Critical 3-hypergraphs
(detailed version)

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Abstract
Given a 3-hypergraph $H$, a subset $M$ of $V(H)$ is a module of $H$ if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$ and for every $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H)$. For example, $\emptyset$, $V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of $H$, called trivial. A 3-hypergraph is prime if all its modules are trivial. Furthermore, a prime 3-hypergraph is critical if all its induced subhypergraphs, obtained by removing one vertex, are not prime. We characterize the critical 3-hypergraphs.

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1 Introduction
Let $H$ be a 3-hypergraph. A tournament $T$, with the same vertex set as $H$, is a realization of $H$ if the edges of $H$ are exactly the 3-element subsets of the vertex set of $T$ that induce 3-cycles. In [2], we characterized the 3-hypergraphs that admit realizations (see [4, Problem 1]). To obtain our characterization, we introduced a new notion of a module for hypergraphs. By using the modular decomposition tree, we demonstrated that a 3-hypergraph is realizable if and only if all its prime (in terms of modular decomposition) induced subhypergraphs are realizable (see [2, Theorem 13]). Moreover, given a realizable 3-hypergraph

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we proved that $H$ is prime if and only if its realizations are prime (see [2, Theorem 12]). These results lead us to study the prime and induced subhypergraphs of a prime 3-hypergraph. Precisely, consider a prime 3-hypergraph $H$. In [3, Theorem 13], we proved that $H$ admits a prime induced subhypergraph obtained by removing 1 or 2 vertices. Similar results were obtained for prime digraphs [4], for prime binary relational structures [11], or for prime 2-structures [5]. Our purpose is to characterize the critical 3-hypergraphs, that is, the prime 3-hypergraphs all the subhypergraphs of which, obtained by removing one vertex, are not prime.

At present, we formalize our presentation. We consider only finite structures. A hypergraph $H$ is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \setminus \{\emptyset\}$. Given $k \geq 2$, a hypergraph $H$ is a $k$-hypergraph if $E(H) \subseteq \binom{V(H)}{k}$. Furthermore, a hypergraph $H$ is a $\{k,k+1\}$-hypergraph if $E(H) \subseteq \binom{V(H)}{k} \cup \binom{V(H)}{k+1}$.

Let $H$ be a hypergraph. With each $W \subseteq V(H)$, we associate the subhypergraph $H[W]$ of $H$ induced by $W$, which is defined on $V(H[W]) = W$ by $E(H[W]) = \{e \in E(H) : e \subseteq W\}$.

**Definition 1.** Let $H$ be a hypergraph. A subset $M$ of $V(H)$ is a module of $H$ if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for every $n \in M$, we have

$$(e \setminus \{m\}) \cup \{n\} \in E(H).$$

Let $H$ be a hypergraph. Clearly, $\emptyset$, $V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of $H$, called trivial modules. A hypergraph $H$ is indecomposable if all its modules are trivial, otherwise it is decomposable. A hypergraph $H$ is prime if it is indecomposable, with $v(H) \geq 3$. In [3], we prove the following result (see [3, Theorem 9]). We need the following notation (see [3, Notation 5]).

**Notation 2.** Let $H$ be a hypergraph. Given $X \subseteq V(H)$ such that $H[X]$ is prime, consider the following subsets of $V(H) \setminus X$

- $\text{Ext}_H(X)$ denotes the set of $v \in V(H) \setminus X$ such that $H[X \cup \{v\}]$ is prime;
- $\langle X \rangle_H$ denotes the set of $v \in V(H) \setminus X$ such that $X$ is a module of $H[X \cup \{v\}]$;
- for each $y \in X$, $X_H(y)$ denotes the set of $v \in V(H) \setminus X$ such that $\{y,v\}$ is a module of $H[X \cup \{v\}]$.

The set $\{\text{Ext}_H(X), \langle X \rangle_H \cup \{X_H(y) : y \in X\}\}$ is denoted by $p_{(H,X)}$. 

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Theorem 3. Let $H$ be a 3-hypergraph. Consider $X \subseteq V(H)$ such that $H[X]$ is prime. Set
\[ X = \{ y \in X : X_H(y) \neq \emptyset \}. \]
If $H$ is prime, then there exist $v, w \in (V(H) \setminus X) \cup X$ such that $H - \{v, w\}$ is prime.

The next result follows from Theorem 3 (see [3, Corollary 10])

Corollary 4. Let $H$ be a prime 3-hypergraph. If $v(H) \geq 4$, then there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime.

Lastly, a prime hypergraph $H$ is critical if $H - v$ is decomposable for each $v \in V(H)$. Our purpose is to characterize the critical 3-hypergraphs.

Corollary 4 leads Ille [8] to introduce the following auxiliary graph.

Definition 5. Let $H$ be a prime 3-hypergraph with $v(H) \geq 5$. The primality graph $P(H)$ associated with $H$ is the graph defined on $V(H)$ as follows. Given distinct $v, w \in V(H)$, $vw \in E(P(H))$ if $H - \{v, w\}$ is prime. When $H$ is critical, it follows from Corollary 4 that $P(H)$ is nonempty.

1.1 Critical and realizable 3-hypergraphs

Let $T$ be a tournament. A subset $M$ of $V(T)$ is a module [12] of $T$ provided that for any $x, y \in M$ and $v \in V(T)$, if $xv, vy \in A(T)$, then $v \in M$. Note that the notions of a module and of a convex subset [17] coincide for tournaments. Moreover, note that the notions of a module and of an interval coincide for linear orders. Given a tournament $T$, $\emptyset, V(T)$ and $\{v\}$, where $v \in V(T)$, are modules of $T$, called trivial modules. A tournament is indecomposable if all its modules are trivial, otherwise it is decomposable. A tournament $T$ is prime if it is indecomposable, with $v(T) \geq 3$. Lastly, a prime tournament $T$ is critical if $T - v$ is decomposable for each $v \in V(T)$. Schmerl and Trotter [11] characterized the critical tournaments. They obtained the tournaments $T_{2n+1}, U_{2n+1}$ and $W_{2n+1}$ defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, as follows.

- The tournament $T_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between even and odd vertices.
- The tournament $U_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between even vertices.
- The tournament $W_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between $2n$ and the even elements of $\{0, \ldots, 2n-1\}$.

Theorem 6 (Schmerl and Trotter [11]). Given a tournament $\tau$, with $v(\tau) \geq 5$, $\tau$ is critical if and only if $v(\tau)$ is odd and $\tau$ is isomorphic to $T_{v(\tau)}$, $U_{v(\tau)}$, or $W_{v(\tau)}$.

A realization of a 3-hypergraph is defined as follows (see [2, Definition 10]). To begin, we associate with each tournament a 3-hypergraph in the following way (see [2, Definition 9]).
**Definition 7.** The 3-cycle is the tournament $C_3 = \{\{0, 1, 2\}, \{01, 12, 20\}\}$. Given a tournament $T$, the $C_3$-structure of $T$ is the 3-hypergraph $C_3(T)$ defined on $V(C_3(T)) = V(T)$ by

$$E(C_3(T)) = \{X \subseteq V(T) : T[X] \text{ is isomorphic to } C_3\}.$$

**Definition 8.** Given a 3-uniform hypergraph $H$, a tournament $T$, with $V(T) = V(H)$, realizes $H$ if $H = C_3(T)$. We say also that $T$ is a realization of $H$.

In [2], we proved the following result (see [2, Theorem 12]).

**Theorem 9.** Consider a realizable and 3-hypergraph $H$. For a realization $T$ of $H$, we have $H$ is prime if and only if $T$ is prime.

The next result follows from Theorems 8 and 9 (see [2, Theorem 49]).

**Corollary 10.** Given a realizable 3-hypergraph $H$, $H$ is critical if and only if $v(H)$ is odd and $H$ is isomorphic to $C_3(T_{v(H)})$, $C_3(T_{v(H)})$, or $C_3(T_{v(H)})$.

**Definition 11.** Given a critical 3-hypergraph $H$, we say that $H$ is circular if $v(H)$ is odd and $H$ is isomorphic to $C_3(T_{v(H)})$.

### 1.2 A construction of 3-hypergraphs

The construction is done from the set of the components of a graph. We use the following notation.

**Notation 12.** Consider a graph $\Gamma$. The set of the components of $\Gamma$ is denoted by $\mathcal{C}(\Gamma)$. Furthermore, set $\mathcal{C}_{\text{even}}(\Gamma) = \{C \in \mathcal{C}(\Gamma) : v(C) \equiv 0 \mod 2\}$ and $\mathcal{C}_{\text{odd}}(\Gamma) = \{C \in \mathcal{C}(\Gamma) : v(C) \equiv 1 \mod 2\}$. Lastly, set $\mathcal{C}_1(\Gamma) = \{C \in \mathcal{C}(\Gamma) : v(C) = 1\}$ and $V_1(\Gamma) = \{v \in V(\Gamma) : \{v\} \in \mathcal{C}_1(\Gamma)\}$. For each $C \in \mathcal{C}(\Gamma)$, set

$$w(C) = \begin{cases} v(C)/2 & \text{if } C \in \mathcal{C}_{\text{even}}(\Gamma) \\ (v(C) - 1)/2 & \text{if } C \in \mathcal{C}_{\text{odd}}(\Gamma). \end{cases}$$

We consider a graph $\Gamma$,

all the components of which are paths. (1)

For each $n \geq 1$, recall that the path $P_n$ is defined on $V(P_n) = \{0, \ldots, n - 1\}$ by

$$E(P_n) = \begin{cases} \emptyset & \text{if } n = 1, \\ \{k(k+1) : 0 \leq k \leq n - 2\} & \text{if } n \geq 2. \end{cases}$$

We suppose that

* $\mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$,
* for each $C \in \mathcal{C}_{\text{odd}}(\Gamma)$, if $V(H) \setminus V(C) \neq \emptyset$, then
  $$(\mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma)) \setminus \{C\} \neq \emptyset,$$
* for each $C \in \mathcal{C}_{\text{even}}(\Gamma)$, if $|V(H) \setminus V(C)| \geq 2$, then
  $$(\mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma)) \setminus \{C\} \neq \emptyset.$$
Consider also a \(\{2,3\}\)-hypergraph \(\mathbb{H}\) defined on \(\mathcal{C}(\Gamma)\) satisfying

- for each \(\varepsilon \in E(\mathbb{H})\), \(|\varepsilon| = 2\) or \(3\),
- for each \(\varepsilon \in E(\mathbb{H})\), if \(|\varepsilon| = 2\), then \(\varepsilon \cap \mathcal{C}_{\text{even}}(\Gamma) \neq \emptyset\) and \(\varepsilon \cap \mathcal{C}_{\text{odd}}(\Gamma) \neq \emptyset\), (3)
- for each \(\varepsilon \in E(\mathbb{H})\), if \(|\varepsilon| = 3\), then \(\varepsilon \subseteq \mathcal{C}_{\text{odd}}(\Gamma)\).

With \(\Gamma\) and \(\mathbb{H}\), we associate the 3-hypergraph \(\Gamma \bullet \mathbb{H}\) defined on \(V(\Gamma)\) in the following manner. For each \(C \in \mathcal{C}(\Gamma)\), we consider an isomorphism \(\varphi_C\) from the path \(P_{v(C)}\) onto \(C\). With each \(C \in \mathcal{C}_{\text{odd}}(\Gamma) \setminus \mathcal{C}_1(\Gamma)\), we associate the set

\[E_C = \varphi_C(E(C_3(U_{v(C)}))).\]

Moreover, consider \(\varepsilon \in E(\mathbb{H})\) such that \(|\varepsilon| = 2\). There exist \(C \in \mathcal{C}_{\text{even}}(\Gamma)\) and \(D \in \mathcal{C}_{\text{odd}}(\Gamma)\) such that \(e = CD\). Associate with \(\varepsilon\) the set

\[E_\varepsilon = \{\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) : 0 \leq i \leq j \leq w(C) - 1, 0 \leq k \leq w(D)\}.

Lastly, consider \(\varepsilon \in E(\mathbb{H})\) such that \(|\varepsilon| = 3\). There exist \(I, J, K \in \mathcal{C}_{\text{odd}}(\Gamma)\) such that \(\varepsilon = IJK\). Associate with \(\varepsilon\) the set

\[E_\varepsilon = \{\varphi_I(2i)\varphi_J(2j)\varphi_K(2k) : 0 \leq i \leq w(I), 0 \leq j \leq w(J), 0 \leq k \leq w(K)\}.

The 3-hypergraph \(\Gamma \bullet \mathbb{H}\) is defined on \(V(\Gamma)\) by

\[E(\Gamma \bullet \mathbb{H}) = \bigcup_{C \in \mathcal{C}_{\text{odd}}(\Gamma) \setminus \mathcal{C}_1(\Gamma)} E_C \cup \bigcup_{\varepsilon \in E(\mathbb{H})} E_\varepsilon.\] (4)

### 1.3 The main results

To begin, we prove the following theorem.

**Theorem 13.** Let \(H\) be a critical and 3-hypergraph such that \(v(H) \geq 5\). If \(H\) is not circular, then there exist a graph \(\Gamma\) satisfying (1) and (2), and a \(\{2,3\}\)-hypergraph \(\mathbb{H}\) defined on \(\mathcal{C}(\Gamma)\) satisfying (3) such that \(H = \Gamma \bullet \mathbb{H}\).

In Theorem 13, the graph \(\Gamma\) is the primality graph of \(H\) (see Definition 3). To state, the second main theorem, we consider a graph \(\Gamma\) satisfying (1) and (2). We consider also a \(\{2,3\}\)-hypergraph \(\mathbb{H}\) defined on \(\mathcal{C}(\Gamma)\) satisfying (3). Furthermore, we use the modules of \(\mathbb{H}\) defined as follows.

**Definition 14.** Given \(\mathbb{W} \subseteq V(\mathbb{H})\), \(\mathbb{W}\) is a module of \(\mathbb{H}\) if \(\mathbb{W}\) is a module of

\[(V(\mathbb{H}), E(\mathbb{H}) \cap \binom{V(\mathbb{H})}{2})\]

and \(\mathbb{W}\) is a module of

\[(V(\mathbb{H}), E(\mathbb{H}) \cap \binom{V(\mathbb{H})}{3})\].

**Theorem 15.** Suppose that \(v(\Gamma \bullet \mathbb{H}) \geq 5\). The 3-hypergraph \(\Gamma \bullet \mathbb{H}\) is critical if and only if the following three assertions hold
(C1) $\mathbb{H}$ is connected;

(C2) for every nontrivial module $\mathcal{M}$ of $\mathbb{H}$, we have $\mathcal{M} \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$;

(C3) for each $v \in V_1(\Gamma)$, if $\mathbb{H} - \{v\}$ is connected, then $\mathbb{H} - \{v\}$ admits a nontrivial module $\mathcal{M}_{\{v\}}$ such that $\mathcal{M}_{\{v\}} \subseteq \mathcal{C}_1(\Gamma) \setminus \{\{v\}\}$.

Before proving Theorem 15, we establish the next proposition.

**Proposition 16.** The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is prime if and only if the following two assertions hold

(C1) $\mathbb{H}$ is connected;

(C2) for every nontrivial module $\mathcal{M}$ of $\mathbb{H}$, we have $\mathcal{M} \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$.

Finally, we improve Theorem 15 by characterizing the 3-hypergraph $\Gamma \bullet \mathbb{H}$ whenever it is critical and satisfies $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$ (see Theorem 45 in Section 6).

### 2 The critical tournaments

The primality graph associated with a prime tournament is defined in the same as that associated with a prime 3-hypergraph (see Definition 5).

**Definition 17.** Let $T$ be a prime tournament. The primality graph $\mathcal{P}(T)$ of $T$ is defined on $V(T)$ as follows. Given distinct $v, w \in V(T)$, $vw \in E(\mathcal{P}(T))$ if $T - \{v, w\}$ is prime.

The basic properties of the primality graph follow. The next lemma is stated in [8] without a proof. For a proof, see [1, Lemma 10].

**Lemma 18** (Ille [8]). Let $T$ be a critical tournament with $v(T) \geq 5$. For every $v \in V(T)$, we have $d_{\mathcal{P}(T)}(v) \leq 2$. Moreover, the next two assertions hold.

1. Given $v \in V(T)$, if $d_{\mathcal{P}(T)}(v) = 1$, then $V(T) \setminus (\{v\} \cup N_{\mathcal{P}(T)}(v))$ is a module of $T - v$.

2. Given $v \in V(T)$, if $d_{\mathcal{P}(T)}(v) = 2$, then $N_{\mathcal{P}(T)}(v)$ is a module of $T - v$.

For each $n \geq 3$, recall that the cycle $C_n$ is defined on $V(C_n) = \{0, \ldots, n-1\}$ by $E(C_n) = E(P_n) \cup \{0(n-1)\}$. The length of $C_n$ is $n$. Given a critical tournament $T$, it follows from Lemma 18 that the connected components of $\mathcal{P}(T)$ are paths or cycles. Boudabbous and Ille [1] characterized the critical tournaments from their primality graphs. To begin, they examine the connected components of the primality graph associated with a critical tournament.

**Lemma 19** (Corollary 17 [1]). If $T$ is a critical tournament, with $v(T) \geq 5$, then $\mathcal{P}(T)$ satisfies one of the following

1. $\mathcal{P}(T)$ is a cycle of odd length.
2. \( \mathcal{P}(T) \) is a path;

3. \( v(T) \) is odd and there is \( v \in V(T) \) such that \( \mathcal{P}(T) - v \) is a path and \( N_{\mathcal{P}(T)}(v) = \emptyset \).

For each of the three shapes described in Lemma \[13\] Boudabbous and Ille \[1\] characterized the corresponding critical tournaments.

**Proposition 20** (Proposition 18 \[1\]). Given a tournament such that \( v(T) \geq 5 \), \( T \) is critical and \( \mathcal{P}(T) = C_{v(T)} \) if and only if \( v(T) \) is odd and \( T = T_{2n+1} \) or \( (T_{2n+1})^* \).

**Proposition 21** (Proposition 19 \[1\]). Given a tournament such that \( v(T) \geq 5 \), \( T \) is critical and \( \mathcal{P}(T) = P_{v(T)} \) if and only if \( v(T) \) is odd and \( T = U_{v(T)} \) or \( (U_{v(T)})^* \).

**Proposition 22** (Proposition 21 \[1\]). Given a tournament defined on \( \{0, \ldots, 2n\} \) with \( 2n+1 \geq 5 \), \( T \) is critical, \( \mathcal{P}(T) - 2n = P_{2n-1} \) and \( N_{\mathcal{P}(T)}(2n) = \emptyset \) if and only if \( T = W_{2n+1} \) or \( (W_{2n+1})^* \).

\section{Proof of Theorem 13}

The purpose of this section is to characterize the non circular and critical 3-hypergraphs. We use the following notation.

**Notation 23.** Let \( H \) be a 3-hypergraph.

\[
\text{For } e, f \in \binom{V(H)}{3}, e \equiv_H f \text{ means } \begin{cases} e, f \in E(H) \\
eq & e, f \notin E(H). \end{cases}
\]

An analogue of lemma \[18\] for prime and 3-uniform hypergraphs follows. For a proof, see \[1\] Lemma 10.

**Lemma 24.** Let \( H \) be a critical and 3-hypergraph with \( v(H) \geq 5 \). For every \( v \in V(H) \), we have \( d_{\mathcal{P}(H)}(v) \leq 2 \). Moreover, the next two assertions hold.

1. Given \( v \in V(H) \), if \( d_{\mathcal{P}(H)}(v) = 1 \), then \( V(H) \cap (\{v\} \cup N_{\mathcal{P}(H)}(v)) \) is a module of \( H - v \).

2. Given \( v \in V(H) \), if \( d_{\mathcal{P}(H)}(v) = 2 \), then \( N_{\mathcal{P}(H)}(v) \) is a module of \( H - v \).

Given a critical 3-hypergraph \( H \), it follows from Lemma \[24\] that the components of \( \mathcal{P}(H) \) are cycles or paths.

**Proposition 25.** Let \( H \) be a critical 3-hypergraph defined on \( \{0, \ldots, p-1\} \), where \( p \geq 5 \). If there exists \( k \in \{3, \ldots, p\} \) such that \( \mathcal{P}(H)\{0, \ldots, k-1\} = C_k \), then \( p = 2n+1 \), \( k = p \), and \( H = C_3(T_{2n+1}) \), where \( n \geq 2 \).
Proof. Suppose that there exists \( k \in \{3, \ldots, p\} \) such that \( \mathcal{P}(H)[\{0, \ldots, k-1\}] = C_k \). First, we show that
\[
k \text{ is odd.} \tag{5}
\]
Otherwise, there exists \( l \geq 2 \) such that \( \mathcal{P}(H)[\{0, \ldots, 2l-1\}] = C_{2l} \). We verify that \( \{0, 2\} \) is a module of \( H \). Consider \( e \in E(H) \) such that \( e \cap \{0, 2\} \neq \emptyset \) and \( e \setminus \{0, 2\} \neq \emptyset \). Suppose for a contradiction that \( 0, 2 \in e \). There exists \( i \in \{1\} \cup \{3, \ldots, 2l-1\} \) such that \( e = 02i \). Since \( N_{\mathcal{P}(H)}(1) = \{0, 2\} \), it follows from Lemma \( 24 \) that \( \{0, 2\} \) is a module of \( H - 1 \). It follows that \( i = 1 \), that is, \( e = 012 \). It follows from Lemma \( 24 \) that \( 01(2l-2) \in E(H) \), which contradicts the fact that \( \{0, 2l-2\} \) is a module of \( H - (2l-1) \). Consequently, \( e \cap \{0, 2\} = 1 \). For a contradiction, suppose that \( 1 \in e \). Since \( e \cap \{0, 2\} = 1 \), there \( j \in \{3, \ldots, 2l-1\} \) such that \( e = 01j \) or \( 12j \). Denote by \( j' \) the unique element of \( \{2l-2, 2l-1\} \) such that \( j' \equiv j \mod 2 \). It follows from Lemma \( 24 \) that \( (e \setminus \{j\}) \cup \{j'\} \in E(H) \). Similarly, by denoting by \( j'' \) the unique element of \( \{3, 4\} \) such that \( j'' \equiv j \mod 2 \), we obtain \( (e \setminus \{j\}) \cup \{j''\} \in E(H) \). We distinguish the following two cases.

1. Suppose that \( e = 01j \). If \( j \) is even, then \( 01(2l-2) \in E(H) \), which contradicts the fact that \( \{0, 2l-2\} \) is a module of \( H - (2l-1) \). If \( j \) is odd, then \( 013 \in E(H) \), which contradicts the fact that \( \{1, 3\} \) is a module of \( H - 2 \).

2. Suppose that \( e = 12j \). If \( j \) is even, then \( 124 \in E(H) \), which contradicts the fact that \( \{2, 4\} \) is a module of \( H - 3 \). If \( j \) is odd, then \( 12(2l-1) \in E(H) \), which contradicts the fact that \( \{1, 2l-1\} \) is a module of \( H - 0 \).

It follows that \( 1 \notin e \). So, \( e \in E(H - 1) \). Since \( N_{\mathcal{P}(H)}(1) = \{0, 2\} \), it follows from Lemma \( 24 \) that \( \{0, 2\} \) is a module of \( H - 1 \). Thus, there exists \( i \in \{0, 2\} \) such that \( e \cap \{0, 2\} = \{i\} \), and \( (e \setminus \{i\}) \cup \{i'\} \in E(H) \) for each \( i' \in \{0, 2\} \). Consequently, \( \{0, 2\} \) is a module of \( H \), which contradicts the fact that \( H \) is prime. It follows that \( 5 \) holds. Set \( k = 2n + 1 \), where \( n \geq 1 \).

Second, we prove that \( \{0, \ldots, 2n\} \) is a module of \( H \). Consider \( e \in E(H) \) such that \( e \cap \{0, \ldots, 2n\} \neq \emptyset \) and \( e \setminus \{0, \ldots, 2n\} \neq \emptyset \). We prove that
\[
|e \cap \{0, \ldots, 2n\}| = 1. \tag{6}
\]
Otherwise, we have \( |e \cap \{0, \ldots, 2n\}| = 2 \). There exist \( i, j \in \{0, \ldots, 2n\} \), with \( i < j \), such that \( e \cap \{0, \ldots, 2n\} = \{i, j\} \). Denote by \( i' \) the unique element of \( \{0, 1\} \) such that \( i' \equiv i \mod 2 \). As previously, we obtain \( (e \setminus \{i\}) \cup \{i'\} \in E(H) \). Denote by \( j' \) the unique element of \( \{2, 3\} \) such that \( j' \equiv j \mod 2 \). We obtain \( (e \setminus \{i, j\}) \cup \{i', j'\} \in E(H) \). Set \( e' = (e \setminus \{i, j\}) \cup \{i', j'\} \). Observe that \( e' \setminus \{0, \ldots, 2n\} = e \setminus \{0, \ldots, 2n\} \), and denote by \( v \) the unique element of \( e' \setminus \{0, \ldots, 2n\} \). If \( v = 0 \) and \( j' = 2 \), then \( 02v \in E(H) \), which contradicts the fact that \( \{0, 2\} \) is a module of \( H - 1 \). If \( i' = 1 \) and \( j' = 3 \), then \( 13v \in E(H) \), which contradicts the fact that \( \{1, 3\} \) is a module of \( H - 2 \). Suppose that \( i' = 0 \) and \( j' = 3 \). We get \( 03v \in E(H) \), and hence \( 0(2n-1)v \in E(H) \), which contradicts the fact that \( \{0, 2n-1\} \) is a module of \( H - (2n) \). Lastly, if \( i' = 1 \) and \( j' = 4 \), then \( 1(2n)v \in E(H) \), which contradicts the fact that \( \{1, 2n\} \) is a module of \( H - 0 \). It follows that \( 6 \) holds.
Denote by $i$ the unique element of $e \cap \{0, \ldots, 2n\}$. For every $l \in \{1, \ldots, n\}$, we obtain
\[
(e \setminus \{i\}) \cup \{i + 2l\} \in E(H),
\]
where $i + 2l$ is considered modulo $2n + 1$. In particular, we have $(e \setminus \{i\}) \cup \{i + 2n\} \in E(H)$, that is, $(e \setminus \{i\}) \cup \{i - 1\} \in E(H)$. Since $\{i - 1, i + 1\}$ is a module of $H - i$, we get $(e \setminus \{i\}) \cup \{i + 1\} \in E(H)$. For each $m \in \{0, \ldots, n - 1\}$, we obtain
\[
(e \setminus \{i\}) \cup \{i + 2m + 1\} \in E(H).
\]
It follows from (7) and (8) that $(e \setminus \{i\}) \cup \{i'\} \in E(H)$ for every $i' \in \{0, \ldots, 2n\}$. Consequently, $\{0, \ldots, 2n\}$ is a module of $H$. Since $H$ is prime, we obtain $V(H) = \{0, \ldots, 2n\}$.

Third, we prove that $H = C_3(T_{2n+1})$. We have
\[
E(C_3(T_{2n+1})) = \{(2i)(2l+1)(2j) : 0 \leq i \leq l \leq j \leq n\}
\cup \{(2i+1)(2l)(2j+1) : 0 \leq i \leq p \leq l \leq n - 1\}.
\]
For a contradiction, suppose that there are $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, $x \equiv y \mod 2$, and $xyz \in E(H)$. It follows from Lemma 24 that $(y - 2)yz \in E(H)$, which contradicts the fact that $\{y - 2, y\}$ is a module of $H - (y - 1)$. Hence, $xyz \notin E(H)$ for $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $x \equiv y \mod 2$. Similarly, $xyz \notin E(H)$ for $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $y \equiv z \mod 2$. It follows that
\[
E(H) \subseteq E(C_3(T)).
\]
Now, consider $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, $x \not\equiv y \mod 2$, and $y \not\equiv z \mod 2$. It follows from Lemma 24 that
\[
xyz \equiv_H x(x + 1)(x + 2).
\]
Now, we prove that the permutation
\[
\theta : \quad \{0, \ldots, 2n\} \quad \mapsto \quad \{0, \ldots, 2n\}
\]
of $\{0, \ldots, 2n\}$ is an automorphism of $H$. Given $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, we have to verify that
\[
xyz \equiv_H (x + 1)(y + 1)(z + 1).
\]
We have
\[
xyz \equiv_H x(y + 2)(z + 2) \equiv_H x(y + 2)(y + 2)(z + 2).
\]
Thus, $xyz \equiv_H (x + 2n + 2)(y + 2n + 2)(z + 2n + 2)$, that is,
\[
xyz \equiv_H (x + 1)(y + 1)(z + 1).
\]
Consequently, $\theta$ is an automorphism of $H$. Since $H$ is prime, there exist $u, v, w \in \{0, \ldots, 2n\}$ such that $u < v < w$ and $uvw \in E(H)$. By (10), $uvw \in E(C_3(T))$. It follows from (12) that $u \not\equiv v \mod 2$ and $v \not\equiv w \mod 2$. It follows from (11) that $u(u + 1)(u + 2) \in E(H)$. Since $\theta$ is an automorphism of $H$, we obtain
\[
012 \in E(H).
\]
Finally, consider $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $xyz \in E(C_3(T))$.

By (9), $x \not\equiv y \mod 2$ and $y \not\equiv z \mod 2$. It follows from (11) that $xyz \equiv_H x(x+1)(x+2)$. Since $\theta$ is an automorphism of $H$, we obtain $x(x+1)(x+2) \equiv_H 012$. Therefore, $xyz \equiv_H 012$. By (12), $xyz \in E(H)$. Consequently, we obtain $E(C_3(T)) \subseteq E(H)$. It follows from (10) that $H = C_3(T)$.

The next corollary is an easy consequence of Lemma 24 and Proposition 25.

**Corollary 26.** Given a critical 3-hypergraph $H$, with $v(H) \geq 5$, $H$ is not circular if and only if all the components of $P(H)$ are paths.

**Proof.** Suppose that $H$ is circular. By Definition 11 $H$ is isomorphic to $C_3(T_{v(H)})$.

Hence, $P(H)$ is isomorphic to $P(C_3(T_{v(H)}))$. It follows from Theorem 9 that

$$P(C_3(T_{v(H)})) = P(T_{v(H)}).$$

By Proposition 20 $P(T_{v(H)}) = C_{v(H)}$. It follows that $P(H)$ contains a cycle among its components.

Conversely, suppose that $P(H)$ contains a cycle among its components. Up to isomorphism, we can assume that $V(H) = \{0, \ldots, p-1\}$, where $p \geq 5$, and there exists $k \in \{3, \ldots, p\}$ satisfying $P(H)[\{0, \ldots, k-1\}] = C_k$. By Proposition 25 $H = C_3(T_{2k+1})$. Therefore, $H$ is circular. □

**Notation 27.** Let $H$ be a non circular and critical 3-hypergraph. For each $C \in \mathcal{C}(P(H)) \setminus \mathcal{C}_1(P(H))$, $C$ is a path. Thus, there exists an isomorphism $\varphi_C$ from $P(v(C))$ onto $C$.

**Lemma 28.** Let $H$ be a critical and 3-hypergraph with $v(H) \geq 5$. Suppose that $H$ is not circular.

1. Let $C \in \mathcal{C}_{\text{odd}}(P(H)) \setminus \mathcal{C}_1(P(H))$. For every $e \in E(H)$, if $e \nabla V(C) \neq \emptyset$ and $e \setminus V(C) \neq \emptyset$, then there exists $i \in \{0, \ldots, w(C)\}$ (see Notation 12) such that $e \nabla V(C) = \{\varphi_C(2i)\}$.

2. Let $C \in \mathcal{C}_{\text{even}}(P(H))$. For every $e \in E(H)$, if $e \nabla V(C) = \emptyset$ and $e \setminus V(C) \neq \emptyset$, then there exist $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$ and $e \nabla V(C) = \{\varphi_C(2i), \varphi_C(2j + 1)\}$.

**Proof.** For the first assertion, consider $C \in \mathcal{C}_{\text{odd}}(P(H)) \setminus \mathcal{C}_1(P(H))$. Let $e \in E(H)$ such that $e \nabla V(C) \neq \emptyset$ and $e \setminus V(C) \neq \emptyset$. We prove that

$$e \nabla \{\varphi_C(2i + 1) : i \in \{0, \ldots, w(C) - 1\}\} = \emptyset. \quad (13)$$

Otherwise, there exists $i \in \{0, \ldots, w(C) - 1\}$ such that $\varphi_C(2i + 1) \in e$. We distinguish the following two cases.

- Suppose that there exists $j \in \{0, \ldots, w(C) - 1\}$ such that $\varphi_C(2j) \in e$. Since $e \setminus V(C) \neq \emptyset$, $e \nabla V(C) = \{\varphi_C(2j), \varphi_C(2j + 1)\}$. By denoting by $v$ the unique element of $e \setminus V(C)$, we obtain $e = \varphi_C(2j)v\varphi_C(2j + 1)v$. It follows from Lemma 24 that $\varphi_C(2j)\varphi_C(2w(C) - 1)v \in E(H)$, which contradicts the fact that $V(H) \setminus \{\varphi_C(2w(C) - 1), \varphi_C(2w(C))\}$ is a module of $H - \varphi_C(2w(C))$. 


Suppose that \( e \cap \{ \varphi_C(2j) : j \in \{0, \ldots, i\} \} = \emptyset \). It follows from Lemma 24 that there exists \( f \in E(H) \) such that \( \varphi_C(1) \in f, \varphi_C(0) \notin f \), and \( f \setminus V(C) \neq \emptyset \), which contradicts the fact that \( V(H) \setminus \{ \varphi_C(0), \varphi_C(1) \} \) is a module of \( H - \varphi_C(0) \).

Consequently, \([13]\) holds.

Now, we prove that there exists \( i \in \{0, \ldots, w(C)\} \) such that

\[
e \cap V(C) = \{ \varphi_C(2i) \}.
\]

Otherwise, it follows from \([13]\) that there exist distinct \( i, j \in \{0, \ldots, w(C)\} \) such that \( 0 \leq i < j \leq w(C) \) such that \( e \cap V(C) = \{ \varphi_C(2i), \varphi_C(2j) \} \). It follows from Lemma 24 that there exists \( f \in E(H) \) such that \( f \cap V(C) = \{ \varphi_C(2i), \varphi_C(2i+2) \} \) and \( f \setminus V(C) \neq \emptyset \), which contradicts the fact that \( \{ \varphi_C(2i), \varphi_C(2i+2) \} \) is a module of \( H - \varphi_C(2i+1) \). Consequently, \([13]\) holds.

For the second assertion, consider \( C \in \mathcal{C}_{\mathit{even}}(P(H)) \). Let \( e \in E(H) \) such that \( e \cap V(C) \neq \emptyset \) and \( e \setminus V(C) \neq \emptyset \). Set

\[
p = \min(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}).
\]

For a contradiction, suppose that \( p \) is odd. It follows from Lemma 24 that \( (e \setminus \{ \varphi_C(p) \}) \cup \{ \varphi_C(1) \} \in E(H) \), which contradicts the fact that \( V(C) \setminus \{ \varphi_C(0), \varphi_C(1) \} \) is a module of \( H - \varphi_C(0) \). It follows that \( p \) is even. Thus, there exists \( i \in \{0, \ldots, w(C) - 1\} \) such that \( p = \varphi_C(2i) \). Similarly, \( \max(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}) \) is odd. Thus there exists \( j \in \{0, \ldots, w(C) - 1\} \) such that \( \max(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}) = \varphi_C(2j + 1) \). We obtain that \( i \leq j \) and \( e \cap V(C) = \{ \varphi_C(2i), \varphi_C(2j + 1) \} \).

**Proposition 29.** Let \( H \) be a critical and 3-hypergraph with \( v(H) \geq 5 \). Suppose that \( H \) is not circular. Consider \( C \in \mathcal{C}_{\mathit{odd}}(P(H)) \setminus \mathcal{C}_{\mathit{1}}(P(H)) \).

1. Suppose that \( V(C) \not\subseteq V(H) \). There exists \( e \in E(H) \), such that \( e \cap V(C) \neq \emptyset \) and \( e \setminus V(C) \neq \emptyset \). Moreover, one of the following two assertions holds
   - there exist distinct \( D, D' \in \mathcal{C}_{\mathit{odd}}(P(H)) \setminus \{ C \} \) such that \( e \subseteq V(C) \cup V(D) \cup V(D') \), and \( e \cap V(D') \neq \emptyset \);
   - there exists \( D \in \mathcal{C}_{\mathit{even}}(P(H)) \setminus \{ C \} \) such that \( e \subseteq V(C) \cap V(D) \), and \( e \cap V(D) \neq \emptyset \).

Furthermore, there exists \( i \in \{0, \ldots, w(C)\} \) such that \( e \cap V(C) = \{ \varphi_C(2i) \} \). Lastly, for each \( j \in \{0, \ldots, w(C)\} \), \( (e \setminus \{ \varphi_C(2i) \}) \cup \{ \varphi_C(2j) \} \in E(H) \).

2. The function \( \varphi_C \) is an isomorphism from \( C_3(U_{w(C)}) \) onto \( H[V(C)] \).

3. If \( V(C) \not\subseteq V(H) \), then there exists \( C' \in \mathcal{C}(P(H)) \setminus \mathcal{C}_1(P(H)) \) such that \( C' \neq C \).
Proof. For the first assertion, suppose that $V(C) \subseteq V(H)$. Since $V(C)$ is not a module of $H$, there exists $e \in E(H)$ such that $e \cap V(C) \neq \emptyset$ and $e \cap V(C) \neq \emptyset$. It follows from Lemma 28 that there exists $i \in \{0, \ldots, w(C)\}$ such that $e \cap V(C) = \{\varphi_C(2i)\}$. It follows from Lemma 24 that $(e \cap \{\varphi(C(2i))\}) \cup \{\varphi_C(2j)\} \in E(H)$ for each $j \in \{0, \ldots, w(C)\}$. Moreover, since $|e \cap V(C)| = 1$, there exists $D \in \mathcal{E}(\mathcal{P}(H)) \setminus \{C\}$ such that $e \cap V(D) \neq \emptyset$. We distinguish the following two cases.

- Suppose that $D \in \mathcal{E}_{\text{even}}(\mathcal{P}(H))$. By Lemma 28, $|e \cap V(D)| = 2$. Thus $e \subseteq V(C) \cup V(D)$.

- Suppose that $D \in \mathcal{E}_{\text{odd}}(\mathcal{P}(H))$. By Lemma 28, $|e \cap V(D)| = 1$. Therefore, there exists $D' \in \mathcal{E}(\mathcal{P}(H)) \setminus \{C, D\}$ such that $e \cap V(D') \neq \emptyset$. It follows from Lemma 28 that $D' \in \mathcal{E}_{\text{odd}}(\mathcal{P}(H))$. Thus $e \subseteq V(C) \cup V(D) \cup V(D')$.

For the second assertion, suppose for a contradiction that for every $e \in E(H)$ such that $e \cap V(C) \neq \emptyset$, we have $e \cap V(C) \neq \emptyset$. By Lemma 28, there exists $i \in \{0, \ldots, w(C)\}$ such that $e \cap V(C) = \{\varphi(C(2i))\}$. Thus, for every $e \in E(H)$, we have $e \cap \{\varphi(C(2i + 1) : i \in \{0, \ldots, w(C) - 1\}\} \neq \emptyset$. Therefore, $\{\varphi(C(2i + 1) : i \in \{0, \ldots, w(C) - 1\}\}$ is a module of $H$, and hence $H$ is decomposable. Consequently, there exists $e \in V(H)$ such that $e \subseteq V(C)$. Set

$$p = \min\{i \in \{0, \ldots, 2w(C)\} : \varphi_C(i) \in e\}.$$ 

For a contradiction, suppose that $p$ is odd. It follows from Lemma 24 that $(e \cap \{\varphi_C(p)\}) \cup \{\varphi_C(1)\} \in E(H)$, which contradicts the fact that $V(C) \cap \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$. Thus, there exists $i \in \{0, \ldots, w(C)\}$ such that $p = \varphi_C(2i)$. Similarly, there exists $k \in \{0, \ldots, w(C)\}$ such that $\max\{i \in \{0, \ldots, 2w(C)\} : \varphi_C(i) \in e\} = \varphi_C(2k)$, since $e \subseteq V(C)$, $i < k$. Consider $q \in \{0, \ldots, 2w(C)\}$ such that $e = \varphi_C(2i)\varphi_C(q)\varphi_C(2k)$. For a contradiction, suppose that $q$ is even. We obtain $2i < q < 2k$. It follows from Lemma 24 that $\varphi_C(2i)\varphi_C(2i + 2)\varphi_C(2k) \in E(H)$, which contradicts the fact that $\{\varphi(C(2i), \varphi_C(2i + 2)\} \subseteq V(H) - \varphi_C(2i + 1)$. It follows that $p$ is odd. Hence, there exists $j \in \{0, \ldots, w(C) - 1\}$ such that $q = \varphi_C(2j + 1)$. We obtain $e = \varphi_C(2i)\varphi_C(2j + 1)\varphi_C(2k)$, where $0 \leq i \leq j < k \leq w(C)$. It follows that

$$E(H[V(C)]) \subseteq \{\varphi_C(2i')\varphi_C(2j' + 1)\varphi_C(2k') : 0 \leq i' < j' < k' \leq w(C)\}. \quad (15)$$

It follows from Lemma 24 that

$$\varphi_C(2i')\varphi_C(2j' + 1)\varphi_C(2k') \equiv H \varphi_C(0)\varphi_C(1)\varphi_C(2) \quad (16)$$

for $0 \leq i' < j' < k' \leq w(C)$. Since $e = \varphi_C(2i)\varphi_C(2j + 1)\varphi_C(2k)$, we obtain $\varphi_C(0)\varphi_C(1)\varphi_C(2)$. It follows from (15) and (16) that

$$E(H[V(C)]) \subseteq \{\varphi_C(2i')\varphi_C(2j' + 1)\varphi_C(2k') : 0 \leq i' < j' < k' \leq w(C)\}.$$ 

In other words, $\varphi_C$ is an isomorphism from $C_{3}(U_v(C))$ onto $H[V(C)]$. 

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For the third assertion, suppose that $V(C) \not\subseteq V(H)$. By Theorem 9, $U_v(C)$ is prime. Hence, $C_3(U_v(C))$ is prime by Theorem 9. It follows from the second assertion above that $H[V(C)]$ is prime. By Theorem 3, there exist $v, w \in (V(H) \setminus V(C)) \cup V(C)$ such that $H - \{v, w\}$ is prime. Since $H$ is critical, we have $v \neq w$. Therefore, there exists $C' \in \mathcal{C}(\mathcal{P}(H)) \setminus \mathcal{C}_1(\mathcal{P}(H))$ such that $v, w \in C'$. Lastly, suppose for a contradiction that $V(C) \neq \emptyset$. There exist $c \in V(C)$ such that $e \in \mathcal{C}(U_v(C))$ by the second assertion above, there exists $e \in E(C)$ such that $c \in e$. Since $\{c, u\}$ is a module of $H[V(C) \cup \{u\}]$, we get $(e \setminus \{c\}) \cup \{u\} \in E(H)$, which contradicts Lemma 28. It follows that $V(C) = \emptyset$. Thus $v, w \notin V(C)$, so $C' \neq C$. □

**Proposition 30.** Let $H$ be a critical and 3-hypergraph such that $v(H) \geq 5$. Suppose that $H$ is not circular. Consider $C \in \mathcal{C}_{\text{even}}(\mathcal{P}(H))$.

1. There exists $D \in \mathcal{C}_{\text{odd}}(\mathcal{P}(H))$ such that $\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k) \in E(H)$, where $i, j \in \{0, \ldots, w(C) - 1\}$, with $i \leq j$, and $k \in \{0, \ldots, w(D)\}$;
2. For each $k \in \{0, \ldots, w(D)\}$, the extension $\psi_C^{2k} : \{0, \ldots, v(C)\} \rightarrow V(C) \cup \{\varphi_D(2k)\}$ of $\varphi_C$ defined by

$$
\begin{align*}
\left(\psi_C^{2k}\right)_{\{0, \ldots, v(C) - 1\}} &= \varphi_C \\
\psi_C^{2k}(v(C)) &= \varphi_D(2k),
\end{align*}
$$

is an isomorphism from $C_3(W_{v(C)+1})$ onto $H[V(C) \cup \{\varphi_D(2k)\}]$.
3. If $|V(H) \setminus V(C)| \geq 2$, then there exists $C' \in \mathcal{C}(\mathcal{P}(H)) \setminus \mathcal{C}_1(\mathcal{P}(H))$ such that $C' \neq C$.

**Proof.** For the first assertion, since $V(C)$ is not a module of $H$, there exists $e \in E(H)$ such that $e \cap V(C) \neq \emptyset$ and $e \setminus V(C) \neq \emptyset$. By Lemma 28 there exist $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$ and $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j + 1)\}$. Since $|e \cap V(C)| = 2$, there exists $D \in \mathcal{C}(\mathcal{P}(H)) \setminus \{C\}$ such that $e \cap V(D) \neq \emptyset$. It follows from Lemma 28 that $D \in \mathcal{C}_{\text{odd}}(\mathcal{P}(H))$ and $e \cap V(D) = \{\varphi_D(2k)\}$, where $k \in \{0, \ldots, w(D)\}$.

For the second assertion, it follows from the first assertion above that there exists $D \in \mathcal{C}_{\text{odd}}(\mathcal{P}(H))$ such that

$$\varphi_C(2i_0)\varphi_C(2j_0 + 1)\varphi_D(2k_0) \in E(H),$$

where $i_0, j_0 \in \{0, \ldots, w(C) - 1\}$, with $i_0 \leq j_0$, and $k_0 \in \{0, \ldots, w(D)\}$. It follows from Lemma 24 that

$$\varphi_C(2i_0)\varphi_C(2j_0 + 1)\varphi_D(2k) \in E(H)$$

for each $k \in \{0, \ldots, w(D)\}$. Let $k \in \{0, \ldots, w(D)\}$. We prove that

$$E(H[V(C) \cup \{\varphi_D(2k)\}]) = \{\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k) : 0 \leq i \leq j \leq w(C) - 1\}.$$
Consider $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$. It follows from Lemma 24 that

$$\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k) \equiv_H \varphi_C(0)\varphi_C(1)\varphi_D(2k). \quad (20)$$

By (18) and (20), we have $\varphi_C(0)\varphi_C(1)\varphi_D(2k) \in E(H)$. It follows from (20) that $\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k) \in E(H)$. Therefore,

$$E(H[V(C) \cup \{\varphi_D(2k)\}]) \geq \{\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k): \quad (21)$$

$$0 \leq i \leq j \leq w(C) - 1\}.$$

Conversely, consider $e \in E(H[V(C) \cup \{\varphi_D(2k)\}])$. For a contradiction, suppose that $\varphi_D(2k) \notin e$. There exist $p, q, r \in \{0, \ldots, 2w(C) - 1\}$, with $p < q < r$, such that $e = \varphi_C(p)\varphi_C(q)\varphi_D(r)$. If $p$ is odd, then $\varphi_C(1)\varphi_C(q)\varphi_D(r) \in E(H)$, which contradicts the fact that $V(H) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$. Hence, $p = 2i$, where $i \in \{0, \ldots, w(C) - 1\}$. Similarly, we have $r = 2l + 1$, where $l \in \{i, \ldots, w(C) - 1\}$. If $q$ is even, then $\varphi_C(2i)\varphi_C(2i + 2)\varphi_D(2l + 1) \in E(H)$, which contradicts the fact that $\{\varphi_C(2i), \varphi_C(2i + 2)\}$ is a module of $H - \varphi_C(2i + 1)$. If $q$ is odd, then $\varphi_C(2i)\varphi_C(2i + 2)\varphi_D(2l + 1) \in E(H)$, which contradicts the fact that $\{\varphi_C(2l - 1), \varphi_C(2l + 1)\}$ is a module of $H - \varphi_C(2l)$. Consequently, we obtain $\varphi_D(2k) \in e$. It follows from Lemma 28 that there exist $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$ and $e \in V(C) = \{\varphi_C(2i), \varphi_C(2j + 1)\}$. Thus,

$$E(H[V(C) \cup \{\varphi_D(2k)\}]) \leq \{\varphi_C(2i)\varphi_C(2j + 1)\varphi_D(2k): \quad (22)$$

$$0 \leq i \leq j \leq w(C) - 1\}.$$

It follows from (21) and (22) that (19) holds. Therefore, the extension $\psi^{2k}_C$ of $\varphi_C$ is an isomorphism from $C_3(W_{v(C)+1})$ onto $H[V(C) \cup \{\varphi_D(2k)\}]$.

For the third assertion, suppose that $|V(H) \setminus V(C)| \geq 2$. Let $k \in \{0, \ldots, w(D)\}$. Set $Y = V(C) \cup \{\varphi_D(2k)\}$. We have $Y \not\subseteq V(H)$. By Theorem 9, $W_{v(C)+1}$ is prime. Hence, $C_3(W_{v(C)+1})$ is prime by Theorem 9. It follows from the second assertion above that $H[Y]$ is prime. By Proposition 3, there exist $v, w \in (V(H) \setminus Y) \cup Y$ such that $H - \{v, w\}$ is prime. Since $H$ is critical, we have $v \neq w$. Therefore, there exists $C' \in \mathcal{E}(\mathcal{P}(H)) \setminus \mathcal{E}_1(\mathcal{P}(H))$ such that $v, w \in C'$. Lastly, suppose for a contradiction that $V(C) \cap Y \neq \emptyset$. There exist $c \in V(C)$ and $u \in V(H) \setminus Y$ such that $\{c, u\}$ is a module of $H[Y \cup \{u\}]$. By the second assertion above, there exists $d \in V(C) \setminus \{c\}$ such that $cd\varphi_D(2k) \in E[H[Y]]$. Since $\{c, u\}$ is a module of $H[V(C) \cup \{u\}]$, we get $ud\varphi_D(2k) \in E[H[Y]]$, which contradicts Lemma 28. It follows that $Y \subseteq V(H) \setminus V(C)$. Thus, $v, w \notin V(C)$, so $C' \neq C$.

**Proof of Theorem 13.** Let $H$ be a non circular and critical and 3-hypergraph such that $v(H) \geq 5$. By Corollary 26, all the components of $\mathcal{P}(H)$ are paths. We associate with $H$ the hypergraph $\mathcal{H}$ defined on $\mathcal{E}(\mathcal{P}(H))$ as follows

1. given distinct $C, D \in \mathcal{E}(\mathcal{P}(H))$, $CD \in E(H)$ if $C \in \mathcal{E}_{even}(\mathcal{P}(H))$, $D \in \mathcal{E}_{odd}(\mathcal{P}(H))$, and there exists $e \in E(H)$ such that $|e \cap V(C)| = 2$ and $|e \cap V(D)| = 1$;
Remark 33. It follows from Propositions 29 and 30 that $H$ is an isomorphism from $C$ from the definition of $\Gamma$. Let $P$ be a 3-hypergraph. Consider $X \subseteq V(H)$ such that $H[X]$ is prime. Let $M$ be a module of $H$. We have $M \cap X = \emptyset$, $M \not\subseteq X$ or $M \cap X = \{y\}$, where $y \in X$. Moreover, the following assertions hold.

1. If $M \cap X = \emptyset$, then all the elements of $M$ belong to the same block of $\mathcal{P}(H\cdot X)$.
2. If $M \not\subseteq X$, then all the elements of $V(H) \setminus M$ belong to $\langle X \rangle_H$.
3. If $M \cap X = \{y\}$, where $y \in X$, then all the elements of $M \setminus \{y\}$ belong to $X_H(y)$.

In this section and the next one, we consider a graph $\Gamma$ satisfying (1) and (2). We consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathcal{C}(\Gamma)$ satisfying (3). We use the following notation.

Notation 32. For $W \subseteq V(\mathbb{H})$, set $\overline{W} = \bigcup_{C \in \mathcal{W}} V(C)$. Conversely, for $W \subseteq V(H_{(\Gamma,\mathbb{H})})$, set $W/\mathcal{C}(\Gamma) = \{C \in \mathcal{C}(\Gamma) : V(C) \cap W \neq \emptyset\}$.

Both next remarks are useful.

Remark 33. Let $C \in \mathcal{C}_{\text{odd}}(\Gamma) \setminus \mathcal{C}_1(\Gamma)$. First, suppose that $w(C) = 1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E((\Gamma \bullet \mathbb{H})[V(C)]) = \{V(C)\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime. Since $v(C) = 3$, the next three assertions are obvious

1. $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \bullet \mathbb{H})[V(C)] - \varphi_C(0)$;
2. $V(C) \setminus \{\varphi_C(1), \varphi_C(2)\}$ is a module of $(\Gamma \bullet \mathbb{H})[V(C)] - \varphi_C(2)$;
3. $\{\varphi_C(0), \varphi_C(2)\}$ is a module of $(\Gamma \bullet \mathbb{H})[V(C)] - \varphi_C(1)$.

Second, suppose that $w(C) \geq 2$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $\varphi_C$ is an isomorphism from $C_3(U_v(C))$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$. By Theorem 8, $U_v(C)$ is critical. By Theorem 3, $C_3(U_v(C))$ is critical and $\mathcal{P}(C_3(U_v(C))) = \mathcal{P}(U_v(C))$. Furthermore, we have $\mathcal{P}(U_v(C)) = P_v(C)$ by Proposition 21. Since $\varphi_C$ is an isomorphism from $C_3(U_v(C))$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$, $(\Gamma \bullet \mathbb{H})[V(H)]$ is critical and

$$\mathcal{P}((\Gamma \bullet \mathbb{H})[V(C)]) = (V(C), \{\varphi_C(i)\varphi_C(i + 1) : 0 \leq i < 2w(C)\}).$$
Lemma 35. Let $C \in \mathcal{C}_{\text{even}}(\Gamma)$ such that $w(C) \geq 1$. Suppose that there exists $e \in E(\mathbb{H})$ such that $C \subseteq e$. We have $e = CD$, where $D \in \mathcal{C}_{\text{odd}}(\Gamma)$. Let $k \in \{0, \ldots, w(D)\}$. First, suppose that $w(C) = 1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E((\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]) = \{V(C) \cup \{\varphi_D(2k)\}\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$ is prime.

Second, suppose that $w(C) \geq 2$. Let $\psi^k : \{0, \ldots, w(C)\} \rightarrow V(C) \cup \{\varphi_D(2k)\}$ satisfying $(\psi^k|_{\{0, \ldots, w(C)\}-1}) = \varphi_C$ and $\psi^k(v(C)) = \varphi_D(2k)$. By Remark 34, $\psi^k$ is an isomorphism from $C_3(W_{w(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$. By Theorem 6, $W_{w(C)+1}$ is critical. By Theorem 9, $C_3(W_{w(C)+1})$ is critical and $\mathcal{P}(C_3(W_{w(C)+1})) = \mathcal{P}(W_{w(C)+1})$. Furthermore, it follows from Proposition 22 that $\mathcal{P}(W_{w(C)+1}) = -2n = P_{w(C)}$ and $N_{\mathcal{P}(W_{w(C)+1})}(2n) = \emptyset$. Since $\psi^k$ is an isomorphism from $C_3(W_{w(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$ is critical and we have

$$\mathcal{P}((\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]) - \varphi_D(2k) = (V(C), \{\varphi_C(i)\varphi_C(i+1) : 0 \leq i < 2w(C) - 1\})$$

and $N_{\mathcal{P}((\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}])}(\varphi_D(2k)) = \emptyset$.

If $\mathbb{H}$ is disconnected, then $\Gamma \bullet \mathbb{H}$ is decomposable, whence the necessity of Assertion (C1) in Proposition 16. Indeed, we have

Lemma 35. If $C$ is a component of $\mathbb{H}$, then $\overline{V(C)}$ and $V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(C)}$ are modules of $\Gamma \bullet \mathbb{H}$.

Proof. For a contradiction, suppose that there exists $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \overline{V(C)} \neq \emptyset$ and $e \cap (V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(C)}) \neq \emptyset$. There exist $C \subseteq V(C)$ and $D \not\subseteq V(C)$ such that $e \cap [V(C) \setminus \partial D] \neq \emptyset$ and $e \cap [V(D) \setminus \partial C] \neq \emptyset$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that there exists $e \in E(\mathbb{H})$ such that $C, D \in e$, which contradicts the fact that $C$ is a component of $\mathbb{H}$. Consequently, each $e \in E(\Gamma \bullet \mathbb{H})$, we have $e \subseteq V(C)$ or $e \subseteq (V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(C)}) \neq \emptyset$. It follows that $\overline{V(C)}$ and $V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(C)}$ are modules of $\Gamma \bullet \mathbb{H}$. Each edge of $\mathbb{H}$ induces a prime subhypergraph of $\Gamma \bullet \mathbb{H}$. Precisely, we have

Lemma 36. For every $e \in E(\mathbb{H})$, $(\Gamma \bullet \mathbb{H})[\overline{e}]$ is prime.

Proof. Let $e \in E(\mathbb{H})$. First, suppose that $|e| = 2$. There exist $C \in \mathcal{C}_{\text{even}}(\Gamma)$ and $D \in \mathcal{C}_{\text{odd}}(\Gamma)$ such that $e = CD$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[[V(C) \cup V(D)]]$ is prime. If $w(D) = 0$, then $(\Gamma \bullet \mathbb{H})[[V(C) \cup V(D)]]$ is prime by Remark 34. Suppose that $w(D) \geq 1$. Set $X = V(D)$. By Remark 34, $(\Gamma \bullet \mathbb{H})[X]$ is prime. We have $V(C) \subseteq (X)$ (see Notation 2). Let $M$ be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ such that $|M| \geq 2$. We have to show that $M = X \cup V(C)$. By Lemma 34, we have $M \cap X = \emptyset$, $M \supseteq X$, or $M \cap X = \{y\}$, where $y \in X$. It follows from the third assertion of Lemma 34 that $|M \cap X| = 1$. Now, suppose that $M \cap X \neq \emptyset$. Hence, we have $M \subsetneq V(C)$. By Remark 34, $(\Gamma \bullet \mathbb{H})[[V(C) \cup \{\varphi_D(0)\}]]$ is prime. Since $|M| \geq 2$ and $M \subseteq V(C)$, we obtain $M = V(C) \cup \{\varphi_D(0)\}$, which contradicts $M \cap X = \emptyset$. Therefore, we have $M \supseteq X$. Let $i, j \in \{0, \ldots, w(C)-1\}$ with $i < j$. We
have $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(0) \in E(H)$. Since $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ and $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(1) \notin E(H)$, we obtain $\varphi_C(2i), \varphi_C(2j+1) \in M$. It follows that $M = X \cup V(C)$.

Second, suppose that $|e| = 3$. There exist $I, J, K \in \mathbb{E}_{odd}(\Gamma)$ such that $e = IJK$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]$ is prime. If $w(I) = w(J) = w(K) = 0$, then $[V(I) \cup V(J) \cup V(K)] = 3$, and hence $E((\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)])$ is prime. It follows that $(\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]$ is prime. Now, suppose that $w(I) > 0$, $w(J) > 0$, or $w(K) > 0$. For instance, assume that $w(I) > 0$. By Remark 33, $(\Gamma \bullet \mathbb{H})[V(I)]$ is prime. Set $X = V(I)$. We have $V(I) \cup V(J) \cup V(K) \subseteq (X)_{\Gamma \bullet \mathbb{H}}$. Let $M$ be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$ such that $|M| \geq 2$. We have to show that $M = X \cup V(J) \cup V(K)$. It follows from Lemma 31 that $M \cap X = \emptyset$ or $X \subseteq M$.

For a contradiction, suppose that $M \cap X = \emptyset$. For $0 \leq j \leq w(J)$ and $0 \leq k \leq w(K)$, we have $\varphi_I(0)\varphi_J(2j)\varphi_K(2k) \in E(\Gamma \bullet \mathbb{H})$. It follows that $M \cap \{\varphi_J(2j) : 0 \leq j \leq w(J)\} = \emptyset$ or $M \cap \{\varphi_K(2k) : 0 \leq k \leq w(K)\} = \emptyset$. For instance, assume that

$$M \cap \{\varphi_J(2j) : 0 \leq j \leq w(J)\} = \emptyset. \quad (23)$$

We distinguish the following two cases.

1. Suppose that $M \cap \{\varphi_J(2j+1) : 0 \leq j \leq w(J) - 1\} = \emptyset$. Hence, there exists $j \in \{0, \ldots, w(J) - 1\}$ such that $\varphi_J(2j+1) \in M$. We have $\varphi_J(2j)\varphi_J(2j+1)\varphi_J(2j+2) \in E(H)$. By (23), $\varphi_J(2j), \varphi_J(2j+2) \notin M$. Since $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_J(2j)\varphi_J(2j+2) \in E(H)$ for every $x \in M$. For each $v \in V(K)$, we have $\varphi_J(2j)v\varphi_J(2j+2) \notin E(H)$. It follows that $M \cap V(K) = \emptyset$. Therefore, $M \subseteq V(J)$, and hence $M$ is a module of $(\Gamma \bullet \mathbb{H})[V(J)]$. Since $|M| \geq 2$, $w(J) \geq 1$. By Remark 33, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Therefore, $M = V(J)$, which contradicts (23).

2. Suppose that $M \cap \{\varphi_J(2j+1) : 0 \leq j \leq w(J) - 1\} = \emptyset$. By (23), $M \cap V(J) = \emptyset$, and hence $M \subseteq V(K)$. As previously for $J$, we obtain $w(k) \geq 1$ and $M = V(K)$, which is impossible because $\varphi_I(0)\varphi_J(0)\varphi_K(0) \in E(H)$ and $\varphi_I(0)\varphi_J(0)\varphi_K(1) \notin E(H)$.

It follows that $X \subseteq M$. Recall that $\varphi_I(0)\varphi_J(2j)\varphi_K(2k) \in E(\Gamma \bullet \mathbb{H})$ for $0 \leq j \leq w(J)$ and $0 \leq k \leq w(K)$. Let $j \in \{0, \ldots, w(J)\}$ and $k \in \{0, \ldots, w(J)\}$. We have $\varphi_I(0) \notin M$ because $X \subseteq M$. Since $\varphi_I(1)\varphi_J(2j)\varphi_K(2k) \notin E(\Gamma \bullet \mathbb{H})$ and $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_I(2j), \varphi_K(2k) \in M$. It follows that $X \cup \{\varphi_J(2j) : 0 \leq j \leq w(J)\} \cup \{\varphi_K(2k) : 0 \leq k \leq w(K)\} \subseteq M$. Clearly, $V(J) \subseteq M$ when $w(J) = 0$. Suppose that $w(J) \geq 1$. By Remark 33, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Since $M \cap V(J)$ is a module of $(\Gamma \bullet \mathbb{H})[V(J)]$ such that $\{\varphi_J(2j) : 0 \leq j \leq w(J)\} \subseteq M \cap V(J)$, we obtain $V(J) \subseteq M$. Similarly, $V(K) \subseteq M$. Consequently, $M = X \cup V(J) \cup V(K)$.

In the next four results, we compare the module of $\mathbb{H}$ with those of $\Gamma \bullet \mathbb{H}$.

**Lemma 37.** Suppose that $\mathbb{H}$ is connected. For each nontrivial module $M$ of $\Gamma \bullet \mathbb{H}$, we have $M = M/\mathbb{E}(\Gamma)$, $|M/\mathbb{E}(\Gamma)| \geq 2$, and $M/\mathbb{E}(\Gamma)$ is a module of $\mathbb{H}$. 

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Proof. First, we prove that \( V(C) \subseteq M \) for every \( C \in \mathcal{M}(\Gamma) \). To begin, suppose that \( C \in \mathcal{C}_{\text{odd}}(\Gamma) \). If \( w(C) = 0 \), then we clearly have \( V(C) \subseteq M \). Hence, suppose that \( w(C) \geq 1 \). By Remark 35 (\( \Gamma \cdot \mathbb{H} \))\([V(C)] \) is prime. For a contradiction, suppose that \( |V(C) \cap M| = 2 \). Denote by \( c \) the unique element of \( V(C) \cap M \). Since \( \Gamma \cdot \mathbb{H} \) is prime, there exist distinct \( d, d' \in V(C) \setminus M \) such that \( cdd' \in E(\Gamma \cdot \mathbb{H}) \). Since \( |M| \geq 2 \), there exists \( v \in M \setminus V(C) \). Since \( M \) is a module of \( \Gamma \cdot \mathbb{H} \), we obtain \( vdd' \in E(H_{\Gamma, \mathbb{H}}) \), which contradicts the definition of \( \Gamma \cdot \mathbb{H} \). It follows that \( |V(C) \cap M| = 2 \). Since \( \Gamma \cdot \mathbb{H} \) is a module of \( \Gamma \cdot \mathbb{H} \), we get \( v \in M \setminus \{ V(C) \cup \{ \varphi_D(0) \} \} \). Since \( M \) is a module of \( \Gamma \cdot \mathbb{H} \), we obtain \( v \varphi_D(0) \in E(\Gamma \cdot \mathbb{H}) \), which contradicts the definition of \( \Gamma \cdot \mathbb{H} \). It follows that \( |V(C) \cap M| = 2 \). Since \( \Gamma \cdot \mathbb{H} \) is a module of \( \Gamma \cdot \mathbb{H} \), we obtain \( V(C) \cap M \neq \emptyset \). Since \( M \) is a module of \( \Gamma \cdot \mathbb{H} \), we get \( \emptyset \neq V(C) \cap M \neq \emptyset \). Thus \( e \in (M \cap V(C)) \setminus \emptyset \), so \( |M \cap V(C)| \geq 2 \).

Second, we prove that \( M \cap V(C) \) is a module of \( \mathbb{H} \). Let \( e \in E(\mathbb{H}) \) such that \( e \cap (M \cap V(C)) \neq \emptyset \) and \( e \cap (M \cap V(C)) \neq \emptyset \). We distinguish the following two cases.

- Suppose that \( |e| = 2 \). There exist \( C \in \mathcal{C}_{\text{even}}(\Gamma) \) and \( D \in \mathcal{C}_{\text{odd}}(\Gamma) \) such that \( \varepsilon = CD \). Thus, there exist distinct \( c, c' \in V(C) \) and \( d, d' \in V(D) \) such that \( cc'dd' \in E(\Gamma \cdot \mathbb{H}) \). For a contradiction, suppose that \( C \in M \cap V(C) \). Since \( e \cap \varepsilon \), \( (M \cap V(C)) \neq \emptyset \), we have \( D \notin M \cap V(C) \). By the first point above, \( V(C) \subseteq M \) and \( V(D) \cap M = \emptyset \). We obtain \( cc'dd' \in E(\Gamma \cdot \mathbb{H}) \), with \( c, c' \in M \) and \( d, d' \in M \), which contradicts the fact that \( M \) is a module of \( \Gamma \cdot \mathbb{H} \). It follows that \( D \in M \cap V(C) \) and \( C \notin M \cap V(C) \). By the first point above, \( V(D) \subseteq M \) and \( V(C) \cap M = \emptyset \). We obtain \( cc'dd' \in E(\Gamma \cdot \mathbb{H}) \), with \( c, c' \notin M \) and \( d, d' \in M \). Since \( M \) is a module of \( \Gamma \cdot \mathbb{H} \), we obtain \( cc'x \in E(H_{\Gamma, \mathbb{H}}) \) for every \( x \in M \).

Consider \( d' \in M \cap V(C) \). By the first point above, \( V(D') \subseteq M \). Let \( d' \in V(D') \). By \( 24 \), \( cc'dd' \in E(\Gamma \cdot \mathbb{H}) \). It follows from the definition of \( \Gamma \cdot \mathbb{H} \) that \( CD' \in E(\mathbb{H}) \).

- Suppose that \( |e| = 3 \). There exist distinct \( I, J, K \in \mathcal{C}_{\text{odd}}(\Gamma) \) such that \( e = IJK \). Moreover, we can assume that \( I \in M \cap V(C) \) and \( K \notin M \cap V(C) \). By the first point above, \( V(I) \subseteq M \) and \( V(K) \cap M = \emptyset \). By definition of \( \Gamma \cdot \mathbb{H} \), we obtain \( \varphi_I(0) \varphi_J(0) \varphi_K(0) \in E(\Gamma \cdot \mathbb{H}) \). Since \( M \) is a module of \( \Gamma \cdot \mathbb{H} \), we have \( \varphi_J(0) \notin M \). It follows from the first point above that

\[ cc'x \in E(H_{\Gamma, \mathbb{H}}) \]
\( V(J) \cap M = \emptyset \). Thus, \( \varphi_J(0) \varphi_J(0) \varphi_K(0) \in E(\Gamma \bullet \mathbb{H}) \), with \( \varphi_J(0) \in M \) and \( \varphi_J(0), \varphi_K(0) \not\in M \). Since \( M \) is a module of \( \Gamma \bullet \mathbb{H} \), we obtain

\[
x \varphi_J(0) \varphi_K(0) \in E(\Gamma \bullet \mathbb{H}) \text{ for every } x \in M.
\] (25)

Consider any \( L \in M/\mathcal{C}(\Gamma) \). By the first point above, \( V(L) \subseteq M \) Let \( d \in V(L) \). By \([23] \), \( d \varphi_J(0) \varphi_K(0) \in E(\Gamma \bullet \mathbb{H}) \). Since \( J, K, L \) are distinct element of \( \mathcal{C}(\Gamma) \), it follows from the definition of \( \Gamma \bullet \mathbb{H} \) that \( JKL \in E(\mathbb{H}) \).

Consequently, \( M/\mathcal{C}(\Gamma) \) is a module of \( \mathbb{H} \).

**Lemma 38.** Suppose that \( \mathbb{H} \) is connected. For a nontrivial module \( M \) of \( \mathbb{H} \), we have \( M \subseteq \mathcal{C}_{\text{even}}(\Gamma) \) or \( M \subseteq \mathcal{C}_{\text{odd}}(\Gamma) \).

*Proof.* Suppose that \( M \setminus \mathcal{C}_{\text{odd}}(\Gamma) \neq \emptyset \). We have to show that \( M \subseteq \mathcal{C}_{\text{even}}(\Gamma) \). Since \( M \setminus \mathcal{C}_{\text{odd}}(\Gamma) \neq \emptyset \), there exists \( C \in M \cap \mathcal{C}_{\text{even}}(\Gamma) \). Since \( \mathbb{H} \) is connected and \( M \subseteq V(\mathbb{H}) \), there exists \( \varepsilon \in E(\mathbb{H}) \) such that \( \varepsilon \cap M \neq \emptyset \) and \( \varepsilon \setminus M \neq \emptyset \). Furthermore, since \( M \) is a module of \( \mathbb{H} \), we have \( |\varepsilon \cap M| = 1 \). It follows that \( \{\varepsilon \cap M\} \cup \{C\} \subseteq E(\mathbb{H}) \). Since \( C \in M \cap \mathcal{C}_{\text{even}}(\Gamma) \), we obtain \( |\varepsilon \cap M| = 1 \) and \( \varepsilon \cap M \subseteq \mathcal{C}_{\text{odd}}(\Gamma) \). Lastly, consider \( D \in M \). Since \( M \) is a module of \( \mathbb{H} \), we have \( \{\varepsilon \cap M\} \cup \{D\} \subseteq E(\mathbb{H}) \). Since \( |\varepsilon \cap M| \leq 2 \) and \( \varepsilon \cap M \subseteq \mathcal{C}_{\text{even}}(\Gamma) \), we obtain \( D \in M \cap \mathcal{C}_{\text{even}}(\Gamma) \). Consequently, \( M \subseteq \mathcal{C}_{\text{even}}(\Gamma) \).

The next result follows from Lemmas [37] and [38].

**Corollary 39.** Suppose that \( \mathbb{H} \) is connected. For each nontrivial module \( M \) of \( \Gamma \bullet \mathbb{H} \), we have \( M \subseteq V_1(\Gamma) \).

*Proof.* It follows from Lemma [37] that \( M/\mathcal{C}(\Gamma) \) is a nontrivial module of \( M \). By Lemma [38], \( M/\mathcal{C}(\Gamma) \subseteq \mathcal{C}_{\text{even}}(\Gamma) \) or \( M/\mathcal{C}(\Gamma) \subseteq \mathcal{C}_{\text{odd}}(\Gamma) \).

For a contradiction, suppose that \( M/\mathcal{C}(\Gamma) \subseteq \mathcal{C}_{\text{even}}(\Gamma) \). Since \( \mathbb{H} \) is connected, there exists \( \varepsilon \in E(\mathbb{H}) \) such that \( \varepsilon \cap (M/\mathcal{C}(\Gamma)) \neq \emptyset \) and \( \varepsilon \setminus (M/\mathcal{C}(\Gamma)) \neq \emptyset \). Since \( M/\mathcal{C}(\Gamma) \subseteq \mathcal{C}_{\text{even}}(\Gamma) \), there exist \( C \in M/\mathcal{C}(\Gamma) \) and \( D \in \mathcal{C}_{\text{odd}}(\Gamma) \) such that \( \varepsilon \cap (M/\mathcal{C}(\Gamma)) = \emptyset \). By Lemma [37], \( M \subseteq M/\mathcal{C}(\Gamma) \). Therefore, we obtain \( V(C) \subseteq M \) and \( V(D) \cap M = \emptyset \). It follows that \( V(C) \) is a nontrivial module of \( \Gamma \bullet \mathbb{H} \). This contradicts Lemma [38]. Consequently, \( M \subseteq \mathcal{C}_{\text{even}}(\Gamma) \).

For a contradiction, suppose that \( M \setminus V_1(\Gamma) \neq \emptyset \). There exists \( D \in (M/\mathcal{C}(\Gamma)) \cap (\mathcal{C}_{\text{odd}}(\Gamma) \setminus \mathcal{C}_1(\Gamma)) \). Since \( \mathbb{H} \) is connected, there exists \( \varepsilon \in E(\mathbb{H}) \) such that \( \varepsilon \cap (M/\mathcal{C}(\Gamma)) \neq \emptyset \) and \( \varepsilon \setminus (M/\mathcal{C}(\Gamma)) \neq \emptyset \). Since \( M/\mathcal{C}(\Gamma) \) is a module of \( M \), there exists \( C \in M/\mathcal{C}(\Gamma) \) such that \( \varepsilon \cap (M/\mathcal{C}(\Gamma)) = \{C\} \) and \( \varepsilon \cap \{C\} \cup \{D\} \subseteq E(\mathbb{H}) \). By Lemma [37], \( V(D) \subseteq M \). We obtain that \( V(D) \) is a nontrivial module of \( \Gamma \bullet \mathbb{H} \). This contradicts Lemma [36]. Consequently, \( M \subseteq V_1(\Gamma) \).

**Lemma 40.** Given a module \( M \) of \( \mathbb{H} \), if \( M \subseteq \mathcal{C}_1(\Gamma) \), then \( \overline{M} \) is a module of \( \Gamma \bullet \mathbb{H} \).
Proof. Let $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \mathbb{M} \neq \emptyset$ and $e \setminus \mathbb{M} \neq \emptyset$. There exist $c \in e \cap \mathbb{M}$ and $x \in e \setminus \mathbb{M}$. We have $\{c\} \in \mathcal{C}(\Gamma)$. Consider $C_x \in \mathcal{C}(\Gamma)$ such that $x \in C_x$. For a contradiction, suppose that $e = cdx$, where $d \in \mathbb{M}$. We have $\{d\} \in \mathcal{C}(\Gamma)$. Since \( cdx \in E(\Gamma \bullet \mathbb{H}) \), we obtain $C_x C_d \in E(\mathbb{H})$, which contradicts the fact that $\mathbb{M}$ is a module of $\mathbb{H}$. It follows that $e = cxy$, where $y \in V(\Gamma \bullet \mathbb{H}) \setminus \mathbb{M}$. Consider $C_y \in \mathcal{C}(\Gamma)$ such that $y \in C_y$. Given $d \in \mathbb{M}$, we have to verify that $dxy \in E(\Gamma \bullet \mathbb{H})$.

We have $\{d\} \in \mathcal{C}(\Gamma)$. We distinguish the following two cases.

- Suppose that $C_x = C_y$. Since $cxy \in E(\Gamma \bullet \mathbb{H})$, we obtain $C_x \in \mathcal{C}_{even}(\Gamma)$, $\{x, y\} = \{\varphi_{C_x}(1)\varphi_{C_x}(2j)\}$, and $\{x, y\} \in \mathcal{C}(\Gamma)$. We obtain $\varphi_{C_x}(1) \varphi_{C_x}(2j) \in E(\mathbb{H})$.

- Suppose that $C_x \neq C_y$. Consider $dxy \in E(\mathbb{H})$. Since $C_x \neq C_y$, $\{d\} \in \mathcal{C}(\Gamma)$, we obtain $dxy \in E(\mathbb{H})$.

Proof of Proposition 14. To begin, suppose that $\Gamma \bullet \mathbb{H}$ is decomposable. If $\mathbb{H}$ is disconnected, then Assertion (C1) does not hold. Hence, suppose that $\mathbb{H}$ is connected. Consider a nontrivial module $M$ of $\Gamma \bullet \mathbb{H}$. It follows from Lemma 37 that $M/\mathcal{C}(\Gamma)$ is a nontrivial module of $\mathbb{H}$. Furthermore, $M \in V_1(\Gamma)$ by Corollary 39. Thus, $M/\mathcal{C}(\Gamma) \in \mathcal{C}_1(\Gamma)$. Consequently, Assertion (C2) does not hold.

Conversely, if $\mathbb{H}$ is disconnected, then it follows from Lemma 35 that $\Gamma \bullet \mathbb{H}$ is decomposable. Hence, suppose that $\mathbb{H}$ is connected. Moreover, suppose that there exists a nontrivial module $\mathbb{M}$ of $\mathbb{H}$ such that $\mathbb{M} \in \mathcal{C}_1(\Gamma)$. It follows from Lemma 40 that $\mathbb{M}$ is a nontrivial module of $\Gamma \bullet \mathbb{H}$. Hence, $\Gamma \bullet \mathbb{H}$ is decomposable.

\section{Proof of Theorem 15}

In the next remark, we describe a simple way to obtain automorphisms of $\Gamma \bullet \mathbb{H}$.

Remark 41. For every $C \in \mathcal{C}(\Gamma)$, consider the function

$$F_C : \begin{cases}
V(\Gamma \bullet \mathbb{H}) \rightarrow V(\Gamma \bullet \mathbb{H}) \\
\varphi_C(m) & (0 \leq m \leq v(C) - 1) \\
v \not\in V(C) & \varphi_C(v(C) - 1 - m) \\
v & v.
\end{cases}$$

Clearly, $F_C$ is an automorphism of $\Gamma$. It is easy to verify that $F_C$ is an automorphism of $\Gamma \bullet \mathbb{H}$. Moreover, consider a permutation $F$ of $V(\Gamma \bullet \mathbb{H})$ such that $F(v) = v$ for each $v \in V_1(\Gamma)$. If for every $C \in \mathcal{C}(\Gamma)$, $F|_{V(C)} = (F_C)|_{V(C)}$ or $\text{Id}_{V(C)}$, then $F$ is an automorphism of $\Gamma \bullet \mathbb{H}$.

The next three results are useful to study the criticality of $\Gamma \bullet \mathbb{H}$. They allow us to prove Theorem 15.
Lemma 42. Given $C \in \mathcal{C}_{\text{odd}}(\Gamma) \setminus \mathcal{C}_1(\Gamma)$, the following assertions hold

1. $V(\Gamma \star \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(0)$;

2. $V(\Gamma \star \mathbb{H}) \setminus \{\varphi_C(2w(C) - 2), \varphi_C(2w(C) - 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2w(C) - 1)$;

3. if $w(C) \geq 1$, then for $m \in \{1, \ldots, 2w(C) - 1\}$, we have $\{\varphi_C(m - 1), \varphi_C(m + 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(m)$.

Proof. We begin with the first assertion. It follows from Remark 43 that $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(0)$. For every $e \in E(\Gamma \star \mathbb{H})$, if $\varphi_C(1) \in e$, then $e \subseteq V(C)$ and $\varphi_C(0) \notin e$. Therefore, $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(0)$.

For the second assertion, recall that the function $F_C$ is an automorphism of $\Gamma \star \mathbb{H}$ (see Remark 11). It follows from the first assertion above that $F_C(V(C) \setminus \{\varphi_C(0), \varphi_C(1)\})$ is a module of $(\Gamma \star \mathbb{H}) - F_C(\varphi_C(0))$, that is, $V(C) \setminus \{\varphi_C(2w(C) - 1), \varphi_C(2w(C))\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2w(C))$.

For the third assertion, suppose that $w(C) \geq 1$. Let $i \in \{0, \ldots, w(C) - 1\}$. It follows from Remark 43 that $\{\varphi_C(2i), \varphi_C(2i + 2)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2i)$. Let $e \in E(\Gamma \star \mathbb{H})$ such that $e \cap \{\varphi_C(2i), \varphi_C(2i + 2)\} \neq \emptyset$ and $e \setminus V(C) \neq \emptyset$. By definition of $\Gamma \star \mathbb{H}$, $|e \cap \{\varphi_C(2i), \varphi_C(2i + 2)\}| = 1$, $e \setminus V(C) = 2$, and $(e \setminus V(C)) \cup \{\varphi_C(2i), \varphi_C(2i + 2)\} \in E(H_{\Gamma \star \mathbb{H}})$. Thus, $\{\varphi_C(2i), \varphi_C(2i + 2)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2i)$. Lastly, suppose that $w(C) \geq 2$, and consider $i \in \{1, \ldots, w(C) - 1\}$. It follows from Remark 43 that $\{\varphi_C(2i - 1), \varphi_C(2i + 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2i)$. For every $e \in E(\Gamma \star \mathbb{H})$, if $e \cap \{\varphi_C(2i - 1), \varphi_C(2i + 1)\} \neq \emptyset$, then $e \subseteq V(C)$. Therefore, $\{\varphi_C(2i - 1), \varphi_C(2i + 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2i)$.

Lemma 43. Given $C \in \mathcal{C}_{\text{even}}(\Gamma)$, the following assertions hold

1. $V(\Gamma \star \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(0)$;

2. $V(\Gamma \star \mathbb{H}) \setminus \{\varphi_C(2w(C) - 2), \varphi_C(2w(C) - 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2w(C) - 1)$;

3. if $w(C) \geq 2$, then for $m \in \{1, \ldots, 2w(C) - 2\}$, we have $\{\varphi_C(m - 1), \varphi_C(m + 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(m)$.

Proof. For the first assertion, consider $e \in E(\Gamma \star \mathbb{H})$ such that $\varphi_C(1) \in e$. By definition of $\Gamma \star \mathbb{H}$, there exists $D \in \mathcal{C}_{\text{odd}}(\Gamma)$ such that $e = \varphi_C(0)\varphi_C(1)\varphi_D(2k)$, where $0 \leq k \leq w(D)$. Consequently, there does not exist $e \in E((\Gamma \star \mathbb{H}) - \varphi_C(0))$ such that $\varphi_C(1) \in e$. It follows that $V(\Gamma \star \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(0)$.

As in the proof of the second assertion of Lemma 42, the second assertion is deduced from the first one by using Remark 11.

For the third assertion, suppose that $w(C) \geq 2$. Consider $i \in \{1, \ldots, w(C) - 1\}$. We prove that $\{\varphi_C(2i - 1), \varphi_C(2i + 1)\}$ is a module of $(\Gamma \star \mathbb{H}) - \varphi_C(2i)$. Let $e \in E((\Gamma \star \mathbb{H}) - \varphi_C(2i))$ such that $e \cap \{\varphi_C(2i - 1), \varphi_C(2i + 1)\} \neq \emptyset$ and
exists $H(C^3)$, consider \( \Gamma \). By definition of \( \Gamma \), there exist $D \in \mathcal{C}_{\text{odd}}(\Gamma)$ and $k \in \{0, \ldots, w(D)\}$ such that

\[
\begin{cases}
  e = \varphi_C(2j)\varphi_C(2i-1)\varphi_D(2k), & \text{where } j \in \{0, \ldots, i-1\} \\
  \text{or} & \\
  e = \varphi_C(2j)\varphi_C(2i+1)\varphi_D(2k), & \text{where } j \in \{0, \ldots, i\}.
\end{cases}
\]

In the second instance, we have $j \in \{0, \ldots, i-1\}$ because $\varphi_C(2i) \notin e$. By definition of $\Gamma \cdot \mathbb{H}$, we have $\varphi_C(2j)\varphi_C(2i-1)\varphi_D(2k)$, $\varphi_C(2j)\varphi_C(2i+1)\varphi_D(2k) \in E((\Gamma \cdot \mathbb{H}) - \varphi_C(2i))$. Thus, \( \{\varphi_C(2i-1), \varphi_C(2i+1)\} \) is a module of $(\Gamma \cdot \mathbb{H}) - \varphi_C(2i)$. Lastly, consider $i \in \{0, \ldots, w(C) - 2\}$. We prove that $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $(\Gamma \cdot \mathbb{H}) - \varphi_C(2i+1)$. Let $e \in E((\Gamma \cdot \mathbb{H}) - \varphi_C(2i+1))$ such that $e \cap \{\varphi_C(2i), \varphi_C(2i+2)\} \neq \emptyset$ and $e \setminus \{\varphi_C(2i), \varphi_C(2i+2)\} \neq \emptyset$. By definition of $\Gamma \cdot \mathbb{H}$, there exist $D \in \mathcal{C}_{\text{odd}}(\Gamma)$ and $k \in \{0, \ldots, w(D)\}$ such that

\[
\begin{cases}
  e = \varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k), & \text{where } j \in \{i, \ldots, w(C) - 1\} \\
  \text{or} & \\
  e = \varphi_C(2i+2)\varphi_C(2j+1)\varphi_D(2k), & \text{where } j \in \{i+1, \ldots, w(C) - 1\}.
\end{cases}
\]

In the first instance, we have $j \in \{i+1, \ldots, w(C) - 1\}$ because $\varphi_C(2i+1) \notin e$. By definition of $\Gamma \cdot \mathbb{H}$, we have $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k), \varphi_C(2i+2)\varphi_C(2j+1)\varphi_D(2k) \in E((\Gamma \cdot \mathbb{H}) - \varphi_C(2i+1))$. Thus, $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $(\Gamma \cdot \mathbb{H}) - \varphi_C(2i+1)$.

The next result is an immediate consequence of Lemmas 12 and 13.

**Corollary 44.** Let $C \in \mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma)$. For every $e \in V(C)$, we have $(\Gamma \cdot \mathbb{H}) - e$ is decomposable. Moreover, we have

1. $N_{\mathcal{P}(\Gamma \cdot \mathbb{H})}(\varphi_C(0)) \subseteq \{\varphi_C(1)\}$;
2. $N_{\mathcal{P}(\Gamma \cdot \mathbb{H})}(\varphi_C(v(C)-1)) \subseteq \{\varphi_C(v(C)-2)\}$;
3. if $v(C) \geq 3$, then for every $m \in \{1, \ldots, v(C)-2\}$, we have $N_{\mathcal{P}(\Gamma \cdot \mathbb{H})}(\varphi_C(m)) \subseteq \{\varphi_C(m-1), \varphi_C(m+1)\}$.

**Proof of Theorem 12.** To begin, suppose that $\Gamma \cdot \mathbb{H}$ is critical. In particular, $\Gamma \cdot \mathbb{H}$ is prime. By Proposition 10, Assertions (C1) and (C2) hold. For Assertion (C3), consider $v \in V_1(\Gamma)$. Since $\Gamma \cdot \mathbb{H}$ is critical, $(\Gamma \cdot \mathbb{H}) - v$ is decomposable. Furthermore, since $v \in V_1(\Gamma)$, we have $\{v\} \in \mathcal{C}_1(\Gamma)$. It follows that $(\Gamma \cdot \mathbb{H}) - v = (\Gamma - v) \cdot (\mathbb{H} - \{v\})$. By Proposition 10 applied to $(\Gamma \cdot \mathbb{H}) - v$, $(\mathbb{H} - \{v\})$ is disconnected or $(\mathbb{H} - \{v\})$ is connected, and there exists a nontrivial module $M_{\{v\}}$ of $(\mathbb{H} - \{v\})$ such that $M_{\{v\}} \subseteq \mathcal{C}_1(\Gamma - v)$. Since $\mathcal{C}_1(\Gamma - v) = \mathcal{C}_1(\Gamma) \setminus \{\{v\}\}$, $M_{\{v\}} \subseteq \mathcal{C}_1(\Gamma) \setminus \{\{v\}\}$.

Conversely, suppose that Assertions (C1), (C2) and (C3) hold. Since Assertions (C1) and (C2) hold, it follows from Proposition 10 that $\Gamma \cdot \mathbb{H}$ is prime. Furthermore, it follows from Assertion (C3) and Proposition 10 that $(\Gamma \cdot \mathbb{H}) - v$ is decomposable for each $v \in V_1(\Gamma)$. Lastly, consider $v \in V(\Gamma) \setminus V_1(\Gamma)$. There exists $C \in \mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma)$ such that $v \in V(C)$. By Corollary 44, $(\Gamma \cdot \mathbb{H}) - v$ is decomposable. Consequently, $\Gamma \cdot \mathbb{H}$ is critical.
6 An improvement of Theorem 15

The purpose of this section is to demonstrate the following result.

**Theorem 45.** Suppose that $v(\Gamma \bullet \mathbb{H}) \geq 5$. The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is critical and $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$ if and only if the following six assertions hold

\begin{enumerate}[(C1)]
\item $\mathbb{H}$ is connected;
\item for every nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$;
\item for each $v \in V_1(\Gamma)$, if $\mathbb{H} \setminus \{v\}$ is connected, then $\mathbb{H} \setminus \{v\}$ admits a nontrivial module $\mathbb{M}_{\{v\}}$ such that $\mathbb{M}_{\{v\}} \subseteq \mathcal{C}_1(\Gamma) \setminus \{\{v\}\}$;
\item for each $C \in \mathcal{C}_{\text{even}}(\Gamma)$ such that $v(C) = 2$, we have

\begin{itemize}
\item $\mathbb{H} - C$ is connected,
\item for every nontrivial module $\mathbb{M}_C$ of $\mathbb{H} - C$, we have $\mathbb{M}_C \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$;
\end{itemize}
\item for each $C \in \mathcal{C}_{\text{odd}}(\Gamma)$ such that $v(C) = 3$, we have

\begin{itemize}
\item $\mathbb{H}_C$ is connected, where $\mathbb{H}_C$ is obtained from $\mathbb{H}$ by replacing $C$ by $\{\varphi_C(0)\}$,
\item for every nontrivial module $\mathbb{M}_C$ of $\mathbb{H}_C$, we have $\mathbb{M}_C \setminus (\mathcal{C}_1(\Gamma) \cup \{\varphi_C(0)\}) \neq \emptyset$;
\end{itemize}
\item for each $C \in \mathcal{C}_1(\Gamma)$, if there exists $D \in \mathcal{C}_1(\Gamma) \setminus \{C\}$ such that $D$ is an isolated vertex of $\mathbb{H} - C$ and $\mathbb{H} \setminus \{C,D\}$ is connected, then $\mathbb{H} \setminus \{C,D\}$ admits a nontrivial module $\mathbb{M}$ such that $\mathbb{M} \subseteq \mathcal{C}_1(\Gamma) \setminus \{C,D\}$.
\end{enumerate}

We use the next four results to prove Theorem 45.

**Proposition 46.** Suppose that $v(\Gamma \bullet \mathbb{H}) \geq 5$. If the 3-hypergraph $\Gamma \bullet \mathbb{H}$ is critical, then for each $C \in \mathcal{C}(\Gamma)$ such that $v(C) \geq 4$, we have

\[
\begin{aligned}
\mathcal{P}(\Gamma \bullet \mathbb{H})[V(C)] &= C, \\
\text{and for each } c \in V(C), \quad &N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(c) \subseteq V(C).
\end{aligned}
\]  

(26)

**Proof.** By Corollary 44 for each $C \in \mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma)$ and for every $c \in V(C)$, we have $N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(c) \subseteq V(C)$. Now, consider $C \in \mathcal{C}(\Gamma)$ such that $v(C) \geq 4$. For instance, suppose that $C \in \mathcal{C}_{\text{odd}}(\Gamma)$. We have $w(C) \geq 2$. Consider $i \in \{1, \ldots, w(C) - 1\}$. We verify that

\[
N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(\varphi_C(2i)) = \{\varphi_C(2i - 1), \varphi_C(2i + 1)\}.
\]  

(27)

We show that $(\Gamma \bullet \mathbb{H}) - \{\varphi_C(2i - 1), \varphi_C(2i)\}$ is prime. To define $\Gamma \bullet \mathbb{H}$, we consider an isomorphism from $P_{2w(C) + 1}$ onto $C$. It follows from the definition
from Remark 41 that $\varphi_N$ is an isomorphism from $C_3(U_{2w(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$. The bijection

$$\psi: \{0, \ldots, 2w(C) - 2\} \rightarrow \{0, \ldots, 2w(C)\} \setminus \{2i - 1, 2i\}$$

$$0 \leq j \leq 2i - 2 \quad \rightarrow \quad j$$

$$2i - 1 \leq j \leq 2w(C) - 1 \quad \rightarrow \quad j + 2,$$

is an isomorphism from $C_3(U_{2w(C)+1})$ onto $C_3(U_{2w(C)+1}) \setminus \{2i - 1, 2i\}$. Therefore, $(\langle \varphi_C \rangle)_{\{0, \ldots, 2w(C) + 1\} \setminus \{i - 1, i\}} \circ \psi$ is an isomorphism from $C_3(U_{2w(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C) \setminus \{\varphi_C(2i - 1), \varphi_C(2i)\}]$. Observe that $\psi$ is also an isomorphism from $P_{2w(C)+1}$ onto the path $C^{-}$ defined on $V(C^{-}) = \{0, \ldots, 2w(C) + 1\} \setminus \{i - 1, i\}$ by $E(C^{-}) = (E(C) \setminus \{\varphi_C(2i - 2)\} \cup \{\varphi_C(2i), \varphi_C(2i + 1)\}) \cup \{\varphi_C(2i - 2), \varphi_C(2i + 1)\}$. Now, consider the graph $\Gamma'$ obtained from $\Gamma$ by replacing the path $C$ by $C^{-}$. Moreover, consider the hypergraph $\mathbb{H}'$ obtained from $\mathbb{H}$ by renaming the vertex $C$ by $C^{-}$. Hence, the bijection

$$\theta: V(\mathbb{H}) \rightarrow (V(\mathbb{H}) \setminus \{C\}) \cup \{C^{-}\}$$

$$D \neq C \rightarrow D$$

$$C \rightarrow C^{-},$$

is an isomorphism from $\mathbb{H}$ onto $\mathbb{H}'$. Since $\psi$ is increasing and preserves the parity, we obtain $(\Gamma \bullet \mathbb{H}') - \{\varphi_C(2i - 1), \varphi_C(2i)\} = (\Gamma \bullet \mathbb{H}').$ Since $\mathbb{H}$ is connected, $\mathbb{H}'$ is as well. Furthermore, consider any nontrivial module $M'$ of $\mathbb{H}'$. We obtain that $\theta^{-1}(M')$ is a nontrivial module of $\mathbb{H}$. Hence, $\Gamma \bullet \mathbb{H}$ is critical, it is prime. By Assertion (C2) of Proposition 16, $\Gamma \bullet \mathbb{H}$ is an isomorphism from $\mathbb{H}$ onto $\mathbb{H}'$. Since $\mathbb{H}$ is connected, $\mathbb{H}'$ is as well. Furthermore, consider any nontrivial module $M'$ of $\mathbb{H}'$. We obtain that $\theta^{-1}(M')$ is a nontrivial module of $\mathbb{H}$. Hence, $\Gamma \bullet \mathbb{H}$ is critical, it is prime. By Proposition 16, $\Gamma \bullet \mathbb{H}'$, that is, $(\Gamma \bullet \mathbb{H}) - \{\varphi_C(2i - 1), \varphi_C(2i)\}$ is prime. Consequently, $\varphi_C(2i - 1) \in N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(2i))$. Similarly, $\varphi_C(2i + 1) \in N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(2i))$. Therefore, (27) follows from Lemma 24.

Now, we verify that

$$N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(0)) = \{\varphi_C(1)\}. \quad (28)$$

As previously, we consider the graph $C^{-}$ defined on $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ by $E(C^{-}) = E(C) \setminus \{\varphi_C(0)\} \cup \{\varphi_C(2i - 1), \varphi_C(2i)\}$. Now, consider the graph $\Gamma'$ obtained from $\Gamma$ by replacing the path $C$ by $C^{-}$. Moreover, consider the hypergraph $\mathbb{H}'$ obtained from $\mathbb{H}$ by renaming the vertex $C$ by $C^{-}$. We obtain $(\Gamma \bullet \mathbb{H}) - \{\varphi_C(0), \varphi_C(1)\} = (\Gamma \bullet \mathbb{H}').$ It follows from Proposition 16 applied with $\Gamma'$ and $\mathbb{H}'$ that $(\Gamma \bullet \mathbb{H}) - \{\varphi_C(0), \varphi_C(1)\}$ is prime. Hence, $\varphi_C(1) \in N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(0))$. Set $X = V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$. Since $\mathbb{H}$ is prime and $\varphi_C(1) \in (X)_{\Gamma \bullet \mathbb{H}}$, it follows from Lemma 24 that (28) holds. It follows from Remark 41 that

$$N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(2w(C))) = \{\varphi_C(2w(C) - 1)\}. \quad (29)$$

Lastly, we prove that

$$N_{\mathcal{D}(\Gamma \bullet \mathbb{H})}(\varphi_C(2i + 1)) = \{\varphi_C(2i), \varphi_C(2i + 2)\}. \quad (30)$$
for each \( i \in \{0, \ldots, w(C)-1\} \). It follows from (27) that \( \varphi_C(2i) \in N_{\mathcal{P}\langle \mathbb{H} \rangle} \langle \varphi_C(2w(C)+1) \rangle \) and \( \varphi_C(2i+2) \in N_{\mathcal{P}\langle \mathbb{H} \rangle} \langle \varphi_C(2w(C)+1) \rangle \). By Lemma 44, (30) holds.

Consequently, it follows from (27), (28), (29), (30), and from Lemma 24 that \( \mathcal{P}\langle \Gamma \cdot \mathbb{H} \rangle[V(C)] = C \).

**Lemma 47.** Suppose that \( \Gamma \cdot \mathbb{H} \) is critical with \( v(\Gamma \cdot \mathbb{H}) \geq 5 \). Given \( C \in \mathcal{C}(\Gamma) \) such that \( v(C) = 2 \), \( C \) satisfies (26) if and only if the following two assertions hold

- \( \mathbb{H} - C \) is connected;
- for every nontrivial module \( M_C \) of \( \mathbb{H} - C \), we have \( M_C \setminus \mathcal{C}_1(\Gamma) \neq \emptyset \).

**Proof.** By Corollary 44, for every \( c \in V(C) \), we have \( N_{\mathcal{P}\langle \mathbb{H} \rangle}(c) \subseteq V(C) \). Consider an isomorphism \( \varphi_C \) from \( P_2 \) onto \( C \). We have \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(0), \varphi_C(1)\} = (\Gamma - V(C)) \cdot (\mathbb{H} - C) \). Moreover, we have \( \mathcal{C}_1(\Gamma - V(C)) \subseteq \mathcal{C}_1(\Gamma) \). To conclude, it suffices to apply Proposition 16 to \( (\Gamma - V(C)) \cdot (\mathbb{H} - C) \).

**Lemma 48.** Suppose that \( \Gamma \cdot \mathbb{H} \) is critical with \( v(\Gamma \cdot \mathbb{H}) \geq 5 \). Given \( C \in \mathcal{C}(\Gamma) \) such that \( v(C) = 3 \), \( C \) satisfies (26) if and only if the following two assertions hold

- \( \mathbb{H}_C \) is connected, where \( \mathbb{H}_C \) is obtained from \( \mathbb{H} \) by replacing \( C \) by \( \{\varphi_C(0)\} \);
- for every nontrivial module \( M_C \) of \( \mathbb{H}_C \), we have \( M_C \setminus (\mathcal{C}_1(\Gamma) \cup \{\varphi_C(0)\}) \neq \emptyset \).

**Proof.** By Corollary 44, for every \( c \in V(C) \), we have \( N_{\mathcal{P}\langle \mathbb{H} \rangle}(c) \subseteq V(C) \). Furthermore, consider an isomorphism \( \varphi_C \) from \( P_3 \) onto \( C \). Denote by \( \mathbb{H}_C \) the \( (2,3) \)-hypergraph obtained from \( \mathbb{H} \) by replacing \( C \) by \( \{\varphi_C(0)\} \). We have

\[
\begin{align*}
\{(\Gamma \cdot \mathbb{H}) - \{\varphi_C(1), \varphi_C(2)\} = (\Gamma - \{\varphi_C(1), \varphi_C(2)\}) \cdot \mathbb{H}_C \\
\text{and} \\
\mathcal{C}_1(\Gamma - \{\varphi_C(1), \varphi_C(2)\}) = \mathcal{C}_1(\Gamma) \cup \{\varphi_C(0)\}.
\end{align*}
\]

To begin, suppose that \( C \) satisfies (26). We obtain that \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(1), \varphi_C(2)\} \) is prime. By Proposition 16 applied to \( H(\Gamma - \{\varphi_C(1), \varphi_C(2)\}, \mathbb{H}_C) \), both assertions above hold.

Conversely, suppose that both assertions above hold. Since (31) holds, it follows from Proposition 16 that \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(1), \varphi_C(2)\} \) is prime. By Corollary 44, \( \{\varphi_C(0), \varphi_C(2)\} \) is a module of \( (\Gamma \cdot \mathbb{H}) - \varphi_C(1) \). Therefore, \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(0), \varphi_C(1)\} \) is prime. Lastly, it follows from Corollary 44 that \( V(\Gamma \cdot \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\} \) is a module of \( \{\Gamma \cdot \mathbb{H} - \varphi_C(0)\} \). Thus, \( V(\Gamma \cdot \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\} \) is a non trivial module of \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(0), \varphi_C(2)\} \). Hence, \( (\Gamma \cdot \mathbb{H}) - \{\varphi_C(0), \varphi_C(2)\} \) is decomposable. It follows that \( \mathcal{P}\langle \Gamma \cdot \mathbb{H} \rangle[V(C)] = C \). It follows that \( C \) satisfies (26). 

□
Lemma 49. Suppose that $\Gamma \bullet \mathbb{H}$ is critical with $v(\Gamma \bullet \mathbb{H}) \geq 5$. For every $v \in V_1(\Gamma)$, we have $d_{\mathcal{G}(\Gamma \bullet \mathbb{H})} = 1$ if and only if there exists $w \in V_1(\Gamma)$ satisfying

- \{w\} is an isolated vertex of $\mathbb{H} - \{v\}$;
- $\mathbb{H} - \{\{v\}, \{w\}\}$ is connected;
- for every nontrivial module $M_{vw}$ of $\mathbb{H} - \{\{v\}, \{w\}\}$, we have $M_{vw} \setminus (\mathcal{C}_1(\Gamma) \setminus \{\{v\}, \{w\}\}) \neq \emptyset$.

Proof. To begin, suppose that $d_{\mathcal{G}(\Gamma \bullet \mathbb{H})}(v) = 1$. Denote by $w$ the unique neighbor of $v$ in $\mathcal{G}(\Gamma \bullet \mathbb{H})$. It follows from Proposition 16, Lemma 47, and Lemma 17 that $w \in V_1(\Gamma)$. We have

$$
\begin{align*}
\left( (\Gamma \bullet \mathbb{H}) - \{v, w\} \right) & = (\Gamma - \{v, w\}) \bullet (\mathbb{H} - \{\{v\}, \{w\}\}) \\
\text{and} \quad \mathcal{C}_1(\Gamma - \{v, w\}) & = \mathcal{C}_1(\Gamma) \setminus \{\{v\}, \{w\}\}.
\end{align*}
$$

(32)

Since $N_{\mathcal{G}(\Gamma \bullet \mathbb{H})}(v) = \{w\}$, $(\Gamma \bullet \mathbb{H}) - \{v, w\}$ is prime. By Proposition 16 applied to $(\Gamma - \{v, w\}) \bullet (\mathbb{H} - \{\{v\}, \{w\}\})$, $\mathbb{H} - \{\{v\}, \{w\}\}$ is connected, and for every nontrivial module $M_{vw}$ of $\mathbb{H} - \{\{v\}, \{w\}\}$, $M_{vw} \setminus (\mathcal{C}_1(\Gamma) \setminus \{\{v\}, \{w\}\}) \neq \emptyset$. Moreover, since $N_{\mathcal{G}(\Gamma \bullet \mathbb{H})}(v) = \{w\}$, it follows from Lemma 24 that $V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$ is a module of $(\Gamma \bullet \mathbb{H}) - v$. We show that

$$
\text{for each } \varepsilon \in E(\mathbb{H}), \text{ if } \{w\} \in \varepsilon, \text{ then } \{v\} \in \varepsilon.
$$

(33)

Indeed, consider $\varepsilon \in E(\mathbb{H})$ such that $\{w\} \in \varepsilon$. For a contradiction, suppose that $|\varepsilon| = 2$. There exists $C \in \mathcal{C}_{\text{even}}(\Gamma)$ such that $\varepsilon = \{w\}C$. We obtain $C \subseteq V(\Gamma \bullet \mathbb{H})\setminus \{v, w\}$ and $w_\varphi(0) \varphi_C(1) \in E(\Gamma \bullet \mathbb{H})$, which contradicts the fact that $V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$ is a module of $(\Gamma \bullet \mathbb{H}) - v$. Consequently, we have $|\varepsilon| = 3$. Thus, there exist distinct $C, D \in \mathcal{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = \{w\}CD$. If $C \neq \{v\}$ and $D \neq \{v\}$, then $\varphi_C(0), \varphi_D(0) \in V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$ and $w_\varphi(0) \varphi_C(1) \varphi_D(1) \in E(\Gamma \bullet \mathbb{H})$, which contradicts the fact that $V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$ is a module of $(\Gamma \bullet \mathbb{H}) - v$. Therefore, $C = \{v\}$ or $D = \{v\}$. Hence, (33) holds. It follows that $\{w\}$ is an isolated vertex of $\mathbb{H} - \{v\}$.

Conversely, suppose that there exists $w \in V_1(\Gamma)$ such that the three assertions above hold. Since (32) holds, it follows from Proposition 16 applied to $(\Gamma - \{v, w\}) \bullet (\mathbb{H} - \{\{v\}, \{w\}\})$ that $(\Gamma \bullet \mathbb{H}) - \{v, w\}$ is prime. Thus, we have

$$
\text{for each } \varepsilon \in E(\mathbb{H}), \text{ if } \{w\} \in \varepsilon, \text{ then } \{v\} \in \varepsilon.
$$

(33)

Furthermore, since $\{w\}$ is an isolated vertex of $\mathbb{H} - \{v\}$, $\mathbb{H}[\{w\}]$ is a component of $\mathbb{H} - \{v\}$. By Lemma 36, $V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$ is a module of $(\Gamma \bullet \mathbb{H}) - v$. Consider any $u \in V(\Gamma \bullet \mathbb{H}) \setminus \{v, w\}$. We obtain that $V(\Gamma \bullet \mathbb{H}) \setminus \{u, v\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \{u, v\}$. Hence, we obtain $u \notin N_{\mathcal{G}(\Gamma \bullet \mathbb{H})}(v)$. It follows from (33) that $N_{\mathcal{G}(\Gamma \bullet \mathbb{H})}(v) = \{w\}$.

\[\square\]
Proof of Theorem 15. To begin, suppose that $\Gamma \bullet \mathbb{H}$ is critical and $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$. Since $\Gamma \bullet \mathbb{H}$ is critical, it follows from Theorem 15 that Assertions (C1), (C2), and (C3) hold. For Assertion (C4), consider $C \in \mathcal{C}_{even}(\Gamma)$ such that $v(C) = 2$. By Corollary 44, for every $c \in V(\Gamma)$, we have $N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(c) \subseteq V(\Gamma)$. Therefore, since $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$, $C$ satisfies (26). Furthermore, since Assertion (C4) holds, it follows from Lemma 47 because $\Gamma$ is critical. It remains to show that $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$. It follows from Proposition 46 that for each $w \in V(\Gamma)$, we have $N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(w) \subseteq V(\Gamma)$. Therefore, Assertions (C1), (C2), (C4) hold. For Assertion (C5), consider $\mathcal{C}, D$. For Assertion (C5), consider $\mathcal{C}, D \in \mathcal{C}_1(\Gamma)$ such that $\mathcal{C}, D$ holds. For Assertion (C6), consider distinct $\mathcal{C}, D \in \mathcal{C}_1(\Gamma)$ such that $\mathcal{C}, D$ holds. For Assertion (C6), consider $\mathcal{C}, D \in \mathcal{C}_1(\Gamma)$ such that $\mathcal{C}, D$ holds.

Conversely, suppose that Assertions (C1), ..., (C6) hold. By Theorem 15, $\Gamma \bullet \mathbb{H}$ is critical. It remains to show that $\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma$. It follows from Proposition 46 that for each $C \in \mathcal{C}(\Gamma)$ such that $v(C) \geq 4$, $C$ satisfies (26). Furthermore, since Assertion (C4) holds, it follows from Lemma 47 that (26) is satisfied by each $C \in \mathcal{C}_{even}(\Gamma)$ such that $v(C) = 2$. Similarly, since Assertion (C5) holds, it follows from Lemma 48 that (26) is satisfied by each $C \in \mathcal{C}_{even}(\Gamma)$ such that $v(C) = 3$. It follows that (26) holds, it remains to prove that $d_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) = 0$ for each $v \in V(\Gamma)$. Set $\mathcal{N} = \{v \in V(\Gamma) : d_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) \neq 0\}$. For a contradiction, suppose that $\mathcal{N} \neq \emptyset$. Since (26) holds, it follows by Lemma 49 that there exists a nontrivial module $M$ of $\mathbb{H} - \{C, D\}$ such that $M \subseteq \mathcal{C}_1(\Gamma) \setminus \{C, D\}$.

Conversely, suppose that Assertions (C1), ..., (C6) hold. By Theorem 15, $\Gamma \bullet \mathbb{H}$ is critical. It remains to show that

$$\mathcal{P}(\Gamma \bullet \mathbb{H}) = \Gamma. \tag{35}$$

It follows from Proposition 46 that for each $C \in \mathcal{C}(\Gamma)$ such that $v(C) \geq 4$, $C$ satisfies (26). Furthermore, since Assertion (C4) holds, it follows from Lemma 47 that (26) is satisfied by each $C \in \mathcal{C}_{even}(\Gamma)$ such that $v(C) = 2$. Similarly, since Assertion (C5) holds, it follows from Lemma 48 that (26) is satisfied by each $C \in \mathcal{C}_{even}(\Gamma)$ such that $v(C) = 3$. It follows that (26) holds, it remains to prove that $d_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) = 0$ for each $v \in V(\Gamma)$. Set $\mathcal{N} = \{v \in V(\Gamma) : d_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) \neq 0\}$. For a contradiction, suppose that $\mathcal{N} \neq \emptyset$. Since (26) holds, it follows by Lemma 49 that there exists $w \in V(\Gamma)$ satisfying

- $\{w\}$ is an isolated vertex of $\mathbb{H} - \{v\}$;
- $\mathbb{H} - \{v\}, \{w\}$ is connected;
- for every nontrivial module $M_{vw}$ of $\mathbb{H} - \{v\}, \{w\}$, we have $M_{vw} \setminus (\mathcal{C}_1(\Gamma) \setminus \{v\}, \{w\}) \neq \emptyset$.

Therefore, Assertion (C6) is not satisfied for $C = \{v\}$ and $D = \{w\}$. It follows from Lemma 24 that $d_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) = 2$ for every $v \in \mathcal{N}$. Moreover, by (36), $N_{\mathcal{P}(\Gamma \bullet \mathbb{H})}(v) \subseteq \mathcal{N}$ for every $v \in \mathcal{N}$. Therefore, $\mathcal{P}(\Gamma \bullet \mathbb{H})$ admits a component $\mathcal{C}$ such that $v(C) \geq 3$, $V(C) \subseteq \mathcal{N}$, and $\mathcal{P}(\Gamma \bullet \mathbb{H})[V(C)]$ is a cycle. Since $\mathcal{C}(\Gamma) \setminus \mathcal{C}_1(\Gamma) \neq \emptyset$ by (2), we obtain $V(C) \subseteq V(\Gamma \bullet \mathbb{H})$, which contradicts Proposition 26 because $\Gamma \bullet \mathbb{H}$ is critical. It follows that $\mathcal{N} = \emptyset$. Therefore, (35) holds.

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