Solving the Ward Identities of Irrational Conformal Field Theory

M.B. Halpern

Department of Physics, University of California
Theoretical Physics Group, Lawrence Berkeley Laboratory
Berkeley, California 94720
USA

N.A. Obers

Physikalisches Institut der Universität Bonn
Nußallee 12, D-53115 Bonn
Germany

Abstract

The affine-Virasoro Ward identities are a system of non-linear differential equations which describe the correlators of all affine-Virasoro constructions, including rational and irrational conformal field theory. We study the Ward identities in some detail, with several central results. First, we solve for the correlators of the affine-Sugawara nests, which are associated to the nested subgroups $g \supset h_1 \supset \ldots \supset h_n$. We also find an equivalent algebraic formulation which allows us to find global solutions across the set of all affine-Virasoro constructions. A particular global solution is discussed which gives the correct nest correlators, exhibits braiding for all affine-Virasoro correlators, and shows good physical behavior, at least for four-point correlators at high level on simple $g$. In rational and irrational conformal field theory, the high-level fusion rules of the broken affine modules follow the Clebsch-Gordan coefficients of the representations.

*e-mail: MBHALPERN@LBL.GOV, THEORY::HALPERN
†e-mail: OBERS@PIB1.PHYSIK.UNI-BONN.DE, 13581::OBERS
1 Introduction

Affine Lie algebra [1,2], or current algebra on $S^1$, was discovered independently in mathematics and physics. Affine-Virasoro constructions are the most general Virasoro operators [3,4]

$$T = L^{ab} J_a J_b$$

(1.1)

which are quadratic in the currents $J_a$ of the affine algebra. The coefficients $L^{ab}$ in the stress tensor $T$ can be any solution of the Virasoro master equation. The solution space of the master equation, called affine-Virasoro space, includes the affine-Sugawara constructions [2,5,6,7], the coset constructions [2,5,8], the affine-Sugawara nests [9,10,11], and a vast number of new constructions, most of which have irrational central charge [10]. As an example, it is known that there are approximately $1/4$ billion solutions of the master equation on each level of affine $SU(3)$, while the value at level 5 [12]

$$c \left( (SU(3))_{D(1)}^# \right) = 2 \left( 1 - \frac{1}{\sqrt{61}} \right) \simeq 1.7439$$

(1.2)

is the lowest unitary irrational central charge yet observed. Partial classification of the solution space and other developments in the Virasoro master equation are reviewed in Ref.[13].

It is clear that the Virasoro master equation is the first step in the study of irrational conformal field theory (ICFT), which includes rational conformal field theory (RCFT) as a small subspace of relatively high symmetry,

$$\text{ICFT} \supset \text{RCFT}$$

(1.3)

The next step is a description of the correlators of irrational conformal field theory, about which we have the intuitive notion that they must involve a generically-infinite number of conformal structures. The organization of these structures must be generically new, since it is unlikely that the generic affine-Virasoro construction supports an extended chiral algebra.

Using null states of the Knizhnik-Zamolodchikov type [7], we recently reported the derivation of dynamical equations, the factorized affine-Virasoro Ward identities [14], which describe the correlators of irrational conformal field theory. For any $K$-conjugate pair [2,5,8,3] of affine-Virasoro constructions, one
may compute a set of affine-Virasoro connections for the pair. The connections are the input data for the Ward identities, which generalize the Knizhnik-Zamolodchikov equations to the broader context of coset constructions and irrational conformal field theory. In form, the Ward identities are a system of coupled non-linear differential equations for the factorized correlators of the K-conjugate pair of conformal field theories. Because the affine-Sugawara constructions are K-conjugate to the trivial theory, the Knizhnik-Zamolodchikov equations are included as the simplest case of the Ward identities.

As a first non-trivial example, we solved the Ward identities for the correlators of the coset constructions, providing a derivation of the coset blocks of Douglas [15]. In the present paper, we go beyond the cosets to study the Ward identities for all affine-Virasoro constructions.

After a brief review of the master equation and the Ward identities, we discuss some general properties of the affine-Virasoro connections, including K-conjugation covariance, crossing symmetry and the high-level form of the general connections. As a first application of the general properties, we construct the connections and solve for the conformal correlators of all the affine-Sugawara nests, which are associated to the nested subgroups

\[ g \supset h_1 \supset \ldots \supset h_n. \]  

(1.4)

Our results explicitly verify the intuition that the nests are tensor-product RCFT’s, constructed by tensoring the relevant coset and subgroup constructions.

Turning to the general affine-Virasoro construction, we find an equivalent algebraic formulation of the system which, given the affine-Virasoro connections, allows us to find global solutions of the Ward identities across all affine-Virasoro space. The solutions involve a generically-infinite number of conformal structures, in accord with intuitive notions about irrational conformal field theory, but many of the solutions are apparently not physical.

Based on a natural eigenvalue problem in the system, we focus on a particular infinite-dimensional global solution with the following properties:

a) The conformal structures are degenerate for the coset constructions and affine-Sugawara nests, and the correct correlators are obtained in these cases.
b) The solution exhibits a braiding for all affine-Virasoro correlators which includes and generalizes the braiding of rational conformal field theory. The origin of the braiding is the linearity of the eigenvalue problem.

c) The solution shows good physical behavior, at least for four-point affine-Virasoro correlators at high level on simple $g$. From the high-level correlators, we determine that the high-level fusion rules of the broken affine modules follow the Clebsch-Gordan coefficients of the representations.

2 The Virasoro Master Equation

In this section, we review the Virasoro master equation and some features of the system which will be useful below.

The general affine-Virasoro construction begins with the currents $J_a$ of untwisted affine $g$ \cite{1,2}

$$J_a(z) = \sum_m J_a^{(m)} z^{-m-1}, \quad a = 1, \ldots, \dim g \quad m, n \in \mathbb{Z} \quad (2.1a)$$

$$J_a(z) J_b(w) = \frac{G_{ab}}{(z-w)^2} + \frac{i f_{ab}^c}{z-w} J_c(w) + O(z-w)^0 \quad (2.1b)$$

where $f_{ab}^c$ and $G_{ab}$ are the structure constants and general Killing metric of $g$. The current algebra (2.1) is completely general since $g$ is not necessarily compact or semisimple. In particular, to obtain level $x_I = 2k_I/\psi_I^2$ of $g_I$ in $g = \oplus_I g_I$ with dual Coxeter number $\tilde{h}_I = Q_I/\psi_I^2$, take

$$G_{ab} = \oplus_I k_I \eta_{ab}^I, \quad f_{ac}^d f_{bd}^e = - \oplus_I Q_I \eta_{ab}^I \quad (2.2)$$

where $\eta_{ab}^I$ and $\psi_I$ are the Killing metric and the highest root of $g_I$.

Next, consider the set of operators quadratic in the currents

$$T(z) = L^{ab} J_a(z) J_b(z) = \sum_m L^{(m)} z^{-m-2} \quad (2.3)$$

where the set of coefficients $L^{ab} = L^{ba}$ is called the inverse inertia tensor, in analogy with the spinning top. The requirement that $T(z)$ is a Virasoro operator

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) T(w) + O(z-w)^0 \quad (2.4)$$
restricts the values of the inverse inertia tensor to those which solve the Virasoro master equation \[3,4\]

\[ L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}^{a}f_{df}^{b} - L^{cd}f_{ce}^{f}f_{df}^{(a}L^{b)e} \quad (2.5a) \]

\[ c = 2G_{ab}L^{ab} \quad . \quad (2.5b) \]

The Virasoro master equation has been identified [16] as an Einstein-like system on the group manifold, with \(L^{ab}\) the inverse metric on tangent space and \(c = \dim g - 4R\), where \(R\) is the Einstein curvature scalar.

Some general features of the Virasoro master equation include:

1. Affine-Sugawara constructions [2,5,6,7]. The affine-Sugawara (A-S) construction \(L_g\) is

\[ L^{ab}_g = \oplus_I \frac{\eta^{ab}_I}{2k_I + Q_I} , \quad c_g = \sum_I \frac{x_I \dim g_I}{x_I + \tilde{h}_I} \quad (2.6) \]

for arbitrary level of any \(g\), and similarly for \(L_h\) when \(h \subset g\).

2. K-conjugation covariance [2,5,8,3]. When \(L\) is a solution of the master equation on \(g\), then so is the K-conjugate partner \(\tilde{L}\) of \(L\),

\[ \tilde{L}^{ab} = L^{ab}_g - L^{ab}_h , \quad \tilde{c} = c_g - c \quad (2.7) \]

and the corresponding stress tensors form a K-conjugate pair of commuting Virasoro operators

\[ \tilde{T}(z) = \sum_m \tilde{L}^{(m)} z^{-m-2} \quad (2.8a) \]

\[ \tilde{T}(z) \tilde{T}(w) = \frac{\tilde{c}/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{1}{z-w} \right) \tilde{T}(w) + \mathcal{O}(z-w)^0 \quad (2.8b) \]

\[ T(z) \tilde{T}(w) = \mathcal{O}(z-w)^0 \quad . \quad (2.8c) \]

The affine-Virasoro stress tensors \(T\) and \(\tilde{T}\) are quasi (2,0) operators under the affine-Sugawara stress tensor \(T_g = T + \tilde{T}\).

3. Cosets and affine-Sugawara nests. The simplest K-conjugate pairs are the subgroup constructions \(L_h\) and the corresponding \(g/h\) coset constructions [2,5,8]

\[ L^{ab}_{g/h} = L^{ab}_g - L^{ab}_h , \quad c_{g/h} = c_g - c_h \quad (2.9) \]
while repeated K-conjugation on subgroup sequences \( g \supset h_1 \supset \ldots \supset h_n \) generates the affine-Sugawara (A-S) nests \([9,10,11]\)

\[
L_{g/h_1/\ldots/h_n} = L_g - L_{h_1/\ldots/h_n} = L_g + \sum_{j=1}^{n} (-)^j L_{h_j} \quad (2.10a)
\]

\[
c_{g/h_1/\ldots/h_n} = c_g - c_{h_1/\ldots/h_n} = c_g + \sum_{j=1}^{n} (-)^j c_{h_j} \quad . \quad (2.10b)
\]

Note that the stress tensors of the affine-Sugawara nests may be written as sums of mutually-commuting Virasoro constructions on \( g/h \) and \( h \)

\[
T_{g/h_1/\ldots/h_{2n+1}} = T_{g/h_1} + \sum_{i=1}^{n} T_{h_{2i}/h_{2i+1}} \quad (2.11a)
\]

\[
T_{g/h_1/\ldots/h_{2n}} = T_{g/h_1} + \sum_{i=1}^{n-1} T_{h_{2i}/h_{2i+1}} + T_{h_{2n}} \quad (2.11b)
\]

so the conformal field theories of the affine-Sugawara nests are expected to be tensor-product theories. By computation of the nest correlators, we will explicitly verify this intuition in Section 5.

Other developments in the Virasoro master equation, including the worldsheet action \([17]\) for the generic affine-Virasoro construction, are reviewed in Ref.[13]

### 3 The Affine-Virasoro Ward Identities

In this section, we review the affine-Virasoro (A-V) Ward identities \([14]\), which generalize the Knizhnik-Zamolodchikov equations \([7]\) to the broader context of coset constructions and irrational conformal field theory.

1. Virasoro biprimary states. Let \(|0\rangle\) be the affine vacuum and \( R_g(\mathcal{T}, z) \) be the affine primary fields corresponding to irreducible matrix representation \( \mathcal{T} \) of \( g \).

Under the stress tensor \( T \), the conformal weights \( \Delta_\alpha(\mathcal{T}) \) of the affine primary states are the eigenvalues of the conformal weight matrix \( L^{ab} \mathcal{T}_a \mathcal{T}_b \) \([18,10]\). In what follows, we choose an \( L \)-basis of \( \mathcal{T} \) \([14]\), in which the conformal weight matrix is diagonal

\[
L^{ab}(\mathcal{T}_a \mathcal{T}_b)_{\alpha}^{\beta} = \Delta_\alpha(\mathcal{T}) \delta_\alpha^{\beta} \quad , \quad \alpha, \beta = 1, \ldots, \dim \mathcal{T} \quad . \quad (3.1)
\]
Then the corresponding eigenstates \( R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle \), called the \( L^{ab}\)-broken affine primary states, satisfy

\[
L^{m \geq 0} R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle = \delta_{m, 0} \Delta_g(\mathcal{T}) R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle \quad (3.2a)
\]

\[
\tilde{L}^{m \geq 0} R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle = \delta_{m, 0} \tilde{\Delta}_g(\mathcal{T}) R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle \quad (3.2b)
\]

\[
\Delta_g(\mathcal{T}) = \Delta_\alpha(\mathcal{T}) + \Delta_\tilde{\alpha}(\mathcal{T}) \quad (3.2c)
\]

where \( \Delta_g(\mathcal{T}) \) and \( \Delta_\tilde{\alpha}(\mathcal{T}) \) are the conformal weights of \( T_g \) and \( \tilde{T} \) respectively. The \( L^{ab}\)-broken affine primary states are examples of Virasoro biprimary states [18,10,14], which are simultaneously Virasoro primary under the K-conjugate stress tensors \( T \) and \( \tilde{T} \).

2. Virasoro biprimary fields. The corresponding \( L^{ab}\)-broken affine primary fields

\[
R^\alpha(\mathcal{T}, \bar{z}, z) = e^{(\bar{z} - z)\tilde{L}^{(-1)}} R^\alpha_g(\mathcal{T}, z) e^{(z - \bar{z})L^{(-1)}} = e^{(\bar{z} - z)L^{(-1)}} R^\alpha_g(\mathcal{T}, \bar{z}) e^{(z - \bar{z})L^{(-1)}}
\]

\[
R^\alpha(\mathcal{T}, z, \bar{z}) = R^\alpha_g(\mathcal{T}, z) \quad , \quad R^\alpha(\mathcal{T}, 0, 0)\langle 0 | 0 \rangle = R^\alpha_g(\mathcal{T}, 0)\langle 0 | 0 \rangle \quad (3.3b)
\]

are examples of Virasoro biprimary fields [18,14], which are simultaneously Virasoro primary under \( T \) and \( \tilde{T} \). The correlators of Virasoro biprimary fields, such as

\[
A^\alpha(\bar{z}, z) = \langle R^{\alpha_1}(\mathcal{T}^1, \bar{z}_1, z_1) \ldots R^{\alpha_n}(\mathcal{T}^n, \bar{z}_n, z_n) \rangle \quad , \quad \alpha = (\alpha_1 \ldots \alpha_n)
\]

(3.4)

are called biconformal correlators.

3. Affine-Virasoro Ward identities [14]. Using null states of the Knizhnik-Zamolodchikov type, one obtains the affine-Virasoro Ward identities for the biconformal correlators. In the case of broken affine primary fields, these read

\[
\partial_{\bar{j}_1} \ldots \partial_{\bar{j}_q} \partial_{i_1} \ldots \partial_{i_p} A^\alpha(\bar{z}, z)|_{\bar{z} = z} = A^\beta_g(z) W_{j_1 \ldots j_q, i_1 \ldots i_p}(z) \beta^\alpha \quad (3.5a)
\]

\[
A^\alpha_g(z) = A^\alpha(z, z) = \langle R^{\alpha_1}_g(\mathcal{T}^1, z_1) \ldots R^{\alpha_n}_g(\mathcal{T}^n, z_n) \rangle \quad (3.5b)
\]

where \( W_{j_1 \ldots j_q, i_1 \ldots i_p} \) are the affine-Virasoro connections and \( A^\alpha_g(z) \) is the affine-Sugawara correlator, which solves the Knizhnik-Zamolodchikov equations [7] and the \( g \)-global Ward identities

\[
A^\beta_g(\sum_{i=1}^n \mathcal{T}_i) \beta^\alpha = 0 \quad , \quad a \in g \quad .
\]

(3.6)
The connections may be computed by standard dispersive techniques from the formula

$$
A^\beta_j(z)W_{j_1 \ldots j_q, i_1 \ldots i_p}(z) = \left[ \prod_{r=1}^q \left( \frac{1}{4\pi i} \int_{z_r} \frac{d\omega_r}{\omega_r} \frac{d\eta_r}{\eta_r - \omega_r} \right) \left[ \prod_{s=1}^p \left( \frac{1}{4\pi i} \int_{\eta_{q+s}} \frac{d\omega_{q+s}}{\omega_{q+s} - \eta_{q+s}} \right) \right] \times \langle J_a(\eta_1)J_b(\omega_1) \ldots J_{a_4}(\eta_q)J_{b_q}(\omega_q)J_{c_1}(\eta_{q+1})J_{d_1}(\omega_{q+1}) \ldots 
\rangle_{\mathcal{R}^1, \bar{z}_1} \ldots \mathcal{R}_g^\alpha(\mathcal{T}^a, z_n) \rangle
$$

since the required averages are in the affine-Sugawara theory on $g$. The first-order connections are

$$W_{i,0} = 2L^{a b} \sum_{j \neq i}^{n} \frac{T_a^i T_b^j}{z_{ij}}, \quad W_0,i = 2L^{a b} \sum_{j \neq i}^{n} \frac{T_a^i T_b^j}{z_{ij}}$$

where $W_i^g$ are the affine-Sugawara connections obtained by Knizhnik and Zamołodchikov. The second-order connections are given for completeness in Appendix A.

4. Invariant correlators. Under $T$ and $\bar{T}$, the biconformal correlators enjoy an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ covariance, and the invariant four-point correlators $Y$ are

$$Y^\alpha(\bar{u}, u) = \left( \prod_{i<j}^{4} z_{ij}^{-\gamma_{ij}} \right) A^\alpha(\bar{z}, z), \quad u = \frac{z_{12}z_{34}}{z_{14}z_{32}}, \quad \bar{u} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}}$$

$$\gamma_{12} = \gamma_{13} = 0, \quad \gamma_{14} = 2\Delta_{a_1}, \quad \gamma_{23} = \Delta_{a_1} + \Delta_{a_2} + \Delta_{a_3} - \Delta_{a_4}, \quad \gamma_{24} = -\Delta_{a_1} + \Delta_{a_2} - \Delta_{a_3} + \Delta_{a_4}, \quad \gamma_{34} = -\Delta_{a_1} - \Delta_{a_2} + \Delta_{a_3} + \Delta_{a_4}$$

$$\bar{\gamma}_{ij} = \gamma_{ij} |_{\Delta \to \bar{\Delta}}$$

where $\alpha = (\alpha_1 \alpha_2 \alpha_3 \alpha_4)$. The invariant correlators satisfy the invariant affine-Virasoro Ward identities

$$\bar{\partial}^\beta \partial^\alpha Y^\alpha(\bar{u}, u)|_{\bar{u}=u} = Y^\beta_g(u)W_{q p}(u)\beta^\alpha$$

$$Y^\alpha_g(u) = Y^\alpha(u, u), \quad Y^\beta_g(\sum_{i=1}^{4} T_i^a)\beta^\alpha = 0, \quad a \in g$$
where $Y_g(u)$ is the invariant affine-Sugawara correlator and $W_{qp}$ are the invariant affine-Virasoro connections. The first-order invariant connections are

$$W_{10} = 2\bar{L}^{ab}\left(\frac{T_a^1 T_b^2}{u} + \frac{T_a^1 T_b^3}{u-1}\right), \quad W_{01} = 2L^{ab}\left(\frac{T_a^1 T_b^2}{u} + \frac{T_a^1 T_b^3}{u-1}\right)$$  \hspace{1cm} (3.11a)

$$W_{10} + W_{01} = W^g = 2L^{ab}\left(\frac{T_a^1 T_b^2}{u} + \frac{T_a^1 T_b^3}{u-1}\right)$$  \hspace{1cm} (3.11b)

and the second-order invariant connections are given in Appendix A.

5. Consistency relations. The affine-Virasoro connections satisfy the consistency relations

$$(\partial + W^g)W_{jq,i_1\ldots i_p} = W_{jq,i_1\ldots i_p} + W_{jq,i_1\ldots i_p}$$  \hspace{1cm} (3.12a)

$$(\partial + W^g)W_{qp} = W_{q+1,p} + W_{q,p+1}$$  \hspace{1cm} (3.12b)

where $W_{00} = 1$. The consistency relations were originally derived [14] from simple properties of the biprimary fields, and the relations are necessary conditions for factorization, discussed below. In fact, the relations also follow directly from the definitions (3.5a) and (3.10) of the connections as derivatives of the biconformal correlators, so the consistency relations are integrability conditions for the existence of the biconformal correlators. To understand this, the reader should begin with the slightly simpler case $f_{qp} \equiv \partial^q \partial^p f(\bar{u}, u)|_{\bar{u}=u}$, which satisfies $\partial f_{qp} = f_{q+1,p} + f_{q,p+1}$ for all $f(\bar{u}, u)$.

6. Factorization. In order to separate the conformal field theories of $\tilde{L}$ and $L$, we assume the abstract factorization

$$A^\alpha(\bar{z}, z) = (\bar{A}(\bar{z}) A(z))^\alpha, \quad Y^\alpha(\bar{u}, u) = (\bar{Y}(\bar{u}) Y(u))^\alpha$$  \hspace{1cm} (3.13)

where the barred and unbarred amplitudes are the proper correlators of the $\tilde{L}$ and the $L$ theories respectively. Then the factorized affine-Virasoro Ward identities

$$(\partial_{j_1\ldots j_q} \partial_{i_1\ldots i_p} A) = A_g^\beta(W_{j_1\ldots j_q,i_1\ldots i_p})^\alpha, \quad A_g^\alpha = (\bar{A} A)^\alpha$$  \hspace{1cm} (3.14a)

$$(\partial^q \partial^p Y)^\alpha = Y_g^\beta(W_{qp})^\beta, \quad Y_g^\alpha = (\bar{Y} Y)^\alpha$$  \hspace{1cm} (3.14b)

are coupled non-linear differential equations for the K-conjugate pair of conformal field theories.
To make the factorization concrete, we must also specify the factorized assignment of the Lie algebra indices. In this paper, we will discuss, at various levels of completeness, four concrete factorization ansätze

\[ A^\alpha(\bar{\nu}, z) = \sum_\nu \bar{\nu} A^\nu(\bar{\nu}) A^\nu(z) \text{ [matrix]} \]  
\[ A^\alpha(\bar{\nu}, z) = \sum_\nu \bar{\nu} A^\nu(\bar{\nu}) A^\nu(z) \text{ [vector]} \]  
\[ A^\alpha(\bar{\nu}, z) = \sum_\nu \bar{\nu} A^\nu(\bar{\nu}) A^\nu(z) \text{ [vector-bar]} \]  
\[ A^\alpha(\bar{\nu}, z) = \sum_\nu \bar{\nu} A^\nu(\bar{\nu}) A^\nu(z) \text{ [symmetric]} \]  

and the corresponding forms for the invariant amplitudes. The first and last of these ansätze were introduced in [14], where a solution for the coset constructions was found in the matrix ansatz. A common feature of these ansätze is the conformal structure index \( \nu \), which labels the conformal structures \( A^\nu \). We shall see that a generically-infinite number of conformal structures is required to factorize the general affine-Virasoro construction.

7. Coset correlators. The biconformal correlators for \( h \) and the \( g/h \) coset constructions are [14]

\[ \bar{L} = L_{g/h} , \quad L = L_h \]  
\[ A^\alpha(\bar{\nu}, z) = A^\alpha_{g/h}(\bar{\nu}, z_0) A_h(\nu, z_0) \]  
\[ A^\alpha_{g/h} = A^\alpha_{g/h}(A^{-1}_h) \]  
\[ A^\alpha_{g/h} = A^\alpha_{g/h}(\sum_{i=1}^n T_a)_{\beta} = 0 , \quad a \in h \]  
\[ Y^\alpha(\bar{\nu}, u) = Y^\alpha_{g/h}(\bar{\nu}, u_0) Y_h(u, u_0) \]  
\[ Y^\alpha_{g/h} = Y^\alpha_g(Y^{-1}_h) \]  
\[ Y^\alpha_{g/h} = Y^\alpha_{g/h}(\sum_{i=1}^4 T_a)_{\beta} = 0 , \quad a \in h \]

where \( A^\alpha_{g/h} \) and \( Y^\alpha_{g/h} \) are the coset correlators and the two-index symbols are the invertible evolution operators of \( h \),

\[ \partial_i A_h(z, z_0) = A_h(z, z_0) W_i^h(z) , \quad \partial_i A_h^{-1}(z, z_0) = -W_i^h(z) A_h^{-1}(z, z_0) \]  
\[ A_h(z_0, z_0)_{\alpha}^\beta = \frac{1}{\Pi_{i<j}(z_0^{ij})^{\gamma_{\alpha}^{ij}}} \delta_{\alpha}^\beta \]  
\[ \partial Y_h(u, u_0) = Y_h(u, u_0) W^h(u) , \quad \partial Y_h^{-1}(u, u_0) = -W^h(u) Y_h^{-1}(u, u_0) \]
\[ Y_h(u_0, u_0)_{\alpha}^\beta = \delta_{\alpha}^\beta . \]  

The solution (3.16) resides in the matrix ansatz (3.15a), with only one conformal structure, and, at the level of conformal blocks, this solution shows the form

\[
Y_g^{\alpha}(u, u_0) = Y_g^{M}(u, u_0) v_\beta^{\alpha}(h) , \quad Y_g^{M}(u, u_0) = d^C(u)_r R \mathcal{F}_h(u_0) R^M
\]

\[
C(u)_r = \mathcal{F}_g(u)_r m \mathcal{F}_h^{-1}(u)_m R
\]

where \( v_\beta^{\alpha}(h) \) are the \( h \)-invariant tensors of \( \mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4 \). The \( u \)-dependent factors \( C(u)_r \) are the coset blocks defined by Douglas [15,14,19].

An important subtlety here is that the evolution operators \( A_h \) and \( Y_h \) are not the \( h \) correlators, because they do not satisfy the \( h \)-global Ward identities. The proper factorization of (3.16) into the correlators of \( g/h \) and \( h \) is [14]

\[
A^\alpha(\tilde{z}, z) = A_g^{M/\alpha}(\tilde{z}, z_0) A_h(z, z_0) M^\alpha
\]

\[
A_g^\alpha = A_g^{M} w_\beta^{\alpha}(h) \quad , \quad (A_h)_M^\alpha \equiv w_\beta^{\alpha}(h)(A_h)_{\beta}^\alpha
\]

\[
Y^\alpha(\bar{u}, u) = Y_g^{M}(\bar{u}, u_0) Y_h(u, u_0) M^\alpha
\]

\[
Y_g^\alpha = Y_g^{M} v_\beta^{\alpha}(h) \quad , \quad (Y_h)_M^\alpha \equiv v_\beta^{\alpha}(h)(Y_h)_{\beta}^\alpha
\]

where \( w_\beta^{\alpha}(h) \) are the \( h \)-invariant tensors of \( \mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4 \). In (3.19), the projected factors \( (A_h)_M^\alpha \) and \( (Y_h)_M^\alpha \) may be identified as \( h \) correlators because they satisfy the \( h \)-global Ward identities. Moreover, the factors \( A_g^\alpha \) and \( Y_g^\alpha \) (see (3.18)) are equivalent representations of the coset correlators \( A_g^\alpha \) and \( Y_g^\alpha \), since the two sets are equal up to constant tensors. We finally note that the factorization (3.19) is in the vector ansatz (3.15b) with \( \nu = M \).

For use below, we also give the known connections for \( h \) and \( g/h \)

\[
W_{g/h}^{j_1\cdots j_q,i_0\cdots i_p} [L = L_{g/h}, L = L_h] = W_{g/h}^{j_1\cdots j_q,i_0\cdots i_p} W_{0,i_0\cdots i_p}^h
\]

\[
W_{g/h}^{j_1\cdots j_q,i_0\cdots i_p} = (\partial_{j_{q+1}} + W_{g/h}^{j_q+1}) W_{j_1\cdots j_q,i_0\cdots i_p}^h W_{j_q+1}^h
\]

\[
W_{0,i_0\cdots i_p}^h = (\partial_{i_0\cdots i_p} + W_{g/h}^h) W_{0,i_0\cdots i_p}^h
\]

\[
W_{g/h}^{q+1,i_0\cdots i_p} = (\partial + W_{g/h}^h) W_{q_0}^h - W_{q_0}^h W_{q_0}^h
\]

\[
W_{0,p+1}^h = (\partial + W_{g/h}^h) W_{0,p+1}^h
\]

where \( W_{00} = 1 \) and the first-order connections of \( g \) and \( h \) are given in (3.8), (3.11). These results are rederived in Section 5 by a method which also generates the connections and conformal correlators of all the affine-Sugawara nests.
4 Some General Properties of the A-V Connections

In this section, we discuss a number of general properties of the affine-Virasoro (A-V) connections.

A. Solution of the consistency relations. It was noted in Ref.[14] that the consistency relations (3.12) can be solved to obtain the general connections in terms of the canonical sets $W_{0p}$ or $W_{0,i_1...i_p}$, or similar sets. The explicit forms of these solutions are easily checked with the binomial identity $(q+1)_r = (q)_r + (q-1)_r$.

In what follows, we refer to $W_{0,i_1...i_p}$, $W_{j_1...j_q,0}$, $W_{0p}$ and $W_{q0}$ as the one-sided connections, and the rest of the connections (e.g. $W_{qp}$, $q,p \geq 1$) as the mixed connections. It is clear from their definition in (3.7) that the one-sided connections are functions only of $\tilde{L}$ or $L$

$$W_{j_1...j_q,0}(\tilde{L}) , \ W_{0,i_1...i_p}(L) , \ W_{q0}(\tilde{L}) , \ W_{0p}(L) \quad (4.2)$$

and so are associated directly to the $\tilde{L}$ or the $L$ theory. This property is not shared by the mixed connections $W_{j_1...j_q,i_1...i_p}(\tilde{L},L)$ and $W_{qp}(\tilde{L},L)$, which are functions of $\tilde{L}$ and $L$.

†In (4.1a,b), $\sum_{p}$ sums over all permutations of the indicated indices and the following rules are included: $\prod_{k=1}^p D^q_{j_k}$ $\equiv$ 1 and $W_{0,j_{r+1}...j_q,i_1...i_p} \equiv W_{0,i_1...i_p}$ for the $r = 0, q$ terms in (4.1a), and similarly for the $r = 0, p$ terms in (4.1b).
B. Connection sum rules. The translation sum rules

\[
\sum_{r,s=0}^{\infty} \frac{1}{r! s!} \sum_{l_1, \ldots, l_r, k_1, \ldots, k_s} \left[ \prod_{\mu=1}^{r} (z_{l_{\mu}} - z_{k_{\mu}}^0) \right] \left[ \prod_{\nu=1}^{s} (z_{k_{\nu}} - z_{l_{\nu}}^0) \right] W_{l_1 \ldots l_r, j_1 \ldots j_q, k_1 \ldots k_s, i_1 \ldots i_p}(z_0) = A_g(z, z_0) W_{j_1 \ldots j_q, i_1 \ldots i_p}(z) \quad (4.3a)
\]

\[
\sum_{r,s=0}^{\infty} \frac{(u - u_0)^{r+s}}{r! s!} W_{r+q, s+p}(u_0) = Y_g(u, u_0) W_{q,p}(u) \quad (4.3b)
\]

relate the connections at different points, where

\[
\partial_{i} A_g(z, z_0)_{\alpha}^{\beta} = A_g(z, z_0)_{\alpha}^{\gamma} W_{q}^{\gamma}(z)_{\gamma}^{\beta} \quad , \quad A_g(z_0, z_0)_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} \quad (4.4a)
\]

\[
\partial_{a} Y_g(u, u_0)_{\alpha}^{\beta} = Y_g(u, u_0)_{\alpha}^{\gamma} W^{\gamma}(u)_{\gamma}^{\beta} \quad , \quad Y_g(u_0, u_0)_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} \quad (4.4b)
\]

are the invertible evolution operators of the affine-Sugawara construction on \( g \). These identities are obtained by repeated application of the consistency relations.

C. K-conjugation covariance. It is clear on inspection of (3.7) that the connections enjoy the K-conjugation covariance

\[
W_{j_1 \ldots j_q, i_1 \ldots i_p}(\tilde{L}, L) = W_{i_1 \ldots i_p, j_1 \ldots j_q}(L, \tilde{L}) \quad (4.5a)
\]

\[
W_{q,p}(\tilde{L}, L) = W_{p,q}(L, \tilde{L}) \quad (4.5b)
\]

under the exchange \( \tilde{L} \leftrightarrow L \) of the \( \tilde{L} \) and \( L \) theories. The simpler covariance of the one-sided connections

\[
W_{j_1 \ldots j_q, 0}(\tilde{L} = L_*) = W_{0, j_1 \ldots j_q}(L = L_*) \quad , \quad W_{0, i_1 \ldots i_p}(L = L_*) = W_{i_1 \ldots i_p, 0}(\tilde{L} = L_*) \quad (4.6a)
\]

\[
W_{0,q}(\tilde{L} = L_*) = W_{0,q}(L = L_*) \quad , \quad W_{0,p}(L = L_*) = W_{p,0}(\tilde{L} = L_*) \quad (4.6b)
\]

follows immediately with (4.2), where \( L_* \) is any particular affine-Virasoro construction.

D. Crossing symmetry. The computations in (3.7) can be performed for a fixed ordering of operators and then again after an exchange \( k \leftrightarrow l \) (including \( T \)'s, \( z \)'s and indices). The crossing symmetry of the connections

\[
W_{j_1 \ldots j_q, i_1 \ldots i_p|k\leftrightarrow l} = W_{j_1 \ldots j_q, i_1 \ldots i_p} \quad (4.7)
\]
then follows from the usual analyticity in $z$ for (derivatives of) the biconformal correlators at $z_i = \bar{z}_i$. We have checked this symmetry for the explicit forms in (3.8), (A.1) and the affine-Sugawara nest connections of Section 5 and Appendix B. A corresponding crossing relation of the invariant connections

$$W_{qp}(1-u) = (-)^{q+p} P_{23} W_{qp}(u) P_{23}$$

(4.8a)

$$P_{23} T^2 P_{23} = T^3 , \quad P_{23}^2 = 1$$

(4.8b)

follows by $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ covariance from (4.7), using (A.3) and the explicit form of $W_{01}$ in (3.11a).

E. High-level connections

To leading non-trivial order in $k^{-1}$ on simple $g$, the currents are effectively abelian and we can evaluate (3.7) by Wick expansion using [12,17]

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2}) , \quad \tilde{L}^{ab} = \frac{\tilde{P}^{ab}}{2k} + \mathcal{O}(k^{-2})$$

(4.9a)

$$P^{ab} + \tilde{P}^{ab} = \eta^{ab} , \quad P^{ab} \eta_{bc} \tilde{P}^{cd} = 0$$

(4.9b)

$$J_a(z) J_b(w) = \frac{k \eta^{ab}}{(z-w)^2} + \mathcal{O}(k^0) .$$

(4.9c)

Here, $\eta^{ab}$ is the Killing metric on $g$ and $\tilde{P}$ and $P$ are the high-level projectors of the $\tilde{L}$ and $L$ theories respectively. The leading term in (3.7) is obtained at $\mathcal{O}(k^0)$ from the maximum number $p + q$ of contractions. This contribution vanishes because no $\omega_i - z_j$ linkage is generated (between the currents and the affine primary fields), so the contour integrals are zero. The next to leading term at $\mathcal{O}(k^{-1})$ can be computed with $p + q - 1$ contractions. As seen in the leading term, the only contractions which survive the contour integrations are those for which no subset of contractions covers a complete subset of $(\omega_i, \eta_i)$ pairs. Then it is not difficult to see that all contributing terms involve a coefficient of the form

$$(P_1 \eta P_2 \eta \cdots \eta P_{p+q})^{ab} ,$$

where $P_i$ may be $P$ or $\tilde{P}$. According to eq.(4.9b), this factor is zero unless all the projectors are the same, which says that the mixed connections are $\mathcal{O}(k^{-2})$.

‡ The following discussion assumes a fixed choice of external representations $\mathcal{T}$ in the biconformal correlators.
The high-level form of the affine-Virasoro connections

\[ W_{j_1...j_q,0} = \left( \prod_{r=1}^{q-1} \partial_{j_r} \right) W_{j_q,0} + {\mathcal O}(k^{-2}) , \quad q \geq 1 \]  

\[ W_{0,i_1...i_p} = \left( \prod_{r=1}^{p-1} \partial_{i_r} \right) W_{0,i_p} + {\mathcal O}(k^{-2}) , \quad p \geq 1 \]  

\[ W_{j_1...j_q,i_1...i_p} = {\mathcal O}(k^{-2}) , \quad q,p \geq 1 \]

is then obtained by solving the consistency relations for the one-sided connections. Similarly, we obtain

\[ W_{00} = \partial^{q-1} W_{10} + {\mathcal O}(k^{-2}) , \quad q \geq 1 \]  

\[ W_{0p} = \partial^{p-1} W_{01} + {\mathcal O}(k^{-2}) , \quad p \geq 1 \]  

\[ W_{qp} = {\mathcal O}(k^{-2}) , \quad q,p \geq 1 \]

for the invariant connections. It follows from (4.10) and (4.11) that both sets may be written in the factorized forms

\[ W_{j_1...j_q,i_1...i_p} = W_{j_1...j_q,0} W_{0,i_1...i_p} + {\mathcal O}(k^{-2}) \]  

\[ W_{qp} = W_{00} W_{0p} + {\mathcal O}(k^{-2}) \]

which, according to eq.(3.20), are exact to all orders for \( \tilde{L} = L_{g/h} \) and \( L = L_h \).

5 The Affine-Sugawara Nests

In this section, we formulate an iterative procedure, using K-conjugation and the solutions (4.1) of the consistency relations, to obtain the connections and conformal correlators of the affine-Sugawara nests \[9,10,11] \]

\[ \tilde{L} = L_{g/h_1/.../h_n} , \quad L = L_{h_1/.../h_n} . \]  

For simplicity, we will follow the argument for the invariant correlators and give the corresponding results for the \( n \)-point correlators at the end.

In this development, we will use two identities repeatedly

\[(\partial + W)^p A = Y^{-1} \partial^p (YA) , \quad \partial Y(u, u_0) = Y(u, u_0) W(u) , \quad Y(u_0, u_0) = 1 \]  

(5.2a)
\[
\sum_{r=0}^{q} (-1)^{r} \binom{q}{r} \partial^{q-r} [A(\partial^{r} B)C] = \sum_{r=0}^{q} \binom{q}{r} (\partial^{q-r} A)B(\partial^{r} C) \quad (5.2b)
\]

which hold for all \(A, B, C\) and \(W\), where \(Y\) is the invertible evolution operator of \(W\).

We begin with the one-sided connections of the trivial theory

\[
W_{0p}(L = 0) = \mathbb{1} \delta_{0,p} \quad .
\]

Substitution of these into the solution (4.1c) gives

\[
W_{qp}(\tilde{L} = L_g, L = 0) = (\partial + W^{g})^{q} \mathbb{1} \delta_{0,p}
\]

which implies the one-sided affine-Sugawara connections on \(g\)

\[
W_{q0}(\tilde{L} = L_g) = (\partial + W^{g})^{q} \mathbb{1} = Y^{-1}\partial^{q}Y_g \quad , \quad \partial Y_g = Y_g W^{g} , \quad Y_g(u_0, u_0) = \mathbb{1} \quad .
\]

Here \(W_{10}(L_g) = W^{g}\) is the invariant affine-Sugawara connection in (3.11b).

Renaming the group, we have for \(h \subset g\)

\[
W_{q0}(\tilde{L} = L_h) = (\partial + W^{h})^{q} \mathbb{1} = Y^{-1}\partial^{q}Y_h \quad , \quad \partial Y_h = Y_h W^{h} , \quad Y_h(u_0, u_0) = \mathbb{1}
\]

(5.6a)

\[
W_{0p}(L = L_h) = (\partial + W^{h})^{p} \mathbb{1} = Y^{-1}\partial^{p}Y_h
\]

(5.6b)

where (5.6b) follows from (5.6a) and (4.6). Then, using (5.6b) in the solution (4.1c), we obtain the connections for \(g/h\) on the left

\[
W_{qp}[\tilde{L} = L_{g/h}, L = L_h] = \sum_{r=0}^{q} (-1)^{r} \binom{q}{r} (\partial + W^{g})^{q-r}W_{0,p+r}(L = L_h)
\]

\[
= \sum_{r=0}^{q} (-1)^{r} \binom{q}{r} (\partial + W^{g})^{q-r}(\partial + W^{h})^{p+r} \mathbb{1}
\]

(5.7)

\[
= Y^{-1}_g \sum_{r=0}^{q} (-1)^{r} \binom{q}{r} \partial^{q-r}(Y_{g/h}\partial^{p+r}Y_h)
\]

\[
= Y^{-1}_g(\partial^{q}Y_{g/h})(\partial^{p}Y_h) \quad , \quad Y_{g/h} = Y_g Y^{-1}_h
\]

where the identities (5.2a,b) were used in the last two steps.

At this point in the iteration, we have regained the known results for the coset constructions, since the one-sided coset connections

\[
W_{q0}(\tilde{L} = L_{g/h}) = Y^{-1}_g(\partial^{q}Y_{g/h})Y_h
\]

(5.8)
solve the recursion relation (3.20e). To see the factorized biconformal correlators, note that (5.7) may be rewritten as:

\[ Y_\beta^\alpha (\bar{u}, u) |_{\bar{u} = u} \]

These forms verify the factorized Ward identities for \( \frac{g}{h} \) and \( h \), where \( Y_\alpha^\alpha \) in (5.9c) is the coset correlator. Note also that \( Y_\alpha^\alpha, \ W_{qp} \) and the biconformal correlators are independent of the reference point \( u_0 \) because

\[ \partial_{u_0} Y(u, u_0) = -W(u_0)Y(u, u_0) \] (5.10)

follows for the evolution operator in (5.2a).

We move on now to the first nontrivial affine-Sugawara nest, defined on \( g \supset h_1 \supset h_2 \). Renaming groups again and using (4.6), we know from the result (5.8) for the cosets that

\[ W_{0p}(L = L_{h_1/h_2}) = Y_{h_1}^{-1}(\partial^pY_{h_1/h_2})Y_{h_2} \] . (5.11)

Then, we obtain the nest connections

\[ W_{qp}[\bar{L} = L_{g/h_1/h_2}, L = L_{h_1/h_2}] = \sum_{r=0}^{q} (-1)^r \binom{q}{r} (\partial + W^p)^{q-r}W_{0,p+r}(L = L_{h_1/h_2}) \]

\[ = \sum_{r=0}^{q} (-1)^r \binom{q}{r} (\partial + W^g)^{q-r}[Y_{h_1}^{-1}(\partial^p+rY_{h_1/h_2})Y_{h_2}] \]

\[ = Y_{g}^{-1} \sum_{r=0}^{q} (-1)^r \binom{q}{r} [\partial^g-rY_{g/h_1}](\partial^pY_{h_1/h_2})(\partial^rY_{h_2}) \] (5.12)

from (4.1c), where the identities (5.2a,b) were used in the last two steps. The nest connections may also be expressed in the form

\[ W_{qp}[\bar{L} = L_{g/h_1/h_2}, L = L_{h_1/h_2}] = \sum_{r=0}^{q} \binom{q}{r} W_{0,q-r}W_{0p}^{r}W_{h_1}^{h_2} \] (5.13)
where $W_{0p}^{g/h} = W_0(L_{g/h})$ and $W_{0p}^h = W_0(L_h)$ are the one-sided connections of $g/h$ and $h$. In this form, we see that the nest connections are independent of the reference point $u_0$.

We may also display the nest connections in the form

$$Y_g W_{qp}[\tilde{L} = L_{g/h_1/h_2}, L = L_{h_1/h_2}] = \left\{ \partial^{\alpha} \partial^p \left[ Y_{g/h_1} (\bar{u}) Y_{h_1/h_2} (u) Y_{h_2} (\bar{u}) \right] \right\}_{u = \bar{u}} \quad (5.14)$$

which verifies the consistency relations (3.12b) on inspection, according to the remarks below (3.12). Moreover, the form (5.14) shows that the unfactorized Ward identities (3.10) are solved by the biconformal nest correlators

$$Y_\alpha (\bar{u}, u) \left[ \tilde{L} = L_{g/h_1/h_2}, L = L_{h_1/h_2} \right] = Y_\beta^{\gamma} \|_{\alpha} \|_{\beta} \quad (5.15)$$

where $Y_{g/h}^\alpha$ is the invariant coset correlator defined in (3.16a). According to (5.10), the biconformal nest correlator is independent of the reference point $u_0$.

In order to factorize the biconformal nest correlator (5.15) into the conformal correlators of $g/h_1/h_2$ and $h_1/h_2$, we need the expansions [14]

$$Y_{g/h_1}^{\beta} = Y_{g/h_1}^{\alpha} v_{m_1}^\beta (h_1) \quad (5.16a)$$

$$(Y_{h_1/h_2})_{m_1}^{\gamma} v_{m_1}^\beta (h_1) (Y_{h_1/h_2})_{\beta}^{\gamma} = (Y_{h_1/h_2})_{m_1}^{m_2} v_{m_2}^{\gamma} (h_2) \quad (5.16b)$$

$$(Y_{h_2})_{m_2}^{\alpha} \equiv v_{m_2}^{\gamma} (h_2) (Y_{h_2})_{\gamma}^{\alpha} \quad (5.16c)$$

$$(Y_{h_1/h_2})_{m_1}^{m_2} (\sum_{i=1}^{4} \mathcal{T}^i_a)_{\beta}^{\alpha} = (Y_{h_2})_{m_2}^{m_2} (\sum_{i=1}^{4} \mathcal{T}^i_a)_{\beta}^{\alpha} = 0 \quad , \quad a \in h_2 \quad (5.16d)$$

where $v_{m_1}^\alpha (h_1)$ are the $h_1$-invariant tensors of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^4$. Then, (5.15) factorizes as follows,

$$Y_\alpha (\bar{u}, u) [\tilde{L} = L_{g/h_1/h_2}, L = L_{h_1/h_2}] = \tilde{Y} (\bar{u}, u_0) [\tilde{L} = L_{g/h_1/h_2}]^{m_1}_{m_2} \alpha \|_{m_2} \alpha \quad (5.17a)$$

$$(Y_{h_1/h_2})_{m_1}^{m_2} \alpha = (Y_{g/h_1} (\bar{u}, u_0) \otimes Y_{h_2} (\bar{u}, u_0))_{m_1}^{m_2} \alpha \quad (5.17b)$$

The evolution operator $\tilde{Y}_{h_1/h_2} (u, u_0)_{\beta}^{\gamma}$ in (5.15) is not a coset correlator because it does not satisfy the $h_2$-global Ward identities [14].
\[ Y(u, u_0)[L = L_{h_1/h_2}]_{m_1}^{m_2} = Y_{h_1/h_2}(u, u_0)_{m_1}^{m_2} \]  

(5.17c)

where (5.17) solves the factorized Ward identities (3.14b). The factorization (5.17) is correct for the \( h_1/h_2 \) theory because the projected factor \( (Y_{h_1/h_2})_{m_1} = (Y_{h_1/h_2})_{m_2} \) satisfies the \( h_2 \)-global Ward identities (5.16d). Moreover, \( Y_{g/h_1}^{m_1} \) and \( (Y_{h_2})_{m_2} \) are equivalent representations of the \( g/h_1 \) and \( h_2 \) correlators respectively.

It is then clear from (5.17b) that the conformal field theory of the nest \( L_{g/h_1/h_2} \) is the tensor-product theory \( (g/h_1) \otimes h_2 \), as anticipated in Section 2. Appendix C gives the explicit form of the nest correlators in terms of conformal blocks.

The procedure followed above can be further iterated to obtain the connections and biconformal correlators of all the affine-Sugawara nests. The general scheme is

\[ W_{0p}(\tilde{L} = L_{g/h_1/\ldots/h_{n+1}}) \rightarrow W_{0p}(L = L_{h_1/\ldots/h_{n+1}}) \rightarrow W_{qp}(\tilde{L} = L_{g/h_1/\ldots/h_{n+1}}, L = L_{h_1/\ldots/h_{n+1}}) \rightarrow W_{0q}(\tilde{L} = L_{g/h_1/\ldots/h_{n+1}}) \]

(5.18)

where the first step uses (4.6b) and a group relabelling, the second step uses the solution (4.1c) of the invariant consistency relations, and the last step uses (4.2) at \( p = 0 \). Continuing the iteration, we find the invariant biconformal nest correlators

\[ Y^\alpha(\bar{u}, u)[\tilde{L} = L_{g/h_1/\ldots/h_{2n+1}}, L = L_{h_1/\ldots/h_{2n+1}}] = \]

\[ Y^\beta_{g/h_1}(\bar{u}, u_0)[Y_{h_1/h_2}(u, u_0)Y_{h_2/h_3}(\bar{u}, u_0) \cdots Y_{h_{2n}/h_{2n+1}}(\bar{u}, u_0)Y_{h_{2n+1}}(u, u_0)]^\alpha_{\beta} \]

(5.19a)

\[ Y^\alpha(\bar{u}, u)[\tilde{L} = L_{g/h_1/\ldots/h_{2n}}, L = L_{h_1/\ldots/h_{2n}}] = \]

\[ Y^\beta_{g/h_1}(\bar{u}, u_0)[Y_{h_1/h_2}(u, u_0)Y_{h_2/h_3}(\bar{u}, u_0) \cdots Y_{h_{2n}/h_{2n}}(u, u_0)Y_{h_{2n}}(\bar{u}, u_0)]^\alpha_{\beta} \]

(5.19b)

which are independent of \( u_0 \) and solve the unfactorized Ward identities (3.10).

The same scheme can be followed to obtain the \( n \)-point biconformal nest correlators

\[ A^\alpha(\bar{z}, z)[\tilde{L} = L_{g/h_1/\ldots/h_{2n+1}}, L = L_{h_1/\ldots/h_{2n+1}}] = \]

\[ A^\beta_{g/h_1}(\bar{z}, z_0)[A_{h_1/h_2}(z, z_0)A_{h_2/h_3}(\bar{z}, z_0) \cdots A_{h_{2n}/h_{2n+1}}(\bar{z}, z_0)A_{h_{2n+1}}(z, z_0)]^\alpha_{\beta} \]

(5.20a)
\[ A^\alpha(\bar{z}, z)[\bar{L} = L_{g/h_1/\ldots/h_{2n}}, L = L_{h_1/\ldots/h_{2n}}] = \]

\[ A^a_{g/h_1}(\bar{z}, z_0)[A_{h_1/h_2}(z, z_0)A_{h_2/h_3}(\bar{z}, z_0) \cdots A_{h_{2n-1}/h_{2n}}(z, z_0)A_{h_{2n}}(\bar{z}, z_0)]^\alpha_{\beta} \]

which are independent of the reference point and solve the unfactorized Ward identities (3.5). Using the general principles of the iteration, the results (5.19) and (5.20) are verified in Appendix B.

Following the discussion of the first nest above, Appendix C discusses the factorization of the biconformal nest correlators (5.19) into the conformal correlators of the general nest. The result

\[ \bar{Y}[\bar{L} = L_{g/h_1/\ldots/h_{2n+1}}] = Y_{g/h_1} \otimes Y_{h_2/h_3} \otimes \cdots \otimes Y_{h_{2n-2}/h_{2n-1}} \otimes Y_{h_{2n}/h_{2n+1}} \] (5.21a)

\[ \bar{Y}[\bar{L} = L_{g/h_1/\ldots/h_{2n}}] = Y_{g/h_1} \otimes Y_{h_2/h_3} \otimes \cdots \otimes Y_{h_{2n-2}/h_{2n-1}} \otimes Y_{h_{2n}} \] (5.21b)

shows that all the affine-Sugawara nests are tensor-product theories. The conformal blocks of the general nest and an explicit example are also worked out in Appendix C.

Beyond the coset constructions and affine-Sugawara nests, it is clear that solution of the factorized Ward identities will be more complex. In the following sections, we develop an algebraic reformulation of the system, which, given the affine-Virasoro connections, allows the construction of global solutions across all affine-Virasoro space.

6 Algebraization of the Ward Identities

Given the affine-Virasoro connections, the factorized affine-Virasoro Ward identities (3.14) are an all-order system of non-linear differential equations. In this section we show that the system has an equivalent algebraic formulation, observed in discussion with E. Kiritsis.

The algebraization may be understood in two ways. In the first viewpoint, we solve the unfactorized Ward identities (3.5) and (3.10) by the partially-
factorized forms of the biconformal correlators

\[ A^\alpha(\bar{z}, z) = \sum_{q,p=0}^{\infty} \frac{1}{q!} \sum_{j_1...j_q}^{n} \frac{1}{p!} \sum_{i_1...i_p}^{n} \prod_{\mu=1}^{q} (\bar{z}_{j_\mu} - z_{0,j_\mu}) [A^\beta_g(z_0)W_{j_1...j_q,i_1...i_p}(z_0)\beta^\alpha] \prod_{\nu=1}^{p}(z_{i_\nu} - z_{0,i_\nu}) \]  

\[ Y^\alpha(\bar{u}, u) = \sum_{q,p=0}^{\infty} \frac{(\bar{u} - u_0)^q}{q!} \frac{(u - u_0)^p}{p!} [Y^\beta_g(u_0)W_{qp}(u_0)\beta^\alpha] Y^\beta_g(u_0)W_{qp}(u_0)\beta^\alpha \]  

\[ (6.1a) \]

\[ (6.1b) \]

where \( z_0 = \{z_i^0\} \) and \( u_0 \) are regular reference points. These forms verify the unfactorized Ward identities by differentiation, as seen explicitly for the invariant case as follows

\[ \partial^\alpha \partial^\beta Y^\alpha(\bar{u}, u)|_{\bar{u} = u} = \sum_{r,s=0}^{\infty} \frac{(u - u_0)^{r+s}}{r! s!} Y^\beta_g(u_0)W_{r+s,p}(u_0)\beta^\alpha \]

\[ = Y^\beta_g(u_0)(Y^\beta_g(u, u_0)W_{qp}(u))\beta^\alpha = Y^\beta_g(u)W_{qp}(u)\beta^\alpha . \]  

\[ (6.2) \]

Here, the translation sum rules (4.3b) were used in the last step, and the same steps with (4.3a) verify the partially-factorized form of the general biconformal correlators in (6.1a).

Note also that the biconformal correlators in (6.1) are independent of the reference point used to define the partial factorization, for example,

\[ \partial_{u_0} Y^\alpha(\bar{u}, u) = \sum_{q,p=0}^{\infty} \frac{(\bar{u} - u_0)^q}{q!} \frac{(u - u_0)^p}{p!} Y^\beta_g(u_0) \]

\[ \times [(\partial_{u_0} + W^g(u_0)W_{qp}(u_0) - W_{q+1,p}(u_0) - W_{q,p+1}(u_0)]\beta^\alpha = 0 \]

\[ (6.3) \]

where the consistency relations (3.12b) were used in the last step. Similarly, \( \partial/\partial z^0 \alpha A^\alpha(\bar{z}, z) = 0 \) is verified with the consistency relations (3.12a).

The meaning of the partially-factorized forms in (6.1) is that the factorized Ward identities can be solved algebraically. More precisely, the biconformal correlators are completely factorized (and hence the factorized Ward identities solved) if we can factorize the connections at the reference point, which is an algebraic problem. For the general connections, the abstract form of this factorization reads

\[ C_{j_1...j_q,i_1...i_p}^\alpha \equiv A^\beta_g(z_0)W_{j_1...j_q,i_1...i_p}(z_0)\beta^\alpha = (A_{j_1...j_q}A_{i_1...i_p})^\alpha \]  

\[ (6.4a) \]

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while for the invariant case we have the simpler problem

\[ C^\alpha_{qp} \equiv Y^\beta_g(u_0) W_{qp}(u_0)_\beta^\alpha = (\bar{Y}_q Y_p)^\alpha \]  

(6.5a)

\[ \bar{Y}_q \equiv \partial^q \bar{Y}(u)|_{u=u_0} \ , \ Y_p \equiv \partial^p Y(u)|_{u=u_0} \]  

(6.5b)

In the following section, we shall return to study the concrete factorization ansätze (3.15) in this algebraic form.

An equivalent statement of the algebraization is as follows. The factorized Ward identities in (3.14) are completely solved if they are solved at the reference point, where they read

\[ (\bar{A}_{j_1...j_q} A_{i_1...i_p})^\alpha = C^\alpha_{j_1...j_q,i_1...i_p} \]  

(6.6a)

\[ (\bar{Y}_q Y_p)^\alpha = C^\alpha_{qp} \]  

(6.6b)

in the notation of (6.4) and (6.5). To check this for the invariant case, assume (6.6b) and follow the steps

\[
(\partial^q \bar{Y} \partial^p Y)^\alpha = \sum_{r=0}^{\infty} \left( \frac{u-u_0)^r}{r!} \right) \partial^q (\partial^q \bar{Y} \partial^p Y)^\alpha |_{u=u_0}
\]

\[ = \sum_{r=0}^{\infty} \left( \frac{u-u_0)^r}{r!} \right) \sum_{s=0}^{r} \left( \frac{r}{s} \right) (\partial^{q+s} \bar{Y} \partial^{p+r-s} Y)^\alpha |_{u=u_0}
\]

\[ = \sum_{r=0}^{\infty} \left( \frac{u-u_0)^r}{r!} \right) \sum_{s=0}^{r} \left( \frac{r}{s} \right) \bar{Y}_{q+s} Y_{p+r-s})^\alpha
\]

\[ = \sum_{r=0}^{\infty} \left( \frac{u-u_0)^r}{r!} \right) \sum_{s=0}^{r} \left( \frac{r}{s} \right) C^\alpha_{q+s,p+r-s}
\]

(6.7)

\[ = Y^\beta_g(u_0) \sum_{r=0}^{\infty} \left( \frac{u-u_0)^r}{r!} \right) \sum_{s=0}^{r} \left( \frac{r}{s} \right) W_{q+s,p+r-s}(u_0)_\beta^\alpha
\]

\[ = Y^\beta_g(u_0) \sum_{r,s=0}^{\infty} \left( \frac{u-u_0)^{r+s}}{r!s!} \right) W_{q+s,p+r}(u_0)_\beta^\alpha
\]

\[ = Y^\beta_g (u) W_{qp}(u)_\beta^\alpha
\]

where \( \sum_{r=0}^{\infty} \sum_{s=0}^{r} f(r, s) = \sum_{r,s=0}^{\infty} f(r+s, s) \) and the translation sum rule (4.3b) were used in the final steps. Similarly, one uses the translation sum rule (4.3a) to see that the factorized n-point Ward identities (3.14a) are solved by the algebraic factorization (6.6a).
7 Factorization

In this section, we factorize the invariant biconformal correlators via concrete realizations of the algebraic factorization (6.5).

In particular, we distinguish four concrete algebraic factorization ansätze

\[ W_{qp}(u_0)_{\beta}^{\alpha} = \sum_{\nu} (\bar{Y}_{q\nu})_{\beta}^{\gamma} (Y_{\nu p})_{\gamma}^{\alpha} \]  

[matrix] (7.1a)

\[ W_{qp}(u_0)_{\beta}^{\alpha} = \sum_{\nu} Y_{q\nu} Y_{\nu p}^{\alpha} \]  

[vector] (7.1b)

\[ W_{qp}(u_0)_{\beta}^{\alpha} = \sum_{\nu} \bar{Y}_{q\nu} Y_{\nu p}^{\alpha} \]  

[vector-bar] (7.1c)

\[ C_{qp}^{\alpha} = \sum_{\nu} \bar{Y}_{q\nu} Y_{\nu p}^{\alpha} \]  

[symmetric] (7.1d)

which correspond to the factorization ansätze listed in (3.15) as follows,

\[ Y^{\alpha}(\bar{u}, u) = \sum_{\nu} Y_{\nu}^{\beta}(\bar{u}) Y_{\nu}^{\alpha}(u) \]  

[matrix] (7.2a)

\[ \bar{Y}_{\nu}^{\alpha}(\bar{u}) = Y_{g}^{\beta}(u_0) \sum_{q=0}^{\infty} \frac{(\bar{u} - u_0)^q}{q!} (\bar{Y}_{q\nu})_{\beta}^{\alpha}, \quad Y_{\nu}^{\alpha}(u) = \sum_{p=0}^{\infty} \frac{(u - u_0)^p}{p!} \]  

\[ Y_{\nu}^{\alpha}(\bar{u}, u) = \bar{Y}_{\nu}(\bar{u}) Y_{\nu}^{\alpha}(u) \]  

[vector] (7.2b)

\[ \bar{Y}_{\nu}(\bar{u}) = Y_{g}^{\beta}(u_0) \sum_{q=0}^{\infty} \frac{(\bar{u} - u_0)^q}{q!} \bar{Y}_{q\nu}^{\beta}, \quad Y_{\nu}^{\alpha}(u) = \sum_{p=0}^{\infty} \frac{(u - u_0)^p}{p!} \]  

\[ \bar{Y}_{\nu}(\bar{u}, u) = \bar{Y}_{\nu}^{\alpha}(\bar{u}) Y_{\nu}^{\alpha}(u) \]  

[vector-bar] (7.2c)

\[ Y^{\alpha}(\bar{u}, u) = \sum_{\nu} Y_{\nu}^{\alpha}(\bar{u}) Y_{\nu}^{\alpha}(u) \]  

[symmetric] (7.2d)

\[ \bar{Y}_{\nu}^{\alpha}(\bar{u}) = \sum_{q=0}^{\infty} \frac{(\bar{u} - u_0)^q}{q!} Y_{q\nu}^{\alpha}, \quad Y_{\nu}^{\alpha}(u) = \sum_{p=0}^{\infty} \frac{(u - u_0)^p}{p!} \]  

The four ansätze share the notion of a conformal structure index \( \nu \), while differing in the assignment of the Lie algebra indices \( \alpha, \beta \). We remind the reader that the solution (3.19) for \( g/h \) and \( h \) resides in the vector ansatz (7.2b) with \( \nu = M \).
Moreover, the factorization (5.17a) of the first non-trivial affine-Sugawara nest is in the vector-bar ansatz with \( \nu = (m_1, m_2) \). For the general nest, the factorization of (5.19) (see eq.(C.3)) is in the vector ansatz for \( \tilde{L} = L_{g/h_1/.../h_{2n+1}} \) and in the vector-bar ansatz for \( \tilde{L} = L_{g/h_1/.../h_{2n}} \).

More generally, the matrices \( W_{qp}(u_0) \) and \( C_{qp} = Y_g(u_0)W_{qp}(u_0) \) are infinite dimensional, so we expect (and will find) that each of the ansätze exhibits infinite-dimensional factorizations, with an infinite number of conformal structures, for any K-conjugate pair of affine-Virasoro constructions. An infinite-dimensional conformal structure is expected in irrational conformal field theory, but the problem is that there are too many solutions, many of which are apparently not physical.

As an example, consider the matrix ansatz (7.1a), whose solutions for any invertible \( \tilde{Y} \) are

\[
Y_{\nu p} = \sum_{q=0}^{\infty} (\tilde{Y}^{-1})_{\nu q} W_{qp}(u_0) \quad , \quad \nu, p = 0, 1, \ldots .
\]

This is a very large class of solutions to the Ward identities, most of which must be unacceptable as they stand. To understand this, consider the simple particular solution

\[
(Y_{\nu p})_{\alpha}^\beta = (Y^{-1})_{\nu q} \delta_{\alpha}^\beta , \quad (Y_{\nu p})_{\alpha}^\beta = W_{\nu p}(u_0)_{\alpha}^\beta \quad (7.4a)
\]

\[
\tilde{Y}_\nu^\alpha (u,u_0) = Y_g^\alpha (u_0) \frac{(u-u_0)^\nu}{\nu!} , \quad Y_{\nu p} (u,u_0)_{\alpha}^\beta = \sum_{p=0}^{\infty} W_{\nu p}(u_0)_{\alpha}^\beta \frac{(u-u_0)^p}{p!} \quad (7.4b)
\]

whose conformal structures \( \tilde{Y}_\nu^\alpha, \nu = 0, 1, \ldots \) do not show the conformal weights of the \( \tilde{L} \) theory. We believe that these conformal structures should be viewed only as a basis for a physical solution, reasoning as follows. Given any particular solution \( \tilde{Y}(u_0), Y(u_0) \) to (7.1a), we also obtain the associated family of solutions

\[
\tilde{Y}_\nu^\alpha (u,u_0) = \tilde{Y}(u,u_0) \Omega(u_0) \quad , \quad Y_{\nu p} (u,u_0)_{\alpha}^\beta = \Omega^{-1}(u_0) Y(u,u_0) \quad (7.5)
\]

where \( \tilde{Y}(u,u_0), Y(u,u_0) \) is the particular solution and \( \Omega(u_0) \) is an arbitrary invertible matrix. It is then clear that the conformal structures \( \tilde{Y}_\nu^\alpha (u,u_0) \) in (7.4) are a basis for the family

\[
(\tilde{Y}_\mu^\alpha)_{\nu}(u,u_0) = \sum_{\nu=0}^{\infty} \tilde{Y}_\nu^\alpha (u,u_0) \Omega(u_0)_{\nu \mu} = \sum_{\nu=0}^{\infty} Y_g^\alpha (u_0) \frac{(u-u_0)^\nu}{\nu!} \Omega(u_0)_{\nu \mu} \quad (7.6)
\]
which is an essentially arbitrary power series in \((u - u_0)\).

Our attention is then focused on the problem of finding a good basis, in which the solution is physical, by paying attention to general principles. In what follows, we study a natural factorization in the vector ansatz which gives a global solution across all affine-Virasoro space. This solution

a) reproduces the correlators (3.16e) and (5.21) of the coset constructions and the affine-Sugawara nests,

b) exhibits braiding for all affine-Virasoro constructions, and

c) shows physical behavior at high level for all affine-Virasoro constructions on simple \(g\).

8 Factorization by Connection Eigenvectors

The invariant affine-Virasoro connections \((W_{qp})_{\alpha}^{\beta}\) define an infinite-dimensional eigenvalue problem

\[
\sum_{q} \psi_{q}^{\beta}(u_0)W_{qp}(u_0)_{\alpha}^{\beta} = E_{\nu}(u_0)\bar{\psi}_{\alpha}^{(\nu)}(u_0) \quad (8.1a)
\]

\[
\sum_{p} \psi_{p}^{\beta}(u_0)W_{qp}(u_0)_{\alpha}^{\beta} = E_{\nu}(u_0)\psi_{\alpha}^{(\nu)}(u_0) \quad (8.1b)
\]

whose eigenvectors provide a natural factorization in the vector ansatz (7.2b),

\[
W_{qp}(u_0)_{\alpha}^{\beta} = \sum_{\nu=0}^{\infty} \bar{\psi}_{\alpha}^{(\nu)}(u_0)E_{\nu}(u_0)\psi_{\beta}^{(\nu)}(u_0) \quad (8.2a)
\]

\[
\bar{Y}_{\nu}(u, u_0) = \sqrt{E_{\nu}(u_0)}Y_{g}^{\alpha}(u_0)\bar{\psi}_{\alpha}^{(\nu)}(u, u_0) , \quad \bar{\psi}_{\alpha}^{(\nu)}(u, u_0) \equiv \sum_{q=0}^{\infty} \frac{(u - u_0)^{q}}{q!} \bar{\psi}_{\alpha q}^{(\nu)}(u_0) \quad (8.2b)
\]

\[
Y_{\nu}^{\alpha}(u, u_0) = \sqrt{E_{\nu}(u_0)}\psi_{\alpha}^{(\nu)}(u, u_0) , \quad \psi_{\alpha}^{(\nu)}(u, u_0) \equiv \sum_{p=0}^{\infty} \frac{(u - u_0)^{p}}{p!} \psi_{\alpha p}^{(\nu)}(u_0) \quad . \quad (8.2c)
\]

More precisely, the spectral resolution (8.2a) holds when \((W_{qp})_{\alpha}^{\beta}\) is diagonalizable, which we shall see is true at least down to some finite level because it is true at high level (see Section 10). In what follows, we refer to the basic
structures $\tilde{\psi}^{(v)}_{\alpha}(u, u_0)$, $\psi^{(v)}_{\nu}(u, u_0)$ in (8.2b,c) as the conformal eigenvectors of the $\tilde{L}$ and $L$ theories respectively. Note also that only the conformal eigenvectors with $E_{\nu} \neq 0$ contribute to the factorized correlators $\tilde{Y}$ and $Y$.

An equivalent form of the factorized correlators (8.2b,c)

\begin{align}
\tilde{Y}_{\nu}(u, u_0) &= \frac{1}{\sqrt{E_{\nu}(u_0)}} \sum_{p=0}^{\infty} \tilde{\psi}^{(v)}_{\mu_\alpha}(u_0) \partial_{u_0}^p \tilde{f}_0^\alpha(u, u_0) \\
Y_{\nu}^\alpha(u, u_0) &= \frac{1}{\sqrt{E_{\nu}(u_0)}} \sum_{q=0}^{\infty} \psi^{\beta}_{q(\nu)}(u_0) [(\partial_{u_0} + W^g(u_0))^q f_0(u, u_0)]_{\beta}^\alpha
\end{align}

(8.3a, 8.3b)

\begin{align}
\tilde{f}_0^\alpha(u, u_0) &\equiv Y_{g}^\beta(u_0) \sum_{q=0}^{\infty} \frac{(u - u_0)^q}{q!} W_{q0}(u_0)_{\beta}^\alpha \\
f_0(u, u_0)_{\beta}^\alpha &\equiv \sum_{p=0}^{\infty} W_{0p}(u_0)_{\beta}^\alpha \frac{(u - u_0)^p}{p!}
\end{align}

(8.3c, 8.3d)

is obtained by using the eigenvalue equations for $E_{\nu} \neq 0$ and the identities

\begin{align}
\sum_{q=0}^{\infty} \frac{(u - u_0)^q}{q!} W_{qp}(u_0) &= (\partial_{u_0} + W^g(u_0))^p \sum_{q=0}^{\infty} \frac{(u - u_0)^q}{q!} W_{q0}(u_0) \\
\sum_{p=0}^{\infty} W_{qp}(u_0) \frac{(u - u_0)^p}{p!} &= (\partial_{u_0} + W^g(u_0))^q \sum_{p=0}^{\infty} W_{0p}(u_0) \frac{(u - u_0)^p}{p!}
\end{align}

(8.4a, 8.4b)

which follow from the consistency relations (3.12b) or their solutions in (4.1c,d). In this form, the factorized correlators are expressed as eigenvector projections of the basic structures $\tilde{f}_0$, $f_0$.

As a first test of the global solution (8.2) and (8.3), we reconsider the familiar case $\tilde{L} = L_{g/h}$ and $L = L_h$, for which the basic structures (8.3c,d) are easily summed,

\begin{align}
\tilde{f}_0^\alpha(u, u_0) &= Y_{g/h}^\alpha(u, u_0) \\
f_0(u, u_0)_{\alpha}^\beta &= Y_{h}^\beta(u, u_0)_{\alpha}^\beta
\end{align}

(8.5)

using the connections in (5.8) and (5.6b). Moreover, we can use (5.10) to evaluate the $u_0$ derivatives in (8.3a), which gives

\begin{align}
\tilde{Y}_{\nu}^{g/h}(u, u_0) &= Y_{g/h}^{\alpha}(u, u_0) d^{(v)}_{\alpha}(u_0) \\
d^{(v)}_{\alpha}(u_0) &\equiv \frac{1}{\sqrt{E_{\nu}(u_0)}} \sum_{p=0}^{\infty} W_{0p}^{h}(u_0)_{\beta}^\alpha \tilde{\psi}_{p\beta}^{(v)}(u_0)
\end{align}

(8.6)
for the coset constructions. In this case, the conformal structures are degenerate in that all the $u$ dependence of each structure is in the same correct coset factor $Y^\alpha_{g/h}(u, u_0)$, defined in (3.16e).

We have also checked that the global solution reproduces the known results in Section 5 for all the affine-Sugawara nests. As an example, the basic structure $\tilde{f}_0$ and the factorized correlators of the first non-trivial nests,

$$\tilde{f}_0^\alpha(u, u_0) = Y_{g/h_1}^\beta(u, u_0)Y_{h_2}(u, u_0)\beta^\alpha$$  

(8.7a)

$$Y_{\nu}^{g/h_1/h_2}(u, u_0) = Y_{g/h_1}^{m_1}(u, u_0)Y_{h_2}(u, u_0)m_2^\alpha D^{(\nu)m_2}_m(u_0)$$  

(8.7b)

$$D^{(\nu)m_2}_m(u_0) \equiv \frac{1}{\sqrt{E_{\nu}(u_0)}} \sum_{p=0}^{\infty} W_{0p}^{h_1/h_2}(u_0)m_1^m \bar{\psi}_{p\alpha}(u_0)$$  

(8.7c)

$$\bar{v}_{m_1}^{\beta}(h_1)(W_{0p}^{h_1/h_2})_\beta^\alpha \equiv (W_{0p}^{h_1/h_2})_m^m \bar{v}_{m_2}^{\alpha}(h_2)$$  

(8.7d)

are obtained with (8.3), (5.12), (5.10), (3.20e) and (5.16) for $\tilde{L} = L_{g/h_1/h_2}$. The conformal structures in (8.7b) are again degenerate, with all $u$ dependence in the correct nest factor $Y_{g/h_1} \otimes Y_{h_2}$.

In what follows, we study two general features of the eigenvectors, which provide some evidence for good physical behavior of these solutions across all affine-Virasoro space.

9 An Origin for Braiding in Irrational CFT

In rational CFT, braiding appears as a property of linear differential equations, but, in the general CFT’s of the Virasoro master equation, it is unlikely that linear differential equations [20,7,14] extend beyond the coset constructions. An important feature of the solution (8.2) is that it is based on a linear (eigenvalue) problem, which, as we shall see, generates braiding in the more general context.

To begin, we write the corresponding eigenvalue problem (8.1) in the more flexible notation

$$W_{qp}(u_0)\bar{\psi}_p^{(\nu(u_0))}(u_0) = E_{\nu(u_0)}(u_0)\bar{\psi}_q^{(\nu(u_0))}(u_0)$$  

(9.1a)

$$\psi_q^{(\nu(u_0))}(u_0)W_{qp}(u_0) = E_{\nu(u_0)}(u_0)\psi_p^{(\nu(u_0))}(u_0)$$  

(9.1b)
to facilitate comparison with the eigenvalue problem at $1 - u_0$,

$$W_{qp}(1 - u_0)\tilde{\psi}_p^{(\nu(1-u_0))}(1 - u_0) = E_{\nu(1-u_0)}(1 - u_0)\tilde{\psi}_q^{(\nu(1-u_0))}(1 - u_0) \tag{9.2a}$$

$$\psi_{q(\nu(1-u_0))}(1 - u_0)W_{qp}(1 - u_0) = E_{\nu(1-u_0)}(1 - u_0)\psi_p^{(\nu(1-u_0))}(1 - u_0) \tag{9.2b}$$

In these forms, we avoid any labelling prejudice by allowing the conformal structure index $\nu$ to depend on $u_0$ or $1 - u_0$.

The eigenvalue problem at $1 - u_0$ may be rewritten as

$$W_{qp}(u_0)\left[(-)^p P_{23}\tilde{\psi}_p^{(\nu(1-u_0))}(1 - u_0)\right] = E_{\nu(1-u_0)}(1 - u_0)\left[(-)^q P_{23}\tilde{\psi}_q^{(\nu(1-u_0))}(1 - u_0)\right] \tag{9.3a}$$

$$\left[(-)^q \psi_{q(\nu(1-u_0))}(1 - u_0)P_{23}\right]W_{qp}(u_0) = E_{\nu(1-u_0)}(1 - u_0)\left[(-)^p \psi_p^{(\nu(1-u_0))}(1 - u_0)P_{23}\right] \tag{9.3b}$$

by using the crossing symmetry (4.8) of the connections. Comparing (9.3) to the eigenvalue problem (9.1) at $u_0$, we learn first that the set of all eigenvalues is closed under $u_0 \rightarrow 1 - u_0$

$$\{E_{\nu(1-u_0)}(1 - u_0)\} = \{E_{\nu(u_0)}(u_0)\} \tag{9.4}$$

and we also learn that the connection eigenvectors enjoy the crossing symmetry

$$(-)^p P_{23}\tilde{\psi}_p^{(\nu(1-u_0))}(1 - u_0) = \sum_{(\mu|E_{\mu(u_0)}(u_0)=E_{\nu(1-u_0)}(1-u_0))} \tilde{\psi}_p^{(\mu(u_0))}(u_0) X_{\mu(u_0)}^{\nu(1-u_0)} \tag{9.5a}$$

$$(-)^p \psi_p^{(\nu(1-u_0))}(1 - u_0)P_{23} = \sum_{(\mu|E_{\mu(u_0)}(u_0)=E_{\nu(1-u_0)}(1-u_0))} X_{\nu(1-u_0)}^{\mu(u_0)} \psi_p^{(\mu(u_0))}(u_0) \tag{9.5b}$$

The sums are over the $u_0$-eigenvectors with eigenvalue in the degenerate subspace labelled by $E_{\nu(1-u_0)}(1 - u_0)$, and $X$, $X$ are braiding matrices, to be determined.

The crossing symmetry (9.5) translates directly to the braiding of the conformal eigenvectors

$$P_{23}\tilde{\psi}_p^{(\nu(1-u_0))}(1 - u, 1 - u_0) = \sum_{(\mu|E_{\mu(u_0)}(u_0)=E_{\nu(1-u_0)}(1-u_0))} \tilde{\psi}_p^{(\mu(u_0))}(u, u_0) X_{\mu(u_0)}^{\nu(1-u_0)} \tag{9.6a}$$

$$\psi_p^{(\nu(1-u_0))}(1 - u, 1 - u_0)P_{23} = \sum_{(\mu|E_{\mu(u_0)}(u_0)=E_{\nu(1-u_0)}(1-u_0))} X_{\nu(1-u_0)}^{\mu(u_0)} \psi_p^{(\mu(u_0))}(u, u_0) \tag{9.6b}$$
according to their definition in (8.2b,c). The braiding (9.6) of the conformal eigenvectors, and the origin of the braiding in an eigenvalue problem, are among the central results of this paper. It remains to study this braiding at the level of conformal blocks, but, because the solution correctly includes the correlators of the coset constructions, there can be little doubt that (9.6) includes and generalizes the braiding of rational conformal field theory.

10 The High-Level Correlators of Irrational CFT

Beyond the coset constructions, it is unlikely that a closed form solution can be obtained for the connection eigenvalue problem (8.1). On the other hand, the problem is tractable by high-level expansion [12,14], which takes the form of a degenerate perturbation theory.

For the expansion, we restrict ourselves to a fixed choice of external representations $T$ on simple $g$, with conformal weights $\Delta(T) = O(k^{-1})$ at high level. Then, the invariant connections exhibit the form in (4.12b),

$$ W_{qp} = W_{q0}W_{0p} + V , \quad V = O(k^{-2}) \quad (10.1) $$

which defines a Hamiltonian perturbation theory with leading-order Hamiltonian $W_{q0}W_{0p}$ and perturbing potential $V$. The result at $V = 0$ is exact to all orders for $\tilde{L} = L_{g/h}$ and $L = L_h$, but, beyond the coset constructions, the form of $V$ will generally violate the factorized form $W_{q0}W_{0p}$ of the leading-order Hamiltonian.

The eigenvalue problem of the leading-order Hamiltonian

$$ W_{q0}(u_0)_\alpha^\gamma [\sum_p W_{0p}(u_0)_\gamma^\beta \tilde{\psi}_{p\beta}^{(\nu)}(u_0)] = E_\nu(u_0)\tilde{\psi}_{q\alpha}^{(\nu)}(u_0) \quad (10.2a) $$

and

$$ [\sum_q \psi_{q(\nu)}^{\beta}(u_0)]W_{q0}(u_0)_\beta^\gamma W_{0p}(u_0)_\gamma^\alpha = E_\nu(u_0)\psi_{q(\nu)}^{\alpha}(u_0) \quad (10.2b) $$

is itself non-trivial, but, according to (8.2a), we need only those eigenvectors with $E_\nu \neq 0$, which are easily characterized by solving (10.2) for the eigenvectors on the right. It follows that all the eigenvectors with $E_\nu \neq 0$ have the form

$$ \tilde{\psi}_{q\alpha}^{(\nu)}(u_0) = W_{q0}(u_0)_\alpha^\beta \tilde{\phi}_{\beta}^{(\nu)}(u_0) , \quad \psi_{p(\nu)}^{\alpha}(u_0) = \phi_{(\nu)}^{\beta}(u_0)W_{0p}(u_0)_\beta^\alpha , \quad E_\nu \neq 0 \quad (10.3) $$
where $\bar{\phi}$ and $\phi$ are the (non-zero eigenvalue) eigenvectors of the reduced eigenvalue problem

$$M(u_0)_{\alpha}^{\beta}\tilde{\phi}_{\beta}^{(\nu)}(u_0) = E_{\nu}(u_0)\tilde{\phi}_{\alpha}^{(\nu)}(u_0), \quad \phi_{(\nu)}^{\beta}(u_0)M(u_0)_{\beta}^{\alpha} = E_{\nu}(u_0)\phi_{(\nu)}^{\alpha}(u_0)$$

(10.4a)

$$M(u_0)_{\alpha}^{\beta} \equiv \sum_{p} (W_{0p}(u_0)W_{p0}(u_0))_{\alpha}^{\beta}$$

(10.4b)

which is defined on the space $T^1 \otimes T^2 \otimes T^3 \otimes T^4$ of the representation matrices.

This result establishes the remarkable fact that only a finite number of conformal structures

$$Y_{\nu}(u, u_0) = \sqrt{E_{\nu}(u_0)}Y_{g,h}^\alpha(u_0)\tilde{\phi}_{\alpha}^{(\nu)}(u, u_0)$$

$$= \sqrt{E_{\nu}(u_0)}Y_{g,h}^\alpha(u_0) \sum_{q=0}^{\infty} \frac{(u - u_0)^q}{q!} W_{q0}(u_0)_{\alpha}^{\beta}\tilde{\phi}_{\beta}^{(\nu)}(u_0) + O(k^{-2})$$

(10.5a)

$$Y_{\nu}^\alpha(u, u_0) = \sqrt{E_{\nu}(u_0)}\phi_{(\nu)}^\alpha(u, u_0)$$

$$= \sqrt{E_{\nu}(u_0)}\phi_{(\nu)}^\alpha(u_0) \sum_{p=0}^{\infty} W_{0p}(u_0)_{\alpha}^{\beta} \frac{(u - u_0)^p}{p!} + O(k^{-2})$$

(10.5b)

$$E_{\nu} \neq 0, \quad \nu = 0, 1, \ldots, D(T) - 1, \quad D(T) \leq \prod_{i=1}^{4} \dim T^i$$

(10.5c)

contribute to the affine-Virasoro correlators at leading order. The counting is a consequence of the factorized form $W_{q0}W_{0p}$ of the leading-order Hamiltonian, a property which will be lost at higher order for the generic construction. In this case, higher-order perturbation theory will generate more non-zero eigenvalue eigenvectors (from the infinite subspace $E_{\nu} = 0$ in (10.2)), leading eventually to an infinite number of contributing conformal structures for the generic affine-Virasoro correlator.

On the other hand, the form (10.1) at $V = 0$, and hence the solution (10.5), is exact to all orders for $\tilde{L} = L_{g/h}$ and $L = L_h$. With (8.5), this gives

$$Y_{\nu}^{g,h}(u, u_0) = Y_{g,h}^\alpha(u, u_0)\sqrt{E_{\nu}(u_0)}\tilde{\phi}_{\alpha}^{(\nu)}(u_0)$$

$$\nu = 0, 1, \ldots, D(T) - 1$$

(10.6)

for the coset constructions, in agreement with (8.6).
More generally, we may use the explicit form of the high-level connections in (3.11), (4.9) and (4.11), for example,

\[ W_{q0}(u) = (-)^{q-1}(q-1)! \frac{\tilde{P}_{ab}}{k} (T^1_a T^2_b u^{-q} + T^1_a T^3_b (u - 1)^{-q}) + O(k^{-2}) \quad , \quad q \geq 1 \]  

(10.7)
to sum the series in (10.5). The result summarizes the high-level affine-Virasoro correlators

\[ Y_{L}(u, u_0) = Y_{\alpha}^\beta(u, u_0) \sqrt{E_{\nu}(u_0)} \tilde{\phi}_{(\nu)}(u_0) + O(k^{-2}) \]  

(10.8a)

\[ Y_{L}^\alpha(u, u_0) = Y_{\alpha}^\beta(u_0)(1 + \frac{\tilde{P}_{ab}}{k} \left[ T^1_a T^2_b \ln \left( \frac{u}{u_0} \right) + T^1_a T^3_b \ln \left( \frac{1 - u}{1 - u_0} \right) \right] )_{\beta}^{\alpha} + O(k^{-2}) \]  

(10.8b)

\[ \tilde{L}^{ab} = \frac{\tilde{P}_{ab}}{2k} + O(k^{-2}) \]  

(10.8c)

for all affine-Virasoro constructions \( \tilde{L} \) on simple \( g \). The form (10.8) is one of the central results of this paper.

As seen above for the coset constructions, the high-level conformal structures (10.8) are degenerate in that all the \( u \) dependence of each structure is in the same factor \( Y_{L}^\alpha(u, u_0) \). We remark that the factor \( Y_{L}^\alpha(u, u_0) \) is the \( n = 4 \) invariant form of the high-level \( n \)-point correlators conjectured for all affine-Virasoro constructions in eq.(14.3) of Ref.[14]. Since the coset correlators (10.6) are correctly included in the exact solution, the form (10.8) with \( \tilde{P} = \tilde{P}_{g/h} = P_g - P_h \) is correct for all high-level coset constructions. More generally, we will argue below that the form shows good physical behavior for all the constructions.

More precisely, we will find that the factor \( Y_{L}^\alpha(u, u_0) \) shows the correct \( \tilde{L}^{ab} \)-broken conformal weights, and hence the correct singularities, for an affine-Virasoro correlator of four broken affine primary fields. To see this, we first expand \( Y_{g}(u_0) \), \( Y_{g}(u_0) \sum_{i=1}^{4} T^i_a = 0 \) in a basis of invariant tensors \( v^\alpha_4 \) of \( T^1 \otimes T^2 \otimes T^3 \otimes T^4 \),

\[ Y_{g}^\alpha(u_0) = \sum_{r, \xi, \xi'} F_g(r, \xi, \xi'; u_0) v^{\alpha}_4(r, \xi, \xi') \quad , \quad v^{\beta}_4(r, \xi, \xi') \sum_{i=1}^{4} (T^i_a)_{\beta}^{\alpha} = 0 \]  

(10.9)
The coefficients \( F_g \) are related to the affine-Sugawara conformal blocks at \( u = u_0 \), whose precise form is not relevant in the present discussion. One choice for \( v^\alpha_4 \)
is the s-channel basis

\[ v_4^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(r, \xi, \xi') = \sum_{\alpha_r \alpha_f} v_3^{\alpha_1 \alpha_2 \alpha_f}(\xi) v_3^{\alpha_3 \alpha_4 \alpha_r}(\xi') \eta_{\alpha_r \alpha_f} \]  
(10.10a)

\[ v_3^{\alpha \alpha_j \alpha_f}(\xi) = \sum_{\alpha_r} \left( \frac{\alpha_i}{i} \frac{\alpha_j}{j} \right) r(\xi) \eta^{\alpha_r \alpha_f} \]  
(10.10b)

\[ v_3^{\beta i \beta j \beta_r}(\xi)(T^i_a + T^j_a + T^r_a)_{\beta i \beta j \beta_r}^{\alpha_1 \alpha_2 \alpha_r} = 0 \]  
(10.10c)

where \( v_3^{\alpha}(\xi) \) are the invariant tensors of \( T^i \otimes T^j \otimes T^r \), \( T^r \) an irreducible representation of \( g \). In (10.10b), these tensors are given in terms of the inverse metric \( \eta^{\alpha_r \alpha_f} \) on the carrier space of \( T^r \) and the Clebsch-Gordan coefficients (\( \cdots \)) for the decomposition \( T^i \otimes T^j = \oplus_r T^r \). The \( \xi \) label in \( v_3^{\alpha} \) is needed when a representation \( T^r \) appears more than once in the decomposition.

Physically, the argument \( r \) in \( v_4^a(r, \xi, \xi') \) labels the irreps \( T^r \) of \( g \) which appear in the \( s \) channel \( (u \to 0) \) of the four-point correlators, while \( \xi, \xi' \) distinguish the different couplings of the various copies of \( T^r \). The basis (10.10) was obtained by studying Haar integration over four representations of \( g \), and corresponding \( t \)- and \( u \)-channel bases are obtained by permutations of \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \) in (10.10a).

Using (3.1) and (10.10c), we verify the exact relation

\[ v_3^{\beta i \beta j \beta_r}(\xi)[2L^{ab} T^1_a T^2_b]_{\beta i \beta j \beta_r}^{\alpha_1 \alpha_2 \alpha_2} = v_3^{\alpha_1 \alpha_2 \alpha_f}(\xi)(\tilde{\Delta}_{\alpha_r}(T^r) - \tilde{\Delta}_{\alpha_1}(T^1) - \tilde{\Delta}_{\alpha_2}(T^2)) \]  
(10.11)

where \( \{\tilde{\Delta}_{\alpha}(T)\} = \text{diag}(\tilde{L}^{ab} T_a T_b) \) are the conformal weights of the broken affine-primary states corresponding to the external representations \( T^1, T^2 \) and the \( s \) channel representation \( T^r \).

Collecting these results, we find that

\[ Y_L^\alpha(u, u_0) \simeq \sum_{r, \xi, \xi', \alpha_r \alpha_f} F_g(r, \xi, \xi', u_0) v_3^{\alpha_1 \alpha_2 \alpha_f}(\xi) \left( \frac{u}{u_0} \right)^{\Delta_{\alpha_r} - \Delta_{\alpha_1} - \Delta_{\alpha_2}} v_3^{\alpha_3 \alpha_4 \alpha_r}(\xi') \eta_{\alpha_r \alpha_f} + \mathcal{O}(k^{-2}) \]  
(10.12)

which shows the correct conformal weight factor \((u/u_0)^{\Delta_{\alpha_r} - \Delta_{\alpha_1} - \Delta_{\alpha_2}}\) for broken affine primaries in the \( s \) channel. A similar analysis in the \( t \) channel \((u \to 1)\) shows the expected factor \((1 - u)/(1 - u_0))^{\Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3}}\), where \( t \) labels the broken affine primaries in the \( t \) channel.
From (10.12), we may also read the high-level fusion rules of the broken affine modules: In rational and irrational conformal field theory, these rules follow the Clebsch-Gordan coefficients (10.10b) of the representations. We remind the reader that the coefficients are computed in the simultaneous $L$-basis of the representations (see Section 3), where all the conformal weight matrices are diagonal. The next step is to study the braiding of the high-level conformal blocks of (10.8).

11 Conclusions

The affine-Virasoro Ward identities [14] are a system of non-linear differential equations which describe the correlators of all affine-Virasoro constructions, including rational and irrational conformal field theory. In Ref.[14], we solved the Ward identities for the coset constructions, providing a derivation of the coset blocks of Douglas [15]. In this paper, we solved for the conformal correlators of the affine-Sugawara nests, and showed that global solutions exist across all affine-Virasoro space, so long as a generically-infinite number of conformal structures is allowed. This is in agreement with intuitive notions about irrational conformal field theory.

We focused on a particular global solution which is based on a natural eigenvalue problem in the system. This solution reproduces the correct coset and nest correlators and exhibits a braiding for all affine-Virasoro correlators which includes and generalizes the braiding of rational conformal field theory. The underlying mechanism of the braiding is the linearity of the eigenvalue problem.

The solution also shows good physical behavior, at least at high level on simple $g$, where we are able to see the high-level correlators and high-level fusion rules of irrational conformal field theory.

In this first look at the correlators of irrational conformal field theory, we have raised as many questions as we have answered. In particular, further work is necessary to be certain that our particular solution is globally physical at higher order and/or finite level.

We are least satisfied in our understanding of the multiplicity of solutions.
to the system, which is associated to various factorization ansätze. In spite of appearances, we have seen some evidence that these solutions are related to each other, sometimes via irrelevant constants and sometimes via a change of basis, as noted in Section 7. Alternately, it is possible that we lack a boundary condition on the system, whose nature could be central in finding the correct solution.

To supplement future discussion of these questions, we have included in Appendix D the results of another natural factorization, in the symmetric ansatz. Although the assignment of the Lie algebra indices is quite different in this solution, it gives the same correct coset and nest correlators, and the same high-level correlators for all affine-Virasoro constructions.

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Appendix A: Second-order connections

For use in the text, we give the known forms of the second-order \((q+p=2)\) affine-Virasoro connections [14]. For the \(n\)-point connections, we have

\[
W_{0,ij} = \partial_i W_{0,j} + \frac{1}{2}(W_{0,i}, W_{0,j})_+ + E_{0,ij}, \quad W_{ij,0} = \partial_i W_{j,0} + \frac{1}{2}(W_{i,0}, W_{j,0})_+ + E_{ij,0}
\]

\[
W_{i,j} = W_{i,0} W_{0,j} + E_{i,j}
\]  

(A.1a)  

(A.1b)
\[ E_{i,j} = -2iL^d a L^c(b_f d e c) \left\{ \frac{T_c^j T_a^i}{z_{ij}^2} + \frac{T_c^j T_b^i}{z_{ij}^2} \right\} - 2 \sum_{k \neq i, j} \frac{T_c^k T_b^i T_a^j}{z_{ij} z_{ik}} \right\}, \quad i \neq j \quad (A.1c) \]

\[ E_{0,ij} = -\frac{1}{2}(E_{i,j} + E_{j,i}) \quad E_{i,0} = E_{0,ij}E_{i,i} = -\sum_{j \neq i} E_{i,j} \quad (A.1d) \]

and the corresponding invariant second-order connections are

\[ W_{02} = \partial W_{01} + W_{01}^2 + E_{02}, \quad W_{20} = \partial W_{10} + W_{10}^2 + E_{20} \quad (A.2a) \]

\[ W_{11} = W_{10}W_{01} - E_{02} = W_{01}W_{10} - E_{20} \quad (A.2b) \]

\[ E_{02} = -2iL^d a L^c(b_f d e c)V_{abc}, \quad E_{20} = E_{02} \big|_{L \rightarrow \tilde{L}} \quad (A.2c) \]

\[ V_{abc} = \frac{1}{u^2}[T_a^1 T_b^2 T_c^2 + T_b^2 T_a^1 T_c^2] + \frac{1}{(u-1)^2} \left[ T_a^1 T_b^3 T_c^3 + T_b^3 T_a^1 T_c^3 \right] + \frac{2}{u(u-1)} T_a^1 T_b^2 T_c^3. \quad (A.2d) \]

More generally, the invariant one-sided connections \( W_{0p} \) can be obtained from the four-point connections by iterating the general \( SL(2, \mathbb{IR}) \times SL(2, \mathbb{IR}) \) relation

\[ W_{0p}(u) = \frac{1}{f_{p,p}(z)} \left( W_{0,1...1}(z) + (-)^{p+1} \frac{\Gamma(2\Delta_1 + p)}{\Gamma(2\Delta_1)} \frac{1}{z_{14}^p} - \sum_{s=1}^{p-1} f_{p,s}(z)W_{0s}(u) \right) \]

\[ - \sum_{r=1}^{p-1} \binom{p}{r} (-1)^r \frac{\Gamma(2\Delta_1 + r)}{\Gamma(2\Delta_1)} \frac{1}{z_{14}^r} \sum_{s=0}^{p-r} f_{p-r,s}(z)W_{0s}(u) \right) \]

\[ \partial_i^p = \left( \frac{\partial u}{\partial z_{1i}} \right)^p = \sum_{s=1}^{p} f_{p,s}(z)\partial_u^s \quad (A.3a) \]

and using the global Ward identity to eliminate the fourth representation \( T^4 \).

Then, the mixed invariant connections \( W_{qp} \) can be obtained from \( W_{0p} \) by using the solution (4.1c) of the invariant consistency relations.

**Appendix B: General A-S nest connections**

The biconformal correlators (5.19) and (5.20) of the affine-Sugawara (A-S) nests were obtained by continuing the iteration schematized in (5.18). In this appendix, we use the general principles of the iteration to prove that these results are correct.
By differentiation of the invariant biconformal correlators (5.19) and comparison with (3.10), we obtain the invariant connections of the general nest

\[ W_{qp}[\tilde{L} = L_{g/h_1/.../h_{2n+1}}, \tilde{L} = L_{h_1/.../h_{2n+1}}] \]

\[ = Y_g^{-1} \left\{ \partial_q \partial_p [Y_{g/h_1}(\bar{u})Y_{h_1/h_2}(u)Y_{h_2/h_3}(\bar{u}) \cdots Y_{h_{2n+1}}(\bar{u})Y_{h_{2n+1}}(u)] \right\} |_{\bar{u}=u} \]

\[ = Y_g^{-1} \sum_{j_1=0}^{q} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{n-1}=0}^{j_{n-2}} \sum_{i_1=0}^{j_{n-2}} \cdots \sum_{i_{n-1}=0}^{j_{n-2}} \left( q \right) \left( j_1 \right) \cdots \left( j_{n-1} \right) \frac{n}{k=2} \left( j_{k-1} \right) \left( i_{k-1} \right) \]

\[ \times \partial^{q-j_1}Y_{g/h_1} \partial^{p-i_1}Y_{h_1/h_2} \partial^{j_1-j_2}Y_{h_2/h_3} \partial^{i_1-i_2}Y_{h_3/h_4} \cdots \]

\[ \times \partial^{j_{n-1}-j_n}Y_{h_{2n-2}/h_{2n-1}} \partial^{i_{n-1}-i_n}Y_{h_{2n-1}/h_{2n}} \partial^{j_n}Y_{h_{2n}/h_{2n+1}} \partial^{i_n}Y_{h_{2n+1}} \quad (B.1a) \]

\[ W_{qp}[\tilde{L} = L_{g/h_1/.../h_{2n}}, \tilde{L} = L_{h_1/.../h_{2n}}] \]

\[ = Y_g^{-1} \left\{ \partial_q \partial_p [Y_{g/h_1}(\bar{u})Y_{h_1/h_2}(u)Y_{h_2/h_3}(\bar{u}) \cdots Y_{h_{2n-1}/h_{2n}}(u)Y_{h_{2n}}(\bar{u})] \right\} |_{\bar{u}=u} \]

\[ = Y_g^{-1} \sum_{j_1=0}^{q} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{n-1}=0}^{j_{n-2}} \sum_{i_1=0}^{j_{n-2}} \cdots \sum_{i_{n-1}=0}^{j_{n-2}} \left( q \right) \left( j_1 \right) \cdots \left( j_{n-1} \right) \frac{n}{k=2} \left( j_{k-1} \right) \left( i_{k-1} \right) \]

\[ \times \partial^{q-j_1}Y_{g/h_1} \partial^{p-i_1}Y_{h_1/h_2} \partial^{j_1-j_2}Y_{h_2/h_3} \partial^{i_1-i_2}Y_{h_3/h_4} \cdots \]

\[ \times \partial^{j_{n-1}-j_n}Y_{h_{2n-2}/h_{2n-1}} \partial^{i_{n-1}-i_n}Y_{h_{2n-1}/h_{2n}} \partial^{j_n}Y_{h_{2n}/h_{2n+1}} \partial^{i_n}Y_{h_{2n+1}} \quad (B.1b) \]

These connections are guaranteed to satisfy the consistency relations (3.12b), which are the integrability conditions for the existence of the biconformal correlators.

Then, we need only check the embedding relations

\[ W_{0p}(L = L_{h_1/.../h_{2n+1}}) = W_{00}(\tilde{L} = L_{h_1/.../h_{2n+1}}) \quad (B.2a) \]

\[ W_{0p}(L = L_{h_1/.../h_{2n}}) = W_{00}(\tilde{L} = L_{h_1/.../h_{2n}}) \quad (B.2b) \]

which are verified from (B.1) by choosing first \( q = 0 \) and then \( p = 0 \), followed by the appropriate renaming of groups.

As another check on these results, note that the form (B.1b) for the even nests can be obtained from the form (B.1a) for the odd nests by setting \( h_{2n+1} = 0 \) and \( Y_{h_{2n+1}} = 1 \).
An alternate expression for the invariant nest connections

\[ W_{qp}[\bar{L} = L_{g/h_1/\ldots/h_{2n+1}}, L = L_{h_1/\ldots/h_{2n+1}}] = \]

\[ \sum_{j_1=0}^{q} \sum_{j_2=0}^{f_1} \cdots \sum_{j_{n-1}=0}^{f_{n-1}} \sum_{p} \sum_{i_1=0}^{\nu_1} \cdots \sum_{i_{n-1}=0}^{\nu_{n-1}} \binom{q}{j_1} \binom{p}{i_1} \prod_{k=2}^{n} \binom{j_{k-1}}{i_{k-1}} \]

\[ \times W_{0,j_1-1}^{h_1/h_2} W_{0,j_1-2}^{h_2/h_3} \cdots W_{0,j_{n-1}-1}^{h_{2n-2}/h_{2n-1}} W_{0,j_{n-1}-2}^{h_{2n-1}/h_{2n}} W_{0,j_{n-1}-1}^{h_{2n}/h_{2n+1}} W_{0,j_{n-1}}^{h_{2n+1}/h_{2n+2}} \]

(B.3)

follows with (5.8), and the corresponding result for even nests is obtained from (B.3) with \( h_{2n+1} = 0, Y_{h_{2n+1}} = 1 \) and \( W_{0i}^{h_{2n+1}} = \delta_{in,0} \). This form of the connections shows that they are independent of the reference point \( u_0 \) of the evolution operators, and it is also the most convenient form to check against the known forms of the first and second-order connections in (3.11) and (A.2). After some algebra, we find that they are in complete agreement.

We turn now to the general \( n \)-point biconformal nest correlators in (5.20). To check this result, we need the general \( n \)-point nest connections

\[ W_{j_1\ldots j_q,i_1\ldots i_p}[\bar{L} = L_{g/h_1/\ldots/h_{2n+1}}, L = L_{h_1/\ldots/h_{2n+1}}] = \]

\[ A_g^{-1} \sum_{p(j_1\ldots j_q)} \frac{1}{q!} \sum_{p(i_1\ldots i_p)} \frac{1}{p!} \sum_{l_1=0}^{l_1} \sum_{l_2=0}^{l_2} \cdots \sum_{l_{n-1}=0}^{l_{n-1}} \sum_{k_1=0}^{k_1} \sum_{k_2=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_{n-1}} \]

\[ \times \binom{q}{l_1} \binom{p}{k_1} \prod_{r=2}^{n} \binom{l_r-1}{k_r-1} \]

\[ \times \left( \prod_{\mu_1=1}^{q-l_1} \partial_{j_{\mu_1}} A_{h_2/h_3} \right) \left( \prod_{\nu_1=1}^{p-k_1} \partial_{i_{\nu_1}} A_{h_{2n-2}/h_{2n-1}} \right) \left( \prod_{\mu_{n-1}+1}^{q-l_{n-1}} \partial_{j_{\mu_{n-1}+1}} A_{h_{2n}/h_{2n+1}} \right) \]

(B.4a)
be obtained from (B.4) by setting the integrability conditions for the biconformal correlators. Finally, the correct evolution operators, and satisfy the consistency relations (3.12a), which are obtained by differentiation of (5.20) and comparison with (3.5). Here, the connections of $\frac{g}{h}$, $0, i = \cdots i_p$ are verified from (B.4) and the corresponding form for the even nests, which are $0, j = \cdots j_q$, $h$. The general nest connections are independent of the reference point $z_0$ of the evolution operators, and satisfy the consistency relations (3.12a), which are the integrability conditions for the biconformal correlators. Finally, the correct embedding relations

$$W_{0,i_1 \cdots i_p}(L = L_{h_1/\ldots/h_{2n+1}}) = W_{i_1 \cdots i_p,0}(\bar{L} = L_{h_1/\ldots/h_{2n+1}}) \quad (B.5a)$$

$$W_{0,i_1 \cdots i_p}(L = L_{h_1/\ldots/h_{2n}}) = W_{i_1 \cdots i_p,0}(\bar{L} = L_{h_1/\ldots/h_{2n}}) \quad (B.5b)$$

are verified from (B.4) and the corresponding form for the even nests, which completes the check of (5.20). We have also checked (B.4) against the known forms of the first and second-order connections in (3.8) and (A.1).

Appendix C: Conformal blocks of the A-S nests

Following the change of basis given for the coset correlators in Ref.[14], we discuss the affine-Sugawara (A-S) nests at the level of conformal blocks.

For four representations $T^i_a$, $i = 1, \ldots, 4$ of $g$ and the subgroup sequence $g \supset h_1 \supset \ldots \supset h_n$, we introduce the $h_j$-invariant tensors $\varphi_{m_j}^\alpha(h_j)$,

$$\varphi_{m_j}^\beta(h_j) \sum_{i=1}^4 (T^i_a)_{\beta}^\alpha = 0 \quad , \quad a \in h_j \quad (C.1a)$$

$$\{\varphi_{m_j}^\alpha(h_j)\} \subset \{\varphi_{m_j}^\alpha(h_{j+1})\} \quad , \quad j = 0, 1, \ldots, n - 1 \quad (C.1b)$$
Here we have introduced $h_0 \equiv g$ for uniformity and the tensors are chosen to satisfy $\nu^\alpha_{m_j}(h_j) = \nu^\alpha_{m_j}(h_{j+1})$, $\{m_{j}\} \subset \{m_{j+1}\}$. Using global Ward identities, we may then expand the operators in (5.19) as [14]

$$Y_{h_0/h_1}(u, u_0) = Y_{h_0/h_1}^{m_1}(u, u_0)\nu^\alpha_{m_1}(h_1) = d^\alpha C_{h_0/h_1}(u) r_j F_{h_1}(u) r_1 m_1 \nu^\alpha_{m_1}(h_1) \quad \text{(C.2a)}$$

$$Y_{h_j}(u, u_0) m_j^\alpha \equiv \nu^\beta_{m_j}(h_j) Y_{h_j}(u, u_0) \nu^\alpha_{m_j}(h_j) = F_{h_j}^{-1}(u_0) m_j^j F_{h_j}(u) r_j \nu^\alpha_{m_j}(h_j) \quad \text{(C.2b)}$$

$$\nu^\beta_{m_j}(h_j) Y_{h_j/h_{j+1}}(u, u_0) \nu^\alpha_{m_j}(h_{j+1}) \equiv Y_{h_j/h_{j+1}}(u, u_0) m_j^{m_j+1} \nu^\alpha_{m_{j+1}}(h_{j+1}) \quad \text{(C.2c)}$$

$$Y_{h_j/h_{j+1}}(u, u_0) m_j^{m_j+1} = F_{h_j}^{-1}(u_0) m_j^j C_{h_j/h_{j+1}}(u) r_j^{r_j+1} F_{h_{j+1}}^{-1}(u_0) r_{j+1}^{m_j+1} \quad \text{(C.2d)}$$

$$C_{h_j/h_{j+1}}(u) r_j^{r_j+1} = F_{h_j}(u) r_j^{m_j} F_{h_{j+1}}^{-1}(u) m_j^{r_j+1} \quad \text{(C.2e)}$$

Here, $d^\alpha$ are constants, $F_{h_j}$ are the conformal blocks of $h_j$, chosen so that the left indices $r_j$ label the blocks by $h_j$ representations in the s channel ($u \to 0$), and $C$ are the coset blocks [15,14].

Using (C.2), the biconformal correlators of the nests (5.19) factorize as follows,

$$Y^\alpha(\bar{u}, u)[\bar{L} = L_{g/h_1/.../h_{2n+1}}, L = L_{h_1/.../h_{2n+1}}]$$

$$= Y_{g/h_1}^{m_1}(\bar{u}, u_0) Y_{h_1/h_2}(u, u_0) m_2 \cdots Y_{h_{2n-1}/h_{2n}}(\bar{u}, u_0) m_{2n+1} Y_{h_{2n+1}}(u, u_0) \quad \text{(C.3a)}$$

$$= Y^\nu(\bar{u}, u_0)[\bar{L} = L_{g/h_1/.../h_{2n+1}}] Y^\alpha(u, u_0)[L = L_{h_1/.../h_{2n+1}}], \quad \nu \equiv (m_1, \ldots, m_{2n+1})$$

$$Y^\alpha(\bar{u}, u)[\bar{L} = L_{g/h_1/.../h_{2n}}, L = L_{h_1/.../h_{2n}}]$$

$$= Y_{g/h_1}^{m_1}(\bar{u}, u_0) Y_{h_1/h_2}(u, u_0) m_2 \cdots Y_{h_{2n-1}/h_{2n}}(u, u_0) m_{2n+1} Y_{h_{2n}}(\bar{u}, u_0) \quad \text{(C.3b)}$$

$$= Y^\mu(\bar{u}, u_0)[\bar{L} = L_{g/h_1/.../h_{2n}}] Y^\mu(u, u_0)[L = L_{h_1/.../h_{2n}}], \quad \mu \equiv (m_1, \ldots, m_{2n})$$

where the factorized correlators of the $\bar{L}$ theories are

$$Y^\nu(\bar{u}, u_0)[\bar{L} = L_{g/h_1/.../h_{2n+1}}]$$

$$= Y_{g/h_1}^{m_1}(\bar{u}, u_0) \prod_{j=1}^n Y_{h_{2j}/h_{2j+1}}(\bar{u}, u_0)m_{2j} \quad \text{(C.4a)}$$

$$= f^\nu(\bar{u}, u_0) \prod_{j=1}^n N_{g/h_1/.../h_{2n+1}}(\bar{u}) \quad \text{(C.4b)}$$
\[ \tilde{Y}_\mu^\alpha(\bar{u}, u_0) [\tilde{L} = L_{g/h_1/\ldots/h_{2n}}] \]
\[ = Y_{g/h_1}^m(\bar{u}, u_0) \prod_{j=1}^{n-1} Y_{h_{2j}/h_{2j+1}}(\bar{u}, u_0)m_{2j} Y_{h_{2n}}(\bar{u}, u_0)m_{2n} \alpha \quad (C.4c) \]
\[ = f_\alpha^\mu(u_0)^{r_0 r_2 \ldots r_{2n-2} r_{2n}} \mathcal{N}_{g/h_1/\ldots/h_{2n}}(\bar{u})^{r_1 r_3 \ldots r_{2n-1} m_{2n}^2} \quad (C.4d) \]

The constant factors \( f(u_0) \) in (C.4) are
\[ f_\nu(u_0)^{r_0 r_2 \ldots r_{2n}} = d^{r_0} \prod_{j=1}^{n} F_{h_{2j}}^{-1}(u_0)m_{2j} \prod_{k=0}^{r_2} F_{h_{2k+1}}(u_0)m_{2k+1} \quad (C.5a) \]
\[ f_\mu(u_0)^{r_0 r_2 \ldots r_{2n-2} r_{2n}} = d^{r_0} \prod_{j=0}^{n-1} F_{h_{2j+1}}^{-1}(u_0)m_{2j} r_{2j+1} m_{2j+1}^\alpha \quad (C.5b) \]

and the nest blocks \( \mathcal{N}_{g/h_1/\ldots/h_{2n}} \) of the conformal field theories \( L_{g/h_1/\ldots/h_{2n}} \) are given by
\[ \mathcal{N}_{g/h_1/\ldots/h_{2n+1}}(\bar{u})^{r_1 r_3 \ldots r_{2n+1}} = \prod_{j=0}^{n} C_{h_{2j}/h_{2j+1}}(\bar{u}) r_{2j} \quad , \quad h_0 = g \quad (C.6a) \]
\[ \mathcal{N}_{g/h_1/\ldots/h_{2n}}(\bar{u})^{r_1 r_3 \ldots r_{2n-1} m_{2n}} = \prod_{j=0}^{n-1} C_{h_{2j}/h_{2j+1}}(\bar{u}) r_{2j+1} \quad (C.6b) \]

Each of the nest blocks \( \mathcal{N} \) is a tensor product of blocks, as expected.

As a consistency check on the factorization, it is not difficult to check that the correlators \( Y(L) \) in (C.3) give the blocks \( \mathcal{N}_{h_1/\ldots/h_{2n+1}} \) and \( \mathcal{N}_{h_1/\ldots/h_{2n}} \), obtained from (C.6) by renaming groups. Note also that eqs. (C.4a,c) are the explicit forms of the tensor products (5.21).

As a concrete example on \( g \supset h_1 \supset \ldots \supset h_n \), we choose the subgroup nest
\[ g_x = (h_0)_x = SU(N)_{x_0} \times SU(N)_{x_1} \times \ldots \times SU(N)_{x_n} \quad (C.7a) \]
\[ h_j = SU(N)_{y_j} \times_{i=j+1}^{n} SU(N)_{x_i} \quad , \quad y_j \equiv \sum_{k=0}^{j} x_k \quad (C.7b) \]

and the integrable representations of \( g \) as \( \mathcal{T}^1 = \mathcal{T}^4 = (\mathcal{T}(N), 0, \ldots, 0), \mathcal{T}^2 = \mathcal{T}^3 = (\mathcal{T}(N), 0, \ldots, 0) \) in the \( N \) and \( \bar{N} \) of \( SU(N)_{x_0} \). Then (C.6) gives the nest blocks
\[ (\mathcal{N}_{g/h_1/\ldots/h_{2n}})^{r_1 r_3 \ldots r_{2n+1}} = \prod_{j=0}^{n} (C_{g_{y_2}y_{2j+1}}) r_{2j+1} \quad , \quad h_0 = g \quad (C.8a) \]

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\( (N_{g/h_1/\ldots/h_{2n}})^{r_1r_3 \ldots r_{2n-1}m_{2n}} = \prod_{j=0}^{n-1} (C_{y_{2j},y_{2j+1}})^{r_{2j}r_{2j+1}} \) \( \mathcal{F}_{y_{2n}}^{m_{2n}} \) \hfill (C.8b)

where the coset blocks \( C \) and the \( SU(N) \) blocks \( \mathcal{F} \) are 2x2 matrices,

\[
(C_{y_j,y_{j+1}})^{r_j} = (\mathcal{F}_{y_j})^{r_j} (\mathcal{F}_{y_{j+1}})^{-1} \] \( \forall \ j \) \hfill (C.9a)

\[
\mathcal{F}_y = \left( \begin{array}{cc} (\mathcal{F}_y)_V^1 & (\mathcal{F}_y)_V^2 \\ (\mathcal{F}_y)_A^1 & (\mathcal{F}_y)_A^2 \end{array} \right), \quad \mathcal{F}_y^{-1} = -\frac{1}{N} (u(1-u))^{4\Delta_y - \Delta_y^2} \left( \begin{array}{cc} (\mathcal{F}_y)_A^2 & -(\mathcal{F}_y)_V^2 \\ -(\mathcal{F}_y)_A^1 & (\mathcal{F}_y)_V^1 \end{array} \right) \] \hfill (C.9b)

\[
\Delta_y = \frac{N^2 - 1}{2N(y+N)}, \quad \Delta_y^2 = \frac{N}{y+N}, \quad \lambda_y = \frac{1}{y+N} \] \hfill (C.9c)

\[
(\mathcal{F}_y(u))_V^1 = u^{-2\Delta_y}(1-u)^{\Delta_y^2-2\Delta_y} F(\lambda_y, -\lambda_y, 1-N\lambda_y; u) \]

\[
(\mathcal{F}_y(u))_A^1 = u^{\Delta_y^2-2\Delta_y}(1-u)^{\Delta_y^2-2\Delta_y} F((N-1)\lambda_y, (N+1)\lambda_y, 1+N\lambda_y; u) \]

\[
(\mathcal{F}_y(u))_V^2 = \frac{1}{y} u^{-2\Delta_y+1}(1-u)^{\Delta_y^2-2\Delta_y} F(1+\lambda_y, 1-\lambda_y, 2-N\lambda_y; u) \]

\[
(\mathcal{F}_y(u))_A^2 = -Nu^{\Delta_y^2-2\Delta_y}(1-u)^{\Delta_y^2-2\Delta_y} F((N-1)\lambda_y, (N+1)\lambda_y, N\lambda_y; u) \] \hfill (C.9d)

Here \( V \) and \( A \) label the vacuum and adjoint blocks [7] in the \( u \rightarrow 0 \) channel and \( F \) is the hypergeometric function. In these examples, the number of nest blocks \( N_{g/h_1/\ldots/h_n} \) is \( 2^{n+1} \). It is interesting to note that, for a fixed choice of external representations, the number of nest blocks grows with the nest depth \( n \). In this sense, the nests may be considered as a prelude to irrational conformal field theory, where a generically-infinite number of blocks is expected.

We have made a spot check of the s-channel singularities of the nest blocks in (C.8), using the known behavior of the subgroup and coset blocks [14] as \( u \rightarrow 0 \). Approximately half of these intermediate states are immediately identifiable as broken affine primary states (with conformal weights \( \Delta = \sum_{j=0}^{n}(-1)^j \Delta_{h_j} \)) and the other states are presumably broken affine secondary.

Finally, the crossing-symmetric non-chiral correlators of these nests

\[
\mathcal{Y}_{g/h_1/\ldots/h_{2n+1}}(u,u^*) = \prod_{j=0}^{n} \mathcal{Y}_{h_{2j}/h_{2j+1}}(u,u^*) , \quad h_0 = g \] \hfill (C.10a)

\[
\mathcal{Y}_{g/h_1/\ldots/h_{2n}}(u,u^*) = \prod_{j=0}^{n-1} \mathcal{Y}_{h_{2j}/h_{2j+1}}(u,u^*) \mathcal{Y}_{h_{2n}}(u,u^*) \] \hfill (C.10b)
\[ \mathcal{Y}_{h_j}(u,u^*) = \mathcal{F}_{y_j}(u) V^1 \mathcal{F}_{y_j}(u^*) V^1 + \mathcal{F}_{y_j}(u) V^2 \mathcal{F}_{y_j}(u^*) V^2 \]
\[ + f(\lambda_{y_j})^{-1} [\mathcal{F}_{y_j}(u) V^1 \mathcal{F}_{y_j}(u^*) V^1 + \mathcal{F}_{y_j}(u) V^2 \mathcal{F}_{y_j}(u^*) V^2] \]
\[ \mathcal{Y}_{h_j/n_{j+1}}(u,u^*) = C_{y_j,y_j+1}(u) V^1 C_{y_j,y_j+1}(u^*) V^1 + f(\lambda_{y_j})^{-1} [C_{y_j,y_j+1}(u) V^2 C_{y_j,y_j+1}(u^*) V^2] \]
\[ + f(\lambda_{y_j})^{-1} [C_{y_j,y_j+1}(u) V^1 C_{y_j,y_j+1}(u^*) V^1 + f(\lambda_{y_j}) C_{y_j,y_j+1}(u) V^2 C_{y_j,y_j+1}(u^*) V^2] \]
\[ f(\lambda) \equiv N^2 \left( \frac{\Gamma(N\lambda)}{\Gamma(1-N\lambda)} \right)^2 \frac{\Gamma(1-(N-1)\lambda) \Gamma(1-(N+1)\lambda)}{\Gamma((N-1)\lambda) \Gamma((N+1)\lambda)} \]

are nothing but the product of the crossing-symmetric non-chiral correlators of the relevant subgroups \([7]\) and cosets \([14]\). At level \(x_0 = 1\), the number of contributing nest blocks in \((C.10)\) is \(2^n\), which corresponds to the usual consistent chiral truncation of the blocks of \(SU(N)_1\).

**Appendix D: Symmetric factorization**

In this appendix, we discuss a second natural factorization of the invariant biconformal correlators, in the symmetric ansatz \((7.2d)\). This gives a second global solution, whose details apparently differ from the solution of the text. Nevertheless, we find that this solution gives the same correct coset and nest correlators, and the same high-level affine-Virasoro correlators found in Section 10.

At each fixed choice of the Lie algebra index \(\alpha = (\alpha_1 \alpha_2 \alpha_3 \alpha_4)\), the matrix \(C_{qp}^\alpha = Y_{\bar{q}}^{\beta}(u_0) W_{pq}(u_0)_{\beta^\alpha}\) defines an infinite-dimensional eigenvalue problem
\[ \sum_p C_{qp}^\alpha \bar{\psi}_{p(\nu)}^\alpha(u_0) = E_{p(\nu)}^\alpha(u_0) \psi_{q(\nu)}^\alpha(u_0) \]  \hspace{1cm} \text{\textit{(D.1a)}}
\[ \sum_q \psi_{q(\nu)}^\alpha(u_0) C_{qp}^\alpha = E_{p(\nu)}^\alpha(u_0) \psi_{p(\nu)}^\alpha(u_0) \]  \hspace{1cm} \text{\textit{(D.1b)}}
\[ C_{qp}^\alpha = \sum_{\nu=0}^\infty \bar{\psi}_{q(\nu)}^\alpha(u_0) E_{p(\nu)}^\alpha(u_0) \psi_{p(\nu)}^\alpha(u_0) \]  \hspace{1cm} \text{\textit{(D.1c)}}

Then, the spectral resolution \((D.1c)\) of \(C_{qp}^\alpha\) gives the factorization
\[ Y_{p}^\alpha(u,u_0) = \sqrt{E_{p}^\alpha(u_0)} \bar{\psi}_{p(\nu)}^\alpha(u,u_0) \quad , \quad \bar{\psi}_{p(\nu)}^\alpha(u,u_0) \equiv \sum_{q=0}^\infty \frac{(u - u_0)^q}{q!} \bar{\psi}_{q(\nu)}^\alpha(u_0) \]  \hspace{1cm} \text{\textit{(D.2b)}}
\[
Y^\alpha_\nu(u, u_0) = \sqrt{E_\nu^\alpha(u_0)} \psi^\alpha_\nu(u, u_0), \quad \psi^\alpha_\nu(u, u_0) \equiv \sum_{p=0}^{\infty} \frac{(u - u_0)^p}{p!} \bar{\psi}^\alpha_\nu(u_0)
\]

where the structures \(\bar{\psi}^\alpha_\nu(u, u_0)\) and \(\psi^\alpha_\nu(u, u_0)\) are the conformal eigenvectors of the \(\tilde{L}\) and \(L\) theories respectively.

An elegant feature of the symmetric factorization is that the conformal structures are symmetric under K-conjugation,

\[
\tilde{L} \leftrightarrow L : \quad \bar{Y}^\alpha_\nu \leftrightarrow Y^\alpha_\nu .
\]

This symmetry, not shared by the solution of the text, follows from the K-conjugation covariance (4.5b) which implies that \(C_{qp} \leftrightarrow C_{pq}\) and hence \(\bar{\psi} \leftrightarrow \psi\) when \(\tilde{L} \leftrightarrow L\). On the other hand, the braiding of the symmetric factorization is further complicated by the factor \(Y^g(u_0)\) in \(C^g_{qp}\).

Corresponding to (8.3), an equivalent form of the factorized correlators

\[
Y^\alpha_\nu(u, u_0) = \frac{1}{\sqrt{E_\nu^\alpha(u_0)}} \sum_{p=0}^{\infty} \bar{\psi}^\alpha_\nu(u_0) \partial_{u_0}^p \bar{f}^\alpha_0(u, u_0)
\]

\[
Y^\alpha_\nu(u, u_0) = \frac{1}{\sqrt{E_\nu^\alpha(u_0)}} \sum_{q=0}^{\infty} \psi^\alpha_\nu(u_0) \partial_{u_0}^q f^\alpha_0(u, u_0)
\]

is obtained from the eigenvalue problem (D.1) when \(E_\nu \neq 0\). The basic structures \(\bar{f}^\alpha_0\) and \(f^\alpha_0 = Y^g_\beta(u_0)(f_0)_\beta^\alpha\) are defined in eqs.(8.3c,d).

As an application of (D.4), we give the results for the coset constructions and the first non-trivial affine-Sugawara nests

\[
(\bar{Y}^g_{gh})^\alpha_\nu(u, u_0) = Y^g_{gh}(u, u_0)d^p_{(\nu)}(u_0)
\]

\[
d^p_{(\nu)}(u_0) \equiv \frac{1}{\sqrt{E_\nu^g(u_0)}} \sum_{p=0}^{\infty} W^h_{0p}(u_0)_{\beta}^\alpha \bar{\psi}^\alpha_\nu(u_0)
\]

\[
(\bar{Y}^{g_{h1/h2}})^\alpha_\nu(u, u_0) = Y^{m_1}_{g_{h1}}(u, u_0)Y^{m_2}_{h2}(u, u_0)m^\alpha_{m_1(\nu)}D^m_{m_1(\nu)}(u_0)
\]

\[
D^m_{m_1(\nu)}(u_0) \equiv \frac{1}{\sqrt{E_\nu^m(u_0)}} \sum_{p=0}^{\infty} W^h_{0p}(u_0)_{m_1}^m \bar{\psi}^\alpha_\nu(u_0)
\]

which correspond to \(\tilde{L} = L_{gh}\) and \(\tilde{L} = L_{gh_{h1/h2}}\) respectively. The symbols in (D.5c,d) with \(m_1\) and/or \(m_2\) indices are defined in eqs.(5.16) and (8.7d).
Although the constants in (D.5) differ from those in the corresponding results (8.6) and (8.7), the same correct $u$ dependence is obtained for these conformal correlators. Simililarly, the global solution (D.2) or (D.4) gives the correct dependence for the conformal correlators of the higher nests.

Finally, we follow the steps in Section 10 to obtain the high-level correlators in the symmetric ansatz. As in the vector ansatz, only a finite number of $E_\nu \neq 0$ eigenvectors are found at leading order

$$
\bar{\psi}_\nu^\alpha(u_0) = (Y_g(u_0)W_{q0}(u_0))^{\beta} \bar{\phi}_\nu^\alpha(u_0) + \mathcal{O}(k^{-2})
$$

$$
\psi_{\nu(p)}^\alpha(u_0) = (Y_g(u_0)W_{0p}(u_0))^{\beta} \phi_{\nu(p)}^\alpha(u_0) + \mathcal{O}(k^{-2})
$$

$$
E_\nu \neq 0 \quad , \quad \nu = 0,1,\ldots,D_s(\mathcal{T})-1 \quad , \quad D_s(\mathcal{T}) \leq \prod_{i=1}^4 \dim \mathcal{T}^i
$$

where the reduced eigenvectors $\bar{\phi}$, $\phi$ are the (non-zero eigenvalue) eigenvectors of the reduced problem

$$
\bar{M}^\alpha(u_0)\bar{\psi}_\nu^\alpha(u_0) = E_\nu^\alpha(u_0)\bar{\phi}_{\nu(p)}^\alpha(u_0)
$$

$$
M^\alpha(u_0)\bar{\phi}_\nu^\gamma = \sum_{p} W_{0p}(u_0)\alpha Y_g(u_0)W_{q0}(u_0) \bar{\phi}_{\nu(p)}^\alpha(u_0)
$$

$$
M^\alpha(u_0)\phi_{\nu(p)}^\gamma = E_\nu^\alpha(u_0)\phi_{\nu(p)}^\alpha(u_0)
$$

$$
M^\alpha(u_0)\bar{\phi}_\nu^\gamma = \sum_{q} W_{q0}(u_0)\beta Y_g(u_0)W_{0q}(u_0) \bar{\phi}_{\nu(p)}^\alpha(u_0)
$$

To obtain this manifestly K-conjugation covariant form, we used the interchange identity $W_{q0}W_{0p} = W_{0p}W_{q0} + \mathcal{O}(k^{-2})$ in the $\psi$ problem.

Summing the series in (D.2), we obtain the high-level form of the affine-Virasoro correlators

$$
Y^\alpha_{\nu}(u, u_0) = Y_g^{\beta}(u_0) \left( \delta^\gamma_{\beta} + \frac{\tilde{p}_{ab}}{k} \left[ \mathcal{T}^a \mathcal{T}^b \ln \left( \frac{u}{u_0} \right) \right]^\gamma_{\beta} \right) \sqrt{E^\alpha_\nu(u_0)} \bar{\phi}_{\gamma(\nu)}^\alpha(u_0) + \mathcal{O}(k^{-2})
$$

$$
Y_{\nu}^\alpha(u, u_0) = Y_g^{\beta}(u_0) \left( \delta^\gamma_{\beta} + \frac{p_{ab}}{k} \left[ \mathcal{T}^a \mathcal{T}^b \ln \left( \frac{u}{u_0} \right) \right]^\gamma_{\beta} \right) \sqrt{E^\alpha_\nu(u_0)} \phi_{\gamma(\nu)}^\alpha(u_0) + \mathcal{O}(k^{-2})
$$
Neglecting the constants $\sqrt{E_\nu}(\bar{\phi}, \phi)$, this result is in complete agreement with the high-level form of the vector ansatz in (10.8)

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