Semi-classical formula for quantum tunneling in asymmetric double-well potentials

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Despite quantum tunneling has been studied since the advent of quantum mechanics, the literature appears to contain no simple (textbook) formula for tunneling in generic asymmetric double-well potentials. In the regime of strong localization, we derive an succinct analytical formula based on the WKB semi-classical approach. Two different examples of asymmetric potentials are discussed: when the two localized levels are degenerate or not. For the first case, we also discuss a time-dependent problem showing quantum Zeno effect.

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I. INTRODUCTION

Quantum tunneling is of continuing interest in many contemporary areas of physics [1]. The simplest problem can be formulated as a degree of freedom whose potential energy $V(x)$ has a double-well shape. In the classical limit and zero temperature one expects that an initial state prepared in one well is stable. Quantum tunneling allows the possibility to escape from one side to the other passing under the classically forbidden region. Canonical examples are an nitrogen atom in ammonia molecule [2] or an electron in a double quantum dot [3].

However, in recent years, there has be a breakthrough in the experimental study of macroscopic quantum tunneling, which is being used to create and study ”Schrödinger-cat” states. Examples include Bose-Einstein condensates in a double trap [4–7] and quantum superconducting circuits based on Josephson junctions [8–14]. Macroscopic quantum tunneling is important to test the validity of the quantum mechanics on scales larger than the atomic one [15]. The investigation of these fundamental issues will be also useful for advanced technological applications, such as the development of devices for quantum information processing [16].

These experiments involve a system tunneling from one macroscopic state to another. Despite the complexity of this process, it is often remarkably well described by the physics of a single particle in a double-well potential in which the variable $x$ corresponds to a collective macroscopic variable.

The double-well needs not be symmetric, and in many experiments the asymmetry can be changed by modifying externally some tunable parameters (see, for instance, Refs. [12,14], Ref. [3] and references therein).

One can formulate the problem as a potential $V_{\eta}(x)$ whose shape depends on a dimensionless parameter $\eta$ that quantifies the asymmetry, i.e. $\eta = 0$ corresponding to the symmetric case. In the limit of high energy barrier, i.e. $V_0 \gg \hbar \omega$ where $V_0$ is the energy scale of the barrier and $\omega$ the typical harmonic frequency in the wells, the low-energy physics reduces to the standard two-level system: $\varepsilon_L(\eta), \varepsilon_R(\eta)$ are the energies of two localized states coupled by quantum tunneling with amplitude $\nu(\eta)$.

The last parameter $\nu(\eta)$ has to be determined from the given double-well potential. Remarkably, the literature appears to contain no simple (textbook) formula for tunneling in generic asymmetric double-well potentials [17–21]. Few exceptions are the works in Refs. [14, 22–25] in which general methods based on sophisticated techniques are discussed but useful analytical formulas are presented only for specific shapes of the potential.

In this work we revisit the problem. By using the standard WKB-approach, we demonstrate that it is possible to express the amplitude $\nu(\eta)$ as the simple formula:

\begin{equation}
\nu(\eta) = A(\eta) \sqrt{\nu_L(\eta)\nu_R(\eta)},
\end{equation}

\begin{equation}
A(\eta) = \frac{1}{2} \left[ \left( \frac{V(0) - \varepsilon_L(\eta)}{V(0) - \varepsilon_R(\eta)} \right)^{1/4} + \left( \frac{V(0) - \varepsilon_R(\eta)}{V(0) - \varepsilon_L(\eta)} \right)^{1/4} \right],
\end{equation}

where $\nu_L(\eta)$ and $\nu_R(\eta)$ are the tunneling amplitudes associated to two symmetric double-well potentials: $V_L(x, \eta) = V_L(-x, \eta)$ and $V_R(x, \eta) = V_R(-x, \eta)$. As shown in Fig. 1, they are defined by the equations $V_{\eta}(x) = V_L(x, \eta)$ for $x < 0$ and $V_{\eta}(x) = V_R(x, \eta)$ for $x > 0$. Fig. 1 $V_0(0)$ is the maximum of the potential at $x = 0$.

Figure 1. (Color on line) The asymmetric potential $V(x, \eta)$, the solid (black) line, is conceived as the result of merging two symmetric double-well potentials: $V_{\eta}(x) \equiv V_L(x, \eta)$ for $x < 0$, the dashed (red) line, whereas $V_{\eta}(x) \equiv V_R(x, \eta)$ for $x > 0$, the dotted (blue) line.

Now $\nu_L(\eta)$ and $\nu_R(\eta)$ can be easily obtained by using the well-known analytic formula for symmetric double-
wells (see Refs. [26–28]) reported in Eqs. (13, 14) of this article for completeness.

Hence the analytical and succinct formula Eqs. (1, 2) together with the Eq. (13) allow a direct calculation of the tunneling amplitude in an asymmetric double-well. To the best of our knowledge, this general formulation has never been proposed although it can be of great use to experimentalists in quantifying their results.

The rest of the paper is organized as follows. In Sec. II we introduce the two-level system and we recall the equivalence of the two semi-classical methods, i.e. the WKB approximation and the instanton technique. Then, by using the simpler WKB method, we derive the formula Eqs. (1, 2) in Sec. III. As an example of application, we discuss two cases in Sec. IV. In the first one we consider an asymmetric double-well potential in which the degeneracy is removed as the asymmetry is introduced: the bias quartic potential with $\varepsilon_L(\eta) \neq \varepsilon_R(\eta)$ for $\eta \neq 0$ (see Fig. 2b). In the second case, we consider a particular situation in which the asymmetry of the potential is introduced without removing the degeneracy $\varepsilon_L(\eta) = \varepsilon_R(\eta) = \varepsilon$ (see Fig. 2b). Now the role of the asymmetry in the tunneling dynamics appears in a clear-cut way as the Rabi frequency is directly related to the tunneling amplitude $\hbar \Omega = 2 \nu(\eta)$. For this second case, we also discuss a time-dependent problem corresponding to a special case of the quantum Zeno effect [29, 30] in Eqs. (13, 14) of this article for completeness.

In the regime defined by Eq. (3), the linear combination of the two wave functions $\phi_L(x)$ and $\phi_R(x)$ is to be the best choice as a (real) two-terms expansion to approximate the exact ground state and the first excited state. The most general linear combination reads

$$\psi_0(x) = \cos(\theta) \phi_L(x) - \sin(\theta) \phi_R(x),$$

$$\psi_1(x) = \sin(\theta) \phi_L(x) + \cos(\theta) \phi_R(x).$$

Eqs. (4, 5) provide the condition of orthonormality between $\psi_0$ and $\psi_1$ assuming that $\phi_L(x)$ and $\phi_R(x)$ are themselves orthogonal and normalized. The above discussion motivates the introduction of the standard two-level Hamiltonian

$$\hat{H} = \sum_{s=L,R} \varepsilon_s(\eta) |s\rangle \langle s| - \nu(\eta) \sum_{s \neq s'} |s\rangle \langle s'|,$$

which describes quantum tunneling between the two localized states in the left well and in the right one and $\nu(\eta)$ is the amplitude for tunneling. The full Hilbert space is thus spanned by only two states. Solving the equation for the two eigenstates $\psi_0$ and $\psi_1$ of the $2 \times 2$ matrix Eq. (6), one obtains the relation between the angle $\theta$ appearing in Eqs. (4, 5) and the parameters $\varepsilon_s, \nu$

$$\sin(2\theta) = \sqrt{\frac{\nu^2(\eta)}{\nu^2(\eta) + (\Delta \varepsilon(\eta)/2)^2}}.$$

II. MODEL AND APPROXIMATIONS

A. Shape of the potential

For the potential $V(x)$, we assume that its maximum is set at the origin $V(0) = 0$, the minima at left $x = a_L < 0$ and at right $x = a_R > 0$, and that the potential increases in the limit $|x| \to \infty$ so that the eigenstates $\{|\psi_n(x)\rangle\}$ decay far away the origin and the energy spectrum $\{E_n\}$ is discrete.

Here we focus on the low-energy regime of tunneling. Then, a priori, only a few eigenstates are involved in the dynamics. Specifically, one can restrict to the two lowest energy states $E_0, E_1$: the ground state $\psi_0(x)$ and the first excited state $\psi_1(x)$.

B. Two-level effective model

Around the two minima $s = L, R$, whose local harmonic frequencies are $\omega_s$, it is possible to solve locally the Schrödinger equation to obtain two localized states $\phi_L(x)$ and $\phi_R(x)$. They are well localized into the wells over a range of order $\sigma_s = [\hbar/(2m\omega)]^{1/2}$ ($m$ is the particle’s mass) when the heights associated to the left and right energy barrier are large compared to the kinetic energies of localization

$$|V(0) - V(a_s)| \gtrsim \frac{\hbar^2}{2m\sigma_s^2} \sim \hbar \omega_s.$$

However $\phi_L(x)$ and $\phi_R(x)$ also decay inside the barrier as we explain below.

Their energies $\varepsilon_s = V(a_s) + \hbar \omega_s/2$ are assumed to be close to the energy range spanned by $E_0, E_1$. We assume that the asymmetry is not too strong to yield resonance between the first (approximate) excited state in one well and the (approximate) ground state of the opposite well [22]. Increasing the asymmetry, the two-level description is no more valid as the resonant condition is approached, for instance when $\varepsilon_L(\eta) \approx \varepsilon_R(\eta) + \hbar \omega_R(\eta)$.

In case of two states, the parameters $\varepsilon_s, \nu$ of Ref. [23] change in a (1 + 1)-dimensional situation. The above discussion motivates the introduction of the standard two-level Hamiltonian

$$\hat{H} = \sum_{s=L,R} \varepsilon_s(\eta) |s\rangle \langle s| - \nu(\eta) \sum_{s \neq s'} |s\rangle \langle s'|,$$

which describes quantum tunneling between the two localized states in the left well and in the right one and $\nu(\eta)$ is the amplitude for tunneling. The full Hilbert space is thus spanned by only two states. Solving the equation for the two eigenstates $\psi_0$ and $\psi_1$ of the $2 \times 2$ matrix Eq. (6), one obtains the relation between the angle $\theta$ appearing in Eqs. (4, 5) and the parameters $\varepsilon_s, \nu$

$$\sin(2\theta) = \sqrt{\frac{\nu^2(\eta)}{\nu^2(\eta) + (\Delta \varepsilon(\eta)/2)^2}}.$$
where $\Delta \varepsilon (\eta ) = \varepsilon _{L}(\eta ) - \varepsilon _{R}(\eta )$. For $\Delta \varepsilon = 0$ the rotating angle is $\theta = \pi /4$. Then the general Hamiltonian for two-level system with degeneracy Eq. (9) has always eigenstates corresponding to symmetric or antisymmetric linear combinations of the two states $L$ and $R$. It is worthwhile to stress that this result holds even if the left and right wave functions $\phi _{L}(x)$ and $\phi _{R}(x)$ are generally different for an arbitrary asymmetric potential. On the other hand, the asymmetry affects the tunneling amplitude $\nu (\eta )$ as we explain below.

Given the parameters entering the effective Hamiltonian Eq. (9), any physical quantities can be evaluated. For instance, solving the time-dependent Schrödinger equation and assuming $|\psi (t = 0)\rangle = |L\rangle$ at the initial time, we obtain $P_{R}(t)$, the probability to have the system at right and at the time $t$. With the boundary condition $P_{R}(0) = 0$, its derivative satisfies the following equation

$$\frac{dP_{R}(t)}{dt} = \nu \frac{\hbar }{m} \sin (2\theta ) \sin (\Omega t),$$

where $\nu$ is the Rabi oscillation frequency is $\hbar \Omega = E_{1} - E_{0}$ and $E_{0}, E_{1} = (\Delta \varepsilon (\eta )/2) \mp [(\Delta \varepsilon (\eta )/2)^2 + \nu^2 (\eta )]^{1/2}$.

### C. WKB approximation and instanton technique for symmetric double-well

To obtain the amplitude $\nu$ for quantum tunneling in the semi-classical regime defined by Eq. (3), the well-known approaches are the instanton technique and the WKB approximation.

The WKB method is based on the Lifshitz-Herring formula [31, 32]. For a symmetric double-well, it reads

$$\nu = \frac{\hbar ^{2}}{m} \left( \phi \frac{d\phi }{dx} \right)_{x=0},$$

in which $\phi (x)$ represents the (approximated) localized solution in the double-well potential ($\phi = \phi _{L} = \phi _{R}$).

In the past the discrepancy of the results given by the two methods has been extensively discussed. For different potentials, they resulted in a difference by the well-known factor $(\epsilon /\pi )^{1/2}$ [33, 34]. For a symmetric potential, Garg showed that this difference is not a failure of the formula Eq. (9) but it was related to the choice of the wave function $\phi$ approximating the local solution [28]. Introducing $\tilde{x}$ as the classical turning point where the local energy $\varepsilon = \varepsilon _{L} = \varepsilon _{R}$ intersects the potential barrier $\varepsilon = V(\tilde{x})$, the corrected WKB solution inside the barrier $(0 < x < \tilde{x})$ for the symmetric double-well reads

$$\phi (x) = \frac{m\omega }{2\pi \epsilon \kappa (x)} e^{-\frac{k}{\hbar } \int _{x}^{x_{0}} d\bar{x}' k(x')} = \sqrt{2m(V(x) - \varepsilon )},$$

where $k(x)$ is now the inverse of the local penetration length. As $dV(x)/dx = 0$ at the maximum at $x = 0$, we have the relation

$$\frac{d\phi (x)}{dx} \bigg|_{x=0} = \frac{1}{\hbar } k(0) \phi (0).$$

When the expression Eq. (10) is inserted in Eq. (9), we obtain the result

$$\nu = \frac{\hbar }{m} k(0) \phi ^2 (0).$$

As discussed in Ref. [28], expanding Eqs. (10), (12) around the singular point $\tilde{x}$, it is possible to recover exactly the instanton solution [24, 27] in which we have the exponential integrals extending from the first $x = -a$ to the second minimum $x = +a$. For the sake of completeness, we recall here the result:

$$\nu = \frac{\hbar \omega }{m} \left( \int _{0}^{a} dx \left( \frac{m\omega a^2}{\sqrt{2m(V(x) - V(a))}} - \frac{1}{a - x} \right) \right).$$

### D. Discussion

Despite their equivalence for the formula of the quantum tunneling amplitude, the instanton technique remains particularly advantageous when the particle is coupled to an external environment [36]. Then the quantum dissipative dynamics can be formulated in terms of the path integral in which the relevant object is the Euclidean action of the full system (and not the wave functions) [37]. In the presence of coupling with a dissipative external bath, a particle moving in an asymmetric double-well potential can even relax towards its minimal configuration [36, 38].

The coupling with the environment is weak as long as $\gamma a^2 /\hbar \ll 1$ where $\gamma$ is the linear friction coefficient [38]. When the coherent quantum dynamics of the particle in the potential is of interest (the decoherence time is much greater than the Rabi frequency), the effective two-level model holds and the tunneling amplitude is an intrinsic quantity which characterizes the quantum isolated system.

As Benderskii et al. demonstrated that the two semi-classical methods are equivalent even for asymmetric double-well potentials [24], one can choose to use the simpler method to tackle the problem, i.e. the WKB approach.

### III. Semi-classical formula for asymmetric potentials

We now formulate the WKB-approach for an arbitrary asymmetric potential. The starting point is the conser-
viation law for the probability density $\rho(x, t) = |\Psi(x, t)|^2$
\[
\frac{d\rho(x, t)}{dt} = -\frac{\hbar}{m} \frac{dJ(x, t)}{dx},
\]
(15)
in which the probability current reads
\[
J(x, t) = -\frac{\hbar}{m} \text{Im} \left( \Psi(x, t) \frac{d\Psi(x, t)^*}{dx} \right).
\]
(16)

Integrating the Eq. (15) from $x = 0$ to $x = +\infty$, at the left hand-side we have the derivative of the probability to have the particle in the right space $x > 0$, $P_R(t)$
\[
\frac{dP_R(t)}{dt} = J(0, t) = -\frac{\hbar}{m} \text{Im} \left( \Psi(x, t) \frac{d\Psi(x, t)^*}{dx} \right)_{x=0}.
\]
(17)

Assuming $\Psi(x, 0) = \phi_L(x)$ at the initial time, by inverting Eqs. (1), (2), we have a simple expression for the evolution of the wave function
\[
\Psi(x, t) = \cos(\theta) e^{-iE_L t} \psi_0(x) + \sin(\theta) e^{-iE_R t} \psi_1(x).
\]
(18)

We now insert Eq. (18) into Eq. (17) and we obtain
\[
\frac{dP_R(t)}{dt} = \frac{\hbar}{2m} \sin(2\theta) \sin(\Omega t) \left( \psi_1 \frac{d\psi_0}{dx} - \psi_0 \frac{d\psi_1}{dx} \right)_{x=0} \]
\[
= \frac{\hbar}{2m} \sin(2\theta) \sin(\Omega t) \left( \frac{d\phi_R}{dx} - \frac{d\phi_L}{dx} \right)_{x=0}
\]
(19)
where we have used again Eqs. (1), (2) in the last line. Comparing Eq. (19) with Eq. (9) we obtain the relation between the tunneling amplitude and the two wave functions:
\[
\nu = \frac{\hbar^2}{2m} \left( \phi_L \frac{d\phi_R}{dx} - \phi_R \frac{d\phi_L}{dx} \right)_{x=0}.
\]
(20)

For a symmetric potential, the left and right states are equal $\phi_R(x) = \phi_L(-x)$ and we recover the standard formula for the tunneling amplitude Eq. (9). We now introduce $x_s$, the crossing point of the left and right energies with the potential barrier $V(x_s) = \varepsilon_L$ and $V(x_0) = \varepsilon_R$. Recalling Eq. (19), the left and right wave functions $(s = L, R)$ inside the barrier are given by
\[
\phi_s(x) = \sqrt{\frac{m\omega_s}{2\pi\varepsilon_s}} \times \left\{ \begin{array}{ll}
e^{-\frac{\pi}{4} \int_{x_{L}}^{x} k_s(x') dx'} & (x < x_L) \\
e^{-\frac{\pi}{4} \int_{x_s}^{x} k_s(x') dx'} & (x > x_R)
\end{array} \right.
\]
(21)
where $k_s(x) = [2m(V(x) - \varepsilon_s)]^{1/2}$. Owing to our choice of the maximum’s position, we have $dV(x)/dx = 0$ at $x = 0$ so that the derivative $d\phi_s/dx$ is related to the function itself $\phi_s(0)$, e.g. Eq. (14). Then the tunneling amplitude Eq. (20) reads
\[
\nu = \frac{\hbar}{2m} \left[ k_R(0)\phi_L(0)\phi_R(0) + k_L(0)\phi_R(0)\phi_L(0) \right]
\]
\[
= \frac{\hbar}{2m} \left( \frac{k_R(0)}{k_L(0)} + \frac{k_L(0)}{k_R(0)} \right) \sqrt{k_R(0)k_L(0)\phi_L(0)\phi_R(0)}
\]
\[
= \frac{1}{2} \sqrt{\frac{k_L(0)}{k_R(0)} + \frac{k_R(0)}{k_L(0)}} \nu_R
\]
(22)

In the last equation, we have used the formula Eq. (12) for the tunneling amplitude $\nu_L$ and $\nu_R$ for two symmetric potentials defined by $V_L(x) \equiv V(x)$ for $x < 0$ and $V_R(x) \equiv V(x)$ for $x > 0$. The Eq. (22) is the result Eqs. (1), (2) as $\kappa_s(0) = \sqrt{2m(V(0) - \varepsilon_s)}$.

IV. APPLICATIONS

A. Asymmetric quartic potential

The first example of asymmetric potential is the quartic potential including a linear term
\[
V(x) = V_0 \left\{ \left[ \frac{x}{a} \right]^2 - 1 \right\}^2 - 1 - \eta \left( \frac{x}{a} \right), \quad V_0 = \frac{m\omega^2 a^2}{8}.
\]
(23)

The positions of the minima for $\eta = 0$ are at $\pm a$, the harmonic frequency at the bottom of the wells is $\omega$. For the symmetric case, the tunnel amplitude reads
\[
\nu = 4 \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \sqrt{(\hbar \omega)V_0} e^{-\frac{\pi}{4} \frac{V_0}{\hbar \omega}}.
\]
(24)

When the asymmetry parameter is introduced $\eta > 0$, a liner term is added in the potential which also removes the energy degeneracy (see Fig. 3). For $0 < \eta < 8/(3\sqrt{3})$ the potential is has still a maximum and two minima which are now shifted from $x = 0$ and $x = \pm a$. They are given by the formula
\[
\frac{x_n}{a} = \frac{2}{\sqrt{3}} \cos \left[ \frac{2\pi}{3} n + \frac{1}{3} \arctan \left( \frac{8}{3\sqrt{3}\eta} \right)^{2} - 1 \right],
\]
(25)

with $n = 2$ for the maximum $a_C$ and $n = 1$ and $n = 0$ for the left $a_L$ and right minimum $a_R$. The harmonic frequency are given by $(\omega_s/\omega)^2 = (3/2)(a_s/a)^2 - 1/2$ with $s = L, R$.

After changing the origin of the $x$ axis, $x' = x - a_C$, and knowing of the position of the two minima, we can use the formulas Eqs. (1), (2). The prefactor Eq. (2) can be easily calculated by observing that it corresponds to $A = (1/2)[(\varsigma_L/\varsigma_R)^{1/2}+(\varsigma_R/\varsigma_L)^{1/2}]$ with $\varsigma_s = V(a_C)-V(a_s)-\hbar \omega_s/2$ for $s = L, R$. We use the formula for the tunneling amplitude in the symmetric double-well Eqs. (13), (14) to obtain $\nu_L$ for and $\nu_R$ from the two potential $V_L(x) = V(x)$ for $x < a_C$ and $V_R(x) = V(x)$ for $x > a_C$.

The result is given in Fig. 3 where the ratio between the tunneling amplitude $\nu(\eta)$ of the asymmetric potential and the amplitude for the symmetric case $\nu(0)$ is shown for different values of $V_0/\hbar \omega$.

Here we summarize the results. First, we observe that the asymmetry of the potential always reduces the tunneling amplitude for the quartic potential. This behavior is not universal but it depends on the specific shape of the potential and how it is deformed (see next section). Second, the resonant condition $\varepsilon_L = \varepsilon_R + \hbar \omega_R$ is
The energy of the localized states corresponds to asymmetry parameter $\eta$ at different values of the ratio $V_0/\hbar\omega$. On the contrary, the energy difference increases linearly leading dependence of $\eta$ looking at expressions similar to the symmetric case $\nu$. We conclude by observing the energy difference $\Delta = \epsilon - V_0$ at small values of $\eta$. The last corrections scale as $\epsilon^2/(2\epsilon^2)$.

A possible choice is $\omega_\eta = \omega(1-\eta)$, $a_\eta/a = \sqrt{\left(1 - \frac{\hbar\omega}{2V_0}\right)/(1-\eta)}$. With the condition that $\eta < 2V_0/\hbar\omega$. The result is shown in Fig. [4].

Let us calculate now the tunneling amplitude $\nu(\eta)$. As the degeneracy is not removed $\epsilon_R = \epsilon_L$, the prefactor $A$ for the tunneling amplitude in Eq. (2) is one. In this case we have simply

$$\nu(\eta) = \sqrt{\nu_L\nu_R} = \sqrt{\nu(0)\nu_R(\eta)}.$$ (30)

We have used $\nu_L = \nu(0)$ as the left part of the potential is unmodified (see Fig. [4]). The tunneling amplitude associated to the left symmetric potential, $V_L(x) \equiv V(x)$ for $x < 0$, corresponds to the symmetric case with $\eta = 0$:

$$\nu_L = \nu(0) = \sqrt{\frac{2}{\pi}} \frac{(\hbar\omega)V_0}{e^{-2V_0/\hbar\omega}}.$$ (31)

As the asymmetry does not change the shape of the potential, we can directly use the previous formula to obtain the tunneling amplitude associate to the right symmetric potential, $V_R(x) \equiv V(x)$ for $x > 0$.

$$\nu_R(\eta) = \sqrt{\frac{2}{\pi}} \frac{(\hbar\omega_\eta)V_0}{e^{-\frac{V_0}{\hbar\omega_\eta}}}.$$ (32)

The result of Eqs. (30), (31), (32) is shown in Fig. [4], where we report again the ratio between $\nu(\eta)/\nu(0)$ for different values of $V_0/\hbar\omega$. We can observe that a small asymmetry of the potential $\eta < 0.2$ can renormalize quantitatively the amplitude $\nu$ of order 20%. As the Rabi frequency equals twice the tunneling amplitude $\hbar\Omega = 2\nu(\eta)$, in this case, the corrections due to the asymmetry appear clearly in the quantum dynamics of the two-level systems.

The behavior of $\nu$ shown in Fig. [4] can be explained by looking at the way we have chosen to deform the potential Eq. (29). For the asymmetric potential, the barrier height at right is modified as $V(0) - V(a_R)$ whereas the right attempt frequency is varied as $\omega_\eta/\omega = 1 - \eta$. The second correction dominates.

**B. Asymmetric parabolic potential**

The second example belongs to the class of asymmetric potentials with degenerate localized states $\epsilon_L = \epsilon_R$. An example of such potentials is the following. For the negative part ($x < 0$) we have $V(x) = V_L(x)$ where $V_L(x)$ reads

$$V_L(x) = V_0 \left[\left(\frac{x}{a_L} + 1\right)^2 - 1\right], \quad V_0 = \frac{m\omega^2a_L^2}{2}.$$ (26)

The energy of the localized states corresponds to $\epsilon_L = -V_0 + \hbar\omega/2$. For the positive part $x > 0$, $V(x)$ corresponds to $V(x) = V_R(x)$ where $V_R(x)$ reads

$$V_R(x) = V_\eta \left[\left(\frac{x}{a_R} - 1\right)^2 - 1\right], \quad V_\eta = \frac{m\omega^2a_R^2}{2}.$$ (27)

Similarly we have $\epsilon_R = -V_\eta + \hbar\omega/2$. This potential is continuous as $V_L(0) = V_R(0)$.

In the next example, we will consider a case in which $\epsilon < 0$.

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owing to the condition $\hbar \omega/(2V_0) < 1$ for the two localized states. When $\eta < 0$, the right attempt frequency increases (hardening) leading to an enhancement of tunneling. On the contrary, for $\eta > 0$ the right attempt frequency decreases (softening) leading to a suppression of tunneling. A choice different from the one of Eq. (29), for instance the one corresponding to increasing or decreasing the height of the barrier in linear way as a function of $\eta$, produces similar results but with a different dependence on $\eta$.

C. A time-dependent problem and quantum Zeno effect

For the asymmetric parabolic potential with degenerate local levels, we want to discuss a simple time-dependent problem.

We assume to tune the asymmetry of the potential in time $\eta = \eta(t)$ so that the tunnel amplitude gets a time dependence $\nu = \nu(t)$. The time scale for the variations of $\eta(t)$ (and therefore $\nu(t)$) are assumed slower enough to avoid excitations of the systems towards higher energy states of the potential $E_n$ with $n \geq 2$. Working under this assumption, the two-level description still holds. As a simple estimate, we consider the time scale for the variations of $\nu(t)$ smaller than the harmonic frequencies $\omega_L$ and $\omega_R$ at the left and right well. Setting the energy level to $\varepsilon = 0$, we write the Schrödinger equation for the two-level system as

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_L(t) \\ c_R(t) \end{pmatrix} = \begin{pmatrix} 0 & -\nu(t) \\ -\nu(t) & 0 \end{pmatrix} \begin{pmatrix} c_L(t) \\ c_R(t) \end{pmatrix},$$

where $c_L(t)$ and $c_R(t)$ are the coefficients of the state at the time $t$. These equations can be easily solved by introducing the sum and the difference $c_{\pm} = c_L \pm c_R$ which satisfy the equation

$$\frac{dc_{\pm}(t)}{dt} = \pm \frac{i}{\hbar} \nu(t)c_{\pm}(t),$$

Assuming that the system is prepared at the initial time $t = 0$ in one localized state, let us say right, we have

$$c_L(t) = \cos \left( \frac{1}{\hbar} \int_0^t dt' \nu(t') \right), \quad c_R(t) = i \sin \left( \frac{1}{\hbar} \int_0^t dt' \nu(t') \right),$$

so that the probability to remain into the initial left state reads

$$P_L(t) = |c_L(t)|^2 = \frac{1}{2} \left[ 1 + \cos \left( \frac{2}{\hbar} \int_0^t dt' \nu(t') \right) \right].$$

An interesting case is the following evolution for $\nu(t)$

$$\nu(t) = \nu_0 + (\nu_1 - \nu_0) \sum_n \chi_n(t),$$

where $\chi_n(t)$ is the characteristic function equals to $\chi_n = 1$ in the time intervals $[n(t_0 + t_1) + t_0] < t < (n+1)(t_0 + t_1)$ and $\chi_n = 0$ elsewhere. $t_0$ is a time interval in which the tunneling amplitude is constant and equals to $\nu = \nu_0$ whereas in a subsequent smaller interval $t_1 < t_0$, the tunneling amplitude is suppressed to $\nu_1 \ll \nu_0$. The signal is repeated many times during a half Rabi period $T/2 = \pi/\Omega = \pi \hbar/(2V_0)$.

The results for evolution of the probability $P_L(t)$ are shown in Fig. 4 both for the case $\nu = \nu(t)$, Eq. (37), and for the case of constant amplitude $\nu = \nu_0$. In Fig. 4 we can observe that the probability $P_L[t, \nu(t)]$ is always higher then $P_L[t, \nu_0]$. Reducing $\nu$ to $\nu_1$ during a short time interval $t_1$ corresponds to deform the potential in time in a way to suppress the tunneling amplitude $\nu_1 < \nu_0$. This corresponds to trap back the particle in the starting well at regular time intervals. As a consequence, a slow-down of the probability to escape from the initial well occurs, a result that is referred as quantum Zeno effect in the literature 29, 31.

V. CONCLUSION

In summary, we have derived a useful and succinct expression for the tunneling amplitude $\nu$ in asymmetric double-well potentials. We applied it to two examples: the quartic potential with a linear force and a kind of
parabolic potential in which the asymmetry does not remove the energy degeneracy of the two localized levels. From these simple examples, one can learn that there are no systematic effects of asymmetry on quantum tunneling. The tunneling amplitude is enhanced or reduced depending on the shape of the potential and how the asymmetry is introduced. However we have illustrated as the formulas Eqs. (1), (2) allow one to obtain analytically and in a direct way the renormalization of the tunneling amplitude in an asymmetric double-well potential in order to discuss its behavior as varying the asymmetry.

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Appendix A: Comparison with exact numerical results

In this appendix we show the comparison between the exact numerical result of the energy splitting \(E_1 - E_0\) and the WKB semi-classical formula for the Rabi frequency \(\hbar \Omega = 2[(\varepsilon_L(\eta) - \varepsilon_R(\eta))^2/4 + \nu^2(\eta)]^{1/2}\) with \(\nu(\eta)\) given by the Eqs. (11), (12) and the Eq. (13).

The first two eigenstates \(\psi_0(x)\), \(\psi_1(x)\) and their energies \(E_0, E_1\) were computed numerically by discretizing the time-independent Schrödinger equation for the two potentials \(V(x)\) discussed in the paper. Using this approach and imposing the boundary conditions \(\psi_0(x) = 0\) at the end points of a finite interval, the Schrödinger equation becomes a linear eigenvalue problem with a tridiagonal matrix assuming that the difference between the exact value of the second derivative \(d^2\psi(x)/dx^2\) and its discretized form \(\psi(x_{i+1}) + \psi(x_{i-1}) - 2\psi(x_i)\)/\(\Delta x^2\) is small. The first low-energy eigenstates are smooth functions which extend over lengths \(\sigma = [\hbar/(m \omega)]^{1/2}\). Then one can choose a spacing \(\Delta x = min(\sigma_L, \sigma_R)/N\) with \(N\) sufficiently large \((N \sim 10^3)\) in order to compute the eigenvalues with an acceptable error \(\delta E_1, \delta E_0 \ll E_1 - E_0\). The end points were set to \(x_{min} \sim -M \sigma_L\) on the left, and to \(x_{max} = M \sigma_R\) on the right \((M \sim 5 - 10)\).

A scaling analysis of \(E_0, E_1\) as functions of \(N, M\) was also carried out to test the convergence.

In Fig. 6 we show two examples for the double-well parabolic potential with two degenerate levels \(\varepsilon_L = \varepsilon_R\) \((h \Omega = 2\nu(\eta))\), i.e. Eqs. (24), (27), (29) of Sec. IVB. For the symmetric case \(\eta = 0\) in Fig. 6a, the analytic semi-classical formula is in agreement with the numerical result for \(V_0/(\hbar \omega) \gtrsim 0.4\) \((error \lesssim 1\%)\) whereas for the asymmetric case \(\eta = -0.5\) in Fig. 6b the agreement is for \(V_0/(\hbar \omega) \gtrsim 0.8\).

In Fig. 6 we compare numerical and analytic solutions for the bias quartic potential Eq. (23) discussed in Sec. IV. A. In this case, the asymmetry due to the linear bias \(\eta > 0\) removes the degeneracy of the localized states. As a consequence, at given \(\eta > 0\), the two-level approximation breaks down at large values of ratio \(V_0/\hbar \omega\) as approaching the resonant condition \(\varepsilon_L = \varepsilon_R + \hbar \omega_R\).
(see inset of Fig. 7b). Thus the upper bound of validity for our analytical approach is given by the condition \( V_0/(\hbar \omega) < (\omega_R/\omega) V_0/(\varepsilon_L - \varepsilon_R) \sim 1/(2\eta) \) to the leading order in \( \eta \) and for \( V_0/\hbar \omega \gg 1 \).

![Diagram](image)

**Figure 7.** (Color on line) Comparison between \( E_1 - E_0 \) numerically computed (the black dots with dashed line) and the semi-classical formula of the Rabi frequency (full red line) for the quartic potential. a) Case \( \eta = 0 \) (\( \hbar \Omega = 2\nu \)). Inset: logarithmic scale for the y axis. b) Case \( \eta = 0.2 \). Inset: a close-up for the range of large ratio \( V_0/\hbar \omega \).

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[39] The derivative is not continuous but it is possible to consider a regularization function which matches the two potentials around the origin. This regularization is important only in an extremely small range around $x = 0$. It does not affect the final result for the tunneling amplitude as the last one depends only on integrals of the potential.

[40] We are considering a square wave signal in which the typical rise time $\tau$ is infinite, i.e. $\tau$ is much higher than the time scales involved in the two-level problem: $\nu_0, \nu_1, t_0$ and $t_1$. Actually, $\tau$ has an upper limit given by the adiabatic condition to avoid higher energy excitations in the double-well.

[41] At given $\Delta x$, this approximation breaks down for the high-energy strongly oscillatory eigenstates $\psi_n(x)$. 