The Fuglede Conjecture holds in $\mathbb{F}_p^3$ for $p = 5, 7$

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Abstract

For $p = 5, 7$, we show that a subset $E \subset \mathbb{F}_p^3$ is spectral if and only if $E$ tiles $\mathbb{F}_p^3$ by translation. Additionally, we give an alternate proof that the conjecture holds for $p = 3$.

1 Introduction

Let $p$ be a prime number, $\mathbb{F}_p$ the finite field with $p$ elements, and $\mathbb{F}_p^d$ the $d$-dimensional vector space over $\mathbb{F}_p$.

Definition 1. A subset $E \subset \mathbb{F}_p^d$ is called spectral if there exists a subset $A \subset \mathbb{F}_p^d$ such that

$$\left\{ \chi_a(x) := e^{\frac{2\pi i}{p} x \cdot a} \mid a \in A \right\}$$

forms an orthogonal basis of the complex vector space $L^2(E)$. We then say $A$ is a spectrum of $E$ and that $(E, A)$ is a spectral pair.

Definition 2. A subset $E \subset \mathbb{F}_p^d$ is said to tile $\mathbb{F}_p^d$ by translations, or more briefly, simply called a tile, if there exists a subset $A \subset \mathbb{F}_p^d$ such that

$$\sum_{a \in A} E(x - a) = 1$$

for all $x \in \mathbb{F}_p^3$. Here, $E(\cdot)$ denotes the indicator function of $E$. We call $A$ a tiling set of $E$ and $(E, A)$ is called a tiling pair.

Interest in the connection between spectral sets and tiles arises from a conjecture of Fuglede in [Fug74], which states that if $\Omega \subset \mathbb{R}^d$ has positive, finite Lebesgue measure, $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if $\Omega$ tiles $\mathbb{R}^d$ by translations. In [Tao04], Tao disproved the Fuglede conjecture in $\mathbb{R}^d$ for $d \geq 5$ by lifting a non-tiling spectral set in $\mathbb{F}_5^3$ to the reals. Examples of non-spectral tiles were found by Kolountzakis and Matolcsi in $\mathbb{R}^5$, $d \geq 5$, [KM06b]. Fuglede’s conjecture was disproved in $\mathbb{R}^4$ and $\mathbb{R}^3$, too, see [Mat05, FMM06, KM06]. The conjecture remains open in $\mathbb{R}^2$ and $\mathbb{R}$. For partial results, see [IKT03, Lab01, Lab02].
In the discrete setting, the examples of Tao from the previous paragraph show that the conjecture is false in $\mathbb{F}_d^p$ for $d \geq 5$. In [AAB+17], it is shown that the conjecture is true when $d = 3$ and $p = 2, 3$, but false when $d = 4$. However, in [IMP17], Iosevich, Mayeli, and Pakianathan proved that the Fuglede conjecture holds in $\mathbb{F}_p^2$. Since the conjecture is trivially true in $\mathbb{F}_p^3$, the only remaining case is $\mathbb{F}_p^4$. The main theorem of this paper extends the results of [AAB+17].

**Theorem 1.** The Fuglede Conjecture holds in $\mathbb{F}_5^5$ and $\mathbb{F}_7^5$.

The methods involved in the proof of Theorem 1 also yield a new proof for the $p = 3$ case.

### 2 Some Background

In this section we record some previous results which will be important for our arguments.

Before continuing, it will be helpful to introduce some notation. A line will be any translate of a one-dimensional subspace of $\mathbb{F}_p^d$. A plane will be any translate of a two-dimensional subspace of $\mathbb{F}_p^3$.

For any subspace $S$, $S^\perp$ will denote the subspace $S^\perp := \{ x \mid x \cdot y = 0 \text{ for all } y \in S \}$.

A direction will mean any one-dimensional subspace and be denoted by $d$. The family of planes orthogonal to $d$ refers to the $p$ distinct translates of $d^\perp$. If $y \in d$ is nonzero, then we abuse notation and use $y^\perp$ and $d^\perp$ interchangeably.

If $E \subset \mathbb{F}_p^d$ and $e_1 - e_2 \in d$ for two distinct $e_1, e_2 \in E$, we say $E$ determines $d$.

Additionally, the Fourier Transform of a set $E \subset \mathbb{F}_p^d$ is given with the following normalization:

$$\hat{E}(\xi) := p^{-3} \sum_{x \in E} e^{-\frac{2\pi i}{p} x \cdot \xi}$$

The following theorem is due to Haessig, Iosevich, Pakianathan, Robins, and Vaicunas. It holds in all dimensions. The language has been changed in order to fit the terminology of the previous paragraph.

**Theorem 2.** [HIP+15] Suppose $E \subset \mathbb{F}_p^d$ and $\xi \in \mathbb{F}_p^d$, $\xi \neq 0$. Then $\hat{E}(\xi) = 0$ if and only if $E$ is equidistributed on the planes orthogonal to $\xi$, meaning $|E \cap \tau_x (\xi^\perp)|$ is constant as a function of $x$. Here $\tau_x (\xi^\perp)$ is translation of $\xi^\perp$ by $x$.

The next theorem is due Iosevich, Mayeli, and Pakianathan also holds in all dimensions. It is an important ingredient of their proof of the Fuglede Conjecture in $\mathbb{F}_p^d$.

**Theorem 3.** [IMP17] If $E \subset \mathbb{F}_p^d$, then $A$ is a spectrum of $E$ if and only if $|A| = |E|$ and $\hat{E}(a_1 - a_2) = 0$ for any two distinct $a_1, a_2 \in A$. If $E$ is spectral and $|E| > p^{d-1}$, then $E = A = \mathbb{F}_p^d$. If $E$ is spectral, then $|E| = 1$ or $|E| = mp$ for some positive integer $m$. 

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The consequences of Theorems 2 and 3 in three dimensions are elegantly summarized by the following result from AAB\textsuperscript{+}17. In fact, this equivalence is one of several in the paper, but will be enough for present purposes.

**Theorem 4.** AAB\textsuperscript{+}17 The following are equivalent

1. The Fuglede Conjecture holds in $\mathbb{F}_p^3$.
2. There exist no spectral sets $E$ satisfying $|E| = mp$, $2 \leq m \leq p - 1$.

So, in order to prove the conjecture in $\mathbb{F}_3^5$, it suffices to verify that no set of size 10, 15 or 20 is spectral. Similarly, for $\mathbb{F}_7^3$, it is enough to show that no set of size 14, 21, 28, 35 or 42 is spectral. This is exactly what we will do. Note that Theorem 4 immediately implies that the conjecture holds in $\mathbb{F}_3^2$.

Finally, we will need one more result from AAB\textsuperscript{+}17.

**Theorem 5.** AAB\textsuperscript{+}17 Suppose $E, A \subset \mathbb{F}_p^d$ and that $(E, A)$ is a spectral pair. Then $(A, E)$ is a spectral pair.

### 3 Direction Non-concentration Lemmas

Before moving on to specific values of $p$, we prove some lemmas that are independent of $p$.

The first lemma and subsequent corollaries say that any non-tiling spectral set can not determine too many directions of any plane. This means, in particular, that no such set can be too concentrated on any plane. The second lemma proves analogous staments for lines.

**Lemma 1.** Suppose $E \subset \mathbb{F}_p^3$, $|E| \leq p^2$, and that there is some plane $P_0$ such that $\hat{E}$ vanishes on $P_0 \setminus \{0\}$. Then $|E| = p^2$.

**Proof.** Since $\hat{E}$ has a zero, by Theorem 2 $|E| = mp$, $1 \leq m \leq p$.

Define $d_0 := P_0^\perp$. Let $\ell$ be an arbitrary translate of $d_0$. For $1 \leq i \leq p + 1$, let $Q_i$ be the distinct planes containing $\ell$. Then, since $\hat{E}$ vanishes on the nonzero points of $Q_i^\perp$ for every $i$, $|E \cap Q_i| = m$.

Notice that

$$\mathbb{F}_p^3 = \bigcup_{i=1}^{p+1} Q_i$$

and for $i \neq j$, $Q_i \cap Q_j = \ell_x$. Define $a := |E \cap \ell|$ and $b := |E \cap Q_i| - a$. Note that $b$ is independent of $i$. Then

$$a + b = m$$
$$a + (p + 1) b = mp.$$  

This implies $m \equiv 0 \mod p$, which forces $m = p$ and $|E| = p^2$. \qed
Corollary 1. Suppose \( E, A \subset \mathbb{F}_p^3 \) and that \((E, A)\) is a spectral pair with \( |E| = |A| = mp, 2 \leq m \leq p - 1 \). Then \( E \) and \( A \) each determine at most \( p \) directions of any plane.

Proof. If \( A \) determines all \( p + 1 \) directions of some plane, then \( \hat{E} \) vanishes on every point of that plane except the origin. By Lemma 1, \( |E| = p^2 \), which is a contradiction. By the symmetry of Theorem 5, \( E \) determining every direction of some plane also delivers a contradiction. \( \blacksquare \)

Corollary 2. Suppose \( E, A \subset \mathbb{F}_p^3 \) and that \((E, A)\) is a spectral pair with \( |E| = |A| = mp, 2 \leq m \leq p - 1 \). Then
\[
\sup_{P} |E \cap P| \leq p \quad \text{and} \quad \sup_{P} |A \cap P| \leq p.
\]
Here the supremum is taken over all translates of all two dimensional subspaces.

Proof. If there is some plane \( P_0 \) with \( |E \cap P_0| \geq p + 1 \), by pigeonholing, \( E \) determines every direction of some two dimensional subspace, which contradicts the previous corollary. By symmetry, the same is true for \( A \). \( \blacksquare \)

Lemma 2. Suppose \( E, A \subset \mathbb{F}_p^3 \) and that \((E, A)\) is a spectral pair with \( |E| = |A| = mp, 2 \leq m \leq p - 1 \). Then
\[
\sup_{\ell} |E \cap \ell| \leq \min\{m, p - m\} \quad \text{and} \quad \sup_{\ell} |A \cap \ell| \leq \min\{m, p - m\}.
\]

Proof. Fix some direction \( d \). Since \( |A| > p \), \( A \) determines some direction of \( d^\perp \). This means any translate of \( d \) is contained in a plane \( P \) with \( |E \cap P| = m \). Since \( d \) is arbitrary, \( \sup_{\ell} |E \cap \ell| \leq m \). By Theorem 5 the same holds for \( A \).

Let \( \ell \) be an arbitrary line, a translation of the arbitrary direction \( d \). Define \( \alpha := |E \cap \ell| \). For \( 1 \leq i \leq p + 1 \), let \( P_i \) be the distinct planes containing \( \ell_0 \) and define \( \beta_i := |E \cap P_i| \). Since \( A \) determines some direction of \( d^\perp \) there is some \( i \) with \( \beta_i = m \), and so, after a relabeling we may assume \( \beta_{p+1} = m \). Observe that \( \beta_i \leq p \) for all \( 1 \leq i \leq p \), by Corollary 2
\[
mp = \alpha + (m - \alpha) + \sum_{i=1}^{p} (\beta_i - \alpha) \leq m + p (p - \alpha).
\]

Rearranging, this gives \( p (m + \alpha - p) \leq m \). Since \( p > m \) and everything in sight is an integer, it follows that \( m + \alpha \leq p \).

Again, by Theorem 5 the same is true for \( A \). \( \blacksquare \)

Corollary 3. If \( E \subset \mathbb{F}_p^3 \) and \( |E| = p (p - 1) \), then \( E \) is not spectral.

Proof. By Lemma 2 any spectral \( E \) of size \( p (p - 1) \) would satisfy \( \sup_{\ell} |E \cap \ell| \leq 1 \), which implies \( E \) has at most one point, a contradiction. \( \blacksquare \)

Corollary 4. The Fuglede Conjecture holds in \( \mathbb{F}_3^3 \).

Proof. By Theorem 4 Corollary 2 and Lemma 2 it is enough to show that if \( E \subset \mathbb{F}_3^3 \) has size six with no three points colinear, then \( \sup_{P} |E \cap P| \geq 4 \). Note
that $\mathbb{F}_3^3$ contains $9 + 3 + 1 = 13$ directions, while $\binom{6}{2} = 15 > 13$. So for any such set, there are two distinct parallel lines $\ell_1$ and $\ell_2$ with

$$|E \cap \ell_1| = |E \cap \ell_2| = 2,$$

and $\sup_P |E \cap P| \geq 4$ follows.

In fact, these arguments can be extended to the case $|A| = |A| = p(p - 2)$.

**Lemma 3.** Suppose $E \subset \mathbb{F}_p^3$ and $|E| = p(p - 2)$. Then $E$ is not spectral.

**Proof.** As before, we will assume such a spectral set exists and derive a contradiction. If $E$ has a spectrum $A$, then $|A| = p(p - 2)$ as well. Define $k := \sup_P |A \cap P|$. There is some line $\ell$ with $|A \cap \ell| = 2$. It follows that

$$2 + (p + 1)(k - 2) \geq (p - 2)p,$$

which implies

$$k \geq p - 1 + \frac{1}{p + 1}.$$

Since $k$ is an integer, $k \geq p$. By Corollary 2 $k = p$. Fix some plane $P_0$ such that $|A \cap P_0| = p$. There are $\frac{p(p - 1)}{2}$ distinct pairs of points of $A \cap P_0$. By Lemma 1 they determine at most $p$ directions. On the other hand, since there are no triples of co-linear points, no direction is determined more than $\frac{p - 1}{2}$ times. So $A \cap P_0$ determines exactly $p$ directions.

Set $d = P_0 \perp$, where $P$ is the translate of $P_0$ that passes through the origin. By Theorem 2 $E$ is equidistributed on all but one family of parallel planes containing $d$. Give each plane in this remaining family a name: $Q_1, \ldots, Q_p$. Let $\ell'$ be a line parallel to $d$ with $|E \cap \ell'| = 0$ and define $\ell'$ such that $\ell' \subset Q$. Since $E$ is equidistributed on the $p$ other planes containing $\ell'$ except $Q$, $|E \cap Q| = 0$.

This means that $|E \cap Q_1| = p$ exactly $p - 2$ times and $|E \cap Q_1| = 0$ the remaining 2 times. And so there is some direction $d_0$ such that there are exactly $(p - 2)\frac{p - 1}{2}$ translations $\ell$ of $d_0$ that satisfy

$$|E \cap \ell| = 2.$$ 

Since

$$(p - 2)\frac{p - 1}{2} = p \left( \frac{p - 1}{2} \right) - p + 1 > p \left( \frac{p - 1}{2} \right) - p = p \left( \frac{p - 3}{2} \right),$$

$E$ cannot be equidistributed on any family of parallel planes containing $d_0$, which is a contradiction.
4 The case \( p = 5 \)

For the case \( p = 5 \), we must exclude the possibility that \(|E| = 10\). First, we show that such a spectral set would have to satisfy \( \sup_{P} |E \cap P| = 4 \), a slight but helpful improvement over \( \sup_{P} |E \cap P| = 5 \).

**Lemma 4.** Suppose \( E \subset \mathbb{F}_5^3 \) is spectral and \(|E| = 10\). Then \( \sup_{P} |E \cap P| = 4 \).

**Proof.** Let \( A \) be a spectrum of \( A \). By symmetry, it will suffice to show that \( \sup_{P} |E \cap P| = 5 \) yields a contradiction.

If there is an plane \( P \) with \(|A \cap P| = 5\), there exists a family of parallel planes \( Q_1, \ldots, Q_5 \) such that

\[
|E \cap Q_1| = |E \cap Q_2| = 5
\]

and

\[
|E \cap Q_3| = |E \cap Q_4| = |E \cap Q_5| = 0
\]

If \( Q \) is the translate of \( Q_1 \) that passes through the origin, there is a direction \( d_0 \subset Q \) such that \( E \) does not determine \( d_0 \). For all other \( d' \subset Q \), there are \( E \) determines \( d' \) exactly four times. And so, if \( d \) is a direction of \( Q \) different from \( d_0 \), \( E \) is equidistributed on at most one family of parallel planes containing \( d \).

So \( E \) is equidistributed on at most ten families of parallel planes. Therefore \( A \) determines at most ten directions.

By symmetry, there is a family of parallel planes \( R_1, \ldots, R_5 \) such that

\[
|A \cap R_1| = |A \cap R_2| = 5
\]

and

\[
|A \cap R_3| = |A \cap R_4| = |A \cap R_5| = 0
\]

As before, let \( R \) denote the translate of \( R_1 \) that passes through the origin. By previous argument \( A \) determines five directions of \( R \). Also, for any \( x \in A \cap R_1 \) the lines joining \( x \) to the points of \( A \cap R_2 \) all lie in different directions, none of which lie in \( R \). Then \( |D(E)| \leq 10 \) implies that for any distinct \( x_1, x_2 \in A \cap R_1 \), \( A \cap R_1 \) is invariant under translation by \( x_1 - x_2 \). This is impossible because \( \mathbb{F}_5 \) has no nontrivial subgroups. \( \square \)

Now we use Lemma 4 to derive the final contradiction. First note that \( \sup_{P} |A \cap P| \leq 4 \) implies \( \sup_{P} |A \cap P| = 4 \): if the supremum were three or less, then by talking any line that contains two points of \( A \), we see that \( A \) has at most six points. So fix some plane \( P_0 \) with \(|A \cap P_0| = 4\). Let \( P \) be the translation of \( P_0 \) that passes through the origin and set \( d_0 := P^\perp \).

Any configuration of four points on a plane, no three of which are colinear determines at least four directions. So \( E \) is equidistributed on at least four families of parallel planes containing \( d_0 \). Combined with the bound \( \sup_{P} |E \cap P| = 4 \), this implies that \( E \) does not determine \( d_0 \).

Additionally, if \( Q \) is a translate of any plane containing \( d_0 \) with \(|E \cap Q| = 0\), \( E \) is in fact equidistributed on five families of parallel planes containing \( d_0 \),
which forces that some plane contains five or more points of $E$, a contradiction. So every plane containing a translate of $d_0$ contains a point of $E$.

Projecting by $d_0$ then gives a subset $B \subset \mathbb{F}_5^2$ of size ten which intersects every line at least once and whose Fourier transform is supported on two lines through the origin. We will show no such subset exists.

Let $L_1, L_2, L_3, L_4, L_5$ and $K_1, K_2, K_3, K_4, K_5$ be the two families of parallel lines on which $B$ is not equidistributed. Let $L$ denote the line $L_i$ that passes through the origin and similarly for $K$. Pick this parameterization so that $L_i \cap K_i$ lies on the same line for all $i$. Without loss of generality $L_i \cap K_i \in E$ for $i = 1, 2$ and $L_i \cap K_i \notin E$ for $i = 3, 4, 5$. Since each line must intersect $E$, this forces, for $i = 1, 2, 3$,

\[ |L_i \cap E| = |K_i \cap E| = 1 \]

and, possibly after switching the labels 1 and 2,

\[ |L_1 \cap E| = |K_2 \cap E| = 3, \quad |L_2 \cap E| = |K_1 \cap E| = 4. \]

Plancherel then gives.

\[ 5 (16 + 9 + 1 + 1 + 1) = 5^4 \sum_{\xi \in L^\perp} |\hat{B}(\xi)|^2 = 5^4 \sum_{\xi \in K^\perp} |\hat{B}(\xi)|^2. \]

Since $\hat{B}$ is supported on $L^\perp \cup K^\perp$

\[ 56 = 2 (16 + 9 + 1 + 1 + 1) = 5^3 \left( \sum_{\xi \in L^\perp} |\hat{B}(\xi)|^2 + \sum_{\xi \in K^\perp} |\hat{B}(\xi)|^2 \right) \]

\[ = 5 \left( 2^5 \left( \sum_{\xi \in \mathbb{F}^2_5} |\hat{B}(\xi)|^2 \right) \right) + 5^3 \left( \frac{10}{25} \right)^2 = 70, \]

which is manifestly untrue.

5 The case $p = 7$

The strategy for $p = 7$ is similar in spirit to the $p = 5$ case, but more complicated. In this section, we rule out the possibility of spectral sets of size 14, 21, and 28. First we give two lemmas each of which brings down the maximum number of points on any plane. These can be though of as analogues of Lemma 4. Observe that two separate lemmas take the place of Lemma 4, which is one way to indicate that these methods become intractable as $p$ gets very large.

Lemma 5. Suppose $E \subset \mathbb{F}_7^2$ has five points, no three of which are collinear. Then $E$ determines at least six directions.
Proof. The collinearity restrictions mean that any direction is realized at most twice. Since $\binom{7}{2} = 10$, $E$ must determine five distinct directions. Suppose for the sake of contradiction the $E$ determines exactly 5 directions. Then each direction realized is realized exactly twice.

If some 4 point subset $E' \subset E$ realizes only 4 directions, that implies that the remaining direction is realized by two lines joined to the same point, which contradicts the collinearity assumption.

So, without loss of generality,

$$E = \{(0, 0), (0, 1), (1, 0), (1, a), (b, 1)\}$$

where $a \neq 1, 6$ and $b \neq 0, 1$. This means $(1, a)$ is a multiple of $(b - 1, 1)$ and $(b, 1)$ is a multiple of $(1, a - 1)$, which implies

$$b^2 - b - 1 = 0$$

which has no solution in $\mathbb{F}_7$.

Lemma 6. Suppose $E \subset \mathbb{F}_7^2$ has seven points, no four of which are collinear. Then $E$ determines at least six directions.

Proof. Suppose, for the sake of contradiction, that $E$ determines five or fewer directions. Since $\binom{7}{2} = 21$, some direction is determined at least 5 times. By Cauchy-Davenport, there can be no two parallel lines that each contain three points. And so, by the equality case of Cauchy-Davenport, without loss of generality

$$E = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (c, d), (c, d + 1)\}$$

where $c \neq 0, 1$ and $d \neq 0$. Matching directions gives

$$(1, -2), (1, -1), (1, 0), (1, 1) = \left\{ \left( 1, \frac{d - 2}{c} \right), \left( 1, \frac{d - 1}{c} \right), \left( 1, \frac{d}{c} \right), \left( 1, \frac{d + 1}{c} \right) \right\}.$$  

Which then gives

$$\Omega := \{-2c, -c, 0, c\} = \{d - 2, d - 1, d, d + 1\}.$$  

Since $0 \in \Omega$, either $1 \in \Omega$ or $-1 \in \Omega$ (or both). This means $c = 1, 3, 4$ or $-1$. Also, $\{3, 4\}$ is not a subset of $\Omega$. This means $c = -1$ and $d = 1$ which then implies $E$ determines six directions.

Next, we rule out the possibility that a spectral set determines six directions of any given plane. We follow the methods of the conclusion of the $p = 5$ argument. In particular, we reduce the problem to a two dimensional problem and then derive a contradiction using Plancherel.

Lemma 7. Suppose $A \subset \mathbb{F}_7^2$ and $|A| = 14$. If $A$ determines six directions of a plane, $A$ is not a spectrum.
Proof. Suppose, for the sake of contradiction, that \((E, A)\) is a spectral pair and \(A\) determines six directions of \(P = d_0\).

Like the \(p = 5\) case, since the equation \(2 + z + x_2 = 14\) implies \(z \geq 6\) or \(w \geq 6\), \(E\) does not determine \(d_0\). Projecting along \(d_0\) gives a set \(B \subseteq \mathbb{F}_7^2\) with \(|B| = 14\) such that \(B\) is supported on the union of two lines through the origin.

For \(1 \leq i \leq 7\) let \(\{L_i\}\) and \(\{K_i\}\) be the two families of parallel planes on which \(B\) may not be equidistributed. Pick this parameterization so that \(\bigcup_i (L_i \cap K_i)\) is a line. Without loss of generality, \(L_i \cap K_i \in B\) if and only if \(i = 1, 2\). Finally, let \(L\) be the translate of \(L_1\) that contains the origin, similarly for \(K\). Identify these two lines with their indicator functions. Define

\[
\bar{L}_i := |L_i \cap E| \quad \bar{K}_i := |K_i \cap E|
\]

This labeling system implies that for \(i = 1, 2\)

\[
(\bar{L}_i, \bar{K}_i) \in \{(7, 2), (6, 3), (5, 4), (4, 5), (3, 6), (2, 7), \}
\]

and for \(3 \leq i \leq 7\)

\[
(\bar{L}_1, \bar{K}_i) \in \{(2, 0), (1, 1), (0, 2)\}.
\]

Maximizing the the following sum of \(L^2\) norms

\[
\sum_{x \in \mathbb{F}_7^2} |B * L(x)|^2 + \sum_{x \in \mathbb{F}_7^2} |B * K(x)|^2
\]

is equivalent to maximizing

\[
\sum_{i=1}^{7} \bar{L}_i^2 + \bar{K}_i^2 = \sum_{i=1}^{7} (\bar{L}_i + \bar{K}_i)^2 - 2\bar{L}_i\bar{K}_i = 2 \cdot 9^2 + 5 \cdot 2^2 - \sum_{i=1}^{7} \bar{L}_i\bar{K}_i,
\]

which is the same as minimizing \(\sum_{i=1}^{7} \bar{L}_i\bar{K}_i\). This happens when, up to a relabeling,

\[
\{(\bar{L}_1, \bar{K}_1), \ldots, (\bar{L}_7, \bar{K}_7)\} = \{(7, 2), (2, 7), (2, 0), (0, 2), (2, 0), (0, 2), (1, 1)\}.
\]

Applying the same Plancherel calculation as the \(p = 5\) case gives

\[
2 \cdot 7 (7^2 + 3 \cdot 2^2 + 1) \geq 7^2 (14) + 14^2,
\]

which is false. \(\square\)

**Lemma 8.** Suppose \(A \subseteq \mathbb{F}_7^3\) and \(|A| = 21\). If \(A\) determines six directions of a plane, \(A\) is not a spectrum.
Proof. Let $P = d_0^\perp$ be such a plane and let $(E, A)$ a proposed spectral pair. By the same calculation, $E$ does not determine $d_0$. Project and get a new $B \subset \mathbb{F}_7^2$. Let $L, K, L_i, K_i, \bar{L}_i, \bar{K}_i$ be as in the last lemma. Then for $1 \leq i \leq 3$  

$$(\bar{L}_i, \bar{K}_i) \in \{(7,3), (6,4), (5,5), (4,6), (3,7)\},$$

and for $4 \leq i \leq 7$,  

$$(\bar{L}_i, \bar{K}_i) \in \{(3,0), (2,1), (1,2), (0,3)\}.$$ 

Then $\sum_{x \in \mathbb{F}_7^2} |B \ast L(x)|^2 + \sum_{x \in \mathbb{F}_7^2} |B \ast K(x)|^2$ is maximized when  

$$\{(\bar{L}_1, \bar{K}_1), \ldots, (\bar{L}_7, \bar{K}_7)\} = \{(7,3), (3,7), (3,7), (3,0), (3,0), (0,3), (2,1)\}$$  

which implies  

$$7(3 \cdot 7^2 + 6 \cdot 3^2 + 2^2 + 1) \geq 7^2(21) + 21^2.$$  

This is again, not true. 

Lemma 9. Suppose $A \subset \mathbb{F}_7^3$ and $|A| = 28$. If $A$ determines six directions of a plane, $A$ is not a spectrum.

Proof. The proof proceeds in exactly the same fashion. Let $(E, A)$ and $P = d_0^\perp$ be as before. For the same reason, $E$ does not determine $d_0$. With $B$ and $L, K, L_i, K_i, \bar{L}_i, \bar{K}_i$ also as before, we have, for $1 \leq i \leq 4$,  

$$(\bar{L}_i, \bar{K}_i) \in \{(7,4), (6,5), (5,6), (4,7)\},$$

and for $5 \leq i \leq 7$  

$$(\bar{L}_i, \bar{K}_i) \in \{(4,0), (3,1), (2,2), (1,3), (0,4)\}.$$ 

In this case, there are, up to relabeling, two ways to maximize the convolution equation:  

$$\{(\bar{L}_1, \bar{K}_1), \ldots, (\bar{L}_7, \bar{K}_7)\} = \{(7,4), (7,4), (4,7), (4,7), (4,0), (0,4), (2,2)\}$$  

and  

$$\{(\bar{L}_1, \bar{K}_1), \ldots, (\bar{L}_7, \bar{K}_7)\} = \{(7,4), (7,4), (5,6), (5,6), (4,0), (0,4), (0,4)\}.$$ 

In either case, the conclusion is  

$$2 \cdot 7(2 \cdot 7^2 + 3 \cdot 4^2 + 2^2) \geq 7^2(28) + 28^2.$$  

Untrue once again. 

With these in lemmas in place, two of the three cases follow quite smoothly. The third is more technical.

Corollary 5. If $E \subset \mathbb{F}_7^3$ has size 28, $E$ is not spectral.
Proof. If \( \sup_{\ell} |E \cap \ell| = 3 \), \( \sup_P |E \cap P| = 7 \) which means \( E \) determines six directions of a plane, which can not be. So \( \sup_{\ell} |E \cap \ell| = 2 \), which implies \( \sup_P |E \cap P| = 6 \), which again means that \( E \) determines six directions of a plane. So \( \sup_{\ell} |E \cap \ell| = 1 \), which is a contradiction. \( \square \)

Corollary 6. If \( E \subset \mathbb{F}_7^3 \) has size 21, \( E \) is not spectral.

Proof. If \( |E \cap \ell| = 3 \) for some line \( \ell \) parallel to \( d \), then \( \sup_P |E \cap P| \leq 6 \) implies \( E \) is equidistributed on at most two families of parallel planes containing \( d \). So if \( A \) is a spectrum of \( E \), then \( A \) determines at most two directions of \( d^\perp \). This means there are two parallel lines \( \ell_1 \) and \( \ell_2 \) such that \( |A \cap \ell_1| = |A \cap \ell_2| = 3 \).

By Cauchy-Davenport, \( A \) determines at least six directions of a plane, which can not be. So \( \sup_{\ell} |E \cap \ell| = 2 \). This means there’s a plane \( P \) with \( |E \cap P| \geq 5 \), which means \( E \) determines six directions of a plane, which again, can not be. So \( \sup_{\ell} |E \cap \ell| = 1 \), another contradiction. \( \square \)

Corollary 7. If \( E \subset \mathbb{F}_7^3 \) has size 14, \( E \) is not spectral.

Proof. We have \( \sup_{\ell} |E \cap \ell| = 2 \) and \( \sup_P |E \cap P| \leq 4 \). Suppose \( E \) determines \( d_0 \). Then \( E \) is equidistributed on at most two families of parallel planes containing \( d_0 \), and so if \( A \) is a spectrum of \( E \), then \( A \) determines at most two directions of \( d_0^\perp \). Call them \( d_1 \) and \( d_2 \).

Up to relabeling the directions there are 4 cases.

1. \( d_1 \) is determined 7 times and \( d_2 \) is determined zero times
2. \( d_1 \) is determined 6 times and \( d_2 \) is determined 1 times
3. \( d_1 \) is determined 5 times and \( d_2 \) is determined 2 times
4. \( d_1 \) is determined 4 times and \( d_2 \) is determined 3 times

The claim is that for all directions \( d \subset d_1^\perp \) except \( d_0 \), \( A \) is not equidistributed on \( d_1^\perp \).

In the first case, projecting by \( d_1 \) gives a 7 element subset of \( \mathbb{F}_7^2 \) with no three points collinear. Such a subset must determine 7 directions, which proves the claim.

In the second case, projecting by \( d_1 \) gives an 8 element subset of \( \mathbb{F}_7^2 \), which will determine all eight elements of the plane. At most one of these directions comes from the two points that determine \( d_2 \), which only eliminates one direction, giving seven directions again.

In the third case, partition \( A \) into two disjoint subsets. Let \( x \in \bar{V} \) if \( x \in A \) and there is a \( y \in A \) different from \( x \) such that \( x - y \in d_1 \). Define \( \bar{W} := A \setminus \bar{V} \). Let \( V \) and \( W \) be the images of \( \bar{V} \) and \( \bar{W} \) respectively after projecting by \( d_1 \). So \( |V| = 5 \) and \( |V| = 4 \). There is some point \( p \) of \( V \) such that there are 3 lines in 3 distinct directions joining \( p \) to \( W \). None of these lines are parallel to the 4 distinct lines joining \( p \) to the other points of \( A \). This proves the claim.
The fourth case comes last. Define $V$ and $W$ as in the third case. Now $|V| = 4$ and $|W| = 6$. If we can prove that some line $\ell_0$ in the plane satisfies $|V \cap \ell_0| = |W \cap \ell_0| = 1$, then

$$(V \cap \ell_0) - (V \cup W)$$

contains at least seven distinct directions, which proves the claim.

If there is a line $\ell'$ that satisfies $|W \cap \ell'| = 3$, then there such an $\ell_0$. Indeed there are 4 lines joining any point of $|W \cap \ell'|$ to the points of $V$. These lines can not intersect the two other points of $W \cap \ell'$. Since there are only three points of $W$ not lying in $\ell'$, one of these lines intersects $W$ only at one point of $\ell'$.

Now the goal is to show that there is some plane $P_0$ containing $d_1$ such that $|A \cap P_0| = 3$. Suppose this were false. Pick some point $q \in W$. If no planes containing $q$ and the direction $d_1$ have three points, then for $1 \leq i \leq 8$, there exist $z_i \in \{0, 1, 3\}$ with

$$\sum z_i = 13.$$

There are only two ways this can happen (up to relabeling). In the first, $z_i = 1$ only once, which means that $E$ is equidistributed on only one family of parallel planes containing $\ell_1$, which is what we ultimately want to show. In the second, $z_i = 3$ only 3 times, which can’t be, since $d_1$ is determined four times.

By spectrality, $E$ determines only one direction of $d_1$: $d_0$ itself. So $d_0$ is realized seven times. Since $d_0$ was arbitrary, any direction determined by $E$ is determined seven times. This forces $\sup_P |E \cap P| = 2$, which is impossible.

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