Some recent trends from research mathematics and their connections to teaching: Case studies inspired by parallel developments in science and technology

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Abstract
We will outline our ideas for teaching in the core mathematics disciplines. They are based on our own experience in teaching at a number of universities in the USA, as well as in Europe. While some of the core ideas stay and have stayed relatively constant over a long period of time, they must be varied in accordance with the needs and the demands of students, and they must constantly updated keeping an eye to current research and to modern international trends in technology. Our thoughts and suggestions on the use of these trends in teaching have been tried out by the author, and they are now in textbooks, some by the author.

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1 Interaction between technology, research in mathematics, and teaching

Mathematics draws ideas and strengths from the outside world, and the connections to parts of engineering have been a boon to mathematics: From signal processing to wavelet analysis! That is true even if we forget about all of the practical applications emerging from these connections. Without inspiration from the neighboring sciences, mathematics would in all likelihood become rather sterile, and overly formal. I see opportunities at crossroads. Mathematics is reaping benefits from trends and topics in engineering and in the sciences. It is witnessed in a striking way by exciting developments in wavelets.

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From wavelets we see how notions of scale-similarity can be exploited in basis computations that use tricks devised for signal processing. This has now all become part of an exciting and fairly recent trend in mathematics and in technology centered around advances in wavelets and some of their many applications. But as we outline below, this is only part of a bigger picture involving also fractal analysis and the use of scale-similarity in a wide area of scientific problems.

At the same time, the key notion of self-similarity, such as the scale-similarity used everywhere for wavelets, is essential to our understanding of fractals: Fern-like pictures that look the same at small and at large scales.

One problem in the generation of wavelet bases is selecting the “nice” (here this means differentiable) wavelets among huge families of fractal-looking (non-smooth, or singular) functions.

Our analysis will take place in a variety of data sets, or in function spaces. Here, for the moment we begin with spaces of square-integrable functions (denoted $L^2$); they are Hilbert spaces, and they have special significance in modeling states in physics, and via the inner product, correlations in statistics. But $L^2$-functions can be very “bad” indeed!! Computers generate the good and the bad, and we are left with the task of sorting them out and making selections. It may be observed (directly from large libraries of pictures [BrJo02, Jo06]) that mathematical wavelet machines are more likely to spit out bad functions unless they are told where to concentrate the search from the intrinsic mathematics. These wavelets, signals, and fractals are things that have caught our attention in recent decades, but the mathematical part of this has roots back at least a hundred years, for example, to Alfred Haar and to Oliver Heaviside at the turn of the last century. From Haar we have the first wavelet basis, and with Heaviside we see the beginning of signal analysis. It is unlikely that either one knew about the other. Ironically, at the time (1909), Haar’s paper had little impact and was hardly noticed, even on the small scale of “notice” that is usually applied to mathematics papers.

Haar’s wonderful wavelet only began to draw attention in the mid-nineteen-eighties when the connections to modern signal processing became much better understood. These connections certainly served as a main catalyst in what are now known as wavelet tools in pure and applied mathematics. But at the outset, the pioneers in wavelets had to “rediscover” a lot of stuff from signal processing: frequency bands, high-pass, low-pass, analysis and synthesis using down-sampling, and up-sampling, reconstruction of signals, resolution of images; all tools that have wonderful graphics representations in the engineering literature. But still, why would we think that Fourier’s basis, and his lovely integral decomposition, are not good enough? Many reasons: Fourier’s method has computational drawbacks. This was less evident before computers became common and began to play important roles in applied and theoretical work.

Expansion of functions or signals into basis decompositions (called “analysis” in signal processing) involves basis coefficients (Fourier coefficients, and so on), and if we are limited to Fourier bases, then the computation of the coefficients must by necessity rely on integration. “Computers can’t integrate!”
Hmmm! Well, not directly. The problem must first be discretized. And there is need for a more direct and algorithmic approach. Hence the wavelet algorithm! This is a trend that has already found its way into textbooks on numerical analysis (see, e.g., [Co03, BCD03], and engineering (see, e.g., [Str00, StrNg96]). In any case, algorithms are central in mathematics even if you do not concern yourself with computers. And it is the engineering connections that inspired the most successful algorithms in our subject. Our thoughts on the use of these trends in teaching have been tried out by the author, and they are now in [Bri02], and more recently in [Jac06].

2 Case studies

2.1 Multiresolutions

While finite or infinite families of nested subspaces are ubiquitous in mathematics, and have been popular in Hilbert-space theory for generations (at least since the 1930s), this idea was revived in a different guise in 1986 by Stéphane Mallat, then an engineering graduate student; see [Mal89]. In its adaptation to wavelets, the idea is now referred to as the multiresolution method. What made the idea especially popular in the wavelet community was that it offered a skeleton on which various discrete algorithms in applied mathematics could be attached and turned into wavelet constructions in harmonic analysis. In fact what we now call multiresolutions have come to signify a crucial link between the world of discrete wavelet algorithms, which are popular in computational mathematics and in engineering (signal/image processing, data mining, etc.) on the one side, and on the other side continuous wavelet bases in function spaces, especially in $L^2(\mathbb{R}^d)$. Further, the multiresolution idea closely mimics how fractals are analyzed with the use of finite function systems.

But in mathematics, or more precisely in operator theory, the underlying idea dates back to work of John von Neumann, Norbert Wiener, and Herman Wold, where nested and closed subspaces in Hilbert space were used extensively in an axiomatic approach to stationary processes, especially for time series. Wold proved that any (stationary) time series can be decomposed into two different parts: The first (deterministic) part can be exactly described by a linear combination of its own past, while the second part is the opposite extreme; it is unitary, in the language of von Neumann.

John von Neumann’s version of the same theorem is a pillar in operator theory. It states that every isometry in a Hilbert space $H$ is the unique sum of a shift isometry and a unitary operator, i.e., the initial Hilbert space $H$ splits canonically as an orthogonal sum of two subspaces $H_s$ and $H_u$ in $H$, one which carries the shift operator, and the other $H_u$ the unitary part. The shift isometry is defined from a nested scale of closed spaces $V_n$, such that the intersection of these spaces is $H_u$. However, Stéphane Mallat was motivated instead by the notion of scales of resolutions in the sense of optics. This in turn is based on a certain “artificial-
intelligence” approach to vision and optics, developed earlier by David Marr at MIT [Mar82], an approach which imitates the mechanism of vision in the human eye.

The connection from these developments in the 1980s back to von Neumann is this: Each of the closed subspaces $V_n$ corresponds to a level of resolution in such a way that a larger subspace represents a finer resolution. Resolutions are relative, not absolute! In this view, the relative complement of the smaller (or coarser) subspace in larger space then represents the visual detail which is added in passing from a blurred image to a finer one, i.e., to a finer visual resolution.

This view became an instant hit in the wavelet community, as it offered a repository for the fundamental father and the mother functions, also called the scaling function $\phi$, and the wavelet function $\psi$. Via a system of translation and scaling operators, these functions then generate nested subspaces, and we recover the scaling identities which initialize the appropriate algorithms.

What results is now called the family of pyramid algorithms in wavelet analysis. The approach itself is called the multiresolution approach (MRA) to wavelets. And in the meantime various generalizations (GMRAs) have emerged.

In all of this, there was a second “accident” at play: As it turned out, pyramid algorithms in wavelet analysis now lend themselves via multiresolutions, or nested scales of closed subspaces, to an analysis based on frequency bands. Here we refer to bands of frequencies as they have already been used for a long time in signal processing.

Even though J. von Neumann and H. Wold had been using nested or scaled families of closed subspaces in representing past and future for time series, in 1989 S. Mallat, an engineering graduate student at the time, found that this same idea applies successfully to the representation of visual resolutions [Mal89]. And even more importantly, it offers a variety of powerful algorithms for processing of digital images.

Now parallel to all of this, pioneers in probability theory had in fact developed versions of the same refinement analysis. For example, in the theory of martingales, consistency relations may naturally be reformulated in the language of nested subspaces in Hilbert space.

One reason for the success in varied disciplines of the same geometric idea is perhaps that it is closely modeled on how we historically have represented numbers in the positional number system; see, e.g., [Knu81]. Analogies to the Euclidean algorithm seem especially compelling; see, e.g., [SzFo70].

2.2 Fractals

Intuitively, think of a fractal as reflecting similarity of scales such as is seen in fern-like images that look “roughly” the same at small and at large scales. While there may not be agreement about a rigorous mathematical definition, Mandelbrot originally defined fractals as sets whose Hausdorff–Besicovich dimension exceeded their topological dimension, but later accepted all self-similar, self-affine, or quasi-self-similar sets as fractals. Moreover, even more generally, the
self-similarity could refer alternately to space, and to time. And further versatility was added, in that flexibility is allowed into the definition of “similar.”

In the book [Jo06], our focus is more narrowly on the self-affine variant (where computations are relatively simple); but in addition, we encounter the fractal concept in other contexts, e.g., for measures, and for probability processes (such as fractal Brownian motion). Further, we have stressed examples more than the general theory.

The fractal concept for measures is especially agreeable in the self-affine case, since affine maps act naturally on measures. So for each fractal dimension $s$, there is a corresponding $s$-fractal probability measure $\mu = \mu_s$ which is the unique solution to a natural fixed-point equation, one which depends on $s$. Moreover, we may then recover this way the spatial fractal set itself as the support of this measure $\mu$.

As for $s$-fractal Brownian motion (fBm), the “$s$-fractal” feature there refers to how the position $X_t$ at time $t$ of the fBm-process transforms under scaling of $t$: If time $t$ scales by $c$, then the respective distributions before and after scaling are related by the power-law $c^s$. Specifically, for all $t$, the finite distributions calculated for $X_{ct}$ and for $c^s X_t$ coincide; see, e.g., [JMR01].

2.3 Data mining

The problem of how to handle and make use of large volumes of data is a corollary of the digital revolution. As a result, the subject of data mining itself changes rapidly. Digitized information (data) is now easy to capture automatically and to store electronically [HTK05]. In science, in commerce, and in industry, data represents collected observations and information: In science, there is data on markets, competitors, and customers [AgKu04a, AgKu04b]. In manufacturing, there is data for optimizing production opportunities, and for improving processes [Ku02, Ku05]. A tremendous potential for data mining exists in medicine [KuLD01, KuDS05], genetics [ShKu04], and energy [KuBu05]. But raw data is not always directly usable, as is evident by inspection. A key to advances is our ability to extract information and knowledge from the data (hence “data mining”), and to understand the phenomena governing data sources.

Data mining is now taught in a variety of forms in engineering departments, as well as in statistics and computer science departments. One of the structures often hidden in data sets is some degree of scale. The goal is to detect and identify one or more natural global and local scales in the data. Once this is done, it is often possible to detect associated similarities of scale, much like the familiar scale-similarity from multidimensional wavelets, and from fractals. Indeed, various adaptations of wavelet-like algorithms have been shown to be useful. These algorithms themselves are useful in detecting scale-similarities, and are applicable to other types of pattern recognition. Hence, in this context, generalized multiresolutions offer another tool for discovering structures in large data sets, such as those stored in the resources of the Internet. Because of the sheer volume of data involved, a strictly manual analysis is out of the question.
Instead, sophisticated query processors based on statistical and mathematical techniques are used in generating insights and extracting conclusions from data sets. But even such an approach breaks down as the quantity of data grows and the number of dimensions increases. Instead there is a new research area (knowledge discovery in databases (KDD)) which develops various tools for automated data analysis.

However, statistics is still at the heart of the problem of inference from the data. The widespread use of statistics, pattern recognition, and machine-learning algorithms is somewhat hindered in many areas by our ability to collect large volumes of data. The next limitation in the subject arises when the data is too large to fit in the main computer memory. As a result, we are faced with new issues, e.g., quality of data, creative data analysis, and data transformation.

Theory and hypothesis formation now becomes critical in our task of deriving insights into underlying phenomena from the raw data. Various adaptations of wavelet-like algorithms have again proved useful in detecting scale-similarities, and in other types of pattern recognition. Hence in this context wavelet ideas offer another tool for discovering structures in vast data sets, such as those in the resources of the Internet. And there are now a variety of such effective Web mining tools in use.

Areas of data mining include problems of representation, search complexity, and automated use of prior knowledge to help in a data search. Thus we see the beginnings of a new science for efficient inference from massive data sets.

3 Technology and the classroom

While information communication technology (ICT) has advanced in leaps and bounds in the past decade, it has not to the same degree impacted the processes of learning and our classroom activities; at least not in striking ways. Sure, you may say there is the Internet and there is Power Point, but their direct effect in the class room has still been relatively modest. There could be good reasons for that. The effect on what we teach and what students learn outside the classroom has been much more striking.

As of yet, the direct benefits of ICT to classroom learning have not been documented in convincing ways. But it is worth keeping in mind that in the past few years, the impact of ICT has referred to both the subjects presented in class, and to the way they are presented.

Examples:

3.1 Technology impacts the form and manner of teaching

The past few years have witnessed a substantial and direct use of the internet as part of classroom presentations. Examples: Java scripts with moving frames illustrating algorithms, direct projection of material in books and in software, Powerpoint, and other such software tools.
3.2 Technology impacting the substance and the subject taught

There is now more emphasis on the teaching of algorithms and approximation. We include more numerical illustrations, more images, more interdisciplinary math (combining ideas from math, from engineering, and from CS, –physics too!) Other trends are two fold:

(a) New topics: e.g., discrete wavelet algorithms.
(b) Old topics in a new light: e.g., Signal and image processing.

3.3 Difficulties in adaptation of various technologies to the classroom

As ICT is adapted to teaching there have been various difficulties, some intrinsic to teaching (a) and some to the infrastructure and organization (b):

(a) Funding shortages, poor understanding of what works and what does not, rigid policies enforced by bureaucrats who have limited knowledge about the subjects affected, a lack of common sense in policies regarding the various implementations. Misuses and overuse of ICT have on occasion had the unintended effect of putting the students to sleep!

It has worked better when teachers have paid close attention to how students in fact use these tools themselves.

(b) Organizational development issues have not always been addressed effectively in implementations of ICT in education. When asked, I tend to warn against models that are too rigid. In fact rigid and centralized policies have often backfired. In my experience, ICT have worked best when teachers listen and pay attention to how students use these tools themselves. Teachers should inspire, and not “force-feed” students! I often notice that the best teachers are also the best listeners!

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