Absence of logarithmic scaling in the ageing behaviour of the 4D spherical model

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Abstract

The non-equilibrium dynamics of the kinetic spherical model with a non-conserved order-parameter, quenched to \( T \leq T_c \) from a fully disordered initial state, is studied at its upper critical dimension \( d = d^* = 4 \). In the scaling limit where both the waiting time \( s \) and the observation time \( t \) are large and the ratio \( y = t/s > 1 \) is fixed, the scaling functions of the two-time autocorrelation and autoresponse functions do not contain any logarithmic correction factors and the typical size of correlated domains scales for large times as \( L(t) \sim t^{1/2} \).
1 Introduction

The study of collective non-equilibrium behaviour has received a lot of interest in recent years [1]. One particular aspect is the study of ageing which for example may arise if a many-body system is rapidly brought out of some initial equilibrium state by the change of external control parameters such as the temperature or an external field. If the change of these variables is such that the equilibrium state of the system is either at an equilibrium critical point or else in a coexistence phase with at least two equivalent but distinct equilibrium states, then one may observe (i) a slow, non-exponential relaxation, (ii) a breaking of time-translation-invariance and (iii) dynamical scaling which are the three defining properties of ageing systems, see [2, 3, 4, 5] for reviews. While ageing was first studied in glassy systems [6], it has been realised more recently that its essential properties can also be found in simple ferromagnets, which may be easier to analyse and still may provide useful clues for the understanding of the more complex glassy systems. In particular, in ferromagnets the form of dynamical scaling is quite simple since the linear domain size scales algebraically according to

$$L(t) \sim t^{1/2}$$

Ageing is conveniently studied through the behaviour of two-time observables. For the two-time autocorrelation and (linear) autoresponse functions, the most simple kind of dynamical scaling behaviour is

$$C(t, s) = \langle \phi(t, r) \phi(s, r) \rangle = s^{-b} f_C(t/s)$$

$$R(t, s) = \left. \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \right|_{h=0} = s^{-1-a} f_R(t/s)$$  (1.1)

where $\phi(t, r)$ is the order-parameter at time $t$ and at the location $r$, $h$ is the conjugate magnetic field, $a$ and $b$ are ageing exponents herewith defined and $f_{C, R}(y)$ are scaling functions. These scaling forms are expected to hold when both times $t, s$ are large and their ratio $y = t/s > 1$ is fixed.

The scaling (1.1) is generally valid for systems with a so-called simple ageing behaviour, but there are exceptions. In certain cases, the above scaling forms are modified by additional logarithmic factors which already manifests themselves in the scaling of the linear domain size, i.e. $L(t) \sim (t/\ln t)^{1/2}$ for a non-conserved order-parameter, and occurs in many systems where topological defects (e.g. vortices) play a role, such as planar magnets, frustrated spin systems, liquid crystals or superconductor arrays, see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Recently, for equilibrium critical phenomena, Kenna, Johnston and Janke [19, 20] have constructed a systematic theory for these logarithmic factors, based on an analysis of the complex zeroes of the partition function, by which they derive scaling relations between the exponents describing eventual logarithmic correction factors to simple scaling. They checked the results of their analysis by comparing with the many results available for logarithmic contributions in equilibrium scaling (see [21] for a very recent example). On the other hand, for non-equilibrium dynamical scaling much less is known on possible logarithmic factors which makes tests of more general ideas difficult. For that reason, we consider in this work the ageing and dynamical scaling of the four-dimensional spherical model whose logarithmic corrections factors to equilibrium scaling are well-established textbook knowledge [22]. While the scaling of its two-time functions has been analysed in great detail, starting with [23], for either $d < 4$ or $d > 4$, to the best of our knowledge, the case $d = 4$ has never been explicitly studied.

In the next section, we define the model and briefly recall those elements of the solution which we need for our analysis which is presented in section 3. We conclude in section 4. Technical details of the calculation are given in an appendix.
2 Model and formalism

The spherical model may be defined in terms of real spin variables $S_{r} \in \mathbb{R}$ attached to the sites $r$ of a hyper-cubic lattice $\Lambda \subset \mathbb{Z}^d$ and which obey the mean spherical constraint

$$\sum_{r \in \Lambda} S(t, r)^2 = N$$

(2.1)

where $N$ is the total number of sites of the lattice. The dynamics is assumed to be given by the stochastic Langevin equation

$$\partial_{t} S(t, r) = \Delta S(t, r) + \hat{\mathfrak{z}}(t) S(t, r) + \eta(t, r)$$

(2.2)

where $\Delta$ is the Laplacian with respect to $r$, $\hat{\mathfrak{z}}(t)$ is a Lagrange multiplier chosen such that the spherical constraint holds and $\eta(t, r)$ is a centred gaussian noise with variance $\langle \eta(t, r) \eta(t', r') \rangle = 2T \delta(t - t') \delta(r - r')$. In writing this, a choice of units was made such that the corresponding kinetic coefficient is set to unity. Throughout, we shall assume a fully disordered initial state (where in particular the order-parameter vanishes) described by a gaussian variable and the variance $\langle S(0, r) S(0, r') \rangle = \delta(r - r')$.

Since the solution of this model is by now standard, see e.g. [23], we shall merely quote those results which we shall need for our analysis of the 4D case. One of the central quantities needed for the analysis is the function $g(t) := \exp(2 \int_{0}^{t}du \hat{\mathfrak{z}}(u))$, which, as a consequence of the spherical constraint, satisfies

$$g(t) = f(t) + 2T \int_{0}^{t} dt' f(t - t') g(t')$$

(2.3)

where the auxiliary function $f(t)$ is given by

$$f(t) = \int_{\mathcal{B}} \frac{dq}{(2\pi)^d} e^{-2\omega(q)t} \left( e^{-4t I_0(4t)} \right)^d \Rightarrow \frac{t \rightarrow \infty}{(8\pi t)^{-d/2}}$$

(2.4)

where $\mathcal{B}$ denotes the first Brillouin zone, $I_0(u) = \frac{1}{\pi} \int_{0}^{\pi} d\theta e^{-u \cos \theta}$ is a modified Bessel function [27] and the dispersion relation is, for a hyper-cubic lattice, $\omega(q) = \sum_{j=1}^{d} (2 - 2 \cos q_j)$. In particular, the critical temperature $T_c(d) > 0$ for $d > 2$ and is given by

$$\frac{1}{2T_c(d)} = \int_{0}^{\infty} dt \left( e^{-4t I_0(4t)} \right)^d$$

(2.5)

from which its numerical value is easily found for any $d$. For example, $T_c(4) = 6.45438\ldots$ Given the function $g(t)$, any observable of interest is readily calculated. In what follows, we shall need the single-time correlation function $\hat{C}_q(t)$ given by

$$\langle \hat{S}_{q}(t) \hat{S}_{q'}(t) \rangle = (2\pi)^d \delta(q + q') \hat{C}_q(t)$$

(2.6)

where the (discrete) Fourier transforms are defined by

$$\hat{f}_q = \sum_{r \in \Lambda} f(r) e^{-iqr} ; \ f(r) = \int_{\mathcal{B}} \frac{dq}{(2\pi)^d} \hat{f}_q e^{iqr}$$

(2.7)

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See [24] for a careful analysis of the mean spherical constraint. In the case of a non-vanishing initial magnetisation, the fluctuations in the Lagrange multiplier $\hat{\mathfrak{z}}(t)$ must be taken into account [25]. This makes the solution of the model, even for $d \neq 4$, a formidable task. In the interest of a relatively simple presentation, we concentrate on the case of a vanishing initial magnetisation only, where the formalism at hand with a non-fluctuating $\hat{\mathfrak{z}}(t)$ is sufficient [25]. See [25] for a detailed discussion of the ageing behaviour when, starting from an ordered state, the quench is made to $T = T_c$.\footnote{See [24] for a careful analysis of the mean spherical constraint. In the case of a non-vanishing initial magnetisation, the fluctuations in the Lagrange multiplier $\hat{\mathfrak{z}}(t)$ must be taken into account [25]. This makes the solution of the model, even for $d \neq 4$, a formidable task. In the interest of a relatively simple presentation, we concentrate on the case of a vanishing initial magnetisation only, where the formalism at hand with a non-fluctuating $\hat{\mathfrak{z}}(t)$ is sufficient [25]. See [25] for a detailed discussion of the ageing behaviour when, starting from an ordered state, the quench is made to $T = T_c$.}
One then has
\[ \tilde{C}_q(t) = \frac{e^{-2\omega(q) t}}{g(t)} \left( 1 + 2T \int_0^t dt' e^{2\omega(q) t'} g(t') \right) \quad (2.8) \]
where the initial condition \( \tilde{C}_q(0) = 1 \) was used. In a similar fashion, one finds for the autocorrelation and autoresponse functions \[23\]
\[ C(t, s) = \langle S(t, r) S(s, r) \rangle = \frac{1}{\sqrt{g(t)g(s)}} \left( f \left( \frac{t+s}{2} \right) + 2T \int_0^s dt' \frac{g(t')}{g(t)} \right) \]
\[ R(t, s) = \left. \frac{\delta \langle S(t, r) \rangle}{\delta h(s, r)} \right|_{h=0} = f \left( \frac{t-s}{2} \right) \sqrt{\frac{g(s)}{g(t)}} \quad (2.9) \]
Finally, we shall also measure the relevant length scale \( L(t) \) of dynamical scaling, which is also a measure of the linear size of correlated clusters, by considering the normalised second moment of the single-time correlator
\[ L^2(t) = \left\{ \begin{array}{ll}
\sum_{r \in \Lambda} r^2 C(t, r) = -\frac{\partial^2 \tilde{C}_q(t)/\partial q^2}{\tilde{C}_q(t)} \bigg|_{q=0} \\
= 4d t + 2T \int_0^t dt'(t-t')g(t') \\
1 + 2T \int_0^t dt' g(t') \end{array} \right. \quad (2.10) \]
We remark that the response function, and also the correlators \( C \) and the typical length scale \( L \) at criticality, where the thermal term dominates, merely depend on ratios \( g(t)/g(t') \) such that the global normalisation of the function \( g(t) \) will disappear from these physical observables.

3 Results

3.1 Solution of the Volterra equation

In order to obtain \( g(t) \) explicitly, one must solve the Volterra integral equation \[23\]. Since its right-hand side has a convolution structure, one considers the Laplace transformation \( \overline{g}(p) = \mathcal{L}(g)(p) = \int_0^\infty dt e^{-pt} g(t) \) and finds
\[ \overline{g}(p) = \frac{\overline{f}(p)}{1 - 2T \overline{f}(p)} \quad (3.1) \]
From standard Tauberian theorems \[28\], the long-time behaviour of \( g(t) \) follows from the asymptotics of \( \overline{g}(p) \) as \( p \to 0 \). This in turn requires the leading behaviour of \( \overline{f}(p) \) which may be obtained in a well-known fashion, see \[28\] \[29\] \[30\], which we outline in the appendix. Together with the known results for \( 2 < d < 4 \) and \( d > 4 \) \[23\], we have
\[ \overline{f}(p) = \frac{1}{2Tc(d)} + \left\{ \begin{array}{ll}
a_1 p^{d/2-1} & ; \text{if } 2 < d < 4 \\
a_0 p(C_E + \ln p) & ; \text{if } d = 4 \\
a_2 p - a_1 p^{d/2-1} + \ldots & ; \text{if } d > 4 \end{array} \right. \quad (3.2) \]
where \( a_0 = (8\pi)^{-2} \), \( a_1 = -(8\pi)^{-d/2} |\Gamma(1 - d/2)| \) for \( d \neq 4 \), \( a_2 = \int_B \frac{\Gamma^2(q)}{(2\pi)^2 (2\omega(q))^2} \) \( \frac{1}{\Gamma(1-d/2)} \) \( = \int_0^\infty du u (e^{-u} I_0(2u))^d \)
and \( C_E = 0.5772 \ldots \) is Euler’s constant.
3.2 Scaling of autocorrelation and autoresponse

We are now ready to discuss the long-time behaviour of our model where the three cases $T > T_c$, $T < T_c$ and $T = T_c$ have to be distinguished. In figure 1(a) we illustrate the long-time behaviour of $g(t)$ for these three cases, obtained by numerically solving eq. (2.3) \[31\], for which $g(t)$ clearly shows different behaviour.

1. $T > T_c$. For quenches into the disordered phase, $g(t)$ is obtained straightforwardly by the method of residues

$$g(t) = T(1/\tau_{eq})e^{t/\tau_{eq}} = \frac{16\pi^2}{T^2} \frac{e^{t/\tau_{eq}}}{\ln(\tau_{eq}) - C_E - 1}$$

where the finite relaxation rate is obtained from $T(1/\tau_{eq}) = 1/2T$ and reads for $T \gtrsim T_c$ $\tau_{eq} \approx \frac{1}{32\pi^2 T_c} \ln \frac{T_c}{T_c - T_c}$. Since the thermal equilibrium correlation length of the $4D$ spherical model behaves as \[19, 22\] $\xi_{eq}^2(T) \sim (T - T_c)^{-1} \ln(T - T_c)$ and one expects that the relaxation time $\tau_{eq} \sim \xi_{eq}^2$, this is consistent with known results. The two-time functions relax within the finite time $\tau_{eq}$ to their time-translation-invariant equilibrium values.

2. $T < T_c$. Now, one has by combining \[31\] and \[3.2\], for $p$ small enough

$$g(p) \approx \begin{cases} 
\frac{1}{2T_c M_{eq}^2} + a_1 p^{d/2-1} & ; \text{if } 2 < d < 4 \\
\frac{1}{2T_c M_{eq}^2} + \frac{a_1}{64\pi^2 M_{eq}^2} p + \frac{p \ln p}{64\pi^2 M_{eq}^4} & ; \text{if } d = 4 \\
\frac{1}{2T_c M_{eq}^2} - \frac{a_1}{M_{eq}^2} p - \frac{a_1}{M_{eq}^2} p^{d/2-1} & ; \text{if } d > 4
\end{cases}$$

where the exact equilibrium result $M_{eq}^2 = 1 - T/T_c$ \[22\] was used. The presence of terms constant in $p$ in \[3.4\] signal that it is not enough to concentrate on the asymptotic long-time behaviour of $g(t)$ but there are also ‘global’ contributions to $g(t)$ which must be taken into account.

Figure 1: (a) Behaviour of the function $g(t)$ in $d = 4$ dimensions as a function of time, above, at and below $T_c(4) = 6.45438 \ldots$. In (b) the behaviour $g(t) \sim t^{-2}$ for $T = 4 < T_c(4)$ and in (c) the behaviour $g(t) \sim \ln(t)^{-1}$ at $T = T_c(4)$ is illustrated.
Formally inverting the Laplace transform in (3.4) then leads to the following form for \( g(t) \),

\[
g(t) \simeq \frac{1}{2T_c M_{eq}^2} \delta(t) + \begin{cases} 
  \frac{f(t)}{M_{eq}^4} & \text{if } 2 < d < 4 \\
  \frac{C_E}{64\pi^2 M_{eq}^4} \delta'(t) + \frac{f(t)}{M_{eq}^4} & \text{if } d = 4 \\
  (-1) \frac{a_d}{M_{eq}^4} \delta'(t) + \frac{f(t)}{M_{eq}^4} & \text{if } d > 4
\end{cases}
\]  

(3.5)

where \( \delta(t) \) is the Dirac delta function. Eq. (3.5) contains two kinds of terms. First, one has the expected ‘regular’ asymptotic form, for \( t \to \infty \), \( g(t) \simeq f(t)M_{eq}^{-4} \) and second, there appear ‘singular’ contributions. While the response functions follow from the ‘regular’ terms alone, the ‘singular’ contributions are important for the correct calculation of the correlation functions, as explained in the appendix. In figure 1b we illustrate the 4D long-time behaviour \( g(t) \sim t^{-2} \). We observe that no logarithmic factors occur, for \( d = 4 \). Therefore, in the scaling limit \( t, s \to \infty \) with \( y = t/s > 1 \) fixed, we recover the simple scaling behaviour

\[
C(t, s) = M_{eq}^2 \left( \frac{4y}{(y+1)^2} \right) \quad \text{and} \quad R(t, s) = \frac{1}{16\pi^2 s^2} \frac{1}{(y-1)^2}
\]

which could also have been obtained from the well-known scaling functions for \( d \neq 4 \) and performing the analytical continuation \( d \to 4 \).

3. \( T = T_c \). In this case, we have from (3.2)

\[
\bar{g}(p) = -\frac{1}{2T_c} + \frac{1}{(2T_c)^2} \begin{cases} 
  -a_1^{-1}p^{1-d/2} & \text{if } 2 < d < 4 \\
  -a_0^{-1}p^{1}(C_E + \ln p)^{-1} & \text{if } d = 4 \\
  a_2^{-1}p^{-1} & \text{if } d > 4
\end{cases}
\]  

(3.7)

As shown in the appendix, for \( d = 4 \) this leads to the long-time behaviour \( g(t) \sim 1/\ln t \). In figure 1c we illustrate that this asymptotic form is indeed compatible with the numerical solution of (2.3). Taking into account also the singular parts of \( g(t) \), we find the scaling behaviour

\[
C(t, s) = \frac{T_c}{8\pi^2 s} \sqrt{\ln t} \frac{1}{\ln s} \frac{1}{y^2 - 1} \quad \text{and} \quad R(t, s) = \frac{1}{16\pi^2 s^2} \sqrt{\ln t} \frac{1}{\ln s} \frac{1}{(y-1)^2}
\]

(3.8)

The calculations are outlined in the appendix.

Do these extra logarithmic factors imply that simple scaling is modified in the 4D spherical model, at least for \( T = T_c \)? Indeed, the answer is negative, since in the scaling limit \( t, s \to \infty \) with \( y = t/s > 1 \) fixed and finite, we can write \( t = ys \), hence \( \sqrt{\ln t/\ln s} = \sqrt{\ln(ys)/\ln s} \simeq 1 + \frac{1}{2} \ln y/\ln s + \ldots \). Therefore, the scaling functions for \( d = 4 \) can be obtained from the known expressions for \( d \neq 4 \) by analytic continuation \( d \to 4 \). The fact that the system is at the upper critical dimension \( d^* = 4 \) of its equilibrium critical point only enters into the additive logarithmic corrections to scaling. In this respect, the 4D spherical models shows a different behaviour from those systems where logarithmic corrections to non-equilibrium dynamical scaling were seen before.

### 3.3 Domain size

In order to understand better the role of the upper critical dimension \( d^* = 4 \) for non-equilibrium dynamical scaling of the spherical model, we now consider the linear domain size \( L(t) \) in order to check

\[ \text{In [23] only the ‘regular’ asymptotic contributions to } g(t) \text{ are explicitly given, while the effect of the ‘singular’ terms in (3.5) is taken into account by non-asymptotic sum rules, with the same end result for the physically observable two-time correlations and responses, if } d \neq 4. \]
for the presence of additional logarithmic factors with respect to the naively expected simple behaviour $L(t) \sim \sqrt{t}$, valid for $T \leq T_c$ and $d \neq 4$. In eq. (2.10) we had already related $L(t)$ to the function $g(t)$ which fixes the spherical model dynamics. From this, it is already clear that for $T = 0$, we simply have $L^2(t) = 4dt$ and since the temperature $T$ is generally thought to be irrelevant for quenched to $T < T_c$ [2], we would expect that this behaviour of $L(t)$ should remain qualitatively correct for all $T < T_c(d)$.

In order to see that this expectation is indeed borne out, we rewrite eq. (2.10) as the ratio of two inverse Laplace transformations, since the two integrals can be seen as Laplacian convolutions

$$L^2(t) = 4d \frac{\mathcal{L}^{-1}(p^{-2} + 2Tp^{-2}g(p))(t)}{\mathcal{L}^{-1}(p^{-1} + 2Tp^{-1}g(p))(t)}$$

On the other hand, for $T < T_c$, we have from (3.4) that $g(p) = (2Tc)^{-1} (1 - T/T_c)^{-1} (1 + o(p))$. Consequently, we easily find for all $T < T_c(d)$, and for all dimensions $d > 2$

$$L^2(t) = 4dt (1 + O(1/t))$$

This is in agreement with our expectation formulated above and also with more indirect conclusions [23] drawn from the scaling of the single-time and two-time correlation functions. In figure 2b we further illustrate this by showing that the ratio $L^2(t)/(td) \to 4$ in the long-time limit in both 3D and 4D (in the 3D case, we have checked explicitly that convergence occurs for times $t \gtrsim 600$).

In a similar way, we now analyse the critical case $T = T_c(d)$. The starting point is again eq. (3.9) where we now have to insert eq. (3.7). This leads for $t$ large to (see the appendix)

$$L^2(t) \simeq t \times \begin{cases} 8 & ; \text{if } 2 < d < 4 \\ 2d (\ln t - 1)/(\ln t - 3/2) & ; \text{if } d = 4 \\ 2d & ; \text{if } d > 4 \end{cases}$$

We see that to leading order as $t \to \infty$, we always have $L^2(t) \sim t$ and that logarithmic contribution at most enter into the additive correction of this scaling behaviour. This shows again the difference...
between the 4D spherical model and the systems with time-dependent logarithmic scaling of the form 
$L(t) \sim (t/\ln t)^{1/2}$ studied in the literature [7, 9, 10, 11, 12, 13, 14, 15, 17, 18]. In figure 2b, we illustrate 
this for the 3D and 4D cases. In the 3D case, a nice convergence towards the amplitude $L^2(t)/(td) = 8/3$ 
is seen, as expected from (3.11). In the 4D case, we extended the calculations up to $t = 10^4$ and find 
$L^2(t)/(td) \simeq 2.06$, not too far from the approximate analytical result (3.11).

We conclude that in all cases considered, we have found clear evidence that the typical length scales 
as $L(t) \sim t^{1/2}$ and that there no evidence for any logarithmic correction factors present.

4 Conclusions

We have considered, as a case study, the ageing behaviour of the four-dimensional spherical model 
quenched to $T \leq T_c$ from a fully disordered initial state. Since the model is at its upper critical dimension, 
we had expected to find modifications of the usual scaling behaviour by logarithmic correction factors which is indeed true when looking at the relaxation time $\tau_{eq}$ as a function of the temperature $T$, for $T > T_c$, 
and fully consistent with existing knowledge of these corrections at equilibrium [19, 20]. Surprisingly, when studying the time-dependent scaling, both for quenches into the ordered phase and for quenches onto the critical point, our main results eqs. (3.6, 3.8, 3.11) show standard simple ageing and we did not see any evidence for logarithmic corrections appearing in the leading scaling behaviour, although additive logarithmic corrections to scaling do occur for quenches to $T = T_c$, see eq. (3.8). We have also found a simple power-law scaling $L(t) \sim t^{1/2}$, without additional logarithmic factors, for the 
typical length scale, see eq. (3.11). Surprisingly, this suggests that the fact of a system being at its upper 
critical dimension should be considerably less relevant for its non-equilibrium dynamical behaviour than it is for its equilibrium critical behaviour. Of course, there may be other reasons for a system to develop logarithmic scaling, such as vortices [10, 11, 12, 13, 14, 15, 17, 18] or a roughening transition of the interfaces between ordered domains [33].

Experimentally, observations of the kind made here could be of relevance for the non-equilibrium dynamics of systems at a tricritical point (where $d^* = 3$) and which arise for example in diluted magnets, 
meta-magnets or solutions of long polymers at a Θ-point. It would be of interest to consider a spherical 
meta-magnet in an external magnetic field and then study its dynamics at the tricritical point at the upper critical dimension $d^* = 3$. Indeed, non-equilibrium relaxation at a tricritical point was studied long ago in the $O(n)$ model by Janssen and Oerding [32]. At the upper critical dimension $d = 3$, starting 
form a small initial magnetisation, they find to one-loop order the habitually expected logarithmic corrections factors, e.g. in the short-time scaling $\langle M(t) \rangle \sim M_0(\ln t)^{-a(n)}$ with $a(n) = \frac{(n+2)(n+4)}{8\pi(3n+22)}$, and similar for the correlation with the initial state $C(t) = \langle M(t)M(0) \rangle \sim t^{-3/2}(\ln t)^{-a(n)}$. However, their study only considered single-time observables and therefore cannot illustrate directly the compensation of logarithmic factors in two-time quantities which was found in the present work. Since $a(n)$ diverges as $n \to \infty$, it appears also possible that the $n \to \infty$ limit might have special properties. It remains an open problem to what extent the results on the two-time quantities in the spherical model at $d = d^*$ reported here are generic.

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Appendix

We outline some details of the calculation for \( d = 4 \) whose results were quoted in the main text [31]. For \( 2 < d < 4 \) and \( d > 4 \) all results can be taken over from [23].

First, we analyse the leading behaviour of \( \mathcal{F}(p) \) for \( p \to 0 \). The first step is to decompose the integral (we set \( d = 4 \) from now on)

\[
\mathcal{F}(p) = \int_{0}^{\infty} du e^{-(p+4d)u} I_0(4u)^d =: \mathcal{F}_{\text{reg}}(p) + \mathcal{F}_{\text{sing}}(p)
\]

into a regular and a singular part by setting \( \int_{0}^{\infty} = \int_{0}^{\eta} + \int_{\eta}^{\infty} \). The integrals are evaluated in the double limit \( \eta \gg 1 \), \( \eta^2 \ll 1 \) and \( \eta^2 \sim O(1) \) kept fixed. The decomposition of the integral into two parts is merely a heuristic device in order to arrive rapidly at the singular terms which will be seen for \( p\eta \) small enough; of course the end result for \( \mathcal{F}(p) \) should be independent of \( \eta \). First, one has, to leading order

\[
\mathcal{F}_{\text{reg}}(p) \to (2T c_4)^{-1}.
\]

Second, we use the identities eqs. (3.351(4)) and (8.214(1)) from [34]

\[
\int_{\eta}^{\infty} du u^{-2} e^{-pu} = p \text{Ei}(-p\eta) + \eta^{-1} e^{-p\eta}
\]

\[
\text{Ei}(-x) = C_E + \ln x + \sum_{k=1}^{\infty} \frac{(-x)^k}{k! k} \; \text{; for } x > 0
\]

where \( \text{Ei}(x) \) is the exponential integral and then obtain, to leading order in the double limit introduced above

\[
\mathcal{F}_{\text{sing}}(p) = \int_{\eta}^{\infty} du e^{-(p+16)u} I_0(4u)^4
\]

\[
= (8\pi)^{-2} \left[ p \text{Ei}(-p\eta) + e^{-p\eta} \right]
\]

\[
\simeq (8\pi)^{-2} [pC_E + p \ln p]
\]

where we dropped an \( \eta \)-dependent term \( \sim p \ln \eta \) which should cancel anyway against a corresponding correction coming from the regular term \( \mathcal{F}_{\text{reg}}(p) \). Then the case \( d = 4 \) of (3.2) follows.

Next, we derive \( g(p) \) for \( d = 4 \) and \( T < T_c \). From the above and (3.1), the leading terms for \( p \to 0 \) are

\[
\overline{g}(p) \simeq \frac{1}{2T_c M_{eq}^4} + \frac{C_E p}{(8\pi)^2 M_{eq}^4} + \frac{p \ln p}{(8\pi)^2 M_{eq}^4}
\]

and \( g(t) \) follows by inverting the Laplace transformations, using the identities \( \mathcal{L}^{-1}(1)(t) = \delta(t) \) and

\[
\mathcal{L}^{-1}(p \ln p)(t) = \frac{d^2}{dt^2} \int_{c-i\infty}^{c+i\infty} dp \frac{e^{pt}}{p} \ln p = \frac{d^2}{dt^2} (\ln C_E - \ln t) = t^{-2}
\]

(see eqs. (17.13.1) and (17.13.12) in [34]). Hence we arrive at the expressions (3.4) and (3.5) for \( d = 4 \).

From eq. (2.9) we get for the correlation function:

\[
C(t, s) = M_{eq}^4 \left[ t^{d/4} s^{d/4} \right]^{1/2} + 2TM_{eq}^4 (ts)^{d/4} \int_{0}^{s} dt_1 f \left( \frac{t+s}{2} - t_1 \right) g(t_1).
\]

in which we estimate the value of the integral by developing the \( f \) function at the first order:
\[
\int_0^s dt_1 f \left( \frac{t + s}{2} - t_1 \right) g(t_1)
\]
\[
\approx f \left( \frac{t + s}{2} \right) \int_0^s dt_1 g(t_1) - f' \left( \frac{t + s}{2} \right) \int_0^s dt_1 t_1 g(t_1)
\]
\[
\approx \left( \frac{t + s}{2} \right)^{-d/2} \frac{1}{2T_c M_{eq}^2} + d \left( \frac{t + s}{2} \right)^{-d/2} \frac{1}{t + s} \left( \frac{8\pi}{M_{eq}^4} \right)^{-d/2} \int_0^s dt_1 t_1^{-d/2+1}
\]
\[
= \left( \frac{t + s}{2} \right)^{-d/2} \frac{1}{2T_c M_{eq}^2} + \text{cste} \cdot (t + s)^{-d/2-1} s^{2-d/2}
\]

(A8)

In the second line we used the sum rule \( \int_0^\infty du g(u) = (2T_c M_{eq}^2)^{-1} \) \eqref{eq:2.39}, which follows in our presentation from the first singular term \( \sim \delta(t) \) in \( (3.5) \) and clarifies the important contributions coming from the singular terms in \( g(t) \). Furthermore it follows from the explicit form of \( g(t) \) that the regular long-time approximation for \( g(t) \) may be used in the second integral in the second line. In addition, we replace the derivative of the function \( f \) by its asymptotic value. In the last line, we see that the second term is negligible in the \( s \to \infty \) limit.

Therefore, using the exact expression \( M_{eq}^2 = 1 - T/T_c \), we find for any dimension \( d \)

\[
C(t, s) \approx \left( \frac{4ts}{(t + s)^2} \right)^{d/4} \left( M_{eq}^4 + \frac{T}{T_c} M_{eq}^2 \right)
\]
\[
= M_{eq}^{-2} \left( \frac{4ts}{(t + s)^2} \right)^{d/4} = M_{eq}^{-2} \left( \frac{4y}{(y + 1)^2} \right)^{d/4}.
\]

(A9)

where the last expression gives the scaling limit \( t, s \to \infty \), with \( y = t/s \) fixed.

The autoresponse function is found directly from \( (2.9) \) by inserting the long-time behaviour for \( g(t) \) from \( (3.5) \).

For the critical case \( T = T_c \) at \( d = 4 \), we have

\[
\bar{g}(p) = -\frac{16\pi^2}{T_c^2} \frac{1}{CEp + p \ln p} - \frac{1}{2T_c}
\]

(A10)

and we now use an approximate method based on the relation \( \ln p = \lim_{n \to 0} (p^n - 1)/n \). In order to find \( g(t) \), at least approximatively, we write for \( p \) small enough

\[
\frac{1}{C_E p + p \ln p} = \lim_{n \to 0} \frac{n}{n C_E - 1} \left( 1 + p^n/(n C_E - 1) \right) \sim \lim_{n \to 0} \frac{n}{n C_E - 1} \left( 1 - \frac{p^n}{n C_E - 1} + \ldots \right)
\]

(A11)

Carrying out the inverse Laplace transform, we find

\[
\mathcal{L}^{-1} \left( \frac{1}{C_E p + p \ln p} \right)(t) \approx \lim_{n \to 0} \frac{n}{n C_E - 1} \left( 1 - \frac{t^{-n}}{(n C_E - 1) \Gamma(1-n)} + \ldots \right)
\]
\[
\approx \lim_{n \to 0} \frac{n}{n C_E - 1} \left( 1 + \frac{e^{-n \ln t}}{(n C_E - 1) \Gamma(1-n)} + \ldots \right)^{-1}
\]
\[
= \frac{1}{\ln t}
\]

(A12)

and hence we expect \( g(t) \sim (\ln t)^{-1} \) for large times. The approximate derivation of \( (A12) \) relies on commut ing several limits. In order to check whether this is justified, we compare in figure \( 1 \) our analytical
result with the direct numerical solution of the Volterra equation and find that the main feature, namely the logarithmic dependence of \( g(t) \) on \( t \), is correctly reproduced, but the corresponding amplitude is not. In addition, since at criticality only ratios \( g(t)/g(t') \) enter into the leading contributions of the physical observables of interest, the corresponding amplitudes cancel and hence will not be required.

Finally, we prove (3.11). For \( d > 4 \), we consider the form (3.9) together with \( g(p) = a_1^{-1} p^{-1} \). Then

\[
L^2(t) = 4d \frac{\mathcal{L}^{-1}(p^{-2} + (2T_c/a_2)p^{-3})(t)}{\mathcal{L}^{-1}(p^{-1} + (2T_c/a_2)p^{-2})(t)} \overset{t \to \infty}{\approx} 2d \frac{(2T_c/a_2) \cdot t^2}{(2T_c/a_2) \cdot t} = 2d t
\]

(A13)

since the second term in both the numerator and the denominator is the dominant one for \( p \to 0 \). Similarly, for \( 2 < d < 4 \) we have

\[
L^2(t) = 4d \frac{\mathcal{L}^{-1}(p^{-2} + (2T_c/a_1)p^{-1-d/2})(t)}{\mathcal{L}^{-1}(p^{-1} + (2T_c/a_1)p^{-d/2})(t)} \overset{t \to \infty}{\approx} 4d \frac{1/\Gamma(d/2 + 1)}{1/\Gamma(d/2)} t = 8t
\]

(A14)

Last, but not least, for \( d = 4 \), we must find the inverse Laplace transforms of \( h_k(p) := p^{-k}(C_E + \ln p)^{-1} \), with \( k = 2, 3 \). Applying the same procedure we used in (A12) in order to derive \( g(t) \), we obtain

\[
\mathcal{L}^{-1}(h_2(p))(t) \simeq -\frac{t}{\ln t - 1}, \quad \mathcal{L}^{-1}(h_3(p))(t) \simeq -\frac{1}{2} \frac{t^2}{\ln t - 3/2}
\]

(A15)

which completes the proof. We expect that this approximation should give the correct leading \( t \)-dependence. Since \( L^2(t) \) is given by the ratio of two such expressions, any inaccuracy in the associated amplitudes should largely cancel.

**Note added in proof:** after this work had been completed, we became aware of the paper [35] which studies the ageing in the spherical model with competing interactions. In particular, a detailed analysis of \( g(t) \) is presented, the results of which are in agreement with our analytical findings, in particular \( g(t) \sim 1/\ln t \) at \( T = T_c \).
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