(Heisenberg-)Weyl Algebras, Segal-Bargmann Transform and Representations of Poincaré Groups

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Abstract. In a recent paper [1] we described a novel treatment of the unitary irreducible representations of the Poincaré groups in 2, 3 and 4 space-time dimensions as unitary operators on the representation spaces of the Schrödinger representation of the Heisenberg-Weyl algebra \( W_r(\mathbb{R}) \) of index \( r = 1, 2, \) and 3, respectively. Here we relate this approach to the usual method of describing the representations of these Poincaré groups, i.e. the Wigner-Mackey construction.

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth.

1. Introduction

In [1] we established the validity of the Gelfand-Kirillov conjecture [2], [3] for the Lie algebras of the Poincaré groups in 2 and 3 space-time dimensions by explicit construction of the isomorphisms guaranteed by the Gelfand-Kirillov conjecture for both of these groups. For the 4 dimensional space-time case, in which the group is the 10 dimensional Poincaré group of ordinary Minkowski space-time, we gave a slightly weaker result, in order to avoid dealing with technical complexities due to spin. We made use of the isomorphisms established in that paper to give a novel description of the unitary irreducible representations (UIR) of the Poincaré groups in 2, 3 and 4 space-time dimensions as unitary operators on the representation spaces of the Schrödinger representation of the Heisenberg-Weyl algebra \( W_r(\mathbb{R}) \) of index \( r = 1, 2, \) and 3, respectively.

Here we establish the equivalence of the UIRs of the Poincaré groups given in [1] with the standard approach using the Wigner-Mackey theory [4], [5], [6]. Our approach leads to realizations of the representations of the Poincaré groups on (Segal-)Bargmann-Fock spaces [7]. In Section 4 we describe in detail the situation for \( n = 2 \) case. In particular, we show how the equivalence between our realization of the representations of the three dimensional Poincaré
group on the Bargmann-Fock space [7] and the usual description obtained from the Wigner-Mackey construction is implemented by the Segal-Bargmann transform [7], [8]. Notation and conventions which we use are for the most part the same as those in [1].

2. Preliminaries

The (Heisenberg-)Weyl Algebra, \( \mathbb{W}_r(\mathbb{R}) \), is an algebra with \( 2r \) generators \( p_1, \ldots, p_r, q_1, \ldots, q_r \) satisfying relations

\[
[p_i, q_j] = \delta_{ij} \mathbb{1} \quad [p_i, p_j] = [q_i, q_j] = 0
\]

for all \( i, j \leq r \). (\( \mathbb{1} \) is the identity in the Heisenberg-Weyl algebra.)

Given a collection of free variables \( y_1, \ldots, y_s \) we define

\[
\mathbb{W}_{(r,s)}(\mathbb{R}) := \mathbb{W}_r(\mathbb{R}) \otimes \mathbb{R}[y_1, \ldots, y_s].
\]

Being a domain (no zero divisors) the algebra \( \mathbb{W}_{(r,s)}(\mathbb{R}) \) admits a skew field of fractions denoted by \( \mathbb{D}_{(r,s)}(\mathbb{R}) \).

The Poincaré group \( \mathcal{P}_n \) is the semidirect product of the connected component of the Lorentz group, \( SO_0(1, n - 1) \), with the abelian group of translations of \( \mathbb{R}^n \). The Lie algebra \( \mathfrak{p}_n \) of \( \mathcal{P}_n \) is a split extension of \( \mathfrak{so}(1, n - 1) \):

\[
0 \rightarrow \mathfrak{t}^n \rightarrow \mathfrak{p}_n \rightarrow \mathfrak{so}(1, n - 1) \rightarrow 0
\]

with \( \mathfrak{p}_n = \mathfrak{so}(1, n - 1) \oplus \mathfrak{t}^n \) and \( \mathfrak{t}^n \) (the Lie algebra of the translation subgroup) an abelian ideal in \( \mathfrak{p}_n \). Basic generators of \( \mathfrak{p}_n \) are \( \mathbf{L}_{ij} \) and \( \mathbf{P}_k \). They satisfy the commutation relations

\[
[\mathbf{L}_{ij}, \mathbf{L}_{jk}] = e_j \mathbf{L}_{ik} \quad (i, j, k = 0, \ldots, n - 1) \quad (e_0 = 1, e_j = -1)
\]

\[
[\mathbf{L}_{0i}, \mathbf{P}_0] = -e_i \mathbf{P}_i \quad [\mathbf{L}_{0i}, \mathbf{P}_j] = -\delta_{ij} \mathbf{P}_0 \quad [\mathbf{L}_{ij}, \mathbf{P}_k] = \delta_{jk} \mathbf{P}_i - \delta_{ik} \mathbf{P}_j
\]

Let \( \mathcal{U}(\mathfrak{p}_n) \) be the universal enveloping algebra of \( \mathfrak{p}_n \). Since \( \mathcal{U}(\mathfrak{p}_n) \) is Noetherian and hence is a domain, it has a skew-field of fractions which we denote by \( \mathbb{D}(\mathfrak{p}_n) \). \( \mathbb{D}(\mathfrak{p}_n) \) is also called the “Lie field” of \( \mathfrak{p}_n \).

\( H_r \) is the \( 2r + 1 \) dimensional Lie group whose Lie algebra \( \mathfrak{h}_r \) consists of elements in the real linear span of

\[-ip_i, q_i \quad (i = 1, \ldots, r) \quad \text{and} \quad \mathbb{Z}. \]

\( \mathfrak{h}_r \) is a trivial central extension of the \( 2r \) dimensional abelian Lie algebra \( \mathbb{R}^{2r} \):

\[
0 \rightarrow \mathbb{R} \rightarrow \mathfrak{h}_r \rightarrow \mathbb{R}^{2r} \rightarrow 0.
\]

Specifically we have

\[
[-ip_i, q_j] = \delta_{ij} \mathbb{Z}
\]

with all other commutators vanishing and

\[
\mathfrak{h}_r = \mathbb{R}^{2r} \oplus \mathbb{R}\mathbb{Z}.
\]
3. Gelfand-Kirillov Conjecture

The Gelfand-Kirillov conjecture is the following [2], [3]: If \( \text{char}(\mathbb{K}) = 0 \) and \( L \) is the Lie algebra of an algebraic \( \mathbb{K} \)-group, then

\[
\mathcal{D}(L) \cong \mathcal{D}_{(r,s)}(\mathbb{K})
\]

with \( s = \) degree of transcendence of \( \mathcal{C}(L) \) over \( \mathbb{K} \) and \( r = \frac{1}{2}(\dim_{\mathbb{K}}(L) - s) \).

The Gelfand-Kirillov conjecture fails for \( \mathfrak{so}(2) \oplus t^a \), the Lie algebra of the Euclidean group of isometries of the Euclidean plane [9]. On the other hand we have the following:

**Theorem I:** The Gelfand-Kirillov conjecture holds for \( \mathfrak{p}_2 \cong \mathfrak{so}(1,1) \oplus t^2 \):

\[
\mathcal{D}(\mathfrak{p}_2) \cong \mathcal{D}_{(1,1)}(\mathbb{R}).
\]

**Proof:** Define \( \varphi : \mathcal{D}(\mathfrak{p}_2) \to \mathcal{D}_{(1,1)}(\mathbb{R}) \) by

\[
\varphi(L_{01}) = qp + \frac{1}{2}1, \quad \varphi(P_0) = \frac{1}{2}(q - yq^{-1}), \quad \varphi(P_1) = -\frac{1}{2}(q + yq^{-1}) \tag{1}
\]

where \( y \) is the free variable in \( \mathcal{D}_{(1,1)}(\mathbb{R}) \) and \( 1 \) is the identity. The inverse map is

\[
\varphi^{-1}(p) = (P_0 - P_1)^{-1}(L_{01} - \frac{1}{2}), \quad \varphi^{-1}(q) = P_0 - P_1, \quad \varphi^{-1}(y) = -(P_0^2 - P_1^2) \tag{2}
\]

\( \varphi \) extends by linearity to a map from \( \mathcal{U}(\mathfrak{p}_2) \) to \( \mathcal{W}_{(1,1)}(\mathbb{R}) \). It follows from the fact that \( \mathcal{W}_1(\mathbb{R}) \) is simple and hence has no nontrivial ideals that \( \varphi^{-1} \) is injective. From this it follows that \( \varphi \) extends to an injective mapping from \( \mathcal{D}(\mathfrak{p}_2) \) onto \( \mathcal{D}_{(1,1)}(\mathbb{R}) \). \( \square \)

**Theorem II:** The Gelfand-Kirillov conjecture holds for \( \mathfrak{p}_3 \cong \mathfrak{so}(1,2) \oplus t^3 \):

\[
\mathcal{D}(\mathfrak{p}_3) \cong \mathcal{D}_{(2,2)}(\mathbb{R}).
\]

**Proof:** The Casimir operators of \( \mathfrak{p}_3 \) are

\[
M^2 = -(P_0^2 - P_1^2 - P_2^2), \quad C = L_{02}P_1 - L_{12}P_0 - L_{01}P_2. \tag{3}
\]

Define \( \varphi : \mathcal{U}(\mathfrak{p}_3) \to \mathcal{D}_{(2,2)}(\mathbb{R}) \) by

\[
\begin{align*}
\varphi(L_{12}) &= -q_1^{-1}\left\{ y_2 + (q_1p_1 + \frac{1}{2})q_2 + \frac{1}{2}(q_2^2 - q_1^2 - y_1)p_2 \right\}, \\
\varphi(L_{01}) &= q_1p_1 + \frac{1}{2}, \quad \varphi(L_{02}) = \varphi(L_{12}) - q_1p_2, \\
\varphi(P_0) &= \frac{1}{2}q_1^{-1}(q_1^2 + q_2^2 - y_1), \quad \varphi(P_1) = \frac{1}{2}q_1^{-1}(q_2^2 - q_1^2 - y_1), \quad \varphi(P_2) = q_2
\end{align*}
\]

\( \tag{4} \)

where \( y_1 \) and \( y_2 \) are the free variables in \( \mathcal{D}_{(2,2)}(\mathbb{R}) \). The inverse map is

\[
\begin{align*}
\varphi^{-1}(p_1) &= (P_0 - P_1)^{-1}(L_{01} - \frac{1}{2}), \quad \varphi^{-1}(q_1) = (P_0 - P_1), \quad \varphi^{-1}(p_2) = (P_0 - P_1)^{-1}(L_{12} - L_{02}) \\
\varphi^{-1}(q_2) &= P_2, \quad \varphi^{-1}(y_1) = P_1^2 + P_2^2 - P_0^2 = M^2, \quad \varphi^{-1}(y_2) = L_{02}P_1 - L_{12}P_0 - L_{01}P_2 = C.
\end{align*}
\]

As in the \( \mathfrak{p}_2 \) case, \( \mathcal{W}_2(\mathbb{R}) \) being simple implies \( \varphi^{-1} \) is injective and \( \varphi \) extends to an injective map from \( \mathcal{D}(\mathfrak{p}_3) \) onto \( \mathcal{D}_{(2,2)}(\mathbb{R}) \). Thus the Gelfand-Kirillov conjecture holds. \( \square \)
4. Wigner-Mackey Theory for the Poincaré Groups

The Poincaré group is \( \mathcal{P}_n = \text{SO}(1, n-1)^\sim \ltimes T^n \). For \( n > 3 \), \( \text{SO}(1, n-1)^\sim \cong \text{Spin}(1, n-1) \); for \( n = 3 \), \( \text{SO}(1, 2)^\sim \cong \text{SL}(2, \mathbb{R})^\sim \), with \( \text{SL}(2, \mathbb{R})^\sim \) being the universal cover of \( \text{SL}(2, \mathbb{R}) \); for \( n = 2 \), \( \text{SO}(1, 1)^\sim \cong \text{SO}(1, 1) \) since \( \text{SO}(1, 1) \) is simply connected.

The translation subgroup \( T^n \) of \( \widetilde{\mathcal{P}}_n \) is an additive vector group and so every unitary irreducible representation (UIR) of \( T^n \) is one-dimensional and of the form

\[ \chi_p : T^n \to \mathbb{C}, \quad a \to \chi_p(a) = \exp(ip \cdot a) \]

where \( p, a \in \mathbb{R}^n \) and \( p \cdot a \) is the \( \text{SO}(1, n-1) \) invariant scalar product of the two vectors \( p \) and \( a \). It follows that we can characterize the equivalence classes of the UIR’s of \( T^n \) by elements \( p \) of the vector space dual \( \widetilde{T}^n \) to \( T^n \).

We have:

\[ \chi_p(a) = \exp \left\{ i \left( p_0a_0 - \sum_{j=1}^{n-1} p_ja_j \right) \right\} \equiv \exp(ip \cdot a), \quad p \in \widetilde{T}^n, a \in T^n, \]

\[ \widetilde{T}^n = \left\{ p = \left( \begin{array}{c} p_0 \\ \vdots \\ p_{n-1} \end{array} \right) | p_k \in \mathbb{R} \right\}. \]

The co-adjoint action of \( \text{SO}(1, n-1) \) on \( \widetilde{T}^n \) is given by \( p \mapsto \Lambda^{-1}p \). Let \( \mathcal{O}_p \) be the orbit in \( \widetilde{T}^n \) of the point \( \tilde{o} \in \widetilde{T}^n \) under the action of \( \text{SO}(1, n-1)^\sim \) and let \( M_p \) be the isotropy subgroup (stabilizer subgroup) of the point \( \tilde{o} \). Clearly \( M_p \) is a closed subgroup of \( \text{SO}(1, n-1)^\sim \) and so \( \mathcal{O}_p \cong \text{SO}(1, n-1)^\sim / M_p \). Let \( \gamma : \mathcal{O}_p \to \text{SO}(1, n-1)^\sim \) be a smooth cross-section such that for any point \( p \in \mathcal{O}_p \) we have \( \gamma(p)\tilde{o} = p \).

Consider a unitary representation \( D^p \) of \( M_p \). The representation space is \( \mathcal{V}^p \) and the action of \( m \in M_p \) on \( \mathcal{V}^p \) is \( D^p(m) \). We extend this to a representation \( \widetilde{D}^p \) of the semidirect product \( B_p = M_p \ltimes T^n \) by requiring \( \widetilde{D}^p((m, a)) = D^p(m) \chi_p(a) \) where \( (m, a) \in B_p \) with \( m \in M_p \) and \( a \in T^n \). It is a remarkable result of Mackey that every UIR of \( \widetilde{\mathcal{P}}_n \) is unitarily equivalent to an induced representation \( U^p = \text{Ind}_{B_p}^{\mathcal{P}_n} (\widetilde{D}^p) \) of \( \widetilde{\mathcal{P}}_n \) induced from the representation \( \widetilde{D}^p \) of the subgroup \( B_p \) and different orbits or inequivalent representations of \( M_p \) for the same orbit induce inequivalent UIR’s of \( \widetilde{\mathcal{P}}_n \).

The induced representation \( U^p \) is defined as follows: (see below for the meaning of \( \sigma(\Lambda, p) \))

\[ U^p((\Lambda, a))\psi(p) = (\sigma(\Lambda, p))^{1/2} \exp(ip \cdot a) D^p(m(\Lambda, a))\psi(\Lambda^{-1}p) \tag{5} \]

where \( (\Lambda, a) \in \widetilde{\mathcal{P}}_n \) and \( \psi : \mathcal{O}_p \to \mathcal{V}^p \) is a square integrable \( \mathcal{V}^p \)-valued function on \( \mathcal{O}_p \) and \( m(\Lambda, p) \) is the unique element of \( M_p \) satisfying

\[ \gamma(p)m(\Lambda, p) = \Lambda \gamma(\Lambda^{-1}p). \tag{6} \]

The representation property

\[ U^p((\Lambda_1, a_1)(\Lambda_2, a_2)) = U^p(\Lambda_1, a_1)U^p((\Lambda_2, a_2) \tag{7} \]
follows from Eq. (2).

By square integrability we mean \( \int_{\mathcal{O}_p} ||\psi(p)||^2_{\mathcal{V}_p} d\mu(p) < \infty \) where, for each \( p \in \mathcal{O}_p \), \( ||\psi(p)||^2_{\mathcal{V}_p} \) is the square of the norm of the vector \( \psi(p) \) in the Hilbert space \( \mathcal{V}_p \). The meaning of \( \sigma(\Lambda, p) \) is as follows: let \( d\mu(p) \) be an \( SO_0(1, n-1) \) quasi-invariant measure on \( \mathcal{O}_p \) and let \( \sigma(\Lambda, p) \) be the unique (up to measure zero) positive real function on \( SO_0(1, n-1) \times \mathcal{O}_p \) such that \( d\mu(\Lambda p) = \sigma(\Lambda, p) d\mu(p) \).

The classification of the orbits for \( n > 2 \) is essentially identical to that given first by Wigner in his 1939 paper for the \( n = 4 \) case [10]. The classification of the orbits for \( n = 2 \) is as follows (cf. [1]):

| Type | Orbit | Stabilizer |
|------|-------|------------|
| 0    | \( p = 0 \) | \( SO_0(1, 1) \) |
| \( I \pm \) | massless, half-lines | \( p \cdot p = 0, \pm p_0 > 0, \pm \varepsilon p_1 > 0 \) \{1\} |
| \( II_{[m]} \) | real mass, positive/negative energy | \( p \cdot p = |m|^2 > 0, \pm p_0 > 0 \) \{1\} |
| \( III_{[m]} \) | imaginary mass | \( p \cdot p = -|m|^2 < 0, \pm p_1 > 0 \) \{1\} |

**Table 1.** Orbits and stabilizer subgroups for \( \mathcal{P}_2: |m| \in \mathbb{R}^+, \varepsilon = \pm 1 \)

5. **Representations of \( H_r \) which take \( Z \) to \(-i\mathbb{I}\), Bargmann-Fock Space and the Bargmann-Bargmann Transform**

The Schrödinger representation of the Heisenberg group \( H_r \) is the unitary irreducible representation \( \pi(H_r) \) on \( L^2(\mathbb{R}^2) \) with representation \( d\pi \) of \( h_r \) satisfying

\[
d\pi(-i\mathbf{p}_i)\psi(x) = -i\frac{d}{dx_i}\psi(x) , \quad d\pi(q_i)\psi(x) = ix_i\psi(x) , \quad d\pi(Z)\psi(x) = -i\psi(x).
\]

Here \( \psi(x) \in \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \) where \( \mathcal{S}(\mathbb{R}^2) \) is Schwartz space. It lifts to a representation of \( \mathcal{W}_r \) on \( \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \). (Note that \( d\pi(-i\mathbf{p}_i) \), \( d\pi(q_i) \) and \( d\pi(Z) \) are skew-symmetric operators on \( L^2(\mathbb{R}^2) \).)

**Theorem III** (Stone-von Neumann) Any irreducible strongly continuous unitary representation \( \pi \) of the group \( H_r \) on a Hilbert space which sends \( Z \) to \( d\pi(Z) = -i\mathbb{I} \), with \( \mathbb{I} \) being the identity operator on the Hilbert space, is unitarily equivalent to the Schrödinger representation on \( L^2(\mathbb{R}^2) \).

The **(Segal-)Bargmann-Fock space** or simply **Bargmann-Fock space** is the Hilbert space \( \mathfrak{F}_r \) of holomorphic functions \( \mathcal{O}(r) \) on \( \mathbb{C}^r \) given by

\[
\mathfrak{F}_r = \left\{ f \in \mathcal{O}(r) \mid \int_{\mathbb{C}^r} \overline{f}(z)f(z)e^{-|z|^2}dz < \infty \right\} \quad (dz = dz_1 \ldots dz_r , \ |z|^2 = z_1 \bar{z}_1 + \cdots + z_r \bar{z}_r).
\]

The **(Segal-)Bargmann-Fock representation** of \( h_r \) is

\[
d\pi(a_i)f(z) = \frac{df}{dz_i} , \quad d\pi(a_i^\dagger)f(z) = z_if(z) , \quad d\pi(Z)f(z) = -if(z)
\]

(9)
where the creation and annihilation operators \( a_i, a_i^\dagger \) are the usual complex linear combinations of the \(-i\mathbf{p}_i\) and \( \mathbf{q}_i\):

\[
-i\mathbf{p}_i = -\frac{1}{2}(a_i + a_i^\dagger), \quad \mathbf{q}_i = \frac{i}{2}(a_i - a_i^\dagger).
\]

\( d\bar{\pi} \) lifts to a representation of \( \mathbb{W}_r \) and also integrates to a strongly continuous unitary irreducible representation of \( H_r \) on \( \mathfrak{f}_r \). The Segal-Bargmann transform is

\[
A: \mathcal{L}^2(\mathbb{R}^r) \to \mathfrak{f}_r \quad ; \quad f(z) = \int_{\mathbb{R}^r} A(z,x)\psi(x)dx
\]

where

\[
A(z,x) = (2\pi\sigma)^{-n/4} \exp\left(\frac{1}{2}z^2 - i\sigma^{-1/2}zx - \frac{1}{4\sigma}x^2\right)
\]

is an isometric isomorphism from \( \mathcal{L}^2(\mathbb{R}^r) \) to \( \mathfrak{f}_r \) which intertwines for the representations \( d\pi \) and \( d\bar{\pi} \). \( \sigma \) is a parameter which physical units is \( \hbar/2m\omega \) with \( \omega = \sqrt{k/m} \) and \( k \) and \( m \) the mass and spring constant for the \( n \) dimensional harmonic oscillator. The Segal-Bargmann transform gives the (infinitesimal) unitary equivalence between \( d\pi \) and \( d\bar{\pi} \) guaranteed by the Stone-von Neumann theorem.

6. The \( n=2 \) case: Representations of \( \mathfrak{p}_2 \) on \( \mathcal{L}^2(\mathbb{R}) \) and \( \mathfrak{f}_1 \)

The Schrödinger representation of \( \mathbb{W}_{(1,1)}(\mathbb{R}) \) is

\[
d\pi(\mathbf{p}) = -i\frac{d}{dx}, \quad d\pi(\mathbf{q}) = ix, \quad d\pi(y) = -m^2 \mathbf{1} \quad (m^2 \in \mathbb{R}).
\]

\( d\pi(\mathbf{p}) \) and \( d\pi(\mathbf{q}) \) are symmetric operators on \( \mathcal{L}^2(\mathbb{R}) \) and \( d\pi(\mathbf{q}) \) is a skew-symmetric operator on \( \mathcal{L}^2(\mathbb{R}) \). From Eqns. (1) and (3) we get for each \( m \in \mathbb{R} \) skew-symmetric representations \( d\pi_m \) of \( \mathfrak{p}_2 \) on \( \mathcal{L}^2(\mathbb{R}) \):

\[
d\pi_m(\text{L}_01) = x\partial_x + \frac{1}{2}, \quad d\pi_m(\text{P}_0) = \frac{i}{2} \left( x + \frac{m^2}{x} \right), \quad d\pi_m(\text{P}_1) = -\frac{i}{2} \left( x - \frac{m^2}{x} \right)
\]

**Theorem IV:** \( d\pi_m \) integrates to a strongly continuous, unitary representation of \( \mathcal{P}_2 \) on \( \mathcal{L}^2(\mathbb{R}) \).

**Proof:** see Ref. I for an alternative proof independent of the Stone-von Neumann theorem.

Using the Segal-Bargmann transform we also obtain skew-symmetric representations of \( \mathfrak{p}_2 \) on \( \mathfrak{f}_1 \) : \( \mathfrak{P}_\pm = \text{P}_0 \pm \text{P}_1 \)

\[
2d\bar{\pi}_m(\text{L}_01) = z^2 - \frac{d^2}{dz^2}, \quad d\bar{\pi}_m(\mathfrak{P}_\pm) = \frac{1}{2} \left\{ \frac{d}{dz} \pm z \right\} \left( \frac{2\sqrt{2}m^2}{2d\bar{\pi}_m(\text{L}_01) - 1} \right)^{\frac{1}{2} \pm \frac{1}{2}}.
\]

We explicitly describe the action of \( \mathfrak{P}_2 \) in the massless representation \( I_{m=+1}^+ \) on \( \mathfrak{f}_1 \). An orthonormal basis for \( \mathfrak{f}_1 \) is:

\[
|k\rangle = \frac{z^k}{\sqrt{k!}} \quad (k = 0, 1, 2, \ldots).
\]

The vacuum vector is given by

\[
|0\rangle = 1.
\]
The action of the generators are:

\[
\begin{align*}
\tilde{d}\tilde{\pi}_0^+(L_0^1)|k\rangle &= \frac{1}{2} \sqrt{(k+1)(k+2)}|k+2\rangle - \frac{1}{2} \sqrt{k(k-1)}|k-2\rangle, \\
\tilde{d}\tilde{\pi}_0^+(P^-)|k\rangle &= \frac{1}{2} \sqrt{k+1}|k+1\rangle - \frac{1}{2} \sqrt{k-1}|k-1\rangle, \\
\tilde{d}\tilde{\pi}_0^+(P^+)|k\rangle &= 0.
\end{align*}
\]

(18)

7. The Case \( \tilde{\mathcal{P}}_2 = SO_0(1,1) \times T^2 \): Equivalence with the Wigner-Mackey Approach

We have \( \tilde{\mathcal{P}}_2 = SO_0(1,1) \times T^2 \) with

\[
T^2 = \left\{ \begin{bmatrix} 1 & 0 & a_0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} | (a_0, a) \in \mathbb{R}^2 \right\}, \quad SO_0(1,1) = \left\{ \begin{bmatrix} p_0 & p & 0 \\ p & p_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} | p_0 = \text{ch}\beta, p = \text{sh}\beta \right\}
\]

where \( \beta \in (-\infty, \infty) \). We consider the cases \( \mathbf{II}_{|m|}^\pm \), which are the real mass representations of \( SO_0(1,1) \times T^1 \), and for simplicity we take \( m = 1 \).

The momentum hyperboloid \( T^1 \) is defined as follows:

\[
T^1 = \{(p_0, p_1) \in \mathbb{R}^2 | p_0^2 - p_1^2 = 1\} = T^1_+ \cup T^1_-.
\]

For \( \mathbf{II}_{|m=1|}^+, \rho_p = (\pm 1, 0, 0) \) and \( \mathcal{O}_p \) is \( T^1_\pm \). The stability subgroup of \( \rho_p \) is \( M_p = I \). Thus \( V^\rho \) is trivial i.e. one-dimensional, and \( \tilde{D}^\rho(I) = \Pi \), so that

\[
\tilde{D}^\rho((m, a)) = D^\rho(I) \chi_p(a) = \chi_p(a).
\]

Futhermore, \( m(\Lambda, p) = 1 \) and \( \sigma(\Lambda, p) = 1 \). We have for the invariant measure on the momentum hyperboloid:

\[
d\mu(p) = d\beta = \frac{dp}{|p_0|}.
\]

The Hilbert space \( \mathcal{H} \) is, in this case, just \( \mathcal{L}^2(T^1_+, d\mu(p)) \) for \( \rho_p = (+1, 0, 0) \) and \( \mathcal{L}^2(T^1_-, d\mu(p)) \) for \( \rho_p = (-1, 0, 0) \). We restrict ourselves to the \( \rho_p = (+1, 0, 0) \) case, since the other case is handled in a completely similar way. For simplicity we denote this representation \( \tilde{U}^\rho(G) \) by \( \tilde{\pi}_1(G) \).

We have:

\[
d\beta = e^{-\beta}dx = \frac{1}{x}dx.
\]

Define the mapping \( \Pi \) from \( \mathcal{L}^2(\mathbb{R}) \) onto \( \mathcal{L}^2(T^1_+, d\mu) \) as follows:

\[
\mathcal{L}^2(T^1_+, d\mu) \ni (\Pi f)(\beta) = x^{\beta}f(x) = (e^\beta)^{\frac{1}{2}}f(e^\beta), \quad f(x) \in \mathcal{L}^2(\mathbb{R}).
\]

(19)

It’s easy to show that this mapping is an isometric isomorphism between the two Hilbert spaces \( \mathcal{L}^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R}, dx) \) and \( \mathcal{L}^2(T^1_+, d\mu) \) and that it intertwines the infinitesimal action \( d\pi_1 \) of \( \mathfrak{g} \) with the following action:

\[
d\pi_1(L_{01}) = \left(x\partial_x + \frac{1}{2}\right), \quad d\pi_1(P_0) = \frac{i}{2}\left(x + \frac{1}{x}\right), \quad d\pi_1(P_1) = -\frac{i}{2}\left(x - \frac{1}{x}\right).
\]

(20)
which agrees with the action on $\mathcal{L}^2(\mathbb{R})$ which we obtained from our proof of the Gelfand-Kirillov conjecture for $p_2$ (cf. Eq. (14)). Clearly these arguments generalize to any $m$ with $|m| \neq 0$. See Ref. I. for the other cases i.e. massless ones. This equivalence with the Wigner-Mackey method affords us with another proof that our representations of $p_2$ are (strongly continuous) unitary group representations, i.e. the representations which we obtain from the Gelfand-Kirillov conjecture for $p_2$ are integrable representations. (It also ensures that our method is exhaustive, i.e. we obtain from our method all unitary irreducible representations of $\mathcal{P}_2$.)

8. Higher Dimensional Cases: $\mathcal{P}_3 \approx SO_0(1,2)^- \ltimes T^3$

The Schrödinger representation of $\mathcal{W}_2(\mathbb{R})$ on $\mathcal{L}^2(\mathbb{R}^2)$ is obtained by the substitutions:

$$q_j \rightarrow ix_j, \quad p_j \rightarrow -i\partial x_j \quad (j = 1, 2).$$

(21)

We further let $M^2 \rightarrow m^2 1$, $C \rightarrow c 1$, where $m^2$, $c \in \mathbb{R}$. Using these substitutions and Eqns. (4) gives us the following skew-symmetric representation of $p_3$ on $\mathcal{L}^2(\mathbb{R}^2)$:

$$d\pi^{(m,c)}(P_0) = i\frac{\partial}{\partial x_1}(m^2 + x_1^2 + x_2^2), \quad d\pi^{(m,c)}(P_1) = i\frac{\partial}{\partial x_1}(m^2 + x_1^2 - x_2^2),$$

$$d\pi^{(m,c)}(P_2) = ix, d\pi^{(m,c)}(L_{01}) = x_1 \partial x_1 + \frac{1}{2}, \quad d\pi^{(m,c)}(L_{12} - L_{02}) = x_1 \partial x_2,$$

$$d\pi^{(m,c)}(L_{12} + L_{02}) = 2x_1^{-1}(ic - (x_1 \partial x_1 + \frac{1}{2})x_2 - \frac{1}{4}(m^2 + x_2^2)\partial x_2).$$

(22)

The momentum hyperboloid $T^2$ is defined as follows:

$$T^2 = \{(p_0,p_1,p_2) \in \mathbb{R}^3 | p_0^2 - p_1^2 + p_2^2 = m^2 \} = T^2_+ \cup T^2_-$$

where $m \in \mathbb{R}$ and $|m|$ is the radius of curvature of $T^2$. We consider the point $p = (|m|, 0, 0)$ for which we have $O_{-p} \cong SO_0(1,2)$ as stability subgroup and $O_{-p} \cong SO_0(1,2)/SO(2)$ is $T^2_+$, the upper branch of the hyperboloid. Similarly we have for the lower branch of the hyperboloid, $T^2_-$, the point $-p = (-|m|, 0, 0)$ with $O_p = SO(2)$ as stability subgroup and $O_p \cong SO_0(1,2)/SO(2)$ is $T^2_-$. The orbits $O_{\pm p}$ give positive and negative energy representations of $\mathcal{P}^3$ which we denote by $\pi^{(|m|,c,\pm)}(P_3)$, respectively, and where $c \in \mathbb{C}$ labels a character of $SO(2)$.

We now consider only the real mass, positive energy representations of $SO_0(1,2) \ltimes T^3$. Some coordinate systems on $T^2_+$ are: ($M = |m|$)

1. Pseudospherical coordinates:

$$p_0 = M \cosh \tau, \quad p_1 = M \sinh \tau \cos \phi, \quad p_2 = M \sinh \tau \sin \phi$$

($\tau \in [0, \infty), \phi \in [0, 2\pi]$.)

2. Horospherical (or Lemaitre) coordinates:

$$p_0 = M \cosh \alpha + \frac{e^\alpha}{2M}u^2, p_1 = e^\alpha u, \quad p_2 = -(M \sinh \alpha - \frac{e^\alpha}{2M}u^2), \quad (\alpha \in [0, \infty), u \in \mathbb{R}).$$

We take for the cross section $\gamma_+(p)$ in the Wigner-Mackey theory (+ for $T^2_+$ i.e. $m > 0$):

$$T^2_+ \ni p \rightarrow \gamma_+(p) = n(u)a(\alpha) := \begin{bmatrix} 1 + \frac{u^2}{2} & u & \frac{u^2}{2} \\ u & 1 & u \\ -\frac{u^2}{2} & -u & 1 - \frac{u^2}{2} \end{bmatrix} \begin{bmatrix} \cosh \alpha & 0 & \sinh \alpha \\ 0 & 1 & 0 \\ \sinh \alpha & 0 & \cosh \alpha \end{bmatrix} \in SO_0(1,2)$$
\[ \gamma_+(p) \hat{p} = \gamma_+(p) \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} M \cosh \alpha + \frac{u^2}{2} e^\alpha \\ Me^\alpha u \\ m \sinh \alpha - \frac{u^2}{2} e^\alpha \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ -p_2 \end{pmatrix} \in \mathbb{T}_+^2. \]

\( m(\Lambda, p) \) (Wigner rotation) is determined by the matrix equation

\[ \gamma_+(p) m(\Lambda, p) = \Lambda \gamma_+(\Lambda^{-1} p) \]

with \( \Lambda \in SO_0(1, 2) \) and \( p' = \Lambda^{-1} p, \quad p' \in \mathbb{T}_+^2 \) (\( m > 0 \)). The \( SO_0(1, 2) \) quasi-invariant measure on the momentum hyperboloid is

\[ d\mu(p) = \frac{dp_1 dp_2}{|p_0|} \]

and \( \sigma(\Lambda p) \), for \( \Lambda \in SO_0(1, 2) \) and \( p \in \mathbb{T}_+^2 \), is determined by

\[ d\mu(\Lambda p) = \sigma(\Lambda, p)d\mu(p). \]

The Hilbert space for \( \hat{p} = (|m|, 0, 0) \) is

\[ \mathcal{H}_+ = L^2(\mathbb{T}_+^2, d\mu(p)). \]

Now let \( y = Me^{-\alpha} \) then

\[ p_0 = \frac{M}{2} (e^{\alpha} + e^{-\alpha}) + \frac{e^{\alpha}}{2M} u^2 = \frac{1}{2y} (M^2 + y^2 + u^2), \]

\[ p_1 = Mu, \quad p_2 = -\frac{M}{2} (e^{\alpha} - e^{-\alpha}) + \frac{e^{\alpha}}{2M} u^2 = \frac{1}{2y} (y^2 + u^2 - M^2). \]

Next let

\[ q_1 = \frac{M^2}{y}, \quad q_2 = \frac{Mu}{y} = \frac{1}{M} q_1 u. \]

Then we have

\[ p_0 = \frac{1}{2q_1} \left\{ q_1^2 + M^2 + q_2^2 \right\} = P_0, \quad p_1 = q_2 = P_2, \]

\[ p_2 = \frac{y}{2M^2} \left( M^2 - \frac{M^4}{y^2} + \frac{M^2 u^2}{y^2} \right) = \frac{1}{2q_1} \left\{ M^2 - q_1^2 + q_2^2 \right\} = P_1 \]

which should be compared with Eqns. (4). In terms of the \( x_i (i = 1, 2) \) defined by \( q_i = ix_i \) we have:

\[ p_0 = -\frac{i}{2x_1} (M^2 - x_1^2 - x_2^2), \quad p_1 = -\frac{i}{2x_1} (M^2 + x_1^2 - x_2^2), \quad p_2 = ix_2. \]  \hspace{1cm} (23)

The \( SO(1, 2) \) quasi-invariant measure on \( \mathbb{T}_+^2 \) in Lemeitre coordinates is, up to inessential constants, given by: \( d\mu(p) = \frac{dudv}{y^2} \). In the same way as in the \( \mathbb{P}_2 \) case we establish \( \mathcal{H}_+ = L^2(\mathbb{T}_+^2, d\mu(p)) \cong L^2(\mathbb{R}^2) \) which is the Hilbert space of the Schrödinger representation.

9. Remarks on the \( \mathbb{P}^4 \) Case:

For this case we outline a method of proof which should establish by explicit contraction the isomorphism

\[ \mathcal{D}(p_4) \cong \mathcal{D}_{(4,2)}(\mathbb{R}) \]

i.e. the validity of the Gelfand-Kirillov conjecture for the Poincaré group of Minkowski spacetime. The proof of this and also of the equivalence with the Wigner-Mackey approach makes use of a mapping similar to that used in [1] for demonstrating the equivalence of the \( P_0 \) and \( P_2 \) induced representations of \( SU(2,2) \). (\( P_0 \) and \( P_2 \) are respectively the minimal and maximal proper parabolic subgroups of \( SU(2,2) \).) In this correspondence, spin is described using a pair of complex numbers [11]. Details for this case will be given at a later time. It should be very similar to the just treated \( \mathbb{P}^3 \) case. (As stated above we demonstrated something much weaker in [1], namely only that the representations of \( \mathbb{P}^4 \) can be realized on \( L^2(\mathbb{R}^3) \times V^s \) for appropriate \( s \).)
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