REPRESENTABILITY OF COHOMOLOGY OF FINITE FLAT ABELIAN GROUP SCHEMES

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ABSTRACT. We prove various finiteness and representability results for cohomology of finite flat abelian group schemes. In particular, we show that if $f: X \to \text{Spec}(k)$ is a projective scheme over a field $k$ and $G$ is a finite flat abelian group scheme over $X$ then $R^i f_* G$ is an algebraic space for all $i$. More generally, we study the derived pushforwards $R^i f_* G$ for $f: X \to S$ a projective morphism and $G$ a finite flat abelian group scheme over $X$. We also develop a theory of compactly supported cohomology for finite flat abelian group schemes, describe cohomology in terms of the cotangent complex for group schemes of height 1, and relate the Dieudonné modules of the group schemes $R^i f_* \mu_p$ to cohomology generalizing work of Illusie. A higher categorical version of our main representability results is also presented.

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1. Introduction

1.1. The results of this article have their origins in an attempt to understand the properties of the flat cohomology sheaves $R^i f_* \mu_p$ for a proper smooth morphism $f: X \to S$ with geometrically connected fibers in positive characteristic $p$. Here, $R^i f_* \mu_p$ denotes the $i$th
higher direct image of the pushforward \( f_* \) from the big flat (fppf) site of \( X \) to that of \( S \), which may be realized as the fppf sheafification of the functor on \( S \) schemes given by

\[
T \mapsto H^i(X \times_S T, \mu_p)
\]

For low degrees the properties of these sheaves are well-understood. For \( i = 0 \) we have \( R^0 f_* \mu_{p,X} \simeq \mu_{p,S} \) (this follows from noting that \( R^0 f_* \mathcal{O}_X \simeq \mathcal{O}_S \)). For \( i = 1 \) one can proceed by considering the exact sequence

\[
0 \longrightarrow \mu_p \longrightarrow G_m \xrightarrow{\times p} G_m \longrightarrow 0,
\]

and observing that \( R^1 f_* G_{m,X} \) is the Picard functor Pic\(_{X/S}\) whose representability by an algebraic space was shown by Artin \[3\]. From this it follows that

\[
R^1 f_* \mu_{p,X} \simeq \text{Ker}(\times p : \text{Pic}_{X/S} \to \text{Pic}_{X/S}),
\]

and in particular \( R^1 f_* \mu_p \) is representable by an algebraic space.

1.2. It is natural to ask about generalizing the representability result for \( R^1 f_* \mu_{p,X} \) in various directions:

(i) Consider all integers \( i \geq 0 \).

(ii) Replace \( \mu_p \) by an arbitrary finite flat abelian group scheme on \( X \).

(iii) Relax the assumptions on \( f \).

The main purpose of this article is to carry out these generalization. Our first main result is the following (in fact, we will prove a slightly more refined statement in \[9.1\]):

**Theorem 1.3.** Let \( k \) be a perfect field, let \( f : X \to S \) be a projective morphism of finite type \( k \)-schemes with \( S \) reduced, and let \( G_X \) be a finite flat abelian group scheme over \( X \). Then there exists a dense open subset \( U \subset S \) such that the sheaf \( R^i f_* G_{m,X} \big|_U \) on the fppf-site of \( U \) is contained in the smallest abelian subcategory of the category of abelian sheaves on \( U_{\text{fppf}} \) containing finite flat abelian group schemes and sheaves which are fppf-locally on \( U \) the underlying abelian sheaves of vector bundles. In particular, \( R^i f_* G_{m,X} \big|_U \) is an algebraic space.

**Corollary 1.4.** Let \( f : X \to \text{Spec}(k) \) be a projective finite type scheme over a field \( k \). If \( G \) is a finite flat abelian group scheme over \( X \), then \( R^i f_* G \) is representable by a finite type group scheme over \( k \). Moreover, this group scheme is an iterated extension of finite commutative group schemes and \( k \)-forms of vector bundles.

For schemes over a finite field we obtain the following consequence, which appears to be new.

**Corollary 1.5.** Let \( X \) be a projective scheme over a finite field. If \( G \) is a finite flat abelian group scheme over \( X \), then for each \( i \geq 0 \) the flat cohomology group \( H^i(X,G) \) is a finite group.

In \[10\] we consider the functors \( \Phi^i(X,G) \) associated to a smooth commutative group scheme \( G \) on a scheme \( X \), which were introduced by Artin–Mazur \[5\]. As a consequence of Theorem 1.3 we show that their fppf sheafifications are always prorepresentable. This result was previously obtained by Raynaud \[35\] using different methods.
**Theorem 1.6.** Let $X$ be a smooth proper scheme over a field $k$. If $G$ is a smooth commutative group scheme on $X$, then for each $i$ the fppf sheafification $\Phi^i_\text{fppf}(X,G)$ of $\Phi^i(X,G)$ is prorepresentable.

1.7. It is natural to ask for strengthening of 1.3 and 1.4 in two directions:

(i) Remove the projectivity assumption on $f$ and only assume $f$ proper. We expect this should be possible, but the method used in this paper, using derived blowups, uses the projection assumption.

(ii) Establish criteria for representability over the entire base scheme $S$, not just after shrinking to an open subset.

In the case of a proper smooth morphism and a finite flat group scheme of height 1 we are able to get these stronger results:

**Theorem 1.8.** Let $f : X \to S$ be a proper smooth morphism of algebraic spaces over a field $k$ of positive characteristic. Let $G$ be a finite flat abelian group scheme on $X$ such that either $G$ or its Cartier dual $G^D$ has height 1. Let $n$ be an integer and assume that the sheaves $R^i f_* G$ are algebraic spaces and flat over $S$ for $i < n$. Then $R^n f_* G$ is an algebraic space.

1.9. Coherent sheaves enter the picture naturally when studying cohomology of finite flat abelian group schemes. For example, the group scheme $\alpha_p$ sits (by definition) in an exact sequence

\begin{equation}
0 \to \alpha_p \to G_a \xrightarrow{F} G_a \to 0
\end{equation}

where $F$ denotes the map $x \mapsto x^p$. Therefore, for a morphism of schemes $f : X \to S$ the cohomology $R^i f_* \alpha_p$ is closely related to the coherent cohomology $R^i f_* \mathcal{O}_X$ and maps between them (note that the maps involved here are not $\mathcal{O}_S$-linear but simply maps of abelian sheaves).

1.10. Let $k$ be an algebraically closed field and let $f : X \to \text{Spec}(k)$ be a smooth projective scheme over $k$. By Corollary 1.4, for any finite flat commutative group scheme $G$ over $k$ the functor $R^i f_* G$ is represented by a group scheme which is an iterated extension of finite flat group schemes and groups of the form $G_a \oplus \mathbb{Z}$. Simple examples show that the vector group component may be nontrivial, and in particular $R^1 f_* \alpha_p$ is positive dimensional. For example, if $X$ is a supersingular elliptic curve, then $R^1 f_* \alpha_p \simeq G_a \oplus \mathbb{Z}$. This phenomenon also occurs for $G = \mu_p$; if $X$ is a supersingular K3 surface, then $R^2 f_* \mu_p$ is an extension of $G_a \oplus \mathbb{Z}$ by a finite étale group scheme (see for example [11, 2.2.4]).

1.11. Our approach to 1.3 is to study stability properties under $Rf_*$ of certain triangulated categories. Namely for a noetherian scheme $X$ let $D(X_{\text{fppf}})$ denote the derived category of abelian sheaves on the fppf-site of $X$. We will also have occasion to consider the derived category of $\mathcal{O}_{X_{\text{fppf}}}$-modules, which we will denote by $D(X_{\text{fppf}}, \mathcal{O}_{X_{\text{fppf}}})$. There is a projection

$$\epsilon : X_{\text{fppf}} \to X_{\text{ét}}$$

from the fppf topos to the étale topos, which is a morphism of ringed topoi. Let

$$D^b_{\text{coh}}(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}) \subset D(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$$
be the subcategory consisting of bounded complexes of $\mathcal{O}_{X_{\text{ét}}}$-modules with coherent cohomology sheaves. We can then consider the composition

$$D^b_{\text{coh}}(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}) \xrightarrow{\text{Le}^*} D(X_{\text{fppf}}, \mathcal{O}_{X_{\text{fppf}}}) \xrightarrow{\text{forget}} D(X_{\text{fppf}}),$$

where the last functor forgets the $\mathcal{O}_{X_{\text{fppf}}}$-module structure. In this way we view coherent sheaves and complexes thereof as objects of $D(X_{\text{fppf}})$. Note that even for a coherent sheaf $F$ on $X_{\text{ét}}$ the associated object $F_{\text{fppf}}$ of $D(X_{\text{fppf}})$ is not a sheaf but a complex: For a morphism $g : T \to X$ the restriction of $F_{\text{fppf}}$ to $T_{\text{ét}}$ is the complex $Lg^*F$.

If $i : Z \hookrightarrow X$ is a closed immersion and $G_Z$ is a finite flat abelian group scheme over $Z$, then, using derived algebraic geometry, we can associate a complex $i_*G_Z \in D(X_{\text{fppf}})$ (see section 3). Roughly the value of this complex on $U \to X$ is given by the value of $G_Z$ on the derived fiber product $Z \times^\text{der} X U$. Let $D^f(X) \subset D(X_{\text{fppf}})$ (resp. $D^{lf}(X) \subset D(X_{\text{fppf}})$) be the smallest thick triangulated subcategory containing such complexes $i_*G_Z$ and complexes given by perfect complexes of $\mathcal{O}_X$-modules (resp. such complexes $i_*\alpha G_Z$ and complexes which are fppf-locally on $X$ given by perfect complexes of $\mathcal{O}_X$-modules). Observe that $D^f(X)$ (resp. $D^{lf}(X)$ contains the triangulated subcategory of $D(X_{\text{fppf}})$ generated by perfect complexes and finite flat group schemes over $X$ (locally perfect complexes and finite flat group schemes over $X$).

We will prove 1.12 by proving the following result:

**Theorem 1.12.** Let $k$ be a perfect field and let $f : X \to S$ be a projective morphism of finite type $k$-schemes with $S$ reduced. Then for any $\mathcal{F} \in D^f(X)$ there exists a dense open subset $U \subset S$ such that the restriction $Rf_*\mathcal{F}|_U$ lies in $D^{lf}(U)$.

1.13. Over an algebraically closed field $k$ of positive characteristic $p$ there is a good understanding of the structure of finite flat group schemes. The category of such group schemes is abelian with simple objects $\mathbb{Z}/(\ell)$, for $\ell$ a prime possibly equal to $p$, $\alpha_p$, and $\mu_p$ (this follows from Dieudonné theory [19, Théorème 1 (iii)]).

We will use devissage to reduce the proof of 1.12 to studying these special cases. This is not straightforward, however, as one cannot directly reduce to these cases when working with families. For example, one can have a finite flat group scheme $G$ over a curve $C$ for which there exists a dense open subset $U \subset C$ for which $G|_U$ is isomorphic to $\mu_p$ but the fiber over $x \in C - U$ is isomorphic to $\alpha_p$. Similarly one can construct degenerations of $\mathbb{Z}/(p)$ to $\alpha_p$.

There are two key technical ingredients, which may be of independent interest, in this devissage:

(i) Compactly supported flat cohomology. This is inspired on the one hand by the theory of compactly supported cohomology for coherent sheaves (developed in [23 Appendix] and [24]) and the systematic use of “animated rings” in [14].

(ii) Calculation of cohomology of finite flat group schemes of height 1 using a derived version of the “Hoobler sequence” (see [25, Theorem 1.3] and [6, 2.4]).

We briefly outline our work on these two aspects.

1.14. **Compactly supported flat cohomology.**
1.15. Let \( k \) be a field and \( X/k \) a finite type proper \( k \)-scheme. Let \( i : Z \hookrightarrow X \) be a closed subscheme with complement \( j : U \hookrightarrow X \). In [23], building on work of Deligne [23, Appendix], one finds a definition of the compactly supported cohomology \( H_c^*(U, \mathcal{F}_U) \) of a coherent sheaf \( \mathcal{F}_U \) on \( U \). Namely choose an extension \( \mathcal{F} \) of \( \mathcal{F}_U \) to \( X \) and define \( R\Gamma_c(U, \mathcal{F}_U) := \text{Cone}(R\Gamma(X, \mathcal{F} \to R\Gamma(\widehat{X}, \widehat{\mathcal{F}}))[-1], \)

where \( \widehat{X} \) denotes the formal completion of \( X \) along \( Z \) (so \( \widehat{X} \) is a formal scheme) and \( \widehat{\mathcal{F}} \) is the pullback of \( \mathcal{F} \) to \( \widehat{X} \). Then it is shown in loc. cit. that \( H_c^*(U, \mathcal{F}_U) := H^*(R\Gamma_c(U, \mathcal{F}_U)) \) is independent of choices and gives a reasonable theory of compactly supported cohomology.

1.16. We will adapt this approach to include also finite flat group schemes. To this end we have to make sense of \( R\Gamma(\widehat{X}, \widehat{G}) \) for a finite flat group scheme \( G/X \). This will be done using derived algebraic geometry in the style of [14, §5]. We won’t explain the precise definitions and details here, referring instead to sections 3-6, but roughly the idea is the following. Let \( H_G \) denote the functor (taking values in complexes of abelian groups) which to any \( T/X \) associates \( R\Gamma(T, G) \).

Let \( i_n : X_n \hookrightarrow X \) be the inclusion of the \( n \)-th infinitesimal neighborhood of \( Z \) in \( X \) and let \( G_n \) be the pullback of \( G \) to \( X_n \). We then define \( i_n^{ani} H_{G_n} \) (the superscript here refers to “animated” as in [14]) by associating to \( T \to X \) the complex of abelian groups \( R\Gamma(T \times^L_X X_n, G_n) \)

where \( T \times^L_X X_n \) denotes a suitably derived fiber product. The complexes \( i_n^{ani} H_{G_n} \) define a projective system of complexes and we set \( H_G := R\lim i_n^{ani} H_{G_n} \).

We can then define the compactly supported cohomology using the cocone of the natural map \( H_G \to H_G \).

To make this all precise, and to state the main results requires some rather technical work with \( \infty \)-categories and pro-objects - we refer the reader to sections 6 and 7.1 for the precise statements of our main results in this regard (see in particular 7.11).

1.17. **Cohomology and Kan extension.**

1.18. Let \( k \) be a perfect ring of positive characteristic \( p \) and let \( X/k \) be a smooth scheme. By [21, Exposé VIIA, Remarque 7.5] there is an equivalence of categories between the category of finite flat abelian group schemes \( G/X \) of height \( \leq 1 \) and the category of pairs \( (\mathcal{V}, \rho) \), where \( \mathcal{V} \) is a locally free sheaf of finite rank on \( X \) and \( \rho : F^1_X \mathcal{V} \to \mathcal{V} \) is a morphism of vector bundles. For such a group scheme \( G \) the corresponding vector bundle \( \mathcal{V} \) is the Lie algebra of \( G \). If \( \epsilon : X_{\text{fppf}} \to X_{\text{et}} \)

is the projection from the fppf topos to the étale topos then it is shown in [6, 2.4] that there is a natural isomorphism \( R\epsilon_* G[1] \simeq \mathcal{V} \otimes F_* Z^1_{X/k} \rho \otimes \text{id} \otimes \Omega^1_{X/k} \)
in the derived category of abelian sheaves on $X_{\text{ét}}$, where $Z^1_{X/k} \subset \Omega^1_{X/k}$ denotes the sheaf of closed 1-forms, $I$ is the inclusion of the closed forms into all forms, and $C$ is the Cartier operator. In particular, if $X = \text{Spec}(A)$ is affine then we have
\begin{equation}
R\Gamma(A, G)[1] \simeq (V \otimes F_s Z^1_{A/k} \xrightarrow{\rho \otimes \text{id} \otimes C} V \otimes \Omega^1_{A/k}).
\end{equation}

1.19. In section 4, building on results and ideas of Bhatt and Lurie, we extend the isomorphism (1.18.1) to all affine schemes over $X$. The main result is 4.8 which identifies the functor (taking values in the $\infty$-category of complexes of abelian groups)
\begin{equation}
(\text{Spec}(A) \to X) \mapsto R\Gamma(A, G)[1]
\end{equation}
with the functor
\begin{equation}
(\text{Spec}(A) \to X) \mapsto (V \otimes F_s L Z^1_{A/k} \xrightarrow{\rho \otimes \text{id} \otimes C} V \otimes L^1_{A/k}),
\end{equation}
where $L Z^1_{A/k}$ denotes the derived functor of closed 1-forms and $L^1_{A/k}$ denotes the cotangent complex. This second functor is the Kan extension of its restriction to smooth algebras.

We will also make use of the fact that the functor (1.19.1) also makes sense more generally for any perfect complex $\mathcal{V}$ on $X$ with a map $\rho : F_s^* \mathcal{V} \to \mathcal{V}$. For example, for a proper smooth morphism $f : X \to S$ a piece of $Rf_* \mathcal{V}$ can be described in terms of $Rf_* \mathcal{O}_X$ with its natural Frobenius structure (see 4.16).

1.20. Relation with prior work.

1.21. We summarize previous work on this topic, specifically with regards to the statement of Corollary 1.4. When $G$ is étale of order prime to the characteristic of $k$ (in particular, when $k$ has characteristic 0), the result follows from [16, Th. finitude, Théorème 1.1]. When the characteristic of $k$ is $p$, the result for $G = \alpha_p$ follows from consideration of the sequence (1.9.1). For $G = \mathbb{Z}/p \mathbb{Z}$, the result similarly follows from consideration of the short exact sequence of sheaves [11 IX, 3.5]
\begin{equation}
0 \longrightarrow \mathbb{Z}/p \mathbb{Z} \longrightarrow G_a \xrightarrow{1-F} G_a \longrightarrow 0,
\end{equation}
More generally, if the Cartier dual $G^D$ has height 1, then Artin-Milne [6 Proposition 1.1] construct an exact sequence similar in form to (1.9.1) and (1.21.1), and deduce the representability result for $G$ as a consequence [6 Corollary 1.4].

When $G = \mu_p$ to the best of our knowledge the representability of $R^i f_* \mu_p$ was known only in a few very special cases. The cases $i = 0, 1$ are obtained in general as described above. When $X$ is a surface, the representability for all $i$ is claimed in [4 Theorem 3.1], with the caveat that the proof would appear elsewhere, although to the best of our knowledge it did not. When $X$ is a K3 surface, the result is shown for $i = 2$ in [5 4.2]. We note that the weaker statement that the perfection of $R^i f_* \mu_p$ (that is, the restriction of $R^i f_* \mu_p$ to the category of perfect schemes over $k$) is representable seems to be well known, and is shown for instance in [32 2.7].

1.22. The article is organized as follows.

We begin in section 2 by discussing a description in terms of coherent sheaves of cohomology of $\alpha_p$, $\mathbb{Z}/p \mathbb{Z}$, and $\mu_p$ for proper smooth morphisms $f : X \to S$ with $S$ smooth over a perfect
field \( k \). In some sense, all the results of this article will be reduced to these calculations by (somewhat complicated) devissage. We also explain how to extend some of these results to height 1 abelian group schemes. In section 3 we review some of the basic theory of animated rings and cohomology from [14]. In section 4 we explain a construction of finite flat group schemes using animated rings. This may be viewed as a version of Dieudonné theory in the animated context and enables us to extend some of the computations in [2] to remove the smoothness assumption on \( S \). In section 5 we use these results to prove 1.3 and 1.8 for height 1 group schemes and smooth proper morphisms. This section uses higher categorical algebraic stacks in the sense of [37]. The reader who is only interested in the proof of 1.3 may skip this section, as the contents are not used elsewhere in the article. Sections 6 through 8 are devoted to developing a theory of compactly supported cohomology for finite flat group schemes. The main result of these sections is 7.11. In section 9 we prove our main finiteness result for flat cohomology, which implies, in particular, 1.12. Having shown our main representability results, we turn in sections 10–12 to the relationship with Dieudonné and Cartier theory. In particular, we study the Dieudonné theory of \( R^i f_* \mu_p \) for a smooth proper morphism \( f : X \to \text{Spec}(k) \). The results of this section reprove and generalize results of Artin-Mazur [5], Illusie [26], Ekedahl [18], Raynaud [35], and Oda [33].

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2. **Cohomology of \( \alpha_p, \mathbb{Z}/p\mathbb{Z}, \) and \( \mu_p \)**

Let \( f : X \to S \) be a proper morphism of schemes over a perfect ring \( k \) of positive characteristic \( p \). The purpose of this section is to elucidate (under various assumptions) the relationship between cohomology of coherent sheaves and the cohomology of \( \mathbb{Z}/p\mathbb{Z}, \alpha_p, \) and \( \mu_p \).

2.1. We begin with the group schemes \( \alpha_p \) and \( \mathbb{Z}/p\mathbb{Z} \), for which our arguments are quite simple. We use the short exact sequences

\[
0 \to \alpha_p \to \mathbb{G}_a^F \to \mathbb{G}_a \to 0
\]

\( \text{(2.1.1)} \)

\[
0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{1-F} \mathbb{G}_a \to 0.
\]

\( \text{(2.1.2)} \)

of group schemes on \( X \).

**Theorem 2.2.** Let \( f : X \to S \) be a proper flat morphism of schemes of characteristic \( p > 0 \). The complexes \( Rf_*\alpha_p \) and \( Rf_*\mathbb{Z}/p\mathbb{Z} \) are in \( D^f(S) \) (defined in 1.11) in fact, here we will show that these complexes lie in the smaller categories generated by perfect complexes and finite flat group schemes so the precise definition of \( D^f(S) \) is not necessary here).

**Proof.** We interpret the sequence \( \text{(2.1.1)} \) as defining a quasi-isomorphism between \( \alpha_p \) and the complex \( [\mathcal{O}_X \xrightarrow{F} \mathcal{O}_X] \). Since \( f \) is proper and flat, the complex \( Rf_*\mathcal{O}_X \) (pushforward for the
fppf-topology) is perfect, which gives the result for $\alpha_p$. Similar reasoning using (2.1.2) gives the result for $\mathbb{Z}/p\mathbb{Z}$.

2.3. There is a natural generalization of the sequences (2.1.1) and (2.1.2). Let $G$ be a finite flat commutative group scheme over $X$, and suppose that the Cartier dual $G^D$ has height 1. This is the case for instance if $G = \alpha_p$ or $G = \mathbb{Z}/p\mathbb{Z}$, but not if $G = \mu_p$. Let $e : X \to G^D$ be the zero section, and put $\omega_{G^D} = e^*\Omega^1_{G^D/X}$. Equivalently, $\omega_{G^D}$ is the dual of the Lie algebra of $G^D$. Write $V(\omega_{G^D})$ for the vector bundle associated to $\omega_{G^D}$, viewed as a group scheme on $X$. Artin and Milne construct [6, Proposition 1.1] a canonical short exact sequence

\begin{equation}
0 \to G \to V(\omega_{G^D}) \to V(\omega_{G^D})' \to 0
\end{equation}

of group schemes on $X$, where $V(\omega_{G^D})' := F^{-1}V(\omega_{G^D})$ denotes the Frobenius twist over $X$. Using this sequence, we obtain the conclusion of Theorem 2.2 more generally for any finite flat commutative group scheme $G$ on $X$ such that $G^D$ has height 1.

2.4. Next we consider the case $G = \mu_p$. We have the defining Kummer exact sequence

\begin{equation}
1 \to \mu_p \to G_m \to G_m \to 1
\end{equation}

which might be viewed as an analog for $\mu_p$ of (2.1.1) and (2.1.2). However, unlike in the latter two cases, $G_m$ is not coherent, and its cohomology will not in general be in $D^f(S)$. Accordingly, the result for $\mu_p$ is more subtle than for $\alpha_p$ and $\mathbb{Z}/p\mathbb{Z}$, and our proof of 1.3 in this case is more involved. We assume for the remainder of this section that $f$ is smooth. Instead of using (2.4.1), we will construct a short exact sequence of group schemes involving $\mu_p$ and two other group schemes $\nu_{X/S}$ and $\omega_{X/S}$, whose definition depends on the morphism $f : X \to S$. We then explain how to compute the flat cohomology of these group schemes in terms of sheaves of differentials in certain situations. We will use this in conjunction with some techniques from derived algebraic geometry to prove 1.3 for $\mu_p$ and proper smooth morphisms (note that in this case we prove 1.3 for proper, and not just projective, morphisms).

2.5. Let $X' = X \times_{S,F_S} S$ be the Frobenius twist of $X$ over $S$, so we have the standard diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/S}} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{F_S} & S,
\end{array}
\]

where $F_{X/S}$ denotes the relative Frobenius of $X/S$, $\pi \circ F_{X/S} = F_X$, and the square is cartesian.

2.6. Let $F_{X/S,*}G_{m,X}$ be the Weil restriction of $G_{m,X}$ along $F_{X/S}$. As $f$ is assumed to be smooth, the relative Frobenius $F_{X/S}$ is flat, and so $F_{X/S,*} : G_{m,X'} \to F_{X/S,*}G_{m,X}$ is injective. We define

\[
\nu_{X/S} := \text{coker}(F_{X/S,*} : G_{m,X'} \to F_{X/S,*}G_{m,X})
\]

to be the cokernel of the adjunction map. Thus, we have a short exact sequence

\[
1 \to G_{m,X'} \to F_{X/S,*}G_{m,X} \to \nu_{X/S} \to 1
\]

of group schemes on $X'$. We let

\[
\omega_{X/S} := F_{X/S,*}\mu_{p,X}
\]

be the Weil restriction of $\mu_{p,X}$ along $F_{X/S}$. 
Lemma 2.7. The group scheme $\nu_{X/S}$ is annihilated by $p$.

Proof. Observe that there exists a finite flat surjection $T \to X'$ such that the base change $q : \tilde{T} := X \times_{F_{X/S}} T \to T$ of $F_{X/S}$ admits a section $s : T \to \tilde{T}$, which is a closed immersion defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\tilde{T}}$ flat over $T$ satisfying $\mathcal{I}^p = 0$. Indeed, we can take $T = X$ with morphism to $X'$ given by $F_{X/S}$ and $s$ the diagonal morphism. From this it follows that the restriction to $T$ of the inclusion $G_{m,X'} \hookrightarrow F_{X/S} \ast G_{m,X}$ is split, and that $\nu_{X/S}|_T$ is the group scheme associated to $1 + \mathcal{I}$. Since $\mathcal{I}^p = 0$ this implies that $\nu_{X/S}|_T$, and therefore also $\nu_{X/S}$, is annihilated by $p$. □

2.8. It follows that there exists a unique surjection $\alpha_{X/S} : F_{X/S} \ast G_{m,X} \to G_{m,X'}$ as in the following commutative diagram of group schemes:

\[ 
\begin{array}{cccccc}
1 & \rightarrow & G_{m,X} & \xrightarrow{F_{X/S}^*} & F_{X/S} \ast G_{m,X} & \rightarrow & \nu_{X/S} & \rightarrow & 1 \\
\downarrow{p} & & \swarrow{\alpha_{X/S}} & \downarrow{p} & & \downarrow{0} & & \\
1 & \rightarrow & G_{m,X'} & \xrightarrow{F_{X/S}^*} & F_{X/S} \ast G_{m,X} & \rightarrow & \nu_{X/S} & \rightarrow & 1 
\end{array}
\] (2.8.1)

Specifically, let $(F_{X/S} \ast G_{m,X})^x \subset F_{X/S} \ast G_{m,X}$ be the image of the middle vertical arrow. The left vertical arrow is surjective, but the middle vertical arrow need not be. The map $F_{X/S}^*$ induces an isomorphism $G_{m,X'} \xrightarrow{\sim} (F_{X/S} \ast G_{m,X})^x$. The map $\alpha_{X/S}$ is the composition $\alpha_{X/S} : F_{X/S} \ast G_{m,X} \xrightarrow{p} (F_{X/S} \ast G_{m,X})^x \xrightarrow{(F_{X/S}^*)^{-1}} G_{m,X'}$. Note that $\alpha_{X/S}$ is surjective, and its kernel is $\omega_{X/S}$.

2.9. Taking kernels of the vertical arrows of (2.8.1), we obtain a short exact sequence

\[ 
1 \rightarrow \mu_{p,X'} \xrightarrow{F_{X/S}^*} \omega_{X/S} \rightarrow \nu_{X/S} \rightarrow 1 
\] (2.9.1)

Thus, $\nu_{X/S}$ may alternatively be defined as the cokernel of the adjunction map $\mu_{p,X'} \rightarrow F_{X/S} \ast \mu_{p,X}$. 

Lemma 2.10. We have $F_{X/S}^* \circ \alpha_{X/S} = [p]$ and $\alpha_{X/S} \circ F_{X/S}^* = [p]$, where we write $[p]$ for multiplication by $p$ on the relevant group schemes.

Proof. This is immediate from the commutativity of (2.8.1). □

2.11. As a consequence of Lemma 2.10, we have commutative diagrams

\[ 
\begin{array}{cccccc}
G_{m,X'} & \xrightarrow{F_{X/S}^*} & F_{X/S} \ast G_{m,X} & \xrightarrow{\alpha_{X/S}} & G_{m,X'} & \\
\downarrow{F_{X/S}} & & \downarrow{p} & & \downarrow{\alpha_{X/S}} & \\
F_{X/S} \ast G_{m,X} & \rightarrow & G_{m,X} & \rightarrow & G_{m,X'} & 
\end{array}
\] (2.11.1)
and

\[
\begin{align*}
F_{X/S}G_{m,X} & \quad F_{X/S}G_{m,X} \xrightarrow{\alpha_{X/S}} G_{m,X'} \\
\downarrow^{\alpha_{X/S}} & \quad \downarrow^{p} & \downarrow^{F_{X/S}} \\
G_{m,X'} & \xrightarrow{F_{X/S}^*} F_{X/S}G_{m,X} & F_{X/S}G_{m,X}
\end{align*}
\]

(2.11.2)

2.12. Our strategy for computing the cohomology of \(\mu_p\) will be to compare it to the cohomology of the group schemes \(\omega_{X/S}\) and \(\nu_{X/S}\). These group schemes are related by the short exact sequence

There is a “dual” version of this sequence which only exists in the derived category. To construct it, we observe that these group schemes are each canonically quasi-isomorphic to a certain complex of smooth group schemes. Specifically, we have quasi-isomorphisms

\[
\begin{align*}
\mu_{p,X'} & \simeq [G_{m,X'} \xrightarrow{p} G_{m,X'}] \\
RF_{X/S}\mu_{p,X} & \simeq [F_{X/S}G_{m,X} \xrightarrow{p} F_{X/S}G_{m,X}] \\
\omega_{X/S} & \simeq [F_{X/S}G_{m,X} \xrightarrow{\alpha_{X/S}} G_{m,X'}] \\
\nu_{X/S}[−1] & \simeq [G_{m,X'} \xrightarrow{F_{X/S}^*} F_{X/S}G_{m,X}]
\end{align*}
\]

where in each case, the complex on the right has its first term placed in degree 0.

In an abuse of notation, we may identify the two objects in (2.12.2), so that \(RF_{X/S}\mu_{p,X}\) denotes the right hand complex. Interpreting (2.11.1) and (2.11.2) as maps of the complexes which form their columns, we obtain distinguished triangles

\[
\begin{align*}
\nu_{X/S}[−1] & \to \mu_{p,X'} \to \omega_{X/S} +1 \\
\omega_{X/S} & \to RF_{X/S}\mu_{p,X} \to \nu_{X/S}[−1] +1
\end{align*}
\]

(2.12.5)

(2.12.6)

Note that (2.12.5) is simply the shift of (2.9.1). Furthermore, we have \(\omega_{X/S} = F_{X/S}\mu_{p,X}\) and \(\nu_{X/S} = R^1F_{X/S}\mu_{p,X}\), and (2.12.6) is the canonical sequence involving the cohomology sheaves of the two-term complex \(RF_{X/S}\mu_{p,X}\).

The sequences (2.9.1) and (2.12.6) fit into a commutative diagram

\[
\begin{array}{ccccccc}
1 & & & & & & 1 \\
\downarrow & & & & & & \downarrow \\
1 & \to & \mu_{p,X'} & \to & \omega_{X/S} & \to & \nu_{X/S} & \to & 1 \\
\downarrow & & & & & & \downarrow \\
1 & \to & RF_{X/S}\mu_{p} & \to & \nu_{X/S} \oplus \nu_{X/S}[−1] & \to & 1 \\
\downarrow & & & & & & \downarrow \\
\nu_{X/S}[−1] & \leftrightarrow & \nu_{X/S}[−1] & \leftrightarrow & 1 & & 1
\end{array}
\]

(2.12.7)
It follows that the triangles (2.12.5) and (2.12.6) fit into the diagram

\[
\begin{array}{ccc}
\nu_{X/S}[-1] & \rightarrow & \mu_{p,X} \\
\downarrow & & \downarrow \\
\nu_{X/S}[-1] & \leftarrow & RF_{X/S} \mu_{p,X} \\
\downarrow & & \downarrow \\
\nu_{X/S}[-1] & \rightarrow & \mu_{p,X} \\
\end{array}
\]

which may be continued indefinitely in the vertical directions. Moreover, either possible composition of the appropriate shift of the +1 maps $\omega_{X/S} \rightarrow \nu_{X/S}$ and $\nu_{X/S}[-1] \rightarrow \omega_{X/S}[1]$ is zero.

Taking cohomology of (2.12.5) and (2.12.6), we find distinguished triangles

(2.12.8) \[ Rf'_*\nu_{X/S}[-1] \rightarrow Rf'_*\mu_p \rightarrow Rf'_*\omega_{X/S} \xrightarrow{+1} \]

(2.12.9) \[ Rf'_*\omega_{X/S} \rightarrow Rf'_*\mu_p \rightarrow Rf'_*\nu_{X/S}[-1] \xrightarrow{+1} \]

2.13. The Frobenius and verschiebung. Consider the diagrams (2.11.1) and (2.11.2) as giving maps of the complexes which are their columns. We get maps

(2.13.1) \[ \alpha_{X/S} : RF_{X/S} \mu_{p,X} \rightarrow \mu_{p,X} \quad \text{and} \quad F_{X/S}^* : \mu_{p,X} \rightarrow RF_{X/S} \mu_{p,X} \]

where $F_{X/S}^*$ is given by composing the right square of (2.11.1) with the left square of (2.11.2) (equivalently, $F_{X/S}^*$ is the adjunction map in the derived category), and $\alpha_{X/S}$ is given by composing the right square of (2.11.2) with the left square of (2.11.1). We now consider the induced maps on cohomology. There is a canonical base change isomorphism

\[ Rf'_*\mu_{p,X} \simeq F_{S}^{-1}(Rf_*\mu_{p,X}) \]

which we will use freely. The maps $\alpha_{X/S}$ and $F_{X/S}^*$ (2.13.1) induce maps

\[ F : Rf_*\mu_{p,X} \rightarrow F_{S}^{-1}(Rf_*\mu_{p,X}) \quad \text{and} \quad V : F_{S}^{-1}(Rf_*\mu_{p,X}) \rightarrow Rf_*\mu_{p,X} \]

We also define maps $F$ and $V$ on $Rf'_*\omega_{X/S}$ and $Rf'_*\nu_{X/S}$ as follows. We define $F : Rf'_*\omega_{X/S} \rightarrow F_{S}^{-1}(Rf'_*\omega_{X/S})$ and $V : F_{S}^{-1}(Rf'_*\nu_{X/S}) \rightarrow Rf'_*\nu_{X/S}$ to be the zero maps, and define

\[ V : F_{S}^{-1}(Rf'_*\omega_{X/S}) \rightarrow F_{S}^{-1}(Rf_*\mu_{p,X}) \rightarrow Rf'_*\omega_{X/S} \]

\[ F : Rf'_*\nu_{X/S} \rightarrow F_{S}^{-1}(Rf_*\mu_{p,X})[1] \rightarrow F_{S}^{-1}(Rf'_*\nu_{X/S}) \]
to be the indicated compositions of the maps in the triangles (2.12.8) and (2.12.9). This situation can be visualized in the commutative diagram

\begin{equation}
Rf^!\omega_{X/S} \xrightarrow{0} F_S^{-1}(Rf^!\omega_{X/S}) \xrightarrow{V} Rf^!\omega_{X/S} \\
Rf^!\mu_p \xrightarrow{F} F_S^{-1}(Rf^!\mu_p) \xrightarrow{V} Rf^!\mu_p \\
Rf^!\nu_{X/S}[-1] \xrightarrow{F} F_S^{-1}(Rf^!\nu_{X/S}[-1]) \xrightarrow{0} Rf^!\nu_{X/S}[-1]
\end{equation}

where the diagonal arrows are those in the triangles (2.12.8) and (2.12.9). Here we have not yet defined the dashed arrows; they exist only when \( S \) is perfect.

**Remark 2.14.** Let \( S \) be a scheme of characteristic \( p \). For any flat commutative group scheme \( G \) over \( S \), there are canonically defined group homomorphisms \( F : G \to G^{(p/S)} \) and \( V : G^{(p/S)} \to G \), called the Frobenius and verschiebung [21 Exposé VIIA, 4.3], which satisfy \( FV = [p] \) and \( VF = [p] \). We will discuss the relationship between these maps and the maps \( F \) and \( V \) of [2.13] in section [10].

**Remark 2.15.** It is straightforward to generalize all of the preceding structures to \( \mu_{p^n} \) for \( n \geq 1 \). Namely, set \( \omega_{X/S,n} = F^n_{X/S,\mu_{p^n},X} \) and let \( \nu_{X/S,n} \) be the cokernel of the map \( \mu_{p^n,X}(p^n/S) \to \omega_{X/S,n} \). In particular, we have \( \nu_{X/S} = R^1F^n_{X/S,\mu_{p^n},X} \).

2.16. **Sheaves of differential forms.** We will relate the group schemes discussed in the previous section to certain sheaves of differential forms. Given a locally free sheaf \( \mathcal{E} \) on a scheme \( S \), we will write \( \mathbf{V}(\mathcal{E}) \) for the corresponding vector bundle, considered as a group scheme over \( S \).

2.17. We will use a certain exact sequence of group schemes, constructed by Hoobler [25]. We modify Hoobler’s notation slightly in the following. Consider a scheme \( Y \) and morphisms \( \varphi : Y \to Z \) and \( \psi : Z \to Y \) such that \( \varphi \circ \psi = F_Z \) and such that \( \varphi \) is a purely inseparable Galois (piG) cover, in the sense of [25 Definition on p. 184]. Hoobler’s result [25 Theorem 1.3] then produces an exact sequence

\begin{equation}
1 \to G_{m,Z} \xrightarrow{\varphi^*} \varphi_*G_{m,Y} \xrightarrow{\text{log}} \mathbf{V}(\varphi_*Z^1_{Y/Z}) \xrightarrow{\psi^* - C} \mathbf{V}(\psi^*\Omega^1_{Y/Z}) \to 0
\end{equation}

of group schemes on \( Z \). Here, \( Z^1_{Y/Z} \) is the kernel of the differential \( d_{Y/Z} : \Omega^1_{Y/Z} \to \Omega^2_{Y/Z} \). Note that, as \( d_{Y/Z} \) is \( \mathcal{O}_Z \)-linear, the sheaf \( \varphi_*Z^1_{Y/Z} \) has a natural \( \mathcal{O}_Z \)-module structure, and \( \mathbf{V}(\varphi_*Z^1_{Y/Z}) \) denotes the corresponding vector bundle. The map \( \psi^* \) is the map of group schemes induced by the \( p \)-linear map of coherent sheaves which is the composition of the inclusion \( \varphi_*Z^1_{Y/Z} \subset \varphi_*\Omega^1_{Y/Z} \) with the canonical (nonlinear) map \( \varphi_*\Omega^1_{Y/Z} \to \psi^*\Omega^1_{Y/Z} \) defined on sections by \( s \mapsto s \otimes 1 \). Finally, the map \( C \) is a certain generalized Cartier operator. In particular, \( C \) is \( \mathcal{O}_Z \)-linear, and \( \psi^* \) is \( p \)-linear, meaning that \( \psi^*(fs) = f^p\psi^*(s) \) for any local sections \( f \in \mathcal{O}_Z \) and \( s \in \varphi_*Z^1_{Y/Z} \).

2.18. If \( f : X \to S \) is a smooth morphism of schemes of characteristic \( p \), then \( F_{X/S} : X \to X' \) is a piG cover, and so the morphisms \( \varphi = F_{X/S} \) and \( \psi = \pi \) satisfy Hoobler’s assumptions.
Consider the exact sequence
\[ F_{X/S}^* \Omega^1_{X'/S} \xrightarrow{dF_{X/S}} \Omega^1_{X/S} \rightarrow \Omega^1_{X/X'} \rightarrow 0 \]
We have \( dF_{X/S} = 0 \), and hence \( F_{X/S}^* \Omega^1_{X/X'} = F_{X/S}^* Z^1_{X/S} \) and \( \pi^* \Omega^1_{X/X'} = \Omega^1_{X'/S} \). With these identifications, Hoobler’s sequence (2.17.1) becomes
\[
(2.18.1) \quad 1 \rightarrow G_m,X' \xrightarrow{F_{X/S}^*} F_{X/S} G_m,X \xrightarrow{\text{dlog}} V(F_{X/S} Z^1_{X/S}) \xrightarrow{\pi^* - C} V(\Omega^1_{X'/S}) \rightarrow 0
\]
Using the definition of \( \nu_{X/S} \), this becomes a short exact sequence
\[
(2.18.2) \quad 1 \rightarrow \nu_{X/S} \xrightarrow{\text{dlog}} V(F_{X/S} Z^1_{X/S}) \xrightarrow{\pi^* - C} V(\Omega^1_{X'/S}) \rightarrow 0
\]
of group schemes on \( X' \). As an immediate consequence we obtain the following.

**Lemma 2.19.** If \( f : X \rightarrow S \) is a proper smooth morphism, then \( Rf'_* \nu_{X/S} \in D^f(S) \).

**Proof.** This follows from the description of \( \nu_{X/S} \) given by the short exact sequence (2.18.2). \( \square \)

**Remark 2.20.** Note that the map
\[
\pi^* - C : F_{X/S}^* \Omega^1_{X'/S} \rightarrow \Omega^1_{X'/S}
\]
is not \( \mathcal{O}_{X'} \)-linear since the map \( \pi^* \) is not linear. It follows that \( Rf'_* \nu_{X/S} \) is not a complex of quasi-coherent sheaves, but nonetheless lies in \( D^f(S) \).

2.21. We will relate the group schemes \( \omega_{X/S} \) and \( \mu_p \) to sheaves of differential forms, analogous to (2.18.2) for \( \nu_{X/S} \). This relationship is however much less direct, involving only the associated étale sheaves. In addition, we will only directly describe this relation only under the assumption that \( S \) is smooth over \( k \). We will then pass to the general case using simplicial resolutions and animated rings to be discussed in following sections.

2.22. Suppose now that \( S \) is smooth over \( k \) and that \( f : X \rightarrow S \) is a smooth morphism. By [25, Proposition 1.2] the map \( F_S : S \rightarrow S \) is a \( \pi \text{G} \)-cover. It follows that the map
\[
\pi : X' \rightarrow X
\]
is also a \( \pi \text{G} \)-cover by [25, Proposition 1.1]. We therefore have three factorizations \((\varphi, \psi)\) of the absolute Frobenius such that \( \varphi \) is a \( \pi \text{G} \) cover, namely \((F_{X/S}, \pi), (F_X, \text{id})\), and \((\pi, F_{X/S})\). We consider the resulting three exact sequences specializing (2.17.1). In order to compare them, we apply \( \pi_* \) to the first. We obtain a commutative diagram
\[
(2.22.1)
\begin{array}{ccc}
1 & \rightarrow & G_m,X' \\
\downarrow & \uparrow & \downarrow \\
1 & \rightarrow & G_m,X \\
\downarrow & \uparrow \pi^* & \downarrow \\
1 & \rightarrow & \pi_* G_m,X' \\
\end{array}
\begin{array}{ccc}
& \xrightarrow{\varphi} & G_m,X' \\
& \xrightarrow{\text{dlog}} & V(f^* F_S Z^1_{S/k}) \\
& \xrightarrow{F_{X/S}^* - C} & V(f^* \Omega^1_{S/k}) \\
& \xrightarrow{F_{X/S}} & V(\Omega^1_{X'/S}) \\
1 & \rightarrow & 0
\end{array}
\begin{array}{ccc}
& \rightarrow & V(F_{X/S} Z^1_{X/k}) \\
& \rightarrow & V(\Omega^1_{X/k}) \\
& \rightarrow & 0
\end{array}
\begin{array}{ccc}
1 & \rightarrow & \pi_* G_m,X' \\
\downarrow & \uparrow \pi_* \pi^* & \downarrow \\
1 & \rightarrow & \pi_* G_m,X \\
\downarrow & \uparrow \pi^* \pi^* & \downarrow \\
1 & \rightarrow & \pi_* G_m,X' \\
\downarrow & \uparrow \pi^* \pi^* & \downarrow \\
1 & \rightarrow & \pi_* G_m,X
\end{array}
\begin{array}{ccc}
& \xrightarrow{\varphi} & G_m,X' \\
& \xrightarrow{\text{dlog}} & V(F_X Z^1_{X/k}) \\
& \xrightarrow{\pi^* - C} & V(\pi_* \Omega^1_{X'/S}) \\
& \xrightarrow{\pi^* \pi^*} & 0
\end{array}
\begin{array}{ccc}
& \rightarrow & V(F_X Z^1_{X/k}) \\
& \rightarrow & V(\pi_* \Omega^1_{X'/S}) \\
& \rightarrow & 0
\end{array}
\begin{array}{ccc}
& \rightarrow & 0
\end{array}
\]
of group schemes on $X$ with exact rows. Here, we have used that there are canonical identifications $\pi_* Z_{X'/X}^1 = f^* F_{S*} Z_{S/k}^1$ and $F_{X/S}^* \Omega_{X'/X}^1 = f^* \Omega_{S/k}^1$. To see this, note that in the exact sequence

$$F_{S*}^* \Omega_{S/k}^1 \xrightarrow{dF_{S*}} \Omega_{S/k}^1 \xrightarrow{\pi_*} \Omega_{F_{S*}}^1 \xrightarrow{0},$$

the map $dF_{S*}$ is zero. It follows that $\Omega_{S/k}^1 = \Omega_{S/F_{S*}S}^1$ and $Z_{S/k}^1 = Z_{S/F_{S*}S}^1$. We then combine these with the base change isomorphisms $f^* \Omega_{S/F_{S*}S}^1 = \Omega_{X'/X}^1$ and $f^* F_{S*} Z_{S/F_{S*}S}^1 = \pi_* Z_{X'/X}^1$.

### 2.23. The group scheme $\pi_* \nu_{X/S}$ is faithfully represented in the diagram (2.22.1) as the cokernel of the lower left horizontal map $F_{X/S*}$. This is not the case however for $\omega_{X/S}$ or $\mu_{\mu_p}$. We will relate these group schemes to the diagram (2.22.1) only after restricting to the étale site.

#### Remark 2.24. As described below, the diagram (2.22.1) is closely related to the distinguished triangle (2.12.6). The analog of (2.22.1) for the factorization $F_{X'} = F_{X/S} \circ \pi$ is similarly related to the triangle (2.12.5).

### 2.25. Restriction to the étale site.

#### 2.26. We consider the (derived) restrictions of the group schemes $\mu_p$, $\nu_{X/S}$, and $\omega_{X/S}$ to the étale site. For a scheme $T$ let

$$\varepsilon_T : T_{\text{fppf}} \to T_{\text{ét}}$$

be the projection from the big fppf site to the small étale site of $T$. In order to keep track of the various topologies involved, let us write $\mathcal{G}_{m,T} = \varepsilon_T^* \mathcal{G}_{m,T}$ for the sheaf on the small étale site of $T$ obtained by restriction from $\mathcal{G}_{m,T}$. Since the cohomology of $\mathcal{G}_{m,T}$ can be computed in the étale topology we also have

$$\mathcal{G}_{m,T} \simeq R\varepsilon_T^* \mathcal{G}_{m,T}.$$

#### 2.27. Let $f : X \to S$ be a smooth morphism of schemes of characteristic $p$. We may compute flat cohomology by first pushing forward to the étale site: if $G$ is any group scheme on $X$, then the derived pushforward $Rf_* G$ (with respect to the fppf topology) is the same as the fppf sheafification (in the sense of $\infty$-categories) of the functor

$$(T \to S) \mapsto R\Gamma(X_{T,\text{ét}}, R\varepsilon_{X_T F*} G).$$

At the level of cohomology, the functor $R^i f_* G$ is the fppf sheafification in the usual (1-categorical) sense of the functor

$$(T \to S) \mapsto H^i(X_{T,\text{ét}}, R\varepsilon_{X_T F*} G)$$

#### 2.28. The restriction of the map $F_{X}^* : \mathcal{G}_{m,X} \to F_{X*} \mathcal{G}_{m,X}$ to the small étale site may be described as follows. If $U \to X$ is an étale morphism, the relative Frobenius $F_{U/X}$ in the diagram

$$U \xrightarrow{F_{U/X}^*} U(p/X) \xrightarrow{\pi_{U/X}} U \xrightarrow{\pi_{U/X}} X$$

is an isomorphism. Pullback along $F_{U/X}$ defines an isomorphism $F_{X*} \mathcal{G}_{m,X} \xrightarrow{\sim} \mathcal{G}_{m,X}$, and the composition

$$\mathcal{G}_{m,X} \xrightarrow{F_{X*}^*} F_{X*} \mathcal{G}_{m,X} \xrightarrow{\sim} \mathcal{G}_{m,X}$$
is multiplication by \( p \). Furthermore, the restriction of \( \alpha_{X/S} \) to the small étale site is isomorphic to the adjunction map

\[ \pi^* : \pi^{-1}\mathcal{G}_{m,X} \to \mathcal{G}_{m,X'} \]

Using the fact that the maps \( \pi_* \) and \( \pi^{-1} \) are mutually inverse equivalences, we obtain quasiisomorphisms

\begin{align*}
(2.28.1) & \quad R\epsilon_{X',\mathbf{m},X'} \simeq [\pi^{-1}\mathcal{G}_{m,X} \xrightarrow{F^*_{X'}} \mathcal{G}_{m,X'}] \\
(2.28.2) & \quad F_{X/S} \cdot R\epsilon_{X',\mathbf{m},X} \simeq [\pi^{-1}\mathcal{G}_{m,X} \xrightarrow{F^*_{X/S}} \mathcal{G}_{m,X}] \\
(2.28.3) & \quad R\epsilon_{X',\omega_{X/S}} \simeq [\pi^{-1}\mathcal{G}_{m,X} \xrightarrow{\pi^*} \mathcal{G}_{m,X'}] \\
(2.28.4) & \quad R\epsilon_{X',\nu_{X/S}} \simeq [\mathcal{G}_{m,X} \xrightarrow{F_{X/S}} \mathcal{G}_{m,X}] 
\end{align*}

where in each case, the complex on the right has its first term placed in degree 0. Because \( \nu_{X/S} \) is a smooth group scheme, we have \( R\epsilon_{X',\nu_{X/S}} = \epsilon_{X',\nu_{X/S}} = (\nu_{X/S})|_{X'} \). The group schemes \( \mathbf{m} \) and \( \omega_{X/S} \) are nonsmooth in general, and accordingly they have nontrivial \( R^1\epsilon_* \).

This can be seen directly: the maps on the right hand side of (2.28.1), (2.28.2), and (2.28.3) are typically not surjective.

2.29. Restricting the distinguished triangle (2.12.6) to the étale site yields

\[ R\epsilon_{X',\omega_{X/S}} \to F_{X/S} \cdot R\epsilon_{X',\mathbf{m},X} \to R\epsilon_{X',\nu_{X/S}}[-1] \xrightarrow{+1} \]

Suppose now that \( S \) is smooth over \( k \). Consider the restriction of the diagram (2.22.1) to the small étale site. We apply the equivalence \( \pi^{-1} \) and combine with (2.28.2), (2.28.3), and (2.28.4) to obtain quasiisomorphisms

\begin{align*}
(2.29.1) & \quad R\epsilon_{X',\omega_{X/S}}[1] \xrightarrow{\sim} \pi^{-1} [f^*F_{S*}Z^1_{S/k} \xrightarrow{F^*_{X/S-C}} f^*\Omega^1_{S/k}] \\
& \quad \downarrow \quad \downarrow \\
& \quad F_{X/S} \cdot R\epsilon_{X',\mathbf{m},X}[1] \xrightarrow{\sim} \pi^{-1} [F_{X*}Z^1_{X/k} \xrightarrow{L-C} \Omega^1_{X/k}] \\
& \quad \downarrow \quad \downarrow \\
& \quad R\epsilon_{X',\nu_{X/S}} \xrightarrow{\sim} [F_{X/S*}Z^1_{X/S} \xrightarrow{\pi^*-C} \Omega^1_{X'/S}] \\
& \quad \downarrow^{+1} \quad \downarrow^{+1} 
\end{align*}

Here, we have used the fact that if \( V \) is a smooth group scheme on \( X' \) then \( \pi^{-1}\epsilon_{X*}(\pi_*V) = \pi^{-1}\pi_*V \simeq \epsilon_{X*}V \).

2.30. We now consider cohomology. Put \( \mathcal{E} = [f^*F_{S*}Z^1_{S/k} \xrightarrow{F^*_{X/S-C}} f^*\Omega^1_{S/k}] \). Applying \( Rf^{\text{ét}}_* \) to the top row of (2.29.1), we have isomorphisms

\begin{align*}
(2.30.1) & \quad (Rf^{\text{ét}}_*\omega_{X/S})|_{S_{\text{ét}}}[1] \simeq F^{-1}_{S}Rf^{\text{ét}}_*\mathcal{E} \simeq Rf^{\text{ét}}_*\mathcal{E}
\end{align*}
where the second is due to the fact that $F_S^{-1}$ defines an equivalence of étale topoi, and where we have written

$$(Rf'_*\omega_{X/S})|_{S_{\text{ét}}} := R\epsilon_{S*}Rf'_*\omega_{X/S} = Rf'_{\text{ét}*}R\epsilon_X^*\omega_{X/S}$$

for the derived restriction to the étale site. We obtain a distinguished triangle

$$(2.30.2) (Rf'_*\omega_{X/S})|_{S_{\text{ét}}} \xrightarrow{\tau} (Rf_*\mathcal{O}_X) \otimes_{\mathcal{O}_S} F_{S*}Z_{S/k}^1 \xrightarrow{\lambda} (Rf_*\mathcal{O}_X) \otimes_{\mathcal{O}_S} \mathcal{O}_{S/k}^1 +1.$$  

of complexes of sheaves on the small étale site of $S$.

**2.31.** The map $\tau$ in (2.30.2) can be described as follows. It is obtained by taking the difference of the two maps

$$\text{id} \otimes C, \rho \otimes \text{id} : (Rf_*\mathcal{O}_X) \otimes_{\mathcal{O}_S} F_{S*}Z_{S/k}^1 \to (Rf_*\mathcal{O}_X) \otimes_{\mathcal{O}_S} \mathcal{O}_{S/k}^1,$$

where

$$\rho : Rf_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$$

is the map on cohomology induced by the $p$th power map $\mathcal{O}_X \to \mathcal{O}_X$.

**Corollary 2.32.** If $S$ is a perfect scheme and $X \to S$ is a smooth morphism, then the maps in the triangles (2.12.8) and (2.12.9) induce isomorphisms

$$R\Gamma(X_{\text{fppf}}, \mu_p) \simeq R\Gamma(X'_{\text{ét}}, \nu_{X/S}[-1]).$$

$$R\Gamma(X'_{\text{ét}}, \nu_{X/S}[-1]) \simeq R\Gamma(X'_{\text{fppf}}, \mu_p).$$

**Proof.** The assertion is étale local on $S$, and therefore it suffices to consider the case when $S$ is affine and equal to the perfect ground ring $k$ in the preceding analysis. In this case the result follows from the description of $\text{Coker}(\alpha_{X/S})$ provided by (2.30.1) and the observation that $Z_{F_S}^1 = 0$. \hfill $\square$

**2.33. Height 1 group schemes and height 1 morphisms.**

**2.34.** In this section we will generalize some of the preceding structures by replacing $\mu_p$ with a finite height 1 group scheme. We first describe some general results concerning the interaction of a height 1 group scheme with a height 1 morphism of schemes.

**2.35.** Let $G$ be a finite flat height 1 commutative group scheme on $S$. The restricted Lie algebra of $G$ is a pair $(\mathcal{V}, \rho)$, where $\mathcal{V}$ is a vector bundle on $S$ and $\rho : F_S^*\mathcal{V} \to \mathcal{V}$ is a map of vector bundles. Moreover, $G$ is determined by $(\mathcal{V}, \rho)$: by [21] Exposé VIIA, Remarque 7.5 the functor sending $G$ to $(\mathcal{V}, \rho)$ is an equivalence of categories between finite flat commutative height 1 group schemes on $S$ and pairs $(\mathcal{V}, \rho)$, where $\mathcal{V}$ is a locally free sheaf on $S$ and $\rho : F_S^*\mathcal{V} \to \mathcal{V}$ is a morphism.

**2.36.** Let $\phi : Y \to Z$ and $\psi : Z \to Y$ be as in [2.17] Let $G = G_Z$ be a flat commutative height 1 group scheme on $Z$. Put $G_Y = \psi^{-1}G_Z$. We define

$$\omega_\phi^G := \phi_*G_Y$$

and let

$$\nu_\phi^G := \text{coker}(G_Z \to \phi_*G_Y)$$

be the cokernel of the adjunction map. Thus, we have a short exact sequence

$$(2.36.1) 1 \to G_Z \to \omega_\phi^G \to \nu_\phi^G \to 1$$
Remark 2.37. As a Weil restriction, the group scheme $\omega^G_\varphi$ can be described explicitly. In particular, one shows that $\omega^G_\varphi$ has relative dimension $\text{rk}(\mathcal{V}_G) \cdot \deg \varphi$ over $Z$, where $\mathcal{V}_G$ is the Lie algebra of $G$. Furthermore, the group scheme $\nu^G_\varphi$ is smooth.

Proposition 2.38. The functors $G \mapsto \nu^G_\varphi$, $G \mapsto \omega^G_\varphi$, and $G \mapsto R^1\varphi_*G_Y$ preserve exactness of short exact sequences of finite flat height 1 group schemes on $Z$, and $R^i\varphi_*G_Y = 0$ for $i > 1$.

Proof. The left exactness of $\omega^-_\varphi$ is immediate from the definition. Furthermore, the statement that $R^i\varphi_*G = 0$ for $i > 1$ and $G$ a finite flat group scheme on $Y$ follows, for example, from [8, 3.1]. This also implies the right exactness of $G \mapsto R^1\varphi_*G_Y$.

Let

$$1 \to K \to G \to H \to 1$$

be a short exact sequence of finite flat height 1 group schemes on $Z$. We then get a commutative diagram

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & K & G & H & 0 \\
0 & \omega^K_\varphi & \omega^G_\varphi & \omega^H_\varphi \\
0 & \nu^K_\varphi & \nu^G_\varphi & \nu^H_\varphi \\
0 & 0 & 0 & 0.
\end{array}
$$

Chasing through this diagram we deduce that $G \mapsto \nu^G_\varphi$ is also left exact. Furthermore, this diagram shows that to complete the proof of the proposition it suffices to show that $\nu^-_\varphi$ is also exact on the right.

For this consider the induced sequence

$$1 \to \mathcal{V}_K \to \mathcal{V}_G \to \mathcal{V}_H$$

of Lie algebras. We claim that $\mathcal{V}_G \to \mathcal{V}_H$ is surjective. By Nakayama’s lemma to verify this we may base change to a field-valued point of $Z$, and so it suffices to consider when $Z$ is the spectrum of a field. Furthermore, in this case it suffices to show that the dual map

$$(2.38.1) \quad \mathcal{V}_H^\vee \to \mathcal{V}_G^\vee$$

is injective. Let $\mathcal{O}_H$ (resp. $\mathcal{O}_G$) be the coordinate ring of $H$ (resp. $G$) and let $\mathcal{J}_H \subset \mathcal{O}_H$ (resp. $\mathcal{J}_G \subset \mathcal{O}_G$) be the ideal of the identity. Then the map $(2.38.1)$ is identified with the map

$$\mathcal{J}_H/\mathcal{J}_H^2 \to \mathcal{J}_G/\mathcal{J}_G^2$$
induced by pullback. Let \( x_1, \ldots, x_r \in \mathcal{I}_H \) be elements mapping to a basis for \( \mathcal{I}_H/\mathcal{I}_H^2 \) and let \( y_1, \ldots, y_t \in \mathcal{O}_G \) be elements that map to a basis for \( \mathcal{I}_G/\mathcal{I}_G^2 \). Then by [21, Théorème 7.4] we have isomorphisms 
\[
\mathcal{O}_H \simeq k[x_1, \ldots, x_r]/(x_1^p, \ldots, x_r^p), \quad \mathcal{O}_G \simeq k[y_1, \ldots, y_t]/(y_1^p, \ldots, y_t^p).
\]
Now observe that an element \( L \in \mathcal{I}_G \) has nonzero linear term in the \( y_j \) if and only if \( L^{p-1} \neq 0 \).

Since \( \mathcal{O}_H \to \mathcal{O}_G \) is an inclusion of rings, it follows that the image of every nonzero linear form in the \( x_i \) maps to a nonzero element in \( \mathcal{I}_G/\mathcal{I}_G^2 \) so the map \( (2.38.1) \) is injective.

We now complete the proof of the proposition by noting that the sequences \( (2.47.1) \), together with the surjectivity of \( \mathcal{V}_G \to \mathcal{V}_H \), imply the surjectivity of \( \nu^G \to \nu^H \). □

2.39. Let \( G = G_Z \) be a finite flat height 1 group scheme on \( Z \), and let \( G' \) denote \( F^{-1}_Z G \) be the Frobenius twist of \( G \) over \( Z \). There is a map
\[
(2.39.1) \quad \nu^G \to R^1(\mathcal{O}_G) \]
defined as follows. Let
\[
0 \to G \to R \to \overline{R} \to 0
\]
be the Bégueri resolution of \( G \) [7, 2.2.1], and let \( R' \) (resp. \( \overline{R} \)) be the Frobenius twist of \( R \) (resp. \( \overline{R} \)). Since the group schemes \( R \) and \( \overline{R} \) are smooth we get an isomorphism
\[
R^1(\mathcal{O}_G) \simeq \text{Coker}(\varphi_* R_y \to \varphi_* \overline{R}y),
\]
and a commutative diagram
\[
\begin{array}{ccc}
0 & \to & G \\
0 & \to & G'
\end{array}
\begin{array}{ccc}
\to & R & \to \overline{R} \\
\to & R' & \to \overline{R}
\end{array}
\begin{array}{ccc}
0 & \to & 0 \\
0 & \to & 0
\end{array}
\]
where the left vertical morphism is 0 since \( G \) has height 1. By the snake lemma we get an induced exact sequence involving the kernels of Frobenius
\[
0 \to G \to R[F] \to \overline{R}[F] \to G' \to 0.
\]
Let \( K \) denote the kernel of \( \overline{R}[F] \to G' \) so we have an exact sequence
\[
0 \to K \to R[F] \to G' \to 0
\]
of height 1 group schemes. By 2.38 we then also get exact sequences
\[
0 \to \varphi_* K_y \to \varphi_* R[F]_y \to \varphi_* G'_y \to 0
\]
and
\[
0 \to \varphi_* G_y \to \varphi_* R[F]_y \to \varphi_* K_y \to 0.
\]
Now define \( (2.39.1) \) to be the morphism induced by the dotted arrow filling in the diagram
\[
\begin{array}{ccc}
0 & \to & \varphi_* G \\
0 & \to & \varphi_* G_y
\end{array}
\begin{array}{ccc}
\to & \varphi_* R & \to \varphi_* \overline{R} \\
\to & \varphi_* R[F]_y & \to \varphi_* \overline{R}[F]_y
\end{array}
\begin{array}{ccc}
\to & R^1(\mathcal{O}_G) & \to 0 \\
\to & \varphi_* G'_y & \to 0
\end{array}
\]
Note here that the image of $G'$ in $R^1\varphi_*G_Y$ is zero since any section of $G'$ over $Z$ lifts locally on $Z$ to a section of $R$.

**Proposition 2.40.** The map (2.39.1) is an isomorphism.

*Proof.* Note that since the Bégueri resolution is functorial in $G$, the preceding discussion defines a morphism of functors from the category of height one abelian finite flat group schemes over $Z$ to abelian sheaves on $Z$. Moreover, these functors are exact by (2.38).

We claim that it suffices to show that the proposition holds for group schemes of the form $A[F]$, the kernel of Frobenius on a smooth group scheme over $Z$. Indeed, by the above for any $G$ we can find an exact sequence of height 1 group schemes

$$1 \to G \to A[F] \to B[F] \to G' \to 1,$$

and from such a diagram we get a commutative diagram with exact rows

$$1 \to \nu_\varphi G' \to \nu_\varphi A'[F] \to \nu_\varphi B'[F] \to \nu_\varphi G'' \to 1 \to 1$$

If we show that the two center vertical arrows are isomorphisms then a diagram chase implies that the outside morphisms are isomorphisms as well.

So suppose that there is a smooth group scheme $A$ on $Z$ such that $G$ is equal to the kernel $A[F]$ of the relative Frobenius $F_{A/Z} : A \to A'$, where $A' = F_Z^{-1}A$ is the Frobenius twist of $A$ over $Z$. We consider the commutative diagram

$$1 \to A \to \varphi_*A_Y \to Q_A \to 1$$

with exact rows. Here, the maps $\eta_A$ and $\eta_{A'}$ are the unit maps for the adjunction $1 \to \varphi_*\varphi^{-1}$ applied to $A$ and $A'$, and we have used the isomorphism $\varphi_*\varphi^{-1}F_{A/Z} = \varphi_*F_{A'/Y}$. As explained in the proof of [23] Theorem 3.2 (see the diagram at the bottom of page 195), the map $\zeta$ is zero. Hence, we have $\nu_\varphi G \cong Q_A$ and $Q_{A'} \cong R^1\varphi_*G_Y$, and we obtain an isomorphism

$$\nu_\varphi G' \cong Q_{A'} \cong R^1\varphi_*G_Y$$

A diagram chase shows that this agrees with our earlier defined map. \qed

2.41. **Projection to the étale topos.**

2.42. Let $S$ be a smooth $k$-scheme and $G$ a finite flat height 1 abelian group scheme over $S$. Since $S$ is smooth the Frobenius morphism $F_S : S \to S$ is a piG cover and we can apply the previous discussion. Let $G'$ denote the pullback $F_S^{-1}G$. Write $\omega_S^G = \omega_{F_S}^G$ and $\nu_S^G = \nu_{F_S}^G$, so that

$$\omega_S^G := F_{S*}G', \quad \nu_S^G := \text{Coker}(G \to F_{S*}G').$$

Let

$$\epsilon_S : S_{\text{fppf}} \to S_{\text{ét}}$$
Lemma 2.43. We have $R\epsilon_S^*\omega_S^G = 0$ and $R^i\epsilon_S^*\nu_S^G = 0$ for $i > 0$.

Proof. We have $R^0\epsilon_S^*\omega_S^G = 0$ since $S$ is reduced. Furthermore, by [8, 3.1] we have $R^i\epsilon_S^*G = 0$ for $i > 1$, and since $\nu_S^G$ is a smooth group scheme we have $R^i\epsilon_S^*\nu_S^G = 0$ for $i > 0$ as claimed. It also follows that the only possibly nonzero cohomology group of $R\epsilon_S^*\omega_S^G$ is $R^1\epsilon_S^*\omega_S^G$ and that the map

$$R^1\epsilon_S^*G \to R^1\epsilon_S^*\omega_S^G$$

is surjective. But this last map is zero by [6, 3.1], and the lemma follows. □

Corollary 2.44. The map

$$\epsilon_S^*\nu_S^G \to R^1\epsilon_S^*G,$$

arising as the boundary map from taking cohomology of the short exact sequence

$$0 \to G \to \omega_S^G \to \nu_S^G \to 0,$$

is an isomorphism, and defines an isomorphism

$$\epsilon_S^*\nu_S^G[-1] \simeq R\epsilon_S^*G.$$

□

2.45. Some distinguished triangles.

2.46. Returning to the general setup of 2.17, let $G = G_Z$ be a finite flat height 1 group scheme on $Z$, and as before write $G_Y = \varphi^*G_Z$ and $G' = F_Z^{-1}G$. The complex $R\varphi_*G_Y$ has cohomology only in degrees zero and one. There is therefore a canonical distinguished triangle

$$\varphi_*G_Y \to R\varphi_*G_Y \to R^1\varphi_*G_Y[-1] \xrightarrow{+1}$$

We apply to this the isomorphism of Proposition 2.40. We also consider the triangle obtained by shifting (2.36.1). This results in distinguished triangles

(2.46.1)

$$\nu_G^G[-1] \to G_Z \to \omega_G^G \xrightarrow{+1}$$

(2.46.2)

$$\omega_G^G \to R\varphi_*G_Y \to \nu_G^{G'}[-1] \xrightarrow{+1}$$

2.47. We now consider sheaves of differentials. As before, we consider morphisms $\varphi : Y \to Z$ and $\psi : Z \to Y$ as in 2.17. Let $G = G_Z$ be a finite flat commutative height 1 group scheme on $Z$, and put $G_Y = \varphi^*G_Z$. Let $(\mathcal{V}, \rho)$ be the restricted Lie algebra of $G_Z$. Hoobler shows [25, Theorem 3.2] that there is an exact sequence

(2.47.1)

$$0 \to G_Z \xrightarrow{\varphi^*} \varphi_*G_Y \xrightarrow{\text{dlog}} \mathcal{V} \otimes \mathcal{V}(\varphi_*Z_Y^1) \xrightarrow{\rho \otimes \psi^* - C} \mathcal{V} \otimes \mathcal{V}(\psi^*\Omega_Y^1) \to 0$$

of group schemes on $Z$. Here, $C$ denotes the map $\text{id}_{\mathcal{V}} \otimes C$.

Remark 2.48. In fact, Hoobler more generally produces the exact sequence (2.47.1) for any flat commutative group scheme $G$ over $Z$ such that $G[F]$ is flat and the Lie algebra $\mathcal{V}$ of $G$ is locally free. Taking $G = \mathbf{G}_{m,Z}$, the above sequence (2.47.1) specializes to (2.17.1).
2.49. We now assume that \( \psi \) (as well as \( \varphi \)) is a \( \pi \text{G} \) cover. It follows that \( F_Z = \varphi \circ \psi \) is a \( \pi \text{G} \) cover.

We consider Hoobler’s sequence (2.47.1) applied to \( \varphi, F_Z, \) and \( \psi \). Reasoning as in Proposition 2.38, we deduce that the latter remains exact after applying \( \pi_\ast \). We obtain a diagram (2.49.1)

\[
1 \longrightarrow G_Z \xrightarrow{\varphi^*} \varphi_\ast G_Y \xrightarrow{\text{dlog}} \vartheta \otimes V(\varphi_\ast Z^1_{Y/Z}) \xrightarrow{\rho \otimes \text{dlog} - C} \vartheta \otimes V(\Omega^1_Y) \longrightarrow 0
\]

\[
1 \longrightarrow G_Z \xrightarrow{F_Z} F_Z \ast G'_Z \xrightarrow{\text{dlog}} \vartheta \otimes V(F_Z \ast Z^1_{Z/k}) \xrightarrow{\rho \otimes I - C} \vartheta \otimes V(\Omega^1_{Z/k}) \longrightarrow 0
\]

\[
1 \longrightarrow \varphi_\ast G_Y \xrightarrow{\psi^*} F_Z \ast G'_Z \xrightarrow{\text{dlog}} \vartheta \otimes V(F_Z \ast Z^1_{Z/Y}) \xrightarrow{\rho \otimes \text{dlog} - C} \vartheta \otimes V(\varphi_\ast Z^1_{Z/Y}) \longrightarrow 0
\]

of group schemes on \( Z \) with exact rows.

2.50. **Cohomology of height 1 group schemes over smooth bases.**

2.51. We now generalize the theory developed earlier for \( \mu_p \) to an arbitrary finite flat height 1 abelian group scheme. We first consider the absolute setting. Let \( S \) be a smooth \( k \)-scheme and let \( G \) be a group scheme on \( S \) as above with corresponding restricted Lie algebra \((\vartheta, \rho)\).

Applying (2.47.1) with \( \varphi = F_S \) and \( \psi = \text{id}_S \), we obtain an exact sequence (2.51.1)

\[
1 \to G \xrightarrow{F_S^\ast} F_S \ast G' \to \vartheta \otimes V(F_S \ast Z^1_{S/k}) \xrightarrow{\rho \otimes I - C} \vartheta \otimes V(\Omega^1_{S/k}) \to 0
\]

where \( G' = F_S^{-1}G \) is the Frobenius twist of \( G \) over \( S \). We obtain the following consequence on the étale site.

**Proposition 2.52.** Let \( S \) be a smooth scheme over \( k \). Let \( G \) be a finite flat commutative group scheme on \( S \) of height 1 with restricted Lie algebra \((\vartheta, \rho)\). There is a canonical isomorphism

\[
R\epsilon_S \ast G[1] \simeq [\vartheta \otimes F_S \ast Z^1_{S/k} \xrightarrow{\rho \otimes I - C} \vartheta \otimes \Omega^1_{S/k}]
\]

of complexes of sheaves on the small étale site of \( S \).

**Proof.** By 2.44 we have

\[
R\epsilon_S \ast G[1] \simeq \epsilon_S \ast \nu_S^G,
\]

which, combined with (2.51.1), yields the result. \( \square \)

**Remark 2.53.** In theorem 4.8 below we describe a generalization of this result to arbitrary base schemes \( S \).

2.54. We now consider the relative setting. Let \( S \) be a \( k \)-scheme (not necessarily smooth) and let \( f : X \to S \) be a smooth morphism. Let \( G = G_X \) be a finite flat commutative height 1 group scheme on \( X' \). Set \( G_X = F_{X/S}^{-1}G_X \). We define

\[
\omega^G_{X/S} := F_{X/S} \ast G_X \quad \text{and} \quad \nu^G_{X/S} := \text{coker}(G_X \to F_{X/S} \ast G_X)
\]

Note that if \( G = \mu_p \), then \( \omega^G_{X/S} \) and \( \nu^G_{X/S} \) agree with the group schemes \( \omega_{X/S} \) and \( \nu_{X/S} \) defined previously (see (2.9.1)). Specializing the distinguished triangles (2.46.1) and (2.46.2) to the
case when \( \varphi = F_{X/S} \), we obtain distinguished triangles

\[
\nu^G_{X/S}[-1] \to G_{X'} \to \omega^G_{X/S} \xrightarrow{+1} \\
\omega^G_{X/S} \to RF_{X/S} \to \nu^G_{X/S}[-1] \xrightarrow{+1}
\]

These generalize (2.12.5) and (2.12.6) for \( G = \mu_p \).

If moreover \( S \) is a smooth \( k \)-scheme, then specializing (2.49.1), we find an analog of (2.22.1).

Note that there are two different possible specializations of (2.49.1), corresponding to the factorizations \( F_{X} = \pi \circ F_{X/S} \) and \( F_{X'} = F_{X/S} \circ \pi \). As in 2.30, we obtain a distinguished triangle

\[
(Rf_*\omega^G_{X/S})|_{S_{et}} \xrightarrow{\tau} (Rf_*\mathcal{V}) \otimes_{O_S}^L \mathcal{F}_{S/k}^1 \xrightarrow{\sigma} (Rf_*\mathcal{V}) \otimes_{O_S}^L \Omega^1_{S/k} \xrightarrow{+1}.
\]

of complexes of sheaves on the small étale site of \( S \). The map \( \tau \) can be described as in 2.31 it is the difference of the maps

\[
id \otimes C, \quad \varpi \otimes \text{id} : (Rf_*\mathcal{V}) \otimes_{O_S}^L F_{S/k}^1 \to (Rf_*\mathcal{V}) \otimes_{O_S}^L \Omega^1_{S/k},
\]

where

\[
\varpi : Rf_*\mathcal{V} \to Rf_*\mathcal{V}
\]

is the map on cohomology induced by the composition of the \( p \)-linear map \( \mathcal{V} \to F_X^*\mathcal{V} \) defined by \( v \mapsto v \otimes 1 \) and the map \( \rho : F_X^*\mathcal{V} \to \mathcal{V} \).

3. Animated rings and their cohomology

To proceed further, we will use various techniques from derived algebraic geometry. In this section we review some of the basic notions following [14, §5].

3.1. Animated rings and sheaves.

3.2. For a ring \( R \) we denote by \( \text{Alg}_R \) the category of \( R \)-algebras. As in [14] we can then consider the associated animated category \( \text{Alg}_R^{\text{ani}} \). This is the \( \infty \)-category obtained from the category of simplicial objects in \( \text{Alg}_R \) by inverting weak equivalences [14, 5.1.6 (3)].

For a morphism of rings \( i : R \to R' \) we have functors

\[
i^* : \text{Alg}_R \to \text{Alg}_{R'}, \quad A \mapsto A \otimes_R R'
\]

and

\[
i_* : \text{Alg}_{R'} \to \text{Alg}_R, \quad (R' \to A) \mapsto (R' \to R \to A).
\]

Here the notation reflects the geometric viewpoint of the morphism \( \text{Spec}(R') \to \text{Spec}(R) \) defined by \( i \). As discussed in [14] these functors have animated versions [14, 5.1.7]

\[
i^*_{\text{ani}} : \text{Alg}_{R}^{\text{ani}} \to \text{Alg}_{R'}^{\text{ani}}, \quad A \mapsto A \otimes_{R}^L R'
\]

and

\[
i_*^{\text{ani}} : \text{Alg}_{R'}^{\text{ani}} \to \text{Alg}_R^{\text{ani}}, \quad (R' \to A) \mapsto (R' \to R \to A).
\]
3.3. For an $\infty$-category $\mathcal{C}$ we can consider (see [30 5.1.0.1]) the category $\mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R)$ of presheaves on $\text{Alg}^{\text{ani}}_R$ taking values in $\mathcal{C}$. This is, by definition, the $\infty$-category of functors (note here that since we are considering rings, rather than affine schemes, we consider covariant functors)

$$\text{Alg}^{\text{ani}}_R \rightarrow \mathcal{C}.$$ 

In the case when $\mathcal{C} = \mathcal{D}(\text{Ab})$, the derived $\infty$-category of abelian groups, we will write simply $\mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R)$ if no confusion seems likely to arise.

If $\mathcal{C}$ is a stable $\infty$-category, such as $\mathcal{D}(\text{Ab})$, then $\mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R)$ is again a stable $\infty$-category by [31 1.1.3.1]. In what follows, we will always be considering sheaves taking values in a stable $\infty$-category.

If $\text{Alg}^{\text{ani}}_R$ is endowed with a Grothendieck topology (as in [14 §5.2.1] we will primarily be interested in the étale or fppf topology) we can then consider the corresponding category of sheaves $\text{Shv}(\text{Alg}^{\text{ani},\text{op}}_R)$. If we wish to emphasize the Grothendieck topology ? we write $\text{Shv}(\text{Alg}^{\text{ani},\text{op}}_R)$ (e.g. $\text{Shv}^\text{ét}(\text{Alg}^{\text{ani},\text{op}}_R)$ or $\text{Shv}^\text{fppf}(\text{Alg}^{\text{ani},\text{op}}_R)$). As before, if we are considering $\mathcal{C} = \mathcal{D}(\text{Ab})$ then we omit the subscript $\mathcal{C}$.

It follows from [30 6.2.2.7] that the inclusion

$$\text{Shv}(\text{Alg}^{\text{ani},\text{op}}_R) \hookrightarrow \mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R)$$

has a left adjoint

$$L : \mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R) \rightarrow \text{Shv}(\text{Alg}^{\text{ani},\text{op}}_R),$$

called sheafification.

For an object $\mathcal{F} \in \mathcal{P}(\text{Alg}^{\text{ani},\text{op}}_R)$ its cohomology (with respect to the Grothendieck topology) is defined by

$$R\Gamma(R, \mathcal{F}) := L(\mathcal{F})(R).$$

More generally for any object $A \in \text{Alg}^{\text{ani},\text{op}}_R$ we can define the cohomology of $\mathcal{F}$ over $A$ by the formula

$$R\Gamma(A, \mathcal{F}) := L(\mathcal{F})(A).$$

3.4. An abelian flat affine group scheme $G/R$ defines a functor

$$G^{\text{ani}} : \text{Alg}^{\text{ani},\text{op}}_R \rightarrow \mathcal{D}(\text{Ab}).$$

This is discussed in [14 5.1.11]. By definition, $G^{\text{ani}}$ is obtained by viewing the coordinate ring of $G$ as a $\mathbb{Z}$-module object in the opposite category of animated rings. The associated fppf-sheafification is the functor

$$H_{G^{\text{ani}}} : \text{Alg}^{\text{ani},\text{op}}_R \rightarrow \mathcal{D}(\text{Ab}), \ A \mapsto R\Gamma(A, G^{\text{ani}}).$$

It follows from the observation that for 0-truncated rings one does not get any new covers in the animated setup (see [14 5.2.1]) that the associated notions of animated cohomology agree with the classical ones.

Similarly, for a complex of $R$-modules $\mathcal{F}^\bullet$ we can consider the associated animation $\mathcal{F}^{\text{ani}}$, taking values in the derived $\infty$-category of $R$-modules, and associated cohomology. For an $R$-algebra $A$ the value of $\mathcal{F}^{\text{ani}}$ on $A$ is given by the derived tensor product

$$\mathcal{F}^\bullet \otimes^L_R A.$$
3.5. We will also consider cohomology of sheaves over non-affine schemes. For a scheme $X$ let $I_X$ be the category of affine $X$-schemes, with $i \in I_X$ corresponding to Spec($R_i$) $\to$ $X$. We then define (limit in the sense of $\infty$-categories)

$$\text{Shv}^{\text{ani}}(X) := \lim_{i \in I_X} \text{Shv}(\text{Alg}^{\text{ani},\text{op}}_{R_i}).$$

If $X = \text{Spec}(R)$ is affine then the category $I_X$ has an initial object and we have

$$\text{Shv}^{\text{ani}}(X) \simeq \text{Shv}(\text{Alg}^{\text{ani},\text{op}}_R).$$

Similarly we define

$$\mathcal{G}^{\text{ani}}(X) := \lim_{i \in I_X} \mathcal{G}(\text{Alg}^{\text{ani},\text{op}}_{R_i}).$$

By the same reasoning as in the previous paragraph, finite flat group schemes $G/X$ as well as complexes of quasi-coherent sheaves $F^\bullet$ on $X$ define objects of $\mathcal{G}^{\text{ani}}(X)$, which we again denote by $G^{\text{ani}}$ and $F^{\bullet \text{ani}}$.

3.6. For a morphism of schemes

$$f : X \to Y$$

we have adjoint functors

$$f^{\text{ani}*} : \text{Shv}^{\text{ani}}(Y) \to \text{Shv}^{\text{ani}}(X)$$

and

$$f^*_\text{ani} : \text{Shv}^{\text{ani}}(X) \to \text{Shv}^{\text{ani}}(Y)$$

defined as follows.

The functor $f^{\text{ani}*}$ is induced by restriction. The functor $f^*_\text{ani}$ is constructed as follows.

Let $\mathcal{F} \in \mathcal{G}^{\text{ani}}(X)$ be an object. We then define $f^*_\text{ani}\mathcal{F} \in \mathcal{G}^{\text{ani}}(Y)$ as follows.

For an affine $Y$-scheme Spec($R$) $\to$ $Y$ there is an induced functor

$$I_{X_R} \times \text{Alg}^{\text{ani}}_R \to \mathcal{G}(\text{Ab}), (B,A) \mapsto \mathcal{F}(B \otimes^L_R A),$$

which upon taking limit over $I_{X_R}$ defines

$$f^*_\text{ani}\mathcal{F}(R,A) := \lim_{B \in I_{X_R}} \mathcal{F}(B \otimes^L_R A).$$

Varying $R$ and $A$ we obtain a functor

$$f^*_\text{ani} : \mathcal{G}^{\text{ani}}(X) \to \mathcal{G}^{\text{ani}}(Y).$$

Restricting this functor to $\text{Shv}^{\text{ani}}(X)$ we get the functor $f^*_\text{ani}$ for sheaves.

3.7. Continuous sheaves.

3.8. As discussed in [14, 5.1.4] for a ring $R$ the category $\text{Alg}^{\text{ani}}_R$ can be characterized using a universal property. Let $\text{Alg}^{\text{sfp}}_R \subset \text{Alg}_R$ be the full subcategory of those $R$-algebras $A$ which are strongly of finite presentation [14, 5.1.1]. Recall that this means that the functor

$$\text{Hom}_R(A,-) : \text{Alg}_R \to \text{Set}$$

commutes with 1-sifted colimits. There is a functor

$$\text{Alg}^{\text{sfp}}_R \to \text{Alg}^{\text{ani}}_R,$$
and as discussed in [14, 5.1.4] this functor is universal from \( \text{Alg}^{\text{ani}}_R \) to \( \infty \)-categories admitting sifted colimits, in the sense that for any such \( \infty \)-category \( \mathcal{D} \) the restriction functor

\[
\text{Fun}_{\text{sifted}}(\text{Alg}^{\text{ani}}_R, \mathcal{D}) \rightarrow \text{Fun}(\text{Alg}^{\text{fpa}}_R, \mathcal{D})
\]

is an equivalence. Here the subscript “sifted” on the left denotes functors that commute with sifted colimits.

**Definition 3.9.** A functor \( \mathcal{F} : \text{Alg}^{\text{ani}}_R \rightarrow \mathcal{D} \) is **continuous** if it commutes with sifted colimits.

**Remark 3.10.** By [30, 5.5.8.15 (2)] a functor \( \mathcal{F} \) is continuous if and only if it commutes with filtered colimits and geometric realizations.

**Example 3.11.** For a perfect complex of \( R \)-modules \( \mathcal{F}^\bullet \) the functor

\[
\mathcal{F}^\bullet : \text{Alg}^{\text{ani}}_R \rightarrow \mathcal{D}(\text{Ab})
\]

is continuous. Indeed by filtering the complex \( \mathcal{F}^\bullet \) one is reduced to considering the case of a projective \( R \)-module, where the result is immediate as the forgetful functor from \( \text{Alg}^{\text{ani}}_R \) to simplicial abelian groups commutes with sifted colimits (see for example [14, 5.1.3]).

**Remark 3.12.** It is immediate that for \( G = G_a \) the functor \( H_{G^{\text{ani}}} \) is continuous. From this and the short exact sequence

\[
0 \rightarrow \alpha_p \rightarrow G_a \xrightarrow{r \rightarrow z^p} G_a \rightarrow 0
\]

it follows that \( \alpha_p \) is also continuous. It is a result of Bhatt and Lurie [9] that the same is true for \( \mu_p \) (or more generally for \( \mathbb{Z}_p(i) \)). This had also been observed previously by Peter Scholze (unpublished). In fact, as we explain in section 4 below, if \( G \) is any finite flat group scheme of height 1 then \( H_{G^{\text{ani}}} \) is continuous.

**3.13.** We denote by

\[
\text{Shv}^{\text{cts}}(\text{Alg}^{\text{ani, op}}_R) \subset \text{Shv}(\text{Alg}^{\text{ani, op}}_R), \quad \mathcal{P}^{\text{cts}}(\text{Alg}^{\text{ani, op}}_R) \subset \mathcal{P}(\text{Alg}^{\text{ani, op}}_R)
\]

the subcategories of continuous objects.

**3.14.** Let \( R \) be a noetherian ring and

\[
f : X \rightarrow \text{Spec}(R)
\]

a finite type separated flat morphism. Let \( G \) be a finite flat abelian group scheme over \( X \).

**Lemma 3.15.** If \( H_{G^{\text{ani}}} \) is continuous, then the object \( Rf^{\text{ani}}_* H_{G^{\text{ani}}} \in \text{Shv}(\text{Alg}^{\text{ani, op}}_R) \) is continuous.

**Proof.** In the case when \( X \) is affine this is immediate.

For general \( X \) as in the proposition, choose a finite cover \( \{U_i\} \) of \( X \) by affine Zariski open subsets. Since \( X \) is separated we then have a finite filtration on both source and target with graded pieces given by cohomology of affines. The general statement therefore follows from the affine case.
4. Finite flat group schemes and animated rings

Let $k$ be a perfect field.

4.1. For a finite flat group scheme $G/k$ for which $H_{G_{\text{ani}}}$ is continuous and a finite type $k$-algebra $A$ we can understand the cohomology $R\Gamma(A,G)$ as follows. Choose a simplicial resolution $P_* \rightarrow A$ with each $P_n$ a finitely generated polynomial ring over $k$. Then the natural map

$$\text{hocolim}_n R\Gamma(P_n, G) \rightarrow R\Gamma(A,G)$$

is an equivalence since $H_{G_{\text{ani}}}$ is continuous. The advantage of this perspective is that $R\Gamma(P_n, G)$ can in some cases be described using differentials as in 2.52.

**Example 4.2.** For $G = \alpha_p$ we see, using the short exact sequence

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{x \mapsto x^p} \mathbb{G}_a \rightarrow 0,$$

that

$$R\Gamma(P_n, \alpha_p) \simeq R\Gamma(S_n, \mathcal{O}_{S_n} \rightarrow F_S^* \mathcal{O}_{S_n}),$$

where we write $S_n$ for $\text{Spec}(P_n)$. On the other hand we have an exact sequence

$$0 \rightarrow \mathcal{O}_{S_n} \rightarrow F_S^* \mathcal{O}_{S_n} \xrightarrow{d} F_S^* Z_{S_n/k}^1 \xrightarrow{C} \Omega_{S_n/k}^1 \rightarrow 0.$$

Note that ordinarily the Cartier operator is viewed as taking values in the differentials of the Frobenius twist of $S_n$, but since $F_k$ is an isomorphism we can view $C$ as having target $\Omega_{S_n/k}^1$. Thus we find that

$$R\Gamma(P_n, \alpha_p) \simeq R\Gamma(P_n, F_S^* Z_{S_n/k}^1 \xrightarrow{C} \Omega_{S_n/k}^1)[-1],$$

and therefore

$$R\Gamma(A, \alpha_p) \simeq \text{hocolim}_n R\Gamma(P_n, F_S^* Z_{S_n/k}^1 \xrightarrow{C} \Omega_{S_n/k}^1)[-1].$$

**Example 4.3.** Similarly we can describe the cohomology of $\mu_p$ as in 2.17 In this case we have the exact sequence

$$0 \rightarrow \mathcal{O}_{S_n}^* \rightarrow F_S^* \mathcal{O}_{S_n}^* \xrightarrow{\text{dlog}} F_S^* Z_{S_n/k}^1 \xrightarrow{I-C} \Omega_{S_n/k}^1 \rightarrow 0,$$

which gives a description (assuming the result that $H_{\mu_p}$ is continuous, which will be shown below)

$$R\Gamma(A, \mu_p) \simeq \text{hocolim}_n R\Gamma(P_n, F_S^* Z_{S_n/k}^1 \xrightarrow{I-C} \Omega_{S_n/k}^1 )[-1].$$

4.4. There is a common generalization of these two examples. Let $S$ be a smooth connected $k$-scheme and let $(\mathcal{V}, \rho)$ be a pair with $\mathcal{V}$ a perfect complex on $S$ and

$$\rho : F_S^* \mathcal{V} \rightarrow \mathcal{V}$$

a morphism of perfect complexes on $S$. Then we can define an object $\mathcal{G}_{(\mathcal{V}, \rho)}$ in the derived category of sheaves on the fppf site of $S$ as follows. This object is defined as the complex associated to an object $\mathcal{G}_{(\mathcal{V}, \rho)}$ of $\text{Shv}_{\text{ani}}(S)$. To define this object it suffices to consider the
case when $S = \text{Spec}(R)$ is affine. In this case we define $\mathcal{G}((\mathcal{V}, \rho))$ by associating to a polynomial algebra $P$ over $R$ the complex

$$V \otimes_R F_{P*}Z^1_{P/k} \xrightarrow{\rho \otimes 1 - \text{id} \otimes C} V \otimes_R \Omega^1_{P/k}.$$

We then extend this to all animated rings using (3.8.1).

Note that by the construction it follows that for any $S$-algebra $A$ we have a distinguished triangle

$$\mathcal{G}((\mathcal{V}, \rho))(A) \longrightarrow V \otimes F_{A*}LZ^1_A \xrightarrow{\rho - C} V \otimes L_A \longrightarrow \mathcal{G}((\mathcal{V}, \rho))(A)[1],$$

where $L_A$ denotes the cotangent complex of $A$ over $k$ and $LZ^1_A$ denotes the value of the Kan extension of the functor $P \mapsto Z^1_{P/k}$. Note that we have a distinguished triangle

$$(A \to F_{A*}A) \longrightarrow F_{A*}LZ^1_A \xrightarrow{C} L_A \longrightarrow (A \to F_{A*}A)[1].$$

From this and [10, 3.1] it follows that $F_{A*}LZ^1_A$ and $\mathcal{G}((\mathcal{V}, \rho))(A)$ are fpqc-sheaves.

4.5. Comparison with cohomology of groups.

4.6. Let $S$ be a smooth $k$-scheme. Let $\mathcal{V}$ be a vector bundle over $S$ and let $\rho : F^*_{S*}\mathcal{V} \to \mathcal{V}$ be a map of vector bundles. Let $G_p(\mathcal{V}, \rho)$ be the corresponding finite flat commutative height 1 group scheme on $S$. By Proposition 2.52, we have an identification

$$R^1\epsilon_S G_p(\mathcal{V}, \rho)[1] \simeq [V \otimes F_{S*}Z^1_{S/k}] \xrightarrow{\rho - C} \mathcal{V} \otimes \Omega^1_{S/k}].$$

By Kan extension, we obtain a morphism

$$\mathcal{G}((\mathcal{V}, \rho)) \longrightarrow H_{G_p(\mathcal{V}, \rho)_{\text{an}}}[1].$$

Remark 4.7. Note that for this we have to lift the isomorphism in Proposition 2.52 from the derived category to its $\infty$-categorical enhancement. This can be done by noting that the truncation map

$$R^1\epsilon_S G_p(\mathcal{V}, \rho)[1] \to R^1\epsilon_S G_p(\mathcal{V}, \rho)$$

is an isomorphism and therefore (4.6.1) is given by an isomorphism of sheaves

$$R^1\epsilon_S G_p(\mathcal{V}, \rho) \simeq \mathcal{H}^0(\mathcal{V} \otimes F_{S*}Z^1_{S/k}) \xrightarrow{\rho - C} \mathcal{V} \otimes \Omega^1_{S/k}).$$

Theorem 4.8. The map (4.6.2) is an equivalence.

The proof occupies the remainder of the section.

Proposition 4.9. Let $A \to S$ be a surjective map of $k$-algebras such that all elements of the kernel are nilpotent and (4.6.2) is an equivalence over $S$, and let $P_* \to A$ be a simplicial resolution by smooth $k$-algebras. Then the natural map

$$\text{hocolim}_n H_{G_p(\mathcal{V}, \rho)}(P_n) \to H_{G_p(\mathcal{V}, \rho)}(A)$$

is an equivalence.
Proof. This argument is essentially due to Bhatt and Lurie, who considered the case of $\mu_p$ (see also [9] for a different argument in that case).

Note first that
\[ R\Gamma(S,G_p(\mathcal{V},\rho)) \simeq \mathcal{G}(\mathcal{V},\rho)(S)[-1] \]
is in $\mathcal{D}^{\leq 2}(\text{Ab})$ by the description in terms of the cotangent complex. Furthermore, by [14, 5.2.10] the cocone of the reduction map
\[ R\Gamma(A,G_p(\mathcal{V},\rho)) \to R\Gamma(S,G_p(\mathcal{V},\rho)) \]
is in $\mathcal{D}^{\leq 1}(\text{Ab})$.

For a $k$-algebra $P$ let $F(P)$ denote the 2-category of $G_p(\mathcal{V},\rho)$-gerbes over $P$. Then we claim that the natural map (4.9.2)
\[ \text{hocolim}_{P \to A} F(P) \to F(A) \]
is an equivalence, where the colimit is taken over smooth $k$-algebras $P$ with a map to $A$. Note that since $H_{G_p(\mathcal{V},\rho)}(A)$ and $H_{G_p(\mathcal{V},\rho)}(P)$ are in $\mathcal{D}^{\leq 2}(\text{Ab})$ showing that (4.9.2) is an equivalence will imply that (4.9.1) is also an equivalence and prove the proposition.

So let $\mathcal{R}$ be a $G_p(\mathcal{V},\rho)$-gerbe over $A$ and let $\mathcal{C}_\mathcal{R}$ be the 2-category of data $(P \to A, \mathcal{R}_P, \sigma)$, where $P$ is a smooth $k$-algebra, $\mathcal{R}_P$ is a $G_p(\mathcal{V},\rho)$-gerbe over $P$, and $\sigma : \mathcal{R}_P \otimes_P A \to \mathcal{R}$ is an equivalence over $A$. We show that $\mathcal{C}_\mathcal{R}$ is contractible.

First note that $\mathcal{C}_\mathcal{R}$ is non-empty. Indeed since (4.6.2) is an equivalence over $S$ we can find a map $P \to S$ with $P$ smooth over $k$ and a gerbe $\mathcal{R}_P$ over $P$ mapping to the reduction of $\mathcal{R}$. Since $P$ is smooth we can lift this to a map $P \to A$ sending $\mathcal{R}_P$ to a gerbe isomorphic to $\mathcal{R}$, since the reduction map is injective on $H^2$.

Lemma 4.10. Let $P$ be a smooth $k$-algebra and let $\mathcal{X}$ be a $G_p(\mathcal{V},\rho)$-gerbe over $P$. Then any $k$-morphism
\[ \alpha : \text{Spec}(A) \to \mathcal{X} \]

admits a factorization
\[ \text{Spec}(R) \xrightarrow{\pi} \text{Spec}(A) \xrightarrow{\alpha} \mathcal{X}, \]

where $\pi$ is smooth.

Proof. Since
\[ R\Gamma(P,G_p(\mathcal{V},\rho))[1] \simeq \mathcal{G}(\mathcal{V},\rho)(P), \]
the class of $\mathcal{X}$ in $H^2(P,G_p(\mathcal{V},\rho))$ can also be viewed as a class in $H^1(\mathcal{G}(\mathcal{V},\rho)(P))$. Now the image of this class in $H^1(\mathcal{G}(\mathcal{V},\rho)(S))$ is zero, and therefore by the continuity property for $\mathcal{G}(\mathcal{V},\rho)$ there exists a smooth algebra $P$-algebra $P'$ and a factorization of the given map $\text{Spec}(A) \to P$ through $P'$ such that the restriction of $\mathcal{X}$ to $P'$ is trivial. Replacing $P$ by $P'$ we are then
reduced to the case when \( X \) is trivial. In this case the result amounts to the statement that for any \( G_p(\mathcal{V}, \rho) \)-torsor \( T \) over \( A \) there exists a smooth \( P \)-algebra \( R \) and a factorization

\[
\begin{array}{ccc}
\text{Spec}(R) & \longrightarrow & \text{Spec}(P) \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & \text{Spec}(P),
\end{array}
\]

such that the \( G_p(\mathcal{V}, \rho) \)-torsor \( T \) lifts to \( R \) and the resulting map to \( BG_p(\mathcal{V}, \rho) \) is smooth. By the preceding argument, after replacing \( P \) by a smooth \( P \)-algebra we can assume that the reduction \( T_S \) admits a lifting \( T_P \) to \( P \). Replacing \( T \) by the difference of \( T \) and the pullback of \( T_P \) to \( A \) we are then reduced to the case when \( T_S \) is trivial. Consider the Bégneri resolution

\[
0 \to G_p(\mathcal{V}, \rho) \to R \to \overline{R} \to 0.
\]

The \( R \)-torsor \( U \) obtained from \( T \) by pushout is a deformation of the trivial \( R \)-torsor over \( S \). Since \( R \) is smooth this implies that \( U \) is trivial. Consideration of the long exact sequence

\[
0 \to G_p(\mathcal{V}, \rho)(A) \to R(A) \to \overline{R}(A) \to H^1(A, G_p(\mathcal{V}, \rho)) \to H^1(A, R),
\]

then shows that \( T \) is isomorphic to the torsor of liftings to \( R \) of a point \( \beta \in \overline{R}(A) \). If \( R \) denotes the coordinate ring of \( \overline{R}_P \) and \( T_R \) the \( G_p(\mathcal{V}, \rho) \)-torsor of liftings to \( R \) of the tautological element \( B \in \overline{R}(R) \) then \( R \) is smooth over \( BG_p(\mathcal{V}, \rho) \). Indeed we have a cartesian diagram

\[
\begin{array}{ccc}
\overline{R}_P & \longrightarrow & \overline{R}_P \\
\downarrow & & \downarrow u \\
\text{Spec}(P) & \longrightarrow & BG_p(\mathcal{V}, \rho)_P,
\end{array}
\]

where the bottom horizontal morphism is given by the trivial torsor over \( P \). We then obtain the proposition by noting that \( T \) is isomorphic to the pullback of \( T_R \) along the map \( \text{Spec}(A) \to \text{Spec}(R) \) characterized by the fact that the pullback of \( B \) is \( \beta \). □

Consider now two objects of \( \mathcal{C} \)

\[
(P_i \to A, \mathcal{R}_P, \sigma_i), \quad i = 1, 2.
\]

To complete the proof of the proposition it suffices to show that the 2-category of diagrams

\[
\begin{array}{ccc}
(P_1 \to A, \mathcal{R}_P, \sigma_1) & \longrightarrow & (P \to A, \mathcal{R}_P, \sigma) \\
\downarrow & & \downarrow \\
(P_2 \to A, \mathcal{R}_P, \sigma_2)
\end{array}
\]

in \( \mathcal{C} \) is contractible. Equivalently, let \( C \) denote \( P_1 \otimes_A P_2 \) and let

\[
\mathcal{X} \to \text{Spec}(C)
\]
denote $G_p(\mathcal{V}, \rho)$-gerbe given by the difference of the pullbacks of the $\mathcal{R}_P$, and let $\mathcal{C}$ be the 2-category of diagrams (4.10.1), with $R$ smooth over $k$. To complete the proof of the proposition it suffices to show that $\mathcal{C}$ is contractible.

By 4.10 the category $\mathcal{C}$ is nonempty. Fix one diagram (4.10.1), with $R$ smooth over $\mathfrak{X}$ (and not just smooth over $k$). For any object

$$
\begin{array}{ccc}
\text{Spec}(S) & \rightarrow & \mathfrak{X} \\
\downarrow & & \\
\text{Spec}(A) & \rightarrow & \text{Spec}(S)
\end{array}
$$

defining an object of $\mathcal{C}$ we get a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(A) & \rightarrow & \text{Spec}(R) \times_{\mathfrak{X}} \text{Spec}(S) \rightarrow \text{Spec}(S) \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \rightarrow & \mathfrak{X}.
\end{array}
$$

Now observe that since the diagonal of $\mathfrak{X}$ is affine the scheme

$$
\text{Spec}(R) \times_{\mathfrak{X}} \text{Spec}(S)
$$

is affine and the projection to $\text{Spec}(S)$ is smooth. We therefore get a canonical path in $\mathcal{C}$ from any object to our fixed object, showing that $\mathcal{C}$ is contractible. This completes the proof of 4.9. \hfill \square

4.11. To complete the proof that (4.6.2) is an isomorphism in general proceed as follows.

First note that $H_{G_p(\mathcal{V}, \rho)}$ is a sheaf for the fpqc topology as is $\mathcal{G}(\mathcal{V}, \rho)$, as noted in 4.4. It follows that if $A$ is an $S$-algebra then to verify that

$$
\mathcal{G}(\mathcal{V}, \rho)(A) \rightarrow R^1(A, G_p(\mathcal{V}, \rho)) \rightarrow 0
$$

is an isomorphism we may replace $A$ by an fpqc cover. This reduces the proof to the case when $A$ is semiperfect (that is, when Frobenius is surjective).

In this case the map

$$
A \rightarrow A_{\text{perf}}
$$

from $A$ to the perfection is surjective with kernel a nil ideal.

Since $A_{\text{perf}}$ is perfect, it follows from [6, 2.4] that the map (4.6.2) is an isomorphism over $A_{\text{perf}}$. Combining this with 4.9 we get that (4.6.2) is also an equivalence over $A$. This completes the proof of 4.8. \hfill \square

4.12. Representability results. We proceed with the notation in 4.4.

Lemma 4.13. There exists a dense open subset $U \subset S$ such that all the cohomology sheaves $\mathcal{H}^i(\mathcal{G}(\mathcal{V}, \rho))$ are representable by finite flat local group schemes.

Proof. By descent theory it suffices to find a dominant morphism $S' \rightarrow S$ of integral schemes such that the restriction of $\mathcal{H}^i(\mathcal{G}(\mathcal{V}, \rho))$ to $S'$ is representable. Fix an algebraic closure $k(S) \hookrightarrow \Omega$ of the function field of $S$. Then over $\Omega$ there is a finite filtration on $(\mathcal{V}, \rho)$ whose successive quotients are isomorphic, up to a shift, to pullbacks from $k$ of objects $(\mathcal{V}, \rho)$ over $k$. By
a standard limit argument, we can then find a dominant morphism $S' \to S$ such that this filtration extends to a filtration over $S'$. Using this the proof is reduced to the case of the restriction of an object $(V, \rho')$ over $k$.

To handle this case, note that given a short exact sequence

$$0 \to (V', \rho') \to (V, \rho) \to (V'', \rho'') \to 0$$

of objects over $k$ we get an induced distinguished triangle

$$\mathcal{G}_{(V', \rho')} \to \mathcal{G}_{(V, \rho)} \to \mathcal{G}_{(V'', \rho'')} \to \mathcal{G}_{(V', \rho')}[1].$$

From this we deduce that it suffices to consider the case when $(V, \rho)$ is a simple object in the category of such pairs. This reduces the proof to the case of $(k, \text{id})$ and $(k, 0)$, where the sheaves are given by $\mu_p$ and $\alpha_p$. □

4.14. Representability results for $\mu_p$. Combining the above discussion with that of section 2 we can now prove the following:

**Theorem 4.15.** Let $f : X \to S$ be a proper smooth morphism of schemes over $k$, with $S$ integral. Then there exists a dense open subset $U \subset S$ such that $Rf_*\mu_p|_U \in Df(U)$ and such that for all $i$ the sheaves $R^i f_*\omega_{X/S}$ are representable.

**Proof.** It suffices to consider the case when $S$ is affine. We consider the distinguished triangle (2.12.9)

$$Rf'_*\omega_{X/S} \to Rf_*\mu_p \to Rf'_*\nu_{X/S}[-1] \to Rf'_*\omega_{X/S}[1]$$

By 2.19 we have that over a dense open in $S$ the sheaves $R^i f'_*\nu_{X/S}$ are representable. It therefore suffices to show that after possibly further shrinking the cohomology sheaves $R^i f'_*\omega_{X/S}$ are representable, which follows from the continuity of $H_{\mu_p, \text{ani}}$ (which implies the continuity of $Rf'_*\omega_{X/S}$, the description in 2.31 and 4.13). □

**Corollary 4.16.** Let $S$ be a smooth $k$-scheme and let $f : X \to S$ be a smooth proper morphism. Assume that for some $i$ the function $s \mapsto H^i(X_s, \mathcal{O}_{X_s})$ on $k$-points of $S$ is locally constant. Let $\mathcal{V}$ be the vector bundle associated to $R^i f_*\mathcal{O}_X$ and let $\rho : F^s_\mathcal{V} \to \mathcal{V}$ be the map induced by the absolute Frobenius of $X$. The flat cohomology group $R^i f'_*\omega_{X/S}$ is representable by the finite flat height 1 group scheme $G_\rho(\mathcal{V}, \rho)$ on $S$ associated to $(\mathcal{V}, \rho)$.

**Proof.** This follows from 2.30 and the proof of 4.15. □

Using the more general distinguished triangles (2.54.1) and (2.54.2), we obtain the statement of Theorem 4.15 also for any finite flat abelian $G$ on $X$ which has height 1. The statement of Corollary 4.16 also extends similarly.

4.17. Projective bundle formula.

4.18. Let $X$ be a $k$-scheme and let $\mathcal{E}$ be a locally free sheaf on $X$ of finite rank with associated projective bundle

$$\pi : P := \text{Proj} \mathcal{E} \to X.$$

Let $(\mathcal{V}, \rho)$ be a perfect complex with semilinear endomorphism as in 4.4 defining $\mathcal{G}_{(\mathcal{V}, \rho)}$ on $X$ and $\mathcal{G}_{(\pi^*\mathcal{V}, \pi^*\rho)}$ on $P$. 
Theorem 4.19. The first chern class of the universal line bundle on $P$ induces an isomorphism

$$R\Gamma(P, \mathcal{G}_{(\pi^*\mathcal{T}, \pi^*\rho)}) \simeq R\Gamma(X, \mathcal{G}_{(\mathcal{V}, \rho)}(\rho-\text{id}) \mathcal{V})[-1].$$

Remark 4.20. Using 4.8 this also gives an analogous decomposition for the cohomology of a finite flat abelian group scheme of height 1.

Example 4.21. (i) For $\alpha_p$ the above implies that the pullback map

$$R\Gamma(X, \alpha_p) \to R\Gamma(P, \alpha_p)$$

is an isomorphism, since in this case $\rho = 0$ and $\rho - \text{id} : \mathcal{V} \to \mathcal{V}$ is an isomorphism. This can also be seen directly from the corresponding result for $G_a$.

(ii) For $G = \mu_p$ we get from the Artin-Schreier sequence \[1, IX, 3.5\] a distinguished triangle

$$R\Gamma(X, \mu_p) \to R\Gamma(P, \mu_p) \to R\Gamma(X, \mathbb{Z}/(p)[-2]) \to R\Gamma(X, \mu_p)[1],$$

from which 4.19 can be obtained directly.

The proof of 4.19 occupies the remainder of this subsection.

4.22. Consider first the case when $X$ is smooth over $k$. Recall that there is a map

$$d\log : \mathcal{O}_P^* \to Z_P^1, \ u \mapsto du/u,$$

which induces a map

$$c_1 : \text{Pic}(P) \to H^1(P, Z_P^1).$$

This map is compatible with the Cartier operator (note here we use the fact that $P$ is perfect to identity $P$ and its Frobenius twist $P \otimes_{k, F_k} F_k$)

$$C : F_{P, k} Z_P^1 \to \Omega_P^1,$$

in the sense that induced diagram

\[
\begin{array}{cccc}
\text{Pic}(P) & \xrightarrow{\gamma} & H^1(P, F_{P, k} \mathcal{O}_P^*) & \xrightarrow{F_{P, k} c_1} & H^1(P, F_{P, k} Z_P^1) & \xrightarrow{C} & H^1(P, \Omega_P^1) \\
\end{array}
\]

commutes. This follows directly from the definition of the Cartier operator.

Lemma 4.23. Let $\mathcal{O}_P(1)$ denote the universal line bundle on $P$. Then the map

$$c_1(\mathcal{O}_P(1)) : \mathcal{O}_X[-1] \to R\pi_* F_{P, k} \mathcal{O}_P(1)$$

induced by the first chern class of $\mathcal{O}_P(1)$ induces an isomorphism

$$F_{X, k} Z_X^1 \oplus \mathcal{O}_X[-1] \to R\pi_* F_{P, k} \mathcal{O}_P(1).$$

Proof. First note that $R\pi_* \mathcal{O}_P \simeq \mathcal{O}_X$ and that $c_1(\mathcal{O}_P(1))$ induces an isomorphism $\Omega_X^1 \oplus \mathcal{O}_X[-1] \simeq R\pi_* \Omega_P^1$. This last statement follows from consideration of the short exact sequence

$$0 \to \pi^* \Omega_X^1 \to \Omega_P^1 \to \Omega_{P/X}^1 \to 0$$

and the fact that $R\pi_* \Omega_{P/X}^1 \simeq \mathcal{O}_X[-1]$, with isomorphism defined by $c_1(\mathcal{O}_P(1))$. 
Next observe that by applying $R\pi_*$ to the exact sequence

$$0 \to \mathcal{O}_P \to F_*\mathcal{O}_P \xrightarrow{d} F_*Z^1_P \xrightarrow{c} \Omega^1_P \to 0$$

we obtain a commutative diagram

which together with preceding calculations shows that $c_1(\mathcal{O}_P(1))$ induces an isomorphism \(\text{(1.23.1)}\).

4.24. Now observe that by the commutativity of (4.22.1) under the isomorphisms

$$R\pi_*(\pi^*\mathcal{V} \otimes F_*Z^1_P) \simeq \mathcal{V} \otimes R\pi_*F_*Z^1_P \simeq \mathcal{V} \otimes (F_*Z^1_X \oplus \mathcal{O}_X[-1]),$$

$$R\pi_*(\pi^*\mathcal{V} \otimes \Omega^1_P) \simeq \mathcal{V} \otimes (\Omega^1_X \oplus \mathcal{O}_X[-1])$$

the map

$$\pi^*\rho - C : R\pi_*(\pi^*\mathcal{V} \otimes F_*Z^1_P) \to R\pi_*(\pi^*\mathcal{V} \otimes \Omega^1_P)$$

occurring in the definition of $R\pi_*G_{(\pi^*\mathcal{V}, \pi^*\rho)}$ is identified with the sum of the maps

$$\rho - C : \mathcal{V} \otimes F_*Z^1_X \to \mathcal{V} \otimes \Omega^1_X, \quad \rho - 1 : \mathcal{V}[-1] \to \mathcal{V}[-1].$$

This implies \(\text{(1.19)}\) in the case when $X$ is smooth.

4.25. The general case of \(\text{(4.19)}\) follows from the smooth case by noting that the preceding arguments give for general $X$ decompositions

$$R\pi_*F_*LZ^1_P \simeq F_*LZ^1_X \oplus \mathcal{O}_X[-1], \quad R\pi_*L \simeq L \oplus \mathcal{O}_X[-1]$$

defining the desired decomposition of $R\pi_*G_{(\pi^*\mathcal{V}, \pi^*\rho)}$.

\[\square\]

5. Cohomology and stacks

In this section we give a proof of \(\text{(1.3)}\) in the case of a projective smooth morphism $f : X \to S$ and a finite flat group scheme $G_X$ on $X$ of height 1. In the following sections we will give an independent proof of \(\text{(1.3)}\) without these additional assumptions on $f$ and $G_X$, and the reader only interested in the proof of \(\text{(1.3)}\) may skip this section. The argument in the present section, however, yields more information about the cohomology groups $R^if_*G_X$ under these additional assumptions, and works more generally for proper smooth morphisms of algebraic spaces, and not just projective morphisms of schemes. In particular, in this section we prove \(\text{(1.8)}\).

Sometimes one can prove the representability of $R^if_*G_X$ directly using Artin’s criteria. This approach, however, introduces additional technical problems and we find it convenient to approach it from a slightly more general perspective.
In this section we will use the notion of algebraic $n$-stack. We will use this in a rather formal way, following [37], but it should be noted that the foundations of this theory has been cemented in various works since the writing of that paper; see, for example, the article [39].

5.1. Let $f : X \to S$ be a proper smooth morphism of algebraic spaces over a field $k$ of positive characteristic $p$. Let $G_X$ be a commutative group scheme over $X$. Fix an integer $n$. By the Dold-Kan correspondence we obtain from the complex $(\tau_{\leq n} Rf_* G_X)[n]$ an $n$-truncated simplicial presheaf $S_n(X, G_X)$ on the category of $S$-schemes.

In the case of $n = 1$, $S_1(X, G_X)$ is the stack whose fiber over a scheme $T/S$ is the groupoid of $G_X$-torsors on $X_T$. In the case $n = 2$, $S_2(X, G_X)$ is the 2-stack associated to the prestack which to $T/S$ associates the 2-category of $G_X$-gerbes over $X_T$. Note that passage to isomorphism classes defines a morphism $S_n(X, G_X) \to R^n f_* G_X$.

The fiber over 0 is given by $B S_{n-1}(X, G_X)$, the $n$-stack associated to the complex $(\tau_{\leq n-1} Rf_* G_X)[n]$.

This corresponds to the observation that the square

$\begin{array}{ccc}
\tau_{\leq n-1} Rf_* G_X & \to & \tau_{\leq n} Rf_* G_X \\
\downarrow & & \downarrow \\
0 & \to & R^n f_* G_X[-n]
\end{array}$

is both a homotopy pushout and homotopy pullback diagram. Geometrically this corresponds to the statement that $R^n f_* G_X$ is the $n$-rigidification (we do not intend to introduce this as a rigorous notion here but the idea is to view this as a suitable generalization of the usual notion of rigidification as discussed for example in [2]) of $S_n(X, G_X)$ with respect to $S_{n-1}(X, G_X)$.

The main results of this section are the following:

Theorem 5.2. Let $C^\bullet$ be a bounded complex of fppf-sheaves on $S$ concentrated in degrees $[0, n]$ and let $S_i$ denote the $i$-stack corresponding, via the Dold-Kan correspondence as above, to $(\tau_{\leq i} C^\bullet)[i]$.

(i) If $S_i$ is an algebraic stack locally of finite presentation over $S$ for all $i$ and the stacks $S_i$ are flat over $S$ for $i < n$, then $\mathcal{H}^n := \mathcal{H}^n(C^\bullet)$ is an algebraic space and the map $S_n \to \mathcal{H}^n$ is faithfully flat.

(ii) If $\mathcal{H}^n$ is an algebraic space and the stacks $S_i$ are algebraic for $i < n$ then $S_n$ is algebraic.

(iii) If $S_n$ is an algebraic stack then $S_i$ is algebraic for $i < n$.

Remark 5.3. Here we say that the morphism $S_n \to \mathcal{H}^n$ is faithfully flat if for some smooth cover $Y \to S_n$, with $Y$ an algebraic space, the induced map $Y \to \mathcal{H}^n$ is faithfully flat. This will suffice for our purposes, though the ambitious reader may wish to develop this in a more formal framework as in the case of 1-stacks [38, Tag 036F].
5.4. Let $(\mathcal{V}, \rho)$ be a perfect complex on $X$ equipped with a morphism $\rho : F^*_{X} \mathcal{V} \to \mathcal{V}$. Let $\mathcal{G}(\mathcal{V}, \rho)$ be defined as in 4.4 and define $\mathcal{S}_{n+1}(X, \mathcal{G}(\mathcal{V}, \rho))$ to be the stack associated to the functor on $S$-algebras given by

$$A \mapsto (\tau_{\leq n} Rf^*_{X} \text{an}(\mathcal{G}(\mathcal{V}, \rho))(A))[n].$$

**Theorem 5.5.** The stack $\mathcal{S}_{n}(X, \mathcal{G}(\mathcal{V}, \rho))$ is an algebraic stack for all $n$.

5.6. We give the proofs of 5.2 and 5.5 below. We first explain a consequence. Consider the special case when $\mathcal{V}$ is a vector bundle, so that $(\mathcal{V}, \rho)$ defines a finite flat group scheme $G_p(\mathcal{V}, \rho)$ over $X$. Using 4.8 we identify $\mathcal{S}_{n}(X, \mathcal{G}(\mathcal{V}, \rho))$ with the stack associated to the functor

$$A \mapsto (\tau_{\leq n} R\Gamma(X_{A}, G_p(\mathcal{V}, \rho)))[n].$$

(note the indexing convention in the definition of $\mathcal{S}_{n}(X, \mathcal{G}(\mathcal{V}, \rho))$). Applied in the case when $\mathcal{V}$ is a vector bundle, and therefore $(\mathcal{V}, \rho)$ corresponds to a finite flat group scheme $G_p(\mathcal{V}, \rho)$, theorems 5.2 and 5.5 imply the conclusion of 1.8 for $G_p(\mathcal{V}, \rho)$.

**Example 5.7.** By 4.2 and 4.3 the stacks $\mathcal{S}_{n}(X, \mu_p)$ and $\mathcal{S}_{n}(X, \alpha_p)$ are examples of such stacks.

**Corollary 5.8.** Let $f : X \to S$ be a smooth proper morphism of schemes of characteristic $p$. Let $G$ be a finite flat commutative group scheme on $X$. Suppose that either $G$ has height 1 or $G^D$ has height 1. If $n$ is an integer such that the sheaves $R^n f_* G$ are representable and flat over $S$ for $i < n$, then $R^n f_* G$ is representable.

**Proof.** If $G$ has height 1, then the stack $\mathcal{S}_n(X, G)$ is algebraic by 5.5. The result then follows from 5.2 (i) and (iii). Suppose that $G^D$ has height 1. Taking cohomology of the short exact sequence (2.3.1), we find a distinguished triangle realizing $Rf_* G$ as the cone of a (possibly non-linear) map of perfect complexes. Arguing as in 5.13 below, we deduce that $\mathcal{S}_n(X, G)$ is algebraic, and the result follows as in the previous case.

**Example 5.9.** Let $f : X \to S$ be a proper smooth morphism of schemes of characteristic $p$. Suppose that $f$ has geometrically connected fibers and that $\text{Pic}_{X/S}[p]$ is flat over $S$. These hypothesis hold, for example, if $f$ is a family of K3 surfaces, abelian varieties, or Enriques surfaces. Corollary 5.8 then implies that $R^2 f_* \mu_p$ is representable. Indeed in this case, we have $R^0 f_* \mu_p|_{X} = \mu_{p,S}$, by our assumptions on the fibers of $f$, and by Kummer theory $R^1 f_* \mu_p|_{X} = \text{Pic}_{X/S}[p]$.

**Example 5.10.** If $f : X \to S$ is a family of K3 surface in characteristic $p$, then $R^2 f_* \mu_p$ is representable, but need not be flat over $S$. For instance, $R^2 f_* \mu_p$ has relative dimension 0 over the finite height locus, and has relative dimension 1 over the supersingular locus. If the family $f$ involves a specialization from the finite height locus to the supersingular locus, then $R^2 f_* \mu_p$ is not representable. Similar phenomenon occur when $f$ is a family of abelian varieties of dimension $\geq 2$.

All of the above assertions can be verified locally on $S$. We therefore make the additional assumption that $S = \text{Spec}(R)$ is an affine scheme in what follows.

**Proof of 5.2.**

We first verify statement (i).
5.11. If \( x, y \in \mathcal{H}^n(T) \) are two sections over an \( S \)-scheme \( T \) then the functor expressing the condition that \( x = y \) is isomorphic to the functor expressing the condition that \( x - y = 0 \). Therefore, to verify the representability of the diagonal it suffices to show that if \( T \) is an \( S \)-scheme and \( x \in \mathcal{H}^n(T) \) is a section then the functor \( F \) given by

\[
F(T' \to T) = \begin{cases} 
\emptyset & \text{if } x|_{T'} \neq 0, \\
\{\ast\} & \text{if } x|_{T'} = 0
\end{cases}
\]

is an algebraic space. For this note that \( F \) is a subfunctor of \( T \) so the diagonal of \( F \) over \( T \) is an isomorphism. To find a flat cover of \( F \) we may replace \( T \) by an fppf cover and therefore may assume that \( x \) lifts to a map \( \tilde{x} : T \to S_n \). Let \( \tilde{F} \) denote the fiber product

\[
\tilde{F} := S_n \times_{\Delta, S_n \times S_n, \tilde{x} \times 0} T,
\]

an algebraic \((n - 1)\)-stack by assumption. There is a natural map

\[
\tilde{F} \to F
\]

which we claim is an fppf surjection.

To prove this, let \( Q_n \) denote the fiber product of the diagram

\[
\begin{array}{ccc}
S_n \times S_n \\
\downarrow \\
S_n \xrightarrow{\delta} S_n \times \mathcal{H}^n,
\end{array}
\]

where \( \delta \) is the map induced by the diagonal. The additive structure on \( C^\bullet \) induces an isomorphism

\[
S_n \times B S_{n-1} \to Q_n, \quad (x, y) \mapsto (x, x + y).
\]

We then have a commutative diagram with cartesian squares

\[
\begin{array}{ccc}
\tilde{F} & \to & F \\
\downarrow & & \downarrow \\
S_n \xrightarrow{id \times 0} S_n \times B S_{n-1} & \to & S_n \times S_n \\
\downarrow & \downarrow & \downarrow \\
S_n & \xrightarrow{id} & S_n \xrightarrow{pr_1} \mathcal{H}^n \times S_n.
\end{array}
\]

Since the map induced by \( 0 \)

\[
S \to B S_{n-1}
\]

is faithfully flat (since \( S_{n-1} \) is faithfully flat over \( S \)). It follows that \( \tilde{F} \to F \) is also faithfully flat. We conclude that \( F \) is an algebraic space and that the diagonal of \( \mathcal{H}^n \) is representable.

5.12. To prove the existence of a flat cover of \( \mathcal{H}^n \) it suffices to show that for any \( S \)-scheme \( T \) and element \( x \in \mathcal{H}^n(T) \) the map

\[
\tilde{T} := S_n \times_{\mathcal{H}^n, x} T \to T
\]
is faithfully flat. This can be verified fppf-locally on $T$, so we may assume that $x$ lifts to a map $\tilde{x} : T \to \mathcal{S}_n$, in which case we have

$$\tilde{T} \simeq B\mathcal{S}_{n-1} \times_S T.$$ 

Indeed, the additive structure on $C^\bullet$ induces a map

$$\mathcal{S}_n \times_S \mathcal{S}_n \to \mathcal{S}_n$$

over the addition map on $\mathcal{H}^n$, and so addition by $\tilde{x}$ identifies the preimage of $0 \in \mathcal{H}^n$, which is $B\mathcal{S}_{n-1}$, with $\tilde{T}$. By our assumption that $\mathcal{S}_{n-1}$ is flat over $S$ we conclude that $\tilde{T} \to T$ is faithfully flat. This completes the proof of 5.2 (i).

**5.13.** For statement (ii), as well as some arguments to follow, it is useful to consider a more general context. Consider a map of complexes

$$C^\bullet \to D^\bullet$$

concentrated in degrees $[0, n]$ and let $K^\bullet$ be the co-cone so we have a distinguished triangle

$$K^\bullet \to C^\bullet \to D^\bullet.$$

Assume further that the map

$$\mathcal{H}^n(C^\bullet) \to \mathcal{H}^n(D^\bullet)$$

is surjective, so that $K^\bullet$ is concentrated in degrees $[0, n]$ as well. Let $\mathcal{S}_{n,C^\bullet}$ (resp. $\mathcal{S}_{n,D^\bullet}$, $\mathcal{S}_{n,K^\bullet}$) be the stack associated by the Dold-Kan correspondence to $C^\bullet[n]$ (resp. $D^\bullet[n]$, $K^\bullet[n]$).

**Lemma 5.14.** If $\mathcal{S}_{n,D^\bullet}$ and $\mathcal{S}_{n,K^\bullet}$ are algebraic, then so is $\mathcal{S}_{n,C^\bullet}$.

**Proof.** To prove the representability of the diagonal of $\mathcal{S}_{n,C^\bullet}$ it suffices, as in the proof of (i), to show that if $T$ is an $S$-scheme and $t : T \to \mathcal{S}_{n,C^\bullet}$ is a morphism then the fiber product

$$T \times_t \mathcal{S}_{n,C^\bullet} \times S$$

is algebraic. We can describe this fiber product in two steps, by noting that we have a homotopy cartesian square

$$\begin{array}{ccc}
\mathcal{S}_{n,K^\bullet} & \rightarrow & S \\
\downarrow & & \downarrow 0 \\
\mathcal{S}_{n,C^\bullet} & \rightarrow & \mathcal{S}_{n,D^\bullet}.
\end{array}$$

Let $\tilde{t} : T \to \mathcal{S}_{n,D^\bullet}$ be the map obtained by composition. Then the fiber product

$$\overline{P} := T \times_{\tilde{t}, \mathcal{S}_{n,D^\bullet}} \times S$$

is algebraic by assumption, and comes equipped with a map

$$p : \overline{P} \to \mathcal{S}_{n,K^\bullet}$$

such that

$$T \times_{t, \mathcal{S}_{n,C^\bullet}} \times S \simeq \overline{P} \times_{\mathcal{S}_{n,K^\bullet}}} \times S.$$ 

Since $\mathcal{S}_{n,K^\bullet}$ is assumed algebraic it follows that this is an algebraic stack as well.

To find a flat cover of $\mathcal{S}_{n,C^\bullet}$ proceed as follows. Let

$$u : U \to \mathcal{S}_{n,D^\bullet}.$$
be a flat surjection, and consider the fiber product
\[ \mathcal{P}_U := \mathcal{S}_{n,C^\bullet} \times_{\mathcal{S}_{n,D^\bullet}, u} U. \]

To prove the existence of a flat cover of \( \mathcal{S}_{n,C^\bullet} \) it suffices to show that \( \mathcal{P}_U \) is algebraic, since a flat cover of \( \mathcal{P}_U \) will then also be a flat cover of \( \mathcal{S}_{n,C^\bullet} \). For this, we may replace \( U \) by a flat cover and therefore, using the surjectivity of \( \mathcal{H}^n(C^\bullet) \to \mathcal{H}^n(D^\bullet) \), may assume that \( u \) lifts to a morphism
\[ \tilde{u} : U \to \mathcal{S}_{n,C^\bullet}. \]

Using the additive structure on \( C^\bullet \) we get a map
\[ \mathcal{S}_{n,C^\bullet} \times \mathcal{S}_{n,C^\bullet} \to \mathcal{S}_{n,C^\bullet}. \]

Precomposing this map with the map
\[ (i, \tilde{u}) : \mathcal{S}_{n,K^\bullet} \times U \to \mathcal{S}_{n,C^\bullet} \times \mathcal{S}_{n,C^\bullet}, \]
where \( i \) is the map induced by \( K^\bullet \to C^\bullet \), we get a map
\[ \mathcal{S}_{n,K^\bullet} \times U \to \mathcal{P}_U, \]
and it follows from the fact that (5.14.1) is cartesian that this map is an isomorphism. \( \square \)

To get statement (ii) in 5.2 we apply the lemma with \( D^\bullet = \mathcal{H}^n(C^\bullet)[-n] \), in which case \( K^\bullet = \tau_{\leq n-1} C^\bullet \).

Finally statement (iii) follows by induction from noting that the fiber product of the diagram
\[ \begin{array}{ccc}
S & \xrightarrow{0} & \mathcal{S}_n \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{S}_n
\end{array} \]
is isomorphic to \( \mathcal{S}_{n-1} \).

This completes the proof of 5.2. \( \square \)

**Example 5.15.** If \( E^\bullet \) is a strictly perfect complex of \( R \)-modules concentrated in degrees \([0, n]\) we can consider \( C^\bullet \) given by
\[ A \mapsto A \otimes_R E^\bullet. \]

Then the above discussion implies that \( \mathcal{S}_n \) is an algebraic stack. For this proceed by induction on the length of \( E^\bullet \). If \( n = 1 \) then the result follows from \([37, 2.10]\). For the inductive step consider the stupid filtration \( \sigma_i E^\bullet \) \([38, \text{Tag 0118}]\) on \( E^\bullet \), so we have exact sequences of complexes
\[ 0 \to \sigma_i E^\bullet \to \sigma_{i+1} E^\bullet \to E^{i+1}[-i-1] \to 0 \]
to which we can apply 5.14. More generally, this implies that if \( E^\bullet \) is a perfect complex on a scheme \( S \) then the associated stack \( \mathcal{S}_n \) is algebraic, since this can be verified locally when \( E^\bullet \) can be represented by a strictly perfect complex.

**Proof of 5.5.**

5.16. By 5.2 (iii) we may assume that \( n \) is chosen sufficiently large so that \( R^i f_* \mathcal{G}_{(Y, \rho)} = 0 \) for \( i \geq n - 1 \).
5.17. Let \( f : X \to Y \) be a quasi-compact and quasi-separated morphism of algebraic spaces over \( k \). We then have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\pi} & Y
\end{array}
\]

where the square is cartesian. From this we get a map

\[
F_Y^* Rf_* \mathcal{V} \longrightarrow Rf'_* \pi^* \mathcal{V} \, \xrightarrow{\text{adj.}} \, Rf'_* RF_{X/Y}^* \pi^* \mathcal{V} \cong Rf_* F_X^* \mathcal{V} \, \longrightarrow \, Rf_* \mathcal{V},
\]

where the map labelled “adj.” is the adjunction map. We thus obtain a pair \((Rf_* \mathcal{V}, f_* \rho)\) consisting of a complex of quasi-coherent sheaves on \( Y \) and a semilinear map. We can then form

\[
\mathcal{G}(Rf_* \mathcal{V}, f_* \rho)
\]
on \( Y \).

5.18. The adjunction map

\[
f^* Rf_* \mathcal{V} \to \mathcal{V}
\]

induces a map

\[
\alpha : \mathcal{G}(Rf_* \mathcal{V}, f_* \rho) \to f_* \text{anic} \mathcal{G}(\mathcal{V}, \rho).
\]
The cone of this map can be described as follows.

We consider the case when \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(C) \) are affine. The general case is obtained by taking limits as in 3.6. For a \( C \)-algebra \( A \) we write \( D := B \otimes_B^L A \) for the animated \( B \)-algebra obtained by forming the derived tensor product. The value of \( \alpha \) on \( A \) then fits into a commutative diagram, using (4.4.2), where the horizontal rows are distinguished triangles,

\[
\begin{array}{ccc}
\mathcal{G}(Rf_* \mathcal{V}, f_* \rho)(A) & \longrightarrow & (Rf_* \mathcal{V})_A \otimes_A F_A, L^1_A \\
& \downarrow{\alpha} & \downarrow{f_* \text{anic} \mathcal{G}(\mathcal{V}, \rho)} \\
\mathcal{G}(\mathcal{V}, \rho)(D) & \longrightarrow & \mathcal{V}_B \otimes_B F_B, L^1_D \\
& & \downarrow{f_* \text{anic} \mathcal{G}(\mathcal{V}, \rho)} \\
& & \mathcal{V}_B \otimes_B L_D +1
\end{array}
\]

We therefore analyze the right two vertical morphisms.

Note that we have a natural isomorphism

\[
(Rf_* \mathcal{V})_A \otimes_A L_A \cong \mathcal{V}_B \otimes_B (D \otimes_A L_A)
\]
so the cone of the map

\[
(Rf_* \mathcal{V})_A \otimes_A L_A \to \mathcal{V}_B \otimes_B L_D
\]
is isomorphic to

\[
\mathcal{V}_B \otimes_B (\text{Cone}(D \otimes_A L_A \to L_D)) \cong (\mathcal{V}_B \otimes_B L_{B/C}) \otimes_B D.
\]
Here we use the distinguished triangle

\[
D \otimes_A L_A \to L_D \to L_{D/A} \to D \otimes_A L_A[1]
\]
and the natural isomorphism
\[ L_{D/A} \simeq D \otimes_B L_{B/C}. \]

Similarly the cone of the middle vertical arrow in (5.18.2) can be described as
\[ \Psi_B \otimes_B \text{Cone}(D \otimes_A F_A \cdot L_{A} \rightarrow F_{D*} L_{D}^1). \]

5.19. Consider now the setting of a proper smooth morphism \( f : X \rightarrow S \) and \((\Psi, \rho)\) on \( X \).

Lemma 5.20. (i) \( \text{Cone}(\alpha) \) is algebraic.

(ii) If \( \rho = 0 \) then \( Rf^\text{ani}_* G_{\Psi,0} \) defines an algebraic stack.

Proof. From the preceding discussion we have a distinguished triangle
\[ Rf_* (\Psi \otimes \Omega^1_{X/S})[-1] \rightarrow \text{Cone}(\alpha) \rightarrow \Sigma_{\Psi,0}, \]
where \( \Sigma_{\Psi,0} \) is described as in (5.18.3). Combining this with (5.14) and (5.15) we see that \( \text{Cone}(\alpha) \) defines an algebraic stack if and only \( \Sigma_{\Psi,0} \) defines an algebraic stack. Now it follows from the definition that
\[ \Sigma_{\Psi,0} = \Sigma_{\Psi,0}. \]
Thus to prove (i) it suffices to consider the case when \( \rho = 0 \).

To understand the case of \( \rho = 0 \), note that the sheaf \( \alpha_p \) has a natural structure of an \( O_X \)-submodule of \( G_a \). Thus it makes sense to form \( \Psi \otimes \alpha_p \). Tensoring the sequences in 4.2 with \( \Psi \) we see that \( G_{\Psi,0} \) is the functor sending \( \text{Spec}(R) \rightarrow X \) to
\[ R\Gamma(R, \Psi \otimes \alpha_p). \]

In particular, this implies that \( G_{(Rf,\Psi,0)} \) defines an algebraic stack. Furthermore, the algebraicity of \( \Sigma_{\Psi,0} \) is then equivalent to the algebraicity of \( Rf^\text{ani}_* G_{\Psi,0} \) so (i) will follow from (ii) in the case when \( \rho = 0 \).

To prove (ii) in this case, we may work locally on \( S \) so may assume that \( S \) is affine. In this case, by our projectivity assumption we can represent \( \Psi \) by a strictly perfect complex. Using the sequences obtained from the “stupid filtration” on \( \Psi \) we are then further reduced to the case when \( \Psi \) is a vector bundle. In this case we proceed by induction on the rank of \( \Psi \).

For the case of \( \Psi = L \) a line bundle on \( X \) have a short exact sequence
\[ 0 \rightarrow L \otimes \alpha_p \rightarrow L \xrightarrow{x \mapsto x^p} L \otimes L \rightarrow 0. \]

Taking cohomology and using the algebraicity of the stacks associated to perfect complexes we deduce that \( Rf^\text{ani}_* G_{L,0} \) is algebraic.

For the inductive step consider the projective bundle
\[ p : P \rightarrow X \]
associated to \( \Psi \). Then the map
\[ G_{a,X} \rightarrow Rp_* G_{a,P} \]
is an isomorphism, as follows from the usual computation of cohomology of projective space, which implies that the map
\[ \alpha_{p,X} \rightarrow Rp_* \alpha_{p,P} \]
is an isomorphism, and from this it follows in turn that the map
\[ V \otimes_{\alpha_p} \alpha_p \rightarrow R p_* (p^* V \otimes_{\alpha_p} p, p) \]
is an isomorphism. Therefore we may replace \( X \) by \( P \), which in conjunction with consideration of the universal quotient over \( P \) reduces the proof to the lower-rank case. \( \square \)

5.21. To complete the proof of 5.5 note that the lemma reduces the proof of the algebraicity of \( R f^* G(V, \rho) \) to showing the algebraicity for \( G(V, \rho) \) (that is, we may consider just the case when \( f \) is the identity).

For this we may work locally on \( S \) and may therefore assume that \( S = \text{Spec}(R) \) is affine and that \( V \) is given by a perfect complex of \( R \)-modules \( V^* \). Using the “stupid filtration” on \( V^* \) we are further reduced to the case when \( V \) is given by a single free \( R \)-module \( V \) with a map \( \rho: F^*_R V \rightarrow V \). If \( G_p(V, \rho) \) denotes the associated finite flat group scheme then by \ref{4.8} and \[37\] 2.10 we get that \( G(V, \rho) \) is algebraic concluding the proof of 5.5. \( \square \)

6. The \( \infty \)-category of pro-objects

6.1. Let \( \mathcal{C} \) be a small \( \infty \)-category which admits finite limits and which is idempotent complete, and let \( \mathcal{S} \) denotes the \( \infty \)-category of spaces \[30\] 1.2.16.1. Then by \[29\] 3.1.1 and 3.1.2 the \( \infty \)-category of pro-objects in \( \mathcal{C} \), denoted by \( \text{Pro}(\mathcal{C}) \), can be defined as the full subcategory of the category \( \text{Fun}(\mathcal{C}, \mathcal{S})^{\text{op}} \) of functors which preserve finite limits. As discussed in \[29\] 3.1.2 this category \( \text{Pro}(\mathcal{C}) \) can also be described as

\[ \text{Pro}(\mathcal{C}) \simeq \text{Ind}(\mathcal{C}^{\text{op}})^{\text{op}}, \]

where \( \text{Ind}(\mathcal{C}^{\text{op}}) \) is defined as in \[30\] 5.3.5.1. By the \( \infty \)-categorical Yoneda lemma \[30\] 5.1.3.1 the natural functor

\[ \mathcal{C} \rightarrow \text{Pro}(\mathcal{C}) \]
is fully faithful, so we often view \( \mathcal{C} \) as a sub-category of \( \text{Pro}(\mathcal{C}) \).

The categories \( \mathcal{C} \) we will be considering are the categories \( \text{Shv}^{\text{ani}}(X) \) and \( \mathcal{P}^{\text{ani}}(X) \) for a scheme \( X \), where sheaves are defined using the fpff topology.

We call an object \( \mathcal{F} \in \text{Shv}^{\text{ani}}(X) \) a \textit{locally perfect complex} if locally in the fpff topology on \( X \) it is isomorphic to the underlying complex of abelian groups of a perfect complex of \( \mathcal{O}_X \)-modules.

Let \( \mathcal{D}^{\text{lp}}(X) \subset \text{Shv}^{\text{ani}}(X) \) (resp. \( \mathcal{D}^{\text{f}}(X) \subset \text{Shv}^{\text{ani}}(X) \), \( \mathcal{D}^{\text{ff}}(X) \subset \text{Shv}^{\text{ani}}(X) \)) be the smallest stable \( \infty \)-subcategory containing the locally perfect complexes (resp. perfect complexes and complexes of the form \( i^*_{\text{ani}} H_{G_{Z_{\text{ani}}}^{\text{ani}}} \) for \( i : Z \hookrightarrow X \) a closed immersion and \( G_Z \) a finite flat group scheme over \( Z \), the locally perfect complexes and complexes of the form \( i^\text{ani}_{\text{ani}} H_{G_{Z_{\text{ani}}}^{\text{ani}}} \) for \( i : Z \hookrightarrow X \) a closed immersion and \( G_Z \) a finite flat group scheme over \( Z \)).

Let \( \mathcal{D}_i(X) \subset \text{Pro}(\mathcal{D}^{\text{ff}}(X)) \) be the smallest stable \( \infty \)-category containing \( \text{Pro}(\mathcal{D}^{\text{lp}}(X)) \) and the image of \( \mathcal{D}(X) \).

6.2. As discussed in \[37\] for a scheme \( X \) a sheaf

\[ \mathcal{F}_0 : \text{Alg}_X \rightarrow \mathcal{D}(\text{Ab}) \]
has an induced animation

\[ \mathcal{F}^{\text{ani}}_0 : \text{Alg}^{\text{ani}}_X \rightarrow \mathcal{D}(\text{Ab}), \]
and an object $\mathcal{F} \in \text{Shv}^{\text{ani}}(X)$ is of this form if and only if it commutes with sifted colimits; in this case, $\mathcal{F}_0$ is given by the restriction of $\mathcal{F}$ to $\text{Alg}_X$.

Passing to categories of pro-objects we also get a functor

$$\text{Pro(sheaves} \mathcal{F}_0 : \text{Alg}_X \to D(\text{Ab})) \to \text{Pro(Shv}^{\text{ani}}(X)).$$

We say that $\mathcal{F} \in \text{Pro(Shv}^{\text{ani}}(X))$ is continuous if $\mathcal{F}$ is in the essential image of this functor. We write $D^\dagger_{\text{cts}}(X) \subset D^\dagger(X)$ for the subcategory of continuous objects. This is a stable $\infty$-subcategory.

**Proposition 6.3.** Let $S$ be a reduced scheme and let $\mathcal{F} \in \text{Pro(Shv}^{\text{ani}}(S))$ be an object. Suppose the following hold:

(i) $\mathcal{F}$ is continuous

(ii) If $\mathcal{F}_0$ denotes the associated pro-complex of sheaves on $S$, then $\mathcal{F}_0$ is bounded and there exists a dense open subset $U \subset S$ such that the restriction of $\mathcal{H}^i(\mathcal{F}_0)$ to $U$ is contained in the smallest stable $\infty$-subcategory of the stable $\infty$-category of abelian sheaves on $U$ containing continuous finite flat group schemes and pro-objects in the category of locally coherent sheaves (sheaves which fppf locally are given by the underlying abelian sheaves of coherent sheaves).

Then there exists a dense open subset $U \subset S$ such that the restriction of $\mathcal{F}$ to $U$ is in $D^\dagger_{\text{cts}}(U)$.

**Remark 6.4.** Here we say that $\mathcal{F}_0$ is bounded if there exists $a < b$ such that $\mathcal{F}_0$ lies in

$$\text{Pro(sheaves} \mathcal{F}_0 : \text{Alg}_X \to D^{[a,b]}(\text{Ab})).$$

**Proof.** This is immediate from the preceding discussion. Condition (i) implies that $\mathcal{F} = \mathcal{F}^{\text{ani}}_0$ over a dense open subset, and shrinking so that $S$ is regular we get that any coherent sheaf on $S$ is perfect. \hfill $\square$

7. Cohomology with compact support

7.1. Formal cohomology.

7.2. Let $X$ be a scheme and let $i : Z \hookrightarrow X$ be a closed subscheme defined by an ideal $I \subset \mathcal{O}_X$. Let

$$i_n : Z_n \hookrightarrow X$$

be the closed subscheme defined by $I^n$ so we have a sequence of closed immersions

$$Z = Z_1 \hookrightarrow Z_2 \hookrightarrow \cdots \hookrightarrow Z_n \hookrightarrow \cdots \hookrightarrow X.$$

For an object $\mathcal{F} \in \text{Shv}^{\text{ani}}(X)$ let $\mathcal{F}_n \in \text{Shv}^{\text{ani}}(Z_n)$ be the restriction of $\mathcal{F}$. Define

$$\mathcal{\hat{F}} := \text{holim}_n i^{\text{ani}}_n \mathcal{F}_n \in \text{Pro(Shv}^{\text{ani}}(X)),$$

where $i^{\text{ani}}_n$ is defined as in 3.6.

7.3. In the case when $X = \text{Spec}(A)$ and $Z$ is defined by an ideal $I = (a_1, \ldots, a_r) \subset A$ we can also consider the derived reduction

$$A^{\text{ani}}_n := A \otimes_{\mathcal{O}^\dagger_{\text{cts}}}^{\mathbf{L}} Z_{[X_1, \ldots, X_r], X_i \to 0} Z$$
and the induced functor

\[ B \mapsto \mathcal{F}(A_n^{\mathrm{ani}} \otimes^L_A B) \]

defining an object \( \mathcal{F}' \in \mathbf{Shv}^{\mathrm{ani}}(X) \). Set

\[ \widehat{\mathcal{F}}' := \text{holim}_n \mathcal{F}_n'. \]

**Lemma 7.4.** Suppose \( \mathcal{F} \) is given by either a bounded complex of coherent sheaves on \( X \) or a finite flat group scheme over \( X \). Then the map

\[ \widehat{\mathcal{F}}' \to \widehat{\mathcal{F}} \]

is an equivalence.

**Proof.** The ring \( A_n^{\mathrm{ani}} \) can be described by a Koszul complex construction as in [28, §3], and therefore its homology groups are given by the homology of the Koszul complex associated to the sequence \((a^n_1, \ldots, a^n_r)\). The functoriality

\[ A_n^{\mathrm{ani}} \to A_{n-1}^{\mathrm{ani}} \]

is described by the functoriality of the Koszul complex using the diagram

\[ \begin{array}{ccc}
A^r & \to & (a^n_1, \ldots, a^n_r) \\
\downarrow & \downarrow & \downarrow \\
(a_1, \ldots, a_r) & \to & A \\
\downarrow & \downarrow & \downarrow \\
A^r & \to & (a^{n-1}_1, \ldots, a^{n-1}_r) \\
\end{array} \]

From this it follows that for \( i > 0 \) the transition maps

\[ \pi_i(A_{2n}^{\mathrm{ani}}) \to \pi_i(A_n^{\mathrm{ani}}) \]

are zero.

In the case when \( \mathcal{F} \) is locally given by a bounded complex of coherent sheaves it follows that the cone of the map of systems

\[ \{ \mathcal{F}'_n \to i_{n*}^{\mathrm{ani}} \mathcal{F}_n \} \]

admits a finite filtration whose associated graded are pro-objects with eventually zero transition maps, and therefore is zero in the pro-cagory. This implies the lemma in this case.

In the case when \( \mathcal{F} \) is given by a finite flat group scheme \( G \) the result follows from the above discussion combined with [14, 5.1.10 (3) and 5.1.13]. \( \square \)

### 7.5. The \( S \)-complex.

We continue with the notation of 7.2.

**7.6.** For an object \( \mathcal{F} \in \mathcal{D}_\dagger(X) \) define

\[ S(\mathcal{F}) := \text{Cocone}(\mathcal{F} \to \widehat{\mathcal{F}}). \]

If we wish to emphasize the dependence on the choice of \( Z \) we write \( S_{(X,Z)}(\mathcal{F}) \) instead of \( S(\mathcal{F}) \).
Theorem 7.7. If
\[ g : \mathcal{F} \to \mathcal{F}' \]
is a morphism in \( \mathcal{D}^{I}(X) \) whose restriction to \( U \) is an isomorphism, then the induced map in \( \mathcal{D}^{\dagger}(X) \)
\[ \mathcal{S}(g) : \mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F}') \]
is an isomorphism.

Proof. The assertion is local on \( X \). It therefore suffices to consider the case when \( \mathcal{F} \) and \( \mathcal{F}' \) lie in the stable subcategory of \( \mathcal{D}^{I}(X) \), which we assume for the remainder of the proof.

It furthermore suffices to show that in the case when \( X = \text{Spec}(A) \) is the spectrum of a strictly henselian local ring and \( Z \) is defined by
\[ I = (a_1, \ldots, a_r) \subset A \]
the natural map
\[ R\Gamma(\text{Spec}(A), \mathcal{S}(\mathcal{F})) \to R\Gamma(\text{Spec}(A), \mathcal{S}(\mathcal{F}')) \]
is an isomorphism.

Let \( \widehat{A} \) be the derived \( I \)-adic completion of \( A \) (in the sense of \( \text{[14, 5.6.1]} \)).

Lemma 7.8. The natural map
\[ R\Gamma(\widehat{A}, \mathcal{F}) \to R\Gamma(A, \widehat{\mathcal{F}}) \]
is an isomorphism.

Proof. It suffices to show that this holds for \( \mathcal{F} \) a perfect complex, where the validity is standard, and \( \mathcal{F} = i_{\text{an}}^* G_Y \) a finite flat group scheme \( G_Y \) over a closed subscheme \( i : Y \hookrightarrow \text{Spec}(A) \), where the result holds by \( \text{[14, 5.6.6]} \). \( \square \)

With this in hand, to prove 7.7 it suffices to show that for \( \mathcal{F} \in \mathcal{D}^{I}(X) \) the square
\[ (7.8.1) \quad \begin{array}{ccc}
R\Gamma(A, \mathcal{F}) & \to & R\Gamma(\widehat{A}, \mathcal{F}) \\
\downarrow & & \downarrow \\
\lim_{f \in I} R\Gamma(A[1/f], \mathcal{F}) & \to & \lim_{f \in I} R\Gamma(\widehat{A}[1/f], \mathcal{F})
\end{array} \]
is homotopy cartesian. Or equivalently, considering the homotopy fibers of the vertical maps, it suffices to show that for \( \mathcal{F} \in \mathcal{D}^{I}(X) \) the natural map on local cohomology
\[ (7.8.2) \quad R\Gamma_1(A, \mathcal{F}) \to R\Gamma_1(\widehat{A}, \mathcal{F}) \]
is an equivalence. Furthermore, it suffices to consider the case when \( \mathcal{F} \) is given by a perfect complex or \( i_{\text{an}}^* G_Y \) for a finite flat group scheme \( G_Y \) over a closed subscheme \( i : Y \hookrightarrow \text{Spec}(A) \).

That (7.8.2) is an equivalence in the case of a perfect complex is the statement that local cohomology is the same for a ring and its completion (see \( \text{[38, Tag 0ALZ]} \)). In the case \( \mathcal{F} = i_{\text{an}}^* G_Y \) the result is \( \text{[14, 5.4.4]} \). \( \square \)
Definition 7.9. For $\mathcal{F} \in \mathcal{D}^{lf}(X)$ we define the compactly supported cohomology of $\mathcal{F}$ with respect to $Z$, denoted by $R\Gamma^c_c((X, Z), \mathcal{F})$, to be

$$R\Gamma^c_c((X, Z), \mathcal{F}) := R\Gamma(X, S(\mathcal{F})).$$

If $f : X \to S$ is a morphism we also consider the relative version, denoted $Rf^c_{(X, Z)}\mathcal{F}$, defined by

$$Rf^c_{(X, Z)} := Rf_{ani}^* S(F).$$

It follows from the definition that there is a distinguished triangle

$$R\Gamma^c_c((X, Z), \mathcal{F}) \to R\Gamma(X, \mathcal{F}) \to R\Gamma(X, \hat{\mathcal{F}}) \to R\Gamma^c_c(X, \mathcal{F})[1].$$

Remark 7.10. Observe that in the case when $\mathcal{F}$ is given by a coherent sheaf our definition recovers the definition of compactly supported cohomology in [24].

The following is our main result about compactly supported cohomology. The proof will be given in the next section.

Theorem 7.11. Let $f : X \to S$ be a projective morphism of schemes of finite type over a field $k$ with $S$ reduced, and let $Z \subset X$ be a closed subscheme. Then for $\mathcal{F} \in \mathcal{D}^{lf}(X)$ there exists a dense open subset $U$ such that the restriction of $Rf^c_{(X, Z)}(\mathcal{F})$ to $U$ lies in $\mathcal{D}^{lf}(U)$.

8. Proof of 7.11

The proof is a somewhat elaborate dévissage to the special cases of $\mathcal{F}$ equal to either a perfect complex, $\mu_p$, $\alpha_p$, or $\mathbb{Z}/(n)$. Before giving the proof we prove some technical results that will be needed.

Note that by 7.7, it suffices to prove 7.11 under the further assumption that $Z$ is nowhere dense. Indeed if $i' : X' \hookrightarrow X$ is the scheme-theoretic closure of $U$ then by 7.7 we can replace $\mathcal{F}$ by $i'^{ani}_{*} \mathcal{F}|_{X'}$.

8.1. Change of subscheme.

8.2. In the setting of 7.11 suppose given a second closed subscheme $Z' \subset X$

containing $Z$, and let $U'$ denote the complement of $Z'$. So we have inclusions

$$Z' \hookrightarrow Z'' \hookrightarrow X \rightarrow U \leftarrow U'.$$

For $\mathcal{F} \in \mathcal{D}^{lf}(X)$ let $\hat{\mathcal{F}}^Z$ (resp. $\hat{\mathcal{F}}^{Z'}$) denote the object defined as in 7.2 using the closed subscheme $Z$ (resp. $Z'$).

There is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} & \rightarrow & \hat{\mathcal{F}}^{Z'} \\
\downarrow & & \downarrow \\
\mathcal{F} & \rightarrow & \hat{\mathcal{F}}^Z
\end{array}
$$
and therefore a map of complexes

\[ S_{(X,Z')}(\mathcal{F}) \to S_{(X,Z)}(\mathcal{F}) \]

**Lemma 8.3.** With notation as in [7.11] assume that \( f \) has generic relative dimension \( d \), \( Z' \) has generic relative dimension \( \leq d - 1 \), and that we have established the theorem for morphisms of generic relative dimension \( \leq d - 1 \). Then there exists a dense open subset \( U \subset S \) such that \( Rf_!(X,Z)(\mathcal{F}) \) lies in \( \mathcal{D}_!(U) \) if and only if there exists a dense open subset \( U' \subset S \) such that \( Rf_!(X,Z')(\mathcal{F}) \) lies in \( \mathcal{D}_!(U') \).

**Proof.** It suffices to prove the lemma in the case when \( \mathcal{F} \) is given by a perfect complex on \( X \) or the cohomology of a finite flat group scheme.

Let \( \widehat{C}^Z \) (resp. \( \widehat{C}'^Z \)) denote the cone of the map

\[ \widehat{F}^Z \to i_{an}^Z F|_Z \] (resp. \( \widehat{F}'^Z \to i_{an}^{Z'} F|_{Z'} \)).

The cone of (8.2.1) is given by the complex

\[ \text{Cone}(\widehat{F}'^Z \to \widehat{F}^Z), \]

and we have a morphism of distinguished triangles

\[
\begin{array}{ccc}
\widehat{C}'^Z[-1] & \rightarrow & \widehat{F}'^Z \\
\downarrow & & \downarrow \\
C^Z[-1] & \rightarrow & \widehat{F}^Z \\
\end{array}
\]

By our induction hypothesis we know that the cohomology of \( F|_Z \) and \( F|_{Z'} \) are in \( \mathcal{D}_!(U) \) (after possibly shrinking on \( S \)), so it suffices to show that \( Rf_!^{an} \widehat{C}^Z \) and \( Rf_!^{an} \widehat{C}'^Z \) lie in \( \mathcal{D}_!(U) \) for suitable \( U \subset S \).

So we show that \( Rf_!^{an} \widehat{C}^Z \) lies in \( \mathcal{D}_!(U) \) for some \( U \) (applying the same argument with \( Z' \) instead of \( Z \) we then get the lemma). Let \( C^Z_n \) denote the cone of the map

\[ i_{an}^{Z_n} \mathcal{F}_n \to i_{an}^{Z_1} \mathcal{F}_1 = \mathcal{F}|_Z. \]

Then \( C^Z_1 \) is 0 and the cone of the natural map

\[ C^Z_n \to C^Z_{n-1} \]

is of the form \( i_{an}^{Z_n} \mathcal{G}_n \) for a perfect complex \( \mathcal{G}_n \) on \( Z \). This is immediate in the case when \( \mathcal{F} \) is a perfect complex and follows from [14, 5.1.13] in the case when \( \mathcal{F} \) is given by a finite flat group scheme. Now shrinking on \( S \) so that \( S \) is regular and \( Z \to S \) is flat. Then by [38, Tag 08EV] each of the complexes \( Rf_!^{an}i_{an}^{Z_n} \mathcal{G}_n \) is given by a perfect complex, which implies the result. \( \square \)

### 8.4. Descent for derived blowups.

**8.5.** Let \( A \) be an animated ring and let

\[ Z[X_1, \ldots, X_r] \to A \]
be a morphism of animated rings corresponding to elements $a_1, \ldots, a_r \in \pi_0(A)$. As in [28] (see also [27]) we can then define the animated blowup

$$\text{Bl}^{\text{ani}} \to \text{Spec}(A)$$

by taking the fiber product, in the derived sense, of the diagram

$$\begin{array}{ccc}
\text{Spec}(A) & \xrightarrow{(a_1, \ldots, a_r)} & \text{Bl} \\
\downarrow & & \downarrow \\
\text{Bl} & \to & \mathbb{A}^r,
\end{array}$$

where the bottom horizontal arrow is the usual blowup of the origin.

Rather than introducing an additional layer of derived schemes, we will work with this derived blowup in a somewhat ad hoc manner. Namely, fix a finite covering of Bl by affines (for example, the one induced by the restriction of the standard affine cover of projective space), so we get an fppf hypercover

$$U_* \to \text{Bl}$$

with each $U_n$ affine. We will, abusively, write simply Bl for the corresponding cosimplicial animated ring, and $\text{Bl}^{\text{ani}}$ for the simplicial object of $\text{Alg}_{A,\text{op}}^{\text{ani}}$ obtained by base change. Taking the coskeleton of the morphism

$$\text{Bl}^{\text{ani}} \to \text{Spec}(A)$$

in the $\infty$-category of simplicial objects of $\text{Alg}_{A,\text{op}}^{\text{ani}}$ we obtain an object

$$B^{\text{ani}}_* \to \text{Spec}(A),$$

which formally is a bisimplicial object in $\text{Alg}_{A,\text{op}}^{\text{ani}}$ though it most naturally should be viewed as a simplicial derived scheme.

**Theorem 8.6.** Let $A$ be a ring and let $a_1, \ldots, a_r \in A$ be elements defining a derived blowup $\text{Bl}^{\text{ani}} \to \text{Spec}(A)$ as above. Then for an object $\mathcal{F} \in \mathcal{D}^f(\text{Spec}(A))$ the natural map

$$R\Gamma(A, \mathcal{F}) \to R\lim_n R\Gamma(B^{\text{ani}}_n, \mathcal{F})$$

is an equivalence, and similarly when $\mathcal{F}$ is replaced by the completion $\widehat{\mathcal{F}}$ of $\mathcal{F}$ with respect to the closed subscheme $Z \hookrightarrow \text{Spec}(A)$ defined by $(a_1, \ldots, a_r)$.

**Proof.** It suffices to consider the two cases when $\mathcal{F}$ is given by a perfect complex of $A$-modules or of the form $i_*^{\text{ani}} G_Y$ for $i : Y \hookrightarrow \text{Spec}(A)$ a closed subscheme. Replacing $\text{Spec}(A)$ by $Y$ in the latter case we see that it suffices to consider the two cases when $\mathcal{F}$ is given by a perfect complex or by a finite flat group scheme $G$ over $A$.

Let $Z \hookrightarrow \text{Spec}(A)$ be the closed subscheme defined by $I$ and let $U \subset \text{Spec}(A)$ be the complement. Define $\widehat{\mathcal{F}}$ and $\mathcal{S}(\mathcal{F})$ with respect to the closed subscheme $Z$.

The proof of [77] and in particular the diagram (7.8.1), shows that the cohomology of $\mathcal{S}(\mathcal{F})$ can be described purely in terms of $U$, and therefore we have that the map

$$R\Gamma(A, \mathcal{S}(\mathcal{F})) \to R\lim_n R\Gamma(B_n, \mathcal{S}(\mathcal{F}))$$

is an equivalence. Using the distinguished triangle

$$\mathcal{F}[-1] \to \mathcal{S}(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}$$
we see that it suffices to show the statement for \( \hat{\mathcal{F}} \). This reduces the statement to the case when \( I = (a_1, \ldots, a_r) \subset A \) is nilpotent.

This case we prove by induction on \( n \) such that \( I^n = 0 \).

For the base case \( n = 1 \), observe that the morphism \( B_1 \to Z \) is an fppf covering (in fact, given by a projective bundle) by \([27, 2.2.4]\), which implies the result in this case.

For the inductive step let \( A' \) denote \( A/I^{n-1} \) and let \( A_0 \) denote \( A/I \) so that we can view \( J := I^{n-1}/I^n \) as an \( A_0 \)-module. Denote by \( B_0^{\text{ani}} \) (resp. \( B_0^{\text{ani}} \)) the base change of \( B_0^{\text{ani}} \) to \( A' \) (resp. \( A_0 \)). In the case when \( \mathcal{F} \) is given by a finite flat group scheme \( G \) the inductive step then follows from \([14, 5.2.8]\) which implies that we have a commutative diagram of fiber sequences

\[
\begin{array}{ccc}
R\Gamma(A, G) & \longrightarrow & R\Gamma(A', G) \\
\downarrow & & \downarrow \\
R\Gamma(B_0^{\text{ani}}, G) & \longrightarrow & R\Gamma(B_0^{\text{ani}}, G) \\
\downarrow & & \downarrow \\
& R\text{Hom}_{A_0}(e^*L_{G_0/A_0}, J)
\end{array}
\]

and in the case when \( \mathcal{F} \) is given by a perfect complex \( E \) the result follows from consideration of the commutative diagram of fiber sequences

\[
\begin{array}{ccc}
R\Gamma(A_0, E_0 \otimes J) & \longrightarrow & R\Gamma(A, E) \\
\downarrow & & \downarrow \\
R\Gamma(B_0^{\text{ani}}, E_0 \otimes J) & \longrightarrow & R\Gamma(B_0^{\text{ani}}, E) \\
\downarrow & & \downarrow \\
& R\Gamma(B_0^{\text{ani}}, E'),
\end{array}
\]

where \( E_0 \) (resp. \( E' \)) denotes the pullback of \( E \) to \( A_0 \) (resp. \( A' \)).

\[\bigtriangleup\]

8.7. We will also consider a more global version of the derived blowup construction. Let \( X \) be a scheme and let \( \{a_i : \mathcal{L}_i \to \mathcal{O}_X\}_{i=1} \) be a collection of lines bundles with \( \mathcal{O}_X \)-linear maps to \( \mathcal{O}_X \). Giving such a collection of line bundles is equivalent to giving a map

\[X \to [A^r/G_m] \cong [A^r/G_m] .\]

Since the origin in \( A^r \) is \( G_m \)-equivariant the \( G_m \)-action on \( A^r \) lifts to \( \text{Bl} \) and we can consider the induced diagram

\[(8.7.1)\]

\[
\begin{array}{ccc}
\text{[Bl}/G_m] & \longrightarrow & [\mathcal{A}^r/\mathcal{G}_m]
\end{array}
\]

The fiber product (in the sense of derived algebraic geometry) \( \text{Bl}^{\text{ani}}_X \to X \) is then the derived blowup of \( X \), which locally (when the line bundles are trivialized) is given by the preceding construction.

Example 8.8. Let \( A \) be a ring and \( a_1, \ldots, a_r \in A \) elements defining an ideal \( I \subset A \). In general it is not the case that the underlying scheme \( \pi_0(\text{Bl}^{\text{ani}}_I) \) of the derived blowup \( \text{Bl}^{\text{ani}}_I \to \text{Spec}(A) \) is equal to the blowup of \( \text{Spec}(A) \) along \( I \). For an explicit example consider \( A = k[t], r = 2, \) and \( a_1 = a_2 = t \). In this case the classical blowup of \( I \) is just \( \text{Spec}(A) \) again, but \( \pi_0(\text{Bl}^{\text{ani}}_I) \) has an extra component. In general there is a closed immersion

\[
\text{Bl}_I(A) \hookrightarrow \pi_0(\text{Bl}^{\text{ani}}_I)
\]
which is an isomorphism away from the closed subscheme of Spec(A) defined by $I$ [27, 4.1.11].

8.9. Consider now a pair $(X, Z)$ consisting of a scheme $X$ with a closed subscheme $Z \subset X$, and let 
$$\pi^{\text{ani}} : B^{\text{ani}} \to X$$
be the derived blowup associated to some collection \{ $a_i : \mathcal{L}_i \to \mathcal{O}_X$ \} defining $Z \subset X$, and let $\pi : B \to X$ be the classical blowup of $Z$ in $X$.

**Theorem 8.10.** Let $\mathcal{F} \in \mathcal{D}^f(X)$ be an object. Then the natural map

$$R\Gamma(X, \mathcal{S}_{(X,Z)}(\mathcal{F})) \to R\Gamma(B, \mathcal{S}_{(B,\pi^{-1}(Z))}(\mathcal{F}|_B))$$

is an isomorphism.

**Proof.** It suffices to verify the theorem after replacing $X$ by an fppf cover. We may therefore assume that $X = \text{Spec}(R)$ is affine and that the animated blowup is defined by an ideal $(a_1, \ldots, a_r) \subset R$. Furthermore, it suffices to consider the two cases when $\mathcal{F}$ is the underlying complex of abelian groups of perfect complex of $R$-modules and the case when $\mathcal{F} = i^*_{\text{ani}} G_Y$ for a finite flat group scheme $G_Y$ over a closed subscheme $i : Y \hookrightarrow X$.

The case when $\mathcal{F}$ is defined by a perfect complex follows from the theorem on formal functions (see also [24, Proof of Proposition 4.1]).

For the case when $\mathcal{F} = i^*_{\text{ani}} G_Y$ for a finite flat abelian group over $i : Y \hookrightarrow X$ proceed as follows. Consider the simplicial derived scheme

$$\pi_{\bullet}^{\text{ani}} : B_{\bullet}^{\text{ani}} \to X$$

obtained by taking coskeleton of $\pi^{\text{ani}}$. For each $n$ we have the unique surjection $[n] \to [0]$ which induces a map

$$B_0^{\text{ani}} \to B_n^{\text{ani}}.$$

Let

$$i_n : B \to B_n^{\text{ani}}$$

be the composition of this map with the inclusion

$$B \hookrightarrow \pi_0(B_0^{\text{ani}}) \hookrightarrow B_0^{\text{ani}}.$$

We show that for all $n$ the induced map

\[(8.10.1) \quad R\Gamma(B_n^{\text{ani}}, \mathcal{S}(i_{\text{ani}}^* G_Y)) \to R\Gamma(B, \mathcal{S}(i_{\text{ani}}^* G_Y))\]

is an isomorphism. This will prove the theorem for $\mathcal{F} = i_{\text{ani}}^* G_Y$, and therefore also complete the proof of the theorem in general, in light of 8.6.

For this we consider the truncations

$$\pi_0(B_n^{\text{ani}}) \hookrightarrow \tau_{\leq 1}(B_n^{\text{ani}}) \hookrightarrow \tau_{\leq 2}(B_n^{\text{ani}}) \hookrightarrow \cdots$$

defined as in [14, 5.1.4]. Note that we have

$$B_n^{\text{ani}} = \tau_{\leq s} B_n^{\text{ani}}$$

for $s \gg 0$. If $U \subset X$ is the complement of $Z$, then the canonical lift $U \hookrightarrow \pi_0(B_n^{\text{ani}})$ factors through $B$ and the complement of $U$ in $\pi_0(B_n^{\text{ani}})$ is the preimage of $Z$. It follows from this and [7,7] that the restriction map

$$R\Gamma(\pi_0(B_n^{\text{ani}}), \mathcal{S}(i_{\text{ani}}^* G_Y)) \to R\Gamma(B, \mathcal{S}(i_{\text{ani}}^* G_Y))$$
is an isomorphism. To prove that (8.10.1) is an isomorphism it therefore suffices to show that for all \( n \) the restriction map
\[
R\Gamma(\tau_{\leq s}(B_n^{ani}), \mathcal{S}(i_s^{ani}G_Y)) \rightarrow R\Gamma(\tau_{\leq s-1}(B_n^{ani}), \mathcal{S}(i_s^{ani}G_Y))
\]
is an isomorphism. By [14, 5.2.8], if \( M \) denotes \( \pi_s(B_n^{ani})[s] \) then the cone of the restriction map (8.10.2) is given by the complex
\[
R\text{Hom}_{R_Y}(e^*L_{GY}, M \otimes L_{ROZ}^r) \rightarrow \lim R\text{Hom}_{R_Y}(e^*L_{GY}, M \otimes L_{ROZ}^r),
\]
where \( Z_r \hookrightarrow X \) denotes the \( r \)-th infinitesimal neighborhood of \( Z \) in \( X = \text{Spec}(R) \) and \( R_Y \) denotes the coordinate ring of \( Y \). Now observe that for \( s > 0 \) the module \( M \) is annihilated by some power of the ideal of \( Z \) in \( X \), since \( M \) is a coherent sheaf supported on preimage of \( Z \). From this it follows that \( M \cong R\lim R \otimes L_{ROZ}^r \) and that (8.10.3) is an isomorphism. \( \Box \)

**Corollary 8.11.** Let \( f : X \rightarrow S \) be a projective morphism with \( S \) reduced and let \( d \) be the generic dimension of the fibers of \( f \), let \( \pi : X' \rightarrow X \) be a blowup (in the ordinary sense), and let \( Z \subset X \) be a closed subscheme with preimage \( Z' \subset X' \). Assume that we have established (8.11) for morphisms of generic dimension \( \leq d - 1 \), and let \( f' : X' \rightarrow S \) be the composition of \( f \) with \( \pi \). Then for \( \mathcal{F} \in \mathcal{D}^f(X) \) there exists a dense open subset \( U \subset S \) such that \( Rf_{i!}^{(X,Z)}(\mathcal{F})|_U \in \mathcal{D}_f(U) \) if and only if there exists a dense open subset \( U' \subset S \) such that \( Rf_{i'}^{(X',Z')}((\mathcal{F}')|_{U'} \in \mathcal{D}_f(U') \), where \( \mathcal{F}' \) denotes the restriction of \( \mathcal{F} \) to \( X' \).

**Proof.** Shrinking on \( S \) we may assume that \( f \) and \( f' \) are flat, and that \( S \) is affine.

Let \( I \subset \mathcal{O}_X \) be the ideal defining the blowup \( \pi \). If \( \mathcal{L} \) is an ample invertible sheaf on \( X \) then we can find a surjection
\[
(\mathcal{L} \otimes \mathcal{O}_X^n)^{\oplus r} \rightarrow I
\]
for \( n \) sufficiently big and some \( r \). Associated to this choice of surjection is a derived blowup
\[
\text{Bl}_{X}^{ani} \rightarrow X
\]
whose underlying scheme contains \( X' \). In a diagram:
\[
\begin{array}{ccc}
X' & \xrightarrow{i} & \text{Bl}_{X}^{ani} \\
\downarrow{\pi} & & \downarrow{\pi^{ani}} \\
X & &
\end{array}
\]

In other words, we may assume that \( \pi \) is the classical blowup underlying a derived blowup and, applying (8.3), we may further assume that \( Z \) is the center of this blowup, in which case the result follows from (8.10). \( \Box \)

8.12. **Finite flat group schemes and blowups.**

8.13. Let \( f : X \rightarrow S \) be a morphism of finite type separated integral \( k \)-schemes and let \( G/X \) be a finite flat group scheme.
Proposition 8.14. There exists a commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow^{f'} & & \downarrow^{f} \\
S' & \longrightarrow & S,
\end{array}
\]

with the following properties:

(i) \( g : S' \to S \) is a dominant quasi-finite morphism of schemes with \( S' \) regular.

(ii) \( f' : X' \to S' \) admits a factorization

\[
X' \xrightarrow{d} X'' \xrightarrow{c} S',
\]

where \( c \) is smooth and quasi-projective and \( d \) is a blowup.

(iii) The map

\[
X' \to X \times_S S'
\]

can be factored as

\[
X' \xrightarrow{a} Y \xrightarrow{b} X \times_S S',
\]

where the map \( a \) is a finite flat surjection and \( b \) is a blowup of a nowhere dense closed subscheme \( Z \subset X \times_S S' \).

(iv) The pullback \( G' \) of \( G \) to \( X' \) admits a filtration

\[
G_0 \subset G_1 \subset \cdots \subset G_s = G'
\]

where each \( G_i \) is a finite flat group scheme and for each successive quotients \( G_i/G_{i+1} \) there exists a dense open subscheme of \( X' \) over which \( G_i/G_{i+1} \) is isomorphic to a group scheme of the form \( \mathbb{Z}/(n) \), \( \alpha_p \), or \( \mu_p \).

Proof. Let \( K \) denote the function field of \( X \), let \( K \hookrightarrow \overline{K} \) be an algebraic closure, and let \( G_{\overline{K}} \) be the pullback of \( G \) to \( \overline{K} \). The simple objects in the category of abelian finite flat group schemes over \( \overline{K} \) are of the form \( \mathbb{Z}/(\ell) \) (\( \ell \) a prime possibly equal to \( p \)), \( \alpha_p \), and \( \mu_p \). We therefore have a filtration

\[
G_{0,\overline{K}} \subset G_{1,\overline{K}} \subset \cdots \subset G_{s,\overline{K}} \subset G_{\overline{K}}
\]

with \( G_{i,\overline{K}}/G_{i-1,\overline{K}} \) isomorphic to \( \mathbb{Z}/(\ell) \), \( \alpha_p \), or \( \mu_p \) for each \( i \). By a standard limit argument this filtration descends to \( G_L \) for a finite extension \( L/K \). Let \( W \to X \) be the normalization of \( X \) in \( L \). Let

\[
G_{i,W} \subset G_W
\]

be the scheme-theoretic closure of \( G_{i,L} \) in \( G_W \). By Raynaud-Gruson [36, I, 5.2.2] there exists a blowup \( W' \to W \) such that the strict transform \( G_{i,W'} \) of \( G_{i,W} \) is a subgroup scheme of \( G_W \), flat over \( W' \). Now using de Jong’s theory of alterations we can find a quasi-finite dominant morphism \( S' \to S \) and a smooth \( S' \)-scheme \( X'' \) together with a proper generically finite surjective morphism

\[
X'' \to W'_{gr}.
\]

Applying another blowup \( Y \to X \) and setting \( X' \) equal to the strict transform of \( X'' \) we then obtain the desired diagram (8.14.1).
8.15. **Proof of 7.11.** We will reduce the proof to the cases when $\mathcal{F}$ is given by a perfect complex, or one of the groups $\mathbb{Z}/(n)$ ($n$ an integer), $\mu_p$, or $\alpha_p$.

8.16. Note first of all that we know that 7.11 holds in the following cases:

(i) $f$ is proper, $Z = \emptyset$, and $\mathcal{F}$ is given by one of $\mathbb{Z}/n\mathbb{Z}$ or $\alpha_p$. Indeed these two cases follow from the results cited in 1.21.

(ii) $f$ is smooth and projective, $Z = \emptyset$, and $\mathcal{F}$ is given by $\mu_p$. This case follows from 4.15.

8.17. Now we turn to the proof in the general case.

Without loss of generality we may assume that $S$ is integral. If we denote by $d$ the dimension of the generic fiber of $f$ we may also, by induction, assume that 7.11 holds for morphisms of dimension $< d$.

By 8.3 it further suffices to consider the case when $Z = \emptyset$, in which case we write $Rf_*(\mathcal{F})$ in place of $Rf_!(\mathcal{F})$.

8.18. Using our inductive hypothesis, we can extrapolate from the special cases in 8.16 to the following. Suppose $f : X \to S$ is a projective flat morphism of schemes with $S$ integral and that $\mathcal{F}$ is given by a finite flat group scheme $G$ over $X$ which admits a finite filtration whose successive quotients are one of $\mathbb{Z}/n\mathbb{Z}$, $\alpha_p$, and $\mu_p$. Then we claim that $Rf_*\mathcal{F}|_U \in \mathcal{D}_I(U)$ for a dense open subset $U \subset S$.

For this, note first of all that by filtering $G$ it suffices to consider $G$ one of the groups $\mathbb{Z}/n\mathbb{Z}$, $\alpha_p$, or $\mu_p$. Furthermore, we have already shown the result in the case of the first two of these group schemes, so we just consider $G = \mu_p$.

By [15, 7.3] we can, after making a quasi-finite dominant base change $S' \to S$, find a proper dominant generically finite morphism $\pi : Y \to X$ with $Y/S$ smooth and projective, and a finite group $\mathcal{G}$ acting on $Y$ over $X$ such that the field extension $k(X) \to k(Y)^\mathcal{G}$ is purely inseparable. Let $W$ be the integral closure of $X$ in $k(Y)^\mathcal{G}$ so we have a factorization

$$Y \xrightarrow{\alpha} W \xrightarrow{\beta} X.$$ 

By [20, I.5.2.2] there exists a blowup $W' \to W$ such that the strict transform $\alpha' : Y' \to W'$ of $\alpha$ is finite and flat. Applying loc. cit. again to $Y' \to X$ and $W' \to X$ we can find a blowup $X' \to X$ such that in the induced diagram of strict transforms

$$Y' \xrightarrow{\alpha'} W' \xrightarrow{\beta'} X',$$ 

the morphisms $\alpha'$ and $\beta'$ are finite and flat. Furthermore, the group $\mathcal{G}$ acts on $Y'$ over $W'$ such that $Y'/\mathcal{G} = W'$. For $n \geq 0$ set

$$Y'_n := Y' \times_{X'} Y' \times_{X'} \cdots \times_{X'} Y'.$$
and consider the morphism
\[ z_n : \prod_{(g_1, \ldots, g_n) \in G^n} Y' \to Y''_n \]
given by taking the coproduct of the maps
\[ Y' \to Y''_n, \quad w \mapsto (w, g_1 w, \ldots, g_n w). \]

Since \( \mathcal{G} \) acts on \( Y' \) over \( W' \) the map \( z_n \) admits a factorization through a map
\[ \tilde{z}_n : \prod_{(g_1, \ldots, g_n) \in G^n} Y' \to \tilde{Y}_n := \underbrace{Y' \times_{W'} Y' \times_{W'} \cdots \times_{W'} Y'}_{n+1}, \]
which is an isomorphism over a dense open subscheme of the target. Furthermore, there is a cartesian diagram
\[ \begin{array}{ccc}
\tilde{Y}_n & \longrightarrow & Y''_n \\
\downarrow & & \downarrow \\
W' & \delta & W''_n,
\end{array} \]
where
\[ W''_n := \underbrace{W' \times_{X'} W' \times_{X'} \cdots \times_{X'} W'}_{n+1}, \]
which implies that the map \( \tilde{Y}_n \to Y''_n \) is a closed immersion defined by a nilpotent ideal \( \mathcal{J} \subset \mathcal{O}_{Y''_n} \).

From this we conclude that \( 7.11 \) holds for \( \mu_p \) on \( Y''_n \). Indeed by \( 4.15 \) and \( 6.3 \) we know the theorem for \( Y \), which implies that it also holds for \( \prod_{g} Y' \) by \( 8.11 \). Using \( 8.11 \) again we conclude from this that the theorem holds for \( \tilde{Y}_n \) and this implies the theorem holds for \( Y''_n \) by \( [13 \ 5.1.13] \). Letting \( Y_n \) denote the \((n+1)\)-fold fiber product of \( Y \) over \( X \) we then also conclude that \( 7.11 \) holds for \( Y_n \) (applying \( 8.11 \) once more to \( Y''_n \to Y_n \)).

From this we then see that \( 7.11 \) also holds for \( f : X \to S \) and \( \mu_p \). Indeed the object \( Rf^\text{an}_* \mu_p \) is animated by \( 3.15 \) and we have a spectral sequence relating \( R^i f_* (\mu_p) \) to the cohomology sheaves \( R^j f_{n*} \mu_p \), where \( f_n : Y_n \to S \) is the structure morphism, which implies that the assumptions of \( 6.3 \) hold.

8.19. Returning now to the proof of \( 7.11 \) in the general case.

Theorem \( 7.11 \) for perfect complexes follows from \( [38 \ Tag 08EV] \). For the remainder of the proof we therefore assume \( \mathcal{F} \) is given by a finite flat group scheme \( G \) over \( X \). Replacing \( S \) by a dominant generically flat cover and applying \( 8.14 \) we may further assume that we have a commutative diagram
\[ (8.19.1) \]
\[ \begin{array}{ccc}
X' & \longrightarrow & Y \\
\downarrow^d & & \downarrow^b \\
X'',
\end{array} \]
where \( b \) and \( d \) are blowups, \( X'' \) is smooth over \( S \), \( a \) is finite and flat, and the pullback of \( G \) to \( X' \) admits a filtration as in 8.14 (iv). By 8.11 theorem 7.11 holds for \( (X, \mathcal{F}) \) if and only if it holds for \( (Y, \mathcal{F}|_Y) \), so we may further assume that \( Y = X \).

8.20. Next note that using the filtration in 8.14 (iv) we obtain that 7.11 holds for \( (X', \mathcal{F}|_{X'}) \) if and only if it holds for \( (X', H) \), for \( H \) one of the group schemes \( \mathbb{Z}/(n) \), \( \alpha \alpha \), or \( \mu \mu \), and the same is true for the fiber products

\[
X'_n = X' \times_X X' \times \cdots \times_X X'.
\]

By 8.18 and the argument at the end of that paragraph and the same argument used in 8.18 we conclude that that 7.11 also holds for \( (X, \mathcal{F}) \).

9. Finiteness of cohomology

**Theorem 9.1.** With notation and assumptions as in 7.11 if \( Z = \emptyset \) then there exists a dense open subset \( U \subset S \) such that \( Rf_*\mathcal{F} \in D^f(U) \).

The proof occupies the remainder of this section.

9.2. It suffices to consider the two cases when \( \mathcal{F} \) is a perfect complex and when \( \mathcal{F} \) is given by a finite flat group scheme over \( X \). The first of these cases follows from [38, Tag 08EV]. So for the remainder of the proof we consider the case when \( \mathcal{F} \) is given by a finite flat group scheme \( G \) over \( X \).

9.3. We will prove 9.1 in this case by reduction to the case of a smooth projective scheme and \( G \) one of \( \mu \mu \), \( \alpha \alpha \), or \( \mathbb{Z}/n\mathbb{Z} \), following the argument proving 7.11.

For this, it is useful to first collect some results about derived blowups in a slightly more general context.

Consider a finite type morphism \( f : X \to S \) and a collection of line bundles and maps \( \{a_i : \mathcal{L}_i \to \mathcal{O}_X\}_{i=1}^r \) as in 8.7 with associated derived blowup

\[
b : X^{ani} \to X.
\]

Recall that \( X^{ani} \) is obtained as the derived fiber product of the diagram (8.7.1). Let \( Z^{ani} \to X \) be the derived fiber product of the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X^{ani} \\
\downarrow & & \downarrow \\
BG^r_m & \hookrightarrow & \mathbb{A}^r//G^r_m,
\end{array}
\]

and let \( E^{ani} \to X^{ani} \) be the derived fiber product of the diagram

\[
\begin{array}{ccc}
Z^{ani} & \longrightarrow & X^{ani} \\
\downarrow & & \downarrow \\
X^{ani} & \longrightarrow & X.
\end{array}
\]
Lemma 9.4. The induced diagram in $\mathcal{D}(S)$

\[
\begin{align*}
RF^\text{ani}_s G & \longrightarrow RF^\text{ani}_Z G|_{Z^\text{ani}} \\
\downarrow & \downarrow \\
RF^\text{ani}_s G|_{X^\text{ani}} & \longrightarrow RF^\text{ani}_E G|_{E^\text{ani}}
\end{align*}
\]

is a homotopy pushout diagram, where $f_Z : Z^\text{ani} \rightarrow S$, $f_E : E^\text{ani} \rightarrow S$, and $f : X^\text{ani} \rightarrow S$ are the structure morphisms.

Proof. The assertion is local on $X$ so it suffices to consider the case when $X = \text{Spec}(A)$ is affine and the line bundles $\mathcal{L}_i$ are trivialized (so we can view the $a_i$ as elements of $A$). In this case the morphism $X \rightarrow [A^r/G_m]$ factors through a morphism

\[\text{Spec}(A) \rightarrow A^r,\]

and $Z^\text{ani}$ is given by the derived tensor product (in the sense of animated rings)

\[A^\text{ani}_Z := A \otimes_k^L \mathcal{L}_{X_1, \ldots, X_r, X_i \rightarrow 0} k.\]

Furthermore, if

\[\text{Spec}(B_\bullet) \rightarrow \text{Bl}\]

is an étale hypercover of the blowup of $A^r$ at the origin, then $X^\text{ani}$ is described by the cosimplicial animated ring

\[A'_\bullet := B_\bullet \otimes_k^L \mathcal{L}_{X_1, \ldots, X_r} A,\]

and if $B_\bullet \rightarrow C_\bullet$ denotes the quotient given by the exceptional divisor then $E^\text{ani}$ is given by

\[D^\text{ani}_\bullet := C_\bullet \otimes_k^L \mathcal{L}_{X_1, \ldots, X_r} A.\]

Now observe that the diagram

\[
\begin{align*}
k[X_1, \ldots, X_r] & \xrightarrow{X_i \rightarrow 0} k \\
\downarrow & \downarrow \\
B_\bullet & \longrightarrow C_\bullet
\end{align*}
\]

is homotopy cartesian and therefore the same is true for the diagram

\[
\begin{align*}
A & \longrightarrow A^\text{ani}_Z \\
\downarrow & \downarrow \\
A'_\bullet & \longrightarrow D^\text{ani}_\bullet.
\end{align*}
\]

Indeed the sequence of $k[X_1, \ldots, X_r]$-modules

\[k[X_1, \ldots, X_r] \rightarrow B_\bullet \oplus k \rightarrow C_\bullet \rightarrow k[X_1, \ldots, X_r][1]\]

is a distinguished triangle which, open derived tensor product with $A$, induces a distinguished triangle

\[A \rightarrow A^\text{ani}_Z \oplus A'_\bullet \rightarrow D^\text{ani}_\bullet \rightarrow A[1].\]
It follows from this that for an affine flat group scheme $G$ over $A$ the induced diagram

$$
tau_{\leq 0}R\Gamma(A, G) \longrightarrow tau_{\leq 0}R\Gamma(A_{\text{ani}}^+, G) \\
\downarrow \quad \quad \downarrow \\
tau_{\leq 0}R\Gamma(A', G) \longrightarrow \tau_{\leq 0}R\Gamma(D_{\text{ani}}^+, G)
$$

is homotopy cartesian. Taking the homotopy limit over étale hypercovers of $A$ and $D_{\text{ani}}^+$ and applying [14, 5.2.9] we obtain that for a smooth $G$ the diagram (9.4.1) for $G$ is a homotopy pushout diagram. The case of a finite flat group scheme then follows from this using the Bégneri resolution [7, 2.2.1].

We will use this in conjunction with the following lemmas.

**Lemma 9.5.** Let $f : X \rightarrow S$ be a proper morphism with $X$ and $S$ reduced and let $G_X$ be a finite flat group scheme over $X$. Let

$$X = X_1 \cup X_2$$

be a decomposition of $X$ into the union of two reduced closed subschemes such that there exists an open subset $U \subset X$ for which the intersections

$$U \cap X_i \hookrightarrow X_i$$

are dense. Let

$$a_i : X_i \hookrightarrow X, \quad i = 1, 2,$$

be the inclusion, let $a_Z : Z \subset X$ be the intersection $X_1 \cap X_2$, with the reduced structure, and let $G_{X_1}$ (resp. $G_Z$) be the restriction of $G_X$ to $X_i$ (resp. $Z$). If [9.1] holds for $(X_1, G_{X_1})$ and $(Z, G_Z)$, then [9.1] also holds for $(X, G_X)$ if and only if it holds for $(X_2, G_{X_2})$.

**Proof.** Let

$$\rho : G_X \rightarrow a_{1*}^\text{ani} G_{X_1} \oplus a_{2*}^\text{ani} G_{X_2}$$

be the map induced by adjunction, and let

$$\hat{\rho} : \hat{G}_X \rightarrow a_{1*}^\text{ani} \hat{G}_{X_1} \oplus a_{2*}^\text{ani} \hat{G}_{X_2}$$

be the induced map on completions with respect to $Z$. By [7.1] we have

$$\text{Cone}(\rho) \simeq \text{Cone}(\hat{\rho}).$$

We show that the cone of the natural map

$$\text{Cone}(\hat{\rho}) \rightarrow a_{Z*}^\text{ani} G_Z$$

lies in the triangulated subcategory of $D(X)$ generated by complexes bounded complexes of coherent sheaves. This will suffice to prove the lemma.

Consider the Bégneri resolution [7, 2.2.1]

$$0 \rightarrow G_X \rightarrow R_X \rightarrow \overline{R}_X \rightarrow 0.$$
and write $R_X$ (resp. $R_Z$) for the restriction of $R_X$ to $X_1$ (resp. $Z$), and similarly for $\overline{R}_X$. We then have a commutative diagram

$$
\begin{array}{c}
G_X \\
\downarrow \rho \\
a^\text{ani}_{1*}G_{X_1} \oplus a^\text{ani}_{2*}G_{X_2} \\
\downarrow a^\text{ani}_{1*}R_{X_1} \oplus a^\text{ani}_{2*}R_{X_2} \\
\downarrow a^\text{ani}_{1*}\overline{R}_{X_1} \oplus a^\text{ani}_{2*}\overline{R}_{X_2},
\end{array}
$$

which identifies $\text{Cone}(\hat{\rho})$ with the cocone of the induced map

$$
\text{Cone}(\hat{\rho}') \to \text{Cone}(\hat{\rho}'').
$$

To prove our claim that the cocone of the map $\text{Cone}(\hat{\rho}) \to a^\text{ani}_{Z*}G_Z$ lies in the triangulated subcategory of $D(X)$ generated by bounded complexes of coherent sheaves it suffices to prove that this is so for the cocones of the maps

$$
\text{Cone}(\hat{\rho}') \to a^\text{ani}_{Z*}R_Z \text{ and } \text{Cone}(\hat{\rho}'') \to a^\text{ani}_{Z*}\overline{R}_Z.
$$

We give the argument for $\text{Cone}(\hat{\rho}')$; the argument for $\text{Cone}(\hat{\rho}'')$ is exactly analogous.

Let $Q$ be the cokernel of the inclusion (note that $X$ is reduced)

$$
\mathcal{O}_X \to a_{1*}\mathcal{O}_{X_1} \oplus a_{2*}\mathcal{O}_X,
$$

and let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal defining $Z$. Let $X_n$ (resp. $X_{j,n}$) denote the closed subscheme of $X$ (resp. $X_j$) defined by $\mathcal{J}^n$. Since $Q$ is coherent and supported on $Z$ there exists an integer $N > 0$ such that $\mathcal{J}^NQ = 0$.

Let $\text{Spec}(A) \to X$ be a flat morphism, write $\text{Spec}(A) \times_X X_j \simeq \text{Spec}(A_j)$, let $A_n$ (resp. $A_{j,n}$) denote the reduction of $A$ (resp. $A_j$) modulo $\mathcal{J}^n$, and let $\hat{A}$ (resp. $\hat{A}_j$) denote the $\mathcal{J}$-adic completion of $A$ (resp. $A_j$). For $n > 0$ we then have a commutative diagram

$$
\begin{array}{c}
R_X(\hat{A}) \\
\downarrow \\
R_X(A_n) \\
\downarrow \\
R_X(\hat{A}_1) \oplus R_X(\hat{A}_2)
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
R_X(A_{1,n}) \oplus R_X(A_{2,n}),
\end{array}
$$

where the vertical morphisms are surjective since $R_X$ is smooth. For $n \geq N$ this diagram is, in fact, cartesian. Indeed considering the exact sequence

$$
0 \to A \to A_1 \oplus A_2 \to Q \otimes_{\mathcal{O}_X} A \to 0
$$

we see that a pair

$$
(\alpha_1, \alpha_2) \in R_X(\hat{A}_1) \oplus R_X(\hat{A}_2)
$$

lies in $R_X(\hat{A})$ if and only if the composition

$$
\mathcal{O}_{R_X,A} \stackrel{(\alpha_1, \alpha_2)}{\longrightarrow} \hat{A}_1 \times \hat{A}_2 \longrightarrow Q \otimes_{\mathcal{O}_X} A
$$

is zero. Now since $\mathcal{J}^NQ = 0$ this map factors through $A_{1,n} \times A_{2,n}$, and therefore vanishes if the image of $(\alpha_1, \alpha_2)$ in $R_X(A_{1,n}) \oplus R_X(A_{2,n})$ is in the image of $R_X(A_n)$. We conclude that $\text{Cone}(\hat{\rho}')$ is isomorphic to the cone of the map

$$
a^\text{ani}_{Z,n}R_Z \to a^\text{ani}_{X_{1,n}}R_{X_{1,n}} \oplus a^\text{ani}_{X_{2,n}}R_{X_{2,n}}.
$$
To conclude the proof of the lemma it then suffices to note that the cones of the maps

\[ a^\text{ani}_{Z,N}^* R_Z \to a^\text{ani}_Z^* R_Z, \quad a^\text{ani}_{X,N}^* R_{X,N} \to a^\text{ani}_{Z,N}^* R_Z \]

admit finite filtrations whose successive quotients are given by bounded complexes of coherent sheaves, as follows from [14 5.2.8].

Lemma 9.6. Let \( f : X \to S \) be a proper morphism with \( S \) reduced and let

\[ \rho : G \to G' \]

be a morphism of finite flat group schemes over \( X \) and assume that \( \rho \) is an isomorphism over the complement of a nowhere dense closed subscheme \( Z \subset X \). Let \( \hat{\rho} : \hat{G} \to \hat{G}' \) be the induced map on complete objects. Then there exists a dense open subscheme \( U \subset S \) over which

\[ Rf_*(\text{Cone}(\hat{\rho})) \]

lies in \( D^f(U) \).

Proof. This is very similar to the proof of [9.5].

We have a commutative diagram of Bégueri resolutions [7 2.2.1]

\[
\begin{array}{cccccc}
0 & \rightarrow & G & \rightarrow & R & \rightarrow \hat{R} & \rightarrow 0 \\
| & & \rho | & & \rho' | & & \rho'' | & & 0 \\
0 & \rightarrow & G' & \rightarrow & R' & \rightarrow \hat{R}' & \rightarrow 0,
\end{array}
\]

where the vertical morphisms are isomorphisms over the complement of \( Z \).

For an integer \( n > 0 \) define \( T_n(\rho) \) to be the cone of the map of complexes

\[ (\hat{G} \xrightarrow{\rho} \hat{G}') \to (\hat{i}_{\text{ns}} G_n \xrightarrow{\rho_n} \hat{i}_{\text{ns}} G'_n), \]

and define \( T_n(\rho') \) and \( T_n(\rho'') \) similarly. We then have a distinguished triangle

\[ T_n(\rho) \to T_n(\rho') \to T_n(\rho'') \to T_n(\rho)[1]. \]

We claim that for \( n \) sufficiently big we have \( T_n(\rho) \in D^f(X) \). To show this it suffices to show that \( T_n(\rho') \) and \( T_n(\rho'') \) are in \( D^f(X) \) for sufficiently large \( n \). We give the argument for \( T_n(\rho') \) (the argument for \( T_n(\rho'') \) is exactly analogous).

Consider the map on coordinate rings

\[ \mathcal{O}_{G'} \rightarrow \mathcal{O}_G. \]

Since \( X \) is reduced, this map is injective. Fix a coherent subsheaf \( \mathcal{M} \subset \mathcal{O}_G \) which generates \( \mathcal{O}_G \) as a \( \mathcal{O}_X \)-algebra and let \( \mathcal{M}' \) denote \( \mathcal{O}_{G'} \cap \mathcal{M} \). After possibly enlarging \( \mathcal{M} \) we may assume that \( \mathcal{M}' \) also generates \( \mathcal{O}_{G'} \) as a \( \mathcal{O}_X \)-algebra. Let \( Q \) denote the quotient \( \mathcal{M}/\mathcal{M}' \), so \( Q \) is a coherent sheaf supported on \( Z \), and let \( m \) be an integer such that \( J^m Q = 0 \), where \( J \subset \mathcal{O}_X \) is the ideal of \( Z \) in \( X \).

Let \( A \) be a flat \( \mathcal{O}_X \)-algebra and let \( A_n \) denote the reduction to \( X_n \). For a homomorphism

\[ a' : \mathcal{O}_{G'} \rightarrow A \]
we then get criterion for \( a' \) to extend to a homomorphism
\[
a : \mathcal{O}_R \to A.
\]
Namely, using the flatness of \( A \) over \( X \) it is necessary and sufficient that the class
\[
[a] \in \text{Ext}^1_{\mathcal{O}_X}(Q, A) \simeq \text{Ext}^1_{\mathcal{O}_X}(Q, \mathcal{O}_X)(\text{Spec}(A))
\]
obtained by pushing out the sequence
\[
0 \to \mathcal{M}' \to \mathcal{M} \to Q \to 0
\]
is 0. Now observe that there exists an integer \( n \), independent of the flat \( A \), such that the reduction map
\[
\text{Ext}^1_{\mathcal{O}_X}(Q, A) \to \text{Ext}^1_{\mathcal{O}_X}(Q, A_n)
\]
is injective. It follows that an element \( a' \in R'(A) \) is in the image of \( R(A) \) if and only if the image \( a'_n \in R'(A_n) \) is in the image of \( R(A_n) \). From this we conclude that
\[
T_n(\rho') \simeq i_{n, s}^* K_n[1],
\]
where \( K_n \) denotes the kernel of \( R_n \to R'_n \). For a flat \( \mathcal{O}_X \)-algebra \( A \) we have
\[
i_{n, s} K_n(A_n) \hookrightarrow \text{Hom}_{\mathcal{O}_X}(Q, A_n).
\]
In fact, the image of this map is contained in
\[
\text{Hom}_{\mathcal{O}_X}(Q, \mathcal{F}^{n-m} A_n).
\]
Define a filtration on \( K_n \) by setting
\[
K_n^i(A) := \text{Ker}(K_n(A) \to G(A_i)), \quad i = 1, \ldots, n.
\]
Then each successive quotient \( K_n^i / K_n^{i-1} \) is given by the kernel of an \( A_1 \)-linear map
\[
\text{Ext}^0(L_{\mathcal{R}'}, \mathcal{F}^{i-1} / \mathcal{F}^i) \otimes_{\mathcal{O}_X} A \to \text{Hom}_{\mathcal{O}_X}(Q, \mathcal{F}^{i-1} / \mathcal{F}^i) \otimes_{\mathcal{O}_X} A.
\]
From this it follows that \( Rf_* i_{n, s}^* K_n[1] \), and therefore also \( Rf_* (T_n(\rho')) \), lies in \( D^f(U) \) for some \( U \subset S \).

**Lemma 9.7.** Let \( X \) be a finite type \( k \)-scheme and let \( \{ L_i \to \mathcal{O}_X \}_{i=1}^r \) be a collection of line bundles and maps defining a derived blowup \( \pi : B^{\text{der}} \to X \). Then the underlying scheme \( \pi_0(B^{\text{der}}) \) can be written as a union
\[
(9.7.1) \quad \pi_0(B^{\text{der}}) = B \cup P_Z \cup T_1 \cup \cdots T_w,
\]
where \( Z \subset X \) is the closed subscheme defined by the ideal generated by the images of the \( L_i \), \( P_Z \) is a projective bundle over \( Z \), \( B \) is the ordinary blowup of \( X \) along \( Z \), and \( T_1, \ldots, T_w \) have dimension \( < \text{dim}(X) \).

**Proof.** We proceed by induction on \( r \). For \( r = 1 \) the result is immediate since the blowup map is an isomorphism.

So assume the result holds for derived blowups associated to fewer than \( r \) line bundles and maps. If \( U \) denotes the complement of \( Z \), then as noted before the restriction of the map \( B^{\text{der}} \to X \) to \( U \) is an isomorphism, and the closure of the resulting inclusion \( U \hookrightarrow \pi_0(B^{\text{der}}) \) is the ordinary blowup \( B \) of \( Z \). This accounts for the first factor in (9.7.1). Furthermore, it is clear from the construction of \( B^{\text{der}} \) that the restriction of \( \pi_0(B^{\text{der}}) \) to \( Z \) is isomorphic to the projective bundle \( P_Z \) associated to the vector bundle \( L_i|_Z \oplus \cdots \oplus L_r|_Z \), accounting for the
second factor. For \( i = 1, \ldots, r \) let \( X_i \subset X \) be the closed subscheme defined by the vanishing of the image of \( \mathcal{L}_i \to \mathcal{O}_X \). In local coordinates when \( X_i \) is affine and the \( \mathcal{L}_i \) are trivial we can described the restriction of \( \pi_0(B^{\text{der}}) \) to \( X_i \) as follows. Write \( X_i = \text{Spec}(R_i) \) and let \( f_j \in R_i \) be a generator for the image of \( \mathcal{L}_j \). Then

\[
\pi_0(B^{\text{der}})|_{X_i} \cong \text{Proj}_{R_i}(R_i[u_1, \ldots, u_r]/(u_i f_j, u_j f_s - u_s f_j)_{j \neq i}).
\]

The closed subscheme of this scheme defined by \( u_i = 0 \) is equal to the underlying scheme of the derived blowup of \( X_i \) associated to the collection \( \{ \mathcal{L}_j \to \mathcal{O}_X \}_{j \neq i} \), and this description shows that \( \pi_0(B^{\text{der}}) \) is the union of this derived blowup and \( \mathbb{P} \mathcal{Z} \). From this and induction we obtain the lemma. \( \square \)

9.8. Consider now the original setup of a projective flat morphism \( f : X \to S \), with \( S \) integral, and \( \mathcal{F} \) given by a finite flat group scheme \( G \) over \( X \). We prove by induction on the generic dimension \( d_X \) of \( X \) that the result holds for \( X \).

So assume that the result holds for projective morphisms of dimension \(< d_X \). We show that for a given integer \( i \) the sheaf \( R^i f_* G \) restricts over some open subset of \( S \) to an object of \( D^f(S) \). By a second induction we may further assume that this holds for \( R^j f_* G \) when \( j < i \).

Let \( X' \to X \) be a blowup along a nowhere dense closed subscheme \( Z \subset X \), and let \( f' : X' \to S \) be the structure morphism. Since \( X \) is projective over \( S \) we can extend this to a derived blowup

\[
X^{\text{fani}} \to X
\]

with \( \pi_0(X^{\text{fani}}) \) a union of \( X' \) together with \( \mathbb{P} \mathcal{Z} \), for some integer \( r \), and schemes \( T_i \) of dimension \(< \dim(X) \). By 9.4 we then find that \( R^i f_* G \) lies in \( D^f(U) \) for some dense open \( U \subset S \) if and only if \( R^i f^{\text{fani}}_* G \) has this property, where \( f^{\text{fani}} : X^{\text{fani}} \to S \) is the structure morphism. Next observe that the cone of the map

\[
R^i f^{\text{fani}}_* G \to R\pi_0(f^{\text{fani}})_* G
\]

admits a finite filtration whose successive quotients are given by the cohomology of a perfect complex \([14, 5.2.8]\). We conclude that \( R^i f_* G \) lies in \( D^f(U) \) for suitable \( U \subset S \) if and only if \( R\pi_0(f^{\text{fani}})_* G \) has this property. Combining this with 9.5 and 9.7 we conclude that \( R^i f_* G \) lies in \( D^f(U) \) for some \( U \subset S \) if and only if this is the case for \( R^i f^{\text{fani}}_* G \).

9.9. Next let us show that 9.1 holds for projective morphisms \( f : X \to S \) and \( G \) admitting a filtration with successive quotients \( \mu_p, \alpha_p \), or \( \mathbb{Z}/n\mathbb{Z} \). For this note that it suffices to show the result for \( G \) one of these three group schemes, and the cases of \( \alpha_p \) and \( \mathbb{Z}/n\mathbb{Z} \) follows from finite of coherent cohomology and étale cohomology. Thus the key case is \( G = \mu_p \). Furthermore, by 14.15 we know the result for \( \mu_p \) in the case when \( f \) is smooth. We reduce to this case following the method of 8.18.

Observe furthermore that if \( i : X_0 \to X \) is a nilpotent closed immersion then 9.1 holds for \( (X, \mu_p) \) if and only if it holds for \( (X_0, \mu_p) \). Using this, and our results about blowups already established, we find that chasing through the same argument as in 8.18 it suffices to prove the result for the schemes \( (Y'_n, \mu_p) \) (notation as in 8.18). Now using the maps \( \tilde{\gamma}_n \), our results about blowups, and \([38, \text{Tag 080P}]\) we see that the finiteness holds for \( (Y'_n, \mu_p) \) if and only if it holds for \( (Y', \mu_p) \). Since \( Y' \) is a blowup of a smooth proper \( S \)-schemes this resolves the case of \( \mu_p \), and therefore also the case of a projective morphism and \( G \) admitting a filtration as above.
9.10. Next the situation when we have a morphism
\[ \rho : G \to G' \]
of finite flat abelian group schemes over a reduced \( X \) which is an isomorphism over the complement of a nowhere dense closed subscheme \( Z \hookrightarrow X \). We then have a distinguished triangle
\[ G \to G' \to \text{Cone}(\rho) \to G[1]. \]
By 7.7 we can also write this triangle as
\[ G \to G' \to \text{Cone}(\hat{\rho}) \to G[1], \]
where \( \hat{\rho} : \hat{G} \to \hat{G}' \) is the map induced by \( \rho \), and 9.6 implies that 9.1 holds for \( G \) if and only if it holds for \( G' \) (assuming throughout that 9.1 has been proven for morphisms of smaller relative dimension).

9.11. Finally we treat the case of general \( G \) and \( f \). For this we consider again a diagram as in (8.19.1). By the discussion of blowups in 9.8 above, we see that 9.1 holds for \((X, G)\) if and only if it holds for \((Y, G)\). Also by 9.10 and the special cases discussed in 9.9, Theorem 9.1 holds for each \((X_n', G|_{X_n'})\), where \( X_n \) denotes the \((n + 1)\)-fold fiber product of \( X' \) with itself over \( Y \). Looking at the spectral sequence associated to the hypercover associated to \( x : X' \to Y \) we conclude that 9.1 also holds for \((Y, G)\).

This completes the proof of 9.1. □

10. The flat Artin-Mazur formal groups

In this section we consider in more detail the formal groups obtained from flat cohomology of a smooth proper variety \( f : X \to \text{Spec}(k) \). In particular, using our results on cohomology of finite flat groups, we will show that the flat sheafifications of the Artin–Mazur formal cohomology functors associated to complexes of smooth group schemes are always prorepresentable (Theorem 10.8). This result was previously obtained by Raynaud using different methods.

10.1. Let \( k \) be a field. Let \( \text{Art}_k \) be the category of Artinian \( k \)-algebras. We consider the fppf topology on \( (\text{Art}_k)^{\text{op}} \), obtained by restricting the fppf topology on the category of all \( k \)-schemes. Note that these covers are all finite locally free.

Lemma 10.2. The restriction functor from presheaves on schemes over \( k \) to presheaves on \( \text{Art}_k \) commutes with sheafification with respect to the fppf topology.

Proof. Let \( F \) be a presheaf on \( k \)-schemes and let \( \overline{F} \) be its restriction to \( \text{Art}_k \). Let \( F^a \) be the sheaf associated to \( F \) and let \( \overline{(F^a)} \) be its restriction to \( \text{Art}_k \). Since the category \( \text{Art}_k \) has the induced topology the restriction \( \overline{(F^a)} \) is a sheaf. To prove that the natural map
\[ \overline{F} \to \overline{(F^a)} \]
is universal for maps to sheaves it suffices to verify the following two properties:

(i) For an object \( A \in \text{Art}_k \) and two elements \( a, b \in F(A) \) with the same image in \( \overline{(F^a)}(A) \) there exists a covering \( A \to B \) such that \( a \) and \( b \) have the same image in \( F(B) \).
(ii) For every $A \in \text{Art}_k$ and $\alpha \in (F^\alpha)(A)$ there exists a covering $A \to B$ in $\text{Art}_k$ and an element $a \in F(B)$ mapping to the restriction of $\alpha$ to $B$.

For (i) observe that there exists an fppf-covering $U \to \text{Spec}(A)$ such that $a$ and $b$ have the same image in $F(U)$. By [38, 0DET], there exists an Artinian $k$-algebra $B$ and a finite locally free morphism $\text{Spec}(B) \to \text{Spec}(A)$ which factors through $U \to \text{Spec}(A)$, which implies that $a$ and $b$ have the same image in $F(B)$. Similarly, given $\alpha$ as in (ii) there exists an fppf-covering $U \to \text{Spec}(A)$ such that $\alpha$ is in the image $F(U)$. Choosing a finite flat covering $\text{Spec}(B) \to \text{Spec}(A)$ which factors through $U$ we then obtained the desired element $a \in F(B)$.

10.3. Given a presheaf $F$ on the category of $k$-schemes with restriction $\overline{F}$ to $\text{Art}_k$, let $\overline{F}_{\text{red}}$ denote the functor on $\text{Art}_k$ given by

$$A \mapsto F(A_{\text{red}}).$$

There is a natural map

$$\overline{F} \to \overline{F}_{\text{red}},$$

and we denote the kernel by $\widehat{F}$. Concretely we have

$$\widehat{F}(A) = \ker(F(A) \to F(A_{\text{red}})).$$

If $F$ is a sheaf in the fppf topology, then so are $\overline{F}$, $\overline{F}_{\text{red}}$, and $\widehat{F}$. By the same argument proving 10.2 the functor $F \mapsto \overline{F}_{\text{red}}$ commutes with sheafification.

Lemma 10.4. (i) The functor $F \mapsto \widehat{F}$ commutes with fppf sheafification.

(ii) For a sheaf $F$ the map $\overline{F} \to \overline{F}_{\text{red}}$ is a surjective map of sheaves on $\text{Art}_k$.

(iii) The functors $F \mapsto \overline{F}$, $F \mapsto \overline{F}_{\text{red}}$, and $F \mapsto \widehat{F}$ from the category of fppf sheaves of abelian groups on the category of $k$-schemes to the category of fppf sheaves on $(\text{Art}_k)^{\text{op}}$ are exact.

Proof. Statement (i) follows from the fact that sheafification is an exact functor and the fact that the functors $F \mapsto \overline{F}$ and $F \mapsto \overline{F}_{\text{red}}$ commute with sheafification.

The second statement follows from the Cohen structure theorem [38, Tag 032A] which implies that for $A \in \text{Art}_k$ the map $A \to A_{\text{red}}$ has a section.

All the functors in (iii) are clearly left exact. The right exactness of the first two functors follows from a quasi-section argument as in the proof of 10.2. Combining this with (ii) and a diagram chase we then obtain the exactness of $F \mapsto \widehat{F}$.

10.5. Let $X$ be a smooth proper scheme over $k$. Let $G$ be a smooth group scheme on $X$ (or more generally a complex of smooth group schemes on $X$). Artin and Mazur [5] studied the functors $\Phi^i(X, G)$ on $\text{Art}_k$, defined by

$$\Phi^i(X, G)(A) = \ker(H^i(X_A, G) \to H^i(X_{A_{\text{red}}}, G)).$$

Equivalently, $\Phi^i(X, G)$ is the completion at the identity of the functor $T \mapsto R^i f_+^* G_{X_T}$ on the category of $k$-schemes. Of particular interest is the case when $G = G_m$. In degree 1, the resulting functor $\Phi^1(X, G_m)$ is the completion of the Picard scheme of $X$, and in degree 2, the formal Brauer group. Artin–Mazur gave conditions under which $\Phi^i(X, G_m)$ is
prorepresentable. If this is the case, then \( \Phi^i(X, G_m) \) is referred to as the *Artin–Mazur formal group* of \( X \) (in degree \( i \)).

**10.6.** Given a complex of smooth group schemes \( G \) on \( X \), we define \( \Phi^i_{fl}(X, G) \) to be the functor given by sheafifying \( \Phi^i(X, G) \) in the fppf topology on \((\text{Art}_k)^{\text{op}}\). By Lemma 10.4, we may equivalently define

\[
\Phi^i_{fl}(X, G) = \widehat{R^i f_* G}
\]

where the derived pushforward on the right hand side is taken with respect to the fppf topology. In particular, we define the *flat Artin–Mazur formal group* of \( X \) in degree \( i \) to be \( \Phi^i_{fl}(X, G_m) \). We will show that the flat sheafifications \( \Phi^i_{fl} \) are always prorepresentable. This is in contrast to functors \( \Phi^i(X, G) \), which may not be representable (in fact, in 10.11 below we “explain” precisely this failure of representability).

Let \( G^{(n)} := G \times_{X, F_n^X} X \) denote the \( n \)th Frobenius twist of \( G \) over \( X \). Set \( G_n = \ker(F^*_n G/X : G \to G^{(n)}) \). These are (complexes of) finite flat group schemes on \( X \), and we have \( \hat{G} = \lim_n G_n \). For example, if \( G = G_m \), then \( G_n = \mu_{p^n} \) and \( \hat{G} = \mu_{p^\infty} \) is the sheaf of units killed by a power of \( p \).

Note also that if \( G \) is a smooth group scheme over \( X \) then the relative Frobenius map \( F^*_n G/X : G \to G^{(n)} \) is faithfully flat so we have a short exact sequence

\[
0 \to G_n \to G \to G^{(n)} \to 0.
\]

**Proposition 10.7.** If \( G \) is a bounded complex of smooth group schemes on \( X \), then the inclusion \( \hat{G} \to G \) induces an isomorphism

\[
\lim_n \widehat{R^i f_* G_n} \xrightarrow{\sim} \Phi^i_{fl}(X, G)
\]

*Proof.* Comparing the defining exact sequences for \( G_n \) and \( G_{n+1} \), taking cohomology, and then completing gives a diagram

\[
\begin{array}{cccccc}
R^{i-1} f_* G^{(n)} & \longrightarrow & R^i f_* G_n & \longrightarrow & R^i f_* G & \longrightarrow & R^i f_* G^{(n)} \\
\downarrow F & & \downarrow & & \downarrow F & & \\
R^{i-1} f_* G^{(n+1)} & \longrightarrow & R^i f_* G_{n+1} & \longrightarrow & R^i f_* G & \longrightarrow & R^i f_* G^{(n+1)}
\end{array}
\]

(10.7.1)

Taking completions is exact, so the rows of this diagram are exact sequences of fppf sheaves. If \( A \) is an Artinian \( k \)-algebra, then for \( N \) sufficiently large the absolute Frobenius of \( X_A \) factors through \( X_{A_{\text{red}}} \). It follows that \( G^{(n)}_{X_A} \to G^{(n+N)}_{X_A} \) factors through \( i_* G_{X_{A_{\text{red}}} \text{red}} \), and so for each \( j \) the limit of the directed system

\[
\ldots \to R^i f_* G^{(n+1)} \to R^i f_* G^{(n+2)} \to R^i f_* G^{(n+3)} \to R^i f_* G^{(n+4)} \to \ldots
\]

is zero. Consider the directed system given by continuing (10.7.1) in the vertical directions. Because filtered colimits commute with taking completions, we conclude that the colimit of the middle horizontal maps of (10.7.1) is an isomorphism, which gives the result. \( \square \)
Combining this with our earlier representability results we obtain the following result, previously obtained by Raynaud [33, Proposition 2.7.5] using different methods. In particular, the following result implies Theorem [1.6]

**Theorem 10.8.** Let $X$ be a smooth proper scheme over a field $k$. Let $G$ be a bounded complex of smooth group schemes on $X$. For each $i$, the functor $\Phi^{i}_\text{fl}(X, G)$ is prorepresentable.

**Proof.** As the functors $R^i f_* G_n$ are representable, the functors $\hat{R}^i f_* G_n$ are prorepresentable, and in particular preserve finite limits. It follows formally that $\lim_{\to} R^i f_* G_n$ preserves finite limits. By a theorem of Grothendieck [22, Proposition 3.1], this is equivalent to being prorepresentable.

**10.9.** We explain the relationship between $\Phi^i$ and $\Phi^{i}_\text{fl}$. Taking the fppf sheafification gives a map

$$\Phi^i(X, G) \to \Phi^{i}_\text{fl}(X, G)$$

of presheaves on $(\text{Art}_k)^{\text{op}}$. Artin–Mazur observed that this map is an isomorphism if $R^{i-1} f_* G$ is formally smooth [3, Remark 1.9]. For example, if $G = G_m$, then because $f_* G_m = G_m$, this condition always holds when $i = 1$. Note that in this case, the conclusion that $\Phi^1(X, G_m)$ is already an fppf sheaf could alternatively be deduced from the fact that the functor $\Phi^1(X, G_m)$ is the completion at the identity of the Picard scheme of $X$. The following results expand on Artin–Mazur’s observation.

**Proposition 10.10.** For each $i$, there is a short exact sequence

$$0 \to R^1 \varepsilon_* \Phi^{i-1}_\text{fl}(X, G) \to \Phi^i(X, G) \to \Phi^{i}_\text{fl}(X, G) \to 0$$

of presheaves on $(\text{Art}_k)^{\text{op}}$, where the right hand map is the map (10.9.1) and $\varepsilon_*$ is the pushforward functor from the category of fppf sheaves on $(\text{Art}_k)^{\text{op}}$ to the category of presheaves on $(\text{Art}_k)^{\text{op}}$.

**Proof.** If $Z$ is a scheme, let $\varepsilon : Z_{\text{fppf}} \to Z_{\text{Et}}$ denote the projection from the big fppf topos to the big étale topos of $Z$. Consider the functor $\Theta = Rf_{1*} \circ R\varepsilon_{X*} = R\varepsilon_{S*} \circ Rf_*$. We compute the completion of $\Theta(G)$ in two ways. On the one hand, because $G$ is smooth, we have $R^j \varepsilon_{X*} G = 0$ for $j \geq 1$, and therefore the completion of $\Theta(G)$ is equal to $\Phi^1(X, G)$. On the other hand, we compute using the Grothendieck spectral sequence for the composition of functors $R\varepsilon_{S*} \circ Rf_*$. We claim that for any $i$ and for any $j \geq 2$, the functor $R^j \varepsilon_{S*} R^i f_* G$ has trivial completion. Note that this will imply the result, for then after taking completions the spectral sequence will degenerate at $E_2$, and the claimed short exact sequences will follow. To show the claim we first observe that the completion of $R^j \varepsilon_{S*} R^i f_* G$ is equal to $R^j \varepsilon_*(\hat{R}^i f_* G)$. By [10.8] we may write $\hat{R}^i f_* G$ as a filtered colimit of completions of sheaves representable by group schemes, say $\hat{R}^i f_* G = \lim_{\to} H_n$. We have

$$R^j \varepsilon_*(\hat{R}^i f_* G) = R^j \varepsilon_*(\lim_{\to} H_n) = \lim_{\to} R^j \varepsilon_{S*} H_n$$

By a theorem of Grothendieck (see for example [8, 3.1]), we have $R^j \varepsilon_{S*} H_n = 0$ for $j \geq 2$, which gives the claim.

**Proposition 10.11.** For each $i$, the following are equivalent.
We may represent elements of $\text{CW}_R$ the group of unipotent Witt covectors of a ring $R$ defined by $(\ldots, a_{-1}, a_0) \mapsto (\ldots, a_{-2}, a_{-1}, a_0)$ and $(\ldots, a_{-2}, a_{-1}, a_0) \mapsto (\ldots, a_{-3}, a_{-2}, a_1)$. The group of unipotent Witt covectors of a ring $R$ is defined by

$$\text{CW}^u(R) = \lim_{\to} W_n(R)$$

We may represent elements of $\text{CW}^u(R)$ by sequences $(\ldots, a_{-2}, a_{-1}, a_0)$ of elements of $R$ such that $a_{-m} = 0$ for $m$ sufficiently large. We have a natural inclusion $\text{CW}^u \subset \text{CW}$ of groups. If $R$ is reduced, then $\text{CW}^u(R) = \text{CW}(R)$. If $R$ is perfect of characteristic $p$, then $\text{CW}(R) = K(R)/W(R)$, where $K(R)$ is the field of fractions of $W(R)$.

11.2. We review the theory of Dieudonné modules of formal groups. We consider a perfect field $k$ of characteristic $p$. We restrict the above functors to the category of schemes over $k$. A formal group over $k$ is a functor from the category of (not necessarily local) Artinian $k$-algebras to abelian groups which is isomorphic to the formal spectrum of a profinite $k$-algebra.
A morphism of formal groups is a morphism of group valued functors. A formal $p$-group over $k$ is a formal group $G$ every section of which is killed by a power of $p$. Equivalently, $G$ is the inductive limit of the kernels of multiplication by $p^n$. If $G$ is a group scheme over $k$, the formal completion of $G$ is the restriction of $G$ to the category of Artinian $k$-algebras. Formal completion defines a functor from the category of group schemes over $k$ to the category of formal groups over $k$. Given a formal group $G$, let $G_{\text{red}} \subset G$ be the reduced subgroup and let $G_{\text{inf}}$ be the quotient $G/G_{\text{red}}$. Thus, $G$ is formally smooth if and only if $G = G_{\text{red}}$. We let $\hat{\mathcal{D}}_k = W(k)_\sigma[F,V]$ denote the Dieudonné ring of $k$, which is the noncommutative polynomial ring over $W(k)$ in the variables $F,V$ subject to the relations

$$FV = VF = p, \quad Fx = \sigma(x)F, \quad V\sigma(x) = xV$$

for $x \in W(k)$. Given a formal group $G$ over $k$ (or more generally a presheaf of abelian groups on the category of Artinian $k$-algebras), the contravariant Dieudonné module of $G$ is defined by

$$M(G) = \text{Hom}(G, \hat{\mathcal{C}}W)$$

where the right hand side denotes the group of homomorphisms of presheaves of groups. The group $M(G)$ has an evident structure of (left) module over the ring $\text{End}(\hat{\mathcal{C}}W)$. There is an inclusion $W(k) \subset \text{End}(\hat{\mathcal{C}}W)$, corresponding to the action

$$x \cdot (\ldots, a_{-2}, a_{-1}, a_0) = (\ldots, \sigma^{-2}(x)a_{-2}, \sigma^{-1}(x)a_{-1}, xa_0)$$

(this is where we use the assumption that $k$ is perfect). Combined with the endomorphisms $F, V$ of $\hat{\mathcal{C}}W$, we find an inclusion $\mathcal{D}_k \subset \text{End}(\hat{\mathcal{C}}W)$ [19, pg. 80]. Using this inclusion, we regard $M(G)$ as a left module over $\mathcal{D}_k$. If $G$ is a group scheme over $k$, we write $M(G)$ for the contravariant Dieudonné module of the formal completion of $G$.

A version of the fundamental theorem of Dieudonné theory states that the functor $G \mapsto M(G)$ defines an antiequivalence between the category of formal $p$-groups which are inductive limits of $p$-groups and the category of profinite $\mathcal{D}_k$-modules (see [19, II Theorem 1] and [19, Remarque, pg. 128]). Here, a $\mathcal{D}_k$-module is profinite if it is the inverse limit of its quotients which are of finite length over $W(k)$.

An equivalent construction is given by Oda [33], who first gives a relatively elementary definition of $M(G)$ for $G$ a group scheme over $k$ [33, Definition 3.12]. This definition is extended to the category of inductive limits of $p$-groups by setting $M(\lim_{\rightarrow} G_n) = \lim_{\leftarrow} M(G_n)$ [33, Definition 3.17].

The association $G \mapsto M(G)$ is contravariant. There is a covariant version of this theory, which we describe. Let $M$ be a left $\mathcal{D}_k$-module. If $M$ is of finite length over $W(k)$, then following Oda [33, Definition 3.5] we define

$$D(M) = \text{Hom}_W(M, K/W(k))$$

where $K$ is the field of fractions of $W(k)$, regarded as a left $\mathcal{D}_k$-module in the natural way. We extend this definition to arbitrary $\mathcal{D}_k$-modules $M$ by setting $D(M) = \lim_{\rightarrow} D(M/N)$, where the limit is taken over all finite length quotients of $M$. If $G$ is a formal group over $k$ (or more
generally a presheaf of abelian groups on the category of Artinian $k$-algebras), we define the $(V$-divided) covariant Dieudonné module of $G$ to be $\text{DM}(G) = D(M(G))$. The most interesting case is when $G$ is a formal group over $k$ which is an inductive limit of $p$-groups, in which case $M(G)$ is a profinite $\mathcal{D}_k$-module. By [33, Proposition 3.9], the functor $D$ defines an antiequivalence from the category of finite length $\mathcal{D}_k$-modules to itself. It follows formally that $D$ induces an antiequivalence from the category of profinite $\mathcal{D}_k$-modules to the Ind category of the category of finite $\mathcal{D}_k$-modules. Thus, we obtain a covariant equivalence of categories from the Ind-category of $p$-groups to the Ind-category of the category of finite $\mathcal{D}_k$-modules.

This theory is compatible with Cartier duality. If $G$ is a finite abelian group scheme, its Cartier dual is the group scheme $\text{D}(G) = \text{Hom}_{\text{grp}}(G, G_m)$. By [33, Theorem 3.19], if $G$ is a finite $p$-group scheme, there is a canonical isomorphism $\text{DM}(G) \cong \text{MD}(G)$.

Example 11.3. We have $\text{DM}(\mu_p) = \mathcal{D}_k/((1 - F)\mathcal{D}_k + V\mathcal{D}_k)$. That is, $\text{DM}(\mu_p) \cong k$ as a $W$-module, with action given by $F = \sigma$ and $V = 0$.

11.4. We now discuss Cartier theory. Let $\widehat{W}$ be the completion of the group scheme $W$ at the origin. Explicitly, $\widehat{W}(A)$ is the set of tuples $(a_0, a_1, \ldots)$ of nilpotent elements of $A$ such that $a_i = 0$ for all $i$ sufficiently large. Let $G$ be a formal group (or more generally a presheaf of abelian groups). We set

$$\text{TC}_n(G) := \text{Hom}_{\text{grp}}(\widehat{W}[F^n], G),$$

where “$\text{grp}$” denotes the category of formal groups. The group of typical curves on $G$ is defined to be

$$\text{TC}(G) := \lim_{\leftarrow n} \text{TC}_n(G) = \text{Hom}(\widehat{W}, G).$$

The Cartier ring $E$ of $k$ is the opposite ring of $\text{End}(\widehat{W})$. We have $E \cong W(k)_\sigma[F][[V]]$. We regard $\text{TC}(G)$ as a module over $E$ in the natural way, and call it the Cartier module of $G$. Note that the formation of the Cartier module of $G$ is covariant. In the literature, $\text{TC}(G)$ is sometimes referred to as the covariant Dieudonné module of $G$ (particularly when $G$ is a $p$-divisible group). The modules $\text{TC}(G)$ and $\text{DM}(G)$ are rarely isomorphic. For instance, if $G$ is a finite connected group scheme, then $\text{TC}(G) = 0$. More generally, if $G$ is a formal group, then $\text{TC}(G_{\text{red}}) = \text{TC}(G)$. Nevertheless, the functors $\text{TC}$ and $\text{DM}$ are closely related, and each can be computed in terms of the other.

The key to this relationship is the Artin–Hasse exponential [33, pg. 93], which is a group homomorphism

$$E : \widehat{W} \to G_m$$

There is a pairing $\widehat{W} \times W \to G_m$, defined by $(u, x) \mapsto E(ux)$. By [33, Proposition 3.21i], the resulting map

$$\widehat{W} \to \text{Hom}_{\text{grp}}(W, G_m)$$

is an isomorphism (this result is also proved by Cartier [13] and Dieudonné [17]). Set $W_{n,m} = W_n[F^m] = \ker(F^m : W_n \to W_n)$. Using this isomorphism, Oda shows [33, Proposition 3.21iii]
that there is a natural isomorphism
\[ W_{m,n} \cong \mathcal{H}om_{\text{gr}}(W_{n,m}, G_m) = D(W_{n,m}) \]

**Proposition 11.5.** Let \( G \) be a formal \( p \)-group. There are canonical isomorphisms
\[
TC_n(G) \cong DM(G)[V^n] \quad \text{and} \quad TC_n(R^1\epsilon_* G) \cong DM(G_{\text{inf}})/V^n
\]

**Proof.** To prove the first isomorphism proceed as follows. We have \( TC_n(G) = TC_n(G[F^n]) \). It will therefore suffice to prove the result under the assumption that \( G \) is finite and connected and killed by \( F^n \). It will further suffice to consider separately the cases when \( G \) is of local–local and local–étale types. Suppose that \( G \) is local–local. We may then take \( m \) sufficiently large so that \( G \) is killed by \( V^m \). Thus, on \( D(G) \), \( V^n \) and \( F^m \) are zero. We note that for any \( n, m \) there is an isomorphism \( \hat{W}[F^n]/V^m \cong W_{m,n} \). By Cartier duality, we obtain
\[
\Hom_{\text{gr}}(\hat{W}[F^n], G) = \Hom_{\text{gr}}(\hat{W}[F^n]/V^m, G)
\]
\[
\cong \Hom_{\text{gr}}(D(G), D(W_{n,m}))
\]
\[
\cong \Hom_{\text{gr}}(D(G), W_{n,m})
\]
\[
\cong \Hom_{\text{gr}}(D(G), \hat{CW})
\]
where the last isomorphism uses that \( D(G) \) is connected and killed by \( V^n \). We conclude the result from the isomorphisms \( MD(G) \cong DM(G) \) and \( DM(G[F^n]) = DM(G)[V^n] \).

Suppose that \( G \) is local–étale. We may further assume \( G = \mu_{pn} \) for some \( m \leq n \). We have a canonical isomorphism
\[
TC(\hat{G}_m) = \Hom_{f\text{gr}}(\hat{W}, \hat{G}_m) \cong W(k)
\]
Thus, we have
\[
TC_n(\mu_{pn}) \cong \Hom_{f\text{gr}}(\hat{W}[F^n], \mu_{pn}) \cong W_m(k)
\]
On the other hand, we have
\[
DM(\mu_{pn}) \cong MD(\mu_{pn}) = \Hom_{f\text{gr}}(\mathbb{Z}/p^m\mathbb{Z}, \hat{CW}) \cong \Hom_{f\text{gr}}(\mathbb{Z}/p^m\mathbb{Z}, W_m) = W_m(k)
\]

Next we turn to the second isomorphism. Smooth formal groups are acyclic for \( \epsilon_* \), so we have \( R^1\epsilon_* G = R^1\epsilon_* G_{\text{inf}} \), and again we may assume \( G \) is finite and connected. By a result of Oort [34 Theorem 4.2c], we have an isomorphism
\[
TC(R^1\epsilon_* G) \cong DM(G)
\]
The presheaf \( R^1\epsilon_* G \) is formally smooth, so the map \( TC(R^1\epsilon_* G) \to TC_n(R^1\epsilon_* G) \) induces an isomorphism \( TC(R^1\epsilon_* G)/V^n \cong TC_n(R^1\epsilon_* G) \). Thus, taking cokernels of \( V^n \) we obtain the result. \( \square \)

**Corollary 11.6.** If \( G \) is a connected formal group, then there is a canonical isomorphism
\[
DM(G_{\text{red}}) \cong \varprojlim V TC_n(G). \quad \text{Furthermore, we have}
\]
\[
DM(G_{\text{red}}) \cong \varprojlim V TC(G)/V^n = TC(G)[V^{-1}]/TC(G) \quad \text{and} \quad DM(G_{\text{inf}}) \cong TC(R^1\epsilon_* G)
\]
Proof. Using (11.5) we have
\[
\lim_{n} TC_{n}(G) = \lim_{n} DM(G)[V^{n}] = \lim_{n} DM(G[F^{n}]) = DM(\lim_{n} G[F^{n}]) = DM(G)
\]
Here, the transition maps in the second directed system are the inclusions \(DM(G)[V^{n}] \subset DM(G)[V^{n+1}]\) (that these are indeed the correct maps follows from examining the isomorphism of (11.5). This gives the first isomorphism. For the second, we use that the inclusion \(G_{\text{red}} \subset G\) induces an isomorphism \(TC(G_{\text{red}}) = TC(G)\). Thus we may assume that \(G\) is itself smooth. In this case, the maps \(TC_{n+1}(G) \to TC_{n}(G)\) are surjective. It follows that \(TC(G) \to TC_{n}(G)\) is surjective and induces an isomorphism \(TC(G)/V^{n} \simeq TC_{n}(G)\). This gives the second isomorphism. The third follows from (11.5) because the inverse system \(\lim_{\longleftarrow n} TC_{n}(R^{1}\epsilon_{*}G) \simeq \lim_{\longleftarrow n} DM(G_{\text{inf}})/V^{n}\) is eventually constant with value \(DM(G_{\text{inf}})\).

Example 11.7. We have \(R^{1}\epsilon_{*}\mu_{p} = G_{m}/G_{m}^{p}\). It follows that \(TC(R^{1}\epsilon_{*}\mu_{p}) = W(k)/pW(k) = k^{*}\), with \(V = 0\) and \(F\) an isomorphism.

Example 11.8. Consider \(G = \widehat{G}_{a}\) (the completion at the identity). We have \(TC(\widehat{G}_{a}) = k_{\sigma}[[V]] = \prod_{n \geq 0} kV^{n}\), with \(F = 0\). On the other hand, \(\widehat{G}_{a} = \lim_{\longleftarrow n} \widehat{G}_{a}[F^{n}] = \lim_{\longleftarrow n} \alpha_{p^{n}}\), so \(DM(\widehat{G}_{a}) = \lim_{\longleftarrow V} (k[V]/V^{n}) = k_{\sigma}((V))/k_{\sigma}[[V]] = k_{\sigma}(V)/k_{\sigma}[V]\) with \(F = 0\).

11.9. So far, we have only considered groups defined over a perfect field \(k\). We briefly discuss extensions of the above theory to the case when \(k\) is a more general scheme of characteristic \(p\). The descriptions of the Dieudonné theory (both contravariant and covariant) as defined above required \(k\) to be perfect. However, the Cartier module \(TC\) makes sense more generally. Let \(S\) be a scheme over a perfect field \(k\) of characteristic \(p\). We let \(\widehat{W}_{S}\) be the formal completion of \(W_{S}\) at the origin. We identify \(\widehat{W}_{S}\) with its Frobenius twist over \(S\), so that we have maps \(F, V : \widehat{W}_{S} \to \widehat{W}_{S}\) of formal groups on \(X\). If \(G\) is a formal group over \(S\), we set \(TC_{S,n}(G) = \mathcal{H}om_{\mathfrak{fGps}_{S}}(\widehat{W}_{S}[F^{n}], G)\) and \(TC_{S}(G) = \lim_{\longleftarrow V} TC_{S,n}(G)\) (regarded in either case as a Zariski sheaf on \(S\)). Using this and the computations of (11.6) we can define a covariant Dieudonné theory for connected formal groups \(G\) over \(S\) by setting \(DM_{S}(G) = \lim_{\longleftarrow V} TC_{S,n,S}(G)\) (this idea goes back at least to Oort [34]).

We note two special cases. If \(G\) is a smooth connected formal group over \(S\), then we have \(DM_{S}(G) = \lim_{\longleftarrow V} TC_{S}(G)/V^{n}\). If \(G\) is a flat local group scheme over \(S\), we may compute \(R^{1}\epsilon_{*}G\) by locally embedding \(G\) in a smooth formal group over \(S\), and so define \(TC_{S}(R^{1}\epsilon_{*}G)\). We have \(DM_{S}(G) = TC_{S}(R^{1}\epsilon_{*}G)\). By (11.6) these definitions agree with our earlier definitions when they may be compared.

Remark 11.10. Consider \(TC_{S}\) as a functor from the category of fpqc sheaves of abelian groups on \(S\) to the category of Cartier modules. This functor is left exact, but not exact. If \(G\) is a flat formal group on \(S\), we may consider the derived Cartier module \(R TC_{S}(G)\). Note that the cohomology sheaves of this object vanish except in degrees 0 and 1, and (as in Proposition (11.6) we have
\[
R^{1} TC_{S}(G) = TC_{S}(R^{1}\epsilon_{*}G) = DM_{S}(G_{\text{inf}})
\]
12. Cartier–Dieudonné modules of the flat Artin–Mazur formal groups

We now compute TC and DM for the flat Artin–Mazur formal groups. Let $X$ be a smooth proper scheme over a perfect field $k$. If $G$ is a complex of smooth connected formal groups over $X$, we define $TC_X(G)$ and $DM_X(G)$ by applying the functors $TC_X$ and $DM_X$ termwise. For each $i$ the cohomology group $H^i(X, TC_X(G))$ has a natural Cartier module structure, induced by the maps $F$ and $V$ on $TC_X(G)$. Similarly, $H^i(X, DM_X(G))$ has a natural $D_k$ module structure.

**Theorem 12.1.** Let $G$ be a bounded complex of smooth connected formal groups over $X$. For each $i$, there are canonical isomorphisms

$$TC(\Phi^i(X, G)) \simeq H^i(X, TC_X(G))$$

$$TC(\Phi^i_\text{fl}(X, G)) \simeq H^i(X, TC_X(G))/V \text{tors}$$

$$DM(\Phi^i_\text{fl}(X, G)) \simeq H^i(X, DM_X(G))$$

of Cartier and Dieudonné modules.

**Remark 12.2.** The computation of $TC(\Phi^i_\text{fl})$ was already shown by Ekedahl [18, Proposition 8.1].

**Proof.** Write $\Phi^i_\text{fl} = \Phi^i(X, G)$ and $\Phi^i = \Phi^i(X, G)$. The first isomorphism is due to Artin–Mazur [5, 2.13]. In more detail, the proof proceeds by noting that we have identifications

$$TC_n(\Phi^i) \simeq H^i(X, TC_{X,n}(G))$$

(a consequence of the projection formula), and then passing to the inverse limit. Consider the short exact sequence

$$(12.2.1) 0 \to R^1\epsilon_* \Phi^{i-1}_\text{fl} \to \Phi^i \to \Phi^i_\text{fl} \to 0$$

of presheaves produced in Proposition [10, 10]. Smooth formal groups are acyclic for $R\epsilon_*$, so $R^1\epsilon_* \Phi^{i-1}_\text{fl} = R^1\epsilon_* \Phi^{i-1}_\text{fl,inf}$. We apply $TC_n$ to obtain a short exact sequence

$$0 \to TC_n(R^1\epsilon_* \Phi^{i-1}_\text{fl,inf}) \to TC_n(\Phi^i) \to TC_n(\Phi^i_\text{fl}) \to 0$$

By Proposition [11, 5] and the above identification of $TC_n(\Phi^i)$, this sequence is isomorphic to the sequence

$$0 \to DM(\Phi^{i-1}_\text{fl,inf})/V^n \to H^i(X, TC_{X,n}(G)) \to DM(\Phi^i[F^n]) \to 0$$

We consider the direct limit over $n$ of this short exact sequence. The resulting left hand term is $\lim V^{-n} DM(\Phi^{i-1}_\text{fl,inf})/V^n$. As $DM(\Phi^{i-1}_\text{fl,inf})$ is killed by $V^n$ for some sufficiently large $n$, this limit vanishes. Furthermore, $\Phi^i_\text{fl}$ is connected, so $\lim \Phi^i_\text{fl}[F^n] = \Phi^i_\text{fl}$. This gives the third claimed isomorphism.

Finally, we construct the second isomorphism. Consider the exact sequence

$$(12.2.2) 0 \to TC(R^1\epsilon_* \Phi^{i-1}_\text{fl,inf}) \to TC(\Phi^i) \to TC(\Phi^i_\text{fl}) \to 0$$

given by applying $TC$ to $\text{inf}$. We will compute the left hand term. Consider the short exact sequences

$$0 \to H^i(X, TC_X(G))/V^n \to H^i(X, TC_{X,n}(G)) \to H^{i+1}(X, TC_X(G))[V^n] \to 0$$
Taking the direct limit over $n$, we find a short exact sequence

\[(12.2.3) \quad 0 \to \lim_{\triangleright} H^i(X, \text{TC}_X(G))/V^n \to H^i(X, \text{DM}_X(G)) \to H^{i+1}(X, \text{TC}_X(G))[V^\infty] \to 0\]

It follows from \((12.2.2)\) that the $V^\infty$-torsion submodule of $H^i(X, \text{TC}_X(G))$ has finite length over $W$. Thus, the left term of \((12.2.3)\) is $V$-torsion free. Furthermore, the right term of \((12.2.3)\) is $V$-torsion. It follows that \((12.2.3)\) is isomorphic to the short exact sequence

\[(12.2.4) \quad 0 \to \text{DM}(\Phi^i_{\text{fl,red}}) \to \text{DM}(\Phi^i_{\text{fl}}) \to \text{DM}(\Phi^i_{\text{fl,inf}}) \to 0\]

By Proposition \([11.5]\) and the above identification of $\text{DM}(\Phi^i_{\text{fl,inf}})$, we have isomorphisms

\[\text{TC}(R^1\varepsilon_*\Phi^i_{\text{fl,inf}}^{-1}) \simeq \text{DM}(\Phi^i_{\text{fl,inf}}^{-1}) \simeq H^i(X, \text{TC}_X(G))[V^\infty]\]

Thus, \((12.2.2)\) is isomorphic to the short exact sequence

\[0 \to H^i(X, \text{TC}_X(G))[V^\infty] \to H^i(X, \text{TC}_X(G)) \to H^i(X, \text{TC}_X(G))/V\text{-tors} \to 0\]

In particular, this gives the second claimed isomorphism. \[\square\]

**Remark 12.3.** That the $V$-torsion in $\Phi^i$ is in the kernel of the sheafification map can be seen directly as follows. Consider $\text{TC}(\Phi^i)$ as a subgroup of the group of curves $C(\Phi^i) = \Phi^i(k[[t]])$. The map $V^n$ on $\text{TC}(\Phi^i)$ is induced by the map $t \mapsto t^p^n$ on $k[[t]]$, which is an fpqc cover.

The failure of TC to commute with taking cohomology might be understood by noting that we have a canonical isomorphism

\[RTC(Rf_*G) \simeq Rf_*\text{TC}_XG\]

for any smooth (possibly formal) group $G$ on $X$ (a consequence of the derived projection formula). If we replace the derived pushforwards with the pushforward in the étale topology, then we may replace the functor $RTC$ with the functor $TC$ on the category of étale sheaves, which is acyclic.

**12.4.** We consider the special case of the above theorem when $G = G_{m,X}$. We have the Zariski sheaf $W\theta_X$ on $X$ which sends an affine open $U = \text{Spec} R$ to $W(R)$. Following the notation of Illusie \([26]\), we set

\[BW\theta_X = \lim_{\triangleright} W\theta_X \quad CW\theta_X = \lim_{\triangleright} W_n\theta_X\]

We have a short exact sequence

\[(12.4.1) \quad 0 \to W\theta_X \xrightarrow{\xi} BW\theta_X \xrightarrow{\xi} CW\theta_X \to 0\]

We have natural identifications $\text{TC}_X(G_{m,X}) = W\theta_X$ and $\text{DM}_X(G_{m,X}) = CW\theta_X$. If $C$ is a sheaf on $X$ and $f : C \to C$ is an endomorphism, we will write $C(f)$ for the complex $[C \xrightarrow{f} C]$, where the first term is placed in degree 0. Also, write $\nu_n = \nu_X/k,n$ and $\omega_n = \omega_X/k,n$. Specializing Theorem \([12.1]\) we obtain the following.
Corollary 12.5. Let $X$ be a smooth proper scheme over a perfect field $k$. For each $i$, there are canonical isomorphisms

\[
\begin{align*}
\text{TC}(\Phi^i(X, G_m)) & \simeq H^i(X, W_0 X) & \text{DM}(R^if_*\mu_p) & \simeq H^i(X, CW_0(p^n)) \\
\text{TC}(\Phi^i(X, G_m)) & \simeq H^i(X, W_0 X)/V - \text{tors} & \text{DM}(R^if_*\nu) & \simeq H^{i+1}(X, CW_0(F^n)) \\
\text{DM}(\Phi^i(X, G_m)) & \simeq H^i(X, CW_0 X) & \text{DM}(R^if_*\omega) & \simeq H^i(X, CW_0(V^n))
\end{align*}
\]

Proof. This is a consequence of [12.1]. The only missing piece is to explain why the functor $\text{TC}_X$ takes the map $\alpha^n_{X/S} \to V^n : CW_0 X \to CW_0 X$. This follows from the relations $F^n_{X/S} \circ \alpha^n_X = \cdot p^n$ and $FV = p$ and the fact that $F$ is injective on $CW_0 X$. □

12.6. The extended Artin–Mazur formal group. The extended Artin–Mazur formal group in degree $i$ is the functor

\[
\Psi^i := R^if_*\mu_p = \lim \limits_n R^if_*\mu_{p^n}
\]

The functors $R^if_*\mu_{p^n}$ are representable, so $\Psi^i$ is a formal group. By Proposition [10.7], the connected component of the identity $(\Psi^i)^c \subset \Psi^i$ is the flat Artin–Mazur formal group $\Phi^i$. Note that our $\Psi^i$ is slightly different from the “enlarged” functor $\Psi^i$ considered by Artin–Mazur [5 IV], which includes only the $p$-divisible part of the étale quotient.

We will compare the extended Artin–Mazur formal group with the large torsion limit of the flat cohomology of $\nu_n$ and $\omega_n$. We have natural inclusions $\mu_{p^n} \subset \mu_{p^{n+1}}$, $\omega_n \subset \omega_{n+1}$, and $\nu_n \subset \nu_{n+1}$ (the latter being induced by $F$). We write $\mu_{p^n}, \omega_{\infty}$, and $\nu_{\infty}$ for the respective direct limits of the resulting systems.

Lemma 12.7. There are natural distinguished triangles

\[
(12.7.1) \quad Rf_*\nu_{\infty}[-1] \to Rf_*\mu_{p^{\infty}} \to \lim \limits_F Rf_*\omega_n \xrightarrow{+1}
\]

\[
(12.7.2) \quad Rf_*\omega_{\infty} \to Rf_*\mu_{p^{\infty}} \to \lim \limits_V Rf_*\nu_n[-1] \xrightarrow{+1}
\]

Proof. This follows by taking the direct limit of pushforwards of the triangles (2.12.8) and (2.12.9) (or rather, their analogs for $n \geq 1$). □

Proposition 12.8. Taking cohomology in degree $i$ of (12.7.1) we obtain an exact sequence

\[
0 \to R^{i-1}f_*\nu_{\infty} \to R^if_*\mu_{p^{\infty}} \to \lim \limits_F R^if_*\omega_n \to 0
\]

which is the unipotent–multiplicative decomposition

\[
0 \to \Psi^i_{\text{uni}} \to \Psi^i \to \Psi^i_{\text{mult}} \to 0
\]

of $\Psi^i$. Taking cohomology in degree $i$ of (12.7.2) gives an exact sequence

\[
0 \to R^if_*\omega_{\infty} \to R^if_*\mu_{p^{\infty}} \to \lim \limits_V R^{i-1}f_*\nu_n \to 0
\]

which is the connected-étale decomposition

\[
0 \to \Phi^i \to \Psi^i \to (\Psi^i)_{\text{ét}} \to 0
\]
of $Ψ^i$.

Proof. The formal group $\lim_{\to} R^i f_* \omega_n$ is connected, because each of the group schemes $R^i f_* \omega_n$ is finite and local. Its Dieudonné modules is the $i$th cohomology of the direct limit $\lim_{\to} F CW\theta_X = CW\theta_X[F^{-1}]$. As $F$ is injective on $CW\theta_X$, it induces an isomorphism in the limit. It follows that $\lim_{\to} R^i f_* \omega_n$ is multiplicative. Moreover, the formal group $R^i f_* \nu_\infty$ is unipotent. This prove the first claim.

The formal group $R^i f_* \omega_\infty$ is connected. Consider the map to $Ψ^i$. On the Dieudonné modules of the completions, this map is induced by the map $\lim_{\to} (id, V) CW\theta_X(V^n) \to CW\theta_X$, which is an isomorphism ($V$ is surjective on $CW\theta_X$). It follows that this map identifies $R^i f_* \omega_\infty$ with the connected component of $Ψ^i$. The result follows. □

12.9. The flat cohomology of $\mu_p$, $\nu_X$, and $\omega_X$. Let $X$ be a smooth proper variety over the perfect field $k$. We examine the structure of the group schemes $R^i f_* \mu_p$, $R^i f_* \nu_X$, and $R^i f_* \omega_X$. We will omit all Frobenius twists over $k$. In Section 2, we equipped these group schemes with maps $F$ and $V$. The relations between these maps are summarized in the diagram

(12.9.1)

The Dieudonné modules of the completions of these cohomology groups are computed in 12.5. Following Illusie [26, II.6A], we now explain how to relate these groups to de Rham cohomology. Applying $TC_X$ to (2.12.5) and (2.12.6) gives distinguished triangles

(12.9.2) $CW\theta_X(F)[-1] \to CW\theta_X(p) \to CW\theta_X(V) \xrightarrow{+1}$

(12.9.3) $CW\theta_X(V) \to CW\theta_X(p) \to CW\theta_X(F)[-1] \xrightarrow{+1}$

Because $W\theta_X$ and $BW\theta_X$ are $p$-torsion free, the short exact sequence (12.4.1) induces a quasi-isomorphism $CW\theta_X(p) \simeq [W\theta_X/p \xrightarrow{\epsilon} BW\theta_X/p]$. Let $\tau_1 : BW\theta_X \to CW\theta_X$ be the direct limit of the maps $W\theta_X \to W_{n+1}\theta_X$ and let $\eta : \theta_X \to CW\theta_X$ be the direct limit of the maps $V^n : \theta_X \to W_{n+1}\theta_X$. As explained by Illusie [26, Lemme II6.9.8, pg. 645], the vertical arrows in the commutative square

$$
\begin{array}{ccc}
W\theta_X/p & \xrightarrow{\epsilon} & BW\theta_X/p \\
\downarrow & & \downarrow \tau_1 \\
\theta_X & \xrightarrow{\eta} & CW\theta_X/F
\end{array}
$$

define a quasi-isomorphism between the rows. Thus, we obtain a quasi-isomorphism $CW\theta_X(p) \simeq [\theta_X \xrightarrow{\eta} CW\theta_X/F]$
Consider the map $F^n d : W_{n+1} \mathcal{O}_X \to \Omega^1_X$. By [26] I.3.11.4, pg. 574, the image of this map is equal to $B_{n+1} \Omega^1_X \subset \Omega^1_X$, and its kernel is equal to $FW_{n+2} \mathcal{O}_X \subset W_{n+1} \mathcal{O}_X$. We have a commuting diagram

$W_{n+1} \mathcal{O}_X \xrightarrow{V} W_{n+2} \mathcal{O}_X \xrightarrow{R} W_{n+1} \mathcal{O}_X$

$\downarrow F^n d \quad \downarrow F^{n+1} d \quad \downarrow F^n d$

$B_{n+1} \Omega^1_X \xrightarrow{C} B_{n+1} \Omega^1_X$

The left hand square commutes because of the relation $FdV = d$, and the right hand square is [26] I.3.11.6, pg. 574. In particular, the maps $F^n d$ induce a map $CW \mathcal{O}_X \to \Omega^1_X$, which induces an isomorphism $d : CW \mathcal{O}_X/F \sim B_\infty \Omega^1_X$. This isomorphism fits into the commutative diagram

$\mathcal{O}_X \xrightarrow{d} \mathcal{O}_X$

$\downarrow \eta \quad \downarrow d$

$CW \mathcal{O}_X/F \sim B_\infty \Omega^1_X$

Furthermore, we have a commutative diagram

$0 \to \mathcal{O}_X/\mathcal{O}_X^p \xrightarrow{\pi} CW \mathcal{O}_X/F \xrightarrow{V} CW \mathcal{O}_X/F \to 0$

$\downarrow d \quad \downarrow i \quad \downarrow d$

$0 \to B_1 \Omega^1_X \xrightarrow{C} B_\infty \Omega^1_X \to 0$

with exact rows.

Consider the diagram

$0 \to \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X$

$\downarrow d \quad \downarrow d$

$B_\infty \Omega^1_X \xrightarrow{C} B_\infty \Omega^1_X \to 0$

which corresponds to a distinguished triangle

(12.9.4) $B_\infty \Omega^1_X[-1] \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{d} B_\infty \Omega^1_X \xrightarrow{\pi} \mathcal{O}_X \xrightarrow{\eta}$

We also have a diagram

$0 \to \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X^p \to 0$

$\downarrow d \quad \downarrow d \quad \downarrow d$

$0 \to B_\infty \Omega^1_X \to B_\infty \Omega^1_X \to 0$

with exact rows. Combining this with the quasi-isomorphisms

$[\mathcal{O}_X/\mathcal{O}_X^p \xrightarrow{d} B_\infty \Omega^1_X] \simeq B_\infty \Omega^1_X/B_1 \Omega^1_X[-1] \xrightarrow{C} B_\infty \Omega^1_X[-1]$

we find a second distinguished triangle

(12.9.5) $\mathcal{O}_X \xrightarrow{F} [\mathcal{O}_X \xrightarrow{d} B_\infty \Omega^1_X] \xrightarrow{C} B_\infty \Omega^1_X[-1] \xrightarrow{\eta}$
Tracing through our constructions, we see that (12.9.4) is quasi-isomorphic to (12.9.2) and (12.9.5) is quasi-isomorphic to (12.9.3).

We summarize our computations in the following result. Consider the long exact sequences

\begin{equation}
\ldots \to \text{DM}(R^if_*\nu_X) \to \text{DM}(\hat{R}^if_*\omega_X) \to \text{DM}(\hat{R}^if_*\psi_X) \to \ldots
\end{equation}

and

\begin{equation}
\ldots \to \text{DM}(\hat{R}^if_*\omega_X) \to \text{DM}(\hat{R}^if_*\psi_X) \to \text{DM}(\hat{R}^if_*\nu_X) \to \ldots
\end{equation}

obtained from the distinguished triangle (2.12.8) (respectively, (2.12.9)) by taking cohomology sheaves, completing at the identity, and then applying the equivalence \(\text{DM}(\omega)\).

**Theorem 12.10.** There are canonical isomorphisms

\[
(\text{DM}(\hat{R}^if_*\mu_y), F, V) \simeq (H^i(X, [O_X \xrightarrow{d} B_\infty \Omega^1_X]), F^*, C^*)
\]

\[
(\text{DM}(\hat{R}^if_*\nu_X), F, V) \simeq (H^i(X, B_\infty \Omega^1_X), 0, C)
\]

\[
(\text{DM}(\hat{R}^if_*\psi_X), F, V) \simeq (H^i(X, O_X), F, 0)
\]

Here, \(F^* = F \circ \pi\) and \(C^* = \iota \circ C\) are the indicated compositions of the maps induced by the maps \(F, \pi, \iota,\) and \(C\) in the distinguished triangles (12.9.4) and (12.9.5).

Furthermore, under these isomorphisms the long exact sequence

\begin{equation}
\ldots \to H^{i-1}(X, B_\infty \Omega^1_X) \xrightarrow{\iota} H^i(X, [O_X \xrightarrow{d} B_\infty \Omega^1_X]) \xrightarrow{\nu} H^i(X, O_X) \xrightarrow{d} H^i(X, B_\infty \Omega^1_X) \to \ldots
\end{equation}

induced by the distinguished triangle (12.9.4) agrees with (12.9.6), and the long exact sequence

\begin{equation}
\ldots \to H^i(X, O_X) \xrightarrow{F} H^i(X, [O_X \xrightarrow{d} B_\infty \Omega^1_X]) \xrightarrow{C} H^{i-1}(X, B_\infty \Omega^1_X) \to H^{i+1}(X, O_X) \to \ldots
\end{equation}

induced by (12.9.5) agrees with (12.9.7).

**12.11.** Write \(\mathcal{G}^* = [O_X \xrightarrow{d} B_\infty \Omega^1_X]\) and \(B_\infty = B_\infty \Omega^1_X\). As in the diagram (12.9.1), the Dieudonné module structure on \(H^i(X, \mathcal{G}^*)\) can be visualized using the diagram

\[
\begin{array}{cccccc}
H^i(O_X) & \xrightarrow{0} & H^i(O_X) & \xrightarrow{F} & H^i(O_X) & \xrightarrow{F} & H^i(O_X) \\
\downarrow F & & \downarrow \pi & & \downarrow \pi & & \downarrow F \\
H^i(\mathcal{G}^*) & \xrightarrow{C^*} & H^i(\mathcal{G}^*) & \xrightarrow{F^*} & H^i(\mathcal{G}^*) & \xrightarrow{F} & H^i(\mathcal{G}^*) \\
\downarrow C & & \downarrow \iota & & \downarrow \iota & & \downarrow C \\
H^{i-1}(B_\infty) & \xrightarrow{C} & H^{i-1}(B_\infty) & \xrightarrow{C} & H^{i-1}(B_\infty) & \xrightarrow{0} & H^{i-1}(B_\infty)
\end{array}
\]

which is compatible with the \(F\) and \(V\) operators on each term. Here, the diagonals are fragments of the long exact sequences (12.10.1) and (12.10.2), and the diagram may be continued in all directions.

**Remark 12.12.** The sequences (12.10.1) and (12.10.2) are sequences of Dieudonné modules. It follows that the image of the boundary map

\[H^{i-1}(X, B_\infty \Omega^1_X) \to H^{i+1}(X, O_X)\]
is contained in $H^{i+1}(X, \mathcal{O}_X)[F] \subset H^{i+1}(X, \mathcal{O}_X)$. In particular, if $F$ acts invertibly on $H^i(X, \mathcal{O}_X)$ for all $i$, then all the boundary maps vanish. Similarly, the image of the boundary map $H^i(X, \mathcal{O}_X) \xrightarrow{d} H^i(X, B_\infty \Omega^1_X)$ is contained in $H^i(X, B_\infty \Omega^1_X)[C] = \mathrm{im}(H^i(X, B_1 \Omega^1_X)) \subset H^i(X, B_\infty \Omega^1_X)$.

The submodules $H^{i+1}(X, \mathcal{O}_X)[F]$ and $H^i(X, B_\infty \Omega^1_X)[C]$ have both $F$ and $V$ zero. Such a Dieudonné module corresponds to a direct sum of copies of $\alpha_p$.

**Remark 12.13.** There is a natural map $[\mathcal{O}_X \xrightarrow{d} B_\infty \Omega^1_X] \to \Omega^*_X$ given by the inclusion in degrees 0 and 1. There is an induced map

$$DM(\hat{R}^i f_* \mu_p) \to H^i_{dR}(X/k)$$

In degree $i = 1$, this gives a map

$$DM(\widehat{\mathrm{Pic}_X[p]} \to H^1_{dR}(X/k)$$

As explained by Illusie [26, II.6.14, pg. 648], this map is an injection, and agrees with Oda’s inclusion [33, Corollary 5.12]. The map (12.13.1) is not injective in general. For instance, if $R^i f_* \mu_p$ is positive dimensional, then the Dieudonné module of its completion will be an infinite dimensional vector space over $k$.

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