The light filament as a new nonlinear polarization state

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We present an analytical approach to the theory of nonlinear propagation in gases of femtosecond optical pulses with broad-band spectrum. The vector character of the nonlinear third-order polarization of the electrical field in air is investigated in details. A new polarization state is presented by using left-hand and right-hand circular components of the electrical field. The corresponding system of vector amplitude equations is derived in the rotating basis. We found that this system of nonlinear equations has $3D + 1$ vector soliton solutions with Lorentz shape. The solution presents a relatively stable propagation and rotation with GHz frequency of the vector of the electrical field in a plane orthogonal to the direction of propagation. The evolution of the intensity profile demonstrates a weak self-compression and a week spherical wave in the first milliseconds of propagation.

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INTRODUCTION

When a femtosecond laser pulse with power above the critical for self-focusing propagates in air, a number of new physical effects are observed, such as long-range self-channeling, coherent and incoherent GHz and THz emission, asymmetric pulse shaping, super-broad spectra, polarization instability of linearly polarized pulse, polarization rotation, self-compression and others. A remarkable effect is also that in lidar experiments the light filaments propagate over distances up to 9-10 kilometers in vertical direction, preserving their spectrum and shape. In a typical experiment in the near zone (up to $1 - 3$ m from the source), when the initial pulse intensity exceeds $I > 10^{12}$ W/cm$^2$, self-focusing and self-compressing start, which makes the $k_z$ spectrum broad band and asymmetric $\Delta k_z \approx k_0$. The process increases the core intensity up to $10^{14}$ W/cm$^2$, where a short non-homogenous plasma column in the nonlinear focus is observed. Usually the standard model describing the propagation in the near zone is a scalar spatio-temporal nonlinear paraxial equation including in addition terms with plasma ionization, higher order Kerr terms, multiphoton ionization and others. The basic model works properly in the near zone because of the fact that the paraxial approximation is valid for pulses with narrow-band spectrum $\Delta k_z << k_0$. At distances longer than 10-20 meters from the laser source, where the stable filament is formed, plasma generation and higher-order Kerr terms are also included as necessary for the balance between the self-focussing and plasma defocussing and for obtaining long range self-channeling in gases. However, the above explanation of the filamentation is difficult to be applied at such distances. As reviewed in [11–17], the plasma density at long distances from the source is too small to prevent self-focusing. There are basically three main characteristics which remain unchanged at these distances - the broad-band spectrum, the coherent GHz generation and the width of the core, while the intensity in a stable filament drops to a value of $10^{12}$ W/cm$^2$. The higher-order Kerr terms for pulses with intensities of the order of $I \sim 10^{12}$ W/cm$^2$ are also too small to prevent self-focussing. The experiments, where observation of long-range self-channeling without ionization was realized [12–14], show the need to change the role of the plasma defocussing at such distances with another effect. In addition, there are difficulties with the physical interpretation of the coherent GHz radiation as a result of plasma generation. The light pulse near the nonlinear focus emits incoherent and non-homogenous plasma, while the coherent GHz radiation requires homogenous plasma with fixed electron density of the order of $10^{15}$ cm$^{-3}$. Only homogenous plasma can generate coherent GHz emission, but such kind of plasma is absent in the process of filamentation. In the real experiments with propagation of a single filament at distances more than 20 $-$ 30 m from the source in air, the following basic characteristics are found:

1. Broad-band spectrum ($\Delta k_z \sim k_0$).
2. Intensity of the order of $I \sim 10^{11-12}$ W/cm$^2$.
3. Absence of plasma at long distances.
4. Asymmetric relatively stable (Lorentz) spectral and longitudinal shapes.
5. Coherent GHz generation.

Recently in [22] we developed a scalar ionization-free non-paraxial nonlinear model, which gives the above characteristics of the stable filament. The analytical and the numerical results describe correctly the linear and nonlinear evolution of narrow-band and broad-band laser pulses. In addition it was found that the equation has exact Lorentz-type soliton solutions in approximation of neglecting the GHz oscillation. Still, this theory cannot resolve some difficulties. The main problems are:

1. Peak instability of the soliton solution under small initial perturbations.
2. The soliton solution is obtained after neglecting the GHz oscillation.

3. The soliton solution has one free parameter.

4. There are problems with the conservation law of the nonlinear operator when we use the GHz oscillation.

To solve the above problems in this paper, we propose a nonlinear vector generalization to the model.

**NONLINEAR POLARIZATION**

The self-action process broadens the pulse spectrum starting from a narrow-band pulse, the stable filament becomes broad-band far from the source. In recent papers [19, 20] it is shown that the evolution of broad-band pulses like filaments can be described correctly by using the generalized nonlinear polarization

\[
\vec{P}_{nl} = n_2 \left( \vec{E} \cdot \vec{E} \right) \vec{E},
\]

which includes additional processes associated with third harmonic generation (THG). A more precise analysis presented in [21] demonstrates that the polarization of the kind [11] is not applicable to a scalar model, because the corresponding Manley-Rowe (MR) conservation laws are not satisfied. That is why we investigate two-component electrical vector field. The generalized nonlinear polarization [11] is quite simple in terms of left-hand and right-hand circular components. Let us now present the electrostatic field of a pulse as a linear decomposition of left- and right-hand circular complex components.

\[
\vec{E}(x, y, z, t) = E_+(x, y, z, t) \sigma_+ + E_-(x, y, z, t) \sigma_-
\]

where the circular-polarization unit vectors are

\[
\sigma_\pm = (\hat{x} \pm i \hat{y})/\sqrt{2}.
\]

If we now represent \( \vec{P}_{nl} \) in terms of its circular components as

\[
\vec{P} = P_+ \sigma_+ + P_- \sigma_-,
\]

we find that the components are given by

\[
P_+ = 2n_2 \left( E_+^2 E_- \right)
\]

\[
P_- = 2n_2 \left( E_-^2 E_+ \right).
\]

**BASIC SYSTEM OF EQUATIONS**

The decomposition [12] allows us to rewrite the nonlinear vector wave equation in the following system of equations.

\[
\Delta E_\pm - \frac{1}{c^2} \frac{\partial^2 E_\pm}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \left\{ \int_0^\infty R^{(1)}(\tau) E_\pm(t - \tau) d\tau \right\} (7)
\]

\[
+ \int_0^\infty \int_0^\infty \int_0^\infty R^{(3)}(\tau, \tau, \tau) [E_\pm(t - \tau)]^3 E_\mp(t - \tau) d\tau^3
\]

To obtain amplitude equations, we use in mind also the causality principles (no negative time \( \tau \) ) to the response functions and their Fourier presentations \( \chi^{(1)}(\omega) \) and \( \chi^{(3)}(\omega, \omega, \omega) \)

\[
\chi^{(1)}(\omega) = \int_0^\infty R^{(1)}(\tau) \exp(i\omega \tau) d\tau
\]

\[
\chi^{(3)}(\omega, \omega, \omega) = \int_0^\infty \int_0^\infty \int_0^\infty R^{(3)}(\tau, \tau, \tau) \exp(3i\omega \tau) d\tau^3.
\]

Reduction of the integrals [8] from 0 to infinity is equal to cosine transforms with the properties

\[
\chi^{(1)}(\omega) = \chi^{(1)}(-\omega)
\]

\[
\chi^{(3)}(\omega, \omega, \omega) = \chi^{(3)}(-\omega, \omega, \omega) = \chi^{(3)}(-\omega, -\omega, \omega)...
\]

The key point in the following transformations is the fact that the circular components of the electrical field, as well as the amplitude functions, are orthogonal in the complex plane. That is why their Fourier presentations are also written in orthogonal basis

\[
E_+(t - \tau) = \int_{-\infty}^{\infty} \tilde{E}_+(\omega) \exp(-i\omega(t - \tau)) d\omega
\]

\[
E_-(t - \tau) = \int_{-\infty}^{\infty} \tilde{E}_-(\omega) \exp(i\omega(t - \tau)) d\omega.
\]

Substituting [10] - [11] into the right-hand side and also to the last term of left-hand side in [7], and using the spectral properties [8] of the response functions [8], the wave system can be written as

\[
\Delta E_+ = -\int_{-\infty}^{\infty} k^2(\omega) \tilde{E}_+(\omega) \exp(-i\omega t) d\omega
\]

\[
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R^{(1)}(\tau, \tau, \tau) \tilde{E}_+(\omega)^2 \tilde{E}_-(\omega) \exp(-i\omega t) d\omega^3
\]

\[
\Delta E_- = -\int_{-\infty}^{\infty} k^2(\omega) \tilde{E}_-(\omega) \exp(i\omega t) d\omega
\]

\[
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R^{(1)}(\tau, \tau, \tau) \tilde{E}_-(\omega)^2 \tilde{E}_+(\omega) \exp(i\omega t) d\omega^3
\]
where \( k^2(\omega) = \omega^2\varepsilon(\omega)/c^2 \); \( k^2_{nl}(\omega) = \omega^2\chi^{(3)}(\omega, \omega, \omega)/c^2 = k^2(\omega)\hat{n}_2(\omega) \) are the square of the linear and nonlinear wave vectors and

\[
\varepsilon(\omega) = 1 + 4\pi\chi^{(1)}(\omega) \quad \text{and} \quad n_2(\omega) = 4\pi\chi^{(3)}(\omega, \omega, \omega)/\varepsilon(\omega). \quad (14, 15)
\]

Lets us now introduce complex amplitude functions using the substitutions

\[
E_+(r, t) = A_+(r, t) \exp \left[ i(k_0 z - \omega_0 t) \right] \quad (16)
\]

\[
E_-(r, t) = A_-(r, t) \exp \left[ -i(k_0 z - \omega_0 t) \right]. \quad (17)
\]

where \( r = (x, y, z) \), \( \omega_0 \) is the carrier frequency and \( k_0 \) is the linear part of the wavevector at the carrier frequency. Applying the translation theorem to the Fourier presentations of the electrical field components we have

\[
\hat{E}_+(r, \omega) = \exp(ik_0 z)\hat{A}_+(r, \omega - \omega_0) \quad (18)
\]

\[
\hat{E}_-(r, \omega) = \exp(-ik_0 z)\hat{A}_-(r, \omega - \omega_0). \quad (19)
\]

Substituting (16)-(19) into Eqs. (13) we obtain

\[
\Delta A_+ + 2ik_0 \frac{\partial A_+}{\partial z} - k_0^2 A_+ =
- \int_{-\infty}^{\infty} k^2(\omega)\hat{A}_+ \exp \left[ -i(\omega - \omega_0) t \right] dt d\omega \quad (20)
\]

\[
+ \int \int_{-\infty}^{\infty} k^2_{nl}(\omega) \left[ \hat{A}_+ \right]^2 \hat{A}_- \exp \left[ -i(\omega - \omega_0) t \right] d\omega d^3 \omega \quad (21)
\]

\[
\Delta A_- - 2ik_0 \frac{\partial A_-}{\partial z} - k_0^2 A_- =
- \int_{-\infty}^{\infty} k^2(\omega)\hat{A}_- \exp \left[ i(\omega - \omega_0) t \right] dt d\omega \quad (21)
\]

\[
- \int \int_{-\infty}^{\infty} k^2_{nl}(\omega) \left[ \hat{A}_- \right]^2 \hat{A}_+ \exp \left[ i(\omega - \omega_0) t \right] d\omega d^3 \omega \quad (21)
\]

where \( \Delta \) is 3D \(- (x, y, z) \) Laplace operator, and \( \beta \) is a number, connected with the dispersion characteristics of the medium \((\beta = k_0 v_{gr}^2 k''/c)\). We note here that in gases the dispersion is weak and the series (22) are strongly converged up to single cycle regime (broad-band pulses). Thus, the non-paraxial system of equations (21) describe correctly the evolution of laser pulses in gases up to a single cycle regime. It is important to mention that from Eqs. (23) paraxial spatio-temporal approximation can be derived for narrow-band laser pulses (22) only. The filamentation experiments demonstrate quite different pulse evolution: the initial laser pulse \((t_0 \geq 50 fs)\) possesses a relatively narrow-band spectrum \((\Delta k_z \ll k_0)\) and during the process the initial self-focusing and self-compression the spectrum broadens significantly. The broad-band spectrum \((\Delta k_z \sim k_0)\) is one of the basic characteristics of the stable filament. That why we do not reduce more Eqs. (24) and try to solve them for the case when the pulse has a large spectrum.

Another standard restriction in the filamentation theory is the use of one-component scalar approximation of the electrical field \(\vec{E} \). This approximation, though, is in contradiction with recent experimental results, where rotation of the polarization vector is observed \([8]\). For this reason in the present paper we use non-paraxial vector model in circular basis (24), in which the nonlinear
effects are described by the nonlinear polarization components \( \mathbf{\Delta} \). The dispersion number in air is very small: \( \beta = k_0v_{gr}^2k'' \approx 2.1 \times 10^{-5} \), so we can solve Eqs. (24) in approximation up to the first order of dispersion. Additionally, we will use the normalized amplitude functions \( A_k = A_0A_k \) to rewrite the system (24) in the form

\[
2ik_0 \left( \frac{\partial A_+}{\partial z} + \frac{1}{v_{gr}} \frac{\partial A_+}{\partial t} \right) = \Delta A_+ - \frac{1}{v_{gr}^2} \frac{\partial^2 A_+}{\partial t^2} + \gamma A_+^2 A_- \\
-2ik_0 \left( \frac{\partial A_-}{\partial z} + \frac{1}{v_{gr}} \frac{\partial A_-}{\partial t} \right) = \Delta A_- - \frac{1}{v_{gr}^2} \frac{\partial^2 A_-}{\partial t^2} + \gamma A_-^2 A_+ ,
\]

where \( \gamma = 2n_2k_0^2A_0^2 \) is a nonlinear coefficient.

**VECTOR SOLUTION AND VECTOR ROTATION. THE FILAMENT AS A WEAK ROUGUE WAVE**

The nonlinear system of equations (24) has exact solitary vector solution when \( \gamma = 2 \) and the spectral width of the pulses reaches the value \( \Delta k_z \approx k_0 \)

\[
A_+(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \exp \left[ i\Delta k_z (z - v_{gr}t) \right] \\
A_-(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \exp \left[ -i\Delta k_z (z - v_{gr}t) \right] ,
\]

where \( \bar{r} = \sqrt{x^2 + y^2 + (z - ik_{cef})^2 - v_{gr}^2(t - ik_{cef}/v_{gr})^2} \) and \( k_{cef} \) is determined below. The solution of the corresponding vector electrical field can be written after multiplying the amplitude functions (26) by the main phases (10)-(17)

\[
E_+(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \exp \left[ i\Delta k_z (v_{ph} - v_{gr})t \right] \\
E_-(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \exp \left[ -i\Delta k_z (v_{ph} - v_{gr})t \right] .
\]

Let us turn from the left-hand and right-hand circular components (3) to the standard Cartesian coordinates

\[
E_x = (E_+ + E_-)/\sqrt{2}, \ E_y = (E_+ - E_-)/(i\sqrt{2}). \tag{28}
\]

The solution (27) written in Cartesian coordinates has the form

\[
E_x(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \sin \left[ \Delta k_z (v_{ph} - v_{gr})t \right] \\
E_y(x, y, z, t) = \frac{2}{1 + \bar{r}^2} \cos \left[ \Delta k_z (v_{ph} - v_{gr})t \right] . \tag{29}
\]

The 3D + 1 Lorentz type solution (29), presented in Cartesian coordinates, gives oscillation of the electrical vector \( \tilde{E} = (E_x, E_y, 0) \) in the \((x, y)\) plane. It can be seen directly that the frequency of oscillation is equal to the carrier to envelope frequency \( \omega_{ce, f} = k_0 (v_{ph} - v_{gr}) \). The corresponding longitudinal spatial carrier to envelope wave number is \( k_{ce, f} = \omega_{ce, f}/v_{gr} \). Detailed investigation on the evolution of the solution (29) at distances more than \( 10 - 20 \) meters from the initial point, reveals weak self-compression. That is why we consider the filament in this zone as closer to a weak Rogue wave. The evolution of the profiles of the electrical field components \( \Re(E_x)-\text{Fig. 1a} \) and \( \Re(E_y)-\text{Fig. 1b} \) are plotted in Fig. 1. The periodical exchange of energy between components, due to nonlinear mechanism, leads to rotation of the electrical vector field in a plane orthogonal to the direction of propagation, with time period \( T_{ce, f} \approx 1 - 2 \times 10^{-10} \) s and spatial period \( \Lambda_{ce, f} \approx 3 - 6 \) cm.

**CONCLUSIONS**

The starting point of our investigation in this paper is the fact, that the generalized nonlinear polarization (1) arise new polarization state in circular basis. This polarization state leads to periodical exchange of energy between the electrical components \( E_x \) and \( E_y \) of a laser pulse. To derive the corresponding amplitude equations associated with this polarization, we take into account the orthogonality in the complex plane of the left-hand and right-hand circular components. We find that the obtained system of amplitude equations (25) has exact (3D+1) Lorentz type soliton solutions (26). Our soliton solution is obtained for pulses which satisfy the additional condition \( \Delta k_z \approx k_0 \). The diffraction of broad-band pulses is not the Fresnel one \( \text{23, 24} \), which leads to the conclusion that the soliton appears as a balance between semi-spherical (Fraunhofer type) diffraction and nonlinear self-focusing. The solution gives also a rotation of the vector of the electrical field with the carrier to envelope frequency.

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FIG. 1. Evolution of the profiles of the electrical field components $\Re(E_x)$-Fig. 1a and $\Re(E_y)$-Fig. 1b of the solution (29). The periodical exchange of energy between components, due to nonlinear mechanism, leads to rotation of the electrical field vector in a plane orthogonal to the direction of propagation.

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