Optical Analogue of the Aharonov-Bohm Effect in a Magneto-Active Medium

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Abstract. We study an optical analogue of the Aharonov-Bohm effect in a magneto-active medium containing a wire with current. It is shown that the phase shift between the light rays propagating around the wire is proportional to the current strength. The current acts as a magnetic flux in the conventional Aharonov-Bohm effect. Therefore the differential cross section of light scattering from the vortex of the magnetic field created by the linear current proves to be a periodic function of the current strength.

1. Introduction
In many cases, electromagnetic waves in a dielectric medium behave like electrons in solids, which is due to the equivalence of the wave equation for classical fields and the Schrödinger equation for particles (see, e.g., [1]). Among the most striking examples confirming this analogy, we can mention the effect of weak localization, which is observed both for light in disordered media [1,2] and for electrons in “dirty” metals [3,4]. Another example is the existence of optical surface states [5], which are analogous to those predicted back in 1932 [6] for electrons localized near the surface of metals (the Tamm states). A significant number of papers are also devoted to the study of the optical Hall effect (see Ref. [7] and references therein).

It was shown [8,9] that the temporal harmonic modulation of the refractive index of a photonic system results in the appearance of a phase that can be interpreted as an effective calibration potential for photons. This potential, according to [8,9], is analogous to the vector potential for charged particles. Based on this analogy, an interference experiment was carried out in [9], in which the so-called the photonic Aharonov-Bohm effect was observed (see also [10]).

In the conventional Aharonov-Bohm effect, two coherent electron waves enveloping a solenoid interfere with each other with a phase shift which is proportional to the magnetic field flux through the surface confined by electron trajectories [11]. In contrast to [8,9] below we consider the possibility of setting up an optical interference experiment which is a direct analogue to the Aharonov-Bohm experiment for electrons. The idea of the experiment is based on the fact that circularly polarized waves enveloping a linear current placed in a magneto-active medium acquire the phase shift proportional to the current strength. Therefore, in scattering of circularly polarized light by a linear current conductor embedded in a magneto-active medium, a combination of two processes (diffraction by the conductor and scattering of light by the magnetic field vortex) should be observed. The latter process is similar to the electron scattering by a vector potential vortex and is a manifestation of the Aharonov-Bohm effect. As a result, the
cross section for light scattering by the linear conductor turns out to be a periodic function of the current strength.

2. Green function of the wave equation in a magnetic field

Consider propagation of light in a magneto-active sample. In this case, the Green function of the electromagnetic field obeys the equation

$$\delta_{ij} \left( \Delta + k_0^2 \right) G_{jk}(r, r') - ik_0^2 e_{ijm} h_m(r) \cdot G_{jk}(r, r') = -4\pi \delta_{ik} \delta(r - r')$$

(1)

where $k_0$ is the wave number, $e_{ijm}$ is the completely anti-symmetric tensor of the third rank; the gyration vector $h$ is proportional to the magnetic induction vector $B$, $h = VB/k_0$ ($V$ is the Verdet constant, the magnetic permeability $\mu$ is assumed to be equal to unity) [12]. We consider a weak gyrotropy in which the light field remains transverse, $\partial G_{ik}(r, r')/\partial x_i = 0$ [13].

In a smoothly inhomogeneous magnetic field ($(|\nabla h|/h) \ll k_0$), we can define the direction of light propagation $n = (r - r')/|r - r'|$ [12,14]. Then in the reference frame associated with the vector $n$ (see, e.g., [15]) the equation for the Green function can be projected onto the plane perpendicular to $n$. In this reference frame the Green function matrix contains only four nonzero elements which obey the equation

$$\left( \Delta + k_0^2 \right) G_{\alpha\beta}(r, r') - ik_0^2 \cdot (nh(r)) e_{\alpha\gamma} G_{\gamma\beta}(r, r') = -4\pi \delta_{\alpha\beta} \delta(r - r')$$

(2)

where $e_{\alpha\gamma}$ is the completely anti-symmetric tensor of the second rank.

The Green function can be presented as superposition of two contributions corresponding to the different helicities (or waves polarized clockwise and counterclockwise) [15]. In the concomitant reference frame, the projection operator onto the states with a given helicity has the form $P^{(\pm)}_{\alpha\beta} = (\delta_{\alpha\beta} \pm i \cdot e_{\alpha\beta})/2$. Using the expansion

$$G_{\alpha\beta}(r, r') = P^{(\pm)}_{\alpha\beta}(n) \cdot G^{(\pm)}(r, r') + P^{(-)}_{\alpha\beta}(n) \cdot G^{(-)}(r, r')$$

(3)

we arrive at two independent scalar wave equations for the circularly polarized components $G^{(\pm)}$

$$\left[ \Delta + k_0^2 \left( 1 \mp (nh(r)) \right) \right] G^{(\pm)}(r, r') = -4\pi \delta(r - r')$$

(4)

Following [14], we can sought for a solution of Eq.(4) in the form

$$G^{(\pm)}(r, r') = G_0(r - r') \exp \left[ i\Phi^{(\pm)}(r, r') \right],$$

(5)

where

$$G_0(r - r') = \frac{1}{|r - r'|} \cdot \exp \left( ik_0 |r - r'| \right),$$

(6)

is the free Green function, and $\Phi^{(\pm)}$ is the phase arising due to the gyrotropy. In the first approximation, $\Phi^{(\pm)}$ obeys the equation

$$\left[ (n \cdot \nabla \Phi^{(\pm)}(r, r')) \right]^2 \pm \frac{k_0}{2} \cdot (nh(r)) = 0$$

(7)

and it is determined by the relation

$$\Phi^{(\pm)}(r, r') = \mp \frac{k_0}{2} \cdot \int_{r'}^{r} dr'' h(r'').$$

(8)
Returning to the laboratory reference frame (the $Oz$ axis is directed along the inner normal to the sample surface), we can present the Green function $G_{ik}$ in the form

$$G_{ik}(r, r') = \left[ P_{ik}^{(+)}(n) \cdot e^{-(ik_0/2) \int_{r'}^{r} dr'' \cdot h(r'')} + P_{ik}^{(-)}(n) \cdot e^{(ik_0/2) \int_{r'}^{r} dr'' \cdot h(r'')} \right] \cdot G_0(r - r')$$

(9)

where $P_{ik}^{(\pm)}(n) = (\delta_{ik} - n_i n_k \pm ie_{ikl}n_l)/2$ is the projection operator onto the states with a given helicity in the laboratory reference frame [13, 16]. The Green function (9) describes the propagation of polarized light in a magneto-active medium in a weakly inhomogeneous magnetic field. In a homogeneous field, Eq.(9) transforms to the well-known result [16].

3. Optical Aharonov-Bohm effect for circularly polarized light

For electrons, the Green function $G^{el}$ of the Schrödinger equation in a weak magnetic field has the form similar to Eq.(5) (see, e.g., [17]) where the phase $\Phi^{el}$ that is obtained from Eq.(8) by substitution of $eA/hc$ ($A(r)$ is the vector potential) for $(k_0 h/2)$. In the Aharonov-Bohm experiment [11], the nonzero phase shift between the interfering electron waves is caused by the vortex nature of the vector potential,

$$\text{rot} \ A = H$$

(10)

In a magneto-active medium, the analogue of Eq.(10) is the equation for the gyration vector

$$\text{rot} \ (k_0 \ h) = \frac{4\pi V}{e} j$$

(11)

which follows directly from the corresponding Maxwell equation. Analogy between the Green functions $G^{(\pm)}$ and $G^{el}$, and the vortex nature of the quantities appearing in the corresponding phases (see Eqs.(10) and (11)) enable us to propose an optical experiment similar to the Aharonov-Bohm experiment for electrons [11].

Consider a two-arm waveguide fabricated by a magneto-active material, the arms of which bend around a linear conductor. We assume that the circular polarization of light is conserved as it propagates along the waveguide. This can be achieved by using the waveguide with a certain radial distribution of the refractive index [18]. Like an electron wave in the experiment [11], circularly polarized light propagating in a two-arm waveguide splits into two coherent beams. The resulting intensity in the photodetector can be written as

$$I = I_0 \cdot \left[ 1 + \cos \left( \frac{k_0}{2} \cdot \int_{\Gamma} dr \cdot h(r) \right) \right]$$

(12)

where $I_0$ is the intensity of light in the absence of a magnetic field, and the closed contour $\Gamma$ is formed by the arms of the waveguide. Substituting Eq.(11) into Eq.(12), we can rewrite Eq.(12) in the form

$$I = 2I_0 \cdot \cos^2 \pi \beta$$

(13)

where $\beta = Vl/c$, and $l$ is the current strength. According to Eqs.(12) and (13), the intensity in the photodetector does not depend on the sign of helicity and it is a periodical function of the current.
4. Scattering of circularly polarized light by a linear conductor embedded in a magneto-active medium

The manifestation of the optical Aharonov-Bohm effect can be exemplified by the scattering of polarized light by a cylindrical conductor inside a magneto-active medium. Consider circularly polarized light incident on a slab of a given thickness $L$. The current is concentrated in a cylindrical region of radius $a$ (see Fig. 1). For the geometry under consideration, the wave equation (4) takes the form

$$\left[ \Delta_{\rho} + k_0^2 (1 + \Delta \varepsilon(\rho) \mp (n_{\rho} \mathbf{h}(\rho))) \right] E^{(\pm)}(\rho) = 0 \quad (14)$$

where $\rho$ is a two-dimensional vector in the plane $(x, z)$, $n_{\rho} = \rho/\rho$. The dielectric constant of the scattering cylinder is assumed to be different from that of the host medium by the value $\Delta \varepsilon$. The boundary condition to Eq.(14) corresponds to a plane wave incident normally to the slab,

$$E^{(\pm)}(\rho) \bigg|_{z \to -\infty} = E^{(\pm)}_0 \exp (ik_0 z) \quad (15)$$

We assume that the conductor radius is great, $k_0 a \gg 1$, and the light scattering occurs predominantly through small angles in the forward direction. When solving the scattering problem, we take advantage of the eikonal approximation [14] (see also [19]). As a result, we obtain the following expression for the components of the field $E^{(\pm)}$:

$$E^{(\pm)}(\rho) = e^{ik_0 z} \cdot \exp \left( \frac{ik_0}{2} \int_{-\infty}^{z} dz' \Delta \varepsilon(x, z') \mp \frac{ik_0}{2} \int_{\rho_0}^{\rho} d\rho' \mathbf{h}(\rho') \right) \quad (16)$$

The phase associated with the magnetic field can be written in the form [19]

$$\frac{k_0}{2} \int_{\rho_0}^{\rho} d\rho' \mathbf{h}(\rho') = \frac{VT}{c} \cdot (\varphi - \varphi_0) \quad (17)$$

where the azimuth angles $\varphi$ and $\varphi_0$ are shown in Fig. 1. The integral (17) is an ambiguous function of the azimuth angle $\varphi$. The same feature is inherent to the Aharonov-Bohm effect for electrons [11,19]. To avoid this uncertainty, we take into account the one-to-one correspondence...
between the input and output points of the light ray (see Fig. 1). As a result, the integral (17) is written as (compare with [19])

\[ \frac{k_0}{2} \int_{\rho_0}^{\rho} d\rho' h(\rho') = \beta \cdot \begin{cases} (\pi - \vartheta - \varphi_0), & x > 0 \\ -(\pi - \vartheta + \varphi_0), & x < 0 \end{cases} \] (18)

where \( \vartheta \) is the angle of deflection from the incident direction.

Using the definition of the scattering amplitude in the 2D geometry [20], we arrive at the following expression for the scattering amplitudes:

\[ f(\pm)(q) = \sqrt{\frac{k_0}{2\pi i}} \cdot \left( \cos \pi \beta \int_{-\infty}^{\infty} dx \cos(qx) \cdot (S(x) - 1) \mp 2 \sin \pi \beta \int_{0}^{\infty} dx \sin(qx) \cdot S(x) \right) \] (19)

where \( q \approx k_0 \vartheta \) and \( S(x) = \exp \left( (ik_0/2) \int_{-\infty}^{x} dz \Delta \varepsilon(x, z) \right) \). The first term in Eq.(19) is proportional to the scattering amplitude

\[ f(q) = \sqrt{\frac{k_0}{2\pi i}} \cdot \int_{-\infty}^{\infty} dx \cos(qx) \cdot (S(x) - 1) \] (20)

in the absence of the current while the second term is due to the optical Aharonov-Bohm effect. The formula Eq.(19) satisfies the relation \( f(\pm)(q; I) = f(\mp)(q; -I) \) which is consequence of time-reversal symmetry.

5. Discussion

In the limiting case of diffraction \( (k_0a|\Delta \varepsilon| \gg 1) \), Eq.(19) takes the form

\[ f(\pm)(q) = -\sqrt{\frac{2k_0}{i\pi}} \cdot \frac{\sin(qa \pm \pi \beta)}{q} \] (21)

Then the differential scattering cross-section is determined by the expression

\[ d\sigma^{(\pm)}(\vartheta) = \frac{2}{\pi k_0} \cdot \sin^2 \left( k_0a \vartheta \pm \pi \beta \right) \cdot \frac{d\vartheta}{\vartheta^2} \] (22)

From Eq.(22) it follows that the Aharonov-Bohm effect shifts the peaks and valleys in the angular profile of the cross-section and becomes dominant in scattering through small angles, \( \vartheta < \pi \beta/k_0a \).

The total scattering cross-section evaluated with Eq.(22) diverges, while the cross section remains finite in the absence of the current. This effect is associated with a slow decay of the magnetic field of the conductor as the distance from it increases. A similar divergence in the total cross section also arises when light is scattered by fluctuations of the dielectric constant in the case of slow attenuation of the fluctuation correlation function with distance (e.g., at the phase transition point [21]).

In the case of alternating current, the cross-section averaged over the current oscillations is written as

\[ d\sigma^{(\pm)}(\vartheta) = \cos^2 \pi \beta \cdot d\sigma^{(diff)}(\vartheta) + \sin^2 \pi \beta \cdot \left( \frac{2}{\pi k_0} \frac{\cos^2(k_0a \vartheta)}{\vartheta^2} d\vartheta \right) \] (23)

where \( d\sigma^{(diff)}(\vartheta) = d\sigma^{(\pm)}(\vartheta; I = 0) \) is the diffraction cross-section. If the current is relatively weak, \( \beta \ll 1 \), the Aharonov-Bohm contribution can be measured as the difference between the scattering cross-section in the presence of the current and without it.
6. Conclusions
Based on the Green function of the wave equation for polarized light in a magneto-active medium, we have studied an optical analogue of the Aharonov-Bohm effect in the medium containing a linear current conductor. It has been shown that the phase shift between the waves enveloping the conductor is proportional to the current strength. The cross-section of light scattering by the magnetic field of the current turns out to be a periodic function of the current strength and prevails at extremely small angles.

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