Supersymmetry and Shape Invariance of exceptional orthogonal polynomials

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Abstract

We discuss the exceptional Laguerre and the exceptional Jacobi orthogonal polynomials in the framework of the supersymmetric quantum mechanics (SUSYQM). We express the differential equations for the Jacobi and the Laguerre exceptional orthogonal polynomials (EOP) as the eigenvalue equations and make an analogy with the time independent Schrödinger equation to define “Hamiltonians” enables us to study the EOPs in the framework of the SUSYQM and to realize the underlying shape invariance associated with such systems. We show that the underlying shape invariance symmetry is responsible for the solubility of the differential equations associated with these polynomials.

Keywords: Exceptional Orthogonal Polynomials, Rational extensions, Supersymmetric Quantum Mechanics, Shape Invariance

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I. INTRODUCTION

In 2009 new families of orthogonal polynomials (known as exceptional orthogonal polynomials (EOP)), related to some of the old classical orthogonal polynomials are discovered [1–3]. Unlike the usual classical orthogonal polynomials, these EOPs start with degree \( n = 1 \) or higher integer values and still form a complete orthonormal set with respect to a positive definite inner product defined over a compact interval. Two of the well known classical orthogonal polynomials, namely the Laguerre orthogonal polynomials and the Jacobi orthogonal polynomials have been extended to the EOPs category. By \( X_m \) Laguerre (Jacobi) EOP means the complete set of Laguerre (Jacobi) orthogonal polynomials with degree \( \geq m \). Here \( m \) is a positive integer and can have values \( 1, 2, 3 \cdots \). Soon after this remarkable discovery, the connection of EOPs with the translationally shape invariant potential was established [4–8]. The list of exactly solvable quantum mechanical potentials thus got enlarged with the corresponding energy eigen functions for the newly obtained exactly solvable systems being expressed in terms of EOPs. Such systems are known as rational extension of the original systems. These developments have greatly helped in extending the list of the exactly solvable potentials over the past decade [9–36]. Attempts have also been made to extend the classical Hermite polynomials to the EOP category [37].

In the present article we would like to study EOPs in the framework of the SUSYQM. The differential equations for the Jacobi and the Laguerre EOPs can be written as eigenvalue equations. We can associate a Hamiltonian operator to each of the EOPs by comparing the associated differential equation with the time independent Schrödinger equation, \( H \psi = E \psi \). Further, we show that these Hamiltonians are factorizable in terms of the raising and the lowering operators. These raising (lowering) operators when act on the EOPs, increase (decrease) the order of the polynomials by one unit. This enables us to construct the supersymmetric partner Hamiltonians for both the EOPs. Supersymmetric techniques are then used to get the solutions corresponding to these EOPs. Further by introducing an arbitrary integer parameter, we construct a sequence of supersymmetric partner Hamiltonians to show the underlying shape invariance associated with such systems. For, simplicity we first present the case of \( X_1 \) EOPs and later generalize to arbitrary \( m \). It is worth adding that a similar study has already been undertaken for the classical orthogonal polynomials [38–40] as well as for hypergeometric function [41].
The plan of the paper is as follows. In Sec. II we briefly introduce the basic ideas of SUSYQM and SI which will be necessary for the analysis of the EOPs. In section III we discuss $X_1$ Jacobi and the $X_1$ Laguerre EOPs within the framework of SUSYQM and bring out the SI associated with these EOPs which is responsible for the solubility of these EOPs. In Sec IV we generalize these results to the $X_m$ EOPs. Finally we summarize the results in Sec V.

### II. SUPERSYMMETRY AND SHAPE INVARIANCE

In this section we briefly review some key features of SUSYQM and shape invariance [42, 43]. One dimensional Hamiltonian (in units of $2m = 1$, $\hbar = 1$) is factorize in terms of first order operators as

$$A = \frac{d}{dx} + W(x)$$
$$A^\dagger = -\frac{d}{dx} + W(x)$$  \hspace{1cm} (1)

where $W(x)$, a real function of $x$, is known as the super potential while $A^\dagger$ and $A$ are raising and lowering operators, analogous to case of simple harmonic oscillator. The supersymmetric partner Hamiltonians are then defined as

$$H_- = A^\dagger A = -\frac{d^2}{dx^2} + V_-(x)$$
$$H_+ = AA^\dagger = -\frac{d^2}{dx^2} + V_+(x)$$  \hspace{1cm} (2)

Where $V_\pm = W^2 \mp W'(x)$. These two Hamiltonians $H_\pm$ have exactly same spectrum except the lowest energy state of $H_-$, which is zero, i.e.

$$E_0^{(-)} = 0, \quad E_n^{(+)} = E_{n+1}^{(-)}.$$  \hspace{1cm} (3)

The eigenfunctions of $H_\pm$, $\psi_\pm^n$, are interrelated through the operations of $A^\dagger$, $A$. In particular,

$$\psi_+^n = A^\dagger \psi_-^{n+1}, \quad \psi_-^{n+1} = A \psi_+^n, \quad \psi_-^0 = Ne\int W(y)dy.$$  \hspace{1cm} (4)

Thus SUSYQM relates the bound state eigenvalues as well as the corresponding eigenfunctions of the partner Hamiltonians without actually finding them.

This analysis can also be extended to the non-Hermitian systems[44]. In the case of non hermitian Hamiltonians the raising and lowering operators are not hermitian conjugate to
each other and hence the supersymmetry partner Hamiltonians are like, $BA$ and $AB$ (instead of $A^tA$ and $AA^t$). In this case the spectrum needs not be bounded from below.

However, in case the two partner potentials are shape invariant, i.e. if they satisfy

$$V_+(x, a_0) = V_-(x, a_1) + R(a_1),$$

(5)

where $a_1$ is some function of $a_0$, i.e. $a_1 = f(a_0)$ and $R(a_1)$ is a constant then one can show that the energy eigenvalues corresponding to $V_+(x)$ are given by

$$E^{(-)}_{n+1} = \sum_{i=1}^{n} R(a_i), \quad E^{(-)}_0 = 0.$$  

(6)

If there is a family of $p$ shape invariant Hamiltonians, one can relate the $p$-th eigenfunction of original Hamiltonian $H^1$ to $H^p$ as

$$\psi^1_p(x; a_1) \propto B(x; a_1)B(x; a_2)...B(x; a_p)\psi^1_0(x; a_{p+1})$$

(7)

where $B(x; a_p)$'s are the raising operators.

III. $X_1$- EXCEPTIONAL ORTHOGONAL POLYNOMIALS

In this section, we discuss the $X_1$ Jacobi and the $X_1$ Laguerre polynomials within the framework of SUSYQM and SI techniques and bring out the reasoning for the solubility of the corresponding $X_1$ Jacobi and the $X_1$ Laguerre differential equations.

A. $X_1$- Jacobi Orthogonal polynomials

The $X_1$-Jacobi orthogonal polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ satisfy the differential equation

$$\frac{d^2}{dx^2} \left[ (x^2-1) \frac{d}{dx} \right] + \frac{2(\beta-\alpha)}{(\beta+\alpha)-(\beta-\alpha)x} \hat{P}_n^{(\alpha,\beta)}(x) = 0$$

(8)

where $n = 1, 2, 3,...$ and the real parameter $\alpha > -1$ and $\beta > -1$.

The Eq. (8) can be expressed as an eigenvalue equation

$$\left[ (x^2-1) \frac{d^2}{dx^2} + (\beta+\alpha+2)x - (\beta-\alpha) + \frac{2(\beta-\alpha)(x^2-1)}{(\beta+\alpha)-(\beta-\alpha)x} \frac{d}{dx} \right] \hat{P}_n^{(\alpha,\beta)}(x) = 0$$

(9)

$$n = 1, 2, 3, ...$$
This can be formally put in the form
\[ [H - (n - 1)(n + \alpha + \beta)]\hat{P}_n^{(\alpha,\beta)}(x) = 0 \] (10)

where
\[ H = (x^2 - 1)\frac{d^2}{dx^2} + \left( (\beta + \alpha + 2)x - (\beta - \alpha) + \frac{2(\beta - \alpha)(x^2 - 1)}{(\beta + \alpha) - (\beta - \alpha)x} \right) \frac{d}{dx} + \left( \beta - \alpha)x + \frac{(\beta - \alpha)^2(x^2 - 1)}{(\beta + \alpha) - (\beta - \alpha)x} \right) \] (11)

The raising and lowering operators for the \( X_1 \)- Jacobi polynomials are
\[ A_{\alpha,\beta} = \frac{(x - c)^2}{x - b} \frac{d}{dx} \left( \frac{1}{x - c} \right) \] (12a)
\[ B_{\alpha,\beta} = (x^2 - 1)\frac{(x - b)}{(x - c)} \left( \frac{d}{dx} + a \right) - a(x^2 - 2bx + 1) \] (12b)

with \( a, b \) and \( c \) being related to \( \alpha, \beta \) by the following relations.
\[ a = \frac{\beta - \alpha}{2} \] (13a)
\[ b = \frac{\beta + \alpha}{\beta - \alpha} \] (13b)
\[ c = b = \frac{1}{a} \] (13c)

These raising and lowering operators hold the following relations
\[ A_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)}(x) = \left( \frac{n + \alpha + \beta}{2} \right)\hat{P}_{n-1}^{(\alpha+1,\beta+1)} \] (14)
\[ B_{\alpha,\beta}\hat{P}_{n+1}^{(\alpha+1,\beta+1)} = 2n\hat{P}_n^{(\alpha,\beta)} \] (15)

and the Hamiltonian in equation (11) can then be factorized in terms of these two operators as
\[ H = B_{\alpha,\beta}A_{\alpha,\beta} = A_{\alpha-1,\beta-1}B_{\alpha-1,\beta-1} - (\alpha + \beta) \] (16)

By using Eqs. (14) and (15) in Eq. (10) we have
\[ B_{\alpha,\beta}A_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)} = E_n\hat{P}_n^{(\alpha,\beta)} \] (17)
\[ A_{\alpha,\beta}B_{\alpha,\beta}(A_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)}) = E_n(A_{\alpha,\beta}\hat{P}_n^{(\alpha,\beta)}) \]

with
\[ E_n = (n - 1)(n + \alpha + \beta) \] (18)
Thus $H^1$ and $H^2$ as defined by

$$H^1 = B_{\alpha,\beta}A_{\alpha,\beta}$$  \hspace{1cm} (19a)
$$H^2 = A_{\alpha,\beta}B_{\alpha,\beta}$$  \hspace{1cm} (19b)

are supersymmetric partner Hamiltonians.

We can generalize this further and show that the pair of Hamiltonians $B_{\alpha+k-1,\beta+k-1}A_{\alpha+k-1,\beta+k-1}$ and $A_{\alpha+k-1,\beta+k-1}B_{\alpha+k-1,\beta+k-1}$ are the supersymmetric partner Hamiltonians for $k=1,2,...$. In particular, by using Eqs. (14) and (15) we can write,

$$B_{\alpha+k,\beta+k}A_{\alpha+k,\beta+k}\hat{P}_n^{(\alpha+k,\beta+k)} = E_n^k \hat{P}_n^{(\alpha+k,\beta+k)} ,$$  \hspace{1cm} (20)

and

$$A_{\alpha+k-1,\beta+k-1}B_{\alpha+k-1,\beta+k-1}\hat{P}_n^{(\alpha+k,\beta+k)} = E_{n+1}^{k-1} \hat{P}_n^{(\alpha+k,\beta+k)} ,$$  \hspace{1cm} (21)

where

$$E_{n}^{k} = (n-1)(n+\alpha+\beta+2k-2) .$$  \hspace{1cm} (22)

Above equations imply

$$B_{\alpha+k,\beta+k}A_{\alpha+k,\beta+k} = E_{n}^{k}$$

$$A_{\alpha+k-1,\beta+k-1}B_{\alpha+k-1,\beta+k-1} - E_{n+1}^{k-1} ,$$  \hspace{1cm} (23)

where (k=1,2,3,...). From Eq. (23) it is clear that the supersymmetric pair of Hamiltonians $B_{\alpha+k-1,\beta+k-1}A_{\alpha+k-1,\beta+k-1}$ and $A_{\alpha+k-1,\beta+k-1}B_{\alpha+k-1,\beta+k-1}$ are in fact shape invariant with only a shift.

By using the above generalize operators we can find the $X_1$ Jacobi polynomials

$$2^{(s-1)}(s-1)!\hat{P}_s^{(\alpha,\beta)} = B_{\alpha,\beta}B_{\alpha+1,\beta+1}\cdots B_{\alpha+s-2,\beta+s-2}\hat{P}_1^{(\alpha+s-1,\beta+s-1)}$$  \hspace{1cm} (24)

where $s = 2, 3, 4 \cdots$. Let us consider,

$$H^k = B_{\alpha+k-1,\beta+k-1}A_{\alpha+k-1,\beta+k-1} + (k-1)(k+\alpha+\beta)$$

It is straight forward to see that this sequence of Hamiltonians satisfy Shape invariant property and can be written as

$$H^{k+1} = H^k + (\alpha + \beta + 2k) = H^k + R_k$$  \hspace{1cm} (25)
where $R_k = (\alpha + \beta + 2k)$

So the $n$th eigenvalue of the original Hamiltonian ($H^1$), as expected, is given by

$$E_n = \sum_{k=1}^{n-1} (\alpha + \beta) + 2k$$

$$= (n - 1)(n + \alpha + \beta).$$

B. $X_1$ Laguerre polynomials

For $n \geq 1$ and the real parameter $k > 0$ the $X_1$ Laguerre polynomials $\hat{L}_n^{(k)}$ satisfy the differential equation

$$-x\frac{d^2}{dx^2} + \frac{(x-k)}{(x+k)}[(x+k+1)\frac{d}{dx}] - \frac{(x-k)}{(x+k)} + (n - 1)\hat{L}_n^{(k)} = 0$$

This differential equation can be written as the eigenvalue equation as

$$\left(-x\frac{d^2}{dx^2} + \frac{(x-k)}{(x+k)}[(x+k+1)\frac{d}{dx}] - 1 - (n - 1)\right)\hat{L}_n^{(k)} = 0,$$

which can be compactly expressed as

$$[H - (n - 1)]\hat{L}_n^{(k)} = 0,$$

where

$$H = -x\frac{d^2}{dx^2} + \frac{(x-k)}{(x+k)}[(x+k+1)\frac{d}{dx}] - 1.$$

One can define the raising and the lowering operators for the $X_1$ Laguerre polynomials as

$$A_k = -\frac{(x+k+1)^2}{x+k}\frac{d^2}{dx}\left(\frac{1}{x+k+1}\right),$$

$$B_k = \frac{x(x+k)}{(x+k+1)}\left(\frac{d}{dx} - 1\right) + k.$$

The operators $A_k$ and $B_k$ satisfy following raising and lowering properties

$$A_k\hat{L}_n^{(k)} = \hat{L}_{n-1}^{(k+1)},$$

$$B_k\hat{L}_n^{(k+1)} = n\hat{L}_{n+1}^{(k)}.$$ 

Hence the Hamiltonian in Eq. (30) can be factorized in terms of these two operators as

$$H = B_kA_k = A_{k-1}B_{k-1} - 1.$$
Using the Eqs. (32a) and (32b) we obtain

\[ B_k A_k \hat{L}_n^{(k)} = E_n^0 \hat{L}_n^{(k)}, \]

\[ A_k B_k (A_k \hat{L}_n^{(k)}) = E_n^0 (A_k \hat{L}_n^{(k)}) \quad \text{or} \quad A_k B_k (\hat{L}_{n-1}^{(k+1)}) = E_n^0 (\hat{L}_{n-1}^{(k+1)}), \]

where,

\[ E_n^0 = (n - 1). \]

This indicates that the Hamiltonians \( A_k B_k \) and \( B_k A_k \) are supersymmetric partner Hamiltonians. Now to establish the shape invariance, we construct the sequence of Hamiltonians \( A_{k+r-1} B_{k+r-1} \) and \( B_{k+r-1} A_{k+r-1} \) with \( r = 1, 2, 3, \ldots \). It is straightforward to check

\[ B_{k+r-1} A_{k+r-1} \hat{L}_n^{(k+r-1)} = E_n^{r-1} \hat{L}_n^{(k+r-1)}, \]

\[ A_{k+r-1} B_{k+r-1} (A_{k+r-1} \hat{L}_n^{(k+r-1)}) = E_n^{r-1} (A_{k+r} \hat{L}_n^{(k+r-1)}) \quad \text{or} \quad A_{k+r-1} B_{k+r-1} \hat{L}_{n-1}^{(k+r)} = E_n^{r-1} \hat{L}_{n-1}^{(k+r)} \]

with \( E_n^{r-1} = (n - 1) \). From the recurrence relations we can write

\[ B_{k+r} A_{k+r} - (n - 1) \]

\[ = A_{k+r-1} B_{k+r-1} - n. \]

The above relation shows that the supersymmetric partner Hamiltonians in Eq. (35) are also shape invariant with a constant shift, independent of \( r \).

We can also relate the \( X_1 \) Laguerre polynomials by using raising operators in the following way

\[ (s - 1)! \hat{L}_s^{(k)} = B_k B_{k+1} B_{k+2} \cdots B_{k+s-2} \hat{L}_1^{(k+s-1)} \]

with \( s = 2, 3, 4, \ldots \). Let us define

\[ H^r = B_{k+r-1} A_{k+r-1} + (r - 1). \]

We can see that the sequence of shape invariant Hamiltonians satisfy

\[ A_{k+r-1} B_{k+r-1} + (r - 1) \]

\[ = B_{k+r} A_{k+r} + 1 + r - 1 \]

\[ = B_{k+r} A_{k+r} + r \]

\[ = H^{r+1} = H^r + 1. \]
Hence the $n^{th}$ eigenvalue of the Hamiltonian $H^1$ is

$$E_n = \sum_{k=1}^{n-1} 1$$

$$= (n - 1).$$

(40)

IV. $X_m$- ORTHOGONAL POLYNOMIALS

The results obtained in the last section are easily generalized for the $X_m$ EOPs as we show now.

A. $X_m$- Jacobi orthogonal polynomials

The $X_m$ Jacobi orthogonal polynomials $\hat{P}_{n,m}^{(\alpha,\beta)}(x)$ satisfy the differential equation

$$\hat{P}_{n,m}^{(\alpha,\beta)}(x) + \left( (\alpha - \beta - m + 1) \frac{P_{m-1}^{(-\alpha,\beta)}(x)}{P_{m-1}^{(-\alpha-1,\beta-1)}(x)} - \left( \frac{\alpha + 1}{1 - x} \right) + \left( \frac{\beta + 1}{1 + x} \right) \right)$$

$$\hat{P}_{n,m}^{(\alpha,\beta)}(x) + \frac{1}{(1 - x^2)} \left[ \beta(\alpha - \beta - m + 1)(1 - x) \frac{P_{m-1}^{(-\alpha,\beta)}(x)}{P_{m-1}^{(-\alpha-1,\beta-1)}(x)} ight.$$  

$$+ m(\alpha - \beta - m + 1) + (n - m)(\alpha + \beta + n - m + 1) \left] \hat{P}_{n,m}^{(\alpha,\beta)}(x) = 0 \right.$$  

(41)

Where $P_{m}^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials.

The $X_m$ Jacobi differential equation can be written as the eigenvalue equation

$$\left[ (x^2 - 1) \frac{d^2}{dx^2} + (x^2 - 1) \left( (\alpha - \beta - m + 1) \frac{P_{m-1}^{(-\alpha,\beta)}(x)}{P_{m-1}^{(-\alpha-1,\beta-1)}(x)} - \left( \frac{\alpha + 1}{1 - x} \right) + \left( \frac{\beta + 1}{1 + x} \right) \right) \right]$$

$$\left[ (x^2 - 1) \frac{d^2}{dx^2} + \left( \beta(\alpha - \beta - m + 1)(1 - x) \frac{P_{m-1}^{(-\alpha,\beta)}(x)}{P_{m-1}^{(-\alpha-1,\beta-1)}(x)} ight.$$  

$$- m(\alpha - \beta - m + 1) - (n - m)(\alpha + \beta + n - m + 1) \left] \hat{P}_{n,m}^{(\alpha,\beta)}(x) = 0 \right.$$  

$$ \tag{42}$$

This can be formally written as

$$[H - (n - m)(\alpha + \beta + n - m + 1)] \hat{P}_{n,m}^{(\alpha,\beta)}(x) = 0,$$  

$$ \tag{43}$$
where
\[
H = (x^2 - 1) \frac{d^2}{dx^2} + (x^2 - 1) \left( (\alpha - \beta - m + 1) \frac{P_m^{-\alpha,\beta}(x)}{P_m^{-\alpha-1,\beta-1}(x)} - \frac{(\alpha + 1)}{1 - x} + \frac{\beta + 1}{1 + x} \right) \ast \frac{d}{dx} + \beta(\alpha - \beta - m + 1)(x - 1) \frac{P_m^{-\alpha,\beta}(x)}{P_m^{-\alpha-1,\beta-1}(x)} - m(\alpha - \beta - m + 1)
\]

The lowering and raising operators for the \(X_m\) Jacobi polynomials are
\[
A_m^{\alpha,\beta} = \frac{P_m^{-(\alpha-2,\beta)}(x)}{P_m^{-(\alpha-1,\beta-1)}(x)} \left[ \frac{d}{dx} - \frac{1}{2} (\alpha - \alpha + m - 1) \frac{P_m^{-(\alpha-1,\beta+1)}(x)}{P_m^{-(\alpha-2,\beta)}(x)} \right], \tag{45}
\]
and
\[
B_m^{\alpha,\beta} = (1 - x^2) \frac{P_m^{-(\alpha-1,\beta-1)}(x)}{P_m^{-(\alpha-2,\beta)}(x)} \left[ \frac{d}{dx} - \left( \frac{1}{2} (\beta - \alpha + m - 1) \frac{P_m^{-(\alpha,\beta)}(x)}{P_m^{-(\alpha-1,\beta-1)}(x)} \right. \\
+ \left. \left( \frac{\alpha + 1}{1 - x} - \left( \frac{\beta + 1}{1 + x} \right) \right) \right] \right] \tag{46}
\]

These lowering and raising operators hold the following relations
\[
A_m^{\alpha,\beta} \hat{P}_n^{(\alpha,\beta)}(x) = \frac{1}{2} (\alpha + \beta + n - m + 1) \hat{P}_{n-1,m}^{(\alpha+1,\beta+1)}(x) \quad n \geq m \tag{47}
\]
and
\[
B_m^{\alpha,\beta} \hat{P}_{n,m}^{(\alpha+1,\beta+1)}(x) = 2(n - m + 1) \hat{P}_{n+1,m}^{(\alpha,\beta)}(x) \quad n \geq m. \tag{48}
\]

Now by using the above recurrence relations, the Hamiltonian in (44) can be factorize as
\[
H = B_m^{\alpha,\beta} A_m^{\alpha,\beta} = A_{m-1,\beta-1}^{\alpha-1,\beta-1} B_{m-1,\beta-1}^{\alpha-1,\beta-1} - (\alpha + \beta). \tag{49}
\]

On using Eqs. (47) and (48), we can write
\[
B_m^{\alpha,\beta} A_m^{\alpha,\beta} \hat{P}_n^{(\alpha,\beta)}(x) = E_{n,m}^0 \hat{P}_n^{(\alpha,\beta)}(x), \tag{50}
\]
\[
A_m^{\alpha,\beta} B_m^{\alpha,\beta} (A_{m-1,\beta-1}^{\alpha-1,\beta-1} \hat{P}_{n,m}^{(\alpha,\beta)}(x)) = E_{n,m}^0 (A_{m-1,\beta-1}^{\alpha-1,\beta-1} \hat{P}_{n,m}^{(\alpha,\beta)}(x)),
\]
where
\[
E_{n,m}^0 = (n - m)(\alpha + \beta + n - m + 1). \tag{51}
\]

Thus the Hamiltonians
\[
H^1 = B_m^{\alpha,\beta} A_m^{\alpha,\beta},
\]
\[
H^2 = A_m^{\alpha,\beta} B_m^{\alpha,\beta}, \tag{52}
\]
are supersymmetric partners. Now we construct a sequence of Hamiltonians by introducing a parameter \( k \) and find that the pair of Hamiltonians \( B_{\alpha+k-1,\beta+k-1}^{m}A_{\alpha+k-1,\beta+k-1}^{m} \) and \( A_{\alpha+k-1,\beta+k-1}^{m}B_{\alpha+k-1,\beta+k-1}^{m} \) are also supersymmetric partner Hamiltonians for \( k = 1, 2, 3, ... \)

\[
B_{\alpha+k-1,\beta+k-1}^{m}A_{\alpha+k-1,\beta+k-1}^{m} P_{n,m}^{(\alpha+k-1,\beta+k-1)}(x) = E_{n,m}^{k-1} P_{n,m}^{(\alpha+k-1,\beta+k-1)}(x)
\]

\[
A_{\alpha+k-1,\beta+k-1}^{m}B_{\alpha+k-1,\beta+k-1}^{m}(A_{\alpha+k-1,\beta+k-1}^{m} P_{n,m}^{(\alpha+k-1,\beta+k-1)}(x)) = E_{n,m}^{k-1}(A_{\alpha+k-1,\beta+k-1}^{m} P_{n,m}^{(\alpha+k-1,\beta+k-1)}(x))
\]

where \( k = 1, 2, 3, ... \) and

\[
E_{n,m}^{k-1} = (n - m)(\alpha + \beta + n - m + 2k - 1)
\]

We can write

\[
B_{\alpha+k,\beta+k}^{m}A_{\alpha+k,\beta+k}^{m} P_{n,m}^{(\alpha+k,\beta+k)}(x) = E_{n,m}^{k} P_{n,m}^{(\alpha+k,\beta+k)}(x),
\]

and

\[
A_{\alpha+k-1,\beta+k-1}^{m}B_{\alpha+k-1,\beta+k-1}^{m} P_{n,m}^{(\alpha+k-1,\beta+k-1)}(x) = E_{n+1,m}^{k-1} P_{n,m}^{(\alpha+k,\beta+k)}(x).
\]

From Eqs. (55) and (56) we can write,

\[
B_{\alpha+k,\beta+k}^{m}A_{\alpha+k,\beta+k}^{m} = A_{\alpha+k-1,\beta+k-1}^{m}B_{\alpha+k-1,\beta+k-1}^{m} + E_{n,m}^{k} - E_{n+1,m}^{k-1},
\]

where \( (k = 1, 2, 3, ...) \). Now from Eq. (57) it is clear that the supersymmetric pairs of Hamiltonians in Eq. (53) are in fact shape invariant with only a shift.

By using the above generalize operators we can write the \( X_{m} \) Jacobi polynomials as

\[
2^{(s-m)}(s-m)! \hat{P}_{s,m}^{(\alpha,\beta)}(x) = B_{\alpha,\beta,\alpha+1,\beta+1,...,\alpha+s-m-1,\beta+s-m-1}^{m} P_{m,m}^{(\alpha+s-m,\beta+s-m)}(x)
\]

Where \( s = m + 1, m + 2, \cdots \). To see that, let us define a family of Hamiltonians

\[
H^{k} = B_{\alpha+k-1,\beta+k-1}^{m}A_{\alpha+k-1,\beta+k-1}^{m} + (k - 1)(k + \alpha + \beta).
\]

We can see that these shape invariant Hamiltonians satisfy

\[
A_{\alpha+k-1,\beta+k-1}^{m}B_{\alpha+k-1,\beta+k-1}^{m} + (k - 1)(k + \alpha + \beta)
\]

\[
= B_{\alpha+k,\beta+k}^{m}A_{\alpha+k,\beta+k}^{m} + (\alpha + \beta + 2k) + (k - 1)(k + \alpha + \beta).
\]

Hence, we can write

\[
H^{k+1} = H^{k} + R_{k}
\]
where $R_k = (\alpha + \beta + 2k)$

So the $n^{th}$ eigenvalue of Hamiltonian $H^1$ is given by

$$E_n^m = \sum_{k=1}^{n-m} (\alpha + \beta) + 2k$$

$$= (n - m)(\alpha + \beta + n - m + 1). \quad (62)$$

**B. $X_m$-Laguerre orthogonal polynomials**

For an integer $m \geq 0$, the $X_m$ Laguerre orthogonal polynomials $\hat{L}_{n,m}^k(x)$ satisfy the differential equation

$$\hat{L}_{n,m}^k(x) + \frac{1}{x} \left[ (k + 1 - x) - 2x \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} \right] \hat{L}_{n,m}^k(x)$$

$$+ \frac{1}{x} \left[ n - 2k \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} \right] \hat{L}_{n,m}^k(x) = 0,$$

where $L_m^k$ are the classical Laguerre polynomials.

The above differential equation can be written in the form of eigenvalue equation as

$$\left[ -x \frac{d^2}{dx^2} + \left( (k - 1) + 2x \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} \right) \frac{d}{dx} \right.$$  
$$+ \left( 2k \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} - n \right) \hat{L}_{n,m}^k(x) = 0.$$

This can be formally put in the form

$$[H - (n - m)] \hat{L}_{n,m}^k(x) = 0,$$  

where

$$H = -x \frac{d^2}{dx^2} + \left( (k - 1) + 2x \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} \right) \frac{d}{dx} + \left( 2k \frac{L_{m-1}^k(-x)}{L_{m-1}^k(-x)} - m \right). \quad (66)$$

The lowering and raising operators for the $X_m$ Laguerre orthogonal polynomials are

$$A_m^k = - \frac{L_m^k(-x)}{L_{m-1}^k(-x)} \left( \frac{d}{dx} - \frac{L_{m-1}^k(-x)}{L_m^k(-x)} \right), \quad (67)$$

and

$$B_m^k = \frac{L_{m-1}^k(-x)}{L_m^k(-x)} \left( x \frac{d}{dx} + (1 + k) \right) - x. \quad (68)$$
These operators hold the following relations

\[ A^m_k \hat{L}^k_{n,m}(x) = \hat{L}^{k+1}_{n-1,m}(x) \]  \quad (69) 

and

\[ B^m_k \hat{L}^{k+1}_{n,m}(x) = (n - m + 1) \hat{L}^k_{n+1,m}(x) \]  \quad (70) 

From the above two recurrence relations we can write the Hamiltonian in Eq. (66) in a factorize form as

\[ H = B^m_k A^m_k = A^m_{k-1} B^m_{k-1} - 1. \]  \quad (71) 

On using Eqs. (69) and (70) we can write

\[ B^m_k A^m_k \hat{L}^k_{n,m}(x) = E^0_{n,m} \hat{L}^k_{n,m}(x) \]

\[ A^m_k B^m_k (A^m_k \hat{L}^k_{n,m}(x)) = E^0_{n,m}(A^m_k \hat{L}^k_{n,m}(x)), \]  \quad (72)

where \( E^0_{n,m} = (n - m). \)

The above conditions show that the Hamiltonians \( A^m_k B^m_k \) and \( B^m_k A^m_k \) are supersymmetric partner Hamiltonians.

Now we introduce a parameter \( r \) and find that the pairs of Hamiltonians \( A^m_{k+r-1} B^m_{k+r-1} \) and \( B^m_{k+r-1} A^m_{k+r-1} \) are supersymmetric partners, for \( r = 1, 2, 3 \cdots \), by satisfying the conditions

\[ B^m_{k+r-1} A^m_{k+r-1} \hat{L}^{k+r-1}_{n,m}(x) = E^{r-1}_{n,m} \hat{L}^{k+r-1}_{n,m}(x) \]

\[ A^m_{k+r-1} B^m_{k+r-1} (A^m_{k+r-1} \hat{L}^{k+r-1}_{n,m}(x)) = E^{r-1}_{n,m}(A^m_{k+r-1} \hat{L}^{k+r-1}_{n,m}(x)), \]  \quad (73)

where \( E^{r-1}_{n,m} = (n - m). \)

We see that the eigenvalues of the Hamiltonians are independent of \( r \). So all \( n \) eigenstates have the degenerate eigenvalues.

From the recurrence relation we can write

\[ B^m_{k+r} A^m_{k+r} - (n - m) = A^m_{k+r-1} B^m_{k+r-1} - (n - m + 1). \]  \quad (74) 

The above relation shows that the supersymmetric pair of Hamiltonians in Eq. (73) are in fact shape invariant with a constant shift.
We can also relate the $X_m$ Laguerre polynomials by using raising operators in the following way

$$(s - m)!\hat{L}_{s,m}^{(k)} = B_k^m B_{k+1}^m B_{k+2}^m \cdots B_{k+s-m-1}^m \hat{L}_{m,m}^{(k+s-m)}$$

(75)

where $s = m, m+1, \cdots$. Let us define

$$H^r = B_k^m A_{k+r-1}^m + (r - 1)$$

(76)

we can see that the sequence of shape invariant Hamiltonian satisfy

$$A_{k+r-1}^m B_{k+r-1}^m + (r - 1) = B_{k+r}^m A_{k+r}^m + r,$$

(77)

or in other words $H^{r+1} = H^r + 1$ so that the $n$'th eigen value of the Hamiltonian is

$$E_n^m = \sum_{k=1}^{n-m} 1 = (n - m).$$

(78)

V. CONCLUSION

Exceptional Laguerre and exceptional Jacobi polynomials are used extensively in the rational extension of many quantum mechanical problems and hence worth investigating them in details. In this paper we have described a new way of looking at the differential equations for these EOPs and their solutions in the framework of supersymmetry and shape invariance. Supersymmetry and Shape invariance of the “Hamiltonians” corresponding to $X_1$-Jacobi and $X_1$-Laguerre polynomials have explicitly been shown in this work. The underlying shape invariance symmetry is responsible for the solubility of differential equations associated with these EOPs. Further these results are then generalised for $X_m$- Jacobi and $X_m$ Laguerre polynomials with $m = 1, 2, 3 \cdots$. We would like to add that classical Legendra and hypergeometric polynomials have been analysed in this fashion in [40, 41]. In the present work we have extended their work for the EOPs. In reference [45] some master conditions are provided to check this for classical polynomials. It will be interesting to investigate EOPs along the line of [45] to obtain similar master condition for EOPs.

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