Projected Estimation for Large-dimensional Matrix Factor Models

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Large-dimensional factor models are drawing growing attention and widely applied to analyze the correlations of large datasets. Most related works focus on vector-valued data while nowadays matrix-valued or high-order tensor datasets are ubiquitous due to the accessibility to multiple data sources. In this article, we propose a projected estimation method for the matrix factor model under flexible conditions. We show that the averaged squared Frobenious norm of our projected estimators of the row (or column) loading matrix have convergence rates \( \min\{(T_p^2)\frac{1}{2}, (T^2)\frac{1}{2}, (p_1^2)\frac{1}{2}\} \) (or \( \min\{(T^2)\frac{1}{2}, (T_p^2)\frac{1}{2}, (p^2)\frac{1}{2}\} \)) where \( T \) is the number of observations. This rate is faster than the typical rates \( T^{-1} \) and \( \min\{(T_p^2)\frac{1}{2}, p^{-1}\} \) (or \( \min\{(T)\frac{1}{2}, p_1^2\frac{1}{2}\} \)) that are conceivable from the literature on vector factor models. An easily satisfied sufficient condition on the projection direction to achieving the given rates for the projected estimators is provided. Moreover, we established the asymptotic distributions of the estimated row and column factor loadings. We also introduced an iterative approach to consistently determine the numbers of row and column factors. Two real data examples related to financial engineering and image recognition show that the projection estimators contribute to explaining portfolio variances and achieving accurate classification of digit numbers.

**Keyword:** Matrix factor model; Vector factor model; Column covariance matrix.

1 Introduction

Factor models have been well studied in multivariate analysis in the past several decades. Such models assume that the dynamics and/or co-movements of a large number of variables are driven by several common factors. It has been widely applied in various research fields such as financial engineering, macroeconomic analysis and gene technology, see Ross (1977); Chamberlain and Rothschild (1983); Fama and French (1993); Stock and Watson (2002); Mayrink and Lucas (2013); Fan et al. (2015). Recently, the low-rank factor structure gains increasing popularity in statistical learning and data science. For instance, the factor-adjusted step in high-dimensional covariance matrix estimation, model selection and multiple testing result in more reliable procedures than the plain versions, see Fan et al. (2013, 2019, 2020).

Flooded with large datasets, researchers focus on large-dimensional factor models recently. The static approximate factor model dates back to Chamberlain and Rothschild (1983). Via principal component analysis, a lot of works later studied extensively on statistical inference of the large-dimensional factor model where the dimension can grow with the sample size and the idiosyncratic errors can be cross-sectionally and/or serially correlated in weak sense, see for example, Stock and Watson (2002); Bai and Ng (2002); Bai (2003); Ahn and Horenstein (2013); Fan et al. (2013, 2016); Bai and Ng (2019); Kong et al. (2019). Except for the principal component analysis method, Fan and Liao (2019) and Kong (2020) proposed a projection approach for inferring the approximate factor models by recovering the factor space in much lower dimension,

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where the factors series could be weakly dependent and the idiosyncratic errors could be serially and cross-sectionally weakly correlated. When the factors can not only explain most of the cross-sectional dependence but also capture most of the temporal dependence structure, Lam et al. (2011) and Lam and Yao (2012) dig out the persistency signal by the eigen-analysis of an auto-cross-covariance matrix assuming white noise type idiosyncratic error matrix. Another promising and completely distinct framework is the dynamic factor model introduced in the seminal work by Forni et al. (2000), and interesting statistical inference works include Hallin and Liška (2007); Bates et al. (2013); Forni et al. (2015). The dynamic factor model allows for more general representations of the factor processes. A continuous-time version of the factor model applied to model the dynamics and co-movements of large-panel high-frequency data sets can be found in Kong (2017) and Kong (2018).

All the factor models in the above papers were designed only for vector-form data, i.e., the observations \( x_t, t = 1, \ldots, T \), are vectors. Recently, we are facing frequently with matrix-variate or high-order tensor data. For illustration, Figure 1a and 1b present two examples of matrix-variate observations in macroeconomic analysis and image recognition. Figure 1a displays a series of data arrays consisting of macroeconomic variables recorded in different countries. Figure 1b shows a collection of hand written digit number pictures with each picture composed by \( 28 \times 28 \) pixels. A naive approach for analyzing matrix-form data is to pile down the columns of each \( p_1 \times p_2 \) data matrix into a single \( p_1 p_2 \) dimensional vector such that the techniques designed for vector factor models can be adopted directly. This naive approach misses spatial correlation patterns and structures, and hence suffers from loss of estimation efficiency. Dealing with the matrix observations directly achieves a better convergence rate in high-dimensional settings than the naive procedure, see for example Chen et al. (2020).

![Economic Indicators](image1.png)  ![Digit dataset](image2.png)

Figure 1: Two real examples of matrix-variate observations. Figure 1a displays a series of data arrays consisting of macroeconomic variables recorded in different countries. Figure 1b shows a collection of hand written digit number pictures with each picture composed by \( 28 \times 28 \) pixels.

Low-rank representation of matrix observations is not a new topic in image processing and machine learning fields, and existing methods usually rely on singular value decomposition (SVD) of the matrices, see for example Yang et al. (2004); Zhang and Zhou (2005); Ye (2005); Crainiceanu et al. (2011). However, almost all the related works focused more on the algorithms and computations of the 2-dimensional principal components. Few theoretical results in econometrics were established. To the best of our knowledge, few works on the matrix factor model appeared in the literature. Wang et al. (2019) is the first to introduce a matrix factor model for matrix-variate time series data. The model exploits two loading matrices and a low-dimensional matrix-variate common factor to achieve two-way dimension reduction. The model was later extended to a constrained matrix factor structure in Chen et al. (2020), and revisited and studied in Chen et al. (2020).

The primary goal of this paper is to reconsider recovering the row and column factor spaces by proposing projection estimators of the row and column loading matrices. Wang et al. (2019) is the first to present
consistent estimators for the loading matrices, along the line of Lam et al. (2011) and Lam and Yao (2012) by implementing the auto-cross-covariance matrix, which relies heavily on the persistency of the factor series. When the serial correlation of the factors is close to zero, their estimators may not work well. Alternatively, in the present paper, to find the projection direction, we will start with the column covariance matrix capturing the contemporaneous correlation of the common components. It is informative even in the case when the serial correlation of the factors is vanishing or weak. Our simulation study shows that our estimators work well when the factor series are independent or even moderately persistent. In this paper, the idiosyncratic errors can be correlated weakly between columns, rows or entries while Wang et al. (2019) assumed white noise idiosyncratic error matrices. Similar settings and the use of the column covariance matrix are found in Virta et al. (2017) and a concurrent interesting working paper of Chen et al. (2020). The former paper considered constructing independent components from matrix observations, but their paper assumed that the model is noiseless. The latter paper also presented estimators of the loading matrices for the same factor model, but their methodology is totally different from the projection approach here.

The projection procedure first projects each $p_1 \times p_2$ data matrix onto a lower dimensional row or column factor space, see Section 3 for more details. While the information on estimating the row or column factor space is kept in the projection process (see (3.1) later), the strengths of the idiosyncratic error entries are decreased in great magnitude if the idiosyncratic error matrix is column-wise or row-wise weakly dependent. (3.1) shows that the transformed idiosyncratic error matrix has asymptotically vanishing entries as $p_1, p_2 \to \infty$. Compared with the factor models with regular idiosyncratic error matrix, one easily sees that the projected estimators have faster convergence rate than $T^{-1}$, a rate obtained in Wang et al. (2019) for matrix factor models and conceivable from Lam et al. (2011) and Lam and Yao (2012) for vector factor models. For the toy example when the columns of all matrix observations are independent, pooling all columns under the typical vector factor model considered in Bai and Ng (2002) and Fan et al. (2013), the effective sample size is $T p_2$ and thus the asymptotic theorems in these two papers indicate a conceivable convergence rate $\min\{T p_2, p_2^2\}^{-1}$ (in terms of the averaged squared Frobenious norm) for recovering the row factor space. Our asymptotic theory below shows that the averaged squared Frobenious norm of our projected estimators of the row (or column) loading matrix has convergence rate $\min\{(T p_2)^{-1}, (T p_1)^{-2}, (p_1 p_2)^{-2}\}$ (or $\min\{(T p_1)^{-1}, (T p_2)^{-2}, (p_1 p_2)^{-2}\}$). To the best of our knowledge, this is the first convergence rate faster than the conceivable rates $T^{-1}$ and $\min\{(T p_2)^{-1}, p_1^{-2}\}$ (or $\min\{(T p_1)^{-1}, p_2^{-2}\}$) for estimating the row factor loading matrix. More detailed explanation is given in Section 3. An easily satisfied sufficient condition on the projection direction is also provided to achieved the claimed convergence rates of the present paper. Finally, based on the projected data matrix, we introduce an iterative approach to determine the sizes of the factor matrices, which is proved to be consistent theoretically and outperforms existing estimates empirically.

One essential difference between our projected estimation method and the auto-cross-covariance-based procedure is the error smoothing dimension. The latter smoothes the idiosyncratic errors along the time of an order-3 tensor and at the same time dig the time persistency information of the common factor matrix. Therefore, a persistence strength condition for time series of the factors and a temporal independence condition for the idiosyncratic errors are usually requested. Our projected method diversifies the error matrix in row-wise or column-wise manner. It is simply like reducing specific risk by constructing portfolios in finance. Hence our essential requisites are the full rank condition of the factor spaces and the row-wise or column-wise weak correlation assumption on the idiosyncratic error matrix. A second difference is that the auto-cross-covariance based procedure does eigen-analysis with a large-sized matrix after smoothing in time domain, while our method first reduces the column or row dimensionality of the data matrix and then manipulates with a projected matrix having low dimensionality column-wise or row-wise.

We introduce some notations used throughout the paper. For a matrix $X_t$ observed at time $t$, $x_{t,i,j}$ denotes its $i$-th row and $j$-th column entry, and $x_{t,i \cdot}$ ($x_{t \cdot j}$) denotes its $i$-th row ($j$-th column) and let vec$(X_t)$ be the vector obtained by stacking the columns of $X_t$. For a matrix $A$, $\|A\|$ and $\|A\|_F$ represent the spectral norm and Frobenious norm, respectively. $\lambda_j(A)$ is the $j$-th eigenvalue of $A$ if $A$ is symmetric. The notations $\to_p$, $\to_d$ and $\to_a$ represent convergence in probability, in distribution and almost surely, respectively. The $o_p$ is
2 Matrix factor model and technical assumptions

2.1 Matrix factor model

In this section, we introduce the matrix factor model invented by Wang et al. (2019) and give a brief review of the auto-cross-covariance-based estimators. The matrix factor model can be written as

\[ X_t = RF_tC^\top + E_t, \quad t = 1, \ldots, T, \]  

(2.1)

where \( X_t \) is a \( p_1 \times p_2 \) matrix observed at time \( t \), \( F_t \) is the unobserved \( k_1 \times k_2 \) matrix-valued common factor, \( R_{p_1 \times k_1} \) and \( C_{p_2 \times k_2} \) are the deterministic row and column factor loading matrices reflecting the interactions between the common factors and the elements in \( X_t \), and \( E_t \) is a \( p_1 \times p_2 \) idiosyncratic error matrix independent of \( \{F_t\}_{t \leq T} \). The parameter \( k_1 \) is the number of row factors and \( k_2 \) is the number of column factors, both of which are assumed to be fixed but unknown. The model factorizes each matrix as a low-rank common component plus idiosyncratic component, which can be regarded as an extension of the vector factor model to the matrix regime. It provides a new framework and interpretation for the analysis of 3-dimensional tensor data.

It’s obvious that the loading matrices \( R \) and \( C \) are not identifiable in model (2.1). In the current paper, only the factor spaces are of interest. Hence, we directly assume that

\[ p_1^{-1}R^\top R = I_{k_1}, \quad \text{and} \quad p_2^{-1}C^\top C = I_{k_2}. \]

If this is not true, there always exist some column-orthogonal bases \( Q_1 \) and \( Q_2 \) such that

\[ R = Q_1W_1, \quad \text{and} \quad C = Q_2W_2, \]

where \( W_1 \) and \( W_2 \) are \( k_1 \times k_1 \) and \( k_2 \times k_2 \) full rank matrices, respectively. Therefore, \( R \) (or \( C \)) lies in the same column space as \( Q_1 \) (or \( Q_2 \)), and

\[ X_t = (\sqrt{p_1}Q_1)\tilde{F}_t(\sqrt{p_2}Q_2)^\top + E_t, \quad \text{with} \quad \tilde{F}_t = \frac{1}{\sqrt{p_1p_2}}W_1F_tW_2^\top, \]

which is a matrix-variate factor model with column-orthogonal row and column loading matrices.

For estimation of \( Q_1 \) and \( Q_2 \), a novel method is the auto-cross-covariance-based approach presented in Wang et al. (2019) which borrowed ideas from Lam et al. (2011) and Lam and Yao (2012). Specifically, given \( k_1, k_2 \) and a predetermined positive lagging parameter \( h_0 \geq 1 \), they defined the auto-cross-covariance matrix as

\[ \hat{M}_w^x = \sum_{h=1}^{h_0} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \hat{\Omega}_{x,ij}(h)\hat{\Omega}_{x,ij}(h)^\top, \quad \text{where} \quad \hat{\Omega}_{x,ij}(h) = \frac{1}{T-h} \sum_{t=1}^{T-h} x_{t,i}x_{t+h,j}^\top. \]

(2.2)
\( \hat{Q}_1 \) is then given by the leading \( k_1 \) eigenvectors of \( \hat{M}^\top \). A parallel step can be applied to \( X_i^\top \) to estimate \( Q_2 \). Wang et al. (2019) proved that

\[
\| \hat{Q}_i - Q_i \|_2 = O_p(T^{-1/2}), \quad i = 1, 2,
\]

under the assumptions that \( \| R \|_2 \approx p_1 \), \( \| C \|_2 \approx p_2 \) and some other technical conditions. The rate \( O_p(T^{-1/2}) \) is the same as that in Theorem 1 of Lam et al. (2011) for vector factor models. For the matrix factor models, the total number of unknown parameters in loading matrices are of order \( O(p_1 + p_2) \) while relatively much more observations (\( TP_2 \) columns) are available than the vector factor model (\( T \) columns). The current paper aims to discover statistically more efficient estimation procedures by fully taking advantage of the matrix factor structure.

### 2.2 Technical assumptions

The matrix factor models are essentially designed for high-order tensor data. The high-order tensor correlations make the theoretical analysis more challenging. Moment constraints are common in the literature to control the correlations, see Bai (2003) and Fan et al. (2016). The following assumptions generalize such moment conditions to the matrix regime.

**Assumption A** **COMMON FACTORS:** there exists a positive constant \( M \) such that \( \mathbb{E}(F_{t,ij}) = 0 \) and \( \mathbb{E}(F_{t,ij}^4) \leq M \) for any \( t, i, j \). \( T^{-1} \sum_{t=1}^{T} F_t F_t^\top \overset{d}{\rightarrow} \Sigma_1, T^{-1} \sum_{t=1}^{T} F_t^\top F_t \overset{d}{\rightarrow} \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are symmetric matrices satisfying \( c_2 \leq \lambda_{k_2}(\Sigma_i) < \cdots < \lambda_1(\Sigma_i) \leq c_1 \) for \( i = 1, 2 \) and positive constants \( c_1 \) and \( c_2 \). The spectral decomposition of \( \Sigma_i \) is \( \Sigma_i = \Gamma_i \Lambda_i \Gamma_i^\top \), \( i = 1, 2 \).

In Assumption A, the common factors are centered with bounded fourth moments, which is standard in the literature. The existence of positive definite \( \Sigma_1 \) and \( \Sigma_2 \) ensures that there are no redundant rows or columns in the common factor matrix, otherwise the smallest eigenvalue will be zero. The eigenvalues are assumed to be distinct so that corresponding eigenvectors are identifiable. Wang et al. (2019) assumed \( \text{vec}(F_t) \) to be serially \( \alpha \)-mixing and had no requirement for stationarity. The factor process is not necessarily stationary in our Assumption A, either. Assumption A also allows for time persistency of \( \text{vec}(F_t) \) such as the AR process designed in the simulation studies. To ensure there are no redundant rows or columns in the common factor matrix, Wang et al. (2019) defined the auto-cross covariance \( \Sigma_f(h) = (T-h)^{-1} \sum_{t=1}^{T-h} \text{Cov}(\text{vec}(F_t), \text{vec}(F_{t+h})) \) and assumed that \( \| \Sigma_f(h) \| = O(1) \approx \sigma_k(\Sigma_f(h)) \times (k \text{-th largest singular value}) \) for some \( h \) and \( k = \max \{k_1, k_2\} \). This assumption is very similar to our Assumption A by taking \( h = 0 \). Under Assumption A, \( \Sigma_1 \) and \( \Sigma_2 \) have the spectral decompositions \( \Sigma_1 = \Gamma_1 \Lambda_1 \Gamma_1^\top \) and \( \Sigma_2 = \Gamma_2 \Lambda_2 \Gamma_2^\top \), respectively.

**Assumption B** **FACTOR LOADINGS:** there exist positive constants \( \bar{r} \) and \( \bar{\epsilon} \) such that \( \| R \|_{\max} \leq \bar{r} \), \( \| C \|_{\max} \leq \bar{\epsilon} \). \( p_1^{-1} R^\top R = I_{k_1} \) and \( p_2^{-1} C^\top C = I_{k_2} \).

Assumption B is standard, cf, Wang et al. (2019) and Chen et al. (2020). For the sizes of loadings, Wang et al. (2019) assumed that \( \| R \|^2 \approx p_1^{\delta_1} \) and \( \| C \|^2 \approx p_2^{\delta_2} \) for some \( \delta_1, \delta_2 \in [0, 1] \). Our Assumption B can be relaxed to \( R^\top R = p_1^{1-\delta_1} I_{k_1} \) and \( C^\top C = p_2^{1-\delta_2} I_{k_2} \), and the convergence rates in later theories can be modified accordingly. But for simplicity of presentation, we assumed strong factor conditions in Assumption B which is mainly used in establishing the central limit theorems.

**Assumption C** **IDIOSYNCRATIC ERRORS:** \( E_t \) are serially independent for \( t = 1, \ldots, T \). \( \{E_t\} \) and \( \{F_t\} \) are
two independent series. \( E_{t,i,j} = 0, \ E_{t,i,j}^2 \leq M \) for any \( t, i, j \).

\[
\max_{t,j} \mathbb{E} \left( \frac{1}{\sqrt{p_1}} R^\top e_{t,j} \right)^4 \leq M, \quad \max_{t,i} \mathbb{E} \left( \frac{1}{\sqrt{p_2}} C^\top e_{t,i} \right)^4 \leq M;
\]

\[
\max_{t,i,j} \sum_{i_1=1}^{p_1} \sum_{j_1=1}^{p_1} |E_{t,i,j} e_{t,i,j}| \leq M; \quad \max_{t,i,j} \sum_{i_1=1}^{p_1} \sum_{j_1=1}^{p_1} |E_{t,i,j} e_{t,i,j}| \leq M;
\]

\[
\max_{t,i} \frac{1}{p_1 p_2} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \sum_{j_1=1}^{p_1} \sum_{j_2=1}^{p_2} |\text{Cov}(e_{t,i,j} e_{t,i,j}, e_{t,i,j} e_{t,i,j})| \leq M;
\]

\[
\max_{t,i} \frac{1}{p_1 p_2} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \sum_{j_1=1}^{p_1} \sum_{j_2=1}^{p_2} |\text{Cov}(e_{t,i,j} e_{t,i,j}, e_{t,i,j} e_{t,i,j})| \leq M;
\]

\[
\max_{t,i,j} \frac{1}{p_1 p_2} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \sum_{j_1=1}^{p_1} \sum_{j_2=1}^{p_2} |\text{Cov}(e_{t,i,j} e_{t,i,j}, e_{t,i,j} e_{t,i,j})| \leq M.
\]

Assumption C exerts moment conditions to control the correlations of idiosyncratic errors across rows and columns. It is an extended version of the conditions for the vector factor model, see for example Bai (2003) and Fan et al. (2016). To ensure the moment constraints in Assumption C, a sufficient condition is \( e_{t,i,j} \perp e_{t,i,j} \), for \( \min\{|i - i_1|, |j - j_1|\} > m \geq 0 \), where \( m \) is a constant. As in Wang et al. (2019) and Chen et al. (2020), we assume that \( E_t \) are serially independent for simplicity of notations in the technical proofs since the observed data are order-3 tensors. We believe the main results below are still correct when the idiosyncratic errors have weak serial correlations by imposing additional cross correlation condition and giving lengthier proofs.

3 Methodology and main results

3.1 The projection estimators

Our projection estimator is well motivated by the finding that \( R \) can be more easily estimated if \( C \) is known in advance. In this case, we can project the data matrices to lower dimensional spaces by setting

\[
Y_t = \frac{1}{p_2} X_t C = \frac{1}{p_2} R F_t C^\top C + \frac{1}{p_2} E_t C := R F_t + \tilde{E}_t.
\]

After transformation, \( Y_t \) is a \( p_1 \times k_2 \) matrix-valued observation while \( F_t \) and \( \tilde{E}_t \) can be regarded as transformed new factors and errors. If \( k_2 = 1 \), it is exactly a vector factor model. One advantage the projection brings is the great decrease of the level of noise entries. For each row of \( \tilde{E}_t \), denoted as \( \tilde{e}_{t,i} \), \( \mathbb{E} \| \tilde{e}_{t,i} \|^2 \leq cp_2^{-1} \) as long as the original errors \( \{e_{t,i,j}\}_{j=1}^{p_2} \) are weakly dependent, a phenomenon also illustrated in Fan and Liao (2019) and Kong (2020) for vector factor models. When \( p_2 \) is large enough, \( Y_t \) can be interpreted as a nearly noise-free factor model with \( O(p_1) \) parameters to be estimated.

Based on \( Y_t \), define

\[
M_1 = \frac{1}{T p_1} \sum_{t=1}^{T} Y_t Y_t^\top,
\]

then the row factor space of \( R \) can be estimated by the leading \( k_1 \) eigenvectors of \( M_1 \). However, the projection matrix \( C \) is usually unavailable in real applications, one has to replace it with an initial estimator \( \hat{C} \). The column factor loading matrix \( C \) can be similarly estimated by projecting \( X_t \) onto the space of \( C \) with transformation \( R \) or its estimated version \( \hat{R} \). Now, we summarize the projection procedure in Algorithm 1.
Algorithm 1 Projected method for estimating the matrix factor spaces

**Input:** Data matrices \{X_t\}_{t\leq T}, numbers of factors \(k_1\) and \(k_2\)

**Output:** The factor loading matrices \(\mathbf{R}\) and \(\mathbf{C}\)

1. obtain the initial estimators \(\hat{\mathbf{R}}\) and \(\hat{\mathbf{C}}\);
2. project the data matrices to lower-dimensions by defining \(\hat{Y}_t = P_2^{-1}X_t\hat{\mathbf{C}}\) and \(\hat{Z}_t = P_1^{-1}X_t\hat{\mathbf{R}}\);
3. based on \(\hat{Y}_t\) and \(\hat{Z}_t\), define \(\hat{M}_1 = (T_{p_1})^{-1}\sum_{t=1}^T \hat{Y}_t\hat{Y}_t^\top\) and \(\hat{M}_2 = (T_{p_2})^{-1}\sum_{t=1}^T \hat{Z}_t\hat{Z}_t^\top\), and estimate the loading spaces by the leading \(k_i\) eigenvectors of \(\hat{M}_i\), denoted as \(\hat{Q}_i, i = 1, 2\);
4. the row and column loading matrices are finally given by \(\mathbf{R} = \sqrt{p_1}\hat{Q}_1\) and \(\mathbf{C} = \sqrt{p_2}\hat{Q}_2\).

The projection method can be implemented recursively by plugging in the newly estimated \(\hat{\mathbf{R}}\) and \(\hat{\mathbf{C}}\) to replace \(\mathbf{R}\) and \(\mathbf{C}\) in **Step 2** and iterating **Step 2**-**Step 4**. Theoretical analysis of the recursive solution is pretty hard due to the complex iterative computational dynamics. The simulation results in Section 4 show that a single iterated projection estimators perform sufficiently well compared with the recursive method. Actually when \(T \approx p_1 \approx p_2\) and the projection matrices \(\hat{\mathbf{C}}\) is chosen by the method in Section 3.2, the convergence rate of the projected estimator \(\hat{\mathbf{R}}\) is of order \(O_{p}(1/\sqrt{T_{p_2}})\) under Frobenius norm, which is the optimal rate even when the loading matrix \(\hat{\mathbf{C}}\) is known.

To ensure a fast convergence rate of the projected estimators, we need the following conditions on the convergence rates of \(\hat{\mathbf{R}}\) and \(\hat{\mathbf{C}}\).

**(Sufficient Condition)** There exist \(k_1 \times k_1\) matrices \(\hat{\mathbf{H}}_1\) satisfying \(\hat{\mathbf{H}}_1\hat{\mathbf{H}}_1^\top \overset{p}{\to} \mathbf{I}_{k_1} \) and for any \(j \leq p_2\),

\[
\begin{align*}
(\text{a). } & \frac{1}{p_1} \| \hat{\mathbf{R}} - \mathbf{R}\hat{\mathbf{H}}_1 \|^2_F = O_{p}(w_1), \\
(\text{b). } & \frac{1}{p_2} \left\| \frac{1}{T_{p_1}} \sum_{s=1}^T E_s^\top (\hat{\mathbf{R}} - \mathbf{R}\hat{\mathbf{H}}_1)F_s \right\|^2_F = O_{p}(w_2),
\end{align*}
\]

where \(w_1, w_2 \to 0\) as \(T, p_1\) and \(p_2\) go to infinity simultaneously. There exist \(k_2 \times k_2\) matrices \(\hat{\mathbf{H}}_2\) satisfying \(\hat{\mathbf{H}}_2\hat{\mathbf{H}}_2^\top \overset{p}{\to} \mathbf{I}_{k_2}\) and for any \(i \leq p_1\),

\[
\begin{align*}
(\text{a). } & \frac{1}{p_2} \| \hat{\mathbf{C}} - \mathbf{C}\hat{\mathbf{H}}_2 \|^2_F = O_{p}(m_1), \\
(\text{b). } & \frac{1}{p_1} \left\| \frac{1}{T_{p_2}} \sum_{s=1}^T E_s^\top (\hat{\mathbf{C}} - \mathbf{C}\hat{\mathbf{H}}_2)F_s \right\|^2_F = O_{p}(m_2),
\end{align*}
\]

where \(m_1, m_2 \to 0\) as \(T, p_1\) and \(p_2\) go to infinity simultaneously.

**Theorem 3.1** (Consistency of the projected estimators). Under Assumptions A, B, C and the sufficient conditions (3.2) and (3.3), there exist matrices \(\hat{\mathbf{H}}_1\) and \(\hat{\mathbf{H}}_2\) satisfying \(\hat{\mathbf{H}}_1\hat{\mathbf{H}}_1/p_1 \overset{p}{\to} \mathbf{I}_{k_1}\) and \(\hat{\mathbf{H}}_2\hat{\mathbf{H}}_2/p_2 \overset{p}{\to} \mathbf{I}_{k_2}\), such that

\[
\begin{align*}
\frac{1}{p_1} \| \hat{\mathbf{R}} - \mathbf{R}\hat{\mathbf{H}}_1 \|^2_F & \lesssim \frac{1}{T_{p_2}} + \frac{1}{p_1^2p_2} + m_1^2 \times \left( \frac{1}{p_1^2} + \frac{1}{T_{p_1}} \right) + m_2, \\
\frac{1}{p_2} \| \hat{\mathbf{C}} - \mathbf{C}\hat{\mathbf{H}}_2 \|^2_F & \lesssim \frac{1}{T_{p_1}} + \frac{1}{p_1^2p_2} + w_1^2 \times \left( \frac{1}{p_2^2} + \frac{1}{T_{p_2}} \right) + w_2,
\end{align*}
\]

as \(T, p_1\) and \(p_2\) go to infinity simultaneously.

It holds that \(w_2 \lesssim T^{-1} w_1\) and \(m_2 \lesssim T^{-1} m_1\) by the Cauchy-Schwartz inequality. Therefore, the projected estimators are always consistent with fast convergence rates under the sufficient conditions. We introduce a
method to construct the initial estimators in Section 3.2 such that
\[ w_1 = \frac{1}{p_1^1} + \frac{1}{T p_2}, \quad w_2 = \frac{1}{T p_1^1} + \frac{1}{T^2 p_2^1}, \quad m_1 = \frac{1}{p_2}, \quad m_2 = \frac{1}{T p_2^2} + \frac{1}{T^2 p_2^1}. \tag{3.4} \]

Hence, a corollary follows directly.

**Corollary 3.1.** Under Assumptions A, B and C, and assume conditions (3.2)-(3.4),
\[ \frac{1}{p_1} \| \tilde{R} - RH_1 \|_F^2 = O_p \left( \frac{1}{T p_1^2} + \frac{1}{p_1^2 p_2^1} + \frac{1}{T^2 p_2^1} \right), \quad \frac{1}{p_2} \| \tilde{C} - CH_2 \|_F^2 = O_p \left( \frac{1}{T p_1} + \frac{1}{p_1^1 p_2^2} + \frac{1}{T^2 p_2^1} \right). \]

Theorem 3.1 and Corollary 3.1 demonstrate that our projected estimators of the row and column factor spaces achieve higher convergence rates than \( T^{-1} \) for the auto-cross-covariance-based estimators by a factor of \( p_2^{-1} \) (or \( p_1^{-1} \)) when \( p_2^1 p_2^2 \) is comparable to or larger than the number of observations. The reason is partly due to the addition of the information contained in the contemporaneous correlation structure of each matrix observation, and the smoothing of noises column-wise within the data matrix in the projection manipulation. The rate is also faster than \( \min \{ (T p_2)^{-1}, p_1^{-2} \} \) (or \( \min \{ (T p_1)^{-1}, p_2^{-2} \} \) which is the rate of the toy example given in the introduction. In the toy example, we pooled all columns of the data matrices to obtain \( T p_2 \) vectors which could be modeled by a vector factor model. The theorems in Bai (2003) and Fan et al. (2013) indicate the convergence rate \( \min \{ (T p_2)^{-1}, p_1^{-2} \} \) (or \( \min \{ (T p_1)^{-1}, p_2^{-2} \} \) as above. But our projection estimators converge even faster than PCA estimators adapted to the vector factor model for the toy example. The reason is that the pooled factor model for matrix series contain \( O(p_1 + T p_2) \) parameters ( \( O(p_1) \) loadings and \( O(T p_2) \) unknown factors) to be estimated while in (3.1) there are only \( O(p_1 + T) \) parameters that are of interest, yet \( E_t \) is asymptotically vanishing.

To further study the entry-wise asymptotic distributions of the estimated loadings, we need the following assumptions.

**Assumption D** For \( i \leq p_1 \),
\[ \frac{1}{\sqrt{T p_1}} \sum_{t=1}^T F_t C^\top e_{t,i} \overset{d}{\to} N(0, V_{1i}), \quad \text{where} \quad V_{1i} = \frac{1}{T p_2} \sum_{t=1}^T EF_t C^\top \text{cov}(e_{t,i}) CF_t^\top. \]

For \( j \leq p_2 \),
\[ \frac{1}{\sqrt{T p_1}} \sum_{t=1}^T F_t^\top R^\top e_{t,j} \overset{d}{\to} N(0, V_{2j}), \quad \text{where} \quad V_{2j} = \frac{1}{T p_1} \sum_{t=1}^T EF_t^\top R^\top \text{cov}(e_{t,j}) RF_t. \]

\( V_{1i} \) and \( V_{2j} \) are positive definite matrices whose eigenvalues are bounded away from 0 and infinity.

Assumption D is satisfied when the row-wise or column-wise correlations of idiosyncratic errors are weak. Similar assumptions are imposed for establishing limiting distributions of the estimated loadings in the literature of vector factor models.

**Theorem 3.2** (Asymptotic normality of the projection estimators). Under Assumptions A, B, C, and D, and assume conditions (3.2)-(3.4),

1. for \( i \leq p_1 \),
\[ \begin{align*}
\sqrt{T p_2} (\tilde{R}_i - H_1^\top R_i) & \overset{d}{\to} N(0, A^{-1}_i \Gamma_1^\top V_{1i} \Gamma_1 A^{-1}_i), \quad \text{if} \quad T p_2 = o_p(\min\{T^2 p_1^2, p_2^2 p_1^2\}), \\
\tilde{R}_i - H_1^\top R_i & = O_p \left( \frac{1}{T p_1^1} + \frac{1}{p_2^2 p_1^1} \right), \quad \text{if} \quad T p_2 \gtrsim \min\{T^2 p_1^2, p_2^2 p_1^2\},
\end{align*} \]

where \( \tilde{R}_i \) and \( R_i \) are the \( i \)-th row vector of \( \tilde{R} \) and \( R \), respectively;
Therefore, we propose to use the leading $k$ are asymptotically negligible under certain conditions. As a consequence, only the leading $k$ of \( T \) satisfy conditions (3.2) and (3.3). The term \( \tilde{R} \) is exactly the sample covariance matrix scaled by \( p_1^{-1} \). When the error matrices are independent of the factor process, it’s easy to see that

\[
\tilde{M}_1 = \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{p_2} x_{i,j} x_{i,j}^\top = \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{p_2} X_i X_i^\top.
\]

Virta et al. (2017) also used this matrix to construct the independent components for noise-free observations of matrix type. We believe other choices of initial estimates of \( R \) are possible as long as conditions (3.2) and (3.3) are fulfilled, but for simplicity we only demonstrate theoretically that an eigen-analysis of the column covariance matrix works. Our simulation studies show that our initial estimate performs well empirically. When \( p_2 = 1 \), \( \tilde{M}_1 \) is exactly the sample covariance matrix scaled by \( p_1^{-1} \). When the error matrices are independent of the factor process, it’s easy to see that

\[
\mathbb{E}(\tilde{M}_1) = \frac{1}{p_1} R \left( \frac{1}{T} \sum_{i=1}^{T} \mathbb{E}(F_i F_i^\top) \right) R^\top + \frac{1}{T} \sum_{i=1}^{T} \mathbb{E}(E_i E_i^\top).
\]

The term \( T^{-1} \sum_{i=1}^{T} \mathbb{E}(F_i F_i^\top) \) typically converges to a symmetric positive definite matrix while the error terms are asymptotically negligible under certain conditions. As a consequence, only the leading \( k_1 \) eigenvalues of \( \tilde{M}_1 \) are “spiky”. Motivated by Davis-Kahan’s \( \sin(\Theta) \) theorem, see for example Davis and Kahan (1970) and Yu et al. (2015), the leading \( k_1 \) eigenvectors of \( \tilde{M}_1 \) lie in the same column space of \( R \) asymptotically. Therefore, we propose to use the leading \( k_1 \) eigenvectors of \( \tilde{M}_1 \) as an estimator of \( Q_1 \), denoted as \( \tilde{Q}_1 \). The row loading matrix is then estimated by \( \tilde{R} = \sqrt{p_1 Q_1} \). The column loading matrix \( C \) can be estimated by parallel steps applied to \( \{X_i^\top\}_{i \leq T} \). The next theorem follows.

**Theorem 3.3.** Under Assumptions A, B and C, the column/row covariance based estimators \( \tilde{R} \) and \( \tilde{C} \) satisfy conditions (3.2) and (3.3) with

\[
w_1 = \frac{1}{p_1^2} + \frac{1}{T p_2}, \quad w_2 = \frac{1}{T p_1^2} + \frac{1}{T^2 p_2}, \quad m_1 = \frac{1}{p_2^2} + \frac{1}{T p_1}, \quad m_2 = \frac{1}{T p_2} + \frac{1}{T^2 p_1}.
\]

The convergence rate \( w_1 \) in Theorem 3.3 match with the typical rate \( O_p(T^{-1} + p_1^{-2}) \) (see Theorem 2 in Bai (2003)) of the vector factor model when \( p_2 = 1 \). When both \( p_1 \) and \( p_2 \) go to infinity simultaneously, the theorem implies that the loading matrices of the matrix-variate factor models can be estimated more accurately, which mainly benefits from the double low rank structure. The initial estimators are also asymptotically normally distributed shown in the next theorem.
Theorem 3.4 (Asymptotic normality of the initial estimators). Under Assumptions A, B, C and D, as $T, p_1, p_2 \to \infty$,

1. for $i \leq p_1$,
   \[
   \sqrt{T} p_2 (\hat{R}_i - \hat{H}_i^\top R_i) \overset{d}{\to} \mathcal{N}(0, \Lambda_1^{-1} \Gamma_1^\top V_1 \Gamma_1 \Lambda_1^{-1}), \quad \text{if} \quad T p_2 = o_p(p_1^2),
   \]
   \[
   \hat{R}_i - \hat{H}_i^\top R_i = O_p(p_1^{-1}), \quad \text{if} \quad T p_2 \gtrsim p_1^2;
   \]
2. for $j \leq p_2$,
   \[
   \sqrt{T} p_1 (\hat{C}_j - \hat{H}_j^\top C_j) \overset{d}{\to} \mathcal{N}(0, \Lambda_2^{-1} \Gamma_2^\top V_2 \Gamma_2 \Lambda_2^{-1}), \quad \text{if} \quad T p_1 = o(p_2^2),
   \]
   \[
   \hat{C}_j - \hat{H}_j^\top C_j = O_p(p_2^{-1}), \quad \text{if} \quad T p_1 \gtrsim p_2^2,
   \]

where $\hat{R}_i$ and $\hat{C}_j$ are the $i$-th row and $j$-th column vectors of $\hat{R}$ and $\hat{C}$, respectively.

Theorem 3.3 and Theorem 3.4 give the consistency and the limiting distributions of the initial estimators. However, if $p_1$ and/or $p_2$ are small or fixed, the initial estimators can be unreliable or inconsistent. This also happens if the correlations of the idiosyncratic errors are moderately strong. Although the initial estimators may not work well, Theorem 3.1 and Theorem 3.2 show that a projection transformation improves the convergence rates by a factor of $T^{-2}$, $p_1^{-2}$ or $p_2^{-2}$.

### 3.3 Determining the factor numbers $k_1, k_2$

The dimensions $k_1$ and $k_2$ of the common factors need to be determined before the procedures can be applied. In this paper, we specify the numbers of row and column factors by borrowing the eigenvalue-ratio technique ever discussed in Lam and Yao (2012) and Ahn and Horenstein (2013). In detail, choose $\hat{R}$ and $\hat{C}$ as the initial projection matrices, then $k_1$ is estimated by

\[
\hat{k}_1 = \arg \max_{j \leq k_{\text{max}}} \frac{\lambda_j(\hat{M}_1)}{\lambda_{j+1}(\hat{M}_1)}, \tag{3.5}
\]

where $k_{\text{max}}$ is a predetermined fixed upper bound for $k_1$. We use $\hat{M}_1$ rather than $\tilde{M}_1$ because $\tilde{M}_1$ is more accurate than $\hat{M}_1$ and then the eigenvalue gaps of $\hat{M}_1$ are not that uncertain than those of $\tilde{M}_1$.

When the signal of the common factors is sufficiently strong, the leading $k_1$ eigenvalues of $\hat{M}_1$ are well separated from the others. Hence, the eigenvalue-ratios in equation (3.5) are asymptotically maximized exactly at $j = k_1$. In real applications, we can add an asymptotically negligible term, for example $c (\min\{T, p_1\})^{-1/2}$ for some small constant $c$, to the denominator in equation (3.5) to avoid vanishing denominator. However, to calculate $\hat{M}_1, \hat{C}$ must be predetermined, which means $k_2$ must be given first. Empirically $k_1$ and $k_2$ are usually both unknown, so we suggest using the following iterative Algorithm 2 to specify the numbers of factors.

**Algorithm 2** Iterative algorithm to specify the numbers of factors

**Input:** Data matrices $\{X_t\}_{t \leq T}$, maximum number $k_{\text{max}}$, maximum iterative step $m$

**Output:** The numbers of row and column factors $\hat{k}_1$ and $\hat{k}_2$

1. initialization: $\hat{k}_1^{(0)} = k_{\text{max}}, \hat{k}_2^{(0)} = k_{\text{max}}$
2. for $t = 1, \ldots, m$, given $\hat{k}_1^{(t-1)}$, estimate $\hat{C}^{(t)}$ by the initial estimator, and calculate $\tilde{M}_1^{(t)}$ using $\hat{C}^{(t)}$, then $\hat{k}_1^{(t)}$ is given by equation (3.5);
3. given $\hat{k}_1^{(t)}$, estimate $\hat{R}^{(t)}$ by the initial estimator, and calculate $\tilde{M}_2^{(t)}$ using $\hat{R}^{(t)}$, then $\hat{k}_2^{(t)}$ is given by a parallel “ER” approach by replacing $\hat{M}_1$ with $\hat{M}_2$ in equation (3.5);
4. repeat Step 2 and 3 until $\hat{k}_1^{(t)} = \hat{k}_1^{(t-1)}$ and $\hat{k}_2^{(t)} = \hat{k}_2^{(t-1)}$, or reach the maximum iterative step.
The consistency of the iterative algorithm is guaranteed by the following theorem.

**Theorem 3.5** (Specifying the numbers of row and column factors). Under Assumptions A, B and C, when \( \min\{k_1, k_2\} > 0, \min\{T, p_1, p_2\} \to \infty \) and \( k_{\max} \) is a predetermined constant no smaller than \( \max\{k_1, k_2\} \), if \( \hat{k}_{2}^{(t-1)} \in [k_2, k_{\max}] \) for some \( t \) in the iterative algorithm 2,

\[
\Pr(\hat{k}_1^{(t)} = k_1) \to 1;
\]

and if \( \hat{k}_1^{(t)} \in [k_1, k_{\max}] \) for some \( t \) in the iterative algorithm 2,

\[
\Pr(\hat{k}_2^{(t)} = k_2) \to 1.
\]

Theorem 3.5 indicates that as long as we start with some \( k_1^{(0)} \) and \( k_2^{(0)} \) larger than the true \( k_1 \) and \( k_2 \), the iterative algorithm can consistently estimate the numbers of factors. The algorithm is computationally very fast because it has a large probability to stop within finite steps. The advantages of this method will be verified by our numerical studies.

4 Simulation studies

4.1 Simulation settings

In this section, we check the numerical performances of the proposed projection procedure. The observed data matrices are simulated according to model (2.1) where the parameters are set similarly to Wang et al. (2019). In detail, we set \( k_1 = 3 \) and \( k_2 = 2 \) for all the simulated cases. The entries of \( \mathbf{R} \) and \( \mathbf{C} \) are generated independently from the uniform distribution \( U(-1,1) \). The common factors \( \mathbf{F}_t \) are simulated by \( k_1k_2 \) independent AR(1) processes such that for any \( i \leq k_1, j \leq k_2 \) and \( t \leq T \),

\[
F_{t,ij} = a_{ij}F_{t-1,ij} + b_{ij}\eta_{ij}, \quad \text{with } a_{ij} \text{ and } b_{ij} \in [0,1], \quad \eta_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0,1).
\]

The error matrices \( \mathbf{E}_t \) are simulated from matrix-variate normal distribution with mean \( \mathbf{0} \) and Kronecker product covariance structure \( \text{Cov}(\text{vec}(\mathbf{E}_t)) = \mathbf{\Gamma}_2 \otimes \mathbf{\Gamma}_1 \) where \( \mathbf{\Gamma}_1 \) and \( \mathbf{\Gamma}_2 \) are \( p_1 \times p_1 \) and \( p_2 \times p_2 \) matrices whose diagonal entries are 1 and off-diagonal entries are \( \rho \). We consider the following 3 cases to comprehensively compare our initial estimator (IE), our projected estimators using IE as initial estimates (PE-IE), the auto-cross-covariance-based estimators (ACCE) presented in Wang et al. (2019), and our projected estimators using ACCE as initial estimates (PE-ACCE).

**Case 1** Set the AR coefficients as \( a = (a_{ij}) = [-0.5, 0.6; 0.8, -0.4; 0.7, 0.3], b_{ij} = 1, \rho = 0.2, \) and \( T, p_1, p_2 \) are chosen from \( \{20, 50, 100\} \).

**Case 2** Set \( \rho = 0.2, T = 100, (p_1, p_2) = (50, 50), a_{ij} = a \in \{0.1, \ldots, 0.9\} \) and \( b_{ij} = \sqrt{1 - a_{ij}^2} \) for all \( i, j \).

**Case 3** Set \( T = 100, (p_1, p_2) = (50, 50), a_{ij} = 0.6, b_{ij} = 0.8 \) for all \( i, j, \) and \( \rho \in \{0, 0.05, \ldots, 0.20\} \).

Case 1 is similar to the setting in Wang et al. (2019), where the factors are serially dependent AR processes and the errors have moderate cross-sectional correlations. The difference lies in that Wang et al. (2019) set the sample size \( T \approx p_1p_2 \) while in Case 1 \( T \) is set relatively smaller. In Case 2, we investigate the performances of these approaches when the serial correlations of the factors gradually grow stronger. The marginal variances of the factors are fixed to control the signal-to-noise ratio. In Case 3, the factors are serially independent while the cross-sectional correlations of the errors are set weak at first and then gradually increase.
4.2 Estimating the loading spaces

To evaluate the performances of these methods, we calculate the column orthogonal matrices $Q_1$ and $Q_2$ by the leading $k_1$ and $k_2$ eigenvectors of $RR^\top$ and $CC^\top$. Therefore, $R$ and $C$ lie in the column spaces of $Q_1$ and $Q_2$ respectively. The distances between the estimated loading spaces and the true loading spaces are defined by

$$D(Q_k^i, Q_1) = \left(1 - \frac{1}{k_i} \text{tr}(Q_k^i Q_k^\top Q_1 Q_1^\top)\right)^{1/2},$$

where $Q_k^i$ are the corresponding estimators for $Q_i$, $i = 1, 2$. The distances $D(Q_k^1, Q_1)$ and $D(Q_k^2, Q_2)$ are always between 0 and 1. When the corresponding matrices lie in the same space, it’s equal to 0. If the two spaces are orthogonal, it’s equal to 1. Hence, we use the two distances to assess the accuracy of these methods. The empirical results are reported in Tables 1-3 based on 200 replications.

Table 1: Means and standard deviations of $D(Q_k^1, Q_1)$ and $D(Q_k^2, Q_2)$ for Case 1 (effects of $T, p_1, p_2$), over 200 replications. “ACCE” is for the approach in Wang et al. (2019), “IE” is for the proposed first-stage method while “PE-IE” is for the projection method using “IE” as initial estimates, “PE-ACCE” is for the projection method using “ACCE” as initial estimates. All the numbers have been multiplied by 10 for better presentation.

| $T$ | $p_1$ | $p_2$ | $D(Q_k^1, Q_1)$ | $D(Q_k^2, Q_2)$ |
|-----|-------|-------|-----------------|-----------------|
|     |       |       | ACCE PE-ACCE IE PE-IE | ACCE PE-ACCE IE PE-IE |
| 20  | 20    | 20    | 3.59(0.63) 1.90(0.93) 4.13(1.37) **1.22(0.68)** | 1.66(1.04) 1.59(0.88) 2.24(1.37) **1.01(0.34)** |
| 50  | 20    | 20    | 1.95(1.29) 1.13(0.44) 3.47(1.50) **0.72(0.34)** | 1.02(0.62) 0.96(0.36) 1.74(1.05) **0.66(0.27)** |
| 100 | 20    | 50    | 0.91(0.47) 0.72(0.22) 3.02(1.39) **0.50(0.19)** | 0.72(0.37) 0.65(0.20) 1.57(1.04) **0.49(0.24)** |
| 100 | 20    | 100   | 0.80(0.62) 0.46(0.13) 2.99(1.34) **0.29(0.08)** | 0.62(0.21) 0.64(0.16) 0.99(0.50) **0.41(0.11)** |
| 100 | 50    | 100   | 0.73(0.61) 0.32(0.08) 2.98(1.32) **0.21(0.06)** | 0.59(0.15) 0.63(0.14) 0.82(0.47) **0.40(0.09)** |
| 100 | 100   | 100   | 0.45(0.20) 0.31(0.06) 2.30(1.17) **0.20(0.05)** | 0.40(0.13) 0.39(0.09) 0.76(0.42) **0.27(0.07)** |

Table 2: Means and standard deviations of $D(Q_k^1, Q_1)$ and $D(Q_k^2, Q_2)$ for Case 2 (effects of the serial correlations on common factors). “ACCE” is for the approach in Wang et al. (2019), “IE” is for the proposed first-stage method while “PE-IE” is for the projection method using “IE” as initial estimates, “PE-ACCE” is for the projection method using “ACCE” as initial estimates. All the numbers have been multiplied by 10 for better presentation.

| $a$ | $T$ | $p_1$ | $p_2$ | $D(Q_k^1, Q_1)$ | $D(Q_k^2, Q_2)$ |
|-----|-----|-------|-------|-----------------|-----------------|
|     |     |       |       | ACCE PE-ACCE IE PE-IE | ACCE PE-ACCE IE PE-IE |
| 0.1 | 4.90(0.91) | 1.03(0.96) | 4.56(1.03) | **0.44(0.14)** | 2.42(1.54) | 1.67(1.19) | 2.05(1.17) | **0.46(0.13)** |
| 0.2 | 4.35(1.35) | 1.46(0.65) | 4.58(1.04) | **0.46(0.20)** | 1.76(1.09) | 1.15(0.55) | 2.10(1.11) | **0.47(0.16)** |
| 0.3 | 3.32(1.58) | 1.64(0.30) | 4.42(1.04) | **0.43(0.14)** | 1.09(0.68) | 0.82(0.25) | 1.97(1.08) | **0.44(0.12)** |
| 0.4 | 2.86(1.61) | 0.85(0.23) | 4.70(0.97) | **0.45(0.19)** | 0.85(0.58) | 0.68(0.18) | 2.11(1.18) | **0.47(0.16)** |
| 0.5 | 1.82(1.33) | 0.69(0.15) | 4.67(0.99) | **0.46(0.18)** | 0.65(0.27) | 0.55(0.14) | 2.12(1.16) | **0.48(0.16)** |
| 0.6 | 1.16(0.83) | 0.57(0.12) | 4.83(0.95) | **0.47(0.18)** | 0.52(0.20) | 0.45(0.10) | 2.25(1.31) | **0.48(0.17)** |
| 0.7 | 0.96(0.92) | **0.49(0.11)** | 4.92(0.93) | 0.51(0.45) | 0.43(0.14) | **0.38(0.08)** | 2.34(1.53) | 0.47(0.16) |
| 0.8 | 0.86(0.81) | **0.44(0.09)** | 5.15(0.73) | 0.66(0.64) | 0.39(0.14) | **0.36(0.08)** | 3.01(1.88) | 0.49(0.20) |
| 0.9 | 0.76(0.64) | **0.42(0.09)** | 5.40(0.51) | 0.99(0.99) | 0.36(0.14) | **0.33(0.09)** | 4.13(2.00) | 0.53(0.23) |

Table 1 shows that all four methods benefit from large $T, p_1$ and $p_2$, while the proposed PE-IE method is the best. IE suffers from moderate cross-sectional correlations of the idiosyncratic errors in this case. PE-ACCE improves the estimation of $Q_1$ but not that significant on $Q_2$ compared with ACCE. Interestingly, although IE works on better than ACCE, its projected version (PE-IE) outperforms PE-ACCE across the board. The reason is that ACCE is specifically designed for mining the dynamic correlation pattern and thus
Table 3: Means and standard deviations of $D(Q_1^1, Q_1)$ and $D(Q_2^1, Q_2)$ for Case 3 (effects of the cross-sectional correlations on idiosyncratic factors). “ACCE” is for the approach in Wang et al. (2019), “IE” is for the proposed first-stage method while “PE-IE” is for the projection method using “IE” as initial estimates, “PE-ACCE” is for the projection method using “ACCE” as initial estimates. All the numbers have been multiplied by 10 for better presentation.

| $\rho$ | $D(Q_1^1, Q_1)$ | $D(Q_2^1, Q_2)$ |
|-------|-----------------|-----------------|
|       | ACCE            | IE              | PE-ACCE | ACCE            | PE-ACCE | IE       | PE-IE |
| 0.20  | 1.27(1.04)      | 4.78(0.97)      | 0.48(0.22) | 0.53(0.20)      | 0.46(0.11) | 2.22(1.22) | 0.48(0.17) |
| 0.15  | 0.98(0.87)      | 3.52(1.35)      | 0.36(0.08) | 0.46(0.18)      | 0.44(0.09) | 1.24(0.69) | 0.36(0.09) |
| 0.10  | 0.61(0.32)      | 1.44(0.85)      | 0.32(0.04) | 0.40(0.07)      | 0.42(0.07) | 0.65(0.28) | 0.27(0.05) |
| 0.05  | 0.50(0.08)      | 0.57(0.21)      | 0.31(0.03) | 0.38(0.05)      | 0.42(0.06) | 0.38(0.11) | 0.25(0.03) |
| 0.00  | 0.48(0.05)      | 0.55(0.08)      | 0.33(0.03) | 0.38(0.04)      | 0.43(0.06) | 0.26(0.03) | 0.25(0.03) |

misses the contemporaneous correlation structure. This makes a following spatial projection onto ACCE less efficient than onto IE which is based on the column covariance matrix. Actually, ACCE itself can be regarded as an estimator already projected onto the time domain of an order-3 tensor.

In Table 2, as the serial correlations of the common factors grow stronger, the performances of ACCE and PE-ACCE become better while IE and PE-IE gradually lose superiority, just as what was expected. But whatever the initial estimates are the projection always improves the accuracy. The projection technique always improves the performance of IE by a large margin. For ACCE, the improvements by projection are impressive when $\rho$ is small but not so significant when $\rho$ is large. In Table 3, all four methods gradually perform better as the cross-sectional correlations of the errors grow weaker. It’s also seen that IE is more sensitive to the correlations of noise compared with other approaches. Once again, the projection improves IE by a large margin. It also works for ACCE in most cases but the improvement is not guaranteed when the errors of ACCE are sufficiently low. A possible reason is that after projection the new idiosyncratic errors are serially dependent. Hence, the time-smoothing method does not always benefit from the projection.

4.3 Evaluation of the recursive procedure

The projection method can be iteratively implemented by setting the newly estimated loadings $\tilde{R}$ and $\tilde{C}$ as initial projection matrices. Algorithm 3 shows the iterative procedure starting with the front loading matrix $R$. It’s easy to construct a parallel algorithm which starts with $C$. 

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Algorithm 3 Iterative Algorithm starting with R

Input: Data matrices \( \{X_t\}_{t=1}^T \), numbers of factors \( k_1 \) and \( k_2 \), maximum iterative steps \( m \)

Output: Estimate of the factor loading spaces at each step, i.e., \( \{\hat{Q}_1^{(k)}\}_{k=1}^m \), \( \{\hat{Q}_2^{(k)}\}_{k=1}^m \)

1: Initialization: \( k = 1 \); Calculate \( \hat{M}_1^{(1)} = (T_1 p_1 p_2)^{-1} \sum_{t=1}^T X_t X_t^\top \), and \( \hat{Q}_1^{(1)} \) is estimated by the leading \( k_1 \) eigenvectors of \( \hat{M}_1^{(1)} \) (exactly \( \hat{R}_1 / \sqrt{p_1} \))

2: \( \hat{Z}_t^{(1)} = p_1^{-1/2} X_t^\top \hat{Q}_1^{(1)}_1 \), \( \hat{M}_2^{(1)} = (T_2 p_2)^{-1} \sum_{t=1}^T Z_t^{(1)} Z_t^{(1)\top} \), \( \hat{Q}_2^{(1)} \) is estimated by the leading \( k_2 \) eigenvectors of \( \hat{M}_2^{(1)} \) (exactly \( \hat{C}_1 / \sqrt{p_2} \))

3: for \( k = 2, \ldots, m \),
   - \( \hat{Y}_t^{(k)} = p_2^{-1/2} X_t^\top \hat{Q}_2^{(k-1)}_2 \), \( \hat{M}_1^{(k)} = (T_1)^{-1} \sum_{t=1}^T Y_t^{(k)} Y_t^{(k)\top} \), \( \hat{Q}_1^{(k)} \) is estimated by the leading \( k_1 \) eigenvectors of \( \hat{M}_1^{(k)} \)
   - \( \hat{Z}_t^{(k)} = p_1^{-1/2} X_t^\top \hat{Q}_1^{(k)}_1 \), \( \hat{M}_2^{(k)} = (T_2)^{-1} \sum_{t=1}^T Z_t^{(k)} Z_t^{(k)\top} \), \( \hat{Q}_2^{(k)} \) is estimated by the leading \( k_2 \) eigenvectors of \( \hat{M}_2^{(k)} \)

4: The front and back loading matrices are given by \( \hat{R}^{(m)} = \sqrt{p_1} \hat{Q}_1^{(m)}_1 \), \( \hat{C}^{(m)} = \sqrt{p_2} \hat{Q}_2^{(m)}_2 \)

In the simulation, we use the setting of Case 1, but set \( T = p_1 = p_2 = 100 \), to study how the estimation error changes with more iterative steps. At each step, the estimation error of the corresponding loading space is recorded. We report the mean error (based on 200 replications) at each step in Figure 3. By Figure 3a, the red real line shows a significant drop at the second step, corresponding to the reduced estimation error compared with \( \hat{R} \). However, the drop of the blue dashed line is not promising, which means we can’t benefit a lot from the iterative procedure compared with \( \hat{C} \). Similar results are found in Figure 3b.
4.4 Selection of the factor numbers

The selection of the row and column factor numbers \((k_1\text{ and } k_2)\) are studied in this part. We use the following simulation settings to investigate the empirical performances of our projection based iterative “ER” approach and the auto-cross-covariance based “ER” approach proposed in Wang et al. (2019). In detail, we set \(p_1 = p_2 = 20\) while \(T\) grows gradually, and choose \(a_{ij}\) from \{0.9, 0.5, 0.1\} and \(\rho\) from \{0.10, 0.05, 0\}. Let \(b_{ij} = \sqrt{1 - a_{ij}^2}\). The frequency of exact estimation \((\hat{k}_1 = 3, \hat{k}_2 = 2)\) over 200 replications are reported in Table 4.

Table 4: Proportion of exact estimation for \((k_1, k_2)\) over 200 replications. “ACCER” is for the auto-cross-covariance based “ER” approach in Wang et al. (2019) and “IterER” is for the iterative “ER” in this paper. \(p_1 = p_2 = 20, k_{max} = 8, \text{ and maximum iterative step is 10.}\)

| \(\rho\) | \(T\) | \(a_{ij} = 0.9\) | \(a_{ij} = 0.5\) | \(a_{ij} = 0.1\) |
|---|---|---|---|---|
| | | ACCER | IterER | ACCER | IterER | ACCER | IterER |
| 0.10 | 100 | 0.420 | 0.950 | 0.280 | 1.000 | 0.105 | 1.000 |
| 0.10 | 150 | 0.655 | 0.970 | 0.425 | 0.995 | 0.140 | 0.995 |
| 0.10 | 200 | 0.765 | 0.990 | 0.540 | 0.995 | 0.120 | 0.995 |
| 0.05 | 100 | 0.590 | 0.990 | 0.515 | 1.000 | 0.485 | 1.000 |
| 0.05 | 150 | 0.805 | 1.000 | 0.690 | 1.000 | 0.525 | 1.000 |
| 0.05 | 200 | 0.855 | 1.000 | 0.740 | 1.000 | 0.530 | 1.000 |
| 0.00 | 100 | 0.680 | 1.000 | 0.700 | 1.000 | 0.735 | 1.000 |
| 0.00 | 150 | 0.870 | 1.000 | 0.805 | 1.000 | 0.740 | 1.000 |
| 0.00 | 200 | 0.915 | 1.000 | 0.845 | 1.000 | 0.815 | 1.000 |

The iterative “ER” estimate performs impressively reliable and stable, even if the factor process is strongly serially correlated. When \(T\) is small or the serial correlations are weak, Wang et al. (2019)’s method loses power quickly, while our approach is still stable.

5 Real data analysis

Two real data examples are given in this section to show the empirical usefulness of the proposed projected method. The first one is related to financial engineering, while the other is about image recognition which is widely used in FinTech.

5.1 Fama-French 10 by 10 portfolios

For ease of comparison, in our first real example we use the same data set as in Wang et al. (2019). It contains monthly returns of 100 portfolios, structured into 10 by 10 matrix according to ten levels of market capital (size) and ten levels of book to equity ratio (BE). Considering the missing rates, the monthly returns from January 1964 to August 2019 are collected, covering 668 months. Detailed information can be found in the website [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).

Following Wang et al. (2019), the return series are first adjusted by subtracting the corresponding monthly market excess returns. In the next step, we impute the missing values by linear interpolation for each of the series. With the standardized monthly returns, our eigenvalue-ratio method suggests that \(k_1 = k_2 = 1\), which is the same as the result in Wang et al. (2019). For better illustration, we also try some other combinations of \(k_1\) and \(k_2 = 2\). The estimated front and back loading matrices after varimax rotation and scaling are reported in Tables 5 and 6.
Table 5: Size loading matrix for Fama-French data set, after varimax rotation and scaling by 30. “ACCE” is for the approach in Wang et al. (2019), while “PE-IE” is for the projected estimator.

| Method | Factor | S1   | S2   | S3   | S4   | S5   | S6   | S7   | S8   | S9   | S10  |
|--------|--------|------|------|------|------|------|------|------|------|------|------|
| ACCE   | 1      | -12  | -14  | -12  | -13  | -10  | -8   | -3   | -1   | 4    | 7    |
|        | 2      | -2   | -1   | -1   | 1    | 5    | 9    | 11   | 18   | 15   | 9    |
| PE-IE  | 1      | -16  | -15  | -12  | -10  | -8   | -5   | -2   | -1   | 4    | 7    |
|        | 2      | -6   | -2   | 3    | 5    | 8    | 10   | 12   | 13   | 15   | 10   |

Table 6: Book-to-Equity (BE) loading matrix for Fama-French data set, after varimax rotation and scaling by 30. “ACCE” is for the approach in Wang et al. (2019), while “PE-IE” is for the projected estimator.

| Method | Factor | BE1  | BE2  | BE3  | BE4  | BE5  | BE6  | BE7  | BE8  | BE9  | BE10 |
|--------|--------|------|------|------|------|------|------|------|------|------|------|
| ACCE   | 1      | 7    | -1   | -4   | -8   | -9   | -8   | -11  | -12  | -14  | -14  |
|        | 2      | 22   | 15   | 11   | 7    | 4    | 3    | 0    | -1   | -2   | 1    |
| PE-IE  | 1      | 7    | 0    | -4   | -8   | -10  | -11  | -13  | -12  | -12  | -10  |
|        | 2      | 21   | 16   | 11   | 8    | 4    | 2    | -1   | -1   | -1   | 0    |

From Tables 5 and 6, we see that the two methods actually lead to very similar estimated loadings. From the perspective of size, the small size portfolios load heavily on the first factor while the large size portfolios load mainly on the second factor. From the perspective of book-to-equity, the small BE portfolios load heavily on the second factor while the large BE portfolios load mainly on the first factor. Clearly, the portfolios tend to perform more similarly if they are constructed by public companies with similar size and book-to-equity ratio. The results of Wang et al. (2019)’s approach here are slightly different from the original paper due to the four years new observations and the preprocessing way of missing values.

To further compare the above two methods, we use a similar rolling-validation procedure as in Wang et al. (2019). For each year from 1996 to 2018, we repeatedly use the \( n \) (bandwidth) observations before the current year to fit the matrix-variate factor model and estimate the two loading matrices. The loadings are then used to estimate the factors and corresponding residuals of the 12 months in the current year. The total predictive sum of squared residuals (SSR) are reported in Figure 3. It’s seen that the projection approach (red real line) always leads to smaller squared residuals, which implies the corresponding factors can explain more variances of the portfolio returns. A potential reason is that the common components have strong contemporary correlations rather than auto-cross correlations, as claimed in Wang et al. (2019). As more factors are added into the model, the residuals tend to become smaller as expected (except \( k_1 = k_2 = 4 \)), but the bandwidth \( n \) seems to have little impact in this example.

### 5.2 MNIST database: handwritten digit numbers

For the second real data example, we use the MNIST database to show that the matrix factor model can be used as a data dimension reduction technique for the classification of handwritten digit numbers. This database is widely applied in machine learning for image classification and other related applications. It has a training set of 60000 examples and a test set of 10000 examples. Each example is a 28 by 28 pixel image of a handwritten digit, from 0 to 9. The digits have been size-normalized and centered in a fixed-size image, see Figure 1b. For detailed information, please see [http://yann.lecun.com/exdb/mnist/](http://yann.lecun.com/exdb/mnist/).

To train a classification model for digit numbers, “naive” methods can first vectorize the pictures and then use the support vector machine (SVM). Since the dimension of the transformed vector is \( 28 \times 28 = 784 \),
directly applying the SVM is more time-consuming. By assuming the matrix factor structure, a more convenient method is to first estimate the common factors with the projected method (using centralized data), and then utilize the low-dimensional factors to train the SVM model. Obviously the classification model only works well when the factors are accurately specified. As for the numbers of factors, our iterative “ER” approach suggests that $k_1 = 5$ and $k_2 = 6$, but we also tried some other combinations of $(k_1, k_2)$ for comparison. The proportion of false classification on the test set is displayed in Figure 4.

Figure 4: False classification rate on test set. “PE-IE+SVM” means we first extract the common factors by “PE-IE” and then train a SVM model, while “ACCE+SVM” means we estimate common factors by Wang et al. (2019)’s method.

Figure 4 indicates that the red surface is lower than the blue one when $k_2$ is no larger than 7 or $(k_1, k_2)$ around (5, 6), which indicates the usefulness of our projected method. For sufficiently large numbers of factors, the two methods tend to perform similarly. This is kind of surprising because there is no clue that these handwritten digit pictures are ordered by certain rules. A possible explanation for the comparable...
performance of Wang et al. (2019)’s method is that the pictures are clean and size-normalized, which makes the classification problem less challenging. Actually even if we estimate the common factors with some randomly generated loading matrices rather than \( \hat{R} \) and \( \hat{C} \), the false classification rate on test set is about 5% when we set \( k_1 = k_2 = 10 \), roughly twice high of the compared two methods in Figure 4. It’s also seen that the numbers of factors are selected reasonably well, because the false classification rate shows no significant decreasing trend when \( k_1 > 5 \) and \( k_2 > 6 \). We conclude that the matrix factor model with projected method is a convenient and reliable feature subtracting technique for digit number classification.

6 Conclusions and discussions

The current paper focuses on the estimation of the matrix factor models. To make the model applicable to serial independent or weakly dependent factor processes, we start with the column sample covariances instead of auto-cross covariances for the estimation of front and back loading matrices. A projected approach is proposed to further improve the estimation accuracy. Statistical convergence rates and asymptotic distributions of the estimated loadings are provided under mild conditions. An iterative approach is introduced to determine the numbers of factors. Thorough numerical studies and real examples indicate that the projected method is accurate and stable. The matrix factor models can be further extended to analyze high-order tensor data, such as video streaming, which are widely applied in recommender systems. We leave it as one of our future works. The projection method will then be modified corresponding to the tensor data of higher order. We are also interested in applying the matrix factor structure to the estimation of covariance matrix and structure break detection.

7 Supplementary Material

The technical proofs of the main results are put into the supplementary material.

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