Multi shocks in reaction-diffusion models

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Abstract

It is shown, concerning equivalent classes, that on a one-dimensional lattice with nearest neighbor interaction, there are only four independent models possessing double-shocks. Evolution of the width of the double-shocks in different models is investigated. Double-shocks may vanish, and the final state is a state with no shock. There is a model for which at large times the average width of double-shocks will become smaller. Although there may exist stationary single-shocks in nearest neighbor reaction diffusion models, it is seen that in none of these models, there exist any stationary double-shocks. Models admitting multi-shocks are classified, and the large time behavior of multi-shock solutions is also investigated.

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1 Introduction

Recently, shocks in one-dimensional reaction-diffusion models have absorbed much interest [1–18]. There are some exact results on shocks in one-dimensional reaction-diffusion models as well as simulations, numeric results [6] and also mean field results [2]. Formation of localized shocks in one-dimensional driven diffusive systems with spacially homogeneous creation and annihilation of particles has been studied in [12]. Recently, in [4], the families of models with traveling wave solutions on a finite lattice have been presented. These models are the Asymmetric Simple Exclusion Process (ASEP), the Branching- Coalescing Random Walk (BCRW) and the Asymmetric Kawasaki-Glauber process (AKGP). In all of these cases the time evolution of the shock measure is equivalent to that of a random walker on a lattice with \(L\) sites with homogeneous hopping rates in the bulk and special reflection rates at the boundary [4]. Shocks have been studied at both the macroscopic and the microscopic levels and there are some efforts on addressing the question that how macroscopic shocks originate from the microscopic dynamics [7]. Hydrodynamic limits are also investigated.

Among the important aspects of reaction-diffusion systems, is the phase structure of the system. The static phase structure concerns with the time-independent profiles of the system, while the dynamical phase structure concerns with the evolution of the system, specially its relaxation behavior. In [19–22], the phase structure of some classes of single- or multiple-species reaction-diffusion systems have been investigated. These investigations were based on the one-point functions of the systems. In a recent article both stationary and also dynamical single-shocks on a one-dimensional lattice have been investigated [18]. It was done for both an infinite lattice and a finite lattice with boundaries. Static and dynamical phase transitions of these models have been studied. It was seen that ASEP has no dynamical phase transition, but both BCRW and AKGP have three phases, and the system may show dynamical phase transitions [18].

The question addressed in this article is that, on a one-dimensional lattice with nearest neighbor interaction, which models possess double shock and also multi shock solutions. By double-shock it is meant an uncorrelated state where the occupation probability has two jumps. All the models have nearest neighbor interactions and are on a one-dimensional lattice. It is shown that, concerning equivalent classes, there are only four independent models possessing double-shocks. For two models, double shock disappears and the final state is a linear combination of Bernoulli measures. There is a model for which At large times the average width of double shock becomes small. stationary state is a state which does not evolve. It can be easily seen that there may exist stationary single shocks in nearest-neighbor reaction diffusion models (BCRW, and AKGP), in other words there are single shock states without any evolution. But in none of these models, there is no stationary double shock. Combining single shocks one may construct multi shocks. There are multi shocks of the type \((0, \rho, 0, \rho, \cdots)\) and \((0, 1, 0, 1, \cdots)\). At large times the final state is a linear combination of single shocks, or a state with no shock.
2 Notation

Consider a one-dimensional lattice, each point of which either is empty or contains one particle. Let the lattice have \( L \) sites. An empty state is denoted by \( |0\rangle \) and an occupied state is denoted by \( |1\rangle \).

\[
|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(1)

If the probability that the site \( i \) is occupied is \( \rho_i \) then the state of that is represented by \( \begin{pmatrix} 1 - \rho_i \\ \rho_i \end{pmatrix} \). The state of the system is characterized by a vector

\[
|\mathcal{P}\rangle \in \mathbb{V} \otimes \cdots \otimes \mathbb{V},
\]

(2)

where \( \mathbb{V} \) is a 2-dimensional vector space. All the elements of the vector \( |\mathcal{P}\rangle \) are nonnegative, and

\[
\langle \mathcal{S} | \mathcal{P} \rangle = 1.
\]

(3)

Here \( \langle \mathcal{S} \rangle \) is the tensor-product of \( L \) covectors \( \langle s \rangle \), where \( \langle s \rangle \) is a covector the components of which \( (s_\alpha)'s \) are all equal to one. The evolution of the state of the system is given by

\[
\dot{|\mathcal{P}\rangle} = \mathcal{H} \cdot |\mathcal{P}\rangle,
\]

(4)

where the Hamiltonian \( \mathcal{H} \) is stochastic, by which it is meant that its nondiagonal elements are nonnegative and

\[
\langle \mathcal{S} | \mathcal{H} \rangle = 0.
\]

(5)

The interaction is nearest-neighbor, if the Hamiltonian is of the form

\[
\mathcal{H} = \sum_{i=1}^{L-1} H_{i,i+1},
\]

(6)

where

\[
H_{i,i+1} := 1 \otimes \cdots \otimes 1 \otimes H \otimes 1 \otimes \cdots \otimes 1.
\]

(7)

Nondiagonal elements of \( H \), shown by \( \omega_{ij} \), are reaction rates, hence nonnegative, and its diagonal elements are nonpositive. \( \omega_{ij} \) is the rate for changes of the configuration of a pair of neighboring sites from the initial state \( j \) to the final state \( i \). We take the state \( |00\rangle \) as the state 1, \( |01\rangle \) as 2, \( |10\rangle \) as 3 and finally \( |11\rangle \) as the fourth state. So, e.g. \( \omega_{23} \) is the rate for change of configuration \( |10\rangle \) to \( |01\rangle \), which is the hoping rate to the right.

Any configuration of the system may be represented by the vector \( |E_a\rangle \). So the system is spanned by \( 2^L \) vectors, \( |E_a\rangle \ (a = 1, 2, \cdots 2^L) \), and any physical state is a linear combination of these vectors

\[
|\mathcal{P}\rangle = \sum_{a=1}^{2^L} P_a |E_a\rangle, \quad \text{where} \quad \sum_{a=1}^{2^L} P_a = 1.
\]

(8)
P\textsubscript{a}s are nonnegative real numbers. \( P_a \) is the probability of finding the system in the configuration \( a \).

It is said that the state of the system is a single-shock at the site \( k \) if there is a jump in the density at the site \( k \) and the state of the system is represented by a tensor product of the states at each site as

\[
|e_k\rangle = u^\otimes k \otimes v^\otimes (L-k),
\]

where

\[
u := \begin{pmatrix} 1 - \rho_1 \\ \rho_1 \end{pmatrix}, \quad v := \begin{pmatrix} 1 - \rho_2 \\ \rho_2 \end{pmatrix}.
\]

It is seen that

\[
\langle S | e_k \rangle = 1.
\]

\(|e_k\rangle\) represents a state for which the occupation probability for the first \( k \) sites is \( \rho_1 \), and the occupation probability for the next \( L - k \) sites is \( \rho_2 \). The set \(|e_k\rangle, k = 0, 1, \cdots, L\) is not a complete set, but linearly independent.

There are three families of stochastic one-dimensional non-equilibrium lattice models, (ASEP, BCRW, AKGP), for which if the initial state of these models is a linear superposition of shock states, at the later times the state of the system \(|\mathcal{P}\rangle\) remains a linear combination of shock states. For these models

\[
\mathcal{H}|e_k\rangle = d|e_{k-1}\rangle + d'|e_{k+1}\rangle - (d + d')|e_k\rangle,
\]

where \( d \) and \( d' \) are some parameters depending on the reaction rates in the bulk, and the densities \( \rho_1 \) and \( \rho_2 \). So the span of \(|e_k\rangle\)'s is an invariant subspace of \( \mathcal{H} \), the Hamiltonian of the above mentioned models. It should be noted that the number of \(|e_k\rangle\)'s are \( L + 1 \), and an arbitrary physical state is not necessarily expressible in terms of \(|e_k\rangle\)'s.

Let’s assume that the initial state of the system is a linear combination of shock states

\[
|\mathcal{P}\rangle(0) = \sum_{k=0}^{L} p_k(0)|e_k\rangle.
\]

\( p_k \)'s, are not necessarily nonnegative, and so any of them may be greater than one. For such an initial state, the system remains in the sub-space spanned by shock measures.

\[
|\mathcal{P}\rangle(t) = \sum_{k=0}^{L} p_k(t)|e_k\rangle.
\]

Using (11), it is seen that

\[
\sum_{k=0}^{L} p_k(t) = 1.
\]

The three models are classified as following [4]
1. **ASEP**. The only non-vanishing rates in the bulk are the rates of diffusion to the right $\omega_{23}$ and diffusion to the left $\omega_{32}$. In this case the densities can take any value between 0 and 1 ($\rho_1$, $\rho_2 \neq 0, 1$). $d$, and $d'$ are

\[
d = \frac{\rho_1(1 - \rho_1)}{\rho_2 - \rho_1}(\omega_{23} - \omega_{32}),
\]

\[
d' = \frac{\rho_2(1 - \rho_2)}{\rho_2 - \rho_1}(\omega_{23} - \omega_{32}).
\]

(16)

It should be noted that the densities $\rho_1$, and $\rho_2$ are also related through

\[
\frac{\rho_2(1 - \rho_1)}{\rho_1(1 - \rho_2)} = \frac{\omega_{23}}{\omega_{32}}.
\]

(17)

So

\[
d = \frac{\rho_1}{\rho_2} \omega_{23}, \quad d' = \frac{\rho_2}{\rho_1} \omega_{32}.
\]

(18)

• 2. **BCRW**. The non-vanishing rates are coalescence ($\omega_{34}$, and $\omega_{43}$), Branching ($\omega_{42}$, and $\omega_{43}$) and diffusion to the left and right ($\omega_{32}$, and $\omega_{23}$). The density $\rho_1$ can take any value between 0 and 1, but $\rho_2$ should be zero. These parameters are related through

\[
\frac{\omega_{23}}{\omega_{43}} = \frac{\omega_{24} + \omega_{34}}{\omega_{42} + \omega_{43}} = \frac{1 - \rho_1}{\rho_1}.
\]

(19)

The parameters $d$, and $d'$ are

\[
d = (1 - \rho_1)\omega_{32} + \rho_1\omega_{34},
\]

\[
d' = \frac{\omega_{43}}{\rho_1}.
\]

(20)

if $\omega_{32} = \omega_{34} = \omega_{43} = \omega_{23} = 0$, and $\omega_{24}/\omega_{42} = (1 - \rho)/\rho$ then $d = d' = 0$, and the model admit stationary single shock.

• 3. **AKGP**. The non-vanishing rates are Death ($\omega_{12}$, and $\omega_{13}$) and Branching to the left and right ($\omega_{42}$, and $\omega_{43}$), and also diffusion to the left $\omega_{32}$, $\rho_1$ should be equal to one, and $\rho_2$ should be zero. The hoping parameters are $d = \omega_{13}$, $d' = \omega_{43}$.

### 3 Double shocks

The state of a double shock may be defined through

\[
|e_{m,k}⟩ = u^\otimes m \otimes v^\otimes k \otimes w^\otimes (L-k-m), \quad m + k \leq L,
\]

(21)

where

\[
u := \begin{pmatrix} 1 - \rho_1 \\ \rho_1 \end{pmatrix}, \quad v := \begin{pmatrix} 1 - \rho_2 \\ \rho_2 \end{pmatrix}, \quad w := \begin{pmatrix} 1 - \rho_3 \\ \rho_3 \end{pmatrix}.
\]

(22)
\(|e_{m,k}\rangle\) represents a state for which the occupation probability for the first \(m\) sites is \(\rho_1\), the occupation probability for the next \(k\) sites is \(\rho_2\), and the occupation probability for remaining sites is \(\rho_3\). We call such state a double shock, with the first shock at the site \(m\), and the other one at the site \(m+k\). \(k\) is the width of double-shock, and \(\rho_i \in [0,1]\). To have a double shock \(\rho_1\) should be different from \(\rho_2\), and \(\rho_2\) also should be different from \(\rho_3\). We search for Hamiltonians for which the span of \(|e_{m,k}\rangle\)’s is an invariant subspace of \(\mathcal{H}\),

\[
\mathcal{H}|e_{m,k}\rangle = d_1|e_{m-1,k+1}\rangle + d'_1|e_{m+1,k-1}\rangle + d_2|e_{m,k-1}\rangle + d'_2|e_{m,k+1}\rangle - (d_1 + d'_1 + d_2 + d'_2)|e_{m,k}\rangle, \quad k \geq 2, \tag{23}
\]

where \(d_i\)’s (\(d'_i\)’s) are parameters depending on the reaction rates and may be considered as the rates of jump of the shock to the left (right). \(\mathcal{H}|e_{m,1}\rangle\) will be discussed later.

As it is seen for single-shocks, one should study cases with different values of \(\rho\) separately. One may divide the region of values for \(\rho\) to \(\rho = 0, 0 < \rho < 1, \text{ and } \rho = 1\). From now on the cases \(\rho = 0, \text{ and } \rho = 1\) will be explicitly stated, and whenever we write \(\rho\), it is meant that \(\rho \neq 0, 1\). To have a double-shock there may be different combinations of densities. There are different models, which may transform to each other through particle-hole, or right-left interchange. We call these models equivalent models. As an example the model admitting the double-shock \((\rho_1, \rho_2, \rho_3) = (0, \rho, 1)\) is related to the model admitting the double-shock \((1, \rho, 0)\) through right-left interchange, and also related to the model admitting the double shock \((1, 1 - \rho, 0)\) through particle-hole interchange. Let’s consider the double shock \((0, 1, \rho)\). A necessary condition for the Hamiltonian for which the span of double-shock measures be an invariant subspace of \(\mathcal{H}\), is that the span of each of single-shock measures \((0, 1)\) and \((1, \rho)\) are separately invariant subspace of \(\mathcal{H}\). The single-shocks \((0, 1)\) form an invariant subspace for the hamiltonian in AKGP. The only interactions which may have nonzero rates are

\[
\emptyset A \rightarrow (\emptyset, \ AA), \ A\emptyset \rightarrow (\emptyset, \ AA, \ \emptyset A). \tag{24}
\]

As far as we consider the single shock \((0, 1)\), there is no extra constraint on the nonzero reaction rates. The single-shocks \((1, \rho)\) form an invariant subspace for the hamiltonian in BCRW, with the following interactions

\[
\emptyset \emptyset \rightarrow (\emptyset A, \ A\emptyset), \ \emptyset A \rightarrow (\emptyset, \ A\emptyset), \ A\emptyset \rightarrow (\emptyset, \ \emptyset A), \tag{25}
\]

whose reaction rates should satisfy

\[
\frac{\omega_{21} + \omega_{31}}{\omega_{12} + \omega_{13}} = \frac{\omega_{23}}{\omega_{13}} = \frac{\rho}{1 - \rho}. \tag{26}
\]

The space of parameters of the model, for a double-shock \((0, 1, \rho)\), is the overlap of the space of parameters of the AKGP and BCRW. Gathering all these together it is seen that all the reaction rates should be zero. So there is no reaction diffusion model with nearest neighbor interaction for which the double shocks \((0, 1, \rho)\) form an invariant subspace.

It can be easily shown that, concerning equivalent classes, there are only four independent cases.
1. \((\rho_1, \rho_2, \rho_3)\),

Among the models possessing shock solution, (and for \(\rho_i \neq 0, 1\)), ASEP is the only model for which double shocks forms an invariant subspace. The only nonvanishing rates are \(\omega_{23}\) and \(\omega_{32}\), and they should satisfy

\[
\frac{\omega_{23}}{\omega_{32}} = \frac{\rho_2(1 - \rho_1)}{\rho_1(1 - \rho_2)} = \frac{\rho_3(1 - \rho_2)}{\rho_2(1 - \rho_3)}. \tag{27}
\]

d_i’s and \(d'_i’s\) are

\[
d_1\frac{\rho_2}{\rho_1} = d_2\frac{\rho_3}{\rho_2} = \omega_{23}, \quad d'_1\frac{\rho_1}{\rho_2} = d'_2\frac{\rho_2}{\rho_3} = \omega_{32} \tag{28}
\]

This model has been studied in [18]. \(\omega_{23}\) and \(\omega_{32}\) are positive nonzero rates. So \(d_i’s\) and \(d'_i’s\) are also nonzero. To have stationary double shock \(H|e_{m,k}\rangle = 0\) leading to \(d_i = d'_i = 0\), which is unacceptable. So It is not possible to have stationary double shock in ASEP.

2. \((0, \rho, 0)\), \((\rho, 0, \rho)\)

The necessary condition for a model possessing double-shocks \((0, \rho, 0)\) (or \((\rho, 0, \rho)\)) is that this model possesses both single shocks \((0, \rho)\), and \((\rho, 0)\). Nonvanishing rates for such a model are \(\omega_{23}, \omega_{24}, \omega_{31}, \omega_{34}, \omega_{42}\) and \(\omega_{43}\). These rates should satisfy

\[
\frac{\omega_{24} + \omega_{34}}{\omega_{42} + \omega_{43}} = \frac{\omega_{31}}{\omega_{42}} = \frac{\omega_{23}}{\omega_{43}} = \frac{1 - \rho}{\rho}. \tag{29}
\]

d_i’s and \(d'_i’s\) are

\[
d_1 = \omega_{34}, \quad d'_1 = (1 - \rho)\omega_{23} + \rho\omega_{24}
\]
\[
d_2 = \omega_{43}, \quad d_2 = (1 - \rho)\omega_{32} + \rho\omega_{34}. \tag{30}
\]

The Hamiltonian with the above mentioned reaction rates also possesses the double-shock \((\rho, 0, \rho)\). The only difference is that the rate of jump to the left (and right) of the first double-shock is the rate of jump to the right (and left) for the second one. To have stationary double shocks \(d_i’s\) should be zero, which needs all the rates to be zero. So, there is no stationary double shock in this model.

3. \((0, 1, 0)\),

Nonvanishing rates are \(\omega_{13}, \omega_{12}, \omega_{42}, \omega_{43}\). This model is an asymmetric generalization of zero temperature Glauber model. \(d_i’s\) and \(d'_i’s\) are

\[
d_1 = \omega_{42}, \quad d'_1 = \omega_{12}
\]
\[
d_2 = \omega_{13}, \quad d_2 = \omega_{43}. \tag{31}
\]

To have stationary double shock \(d_i’s\) and \(d'_i’s\) should be zero, which leads to vanishing all the reaction rates. So, this model does not have any stationary double shock either.
4. \((0, \rho, 1)\).

The only nonvanishing rate is \(\omega_{23}\). \(d_i\)'s and \(d'_i\)'s are

\[
d_1 = 0, \quad d'_1 = (1 - \rho)\omega_{23}
\]

\[
d_2 = \rho \omega_{23}, \quad d'_2 = 0.
\] (32)

This model does not have any stationary double shock either.

If the initial state is a linear combination of double shocks, then

\[
|\Psi(t)\rangle = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} p_{mk}(t)|e_{mk}\rangle,
\] (33)

where \(p_{mk}\) is the contribution of the double-shock, \((mk)\) in the state of the system. Using (4), (33), and also the linear independency of \(|e_{mk}\rangle\)'s, one can obtain the evolution equation for \(p_{mk}\)'s. It is a difficult task to solve difference equations of this type. \(p_{m,1}\) has two indices, \(m\) representing the position of the first shock, and \(k\) the width of the double shock. We may forget about the position of the first shock and ask only about the width of double shock. Then one may encounter with a difference equation which can be solved more easier.

Let’s consider the general case where the span of \(|e_{mk}\rangle\)'s is an invariant subspace of \(H\)

\[
H|e_{m,k}\rangle = \sum_{m',k'} H_{mk}^{m',k'} |e_{m',k'}\rangle.
\] (34)

If the Hamiltonian has the property that \(\sum_{m'} H_{mk}^{m',k'}\) is independent of \(m\), then one may define a new Hamiltonian \(\hat{H}\) through

\[
\hat{H}_{k'} := \sum_{m'} H_{mk}^{m',k'}
\] (35)

It can be easily shown that \(\hat{H}\) is stochastic, it is meant that

\[
\hat{H}_{k'} > 0, \quad \text{for} \quad k' \neq k
\]

\[
\sum_{k'} \hat{H}_{k'} = 0.
\] (36)

Then one may forget about \(m\), position of the first shock, and only ask about the contribution of double shocks with the width \(k\). It is obvious that some part of information about the position of the first shock will be lost. Now one may define \(|f_k\rangle\) as the state of a double shock with the width \(k\). Identifying all \(|e_{m,k}\rangle\) with the same \(m\) to each other in the state (33), one may define another state \(|\tilde{\Psi}\rangle\) where the information of the position of the first shock has being lost

\[
|\tilde{\Psi}(t)\rangle = \sum_{k=1}^{\infty} q_k(t)|f_k\rangle.
\] (37)
Here \( q_k(t) \) is defined through
\[
q_k := \sum_{m=-\infty}^{\infty} p_{mk}. \tag{38}
\]
and it is the contribution of all double shocks with the width \( k \). Then instead of (23), one may obtain
\[
\tilde{H}|f_k\rangle = D|f_{k+1}\rangle + D'|f_{k-1}\rangle - (D + D')|f_k\rangle, \quad k \geq 2. \tag{39}
\]
where
\[
D := d_1 + d'_2, \quad D' := d'_1 + d_2. \tag{40}
\]

### 3.1 Double-shocks \((0, \rho, 0)\) and \((0, 1, 0)\) on a periodic lattice

Let’s consider a lattice with \( L \) sites and with periodic boundary conditions. Then, the only double shocks which could exist, are \((0, 1, 0)\) and \((0, \rho, 0)\). Let’s sum up the contributions of all double shocks with the same width. The position of double shocks will be again washed out. Then one should work with \(|f_k\rangle\), which stands for the state of a double-shock with the width \( k \). \(|f_0\rangle\) and \(|f_L\rangle\) are Bernoulli measures corresponding to an empty lattice and a full lattice, respectively. It can be easily shown that
\[
\tilde{H}|f_0\rangle = 0, \\
\tilde{H}|f_k\rangle = D|f_{k+1}\rangle + D'|f_{k-1}\rangle - (D + D')|f_k\rangle \quad k \neq 0, L, \\
\tilde{H}|f_L\rangle = 0, \tag{41}
\]
Here \( D \) stands for the rate of increasing the width of double-shock, and \( D' \) stands for the rate of decreasing its width. One can map this model to a model with one particle on lattice with boundaries at \( k = 0 \), and \( k = L \). This particle hops to the right and left with the rates \( D \) and \( D' \), and there are traps at the boundaries. The system has only two stationary state, \(|f_0\rangle\), and \(|f_L\rangle\), means that at large times there is no shock, and the final state is a linear combinations of the Bernoulli measures.
\[
|\tilde{\mathcal{P}}\rangle = q_0|f_0\rangle + q_L|f_L\rangle. \tag{42}
\]
If the initial state is a linear combination of \(|f_k\rangle\)’s then
\[
|\tilde{\mathcal{P}}\rangle(t) = \sum_{k=0}^{L} q_k(t)|f_k\rangle. \tag{43}
\]
Using (40), one arrives at
\[
\dot{q}_0 = D' q_1, \\
\dot{q}_1 = D' q_2 - (D + D') q_1. 
\]
\[ \dot{q}_k = D' q_{k+1} + D q_{k-1} - (D + D') q_k \quad k \neq 0, 1, L - 1, L, \]
\[ \dot{q}_{L-1} = D q_{L-2} - (D + D') q_{L-1}, \]
\[ \dot{q}_L = D q_{L-1}. \]  
(44)

$q_k(t)$’s in the bulk, $(k \neq 0, L)$, can be obtained. They are
\[ q_k(t) = 2 L (D D')^{k/2} e^{- (D + D') t} \sum_{s=1}^{L-1} \sum_{m=1}^{L-1} q_m(0) (D D')^{m/2} \sin \left( \frac{s \pi m}{L} \right) \sin \left( \frac{s \pi k}{L} \right) e^{2 t \sqrt{D D'} \cos (s \pi / L)}. \]  
(45)

One may integrate $q_1(t)$, and $q_{L-1}(t)$ to obtain $q_0(t)$, and $q_L(t)$, which are the only terms surviving at large times. There is also another way to obtain the $q_0$, and $q_L$ at infinitely large times. In fact, there are two constants of motion $I_1$ and $I_2$. $I_1$ is related to the conservation of probability
\[ \langle S | \tilde{P} \rangle = 1 \quad \Rightarrow \quad I_1 := \sum_{k=0}^{L} q_k(t) = 1, \]  
(46)

and
\[ I_2 := \sum_{k=0}^{L} q_k(t) (D')^k. \]  
(47)

It should be noted that the system has two stationary states, so there are two right eigenvectors corresponding to zero eigenvalue for the Hamiltonian $\mathcal{H}$. Therefore there are also two left eigenvectors corresponding to zero eigenvalue for $\mathcal{H}$. These are
\[ \langle S | (\mathcal{S})' \rangle = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \]  
(48)
\[ \langle S' | (\mathcal{S}')' \rangle = \begin{pmatrix} 1 & D' & (D')^2 & (D')^3 & \cdots & (D')^L \end{pmatrix}. \]  
(49)

The second constant of motion can be obtained using $\langle S' | \tilde{P} \rangle(t)$. As long as $D \neq D'$, the constants of motion $I_1$ and $I_2$ are two independent quantities. For $D = D'$, $I_1$ and $I_2$ are the same. But as the stationary state has two-fold degeneracy, there should exist another constant of motion. The second independent constant of motion is $I'_2 := \sum_{k=0}^{L} k q_k(t) = \langle k \rangle$. So, for $D = D'$, the average width of the shock, $\langle k \rangle$, is a constant of motion. One should expect this, because $D$ and $D'$ are the rates for increasing the width of the double shock and decreasing it respectively.

For the double-shock $(0, \rho, 0)$, $D'/D = 1 - \rho < 1$. So, the constant of motions are $I_1$ and $I_2$. $I_1$ is the summation of probabilities for finding a double shock with any width, so it should be equal to one. The second constant of motion also has a physical interpretation. The rate for changing any configuration of a pair of neighboring sites to the state $|00\rangle$ is zero. So the probability for
finding a completely empty lattice does not change with time. $I_2$ is exactly the probability of finding an empty lattice in the initial state

$$I_2 = \sum_{k=0}^{L} q_k(t)(1 - \rho)^k = \sum_{k=0}^{L} q_k(0)(1 - \rho)^k.$$  \hspace{1cm} (50)

Using constant of motions, for $D \neq D'$, at infinitely large times, we have

$$q_0 + q_L = 1,$$

$$q_0 + \left(\frac{D'}{D}\right)^L q_L = \sum_{k=0}^{L} q_k(0)\left(\frac{D'}{D}\right)^k.$$  \hspace{1cm} (51)

Solving these equations one obtains

$$q_0(\infty) = \left[\sum_{k=0}^{L} q_k(0)\left(\frac{D'}{D}\right)^k - \left(\frac{D'}{D}\right)^L\right]/\left[1 - \left(\frac{D'}{D}\right)^L\right],$$

$$q_L(\infty) = \left[1 - \sum_{k=0}^{L} q_k(0)\left(\frac{D'}{D}\right)^k\right]/\left[1 - \left(\frac{D'}{D}\right)^L\right].$$  \hspace{1cm} (52)

As it is seen the contribution of $|f_0\rangle$ and $|f_L\rangle$ in the final state depends on both reaction rates and initial conditions.

The Hamiltonian for the model possessing the double-shock (010), with $D = D'$ is the Hamiltonian for zero temperature Glauber model. This model have been studied in [19, 23, 24]. The average density at each site $\langle n_i \rangle(t)$ at the time $t$ and also all the correlation functions at large times for an infinite lattice have been calculated in [23]. Static and dynamical phase transitions of this model have been also studied in [19]. Here, $D$ is not necessarily equal to $D'$. For $D' > D$, and large $L$, one arrives at

$$q_0(\infty) = 1 - \sum_{k=0}^{L} q_k(0)\left(\frac{D'}{D}\right)^k,$$

$$q_L(\infty) = 1 - q_0(\infty).$$  \hspace{1cm} (53)

If initially only double shocks with finite widths have contributions, in the thermodynamic limit ($L \to \infty$) the system will finally fall in the state $f_0$, but if $D' < D$ it can be seen that both stationary states have contributions in the final state.

For the case $D = D'$, one obtains

$$q_0(\infty) = 1 - \frac{1}{L} \sum_{k=0}^{L} k \ q_k(0) = 1 - \frac{1}{L} \langle k \rangle,$$

$$q_L(\infty) = \frac{1}{L} \sum_{k=0}^{L} k \ q_k(0) = \frac{1}{L} \langle k \rangle.$$  \hspace{1cm} (54)
which means that at large times the system is fully occupied or empty. The probability of finding a fully occupied lattice at large times is equal to the initial average width of the double-shock divided by the size of the lattice.

### 3.2 Double-shock \((0, \rho, 1)\)

Let’s consider the double-shock \((0, \rho, 1)\) on an infinite lattice. The only non-vanishing rate is \(\omega_{23}\), which can be set equal to 1, by a suitable redefinition of time. Direct calculation gives

\[
\mathcal{H}|e_{m,1}\rangle = 0, \\
\mathcal{H}|e_{m,k}\rangle = (1 - \rho)|e_{m+1,k-1}\rangle + \rho|e_{m,k-1}\rangle - |e_{m,k}\rangle, \quad k \neq 1. \tag{55}
\]

It is seen that there is no probability for width increase. If there is initially a shock \(|e_{mk}\rangle\), at later times its width becomes smaller, and at large times there are only double shocks with the width 1. Starting with a linear combination of the shocks, the evolution equation for \(p_{mk}\)’s, can be obtained to be

\[
\dot{p}_{m,1} = (1 - \rho)p_{m-1,2} + \rho p_{m,2}, \\
\dot{p}_{m,k} = (1 - \rho)p_{m-1,k+1} + \rho p_{m,k+1} - p_{m,k} \quad k \neq 1. \tag{56}
\]

Defining \(q_k\), through (38), one arrives at

\[
\dot{q}_1 = q_2, \\
\dot{q}_k = q_{k+1} - q_k \quad k \neq 1. \tag{57}
\]

If initially the state of the system is a double shock, e.g. \(|e_{MK}\rangle\). Then, it is obvious that at later times there are only double shocks with the position of the first shock in the range \(M \leq m \leq M + K - 1\), and with the width \(1 \leq k \leq M + K - m\). Let’s assume the initial state is

\[
|\hat{\mathcal{P}}\rangle = \sum_{k=0}^{L} q_k(0)|f_k\rangle, \tag{58}
\]

where \(|f_k\rangle\) is the state of double shocks with the width \(k\). Then

\[
\dot{q}_0 = 0, \\
\dot{q}_1 = q_2, \\
\dot{q}_k = q_{k+1} - q_k, \quad 2 \leq k \leq L - 1, \\
\dot{q}_L = -q_L, \\
\dot{q}_k = 0, \quad L + 1 \leq k. \tag{59}
\]

The above equations show that at large times there are only contributions of the double shocks with the width 1. These set of equation can be solved leading to

\[
q_k(t) = \sum_{n=0}^{L-k} q_{k+n}(0) \frac{t^n}{n!} e^{-t} \quad 2 \leq k \leq L. \tag{60}
\]
This together with \( \dot{q}_1 = q_2 \) can be used to obtain \( q_1(t) \).

\[
q_1(t) = q_1(0) + \sum_{n=0}^{L-2} q_{n+2}(0) \int_0^t \frac{t^n}{n!} e^{-t'} dt' = \sum_{n=0}^L q_n(0) - \sum_{n=0}^{L-2} \sum_{m=0}^n q_{n+2}(0) \frac{t^m}{m!} e^{-t}.
\]

(61)

As it is expected at large times all the double shocks changes to the double-shock with the width 1.

Let’s study the distribution of these double shocks at large times. Using (55), and defining \( A_{m,k} := \exp(tH) |e_{mk}\rangle \), it is seen that

\[
\frac{\partial A_{m,k}}{\partial t} + A_{m,k} = (1 - \rho) A_{m+1,k-1} + \rho A_{m,k-1} \quad k \neq 1, \\
A_{m,1} = |e_{k1}\rangle.
\]

(62)

At large times this equation recasts to

\[
A_{m,k}(\infty) = (1 - \rho) A_{m+1,k-1}(\infty) + \rho A_{m,k-1}(\infty) \quad k \neq 1,
\]

(63)

whose solution is obtained to be

\[
A_{m,k}(\infty) = \lim_{t \to \infty} \exp(tH) |e_{mk}\rangle = \sum_{j=0}^{k-1} \binom{k-1}{j} (1 - \rho)^j \rho^{k-1-j} |e_{m+j,1}\rangle.
\]

(64)

So, at large times the state of the system is a linear combination of double shocks with the width 1. The distribution of the position of these double shocks is a binomial distribution. Let’s consider the initial state to be a double shock with the width \( k \), \( |e_{0,k}\rangle \), then the average position of the first shock at large times is

\[
\langle j \rangle = (k - 1)(1 - \rho),
\]

(65)

and the width of the binomial distribution is \( \sqrt{\rho(1 - \rho)(k - 1)} \).

4 Multi shocks

Combining single shocks one may construct multi shocks. The only models with multi shocks are as following

\begin{itemize}
  \item 1. \((\rho_1, \rho_2, \rho_3, \cdots)\)

The span of multi shocks \((\rho_1, \rho_2, \rho_3, \cdots)\) is an invariant subspace of Hamiltonian of ASEP provided the densities satisfy

\[
\frac{\rho_{i+1}(1 - \rho_i)}{\rho_i(1 - \rho_{i+1})} = \frac{\omega_{23}}{\omega_{12}}.
\]

(66)
\end{itemize}
It can be easily seen that the rate of hoping of the $i$th shock to the left, $d_i$, and also the rate of hoping of the $i$th shock to the right, $d'_i$, is given by

\[
d_i = \frac{\omega_{23} \rho_i}{\rho_{i+1}}
\]

\[
d'_i = \frac{\omega_{32} \rho_{i+1}}{\rho_i}.
\] (67)

This result are obtained in [3].

- 2. $(0, \rho, 0, \rho, \cdots)$ and $(0, 1, 0, 1, \cdots)$

The model admitting multi shock of the type $(0, \rho, 0, \rho, \cdots)$ are that admitting double shocks $(0, \rho, 0)$, or $(\rho, 0, \rho)$. The model Possessing multi shocks of the type $(0, 1, 0, 1, \cdots)$ is an asymmetric generalization of the zero temperature Glauber model. In both of these multi shocks there are edges at shock points. The edges are destroyed two by two. Let’s consider a multi shock of order $N$ with the first shock at the site $m$. It is seen that the action of Hamiltonian on such state is

\[
\mathcal{H}|e_{m,k_1,\cdots,k_{N-1}}\rangle = d_1|e_{m-1,k_1+1,\cdots,k_{N-1}}\rangle + d'_1|e_{m+1,k_1-1,\cdots,k_{N-1}}\rangle + d_2|e_{m,k_1-1,\cdots,k_{N-1}}\rangle + d'_2|e_{m,k_1+1,\cdots,k_{N-1}}\rangle + \cdots + d_2|e_{m,k_1,\cdots,k_{N-1}-1}\rangle + d'_2|e_{m,k_1,\cdots,k_{N-1}+1}\rangle
\] (68)

If any of the $k_i$s in the left hand side is equal to one, at the right hand side there will be a multi shock of order $N - 2$. So there is a finite probability that the system falls in a state with lower shocks, and there is no probability for increasing the number of shocks. In fact if there exists a state for which any state can transform directly or even indirectly to it, and that state has no evolution, then that state is the final stationary state. Let’s consider a periodic lattice. The number of shocks, $N$, should be even. So at large times the state of system is a state with no shock. For the models on an infinite lattice number of shocks, $N$, may be even or odd. Then For odd $N$ the final state is a linear combination of single shocks.

5 Summary

There are three types of models with travelling wave solutions on a one-dimensional lattice. These are classified in [4]. It is seen that there are four type of models admitting double shocks. Double shocks and the models admitting these double shocks are as following

- $(\rho_1, \rho_2, \rho_3)$. Nonvanishing rates are $\omega_{23}$, and $\omega_{32}$.
- $(0, \rho, 0)$, (and also $(\rho, 0, \rho)$). Nonvanishing rates are $\omega_{23}, \omega_{24}, \omega_{32}, \omega_{34}, \omega_{42}$ and $\omega_{43}$. 

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• (0, 1, 0). Nonvanishing rates are $\omega_{13}, \omega_{12}, \omega_{42}, \omega_{43}$.

• (0, $\rho$, 1). The only nonvanishing rate is $\omega_{23}$.

There are three type of models admitting multi shocks. The multi shocks are of the type

• $(\rho_1, \rho_2, \rho_3, \cdots)$. Nonvanishing rates are $\omega_{23}$, and $\omega_{32}$.

• (0, $\rho$, 0, $\rho$, $\cdots$). Nonvanishing rates are $\omega_{23}, \omega_{24}, \omega_{32}, \omega_{34}, \omega_{42}$ and $\omega_{43}$.

• (0, 1, 0, 1, $\cdots$). Nonvanishing rates are $\omega_{13}, \omega_{12}, \omega_{42}, \omega_{43}$.

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References

[1] B. Derrida, L. Lebowitz, and E. R. Speer, Journal of Statistical Physics 89 (1997) 135.

[2] V. Popkov, and G. M. Schütz; Journal of Statistical Physics 112, (2003) 523.

[3] V. Belitsky, and G.M. Schütz; Electronic Journal of Probability 7 (2002) 1.

[4] K. Krebs., F.H. Jafarpour, and G.M. Schütz; New Journal of physics 5 (2003) 145.1.

[5] C. Pigorsch and G. M. Schütz; Journal of physics A33 (2000) 7919.

[6] F.H. Jafarpour; Physics Letters A326 (2004) 14.

[7] M. Paessens and G. M. Schütz; New Journal of physics 6(2004) 120.

[8] F.H. Jafarpour, F. E. Ghafari, and S. R. Masharian; Journal of physics A38 (2005) 4579-4588

[9] F.H. Jafarpour; Physica A 358 (2005) 413.

[10] F.H. Jafarpour, and S. R. Masharian; Physical Review E70 (2004) 056121.

[11] A. Rákos and G. M. Schütz; Journal of Statistical Physics 117 (2004) 55.

[12] V. Popkov 1, A. Rakos, R.D. Willmann, A.B. Kolomeisky, and G.M. Schütz; Physical Review E67, (2003) 066117.

[13] S. Mukherji, and S. M. Bhattacharjee; Journal of physics A38 (2005) L285.

[14] A. Parmeggiani, T. Franosch, and E. Fraey; Physical Review Letters 90, (2003) 086601.

[15] A. Rakos, M. Paessens, and G.M. Schütz; Physical Review Letters 91, (2003) 238302.

[16] M. R. Evans, R. Juhasz, and L. Santen; Physical Review E68, (2003) 026117.

[17] S. M. Bhattacharjee; cond-mat/0604444.

[18] M. Arabsalmani and A. Aghamohammadi; Physical Review E74 (2006) 011107.

[19] M. Khorrami and A. Aghamohammadi; Physical Review E63 (2001) 042102.

[20] N. Majd, A. Aghamohammadi, and M. Khorrami; Physical Review E64 (2001) 046105.
[21] M. Khorrami and A. Aghamohammadi; Physical Review E65 (2002) 056129.

[22] A. Aghamohammadi and M. Khorrami; Journal of physics A34 (2001) 7431.

[23] A. Aghamohammadi and M. Khorrami; Journal of physics A33 (2000) 7843.

[24] F. Roshani, A. Aghamohammadi, and M. Khorrami; Physical Review E70 (2004) 056128.

[25] F. C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg; Annals of Physics 230 (1994) 250.

[26] K. Krebs, M. P. Pfannmüller, B. Wehefritz, and H. Hinrichsen; Journal of Statistical Physics 78[FS] (1995) 1429.

[27] G. M. Schütz and E. Domany; Journal of Statistical Physics 72(1993) 277; S. Sandow; Physical Review E50 (1994) 2660; K. Mallick and S. Sandow; Journal of Physics A30 (1997) 4513.

[28] G. Schütz; Journal of Statistical Physics 79 (1995) 243.

[29] A. Aghamohammadi, A. H. Fatollahi, M. Khorrami, and A. Shariati; Physical Review E62 (2000) 4642.

[30] A. Shariati, A. Aghamohammadi, and M. Khorrami; Physical Review E64 (2001) 066102.