THE BROWN-YORK MASS OF REVOLUTION SURFACE IN ASYMPTOTICALLY SCHWARZSCHILD MANIFOLD

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ABSTRACT. In this paper, we will show that the limit of the Brown-York mass of a family of convex revolution surfaces in an asymptotically Schwarzschild manifold is the ADM mass.

1. Introduction

Throughout this paper, we will denote \((\mathbb{R}^3, \delta_{ij})\) as the 3-dimensional Euclidean space, \(x^1, x^2, x^3\) as the standard coordinates, \(r\) and \(\partial\) as the Euclidean distance and the standard derivative operator on \(\mathbb{R}^3\) respectively. Let us first recall some definitions. First of all, we will adopt the following definition of asymptotically flat manifolds.

Definition 1.1. A complete three dimensional manifold \((M, \lambda)\) is said to be asymptotically flat (AF) of order \(\tau\) (with one end) if there is a compact subset \(K\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B_R(0)\) for some \(R > 0\) and in the standard coordinates in \(\mathbb{R}^3\), the metric \(\lambda\) satisfies:

\[
\lambda_{ij} = \delta_{ij} + \sigma_{ij}
\]

with

\[
|\sigma_{ij}| + r |\partial \sigma_{ij}| + r^2 |\partial \partial \sigma_{ij}| + r^3 |\partial \partial \partial \sigma_{ij}| = O(r^{-\tau}),
\]

for some constant \(1 \geq \tau > \frac{1}{2}\).

A coordinate system of \(M\) near infinity so that the metric tensor in this system satisfy the above decay conditions is said to be admissible. In such a coordinate system, we can define the ADM mass as follows.

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**Definition 1.2.** The Arnowitt-Deser-Misner (ADM) mass (see [1]) of an asymptotically flat manifold \((M, \lambda)\) is defined as:

\[
m_{\text{ADM}}(M, \lambda) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (\lambda_{ij,i} - \lambda_{ii,j}) \nu^j d\Sigma^0_r,
\]

where \(S_r\) is the Euclidean sphere, \(d\Sigma^0_r\) is the volume element induced by the Euclidean metric, \(\nu\) is the outward unit normal of \(S_r\) in \(\mathbb{R}^3\) and the derivative is the ordinary partial derivative.

We always assume that the scalar curvature is in \(L^1(M)\) so that the limit exists in the definition. In [2], Bartnik showed that the ADM mass is a geometric invariant. More precisely, he proved the following theorem (see [2, Proposition 4.1] for a more general setting):

**Theorem 1.1.** Suppose \((M, \lambda)\) is an AF manifold with scalar curvature \(R(\lambda) \in L^1(M)\). Let \(\{D_k\}_{k=1}^\infty\) be an exhaustion of \(M\) by closed sets such that the set \(S_k = \partial D_k\) are connected \(C^1\) surfaces without boundary in \(\mathbb{R}^3\) such that

\[
r_k = \inf \{|x|, x \in S_k\} \to \infty \text{ as } k \to \infty
\]

\[
r_k^{-2} \text{Area}(S_k) \text{ is bounded as } k \to \infty.
\]

Then

\[
m_{\text{ADM}}(M, \lambda) = \lim_{k \to \infty} \frac{1}{16\pi} \int_{S_k} (\lambda_{ij,i} - \lambda_{ii,j}) \nu^j d\Sigma^0_r.
\]

That is, the ADM mass is independent of the sequence of \(\{S_k\}\).

On the other hand, there have been many studies on the relation between the ADM mass of an AF manifold and the so called quasi-local mass. Let us recall the definition of the Brown-York quasi-local mass. Suppose \((\Omega, \mu)\) is a compact three dimensional manifold with smooth boundary \(\partial \Omega\), if moreover \(\partial \Omega\) has positive Gauss curvature, then the Brown-York mass of \(\partial \Omega\) is defined as (see [5, 6]):

**Definition 1.3.**

\[
m_{\text{BY}}(\partial \Omega) = \frac{1}{8\pi} \int_{\partial \Omega} (H_0 - H) d\sigma
\]

where \(H\) is the mean curvature of \(\partial \Omega\) with respect to the outward unit normal and the metric \(\mu, d\sigma\) is the volume element induced on \(\partial \Omega\) by \(\mu\) and \(H_0\) is the mean curvature of \(\partial \Omega\) when embedded in \(\mathbb{R}^3\).

The existence of an isometric embedding in \(\mathbb{R}^3\) for \(\partial \Omega\) was proved by Nirenberg [15], the uniqueness of the embedding was given by [11, 17, 16], so the Brown-York mass is well-defined.
It can be proved that the Brown-York mass and the Hawking quasi-local mass of the coordinate spheres tends to the ADM mass in some AF manifolds, see [6, 10, 4, 3, 20, 8], and even of nearly round surfaces [19]. It is therefore natural to ask whether the quasi-local mass of a more general class of surfaces tends to the ADM mass.

In this paper, we will consider a special class of AF manifolds, called asymptotically Schwarzschild manifolds, which is defined as follows:

**Definition 1.4.** \((N, \tilde{g})\) is called an asymptotically Schwarzschild manifold if \(N = \mathbb{R}^3 \setminus K\), \(K\) is a compact set containing the origin, and

\[
\tilde{g}_{ij} = \phi^4 \delta_{ij} + b_{ij}, \quad \phi = \left(1 + \frac{2m}{r}\right), \quad m > 0,
\]

where \(|b_{ij}| + r|\partial b_{ij}| + r^2|\partial^2 b_{ij}| + r^3|\partial^3 b_{ij}| = O\left(r^{-2}\right)\).

Clearly, \((N, \tilde{g})\) is an AF manifold. For \(b = 0\), \((N, \tilde{g})\) is called a Schwarzschild manifold. In this case, we always denote \(\tilde{g}\) as \(g\). Note that the scalar curvature of \((N, g)\) is zero [13] (page 283) and that of \((N, \tilde{g})\) is in \(L^1(N)\), so in both cases the ADM mass is well defined.

We will study the limiting behaviors of Brown-York mass on some family of convex revolution surfaces in an asymptotically Schwarzschild manifold. We will first prove the following:

**Theorem 1.2.** Let \((N, g)\) be a Schwarzschild manifold, \(\{D_a\}_{a>0}\) be an exhaustion of \(N\). Suppose \(\{S_a = \partial D_a\}_{a>0}\) is a family of closed convex surfaces of revolution in \((\mathbb{R}^3, \delta_{ij})\) with the rotation axis passing through the origin, satisfying the following conditions:

(i)

\[
K \geq \frac{C_1}{a^2}
\]

where \(K\) is the Gaussian curvature of \(S_a\) with induced Euclidean metric, \(C_1 > 0\) is independent of \(a\).

(ii)

\[
0 < \bar{H} \leq \frac{C_2}{a}
\]

where \(\bar{H}\) is the mean curvature of \(S_a\) with induced Euclidean metric, \(C_2 > 0\) is independent of \(a\).

(iii)

\[
C_3 a \leq \min_{x \in S_a} r(x) \leq \max_{x \in S_a} r(x) \leq C_4 a
\]

where \(C_i > 0\) are independent of \(a\). Then

\[
\lim_{a \to \infty} m_{BY}(S_a) = m_{ADM}(N, g).
\]
One example of surfaces satisfying the conditions in Theorem 1.2 (and also Theorem 1.3 below) is the family of ellipsoids:

\[ S_a = \left\{ (x^1)^2 + (x^2)^2 + \frac{(x^3)^2}{4} = a^2 \right\}, \]

which is not nearly round [19]. In contrast, the Hawking mass of this family of ellipsoids in \((N, g)\) does not tend to the ADM mass of \((N, g)\), indeed one can check that the Hawking mass of this family tends to negative infinity as \(a \to \infty\).

**Remark 1.1.** The above conditions (i) and (ii) imply that the principal curvature \(\lambda\) of \(S_a\) in \((\mathbb{R}^3, \delta)\) satisfy

\[ \frac{C_1}{C_2 a} \leq \lambda \leq \frac{C_2}{a}, \]

for any \(a\). For if \(0 < \lambda_1 \leq \lambda_2\) are the principal curvatures, then (1.6) implies

\[ \lambda_2 \leq \frac{C_2}{a}. \]

Together with (1.5),

\[ \lambda_1 \geq \frac{C_1}{\lambda_2 a^2} \geq \frac{C_1}{C_2 a}. \]

**Remark 1.2.** By condition (i) of Theorem 1.2 and the Gauss-Bonnet theorem, the Euclidean area of \(S_a\) is of order \(O(a^2)\).

Then we will generalize the result of Theorem 1.2 to an asymptotically Schwarzschild manifold:

**Theorem 1.3.** Let \((N, \tilde{g})\) be an asymptotically Schwarzschild manifold and \(S\) be a \(C^{6,\alpha}\) \((0 < \alpha < 1)\) closed convex revolution surface parametrized by

\[ (\bar{w}(\varphi) \cos \theta, \bar{w}(\varphi) \sin \theta, \bar{h}(\varphi)), \quad 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq l. \]

Then there exists \(\varepsilon > 0\) such that for any family of \(C^{5,\alpha}\) closed convex revolution surfaces \(S_a\) in \((\mathbb{R}^3, \delta)\) satisfying (1.5)-(1.7) and is parametrized by

\[ (aw_a(\varphi) \cos \theta, aw_a(\varphi) \sin \theta, ah_a(\varphi)), \quad 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq l \]

such that

\[ |w_a - \bar{w}|_{C^4} + |h_a - \bar{h}|_{C^4} \leq \varepsilon \quad \text{for } a >> 1, \]

we have

\[ \lim_{a \to \infty} m_{BY}(S_a) = m_{ADM}(N, \tilde{g}). \]
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From this result, one has

**Corollary 1.1.** Let \((N, \tilde{g})\) be an asymptotically Schwarzschild manifold. Let \(\{S_i\}\) be a family of \(C^7\) closed convex revolution surfaces in \((\mathbb{R}^3, \delta)\) satisfying (1.5)-(1.7) and is parametrized as:

\[
(a_iw_i(\varphi) \cos \theta, a_iw_i(\varphi) \sin \theta, a_i h_i(\varphi)), \quad 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq l
\]

for some constant \(l > 0\), here \(a_i\) are positive numbers with \(\lim_{i \to \infty} a_i = +\infty\). If there is a constant \(c\) such that

(1.13) \[|w_i|_{C^7} + |h_i|_{C^7} \leq c\]

for all \(i\), then there is a subsequence \(\{S_{i_k}\}\) of \(\{S_i\}\) such that

\[
\lim_{k \to \infty} m_{BY}(S_{i_k}) = m_{ADM}(N, \tilde{g}).
\]

This paper is organized as follows. In section 2, we will prove Theorem 1.2. Theorem 1.3 will be proved in section 3.

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2. Proof of Theorem 1.2

For simplicity, from now on to the end of the paper, we use \(O(a^{-1})\) to denote a quantity which is bounded by \(Ca^{-1}\) for some constant \(C\) independent of \(a\) as \(a\) is big enough. Similar for \(O(1)\), \(O(a^{-2})\) and \(O(a^{-3})\). We will first compute the mean curvature of \(S_a\) in \((N, g)\) and of the embedded surface of the Euclidean space respectively.

From the assumptions of \(S_a\), we can assume that \(S_a\) is parametrized by

(2.1) \[(aw_a(\varphi) \cos \theta, aw_a(\varphi) \sin \theta, ah_a(\varphi)), \quad 0 \leq \varphi \leq l_a, 0 \leq \theta \leq 2\pi,\]

\(w_a(\varphi), h_a(\varphi)\) being smooth functions for \(\varphi \in [0, l_a]\) (i.e. \(w_a, h_a\) can be extended smoothly on a slightly larger interval ). Moreover,

(i)

\[
h_a(0) > h_a(l_a)
\]

(2.2) \[C_3 \leq \sqrt{w_a^2 + h_a^2} \leq C_4
\]

\(w_a > 0\) on \((0, l_a)\),

(ii) The generating curve \((w_a(\varphi), h_a(\varphi))\) is parameterized by arc length. i.e.

(2.3) \[w_a^2 + h_a^2 = 1.
\]
(iii) $w_a$ is anti-symmetric about 0 and $l_a$, $h_a$ is symmetric about 0 and $l_a$, i.e.

\begin{align}
&w_a(-\varphi) = -w_a(\varphi), \quad w_a(l_a + \varphi) = -w_a(l_a - \varphi), \\
h_a(-\varphi) = h_a(\varphi), \quad h_a(l_a + \varphi) = h_a(l_a - \varphi) \quad \text{for } \varphi \in [0, \varepsilon).
\end{align}

This implies

\begin{align}
w_a(0) = w_a(l_a) = h_a'(0) = h_a'(l_a) = 0.
\end{align}

Since $S_a$ is convex in $(\mathbb{R}^3, \delta)$ and the Gaussian curvature $\bar{K}$ of $S_a$ with the induced metric $d\bar{s}^2$ is

\begin{align}
\bar{K} = \frac{h_a''(w_a' h_a'' - w_a'' h_a')}{a^2 w_a} \quad \text{for } \varphi \in (0, l_a).
\end{align}

So $h_a' < 0$ for $\varphi \in (0, l_a)$ by (2.2).

Let $\phi_a$ be the function $\phi$ restricted on $S_a$, note that in $(\varphi, \theta)$ coordinates, $\phi_a = \phi_a(\varphi)$ is independent of $\theta$. We have the following lemma:

**Lemma 2.1.** The functions $\frac{w_a}{h_a'}$ and $\frac{\phi_a'}{h_a'}$ can be extended continuously to the whole $[0, l_a]$. Moreover there exists a constant $C$ independent of $a$ such that

\begin{align}
\left| \frac{w_a}{h_a'} \right| \leq C, \quad \left| \frac{\phi_a'}{h_a'} \right| \leq \frac{C}{a}
\end{align}

for all $a$.

**Proof.** We first show that the limits

\begin{align}
\lim_{\varphi \to 0} \frac{w_a}{h_a'}, \lim_{\varphi \to l_a} \frac{w_a}{h_a'}
\end{align}

exist and are bounded by $C$.

The Gaussian curvature $\bar{K}$ of the point $(0, 0, ah_a(0))$ on $S_a$ with induced Euclidean metric is equal to

\begin{align}
\bar{K} = \frac{h_a''(0)^2}{a^2}.
\end{align}

This can be shown by the fact that for an arc-length parametrized plane curve $(w_a(\varphi), h_a(\varphi))$, its curvature is given by $-w_ah_a' + h_a''w_a'$. So at $(0, h_a(0))$, its curvature is $h_a''(0)$.

As $\bar{K} \geq \frac{C_1}{a^2}$ by $1.5$, $|h_a''(0)| \geq \sqrt{C_1} > 0$. By L’Hospital rule,

\begin{align}
\lim_{\varphi \to 0} \frac{w_a}{h_a'} = \frac{w_a'(0)}{h_a''(0)}
\end{align}
which is finite and is bounded by some $C > 0$ by \[\text{(1.5)}\] and \[\text{(2.3)}\]. The same applies to $\lim_{\varphi \to \omega} \frac{w_a}{h'_a}$.

Next, observe that one of the principal curvatures of $S_a$ in $(\mathbb{R}^3, \delta)$ is $-\frac{h'_a}{aw_a}$ (\[\text{[7]}\) p.162, (10)). So by Remark 1.1 we have

\[\text{(2.11)} \quad \left| \frac{w_a}{h'_a} \right| \leq C\]
on the whole $[0, l_a]$ for all $a$.

By differentiating $\phi_a = 1 + \frac{m}{2a\sqrt{w_a^2 + h_a^2}}$,

\[\text{(2.12)} \quad \frac{\phi'_a}{h'_a} = -\frac{m}{2a(w_a^2 + h_a^2)\frac{1}{2}}(w_a'h_a' + h_a)\]

which can be extended to $[0, l_a]$ by the above, and is of order $O(a^{-1})$ by \[\text{(2.2)}\], \[\text{(2.3)}\].

We have the following estimates

**Lemma 2.2.** Regarding $\phi_a = \phi_a(\varphi)$ as functions on $S_a$, we have

\[\text{(2.13)} \quad \phi'_a = O(a^{-1}), \quad \phi''_a = O(a^{-1}).\]

**Proof.** Let $A = w_a^2 + h_a^2$. As $\phi_a = 1 + \frac{m}{2a\sqrt{A}}$, we only have to prove

\[\text{(2.14)} \quad (A^{-\frac{1}{2}})' = O(1), \quad (A^{-\frac{1}{2}})'' = O(1).\]

By direct computations and \[\text{(2.2)}\], \[\text{(2.3)}\],

\[\text{(2.15)} \quad \left| (A^{-\frac{1}{2}})' \right| = \left| A^{-\frac{1}{2}}(w_a'w_a' + h_a'h_a') \right| \leq A^{-\frac{1}{2}}(w_a^2 + h_a^2)^{\frac{1}{2}}(w_a^2 + h_a^2)^{\frac{1}{2}} = O(1)\]

and

\[\text{(2.16)} \quad \left| (A^{-\frac{1}{2}})'' \right| = \left| \frac{3}{2} A^{-\frac{3}{2}}(w_a'w_a' + h_a'h_a')^2 - A^{-\frac{3}{2}}(1 + w_a'w_a' + h_a'h_a') \right| \leq \frac{3}{2} A^{-\frac{3}{2}}(w_a^2 + h_a^2) + A^{-\frac{3}{2}}(1 + (w_a^2 + h_a^2)^{\frac{1}{2}}(w_a'^2 + h_a'^2)^{\frac{1}{2}}).\]

The two principal curvatures of $S_a$ with induced Euclidean metric are $-\frac{h_a'}{aw_a}$ and $a^{-1}(w_a'^2 + h_a'^2)^{\frac{1}{2}}$ (\[\text{[7]}\) p.162, (10)), hence by Remark 1.1

\[\left| (A^{-\frac{1}{2}})'' \right| = O(1).\]
From now on, we will drop the subscript $a$ and denote $w_a$ by $w$, $h_a$ by $h$, $\phi_a$ by $\phi$ and $l_a$ by $l$. We also denote $ds^2$ to be the metric on $S_a$ induced from $g$.

**Lemma 2.3.** The Gaussian curvature $K$ of $(S_a, ds^2)$ is positive for large enough $a$. In particular, there exists a unique isometric embedding of $(S_a, ds^2)$ into $(\mathbb{R}^3, \delta)$ for sufficiently large $a$.

**Proof.** Let $ds^2$ be the metric on $S_a$ induced by $\delta$ and $ds^2$ be that induced by $g$. Since

\[
d s^2 = a^2(d\varphi^2 + w^2d\theta^2) = \bar{E}d\varphi^2 + \bar{G}d\theta^2 \quad \text{and} \quad ds^2 = \phi^4ds^2 = Ed\varphi^2 + Gd\theta^2,
\]

one has

\[
E = \bar{E} + O(a), E_\varphi = \bar{E}_\varphi + O(a), E_{\varphi\varphi} = \bar{E}_{\varphi\varphi} + O(a)
\]

and

\[
E_\theta = \bar{E}_\theta + O(a), E_{\theta\theta} = \bar{E}_{\theta\theta} + O(a).
\]

Similar result holds for $G$. By the formula

\[
(2.17) \quad K = -\frac{1}{\sqrt{EG}} \left( \frac{E_\theta}{\sqrt{EG}} \right)_\theta + \frac{G_\phi}{\sqrt{EG}} \right)_{\phi}
\]

and the corresponding formula for $\bar{K}$, one can get $K = \bar{K} + O(a^{-3})$. Hence the lemma holds. \hfill \Box

Now let us compute the mean curvature of a revolution surface in $(\mathbb{R}^3, \delta)$.

**Lemma 2.4.** For a smooth revolution surface $S$ in $(\mathbb{R}^3, \delta)$ parametrized by

\[
(2.18) \quad (au(\varphi) \cos \theta, au(\varphi) \sin \theta, av(\varphi)), \quad 0 < \varphi < l, 0 < \theta < 2\pi,
\]

its mean curvature $\bar{H}$ with respect to $\delta$ is

\[
(2.19) \quad \bar{H} = \frac{w''}{aTv'} - \frac{T'u'}{aTv'} - \frac{v'}{aTu} \quad \text{where} \quad T = \sqrt{u'^2 + v'^2}.
\]

**Proof.** The mean curvature $\bar{H}$ of $S$ with respect to $\delta$ is computed to be

\[
(2.20) \quad \bar{H} = \frac{v'u'' - u'v''}{aT^3} - \frac{v'}{aTu}.
\]

By differentiating $T^2$,

\[
(2.21) \quad u'u'' + v'v'' = TT'.
\]
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which implies

\[
    v'u'' - u'v'' = v'u'' + \frac{u^2u'' - u'T'T'}{v'} = \frac{(u^2 + v^2)u'' - u'T'T'}{v'} = \frac{T^2u'' - TT'u'}{v'}.
\]

So we have

\[
    \bar{H} = \frac{w''}{aTv'} - \frac{T'u'}{aT^2v'} - \frac{v'}{aTu}.
\]

The lemma is proved. \(\square\)

**Lemma 2.5.** The mean curvature \(H\) of \(S_a\) with respect to \(g\) is

\[
    H = \frac{w''}{a\phi^2h'} - \frac{h'}{a\phi^2w} + 4\phi^{-3}n(\phi)
\]

where \(n\) is the outward unit normal vector of \(S_a\) with respect to \(\delta\).

**Proof.** By Lemma 2.4 the mean curvature of \(S_a\) with respect to \(\delta\) is

\[
    \bar{H} = \frac{w''}{ah'} - \frac{h'}{aw}.
\]

The mean curvature \(H\) of \(S_a\) with respect to \(g\) is ([18], page 72):

\[
    H = \frac{w''}{a\phi^2h'} - \frac{h'}{a\phi^2w} + 4\phi^{-3}n(\phi).\]

Therefore

\[
    H = \frac{w''}{a\phi^2h'} - \frac{h'}{a\phi^2w} + 4\phi^{-3}n(\phi).
\]

\(\square\)

**Lemma 2.6.** For sufficiently large \(a\), there is an isometric embedding of \((S_a, ds^2)\) into \((\mathbb{R}^3, \delta)\) which is given by

\[
    x^1 = au(\varphi)\cos\theta, \quad x^2 = au(\varphi)\sin\theta, \quad x^3 = av(\varphi), \quad 0 \leq \varphi \leq l, 0 \leq \theta \leq 2\pi
\]

where

\[
    u = \phi^2w
\]

\[
    v' = \phi^2 h' \left( 1 - \frac{2\phi'ww'}{h'^2} + O(a^{-2}) \right)
\]

\[
    u^2 + v'^2 = \phi^4.
\]
The existence has already been proved in Lemma 2.3. In \((\varphi, \theta)\) coordinates, the metric on \(S_a\) induced by \(g\) can be written as:

\[
ds^2 = a^2 \phi^4 d\varphi^2 + a^2 \phi^4 w^2 d\theta^2.
\]

We can regard \((S_a, ds^2)\) as \(S^2\), the sphere with the metric \(ds^2\). Now we want to find two functions \(u, v\) such that the surface written as the form (2.26) is an embedded surface \(S^e_a\) of \(S_a\) into \((\mathbb{R}^3, \delta)\). First of all, the induced metric by the Euclidean metric on the surface which is of the form (2.26) can be written as:

\[
ds^2 = a^2 (u'^2 + v'^2) d\varphi^2 + a^2 u^2 d\theta^2.
\]

Comparing this with (2.28), one can choose

\[
u = \phi^2 w.
\]

Consider

\[
\phi^4 - u'^2 = \phi^2 (\phi^2 - (2\phi'w + \phi w')^2)
\]

\[
= \phi^2 (\phi^2 (w'^2 + h'^2) - (2\phi'w + \phi w')^2)
\]

\[
= \phi^2 (\phi^2 h'^2 - 4\phi\phi'ww' - 4\phi'^2w^2
\]

\[
= \phi^4 h'^2 \left(1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2 h'^2}\right).
\]

By Lemma 2.1 and Lemma 2.2, the functions \(\frac{\phi'ww'}{\phi h'^2}, \frac{\phi'^2w^2}{\phi^2 h'^2}\) can be extended continuously on \([0, l]\) with \(\frac{\phi'ww'}{\phi h'^2} = O(a^{-1}), \frac{\phi'^2w^2}{\phi^2 h'^2} = O(a^{-2})\).

So \(1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2 h'^2} > 0\) for sufficiently large \(a\). For these \(a\), we can take

\[
v' = \phi^2 h' \left(1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2 h'^2}\right)^\frac{1}{2},
\]

so that

\[
u'^2 + v'^2 = \phi^4.
\]

Note that by (2.4), \(v'\) is an odd function for \(\varphi \in [-l, l]\). By choosing an initial value, one can get an even function \(v\). By the above argument, one has

\[
v' = \phi^2 h' \left(1 - \frac{2\phi'ww'}{h'^2} + O(a^{-2})\right).
\]
From (2.29) and (2.30), near $\varphi = 0$, $u$, $v$ can be extended naturally to $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Since $u$ is an odd function in $\varphi$, $v$ is an even function in $\varphi$, and $u'^2 + v'^2 = T^2 > 0$, the generating curve in $\{x^2 = 0\}$ is symmetric with respect to $x^3$-axis, and is smooth at $\varphi = 0$. Similarly, it is also smooth at $\varphi = l$. Hence the revolution surface determined by the choice of $u$, $v$ as above, can be extended smoothly to a closed revolution surface, which is an embedded surface of $S_a$. This completes the proof of the lemma. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $u$, $v$ be defined as in Lemma 2.6. Recall that
\begin{equation}
\begin{aligned}
u' &= \phi^2 h' \left(1 - \frac{2\phi' \omega w'}{h'^2} + O\left(a^{-2}\right)\right) \\
u'^2 + v'^2 &= \phi^4 = T^2 \quad \text{where } T = \phi^2.
\end{aligned}
\end{equation}

By Lemma 2.2 we have
\begin{equation}
\begin{aligned}
T' &= 2\phi' + O\left(a^{-2}\right) \\
u' &= \phi^2 w' + O\left(a^{-1}\right) \\
u'' &= \phi^2 w'' + 4\phi' w' + 2\phi'' w + O\left(a^{-1}\right).
\end{aligned}
\end{equation}

By Lemma 2.4 and Lemma 2.6
\begin{equation}
\begin{aligned}
H_0 &= \left(\frac{u''}{aTv'} - \frac{w''}{a\phi^2 h'}\right) - \frac{T'u'}{aT^2 v'} - \frac{v'}{aTu}.
\end{aligned}
\end{equation}

Combining with Lemma 2.5
\begin{equation}
\begin{aligned}
H_0 - H = \left(\frac{u''}{aTv'} - \frac{w''}{a\phi^2 h'}\right) - \frac{T'u'}{aT^2 v'} - \left(\frac{v'}{aTu} - \frac{h'}{a\phi^2 w}\right) - 4\phi^{-3} h(\phi).
\end{aligned}
\end{equation}

Using (2.34) and (2.35),
\begin{equation}
\begin{aligned}
\frac{u''}{aTv'} - \frac{w''}{a\phi^2 h'} &= \frac{w''}{a\phi^2 h'} + \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} + \frac{2\phi' w w''}{ah'^3} + \frac{w''}{a\phi^2 h'} + O\left(a^{-3}\right) \\
&= \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} + \frac{2\phi' w w''}{ah'^3} + O\left(a^{-3}\right).
\end{aligned}
\end{equation}

By (2.34) and (2.35),
\begin{equation}
\begin{aligned}
-\frac{T'u'}{aT^2 v'} &= -\frac{2\phi' w'}{ah'} + O\left(a^{-3}\right).
\end{aligned}
\end{equation}
By (2.34),

\[
-\frac{v'}{aTu} + \frac{h'}{a\phi'^2w} = -\frac{h'}{a\phi'^2w} + \frac{2\phi'w'}{ah'} + \frac{h'}{2a\phi'^2w} + O\left(a^{-3}\right)
\]

\[= \frac{2\phi'w'}{ah'} + O\left(a^{-3}\right).\] \hfill (2.40)

Summing (2.38), (2.39) and (2.40) and comparing with (2.37), we have

\[
(H_0 - H) = 4 \frac{\phi'w'}{ah'} + 2 \frac{\phi''w}{ah'} - 2 \frac{\phi'wh'}{ah'^2} - 4\phi^{-3}n(\phi) + O\left(a^{-3}\right).
\] \hfill (2.41)

As \(w'w'' = -h'h''\) by (2.3), so

\[
(H_0 - H) = 4 \frac{\phi'w'}{ah'} + 2 \frac{\phi''w}{ah'} - 2 \frac{\phi'wh'}{ah'^2} - 4\phi^{-3}n(\phi) + O\left(a^{-3}\right).
\] \hfill (2.42)

Denote the Euclidean area element of \(S_a\) by \(d\sigma_0\), the area element of \((S_a, ds^2)\) by \(d\sigma\). Note that \(H_0 - H = O\left(a^{-2}\right)\), \(d\sigma - d\sigma_0 = O\left(a^{-1}\right)d\sigma_0\) and \(\int_{S_a} d\sigma_0 = O\left(a^2\right)\). To prove the result, it suffices to show

\[
\lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0 = m.
\]

The Euclidean area element is computed to be

\[
d\sigma_0 = a^2 w d\varphi d\theta.
\] \hfill (2.43)

By (2.3) and Lemma 2.1

\[
\int_{S_a} \frac{4\phi'w'}{ah'} + 2 \frac{\phi''w}{ah'} - 2 \frac{\phi'wh'}{ah'^2} d\sigma_0
\]

\[= 2\pi a \int_0^l \left( \frac{4\phi'w'}{h'} + 2 \frac{\phi''w^2}{h'} - 2 \frac{\phi'w^2h''}{h'^2} \right) d\varphi
\]

\[= 2\pi a \int_0^l \frac{d}{d\varphi} \left( \frac{2\phi'w^2}{h'} \right) d\varphi
\]

\[= 0.
\] \hfill (2.44)

Since the norm of the Euclidean gradient of \(\phi\) has \(|\nabla_0 \phi| = O(r^{-2})\), one has \(n(\phi) = O(a^{-2})\). So

\[
\frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0 = -\frac{1}{2\pi} \int_{S_a} \phi^{-3}n(\phi)d\sigma_0 + O\left(a^{-1}\right)
\]

\[= -\frac{1}{2\pi} \int_{S_a} n(\phi)d\sigma_0 + O\left(a^{-1}\right).
\] \hfill (2.45)
By Theorem 1.1 or Proposition 4.1 in [2], the definition of the ADM mass of \( N \) can be taken as

\[
(2.46) \quad \lim_{a \to \infty} \frac{1}{16\pi} \int_{S_a} \left( \sum_{i,j} (g_{ij,i} - g_{ii,j}) n^j \right) d\sigma_0 = m.
\]

where \( n \) is the unit outward normal of \( S_a \) with respect to \( \delta \). By a direct computation,

\[
(2.47) \quad \sum_{i,j} (g_{ij,i} - g_{ii,j}) n^j = -8\phi^3 n^j \frac{\partial \phi}{\partial x^j} = -8n(\phi) + O(a^{-3}).
\]

Combining (2.46) and (2.47), we have

\[
(2.48) \quad m = -\lim_{a \to \infty} \frac{1}{2\pi} \int_{S_a} n(\phi) d\sigma_0.
\]

Therefore

\[
(2.49) \quad \lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma = \lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0 = m.
\]

We are done. \( \square \)

3. PROOF OF THEOREM 1.3

We will reduce the case of Theorem 1.3 to the Schwarzschild manifold. Let us compare the mean curvature before embedding.

**Lemma 3.1.** For the surfaces \( S_a \) satisfying the conditions in Theorem 1.2, we have

\[
|\tilde{H} - H| \leq Ca^{-3}
\]

for some constant \( C \) independent of \( a \), where \( \tilde{H} \) and \( H \) are the mean curvatures of \( S_a \) with respect to \( \tilde{g} \) and \( g \) respectively.

**Proof.** We claim that

\[
(3.1) \quad |\tilde{A} - A|_g = O(a^{-3})
\]

where \( A \) and \( \tilde{A} \) are the second fundamental forms with respect to \( g \) and \( \tilde{g} \) respectively.

Let \( \rho(x) \) defined on \( N \) to be the distance from \( x \) to \( S_a \) with respect to \( \tilde{g} \). We will use the fact [12] (7.10)]:

\[
(3.2) \quad \tilde{A}(X,Y) - |\nabla \rho|_g A(X,Y) = \left( \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij} \right) X^i Y^j \rho_k
\]
for any tangent vectors $X, Y$ of $S_a$. For completeness, we prove it here. We proceed as in [19] Lemma 2.6. First of all, we have

$$A(X, Y) = g \left( \nabla_X \left( \frac{\nabla \rho}{|\nabla \rho|_g} \right), Y \right)$$

$$= \frac{g(\nabla_X(\nabla \rho), Y)}{|\nabla \rho|_g}$$

$$= \frac{\left( X(Y(\rho)) - (\nabla_X Y)(\rho) \right)}{|\nabla \rho|_g}$$

$$= \frac{X^i Y^j \rho_{ij} - X^j Y^i \Gamma_k \rho_k}{|\nabla \rho|_g};$$

(3.3)

here the subscript denotes ordinary derivative and $\Gamma_k \rho_{ij}$ are the Christoffel symbols with respect to $g$, with the indices $i, j, k = 1, 2, 3$. Denote $\tilde{\Gamma}_{ij}^k$ to be the Christoffel symbols with respect to $\tilde{g}$. Then since the $\tilde{g}$ gradient $|\tilde{\nabla} \rho|_{\tilde{g}} = 1$, we also have

$$\tilde{A}(X, Y) = X^i Y^j \rho_{ij} - X^j Y^i \tilde{\Gamma}_{ij}^k \rho_k.$$ (3.4)

Combining this with (3.3), we can get (3.2).

Note that $|\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k| = O(r^{-3})$ by the assumptions of the metrics. By asymptotic flatness, $1 = \tilde{g}^{ij} \rho_i \rho_j \geq C \sum \rho_i^2$, so $|\rho_i|$ is uniformly bounded. The condition $\tilde{g}_{ij} = g_{ij} + b_{ij}$ implies $|\tilde{g}^{ij} - g^{ij}| = O(r^{-2})$, so

$$||\nabla \rho|_g^2 - 1| = |(g^{ij} - \tilde{g}^{ij}) \rho_i \rho_j| = O(r^{-2})$$

which implies

$$|\nabla \rho|_g = 1 + O(r^{-2}).$$

(3.6)

Finally, the principal curvatures $\bar{\lambda}_i$ in Euclidean metric are of order $O(a^{-1})$ by Remark [14], the principal curvatures $\lambda_i$ with respect to $g$ are related to $\bar{\lambda}_i$ by ([13] Lemma 1.4):

$$\lambda_i = \phi^{-2} \bar{\lambda}_i + 2\phi^{-3} n(\phi)$$

(3.7)

where $n$ is the unit outward normal with respect to $\delta$. In particular, as $n(\phi) = O(a^{-2}),$

$$|A|_g = O(a^{-1}).$$

(3.8)

Combining all these together with (3.2), it is easy to see that (3.1) holds. Combining (3.1) and the metric conditions of $g$ and $\tilde{g}$ in Definition [14], this implies the lemma. □
Let \((S_a, d\tilde{s}^2), (S_a, ds^2)\) denote the surface \(S_a\) with metric \(d\tilde{s}^2, ds^2\) induced from \(\tilde{g}, g\) respectively. By Lemma 2.3, for \(a \gg 1\), the Gaussian curvatures on \((S_a, d\tilde{s}^2)\) and \((S_a, ds^2)\) are both positive, which implies that they can be isometrically embedded into \((\mathbb{R}^3, \delta)\) uniquely. Now let us compare the mean curvature after embedding:

**Lemma 3.2.** Under the same notations and conditions of Theorem 1.3. Let \(\bar{H}_0, H_0\) be the mean curvature of the embedded surfaces of \((S_a, d\tilde{s}^2)\) and \((S_a, ds^2)\) in \((\mathbb{R}^3, \delta)\) respectively, as \(a \gg 1\), we have \(|\bar{H}_0 - H_0| \leq C_5 a^{-3}\) for some constant \(C_5(S)\).

**Proof.** We can set \(\bar{\varphi} = \pi l \varphi\), so it suffices to show that the lemma holds for \(l = \pi\). Also, by identifying \(S\) and \(S_a\) with the sphere \(S^2\), we can regard all the metrics here \((ds^2, \text{etc.})\) to be metrics on \(S^2\). We will denote \(w_a\) as \(w\) and \(h_a\) as \(h\). Similar to (2.28), one has

\[
    ds^2 = a^2 \left( \left( (w')^2 + (h')^2 \right) d\varphi^2 + w^2 d\theta^2 \right) \quad \text{and} \quad ds_a^2 = \left( \left( (\bar{w}')^2 + (\bar{h}')^2 \right) d\bar{\varphi}^2 + \bar{w}^2 d\bar{\theta}^2 \right)
\]

which are the metrics on \(S_a\) and \(S\) induced from the Euclidean metric respectively. By definition,

\[
    ds^2 = \phi^4 ds_a^2, \quad ds^2 = ds^2 + b_{ij} dx^i dx^j |_{S_a}.
\]

From (1.12), \(w\) and its derivatives up to forth order are uniformly bounded for \(a \gg 1\), the same holds for \(h\). By the conditions of \(b_{ij}\), it is easy to see that the followings hold:

\[
    \|a^{-2} ds^2 - a^{-2} ds_a^2\|_{C^3} = \|b_{ij} dx^i dx^j |_{S_a}\|_{C^3} \leq C_6 a^{-2}, \quad (3.11)
\]

\[
    \|a^{-2} ds^2 - a^{-2} ds_a^2\|_{C^3} = \| (\phi^4 - 1) ds^2 \|_{C^3} \leq C_6 a^{-1} \quad (3.12)
\]

for some constant \(C_6(S)\). By (1.12), we have

\[
    \|a^{-2} ds^2 - ds_a^2\|_{C^3} \leq C_7 \varepsilon \quad (3.13)
\]

for some constant \(C_7(S)\). So for \(a \gg 1\), by (3.12) and (3.13), we have

\[
    \|a^{-2} ds^2 - ds_a^2\|_{C^3} \leq (C_6 + C_7) \varepsilon.
\]

By the result of [14, Lemma 5.3], if we choose some \(0 < \varepsilon < \frac{\delta}{\pi^{1-\alpha}(C_6 + C_7)}\) such that

\[
    \|a^{-2} ds^2 - ds_a^2\|_{C^2, \alpha} < \delta
\]
for $a$ big enough, where $\delta$ is the one given by [14, Lemma 5.3], then there are isometric embeddings $\tilde{X}$ and $X$ of $(S^2, a^{-2}d\tilde{s}^2)$ and $(S^2, a^{-2}ds^2)$ respectively, such that by (3.11), for sufficiently large $a$,
$$
\|\tilde{X} - X\|_{C^{2,\alpha}} \leq C_8 \|a^{-2}d\tilde{s}^2 - a^{-2}ds^2\|_{C^{2,\alpha}} = O(a^{-2})
$$
for some constant $C_8(S)$. Since $a\tilde{X}, aX$ are the isometric embeddings of $(S^2, d\tilde{s}^2)$ and $(S^2, ds^2)$ respectively. Hence $|\tilde{H}_0 - H_0| = O(a^{-3})$. The lemma holds. □

Now we can prove Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 1.2, we know that
$$
\lim_{a \to \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) \, d\sigma = m_{ADM}(N, g).
$$
Since the ADM mass of $(N, g)$ is equal to that of $(N, \tilde{g})$, combining with Lemma 3.1 and Lemma 3.2, we can get the result. □

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