The Convergence Analysis of Parallel Alternating Two-stage Iterative Algorithm for Linear Complementarity Problem

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ABSTRACT

In this paper, the authors first present parallel alternating two-stage iterative Algorithm some new relaxation algorithms for solving the linear complementarity problem. And then, when the coefficient matrices are monotone or H-matrices, they establish the global convergence theory of the algorithm. The algorithm has less computational complexity and quicker velocity and is especially suitable for parallel computation of large-scale problem.

KEYWORDS

linear complementarity problem; alternating two-stage method; parallel computation; two-stage iterative; convergence

INTRODUCTION

This paper focuses on the linear complementarity problem, which is to find a pair of real vectors \( r \) and \( z \in \mathbb{R}^n \) such that

\[
0 = \begin{pmatrix} r^T \end{pmatrix} = \begin{pmatrix} q + Az \end{pmatrix} \geq 0, \quad z \geq 0, \quad z^T(q + Az) = 0.
\] (1)

Where \( A \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \) are given real matrix and vector, respectively.

With the fast development of computing technology, parallel computation technology of numeric calculation has gradually become a hot research direction of scientific computing. Much attention has recently been paid on a class of iterative methods called the matrix-splitting methods.

In this article we will make a further promotion of these methods. And also consider about the multi-splitting two-stage relaxation algorithm, alternating iterative algorithm and parallel iterative algorithm.

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 Constructed a parallel alternating two-stage method to solve linear complementarity problem, and received a convergence theorems when coefficient matrix are monotone or H-matrices. Finally we introduced relaxing factor, presented relaxation iterative algorithms and convergence theorem.

Define 1.1 Given $A$ is a non-singular real matrix as: $n \times n$, $M_i, N_i, P_i, Q_i, R_i, S_i, E_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, \alpha$, To satisfy: 
1. $M_i = M_i - N_i$, $M_i(i = 1, 2, \ldots \alpha)$ Non-singular; 
2. $M_i = P_i - Q_i = R_i - S_i$, $P_i, R_i(i = 1, 2, \ldots, \alpha)$ Non-singular; 
3. $\sum_{i=1}^{\alpha} E_i = I$ 
($n \times n$ is unique matrix) Called these splittings are two-stage multi-splitting of A-matrix.

Lemma 1.1 Suppose $A$ is H-matrix when Diagonal element is positive. For any given $q \in \mathbb{R}^n$, linear complementarity problem has a unique solution.

Lemma 1.2 Suppose $G_1, G_2, \ldots, G_k, \ldots$ is a non-negative matrix sequence $n \times n$, $B_1, B_2, \ldots, B_k, \ldots$ is another non-negative matrix sequence $n \times n$, IF there is a vector $x > 0$ and a constant $0 \leq \beta < 1$, to make $G_k x \leq \beta x, k = 1, 2, 3, \ldots$ And also there is a non-singular matrix $(n \times n) S$, to make $G_k = SB_k S^{-1}$. Then we have 

\[ \lim_{k \to \infty} (B_k \cdots B_2 B_1) = 0. \]

PARALLEL ALTERNATING TWO-STAGE ITERATIVE ALGORITHM

We definite the parallel alternating two-stage iterative algorithm for the linear complementarity problem as follows:

Algorithm 2.1 (parallel alternating two-stage iterative algorithm)

Step 1. Initialization. Let $z^0, y^{0,0} \in \mathbb{R}^n$ be any given initial value, set $k = 0$.

Step 2. General iteration. Given $z^k, y^{k,0} \in \mathbb{R}^n$, For the multi-splitting of $A, A = M_i - N_i$, $M_i = P_i - Q_i = R_i - S_i$ ($i = 1, 2, \ldots, \alpha$), Parallel and alternating solving, Created Sequence:

\[
\begin{align*}
y^{k,0}, y^{k,1}, \ldots, y^{k,1/2}, y^{k,3}, \ldots, y^{k,1/2,1}, y^{k,1,1}, y^{k,1+1/2}, y^{k,1+1} \\
y^{k,1+1/2} &\geq 0, \\
q - N_i z^k + P_i y^{k,1+1/2} - Q_i y^{k,1} &\geq 0, \\
(y^{k,1+1/2})^T (q - N_i z^k + P_i y^{k,1+1/2} - Q_i y^{k,1}) &= 0, \\
y^{k,1+1} &\geq 0, \\
q - N_i z^k + R_i y^{k,1+1} - S_i y^{k,1+1/2} &\geq 0, \\
(y^{k,1+1})^T (q - N_i z^k + R_i y^{k,1+1} - S_i y^{k,1+1/2}) &= 0.
\end{align*}
\]

After $s(k)$ times of inner iterative, Let $z^{k,i} := y^{k,i+1}$,
Step 3. Let $z^{k+1} = \sum_{i=1}^{g} E_i z^{k,i}$.  \hfill (4)

Step 4. Test for termination. If $z^{k+1}$ satisfies a prescribed stopping rule, terminate. Otherwise, return to Step 2 with $k$ replaced by $k + 1$.

Notice 2.1 Let $S = \{1, 2, \cdots, n\}$ as some subsets union: $S_i \subseteq S(i = 1, 2, \cdots, \alpha)$, $S=S_i \cup S_j \cup \cdots \cup S_k$, those subsets can be overlap or non-overlap, Define $M_i = (m_{ij}^i), P_i = (p_{ij}^i), R_i = (r_{ij}^i), E_i = \text{diag}(e_i)$:

\[
e_j = \begin{cases} \alpha, & j \in S_i \\ 0, & \text{others} \end{cases}, \quad m_{ij}^k = \begin{cases} a_{ik} I_{jk}, & j, \alpha x, j \in S_i \\ 0, & \text{others} \end{cases},
\]

\[
p_{ij}^k = \begin{cases} a_{ij} I_{jk}, & j, \alpha x, j \in S_i \\ 0, & \text{others} \end{cases}, \quad r_{ij}^k = \begin{cases} a_{ij} I_{jk}, & j, \alpha x, j \in S_i \\ 0, & \text{others} \end{cases}
\]

In computing, $\alpha_j > 0, \sum_{i=1}^{g} E_i = I$, we only need to compute corresponding the component $(z^{k,i})_j$ of $(E_i)_j \neq 0$, And because this algorithm is suitable for parallel computation, It will have a higher efficient.

Notice 2.2 Generally, To any Outer iteration $k$, If inside iteration $(s(k))$ is along with Outer iteration $k$, then this method is unsteady. So use SPATS and NSPATS to define multi-splitting parallel iterative algorithm. And unsteady multi-splitting parallel iterative algorithm.

THE CONVERGENCE OF PARALLEL ALTERNATING TWO-STAGE ITERATIVE ALGORITHM

In this section, we establish the convergence theory for algorithm 2.1.

Lemma 3.1 Suppose $A = M_j - N_j$, $M_i = P_i - Q_i = R_i - S_i (i = 1, 2, \cdots, \alpha)$ is the two-stage multiple division of $A, A, M_j, P_i, R_i (i = 1, 2, \cdots, \alpha)$ are H-matrix which has a positive diagonal elements. $z^*$ is the only solution of question (1), Then to any vector $q \in R^n$ and any initial vector, $z^0, y^{0,0} \in R^n, z^0 \geq 0, \text{Series } \{y^{k,i+1/2}\}$ and \{yr^{k,i+1}\} created by algorithm 3.1 will respectively satisfy

\[
< P_i \gg |y^{k,i+1/2} - z^*| \\
\leq N_j \quad |z^k - z^*| + |Q_j| \quad |y^{k,i} - z^*|
\]

\[
< R_i \gg |y^{k,i+1} - z^*| \\
\leq N_j \quad |z^k - z^*| + |S_j| \quad |y^{k,i+1/2} - z^*|
\]  \hfill (5)

Prove. Since $P_i (i = 1, 2, \cdots, \alpha)$ is a diagonal element for positive H-matrix. From Lemma 1.1: branch question (2) has only solution that $y^{k,i+1/2}$ is unique defined. We will prove inequation (5) by each component.

To any $j$, Suppose $|y^{k,i+1/2} - z^*| = (y^{k,i+1/2} - z^*). Under the above assumptions, If $y_j^{k,i+1/2} = 0$, Then because the $j$-th component of the left-hand vector in (5) is then nonpositive and the right-hand component is always nonnegative. equation (5) is right. Now suppose $y_j^{k,i+1/2} > 0$, Then according to algorithm 2.1, can have
\((q - N_j z^k + P_i y^{k,i+1/2} - Q_i y^{k,i})_j = 0\). On the other hand, \((q - N_j z^* + P_i z^* - Q_i z^*)_j \geq 0\), subtraction, we have
\[
(q - N_j z^k + P_i y^{k,i+1/2} - Q_i y^{k,i})_j - (q - N_j z^* + P_i z^* - Q_i z^*)_j \leq 0
\]
(Cause the first equation is 0, and the second is greater than or equal to 0).

To put in short, \((P_j(y^{k,i+1/2} - z^*)_j \leq (N_j(z^k - z^*) + Q_j(y^{k,i} - z^*)_j\), Then \(< P_j \rangle \parallel y^{k,i+1/2} - z^* \rangle_j \leq (\parallel N_j \parallel \parallel z^k - z^* \parallel + \parallel Q_j \parallel \parallel y^{k,i} - z^* \parallel)_j\).

(Cause the diagonal element of \(P_j\) is positive. \(|y^{k,i+1/2} - z^*_j| = (y^{k,i+1/2} - z^*)_j\).

Same as when \(|y^{k,i+1/2} - z^*_j| = (z^* - y^{k,i+1/2})_j\), then we can get equation (5).

Same as equation (6). So Lemma 3.1 has been proved.

**Theorem 3.1**

Suppose \(A > \alpha \geq 0\), \(A = M_j - N_j\), \(M_j = P_i - Q_i = R_i - S_i\) \((i = 1, 2, \cdots, \alpha)\) is the two-stage multiple division of \(A\), \(A, M_j, P_i, R_i\) \((i = 1, 2, \cdots, \alpha)\) are diagonal element for positive H-matrix. and \(<A> = \langle d_j \rangle > \parallel N_j \parallel < M_j > = \langle d_j \rangle > \parallel Q_j \parallel \langle R_j \parallel \parallel S_j \parallel \). then any initial value \(z^0 \in R^\alpha\), According to the unique solution \(z^*\) of series \(\{z^k\}\) which is created by algorithm 2.1.

**Prove.** Since \(A\) is a diagonal element for positive H-matrix, so \(z^*\) is the unique solution of the linear complementarity problem. According to equation (5) from Lemma 4.1:
\[
|y^{k,i+1/2} - z^*| \leq < P_i >^{-1} \parallel Q_i \parallel \parallel y^{k,i} - z^* \parallel + < P_i >^{-1} \parallel N_i \parallel \parallel z^k - z^* \parallel
\]
And according to equation (6),
\[
|y^{k,i+1} - z^*| \leq < R_i >^{-1} \parallel S_i \parallel \parallel y^{k,i+1/2} - z^* \parallel + < R_i >^{-1} \parallel N_i \parallel \parallel z^k - z^* \parallel
\]
\< R_i >^{-1} \parallel Q_i \parallel \parallel y^{k,i} - z^* \parallel + < R_i >^{-1} \parallel S_i \parallel \parallel z^k - z^* \parallel
\]
\(\square\)

2.1 After \(s(k)\) times of inner iteration,
\[
|Q_i| \parallel y^{k,0} - z^* \parallel + \sum_{j=0}^{s(k)-1} (< R_i >^{-1} \parallel P_i >^{-1} | \parallel S_i \parallel \parallel Q_i \parallel) \]
\[
\sum_{j=0}^{s(k)-1} (< R_i >^{-1} \parallel S_i < P_i >^{-1} \parallel Q_i \parallel) \]
\< R_i >^{-1} \parallel S_i \parallel \parallel Q_i \parallel \]
\[
|z^k - z^*|
\]

199
Suppose
\[ T(k) = |< R_i >^{-1} | S_j |< P_i >^{-1} | Q_j | x^{(k)} \]
+ \[ \sum_{j=0}^{x^{(k)}-1} |< R_i >^{-1} | S_j |< P_i >^{-1} | Q_j | j < R_i >^{-1} \]
\[ (| S_j |< P_i >^{-1} + I) | N_j | \]

According to the third step of algorithm 2.1,
\[ | z^{k,i} - z^* | \leq | z^{k, i+1} - z | \leq T(k) | z^k - z^* | \quad (7) \]

According to equation (4) from algorithm, \[ z^{k+1} = \sum_{i=1}^{d} E_i z^{k,i} \]

Then \[ T(k) \geq 0, H(k) \geq 0 \]

By the known conditions
\[ < A ><< M_i > - | N_j |, < M_i >== P_i > - | Q_j | == R_i > - | S_j |, \]

\[ T(k) = |< R_i >^{-1} | S_j |< P_i >^{-1} | Q_j | x^{(k)} \]
+ \[ \sum_{j=0}^{x^{(k)}-1} |< R_i >^{-1} | S_j |< P_i >^{-1} | Q_j | j \]
\[ < R_i >^{-1} (| S_j || \]
\[ < P_i >^{-1} + I) | N_j | = (| R_i >^{-1} S_j |< P_i >^{-1} \]
\[ | Q_j | x^{(k)} + \sum_{j=0}^{x^{(k)}-1} (| R_i >^{-1} S_j |< P_i >^{-1} | Q_j | j \]
\[ < R_i >^{-1} (| S_j |< P_i >^{-1} + I) (| P_i | - | Q_i |) \]
\[ < M_i >^{-1} | N_j | = (| R_i >^{-1} S_j |< P_i >^{-1} | Q_j | x^{(k)} \]
+ \[ \sum_{j=0}^{x^{(k)}-1} (| R_i >^{-1} S_j |< P_i >^{-1} | Q_j | j \]
\[ < R_i >^{-1} | S_j | (I - | P_i >^{-1} | Q_j |) < M_i >^{-1} \]
\[ N_j | \]
+ \[ \sum_{j=0}^{x^{(k)}-1} (| R_i >^{-1} S_j |< P_i >^{-1} | Q_j | j \]
\[ (I - | R_i >^{-1} S_j |< M_i >^{-1} \]
\[ N_j | \]
\[ = (| R_i >^{-1} S_j |< P_i >^{-1} | Q_j | x^{(k)} + (I - | R_i >^{-1} \]
\[ | S_j |< P_i >^{-1} | Q_j | x^{(k)}) < M_i >^{-1} | N_j | \]

Consider Vector \( e = (1, \ldots, 1) \), Given \( x = < A >^{-1} e \), Cause \( < A >^{-1} \geq 0 \), and \( < A >^{-1} \)
not all is 0, then \( x > 0 \). Same we can get \( < R_i >^{-1} (| S_j |< P_i >^{-1} + I) > 0 \).
Through

\[ (I - < M_i >^2 | N)_x = < M_i >^2, Ax = < M_i >^2 e \]

\[ T(k)x = \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k) \]

\[ + \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ (\langle S_j < P_j >^{-1} + I > | N_j | x = \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k + (I - < R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ (\langle S_j < P_j >^{-1} | N_j | x = \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k) + (I - < R_j >^{-1} | N_j | x = [I - (I - < M_i >^{-1} | N_i | x = \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

\[ \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} \]

So

\[ H(k)x = \sum_{j=1}^{n} E_jT(k)x \]

\[ = x - \sum_{j=1}^{n} E_j < R_j >^{-1} (\langle S_j < P_j >^{-1} + I > e \]

\[ - \sum_{j=1}^{n} E_j \sum_{j=0}^{s(k)-1} (\langle R_j >^{-1} | S_j < P_j >^{-1} | Q_j^k)j < R_j >^{-1} e \]

\[ < R_j >^{-1} (\langle S_j < P_j >^{-1} + I > e \]

\[ < x - \sum_{j=1}^{n} E_j < R_j >^{-1} (\langle S_j < P_j >^{-1} + I > e \]

Since

\[ T(k)x \geq 0, \text{ then} \]

\[ x - \sum_{j=1}^{n} E_j < R_j >^{-1} (\langle S_j < P_j >^{-1} + I > e \]

\[ x - \sum_{j=1}^{n} E_j < R_j >^{-1} (\langle S_j < P_j >^{-1} + I > e \]

\[ \lim_{k \to \infty} H(k) = H(0) = 0. \]

Then this lemma have been proved.

Now we talk about the convergence of the parallel alternating two-stage iterative algorithm.

**Theorem 3.2** Given \( A = M_i - N_i, M_i = P_i - Q_i = R_i - S_i (i = 1, 2, \cdots, \alpha) \) as a two-stage multiple division of \( A, A, M_i, P_i, R_i \) are diagonal element for positive H-matrix. and \( < A > < M_i > < N_i >, < M_i > < Q_i > < R_i > < S_i > \), Then for any initial value \( z^0 \in R^n \), Series \( \{z^k\} \) created by (NPATS) 2.1 is convergent \( z^* \). Which is the unique solution of linear complementarity problem.

Prove. \( P_i \) and \( R_i \) are H-matrix by lemma 1.2, then,
Since $\overline{H}(k)$ is a iterative matrix of a method of the parallel alternating two-stage iterative algorithm about the matrix $<M_i> << P_j > - |Q_j | << R_i > - |S_i |$, $i = 1, 2, \ldots, \alpha$ are all positive splitting, we have

$$\lim_{k \to \infty} |H(k)| \leq \lim_{k \to \infty} |\overline{H}(k)| = 0$$

This completes the proof.

**RELAXED PARALLEL ALTERNATING TWO-STAGE ITERATIVE ALGORITHM**

Now we introduce the relaxation factor in algorithm 3.1, then we can get a new algorithm below.

**Algorithm 4.1 (Relaxed parallel alternating two-stage iterative algorithm)**

**Step 1. Initialization.** Let $z_0, y_0^0 \in R^n$ be any given initial value, set $k = 0$.

**Step 2. General iteration.** Set $z^k, y_0^k \in R^n$, For the multi-splitting of $A$, $A = M_i - N_i$, $M_i = P_i - Q_i = R_i - S_i$ ($i = 1, 2, \ldots, \alpha$), Parallel and alternating solving, Created Sequence: $y_0^{k,0}, y_0^{k,1/2}, y_0^{k,1}, \ldots, y_0^{k,1/2}, y_0^{k,1}$, and $y_i^{k,1/2}, y_i^{k,1}$ is the solution of complement subset question (2) and (3), After $s(k)$ times of inner iterative, Let $z_i^{k,i} := y_i^{k,i+1}$.

**Step 3. Let**

$$z^{k+1} = \omega \sum_{i=1}^n E_i z^{k,i} + (1 - \omega)z^k.$$

**Step 4. Test for termination.** If $z^{k+1}$ satisfies a prescribed stopping rule, terminate. Otherwise, return to Step 2 with $k$ replaced by $k + 1$. is relaxation factor.

On relaxed Parallel Multi-splitting alternating two stage iterative method, we have the following two convergence theorem:

**Theorem 4.1** Given $< A >^{-1} \geq 0$, $A = M_i - N_i$, $M_i = P_i - Q_i = R_i - S_i$ ($i = 1, 2, \ldots, \alpha$) as a two-stage multiple division of $A$, $A, M_i, P_i, R_i$ are diagonal element for positive H-matrix respectively. and $< A > << M_i > - |N_i |, < M_i > << Q_i > - |R_i |, < Q_i > - |S_i |, 0 < \omega < 1$. For any initial value $z_0 \in R^n$, According to the unique solution $z^*$ of series $\{z^k\}$ which is created by algorithm 4.1 of (PATS).
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