Flexible Time and the Evolution of One-Dimensional Cellular Automata

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Abstract
Here I describe a view of the evolution of cellular automata that allows to operate on larger structures. Instead of calculating the next state of all cells in one step, the method here developed uses a time slice that can proceed at different places differently. This allows to “jump” over the evolution of known structures in a single step.

1 Introduction
In this text I introduce a generalised time concept which is helpful for the study of cellular automata. It is motivated by the view of a cellular automaton as a parallel computer which executes a number of tasks that require different amounts of time. The relevant time concept for the description of such computations is therefore not clock time or the number of executed machine instructions but the moment when a certain part of the computation has been completed.

This “moment” is a snapshot of the computation and contains different parts of the machine at different times. For one-dimensional cellular automata we get a snake-line picture like, e.g., ■□ □ □ □ ■■□□■ □ □■■□. In this text I describe a system that allows one to work with such objects. I will introduce three related variants; the last one is an example of how the behaviour of several cells over a number of time steps can be described as a single operation: In the computation analogy, we have identified a very simple subroutine.

2 Cellular Processes
The first concept I introduce is that of a cellular process, which describes the behaviour of some cells at some time.

Let Σ be the set of the states that a cell may have. Then a cellular process with states in Σ is simply a function

\[ \pi: W \rightarrow \Sigma \] (1)

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The earliest form of a similar idea of which I know occurs in [1]. Or you may view it as an application of the concept of “many-fingered time” in General Relativity [2, p. 714] to cellular automata.
with \( W \subseteq \mathbb{Z}^2 \). The set of all such processes is \( \mathcal{P}(\Sigma) \).

\( W \) is the usually finite observation window to the behaviour of an infinite line of cells. Its elements are space-time points of the form \((t, x)\), where \( x \) is the position of a cell and \( t \) the time at which it is observed. Its state at this time is then \( \pi(t, x) \).

A cellular process is also a formalisation of diagrams like □■■□■■□□■□■□□■□. This one has \( \Sigma = \{\square, \blacksquare\} \), and \( W \) may be \( \{0, 1, 2, 3\} \times \{0, 1, 2, 3, 4\} \). I use in this text the convention that time runs upwards, therefore the line at the bottom contains the earliest generation of cells and the whole diagram shows their behaviour over four time steps. Equations for \( \pi \) can then be read directly from the diagram, \( \pi(0, 0) = \blacksquare, \pi(1, 0) = \square, \pi(2, 0) = \square, \pi(3, 0) = \blacksquare \), and more. The four values of \( \pi \) that I just wrote down describe the states of the cell at \( x = 0 \) over four time steps.

The reason for definition (1) is that it gives us automatically an arithmetic for cellular processes. We must only view processes like \( \pi \) set-theoretically, as a set of pairs \( (p, \pi(p)) \), then expressions like \( \pi \subseteq \theta, \pi \cap \theta \), and \( \pi \cup \theta \) have obvious meanings for all cellular processes \( \pi \) and \( \theta \). This also means that \( \emptyset \) is the process with empty domain and that the set \( W \) in equation (1) needs no name of its own but can be written as \( \text{dom } \pi \).

However, the union \( \pi \cup \theta \) of two cellular processes is not always a function: there might be a \( p \in \text{dom } \pi \cap \text{dom } \theta \) with \( \pi(p) \neq \theta(p) \). If there is no such \( p \), then \( \pi \) and \( \theta \) are compatible. Operations on cellular processes will usually be restricted to compatible ones.

3 Transition rules

A special kind of cellular processes describe the behaviour of cellular automata. In them, the state of a cell at time \( t + 1 \) depends on its own state and that of a finite number of neighbours at time \( t \).

If the state of a cell is determined by its \( r \) nearest neighbours at each side, we have a transition function

\[ \varphi: \Sigma^{2r+1} \longrightarrow \Sigma, \quad (2) \]

and \( r \) is the radius of the automaton.

We have then for every element of \( \Sigma^{2r+1} \) a cellular process

\[ \tau \]

\[ \sigma_{-r} \ldots \sigma_0 \ldots \sigma_r \quad (3) \]

which expresses that a cell in state \( \sigma_0 \) is in the next time step in state \( \tau \) if its left neighbours are in states \( \sigma_{-r}, \ldots, \sigma_{-1} \) and its right neighbours are in states \( \sigma_1, \ldots, \sigma_r \). With \( \varphi \) this would be written as \( \varphi(\sigma_{-r} \ldots \sigma_r) = \tau \).

A list of processes in the form (3) gives therefore a very visual way to write \( \varphi \). But it tells almost nothing about the global behaviour of the cellular automaton; we will have to rewrite it to get something understandable.

3.1 Rule 110 and a computation

The concrete question that motivates this all is about the behaviour of the elementary cellular automaton 110 in Stephen Wolfram’s numbering scheme [3].
commonly called Rule 110. It has \( r = 1, \Sigma = \{0,1\} \), and

\[
\varphi(w) = \begin{cases} 0 & \text{for } w \in \{000, 100, 111\}, \\ 1 & \text{for all other } w \in \Sigma^3. \end{cases}
\]  

(4)

The diagrams for them in the style of (3) look much clearer if we write them with \( \square \) and \( \blacksquare \) instead of 0 and 1:

\[
\square\square\square, \quad \blacksquare\square\square, \quad \square\square\square, \quad \square\square\square, \quad \square\square\square, \quad \square\square\square, \quad \square\square\square, \quad \square\square\square.
\]  

(5)

With them we can compute the evolution of a cell configuration directly. We may start, e.g., with the line \( \blacksquare\square\square\square\square \) and extend it to \( \square\square\square\square\square\square \) by placing over every subsequence of three cells the top cell in the corresponding diagram in (5). The new cells form another line, \( \square\square\square\square\square \), which can be extended in the same way. We stop here and get a computation in three steps,

\[
\blacksquare\square\square\square\square \rightarrow \square\square\square\square\square \rightarrow \square\square\square\square\square, \quad (6)
\]

Each step in (6) is itself a partial computation of the cellular automaton and extends the previous one. Now remove the repeated parts so that only the end situations are left,

\[
\square\square\square\square\square \rightarrow \square\square\square\square\square \rightarrow \square\square\square\square\square, \quad (7)
\]

The resulting cellular processes are almost linear sequences of cells, only a little bit bent. With an appropriate notation they can even be written as lines of cells, namely as

\[
10001001 \rightarrow 10\oplus001101\oplus01 \rightarrow 10\oplus00\oplus1111\oplus01\oplus01, \quad (8)
\]

where I have written the states of the cells once again as numbers to let it look even more like a formula.

This is the system for the description of cellular automata which I will now develop in detail. We will call (8) a sequence of two reactions (written as arrows) between three situations. These terms are explained in the next section. To distinguish between situations and cellular processes, I will use the symbols \( \square \) and \( \blacksquare \) only for processes and the digits 0 and 1 only for situations.

4 Situations

A situation is a sequence of cell states and certain elements \( [p] \), which represent gaps between the cell states.

I will now describe these two kinds of situations and then define a product between situations: the set of all situations consists of all finite products of the elementary ones.

4.1 Elementary Situations

To every situation \( a \) belongs a size, \( \delta(a) \in \mathbb{Z}^2 \), and a cellular process \( \pi_a \). Conceptually, \( a \) is a sequence of cells that reaches from \((0,0)\) at the left to \( \delta(a) \) at the right (even then if \( \pi_a \) has no cell at \((0,0)\) or \( \delta(a) \)). \( \delta(a) \) is the difference between the end and the start of the cell sequence \( a \), therefore the symbol \( \delta \).

We start with the two kinds of elementary situations.
1. Every cell state $\sigma \in \Sigma$ is a situation, with

$$
\pi_{\sigma} : \{(0, 0)\} \rightarrow \Sigma \quad \text{and} \quad \delta(\sigma) = (0, 1). \quad (9)
$$

It describes the case that the cell at $x = 0$ at time $t = 0$ is in state $\sigma$.

2. There is for every $p \in \mathbb{Z}^2$ a different situation $[p]$, a displacement. The set of displacements is also disjoint from $\Sigma$, and we have

$$
\pi_{[p]} : \emptyset \rightarrow \Sigma \quad \text{and} \quad \delta([p]) = p. \quad (10)
$$

Since $\pi_{[p]}$ is empty, a displacement tells nothing about the cells, but it is useful to manipulate $\delta$ values. We may abbreviate $[(t, x)]$ as $[t, x]$ and $[0, x]$ as $[x]$.

Displacements of this general form appear seldom in this text. We will mostly use the definitions

$$
\ominus_i = [-1, -i] \quad \text{and} \quad \oplus_i = [1, -i]. \quad (11)
$$

With a cellular automaton of radius $r$ we will abbreviate further and use the forms $\ominus = \ominus_r$ and $\oplus = \oplus_r$.

4.2 Products

All situations are products of the elementary ones. The product is subject to a compatibility condition.

3. If $[p]$ is a displacement and $a$ an arbitrary situation, their product $[p]a$ has

$$
\pi_{[p]a} : p + \text{dom } \pi_a \rightarrow \Sigma \quad \text{and} \quad \delta([p]a) = p + \delta(a) \quad (12)
$$

and is a copy of $a$ that is shifted by $p$.

4. The product $ab$ of two situations $a$ and $b$ is then defined by

$$
\pi_{ab} = \pi_a \cup \pi_{[\delta(a)]}b \quad \text{and} \quad \delta(ab) = \delta(a) + \delta(b) \quad (13)
$$

and exists if $\pi_a$ and $\pi_{[\delta(a)]}b$ are compatible. It consists of a shifted version of $b$ attached to the right end of $a$.

5. The set of all finite products of the elementary situations, with the empty product written as $\lambda$, is $S(\Sigma)$.

Two situations $a$ and $b$ are compatible if $\pi_a$ and $\pi_b$ are compatible and $\delta(a) = \delta(b)$. We can therefore say that $ab$ exists if $a[\delta(b)]$ and $[\delta(a)]b$ are compatible.
4.3 Sets of Situations

Now that we have products, all conventions for them can be used. Therefore $a^k$ is the $k$-th power of $a$, and $a^0 = \lambda$. The set of all powers of $a$, with or without $\lambda$, is $a^* = \{ a^k : k \geq 0 \}$ or $a^+ = \{ a^k : k > 0 \}$, respectively. For a set $S$ of situations exist the multiplicative closures $S^*$ and $S^+$, where $S^*$ is the set of all products of elements of $S$, and $S^* = S^+ \cup \{ \lambda \}$. Note that all the products involved are subject to a compatibility condition; it is therefore, e.g., possible that $a^*$ is finite.

Because the elementary situations in an $a \in S(\Sigma)$ have a fixed order, it is meaningful to speak of the factors of $a$. If there are situations $a_1$ and $a_2$ with $a = a_1b_2a_2$, then $b$ is a factor of $a$. This will be used to define situations in terms of forbidden factors.

A third concept that has turned out to be very useful for the definition of situations are extension rules. Let $a$ and $m$ be situations and $M$ a set of situations. Then we say that in $a$, $m$ extends to $M$ if for every decomposition $a = a_1ma_2$ (14) we have a decomposition $a = b_1m_1mm_2b_2$ (15) with $a_1 = b_1m_1$, $a_2 = m_2b_2$, and $m_1mm_2 \in M$.

In this text I use the convention that $\rho, \sigma, \tau, \ldots$ are elements of $\Sigma$, while $u, v, w, \ldots$ are elements of $\Sigma^*$, and $a, b, c, \ldots$ are elements of $S(\Sigma)$.

5 Reactions

A reaction is simply a pair $(a, a')$ of compatible situations. It expresses the fact that in a certain cellular automaton the situation $a$ is a result of the initial condition $a'$. In the extended time concept of the introduction, $a'$ is “later” than $a$.

The behaviour of a cellular automaton is then described by a reaction system $(S, \rightarrow)$ that consists of a set $S$ of situations and the reactions between them. The reactions, a subset of $S \times S$, form a binary relation $\rightarrow$, and if there is a reaction $(a, a')$ that belongs to the system $(S, \rightarrow)$, it is written as $a \rightarrow a'$.

A reaction system $(S, \rightarrow)$ must obey the following rules:

1. if $a \in S$ then $a \rightarrow a$, (Reflectivity)
2. if $a \rightarrow b$ and $b \rightarrow c$ then $a \rightarrow c$, (Transitivity) and
3. if $b \rightarrow b'$ and $abc \in S$, then $abc \rightarrow ab'c$ and $ab'c \in S$. (Extension Rule)

As with other mathematical structures, I will write $(S, \rightarrow)$ as $S$ if the context is unambiguous.

The first two conditions make $\rightarrow$ a quasiorder on $S$. The third one allows to define a reaction system by a small set of generator reactions and some initial situations. In the simplest cases, the generator reactions are derived directly from $\varphi$ and describe the computation of exactly one new cell state.

If a reaction $abc \rightarrow ab'c$ has been derived with rule 3, we say that $b \rightarrow b'$ has been applied to $abc$. Note however that a product like $ab'c$ needs not to exist and that therefore the extension rule places an implicit condition on the reaction system and it must be proved to be consistent.
5.1 A General Reaction System

The rewriting of $\varphi$ to get something better understandable can now begin with the construction of the reaction system $\Phi$. Its generator reactions are

\[
\begin{align*}
\ominus \sigma w & \rightarrow \varphi(\sigma w) \ominus w, \\
w & \rightarrow w \ominus \ominus w, \\
\ominus w \ominus w & \rightarrow \ominus w, \\
\ominus w \ominus \ominus w & \rightarrow \lambda.
\end{align*}
\] (16)

for every $w \in \Sigma^{2r}$ and $\sigma \in \Sigma$. The set of all situations contains all $b \in (\Sigma \cup \{\ominus, \ominus\})^*$ such that

1. in $b$, $\ominus$ extends to $\ominus \Sigma^{2r}$ and $\ominus$ extends to $\Sigma^{2r} \ominus$, and
2. there are $a, c \in S(\Sigma)$ and $w \in \Sigma^*$ with $w \rightarrow abc$.

One can see that the generator reactions (and therefore all reactions) preserve these conditions. The second condition is not absolutely necessary for a consistent reaction system, but it makes the consistency proof more easily generalisable.

5.2 How it is Used

As an example how this system works, I will now show how the first reaction in (8) is derived. The initial situation, $10001001$, contains neither $\ominus$ nor $\ominus$, therefore we must first apply a reaction of the type $w \rightarrow w \ominus \ominus w$ to it, e.g. with $w = 10$. We get then

\[10001001 \rightarrow 10 \ominus 10001001\] (17)

among other possibilities. One reaction from the top left of (16), namely $\ominus 100 \rightarrow 0 \ominus 00$, can then be applied to it, resulting in

\[10 \ominus 10001001 \rightarrow 10 \ominus 0 \ominus 001001,\] (18)

and then others until we reach $10 \ominus 001101 \ominus 01$.

The last reaction type in (16), $\ominus w \ominus \rightarrow \lambda$, is needed after reactions have been started from different places. As an example, we could have continued after (17) with a reaction to $10 \ominus 10001001 \ominus 001$, and later reached $10 \ominus 001101 \ominus 01 \ominus 01$ in the same way as before. The reaction $\ominus 01 \ominus \rightarrow \lambda$ then removes the extra factor $\ominus 01 \ominus$ and we get $10 \ominus 001101 \ominus 01$ again.

5.3 The Unique Future

We still have to answer the question whether the extension rule holds for $\Phi$.

To do this, I define now for each $b \in \Phi$ a cellular process $\bar{\pi}_b$, the future of $b$. It contains $\pi_b$ and all the cell states that are influenced by it through $\varphi$. More precisely, $\bar{\pi}_b$ is the smallest process (in the set-theoretic sense) that has $\pi_b$ as a subset and where for every $p \in \mathbb{Z}^2$ and $w \in \Sigma^{2r+1}$ we have

\[
\text{if } \pi_{[p] \ominus w} \subseteq \bar{\pi}_b \text{ then } \pi_{[p] \varphi(w)} \subseteq \bar{\pi}_b.
\] (19)

Each cell state in $\bar{\pi}_b \setminus \pi_b$ depends therefore uniquely on $2r+1$ cells in the previous step; when $\bar{\pi}_b$ exists, it is by induction unique.
The following diagram is an example for $r = 1$, with $\text{dom } \pi_b$ shown as $\circ$ and $\text{dom } \bar{\pi}_b$ as $\cdot$ and $\circ$:

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  · · · · · · · · · · ·
  ◦ ◦ ◦ ◦ ◦ ◦ ◦ ◦ ◦ ◦
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The future of a situation is defined in this way because we have then

$$
\text{if } b \rightarrow b' \text{ then } \bar{\pi}_b \supseteq \bar{\pi}_{b'}
$$

(20)

provided that $\pi_b$ exists; in this case $\pi_{b'}$ exists, too. The proof of this begins with the generator reaction in (16), which fulfill (20). Now if $b \rightarrow b'$ is a generator reaction, $abc$ is an element of $\Phi$, and $\bar{\pi}_{abc}$ exists, then $\bar{\pi}_{abc} \supseteq \bar{\pi}_{\delta(a)\bar{b}}$ and $\bar{\pi}_{\delta(a)\bar{b}} \supseteq \bar{\pi}_{\delta(a)\bar{b'}}$ (by (20)), therefore $\bar{\pi}_{abc} \supseteq \bar{\pi}_{abc'}$. This means that (20) is true for reactions where a single generator reaction is applied to a situation. By induction it is therefore true for all reactions in $\Phi$.

With (20) we can now see that in fact every $b \in \Phi$ has a future. This is because every $w \in \Sigma^*$ has a future and there is for every $b \in \Phi$ a reaction $w \rightarrow abc$. So we have $\bar{\pi}_w \supseteq \bar{\pi}_{abc} \supseteq \bar{\pi}_{\delta(a)\bar{b}}$, therefore $\bar{\pi}_{\delta(a)\bar{b}}$ has a future. But this is only a shifted version of $\pi_b$. (Instead of $w \in \Sigma^*$ I could have used in the definition of $\Phi$ elements of a larger set for which a future exists, but for the present purpose $\Sigma^*$ is enough.)

The extension rule, finally, is a side effect of the proof of (20). We have already seen that it is valid for generator reactions: If $b \rightarrow b'$ is a generator reaction and $abc \in \Phi$, then $\bar{\pi}_{abc} \supseteq \bar{\pi}_{abc'}$ and therefore $\bar{\pi}_{abc'}$ exists. But then it exists by induction for every reaction $b \rightarrow b'$.

Other facts that follow from (20) are: If $a \rightarrow b$ and $a \rightarrow b'$ then $b$ and $b'$ are compatible, and if also $\text{dom } \pi_b = \text{dom } \pi_{b'}$, then $\pi_b = \pi_{b'}$. This means that different reaction paths, as in Section 5.2, lead to essentially the same result.

## 6 Narrow Rules

The definition of $\bar{\pi}_b$ imitates the computation of a cell state in a cellular automaton from the $2r + 1$ states in its neighbourhood one time step earlier.

But often not all of them are actually needed. In Rule 110, e. g., we have both $\varphi(000) = 0$ and $\varphi(100) = 0$, so we need to know only the two cells at the left to compute the next cell state. In other words, the situation $00$, which has under definition (19) only the trivial future $\Box\Box$, “should have” the future $\Box\Box$.

### 6.1 A Better Future

For this we need a more complex definition than (19). In the new kind of future, $\hat{\pi}_b$, the variable $w$ of (19), which represents the predecessors of the cell state at $p$, is replaced with a whole set

$$
W_p = \{ w \in \Sigma^{2n+1} : \pi_{[p] \cup w} \text{ is compatible to } \hat{\pi}_b \}
$$

(21)

of possible predecessor sequences. With them I define $\hat{\pi}_b$ as the smallest process containing $\pi_b$ where for every $p \in \mathbb{Z}^2$ we have

$$
\text{if } \exists \sigma \in \Sigma \forall w \in W_p : \varphi(w) = \sigma \text{ then } \pi_{[p] \cup \sigma} \subseteq \hat{\pi}_b.
$$

(22)
Then every cell state in $\tilde{\pi}_b \setminus \pi_b$ is uniquely determined by the known part of its $2r + 1$ predecessors. Therefore, as in the case of $\pi_b$, if $\tilde{\pi}_b$ exists, it is uniquely determined by $b$.

Definition (22) can behave differently from (19) only on the boundaries of $\tilde{\pi}_b$; we get therefore longer and narrower diagrams than with $\pi_b$. For example, $\tilde{\pi}_{16}$ under Rule 110 is now

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□□□□□
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The corresponding reaction system $\Psi$, which I describe next, is therefore called the narrow form of $\Phi$.

### 6.2 Defining Reactions

The reaction system $\Psi$ is a generalisation of $\Phi$. Its definition is more complex than that of $\Phi$, but its properties are already given by a subset of its generator reactions. This subset, the defining reactions of $\Psi$, contains reactions of the form

$$u \oplus_i u \rightarrow v \ominus_j v \quad \text{or} \quad u' \oplus_i v \rightarrow u' \ominus_j v',$$

with $u, u', \ldots, w' \in \Sigma^*$ and $i, i', j, j' \in \mathbb{Z}$.

Before $\Psi$ can be described, two sets of boundary terms, $B_-$ and $B_+$, must be introduced. To define $B_-$, we use temporarily a set $\tilde{B}_-$ that contains all terms $\ominus_j w$ for which there is a defining reaction of the form $\ominus_i u \rightarrow v \ominus_j w$. Then $B_-$ is the set of “shortest” elements of $\tilde{B}_-$:

$$B_- = \tilde{B}_- \setminus \{ \ominus_j wx : \ominus_j w \in \tilde{B}_-, x \in \Sigma^+ \}.$$  \hspace{1cm} (24)

The set $B_+$ is constructed in the same way from the terms $w' \oplus_j$ that occur in the defining reactions at the right side of (23).

Now we can define $\Psi$. Its situations are all $b \in (\Sigma \cup \{ \ominus_i, \oplus_i : i \in \mathbb{Z} \})^*$ where

1. in $b$, all $\ominus_i$ extend to $B_-$ and all $\oplus_i$ extend to $B_+$, and

2. there are $a, c \in S(\Sigma)$ and $w \in \Sigma^*$ with $w \rightarrow abc$.

The generator reactions of $\Psi$ are then those in (23) together with all minimal reactions of the form

$$w \rightarrow u \ominus_i \ominus_j v$$

with $u, v, w \in \Sigma^*$ and certain reactions of the form

$$\ominus_i u \oplus_j v \rightarrow v$$

with $u, v, w \in \Sigma^*$.

A reaction (25) is minimal if no reaction of the same type can be applied to a part of $w$. The reactions of (26) are constructed from all pairs of reactions, $w \rightarrow x \ominus_k \ominus_i u \ominus_j \ominus_k' x'$ and $w \rightarrow x \oplus_k v \ominus_k' x'$ with $w, x, x' \in \Sigma^*$, where either no reaction of (23) can be applied to $\ominus_i u \ominus_j$ or it would lead to a result with overlappings (like $v' \ominus_i u' \oplus_j$ with $u' \in \Sigma^\ell$ and $\ell < i' + j$).
6.2.1 Subsystems

Two subsystems of $\Psi$ are sometimes useful, $\Psi_-$ and $\Psi_+$. $\Psi_-$ is the system that has only the reactions at the left side of (23) as defining reactions. Therefore no reactions of the types described in (25) and (26) are possible for it. The situations in $\Psi_-$ are those elements of $\Psi$ that contain no $\oplus_i$:

$$\Psi_- = \Psi \cap (\Sigma \cup \{ \oplus_i : i \in \mathbb{Z} \})^*.$$  

Similarly, $\Psi_+$ is the reaction system that we get when we start with the reactions at the right side of (23).

6.3 Narrowing

Now we construct the defining reactions of $\Psi$. The construction uses a sequence of intermediate reaction systems $\Psi_i = (\Psi_i, \rightarrow_i)$ with $\Psi_0 = \Phi$ and $\Psi_4 = \Psi$. It can be described by transformations of the defining reactions because everything else follows from them. I will only show the transformations of the “$\ominus$” reactions since the transformations of the “$\oplus$” reactions are their mirror images.

Our starting point, $\Psi_0$, is the reaction system with the defining reactions

$$\ominus w \rightarrow_0 \varphi(w) \ominus w, \quad \sigma \in \Sigma, w \in \Sigma_{2r},$$  

(and their “$\oplus$” versions). Since $\Psi_0 = \Phi$ and $\hat{\pi}_b \supseteq \bar{\pi}_b$, condition (19) is true for $\hat{\pi}$ too. All other properties of $\bar{\pi}$ in Section 5.3 follow from (19) and are therefore also valid for $\hat{\pi}$. The most important of them is that for all $a \in \Psi_0$ the process $\hat{\pi}_a$ exists and so $\Psi_0$ is actually a reaction system. The following transformations are defined in such a way that they preserve property (19), which amounts to a proof that $\Psi$ is a valid reaction system.

1. To describe the first transformation we write $w \in \Sigma_{2r+1}$ as $\omega_0 \ldots \omega_{2r}$ and define $w_{i,j} = \omega_i \ldots \omega_j$. Then a typical defining reaction of $\Psi_0$ can be written as

$$\ominus w_0 \rightarrow_0 \sigma \ominus w_{1,2r}.$$  

In $\Psi_1$ it is replaced with reactions

$$\ominus \omega_{2r} \rightarrow_1 \sigma \ominus w_{1,2r},$$  

$$\ominus r \ominus \omega_k \rightarrow_1 \sigma \ominus r \ominus w_{k+1,2r},$$  

for appropriate values of $k$ and $\ell$. We say then that the left side of (28) has been reduced by $k$ cells and the right side by $\ell$ cells.

We can also say that these reductions have been achieved by applying the reaction $\ominus w_{0,2r-1} \rightarrow_1 \ominus r \ominus w_{k,2r-1}$ to the left side and $\ominus w_{1,2r} \rightarrow_1 \ominus r \ominus w_{k,2r}$ to the right side of (28). Both reactions are of type (30), and they will be noted in the calculations of Section 6.4 to show what has been done.

It remains to find values for $k$ and $\ell$. The transformation of (28) is of course only sensible if the value of $w_{0,k-1}$ has no influence on $\sigma$. This means that there must be for all $\xi \in \Sigma$, $x \in \Sigma^{k-1}$ a reaction

$$\ominus x w_{k,2r} \rightarrow_0 \sigma \ominus x w_{k,2r}.$$  


and (30) is a common replacement for all of them, or else \( k = 0 \). But it is also necessary that no cell is removed that is needed in following reactions. Therefore we must define \( k \) and \( \ell \) recursively:

(a) \( \ell \) is the largest value such that for every \( \tau \in \Sigma \) the left side of the reaction in \( \Psi_0 \) that starts with \( \ominus w_{1,2r}\tau \) can be reduced by at least \( \ell \) cells, and

(b) \( k \) is the largest value \( \leq \ell - 1 \) that fulfils (31).

This induction can begin because a value of \( \ell = 0 \) is always possible.

The reactions of (30) are necessary because those of (29) do not always match correctly: one of the reactions that start with \( \ominus w_{1,2r}\tau \) may have been reduced by more than \( \ell \) cells. In this case one of the reactions of (30) removes the superfluous cells.

2. Next we unify reactions that differ only on the right side. If \( k \) is maximal so that

\[
\ominus j_i u \rightarrow \sigma_1 \ominus j_v w
\]

for all \( w \in \Sigma^k \), then

\[
\ominus j_i u \rightarrow \sigma_2 \ominus j_v .
\]  

(33)

Since \( k \) can be 0, every defining reaction in \( \Psi_1 \) has its counterpart in \( \Psi_2 \).

3. It is possible that a defining reaction of \( \Psi_2 \) can be applied to the result of another defining reaction, and to its result possibly others. Then we have a sequence

\[
\ominus j_0 u_0 \rightarrow \sigma_1 \ominus j_1 u_1 \rightarrow \ldots \rightarrow \sigma_1 \ldots \sigma_k \ominus j_k u_k .
\]  

(34)

Every defining reaction in \( \Psi_2 \) is the start of such a sequence, possibly of length 1. The length is always finite because the \( u_i \) never become longer and the maximal value of each \( j_i \) is \( r \).

Therefore we can extend every reaction \( \ominus j_0 u_0 \rightarrow \sigma_1 \ominus j_1 u_1 \) to a maximal sequence (34) and set

\[
\ominus j_0 u_0 \rightarrow \sigma_1 \ldots \sigma_k \ominus j_k u_k .
\]  

(35)

This gives the defining reactions of \( \Psi_3 \).

4. As a result of this and of (30), some defining reactions in \( \Psi_3 \) can never be applied to the result of another reaction. They have no influence on the long-term behaviour of the system and are therefore left out. The rest (and the corresponding “\( \oplus \)” reactions) are the defining reactions of \( \Psi \).

This procedure has been defined in such a way that it always ends in a finite number of steps, so questions of halting and computability do not arise here.
Figure 1: Rule 110 as a graph. Every vertex describes a transition $\varphi(w) = \sigma$. Vertices that only differ by one cell in $w$ are connected by a line. They are connected by a double line if their values of $\sigma$ are equal.

6.4 The Narrow Form of Rule 110

We can now read from Figure 1 the following cases where the value of $\varphi$ does not depend on all cells:

$$\forall \sigma \in \Sigma: \varphi(\sigma00) = 0 \land \varphi(\sigma01) = 1 \land \varphi(\sigma10) = 1 \land \varphi(01\sigma) = 1$$  \hspace{1cm} (36)

These are the cases where the ignored cells are at the end, because only they can be used here. They influence the constructions of $\Psi_-$ and $\Psi_+$ in different ways.

In the case of $\Psi_-$, the first two terms in (36) lead to the reactions $0\oplus00 \rightarrow 0\oplus00$ and $0\oplus01 \rightarrow 1\oplus01$. It is therefore possible to reduce situations of the form $0\oplus\sigma$ to $0\oplus0$. The last term in (36), written as a reaction, is $0\oplus01 \rightarrow 0\oplus11\sigma$: these two reactions can be unified by committing the $\sigma$.

The computation that follows from these ideas can be summarised in the following scheme:

$$\begin{align*}
\text{\oplus00} \rightarrow_a \text{\oplus00} \\
\text{\oplus10} \rightarrow_b \text{\oplus00} \\
\text{\oplus01} \rightarrow \text{\oplus01} & \rightarrow_a \text{\oplus01} \rightarrow_c \text{\oplus01} \\
\text{\oplus10} \rightarrow \text{\oplus01} & \rightarrow_c \text{\oplus01} \rightarrow_b \text{\oplus01} \\
\text{\oplus11} \rightarrow \text{\oplus01} & \rightarrow_c \text{\oplus01} \rightarrow_b \text{\oplus01} \\
\text{\oplus11} \rightarrow \text{\oplus10} & \rightarrow_b \text{\oplus01} \\
\text{\oplus11} \rightarrow \text{\oplus11} & \rightarrow_b \text{\oplus01}
\end{align*}$$

This diagram must be read from left to right. The first column contains those generator reactions that can be transformed. The middle column has, from top to bottom, the new reduction reactions, the transformed reactions, and the untransformed reactions. If a reaction can be continued, the result is appended at the right. The indices on some of the reaction arrows are used only in this...
States: 0, 1, ⊗, ⊗₀, ⊗₁.

Situations:
⊗ extends to {⊗¹}
⊗₀ extends to {⊗₀₀}
⊗₁ extends to {⊗₁₀, 1⊗₁}

Reactions:
⊗₀₀₀ → 0⊗₀₀
do not occur in any reaction result because there is no reaction that creates ⊗₀.

Table 1: Rule 110, narrow form

| Reactions | Diagram |
|-----------|---------|
| ⊗₀₀₀ → 0⊗₀₀ | 00 ⊗ → 0⊗₀ |
| ⊗₀₀₁ → 1⊗₀₁ | 10 ⊗ → 1⊗₁ |
| ⊗₁₀ → ⊗₀₀ | 01 ⊗ → 0⊗₁ |
| ⊗₁₁₀ → 1⊗₀₀ | 011 ⊗ → 0⊗₁₁ |
| ⊗₁₁₁ → 0⊗₁₁ | 111 ⊗ → 11⊗₀ |

From these reactions all other reactions of Ψ are derived. The result is shown in Table 1.

7 A Rule for Flexible Time

Now we can return to the generalised time concept of the introduction. I will choose a very simple kind of “subroutines” and show how to take snapshots of the system when they have stopped.

These “subroutines” are all finite sequences of zeroes and have a common behaviour,

\[ 0^k \rightarrow (0\oplus)^k (\ominus_0)^k \quad \text{for } k \geq 0. \]  

They can be interpreted as a kind of timer which lasts \( k \) time steps, as many as the initial sequence is long. Geometrically, (37) traces a triangle of zeros, with its base at the left side and the other edges at the right side of the reaction.

We can simplify the formulas by introducing the abbreviations

\[ \varepsilon_- = \ominus_0 0, \quad \varepsilon_+ = 0\oplus, \]  

(38)
States: \( \varepsilon_-, \varepsilon_+, 1 \).

Situations: No subsequence \( \varepsilon_-\varepsilon_+ \).

Reactions:

\[
\begin{align*}
\varepsilon_-1\varepsilon_+ & \rightarrow 1^2, \\
\varepsilon_-1k^2\varepsilon_+ & \rightarrow 12\varepsilon_+^k\varepsilon_-1, \quad k \geq 0.
\end{align*}
\]

Table 2: Rule 110, selective evolution

then \( (37) \) becomes

\[
0^k \rightarrow \varepsilon_k^k\varepsilon_-^k \quad \text{for } k \geq 0. \quad (39)
\]

In the new, third, reaction system we will have only situations where all subprocesses \( (39) \) have ended. This means that the elements of \( \Sigma^+ \) are not among the situations. But if we apply \( (39) \) to all maximal subsequences of zeros then we get from an element of \( \Sigma^* \) a situation of the form

\[
1^\ell_0\varepsilon_+^{k_1}1\varepsilon_+^{k_2}1\varepsilon_+^{k_3}1\ldots \varepsilon_+^{k_n}1\varepsilon_-^n \quad (40)
\]

with \( \ell_0, \ell_n \geq 0 \), and all other \( k_n, \ell_i \geq 1 \). It has the additional property that it consists only of \( 1, \varepsilon_- \) and \( \varepsilon_+ \), and that it does not contain \( \varepsilon_-\varepsilon_+ \). We can now introduce reactions that preserve this.

To get them, we evolve the situations \( \varepsilon_-1^k\varepsilon_+ \) with \( k \geq 1 \). Since the equivalent of \( (39) \) for ones is only valid for sequences longer than one cell,

\[
1^{k+2} \rightarrow 11\oplus0^k\ominus11 \rightarrow 11\oplus\varepsilon_+^k\ominus\varepsilon_-^k\ominus11 \quad \text{for } k \geq 0, \quad (41)
\]

we have to distinguish two cases:

\[
\begin{align*}
\varepsilon_-1\varepsilon_+ = \ominus0010\oplus & \rightarrow 11, \\
\varepsilon_-1^{k+2}\varepsilon_+ = \ominus01^{k+2}0\ominus & \rightarrow \ominus0011\oplus\varepsilon_+^k\ominus\varepsilon_-^k\ominus110\ominus & \rightarrow 11\varepsilon_+^k1\ominus1. \quad (42)
\end{align*}
\]

The resulting reaction system is summarised in Table 2. A way for further research into Rule 110 will be to search for other, more complex, subprocesses and incorporate them into the reaction system as well.

7.0.1 Acknowledgement

I want to thank Genaro Juárez Martínez for encouragement and discussion.

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