Conserving the lepton number $L_e - L_{\mu} - L_{\tau}$ in the exact solution of a 3-3-1 gauge model with right-handed neutrinos

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Abstract

In this paper we consider a plausible scenario with conserved lepton number $L = L_e - L_{\mu} - L_{\tau}$ within the framework of the exact solution of a particular 3-3-1 gauge model. We discuss the consequences of conserving this global leptonic symmetry from the viewpoint of the neutrino mass matrix constructed via special Yukawa terms (involving tensor products among Higgs triplets). We prove that the actual experimental data can naturally be reproduced by our scenario since soft breaking terms with respect to this lepton symmetry are properly introduced. As a consequence, our solution predicts for the neutrino sector the correct mass splitting ratio ($\Delta m_{12}^2/\Delta m_{23}^2 \approx 0.033$), the inverted mass hierarchy, the correct values for the observed mixing angles ($\sin^2 \theta_{23} \approx 0.5$ and $\sin^2 \theta_{12} = 0.31$) and the absolute mass of the lightest neutrino ($m_0 \sim 0.001$eV) independent of the breaking scale of the model.

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1 Introduction

In a recent series of papers (Refs. [1, 2, 3]), the author has developed an original method for contructing the neutrino mass matrix within the framework of a particular 3-3-1 gauge model with no other restrictive additional symmetries (such as the lepton number). A proper tensor product among the Higgs triplets - designed to recover (successively the spontaneous symmetry breakdown) the well known mass-generating Yukawa terms in the unitary gauge - is exploited within the exact solution of a model based on the gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ that does not contain particles with exotic electric charges. Here we mean by the exact solution (see for further details the general prescriptions in Ref. [4]) a specific algebraical method relying on an appropriate parametrization in the scalar sector of the model. It provides us with the exact mass eigenstates and mass eigenvalues for the gauge bosons and the charges (both the electric and neutral ones) of all the involved particles, just by exactly solving certain equations.
The charged fermions acquire their masses through traditional Yukawa couplings. At the same time, they essentially determine the texture of the neutrino mass matrix, since the same coupling coefficient acts for both the charged lepton and its neutrino partner. All these results can be achieved [1] just by tuning a sole free remaining parameter in the model. Regarding the neutrino sector, the predictions claimed by this method include [2]: the inverted mass hierarchy, the bi-maximal mixing and - for the first time in the literature, as we know - the minimal absolute value in the neutrino mass spectrum $m_0 \simeq 0.0035\text{eV}$.

The price of dealing with only one free parameter resided in a very large breaking scale of the model and, consequently, in some very heavy new gauge bosons that had largely overtaken their lower experimental mass limit [5] (around 1 TeV for non-SM bosons). In order to improve the phenomenological consequences regarding the breaking scale, a new approach was proposed: a canonical seesaw mechanism can arise in the model, just by altering the parameter matrix of the scalar sector with a small amount (a fine-tuning parameter) [3]. This second parameter allows for decoupling the neutrino phenomenology from the breaking scale issue (generating thus reasonable masses in the boson sector), while keeping unaltered all the neutrino masses and mixing angles achieved in the one-parameter version [2].

Here we intend to improve the original one-parameter solution of the 3-3-1 model of interest (briefly presented in Sec. 2) in a different way. This time, a new symmetry is taken into consideration, namely the global lepton number $L = L_e - L_\mu - L_\tau$ (Sec. 3) and its implications for our model are discussed. Then, some new small parameters ($\alpha, \beta, \gamma$) are introduced in the special Yukawa terms in order to softly break this global lepton symmetry. When the $\mu - \tau$ interchange symmetry is invoked the parameters get a particular ratio, namely $\gamma/\beta = m(\mu)/m(\tau)$. Under these circumstances, the diagonalization of the mass matrix leads to predictions in good agreement with the experimental values for mass splitting ratio and mixing angles. We also consider as a big success of our scenario the minimal absolute neutrino mass computed independent of any parameter, depending only on the precise account for the mixing angles. A few remarks concerning phenomenological aspects of our method are sketched in the final section (Sec. 4).

2 Preliminaries

In order to get to the very essence of the resulting phenomenology of introducing the global lepton symmetry $L = L_e - L_\mu - L_\tau$ into the 3-3-1 model without exotic electric charges, we start by presenting the particle content of this model and the original manner to generate neutrino mass matrix via special Yukawa terms.

2.1 Fermion content of the model

The fermion sector of the pure left (subscript $L$) 3-3-1 gauge model with right-handed neutrinos (namely model D in Ref. 6) consists of two distinct sectors: lepton families and quark families. All the lepton generations obey the same representation with respect to the gauge group:
The quarks come (according to the anomaly cancellation requirement that preserves the renormalizability) in three distinct generations as follows:

\[ Q_{iL} = \begin{pmatrix} u_i \\ d_i \\ D_i \end{pmatrix}_L \sim (3, 3, 0) \quad Q = \begin{pmatrix} d \\ u \\ U \end{pmatrix}_L \sim (3, 3^*, 1/3) \]

\[ (d_L^c)^c, (d_L^c)^c \sim (3^*, 1, 1/3) \quad (u_L^c)^c, (u_L^c)^c \sim (3^*, 1, -2/3) \]

\[ (U_L^c)^c \sim (3^*, 1, -2/3) \quad (D_L^c)^c \sim (3^*, 1, 1/3) \]

with \( i = 1, 2 \) and capital letters denoting exotic quarks. The numbers in brackets following the left-handed fermion fields (Eqs. (1) - (4)) label the representations and their characters with respect to the gauge group \( SU(3)_c \otimes SU(3)_L \otimes U(1)_X \). Note that the third quark family has to obey a different representation compared to the other two families.

The main phenomenology of the exact solution of this model can be found in Ref. [1] where a special generalized Weinberg transformation is also given in order to separate the neutral bosons (\( A_e \) - the electromagnetic one, \( Z^\prime_\mu \) - the Weinberg boson from Standard Model, and \( Z'^{\prime}_e \) - the new neutral boson of the present theory) that get their exact eigenstates. Here we are concerned only with the neutrino mass issue since some special coupling terms are introduced in the Yukawa sector and a global symmetry is added as well.

### 2.2 The resulting neutrino mass matrix

The masses of the fermions in the model are generated by the Yukawa Lagrangian. In our approach [1] it looks like:

\[ G_{\alpha\beta} \bar{f}_{\alpha L} (\phi^{(\rho)} e^c_{\beta L} + S f^c_{\beta L}) + h.c. \]

with \( S = \phi^{-1} (\phi^{(n)} \otimes \phi^{(x)} + \phi^{(x)} \otimes \phi^{(n)}) \sim (1, 6, -2/3) \) and \( G_{\alpha\beta} \) as the coupling coefficients of the lepton sector. The Yukawa sector relies on the scalar triplets \( \{ \phi^{(\rho)} \sim (1, 3^*, 2/3), \phi^{(x)}, \phi^{(n)} \sim (1, 3^*, -1/3) \} \) (see for details of constructing the Higgs sector in 3-3-1 models with no exotic electric charges Ref. [1]). We recall that the exact solution for the 3-3-1 model of interest here leads to the one-parameter matrix \( \eta = (1 - \eta^2_0) \text{diag} [a/2 \cos^2 \theta_W, 1 - a, a (1 - \tan^2 \theta_W)/2] \) which determines
the VEVs alignment in the Higgs sector \( \langle \phi^{(i)} \rangle = \eta^i \langle \phi \rangle, i = 1, 2, 3 \) (successively the SSB), according to the general prescriptions of the method shown in Sec. 4.1 of Ref. [4]. Obviously, \( \theta_W \) is the Weinberg angle from SM.

Note that the Higgs field \( \phi \) plays the role of the "norm" for the geometrized scalar sector and it is therefore a main factor in all the VEVs. It is appropriate for one to consider it as the homologue of the neutral scalar field of the Standard Model, since their vacuum expectation values supply the masses for the particle of the model they act in.

The charged leptons get their masses via traditional Yukawa couplings as one can easily observe in Eq.(5): 
\[ m(e) = A \langle \phi^{(\rho)} \rangle, \quad m(\mu) = B \langle \phi^{(\rho)} \rangle, \quad m(\tau) = C \langle \phi^{(\rho)} \rangle. \]

Obviously, \( G_{ee} = A, \quad G_{\mu\mu} = B, \quad G_{\tau\tau} = C \), but we specify in advance that our notations include: \( G_{e\mu} = D, \quad G_{e\tau} = E, \quad G_{\mu\tau} = F \) for the off-diagonal terms in the flavor basis.

Neutrino mixing is expressed by 
\[ \nu_{\alpha L}(x) = \sum_{i=1}^{3} U_{\alpha i} \nu_{iL}(x), \]
where \( \alpha = e, \mu, \nu \) label the flavor space (flavor eigenstates) while \( i = 1, 2, 3 \) denote the massive physical eigenstates. We consider throughout the paper the physical neutrinos as Majorana fields, i.e. \( \nu_{iL}(x) = \nu_{iL}(x) \). The neutrino mass term in the Yukawa sector yields then:
\[ -L_Y = \frac{1}{2} \bar{\nu}_L M \nu_L^c + H.c \quad (6) \]

with \( \nu_L = (\nu_e \quad \nu_\mu \quad \nu_\nu)^T \) where the superscript \( T \) stands for "transposed". The mixing matrix \( U \) that diagonalizes the mass matrix in the manner \( U^+ M U = m_{ij} \delta_{ij} \) has in the standard parametrization the form:
\[ U = \begin{pmatrix}
  c_2 c_3 & c_1 s_3 e^{i \delta} & s_3 e^{-i \delta} \\
  -s_2 c_1 - c_2 s_1 s_3 e^{i \delta} & c_1 c_2 - s_2 s_3 s_1 e^{i \delta} & c_3 s_1 \\
  s_2 s_1 - c_1 c_2 s_3 e^{i \delta} & -s_1 c_2 - s_2 s_3 c_1 e^{i \delta} & c_3 c_1
\end{pmatrix} \quad (7) \]

with natural substitutions: \( \sin \theta_{23} = s_1 \), \( \sin \theta_{12} = s_2 \), \( \sin \theta_{13} = s_3 \), \( \cos \theta_{23} = c_1 \), \( \cos \theta_{12} = c_2 \), \( \cos \theta_{13} = c_3 \) for the mixing angles, and \( \delta \) for the CP phase.

Our procedure provides (after the SSB) the following new symmetric mass matrix for the neutrinos involved in the model:
\[ M = 4 \begin{pmatrix}
  A & D & E \\
  D & B & F \\
  E & F & C
\end{pmatrix} \begin{pmatrix}
  \langle \phi^{(\rho)} \rangle \\
  \langle \phi^{(\chi)} \rangle \\
  \langle \phi \rangle
\end{pmatrix} \quad (8) \]

The next task is to diagonalize the matrix (8) in order to get the physical eigenstates of the massive neutrinos. This procedure will lead (within the Case 1 in Ref. [1], which is the only acceptable one from the phenomenological point of view) to the following generic solution:
\[ m_i = f_i [\theta_{12}, \theta_{23}, \theta_{13}, m(e), m(\mu), m(\tau), \frac{2a}{\sqrt{1 - a}} \sqrt{1 - 2 \sin^2 \theta_W} \cos^2 \theta_W] \quad (9) \]

with \( i = 1, 2, 3 \). In these expressions \( f, s \) are analytical functions depending on the mixing angles and the charged lepton masses in a particular way to be determined.
3 Conserving the lepton number $L = L_e - L_\mu - L_\tau$

In the following we assume that the lepton number $L = L_e - L_\mu - L_\tau$ is conserved (or approximately conserved, as suggested by a lot of papers concerning this issue [7] - [21]) in the 3-3-1 model presented above. We first analyze the consequences of this global symmetry for the neutrino sector and then the modifications provided by soft-breaking terms with respect to this symmetry. The modifications occur in the mass matrix due to some new small parameters introduced in the Yukawa sector.

3.1 Neutrino mass spectrum

At this point, if one imposes for the lepton sector an additional global symmetry given by the lepton number $L = L_e - L_\mu - L_\tau$, certain coupling coefficients in the Yukawa Lagrangian get suppressed, namely $A = B = C = F = 0$, since the following assignment for the lepton number holds: $L(f_{eL}) = L(e_R) = 1$, $L(f_{\mu L}) = L(\mu_R) = -1$ and $L(f_{\tau L}) = L(\tau_R) = -1$, while all the scalar fields carry zero lepton number.

Hence, the mass matrix with the exact global $L$ symmetry becomes:

$$M = \begin{pmatrix} 0 & D & E \\ D & 0 & 0 \\ E & 0 & 0 \end{pmatrix} \begin{pmatrix} 2a \\ \sqrt{1-a} \sin 2\theta_w \cos^2 \theta_W \langle \phi \rangle \end{pmatrix} \tag{10}$$

The concrete forms of $f_i$s remain to be computed by solving the following set of equations:

$$\begin{cases} 
  f_1 = -2c_1c_2s_2D + 2s_1c_2s_2E \\
  0 = -c_1s_2^2D + c_1s_2^2D - s_1c_2^2E + s_1s_2^2E \\
  0 = c_2s_1D + c_2c_1E \\
  0 = c_1c_2^2D - c_1s_2^2D + s_1s_2^2E - s_1c_2^2E \\
  f_2 = 2c_1c_2s_2D - 2s_1c_2s_2E \\
  0 = s_1s_2D + c_1s_2E \\
  0 = s_1c_2D + c_1c_2E \\
  0 = s_1s_2D + c_1s_2E \\
  f_3 = 0 
\end{cases} \tag{11}$$

obtained straightforwardly from Eq. (10) via $U^+MU = \text{diag}(m_1, m_2, m_3)$. The lines 3, 6, 7 and 8 in Eqs. (11) express the same condition, namely: $D = -E \cot \theta_{23}$. The lines 2 and 4 in the set of equations (11) are fulfilled simultaneously if and only if $\cos^2 \theta_{12} = \sin^2 \theta_{12}$ (maximal atmospheric mixing angle).

Under these circumstances, taking into consideration the maximal atmospheric mixing angle too, the solution reads:

$$m_1 = |m_2| = \sqrt{2D} \begin{pmatrix} 2a \\ \sqrt{1-a} \end{pmatrix} \begin{pmatrix} \sqrt{1-2\sin^2 \theta_W} \cos^2 \theta_W \langle \phi \rangle \end{pmatrix} \tag{12}$$

$$m_3 = 0 \tag{13}$$
giving rise to a $\mu - \tau$ interchange symmetry \[22\] - \[30\].

### 3.2 Introducing soft breaking terms

We exploit in the following the consequences of assuming some soft breaking terms with respect to the global lepton number $L$, by introducing the free parameters $\alpha$, $\beta$, $\gamma$ in the Yukawa Langrangian:

\[
\begin{align*}
G_{ee} &\bar{e}_e L (\phi(\rho) e^c_L + \alpha S f_{eL}^c) + \text{h.c.} \\
G_{\mu\mu} &\bar{\mu}_\mu L (\phi(\rho) \mu^c_L + \beta S f_{\mu L}^c) + \text{h.c.} \\
G_{\tau \tau} &\bar{\tau}_\tau L (\phi(\rho) \tau^c_L + \gamma S f_{\tau L}^c) + \text{h.c.}
\end{align*}
\]  

(14)

Obviously, $\alpha, \beta, \gamma \in [0, 1]$. A small nonzero $F = \beta G_{\mu \tau}$ coupling coefficient is also allowed. Now, the mass matrix of the neutrino sector stands:

\[
M = \begin{pmatrix}
\begin{array}{ccc}
\alpha A & D & D \\
D & \beta B & \beta F \\
D & \beta F & \gamma C
\end{array}
\end{pmatrix}
\left(\frac{2a}{\sqrt{1-a}}\right) \frac{\sqrt{1 - 2 \sin^2 \theta_W}}{\cos^2 \theta_W} \langle \phi \rangle
\]  

(15)

In order to remain in the spirit of the lepton symmetry $L = L_e - L_\mu - L_\tau$ which also favors the interchange symmetry $\mu - \tau$, one can approximate:

\[
\beta B \approx \gamma C
\]  

(16)

Hence, one of the three new parameters has disappeared. The two remaining ones finally will have to fulfill a particular ratio in order to be compatible with the phenomenological data, so one remains with only one parameter to be tuned apart from the main one ($a$ in our model).

The new resulting mass matrix

\[
M(\nu) = \begin{pmatrix}
\begin{array}{ccc}
\alpha A & D & D \\
D & \beta B & \beta F \\
D & \beta F & \beta B
\end{array}
\end{pmatrix}
\left(\frac{2a}{\sqrt{1-a}}\right) \frac{\sqrt{1 - 2 \sin^2 \theta_W}}{\cos^2 \theta_W} \langle \phi \rangle
\]  

(17)

is now to be diagonalized. This task was already carried out in Ref. \[2\] for the general case, now we just have to insert the new diagonal entries in the solution obtained there (Eqs. (13) in Ref. \[2\]) and to impose the maximal atmospheric mixing angle at least in the numerators, while the denominators will be computed in the final stage of solving the mass issue by inserting a suitable approximation for the maximal atmospheric mixing angle. Hence, an interesting mass split is obtained:

\[
f_1 = -\frac{\beta B}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} \sin^2 \theta_{12} + \frac{\alpha A}{(1 - 2 \sin^2 \theta_{12})} \left(1 - \sin^2 \theta_{12}\right),
\]

\[
f_2 = \frac{\beta B}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} \sin^2 \theta_{12} + \frac{\alpha A}{(1 - 2 \sin^2 \theta_{12})} \sin^2 \theta_{12},
\]

\[
f_3 = \beta B.
\]  

(18)
Evidently, it is required a $\gamma^5$ transformation performed on the first neutrino field in order to get the sign change for its mass ($m_1$).

The mass spectrum in the neutrino sector becomes:

$$|m_1| = \left[ \frac{\beta m(\mu) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} - \frac{\alpha m(e) (1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{12})} \right] \left( \frac{2a}{\sqrt{1 - a}} \right) \sqrt{1 - 2 \sin^2 \theta_W} \frac{\cos^2 \theta_W}{\cos^2 \theta_W},$$

$$m_2 = \left[ \frac{\beta m(\mu) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} + \frac{\alpha m(e) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})} \right] \left( \frac{2a}{\sqrt{1 - a}} \right) \sqrt{1 - 2 \sin^2 \theta_W},$$

$$m_3 = \beta m(\mu) \left( \frac{2a}{\sqrt{1 - a}} \right) \sqrt{1 - 2 \sin^2 \theta_W} \frac{\cos^2 \theta_W}{\cos^2 \theta_W}$$

(19)

### 3.3 Phenomenological predictions

The physical relevant magnitudes for the neutrino oscillations are the mass squared differences for solar and atmospheric neutrinos, defined as: $\Delta m_{12}^2 = m_2^2 - m_1^2$ and $\Delta m_{23}^2 = m_3^2 - m_2^2$. They result from the above expressions (Eqs. (19)):

$$\Delta m_{12}^2 \approx 2\alpha\beta \frac{m(e)m(\mu) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})^2 (1 - 2 \sin^2 \theta_{23})} \left( \frac{4a^2}{1 - a} \right) \frac{1 - 2 \sin^2 \theta_W}{\cos^2 \theta_W}$$

(20)

$$\Delta m_{23}^2 \approx \beta^2 \frac{m(\mu) \sin^4 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})^2} \left( \frac{4a^2}{1 - a} \right) \frac{1 - 2 \sin^2 \theta_W}{\cos^2 \theta_W}$$

(21)

The mass splitting ratio defined as $r_\Delta = \Delta m_{12}^2/\Delta m_{23}^2$ yields in our scenario:

$$r_\Delta = 2 \left( \frac{\alpha}{\beta} \right) \frac{m(e)}{m(\mu)} \frac{1 - 2 \sin^2 \theta_{23}}{\sin^2 \theta_{12}}$$

(22)

As in all the cases where our method of generating neutrino masses is employed, the mass splitting ratio comes out independently of the breaking scale in the theory. The latter is determined by the parameter $a$ which is not involved in formula (22). Once again, we have got an important conclusion: the breaking scale of the model (and consequently, the boson mass spectrum) and the neutrino phenomenology do not influence one another (the same happens in Ref. [3], but a different strategy based on a seesaw mechanism was employed therein). Therefore, as soon as new data regarding the precision measurements for the mass of the new neutral boson (other than the Weinberg boson from SM) are available, one can establish the parameter $a$. As far as we know [5], the lower limit is $m(Z') \geq 1.5$TeV, that claims in our solution for the 3-3-1 model: $a \leq 0.06$ and $\langle \phi \rangle \geq 1$TeV [1][3].

The next step is to implement the specific values for the atmospheric and solar mixing angles into formula (22) and estimate how they can influence the mass splitting.
ratio. For instance, if the plausible $\sin^2 \theta_{23} = 0.499$ and $\sin^2 \theta_{12} = 0.31$ are taken into account, and assuming the charged lepton masses $m(e) = 0.511$ MeV and $m(\mu) = 106$ MeV, one obtains $r_\Delta \simeq 0.033$ (very close to the value supplied by data) if

$$\frac{\alpha}{\beta} = 530.5$$

(23)

This estimate suggests that the added matrix responsible for the soft violation of the initial conserved $L$ symmetry can be actually considered as a small perturbation of the form: $\delta M = \varepsilon \text{diag}(1, 1, 1)$.

The method presented above allows one to estimate the sum of the absolute masses in the neutrino sector:

$$\sum_{i=1}^{3} m_i \simeq \frac{2\beta m(\mu) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})} \left( \frac{2\alpha}{\sqrt{1 - 2 \sin^2 \theta_W}} \right) \sqrt{1 - 2 \sin^2 \theta_W}$$

(24)

This is experimentally restricted to: $\sum_{i=1}^{3} m_i \sim 1$ eV, if we take into consideration the Troitsk [31] and Mainz [32, 33] experiments. On the other hand, combining Eqs. (19) and (23) one obtains:

$$\sum_{i=1}^{3} m_i = \frac{2 \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})} m_0$$

(25)

with minimal neutrino mass $m_0 = m_3$. This leads (with the above considered values for mixing angles) to $m_0 \simeq 0.001$ eV.

### 4 Concluding remarks

In this paper we have developed a strategy that combines the exact solution of a particular 3-3-1 gauge model with possible global leptonic symmetry $L = L_e - L_\mu - L_\tau$. When this additional symmetry is rigourously conserved, the neutrino mass spectrum exhibits an inverted hierarchy with two degenerate masses and the third one equal to zero. At the same time this symmetry restricts the mixing angles to the bi-maximal setting only. This state of affairs can be naturally overtaken if one deals with the approximate global leptonic symmetry, by introducing some small terms that softly violate this symmetry in the Yukawa sector. The results are amazing: the correct predictions regarding the mass splitting ratio $\Delta m^2_{12}/\Delta m^2_{23} \simeq 0.033$ and the observed values for the mixing angles $\sin^2 \theta_{23} \approx 0.5$ and $\sin^2 \theta_{12} = 0.31$ arise independently of the main parameter $\alpha$ in the 3-3-1 model of interest. At the same time, the minimal absolute mass in the neutrino sector can be computed based on an exact formula depending only on the accurate account for the mixing angles. In our approximation, the scenario leads to the minimal mass in the neutrino spectrum: $m(\nu_3) = 0.001$ eV, but this order of magnitude can decrease if the atmospheric angle gets closer to maximal values.

All the results regarding the mass squared differences and the mixing angles are achieved just by tuning a second small parameter, let it be $\alpha$ or $\beta$. There are many
advantages of the present method over the previous ones followed by the author in Refs. 2 and 3 respectively. In the one-parameter case 2 with no lepton number conserved and no $\mu - \tau$ interchange symmetry, the neutrino phenomenology required a very large breaking scale and a radiative mechanism was set to supply possible deviations from the bi-maximal mixing. In Ref. 3 a second parameter was introduced (that time in the Higgs sector) accompanied by seek for a canonical seesaw interpretation in order to get a considerable autonomy for the neutrino phenomenology from the breaking scale issue. Here, although a second parameter is introduced too, there is no need for any seesaw mechanism. Instead, finally a $\mu - \tau$ interchange symmetry remains. Along with all the old valuable predictions supplied by the exact solution of our model, the approach based on this second parameter explains the neutrino mass phenomenology and, in addition, successfully passes the difficult challenge of predicting the correct solar and atmospheric mixing angles.

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