Abstract—We propose a verification-based Interval-Passing (IP) algorithm for iteratively reconstruction of nonnegative sparse signals using parity check matrices of low-density parity check (LDPC) codes as measurement matrices. The proposed algorithm can be considered as an improved IP algorithm by further incorporation of the mechanism of verification algorithm. It is proved that the proposed algorithm performs always better than either the IP algorithm or the verification algorithm. Simulation results are also given to demonstrate the superior performance of the proposed algorithm.

Index Terms—Compressed sensing, interval-passing algorithm, verification algorithm, sparse measurements, low-density parity-check (LDPC) codes.

I. INTRODUCTION

COMPRESSED Sensing (CS) problem in the noiseless setting considers the estimation of an unknown and sparse signal vector \( x \in \mathbb{R}^N \) from a vector of linear observations \( y \in \mathbb{R}^M \), i.e.,

\[
y = H \cdot x,
\]

where \( H \in \mathbb{R}^{M \times N} \) is often referred as measurement matrix or sensing matrix and only a small number (the sparsity index), \( K << N \), of elements of \( x \) are non-zero. The set containing the positions of these elements is known as the support set, defined as \( S = \{ i \in [1, N] : x_i \neq 0 \} \), with cardinality \( |S| = K \). The sparsity is often defined as \( k = \frac{K}{N} \).

The solution to this system of equations is known to be given by the vector that minimizes \( \| x_0 \|_0 \) (\( \ell_0 \)-norm) subject to \( y = H \cdot x_0 \), which is a non-convex optimization problem. In [1], it was established that the vector \( x_1 \) with minimum \( \ell_1 \)-norm subject to \( y = H \cdot x_1 \) coincides with \( x_0 \) whenever the measurement matrix satisfies the well-known restricted isometry property (RIP) condition.

The connection between CS and Channel Coding (CC) has been explored extensively in recent years [2]–[6]. The sensing process in CS, i.e., \( y = H \cdot x \), is very similar to encoding in CC if the \( H \) is considered as a generator matrix while the reconstruction process in CS looks similar to decoding in CC. Therefore, the sensing process in CS is often referred as encoding and the recovery process as decoding.

In general, the sensing matrix in compressed sensing can be either dense or sparse according to the density of the nonzero entries in the sensing matrix. As a special class of sparse matrices, the parity-check matrices of Low-Density Parity-Check (LDPC) codes have the favorite feature of a constant (average) number of non-zero entries in each row when the code length may increase without bound. Due to the potential advantages on both encoding and decoding, sparse sensing matrices have gained increasing interest in CS. With the bipartite-graph representation of any sparse measurement, various message-passing algorithms originally developed for decoding sparse-graph codes have been introduced for reconstruction of sparse signals [4], [5], [7]–[9]. The connection between LDPC codes and CS was addressed in detail in [2].

Among various message-passing algorithms, the verification algorithm was first introduced for compressed sensing in [7]. It is essentially identical to the earlier idea of verification decoding of packet-based LDPC codes [10] as recognized by Zhang and Pfister [4]. This observation allowed a rigorous analysis of the verification decoding for compressed sensing via density evolution [6]. The verification algorithm can be efficiently implemented with the complexity \( O(N) \). In the literature, there are two categories of verification algorithms: node-based and message-based [4], [6]. In general, the node-based algorithms have better performance and lower complexity.

For non-negative sparse signals, a new message-passing algorithm, referred as the Interval-Passing (IP) algorithm, was proposed in [11], which can perform better than the verification algorithm for some LDPC-based sensing matrices [12]. The IP algorithm can be implemented efficiently with the complexity \( O(N(\log(N))^2 \log(K)) \).

In this paper, we focus on the two iterative message-passing reconstruction algorithms, i.e., the IP algorithm and the node-based verification algorithm. As the basic mechanism is somewhat different for the two message-passing algorithms, it remains open if one can find a message-passing algorithm taking advantages of both. Indeed, we propose a novel verification-based interval-passing algorithm for recovery of nonnegative sparse signals, in which the mechanism of verification decoding can be concisely incorporated.

II. PRELIMINARIES

A. LDPC-Based Sensing Matrix and Sensing Graph

In this paper, we consider a special class of sparse measurement matrices, the parity check matrices of binary LDPC codes.

It is well known that an LDPC code can be well defined by the null space of its sparse parity-check matrix. For any given sparse parity-check matrix, it can be efficiently represented

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by a bipartite graph. A bipartite graph has two sets of nodes
representing the code symbols and the parity-check equations.
These are called “variable” and “check” nodes, respectively.
An edge connects a variable node $n$ to a check node $m$ if and
only if $n$ appears in the parity-check equation associated with
$m$.

Similarly, a sparse sensing matrix has its own bipartite graph,
which is referred to as a sensing graph. Due to the close
relation between CS and CC [2], it is very helpful to describe
iterative reconstruction algorithms over the sensing graph.

Let $G = (V \cup C, E)$ be the bipartite graph of a sensing
matrix $H = [h_{m,n}]$, where the set of variable nodes $V$
represents the signal components (or columns of $H$) and the
set of check nodes $C$ represents the set of sensing constraints
(or rows of $H$) satisfied by the signal components.

Throughout this paper, we denote the set of variable nodes
that participate in check $m$ by $N'(m) = \{n : h_{m,n} \neq 0\}$. Similarly, we denote the set of checks in which variable node
$n$ participates as $M(n) = \{m : h_{m,n} \neq 0\}$.

### B. Interval-Passing Algorithm

For nonnegative sparse signals, the IP algorithm iteratively
computes the simple bounds of each signal component, i.e.,
both lower and upper bounds. With sufficient expansion guarantee for the sensing graph, it was proved in [11] that either
the lower or the upper value can converge to the same value
for each variable node. Therefore, it can successfully estimate
the sparse signal with a well-designed sensing graph.

The IP algorithm can be described in a rigorous message-
passing form. It consists of two alternative update rules for
messages along edges, one for check nodes and the other for
variable nodes. In general, there are two bounding messages
for any directed edge from $m$ to $n$, i.e., $L_{m\to n}$ for the lower
bound and $U_{m\to n}$ for the upper bound.

Initially, $L_{m\to n} = 0, U_{m\to n} = y_m, \forall m \in [1,M], n \in [1,N]$. Then, two alternative update rules can be described
as follows.

**Check node update:**

$$L_{m\to n} = \max \left\{ 0, y_m - \sum_{n' \in N'(m) \setminus n} U_{n'\to m} \right\}$$

$$U_{m\to n} = y_m - \sum_{n' \in N'(m) \setminus n} L_{n'\to m}.$$  

**Variable node update:**

$$L_{n\to m} = \max_{m' \in M(n)} L_{m'\to n} \triangleq L_n,$$

$$U_{n\to m} = \min_{m' \in M(n)} U_{m'\to n} \triangleq U_n.$$  

When it converges or the maximum number of iterations
reaches, the IP algorithm finally outputs the decision vector
$x = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N]$, where $\hat{x}_n = L_n, \forall n \in [1,N]$.

### C. Verification Algorithm

For compressed sensing, the node-based verification algo-
rithm works iteratively in a progressive way for recovering the
signal components (variable nodes). Once a variable node is
recovered, it is attached to the state of verified; otherwise,
it is unverified. At the beginning of decoding, all the
variable nodes are initialized as unverified. In each decoding
iteration, it employs three simple rules as follows.

- **S1:** If a measurement at “check node” $m$ is zero, all the
  variable nodes neighbor to $m$ are verified as a zero value
  and labeled as verified.
- **S2:** For a degree-1 measurement $y_m$, the variable node
  neighbor to $m$ is verified as the value of $y_m$ and labeled as
  verified.
- **S3:** For a variable node $n$ subject to which there are
  $P \geq 2$ identical measurements ($y_m = \eta, \forall m \in M_0(n)$
  with $|M_0(n)| = P$ for some $\eta \in \mathbb{R}$), all variable nodes
  neighboring a subset of these check nodes $M_0(n)$ (not
  all of them) are verified with the value zero and labeled as
  verified.

Then, the verified variable node propagates its state to the other
check nodes in its neighborhood so that these check nodes can
remove the contribution of this variable node from their
respective measurement and try to infer, in the next round, the
values of the remaining variable nodes connected to them.

The above three rules in their rigorous form were reported
in [6]. The only difference here lies in the rule S3. In [6],
it includes the mechanism that the unique variable node
neighboring these check nodes is verified with the common
value of the check nodes. This additional mechanism, however,
is not required considering the rule S2. This rigorous form
of S3 may not be fully recognized in [4], [9] and the authors
in [9] rediscovered this mechanism to cope with cycles of
length-4 in the sensing graph.

In what follows, we provide a precise implementation of the
above three rules thanks to various formulations in [6], [9].
The node-based verification algorithm consists of two alternative
update rules for node-based messages, one for check nodes
and the other for variable nodes. The state of variable node $n$
is denoted as $s_n$. Then, $s_n = 1$ is used to denote the variable
node $n$ in a verified state, and $s_n = 0$ means unverified.
For an efficient implementation of the rule S3, we introduce
the state of coincidence [9] with $s_n = -1$ for identifying the
variable node $n$ when it accepts $P \geq 2$ identical measurements
and $\xi_m = 1$ for signalling the check node $m, m \in M_0(n)$.
Let $\delta(x) = 1$ if $x = 0$ and zeros otherwise.

Initially, $\hat{x}_n = 0, s_n = 0, \forall n \in [1,N]$; $\xi_m = 0, \forall m \in [1,M]$. Then, two alternative update rules for the node-based
verification algorithm can be stated as follows.

**Check node update:**

$$\eta_m = y_m - \sum_{n \in N(m)} \delta(s_n - 1)\hat{x}_n,$$

$$d_m = |N(m)| - \sum_{n \in N(m)} \delta(s_n - 1),$$

$$N_0(m) = \{n : s_n = 0, n \in N(m)\}.$$  

- **IF** ($\xi_m = 1$) and ($|N_0(m)| \geq 1$)

  $$\forall n \in N_0(m) : \hat{x}_n = 0, s_n = 1;$$

  (Signal release) : $\xi_m = 0.$
Variable node update (only for $n : s_n \neq 1$):
\[ \mathcal{M}_\eta(n) = \{ m : m \in \mathcal{M}(n), \eta_m = \eta \} . \]

- IF ($\exists m \in \mathcal{M}(n), d_m = 1$)
  \[ \hat{x}_n = \eta_m, \quad s_n = 1. \]
- IF ($\exists m \in \mathcal{M}(n), \eta_m = 0$)
  \[ \hat{x}_n = 0, \quad s_n = 1. \]
- IF ($\exists \eta \in \mathbb{R}, s.t.|\mathcal{M}_\eta(n)| \geq 2$) and ($s_n = 0$)
  \[ \begin{align*}
  s_n &= -1, \\
  \xi_m &= 1, \forall m \in \mathcal{M}_\eta(n).
  \end{align*} \]

If all the variable nodes are verified, the algorithm outputs the recovered vector $\hat{x}$.

D. Remarks

As shown, both the IP algorithm and the verification algorithm can be described in the form of message-passing decoding. Although both algorithms require the expansion property of the sensing graph, their mechanisms in recovering the signal are somewhat different and simulations show their difference in recovery patterns.

Therefore, it is natural to ask if there is a message-passing algorithm in favor of the recovery mechanisms of both algorithms.

III. VERIFICATION-BASED INTERVAL-PASSING ALGORITHM

In this section, we can answer the question in an affirmative manner. In what follows, we propose a verification-based interval-passing algorithm, which keeps all the messages appeared in the IP algorithm. Besides, it also introduces a new kind of message, $\eta_m$, attached to each check node $m$, which do the same role as in the verification algorithm.

Initially, $s_n = 0, \hat{x}_n = 0, \forall n \in [1, N]$ and $\xi_m = 0, \forall m \in [1, M]$. The messages along any edge from variable node $n$ to check node $m$ are initialized as $L_{n \rightarrow m} = 0, U_{n \rightarrow m} = y_m$.

Then, the new algorithm iteratively does the following update operations in an alternative manner.

Check node update:
\[
\begin{align*}
L_{m \rightarrow n} &= \max \left\{ 0, y_m - \sum_{n' \in \mathcal{N}(m) \setminus n} U_{n' \rightarrow m} \right\}, \\
U_{m \rightarrow n} &= y_m - \sum_{n' \in \mathcal{N}(m) \setminus n} L_{n' \rightarrow m}, \\
\eta_m &= y_m - \sum_{n \in \mathcal{N}(m)} \delta(s_n - 1)\hat{x}_n, \\
\mathcal{N}_0(m) &= \{ n : s_n = 0, n \in \mathcal{N}(m) \}.
\end{align*}
\]

Variable node update (only for $n : s_n \neq 1$):
\[
\forall n \in \mathcal{N}_0(m) : L_{m \rightarrow n} = U_{m \rightarrow n} = 0, \\
\hat{x}_n = 0, s_n = 1; \\
(Signal \ release) : \xi_m = 0.
\]

1) IF ($\exists \eta \in \mathbb{R}, s.t.|\mathcal{M}_\eta(n)| \geq 2$) and ($s_n = 0$)
   \[ \begin{align*}
   s_n &= -1, \\
   \xi_m &= 1, \forall m \in \mathcal{M}_\eta(n).
   \end{align*} \]

2) IF ($L_n = U_n$)
   \[ \begin{align*}
   \hat{x}_n &= L_n, \\
   s_n &= 1.
   \end{align*} \]

When it converges or the maximum number of iterations reaches, the algorithm finally outputs the decision vector $\hat{x} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N]$, where $\hat{x}_n = L_n, \forall n \in [1, N]$.

Theorem 1: For recovery of nonnegative sparse signals with continuously-distributed nonzero entries, the new algorithm performs always better than either the IP algorithm or the verification algorithm.

Proof: Given a nonnegative sparse signal $x$ and any measurement vector $y = Hx$. Let $\mathcal{V}_l = \{ n : s_n = 1 \}$ denote the set of verified nodes at the $l$th decoding iteration. Hence, it is enough to prove that $\mathcal{V}_l^{new} \supseteq \mathcal{V}_l^{VB}$ and $\mathcal{V}_l^{new} \supseteq \mathcal{V}_l^{IP}$, where the subscripts $new, VB$ and $IP$ mean the new algorithm, the verification algorithm and the IP algorithm, respectively.

Firstly, the probability of false verification has been demonstrated to be zero for the verification algorithm if the nonzero entries of the source signal are continuously distributed \[9\]. For the IP algorithm, it always output the correct decision $\hat{x}_n = x_n$ if it converges at some variable node $n$ since $L_n \leq x_n \leq U_n$ always holds in the decoding process. Hence, it follows that the probability of false verification is zero for the new algorithm. Now, it is clear that $\mathcal{V}_l^{new} \supseteq \mathcal{V}_l^{IP}$ as no error propagation occurs.

Secondly, the two rules (S1/S3) employed by the verification algorithm still hold for the new algorithm. Hence, it is enough to show that the rule S2 still holds. Indeed, whenever $d_m = 1$ for any check node $m$, it means that there are $|\mathcal{N}(m)| - 1$ verified variable nodes neighbouring to $m$. Let us denote $n^*$ as the remaining unverified node (only one) neighbouring to $m$. Hence, $\forall n \in \mathcal{N}(m)/n^*, L_n = U_n$ as it is verified. With the update operations at check node $m$, $L_n^{\ast} = U_n^{\ast}$ holds surely. It follows that $\mathcal{V}_l^{new} \supseteq \mathcal{V}_l^{VB}$.

Comparing the proposed algorithm with the IP algorithm, it is straightforward to show that the complexity of the proposed algorithm scales the same as that of the IP algorithm. For the quasi-cyclic (QC) LDPC-based sensing matrices, it may be implemented in a hardware-friendly manner as shown in decoding of QC-LDPC codes \[13\].
IV. SIMULATION RESULTS

In this section, we provide the simulation results for three iterative reconstruction algorithms. The parity-check matrices of binary LDPC codes are adopted as sparse measurement matrices. The number of maximum decoding iterations is set to 50 for all simulations. In all simulations, sparse signals are generated as follows. First, the support set of cardinality $K$ is randomly generated. Then, each nonzero signal element is drawn according to a standard Gaussian distribution. If the generated element is negative, it is simply inverted for ensuring a nonnegative sparse signal model with continuously-distributed nonzero entries. For getting a stable point in all figures (except that the probability of correct reconstruction is approaching 1), at least 100 reconstruction fails are counted.

Firstly, the binary MacKay-Neal LDPC matrix [14] of size $M \times N = 252 \times 504$ is adopted. The probability of correct reconstruction is plotted against the sparsity of the source signals. As shown in Fig. 1, the new algorithm performs always better than either the IP algorithm or the verification algorithm. It should be noted that this sensing matrix is constructed without length-4 cycles.

Secondly, we adopt the QC-LDPC code from the IEEE 802.16e standard, referred as the WiMax code. The parity-check matrix is of size $M \times N = 384 \times 768$ and it has cycles of length-4. The same phenomena has been observed in Fig. 2 which is clearly different with the results reported in [12], where the verification algorithm performs poorly due to lack of mechanism to cope with cycles of length-4.

V. CONCLUSION

We have proposed a new message-passing reconstruction algorithm for nonnegative sparse signals. This new message-passing algorithm, as a combination of both the IP algorithm and the verification algorithm, has been shown to perform better than either the IP algorithm or the verification algorithm for LDPC-based sensing matrices. The complexity of the new algorithm scales the same as that of the IP algorithm.

APPENDIX

RESPONSE TO ONE OF THE REVIEWERS FOR THE IMPLEMENTATION OF THE S3 RULE

One of the reviewers has the following comments. “The verification algorithm discussed is only equivalent to the S3 rule if the edge weight of the sensing graph comes from a continuous distribution (or its sampled version with infinitely many elements). Otherwise, it is very easy to come up with examples that the algorithm proposed there does not perform like the S3 rule. Indeed, it is very easy to show that in order to capture the S3 rule, one needs a memory state with 2 levels (variable nodes knowing about other variable nodes and check nodes knowing about other check nodes). The only way that you can implement the S3 rule with one level of memory (message passing) is to have continuous distribution on the edges of the sensing graph.”

In this paper, we always assume a binary edge weight for LDPC-based sensing graph. Consider the case of Fig. 3 with LDPC-based sensing graph. For the variable node $n = 2$, there are $P = 3$ identical measurements $\eta_1 = \eta_2 = \eta_3 = \eta$. It is clear that the verification algorithm proposed in Section-II can implement the S3 rule precisely if the variable node $n = 2$ is firstly visited. As shown, $M_q(n) = \{1, 2, 3\}$. Then, according
to the S3 rule, the variables $n = 1, 3, 4$ are verified with the value zero. This can be precisely implemented. However, it requires that the variable nodes are visited according to a descending order of the number of identical measurements inherited to each variable node. In practice, this can be omitted as the occurrence of $P > 2$ identical measurements is rare.

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