PLURICANONICAL MAPS OF STABLE LOG SURFACES

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Abstract. Stable surfaces and their log analogues are the type of varieties naturally occurring as boundary points in moduli spaces. We extend classical results of Kodaira and Bombieri to this more general setting: if \((X, \Delta)\) is a stable log surface with reduced boundary (possibly empty) and \(I\) is its global index, then \(4I(K_X + \Delta)\) is base-point-free and \(8I(K_X + \Delta)\) is very ample.

These bounds can be improved under further assumptions on the singularities or invariants, for example, \(5(K_X + \Delta)\) is very ample if \((X, \Delta)\) has semi-canonical singularities.

1. Introduction

It is a general fact that moduli spaces of nice objects in algebraic geometry, say smooth varieties, are often non-compact. But usually there is a modular compactification where the boundary points correspond to related, but more complicated objects.

Such a modular compactification has been known for the moduli space \(\mathcal{M}_g\) of smooth curves of genus \(g\) for a long time and in [KSB88] Kollár and Shepherd-Barron made the first step towards the construction of a modular compactification \(\overline{\mathcal{M}}\) for the moduli space \(\mathcal{M}\) of surfaces of general type. Even though the actual construction of the moduli space was delayed for several decades because of formidable technical obstacles to be overcome, it was clear from the beginning that the objects parametrised by \(\overline{\mathcal{M}}\) should be surfaces with semi-log-canonical singularities and ample canonical divisor, for short stable surfaces.

A more general version also incorporates the possibility of a (reduced) boundary divisor (see Section 2 for the precise definitions): this is the higher dimensional analogue of pointed stable curves and was worked out by Alexeev [Ale96, Ale06].

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In recent years, several components of the moduli space of stable varieties or pairs have been investigated in detail\(^1\). On the other hand, many of the standard tools to study, for example, smooth surfaces of general type are not yet available for stable surfaces. In this paper, we make the first steps in the understanding of pluricanonical maps of such surfaces.

Pluricanonical maps are one of the main tools in the study of smooth surfaces of general type and their canonical models. They have been an active subject of research ever since Bombieri’s seminal paper \([\text{Bom}73]\). Recall that the \(m\)-canonical map of a variety \(X\) is the rational map \(\varphi_m : X \dashrightarrow \mathbb{P}^N\) associated to the linear system \(|mK_X|\). Then the roughest version of Bombieri’s results says that on a surface with canonical singularities and ample canonical divisor \(\varphi_m\) is an embedding for \(m \geq 5\); it had been proved earlier by Kodaira that \(\varphi_m\) is a morphism for \(m \geq 4\) \([\text{Kod}68]\). These results are sharp but can be much refined and we refer to [BHPV04, Sect. VII] or the recent survey [BCP06] for more information.

The singularities of a stable surface can be much worse than canonical singularities: in general they are non-normal, not Gorenstein and not (semi-)rational. Thus many of the techniques which one can use to prove Bombieri-type theorems do not carry over directly. The following theorem is proved in Section 4 by applying a Reider-type result due to Kawachi on the normalisation combined with a detailed analysis of the non-normal locus.

**Theorem 4.1** — Let \((X, \Delta)\) be a connected stable log surface with global index \(I\).\(^2\)

(i) The line bundle \(\omega_X(\Delta)^{|mI|}\) is base-point-free for \(m \geq 4\).

(ii) The line bundle \(\omega_X(\Delta)^{|mI|}\) is base-point-free for \(m \geq 3\) if one of the following holds:
   a) \(I \geq 2\).
   b) There is no irreducible component \(\bar{X}_i\) of the normalisation such that \((\pi^*(K_X + \Delta)|_{\bar{X}_i})^2 = 1\), and the union of \(\Delta\) and the non-normal locus is a nodal curve.
   c) \(X\) is normal and we do not have \(I = (K_X + \Delta)^2 = 1\).

For normal stable surfaces without boundary this recovers [KM98a, Cor. 3, Cor. 4].

Our results on pluri-log-canonical embeddings are somewhat more involved. We follow an approach due to Catanese and Franciosi [CF96], later refined in collaboration with Hulek and Reid [CFHR99]: for every subscheme of length two find a pluri-log-canonical curve containing it and then prove that this curve is embedded by \(|mI(K_X + \Delta)|\). Without further assumptions on singularities and invariants we get:

**General bounds (Theorem 5.1)** — Let \((X, \Delta)\) be a connected stable log surface of global index \(I\).

(i) The line bundle \(\omega_X(\Delta)^{|mI|}\) is very ample for \(m \geq 8\).

(ii) The line bundle \(\omega_X(\Delta)^{|mI|}\) defines a birational morphism for \(m \geq 6\).

(iii) The line bundle \(\omega_X(\Delta)^{|m|}\) is very ample for \(m \geq 6\) if \(I \geq 2\).

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\(^1\)The following is a probably incomplete list of results in this direction: [Has99, Lee00, Hac04, vO05, vO06b, vO06a, AP09, HKT09, Rol10, Lin12, Laz12, BHPS12, Pat12].

\(^2\)Multiplication with the index is clearly necessary since \(\varphi_m\) cannot be a morphism if \(m(K_X + \Delta)\) is not a Cartier divisor. However, it does make sense to ask which is the first pluri-log-canonical map to be birational. In the normal case Langer has given an explicit but still unrealistically large bound [Lan01, Sect. 9].
We do not believe all of these bounds to be sharp. The main obstacles in our proof are the extra contributions from the worse than canonical singularities and the fact that curves containing irreducible components of the non-normal locus and the boundary do not behave well under normalisation. We explain this more in detail in Section 5.1, see also Remark 5.8.

Under additional assumptions we can improve the bounds obtained above. In particular, if $X$ is semi-canonical then we obtain the same bound as in the classical case.

Bounds for milder singularities (Theorem 5.2) — Let $(X, \Delta)$ be a connected stable log surface of global index $I$ and let $D$ be the non-normal locus of $X$.

(i) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 7$ if one of the following holds:

a) There is no irreducible component $\bar{X}_i$ of the normalisation such that $(\pi^*(K_X + \Delta)|_{\bar{X}_i})^2 = 1$, and the union of $\Delta$ and the non-normal locus is a nodal curve.

b) $X$ is normal and not $(K_X + \Delta)^2 = 1$.

(ii) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 6$ if the normalisation $\bar{X}$ is smooth along the conductor divisor and has at most canonical singularities elsewhere.

(iii) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 5$ if $D \cup \Delta$ is a nodal curve, $\bar{X}$ is smooth along the conductor divisor, and $X \setminus D$ has at most canonical singularities.

In particular these conditions are satisfied, if $(X, \Delta)$ has semi-canonical singularities.

For a connected stable surface $X$ with canonical singularities the bi-canonical map is a morphism as soon as $K_X^2 \geq 5$ and the tri-canonical map is an embedding as soon as $K_X^3 \geq 6$ (see [Cat87]). Such behaviour cannot be expected for stable surfaces: in Example 7.2 we construct an irreducible, Gorenstein stable surface with $K_X^2$ arbitrarily large such that the bi-canonical map not a morphism and neither the tri-canonical nor the 4-canonical map is an embedding.

A natural extension of the aforementioned results is the study of the log-canonical ring. We do not engage in a detailed study but only state the results that follow by standard methods from Theorem 4.1.

Theorem 6.1 — Let $(X, \Delta)$ be a stable log surface of index $I$. Then the log-canonical ring,

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, \omega_X(\Delta)^{[mI]}),$$

is generated in degree at most $12I + 1$ and in degree at most $9I + 1$ under the same assumptions as in Theorem 4.1(ii).

All the results should only be regarded as a first step towards a precise understanding of pluri-log-canonical maps and log-canonical rings of stable log surfaces.

Our method relies on the rough classification of semi-log-canonical singularities and therefore does not generalise to higher dimensions at the moment.

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1.1. Notations and conventions. We work exclusively with schemes of finite type
over the complex numbers.

- The singular locus of a scheme $X$ will be denoted by $X_{\text{sing}}$.
- A surface is a reduced, projective scheme of pure dimension 2 but not neces-
sarily irreducible or connected.
- A curve is a purely 1-dimensional scheme that is Cohen–Macaulay. A curve
is not assumed to be reduced, irreducible or connected. For a point $p \in C$
we denote by $\mu_p(C)$ its multiplicity.
- For a sheaf $\mathcal{F}$ on $X$ we denote by $\mathcal{F}^m = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes m, \mathcal{O}_X), \mathcal{O}_X)$
the reflexive powers.
- We switch back and forth between multiples of a canonical divisor, $mK_X$, 
and reflexive powers of the canonical sheaf $\omega^m_X = \mathcal{O}_X(mK_X)$. See Section
2.3 for a discussion of divisors and associated divisorial sheaves.

Some further notation on demi-normal schemes or semi-log-canonical pairs will be
fixed in Notation 2.3.

2. Preliminaries

In this section we recall some necessary notions as well as constructions that we
need throughout the text. Most of these are available in all dimensions, but for our
purpose it suffices to focus on the case of surfaces. Our main reference is [Kol13,
Sect. 5.1–5.3].

2.1. Stable log surfaces. Let $X$ be a demi-normal surface, that is, $X$ satisfies $S_2$
and at each point of codimension 1, $X$ is either regular or has an ordinary double
point. We denote by $\pi: \tilde{X} \rightarrow X$ the normalisation of $X$. The conductor ideal
$\mathcal{H}om_{\mathcal{O}_X}(\pi_*, \mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$ is an ideal sheaf in both $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$ and as such defines
subscheme $D \subset X$ and $\tilde{D} \subset \tilde{X}$, both reduced and of pure codimension 1; we often
refer to $D$ as the non-normal locus of $X$.

Let $\Delta$ be a reduced curve on $X$ whose support does not contain any irreducible
component of $D$. Then the strict transform $\tilde{\Delta}$ in the normalisation is well defined.

Definition 2.1 — We call a pair $(X, \Delta)$ as above a log surface; $\Delta$ is called the
(reduced) boundary.\(^3\)

A log surface $(X, \Delta)$ is said to have semi-log-canonical (slc) singularities if it
satisfies the following conditions:

(i) $K_X + \Delta$ is $\mathbb{Q}$-Cartier, that is, $m(K_X + \Delta)$ is Cartier for some positive integer
$m$; the minimal such $m$ is called the (global) index of $(X, \Delta)$.

(ii) The pair $(\tilde{X}, \tilde{D} + \tilde{\Delta})$ has log-canonical singularities.

The pair $(X, \Delta)$ is called stable log surface if in addition $K_X + \Delta$ is ample. A stable
surface is a stable log surface with empty boundary; these are the surfaces relevant
for the compactification of the Gieseker moduli space.

\(^3\)In general one can allow rational coefficients in $\Delta$, but we will not use this here.
By abuse of notation we say \((X, \Delta)\) is a Gorenstein stable log surface if the index is equal to one, i.e., \(K_X + \Delta\) is an ample Cartier divisor.

Let \((X, \Delta)\) be a log surface. Since \(X\) has at most double points in codimension 1 the map \(\pi: \bar{D} \to D\) on the conductor divisors is generically a double cover and thus induces a rational involution on \(D\). Normalising the conductor loci we get an honest involution \(\tau: D^\nu \to D^\nu\) such that \(D^\nu = \bar{D}^\nu/\tau\).

**Theorem 2.2** ([Kol13, Thm. 5.13]) — Associating to a log-surface \((X, \Delta)\) the triple \((\bar{X}, \bar{D}, \tau): \bar{D}^\nu \to \bar{D}^\nu\) induces a one-to-one correspondence
\[
\begin{align*}
\{\text{stable log surfaces} \} & \leftrightarrow \{ \text{log-canonical pair} \} \\
(X, \Delta) & \leftrightarrow (\bar{X}, \bar{D}, \tau) \\
& \quad \text{with } K_{\bar{X}} + \bar{D} + \bar{\Delta} \text{ ample,} \\
& \quad \tau: D^\nu \to D^\nu \text{ an involution} \\
& \quad \text{s.th. } \text{Diff}_{D^\nu}(\Delta) \text{ is } \tau\text{-invariant.}
\end{align*}
\]

For the definition of the different see Definition 2.12 below.

**Notation 2.3** — In the rest of the article we continue to use the notation above, repeated here as a diagram:

\[
\begin{array}{ccc}
(1) & & \\
\bar{X} & \xleftarrow{\pi} & D & \xleftarrow{\nu} & D^\nu \\
\downarrow & & \downarrow & & \downarrow/\tau \\
X & \xleftarrow{\pi} & D & \xleftarrow{\nu} & D^\nu.
\end{array}
\]

An important consequence of Theorem 2.2 and its proof is that both squares in the diagram are pushouts.

2.2. **Semi-resolutions.** It is sometimes useful to resolve stable surfaces as much as possible while keeping the singularities in codimension 1.

**Definition 2.4** ([KSB88, Kol13]) — A surface \(Y\) is called semi-smooth if every point of \(Y\) is either smooth or double normal crossing or a pinch point.\(^4\)

A smooth rational curve \(E\) on a semi-smooth surface \(Y\) which is not contained in the non-normal locus is called a \((-1)\)-curve if \(E^2 = -1\) and \(\deg K_Y|_E \leq 0\).

A morphism of semi-normal surfaces \(f: Y \to X\) is called a semi-resolution if the following conditions are satisfied:

(i) \(Y\) is semi-smooth;

(ii) there is a semi-smooth open subscheme \(U\) of \(X\) such that the codimension of \(X \setminus U\) is two and \(f\) is an isomorphism over \(U\);

(iii) \(f\) maps the singular locus of \(Y\) birationally onto the non-normal locus of \(X\).

A semi-resolution is called minimal if it does not contract \((-1)\)-curves.

**Theorem 2.5** ([vS87, Kol13, Thm. 10.54]) — Let \(X\) be a semi-normal surface. Then \(X\) has a unique minimal semi-resolution.

The possible configurations of exceptional divisors on the minimal semi-resolution of an slc point will be discussed in Section A.4. Looking at these possibilities it is easy to see how to incorporate a reduced boundary into the resolution process for an slc pair: if \((X, \Delta)\) is an slc pair and \(f: Y \to X\) is the minimal semi-resolution of \(X\) then blowing up all intersection points of the non-normal locus of \(Y\) and the strict transform \(\Delta_Y = (f^{-1})_*\Delta\) we get a semi-resolution of \(X\) such that the strict

\(^4\)A local model for the pinch point in \(\mathbb{A}^3\) is given by the equation \(x^2 + yz^2 = 0\).
transform of the boundary is contained in the normal locus and it is minimal with this property. We call this the minimal log-semi-resolution of \((X, \Delta)\). The general case in all dimensions is treated in \([\text{Kol13}, \text{Sect. 10.4}]\).

2.3. Divisors and restrictions to curves. Let \(X\) be a demi-normal surface. In particular, \(X\) is Gorenstein in codimension 1 and \(S_2\) and the theory of generalised divisors from \([\text{Har94}]\) applies to \(X\).

A divisorial sheaf on \(X\) is a reflexive coherent \(\mathcal{O}_X\)-module that is locally free of rank 1 at the generic points of \(X\) \([\text{Har94}, \text{Prop. 2.8}]\) and there is a one-to-one correspondence between divisorial subsheaves of the sheaf of total quotient rings which are contained in \(\mathcal{O}_X\) and closed subschemes of codimension 1 without embedded points \([\text{Har94}, \text{Prop. 2.4}]\). A divisorial sheaf is called almost Cartier if it is invertible in codimension 1.

A Weil divisor (resp. \(\mathbb{Q}\)-Weil divisor) on \(X\) is a finite, formal, \(\mathbb{Z}\)-linear (resp. \(\mathbb{Q}\)-linear) combination \(D = \sum_i m_i D_i\) of irreducible and reduced subschemes of codimension 1 \([\text{Kol13}, \text{Sect. 1.1}]\). By a (\(\mathbb{Q}\)-)divisor we mean a (\(\mathbb{Q}\)-)Weil divisor. We call a \(\mathbb{Q}\)-Weil divisor reduced if all non-zero coefficients are equal to 1.

Arbitrary Weil divisors containing a component of the non-normal locus do not behave well in many respects, so we often need to exclude them. This is encoded in the following definition.

Definition 2.6 — A \(\mathbb{Q}\)-Weil divisor \(B\) on a log surface \((X, \Delta)\) is called well-behaved (resp. log-well-behaved) if its support does not contain any irreducible component of \(D\) (resp. \(D \cup \Delta\)).

For a well-behaved Weil divisor \(B\), the corresponding divisorial sheaf \(\mathcal{O}_X(B)\) is obtained as follows: let \(Z\) be the locus where \(B\) is not a Cartier divisor and \(U = X \setminus Z \overset{\iota}{\rightarrow} X\). Then \(Z\) is of codimension 2 since \(B\) does not contain a component of the non-normal locus and \(\mathcal{O}_X(B) := \iota_* \mathcal{O}_U(B)\) is an almost Cartier divisorial sheaf. If, in addition, \(B\) is effective, then the inclusion \(\mathcal{O}_X(-B) = \iota_* \mathcal{O}_U(-B) \hookrightarrow \iota_* \mathcal{O}_U = \mathcal{O}_X\) defines a subscheme structure on \(B_{\text{red}}\). With this subscheme structure \(B\) satisfies \(S_1\), i.e., is Cohen–Macaulay, since \(\mathcal{O}_X(-B)\) and \(\mathcal{O}_X\) satisfy \(S_2\). We will not distinguish the subscheme and the divisor in the notation.

On the other hand, given an almost Cartier divisorial sheaf \(A\) we can find a well-behaved Weil divisor \(A\) such that \(A = \mathcal{O}(A)\) \([\text{Har94}, \text{Prop. 2.11}]\).

Let \(\omega_X\) be the dualising sheaf which coincides with the pushforward of the canonical bundle on the Gorenstein locus. Note that \(\omega_X\) is almost Cartier, so there is a well-behaved canonical divisor \(K_X\), defined up to linear equivalence, such that \(\mathcal{O}_X(K_X) = \omega_X\). By a local computation we have \((\pi^* \omega_X)^{[1]} \cong \omega_Y(D)\).

Restricting divisors and divisorial sheaves to curves requires some extra care, if the divisor is not Cartier.

Definition 2.7 — Let \(B \subset X\) be a curve, that is, a Cohen–Macaulay subscheme of pure codimension 1, and let \(A\) be well-behaved divisor. Then we define

\[
\mathcal{O}_B(A) = \mathcal{O}_X(A) \otimes \mathcal{O}_B/\text{torsion}.
\]

Note that modding out the torsion subsheaf is in general not equal to taking the double dual if the curve is not Gorenstein \([\text{Kas13}, \text{Example 4.1.9}]\).

On the set of torsion-free sheaves of rank 1 on a curve \(B\) we can also define a multiplication

\[
\mathcal{F} \otimes \mathcal{F}' = \mathcal{F} \otimes \mathcal{F}'/\text{torsion}.
\]
This product is well-behaved only if one of the sheaves is a line bundle. For example, it may well happen that the restriction map from divisorial sheaves on $X$ to torsion-free sheaves on $B$ is not multiplicative, that is, in general
\[ \mathcal{O}_B(mA) \not\cong \mathcal{O}_B(A)^\otimes \cdots \mathcal{O}_B(a) \]

As a concrete example one may consider $A$ and $B$ to be a ruling of the cone over a twisted cubic. However, the usual short exact sequences suggested by the notation still work.

**Lemma 2.8** — Let $X$ be a demi-normal surface and $B$ a well-behaved curve. Let $A$ be a well-behaved divisor.

(i) There is an exact sequence
\[ 0 \to \mathcal{O}_X(A - B) \to \mathcal{O}_X(A) \to \mathcal{O}_B(A) \to 0. \]

(ii) If $B = B_1 + B_2$ is a decomposition of $B$ into a sum of (non-empty) subcurves then there is an exact sequence
\[ 0 \to \mathcal{O}_{B_1}(A - B_2) \to \mathcal{O}_B(A) \to \mathcal{O}_{B_2}(A) \to 0. \]

**Proof.** The composition $\varphi : \mathcal{O}_X(A) \to \mathcal{O}_X(A)|_B \to \mathcal{O}_B(A)$ is surjective. By the depth lemma $\ker \varphi$ is an $S_2$-sheaf. By looking at the open subscheme of $X$ where $\mathcal{O}_X(A)$ is Cartier, one sees easily that $\ker \varphi = \mathcal{O}_X(A - B)$ which implies (i).

For (ii) consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \downarrow & \downarrow & \downarrow & & 0 \\
0 & \longrightarrow & \mathcal{O}_X(A - B) & \longrightarrow & \mathcal{O}_X(A) & \longrightarrow & \mathcal{O}_B(A) & \longrightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(A - B_2) & \longrightarrow & \mathcal{O}_X(A) & \longrightarrow & \mathcal{O}_{B_2}(A) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
\mathcal{O}_{B_1}(A - B_2) & & & & & & \\
& & \downarrow & & \downarrow & & \\
0 & & & & & & \\
\end{array}
\]

where the exactness of the column and rows follows from (i). The assertion follows by the Snake Lemma. \hfill \Box

2.4. Intersection pairing.

**Definition 2.9** — Let $X$ be a demi-normal surface.

(i) We define a $\mathbb{Q}$-valued intersection pairing for well-behaved Weil divisors in the following way: let $A, B$ be well-behaved Weil divisors on $X$ and $\tilde{A}, \tilde{B}$ their strict transforms the normalisation $\tilde{X}$. Then the intersection number is
\[ AB := \tilde{A}\tilde{B} \]
where we use Mumford’s intersection pairing for normal surfaces. (see e.g. [Sak84]).
(ii) For a well-behaved $\mathbb{Q}$-Cartier Weil divisor $F$ and a curve $B$ on $X$ we denote by
\[
\deg F|_B := \frac{1}{m} \deg \mathcal{O}_X(mF)|_B,
\]
where $m$ is a positive integer such that $mF$ is Cartier. This could be called the numerical degree of $F$ on $B$. Note that $\deg F|_B = FB$ always holds.

(iii) For a torsion-free sheaf $\mathcal{F}$ on a Cohen–Macaulay curve $B$ we define its degree as in [CFHR99] through the Riemann–Roch formula
\[
\deg \mathcal{F} = \chi(\mathcal{F}) - \chi(\mathcal{O}_B).
\]

Remark 2.10 — In Definition 2.7 we defined for a well-behaved divisor $A$ the sheaf $\mathcal{O}_B(A)$ on a curve $B$. Unfortunately, if $A$ is $\mathbb{Q}$-Cartier but not Cartier, then the degree of the divisor on $B$ as in (ii) and the degree of the associated sheaf as in (iii) behave differently: the former may be rational while the latter is always an integer. This happens precisely because the restriction of divisorial sheaves to a curve is not multiplicative in general.

We will mostly work with the numerical definition (ii) and try to make it clear when we need to consider torsion-free sheaves. In some special situations, for example when $F = m(K_X + \Delta)$ and $B \subset \Delta$ on a stable log surface $(X, \Delta)$, a comparison between $\deg F|_B$ and $\deg \mathcal{O}_B(F)$ is possible (see Lemma 3.8).

Remark 2.11 — The intersection form defined in this way has some unexpected properties: for example, if $A$ and $B$ are contained in different irreducible components of $X$ then their intersection number is zero even if they intersect in the non-normal locus.

2.5. Descending pluri-log-canonical sections and invariants. Since we are especially interested in pluri-log-canonical maps and thus sections of pluri-log-canonical bundles, the following will play a role.

Definition 2.12 ([Kol13, 5.11]) — Let $B$ be a well-behaved reduced curve on $X$ and $B^\nu$ the normalisation of $B$. Suppose $\omega_X(\Delta + B)^{\lfloor m \rfloor}$ is a line bundle for some positive integer $m$. Then the different $\Diff_{B^\nu}(\Delta)$ is a $\mathbb{Q}$-divisor on $B^\nu$ such that $m\Diff_{B^\nu}(\Delta)$ is integral and
\[
\omega_X(\Delta + B)^{\lfloor m \rfloor}|_{B^\nu} \cong \omega_{B^\nu}^{\lfloor m \rfloor}(m\Diff_{B^\nu}(\Delta)).
\]

Proposition 2.13 ([Kol13, Prop. 5.8]) — Let $(X, \Delta)$ be a stable log surface and $m \geq 1$ an integer. A section $s \in H^0(\bar{X}, \omega_X(\bar{D} + \Delta)^{\lfloor m \rfloor})$ descends to a section in $H^0(X, \omega_X(\Delta)^{\lfloor m \rfloor})$ if and only if its residue at the generic points of $\bar{D}^\nu$ is $\tau$-invariant if $m$ is even respectively $\tau$-anti-invariant if $m$ is odd.

If $\omega_X(\bar{D} + \Delta)^{\lfloor m \rfloor}$ is a line bundle then this is equivalent to the image of $s$ in $H^0(D^\nu, \omega_{D^\nu}^{\lfloor m \rfloor}(m\Diff_{D^\nu}(\Delta)))$ being $\tau$-invariant if $m$ is even respectively $\tau$-anti-invariant if $m$ is odd.

The alternating signs are related to the Poincaré residue map: localising at a codimension 1 nodal point we look at the local model $\mathbb{A}^2 \subseteq X = (xy = 0) = L_x \cup L_y$ so that a local generator for $\omega_{\mathbb{A}^2}(X)$ is $dx \wedge dy/xy$. Taking residues along the two lines we have
\[
\Res_{L_x} \left( \frac{dx \wedge dy}{xy} \right) = \frac{dy}{y}, \quad \Res_{L_y} \left( \frac{dx \wedge dy}{xy} \right) = -\frac{dx}{x},
\]
so they differ in sign at the node.

For later reference we also state
Proposition 2.14 — Let $X$ be a stable surface with normalisation $\bar{X}$. In the notation above we have $K_X^2 = (K_X + D)^2$ and $\chi(O_X) = \chi(O_{\bar{X}}) + \chi(O_D) - \chi(O_{\bar{D}})$.

Proof. The first part is clear. For the second note that the conductor ideal defines $\bar{D}$ on $\bar{X}$ and the non-normal locus $D$ on $X$. In particular, $\pi_*O_X(-\bar{D}) = I_D$ and additivity of the Euler characteristic for the two sequences

$$0 \to O_X(-\bar{D}) \to O_X \to O_D \to 0,$$

$$0 \to \pi_*O_X(-\bar{D}) \to O_X \to O_D \to 0$$

gives the claimed result. $\square$

2.6. The curve embedding theorem. The technique of restriction to curves will play a major role in our approach and thus we will often need the following numerical criterion due to Catanese, Franciosi, Hulek and Reid. We state it in a slightly weaker version, that suffices for our purpose.

Theorem 2.15 ([CFHR99, Thm. 1.1]) — Let $C$ be a projective curve (over an algebraically closed field) which is Cohen–Macaulay but not necessarily irreducible or reduced and $L$ a line bundle on $C$. Then $L$ is base-point-free if for every generically Gorenstein subcurve $B \subset C$

$$\deg L|_B = \chi(L|_B) - \chi(O_B) \geq 2p_a(B) = 2(1 - \chi(O_B))$$

and $L$ is very ample if the inequality is strict.

Note that for an irreducible and smooth curve this gives the classical bounds.

3. Some vanishing results

We will need the following basic vanishing result, which is a variant of [KSS10, Cor. 6.6], and some consequences. All of these results follow from general vanishing theorems in [Fuj12] but the surface case is technically much simpler.

Proposition 3.1 (Generalised Kodaira vanishing) — Let $X$ be an slc surface and $A$ a well-behaved ample divisor on $X$. Then

$$H^i(X, O_X(-A)) = 0 \quad \text{for all } i < 2.$$

Proof. Choose $k \in \mathbb{N}$ such that $O_X(kA)$ is a very ample line bundle and let $B$ be the divisor of a general section. In particular, $B$ is a reduced divisor contained in the locus where $X$ has at most normal crossing singularities and $B$ is non-singular where $X$ is smooth. Now consider the ramified simple cyclic cover

$$\pi: Y = \text{Spec}_X \left( \bigoplus_{i=0}^{k-1} O_X(-kA) \right) \to X$$

with the following properties:

(i) $Y$ is Cohen–Macaulay by construction since each $O_X(-kA)$ is $S_2$ and $Y$ is a surface.

(ii) $Y$ has Du Bois singularities by [Kol13, Cor. 6.21].

(iii) $(\pi^*O_X(A))^{[1]}$ is locally free [Kol13, 2.44.6] and ample because the pullback of an ample divisor via a finite map is ample.
(iv) Let $Z \subset X$ be a codimension 2 subset such that $X \setminus Z$ is Gorenstein. For every reflexive sheaf $\mathcal{F}$ on $X$ we have

$$\pi_* (\pi^* \mathcal{F})^1 \cong (\pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F})^1 \cong \bigoplus_{i=0}^{k-1} (\mathcal{O}_X(-kA)) \otimes_{\mathcal{O}_X} \mathcal{F}^1$$

because all sheaves above are $S_2$ and isomorphic over $X \setminus Z$ where $\pi$ is flat.

Now by (iii) the sheaf $\pi^* (\mathcal{O}_X(-A))^1$ is a line bundle on $Y$ whose inverse is ample. Since $Y$ is Cohen–Macaulay and Du Bois, the Du Bois version of Kodaira vanishing \[Kol13\,\text{Thm. 10.42}\] implies for $i < 2$

$$H^i (Y, (\pi^* \mathcal{O}_X(-A))^1) = H^i (X, \pi_* ((\pi^* \mathcal{O}_X(-A))^1)) = H^i (X, \bigoplus_{i=0}^{k-1} (\mathcal{O}_X(-kA)) \otimes_{\mathcal{O}_X} (\mathcal{O}_X(-A))^1)$$

by (iv)

$$\supset H^i (X, \mathcal{O}_X(-A))$$

This concludes the proof. \qed

Remark 3.2 — The reason for the restriction to dimension 2 in the above theorem is that the index-1-cover constructed in the proof may well fail to be Cohen–Macaulay in higher dimensions, which is needed to apply the Du Bois version of Kodaira vanishing.

Lemma 3.3 — Let $X$ be a demi-normal surface and $\mathcal{F}, \mathcal{G}$ reflexive sheaves on $X$ that are locally free outside codimension 2. Then $\mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G}) = 0$. In particular, the local-to-global Ext-spectral sequence induces isomorphisms

$$H^0 (X, \mathcal{H}om_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G})) \cong \text{Hom}_X (\mathcal{F}, \mathcal{G}) \text{ and } H^1 (X, \mathcal{H}om_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G})) \cong \text{Ext}^1_X (\mathcal{F}, \mathcal{G}).$$

Proof. The first isomorphism is clear. The second sits in an exact sequence

$$0 \to H^1 (X, \mathcal{H}om_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G})) \to \text{Ext}^1_X (\mathcal{F}, \mathcal{G}) \to H^0 (X, \mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G}))$$

so it is sufficient to show that $\mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G}) = 0$.

To see this, let $j : U \hookrightarrow X$ be the inclusion of an open subset where all sheaves in question are locally free and such that the complement is of codimension 2. Then by [Har66, Ch. 2, Prop. 5.8]

$$j^* \mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G}) = \mathcal{E}xt^1_{\mathcal{O}_U} (j^* \mathcal{F}, j^* \mathcal{G}) = 0$$

so that $\mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G})$ is torsion supported in codimension 2.

Therefore, to study $\mathcal{E}xt^1_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G})$ we may assume that $X$ is affine. Since both $\mathcal{F}$ and $\mathcal{G}$ are reflexive, in any extension $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ the sheaf $\mathcal{E}$ is also reflexive. Thus the extension is determined outside codimension 2 as can be seen
from the diagram

\[\begin{array}{cccc}
0 & \rightarrow & \mathcal{G} & \rightarrow \mathcal{E} & \rightarrow \mathcal{F} & \rightarrow 0 \\
0 & \rightarrow & j_*(j^* \mathcal{G}) & \rightarrow j_*(j^* \mathcal{E}) & \rightarrow j_*(j^* \mathcal{F}) & \rightarrow 0.
\end{array}\]

Thus the extension splits if it splits outside codimension 2 and \(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\) has no torsion supported in codimension 2. By the above it is actually zero. \(\Box\)

**Corollary 3.4** — Let \((X, \Delta)\) be a stable log surface. Then for all \(i > 0\) and all integers \(m \geq 2\), we have \(H^i(X, \omega_X^m((m - 1)\Delta)) = 0\).

**Proof.** By Serre duality on the Cohen–Macaulay scheme \(X\) and Lemma 3.3, we have

\[
H^i \left( X, \omega_X^m((m - 1)\Delta) \right)^* \cong \text{Ext}^{2-i}_{\mathcal{O}_X}(\omega_X^m((m - 1)\Delta), \omega_X) \\
\cong H^{2-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\omega_X^m((m - 1)\Delta), \omega_X)) \\
\cong H^{2-i}(X, \omega_X(\Delta)^{1-m}),
\]

where in the last step we used the fact that \(\mathcal{H}om_{\mathcal{O}_X}(\omega_X^m((m - 1)\Delta), \omega_X)\) and \(\omega_X(\Delta)^{1-m}\) are both reflexive and coincide outside codimension 2. So by Proposition 3.1, \(\omega_X^m((m - 1)\Delta)\) has no higher cohomology for \(m \geq 2\) which proves the claim. \(\Box\)

**Corollary 3.5** — Let \((X, \Delta)\) be a stable log surface, \(\mathcal{I}\) the ideal sheaf of \(D \cup \Delta \subset X\) and \(\mathcal{I}(m(K_X + \Delta)) = (\mathcal{I} \otimes \omega_X(\Delta)^{[m]})\). In our standard notation 2.3 we have

(i) \(H^0(X, \mathcal{I}(m(K_X + \Delta))) = H^0(X, \omega_X(D + \Delta)^m(-D - \Delta))\) for \(m \geq 1\),

(ii) \(H^i(X, \mathcal{I}(m(K_X + \Delta))) = 0\) for all \(i > 0\) and all integers \(m \geq 2\).

**Proof.** Let \(\tilde{\mathcal{I}}\) be the ideal sheaf of \(D \cup \Delta\) in \(\tilde{X}\). Then we have \(\pi_* \tilde{\mathcal{I}} = \mathcal{I}\). Let \(U\) be the subset of \(X\) where \(K_X + \Delta\) is Cartier, whose complement has codimension at least 2, and \(\tilde{U} = f^{-1}(U)\). Then over \(U\) we can use the projection formula to obtain

\[
\pi_* \left( \omega_X(D + \Delta)^m(-D - \Delta) \right) \cong \pi_* \left( \tilde{\mathcal{I}} \otimes \pi^* \omega_X(\Delta)^{[m]} \right) \cong \pi_* \tilde{\mathcal{I}} \otimes \omega_X(\Delta)^{[m]} |_{\tilde{U}} \cong \mathcal{I} \omega_X(\Delta)^{[m]} |_{\tilde{U}}.
\]

Because all sheaves in question are \(S_2\) this extends to an isomorphism

\[
\pi_* \omega_X(D + \Delta)^m(-D - \Delta) \cong \mathcal{I}(m(K_X + \Delta)).
\]

To conclude we use that \(\pi\) is a finite morphism and thus

\[
H^i(X, \mathcal{I}(m(K_X + \Delta))) = H^i(X, \omega_X(D + \Delta)^m(-D - \Delta))\) for \(i \geq 0\).

In particular (i) is proved. By applying Corollary 3.4 to the pair \((\tilde{X}, D + \tilde{\Delta})\), the second item also follows. \(\Box\)

The first item of Corollary 3.5 is also a direct consequence of Proposition 2.13.

The next proposition is much easier to prove if \(m\) is a multiple of the index, but we need this strong form for the result about the log-canonical-ring in Section 6. It is actually identical to the much easier Corollary 3.4 if \(\Delta = 0\).
Proposition 3.6 — Let $(X, \Delta)$ be a stable log surface. Then for all $i > 0$ and all integers $m \geq 2$, we have $H^i(X, \omega_X(\Delta)^{[m]}) = 0$.

The idea of the proof is to use the restriction sequence (Lemma 2.8) together with vanishing on a curve and the vanishing results proved above. Due to the singularities one has to take extra care. We begin with some preliminary results.

We consider the minimal log-semi-resolution $f: Y \to X$ (see remarks after Definition 2.4) and the respective normalisations $\pi: \tilde{X} \to X$ and $\mu: \tilde{Y} \to Y$. In particular, $\Delta_Y = (f^{-1})_* \Delta$ and $D_Y$, the conductor of $Y$ are disjoint and hence $\Delta_Y$ is contained in the smooth locus of $Y$. Obviously, there is a birational morphism $\bar{f}: \bar{Y} \to \tilde{X}$ such that $\pi \circ \bar{f} = f \circ \mu$ (cf. diagram (15) in the appendix).

Lemma 3.7 — For any well-behaved divisor $M$ on $X$, one has

$$R^1 f_* \mathcal{O}_Y(K_Y + [f^* M]) = 0.$$  

Proof. Let $\bar{M}$ be the strict transform of $M$ on $\tilde{X}$. Then $\bar{f}^* \bar{M}$ is an $\bar{f}$-nef divisor on $\bar{Y}$. By [Sak84, 2.2] we have $R^1 f_* \mathcal{O}_Y(K_Y + [\bar{f}^* \bar{M}]) = 0$. Now consider the following exact sequence

$$0 \to \mu_* \mathcal{O}_Y(K_Y + [\bar{f}^* \bar{M}]) \to \mathcal{O}_Y(K_Y + [f^* M]) \to \mathcal{Q} \to 0$$

where the quotient $\mathcal{Q}$ is supported on the conductor divisor $D_Y$. Applying $f_*$ to (2) we obtain an exact sequence

$$R^1 f_* \mu_* \mathcal{O}_Y(K_Y + [\bar{f}^* \bar{M}]) \to R^1 f_* \mathcal{O}_Y(K_Y + [f^* M]) \to R^1 f_* \mathcal{Q}.$$ 

Since $\mu: \tilde{Y} \to \tilde{X}$, $\pi: \tilde{X} \to X$ and $f|_{D_Y}$ are all finite morphisms, one sees easily that $R^1 f_* \mathcal{Q} = 0$ and $R^1 f_* \mu_* \mathcal{O}_Y(K_Y + [\bar{f}^* \bar{M}]) = \pi_* R^1 f_* \mathcal{O}_Y(K_Y + [f^* M]) = 0$. The lemma follows now from (3). \qed

Lemma 3.8 —

(i) The torsion free sheaf $\mathcal{O}_\Delta(m(K_X + \Delta))$ is a line bundle for any positive integer $m$.

(ii) Let $\nu: \Delta^\nu \to \Delta$ be the normalisation of $\Delta$ and $\text{Diff}_{\Delta^\nu}(0) = \sum a_p p$ the different. For $m \geq 2$ we have an inclusion of line bundles

$$\omega^\otimes_m \left( \sum_{\nu(p) \in \Delta_{sm}} [(m-1)a_p]\nu(p) \right) \subset \mathcal{O}_\Delta(m(K_X + \Delta)),$$

where $\Delta_{sm}$ denotes the smooth locus of $\Delta$.

Proof. First we prove that, if $p \in \Delta$ is singular then $K_X + \Delta$ is Cartier at $p$. Note that $\Delta$ has at most ordinary nodes as singularities and the resolution graph of $p \in X$ is a chain of analytic irreducible components of type (C2) in Section A.3. Let $E \subset Y$ be the (whole) reduced exceptional divisor over $p$. Then it is easy to see that the divisor $K_Y + \Delta_Y + E$ is Cartier in a neighbourhood of $E$. Using the same argument as [SB83, Lemma 1.1] an isomorphism $\mathcal{O}_Y(K_Y + \Delta_Y + E) \cong \mathcal{O}_Y$ in an analytic neighbourhood of $E$ can be established and consequently $K_X + \Delta$ is Cartier at $p$.

So the rank one torsion free sheaf $\mathcal{O}_\Delta(m(K_X + \Delta))$ is locally free at the singular points of $\Delta$ and hence is locally free everywhere. This proves (i).

On the open subset $U$ where $K_X + \Delta$ is Cartier the residue map induces a canonical isomorphism $\mathcal{R}_{U\cap \Delta}: \omega_X(\Delta)^{[m]}|_{U\cap \Delta} \cong \omega^\otimes_{U\cap \Delta}$. Viewing the residue map $\mathcal{R}_{U\cap \Delta}$ as a rational section of $\mathcal{H}om(\omega_X(\Delta)^{[m]}|_{\Delta}, \omega^\otimes_{\Delta})$ we obtain a natural isomorphism

$$\mathcal{R}_\Delta: \mathcal{O}_\Delta(m(K_X + \Delta)) \cong \omega^\otimes_{\Delta}(\sum_q b_q q)$$.
where \( \sum_q b_q q \) is a uniquely determined integral divisor supported on those smooth points of \( \Delta \) where the different is non-zero (see [Kol13, Sect. 4.1]). The integers \( b_q \) only depend on what happens in a neighbourhood of \( q \in X \).

Now we will establish the inequality \( b_q \geq [(m-1)a_{\nu-1}(q)] \) for \( q \in \Delta_{\text{sm}} \).

Recall that \( f : Y \to X \) is the minimal semi-log-resolution, so that \( \Delta_Y \) and \( D_Y \) are disjoint. We choose a well-behaved semi-log-resolution \( K_X \) and let \( M = (m-1)(K_X + \Delta) \) (cf. Section 2.3). Applying \( f_* \) to the following exact sequence

\[
0 \to \mathcal{O}_Y(K_Y + [f^*M]) \to \mathcal{O}_Y(K_Y + \Delta_Y + [f^*M]) \to \mathcal{O}_{\Delta_Y}(K_Y + \Delta_Y + [f^*M]) \to 0.
\]

we obtain by Lemma 3.7

\[
0 \to f_*\mathcal{O}_Y(K_Y + [f^*M]) \to f_*\mathcal{O}_Y(K_Y + \Delta_Y + [f^*M]) \to f_*\mathcal{O}_{\Delta_Y}(K_Y + \Delta_Y + [f^*M]) \to 0.
\]

The identification of the sheaves on semi-log-smooth locus of \( X \) yields natural vertical morphisms in the following diagram

\[
\begin{array}{ccc}
\mathcal{O}_Y(K_Y + [f^*M]) & \longrightarrow & \mathcal{O}_Y(K_Y + \Delta_Y + [f^*M]) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(K_X + M) & \longrightarrow & \mathcal{O}_X(K_X + \Delta + M).
\end{array}
\]

This induces a morphism between the quotients of the horizontal arrows, namely,

\[
\varphi : f_*\mathcal{O}_{\Delta_Y}(K_Y + \Delta_Y + [f^*M]) \to \mathcal{O}_\Delta(\mathcal{O}_X(K_X + \Delta + M)),
\]

which is isomorphism at the generic points of \( \Delta \) and in particular injective. For \( \Delta_Y \), which lies in the smooth locus of \( Y \), we have a natural identification induced by residue map

\[
\mathcal{O}_{\Delta_Y}(K_Y + \Delta_Y + [f^*M]) = \mathcal{O}_{\Delta_Y}(m(K_Y + \Delta_Y)) \otimes \mathcal{O}_{\Delta_Y}(\sum_p [(m-1)a_p]p)
\]

\[
= \omega_{\Delta_Y}^{\otimes m}(\sum_p [(m-1)a_p]p).
\]

As indicated before, for a smooth point \( q \) of \( \Delta \) we can focus on a neighbourhood of \( q \in X \) and consequently obtain a comparison \( b_q \geq [(m-1)a_{\nu-1}(q)] \) by the injectivity of \( \varphi \) at \( q \). This finishes the proof of (ii). \( \square \)

**Proof of Proposition 3.6.** Consider the exact sequence

\[
0 \to \omega_X^{[m]}((m-1)\Delta) \to \omega_X(\Delta)[m] \to \mathcal{O}_\Delta(m(K_X + \Delta)) \to 0,
\]

whose long exact cohomology sequence gives, using Corollary 3.4,

\[
(4) \quad H^1(X, \omega_X(\Delta)[m]) \cong H^1(\Delta, \mathcal{O}_\Delta(m(K_X + \Delta))), \quad H^2(X, \omega_X(\Delta)[m]) = 0.
\]

It remains to show that \( H^1(\Delta, \mathcal{O}_\Delta(m(K_X + \Delta))) = 0 \) for \( m \geq 2 \). We argue by contradiction, so assume \( H^1(\Delta, \mathcal{O}_\Delta(m(K_X + \Delta))) \neq 0 \). By Lemma A.1 there is a subcurve \( B \subset \Delta \) with a generically onto morphism

\[
\lambda_B : \mathcal{O}_B(m(K_X + \Delta)) \to \omega_B.
\]

On the other hand, one sees by Lemma 3.8 that \( \mathcal{O}_B(m(K_X + \Delta)) = \mathcal{O}_\Delta(m(K_X + \Delta))|_B \) has degree strictly larger than \( \deg \omega_B \). This is a contradiction. \( \square \)
4. Base-point-freeness of pluri-log-canonical maps

Many of the standard techniques do not work directly on non-normal and possibly reducible surfaces. Thus to prove base-point-freeness we first use a Reider-type result of Kawachi on the normalisation to produce pluri-log-canonical sections vanishing along the non-normal locus. Then we analyse the restriction of the pluri-log-canonical bundle to the non-normal locus directly. The overall result is

**Theorem 4.1** — Let \((X, \Delta)\) be a connected stable log surface of global index \(I\).

(i) The line bundle \(\omega_X(\Delta)^{[mI]}\) is base-point-free for \(m \geq 4\).

(ii) The line bundle \(\omega_X(\Delta)^{[mI]}\) is base-point-free for \(m \geq 3\) if one of the following holds:
   a) \(I \geq 2\).
   b) There is no irreducible component \(\bar{X}_i\) of the normalisation such that \((\pi^*(K_X + \Delta)|_{\bar{X}_i})^2 = 1\), and the union of \(\Delta\) and the non-normal locus is a nodal curve.
   c) \(X\) is normal and we do not have \(I = (K_X + \Delta)^2 = 1\).

This result is sharp in the sense that there are examples of smooth surfaces such that \(\omega_X^{\otimes 3}\) has base-points [BHPV04, Remark on p.287]. However, the conditions in the theorem are by no means necessary for base-point-freeness.

In the case of surfaces with canonical singularities the bi-canonical map is a morphism as soon as \(K_X^2 \geq 5\). We will show in Example 7.2 that this does not generalise to stable surfaces, even irreducible ones.

**Remark 4.2** — If \([mI(K_X + \Delta)]\) is base-point-free, the fact that \(K_X\) is ample implies that the pluri-log-canonical map \(\varphi_{mI} : X \to \mathbb{P}^N\) does not contract any curve on \(X\), hence defines a finite morphism from \(X\) to its reduced image.

4.1. Base-point-freeness on the complement of boundary and non-normal locus. We now analyse the base-point-freeness of pluri-log-canonical maps outside the non-normal locus and the boundary of a stable log surface, starting with the following auxiliary result.

**Proposition 4.3** — Let \((\bar{X}, \bar{\Delta})\) be an irreducible (hence connected) log surface with log-canonical singularities, \(K_{\bar{X}} + \bar{\Delta}\) ample and \(I\) the global index. Assume \(m \geq 3\). If \(I = 1\) assume in addition \((m - 1)^2(K_{\bar{X}} + \bar{\Delta})^2 > 4\).

Then for every \(x \in X \setminus \Delta\) there is a section of \(\omega_X(\bar{\Delta})^{[mI]}(-\bar{\Delta})\) not vanishing at \(x\). In particular, the rational map associated to \(\omega_X(\bar{\Delta})^{[mI]}\) is a morphism on \(\bar{X} \setminus \bar{\Delta}\).

**Proof.** Let \(\bar{f} : \bar{Y} \to \bar{X}\) be the minimal resolution of those singularities of \(\bar{X}\) that are contained in \(\bar{\Delta}\).

Fix once for all a Weil divisor \(K_{\bar{Y}}\) representing the canonical class of \(\bar{Y}\) and a Weil divisor \(M\) on \(\bar{Y}\) such that \(O_{\bar{Y}}(M) = \bar{f}^*(\omega_{\bar{X}}(\bar{\Delta})^{[mI]})\). Let \(\Delta_{\bar{Y}}\) be the strict transform of \(\bar{\Delta}\). We can write

\[
K_{\bar{Y}} + \Delta_{\bar{Y}} + E \sim_{\mathbb{Q}} \bar{f}^*(K_{\bar{X}} + \bar{\Delta}) \sim_{\mathbb{Q}} \frac{1}{mI} M
\]

where \(\sim_{\mathbb{Q}}\) denotes \(\mathbb{Q}\)-linear equivalence and \(E\) is a \(\mathbb{Q}\)-divisor supported on the exceptional locus, effective since \(\bar{f}\) was chosen to be minimal.

With these choices \(M - K_{\bar{Y}} - \Delta_{\bar{Y}} - E\) is a \(\mathbb{Q}\)-Weil divisor such that

\[
M - K_{\bar{Y}} - \Delta_{\bar{Y}} - E \sim_{\mathbb{Q}} (mI - 1)\bar{f}^*(K_{\bar{X}} + \bar{\Delta})
\]
and hence big and nef. In addition
\[ K_Y + [M - K_Y - \Delta_Y - E] = [K_Y + M - K_Y - \Delta_Y - E] \]
\[ = M - \Delta_Y - [E] \]
is Cartier because \( \bar{Y} \) is smooth near \( \Delta_Y \cup E \).
Thus \[ \text{[Kaw00, Thm. 2]} \] implies that \( K_Y + [M - K_Y - \Delta_Y - E] \) is base-point-free outside the exceptional locus of \( \bar{f} \) as soon as the numerical conditions (using (5))
\[(mI - 1)\bar{f}^*(K_X + \bar{\Delta})^2 > 4 \text{ and } (mI - 1)\bar{f}^*(K_X + \bar{\Delta})C \geq 2\]
hold for all curves \( C \) not contained in the exceptional locus of \( \bar{f} \).
Assuming this for a moment we see that
\[ H^0(\bar{Y}, \mathcal{O}_Y(K_Y + [M - K_Y - \Delta_Y - E])) = H^0(\bar{Y}, \mathcal{O}_Y(M - \Delta_Y - [E])) \]
\[ \subset H^0(\bar{Y}, \mathcal{O}_Y(M - \Delta_Y)) \]
\[ \equiv H^0(\bar{X}, \bar{f}_*\mathcal{O}_Y(M - \Delta_Y)) \]
\[ = H^0(\bar{X}, \omega_X(\bar{\Delta})^{[mI]}(-\bar{\Delta})). \]

Thus all sections of \( K_Y + [M - K_Y - \Delta_Y - E] \) descend to sections of \( \omega_X(\bar{\Delta})^{[mI]}(-\bar{\Delta}) \). Via the inclusion \( \omega_X(\bar{\Delta})^{[mI]}(-\bar{\Delta}) \hookrightarrow \omega_X(\bar{\Delta})^{[mI]} \) we can also interpret these as section of \( \omega_X(\bar{\Delta})^{[mI]} \) vanishing along \( \bar{\Delta} \). The restriction \( \bar{f}: \bar{Y} \setminus \bar{f}^{-1}\bar{\Delta} \rightarrow \bar{X} \setminus \bar{\Delta} \) is an isomorphism and thus base-point-freeness holds on \( \bar{X} \setminus \bar{\Delta} \) under the assumption (6).

It remains to show that (6) holds under the assumptions of the proposition. First we note that for \( m \geq 3 \) and for every non-exceptional curve \( C \subset \bar{Y} \) we have
\[(mI - 1)\bar{f}^*(K_X + \bar{\Delta})C = (mI - 1)(K_X + \bar{\Delta})\bar{f}_*C \]
\[= 2I(K_X + \bar{\Delta})\bar{f}_*C + ((m - 2)I - 1)(K_X + \bar{\Delta})\bar{f}_*C \geq 2 \]
because \( I(K_X + \bar{\Delta}) \) is an ample Cartier divisor.

Noting that \( I(K_X + \bar{\Delta})^2 \) is a positive integer, because it is the intersection of an ample Cartier divisor with an integral divisor, and writing
\[ (Im - 1)^2(K_X + \bar{\Delta})^2 > 4 \Leftrightarrow \left(m - \frac{1}{I}\right)^2 I(K_X + \bar{\Delta})^2 > \frac{4}{I} \]
we see that if \( I \geq 2 \) and \( m \geq 2 \) the inequality is satisfied. If \( I = 1 \) we need that \( (m - 1)^2(K_X + \bar{\Delta})^2 > 4 \) in addition to \( m \geq 3 \). \( \square \)

We now descend the above result to a possibly non-normal stable log surface.

**Corollary 4.4** — Let \((X, \varDelta)\) be a stable log surface with global index \( I \). Then the base-points of \( \omega_X(\varDelta)^{[mI]} \) are contained in the union of the non-normal locus \( D \) and the boundary \( \varDelta \) if
\[(i) \ m \geq 4, \]
\[(ii) \ m \geq 3 \text{ unless the index } I = 1, \text{ and there is an irreducible component } \bar{X}_i \text{ of the normalisation such that } \left(\pi^*(K_X + \varDelta)|_{\bar{X}_i}\right)^2 = 1. \]

**Proof.** We use our standard notation 2.3. On every irreducible component \( \bar{X}_i \) of the normalisation \( \bar{X} \), we apply Proposition 4.3 to the pair \((\bar{X}_i, (\bar{\Delta} + \bar{D})|_{\bar{X}_i})\) which, under our assumptions, gives for every point \( \bar{x} \in \bar{X} \setminus (\bar{\Delta} \cup \bar{D}) \) a section of \( \omega_{\bar{X}}(\bar{D} + \bar{\Delta})^{[mI]}(\bar{D} - \bar{\Delta}) \) not vanishing at \( \bar{x} \).
Via the inclusion
\[ \omega_X(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta}) \rightarrow \omega_X(\bar{D} + \bar{\Delta})^{[mI]}, \]
the sections of \( \omega_X(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta}) \) are mapped to sections of \( \omega_X(\bar{D} + \bar{\Delta})^{[mI]} \) that vanish along \( \bar{D} \cup \bar{\Delta} \) and thus descend to sections of \( \omega_X(\Delta)^{[mI]} \) by Proposition 2.13. Consequently, \( \omega_X(\Delta)^{[mI]} \) has no base-points outside \( \Delta \cup D \). \( \square \)

4.2. Restricting to non-normal locus and boundary. In this section we concentrate on the geometry of the non-normal locus \( D \) of a stable log surface \((X, \Delta)\), using the same notation 2.3 as always. We start with a definition that will turn out to describe all possible singularities of the non-normal locus.

**Definition 4.5** — Let \( B \) be a reduced curve, \( p \in B \) and \( \mu = \mu_p(B) \) the multiplicity of \( p \) in \( B \). We call \( p \) a \( \mu \)-multi-node, if locally analytically at \( p \) the curve is isomorphic to the union of the coordinate axes in \( \mathbb{A}^\mu \).

Thus a 1-multi-nodal point is smooth and a 2-multi-node is just an ordinary nodal point.

By the correspondence between stable log surfaces and their normalisation explained in Theorem 2.2 and Notation 2.3, the non-normal locus \( D \) is quotient by the finite equivalence relation on \( D \) induced by \( \tau \). In other words, as a set \( D \) is the set of equivalence classes of the equivalence relation on \( D^\nu \) generated by
\[
\bar{p} \sim \bar{q} \text{ if } \tau(\bar{p}) = \bar{q} \text{ or } \nu(\bar{p}) = \nu(\bar{q}),
\]
and the scheme structure on \( D \) is determined by the requirement that the diagram
\[
\begin{array}{ccc}
\bar{D} & \leftarrow^\bar{\nu} & D^\nu \\
\downarrow^\pi & & \downarrow_/^{/\tau} \\
D & \leftarrow^{\nu} & D^\nu
\end{array}
\]
is a pushout diagram.

Recall that \( \bar{D} \) is a nodal curve by the classification of log-canonical singularities. If \( \bar{D} \) is smooth then \( \bar{\nu} \) is an isomorphism, so \( D^\nu = \bar{D}/\tau \) satisfies the pushout property and \( D = D^\nu \). Applying the same argument locally, it follows that if \( p \in D \) is a point such that \( \tau^{-1}(p) \) contains only smooth points of \( \bar{D} \) then \( p \) itself is smooth.

Now let \( p \) be a singular point of \( D \). Write the preimage of \( p \) in \( \bar{D}^\nu \) as
\[
(\pi \circ \bar{\nu})^{-1}(p) = \{a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_l\}
\]
such that \( \bar{\nu}(a_i) = \bar{\nu}(b_i) \) is a node of \( \bar{D} \) and \( \bar{\nu}(c_j) \) is a smooth point of \( \bar{D} \). By the above there is at least one node of \( \bar{D} \) mapping to \( p \), so \( k \geq 1 \). Since the preimage of \( p \) in \( D^\nu \) is an equivalence class with respect to the relation generated by (7), we have \( l \in \{0, 1, 2\} \) and, up to reordering, the following cases can occur (compare also [Kol12, 17.4]):

**Type I:** The preimage of \( p \) in \( \bar{D} \) consists only of nodes and we glue the normalisation \( \bar{D}^\nu \) in a circular fashion: \( \tau(b_i) = a_{i+1} \) (\( i = 1, \ldots, k - 1 \)), \( \tau(b_k) = a_1 \). If \( k = 1 \), then \( p \) is a smooth point of \( D \), so we may assume \( k \geq 2 \).

**Type II:** We glue the preimages of the nodes of \( \bar{D} \) in a chain \( \tau(b_i) = a_{i+1} \) (\( i = 1, \ldots, k - 1 \)) and have two remaining points \( a_1 \) and \( b_k \) at the ends. At each end we have two possibilities which we spell out only for \( a_1 \): either \( \tau(a_1) = c_j \) or \( \tau(a_1) = c_j \) for a point \( c_j \) mapping to a smooth point of \( \bar{D} \).
To determine the local structure of $D$ at $p$, we replace $D$ by a small analytic neighbourhood of $p$ such that $p$ is the only singular point and $\tau$ has no fixed points on $D^\nu \setminus (\pi \circ \nu)^{-1}(p)$. Then, possibly shrinking $D$ further, the normalisation $D^\nu = D^\nu / \tau$ consists of $k$ (Type I) or $k + 1$ (Type II) branches, each containing a unique point that maps to $p$, and $\bar{D}$ consists of $k$ nodal and possibly one or two smooth branches for Type II.

Thus there are maps from $\bar{D}$ and $D^\nu$ to a neighbourhood of a multi-nodal point compatible with the equivalence relation. These satisfy the pushout conditions because if the tangent directions were not independent then the map to $D$ would factor over the multi-nodal point. So every singular point $p \in D$ is a $\mu$-multi-nodal point for some $\mu \geq 2$.

By Theorem 2.2 the different $\text{Diff}_{D^\nu}(\bar{\Delta})$ is $\tau$-invariant and thus has the same coefficient for each point in an equivalence class of the relation generated by (7). In particular, if $p \in D$ is singular then the preimage contains at least one node and the different has coefficient 1 at each point mapping to $p$. This restricts the possibilities for the smooth points $\nu(c_i) \in \bar{D}$ occurring in Type II: by the classification of log-canonical singularities they are either dihedral quotient singularities (see [Kol12, 16.3]) of $\bar{X}$ or smooth points of $D$ where $\bar{D}$ intersects $\Delta$.

The first two items of the next lemma sum up the discussion so far.

**Lemma 4.6** — Let $(X, \Delta)$ be a log surface with slc singularities and $D \subset X$ the non-normal locus. Let $p \in D \cup \Delta$. Then the following holds.

1. $p$ is a $\mu$-multi-node of $D \cup \Delta$ for $\mu = \mu_p(D \cup \Delta) \geq 1$.
2. If $p$ is a singularity of $D \cup \Delta$ then the inverse image $\pi^{-1}(p)$ contains at least one node of $D \cup \Delta$ and at each point of $D^\nu$ mapping to $p$ the different $\text{Diff}_{D^\nu}(\bar{\Delta})$ has coefficient 1.
3. Let $B \subset D \cup \Delta$ be a subcurve. Then $p_a(B) = p_a(B^\nu) + \sum_p(\mu_p(B) - 1)$.

**Proof.** It remains to prove the last item. We compare the arithmetic genera of $B$ and $B^\nu$ by taking Euler characteristics in the short exact sequence

$$0 \to \mathcal{O}_B \to \nu_*\mathcal{O}_{B^\nu} \to \nu_*\mathcal{O}_{B^\nu}/\mathcal{O}_B \to 0.$$ 

Since all singular points are $\mu$-multi-nodes the length of the subsheaf of $\nu_*\mathcal{O}_{B^\nu}/\mathcal{O}_B$ supported at the point $p$ is exactly $\mu_p(B) - 1$, so $p_a(B) = p_a(B^\nu) + \sum_p(\mu_p(B) - 1)$. \qed

Now we analyse the restriction of the log-canonical divisor to the non-normal locus and the boundary.

**Lemma 4.7** — Let $(X, \Delta)$ be a stable log surface with normalisation $(\bar{X}, \bar{D} + \bar{\Delta})$. Consider a subcurve $B \subset D \cup \Delta$, with normalisation $B^\nu$. Let $s$ be the number of smooth points of $B$ that are singular points of $D \cup \Delta$. Then

$$\deg(K_X + \Delta)|_{B^\nu} \geq 2p_a(B^\nu) - 2 + \sum_{p \in (D \cup \Delta)_{\text{sing}}} \mu_p(B)$$

$$\geq 2p_a(B) - 2 + \sum_{p \in B_{\text{sing}}} (2 - \mu_p(B)) + s$$

$$\geq 2p_a(B) - 2 + \sum_{p \in B_{\text{sing}}} (2 - \mu_p(B)).$$
Proof. Let $B_1 = B \cap D$ and $B_2 = B \cap \Delta$, that is, $B_1$ (resp. $B_2$) is the subcurve of $B$ contained in $D$ (resp. $\Delta$). As Weil divisors we can write $D + \Delta = A + B = A_i + B_i$ with $A$ (resp. $A_i$) being the complement curve of $B$ (resp. $B_i$). We adopt our usual notation for strict transforms in $\bar{X}$ and normalisation: for example, $\bar{B}$ is the strict transform of $B$ in $\bar{X}$, $B^\nu$ is the normalisation of $B$ while $\bar{B}^\nu$ is the normalisation of $\bar{B}$.

Note that $\pi_1 : \bar{B}_1^\nu \to B_1^\nu$ is a double cover and $\pi_2 : \bar{B}_2^\nu \to B_2^\nu$ is an isomorphism. Thus by Hurwitz formula

$$K_{\bar{B}_1^\nu} = \pi^* K_{B_1^\nu} + R \text{ and } K_{\bar{B}_2^\nu} = \pi^* K_{B_2^\nu}$$

where $R$ is the (reduced) ramification divisor on $\bar{B}_1^\nu$. Now we compute the degree of $K_X + \Delta$ restricted to $B$:

$$\deg(K_X + \Delta)|_B = \deg(K_X + \Delta)|_{B_1} + \deg(K_X + \Delta)|_{B_2}$$

$$= \frac{1}{2} \deg(K_X + \bar{D} + \bar{\Delta})|_{\bar{B}_1} + \deg(K_X + \bar{D} + \bar{\Delta})|_{\bar{B}_2}$$

$$= \frac{1}{2} \deg(K_{\bar{B}_1} + \text{Diff}_{\bar{B}_1}(\bar{A}_1)) + \deg(K_{\bar{B}_2} + \text{Diff}_{\bar{B}_2}(\bar{A}_2))$$

(8)

$$= \deg \left( \frac{1}{2} \left( \pi_1^* K_{B_1} + R + \text{Diff}_{B_1}(\bar{A}_1) \right) + \pi_2^* K_{B_2} + \text{Diff}_{B_2}(\bar{A}_2) \right)$$

$$= \deg(K_{B_1} + K_{B_2}) + \deg \left( \frac{1}{2} \left( \text{Diff}_{B_1}(\bar{A}_1) + R \right) + \text{Diff}_{B_2}(\bar{A}_2) \right)$$

= $\deg K_{B^\nu} + \deg \left( \frac{1}{2} \left( \text{Diff}_{B_1}(\bar{A}_1) + R \right) + \text{Diff}_{B_2}(\bar{A}_2) \right)$

(9)

Let $p$ be a point in $B$ such that $\mu_p(D \cup \Delta) \geq 2$. Then we can decompose the multiplicity $\mu_p(B) = \mu_p(B_1) + \mu_p(B_2)$. Taking the degree of the maps into account we have

$$\mu_p(B_1) = \frac{1}{2} \#(\nu_1 \circ \pi_1)^{-1}(p) + \frac{1}{2} \# ((\nu_1 \circ \pi_1)^{-1}(p) \cap R),$$

$$\mu_p(B_2) = \#(\nu_2 \circ \pi_2)^{-1}(p),$$

(10)

where $\nu_i : B_i^\nu \to B_i$, $i = 1, 2$, are the normalisations and $R$ is the set of ramification points of $\pi_1$.

The different $\text{Diff}_{B^\nu}(\bar{A}) = \text{Diff}_{\bar{B}_1}(\bar{A}_1) + \text{Diff}_{\bar{B}_2}(\bar{A}_2)$ is effective and has coefficient 1 over every singular point of $D \cup \Delta$ which lies in $B$ (Lemma 4.6). Combining this with (9) and (10), we have

$$\deg(K_X + \Delta)|_B \geq \deg K_{B^\nu} + \sum_{p \in (D \cup \Delta)_{\text{sing}}} (\mu_p(B_1) + \mu_p(B_2))$$

$$= 2p_a(B^\nu) - 2 + \sum_{p \in (D \cup \Delta)_{\text{sing}}} \mu_p(B)$$
and using Lemma 4.6(iii)

\[
\geq 2p_a(B) - 2 + s + \sum_{p \in B_{\text{sing}}} \mu_p(B) - 2(\mu_p(B) - 1)
\]

\[
= 2p_a(B) - 2 + s + \sum_{p \in B_{\text{sing}}} 2 - \mu_p(B)
\]

\[
\geq 2p_a(B) - 2 + \sum_{p \in B_{\text{sing}}} 2 - \mu_p(B).
\]

This concludes the proof. \[\square\]

**Proposition 4.8** — Let \((X, \Delta)\) be a stable log surface with global index \(I\).

(i) If \(m \geq 4\) then the line bundle \(\omega_X(\Delta)^{[mI]}|_{D \cup \Delta}\) is base-point-free and the associated morphism is birational.

(ii) If \(m \geq 3\) and \(I \geq 2\) then the line bundle \(\omega_X(\Delta)^{[mI]}|_{D \cup \Delta}\) is base-point-free.

(iii) If \(m \geq 2\) and \(D \cup \Delta\) is a nodal curve then the line bundle \(\omega_X(\Delta)^{[mI]}|_{D \cup \Delta}\) is base-point-free.

In each case the line bundle is very ample if the inequality for \(m\) is strict.

**Proof.** By the curve embedding theorem 2.15 it suffices to check that for every (Cohen–Macaulay) subcurve \(B \subset D \cup \Delta\) we have \(\deg mIK_X|_B \geq 2p_a(B)\) for base-point-freeness respectively \(\deg mIK_X|_B \geq 2p_a(B) + 1\) for very ampleness. We concentrate on the base-point-freeness now, the proof for very ampleness being the same.

We start with (iii) where \(B\) is a nodal curve. Recall that \(I(K_X + \Delta)\) is an ample Cartier divisor and thus has degree at least one on each irreducible component of \(B\). If \(p_a(B) \leq 1\) the inequality \(\deg mIK_X|_B \geq 2p_a(B)\) is trivially satisfied for \(m \geq 2\). If \(p_a(B) \geq 2\) then by Lemma 4.7

\[
\deg I(K_X + \Delta)|_B \geq \deg(K_X + \Delta)|_B \geq 2p_a(B) - 2 + \sum_{p \in B_{\text{sing}}} 2 - \mu_p(B) = 2p_a(B) - 2 \geq 2
\]

and thus \(\deg 2I(K_X + \Delta)|_B \geq 2p_a(B)\).

Now suppose \(B\) is singular. We compute, using Lemma 4.7,

\[
\deg(K_X + \Delta)|_{B^\nu} = \deg(K_X + \Delta)|_{B^\nu} + \deg(K_X + \Delta)|_{B^\nu}
\]

\[
\geq 2p_a(B^\nu) - 2 + \sum_{p \in B_{\text{sing}}} \mu_p(B) + 2p_a(B) - 2 + \sum_{p \in B_{\text{sing}}} (2 - \mu_p(B))
\]

\[
\geq 2p_a(B) + 2p_a(B^\nu) - 4 + 2\# \{p \in B \mid \mu_p(B) \geq 2\}
\]

\[
\geq 2p_a(B) + 2(p_a(B^\nu) - 1) \quad \text{(because } B \text{ is singular)}
\]

If \(B\) has \(a\) irreducible components then \(p_a(B^\nu) - 1 = -\chi(O_{B^\nu})\) is at least \(-a\) and thus we continue the computation to get

\[
\deg 2(K_X + \Delta)|_{B^\nu} \geq 2p_a(B) - 2a \geq 2p_a(B) - 2\deg I(K_X + \Delta)|_{B^\nu},
\]

where the second inequality follows because the degree of the line bundle \(I(K_X + \Delta)\) is at least 1 on each irreducible component of \(B^\nu\).

Therefore we have \(\deg (2I + 2)(K_X + \Delta)|_{B^\nu} \geq 2p_a(B)\). Consequently if \(m \geq \frac{2 + 2I}{I}\) then \(\deg mI(K_X + \Delta)|_B \geq 2p_a(B)\). This proves the base-point-freeness in case (i) and (ii).
It remains to prove that $\omega_X(\Delta)^{[mI]}|_{D \cup \Delta}$ gives a birational map for $m \geq 4$. Since this map is very ample if $I \geq 2$ or $m > 4$ by the above, we may assume that $I = 1$ and $m = 4$.

Let $p, q$ be two smooth points of $D \cup \Delta$ and let $\mathcal{I} = \mathcal{O}_{D \cup \Delta}(-p - q) \subset \mathcal{O}_{D \cup \Delta}$ be the corresponding (invertible) ideal sheaf. To show that $\omega_X(\Delta)^{[4]}$ separates $p$ and $q$ is suffices to prove $H^1(D \cup \Delta, \omega_X(\Delta)^{[4]} \otimes \mathcal{I}) = 0$. Assume on the contrary $H^1(D \cup \Delta, \omega_X(\Delta)^{[4]} \otimes \mathcal{I}) \neq 0$. Then, by Lemma A.1, there is non-empty subcurve $B \subset D \cup \Delta$ such that

\[(11) \quad \chi(B, \omega_X(\Delta)^{[4]} \otimes \mathcal{I}|_B) \leq \chi(B, \omega_B)\]

with equality if and only if $\omega_X(\Delta)^{[4]} \otimes \mathcal{I}|_B \cong \omega_B$. On the other hand, we have shown above that $\deg \omega_X(\Delta)^{[4]} \otimes \mathcal{I}|_B \geq 2p_\mu(B) - 2$ with strict inequality if $B$ is nodal.

Hence $B$ is not a nodal curve, (11) is in fact an equality and $\omega_B \cong \omega_X(\Delta)^{[4]} \otimes \mathcal{I}|_B$ is a line bundle. But the dualising sheaf a curve with a $\mu$-multi-node with $\mu \geq 3$ is not a line bundle by [Kol13, Aside 5.9.3] — a contradiction. □

Remark 4.9 — Examining the proof closely the bounds can be improved under additional assumptions, for example if there are no rational irreducible components of $D \cup \Delta$ on which $\omega_X$ has small degree.

Numerically, the worst case occurs if $X$ is Gorenstein, the non-normal locus $D$ has just one singular point, and every irreducible component $B$ of $D$ is rational with a 3-multi-nodal point such that $\deg(\omega_X|_B) = 1$. In this case, denoting by $k$ the number of irreducible components of $D$, we have $p_\mu(D) = 2k$ and hence $\deg(\omega_X^{[3]}|_D) = 2p_\mu(D) - k$. Such examples can be constructed explicitly, see [LR13]. We will analyse the simplest such curve, a rational curve with a single 3-multi-node in Example 7.3. We will see that the conditions of Proposition 4.8 are not necessary for base-point-freeness on such a curve but the bound for very ampleness is sharp: $\omega_X(\Delta)^{[4]}|_{D \cup \Delta}$ need not be an embedding.

The following technical result will be used later on.

Corollary 4.10 — Let $(X, \Delta)$ be a stable log surface with global index $I$. Assume $m \geq 3$ if $D \cup \Delta$ is a nodal curve and $m \geq 4$ otherwise.

Then for each irreducible component $B \subset D \cup \Delta$ the image of the restriction map $\rho_B : H^0(D \cup \Delta, \omega_X(\Delta)^{[mI]}|_{D \cup \Delta}) \to H^0(B, \omega_X(\Delta)^{[mI]}|_B)$ has dimension at least 3.

We will see in Example 7.3 that the bound cannot be improved if $D \cup \Delta$ is not a nodal curve: it is possible that $h^0(B, \omega_X(\Delta)^{[mI]}|_B) = 2$ for $m = 3$.

Proof. Let $B$ be an irreducible component of $D$ and $n$ the dimension of the image of the restriction map $\rho_B$. By Proposition 4.8 the map induced by $\omega_X(\Delta)^{[mI]}|_{D \cup \Delta}$ is a birational morphism if $m \geq 4$ and an embedding if $m = 3$ and $D \cup \Delta$ is nodal. Thus if $\rho_B$ defines a birational morphism $\varphi : B \to \varphi(B) \subset \mathbb{P}^{n-1}$ and $\varphi(B)$ is a non-degenerate curve of degree $\deg \omega_X(\Delta)^{[mI]}|_B \geq 3$. Therefore $\dim_{\mathbb{C}} \im \rho_B \geq 3$ as claimed. □

4.3. Proof of Theorem 4.1. Let $m \geq 4$, or $m \geq 3$ if one of the conditions in Theorem 4.1(ii) holds. Note that b) implies c) by the classification of log-canonical singularities with reduced boundary.
The restriction map $\gamma: H^0(X, \omega_X(\Delta)^{[mI]}) \to H^0(D \cup \Delta, \omega_X(\Delta)^{[mI]}|_{D \cup \Delta})$ is surjective by the vanishing in Corollary 3.5. By Proposition 4.8 and the surjectivity of $\gamma$, we know that $\omega_X(\Delta)^{[mI]}$ has no base-points on $D \cup \Delta$ under our assumptions.

Combining this with Corollary 4.4 we conclude that $\omega_X(\Delta)^{[mI]}$ has no base-points for $m \geq 4$ and for $m \geq 3$ under the conditions given in Theorem 4.1 (ii).

**Remark 4.11** — According to the proof of Theorem 4.1, $\varphi_{mI}$ ($m \geq 4$) already separates the points on different irreducible components of $X$, that is, the image has the same number of irreducible components; in addition no normal point of $X$ is mapped to the image of $D \cup \Delta$.

### 5. Pluri-log-canonical embeddings

In this section we prove our results on pluri-log-canonical embeddings. For a general stable log surface we can prove the following:

**Theorem 5.1** — Let $(X, \Delta)$ be a connected stable log surface of global index $I$.

(i) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 8$.

(ii) The line bundle $\omega_X(\Delta)^{[mI]}$ defines a birational morphism for $m \geq 6$.

(iii) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 6$ if $I \geq 2$.

Under further assumptions on singularities and invariants we can improve these bounds.

**Theorem 5.2** — Let $(X, \Delta)$ be a connected stable log surface of global index $I$.

(i) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 7$ if one of the following holds:
   
   a) There is no irreducible component $\bar{X}_i$ of the normalisation such that $(\pi^*(K_X + \Delta)|_{\bar{X}_i})^2 = 1$, and the union of $\Delta$ and the non-normal locus is a nodal curve.

   b) $X$ is normal and not $(K_X + \Delta)^2 = 1$.

(ii) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 6$ if the normalisation $\bar{X}$ is smooth along the conductor divisor and has at most canonical singularities elsewhere.

(iii) The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 5$ if $D \cup \Delta$ is a nodal curve, $\bar{X}$ is smooth along the conductor divisor, and $X \setminus D$ has at most canonical singularities. In particular these conditions are satisfied, if $(X, \Delta)$ has semi-canonical singularities.

We cannot prove and do not believe all bounds given in Theorems 5.1 and 5.2 to be sharp; we will discuss some evidence for this in Remark 5.8.

**Remark 5.3** — One should resist the temptation to believe that $X$ is Gorenstein if the normalisation is smooth: for example if we take $\bar{D} \subset \bar{X}$ to be the coordinate axes in the plane and pinch both lines via the restriction of $p \mapsto -p$ then the resulting slc surface has index 2. A local trivialising section of $\omega_{\bar{X}}(\bar{D})$ is $\frac{dx}{xy}$ and the residue along the $x$-axis $-\frac{dx}{x}$ is invariant under the involution and thus does not descend to $X$ by Proposition 2.13.

However, in this case the index is always at most 2.
5.1. Outline of the proof and preliminary results. The strategy of our proof is classical: for every subscheme $\xi$ of length two, find an appropriate pluri-log-canonical curve $C$ containing it, and then prove that $mI(K_X + \Delta)$ embeds this curve, and hence $\xi$, by the numerical criterion of Theorem 2.15. There are two obstacles that restrict the choice of a curve $C$ that we can handle:

- If $C$ has a common irreducible component with $D \cup \Delta$, i.e., $C$ is not log-well-behaved (Definition 2.6), then it is not under control when we pull it back to the normalisation to compute intersection numbers and the adjunction.
- If $(\bar{X}, \bar{\Delta} + \bar{D})$ has worse than canonical singularities then some of the connectedness properties of ample curves are needed to counter-weight the contributions from singularities, so in this case we can only effectively handle reduced curves $C$.

We start with some preliminary considerations how to construct log-well-behaved curves and how to get around the failure of the adjunction formula. At the end we include a version of the connectedness lemma for ample Cartier divisors on normal surfaces.

5.1.1. Construction of log-well-behaved curves.

**Lemma 5.4** — Let $(X, \Delta)$ be a stable log surface and $\xi \subset X$ a subscheme of length 2.

(i) If $\omega_X(\Delta)^{[mI]}$ has no base-points but $\varphi_{mI}|_{\xi}$ is not an embedding then there exists a log-well-behaved reduced curve $C \in |mI(K_X + \Delta)|$ containing $\xi$.

(ii) Assume $m \geq 3$ and $m \geq 4$ if $D \cup \Delta$ is not a nodal curve. Then there exists a log-well-behaved curve $C$ in $|mI(K_X + \Delta)|$ containing $\xi$.

In addition, if $|mI(K_X + \Delta)|$ has no base-points then $C$ can be chosen to be reduced unless $\varphi_{mI}|_{\xi}$ is an embedding and the line spanned by $\varphi_{mI}(\xi)$ is contained in the branch locus of $\varphi_{mI}$.

**Proof.** Let $\xi \subset X$ be an arbitrary subscheme of length 2 and $\mathcal{I}_\xi$ its ideal sheaf. Recall that if $\varphi_{mI}$ is a morphism then it is automatically finite (Remark 4.2).

In case (i) let $Z \subset \mathbb{P}^{h^0(\omega(\Delta)^{[mI]})-1}$ be the image of $\varphi_{mI}$ and $p$ be the image of $\xi$, a reduced point. Then the preimage of every hyperplane containing $p$ contains $\xi$ and the base locus of this linear system of hyperplanes is exactly the point $p$. Thus a general hyperplane section of $Z$ containing $p$ is reduced by Bertini and does not contain any irreducible component of the image of the non-normal locus or of the branch locus of $\varphi_{mI}$. Pullback gives the required curve $C$.

We now prove (ii). First assume that $D \cup \Delta$ is empty, i.e., we are on a normal surface without boundary. Then by Blache’s version of Riemann–Roch for normal surfaces [Bla95, 3.4, 3.3(c), 2.1(d)] and Proposition 3.6, which applies as $m \geq 3$, we have

$$h^0(X, \omega_X^{[mI]}) = \chi(\mathcal{O}_X(mIK_X)) = \chi(\mathcal{O}_X) + \frac{m(mI-1)}{2}IK_X^2$$

$$\geq \chi(\mathcal{O}_X) + \frac{m(mI-1)}{2} \geq \chi(\mathcal{O}_X) + 3 \geq 4$$

where in the last step we used $\chi(\mathcal{O}_X) \geq 1$ from [Bla94, Theorem 2]. Thus there are at least 4 sections and at least a 2-dimensional space of sections vanishing on $\xi$. 

Now assume $D \cup \Delta$ is non-empty. Consider, for an irreducible component $B$ of $D \cup \Delta$, the diagram with exact rows and columns

\[
\begin{array}{c}
0 \\
\downarrow \\
H^0(X, \omega_X(\Delta)^{[mI]} \otimes I) \\
\downarrow \psi \\
H^0(X, \omega_X(\Delta)[mI]) \to H^0(D, \omega_X(\Delta)^{[mI]}|_D) \to 0 \\
\downarrow \rho_B \\
H^0(B, \omega_X(\Delta)^{[mI]}|_B).
\end{array}
\]

By Corollary 4.10 the kernel of $\rho_B$ has codimension at least 3 under our assumptions while the image of $\psi$ has codimension at most 2. Thus a general section in $H^0(X, \omega_X(\Delta)^{[mI]} \otimes I)$ does not restrict to zero on any irreducible component of $D \cup \Delta$.

Now assume in addition that $|mI(K_X + \Delta)|$ has no base-points. Because of (i), to get a reduced curve we only need to consider the case where $\varphi_{mI}\xi$ is an embedding. In that case $\varphi_{mI}(\xi)$ spans a line which is the base locus of the linear system of hyperplanes whose preimage contains $\xi$. Since a general curve $C \in |mI(K_X + \Delta)|$ that contains $\xi$ is log-well-behaved, as is shown above, we can choose a hyperplane with reduced preimage if and only if the line is not contained in the branch locus of $\varphi_{mI}$. □

5.1.2. **Corrections to adjunction.** Now our aim is to bound the correction terms occurring in the adjunction formula for a log-well-behaved curve on a stable log surface.

Let $(X, \Delta)$ be a stable log surface. We consider the minimal semi-resolution $f: Y \to X$. Let $\eta: \bar{Y} \to Y$ be the normalisation whose conductor divisor is denoted by $D_{\bar{Y}}$. The map $\bar{Y} \to X$ factors through $\bar{X}$ the normalisation of $X$, whose conductor divisor will now be denoted $D_{\bar{X}}$ (instead of $\bar{D}$); we get a commutative diagram

\[
\begin{array}{ccc}
Y & f & \to \bar{X} \\
\downarrow \eta & & \downarrow \pi \\
\bar{Y} & f & \to X.
\end{array}
\]

Let $B \subset X$ be a log-well-behaved curve. We fix some notation to formulate the first result:

- $B_{\bar{Y}} \subset \bar{Y}$ and $B_{\bar{X}} \subset \bar{X}$ are the strict transforms of $B$.
- $\hat{B}_Y \subset Y$ is the hat transform of $B$, defined in Appendix A.6.
- $\hat{\Gamma}_{B_{\bar{X}}} = \hat{B}_Y - B_Y$ and $\Gamma_{B_{\bar{X}}} = \hat{f}^*B_{\bar{X}} - B_Y$.
- $\Delta_{\bar{X}} \subset \bar{X}$ the strict transform of $\Delta$.
- $\Lambda = \hat{f}^*(K_{\bar{X}} + D_{\bar{X}}) - (K_{\bar{Y}} + D_{\bar{Y}})$ is the codiscrepancy of the pair $(\bar{X}, D_{\bar{X}})$.

Note that $\Lambda$ is effective because $f: Y \to X$ is the minimal semi-resolution [KSB88, Prop. 4.12].
In the next lemma we estimate the failure of adjunction on the singular surface $X$ in terms of data on the normalisation of the minimal semi-resolution.

**Lemma 5.5** — Let $(X, \Delta)$ be a stable log surface and let $B \subset X$ be a log-well-behaved, not necessarily reduced curve. Then with the above notation we have

$$(K_X + \Delta + B)B \geq (K_X + B)B \geq 2p_a(B) - 2 + (\Lambda - \hat{\Gamma}_{B_X} + \Gamma_{B_X}^*)(\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*).$$

**Proof.** Since $B$ is log-well-behaved, $\Delta B = \Delta_X B_X \geq 0$ and it suffices to prove the second inequality. By Proposition A.22, $p_a(B) \leq p_a(\hat{B}_Y) + \frac{\hat{B}_Y \hat{B}_X}{2}$, and hence, by adjunction on $Y$, we have

$$2p_a(B) - 2 \leq (K_Y + D_Y + \hat{B}_Y)\hat{B}_Y.$$

It follows that

$$(K_X + B)B - (2p_a(B) - 2) = (K_X + D_X + \hat{B}_X)B_X - (2p_a(B) - 2) \geq (K_X + D_X + B_X)B_X - (K_Y + D_Y + \hat{B}_Y)\hat{B}_Y = (K_Y + D_Y + \Lambda + B_Y + \Gamma_{B_X}^*)B_Y - (K_Y + D_Y + B_Y + \hat{B}_X)(B_Y + \hat{B}_X) = (\Lambda + \Gamma_{B_X}^* - \hat{\Gamma}_{B_X} B_Y - (K_Y + D_Y + B_Y + \hat{B}_X)\hat{\Gamma}_{B_X}$$

and since $B_Y \hat{E}_i = -\Gamma_{B_X}^* \hat{E}_i$ and $(K_Y + D_Y + \Lambda)\hat{E}_i = \pi^* K_X \hat{E}_i = 0$ for any $i$

$$= (\Lambda + \Gamma_{B_X}^* - \hat{\Gamma}_{B_X})(-\Gamma_{B_X}^*) - (-\Lambda - \Gamma_{B_X}^* + \hat{\Gamma}_{B_X})\hat{\Gamma}_{B_X} = (\Lambda - \hat{\Gamma}_{B_X} + \Gamma_{B_X}^*)(\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*),$$

Bringing $2p_a(B) - 2$ to the other side gives the second inequality. 

**Lemma 5.6** — Let $(X, \Delta)$ be a stable log surface of global index $I$, $m \geq 1$ and let $C \subset mI(K_X + \Delta)$ be a log-well-behaved reduced curve. Then for every subcurve $B \subset C$ we have, in the notation introduced in Section 5.1.2,

$$(mI + 1)(K_X + \Delta)B = (mI + 1)(K_X + D_X + \Delta_X)B_X \geq 2p_a(B) - 2.$$ 

**Proof.** The first equality is clear and we only prove the second. We decompose $C = A + B$ as a Weil divisor and let $A_X$, $A_Y$, $\hat{C}_X$, and $C_Y$ be the strict transform of $A$ and $C$ in $\hat{X}$ resp. $\hat{Y}$.

We have

$$mI(K_X + D_X + \Delta_X)B_X - B_X^2 = (C_X - B_X)B_X$$

$$= A_X B_X$$

$$= (\hat{f}^* A_X)(\hat{f}^* B_X)$$

$$= (A_Y + \Gamma_{A_X})B_Y$$

$$\geq \Gamma_{A_X}^* B_Y$$

(since $C_Y$ is reduced, $A_Y B_Y \geq 0$)

$$= -(\Gamma_{B_X}^* - \Gamma_{B_X}^* \Gamma_{B_X}^*)$$

where in the last line we use that $\hat{E}(B_Y + \Gamma_{B_X}^*) = 0$ for every $\hat{f}$-exceptional curve $\hat{E}$. 

Adding this to the equation resulting from Lemma 5.5 we get
\begin{equation}
(12) \quad (mI + 1)(K_X + D_X + \Delta_X)B_X \\
\geq 2p_a(B) - 2 - (\hat{\Gamma}_{B_X} - \Gamma^*_{B_X})(\hat{\Gamma}_{B_X} - \Lambda) - \Gamma^*_{B_X}(\Gamma^*_{C_X} - \hat{\Gamma}_{B_X}).
\end{equation}
By the definition of hat transform (Definition A.18) the intersection numbers of \(\hat{\Gamma}_{B_X} - \Gamma^*_{B_X}\) and \(\Gamma^*_{B_X}\) with any exceptional divisor of \(f\) are non-positive. On the other hand \(\Gamma^*_{C_X} - \hat{\Gamma}_{B_X} \geq 0\) by Lemma A.19(iii). Also \(\hat{\Gamma}_{B_X} - \Lambda\) has non-negative coefficients at every exceptional divisors mapped to \(B_X\) because at each of those \(\hat{\Gamma}_{B_X}\) has coefficients at least 1 while the coefficients of \(\Lambda\) are at most 1 for the log-canonical pair \((\bar{X}, D_X)\). So
\[-(\hat{\Gamma}_{B_X} - \Gamma^*_{B_X})(\hat{\Gamma}_{B_X} - \Lambda) - \Gamma^*_{B_X}(\Gamma^*_{C_X} - \hat{\Gamma}_{B_X}) \geq 0\]
and the claim follows from (12). \(\square\)

5.1.3. Connectedness of ample Cartier divisors on normal surfaces. This section provides a connectedness result about ample Cartier divisors on normal surfaces.

**Lemma 5.7** — Let \(X\) be a projective normal surface and \(M\) an ample Cartier divisor on \(X\). Let \(n \in \mathbb{N}_{\geq 2}\) and \(C \in |nM|\). Suppose \(C = C_1 + C_2\) is a decomposition into two (non-empty) curves. Then
\[C_1C_2 \geq n - \frac{1}{M^2} \geq n - 1,\]
and \(C_1C_2 = n - 1\) if and only if \(M^2 = 1\) and one of the \(C_i\) is numerically equivalent to \(M\).

**Proof.** We can numerically write ([Bom73, §4, Lem. 1])
\[C_1 \equiv aM + \varepsilon, \quad C_2 \equiv (n - a)M - \varepsilon,\]
where \(a = \frac{MC_1}{M^2}\) and \(M\varepsilon = 0\).
If \(f: Y \rightarrow X\) is a resolution then, by [Sak84, p. 878], the Picard lattice of \(Y\) contains the subspace of \(f\)-exceptional curves as a direct summand on which the intersection form is negative definite. Thus the Hodge index theorem on \(Y\) implies that the intersection form on \(X\) has signature \((1, k)\) for some \(k \geq 0\). Hence \(-\varepsilon^2 \geq 0,\) with equality if and only if \(\varepsilon \equiv 0\).
Since \(M\) is an ample Cartier divisor, we have \(MC_i > 0\) for \(i = 1, 2\), and both of the intersection numbers are integers. Therefore
\[a = \frac{MC_1}{M^2} \geq \frac{1}{M^2}\]
and also
\[\frac{1}{M^2} \leq \frac{MC_1}{M^2} = a = n - \frac{MC_2}{M^2} \leq n - \frac{1}{M^2}.
\]
The expression \(a(n - a)M^2\), considered as a quadratic function in \(a\), attains its minimum for the smallest (or biggest) possible value of \(a\) and thus
\[C_1C_2 = a(n - a)M^2 - \varepsilon^2 \geq n - \frac{1}{M^2} - \varepsilon^2 \geq n - \frac{1}{M^2} \geq n - 1.\]
The inequalities in (13) are all equalities if and only if \(M^2 = 1, \varepsilon \equiv 0,\) and \(a = aM^2 = 1\) or \(a = aM^2 = n - 1\). This is possible if and only if one of the curves is numerically equivalent to \(M\). \(\square\)
5.2. Proof of Theorem 5.1(i). Let $\xi \subset X$ be a subscheme of length 2. By Theorem 4.1 the 4-canonical map $\varphi_{4I}$ is a morphism. If $\varphi_{4I}|_{\xi}$ is an embedding then $\varphi_{mI}|_{\xi}$ is also an embedding for $m \geq 8$, because $|(m - 4)I(K_X + \Delta)|$ is base-point-free, again by Theorem 4.1. If $\varphi_{4I}|_{\xi}$ is not an embedding then by Lemma 5.4 there exists a log-well-behaved reduced curve $C$ containing $\xi$. Proposition 3.6 yields a surjection of global sections

$$H^0(X, \omega_X(\Delta)[mI]) \to H^0(C, \omega_X(\Delta)[mI]|_C)$$

for $m \geq 6$.

Therefore to show that $\varphi_{mI}$ ($m \geq 8$) is an embedding, it suffices to show that $\omega_X(\Delta)[mI]|_C$ defines an embedding for any subscheme $\xi$ of length two that is contracted by $\varphi_{4I}$. By Theorem 2.15 it suffices to show that, for any subcurve $B \subset C$, we have $8I(K_X + \Delta)B \geq 2p_a(B) + 1$.

By Lemma 5.6, applied to $B$, we have $(4I + 1)(K_X + \Delta)B \geq 2p_a(B) - 2$. Since $I(K_X + \Delta)$ is an ample Cartier divisor and thus has degree at least 1 on $B$, we obtain

$$8I(K_X + \Delta)B \geq 2p_a(B) - 2 + (4I - 1)(K_X + D_X + \Delta_X)B_X \geq 2p_a(B) + 1,$$

which concludes the proof.

Remark 5.8 — Employing a trick used below in the proof of Theorem 5.1(iii) one could get a better bound of $7I$ for those $\xi \in X$ that are not embedded by $\varphi_{4I}$. This does not allow us to conclude that $\varphi_{7I}$ is very ample in general: let $\xi$ be a subscheme of length two such that $\varphi_{4I}|_{\xi}$ is an embedding. Then $\varphi_{7I}|_{\xi}$ is an embedding at $\xi$ unless $\xi$ is supported on a base-point of $3I K_X$.

However, in the latter case we do not know how to find a log-well-behaved reduced curve in $|3I(K_X + \Delta)|$ or $|4I(K_X + \Delta)|$ containing $\xi$. This seems to be an artefact of our method and we are led to believe that the bound in Theorem 5.1(i) is not sharp.

5.3. Proof of Theorem 5.1(iii) and Theorem 5.2(i). Let $\xi \subset X$ be a subscheme of length 2 and assume we are in the case of Theorem 5.1(iii) or Theorem 5.2(i). Then by Theorem 4.1 the tri-canonical map $\varphi_{3I}$ is a morphism. If $\varphi_{3I}|_{\xi}$ is an embedding then $\varphi_{mI}|_{\xi}$ is also an embedding for $m \geq 6$, because $|(m - 3)I(K_X + \Delta)|$ is base-point-free, again by Theorem 4.1. If $\varphi_{3I}|_{\xi}$ is not an embedding then by Lemma 5.4 there exists a log-well-behaved reduced curve $C$ containing $\xi$.

Proposition 3.6 yields a surjection of global sections

$$H^0(X, \omega_X(\Delta)[mI]) \to H^0(C, \omega_X(\Delta)[mI]|_C)$$

for $m \geq 5$.

Therefore to show that $\varphi_{mI}$ ($m \geq 6$) is an embedding, it suffices to show that $\omega_X(\Delta)[mI]|_C$ defines an embedding. By Theorem 2.15 it suffices to show that, for any subcurve $B \subset C$, we have $mI(K_X + \Delta)B \geq 2p_a(B) + 1$. As $m \geq 5$, this is trivial if $p_a(B) \leq 2$, so we assume $p_a(B) \geq 3$.

By Lemma 5.6, applied to $B$, we have $(3I + 1)(K_X + \Delta)B \geq 2p_a(B) - 2$ and thus

$$6I(K_X + \Delta)B \geq 2p_a(B) - 2 + \frac{3I - 1}{3I + 1}(2p_a(B) - 2)$$

$$\geq 2p_a(B) - 2 + \frac{4(3I - 1)}{3I + 1} (2p_a(B) - 2) \quad \text{(since } p_a(B) \geq 3 \text{)}$$

$$\geq \begin{cases} 2p_a(B) + \frac{4}{3I} & I \geq 2 \\ 2p_a(B) & I = 1 \end{cases}.$$

Since $6I(K_X + \Delta)B$ is an integer, we have the required $6I(K_X + \Delta)B \geq 2p_a(B) + 1$ if $I \geq 2$. If $I = 1$ then $7I(K_X + \Delta)B \geq 2p_a(B) + 1$ because $(K_X + \Delta)B \geq 1$. □
5.4. **Proof of Theorem 5.2**(ii). For any subscheme $\xi \subset X$ of length two we have a log-well-behaved curve $C \in |4K_X|$ (not necessarily reduced) containing $\xi$ by Lemma 5.4(ii).

Proposition 3.6 yields a surjection of global sections
$$H^0(X, \omega_X(\Delta)[mI]) \twoheadrightarrow H^0(C, \omega_X(\Delta)[mI]|_C) \quad \text{for } m \geq 6.$$ Therefore to show that $\varphi_{mI}$ ($m \geq 6$) is an embedding, it suffices to show that $\omega_X(\Delta)[mI]|_C$ defines an embedding for any subscheme $\xi$ of length two. By Theorem 2.15 it suffices to show that, for any subcurve $B \subset C$, we have $6I(K_X + \Delta)B \geq 2p_a(B) + 1$.

We continue to use the notation from Section 5.1.2. By assumption, $\tilde{X}$ is smooth along $D_X$ and has canonical singularities elsewhere. Thus $\Lambda = \tilde{f}^*(K_X + D_X) - (K_Y + D_Y)$ is supported on the preimages of the nodes of $D_X$. On the other hand the divisor $B_X$, the strict transform of $B$ in the normalisation, is Cartier in a neighbourhood of $D_X$. The hat transform was defined in terms of intersection numbers, which are defined via the normalisation, and thus $\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*$ is trivial on those exceptional divisors mapping to the nodes of $D_X$. Therefore $\Lambda$ and $\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*$ have disjoint support on $\tilde{Y}$ and the intersection number $\Lambda(\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*) = 0$.

Lemma 5.5 now implies
$$\begin{align*}
(K_X + \Delta + B)B &= 2p_a(B) - 2 - (\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*)(\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*) \\
&\geq 2p_a(B) - 2 - (\hat{\Gamma}_{B_X} - \Gamma_{B_X}^*)^2 \\
&\geq 2p_a(B) - 2
\end{align*}\tag{14}$$

because the intersection form is negative definite on the exceptional divisors of $\tilde{f}$.

If $B = C \in |4I(K_X + \Delta)|$ then $B$ is a well-behaved Carter divisor and adjunction gives
$$6I(K_X + \Delta)B = (K_X + B)B + \Delta B + (2I - 1)(K_X + \Delta)B \geq 2p_a(B) - 2 + 4(2I - 1)I(K_X + \Delta)^2 > 2p_a(B) + 1.$$

If $B < C$ then there is at least one irreducible component $\tilde{X}_i$ of $\tilde{X}$ such that $B_{\tilde{X}_i} := B_X \cap \tilde{X}_i \subset C_X \cap \tilde{X}_i =: C_{\tilde{X}_i}$.

Now Lemma 5.7 says that $C_{\tilde{X}_i}B_{\tilde{X}_i} - B_{\tilde{X}_i}^2 \geq 3$. Hence
$$4IK_XB - B_X^2 = C_XB_X - B_X^2 \geq C_{\tilde{X}_i}B_{\tilde{X}_i} - B_{\tilde{X}_i}^2 \geq 3.$$

Combining with (14), we have
$$6I(K_X + \Delta)B \geq 2p_a(B) + 1.$$ As before $\varphi_{6I}$ embeds $C$ by Theorem 2.15. This implies that $\varphi_{6I}$ embeds $\xi$. Since $\xi$ is an arbitrary subscheme of length two, $\varphi_{6I}$ embeds $X$. 

5.5. **Proof of Theorem 5.2**(iii). The proof is exactly the same as for the previous case with the twist that, under our assumptions, we can choose the curve $C$ to be contained in $|3I[K_X]|$ by Lemma 5.4. Even though we get weaker connectedness from Lemma 5.7, the numerical criterion is still satisfied for $m \geq 5$. 

\[\square\]
5.6. Proof of Theorem 5.1(ii). Let $S$ be the finite subset of $X$ defined as the union of

$$\{P \in D \mid \pi^{-1}(P) \text{ contains a singular point of } \bar{X}\}$$

and

$$\{P \in X \setminus D \mid P \text{ is worse than a canonical singularity}\}.$$  

Let $\xi$ be the length 2 subscheme consisting of two general points in $X$. Then there exists a log-well-behaved curve $C \in |4IK_X|$ containing $\xi$ and choosing $C$ general we can assume that $C$ does not intersect $S$. Repeating the argument from the proof of Theorem 5.2(ii) in Section 5.4 we conclude that $\varphi_6 I$ is an embedding. Thus $\varphi_6 I$ is a morphism that separates every two general points in $X$, hence is birational. □

6. The log-canonical ring

For a log surface $(X, \Delta)$ the log-canonical ring is

$$R(X, K_X + \Delta) = \bigoplus_{m \geq 0} H^0(X, \omega_X(\Delta)^{[m]}) .$$

In this section we study the implications of our results so far for this ring.

**Theorem 6.1** — Let $(X, \Delta)$ be a stable surface of index $I$. Assume that $\omega_X(\Delta)^{[aI]}$ is generated by global sections. Then for $k \geq 2 + 2aI$ the multiplication maps

$$H^0(X, \omega_X(\Delta)^{[k]}) \otimes H^0(X, \omega_X(\Delta)^{[aI]}) \to H^0(X, \omega_X(\Delta)^{[k+aI]})$$

are surjective and the log-canonical ring is generated in degree at most $3aI + 1$.

In particular,

(i) $R(X, K_X + \Delta)$ is generated in degree at most $12I + 1$, and

(ii) $R(X, K_X + \Delta)$ is generated in degree at most $9I + 1$ if one of the following holds:
   a) $I \geq 2$,
   b) There is no irreducible component $\bar{X}_i$ of the normalisation such that $(\pi^*(K_X + \Delta)|_{\bar{X}_i})^2 = 1$, and the non-normal locus is a nodal curve.
   c) $X$ is normal and we do not have $I = (K_X + \Delta)^2 = 1$.

**Proof.** The line bundle $M := \omega_X(\Delta)^{[aI]}$ is generated by global sections and ample. By Proposition 3.6 we have

$$H^i(X, \omega_X(\Delta)^{[k]}(-iM)) = H^i(X, \omega_X(\Delta)^{[k-iaI]}) = 0$$

for $i > 0$ and $k \geq 2 + 2aI$; we say $\omega_X(\Delta)^{[k]}$ is 0-regular. Thus the multiplication map is surjective for $k \geq 2 + 2aI$ by Mumford’s Lemma [Laz04, Thm. 1.8.5], which is also valid for reducible varieties. The statement on the generators of the ring follows.

For the second part note that we can always choose $a = 4$ and $a = 3$ under the stronger assumptions given in (ii) by Theorem 4.1. □

**Remark 6.2** — One can also deduce from [Mum70, Thm. 3] that the line bundles $\omega_X(\Delta)^{[12I]}$ in case (i) respectively $\omega_X(\Delta)^{[9I]}$ in case (ii) satisfies property $N_1$, that is, the image of $\varphi_{12I}$ respectively $\varphi_{9I}$ is projectively normal and cut out by quadrics.
7. Examples

In this section we construct some examples of stable surfaces and analyse line bundles on a rational curve with a single 3-multi-node.

We concentrate on examples strictly related to the topic of this article; for further constructions and observations we refer to [LR13].

Example 7.1 (Very ampleness of $K_X + D$ does not descend) — Let $D$ be a smooth plane quartic curve invariant under the involution of $\tau(x, y, z) = (x, -y, z)$ on $\mathbb{P}^2$; to be concrete set $D = \{f = x^4 + y^4 + z^4 = 0\}$. Then let $X$ be the (semi-smooth) stable surface corresponding to the triple $(\mathbb{P}^2, \bar{D}, \tau|_D)$, that is, we glue $\bar{D}$ to itself via $\tau$. The quotient $D = \bar{D}/\tau$ is an elliptic curve and thus by Proposition 2.14 the invariants of $X$ are $K_X^2 = 1$ and $\chi(O_X) = 3$.

We will now study the canonical ring $R = \bigoplus_k H^0(X, \omega_X^\otimes k)$ of $X$ and show that while $\pi^*\omega_X^\otimes k \cong \mathcal{O}_{\mathbb{P}^2}(k)$ is very ample for $k \geq 1$ the line bundle $\omega_X^\otimes k$ is very ample only for $k \geq 5$.

Consider the residue sequence $0 \to \omega_{\mathbb{P}^2}(\bar{D})^\otimes k(-\bar{D}) \to \omega_{\mathbb{P}^2}(\bar{D})^\otimes k \to \omega_D^\otimes k \to 0$ which gives

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}(k - 4)) \to H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}(\bar{D})^\otimes k) \xrightarrow{R} H^0(\bar{D}, \omega_D^\otimes k) \to 0.$$ 

It turns out that, if we identify $H^0(\mathbb{P}^2, \omega_{\bar{X}}(\bar{D})^\otimes k)$ with elements of degree $k$ in $S = \mathbb{C}[x, y, z]$ then the residue of a section is (anti)-invariant if and only if its residue is zero or the section is (anti)-invariant under the induced action of $\tau$ on the polynomial ring $S$. Thus, by Proposition 2.13, $R_k = f \cdot S_{k-4} + S_k^\pm \tau$ where $S_k^\pm \tau$ are the invariant or anti-invariant polynomials of degree $k$ according to the parity of $k$.

Writing out the first degrees explicitly it is easy to see that as a subring we have

$$R = \mathbb{C}[x, y, z^2, zf] \subset S,$$

and $X$ is the hypersurface in $\mathbb{P}(1, 1, 2, 5)$ given by the equation $w_2^2 - w_3(w_1^2 + w_4^2 + w_5^2)^2 = 0$. In particular, as long as $k \leq 4$ the $k$-canonical map factors over the quotient $\mathbb{P}^2/\tau = \mathbb{P}(1, 1, 2)$ and it is very ample for $k \geq 5$. So while $\omega_X(\bar{D})^\otimes k$ is very ample on $\bar{X} = \mathbb{P}^2$ for every $k \geq 1$ this very ampleness does not descend to $X$.

Incidentally the canonical ring of a smooth surface of general type with $p_g = 2$ and $K^2 = 1$ is known to be of the same form [BHPV04, VII.(7.1)], so we have constructed a surface in the boundary of that irreducible component of the moduli space of smooth surfaces.

Note that this example also shows that our Ansatz to prove base-point-freeness is sharp: the space of sections of $\omega_X^\otimes k$ vanishing along the non-normal locus might be empty for $k < 4$.

Example 7.2 (Large $K_X^2$ is not enough for non-normal surfaces) — For a (connected) stable surface $X$ with canonical singularities, the bi-canonical map is a morphism as soon as $K_X^2 \geq 5$ and the tri-canonical map is an embedding as soon as $K_X^3 \geq 6$ (see [Cat87]).

We will now construct examples of non-normal stable surfaces (Gorenstein and irreducible) with $K_X^2$ arbitrarily large such that the bi-canonical map not a morphism, and neither the tri-canonical not the 4-canonical map is an embedding. Morally, the obstructions to being base-point-free as well as the increase of $K_X^2$ happen locally, so that they cannot affect each other.
Fix once for all an inhomogeneous coordinate \( z \) on \( \mathbb{P}^1 \) and let \( \tau_0(z) = -z \). On \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) let \( H_x = \mathbb{P}^1 \times \{ x \} \) and \( V_x = \{ x \} \times \mathbb{P}^1 \) and consider for \( k \geq 2 \) the divisor

\[
\tilde{D}_k = H_0 + H_1 + H_\infty + \sum_{j=1}^k (V_j + V_{-j}).
\]

We specify an involution \( \tau \) on the normalisation \( \tilde{D}_k' = H_0 \sqcup H_1 \sqcup H_\infty \sqcup \bigsqcup_{j=1}^k (V_j \sqcup V_{-j}) \) by

\[
\tau|_{H_0} = \tau_0, \quad \tau: H_1 \xrightarrow{\text{id}} H_\infty, \quad \tau: V_k \rightarrow V_{-k}, \; z \mapsto \frac{1}{1 - z}.
\]

Because \( \tau \) preserves the preimages of the nodes of \( \tilde{D}_k \) it preserves the different \( \text{Diff}_{\tilde{D}_k'}(0) \) and thus by Theorem 2.2 the triple \((X, \tilde{D}_k, \tau)\) determines uniquely a stable surface \( X_k \).

We determine the singular points of the non-normal locus as described in Section 4.2: for all \( j = 1, \ldots, k \) the points \((\pm j, 0), (\pm j, 1), (\pm j, \infty)\) are mapped to a single point \( P_j \in X_k \) and every \( P_j \) is a 6-multi-node of \( D_k \). The non-normal locus \( D_k \) has \( k + 2 \) irreducible components: a smooth rational curve containing all \( P_j \), which is the image of \( H_0 \), a nodal rational curve with a node at each \( P_j \), which is the image of \( H_1 \cup H_\infty \), and for \( j = 1, \ldots, k \) rational curves \( C_j = \pi(V_j \cup V_{-j}) \) with a single 3-multi-node at \( P_j \).

The only non-semi-smooth singularities of \( X_k \) are degenerate cusps at the points \( P_j \), where \( X_k \) locally looks like the cone over a cycle of 6 independent lines. Thus \( X_k \) is a Gorenstein stable surface.

We have \( \chi(\mathcal{O}_{D_k}) = \chi(\mathcal{O}_{D_k'}) - \chi(\nu_* \mathcal{O}_{D_k}/\mathcal{O}_{D_k}) = k + 2 - 5k = 2 - 4k \). On the other hand it is easy to calculate \( \chi(\mathcal{O}_{D_k}) = 3 - 4k \), so by Proposition 2.14 the invariants of \( X_k \) are \( \chi(\mathcal{O}_{X_k}) = 1 + (2 - 4k) - (3 - 4k) = 0 \) and \( K_{X_k}^2 = (K_X + \tilde{D}_k)^2 = 4k - 4 \).

To prove that \( |2K_X| \) has base points and \( |3K_X| \) and \( |4K_X| \) are not very ample we analyse the restriction the curves \( C_j \). Its degree is

\[
\deg(\omega_{X_k}|_{C_j}) = \frac{1}{2}(K_{X_k} + \tilde{D}_k)(V_j + V_{-j}) = 1.
\]

Our claim now follows from the properties of line bundles of low degree on rational curves with a single 3-multi-node analysed in Example 7.3 below.

**Example 7.3** (A special curve) — Let \( B \) be a rational curve with a single 3-multi-node, \( \nu: B' \rightarrow B \) its normalisation. Then \( \chi(\mathcal{O}_B) = \chi(\mathcal{O}_{B'}) - 2 = -1 \) and \( B \) has arithmetic genus \( p_a(B) = 2 \).

For any line bundle \( \mathcal{L} \) on \( B \) the following properties hold.

\( (i) \) If \( \deg \mathcal{L} \geq 2 \) then \( H^1(B, \mathcal{L}) = 0 \) and \( h^0(B, \mathcal{L}) = \deg \mathcal{L} - 1 \).

\( (ii) \) If \( \deg \mathcal{L} = 2 \) then \( h^0(B, \mathcal{L}) = 1 \) and \( \mathcal{L} \) does not define a morphism.

\( (iii) \) If \( \deg \mathcal{L} = 3 \) or \( \deg \mathcal{L} = 4 \) then \( |\mathcal{L}| \) is base-point-free but not an embedding.

\( (iv) \) If \( \deg \mathcal{L} = 4 \) then \( |\mathcal{L}| \) defines a birational morphism, which is an embedding on the smooth locus.

\( (v) \) If \( \deg \mathcal{L} \geq 5 \) then \( |\mathcal{L}| \) defines an embedding.

**Proof.** By Serre duality \( H^1(B, \mathcal{L}) = \text{Hom}_{\mathcal{O}_B}(\mathcal{L}, \omega_B) \). If \( H^1(B, \mathcal{L}) \neq 0 \), then there is a non-zero \( \lambda: \mathcal{L} \rightarrow \omega_B \). As \( \lambda \) is an isomorphism at the generic point and \( \mathcal{L} \) is
torsion-free \( \lambda \) is automatically injective and the cokernel is supported on a finite set of points. Thus

\[ 1 = \chi(\omega_B) \geq \chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O}_B) = \deg \mathcal{L} - 1 \iff \deg \mathcal{L} \leq 2. \]

As \( \deg \mathcal{L} \geq 2 \) by the assumptions, we have \( \deg \mathcal{L} = 2 \). Then \( \lambda \) is an isomorphism. On the other hand, since \( B \) has a 3-multi-node, \( \omega_B \) is not locally free—a contradiction.

So there is no non-zero \( \lambda \) and \( H^1(B, \mathcal{L}) = 0 \). This implies the formula for \( h^0(B, \mathcal{L}) \) by Riemann–Roch and we get (i). The second item is an immediate consequence.

For (iii), note that the embedding dimension of the 3-multi-node is 3 while \( \mathcal{L} \) has at most 3 sections so we cannot have an embedding.

Note that, for \( p \) a smooth point of \( B \), part (i) applies to \( \mathcal{L}(−p) \) so \( H^1(B, \mathcal{L}(−p)) = 0 \) and \( p \) is not a base-point. If \( p \) is the 3-multi-node then the ideal sheaf \( \mathcal{I}_p \) of \( p \) is 

\[ \nu_*\mathcal{O}_{B^\nu}(−q_1−q_2−q_3), \]

so

\[ H^1(B, \mathcal{L} \otimes \mathcal{I}_p) = H^1(B^\nu, \nu^*\mathcal{L}(−q_1−q_2−q_3)) = 0, \]

because \( \nu^*\mathcal{L}(−q_1−q_2−q_3) \) is a line bundle of non-negative degree on \( \mathbb{P}^1 \cong B^\nu \).

Thus also the 3-multi-nodal point is not a base-point.

If \( p, p' \) are smooth points of \( B \) then \( H^1(B, \mathcal{L}(−p−p')) = 0 \) if \( \deg \mathcal{L} \geq 4 \) by (i) and \( \mathcal{L} \) separates smooth points and tangent vectors at smooth points. This proves (iv). The last item follows from Theorem 2.15. \( \square \)

Remark 7.4 (Consequences of Example 7.3) — Assume that \( X \) is a stable surface such that the non-normal locus \( D \) contains a rational curve with a single 3-multi-node. If \( \deg IK_X|_B = 1 \) then \( \varphi_{3l} \) is not a morphism (by Example 7.3 (ii)) and \( \varphi_{3ll} \) and \( \varphi_{4l} \) are not an embedding (by Example 7.3 (iii)), because the respective restriction to \( B \) has this property. In particular, this applies to Example 7.2.

Appendix A. Curves on surfaces with slc singularities

We fix some notation for this section: let \( X \) be a surface with slc singularities and (possibly empty) non-normal locus \( D \). Let \( f : Y \to X \) be the minimal semi-resolution (Definition 2.4) with conductor divisor \( D_Y \). In some instances when working near \( p \in X \) we replace \( X \) by a small affine or analytic neighbourhood of \( p \).

Let \( \nu : Y \to Y \) and \( \eta : X \to X \) be the normalisations. We have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow \eta & & \downarrow \pi \\
Y & \xrightarrow{f} & X.
\end{array}
\]

Denote by \( D_Y \subset \bar{Y} \) and \( D_X \subset \bar{X} \) the conductor divisors. Then

(i) \( K_X + D \) is a \( \mathbb{Q} \)-Cartier divisor and \( K_Y + D_Y \) is a Cartier divisor.

(ii) \( D_Y \) and \( D_Y \) are smooth and \( \eta|_{D_Y} : D_Y \to D_Y \) is a double cover.

(iii) \( D \) has at most \( \mu \)-multi-nodes (see Definition 4.5), \( \bar{D} \) has at most nodes and \( \pi|_{\bar{D}} : \bar{D} \to D \) is generically two to one.

Now let \( B \subset X \) be a well-behaved curve. Our aim is to construct a curve \( \hat{B}_Y \subset Y \), the hat transform of \( B \), such that we control the difference of the arithmetic genera \( p_a(B) \) and \( p_a(\hat{B}_Y) \), where \( \hat{B}_Y \subset \bar{Y} \) is the strict transform of \( \hat{B}_Y \). This will be achieved in Proposition A.22. The same idea has been used for surfaces
with canonical singularities in [CFHR99], but we have to work harder because our singularities are a lot worse.

Recall also that for a well-behaved divisor $A$ on $X$ and $C \subset X$ a curve, the sheaf $\mathcal{O}_C(A)$ is the restriction $\mathcal{O}_X(A)|_C$ modulo torsion (Definition 2.7).

A.1. **Automatic adjunction lemma.** The following technical result will be used several times so we state it here for further reference.

**Lemma A.1** — Let $C \subset X$ be a well-behaved curve and $A$ a well-behaved divisor on $X$. Then $H^1(C, \mathcal{O}_C(A)) \neq 0$ if and only if there is a non-empty subcurve $B \subset C$ with a generically onto map $\lambda_B : \mathcal{O}_B(A) \to \omega_B$. For such a subcurve $B$, we have $\chi(B, \mathcal{O}_B(A)) \leq \chi(B, \omega_B)$ and equality holds if and only if $\mathcal{O}_B(A) \cong \omega_B$. We can choose $B$ to be connected.

**Proof.** By Serre duality, $H^1(C, \mathcal{O}_C(A)) \neq 0$ if and only if there is a non-zero homomorphism $\lambda : \mathcal{O}_C(A) \to \omega_C$ in the dual space $\text{Hom}(\mathcal{O}_C(A), \omega_C)$. By automatic adjunction [CFHR99, Lem. 2.4], there is a subcurve $B$ of $C$ such that $\lambda$ restricts to a generically onto map $\lambda_B : \mathcal{O}_B(A) \to \omega_B$.

On the other hand, if $B$ is a subcurve with a generically onto map $\lambda_B : \mathcal{O}_B(A) \to \omega_B$, then the composition

$$\mathcal{O}_C(A) \to \mathcal{O}_B(A) \xrightarrow{\lambda} \omega_B \xrightarrow{\omega} \omega_C$$

is the corresponding non-zero morphism from $\mathcal{O}_C(A)$ to $\omega_C$.

Since $\mathcal{O}_B(A)$ is torsion free, the morphism $\lambda_B$ is injective. Being generically onto, $\lambda_B$ has a finite cokernel $Q$. So we have the following short exact sequence

$$0 \to \mathcal{O}_B(A) \to \omega_B \to Q \to 0,$$

which yields

$$\chi(B, \omega_B) = \chi(B, \mathcal{O}_B(A)) + \text{length}(Q) \geq \chi(B, \mathcal{O}_B(A)).$$

This is an equality if and only if the length of $Q$ is 0 which is in turn equivalent to $\lambda_B$ being an isomorphism. \qed

A.2. **Holomorphic Euler characteristics of well-behaved curves.**

**Definition A.2** — Let $F$ be a well-behaved curve on the semi-smooth surface $Y$ and $F_{\bar{Y}}$ its strict transform in $\bar{Y}$. We denote by $I_t(F_{\bar{Y}}, D_{\bar{Y}})$ the intersection number of $F_{\bar{Y}}$ and $D_{\bar{Y}}$ at a point $t \in Y$.

For a point $q \in Y$ we will define the local genus correction $n_q(F)$ of $F$ at $q$ and the local intersection difference $d_q(F)$ of $F$ at $q$ such that the relation

$$2n_q(F) + d_q(F) = \sum_{t \in \eta^{-1}(q)} I_t(F_{\bar{Y}}, D_{\bar{Y}})$$

holds. If $q$ is a normal crossing point with preimages $t_1, t_2$ then

$$n_q(F) = \min\{I_{t_1}(F_{\bar{Y}}, D_{\bar{Y}}), I_{t_2}(F_{\bar{Y}}, D_{\bar{Y}})\},$$

$$d_q(F) = I_{t_1}(F_{\bar{Y}}, D_{\bar{Y}}) - I_{t_2}(F_{\bar{Y}}, D_{\bar{Y}}).$$

If $q$ is a pinch point with preimage $t$ then

$$n_q(F) = \frac{1}{2} I_{\eta^{-1}(q)}(F_{\bar{Y}}, D_{\bar{Y}}),$$

$$d_q(F) = I_t(F_{\bar{Y}}, D_{\bar{Y}}) - 2 \frac{1}{2} I_t(F_{\bar{Y}}, D_{\bar{Y}}).$$

At a smooth point $q \in Y$ we set $n_q(F) = d_q(F) = 0.$
Remark A.3 — Let \( F, G \) be two well-behaved curves on a semi-smooth surface \( Y \). Elementary arithmetics with minimum and round down show that, for \( q \in Y_{\text{sing}} \),

\[
- \min\{d_q(F), d_q(G)\} \leq n_q(F) + n_q(G) - n_q(F + G) \leq 0
\]

and the inequality on the right hand side is an equality if one of \( F \) and \( G \) is Cartier at \( q \).

We call \( q \) a bad point with respect to \( F \) and \( G \) if \( n_q(F) + n_q(G) - n_q(F + G) < 0 \); we have \( d_q(F), d_q(G) \geq 1 \) for bad points \( q \).

We now use these locally defined corrections to prove global identities for the holomorphic Euler-characteristic of well-behaved curves on semi-smooth surfaces.

**Proposition A.4** — Let \( F \) and \( G \) be well-behaved curves on \( Y \) and \( F_Y \subset \bar{Y} \) (resp. \( G_Y \)) the strict transforms of \( F \) (resp. \( G \)). Then

(i) \( \chi(Y, \mathcal{O}_Y(-F)) = \chi(Y, \mathcal{O}_Y) - \chi(F_Y, \mathcal{O}_{F_Y}) + \sum_{q \in Y_{\text{sing}}} n_q(F) \).

(ii) \( \chi(F, \mathcal{O}_F) = \chi(F_Y, \mathcal{O}_{F_Y}) - \sum_{q \in Y_{\text{sing}}} n_q(F) \).

(iii) \( \chi(G, \mathcal{O}_G(-F)) = \chi(G, \mathcal{O}_G) - FG + \sum_{q \in Y_{\text{sing}}} n_q(F) + n_q(G) - n_q(F + G) \).

**Proof.** Recall that the map

\[ \eta|_{D_Y} : D_Y \to D_Y \]

is a double cover between smooth curves; the branch locus of \( \eta|_{D_Y} \) consists exactly of the pinch points of \( Y \). Thus

\[ \eta_* \mathcal{O}_{D_Y} = \mathcal{O}_{D_Y} \oplus \mathcal{L}^{-1} \]

where \( \mathcal{L} \) is a line bundle on \( D_Y \) with \( \mathcal{L}^{\otimes 2} = \mathcal{O}_{D_Y}(\sum_p \text{pinch point } p) \).

There is a commutative diagram of sheaves with exact rows and columns:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{O}_Y(-F) & \mathcal{O}_Y & \mathcal{O}_F & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \eta_* \mathcal{O}_Y(-F_Y) & \eta_* \mathcal{O}_Y & \eta_* \mathcal{O}_{F_Y} & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{M} & \mathcal{L}^{-1} & \mathcal{R} & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

where \( \mathcal{M} \) is cokernel of the natural morphism \( \mathcal{O}_Y(-F) \to \eta_* \mathcal{O}_Y(-F_Y) \). Here, because of the Snake Lemma, the last row is exact at \( \mathcal{M} \). Using the additivity of the Euler characteristic, we have

\[
\chi(Y, \mathcal{O}_Y(-F)) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}_F)
\]

\[
= \chi(Y, \mathcal{O}_Y) - \chi(Y, \eta_* \mathcal{O}_{F_Y}) + \chi(Y, \mathcal{R})
\]

\[
= \chi(Y, \mathcal{O}_Y) - \chi(F_Y, \mathcal{O}_{F_Y}) + \chi(Y, \mathcal{R}) \quad (\text{since } \eta \text{ is finite}).
\]

Note that \( \mathcal{R} \) is finite with support in \( F \cap D_Y \). For (i) it suffices to prove the following claim.
Claim. For a point $q \in Y_{\text{sing}}$ we have $\dim \mathcal{R}_q = n_q(F)$, where $n_q$ is the local genus correction defined above.

Proof of the claim. We can calculate $\mathcal{R}$ analytically locally around $q$.

If $q$ is a double normal crossing point of $Y$, then analytically locally $Y$ is $(xy = 0) \subset \mathbb{C}^3$ with $q = (0,0,0)$ and $D_Y = (x = y = 0)$. The normalisation is $\bar{Y} = \mathbb{C}^2_{x,z_1} \cup \mathbb{C}^2_{y,z_2}$ and the preimages $t_1, t_2$ are the origins in the components. The cokernel of inclusion of the coordinate rings $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[\bar{Y}]$ is isomorphic to $\mathbb{C}[D_Y] = \mathbb{C}[z]$. If $F$ is defined by $f(x,z_1)$ in one irreducible component $\mathbb{C}^2_{x,z_1}$ and by $g(y,z_2)$ in the other irreducible component $\mathbb{C}^2_{y,z_2}$, then the image of its defining ideal $\mathcal{I}_{F_C} = (f(x,z_1), g(y,z_2))$ in the localised ring $\mathbb{C}[D_Y] = \mathbb{C}[z]$ is the ideal of $\mathbb{C}[z]$ generated by $f(0,z)$ and $g(0,z)$. Note that the orders of $f(0,z)$ and $g(0,z)$ in $z$ are just the intersection numbers $I_i(F, D_Y)$, $i = 1, 2$. Therefore we have

$$\dim \mathcal{R}_q = \dim \mathbb{C}[z]/(f(0,z), g(0,z)) = \min \{I_{i_1}(F, D_Y), I_{i_2}(F, D_Y)\}$$

If $q$ is a pinch point of $Y$, then analytically locally $Y$ is $(x^2 - y^2z = 0) \subset \mathbb{C}^3$ with $p = (0,0,0)$ and $D_Y = (x = y = 0)$. The normalisation $\bar{Y}$ of $Y$ is $\mathbb{C}^2_{u,y}$ with normalisation map

$$\bar{Y} = \mathbb{C}^2_{u,y} \ni (u,y) \mapsto (u^2,y,u^2).$$

The preimage $t$ of $q$ is the origin in $\bar{Y} = \mathbb{C}^2_{u,y}$ and the conductor divisor $D_{F_C}$ is defined by $y = 0$ in $\bar{Y}$. The cokernel of the inclusion of coordinate rings $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[\bar{Y}]$ is naturally isomorphic to $u\mathbb{C}[D_{F_C}] = u\mathbb{C}[z]$. (Note that in $\mathbb{C}[\bar{Y}]$ we have $z = u^2$.) Now suppose $F$ is defined by $f(u,y) = 0$ locally around $q = (0,0)$. Then $\mathcal{I}_{F_C}$ is the ideal of $\mathbb{C}[u,y]$ generated by $f(u,y)$. Note that the order of $f(u,0)$ is just the intersection number $I_i(F, D_Y)$. Then the image of $\mathcal{I}_{F_C}$ in the localised module $u\mathbb{C}[z]$ is the submodule generated by $uz^k$, where $\dim \mathcal{R}_q = k = \lceil \frac{1}{2}I_i(F, D_Y) \rceil$. This concludes the proof. □

The first row of (16) gives $\chi(F, \mathcal{O}_F) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}_Y(-F))$, so (ii) is implied by (i).

The short exact sequence $0 \rightarrow \mathcal{O}_Y(-F - G) \rightarrow \mathcal{O}_Y(-F) \rightarrow \mathcal{O}_G(-F) \rightarrow 0$ from Lemma 2.8 gives $\chi(G, \mathcal{O}_G(-F)) = \chi(Y, \mathcal{O}_Y(-F)) - \chi(Y, \mathcal{O}_Y(-F - G))$. Applying (i) to both terms on the right hand side and then substituting (ii) gives (iii). □

A.3. Resolution graphs and semi-rationality. We will now recall some more of the classification of slc singularities. The resolution graphs of log-canonical surface singularities are well known (e.g. [KM98b, Ch. 4]) so we concentrate on the non-normal case; our sources are [Ko12, 17] and [KSB88, Sect. 4].

Over a non-normal point $p \in X$ we can write $\bar{Y}$ analytically locally $\bar{Y} = \bigcup \bar{Y}_\alpha$ as the union of local irreducible components. On each component the $f$-exceptional divisors together with the components of the double locus give rise to an (extended) dual graph: every $f$-exceptional component, which are all rational because we are over a non-normal point of $X$, gives a vertex which is either marked with “●” or with the negative self-intersection; we add a “★” for every component of the conductor divisor $D_Y$ and connect two vertices if the corresponding curves intersect.
The edges connecting the resolution graph to the boundary components are marked with the coefficient of the different $\text{Diff}_{\overline{X}}(0)$ at the corresponding point of the conductor divisor on $\overline{X}$.

The following three cases can occur:

(C1) \[ \bullet \frac{1-\frac{1}{\delta}}{c_1 - \cdots - c_n} (c_i \geq 2) \]

where $-\delta$ is the determinant of the intersection form of the exceptional divisors.

(C2) \[ \bullet \frac{1}{c_1 - \cdots - c_n - \bullet} (c_i \geq 1) \]

and if some $c_j = 1$, then $n = 1$, because we consider the minimal semi-resolution.

\[
\begin{array}{c}
\begin{array}{c}
\bullet \frac{1}{c_1 - \cdots - c_n - \bullet}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\bullet \frac{1}{c_1 - \cdots - c_n - \bullet}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\bullet \frac{1}{c_1 - \cdots - c_n - \bullet}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\bullet \frac{1}{c_1 - \cdots - c_n - \bullet}
\end{array}
\end{array}
\]

According to the type of extended dual graph associated to the curves in an irreducible component $Y_\alpha$ we say $Y_\alpha := \eta(Y_\alpha)$ is of type (C1) (resp. (C2), (Dh)).

The whole extended dual graph of an slc singularity is obtained by attaching graphs of the types (C1), (C2), and (Dh) along the boundary components, with the restriction that the different should match (see Theorem 2.2). If the exceptional divisor intersects the conductor on $Y$ in a pinch point, then there is a boundary component in the resolution graph which is not glued to any other component (compare [KSB88], Prop. 4.27 for more details). In total we see that the exceptional divisors form a tree of rational curves unless we glue a number of components of type (C2) in a circle, that is, the singularity is a degenerate cusp.

The following is an important property of a singularity.

**Definition A.5** ([KSB88, Def. 4.14], [vS87, Def. 4.1.1]) — If $R^1f_*\mathcal{O}_Y = 0$ then we say $X$ has semi-rational singularities.

Morally all results valid for rational singularities hold also in the semi-rational case.

The following makes the connection to slc singularities.

**Lemma A.6** — Let $p \in X$ be an slc surface singularity and $f: Y \to X$ the minimal semi-resolution.

The point $p$ is not semi-rational if and only if $p$ is simple elliptic, a cusp, or a degenerate cusp. In this case $\dim_\mathbb{C}(R^1f_*\mathcal{O}_Y)_p = 1$. (This number is sometimes called the geometric genus of a singularity.)

**Proof.** In the normal case this is well known, see for example [Kaw88, Lem. 9.3]. In lack of an appropriate reference we sketch a proof in the non-normal case.

The statement that $\dim_\mathbb{C}(R^1f_*\mathcal{O}_Y)_p = 1$ for a degenerate cusp is contained in [vS87, Thm. 4.3.6]. So it remains to prove that in all other cases the singularity is semi-rational.

Suppose first $p \in X$ is a non-normal Gorenstein point but not a degenerate cusp. By [KSB88, Thm. 4.21] $p$ is a normal crossing or a pinch point and thus semi-rational. So it remains to show that a non-Gorenstein slc singularity $p \in X$ is semi-rational.
Let \( \tilde{p} \in \tilde{X} \) be the canonical index one cover of \( p \in X \). Then \( \tilde{X} \to X \) is a \( G \)-covering branched only at the single point \( p \) ([KSB88, Thm. 4.24]) for some finite group \( G \).

If \( \tilde{p} \in \tilde{X} \) is semi-canonical, i.e., a normal crossing or pinch point, then we can argue as in [Kov00, Thm. 1], which works for semi-rationality as well.

Otherwise \( \tilde{p} \in \tilde{X} \) is a degenerate cusp. Let \( \tilde{g} : \tilde{W} \to \tilde{X} \) the minimal semi-resolution. Then the \( G \)-action lifts to \( \tilde{W} \) and we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{W} & \xrightarrow{\lambda} & W := \tilde{W}/G \\
\downarrow{\tilde{g}} & & \downarrow{g} \\
\tilde{X} & \xrightarrow{\varphi} & X = \tilde{X}/G
\end{array}
\]

Arguing again as in [Kov00, Thm. 1], we see that \( W \) has semi-rational singularities.

Denote by \( \varphi_*^G \) (resp. \( \lambda_*^G \)) be the composite functor of pushforward and taking the \( G \)-invariant part. Then \( \varphi_*^G \) (resp. \( \lambda_*^G \)) is an exact functor from the category of \( G \)-equivariant \( \mathcal{O}_X \)-modules (resp. \( \mathcal{O}_W \)-modules) to the category of \( \mathcal{O}_X \)-modules (resp. \( \mathcal{O}_W \)-modules). Then \( g_* \circ \lambda_*^G = \varphi_*^G \circ g_* \), and by the Grothendieck spectral sequence we get

\[
R^1g_*\mathcal{O}_W \cong R^1(g_* \circ \lambda_*^G)(\mathcal{O}_W) \cong R^1(\varphi_*^G \circ \tilde{g})(\mathcal{O}_\tilde{W}) \cong \varphi_*^G R^1\tilde{g}_*(\mathcal{O}_\tilde{W}) \cong (R^1\tilde{g}_*(\mathcal{O}_\tilde{W}))^G.
\]

With the same argument as in [Kaw88, Proof of Thm. 9.6, p. 143] one proves that \( G \) acts effectively on \( R^1\tilde{g}_*(\mathcal{O}_\tilde{W}) \cong \mathbb{C} \) thus \( R^1\tilde{g}_*(\mathcal{O}_\tilde{W}))^G = 0 \) and \( p \in X \) is semi-rational also in this case. This concludes the proof.

An alternative approach to this result is to compute the fundamental cycle on a stable improvement in the sense of [vS87] and then use [vS87, Thm. 4.1.3].

**Remark A.7** — Let \( p \in X \) be an slc point and \( Y \) the semi-resolution of a sufficiently small neighbourhood of \( p \). Then for every effective divisor \( E \) supported on the exceptional divisors we have a surjection

\[
(R^1f_*\mathcal{O}_Y)_p \cong H^1(Y, \mathcal{O}_Y) \to H^1(E, \mathcal{O}_E).
\]

Thus if \( p \) is semi-rational then \( h^1(E, \mathcal{O}_E) = 0 \) and \( \chi(E, \mathcal{O}_E) = h^0(E, \mathcal{O}_E) \geq 1 \); if \( p \) is not semi-rational then \( h^1(E, \mathcal{O}_E) \leq 1 \) and \( \chi(E, \mathcal{O}_E) \geq h^0(E, \mathcal{O}_E) - 1 \geq 0 \).

More precisely, in the non-semi-rational case equality can only occur if the support of \( E \) is the full exceptional locus since otherwise \( E \) is supported on the exceptional divisor of a semi-rational singularity.

**A.4. Semi-numerical cycle.**

**Definition A.8** — Let \( p \in X \) be a non-semi-smooth point and \( E_i \) the exceptional divisors over \( p \). The semi-numerical cycle \( Z \) over \( p \) is a minimal Weil divisor \( Z = \sum \alpha_i E_i \) of \( Y \) such that

1. \( \alpha_i \in \mathbb{Z} \) for any \( i \);
2. \( (Z^Y + D^Y)E_{i^Y} \leq 0 \) for any \( i \).

**Remark A.9** — If \( X \) is normal, then \( D^Y \) is empty and Definition A.8 coincides with the usual definition of the numerical cycle ([Rei97, Sect. 4.5]); the existence and the uniqueness of a semi-numerical cycle is proved in the same fashion as for the normal singularities, using the negative definiteness of the intersection form on the
exceptional curves. The semi-numerical cycle turns out to carry the cohomology $R^1 f_*\mathcal{O}_Y$ (cf. Remark A.12).

See [vS87, 3.4] for a discussion of the notion of fundamental cycle for a more general class of non-normal surfaces singularities.

In the case where $p \in X$ is normal, the numerical cycle is nicely elaborated in [Rei97, Section 4] (see also [KM98b, Thm. 4.7]), so we concentrate on the non-normal case.

**Remark A.10 (Semi-numerical cycle on non-normal slc singularities)** — Let $p \in X$ be a non-normal slc point. Locally analytically around the preimage of $p$ we decompose the resolution $Y$ into irreducible components of the type presented in Section A.3 and correspondingly the semi-numerical cycle $Z = \bigcup \alpha Z_\alpha$ where $Z_\alpha \subset Y_\alpha$. Since the intersection form is defined via the normalisation the divisors $Z_\alpha$ are uniquely determined by the configuration of exceptional curves and boundary components on $Y_\alpha$.

Computing in each of the different cases we see that $Z_\alpha$ is the reduced sum of exceptional divisors except in the following cases:

(i) The component $Y_\alpha$ is of type (C2) with extended dual graph

```
  • - 1 - •
```

and $Z_\alpha = 2E$, where $E$ is the exceptional curve.

(ii) The component is of type (Dh) with dual graph

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  2

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and, denoting by $E'$ and $E''$ the two exceptional curves on the right of the fork and by $E_j$ the exceptional curves in the chain to the left of the fork, the restriction of the semi-numerical cycle is $Z_\alpha = E' + E'' + 2 \sum_{j=1}^n E_j$.

In particular, $Z$ has multiplicity at most 2 at each irreducible exceptional curve.

The next result shows the importance of the semi-numerical cycle for the computation of higher pushforward sheaves.

**Lemma A.11** — Let $C$ and $F$ be well-behaved curves on $X$ and $Y$ respectively such that $f_*F = C$ as Weil divisors. Suppose moreover $FE \leq 0$ for any effective exceptional divisor $E$ over $p$. Then $R^1 f_*\mathcal{O}_Y(-F)_p \cong H^1(Z, \mathcal{O}_Z(-F))$ where $Z$ is the semi-numerical cycle over $p$.

**Remark A.12** — Applying the above to an empty curve we see that $H^1(Z, \mathcal{O}_Z) = (R^1 f_*\mathcal{O}_Y)_p$ where $p \in X$ and $Z$ is the semi-numerical cycle over $p$.

**Proof.** If $p$ is semi-smooth then the map $f$ is finite in a neighbourhood of $p$ and both sides are 0. So assume $p$ to be a non-semi-smooth point of $X$. Let $E$ be any effective divisor supported on the exceptional locus over $p$. The restriction sequence from Lemma 2.8 yields an exact sequence in cohomology:

\[(18) \quad H^1(E, \mathcal{O}_E(-Z - F)) \to H^1(Z + E, \mathcal{O}_{Z+E}(-F)) \to H^1(Z, \mathcal{O}_Z(-F)) \to 0.\]

**Claim.** $H^1(E, \mathcal{O}_E(-Z - F)) = 0$. 

Proof of the claim. Suppose on the contrary that $H^1(E, \mathcal{O}_E(-Z - F)) \neq 0$. By Lemma A.1, there is a subcurve $E' \subset E$ such that
\begin{equation}
\chi(E', \mathcal{O}_{E'}(-Z - F)) \leq \chi(E', \omega_{E'}) = -\chi(E', \mathcal{O}_{E'}). \tag{19}
\end{equation}
Recall that by Remark A.7
\begin{align}
\chi(E', \mathcal{O}_{E'}) &\geq 0 \quad \text{and} \\
\chi(E', \mathcal{O}_{E'}) &\geq 1 \text{ unless } p \text{ is not semi-rational and } \text{supp } E' = f^{-1}(p). \tag{20}
\end{align}
Applying Proposition A.4(iii) to $\mathcal{O}_{E'}(-Z - F)$ equation (19) becomes
\begin{equation}
2\chi(E', \mathcal{O}_{E'}) + \sum_{q \in Y_{\text{sing}}} n_q(E') + n_q(Z + F) - n_q(E' + Z + F) \leq (Z + F)E'. \tag{21}
\end{equation}
We treat the case $p \in X$ a normal point first. Then $Y$ is smooth and the above equation becomes
\begin{equation}
2\chi(E', \mathcal{O}_{E'}) \leq (Z + F)E' \leq 0.
\end{equation}
If $p \in X$ is rational or $\text{supp}(E') \neq f^{-1}(p)$, then $2\chi(E', \mathcal{O}_{E'}) \geq 2$ by (20) — a contradiction. If $p \in X$ is not rational and $\text{supp}(E') = f^{-1}(p)$ then $E' = Z + E''$ for some effective $E''$. Hence $(Z + F)E' \leq ZE' = Z^2 + ZE'' \leq Z^2 < 0$ while $2\chi(E', \mathcal{O}_{E'}) \geq 0$ which again contradicts (20).

Now we can assume $p \in X$ is non-normal. Let $E'_Y \subset \tilde{Y}$ be the strict transform of $E'$. Then
\begin{equation}
n_q(Z + F) - n_q(E' + Z + F) \geq - \sum_{t \in \eta^{-1}(q)} I_t(E'_Y, D_Y) \geq ZE'. \tag{22}
\end{equation}
where the last inequality is because of the definition of the semi-numerical cycle. Combining (21) and (22), we have
\begin{equation}
2\chi(E', \mathcal{O}_{E'}) + \sum_{q \in Y_{\text{sing}}} n_q(E') \leq FE' \leq 0.
\end{equation}
where the last inequality is by our assumption on $F$.

If $p \in X$ is non-semi-rational with $\text{supp}(E') = f^{-1}(p)$, then $\sum_{q \in Y_{\text{sing}}} n_q(E') > 0$ and $\chi(E', \mathcal{O}_{E'}) \geq 0$ by (20) — contradiction. Otherwise $\chi(E', \mathcal{O}_{E'}) \geq 1$ and again we get a contradiction.

Thus a subcurve $E'$ as in (19) cannot exist and $H^1(E, \mathcal{O}_E(-Z - F)) = 0$ as claimed. \qed

Therefore the sequence (18) yields $H^1(Z, \mathcal{O}_Z(-F)) \cong H^1(Z + E, \mathcal{O}_{Z+E}(-F))$ for every effective exceptional divisor $E$ over $p$. Moreover, the surjection $\mathcal{O}_Y(-F)|_{Z+E} \to \mathcal{O}_{Z+E}(-F)$ induces an isomorphism
\begin{equation}
H^1(Z + E, \mathcal{O}_{Z+E}(-F)) \cong H^1(Z + E, \mathcal{O}_Y(-F)|_{Z+E})
\end{equation}
because the kernel is supported on points. By the theorem on formal functions ([Har77, Thm. III.11.1]) we have $R^1f_*(\mathcal{O}_Y(-F))_p = H^1(Z, \mathcal{O}_Z(-F))$ as claimed. \qed

We now give some lower bounds on the Euler characteristic of subcurves of semi-numerical cycles.
Lemma A.13 — Let $Z$ be the semi-numerical cycle over $p \in X$ and $E \subset Z$ a connected subcurve. Then

$$2\chi(E, \mathcal{O}_E) \geq \sum_{q \in Y_{\text{sing}}} d_q(E)$$

with equality if and only if $E$ satisfies one of the following

(i) $p \in X$ is simple elliptic singularity or a cusp and $E = Z$.

(ii) $p \in X$ is non-normal, $E$ is reduced and every connected component of $E$ is a chain of smooth rational curves intersecting $D_Y$ in two points.

Proof. If $p \in X$ is normal, then $d_q(E) = 0$ for all $q$ and the assertions follow from Remark A.7. So in the following we assume $p \in X$ is non-normal and that $E$ is connected.

Denote by $\bar{E}$ the strict transform of $E$ in $\bar{Y}$. Let $E^{(\alpha)}$ $(1 \leq \alpha \leq n)$ be the connected components of $\bar{E}$ and $E^{(\alpha)} := \eta_\ast \bar{E}^{(\alpha)}$ the pushforward as Weil divisors. We arrange the labels in such a way that $E^{(\alpha)} \cap E^{(\alpha+1)} \neq \emptyset$ for $1 \leq \alpha \leq n - 1$. By Proposition A.4, we have

$$\chi(E, \mathcal{O}_E) = \chi(\bar{E}, \mathcal{O}_{\bar{E}}) - \sum_{q \in Y_{\text{sing}}} n_q(E) = \sum_{\alpha=1}^{n} \chi(E^{(\alpha)}, \mathcal{O}_{E^{(\alpha)}}) - \sum_{q \in Y_{\text{sing}}} n_q(E),$$

and similarly

$$\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) = \sum_{\alpha=1}^{n} \chi(E^{(\alpha)}_{\text{red}}, \mathcal{O}_{E^{(\alpha)}_{\text{red}}}) - \sum_{q \in Y_{\text{sing}}} n_q(E_{\text{red}}).$$

First we look at the reduction $E_{\text{red}}$ of $E$, which we assumed to be connected. If $p \in X$ is either semi-rational or non-semi-rational but $\text{supp}(E) \neq f^{-1}(p)$, then $E_{\text{red}}$ is a reduced tree of rational curves; we have $\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) = 1$ and since a local intersection difference can only occur at the end points $\sum_{q \in Y_{\text{sing}}} d_q(E_{\text{red}}) \leq 2$.

If $p \in X$ is a non-semi-rational non-normal point then it is a degenerate cusp, and if $\text{supp}(E) = f^{-1}(p)$ the divisor $E_{\text{red}}$ is a cycle of rational curves so that $\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) = 0$ and $\sum_{q \in Y_{\text{sing}}} d_q(E_{\text{red}}) = 0$. In both cases we have

$$2\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) - \sum_{q \in Y_{\text{sing}}} d_q(E_{\text{red}}) \geq 0$$

with equality if and only if $E_{\text{red}}$ is as described in (ii).

In general, $E$ is obtained from $E_{\text{red}}$ by adding some irreducible components of $E_{\text{red}}$. More precisely, let $F_1, \ldots, F_k$ be the irreducible components of $E - E_{\text{red}}$ so that we can write $E = E_{\text{red}} + \sum_{1 \leq j \leq k} F_j$. We order in such a way that $F_1, \ldots, F_{k'}$ have non-empty intersection with $D_Y$ while $F_{k'+1}, \ldots, F_k$ do not intersect $D_Y$.

Using the computation of semi-numerical cycles from Remark A.10 we distinguish three possible cases for $F_j \subset Y_{(\alpha_j)}$.

a) $F_j$ is a $(-1)$-curve. Then $E^{(\alpha_j)} = 2F_j$ because the only possibility is type (C2) of length 1. Then by the adjunction formula on $Y$, we have $\chi(E^{(\alpha_j)}, \mathcal{O}_{E^{(\alpha_j)}}) = 3$. Also $\sum_{q \in Y_{\text{sing}}} \sum_{t \in \eta^{-1}(q)} I_t(F_j, D_Y) = 2$ for such a curve $F_j$.

b) $F_j$ lies in an irreducible component $Y_{\alpha_j} \subset Y$ of type (Dh) and intersects $D_Y$. Write $E^{(\alpha_j)} = \bar{E}^{(\alpha_j)} + F_j$. Computing Euler characteristics for the structure sequence of $\bar{E}^{(\alpha_j)} \subset \bar{E}^{(\alpha_j)}$ in the explicit situation of Remark
A.10 one obtains that $\chi(\bar{E}^{(\alpha_i)}, \mathcal{O}_{E^{(\alpha_i)}}) \geq 2$. For such an $F_j$, we have
\[ \sum_{q \in Y_{\text{sing}}} \sum_{i \in \nu^{-1}(q)} I_i(F_j, D_Y) = 1. \]
c) $F$ lies in an irreducible component $Y_{\alpha} \subset Y$ of type (Dh) and there is no non-reduced irreducible component of $E_{\alpha}$ intersecting the conductor. Thus $\bar{E}^{(\alpha_i)}$ is supported on the exceptional divisor of a rational surface singularity and $\chi(\bar{E}^{(\alpha_i)}, \mathcal{O}_{E^{(\alpha_i)}}) \geq 1$ by [Rei97, Prop. 4.12].

Let $r_1 := \# \{ j \mid 1 \leq j \leq k', F_j \text{ is a } (-1)\text{-curve} \}$ and $r_2 := k' - r_1$. Then since by classification $\chi(\bar{E}_{\text{red}}^{(\alpha)}, \mathcal{O}_{E^{(\alpha)}}) = 1$ for every connected component of $\bar{E}$ in total we get
\[
(26) \quad 2 \sum_{\alpha=1}^n \chi(\bar{E}^{(\alpha)}, \mathcal{O}_{E^{(\alpha)}}) - 2 \sum_{\alpha=1}^n \chi(\bar{E}_{\text{red}}^{(\alpha)}, \mathcal{O}_{E_{\text{red}}^{(\alpha)}}) \geq 2(2r_1 + r_2).
\]

On the other hand, we have
\[
\sum_{q \in Y_{\text{sing}}} 2n_q(E_{\text{red}}) + d_q(E_{\text{red}}) - \sum_{q \in Y_{\text{sing}}} 2n_q(E) + d_q(E) = E_{\text{red}} D_Y - \bar{E} D_Y \quad \text{(by Definition A.2)}
\]
\[
= - \sum_{j=1}^{k'} \bar{F}_j D_Y
\]
\[
= - (2r_1 + r_2)
\]
Adding these equations and using (23), (24) and (25), we have
\[
2\chi(E, \mathcal{O}_E) - \sum_{q \in Y_{\text{sing}}} d_q(E) \geq 2\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) - \sum_{q \in Y_{\text{sing}}} d_q(E) + 2r_1 + r_2 \geq 2r_1 + r_2 \geq 0.
\]
If equality holds then $r_1 = r_2 = 0$ and $2\chi(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}}) = \sum_{q \in Y_{\text{sing}}} d_q(E)$ so $E_{\text{red}}$ is as described in (ii). As a consequence, no irreducible component of $E_{\text{red}}$ is contained in a component of type (Dh) and there cannot be non-reduced irreducible components not intersecting the conductor. Thus $E = E_{\text{red}}$ and $E$ is as in (ii).

On the other hand it is easy to see that equality holds if $E$ is as in (ii). \hfill \Box

Lemma A.14 — Let $F \subset Y$ be a well-behaved curve such that $FE_i \leq 0$ for any exceptional curve $E_i$ over $p \in X$. Then we have
\[
\dim_{\mathbb{C}} R^1 f_* \mathcal{O}_Y(-F)_p \leq \dim_{\mathbb{C}}(R^1 f_* \mathcal{O}_Z)_p + \frac{1}{2} \# \{ q \in f^{-1}(p) \mid d_q(F) > 0 \},
\]
where the function $d_q$ is as in Definition A.2.

Proof. If $p \in X$ is semi-smooth then the map $f$ is an isomorphism in a neighbourhood of $p$ and $\dim_{\mathbb{C}} R^1 f_* \mathcal{O}_Y(-F)_p = 0$. So we may assume that $p$ is a non-semi-smooth singularity of $X$. Let $Z \subset Y$ the semi-numerical cycle over $p$. By Lemma A.11 we have
\[
R^1 f_* (\mathcal{O}_Y(-F))_p = H^1(Z, \mathcal{O}_Z(-F)).
\]
and it remains to estimate the dimension of the right hand side.

Suppose $H^1(Z, \mathcal{O}_Z(-F)) \neq 0$. Then, by Lemma A.1, there is a connected subcurve $E \subset Z$ such that $\chi(E, \mathcal{O}_E(-F)) \leq \chi(E, \omega_E) = -\chi(E, \mathcal{O}_E)$ with equality if and only if $\mathcal{O}_E(-F) \cong \omega_E$. Combining with Proposition A.4(iii) yields
\[
(27) \quad 2\chi(E, \mathcal{O}_E) + \sum_{q \in Y_{\text{sing}}} n_q(E) + n_q(F) - n_q(E + F) \leq FE,
\]
This completes the proof of the lemma. By Remark A.3
\[ n_q(E) + n_q(F) - n_q(E + F) \geq -\min\{d_q(E), d_q(F)\} \geq -d_q(E) \]
and hence
\[ 2\chi(E, \mathcal{O}_E) + \sum_{q \in Y_{\text{sing}}} n_q(E) + n_q(F) - n_q(E + F) \geq 2\chi(E, \mathcal{O}_E) - \sum_{q \in Y_{\text{sing}}} d_q(E) \geq 0 \]
where the last inequality comes from Lemma A.13. Since \( FE \leq 0 \) by assumption we have equality in (27) and \( \mathcal{O}_E(-F) \cong \omega_E \). This implies \( FE = 0 \),
\[ 2\chi(E, \mathcal{O}_E) = \sum_{q \in Y_{\text{sing}}} d_q(E), \]
and \( n_q(E) + n_q(F) - n_q(E + F) = -\min\{d_q(E), d_q(F)\} = -d_q(E) \) for all \( q \in Y_{\text{sing}} \).
In particular,
\[ (28) \quad d_q(F) \geq d_q(E). \]

**Case 1:** \( p \in X \) is normal. By Lemma A.13, \( p \in X \) is either a simple elliptic singularity or a cusp and \( E = Z \). In particular
\[ h^1(Z, \mathcal{O}_Z(-F)) = h^1(Z, \omega_Z) = h^1(Z, \mathcal{O}_Z) = 1, \]
and we have equality in the claim of the Lemma.

**Case 2:** \( p \in X \) is non-normal. Let \( \mathcal{E} \) be set of connected subcurves \( E \) of \( Z \) such that there is a generically onto homomorphism \( \lambda_E: \mathcal{O}_E(-F) \to \omega_E \).

**Claim:** If \( E \neq E' \in \mathcal{E} \) then \( E \) and \( E' \) have disjoint support.

*Proof.* Interpreting the morphisms \( \lambda: \mathcal{O}_E(-F) \to \omega_E \) and \( \lambda': \mathcal{O}_{E'}(-F) \to \omega_{E'} \) as elements in \( H^1(Z, \mathcal{O}_Z(-F)) \) a general linear combination will give a generically onto morphism supported on \( E \cup E' \). Thus our claim follow if we can show that for a curve \( E \in \mathcal{E} \) no connected proper subcurve can be contained in \( \mathcal{E} \).

By Lemma A.13, \( E \) lies completely in the union of irreducible components of type \((C2)\), so \( E \) is a reduced nodal curve of arithmetic genus 0 or 1. By the above \( \mathcal{O}_E(-F) \cong \omega_E \). For every connected proper subcurve \( E' \subset E \) we have \( \deg \mathcal{O}_E(-F)|_{E'} = \deg \omega_E|_{E'} \geq -1 > -2 = \deg \omega_{E'} \) and thus there is no generically onto morphism from \( \mathcal{O}_{E'}(-F) \to \omega_{E'} \).

Since different curves in \( \mathcal{E} \) are disjoint, we have
\[ (29) \quad H^1(Z, \mathcal{O}_Z(-F)) = \bigoplus_{E \in \mathcal{E}} H^1(E, \mathcal{O}_E(-F)) = \bigoplus_{E \in \mathcal{E}} H^1(E, \omega_E) \cong \mathbb{C}^{|\mathcal{E}|}. \]

If \( p_a(E) = 1 \) for some \( E \in \mathcal{E} \) then \( p \) is a degenerate cusp and \( E \) is the reduced preimage of \( p \). Thus \( \mathcal{E} = \{E\} \) and \( H^1(Z, \mathcal{O}_Z(-F)) = H^1(E, \omega_E) \) and the claimed inequality holds.

Otherwise by Lemma A.13, every \( E \in \mathcal{E} \) is a chain of rational curves such that the end(s) of the chain intersect the conductor in two (different) points \( q_1 \) and \( q_2 \); at these intersection points \( q_i \) we have \( 1 = d_{q_i}(E) \leq d_{q_i}(F) \) by (28). Since every two different curves in \( \mathcal{E} \) are disjoint the following inequality holds
\[ \#\mathcal{E} \leq \frac{1}{2} \#\{q \in f^{-1}(p) | d_q(F) > 0\}. \]
Together with equation (29) and Lemma A.11 this completes the proof of the lemma. \( \square \)
A.5. A relative duality. As a preparation for the relative duality result we need the following lemma.

Lemma A.15 — Let $C$ be a well-behaved curve and $A$ a well-behaved divisor on $X$. Then $\text{Ext}^1_{O_X}(O_C, \omega_X(A)) \cong \text{Ext}^1_{O_X}(O_C(-A), \omega_X)$.

Proof. We look at the structure sequence

\[ 0 \to O_X(-C) \to O_X \to O_C \to 0 \]

and, applying $\mathcal{H}om_{O_X}(\cdot, \omega_X(A))$, obtain an exact sequence

\[ 0 \to \omega_X(A) \to \mathcal{H}om_{O_X}(O_X(-C), \omega_X(A)) \to \text{Ext}^1_{O_X}(O_C, \omega_X(A)) \to 0. \]

Since $\omega_X(A)$ is $S_2$, $\mathcal{H}om_{O_X}(O_X(-C), \omega_X(A))$ is also $S_2$ by [AH11, Lem. 5.1.1]. Therefore we have $\mathcal{H}om_{O_X}(O_X(-C), \omega_X(A)) \cong \omega_X(C + A)$, since the two coincide outside a finite set of points and both are $S_2$. So there is a short exact sequence

\[ (30) \quad 0 \to \omega_X(A) \to \omega_X(C + A) \to \text{Ext}^1_{O_X}(O_C, \omega_X(A)) \to 0. \]

Applying $\mathcal{H}om_{O_X}(\cdot, \omega_X)$ to the restriction sequence

\[ 0 \to O_X(-C - A) \to O_X(-A) \to O_C(-A) \to 0 \]

from Lemma 2.8, we get

\[ 0 \to \mathcal{H}om_{O_X}(O_X(-A), \omega_X) \to \mathcal{H}om_{O_X}(O_X(-C - A), \omega_X) \to \text{Ext}^1_{O_X}(O_C(-A), \omega_X(A)) \to 0. \]

As before, by the $S_2$ property of the relevant sheaves, there are isomorphisms

\[ \mathcal{H}om_{O_X}(O_X(-A), \omega_X) \cong \omega_X(A), \]

\[ \mathcal{H}om_{O_X}(O_X(-C - A), \omega_X) \cong \omega_X(C + A). \]

Also, we have $\text{Ext}^1_{O_X}(O_X(-A), \omega_X(A)) = 0$ by Lemma 3.3, which leaves us with a short exact sequence

\[ (31) \quad 0 \to \omega_X(A) \to \omega_X(C + A) \to \text{Ext}^1_{O_X}(O_C(-A), \omega_X) \to 0. \]

Comparing (30) and (31) gives $\text{Ext}^1_{O_X}(O_C, \omega_X(A)) \cong \text{Ext}^1_{O_X}(O_C(-A), \omega_X)$. Now, $\mathcal{H}om_{O_X}(O_C, \omega_X(A))$ and $\mathcal{H}om_{O_X}(O_C(-A), \omega_X)$ both being zero, the claim follows from the local-to-global-Ext-spectral-sequence. \qed

Proposition A.16 — Let $C$ and $F$ be well-behaved curves on $X$ and $Y$ respectively such that $f_*F = C$ as Weil divisors. Then, for any $p \in X$, the vector spaces $R^1f_*O_Y(-F)_p$ and $(\omega_X(C)/f_*\omega_Y(F))_p$ are dual to each other.

Proof. Let $E$ be the reduced exceptional divisor over $p$. As in [Kol85, Lem. 3.3.3] we have, using Lemma A.15,

\[ (\omega_X(C)/f_*\omega_Y(F))_p = H^1_E(\omega_Y(F)) \]

\[ = \lim_{\rightarrow} \text{Ext}^1_{O_Y}(O_{nE}, \omega_Y(F)) \cong \lim_{\rightarrow} \text{Ext}^1_{O_Y}(O_{nE}(-F), \omega_Y) \]

The surjection $O_Y(-F)|_{nE} \to O_{nE}(-F)$ has torsion kernel and thus by Serre duality

\[ \text{Ext}^1_{O_Y}(O_{nE}(-F), \omega_Y) = H^1(Y, O_{nE}(-F)) = H^1(Y, O_Y(-F))|_{nE}). \]

Combining these equations we have by the theorem of formal functions ([Har77, Thm. III.1.11])

\[ (\omega_X(C)/f_*\omega_Y(F))_p \cong \lim_{\rightarrow} H^1(Y, O_Y(-F)|_{nE}) = R^1f_*O_Y(-F)_p \]
which concludes the proof.

A.6. The hat transform. Since the intersection form is negative definite on the exceptional divisors of \( f : Y \to X \), we have (cf. [KM98b, Lemma 3.41])

Lemma A.17 — Let \( \hat{B} \) and \( \hat{C} \) be two well-behaved divisors on \( Y \). Assume that \( f_*\hat{B} = f_*\hat{C} \) and \( \hat{C}E \leq BE \) for any exceptional divisor \( E \) of \( f \). Then \( \hat{B} \leq \hat{C} \).

Proposition/Definition A.18 — Let \( B \subset X \) be a well-behaved curve. Then there exists a unique well-behaved curve \( \hat{B}_Y \subset Y \) which is is minimal with respect to the properties \( f_*\hat{B}_Y = B \) and for all exceptional divisors \( E \) of \( f \)

\[ \hat{B}_Y E \leq 0. \]

We call \( \hat{B}_Y \) the hat transform of \( B \) with respect to \( f \).

Proof. We can take a well-behaved very ample Cartier divisor \( H \) of \( X \) that contains \( B \). Then \( f^*H - (f^{-1})_* (H - B) \) contains the strict transform of \( B \) and has non-positive intersection with any exceptional divisor \( E_i \) for any \( i \). The existence follows.

Suppose \( \hat{B}_1 \) and \( \hat{B}_2 \) are two hat transforms of \( B \) under \( f \). Then we have inequalities \( \hat{B}_i E \leq 0 \leq B_Y E \) (\( i = 1, 2 \)), where \( B_Y \) is the strict transform of \( B \) on \( Y \). So both \( \hat{B}_1 \) and \( \hat{B}_2 \) are effective divisors by Lemma A.17. Write \( \hat{B}_1 = \hat{B}_3 + A_1, \hat{B}_2 = \hat{B}_3 + A_2 \), where \( A_1 \) and \( A_2 \) are two well-behaved effective divisors with no common irreducible components. Let \( E \) be a reduced and irreducible exceptional divisor of \( f \). If \( E \subset A_1 \) then \( E \not\subset A_2 \), and \( \hat{B}_3 E = (\hat{B}_2 - A_2) E \leq 0 \); if \( E \subset A_2 \) then \( E \not\subset A_1 \), and \( \hat{B}_3 E = (\hat{B}_1 - A_1) E \leq 0 \). By the minimality of a hat transform we have \( \hat{B}_1 = \hat{B}_2 = \hat{B}_3 \). The uniqueness is proved.

We start to gather some properties of the hat transform.

Lemma A.19 — Let \( \hat{B}_Y \subset Y \) be the hat transform of \( B \). Then the following holds.

(i) \( \hat{B}_Y - B_Y \) contains only exceptional curves of \( f : Y \to X \).

(ii) Let \( \hat{B}^*_Y := f^*B_Y = B_Y + \Gamma^* \) be the numerical pullback of \( B_Y \), so that \( \hat{B}^*_Y E_i = 0 \) for any exceptional curve \( E_i \) of \( f : Y \to X \). Then \( \hat{B}_Y \geq \hat{B}^*_Y \).

(iii) If \( C \subset X \) is an effective Cartier divisor such that \( B \leq C \) then \( \hat{B} \leq f^*C \).

In particular, if \( B \) is Cartier then \( \hat{B}_Y = B^*_Y \).

Proof. Recall that \( \hat{B}_Y \) is well-behaved.

For (i), if \( \hat{B}_Y - B \) contains some curve \( A \) that is not exceptional then \( (\hat{B}_Y - A) E_i \leq 0 \) for any exceptional \( E_i \), contradicting the minimality of \( \hat{B}_Y \).

For (ii), note that \( \hat{B}^*_Y E = 0 \) for any exceptional divisor \( E \). So \( \hat{B}_Y E \leq \hat{B}^*_Y E \) for any exceptional curve \( E \). Since \( \hat{B}_Y - \hat{B}^*_Y \) is supported only on exceptional divisors, we have \( \hat{B}_Y \geq \hat{B}^*_Y \) by Lemma A.17.

For (iii), note that \( f^*C \) is an integral divisor, since \( C \) is Cartier. Moreover \( f^*C E = 0 \) for any exceptional curve \( E \), and the strict transform \( B_Y \) of \( B \) is contained in \( f^*C \). By the minimality of \( B_Y \), the inequality \( B_Y \leq f^*C \) follows.

Remark A.20 — Note that since we used normalisation to define intersection numbers both \( \hat{B}_Y \) and \( \hat{B}^*_Y \) can behave unexpectedly: they might not contain all exceptional curves mapping to \( B \). See also Remark 2.11.

Proposition A.21 — In the situation above we have
(i) \( R^1 f_* \omega_Y(\hat{B}_Y) = 0; \)
(ii) \( R^1 f_* \omega_{\hat{B}_Y} = 0; \)
(iii) \( \chi(\omega_{\hat{B}_Y}) = \chi(f_* \omega_{\hat{B}_Y}). \)

Proof. For (i) we look at the diagram (15) and consider the exact sequence

\[
0 \to \eta_* \omega_Y(\hat{B}_Y) \to \omega_Y(\hat{B}_Y) \to Q_Y \to 0
\]

where \( \hat{B}_Y \) is the strict transform of \( B_Y \) on \( \hat{Y} \) and \( Q_Y \) is supported on \( D_Y \). Applying \( f_* \) we have

\[
R^1 f_* \eta_* \omega_Y(\hat{B}_Y) \to R^1 f_* \omega_Y(\hat{B}_Y) \to R^1 f_* Q_Y.
\]

Since \( f|_{D_Y} \) is finite, \( R^1 f_* Q_Y = 0 \). On the other hand, using the Leray spectral sequence and finiteness of \( \eta \) and \( \pi \), we have

\[
R^1 f_* \eta_* \omega_Y(\hat{B}_Y) = R^1 (f \eta)_* \omega_Y(\hat{B}_Y)
= R^1 (\pi f)_* \omega_Y(\hat{B}_Y)
= \pi_* R^1 f_* \omega_Y(\hat{B}_Y).
\]

The argument for [CFHR99, Claim 4.3 (iv)] gives \( R^1 f_* \omega_Y(\hat{B}_Y) = 0 \) and hence \( R^1 f_* \eta_* \omega_Y(\hat{B}_Y) = 0 \). Now the vanishing of \( R^1 f_* \omega_Y(\hat{B}_Y) \) follows from the exact sequence (32).

For (ii), we apply \( \text{Hom}_{\mathcal{O}_Y}(\cdot, \omega_Y) \) to the short exact sequence

\[
0 \to \mathcal{O}_Y(-\hat{B}_Y) \to \mathcal{O}_Y \to \mathcal{O}_{\hat{B}_Y} \to 0,
\]

and get

\[
0 \to \omega_Y \to \omega_Y(\hat{B}_Y) \to \omega_{\hat{B}_Y} \to 0.
\]

Applying \( f_* \) to the above sequence and using (i) gives \( R^1 f_* \omega_{\hat{B}_Y} \cong R^1 f_* \omega_Y(\hat{B}_Y) = 0 \).

The last item is a direct consequence of (ii) and the Leray spectral sequence. \( \square \)

We now prove the main result of the section, an estimate for the change in arithmetic genus of the strict transform of the hat transform.

**Proposition A.22** — Let \( B \) be a well-behaved curve on \( X \) and \( \hat{B}_Y \subset \hat{Y} \) the strict transform of the hat transform of \( B \). Then

\[
p_a(B) \leq p_a(\hat{B}_Y) + \frac{\hat{B}_Y D_Y}{2}.
\]
Proof. The short exact sequence $0 \to \omega_X \to \omega_X(B) \to \omega_B \to 0$ fits into the diagram

\[
\begin{array}{ccc}
0 & \to & \omega_X \\
\downarrow & & \downarrow \\
0 & \to & f_*\omega_Y \\
\downarrow & & \downarrow \\
\omega_X/f_*\omega_Y & \to & \omega(X)(B)/f_*\omega_Y(\hat{B}_Y) \\
\downarrow & & \downarrow \\
0 & \to & \omega_B \\
\end{array}
\]

The first row is exact by Proposition A.21 and the rest of the diagram is exact by defining $K_B$ (resp. $Q_B$) to be the kernel (resp. cokernel) of $f_*\omega_{\hat{B}_Y} \to \omega_B$. The Snake Lemma together with the duality from Lemma A.16 gives

\[
\dim \mathbb{C} K_B - \dim \mathbb{C} Q_B = h^0(R^1f_*\mathcal{O}_Y) - h^0(R^1f_*\mathcal{O}_Y(-\hat{B}_Y)).
\]

Now we have

\[
p_a(\hat{B}_Y) = 1 - \chi(\mathcal{O}_{\hat{B}_Y}) \\
= 1 + \chi(\omega_{\hat{B}_Y}) \\
= 1 + \chi(f_*\omega_{\hat{B}_Y}) \quad \text{(by Prop. A.21(iii))} \\
= 1 + \chi(\omega_B) + \dim \mathbb{C} K_B - \dim \mathbb{C} Q_B \\
= 1 + \chi(\omega_B) + h^0(R^1f_*\mathcal{O}_Y) - h^0(R^1f_*\mathcal{O}_Y(-\hat{B}_Y)) \\
\geq p_a(B) - \frac{1}{2} \sum_{q \in Y_{\text{sing}}} d_q(\hat{B}_Y) \\
\geq p_a(B) - \frac{1}{2} \sum_{q \in Y_{\text{sing}}} d_q(\hat{B}_Y)
\]

Thus, using Proposition A.4(ii) for $\hat{B}_Y$, we get

\[
p_a(B) \leq p_a(\hat{B}_Y) + \frac{1}{2} \sum_{q \in Y_{\text{sing}}} d_q(\hat{B}_Y) \\
= p_a(\hat{B}_Y) + \sum_{q \in Y_{\text{sing}}} n_q(\hat{B}_Y) + \frac{1}{2} \sum_{q \in Y_{\text{sing}}} d_q(\hat{B}_Y) \\
= p_a(\hat{B}_Y) + \frac{\tilde{B}_Y \cdot D_Y}{2},
\]

where the last equality is by Definition A.2. This concludes the proof. \qed
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