Mixed incremental $H_{\infty}$ and incremental passivity analysis for Markov switched stochastic nonlinear systems

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ABSTRACT The current study addresses the mixed incremental $H_{\infty}$ and incremental passivity analysis for Markov switched stochastic (MSS) nonlinear systems. The multiple Lyapunov functions approach and the structure of Markov framework are utilized to establish some sufficient conditions for the MSS nonlinear systems, which will be used for the incrementally globally asymptotically stable in the mean (IGASiM) and performance index analysis. Then, the mixed incremental $H_{\infty}$ and incremental passivity performance issues are solved for two instances: in the first case, all subsystems are not IGASiM, while in the second one, both of IGASiM and unstable subsystems exist. Hence, it is shown that when none of the subsystems is IGASiM, the MSS nonlinear systems are IGASiM and possess the mixed incremental $H_{\infty}$ and incremental passivity performance metric in the presence of specified conditions. The mathematical induction is selected to guarantee the robust incremental stability of MSS systems with IGASiM and non-IGASiM subsystems and the performance index can be exhibited a prescribed decay rate. The effectiveness of the proposed results is demonstrated by two simulation examples.

INDEX TERMS Markov switched stochastic systems; Incrementally globally asymptotically stable; Nonlinear performance.

I. INTRODUCTION

Stochastic hybrid systems include a group of dynamic systems consisting of continuous-time systems combined with discrete-time parts influencing by the measurement noise and discrete random events. These systems can be described by a variety of models, including stochastic switched systems [1], Markov jump systems [2]–[4], impulsive stochastic systems [5], [6]. MSS systems include a category of different active subsystems under actions governing a continuous-time Markov system, which can take values in defined state space. Based on the Markov switching principle, a subsystem activation along the system trajectory can be realized at defined samples. All subsystems will be endowed with continuous stochastic dynamics describing with a control-dependent stochastic differential equation. The Markov switching phenomenon may be due to switching events among various subsystems in dynamical systems caused by sudden changes in parameters or environmental features. Besides, by Markov process structural changes, the MSS systems are usually utilized to model some industrial hybrid systems with multiple or failure modes, including Hamiltonian systems, multi-agent systems, manufacturing systems, and communication systems [7]–[11].

Incremental stability [12] deals with the mutual convergence of trajectories, rather than a certain equilibrium point or a definite path. More recently, incremental stability has been presented for various types of stochastic nonlinear systems, like stochastic switched systems [13], stochastic control systems [14], randomly switched stochastic systems [15], and their representation based on several notions of incremental multiple Lyapunov functions [16]. By resorting to the incremental multiple Lyapunov function method, the IGASiM for switched stochastic nonlinear systems [17] are investigated, and the incremental stability criteria was also presented for the feedback coupled switched stochastic nonlinear systems. Moreover, the contraction metric concept has been utilized to verify the incremental stability of stochastic nonlinear systems [18], [19].
In many practical control systems, there exist unstable subsystems, leading to various problems, including controller fault, sensor fault, and external random disturbance. Therefore, switched systems should be considered in both stable and unstable subsystems [20]-[23], where the active time for stable ones must be higher than a threshold to compensate unstable ones. Stabilization of such unstable systems may be very complicated and costly. Accordingly, the switching signal and the control protocols should be reconsidered to ensure the stability of the overall switched system. As an intuitive fact, if the signal rarely switches at stable subsystems, while often switches at unstable ones, the stability for the overall switched system can be preserved. It has been demonstrated that this idea could be reasonable using the dwell-time principle [20], [21], [24]. The use of incremental $H_\infty$ performance issue is introduced in [24], where a performance metric has been obtained for unstable systems by resorting to a switching rule depending on states and the dwell time technique. It is popular that the stationary distribution of Markovian chains is unique that the stationary distribution of Markovian chains is increasing and right-continuous while $\pi_i$ is decreasing to zero for any fixed $r \geq 0$.

### II. PRELIMINARIES AND PROBLEM STATEMENT

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0 = 0})$ and $\Omega$ denote complete probability and sample spaces, respectively, $\mathcal{F}$ is a $\sigma$-field, $\{\mathcal{F}_t\}_{t \geq t_0 = 0}$ indicates a filtration satisfying the usual conditions (i.e., it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets), which $\mathcal{P}$ being a probability measure. $\mathbb{E}[\cdot]$ denotes the expectation operator with respect to probability measure $\mathcal{P}$. Let $\sigma(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\Gamma = \{1, 2, \ldots, M\}$ with generator $Q = (q_{ij})_{M \times M}$ given by

$$
\mathbb{P}(\sigma(t + h) = j|\sigma(t) = i) = \begin{cases} q_{ij}h + o(h), & i \neq j, \\ 1 + q_{ii}h + o(h), & i = j, 
\end{cases}
$$

where $h > 0$ and $\frac{o(h)}{h} \to 0$ as $h \to 0$. Here, if $i \neq j$, then $q_{ij} \geq 0$ and $q_{ii} = -\sum_{i \neq j} q_{ij}$. The Markov chain $\sigma(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_M)$. Let $\{t_k, k \in \mathbb{N}\}$ be the switching moments. Let $N_i(t_0, t)$ be the occurrence number for the $i$-th subsystem over the interval $[t_0, t]$ and $S_i$ be the sojourn time at any state $i \in \Gamma$ with an exponential distribution, i.e., $\mathbb{P}(S_i \leq x) = 1 - e^{-q_{ii}x}, x > 0, q_{ii} = q_{ii}, i \in \Gamma$.

Assume a stochastic nonlinear system with Markovian switching described by the following equations:

$$
\begin{align*}
\dot{x}(t) &= [F_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))\omega_{\sigma(t)}(t)]dt \\
&+ D_{\sigma(t)}(x(t))d\mathcal{W}(t) \\
(y(t) &= H_{\sigma(t)}(x(t)), x(t_0) \overset{q.s.}{=} x_0, t \geq t_0 = 0, \\
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the system state vector; $\omega_{\sigma(t)}(t) \in \mathbb{R}^m$ which belongs to $L_2[0, \infty)$, is either a disturbance input or a reference signal; $y(t) \in \mathbb{R}^m$ indicates the controlled output; $\mathcal{W}(t)$ indicates an $r$-dimensional Brownian motion satisfying $\mathbb{E}[d\mathcal{W}(t)] = 0$ and $\mathbb{E}[d\mathcal{W}^2(t)] = dt$. Moreover, let $\mathcal{W}(t)$ be independent of $\sigma(t)$ in this paper.

In what follows, we impose some additional Lipschitz and linear growth conditions to ensure the existence and uniqueness of solutions. Then, we assume that these functions satisfy the following:
1) Lipschitz: \( \forall x, \tilde{x} \in \mathbb{R}^n, \exists C > 0 \)
\[
\| F_{\sigma(t)}(x(t)) - F_{\sigma(t)}(\tilde{x}(t)) \| + \| G_{\sigma(t)}(x(t)) - G_{\sigma(t)}(\tilde{x}(t)) \| \\
+ \| D_{\sigma(t)}(x(t)) - D_{\sigma(t)}(\tilde{x}(t)) \|_F \leq C \| x(t) - \tilde{x}(t) \|.
\]
2) Linear growth: \( \forall x \in \mathbb{R}^n, \exists K > 0 \)
\[
\| F_{\sigma(t)}(x(t)) \|^2 + \| G_{\sigma(t)}(x(t)) \|^2 + \| D_{\sigma(t)}(x(t)) \|^2_R \\
\leq K (1 + \| x(t) \|^2).
\]

where \( \| \cdot \|_F \) is the Frobenius norm.

In addition, suppose that the external disturbance \( \omega_i(t) \) is bounded. Besides, \( \Delta \omega_i(t) = \omega_i(t) - \hat{\omega}_i(t) \), \( \Delta x(t) = x(t) - \hat{x}(t) \).

The following assumption for system (1) is adopted, which first appears in [2].

**Assumption 2.1:** The time series \( \{ t_{n+1} - t_n, n \in \mathbb{N} \} \) consists of a set of independent random variables and independent of \( \{ \sigma(t_n), n \in \mathbb{N} \} \).

The following key technical assumption will be proposed to handle mixed incremental performance index problem for system (1).

**Assumption 2.2:** Consider that there exists some \( \lambda \in \mathbb{R}^+ \) such that for each stochastic process \( \sigma \), the probability of sojourn (remaining in a mode) within an infinitesimal time interval \( h \) is lower-bounded by the following, for any \( i \in \Gamma \):
\[
\mathbb{P}(\sigma(t + h) = i|\sigma(t) = i) \geq 1 - \lambda h.
\]

In order to simultaneously analyze performance problem and IGASI stability theory of stochastic nonlinear systems, incremental stability theory is widely used, which has its roots in [25] for stochastic systems.

**Definition 2.1:** [25] A system of the form (1) is said to be IGASI, if there exists \( \beta(\cdot, \cdot) \in KL \) such that for all \( t \geq t_0 \), \( x(t_0) = x_0, \hat{x}(t_0) = \hat{x}_0 \) and \( u = \hat{u} \in \mathbb{R}^n \),
\[
\mathbb{E}[\| x(t, x_0, u) - \hat{x}(t, \hat{x}_0, u) \|] \leq \beta(\mathbb{E}[\| x_0 - \hat{x}_0 \|], t - t_0).
\]

The following technical lemma will be given, it enables us to estimate an upper bound of switching number in probability. The estimation is based on the transition rate \( q_{ij} \).

**Lemma 2.1:** [2] Let \( N_\sigma(t_0, t) \) be the switching number of \( \sigma(t) \) on the interval \( (t_0, t] \), now, the following relation can be written
\[
\mathbb{P}(N_\sigma(t_0, t] = k) \leq e^{-\hat{q}(k/t^k)k!}, k \geq 0,
\]
where \( \hat{q} = \max \{ q_{ij} : i, j \in \Gamma \} \), \( \hat{q} = \max \{ |q_{ij}| : i, j \in \Gamma \} \).

Now, for the system (1), let \( \gamma > 0 \), \( \theta \in [0, 1] \) and \( \sigma(t) \in \Gamma \), the mixed incremental \( H_\infty \) and incremental passivity performance issue can be addressed in this paper are formulated as follows:

(i) In the case that \( \Delta \omega_i \equiv 0 \), the system (1) is IGASI.

(ii) If \( x_0 \neq \hat{x}_0 \), the \( \Delta \omega_i \) and \( \Delta y \) satisfy the following inequality
\[
\mathbb{E} \left\{ \int_0^\infty | \Delta y^T(s) \Delta y(s) \theta - 2(1 - \theta) \Delta y^T(s) \Delta \omega_i(s) | ds \right\} \\
\leq \gamma^2 \int_0^\infty \Delta \omega_i^T(s) \Delta \omega_i(s) ds + \mathbb{E} [ \beta(\| x_0 - \hat{x}_0 \|) ],
\]
for some real valued function \( \beta(\cdot) \) and \( \Delta \omega_i \in L_2[0, \infty) \).

**III. MIXED INCREMENTAL \( H_\infty \) AND INCREMENTAL PASSIVITY PERFORMANCE ANALYSIS**

This section aims at discussing two types of MSS nonlinear systems: (a) Each subsystem is non-IGASI, (b) Simultaneous existence of both IGASI and non-IGASI subsystems. Just for the sake of notation, let
\[
\Theta = \frac{\partial V_i}{\partial x} F_i(x) + \frac{\partial V_i}{\partial u} f_i(x) + \gamma^T G_i(x) \frac{\partial V_i}{\partial x} \frac{\partial x}{\partial x} G_i(x) + \frac{1}{2} T \left( D_i(x) - D_i(\hat{x}) \right) \left( \frac{\partial x}{\partial x} V_i - \frac{\partial \hat{x}}{\partial x} V_i \right) \left( \frac{\partial x}{\partial x} V_i - \frac{\partial \hat{x}}{\partial x} V_i \right).
\]

For brevity, let \( V_{\sigma(t)}(t) = V_{\sigma(t)}(x(t), \hat{x}(t)) \).

Assume that
\[
\alpha_1(\| x - \hat{x} \|) \leq V_{\sigma(t)}(t) \leq \alpha_2(\| x - \hat{x} \|), \forall x, \hat{x} \in \mathbb{R}^n,
\]
where the convex function \( \alpha_1 \in KL \), the concave function \( \alpha_2(\cdot) \in KL \).

**A. CASE (A): EACH SUBSYSTEM IS NON-IGASI**

**Theorem 3.1:** For the given system (1) and constants \( 0 < \mu < 1, \lambda_i > 0, \gamma > 0, 0 \leq \theta \leq 1 \), if there exist \( V_i(x, \hat{x}) \in C^2, i, j \in \Gamma \), and such that (5)
\[
H_\infty(V_i(x, \hat{x})) = \Theta + \theta \Delta y^T \Delta y - 2(1 - \theta) \Delta y^T \Delta \omega_i \\
- \lambda_i V_i(x, \hat{x}) \leq 0,
\]
\[
\frac{\partial V_i}{\partial x} G_i(x) + \frac{\partial V_i}{\partial x} G_i(\hat{x}) = 0,
\]
\[
V_i(x, \hat{x}) \leq \mu V_j(x, \hat{x}), \mu \neq j; \tilde{\lambda} = \min_{i \in \Gamma} \lambda_i, \bar{\lambda} = \max_{i \in \Gamma} \lambda_i,
\]
\[
\mu \tilde{\lambda} - \tilde{\lambda} < 0,
\]
hold, where \( (x(t), \hat{x}(t)) \in \mathbb{R}^n x \mathbb{R}^n \Delta \), \( \tilde{q} = \max \{ q_{ij} : i, j \in \Gamma \} \), \( \tilde{q} = \max \{ |q_{ij}| : i, j \in \Gamma \} \). Then, the system (1) is IGASI for \( \Delta \omega_i \equiv 0 \) with performance index \( \gamma \).

**Proof:** From (6) and (7), one obtains
\[
\mathcal{L} V_i(x, \hat{x}) \leq \Theta + \gamma^2 \Delta \omega_i^T(t) \Delta \omega_i(t) \\
\leq \gamma^2 \Delta \omega_i^T(t) \Delta \omega_i(t) - \theta \Delta y^T(t) \Delta y(t) \\
+ 2(1 - \theta) \Delta y^T(t) \Delta \omega_i(t) + \lambda_i V_i(x, \hat{x}) \\
= \lambda_i V_i(x, \hat{x}) - \phi(t),
\]
where \( \phi(t) = \theta \Delta y^T(t) \Delta y(t) - 2(1 - \theta) \Delta y^T(t) \Delta \omega_i(t) - \gamma^2 \Delta \omega_i^T(t) \Delta \omega_i(t) \).

When \( \Delta \omega_i(t) \equiv 0 \) and (10), one gets
\[
\mathcal{L} V_i(x, \hat{x}) \leq \lambda_i V_i(x, \hat{x}).
\]
Next, we aim at showing that for any $k > 0$

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=k)]$$

$$\leq \mathbb{E}\left[\mu^k e^{-\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k)\right],$$

(12)

where $T_{t_0,t} = \sum_{i=k}^{t_0}(t_0-1)I(\sigma(t_0,t)\in \Gamma_i)$.

The above result can be obtained by the mathematical inductive method, the switching number $N_\sigma(t)(t_0,t) = k$ is equivalent to the $(t_0,t) = i_k \in \Gamma_i$, $\forall t \in [t_0, t_{k+1})$.

(I) For $t \in [t_0, t_1)$ and $N_\sigma(t)(t_0,t) = k = 0$, by utilizing Itô’s formula, multiplied by $\lambda k$, one obtains

$$V_{i_0}(x(t), \dot{x}(t)) = e^{\lambda t_0}v_{i_0}(t-t_0)V_{i_0}(t_0)$$

$$+ \int_t^{t_0} e^{\lambda s(t-s)}\left(LV_{i_0}(x(s), \dot{x}(s) - \lambda x_0V_{i_0}(x(s), \dot{x}(s)))dsight)$$

$$+ \int_t^{t_0} e^{\lambda s(t-s)}\left(\frac{\partial V_{i_0}}{\partial x}D_0(x) + \frac{\partial V_{i_0}}{\partial \dot{x}}D_0(\dot{x})\right)d\sigma(s).$$

(13)

Multiplying $I(N_\sigma(t)(t_0,t)=0)$ and taking expectation about (13), $\dot{\lambda} = \max_{i \in \Gamma} \lambda_i$, one obtains

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=0)]$$

$$\leq \mathbb{E}\left[e^{\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=0)\right].$$

(14)

(II) Now, suppose that for the case of $k-1$ the implication (12) holds, where $t \geq t_0$. Obviously,

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=k-1)]$$

$$\leq \mathbb{E}\left[\mu^{k-1}e^{\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k-1)\right].$$

(15)

(III) Finally, the result of (12) is formulated for the case of $k$. Recall that $N_\sigma(t)(t_0,t) = k$ for all $t \in [t_k, t_{k+1})$, $\sigma(t_k) = i_k \in \Gamma$, one has

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=k)]$$

$$\leq \mathbb{E}\left[e^{\lambda(t-t_k)}V_{\sigma(t_k)}(t_k)I(\sigma(t_0,t)=k)\right].$$

(16)

Noting that $(x(t_k), \dot{x}(t_k)) \in \mathcal{F}_k$ for any $k \geq 0$, applying the property of conditional expectation, we obtain

$$\mathbb{E}\left[e^{\lambda(t-t_k)}V_{\sigma(t_k)}(t_k)I(\sigma(t_0,t)=k)\right]$$

$$= \mathbb{E}\left[e^{\lambda(t-t_k)}V_{\sigma(t_k)}(t_k)I(\sigma(t_0,t)=k)\mid \mathcal{F}_k\right]$$

$$= \mathbb{E}\left[V_{\sigma(t_k)}(t_k)\mathbb{E}\left[e^{\lambda(t-t_k)}I(\sigma(t_0,t)=k)\mid \mathcal{F}_k\right]\right].$$

(17)

By Assumption 2.1, $t - t_k$ is independent of $\mathcal{F}_k$, the Eq.(17) can be rewritten as follow

$$\mathbb{E}\left[V_{\sigma(t_k)}(t_k)\mathbb{E}\left[e^{\lambda(t-t_k)}I(\sigma(t_0,t)=k)\mid \mathcal{F}_k\right]\right]$$

$$= \mathbb{E}\left[V_{\sigma(t_k)}(t_k)\mathbb{E}\left[e^{\lambda(t-t_k)}I(\sigma(t_0,t)=k)\right]\right]$$

$$= \mathbb{E}\left[V_{\sigma(t_k)}(t_k)\mathbb{E}\left[e^{\lambda(t-t_k)}I(\sigma(t_0,t)=k)\right]\right].$$

(18)

Using (8) and (15), we can deduce that

$$\mathbb{E}[V_\sigma(t_k)(t)] \leq \mathbb{E}\left[\mu^k e^{\lambda(t-t_k)}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k)\right]$$

$$\leq \mu^k \mathbb{E}\left[e^{\lambda(t-t_k)}V_{\sigma(t_0)}(t_0)\right].$$

(19)

From (16), (18) and (19), it is not difficult to calculate that

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=k)]$$

$$\leq \mu^k V_{\sigma(t_0)}(t_0)e^{\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k)$$

$$= \mu^k V_{\sigma(t_0)}(t_0)e^{\lambda t}T_{t_0,t}I(N_\sigma(t_0,t)=k).$$

(20)

Now, for each $t \geq t_0$, by (9), (20) and Lemma 2.1, one deduces that

$$\mathbb{E}[V_\sigma(t)(t)] \geq \sum_{k=0}^{\infty} \mathbb{E}[V_\sigma(t)(t)I(t_k \leq t < t_{k+1})]$$

$$\leq \sum_{k=0}^{\infty} \mu^k e^{\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k)$$

$$= \sum_{k=0}^{\infty} \mu^k e^{\lambda t}T_{t_0,t}I(N_\sigma(t_0,t)=k)$$

$$= \sum_{k=0}^{\infty} \mu^k e^{\lambda t}T_{t_0,t}I(N_\sigma(t_0,t)=k)$$

$$\leq \mu^k e^{\lambda t_{k+1}}V_{\sigma(t_0)}(t_0),$$

(21)

where $c = e^{-\lambda t_0}$.

Considering the Jensen’s inequality [26] and (5, 21), one obtains

$$\alpha_1(\mathbb{E}[\|x(t) - \dot{x}(t)\|]) \leq c^{\mu q - \dot{\lambda}}\alpha_2(\|x_0 - \dot{x}_0\|)$$

$$\leq c^{\mu q - \dot{\lambda}}\alpha_2(\|x_0 - \dot{x}_0\|).$$

(22)

From (9), it follows that

$$\mathbb{E}[\|x(t) - \dot{x}(t)\|] \leq \alpha_1^{-1}(c^{\mu q - \dot{\lambda}}\alpha_2(\|x_0 - \dot{x}_0\|)).$$

Now, the system (1) is IGASIM for $\Delta \omega_{\nu} = 0$.

Applying the expectation to (10), one leads to

$$\mathbb{E}\left[\frac{dV_\sigma(t)}{dt}\right] + \mathbb{E}[\phi(t)] \leq \lambda_\sigma(t)\mathbb{E}[V_\sigma(t)(t)].$$

(23)

Next, we aim at showing that for any $k > 0$

$$\mathbb{E}[V_\sigma(t)(t)I(N_\sigma(t)(t_0,t)=k)]$$

$$\leq \mathbb{E}\left[\mu^k e^{\lambda t}T_{t_0,t}V_{\sigma(t_0)}(t_0)I(\sigma(t_0,t)=k)\right]$$

$$- \int_{t_0}^{t} \mathbb{E}[e^{\lambda s}I(\sigma(t_0,s)=k)] ds.$$

(24)

The above result can be obtained by the mathematical inductive method, the switching number $N_\sigma(t)(t_0,t) = k$ is equivalent to the $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, in the sense that $\sigma(t) = i_k \in \Gamma$. 

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(I) For \( t \in [t_0, t_1) \) and \( N_{\sigma(t)}(t_0, t) = k = 0 \), by integrating from both sides of (23), we can get
\[
\begin{align*}
E \left[ V_{\sigma(t)}(t) I_{(N_{\sigma(t)}(t_0, t)=0)} \right] \\
& \leq E \left[ e^{\lambda T(t_0,t)} V_{\sigma(t_0)}(t_0) I_{(N_{\sigma(t)}(t_0, t)=0)} \right] \\
& - \int_{t_0}^{t} E \left[ e^{\lambda T[s,t]} \phi(s) I_{(N_{\sigma(t)}(s,t)=0)} \right] ds. 
\end{align*}
\] (25)

(II) Now, suppose that for the case of \( k - 1 \) the implication (24) holds, where \( t \geq t_0 \). Obviously,
\[
\begin{align*}
E \left[ V_{\sigma(t)}(t) I_{(N_{\sigma(t)}(t_0, t)=k-1)} \right] \\
& \leq E \left[ e^{k-1 \lambda T(t_0,t)} V_{\sigma(t_0)}(t_0) I_{(N_{\sigma(t)}(t_0, t)=k-1)} \right] \\
& - \int_{t_0}^{t} E \left[ e^{k-1 \lambda T[s,t]} \phi(s) I_{(N_{\sigma(t)}(s,t)=k-1)} \right] ds. 
\end{align*}
\] (26)

(III) Finally, the result of (24) is formulated for the case of \( k \). Recall that \( N_{\sigma(t)}(t_0, t) = k \) for all \( t \in [t_k, t_{k+1}) \), \( \sigma(t_k) = i_k \in \Gamma \), we have
\[
\begin{align*}
E \left[ V_{\sigma(t)}(t) I_{(N_{\sigma(t)}(t_0, t)=k)} \right] \\
& \leq E \left[ e^{k \lambda T(t_0,t)} V_{\sigma(t_0)}(t_0) I_{(N_{\sigma(t)}(t_0, t)=k)} \right] \\
& - \int_{t_k}^{t} E \left[ e^{k \lambda T[s,t]} \phi(s) I_{(N_{\sigma(t)}(s,t)=k)} \right] ds. 
\end{align*}
\] (27)

For each \( t \in [t_k, t_{k+1}) \), \( x(t_k) = x(t_{k}^{-}) \), \( V_{\sigma(t_k)}(t_k) \leq \mu V_{\sigma(t_k^{-})}(t_{k}^{-}) \). From (18) and (26), the following relation could be obtained
\[
\begin{align*}
E \left[ e^{k \lambda T(t_k,t)} V_{\sigma(t_k)}(t_k) I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& \leq E \left[ \mu V_{\sigma(t_{k}^{-})}(t_{k}^{-}) \right] E \left[ e^{k \lambda T(t_k,t)} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& \leq E \left[ \mu e^{k \lambda T(t_k,t)} V_{\sigma(t_0)}(t_0) \right] E \left[ e^{k \lambda T(t_k,t)} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& - \int_{t_0}^{t_k} E \left[ e^{k \lambda T[s,t_k]} \mu \phi(s) \right] ds E \left[ e^{k \lambda T(t_k,t)} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right], 
\end{align*}
\] (28)

which implies that
\[
\begin{align*}
E \left[ e^{k \lambda T(t_k,t)} V_{\sigma(t_k)}(t_k) I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& \leq \mu^k V_{\sigma(t_0)}(t_0) E \left[ e^{k \lambda T(t_0,t)} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& - \int_{t_0}^{t_k} \mu^k E \left[ e^{k \lambda T[s,t]} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \phi(s) \right] ds. 
\end{align*}
\] (29)

From (27) and (29), it follows that
\[
\begin{align*}
E \left[ V_{\sigma(t)}(t) I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& \leq \mu^k V_{\sigma(t_0)}(t_0) E \left[ e^{k \lambda T(t_0,t)} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \right] \\
& - \int_{t_0}^{t} \mu^k E \left[ e^{k \lambda T[s,t]} I_{(N_{\sigma(t_0, t_{k+1})}=k)} \phi(s) \right] ds. 
\end{align*}
\] (30)

By (30), let \( t_0 = 0 \), one has
\[
\begin{align*}
\sum_{k=0}^{\infty} \int_{0}^{t} e^{\lambda(t-s)} \mu^k \mathbb{P}(N_{\sigma}(s, t) = k) E \left[ \psi(s) \right] ds \\
& \leq \sum_{k=0}^{\infty} \mu^k E \left[ V_{\sigma(t_0)}(t_0) \right] \\
& + \gamma^2 \sum_{k=0}^{\infty} \int_{0}^{t} \mu^k \mathbb{P}(N_{\sigma}(s, t) = k) e^{\lambda(t-s)} \Delta \omega_t^\sigma(s) \Delta \omega_i(s) ds.
\end{align*}
\] (31)

where \( \psi(s) = \theta \Delta y_t^\sigma(s) \Delta y_i(s) - 2(1 - \theta) \Delta y_t^\sigma(s) \Delta \omega_i(s) \).

From Lemma 2.1, (31) can be expressed as
\[
\begin{align*}
\sum_{k=0}^{\infty} \int_{0}^{t} e^{(\mu q - \tilde{q} + \tilde{\lambda})(t-s)} \mathbb{E} \left[ \psi(s) \right] ds \\
& \leq e^{(\mu q - \tilde{q} + \tilde{\lambda})t} \mathbb{E} \left[ V_{\sigma(t_0)}(t_0) \right] \\
& + \gamma^2 \sum_{k=0}^{\infty} \int_{0}^{t} e^{(\mu q - \tilde{q} + \tilde{\lambda})(t-s)} \Delta \omega_t^\sigma(s) \Delta \omega_i(s) ds.
\end{align*}
\] (32)

From (9), integrating from the inequality (32) from \( t = 0 \) to \( \infty \) gives
\[
\begin{align*}
\int_{0}^{\infty} \mathbb{E} \left[ (\theta \Delta y_t^\sigma(s) \Delta y_i(s) - 2(1 - \theta) \Delta y_t^\sigma(s) \Delta \omega_i(s)) \right] ds \\
& \leq \frac{-1}{\mu q - \tilde{q} + \tilde{\lambda}} \int_{0}^{\infty} \mathbb{E} \left[ V_{\sigma(t_0)}(t_0) \right] \\
& + \gamma^2 \int_{0}^{\infty} \Delta \omega_t^\sigma(s) \Delta \omega_i(s) ds,
\end{align*}
\] (33)

which is equivalent to
\[
\begin{align*}
\int_{0}^{\infty} \mathbb{E} \left[ \theta \Delta y_t^\sigma(s) \Delta y_i(s) - 2(1 - \theta) \Delta y_t^\sigma(s) \Delta \omega_i(s) \right] ds \\
& \leq \frac{-1}{\mu q - \tilde{q} + \tilde{\lambda}} \mathbb{E} \left[ V_{\sigma(t_0)}(t_0) \right] + \gamma^2 \int_{0}^{\infty} \Delta \omega_t^\sigma(s) \Delta \omega_i(s) ds.
\end{align*}
\] (34)

From (5), one can get
\[
\begin{align*}
\mathbb{E} \left\{ \int_{0}^{\infty} \left[ \theta \Delta y_t^\sigma(s) \Delta y_i(s) - 2(1 - \theta) \Delta y_t^\sigma(s) \Delta \omega_i(s) \right] ds \right\} \\
& \leq \gamma^2 \int_{0}^{\infty} \Delta \omega_t^\sigma(s) \Delta \omega_i(s) ds + \mathbb{E} \left[ \beta(||x_0 - \tilde{x}_0||) \right],
\end{align*}
\] (35)

where \( \mathbb{E} \left[ \beta(||x_0 - \tilde{x}_0||) \right] = \frac{\mu q - \tilde{q} + \tilde{\lambda}}{\mu t - \tilde{q} + \tilde{\lambda}} \mathbb{E} [\sigma_0(||x_0 - \tilde{x}_0||)]. \)

Now, the system (1) has the performance index \( \gamma \). \( \square \)

Remark 3.1: Condition (6) is a set of coupled incremental Hamilton-Jacobi inequalities. It can be concluded from the condition \( \lambda_i > 0 \) that the IGASiM condition is not necessarily true for the continuous dynamics of all subsystems. In fact, the corresponding proofs will not be affected if the condition (6) is replaced by \( HT(V_\gamma(x, \tilde{x})) = 0 \). Condition (9) is introduced mainly to make sure that IGASiM of the whole system can be obtained by the framework of Markov switching process. According to inequality (9), the large sojourn time of non-IGASiM subsystems can be compensated for by

Yuanhong Ren et al.: Mixed incremental \( H_\infty \) and incremental passivity analysis
a lower probability of the switching process, which activates the corresponding subsystem. Theorem 3.1 shows that if each subsystem is non-IGASIM, the IGASIM of all systems can be realized through the Markov switching process.

**Remark 3.2:** Note that the switching time $t_k$ is random, so that the interval $[t_k, t_{k+1})$ is a random time interval. The method of iterating the time was used to estimate the $\mathbb{E}\mathcal{V}(t)$, which is not correct. Hence, we adopt the mathematical induction to estimate the $\mathbb{E}\mathcal{V}(t)$ and the performance index.

**B. CASE(B): THE SIMULTANEOUS EXISTENCE OF BOTH IGASIM AND NON-IGASIM SUBSYSTEMS**

Consider that all subsystems are not necessarily IGASIM, let $\Gamma_s$ describe an appropriate nonempty subset of $\Gamma$, which its complement with respect to $\Gamma$ is indicated by $\Gamma_u$, where $\Gamma_s = \{i \in \Gamma : \lambda_i \geq 0\}$ and $\Gamma_u = \{i \in \Gamma : \lambda_i < 0\}$. Let $T^s_i [t_0, t]$ and $T^u_i [t_0, t]$ represent the overall activation times of non-IGASIM subsystems and IGASIM subsystems in $[t_0, t)$, respectively. Let $\lambda_u = \max_{i \in \Gamma_u} \{-\lambda_i | \lambda_i < 0\}$, $\lambda_s = \min_{i \in \Gamma_s} \{\lambda_i | \lambda_i \geq 0\}$.

**Theorem 3.2:** Consider system (1) and $\gamma > 0$, $\mu > 1$, $\lambda_i \in \mathbb{R}$, $0 \leq \theta \leq 1$, if there exist $V_i(x, \hat{x}) \in C^2$, $\gamma, \lambda_i$ in $[t_k, t_{k+1})$, similar to Theorem 3.1, one has

$$\mathbb{E}\left[ V_i(t_{k+1}) - V_i(t_k) \right] \leq \lambda_i V_i(x, \hat{x}) - \mu \gamma T^s_i [t_0, t],$$

(36)

where

$$\mu \gamma T^s_i [t_0, t] = \sum_{k=0}^{\infty} \mathbb{E}\left[ V_i(t_{k+1}) - V_i(t_k) \right].$$

(42)

From (37) and (42), $\forall t \in [t_k, t_{k+1})$, by the mathematical induction, one can get

$$\mathbb{E}\left[ V_i(t_{k+1}) - V_i(t_k) \right] \leq \lambda_i V_i(x, \hat{x}) - \mu \gamma T^s_i [t_0, t].$$

(44)

For every $t \in [t_k, t_{k+1})$, $k \geq 0$, $T^s_i [t_0, t] + T^u_i [t_0, t] = t - t_0$, one obtains

$$\mathbb{E}\left[ e^{\lambda_s T^s_i [t_0, t]} - \lambda_s T^s_i [t_0, t] V_i(t_0) \right].$$

(43)

For any $t \geq t_0$, one has

$$\mathbb{E}\left[ V_i(t) \right] = \sum_{k=0}^{\infty} \mathbb{E}\left[ V_i(t_{k+1}) - V_i(t_k) \right].$$

(43)

where $T^u_i [t_0, t] = \sum_{k=0}^{\infty} (t_{k+1} - t_k) I(\sigma(t_i) \in \Gamma_u).$

(44)

For every $t \in [t_k, t_{k+1})$, $t \geq t_0$, $T^s_i [t_0, t] = t - t_0$, one obtains

$$\mathbb{E}\left[ e^{\lambda_s T^s_i [t_0, t]} - \lambda_s T^s_i [t_0, t] \right] = e^{\lambda_s (t-t_0)} \mathbb{E}\left[ e^{\lambda_s T^s_i [t_0, t]} \right].$$

(44)

By the ergodicity of $\sigma(t)$ [27], one obtains

$$\lim_{k \to \infty} \mathbb{P}(\sigma(t_k) \in \Gamma_u) = p, \quad a.s.$$
From (45) and (47), one leads to
\[
\Pi_{t=0}^K \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \mathbb{P}(\sigma(t_i) \in \Gamma_u) + 1 \right] \\
= \Pi_{t=0}^k \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \mathbb{P}(\sigma(t_i) \in \Gamma_u) + 1 \right] I_{(k \leq K)} \\
+ \Pi_{t=k+1}^K \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \mathbb{P}(\sigma(t_i) \in \Gamma_u) + 1 \right] I_{(k > K)} \\
\leq \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} + 1 \right]^k I_{(k \leq K)} \\
+ \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} (p + \varepsilon) + 1 \right]^{k-K} I_{(k > K)}. 
\]

By Lemma 2.1, (44),(45) and (48), for every \( t \geq t_0 = 0 \), one has
\[
\mathbb{E}[\mathbb{V}(t)] \leq e^{-\lambda_u(t-t_0)} \mathbb{E}[\mathbb{V}(t_0)] \mathbb{P}(N_0(0, t) = k) \\
\leq \sum_{k=0}^\infty \mu^k \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} + 1 \right]^k \\
+ \sum_{k=K+1}^\infty \mu^k \left[ \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} + 1 \right]^k \\
\leq e^{-\lambda_u(t-t_0)} \mathbb{E}[\mathbb{V}(t_0)] e^{-\lambda_s t} \left( \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \right) \frac{1}{k!}.
\]

Thus, by letting \( \varepsilon \to 0^+ \), from (5), (38) and (49), one obtains
\[
\alpha_1(\mathbb{E}[|x(t) - \hat{x}(t)|]) \leq e^{\mu_s (\frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} p + 1)} \left( \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \right)^t, \\
\alpha_2(\mathbb{E}[|x_0 - \hat{x}_0|]) \leq e^{\mu_s (\frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} p + 1)} \left( \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \right)^t. 
\]

From (38), we obtain that
\[
\mathbb{E}[|x(t) - \hat{x}(t)|] \leq \alpha_1^{-1}(e^{\mu_s (\frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} p + 1)} \left( \frac{\lambda_u + \lambda_s}{q - (\lambda_u + \lambda_s)} \right)^t), \\
\alpha_2(\mathbb{E}[|x_0 - \hat{x}_0|]).
\]

Then, the system (1) is IGASIM for \( \Delta \omega_i(0) \equiv 0 \).

Next, we show that the system (1) has a mixed incremental \( H_\infty \) and incremental passivity performance index \( \gamma_0 \).

It can be easily seen from (40) that for \( \forall t \in [t_k, t_{k+1}) \), one can get
\[
\mathbb{E}[\mathbb{V}(t)] \leq \sum_{k=0}^\infty \mathbb{E}[\mathbb{V}(t_0)] \mathbb{P}(N_0(0, t) = k) \\
\leq \sum_{k=0}^\infty \mu^k \mathbb{E}[e^{\lambda_u T_k^u(0,t) - \lambda_s T_k^s(0,t)} \mathbb{V}(t_0)] I_{(N_0(0, t) = k)} \\
- \int_0^t \mathbb{E}[e^{\lambda_u T_k^u(s,t) - \lambda_s T_k^s(s,t)} \mu^k \phi(s)] I_{(N_0(s, t) = k)} ds. 
\]

From (37), by the mathematical induction, we can obtain
\[
\mathbb{E}[\mathbb{V}(t)] I_{(N_0(t_0, t) = k)} \\
\leq \mu^k \mathbb{E}[e^{\lambda_u T_k^u(0,t) - \lambda_s T_k^s(0,t)} \mathbb{V}(t_0)] I_{(N_0(t_0, t) = k)} \\
- \int_0^t \mathbb{E}[e^{\lambda_u T_k^u(s,t) - \lambda_s T_k^s(s,t)} \mu^k \phi(s)] I_{(N_0(s, t) = k)} ds. 
\]

Then, for any \( t \geq t_0 \), one has
\[
\mathbb{E}[\mathbb{V}(t)] \leq \sum_{k=0}^\infty \mathbb{E}[\mathbb{V}(t_0)] I_{(N_0(t_0, t) = k)} \\
\leq \sum_{k=0}^\infty \mu^k \mathbb{E}[e^{\lambda_u T_k^u(0,t) - \lambda_s T_k^s(0,t)} \mathbb{V}(t_0)] I_{(N_0(t_0, t) = k)} \\
- \int_0^t \sum_{k=0}^\infty \mu^k \mathbb{E}[e^{\lambda_u T_k^u(s,t) - \lambda_s T_k^s(s,t)} \mu^k \phi(s)] I_{(N_0(s, t) = k)} ds. 
\]
On the other hand, we show an inequality as follows
\[
\mathbb{E}\left[ e^{\lambda_u T_u^s(s,t)} - \lambda_s T_s^u(s,t) \right] := e^{-\lambda_s (t-s)} \mathbb{E}\left[ e^{(\lambda_u + \lambda_s) T_u^s(s,t)} \right] \\
\geq e^{-\lambda_s (t-s)} \mathbb{E}\left[ e^{(\lambda_u + \lambda_s) T_u^s(t)} \right] \\
= e^{-\lambda_s (t-s)}. \tag{56}
\]

By Lemma 2.1, from (49) and (54-56), we have that for every \( t \geq t_0 \),
\[
\int_0^t \sum_{k=0}^{\infty} \mu^k \mathbb{P}(N_\sigma(s,t) = k)e^{-\lambda_s (t-s)} \mathbb{E}\left[ \psi(s) \right] ds \\
\leq e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \mathbb{E}\left[ V_{\sigma(t_0)}(t_0) \right] \\
+ \gamma^2 \int_0^t e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \rangle \\
\Delta \hat{w}_T^s(s) \Delta \hat{w}_i(s) ds, \tag{57}
\]
which implies that
\[
\int_0^t \mathbb{P}(N_\sigma(s,t) = 0)e^{-\lambda_s (t-s)} \mathbb{E}\left[ \psi(s) \right] ds \\
\leq e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \mathbb{E}\left[ V_{\sigma(t_0)}(t_0) \right] \\
+ \gamma^2 \int_0^t e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \rangle \\
\Delta \hat{w}_T^s(s) \Delta \hat{w}_i(s) ds. \tag{58}
\]
Note that Assumption 2.2 implies \( \mathbb{P}(\sigma(s + h) = i|\sigma(s) = i) \geq 1 - \lambda h \) for \( s + h = t, \lambda \in \mathbb{R}^+ \), it follows that
\[
\mathbb{P}(N_\sigma(s,t) = 0) = \mathbb{P}(N_\sigma(0,t) - N_\sigma(0,s) = 0) \\
\geq 1 - \lambda h, t = s + h. \tag{59}
\]
Hence,
\[
\int_0^t \left[ 1 - \lambda(t-s) \right] e^{-\lambda_s (t-s)} \mathbb{E}\left[ \psi(s) \right] ds \\
\leq e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \mathbb{E}\left[ V_{\sigma(t_0)}(t_0) \right] \\
+ \gamma^2 \int_0^t e^{\left[ \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \right] (t-s)} \rangle \\
\Delta \hat{w}_T^s(s) \Delta \hat{w}_i(s) ds. \tag{60}
\]
If both sides of (60) can be integrated from \( t = 0 \) to \( c \), from (38), we have
\[
\frac{\lambda_u - \lambda}{\lambda^2} \int_0^c \mathbb{E}\left[ \psi(s) \right] ds \leq \mathbb{E}\left[ V_{\sigma(t_0)}(t_0) \right] \\
- 1 \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \\
+ \gamma^2 \mu \bar{q}_\sigma \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} \right) \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \\
\int_0^\infty \Delta \hat{w}_T^s(s) \Delta \hat{w}_i(s) ds, \tag{61}
\]
where \( \lambda_u > \lambda \), and thus,
\[
\mathbb{E}\left\{ \int_0^\infty [\theta \Delta y^T(s) \Delta y(s) - 2(1 - \theta) \Delta y^T(s) \Delta \hat{w}_i(s)] ds \right\} \\
\leq \gamma_0^2 \int_0^\infty \Delta \hat{w}_T^s(s) \Delta \hat{w}_i(s) ds + \mathbb{E}[\beta(\|x_0 - \hat{x}_0\|)], \tag{62}
\]
where \( \gamma_0^2 = \left( \frac{\lambda_u - \lambda}{\lambda_u + \lambda_s + \lambda_s^2} \right) + 1 \left( \frac{\lambda_u + \lambda_s}{\lambda_u + \lambda_s + \lambda_s^2} (p+e) + 1 \right) - \tilde{q}_u - \lambda_s \\
\mathbb{E}[\delta_2(\|x_0 - \hat{x}_0\|)]. \]

Now, the system (1) has the performance index \( \gamma_0 \). \( \square \)

Remark 3.3: Condition (38) represents the quantitative index of the unstable subsystem characterized by \( p, \lambda_u, \tilde{q}_u, \tilde{q} \). The performance index in Theorem 3.2 is more extensive than [24], and when \( \theta = 1 \), it can be degenerated into the incremental \( H_\infty \) performance index. Due to the sequence \( \{t_{n+1} - t_n, n \in \mathbb{N}\} \) is a collection of independent random variables, it is observed that the proof of the IGASIM results is more complicated than the corresponding proof in [24]. It is not too difficult to handle performance index problem by applying lower-bounded on the probability of sojourn time.

IV. EXAMPLE

In the current section, two examples of the MSS nonlinear systems are presented to evaluate the established results. Assume that a one-dimensional Brownian motion is described by \( \sigma(t) \). Consider a right-continuous Markov chain \( \sigma(t) \) with values in \( \Gamma = \{1, 2\} \) with generator.

**Example 4.1.** Assume that the generator is described by
\[
Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.
\]

Assume a MSS system represented by (1) with the following two subsystems
\[
F_1(x) = \begin{pmatrix} -3x_1^3 - x_2 \\ x_1 - \frac{3}{4} x_2 \end{pmatrix}, G_1(x) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \\
D_1(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, y_1 = \begin{pmatrix} \sqrt{2} x_2 \\ \frac{\sqrt{2}}{8} x_1 \end{pmatrix}. \tag{63}
\]
\[
F_2(x) = \begin{pmatrix} \frac{1}{4} x_1 + \frac{1}{4} x_2 \\ \frac{1}{4} x_1 + \frac{1}{4} x_2 \end{pmatrix}, G_2(x) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \\
D_2(x) = \begin{pmatrix} -\frac{\sqrt{2}}{4} x_1 \\ \frac{\sqrt{2}}{4} x_2 \end{pmatrix}, y_2 = \begin{pmatrix} \frac{1}{4} x_1 \\ \frac{1}{4} x_2 \end{pmatrix}. \tag{64}
\]

The storage function of each subsystem is assumed to be
\[
V_1(x, \dot{x}) = \frac{1}{4} (x - \dot{x})^2, V_2(x, \dot{x}) = 2 (x - \dot{x})^2.
\]
It can be simply verified that each subsystem is non-IGASIM for \( \Delta \hat{w}_i = 0, i = 1, 2 \). Let \( \theta = 1, \gamma = \sqrt{2}, \lambda_1 = \frac{9}{2}, \lambda_2 = \frac{2}{2}, \mu = \frac{1}{2}, V_1 \leq \frac{1}{2} V_2 \), one obtains
\[
VOLUME 4, 2016
\]
$$\Theta|_{V_1} + \theta \Delta y_1^T \Delta y_1 - \lambda_1 V_1(x, \hat{x})$$

$$= -\frac{3}{160} (x_1 - \hat{x}_1)^2 - \frac{29}{80} (x_2 - \hat{x}_2)^2$$

$$- \frac{1}{2} (x_1 - \hat{x}_1)^2 (x_1^2 + x_1 \hat{x}_1 + \hat{x}_1^2) \leq 0,$$

(65)

$$\Theta|_{V_2} + \theta \Delta y_2^T \Delta y_2 - \lambda_2 V_2(x, \hat{x})$$

$$= -\frac{19}{80} (x_1 - \hat{x}_1)^2 - \frac{199}{480} (x_2 - \hat{x}_2)^2 \leq 0,$$

(66)

where $x_1 \hat{x}_1 \geq 0$.

Let $\mu = \frac{1}{4}, \bar{q} = 2, \bar{\lambda} = \frac{6}{5}, \bar{q} = 2$, it can be checked that $\mu \bar{q} + \bar{\lambda} - \bar{q} = -0.3 < 0$.

Consider that the initial states are given by $x_1(0) = 1, x_1(0) = -4, x_2(0) = 3, \hat{x}_2(0) = -3$. Now, based on Theorem 3.1, the subsystems (63) and (64) are non-IGASiM (See Fig.1). According to Fig.2, the MSS nonlinear system is IGASiM. The Markov switching signal is described by Fig.2.

**Example 4.2.** Assume that the generator can be shown by

$$Q = (q_{ij}) = \begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix}.$$  

Assume a MSS system represented by (1) with the following two subsystems

$$F_1(x) = \begin{pmatrix} -3x_1^3 + \frac{1}{4}x_2 - 4x_1 \\ -5x_2^3 + \frac{1}{4}x_1 - 4x_2 \end{pmatrix}, G_1(x) = \begin{pmatrix} \frac{1}{2}x_2 \\ \frac{1}{2} \end{pmatrix},$$

$$D_1(x) = \begin{pmatrix} \frac{1}{4}x_2 \\ \frac{1}{4} \end{pmatrix}, y_1 = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}. \quad (67)$$

$$F_2(x) = \begin{pmatrix} -\frac{1}{10}x_1 + \frac{1}{10}x_2 \\ -\frac{1}{10}x_1 - \frac{1}{10}x_2 - \frac{1}{10}x_2 \end{pmatrix}, G_2(x) = \begin{pmatrix} \frac{1}{4} \end{pmatrix},$$

$$D_2(x) = \begin{pmatrix} \frac{1}{4} \end{pmatrix}, y_2 = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}. \quad (68)$$

The storage function of each subsystem is chosen as $V_1(x, \hat{x}) = (x - \hat{x})^2, V_2(x, \hat{x}) = 2(x - \hat{x})^2$. Based on Theorem 3.2, in the case that $\Delta \omega_i = 0, i = 1, 2$, the subsystem (67) is IGASiM and the subsystem (68) is non-IGASiM (See Fig.3). Let $\theta = 1, \gamma = \sqrt{2}, \lambda_1 = \frac{1}{10}, \lambda_2 = -\frac{7}{10}$, then we have $\lambda_s = \frac{1}{10}, \lambda_u = \frac{7}{10}$. From (36), one obtains

$$\Theta|_{V_1} + \theta \Delta y_1^T \Delta y_1 + \lambda_1 V_1(x, \hat{x})$$

$$= -\frac{123}{20} (x_1 - \hat{x}_1)^2 - 6(x_1 - \hat{x}_1)^2(x_1^2 + x_1 \hat{x}_1 + \hat{x}_1^2)$$

$$- \frac{123}{20} (x_2 - \hat{x}_2)^2 - 10(x_2 - \hat{x}_2)^2(x_2^2 + x_2 \hat{x}_2 + \hat{x}_2^2) \leq 0,$$

(69)

$$\Theta|_{V_2} + \theta \Delta y_2^T \Delta y_2 + \lambda_2 V_2(x, \hat{x})$$

$$= -\frac{171}{240} (x_1 - \hat{x}_1)^2 - \frac{1}{4} (x_2 - \hat{x}_2)^2(x_2^2 + x_2 \hat{x}_2 + \hat{x}_2^2)$$

$$- \frac{211}{240} (x_2 - \hat{x}_2)^2 \leq 0,$$

(70)

where $x_1 \hat{x}_1 \geq 0, x_2 \hat{x}_2 \geq 0$. 

**FIGURE 1.** The state trajectories of subsystems (63) and (64)

**FIGURE 2.** IGASiM of the system trajectories (63-64) with $\Delta \omega_i = 0$ and Markov switching signal
In the case that $\Delta \omega_i = 0$, $i = 1, 2$, to verify whether the system is IGASiM, considering there specific parameter $\mu = 2$, $q_s = 1$, $\tilde{q} = 3$, $q_u = 3$. Then, the Markov chain has a unique stationary distribution $\pi = (0.75, 0.25)$, $p = 0.25$. It can be checked that

$$\mu \hat{q}_s \left( \frac{\lambda_u + \lambda_s}{\hat{q} - (\lambda_u + \lambda_s)} + 1 \right) \left( \frac{\lambda_u + \lambda_s}{\hat{q} - (\lambda_u + \lambda_s)} p + 1 \right) - \lambda_s - \hat{q}_u = -0.1248 < 0$$

Assume the initial states as $(x_1(0), x_2(0)) = (2, -5)$, $(\dot{x}_1(0), \dot{x}_2(0)) = (-3, 6)$. This is also to be expected, the whole system has a faster convergence response and less time. As shown in Fig.4, the MSS system is IGASiM.

V. CONCLUSION

In the current research, the incremental Lyapunov functions and the stochastic analysis approaches are employed to solve the mixed incremental $H_\infty$ and incremental passivity issue for MSS nonlinear systems. Based on the stationary distribution of Markovian switching procedure, some sufficient conditions represented by inequalities are established and the incremental $H_\infty$ and incremental passivity performance problem to be solvable can be ensured. It is also suggested that via addressing mixed incremental $H_\infty$ and incremental passivity performance problem of MSS nonlinear systems that the mixed performance index problem developed in this work is more powerful than the general incremental $H_\infty$ control problem in those existing work. The presented theory is validated using a couple of examples and numerical simulations.

ACKNOWLEDGMENT

Conflict of Interest: The authors declare that they have no conflict of interest.

Availability of data and material: The authors declare that the manuscript has no associated data.

REFERENCES

[1] P. Zhao, W. Feng, Y. Kang, Stochastic input-to-state stability of switched stochastic nonlinear systems,” Automatica vol. 48, no. 10, pp. 2569-2576, 2012.
[2] B. Wang, Q.X. Zhu, “Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems,” Syst. Control Lett. vol.105 , pp. 55-61, 2017.
[3] Y.Z. Zhu, Z.X. Zhong, W. X. Zheng, D.H. Zhou, “HMM-based $H_\infty$ filtering for discrete-time markov jump LPV systems over unreliable communication channels,” IEEE Trans. Systems, Man, Cyber: Syst. vol. 48, no. 12, pp. 2035-2046, 2018.
[4] X.T. Wu, Y. Tang, J.D. Cao, X.R. Mao, “Stability Analysis for Continuous-Time Switched Systems with Stochastic Switching Signals,” IEEE Trans. Autom. Control vol. 63, no. 9, pp. 3083-3090, 2018.
[5] W. Ren, L.J. Xiong, “Stability Analysis of Impulsive Stochastic Nonlinear Systems,” IEEE Trans. Autom. Control vol. 62, no. 9, pp. 4797-4797, 2017.
[6] C. Sowmiya, R. Raja, J.D. Cao, G. Rajchakit, “Impulsive discrete-time bam neural networks with random parameter uncertainties and time-varying leakage delays: an asymptotic stability analysis,” Nonlinear Dyn. vol.91, no. 2, pp. 1-22, 2018.
[7] P. Jagtap, M. Zamani, “Backstepping design for incremental stability of stochastic Hamiltonian systems with Jumps,” IEEE Trans. Autom. Control vol. 63, no. 1, pp. 255-261, 2018.
[8] X.T. Wu, Y. Tang, W.B. Zhang, “Stability Analysis of Stochastic Delayed Systems With an Application to Multi-Agent Systems,” IEEE Trans. Autom. Control vol.61, no. 12, pp. 4143-4149, 2016.
[9] X. Yang, J. Lu, “Finite-time synchronization of coupled networks with Markovian topology and impulsive effects,” IEEE Trans. Autom. Control vol. 61, no. 8, pp. 2256-2261, 2016.
Mixed incremental $H_{\infty}$ and incremental passivity analysis

[10] Y. Tang, H. Gao, J. Lu, J. Kurths, “Pinning distributed synchronization of stochastic dynamical networks: a mixed optimization approach,” IEEE Trans. Neural Netw. Learn. Syst. vol. 25, no. 10, pp. 1804-1815, 2014.

[11] D. Antunes, J. Hespanha, C. Silvestre, “Stability of networked control systems with asynchronous renewal links: An impulsive systems approach,” Automatica vol. 49, no. 2, pp. 402-413, 2013.

[12] D. Angeli, “A Lyapunov approach to incremental stability properties,” IEEE Trans. Autom. Control vol. 47, no. 3, pp. 410-421, 2002.

[13] M. Zamani, A. Abate, A. Girard, “Symbolic models for stochastic switched systems: A discretization and a discretization-free approach,” Automatica vol.55, pp. 183-196, 2015.

[14] M. Zamani, P. M. Esfahani, R. Majumdar, A. Abate, J. Lygeros, “Symbolic control of stochastic systems via approximately bisimilar finite abstractions,” IEEE Trans. Autom. Control vol. 59, no. 12, pp. 3135-3150, 2014.

[15] M. Zamani, N. van de Wouw, R. Majumdar, “Backstepping controller synthesis and characterizations of incremental stability,” Syst. Control Lett. vol. 62, no.10, pp. 949-962, 2013.

[16] M. Zamani, P. Tabuada, “Backstepping design for incremental stability,” IEEE Trans. Autom. Control vol. 56, no.9, pp. 2184-2189, 2011.

[17] Y.H. Ren, W.Q. Wang, Y.X. Wang, W.S. Zhou, “Exponentially incremental dissipativity for nonlinear stochastic switched systems,” Int. Jour. Cont. vol. 93, no. 5, pp. 1074-1087, 2020.

[18] B.G. Zhang, L. Chen, K. Aiha, “Incremental stability analysis of stochastic hybrid systems,” Nonlinear Anal. Real World Appl. vol. 14, no. 2, pp. 1225-1234, 2013.

[19] Q.C. Pham, N. Tabadeau, J.J. Slotine, “A contraction theory approach to stochastic incremental stability,” IEEE Trans. Autom. Control vol. 54, no. 4, pp. 816-820, 2009.

[20] X. Xing, Y. Liu, B. Niu, “$H_{\infty}$ control for a class of stochastic switched nonlinear systems: An average dwell time method,” Nonlinear Anal. Hybr. Syst. vol. 19, pp. 198-208, 2016.

[21] Y.H. Ren, W.Q. Wang, Y.X. Wang, “Incremental $H_{\infty}$ control for switched nonlinear systems,” Appl. Math. Comput. vol. 331, pp. 251-263, 2018.

[22] W. Feng, J. Tian, P. Zhao, “Stability analysis of switched stochastic systems,” Automatica, vol. 47, no. 1, pp. 148-157, 2011.

[23] L. Zhang, H. Gao, “Asynchronously switched control of switched linear systems with average dwell time,” Automatica vol. 46, no. 5, pp. 953-958, 2010.

[24] Y.H. Ren, W.Q. Wang, Y.X. Wang, M.X. Shen, “Incremental $H_{\infty}$ performance for a class of stochastic switched nonlinear systems,” Jour. Frank. Inst. vol. 355, pp. 7134-7157, 2018.

[25] M. Zamani, A. Abate, “Approximately bisimilar symbolic models for randomly switched stochastic systems,” Syst. Control Lett. vol. 69, pp. 38-46, 2014.

[26] Øksendal, B. Stochastic differential equations: An introduction with applications Springer, 6th edition, 2007.

[27] H. Kobayashi, B.L. Mark, W. Turin, Probability, Random Processes, and Statistical Analysis Cambridge University Press, New York, 2011.

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VOLUME 4, 2016

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