Flow dynamics of a time-dependent non-Newtonian and non-isothermal fluid between coaxial squeezing disks

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Abstract
The goal of this research is to investigate the behaviours of porosity and squeezing phenomena in the presence of time-dependent heat flow that affect the flow rate and improve the system’s heating/cooling mechanism, reduce non-Newtonian fluid turbulence and scale-up flow tracers. Squeezing discs in the presence of no-slip velocity and convective surface boundary conditions induces a laminar, unstable and incompressible non-Newtonian fluid. The convective form of the momentum, concentration and energy equations are modelled for smooth discs to evaluate and offer an analytical and numerical examination of the flow for heat and mass transfer, which are further transformed to a highly non-linear system of ordinary differential equation using similarity transformations. In the case of smooth disks, the self-similar equations are solved using Homotopy Analysis Method (HAM) with appropriate initial guesses and auxiliary parameters to produce an algorithm with an accelerated and assured convergence. The comparison of HAM solutions with numerical solver programme BVP4c proves the validity and correctness of HAM results. It is found that increasing or bypassing the Hartman number reduces the capillary region, making the Lorentz force effect more visible for small values of non-Newtonian parameter. The concentration rate at the bottom disc rises rapidly as the thermal diffusivity rises. In addition, because the rate of outflow from the flow domain increases, the suction/injection parameter lowers the radial velocity. Additionally, as the non-Newtonian parameter is increased, skin friction and heat/mass flux rise. In the suction/injection situation, all physical characteristics have the opposite effect on flow field profiles.

Keywords
Squeeze flow, contracting channels, second-grade viscoelastic fluid, Lorentz force, HAM, BVP4c

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Introduction
Exploring the behaviour and diverse uses of the non-Newtonian fluid, including the reduction non-Newtonian fluid friction, friction reduction of oil pipelines, large-scale cooling and heating applications, escalation and owing tracers. Surfactant applications for heating and refrigeration devices with large dimensions. The use of surface substances in quarters and buildings is a new technique that is very sensational for substantial cant energy savings. A highly efficient flow tracer has been developed by integrating non-Newtonian

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properties into a colouring track. It is being utilized to design a tracer fluid which may be evicted as a dye-streak into the turbulent flow that prevents dispersion and breakdown as it follows the flow direction. Squeezing flow is a form of flow in which the fluid is compacted amongst two parallel discs reaching to one another and hence radially compressed.

Squeeze flow has various uses in automotive technologies such as compaction and injection, blood attributable to contraction and relaxation of tubes, rotating pistons in motors, hydraulic frequencies, lubrications and handling of materials etc. In the late 20th century, the geometry of squeezing flow attracted scientists’ interest and a lot of research is being done on this. Ali\(^2\) studied the oscillating flow problem in a porous half-space of an incompressible magneto-hydrodynamic (MHD) second-grade fluid. To gradually evolve the solutions of sine and cosine, the Laplace transformation method is applied. Considering here that flow is symmetric at \(y = 0\) and satisfies the no-slip condition at the top layer, approximate results are found up to first order. Domairry\(^3\) also explored the detail study of oscillating fluid flow of second grade incompressible magneto-hydrodynamic (MHD) second-grade fluid. To gradually evolve the solutions of sine and cosine, the Laplace transformation method is applied. Considering here that flow is symmetric at \(y = 0\) and satisfies the no-slip condition at the top surface, approximate results are found up to first order. Khan et al.\(^4\) used laplacian and Fourier transform methods for the fluid substance. For fixed amplitude disruptions, the stability of above flow is studied. Hayat\(^6\) studied the effect of second grade fluid using the technique laplacian transformation. Serth was the first to produce the BVP solution given in Beard and Walters\(^7\) for data simulation the HAM BVPh 2.0 and BVP4c Kits are used. The residual error has been set to \(10^{-40}\), and the analysis are done using a 40th order approximation to minimize the error and to obtain feasible outcomes.

**Mathematical formulation**

Suggesting a second-grade fluid squeezed in the gap of two parallel disks split by a path length \(h(t) = H(1 − \zeta t)^{0.5}\), both disks are squeezed unless they meet at, \(t = \xi^{-1}\) for +ve value of \(\xi\) and separated for −ve value of \(\xi\) as shown in Figure 1. The fluid will be conducted out by electric currents with the insertion of the magnetic-field \(B(t) = B_0(1 − \zeta t)^{−0.5}\) and no induced-magnetic-field will be considered. The polar co-ordinate system \((r, \theta, z)\) is selected for examination of the fluid. The azimuthal component \(V_\theta\) of the velocity field \(\vec{U} = (V_r, V_\theta, V_z)\) is taken zero because of the absence of rotational movement of the disks that is, \(\frac{\partial V_\theta}{\partial r} = 0\). The central point of the bottom disk is fixed as the origin with the use of cylindrical coordinates. At uniform temperatures \(T_1\) and \(T_2\), both the top and bottom disks are preserved. The turbulent, axisymmetric, inconsistent governing equations in polar coordinates system are:

Conservation of mass equation\(^3−5\):

\[
\frac{1}{r} \frac{\partial (r V_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0
\]

Radial-component of the momentum equation\(^3−5\):

\[
\frac{1}{r} \frac{\partial (r V_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0
\]
\[
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_{\theta} \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r} \\
+ \frac{\partial}{\partial r} \left[ \frac{1}{\rho} \left( \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial \tau_{rz}}{\partial z} - \frac{1}{r^2} \tau_{\theta \theta} \right) \right]
\]

Azimuthal-component of the momentum equation:\n
\[
\frac{\partial V_{\theta}}{\partial t} + V_r \frac{\partial V_{\theta}}{\partial r} + V_{\theta} \frac{\partial V_{\theta}}{\partial \theta} + V_z \frac{\partial V_{\theta}}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial \theta} \\
+ \frac{\partial}{\partial \theta} \left[ \frac{1}{\rho} \left( \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \right) \right]
\]

Axial-component of the momentum equation:\n
\[
\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_{\theta} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} \\
+ \frac{\partial}{\partial z} \left[ \frac{1}{\rho} \left( \frac{\partial \tau_{r z}}{\partial r} \right) \right]
\]

The energy equation:\n
\[
\frac{\partial T}{\partial t} + V_r \frac{\partial T}{\partial r} + V_{\theta} \frac{\partial T}{\partial \theta} + V_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right)
\]

The transport equation:\n
\[
\frac{\partial C}{\partial t} + V_r \frac{\partial C}{\partial r} + V_{\theta} \frac{\partial C}{\partial \theta} + V_z \frac{\partial C}{\partial z} = D \left( \frac{\partial^2 C}{\partial r^2} + \frac{\partial^2 C}{\partial \theta^2} + \frac{\partial^2 C}{\partial z^2} \right)
\]

The constitutive equation of second-grade viscoelastic fluid is\n
\[
\tau = \mu A_1 + \beta_1^2 A_2 + \beta_2^2 A_1^2
\]

where \( \beta_1, \beta_2 \) are material constants and \( A_1 \) and \( A_2 \) are Rivlin-Ericksen tensors, such that\n
\[
A_1 = L + L^T, \quad L = \nabla \cdot \vec{u} \quad \text{and} \quad A_2 = \frac{dA_1}{dt} + A_1L + L^TA_1
\]

**Physical BCs**\n
The following physical boundary conditions are considered\n
\[
V_r = 0, \quad V_\theta = 0, \quad V_z = \frac{\partial h}{\partial t}, \quad T = T_e, \quad C = C_e, \quad \text{at} \quad z = h(t) \\
V_r = 0, \quad V_\theta = 0, \quad V_z = \frac{\partial h}{\partial t}, \quad T = T_e, \quad C = C_e, \quad \text{at} \quad z = 0
\]

where \( \tau_q \) are the stress components, \( \rho \) is fluid density, \( V_r, V_\theta, V_z \) are velocity components, similarly \( \kappa, p, C, T, \)

\( C_1, C_\eta, T_l, T_u, D, T_m \) and \( \mu \) are the thermal diffusivity, pressure, concentration variable, temperature variable, concentration and heat at lower and upper disks, diffusion coefficient, mean fluid temperature and dynamic viscosity of the fluid respectively.

Using the shear stress components and velocity field, the components of the Momentum equation are reduce to the following form:

**Radial-component:**\n
\[
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_\theta \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu [\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r}] \\
+ \frac{\partial^2 V_r}{\partial z^2} + \frac{\partial^2 V_r}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial V_r}{\partial \theta}
\]

**Axial-component:**\n
\[
\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_\theta \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu [\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r}] \\
+ \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_z}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial V_z}{\partial \theta}
\]

To convert the above system of partial differential equations into a system of ordinary differential equations the similarity transformations are applied.\n
\( \frac{\partial C}{\partial t} + V_r \frac{\partial C}{\partial r} + V_\theta \frac{\partial C}{\partial \theta} + V_z \frac{\partial C}{\partial z} = D \left( \frac{\partial^2 C}{\partial r^2} + \frac{\partial^2 C}{\partial \theta^2} + \frac{\partial^2 C}{\partial z^2} \right) \)
The mass conservation equation is identically satisfied and the Momentum equation, the heat equation and the mass equation are converted into the following form

\[ f'''(\eta) - S_q[\eta f'''(\eta)] + 3f''(\eta) - 2f(\eta)f''(\eta) - M^2f'''(\eta) + \beta[\eta f'''(\eta)] + 6f''(\eta)f''(\eta) - 4f'(\eta)f'''(\eta) - 4f''(\eta) = 0 \]

(12)

and boundary conditions are reduced to

\[ f(0) = \xi_a, \quad f'(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1 \]
\[ f(1) = \frac{1}{2}, \quad f'(1) = 0, \quad \theta(1) = 0, \quad \phi(1) = 1 \]

(15)

where \( M \) is the Hartman number, \( S_q \) is squeeze number, \( \xi_a \) is suction/blowing parameter, \( Pr \) is Prandtl number and \( L \) is the Lewis number given as

\[ M = \sqrt{\frac{\sigma B_0^2 H^2}{\nu}}; \quad S_q = \frac{\xi H^2}{2 \nu}; \quad \xi_a = \frac{w_0}{\xi H}; \quad Pr = \frac{\nu}{\beta}; \quad L = \frac{\nu}{D} \]

**Solution of the problem by Homotopy Analysis Method**

HAM is applied for the analytical solution of equations (13) to (16). To apply HAM, \( f(\eta), \theta(\eta) \) and \( \phi(\eta) \) could be written as: \(^{14}\)

\[ f_m(\eta) = \sum_{k=0}^{\infty} a_k \eta^k \]

(16)

\[ \theta_m(\eta) = \sum_{k=0}^{\infty} b_k \eta^k \]

(17)

\[ \phi_m(\eta) = \sum_{k=0}^{\infty} c_k \eta^k \]

(18)

The required constants for solution are \( a_k, b_k \) and \( c_k \). The chosen initial guesses are:

\[ f_0(\eta) = (2\xi_a - 1)\eta^3 - \frac{3}{2}(2\xi_a - 1)\eta^2 + \xi_a \]

(19)

\[ \theta_0(\eta) = 1 - \eta \]

(20)

\[ \phi_0(\eta) = 1 - \eta \]

(21)

To choose the auxiliary operators, the following selection is made

\[ \xi_f = \frac{\partial^4}{\partial \eta^4}, \xi_\theta = \frac{\partial^3}{\partial \eta^3}, \xi_\phi = \frac{\partial^2}{\partial \eta^2} \]

(22)

along with

\[ \xi_f(k_1 \eta^3 + k_2 \eta^2 + k_3 \eta + k_4) = 0 \]

(23)

\[ \xi_\phi(k_5 \eta + k_6) = 0 \]

(24)

\[ \xi_\phi(k_7 \eta + k_8) = 0 \]

(25)

where \( k_i \) constants of required solution.

For the solution, the 0th order deformation problems, is selected as:

\[ (1 - q)\xi_f[f'(\eta); q - f_0(\eta)] = q h_f \xi_f[f'(\eta); q] \]

(26)

\[ (1 - q)\xi_\theta[\theta'(\eta); q - \theta_0(\eta)] = q h_\theta \xi_\theta[\theta'(\eta); q] \]

(27)

\[ (1 - q)\xi_\phi[\phi'(\eta); q - \phi_0(\eta)] = q h_\phi \xi_\phi[\phi'(\eta); q] \]

(28)

for \( q = 0 \) and 1, we have

\[ f'(\eta, 0) = f_0(\eta), \quad f'(\eta, 1) = f(\eta), \quad \theta'(\eta, 0) = \theta_0(\eta) = \theta(\eta), \quad \phi'(\eta, 0) = \phi_0(\eta) = \phi(\eta) \]

(29)

therefor as \( q \) vary from 0 to 1, \( f'(\eta, 0), \theta'(\eta, 0), \phi'(\eta, 0) \) vary from initial value \( f_0(\eta), \theta_0(\eta) \) and \( \phi_0(\eta) \) to exact solution \( f(\eta), \theta(\eta) \) and \( \phi(\eta) \) respectively.

So by Taylor’s series:

\[ f(\eta; q) = f_0(\eta) + \sum_{m=1}^{\infty} q^m f_m(\eta) \]

(30)

\[ \theta(\eta; q) = \theta_0(\eta) + \sum_{m=1}^{\infty} q^m \theta_m(\eta) \]

(31)

\[ \phi(\eta; q) = \phi_0(\eta) + \sum_{m=1}^{\infty} q^m \phi_m(\eta) \]

(32)

\[ f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; q)}{\partial q^m} \bigg|_{q=0}, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; q)}{\partial q^m} \bigg|_{q=0}, \quad \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta; q)}{\partial q^m} \bigg|_{q=0} \]

(33)

\( h_f, h_\theta \) and \( h_\phi \) are strongly important for the convergence of above series.

The following auxiliary parameters are selected for the convergence of equations (27) to (29) at \( q = 1. \)
Differentiating the deformation equations (27) to (29) and so the exact solution

The general solution can be written as

\[
 f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \quad (34)
\]

\[
 \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) \quad (35)
\]

\[
 \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) \quad (36)
\]

Differentiating the deformation equations (27) to (29) m-times with respect to \( q \) and putting \( q = 0 \), we have

\[
 L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_{f,m}(\eta) \quad (37)
\]

\[
 L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_{\theta,m}(\eta) \quad (38)
\]

\[
 L_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R_{\phi,m}(\eta) \quad (39)
\]

subject to the boundary conditions

\[
 f_m(0) = \xi_\alpha, \quad f'_m(0) = 0, \quad \theta_m(0) = 0, \quad \phi_m(0) = 1
\]

\[
 f_m(1) = 0.5, \quad f'_m(1) = 0, \quad \theta_m(1) = 0, \quad \phi_m(1) = 0 \quad (40)
\]

The general solution can be written as

\[
 f_m(\eta) = \int_{0}^{\eta} \int_{0}^{\eta} \int_{0}^{\eta} h_f R_{f,m}(z) dz dz \quad (41)
\]

\[
 \theta_m(\eta) = \int_{0}^{\eta} h_\theta R_{\theta,m}(z) dz + \chi_m \theta_{m-1} + k_5 \eta + k_6
\]

\[
 \phi_m(\eta) = \int_{0}^{\eta} h_\phi R_{\phi,m}(z) dz + \chi_m \phi_{m-1} + k_7 \eta + k_8 \quad (43)
\]

and so the exact solution \( f(\eta), \theta(\eta) \) and \( \phi(\eta) \) becomes

\[
 f(\eta) = \sum_{n=0}^{m} f_n(\eta)
\]

\[
 \theta(\eta) = \sum_{n=0}^{m} \theta_n(\eta) \quad (44)
\]

\[
 \phi(\eta) = \sum_{n=0}^{m} \phi_n(\eta)
\]

Error analysis

To numerically analyze the current problem, an error analysis is made by HAM BVPh 2.0 kit, with a maximum 10^{-40} residual-error is used. Analysis is performed via approximations of the 40th order. To achieve the respective optimum convergence the minimize command is used. Table 1 provides the optimum values of controlling parameters and the minimum values for the over-all average residual-error according to the various orders of approximation. Table 2 demonstrates the specific average residual-error at various orders of approximations utilizing the optimum outputs of Table 1. Moreover, the errors curves for the various values of \( m \) and fixed values of \( M, L, \xi_\alpha, Pr \) and \( S_q \) are exposed in Figure 2. It is apparent because, as the approximation order rises, the average squared-errors and the overall average squared-errors are reduced. Authentications of boundary conditions and the comparison of analytical and numerical solutions is shown through Table 3. For further validation, the numerical results of \( f''(0), -\theta'(0) \) and \( -\phi'(0) \) are added through Tables 4 to 6 which shows that the convergence of the Homotopy solution for different orders of approximation for \( f''(0), -\theta'(0) \)
Table 4. HAM solution for different orders of approximation for $f''(0)$, $-\phi'(0)$ and $-\phi(0)$ when $S_q = 0.01$, $M = 0.1$, $Pr = 2$, $L = 1$, $\xi_a = 2$ and different values of $\beta = 0.001$, 0.01, 0.05 are depicted, through these tables, it would seem that the results are nearly in the 10th order of approximations.

| $m$ | $f''(0)$  | $-\phi'(0)$ | $-\phi(0)$ |
|-----|-----------|-------------|------------|
| 4   | -9.05372854 | 1.02760954  | 1.01376504 |
| 8   | -9.05374433 | 1.02796615  | 1.01390527 |
| 12  | -9.05374433 | 1.02796615  | 1.01390527 |
| 16  | -9.05374433 | 1.02796615  | 1.01390527 |
| 20  | -9.05374433 | 1.02796615  | 1.01390527 |
| 24  | -9.05374433 | 1.02796615  | 1.01390527 |
| 28  | -9.05374433 | 1.02796615  | 1.01390527 |
| 32  | -9.05374433 | 1.02796615  | 1.01390527 |
| 36  | -9.05374433 | 1.02796615  | 1.01390527 |
| 40  | -9.05374433 | 1.02796615  | 1.01390527 |

It is also observed that by increase in the visco-elastic parameter, the skin-friction, heat and mass fluxes are also increases. The effects of physical parameters on the skin friction, heat flux and mass flux are also studied and depicted in Tables 7 to 9.

Results and discussions

To investigate and examine the effects of porosity and squeezing phenomena in the existence of unsteady temperature that change the flow rate and bring improvements in the system’s heating/cooling mechanism, reduce the turbulence of non-Newtonian fluid and scale-up flow tracers. By taking laminar, unsteady and incompressible non-Newtonian flow which is induced by squeezing disks in the presence of no-slip velocity and convective surface boundary conditions, to investigate and provides an analytical and numerical study of the flow for heat and mass transfer. The effect of the various flow parameters are addressed visually for the
Table 5. HAM solution for different orders of approximation for \( f''(0), -\theta'(0) \) and \( -\phi'(0) \) at \( S_q = 0.01, M = 0.1, Pr = 2, \beta = 0.01, L = 1 \) and \( \xi_a = 2.0 \).

| \( m \) | \( f''(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) |
|--------|-------------|----------------|-------------|
| 4      | -9.03829731| 1.15231781     | 1.14532672  |
| 8      | -9.03835429| 1.15251239     | 1.14571876  |
| 12     | -9.03835429| 1.15251239     | 1.14571876  |
| 16     | -9.03835429| 1.15251239     | 1.14571876  |
| 20     | -9.03835429| 1.15251239     | 1.14571876  |
| 24     | -9.03835429| 1.15251239     | 1.14571876  |
| 28     | -9.03835429| 1.15251239     | 1.14571876  |
| 32     | -9.03835429| 1.15251239     | 1.14571876  |
| 36     | -9.03835429| 1.15251239     | 1.14571876  |
| 40     | -9.03835429| 1.15251239     | 1.14571876  |

Table 6. HAM solution for different orders of approximation for \( f''(0), -\theta'(0) \) and \( -\phi'(0) \) at \( S_q = 0.01, M = 0.1, Pr = 2, \beta = 0.05, L = 1 \) and \( \xi_a = 2.0 \).

| \( m \) | \( f''(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) |
|--------|-------------|----------------|-------------|
| 4      | -9.00956247| 1.38725923     | 1.45889274  |
| 8      | -9.00998752| 1.38762527     | 1.45899982  |
| 12     | -9.00998752| 1.38762527     | 1.45899982  |
| 16     | -9.00998752| 1.38762527     | 1.45899982  |
| 20     | -9.00998752| 1.38762527     | 1.45899982  |
| 24     | -9.00998752| 1.38762527     | 1.45899982  |
| 28     | -9.00998752| 1.38762527     | 1.45899982  |
| 32     | -9.00998752| 1.38762527     | 1.45899982  |
| 36     | -9.00998752| 1.38762527     | 1.45899982  |
| 40     | -9.00998752| 1.38762527     | 1.45899982  |

Table 7. HAM and BVP4c result comparison for the computations of \( f''(0), -\theta'(0) \) and \( -\phi'(0) \) at \( \beta = 0.1, M = 1, L = 2, \xi_a = 2, S_q = 0.2 \) and different values of \( Pr \).

| \( Pr \) | HAM \( f''(0) \) | -\( \theta'(0) \) | -\( \phi'(0) \) | BVP4c \( f''(0) \) | -\( \theta'(0) \) | -\( \phi'(0) \) |
|--------|----------------|----------------|-------------|----------------|----------------|-------------|
| 0.1    | 3.40455151     | 0.980413       | 1.153067    | 3.404550       | 0.980413       | 1.153067    |
| 1      | 3.40455151     | 0.918726       | 1.159177    | 3.404550       | 0.918726       | 1.159177    |
| 2      | 3.40455151     | 0.844146       | 1.166511    | 3.404550       | 0.844141       | 1.166563    |
| 2.5    | 3.40455151     | 0.804209       | 1.170511    | 3.404550       | 0.804192       | 1.170517    |
| 3      | 3.40455151     | 0.763560       | 1.173853    | 3.404550       | 0.762300       | 1.174664    |

Table 8. HAM and BVP4c result comparison for the computations of \( f''(0), -g'(0), -\Theta'(0) \) and \( -\phi'(0) \) at \( \beta = 0.1, M = 1, L = 2, \xi_a = 2, Pr = 2 \) and different values of \( S_q \).

| \( S_q \) | HAM \( f''(0) \) | -\( \theta'(0) \) | -\( \phi'(0) \) | BVP4c \( f''(0) \) | -\( \theta'(0) \) | -\( \phi'(0) \) |
|--------|----------------|----------------|-------------|----------------|----------------|-------------|
| 0.1    | 5.73132151     | 0.998768       | 1.009664    | 5.731321       | 0.998768       | 1.009664    |
| 1      | 3.42539851     | 0.990088       | 1.077669    | 3.425369       | 0.990088       | 1.077668    |
| 2      | 3.40455151     | 0.980413       | 1.153067    | 3.404550       | 0.980413       | 1.153067    |
| 3      | 3.47472551     | 0.970731       | 1.227853    | 3.474725       | 0.970730       | 1.227853    |
| 4      | 3.56573751     | 0.961063       | 1.301666    | 3.565737       | 0.961063       | 1.301666    |
increase in the suction/injection parameter will increase the axial velocity due to the suction of the fluid from the lower disk, same but opposite behaviour is noted during injection of the fluid between the gap of the two disks. This phenomenon could be verified from Figures 5 and 6. The influence of \( \xi_a \) on heat and mass transfer is depicted in Figures 7 to 10. Firstly, the effect of suction and injection parameter on \( f'(0) \) and \( \phi'(0) \) are observed identical from Figures 7 and 9. Secondly the variation for higher values are smooth in \( f'(0) \) and \( u'(0) \). An increase in the rate of suction \( \xi_a = 0.5, 1, 1.5 \) increases the rate of outflow from the flow domain due to which the fluid temperature and mass transfer fall. The same but opposite behaviour is noted in the case of fluid injection. Figures 8 and 10 are drawn to observe this phenomenon in 3D shape.

| L | HAM | BVP4c |
|---|-----|-------|
| \( f''(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) | \( f''(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) |
|---|-----|-----|-----|-----|-----|
| 1 | 4.354185 | 0.999448 | 1.004323 | 4.354185 | 0.999448 | 1.004323 |
| 2 | 8.388572 | 0.999249 | 1.005891 | 8.388572 | 0.999249 | 1.005891 |
| 3 | 15.08649 | 0.998920 | 1.008476 | 15.08649 | 0.998920 | 1.008476 |
| 4 | 24.40871 | 0.998467 | 1.012038 | 24.40871 | 0.998467 | 1.012038 |
| 5 | 36.30043 | 0.997897 | 1.016526 | 36.30045 | 0.997897 | 1.016526 |

**Figure 3.** Impact of suction/injection parameter \( \xi_a = \pm 5, \pm 10, \pm 15 \) and \( \beta = 0.2, 0.4, 0.6, 0.8 \) on \( f'(0) \) with \( S_q = 5, M = 10, Pr = 1, L = 1.5 \).
The impact of $M$ and $b$ on $f_0(h)$ and $f(h)$ are seen at Figures 11 to 14. Hartmann number is the ratio of electromagnetic force to viscous force, by increasing $M$ and skipping Hartman number a vascular area is noted. It could be observed that during the injection of the fluid, radial velocity increase near the two disks due to the

![Figure 4. 3D-representation of $f'(\eta)$ for $\xi_0 = \pm 5$, $\pm 10$, $\pm 15$, $\beta = 0.6, 0.8, S_q = 5$, $M = 10$, $Pr = 1$, $L = 1.5$.](image)

![Figure 5. Impact of suction/injection parameter $\xi_a = \pm 5$, $\pm 10$, $\pm 15$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f(\eta)$ with $S_q = 5$, $M = 10$, $Pr = 1$, $L = 1.5$.](image)
decrease in the viscous force. This phenomenon is opposite near the centre of the fluid domain. The impact of $M$ during the suction of the fluid is negligible,

It also describes that Lorentz-force has a smoother effect on $b = 0.2, 0.4, 0.6, 0.8$ and greater values. For the axial velocity, the velocity increase near the lower

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Figure 6. 3D-representation of $f(\eta)$ for $\xi_\alpha = \pm 5, \pm 10, \pm 15, \beta = 0.6, 0.8, S_q = 5, M = 10, Pr = 1, L = 1.5$.

Figure 7. Impact of suction/injection parameter $\xi_\alpha = \pm 0.5, \pm 1, \pm 1.5$ on $\theta(\eta)$ with $\beta = 0.2, 0.4, 0.6, 0.8$ and $S_q = 1, M = 5, Pr = 1, L = 1.5$. 

disk because of the injection of the fluid which mobilizes the fluid to move in the axial direction. This effect gradually decreases due to a decrease in the viscous force and hence from the middle region, the velocity starts to decrease as shown in Figures 11 and 13. Figures 12 and 14 are made to observe the effect of $M$
in 3D geometries. Figure 15 present the impact of the Prandtl number on the heat transfer. Prandtl number is the ratio of momentum diffusivity to thermal diffusivity. Figure 15 explains the impact of Prandtl number $Pr$ on temperature. It is noted that while in suction, the flow temperature increases with an increase
in the amount of Prandtl and the visco-elastic parameter because of the increase in momentum diffusivity, although this behaviour is inversely proportional during injection of the fluid between the disks. For variable $\beta$ and $S_q$, Figures 16 to 23 are plotted. The profiles of velocity components plotted in Figure 16 show

Figure 12. 3D-representation of Hartmann number $M = 0, 5, 10$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f'(\eta)$ with $S_q = 0.1, Pr = 5, L = 1, \xi_a = \pm 2$.

Figure 13. Impact of Hartmann number $M = 0, 5, 10$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f'(\eta)$ with $S_q = 0.1, Pr = 5, L = 1, \xi_a = \pm 2$. 
that initially, the redial part of the velocity rises with the increase of $S_q$ when the fluid is injected, but, as the fluid flow reached the central area it gradually falls.

The maximal decrements for the lower values of $S_q$ can be observed on the radial-velocity. Velocity increases with the increase in $S_q$ on the left of the bottom disk.

Figure 14. 3D-representation of Hartmann number $M = 0, 5, 10$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f(\eta)$ with $S_q = 0.1, Pr = 5, L = 1, \xi_{\omega} = \pm 2$.

Figure 15. Impact of Prandtl number $Pr = 1, 5, 9$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $\theta'\eta$ with $S_q = 0.1, M = 5, L = 1, \xi_{\omega} = \pm 2$. 
Figure 16. Impact of squeezing parameter $S_q = -0.1, -1, -2$ with $\beta = 0.2, 0.4, 0.6, 0.8$ on $f'(\eta)$ and $M = 5, Pr = 2, L = 1, \xi_a = \pm 2$.

Figure 17. 3D-representation of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f'(\eta)$ with $M = 5, Pr = 2, L = 1, \xi_a = \pm 2$. 
Figure 18. Impact of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f(\eta)$ with $M = 5$, $Pr = 2$, $L = 1$, $\xi_n = \pm 2$.

Figure 19. 3D-representation of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $f(\eta)$ with $M = 5$, $Pr = 2$, $L = 1$, $\xi_n = \pm 2$. 
Figure 20. Impact of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $\theta(\eta)$ and $M = 5, \text{Pr} = 2, L = 1, \xi_o = \pm 2$.

Figure 21. 3D-representation of squeezing parameter $S_q = -0.1, -1, -2$ with $\beta = 0.2, 0.4, 0.6, 0.8$ on $\theta(\eta)$ with $M = 5, \text{Pr} = 2, L = 1, \xi_o = \pm 2$. 
Figure 22. Impact of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $\phi(\eta)$ with $M = 5, Pr = 2, L = 1, \xi_o = \pm 2$.

Figure 23. 3D-representation of squeezing parameter $S_q = -0.1, -1, -2$ and $\beta = 0.2, 0.4, 0.6, 0.8$ on $\phi(\eta)$ with $M = 5, Pr = 2, L = 1, \xi_o = \pm 2$. 
whereas velocity decreases for the same $S_q$ upon the right of the top disk. Furthermore, the squeeze-parameter effect is not visible for the large $\beta$, but for tiny $\beta$ is noticeable as the top of the disk is heading away from the bottom disk. In Figure 16 also demonstrated that velocity changes are negligible by increasing space between disks during injection of the fluid. In a suction scenario, it would be noticeable that the effects of $f'(\eta)$ are relatively greater when $\beta$ is small. It may also be observed that for small squeezing parameters, $f'(\eta)$ abruptly rises close to the bottom disk afterward begins to decrease to meet boundary conditions. The effect of $S_q = -0.1, -1, -1$ on axial velocity profile could also be seen its graphical representation in Figure 18. Figures 17 and 19 show the 3D geometry of $f'(\eta)$ and $f(\eta)$. The minor impact of $S_q$ observed on the velocity profile for the minimal amount of $\beta$ but for larger $\beta$ this effect is negligible. The effect of suction on velocity is the opposite of injection. Figures 20 and 21 depicts that as the fluid slips from the disk’s region the temperature of the fluid decrease abruptly due to suction, however, this behaviour is the opposite in the case of injection. A similar scenario could be seen for the mass transfer in Figures 22 and 23.

The impact of Lewis number is presented in Figures 24 and 25. Lewis number is the ratio of thermal diffusivity to mass diffusivity. An increase in the Lewis number means an increase in the thermal diffusivity of the fluid. It Could be observed that for greater Lewis number, $\phi(\eta)$ abruptly falls close to the bottom disk afterward begins to decrease to meet boundary conditions. The minor impact of $L$ was observed on the concentration profile for the minimal amount of $\beta$ but for larger $\beta$ this effect is negligible. Figure 25 presents the 3D view of the effect of Lewis number on mass transfer.

**Conclusion**

In this paper, unsteady squeezing flow between parallel disks is considered for investigation of the non-
Newtonian fluid. The modeled differential equations (13) to (15) are solved by HAM using mathematica package BVPh 2.0 and numerical solver BVP4c. An excellent averaged residual error is obtained at only 10th-order Homotopy solution. An excellent agreement is found between analytical and numerical solutions. The effect of the various flow parameters are addressed visually for the case of suction and injection respectively. The effect of the flow parameters are shown for velocity field, heat and mass transfer respectively. The following conclusions are made during this investigation:

- It is concluded that increase in the squeezing Reynolds number and non-Newtonian parameter have a negligible impact on velocity distribution in the case of suction from the flow domain.
- Rise in the value of the non-Newtonian parameter has a negligible effect on the radial and axial components of the flow field.
- A rise in the suction/injection parameter falls the radial velocity due to the increase in the rate of outflow from the flow domain. It could also be observed that by rising the non-Newtonian parameter, parabolic curves are found for both suction and injection of the fluid.
- The impact of all physical parameters is opposite on the flow field profiles in the scenario of suction and injection. On the other hand the effect of parameters remains similar on temperature profiles.
- Temperature is directly proportional to Prandtl number and non-Newtonian parameter during suction and inversely proportional in case of injection.
- By rising or skipping Hartman number, a vascular area is noted. It could be observed that during the injection of the fluid, radial velocity increase near the two disks due to the decrease in the viscous force. The impact of $M$ during the suction of the fluid is negligible. Its also describe that Lorentz-force has a smoother effect for large values of non-Newtonian parameter.
- The present problem is limited to 2D-geometry which could be converted to 3D-geometry by rotating the disks in a specific direction.
- The governing equations of this problem are converted from PDEs to the ODEs because of the boundary conditions. One could find appropriate boundary conditions and solve the PDEs by a suitable numerical technique.
- Selection of the values of physical parameters depend on the convergence of the system of equations.

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Appendix

**Nomenclature**

| Symbol | Description |
|--------|-------------|
| $p$    | pressure (N·m$^{-2}$) |
| $r$, $\theta$, $z$ | cylindrical polar coordinates |
| $V_r$  | radial velocity (m·s$^{-1}$) |
| $V_\theta$ | azimuthal velocity (m·s$^{-1}$) |
| $V_z$  | axial velocity (m·s$^{-1}$) |
| $t$    | time (s) |
| $T_u$  | temperature at upper disc (K) |
| $T_l$  | temperature at lower disc (K) |
| $C_u$  | concentration at upper disc |
| $C_l$  | concentration at lower disc |
| $Pr$   | Prandtl number ($\nu/k$) |
| $M$    | Hartman Number |
| $S_d$  | squeezing parameter |
| $h(t)$ | distance between the two discs (m) |
| $L$    | Lewis number |
| $B$    | induced magnetic field |
| $\delta_o$ | Soret number |
| $\xi_o$ | suction/blowing parameter |
| $\tau_{ij}$ | stress components |
| $A_1, A_2$ | Rivlin-Ericksen tensors |
| $C$    | dimensional concentration |
| $\vec{V}$ | velocity vector |

**Greek symbols**

| Symbol | Description |
|--------|-------------|
| $\omega$ | rotation vector |
| $\Omega_l$ | lower disc angular velocity |
| $\kappa$ | thermal conductivity (W/m·K) |
| $\mu$   | dynamic viscosity (Pa·s) |
| $\nu$   | kinematic viscosity (kg/ms) |
| $\sigma$ | relative angular velocity |
| $\rho$  | fluid density (kg/m$^3$) |
| $\alpha$ | positive constant |
| $\eta$  | similarity variable |
| $\Gamma$ | transformed fluid temperature |
| $\sigma_t$ | Stefan-Boltzmann constant |
| $N_s$  | constant number |
| $\Psi$  | transformed fluid concentration |
| $\kappa^*$ | mean absorption co-efficient |

**Subscript**

| Symbol | Description |
|--------|-------------|
| $u$    | fluid condition on upper disc |
| $l$    | fluid condition on lower disc |

**Superscript**

| Symbol | Description |
|--------|-------------|
| $*$    | dimensionless variable |
| $r$    | derivative w.r.t $\eta$ |