RELATING SMALL FEYNMAN AND BJOKEN $x$

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This is a working progress report on the attempt by Yuri Dokshitzer, Gavin Salam and myself to relate the small-$x$ behaviour of the anomalous dimensions in the time- and space-like cases. This relation is based on a *reciprocity respecting equation* we propose.

1 Motivations

It is challenging to try to extract physics features from the 10²-page long formula of the three loops space-like anomalous dimension. That these coefficients could reveal general features of QCD can be illustrated by the example of coherence of QCD radiation. Consider the formula for the small-$N$ limit of the time-like anomalous dimension which, in double logarithmic approximation (DLA), is given by ($N$ is the Mellin moment conjugate to $x$, the limit $x \to 0$ corresponds to $N \to 0$)

$$
\gamma_{\text{DLA}}^+(N, \alpha_s) = \frac{1}{4} \left( \sqrt{N^2 + 8\alpha_s} - N \right) = \frac{\bar{\alpha}_s}{N} - \frac{2 \bar{\alpha}_s^2}{N^3} + \cdots \quad \bar{\alpha}_s = \frac{C_A \alpha_s}{\pi} .
$$

(1)

This formula was derived by studying multi-soft gluon distributions and the evolution of jets. Here one discovered QCD coherence, i.e. cancellation in part of phase space due to destructive interference leading to angular ordering. Actually, this important feature of QCD radiation is implicit in the formula of the two loops time-like anomalous dimension. Indeed, if in the jet evolution one would take into account the full kinematically available phase space, instead of the correct two-loop coefficient $-2/N^3$ in (1) one would obtain $-1/N^3$ and this is the signal of the need of cancellations in part of phase space.

The attempt to recognise physics features from the high order coefficients of the space- and time-like anomalous dimensions is a long standing Yuri’s project that, after the publication of the three loop space-like anomalous dimension, we revisited. We studied the large-$x$ region and...
analysed to what extent classical soft emission and the reciprocity relation can explain the structure of the known expansion coefficients. In this talk I report some further considerations in this project and I focus on the small-$x$ region.

2 Reciprocity respecting equation

The fact that the Bjorken and Feynman variables in DIS and $e^+e^-$ inclusive fragmentation

$$x_B = \frac{-q^2}{2(Pq)}, \quad x_F = \frac{2(Pq)}{q^2},$$

(2)

are indicated by the same letter is certainly not accidental. These variables are mutually reciprocal: after the crossing operation $P \rightarrow -P$ one $x$ becomes the inverse of the other (although in both channels $0 \leq x \leq 1$ thus requiring the analytical continuation).

Such a reciprocity property can be extended to the Feynman diagrams for the two processes and, in particular, to the contributions from mass-singularities. Consider, for DIS (S-case) and $e^+e^-$ annihilation (T-case), the skeleton structure of Feynman graphs in axial gauge and the mass-singularities phase space ordering:

The same Feynman graphs are contributing and, going from S- to T-channel, the mass singularities are obtained by reciprocity: change $z$ into $1/z$ and the momentum $k$ from space-like to time-like.

This fact is at the origin of the Drell-Levy-Yan relation which has been largely used in order to obtain the time-like anomalous dimensions from the space-like ones. When dimensional regularization is used, this simple kinematical reciprocity relation is corrupted: the coefficient functions and parton distributions in the S- and T-channel differ by factors $z^{-2\epsilon}$ (from the phase space) and $(1-\epsilon)$ from spin averages. However these corrections do not lead to really new structures but are mostly related to the anomalous dimensions to lower order.

Could this $\epsilon$-corruption be a peculiar artifact of the calculation in dimensional regularization so that, at the end of the calculation, reciprocity is restored? This question was also raised by Stratmann and Vogelsang. In the following we explore this. More precisely we assume that parton distributions in the two channels simply resum mass-singularities from the ordered phase space and we neglect regularization subtleties.

We introduce the probability $D_\sigma(N, \kappa^2)$ to find a parton with virtuality $\sigma k^2$ up to $\kappa^2$ with $\sigma = -1$ for the S-case and $\sigma = 1$ for the T-case. The ordering gives rise to the following reciprocity respecting equation

$$\kappa^2 \partial_{\kappa^2} D_\sigma(N, \kappa^2) = \gamma_\sigma(N) D_\sigma(N, \kappa^2) = \int_0^1 \frac{dz}{z} z^N P(z, \alpha_s) D_\sigma(N, \kappa^2 z^\sigma), \quad \sigma = \pm 1.$$  

(5)

The difference between the two channels is simply in the fact that the virtuality of the integrated parton distribution is $\kappa^2 z^\sigma$, see [11]. The splitting function $P(z, \alpha_s)$ does not depend on the S-
or T-channel (its Mellin moments are not the anomalous dimensions). The running coupling in the splitting function depends on the virtuality in a reciprocity respecting form. This equation (in general a matrix equation) is non-local: the S-case with \( \sigma = -1 \) (the T-case with \( \sigma = 1 \)) involves the parton distribution with virtualities larger (smaller) than \( \kappa \). So it is not suitable for explicit calculations of the anomalous dimensions, but to relate them.

Here I discuss some of the consequences of \( \langle 5 \rangle \). I neglect the running coupling dependence so that in \( \gamma(\sigma) \) all beta-function dependent contributions are missed. In this case \( \langle 5 \rangle \) can be solved as (see also \( \langle 9 \rangle \))

\[
\gamma(\sigma) = \mathcal{P}(N + \sigma \gamma(\sigma)), \quad \mathcal{P}(n) = \int_0^1 \frac{dz}{z^2} z^n P(z, \alpha_s) .
\]  

(6) The case of large-\( x \) (corresponding to large-\( N \)) has been discussed in \( \langle 4 \rangle \). Here the Mellin transformed \( z \)-dependent anomalous dimensions \( \tilde{\gamma}(x) \) have the expansion

\[
\tilde{\gamma}(x) = A(1 - x) + B\delta(1 - x) + C \ln(1 - x) + D + \cdots
\]  

(7) These four coefficients are non-vanishing only for the diagonal matrix-elements and are given by expansions in \( \alpha_s \). By using the reciprocity respecting equation \( \langle 6 \rangle \) one obtains, for the quark and gluon matrix elements

\[
A = A, \quad B = B, \quad C = -\sigma A^2, \quad D = -\sigma AB .
\]  

(8) For both the quark and gluon channel, these relations are satisfied (neglecting the beta-function contribution) in the S- and T-case at 2-loop level \( \langle 3 \rangle \) and in the S-case \( \langle 1 \rangle \) at 3-loop.

Recently it has actually been shown \( \langle 10 \rangle \) that \( \langle 5 \rangle \) holds for the whole of the \( \sigma = 1 \) 3-loop non-singlet splitting function.

3 Smal- \( x \) case

For \( N \to 0 \) we consider a single anomalous dimension for the S- and T-channel, essentially the gluon-gluon case. The most singular terms of \( \gamma_-(N) \) are given by the BFKL formula \( \langle 11 \rangle \), an expansion in \( (\bar{\alpha}_s/N)^p \). Next-to-BFKL contributions \( \bar{\alpha}_s (\bar{\alpha}_s/N)^p \) are also known \( \langle 12 \rangle \). Therefore, in the S-case, all singular terms \( \bar{\alpha}_s^p/N^k \) with \( k > p \) fully cancel. To recall the physics behind this result, consider the leading order term.

DLA. There is a single DLA term in the space-like anomalous dimension

\[
\gamma_{DLA}^-(N) = \frac{\bar{\alpha}_s}{N} .
\]  

(9) The fact that all other singular terms vanish, results from cancellations (coherence) in the mass-singularity phase space \( \langle 3 \rangle \) leaving, to this order, transverse momentum ordering.

Given \( \gamma_{DLA}^-(N) \) one derives, from the reciprocity equation \( \langle 5 \rangle \), the corresponding time-like anomalous dimension \( \gamma_{DLA}^+(N) \) given in \( \langle 11 \rangle \) which, as mentioned, results from angular ordering. Therefore coherence in the S-case (\( k_t \)-ordering) implies coherence in the T-case (ordering in the angle \( k_t/k_\perp \)) and this is just the kinematical reciprocity transformation.

\(^6\)Other beta-function dependent contributions to the anomalous dimensions are generated by changing coefficient functions.
MLLA. The small-$x$ gluon-gluon space-like anomalous dimension at one-loop order is

$$\gamma_{-}(N) = \frac{\bar{\alpha}_s}{N} - a \bar{\alpha}_s, \quad a = \frac{11}{12} + \frac{n_f}{6N_c^3}. \quad (10)$$

Given (10), the reciprocity respecting equation (5) gives the time-like anomalous dimension

$$\gamma_{+}(N) = \frac{1}{4} \left( -(N + 2a \bar{\alpha}_s) + \sqrt{(N - 2a \bar{\alpha}_s)^2 + 8 \bar{\alpha}_s} \right). \quad (11)$$

The first subleading correction $a \bar{\alpha}_s$ to the DLA expression (1) corresponds to the MLLA result (modulo the missing term proportional to $\beta_0$) obtained by using the fact that exact angular ordering does not acquire specific soft corrections from 2-gluon configurations.

Higher order terms. The expansion for the S- and T-case can be organized as follows according to increasing singularities for $N \to 0$

$$\gamma_{-}(N) = \sum_{p=1}^{\infty} \sum_{k=0}^{p} s_{pk} \frac{\bar{\alpha}_s^p}{N^k}, \quad \gamma_{+}(N) = \sum_{p=1}^{\infty} \sum_{k=0}^{2p-1} t_{pk} \frac{\bar{\alpha}_s^p}{N^k}, \quad (12)$$

The coefficients $s_{pk}$ and $t_{pk}$ are then arranged into lines as depicted in the following chart.

Solid line with black circles shows the BFKL terms ($\bar{\alpha}_s^p/N^p$). Here empty circles stand for BFKL terms that are “accidentally” zero.

Line with green circles shows the next-to-BFKL terms ($\bar{\alpha}_s^p/N^{p-1}$).

Light blue circles mark the other terms known from exact calculations up to three loops.

Dashed lines with yellow circles mark series in $\gamma_{+}$ that are generated by terms of $\gamma_{-}$ on the same lines. Taking $N \sim \sqrt{\alpha_s}$ all terms on the same line are of order $(\sqrt{\alpha_s})^\tau$ with $\tau = 1, 2, \ldots$ moving to right.

The first two series for $\gamma_{+}(N)$ with $\tau = 1, 2$ are generated by the $\bar{\alpha}_s/N$ and $\bar{\alpha}_s$ terms in $\gamma_{-}(N)$, see (9,11). Notice that this results from the fact that the BFKL term of order $\bar{\alpha}_s^2/N^2$ is “accidentally” zero.

Predictions. The known terms of $\gamma_{-}(N)$ allow us to obtain the series of $\gamma_{+}(N)$ corresponding to the lines $(\sqrt{\alpha_s})^\tau$ with $\tau$ up to $\tau = 5$ as follows:

- the series with $\tau = 3$ is generated accounting only for next-to-BFKL term $c \bar{\alpha}_s^2/N$ (the $\bar{\alpha}_s^3/N^3$ BFKL-coefficient vanishes). The corresponding expression for $\gamma_{+}(N)$ is again given by (11) with $8 \bar{\alpha}_s$ in the square bracket replaced by $8(\bar{\alpha}_s + c \bar{\alpha}_s^2)$ and this is just a redefinition of the coupling;

- from the previous cases with $\tau = 2, 3$, we conclude that the fact that both $\bar{\alpha}_s^2/N^2$ and $\bar{\alpha}_s^3/N^3$ BFKL-coefficient are “accidentally” zero (first two white circles in the chart) translates into exact angular ordering according to which no specific soft corrections from 2- and 3-soft gluon configurations emerge;
• the series with $\tau = 4$ can be computed from terms in $\gamma_-(N)$ of order $\bar{\alpha}_s^2/N^4$ (from exact two loop results$^3$), $\bar{\alpha}_s^3/N^4$ and $\alpha_s^3/N^2$ from BFKL and next-to-BFKL respectively;

• the series with $\tau = 5$ can be obtained including $\bar{\alpha}_s^3/N$ (from exact three loop results$^1$) and $\bar{\alpha}_s^4/N^3$ from next-to-BFKL (BFKL term $\bar{\alpha}_s^5/N^5$ is accidentally zero);

• the first series in $\gamma_+(N)$ which cannot be computed from known space-like results corresponds to $\tau = 6$ since the coefficient $\bar{\alpha}_s^4/N^2$ of $\gamma_-(N)$ is not known (red circle in the chart).

4 Final considerations

This is a working progress report on the attempt by Yuri Dokshitzer, Gavin Salam and myself to relate the small-$x$ behaviour of anomalous dimensions in the time- and space-like cases. I reported here only the case in which the scale of the running coupling is neglect (no beta-function contributions$^6$ in $\gamma_\pm(N)$). The basis is the reciprocity respecting equation$^5$ which is deduced by taking into account simply the reciprocity relation in the mass-singularity phase space$^3$ for the S- and T-case. There may be subtleties coming from regularization and factorization prescriptions used in NLO calculations which go beyond the analysis of mass-singularities phase space. This is what happens in the $\overline{\text{MS}}$ calculation, such as extra factors $z^{-2\epsilon}$ or $(1 - \epsilon)$. On the other hand, these factors are absent if one uses the Wilson-Polchinski regularization scheme (one works in four dimension and introduces a momentum cutoff and counter-terms to ensure$^{16}$ gauge symmetry). How to rescue reciprocity in terms of renormalization scheme transformation has been discussed also in$^8$. An example could be the scheme used in$^7$ where, in dimensional regularization, the $\epsilon \rightarrow 0$ limit is taken before virtuality integration and then the role of mass-singularity ordering is more clear.

Our study should allow us to check whether regularization subtleties could at the end leave uncorrupted the reciprocity relation at the level of relating physical observables.

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$^6$This limit might be most easily be seen in the N=4 SUSY limit.
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