Research Article

A New General Decay Rate of Wave Equation with Memory-Type Boundary Control

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Of interest is a wave equation with memory-type boundary oscillations, in which the forced oscillations of the rod is given by a memory term at the boundary. We establish a new general decay rate to the system. And it possesses the character of damped oscillations and tends to a finite value for a large time. By assuming the resolvent kernel that is more general than those in previous papers, we establish a more general energy decay result. Hence the result improves earlier results in the literature.

1. Introduction

It is well-known that if we add a damping to a system, the amplitude of the oscillations can be reduced very fast. The memory term can be as a damping (viscoelastic damping) which is weaker than frictional damping. For viscoelastic materials, Boltzmann theory gives us that the stress-strain viscoelastic law depending on a relaxation measure, see Prüss [1] and Eden et al. [2]. Based on the Boltzmann principle, the viscoelastic stress-strain relations can be generally given by a convolution term, which can be regarded as a lower order perturbation and can also be regarded as a kind of memory effect, for instance, \( g \ast u \). And we call the function \( g(t) \) memory kernel. One can find a detail derivation on some systems with memory in [3].

To motivate our work, we start with some known results on wave equation with memory-type oscillations. The general wave equation with viscoelastic term in the internal feedback

\[
\ddot{u} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = F(u). \tag{1}
\]

Messaoudi and Messaoudi [4, 5] studied \( F(u) = 0 \) and \( F(u) = |u|^\rho u \), by introducing the assumption \( g'(t) \leq -\xi(t)g(t) \), and obtained the energy decays exponentially (polynomially) as \( g \) decays exponentially (polynomially), respectively.

Lasiecka et al. [6] considered the general assumption on \( g; g'(t) \leq -H(g(t)) \) to establish general decay of energy. Here \( H \), which was introduced by Alabau-Boussouira and Cannarsa [7], is strictly convex and increasing function. Cavalcanti et al. [8, 9], Lasiecka and Wang [10], Mustafa and Messaoudi [11], and Xiao and Liang [12] also used this
assumption to obtain some general decay rates of corresponding models. In recent papers [13–15], the authors investigated three classes of viscoelastic wave equation as in [4, 5] and established optimal and explicit decay results of energy by adopting the assumption on $g$: $g' (t) \leq -\xi (t) H (g(t))$.

In this paper, we considered the following wave equation with boundary oscillations of memory type:

$$
\begin{align*}
    & u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\
    & u = 0, \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \\
    & u + \int_0^t g(t-s) \frac{\partial u}{\partial \nu} (s) ds = 0, \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\
    & u(x, 0) = u_0 (x), \\
    & u_t (x, 0) = u_1 (x), \quad x \in \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\Gamma$, $\Gamma = \Gamma_0 \cup \Gamma_1$, and $\Gamma_0$ and $\Gamma_1$ are closed and disjoint with measure $(\Gamma_0') > 0$. $\nu$ is the unit outward normal to $\Gamma$.

For wave equation with memory-type boundary oscillations, it can be regarded as a wave equation with viscoelastic damping at the boundary. Santos [16] considered a one-dimensional wave equation with memory conditions at the boundary, respectively. He proved that the energy of solutions decays exponentially (polynomially) as $k$ and $k'$ decay exponentially (polynomially). Here $k$ is the resolvent kernel of $(-g'/g(0))$. Santos et al. [17] extended the results in [16] to an n-dimensional wave equation of Kirchhoff type with memory-type boundary. They proved the global existence of solutions and obtained that the energy of solution decays uniformly with the same rate of decay $k$ under the same conditions on $k$ and $k'$, which improves the results in [18] by Park et al. Santos and Junior [19] obtained a similar result for plate equation with memory-type boundary. We also mention the work of Cavalcanti et al. [20], where the authors showed the global existence and the uniform decay of solutions to a semilinear wave equation with memory-type boundary condition and a nonlinear boundary source. Messaoudi and Soufyane [21] considered a general assumption on $k'$: $k' \geq -\xi (t) k' (t)$ and established a general decay result. Wu [22] used this assumption to study a wave Kirchhoff-type wave equation with a boundary control of memory type. For nonlinear wave equations with memory-type boundary condition, we refer to Cavalcanti and Guesmia [23], Feng [24], Feng et al. [25–27], Muñoz Rivera and Andrade [28], and Zhang [29].

Concerning the system (2), Mustafa [30], by assuming the function $k$: $k' (t) \geq H (-k' (t))$, where $k$ is the resolvent kernel of $(-g'/g(0))$, established a general decay of solutions of the form

$$
E (t) \leq k_3 H_1^{-1} (k_1 t + k_2), \quad \forall t \geq 0.
$$

Here

$$
H_1 (t) = \int_1^t \frac{1}{s H_0 (\xi_0 s)} ds,
$$

$H_0 (t) = H (D (t))$, and $D$ is a positive $C^1$ function with $D (0) = 0$, and $H_0$ is strictly increasing and strictly convex $C^2$ function on $(0, r]$. In particular, for $H(t) = t^p$, i.e., $k' \geq c (-k')^p$, the author proved the energy decay holds for $1 \leq p < (3/2)$. Whether can the range be extended to a more larger range? In this paper, we give a positive answer to study problem (2) and extend the result to get a more general decay rate. In particular, we obtain that the energy result holds for $H(t) = t^p$ with the full admissible range $1 \leq p < 2$. More exactly, by assuming the relaxation function $k$ with minimal conditions on $L^1 (0, \infty)$, i.e., $k' (t) \geq \eta (t) H (-k' (t))$, where $H$ is linear or strictly increasing and strictly convex functions of class $C^2 (\mathbb{R}^+)$, we establish an optimal explicit and general energy decay result. In particular, the energy result holds for $H(t) = t^p$ with the range $p \in [1, 2)$ instead of $p \in [1, (3/2)]$ in [30]. Hence our results extend and improve the stability results in [30] and also in [16–18, 21]. We mainly adopt the idea of [14, 15, 31] and some properties of convex function developed in [7, 32].

The remaining of the paper is organized as follows: in Section 2, we propose some preliminaries. In Section 3, main results are given. Section 4 is devoted to proving the general decay result.

## 2. Preliminaries

Taking the derivative of (2) with respect to $t$, we shall see that

$$
\frac{\partial u}{\partial t} = -\frac{1}{g(0)} \left[ u_t + g' \ast \frac{\partial u}{\partial \nu} \right].
$$

We denote the resolvent kernel of $(-g'/g(0))$ by $k$ satisfying for $t \geq 0$:

$$
k (t) + \frac{1}{g(0)} (g' \ast k) (t) = -\frac{1}{g(0)} g' (t).
$$

Using Volterra’s inverse operator and taking $\alpha = (1/g(0))$, we have

$$
\frac{\partial u}{\partial \nu} = -\alpha [u_t + k_2 \ast u_t].
$$

Assume $u_0 = 0$ on $\Gamma_1$ in this paper, we get

$$
\frac{\partial u}{\partial \nu} = -\alpha [u_t + k (0) + k \ast u_t], \quad \text{on } \Gamma_1 \times \mathbb{R}^+.
$$

In the following, we use boundary conditions (8) instead of (2).

As in [30], we consider the following assumption:
(A1) There exists a fixed point $x_0 \in \mathbb{R}^2$ and some constant $\delta_0 > 0$ such that for $m(x) = x - x_0$,
\[ \Gamma_0 = \{ x \in \Gamma : m(x) \cdot v(x) \leq 0 \}, \]
\[ \Gamma_1 = \{ x \in \Gamma : m(x) \cdot v(x) \geq \delta_0 \}. \tag{9} \]

For the kernel $k$, we assume
(A2) $k: \mathbb{R}^+ \to \mathbb{R}^+$ is nonincreasing and twice differentiable function satisfying for any $t \geq 0$,
\[ k(0) > 0, \]
\[ k'(t) \leq 0. \tag{10} \]

(A3) There exist a $C^1$ function $H: \mathbb{R}^+ \to \mathbb{R}^+$, with $H(0) = H'(0) = 0$, which is linear or is strictly increasing and strictly convex function of class $C^2(\mathbb{R}^+)$ on $(0, r]$, $r \geq k'(0)$ such that
\[ k''(t) \geq \eta(t)H(-k'(t)), \quad \forall t \geq 0, \tag{11} \]
where $\eta(t)$ is $C^1$ nonincreasing continuous function.

Remark 2.1. If assuming further $\lim_{t \to \infty} k'(t) = 0$, then $\lim_{t \to \infty} (-k'(t)) = 0$. Since $\lim_{t \to \infty} k'(t) = 0$ and $(-k'(t))$ is nonincreasing and non-negative, we can get
\[ \lim_{t \to \infty} (-k'(t)) = 0. \tag{12} \]

Then for some $t_1 \geq 0$ large,
\[ -k'(t_1) = r \implies -k'(t) \leq r, \quad \forall t \geq t_1. \tag{13} \]

Noting that $(-k')$ is nonincreasing, $-k'(0) > 0$, and $-k'(t_1) > 0$ for any $t \in [0, t_1]$, and for any $t \in [0, t_1]$,
\[ 0 < -k'(t_1) \leq -k'(t) \leq -k'(0), \tag{14} \]
\[ 0 < \eta(t_1) \leq \eta(t) \leq \eta(0). \]

Therefore we obtain that there exist two positive constants $a$ and $b$ such that for any $t \in [0, t_1]$,
\[ a \leq \eta(t)H(-k'(t)) \leq b. \tag{15} \]

Then for any $t \in [0, t_1]$,
\[ k''(t) \geq \eta(t)H(-k'(t)) \geq \frac{a}{k'(0)}k'(0) \geq \frac{a}{k'(0)}k'(t). \tag{16} \]

This implies that there exists a constant $d > 0$ such that for any $t \in [0, t_1]$,
\[ k''(t) \geq -dk'(t). \tag{17} \]

The proof is done.

3. Main Results

The well-posedness result is given in [30] proved by using the Faedo–Galerkin method as in [17].

**Theorem 1.** Assume that (A1) and (A2) hold. Let $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$, and then problem (2) admits a unique solution $u$ satisfying
\[ u \in L^\infty(0, T; H^1(\Omega) \cap V) \cap W^{1,\infty}(0, T; V) \cap W^{2,\infty}(0, T; L^2(\Omega)), \tag{18} \]
where $V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0 \}$.

The total energy of the system is defined by
\[ E(t) = \frac{1}{2} \| u \|_V^2 + \frac{1}{2} \| \nabla u \|_V^2 + \frac{\alpha}{2} \left[ k(t) \| u \|_{H^1}^2 - \int_{\Gamma_1} k''u \, d\Gamma \right], \tag{19} \]
where
\[ (k \ast u)(t) = \int_0^t k(t-s) |u(t)-u(s)|^2 \, ds. \tag{20} \]

We can get the following stability result.

**Theorem 2.** Assume $k$ satisfies (A1)–(A3) and further $\lim_{t \to \infty} k(t) = 0$. Then there exist $\lambda_1, \lambda_2 > 0$ such that
\[ \mathcal{E}(t) \leq \lambda_2 H_4^{-1} \int_{K^{-1}(r)}^r \eta(s) \, ds, \quad \forall t > K^{-1}(r), \tag{21} \]
where
\[ H_4(t) = \int_t^\infty \frac{1}{s H_0(s)} \, ds, \]
and $K(t) = -k'(t)$. In particular, if $H(t) = t^p$, then for any $t > 0$,
\[ \mathcal{E}(t) \leq \begin{cases} c_1 e^{-c_2 \int_0^t \eta(s) \, ds}, & \text{if } p = 1, \\ c_3 \left( 1 + \int_0^t \eta(s) \, ds \right)^{-1/(p-1)}, & \text{if } 1 < p < 2, \end{cases} \tag{23} \]
where $c_1, c_3$, and $c_2 \leq 1$ are positive constants.

Remark 3.1. From (23), the energy result holds for $H(t) = t^p$ with the full admissible range $p \in (1, 2)$ instead of $p \in [1, (3/2)]$. If the viscoelastic term is as internal feedback, Lasiecka and Wang [10] provided the proof for optimal decay rates of second-order systems in the full admissible range [1, 2].

At last, we show two examples to illustrate explicit formulas for the decay rates of the energy, which can be found in the studies of Mustafa and Mustafa [14, 15].

**Example 1.** Take $k'(t) = -e^{-rt}$ with $0 < q < 1$, we get
\[ k''(t) = H(-k'(t)), \quad H(t) = \frac{1}{(t+1)^{(1/|q|-1)}}. \]

Since
\[ H'(t) = \frac{(1 - q) + q \ln(1/t)}{\ln(1/t)^{(1/q)}} \]
\[ H''(t) = \frac{(1 - q)[\ln(1/t) + \ln(1/q)]}{\ln(1/t)^{(1/(1+q))}} \]  

we can deduce that the function \( H \) satisfies (A3) on \((0, r]\) for any \( 0 < r < 1 \). Then,
\[ \mathcal{F}(t) \leq c_1 e^{-c_1 r^2}. \]  

**Example 2.** Consider \( k'(t) = (-1/((t + e)[\ln(t + e)]^p)) \) with \( p > 1 \), we get \( k''(t) = ((\ln(t + e) + p)/(e + e^2[\ln(t + e)]^{p+1})) \). Clearly,
\[ k''(t) = \frac{[\ln(t + e) + p]}{(t + e)[\ln(t + e)]} [-k'(t)]. \]  

By part 1 of (23), we get
\[ \mathcal{F}(t) \leq c_1 \exp \left( -c_2 \int_0^t \frac{\ln(t + e) + p}{(t + e)[\ln(t + e)]} ds \right) \]
\[ = \frac{c_1}{[(t + e)[\ln(t + e)]^p]^{0.5}}. \]  

As \( c_2 \leq 1 \), this is slower rate than \([-k'(t)] \). In addition,
\[ k''(t) = \frac{\ln(t + e) + p}{(t + e)^{1-(1/p)}} [-k'(t)]. \]  

From part 2 of (23), we infer that for large \( t \)
\[ \mathcal{F}(t) \leq c_3 \left( 1 + \int_0^t \frac{\ln(t + e) + p}{(t + e)^{1-(1/p)}} ds \right)^p \leq \frac{c_3}{(t + e)[\ln(t + e)]^p} \]  

which is the same rate as \([-k'(t)] \).

## 4. Proof of Main Result

To prove Theorem 2, we need the following lemmas.

### 4.1. Technical Lemmas

**Lemma 1.** The total energy functional \( E(t) \) satisfies for any \( t \geq 0 \),
\[ E'(t) \leq -\frac{\alpha}{2} \left( \|u_0\|^2_{L_2} + \int_{\Gamma_1} k'' u d\Gamma \right) \leq 0. \]  

**Proof.** See [30].

As in [31], for \( 0 < \delta < 1 \), we introduce
\[ C_\delta = \int_0^\infty \frac{[k'(s)]^2}{k''(s) - \delta k'(s)} ds, \]  

**Lemma 2.** Define the functional \( \Phi(t) \) by
\[ \Phi(t) = \int_\Omega [2m \cdot \nabla u + (n-1)u] u dx. \]  

Then we can get for any \( t \geq t_1 \),
\[ \Phi'(t) \leq -\|u_0\|^2_{L_2} - \frac{1}{2} \|\nabla u\|^2_{L_2} + c \|u_0\|^2_{L_2} + C_\delta \int_{\Gamma_1} h u d\Gamma. \]  

**Proof.** From the same arguments as in the study of Mustafa [30], we can obtain
\[ \Phi'(t) \leq -\|u_0\|^2_{L_2} - \|\nabla u\|^2_{L_2} - \delta_0 \|\nabla u\|^2_{L_2} + \int_{\Gamma_1} (m \cdot \nabla u) u |u|^2 d\Gamma \]
\[ + \int_{\Gamma_1} (2m \cdot \nabla u) \frac{\partial u}{\partial v} d\Gamma + (n-1) \int_{\Gamma_1} \frac{\partial u}{\partial v} d\Gamma. \]  

It follows from Young’s inequality that for any \( \varepsilon > 0 \),
\[ \int_{\Gamma_1} (2m \cdot \nabla u) \frac{\partial u}{\partial v} d\Gamma + (n-1) \int_{\Gamma_1} \frac{\partial u}{\partial v} d\Gamma \]
\[ \leq \delta_0 \|\nabla u\|^2_{L_2} + \varepsilon \|u\|^2_{L_2} + c \frac{\|u\|^2_{L_2}}{\|\nabla u\|_{L_2}^2}. \]  

Recalling \( k' \cdot u = (-k' \odot u) + [k(t) - k(0)]u \), where \( k \odot u = \int_0^t k(t-s)(u(t) - u(s)) ds \); then we have from (8),
\[ \frac{\partial u}{\partial v}(t) = -\alpha [u_0(t) + k(t)u(t) + (-k' \odot u)(t)]. \]  

By using Young’s inequality, we obtain
\[
\left\| \frac{\partial u}{\partial y} (t) \right\|^2_{\Gamma_1} \leq 4\alpha^2 \left[ \| u_t \|^2_{\Gamma_1} + k^2 (t) \| u \|^2_{\Gamma_1} + \int_{\Gamma_1} (-k' \circ u)^2 \, dl' \right].
\]

(37)

Hölder’s inequality implies

\[
(-k' \circ u)^2 = \left( \int_0^t (-k' (t - s)) (u(t) - u(s)) ds \right)^2
\]

\[
= \left( \int_0^t \frac{-k' (t - s)}{\sqrt{k'' (t - s) - \delta k' (t - s)}} \sqrt{k'' (t - s) - \delta k' (t - s)} (u(t) - u(s)) ds \right)^2
\]

\[
\leq \left( \int_0^t \frac{[k' (s)]^2}{\sqrt{k'' (s) - \delta k' (s)}} ds \right)^{\frac{1}{2}} \left( \int_0^t (k'' (t - s) - \delta k' (t - s)) (u(t) - u(s))^2 ds \right)^{\frac{1}{2}}
\]

\[
\leq C_d (h \circ u),
\]

which, together with (37), gives us that

\[
\left\| \frac{\partial u}{\partial y} (t) \right\|^2_{\Gamma_1} \leq 4\alpha^2 \left[ \| u_t \|^2_{\Gamma_1} + k^2 (t) \| u \|^2_{\Gamma_1} + C_d \int_{\Gamma_1} (h \circ u) \, dl' \right].
\]

(39)

Inserting (39) into (35), we obtain for any \( \varepsilon > 0, \)

\[
\int_{\Gamma_1} (2m \cdot \nabla u) \frac{\partial u}{\partial y} \, dl' + (n - 1) \int_{\Gamma_1} u \frac{\partial u}{\partial y} \, dl'
\]

\[
\leq \delta_0 \| \nabla u \|^2_{\Gamma_1} + (\varepsilon + 4\alpha^2 k^2 (t)) \| u \|^2_{\Gamma_1}
\]

\[
+ 4\alpha^2 \| u_t \|^2_{\Gamma_1} + C_d \int_{\Gamma_1} (h \circ u) \, dl'.
\]

(40)

Noting that

\[
\| u \|^2_{\Gamma_1} \leq \| \nabla u \|^2_{\Gamma_1},
\]

using \( \lim_{t \to -\infty} k(t) = 0 \) and taking \( \varepsilon > 0 \) small enough, we can get (33) from (34) and (40). The proof is done. \( \square \)

To get the optimal energy decay, we need the following estimate.

\[
2 \int_{\Gamma_1} u(t) \int_0^t k' (t - s) \left[ u(s) - u(t) \sqrt{a^2 + b^2} \right] ds \, dl'
\]

\[
\leq 2k(0) \int_{\Gamma_1} u^2(t) \, dl' + \frac{1}{2k(0)} \int_{\Gamma_1} \left( \int_0^t \sqrt{-k''(t - s)} \sqrt{-k'(t - s)} [u(s) - u(t)] ds \right)^2 \, dl'
\]

(45)

In view of Young’s and Hölder’s inequalities, we obtain

\[
\Psi(t) = \int_0^t k(t - s) \| u(s) \|^2_{\Gamma_1} ds,
\]

which satisfies for any \( t > 0, \)

\[
\Psi'(t) \leq \frac{1}{2} \int_{\Gamma_1} \left( k' u \, dl' + 3k(0) \| u(t) \|^2_{\Gamma_1} \right).
\]

(43)

**Lemma 3.** The functional \( \Psi(t) \) is defined by

\[
\Psi(t) = \int_0^t k(t - s) \| u(s) \|^2_{\Gamma_1} ds,
\]

(42)

**Proof.** Differentiating \( \Psi(t) \) with respect to \( t, \) we get

\[
\Psi'(t) = k_2 (0) \| u(t) \|^2_{\Gamma_1} + \int_0^t k_2 (t - s) \| u(s) \|^2_{\Gamma_1} ds
\]

\[
= \int_0^t k'(t - s) \int_{\Gamma_1} [u(s) - u(t)]^2 ds + k(t) \| u(t) \|^2_{\Gamma_1}
\]

\[
+ 2 \int_{\Gamma_1} u(t) \int_0^t k' (t - s) [u(s) - u(t)] ds \, dl'.
\]

(44)
Then we can get (43) following from the fact
\[ k(t) \leq k(0), \]
\[ \int_0^t k'(s) ds \leq \frac{1}{2k(0)} \geq \frac{1}{2} \tag{48} \]
\[ \int_0^t k'(s) ds \leq \frac{1}{2k(0)} \geq \frac{1}{2} \]
The proof is complete. \[ \Box \]

4.2. Proof of Theorem 2

Proof. Define the functional \( L(t) \) by
\[ L(t) := N \mathcal{E}(t) + \Phi(t), \tag{49} \]
where \( N > 0 \) is a constant that will be taken later. Clearly we can take \( N \) a large value to get
\[ L(t) \sim \mathcal{E}(t). \tag{50} \]
Recalling \( k'' = \delta k' + h \), combining (30) and (33), we conclude that for any \( t > t_1 \),
\[ L'(t) \leq -\left( \frac{\alpha}{2} N - c \right) \| u_t \|_{L_1}^2 - \frac{1}{2} \| u \|^2 - \frac{1}{2} \| \nabla u \|^2 \]
\leq -\frac{\alpha}{2} N \int_{\Gamma} k'' u d\Gamma + \left( \frac{\alpha}{2} N - c \right) \int_{\Gamma} h u d\Gamma. \tag{51} \]
Noting \(-k' > 0 \) and \( k'' > 0 \), for each \( s \in [0, \infty) \), we shall see below,
\[ \lim_{\delta \to 0} \frac{\delta[k'(s)]^2}{[k'(s) - \delta k'(s)]} = 0, \tag{52} \]
\[ \frac{\delta[k'(s)]^2}{[k'(s) - \delta k'(s)]} < -k'(s). \]
It follows from Lebesgue dominated convergence theorem that
\[ \lim_{\delta \to 0} \delta C_\delta = \lim_{\delta \to 0} \int_0^\infty \frac{\delta[k'(s)]^2}{[k'(s) - \delta k'(s)]} ds = 0. \tag{53} \]
Therefore there exist \( 0 < \gamma < 1 \) such that if \( \delta < \gamma \), then
\[ \delta C_\delta < \frac{1}{4c}. \tag{54} \]
And then we choose \( N \) a larger value that
\[ \frac{\alpha}{2} N - c > 4k(0), \tag{55} \]
and take \( \delta > 0 \) satisfying
\[ \delta = \frac{1}{2a \gamma} < \gamma. \tag{56} \]
This implies
\[ \frac{\alpha}{2} N - c C_\delta > 0. \tag{57} \]
Then there exists a positive constant \( \beta \) such that for large \( t_1 > 0 \),
\[ L'(t) \leq -\beta \left( \| u_t \|^2 + \| \nabla u \|^2 \right) - 4k(0) \| u_t \|^2 \]
\[ - \frac{1}{4} \int_{\Gamma} k'' u d\Gamma, \quad \forall t \geq t_1. \tag{58} \]
By (17) and (30), we get
\[ \int_0^{t_1} (-k'(s)) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds \]
\[ \leq \frac{1}{d} \int_0^{t_1} k''(s) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds \leq -c \mathcal{E}'(t). \tag{59} \]
Then from (56), we infer that there exists a constant \( \chi > 0 \) such that
\[ L'(t) \leq -\chi \mathcal{E}(t) - c \int_0^{t_1} k'(s) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds \]
\[ \leq -\chi \mathcal{E}(t) - c \int_0^{t_1} k'(s) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds. \tag{60} \]
\[ \leq -\chi \mathcal{E}(t) - c \int_0^{t_1} k'(s) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds. \tag{61} \]
Denoting \( F(t) = L(t) + c \mathcal{E}(t) - E(t) \), and using (58), we know that
\[ F'(t) \leq -\chi \mathcal{E}(t) - c \int_0^{t_1} k'(s) \int_{\Gamma} [u(t) - u(t - s)]^2 d\Gamma ds. \tag{62} \]
In the sequel, we consider two cases.

Case 1. The particular case \( H(t) = t^p \).

(I) \( p = 1 \).
Multiplying (59) by \( \eta(t) \), and using (19) and (A2)-(A3), we have
\[ \eta(t) F'(t) \leq -\chi \eta(t) \mathcal{E}(t) - c \mathcal{E}'(t), \quad \forall t \geq t_1. \tag{63} \]
Since \( \eta(t) \) is a nonincreasing continuous function and \( \eta'(t) \leq 0 \) for a.e. \( t \), then
\[ \eta(t) F'(t) \leq -\eta(t) \mathcal{E}'(t) + c \mathcal{E}'(t) \]
\[ \leq -m \eta(t) \mathcal{E}(t), \quad \text{a.e. } t \geq t_1. \tag{64} \]
In view of \( \eta(t) \mathcal{E} + c \mathcal{E} \sim \mathcal{E} \), we obtain that there exist two positive constants \( c_1, c_2 > 0 \),
\[ \mathcal{E}(t) \leq c_1 e^{-c_2 \int_0^t \eta(s) ds} \tag{65} \]
(II) \( 1 < p < 2 \).
Define \( \mathcal{E}(t) \) by
\[ \mathcal{E}(t) = L(t) + \Psi(t). \tag{66} \]
It follows from (43) and (56) that \( \mathcal{E}(t) \geq 0 \), and for any \( t \geq t_1 \),
\( \mathcal{G}^\prime (t) \leq - \beta \left( \| u_t \|^2 + \| u \|^2 \right) - k \| u_t \|^2 + \frac{1}{4} \int_{\Gamma_1} k \circ u d\Gamma. \)  
(64)

Then there exists a certain constant \( \beta_1 > 0, \)
\( \mathcal{G}^\prime (t) \leq - \beta_1 \mathcal{G} (t), \quad \forall t \geq t_1. \)  
(65)

This gives us
\[ \beta_1 \int_{t_1}^t E(s) ds \leq \mathcal{G} (t) - \mathcal{G} (t_1). \]  
(66)

Hence
\[ \int_0^\infty \mathcal{G} (s) ds < \infty. \]  
(67)

Define
\[ I(t) = \int_0^t \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds, \]  
(68)

we know that
\[ I(t) \leq c \int_0^t \mathcal{E} (s) ds. \]  
(69)

Without loss of generality assuming \( t_1 \) so large that \( I(t_1) > 0, \) then
\[ 0 < I(t_1) \leq I(t) < \infty, \quad \forall t \geq t_1. \]  
(70)

Using Jensen’s inequality and by (30) and (A2)-(A3), we can derive from (56) that for some constant \( q > 0, \)
\[ L^\prime (t) \leq - q \mathcal{E} (t) + c I(t) \int_{\Gamma_1} (-k')^{(1/p)} u d\Gamma \]
\[ \leq - q \mathcal{E} (t) + c I(t) \left[ \frac{1}{I(t)} \int_{\Gamma_1} (-k')^{(1/p)} u d\Gamma \right]^{(1/p)} \]
\[ \leq - q \mathcal{E} (t) + c \left[ \frac{1}{\eta (t)} \int_{\Gamma_1} k'' u d\Gamma \right]^{(1/p)} \]
\[ \leq - q \mathcal{E} (t) + \left[ \frac{c}{\eta (t)} \right]^{(1/p)} \left[ \int_{\Gamma_1} k'' u d\Gamma \right]^{(1/p)} \]
\[ \leq - q \mathcal{E} (t) + \left[ \frac{c}{\eta (t)} \right]^{(1/p)} [- \mathcal{E} (t)]^{(1/p)}. \]  
(71)

We multiply (71) by \( \mathcal{E}^{(1/p)} (t) \) and use (19) to deduce
\[ (L\mathcal{E}^{(1/p - 1)})^\prime (t) \leq L^\prime (t) \mathcal{E}^{(1/p - 1)} (t) \leq - q \mathcal{E} (t) \]
\[ + c \left[ \frac{\mathcal{G}^\prime (t)}{\eta (t)} \right]^{(1/p)} \mathcal{E}^{(1/p - 1)} (t). \]  
(72)

By Young’s inequality, we have for any \( \epsilon_1 > 0, \)
\[ (L\mathcal{E}^{(1/p - 1)})^\prime (t) \leq - q \mathcal{E} (t) + \epsilon_1 \mathcal{E} (t) + c \left[ \frac{\mathcal{G}^\prime (t)}{\eta (t)} \right]. \]  
(73)

Taking \( \epsilon_1 < (1/2)q, \) we conclude
\[ (L\mathcal{E}^{(1/p - 1)})^\prime (t) \leq - q \mathcal{E} (t) - c \left[ \frac{\mathcal{G}^\prime (t)}{\eta (t)} \right]. \]  
(74)

Define \( F(t) = \eta L \mathcal{E}^{(1/p - 1)} + c \mathcal{E} \sim \mathcal{G}. \) Multiplying (74) by \( \eta (t), \) we have
\[ F^\prime (t) \leq - q \mathcal{E} (t)^{(1/p - 1)} \mathcal{E} (t) \]  
(75)

Then there exists a certain constant \( q_0 > 0 \) such that
\[ F^\prime (t) \leq - q_0 \eta (t) F(t), \]  
(76)

from which we obtain
\[ \mathcal{E} (t) \leq c_3 \left( 1 + \int_0^t \eta (s) ds \right)^{(1/(1-p))}, \]  
(77)

where \( c_3 \) is a positive constant.

Combining (I) and (II) and using the boundedness of \( \eta (t) \) and \( \mathcal{E} (t), \) we can get (23).

Case 2. The general case.
Define
\[ I(t) := q \int_{t_1}^t \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \]  
(78)

In view of (67), we can take \( 0 < q < 1 \) such that
\[ I(t) < 1, \quad \forall t \geq t_1. \]  
(79)

Without loss of generality, we assume that \( I(t) > 0 \) for all \( t \geq t_1. \) On the other hand, we define
\[ \pi (t) = \int_{t_1}^t k'' (s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \]  
(80)

From (30), we can easily get \( \pi (t) \leq - c \mathcal{E}^2 (t). \) As \( H(t) \) is strictly convex on \( (0, r] \) and \( H(0) = 0, \) we see that
\[ H(\lambda x) \leq \lambda H(x), \quad i = 1, 2, 0 \leq \lambda \leq 1, x \in (0, r]. \quad (81) \]

It follows from Jensen’s inequality and (11) and (79) that

\[ \pi_i (t) = \frac{1}{qI(t)} \int_{t_1}^{t} I(t)(k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \]

\[ \geq \frac{1}{qI(t)} \int_{t_1}^{t} I(t)q(s)H(-k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \]

\[ \geq \frac{\eta(t)}{qI(t)} \int_{t_1}^{t} H(I(t)(-k'(s)))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \]

\[ \geq \frac{\eta(t)}{qH} \left( \int_{t_1}^{t} I(t)(-k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \right) \]

\[ = \frac{\eta(t)}{qH} \left( q \int_{t_1}^{t} (-k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \right) \]

\[ = \frac{\eta(t)}{qH} \left( q \int_{t_1}^{t} (-k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \right) \]

where \( \overline{H} \), which is strictly convex and increasing function on \((0, \infty)\) of class \( C^2 \), is called the extension of \( H \). We infer from (82) that

\[ \int_{t_1}^{t} (-k'(s))q \int_{\Gamma_1} [u(t) - u(t-s)]^2 \, d\Gamma \, ds \leq \frac{1}{qH} \left( \frac{\eta(t)}{\eta(t)} \right). \]

(83)

Then we can get from (59) that for any \( t \geq t_1 \),

\[ F'(t) \leq -c\overline{E}(t) + cH^{-1} \left( \frac{q\pi(t)}{\eta(t)} \right). \quad (84) \]

Denote

\[ H_0(t) = \overline{H}'(t). \quad (85) \]

For \( r_0 < r \), we define \( \mathcal{K}_1(t) \) by

\[ \mathcal{K}_1(t) = H_0 \left( r_0 \overline{E}(t) \right) F(t) + \overline{E}(t) - \overline{E}(t). \quad (86) \]

Since \( E'(t) \leq 0, \overline{H}' > 0 \), and \( H'' > 0 \), we get from (84) that

\[ \mathcal{K}_1(t) = \frac{\mathcal{K}'(t)}{\mathcal{K}(t)} = \left( r_0 \overline{E}(t) \right) H_0 \left( r_0 \overline{E}(t) \right) F(t) + H_0 \left( r_0 \overline{E}(t) \right) F(t) + \overline{E}'(t) \]

\[ \leq -m\overline{E}(t)H_0 \left( r_0 \overline{E}(t) \right) + cH_0 \left( r_0 \overline{E}(t) \right) \overline{H}^{-1} \left( \frac{q\pi(t)}{\eta(t)} \right). \quad (87) \]

We denote by \( \overline{H}' \) the conjugate function of the convex function \( \overline{H} \) (see Arnold [33]), and then

\[ \overline{H}'(s) = s(\overline{H})^{-1}(s) - \overline{H}'(\overline{H})^{-1}(s) \quad (88) \]

satisfies Young’s inequality,

\[ AB \leq \overline{H}'(A) + \overline{H}(B). \quad (89) \]
Taking \( A = \overline{H}_0(r_0(E(t)/E(0))) \) and \( B = \overline{H}^{-1}(q\pi(t)/\eta(t)) \), and using \( \overline{H}^*(s) \leq s(\overline{H}')^{-1}(s) \) and (87), we have

\[
\mathcal{K}_1(t) \leq -\chi \mathcal{E}(t) H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c \overline{H}'\left( H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q\pi(t)}{\eta(t)}
\]

\[
\leq -\chi \mathcal{E}(t) H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \left( \overline{H}' \right)^{-1}\left( H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q\pi(t)}{\eta(t)}
\]

\[
\leq -\chi \mathcal{E}(t) H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \left( \overline{H}' \right)^{-1}\left( H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q\pi(t)}{\eta(t)}
\]

\[
\leq -\left( \chi \mathcal{E}(0) - cr_0 \right) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + cq \frac{\pi(t)}{\eta_1(t)}
\]

We multiply (90) by \( \eta(t) \) to arrive at

\[
\eta(t) \mathcal{K}_1(t) \leq -\left( \chi \mathcal{E}(0) - cr_0 \right) \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + cq \pi(t)
\]

\[
\leq -\left( \chi \mathcal{E}(0) - cr_0 \right) \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) - c \mathcal{E}'(t).
\]

The functional \( \mathcal{K}_2(t) \) is defined by

\[
\mathcal{K}_2(t) = \eta(t) \mathcal{K}_1(t) + c \mathcal{E}(t).
\]

Then we can easily obtain that there exist constants \( \beta_2 > 0 \) and \( \beta_\eta > 0 \) such that

\[
\beta_2 \mathcal{K}_2(t) \leq E(t) \leq \beta_\eta \mathcal{K}_2(t).
\]

Choosing a suitable \( r_0 > 0 \), and defining \( H_3(t) = t H_0(r_0 t) \), from (91), we infer that for a constant \( \gamma > 0 \),

\[
\mathcal{K}_2(t) \leq -\gamma \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0\left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) = -\gamma \eta(t) H_3\left( \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right).
\]

(94)

It follows from \( 0 \leq r_0(\mathcal{E}(t)/\mathcal{E}(0)) < r \) that for any \( t > 0 \),

\[
\int_{t_1}^t -\frac{R'(s)}{H_3(R(s))} \frac{ds}{\eta(s)} \geq \gamma_1 \int_{t_1}^t \frac{ds}{\eta(s)} \Rightarrow \int_{t_1}^t \frac{ds}{\eta(s)} \geq \gamma_1 \int_{t_1}^t \frac{ds}{\eta(s)}.
\]

(98)

Define

\[
H_4(t) = \int_t^{R(t)} \frac{1}{s H_0(s)} ds.
\]

(99)

It is to verify that \( H_4 \) is strictly decreasing on \( (0, r] \) and \( \lim_{t \to 0} H_4(t) = +\infty \). It follows that

\[
R(t) \leq \frac{1}{r_0} H_4^{-1}\left( \chi_1 \int_{t_1}^t \frac{ds}{\eta(s)} \right).
\]

(100)
Combining (96) and (100), we can obtain (21). This finishes the proof of Theorem 2

**Data Availability**

No data were used during this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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