RULED AND QUADRIC SURFACES OF FINITE CHEN-TYPE

HASAN AL-ZOUBI, STYLIANOS STAMATAKIS, AND HANI ALMIMI

ABSTRACT. In this paper, we study ruled surfaces and quadrics in the 3-dimensional Euclidean space which are of finite III-type, that is, they are of finite type, in the sense of B.-Y. Chen, with respect to the third fundamental form. We show that helicoids and spheres are the only ruled and quadric surfaces of finite III-type, respectively.

1. INTRODUCTION

Let $M^n$ be a (connected) submanifold in the $n$-dimensional Euclidean space $E^m$. Let $x, H$ be the position vector field and the mean curvature field of $M^n$ respectively. Denote by $\Delta^I$ the second Beltrami-Laplace operator corresponding to the first fundamental form $I$ of $M^n$. Then, it is well known that \[\Delta^I x = -nH.\]

From this formula one can see that $M^n$ is a minimal submanifold if and only if all coordinate functions, restricted to $M^n$, are eigenfunctions of $\Delta^I$ with eigenvalue $\lambda = 0$. Moreover in [32] T. Takahashi showed that the submanifold $M^n$ for which $\Delta^I x = \lambda x$, i.e., for which all coordinate functions are eigenfunctions of $\Delta^I$ with the same eigenvalue $\lambda \in \mathbb{R}$, are precisely either the minimal submanifold with eigenvalue $\lambda = 0$ or the minimal submanifold of hyperspheres $S^{m-1}$ with eigenvalue $\lambda > 0$. Although the class of finite type submanifolds in an arbitrary dimensional Euclidean spaces is very large, very little is known about surfaces of finite type in the Euclidean 3-space $E^3$. Actually, so far, the only known surfaces of finite type corresponding to the first fundamental form in the Euclidean 3-space are the minimal surfaces, the circular cylinders and the spheres. So in [23] B.-Y. Chen mentions the following problem

**Problem 1.** Determine all surfaces of finite Chen I-type in $E^3$.

In order to provide an answer to the above problem, important families of surfaces were studied by different authors by proving that finite type ruled surfaces, finite type quadrics, finite type tubes, finite type cyclides of Dupin and finite type spiral surfaces are surfaces of the only known examples in $E^3$. However, for another classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet. For a more details, the reader can refer to [24].

In this context, Chen and Piccini in [25], introduced in the same way the theory of submanifolds of finite type Gauss map. A special case for $E^3$ one can ask
Problem 2. Classify all surfaces in $E^3$ with finite type Gauss map.

Results concerning this problem can be found in (2, 13, 16, 17).

Later in 28 O. Garay generalized T. Takahashi’s condition studied surfaces in $E^3$ for which all coordinate functions $(x_1, x_2, x_3)$ of $x$ satisfy $\Delta^i x_1 = \lambda_i x_i, i = 1, 2, 3$, not necessarily with the same eigenvalue. Another generalization was studied in 27 for which surfaces in $E^3$ satisfy the condition $\Delta^i x = Ax + B (\dagger)$ where $A \in \mathbb{R}^{3 \times 3}, B \in \mathbb{R}^{3 \times 1}$. It was shown that a surface $S$ in $E^3$ satisfies $(\dagger)$ if and only if it is an open part of a minimal surface, a sphere, or a circular cylinder. Surfaces satisfying $(\dagger)$ are said to be of coordinate finite type.

In the thematic circle of the surfaces of finite type in the Euclidean space $E^3$, S. Stamatakis and H. Al-Zoubi in 30 restored attention to this theme by introducing the notion of surfaces of finite type corresponding to the second or the third fundamental forms of $S$ in the following way:

A surface $S$ is said to be of finite type corresponding to the fundamental form $J$, or briefly of finite $J$-type, $J = II, III$, if the position vector $x$ of $S$ can be written as a finite sum of nonconstant eigenvectors of the operator $\Delta^J$, that is if

$$
x = x_0 + \sum_{i=1}^{k} x_i, \quad \Delta^J x_i = \lambda_i x_i, \quad i = 1, \ldots, k,
$$

where $x_0$ is a fixed vector and $x_1, \ldots, x_k$ are nonconstant maps such that $\Delta^J x_i = \lambda_i x_i, i = 1, \ldots, k$. If, in particular, all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are mutually distinct, then $S$ is said to be of $J$-type $k$, otherwise $S$ is said to be of infinite type. When $\lambda_i = 0$ for some $i = 1, \ldots, k$, then $S$ is said to be of null $J$-type $k$.

In general when $S$ is of finite type $k$, it follows from 21 that there exist a monic polynomial, say $R(x) \neq 0$, such that $R(\Delta^J)(x - c) = 0$. Suppose that $R(x) = x^k + \sigma_1 x^{k-1} + \ldots + \sigma_{k-1} x + \sigma_k$, then coefficients $\sigma_i$ are given by

$$
\sigma_1 = - (\lambda_1 + \lambda_2 + \ldots + \lambda_k), \\
\sigma_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \ldots + \lambda_1 \lambda_k + \lambda_2 \lambda_3 + \ldots + \lambda_2 \lambda_k + \ldots + \lambda_{k-1} \lambda_k), \\
\sigma_3 = - (\lambda_1 \lambda_2 \lambda_3 + \ldots + \lambda_{k-2} \lambda_{k-1} \lambda_k) \\
\text{.........................} \\
\sigma_k = (-1)^k \lambda_1 \lambda_2 \ldots \lambda_k.
$$

Therefore the position vector $x$ satisfies the following equation, (see 21)

$$
(\Delta^J)^k x + \sigma_1 (\Delta^J)^{k-1} x + \ldots + \sigma_k (x - c) = 0.
$$

In 4 Ruled surfaces were studied regarding the second fundamental form, another classes of surfaces were investigated in (5, 8, 9), meanwhile similar study was done but for the Gauss map of the surface as one can see in 6. In this paper we contribute to the solution of this problem by investigating the ruled surfaces and the quadric surfaces in $E^3$. On the other hand it is also interesting studying surfaces in the three-dimensional Euclidean space of coordinate finite type or coordinate finite type Gauss map with respect to the second or third fundamental form, result concerning this can be found in (1, 3, 7, 10, 11, 12, 14, 29, 31).

Our main results are the following
Theorem 1. The only ruled surfaces of finite III-type in the three-dimensional Euclidean space are the helicoids.

Theorem 2. The only quadric surfaces of finite III-type in the three-dimensional Euclidean space are the spheres.

2. Proof of Theorem 1

In the three-dimensional Euclidean space $E^3$ let $S$ be a ruled $C^r$-surface, $r \geq 3$, of nonvanishing Gaussian curvature defined by an injective $C^r$-immersion $x = x(s,t)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$.

The surface $S$ can be expressed in terms of a directrix curve $\Gamma$: $\sigma = \sigma(s)$ and a unit vector field $\rho(s)$ pointing along the rulings as follows

$$S: x(s,t) = \sigma(s) + t\rho(s), \quad s \in I, t \in \mathbb{R}. \quad (2.1)$$

Moreover, we can take the parameter $s$ to be the arc length along the spherical curve $\rho(s)$. Then we have

$$\langle \sigma', \rho \rangle = 0, \quad \langle \rho, \rho \rangle = 1, \quad \langle \rho', \rho' \rangle = 1,$$

where the differentiation with respect to $s$ is denoted by a prime and $\langle , \rangle$ denotes the standard scalar product in $E^3$. It is easily verified that the first and the second fundamental forms of $S$ are given by

$$I = n \, ds^2 + dt^2,$$
$$II = \frac{m}{\sqrt{n}} \, ds^2 + \frac{2A}{\sqrt{n}} \, ds \, dt,$$

where

$$n = \langle \sigma', \sigma' \rangle + 2\langle \sigma', \rho' \rangle t + t^2,$$
$$m = \langle \sigma', \rho, \sigma'' \rangle + [(\sigma', \rho', \sigma''') + (\rho', \rho, \sigma'')]' + (\rho', \rho, \rho'') t^2,$$
$$A = \langle \sigma', \rho, \rho' \rangle.$$

If, for simplicity, we put

$$\zeta := \langle \sigma', \sigma' \rangle, \quad \eta := \langle \sigma', \rho' \rangle,$$
$$\mu := \langle \rho', \rho, \rho'' \rangle, \quad \nu := \langle \sigma', \rho, \rho'' \rangle + (\rho', \rho, \sigma'') + (\rho', \rho, \sigma''), \quad \xi := \langle \sigma', \rho, \sigma'' \rangle,$$

we have

$$n = t^2 + 2\eta t + \zeta, \quad m = \mu t^2 + \nu t + \xi.$$

For the Gauss curvature $K$ of $S$ we find

$$K = -\frac{A^2}{n^2}.$$

The second Beltrami differential operator with respect to the third fundamental form is defined by

$$\triangle^III f = -\frac{1}{\sqrt{e}} \frac{\partial}{\partial u^i} \left( \sqrt{e} e^{i j} \frac{\partial f}{\partial u^j} \right),$$

\[\text{The reader is referred to [?] for definitions and formulae on ruled surfaces.}\]
where \( f \) is a sufficient differentiable function on \( S \) and \( e := \det(e_{ij}) \). After a long computation it can be expressed as follows:

\[
\Delta_{III} = -\frac{n}{A^2} \frac{\partial^2}{\partial s^2} + \frac{2nm}{A^3} \frac{\partial^2}{\partial s \partial t} - \left( \frac{n^2}{A^2} + \frac{nm^2}{A^3} \right) \frac{\partial^2}{\partial t^2}
+ \left( \frac{n_s}{2A^2} + \frac{nm_t}{A^3} - \frac{mn_t}{2A^3} \right) \frac{\partial}{\partial s}
+ \left( \frac{n m_s}{A^3} - \frac{mm_A}{2A^3} - \frac{mn_A}{A^4} - \frac{n n_A}{2A^2} + \frac{m^2 n_t}{2A^4} - \frac{2nm m_t}{A^4} \right) \frac{\partial}{\partial t}
= P_1 \frac{\partial^2}{\partial s^2} + P_2 \frac{\partial^2}{\partial s \partial t} + P_3 \frac{\partial}{\partial s} + P_4 \frac{\partial}{\partial t} + P_5 \frac{\partial^2}{\partial t^2},
\]

where

\[
\begin{align*}
n_t &= \frac{\partial n}{\partial t}, \quad n_s = \frac{\partial n}{\partial s}, \quad m_t = \frac{\partial m}{\partial t}, \quad m_s = \frac{\partial m}{\partial s}
\end{align*}
\]

and \( P_1, \ldots, P_5 \) are polynomials in \( t \) with functions in \( s \) as coefficients and \( \deg(P_t) \leq 6 \). More precisely we have

\[
\begin{align*}
P_1 &= -\frac{1}{A^2} [t^2 + 2\eta t + \zeta], \\
P_2 &= \frac{2}{A^3} [\mu t^4 + (2\eta + \nu + \xi + \zeta + \mu) t^3 + (2\eta + \xi + \zeta + \mu) t^2 + (2\eta + \zeta + \nu) t + \zeta], \\
P_3 &= \frac{1}{A^3} [\mu t^3 + 3\eta + \mu + \nu] t^2 + (\eta + \xi + \zeta + \mu + \eta + \nu + \eta + \zeta + \nu), \\
P_4 &= \frac{1}{A^4} [-3\mu^2 t^5 + (\mu + \mu^2 A - \mu A^2) t^4 \quad + (\nu + A - \eta + \nu + \eta + \mu) t^3 \quad + (\xi + A - \zeta + \mu + \nu) t^2 \quad + (\zeta A + \xi + A - \nu + \zeta + \mu) t^1 \quad + (\zeta + \xi + A - \eta + \zeta + \mu)]
\end{align*}
\]

\[
P_5 = -\frac{1}{A^4} [\mu^2 t^6 + (2\nu + 2\eta + \mu^2) t^5 + (2\mu + \xi + \nu + \eta + \mu + \zeta) t^4 \quad + (2\nu + 4\eta + \mu + \zeta + \nu + 4\eta + A) t^3 \quad + (\zeta + 3\nu + \eta + \zeta + \nu) t^2 \quad + (\zeta + 2\mu + 4\nu + \eta + \zeta + \nu)]
\]

Applying (2.2) on the position vector (2.1) of the ruled surface \( S \) we find

\[
\Delta_{III} \mathbf{x} = P_1 \sigma'' + P_2 \rho' + P_3 \sigma' + P_4 \rho + (P_1 \rho' + P_3 \rho') t.
\]
We write this last expression of $\triangle^{III} \mathbf{x}$ as a vector $Q_1(t)$ whose components are polynomials in $t$ with functions in $s$ as coefficients as follows:

\[
Q_1(t) = \frac{1}{A^4} \left[ -3 \mu^2 \rho t^5 + \left( (\mu' A - \mu A^2) \rho + 3 \mu A \rho \right) t^4 + \left( \mu A \sigma^2 \rho'' + (2 \nu A + 7 \eta A) \rho' \right.ight.
\]
\[
+ (\nu' A - \nu A' + 2 \eta \mu' A - 2 \eta \mu A' - \eta' \mu A - A^2 - 10 \eta \mu \nu - 2 \mu \xi - \nu^2 - 4 \zeta \mu^2) \rho \right]
\]
\[
+ \left( (\zeta \mu' A - \zeta \mu A' - \frac{1}{2} \zeta' \mu A + 2 \eta \zeta' A - 2 \eta \xi A' - \eta' \xi A
\]
\[
- \zeta' A' + \xi - 3 \eta \nu - 6 \eta \mu \xi - 6 \zeta \mu \nu \right) \rho
\]
\[
+ 3 \eta \mu A \sigma^2 \rho'' - A^2 \sigma'' + (\eta' A + 5 \eta \nu + 4 \zeta \mu + \xi) A \sigma') t^2
\]
\[
+ \left( (\zeta \nu' A - \zeta \nu A' - \frac{1}{2} \zeta' \nu A + 2 \eta \zeta' A - 2 \eta \xi A' - \eta' \xi A
\]
\[
- \zeta' A' + \xi - 2 \eta \nu^2 + \xi^2 - 2 \eta \nu \xi - 4 \zeta \mu \xi) \rho - 2 \eta A^2 \sigma''
\]
\[
+ \left( \frac{1}{2} \zeta A + 3 \zeta \nu + 3 \eta \xi \right) A \rho^2 \rho'' + (\eta \nu - \xi + 2 \zeta \mu + \xi) A \sigma') t
\]
\[
+ (\zeta \xi' A - \zeta \xi A' - \frac{1}{2} \zeta' \xi A - \zeta \eta A^2 - 2 \zeta \nu \xi) \rho
\]
\[
+ \left( \frac{1}{2} \zeta A - \eta \xi + \zeta \nu \right) A \sigma' + 2 \zeta \xi A \rho^2 \sigma'' \right].
\]

Notice that $\deg(Q_1) \leq 5$. Furthermore $\deg(Q_1) = 5$ if and only if $\mu \neq 0$, otherwise $\deg(Q_1) \leq 3$.

Before we start the proof of the first theorem we give the following Lemma which can be proved by a straightforward computation.

**Lemma 1.** Let $g$ be a polynomial in $t$ with functions in $s$ as coefficients and $\deg(g) = d$. Then $\triangle^{III} \bar{g} = \hat{g}$, where $\hat{g}$ is a polynomial in $t$ with functions in $s$ as coefficients and $\deg(\bar{g}) \leq d + 4$.

We suppose that $S$ is of finite $III$-type $k$. It is well known that there exist real numbers $c_1, \ldots, c_k$ such that

\[
(\triangle^{III})^{k+1} \mathbf{x} + c_1 (\triangle^{III})^k \mathbf{x} + \cdots + c_k \triangle^{III} \mathbf{x} = 0,
\]

see [21]. By applying Lemma 1 we conclude that there is an $\mathbb{E}^3$-vector-valued function $Q_k$ in the variable $t$ with some functions in $s$ as coefficients, such that

\[
(\triangle^{III})^k \mathbf{x} = Q_k(t),
\]

where $\deg(Q_k) \leq 4k + 1$. Now, if $k$ goes up by one, the degree of each component of $Q_k$ goes up at most by $4$. Hence the sum $Q_k$ can never be zero, unless of course

\[
\triangle^{III} \mathbf{x} = Q_1 = 0.
\]

On account of the well known relation

\[
\triangle^{III} \mathbf{x} = \nabla^{III} \left( \frac{2H}{K} \mathbf{n} \right) - \frac{2H}{K} \mathbf{n},
\]
where $H, n$ and $\nabla^{III}$ denote the mean curvature, the unit normal vector field and the first Beltrami-operator with respect to $III$, see [30], from (2.4) we result that $S$ is minimal, and that $S$ is a helicoid.

3. Proof of Theorem 2

Let now $S$ be a quadric in $\mathbb{E}^3$. Then $S$ is either a ruled surface or one of the following two kinds, see [26],

$$z^2 - ax^2 - by^2 = c, \quad a, b, c \in \mathbb{R}, \quad a b \neq 0, \quad c > 0, \quad (3.1)$$

or

$$z = \frac{a}{2} x^2 + \frac{b}{2} y^2, \quad a, b \in \mathbb{R}, \quad a, b > 0. \quad (3.2)$$

If $S$ is a ruled surface of finite $III$-type, then, according to theorem 1, $S$ is a helicoid.

In this section we will first show that a quadric of the kind (3.1) is of finite $III$-type if and only if $a = -1$ and $b = -1$, that is, if and only if $S$ is a sphere. Next we will show that a quadric of the kind (3.2) is of infinite type.

3.1. Quadrics of the first kind.

A part of a quadric of this kind can be parametrized by

$$x(u, v) = (u, v, \sqrt{c + a u^2 + b v^2}), \quad \sqrt{c + a u^2 + b v^2} > 0. \quad (3.3)$$

We put for simplicity

$$c + a u^2 + b v^2 = \omega.$$

The third fundamental form of $S$ becomes

$$III = \frac{a^2}{\omega T^2} C(u, v) du^2 - 2 \frac{a b}{\omega T^2} B(u, v) du dv + \frac{b^2}{\omega T^2} A(u, v) dv^2,$$

where

$$T = c + a(a + 1)u^2 + b(b + 1)v^2,$$

$$A(u, v) = a^2 u^2 v^2 + (a u^2 + c)^2 + a^2 u^2 \omega,$$

$$B(u, v) = u v [c(a + b) + a b(v^2 + \omega)] ,$$

$$C(u, v) = b^2 u^2 v^2 + (b v^2 + c)^2 + b^2 v^2 \omega.$$

Then the second Beltrami operator $\triangle^{III}$ of $S$ can be expressed as follows:

\[
\triangle^{III} = -\frac{T}{a^2 b^2 c^2} \left[ b^2 A \frac{\partial^2}{\partial u^2} + 2 a b B \frac{\partial^2}{\partial u \partial v} + a^2 C \frac{\partial^2}{\partial v^2} \right] \\
- \frac{T}{a^2 b^2 c^2} \left[ \frac{b}{\omega} \left( b \frac{\partial A}{\partial u} + a \frac{\partial B}{\partial v} \right) \frac{\partial}{\partial u} + a \left( a \frac{\partial C}{\partial v} + b \frac{\partial B}{\partial u} \right) \frac{\partial}{\partial v} \right] \\
+ \frac{T}{a^2 b^2 c^2} \left[ \frac{a b^2}{\omega} (u A + v B) \frac{\partial}{\partial u} + \frac{a^2 b}{\omega} (u B + v C) \frac{\partial}{\partial v} \right] \\
+ \frac{1}{a^2 b^2 c^2} \left[ a b^2 ((a + 1) u A + (b + 1) v B) \frac{\partial}{\partial u} + a^2 b ((b + 1) v C + (a + 1) u B) \frac{\partial}{\partial v} \right]. \quad (3.4)
\]
We note that

\[ \frac{b}{\partial u} \frac{\partial A}{\partial u} + \frac{a}{\partial v} \frac{\partial B}{\partial v} = a \left[ 5ab(a+1)u^2 + 5ab(b+1)v^2 + c(3a + 5b + a) \right], \]
\[ \frac{a}{\partial v} \frac{\partial C}{\partial v} + \frac{b}{\partial u} \frac{\partial B}{\partial u} = b \left[ 5ab(a+1)u^2 + 5ab(b+1)v^2 + c(3a + 5b + a) \right], \]
\[ uA + vB = \left[ c + a(a+1)u^2 + a(b+1)v^2 \right] u\omega, \]
\[ uB + vC = \left[ c + b(a+1)u^2 + b(b+1)v^2 \right] v\omega, \]
\[ (a + 1)uA + (b + 1)vB = \left[ c(a + 1) + a(a+1)u^2 + a(b+1)v^2 \right] uT, \]
\[ (b + 1)vC + (a + 1)uB = \left[ c(b + 1) + b(a+1)u^2 + b(b+1)v^2 \right] vT. \]

Hence (3.4) becomes

\[ \nabla^{III} \frac{b}{c^2} \left( \mu \frac{\partial^2}{\partial u^2} + 3 \frac{\partial}{\partial u} \right) + \frac{b(b + 1)}{c^2} \left( \nu \frac{\partial^2}{\partial v^2} + 3 \frac{\partial}{\partial v} \right) \]
\[ + f_1(u, v) \frac{\partial^2}{\partial u \partial v} + f_2(u, v) \frac{\partial^2}{\partial u^2} + f_3(u, v) \frac{\partial^2}{\partial v^2} \]
\[ + f_4(u, v) \frac{\partial}{\partial u} + f_5(u, v) \frac{\partial}{\partial v}, \] (3.5)
where

\[ f_1(u, v) = -2uv \left( \frac{(a+1)^2}{c^2} u^4 + \frac{(a+1)(a+ab+2b)}{bc} u^2 + \frac{a+b+ab}{ab} \right) \]
\[ -2uv \left( \frac{b(b+1)^2}{c^2} v^4 + \frac{(b+1)(b+ab+2a)}{ac} v^2 \right) \]
\[ -2uv \left( \frac{(a+1)(b+1)(a+b)}{ac} u^2 v^2 \right), \]
\[ f_2(u, v) = -\frac{(a+1)(a+3)}{c} u^4 - \frac{(2a+3)}{a} u^2 - \frac{c}{a^2} \]
\[ -\frac{(a+1)(b+1)(a+b)}{ac} u^2 v^2 - \frac{b(b+1)^2}{c^2} u^2 v^4 \]
\[ -\frac{(a+1)(a+ab+2b)}{ac} u^2 v^2 - \frac{b(b+1)}{a^2} v^2, \]
\[ f_3(u, v) = -\frac{(b+1)(b+3)}{c} v^4 - \frac{(2b+3)}{b} v^2 - \frac{c}{b^2} \]
\[ -\frac{(a+1)(b+1)(a+b)}{ac} u^2 v^2 - \frac{a(a+1)^2}{c^2} u^2 v^2 \]
\[ -\frac{(a+1)(2a+ab+b)}{bc} u^2 v^2 - \frac{a(a+1)}{b^2} u^2, \]
\[ f_4(u, v) = -\frac{(a+1)(a+6b+2ab)}{bc} u^3 - \frac{(2ab+a+3b)}{ab} u \]
\[ -\frac{3(a+1)(b+1)(a+b)}{c^2} u^3 v^2 - \frac{3b(b+1)^2}{c^2} u v^4 \]
\[ -\frac{(b+1)(4a+2ab+3b)}{ac} u^3 v^2, \]
\[ f_5(u, v) = -\frac{(b+1)(6a+b+2ab)}{ac} v^3 - \frac{(2ab+3a+b)}{ab} v \]
\[ -\frac{3(a+1)(b+1)(a+b)}{c^2} u^3 v^3 - \frac{3a(a+1)^2}{c^2} u^4 v \]
\[ -\frac{(a+1)(3a+2ab+4b)}{bc} u^2 v. \]

Here again the functions \( f_i, i = 1, \ldots, 5, \) are polynomials in \( u \) and \( v \) with \( \text{deg}(f_i) \leq 6 \).

We consider a function \( g(u) \in C^\infty(U) \). By means of (3.5), we find

\[ \triangle^{III} g = -\frac{a(a+1)^2}{c^2} u^5 \left( u \frac{\partial^2 g}{\partial u^2} + 3 \frac{\partial g}{\partial u} \right) + f_2(u, v) \frac{\partial^2 g}{\partial u^2} + f_4(u, v) \frac{\partial g}{\partial u}. \]  
(3.6)

If we put \( v = 0 \), then the functions \( f_2 \) and \( f_4 \) are polynomials in \( u \) of degree less than or equal 4. Now we prove the following

**Lemma 2.** The relation

\[ (\triangle^{III})^k u = (-1)^k \left( \prod_{i=1}^{2k} (2i-1) \right) \left( \frac{a^k (a+1)^{2k} u^{4k+1}}{c^{2k}} \right) + P_{4k}(u, v), \]

holds true, where \( \text{deg}(P_{4k}(u, 0)) \leq 4k \).
Proof. The proof goes by induction on \( k \). For \( k = 1 \) the formula follows immediately from \((3.6)\) applied to \( g = u \). Suppose the Lemma is true for \( k - 1 \). Then

\[
(\triangle^{III})^{k-1} u = (-1)^{k-1} \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) + P_{4k-4}(u, v).
\]

Taking into account \((3.6)\) we obtain

\[
(\triangle^{III})^{k} u = \triangle^{III} \left( (\triangle^{III})^{k-1} u \right) = - \frac{a(a+1)^2u^5}{c^2} (-1)^{k-1} \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) \\
- \frac{a(a+1)^2u^5}{c^2} \left( u \frac{\partial^2}{\partial u^2} (P_{4k-4}) + 3 \frac{\partial}{\partial u} (P_{4k-4}) \right) \\
- (-1)^k \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) f_2(u, v) \frac{\partial^2}{\partial u^2} (u^{4k-3}) \\
- (-1)^k \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) f_4(u, v) \frac{\partial}{\partial u} (u^{4k-3}) \\
+ f_2(u, v) \frac{\partial^2}{\partial u^2} (P_{4k-4}) + f_4(u, v) \frac{\partial}{\partial u} (P_{4k-4}) \\
= (-1)^k \left( \prod_{i=1}^{2k} (2i-1) \right) \left( \frac{a^{k} (a + 1)^{2k} u^{4k+1}}{c^{2k}} \right) + P_{4k}(u, v),
\]

where

\[
P_{4k}(u, v) = - \frac{a(a+1)^2u^5}{c^2} \left( u \frac{\partial^2}{\partial u^2} (P_{4k-4}) + 3 \frac{\partial}{\partial u} (P_{4k-4}) \right) \\
- (-1)^k \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) f_2(u, v) \frac{\partial^2}{\partial u^2} (u^{4k-3}) \\
- (-1)^k \left( \prod_{i=1}^{2k-2} (2i-1) \right) \left( \frac{a^{k-1} (a + 1)^{2k-2} u^{4k-3}}{c^{2k-2}} \right) f_4(u, v) \frac{\partial}{\partial u} (u^{4k-3}) \\
+ f_2(u, v) \frac{\partial^2}{\partial u^2} (P_{4k-4}) + f_4(u, v) \frac{\partial}{\partial u} (P_{4k-4}). \tag{3.7}
\]

Since

\[
\deg(P_{4k-4}(u, 0)) \leq 4k - 4, \quad \deg(f_2(u, 0)) \leq 4 \quad \text{and} \quad \deg(f_4(u, 0)) \leq 4,
\]

from \((3.7)\) we find that \( \deg(P_{4k}(u, 0)) \leq 4k \). \( \square \)

By applying now \((3.5)\) on a function \( h(v) \in C^\infty(U) \) we find

\[
\triangle^{III} h = - \frac{b(b+1)^2v^5}{c^2} \left( v \frac{\partial^2 h}{\partial v^2} + 3 \frac{\partial h}{\partial v} \right) + f_3(u, v) \frac{\partial^2 h}{\partial v^2} + f_5(u, v) \frac{\partial h}{\partial v}.
\]

If we put \( u = 0 \), then the functions \( f_3 \) and \( f_5 \) are polynomials in \( v \) of degree less than or equal 4. Proceeding analogously as in Lemma 2 we prove the following
Lemma 3. The relation
\[
(\Delta^{III})^k v = (-1)^k \left( \prod_{i=1}^{2k} (2i-1) \right) \left( \frac{b^{k+1}}{c^{2k+2}} (b + 1)^{2k+2} v^{4k+1} \right) + Q_{4k}(u, v)
\]
holds true, where \(\deg(Q_{4k}(0, v)) \leq 4k\).

We suppose now that \(S\) is of finite III-type \(k\). Then there exist real numbers \(c_1, \ldots, c_k\) such that
\[
(\Delta^{III})^{k+1} x + c_1 (\Delta^{III})^k x + \ldots + c_k \Delta^{III} x = 0.
\]
(3.8)

Applying (3.8) on the coordinate functions \(x_1 = u\) and \(x_2 = v\) of the position vector \(\mathbf{r}^{III}\) of the quadric \(S\) we obtain
\[
(\Delta^{III})^{k+1} u + c_1 (\Delta^{III})^k u + \ldots + c_k \Delta^{III} u = 0, \quad (3.9)
\]
\[
(\Delta^{III})^{k+1} v + c_1 (\Delta^{III})^k v + \ldots + c_k \Delta^{III} v = 0. \quad (3.10)
\]

From Lemma 2 and the relation (3.9) we obtain that there exists a polynomial \(P_{4k+4}(u, v)\) of degree at most \(4k+4\) such that
\[
(-1)^{k+1} \left( \prod_{i=1}^{2k+2} (2i-1) \right) \left( \frac{a^{k+1}}{c^{2k+2}} (a + 1)^{2k+2} u^{4k+5} \right) + P_{4k+4}(u, v) = 0. \quad (3.11)
\]

If we put \(v = 0\) in (3.11), then we get a non-trivial polynomial in \(u\) with constant coefficients. Since \(a \neq 0\) the relation (3.11) implies \(a = -1\).

Similarly, from Lemma 3 and the relation (3.10) we obtain that there exists a polynomial \(Q_{4k+4}(u, v)\) of degree at most \(4k+4\) such that
\[
(-1)^{k+1} \left( \prod_{i=1}^{2k+2} (2i-1) \right) \left( \frac{b^{k+1}}{c^{2k+2}} (b + 1)^{2k+2} v^{4k+5} \right) + Q_{4k+4}(u, v) = 0. \quad (3.12)
\]

Putting \(u = 0\) in (3.12), we get again a non-trivial polynomial in \(v\) with constant coefficients. Since \(b \neq 0\), we obtain from (3.12) \(b = -1\). Hence \(S\) must be a sphere.

3.2. Quadrics of the second kind. A quadric surface of this kind can be parametrized by
\[
\mathbf{x}(u, v) = \left( u, v, \frac{a^2}{2} u^2 + \frac{b}{2} v^2 \right). \quad (3.13)
\]

Then the third fundamental form of \(S\) is the following
\[
III = \frac{a^2}{g^2} (1 + v^2 u^2) du^2 - 2 \frac{a^2 b^2}{g^2} u v du dv + \frac{b^2}{g^2} (1 + a^2 u^2) dv^2,
\]
where
\[
g := \det (g_{ij}) = 1 + a^2 u^2 + b^2 v^2
\]
is the discriminant of the first fundamental form
\[
I = (1 + a^2 u^2) du^2 + 2 a b u v du dv + (1 + b^2 v^2) dv^2
\]
of \(S\). Hence the Beltrami operator \(\Delta^{III}\) of \(S\) takes the following form
\[
\Delta^{III} = -\frac{g(1 + a^2 u^2)}{a^2} \frac{\partial^2}{\partial u^2} - \frac{g(1 + b^2 v^2)}{b^2} \frac{\partial^2}{\partial v^2} - 2 u v g \frac{\partial^2}{\partial u \partial v} - 2 u g \frac{\partial}{\partial u} - 2 v g \frac{\partial}{\partial v}.
\]
which can be written as

\[ \Delta^{III} = -a^2u^3 \left( u \frac{\partial^2}{\partial u^2} + 2 \frac{\partial}{\partial u} \right) - b^2v^3 \left( v \frac{\partial^2}{\partial v^2} + 2 \frac{\partial}{\partial v} \right) 
- f_1(u,v) \frac{\partial^2}{\partial u \partial v} - f_2(u,v) \frac{\partial^2}{\partial u^2} - f_3(u,v) \frac{\partial^2}{\partial v^2} 
- f_4(u,v) \frac{\partial}{\partial u} - f_5(u,v) \frac{\partial}{\partial v}, \]  

(3.14)

where

\[ f_1(u,v) = 2uvg, \]
\[ f_2(u,v) = 2u^2 + b^2u^2v^2 + \frac{1}{a^2} \left( 1 + b^2v^2 \right), \]
\[ f_3(u,v) = 2v^2 + a^2u^2v^2 + \frac{1}{b^2} \left( 1 + a^2u^2 \right), \]
\[ f_4(u,v) = 2u \left( 1 + b^2v^2 \right), \]
\[ f_5(u,v) = 2v \left( 1 + a^2u^2 \right). \]

Notice that the functions \( f_i, i = 1, \ldots, 5, \) are polynomials in \( u \) and \( v \) with \( \deg(f_i) \leq 4. \) By applying the operator \( \Delta^{III} \) on a function \( g(u) \in C^\infty(U) \) we find by means of (3.14)

\[ \Delta^{III} g = -a^2u^3 \left( u \frac{\partial^2 g}{\partial u^2} + 2 \frac{\partial g}{\partial u} \right) - f_2(u,v) \frac{\partial^2 g}{\partial u^2} - f_4(u,v) \frac{\partial g}{\partial u}. \]

(3.15)

If we put \( v = 0, \) then the functions \( f_2 \) and \( f_4 \) are polynomials in \( u \) of degree less than or equal 2.

Using (3.15) and by induction on \( k \) we can prove the following

**Lemma 4.** The relation

\[ (\Delta^{III})^k u = (-1)^k(2k)!a^{2k}u^{2k+1} + P_{2k}(u,v), \]

holds true, where \( \deg(P_{2k}(u,0)) \leq 2k. \)

By applying (3.14) on a function \( h(v) \in C^\infty(U) \) we get

\[ \Delta^{III} h = -b^2v^3 \left( v \frac{\partial^2 h}{\partial v^2} + 2 \frac{\partial h}{\partial v} \right) - f_3(u,v) \frac{\partial^2 h}{\partial v^2} - f_5(u,v) \frac{\partial h}{\partial v}. \]

(3.16)

If we put \( u = 0, \) then the functions \( f_3 \) and \( f_5 \) are polynomials in \( v \) of degree less than or equal 2. In the same way the following Lemma can be proved

**Lemma 5.** The relation

\[ (\Delta^{III})^k v = (-1)^k(2k)!b^{2k}v^{2k+1} + Q_{2k}(u,v), \]

holds true, where \( \deg(Q_{2k}(0,v)) \leq 2k. \)

Now, if the quadric \( S \) is of finite III—type \( k, \) then again the relations (3.8), (3.9) and (3.10) are valid. Combining the equations (3.15) and (3.16) with Lemma 4 and Lemma 5 respectively, we conclude that there exist two polynomials \( P_{2k+2}(u,v) \) and \( Q_{2k+2}(u,v) \) of degree at most \( 2k + 2 \) such that

\[ (-1)^{k+1}(2k+2)!a^{2k+2}u^{2k+3} + P_{2k+2}(u,v) = 0, \]

(3.17)

\[ (-1)^{k+1}(2k+2)!b^{2k+2}v^{2k+3} + Q_{2k+2}(u,v) = 0. \]

(3.18)
We put \( v = 0 \) in (3.17). Then the left member of the equation (3.17) is a nontrivial polynomial in \( u \) with constant coefficients. This polynomial can never be zero, unless \( a = 0 \). Similarly, if we put \( u = 0 \) in (3.18), then the left member of (3.18) is a nontrivial polynomial in \( v \) with constant coefficients. This implies \( b = 0 \). This is clearly impossible since \( a, b > 0 \).

References

[1] H. Al-Zoubi, H. Alzaareer, T. Hamadneh, M. Al Rawajbeh, Tubes of coordinate finite type Gauss map in the Euclidean 3-space, Indian J. Math. 62 (2020), 171-182.
[2] H. Al-Zoubi, M. Al-Sabbagh, Anchor rings of finite type Gauss map in the Euclidean 3-space, International Journal of Mathematical and Computational Methods 5 (2020), 9-13.
[3] H. Al-Zoubi, On the Gauss map of quadric surfaces, arxiv 1905.00962v1, (2019).
[4] H. Al-Zoubi, A. Dababneh, M. Al-Sabbagh, Ruled surfaces of finite \( II \)-type, WSEAS Trans. Math. 18 (2019), 1-5.
[5] H. Al-Zoubi, Tubes of finite \( II \)-type in the Euclidean 3-space, WSEAS Trans. Math. 17 (2018), 1-5.
[6] H. Al-Zoubi, T. Hamadneh, M. Abu Hammad, and M. Al-Sabbagh, Tubular surfaces of finite type Gauss map, J. Geom. Graph. 25 (2021), 45-52.
[7] H. Al-Zoubi, T. Hamadneh, Surfaces of coordinate finite \( II \)-type, arXiv: 2005.05120v1, May (2020).
[8] H. Al-Zoubi, K. M. Jaber, S. Stamatakis, Tubes of finite Chen-type, Comm. Korean Math. Soc. 33 (2018), 581-590.
[9] H. Al-Zoubi, W. Al Mashaleh, Surfaces of finite type with respect to the third fundamental form, IEEE Jordan International Joint Conference on Electrical Engineering and Information Technology (JEEIT), Amman, April 9-11, (2019).
[10] H. Al-Zoubi, T. Hamadneh, Surfaces of revolution of finite \( III \)-type, arXiv: 1907.12390v2, Oct (2019).
[11] H. Al-Zoubi, S. Stamatakis, Ruled and Quadric surfaces satisfying \( \triangle^{III}\mathbf{x} = A\mathbf{x} \), J. Geom. Graph. 20 (2016), 147-157.
[12] H. Al-Zoubi, S. Stamatakis, W. Al Mashaleh and M. Awadallah, Translation surfaces of coordinate finite type, Indian J. Math. 59 (2017), 227-241.
[13] H. Al-Zoubi, T. Hamadneh, H. Alzaareer, M. Al-Sabbagh, Tubes in the Euclidean 3-space with coordinate finite type Gauss map, IEEE Jordan The 10th International Conference on Information Technology (ICTIT 2021), Amman, July 14-15, (2021).
[14] H. Al-Zoubi, F. Abdel-Fattah, M. Al-Sabbagh, Surfaces of finite \( III \)-type in the Euclidean 3-space, WSEAS Trans. Math. 20 (2021), 729-735.
[15] Ch. Baikoussis, L. Verstraelen, The Chen-type of the spiral surfaces, Results. Math. 28 (1995), 214-223.
[16] Ch. Baikoussis, D. E. Blair, On the Gauss map of Ruled Surfaces, Glasgow Math. J. 34 (1992), 355-359.
[17] Ch. Baikoussis, F. Denever, P. Emprechts, L. Verstraelen, On the Gauss map of the cyclides of Dupin, Soochow J. Math., 19 (1993), 417-428.
[18] Ch. Baikoussis, B.-Y. Chen, L. Verstraelen, Ruled Surfaces and tubes with finite type Gauss map, Tokyo J. Math. 16 (1993), 341-349.
[19] Ch. Baikoussis, L. Verstraelen, On the Gauss map of translation surfaces, Rend. Semi. Mat. Messina Ser II (in press).
[20] Ch. Baikoussis, L. Verstraelen, On the Gauss map of helicoidal surfaces, Rend. Semi. Mat. Messina Ser II 16
[21] B.-Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific Publisher, 1984.
[22] B.-Y. Chen, Surfaces of finite type in Euclidean 3-space, Bull.Soc. Math. Belg., 39 (1987), 243-254.
[23] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, Soochow J. Math., 17 (1991), 169-188.
[24] B.-Y. Chen, A report on submanifolds of finite type, Soochow J. Math., 22 (1996), 117-337.
[25] B.-Y. Chen, P. Piccini, submanifolds of finite type Gauss map, Bull. Austral. Math. Soc. 35 (1987), 161-186.
[26] B.-Y. Chen, F. Dillen, Quadrics of finite type, J. of Geom., 38 (1990), 16-22.
[27] F. Dilen, J. Pas, L. Verstraelen, On surfaces of finite type in Euclidean 3-space, Kodai Math. J. 13 (1990), 10-21.
[28] O. Garay, Finite type cones shaped on spherical submanifolds, Proc. Amer. Math. Soc. 104 (1988), 868-870.
[29] B. Senoussi, H. Al-Zoubi, Translation surfaces of finite type in Sol3, Comment. Math. Univ. Carolin. 61 (2020), 237-256.
[30] S. Stamatakis, H. Al-Zoubi, On surfaces of finite Chen-type, Results. Math., 43 (2003), 181-190.
[31] S. Stamatakis, H. Al-Zoubi, Surfaces of revolution satisfying $\triangle^{III}x = Ax$, Journal for Geometry and Graphics, 14 (2010), 181-186.
[32] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan, 18 (1966), 380-385.

DEPARTMENT OF MATHEMATICS, AL-ZAYTOONAH UNIVERSITY OF JORDAN, P.O. BOX 130, AMMAN, JORDAN 11733

Email address: dr.hassanz@zuj.edu.jo

DEPARTMENT OF MATHEMATICS, ARISTOTLE UNIVERSITY OF THESSALONIKI

Email address: stamata@math.auth.gr

DEPARTMENT OF COMPUTER SCIENCE, AL-ZAYTOONAH UNIVERSITY OF JORDAN, P.O. BOX 130, AMMAN, JORDAN 11733

Email address: Hani.Mimi@zuj.edu.jo