Anharmonic oscillator potentials in the prolate γ-rigid regime of the collective geometrical model

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Abstract. Based on higher order JWKB approximation, an analytical formula is derived for the energy spectrum of the prolate γ-rigid Bohr-Mottelson Hamiltonian with an oscillator potential and a quartic or sextic anharmonicity in β shape variable. The model is found to provide good description for near vibrational nuclei situated in vicinity of a closed neutron shell.

1. Introduction
An analytical expression for the energy spectrum of the ground and β bands was obtained in the axially symmetric γ-rigid regime of the Bohr Hamiltonian with a harmonic oscillator potential in the β shape variable amended with a higher order anharmonicity term. The anharmonic terms are considered of quartic [1] and sextic [2] types. The particular structure of the model space facilitates the exact separation of angular variables from the β variable, leading to a differential Schrödinger equation with an anharmonic potential and a centrifugal-like barrier. The Schrödinger equation for such potentials is not exactly solvable. Thus, the corresponding eigenvalues are approximated by an analytical formula derived on the basis of the JWKB approximation [3, 4], which depends on a single parameter except the scale. The experimental realization of the model is found in a variety of vibrational-like nuclei exhibiting some regularity regarding the order of the considered anharmonicity.

2. Prolate γ-rigid collective Hamiltonian
The Hamiltonian associated to a prolate γ-rigid nucleus is [1, 2, 5, 6, 7]:

\[ H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} - \frac{\hat{I}^2}{3\hbar^2\beta^2} \right] + U(\beta), \]

where \( \hat{I} \) is the angular momentum operator from the intrinsic frame of reference, while \( B \) is the mass parameter. The Schrödinger equation associated to such a Hamiltonian is solved by separating the β variable from the angular ones which is achieved through the factorization \( \Psi(\beta, \theta_1, \theta_2) = F(\beta)Y_{JM}(\theta_1, \theta_2) \), where the angular factor state is a spherical harmonic function. With this, the Schrödinger equation is reduced to a second order differential equation in variable...
β, which is brought to a canonical form through the change of function $F(β) = \frac{I(β)}{β}$:

$$\left[\frac{d^2}{dβ^2} - \frac{I(I+1)}{3β^2} + 2(ε - u(β))\right] f(β) = 0,$$

(2)

where $ε = BE/h^2$, $u(β) = BU(β)/h^2$ denote the reduced energy and potential.

3. Quartic and sextic anharmonicities

The eigenvalue problem for potentials:

$$u(β) = \frac{1}{2} \alpha_1 β^2 + \alpha_2 β^{2m}, \quad \alpha_1 ≥ 0, \alpha_2 > 0 \quad \text{and} \quad m = 2, 3,$$

has the following scaling property:

$$ε(α_1, α_2) = α_2^{\frac{1}{2m+1}} ε(λ_m, 1) = α_2^{\frac{1}{2m+1}} W^m(λ_m),$$

(4)

where $λ_m = α_1α_2^{\frac{m}{2m+1}} ≥ 0$. This follows from the change of variable $β’ = α_2^{\frac{1}{2m+1}}β$ in the differential equation (2). Similar scaling was used in Refs.[7, 8] in the case of a quasi exactly solvable sextic potential. The problem is then reduced to a radial Schrödinger equation with a modified centrifugal term for the scaled potential $\hat{u}(β’) = \frac{1}{2}λ_m β’^2 + β’^{2m}$. Adapting the procedure of Refs.[3, 4] one can write the eigenvalue $W^m$ as function of $λ_m$, $n$ - the β vibration quantum number and the intrinsic angular momentum $I$, as follows:

$$W^2_{nI}(λ_2) = (C_2N_{nI})^{\frac{1}{2}} \sum_{k=0}^{8} G_k^2(λ_2, I) (C_2N_{nI})^{-\frac{2k}{2}},$$

(5)

$$W^3_{nI}(λ_3) = (C_3N_{nI})^{\frac{1}{2}} \sum_{k=0}^{5} G_k^3(λ_3, I) (C_3N_{nI})^{-k},$$

(6)

where $N_{nI} = 2n + 1 + (\sqrt{1 + 4I(I+1)/3})/2$. The functions $G_k^m(λ_m, I)$ represent polynomials in $λ_m$ of order $k$ with angular momentum dependent coefficients and are given explicitly in [1, 2] together with the constants $C_m$.

The energy spectrum of a nucleus described by the Hamiltonian (1) with an oscillator potential with quartic or sextic anharmonicities is then given by:

$$E^m_{nI} = \frac{h^2}{B} α_2^{\frac{1}{2m+1}} [W^m_{nI}(λ_m) - W^m_{00}(λ_m)].$$

(7)

4. Numerical results

The energy formula depends up to an overall factor on a single free parameter which is bounded by the convergence radius $λ^*_m$ of the considered approximation. The acting space of the model is then restricted to the interval $[0, λ^*_m]$ which put in terms of the ratios $R_{4/2}$ and $R_{0/2}$ lies between $[2.29, 2.00]$ and $[2.37, 1.81]$ for quartic and respectively $[2.34, 2.15]$ and $[2.56, 2.11]$ for sextic anharmonicities (See Figure 1). The upper limit corresponds to $λ_m = 0$ which is associated to the free parameter models $X(3)-β^{2m}$, while the lower limit is close to the $X(3)-β^2$ predictions [1].

As can be seen from Figure 2, the quartic model is experimentally realized for $^{100}$Pd, $^{148}$Sm, and $^{222}$Th nuclei, while the sextic one describes their next heavier neighbours. The three isotopic pairs also have the same number of neutrons above the corresponding shell closure at magic numbers $N = 50, 82 \text{ and } 126$. 

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Figure 1. Theoretical low lying energy spectrum for ground and two $\beta$ excited bands.

Figure 2. Quartic model description of the lighter isotopes of Pd, Gd and Th [9] in comparison to the heavier ones treated by means of a sextic term.

5. Outlook
The present approximate formulas can be easily translated to the case of four and five dimensional Bohr-Mottelson Hamiltonians with a similar separated potential for the $\beta$ shape variable.

References
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