1-Loop improved lattice action for the nonlinear $\sigma$-model *

M. Bartels, G. Mack$^a$, and G. Palma$^b$

$^a$II. Institut für theoretische Physik, Universität Hamburg
D-22761 Hamburg, Luruper Chaussee 149, Germany

$^b$Departamento de Física, Universidad de Santiago de Chile,
Casilla 307, Correo 2, Santiago, Chile

In this paper we show the Wilson effective action for the 2-dimensional $O(N + 1)$-symmetric lattice nonlinear $\sigma$-model computed in the 1-loop approximation for the nonlinear choice of blockspin $\Phi(x)$, $\Phi(x) = C\phi(x)/|C\phi(x)|$, where $C$ is averaging of the fundamental field $\phi(z)$ over a square $x$ of side $\tilde{a}$.

The result for $S_{\text{eff}}$ is composed of the classical perfect action with a renormalized coupling constant $\beta_{\text{eff}}$, an augmented contribution from a Jacobian, and further genuine 1-loop correction terms. Our result extends Polyakov’s calculation which had furnished those contributions to the effective action which are of order $\ln \tilde{a}/a$, where $a$ is the lattice spacing of the fundamental lattice. An analytic approximation for the background field which enters the classical perfect action will be presented elsewhere [1].

1. Introduction

Effective lattice actions $S_{\text{eff}}$ in the sense of Wilson are perfect actions in the sense that they reproduce the long distance behaviour of a theory with a much larger UV cutoff. Different approximations have been considered before [2–4].

Our computation of $S_{\text{eff}}$ in 1-loop approximation identifies genuine 1-loop corrections beyond the appearance of a running coupling constant in the classical perfect action, when terms are included which are not $O(\ln \tilde{a}/a)$. Details and an analytical approximation for $\Psi$ as a function of $\Phi$ are found in ref. [1].

2. Definitions

The model lives on a square lattice $\Lambda$ of lattice spacing $a$ with points typically denoted $z, w, \ldots$. We use lattice notations so that $\int_z (\ldots) \equiv a^2 \sum_z (\ldots) \rightarrow \int d^2 z (\ldots)$ in the continuum limit $a \rightarrow 0$; $\mu$ is the lattice vector of length $a$ in $\mu$-direction ($\mu = 1, 2$). The field $\phi(z) \in S^N$ is a $(N + 1)$-dimensional unit vector. The action of the model is

$$S[\phi] = \frac{\beta}{2} \int_z |\nabla_\mu \phi(z)|^2 = -\frac{\beta}{2} \int_z \phi \Delta \phi. \quad (1)$$

A block lattice $\hat{\Lambda}$ of lattice spacing $\hat{a} = s \cdot a$ is superimposed ($s$, a positive integer). Its points are typically denoted $x, y, \ldots$. They are identified with squares of sidelength $\tilde{a}$ in $\Lambda$.

We define a blockspin $\Phi(x)$ which lives on the block lattice as a function $\Phi(x) = C\phi(x)$ of the fundamental field. $\Phi(x)$ is also a $(N+1)$-unit vector; therefore the operator $C$ is necessarily nonlinear. We choose

$$\Phi(x) = C\phi(x) \equiv C\phi(x)/|C\phi(x)|. \quad (2)$$

The linear operator $C$ averages over blocks,

$$C\phi(x) = \text{av}_{z \in x} \phi(z) \equiv \hat{a}^{-2} \int_{z \in x} \phi(z). \quad (3)$$

The Wilson effective action is defined by

$$e^{-S_{\text{eff}}[\Phi]} = \int D\phi \prod_x \delta(C\phi(x), \Phi(x)) e^{-S[\phi]}; \quad (4)$$

$$D\phi = \prod_z d\phi(z),$$

where $d\phi$ is the uniform measure on the sphere $S^N$, and $\delta$ is the $N$-dimensional $\delta$-function on the sphere.
We consider a $\delta$-function constraint because computation of expectation values of observables which depend on $\phi$ only through the blockspin $\Phi$ must then be identical whether computed with $S$ or $S_{\text{eff}}$. This prepares best for stringent tests of the accuracy of the result.

Hasenfratz and Niedermayer showed numerically that much better locality properties of effective actions are obtained when a Gaussian is used in the definition of the effective action instead of a sharp $\delta$-function. Therefore we admit the substitution $d\delta(C\phi(x), \Phi(x)) \Rightarrow J_0(C\phi(x))e^{-\frac{1}{2}\kappa ||C^-\Phi||^2}$ (5)

with $C^-\Phi(x) = C\phi(x) - \Phi(x) \cdot C\phi(x) = C\phi^\perp(x)$ and $J_0$ as in eq. (11) below. The $\delta$-function is recovered for $\kappa = \infty$.

3. Background field and classical action

Given a blockspin configuration $\Phi$, let $\Psi = \Psi[\Phi]$ be that field on the fine lattice $\Lambda$ which extremizes $S$, resp. $S(\phi) + \frac{1}{2} \kappa \sum_x |C\phi^\perp(x)|^2$ for $\kappa < \infty$ subject to the constraints $|\Psi(z)|^2 = 1$ and $C\Psi = \Phi$.

$\Psi$ is called the background field. The classical perfect action is

$$S_{\text{cl}}[\Phi] = S(\Psi[\Phi]) + \frac{\beta K}{2} \sum_x (C\phi^\perp(x))^2 \Rightarrow S(\Psi).$$

(7)

Here we wish to compute the 1-loop corrections. It is convenient to regard the full effective action as a function of $\Psi$. This is possible because $\Phi$ is determined by $\Psi$ according to eq. (6).

For large enough blocks, the background field $\Psi$ is smooth.

4. The 1-loop approximation

A perturbative calculation of the functional integral (11) for the effective action is not straightforward because the argument of the $\delta$-function is a nonlinear function of the field.

To solve this problem, we find a parametrization $\phi(z) = \phi[\Psi, \zeta](z)$ of an arbitrary field $\phi$ on $\Lambda$ in terms of the background field $\Psi = \Psi[\Phi]$ and a fluctuation field obeying $\zeta(z) \perp \Phi(x)$ for $z \in x$ such that the constraint becomes a linear constraint on $\zeta$, viz. $C\zeta = 0$ for $\kappa = \infty$.

The background field is a smooth field. It represents the low frequency part of $\phi$, while $\zeta$ adds the high frequency contributions. $\zeta$ takes its values in a linear space. We decompose $\phi(z)$ in components $\perp$ and $\parallel$ to $\Phi(x), (x \ni z)$ and put $\phi^\perp(z) = \Psi^\perp(z) + \zeta(z)$. Balaban has shown how to find a suitable parametrization for lattice gauge fields.

There is a jacobian $J$ to the transformation, and the result has the form

$$e^{-S_{\text{cl}}[\Psi]} = \prod_z d\zeta(z) \delta(C\zeta) J(\Psi, \zeta) e^{-S(\Psi[\Psi])}$$

(8)

with a Gaussian in place of $\delta$ if $\kappa < \infty$.

The 1-loop approximation yields the effective action to order $\beta^0$. It is obtained by expanding the action to second order and the Jacobian to zeroth order in the fluctuation field. This approximates expression (11) by a Gaussian integral. The resulting $Tr \log$ formula is not particularly useful, though.

A first simplification is achieved by exploiting the fact that the background field $\Psi$ is smooth. This is always true, for large enough blocks, because a 2-dimensional Heisenberg ferromagnet has no domain walls. Because of the smoothness of $\Psi$ one can neglect terms of higher order than second in $\nabla \Psi$.

The exact 1-loop perfect action to this order is as follows:

$$S_{\text{eff}} = S_{\text{cl}} - \sum_x \ln J_0(C\Psi(x)) - \frac{1}{2} Tr \ln \Gamma_Q +$$

$$\frac{1}{2} \int \left( \nabla_\mu \Psi^T(z) \beta_{\text{eff}}^2(z) \nabla_\mu \Psi(z) + \beta_{\text{eff}}^2(z) \frac{\Phi(z)^T(\Delta)\Psi(z)}{\cos \theta(z)} \right)$$

$$+ \frac{S_{\text{eff}}(2)}{2} + \int_z tr J_\mu(z) \nabla_\mu \Psi^T(z) + \nabla_\mu \Psi(z) \Psi^T(z + \hat{\mu}) + \Psi^T \nabla_\mu \Psi^T(z + \hat{\mu}) \Psi(z)$$

(9)

$$j_\mu(z) = \Psi(z) \nabla_\mu \Psi^T(z) + \nabla_\mu \Psi(z) \Psi^T(z + \hat{\mu}) + \Psi^T \nabla_\mu \Psi(z + \hat{\mu}) \Psi(z)$$

(10)

$^2$To save brackets, we adopt the notational convention that derivatives acts only on the factor immediately following it. We used vector notation, $\Psi^T$ is the row vector transpose to $\Psi$. Note that $j_\mu(z)$ is a matrix.
where in the expression $\nabla_{\mu} \Gamma_Q(z, z+\hat{\mu})$ the derivative acts only on the first argument, $[z]$ is the block containing $z$, the Jacobian is
\[ J_0(\mathcal{C}(x)) = \left( |\mathcal{C}(x)|^2 - \frac{1}{\beta_K} + \ldots \right)^{\frac{1}{2}} \] (11)
and $S_{\text{eff}}^{(2)}$ is a contribution from a renormalized 1-loop graph with 2 vertices as follows
\[ S_{\text{eff}}^{(2)} = -\frac{1}{2} \int_z w \text{ tr} \left( \nabla_{\mu} \Gamma_Q(z, w) - \nabla_{\nu} j_{\mu}^T(w) \Gamma_Q(w, z) j_{\mu}(z) + \nabla_{\mu} \Gamma_Q(z, w) j_{\nu}(w) \nabla_{\nu} \Gamma_Q(w, z) j_{\mu}(z) + \delta_{\mu\nu} \delta_{z,w} j_{\mu}(z) \Gamma_Q(z, w) j_{\nu}(w) \right). \] (12)

We used the notation $z_\mu = z + \hat{\mu}$. The $\delta_{\mu\nu} \delta_{z,w} j_{\mu}$ term subtracts the part which diverges in the limit $a \to 0$. The last term in the definition (10) of $j_{\mu}$ can be dropped inside eq. (12) because its contribution is of higher order in $\nabla \Psi$.

$\Gamma_Q$ is an $(N + 1) \times (N + 1)$ matrix propagator,
\[ \Gamma_Q = (\mathcal{Q} + \kappa Q^T C^T C Q)^{-1}; \] (13)
\[ Q(z) = 1 - \Psi(z) \Phi^T(z) + \Phi(z) \]
\[ (\Psi^T(z)[1 + \cos \theta(z)] - \Phi^T(x)) \] (14)
with $\cos \theta(z) = \Psi(z) \cdot \Phi(x)$, $(x \ni z)$. Both coupling constant renormalizations $\beta_1^{\text{eff}}$ and $\beta_2^{\text{eff}}$ have a residual dependence on $\Psi$ through $Q$, so they fluctuate somewhat with $\Psi$; to leading order the dependence is through $\cos \theta$. Note that $\beta_1^{\text{eff}}$ is a $(N + 1) \times (N + 1)$ matrix, while $\beta_2^{\text{eff}}$ is a scalar.

$\beta_1^{\text{eff}}(z) = \Gamma_Q(z, z)$, \hspace{1cm} (15)
$\beta_2^{\text{eff}}(z) = -\text{tr} \left[ 1 - \Psi \Phi^T(z) \right] \Gamma_Q(z, z)$. \hspace{1cm} (16)

Finally, the last term in eq. (10) is a lattice artifact.

If $\Phi$ is smooth enough, one may expand
\[ -\frac{1}{2} \text{ Tr } \ln \Gamma_Q = \int_z w \mathcal{A}(z, w) \mathcal{C}(x, z) \mathcal{B}[\cos \theta(z) - 1] \]
\[ + \int_z w \int_{x,y} \Psi^T(z) \Phi^T(w) \]
\[ - \Gamma_{KG}(z, w) \Delta \mathcal{A}(w, y) \mathcal{C}(y, z) \]
\[ + \mathcal{A}(z, x) \mathcal{C}(x, w) \mathcal{A}(w, y) \mathcal{C}(y, z) + \ldots \]
and substitute $\Gamma_{Qz} = 1 \equiv \Gamma_{KG} \mathbf{1}$ elsewhere.
\[ \mathcal{A} = \kappa \Gamma_{KG} \mathcal{C}^{*} \]
has a finite limit as $\kappa \to \infty$.

5. Recovery of Polyakov’s result

Polyakov determined the contributions to the effective action which are of order $\ln a/a$. They do not depend on the detailed form of the block spin which fixes the infrared cutoff in the auxiliary theory with fields $\zeta$. The presence of $\mathcal{M}^2 = \kappa Q^T C^T C Q$ in the high frequency propagator has the effect of an infrared cutoff. To get the result modulo details of the choice of infrared cutoff, we may therefore replace $\mathcal{M}$ by a mass term $\mathcal{M}^2$, where $\mathcal{M} = O(\tilde{a}^{-1})$. $\Gamma_Q$ then becomes translation invariant. One shows that $(\Phi(z)^T \Delta \Psi(z)) / \cos \theta(z) = \Psi^T(z) \Delta \Psi(z)$ by the extremality condition for $\Psi$. Polyakov’s result is now recovered because $\text{Tr } \ln \Gamma_Q$ becomes constant and neither in $J_0$ nor the last two terms in eq. (10) are of order $\ln a/a$.

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