GENERALIZATION OF BERTRAND’S POSTULATE FOR GAUSSIAN PRIMES

ABSTRACT. In this paper, we have proved the Generalization of Bertrand’s Postulate for Gaussian primes, which is an extension of the work by Das et.al [Arxiv 2018].

1. INTRODUCTION

Prime numbers are always an interesting topic for mathematicians. There is no generating formula for prime numbers for their irregular distribution. In number theory, the prime number theorem (PNT) describes the asymptotic distribution of the prime numbers among the positive integers. It gives the idea that primes become less common as they become larger. The first such distribution found is \( \pi(n) \sim \frac{n}{\log n} \), where \( \pi(n) \) is the prime-counting function and \( \log(n) \) is the natural logarithm of \( n \).

There are many results about the density of the prime numbers and it is a most interesting topic for mathematicians, especially for number theorists. In 1845 by Joseph Bertrand has postulated that there exists a prime between \( n \) and \( 2n \), where \( n \) is a natural number. There are eighteen different proofs of this result but S. Ramanujan’ [10] has proved it by the method of mathematical induction which is more efficient. In the paper by Mitra et.al [11], they have generalized the idea for the interval \([n, kn]\) instead of \([n, 2n]\). In, 2018, Das et.al [3] has proved the result.

Now let us move to some other idea about the prime numbers. In number theory, a Gaussian integer is a complex number whose real and imaginary parts are both integers. So, we can think about the Gaussian prime number and their distribution. In this paper we have formalizes this idea for the generalized Bertrand Postulate for Gaussian primes and also proved it.

One important result about the distribution of Gaussian Prime is the “Gaussian Moat” problem [12], which is also a famous unsolved problem in number theory. Note that the results of this paper are not related to the Gaussian Moat problem.

2. PRELIMINARIES

In this section, we are going to study the properties of Gaussian Primes. In the paper, “Bertrands Postulate over the Gaussian integers” [2] by Steven Klee et.al, they have proved Bertrand’s Postulate for Gaussian Primes. In the paper “A Short Note on Prime Gaps” Das et.al have proved the generalization of Bertrand’s postulate for all primes greater than equal to 2. In this paper, we will prove the same result for Gaussian Primes. So let us define the Gaussian primes.

Definition 2.1. Gaussian Primes: Gaussian primes [7] are Gaussian integers \( z = a + bi \) satisfying one of the following properties.

1. If both \( a \) and \( b \) are nonzero then, \( a + bi \) is a Gaussian prime iff \( a^2 + b^2 \) is an ordinary prime.
2. If \( a = 0 \), then \( bi \) is a Gaussian prime iff \( |b| \) is an ordinary prime and \( |b| \equiv (\mod 4) \).

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(3) If \( b = 0 \), then \( a \) is a Gaussian prime iff \(|a|\) is an ordinary prime and \(|a| \equiv 3 \pmod{4}\).

To prove our main result we will use some properties of lattice points.

**Definition 2.2. Point:** A point \([8]\) is a 0-dimensional mathematical object which can be specified in \( n \)-dimensional space using an \( n \)-tuple \((x_1, x_2, ..., x_n)\) consisting of \( n \) coordinates.

**Definition 2.3. Lattice Point:** A point at the intersection of two or more grid lines \([1]\) in a point lattice.

A point lattice is a regularly spaced array of points. In the plane, point lattices can be constructed having unit cells in the shape of a square, rectangle, hexagon, etc. Unless otherwise specified, point lattices may be taken to refer to points in a square array, i.e., points with coordinates \((m, n, \ldots)\), where \( m, n, \ldots \) are integers. Such an array is often called a grid or a mesh. Point lattices are frequently simply called “lattices,” which unfortunately conflicts with the same term applied to ordered sets treated in lattice theory. Every “point lattice” is a lattice under the ordering inherited from the plane, although a point lattice may not be a sublattice of the plane since the infimum operation in the plane need not agree with the infimum operation in the point lattice. On the other hand, many lattices are not pointed lattices.

Now we have a clear picture of a lattice point. Now we will describe some known facts about point lattice.

**Definition 2.4. Mutually visible lattice points:** Two lattice points \( P \) and \( Q \) are said to be mutually visible if the line segment which joins them contains no lattice points other than the endpoints \( P \) and \( Q \).

**Lemma 2.5.** Two lattice points \((a, b)\) and \((m, n)\) are mutually visible \([5]\) if, and only if, \( a - m \) and \( b - n \) are relatively prime.

**Proof:** It is clear that \((a, b)\) and \((m, n)\) are mutually visible if and only if \((a - m, b - n)\) is visible from the origin. Hence it suffices to prove the theorem when \((m, n) = (0, 0)\). Assume \((a, b)\) is visible from the origin, and let \( d = (a, b) \). We wish to prove that \( d = 1 \). If \( df > 1 \) then \( a = da', b = db' \) and the lattice point \((a', b')\) is on the line segment joining \((0, 0)\) to \((a, b)\). This contradiction proves that \( d = 1 \).

Conversely, assume \((a, b) = 1 \). If a lattice point \((a', b')\) is on the line segment joining \((0, 0)\) to \((a, b)\) we have

\[
a' = ta, b' = tb, \text{ where } 0 < t < 1.
\]

Hence \( t \) is rational, so \( t = r/s \) where \( r, s \) are positive integers with \((r, s) = 1 \). Thus

\[
sa' = ar \text{ and } sb' = br,
\]

so \( s|ar, s|br \). But \((s, r) = 1 \) so \( s|a, s|b \). Hence \( s = 1 \) since \((a, b) = 1 \). This contradicts the inequality \( 0 < t < 1 \). Therefore the lattice point \((a, b)\) is visible from the origin.

From the work of Das and Paul \([3]\), they have proved “Generalization of Bertrand Postulate” for natural prime numbers, under certain conditions. In this note, we study the behavior of primes in higher dimensions. In particular, some facts are already known about Gaussian Primes. So, we have state and prove the generalization of Bertrand’s postulate for Gaussian primes. There are many interesting results on Gaussian Primes \([4]\).
3. The main result

In this section, we are going to prove the main result of this article which is the generalization of Bertrand’s Postulate for Gaussian primes. First let us state the Bertrand’s postulate [9, 10].

**Proposition 1.** For every $n > 1$ there is always at least one prime $p$ such that

$$n < p < 2n.$$

Now let us state it’s generalization which is conjectured by Mitra et al. [11] and proved by Das et al. [3].

**Proposition 2.** For any integers $n$ and $k$, where $f(k) = \lceil 1.1\ln(2.5k) \rceil$, then there are at least $(k−1)$ primes between $n$ and $kn$, where $n ≥ f(k)$.

Now we will prove this result for Gaussian primes.

**Theorem 3.1.** Let $z$ be an gaussian inetger, $z = z_1 + iz_2$ where $z_1, z_2 ∈ \mathbb{Z}$ and $z_1 ≥ 0, z_2 ≥ 0$. We define the function $f(k) = \lceil 1.1\ln(2.5k) \rceil$ for all positive integer $k ≥ 2$ and if $(z_1, z_2) = f(k)$ then there exist at least $\frac{k-1}{2}$ gaussian primes between the gap $(z, kz)$.

**Proof:** There are three possible cases.

**Case (i)** Let $z$ be an gaussian integer and $z = z_1 + iz_2$ where $z_1, z_2 ∈ \mathbb{Z}$ and $z_1 > 0, z_2 > 0$. It is given that $\gcd(z_1, z_2) = f(k)$. Now for the gap $(z, kz)$ we can write it,

$$(z, kz) = (z_1 + iz_2, k(z_1 + iz_2)).$$

So we have to prove that there exist at least $\frac{k-1}{2}$ gaussian primes between the gap $(z_1 + iz_2, k(z_1 + iz_2))$. Now observe that,

$$kz - z = (k - 1)z = ((k - 1)z_1, (k - 1)z_2) = (k - 1)(z_1, z_2).$$

But it is given that $\gcd(z_1, z_2) = f(k)$ So, we can say by the properties of lattice point that there exist $(k - 1)f(k)$ many gaussian integer between the gaussian integer $(z_1 + iz_2, k(z_1 + iz_2)) = (z, kz)$. Suppose $z_1 = af(k)$ and $z_2 = bf(k)$ where $(a, b) = 1$ and $a > 0$ and $b > 0$ are positive integers.

$$|z| = \sqrt{z_1^2 + z_2^2} = \sqrt{a^2f(k)^2 + b^2f(k)^2} = f(k)\sqrt{a^2 + b^2}$$

Clearly, $\min\{\sqrt{a^2 + b^2}\} = 1$. Then we have to prove that there exist at least $\frac{k-1}{2}$ primes of the form $4n + 1$ (where $n ≥ 1$) between the gap $(f(k), kf(k))$.

In our previous work [3] we have proved that there exist at least $k-1$ primes between the gap $(f(k), kf(k))$ for all $k ≥ 2$. Now by Cheveschiv Bias we can say that among that $k-1$ primes between the gap $(f(k), kf(k)), \frac{k-1}{2}$ of the are in the form $4n + 1$. Hence, we have proved that there exist at least $\frac{k-1}{2}$ gaussian primes between the gap $(z, kz)$ where $z$ is a Gaussian integer and $k ≥ 2$.

**Case (ii)** The left posibilities are when $Re(z) = 0$ or $Im(z) = 0$. Both cases are symmetric. So we are proving for $Re(z) = 0$. $Re(z) = 0$ implies $z = z_1 + i·0 = z_1$. By the definition of gaussian primes we have to show that there exist at least $\frac{k-1}{2}$ primes of the form $4n + 3$, (where $n ∈ \mathbb{Z}$) in the gap $(z, kz) = (z_1, kz_1)$. Again by the Cheveschiv Bias which is true. □

4. Conclusion

In this paper, we have formalizes the idea of Bertrand’s Postulate for the Gaussian primes. Also, we have proved the generalization of Bertrand’s Postulate for the Gaussian prime number using the properties of lattice points. This work is an extension of the paper done by das et.al [3] in 2018.
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