Reconstruction of weak bialgebra maps and its applications

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Abstract

In this note we give a precise statement and a detailed proof for reconstruction problem of weak bialgebra maps. As an application we characterize indecomposability of weak algebras in categorical setting.

1 Introduction

The concept of weak bialgebras and weak Hopf algebras was introduced by Böhm, Nill and Szlachányi [1] as a generalization of bialgebras and Hopf algebras. In some literatures they are called quantum groupoids [13]. As shown by Etingof, Nikshych and Ostrik [5] the language of weak Hopf algebras is convenient to visualize various categorical construction, including (multi-)fusion categories. At present, a lot of concepts and results for the ordinal bialgebras and Hopf algebras are generalized or extended to weak versions.

By the classical Tannaka-Krein reconstruction theorem, it is well known that any finite-dimensional bialgebra over a fixed field $k$ is determined up to isomorphism by its comodule category $\mathcal{M}_A$ whose objects are of finite dimension [4, 9, 15, 19]. More precisely, let $A$ and $B$ be two finite-dimensional bialgebras over $k$, and $F: \mathcal{M}_A \to \mathcal{M}_B$ be a $k$-linear monoidal functor. If $F$ is fibered, then there is a bialgebra map $\varphi: A \to B$ such that $F = M_\varphi$, where $M_\varphi$ is the induced monoidal functor from $f$. This statement can be found in Majid’s book [10, Theorem 2.2] and a detailed proof is given in Franco’s lecture note [6, p.80–84].

In this note we treat the weak bialgebra version of the above result. It is noted that as is well known, the reconstruction theorem for weak bialgebras has been established by several researchers [3, 7, 11, 2] (see [20] for other variants). However, it seems that there is no statement on a reconstruction theorem for “weak bialgebra map” in any papers though it is very fundamental. This note is devoted to give a precise statement of it and prove it. In fact, although the same statement with the classical one holds, the proof is a rather complicated since the unit object in the comodule category over a weak bialgebra $A$ is not necessary to the base field $k$. Actually, it is a subalgebra, which is called the source counit subalgebra of $A$. For that reason, the proof of the reconstruction theorem for weak bialgebra maps is accomplished by re-examining the method used in the proof of the reconstruction of coalgebra map.

Recently, the author study on indecomposability of weak bialgebras, and find some interesting results and unsolved problems [21]. As an application of the reconstruction theorem of weak bialgebra maps, we derive a categorical interpretation of indecomposability of a weak bialgebra.
This paper is organized as follows. In Section 2 we recall the definition and basic properties of weak bialgebras, and also the comodule structure over them. In Section 3 after we overview the proof of the reconstruction theorem of coalgebra maps, we state and prove the reconstruction theorem of bialgebra maps. In Section 4, that is the final section, we apply the theorem to characterize indecomposability of finite-dimensional weak bialgebras.

Throughout this note, \( k \) denotes a field, and \( \text{Vect}_{\text{f.d.}}^k \) stands for the monoidal category whose objects are finite-dimensional vector spaces over \( k \) and morphisms are \( k \)-linear maps between them. For a weak bialgebra or a weak Hopf algebra \( H \), we denote by \( \Delta_H, \varepsilon_H \) and \( S_H \) the comultiplication, the counit and the antipode of \( H \), respectively. When it is clear that they are for \( H \), they are simply denoted by \( \Delta, \varepsilon \) and \( S \), respectively. The notation \( \Delta^{(2)} \) means the composition \( (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \). We use Sweedler's notation such as \( \Delta(x) = x^{(1)} \otimes x^{(2)} \) for \( x \in H \), and for a right \( H \)-comodule \( M \) with coaction \( \rho \) we also use the notation \( \rho(m) = m^{(0)} \otimes m^{(1)} \) for \( m \in M \).

A monoidal category is described as \( C = (C, \otimes, I, a, l, r) \), as in MacLane’s book [8], and a monoidal functor between monoidal categories \( C \) and \( D \) is described as \( F = (F, \Phi^F, \omega^F) \), where \( F \) is a covariant functor from \( C \) to \( D \), and \( \Phi^F \) is a natural transformation obtained by collecting morphisms \( \phi^F_{M,N} : F(M) \otimes F(N) \rightarrow F(M \otimes N) \) for all \( M,N \in C \), and \( \omega^F : F(I_C) \rightarrow I_D \) is a morphism, where \( I_C \) and \( I_D \) are the unit objects in \( C \), and \( D \), respectively. If \( \Phi^F \) is a natural equivalence and \( \omega^F \) is an isomorphism, then the monoidal functor \( (F, \Phi^F, \omega^F) \) is called strong. As in the classical case, every weak bialgebra map \( \varphi : H \rightarrow K \) induces a covariant functor \( M^\varphi : M^H \rightarrow M^K \). We note that the functor \( M^\varphi \) is not monoidal but is comonoidal. By a comonoidal functor we mean a triplet \( F = (F, \Phi^F, \omega^F) \), which all arrows in \( \Phi^F \) and \( \omega^F \) are in reverse in monoidal categories, that is, \( \Phi^F \) is a natural transformation obtained by collecting morphisms \( \phi^F_{M,N} : F(M \otimes N) \rightarrow F(M) \otimes F(N) \) for all \( M,N \in C \), and \( \omega^F : I_D \rightarrow F(I_C) \) is a morphism. By the same condition for a monoidal functor, the concept called strong is defined for a comonoidal functor. In some literatures terminologies “colax” or “op-monoidal” are used for “comonoidal”.

For general facts on Hopf algebras, we refer the reader to Montgomery’s book [12].

2 Definition of weak bialgebras and structures of their comodule categories

In this section we recall the definition and basic properties of weak bialgebras, and also the comodule structure over them mainly following [1] and [13].

Let \( H \) be a vector space over \( k \), and \( (H, \mu, \eta) \) is an algebra and \( (H, \Delta, \varepsilon) \) is a coalgebra over \( k \). The 5-tuple \( (H, \mu, \eta, \Delta, \varepsilon) \) is said to be a weak bialgebra over \( k \) if the following three conditions are satisfied.

\[(WH1) \quad \Delta(xy) = \Delta(x)\Delta(y) \text{ for all } x,y \in H.\]

\[(WH2) \quad \Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \text{ where } 1 = \eta(1) \text{ is the identity element of the algebra } (H, \mu, \eta).\]

\[(WH3) \quad \text{For all } x,y,z \in H \]
\( \varepsilon(xyz) = \varepsilon(xy(1))\varepsilon(y(2)z). \)

(ii) \( \varepsilon(xyz) = \varepsilon(xy(2))\varepsilon(y(1)z). \)

Let \( S : H \to H \) be a \( k \)-linear map. The 6-tuple \((H, \mu, \eta, \Delta, \varepsilon, S)\) is said to be a weak Hopf algebra over \( k \) if the above three conditions and the following additional condition are satisfied.

(WH4) For all \( x \in H \)

(i) \( x(1)S(x(2)) = \varepsilon(1(1)x)1(2). \)

(ii) \( S(x(1))x(2) = 1(1)\varepsilon(x1(2)). \)

(iii) \( S(x(1))x(2)S(x(3)) = S(x). \)

The above \( S \) is called the antipode of \((H, \mu, \eta, \Delta, \varepsilon)\) and \((H, \mu, \eta, \Delta, \varepsilon, S)\). We note that it is unique if it exists.

For a weak bialgebra \( H = (H, \mu, \eta, \Delta, \varepsilon) \), by the condition (WH4)(i),(ii), the following two \( k \)-linear maps \( \varepsilon_t, \varepsilon_s : H \to H \) are defined:

\[
\varepsilon_t(x) = \varepsilon(1(1)x)1(2), \tag{2.1}
\]

\[
\varepsilon_s(x) = 1(1)\varepsilon(x1(2)). \tag{2.2}
\]

The maps \( \varepsilon_t \) and \( \varepsilon_s \) are called the target counital map and the source counital map, respectively. These maps satisfy the following properties.

**Lemma 2.1.**

(1) \( \varepsilon_t^2 = \varepsilon_t, \varepsilon_s^2 = \varepsilon_s. \)

(2) For all \( x \in H \)

(i) \( (\text{id} \otimes \varepsilon_t) \circ \Delta(x) = 1(1)x \otimes 1(2). \)

(ii) \( (\varepsilon_s \otimes \text{id}) \circ \Delta(x) = 1(1) \otimes x1(2). \)

In particular,

\[
1(1) \otimes \varepsilon_t(1(2)) = 1(1) \otimes 1(2) = \varepsilon_s(1(1)) \otimes 1(2). \tag{2.3}
\]

(3) For all \( x \in H \)

(i) \( \varepsilon_t(x) = x \iff \Delta(x) = 1(1)x \otimes 1(2). \)

(ii) \( \varepsilon_s(x) = x \iff \Delta(x) = 1(1) \otimes x1(2). \)

Especially, by Part (2)(i)

\[
1(1)1[1] \otimes 1(2) \otimes 1[2] = 1(1) \otimes \varepsilon_t(1(2)) \otimes 1(3),
\]

\[
1(1) \otimes 1[1] \otimes 1(2)1[2] = 1(1) \otimes \varepsilon_s(1(2)) \otimes 1(3),
\]

where \( \Delta^{(2)}(1) = 1(1) \otimes 1(2) \otimes 1(3) \) and \( \Delta(1) = 1[1] \otimes 1[2]. \)
Lemma 2.2. Let $H$ be a weak bialgebra over $k$. For all $x, y \in H$

(1) $\varepsilon(x\varepsilon(y)) = \varepsilon(xy)$, $\varepsilon(\varepsilon_s(x)y) = \varepsilon_s(xy)$.

(2) $\varepsilon(x\varepsilon(y)) = \varepsilon(xy)$, $\varepsilon(\varepsilon_s(x)y) = \varepsilon(xy)$.

(3) $\varepsilon \circ \varepsilon_t = \varepsilon = \varepsilon \circ \varepsilon_s$.

(4) $x = \varepsilon_t(x(1))x(2) = x(1)\varepsilon_s(x(2))$.

(5) $x\varepsilon_t(y) = \varepsilon_t(x(1)y)x(2)$, $\varepsilon_s(x)y = y(1)\varepsilon_s(xy(2))$.

Lemma 2.3. Let $H$ be a weak bialgebra over $k$, and set $H_t := \varepsilon_t(H)$, $H_s := \varepsilon_s(H)$. Then

(1) $x\varepsilon_t(y) = \varepsilon_t(xy)$ for all $x \in H_t$ and $y \in H$.

(2) $\varepsilon_s(xy) = \varepsilon_s(xy)$ for all $x \in H$ and $y \in H_s$.

(3) (i) An element of $H_t$ and an element of $H_s$ commute, and

(ii) $H_t$ and $H_s$ are a left coideal and a right coideal subalgebras of $H$, respectively.

The subalgebras $H_t$ and $H_s$ are called the target and source subalgebras of $H$, respectively. By (2.3) we have

\[ \Delta(1) \in H_s \otimes H_t. \] (2.4)

(4) For all $x \in H$, $z \in H_t$ and $y \in H_s$

\[ \Delta(xz) = z(1)x \otimes x(2), \quad \Delta(zx) = z(1) \otimes x(2), \]
\[ \Delta(xy) = x(1) \otimes x(2)y, \quad \Delta(yx) = x(1) \otimes yx(2). \]

In particular

\[ xz = \varepsilon(x(1))z(2), \quad zx = \varepsilon(zx(1))x(2), \]
\[ xy = \varepsilon(x(1))y, \quad yx = x(1)\varepsilon_s(yx(2)). \]

For a weak bialgebra $H$, one can also consider two $k$-linear maps $\varepsilon'_t, \varepsilon'_s : H \rightarrow H$ defined by

\[ \varepsilon'_t(x) = \varepsilon(x1(1))1(2), \] (2.5)
\[ \varepsilon'_s(x) = 1(1)\varepsilon(1(2)x) \] (2.6)

for all $x \in H$. Then, $H^{\text{op}} = (H, \mu^{\text{op}}, \eta, \Delta, \varepsilon)$, $H^{\text{cop}} = (H, \mu, \eta, \Delta^{\text{cop}}, \varepsilon)$, $H^{\text{opcop}} = (H, \mu^{\text{cop}}, \eta, \Delta^{\text{cop}}, \varepsilon)$ are weak bialgebras over $k$, where $\mu^{\text{op}}$ and $\Delta^{\text{cop}}$ are the opposite multiplication and comultiplication, respectively. The target and the source subalgebras of them are given by $(H^{\text{op}})_t = H_t$, $(H^{\text{op}})_s = H_s$, $(H^{\text{cop}})_t = H_s$, $(H^{\text{cop}})_s = H_t$, $(H^{\text{opcop}})_t = H_s$, $(H^{\text{opcop}})_s = H_t$. The target and the source counital maps of them are given by $(\varepsilon_{H^{\text{op}}})_t = \varepsilon'_t$, $(\varepsilon_{H^{\text{op}}})_s = \varepsilon'_s$, $\varepsilon^{\text{cop}}_{H^{\text{cop}}}$.
\((\varepsilon_{\text{H\text{-cop}}})_t = \varepsilon'_s, (\varepsilon_{\text{H\text{-cop}}})_s = \varepsilon'_t, (\varepsilon_{\text{H\text{-cop}}}^\circ_p)_t = \varepsilon_s, (\varepsilon_{\text{H\text{-cop}}}^\circ_p)_s = \varepsilon_t\). If \(S\) is the antipode of \(H\), then it is also of \(H^{\text{op\text{-cop}}}\).

For a weak bialgebra \(H\) over \(k\), we denote by \(\text{M}^H\) the \(k\)-linear category whose objects are right \(H\)-comodules and morphisms are \(H\)-comodule maps between them. The comodule category \(\text{M}^H\) has a monoidal structure [13, Section 4] as the following lemma.

**Lemma 2.4 ([14, Lemma 4.2]).** Let \(H\) be a weak bialgebra over \(k\).

1. For two right \(H\)-comodules \((M, \rho_M), (N, \rho_N)\), the pair \((M, \rho_M) \oplus (N, \rho_N) := (M \otimes_{\text{H}} N, \rho)\) is also a right \(H\)-comodule, where \(\rho : M \otimes_{\text{H}} N \rightarrow (M \otimes_{\text{H}} N) \otimes H\) is a \(k\)-linear map defined by

\[
\rho(m \otimes_{\text{H}} n) = (m_{(0)} \otimes_{\text{H}} n_{(0)}) \otimes_{\text{H}} m_{(1)}n_{(1)} \quad (m \in M, n \in N).
\] (2.7)

The source algebra \(H_s\) can be regarded as a right \(H\)-comodule with the coaction \(\Delta_s := \Delta|_{H_s} : H_s \rightarrow H_s \otimes H\), and for all right \(H\)-comodules \(L, M, N\) there are natural isomorphisms

1. \(H_s \otimes M \cong M \cong M \otimes H_s\) as right \(H\)-comodules,
2. \((L \otimes M) \otimes N \cong L \otimes (M \otimes N)\) as right \(H\)-comodules.

Here the natural isomorphisms in (i) are given as follows: For a right \(H\)-comodule \(M\)

\[
l_M : H_s \otimes_{\text{H}} M \rightarrow M, \quad l_M(y \otimes_{\text{H}} m) = y \cdot m \quad (y \in H_s, m \in M),
\]

\[
l^{-1}_M : M \rightarrow H_s \otimes_{\text{H}} M, \quad l^{-1}_M(m) = 1 \otimes_{\text{H}} m \quad (m \in M),
\]

\[
r_M : M \otimes_{\text{H}} H_s \rightarrow M, \quad r_M(m \otimes_{\text{H}} y) = m \cdot y \quad (m \in M, y \in H_s),
\]

\[
r^{-1}_M : M \rightarrow M \otimes_{\text{H}} H_s, \quad r^{-1}_M(m) = m \otimes_{\text{H}} 1 \quad (m \in M).
\]

The isomorphism in (ii) is induced from a usual isomorphism between vector spaces.

2. Let \(f : (M, \rho_M) \rightarrow (N, \rho_N), g : (M', \rho_M') \rightarrow (N', \rho_N')\) be right \(H\)-comodule maps. Then \(f \otimes_{\text{H}} g : M \otimes_{\text{H}} M' \rightarrow N \otimes_{\text{H}} N'\) is also a right \(H\)-comodule map with respect to the comodule structure given in (1). We denote the map \(f \otimes_{\text{H}} g\) by \(f \oplus g\).

By Parts (1), (2) the abelian category \(\text{M}^H\) becomes a \(k\)-linear monoidal category whose unit object is \((H_s, \Delta_s)\).

**Lemma 2.5 ([14, Proposition 4.1]).** Let \(H\) be a weak bialgebra over \(k\), and \((M, \rho_M)\) be a right \(H\)-comodule. For \(y \in H_s\) and \(m \in M\), the elements \(y \cdot m\) and \(m \cdot y\) in \(M\) are defined as follows:

\[
y \cdot m := m_{(0)}\varepsilon(ym_{(1)}), \quad (2.8)
\]

\[
m \cdot y := m_{(0)}\varepsilon(m_{(1)}y). \quad (2.9)
\]

1. \(M\) becomes an \((H_s, H_s)\)-bimodule equipped with the above actions.
(2) $M \otimes H$ becomes an $(H_s, H_s)$-bimodule equipped with the following actions: For $y \in H_s$, $m \in M$, $x \in H$,

$$
y \cdot (m \otimes x) := (1_{(1)} \cdot m) \otimes (y1_{(2)}x) = m_{(0)} \otimes \varepsilon_s(m_{(1)})yx, \quad (2.10)
$$

$$
(m \otimes x) \cdot y := (m \cdot 1_{(1)}) \otimes (xy1_{(2)}) = m_{(0)} \otimes xy\varepsilon'_s(m_{(1)}). \quad (2.11)
$$

(3) The following equations hold for all $y \in H_s$ and $m \in M$:

(i) $\rho_M(y \cdot m) = m_{(0)} \otimes ym_{(1)} = y \cdot \rho_M(m)$.

(ii) $\rho_M(m \cdot y) = m_{(0)} \otimes m_{(1)}y = \rho_M(m) \cdot y$.

(iii) $\varepsilon'_s(m_{(1)}) \cdot m_{(0)} = m = m_{(0)} \cdot \varepsilon_s(m_{(1)})$.

In particular, $\rho_M : M \rightarrow M \otimes H$ is an $(H_s, H_s)$-bimodule map by (i) and (ii).

(4) Let $(N, \rho_N)$ be a right $H$-comodule and $f : (M, \rho_M) \rightarrow (N, \rho_N)$ be an $H$-comodule map. Then $f$ becomes an $(H_s, H_s)$-bimodule map with respect to the bimodule structures given by (1).

Let $H$ be a weak bialgebra over $k$, and denote by $H_s \mathbf{M}_{H_s}$ the $k$-linear category consisting of whose objects are $(H_s, H_s)$-bimodules and morphisms are $(H_s, H_s)$-bimodule maps between them. By Lemma 2.5 for a right $H$-comodule $(M, \rho_M)$, the underlying vector space $M$ has an $(H_s, H_s)$-bimodule structure, and $\rho_M : M \rightarrow M \otimes H$ becomes an $(H_s, H_s)$-bimodule map. Thus $(M, \rho_M)$ can be regarded as a right $H$-comodule in $H_s \mathbf{M}_{H_s}$. Then any $H$-comodule map $f : M \rightarrow N$ always an $(H_s, H_s)$-bimodule map.

We denote by $H_s \mathbf{M}^H_{H_s}$ the $k$-linear category whose objects are right $H$-comodules in $H_s \mathbf{M}_{H_s}$ and morphisms are right $H$-comodule and $(H_s, H_s)$-bimodule maps between them. The category $H_s \mathbf{M}^H_{H_s}$ has a $k$-linear monoidal structure whose tensor product is given by $\otimes_{H_s}$.

As a special case of [18, Theorem 2.2] we have:

**Lemma 2.6.** Let $H$ be a weak bialgebra over $k$. By Lemma 2.5 for a right $H$-comodule $(M, \rho_M)$, there is an $(H_s, H_s)$-bimodule structure on the underlying vector space $M$, and $\rho_M : M \rightarrow M \otimes H$ becomes an $(H_s, H_s)$-bimodule map. Thus $(M, \rho_M)$ can be regarded as a right $H$-comodule in $H_s \mathbf{M}_{H_s}$. Then any $H$-comodule map $f : M \rightarrow N$ is always an $(H_s, H_s)$-bimodule map. This correspondence gives rise to a $k$-linear monoidal equivalence $\Xi^H : \mathbf{M}^H \rightarrow H_s \mathbf{M}^H_{H_s}$ between $k$-linear monoidal categories.

We set

$$
\hat{U}^H := U^H \circ \Xi^H : \mathbf{M}^H \rightarrow H_s \mathbf{M}_{H_s},
$$

where $U^H : H_s \mathbf{M}^H_{H_s} \rightarrow H_s \mathbf{M}_{H_s}$ is the forgetful monoidal functor. The composition $\hat{U}^H$ of monoidal functors becomes a monoidal functor, whose structure is given as follows:

- $\phi_{M, N}^H = \text{id}_{M \otimes H_s N}$ for each $M, N \in \mathbf{M}^H$,

- $\omega^H : H_s \rightarrow \hat{U}^H(H_s) = H_s$ is the identity map.
Lemma 2.7. Let $H, K$ be weak bialgebras over $k$, and $\varphi : H \rightarrow K$ be a weak bialgebra map, namely, an algebra map and coalgebra map. Then $\varphi(H_s) \subset K_s$ and $\varphi(H_t) \subset K_t$. The former inclusion induces an algebra map $\varphi_s := \varphi|_{H_s} : H_s \rightarrow K_s$.

(1) For a right $H$-comodule $(M, \rho_M)$

$$M^\varphi(M, \rho_M) := (M, (id_M \otimes \varphi) \circ \rho_M)$$

is a right $K$-comodule, and for a right $H$-comodule map $f : (M, \rho_M) \rightarrow (N, \rho_N)$

$$M^\varphi(f) := f : M^\varphi(M, \rho_M) \rightarrow M^\varphi(N, \rho_N)$$

is a right $K$-comodule map. These correspondences define a covariant functor $M^\varphi : M^H \rightarrow M^K$.

(2) The functor $M^\varphi$ is a $k$-linear comonoidal functor. If $\varphi_s$ is bijective, then the $k$-linear comonoidal functor $M^\varphi$ is strong. In this case it can be regarded as a $k$-linear monoidal functor.

(3) The algebra map $\varphi_s$ induces a $k$-linear monoidal functor $\varphi_* M_{\varphi_s} : K_s M_{K_s} \rightarrow H_s M_{H_s}$, and if $\varphi_s$ is bijective, then $U^K \circ M^\varphi = \varphi_* M^\varphi^{-1} \circ U^H : M^H \rightarrow K_s M_{K_s}$ as monoidal functors.

Proof. (2) For right $H$-comodules $(M, \rho_M), (N, \rho_N)$ we set

$$M^\varphi(M, \rho_M) \otimes M^\varphi(N, \rho_N) = (M \otimes K, N, \rho),
\quad M^\varphi((M, \rho_M) \otimes (N, \rho_N)) = (M \otimes H, N, \rho'),$$

where $\rho$ and $\rho'$ are given as follows:

$$\rho : M \otimes K, N \rightarrow (M \otimes K, N) \otimes K,
\quad \rho(m \otimes K, n) = m(0) \otimes K, n(0) \otimes \varphi(m(1)) \varphi(n(1)),$$

$$\rho' : M \otimes H, N \rightarrow (M \otimes H, N) \otimes K,
\quad \rho'(m \otimes H, n) = m(0) \otimes H, n(0) \otimes \varphi(m(1)) \varphi(n(1)).$$

Since $\varphi$ is an algebra map, the identity map $id_{M \otimes N}$ induces a surjection $\iota_{M,N} : M \otimes H, N \rightarrow M \otimes K, N$ which is a right $K$-comodule map.

The restriction $\varphi_s : H_s \rightarrow K_s$ can be regarded as a right $K$-comodule map from $M^\varphi(H_s, (\Delta_H)_s) = (H_s, (id_{H_s} \otimes \varphi) \circ (\Delta_H)_s)$ to $(K_s, (\Delta_K)_s)$. Thus, the triplet $(M^\varphi, \iota, \varphi_s)$ is a comonoidal functor from $M^H$ to $M^K$, where $\iota := \{\iota_{M,N}\}_{M, N \in M^H}$.

If $\varphi_s$ is bijective, then $\iota_{M,N}$ for all $M, N \in M^H$ is an isomorphism. Thus, $(M^\varphi, \iota^{-1}, \varphi_s^{-1}) : M^H \rightarrow M^K$ is a strong monoidal functor.

(3) For a $(K_s, K_s)$-bimodule $(M, \alpha_l, \alpha_r)$ we set

$$\varphi_* M_{\varphi_s}(M, \alpha_l, \alpha_r) = (M, \alpha_l \circ (\varphi_s \otimes id_M), \alpha_r \circ (id_M \otimes \varphi_s)),$$

and a $(K_s, K_s)$-bimodule map $f : M \rightarrow N$ we set

$$\varphi_* M_{\varphi_s}(f) = f.$$

Then, $\varphi_* M_{\varphi_s}$ is a $k$-linear covariant functor.
For \((K_s, K_s)\)-bimodules \(M = (M, \alpha^M_i, \alpha^M_r)\) and \(N = (N, \alpha^N_i, \alpha^N_r)\), let

\[ J_{M,N} : \varphi_s M \otimes_{H_s} \varphi_s N \rightarrow \varphi_s M \otimes_{K_s} N \]

be the induced \(k\)-linear map from the identity map \(\text{id}_{M \otimes N}\). This is an \((H_s, H_s)\)-module map which is natural with respect to \(M, N\). So, we have a monoidal functor \((\varphi_s, \varphi_s, \varphi_s)\) by setting \(\varphi_s = \{ \varphi_s \}_{M,N} \in (K_s)^{M \otimes N}\). If \(\varphi_s\) is bijective, then one can show that \(\hat{U}_K \circ \varphi_s = \varphi_s^{-1} \circ \hat{U}_H\) as \(k\)-linear monoidal functors.

### 3 Reconstruction of a weak bialgebra map

First of all, we will recall the proof of the reconstruction theorem of coalgebra maps, that is a classical and fundamental theorem for Tannakian reconstruction theorem.

For a coalgebra \(C\) over \(k\) we denote by \(M^C\) the \(k\)-linear category consisting of whose objects are right \(C\)-comodules of finite dimension and morphisms are \(C\)-comodule maps between them, and denote by \(U^C\) the forgetful functor from \(M^C\) to \(\text{Vect}_k\).

**Theorem 3.1.** Let \(C, D\) be coalgebras over \(k\), and \(F : M^C \rightarrow M^D\) be a \(k\)-linear covariant functor. If \(U^D \circ F = U^C\), then there is a unique coalgebra map \(\varphi : C \rightarrow D\) such that \(F = M^\varphi : M^C \rightarrow M^D\). Here, \(M^\varphi\) denotes the \(k\)-linear functor induced from \(\varphi\).

**Proof.** The proof follows from Franco’s lecture note [6, p.81–84].

Let \((M, \rho_M)\) be a finite-dimensional \(C\)-comodule. By the assumption \(U^D \circ F = U^C\), one can set \(F(M, \rho_M) = (M, \rho^F_M)\), where \(\rho^F_M : M \rightarrow M \otimes D\) is a right coaction of \(D\).

Let \(P\) be a finite-dimensional subcoalgebra of \(C\). It can be regarded as a right \(C\)-comodule by the coaction

\[ \rho_P : P \xrightarrow{\Delta_P} P \otimes P \xrightarrow{id \otimes \iota_P} P \otimes C, \]

where \(\iota_P\) stands for the inclusion. Since \(P\) is finite-dimensional, it gives an object of \(M^C\), and therefore we have \(F(P, \rho_P) = (P, \rho^F_P) \in M^D\).

Let us consider the composition

\[ \varphi_P : P \xrightarrow{\rho^F_P} P \otimes D \xrightarrow{\varepsilon_P \otimes \text{id}} k \otimes D \cong D. \]

Then \(\varphi_P : P \rightarrow D\) is a coalgebra map. This fact can be verified as follows.

- The equation \(\varepsilon_D \circ \varphi_P = \varepsilon_P\) comes from the following commutative diagram.

\[
\begin{array}{ccc}
P & \xrightarrow{\rho^F_P} & P \otimes D \\
\downarrow{id} & & \downarrow{id} \\
\cong & & \cong \\
P & \xrightarrow{\varepsilon_P \otimes \text{id}} & k \otimes D \\
\downarrow{id} & & \downarrow{id} \\
P \otimes k & \xrightarrow{\varepsilon_P \otimes \text{id}} & k \otimes k \\
\downarrow{\varepsilon_D} & & \downarrow{\varepsilon_D} \\
& & k
\end{array}
\]


To show the equation $\Delta_D \circ \varphi_P = (\varphi_P \otimes \varphi_P) \circ \Delta_P$, it is enough to verify the following diagram commutes:

Part (A) is commutative since $\rho^F_P$ is a right $D$-coaction, and Part (C) is commutative since the following diagram is so.

The proof the commutativity of Part (B) is a little technical. For any $k$-linear map $\gamma : P \longrightarrow k$, the composition

$$P \xrightarrow{\Delta_P} P \otimes P \xrightarrow{\gamma \otimes \text{id}_P} P$$

is a right $P$-comodule map, and hence it is also a right $C$-comodule map. Sending it by $F$ we have a right $D$-comodule map $(\gamma \otimes \text{id}_P) \circ \Delta_P : P \longrightarrow P$. Thus the following diagram commutes:

Combining the above diagram with $\rho^F_P \circ (\gamma \otimes \text{id}_P) = (\gamma \otimes \text{id}_{P \otimes D}) \circ (\text{id}_P \otimes \rho^F_P)$, we have

$$(\gamma \otimes \text{id}_{P \otimes D}) \circ (\text{id}_P \otimes \rho^F_P) \circ \Delta_P = (\gamma \otimes \text{id}_{P \otimes D}) \circ (\Delta_P \otimes \text{id}_P) \circ \rho^F_P.$$
Since this equation holds for all $k$-linear maps $\gamma$, we see that $(\text{id}_P \otimes \rho_P^F) \circ \Delta = (\Delta \otimes \text{id}_P) \circ \rho_P^F$. This implies the commutativity of Part (B).

By the fundamental theorem for coalgebras, $C$ is a union of finite-dimensional subcoalgebras. Based on the fact, it can be shown that the coalgebra maps $\varphi_P : P \rightarrow D$ for all finite-dimensional subcoalgebras $P$ induce a well-defined coalgebra map $\varphi : C \rightarrow D$. In fact it is easily verified that $(\varphi_Q)|_P = \varphi_P$ for two finite-dimensional subcoalgebras $P$ and $Q$ satisfying $P \subset Q$. In this way, it is proved that there is a coalgebra map $\varphi : C \rightarrow D$ such that $\varphi|_P = \varphi_P$ for all finite-dimensional subcoalgebra $P$ of $C$.

The coalgebra map $\varphi$ satisfies $F = M\varphi$. This fact can be derived as follows.

Let $(M, \rho_M)$ be a finite-dimensional right $C$-comodule. Then there is a finite-dimensional subcoalgebra $P$ of $C$ such that $\rho_M(M) \subset M \otimes P$. Since $(M \otimes P, \text{id}_M \otimes \rho_P)$ is a right $C$-comodule, we have a right $D$-comodule $(M \otimes P, (\text{id}_M \otimes \rho_P)^F)$. Since $M \otimes P$ is decomposed to a direct sum of finite copies of $P$ as a right $C$-comodule and $F$ is a $k$-linear functor satisfying $U^C = U^D \circ F$, it follows that $(\text{id}_M \otimes \rho_P)^F = \text{id}_M \otimes \rho_P^F : M \otimes P \rightarrow M \otimes P \otimes D$.

Let $\rho'_M : M \rightarrow M \otimes P$ be the restriction of $\rho_M$. Then the map $\rho'_M$ is a right $C$-comodule map from $(M, \rho_M)$ to $(M \otimes P, \text{id} \otimes \rho_P)$. Thus, $F(\rho'_M) = \rho'_M : M \rightarrow M \otimes P$ is a right $D$-comodule map from $(M, \rho'_M)$ to $(M \otimes P, (\text{id} \otimes \rho_P)^F) = (M \otimes P, \text{id}_M \otimes \rho_P^F)$, and hence the following diagram commutes.

It follows that we have the commutative diagram:

This implies that $\rho_M^F = (\text{id} \otimes \varphi) \circ \rho_M$, and hence we see that $F(M, \rho_M) = M\varphi(M, \rho_M)$. 
Let \( f : (M, \rho_M) \rightarrow (N, \rho_N) \) be a right \( C \)-comodule map between finite-dimensional right \( C \)-comodules. Then
\[
F(f) = f : (M, \rho_M^F) \rightarrow (N, \rho_N^F),
\]
\[
M^p(f) = f : (M, (id \otimes \varphi) \circ \rho_M) \rightarrow (N, (id \otimes \varphi) \circ \rho_N).
\]
As shown that \( \rho_M^F = (id \otimes \varphi) \circ \rho_M \) and \( \rho_N^F = (id \otimes \varphi) \circ \rho_N \), we see that \( F(f) = M^p(f) \). Thus \( F = M^p \) as \( k \)-linear functors.

Finally, we show the uniqueness of \( \varphi \). Suppose that a coalgebra map \( \psi : C \rightarrow D \) satisfies \( F = M^\psi \), too. Let \( P \) be a finite-dimensional subcoalgebra of \( C \), and regard it as the right \( C \)-comodule \( (P, \rho_P) \). Since \( \bigcap \rho_P = \bigcap \rho_P \), the following diagram commutes.

Since the diagram

commutes, and the same commutative diagram holds for \( \psi \), we have \( \varphi \circ \iota_P = \psi \circ \iota_P \). This implies that \( \varphi = \psi \) since \( C \) can be regarded as a union of finite-dimensional subcoalgebras. \( \square \)

**Remark 3.2.** The coalgebra map \( \varphi_P : P \rightarrow D \) in the above proof is a right \( D \)-comodule map from \( (P, \rho_P^F) \) to \( (D, \Delta_D) \). It follows from the following commutative diagram.
Here, commutativity of Parts (1) and (2) come from the definition of $\varphi_P$, and Part (3) comes from what $(P, \rho_P)$ is a right $D$-comodule.

As an application of Theorem 3.1 we have:

**Corollary 3.3.** Let $C, D$ be two coalgebras over $k$, and $F : M^C \longrightarrow M^D$ be a $k$-linear functor. If $F$ is an equivalence of $k$-linear categories and $U^D \circ F = U^C$ is satisfied, then the coalgebra map $\varphi : C \longrightarrow D$ determined by $F = M^{\varphi} : M^C \longrightarrow M^D$ in Theorem 3.1 is an isomorphism.

The above corollary can be easily proved by taking a quasi-inverse $G$ of $F$ such as $U^C \circ G = U^D$, $G \circ F = 1_{M^C}$ and $F \circ G = 1_{M^D}$.

Dualizing of Theorem 3.1 we have also the following corollary.

**Corollary 3.4.** Let $A, B$ be two finite-dimensional algebras over $k$, and $F : B^M \longrightarrow A^M$ be a $k$-linear functor. If $A_U \circ F = B_U$ is satisfied for the forgetful functors $A_U, B_U$ to Vect$_k$, then

1. there is a unique algebra map $\varphi : A \longrightarrow B$ such that $F = \varphi^M : B^M \longrightarrow A^M$, where $\varphi^M$ stands for the $k$-linear coalgebra induced from $\varphi$,

2. if $F$ is an equivalence of $k$-linear categories, then the algebra map $\varphi : A \longrightarrow B$ given by (1) is an isomorphism.

Now, we show the main theorem in the present paper:

**Theorem 3.5.** Let $A, B$ be weak bialgebras over $k$, and $F = (F, \tilde{\varphi}^F, \tilde{\phi}^F) : M^A \longrightarrow M^B$ be a strong $k$-linear comonoidal functor. Suppose that $U^B \circ F = U^A$ as $k$-linear monoidal categories. Then there is a unique weak bialgebra map $\varphi : A \longrightarrow B$ such that $F = M^{\varphi}$ as $k$-linear monoidal categories and $\bar{\varphi}^F = \varphi|_{A_s} : A_s \longrightarrow B_s$ is an algebra isomorphism. Furthermore, $\hat{U}^B \circ F = \varphi^{A_s-1} \bar{\varphi}^{A} \circ \hat{U}^A$ is satisfied, where $\hat{U}^A : M^A \longrightarrow A_s A_s$, $\hat{U}^B : M^B \longrightarrow B_s B_s$ are forgetful functors.

**Proof.** By Theorem 3.1 there is a unique coalgebra map $\varphi : A \longrightarrow B$ such that $F = M^{\varphi}$ as $k$-linear functors. Since $U^B \circ F = U^A$ as $k$-linear monoidal functors, for all $M, N \in M^A$ the composition

$$U^B(F(M)) \otimes U^B(F(N)) \xrightarrow{\varphi_{F(M),F(N)}^{U_B}} U^B(F(M) \otimes_{A_s} F(N)) \xrightarrow{U^B((\bar{\varphi}^{F, \varphi}_{M,N})^{-1})} U^B(F(M \otimes_{A_s} N))$$

coincides with the natural projection $\varphi_{M,N}^U : U^A(M) \otimes U^A(N) \longrightarrow U^A(M \otimes A_s N)$. This implies that the diagram

$$\begin{array}{ccc}
M \otimes A & \xrightarrow{\varphi_{M,N}} & M \otimes B_s N \\
\downarrow \text{natural proj.} & & \downarrow \text{natural proj.} \\
A \otimes_{A_s} M \otimes N & \xrightarrow{id_{M \otimes N}} & M \otimes N
\end{array}$$
commutes. This means that the map \( \bar{\varphi}_{M,N}^F : F(M \otimes A_s N) \to F(M) \otimes_B F(N) \) is induced from the identity map \( \text{id}_{M \otimes N} \).

Let us show that \( \bar{\omega}^F = \varphi|_{A_s} : A_s \to B_s \) is an algebra isomorphism. Since \( U^B \circ F = U^A \) as \( k \)-linear monoidal functors, the composition

\[
U^B \left((\bar{\omega}^F)^{-1}\right) \circ \omega^{U^B} : k \to (U^B \circ F)(A_s) = A_s
\]

coincides with \( \omega^{U^A} : k \to A_s \). Thus \( (\bar{\omega}^F)^{-1} \circ \omega^{U^B} = \omega^{U^A} \) as maps, and hence \( \bar{\omega}^F(1) = 1 \).

Since \( \bar{\omega}^F : F(A_s) \to B_s \) is a right \( B \)-comodule map and \( F(A_s) = \overline{M}^p(A_s) \), the following diagram commutes.

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\bar{\varphi}^F} & M \otimes N \\
\Delta_A|_{A_s} & & \Delta_B|_{B_s} \\
\downarrow & & \downarrow \\
M \otimes A_s A & \xrightarrow{\bar{\varphi}^F \otimes \text{id}} & M \otimes B_s N \\
\text{id} \otimes \varphi & & \\
M \otimes A_s A & \xrightarrow{\bar{\varphi}^F \otimes \text{id}} & M \otimes B_s N
\end{array}
\]

Thus \( \Delta_B(\bar{\omega}^F(y)) = \bar{\omega}^F(y_{(1)}) \otimes \varphi(y_{(2)}) \) for all \( y \in A_s \). Applying \( \varepsilon_B \otimes \text{id} \) to both sides, we have

\[
\bar{\omega}^F(y) = \varepsilon_B(\bar{\omega}^F(y_{(1)})) \varphi(y_{(2)})
= \varepsilon_A(y_{(1)}) \varphi(y_{(2)})
= \varphi(y).
\]

This implies that \( \bar{\omega}^F = \varphi|_{A_s} : A_s \to B_s \), and hence \( \varphi(1) = \bar{\omega}^F(1) = 1 \).

Next we show that \( \bar{\omega}^F \) preserves products. Since \( F = (F, \bar{\varphi}^F, \bar{\omega}^F) \) is a comonoidal functor, the following diagram commutes for all \( M \in M^A \):

\[
\begin{array}{ccc}
F(A_s \otimes A_s M) & \xrightarrow{F(I_M^A)} & F(M) \\
\bar{\varphi}_{A_s,M}^F & & I_M^B \\
F(A_s) \otimes_{B_s} F(M) & \xrightarrow{\bar{\omega}^F \otimes_{B_s} \text{id}} & B_s \otimes_{B_s} F(M)
\end{array}
\]

In particular, the following diagram commutes:
preserves products. For two subspaces \( a \leq A \) and \( \rho \), let \( \mu \) denote the subspace of \( A \) spanned by the set \{ \( pp' \mid p \in P, p' \in P' \) \}. Let \( \mu_{P,P'} : P \otimes P' \to PP' \) be the restriction of the product \( \mu_A \) of \( A \). Then \( \mu_{P,P'} \) induces a \( k \)-linear map \( \bar{\mu}_{PP'} : P \otimes A \to PP' \) since the equation \((p \cdot y)p' = p(y \cdot p')\) holds for \( p \in P, y \in A, p' \in P' \).

Next let us show that \( \varphi \) preserves products. For two subspaces \( P, P' \) of \( A \), let \( PP' \) denote the subspace of \( A \) spanned by the set \{ \( pp' \mid p \in P, p' \in P' \) \}. Let \( \mu_{P,P'} : P \otimes P' \to PP' \) be the restriction of the product \( \mu_A \) of \( A \). Then \( \mu_{P,P'} \) induces a \( k \)-linear map \( \bar{\mu}_{PP'} : P \otimes A \to PP' \) since the equation \((p \cdot y)p' = p(y \cdot p')\) holds for \( p \in P, y \in A, p' \in P' \).

Let \( a, a' \in A \), and let \( P, P' \) be finite-dimensional subcoalgebras of \( A \) such that \( a \in P, a' \in P' \). Then \( PP' \) is also a finite-dimensional subcoalgebra of \( A \) containing \( aa' \), and the map \( \bar{\mu}_{PP'} \) is a right \( A \)-comodule map. Let \( \varphi_P, \varphi_{P'} \) be the coalgebra maps defined in the proof of Theorem 3.1. We will show that the following diagram commutes.

\[
\begin{array}{ccc}
F(P \otimes A) & \xrightarrow{\tilde{\varphi}_{P,P'}} & F(P') \\
\downarrow{\varphi_P} & & \downarrow{\varphi_{P'}} \\
B & \xrightarrow{\bar{\mu}_B} & B
\end{array}
\]

Here, \( \mu_B \) is the induced map from the product \( \mu_B \) of \( B \). Since \( \varphi_P, \varphi_{P'} \) are right \( B \)-comodule maps by Remark 3.2, the map \( \varphi_P \otimes B, \varphi_{P'} \) is well-defined. Since \( F = \mathbf{M}^P \) and \( \tilde{U}^B \circ \mathbf{M}^P = \varphi^{-1} \mathbf{M}_P \circ \tilde{U}^A \) as \( k \)-linear monoidal functors, one can verify that \( \tilde{\varphi}_{P,P'}^F : F(P \otimes A) \to F(P') \) is a \( k \)-linear map given by \( \tilde{\varphi}_{P,P'}^F(p \otimes A, p') = p \otimes B, p' \) such that \( \bar{\varphi}_{P,P'}^F \) is the right coaction given by \( \bar{\varphi}_{P,P'}^F(p \otimes B, n) = m(n_0) \otimes m_1 n_1 \) for \( m \in F(P), n \in F(P') \).
Then it can be shown that the following diagram commutes.

\[
\begin{array}{ccc}
F(P \otimes_{A_s} P') & \xrightarrow{\rho^F_{P \otimes_{A_s} P'}} & F(P \otimes_{A_s} P') \otimes B \\
\tilde{\phi}^F_{P,P'} & & \tilde{\phi}^F_{P,P'} \otimes \text{id}_B \\
F(P) \otimes_{B_s} F(P') & \xrightarrow{\rho^F_{P,P'}} & (F(P) \otimes_{B_s} F(P')) \otimes B
\end{array}
\]

On the other hand, there is a \((B_s, B_s)\)-bimodule map \(\chi_{P, P'} : (P \otimes B) \otimes_{B_s} (P' \otimes B) \to (P \otimes_{B_s} P') \otimes B\), which is defined by

\[
\chi_{P, P'}((p \otimes b) \otimes_{B_s} (p' \otimes b')) = (p \otimes_{B_s} p') \otimes b b' \quad (p \in P, p' \in P', b, b' \in B).
\]

We also define a \(k\)-linear map \(\varepsilon_{P, P'} : P \otimes_{A_s} P' \to k\) by

\[
\varepsilon_{P, P'}(p \otimes_{A_s} p') = \varepsilon(p_1(1)) \varepsilon(1_{(2)} p') \quad (p \in P, p' \in P').
\]

Then it can be shown that the following diagram commutes.

\[
\begin{array}{ccc}
F(P \otimes_{A_s} P') & \xrightarrow{\tilde{\phi}^F_{P,P'}} & F(P) \otimes_{B_s} F(P') \\
\rho^F_{P \otimes_{A_s} P'} & & \chi_{P, P'} \quad (F(\varepsilon_P) \otimes \text{id}) \rho, (F(\varepsilon_{P'}) \otimes \text{id}) \\
F(P) \otimes_{B_s} F(P') & \xrightarrow{\varepsilon_{P, P'}} & k \otimes B \quad (*) \\
\varphi_{P \otimes_{B_s} P'} \otimes \text{id} & & \\
F(\mu_{P, P'}) & \xrightarrow{\varepsilon_{P, P'}} & F(\mu_{P, P'}) \otimes \text{id} \quad \equiv \\
F(PP') & \xrightarrow{\varphi_{P'}} & B
\end{array}
\]

In fact, the commutativity of \((*)\) in the above diagram comes from the following three facts:

(i) \(\tilde{\phi}^F_{M, N} : F(M \otimes_{A_s} N) \to F(M) \otimes_{B_s} F(N)\) is an isomorphism,

(ii) the equation (3.2) holds,

(iii) for \(p \in P, p' \in P'\),

- \(p \otimes_{A_s} p' = (p \cdot 1) \otimes_{A_s} p' = (p \cdot 1) \varepsilon_s(1_{(2)}) \otimes_{A_s} p' = p_1(1) \otimes_{A_s} \varepsilon_s(1_{(2)}) p'\),
- \(\varepsilon(\varepsilon_s(1_{(2)}) p') = \varepsilon(1_{(2)} p')\) by Lemma 2.2(2).

Thus, the diagram (3.1) commutes, and therefore \(\varphi\) preserves products.

To complete the proof we need to verify that \(F = M^p\) as \(k\)-linear comonoidal functors. This is an easy consequence of the proof of Lemma 2.7(2) since \(\tilde{\phi}^F_{M, N}\) is induced from the identity map \(\text{id}_{M \otimes N}\) for all \(M, N \in M^A\). \(\square\)
By Corollary 3.3 and Theorem 3.5 we have:

**Corollary 3.6.** Let \( A, B \) be weak bialgebras over \( k \), and \( F = (F, \bar{\phi}^F, \bar{\omega}^F) : M^A \longrightarrow M^B \) be a strongly \( k \)-linear comonoidal functor satisfying \( U_B \circ F = U_A \) as \( k \)-linear monoidal categories. If \( F \) is a \( k \)-linear monoidal equivalence, then the weak bialgebra map \( \varphi : A \longrightarrow B \) determined in Theorem 3.5 is an isomorphism of weak bialgebras.

### 4 Categorical aspects of indecomposable weak bialgebras

Let \( A = (A, \Delta_A, \varepsilon_A) \) and \( B = (B, \Delta_B, \varepsilon_B) \) be two weak bialgebras over \( k \), and set \( H = A \oplus B \). Then, \( H \) has a weak bialgebra structure such that the algebra structure is the product and the coalgebra structure is the direct sum of \( A \) and \( B \). The target and source counital maps \( \varepsilon_t \) and \( \varepsilon_s \) are given by

\[
\varepsilon_t(x) = (\varepsilon_A)_t(a) + (\varepsilon_B)_t(b), \\
\varepsilon_s(x) = (\varepsilon_A)_s(a) + (\varepsilon_B)_s(b)
\]

for all \( x = a + b \in H, \ a \in A, \ b \in B \), where \((\varepsilon_A)_t, (\varepsilon_A)_s\) are the target and source counital maps of \( A \), and \((\varepsilon_B)_t, (\varepsilon_B)_s\) are that of \( B \), respectively. Moreover, the target and source subalgebras are given as follows:

\[
H_t = \varepsilon_t(H) = (\varepsilon_A)_t(A) + (\varepsilon_B)_t(B) = A_t + B_t, \\
H_s = \varepsilon_s(H) = (\varepsilon_A)_s(A) + (\varepsilon_B)_s(B) = A_s + B_s.
\]

We call the above weak bialgebra \( H \) the direct sum of \( A \) and \( B \).

A weak bialgebra \( H \) is called **indecomposable** if there are no weak bialgebras \( A \) and \( B \) such that \( H \cong A \oplus B \). Any finite-dimensional weak bialgebra can be decomposed into the finitely many direct sum of indecomposable ones. More precisely, we have:

**Theorem 4.1.** Let \( H \) be a finite-dimensional weak bialgebra over \( k \). Then

1. there are finitely many indecomposable weak bialgebras \( H_i \) \((i = 1, \ldots, k)\) such that \( H = H_1 \oplus \cdots \oplus H_k \).
2. If indecomposable weak bialgebras \( H_1, \ldots, H_k \) and \( H'_1, \ldots, H'_l \) satisfy
   \[
   H_1 \oplus \cdots \oplus H_k = H = H'_1 \oplus \cdots \oplus H'_l,
   \]
   then \( k = l \), and there is a permutation \( \sigma \in \mathfrak{S}_k \) such that \( H'_i = H_{\sigma(i)} \) for all \( i = 1, \ldots, k \).

The above theorem can be proved by decomposability and uniqueness of finite-dimensional algebras into indecomposable ones (see [21] for detail).

A \( k \)-linear monoidal category \( \mathcal{C} \) is called **indecomposable** if there are no \( k \)-linear monoidal categories \( \mathcal{C}_1, \mathcal{C}_2 \) such that \( \mathcal{C} \cong \mathcal{C}_1 \times \mathcal{C}_2 \) as \( k \)-linear monoidal categories.
Proposition 4.2. Let $H$ be a weak bialgebra over $k$. If $H$ is decomposable, then

(1) the left $H$-module category $H\mathcal{M}$ and the finite-dimensional left $H$-module category $H\mathcal{M}$ are decomposable.

(2) the right $H$-comodule category $\mathcal{M}^H$ and the finite-dimensional right $H$-comodule category $\mathcal{M}^H$ are decomposable.

Proof. Suppose that $H = A \oplus B$ for some weak bialgebras $A, B$ over $k$.

(1) The left $H$-module category $H\mathcal{M}$ is equivalent to the Cartesian product $\mathcal{A}\mathcal{M} \times \mathcal{B}\mathcal{M}$ as $k$-linear monoidal categories. An equivalence is given by the following monoidal functors, which are quasi-inverse each other:

$$(F, \phi^F, \omega^F) : H\mathcal{M} \rightarrow \mathcal{A}\mathcal{M} \times \mathcal{B}\mathcal{M},$$

with $F(X) = (1_A \cdot X, 1_B \cdot X)$,

$$\phi^F_{X,Y} = \text{id}_{F(X) \otimes F(Y)} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y),$$

$$\omega^F = \text{id}_{(A_t, B_t)} : (A_t, B_t) \rightarrow F(H_t),$$

$$(G, \phi^G, \omega^G) : \mathcal{A}\mathcal{M} \times \mathcal{B}\mathcal{M} \rightarrow H\mathcal{M},$$

$$G(U, V) = U \times V,$$

$$\phi^G_{(U_1, V_1), (U_2, V_2)} = \text{id}_{G(U_1, V_1) \otimes G(U_2, V_2)} : G(U_1, V_1) \otimes G(U_2, V_2) \rightarrow G((U_1, V_1) \otimes (U_2, V_2)),$$

$$\omega^G : H_t \rightarrow G(A_t, B_t) = A_t \times B_t,$$

where $\omega^G(z) = (1_A \cdot z, 1_B \cdot z)$ (for some weak bialgebras $A, B$)

By restricting the above equivalence $H\mathcal{M} \simeq \mathcal{A}\mathcal{M} \times \mathcal{B}\mathcal{M}$ to finite dimension a $k$-linear monoidal equivalence $H\mathcal{M} \simeq \mathcal{A}\mathcal{M} \times \mathcal{B}\mathcal{M}$ is obtained.

(2) The right $H$-comodule category $\mathcal{M}^H$ is equivalent to the Cartesian product $\mathcal{A}\mathcal{M}^A \times \mathcal{B}\mathcal{M}^B$ as $k$-linear monoidal categories. An equivalence is given by the following monoidal functors, which are quasi-inverse each other:

$$(F, \phi^F, \omega^F) : \mathcal{M}^H \rightarrow \mathcal{A}\mathcal{M}^A \times \mathcal{B}\mathcal{M}^B,$$

with $F((X, \rho)) = \left((\varepsilon_A \circ \pi_A) \cdot X, (\text{id} \otimes \pi_A) \circ \rho_{(\varepsilon_A \circ \pi_A) \cdot X}, ((\varepsilon_B \circ \pi_B) \cdot X, \text{id} \otimes \pi_B) \circ \rho_{(\varepsilon_B \circ \pi_B) \cdot X})\right)$,

$$\phi^F_{X,Y} = \text{id}_{F(X) \otimes F(Y)} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y),$$

$$\omega^F = \text{id}_{(A_s, B_s)} : (A_s, B_s) \rightarrow F(H_s),$$

$$(G, \phi^G, \omega^G) : \mathcal{A}\mathcal{M}^A \times \mathcal{B}\mathcal{M}^B \rightarrow H\mathcal{M},$$

$$G(U, V) = U \times V,$$

$$\phi^G_{(U_1, V_1), (U_2, V_2)} = \text{id}_{G(U_1, V_1) \otimes G(U_2, V_2)} : G(U_1, V_1) \otimes G(U_2, V_2) \rightarrow G((U_1, V_1) \otimes (U_2, V_2)),$$

$$\omega^G : H_s \rightarrow G(A_s, B_s) = A_s \times B_s,$$

where $\pi_A$ and $\pi_B$ are natural surjections associated with the direct sum decomposition $H = A \oplus B$.

By restricting the above equivalence $\mathcal{M}^H \simeq \mathcal{A}\mathcal{M}^A \times \mathcal{B}\mathcal{M}^B$ to finite dimension a $k$-linear monoidal equivalence $\mathcal{M}^H \simeq \mathcal{A}\mathcal{M}^A \times \mathcal{B}\mathcal{M}^B$ is obtained. \qed
The converse of the above proposition is true. To prove it we need the following reconstruction theorem of bialgebras.

**Theorem 4.3** (Ulbrich [19], Schauenburg [17, Theorem 5.4]). Let \( C \) be a \( k \)-linear monoidal category, and \( \omega : C \to \text{Vect}^f_k \) be a faithful and exact \( k \)-linear monoidal functor. Then, there are a bialgebra \( B \) and a monoidal equivalence \( F : C \to M_B^* \) such that \( U^B \circ F = \omega \), where \( U^B : M_B^* \to \text{Vect}^f_k \) is the forgetful functor.

Combining Theorems 3.5 and 4.3 one can show the following.

**Theorem 4.4.** Let \( H \) be a finite-dimensional weak bialgebra over \( k \). Then \( H \) is indecomposable as a weak bialgebra if and only if the finite-dimensional left \( H \)-module category \( H^* \) is indecomposable as a \( k \)-linear monoidal category.

**Proof.** By the contraposition of Proposition 4.2 “if part” holds. We will show that “only if part”. Suppose that \( H \) is indecomposable whereas \( H^* \) is not. Then, there are \( k \)-linear monoidal categories \( C_1, C_2 \) such that \( H^* \simeq C_1 \times C_2 \) as \( k \)-linear monoidal categories. Let \( U^H : H^* \to \text{Vect}^f_k \) be the forgetful functor, and \( F : C_1 \times C_2 \to H^* \) be a \( k \)-linear monoidal category equivalence. Since the two \( k \)-linear monoidal functors

\[
\begin{align*}
\omega_1 : C_1 &\cong C_1 \times 0 \xrightarrow{F} H^* \xrightarrow{U^H} \text{Vect}^f_k, \\
\omega_2 : C_2 &\cong 0 \times C_2 \xrightarrow{F} H^* \xrightarrow{U^H} \text{Vect}^f_k,
\end{align*}
\]

are faithful and exact, there are bialgebras \( A, B \) and \( k \)-linear monoidal equivalences \( G_1 : C_1 \cong M^A \), \( G_2 : C_2 \cong M^B \) such that \( U^A \circ G_1 = \omega_1 \), \( U^B \circ G_2 = \omega_2 \) by Theorem 4.3. Then

\[
C_1 \times C_2 \cong M^A \times M^B \cong M^{A \oplus B}
\]

as \( k \)-linear monoidal categories. Thus we have \( k \)-linear monoidal equivalence \( G : M^{H^*} \to M^{A \oplus B} \). This equivalence satisfies \( U^{A \oplus B} \circ G = U^{H^*} \). So, by Corollary 3.6, there is a weak bialgebra isomorphism \( g : A \oplus B \to H^* \). Therefore, \( H \cong H^{**} \cong (A \oplus B)^* \cong A^* \oplus B^* \). This contradicts the fact that \( H \) is indecomposable as a weak bialgebra.

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