We analyze previously proposed order parameters for the confinement - deconfinement transition in lattice SU(2) Yang–Mills theory, defined as vacuum expectation value (v.e.v.) of monopole fields in abelian projection gauges. We show that they exhibit some inconsistency in the treatment of small scales, due to a violation of Dirac quantization condition for fluxes. We propose a new order parameter avoiding this inconsistency. It can be interpreted as v.e.v. of the field of a regular monopole in any abelian projection gauge, but it is independent of the choice of the abelian projection. Furthermore, being constructed in terms of surfaces of center vortices, it has also a natural interpretation in the approach of center dominance.

1. Introduction

There now appears to be a general consensus about the idea that color confinement in Yang Mills theories is caused by the condensation of some topological defects. However it is still debated whether the important defects are center vortices or magnetic monopoles.
The first proposal was brought forward, in the ’70, by ’t Hooft, Polyakov, Mack, Nielsen and Olesen [1] and others and has received renewed interest in recent times, stimulated by numerical simulations [2]; presently it is usually named “center dominance”.

The basic idea is drawn from an analogy with ferromagnets and may be roughly phrased as follows. The expectation value of a temporal Wilson loop in four dimensions can be seen, by fixing the temporal gauge, as a product of two–point spin correlation functions of three dimensional non linear $\sigma$ models coupled to gauge fields, one for each time involved in the Wilson loop. If the non–linear $\sigma$–models are disordered, their spin correlation functions have an exponential decay. This implies an area law for the Wilson loop of the four–dimensional gauge theory and hence, according to Wilson’s criterion, confinement. (Actually, a rigorous proof of a mathematically precise version of this idea has been established in [3]). The defects responsible for disorder in the non linear $\sigma$ models are vortices. For $SU(N)$ theories they are $\mathbb{Z}_N$ center vortices.

The alternative with monopoles as the relevant defects [4] has been put on a concrete basis by ’t Hooft [5], who suggested to make explicit the monopoles in $SU(N)$ Yang–Mills theories, by performing a gauge fixing that leaves a maximal Cartan torus, $U(1)^{N-1}$, unbroken. These gauges are called abelian projection gauges. The resulting abelian gauge theories can be rewritten in terms of “photons”, gauge fields of a theory with gauge group the decompactification of the residual gauge group, and monopoles. The monopoles corresponding to this decompactification are those expected to condense in the confinement phase.

To define in $SU(2)$ Yang-Mills theory, to which this paper is addressed explicitly, an abelian projection gauge, ’t Hooft suggested to construct some scalar field $X$ taking values in $su(2)$, as a function of the gauge connection and transforming in the adjoint representation of the gauge group, $SU(2)$. To perform the gauge fixing one imposes the constraint that $X$ is diagonal.

The diagonal component of the $SU(2)$–connection in this gauge plays the role of the “photon field”, the off–diagonal components are charged with respect to the residual gauge group $U(1)$.

The points in space–time where two eigenvalues of the matrix $X$ coincide are the positions of the monopoles in this gauge. Confinement is then believed to emerge as a consequence of monopole condensation in the form of a “dual–Meissner” effect.

Together with the assumption that the effect of the charged off–diagonal degrees of freedom are qualitatively irrelevant for the description of the low–energy physics, the above scenario is usually called “abelian dominance”.

Two natural questions arise in the “abelian dominance” scenario:
1) Is there a “monopole field operator” which plays the rôle of an order parameter for the confinement–deconfinement transition, i.e. with vanishing expectation value in the deconfined phase and non–vanishing expectation value in the confined phase, so that the transition can be interpreted as due to a “monopole condensation”?

2) does the choice of the field $X$ influence the behaviour of the expectation value of the monopole field operator?

In [6,7] an attempt has been made to give a positive answer to the first question on the basis of a circle of ideas which can be presented as follows.

In electrodynamics with charged scalar field $\phi$ one can construct gauge–invariant charged field operators and their correlation functions adapting the Dirac recipe [8], dressing the local non–gauge invariant field $\phi(x)$ with a cloud of soft photons, represented by multiplication by a phase factor with argument given by the gauge field $\vec{A}$ weighted by a classical Coulomb field $\vec{E}(x)$.

In abelian gauge theories there is a natural notion of duality exchanging the rôle of charges and monopoles. One can obtain monopole correlation functions from gauge–invariant charged correlation functions by a duality transformation; in particular this applies to monopoles in $U(1)$ lattice gauge theory.

In [6,7] it was suggested to apply this construction to the $SU(2)$ gauge theory in the abelian projection gauge and it was shown [7] how to render this construction gauge–invariant.

Starting from the expectation value of the monopole operator constructed in this manner, Montecarlo simulations show that

1) this monopole field operator indeed behaves as a good order parameter signaling the confinement – deconfinement transition;

2) the (physical) temperature of the transition is independent of the choice of the scalar field $X$ [9].

This approach however presents a foundational problem in spite of its great numerical success: it is inconsistent with Dirac’s quantization condition of fluxes. This, in turn, implies an inconsistency of the treatment of small scales.

To understand the origin of the Dirac quantization condition, let us consider the electrodynamics of electrically and magnetically charged point–like particles. The equations of motion proposed by Dirac read:

\[ \partial^\mu F_{\mu\nu}(x) = q_e j^e_\nu(x), \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}(x) = q_m j^m_\nu(x) \]  \hspace{1cm} (1.1)

where $q_e j^e_\nu$ and $q_m j^m_\nu$ are the electric and magnetic currents generated by the particles, $q_e$ and $q_m$ being their electric and magnetic charges.

Since equation (1.1) implies current conservation
\[ \partial^\nu j^e_\nu = 0 = \partial^\nu j^m_\nu, \]
Poincaré’s lemma ensures the existence of antisymmetric tensor fields \( n^e_\rho\sigma, n^m_\rho\sigma, \) such that
\[ j^e_\mu = \epsilon_{\mu\nu\rho\sigma} \partial_\nu n^e_\rho\sigma, \quad j^m_\mu = \epsilon_{\mu\nu\rho\sigma} \partial_\nu n^m_\rho\sigma. \] (1.2)
However these fields are determined only modulo the transformation
\[ n^\#_\rho\sigma \rightarrow n^\#_\rho\sigma + \partial_\rho \lambda^\#_\sigma \] (1.3)
where \( \# = e, m \) and \( \lambda^\#_\sigma \) is a vector field.
Schwinger proposed an action leading to the Dirac equations (1.1):
\[ S(A, j^e, n^m) = \int \left[ \frac{1}{2} (\partial_\mu A_\nu + q_m n^m_\mu\nu)^2 (x) + q_e (A_\mu j^e_\mu)(x) \right] d^4 x, \]
where if the support of the current \( j^m_\mu \) lies on a curve \( \gamma \), then \( n^m_\mu\nu \) should be taken as a surface current whose support is a surface \( \Sigma \), with boundary \( \gamma \). The current \( n^m \) describes the Dirac strings attached to the magnetic monopoles whose worldlines are given by \( \gamma \).

The quantum theory corresponding to the classical action \( S(A, j^e, n^m) \) is well defined, provided the partition function
\[ \int \mathcal{D}A e^{iS(A, j^e, n^m)} \]
is independent of the choice of the surface \( \Sigma \), with \( n^m \) satisfying (1.2), i.e. invariant under the gauge transformations (1.3).

If one chooses \( n^m_\rho\sigma \) and \( n'^m_\rho\sigma \) corresponding to the surfaces \( \Sigma \) and \( \Sigma' \), then the parameter \( \lambda^m_\sigma \) appearing in (1.3) is dual to a volume-current with support in the volume \( V \) whose boundary is given by the closed surface difference of \( \Sigma' \) and \( \Sigma \).

The consistency condition turns out to be given by
\[ q_m q_e \int \lambda^m_\mu j^e_\mu d^4 x \in 2\pi \mathbb{Z}. \] (1.4)
Since \( j_\mu \) are line currents with integral coefficients one can recognize in (1.4) the Dirac quantization condition:
\[ q_e q_m \in 2\pi \mathbb{Z}. \] (1.5)
From equation (1.4) it follows as consistency requirement that the integral over an arbitrary volume of any electric current appearing in the partition function,
multiplied by $\frac{q_m}{2\pi}$ must be an integer. One can easily prove that this condition extends to all physical correlation functions.

However in the construction of the gauge invariant correlation functions for charged fields following the Dirac recipe, one introduces the electric smooth current $E$, whose integral over a generic volume $V$ is a real number, thus violating Dirac quantization condition. One may say that in the presence of the electric current $E$ the position of the Dirac strings (of monopoles) crossing the support of $E$ become visible, thus introducing a physical inconsistency.

The monopole correlation functions may be obtained from the charged correlation functions by duality, therefore in the presence of dynamical charges their construction following Dirac’s recipe encounter the same inconsistency. This extends also to the constructions of monopoles for $SU(2)$ presented in [6,7], since it involves charged degrees of freedom corresponding to the off–diagonal components of the Yang–Mills gauge field.

Nevertheless, since in $U(1)$ gauge theories the constraint of Dirac quantization is expected to become irrelevant at large distances, i.e. in the scaling limit, one might still imagine that the low–energy physics, (e.g. the deconfinement transition temperature), should not be affected by the above problems.

In this paper we propose a new order parameter for the confinement – deconfinement transition, defined in terms of the expectation value of a (regular) monopole field operator, which avoids the inconsistency discussed above.

Our construction is a variant of that proposed for the ’t Hooft–Polyakov monopoles in the Georgi–Glashow model [10] and, as a basic difference with respect to previous constructions, the correlation functions of monopoles are obtained in terms of sheets of center vortices.

The new order parameter exhibits the following features:

1) it respects the Dirac quantization condition
2) it is naturally independent of the choice of a $U(1)$ subgroup of $SU(2)$, needed in the abelian projection
3) if one chooses an abelian projection gauge we argue that in the scaling limit it approaches the order parameter constructed in [7] in correspondence to that projection gauge
4) it creates a bridge between “abelian” and “center dominance” suggesting how one can reconcile the two approaches.

Although our discussion is heuristic and partly conjectural, the overall picture that emerges is consistent with known mathematical estimates and numerical lattice simulations.

In order to make the paper selfcontained, we start by recalling in section 2 Dirac
recipe and its dual, sketching the corresponding construction of charged and monopole Green function in the euclidean formalism.

In section 3 we review the modification of Dirac’s ansatz proposed in order to satisfy the Dirac quantization condition if dynamical charges and monopoles coexist.

In sect. 4 we sketch how one defines the magnetic charge in $SU(2)$ theory.

In sect. 5 we review, in the euclidean formalism, earlier constructions of monopole field in $SU(2)$, based on the dual of Dirac’s recipe.

In sect. 6 we define our new monopole field and the corresponding order parameter and discuss the link to previous constructions.

In sect. 7 we outline the connection between “abelian” and “center dominance” suggested by our order parameter.

In order to simplify our formulas in the rest of the paper we use the language of forms and currents, both in the continuum and on the lattice. The basic definitions of this formalism are reviewed in an appendix.

2. Dirac’s ansatz and its dual

We start by discussing a simple model, scalar QED, where only electric dynamical charges appear, but no magnetic monopoles. We show how to construct gauge–invariant, charged field operators following Dirac’s ansatz.

Let $\phi$ be a massive scalar field with charge $e$ coupled to an abelian gauge field, $A_\mu$, described in terms of a real 1–form $A$, with classical action

$$S(A, \phi) = \int \left[ \frac{1}{2} (\partial_\mu A_\nu)^2(x) + \frac{1}{2} |\partial_\mu - ieA_\mu| \phi|^2(x) + \frac{m^2}{2} \bar{\phi}\phi(x) \right] d^4x$$

or, in the notation of differential forms,

$$S(A, \phi) = \frac{1}{2} ||dA||^2 + \frac{1}{2} ||(d - ieA)\phi||^2 + \frac{m^2}{2} ||\phi||^2.$$  \hspace{1cm} (2.1)

Dirac’s ansatz can be formulated as follows: Let $\hat{\phi}$ and $\hat{A}$ denote the quantum field operators corresponding to the classical fields $\phi$ and $A$ and let $E_\mu^\vec{x} dx^\mu \equiv E^\vec{x}$ denote the 1–form corresponding to the classical electromagnetic field generated by a pointlike unit charge located at $\vec{x}$ in $\mathbb{R}^3$. Then Dirac’s charged field is defined, heuristically, by the formula:

$$\hat{\phi}(E^\vec{x}) = \hat{\phi}(\vec{x}) e^{i \int_{\mathbb{R}^3} \hat{A} A^* E^z}.$$ \hspace{1cm} (2.2)

This construction has been rendered rigorous (in the presence of an ultraviolet regulation) in [11], within the indefinite metric approach (Gupta–Bleuler gauge).
There is however an alternative route to construct charged fields. One starts from euclidean Green functions and then invokes the Osterwalder–Schrader reconstruction theorem [12]. In approximate terms it works as follows: One constructs gauge–invariant euclidean correlation functions for charged fields obeying the O.S. axioms (essentially translation invariance, reflection (O.S.) positivity and clustering) from which one can reconstruct a Hilbert space of physical states containing the vacuum vector $\Omega$, a unitary representation of space–time translations, whose generators satisfy the spectral condition, and which leaves $\Omega$ invariant, and quantum field operators.

There is also a version of the reconstruction theorem that applies to lattice theories.

A euclidean version of Dirac’s ansatz is then obtained by replacing the quantum field (2.2) by a euclidean field

$$\phi(E^x) = \phi(x)e^{i\int_{\mathbb{R}^4} A^{\wedge^*}E^x} = \phi(x)e^{i(A,E^x)} \quad (2.3)$$

where $E^x$ is the 1–current in $\mathbb{R}^4$ given by

$$E^x(y) = E^x(\bar{y})\delta(y^0 - x^0). \quad (2.4)$$

With a lattice regularisation euclidean correlation functions of these fields, $\langle \prod_i \phi(E^x_i) \prod_j \bar{\phi}(E^y_j) \rangle$, have been proved to satisfy the O.S. axioms [13]. Here $\langle \cdot \rangle$ denotes the euclidean expectation value corresponding to the action (2.1) and $\bar{\phi}(E^x)$ is the complex conjugate of $\phi(E^x)$.

It is useful for later purposes to notice that one can obtain a representation of the correlation functions of $\phi(E)$ as partition functions of a gas of closed electric currents coupled to $A$, by integrating out $\phi$.

As an example consider the two-point function

$$\langle \phi(E^x)\bar{\phi}(E^y) \rangle = \frac{1}{Z} \int DAe^{-\frac{1}{2}\|dA\|^2}\det(-\Delta_{eA} + m^2) e^{ie(A,(E^x-E^y))} \langle \phi(x)\bar{\phi}(y) \rangle(A) \quad (2.5)$$

where $\langle \cdot \rangle(A)$ denotes the (normalised) expectation value corresponding to the action of $\phi$ coupled to $A$, viewed as an “external” field, and $\Delta_{eA}$ is the covariant Laplacian.

Using a euclidean version of Feynman’s path–integral formula for the quantum–mechanical time evolution kernel $(e^{s\Delta_{eA}})(x,y)$, one obtains formally

$$\langle \phi(x)\bar{\phi}(y)(A) \rangle = (-\Delta_{eA} + m^2)^{-1}(x,y) = \int_0^\infty ds e^{-sm^2} (e^{s\Delta_{eA}})(x,y) = \int_0^\infty ds e^{-sm^2} \langle \phi(x)\bar{\phi}(y) \rangle(A)$$
\[ \int_{0}^{\infty} dse^{-sm^2} \int Dq(t)e^{-\int_{0}^{s}} \left[ \frac{1}{2}\dot{q}^2(t) + eA_{\mu}(q(t))\dot{q}\mu(t) \right] dt. \]

If with every trajectory \( \{q_{\mu}(t), t \in [0, s]\}, s \in \mathbb{R}_+ \), we associate the 1–current

\[ j_{xy}(y) = \int dt\dot{q}_{\mu}(t)\delta(q(t) - y)dy^\mu, \]

(* \( j_{xy} \) is Poincaré dual of the trajectory, see appendix), then, one finds

\[ \langle \phi(E_x)\bar{\phi}(E_y) \rangle = \int D\mu(j_{xy})e^{ie(A,j_{xy})}, \]

for a suitable measure \( D\mu(j_{xy}) \).

One can also express the determinant in (2.5) in terms of a sum over closed current networks \( j \), so that, for a suitable measure \( D\mu(j) \) on the current networks \( j \), one obtains

\[ \langle \phi(E_x)\bar{\phi}(E_y) \rangle = \int D\mu(j)D\mu(j_{xy})e^{ie(A,E_x-E_y+j_{xy}+j)} \int D\mu(j)e^{ie(A,j)} \]^{-1}. \hfill (2.6)

We notice that, while the current networks \( j \) in the partition function (the denominator of (2.6)) are all integer–valued, the currents appearing in the numerator of (2.6) also involve a real–valued term \( (E_x - E_y) \).

We now sketch how one can obtain monopole correlation functions in a dual theory. For \( U(1) \)–gauge theories in \( d=4 \) dimensions, S–duality is a transformation mapping the correlation functions of the original gauge theory onto those of a dual gauge theory, exchanging the role of charges and monopoles. The underlying idea can be presented as follows.

Let \( S(dA) \) denote a gauge–invariant action for the gauge field \( A \), written in terms of its curvature 2–form \( dA \). Introducing an auxiliary 2–form \( F \) we can write the partition function as

\[ Z = \int D\mu(j)D\mu(j_{xy})e^{ie(A,(E_x-E_y)+j_{xy}+j)} \int D\mu(j)e^{ie(A,j)} \]^{-1}. \hfill (2.6)

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We now express the constraint in (2.8) by a Fourier representation of the δ functional:
\[
\delta(dF) = \int \mathcal{D}\tilde{A}e^{i\int F \wedge d\tilde{A}}
\]
(2.9)
where \(\tilde{A}\) is a new gauge field, the “dual of \(A\)”. We define the dual action \(\tilde{S}(d\tilde{A})\) through the functional integral Fourier transform
\[
e^{-\tilde{S}(d\tilde{A})} \equiv \int \mathcal{D}F e^{-S(F)} e^{i\int F \wedge d\tilde{A}}.
\]
(2.10)
Plugging (2.9) and (2.10) in to (2.7) we obtain
\[
Z = \int \mathcal{D}F e^{-S(F)} \int \mathcal{D}\tilde{A}e^{i\int F \wedge d\tilde{A}} = \int \mathcal{D}\tilde{A}e^{-\tilde{S}(d\tilde{A})}
\]
where the last term gives the partition function of the dual theory.
The same procedure proves that duality exchanges the Wilson loop, which can be related to worldlines of a charged particle–antiparticle pair, with the Wegner – ’t Hooft disorder operator [14], related to worldlines of a monopole–antimonopole pair.
Let \(\Sigma\) be a 2–dimensional surface; the Poincaré–dual current is also denoted by \(\Sigma\). The Wilson loop \(W_\alpha(\Sigma), \alpha \in \mathbb{R}\), is defined in terms of the 2–form \(F\) appearing in (2.7) as
\[
W_\alpha(\Sigma) = e^{i\alpha \int \Sigma dA} = e^{i\alpha \int \Sigma F} = e^{i\alpha \int F \wedge \Sigma}.
\]
(2.11)
In the same model the Wegner–’t Hooft disorder field is given by
\[
D_\alpha(\Sigma) = e^{-[S(F+\alpha \Sigma) - S(F)]}.
\]
(2.12)
The duality transformation acts on such fields as follows:
\[
\langle W_\alpha(\Sigma) \rangle = \int \mathcal{D}F e^{-S(F)} \int \mathcal{D}\tilde{A}e^{i\int F \wedge d\tilde{A}} e^{i\alpha \int F \wedge \Sigma} = \int \mathcal{D}\tilde{A}e^{-\tilde{S}(d\tilde{A}+\alpha \Sigma)}
\]
\[
= \int \mathcal{D}\tilde{F} e^{-\tilde{S}(\tilde{F}+\alpha \Sigma)} \int \mathcal{D}A e^{i\int \tilde{F} \wedge dA} = \langle D_\alpha(\Sigma) \rangle^\sim
\]
(2.13)
where \(\langle \cdot \rangle^\sim\) denotes the expectation value in the dual theory.
To apply duality to scalar QED, we notice that the representation of the partition function in terms of the current networks appearing in (2.6) can be viewed as the expectation value of a weighted sum of Wilson loops, \(e^{ie \int_{\Sigma(j)} dA}\), with a weighting measure \(\mathcal{D}\mu(j)\) in a gauge theory with action
\[
S(dA) = \frac{1}{2} ||dA||^2,
\]
if we associate to every current configuration \( \mathbf{j} \) a 2–current \( \Sigma(\mathbf{j}) \) satisfying
\[
d\Sigma(\mathbf{j}) = ^* \mathbf{j}. \tag{2.14}
\]
As a result the partition function of the dual theory can be written as
\[
\tilde{Z} = \int \mathcal{D}\mu(\mathbf{j}) \int \mathcal{D}\tilde{A} e^{-\frac{1}{2} ||d\tilde{A} + e\Sigma(\mathbf{j})||^2}. \tag{2.15}
\]
Obviously it corresponds to a Maxwell theory with gauge potential \( \tilde{A} \) coupled to monopoles, whose worldlines are described by \( \mathbf{j} \); \( \Sigma(\mathbf{j}) \) can then be identified as the surface spanned by the Dirac strings of the monopoles.

The partition function is independent of the choice of the Dirac strings, since a different choice \( \Sigma'(\mathbf{j}) \) also satisfying
\[
d\Sigma'(\mathbf{j}) = ^* \mathbf{j},
\]
differs from \( \Sigma(\mathbf{j}) \) by an exact 2–form \( dV \) which can be absorbed by a change of variables \( \tilde{A} \rightarrow \tilde{A} + V \); (Poincaré’s lemma).

By performing the shift
\[
A \rightarrow A + e\delta\Delta^{-1}\Sigma(\mathbf{j})
\]
and using the Hodge decomposition for \( \Sigma(\mathbf{j}) \) (see equation (A.4) in the appendix) one can alternatively rewrite the partition function (2.15) as
\[
\tilde{Z} = \int \mathcal{D}\mu(\mathbf{j}) \int \mathcal{D}\tilde{A} e^{-\frac{1}{2} ||d\tilde{A} + e\delta\Delta^{-1}^* \mathbf{j}||^2}. \tag{2.16}
\]
The term \( \delta\Delta^{-1}^* \mathbf{j} \) can be interpreted as the magnetic field generated by the magnetic current networks \( \mathbf{j} \). One can obtain the two–point monopole correlation function, \( \langle m(B^x)\bar{m}(B^y) \rangle \), from (2.6) by applying the duality transformation. Setting \( B = ^* E \), and \( e = \tilde{g} \) (the magnetic charge in the dual theory), one finds
\[
\langle m(B^x)\bar{m}(B^y) \rangle \sim = \langle \phi(E^x)\bar{\phi}(E^y) \rangle = \frac{1}{\tilde{Z}} \int \mathcal{D}\mu(\mathbf{j}) \mathcal{D}\mu(\mathbf{j}_{xy}) \mathcal{D}\tilde{A} e^{-\frac{1}{2} ||d\tilde{A} + \tilde{g}\delta\Delta^{-1}(^* \mathbf{j} + ^* \mathbf{j}_{xy} + B^x - B^y)||^2}. \tag{2.17}
\]
The magnetic 3–current \( B^x(y) \) can be related to the magnetic field strength (2–form) \( B^x \) of a classical monopole located at \( \vec{x} \) in \( \mathbb{R}^3 \) by
These ideas can be applied to general models of gauge theories with only dynamical charges or monopoles, and they can be rendered mathematically rigorous for lattice theories. As an example of a lattice theory with monopoles, one may consider the \(U(1)\) gauge theory in the Villain or Wilson formulation. The basic field is a \(U(1)\) lattice gauge field \(\theta\), i.e. a \(U(1)\)-valued 1-form, and in the Villain formulation one must introduce a \(2\pi\mathbb{Z}\)-valued two-form field \(n\). The actions are given by

\[
S_V(\theta, n) = \beta ||d\theta + n||^2 \\
S_W(\theta) = \beta \sum_p (1 - \cos(d\theta)_p)
\]

(2.19)

where the subscript \(V\) stands for “Villain” and \(W\) for “Wilson”. The monopole two-point functions are given by

\[
G_V(x, y) = \frac{1}{Z_V} \int \prod_{\langle xy \rangle} d\theta_{<xy>} \sum_n e^{-\beta ||d\theta + n + 2\pi \delta \Delta^{-1}(\omega + B_x - B_y)||^2} \\
G_W(x, y) = \frac{1}{Z_W} \int \prod_{\langle xy \rangle} d\theta_{<xy>} e^{-\beta \sum_p (1 - \cos(d\theta + 2\pi \delta \Delta^{-1}(\omega + B_x - B_y)_p))},
\]

(2.20)

where \(Z\) denotes the partition function of the model, the summation in the Villain model is over the \(2\pi\mathbb{Z}\)-valued \(n\)-configurations and \(\omega\) is a 3-form Poincaré dual of a path joining \(\{x\}\) to \(\{y\}\). The Green function for monopoles in the Villain model can be recast in a form similar to that appearing in eq. (2.17) by defining a real-valued 1-form \(A\) and a \(2\pi\mathbb{Z}\)-valued 3-form \(m\), with

\[
A = \theta + \delta \Delta^{-1} n, \quad m = dn.
\]

Then

\[
G_V(x, y) = \frac{1}{Z_V} \sum_{m; dm = 0} \int \prod_{\langle xy \rangle} dA_{<xy>} e^{-\beta ||dA + \delta \Delta^{-1}(m + 2\pi(B_x + B_y + \omega))||^2}.
\]

(2.21)

Gauge-fixing for \(A\) (or quotienting w.r.t. gauge transformations) is understood in the \(A\)-measure in (2.21).
Finally, by defining, e.g. in the Wilson formulation, a disorder field
\[ D_\omega(B^x, B^y) = e^{-[S_W(d\theta + 2\pi \delta \Delta^{-1}(B^x + \omega - B^y)) - S_W(d\theta)]}, \] (2.22)
on one can express the monopole two-point function as an expectation value of the disorder field:
\[ G_W(x, y) = \langle D_\omega(B^x, B^y) \rangle. \]

It has been rigorously shown in [13,15] (see also [16]) that this disorder field is a good order parameter for the confinement–deconfinement transition in \( d = 4 \) \( U(1) \) gauge theories, \( G(x, y) \) approaches a finite value in the confining phase and vanishes in the deconfined phase, as \( |x - y| \to \infty \).

One can interpret the non-vanishing asymptotic value of \( G(x, y) \) as a signal of monopole–condensation.

3. A modified Dirac ansatz consistent with Dirac’s quantization condition

In this section we discuss the modification of the construction of the previous section needed when dynamical charges and monopoles coexist. We start by showing that, as it stands, the above construction becomes inconsistent in this enlarged setting.

Consider, for example, a model of “compact” scalar QED with a scalar field \( \phi \) of electric charge \( e \) and monopoles of magnetic charge \( g \), whose partition function is given by
\[
Z = \int \mathcal{D}\tilde{\mu}(j) \int \mathcal{D}A e^{-\frac{\beta}{2} \|dA + g\Sigma(j)\|^2} \int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{-\frac{1}{2} \int \bar{\phi}(-\Delta_e A + m^2)\phi} \int \mathcal{D}\tilde{\mu}(j) e^{ie(j,A)}, \] (3.1)

where \( \Sigma(j) \) are the \( \mathbb{Z} \)-valued Dirac strings of the monopoles, and the measure \( \mathcal{D}\tilde{\mu}(j) \) is derived from the action of a matter field coupled to \( A \), e.g. a complex scalar field \( \tilde{\phi} \) of mass \( \tilde{m} \), through an equation like
\[
\frac{\int \mathcal{D}\tilde{\phi} \mathcal{D}\bar{\phi} e^{-\frac{1}{2} \int \bar{\phi}(-\tilde{\Delta}_e A + \tilde{m}^2)\phi} \int \mathcal{D}\tilde{\phi} \mathcal{D}\bar{\phi} e^{-\frac{1}{2} \int \bar{\phi}(-\Delta + m^2)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{-\frac{1}{2} \int \bar{\phi}(-\Delta + m^2)\phi}} = \int \mathcal{D}\tilde{\mu}(j) e^{ie(j,A)} \] (3.2)

In order for \( Z \) to be physically well defined, it should be independent of the choice of the Dirac strings satisfying \( *d\Sigma(j) = j \).

It is convenient to rewrite the integral over \( \phi \) and \( \bar{\phi} \) in terms of \( \mathbb{Z} \)-valued electric current networks \( l \), as
\[
\int D\phi D\bar{\phi} e^{-\frac{1}{2} \int \bar{\phi} (-\Delta + m^2) \phi} = \int D\mu(l) e^{ie(A,l)}. \tag{3.3}
\]

Then the situation analysed in the introduction emerges, and the consistency condition that guarantees that Dirac strings are invisible, in the sense that \(\Sigma(j) \to \Sigma'(j)\) is a symmetry of the theory, becomes invariance of \(Z\) under the shift \(A \to A + gV(j)\), for \(V(j)\) a \(Z\)-valued 1–current satisfying \(dV(j) = \Sigma'(j) - \Sigma(j)\), i.e.

\[
e^{ieg(V(j),l)} = 1. \tag{3.4}
\]

The condition (3.4) is the Dirac quantization condition at the level of currents, and it is satisfied, provided the Dirac quantization condition for charges

\[
eg \in 2\pi Z \tag{3.5}
\]

holds.

If one tries to construct the 2–point function of the charged field according to Dirac’s ansatz, as in the previous section, one meets an inconsistency, since, in contrast to (3.4),

\[
e^{ige(V(j),(l + l_{xy} + E^x - E^y))} \neq 1, \tag{3.6}
\]

even if (3.5) holds.

The origin of this problem is a violation of Dirac’s quantization condition at the level of currents, due to the introduction of the real–valued current \(E\).

Let us first consider this problem for minimal charges: \(eg = 2\pi\). One might envisage avoiding the difficulty encountered above by replacing the electric current \(E^x\) by an electric “Mandelstam string” [17] \(\gamma^x\) carrying a unit flux along a path starting from \(x\) and reaching infinity, at fixed time.

Such a current would still satisfy \(\delta \gamma^x = \delta_x\), and it does not violate Dirac’s quantization condition. However, the current \(\gamma^x\) does not decay at infinity (in contrast to \(E^x\)) and, as a consequence, infrared divergences appear in the construction of charged correlation functions based on this ansatz.

In fact, lattice calculations [13] suggest that an abelian gauge theory with massive monopoles and charges scales to a gaussian gauge theory at large distances, in the Coulomb phase.

Hence, to every Mandelstam string \(\gamma^x\) is associated an infinite positive self–energy, \(\sim (\gamma^x, \Delta^{-1}\gamma^x)\), and the interaction energy between two Mandelstam strings \(\gamma^x, \gamma^y\) of opposite charge is infinite and negative, \(\sim -(\gamma^x, \Delta^{-1}\gamma^y)\), because the strings have infinite length.
Even if the selfenergies are subtructed off, via a multiplicative renormalisation, the interaction between the two strings cannot be removed without violating reflection positivity, because it depends on the distance between $x$ and $y$. A violation of reflection positivity would, however, render impossible the reconstruction of charged quantum fields.

A possible way to circumvent this infrared divergence was suggested in [10]: one has to replace a fixed Mandelstam string $\gamma^x$ by a sum over fluctuating Mandelstam strings weighted by a measure $\mathcal{D} \nu(\gamma^x)$ supported on strings fluctuating so strongly that their interaction energy remains finite even in the limit of infinite length.

The strings appearing in the construction of the correlation functions of charged fields should then converge to a common point at infinity.

As will be reviewed later, it has been shown in [10] that there exists a lattice regularised complex measure $\mathcal{D} \nu_E(\gamma^x)$ on $\mathbb{Z}$–valued 1–currents $\gamma^x$ satisfying

\[ \delta \gamma^x = \delta_x \]

such that

i) the correlation functions for the euclidean fields

\[ \phi(x) \int \mathcal{D} \nu_E(\gamma^x) e^{ie(A,\gamma^x)}, \]

\[ \bar{\phi}(y) \int \mathcal{D} \nu_E(\gamma^y) e^{-ie(A,\gamma^y)} \] (3.8)

satisfy a lattice version of the O.S. axioms; and

ii) in the scaling limit,

\[ \int \mathcal{D} \nu_E(\gamma^x) e^{ie(\gamma^x,A)} \sim e^{ie(E^x,A)}, \] (3.9)

up to a multiplicative renormalisation, where $E$ is the electric “Coulomb” field. Thus on large scales, the sum over fluctuating Mandelstam strings reproduces the behaviour of phase factor appearing in the Dirac ansatz. (This has been verified in [10], in a gaussian approximation). Equation (3.9) suggests that, at large scales, the measure $\mathcal{D} \nu_E(\gamma^x)$ mimics an approximate $\delta$–function peaked around $E^x$.

Let us suppose that an appropriate variant of the O.S. axioms is satisfied by expectation values of the euclidean fields (3.8), (as, follows formally, from their definition). Then from their correlation functions one can reconstruct quantum field operators

\[ \hat{\phi}(E^x), \quad \hat{\phi}(E^y). \]
See [10,13] for details.

If we consider compact scalar QED and the Dirac quantization condition (3.5) is satisfied in the more general form \( g = \frac{2\pi}{e} q \), \( q \in \mathbb{Z} \) \( (q \neq 0) \), then we can repeat the above construction of charged quantum fields, replacing the \( \mathbb{Z} \)-valued currents \( \gamma^x \) with \( \mathbb{Z}/q \)-valued currents \( \gamma^x \) (satisfying (3.7)).

Let us describe how to construct a monopole–monopole Green function. We associate a \( \mathbb{Z}/q \)-valued 2–current \( \Sigma(\gamma^x - \gamma^y + j_{xy}) \) to an integral 1–current \( j_{xy} \) satisfying \( \delta j_{xy} = \delta_y - \delta_x \) and a pair of \( \mathbb{Z}/q \)-valued currents \( \gamma^x \) and \( \gamma^y \) (with \( \delta \gamma^z = \delta_z, z = x, y \)), such that

\[
* d\Sigma(\gamma^x - \gamma^y + j_{xy}) = \gamma^x - \gamma^y + j_{xy}.
\] (3.10)

We then define a disorder field by setting

\[
D(\Sigma(\gamma^x - \gamma^y + j_{xy})) = e^{-\frac{1}{2} \left( ||dA + g(\Sigma(\gamma^x - \gamma^y + j_{xy}) + \Sigma(j))||^2 - ||dA + \Sigma(j)||^2 \right)}. \tag{3.11}
\]

One can easily verify that \( \langle D(\Sigma(\gamma^x + \gamma^y + j_{xy})) \rangle \) depends only on \( \gamma^x + \gamma^y + j_{xy} \) and not on a specific choice of \( \Sigma \), because different choices can be mapped onto one another by a shift of \( A \).

The monopole 2–point function is given by

\[
\int D\nu_E(\gamma^x)D\nu_E(\gamma^y)D\tilde{\mu}(j_{xy})\langle D(\Sigma(\gamma^x + \gamma^y + j_{xy})) \rangle,
\] (3.12)

where \( \tilde{\mu}(j_{xy}) \) is the measure defined through the equation

\[
\int D\tilde{\phi}D\phi e^{-\frac{1}{2} \int \tilde{\phi}(-\Delta_{cA} + \tilde{m}^2)\phi} \tilde{\phi}(y)\phi(x) = \int D\tilde{\mu}(j_{xy})e^{ie(j_{xy}, A)},
\]

for arbitrary \( A \).

Denoting by \( \langle \cdot \rangle^\sim \) the expectation value in the dual model, one can easily verify that (3.12) equals

\[
\langle \tilde{\phi}(E^y)\phi(E^x) \rangle^\sim.
\]

From correlation functions of disorder fields such as (3.11) one can obtain, via O.S. reconstruction, monopole field operators, \( \hat{m}(B^x) \), where \( B^x = * E^x \) encodes the infrared behaviour of the soft photon cloud accompanying the monopole, (which is an “infra–particle”).

In particular, for \( y^0 < 0 < x^0 \), the correlation function (3.12) is equal to

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\[ \langle \hat{m}(B^y)\Omega | \hat{m}(B^x)\Omega \rangle. \]

The modification of Dirac’s recipe and its dual suggested above can be adapted to any abelian theory of coexisting charges and monopoles, and it can be made precise for lattice theories. The measure \( \mathcal{D}\nu_E \) can then be constructed as follows; (we omit some technical details about b.c., see [10]): We introduce a \( U(1) \)-valued scalar field \( \chi \) of period \( 2\pi q \) defined on a three-dimensional hyperplane \( \Lambda_{x^0} \), at fixed time \( x^0 \), minimally coupled to the gauge field \( A \). An action is given e.g. by

\[
S(\chi, A) = \frac{\xi}{2} \sum_{<xy> \in \Lambda_{x^0}} \left[ 1 - \cos \left( \frac{d\chi + A}{q} <xy> \right) \right],
\]

and we denote the corresponding expectation value by \( \langle \cdot \rangle_{x^0}(A) \).

Then the two-point function, \( \langle e^{i\chi_x} e^{-i\chi_y} \rangle_{x^0}(A) \), can be expressed in terms of \( \mathbb{Z}/q \)-valued 1–currents \( \gamma_{xy} \) satisfying \( \delta \gamma_{xy} = \delta_x - \delta_y \) as

\[
\langle e^{i\chi_x} e^{-i\chi_y} \rangle_{x^0}(A) = \int \mathcal{D}\nu(\gamma_{yx}) e^{i(A,\gamma_{xy})},
\]

where \( x^0 = y^0 \).

If \( \xi \) is sufficiently large (and if the field strength \( \{(dA)_p\}_{p \in \Lambda_{x^0}} \) does not fluctuate much) the system described by \( \chi \) is in a phase where the symmetry

\[
\chi \rightarrow \chi + \text{const}
\]

is spontaneously broken [3]. The associated Goldstone boson is a real field, \( \lambda \), describing a spin wave. It corresponds to the decompactification of the range of \( \chi \). Deviations from the theory described by non–interacting spin waves are due to vortices. They are believed to be irrelevant at large scales (in the sense of the renormalization group), provided the symmetry (3.15) is spontaneously broken.

We propose to evaluate (3.14) at large scales, neglecting vortices. The result is

\[
\frac{1}{Z} \int \mathcal{D}\lambda e^{i\lambda_x} e^{-i\lambda_y} e^{-\xi_{\text{ren}}||d\lambda + A||^2} e^{i(\Delta_{x^0}(\delta_x - \delta_y),A)} = e^{-\frac{1}{4}((\delta_x - \delta_y),\Delta_{x^0}^{-1}(\delta_x - \delta_y))} e^{i(\Delta_{x^0}^{-1}(\delta_x - \delta_y),A)},
\]

where \( \xi_{\text{ren}} \) denotes a renormalised coupling, and \( \Delta_{x^0} \) is the 3D laplacian on \( \Lambda_{x^0} \).

One can easily verify that

\[
E^x - E^y = d \Delta_{x^0}^{-1}(\delta_x - \delta_y).
\]

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Sending $y$ to $\infty$ and eliminating the first factor in (3.16) by a multiplicative renormalization, one obtains

$$
\left(\langle e^{i\chi x} e^{-i\chi \infty}\rangle(A)\right)_{\text{ren}} \sim e^{i(E^x.A)}
$$

This suggests that the desired measure is of the form

$$
D\nu_E(\gamma^x) \sim \lim_{y \to \infty} c_y D\nu(\gamma_{xy}),
$$

where $c_y$ denotes a suitable multiplicative renormalisation constant, that we expect to be finite, on the basis of a gaussian computation performed in [10].

As example of lattice theories with dynamical charges and monopole one may consider models where a $U(1)$–gauge field $\theta$ is coupled to a $U(1)$–valued scalar matter field $\phi$ of charge $q = 1, 2, \ldots$; (here we set the elementary charge equal to unity).

In Wilson’s formulation, the action is given by

$$
S = \sum_{p} \beta (1 - \cos(d\theta)_p) + \kappa \sum_{<ij>} (1 - \cos(q\theta + d\phi)_{<ij>}).
$$

According to the recipe explained above, the monopole two–point Green function (for monopoles of unit charge) is given by

$$
\frac{1}{Z} \int \prod_{<ij>} d\theta_{<ij>} \prod_{i} d\phi_i \int D\nu_E(\gamma^x) D\nu_E(\gamma^y) e^{-\beta \sum_{p} (1 - \cos(2\pi \Sigma(\gamma^x - \gamma^y + j_{xy}))_p) - \kappa \sum_{<ij>} (1 - \cos(q\theta + d\phi)_{<ij>})},
$$

where $\Sigma(\gamma^x - \gamma^y + j_{xy})$ is a $\mathbb{Z}/q$–valued 2–form satisfying (3.10), and $j_{xy}$ is an integral 1–current with support on a path connecting $x$ to $y$ and $\delta j_{xy} = \delta x - \delta y$.

The Green function (3.18) is independent of the choice of $j_{xy}$ since a change in $j_{xy}$ can be compensated by a shift of $\theta$. Hence, in contrast to the continuum models previously discussed, no integration over $j_{xy}$ is needed.

4. The magnetic charge in $SU(2)$ gauge theory

We turn to the $SU(2)$ Yang–Mills theory, identifying its magnetic monopoles and defining their magnetic charge.

We first discuss these matters in the continuum, where we denote by $A$ the $SU(2)$–connection form.

Let us consider a cube $c$ in $\mathbb{R}^3$; its boundary, $\partial c$ is homeomorphic to a 2–sphere. The restriction of $A$ to $\partial c$ can be viewed as a connection form $A \equiv A|_{\partial c}$ of an $SO(3)$–bundle over $\partial c$. 

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SO(3)–bundles over \( S^2 \simeq \partial c \) are classified by \( \pi_1(SO(3)) \simeq \mathbb{Z}_2 \). The following relation holds between \( \tilde{A} \) and the integer class \( n \mod 2 \in \mathbb{Z}/2\mathbb{Z} \simeq \pi_1(SO(3)) \) classifying the corresponding SO(3) bundle:

\[
e^{i\pi n} = e^{i \arg \sum_{p \in \partial c} Tr \left( P e^{i \oint_{\partial p} A^{(p)}} \right)}, \tag{4.1}
\]

where \( A^{(p)} \) is the connection 1–form of an SU(2)–bundle over the face (“plaque-tte”) \( p \) of the cube \( c \), obtained by lifting the SO(3)–bundle over \( p \).

If \( e^{i\pi n} \neq 1 \) one cannot extend the SO(3)–bundle to the interior of the cube \( c \); this signals the presence of an odd number of \( \mathbb{Z}_2 \)–monopoles of SO(3) inside \( c \) and we identify \( e^{i\pi n} \) as the \( \mathbb{Z}_2 \)–charge contained in \( c \).

An abelian projection gauge, defined as in the introduction, selects a \( U(1) \) residual gauge group. By projecting \( \tilde{A} \) to the Cartan subalgebra of \( su(2) \) corresponding to the residual gauge group one obtains a \( U(1) \) connection \( a \) on \( \partial c \).

\( U(1) \)–bundles over \( S^2 \simeq \partial c \) are classified by \( \pi_1(U(1)) \simeq \mathbb{Z} \). The relation between the integer \( n \) classifying the bundle and the connection \( a \) is given by

\[
n = \frac{1}{2\pi} \sum_{p \in \partial c} \int F(a), \tag{4.2}
\]

where \( F(a) \) is the curvature of \( a \).

We identify the magnetic charge contained inside \( c \) in the abelian projection gauge with the integer \( n \), corresponding to the first Chern number of \( F(a) \).

If \( a \) is derived from \( \tilde{A} \) by abelian projection, the integer \( n \) appearing in (4.2) is the same appearing (4.1).

According to a general theorem [18], the classical monopoles associated to a \( U(1) \) subgroup SO(3) are regular if they carry even magnetic charge. For these monopoles there are gauge choices for which no singularity of Yang–Mills curvature occurs at the monopole position. Monopoles with odd magnetic charge are singular, i.e. in every gauge there is a singularity of the Yang–Mills curvature where the monopole is located. The position of the singularity is then independent of the choice of the \( U(1) \) subgroup, or equivalently of the abelian gauge projection, used to define the magnetic charge and it identifies the position of a \( \mathbb{Z}_2 \)–monopole.

The definition of magnetic charge and \( \mathbb{Z}_2 \)– charge can be easily adapted to the lattice formulation as follows [19].

Let \( g \) denote the \( SU(2) \) lattice gauge field. The Yang–Mills action on the lattice is given by
\[ S_{YM}(g) = \beta \sum_p \left( 1 - \chi(g\partial p) \right) \]  

(4.3)

where \( \chi \) is the character of the fundamental representation.

Denote by \( X(g) \) the scalar field with values in \( su(2) \) transforming under the adjoint representation, identifying the abelian projection gauge. Let \( W \) be the \( SU(2) \) gauge transformation such that for every site \( i \) on the lattice one has

\[ W_i X_i(g) W_i^+ = |X_i(g)| \sigma_3, \]  

(4.4)

where \( |X_i| = \sqrt{Tr X_i^a X_i^a} \) and define

\[ \tilde{g}_{ij} = W_i g_{ij} W_j^+. \]  

(4.5)

\( \tilde{g} \) denotes the \( SU(2) \)-gauge field in the abelian projection gauge defined by \( X \).

\( \tilde{g}_{ij} \) can be decomposed as a product of two matrices \( C_{ij} \) and \( u_{ij}(\theta) \), where

\[ C_{ij} = \begin{pmatrix} (1 - |c_{ij}|^2)^{\frac{1}{2}} & -\bar{c}_{ij} \\ c_{ij} \end{pmatrix}, \quad u_{ij} = e^{i\frac{1}{2}\theta_{ij}} \sigma_3, \]  

(4.6)

with \( c_{ij} \in \mathbb{C}, \bar{c}_{ij} \) denoting its complex conjugate, and \( \frac{1}{2}\theta_{ij} = \text{arg} (\tilde{g}_{ij})_{11} \).

Hence \( \theta \) is a \( U(1) \) gauge field with range \((-2\pi, 2\pi)\) and \( c \) is a charged field of charge 1.

A similar decomposition holds for the coset variable given on a link \( <ij> \) by \( g_{ij}\Gamma \equiv U_{ij} \), where \( \Gamma \) is the centre of \( SU(2) \), isomorphic to \( \mathbb{Z}_2 \). \( U_{ij} \) can be viewed as an \( SO(3) \)-gauge field and in the decomposition (4.6) \( \theta \) is now a \( U(1) \) field with range \((-\pi, \pi)\).

We define the magnetic charge in a lattice cube \( c \) by

\[ m_c(\theta) = \frac{1}{2\pi} \sum_{p \in \partial c} (d\theta)_p \]  

(4.7)

where \( (d\theta)_p \) is restricted to the range \((-\pi, \pi)\).

The \( \mathbb{Z}_2 \)-charge in \( c \) is defined by

\[ e^{i\pi z_c(g)} = e^{i} \sum_{p \in \partial c} \text{arg} \chi(g\partial p). \]  

(4.8)

A plaquette \( p \) where

\[ e^{i\text{arg} \chi(g\partial p)} = -1 \]  

(4.9)
can be identified as the location of a Dirac string of a $\mathbb{Z}_2$–monopole intersecting the plane containing $p$.

In the lattice formulation the relation discussed in the continuum between $\mathbb{Z}_2$– and magnetic charge becomes

$$e^{i\pi z_c(g)} = e^{i\pi m_c(\theta)} \quad (4.10)$$

5. The $U(1)$–monopole order parameter for $SU(2)$ Yang–Mills theory

In this section we recall some attempts made to derive an order parameter in $SU(2)$ Yang–Mills theory on the lattice, using a construction directly inspired by Dirac ansatz.

Integrating out the charged field $c$ defined in (4.6) one can view the $SU(2)$ gauge theory in an abelian projection gauge as a $U(1)$ theory with gauge field $\theta$ and an effective action of the form $S_{\text{eff}}(d\theta)$, where we used the residual $U(1)$ gauge invariance to deduce the dependence on the curvature $d\theta$.

It is then natural, following the dual of Dirac ansatz, to try to define the two–point Green function of the magnetic monopole of the abelian projection by performing the shift

$$d\theta \to d\theta + 4\pi \delta \Delta^{-1}(B^x - B^y + \omega) \quad (5.1)$$

where $\omega$ is a 3–form Poincaré dual of a path connecting $x$ to $y$.

This is basically the attempt made in [6] in the Maximal Abelian Gauge [19] and in [10] in the Spatial Maximal Abelian Gauge.

However, a closer look shows some inconsistency in this approach. In fact, it is easy to prove that the effective action is given by an expression of the form

$$\sum_L C_L e^{i\frac{1}{2} \sigma_3 \sum_{<ij> \in L} \theta_{<ij>}} \quad (5.2)$$

where $L$ is a loop and $C_L$ a complex coefficient independent of $\theta$.

Choosing, for each $L$, a surface $\Sigma(L)$ such that $\partial \Sigma(L) = L$, one can rewrite (5.2) as

$$\sum_L C_L e^{i\frac{1}{2} \sigma_3 \sum_{p \in \Sigma(L)} (d\theta)_p},$$

i.e. in the form $S_{\text{eff}}(d\theta)$ and the choice of the surfaces $\Sigma(L)$ is obviously irrelevant.

However, when the shift (5.1) is performed the term

$$e^{i\sigma_3 \sum_{p \in \Sigma(L)} 2\pi \delta \Delta^{-1}(B^x - B^y + \omega)} \quad (5.3)$$
is no more independent of the choice of \( \Sigma(L) \). This inconsistency is due to the violation of Dirac quantization condition at the level of currents related to \( B^x - B^y \).

The approach of [7,9] is slightly different and monopole Green functions are defined directly by modifying the Yang–Mills action.

Let us fix a convention assigning to every plaquette \( p \) a site \( j(p) \) on its boundary (see [9] for more details) and let \( X(g) \) denote the scalar field defining the abelian projection.

Within our setting the proposal made in [7,9] to obtain the monopole–monopole Green function is to replace the plaquette term \( \chi(g \partial_p) \) in the Yang–Mills action (4.3) by

\[
\chi\left(g \partial_p e^{\frac{N}{|L|} j(p) 2\pi \delta \Delta^{-1} (B^x - B^y + \omega)}\right).
\]

Actually, in [7,9] the support of \( \omega \) consists of a sum of the dual of two straight lines, \( \omega^x \) and \( -\omega^y \) each at constant time in the 3–direction. With this choice \( B^x + \omega^x \) can be related, as in (2.18), to the magnetic field of a Dirac monopole with its Dirac string along the 3–direction.

In the abelian projection gauge defined by \( X \), using the decomposition (4.6), one can rewrite the argument of \( \chi \) in (5.4) as

\[
\prod_{<ij> \in \partial p} \left( e^{i\vec{c} \cdot \vec{\sigma}} e^{i\Delta^{-1} (B^x - B^y + \omega)}\right),
\]

where

\[
c_{<ij>} = c^1_{<ij>} + ic^2_{<ij>}, \quad c^\alpha_{<ij>}, \in \mathbb{R}, \alpha = 1, 2 \quad \vec{c} = (c^1, c^2).
\]

We insert the lattice constant \( \epsilon \) and, since we are really interested in the continuum limit, we consider an expansion in \( \epsilon \) up to \( 0(\epsilon^2) \): One finds

\[
(5.5) \sim \prod_{<ij> \in \partial p} \left( e^{i\vec{c} \cdot \vec{\sigma}} e^{i\sigma_3 2\pi \delta \Delta^{-1} (B^x - B^y + \omega)}\right) (1 + O(\epsilon^2)).
\]

Therefore to \( O(\epsilon^2) \) this recipe seems to give a consistent prescription; however problems arise at order \( \epsilon^2 \) because in \( g \partial_p \) there appear terms depending on \( \theta \) which cannot be rewritten in terms of \( (d\theta)_p \):

\[
O(\epsilon^2) = \epsilon^2 \left\{ [(d\theta)_p \sigma_3, \widetilde{\gamma}(c) \cdot \vec{\sigma}] + (\theta \sigma_3 \land \widetilde{\gamma}(c) \cdot \vec{\sigma})_p \right\} + O(\epsilon^3),
\]

where \( \land \) is the wedge product on the lattice (see e.g. [10]) and \( \widetilde{\gamma} = (\gamma^1, \gamma^2) \) are functions of \( c \).
As a consequence of the last term in (5.6), a change in the choice of the “Dirac string” \( \omega \) cannot be eliminated by a shift in \( \theta \), and again this can be traced back to a violation of Dirac quantization condition at the level of currents.

The consistency to order \( \epsilon^2 \) of the recipe in [7,9] suggests however that this disorder parameter (and the one of [6]) could be meaningful at large scales, as supported by numerical results for the (physical) temperature of the deconfinement transition.

6. Green functions for regular monopoles in SU(2) lattice gauge theory

Combining the ideas of the last three sections, one is lead to propose a definition of Green functions for regular monopoles of any abelian projection, as follows.

Imagine that a regular charge–2 monopole in an abelian projection gauge is created at a site \( x \) and annihilated at a site \( y \). We propose to construct the corresponding two–point function summing over a pair of fluctuating strings carrying magnetic flux 1 with end points at the position of the monopole.

As discussed in section 4, strings of odd magnetic flux in every abelian projection can be identified as Dirac strings of \( \mathbb{Z}_2 \)–monopoles, and hence are independent of the choice of the projection.

In turn, \( \mathbb{Z}_2 \)–monopoles can be introduced by means of a ’t Hooft disorder field which, in \( SU(2) \) lattice gauge theory, is defined as follows.

Let \( \Sigma \) be the Poincaré dual of a surface bounded by a loop \( \mathcal{L} \). Then the corresponding disorder field is defined by

\[
D(\Sigma) = e^{-[S_{YM}(\{g_\partial e^{i\Sigma_\sigma \Sigma_3}\}) - S_{YM}(\{g_\partial \}]} \tag{6.1}
\]

where \( S_{YM} \) is the action (4.3). The expectation value of \( D(\Sigma) \) depends only on \( \mathcal{L} \) and describes the worldlines of a \( \mathbb{Z}_2 \) monopole–antimonopole pair.

Since the regular monopoles have even magnetic charge and the charged field, \( c \), of the abelian projection has integer electric charge, one can adapt to the present setting the construction presented in section 3 for \( q = 2 \).

Let \( j_{xy} \) be an integer 1–current satisfying \( \delta j_{xy} = \delta y - \delta x \) with support on a path connecting \( x \) to \( y \), and let \( \mathcal{D}_\nu E(\gamma^x), \mathcal{D}_\nu E(\gamma^y) \) denote normalized, signed measures over \( \mathbb{Z}/2 \)–valued currents \( \gamma^x, \gamma^y \), constrained by \( \delta \gamma^z = \delta z, z = x, y \) as defined in section 3.

To a configuration \( \{\gamma^x, \gamma^y, j_{xy}\} \) we associate a 2–current \( \Sigma(\gamma^x - \gamma^y + j_{xy}) \) satisfying

\[
* d\Sigma(\gamma^x - \gamma^y + j_{xy}) = 2(\gamma^x - \gamma^y + j_{xy}),
\]
where the 2 in the h.s. appears for agreement with definition (6.1). \( \Sigma \) is then the Poincaré dual of a two-sheet surface, with boundary given by the support of \( \gamma^x - \gamma^y \) and the two sheets joining each other along the support of \( j_{xy} \).

We define the monopole two-point correlation function by

\[
G(x, y) = \int D\nu_E(\gamma^x) D\nu_E(\gamma^y) \langle D(\Sigma(\gamma^x - \gamma^y + j_{xy})) \rangle.
\]

We claim that long-range order in this correlation function characterizes the confinement phase.

From the \( n \)-point monopole correlation functions defined by generalizing eq. (6.2) one can (at least formally) reconstruct, via O.S. lattice reconstruction, a monopole field operator \( \hat{m}(B^x), B = \ast E \). The long-range order for \( G(x, y) \) then corresponds to a non-vanishing vacuum expectation value of \( \hat{m} \).

Since equation (6.2) does not involve any abelian projection, it follows that, while the definition of the trajectory of a regular monopole requires choosing an abelian projection, the locations of creation and annihilation of a monopole are independent of that choice, hence intrinsic to the \( SU(2) \) theory.

From numerical simulations one may gain some indirect support for the conjecture that \( \hat{m} \) is a good order parameter for the confinement–deconfinement transition by noticing that the large distance behaviour of \( G(x, y) \) appears to approach that of the \( U(1) \)-monopole Green functions of [7].

In fact, since the group manifold of \( SU(2) \) is isomorphic, via the exponential map, to a 3-ball of radius \( \pi \) with boundary points identified, one may replace \( e^{i\pi \Sigma_p \sigma_3} \) by \( e^{i\pi (\frac{X}{|X|}) j(p) \Sigma_p} \) in the definition (6.2) of the disorder field for any choice of scalar \( X \) defining an abelian projection. This is because \( (\frac{X}{|X|}) j(p) \) defines a unit vector in \( su(2) \).

As remarked in section 3, at large scales the measure \( D\nu_E(\gamma^x) \) behaves as an approximate Dirac measure peaked around the current configuration \( E^x \), and this, in turn, implies that in (6.2) the configurations of \( d\Sigma(\gamma^x - \gamma^y + j_{xy}) \) are peaked around \( 2(B^x - B^y + \omega) \), with \( B = \ast E, \omega = \ast j_{xy} \).

Therefore, in the scaling limit, one expects that

\[
\int D\nu_E(\gamma^x) D\nu_E(\gamma^y) e^{-S_{YM}[\{g_{\alpha\beta} e^{i\pi (\frac{X}{|X|}) j(p) \Sigma_p} \}]} \\
\sim e^{-S_{YM}[\{g_{\alpha\beta} e^{i(\frac{X}{|X|}) j(p) \Sigma_p} e^{2\pi i [\Delta^{-1}(B^x - B^y + \omega)] p} \}]}.
\]

(6.3)
This would reproduce the behaviour of the disorder field defined via eq. (5.4). Numerically, its expectation value provides a clear signal of the confinement–deconfinement transition [7]. Eq. (6.3) yields an explanation of the numerical evidence that, from order parameters corresponding to different abelian projections, one obtains the same transition temperature [9].

In fact, this transition is governed by the low–energy physics, correctly captured by the scaling limit. Hence, assuming that eq. (6.3) holds, the order parameters defined via eq. (5.4), unphysically dependent on the choice of the “Dirac string $\omega$” at small scales, are just the scaling limit of the order parameter defined via (6.2), which is manifestly independent of the choice of an abelian projection and of the “Dirac string $j_{xy}$”.

Expressing $\langle D\left(\Sigma(\gamma^x - \gamma^y + j_{xy})\right)\rangle$ in terms of magnetic currents, through a duality transformation, one can exhibit more explicitly the non–vanishing vacuum expectation value of $\hat{m}$ as a dual Higgs mechanism (in the spirit of the “abelian dominance” scenario).

Since, however, $\Sigma(\gamma^x - \gamma^y + j_{xy})$ can be interpreted as a (double) sheet of center vortices, a connection with the “center dominance” scenario emerges, as discussed in next section.

To derive the duality transformation, we express the correlation function $\langle D(\Sigma)\rangle$, where $\Sigma \equiv \Sigma(\gamma^x - \gamma^y + j_{xy})$, in terms of the variable $\theta$ and $c$ defined in eq. (4.6), and insert in the integration measure an abelian projection gauge fixing

$$\delta(\frac{X}{|X|}(c, \theta) - \sigma_3).$$

First we integrate out $c$. As a consequence of the residual $U(1)$ gauge invariance we can expand

$$\int \prod_{<ij>} dc_{<ij>} d\bar{c}_{<ij>} e^{-S_{YM}\{g_{ab}(\theta, c)e^{i\pi\Sigma} \sigma_3\}} \prod_j \delta\left(\frac{X_j}{|X_j|}(\theta, c) - \sigma_3\right)$$

as a Fourier series in $d\theta + 2\pi\Sigma$.

The coefficients are denoted by $F(n)$, where $n$ is a $\mathbb{Z}/2$–valued 2–form.

We define a 1–form $\ell$ by

$$\delta n = \ell,$$

and decompose the 2–form $n$ as

$$n = n[\ell] + ^*d\xi,$$
where \( n[\ell] \) is a \( \mathbb{Z}/2 \)-valued solution of (6.5) and \( \xi \) is a \( \mathbb{Z}/2 \)-valued 1–form in the dual lattice. Then one obtains

\[
\langle D(\Sigma) \rangle = \frac{1}{\mathbb{Z}} \sum_{[\xi]} \sum_{\ell : \delta \ell = 0} F(n[\ell] + * d\xi) \int \prod_{<ij>} d\theta_{<ij>} e^{i(\ell, \theta) e^{i2\pi (\Sigma, * d\xi)}}, \tag{6.7}
\]

where \([\xi]\) denotes a gauge equivalence class of \( \xi \), and the equation

\[
(n[\ell], d\theta) = (\ell, \theta)
\]

has been used.

Integrating over \( \theta \) imposes the constraint \( \ell = 0 \). Hence, in particular, one can choose \( n[\ell] = 0 \). Furthermore we can replace \( \xi \) by a real–valued 1–form \( A \) by inserting the term \( \sum_{\rho : \delta \rho = 0} e^{i4\pi(A, \rho)} \), where \( \rho \) is a \( \mathbb{Z} \)–valued 1–form (see e.g. [21]).

As a result we obtain

\[
\langle D\left(\Sigma(\gamma^x - \gamma^y + j^{xy})\right) \rangle = \frac{1}{\mathbb{Z}} \int d[A] \sum_{\rho : \delta \rho = 0} F(* dA) e^{i4\pi(\gamma^x - \gamma^y + j^{xy} + \rho, A)}, \tag{6.8}
\]

where \( d[A] \) denotes formal integration over gauge equivalence classes of \( A \).

In (6.8), worldlines of regular monopoles of the abelian projection are described by the currents \( j^{xy} + \rho \), and they exhibit sources at \( \{x\} \) and \( \{y\} \).

The representation (6.8) explicitly proves independence of the choice of \( j^{xy} \) in the construction of Green functions.

By setting

\[
\tilde{S}(A) \equiv -\ln F(* dA)
\]

one can view the dual model appearing in (6.8) as a Higgs model with gauge action \( \tilde{S}(A) \) and the correlation function

\[
\int \mathcal{D}\nu_E(\gamma^x) \mathcal{D}\nu_E(\gamma^y) \langle D(\Sigma(\gamma^x - \gamma^y + j^{xy})) \rangle
\]

can be viewed as the two–point function of the charged field, \( \langle \phi(E^x) \bar{\phi}(E^y) \rangle \), of that dual model, where \( \langle \cdot \rangle \) denotes the corresponding expectation value.

The abelian Higgs model in four dimensions has two phase, the Coulomb and the Higgs phase. If the dual model is in the Higgs phase one expects that \( \langle \phi(E^x) \rangle \neq 0 \). (In fact, for the standard gauge action and with the Dirac recipe for the charged field, this has been proved in [13], [16].) In the original model this non–vanishing expectation value corresponds to \( \langle \Omega | \hat{m}(B^x) \Omega \rangle \neq 0 \), thus describing monopole condensation.
The suggestion that the dual model is in the Higgs phase comes from numerical simulations, as previously discussed. The above construction makes the relation between the original $SU(2)$ gauge theory and a dual abelian Higgs model more precise, as advocated by many authors.

We end this section with a remark about about monopole Green functions in the formal continuum limit. Our discussion of monopole in $SU(2)$–Yang Mills theory was performed in the lattice, because it heavily relies upon the ’t Hooft disorder field which has no simple continuum analog.

Presumably one can construct a disorder field with the desired properties in the continuum using the loop space formalism developed in [22], but this will be discussed elsewhere.

7. Relation with center dominance

The representation (6.8) does not exhibit center vortex sheets; they are hidden in the definition of $F(\ast dA)$. To exhibit them explicitly one starts, following [23], by replacing the $SU(2)$–gauge field $g$ with a couple of new variables $\{U, \sigma\}$ where $U$ is the gauge coset variable, introduced in sect.4, which can be viewed as an $SO(3)$ gauge field and $\sigma$ is a 2–form with values in $\{0, 1\} \simeq \mathbb{Z}_2$.

It has been shown in [22] that the two fields $U$ and $\sigma$ are not independent. It is easy to show that $\sum_{p \in \partial c} \text{arg} \chi(g_{\partial p})$ is only a function of the coset field $U$, which we denote by $z_c(U)$ and the following constraints holds:

$$e^{i\pi(d\sigma)_c} = e^{iz_c(U)}$$  \hspace{1cm} (7.1)

for each cube $c$.

Let us discuss the relation between $\sigma$ and the ’t Hooft disorder field.

The plaquette term $\chi(g_{\partial p} e^{i\pi \sigma_3 \Sigma_p})$ appearing in $\langle D(\Sigma) \rangle$ is rewritten in the new variables, as

$$|\chi|(U_{\partial p}) e^{i\pi(\sigma_p + \Sigma_p)},$$  \hspace{1cm} (7.2)

where $|\chi|(U) = |\chi(g)|$.

Hence the introduction of the disorder field $D(\Sigma)$ induces a shift of $\sigma_p$ by $\Sigma_p$.

The constraint (7.1) can be solved by

$$e^{i\pi \sigma_p} = \text{sign} \chi(U_{\partial p}) \prod_{<ij> \in \partial p} \text{sign} \chi(U_{<ij>})$$

$$= \text{sign} \chi(g_{\partial p}) \prod_{<ij> \in \partial p} \text{sign} \chi(g_{<ij>})$$  \hspace{1cm} (7.3)
This solution is gauge–dependent but does not involve any choice of abelian projection.

In the center dominance scenario one defines the maximal center gauge in $SU(2)$ gauge theory as the gauge which brings the link variables $\{g_{<ij>}\}$ as close as possible to the center, $\Gamma \simeq \mathbb{Z}_2$, of $SU(2)$, by maximizing the quantity

$$R = \sum_{<ij>} Tr(g_{<ij>}^2).$$

In the maximal center gauge, a plaquette $p$ where

$$\prod_{<ij>} \text{sign} \chi(g_{<ij>}) = -1 \quad (7.4)$$

is the location of a P–vortex.

It has been rigorously proven in [23] that, for large $\beta$, i.e., close to the continuum limit, the set of plaquettes where $\text{sign} \chi(g_{\partial p}) = -1$ is dilute.

Therefore, since $\text{sign} \chi(g_{\partial p})$ is gauge–invariant, for large $\beta$, the identification of a P–vortex location as the set of plaquettes where $\sigma_p \neq 0$ in the center projection gauge, should be equivalent to the standard definition, equation (7.4), from the point of view of discussing the deconfinement transition.

Numerical simulations shows that, in the confining phase, P–vortex sheets percolate [2]. This suggests that in this phase even the introduction of an additional infinite P–vortex sheet $\Sigma$, like the one involved in the construction of the monopole Green function, should be a small perturbation and should not lead to a clustering behaviour.

In other words one could interpret a non–vanishing expectation value of the monopole operator $\hat{m}$ in the maximal center gauge as due to a condensation of P–vortex sheets, in the spirit of center dominance.

In the deconfinement phase at positive temperature, numerical simulations shows that P–vortex sheets are dilute [2]. Hence, for large $\beta$, it is natural to conjecture that the introduction of an infinite P–vortex sheet $\Sigma$ leads to clustering and, as a consequence, to a vanishing expectation value for $\hat{m}$.

Finally we remark that an approximate relation between P–vortex sheets and regular monopole worldlines can be established following [24].

The double–sheet P–vortex structure associated to monopole worldlines appearing in [24] is a natural counterpart of our construction of monopoles in terms of the double–sheet surface $\Sigma$. 

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Appendix

Forms and currents in the continuum and on the lattice

In the continuum we consider as “space–time” the euclidean space $\mathbb{R}^d$. Given an antisymmetric tensor field of rank $k$ on $\mathbb{R}^d$, $a_{\mu_1...\mu_k}(x)$ one defines the associated $k$–form by setting

$$a^{(k)}(x) = \frac{1}{k!}a_{\mu_1...\mu_k}(x)dx^\mu_1 \wedge ... \wedge dx^\mu_k$$  \hspace{1cm} (A.1)

where $\wedge$ is the wedge (antisymmetric tensor) product. The space of $k$–forms is a group $\Lambda^k(\mathbb{R}^d)$ under pointwise addition. We denote by

$$d : \Lambda^k(\mathbb{R}^d) \to \Lambda^{k+1}(\mathbb{R}^d)$$

the exterior differential defined through

$$da^{(k)}(x) = \frac{1}{(k+1)!}\partial_\mu a_{\mu_1...\mu_k}(x)dx^\mu \wedge dx^\mu_1 \wedge ... dx^\mu_k,$$  \hspace{1cm} (A.2)

by

$$* : \Lambda^k(\mathbb{R}^d) \to \Lambda^{d-k}(\mathbb{R}^d)$$

the Hodge star defined through

$$*(a^{(k)}(x)) = \frac{1}{k!(d-k)!}\epsilon_{\mu_1...\mu_{d-k+1}...\mu_d}a^{\mu_{d-k+1}...\mu_d}(x)dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{d-k}},$$  \hspace{1cm} (A.3)

by $\delta = *d(-1)^{d(k+1)}$ the codifferential and by

$$\Delta = \delta d + d\delta$$  \hspace{1cm} (A.4)

the Laplacian.

An inner product between $k$–forms is defined by setting

$$(a^{(k)}, b^{(k)}) = \int d^dx \ a_{\mu_1...\mu_k}(x)b^{\mu_1...\mu_k}(x) = \int a^{(k)} \wedge *b^{(k)}$$  \hspace{1cm} (A.5)

and it satisfies

$$(a^{(k)}, db^{(k-1)}) = (\delta a^{(k)}, b^{(k-1)}).$$  \hspace{1cm} (A.6)

The $L^2$ norm corresponding to the inner product (A.5) is denoted by $|| \ ||$, i.e. $(a^{(k)}, a^{(k)}) \equiv ||a^{(k)}||^2$.  \hspace{1cm} 28
The Poincaré lemma states that if \( da^{(k)} = 0 \), then there exist \( a^{(k-1)} \) such that \( a^{(k)} = da^{(k-1)} \).

A \( k \)-current in \( \mathbb{R}^d \) is a linear functional in the space of \( d - k \) forms with compact support, continuous in the sense of distributions, i.e. \( k \)-current are \( k \)-forms with distribution-valued components [25].

In the space of currents there exist a map, Poincaré duality, associating to a \( k \)-dimensional surface \( \Sigma_k \) a \((d - k)\)-current, \( PD(\Sigma_k) \), according to
\[
\int_{\Sigma_k} a^{(k)} = \int_{\mathbb{R}^d} a^{(k)} \wedge PD(\Sigma_k), \tag{A.7}
\]
for any \( k \)-form \( a^{(k)} \) of compact support. The following property holds:
\[
PD(\partial \Sigma_k) = dPD(\Sigma_k) \tag{A.8}
\]
where \( \partial \) denotes the boundary operator.

A basic consequence of Poincaré’s duality is that, whenever well defined,
\[
\int_{\mathbb{R}^d} PD(\Sigma_k) \wedge PD(\Sigma_{d-k})
\]
is an integer counting the intersection with sign of \( \Sigma_k \) with \( \Sigma_{d-k} \).

Linear combinations of such \( k \)-currents \( PD(\Sigma_k) \) with integer coefficients are called integral \( k \)-currents. Poincaré’s lemma holds also for currents.

We now turn to the lattice.

Our lattice is \( \mathbb{Z}^d_{1/2} \), where the subscript \( 1/2 \) indicate that the coordinates of the sites are half-integer.

If \( W \) is an additive abelian group, one can define \( k \)-forms with values in \( W \) as maps, \( a^{(k)} \), from oriented \( k \)-cells, \( c_k \), of the lattice to \( W \) satisfying \( a^{(k)}(-c_k) = -a^{(k)}(c_k) \), where \( -c_k \) denotes the cell obtained from \( c_k \) reversing the orientation.

We denote by \( d \) the lattice exterior differential:
\[
da^{(k)}(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} a^{(k)}(c_k) \tag{A.9}
\]
and by * the Hodge star. Let \( c^*_{d-k} \) denote the cell in the dual lattice, \( \mathbb{Z}^4 \), dual to \( c_k \). Then
\[
*(a^{(k)})(c^*_{d-k}) = a^{(k)}(c_k). \tag{A.10}
\]
We also introduce the (lattice) codifferential \( \delta = (-)^{d(k+1)}d^* \) and Laplacian \( \Delta = d\delta + \delta d \).
If $W$ is a Hilbert space with inner product $(\ , \ )$ one can define a inner product among $W$–valued $k$–forms $a^{(k)}$ and $b^{(k)}$ by

$$ (a^{(k)}, b^{(k)}) = \sum_c c_k (a^{(k)}, b^{(k)}). \quad (A.11) $$

The $\ell_2$–norm corresponding to this scalar product is denoted by $|| \ ||$. Equation (A.6) and the Poincaré lemma hold also on the lattice.

If $W$ is a discrete group we call the $W$–valued $k$–forms also $k$–currents, in analogy with the continuum definition.

If $\Sigma_k$ is a $k$–dimensional surface in the lattice one defines its Poincaré dual as the $d - k$ current $PD(\Sigma_k)$ in the dual lattice such that

$$ PD(\Sigma_k)(c^*_k) = \begin{cases} 1 & \text{if } c_k \in \Sigma_k \\ 0 & \text{otherwise.} \end{cases} \quad (A.12) $$

In the paper we do not make use of the symbol $PD$, introduced in this appendix for sake of clarity, and often identify a $k$–surface with its Poincaré dual defined above.

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References
[1] G. 't Hooft, Nucl. Phys. B 153, 141 (1979);
G. Mack in “Recent Developments in Gauge Theories”, (Cargése 1979), 't Hooft ed. Plenum Press 1980;
A.M. Polyakov, Phys. Lett. 72B, 477 (1978)
H. Nielsen, P. Olesen, Nucl. Phys. B160, 360 (1979).
[2] L. Del Debbio, M. Faber, J. Greensite, S. Oleynik, Phys. Rev. D55, 2298 (1997).
[3] D. Durhuus, J. Fröhlich, Commun. Math. Phys., 75, 103 (1983).
[4] S. Mandelstam, Phys. Rep. 23C, 245 (1976).
[5] G. 't Hooft, Nucl. Phys. B190, 455 81981)
[6] M. Chernodub, M. Polikarpov, M. Zubkov, Nucl. Phys. Proc. Suppl. 34, 256 (1994);
[7] L. Del Debbio, A. Di Giacomo, G. Paffuti, P. Pieri, Phys. Lett. B355, 255 (1995); G. Di Cecio, A. Di Giacomo, G. Paffuti, M. Trigiante, Nucl. Phys. B489, 739 (1997); M. Chernodub, M. Polikarpov, A. Veselov, Phys. Lett. B 399, 267 (1997).
[8] P.A.M. Dirac, Can. J. Phys. 33, 650 (1955).
[9] A. Di Giacomo, B. Lucini, L. Montesi, G. Paffuti, Nucl. Phys. Proc. Suppl. 63, 540 (1998).
[10] J. Fröhlich, P.A. Marchetti, Nucl. Phys. B511, 770 (1999).
[11] G. Morchio, F. Strocchi, Nucl. Phys. B211, 471 (1983).
[12] K. Osterwalder, R. SCHRADER, Commun Math. Phys. 31(1973); 42, 281 (1975).
   E. Seiler “Gauge Theories as a Problem in Constructive Quantum Field Theory and Statistical Mechanics”. Lecture Notes in Physics 159, Springer Verlag 1982.
[13] J. Fröhlich, P.A. Marchetti, Europhys. Lett. 2, 933 (1986); in “Algebraic Theory of Superselection Sectors. Introduction and Recent results”. ed. D. Kastler, World Scientific 1990.
[14] F. Wegner, J. Math. Phys. 12, 2254 (1971);
   G. ’t Hooft, Nucl. Phys. B138, 1 (1978).
[15] J. Fröhlich, P.A. Marchetti, Commun. Math. Phys. 112, 343 (1987).
[16] T. Kennedy, C. King, Phys. Rev. Lett. 55, 776 (1985)
[17] S. Mandelstam, Ann. Phys. 19, 1(1962).
[18] S. Coleman in “The Unity of the Fundamental Interactions” (Erice 1981), A. Zichichi ed. Plenum 1983.
[19] A.S. Kronfeld, G. Schierholz, U.J. Wiese, Nucl. Phys. B292, 461 (1987).
[20] M. Chernodub, M. Polikarpov, A. Veselov, JETP 69, 174 (1999).
[21] J. Fröhlich, T. Spencer, Commun. Math. Phys. 83, 411 (1982).
[22] Chen Hong–Mo, J. Faridani, Tsou Sheung Tsun, Phys. Rev. D53, 7293 (1996).
[23] G. Mack, V. Petkova, Z. Phys. C 12, 177 (1982)
[24] M. Engelhardt, H. Reinhardt, Nucl. Phys. B567, 249 (2000);
   C. Alexandrou, M. D’Elia, Ph. de Forcrand, Nucl. Phys. Proc. Suppl. 83, 437 (2000).
[25] G. de Rham “Differential Manifolds. Forms, Currents, Harmonic Forms”, Springer Verlag 1984.