Second- and third-order moment budgets in a turbulent patch resulting from internal gravity wave breaking

S N Yakovenko$^{1,2}$

$^1$ Institute of Theoretical and Applied Mechanics SB RAS, Novosibirsk 630090, Russia
$^2$ School of Engineering Sciences, University of Southampton, Southampton SO17 1BJ, UK

E-mail: yakovenk@itam.nsc.ru

Abstract. To study second- and third-order statistical moments in a quasi-steady turbulent region observed in stratified flows over obstacles after the wave breaking, the direct simulation method developed for variable density fields is used. The data of runs at $\text{Re} = 4000$, $\text{Fr}_h = 0.6$ (based on obstacle height and inflow velocity) are used to obtain time dependence and spatial distributions of Reynolds stress tensor components, turbulent kinetic energy, scalar variance, triple correlations, and terms of budgets of their transport equations. The analysis shows that the global balance for both scalar variance and turbulent kinetic energy is between mean-shear production, advection and dissipation, whereas locally buoyant production and turbulent transport due to third-order moments are significant. For normal Reynolds stresses, the pressure-strain term is important too, providing the redistribution of stress components. The studies allow us not only to explore the turbulent patch by means of statistical moments, but also to examine closure assumptions for separate items and evaluate geophysically interesting quantities produced from the averaged data.

1. Introduction

Breaking internal waves generated topographically can lead at some flow parameters to turbulent patches observed in towing tank experiments (Castro & Snyder, 1993) and various environmental applications. Such a problem has been recently studied (Yakovenko et al., 2011) by means of well-resolved Navier–Stokes DNS methods using parallel multi-block architecture with the Boussinesq approximation and a sponge layer treatment to avoid wave reflection from upstream and downstream boundaries. Internal waves were produced by a 2D cosine-shaped bump of height $h$ in a flow with constant values of inflow density gradient and velocity $U$. With the uniform grid resolution $\Delta x = 0.02 h$ in DNS runs at $\text{Re} = Uh/\nu = 4000$ and $\text{Sc} = \nu/\kappa = 1$, we had $\Delta x/\eta < 3$ (where $\eta$ is the Kolmogorov micro-scale estimated from the dissipation rate). The wave breaking is captured if the Froude number $\text{Fr}_h$ (based on $h$) is quite low as in (Castro & Snyder, 1993). The results of our previous studies (Yakovenko et al., 2011) at $\text{Fr}_h = 0.6$ have been presented by instantaneous and averaged pathlines and density contours, temporal and spatial spectra, turbulent kinetic energy (TKE) balance terms obtained from the data averaged over the span in the mixed zone. It was shown that the arising turbulent patch is fully developed and quasi-steady during a relatively long period ($35 \leq t \leq 55$) which was also used for time averaging.
The objective of the present study is to further explore the quasi-steady turbulent patch in stably stratified flows past obstacles in terms of time dependence and spatial distributions for Reynolds stress tensor components, TKE, scalar (density) variance (SV), some triple correlations, and budgets of their transport equations. Such an analysis (as was performed e.g. by Mansour et al. (1988) for channel flow) is instructive to explore phenomena and energetics in the patch in terms of statistical moments, to validate turbulence models and evaluate geophysically interesting quantities.

2. Transport equations for statistical moments

Non-dimensional transport equations for the second-order moments (Reynolds stress \( \langle u'_i u'_j \rangle \), scalar variance \( \langle f' f' \rangle \)) are

\[
\frac{\partial \langle \Phi' \rangle}{\partial t} = -A + M + T + D + P + G + F - E, \quad \Phi' = \{u'_i u'_j, \ f' f'\}
\]

where the time rate is balanced by advection \( (A) \), turbulent transport \( (T) \) by third-order moments, molecular diffusion \( (M) \), turbulent transport by pressure fluctuations \( (D) \), buoyant production \( (G) \), mean-shear production \( (P) \), pressure-strain correlation \( (F) \), dissipation \( (E) \) which are (for \( Sc = 1 \)):

\[
A(\Phi') = \langle u_k \rangle \frac{\partial \langle \Phi' \rangle}{\partial x_k}, \quad M(\Phi') = Re^{-1} \frac{\partial^2 \langle \Phi' \rangle}{\partial x_k^2}, \quad T(\Phi') = -\frac{\partial \langle u'_i \Phi' \rangle}{\partial x_k},
\]

\[
D(u'_i u'_j) = -\frac{\partial \langle \hat{p} u'_i \rangle}{\partial x_j} - \frac{\partial \langle \hat{p} u'_j \rangle}{\partial x_i},
\]

\[
P(u'_i u'_j) = -\langle u'_i u'_k \rangle \frac{\partial \langle u_j \rangle}{\partial x_k} - \langle u'_j u'_k \rangle \frac{\partial \langle u_i \rangle}{\partial x_k},
\]

\[
G(u'_i u'_j) = -\beta_i \langle u'_j f' \rangle - \beta_j \langle u'_i f' \rangle,
\]

\[
F(u'_i u'_j) = \left\langle \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right\rangle, \quad E(u'_i u'_j) = \frac{2}{Re} \left\langle \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_k} \right\rangle,
\]

\[
P(f' f') = -2 \langle u'_i f' \rangle \frac{\partial \langle f \rangle}{\partial x_i}, \quad E(f' f') = \frac{2}{Re} \left\langle \frac{\partial f'}{\partial x_i} \frac{\partial f'}{\partial x_j} \right\rangle,
\]

\[
F(f' f') = G(f' f') = D(f' f') = 0.
\]

where \( \beta_i = \delta_i^2 / F_b^2 \). Note, pressure \( \hat{p} \) and scalar \( f = \hat{\rho} \) represent normalized deviations from the (hydrostatic) reference values of pressure and density, respectively, as used by Yakovenko et al. (2011).

The equation for triple velocity correlations has the similar form as the Reynolds-stress one, but with three production terms, namely buoyant production \( (G) \), mean-shear production \( (P) \), and second-moment production \( (S) \), which are:

\[
G(k'_j) = -\beta_i \langle u'_j u'_j f' \rangle - \beta_j \langle k' f' \rangle,
\]

\[
P(k'_j) = -\langle k'_j m' \rangle \frac{\partial \langle u_j \rangle}{\partial x_m} - \langle u'_j u'_j m' \rangle \frac{\partial \langle u_i \rangle}{\partial x_m},
\]

\[
S(k'_j) = -\langle u'_j u'_j m' \rangle \frac{\partial k}{\partial x_m} - \langle u'_j m' \rangle \frac{\partial \langle u'_i u'_j \rangle}{\partial x_m}.
\]

(Only production terms for budget of third moments which are responsible for the TKE turbulent transport are given above since some algebraic models for \( \langle k' u'_j \rangle \) are discussed below.)
3. Results of computations and budgets of transport equations

Figure 1(a) presents a flow view as a snapshot on the spanwise centre-plane ($y = 0$) taken during the quasi-steady period of the first elevated turbulent patch arising from internal wave breaking (the patch area is shown by orange rectangle). The corresponding vorticity and velocity contours within the mixed region are given by Figures 1(b-d). Note that due to homogeneity in spanwise direction (Yakovenko et al., 2011) some velocity and vorticity components become negligible after averaging along $y$, i.e. $\langle v \rangle = \langle \omega_x \rangle = \langle \omega_z \rangle = 0$.

Figure 1. Instantaneous pathlines around obstacle at $t = 37.5$ (a); contours of mean vorticity $\langle \omega_y \rangle$ (b), mean velocity components, $\langle u \rangle$ (c) and $\langle v \rangle$ (d), averaged over all 512 nodes along $y$ and over $(9 \times 9)$ cells locally in $(x, z)$ plane, then during the quasi-steady period ($35 \leq t \leq 55$).

The mean velocity is close to zero inside the considered patch (Figure 1) contrary to turbulence characteristics discussed below, and the recirculation structure formed during the laminar development of wave breaking is destroyed completely by turbulent activity. The mean vorticity has the local positive peak at $z \approx 3$ just in the area of peaks for TKE and dissipation.

The second-moment budgets (Figures 2–4) suggest the insignificant time rates and confirm the steady behavior of the turbulent patch. Molecular diffusion $M$ is everywhere negligible (within 3% of turbulent transport), so not shown separately. Except for the residual determination of the time rate, all items of the transport equations were directly calculated from the computational data. The residual time-rate value denoted by $\langle \Phi' \rangle_i$ includes implicitly numerical errors (of the computed balance) which in fact were very small (Yakovenko et al., 2011).

The buoyant production $G_{ww}$ appears first at $t \sim 22$, giving the fast growth of $\langle w'w' \rangle$, but then becomes globally small and positive. The pressure-fluctuation turbulent transport has a negative contribution of the same order as buoyancy for normal Reynolds stresses and positive
Figure 2. Reynolds-stress equation budgets: (a) time dependence of balance terms averaged along $y$ and locally in $(x, z)$-plane, then within the region $(1.41 < x < 4.84$ and $1.41 < z < 3.59)$ containing the turbulent patch; (b) vertical profiles at $x = 2.5$ averaged as in Figure 1(b-d).
plus molecular diffusion, $P_{\theta\theta}$. Figure 1(b-d): mean-shear production, $P_{ff}$, advection, $-A_{ff}$, third-moment turbulent transport plus molecular diffusion, $T_{ff} + M_{ff}$, dissipation, $-E_{ff}$, time rate plus numerical residual, $(f'f')_t$. Figure 3. SV equation budget: (a) time dependence; (b) vertical profiles, $x = 2.5$. Figure 4. Contours of scalar variance $(f'f')$ and balance terms (in its equation) averaged as in Figure 1(b-d): mean-shear production, $P_{ff}$, advection, $-A_{ff}$, third-moment turbulent transport plus molecular diffusion, $T_{ff} + M_{ff}$, dissipation, $-E_{ff}$, time rate plus numerical residual, $(f'f')_t$. 

13th European Turbulence Conference (ETC13) 
Journal of Physics: Conference Series 318 (2011) 072022 doi:10.1088/1742-6596/318/7/072022
one for shear stress, whereas the triple-product turbulent transport stays near zero (Figure 2a). Globally, dissipation and advection roughly balance mean-shear production not only for TKE (Yakovenko et al., 2011), but also for SV budgets. Locally (Figures 2–4), the spatial distributions of buoyant production and/or turbulent transport have peaks compatible with those of $F$, $P$ and $A$. For shear Reynolds stress, however, the buoyant production $G_{uw}$ is globally the largest contribution (because the mean-shear production $P_{uw}$ changes sign inside the patch).

![Figure 5](image-url) Contours of pressure-strain correlations averaged as in Figure 1(b-d).

For Reynolds stress components, the pressure-strain term $F$ (Figures 2, 5) plays a significant role, redistributing the normal stresses between each other ($F_{uu} + F_{vv} + F_{ww} = 0$), mostly from the largest value, $\langle w' w' \rangle$ (where $F_{ww}$ is negative and consumes a part of production), to the smallest one, $\langle v' v' \rangle$ (where $F_{vv}$ is positive and stands as the only source since $P_{vw} = G_{vw} = 0$). Anisotropy of normal Reynolds stresses is shown in Figure 6. Thus, behavior of $F$ lays within the linear return-to-isotropy assumption (Gibson & Launder, 1978) as approximately confirmed by Figure 7, whereas the dissipation tensor (Figure 8) roughly follows the local-isotropy hypothesis, $E_{ij} = E(u'_i u'_j) = (2/3) \delta_{ij} E_k$, where $E_k = E(u'_i u'_j)/2$ is the TKE dissipation. Although the dissipation tensor component $E_{ww}$ for the leading Reynolds-stress component $\langle w' w' \rangle$ is slightly larger than $E_{uu}$ and $E_{vv}$, and pressure-strain terms are somewhat under-estimated. The latter can be corrected, e.g. by taking $C_2 = C_3 \simeq 0.75$ instead of 0.6 in the production-isotropization item or by appropriate non-linear modifications for the $F(\langle u'_i u'_j \rangle)$ model. The distributions of $F$ and $E$ terms in the equation for shear Reynolds stress $\langle u' w' \rangle$ confirm the above-mentioned models: the pressure-strain term $F_{ww}$ is mostly opposite to the stress value and its full production, $P_{uw} + G_{uw}$ (Figure 9), and proportional to it, whereas the dissipation item $E_{uw}$ is insignificant in comparison with $E_k$.

According to Figures 2–4, 8 and 9, the mixing efficiency ratio $\Gamma = G_k/E_k$ (where the buoyant production of TKE is $G_k = G_{ww}/2$) varies significantly with time and space, but its global value is close to 0.2 as in oceanic applications (Pham et al., 2009), whereas the scalar mixing efficiency ($\Gamma_\rho = E_{ffj}/E_k \sim 0.13$) is smaller and more steady. On the other hand, for a stably stratified
Contours of pressure-strain correlations approximated by linear algebraic expression

\[
F(u'u_i') = -C_1[(u'_i u'_j') - (2/3)\delta_{i,j}] / \tau - C_2[P(u'_i u'_j') - (2/3)P_k] - C_3[G(u'_i u'_j') - (2/3)G_k], \quad \tau = k/E_k,
\]

\[
P_k = P(u'_i u'_j')/2 = (P_{uu} + P_{ww})/2,
\]

\[
G_k = G(u'_i u'_j')/2 = G_{ww}/2,
\]

\[
C_1 = 1.8, \quad C_2 = C_3 = 0.6.
\]

**Figure 6.** Contours of Reynolds stress tensor components averaged as in Figure 1(b-d).

**Figure 7.** Contours of pressure-strain correlations approximated by linear algebraic expression
(Figure 10), the popular tensor-invariant model (Hanjalic & Launder, 1972), deduced from the $k$ both
the (non-dimensional) length and time scales of large turbulent eddies at
form of the pressure-containing term $E$
approximations are
shear layer both values are larger, $\Gamma = \Gamma_\rho = 0.44$ (Pham et al., 2009). The smaller values of mixing efficiencies in the present studies may be related with weak values of mean velocity and
density shears in the lee-wave turbulent patch as is evident from Figures 1(c,d).

The TKE dissipation (Figure 8) follows roughly the $k$ behavior (Figure 10), so acceptable
approximations are $E_k = k^{3/2}/L^* \text{ or } E_k = k/\tau^*$ where $L^* = 1.45$ and $\tau^* = 6.4$ are, respectively, the (non-dimensional) length and time scales of large turbulent eddies at $x \sim 2.2$, $z \sim 3.0$ where both $k$ and $E_k$ have maximum values. For triple-product turbulent transport $T_k = T(u'u'_j)/2$ (Figure 10), the popular tensor-invariant model (Hanjalic & Launder, 1972), deduced from the third-moment equation taking the turbulent production term $S(k'u'_j)$, assuming a relaxation form of the pressure-containing term $F(k'u'_j)$ and neglecting other items in this equation, follows evidently the distribution of $\partial S(k'u'_j)/\partial x_j$ and yields an unacceptable result, as well as the simpler model of Daly & Harlow (1970) with isotropic effective diffusion (Figure 11). Since the full-production gradient $\partial[P(k'u'_j) + S(k'u'_j) + G(k'u'_j)]/\partial x_j$ represents correctly the $T_k$ behavior (Figures 10, 11c), an adequate algebraic expression for triple-product turbulent transport of
scalar (mass) flux and different triple correlations using the available DNS data which allows us
equations. The similar transport-equation-budget analysis can be also performed for turbulent
variance, TKE and triple correlations responsible of its transport) and budgets of their transport
surface-placed obstacle (Yakovenko et al., 2011), the arising turbulent patch has been studied
Based on the direct numerical simulation results for the stably stratified flow overcoming the

4. Conclusions

TKE should thus take into account all third-moment production terms with corresponding
contributions from $F(k'u'_j)$. Those from dissipation can indeed be skipped due to its relative
insignificance in comparison with full production, i.e. $|E(k'u'_j)| \ll |P(k'u'_j) + S(k'u'_j) + G(k'u'_j)|$.

Figure 9. Contours of Reynolds-stress production terms averaged as in Figure 1(b-d).

Figure 10. Contours of TKE and its turbulent transport, $T_k$, averaged as in Figure 1(b-d).

Based on the direct numerical simulation results for the stably stratified flow overcoming the
surface-placed obstacle (Yakovenko et al., 2011), the arising turbulent patch has been studied
by means of the analysis of statistical moments (Reynolds-stress tensor components, scalar
variance, TKE and triple correlations responsible of its transport) and budgets of their transport
equations. The similar transport-equation-budget analysis can be also performed for turbulent
scalar (mass) flux and different triple correlations using the available DNS data which allows us
to validate appropriate models for separate items.
Figure 11. Comparison of contours for $T_k$ approximations and leading terms of the $(k'u'_j)$ equation: (a) gradient-transport model (Hanjalic & Launder, 1972), $T_k = -\partial[0.11\tau S(k'u'_j)]/\partial x_j$; (b) isotropic-diffusion model (Daly & Harlow, 1970), $T_k = -\partial[0.22\tau (u'_i u'_j)]\partial k/\partial x_i$; gradients of turbulent (c) and total (d) third-moment productions averaged as in Figure 1(b-d).

Acknowledgments

The author is thankful to Prof. I. P. Castro and Dr. T. G. Thomas for support in the studies and useful discussions.

References

CASTRO, I. P. & SNYDER, W. H. 1993 Experiments on wave breaking in stratified flow over obstacles. J. Fluid Mech. 255, 195–211.

Daly, B. J. & Harlow, F. H. 1970 Transport equations of turbulence. Phys. Fluids 13, 2634-2649.

Hanjalic, K. & Launder, B. E. 1972 A Reynolds stress model of turbulence and its application to thin shear flows. J. Fluid Mech. 52, 609-638.

Gibson, M. M. & Launder, B. E. 1978 Ground effects on pressure fluctuations in the atmospheric boundary layer. J. Fluid Mech. 86, 491-511.

Mansour, N. N., Kim, J. & Moin, P. 1988 Reynolds-stress and dissipation-rate budgets in a turbulent channel flow. J. Fluid Mech. 194, 15-44.

Pham, H. T., Sarkar, S. & Brucker, K. A. 2009 Dynamics of a stratified shear layer above a region of uniform stratification. J. Fluid Mech. 630, 191-223.

Yakovenko, S. N., Thomas, T. G. & Castro, I. P. 2011 A turbulent patch arising from a breaking internal wave. J. Fluid Mech. 677, 103-133.