Cauchy Formulas and Billey’s Formulas for Generalized Grothendieck polynomials

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Abstract  We study the generalized double $\beta$-Grothendieck polynomials for all types. We study the Cauchy formulas for them. Using this, we deduce the K-theoretic version of the comodule structure map $\alpha^* : K(G/B) \to K(G) \otimes K(G/B)$ induced by the group action map for reductive group $G$ and its flag variety $G/B$. Furthermore, we give a combinatorial formula to compute the localization of Schubert classes as a generalization of Billey’s formula.

I would politely express my gratitude to Victor Petrov, Neil JiuYu Fan, Peter Long Guo and for discussion.

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1 Main Results

1.1 Let $G$ be a connected complex reductive group, and $B$ its Borel subgroup. The homogenous variety $G/B$ is called the flag variety of $G$.

Let $W = N_G(T)/T$ be the Weyl group and $\ell$ the standard length function. For elements $u, v, w \in W$, we write the reduced decomposition $w = u \circ v$ if $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.

We also introduce the product $*$ over $W$ which is uniquely characterized by

$\forall i \in I, \ s_i * s_i = s_i \ \ \ \ \ uv = u \circ v \implies uv = u * v,$

with $\{s_i : i \in I\}$ the set of simple reflections. Geometrically, $\Sigma_u \cdot \Sigma_v = \Sigma_{uv}$ for any $u, v \in W$ by the Tits system.
1.2 For an element $w \in W$, we define the lower Schubert variety $\Sigma_w$ to be the Zariski closure of $BwB/B \subseteq G/B$, and the (upper) Schubert variety $\Sigma^w$ to be the Zariski closure of $B^-wB/B \subseteq G/B$, where $B^-$ is the opposite Borel subgroup. It is well-known [Spr09] that the dimension $\dim \Sigma_w = \operatorname{codim} \Sigma_w = \ell(w)$. Furthermore, $\Sigma_w$ and $\Sigma^w$ are both isomorphic to affine linear spaces.

1.3 Let $\beta$ be a parameter. Denote $K(\beta) = K^{\beta}(\beta)$ the $\beta$-Grothendieck group. It is the oriented cohomology theory universal with respect to the multiplicative form group law $x \oplus_\beta y = x + y - \beta xy$. See for example [LM07]. After specialization at $\beta = 1$, we get the usual Grothendieck group $K_0$. We also use the equivariant form $K_B(\beta)$. For the case $\beta = 1$, the basic definition can be found in [CG09]. The general case can be easily established similarly.

Over the flag variety $G/B$, we denote $\mathcal{O}w$ the push forward of $\mathcal{O} \Sigma_w$ to $G/B$. We denote $[\mathcal{O}w]$ the class of it in $K(G/B)$ and $[\mathcal{O}^w]$ the equivariant analogy. Then $K(B/G/B)$ is freely generated by $[\mathcal{O}w]$ over $\mathbb{Z}[\beta, \beta^{-1}]$, and $K_B(G/B)$ is freely generated by $[\mathcal{O}^w]$ over $K_B(pt)$.

1.4 Let $K^{\top}(\beta)$ be the complex topological K-theory. We have a map $\eta_X : K^{\beta=1}(X) \to K^{\top}(X)$. By topological K-theory, $\eta_{G/B}$ is an isomorphism. The equivariant version is also true but after completion, by the Atiyah completion theorem [AS+69] and the Atiyah–Hirzebruch spectral sequence.

1.5 We have the following analogy of the main result of [Xio20].

**Theorem 1.1** The map induced by the left action $\alpha : G \times G/B \to G/B$

$$\alpha^* : K(G/B) \longrightarrow K(G) \otimes K(G/B)$$

is given by

$$[\mathcal{O}^w] \longmapsto \sum_{w = u * v} (-\beta)^{\ell(v) + \ell(u) - \ell(w)} \pi^* [\mathcal{O}^u] \otimes [\mathcal{O}^v].$$

The topological K-theory has the same formula after specialization at $\beta = 1$.

1.6 Other than consider the Borel construction, we think $G$ with $B$ acting on both sides, and $G \times G$ with $B$ acting on left, right, and middle

$$(b_1, b_2, b_3) \cdot (g, h) = (b_1gb_2^{-1}, b_2hb_3^{-1}).$$

Then the multiplication map $\mu : G \times G \to G \times B$ is $(B \times B \times B)$-equivariant. We denote the obvious map

$$K_{B \times B \times B}(G \times G) \xleftarrow{\mu_1} K_{B \times B \times B}(G \times G) \cong K_{B \times B}(G) \xrightarrow{\mu_2} K_{B \times B \times G}(G \times G) \cong K_{B \times B}(G).$$
Theorem 1.2 The map induced by the multiplication $\mu$

$$\mu^*: K_B \times B(G; \mathbb{Q}) \to K_B \times B \times B(G \times G; \mathbb{Q})$$

is given by

$$[O \omega]_B \mapsto \sum_{\omega = u \circ v} (-\beta)^{\ell(v) + \ell(u) - \ell(\omega)} \pi_1^*[O \omega]_B \cdot \pi_2^*[O \omega]_B.$$

The topological $K$-theory has the same formula after specialization at $\beta = 1$.

1.8 Denote $R^G_\beta = \mathbb{Q}[e^{\beta \Lambda}] \otimes \mathbb{Q}[\beta, \beta^{-1}]$, obtained by adjoining a new variable $\beta$ to the group ring of $\beta \Lambda \cong \Lambda$. Let $R_G = (R_T)^W$ be the invariant subalgebra of $R_T$. Then Borel’s presentation still holds

$$K(G/B; \mathbb{Q}) = R_T \otimes_{R_G} \mathbb{Q}, \quad K_B(G/B; \mathbb{Q}) = R_T \otimes_{R_G} R_T.$$

Actually, the above map is compatible with the Borel’s presentation of cohomology under the Chern character.

To be precise, $O(\lambda)$ is presented by $e^{\beta \lambda}$. Note that its first Chern class is $e^{\beta \lambda} - 1$ by definition.

1.9 Then the Schubert class $[O^w] \in K(G/B; \mathbb{Q})$ corresponds to some element in $R_T \otimes_{R_G} \mathbb{Q}$, which we will denote by $\mathcal{G}_w(X)$ and call it the generalized $\beta$-Grothendieck polynomial. On the other hand, the equivariant analogy for $[O^w]_B \in H^*_B(G/B; \mathbb{Q})$ is denoted by $\mathcal{G}_w(X, T)$ and is called the generalized double $\beta$-Grothendieck polynomial. Then $\mathcal{G}(X, 1) = \mathcal{G}(X)$. To be precise, $T \mapsto 1$ is the map sending all $1 \otimes e^{\beta \lambda}$ to $1$.

1.10 We have the following analogy of Cauchy formulas proved in [Xio20].

Theorem 1.3 We have

$$\mathcal{G}_w(X, Z) = \sum_{u \star v = w} (-\beta)^{\ell(u) + \ell(v) - \ell(w)} \mathcal{G}_v(X, Y) \mathcal{G}_u(Y, Z)$$

in $R_T \otimes_{R_G} R_T \otimes_{R_G} R_T$.

1.11 In the case $G = GL_n$, the generalized Grothendieck polynomials has a stable choice as in the cohomology case. This coincides with the definition of [FK93] after replacing $\beta$ by $-\beta$. In particular, when $\beta = 1$, it recovers the usual Grothendieck polynomials. We will shortly review the combinatorics in the appendix.

1.12 Actually we introduce a generalized double dual $\beta$-Grothendieck polynomial with plenty of good properties. They have an inversion formula (Theorem 4.1 and Theorem 6.4). The Cauchy formula (Theorem 4.3 and Theorem 6.6). They are the dual basis of the Demazure operators (Theorem 4.2 and Theorem 6.5).
1.13 Let $T$ be the maximal torus $T$ of $G$ contained in $B$. For the point $w \cdot B/B$, there is a localization map

$$K_B^\beta(G/B) \cong K_T^\beta(G/B) \hookrightarrow K_T^\beta(wB/B) \cong K_T^\beta(\text{pt}).$$

We give a combinatorial description of the image of $[O^\alpha]$ under this localization map.

1.14 Assume that $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ (not necessarily reduced). For any $j = 1, \ldots, r$, denote

$$d_j = s_{i_1} \cdots s_{i_{j-1}} \tilde{a}_j,$$

where $a_j = \frac{1-e^{-\beta s_i}}{\beta}$, and $\tilde{a}_j = \bigoplus_{\beta} a_i := \frac{-a_j}{1-\beta a_i}$. For any subset $J = \{j(1) < \cdots < j(k)\} \subseteq \{1, \ldots, r\}$, denote

$$w(J) = s_{i_{j(1)}} \cdots s_{i_{j(k)}}.$$

**Theorem 1.4** The image of $[O^\alpha]|_B$ under this localization map to $wB/B$ is given by

$$\mathfrak{G}_w(wx, x) = \sum_{w(J) = \alpha} (-\beta)^{|J|-\ell(u)} \prod_{j \in J} d_j \in K_T^\beta(\text{pt}).$$

Note that this expression does not depends on the choice of the decomposition of $w$.

1.15 Taking $\beta \to 0$, this recovers Billey’s formula \cite{B99}. But actually, this formula has been discovered in \cite{AJS94}. See \cite{Tym13} for remarks and applications. This formula also generalizes the Buch–Rimányi formula \cite{BR04} in $A$-Type Grothendieck polynomial ($\beta = 1$).

## 2 Demazure Operators and Localization

To simplify notations, all Grothendieck groups are of coefficients in $\mathbb{Q}$ in this section.

2.1 For the standard parabolic subgroup $P_i = B \cup Bs_i B$, the group $K_T^\beta(\text{pt})$ can be computed to be $R_T^\beta$ the subalgebra of $R_T$ fixed by $\{1, s_i\}$. Note that the $G$-equivariant K-theory $K_G(G/H) = K_H(\text{pt})$ for any closed subgroup $H$. We introduce the Demazure operator $\pi_i : R_T \to R_T$ to be the composition of

$$\sigma_* : R_T = K_G(G/B) \to H_G^{*-2}(G/P_i) = R_T^\beta,$$

$$\sigma^* : R_T^\beta = K_G(G/P_i) \to K_B(G/B) = R_T$$

where $\sigma : G/B \to G/P_i$ is the natural projection, inducing the Gysin push forward $\sigma_*$ and pull back $\sigma^*$.

2.2 By definition, the $B$-equivariant cohomology Demazure operator $R_T \otimes_{R_G} R_T \to R_T \otimes_{R_G} R_T$, the composition

$$\sigma_* : R_T \otimes_{R_G} R_T = K_B(G/B) \to H_B^{*-2}(G/P_i) = R_T^\beta \otimes_{R_G} R_T,$$

$$\sigma^* : R_T^\beta \otimes_{R_G} R_T = K_B(G/P_i) \to K_B(G/B) = R_T \otimes_{R_G} R_T$$

is given by $\pi_i \otimes 1$.  

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2.3 It is well-known that over $K_B(G/B)$,
\[\ell(ws_i) = \ell(w) - 1 \implies \pi_i[O^w]_B = [O^{ws_i}]_B.\]
In terms of generalized double $\beta$-Grothendieck polynomials, over $R_T^\beta \otimes R_G^\beta R_T^\beta$,
\[\ell(ws_i) = \ell(w) - 1 \implies \pi_i^X \mathfrak{G}_w(X,Y) = \mathfrak{G}_{ws_i}(X,Y)\]
where $\pi_i^X = \pi_i \otimes 1$. At the present stage, we cannot obtain the case $\ell(ws_i) = \ell(w) + 1$ directly, but it will be described later.

2.4 To obtain the K-theory version of Demazure operators, we use the Grothendieck–Riemann–Roch theorem for $\beta$-Grothendieck group [LM07]. To be precise, the relative tangent bundle of $G/B \to G/P_i$ is $O(\alpha_i)$ whose $\beta$-Todd class is $\frac{-\alpha_i}{1 - e^{-\beta\alpha_i}}$. Thus we have
\[
\pi_i f = \frac{\beta_\alpha_i f - s_i \left( \frac{\beta_\alpha_i f}{1 - e^{-\beta\alpha_i}} \right)}{\alpha_i} = \beta \frac{f - \frac{\alpha_i}{1 - e^{-\beta\alpha_i}} s_i f}{\alpha_i} = \beta \frac{f - e^{-\beta\alpha_i} s_i f}{1 - e^{-\beta\alpha_i}}.
\]
Actually, when $\beta = 1$, this is what Demazure originally obtained in [Dem74]. Taking into account of the action of $\pi_i$ on $[O^w]_B$, we can easily see that $\pi_i^X = \pi_i \otimes 1$.

2.5 As a corollary,
\[
\pi_i[O^w]_B = \begin{cases} [O^{ws_i}]_B, & \ell(ws_i) = \ell(w) - 1, \\ \beta[O^w]_B, & \ell(ws_i) = \ell(w) + 1. \end{cases}
\]
In terms of generalized double $\beta$-Grothendieck polynomials, $R_T^\beta \otimes R_G^\beta R_T^\beta$,
\[
\pi_i^X \mathfrak{G}_w(X,Y) = \begin{cases} \mathfrak{G}_{ws_i}(X,Y), & \ell(ws_i) = \ell(w) - 1, \\ \beta \mathfrak{G}_{ws_i}(X,Y), & \ell(ws_i) = \ell(w) + 1. \end{cases}
\]
Here $\pi_i^X = \pi_i \otimes 1$.

2.6 Note that the lifting of $W$ to $G/B$ is exactly $(G/B)^T$, the $T$-fixed point of $G/B$. By the K-theoretic localization theorem [CG09], the map
\[K_B(G/B) \to \bigoplus_{w \in W} K_B(w \cdot B/B)\]
induced by $(G/B)^T \subseteq G/B$ is injective. By a simple computation, the corresponding map $R_T \otimes R_G R_T \to \bigoplus_{w \in W} R_T$ is given by $x \otimes y \mapsto ((wx) \cdot y)_{w \in W}$.
2.7 For a closed $B$-subvariety $Y$ in $G/B$. Denote the image of $[O_Y]_B$ under the above localization map to be $(a_w)_{w \in W}$. If the fixed point $w$ is not contained in $Y$, then $a_w = 0$.

For Schubert varieties, the fixed point $u \in \Sigma^w$ if and only $w_0u \leq w_0w$ in the Bruhat order \cite{Spr09} where $w_0$ is the unique longest element of $W$. Equivalently, $w \leq u$. In terms of generalized double $\beta$-Grothendieck polynomials,

$$\mathfrak{S}_w(uT, T) \neq 0 \implies w \leq u.$$ 

We will use the case $\mathfrak{S}_w(T, T) = \begin{cases} 1, & w = \text{id}, \\ 0, & \text{otherwise}. \end{cases}$

3 Leibniz Rules

3.1 Let us denote the $\beta$-affine nil-Hecke algebra $NH^\beta(W)$ the algebra generated by left multiplications of elements of $R^n_\beta$ and $\pi_i$ with $i \in \mathbb{I}$ over $R^n_\beta$.

We introduce the inverse $\beta$-Demazure operator $\hat{\pi}_i = \pi_i - \beta \in NH^\beta(W)$, that is

$$\hat{\pi}_i f = \beta \frac{f - s_i f}{e^{\beta \alpha_i} - 1},$$

for $f \in R_T$. Consider the involution $D : f \mapsto e^{-\beta \rho} f$, where $e^\lambda = e^{\lambda}$, and $\rho$ is the half sum of positive roots. By a direct computation

$$\pi_i(Df) = \beta \frac{e^{-\beta \rho} f - e^{-\beta \rho} s_i(e^{-\beta \rho} f)}{1 - e^{-\beta \rho}} = \beta \frac{e^{-\beta \rho} - e^{-\beta \rho} s_i f}{1 - e^{-\beta \rho}} = e^{-\beta \rho} \beta \frac{f - s_i f}{1 - e^{-\beta \rho}} = -D(\hat{\pi}_i f)$$

As a result, $\hat{\pi}_i = -D \pi_i D$, thus must satisfy braid relation with $\hat{\pi}_i^2 = -\beta \hat{\pi}_i$.

3.2 Finally, the computation

$$\pi_i(fg) = \beta \frac{f - e^{-\beta \alpha_i} s_i f}{1 - e^{-\beta \alpha_i}} g + \beta e^{-\beta \alpha_i} \frac{g - s_i g}{1 - e^{-\beta \alpha_i}} s_i f = (\pi_i(f)g + (s_i f)(\hat{\pi}_i g))$$

$$\hat{\pi}_i(fg) = \beta \frac{f - s_i f}{e^{\beta \alpha_i} - 1} g + \beta \frac{g - s_i g}{e^{\beta \alpha_i} - 1} s_i f = (\hat{\pi}_i f)g + (s_i f)(\hat{\pi}_i g)$$

proves the Leibnize rule

$$\begin{cases} 
\pi_i(fg) = (\pi_i)f g + (s_i f)(\hat{\pi}_i g), \\
\hat{\pi}_i(fg) = (\hat{\pi}_i f)g + (s_i f)(\hat{\pi}_i g). 
\end{cases}$$

3.3 The next lemma is a generalization of author’s previous work \cite{Xio20}. Note that $\pi_i \hat{\pi}_i = \hat{\pi}_i \pi_i = 0$, thus

$$\forall w \in W, \quad \hat{\pi}_i \pi_{w} = \begin{cases} 0 & \ell(s_i w) = \ell(w) - 1, \\
\pi_{s_i w} - \beta \pi_{w} & \ell(s_i w) = \ell(w) + 1. \end{cases}$$

Lemma 3.1 (Top Leibniz rule) We have the following

$$\hat{\pi}_w((w_0 f)g) = \sum_{w \in W} (-1)^{\ell(w)} (\pi_{ww_0} f) \cdot \pi_w g,$$

for all $f, g \in R_T$. 

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By induction, we have 

$$c_i$$ is easy to see that

$$3.4$$ Let us denote the Demazure operator $$\Pi_i \in NH^\beta(W)$$ by

$$\Pi_i f := \frac{\pi_i f}{1 - e^{-\beta f}}.$$ 

They also satisfy the braid relations and $$\Pi_i^2 = -\beta \Pi_i$$, thus we can write $$\Pi_w$$ for $$w \in W$$. Note that

$$e^{\beta \pi_w f} = e^{-\beta \pi_w f} = D(\pi_w f) = (-1)^{\ell(w)} \pi_w Df = (-1)^{\ell(w)} \Pi_w Df.$$ 

Corollary 3.3 For any $$f, g \in RT$$,

$$\sum_{w \in W} (\Pi_{w0} f) \cdot \pi_w g = \sum_{w \in W} (\Pi_w f) \cdot \pi_w g,$$

is symmetric under $$W$$. 

Corollary 3.2 For any $$f, g \in RT$$,

$$\sum_{w \in W} (-1)^{\ell(u)} e^{\beta (\pi_{w0} f)} \cdot \pi_u g,$$

is symmetric under $$W$$. 

Proof. For $$w \in W$$, $$s_i \rho = \rho - \alpha_i$$, thus $$\pi_i e^{-\beta \rho} = 0$$. So

$$\forall f \in RT, \quad s_i f = f \iff \pi_i(e^{-\beta f}) = 0$$

Then $$\pi_i g \cdot e^{\beta \rho}$$ is always symmetric for any $$g \in RT$$. Q.E.D.
3.5 We can take the limit $\beta \to 0$ to get

$$
\lim_{\beta \to 0} \pi_i = \partial_i,
\lim_{\beta \to 0} \hat{\pi}_i = \partial_i,
\lim_{\beta \to 0} \Pi_i = -\partial_i.
$$

Here $\partial_i$ is the cohomology Demazure operator.

4 Cauchy Formulas

4.1 The K-theory analogy is not absolutely direct from the proof of cohomology as in [Xio20], since $\pi_i \neq -\Pi_i$.

**Theorem 4.1 (Inversion Formula)** For any $f \in R^\beta_T$, we have the following identity in $R^\beta_T \otimes_{R^\beta_G} R^\beta_T$,

$$
f(Y) = \sum_{w \in W} \mathcal{G}_w(X, Y) \Pi^X_w f(X)
$$

**Proof.** Note that the right-hand-side is

$$
\pi^X_{w^{-1}w_0} \mathcal{G}_{w_0}(X, Y) \Pi^X_{w^{-1}} f(X)
$$

which is symmetric in $X$ by Corollary 3.3 so that we can take $X = Y$. Q.E.D.

**Corollary 4.2 (Dual Basis)** The operator $\Pi_{w^{-1}}$ and $\mathcal{G}_w$ are dual to each other, that is,

$$
\Pi^Y_{w^{-1}} \mathcal{G}_u(X, Y) \bigg|_{X=Y} = \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}
$$

**Proof.** This is a standard computation

$$
\mathcal{G}_u(X, X) \bigg|_{Y=X} = \sum_{u \in W} \mathcal{G}_u(\bullet, Y) \Pi^Y_{w^{-1}} \mathcal{G}_u(X, \bullet) \bigg|_{Y=\bullet} \bigg|_{\bullet=Y=X}.
$$

Change the variables, we get the assertion. Q.E.D.

4.2 We can get more interesting combinatorial identities.

**Theorem 4.3 (Star-Cauchy Formula=Theorem 1.3)** We have the following identity

$$
\mathcal{G}_w(X, Z) = \sum_{u+v=w} (-\beta)^{\ell(u)+\ell(v)-\ell(w)} \mathcal{G}_u(X, Y) \mathcal{G}_v(Y, Z)
$$

**Proof.** Apply the inversion formula twice,

$$
f(Z) = \sum_{w \in W} \mathcal{G}_w(X, Z) \Pi^X_{w^{-1}} f(X)
$$

$$
= \sum_{u \in W} \mathcal{G}_u(Y, Z) \Pi^Y_{w^{-1}} f(Y)
$$

$$
= \sum_{u \in W} \mathcal{G}_u(Y, Z) \sum_{v \in W} \mathcal{G}_v(X, Y) \Pi^X_{w^{-1}} \Pi^X_{v^{-1}} f(X)
$$

Since $f$ is arbitrary, we can compare the coefficients of $\Pi^Y_{w^{-1}} f(X)$. Q.E.D.
Here we use the fact $\pi$ is symmetric in $Z$. We apply Corollary 3.2 to

Proof. We apply Corollary 3.2 to

Theorem 4.4 (Reduced Cauchy Formula) In $R_T^\beta \otimes_{R_G} R_T^\beta \otimes_{R_G} R_T^\beta$,

$$X^{\beta \rho} \mathfrak{G}_w(Y, X) = (-1)^{\ell(u)} Y^{\beta \rho} \mathfrak{G}_w(Y, X)$$

$$= \sum_{u \subseteq v = u \circ v} (-1)^{\ell(v)} Z^{\beta \rho} \mathfrak{G}_{v^{-1}}(Z, X) \mathfrak{G}_w(Z, Y).$$

Proof. We apply Corollary 3.2 to $f = \mathfrak{G}_w(Z, X), g = \mathfrak{G}_w(Z, Y)$. We get

$$\sum_{w_0 = w \circ v} (-1)^{\ell(v)} Z^{\beta \rho} \mathfrak{G}_{v^{-1}}(Z, X) \mathfrak{G}_w(Z, Y)$$

is symmetric in $Z$. Thus we can exchange $Z$ to $X$ or $Y$

$$\sum_{w_0 = w \circ v} (-1)^{\ell(v)} Z^{\beta \rho} \mathfrak{G}_{v^{-1}}(Z, X) \mathfrak{G}_w(Z, Y)$$

$$= \sum_{w_0 = w \circ v} (-1)^{\ell(v)} X^{\beta \rho} \mathfrak{G}_{v^{-1}}(X, X) \mathfrak{G}_w(X, Y) = X^{\beta \rho} \mathfrak{G}_w(X, Y)$$

$$= \sum_{w_0 = w \circ v} (-1)^{\ell(v)} Y^{\beta \rho} \mathfrak{G}_{v^{-1}}(Y, X) \mathfrak{G}_w(Y, Y) = (-1)^{\ell(v)} Y^{\beta \rho} \mathfrak{G}_w(Y, X)$$

is what claimed in the proposition. Q.E.D.

Proposition 4.5 We have

$$X^{\beta \rho} \mathfrak{G}_w(X, Y) = \sum_{v \geq w} (-1)^{\ell(v)} \beta^{\ell(v) - \ell(w)} Y^{\beta \rho} \mathfrak{G}_{v^{-1}}(Y, X).$$

Proof. We apply Corollary 3.2 to $f = \mathfrak{G}_w(Z, X), g = \mathfrak{G}_w(Z, Y)$. The element

$$\sum_{v \in W} (-1)^{\ell(v)} Z^{\beta \rho} \mathfrak{G}_{v^{-1}}(Z, X) \pi_v^Z \mathfrak{G}_w(Z, Y)$$

is symmetric in $Z$. Thus we can exchange $Z$ to $X$ or $Y$

$$\sum_{v \in W} (-1)^{\ell(v)} Z^{\beta \rho} \mathfrak{G}_{v^{-1}}(Z, X) \pi_v^Z \mathfrak{G}_w(Z, Y)$$

$$= \sum_{v \in W} (-1)^{\ell(v)} X^{\beta \rho} \mathfrak{G}_{v^{-1}}(X, X) \pi_v^X \mathfrak{G}_w(X, Y) = X^{\beta \rho} \mathfrak{G}_w(X, Y)$$

$$= \sum_{v \in W} (-1)^{\ell(v)} Y^{\beta \rho} \mathfrak{G}_{v^{-1}}(Y, X) \pi_v^Z \mathfrak{G}_w(Z, Y)$$

Here we use the fact $\pi_u \mathfrak{G}_w$ is a constant if and only if $u \geq w$. Q.E.D.
5 Generalized Billey formula

5.1 Taking in $Y = uX$ in Theorem 4.1, we get

**Theorem 5.1** The value of the localization $G_w(X, uX)$ is determined by the following properties in $\text{NH}^\beta(W)$

$$u = \sum_{w \in W} G_w(X, uX) \Pi_{w^{-1}},$$

for any $u \in W$.

5.2 By the description of the localization map, Theorem 1.4 is equivalent to

**Theorem 5.2 (Generalized Billey Formula)** For the localizations of the generalized double $\beta$-Grothendieck polynomials,

$$G_u(wx, x) = \sum_{w(J) = u} (-\beta)^{|J| - \ell(u)} \prod_{j \in J} d_j.$$ 

The notation is introduced before Theorem 1.4.

5.3 For example, we consider the case of $A$-type. Now, $a_i = \frac{x_i - x_{i+1}}{1 - \beta x_{i+1}} = \beta^{-1}(1 - \frac{x_i}{x_{i+1}})$. Thus $\bar{a}_i = \frac{x_i - x_{i+1}}{1 - \beta x_i} = \beta^{-1}(1 - \frac{x_i}{x_{i+1}})$.

Consider the case $w = s_2 s_1$ and $u = s_2$,

| $w = s_2 s_1$ | $|J|$ | $d_j$ | $\Pi d_j$ |
|----------------|------|------|----------|
| $s_2 = s_2$    | 1    | $\bar{a}_2$ | $\beta^{-1}(1 - \frac{X_2}{X_3})$ |

So $G_u(wx, x) = \beta^{-1}(1 - \frac{X_2}{X_3})$. This coincides with our computation before.

Similarly, for the case $w = s_1 s_2$ and $u = s_2$,

| $w = s_1 s_2$ | $|J|$ | $d_j$ | $\Pi d_j$ |
|----------------|------|------|----------|
| $s_2 = s_2$    | 1    | $s_1 \bar{a}_2$ | $\beta^{-1}(1 - \frac{X_1}{X_3})$ |

So $G_u(wx, x) = \beta^{-1}(1 - \frac{X_1}{X_3})$.

5.4 Consider $w = s_2 s_1 s_2$, and $u = s_2$.

| $w = s_2 s_1 s_2$ | $|J|$ | $d_j$ | $\Pi d_j$ |
|---------------------|------|------|----------|
| $s_2 = s_2$         | 1    | $\bar{a}_2$ | $\beta^{-1}(1 - \frac{X_2}{X_3})$ |
| $s_2 = s_2$         | 1    | $s_2 s_1 \bar{a}_2$ | $\beta^{-1}(1 - \frac{X_2}{X_3})$ |
| $s_2 = s_2 * s_2$   | 2    | $2 \bar{a}_2$ | $\beta^{-2}(1 - \frac{X_2}{X_3})(1 - \frac{X_1}{X_2})$ |

Thus

$$G_u(wx, x) = \beta^{-1}(1 - \frac{X_2}{X_3}) + \beta^{-1}(1 - \frac{X_1}{X_2}) - \beta^{-1}(1 - \frac{X_1}{X_2})(1 - \frac{X_2}{X_3}).$$

This coincides with our computation before. On the other hand, we can take $w = s_1 s_2 s_1$. Then

| $w = s_1 s_2 s_1$ | $|J|$ | $d_j$ | $\Pi d_j$ |
|---------------------|------|------|----------|
| $s_2 = s_2$         | 1    | $s_1 \bar{a}_2$ | $\beta^{-1}(1 - \frac{X_1}{X_3})$ |
So
\[
\mathcal{G}_u(wx, x) = \beta^{-1}(1 - \frac{x_3}{x_2}x_1).
\]

One can check that they are equal.

5.5 For $A$-type, above process can be interpolated to the language of pipe dreams when the decomposition of $w$ is reduced. For two pipe dreams $\pi_1$ and $\pi_2$, we write $\pi_1 \leq \pi_2$ if they are the same at all the positions where $\pi_2$ is not tiled by $\bigcirc$

Assume further that $\pi_2$ does not contain $\bigcirc$. Then we define the following weights for $(\pi_1, \pi_2)$.

\[
\begin{align*}
\text{wt}(\bigcirc \bigcirc) &= \text{wt}(\bigcirc \bigcirc) = 1, \\
\text{wt}(\bigcirc \bigcirc) &= \text{wt}(\bigcirc \bigcirc) = \beta^{-1}(1 - \frac{x_1}{x_2}), \\
\text{wt}(\bigcirc \bigcirc) &= = -\beta^{-1}(1 - \frac{x_1}{x_2}).
\end{align*}
\]

where the two pipes in the second entry go to the $i$-th and $j$-th endings in $\pi_2$ with $i > j$ (NOT in $\pi_1$!). Then define $\text{wt}(\pi_1, \pi_2)$ by the product of weight at each position. Then for a fix one pipe dream $\pi_0$ consisting only $\bigcirc$ and $\bigcirc$ with $w(\pi_0) = w$, and permutation $u$,

\[
\mathcal{G}_u(wx, x) = \sum_{\pi \leq \pi_0, w(\pi) = w} \text{wt}(\pi, \pi_0).
\]

5.6 For example, let $w = s_2s_3s_2$, and $u = s_3$.

Thus
\[
\mathcal{G}_u(wx, x) = \beta^{-1}(1 - \frac{x_3}{x_4}) + \beta^{-1}(1 - \frac{x_4}{x_5}) - \beta^{-1}(1 - \frac{x_4}{x_5})(1 - \frac{x_5}{x_3}).
\]

On the other hand, we can pick

Thus
\[
\mathcal{G}_u(wx, x) = \beta^{-1}(1 - \frac{x_4}{x_5}).
\]

This coincides with the Buch–Rimányi formula introduced in [BR04]. See [FG20] for the combinatorial model of it.
5.7. Proof of Theorem 5.2 For any \( f(x) \in R^a \), we denote \( L_f(x) : NH^a(W) \to NH^a(W) \) the left multiplication by \( f(x) \). Denote \( T_w : NH^a(W) \to NH^a(W) \) the right multiplication by \( \Pi_{w^{-1}}^X \), and \( T_i = T_{s_i} \) for simplicity. We also denote \( a_i = \frac{1 + s_i}{a_i} \). We denote \( h_i(x) = 1 + L_{e_i}T_i \). Then

\[
w_i = w(1 + \frac{1 + s_i}{a_i} \Pi_i) = w(1 + a_i \Pi_i) = w + (w a_i) w \Pi_i = (1 + L_w a_i T_i)(w) = h_i(w a_i)(w).
\]

Since \( L_w \) commutes with \( T_i \)'s, \( h_i \)'s satisfy the Yang–Baxter equation in \( A \)-type above (while we will not use this fact). Recall that \( w = s_{i_1} s_{i_2} \cdots s_{i_r} \). Let us denote \( w^{(j)} = s_{i_1} \cdots s_{i_j} \). Then

\[
w = w^{(r-1)} s_{i_r} = h_{i_r}(w^{(r-1)} a_{i_r})(w^{(r-1)})
\]

\[
= (h_{i_r}(w^{(r-1)} a_{i_r}) \circ h_{i_{r-1}}(w^{(r-2)} a_{i_{r-1}}))(w^{(r-2)})
\]

\[
= (h_{i_r}(w^{(r-1)} a_{i_r}) \circ \cdots \circ h_i(w^{(0)} a_i))(id)
\]

Assume

\[
h_{i_r}(w^{(r-1)} a_{i_r}) \circ \cdots \circ h_i(w^{(0)} a_i) = \sum_{u \in W} L_u \tau(x) T_u
\]

Then \( w = \sum c_w^u(x) \Pi_{u^{-1}} \). By Theorem 5.1, \( c_w^u(x) = \mathcal{G}_u(x, wx) \).

Under the notation of the previous subsection, we get the following identity

\[
h_{i_r}(w^{(r-1)} a_{i_r}) \cdots h_i(w^{(0)} a_i) = \sum_{u \in W} \mathcal{G}_u(x, wx) T_u,
\]

By applying \( w^{-1} \) both side and replace \( w \) by \( w^{-1} \), we get

\[
h_{i_r}(w^{(0)} a_{i_r}) \cdots h_i(w^{(r-1)} a_i) = \sum_{u \in W} \mathcal{G}_u(wx, x) T_u.
\]

Here \( \tilde{a}_i = \bigwedge_{x} a_i := \frac{\bar{a}_i}{1 - \bar{a}_i} \). That is,

\[
h_{i_1}(d_1) \cdots h_{i_r}(d_r) = \sum_{u \in W} \mathcal{G}_u(wx, x) T_u.
\]

The expansion of the left-hand-side gives the formula in Theorem 5.2 Q.E.D.

6 Dual Grothendieck Polynomials

6.1 We define the dual Grothendieck polynomial

\[
\mathfrak{G}_{w^{-1}}(X, Y) = \Pi_{w^{-1}, u_0}^Y \mathcal{G}_{u_0}(X, Y).
\]

Note that \( \lim_{\beta \to 0} \mathfrak{G}_{w}(X, Y) = \mathcal{G}_{w}(x, y) \).
6.2 The following identity is the analogy of the second identity of Theorem 4.3.

**Theorem 6.1** We have

\[ X^{\beta_\rho}G_w(X, Y) = (-1)^{\ell(w)}Y^{\beta_\rho}g_{w^\perp}(Y, X). \]

In particular,

\[ u \leq w \implies g_w(uX, X) = 0, \quad g_{id}(X, Y) = X^{\beta_\rho}Y^{-\beta_\rho}. \]

**Proof.** We can compute

\[
X^{\beta_\rho}G_w(X, Y) = X^{\beta_\rho\pi_{w^\perp}}_w (X, Y) \\
= (-1)^{\ell(w)} X^{\beta_\rho\pi_{w^\perp}}_w X^{-\beta_\rho}Y^{\beta_\rho}G_w(Y, X) \\
= (-1)^{\ell(w)} X^{\beta_\rho\pi_{w^\perp}}_w D(Y^{\beta_\rho}G_w(Y, X)) \\
= (-1)^{\ell(w)} \Pi_{w^\perp} Y^{\beta_\rho}G_w(Y, X) = (-1)^{\ell(w)}Y^{\beta_\rho}g_{w^\perp}(Y, X).
\]

The localization condition is trivial. Q.E.D.

6.3 We can rewrite the reduced Cauchy formula (4.4) and Proposition 4.5.

**Corollary 6.2 (Reduced Cauchy Formula)** In \( K \),

\[ G_w(X, Y) = \sum_{u \subseteq v = w} g_v(X, Z)G_u(Z, Y). \]

**Corollary 6.3 (Möbius Inversion)** We have

\[ G_w(X, Y) = \sum_{v \geq w} \beta_{\ell(v) - \ell(w)}g_v(X, Y), \]

\[ g_w(X, Y) = \sum_{v \geq w} (-\beta)^{\ell(v) - \ell(w)}G_v(X, Y). \]

6.4 Dual to Theorem 4.1 and Corollary 4.2 above, we have

**Theorem 6.4 (Inversion Formula)** For any \( f \in R_T \), we have the following identity in \( K \)

\[ f(X) = \sum_{w \in W} g_w(X, Y)\pi_w^Y f(Y) \]

**Proof.** Note that the right-hand-side is

\[ \Pi_{w^\perp}^Y G_w(X, Y)\pi_w^Y f(Y) \]

which is symmetric in \( Y \) by Corollary 3.3, so that we can take \( Y = X \). Q.E.D.
Corollary 6.5 (Dual Basis) The operator $\pi_w$ and $g_w$ are dual to each other, that is,
\[
\pi_w^X g_z^O(X, Y)|_{Y=X} = \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}
\]

Theorem 6.6 (Star-Cauchy Formula) We have the following identity
\[
g_w(X, Z) = \sum_{u+z=w} \beta^{\ell(x)+\ell(y)-\ell(u)} g_v(X, Y) g_u(Y, Z)
\]

Proof. The proof is the same to Theorem [FK93] Q.E.D.

7 Appendix: Pipe Dream

7.1 We will shortly review the combinatorics in $A$-type which provides examples for the next subsection. In the case $G = GL_n$. We recognize $\Lambda = \mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n$. In cohomology, we denote $x_i = t_i$ (not confusing with the K-theory notation). The Demazure operator
\[
\partial_i f = \frac{f - s_i f}{x_i - x_j}.
\]

Other than the Euler class, we have another choice [Las07], $\mathcal{G}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_n^{-1}$, and $\mathcal{G}_{w_0}(x, y) = \prod_{i+j \leq n} (x_i - y_j)$. This choice is stable.

7.2 We denote $X_i = e^{\beta t_i} \otimes 1$, $Y_i = 1 \otimes e^{\beta t_i}$, and $x_i = \frac{1-e^{-\beta t_i}}{\beta} \otimes 1$, $y_i = 1 \otimes \frac{1-e^{\beta t_i}}{\beta}$. Then in terms of $X_i$,
\[
\pi_i f = \beta f \frac{e^{-\beta t_i} s_i f}{1-e^{-\beta t_i}} = \beta f \frac{X_{i+1} s_i f}{X_i - X_{i+1}} = \beta X_i f - X_{i+1} s_i f = \beta \partial_i X_i f.
\]

In terms of $x_i$,
\[
\pi_i f = \beta \frac{1-e^{\beta x_i}}{1-e^{\beta x_{i+1}}} \frac{1-s_i f}{1-s_i} = \beta \frac{(1-\beta x_{i+1}) f - (1-\beta x_i) s_i f}{x_i - x_{i+1}} = \partial_i((1-\beta x_{i+1}) f).
\]

We have the stable choice [FK93] (after replacing $\beta$ by $-\beta$)
\[
\mathcal{G}_{w_0}(X, Y) = \beta^{-\ell(w_0)} \prod_{i+j \leq n} \left(1 - \frac{x_i - y_j}{1-\beta y_j} \right).
\]

7.3 We denote $h_i(x) = (1 + x T_i)$, where $T_i$'s satisfy $T_i^2 = -\beta T_i$ and the Braid relations.

In $A$-type, one can check the following Yang–Baxter equations
\[
\begin{align*}
h_i(x) h_i(y) &= h_i(x \oplus \beta y) \\
h_i(x) h_j(y) &= h_j(y) h_i(x) \quad |i-j| \geq 2 \\
h_i(x) h_{i+1}(x \oplus \beta y) h_i(y) &= h_{i+1}(y) h_i(x \oplus \beta y) h_{i+1}(x)
\end{align*}
\]

Here the variables $x$ and $y$ commute with $T_i$'s. There are also Yang–Baxter equations for non-simply-laced cases, see [Kir15] and [BY99].
7.4 We consider the generating function

\[ G^\beta(x, y) = \sum G_w^\beta(x, y) T_w^\beta. \]

It is amazing that it factors into

\[
\begin{align*}
    h_{n-1}(x_1 \ominus y_{n-1}) & \quad h_{n-2}(x_1 \ominus y_{n-2}) & \quad \cdots \quad h_1(x_1 \ominus y_2) & \quad h_1(x_1 \ominus y_1) \\
    h_{n-1}(x_2 \ominus y_{n-1}) & \quad h_{n-2}(x_2 \ominus y_{n-2}) & \quad \cdots \quad h_3(x_2 \ominus y_2) & \quad h_2(x_1 \ominus y_1) \\
    \vdots & \quad \vdots & \quad \ddots & \quad \vdots \\
    h_{n-1}(x_{n-2} \ominus y_{n-1}) & \quad h_{n-2}(x_{n-2} \ominus y_{n-1}) & \quad \cdots & \quad h_1(x_1 \ominus y_1) \\
    h_{n-1}(x_{n-1} \ominus y_{n-2}) & \quad h_{n-2}(x_{n-1} \ominus y_{n-1}) & \quad \cdots & \quad h_2(x_1 \ominus y_1)
\end{align*}
\]

where \( x \ominus \beta y := \frac{x - \beta y}{1 - \beta y} \), see [FK96], it is the proof for \( \beta = 0 \), but the proof works in general. This observation originally appeared in [FS94], the case \( \beta = 0, y_i = 0 \).

7.5 The expansion of this expression gives the model known as pipe dreams [BB93] (the case \( \beta = 0 \) and \( y = 0 \)), and [Knu19] (the case \( \beta = 0 \)). To be exact, a pipe dream is a tiling of the type \((n-1, \ldots ,1,0)\) tableau by three kinds of tiles \( \square \) and \( \square \) in such that

- any pair of pipes intersect at most once and
- the two pipes in any appearance \( \square \) intersects at the northeast of it.

For each pipe dream \( \pi \), one can read a permutation \( w(\pi) \) from the left boundary to the up boundary. We define the weight of tiles at \((i, j)\)-position by

\[
\text{wt}(\square) = 1,
\]

\[
\text{wt}(\square) = \frac{x_i - y_j}{1 - \beta y_j} = \beta^{-1} \left( 1 - \frac{y_j}{x_i} \right),
\]

\[
\text{wt}(\square) = -\beta \frac{x_i - y_j}{1 - \beta y_j} = -\left( 1 - \frac{y_j}{x_i} \right).
\]

The weight \( \text{wt}(\pi) \) of a pipe dream \( \pi \) is defined to be the product of weights of all tiles. Then

\[ G_w(x, y) = \sum_{u(\pi) = w} \text{wt}(\pi). \]
For example, for the permutation \( w = (123)_{(132)} \in S_3 \),

\[
\begin{align*}
\text{wt} & \quad \frac{x_1 - y_1}{1 - \beta y_1} \quad \frac{x_2 - y_2}{1 - \beta y_2} \quad -\beta \left( \frac{x_1 - y_2}{1 - \beta y_2} \right) \left( \frac{x_2 - y_1}{1 - \beta y_1} \right) \quad \text{Not allowed} \\
\text{wt} & \quad \beta^{-1}(1 - \frac{y_1}{x_2}) \quad \beta^{-1}(1 - \frac{y_2}{x_1}) \quad -\beta^{-1}(1 - \frac{y_1}{x_2})(1 - \frac{y_2}{x_1}) \quad \text{Not allowed}
\end{align*}
\]

Thus

\[
\mathfrak{g}_w(x, y) = \frac{x_2 - y_2}{1 - \beta y_2} + \frac{x_1 - y_1}{1 - \beta y_1} - \beta \left( \frac{x_1 - y_2}{1 - \beta y_2} \right) \left( \frac{x_2 - y_1}{1 - \beta y_1} \right) \\
= \beta^{-1}(1 - \frac{y_1}{x_2}) + \beta^{-1}(1 - \frac{y_2}{x_1}) - \beta^{-1}(1 - \frac{y_1}{x_2})(1 - \frac{y_2}{x_1}).
\]

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