Benefits and Pitfalls of the Exponential Mechanism with Applications to Hilbert Spaces and Functional PCA

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Abstract

The exponential mechanism is a fundamental tool of Differential Privacy (DP) due to its strong privacy guarantees and flexibility. We study its extension to settings with summaries based on infinite dimensional outputs such as with functional data analysis, shape analysis, and nonparametric statistics. We show that one can design the mechanism with respect to a specific base measure over the output space, such as a Gaussian process. We provide a positive result that establishes a Central Limit Theorem for the exponential mechanism quite broadly. We also provide an apparent negative result, showing that the magnitude of the noise introduced for privacy is asymptotically non-negligible relative to the statistical estimation error. We develop an $\epsilon$-DP mechanism for functional principal component analysis, applicable in separable Hilbert spaces. We demonstrate its performance via simulations and applications to two datasets.

1 Introduction

Data privacy and security have become increasingly critical to society as we continue to collect troves of highly individualized data. In the last decade, we have seen the emergence of new tools and perspectives on data privacy such as Differential Privacy (DP), introduced by [Dwork et al.] (2006), which provides a rigorous and interpretable definition of privacy. Within the DP framework, numerous tools have been developed that achieve DP in a variety of applications and contexts, such as empirical risk minimization ([Chaudhuri et al., 2011; Kifer et al., 2012]), linear and logistic regression ([Chaudhuri & Monteleoni, 2009; Zhang et al., 2012; Yu et al., 2014; Sheffet, 2017; Awan & Slavkovic, 2018]), hypothesis testing ([Yu & Slavkovic, 2009; Wang et al., 2015; Gaboardi et al., 2015; Awan & Slavkovic, 2018; Canonne et al., 2018]), network data ([Karwa et al., 2016; Karwa & Slavkovic, 2016]), and density estimation ([Wasserman & Zhou, 2010]), to name a few.

One of the most flexible and convenient DP tools is the exponential mechanism, introduced by [McSherry & Talwar, 2007], which often fits in naturally with estimation techniques from statistics and machine learning. Many estimation procedures can be described as maximizing a particular objective or utility function:

$$\hat{b} = \arg \max_{b \in \mathcal{Y}} \xi(b), \quad \text{where } \xi : \mathcal{Y} \to \mathbb{R},$$

or, equivalently, minimizing a loss function such as least squares or the negative log-likelihood. The exponential mechanism provides a sanitized version of $\hat{b}$ by using the objective function directly to add noise. The sanitized estimate, $\tilde{b}$, is drawn from a density, $f(b)$, that is proportional to

$$f(b) \propto \exp \left\{ \frac{\epsilon}{2\Delta} \xi(b) \right\},$$

where $\Delta$ captures the sensitivity of the objective function to small perturbations in the data, and $\epsilon$ is the desired privacy budget (details in Sections 2 and 3). The idea behind this mechanism is to assign higher density values to regions with higher utility. The constant $\Delta/\epsilon$ adjusts the spread of the density; as the sensitivity increases or as the privacy budget decreases (meaning a decreased disclosure risk), the variability of $\tilde{b}$ increases. A major advantage of such an approach is its use of the objective function from the non-private estimate, $\hat{b}$, which naturally promotes perturbations with higher utility and discourages those with poor utility.
In this paper we study the exponential mechanism, especially as it pertains to functional data analysis, shape analysis, and nonparametric statistics, where one has a (potentially) infinite dimensional output. We show that the exponential mechanism can be applied in such settings, but requires a specified base measure over the output space $\mathcal{Y}$. We propose using a Gaussian process as the base measure, as these distributions are well studied and easy to implement. We derive a Central Limit Theorem (CLT) for the exponential mechanism quite broadly, however, this result also implies that the magnitude of the noise introduced for privacy is of the same order as the statistical estimation error. In particular, we show that in most natural settings the exponential mechanism does not add an asymptotically negligible noise, even in finite dimensions.

Using our approach, we develop an $\epsilon$-DP mechanism for functional principal component analysis (FPCA), which extends the method of Chaudhuri et al. (2013) to separable Hilbert spaces. We show that a Gaussian process base measure enables us to modify the Gibbs sampling procedure of Chaudhuri et al. (2013) to this functional setting. We illustrate the performance of our private FPCA mechanism through simulations, and apply our mechanism to both the Berkeley growth study from the fda package Ramsay et al. (2018) and the Diffusion Tensor Imaging (DTI) dataset from the refund package Goldsmith et al. (2018).

**Related Work:** This work most directly builds off of Hall et al. (2013) and Mirshani et al. (2017), which develop the first techniques for producing fully functional releases under DP. Another work in this direction is Alda & Rubinstein (2017), in which they use Bernstein polynomial approximations to release functions. Recently, Smith et al. (2018) applied the techniques of Hall et al. (2013) to privatize gaussian process regression. In their setup, they assume that the predictors are public knowledge, and use this information to carefully tailor the sanitization noise.

There have been a few accuracy bounds regarding exponential mechanism, which can be found in Section 3.4 of Dwork & Roth (2014). However, these results bound the loss in terms of the objective function, rather than in terms of the private release. Wasserman & Zhou (2010) also develop some accuracy bounds for the exponential mechanism, focusing on mean and density estimation. They show that in the mean estimation problem, the exponential mechanism introduces $O(1/\sqrt{n})$ noise. Our asymptotic analysis of the exponential mechanism agrees in this setting, and extends this result to a large class of objective functions.

Our application to FPCA extends the private PCA method proposed in Chaudhuri et al. (2013). There have been other approaches to private multivariate PCA. Blum et al. (2005) were one of the first to develop a DP procedure for principal components, which is a postprocessing of a noisy covariance matrix. Dwork et al. (2014) follow the same approach and develop bounds for this algorithm; they also develop an online algorithm for private PCA. Jiang et al. (2013) modify this approach by both introducing noise in the covariance matrix as well as to the projection. Imtiaz & Sarwate (2016) also add noise to the covariance matrix, but use a Wishart distribution rather than normal or Laplace noise.

**Organization:** In Section 2 we review the necessary background of Differential Privacy. In Section 3 we recall the exponential mechanism and give asymptotic results for the performance of the exponential mechanism in both finite and infinite dimensional settings. In Section 4 we show how the exponential mechanism can be applied to produce Functional Principal Components, and in Section 5 we give a Gibbs sampler for this mechanism. In Section 6 we study the performance of the private principal components on both simulated data and on the Berkeley and DTI datasets. Finally, we give our concluding remarks in Section 7.

## 2 Differential Privacy

In this section we provide a brief overview of differential privacy (DP). Throughout, we let $\mathcal{X}$ denote an arbitrary set, which represents a particular population, and let $\mathcal{X}^n$ be the $n$-fold Cartesian product, which represents the collection of all possible samples that could be observed. We begin by defining the Hamming Distance between two databases.

**Definition 2.1 (Hamming Distance).** The bivariate function $\delta : \mathcal{X}^n \times \mathcal{X}^n \to \mathbb{Z}$, which maps $\delta(X, Y) := \# \{ i \mid X_i \neq Y_i \}$, is called the Hamming Distance on $\mathcal{X}^n$.

It is easy to verify that $\delta$ is a metric on $\mathcal{X}^n$. If $\delta(X, Y) = 1$ we call $X$ and $Y$ adjacent.

Since we are focused on infinite dimensional objects, we define Differential Privacy broadly for any statistical summary. In particular, suppose that $f : \mathcal{X}^n \to \mathcal{Y}$ represents a summary of $\mathcal{X}^n$, and let $\mathcal{F}$ be a
\( \sigma \)-algebra of subsets of \( \mathcal{Y} \) so that the pair \( (\mathcal{Y}, \mathcal{F}) \) is a measurable space. From a probabilistic perspective, a privacy mechanism is a family of probability measures \( \{\mu_X : X \in \mathcal{X}^n \} \) over \( \mathcal{Y} \). We can now define what we mean when we say the mechanism satisfies \( \epsilon \)-DP. While DP was originally introduced in [Dwork et al., 2006], Definition 2.2 is similar to the versions given in [Wasserman & Zhou, 2010] and [Kifer & Lin, 2010].

**Definition 2.2 (Differential Privacy: [Dwork et al., 2006].** A privacy mechanism \( \{\mu_X : X \in \mathcal{X}^n \} \) satisfies \( \epsilon \)-Differential Privacy (\( \epsilon \)-DP) if for all \( B \in \mathcal{F} \) and adjacent \( X, X' \in \mathcal{X}^n \),

\[
\mu_X(B) \leq \mu_{X'}(B) \exp(\epsilon).
\]

From Definition 2.2 we see that, for an \( \epsilon \)-DP mechanism, \( \mu_X \) and \( \mu_{X'} \) must be equivalent measures (i.e. they agree on sets of measure zero) if \( \delta(X, X') = 1 \). By transitivity, it follows that \( \mu_X \) and \( \mu_Y \) are equivalent measures for any \( X, Y \in \mathcal{X}^n \). By the Radon-Nikodym Theorem, we can always therefore interpret DP in terms of densities with respect to a common base measure, \( \nu \) (if needed, one can always take \( \nu = \mu_X \) for an arbitrary \( X \in \mathcal{X}^n \)).

**Proposition 2.3.** Let \( \mathcal{M} = \{\mu_X : X \in \mathcal{X}^n \} \) be a privacy mechanism over a measurable space \( (\mathcal{Y}, \mathcal{F}) \). Then \( \mathcal{M} \) achieves \( \epsilon \)-DP if and only if there exists a base measure \( \nu \) such that \( \mu_X \ll \nu \) for all \( X \in \mathcal{X}^n \) and the densities \( \{f_X : X \in \mathcal{X}^n \} \) (Radon-Nikodym derivatives) of the \( \mu_X \) (with respect to \( \nu \)) satisfy

\[
f_X(b) \leq f_{X'}(b) \exp(\epsilon),
\]

\( \nu \)-almost everywhere and for all adjacent \( X, X' \in \mathcal{X}^n \).

**Proof.** The reverse direction is given in Remark 1 from [Hall et al., 2013], though we provide the argument here again for completeness. Let \( B \in \mathcal{F} \) and \( X, X' \in \mathcal{X}^n \) be adjacent elements. Then

\[
\mu_X(B) = \int_B f_X(b) \, d\nu(b) = \int_B \frac{f_X(b)}{f_{X'}(b)} f_{X'}(b) \, d\nu(b) \leq \int_B \exp(\epsilon) f_{X'}(b) \, d\nu(b) = \exp(\epsilon) \mu_{X'}(B).
\]

Going in the other direction we will use a proof by contradiction. Assume that \( \mathcal{M} \) is an \( \epsilon \)-DP mechanism. Recall that two measures are equivalent if they agree on the zero sets, thus, as we have said, the measures in a DP mechanism must all be equivalent. So, we can assume that all of the measures have a density with respect to some common base measure, \( \nu \), which, without loss of generality, we can take to be one of the elements of \( \mathcal{M} \). Now assume that there exists a set \( B \) and some adjacent databases \( X, X' \) such that \( f_X(b) > f_{X'}(b) \exp(\epsilon) \) for all \( b \in B \) and that \( \nu(B) > 0 \). Then this would imply the strict inequality

\[
\mu_X(B) = \int_B f_X(b) \, d\nu(b) > \exp(\epsilon) \int_B f_{X'}(b) \, d\nu(b) = \exp(\epsilon) \mu_{X'}(B),
\]

which is a contradiction, and thus the claim holds. \( \square \)

Interpreting DP in terms of densities is common in the DP literature (e.g. [Dwork & Roth, 2014] [Kifer & Hall et al. 2012]), however, we could not find a reference for the precise statement and proof, especially for the reverse implication.

### 3 Exponential Mechanism

One of the earliest mechanisms designed to satisfy \( \epsilon \)-DP, is the exponential mechanism, introduced by [McSherry & Talwar, 2007]. It uses an objective or utility function, which in practice, can be the same objective function used for a (non-private) statistical or machine learning analysis, thus making it especially easy to link DP with existing inferential tools. A simple proof for Proposition 3.1 can be found in [McSherry & Talwar, 2007].

**Proposition 3.1 (Exponential Mechanism: [McSherry & Talwar, 2007].** Let \( (\mathcal{Y}, \mathcal{F}, \nu) \) be a measure space. Let \( \{\xi_X : \mathcal{Y} \rightarrow \mathbb{R} \mid X \in \mathcal{X}^n \} \) be a collection of measurable functions. We say that this collection has a finite sensitivity \( \Delta_\xi \), if

\[
|\xi_X(b) - \xi_{X'}(b)| \leq \Delta_\xi < \infty,
\]

\( \forall b \in \mathcal{Y} \).
The first term will be absorbed into the constants, since it does not depend on $z$, while the second term is zero for $n$ large, leaving only the third term to contribute to the form of the density. Obviously $|g(\hat{b} + z/\sqrt{n}) - g(b^*)| \to 0$, so the only remaining task is to show that the combined constants behave appropriately. Recall that

$$c_n n^{1/2} \exp \left\{ -\frac{\epsilon}{2\Delta} \xi_n(\hat{b}) \right\} = \int_{B_n} g(\hat{b} + z/\sqrt{n}) \exp \left\{ \frac{\epsilon}{2\Delta} [\xi_n(\hat{b} + z/\sqrt{n}) - \xi_n(\hat{b})] \right\} \, dz.$$
By Assumption (1) we have that
\[ \xi_X(b + z/\sqrt{n}) - \xi_n(b) \leq -\frac{\alpha}{2} ||z||^2. \]

Since \( \exp\{-||z||^2\} \) is integrable, we can apply the dominated convergence theorem to conclude that the constants converge to something nonzero as well.

Putting everything together, we can conclude that
\[ f_n(z) \to f(z) \propto \exp \left\{ -\frac{\epsilon}{2\Delta} z^\top \Sigma^{-1} z/2 \right\}, \]

which, is the density of the multivariate normal. Applying Scheffe’s Theorem, we thus have both convergence in distribution as well as convergence in total variation:
\[ \sqrt{n}(\tilde{b} - \hat{b}) \overset{D}{\to} N_p \left( 0, \frac{\epsilon}{2\Delta} \Sigma \right). \]

The previous result shows that under common conditions, the noise added by the exponential mechanism is of order \( O(1/\sqrt{n}) \). We know by the theory of M-estimators that the non-private solution to the objective functions \( \hat{b} \) also converges at rate \( O(1/\sqrt{n}) \). So, we have that the use of the exponential mechanism in such cases preserves the \( 1/\sqrt{n} \) convergence rate, but with a sub-optimal asymptotic variance. This means that asymptotically, to achieve the same performance as the non-private estimator, the exponential mechanism requires \( k \) times as many samples, where \( k \) is some constant larger than 1, which depends on \( \epsilon \) and \( \Delta \). However, we know that for many problems, it is possible to construct DP mechanisms which only introduce \( O(k) \) noise, thus having equivalent asymptotics to the non-private estimator (e.g. Smith, 2011; Awan & Slavkovic, 2018). Even though in these settings, the noise is asymptotically negligible, developing accurate approximations is still a challenge, which Wang et al. (2018) recently tackled.

In the next result, we extend Theorem 3.2 from \( \mathbb{R}^p \) to Hilbert spaces. However, we currently only consider base measures which are Gaussian processes.

**Theorem 3.3 (Utility of Exp Mech).** Suppose that the observed record, \( X_1, \ldots, X_n \), and objective function \( \xi_X(b) \), for \( b \in \mathcal{H} \) satisfy

1. \( -n^{-1} \xi_n(b) \) are twice differentiable convex functions and there exists a finite \( \alpha > 0 \) such that the eigenvalues of \( -n^{-1} \xi_n(b)^{''} \) are greater than \( \alpha \) for all \( n \) and \( b \in \mathcal{H} \);
2. the minimizers satisfy \( \hat{b} \to b^* \in \mathcal{H} \) and \( -n^{-1} \xi_n(\hat{b})^{''\prime\prime} \to \Sigma \) where \( \Sigma \) is positive definite trace class operator (and convergence is wrt this space);
3. \( \xi_n \) has finite sensitivity \( \Delta \), which is constant in \( n \).

Assume the base measure is taken to be a Gaussian process, \( \nu \sim N_\mathcal{H}(0, C) \), such that \( C^{-1} \Sigma \) is bounded. Then the sanitized estimate \( \hat{b} \) is asymptotically normal
\[ \sqrt{n}(\hat{b} - \tilde{b}) \overset{D}{\to} N_\mathcal{H} \left( 0, \frac{2\Delta}{\epsilon} \Sigma \right). \]

**Proof.** The proof would be essentially the same as before, however when changing variables via standardizing, the base measure is no longer Lebesgue and thus the effects of rescaling the base measure cannot be ignored. Recall there is no translation invariant \( \sigma \)-finite measure in infinite dimensions. Consider \( Z = \sqrt{n}(\hat{b} - \tilde{b}) \) and
\[ P(\sqrt{n}(\hat{b} - \tilde{b}) \in A) = \int_{\mathcal{H}} f_X(b) \, d\nu(b) = n^{-1/2} \int_A f_X(b + z/\sqrt{n}) \, d\nu(b + z/\sqrt{n}). \]

The same Taylor expansion arguments from before still apply, however the base measure has now been shifted and scaled. In particular, if \( d\tilde{\nu}(z) = d\nu(b + z/\sqrt{n}) \), then \( \tilde{\nu} \) is the measure of a Gaussian process with mean \( -\sqrt{n} b \) and covariance operator \( nC \). So, we have that, up to a normalizing constant
\[ P(\sqrt{n}(\hat{b} - \tilde{b}) \in A) \approx \int_A \exp \left\{ -\frac{\epsilon}{2\Delta} \xi_X(b)z/2n \right\} \, d\tilde{\nu}(z) \approx \int_A \exp \left\{ -\frac{\epsilon}{2\Delta} \Sigma^{-1} z/2 \right\} \, d\tilde{\nu}(z). \]
However, this is a Gaussian measure with covariance operator \((\frac{\epsilon}{2\Delta} \Sigma^{-1} + C^{-1}/n)^{-1}\) and mean \(-n^{-1/2}(\frac{\epsilon}{2\Delta} \Sigma^{-1} + C^{-1}/n)^{-1} C^{-1} \hat{b}\). Since \(C\) is fixed, the following limits hold

\[
\left(\frac{\epsilon}{2\Delta} \Sigma^{-1} + C^{-1}/n\right)^{-1} \rightarrow \frac{2\Delta}{\epsilon} \Sigma
\]

\[-n^{-1/2} \left(\frac{\epsilon}{2\Delta} \Sigma^{-1} + C^{-1}/n\right)^{-1} C^{-1} \hat{b} \rightarrow 0.
\]

\[\Box\]

**Remark 3.4.** The requirement that \(C^{-1} \Sigma\) is bounded implies that the base measure is “rougher” than the asymptotic distribution of \(\hat{b}\). One way to view the assumption on \(\xi''_X\) is through tightness. In particular, if one assumed only that \(\Sigma\) was bounded, then the sequence of measures need not be tight and thus one does not get convergence in the “strong topology” in \(\mathcal{H}\) [Billingsley 2013, Chen & White 1998, Remark 3.3]. However, one could still obtain convergence of properly normalized continuous linear functionals.

**Example 3.5.** Consider \(X_1, \ldots, X_n \in \mathcal{H}\) are drawn from a Gaussian process with mean \(\mu_X\) and covariance operator \(C_X\). Consider estimating \(\mu_X\) using the target function

\[-\xi_X(b) = \sum_{i=1}^{n} \|X_i - b\|^2.
\]

Assume that the \(\|X_i\| \leq 1\) and thus we need only consider \(\|b\| \leq 1\). In that case, the sensitivity is bounded by \(4\). However, for this target function the exponential mechanism will not be asymptotically Gaussian (in the strong topology). If we consider the second derivative we have

\[-\xi''_X(b) = 2nI,
\]

and thus \((-\xi''_X(b)/n)^{-1} = (1/2)I\), which is not a nuclear operator in infinite dimensions. However, if instead we consider the penalized version

\[-\xi_X(b) = \sum_{i=1}^{n} \|X_i - b\|^2 + n\lambda \|b\|^2_C,
\]

where \(\|b\|^2_C = \langle b, C^{-1} b \rangle\), then the sensitivity is the same, but the second derivative is now

\[-\xi''_X(b) = 2nI + 2n\lambda C^{-1},
\]

which satisfies the assumptions of Theorem 3.3. In this case, we can now take our Gaussian mechanism to be a mean zero Gaussian process with covariance \(C\).

We stress that, in finite samples, there is no issue related to privacy even when \(\xi''(b)^{-1}\) is not nuclear since we are assuming the mechanism is defined using a value probability distribution as the base measure. What the previous results and this example illustrate is that there is a price to pay for using such a flexible mechanism. In the “good” case, when the assumptions of Theorem 3.3 are met, one has an asymptotically non-negligible noise, but in the ”bad” case, the noise can be even larger, since the covariance operator can blow up.

### 4 DP Functional Principal Components

In this section, we apply the exponential mechanism to the problem of producing private functional principal component analysis (FPCA).

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert Space. Let \(X \in \mathcal{H}^n\) be such that its components satisfy \(\|X_i\| \leq 1\) for all \(i = 1, \ldots, n\). Call \(S(X)\) the \(k\)-dimensional subspace of \(\mathcal{H}\) given by the span of the first \(k\) principal components
of $X$. Let $P_{\hat{S}(X)} : \mathcal{H} \to \mathcal{H}$ denote the projection operator of $\mathcal{H}$ onto $\hat{S}(X)$. We can write $P_{\hat{S}(X)}$ as the solution to the optimization problem

$$P_{\hat{S}(X)} = \arg \min_{P \in \mathcal{P}_k} \sum_{i=1}^n \|X_i - PX_i\|^2,$$

where $\mathcal{P}_k$ is the set of projection operators $P : \mathcal{H} \to \mathcal{H}$ of rank $k$. Equivalently, we can write

$$P_{\hat{S}(X)} = \arg \max_{P \in \mathcal{P}_k} \sum_{i=1}^n \|PX_i\|^2.$$

More specifically, in this section, we develop a set of probability measures $\mathcal{M}$ on $\mathcal{P}_k$, indexed by $\mathcal{H}^n$, which satisfy $\epsilon$-DP, and such that a random element $P$ from $\mu_X \in \mathcal{M}$ is “close” to $P_{\hat{S}(X)}$.

Our approach follows that of Chaudhuri et al. (2013). We take our objective function to be $\xi : X^n \times \mathcal{P}_k \to \mathbb{R}$ defined by $\xi_X(P) = \sum_{i=1}^n \|PX_i\|^2$. Note that $\Delta_\xi = 1$, since $\|PX_i\|^2 \leq \|X_i\|^2 \leq 1$ for any $P \in \mathcal{P}_k$ and any $i = 1, \ldots, n$. Since $\sum_{i=1}^n \|PX_i\|^2 \leq n$, for any probability measure $\nu$ on $\mathcal{P}_k$, the class of densities on $\mathcal{P}_k$ with respect to $\nu$ given by

$$f_X(P) \propto \exp \left( \frac{\xi}{2} \sum_{i=1}^n \|PX_i\|^2 \right),$$

satisfies $\epsilon$-DP.

If $\mathcal{H}$ is finite dimensional, then $\mathcal{P}_k$ is a compact subset of the space of linear operators (e.g. matrices when $\mathcal{H} = \mathbb{R}^p$). In that case, there exists a uniform distribution on $\mathcal{P}_k$. In Chaudhuri et al. (2013), they implement the exponential mechanism as above, with respect to the uniform distribution on $\mathcal{P}_k$.

For arbitrary $\mathcal{H}$, $\mathcal{P}_k$ is not compact, so we must find another base measure on $\mathcal{P}_k$. To understand our proposed construction, we again consider the finite dimensional $\mathcal{H}$. Let $P \sim \text{Unif}(\mathcal{P}_k)$, that is $P$ is drawn from the uniform distribution on $\mathcal{P}_k$. Let $V_1, \ldots, V_k \overset{\text{iid}}{\sim} N(0, I)$, be iid multivariate normal with mean zero and identity covariance matrix. Then $P \overset{\text{d}}{=} \text{Projection}(\text{span}(V_1, \ldots, V_k))$ (since $V_k$ is invariant under rotations).

From this factorization, a natural extension for arbitrary $\mathcal{H}$ becomes clear. Let $V_1, \ldots, V_k \overset{\text{iid}}{\sim} N_H(0, \Sigma)$, be iid Gaussian processes in $\mathcal{H}$ with zero mean and covariance operator $\Sigma$. Note that $\Sigma$ must be positive definite and trace class, which excludes the identity when $\mathcal{H}$ is infinite dimensional (Bogachev, 1998). We can also tailor $\Sigma$ to instill certain properties such as smoothness or periodicity. Then set $P = \text{Projection}(\text{span}(V_1, \ldots, V_k))$. This procedure induces in a probability measure on $\mathcal{P}_k$, which we call $\nu$.

**Theorem 4.1.** Let $\mathcal{H}$ be a real separable Hilbert Space and $\mathcal{P}_k$ the collection of all $k$-dimensional projection operators over $\mathcal{H}$. Let $\nu$ be the probability measure over $\mathcal{P}_k$ induced by the transformation $\text{Projection}(\text{span}(V_1, \ldots, V_k))$, where $V_i \in \mathcal{H}$ are iid Gaussian process with mean 0 and covariance operator $\Sigma$. Let $\mathcal{M}$ be the class of probability measures on $\mathcal{P}_k$ with densities

$$f_X(P) \propto \exp \left( \frac{\xi}{2} \sum_{i=1}^n \|PX_i\|^2 \right)$$

with respect to $\nu$. Then $\mathcal{M}$ satisfies $\epsilon$-DP.

**Theorem 4.2.** Let $\mathcal{H}$ be a Hilbert Space, $k < n$ be two positive integers, and $\mathcal{M}$ be the class of probability measures on $\mathcal{H}^k$ with densities $f_X(V_1, \ldots, V_k)$ proportional to

$$\exp \left( \frac{\xi}{2} \sum_{i=1}^n \|\text{Projection}(\text{span}(V_1, \ldots, V_k))X_i\|^2 \right)$$

with respect to $\nu$ (the measure induced by the Gaussian distribution $N^k(0, \Sigma)$) on $\mathcal{H}^k$. Then $\mathcal{M}$ satisfies $\epsilon$-DP.

As we see in the next section, we will represent the output of Theorem 4.2 as an arbitrary basis for a $k$-dimensional subspace $\hat{S}$ of $\mathcal{H}$, which can then be assembled into a projection operator as needed.

**Remark 4.3.** In many cases we can interpret $\Sigma$ as instilling some particular structure on $P$ or the $V_i$. For example, if $\mathcal{H} = L^2[0, 1]$, then we could define $\Sigma$ using the kernel of an RKHS. The kernel could then be chosen so that $\hat{S}$ have a certain number of derivatives as many Sobolev spaces are RKHS as well (Berlinet & Thomas-Agnan, 2011), which is often a natural assumption.
5 PCA continued: Sampling

In the previous section, we developed a set of \(\epsilon\)-DP probability measures for arbitrary Hilbert Spaces. However, for these to be of use to us, we need to be able to sample from these probability distributions. As is common in FDA (Ramsay & Silverman, 2005; Kokoszka & Reimherr, 2017) we use finite dimensional approximations via basis expansions for computation.

Let \(b_1, b_2, \ldots\) be an orthonormal basis for \(H\). We will work in the \(m\)-dimensional subspace \(H_m = \text{span}(b_1, \ldots, b_m)\). Given our observed values \(X_i \in H\), call \(X_{ij} = \langle X_i, b_j \rangle\) for \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). We arrange these real values in an \(n \times m\) matrix \(X = (X_{ij})\). Next, let \(\Sigma\) be a trace class covariance operator on \(H\). Write \(\Sigma_{ij} = \langle b_i, \Sigma b_j \rangle\) for \(i, j = 1, \ldots, m\). We put these values in an \(m \times m\) matrix \(\Sigma = (\Sigma_{ij})\), which is a positive definite matrix in \(\mathbb{R}^{m \times m}\). In this setup, we then draw \((V_1, \ldots, V_k) \in H_m\). Call \(V_{ij} = \langle V_i, b_j \rangle\), and arrange these values into a real-valued matrix \(V = (V_{ij})\). We then draw from the density \(f(V)\), with respect to Lebesgue measure on \(\mathbb{R}^{k \times m}\), which is proportional to

\[
\exp \left( \frac{\epsilon}{2} \text{tr}(X^\top X (V^\top V)^{-1} V^\top V - V^\top \Sigma^{-1} V) \right).
\]

In fact, we can obtain a more convenient form for sampling. Since we only need the span of \(V\), we can condition on the columns of \(V\) being orthonormal. The density \(f(V | \text{orthonormal})\), with respect to the uniform measure on the set of orthonormal matrices in \(\mathbb{R}^{m \times k}\), is proportional to

\[
\exp \left( \frac{\epsilon}{2} \text{tr} \left( V^\top (X^\top X - \Sigma^{-1}) V \right) \right),
\]

which is an instance of the Matrix-Bingham-Von-Mises distribution, for which an efficient Gibbs sampler is known (Hoff, 2009; Hoff & Franks, 2018).

6 Numerical Studies

In this section we assess the performance of the exponential mechanism for private FPCA, as developed in Sections 4 and 5 on both simulated and real data.

6.1 Simulation Study

For our simulation study, we generated data on a grid of 100 evenly spaced points on \([0, 1]\) using the Karhunen-Loeve expansion with Gaussian noise added:

\[
X_i(t_{ik}) = \mu(t_{ik}) + \sum_{j=1}^{p} \frac{1}{\sqrt{j}} U_{ij} u_j(t_{ik}) + \varepsilon_{ik},
\]

for \(i = 1, \ldots, n, k = 1, \ldots, 100\). The \(u_j(t)\) are the true functional principal components, \(\varepsilon_{ik}\) are independent errors sampled from the Gaussian distribution \(N(0, 1)\), and scores \(U_{ij}\) are sampled from \(N(0, 0.1)\). Note that for each scenario we re-scale the \(X_i\) so that \(||X_i||^2 < 1\) for \(i = 1, \ldots, n\).

The \(u_j(t)\) are comprised of Fourier basis functions and to fully explore the effectiveness of this approach, we vary the sample size \(n\), privacy budget \(\epsilon\), and repeat each scenario 10 times. Data is generated using \(p = 21\) true components and additional weights were placed on the fourth term in the Fourier expansion, creating the overall shape shown in Figure 1. We release only \(k = 1\) components.
We also specify $m$, the number of orthonormal basis functions $b_i$, when restricting the functional observations to a finite dimensional space and $\Sigma$, a trace class covariance operator on $\mathcal{H}$. It is common to take $m$ to be some sufficiently large value, usually around 40-50, and for our simulation scenarios we have $m = 40$. The choice of $\Sigma$ can vary depending on what structures one may want to induce on the functions (i.e. the number of derivatives). For our choice of $\Sigma$, we take it to be a diagonal matrix with $\Sigma_{ii} = i^{-3}$, which corresponds to requiring that the $V_i$ are continuous. Given that the data is periodic, we chose to use the Fourier basis functions as $b_i$. Finally, recall there is an efficient Gibbs sampler for this approach [Hoff (2009)], which has been implemented in the \texttt{rstiefel} package [Hoff & Franks (2018)] in R. This also requires a fixed number of iterations as burn-in prior to starting the procedure. Following the computational experiments in Chaudhuri et al. (2013), we used 20,000 iterations and had similar convergence results.

We provide two measurements of performance to compare the resulting space of orthogonal projection operators. The first compares the ratio of variability accounted for between the private and non-private estimates of the $k$ functional principal components. More explicitly,

$$0 \leq \frac{\| X^T \tilde{P} X \|_F^2}{\| X^T \hat{P} X \|_F^2} \leq 1,$$

where $\| \cdot \|_F$ is the Frobenius norm, $\tilde{P}$ is the projection onto the span of $V$ drawn from the mechanism in Theorem 4.2, and $\hat{P}$ the the non-private solution to (1).

The second measure gives an indication of how close the range of $\tilde{P}$ is to $\hat{P}$:

$$0 \leq \frac{1}{2} \| \tilde{P} - \hat{P} \|_F^2 \leq k.$$ 

If the range of $\tilde{P}$ and $\hat{P}$ agree in $h$ dimensions and are orthogonal in $k - h$ dimensions, then this measure gives the value $k - h$. So this can be interpreted as roughly the number of dimensions that $\tilde{P}$ and $\hat{P}$ disagree.

We summarize the results in Figures 2a and 2b over a range of sample size $n$ and privacy budget $\epsilon$. Note that, as expected, larger sample sizes can preserve utility (in terms of the two measurements described.
Table 1: Average performance for private principal components from the Berkeley growth and DTI data sets. Standard errors are provided in parenthesis for reference.

| No. of Components (k) | Variance Ratio |   |
|-----------------------|----------------|---|
|                       | Berkeley       | DTI (cca)     |
|                       | 1/8            | 1/8            |
| 1                     | 0.264 (.024)   | 0.372 (.025)  |
| 2                     | 0.494 (.023)   | 0.569 (.024)  |
| 3                     | 0.672 (.020)   | 0.727 (.018)  |
|                       | 1/4            | 1/4            |
| 1                     | 0.343 (.024)   | 0.497 (.026)  |
| 2                     | 0.523 (.023)   | 0.676 (.021)  |
| 3                     | 0.681 (.020)   | 0.811 (.011)  |
|                       | 1/2            | 1/2            |
| 1                     | 0.408 (.025)   | 0.726 (.020)  |
| 2                     | 0.523 (.022)   | 0.812 (.014)  |
| 3                     | 0.729 (.019)   | 0.876 (.009)  |
|                       | 1               | 1               |
| 1                     | 0.550 (.025)   | 0.879 (.009)  |
| 2                     | 0.680 (.018)   | 0.885 (.007)  |
| 3                     | 0.775 (.015)   | 0.910 (.005)  |
|                       | 2               | 2               |
| 1                     | 0.743 (.018)   | 0.933 (.006)  |
| 2                     | 0.787 (.012)   | 0.928 (.004)  |
| 3                     | 0.855 (.010)   | 0.939 (.003)  |

previously) for stricter privacy requirements. Additionally, we provide the sanitized curve for the first principal component for one instance of a scenario with a sample size of $n = 500$, and $n = 5000$, seen in Figures 3a and 3b. The last 100 Gibbs updates are given as well, demonstrating the variability in each sample size. Note that even with a privacy budget of $\epsilon = 1$ and relatively low sample size, the overall shape is captured, but the variance is reduced when $n = 5000$.

6.2 Applications

For the real data application, we applied our procedure to two data sets, the Berkeley growth study from the fda package Ramsay et al. (2018), and Diffusion Tensor Imaging (DTI) from the refund package Goldsmith et al. (2018). The Berkeley data has the heights of 93 children at 31 time points with ages ranging from 1-18. DTI gives fractional anisotropy (FA) tract profiles for the corpus callosum (CCA) the right corticospinal tract (RCST) for patients with Multiple Sclerosis as well as controls. We focus on the cca data, which includes 382 patients measured at 93 equally spaced locations along the CCA.

Results are summarized in Tables 1 and 2 when releasing 1-3 principal components across a range of privacy budgets and averaging the performance measurements over 100 repetitions of our procedure. For each data set we selected the Gaussian kernel for $\Sigma$ with a smoothness parameter that requires $m = 5$ eigenvalues to explain $>99\%$ of variation. Its corresponding eigenfunctions were selected for the orthonormal basis $b_i$. Our approach is more effective over the DTI data set, which may be due to the true variation explained by the non-private components. For DTI the cumulative variation is .77, .86, and .93 for the top 1, 2, and 3 components respectively, while Berkeley has 0.82, 0.95, and 0.98. When things are too “simple”, necessary deviations for privacy show more loss in variation explained compared to the non-private estimates. Overall, this still demonstrates the effectiveness of our procedure under different types of real data with smaller sample sizes.

7 Discussion

In this paper, we studied the exponential mechanism in the setting of separable Hilbert spaces. We showed that generally when the objective is an empirical risk function, the exponential mechanism has a CLT implying that asymptotically non-negligible noise is introduced. Since the exponential mechanism is popularly used, this result demands the following question: what properties of the objective function guarantee asymptotically negligible noise?

Through our simulations and applications, we found that the choice of $\Sigma$ can have a significant impact on
Table 2: Average performance for private principal components from the Berkeley growth and DTI data sets. Standard errors are provided in parenthesis for reference.

| No. of Components ($k$) | Berkeley Subspace Norm | DTI (cca) Subspace Norm |
|-------------------------|-------------------------|-------------------------|
|                         | 1/8                     | 1/4                     |
|                         | 1/2                     | 1  | 2 |

| No. of Components ($k$) | Berkeley Subspace Norm | DTI (cca) Subspace Norm |
|-------------------------|-------------------------|-------------------------|
|                         | 1/8                     | 1/4                     |
|                         | 1/2                     | 1  | 2 |

the result of the private FPCA analysis. In particular, $\Sigma$ can be rescaled by any positive constant, which affects the smoothing but does not change the interpretation in terms of number of derivatives. While our approach requires that $\Sigma$ is chosen before seeing the data, it would be preferable to have a method of learning $\Sigma$ within the DP procedure. Future researchers should investigate effective methods of tuning parameters under DP.

In the data applications, we found that counter-intuitively, our DP FPCA approach performs better when there is more variability in the data. Perhaps this is because our measures of performance are comparing the DP estimates to the non-private estimates, and the variability hurts both. It would be worth while to investigate this further to better understand how variability in the data affects the performance of DP methods.

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