Rigorous Covariant Path Integrals

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Abstract

Our rigorous path integral is extended to quantum evolution on metric-affine manifolds.

1 Path Integrals on Euclidean Spaces.

Consider the evolution operator

\[
U(t', q) = \psi(t'', q), \quad t'' \geq t',
\]

of the quantum evolution equation on \(L^2(\mathbb{R}^d)\):

\[
\frac{\hbar}{i} \frac{\partial}{\partial t} \psi(t, q) + f(t, q, \frac{\hbar}{i} \frac{\partial}{\partial q}) \psi(t, q) = 0.
\]

Here the operator \(f(t, q, \frac{\hbar}{i} \frac{\partial}{\partial q})\) is the standard quantization of a classical time-dependent quasi-Hamiltonian \(f(t, q, p)\) on the phase space \(\mathbb{R}^{2d}\).

The formal Hamiltonian functional integral

\[
\int \prod_{t'\geq t} \frac{dq(t)dp(t)}{(2\pi\hbar)^d(t)} \exp \frac{i}{\hbar} \int_{t'}^{t''} \left[p(t)\dot{q}(t) - f(t, q(t), p(t))\right] dt,
\]

presumably represents the standard symbol \(U^{(st)} = \langle q|U|p\rangle\langle p|q\rangle\) of the evolution operator \(U\). In practice, such formal integral is just a convenient notation subjected to all standard algorithms of the integral calculus, including the stationary phase approximations and analytic continuation to the imaginary time. Application of other ordering rules of canonical quantization, such as Weyl, Wick and more general \(\Omega\)-rules [1], leads to so called \(\Omega\)-symbols [2].

Actually, \(\Omega = \Omega(q, p)\) is a real analytic function on the phase space \(\mathbb{R}^{2d}\). Suppose, to start with, that \(\Omega(q, p)\) has no zeroes. Then, by definition, the \(\Omega\)-symbol of the operator \(f(t, q, \frac{\hbar}{i} \frac{\partial}{\partial q})\) is

\[
f^{\Omega}(q, p) = \frac{1}{\Omega}(-i\hbar \partial_q, -i\hbar \partial_p) f^{(we)}(q, p)
\]
on \(\mathbb{R}^{2d}\). Here \(f^{(we)}(q, p)\) is the Weyl symbol of that operator.
For example, $\Omega$ is $\exp(\frac{i}{\hbar}qp)$ for the standard quantization, and for the Wick quantization $\Omega$ is $\exp(-\frac{1}{\hbar}(p^2 + q^2))$.

In general, $\Omega(q, p)$ has real zeroes. Then the $\Omega$-symbol exists only $h$-asymptotically. An $\Omega$-symbol $f(q, p)$ is called a **quasi-Hamiltonian** if, uniformly in $t' \leq t \leq t''$, it is

(i) **quasi-polynomial** of order $m > 0$: for any multi-index $\alpha$ there is a constant $C$ such that $|\partial^\alpha f(q, p)| \leq C(1 + \sqrt{q^2 + p^2})^m - |\alpha|$.

(ii) **quasi-dissipative**: $\Re(if) > \delta$, where $\delta$ a constant.

(iii) **hypo-elliptic**.

(iv) continuous in $t$ along with all its derivatives $\partial_{q, p}$.

Simple examples are the Schrödinger Hamiltonians with arbitrary quasi-polynomial scalar and vector potentials.

The path integral construction is based on the following **Convergence Theorem**:

Consider the ordered products $U_P = \prod_{n=1}^{N} U_n$, where $U_n$ are the operators with the Weyl symbols $[1 + \frac{i}{\hbar}f(t_n, q, p)(t_n - t_{n-1})]^{-1}$, associated with partitions $\mathcal{P}: t' = t_0 < t_1 < \ldots < t_{n-1} < t_N = t''$ of the time-interval $[t', t'']$. Then the strong (i.e. in expectation) operator limit $U = \lim_{|P| \to \infty} U_P$ exists at least for small $h > 0$.

The existence of the operator limit entails the representation of the (not quasi-polynomial) symbols $U^{\Omega}$ as the limits of the quasi-polynomial $\Omega$-symbols of $U_P^\Omega$. According to the $\Omega$-symbols product rules the latter are multiple distributional integral transforms of the products of the $U_n^\Omega$.

For example, the standard symbol $U^{(st)}_P(q, p)$ is the limit of the multiple distributional integrals

$$\int_{\mathbb{R}^{2N}} \prod_{n=1}^{N} \frac{dq_n dp_n}{(2\pi \hbar)^2} \prod_{n=1}^{N} U_n^{(st)}(q_n, p_n-1) \exp \left[ \frac{i}{\hbar} \sum_{n=1}^{N} p_n - 1 \cdot (q_n - q_{n-1}) \right].$$

with the boundary conditions $q_0 = q_N = q$, $p_0 = p_N = p$.

If $\Omega(q, p)$ has zeroes, then the functional integral is $h$-asymptotic.

By our definition, the path integrals are such limits. The integral calculus algorithms for such path integrals are justified by such approximations.

For example, the spectacular reappearance of the classical Hamiltonian equations from the semi-classical Feynman functional integral takes the following form: the principal terms in the $h$-expansions of the partial products symbols are the backward Euler approximations of the corresponding classical Hamilton equations.

### 2 Covariant Path Integrals on Manifolds.

The proof of the Convergence Theorem is based on the theory of the finite difference solutions of the Cauchy problem with necessary estimates, provided by the calculus of the $\Omega$-symbols. Consequently, an extension of the functional
integral to quantum evolution on manifolds requires a covariant extension of the \( \Omega \)-calculus.

Let \( Q \) be a finite-dimensional configuration manifold with a Riemannian metric \( g^{ij} \), and \( \nabla \) a (possibly non-symmetric) metric connection on \( Q \). Such metric-affine manifolds appear, e.g., in the Kleinert’s functional integral for the hydrogen atom via a non-holonomic change of variables [4].

We assume that the connection has \textit{bounded geometry}, i.e. the curvature and torsion tensors and their covariant derivative of any order are uniformly bounded relative to the given metric \( g^{ij} \).

Let \( S(Q) \) and \( S'(Q) \) be the Schwartz spaces of the test functions and temperate distributions (alias ket- and bra-vectors), defined as for the Euclidean \( Q = \mathbb{R}^d \) [6]: \( S(Q) \) is the space of the functions with all covariant derivatives falling off faster than any negative power of the geodesic distance from a fixed observation point \( o \), and \( S'(Q) \) is the anti-dual space of the anti-linear functionals on it, which are continuous relative to the natural Frechet topology of the \( S(Q) \). The anti-duality is assumed in the form 

\[
\langle \Psi | \psi \rangle = \int_Q dq \sqrt{g(q)} \Psi(q)^* \psi(q).
\]

The \( S'(Q) \) is supplied with the strong topology of uniform convergence on bounded subsets of \( S(Q) \). Actually the Schwartz spaces do not depend on the connection of bounded geometry and the observation point.

A continuous linear operator \( A : S(Q) \to S'(Q) \) is called a \textit{Schwartz operator}. Any differential operator on \( Q \) with coefficients from \( S'(Q) \) is such an operator. The celebrated \textit{Schwartz Kernel Theorem} states that

(i) there is one to one correspondence between the Schwartz operators \( A \) and their Schwartz kernels \( A(q'', q') \in S'(Q'' \times Q') \) such that 

\[
\langle A\psi(q'')|\psi(q') \rangle = \langle A(q'', q')|\psi(q'')\psi(q') \rangle,
\]

and vice versa.

(ii) The strong convergence of Schwartz operators is equivalent to the strong convergence of their Schwartz kernels.

Let \( N \) be a closed \textit{symmetric convex normal neighborhood} of the diagonal of the square \( Q \times Q \) with the induced metric and connection, so that for all \( (q', q'') \in N \)

(i) \( (q'', q') \in N \),

(ii) there is a unique \( \nabla \)-geodesic \( \gamma_{q'q''} \) from \( q' \) to \( q'' \),

(iii) the inverse exponential mapping \( \exp^{-1}(q'') \) defines the normal coordinates on the normal neighborhood \( N_{q'} = \{q'' : (q', q'') \in N \} \).

We require our Schwartz operators to be \( N \)-proper: their kernels should vanish outside of \( N \). Such are, for example, the differential operators with quasi-polynomial coefficients.

The \textit{covariant Weyl symbol} of \( A \) is defined (cf. [3])

\[
A^{(wc)}(q, p) = \int_{T_q} dv \sqrt{g(q)} e^{-ip\cdot v} A(\exp_q(v/2), \exp_q(-v/2)), \quad p \in T_q^*. \]

The covariant \( \Omega \)-symbol is defined via the Weyl symbol as

\[
A^{\Omega}(q, p) = (1/\Omega)\left(\frac{\hbar}{i}\right)(\nabla_q^{sym}, \partial_p)A^{(wc)}(q, p),
\]

where the order of the differentiation is such that the symmetric covariant \( \nabla_q^{sym} \) are always prior to the partial \( \partial_p \).
The definition of the covariant $\Omega$-symbol is compatible with that of Safarov \cite{5} for his $s$-symbols, $0 \leq s \leq 1$. In particular, $s = 1$ corresponds to the standard symbol $A^{(st)}(q, p) = \int_{T^*q} dv \sqrt{g(q)} \cdot e^{-i p \cdot v} \cdot A(q, \exp_q v)$. The quasi-dissipativity, hypo-ellipticity and $t$-continuity for quasi-Hamiltonian symbols are covariantly defined as in the Euclidean case.

The convergence theorem is valid on the metric-affine manifolds $Q$ as well. The covariant $\Omega$-symbols of an evolution operator are the limits of covariant multiple integrals over $(T^*Q)^N$. In particular, the standard symbol $U^{(st)}(q, p)$ is the strong limit of the standard symbols $U_p^{(st)}(q, p)$, which are

$$\int_{(T^*Q)^N} d\lambda_N \left[ \prod_{n=1}^N U_n^{(st)}(q_n, \exp_{q_n}^{-1}(p_{n-1})) \right] \cdot \exp \left( \frac{i}{\hbar} \sum_{n=1}^N p_{n-1} \cdot \exp_{q_n}^{-1}(q_n) \right).$$

Here $d\lambda_N = \prod_{n=1}^{N-1} \frac{dq_n dp_n}{(2\pi\hbar)^{2N-2}}$, $p_n \in T^*_{q_n}$, $q_0 = q_N = q$, $p_0 = p_N = p$.

For the functional integrals of evolution operators in metric vector bundles $E$ over $Q$ with connections $\nabla^E$ of bounded geometry (such as spinor bundles over compact manifolds), the covariant kernels $A(q', q')$ and their $\Omega$-symbols are linear transformations of the fibers $E_q \to E_{q'}$ (as for the Dirac operators). Accordingly, all corresponding formulas involve the parallel transport $\tau_{q'}^q$ along $\gamma_{q'q}$ in $E$.

3 Conclusion.

Our rigorous path integral is extended to quantum evolution on metric-affine manifolds.

References

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