A functional equation related to generalized entropies and the modular group

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Abstract. We solve a functional equation connected to the algebraic characterization of generalized information functions. To prove the symmetry of the solution, we study a related system of functional equations, which involves two homographies. These transformations generate the modular group, and this fact plays a crucial role in solving the system. The method suggests a more general relation between conditional probabilities and arithmetic.

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1. Motivation and results
In this paper, we study the measurable solutions $u : [0, 1] \to \mathbb{R}$ of the functional equation

$$u(1 - x) + (1 - x)^\alpha u \left( \frac{y}{1 - x} \right) = u(y) + (1 - y)^\alpha u \left( \frac{1 - x - y}{1 - y} \right) \quad (1.1)$$

for all $x, y \in [0, 1)$ such that $x + y \in [0, 1]$. The parameter $\alpha$ can take any positive real value.

This equation appears in the context of algebraic characterizations of information functions. Given a random variable $X$ whose range is a finite set $E_X$, a measure of its “information content” is supposed to be a function $f[X] : \Delta(E_X) \to \mathbb{R}$, where $\Delta(E_X)$ denotes the set of probabilities on $E_X$,

$$\Delta(E_X) = \left\{ \ p : E_X \to [0, 1] \ \big| \ \sum_{x \in E_X} p(x) = 1 \right\}. \quad (1.2)$$

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The most important example of such a function is the Shannon-Gibbs entropy

\[ S_1[X](p) := - \sum_{x \in E_X} p(x) \log p(x), \quad (1.3) \]

where \(0 \log 0 = 0\) equals 0 by convention.

Shannon entropy satisfies a remarkable property, called the chain rule, that we now describe. Let \(X\) (resp. \(Y\)) be a variable with range \(E_X\) (resp. \(E_Y\)); both \(E_X\) and \(E_Y\) are supposed to be finite sets. The couple \((X, Y)\) takes values in a subset \(E_{XY}\) of \(E_X \times E_Y\), and any probability \(p\) on \(E_{XY}\) induce by marginalization laws \(X_*p\) on \(E_X\) and \(Y_*p\) on \(E_Y\). For instance,

\[ X_*p(x) = \sum_{y: (x, y) \in E_{XY}} p(x, y). \quad (1.4) \]

The chain rule corresponds to the identities

\[ S_1[(X, Y)](p) = S_1[X](X_*p) + \sum_{x \in E_X} X_*p(x) S_1[Y](Y_*p|X=x), \quad (1.5) \]
\[ S_1[(X, Y)](p) = S_1[Y](Y_*p) + \sum_{y \in E_Y} Y_*p(y) S_1[X](X_*p|Y=y), \quad (1.6) \]

where \(p|X=x\) denotes the conditional probability \(y \mapsto p(y, x)/X_*p(x)\). These identities reflect the third axiom used by Shannon to characterize an information measure \(H\): “if a choice be broken down into two successive choices, the original \(H\) should be the weighted sum of the individual values of \(H\)” [7].

There is a deformed version of Shannon entropy, called generalized entropy of degree \(\alpha\) [1, Ch. 6]. For any \(\alpha \in (0, \infty) \setminus \{1\}\), it is defined as

\[ S_\alpha[X](p) := \frac{1}{1 - \alpha} \left( \sum_{x \in E_X} p(x)^\alpha - 1 \right). \quad (1.7) \]

This function was introduced by Havrda and Charvát [4]. Constantino Tsallis popularized its use in physics, as the fundamental quantity of non-extensive statistical mechanics [8], so \(S_\alpha\) is also called Tsallis \(\alpha\)-entropy. It satisfies a deformed version of the chain rule:

\[ S_\alpha[(X, Y)](p) = S_\alpha[X](X_*p) + \sum_{x \in X} (X_*p(x))^\alpha S_\alpha[Y](Y_*p|X=x). \quad (1.8) \]

Suppose now that, given \(\alpha > 0\), we want to find the most general functions \(f[X]\)—for a given collection of finite random variables \(X\)—such that

A. \(f[X](\delta) = 0\) whenever \(\delta\) is any Dirac measure—a measure concentrated on a singleton—, which means that variables with deterministic outputs do not give (new) information when measured;
B. the generalized $\alpha$-chain rule holds, i.e. for any variables $X$ and $Y$ with finite range,$^1$

\[ f[(X,Y)](p) = f[X](X,p) + \sum_{x \in E_X} (X,p(x))^{\alpha} f[Y](Y,(p|_{X=x})), \quad (1.9) \]

\[ f[(X,Y)](p) = f[Y](Y,p) + \sum_{y \in E_Y} (Y,p(y))^{\alpha} f[X](X,(p|_{Y=y})). \quad (1.10) \]

The simplest non-trivial case corresponds to $E_X = E_Y = \{0,1\}$ and $E_{XY} = \{(0,0), (1,0), (0,1)\}$; a probability $p$ on $E_{XY}$ is a triple $p(0,0) = a$, $p(1,0) = b$, $p(0,1) = c$, such that $X,p = (a + c, b)$ and $Y,p = (a + b, c)$. The equality between the right-hand sides of (1.9) and (1.10) reads

\[ f[X](a + c, b) + (1 - b)^{\alpha} f[Y]\left(\frac{a}{1 - b}, \frac{c}{1 - b}\right) = \]

\[ f[Y](a + b, c) + (1 - c)^{\alpha} f[X]\left(\frac{a}{1 - c}, \frac{b}{1 - c}\right), \quad (1.11) \]

for any triple $(a,b,c) \in [0,1]^2$ such that $a + b + c = 1$. Setting $a = 0$ and using assumption A, we conclude that $f[X](c, 1 - c) = f[Y](1 - c, c) =: u(c)$ for any $c \in [0,1]$. Therefore, (1.11) can be written in terms of this unique unknown $u$; if moreover we set $c = y$, $b = x$ and consequently $a = 1 - x - y$, we get the functional equation (1.1), with the stated boundary conditions.

The main result of this article is the following.

**Theorem 1.1.** Let $\alpha$ be a positive real number. Suppose $u : [0,1] \to \mathbb{R}$ is a measurable function that satisfies (1.1) for every $x, y \in [0,1]$ such that $x + y \in [0,1]$. Then, there exists $\lambda \in \mathbb{R}$ such that $u(x) = \lambda s_\alpha(x)$, where

\[ s_1(x) = -x \log_2 x - (1 - x) \log_2(1 - x) \]

and

\[ s_\alpha(x) = \frac{1}{1 - \alpha}(x^\alpha + (1 - x)^\alpha - 1) \]

when $\alpha \neq 1$.

By convention, $0 \log_2 0 := \lim_{x \to 0} x \log_2 x = 0$. For $\alpha = 1$, Theorem 1.1 is essentially Lemma 2 in [5]. Our proof depends on two independent results.

**Theorem 1.2 (Regularity).** Any measurable solution of (1.1) is infinitely differentiable on the interval $(0,1)$.

**Theorem 1.3 (Symmetry).** Any solution of (1.1) satisfies $u(x) = u(1 - x)$ for all $x \in \mathbb{Q} \cap [0,1]$.

The first is proved analytically, by means of standard techniques in the field of functional equations (cf. [1, 9, 5]), and the second by a novel

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$^1$Assumption A can be deduced from B if one identifies $X$ with $(X, X)$ through the diagonal map $E_X \to E_X \times E_X$, $x \mapsto (x, x)$ and then evaluates (1.9) at $Y = X$ and $p = (s_{x_0})$, for any $x_0 \in E_X$. 
Theorems 1.2 and 1.3 above imply that any measurable solution $u$ of (1.1) must be symmetric, i.e. $u(x) = u(1-x)$ for all $x \in [0, 1]$, and therefore
$$u(x) + (1-x)^\alpha u \left( \frac{y}{1-x} \right) = u(y) + (1-y)^\alpha u \left( \frac{x}{1-y} \right)$$
whenever $x, y \in [0, 1)$ and $x + y \in [0, 1]$. When $\alpha = 1$, this equation is called “the fundamental equation of information theory”; it first appeared in the work of Tverberg [9], who deduced it from a characterization of an “information function” that not only supposed a version of the chain rule, but also the invariance of the function under permutations of its arguments. Daróczy introduced the fundamental equation for general $\alpha > 0$, and showed that it can be deduced from an axiomatic characterization analogue to that of Tverberg, that again supposed invariance under permutations along with a deformed chain rule akin to (1.8), see [3, Thm. 5].

For $\alpha = 1$, Tverberg [9] showed that, if $u : [0, 1] \to \mathbb{R}$ is symmetric, Lebesgue integrable and satisfies (1.12), then it must be a multiple of $s_1(x)$. In [5], Kannappan and Ng weakened the regularity condition, showing that all measurable solutions of (1.12) have the form $u(x) = As_1(x) + Bx$ (where $A$ and $B$ are arbitrary real constants), which reduces to $u(x) = As_1(x)$ when $u$ is symmetric. In fact, they solved some generalizations of the fundamental equation, proving among other things that, when $\alpha = 1$, the only measurable solutions of (1.1) are multiples of $s_1(x)$.

For $\alpha \neq 1$, Daróczy [3] established that any $u : [0, 1] \to \mathbb{R}$ that satisfies (1.12) and $u(0) = u(1)$ has the form
$$u(x) = \frac{u(1/2)}{2^{-1/\alpha} - 1} \left( x^\alpha + (1-x)^\alpha - 1 \right),$$
without any hypotheses on the regularity of $u$. The proof starts by proving that any solution of (1.12) must satisfy $u(0) = 0$ (setting $x = 0$), and hence be symmetric (setting $y = 1-x$). Since we are able to prove symmetry of the solutions of (1.1) restricted to rational arguments without any regularity hypothesis, we also get the following result.

**Corollary 1.4.** For any $\alpha \in (0, \infty) \setminus \{1\}$, the only functions $u : \mathbb{Q} \cap [0, 1] \to \mathbb{R}$ that satisfy equation (1.1) are multiples of $s_\alpha$.

**Proof.** Set $x = 0$ in (1.1) to conclude that $u(1) = 0$, and $y = 0$ to obtain $u(0) = 0$. Moreover, $u$ must be symmetric (Theorem 1.3), hence it must fulfill (1.12) when the arguments are rational. Given these facts, Daróczy’s proof in [3, p. 39] applies with no modifications when restricted to $p, q \in \mathbb{Q}$.

More details on the characterization of information functions by means of functional equations can be found in the classical reference [1], which gives a detailed historical introduction. Reference [2] summarizes more recent developments in connection with homological algebra.

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2In fact, he supposes $u(1/2) = 1$, but the argument works in general.
It is quite remarkable that Theorem 1.1 serves as a fundamental result to prove that, up to a multiplicative constant, \( \{ S_\alpha[X] \} \) for any generic set of random variables \( S \) is the only collection of measurable functionals (not necessarily invariant under permutations) that satisfy the corresponding \( \alpha \)-chain rule. To do this, one introduces an adapted cohomology theory, called information cohomology \[2\], where the chain rule corresponds to the 1-cocycle condition and thus has an algebro-topological meaning. The details can be found in the dissertation \[10\].

2. The modular group

The group \( G = \text{SL}_2(\mathbb{Z})/\{ \pm I \} \) is called the modular group; it is the image of \( \text{SL}_2(\mathbb{Z}) \) in \( \text{PGL}_2(\mathbb{R}) \). We keep using the matrix notation for the images in this quotient. We make \( G \) act on \( \mathbb{P}^1(\mathbb{R}) \) as follows: an element \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \) acts on \( [x : y] \in \mathbb{P}^1(\mathbb{R}) \) (homogeneous coordinates) gives

\[
g[x : y] = [ax + by : cx + dy].
\]

Let \( S \) and \( T \) be the elements of \( G \) defined by the matrices

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.1}
\]

The group \( G \) is generated by \( S \) and \( T \) \[6, \text{Ch. VII, Th. 2}\]; in fact, one can prove that \( \langle S, T; S^2, (ST)^3 \rangle \) is a presentation of \( G \).

3. Regularity: proof of Theorem 1.2

Lemma 3 in \[5\] implies that \( u \) is locally bounded on \( (0, 1) \) and hence locally integrable. Their proof is for \( \alpha = 1 \), but the argument applies to the general case with almost no modification, just replacing

\[
|u(y)| = \left| u(1 - x) + (1 - x)u \left( \frac{y}{1 - x} \right) - (1 - y)u \left( \frac{1 - x - y}{1 - y} \right) \right| \leq 3N,
\]

where \( x, y \) are such that \( u(1 - x) \leq N \), \( u \left( \frac{y}{1 - x} \right) \leq N \) and \( u \left( \frac{1 - x - y}{1 - y} \right) \leq N \), by

\[
|u(y)| = \left| u(1 - x) + (1 - x)^\alpha u \left( \frac{y}{1 - x} \right) - (1 - y)^\alpha u \left( \frac{1 - x - y}{1 - y} \right) \right| \leq 3N,
\]

which is evidently valid too whenever \( x, y \in (0, 1) \).

To prove the differentiability, we also follow the method in \[5\]—already present in \[9\]. Let us fix an arbitrary \( y_0 \in (0, 1) \); then, it is possible to chose \( s, t \in (0, 1), s < t \), such that

\[
\frac{1 - y - s}{1 - y}, \frac{1 - y - t}{1 - y} \in (0, 1),
\]
for all $y$ in certain neighborhood of $y_0$. We integrate (1.1) with respect to $x$, between $s$ and $t$, to obtain

$$ (s-t)u(y) = \int_{1-t}^{1-s} u(x)x+y^{1+\alpha} \int_{\frac{y}{1-s}}^{\frac{y}{1-t}} u(z)z + (1-y)^{1+\alpha} \int_{\frac{y}{1-t}}^{\frac{y}{1-s}} u(z)z. \quad (3.1) $$

The continuity of the right-hand side of (3.1) as a function of $y$ at $y_0$, implies that $u$ is continuous at $y_0$ and therefore on $(0,1)$. The continuity of $u$ in the right-hand side of (3.1) implies that $u$ is differentiable at $y_0$. An iterated application of this argument shows that $u$ is infinitely differentiable on $(0,1)$.

4. Symmetry: proof of Theorem 1.3

Define the function $h : [0,1] \to \mathbb{R}$ through

$$ \forall x \in [0,1], \quad h(x) = u(x) - u(1-x). \quad (4.1) $$

Observe that $h$ is anti-symmetric around $1/2$, that is, we have

$$ \forall x \in [0,1], \quad h(x) = -h(1-x). \quad (4.2) $$

Let now $z \in \left[\frac{1}{2}, 1\right]$ be arbitrary and use the substitutions $x = 1-z$ and $y = 1-z$ in (1.1) to derive the identity

$$ \forall z \in \left[\frac{1}{2}, 1\right], \quad h(z) = z^\alpha h(2-z^{-1}). \quad (4.3) $$

Using the anti-symmetry of $h$ to modify the right-hand side of the previous equation, we also deduce that

$$ \forall z \in \left[\frac{1}{2}, 1\right], \quad h(z) = -z^\alpha h(z^{-1}-1). \quad (4.4) $$

Setting $x = 0$ (respectively $y = 0$) in (1.1), we conclude that $u(1) = 0$ (resp. $u(0) = 0$). Hence, the function $h$ is subject to the boundary conditions $h(0) = h(1) = 0$. From (4.3), it follows that $h(1/2) = h(0)/2^\alpha = 0$. If the domain of $h$ is extended to the whole real line imposing 1-periodicity:

$$ \forall x \in \mathbb{R}, \quad h(x+1) = h(x), \quad (4.5) $$

a similar argument can be used to determine the value of $h$ at any rational argument. To that end, it is important to establish first that (4.3) and (4.4) hold for the extended function.

**Theorem 4.1.** The function $h$, extended periodically to $\mathbb{R}$, satisfies the equations

$$ \forall x \in \mathbb{R}, \quad h(x) = \left|x\right|^\alpha h\left(\frac{2x-1}{x}\right), \quad (4.6) $$

$$ \forall x \in \mathbb{R}, \quad h(x) = -\left|x\right|^\alpha h\left(\frac{1-x}{x}\right). \quad (4.7) $$
We establish first the anti-symmetry around $1/2$ of the extended $h$ (Lemma 4.2), which implies that (4.7) follows from (4.6); the latter is a consequence of Lemmas 4.3, 4.7.

**Lemma 4.2.**
\[
\forall x \in \mathbb{R}, \quad h(x) = -h(1 - x).
\]

**Proof.** We write $x = [x] + \{x\}$, where $\{x\} := x - [x]$. Then,

\[
h(x) \overset{(4.5)}{=} h(\{x\}) \overset{(4.2)}{=} -h(1 - \{x\}) \overset{(4.5)}{=} -h(1 - \{x\} - [x]) = -h(1 - x).
\]

\[
\square
\]

**Lemma 4.3.**
\[
\forall x \in [1, 2], \quad h(x) = x^\alpha h(2 - x^{-1}). \tag{4.8}
\]

**Proof.** For $h$ is periodic, (4.8) is equivalent to

\[
\forall x \in [1, 2], \quad h(x - 1) = x^\alpha h(1 - x^{-1}), \tag{4.9}
\]

and the change of variables $u = x - 1$ gives

\[
\forall u \in [0, 1], \quad h(u) = (u + 1)^\alpha h \left( \frac{u}{u + 1} \right). \tag{4.10}
\]

Note that $1 - \frac{u}{u + 1} = \frac{1}{u + 1} \in [1/2, 1]$ whenever $u \in [0, 1]$. Therefore,

\[
h \left( \frac{u}{u + 1} \right) \overset{(\text{Lemma 4.2})}{=} -h \left( \frac{1}{u + 1} \right) \overset{(4.3)}{=} \left( \frac{1}{u + 1} \right)^\alpha h(u).
\]

This establishes (4.10). \(\square\)

**Lemma 4.4.**
\[
\forall x \in [2, \infty[, \quad h(x) = x^\alpha h(2 - x^{-1}). \tag{4.11}
\]

**Proof.** If $x \in [2, \infty[$, then $1 - \frac{1}{x} \in \left[ \frac{1}{2}, 1 \right]$ and we can apply equation (4.3) to obtain

\[
h \left( 1 - \frac{1}{x} \right) \overset{(4.3)}{=} \left( 1 - \frac{1}{x} \right)^\alpha h \left( 2 - \left( 1 - \frac{1}{x} \right)^{-1} \right) = \left( \frac{x - 1}{x} \right)^\alpha h \left( 1 - \frac{1}{x - 1} \right). \tag{4.12}
\]

We prove (4.11) by recurrence. The case $x \in [1, 2]$ corresponds to Lemma 4.3. Suppose it is valid on $[n - 1, n]$, for certain $n \geq 2$; for $x \in [n, n + 1]$,

\[
h(x) \overset{(\text{rec})}{=} h(x - 1)
\]

\[
\overset{(4.7)}{=} (x - 1)^\alpha h(2 - (x - 1)^{-1})
\]

\[
\overset{(4.3)}{=} (x - 1)^\alpha h(1 - (x - 1)^{-1})
\]

\[
\overset{(4.3)}{=} x^\alpha h(1 - x^{-1})
\]

\[
\overset{(4.3)}{=} x^\alpha h(1 - x^{-1}).
\]

\(\square\)
Lemma 4.5.

\[ \forall x \in \left[0, \frac{1}{2}\right], \quad h(x) = -x^\alpha h(x^{-1} - 1). \quad (4.13) \]

**Proof.** The previous lemma and periodicity imply that

\[ h(x - 1) = x^\alpha h(1 - x^{-1}) \]

for all \( x \geq 2 \), i.e.

\[ \forall u \geq 1, \quad h(u) = (u + 1)^\alpha h \left( 1 - \frac{1}{u + 1} \right). \quad (4.14) \]

Then, for \( u \geq 1 \),

\[ h \left( \frac{1}{u + 1} \right) = -h \left( 1 - \frac{1}{u + 1} \right) = -\left( \frac{1}{u + 1} \right)^\alpha h(u). \quad (4.15) \]

We set \( y = (u + 1)^{-1} \in (0, \frac{1}{2}) \). Equation (4.15) reads

\[ \forall y \in \left(0, \frac{1}{2}\right], \quad h(y) = -y^\alpha h(y^{-1} - 1). \quad (4.16) \]

Since \( h(0) = 0 \), the lemma is proved. \( \square \)

Lemma 4.6.

\[ \forall x \in \left[0, \frac{1}{2}\right], \quad h(x) = x^\alpha h(2 - x^{-1}). \quad (4.17) \]

**Proof.** Immediately deduced from the previous lemma using the anti-symmetric property in Lemma 4.2. \( \square \)

Lemma 4.7.

\[ \forall x \in ]-\infty, 0], \quad h(x) = -x^\alpha h(2 - x^{-1}). \]

**Proof.** On the one hand, periodicity implies that

\[ h(x) = h(x + 1) = h(1 - (x + 1)) = -h(-x). \]

On the other, for \( x \leq 0 \), the preceding results imply that

\[ h(-x) = (-x)^\alpha h(2 - (-x)^{-1}) = |x|^\alpha h(2 - (-x)^{-1}). \]

Therefore,

\[ h(x) = -h(-x) = -|x|^\alpha h \left( 2 + \frac{1}{x} \right) \]

\[ = |x|^\alpha h \left( 1 - \left( 2 + \frac{1}{x} \right) \right) = |x|^\alpha h \left( 2 - \frac{1}{x} \right). \]

\( \square \)

The transformations \( x \mapsto \frac{2x - 1}{x} \) and \( x \mapsto \frac{1-x}{x} \) in Equations (4.6) and (4.7) are homographies of the real projective line \( \mathbb{P}^1(\mathbb{R}) \), that we denote respectively by \( \alpha \) and \( \beta \). They correspond to elements

\[ A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \]

in \( G \), that satisfy

\[ B^2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad BA^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}. \]

This last matrix corresponds to \( x \mapsto 1 - x \).
Lemma 4.8. The matrices $A$ and $B^2$ generate $G$.

Proof. Let

$$P = S^{-1}T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$ 

One has

$$PAP^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{4.20}$$

and

$$PB^2P^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.21}$$

Therefore, $PAP^{-1} = T^{-1}$ and $S = T^{-3}PB^{-2}P^{-1}$. Inverting these relations, we obtain

$$T = PA^{-1}P^{-1}, \quad S = PA^3B^{-2}P^{-1}. \tag{4.22}$$

Let $X$ be an arbitrary element of $G$. Since $Y = PXP^{-1} \in G$ and $G$ is generated by $S$ and $T$, the element $Y$ is a word in $S$ and $T$. In consequence, $X$ is a word in $P^{-1}SP$ and $P^{-1}TP$, which in turn are words $A$ and $B^2$. □

It is possible to find explicit formulas for $S$ and $T$ in terms of $A$ and $B^2$. Since $P = S^{-1}T^{-1}$, we deduce that $PSP^{-1} = S^{-1}T^{-1}STS$ and $PTP^{-1} = S^{-1}T^{-1}TTS = S^{-1}TS$. Hence, in virtue of $(4.22)$,

$$S = P^{-1}S^{-1}T^{-1}STS$$

$$= (P^{-1}S^{-1}P)(P^{-1}T^{-1}P)(P^{-1}SP)(P^{-1}TP)(P^{-1}SP)$$

$$= B^2AB^{-2}A^2B^{-2}$$

and

$$T = P^{-1}S^{-1}TSP$$

$$= (P^{-1}S^{-1}P)(P^{-1}TP)(P^{-1}SP)$$

$$= B^2A^{-1}B^{-2}.$$

To finish our proof of Proposition 1.3, we remark that the orbit of 0 by the action of $G$ on $P^1(\mathbb{R})$ is $\mathbb{Q} \cup \{\infty\}$, where $\mathbb{Q} \cup \{\infty\}$ has been identified with $\{p : q \in P^1(\mathbb{R}) \mid p, q \in \mathbb{Z}\} \subset P^1(\mathbb{R})$. This is a consequence of Bezout’s identity: for every point $[p : q] \in P^1(\mathbb{R})$ representing a reduced fraction $\frac{p}{q} \neq 0$ ($p, q \in \mathbb{Z} \setminus \{0\}$ and coprime), there are two integers $x, y$ such that $xq - yp = 1$. Therefore

$$g' = \begin{pmatrix} x & p \\ y & q \end{pmatrix}$$

is an element of $G$ and $g'[0 : 1] = [p : q]$. The case $q = 0$ is covered by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} [0 : 1] = [1 : 0].$$

The extended equations $(4.6)$ and $(4.7)$ are such that

1. for all $x \in \mathbb{R}$, if $h(x) = 0$ then $h(\alpha x) = 0$ and $h(\beta x) = 0$;
2. for all $x \in \mathbb{R} \setminus \{0\}$, if $h(x) = 0$ then $h(\alpha x) = 0$ and $h(\beta x) = 0$. 

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Since $h(1/2) = 0$, the following lemma is the missing piece to establish that the extended $h$ vanishes on $\mathbb{Q}$ (and hence the original $h$ necessarily vanishes on $[0, 1] \cap \mathbb{Q}$).

**Lemma 4.9.** For any $r \in \mathbb{Q} \setminus \{0\}$, there exists a finite sequence

$$w = (w_i)_{i=1}^n \in \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}^n$$

such that $r = w_n \circ \cdots \circ w_1(1/2)$ and, for all $i \in \{1, ..., n\}$, the iterate $x_i := w_i \circ \cdots \circ w_1(1/2)$ does not equal 0 or $\infty$.

**Proof.** Since the orbit in $P^1(\mathbb{R})$ of $1/2$ by the group of homographies generated by $A$ and $B^2$ (i.e. $G$ itself) contains the whole set of rational numbers $\mathbb{Q}$, there exists a $w$ such that $r = w_n \circ \cdots \circ w_1(1/2)$, where each $w_i$ equals $\alpha$, $\beta$ or one of their inverses.

If some iterate equals 0 or $\infty$, the sequence $w$ can be modified to avoid this. Let $i \in \{0, ..., n\}$ be the largest index such that $x_i \in \{0, \infty\}$; in fact, $i < n$ because $r \neq 0, \infty$.

- If $x_i = 0$, then $x_{i+1} \in \{1/2, 1\}$ (the possibility $x_{i+1} = \infty$ is ruled out by the choice of $i$). In the case $x_{i+1} = 1/2$, the equality $r = w_n \circ \cdots \circ w_{i+2}(1/2)$ holds, and when $x_{i+1} = 1$, we have $r = w_n \circ \cdots \circ w_{i+2} \circ \beta(1/2)$.

- If $x_i = \infty$, then $x_{i+1} \in \{2, -1\}$ (again, $x_{i+1} = 0$ is ruled out). When $x_{i+1} = 2$, we have $r = w_n \circ \cdots \circ w_{i+2} \circ \beta \circ \alpha \circ \beta^{-1}(1/2)$, and when $x_{i+1} = -1$, it also holds that $r = w_n \circ \cdots \circ w_{i+2} \circ \alpha \circ \beta^{-1}(1/2) \circ \beta^{-1}(1/2)$.

$\square$

**References**

[1] J. Aczél and Z. Daróczy. *On Measures of Information and Their Characterizations*. Mathematics in Science and Engineering. Academic Press, 1975.

[2] P. Baudot and D. Bennequin. The homological nature of entropy. *Entropy*, 17(5):3253–3318, 2015.

[3] Z. Daróczy. Generalized information functions. *Information and control*, 16(1):36–51, 1970.

[4] J. Havrda and F. Charvát. Quantification method of classification processes. Concept of structural $\alpha$-entropy. *Kybernetika*, 3(1):30–35, 1967.

[5] P. Kannappan and C. T. Ng. Measurable solutions of functional equations related to information theory. *Proceedings of the American Mathematical Society*, 38(2):pp. 303–310, 1973.

[6] J. Serre. *A Course in Arithmetic*. Graduate texts in mathematics. Springer, 1973.

[7] C. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656, 1948.

[8] C. Tsallis. *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*. Springer New York, 2009.

[9] H. Tverberg. A new derivation of the information function. *Mathematica Scandinavica*, 6:297–298, 1958.
[10] J. P. Vigneaux. *Topology of Statistical Systems: A Cohomological Approach to Information Theory*. PhD thesis, Université Paris Diderot, 2019.

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