Is the right-angled building associated to a universal group unique?

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Abstract. A universal group is a subgroup of the group of type-preserving automorphisms of a right-angled building and hence associated to this building. A question is then whether this universal group can act chamber-transitively and with compact open stabilisers on a different right-angled building of the same type. We answer this question and define two universal groups associated to different right-angled buildings which are isomorphic as topological groups. Moreover, we show that different right-angled buildings can have the same universal group.

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1. Introduction. A universal group associated to a right-angled building is a subgroup of the group of type-preserving automorphisms of the building such that the local actions are given by prescribed permutation groups. These groups have been introduced for trees by Burger and Mozes in [3] and generalised to right-angled buildings by De Medts, Silva, and Struyve in [4].

When studying universal groups associated to right-angled buildings, the question arises whether two universal groups associated to different right-angled buildings can be isomorphic.

We will answer this question and describe two universal groups associated to different right-angled buildings that are isomorphic as topological groups. By choosing the thickness the right way, we obtain two different buildings of the same type but in such a way that we also obtain a bijection of chambers. To
get this bijection, we use the so-called tree-wall tree. Then we use this bijection of chambers to obtain a bijection of the associated universal groups. For this, we choose the local groups in the right way. This bijection is then continuous and open and hence the universal groups are isomorphic as topological groups.

We start with defining universal groups. A building of type \((W, I)\) is right-angled if the Coxeter system \((W, I)\) is right-angled, i.e. \(m_{ij} \in \{2, \infty\}\) for all \(i, j \in I\) with \(i \neq j\).

Let \(\Delta\) be a right-angled building of type \((W, I)\) with prescribed thickness \((q_i)_{i \in I}\) such that the \(q_i\) are finite but at least 3 for every \(i \in I\). Therefore every \(i\)-panel contains exactly \(q_i\) many chambers. The set of chambers of \(\Delta\) is denoted by \(\text{Ch}(\Delta)\). We view automorphisms of \(\Delta\) as bijections from \(\text{Ch}(\Delta)\) to \(\text{Ch}(\Delta)\) such that \(i\)-adjacency is preserved for all \(i \in I\). For every \(i \in I\), let \(X_i\) be a set with cardinality \(q_i\); we call \(X_i\) the set of \(i\)-colours. Let \(\lambda : \text{Ch}(\Delta) \rightarrow \prod_{i \in I} X_i, c \mapsto (\lambda_i(c))_{i \in I}\), be a map such that \(\lambda_i|_P : P \rightarrow X_i\) is bijective and \(\lambda_j|_P : P \rightarrow X_j\) is constant for every \(i \in I\), every \(j \in I\) with \(j \neq i\), and every \(i\)-panel \(P\). We call \(\lambda\) then a legal colouring of \(\Delta\). If \(m_{ij} = 2\) for \(i, j \in I\), then every residue of type \(\{i, j\}\) is finite. Observe that every residue of this type contains \(q_i\) many \(j\)-panels and \(q_j\) many \(i\)-panels. By the definition of the colouring, all chambers in a single \(j\)-panel get assigned the same \(i\)-colour and all chambers in an \(i\)-panel have the same \(j\)-colour. Hence inside such a residue of type \(\{i, j\}\) every tuple of colours in \(X_i \times X_j\) occurs exactly once.

The map

\[
\sigma_\lambda(g, P) = \lambda_i|_{g(P)} \circ g|_P \circ (\lambda_i|_P)^{-1} \in \text{Sym}(X_i)
\]

is called the local action of the automorphism \(g \in \text{Aut}(\Delta)\) on the \(i\)-panel \(P\).

The universal group is the subgroup of the automorphism group such that all local actions are contained in prescribed permutation groups. Let \(F_i \subseteq \text{Sym}(X_i)\) be a permutation group for every \(i \in I\). We refer to these groups as the local groups. Then the universal group of \(\Delta\) with respect to the groups \((F_i)_{i \in I}\) is defined as

\[
\mathcal{U} = \{g \in \text{Aut}(\Delta) \mid \sigma_\lambda(g, P) \in F_i \text{ for every } i\text{-panel } P \text{ and every } i \in I\}.
\]

These universal groups have been introduced and studied in [4] and have been further studied in [1,2]. The properties of the universal group depend on and correspond to the properties of the local groups. By [4, Proposition 3.7], the universal groups for different legal colourings are conjugate in \(\text{Aut}(\Delta)\) and hence the structure of \(\mathcal{U}\) does not depend on the choice of the colouring. Furthermore, by [2, Corollary 3.2], a universal group acts chamber-transitively if and only if all local groups act transitively.

We equip the universal group with a topology defined by a neighbourhood basis of the identity that is given by the pointwise stabilisers of finite sets of chambers in the universal group. Since \(q_i\) is finite for every \(i \in I\), every pointwise stabiliser of a finite set of chambers has finite orbits and thus is
compact. Hence, the universal group is a locally compact totally disconnected group (cf. [2, Prop. 2.2 and 3.8]).

2. Two isomorphic universal groups. We answer the question of whether a universal group can be associated to two different right-angled buildings of the same type. For this, we construct two universal groups associated to different buildings of the same type and show that they are isomorphic as topological groups. Then a universal group can act on a different building of the same type as the original one such that the action is chamber-transitive and with compact open stabilisers.

So let $\Delta$ and $\tilde{\Delta}$ be two locally finite thick buildings of type $(W, I)$ where $I = \{i, j, k\}$ and $W = \langle i, j, k \mid i^2 = j^2 = k^2 = (ij)^2 = 1 \rangle$. Moreover, we prescribe the thickness $(q_i, q_k, q_j)$ for $\Delta$ and $(\tilde{q}_i, \tilde{q}_k, \tilde{q}_j)$ for $\tilde{\Delta}$ such that the equations

$q_i q_j = \tilde{q}_k$ and $\tilde{q}_i \tilde{q}_j = q_k$

are satisfied. For every right-angled Coxeter system $(W, I)$ and every family $(q_i)_{i \in I}$ with $q_i \geq 2$ for all $i \in I$, these exists a unique, up to isomorphism, building of type $(W, I)$ with prescribed thickness $(q_i)_{i \in I}$ by Proposition 1.2 of Haglund and Paulin in [5]. Hence, the buildings $\Delta$ and $\tilde{\Delta}$ exist and are unique up to isomorphism. Parts of the Davis realisations of two buildings of this type, which satisfy the assumptions, are pictured in Fig. 1.

At first, we want to construct a bijection of chambers from $Ch(\Delta)$ to $Ch(\tilde{\Delta})$. For this, we use tree-wall trees, which have been studied for universal groups in [4, Section 2.3]. The $k$-tree-wall tree associated to a building of this type is the following infinite graph. The vertices are the $k$-panels, which are residues of type $k$, and the residues of type $\{i, j\}$, and there is an edge whenever the intersection of the residues is not empty. So there are no edges between different $k$-panels and between different residues of type $\{i, j\}$. The intersection of a $k$-panel and a residue of type $\{i, j\}$ is either empty or a single chamber. Moreover, every chamber is contained in exactly one $k$-panel and one residue of type $\{i, j\}$. Hence, the edges are in bijection with the chambers of the building. The graph is a biregular tree with valencies the cardinality of the $k$-panels and of the residues of type $\{i, j\}$ and by [4, Proposition 2.39], it is indeed a tree. An example of a $k$-tree-wall tree is pictured in Fig. 2.

The $k$-tree-wall trees associated to $\Delta$ and $\tilde{\Delta}$ are both $(q_k, \tilde{q}_k)$-regular trees, hence they are isomorphic and so we get a bijection of chambers $\psi : Ch(\Delta) \to Ch(\tilde{\Delta})$, that maps $k$-panels of $\Delta$ to residues of type $\{i, j\}$ in $\tilde{\Delta}$ and the residues of type $\{i, j\}$ in $\Delta$ to the $k$-panels of $\tilde{\Delta}$. We get the equivalences

$c \sim_k d \iff \psi(c) \in Res_{\{i, j\}}(\psi(d)),  

\iff c \in Res_{\{i, j\}}(d) \Rightarrow \psi(c) \sim_k \psi(d)$

for all chambers $c, d \in Ch(\Delta)$.

Now we need colourings for both buildings. Let $X_i$ and $X_j$ be sets of cardinality $q_i$ and $q_j$ respectively, and $\tilde{X}_i$ and $\tilde{X}_j$ be sets of cardinality $\tilde{q}_i$ and $\tilde{q}_j$ and
First, a part of the Davis realisation of Δ with thickness $q_i = 4, q_j = 3$, and $q_k = 9$. Let

\[ \lambda_i : Ch(\Delta) \to X_i \]

be a map such that the restriction of the map to every $i$-panel in $\Delta$ is a bijection and to every $j$-panel.

Second, a part of the Davis realisation of $\tilde{\Delta}$ with thickness $\tilde{q}_i = 3, \tilde{q}_j = 3$, and $\tilde{q}_k = 12$.

In both pictures the $\circ$-vertices correspond to $k$-panels, the $\blacksquare$-vertices to $i$-panels, the $\Delta$-vertices to $j$-panels, and the $\blacklozenge$-vertices to residues of type $\{i,j\}$. The $\blacktriangle$-vertices correspond to chambers. All squares are filled.

**Figure 1.** An example of a pair of buildings satisfying the assumptions

define $X_k = \tilde{X}_i \times \tilde{X}_j$ and $\tilde{X}_k = X_i \times X_j$. Then $X_k$ has cardinality $q_k = \tilde{q}_i \tilde{q}_j$ and $\tilde{X}_k$ has cardinality $\tilde{q}_k = q_i q_j$. Let $\lambda_i : Ch(\Delta) \to X_i$ be a map such that the restriction of the map to every $i$-panel in $\Delta$ is a bijection and to every
Figure 2. Example of a $k$-tree-wall tree

A part of the $k$-tree-wall tree associated to $\Delta$ with thickness $q_i = 4, q_j = 3,$ and $q_k = 9.$ The $\circ$-vertices correspond to $k$-panels and the $\blacklozenge$-vertices to residues of type $\{i,j\}.$

panel of different type is constant. Define $\lambda_j: Ch(\Delta) \rightarrow X_j, \tilde{\lambda}_i: Ch(\tilde{\Delta}) \rightarrow \tilde{X}_i,$ and $\tilde{\lambda}_j: Ch(\tilde{\Delta}) \rightarrow \tilde{X}_j$ in the same way. We use these maps and the bijection of chambers to define the $k$-colourings in the following way

$$\lambda_k: Ch(\Delta) \rightarrow X_k, c \mapsto (\tilde{\lambda}_i(\psi(c)), \tilde{\lambda}_j(\psi(c))) ,$$

$$\tilde{\lambda}_k: Ch(\tilde{\Delta}) \rightarrow \tilde{X}_k, \tilde{c} \mapsto (\lambda_i(\psi^{-1}(\tilde{c})), \lambda_j(\psi^{-1}(\tilde{c}))).$$

Since $\lambda_i$ and $\tilde{\lambda}_j$ are bijective, and the image of a $k$-panel is a residue of type $\{i,j\}$, which is finite, the restriction of $\lambda_k$ to a $k$-panel is a bijection. The image of a panel of type $i$ or $j$ is contained in a $k$-panel and since $\lambda_i$ and $\lambda_j$ are constant on $k$-panels, the restriction of $\lambda_k$ to panels of type $i$ and $j$ is constant. Analogously, it follows that the restriction of $\tilde{\lambda}_k$ to a $k$-panel is bijective and to a panel of type $i$ or $j$ constant. Hence, we get legal colourings $\lambda: Ch(\Delta) \rightarrow X_i \times X_j \times X_k$ of $\Delta$ and $\tilde{\lambda}: Ch(\tilde{\Delta}) \rightarrow \tilde{X}_i \times \tilde{X}_j \times \tilde{X}_k$ of $\tilde{\Delta}$.

Let $F_i \subseteq Sym(X_i), F_j \subseteq Sym(X_j), \tilde{F}_i \subseteq Sym(\tilde{X}_i),$ and $\tilde{F}_j \subseteq Sym(\tilde{X}_j)$ be transitive subgroups. It is not necessary to assume transitivity, however we want to consider chamber-transitive universal groups and therefore we need to consider transitive local groups. Define $F_k = \tilde{F}_i \times \tilde{F}_j$ and $\tilde{F}_k = \tilde{F}_i \times \tilde{F}_j,$ then $F_k$ acts transitively on $X_k$ and $\tilde{F}_k$ transitively on $\tilde{X}_k.$ Let $U$ be the universal group of $\Delta$ with respect to $(F_i, F_j, F_k)$ and let $\tilde{U}$ be the universal group of $\tilde{\Delta}$ with respect to $(\tilde{F}_i, \tilde{F}_j, \tilde{F}_k).$

Remark 2.1. The universal group $U$ acts also on the $k$-tree-wall tree associated to $\Delta$ (cf. [4, Section 3.5]). This action is type-preserving and the local actions are contained in $F_i \times F_j$ and $F_k.$ We can associate to the $k$-tree-wall tree itself a universal group with respect to $(F_i \times F_j, F_k).$ The universal group $U$ is then isomorphic to this universal group (cf. [1, Theorem 3.5.1]). Moreover, a finite set of chambers in $\Delta$ is a finite set of edges in the $k$-tree-wall tree, which
are the chambers when viewing the tree as a building, and thus the pointwise stabilisers of finite sets of chambers coincide. Hence, $\mathcal{U}$ and the universal group associated to the $k$-tree-wall tree are topologically isomorphic.

Since the $k$-tree-wall trees are isomorphic and hence the universal groups associated to them are topologically isomorphic, we get also a topological isomorphism between the universal groups over the two buildings. However, our approach is more direct and constructs a concrete topological isomorphism between the two universal groups.

We will prove that the universal groups $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are isomorphic as topological groups. We start with setting the automorphisms of $\Delta$ in relation to those of $\tilde{\Delta}$ by using the bijection of chambers.

**Lemma 2.2.** For every $g \in \mathcal{U}$, the conjugate $\psi g \psi^{-1}$ is an automorphism of $\tilde{\Delta}$ and contained in the universal group $\tilde{\mathcal{U}}$.

**Proof.** First, we need to prove that $\psi g \psi^{-1}$ is an automorphism of $\tilde{\Delta}$. Let $\tilde{c}, \tilde{d} \in \text{Ch}(\tilde{\Delta})$. Assume that $\psi g \psi^{-1}(\tilde{c}) = \psi g \psi^{-1}(\tilde{d})$, since $\psi$ and $g$ are bijections of chambers, it follows that $\tilde{c} = \tilde{d}$. Furthermore, $\psi g^{-1} \psi^{-1}(\tilde{c})$ is a chamber of $\tilde{\Delta}$ and a preimage of $\tilde{c}$. Hence, $\psi g \psi^{-1}$ is a bijection of chambers.

Further, we need to prove that $\psi g \psi^{-1}$ acts type-preservingly. For this, let $\tilde{c} \sim_i \tilde{d}$, i.e., they have the same $j$-colour. Furthermore, by the definition of the colouring, their preimages $\psi^{-1}(\tilde{c})$ and $\psi^{-1}(\tilde{d})$ lie in the same $k$-panel of $\Delta$. This adjacency is preserved by $g$ and again by the definition of $\psi$, it follows that $\psi g \psi^{-1}(\tilde{c})$ and $\psi g \psi^{-1}(\tilde{d})$ are in the same residue of type $\{i, j\}$ in $\tilde{\Delta}$.

By the definition of the universal group, there exist $\sigma_1 \in \tilde{F}_i$ and $\sigma_2 \in \tilde{F}_j$ such that $\sigma_3(g, \mathcal{P}_{k, \psi^{-1}(\tilde{c})}) = (\sigma_1, \sigma_2)$. Since $\psi^{-1}(\tilde{c})$ and $\psi^{-1}(\tilde{d})$ are $k$-adjacent, it follows that

$$(\sigma_1, \sigma_2)(\lambda_k(\psi^{-1}(\tilde{c}))) = \lambda_k(g \psi^{-1}(\tilde{c}))$$

and moreover, by the definition of the $k$-colouring of $\Delta$, it follows that

$$\left(\sigma_1(\tilde{\lambda}_i(\tilde{c})), \sigma_2(\tilde{\lambda}_j(\tilde{c}))\right) = \left(\tilde{\lambda}_i(\psi g \psi^{-1}(\tilde{c})), \tilde{\lambda}_j(\psi g \psi^{-1}(\tilde{c}))\right)$$

and

$$\left(\sigma_1(\tilde{\lambda}_i(\tilde{d})), \sigma_2(\tilde{\lambda}_j(\tilde{d}))\right) = \left(\tilde{\lambda}_i(\psi g \psi^{-1}(\tilde{d})), \tilde{\lambda}_j(\psi g \psi^{-1}(\tilde{d}))\right).$$

By using that $\tilde{c}$ and $\tilde{d}$ have the same $j$-colour, we conclude that

$$\tilde{\lambda}_j(\psi g \psi^{-1}(\tilde{c})) = \sigma_2 \lambda_j(\tilde{c}) = \sigma_2 \lambda_j(\tilde{d}) = \lambda_j(\psi g \psi^{-1}(\tilde{d}))$$

and hence $\psi g \psi^{-1}(\tilde{c})$ and $\psi g \psi^{-1}(\tilde{d})$ have the same $j$-colour and are contained in the same residue of type $\{i, j\}$. Every residue of type $\{i, j\}$ is finite, it contains $q_i q_j$ many chambers, and by the definition of the colouring, every $i$-panel gets assigned one $j$-colour and every $j$-panel gets one $i$-colour. Thus if two chambers inside a common residue of type $\{i, j\}$ have the same $j$-colour, they must be
contained in a common $i$-panel. Hence, it follows that the chambers $\psi g \psi^{-1}(\bar{c})$ and $\psi g \psi^{-1}(\bar{d})$ are contained in the same $i$-panel.

If $\bar{c}$ and $\bar{d}$ are contained in the same $j$-panel, it follows analogously that their images $\psi g \psi^{-1}(\bar{c})$ and $\psi g \psi^{-1}(\bar{d})$ are also contained in the same $j$-panel.

Let now $\bar{c} \sim_k \bar{d}$. Then by definition of $\psi$, the preimages $\psi^{-1}(\bar{c})$ and $\psi^{-1}(\bar{d})$ lie in the same residue of type $\{i, j\}$ in $\Delta$. Since adjacency is preserved by $g$ and again by the definition of $\psi$, we get that $\psi g \psi^{-1}(\bar{c}) \sim_k \psi g \psi^{-1}(\bar{d})$. So we conclude that $\psi g \psi^{-1}$ is an automorphism of $\tilde{\Delta}$.

Second, we need to prove that $\psi g \psi^{-1}$ is contained in the universal group $\tilde{\mathcal{U}}$. We consider now the local actions and show that they are contained in the local groups. Let $\tilde{\mathcal{P}}$ be an $i$-panel of $\tilde{\Delta}$, then we get

$$\sigma_{\tilde{\lambda}} \left( \psi g \psi^{-1}, \tilde{\mathcal{P}} \right) = \tilde{\lambda}_i \circ \psi g \psi^{-1} \circ (\tilde{\lambda}_i)^{-1} |_{\tilde{\mathcal{P}}}$$

$$= pr_1 \lambda_k g(\lambda_k)^{-1} |_{\psi^{-1}(\tilde{\mathcal{P}})}$$

$$= pr_1 \sigma_{\lambda} \left( g, Res_k(\psi^{-1}(\tilde{\mathcal{P}})) \right) \in \tilde{F}_i.$$

With the same arguments, it holds for a $j$-panel $\tilde{\mathcal{P}}$ that $\sigma_{\tilde{\lambda}}(\psi g \psi^{-1}, \tilde{\mathcal{P}}) \in \tilde{F}_j$.

Assume now that $\tilde{\mathcal{P}}$ is a $k$-panel, then we get

$$\sigma_{\tilde{\lambda}} \left( \psi g \psi^{-1}, \tilde{\mathcal{P}} \right) = \tilde{\lambda}_k \circ \psi g \psi^{-1} \circ (\tilde{\lambda}_k)^{-1} |_{\tilde{\mathcal{P}}}$$

$$= (\lambda_i g(\lambda_i)^{-1}, \lambda_j g(\lambda_j)^{-1}) |_{\psi^{-1}(\tilde{\mathcal{P}})} \in F_i \times F_j = \tilde{F}_k.$$

Hence we conclude that $\psi g \psi^{-1} \in \tilde{\mathcal{U}}$. 

Completely analogously, we get that $\psi^{-1} \tilde{g} \psi \in \mathcal{U}$ for every automorphism $\tilde{g} \in \tilde{\mathcal{U}}$. So we can define the following map

$$\varphi : \mathcal{U} \rightarrow \tilde{\mathcal{U}}, \ g \mapsto \psi g \psi^{-1}.$$

It follows immediately that $\varphi$ is a group homomorphism.

**Proposition 2.3.** The group homomorphism $\varphi : \mathcal{U} \rightarrow \tilde{\mathcal{U}}, g \mapsto \psi g \psi^{-1}$ is a bijection.

**Proof.** For every $\tilde{g} \in \tilde{\mathcal{U}}$, the automorphism $\psi^{-1} \tilde{g} \psi$ is contained in $\mathcal{U}$ and moreover $\varphi(\psi^{-1} \tilde{g} \psi) = \tilde{g}$. Hence, $\varphi$ is surjective.

Since $\psi$ is a bijection of chambers, it follows for any two automorphisms $g, h \in \mathcal{U}$ with $\varphi(g) = \varphi(h)$ that $g = h$ and hence $\varphi$ is injective. 

So we conclude that $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are isomorphic as abstract groups. Moreover, we want to prove that the topologies of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ also coincide and hence we need to prove first that $\varphi$ is continuous and then that it is open.

**Proposition 2.4.** The isomorphism $\varphi$ is continuous.

**Proof.** Let $W \subseteq \tilde{\mathcal{U}}$ be a neighbourhood of the identity. Hence, there exists a finite set of chambers $D \subseteq Ch(\tilde{\Delta})$ such that the pointwise stabiliser of $D$ in $\tilde{\mathcal{U}}$ is contained in $W$. Let $g \in Fix_\mathcal{U}(\psi^{-1}(D))$, then it follows that $\psi g \psi^{-1}(d) = d$ for every $d \in D$ and hence $\varphi(g) \in Fix_{\tilde{\mathcal{U}}}(D)$. 
It follows that the pointwise stabiliser of $\psi^{-1}(D)$ is contained in the preimage of the pointwise stabiliser of $D$ and since $\psi^{-1}(D)$ is finite, the pointwise stabiliser $Fix_U(\psi^{-1}(D))$ is an identity neighbourhood and contained in $\varphi^{-1}(W)$. Hence, $\varphi$ is continuous at the identity and thus everywhere. \hfill \Box

**Proposition 2.5.** The continuous isomorphism $\varphi$ is open and hence a homeomorphism.

**Proof.** Completely analogously to the proof of Proposition 2.4, it follows that $\varphi^{-1}: \tilde{U} \to U, \tilde{g} \mapsto \psi^{-1}\tilde{g}\psi$, is continuous. Hence, it follows that $\varphi$ is open and a homeomorphism. \hfill \Box

Since all local groups are assumed to act transitively, the universal groups $U$ and $\tilde{U}$ act chamber-transitively on $\Delta$ and $\tilde{\Delta}$, respectively. So all in all, we proved that $U$ acts via $\varphi$ chamber-transitively and with compact open stabilisers on the building $\tilde{\Delta}$. From this, we can conclude that the building that is associated to a universal group may not be the unique right-angled building on which $U$ acts chamber-transitively and with compact open stabilisers.

Note that the topological properties of the universal groups depend on the properties of the local groups. If at least one local group does not act freely, then by [2, Proposition 3.14] the universal group is not discrete. Hence, by assuming at the beginning that at least one local group does not act freely, we obtain in this way topologically isomorphic non-discrete locally compact universal groups.

This construction can be generalised to every thick semi-regular locally finite right-angled building of type $(W, I)$ if there exists $k \in I$ such that $I - \{k\}$ is spherical and $kj \neq jk$ for every $j \in I - \{k\}$. The vertices of the $k$-tree-wall tree correspond then to the $k$-panels and the residues of type $I - \{k\}$. Hence, the edges are still in bijection with the chambers and thus this can be used to construct the bijection of chambers. Moreover, by assumption, the residues of type $I - \{k\}$ are finite and hence the same arguments apply.

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