( \Omega \) such that \( T^{-1}(x_1) \) has infinitely many components. The basic idea is as follows. \( \partial H^2_1 \) is a straight line. Everything on the left is transported to \( x_2 \) everything on the right to \( x_1 \). Just cut out a set with infinitely many components on the right hand side of \( \partial H^2_1 \).

Cut a nice set \( \Omega_0 \) out of \( H^2_1 \) such that some segment of the straight line \( \partial H^2_1 \) is a connected subset of the boundary of \( \Omega_0 \), say \( G = \partial H^2_1 \cap \partial H^2_0 \). Then, take a nice subset \( W_1 \) of \( H^2_1 \) such that \( \partial W_1 \cap \partial H^2_1 = g_1 \subset G \). Set \( \Omega_1 = \Omega_0 \cup W_1 \).

Let \( W_2 \) be a version of \( W_1 \) scaled down by some factor \( \kappa/2 \) and translated such that \( g_2 = \partial W_2 \cap \partial H^2_1 \subset G \) and \( \inf_{x \in W_{2}, y \in W_1} d(x, y) \geq \kappa/2 \). Then set \( \Omega_2 = \Omega_1 \cup W_2 \).

Let \( \Omega_n \) be constructed and let \( W_{n+1} \) be a version of \( W_n \) scaled down by the factor \( \kappa/2 \) and translated such that \( g_{n+1} = \partial W_{n+1} \cap \partial H^2_1 \subset G \), \( \inf_{x \in W_{n+1}, y \in W_n} d(x, y) \geq (\kappa/2)^n \) and \( \inf_{x \in W_{n+1}, y \in W_n} d(x, y) \geq \sum_{k=j}^{n} 2(\kappa/2)^k \). Then set \( \Omega_{n+1} = \Omega_n \cup W_{n+1} \).

Following this procedure inductively, we set \( \Omega = \Omega_\infty \), which will look like figure 3.

The red area is transported inductively, we set \( \Omega = \Omega_\infty \), which will look like figure 3.

![Figure 3. Example](image-url)
3. The sphere

In the case of $M = S^n$, the n-dimensional sphere, the picture changes rather drastically. It seems that there is just not enough space on the sphere to produce disconnected sets. We have

**Theorem 2.** Let $M = S^n$ and consider the optimal transport problem from $\mu$ to $\nu$ with cost function $c_p(x, y) = \frac{1}{p}d^p(x, y)$. Then, for any $i$ with $\lambda_i > 0$ and any $p \geq 1$ the set $T^{-1}(x_i)$ is simply connected.

For the proof we need two lemmas. Let us fix some notation first. For $z \in S^n$ we denote by $z'$ its antipode. For $z, y \in S^n$ we denote by $C_{z, y}$ the closed geodesic connecting $z$ and $y$. Then we have

**Lemma 2.** Let $C$ be any circle in $S^n$, i.e. $C = \partial B(q, r)$ for some $r > 0$ and $q \in S^n$. Then, for any $i$ there exist two points on $C$, say $x_i^n$ and $x_i^f$, such that the function $\Phi_i(z)$ attains its maximum and minimum on $C$ in $x_i^n$ respectively $x_i^f$. Moreover, $\{x_i^n, x_i^f\} = C \cap C_{x_i, q}$. In particular, $C \cap H_j^i$ is either empty or connected for any choice of $i$ and $j$.

**Proof.** If $\Phi_i$ is constant on $C$ the statement is clear, as we can choose any two points on $C$. If $\Phi_i$ is not constant on $C$, by continuity of the distance function, it is clear that there are two such points as claimed. The great circle $C_{x_i, q}$ divides $C$ into two halves on which $\Phi_i$ is symmetric. Thus, it suffices to show that $\Phi_i$ is monotonely decreasing from $x_i^n$ to $x_i^f$, otherwise said, that the function $D_i(z) = d(x_i, z)$ is monotonely increasing from $x_i^n$ to $x_i^f$. However, this is a direct consequence of the spherical law of cosine for the geodesic triangle $\Delta(x_i, q, z)$ for some $z \in C$. Parametrize the points $z$ on $C$ by the angle $\alpha_z$ relative to the geodesic from $q$ to $x_i$. Then we have

$$\cos(d(z, x_i)) = \cos(r) \cos(d(x_i, q)) + \sin(r) \sin(d(x_i, q) \cos(\alpha_z)).$$

If we now walk on $C$ from $x_i^n$ to $x_i^f$ the angle $\alpha_z$ is increasing from 0 to $\pi$. Therefore, $D_i(z)$ has to be increasing as well. The last statement of the lemma is now a direct consequence of the monotonicity of the two functions $\Phi_i$ and $\Phi_j$ between its respective extremal points. Indeed, parametrize $C$ by $S^1$. Then, on $C$ both $\Phi_i$ and $\Phi_j$ are $2\pi$-periodic and symmetric with respect to $x_i$ respectively $x_j$ and have the same shape. They are shifted versions of each others. Thus, proving the claim.

**Lemma 3.** If $x_i' \in H_j^i$ then $H_j^i = S^n$.

**Proof.** $x_i' \in H_j^i$ means that $\Phi_i(x_i') > \Phi_j(x_i')$. We have to show that for arbitrary $z \in S^n$ we have $\Phi_i(z) > \Phi_j(z)$. For given $z$ we consider the circle $C = \partial B(x_j, d(x_j, z))$ around $x_j$, where $\Phi_j$ has constant value $\Phi_j(z)$. From the last lemma we know that there is a point $y \in C$ such that $\Phi_i(y) \leq \Phi_i(w)$ for all $w \in C$. Furthermore, $y \in C \cap C_{x_j, x_i}$. Hence, it suffices to show that $\Phi_i(y) > \Phi_j(y)$. For this, it is sufficient to show that $\Phi_i(q) > \Phi_j(q)$ for all $q \in C_{x_j, x_i} =: C$. Parametrizing $C$ as $S^1$ we see that both $\Phi_i$ and $\Phi_j$ are $2\pi$-periodic. Moreover, $\Phi_j$ is just a shift of $\Phi_i$, that is

$$\Phi_j(q) = \Phi_i(q + d(x_i, x_j)) + b_j - b_i.$$
implies that \( C \) by a segment of \( C \) be mapped to \( x \) manifold. For the case \( H \) a good geometric description of the cells of the Laguerre tessellation. Due to the Theorem 3. Theorem 3. Let \( M \) be a compact connected complete Riemannian manifold without boundary. Consider the optimal transport problem from \( \mu \) to \( \nu \) with cost function \( c_1(x, y) = d(x, y) \), the geodesic distance. Then, for any \( i \) with \( \lambda_i > 0 \) the set \( T^{-1}(x_i) \) is simply connected and starlike with respect to \( x_i \).

Proof. We claim that the halfspace \( H^j \) is starlike with respect to \( x_i \). Then, if \( z \in T^{-1}(x_i) = \bigcap_{j \neq i} H^j \) we have \( z \in H^j \) for all \( j \). As \( H^j \) is starlike with respect to \( x_i \), the geodesic curve connecting \( z \) and \( x_i \) lies entirely inside \( H^j \). Therefore, \( \gamma \) is not simply connected. Then there are \( x_i \) and \( x_i' \) without passing through \( x_i' \) is transported to \( x_i \). As this is independent of \( j \) this shows that \( T^{-1}(x_i) \) is pathwise connected and therefore connected.

Assume that \( T^{-1}(x_i) \) is not simply connected. Then there are \( x_j, x_k \) with \( k \neq j \neq i \) such that any path joining \( T^{-1}(x_j) \) and \( T^{-1}(x_k) \) has to pass through \( T^{-1}(x_i) \). Without loss of generality we can assume that \( x_i' \in T^{-1}(x_j) \) (if there is no such \( j \) the whole sphere is transported to \( x_i \)). Take any \( z \in T^{-1}(x_k) \) and consider \( C_{z,x_i'} \). By assumption, on either segment of \( C_{z,x_i'} \) connecting \( z \) and \( x_i' \) there are subsegments mapped to \( x_i \), say \( G_1 \) and \( G_2 \). From the lemma above we know that \( H^j \cap C_{z,x_i'} \) is connected. As \( G_1 \) and \( G_2 \) are mapped to \( x_i \) they have to be connected by a segment of \( C_{z,x_i'} \), say \( G_3 \). Then either \( z \in G_3 \) or \( x_i' \in G \). The latter again implies that \( C_{z,x_i'} = G_3 \) and, in particular, we thus have \( z \in G_3 \). But then \( z \) cannot be mapped to \( x_i \), a contradiction.

This enables us to prove the theorem.

Proof of Theorem 4. Assume \( y, z \in T^{-1}(x_i) \), i.e. \( \Phi_i(y) > \Phi_j(y) \) for all \( j \neq i \) and for \( z \) likewise. Consider the circle \( C \) going through \( y, z \) and \( x_i' \). As \( H^i \cap C =: G \) is not empty (\( y, z \in H^i \cap C \)) it is connected and \( y \) and \( z \) have to lie in this connected segment of \( C \). Thus, either \( x_i' \in G \) or \( x_i' \notin G \). In the former case however we have \( \Phi_i(x_i') > \Phi_j(x_i') \). By the previous lemma, this means \( H^j = S^n \), in particular \( C \subset H^j \). Therefore, in either case the segment of \( C \) connecting \( z \) and \( y \) without passing through \( x_i \) is transported to \( x_i \). As this is independent of \( j \) this shows that \( T^{-1}(x_i) \) is pathwise connected and therefore connected.

Thus, \( \Pi \) is connected and \( \nu \) is connected.

4. General manifolds and “\( p = 1 \)”

We now want to consider the problem on a general complete compact Riemannian manifold. For the case \( p = 1 \) the situation is very nice for all manifolds and we get a good geometric description of the cells of the Laguerre tessellation. Due to the triangle inequality, we have the following

Theorem 3. Let \( M \) be a compact connected complete Riemannian manifold without boundary. Consider the optimal transport problem from \( \mu \) to \( \nu \) with cost function \( c_1(x, y) = d(x, y) \), the geodesic distance. Then, for any \( i \) with \( \lambda_i > 0 \) the set \( T^{-1}(x_i) \) is simply connected and starlike with respect to \( x_i \).
Assume $T(x_i) \neq x_i$. Then, there is a $j \neq i$ such that $T(x_i) = x_j$, i.e. $\Phi_j(x_i) = -d(x_i, x_j) + b_j > b_i = \Phi_i(x_i)$. Then, we have for any $q \in M$

$$-d(q, x_i) + b_i \geq -d(q, x_i) - d(x_i, x_j) + b_j > -d(q, x_i) + b_i.$$

This implies, that $T^{-1}(x_i) = \emptyset$ contradicting the assumption of $\lambda_i > 0$. Thus, $T(x_i) = x_i$.

Take any $w \in T^{-1}(x_i)$ (hence, $\Phi_i(w) > \Phi_j(w)$ for all $j \neq i$) and $q \in M$ such that $d(x_i, w) = d(x_i, q) + d(q, w)$, i.e. $q$ lies on the minimizing geodesic from $x_i$ to $w$. Then, we have for any $j \neq i$ by using the triangle inequality once more

$$-d(q, x_i) + b_i = -d(x_i, w) + d(q, w) + b_i \geq -d(x_i, w) + b_i + d(w, x_j) - d(q, x_j) > -d(q, x_j) + b_j,$$

which means that $\Phi_i(q) > \Phi_j(q)$ for all $j \neq i$. Hence, $q \in \bigcap_{j \neq i} H_j^i$ proving the claim. □

**Remark 1.** Note that in this case the transport map is unique even though we are working with the cost function $c_1(x, y) = d(x, y)$. The reason is that we transport the mass to a discrete measure. The sets $\partial H_j^i = \{x \in M : \Phi_i(x) = \Phi_j(x)\}$ are null sets. Thus, their (countable) union is a null set and for any $z \in M \setminus \bigcup_{i,j} \partial H_j^i$ there is a unique transport plan.

However, as mentioned above, already for the case $p = 2$ on a compact subset of the hyperbolic space the cells $T^{-1}(x_i)$ can be disconnected (see Example 3.8 in [KM07]). Results of various simulations hint at this not being special for $p = 2$. We believe, that for the hyperbolic space, this holds for all $p > 1$. In the case of a compact manifold with positive and negative curvature, we can be even worse off.

**Example 2.** Let $p > 1$ be given. Consider two points on the sphere which are the antipodes of each other, say $x$ and $x'$. Fix the transport map $T = \exp(\nabla \Phi)$ with $\Phi = \max\{\Phi_1, \Phi_2\}$ and $\Phi_1(y) = -\frac{1}{p} d^p(x, y) + b_1$ and $\Phi_2(y) = -\frac{1}{p} d^p(x', y)$, as above. Assume, that $b_1$ is such that $\Phi_1(x') = \Phi_2(x')$. In particular, we have $T(S^n) = x$. We now want to deform the sphere into a manifold $M$ with positive and negative curvature while fixing the transport map $T$ (the metric of the manifold will change and therefore we will end up with another distance function, but we will suppress that in the notation).

Fix a geodesic connecting $x$ and $x'$ (that is one half of a great circle connecting $x$ and $x'$). The $x'$-equator divides the geodesic into two segments. One of the segments, say $G$, has the property that all of its points are closer to $x'$ than to $x$. Let us fix a point $g \in G$ not equal to $x'$. By adding a very thin spike to $S^n$ around $g$ and making it, if necessary, very long, we can achieve, that the top of the spike is transported to $x'$. The reason is, that the cost function $c(z, y) = \frac{1}{p} d^p(z, y)$ is strictly convex and the points on the spike are closer to $x'$ than to $x$ (take $\kappa$ arbitrary, $p > 1$, then, there is $R$ such that for all $r > R$ we have $-(r + 1)^p + \kappa < -(r - 1)^p$). Then, in a similar manner to Example 1 we can add a sequence of spikes not touching each other, getting smaller and smaller and converging to a “zero-spike” at $x'$ (see figure 4). In order that this yields infinitely many components of the cell $T^{-1}(x')$ we need the assumption $\Phi_1(x') = \Phi_2(x')$. 

Thus, a natural question is, what is the geometry of the cells $T^{-1}(x_i)$ in an arbitrary compact connected Riemannian manifold with curvature bounded below by a strictly positive constant. Rough estimates on deformation of the sphere suggest that the cells should at least be connected. However, so far we were just able to prove connectedness in the case of $p = 2$, curvature bounded below by zero under the additional assumption that $T(x_i) = x_i$.

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