Mean velocity equation
for turbulent flow

Jürgen Piest
May 11, 2014

Abstract

The hydrodynamic equation derived by Zwanzi-Mori technique of N-particle statistical mechanics is investigated. This is an attempt to provide additional information concerning the closure problem of turbulence theory. The equation is interpreted as mean velocity equation for turbulent fluid flow. The third-order term of the friction force is calculated. Multilinear mode coupling theory is applied in order to obtain formulas for the third- and fourth-order equilibrium time correlation functions appearing in the expression. The force term is obtained as a convolution integral containing higher order gradients of the velocity field.

1 Introduction

The paper deals with the hydrodynamic equation derived by N-particle statistical mechanics; especially with the calculation of the third order term of the friction force.

The motive for the investigation is the closure problem of classical turbulence theory. It is well known that the problem arises when one declares, in case of turbulent fluid motion, the hydrodynamic velocity $u$ to be a random variable, and attempts to obtain an equation for its expectation by just building the expectation over the Navier-Stokes equation. The non-linear term then gives rise to an additional variable; when one formulates an additional equation for this, another variable is obtained, and so on. Apparently, if the basic equation of motion is non-linear, this method does not work. Instead, from the point of probability theory, it would be necessary to find the probability density $f(u,t)$ of the turbulent process, and to calculate the expectation as:

$$\langle u \rangle = \int u f(u,t) du$$

(1.1)

$f$ is functionally dependent on $u$; thus, (1.1) is a functional integral. As early as 1952, Hopf has formulated this theoretical approach [1]; a short account can be found in [2], ch. 4. - It became apparent that it is extremely difficult to provide an equation for $\langle u \rangle$ by elaborating this theory. On the other hand, the success of direct numerical simulation (see, e. g. [3]) leads to the conclusion that it must be possible to calculate the mean velocity on the basis of the Navier-Stokes equation.

In this situation, it is perhaps reasonable to attempt the derivation of the mean velocity equation by N-particle statistical mechanics, in order to provide further information on the nature of the problem. It is obvious that for investigation of turbulent processes it is not necessary to deal with the molecules of the fluid. It is the probabilistic structure of statistical mechanics which constitutes the difference to Navier-Stokes theory. Statistical mechanics is the basis for describing classical processes including fluid dynamics; it would be a surprise if turbulent motion would be an exception. Thus, it is assumed here that it is principally possible to derive the mean velocity equation by this approach. The complication is that in a statistical mechanics analysis it is necessary to distinguish between macroscopic and microscopic parts of the motion, and to handle the latter in a suitable way. - The Navier-Stokes equation can be derived from statistical mechanics via the Boltzmann equation by Chapman-Enskog method; see e. g. [4]. But in the course of the derivation of the Boltzmann equation the molecular chaos assumption is introduced, which is correct for many applications including laminar fluid motion, but seems problematic for a turbulence investigation.

An essential step is defining the mean velocity in the frame of statistical mechanics. In general, the fluid velocity is defined:

$$u = \frac{1}{\rho} \langle p \rangle$$

(1.2)
\( \rho \) is the mass density, \( p \) the microscopic momentum density. In this paper, we consider incompressible constant density and temperature processes. Then \( u \) is, up to a constant factor, equal to the expectation of \( p \). Under rather general conditions, the expectation is equal to the arithmetic mean of a time or space series of the microscopic quantity. Since for laminar flow fluctuations are microscopic, \( u \) then is the “point” velocity of the flow as we usually understand it. On the other hand, in turbulent flow there appear macroscopic fluctuations; the arithmetic mean by definition averages over these also; thus, in this case \( u \) is already the mean velocity. We have the peculiar situation that the statistical mechanics \( u \) is equal the classical hydrodynamic \( u \) or \( \langle u \rangle \), depending on whether the flow is laminar or turbulent. Therefore, the hydrodynamic equation in the form derived by N-particle statistical mechanics will, by way of trial, be considered the mean velocity equation.

The derivation by means of Zwanzig-Mori projection operator technique of statistical mechanics (POT) is utilized, in the presentation of Grabert [5]. For the convenience of the reader, some introductory material from an earlier paper of the author [6] is repeated. The hydrodynamic equation obtained by this technique shows a formula for the friction force containing a local equilibrium time correlation, which is a non-linear functional of the velocity; presently, there exists no theory from which it could be calculated. Therefore, it is necessary to develop it into a functional power series in the velocity; this presently restricts the applicability of the equation to low Reynolds number flow, not very far from the Navier-Stokes regime. The coefficients of the series now contain total equilibrium correlation functions, which can be calculated by employing the multilinear mode coupling theory of Schofield and co-workers [8], [9].

The third order term is the first term which furnishes an equation beyond the Navier-Stokes equation. Time correlation functions of third and fourth order appearing in this term are calculated. Finally one obtains an convolution integral which contains gradients of the velocity up to sixth order. Some preliminary calculations concerning the design of a test of the theory are reported.

2 Hydrodynamic equation

This section is a somewhat altered version of section 2 of [6]. There, the description was restricted to stationary processes. In this paper, I provide the usual time-dependent formulas. The fluid is considered to be a system of \( N \) particles of mass \( m \) with positions \( y_j \) and velocities \( v_j \) which are combined to the phase space matrix \( z \). Vector components are described by Latin indices, e. g., \( y_j = \{ y_{ja} \} \). The system is enclosed in a box of Volume \( V \). Later on, the thermodynamic limit is performed. - A function \( g(z) \) is called a phase space function, or microscopic variable. Especially, we need the space densities of the conserved quantities mass, energy and momentum \( n, e, p \) which are collected to a 5-element linear matrix \( a \). They are functions of an additional space variable \( x \):

\[
a = \sum_{j=1}^{N} \tilde{a}_j \delta(x - y_j) \tag{2.1}
\]

For the particle functions \( \tilde{a}_j \) we have \( \tilde{n}_j = m \), \( \tilde{p}_j = mv_j \), while the energy function contains the interparticle potential. \( m \) ist the particle mass. Quantities (like \( a \)) which are lists of 5 elements are denoted by normal letters while three-component vectors and tensors are bold. - The quantities \( a \) obey the conservation relations:

\[
a = -\nabla \cdot s \tag{2.2}
\]

The fluxes \( s \) have the same general structure as the \( a \) (2.1); especially, we have \( s_1 = p \). The time evolution of any phase space function \( A \) is described by the Liouville equation:

\[
\dot{A} = \mathcal{L} A \tag{2.3}
\]

\( \mathcal{L} \) (often defined as \( i\mathcal{L} \)) is the Liouville operator. From (2.3), the formal solution for \( A(t) \) given the initial value \( A \) is:

\[
A(t) = e^{\mathcal{L} t} A \tag{2.4}
\]

In the statistical model, \( z \) and \( N \) are considered random variables; that is, the probability density \( f(z, N) \) is of grand canonical type. The ensemble mean value (expectation) of a phase space function \( A \)
is defined in the ‘Heisenberg’ picture:

\[(A)(t) = \sum_{N=1}^{\infty} \int dz A(z, N, t)f(z, N) \tag{2.5}\]

In this formula, \(f(z, N)\) is the initial probability distribution, and \(A(z, N, t)\) is the value of \(A\) at time \(t\) if the initial positions and velocities of the particles are described by \(z\). The operation (integration + Summation) is sometimes indicated by the symbol ‘\(\text{tr}\)’:

\[\text{tr}\{\Omega\} = \sum_{N=1}^{\infty} \int dz\Omega(z, N) \tag{2.6}\]

Certain probability densities (also called distributions here) are frequently used in the analysis. One of them is the (total) equilibrium distribution which corresponds to macroscopic rest:

\[f_0 = \psi(N)\exp(\Phi_0 + \beta(\mu N - H(z)))\]

\[\psi(N) = \frac{1}{N!} \left(\frac{m}{h^2}\right)^3 N \tag{2.7a}\]

Here, \(h\) is Planck’s constant, \(\beta = 1/(k_B T)\), \(k_B\) being Boltzmann’s constant and \(T\) the temperature, \(\mu\) is the chemical potential which is a function of mass density \(\rho = \langle n \rangle\) and temperature, and \(H(z)\) is Hamilton’s function which describes the total energy of the fluid. For the normalization constant, we have \(\Phi_0 = -\beta PV\), \(P\) being the equilibrium pressure. Expectations with respect to the equilibrium distribution are denoted by \(\langle \rangle\). - In case of a simple fluid, the ‘relevant probability distribution’ of Grabert’s formalism (see [5], sec. 2.2) is the local equilibrium distribution:

\[f_L(t) = \psi(N)\exp(\Phi(t) - a(z) \circ b(t)), \tag{2.8a}\]

\[b = \{\beta\left(\frac{1}{2} u^2 - \frac{\mu}{m}\right), \beta, -\beta u\}, \tag{2.8b}\]

\[\Phi(t) = - \log(\text{tr}(\psi\exp(-a \circ b(t))))\]. \(\tag{2.8c}\]

Here the symbol \(\circ\) is introduced for the operation: Multiplication, plus Summation over the 5 elements of the linear matrices \(a\), \(b\), plus Integration over geometrical space. The elements of \(b\) are called the conjugate parameters; they are functions of the quantities \(\beta\), \(\mu\) and \(u\) which we will sometimes call the thermodynamic parameters, and which will be considered to be slowly varying functions of space and time. The \(b\) are defined such that the expectations of the \(a\) are identical to their expectations in local equilibrium:

\[\langle a \rangle = \langle a \rangle_L \tag{2.9}\]

The POT is a means for separating macroscopic and microscopic parts of the random variables. It starts by defining the set of phase space functions which are relevant for the description of the process. For simple fluids, this set is identified with the densities of conserved variables, \(a\). A projection operator is defined which projects out of any microscopic variable \(A\) the part which is proportional to the relevant variables. It reads:

\[\mathcal{P}A = \langle A \rangle_L + \langle A \delta a \rangle_L \circ \langle \delta a \delta a \rangle_L^{-1} \circ \delta a \tag{2.10}\]

Here, \(\delta a = a - \langle a \rangle_L\); \(\langle \rangle^{-1}\)denotes the inverse of the expectation matrix in the formula. For general non-stationary flow, the local equilibrium distribution \([2.5a]\), and therefore \(\mathcal{P}\), are time-dependent. Then, instead of \([6]\) (2.11), for the decomposition of \(e^{\mathcal{L}t}\) the formula \([5]\) (2.4.1) is obtained:

\[e^{\mathcal{L}t} = e^{\mathcal{L}t} \mathcal{P}(t) + \int_0^t dt' e^{\mathcal{L}t'} \mathcal{P}(t')(\mathcal{L} - \mathcal{P}(t'))(1 - \mathcal{P}(t')) \mathcal{G}(t', t) + (1 - \mathcal{P}(t)) \mathcal{G}(0, t) \tag{2.11}\]

\[\mathcal{G}(t', t) = \exp(-\int_0^t dt'' \mathcal{L}(1 - \mathcal{P}(t''))) \tag{2.12}\]

\(\mathcal{G}(t', t)\) is a time-ordered exponential Operator which describes the time dependence of the dissipative part of the equation of motion. The analysis in \([5]\) consists in applying \([2.4]\) to \(a\), using \([2.11]\). By averaging over the initial probability density, and after some manipulations, Grabert’s generalized transport equation \([5]\), (2.5.17) is obtained. It is postulated in POT that the initial probability density is of the
form of the ‘relevant probability density’, which for simple fluids is defined to be the local equilibrium density \(\rho \delta u / \delta t + u \cdot \nabla u = -\nabla P + \nabla \cdot R\) \(2.13\).

Grabert states that this should not be considered a general restriction of the method but a means to form the general particle system into the type specially considered (the simple fluid here); see \[5\], sec. 2.2. It is shown that in this case the last term in \(2.11\) vanishes after averaging. Finally the part of the formula pertaining to the momentum density is taken, and \(1.2\) is used. The result, \[5\] \((8.1.12), (8.1.13)\), reads with the denotations employed here:

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} + u \cdot \nabla u &= -\nabla P + \nabla \cdot R \\
R(x, t) &= \int_0^t dt' \int dx' S(x, x', t, t') \nabla' u(x', t') \\
S(x, x', t, t') &= \beta \langle [G(t', t)(1 - \mathcal{P}(t)\rho(x))(1 - \mathcal{P}(t'))\rho(x')] \rangle_{L,t} \\
\n\end{align*}
\]

The microscopic definition of the pressure \(P(x, t)\) \([5]\) \((8.4.7)\) will not be repeated here. \(\langle \rangle_{L,t}\) denotes an expectation with respect to the local equilibrium distribution at time \(t\). Equation \(2.13\) has the formal structure of the hydrodynamical equation; but at the present state of the analysis, it is still an exact equation. It is the approximations performed later on in the formula for the stress tensor \(R\) which will transform it into an irreversible equation. - In addition to \(2.13\), the equation for the particle number density component of \(\langle a \rangle\) yields the continuity equation:

\[
\nabla \cdot u = 0
\]

The kernel function \(S\) of the stress tensor \(R\) is a time correlation function in local equilibrium which is a nonlinear functional of \(u\). As far as the author knows, presently it is possible to calculate correlation functions for total equilibrium only. Therefore, it has been necessary to expand \(S\) into a functional power series in \(u\). As has been explained in section 3 of \([6]\), the expansion can be done with the set \(b(t)\) of conjugate parameters of the local equilibrium formula \(2.83\). The expansion is performed at the point \(b = b_0\), which corresponds to \(u = 0\):

\[
\begin{align*}
b_0 &= \{-\beta \frac{\mu}{m}, \beta, 0\} \\
b - b_0 &= \{\beta \frac{1}{2} a^2, 0, -\beta u\}
\end{align*}
\]

The power expansion of \(S\) reads:

\[
S = S^{(0)} + \delta S \big|_{b_0} \ast \ast (b - b_0) + \frac{1}{2!} \frac{\delta^2 S}{\delta b \delta b} \big|_{b_0} \ast \ast \ast (b - b_0, (b - b_0)) + \cdots
\]

\[
S = S^{(0)} + S^{(1)} + S^{(2)} + \cdots
\]

The designation of terms in the second row is for later reference. The \(\ast\) indicates multiplication, summation over five elements and integration over space and time. For the present purpose, the expansion is cut after \(S^{(2)}\). When these terms are inserted into \(2.14\), one obtains corresponding parts \(R^{(1)}, R^{(2)}, R^{(3)}\) of \(R\); the upper index again describing the order in \(u\). The elements of \(b_0\) are the conjugate parameters of the total equilibrium ensemble. Thus, when \(S\) and its derivatives are taken at \(b = b_0\), the quantities in the integrand resemble total equilibrium space-time correlation functions. In \([6]\), these quantities have been calculated by Kawasaki technique \([7]\); though this theory still contains certain intuitive elements. In the present state of the project, for correlation functions the multilinear mode-coupling theory (MCT) of Schofield and co-workers has been used \([8, 9]\). For the 3-point correlations in \(S^{(1)}\), the results of the two methods coincide; for 4-point quantities, there are differences. - In order to calculate \(S^{(1)}\), it is sufficient to restrict \(b - b_0\) to the last element of \(2.18\), since the first element is of second order in \(u\); but for \(S^{(2)}\), this element of \(S^{(1)}\) has to be added.

It is an important test for the calculation method that \(2.13\), correctly approximated, should yield the Navier-Stokes equation. The linear part \(R^{(1)}\) of the stress tensor has been calculated by several authors including Grabert \([5]\), sec. 4; the Stokes form of the stress tensor is obtained, with a microscopic definition of the friction matrix. There remains another detail of the argumentation: The left-hand side of equation \(2.13\) is of second order in the velocity. Therefore, in order for the equation accurately derived, it must be shown that the second-order part \(R^{(2)}\) of the stress tensor vanishes. The present
author published two papers in ArXiv [10] dealing with this topic, and finally could show that $R^{(2)} = 0$, so that the Navier-Stokes equation derives correctly.

3 Third-order term of the stress tensor

Derivation of the third-order term provides considerable effort; a computer algebra system (Mathematica) has been used for most of the calculations. In order to describe the third-order term, we switch from the stress tensor $R$ to the friction force $D = \nabla \cdot R$. By introducing the corresponding term of (2.14) into (2.13), we obtain a formula for the third-order term $D_3$ of $D$. Again, only the $u$-part of (2.18) has to be considered. For the detailed investigation, we need the indexed form of the formula:

\[
(D_3)_{u}(x, t) = \frac{\beta^2}{2} \nabla_c \int_0^t dt' \int dx' \int_0^\infty dt'' \int dx'' \int_0^\infty dt''' \int dx''' \frac{\delta^2 S_{abcd}(x, x', t, t')}{\partial b_c(x', t') \partial b_d(x''', t''')} |_{b_c} \times \nabla_d u_b(x', t') u_c(x''', t'') u_f(x''', t''') \quad (3.1)
\]

Latin letters denote indices which run over 3 elements; by contrast, greek indices run from 1 to 5. - In (3.1), we need the general form:

\[
\left[ \frac{\delta S_{abcd}}{\partial b_c(x'', t''')} \right]_p = -\delta(t'' - t') \beta \langle [\mathcal{G}(t', t) \mathcal{Q}(t)] s_{ac}(x) \rangle \mathcal{Q}(t') \delta a_c(x''', t') \mathcal{Q}(t') s_{bd}(x') \rangle_{L,t} \quad (3.2)
\]

\[
\mathcal{Q}(t) = 1 - \mathcal{P}(t) \quad (3.3)
\]

\[
\delta a_c(x, t) = a_c(x) - \langle a_c(x) \rangle_{L,t} \quad (3.4)
\]

The functional part is [10] (second paper) (A14):

\[
\left[ \frac{\delta S_{abcd}}{\partial b_c(x'', t'')} \right]_f = \beta \Theta(t'' - t') \Theta(t - t'') \langle [\mathcal{G}(t', t'') \mathcal{L} \mathcal{P}(t'')] \delta a_c(x''', t''') \mathcal{Q}(t''') \mathcal{G}(t'', t) \mathcal{Q}(t) \mathcal{Q}(t) s_{ac}(x) \rangle \mathcal{Q}(t') s_{bd}(x') \rangle_{L,t} \quad (3.5)
\]

For later reference, the first-order derivatives are formulated with respect to all five conjugated parameters (the greek index $\epsilon$ is used). - These formulas, as well as the starting formula (2.15), contain operator chains which are, as usual, written like products. The question arises whether I can, when differentiating a chain, use the product rule; in other words, if $A$ and $B$ are operators, whether it is correct to write:

\[
\frac{\delta A B}{\delta b} = \frac{\delta A}{\delta b} B + A \frac{\delta B}{\delta b} \quad (3.6)
\]

Since I could not find a general proof in the literature, I checked (3.6) for any two consecutive operators appearing in (3.2), (3.5), and found it correct in all cases. The calculations are partly long and tedious and will not be shown here. - From (3.2), (3.5), the second-order functional derivatives are to be calculated.

The formulas depend on $b$ by 6 and 8 'factors', respectively; including, in each formula, the dependence of the local equilibrium probability density. The corresponding terms are given in Appendix A, (A.12) to (A.15). In the course of the calculation, for $\mathcal{G}(t', t)$ an identity formula is needed which can be checked by discretizing it:

\[
\mathcal{Q}(t') \mathcal{G}(t', t) = \mathcal{Q}(t') \mathcal{G}(t', t) \mathcal{Q}(t) \quad (3.7)
\]

Three of the terms are found to vanish; thus, the second-order derivative of $S$, and therefore the third-order term of $R$ consist of 11 terms. Finally to be added are parts of the first-order derivative (3.2), (3.5), which stem from $\epsilon = 1$. - For application in (3.1), all terms have to be taken at $b = b_0$. This has effects listed below:

- expectations in local equilibrium change into those in total equilibrium, denoted by $\langle \rangle_0$;
- projection operators $\mathcal{P}(t)$, $\mathcal{Q}(t)$ change to time-independent operators $\mathcal{P}$, $\mathcal{Q}$;
- $\delta a_c(x, t)$ changes to $a_c(x)$;

5
\[ g(t) = e^{\mathcal{Q} \mathcal{L} t} \]  
(3.8)

This operator has to be substituted by the similar operator \( g(t) \):

\[ g(t) = e^{\mathcal{L} t} \]  
(3.9)

The reason is that for \( g(t) \) there is an identity which substitutes it further to the non-projected exponential operator \( f(t) = e^{\mathcal{L} t} \). - The substitution succeeds for all 11 terms with the aid of one of the following identities:

\[
\begin{align*}
\mathcal{Q} \tilde{g}(t) &= g(t) \mathcal{Q} \\
\tilde{g}(t) \mathcal{L} &= \mathcal{L} g(t) \\
\tilde{g}(t) \mathcal{Q} \mathcal{L} &= -\mathcal{P} \mathcal{L} + \mathcal{L} g(t)
\end{align*}
\]  
(3.10)

Next, we change from the microscopic variables \( a, s \) to the corresponding orthonormalized quantities \( h, r \); this extracts from each term the factor \((\frac{\pi}{2})^2\). Together with the factor \( \beta \) which appears explicitly in each factor, the pre-factor in \((3.1)\) changes to \( \frac{1}{\pi \rho^2} \beta \). The gradient operators appearing in \((3.1)\) are transferred to the kernel function; by partial integration, this alters the sign of the expression. In the formula, we then have the expressions \( \nabla_c r\alpha(x), \nabla_d r\beta(x') \) which we denote \( r_\alpha(x), r_\beta(x') \) respectively; the distinction to the original \( r\)'s is by the number of indices appended. Finally, we change to Fourier space. The expression now reads:

\[
(D_0)_{0}(t) = -\frac{1}{2} \rho^2 \beta \int_0^t dt' \int_0^\infty dt'' \int_0^\infty dt''' \mathcal{P}(r_0, r_1', h_2', h_3', t, t', t'', t''') \mathbf{u}_1(t') \mathbf{u}_2(t'') \mathbf{u}_3(t''')
\]
(3.11)

Here we have introduced number indices. A number index is a combined index which contains a component index and a wave number variable. Moreover, double indices contain a summation over the indexed elements and an integration over the corresponding wave number space, together with the factor \((\frac{1}{2\pi})^2\). The kernel function \( K \) consists of the 11 Terms presented in Appendix A, \((A.10)\) to \((A.26)\). For shortness, I used the abbreviation \( \hat{r} = q r \).

### 4 Preparation of the kernel function

Starting with this section all expectations are with respect to total equilibrium. We can therefore omit the \( 0 \) at the expectation symbol. - We begin by mentioning a well known property of equilibrium correlation functions. The general form of such a function is:

\[
\langle B_1^{(1)} B_2^{(2)} \cdots B_n^{(n)} \rangle
\]
(4.1)

The upper indices just number the microscopic functions \( B^{(i)} \), while the lower ones are the number indices defined in the preceding section. Since this is a theory with unlimited geometric space, in wave number space the quantity contains a factor:

\[
(2\pi)^3 \delta(k_1 + k_2 + \cdots + k_n)
\]
(4.2)

This fact simplifies some formulas considerably and will be used several times in the next sections. - The kernel function of \((3.11)\) has to be brought into a form so that it contains time correlation functions which can be evaluated by MCT. These are of the general form \( \langle f(t)A[B^*] \rangle_0 \) mit \( f(t) = e^{\mathcal{L} t} \). As a first step, the projection operators \( \mathcal{P} \) and \( \mathcal{Q} \) are to be substituted. As can be seen in the formulas in the appendix, the terms contain 0 to 4 of these operators (the operator contained in \( \hat{r} \) remains unaltered). With the reformulations done in the last section, the operators now read:

\[
\mathcal{P} A = \langle A \rangle + \langle A h_1^* \rangle h_4
\]
(4.3)

\[
\mathcal{Q} A = A - \langle A \rangle - \langle A h_1^* \rangle h_4
\]
(4.4)

Generally, the component indices contained in the number indices in these formulas must be greek (running over the full list of 5 elements); it then would follow that I have to insert \( \delta h_4 \) instead of \( h_4 \).
But as will be seen later, finally all indices reduce to be "latin", i.e. will run over the 3 momentum components. Then, ∥h4∥ = 0, so that formulas (4.3), (4.4) are correct. - The formulas are applied in several consecutive steps. After each step, certain auxiliary formulas are applied in order to simplify the expressions:

\[ \langle h_1 \dot{r}_2 \rangle = \langle (Qh_1)r_2 \rangle = 0 \]
\[ \langle \mathcal{L}A \rangle = 0 \]
\[ \mathcal{L}g(t)h_i = 0 \]
\[ \langle r \rangle = -\langle Lh \rangle = 0 \]
\[ \mathcal{L}_h = -\dot{r}_1 - \langle r_1 h_4 \rangle h_4 \]  

(4.5)

In this set, the third and fourth formula are special cases of the second. - The result for the 11 terms, now denoted \( K_1, \cdots, K_{11} \), is presented in the appendix, (A.27). The \( \delta \) and \( \theta \)-factors are temporarily omitted; when the time integrations start they will be added again.

It is seen that the time correlations in these formulas belong to one of two groups described by:

\[ \langle [g(t)\dot{r}_0]B^* \rangle \]
\[ \langle [\mathcal{L}g(t)\dot{r}_0]B^* \rangle \]  

(4.6a, b)

\( B \) is a certain microscopic function. The formulas which are still built with the operator \( g(t) \) are to be transformed into those with \( f(t) \). This will be done in a number of steps. We start with (4.6a). First, we have:

\[ \langle [g(t)\dot{r}_0]\dot{h}_t \rangle = 0 \]  

(4.7)

(4.10a) and \( Qh_1 = 0 \) have been used. Next, we introduce a symbol for the memory function of the process:

\[ \Gamma_{01}(t) \equiv \langle [g(t)\dot{r}_0]\dot{r}_1^* \rangle \]  

(4.8)

The memory function is, as usual, "localized" in time, i.e. approximated by:

\[ \Gamma_{01}(t) = \delta(t)(2\pi)^3\delta(k_0 - k_1)\gamma_{01} \]  

(4.9)

\( \gamma_{01} \), when written with component indices \( a, b \), for incompressible fluid reduces to:

\[ \gamma_{ab} = \nu k_0^2 \delta_{ab} \]  

(4.10)

\( \nu \) is the kinematic viscosity, and \( \delta_{ab} \) is the Kronecker symbol. - Next, we need an operator identity described in (11), Appendix B2; in the time-dependent form, and applied to the operators defined in this paper, it reads:

\[ g(t) = f(t) - \int_0^t dt' f(t')\mathcal{P}Lg(t - t') \]  

(4.11)

We obtain the correlation function identity, for any microscopic function \( B \):

\[ \langle [g(t)\dot{r}_0]B^* \rangle = \langle [f(t)\dot{r}_0]B^* \rangle - \int_0^t dt' \Gamma_{04}(t - t')\langle [f(t')h_4]B^* \rangle \]  

(4.12)

After localization of the memory function, we get:

\[ \langle [g(t)\dot{r}_0]B^* \rangle = \langle [f(t)\dot{r}_0]B^* \rangle - \gamma_{04}\langle [f(t)h_4]B^* \rangle \]  

(4.13)

The first term on the rhs of (4.12) is still to be transformed. The fifth formula (4.5) together with (4.8) and \( \omega_{04} = \langle r_0 h_4^* \rangle \) yields:

\[ \dot{r}_0 = -\partial_t h_0 - \omega_{04} h_4 \]  

(4.14)

We obtain:

\[ \langle [g(t)\dot{r}_0]B^* \rangle = -\partial_t \langle [f(t)h_0]B^* \rangle - \kappa_{04}\langle [f(t)h_4]B^* \rangle \]
\[ \kappa_{04} = \omega_{04} + \gamma_{04} \]  

(4.15, 16)
We denote with \( C_n \) die time correlation functions which can be calculated by MCT:

\[
(C_n)_{01\ldots n-1}(t) = \langle [f(t) h_0] h_1^* \cdots h_{n-1}^* \rangle \tag{4.17}
\]

Then, we have the following applications of \((4.15)\):

\[
\langle [g(t) \hat{r}_0] h_1^* h_2^* \rangle = -\partial_t (C_3)_{012}(t) - \kappa_{08}(C_3)_{s12}(t) \tag{4.18}
\]

\[
\langle [g(t) \hat{r}_0] h_1^* h_2^* \rangle = -\partial_t (C_4)_{0123}(t) - \kappa_{08}(C_4)_{s123}(t) \tag{4.19}
\]

Somewhat more complicated is the transformation if \( B \) contains \( \hat{r} \) as a factor. Again, \((4.14)\) is applied, together with \( \omega_{04}^* = -\omega_{04} \):

\[
\langle [g(t) \hat{r}_0] h_1^* h_2^* \rangle = -\partial_t (\langle [f(t) h_0] h_1^* h_2^* \rangle - \kappa_{04} (\langle [f(t) h_4] h_1^* h_2^* \rangle)
\]

\[
= \partial_t (\langle [f(t) h_0] (\partial_t h_1^*) h_2^* \rangle - \omega_{17} \partial_t (\langle [f(t) h_0] h_1^* h_2^* \rangle + \kappa_{04} (\langle [f(t) h_4] (\partial_t h_1^*) h_2^* \rangle) - \omega_{17} (\langle [f(t) h_4] h_1^* h_2^* \rangle) \tag{4.20}
\]

The first and the third term need further transformation. Because of the symmetries of \((3.11)\), the following formula is correct if it is used only within this expression:

\[
(\partial_t h_1^*) h_2^* = \frac{1}{2} (\partial_t h_1^*) h_2^* + h_1^* \partial_t h_2^* = \frac{1}{2} \partial_t (h_1^* h_2^*) \tag{4.21}
\]

We obtain, for the correlation function \((4.20)\):

\[
\langle [g(t) \hat{r}_0] h_1^* h_2^* \rangle = \frac{1}{2} \partial_t (h_1^* h_2^*) - \omega_{17} \partial_t (h_1^* h_2^*) + \kappa_{04} (\langle [f(t) h_4] (\partial_t h_1^*) h_2^* \rangle) - \omega_{17} (\langle [f(t) h_4] h_1^* h_2^* \rangle) \tag{4.22}
\]

We now switch to time correlations belonging to group \((4.6b)\). We need an operator identity similar to \((4.11)\) which again can be found in \([11]\), Appendix 2B. Applied to this paper, it reads:

\[
g(t) = f(t) - \int_0^t dt' g(t') \mathcal{L} f(t - t') \tag{4.23}
\]

We obtain a formula similar to \((4.12)\):

\[
\langle [\mathcal{L} g(t) h_0] B^* \rangle = \langle [\mathcal{L} f(t) h_0] B^* \rangle - \int_0^t dt' \langle [\mathcal{L} f(t - t') h_0] h_4^* \rangle (\langle [\mathcal{L} g(t') h_4] B^* \rangle \tag{4.24}
\]

For shortness, we introduce some new denotations:

\[
\Psi_0(t) = \langle [\mathcal{L} g(t) h_0] B^* \rangle \tag{4.25}
\]

\[
\Phi_0(t) = \langle [\mathcal{L} f(t) h_0] B^* \rangle = \partial_t (\langle [f(t) h_0] B^* \rangle \tag{4.26}
\]

Then, applying \((4.14)\), \((4.24)\) reads:

\[
\Psi_0(t) = \Phi_0(t) - \int_0^t dt' \partial_t (C_2)_{04}(t - t') \Psi_4(t') \tag{4.27}
\]

This is an integral equation for \( \Psi \) which can be formally solved by temporarily switching to Laplace space. The result is:

\[
\Psi_0(t) = \Phi_0(t) + \kappa_{04} \int_0^t dt' \Phi_4(t') \tag{4.28}
\]

Taking \((4.26)\) into account, we can integrate \((4.24)\):

\[
\langle [\mathcal{L} g(t) h_0] B^* \rangle = \partial_t (\langle [f(t) h_0] B^* \rangle + \kappa_{04} (\langle [f(t) h_4] B^* \rangle - \langle h_4 B^* \rangle) \tag{4.29}
\]

This is the basic formula for time correlations of form \((4.6b)\). Taking \( B = h_1 \), we have:

\[
\langle [\mathcal{L} g(t) h_0] h_1^* \rangle = \partial_t (C_2)_{01}(t) + \kappa_{07} ((C_2)_{71}(t) - \delta t) \tag{4.30}
\]
The lowest-order non-projected time correlation function $C_2$ obeys a simple differential equation (10), second paper, (4.6)):

$$\partial_t (C_2)_{01}(t) = -\kappa_{07}(C_2)\tau_1(t)$$ (4.31)

This is introduced into (4.30):

$$\langle [Lg(t)h_0]h_1^* \rangle = -\kappa_{01}$$ (4.32)

The lowest-order time correlation of type (4.6) is a constant. For the next-higher order, we find from (4.29):

$$\langle [Lg(t)h_0]h_1^* h_2^* \rangle = \partial_t \langle [f(t)h_0]h_1^* h_2^* \rangle + \kappa_{04} \langle [f(t)h_4]h_1^* h_2^* \rangle - \langle h_4 h_1^* h_2^* \rangle$$ (4.33)

$$\langle (j_3)_{01}(t = 0) = \langle h_0 h_1^* h_2^* \rangle$$ (4.34)

Again the situation is somewhat more complicated if $B$ contains the projected flux $\dot{r}$. For the lowest order, we obtain, by applying (4.29):

$$\langle [Lg(t)h_0]h_1^* h_2^* \rangle = \partial_t \langle [f(t)h_0]h_1^* h_2^* \rangle + \kappa_{04} \langle [f(t)h_4]h_1^* h_2^* \rangle - \langle h_4 h_1^* h_2^* \rangle$$ (4.35)

We apply (4.11):

$$\langle [Lg(t)h_0]h_1^* h_2^* \rangle = -\partial_t \langle [f(t)h_0]h_1^* h_2^* \rangle + \omega_{17} \partial_t \langle [f(t)h_0]h_1^* h_2^* \rangle$$

$$+ \kappa_{04} \langle [f(t)h_4]h_1^* h_2^* \rangle + \omega_{17} \langle [f(t)h_4]h_1^* h_2^* \rangle - \langle h_4 h_1^* h_2^* \rangle$$ (4.36)

We introduce a notation for the static correlation appearing in the formula:

$$(s_3)_{124} = \langle \hat{r}_1 h_2 h_4^* \rangle = -\langle \hat{r}_1 h_2 h_4^* \rangle^* = -\langle h_4 h_1^* h_2^* \rangle$$ (4.37)

Finally, we can apply a symmetry consideration as in (4.20); by using (4.21) we obtain:

$$\langle [Lg(t)h_0]h_1^* h_2^* \rangle = \frac{1}{2} \partial_t (C_3)_{012}(t) + \omega_{17} \partial_t (C_3)_{072}(t)$$

$$+ \kappa_{04} \left( \frac{1}{2} \partial_t (C_3)_{112}(t) + \omega_{17} (C_3)_{472}(t) + (s_3)_{124} \right)$$ (4.38)

For the next-higher order correlation we have, by (4.29):

$$\langle [Lg(t)h_0]h_1^* h_2^* h_3^* \rangle = \partial_t \langle [f(t)h_0]h_1^* h_2^* h_3^* \rangle + \kappa_{04} \langle [f(t)h_4]h_1^* h_2^* h_3^* \rangle - \langle h_4 h_1^* h_2^* h_3^* \rangle$$ (4.39)

Introduction of (4.11) yields:

$$\langle [Lg(t)h_0]h_1^* h_2^* h_3^* \rangle = -\partial_t \langle [f(t)h_0]h_1^* h_2^* h_3^* \rangle + \omega_{17} \partial_t \langle [f(t)h_0]h_1^* h_2^* h_3^* \rangle$$

$$+ \kappa_{04} \langle [f(t)h_4]h_1^* h_2^* h_3^* \rangle + \omega_{17} \langle [f(t)h_4]h_1^* h_2^* h_3^* \rangle - \langle h_4 h_1^* h_2^* h_3^* \rangle$$ (4.40)

The symmetry consideration which is correct if it is applied within the integral formula (5.11) gives:

$$\langle \partial_t h_1^* h_2^* h_3^* \rangle = \frac{1}{4} \langle [\partial_t h_1^*] h_2^* h_3^* + h_1^* [\partial_t h_2^*] h_3^* + h_1^* h_2^* [\partial_t h_3^*] \rangle = \frac{1}{2} \partial_t (h_1^* h_2^* h_3^*)$$ (4.41)

Again a denotation for the static correlation is introduced:

$$(s_4)_{1234} = \langle \hat{r}_1 h_2 h_3 h_4^* \rangle = -\langle \hat{r}_1 h_2 h_3 h_4^* \rangle^* = -\langle h_4 h_1^* h_2^* h_3^* \rangle$$ (4.42)

The final form for (4.39) is:

$$\langle [Lg(t)h_0]h_1^* h_2^* h_3^* \rangle = \frac{1}{2} \partial_t (C_4)_{0123}(t) + \omega_{17} \partial_t (C_4)_{0723}(t)$$

$$+ \kappa_{04} \left( \frac{1}{2} \partial_t (C_4)_{1123}(t) + \omega_{17} (C_4)_{4723}(t) + (s_4)_{1234} \right)$$ (4.43)
The formulas obtained in this section are introduced into the terms (A.27). The Dirac and Heaviside functions omitted are added again. The kernel function \( K \) in (3.11) now consists of 5 main parts:

\[
K = -\delta(t'' - t')\delta(t'' - t)M_1 + \delta(t'' - t')\Theta(t'' - t)\Theta(t'' - t)M_2 + \delta(t'' - t')\Theta(t'' - t)\Theta(t'' - t)M_3 + \Theta(t'' - t')\Theta(t'' - t)\Theta(t'' - t)\Theta(t'' - t)\delta(t'' - t')M_4 + \Theta(t'' - t')\Theta(t'' - t)\Theta(t'' - t)\Theta(t'' - t)\delta(t'' - t')M_5
\]  

(4.44)

The coefficients \( M_1, \cdots, M_5 \) are presented in the appendix, (A.28). We denote:

\[
j(j_2)_{12} = \langle \delta h_1^* h_2^* \rangle
\]  

(4.45)

When (A.28) is inserted into (3.11), some of the time integrals can be executed. Moreover, we can apply an approximation since the time scale of the \( u \)-factors is much larger than that of the equilibrium time correlations. The upper limits of integrals over correlation functions can then be extended to infinity. The formula now reads:

\[
(D_3)_0(t) = -\frac{1}{2} \rho^2 \beta \bar{t} \left( \frac{\partial \bar{t}}{\partial \rho} \right)^{-\frac{1}{2}} \int_0^t dt' \int_0^\infty dt'' (K_2)_{0123}(t, t', t'') u_1(t') u_q(t'')
\]  

(5.1)

In this section, the number index 2 contains a component index which has the fixed value 1, if not stated otherwise. The pre-factor contains a part which results from switching to normalized variables in the correlation functions. \( u_q \) is the Fourier transform of \( u^2 \) for the wave number variable \( k'' \). The kernel function \( K_2 \) is the sum of (5.2) and (5.5), taken at \( b = b_0 \) for \( \epsilon = 1 \); (5.10a) has been used to substitute the operator \( \tilde{g} \):

\[
(K_2)_{0123}(t, t', t'') = -\delta(t'' - t') \langle [g(t - t') \tilde{r}_0] \bar{r}_1^* \delta h_2^* \rangle + \Theta(t'' - t')\Theta(t'' - t') \langle [g(t'' - t') \mathcal{L} \mathcal{P} \delta h_2^* g(t - t') \tilde{r}_0] \bar{r}_1^* \rangle
\]  

(5.2)

For the first term, we have:

\[
\langle [g(t - t') \tilde{r}_0] \bar{r}_1^* \delta h_2^* \rangle = \langle [g(t - t') \tilde{r}_0] \bar{r}_1^* h_2^* \rangle - \langle [g(t - t') \tilde{r}_0] \bar{r}_1^* \rangle \langle h_2^* \rangle
\]  

(5.3)

The operator \( \mathcal{P} \) in the second term in (5.2) is evaluated:

\[
\langle [g(t'' - t') \mathcal{L} \mathcal{P} \delta h_2^* g(t - t') \tilde{r}_0] \bar{r}_1^* \rangle
\]

\[
= -\langle [g(t'' - t') \tilde{r}_0] h_2^* \rangle \langle [g(t'' - t') \tilde{r}_3] \bar{r}_1^* \rangle
\]  

(5.4)

The identity \( \mathcal{L} \delta h_2 = -\mathcal{L} r_3 = -\tilde{r}_3 \) has been used. Originally, the full form (4.23) with \( \delta h \) instead of \( h \) had to be used in the first factor; but it is found that the terms with the constant parts \( \langle h_2 \rangle, \langle h_3 \rangle \) vanish,
so that the final form is \(5.4\). - For further preparation the formulas \([4,8]\) and \([10,11,13,12,22]\) are used, integration over \(t''\) and an approximation because of different time scales is performed:

\[
(D_2)_{0}(t) = -\frac{1}{2}\rho^2 \beta \left( \partial_p \rho |\beta\right) - \frac{1}{4} \ u_1(t) u_2(t) \int_0^\infty dt' (K_2)_{012} (t')
\]  
\(5.5\)

\(K_2\), now a function of \(t'\) only, is given in the appendix, \([A,30]\). \(5.5\) has to be added to \([4,46]\) in order to obtain the full third-order term \(D_3\) of the friction force:

\[
(D_3)_{0}(t) = -\frac{1}{2}\rho^2 \beta (u_1(t) u_2(t) u_3(t)) \times 
\]

\[
\times (\int_0^\infty dt' \Phi_{0123}(t') + \lim_{t' \to \infty} \int_0^t dt' \int_0^{t-t'} dt'' \Phi_{0123}(t', t'')) 
\]

\[
+ (\rho \beta \partial_p \rho |\beta\) - \frac{1}{4} \ u_1(t) u_2(t) \int_0^\infty dt' (K_2)_{012} (t')
\]  
\(5.6\)

To complete the derivation, the equilibrium correlation functions \(C_3\) and \(C_4\) appearing in the coefficients have to be calculated. The MCT has been applied for this purpose \([8,9]\). A formula for \(C_3\) is explicitly given in \([8]\), while \(C_4\) is somewhat simpler than \([9]\) (15). In the correlations appearing in \(D_3\) and \(D_2\), we have \(\alpha = 1\); for \(\beta = 1\), \(6.4\) is directly the two-point correlation \(C_2\), while the higher-order correlations \(C_3\) and \(C_4\) are obtained from \(6.4\) for \(\beta = 2\) and 3, respectively. In the latter cases, formula \([9]\) (26) (given in Laplace space) has to be applied. Since \(\beta \neq \alpha\), the first term vanishes. The case \(\beta = 2\) is analyzed in \([9]\) as an example. The \(N\)-ordering analysis shows that only the second term of the formula survives. For \(\beta = 3\), the second and certain parts of the third term survive. But the detailed analysis

### 6 Multilinear mode coupling theory

We refer to the paper by van Zon and Schofield \([9]\). We continue to use the orthonormalized microscopic variables \(h\) defined in sec. 3. In \([9]\), multiple variables \(A_\alpha\) are employed which are products of the single variables:

\[
A_\alpha = h_1 h_2 \cdots h_\alpha
\]  
\(6.1\)

Greek indices, in addition to describing a component index running over 5 elements, are used in this section in connection with the multiple variables. Depending on the context, they denote either the order of the multiple variable or the full set of number indices:

\[
\alpha = \{1, 2, \cdots, \alpha\}
\]  
\(6.2\)

Orthogonal multiple variables \(Q_\alpha\) and projection operators \(P_\alpha\) are successively defined (do not mix up these symbols with the (non-indexed) \(P\) and \(Q\) used in earlier sections):

\[
\begin{aligned}
Q_0 &= 1 \\
Q_\alpha &= (1 - \sum_{\nu=0}^{\alpha-1} P_\nu) A_\alpha \\
P_\alpha X &= \langle X Q_\alpha \rangle Q_\alpha
\end{aligned}
\]  
\(6.3\)

A greek symbol appearing twice includes summation over each of the component indices and integration over each of the wave numbers; for each wave number including a factor \((2\pi)^{-3}\). The last of the formulas \(6.3\) is somewhat simpler than \([9]\) (6), owing to the fact that our basic variables \(h\) are normalized. - An important property of the theory is that it is formulated before performing the thermodynamic limit; that is, there is a finite number of particles, \(N\). Finally, the limit \(N \to \infty\) is taken. In the formulas appear terms which are proportional to different powers of \(N\); in the limit, only those with the highest power of \(N\) 'survive'. In this paper, we will provide only the final formulas obtained after performing the thermodynamic limit. - The object of the theory are time correlations of the \(Q\):

\[
G_{\alpha\beta} = \langle Q_\alpha(t) Q_\beta^* \rangle
\]  
\(6.4\)

Again, formula \(6.4\) is somewhat simpler than \([9]\) (15). In the correlations appearing in \(D_3\) and \(D_2\), we have \(\alpha = 1\); for \(\beta = 1\), \(6.4\) is directly the two-point correlation \(C_2\), while the higher-order correlations \(C_3\) and \(C_4\) are obtained from \(6.4\) for \(\beta = 2\) and 3, respectively. In the latter cases, formula \([9]\) (26) (given in Laplace space) has to be applied. Since \(\beta \neq \alpha\), the first term vanishes. The case \(\beta = 2\) is analyzed in \([9]\) as an example. The \(N\)-ordering analysis shows that only the second term of the formula survives. For \(\beta = 3\), the second and certain parts of the third term survive. But the detailed analysis
shows that the third term contains a dimensionless combination of thermodynamic parameters which is \(\ll 1\), so that finally both cases are described by the same formula:

\[
G_{\alpha\beta}(\zeta) = G_{\alpha\alpha}(\zeta)M_{\alpha'\beta'}(\zeta)G_{\beta'\beta}(\zeta)
\]

(6.5)

\[
M_{\alpha\beta}(\zeta) = (Q_\alpha Q_\beta^*) - \Gamma_{\alpha\beta}(\zeta)
\]

(6.6)

Certain simplifications (compared with [2] (26)) stem from the fact that for the quantities considered \(\beta \neq \alpha\). \(\zeta\) is the independent variable of Laplace space. \(\alpha’\) describes an index set of the order \(\alpha\) with the same wave numbers as \(\alpha\) but different component indices.

\(\Gamma_{\alpha\beta}\) is the time correlation of the fluctuating forces defined in [2] (11). These quantities describe the friction effects of the fluid. Generally, \(\Gamma_{\alpha\beta}(z)\) is replaced by its value at \(z = 0\) (‘localization’ in time).

The lowest order quantity, \(\alpha = \beta = 1\), for small wave numbers resembles the Green-Kubo expression. In [9], there is no prescription for the higher order \(\Gamma_{\alpha\beta}\). We refer to the older correlation function theory by Kawasaki [12], where these quantities are neglected. - In addition to (6.5), (6.6), it is shown ([9] (25)) that, after transforming back to the time domain, for any \(\alpha\), in highest \(N\)-order \(G_{\alpha\alpha}(t)\) factorizes into the product of the corresponding order-one quantities. - Essentially, (6.5), (6.6) express the higher order correlations in terms of \(C_2\); to calculate \(C_2\), an additional formula is used (see below).

For \(C_3\), the formulas derived in [9] are (30), (31); with the denotations of the present paper and in terms of number indices, they read:

\[
(C_3)_{123}(t) = (C_2)_{14}(t)j_{123} + (G_{12})_{123}(t)
\]

(6.7)

\[
(G_{12})_{123}(t) = -\int_0^t dt' (C_2)_{14}(t - t') (s_1)_{156}(C_2)_{25}(t') (C_2)_{36}(t')
\]

(6.8)

\[
(s_3)_{123} = \{\hat{r}_1 h_1^* h_3^*\}
\]

(6.9)

\(j_3\) is defined in (4.3). Flux densities \(r\) and projected flux densities \(\hat{r}\) are defined in (4.3). \(s_3\) is, by definition, different from the quantity defined in (4.36). But after the factor extraction described at the beginning of the next section, the difference will vanish. We will therefore use the same notation. - The formulas for \(C_4\) obtained from (6.5), (6.6) for \(\beta = 3\) read, after some manipulations:

\[
(C_4)_{1234}(t) = ((C_3)_{156}(t) - (j_3)_{156})\Phi_{23456} + (C_2)_{17}(t)j_{234} + (G_{13})_{1234}(t)
\]

(6.10)

\[
(G_{13})_{1234}(t) = -\int_0^t dt' (C_2)_{15}(t - t') (s_4)_{156}(C_2)_{26}(t') (C_2)_{37}(t') (C_2)_{48}(t')
\]

(6.11)

\[
\Phi_{12345} = (j_4)_{1234} - (j_4)_{12}(j_4)_{15} - (j_4)_{13}(j_4)_{17} - (j_4)_{123}(j_4)_{1745}
\]

(6.12)

\[
(s_4)_{1234} = \{\hat{r}_1 h_1 h_3^* h_4^*\}
\]

(6.13)

The choice of conjugate complex variables in (6.14) looks somewhat arbitrary; the quantities are defined as they appear in (6.10) to (6.13). \(j_3\) is different from (4.33), but again the difference vanishes after factor extraction. - \(C_2(t)\) is, in [9], the quantity \(G_{11}\) called the propagator. For the calculations described in this paper, the formula [9] (34) is approximated to lowest order, which in the time domain leads to the equation:

\[
\partial_t (C_2)_{12}(t) = -\omega_{13} (C_2)_{32}(t) - \int_0^t dt' (\Gamma_{11})_{13}(t - t') (C_2)_{32}(t')
\]

(6.15)

\[
\omega_{12} = \{-\hat{h}_1 h_2^*\}
\]

(6.16)

\(\Gamma_{11}\) is localized in time, as in (4.9). But then \(\partial_t C_2\) is discontinuous at \(t = 0\) since \(\partial_t (C_2)_{12}(0) = -\omega_{12}\); therefore, the formula resulting from (6.15) must be written:

\[
\partial_t (C_2)_{12}(t) = -\kappa_{13}(t) (C_2)_{32}(t)
\]

(6.17)

\[
\kappa_{13}(t) = \omega_{13} + \theta(t)_{13}
\]

\[
\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}
\]
The time dependence of \( \kappa_{13} \) is important when one needs to calculate the second derivative of \( C_2 \); otherwise, it can be neglected. We have:

\[
\partial_t \kappa_{13}(t) = \delta(t) \gamma_{13} \quad (6.18)
\]

\[
\partial_t (C_2)_{12}(t) = -\delta(t) \gamma_{13} (j_2)_{32} + \kappa_{13} \kappa_{34} (C_2)_{42}(t) \quad (6.19)
\]

In (6.19) the denotation \((C_2)_{12}(0) = \langle h_1 h_2 \rangle = (j_2)_{12}\) has been used.

### 7 Explicit formulation

In order to keep the formulation reasonably short, in the past sections of this paper, a rather formal description has been used employing number indices; double indices indicate a component summation and a wave number integration, together with a factor \((2\pi)^{-3}\). While this is reasonable for formal calculations, for the rest of this paper we will switch to an easily readable form where numbers are component indices, and the wave numbers are explicitly shown. For all correlation functions, we will extract the factor described in (4.2), denoting the remaining part with the same symbol as before. It then becomes apparent that the additional wave number integrals introduced via the definition of the projection operators (4.3), (4.4), all can be executed. The formula for \(\Phi_1, \Phi_2, \Phi_3\) is given in the appendix, (A.31) to (A.33). - The equation for \( C_2 \) (6.17) now reads:

\[
(D_3)_{0}(k,t) = -\frac{1}{2} \beta^2 \frac{1}{(2\pi)^6} \int d\mathbf{k}' d\mathbf{k}'' d\mathbf{k}''' u_1(k', t) u_2(k'', t) u_3(k''', t) \delta(k-k''-k''') \times
\]

\[
\times (\Delta_{3})_{0123}(k', k'', k''')
\]

\[
+ \xi^{-\frac{1}{2}} \frac{1}{(2\pi)^3} \int d\mathbf{k}' d\mathbf{k}'' u_1(k', t) u_2(k', t) \delta(k-k'')(\Delta_{2})_{01}(k', k'')
\]

\[
(\Delta_{3})_{0123}(k', k'', k''') = \int_0^\infty dt' \Phi_{123}(k', k'', k''', t') + \lim_{t \to \infty} \int_0^t \int_0^{t-t'} dt'' \Phi_{20123}(k', k'', k''', t, t'')
\]

\[
(\Delta_{2})_{01}(k', k'') = \int_0^\infty dt' (K_{2})_{01}(k', k'', t')
\]

\[
\xi = \rho \beta \partial_\mu P |_\beta
\]

The factor \((2\pi)^3 \delta(k-k'-k''-k''')\) has been extracted from the quantities \(\Phi_1, \Phi_2\) in (A.29); also, \((2\pi)^3 \delta(k-k'-k'')\) has been extracted from \(K_2\) in (A.30). \(u_2(k)\) is the Fourier transform of \((u(x))^2\). The resulting explicit formulas for \(\Phi_1, \Phi_2, K_2\) are given in the appendix, (A.31) to (A.33). - The equation for \( C_2 \) (6.17) now reads:

\[
\partial_t (C_2)_{12}(k,t) = -\kappa_{13}(k,t)(C_2)_{32}(k,t)
\]

We have, after factor extraction, \((j_2)_{12} = \delta_{12}\). Therefore, for the second derivative, one obtains from (6.19):

\[
\partial_t (C_2)_{12}(k,t) = -\delta(t) \gamma_{12}(k) + \kappa_{13} \kappa_{34} (C_2)_{42}(k,t)
\]

\[
(6.7)
\]

\[
(6.8)
\]

Instead of (6.7), (6.8) we have:

\[
(C_3)_{123}(k,k',k'',t) = (C_2)_{14}(k,t)(j_3)_{423} + (G_{12})_{123}(k,k',k'',t)
\]

\[
(7.7)
\]

\[
(G_{12})_{123}(k,k',k'',t) = -\int_0^t dt' (C_2)_{14}(k,t-t')(s_3)_{456}(k)(C_2)_{25}(k',t')(C_2)_{36}(k'',t')
\]

We write \(C_3\) and \(G_{12}\) as functions of three wave numbers which correspond to the wave numbers of the three \(h\)-factors in the definition formula of \(C_3\); though, because of \(k'' = k - k'\), they are actually functions of \(k, k'\) only. Corresponding arguments apply for \(C_4, G_{13}, \zeta_3, \zeta_4\). - Finally, (6.13), (6.11) turn to:
\[(C_4)_{1234}(k, k', k'', k''', t) = ((C_3)_{156}(k, k', k'') - (j_3)_{156})\Phi_{23456} + (C_2)_{17}(k, t)(j_4)_{7234} + (G_{13})_{1234}(k, k', k'', k''', t)\]  \[\text{(7.9)}\]

\[(G_{13})_{1234}(k, k', k'', k''', t) = -\int_0^t dt' (C_2)_{15}(k, t - t')(\sigma_5)_{678}(k)(C_2)_{26}(k, t') (C_2)_{37}(k''', t') (C_2)_{48}(k''', t')\]  \[\text{(7.10)}\]

with \(k = k' + k'' + k'''\).

8 Substitution of time derivatives; time integration

The results of sec. 6, in the form given in sec. 7, have to be introduced into the formulas for \(\Phi_1\), \(\Phi_2\), \(K_2\). Here, time derivatives of \(C_3\), \(C_4\) up to the second order appear. From \[\text{(7.4)}\], we have, with \[\text{(7.5)}\], \[\text{(7.6)}\]:

\[\partial_t(C_3)_{1234}(k, k', k'', t) = -\kappa_{15}(k, t)(C_2)_{54}(k, t)(j_3)_{423} + \partial_t(G_{12})_{1234}(k, k', k'', t)\]  \[\text{(8.1)}\]

\[\partial_t(G_{12})_{1234}(k, k', k'', t) = -s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) + \int_0^t dt' \kappa_{17}(k, t - t')(C_2)_{74}(k, t - t')(s_3)_{156}(k)(C_2)_{25}(k', t')(C_2)_{36}(k''', t') - s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) - \kappa_{14}(k)(C_2)_{1234}(k, k', k'', t)\]  \[\text{(8.2)}\]

From \[\text{(7.8)}\], we obtain:

\[\partial_t(G_{12})_{1234}(k, k', k'', t) = -s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) + \int_0^t dt' \kappa_{17}(k, t - t')(C_2)_{74}(k, t - t')(s_3)_{156}(k)(C_2)_{25}(k', t')(C_2)_{36}(k''', t') - s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) - \kappa_{14}(k)(C_2)_{1234}(k, k', k'', t)\]  \[\text{(8.3)}\]

We remember that \(k'' = k - k'\). The first step is for calculation of the second-order derivative, where it is essential that \(\kappa\) depends on time; the second step is for insertion into the main formula. For \(\partial_t(G_{12})\) we obtain, after some manipulations:

\[\partial_t(G_{12})_{1234}(k, k', k'', t) = (C_3)_{156}(k, k', k'') + (C_2)_{25}(k', t)(C_2)_{36}(k''', t) + \kappa_{14}(k)(C_2)_{1234}(k, k', k'', t)\]  \[\text{(8.4)}\]

\[\kappa_{14}(k, k', k'') = \kappa_{14}(k)\delta_{25}\delta_{36} + \delta_{14}\kappa_{25}(k')\delta_{36} + \delta_{14}\delta_{25}\kappa_{36}(k'')\]  \[\text{(8.5)}\]

By a similar calculation, we obtain for \(C_4\), from \[\text{(7.9)}\]:

\[\partial_t(C_4)_{1234}(k, k', k'', t) = (C_3)_{156}(k, k', t)\Phi_{23456} + \partial_t(C_2)_{17}(k, t)(j_4)_{7234} + \partial_t(G_{13})_{1234}(k, k', k'', t)\]  \[\text{(8.6)}\]

\[\partial_t(G_{13})_{1234}(k, k', k'', t) = -s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) - \kappa_{15}(k)(G_{13})_{5234}(k, k', k'', t)\]  \[\text{(8.7)}\]

For the time derivatives of \(C_2\) and \(C_3\), \[\text{(7.5)}\], \[\text{(7.6)}\], \[\text{(8.1)}\], \[\text{(8.2)}\] can be used for substitution. From \[\text{(7.10)}\], we find:

\[\partial_t(G_{13})_{1234}(k, k', k'', k''', t) = -s_{15}(k)(C_2)_{25}(k', t)(C_2)_{36}(k''', t) - \kappa_{15}(k)(G_{13})_{5234}(k, k', k'', k''', t)\]  \[\text{(8.8)}\]
\[ \partial_t (G_{13})_{1234} = \zeta_4 \sigma_{567} (k, k', k'', k'''(t) = (\zeta_4)_{15678234} (k, k', k'', k''')(t) \delta_{g} + \kappa_{15}(k) \delta_{26}(k) + \kappa_{26}(k) \delta_{37}(k) \delta_{48} + \kappa_{15} \delta_{26} \delta_{37} \delta_{48}(k''') + \kappa_{15} \delta_{26} \delta_{37} \delta_{48}(k''') \] (8.9)

\[ (\zeta_4)_{12345678} = \kappa_{15}(k) \delta_{26} \delta_{37} \delta_{48} + \delta_{15} \kappa_{26}(k') \delta_{37} \delta_{48} + \delta_{15} \kappa_{26}(k') \delta_{37} \delta_{48} + \delta_{15} \delta_{26} \delta_{37} \delta_{48}(k''') \] (8.10)

The formulas derived in this section are inserted into \( \Phi_1, \Phi_2 \). It is seen that terms still containing \( G_{12}, G_{13} \) cancel each other. The remaining terms are products of \( C_2 \) of different order (in \( \Phi_2 \) with different time variables) multiplied by certain coefficients. Now we are able to perform the time integrations in (7.2). The solution of (7.5) can be expressed by the matrix exponential function. The time dependence of \( \kappa \) can now be neglected:

\[ (C_2)_{12} = e^{-\kappa_{12}(k)t} \] (8.11)

The integrations in (7.2) result in inverses of \( \zeta_3, \zeta_4 \) and of:

\[ (\zeta_4)_{1234} = \kappa_{13}(k) \delta_{24} + \kappa_{13}\kappa_{24}(k') \] (8.12)

\[ (\zeta_3)_{123456} = -\kappa_{14}(k) \delta_{25} \delta_{36} + \kappa_{14}\kappa_{25}(k') \delta_{36} + \kappa_{14}\kappa_{25}(k') \delta_{36} \] (8.13)

For instance, we have:

\[ \int_0^\infty dt (C_2)_{12}(k,t)(C_2)_{34}(k',t) = (\zeta_2)_{1324}(k,k') \] (8.14)

After integration, the parts from \( \Phi_1 \) and \( \Phi_2 \) can be united. Some terms cancel, others can be simplified. For instance, the relation applies:

\[ (\zeta_3)_{123456} = (\zeta_2)_{7896}^{-1}(k,k') = \kappa_{14}(k) (\zeta_2)_{2356}^{-1}(k,k') + \kappa_{14}\kappa_{25}(k') \delta_{36} \] (8.15)

From this calculation, we obtain an expression for the kernel function \( \Delta_3 \) (7.2), which is presented in the appendix, (A.34). After combining some similar quantities, it consists of 10 terms. A problem arises with terms no. 7 and 9 where wave number integrals appear which contain static correlations. Within the integral, all external wave numbers have been set equal to zero; the appearing wave number arguments pertain to the indices 5 and 6. Here it is not possible to localize the integrand factors totally since then the integral will diverge. We will discuss this in the next section.

### 9 Modes; static correlations; 2\textsuperscript{nd} order term

During the investigation, I usually denoted a component index which runs from 1 ot 5 with a greek symbol; and one running over 3 values with a latin. In order to have a short notation, I sometimes speak of greek and latin indices; and, describing an indexed quantity, I call it greek or latin, or I speak of its greek or latin values. - Component indices generally are greek; but indices 0 to 3 in (7.1), are latin. - To proceed further, we make use of the mode analysis of hydrodynamics. The (greek) matrix \( \kappa \) is identical to the matrix of linearized hydrodynamics; see, e. g., [13]. It is symmetric in its indices and therefore can be decomposed into hydrodynamic modes \( \hat{\kappa} \):

\[ \kappa_{12} = \chi_{13}\chi_{24}\hat{\kappa} \] (9.1)

\( \chi \) is the modal matrix which facilitates the decomposition. The modes are: two sound modes, a heat mode and two identical shear modes. Since in this investigation we are not interested in sound and heat effects, we neglect all parts of \( \kappa \) connected to the first three modes. Then \( \kappa \) simplifies considerably: It now is latin, and it reads:

\[ \kappa_{12}(k) = -\varepsilon_{12}(k) \nu \]

\[ \varepsilon_{12}(k) = k^2 \delta_{12} - k_1 k_2 \] (9.2)

\( \nu \) is the kinematic shear viscosity. - Now that \( \kappa \) is latin, this also applies for matrices \( \zeta_2, \zeta_3, \tilde{\zeta}_3, \zeta_4 \) and their inverses. One then finds that all component indices of (A.34) are latin, with the exception of indices 5, 6 in the wave number integrals in terms 7 and 9.
Static correlations can be calculated with a method described in [14]. A modified equilibrium probability density is used which describes a fluid which moves with a uniform velocity \( \mathbf{u} \). Expectations of the conserved quantity densities and their fluxes are formulated; derivatives of several order with respect to the the conjugate parameters are taken; it is shown that the resulting formulas for \( \mathbf{u} = \mathbf{0} \) are the static correlations.

Many elements of the static correlation matrices are zero. As a first step of application of the described technique, one needs a means for finding these zero elements. The rule is this: Let \( n \) be the order of the static correlation. Let \( \eta \) be the number of (generally greek) indices whose values are not latin. Build \( n + \eta \). Write \( e \) for the case that this number is even and \( o \) that it is odd. Write, for short, \( j_n \) and \( s_n \) for the two types of static correlations which appear in the formulas. Then:

\[
\begin{array}{c|cc}
  n + \eta & e & o \\
  \hline
  j_n & \neq 0 & 0 \\
  s_n & 0 & \neq 0
\end{array}
\]  

\( \neq 0 \) means 'generally not zero'. For the majority of coefficients in [A.34] all indices are latin, so \( \eta = 0 \). We find that the latin elements of \( s_4 \) and \( j_3 \) are zero; so, in [A.34], the terms 1 an 4 as well as the second parts of terms 7 and 9 are zero. The wave number integrals in terms 7 and 9 remain being a problem. We will assume here that they are somehow related to the localized value of the integrand (i. e. the integrand with all wave numbers equal to zero). In term 9, all indices of \( \Phi \) are latin. With \( j_3 \) and \( j_3 \) equal to zero, \( \Phi \) is zero too. In term 7, indices 4, 7, 8 are latin while 5, 6 are either both latin or both non-latin (1 oder 2), since otherwise the second factor of the integrand will be zero. In the first case, \( \Phi = 0 \). In the second case, in (6.12), the second term is zero; in the first term, for \( j_3 \), \( \eta = 2 \), and for the third term, for \( j_3 \), \( \eta = 2 \) also; then, from [A.33] both coefficients are zero. Therefore, in the wave number integrals in terms 7 and 9, the localized value of the integrand is zero. This does not necessarily include that the integrals vanish. But, with this argument, we will neglect them. - The remaining formula for \( \Delta_3 \) is given in the appendix (A.33).

The non-zero static correlation components are calculated with the technique outlined above. One obtains (all indices latin):

\[
(j_4)_{1234} = \xi^{-1} (\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23})
\]  

\[
(s_3)_{123}(k) = ik_4 \frac{1}{\sqrt{\rho \beta}} (\delta_{12}\delta_{34} + \delta_{13}\delta_{24} - \lambda \delta_{14}\delta_{23})
\]

\[
\lambda = \frac{\partial \rho \beta_\nu}{\partial \beta \varepsilon |_\rho}
\]

\( p \) is the static pressure and \( \varepsilon \) the energy density, both as functions of mass density \( \rho \) and inverse kinetic temperature \( \beta \). Via \( \lambda \), the formulas are dependent on the physical properties of the fluid.

We will show that the second-order term (the last row of (7.1)) just cancels the first term of (A.35). The part of \( D_3 \) which contains this term simplifies somewhat when (7.4) is introduced:

\[
-\frac{1}{2} \rho^2 \beta \frac{1}{(2\pi)^6} \int dk' dk'' dk''' u_1(k', t) u_4(k'', t) u_4(k''', t) \delta(k - k' - k'' - k''') \times
\]

\[
\gamma_{01}(k)(\xi^{-1} - (2\pi)^3 \delta(k - k'))
\]

The quantity \( K_2 \) (A.33) has been processed by introducing the formulas for \( C_3 \) (6.7), (6.8) and its derivatives (8.1) to (8.4). Then, \( G_{12} \) does not explicitly show up any more, and \( K_2 \) now reads:

\[
(K_2)_{01}(k', k'', t) = (\gamma_{01}(k)(2\pi)^3 \delta(k'')(h_2) - \gamma_{04}(k) j_{412}(t) \delta(t)
\]

\[
- i k''(C_2)_{56}(k'', t)(C_2)_{41}(k', t) s_{045}(k)
\]

\[
+ (C_2)_{52}(k', t) \left( -(C_2)_{43}(k', t) s_{045}(k) \gamma_{31}(k') + (C_2)_{41}(k', t) s_{067}(k) (\zeta_2)_{6145}(k', k'') \right)
\]

Remember that the component index 2 has the fixed value 1. The formula now consists of 3 terms. The second term, after time integration, contains the factor \( k''(\zeta_2)_{4156}(k', k'') \) which is zero. The third term shows a \( C_2 \) factor one index of which has the fixed value 1. When the analysis is restricted to shear modes, as is introduced in the previous section, this term does not contribute. Thus, in (9.3), the first term remains. - We can write \( h_2 = h_1 \), where on the left the index variable 2 appears and on the right
its fixed value 1; we have \( \langle h_1 \rangle = \xi^\perp \). For the coefficient of the second partial term we find \( j_{112} = \xi^{-\perp} \delta_{41} \).

The last row of (7.1) now reads:

\[
\langle 10.3 \rangle. \quad \text{We now have:}
\]

\[
\rho \frac{1}{(2\pi)^{5}} \int dk' dk'' u_1(k', t) u_4(k', t) \delta(k - k' - k'') \gamma_{01}(k)(\xi^{-1} - (2\pi)^3 \delta(k'')) \quad (9.9)
\]

The main difference to (9.7), apart from the order of integration, is the appearance of \( u_q \). The relation to the \( u \)'s is:

\[
u(k, t) = \int dxe^{-ik \cdot x} u_4(x, t) u_4(x, t)
\]

\[
= \frac{1}{(2\pi)^{3}} \int dk' \int dk'' \delta(k - k' - k'') u_4(k', t) u_4(k', t) \quad (9.10)
\]

When this is introduced into (9.9) (and some wave numbers are renamed), it is seen that (9.9) actually cancels (9.7), as is stated in the beginning of the section.

10 Final form of the 3\textsuperscript{rd} order term

In (8.35), in addition to eliminating the first term, the reduction to shear modes has to be introduced. For \((\xi_2)^{-1}, (\xi_3)^{-1}\) we obtain:

\[
(\xi_2)^{-1} \xi_{234} = \frac{\varepsilon_{13}(k') \varepsilon_{24}(k'')}{k'^2 k''^2 \rho(k'^2 + k''^2)}
\]

(10.1)

\[
(\xi_3)^{-1} \xi_{123456} = \frac{\varepsilon_{14}(k') \varepsilon_{25}(k'') \varepsilon_{36}(k''')}{k'^2 k''^2 k'''^2 \rho(-k'^2 + k''^2 + k'''^2)}
\]

(10.2)

When this is inserted into (8.35) and then in (8.41), after expansion some terms which stem from the second term of (9.2) vanish because of the continuity condition (2.16). Then, it becomes apparent that the last three terms of (8.35) cancel each other, while the two remaining terms can be somewhat simplified. The final form in wave number space is:

\[
(D_3)_{0}(k, t) = \rho \frac{1}{(2\pi)^{5}} \int dk' dk'' d\kappa u_1(k', t) u_2(k', t) u_3(k'', t) \delta(k - k' - k' - k'') \times \Delta_{1234}(k', k'', k''')
\]

(10.3)

\[
\Delta_{1234}(k', k'', k''') = (s_3)_{025}(k' + k'' + k''') \varepsilon_{56}(k' + k''') \times \left( - (s_3)_{136}(k') \right) \frac{1}{(k' + k''')^2 (k'^2 + (k' + k''')^2)^2} + (s_3)_{613}(k' + k''') \frac{k'''}{(k' + k''')^2 (k'^2 + (k' + k'''')^2)(k'^2 + k''')^2}
\]

(10.4)

We extracted the thermodynamic factors from the \( s_3 \) coefficients in order to collect them in front of (10.3). We now have:

\[
(s_3)_{123}(k) = ik_4(s_3)_{1234}
\]

\[
(s_3)_{1234} = \delta_{12} \delta_{34} + \delta_{13} \delta_{24} - \lambda \delta_{14} \delta_{23}
\]

(10.5)

The quantities on the left are distinguished by the number of indices. - In order to transform (10.3), (10.4) back to geometric space, is seems reasonable to switch to a different notation:

\[
(D_3)_{0}(k, t) = \frac{2}{\rho} \int dk' dk'' dk''' \delta(k - k' - k' - k'') \sum_{\mu=1}^{2} M_{\mu}(k', k'', k''')(Q_{\mu})_{0}(k', k'', k''')
\]

(10.6)
for a point on the jet axis. When the Schlichting formulas are inserted, some of the space integrals in (Q1) may be rather small. As a first attempt, the self-similar jet flow formulas by Schlichting [15] have been used. This form is unable to describe turbulent tube flow. Presently, it is intended to test the theory with the tube flow. The formulas have been applied to circular tube flow (10.10) to (10.15).

The result in geometric space are formulas (10.13) and (10.14) leads to lengthy expressions which will not be given here.

\[ M_1 = \frac{1}{(k' + k'')^2(k'^2 + (k' + k'')^2)} \]
\[ M_2 = \frac{1}{(k' + k'')^2(k'^2 + (k' + k'')^2)(k'^2 + k''^2)} \]  (10.7)

\[ (Q_1)_0 = -i(k'_0 + k''_0)\rho_{257}(i\delta_{3})_{1368}\varepsilon_{266}(k' + k'')u_1(k', t)u_2(k'', t)u_3(k'', t) \]  (10.8)

\[ (Q_2)_0 = i(k'_0 + k''_0)\rho_{257}(i\delta_{3})_{6138}k'^2\varepsilon_{266}(k' + k'')u_1(k', t)u_2(k'', t)u_3(k'', t) \]  (10.9)

Then, the formal expression for the transformed quantity is:

\[ (D_3)_0(x, t) = \frac{\rho}{\nu} \int dx' dx'' dx''' \sum_{\mu=1}^{2} M_\mu(x - x', x - x'', x - x''') (Q_\mu)_0(x', x'', x''') \]  (10.10)

The functions in this formula are the (backwards) Fourier Transforms of the equally denoted functions in (10.6). The transform of the kernel functions (10.7) can be found in closed form:

\[ M_1(x', x'', x''') = \frac{1}{2\pi^2 k' x'} \arctan\left(\frac{x'}{k'}\right) \delta(x' - x''') \]
\[ M_2(x', x'', x''') = \frac{1}{2\pi^2 k' x' x''} \arctan\left(\frac{x'}{k'}\right) \]  (10.11)

\[ \hat{x} = \frac{1}{2}(x' - x''') \]
\[ x^* = \frac{1}{2}(x' + x''') \]
\[ \hat{x} = x'' + x^* \]  (10.12)

As usual, vectors are in bold letters, while their magnitude is in normal letters. The velocity functions (10.9) are transformed straightforwardly:

\[ (Q_1)_0 = (s_3)_{0257}(s_3)_{1368}(\nabla' + \nabla'' + \nabla''')\nabla'\varepsilon_{266}(\nabla' + \nabla''')u_1(x', t)u_2(x'', t)u_3(x''', t) \]  (10.13)

\[ (Q_1)_0 = (s_3)_{0257}(s_3)_{6138}(\nabla' + \nabla'' + \nabla''')\nabla'\varepsilon_{266}(\nabla' + \nabla''')u_1(x', t)u_2(x'', t)u_3(x''', t) \]  (10.14)

\[ \varepsilon_{12}(\nabla) = \nabla'^2 \delta_{12} - \nabla_1 \nabla_2 \]  (10.15)

Expansion of (10.13), (10.14) leads to lengthy expressions which will not be given here.

11 Summary and outlook

The equation for the hydrodynamic velocity derived by N-particle statistical mechanics has been investigated. It is argued that in case of turbulent fluid motion this quantity represents the mean velocity. The dissipative force which is nonlinear in the velocity has been expanded and developed up to the third-order term. The resulting formula contains equilibrium time correlation functions up to fourth order. Multilinear mode coupling theory has been applied to calculate these quantities. Moreover, the formalism is restricted to shear motion (no sound and heat flow effects). The result in geometric space are formulas (10.10) to (10.14).

A major restriction of the investigation is that the N-particle theory in its present form is with no boundaries. This is not just a theoretical argument. The formulas have been applied to circular tube flow with the result that the third order friction term is zero. This probably means that the theory in this form is unable to describe turbulent tube flow. Presently, it is intended to test the theory with the empirical data of the circular jet flow; since this is a configuration where the influence of the boundary may be rather small. As a first attempt, the self-similar jet flow formulas by Schlichting [15] have been used. A boundary layer approximated form of the theoretical formulas has been explicitly calculated for a point on the jet axis. When the Schlichting formulas are inserted, some of the space integrals in
diverge in the region far from the nozzle. Thus, the theory is in contradiction with these empirical formulas.

An explanation can perhaps be found in the observation that at least for laminar flow of low Reynolds number in the region far from the nozzle the round jet transforms into a recirculation current. This has been experimentally shown by Zauner [16] and theoretically calculated by Schneider and co-workers [17, 18]. Since, by this configuration, the flow essentially extends over a finite space region, the divergence mentioned above cannot occur. As a next step, therefore, it is intended to investigate in which way the flow characteristics change when the third order friction part is included into the calculation.

A Appendix: Detailed formulas

In this appendix, formulas are presented which appear in the course of the analysis in the main part of the paper, but are too voluminous to be given there. - First, the terms of the second order functional derivative of the stress tensor kernel function \( S \) are written, as calculated from (3.2), (3.3). The detailed formula for the second derivative reads:

\[
\frac{\delta^2 S_{abcd}(x', t')}{\delta b_c(x'', t'') \delta b_f(x'', t''')} \quad (A.1)
\]

For shortness, on the left hand side, indices and variables are omitted. Second order derivatives with respect to the momentum part of the conjugated parameters are needed only (that is, the greek index \( \epsilon \) is switched to \( c \)). For each term, the first step is just the definition of the quantity.

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,1)} = -\delta(t'' - t') \beta \tr \left\{ \frac{\delta f_L(t')}{\delta b_f(x'', t'')} [\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') s_{bd}(x') \delta a_c(x', t') \right\} = \delta(t'' - t') \delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x'', t'))_{L,t'} = 0 \quad (A.2)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,2)} = -\delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} = -\delta(t'' - t') \delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} \quad (A.3)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,3)} = -\delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} = -\delta(t'' - t') \delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} \quad (A.4)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,4)} = -\delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} = -\delta(t'' - t') \delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} \quad (A.5)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,5)} = -\delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} = -\delta(t'' - t') \delta(t'' - t') \beta ([\tilde{G}(t', t) Q(t) s_{ac}(x)] Q(t') Q(t') s_{bd}(x') \delta a_c(x', t') \delta a_f(x''', t'))_{L,t'} \quad (A.6)
\]

The last step is correct because of (3.4) and \( Q P = 0 \).
\[
\frac{\delta^2 S}{\delta b \delta b}^{(p,6)} = -\delta(t'' - t')\beta \langle \frac{\delta \tilde{G}(t'')}{\delta b_f(x'', t'')} \rangle \Theta(t'' - t') \Theta(t - t') \times \beta \langle [\tilde{G}(t''), Q(t'')] \delta a_f(x'', t'') \rangle L \nu
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,1)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.7)
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,2)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.8)
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,3)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.9)
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,4)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.10)
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,5)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.11)
\]

\[
\frac{\delta^2 S}{\delta b \delta b} \text{(f,6)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta \langle [\tilde{G}(t', t'') \mathcal{L} P(t'') \delta a_c(x', t'') \rangle \Theta(t'' - t') \times \\
\beta \langle [\tilde{G}(t'', t') \mathcal{L} P(t') \delta a_f(x'', t'') \rangle L \nu \rangle (A.12)
\]
\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,6)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial b_f(x'', t'')} \delta_{ac}(x'', t'') Q(t'') \tilde{G}(t'', t) Q(t) \tilde{s}_{ac}(x) Q(t') \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
= -\delta(t'' - t') \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial f}(x'', t'') Q(t'') \delta_{ac}(x'', t'') Q(t'') \tilde{G}(t'', t) Q(t) \tilde{s}_{ac}(x) \times \\
\times Q(t') \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
(A.13)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,7)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial ac}(x'', t'') Q(t'') \tilde{G}(t'', t) Q(t) \tilde{s}_{ac}(x) Q(t') \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
= \Theta(t'' - t') \Theta(t'' - t') \Theta(t - t'') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial f}(x'', t'') Q(t'') \times \\
\times \tilde{G}(t'', t''') \frac{\delta P(t''')}{\partial ac}(x', t''') Q(t''') \tilde{G}(t'', t') Q(t) \tilde{s}_{ac}(x) \times \\
\times \tilde{G}(t'', t) Q(t) \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
(A.14)
\]

\[
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,8)} = \Theta(t'' - t') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial ac}(x'', t'') Q(t'') \tilde{G}(t'', t) Q(t) \tilde{s}_{ac}(x) Q(t') \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
= \Theta(t'' - t') \Theta(t'' - t') \Theta(t - t'') \Theta(t - t'') \times \\
\times \beta(\tilde{G}(t'', t')) \frac{\delta P(t'')}{\partial f}(x'', t'') Q(t'') \times \\
\times \tilde{G}(t'', t''') \frac{\delta P(t''')}{\partial ac}(x', t''') Q(t''') \tilde{G}(t'', t') Q(t) \tilde{s}_{ac}(x) \times \\
\times \tilde{G}(t'', t) Q(t) \tilde{s}_{bd}(\tilde{x}'))_{L,t'}
\]

\[
(A.15)
\]

Results (A.10), (A.12) are found the same way as (A.6). Below, the non-zero terms are presented in the form they assume after taking \( b = b_0 \), and reformulation, in sec. 3:

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(p,1)}_{0} = -\delta(t'' - t') \delta(t'' - t') \langle [\mathcal{P} r_0 + \mathcal{L} g(t - t') h_0] \rangle \hat{Q} \tilde{r}_1^2 h_2^3 \hat{h}_3 h_4
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(p,2)}_{0} = -\delta(t'' - t') \delta(t'' - t') \langle [g(t - t') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(p,3)}_{0} = -\delta(t'' - t') \delta(t'' - t') \langle [g(t - t') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(p,4)}_{0} = \delta(t'' - t') \delta(t'' - t') \langle [\mathcal{Q} \mathcal{P} r_0 + \mathcal{L} g(t - t') h_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(p,6)}_{0} = -\delta(t'' - t') \Theta(t'' - t') \Theta(t'' - t') \langle [g(t'' - t') \mathcal{Q} \mathcal{P} h_3 g(t - t'') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(f,1)}_{0} = -\delta(t'' - t') \Theta(t'' - t') \Theta(t'' - t') \langle [\mathcal{L} g(t'' - t') \mathcal{P} h_2^3 g(t - t'') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(f,2)}_{0} = \delta(t'' - t') \Theta(t'' - t') \Theta(t'' - t') \langle [\mathcal{L} g(t'' - t') \mathcal{P} h_2^3 g(t - t'') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]

\[
\left[ \frac{\delta^2 R}{\delta b \delta b} \right]^{(f,4)}_{0} = \delta(t'' - t') \Theta(t'' - t') \Theta(t'' - t') \langle [g(t'' - t') \mathcal{Q} \mathcal{P} h_2^3 \mathcal{P} h_2^3 g(t - t'') \tilde{r}_0] \rangle \hat{h}_2^3 \hat{h}_3
\]
\[
\begin{align*}
\frac{\delta^2 R}{\delta b \delta b}^{(f,6)}_0 &= -\delta(t'' - t')\Theta(t'' - t')\Theta(t - t'')\langle [g(t'' - t') QLP h_2^* Qh_2^* g(t - t'')] \tilde{r}_0 | \tilde{r}_1 \rangle_0 \\
\frac{\delta^2 R}{\delta b \delta b}^{(f,7)}_0 &= \Theta(t'' - t')\Theta(t - t''')\Theta(t'' - t')\Theta(t - t'') \\
&\times \langle [g(t'' - t') QLP h_2^* g(t'' - t''')] QLP h_2^* g(t - t''') \tilde{r}_0 | \tilde{r}_1 \rangle_0 \\
\frac{\delta^2 R}{\delta b \delta b}^{(f,8)}_0 &= \Theta(t'' - t')\Theta(t'' - t')\Theta(t'' - t')\Theta(t - t'') \\
&\times \langle [g(t'' - t') QLP h_2^* g(t'' - t'')] QLP h_2^* g(t - t''') \tilde{r}_0 | \tilde{r}_1 \rangle_0
\end{align*}
\]

After evaluation of the projection operators in section 4, we obtain for the 11 terms, now called \( K_1, \ldots, K_{11} \) (Symbol 0 omitted from \( \langle \rangle \)):

\[
\begin{align*}
(K_1)_{0123} &= \langle [Lg(t - t_1) h_0 | h_3^* h_3^*] \langle \tilde{r}_1^* h_2^* h_4^* \rangle - \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* h_4^* \rangle \\
(K_2)_{0123} &= -\langle [g(t - t_1) \tilde{r}_0] \tilde{r}_1^* | h_2^* h_3^* \rangle \\
(K_3)_{0123} &= -\langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [Lg(t - t_1) h_4^*] \tilde{r}_1^* h_2^* h_4^* \rangle \\
(K_4)_{0123} &= \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* h_4^* \rangle - \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* h_4^* \rangle \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* \rangle \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* h_4^* \rangle \\
(K_5)_{0123} &= \langle [g(t - t_3) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_3) \tilde{r}_0] \tilde{r}_1^* | h_2^* h_3^* \rangle \\
(K_6)_{0123} &= -\langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [Lg(t - t_1) h_4^*] \tilde{r}_1^* h_2^* h_4^* \rangle \\
(K_7)_{0123} &= \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [Lg(t - t_1) h_0] \tilde{r}_1^* h_2^* h_4^* \rangle \\
(K_8)_{0123} &= \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle - \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_3^* h_5^* \rangle \\
(K_9)_{0123} &= \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle + \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle + \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle \\
(K_{10})_{0123} &= \langle [g(t - t_3) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle + \langle [g(t - t_3) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle \\
(K_{11})_{0123} &= \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle + \langle [g(t - t_2) \tilde{r}_0] h_2^* h_3^* \rangle \langle [g(t - t_1) \tilde{r}_3] \tilde{r}_1^* | h_2^* h_3^* \rangle
\end{align*}
\]

In sec. 4, these terms are combined into 5 main parts:
When the $M$-quantities are inserted into (5.11), some of the time integrations can be performed; finally, two parts of the kernel function remain:

$$
\Phi_{1012}\left(t', t''\right) = -\left(j_2\right)_{23} \gamma_{01} \delta(t') - 2(s_1)_{124} (\kappa_0 \delta(C_3)_{534}(t') + \partial_t(C_3)_{034}(t')) + 2(j_2)_{25} \gamma_{51} (\kappa_0 \delta(C_3)_{36}(t') + \partial_t(C_3)_{036}(t')) - \kappa_1 (\kappa_1)_{2435} (t')_{08} + \partial_t(C_4)_{0235}(t')) - \frac{1}{3} (\kappa_7 \partial_t(C_4)_{123}(t') + \partial_t(C_4)_{123}(t'))
$$

$$
\Phi_{2012}\left(t', t''\right) = 2 \gamma_{51} (\kappa_0 \delta(C_3)_{837}(t') + \partial_t(C_3)_{037}(t')) \times (\kappa_7 \delta(C_3)_{925}(t') + \partial_t(C_3)_{725}(t'')) + ((\kappa_8 \delta(C_3)_{924}(t') + \partial_t(C_3)_{024}(t')) \times (\kappa_9 \delta(C_3)_{513} - (s_3)_{135}) + 2 \omega_{16}((\kappa_3)_{763}(t''))_{\kappa_47} + \partial_t(C_3)_{1063}(t'')) + \kappa_7 \partial_t(C_3)_{713}(t'') + \frac{1}{2} \partial_t(C_3)_{413}(t'')
$$

In sec. 5, the part of the second-order (in $b - b_0$) term which adds to the third-order velocity term is calculated. The kernel function reads:

$$
(K_{2})_{012}(t') = \gamma_{01}(t') (h_2) + \gamma_{31} (\partial_t(C_3)_{302}(t') + \kappa_4(C_3)_{32}(t')) + \partial_t(C_3)_{3012}(t') + i k_7 \partial_t(C_3)_{012}(t') + \omega_{14} \partial_t(C_3)_{42}(t') + \omega_{03} (\partial_t(C_3)_{312}(t') + \partial_t(C_3)_{312}(t''))
$$

The index $\sim_{e}$ means that the component index of the number index 2 runs over the three values $c = 3, 4, 5$; in contrast to the general prescription of this section where it has the fixed value 1. - In the explicit formulation in sec. 7, we obtain for the expression in (7.2), (7.3):

$$
\Phi_{1012}^{(k', k'', k''', t', t'')} = -\left(2\pi\right)^3 \delta_{23} \delta(k - k') \gamma_{01}(k) \delta(t') - 2(s_1)_{124} (\kappa_0 \delta(C_3)_{534}(k, k'', k' + k'', t') + \partial_t(C_3)_{034}(k, k'', k' + k'', t')) + 2(j_2)_{25} \gamma_{51} (\kappa_0 \delta(C_3)_{36}(k, k'', k' + k'', t') + \partial_t(C_3)_{036}(k, k'', k' + k'', t')) - \kappa_1 (\kappa_1)_{2435} (k, t')_{08} + \partial_t(C_4)_{0235}(k, k'', k', t')) - \frac{1}{3} (\kappa_7 \partial_t(C_4)_{123}(k, k', k'', k''', t') + \partial_t(C_4)_{0123}(k, k', k'', k''', t'))
$$

$$
\Phi_{2012}^{(k', k'', k''', t', t'')} = 2 \gamma_{51} (\kappa_0 \delta(C_3)_{837}(k, k'', k' + k'', t') + \partial_t(C_3)_{037}(k, k'', k' + k'', t')) \times (\kappa_7 \delta(C_3)_{925}(k, k'', k' + k'', t') + \partial_t(C_3)_{725}(k, k'', k' + k'', t'')) + 2(j_2)_{25} \gamma_{51} (\kappa_0 \delta(C_3)_{36}(k, k'', k' + k'', t') + \partial_t(C_3)_{036}(k, k'', k' + k'', t'')) \times (\kappa_7 \delta(C_3)_{925}(k, k'', k' + k'', t') + \partial_t(C_3)_{725}(k, k'', k' + k'', t'')) + \kappa_7 \partial_t(C_3)_{713}(k', k'', k''', t'') + \frac{1}{2} \partial_t(C_3)_{413}(k', k'', k''', t'')
$$

$$
(K_{2})_{012}(k', k'', t') = \gamma_{01}(k) (2\pi)^3 \delta(k''') \delta(t') + \gamma_{31}(k') (\partial_t(C_3)_{302}(k, k', k', t') + \kappa_4(C_3)_{32}(k, k', k', t')) + \partial_t(C_3)_{3012}(k, k', k', t') + i k_7 \partial_t(C_3)_{012}(k, k', k', t') + \omega_{14}(k') \partial_t(C_3)_{024}(k, k', k', t') + \omega_{03}(k) (\partial_t(C_3)_{312}(k, k', k', t') + i k_7 \partial_t(C_3)_{312}(k, k', k', t') + \omega_{14}(k') \partial_t(C_3)_{342}(k, k', k', t'))
$$
The wave number \( \mathbf{k} \) is, in (A.31) and (A.32), to be substituted by \( \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' \), and in (A.33) by \( \mathbf{k}' + \mathbf{k}'' \). - The calculation in sec. 8 of the kernel function \( \Delta_3 \) appearing in (7.1) yields the expression (again \( \mathbf{k} = \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' \)):

\[
\begin{align*}
\Delta_3(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''') &= -\frac{1}{3}(s_4)_{0123}(\mathbf{k}) \\
&+ \gamma_{01}(\mathbf{k})\left(\frac{1}{7}(\mathbf{j}_4)_{123} - 2(2\pi)^3\delta(\mathbf{k} - \mathbf{k}'')\delta_{14}\delta_{23}\right) \\
&+ 2(s_4)_{056}(\mathbf{k})(s_3)_{134}(\mathbf{k}' - (s_3)_{413}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{5624}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}''')) \\
&- 2(s_4)_{056}(\mathbf{k})(\mathbf{j}_4)_{143}(\mathbf{k}') - \frac{1}{7}(\mathbf{j}_4)_{143}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{5624}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}'''')) \\
&+ 2(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{4738}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}'''))((s_2)_{8921}(\mathbf{k}'', \mathbf{k}'))\kappa_{56}(\mathbf{k}'' + \mathbf{k}''') \\
&+ \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{19}(\mathbf{k}' + \mathbf{k}''') \\
&+ \left(-\frac{1}{(2\pi)^3}\int d\mathbf{k}'\Phi_{47856}(\mathbf{k}, -\mathbf{k})(s_3)_{056}(\mathbf{k}, -\mathbf{k}) + (s_4)_{047}(\mathbf{k})\right)\times (\zeta_3)_{47829}^{-1}(\mathbf{k}'', \mathbf{k}'', \mathbf{k}')\kappa_{19}(\mathbf{k}') \\
&- \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{96}(\mathbf{k}'' + \mathbf{k}''') \\
&+ (s_4)_{047}(\mathbf{k})\left(\kappa_{19}(\mathbf{k}')\frac{1}{(2\pi)^3}\int d\mathbf{k}'\Phi_{23956}(\mathbf{k}, -\mathbf{k})(\zeta_2)_{4756}^{-1}(\mathbf{k}, -\mathbf{k}) + \frac{1}{2}(\mathbf{j}_3)_{813}(\zeta_2)_{4729}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}''')\kappa_{96}(\mathbf{k}'' + \mathbf{k}''')\right) \\
&+ \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{94}(\mathbf{k}' + \mathbf{k}''') \\
&\times (\zeta_3)_{48956}^{-1}(\mathbf{k}' + \mathbf{k}'', \mathbf{k}'', \mathbf{k}'')\kappa_{89}(\mathbf{k}' + \mathbf{k}''') (A.34)
\end{align*}
\]

With, in sec. 9, several of the static correlations found to be zero, the formula simplifies:

\[
\begin{align*}
\Delta_3(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''') &= \gamma_{04}(\mathbf{k})\left(\frac{1}{7}(\mathbf{j}_4)_{123} - 2(2\pi)^3\delta(\mathbf{k} - \mathbf{k}'')\delta_{14}\delta_{23}\right) \\
&+ 2(s_4)_{056}(\mathbf{k})(s_3)_{134}(\mathbf{k}') - (s_3)_{413}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{5624}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}''')) \\
&+ 2(s_4)_{056}(\mathbf{k})(\mathbf{j}_4)_{143}(\mathbf{k}') - \frac{1}{7}(\mathbf{j}_4)_{143}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{5624}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}'''')) \\
&+ 2(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))((\zeta_2)_{4738}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}'''))((s_2)_{8921}(\mathbf{k}'', \mathbf{k}'))\kappa_{56}(\mathbf{k}'' + \mathbf{k}''') \\
&+ \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{19}(\mathbf{k}' + \mathbf{k}''') \\
&+ \left(-\frac{1}{(2\pi)^3}\int d\mathbf{k}'\Phi_{47856}(\mathbf{k}, -\mathbf{k})(s_3)_{056}(\mathbf{k}, -\mathbf{k}) + (s_4)_{047}(\mathbf{k})\right)\times (\zeta_3)_{47829}^{-1}(\mathbf{k}'', \mathbf{k}'', \mathbf{k}')\kappa_{19}(\mathbf{k}') \\
&- \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{96}(\mathbf{k}'' + \mathbf{k}''') \\
&+ (s_4)_{047}(\mathbf{k})\left(\kappa_{19}(\mathbf{k}')\frac{1}{(2\pi)^3}\int d\mathbf{k}'\Phi_{23956}(\mathbf{k}, -\mathbf{k})(\zeta_2)_{4756}^{-1}(\mathbf{k}, -\mathbf{k}) + \frac{1}{2}(\mathbf{j}_3)_{813}(\zeta_2)_{4729}^{-1}(\mathbf{k}'', \mathbf{k}' + \mathbf{k}''')\kappa_{96}(\mathbf{k}'' + \mathbf{k}''')\right) \\
&+ \frac{1}{2}(s_4)_{047}(\mathbf{k})(s_3)_{566}(\mathbf{k}' + \mathbf{k}'''))\kappa_{94}(\mathbf{k}' + \mathbf{k}''') \\
&\times (\zeta_3)_{48956}^{-1}(\mathbf{k}' + \mathbf{k}'', \mathbf{k}'', \mathbf{k}'')\kappa_{89}(\mathbf{k}' + \mathbf{k}''') (A.35)
\end{align*}
\]

References

[1] E. Hopf, J. rathl. mech. Anal. 1, p. 87, 1952

[2] W. D. McComb, The physics of fluid turbulence, Oxford Science Pub., Oxford 1990

[3] P. Moin, K. Mahesh: Direct numerical simulation. A tool in turbulence research. In: Annual Review of Fluid Mechanics. Vol. 30, 1998, S. 539–578

[4] K. Huang, Statistical Mechanics. Wiley, N. Y. 1963

[5] H. Grabert, Projection operator techniques in nonequilibrium statistical mechanics. Springer, Berlin, Heidelberg, New York (1982)

[6] J. Piest, arXiv [physics/0310054] 2003
[7] K. Kawasaki, Ann. Phys. 61, p. 1, 1970;
    P. C. Martin, E. D. Siggia, H. A. Rose, Phys. Rev. A8, p. 423, 1973;
    U. Deker, F. Haake, Phys. Rev. A11, 2043, 1975
[8] J. Schofield, R. Lim, I. Oppenheim, Physica A181, p. 89, 1992;
[9] R. van Zon, J. Schofield, Phys. Rev. E65, 011106, 2001
[10] J. Piest, arXiv 0711.2790v1, 2007;
    - , arXiv 0803.3972, 2008
[11] Zubarev, D.; Mozorov, V.; Röpke, G.: Statistical mechanics of nonequilibrium processes. Vol. 1,
    Akademie Verlag Berlin 1996.
[12] K. Kawasaki, Ann. Phys. 61, (1970), 1
[13] M. H. Ernst, E. H. Hauge, J. M. J. van Leeuwen, Phys. Rev. A 4, 5, (1971), 2055
[14] A. Münster: Statistische Thermodynamik. Springer, Berlin 1956
[15] H. Schlichting, Boundary layer theory, 7th ed., McGraw Hill 1979
[16] E. Zauner: Visualization of the viscous flow induced by a round jet. J. Fluid Mech. 154, 111-119,
    1985
[17] W. Schneider: Decay of momentum flux in submerged jets. J. Fluid Mech. 154, 91-110, 1985
[18] W. Schneider, E. Zauner, H. Böhm: The recirculatory flow induced by a laminar axisymmetric jet
    issuing from an wall. J. Fluids Eng. 109, 237-241, 1987