State complexity of the star of a Boolean operation

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Abstract

Monsters and modifiers are two concepts recently developed in the state complexity theory. A monster is an automaton in which every function from states to states is represented by at least one letter. A modifier is a set of functions allowing one to transform a set of automata into one automaton. The paper describes a general strategy that can be used to compute the state complexity of many operations. We illustrate it on the problem of the star of a Boolean operation. After applying modifiers on monsters, the states of the resulting automata are assimilated to combinatorial objects: the tableaux. We investigate the combinatorics of these tableaux in order to deduce the state complexity. Specifically, we recover the state complexity of star of intersection and star of union, and we also give the exact state complexity of star of symmetrical difference. We thus harmonize the search strategy for the state complexity of star of any Boolean operations.

1 Introduction

The (deterministic) state complexity is a measure on regular languages defined as the size of the minimal automaton. This measure extends to regular operations as a function of the state complexity of inputs. The state complexity of an operation measures the number of states needed in the worst case to encode the resulting language in an automaton. The classical approach consists in computing an upper bound and providing a witness, that is a specific example showing that the bound is tight.

Pioneered in the 70s, the state complexity has been investigated for numerous unary and binary operations. See, for example, \cite{13,15,19,20,21,25} for a survey. More recently, the state
complexity of compositions of operations has also been studied. In most cases, the state complexity of a composition is strictly lower than the composition of the state complexities of the individual operations. The studies lead to interesting situations, see e.g. [8, 9, 11, 16, 22, 24].

In some cases, the classical method has to be enhanced by two independent approaches. The first one consists in describing states by combinatorial objects. Thus the upper bound is computed using combinatorial tools. For instance, in [7], the states are represented by tableaux representing Boolean matrices and an upper bound for the catenation of symmetrical difference is given. The second one is an algebraic method consisting in building a witness for a certain class of regular operations by searching in a set of automata with as many transition functions as possible. This method has the advantage of being applied to a large class of operations. This approach has been described independently by Caron et al. in [4] as the monster approach and by Davies in [12] as the OLPA (One Letter Per Action) approach but was implicitly present in older papers like [2, 14].

In the authors’ formalism, the algebraic aspects are divided into two distinct notions. The first one, called modifiers (see Section 2.3), allows to encode an operation as a transformation on automata. The second one allows one to encapsulate the set of possible transitions in a single object, a monster (see Section 2.2), which is a k-tuple of automata on a huge alphabet. We apply this strategy to compute the state complexity of the star of a Boolean operation. After applying the modifier of these operations to the monsters, states of the resulting automata are encoded by combinatorial objects. In our case, the objects are tableaux. We then investigate the combinatorics of these tableaux in order to understand the properties of accessibility and Nerode equivalence (see Section 3). To be more precise, in Section 3.1, we define the validity property on tableaux which is a combinatorial property indicating the presence of a cross at the coordinate (0,0) under certain conditions. This property is related to the accessibility of the states in the resulting automaton. We also define, in Section 3.4, a combinatorial operation, called local saturation, on tableaux. This operation consists in adding crosses in an array until some patterns are avoided that are defined in Section 3.3. By using preliminary results obtained in Section 3.2, the local saturation operation allows us to describe the Nerode classes in a combinatorial way. Finally, we exhibit witnesses among the set of the monsters in Section 4.

Through this method, we harmonize the search strategy for the state complexity of star of any Boolean operations. Specifically, we recover the state complexity of star of intersection and star of union, and we also give the exact state complexity of star of symmetrical difference.

2 Preliminaries

2.1 Languages, automata and algebraic tools

The cardinality of a finite set E is denoted by #E, the set of subsets of E is denoted by 2^E and the set of mappings of E into itself is denoted by E^E.

Any binary Boolean function • : B^2 → B is extended to a binary operation on sets as F • G = {x | ([x ∈ F] • [x ∈ G]) = 1} for any sets F and G. For instance, the symmetric difference of two sets F and G denoted by ⊕ and defined by F ⊕ G = (F ∪ G) \ (F ∩ G) is the extension of the Boolean function xor.

For any positive integer n, let us denote the set {0, . . . , n − 1} by [n].

Let Σ be a finite alphabet. A word w over Σ is a finite sequence of symbols of Σ. The length of w, denoted by |w|, is the number of occurrences of symbols of Σ in w. The set of all finite words
where, for each $a$ from $\Sigma$, $\Gamma(a)$ is a finite set of states, $\Gamma_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states and $\delta$ is the transition function from $Q \times \Sigma$ to $Q$ extended in a natural way from $Q \times \Sigma^*$ to $Q$. The cardinal of $A$ is the cardinal of its set of states, i.e. $\# A = \# Q$. The automaton $B = (\Sigma, Q_B, i_B, F_B, \delta_B)$ is a sub-automaton of $A$ if $Q_B \subseteq Q_A$, $i_B = i_A$, $F_B \subseteq F_A$ and $\delta_B = (\delta_B \circ \delta) \mid_{\delta_B} \in \delta_B \in Q_B$.

Let $A = (\Sigma, Q, q_0, F, \delta)$ be a DFA. A word $w \in \Sigma^*$ is recognized by the DFA $A$ if $\delta(q_0, w) \in F$. The language recognized by a DFA $A$ is the set $L(A)$ of words recognized by $A$. Two DFAs are equivalent if they recognize the same language. The set $\text{Rec}(\Sigma^*)$ of all languages recognized by DFAs over $\Sigma$ is the set of recognizable languages. For any word $w$, we denote by $\delta^w$ the function $q \rightarrow \delta(q, w)$. A state $q$ is accessible if there exists a word $w \in \Sigma^*$ such that $q = \delta(q_0, w)$. We denote by $\text{Acc}(A) = (\Sigma, \text{Acc}(Q), q_0, \text{Acc}(F), \delta)$ the sub-automaton of $A$ containing exactly accessible states of $A$. By abuse of languages, we denote also by $\text{Acc}(A)$ the set of accessible states.

Two states $q_1, q_2$ of $A$ are equivalent if for any word $w$ of $\Sigma^*$, $\delta(q_1, w) \in F$ if and only if $\delta(q_2, w) \in F$. This equivalence relation is called the Nerode equivalence and is denoted by $q_1 \sim q_2$. If two states are not equivalent, then they are called distinguishable. A DFA is minimal if there does not exist any equivalent DFA with less states. It is well known that for any DFA, there exists a unique minimal equivalent DFA $([18])$. Such a minimal DFA can be obtained from $A$ by computing

$$\text{Min}(A) = \text{Acc}(A)/\sim = (\Sigma, \text{Acc}(Q)/\sim, [q_0], \text{Acc}(F)/\sim, \delta/\sim)$$

where, for any $q \in \text{Acc}(Q)$, $[q]$ is the $\sim$-class of the state $q$ and satisfies the property $\delta(\sim([q]), a) = [\delta(q, a)]$, for any $a \in \Sigma$. In a minimal DFA, any two distinct states are pairwise distinguishable.

The set $\text{Rat}(\Sigma^*)$ of regular languages defined over an alphabet $\Sigma$ is the smallest set containing $\{a\}$ for each $a \in \Sigma$ and $\emptyset$ and closed by union, catenation and Kleene star. We recall that the Kleene star of the language $L$ is defined by $L^* = \{w = u_1 \cdots u_n \mid u_i \in L \land n \in \mathbb{N}\}$. Kleene theorem [23] asserts that $\text{Rat}(\Sigma^*) = \text{Rec}(\Sigma^*)$.

A $k$-ary regular operation is an operation sending for each alphabet $\Sigma$ any $k$-tuples of regular languages over $\Sigma$ to a regular language over $\Sigma$. The state complexity of a regular language $L$ denoted by $\text{sc}(L)$ is the number of states of its minimal DFA. This notion extends to regular operations: the state complexity of a unary regular operation $\otimes$ is the function $\text{sc}_{\otimes}$ such that, for all $n \in \mathbb{N} \setminus \{0\}$, $\text{sc}_{\otimes}(n)$ is the maximum of all the state complexities of $\otimes(L)$ when $L$ is of state complexity $n$, i.e. $\text{sc}_{\otimes}(n) = \max\{\text{sc}(\otimes(L)) \mid L \in \Gamma\} = \text{sc}(L) = n$.

This can be generalized, and the state complexity of a $k$-ary operation $\otimes$ is the $k$-ary function $\text{sc}_{\otimes}$ such that, for all $(n_1, \ldots, n_k) \in (\mathbb{N} \setminus \{0\})^k$,

$$\text{sc}_{\otimes}(n_1, \ldots, n_k) = \sup\{\text{sc}(\otimes(L_1, \ldots, L_k)) \mid \text{ for all } i \in \{1, \ldots, k\}, \text{sc}(L_i) = n_i\}. \quad (1)$$

When the state complexity is finite, a witness for $\otimes$ is a a way to assign to each $(n_1, \ldots, n_k)$, assumed sufficiently big, a $k$-tuple of languages $(L_1, \ldots, L_k)$ with $\text{sc}(L_i) = n_i$, for all $i \in \{1, \ldots, k\}$, satisfying $\text{sc}_{\otimes}(n_1, \ldots, n_k) = \text{sc}(\otimes(L_1, \ldots, L_k))$. Let $\Sigma$ and $\Gamma$ be two alphabets. A morphism is a function $\phi$ from $\Sigma^*$ to $\Gamma^*$ such that, for all $v, w \in \Sigma^*$, $\phi(vw) = \phi(v)\phi(w)$. Notice that $\phi$ is completely defined by its value on letters.

Let $L$ be a regular language recognized by a DFA $A = (\Gamma, Q, q_0, F, \delta)$ and let $\phi$ be a morphism from $\Sigma^*$ to $\Gamma^*$. Then, $\phi^{-1}(L)$ is the regular language recognized by the DFA $B = (\Sigma, Q, q_0, F, \delta')$ where, for each $a \in \Sigma$ and $q \in Q$, $\delta'(q, a) = \delta(q, \phi(a))$. Therefore, notice that we have
Property 1 Let $L$ be a regular language and $\phi$ be a morphism. We have $\text{sc}(\phi^{-1}(L)) \leq \text{sc}(L)$.

A morphism $\phi$ is 1-uniform if the image by $\phi$ of any letter is a letter. In other words, a 1-uniform morphism is a (not necessarily injective) renaming of the letters and the only complexity of the mapping stems from mapping $a$ and $b$ to the same image, i.e., $\phi(a) = \phi(b)$.

A transformation of a set $Q$ is a map from $Q$ into itself. The set of transformations endowed with the composition is a monoid where $\text{Id}$, the identity map, is its neutral element. Any transformation $t$ of $Q$ induces a transformation of $2^Q$ defined by $E \cdot t = \{ q \cdot t \mid q \in E \}$ for any $E \in 2^Q$. By extension, any transformation $t_1$ of $Q_1$ and any transformation $t_2$ of $Q_2$ induce a transformation of $2^{Q_1 \times Q_2}$ defined by $E \cdot (t_1, t_2) = \{(q_1 \cdot t_1, q_2 \cdot t_2) \mid (q_1, q_2) \in E \}$ for any $E \in 2^{Q_1 \times Q_2}$.

We consider the special case where $Q = \llbracket n \rrbracket$. If $t$ is a transformation and $i$ an element of $\llbracket n \rrbracket$, we denote by $i \cdot t$ the image of $i$ under $t$. A transformation of $\llbracket n \rrbracket$ can be represented by $t = [i_0, i_1, \ldots, i_{n-1}]$ which means that $i_k = k \cdot t$ for each $0 \leq k \leq n-1$. A permutation is a bijective transformation on $\llbracket n \rrbracket$. A cycle of length $\ell \leq n$ is a permutation $t$, denoted by $(i_0, i_1, \ldots, i_{\ell-1})$, on a subset $J = \{i_0, \ldots, i_{\ell-1}\}$ of $\llbracket n \rrbracket$ where $i_k \cdot t = i_{k+1}$ for $0 \leq k < \ell - 1$, $i_{\ell-1} \cdot t = i_0$ and for every elements $j \in \llbracket n \rrbracket \setminus I$ $t \cdot j = j$. A transposition $t = (i, j)$ is a permutation on $\llbracket n \rrbracket$ where $i \cdot t = j$ and $j \cdot t = i$ and for every elements $k \in \llbracket n \rrbracket \setminus \{i, j\}$, $k \cdot t = k$. A contraction $t = \begin{pmatrix} i & j \end{pmatrix}$ is a transformation where $i \cdot t = j$ and for every elements $k \in \llbracket n \rrbracket \setminus \{i\}$, $k \cdot t = k$. We use both notations $t(i)$ or $i \cdot t$ depending on the context.

### 2.2 Monsters and state complexity

In $[1]$, Brzozowski gives a series of properties that would make a language $L_n$ of state complexity $n$ sufficiently complex to be a good candidate for constructing witnesses for numerous classical regular operations. One of these properties is that the size of the syntactic semigroup is $n^n$, which means that each transformation of the minimal DFA of $L_n$ can be associated to a transformation by some non-empty word. This upper bound is reached when the set of transition functions of the DFA is exactly the set of transformations from state to state. We thus consider the set of transformations of $\llbracket n \rrbracket$ as an alphabet where each letter is simply named by the transition function it defines. This leads to the following definition:

**Definition 1** A 1-monster is an automaton $\text{Mon}_n^1 = (\Sigma, \llbracket n \rrbracket, 0, F, \delta)$ defined by

- the alphabet $\Sigma = \llbracket n \rrbracket^{\llbracket n \rrbracket}$,
- the set of states $\llbracket n \rrbracket$,
- the initial state $0$,
- the set of final states $F$,
- the transition function $\delta$ defined for any $a \in \Sigma$ by $\delta(q, a) = a(q)$.

The language recognized by a 1-monster DFA is called a 1-monster language.

**Example 1** The 1-monster $\text{Mon}_2^{[1]}$ is
where, for all $i, j \in \{0, 1\}$, the label $[ij]$ denotes the transformation sending $0$ to $i$ and $1$ to $j$, which is also a letter in the DFA above.

Let us notice that some families of $1$-monster languages are witnesses for the Star and Reverse operations ([4]). The following claim is easy to prove and captures a universality-like property of $1$-monster languages:

**Property 2** Let $L$ be any regular language recognized by a DFA $A = (\Sigma, \llbracket n \rrbracket, 0, F, \delta)$. The language $L$ is the preimage of $L(Mon_n^1)$ by the $1$-uniform morphism $\phi$ such that, for all $a \in \Sigma$, $\phi(a) = \delta^a$, i.e.

$$L = \phi^{-1}(L(Mon_n^1)).$$

This is an important and handy property that we should keep in mind. We call it the restriction-renaming property.

We can wonder whether we can extend the notions above to provide witnesses for $k$-ary operators. In the unary case, the alphabet of a monster is the set of all possible transformations we can apply on the states. In the same mindset, a $k$-monster DFA is a $k$-tuple of DFAs, and its construction must involve the set of $k$-tuples of transformations as an alphabet. Indeed, the alphabet of a $k$-ary monster has to encode all the transformations acting on each set of states independently one from the others. This leads to the following definition:

**Definition 2** A $k$-monster is a $k$-tuple of automata $Mon_{n_1, \ldots, n_k}^{F_1, \ldots, F_k} = (M_1, \ldots, M_k)$ where, for any $j \in \{1, \ldots, k\}$, $M_j = (\Sigma, \llbracket n_j \rrbracket, 0, F_j, \delta_j)$ is defined by

- the common alphabet $\Sigma = \llbracket n_1 \rrbracket^{\llbracket n_1 \rrbracket} \times \ldots \times \llbracket n_k \rrbracket^{\llbracket n_k \rrbracket}$,
- the set of states $\llbracket n_j \rrbracket$,
- the initial state $0$,
- the set of final states $F_j$,
- the transition function $\delta_j$ defined for any $(a_1, \ldots, a_k) \in \Sigma$ by $\delta_j(q, (a_1, \ldots, a_k)) = a_j(q)$.

A $k$-tuple of languages $(L_1, \ldots, L_k)$ is called a monster $k$-language if there exists a $k$-monster $(M_1, \ldots, M_k)$ such that $(L_1, \ldots, L_k) = (L(M_1), \ldots, L(M_k))$.

**Remark 1** When $F_j$ is different from $\emptyset$ and $\llbracket n_j \rrbracket$, $M_j$ is minimal.

Definition 2 allows us to extend the restriction-renaming property in a way that is still easy to check.
Proof: Let \((L_1,\ldots,L_k)\) be a \(k\)-tuple of regular languages over the same alphabet \(\Sigma\). We assume that each \(L_j\) is recognized by the DFA \(A_j = (\Sigma, [n_j], 0, F_j, \delta_j)\). Let \(\text{Mon}_{n_1,\ldots,n_k} = (M_1,\ldots,M_k)\). For all \(j \in \{1,\ldots,k\}\), the language \(L_j\) is the preimage of \(L(M_j)\) by the 1-uniform morphism \(\phi\) such that, for all \(a \in \Sigma\), \(\phi(a) = (\delta^a_j,\ldots,\delta^a_k)\), i.e.

\[
(L_1,\ldots,L_k) = (\phi^{-1}(L(M_1)),\ldots,\phi^{-1}(L(M_k))).
\]

It has been shown that some families of 2-monsters are witnesses for binary Boolean operations and for the catenation operation [4]. Many papers concerning state complexity actually use monsters as witnesses without naming them (e.g. [2]). Therefore, a natural question arises: can we define a simple class of regular operations for which monsters are always witnesses? This class should ideally encompass some classical regular operations, in particular the operations studied in the papers cited above. The objects that allow us to answer this question are 1-uniform operations and are defined in the next section.

2.3 Modifiers and 1-uniform operations

A regular operation \(\otimes\) is 1-uniform if it commutes with any 1-uniform morphism, i.e. for any \(k\)-tuple of regular languages \((L_1,\ldots,L_k)\), for any 1-uniform morphism \(\phi\), \(\phi(\otimes(L_1),\ldots,\otimes(L_k)) = \otimes(\phi^{-1}(L_1),\ldots,\phi^{-1}(L_k))\). For example, it is proven in [12] that Kleene star and mirror operations are 1-uniform.

Property 4 Let \(\phi\) (resp. \(\psi\)) be a \(j\)-ary (resp. \(k\)-ary) 1-uniform operation. Then, for any integer \(1 \leq p \leq j\), the \((j + k - 1)\)-ary operator \(\phi \circ_p \psi\) such that

\[
\phi \circ_p \psi(L_1,\ldots,L_{j+k-1}) = \phi(L_1,\ldots,L_{p-1},\psi(L_p,\ldots,L_{p+k-1}),L_{p+k},\ldots,L_{j+k-1})
\]

is 1-uniform.

Monsters are relevant for the study of state complexity of 1-uniform operations as shown in the theorem below.

Theorem 1 Any \(k\)-ary 1-uniform operation admits a family of monster \(k\)-languages as a witness.

Proof: Suppose now that \(\otimes\) is a \(k\)-ary 1-uniform operation. Then, if \((L_1,\ldots,L_k)\) is a \(k\)-tuple of regular languages over \(\Sigma\), \((A_1,\ldots,A_k)\) the \(k\)-tuple of DFAs such that each \(A_j = (\Sigma, Q_j, i_j, F_j, \delta_j)\) is the minimal DFA of \(L_j\), and \(\phi\) the 1-uniform morphism such that, for all \(a \in \Sigma\), \(\phi(a) = (\delta^a_1,\ldots,\delta^a_k)\), and if \(\text{Mon}_{n_1,\ldots,n_k} = (M_1,\ldots,M_k)\), then \(\otimes = \otimes(\phi^{-1}(L(M_1)),\ldots,\phi^{-1}(L(M_k))) = \phi^{-1}(\otimes(L(M_1),\ldots,L(M_k)))\). It follows that \(\text{sc}(\otimes(L)) = \text{sc}(\phi^{-1}(\otimes(L(M_1),\ldots,L(M_k)))) \leq \text{sc}(\otimes(L(M_1),\ldots,L(M_k)))\) by Property 1. In addition, each \(L(M_j)\) has the same state complexity as \(L_j\). \(\square\)

Each 1-uniform operation corresponds to a construction over DFAs, which is handy when we need to compute the state complexity of its elements. Such a construction on DFAs has some constraints that are described in the following definitions.

Definition 3 A state configuration is a 3-tuple \((Q, i, F)\) such that \(Q\) is a finite set, \(i \in Q\) and \(F \subseteq Q\). A transition configuration is a 4-tuple \((Q, i, F, \delta)\) such that \((Q, i, F)\) is a state configuration and \(\delta \in Q^Q\). If \(A = (\Sigma, Q, i, F, \delta)\) is a DFA, the transition configuration of a letter \(a \in \Sigma\) in \(A\) is the 4-tuple \((Q, i, F, \delta^a)\). The state configuration of \(A = (\Sigma, Q, i, F, \delta)\) is the triplet \((Q, i, F)\).
Definition 4 A modifier \( m \) is a \( k \)-ary operation over transition configurations such that the state configuration of the output depends only on the state configurations of the inputs.

It means that if \( t_1, \ldots, t_k \) and \( t'_1, \ldots, t'_k \) are two \( k \)-tuples of transition configurations such that, for any \( j \in \{1, \ldots, k\} \), the state configurations of \( t_j \) and \( t'_j \) are equal, then the state configuration of \( m(t_1, \ldots, t_k) \) is equal to the state configuration of \( m(t'_1, \ldots, t'_k) \).

We thus introduce the following abuse of notation. For any \( k \) DFAs \( A_1, \ldots, A_k \) over the alphabet \( \Sigma \), we let \( m(A_1, \ldots, A_k) \) denote the DFA \( A \) over \( \Sigma \) such that for any letter \( a \) of \( \Sigma \), the transition configuration of \( a \) in \( A \) is equal to

\[
m((Q_1, i_1, F_1, \delta'_1), \ldots, (Q_k, i_k, F_k, \delta'_k)).
\]

Example 2 (The unary modifier \( \text{Star} \)) For any DFA \( A = (\Sigma, Q, i, F, \delta) \), let us define

\[
\text{Star}(A) = (\Sigma, 2^Q, \emptyset, \{E | E \cap F \neq \emptyset \} \cup \{\emptyset\}, \delta_1),
\]

where \( \delta_1 \) is as follows:

\[
\text{for all } a \in \Sigma, \quad \delta_1^a(\emptyset) = \begin{cases} \{\delta^a(i)\} & \text{if } \delta^a(i) \notin F \\ \{\delta^a(i), i\} & \text{otherwise} \end{cases} \quad \text{for all } E \neq \emptyset, \quad \delta_1^a(E) = \begin{cases} \delta^a(E) & \text{if } \delta^a(E) \cap F = \emptyset \\ \delta^a(E) \cup \{i\} & \text{otherwise} \end{cases}
\]

The modifier \( \text{Star} \) describes the classical construction associated to the star operation on languages, i.e. for all DFA \( A \), \( L(A)^* = \text{L(Star}(A)) \).

Example 3 (The binary modifier \( \text{Bool}_\bullet \)) Consider a Boolean binary operation denoted by \( \bullet \). For all DFAs \( A = (\Sigma, Q_1, i_1, F_1, \delta_1) \) and \( B = (\Sigma, Q_2, i_2, F_2, \delta_2) \), define

\[
\text{Bool}_\bullet(A, B) = (\Sigma, Q_1 \times Q_2, (i_1, i_2), (F_1 \times Q_2) \bullet (Q_1 \times F_2), (\delta_1, \delta_2))
\]

The modifier \( \text{Bool}_\bullet \) describes the classical construction associated to the operation \( \bullet \) on couples of languages, i.e. for all DFAs \( A \) and \( B \), \( L(A) \bullet L(B) = L(\text{Bool}_\bullet(A, B)) \).

Definition 5 A \( k \)-ary modifier \( m \) is coherent if, for every pair of \( k \)-tuples of DFAs \( (A_1, \ldots, A_k) \) and \( (B_1, \ldots, B_k) \) such that \( L(A_j) = L(B_j) \) for all \( j \in \{1, \ldots, k\} \), we have \( L(m(A_1, \ldots, A_k)) = L(m(B_1, \ldots, B_k)) \).

If \( m \) is a coherent modifier then we denote \( \otimes_m \) the operation such that for any \( k \)-tuple of regular languages \( (L_1, \ldots, L_k) \) we have \( \otimes_m(L_1, \ldots, L_k) = L(m(A_1, \ldots, A_k)) \) for any \( k \)-tuple of DFAs \( (A_1, \ldots, A_k) \) such that each \( A_i \) recognizes \( L_i \).

Theorem 2 A \( k \)-ary regular operation \( \otimes \) is \( 1 \)-uniform if and only if \( \otimes = \otimes_m \) for some coherent modifier \( m \).

Proof: Let \( \otimes \) be a \( 1 \)-uniform operation. For any \( k \)-tuple \( C = (n_1, F_1), \ldots, (n_k, F_k) \) with \( n_i \in \mathbb{N} \) and \( F_i \subseteq \llbracket n_i \rrbracket \) for any \( i \in \{1, \ldots, k\} \), we define

\[
B_C = (\llbracket n_1 \rrbracket^{[n_1]} \times \cdots \times \llbracket n_k \rrbracket^{[n_k]}, Q_C, i_C, F_C, \delta_C)
\]

as the minimal automaton that recognizes \( \otimes(L(M_1), \ldots, L(M_k)) \) with \( \text{Mon}_{n_1, \ldots, n_k} = (M_1, \ldots, M_k) \).

Let \( T = ((Q_1, i_1, F_1, \delta_1), \ldots, (Q_k, i_k, F_k, \delta_k)) \) be a \( k \)-tuple of transition configurations. By the axiom choice, we associated to any \( (Q, i), i \in Q \), a bijection \( \varphi_Q,i : Q \rightarrow \llbracket \#Q \rrbracket \)
sending \( i \) to 0. To any \( f : Q \rightarrow Q \), we associate the unique map \( \hat{f} : \#Q \rightarrow \#Q \) such that 
\[ \hat{f}(\varphi_{Q,q}(j)) = \varphi_{Q,f(q)}(j) \] for any \( q \in Q \). When there is no ambiguity, we omit the subscript and denote \( f = \hat{f} \).

We associate to \( T \), the automaton
\[ A_T^\otimes = B((\#Q_1\varphi_{Q_1,i_1}(F_1)), \ldots, (\#Q_k\varphi_{Q_k,i_k}(F_k))) = (\#Q_1, \ldots, \#Q_k, Q_T, i_T, F_T, \delta_T), \]
and we set
\[ m(T) = (Q_T, i_T, F_T, \delta) \]
where \( \delta(q) = \delta^\otimes_{T(i_1, \ldots, i_k)}(q) \).

Obviously \((Q_T, i_T, F_T)\) depends only on the \( k \)-tuple \((Q_1, i_1, F_1), \ldots, (Q_k, i_k, F_k)\) and so \( m \) is a modifier. Furthermore, by construction, for any \( k \)-tuple of automata \((A_1, \ldots, A_k)\) with \( A_j = (\Sigma, Q_j, i_j, F_j, \delta_j) \) we have
\[ L(m(A_1, \ldots, A_k)) = \phi^{-1}(L(B((\#Q_1\varphi_{Q_1,i_1}(F_1)), \ldots, (\#Q_k\varphi_{Q_k,i_k}(F_k)))), \]
where \( \phi \) is the 1-uniform morphism such that \( \phi(a) = (\hat{\delta}_j, \ldots, \hat{\delta}_k) \) for all \( a \in \Sigma \). Therefore, we have
\[ L(m(A_1, \ldots, A_k)) = \otimes(\phi^{-1}(L(M_1), \ldots, L(M_k))) \]
where \( (M_1, \ldots, M_k) = \text{Mon}_{\#Q_1, \ldots, \#Q_k}^{\varphi_{Q_1,i_1}(F_1), \ldots, \varphi_{Q_k,i_k}(F_k)} \). Since \( \otimes \) is 1-uniform, we obtain
\[ L(m(A_1, \ldots, A_k)) = \otimes(\phi^{-1}(L(M_1), \ldots, L(M_k))) = \otimes(L(A_1), \ldots, L(A_k)). \]

Conversely, let \( \otimes \) be a \( k \)-ary regular operation and let \( m \) be a \( k \)-ary modifier such that for any \( k \)-tuple of regular languages \((L_1, \ldots, L_k)\) and any \( k \)-tuple of DFA \((A_1, \ldots, A_k)\) such that each \( A_i \) recognizes \( L_i \otimes L(L_1, \ldots, L_k) = L(m(A_1, \ldots, A_k)) \). Let us prove that \( \otimes \) is 1-uniform. Let \( \Gamma \) and \( \Sigma \) be two alphabets. Consider a 1-uniform morphism \( \phi \) from \( \Gamma^* \) to \( \Sigma^* \) and \((L_1, \ldots, L_k)\) a \( k \)-tuple of languages over \( \Sigma \). For any \( j \), we consider an automaton \( A_j = (\Sigma, Q_j, i_j, F_j, \delta_j) \) that recognizes \( L_j \) and let \( B_j = (\Gamma, Q_j, i_j, F_j, \delta_j) \) be the DFA such that \( \delta^\phi_j(a) = \delta^\phi_{i_j}(a) \) for any letter \( a \in \Gamma \). We have \( L(B_j) = \phi^{-1}(L(A_j)) \).

Let \( m(A_1, \ldots, A_k) = (\Sigma, Q, i, F, \delta) \) and \( m(B_1, \ldots, B_k) = (\Gamma, Q', i', F', \delta') \). Since for any \( j \) the state configuration of \( A_j \) is the same as the state configuration of \( B_j \), we have \((Q, i, F) = (Q', i', F')\). Furthermore, because the transition function of any letter \( a \in \Gamma \) in \( B_j \) is also the same as the transition function of \( \phi(a) \) in \( A_j \), we have \( \delta^\phi_j(a) = \delta^\phi_{i_j}(a) \). Hence, \( L(m(B_1, \ldots, B_k)) = \phi^{-1}(L(m(A_1, \ldots, A_k))) \), which implies that \( \otimes(L(B_1), \ldots, L(B_k)) = (\phi^{-1}(\otimes(A_1)), \ldots, \phi^{-1}(\otimes(A_k))) \). Therefore,
\[ \otimes(\phi^{-1}(L_1, \ldots, L_k)) = \otimes(\phi^{-1}(L(A_1)), \ldots, \phi^{-1}(L(A_k))) = \phi^{-1}(\otimes(L(A_1)), \ldots, L(A_k))) = \phi^{-1}(\otimes(L_1, \ldots, L_k)) \]
as expected.

\[ \square \]

3 The combinatorics of the star of a binary Boolean operation

We list in the Table below the 16 possible Boolean functions and we write what they correspond to in terms of operation on languages.

\[ L_1 \cdot L_2 = \{ z \mid [z \in L_1] \cdot [z \in L_2] = 1 \}. \]
From Table 1 we observe that there are only 10 non degenerated operations that depend truly of the two operands \( L_1 \) and \( L_2 \). These operations are the only ones that we have to consider. Table 1 also shows that each of the 10 non degenerated operations can be obtained by acting with union, intersection or xor on two languages or their complementary.

We use the symbol \( \oplus \) to denote any operation defined by

\[
L_1 \oplus L_2 = (L_1 \bullet L_2)'.
\]

Examples 2 and 3 together with Theorem 2 show that any \( \oplus \) is 1-uniform. We define the modifier

\[
\exists \mathsf{Bool}_* = \exists \mathsf{star} \circ \mathsf{Bool}_*.
\]

To be more precise, let us describe how the modifier \( \exists \mathsf{Bool}_* \) acts on automata: if \( A_1 = (\Sigma, Q_1, i_1, F_1, \delta_1) \) and \( A_2 = (\Sigma, Q_2, i_2, F_2, \delta_2) \), then

\[
\exists \mathsf{Bool}_*(A_1, A_2) = (\Sigma, 2^{Q_1 \times Q_2}, \emptyset, \{ E \in 2^{Q_1 \times Q_2} \mid E \cap F \neq \emptyset \} \cup \{ \emptyset \}, \delta)
\]

where \( F = (F_1 \times Q_2) \bullet (Q_1 \times F_2) \) and, for all \( a \in \Sigma,

\[
\delta^a(\emptyset) = \begin{cases} 
\{ (\delta^a_1(i_1), \delta^a_2(i_2)) \} & \text{if } (\delta^a_1(i_1), \delta^a_2(i_2)) \in E \\
\{ (\delta^a_1(i_1), \delta^a_2(i_2)), (i_1, i_2) \} & \text{otherwise}
\end{cases}
\]

and, for all \( E \neq \emptyset, \delta^a(E) = \begin{cases} 
E \cdot (\delta^a_1, \delta^a_2) & \text{if } E \cdot (\delta^a_1, \delta^a_2) \cap F = \emptyset \\
E \cdot (\delta^a_1, \delta^a_2) \cup \{ (i_1, i_2) \} & \text{otherwise}
\end{cases}
\]

Theorem 1 states that any \( \oplus \) admits a family of 2-monsters as witness. For any positive integers \( m, n \), let \( (M_1, M_2) = \text{Mon}_{m,n}^{\overline{m-1},\overline{0}} \). We are going to show that, for all \( (m, n) \in (\mathbb{N} \setminus \{0\})^2 \), \((L(M_1)), (L(M_2))\) is indeed a witness for any \( \oplus \). For any positive integers \( m, n \), any \( F_1 \subseteq \llbracket m \rrbracket \), and \( F_2 \subseteq \llbracket n \rrbracket \), let us denote by \( \text{M}_{F_1,F_2}^* \) the DFA \( \exists \mathsf{Bool}_*(\text{Mon}_{m,n}^{F_1,F_2}) \).

We identify elements of \( \mathbb{2}^{\llbracket m \rrbracket \times \llbracket n \rrbracket} \) to Boolean matrices of size \( m \times n \). Such a matrix is called a tableau when crosses are put in place of 1s, and 0s are erased. The set of such tableaux is denoted by \( \mathcal{T}_{m,n} \). Each tableau of \( \mathcal{T}_{m,n} \) labels a state of \( \text{M}_{F_1,F_2}^* \).

If \( T \) is an element of \( \mathcal{T}_{m,n} \), we denote by \( T_{x,y} \) the value of the Boolean matrix \( T \) at row \( x \) and column \( y \). Therefore, the three following assertions mean the same thing : a cross is at the coordinates \((x, y)\), \( T_{x,y} = 1 \), and \((x, y) \in T \). We denote by \( \text{IsFinal}(x, y) \) the Boolean value \([i \in F_1] \bullet [j \in F_2] = [(i, j) \in F] \). The cell \((x, y)\) of \( T \) is \( \text{final} \) if and only if \( \text{IsFinal}(x, y) = 1 \). The tableau \( T \) is \( \text{final} \) if there exists \((i, j) \in \llbracket m \rrbracket \times \llbracket n \rrbracket \) such that \( T_{x,y} = 1 \) and \( \text{IsFinal}(x, y) = 1 \). The \( \text{final zone} \) of the tableau \( T \) is the set of all his final cells, \( i.e. \) \( \text{FinalZone}(T) = \{(i, j) \in \llbracket m \rrbracket \times \llbracket n \rrbracket \mid \text{IsFinal}(i, j) \} \). When

| 0 \( \bullet \) 0 | 0 0 0 0 0 0 1 1 1 | 0 0 0 0 1 1 1 1 1 | 0 1 1 1 0 0 0 1 1 | 0 0 0 0 0 0 0 0 1 | 0 0 0 0 0 0 0 0 1 |
| 0 \( \bullet \) 1 | 0 0 0 0 1 1 0 0 0 | 0 0 0 0 1 1 0 0 0 | 0 0 0 0 1 1 0 0 0 | 0 0 0 0 1 1 0 0 0 |
| 1 \( \bullet \) 0 | 0 0 1 1 0 0 0 1 1 | 0 0 1 1 0 0 0 1 1 | 0 0 1 1 0 0 0 1 1 | 0 0 1 1 0 0 0 1 1 |
| 1 \( \bullet \) 1 | 0 1 0 1 0 1 0 1 0 | 0 1 0 1 0 1 0 1 0 | 0 1 0 1 0 1 0 1 0 | 0 1 0 1 0 1 0 1 0 |

Table 1: The 16 binary Boolean functions.
there is no ambiguity, we refer to final zone for the final zone of any tableau labelling a state of the automaton we consider.

Notice that the final zone of $T$ is exactly the set $F = (F_1 \times \mathbb{N}) \bullet (\mathbb{M} \times F_2)$ of the modifier $\exists \mathbb{S} \mathbb{B} \mathbb{O} \mathbb{L}$. Notice also that a tableau $T \neq \emptyset$ is final if and only if it labels a non empty final state in the automaton $M_{F_1,F_2}$.

3.1 Valid states and accessibility

We refer to the final zone for both the final zone of $T$ and of $T'$ because these two tableaux label a state of the same automaton.

**Definition 6** A tableau $T$ labelling a state of $M_{F_1,F_2}$ is valid if either its final zone is empty or if $T_{(0,0)} = 1$.

Let $\text{Val}_{F_1,F_2}$ be the set of valid tableaux labeling states of $M_{F_1,F_2}$ and $\text{Val}(M_{F_1,F_2})$ be the restriction of the DFA $M_{F_1,F_2}$ to the states of $\text{Val}_{F_1,F_2}$. The aim of this section is to compare the set $\text{Val}_{F_1,F_2}$ to the set, denoted by $\text{Acc}_{F_1,F_2}$, of accessible states of $M_{F_1,F_2}$. Straightforward from the definition of the modifier $\exists \mathbb{S} \mathbb{B} \mathbb{O} \mathbb{L}$, the set of accessible states are valid. Nevertheless, we show that the accessibility of the valid states depends on the final zone.

For any tableau $T$, we define the finest equivalence relation $\Delta$ on $T$ satisfying $(i,j)\Delta(i',j')$ and $(i,j)\Delta(i',j)$.

**Lemma 1** Let $T \neq \emptyset$ be a tableau representing a state of $M_{F_1,F_2}$. Let us suppose that for all $i \neq 0$ and for all $j \neq 0$, $(i,j)$ is not in the final zone of $T$. Then, we have

$$T \text{ is accessible implies } \#T/\Delta = 1.$$ 

**Proof:** Let us prove the result by induction on the minimal words $w$ such that $\delta^w(\emptyset) = T$. If $w \in \Sigma$ then either $\#T = 1$ or $T = \{(i,0),(0,0)\}$ for some $i$ or $T = \{(0,j),(0,0)\}$ for some $j$. Obviously, in the three cases $\#T/\Delta = 1$.

Suppose now the result true for any strict prefix of $w = w'a$ with $a \in \Sigma$. Let $T' = \delta^w(\emptyset)$. By induction $\#T'/\Delta = 1$. We also have $T = \delta^a(T')$.

First consider the tableau $\hat{T} = T',(\delta_i^a,\delta_j^a)$. If $(i,j)\Delta(i',j')$ in $T'$ then $(\delta_i^a(i),\delta_j^a(i))\Delta(\delta_i^a(i),\delta_j^a(i))$. Indeed, this is obtained by transitivity considering that

$$(i,j),(i,j') \in T' \Rightarrow (\delta_i^a(i),\delta_j^a(j)),(\delta_i^a(i),\delta_j^a(j')) \in T'$$

and

$$(i,j),(i',j) \in T' \Rightarrow (\delta_i^a(i),\delta_j^a(j)),(\delta_i^a(i'),\delta_j^a(j)) \in T'.$$

So by induction $\#T'/\Delta = \#\hat{T}/\Delta = 1$.

If $T \neq \hat{T}$ then $T = \hat{T} \cup \{(0,0)\}$ and there exists $(i,j) \in T$ in the final zone. But this implies that $i = 0$ or $j = 0$, so $\Delta(i,j)$ and so $\#T/\Delta = 1$.

**Proposition 1** When the final zone is included in $(\mathbb{M} \times \{0\}) \cup \{(0) \times \mathbb{N}\}$, we have

$$\text{Acc}_{F_1,F_2} \subseteq \text{Val}_{F_1,F_2}.$$
**Proof:** Straightforward from the definition of the modifier $\mathcal{St BOOL}_\bullet$, we have $\text{Acc}^{F_1,F_2} \subseteq \text{Val}^{F_1,F_2}$. In the context of the proposition, the tableau $\{(0,0),(1,1)\}$ is valid but $\Delta$ has two distinct classes. Hence, from Lemma 1, it is not accessible. □

Next definition defines an order $<$ on tableaux of $\mathcal{T}_{m,n}$ labelling the states of $M^{F_1,F_2}_\bullet$.

**Definition 7** Let $T$ and $T'$ be two tableaux of $\mathcal{T}_{m,n}$. The number of cross in the non-final zone of $T$ is denoted by $\text{nf}(T)$.

We write $T < T'$

1. if $\#T < \#T'$.

![Diagram 1]

2. if $\#T = \#T'$ and if $\text{nf}(T) < \text{nf}(T')$.

![Diagram 2]

3. if $\#T = \#T'$, $\text{nf}(T) = \text{nf}(T')$ and $T_{0,0} = 1$ while $T'_{0,0} = 0$.

![Diagram 3]

**Proposition 2** If the final zone of $M^{F_1,F_2}_\bullet$ is not a subset of $((0) \times \llbracket n \rrbracket) \cup (\llbracket m \rrbracket \times \{0\})$ then any of its non-empty state is accessible if and only if it is a valid tableau.

**Proof:** From the definition of the transition function of $\mathcal{St BOOL}_\bullet$, every non-empty tableau $T$ which is not valid, i.e. with $T_{0,0} = 0$ and a cross in the final zone, is not accessible.

Now we prove the converse by induction on the order $<$ defined above.

The initial case is obvious since the tableau $\{(0,0)\}$ is accessible from $\emptyset$ by reading the letter $(\text{Id}, \text{Id})$. Let $T' > \{(0,0)\}$ be a valid tableau. We exhibit a letter $(f,g)$ together with a valid tableau $T < T'$ such that $\delta^{(f,g)}(T) = T'$. Assuming, by induction, that $T$ is accessible, this shows the accessibility of $T$.

Suppose first that $T'$ is not final. We have to consider two cases:

1. If $T'_{0,0} = 0$ then we consider a cross $(i,j) \in T'$ and we set $T = T' \cdot ((0i),(0j))$. We notice that $T$ is a valid tableau that has the same number of crosses as $T'$, at most as many non-final crosses as $T'$ because all the crosses in $T'$ are non-final, and $T_{0,0} = 1$. Hence, $T < T'$ and by induction $T$ is accessible. Since $T \cdot ((0i),(0j)) = T'$ is not final, we have $\delta^{(0i),(0j)}(T) = T'$. We deduce that $T'$ is accessible.
2. If \( T'_{0,0} = 1 \) then, since \( T' \) is not final, the cell \((0, 0)\) is not final. Furthermore \( \{(0, 0)\} < T' \) implies there exists \((i, j) \neq (0, 0)\) such that \( T'_{i,j} = 1 \). Let \((i', j')\) be a cell in the final zone with \( i' \neq 0 \) and \( j' \neq 0 \). We have to consider the following two sub-cases:

(a) Suppose that we can choose \((i, j)\) such that both \( i \) and \( j \) are not zero. We set \( T = T' \cdot ((i, i'), (j, j')) \). The tableau \( T \) is valid and has the same number of crosses as \( T' \) but has less non-final crosses than \( T' \) because \( T \) contains at least one final cross while all the crosses of \( T' \) are non-final. So we have \( T < T' \) and \( \delta(i, j)(T) = T \cdot ((i, i'), (j, j')) = T' \) because \( T \) is not final. By induction, we deduce that \( T' \) is accessible.

(b) If the tableau \( T' \) contains crosses only in its first row and in its first column then we suppose that \( i = 0 \) (the other case is symmetric) which means that there exists \( j \neq 0 \) such that \( T'_{0,j} = 1 \). For each of the following cases we check that \( T < T' \) and \( \delta(f, g)(T) = T \cdot (f, g) = T' \):

i. If \( T'_{i',0} = T'_{0,j} = 0 \) then we set \( T = T' \cup \{(i', j')\} \setminus \{(0, j)\} \) and \((f, g) = \left(\left( \begin{array}{c} f' \\
0 \end{array} \right), \left( \begin{array}{c} j' \\
0 \end{array} \right) \right) \).

ii. If \( T'_{i',0} = 0 \) and \( T'_{0,j'} = 1 \) then we set \( T = T' \cup \{(i', j')\} \setminus \{(0, j')\} \) and \((f, g) = \left(\left( \begin{array}{c} f' \\
0 \end{array} \right), \text{Id} \right) \).

iii. If \( T'_{i',0} = 1 \) and \( T'_{0,j'} = 0 \) then, symmetrically to the previous case, we set \( T = T' \cup \{(i', j')\} \setminus \{(i', 0)\} \) and \((f, g) = \left(\left( \begin{array}{c} f' \\
0 \end{array} \right), \text{Id} \right) \).

iv. If \( T'_{i',0} = T'_{0,j'} = 1 \) then we set \( T = T' \cdot (\text{Id}, (0, j')) \) and \((f, g) = (\text{Id}, (0, j')) \).
Now suppose that \( T' \) is final. This implies that \( T'_{0,0} = 1 \) and there exists a final cell \((i, j)\) such that \( T'_{i,j} = 1 \). We have two cases to consider.

1. If \((i, j) \neq (0, 0)\) then we set \( T = T' \cdot ((0 i), (0 j)) \setminus \{(i, j)\} \). Since \( T \) has one cross less than \( T' \), we have \( T < T' \). Furthermore, \( \delta(\text{Id},(0 i), (0 j)) \cdot T = T' \cdot ((0 i), (0 j)) \cup \{(0,0)\} = T' \). By induction, the state \( T' \) is accessible.

2. If \((0,0)\) is the only final cross in \( T' \) then we consider \( i \neq 0 \) and \( j \neq 0 \) be such that \((i, j)\) is a final cell. Since \([(0,0)] < T', \) there exists \((i', j') \in T' \setminus [(0,0)]\). We consider the following two cases:

(a) Suppose that we can choose \( i' \neq 0 \) and \( j' \neq 0 \). We set \( T = T' \cdot ((i i'), (j j')) \). Since \( T_{0,0} = 1 \), the tableau \( T \) is final and valid. Furthermore, \( T \) has the same number of crosses as \( T' \) but \( T \) has more final crosses than \( T' \) because \( T_{i,j} = 1 \) and \( T'_{i,j} = 0 \). This implies \( T < T' \).

We check that \( \delta(i i', (i', j')) \cdot T = T' \cdot ((i i'), (j j')) = T' \). Hence, by induction \( T' \) is accessible.

(b) Suppose that \( T'_{k,j} = 1 \) implies \( k = 0 \) or \( j = 0 \). We assume that \( i' = 0 \) and \( j' \neq 0 \) (the other case is obtained symmetrically). Let us assume first that there exists \( i' \neq 0 \) such that \( T'_{i',0} = 1 \). In that case we define \( T^{(2)} = T' \cdot ((i i'), (j j')) \) and \( T = T^{(2)} \cdot ((0,0), \text{Id}) \). We observe that \( \delta((i,0), \text{Id}(T) = T' \cdot ((i,0), \text{Id})) = T^{(2)} \).
We check that $T < T'$. Hence, by induction $T'$ is accessible.

Now consider the case where, for every $i'' \neq 0$, $T'_{i'',0} = 0$. The only crosses in $T'$ are in the line 0, otherwise we apply the previous case. We construct $T(2) = T' \cdot (\text{Id}, (j')')$ and $T = T(2) \cup (i, j) \setminus \{(0, j)\}$. We have $T(2) = \delta((\delta_0), \text{Id})(T)$.

Furthermore $\delta = (\text{Id}, (j')')(T(2)) = T(2) \cdot (\text{Id}, (j')') = T'$.

We check that $T < T'$ and by induction $T'$ is accessible.

\[\square\]

The results of the section are summarized in the following theorem.

**Theorem 3** The automaton $\text{Acc}(M_{F_1,F_2})$ is a sub-automaton of $\text{Val}(M_{F_1,F_2})$. Moreover, the two following assertions are equivalent

1. The final zone is not empty and not included in $(\llbracket m \rrbracket \times \{0\}) \cup \{0\} \times \llbracket n \rrbracket)$.

2. We have $\text{Acc}(M_{F_1,F_2}) = \text{Val}(M_{F_1,F_2})$. 

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3.2 Saturation and Nerode equivalence

Let \( \hat{M}_{F_1,F_2} \) be the automaton with the same alphabet, the same states, the same initial state and the same final states as \( M_{F_1,F_2} \) but with the transition function \( d \) defined by \( d(f,g)(T) = T \cdot (f,g) \).

The \( \hat{M} \) automata are simpler than the \( M \) ones as the transition function is only defined by the composition of functions. Therefore we will show that two states are equivalent in \( \hat{M}_{F_1,F_2} \) if and only if they are equivalent in \( M_{F_1,F_2} \). Thus, we compute the Nerode equivalence on \( \hat{M}_{F_1,F_2} \) which is easier than to compute it on \( M_{F_1,F_2} \).

Let us denote by \( E_{i,j} = \{(i, j)\} \) the tableau with only one cross at position \( (i, j) \)

Lemma 2 Let \( T \) be a non empty tableau and \( w \in \left(\left[ m \right]^{\left[ m \right]} \times \left[ n \right]^{\left[ n \right]}\right) \ast \). The tableau \( \delta^w(T) \) is final in \( \hat{M}_{F_1,F_2} \) if and only if at least one of the following assertion is true:

1. There exists \( (i, j) \in T \) such that \( d^w(E_{i,j}) \) is final.

2. There exists \( w = ps \in \left(\left[ m \right]^{\left[ m \right]} \times \left[ n \right]^{\left[ n \right]}\right) \ast \), with \( s \neq \varepsilon \) such that \( \delta^p(T) \) is final and \( d^s(E_{0,0}) \) is final.
Proof: Notice that for any \( w = ps \), if \( (i, j) \in \delta^w(T) \) and \( d^w(E_{i,j}) \) is final then the tableau \( \delta^w(T) \) is final. The two assertions are special cases of this property. This shows the if part of the lemma.

Now we prove the converse. Let \( w \in (\mathbb{m}^{[m]} \times \mathbb{n}^{[n]})^* \) such that \( \delta^w(T) \) is final. We prove the result by induction on \( w \). If \( w = \epsilon \) then the result is obvious. If \( |w| = 1 \) then by construction \( d^w(T) \) is final and this means that there exists \( (i, j) \in T \) such that \( d^w(E_{i,j}) \) is final. Now we assume that the length of \( w \) is at least 2. We have \( w = a \cdot w' \) for some \( a \in \mathbb{m}^{[m]} \times \mathbb{n}^{[n]} \) and we set \( T' = \delta^a(T) \).

By construction, we have \( T' = d^a(T) \) or \( T' = d^a(T) \cup \{(0, 0)\} \) depending on the finality of \( d^a(T) \). By induction one of the following property is true:

1. There exists \( (i', j') \in T' \) such that \( d^a(E_{i',j'}) \) is final. If \( (i', j') = (0, 0) \) and \( T' \) is final then the second assertion is immediately true for \( T \). If \( (i', j') \neq (0, 0) \) or if \( T' \) is non final then there exists \( (i, j) \in T \) such that \( d^a(E_{i,j}) = E_{i',j'} \) and \( d^w(E_{i,j}) = d^a(E_{i',j'}) \) is final; so, the first assertion is true for \( T \).

2. There exists \( w' = p's \) with \( s \neq \epsilon \) such that \( \delta^w(T') \) is final and \( d^w(E_{0,0}) \) is final. Hence, \( \delta^w(T') \) is final, and the second assertion is true for \( T \).

\[ \square \]

Lemma 3 For any \( w \in (\mathbb{m}^{[m]} \times \mathbb{n}^{[n]})^* \), there exists \( z \in \mathbb{m}^{[m]} \times \mathbb{n}^{[n]} \) such that for any non empty tableau \( T \) we have \( d^w(T) = d^z(T) \).

Proof: The result comes directly from the fact that any composition of functions of \( \mathbb{m}^{[m]} \times \mathbb{n}^{[n]} \) is a function of \( \mathbb{m}^{[m]} \times \mathbb{n}^{[n]} \) i.e. \( d^{f_1 \circ \ldots \circ f_n} = d^{f_n \circ \ldots \circ f_1} \).

\[ \square \]

The following result characterizes Nerode classes of \( \mathbb{M}^{F_1,F_2} \).

Proposition 3 Any two states represented by non-empty tableaux are Nerode equivalent in \( \mathbb{M}^{F_1,F_2} \) if and only if they are Nerode equivalent in \( \mathbb{M}^{F_1,F_2} \). Furthermore, in \( \mathbb{M}^{F_1,F_2} \), the empty tableau is Nerode equivalent to the state \( (0, 0) \) if and only if \( (0, 0) \) is in the final zone.

Proof: Let \( T \) and \( T' \) be two non-empty tableaux. Suppose first that \( T \) and \( T' \) are not Nerode equivalent in \( \mathbb{M}^{F_1,F_2} \). Without loss of generalities, we assume there exists \( w \in (\mathbb{m}^{[m]} \times \mathbb{n}^{[n]})^* \) such that \( \delta^w(T) \) is final while \( \delta^w(T') \) is not final. We assume that \( w \) is minimal in the sense that for any prefix \( p \neq w \) of \( w \), \( \delta^p(T) \) and \( \delta^p(T') \) have the same finality. From Lemma 2 one has to consider two cases:

1. There exists \( (i, j) \in T \) such that \( d^w(E_{i,j}) = E_{i',j'} \) is final. We have \( (i, j) \notin T' \) because, in that case, the tableau \( T' \) would be final. Hence, \( d^w(T) \) is final because \( (i', j') \in d^w(T) \) while \( d^w(T') \) is not final because \( d^w(T') \subset \delta^w(T') \). So the tableaux \( T \) and \( T' \) are not Nerode equivalent in \( \mathbb{M}^{F_1,F_2} \).

2. There exists \( w = ps \) with \( s \neq \epsilon \) such that \( \delta^w(T) \) is final and \( d^w(E_{0,0}) \) is final. In that case, since \( \delta^w(T') \) is not final, we have \( (0, 0) \notin \delta^p(T') \) which is also non final. This contradicts the minimality of \( w \).

Conversely, from Lemma 3 it suffices to prove that if \( a \in \mathbb{m}^{[m]} \times \mathbb{n}^{[n]} \) is such that \( d^a(T) \) is final while \( d^a(T') \) is not final then \( \delta^a(T) \) is final while \( \delta^a(T') \) is not final. This property comes directly from the definition of \( \delta \) since the addition of \( (0, 0) \) does not change the finality of a final state.
The case of the empty tableau is straightforward from the construction. □

This proposition means that to describe Nerode equivalence on non-empty tableaux in \( M_{F_1,F_2} \) it suffices to describe it in the simpler automaton \( \hat{M}_{F_1,F_2} \).

From now we only consider non-empty tableaux. Recall that we denote Nerode equivalence by \( \sim \).

**Proposition 4** For any two states \( T_1 \) and \( T_2 \) of \( \hat{M}_{F_1,F_2} \), we have \( T_1 \sim T_2 \) implies \( T_1 \sim T_1 \cup T_2 \).

Equivalently each Nerode class in \( \hat{M}_{F_1,F_2} \) is a join lattice for the inclusion order.

**Proof:** Since for any \( w \in (\{m\}^{[m]} \times \{n\}^{[n]})^* \) there exists \( a \in \{m\}^{[m]} \times \{n\}^{[n]} \) such that \( d^a = d^w \), we have just proved that \( d^a(T_1) \) is final if and only if \( d^a(T_1 \cup T_2) \) is final for any \( a \in \{m\}^{[m]} \times \{n\}^{[n]} \).

If \( d^a(T_1) \) is final then there exists \((i,j) \in T_1 \) such that \( d^a(E_{i,j}) \) is final and so, since \((i,j) \in T_1 \cup T_2 \) the tableau \( d^a(T_1 \cup T_2) \) is also final.

Conversely, if \( d^a(T_1 \cup T_2) \) is final then there exists \((i,j) \in (T_1 \cup T_2) \) such that \( d^a(E_{i,j}) \) is final. If \((i,j) \in T_1 \) then \( d^a(T_1) \) is obviously final. If \((i,j) \in T_2 \) then \( d^a(T_2) \) is final and, since \( T_1 \) and \( T_2 \) are Nerode equivalent, the tableau \( d^a(T_1) \) is also final.

From this proposition, in each Nerode class there exists a unique state represented by a tableau with a maximal number of crosses. This allows us to give the following definition.

**Definition 8** A tableau is saturated if it is the unique tableau having the maximal number of crosses in its Nerode class, i.e. \( T \) is saturated if and only if \( T \sim T' \) implies \( T' \subset T \).

The set of saturated tableaux is representative of Nerode classes. We denote by \( \text{Sat}(T) \) the unique saturated tableau in the Nerode class of \( T \) and by \( \text{Sat}(A) \) the automaton isomorphic to \( A/\sim \), the states of which are labelled by saturated tableaux, for any sub-automaton \( A \) of \( M_{F_1,F_2} \).

**Corollary 1** For any tableau \( T \), the two following assertions are equivalent

1. There exists a valid tableau in the class of \( T \).
2. The tableau \( \text{Sat}(T) \) is valid.

**Proof:** We have only to prove that \( T \) is valid implies \( \text{Sat}(T) \) is valid. The tableaux \( T \) and \( \text{Sat}(T) \) have the same finality. If \( T \) and \( \text{Sat}(T) \) are both final then \( T_{0,0} = 1 \) because \( T \) is valid. From Proposition 3 this implies that \( \text{Sat}(T)_{0,0} = 1 \) and so \( \text{Sat}(T) \) is valid. If \( T \) and \( \text{Sat}(T) \) are both non final then \( \text{Sat}(T) \) have no cross in the final zone, so it is valid. □

**Corollary 2** If \( T \) and \( T' \) are Nerode equivalent in \( M_{F_1,F_2} \) then for any pair \((f,g) \in \{m\}^{[m]} \times \{n\}^{[n]} \) the tableaux \( d^{(f,g)}(T) \) and \( d^{(f,g)}(T') \) are Nerode equivalent.

**Furthermore, when \( f \) and \( g \) are two permutations, the converse is true and \( d^{(f,g)}(\text{Sat}(T)) = \text{Sat}(d^{(f,g)}(T)) \).**

We denote by

\[
\text{SV}_{F_1,F_2} = \{ \text{Sat}(T) \mid T \in \text{Val}_{F_1,F_2} \}
\]

the set of valid saturated tableaux.
Theorem 4 For any $F_1, F_2$ and any operation $\bullet$, we have $\#\text{Min}(M_{\bullet}^{F_1,F_2}) \leq \#\text{SV}_{\bullet}^{F_1,F_2}$. Furthermore, the equality holds when there exist $i \neq 0$ and $j \neq 0$ such that $(i, j)$ is in the final zone.

Proof: The automaton $\text{Sat}(\text{Val}(M_{\bullet}^{F_1,F_2}))$ is the Nerode quotient of a sub-automaton of $M_{\bullet}^{F_1,F_2}$ containing at least all its accessible states. This implies $\#\text{Min}(M_{\bullet}^{F_1,F_2}) \leq \#\text{SV}_{\bullet}^{F_1,F_2}$. Furthermore Proposition 2 means $\text{Val}(M_{\bullet}^{F_1,F_2}) = \text{Acc}(M_{\bullet}^{F_1,F_2})$ when there exist $i \neq 0$ and $j \neq 0$ such that $(i, j)$ is in the final zone. So by construction, the equality holds. □

As a consequence, the state complexity of $\oplus$ satisfies the inequality below:

$$\text{sc}(m, n) \leq \max \left\{ \#\text{SV}_{\bullet}^{F_1,F_2} \mid F_1 \subset \{m\}, F_2 \subset \{n\} \right\}. $$

The main implication of Theorem 4 is that in order to compute the state complexity, we need to study the combinatorics of saturated tableaux. This is what we do in the following sections. First, we study the (simplest) $2 \times 2$ case and then we generalize it for bigger tableaux, via the notion of local saturation.

3.3 Computing the Nerode equivalence: the $2 \times 2$ case

Let us set $m = n = 2$ and investigate in more details Nerode equivalence of the automaton $M_{\bullet}^{F_1,F_2}$ in that case. We assume that $\#F_1 = \#F_2 = 1$ and that the operation $\bullet$ is not degenerated, otherwise at most one of the automata of $\text{Mon}_{n,n}^{F_1,F_2}$ is not minimal. Nerode classes depend only on the set $(F_1 \times \{2\}) \bullet (\{2\} \times F_2)$ which can be either a singleton $\{(\alpha, \beta)\}$, or the complementary of a singleton $(\{2\} \times \{2\}) \setminus \{(\alpha, \beta)\}$, or a diagonal set $\{(\alpha, \beta), (1-\alpha, 1-\beta)\}$. In each case, acting on languages or complements does not change the global form of $(F_1 \times \{2\}) \bullet (\{2\} \times F_2)$. According, from Table 1 we have to consider only three cases

A If $\bullet = \cap$ then the final zone is $\{(\alpha, \beta)\}$ for some $\alpha, \beta \in \{0, 1\}$. We check that all Nerode classes are singletons.

X If $\bullet = \oplus$ then the final zone is $\{(\alpha, \beta), (1-\alpha, 1-\beta)\}$ for some $\alpha, \beta \in \{0, 1\}$. We check that the only Nerode class containing at least two tableaux is the class

$$\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times 
\end{bmatrix}.$$

O If $\bullet = \cup$ then the final zone is $(\{2\} \times \{2\}) \setminus \{(\alpha, \beta)\}$ for some $\alpha, \beta \in \{0, 1\}$. We check that the only Nerode class containing at least two tableaux is the class

$$\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times 
\end{bmatrix}.$$

As a consequence of the classification above, for any $2 \times 2$-tableau $T$, we have $\text{Sat}(T) \in \left\{ \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \right\}$.

In Case A, we have $\text{Sat}(T) = T$ for any $T$.

In Case X, we have $\text{Sat}(T) = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$ if and only if $\#T > 2$.

In Case O, we have $\text{Sat}(T) = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$ if and only if $T$ contains a diagonal, i.e. there exist $\alpha, \beta \in \{0, 1\}$ such that $\{(\alpha, \beta), (1-\alpha, 1-\beta)\} \subseteq T$. 18
3.4 Computing the Nerode equivalence: from the $2 \times 2$ case to the general case

We assume that $m, n \geq 2$, $#F_1 \not\in \{0, m\}$, $#F_2 \not\in \{0, n\}$ and that the law $\bullet$ is not degenerated. According to Table [1] without loss of generality, we assume that $\bullet \in \{\cup, \cap\}$.

**Remark 2** The above condition implies that there exists at least one row (resp. column) containing a final cell and a non final cell.

**Definition 9** Let $T \in 2^{[m] \times [n]}$. Let $i_0, i_1 \in [m]$, $j_0, j_1 \in [n]$ and $I = \{i_0, i_1\} \times \{j_0, j_1\}$. We denote by $T[I] = T \cap I$ the restriction of $T$ to the cells $\{i_0, i_1\} \times \{j_0, j_1\}$. A tableau $T$ is called simple if $T = T[I]$ for some $I = \{i_0, i_1\} \times \{j_0, j_1\}$.

Let $T$ be a simple tableau and $I = \{i_0, i_1\} \times \{j_0, j_1\}$ with $i_0 \leq i_1$ and $j_0 \leq j_1$ such that $T = T[I]$. We define the $I$-reduced (or simply reduced when the context does not induce ambiguity) of $T$ as the $2 \times 2$ tableau $\text{Red}_I(T) = \{(a, \beta) \mid a, \beta \in [2]\}$ and $(i_0, j_0) \in T$. When there is no ambiguity we omit the subscript $I$.

Symmetrically, for any $2 \times 2$-tableau $X$, $m, n \geq 2$, and $I = \{i_0, i_1\} \times \{j_0, j_1\}$ with $0 \leq i_0 < i_1 \leq m - 1$ and $0 \leq j_0 < j_1 \leq n - 1$, we define its $I$-induced as the tableau $\text{Ind}_I(X)$ which is the unique $m \times n$-simple tableau the $I$-reduced of which is $X$, i.e. $\text{Red}_I(\text{Ind}_I(X)) = X$.

Let symbol $F$ refer to the final zone of any tableau $T$. More precisely, we have $F = (F_1 \times [n]) \bullet ([m] \times F_2)$. We consider that the final zone of any reduced tableau is $F_R = ([1] \times [2]) \bullet ([2] \times [1])$. By this way, the notion of saturated reduced tableau makes sense. If $I = \{i_0, i_1\} \times \{j_0, j_1\} \subset [m] \times [n]$ then the set $F_I = I \cap F$ refers to the restricted final zone associated to $I$.

**Remark 3** The restricted final zone satisfies one of the following identities

(a) $F_I = \emptyset$,

(b) $j_0 \neq j_1$ and $F_I = \{(i_0, j_0), (i_0, j_1)\}$ for some $a \in [2]$,

(c) $i_0 \neq i_1$ and $F_I = \{(i_0, j_0), (i_1, j_0)\}$ for some $b \in [2]$,

(d) $i_0 \neq i_1$, $j_0 \neq j_1$, and $F_I = \{(i_0, j_0), (i_1, j_1)\} \bullet \{(i_0, i_1) \times \{j_0, j_1\}\}$ for some $a, b \in [2]$,

(e) $F_I = I$.

The following lemma shows that every simple tableau can be projected to its reduced tableau preserving the property of finality of the cells.

**Lemma 4** Let $I = \{i_0, i_1\} \times \{j_0, j_1\} \subset [m] \times [n]$. There exist two maps $f_I, g_I \in [2]^{[2]}$ such that for any $a, \beta \in [2]$, $(i_0, j_0) \in F_I$ if and only if $(f_I(a), g_I(\beta)) \in F_R$.

**Proof:** The construction is summarized in Table 2 according to the type $A$, $X$ or $O$ of $\bullet$ (see Section 3.3) and the cases listed in Remark 3.

| Types($\bullet$) \ Cases($I$) | (a) | (b) | (c) | (d) | (e) |
|-------------------------------|-----|-----|-----|-----|-----|
| $A$                           | $f_I$ | 0 | $\alpha \rightarrow [\alpha = a]$ | 1 | $\alpha \rightarrow [\alpha = a]$ | 1 |
|                              | $g_I$ | 0 | 1 | $\beta \rightarrow [\beta = b]$ | $\beta \rightarrow [\beta = b]$ | 1 |
| $X$                           | $f_I$ | 0 | $\alpha \rightarrow [\alpha = a]$ | 0 | $\alpha \rightarrow [\alpha = a]$ | 1 |
|                              | $g_I$ | 0 | 0 | $\beta \rightarrow [\beta = b]$ | $\beta \rightarrow [\beta = b]$ | 0 |
| $O$                           | $f_I$ | 0 | $\alpha \rightarrow [\alpha = a]$ | 0 | $\alpha \rightarrow [\alpha = a]$ | 1 |
|                              | $g_I$ | 0 | 0 | $\beta \rightarrow [\beta = b]$ | $\beta \rightarrow [\beta = b]$ | 1 |

Table 2: The maps $f_I$ and $g_I$. 19
Example 4 The following picture illustrates the various items of Remark 3 for $\bullet = \oplus$. The red cells correspond to the final zone $F$ of the tableau.

The pair of maps $(f_1, g_1)$ of Table 2 is illustrated below with respect to each final zone $F_I$, (a) to (e) drawn in the previous figure.

Next result states that the operations of saturation and reduction commute for simple tableaux. This is the first step to compute the saturation of any tableau.

Proposition 5 Let $I = \{i_0, i_1\} \times \{j_0, j_1\}$ with $i_0 < i_1$ and $j_0 < j_1$. Let $T, T'$ be two simple tableaux such that $T|_I = T$ and $T'|_I = T'$ and let $X$ and $X'$ be two $2 \times 2$-tableaux. We have the following properties

1. The tableau $\text{Sat}(T)$ is simple and $\text{Sat}(T) = \text{Sat}(T)|_I$.
2. If $T$ and $T'$ are Nerode equivalent then $\text{Red}_I(T)$ and $\text{Red}_I(T')$ are Nerode equivalent.
3. If $X$ and $X'$ are Nerode equivalent then $\text{Ind}_I(X)$ and $\text{Ind}_I(X')$ are Nerode equivalent.
4. We have $\text{Sat}(\text{Red}_I(T)) = \text{Red}_I(\text{Sat}(T))$.

Proof:
1. We have to prove that $\text{Sat}(T) \subseteq I$.
Let $(i, j) \in \text{Sat}(T)$. If $(i, j) \notin I$ then $(i, j)$ matches with one of the following three cases

(a) If $i \notin [i_0, i_1]$ and $j \notin [j_0, j_1]$ then we consider two crosses: a first one, $(\alpha, \beta)$, belonging to the final zone, and a second one, $(\alpha', \beta')$ not belonging to the final zone. Let $f$ and $g$ be such that $f(i_0) = f(i_1) = \alpha'$, $g(j_0) = g(j_1) = \beta'$, $f(i) = \alpha$, and $g(j) = \beta$. We have $d^{(\mathcal{J}, \mathcal{S})}(T) = \{(\alpha', \beta')\}$ which is not final, while $d^{(\mathcal{J}, \mathcal{S})}(\text{Sat}(T))$ is final because $(\alpha, \beta) \in d^{(\mathcal{J}, \mathcal{S})}(\text{Sat}(T))$. This contradicts the fact that $T$ and $\text{Sat}(T)$ are in the same Nerode class.

(b) If $i \in [i_0, i_1]$ and $j \notin [j_0, j_1]$ then, according to Remark 2, there exist $\alpha, \beta, \beta'$ such that $(\alpha, \beta)$ is in the final zone while $(\alpha, \beta')$ is not. Let $f$ and $g$ be such that $f(i_0) = f(i_1) = \alpha$, $g(j_0) = g(j_1) = \beta$ and $g(j) = \beta'$. The tableau $d^{(\mathcal{J}, \mathcal{S})}(T) = \{(\alpha, \beta)\}$ is not final but, since $(\alpha', \beta') \in d^{(\mathcal{J}, \mathcal{S})}(\text{Sat}(T))$, the tableau $d^{(\mathcal{J}, \mathcal{S})}(\text{Sat}(T))$ is final. This contradicts the Nerode equivalence of $T$ and $\text{Sat}(T)$.

(c) The remaining case $(i \notin [i_0, i_1]$ and $j \in [j_0, j_1]$) is obtained symmetrically to the previous one.

Accordingly to the previous enumeration, we have necessarily $(i, j) \in I$. So $\text{Sat}(T) \subseteq I$, i.e. $\text{Sat}(T)$ is simple.

2. Since we are not in a degenerated case, there exist $J = [k_0, k_1] \times [\ell_0, \ell_1]$ with $k_0 \neq k_1$ and $\ell_0 \neq \ell_1$ such that for any $(\alpha, \beta) \in [2] \times [2]$ we have $(k_0, \ell_0) \in F_J$ if and only if $(\alpha, \beta) \in F_R$. Suppose that $\text{Red}(T) \neq \text{Red}(T')$ then, without loss of generality, there exists $(f, g) \in [2]^{21} \times [2]^{21}$ such that $d^{(\mathcal{J}, \mathcal{S})}(\text{Red}(T))$ is final while $d^{(\mathcal{J}, \mathcal{S})}(\text{Red}(T'))$ is non final. This implies that there exists $(a, b) \in \text{Red}(T)$ such that $(f(a), g(b)) \in F_R$ while for any $(\alpha, \beta) \in \text{Red}(T')$ we have $(f(\alpha), g(\beta)) \notin F_R$. Consider the two following maps

$$f^l(i) = \begin{cases} k(f(i)) & \text{if } i = i_\alpha \text{ for some } \alpha \in [2] \\ i & \text{otherwise} \end{cases} \quad \text{and} \quad g^l(j) = \begin{cases} \ell(g(j)) & \text{if } j = j_\beta \text{ for some } \beta \in [2] \\ j & \text{otherwise} \end{cases}$$

We check that $d^{(\mathcal{J}, \mathcal{S})}(T)$ is final while $d^{(\mathcal{J}, \mathcal{S})}(T')$ is non final. This shows that $T \neq T'$ and proves the result.

3. Suppose that $\text{Ind}_I(X) \neq \text{Ind}_I(X')$. This means that there exists $(f, g) \in [m]^{[m]} \times [n]^{[n]}$ such that $d^{(\mathcal{J}, \mathcal{S})}(\text{Ind}_I(X))$ and $d^{(\mathcal{J}, \mathcal{S})}(\text{Ind}_I(X'))$ have not the same finality. Let $J = \{f(i_0), f(i_1)\} \times [g(j_0), g(j_1)] = [k_0, k_1] \times [\ell_0, \ell_1]$ with $k_0 \leq k_1$ and $\ell_0 \leq \ell_1$. Lemma 3 implies that there exists a pair of maps $(f_j, g_j)$ such that for any $\alpha, \beta \in [2]$, $(f_j(k_\alpha), g_j(\ell_\beta)) \in F_J$ if and only if $(\alpha, \beta) \in F_R$. We set

$$\vec{f} = \begin{cases} f_j & \text{if } k_0 = f(i_0) \\ 1 - f_j(\alpha) & \text{if } k_0 \neq f(i_0) \end{cases} \quad \text{and} \quad \vec{g} = \begin{cases} g_j & \text{if } \ell_0 = g(j_0) \\ 1 - g_j(\beta) & \text{if } \ell_0 \neq g(j_0) \end{cases}$$

We check that $(\vec{f}(\alpha), \vec{g}(\beta)) \in F_R$ if and only if $(f(i_\alpha), g(j_\beta)) \in F_J$. Hence, $d^{(\mathcal{J}, \mathcal{S})}(X)$ and $d^{(\mathcal{J}, \mathcal{S})}(X')$ have not the same finality and so $X \neq X'$. This proves the result.

4. Let $R = \text{Ind}_I(\text{Sat}(\text{Red}(T)))$. Recall that we have

$$\text{Red}(R) = \text{Sat}(\text{Red}(T)), \quad (5)$$

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see Figure 1 for an illustration.

From Property 3 of the statement, we have $T \sim R$ because $T = \text{Ind}_I(\text{Red}(T))$, $R = \text{Ind}_I(\text{Sat}(\text{Red}(T)))$ and $\text{Red}(T) \sim \text{Sat}(\text{Red}(T))$.

Let $R' = \text{Sat}(T)$. From Property 1, the tableau $R'$ is also simple for $I$. Since $T \sim R$ we have also $\text{Sat}(R) = R'$ and then $\#R \leq \#R'$ which implies $\#\text{Red}(T) \leq \#\text{Red}(R')$. But from Property 2, the tableaux $\text{Red}(R) = \text{Sat}(\text{Red}(T))$ and $\text{Red}(R')$ are Nerode equivalent. Hence $\text{Sat}(\text{Red}(R')) = \text{Red}(R)$ and so $\#\text{Red}(R') \leq \#\text{Red}(R)$. It follows that $\#\text{Red}(R') = \#\text{Red}(R)$ and, as a consequence of Proposition 4, we have $\text{Sat}(\text{Red}(T)) = \text{Red}(R) = \text{Red}(R') = \text{Red}(\text{Sat}(T))$.

The meaning of this result is that, in the case of simple tableaux, the saturation reduces to the $2 \times 2$ case. We now investigate the general case and prove that the saturated of any tableau can be obtained by iterated saturation on $2 \times 2$ sub-tableaux.

**Definition 10** Let $T \in 2^{[m] \times [n]}$. We write $T \rightarrow T'$ if $T' = T \cup \text{Sat}(T|_I)$ for any $I = \{i_0, i_1\} \times \{j_0, j_1\} \subseteq [m] \times [n]$. We denote by $\rightarrow^*$ the transitive closure of $\rightarrow$.

We say that a tableau $T$ is **locally saturated** when $T \rightarrow^* T'$ implies $T = T'$. We denote by $S_{V_{\bullet},F_2}$ the set of the locally saturated valid tableaux.

**Lemma 5** If $T \rightarrow T'$ then $T \sim T'$.

**Proof:** Let $I = \{i_0, i_1\} \times \{j_0, j_1\} \subseteq [m] \times [n]$ such that $\text{Sat}(T|_I) = T'|_I$ and $T \setminus I = T' \setminus I$. Let $(f, g) \in [m]^{[m]} \times [n]^{[n]}$. Suppose first that $d(f, g)(T)$ is final. Since $T \subset T'$, the tableau $d(f, g)(T')$ is
also final. Conversely suppose that \(d^{(f,g)}(T)\) is not final and assume that \(d^{(f,g)}(T')\) is final. This implies that there exists \((i, j) \in \text{Sat}(T_1) \setminus T\) such that \((f(i), g(j))\) is in the final zone. This implies that \(d^{(f,g)}(\text{Sat}(T_1))\) is final. But since \(T_1 \subset T\), the tableau \(d^{(f,g)}(T_1)\) is necessarily not final. The two previous results contradict the fact that \(T_1 \sim \text{Sat}(T_1)\). Hence, the tableau \(d^{(f,g)}T'\) is not final and \(T \sim T'\). □

As a consequence of Lemma 5 if a tableau is saturated then it is also locally saturated. Now, let us prove the converse.

Lemma 6 Let \(T \sim T''\) with \(T \subset T''\). For any \(T \subset T' \subset T''\), we have \(T \sim T'\). In other words, each Nerode class is a union of intervals for the inclusion order.

Proof: First, suppose that \(d^{(f,g)}(T)\) is final. Since \(T \subset T'\), we have \(d^{(f,g)}(T) \subset d^{(f,g)}(T')\) and the tableau \(d^{(f,g)}(T')\) is also final.
Conversely, suppose that \(d^{(f,g)}(T')\) is final then, since \(T' \subset T''\), the tableau \(d^{(f,g)}(T'')\) is also final. But \(T \sim T''\) implies that \(d^{(f,g)}(T)\) is final. We deduce that \(T \sim T'\). □

According to the Proposition 5 each step of local saturation works as in the \(2 \times 2\) case. So we have to investigate the different possibilities with respect to the type of the operation \(\bullet\) as described in Section 3.3. We first examine the case where \(\bullet\) has the type (A).

Lemma 7 For any type (A) operation \(\bullet\), any tableau \(T\) and any \((i_c, j_c) \notin T\), the tableaux \(T\) and \(T \cup \{(i_c, j_c)\}\) are not Nerode equivalent.

Proof: First we remark that, since \(\bullet\) is a non degenerated type A operation there exist \(\alpha, \beta \in [m]\) and \(\gamma, \delta \in [n]\) such that \((\alpha, \gamma) \in F\) and \(\{(\alpha, \delta), (\beta, \gamma), (\beta, \delta)\} \cap F = \emptyset\). We consider the two maps

\[
f(i) = \begin{cases} \alpha & \text{if } i = i_c \\ \beta & \text{otherwise} \end{cases} \quad \text{and} \quad g(j) = \begin{cases} \gamma & \text{if } j = j_c \\ \delta & \text{otherwise} \end{cases}.
\]

Obviously \(d^{(f,g)}(T) \subset \{(\alpha, \delta), (\beta, \gamma), (\beta, \delta)\}\) is non final while \(d^{(f,g)}(T \cup \{(i_c, j_c)\})\) is final since it contains \((\alpha, \gamma)\). It follows that the states \(T\) and \(T \cup \{(i_c, j_c)\}\) are not Nerode equivalent. □

Example 5 Let us illustrate the proof of Lemma 7 Consider the two following tableaux:

\[
\begin{array}{cccccc}
2 & 2 & 2 & ? & ? & ? \\
2 & 2 & 2 & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
\end{array}
\quad
\begin{array}{cccccc}
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
\end{array}
\]

\[T \quad T' = T \cup \{(i_c, j_c)\}\]

In these pictures, the red zones illustrate the common final zone of \(T\) and \(T'\), and the question marks denote the same values at the same cells in both tableaux. Notice that, as specified in the statement of Lemma 7, the tableaux \(T\) and \(T'\) only differ at the cell \((i_c, j_c)\). We choose a \(2 \times 2\) zone that behaves as a non degenerate \(2 \times 2\) tableau. This zone is drawn in green in the following picture:
Proposition 6 If \( \bullet \) is a non degenerated type A operation then any Nerode class of \( \tilde{M}^{F_1,F_2}_\bullet \) is a singleton.

Proof: Suppose that \( T \sim T' \) and \( T \not= T' \). From Proposition 4 we have \( T \sim T \cup T' \). Without loss of generality, we assume that there exists \((i, j) \in T' \) such that \((i, j) \not\in T \). From Lemma 6 \( T \) is equivalent to any tableau \( T'' \) satisfying \( T \subset T'' \subset T' \). In particular we have \( T \equiv T \cup \{(i, j)\} \). But this contradicts Lemma 7. So we deduce that \( T \not\sim T' \) or \( T = T' \). \( \square \)

Lemma 8 generalizes to other types as follows:

Lemma 8 For any non degenerated operation \( \bullet \) we have \( T \not\sim T \cup \{(i_c, j_c)\} \) for any locally saturated tableau \( T \) and any \((i_c, j_c) \not\in T \).

Proof: Proposition 6 implies the assertion for the type A. Let us now consider two sets \( R = (\|m\| \times \{j_c\}) \cap T \) and \( C = (\{i_c\} \times \|n\|) \cap T \).

1. If \( \bullet \) has the type O then either \( R = \emptyset \) or \( C = \emptyset \). Indeed, if there exist \( j \) such that \((i_c, j) \in C \) and \( i \) such that \((i, j_c) \in R \) then the restriction \( T_{\|\{i_c\}\| \times \{j_c\}} \) is not saturated because, from Proposition 6, \( \text{Red}(T_{\|\{i_c\}\| \times \{j_c\}}) \) is not saturated, and so \( T \not\sim T \cup \{(i_c, j_c)\} \not\equiv T \) and this contradicts the fact that \( T \) is locally saturated. Without loss of generality, we assume that \( R = \emptyset \), the other case being obtained symmetrically. We recall that there exist \( I, J \) such that \( F = (I \times \|n\|) \cup (\|m\| \times J) \). Since \( F \not\equiv \emptyset \) and \( F \not\equiv \|m\| \times \|n\| \), there exist \( \alpha, \gamma, \delta \) such that \((\alpha, \gamma) \in F \) and \((\alpha, \delta) \not\in F \). Indeed, it suffices to choose \((\alpha, \gamma) \in (\|m\| \setminus I) \times J \) and \( \delta \in (\|n\| \setminus J) \). Let \( f : \|m\| \to \|m\| \) and \( g : \|n\| \to \|n\| \).
such that
\[ f(i) = \alpha \text{ and } g(j) = \begin{cases} 
\gamma & \text{if } j = j_c \\
\delta & \text{otherwise}
\end{cases} \]

Since \( R = \emptyset \), for any \((i, j) \in T\) we have \((f(i), g(j)) = (\alpha, \delta) \notin E\). This implies that \( d^{(f, g)}(T) \) is not final. But since \((f(i_c), g(j_c)) = (\alpha, \gamma) \in E\), we obtain also that \( d^{(f, g)}(T \cup \{(i_c, j_c)\}) \) is final. So \( T \) and \( T \cup \{(i_c, j_c)\} \) are not Nerode equivalent.

2. Suppose that \( \bullet \) has the type \( X \). If \( R = \emptyset \) or \( C = \emptyset \) then we prove our result by applying the same strategy as in case 1. So we suppose that \( R \neq \emptyset \) and \( C \neq \emptyset \). Let us first prove that the result reduces to the case where \#R = \#C = 1. We remark that for any \( i, j \) such that \((i, j_c) \in C\) and \((i_c, j) \in R\) we have \((i, j) \notin T\), otherwise, as \( T \) is saturated, \((i_c, j_c)\) is in \( T\). Let \( i', j' \) such that \((i', j_c) \in C\) and \((i_c, j') \in R\). Let \( f_1 : [m] \rightarrow [m] \) and \( g_1 : [n] \rightarrow [n] \) such that
\[ f_1(i) = \begin{cases} 
\ i' & \text{if } (i, j_c) \in C \\
\ i & \text{otherwise}
\end{cases} \quad \text{and} \quad g_1(j) = \begin{cases} 
\ j' & \text{if } (i_c, j) \in R \\
\ j & \text{otherwise}
\end{cases} \]

We set \( T_2 = d^{(f_1, g_1)}(T) \) and \( T_2' = d^{(f_1, g_1)}(T \cup \{(i_c, j_c)\}) \), \( C_2 = ([m] \times j_c) \cap T_2 \), and \( R_2 = (i_c \times [n]) \cap T_2 \). The tableau \( T_2 \) is obtained by removing each row identical to the \( j_c \) one and each column identical to the \( i_c \) one. As a consequence, \( T_2 \) is locally saturated and \((i_c, j_c) \notin T_2\). We check also that \( T_2' = T_2 \cup \{(i_c, j_c)\} \), and \#R_2 = \#C_2 = 1.

Consider now \( g_2 : [n] \rightarrow [n] \) such that
\[ g_2(j) = \begin{cases} 
\ j_c & \text{if } (i', j) \in T_2 \\
\ j' & \text{otherwise}
\end{cases} \]

and set \( T_3 = d^{(\text{Id}, g_2)}(T_2) \) and \( T_3' = d^{(\text{Id}, g_2)}(T_2') \). The tableau \( T_3 \) contains crosses only in the two columns \( j' \) and \( j_c \).

More precisely we have
\[ T_3 \cap ([m] \times \{j_c\}) = \{(i, j_c) \mid \exists j \in [n] \text{ such that } (i, j, (i', j) \in T_2\} \tag{6} \]
and
\[ T_3 \cap ([m] \times \{j'\}) = \{(i, j') \mid \exists j \in [n] \text{ such that } (i, j) \in T_2 \text{ and } (i', j) \notin T_2\} \tag{7} \]

Suppose that \((i_c, j_c) \in T_3\) then, by Formula \(\text{6}\), there exists \( j \in [n] \) such that \((i_c, j) \in T_2\) and \((i', j) \notin T_2\). Since \((i', j_c) \in T_2\) and \( T_2 \) is locally saturated, this implies \((i_c, j_c) \in T_2\). This contradicts the definition of \( T_2 \). Hence, \((i_c, j_c) \notin T_3\). Furthermore, straightforwardly from \(\text{7}\), \((i', j') \notin T_3\).

We have also
\[ T_3' \cap ([m] \times \{j_c\}) = \{(i, j_c) \mid \exists j \in [n] \text{ such that } (i, j) \in T_2' \text{ and } (i', j) \in T_2\}, \]
and
\[ T_3' \cap ([m] \times \{j'\}) = \{(i, j') \mid \exists j \in [n] \text{ such that } (i, j) \in T_2' \text{ and } (i', j) \notin T_2\}. \]

Hence we check that \((i_c, j'), (i', j_c), (i_c, j_c) \in T_3'\). Consider now the map \( f_2 : [m] \rightarrow [m] \) such that
\[ f_2(i) = \begin{cases} 
\ i_c & \text{if } (i, j) \in T_3 \\
\ i' & \text{otherwise}
\end{cases} \]

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and set $T_4 = d(f_2,i)(T_3)$ and $T'_4 = d(f_2,i)(T'_3)$. By construction, $T_4, T'_4 \subset \{i', i_c\} \times \{j', j_c\}$. From the definition of $f_2$, $(i', j') \not\in T_4$. If we suppose that $(i_c, j_c) \in T_4$ then there exists $i \in [m]$ such that $(i, j_c) \in T_3$ and $(i, j') \in T_3$. Since $(i, j_c) \in T_3$, Formula (5) implies that there exists $j_1 \in [n]$ such that $(i, j_1) \in T_2$ and $(i', j_1) \in T_2$. Since $(i, j') \in T_3$, Formula (7) implies that there exists $j_2 \in [n]$ such that $(i, j_2) \in T_2$ and $(i', j_2) \not\in T_2$. To summarize we have $\{(i, j_1), (i', j_1), (i, j_2)\} \subset T_2$ but $(i', j_2) \not\in T_2$. This contradicts the fact that $T_2$ is saturated. Hence, $(i_c, j_c) \not\in T_4$. On the other hand, we have $\{(i_c, j'), (i', j_c), (i_c, j_c)\} \subset T'_4$ because $\{(i_c, j'), (i', j_c), (i_c, j_c)\} \subset T'_3$, $f_2(i_c) = i_c$ and $f_2(i') = i'$.

From Proposition 5 and Section 3.2, since $T_4 \subset \{(i_c, j'), (i', j_c)\}$, the two tableaux $T_4$ and $T'_4$ are clearly non-equivalent.

**Example 6** We illustrate first the case of the type (O). Let us consider the two following tableaux

\[
\begin{array}{cccccc}
2 & ? & ? & ? & ? \\
2 & ? & ? & ? & ? \\
2 & ? & ? & ? & ? \\
2 & ? & ? & ? & ? \\
\times & \times & & & \\
? & ? & ? & ? & \\
\end{array}
\quad
\begin{array}{cccccc}
? & ? & ? & ? & ? \\
? & ? & ? & ? & ? \\
? & ? & ? & ? & ? \\
? & ? & ? & ? & ? \\
\times & \times & & & \\
? & ? & ? & ? & \\
\end{array}
\]

$T$ \quad $T' = T \cup \{(i_c, j_c)\}$

In this figure, the red zone corresponds to the final zone of the tableaux, the red cross indicates the cell $(i_c, j_c)$ and the set of the blue crosses is the set $R$. Since $R \neq \emptyset$ we have $C = \emptyset$ as pictured above. From the definition of $(f, g)$, the value of any cell which is on the same column as the red cross is sent to the cell $A$ while the other values are sent to the cell $B$. Notice that the image of $(i_c, j_c)$ is the letter colored in red.

\[
\begin{array}{cccc}
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
\end{array}
\quad
\begin{array}{cccc}
\gamma & \delta \\
A & B \\
\end{array}
\]

\[
\begin{array}{cccc}
\begin{array}{cccc}
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
B & B & B & A \ B \\
\end{array}
\end{array}
\quad
\begin{array}{cccc}
\begin{array}{cccc}
\times & \times \\
\end{array}
\end{array}
\]

Hence the image of the tableaux $T$ and $T'$ are respectively

$\delta(f, g)(T)$ and $\delta(f, g)(T')$.

The left tableau being non final while the right one is final shows that $T$ and $T'$ are not Nerode equivalent.

Notice that the same strategy works for the same pair of tableaux even in the case of the type (X), as shown in the figure below.
Example 7  Now let us illustrate the case of the type (X) and consider the two following tableaux with \((i_c, j_c)\) denoted by the red cross on the right tableau
\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T' = T \cup \{(i_c, j_c)\}
\end{array}
\]

We first choose the column \(i'\) and the row \(j'\) (in green in the following figure)
\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T' = T \cup \{(i_c, j_c)\}
\end{array}
\]

We apply the transformation \((f_1, g_1)\) in order to obtain the tableaux \(T_2\) and \(T'_2\)
\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T_2 \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T'_2 = T \cup \{(i_c, j_c)\}
\end{array}
\]

Applying \((\text{Id}, g_2)\), we obtain
\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T_3 \\
? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? \\
T'_3 = T_3 \cup \{(i_c, j_c)\}
\end{array}
\]

Finally, applying \((f_2, \text{Id})\), one obtains the tableaux
which are not Nerode equivalent. Indeed, a permutation of the columns sends these tableaux respectively to

The tableau $T_5$ is non final while the tableau $T'_5$ is final.

**Proposition 7** A non-empty tableau $T$ is saturated if and only if it is locally saturated. Furthermore the locally saturated empty tableau is saturated if and only if $(0, 0)$ is not in the final zone.

**Proof:** Let $T$ be a non-empty locally saturated tableau and suppose that $T \neq \text{Sat}(T)$. From Lemma 6, $T$ is Nerode equivalent to any $T'$ such that $T \subset T' \subset \text{Sat}(T)$. In particular, if $(i, j) \in \text{Sat}(T) \setminus T$ then $T$ is Nerode equivalent to $T \cup \{(i, j)\}$. But this contradicts Lemma 8. Hence, $T = \text{Sat}(T)$.

By construction, if $(0, 0) \in F$ the tableau $\emptyset$ is Nerode equivalent to $\{(0, 0)\}$ so it is not saturated. Conversely, if $(0, 0) \notin F$ then the tableau $\emptyset$ is the only final state with no cross in the final zone. So the empty tableau is the only one in its Nerode class and it is saturated. $\square$

### 4 Monster witnesses

Let us denote by $\text{AS}_{F_1, F_2}^*$ the set of the accessible saturated tableaux. From Section 3.2, the set of saturated tableaux is always a representative set of the Nerode classes of $M_{F_1, F_2}^*$. Hence, the tableaux of $\text{AS}_{F_1, F_2}^*$ are in one to one correspondence with the states of $\text{Min}(M_{F_1, F_2}^*)$ and this implies

$$\text{sc}(m, n) = \max(\#\text{AS}_{F_1, F_2}^* | F_1 \subset \lfloor m \rfloor, F_2 \subset \lfloor n \rfloor).$$

To accurately calculate the state complexity, we have to compute the value of $\#\text{AS}_{F_1, F_2}^*$. This value depends both on the number of elements in $F_1$ and $F_2$ and of some properties of the final zone as shown in Table 3.

| $F \subset \{(0) \times \lfloor n \rfloor\} \cup (\lfloor m \rfloor \times \{(0)\})$ | $F \not\subset \{(0) \times \lfloor n \rfloor\} \cup (\lfloor m \rfloor \times \{(0)\})$ |
|---|---|
| $(0, 0) \in F$ | $\#\text{AS}_{F_1, F_2}^* \leq \#\text{SV}_{F_1, F_2}^* = \#S_{\ell}^*V_{F_1, F_2}^* - 1$ |
| $(0, 0) \notin F$ | $\#\text{AS}_{F_1, F_2}^* \leq \#\text{SV}_{F_1, F_2}^* = \#S_{\ell}^*V_{F_1, F_2}^*$ |

Table 3: Relations between the number of accessible saturated states $\#\text{AS}_{F_1, F_2}^*$, saturated valid states $\#\text{SV}_{F_1, F_2}^*$, and local-saturated valid states $\#S_{\ell}^*V_{F_1, F_2}^*$. 28
The condition on the columns of Table 3 means that the final zone is either contained in the union of row 0 and column 0 or not. By construction, this condition does not affect the number of elements of $S_V^{F_1,F_2}$, thus nor the number of elements of $\#SV_{F_1,F_2}$. From Table 3 we deduce that to maximize $\#AS_{F_1,F_2}$, it is sufficient to maximize $\#SV_{F_1,F_2}$ when $F \not\subseteq (\{0\} \times \{n\}) \cup (\{m\} \times \{0\})$. In other words,

**Theorem 5** We have

$$sc_{\emptyset}(m,n) = \max \left\{ \#SV_{F_1,F_2} \mid F_1 \subseteq \{m\}, F_2 \subseteq \{n\}, F_1,F_2 \neq \emptyset, (F_1 \times \{m\}) \bullet (\{m\} \times F_2) \not\subseteq (\{0\} \times \{n\}) \cup (\{m\} \times \{0\}) \right\}.$$ 

As a preliminary result, we give an expression of each $\#SV_{F_1,F_2}$ depending on the fact that the cell $(0,0)$ belongs or not to the final zone. According to Table 1, without loss of generality, we consider $\bullet \in \{\cap, \cup, \oplus\}$, the other cases being recovered by replacing $F_1$ or $F_2$ by its complementary.

### 4.1 Counting saturated valid tableaux

#### 4.1.1 Type A ($\bullet = \cap$)

From Proposition 7 and Section 3.3 any valid tableau is saturated. So valid tableaux are the only ones we have to count. We consider two cases.

1. Suppose $(0,0) \in F$. There are $2^{mn-1}$ local saturated valid tableaux containing $(0,0)$ because any tableau containing $(0,0)$ is valid (see Fig. 2 for an illustration).

![Figure 2: Example of valid tableau containing (0,0) for the type A.](image)

Furthermore, the set of the valid tableaux that do not contain $(0,0)$ is the set of tableaux having no cross in $F_1 \times F_2$ (see an example in Fig. 3). There are $2^{mn-\#F_1 \#F_2}$ such tableaux.

![Figure 3: Example of valid tableau that does not contain (0,0) for the type A when (0,0) \notin F](image)

We deduce

$$\#S_V^{F_1,F_2} = 2^{mn-1} + 2^{mn-\#F_1 \#F_2} \quad \text{and} \quad \#SV_{\cap}^{F_1,F_2} = 2^{mn-1} + 2^{mn-\#F_1 \#F_2} - 1. \quad (8)$$
2. Suppose \((0, 0) \notin F\). For the same reason as in the previous case, there are still \(2^{mn-1}\) valid tableaux having a cross at \((0, 0)\). But there are only \(2^{mn-\#F_1\#F_2-1}\) valid tableaux with no cross at \((0, 0)\) (see an example in Fig. 4).

![Figure 4: Example of valid tableau that does not contain \((0, 0)\) for the type A when \((0, 0) \notin F\).](image)

We deduce that in this case we have

\[
\#SV^{F_1, F_2}_{\ell} = \#S_{\ell}^{F_1, F_2} = 2^{mn-1} + 2^{mn-\#F_1\#F_2-1}.
\] (9)

### 4.1.2 Type O (\(\bullet = \cup\))

According to Proposition 7 and Section 3.3, for any non-empty saturated tableau \(T\), if \((i, j)\) and \((i', j')\) belong to \(T\) then \((i', j)\) and \((i, j')\) belong to \(T\). In another words, for any saturated tableau \(T\), there exist \(A \subset \llbracket m \rrbracket\) and \(B \subset \llbracket n \rrbracket\) such that \(T = A \times B\). So we have to enumerate the pairs \((A, B)\) such that if \((A \times B) \cap F \neq \emptyset\) then \(0 \in A\) and \(0 \in B\). We consider two cases

1. Suppose \((0, 0) \in F\). The configuration of the final zone is illustrated in Fig. 5. Remark first that any tableau \(T = A \times B\) containing \((0, 0)\) is valid. There are \(2^{m+n-2}\) such tableaux. If \((0, 0) \notin A \times B\) then the condition of validity implies \(A \subset \llbracket m \rrbracket \setminus F_1\) and \(B \subset \llbracket n \rrbracket \setminus F_2\). Hence, there are \((2^{m-\#F_1} - 1)(2^{n-\#F_2} - 1) + 1\) such tableaux.

![Figure 5: Final zone for the type O when \((0, 0) \in F\).](image)

Hence, we obtain

\[
\#S_{\ell}^{F_1, F_2} = 2^{m+n-2} + (2^{m-\#F_1} - 1)(2^{n-\#F_2} - 1) + 1
\]

\[
= 2^{m+n-2} + 2^{m+n-\#F_1\#F_2} - 2^{m-\#F_1} - 2^{n-\#F_2} + 2,
\]

and then

\[
\#SV^{F_1, F_2}_{\ell} = 2^{m+n-2} + 2^{m+n-\#F_1\#F_2} - 2^{m-\#F_1} - 2^{n-\#F_2} + 1.
\] (10)

2. Suppose \((0, 0) \notin F\). The configuration of the final zone is illustrated in Fig. 6. The same reasoning as in the previous case shows that there are \(2^{m+n-2}\) saturated valid tableaux having
Figure 6: Final zone for the type O when $(0, 0) \notin F$

a cross at $(0, 0)$. In order to count the tableaux that do not contain $(0, 0)$, we apply the inclusion-exclusion principle. Indeed, we sum the number of locally saturated tableaux which are subsets of $\left([m] \setminus \{0\}\right) \times [n]$, the number of locally saturated tableaux which are subsets of $[m] \times ([n] \setminus \{0\})$, and we subtract to the number of locally saturated tableaux which are subset of $([m] \setminus \{0\}) \times ([n] \setminus \{0\})$ because they were counted twice. So we obtain

\[
\left(2^{m-\#F_1} - 1\right)\left(2^{n-\#F_2} - 1\right) + \left(2^{m-\#F_1} - 1\right)\left(2^{n-\#F_2} - 1\right) + 1 - \left(2^{m-\#F_1} - 1\right)\left(2^{n-\#F_2} - 1\right) - 1 + 1 = \frac{3}{4}2^{m+n-\#F_1-\#F_2} - 2^{m-\#F_1} - 2^{n-\#F_2} + 2 \text{ such tableaux.}
\]

Hence,

\[
\#SV_{F_1,F_2} = \#S_{\ell}V_{F_1,F_2} = 2^{m+n-2} + \frac{3}{4}2^{m+n-\#F_1-\#F_2} - 2^{m-\#F_1} - 2^{n-\#F_2} + 2.
\]  (11)

4.1.3 Type X ($\bullet = \oplus$)

In this case the valid saturated tableaux are more complicated to enumerate. Indeed, from Proposition [7] and Section 3.3, the locally saturated tableaux are the tableaux that avoid one of the following $2 \times 2$ motives:

Let us denote by $\alpha_{m,n}$ the number of $m \times n$ locally saturated tableaux and by $\alpha'_{m,n}$ the number of $m \times n$ locally saturated tableaux having a cross at $(0, 0)$. In a previous paper [7], we give formulas allowing to compute these values. Nevertheless, the precise knowledge of these formulas is not useful for our purpose, we do not recall it here but we invite the reader interested by these enumeration results to read this paper.

Again, we have to consider two cases:

1. Suppose $(0, 0) \in F$. The configuration of the final zone is illustrated in Fig. 7

![Figure 7: Final zone for the type X when $(0, 0) \in F$.](image)

As any locally saturated tableaux having a cross at $(0, 0)$ is valid, the number of such tableaux equals to $\alpha'_{m,n}$. Notice also that a locally saturated tableau $T$ having no cross at $(0, 0)$ is valid...
Obviously, from (8) and (9) the maximal value of $\#S$.

2. Suppose $(0, 0) \notin F$. The configuration of the final zone is illustrated in Fig. 8.

As in the previous case, the number of tableaux with a cross in $(0, 0)$ is $\alpha'_{m,n}$. For the tableaux not containing a cross in $(0, 0)$, we have to consider tableaux in the non-final zone containing $(0, 0)$ ($\alpha_{\#F_1,\#F_2} - \alpha'_{\#F_1,\#F_2}$ or $\alpha_{m,\#F_1,\#F_2} - \alpha'_{m,\#F_1,\#F_2}$) depending on the fact that $(0, 0)$ is in $F_1 \times F_2$ or in $\overline{F_1} \times F_2$) and tableaux in the non-final zone not containing $(0, 0)$, which gives us

$$\#SV_{\oplus}^{F_1,F_2} = \#SV_{\oplus}^{F_2,F_1} = \left\{ \begin{array}{ll} \alpha'_{m,n} & \text{if } (0,0) \in F_1 \times F_2 \\ \alpha'_{m,n} + \alpha_{m,\#F_1,\#F_2} - \alpha'_{m,\#F_1,\#F_2} & \text{otherwise.} \end{array} \right. \tag{13}$$

4.2 Computing witnesses

In this section, we compute a final zone allowing us to obtain the tight bound for the complexity of each studied operation. This computation allows us to recover the tight bound for the star of intersection due to Jirásková and Okhotin [22] and that of the star of union due to Salomaa et al. [24]. This also allows us to give an expression of the tight bound for the star of xor.

4.2.1 Type A

Obviously, from (8) and (9) the maximal value of $\#SV_{\oplus}^{F_1,F_2}$ is reached when $F_1 = F_2 = 1$ in both cases. Notice that if $(0,0) \in F$ then $F_1 = F_2 = 1$ implies $F = \{(0,0)\}$. But, in this case a tableau is accessible if and only if it contains at most one cross. The states $\emptyset$ and $\{(0,0)\}$ being Nerode equivalent, we obtain

$$\#AS_{\cap}^{(0),(0)} = mn. \tag{14}$$

If $(0,0) \in F$ and $(\#F_1 > 1$ or $\#F_2 > 1)$ then, by Table 3 and (8), we have

$$\#AS_{\cap}^{F_1,F_2} \leq \#SV_{\cap}^{F_1,F_2} - 1 \leq \#SV_{\cap}^{(0),(0)} - 1 = 2^{mn-1} + 2^{mn-2} - 1 = \frac{3}{4} 2^{mn} - 1. \tag{15}$$
Suppose now $(0, 0) \not\in F$, from Table 3 and the discussion above, the maximal values for $\#\text{AS}_{F_1 \cap F_2}$ is reached when $F_1 = \{f_1\}$ and $F_2 = \{f_2\}$ with $f_1 \in \llbracket m \rrbracket \setminus \{0\}$ and $f_2 \in \llbracket n \rrbracket \setminus \{0\}$. In this case we have

$$\#\text{AS}_{F_1 \cap F_2} = \#\mathcal{S}_V^{f_1, f_2} = 2^{m-1} + 2^{n-2} = \frac{3}{4} 2^{mn}.$$  \hspace{1cm} (16)

We summarize the results contained in (14), (15) and (16) in the following theorem:

**Theorem 6 (Jirásková and Okhotin [22])** When $m, n > 1$ we have

$$\text{sc}(m, n) = \frac{3}{4} 2^{mn}$$

and 2-monster $\text{Mon}_{m,n}^{f_1, f_2}$ with $f_1 \in \llbracket m \rrbracket \setminus \{0\}$ and $f_2 \in \llbracket n \rrbracket \setminus \{0\}$ is a witness.

**Example 8** We list in Table 4 the first values of the state complexity. The valid saturated tableaux illustrating the case $m = n = 2$ are pictured in Figure 9.

| $m \setminus n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|
| 2               | 12 | 48 | 192 | 768 | 3072 | 12288 |
| 3               | 48 | 384 | 3072 | 24576 | 196608 | 1572864 |
| 4               | 192 | 3072 | 49152 | 786432 | 12582912 | 201326592 |
| 5               | 768 | 24576 | 786432 | 25165824 | 805306368 | 25769803776 |
| 6               | 3072 | 196608 | 12582912 | 805306368 | 3298534883328 |
| 7               | 12288 | 1572864 | 201326592 | 3298534883328 | 422212465065984 |
| 8               | 49152 | 12582912 | 3221225472 | 824633720832 | 5404319552844592 |

Table 4: First values of $\text{sc}(m, n)$ for the type A.

![Figure 9: The 12 saturated valid 2 × 2-tableaux for the type A, $F_1 = F_2 = \{1\}$](image)

4.2.2 Type O

We assume that $m, n \geq 2$. From (10) and (11) the maximal value of $\#\mathcal{S}_V^{F_1, F_2}$ is reached when $F_1 = F_2 = 1$ in both cases. Let $F_1 = \{f_1\}$ and $F_2 = \{f_2\}$ be such that $(0, 0)$ is in the final zone of $M^{F_1, F_2}_U$. From (10) and (11), if $F'_1$ and $F'_2$ are such that $F'_1 = F_1$ and $F'_2 = F_2$ and $(0, 0)$ is not in the final zone of $M^{F'_1, F'_2}_U$ then $\#\mathcal{S}_V^{F'_1, F'_2} > \#\mathcal{S}_V^{F_1, F_2}$. So, as the final zone must not be completely included in the 0-row and the 0-column and as $(0, 0)$ is in the final zone of $M^{F_1, F_2}_U$ one of the two state $f_1$ or $f_2$ is 0 while the other is not.

Hence, from Table 3 (10) and (11) we have

$$\#\text{AS}_U^{f_1, f_2} = \#\text{AS}_U^{f_1, [0]} = 2^{m+n-1} + 2^{m-1} - 2^{n-1} + 1.$$  \hspace{1cm} (16)

We summarize the results above in the following theorem:
Theorem 7 (Salomaa et al. [24]) When \( m, n > 1 \) we have

\[
\text{sc}_\oplus(m, n) = 2^{m+n-1} - 2^{m-1} - 2^{n-1} + 1,
\]

and 2-monster \( \text{Mon}_{m,n}^{\{f_1\},\{0\}} \) or \( \text{Mon}_{m,n}^{\{0\},\{f_2\}} \), with \( f_1 \in \llbracket m \rrbracket \setminus \{0\} \) and \( f_2 \in \llbracket n \rrbracket \setminus \{0\} \), is a witness.

Example 9 We list in Table 5 the first values of the state complexity. The valid saturated tableaux illustrating the case \( m = n = 2 \) are pictured in Figure 10.

| \( m \setminus n \) | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|----------------------|-----|-----|-----|-----|-----|-----|-----|
| 2                    | 5   | 11  | 23  | 47  | 95  | 191 | 383 |
| 3                    | 11  | 25  | 53  | 109 | 221 | 445 | 893 |
| 4                    | 23  | 53  | 113 | 233 | 473 | 953 | 1913|
| 5                    | 47  | 109 | 233 | 481 | 977 | 1969| 3953|
| 6                    | 95  | 221 | 473 | 977 | 1985| 4001| 8033|
| 7                    | 191 | 445 | 953 | 1969| 4001| 8065| 16193|
| 8                    | 383 | 893 | 1913| 3953| 8033| 16193| 32513|

Table 5: First values of \( \text{sc}_\oplus(m, n) \) for the type O.

![Figure 10: The 5 saturated valid 2 × 2-tableaux for the type O, \( F_1 = \{1\}, F_2 = \{0\} \).](image)

4.2.3 Type X

We assume that \( m, n \geq 2 \). From (12) and (13) the maximal value of \( \#S_\oplus V_{F_1,F_2}^{F} \) is reached when \((0,0)\) is in the final zone. We now show that the state complexity of star of xor is reached when the size of the final zone is minimal, i.e. final sets are both minimal (\( \#F_1 = 1 \) and \( \#F_2 = 1 \)) or both maximal (\( \#F_1 = m - 1 \) and \( \#F_2 = n - 1 \)). Equality (12) implies that we have to show that \( \alpha_{p,q} \alpha_{m-p,n-q} \leq \alpha_{1,1} \alpha_{m-1,n-1} \) for any \( 1 \leq p \leq m - 1 \) and \( 1 \leq q \leq n - 1 \). Rather than proving this inequality directly, which is quite difficult, we will instead examine the combinatorics of the objects that are counted. The number \( \alpha_{p,q} \alpha_{m-p,n-q} \) counts the \( m \times n \) non final locally saturated tableaux having a final zone \( F = (F_1 \times \llbracket n \rrbracket) \oplus (\llbracket m \rrbracket \times F_2) \) with \( \#F_1 = p \) and \( \#F_2 = q \). Without loss of generality, we assume that \( F_1 = \llbracket p \rrbracket \) and \( F_2 = \llbracket n \rrbracket \setminus \{n-q\} \) and so \( F = (\llbracket m-p \rrbracket \times \llbracket q \rrbracket) \cup (\{m-p,m-p+1,\ldots,m-1\} \times \{q,q+1,\ldots,n-1\}) \). An illustration is give in Figure 11.
Let \( \text{LST}_{m,n}(p,q) \) be the set of these tableaux. In other words, the set \( \text{LST}_{m,n}(p,q) \) is the set of locally saturated \( m \times n \)-tableaux \( T \) satisfying \( T \cap F = \emptyset \).

Our proof is constructive and consists in exhibiting a map \( \phi : \text{LST}_{m,n}(p,q) \to \text{LST}_{m,n}(1,1) \) and proving it is an injection (see an illustration in Fig.12).

Before describing \( \phi \), we need to introduce some tools on locally saturated tableaux.

**Definition 11** For any tableau \( T \), the set of indices of the crosses belonging to the \( i^{th} \) row (resp. \( j^{th} \) column) is denoted by

\[
\text{row}_i(T) = \{ j \mid (i, j) \in T \} \quad (\text{resp.} \quad \text{col}_j(T) = \{ i \mid (i, j) \in T \}).
\]

Thus defined, the \( i^{th} \) row (resp. \( j^{th} \) column) of \( T \) is the set \( [i] \times \text{row}_i(T) \) (resp. \( \text{col}_j(T) \times [j] \)).

The following proposition is a reformulation of a result proved in [7] and initially stated in terms of words over an alphabet whose letters are indexed by subsets of \( \llbracket n \rrbracket \).

**Proposition 8** The three following assertions are equivalent

1. The tableau \( T \) is locally saturated.
2. For any \( i_1, i_2 \in \llbracket m \rrbracket \) either \( \text{row}_{i_1}(T) = \text{row}_{i_2}(T) \) or \( \text{row}_{i_1}(T) \cap \text{row}_{i_2}(T) = \emptyset \).
3. For any \( j_1, j_2 \in \llbracket n \rrbracket \) either \( \text{col}_{j_1}(T) = \text{col}_{j_2}(T) \) or \( \text{col}_{j_1}(T) \cap \text{col}_{j_2}(T) = \emptyset \).
Definition 13 Let $T$ be a locally saturated $m \times n$-tableau and $(i, j) \in [m] \times [n]$. We define 

$$\text{Supp}_{R_i}(T) = T \setminus \{i \times \text{row}_i(T)\}$$ and 
$$\text{Supp}_{C_j}(T) = T \setminus \{\text{col}_j(T) \times \{j\}\}.$$ 

as the tableau $T$ in which row $i$ (resp. column $j$) has been emptied of its crosses.

Definition 12 Let $T$ be a locally saturated $m \times n$-tableau and $i_1, i_2 \in \{m\}$. We define the tableau $\text{Merge}_{R_{i_1,i_2}}(T)$ as the unique $m \times n$-tableau $T'$ such that for any $i \in \{m\}$, 

$$\text{row}_i(T') = \begin{cases} \text{row}_i(T) \cup \text{row}_{i_2}(T) & \text{if } \text{row}_i(T) \in \{\text{row}_{i_1}(T), \text{row}_{i_2}(T)\} \\
\text{row}_i(T) & \text{otherwise.} \end{cases}$$

Symmetrically, we define the tableau $\text{Merge}_{C_{j_1,j_2}}(T)$ as the unique $m \times n$-tableau $T'$ such that for any $j \in \{n\}$, 

$$\text{col}_j(T') = \begin{cases} \text{col}_j(T) \cup \text{col}_{j_2}(T) & \text{if } \text{col}_j(T) \in \{\text{col}_{j_1}(T), \text{col}_{j_2}(T)\} \\
\text{col}_j(T) & \text{otherwise.} \end{cases}$$

Example 10 The following picture illustrates the action of $\text{Merge}_{R_{0,1}}$ on a tableau:

![Tableau Illustration]

In this example $\text{row}_0(T) = \{0, 4\} = \text{row}_2(T)$, $\text{row}_1(T) = \{2\} = \text{row}_4(T)$, and 

$$\text{row}_0(T') = \text{row}_1(T') = \text{row}_2(T') = \text{row}_4(T') = \{0, 2, 4\}.$$ 

Lemma 9 Let $T$ be a locally saturated $m \times n$-tableau. The following assertions hold:

1. For any $i_1, i_2 \in \{m\}$, the tableau $\text{Merge}_{R_{i_1,i_2}}(T)$ is locally saturated.
2. For any $j_1, j_2 \in \{n\}$, the tableau $\text{Merge}_{C_{j_1,j_2}}(T)$ is locally saturated.
3. For any $i \in \{m\}$, the tableau $\text{Supp}_{R_i}(T)$ is locally saturated.
4. For any $j \in \{n\}$, the tableau $\text{Supp}_{C_j}(T)$ is locally saturated.

Proof: Assertions 1 and 3 are obtained by using assertion 2 of Proposition\textsuperscript{8}. Assertions 2 and 4 are obtained by using assertion 3 of Proposition\textsuperscript{8}.

Let $\phi : \text{LST}_{m,n}(p, q) \rightarrow \text{LST}_{m,n}(1, 1)$ defined by

1. If $(\{m - 1\} \times \text{row}_{m-1}(T)) \cup (\text{col}_0(T) \times \{0\}) \subset \{(m - 1, 0)\}$ then we set $\phi(T) = T$.
   The set of images by $\phi$ of the tableaux $T$ that satisfy the condition of this case are those that have no crosses in either the final zone of $T$ or the final zone of $\phi(T)$.
2. If $(\{m - 1\} \times \text{row}_{m-1}(T)) \cup (\text{col}_0(T) \times \{0\}) \not\subset \{(m - 1, 0)\}$ then we have to consider the two following cases:
(a) If \((m - 1, 0) \notin T\) then we have to consider two cases

i. If there exists \((i, j) \in \llbracket m - p - 1 \rrbracket \times \llbracket q, \ldots, n - 1 \rrbracket\) such that \((i, j) \notin T\) then we choose \((i, j)\) minimal for the lexicographic order. We set

\[
\phi(T) = \text{Supp}R_{m-1} \left( \text{Merge}R_{i,m-1} \left( \text{Supp}C_0 \left( \text{Merge}C_{0,j}(T) \right) \right) \right).
\]

From Lemma 9, the tableau \(\phi(T)\) is locally saturated and the use of the functions \(\text{Supp}R_{m-1}\) and \(\text{Supp}C_0\) implies that \(\phi(T)\) belongs to \(\text{LST}_{m,n}(1, 1)\) (see an example in Fig. 13).

The set of images by \(\phi\) of the tableaux \(T\) that satisfy the condition of this case are those that do not have a cross at \((m - 1, 0)\), have at least a cross in the final zone of \(T\) and that do not contain \(\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\}\) as a subset.

\[0 1 \cdots q \ q + 1 \cdots n - 1\]
\[0 1 \vdots\]
\[m - p \ x \ x \vdots x\]
\[m - p + 1 \ x \ x \vdots x\]
\[\vdots \ x \ x \vdots x\]
\[m - 1 \ x \ x \vdots x\]

The green dot corresponds to the point \((i, j)\). The blue lines symbolize the operation \(\text{Merge}R_{i,m-1}\). The red lines symbolize the operation \(\text{Merge}C_{0,j}\).

Figure 13: Computation of \(\phi\) in the case 2a.

ii. If \(\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\} \subset T\) then the tableau \(\tilde{T} = T \setminus \llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\}\) is locally saturated. From Lemma 9, the tableau

\[
\tilde{T}' = \text{Supp}R_{m-1} \left( \text{Merge}R_{0,m-1} \left( \text{Supp}C_0 \left( \text{Merge}C_{0,q}(\tilde{T}) \right) \right) \right)
\]

is also locally saturated and belongs to \(\text{LST}_{m,n}(1, 1)\). We set \(\phi(T) = \tilde{T}' \cup \{(m - 1, 0)\}\).

Since \(\tilde{T}' \in \text{LST}_{m,n}(1, 1)\), we have also \(\phi(T) \in \text{LST}_{m,n}(1, 1)\) (see Figure 14 for an example).

The set of images by \(\phi\) of the tableaux \(T\) that satisfy the condition of this case are those that have a cross in \((m - 1, 0)\), have at least a cross in the final zone of \(T\) and that have no cross in the \(\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\}\) zone.
The green cross corresponds to the added cross \((m - 1, 0)\). The blue lines symbolize the operation \(\text{MergeR}_{0,m-1}\). The red lines symbolize the operation \(\text{MergeC}_{0,q}\).

Figure 14: Computation of \(\phi\) in the case 2ai.

(b) If \((m - 1, 0) \in T\) then we have to consider two cases

i. If there exists \((i, j) \in (\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\}) \cap T\) then we consider such a pair \((i, j)\) minimal for the lexicographic order. From Lemma 9 the tableau

\[
\tilde{T} = \text{SuppR}_{m-1} \left( \text{MergeR}_{i,m-1} \left( \text{SuppC}_0 \left( \text{MergeC}_{0,j}(T) \right) \right) \right)
\]

belongs to \(\text{LST}_{m,n}(1, 1)\). Hence, we set

\[
\phi(T) = \tilde{T} \cup \{(m - 1, 0)\} \in \text{LST}_{m,n}(1, 1).
\]

See an example in Figure 15.

The set of images by \(\phi\) of the tableaux \(T\) that satisfy the condition of this case are those that have a cross at \((m - 1, 0)\), have at least a cross in the final zone of \(T\) and at least a cross in the \(\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n - 1\}\) zone.

The green circle corresponds to the point \((i, j)\). The blue lines symbolize the operation \(\text{MergeR}_{i,m-1}\). The red lines symbolize the operation \(\text{MergeC}_{0,j}\).

Figure 15: Computation of \(\phi\) in the case 2bi.
ii. If \((\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n-1\}) \cap T = \emptyset\) then the tableau \(\widetilde{T} = T \cup \llbracket m - p - 1 \rrbracket \times \{q, \ldots, n-1\}\) is locally saturated. We set

\[
\phi(T) = \text{Supp}_{m-1} \left( \text{Merge}_{m-p,m-1} \left( \text{Supp}_{C_0} \left( \text{Merge}_{0,n-1} (\widetilde{T}) \right) \right) \right).
\]

From Lemma 9, the tableau \(\phi(T)\) belongs to \(\text{LST}_{m,n}(1,1)\) (see Fig. 16 for an example).

The set of images by \(\phi\) of the tableaux \(T\) that satisfy the condition of this case are those that have no cross in \((m - 1, 0)\), have at least a cross in the final zone of \(T\) and contains the \(\llbracket m - p - 1 \rrbracket \times \{q, \ldots, n-1\}\) zone as a subset.

![Figure 16: Computation of \(\phi\) in the case 2bi.](image)

Figure 16 shows that sets of images by \(\phi\) for each of the previous cases are pairwise disjoint.

| Case      | \(E \cap \phi(T)\) | \(F \cap \phi(T)\) | \(G \cap \phi(T)\) |
|-----------|---------------------|---------------------|---------------------|
| 1         |                     |                     |                     |
| 2ai       | 0                   | \(\neq 0\)          | \(\neq G\)          |
| 2aii      | \(E\)               | \(\neq 0\)          | 0                   |
| 2bi       | \(E\)               | \(\neq 0\)          | \(\neq 0\)          |
| 2bii      | 0                   | \(\neq 0\)          | \(G\)               |

\(E = \{(m - 1, 0)\},\)

\(F = (\llbracket m - p \rrbracket \times \{q\}) \cup (\llbracket m - p, \ldots, m - 1 \rrbracket \times \{q, \ldots, n - 1\}),\)

\(G = \llbracket m - p \rrbracket \times \{q, \ldots, n - 1\}.)\)

![Figure 17: Configurations of three pairwise disjoint zones F, G, and E in \(\phi(T)\) with respect to cases 1, 2a, 2ai, 2b, and 2bi.](image)

Figure 17: Configurations of three pairwise disjoint zones \(E, F,\) and \(G\) in \(\phi(T)\) with respect to cases [1], [2a], [2ai], [2b], and [2bi].

So it remains to check that there is a \(\psi\) function that allows us to go back from \(\phi(T)\) to \(T\) in each case. Let \(T'\) be any tableau in the image of \(\phi\). Recall that \(F\) is the final zone of \(T\).

The definition of \(\psi\) is as follows:
1. We set $\psi(T') = T'$.

2. (a) i. Let $(i, j)$ be the minimal element for the lexicographic order in $T'$. We set

$$\psi(T') = \text{MergeC}_{0,j}(\text{MergeR}_{i,m-1}(T')) \setminus F.$$  

(21)

ii. We set

$$\psi(T') = \left(\text{MergeC}_{0,j}(\text{MergeR}_{i,m-1}(T')) \cup (\mathbb{I} \setminus \{q, \ldots, n-1\})\right) \setminus (F \cup \{(m-1,0)\}).$$  

(22)

(b) i. Let $(i, j)$ be the minimal element for the lexicographic order in $T'$. We set

$$\psi(T') = \text{MergeC}_{0,j}(\text{MergeR}_{i,m-1}(T')) \setminus F.$$  

(23)

ii. We set

$$\psi(T') = \left(\{0, m-1\} \cup (\text{MergeC}_{0,q}(\text{MergeR}_{0,m-1}(T'))) \setminus (F \cup (\mathbb{I} \setminus \{q, \ldots, n-1\})\right) \setminus F.$$  

(24)

Consider also the zone $D = \{(m-1,0)\} \cup ((\mathbb{I} \setminus \{0\} \setminus D)$ (see Figure 18).

Figure 18: Illustration of the zone $D$ (in blue in the picture).

**Lemma 10** For any $T \in \text{LST}_{m,n}(p,q)$ we have

$$T \cap D = \psi(\phi(T)) \cap D.$$  

(25)

**Proof:** In what follows, we check that Equality (25) holds for all cases of the definition of $\phi$:

1. In that case, both $\phi$ and $\psi$ equal the identity. Hence, the result is trivial.

2. (a) i. Since $T \cap F = \emptyset$, Formula (17) and definitions of MergeC and MergeR imply that $T \cap D = \phi(T) \cap D$. Since $\phi(T) \cap ((\mathbb{I} \setminus \{0\} \setminus (\mathbb{I} \setminus \{0\}) = \emptyset$, Equality (21) and definition of MergeC and MergeR imply that $\phi(T) \cap D = \psi(\phi(T)) \cap D$. Hence, $T \cap D = \psi(\phi(T)) \cap D$. 

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ii. For any tableau \( T \) satisfying condition (2a), we have \( T \cap D = \{ m-p \} \times \{ q, \ldots, n-1 \} \). But from Formula (22), we also have \( \phi(\psi(T)) \cap D = \{ m-p \} \times \{ q, \ldots, n-1 \} \). Hence, \( T \cap D = \psi(\phi(T)) \cap D \).

(b) i. Since \( T \cap F = \emptyset \), Formula (19) and definition of MergeC and MergeR imply that \( T \cap D = \psi(\phi(T)) \cap D \). Since \( \phi(T) \cap (\{ m \} \times \{ 0 \}) \cup (\{ m-1 \} \times \{ n \}) = \emptyset \), Equality (23) and definition of MergeC and MergeR imply that \( \phi(T) \cap D = \psi(\phi(T)) \cap D \). Hence, \( T \cap D = \psi(\phi(T)) \cap D \).

ii. For any tableau \( T \) satisfying condition (2bii), we have \( T \cap D = \emptyset \). But from Formula (24), we also have \( \phi(\psi(T)) \cap D = \emptyset \). Hence, \( T \cap D = \psi(\phi(T)) \cap D \).

\[ \square \]

We shall now prove that \( \psi \) is the reciprocal function of \( \phi \).

**Proposition 9** For any \( T \in \text{LST}_{m,n}(p,q) \), we have \( \psi(\phi(T)) = T \).

**Proof:** From the definition of \( \psi \), it is easy to check that \( \psi(\phi(T)) \cap F = \emptyset = T \cap F \). So using Lemma 10 it suffices to prove that

\[ T \cap R = \psi(\phi(T)) \cap R, \]

where

\[ R = \{ m \} \times \{ n \} \setminus (D \cup F) = R_1 \cup R_2, \]

with \( R_1 = \{ m-p, \ldots, m-2 \} \times \{ 0 \} \) and \( R_2 = \{ m-1 \} \times \{ 1, \ldots, p-1 \} \). Hence, it suffices to prove that

\[ \psi(\phi(T)) \cap R_1 = T \cap R_1, \tag{26} \]

the other equality \( (\psi(\phi(T)) \cap R_2 = T \cap R_2) \) being obtained symmetrically. In what follows, we check Equality (26) for every case occurring in the definition of \( \phi \):

1. If \( T \) matches with case (1) of definition of \( \phi \) then both \( \phi \) and \( \psi \) equal the identity. So the result is obvious.

2. (a) i. Let \( T \in \text{LST}_{m,n}(p,q) \) be any tableau satisfying the condition of case (2a) of the definition of \( \phi \). Let \( (i,j) \) be the smallest pair of \( (\{ m-p \} \times \{ q, \ldots, n-1 \}) \setminus T \) for the lexicographic order. From Equality (17) and the definition of MergeC, we have \( (i',0) \in T \cap R_1 \) if and only if \( (i',j) \in \phi(T) \) for any \( i' \in \{ m-p, \ldots, m-2 \} \). Furthermore from Equality (21) and the definition of MergeC, we have also for any \( i' \in \{ m-p, \ldots, m-2 \} \), \( (i',j) \in \phi(T) \) if and only if \( (i',0) \in \psi(\phi(T)) \cap R_1 \) because \( \phi(T) \cap (\{ m-p \} \times \{ 0 \}) = \emptyset \). Hence, \( (i',0) \in T \cap R_1 \) if and only if \( (i',0) \in \psi(\phi(T)) \cap R_1 \). This proves Equality (26).

ii. Let \( T \in \text{LST}_{m,n}(p,q) \) be any tableau satisfying the condition of case (2a) of the definition of \( \phi \). From Equality (18) and the definition of MergeC, for any \( m-p-1 < i < m-1 \), we have \( (i,0) \in T \cap R_1 \) if and only if \( (i,p) \in \phi(T) \). Furthermore, since \( \phi(T) \cap (\{ m-p \} \times \{ 0 \}) = \emptyset \), equality (22) and the definition of MergeC imply that for any \( i \in \{ m-p, \ldots, m-2 \} \), we have \( (i,p) \in \phi(T) \) if and only if \( (i,0) \in \psi(\phi(T)) \cap R_1 \). Hence, we have \( (i,0) \in T \cap R_1 \) if and only if \( (i,0) \in \psi(\phi(T)) \cap R_1 \). This proves Equality (26).
(b) i. Let \( T \in \text{LST}_{m,n}(p,q) \) be any tableau satisfying the condition of case \([2b]\) of the definition of \( \phi \). Let \((i,j)\) be the smallest pair of \((\{m-p\} \times \{q, \ldots, n-1\}) \cap T\) for the lexicographic order. From Equality \([19]\) and the definition of MergeC, for any \( i' \in \{m-p, \ldots, m-2\} \), we have \((i',0)\) \( T \cap R_1 \) if and only if \((i',j)\) \( \phi(T) \). Furthermore, since \( \phi(T) \cap (\{m-p\} \times \{0\}) = \{(m-1,0)\} \), Equality \([25]\) and the definition of MergeC imply that for any \( i' \in \{m-p, \ldots, m-2\} \), we have \((i',j)\) \( \phi(T) \) if and only if \((i',0)\) \( \psi(\phi(T)) \cap R_1 \). Hence, \((i',0)\) \( T \cap R_1 \) if and only if \((i',0)\) \( \psi(\phi(T)) \cap R_1 \). This proves Equality \([26]\).

ii. Let \( T \in \text{LST}_{m,n}(p,q) \) be any tableau satisfying the condition of case \([2b]\) of the definition of \( \phi \). From Equality \([20]\) and the definition of MergeC, for any \( i \in \{m-p, \ldots, m-2\} \), we have \((i,0)\) \( T \cap R_1 \) if and only if \((i,q)\) \( \phi(T) \). Furthermore, since \( \phi(T) \cap (\{m-p\} \times \{0\}) = \emptyset \), Equality \([24]\) and the definition of MergeC imply that for any \( i \in \{m-p, \ldots, m-2\} \), we have \((i,q)\) \( \phi(T) \) if and only if \((i,0)\) \( \psi(\phi(T)) \cap R_1 \). Hence, we have \((i,0)\) \( T \cap R_1 \) if and only if \((i,0)\) \( \psi(\phi(T)) \cap R_1 \). This proves Equality \([26]\).

Hence, Equality \([26]\) holds in any case and this implies the proposition. \( \Box \) Proposition\([9]\) implies that \( \phi \) is an injection and, since \( \alpha_{1,1} = 2 \), we obtain

\[
\max(\alpha_{p,q} \alpha_{m-p,n-q} | p \in \{1, \ldots, m-1\}, q \in \{1, \ldots, n-1\}) = 2 \alpha_{m-1,n-1}.
\]

We deduce the following theorem:

**Theorem 8** When \( m, n > 1 \) we have

\[
\text{sc}_{\boxtimes}(m,n) = \alpha'_{m,n} + 2 \alpha_{m-1,n-1} - 1
\]

and the 2-monsters \( \text{Mon}_{m,n}^{\{f_1\},\{0\}} \) and \( \text{Mon}_{m,n}^{\{0\},\{f_2\}} \), for \( f_1 \in \mathbb{I}[m] \setminus \{0\} \) and \( f_2 \in \mathbb{I}[n] \setminus \{0\} \) are witnesses.

**Example 11** We use formulas of \([7]\) to compute the first values that are listed in Table\([6]\). The valid saturated tableaux illustrating the case \( m = n = 2 \) are pictured in Figure\([19]\).

| \( m \setminus n \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|
| 2               | 8 | 20 | 50 | 128 | 338 | 920 | 2570 |
| 3               | 20 | 66 | 212 | 690 | 2300 | 7866 | 27572 |
| 4               | 50 | 212 | 848 | 3368 | 13520 | 55232 | 230168 |
| 5               | 128 | 690 | 3368 | 15930 | 75008 | 355890 | 1711208 |
| 6               | 338 | 2300 | 13520 | 75008 | 407528 | 2206880 | 12020360 |
| 7               | 920 | 7866 | 55232 | 355890 | 2206880 | 13482546 | 82181312 |
| 8               | 2570 | 27572 | 230168 | 1711208 | 12020360 | 82181312 | 555813728 |

Table 6: First values of \( \text{sc}_{\boxtimes}(m,n) \) for the type X.
Hence, there is $2^n$ shows that the number of valid tableaux is maximal when the final zone is $\delta$. In that case, a tableau is valid if and only if it contains $(0,0)$ or it contains neither $(0,0)$ nor $(0,f)$. Hence, there is $2^{n-1} + 2^{n-2} = \frac{3}{2}2^n$ valid tableaux. We first check that all valid tableaux are accessible. We proceed by induction and consider the order $< of Definition 7$. The tableaux $\{\}$ and $\{(0,0)\}$ are obviously accessible. Let $T \neq \{\}$ and $T \neq \{(0,0)\}$. We consider three cases

1. If neither $(0,0)$ nor $(0,f)$ are in $T$ then we choose $(0,j) \in T$ and we set $T' = T \cup \{(0,0)\} \setminus \{(0,j)\}$. Since $T' < T$, by induction hypothesis, the tableau $T'$ is accessible and so $T = \delta^{(Id,0,f)}T'$ is accessible too.

2. If $(0,0) \in T$ and $(0,f) \notin T$ then we choose $(0,j) \in T$ and we set $T' = T \cup \{(0,f)\} \setminus \{(0,j)\}$. The tableau $T'$ is valid and since $T' < T$, the induction hypothesis show that $T'$ is accessible. Hence, the tableau $T = \delta^{(Id,0,f)}$ is accessible too.

3. If both $(0,0)$ and $(0,f)$ are in $T$ then we set $T' = T \setminus \{(0,f)\}$. Since $T' < T$, by induction hypothesis, the tableau $T'$ is accessible and so $T = \delta^{(Id,0,f)}T'$ is accessible too.

This proves that all valid tableaux are accessible. Now we show that they are pairwise non-equivalent. Let $T \neq T'$ be two valid tableaux. Without restriction, we assume that there exists $j$ such that $(0,j) \in T$ and $(0,j) \notin T'$. Let $g \in \mathbb{B}[n]^{1\rightarrow 1}$ defined by $g(j') = 0$ if $j' \neq j$ and $g(j) = f$. The tableau $\delta^{(Id,0,f)}T$ is final while the tableau $\delta^{(Id,0,f)}T'$ is non-final. This shows that valid tableaux are pairwise non-equivalent.

It remains to find $F_1$ and $F_2$ such that $F_1 \bullet F_2 = \{f\}$ whatever the Boolean operation $\bullet$ is. For $\bullet = \oplus$ or $\bullet = \cup$, we set $F_1 = \emptyset$ and $F_2 = \{f\}$ while for $\bullet = \cap$, we set $F_1 = \{0\}$ and $F_2 = \{f\}$.

It follows that the state complexity is $\frac{3}{4}2^n$ and a witness is $\text{Mon}_{1,n}^{F_1,F_2}$ with $F_1$ and $F_2$ as above.

### 5 Conclusion

The interest of this discussion does not only lie in the fact that a new state complexity is calculated but especially because it illustrates methods and strategies that can be implemented to compute state complexities. The first lesson is that it is often easier to compute state complexities of families of operators than of isolated operations. Indeed, identifying common points between different operations leads us to develop theoretical tools that are relevant in the context. We have already illustrated this approach in previous papers. The most telling example is the family of 1-uniform operations that led us to develop the theory of modifiers and monsters [4] (also called OLPA in [12]). This rather general theory does not directly give formulas but offers a mathematical framework to
calculate them. Among other examples that we have studied, let us mention the family of multiple catenation operators [9], the family of friendly operators [5], and the quasi-Boolean operations [6].

We are specifically interested in classes of operators, called 1-uniforms, that can be encoded using modifiers. For this range of problems, a three-step strategy emerged and this is what the paper illustrates. For many 1-uniform operations, we know algorithms that encode them on automata and behind many of these algorithms are in fact modifiers. Modifiers explain how states and transitions of the input automata are transformed and make combinatorial objects naturally appear as states of the output automaton (in our case the combinatorial objects are tableaux). Transitions of the output automaton are also described combinatorially through an action of the transformation monoid on the combinatorial objects. This is the first step in the strategy and is illustrated in the preliminary paragraph of Section 3. It allows to (re)encode the problem initially stated in terms of language theory into a combinatorial language through algebraic tools (modifiers). The second step is to study the combinatorics of the objects and to deduce properties of the output automaton. In the present paper, the combinatorial notion of validity is linked to the notion of accessibility (see Section 3.1) and the notion of local saturation is linked to the Nerode equivalence (see Sections 3.2, 3.3, and 3.4). Finally, the third step is to find witnesses. We choose witnesses among the monsters by using combinatorics as a guide. Section 4 illustrates this step.

One of the main interest of this method is that even when it does not allow to obtain a closed formula for the state complexity, it allows to express it as the number of elements of a set having a combinatorial description. Let’s take the example of the shuffle product: although the value of its state complexity is not known (only conjectured [2]), it can be expressed in terms of tableaux of set partitions [10]. Regarding the complexities studied in this paper, the one of star of union (more generally type O operations, see Theorem 7) and the one of star of intersection (more generally, type A operations, see Theorem 6) admit a closed form. We can also consider formula of Theorem 8 as a quasi-closed form for the complexity of star of symmetric difference and, more generally, of type X operations. Indeed, although the asymptotic properties of the $\alpha_{n,k}$ and $\alpha'_{n,k}$ numbers are not well known, we can compute these numbers far enough and relate them to other combinatorial numbers (see [7]).

Throughout this paper, we can see that the main difficulty comes from the combinatorics of tableaux that constitutes the bulk of the calculations. More generally, a deep understanding of the theory of modifiers requires to study the actions of the transformation monoid of a finite set on combinatorial objects as vectors, tableaux, sets, functions etc.

At this stage, we have not developed tools to determine alphabetic simplicity that is the minimum size of the alphabet for the state complexity to be reached. Notice that, for an alphabet of a given size, the higher the alphabetic simplicity, the smaller the state complexity. The alphabetic simplicity is not necessarily a constant but depends on state complexities of the input languages. Although in his thesis, Edwin Hamel-de-le Court [17] has shown that, alphabetic simplicity for star of symmetrical difference is bounded by 17, we prefer to defer this study to a future work in which we will develop different tools and concepts related to the notion of alphabetic simplicity.

We have investigated the computation of state complexity with respect to complete deterministic automata. Changing, even slightly, characteristics of the automata encoding input and/or output languages could have significant consequences on state complexities. For instance, the state complexity of the shuffle product with respect to (not necessarily complete) deterministic automata is well known [3] whereas we have only a conjecture for complete deterministic automata [2, 10]. We can also define the state complexity with respect to other types of automata (non-
deterministic, 2-ways etc). We think that there could exist, for each case, tools similar to modifiers. In order to define them in a very general framework, it will probably be necessary to use notions from category theory. This is an ambitious scientific project which is only at the beginning and which will require the accumulation of many examples before leading to a consistent theory.

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A Notations

\begin{itemize}
  \item \textbf{Acc}(A) : Restriction of the automaton \( A \) to its accessible. See Section \ref{sec:accessible-states}.
  \item \textbf{Acc}\(_{F_1,F_2}\) : Set of accessible states of \( M_{F_1,F_2}^\bullet \).
  \item \textbf{AS}\(_{F_1,F_2}\) : Set of accessible saturated tableaux with respect to the final sets \( F_1 \) and \( F_2 \) and the boolean operation \( \bullet \).
\end{itemize}
• : A Boolean operation or a Boolean function. See Section 2.1 and Table 1.

\[ \text{col}_j(T) \] : Indices of the rows in which there are crosses in the column \( i \) of the tableau \( T \). See Definition 11.

\( M_{F_1,F_2}^{*} \) : The DFA \( \mathcal{A}_{\text{Bool}} \text{(Mon}_{F_1,F_2}) \).

\( \overline{M}_{F_1,F_2}^{*} \) : The DFA with the same alphabet, the same states, the same initial state and the same final states as \( M_{F_1,F_2}^{*} \) but with the transition function \( d \) defined by \( d(f,g)(T) = T \cdot (f,g) \). See Section 3.2.

\( E_{i,j} = \{(i,j)\} \) : Tableau with only one cross at position \((i,j)\). See Section 3.1.

\( \bigstar \) : Kleene star of a Boolean operation. See formula (2).

\( T \rightarrow T' \) : One step of the local saturation. See Definition 10.

\( \Rightarrow \) : Transitive closure of \( \rightarrow \). See Definition 10.

\( S_{/V_{F_1,F_2}}^{*} \) : Set of local saturated tableaux with respect to the final states \( F_1 \) and \( F_2 \) and the Boolean operation \( \cdot \). See Definition 10.

\( \text{LST}_{m,n}(p,q) \) : Set of the locally saturated \( m \times n \) tableaux \( T \), in the type \( X \), satisfying \( T \cap F = \emptyset \) for \( F = \{(m-p) \times \{q\}\} \cup \{(m-p,m-p+1,\ldots,m-1) \times \{q,q+1,\ldots,n-1\}\} \). See Section 4.2.3.

\( \text{Merge}_{C_{j_1},j_2}(T) \) : Tableau \( T \) in which columns \( j_1 \) and \( j_2 \) have been merged with respect to the operation of saturation. See Definition 13.

\( \text{Merge}_{R_{i_1},i_2}(T) \) : Tableau \( T \) in which rows \( i_1 \) and \( i_2 \) have been merged with respect to the operation of saturation. See Definition 13.

\( \text{Supp}_{C_{j}}(T) \) : The tableau \( T \) in which all crosses have been removed from column \( j \). See Definition 12.

\( \text{Supp}_{R_{i}}(T) \) : The tableau \( T \) in which all crosses have been removed from row \( i \). See Definition 12.

\( \text{Bool}_{\cdot} \) : Modifier of the Boolean operation \( \cdot \). See Example 3.

\( \overline{\bigstar} \) : Modifier of the Kleene star operation. See Example 2.

\( \overline{\bigstar^{\text{Bool}}} \) : Modifier of the Kleene star of the Boolean operation \( \cdot \). See Formula (3).

\( \text{Mon}_{n_1,\ldots,n_k} \) : \( k \)-tuple of automata over a big alphabet encoding all the possible transitions. See Definition 12.

\( T < T' \) : Partial order on tableaux. See Definition 7.

\( F_I \) : Restricted final zone associated to \( I \). See Definition 9.

\( \text{Ind}_I(T) \) : Inducted of \( T \). See Definition 9.

\( \text{Red}(T) \) : Reduced of \( T \). See Definition 9.

\( T|_I \) : Restriction of the tableau \( T \) to \( I \). See Definition 9.

\( \text{row}_i(T) \) : Indices of the columns in which there are crosses in the row \( i \) of the tableau \( T \). See Definition 11.

\( \text{Sat}(T) \) : Unique saturated tableau in the Nerode class of \( T \). See Definition 8.

\( \text{Sat}(A) \) : Automaton labelled by saturated tableaux. See Definition 8.

\( \text{SV}_{F_1,F_2}^{*} \) : Set of saturated valid tableaux with respect to the final sets \( F_1 \) and \( F_2 \) and the Boolean operation \( \cdot \). See Formula (4).

\( \text{sc}_{\otimes} \) : State complexity of the operation \( \otimes \). See Formula (1).

\( \text{Val}_{F_1,F_2}^{*} \) : Set of valid tableaux labelling states of \( M_{F_1,F_2}^{*} \). See Section 3.1.
$\text{Val}(M_{F_1,F_2}^*) \; : \; \text{Restriction of the DFA } M_{F_1,F_2}^* \text{ to its valid states. See Section 3.1}$