The internal-environment model of the Stern-Gerlach experiment

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Abstract: The standard interpretation of the Stern-Gerlach experiment assumes that the atomic center-of-mass plays the role of “quantum apparatus” for the atomic spin. Following a recent, decoherence-based, model fitting with this interpretation, we investigate whether or not such model can be constructed. Our conclusions are somewhat surprising: only if the screen capturing the atoms in the experiment brings the information about the atomic-nucleus center-of-mass, one may construct the model desired. The nucleus $CM$ system is monitored by the nucleus “relative system ($R$)”. There appear the effective (the electrons-mediated) interaction between $CM$ and $R$ that is possibly responsible for decoherence. For larger atoms, the interaction scales as $Z^2$ ($Z$ is the “atomic number”), being totally independent on the atomic mass. The interaction selects the $CM$-wave-packet states as the approximate pointer basis. Interestingly enough, the model stems nonoccurrence of decoherence due to the internal environment for the larger systems (such as the macromolecules and the macroscopic systems). Certainly, disproving this model (e.g. in an experiment) stems the active role of the screen, which becomes responsible for the “$CM$ + spin”-state “reduction” (“collapse”), i.e. for the irreversible retrieval of the classical information from the quantum world.

1 Introduction

The Stern-Gerlach experiment is a paradigm of the quantum measurement of spin $\frac{1}{2}$. The standard interpretation of the experiment (cf. e.g. [1, 2, 3]) distinguishes the atomic center-of-mass system as playing the role of the quantum “apparatus” that brings the (classical) information about the atomic
spin. The interpretation assumes the definite trajectory of every single atom (in an ensemble of atoms) that is in one-to-one correspondence (classical correlation) with the spin-projection along the axis of the external magnetic field. However, in the more refined analysis, this interpretation calls for the additional arguments.

The point is the role of the final screen capturing the atoms in the experiment. First, in the case where the screen is a passive element simply recording the objectively existing trajectories (as distinguished above) the screen serves for the so-called quantum measurement of the second kind (the “retrospective” measurement) [3]—the screen does not bring any decision, it just records the objectively present trajectories. On the other side, having the “active role” in the experiment, the screen provides the information about the trajectories, which have never objectively existed before the screen. In the terms of the von Neumann’s quantum measurement theory [3]: the screen plays an active role in the “reduction” (“collapse”) of the center-of-mass state.

Formally, the first scenario calls for the “mixed” state of the “center-of-mass+spin (CM+S)” composite system in front of the screen, e.g.:

$$\hat{\rho}_{CM+S} = \frac{1}{2}|\uparrow\rangle_{CM}\langle\uparrow| \otimes |-\rangle_S\langle-| + \frac{1}{2}|\downarrow\rangle_{CM}\langle\downarrow| \otimes |+\rangle_S\langle+|$$  \hspace{1cm} (1)

where $|\uparrow\rangle_{CM}$ and $|\downarrow\rangle_{CM}$ are the states of the “center-of-mass” system (up and down trajectories, respectively) and $|\rangle_S$ and $|+\rangle_S$ are the eigenstates of the spin projection along the axis of the external field. The second scenario, however, assumes the pure, entangled state in the composite system in front of the screen e.g. of the simplified form (but see [4] for more details):

$$|\Psi\rangle_{CM+S} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{CM}|+\rangle_S + |\downarrow\rangle_{CM}|-\rangle_S).$$  \hspace{1cm} (2)

Interestingly enough, there does not seem to appear a room for another interpretation of the experiment: the trajectories are either objectively present in front of the screen, or they are not. In the terms of the quantum measurement theory [3], one may say that the quantum-state “reduction” (“collapse”) occurs—if at all—either in-front-of or just on the screen.

Deciding which interpretation is correct is a subtle task, indeed. To this end, recently, a qualitative model (yet, in principle, allowing experimental test) has been proposed [5]. Actually, following the first scenario (stemming yet from the traditional wisdom), a need for the decoherence-like process for
the $CM$ system has been elevated. Namely, the mixed state eq. (1) requires the decoherence-like process [6, 7, 8], which might be responsible for the appearance of the “objective” trajectories in front of the screen. Interestingly enough, in the context of this (standard) interpretation, it seems that the occurrence of decoherence does not have any alternative.

Following this finding, the so-called “relative coordinates” (the subsystem $R$) of an atom has been proposed to play the role of the “environment”, which might be responsible [5-8] for the appearance of the trajectories of the atom in front of the screen. This qualitative picture offers a consistent interpretation of the Stern-Gerlach experiment. First, it completes the standard (and widely used) interpretation of the experiment, and, second it implies non-existence of the definite trajectories (i.e. implies appearance of the spatial interference) for the quantum particles not bearing any internal structure (the subsystem $R$), such as e.g. the free electrons.

In this paper, we elaborate the model of Ref. [5]. Actually, we start an “ab initio” analysis of the atomic system in search for the interaction between $CM$ and $R$, denoted as $\hat{H}_{CM+R}$, that might be responsible for the decoherence [5, 7, 8], i.e. for the appearance of the $CM$’s “objective” trajectories in front of the screen. To this end, we find that “$CM$” is the nuclear-system center-of-mass, not yet the center of mass of the atom as a whole (in contrast to the wide-spread believing [1, 2, 3]), while the corresponding interaction is a reminiscent of the Coulomb interaction between the electrons and the protons in the atom. Our quantitative estimations stem the relative increase of the strength of $\hat{H}_{CM+R}$ for the larger atoms. Interestingly enough, the strength scales as $Z^2$ ($Z$ being the so called atomic number) for larger atoms, being yet totally independent on the mass number $A$. In other words: as distinct from the traditional expectation, the atomic mass does not appear as a parameter of importance for the interpretation of the SG experiment, i.e. for the occurrence of decoherence in $CM$ system.

Of course, in the case of the physical (e.g. the experimental) disproving of our conclusions (i.e. rejecting the decoherence-based model), there remains the alternative scenario–cf. eq. (2)–as the only remaining candidate for interpretation of the Stern-Gerlach experiment. Needless to say, this possibility might reinforce the interpretation of the non-repeatable quantum measurements as the basic source of the irreversible retrieval of the classical information from the quantum world.
2 The atomic subsystems

“Atom” is a complex system—it consists of the electrons, the protons and in general the neutrons. This fundamental definition is however usually inaccessible experimentally. Actually, likewise for the macroscopic systems, the atoms (but also the molecules) are usually observed by observing their center-of-mass degrees of freedom. This is actually the case in the SG experiment: the screen captures the atoms thus revealing their center-of-mass positions, while usually not providing us with any information about the internal state; this is the reason one may assume the ground internal-state of the atom.

In general, the linear canonical transformations

\[
\hat{\mathbf{R}}_{CM} = \sum_{i=1}^{N} m_i \hat{\mathbf{r}}_i / \sum_{i=1}^{N} m_i \tag{3}
\]

introduce the center-of-mass (CM) of an \(N\)-particle system, likewise the “relative coordinates” \(\hat{\mathbf{r}}^R\)

\[
\hat{\mathbf{r}}^R = \hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j \tag{4}
\]

where \(\hat{\mathbf{r}}_m\) represents the position of the \(m\)-th constituent particle of the composite system. The inverse transformations also hold

\[
\hat{\mathbf{r}}_i = \hat{\mathbf{R}}_{CM} + \sum_{\alpha=1}^{N-1} \omega_\alpha \hat{\mathbf{r}}^{(i)}_R \tag{5}
\]

where \(\omega_\alpha = m_\alpha / M\) (\(m_\alpha\)'s are the masses of the constituent particles, while \(M\) is the total mass of the system). Bearing in mind eqs. (3), (4), (5), we point out the first observation as made in [5]: introducing the CM system necessarily calls for the “relative system” \(R\) as formally described (non-uniquely) by the relative coordinates eq. (4). The standard model of the Stern-Gerlach experiment fully discards the (sub)system \(R\) from the considerations, while assuming \(R\) to be in the ground state, and the atomic spin being describable solely by the state of the valent electron. And so appears the basic idea of the decoherence-based approach [5]: introducing the system \(R\) as the CM’s environment might complete the physical picture as described in Introduction.

2.1 The general task

We search for the possible physical origin of the interaction \(\hat{H}_{CM+R}\), which is necessary [6, 7, 8] for the occurrence of the CM’s decoherence. More pre-
cisely: our starting point is the fundamental model of an atom—defined as a collection of the electrons, protons and neutrons—and we search for the (effective) interaction between the properly defined $CM$ and $R$ systems. Once having a proper model of interaction, we investigate the possible “pointer basis (states)” and the relative strength of the interaction as essential for the description of the desired decoherence effect. The analysis bears some subtlety yet, as apparent in the sequel.

In order to elaborate this decoherence–based model, we begin from the very definition of the atomic Hamiltonian, for the atom defined as a collection of the electrons, protons and the neutrons. Then, the atomic Hamiltonian reads:

$$\hat{H} = \sum_{i=1}^{Z} \hat{T}_{Ei} + \sum_{j=1}^{Z} \hat{T}_{pj} + \sum_{k=1}^{A-Z} \hat{T}_{ nk} + \hat{V}_{Coul}^{ee} + \hat{V}_{Coul}^{ep} + \hat{V}_{Coul}^{pp} + \hat{V}_{nucl}$$

(6)

with the following meaning for each term: $\hat{T}$ stands for the kinetic terms, $\hat{V}_{Coul}$ for the Coulomb interaction of the pairs of particles ($ee$-the electrons, $ep$-the electron-proton, $pp$-the protons pairs), and the nucleon interaction for a pair $(n, n')$ of nucleons is given e.g. by [9]:

$$\hat{V}_{nucl}^{nmn'} \equiv -\frac{\gamma^2 \exp(-\mu |\hat{r}_n - \hat{r}_{n'}|)}{\vert \hat{r}_n - \hat{r}_{n'} \vert}$$

(7)

where $\gamma$ is a constant and $r = \frac{1}{\mu}$ is the range of the nuclear forces. For simplicity, we omit the comparatively weak interactions, such as the spin-spin or spin-orbit interactions in the atom.

(a) The atomic center of mass

We apply the linear canonical transformations eqs. (3), (4) to the fundamental model of “atom”, eqs. (6), (7). Instead of the model “electrons + protons + neutrons”, we introduce the center of mass $CM$ and the “relative system” $R$ of the atom as a whole, thus defining the following tensor-product structure of the atomic Hilbert state space: $\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_{R} \otimes \mathcal{H}_{S}$, where the index $S$ refers to the atomic spin.

Bearing in mind the standard model of the SG experiment, the atom exposed to the external magnetic field is described by the following Hamiltonian (that is a straightforward transform of eq. (6)):

$$\hat{H} = \hat{T}_{CM} + \hat{H}_{R} + \hat{H}_{CM+S},$$

(8)
where $\hat{H}_{CM+S} = \mu_B B(\hat{z}_{CM}) \otimes \hat{S}_z$ is the standard term [5] coupling the external magnetic field (for simplicity lying along the z-axis) and the atomic spin \( \hat{S}_z \), while the \( R \)-system’s self-Hamiltonian reads:

$$
\hat{H}_R = \sum_{\alpha=1}^{Z+A-1} \hat{T}_{\alpha} + \hat{V}_{\text{nucl}}^{(R)} + \hat{V}_{\text{Coul}}^{(R)} + \hat{M}_{\eta\nu}^{(R)},
$$

(9)

in analogy with eq. (8), and $\hat{M}_{\eta\nu}^{(R)}$ is the internal interaction (cf. Appendix 1).

Regarding eq. (9), it is important to note: being the distance-dependent, all the original interactions (the Coulomb interactions and the nuclear interactions in eq. (6)) transform into the “external fields” (the one-particle potentials $V(\hat{\rho}_R)$) for the “relative particles” system $R$ (cf. Appendix 1). These effective potentials are the terms of the $R$’s self-Hamiltonian $\hat{H}_R$. Needless to say, this gives the exact separation of $CM$ and $R$ that does not leave a room for the desired decoherence of the $CM$ states.

In the terms of the quantum states, the initial state, e.g.,

$$
|\Psi\rangle_{CM}|0\rangle_R \frac{1}{\sqrt{2}} (|+\rangle_S |−\rangle_S)
$$

(where we omit the symbol of the tensor product) dynamically transforms as presented by the following simplified expression:

$$
\hat{U} |\Psi\rangle_{CM}|0\rangle_R \frac{1}{\sqrt{2}} (|+\rangle_S + |−\rangle_S) = \frac{1}{\sqrt{2}} (|\uparrow\rangle_{CM}|0\rangle_R |−\rangle_S + |\downarrow\rangle_{CM}|0\rangle_R |+\rangle_S).
$$

(10)

$\hat{U}$ is the unitary operator of evolution in time generated by the Hamiltonian eq. (8)–recall that the system $CM + R + S$ (plus the external classical magnetic field) is an isolated quantum system. By tracing out the system $R$, one obtains the entangled state eq. (2).

So, we conclude: the introduction of $CM$ and $R$ of the atom as a whole does not provide the occurrence of decoherence as desired in Section 1. Interestingly enough, this is not the end of the program, as it is shown in the sequel.

(b) The nucleus center of mass

More than 99.99 percents of the atomic mass is placed in the atomic nucleus. Practically, it is truly hard to distinguish between the atomic and the nucleus
center-of-mass systems. So, we investigate another application of eqs. (3), (4): we introduce the center-of-mass system and the “relative system” for the atomic nucleus while leaving the electrons variables intact.

Introducing the collective degrees of freedom of the atomic nucleus is the standard procedure in nuclear physics [10, 11]. On the other side, the similar idea appears in certain models of the quantum measurement theory, unfortunately not yet being fully elaborated [12]. So, introducing the center of mass of the atomic nucleus not yet involving the electrons is physically legitimate a procedure. Its relation to the procedure presented in (a) will be discussed in Section 4.

Then, “atom” is a composite system defined as “E+CM+R+S”, where E stands for the electrons-system, CM and R for the nucleus center-of-mass and the “relative” systems, respectively, while S is the atomic spin. The corresponding Hilbert state space of “atom” now reads: $\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_{CM} \otimes \mathcal{H}_R \otimes \mathcal{H}_S$; we hope the use of the same notation for the center-of-mass and the relative system as in (a) will not produce any confusion–further on, we are only concerned with the case (b) under consideration.

Now, the standard model of the SG experiment is defined by the following form of the atomic Hamiltonian (in analogy with eq. (8)):

$$\hat{H} = \hat{H}_E + \hat{T}_{CM} + \hat{H}_R + \hat{H}_{CM+S} + \hat{H}_{E+CM+R}. \quad (11)$$

Certainly, the Hamiltonian $\hat{H}$ in eq. (8) and eq. (11) is the one and the same observable–it is just written in the different forms, yet in eq. (11) appearing the interaction term for $E$, $CM$ and $R$ systems (cf. Appendix 1):

$$\hat{H}_{E+CM+R} = k \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{1}{|\vec{r}_{Ei} - \vec{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}(j) \hat{\rho}_{R\alpha}(j)|}. \quad (12)$$

Physically, the tripartite interaction $\hat{H}_{E+CM+R}$ is a particular form of the Coulomb interaction between the atomic electrons and the protons. Fortunately enough, this tripartite interaction can be reduced to a bipartite interaction coupling $CM$ and $R$ systems–cf. below.

The close inspection of the rhs of eq. (11) justifies the application of the adiabatic approximation that in its zeroth order separates the electrons system from the rest (cf. Appendix 2). More precisely: the electrons are too light relative to both the $CM$- and $R$-mass, thus allowing the standard
procedure of the adiabatic approximation [2, 13, 14]. On the other side, for the realistic atoms (Appendix 2), the CM and \( R \) mass-ratio does not allow the application of the adiabatic approximation. So, we expect the approximate separation of the electrons-state from the rest, \( CM + R + S \), as well as non-negligible entanglement between \( CM \) and \( R \). Formally, the state of this decomposition of “atom” in subsystems reads:

\[
|\chi\rangle_E|\Phi\rangle_{CM+R+S} + |O(\kappa)\rangle_{E+CM+R+S},
\]

where the small term (that bears entanglement, in general, of all of the subsystems) is of the norm \( \kappa^{3/4} \), where \( \kappa = \max\{\kappa_1, \kappa_2\} \), and \( \kappa_i \) are the corresponding mass ratios, cf. Appendix 2.

In order to obtain the dynamics of the ”slow” system \( CM + R \) (i.e. of \( CM + R + S \)), one should discard the electrons system as (cf. Appendix 2):

\[
\hat{H}_{CM+R+S} \equiv E \langle \chi|\hat{H}|\chi\rangle_E \approx \hat{T}_{CM} + \hat{H}_R + \hat{H}_{CM+S} + \hat{H}_{CM+R},
\]

where

\[
\hat{H}_{CM+R} \equiv E \langle \chi|\hat{H}_{E+CM+R}|\chi\rangle_E
\]

represents the effective (the electrons–system mediated) interaction between \( CM \) and \( R \).

Now, due to the two interaction terms, \( \hat{H}_{CM+S} \) and \( \hat{H}_{CM+R} \) in eq. (14), it is straightforward dynamically to obtain entanglement in the dominant term of the state in eq. (13), \( |\Phi\rangle_{CM+R+S} \). Actually, for the initial state

\[
|\Psi\rangle_{CM}|0\rangle_R \frac{1}{\sqrt{2}}(|+\rangle_S + |−\rangle_S),
\]

it is easy to obtain entanglement (compare to eq. (10)) as:

\[
\hat{U}|\Psi\rangle_{CM}|0\rangle_R \frac{1}{\sqrt{2}}(|+\rangle_S + |−\rangle_S) \approx \frac{1}{\sqrt{2}}(|\uparrow\rangle_{CM}|1\rangle_R|−\rangle_S + |\downarrow\rangle_{CM}|2\rangle_R|+\rangle_S),
\]

where \( \hat{U} \) is generated by \( \hat{H} \) represented in eq. (14). Now, assuming the orthogonality \( R\langle 1|2\rangle_R \approx 0 \), by tracing out the “environment” \( R \) from the rhs of eq. (16) follows the mixed state eq. (1) for \( CM + S \) system.

3. The occurrence of decoherence in SG experiment

In general, the interaction between the open system and its environment is a necessary, not yet the sufficient condition for decoherence. The occurrence of
decoherence is a subtle an issue, indeed [7, 8], formally depending on the (at least approximate) orthogonality of the environmental states–e.g., the states $|i\rangle_R$ ($i = 1, 2$) on the rhs of eq. (16). So, the presence of the interaction $\hat{H}_{CM+R}$ opens in principle a room for decoherence of $CM$’s states that are yet to be determined.

The occurrence of decoherence generated by $\hat{H}_{CM+R}$ competes with the recurrence that bears the two origins. First, the dynamics generated by the open system’s self-Hamiltonian ($\hat{H}_{CM}$) produces coherence as–cf. below–not commuting with the interaction term ($\hat{H}_{CM+R}$). Second, the overall dynamics (generated by the total Hamiltonian $\hat{H}$) is unitary, and for the small environment, the recurrence of the initial coherence may be effective. Unfortunately, the method of the unitary operator $\hat{U}$ applied here does not in general allow the estimations of the decoherence time and of the recurrence time. On the other side, by investigating solely the interaction term, one may conclude about the possible “pointer basis” of the open system $CM$ [7, 8].

Therefore, the task for this section reads: to calculate the interaction term $\hat{H}_{CM+R}$ and to estimate both the possible “pointer basis” states (within the context of the standard model of SG experiment) as well as the dependence of the interaction on the size of the atom; following eq. (12), the “size of the atom” is determined by the number of the electrons (protons), $Z$, not yet by the “mass number” $A$. This way, we estimate the relative strength of the interaction and its efficiency in producing decoherence for the $CM$’s pointer basis states. Investigating the decoherence- and the recurrence-time requires the different methods (e.g. certain types of the master equations, cf. e.g. [15]) and comes out of the scope of the present paper.

3.1 The interaction $\hat{H}_{CM+R}$

The dynamics of the slow system $CM + R + S$ is generated by $\hat{H}_{CM+R+S}$, eq. (14). However, essential for our program is the interaction term eq. (15). Substitution of eq. (12) in eq. (15) directly gives:

$$\hat{H}_{CM+R} = k\sum_{i=1}^{Z} \sum_{j=1}^{Z} \langle \chi | \hat{r}_{Ei} - \hat{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}^{(j)} \hat{p}_{R\alpha}^{(j)} | \chi \rangle_E$$

where $|\chi\rangle_E$ are the eigenstates of the adiabatically-defined electrons-system self-Hamiltonian $\hat{H}_E$. Being interested in the $Z$-dependence of $\hat{H}_{CM+R}$, we
introduce the following choice: for $|\chi\rangle_E$, we take the $Z$-electrons Slater determinant constructed from the hydrogen-atom states. So, we neglect both, the adiabatic approximation as well as the corrections for the many-electrons atoms. We believe that these simplifications will not significantly change the result, while bearing in mind yet that the states appearing in $|\chi\rangle_E$ refer to the point-like nucleus.

The expression eq. (17) bears formally identical terms for all the protons, so we can write:

$$\hat{H}_{CM+R} = Z\hat{H}'_{CM+R}$$

where

$$\hat{H}'_{CM+R} = k \sum_{i=1}^{Z} \frac{1}{|\tilde{r}_{Ei} - \tilde{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}\tilde{\rho}_{R\alpha}|}|\phi_i\rangle_E, \forall j.$$  

(19)

Due to the exact orthogonality of the one-electron states $|\phi_i\rangle_E$ appearing in $|\chi\rangle_E$, one easily obtains:

$$\hat{H}'_{CM+R} = k \sum_{i=1}^{Z} \frac{1}{|\tilde{r}_{Ei} - \tilde{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}\tilde{\rho}_{R\alpha}|}|\phi_i\rangle_E,$$  

(20)

where the $i$ enumerates the electrons, i.e. the states $|\phi_i\rangle_E$ appearing in the Slater determinant $|\chi\rangle_E$.

Measuring (as usual in quantum mechanics of molecules [13, 14]) the electrons positions from the (point-like) atomic nucleus, $\tilde{r}_{Ei} \rightarrow \tilde{r}_{Ei} - \tilde{r}_{CM}\hat{I}_E$, the expression (20) obtains the following form in the position-representation:

$$\hat{H}_{CM+R} = kZ \sum_{i=1}^{Z} \int \frac{|\phi_i(\tilde{r}_{Ei} - \tilde{r}_{CM})|^2}{|\tilde{r}_{Ei} - \tilde{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}\tilde{\rho}_{R\alpha}|} d^3\tilde{r}_{Ei}.$$  

(21)

By introducing the “shift”: $\tilde{r}_{Ei} - \tilde{r}_{CM} \equiv \tilde{\xi}_i$, one obtains the following form for $\hat{H}_{CM+R}$ (while omitting the unnecessary index $i$ in the vector $\tilde{\xi}_i$):

$$\hat{H}_{CM+R} = kZ \sum_{i=1}^{Z} \int \frac{|\phi_i(\tilde{\xi})|^2}{|\tilde{\xi} - \tilde{\Omega}_{CM+R}|} d^3\tilde{\xi}.$$  

(22)

where $\tilde{\Omega}_{CM+R} \equiv -\tilde{r}_{CM}\hat{I}_E + \tilde{R}_{CM} + \sum_{\alpha=1}^{A-1} \omega_{\alpha}\tilde{\rho}_{R\alpha}$. 

10
The method for calculating (22) can be found in Ref. [16]. The details are presented in Appendix 3, and the result for the atoms with the “closed shells”, reads:

\[
\hat{H}_{CM+R} = kZ \sum_{n} \sum_{\ell=0}^{n-1} \sum_{g=0}^{\ell} \sum_{t=0}^{g} \frac{(2\ell + 1)}{2n2^{n-1}} \left( \frac{2(n - \ell - 1) - 2g}{n - \ell - 1 - g} \right) \times \\
\times \frac{(2g)!}{g!(2\ell + 1 + g)!} \left( \frac{2g + 2(2\ell + 1)}{2g - t} \right) \frac{(-2)^t}{t!} \\
\left\{ (2\ell + t + 2)! \left( 1 - \exp \left( -\frac{2Z\hat{\Omega}}{na_\mu} \right) \right) \sum_{f=0}^{2\ell + t + 2} \frac{(2\hat{\Omega})^f}{f!} \right\} \hat{\Omega}^{-1} \\
+ \frac{2Z}{na_\mu} (2\ell + t + 1)! \exp \left( -\frac{2Z\hat{\Omega}}{na_\mu} \right) \sum_{f=0}^{2\ell + t + 1} \frac{(2\hat{\Omega})^f}{f!} \right\}, 
\]

where \( \hat{\Omega} \equiv |\hat{\Omega}_{CM+R}| \) and \( a_\mu \) is the analog of the Bohr radius. In eq. (22), the index \( i \) enumerates the electrons, i.e. the one-electron quantum states determined uniquely by the hydrogen-atom quantum numbers, \( n, \ell \) and \( m \). This is the origin and the meaning of the indices \( n, \ell \) and \( m \) in eq. (23).

### 3.2 The pointer basis and the relative strength of interaction

The expression (23) is the effective interaction of \( CM \) and \( R \) subsystems of the atomic nucleus. The interaction stems directly from the couplings present in the exponential-, polynomial- as well from the \( \hat{\Omega}^{-1} \)–terms.

More precisely, due to definition of \( \hat{\Omega} \) (cf. above) one directly infers the information about the possible pointer basis [7, 8] for \( CM \) system. It is rather apparent: \( \hat{H}_{CM+R} \) is exactly diagonalizable in the continuous eigenbasis of the observable \( \hat{R}_{CM} \): \( \langle \hat{R}_{CM} | \hat{H}_{CM+R} | \hat{R}_{CM} \rangle = 0 \) for \( \hat{R}_{CM} \neq \hat{R}'_{CM} \), thus promoting [7, 8] the basis \( |\hat{R}_{CM}\rangle \) as the possible pointer basis. However, there is an even better choice of the pointer basis, i.e. of the “preferred set” of (non-orthogonal) yet normalizable states.

Actually, the minimal–uncertainty (the “coherent”) states \( |\Psi_{\mu\nu}\rangle_{CM} \) modeling the \( CM \) wave-packets approximately diagonalize \( \hat{H}_{CM+R} \)–as presented in [3, 17]. These states approximate both \( \hat{R}_{CM} \) and its conjugate momentum-operator \( \hat{P}_{CM} \), thus allowing one to define the semi-classical trajectories of
the wave packet modeled by $|\Psi_{\mu\nu}\rangle_{CM}$. The transition from the exact (non-normalizable) states $|\vec{R}_{CM}\rangle$ (or $|\vec{P}_{CM}\rangle$) to the “coherent states” $|\Psi_{\mu\nu}\rangle_{CM}$ is generally established by von Neumann [3] (cf. ch. 5 therein). So, we may promote the “coherent states” $|\Psi_{\mu\nu}\rangle_{CM}$ as the approximate pointer basis that, fortunately enough, fits with the wave-packet model [1, 2, 3, 5] of the atoms in the SG experiment.

Finally to this section, our aim is to determine the relative strength of $\hat{H}_{CM+R}$, i.e. to determine its $Z$–dependence. This is certainly essential for our program: the cases $Z = 0, 1$ can not be described by the model (b), so we want to estimate the strength of $\hat{H}_{CM+R}$ for the different values of $Z > 1$.

To this end, we first want to stress: in eq. (23) appear the multiplications of the exponential functions of $Z$ with the polynomials of $Z$. As it can be easily shown, already for $Z \sim 10$, one may neglect all such multiplications, thus obtaining the “linear” dependence of $\hat{H}_{CM+R}$ of $Z$ (for $Z \sim 10$):

$$\|\hat{H}_{CM+R}\| \approx \beta Z \|\hat{\Omega}^{-1}\|$$

where

$$\beta \equiv k \sum_{n} \frac{n-1}{n} \sum_{\ell=0}^{n-1} \sum_{g=0}^{2g} \frac{(2\ell + 1)}{2n2^{2(n-\ell-1)}} \left( \frac{2(n-\ell-1) - 2g}{n-\ell-1-g} \right) \times$$

$$\times \frac{(2g)!}{g!(2\ell + 1 + g)!} \left( \frac{2g + 2(2\ell + 1)}{2g-t} \right) \frac{(-2)^t}{t!} [((2\ell + t + 2)!].$$

The appearance of the quantum numbers $n, \ell$ and $m$ in eq. (25) apparently points the $Z$–dependence of the parameter $\beta$. Now, bearing in mind that eq. (23) refers to the atoms with the “closed shells”, we prepared a numerical estimation for the $Z$–dependence of $\beta$, yet for the particular values: $Z=10, 28, 60, 110, 182$ (i.e. for $n=2, 3, 4, 5, 6$). The result is presented in Fig. 1, clearly stemming the linear dependence of $\beta$ on $Z$.

Therefore, we emphasize: except for the small atoms, i.e. for the small values of $Z$ ($Z < 10$), one obtains that the strength of interaction $\hat{H}_{CM+R}$ approximately scales as $Z^2$, thus clearly giving advantage to the larger atoms (with the larger $Z$) in the sense of the possible occurrence of decoherence, which may select (cf. above) the “coherent states” as the approximate pointer basis.

### 3.3 The physical picture
Figure 1: Dependence of the factor $\beta$ on $Z$ for the atoms with the closed shells.

The possible occurrence of decoherence regarding the model (b) gives the following physical description of the SG experiment.

The free moving of the nucleus center-of-mass wave-packet is adiabatically followed by the electrons. The external magnetic field quickly induces entanglement between the $CM$ system and the atomic effective spin $S$ [5]. The fast formation of entanglement in $CM + S$ system can be easily shown to be followed by the formation of entanglement between $CM$ and $R$ system (as generated by $\hat{H}_{CM+R}$), the later also being adiabatically decoupled from the electrons system $E$. The subsequent decoherence gives rise to the semi-classically well-defined trajectories of the $CM$ system, thus giving rise to the mixed state eq. (1). Certainly, the screen is supposed to serve as a passive recorder of the “objectively” existing trajectories of the nucleus center-of-mass—the atomic-nucleus center-of-mass serves as a “quantum apparatus” for the atomic spin, that fits essentially with the standard interpretation of the SG experiment.

However, as distinct from the standard interpretation, our model of the SG experiment removes the focus from both the atomic center-of-mass as
well as from the *atomic mass*. As to the later, the parameter of interest is *not the atomic mass* but the number of the protons (electrons) in the atom, $Z$. The interaction $\hat{H}_{CM+R}$, that is responsible for the possible occurrence of decoherence, scales (approximately) as $Z^2$, thus clearly giving advantage to the larger atoms as possibly the decoherent systems.

Interestingly enough, the presence of both $R$ and $E$ systems is necessary for the occurrence of decoherence. First, nonexistence of $R$ stems nonexistence of the interaction $\hat{H}_{CM+R}$. On the other side, nonexistence of the electrons system would imply nonexistence of the tripartite interaction-term $\hat{H}_{E+CM+R}$ and thus—cf. eq. (15)—again, nonexistence of the interaction $\hat{H}_{CM+R}$ (in some analogy with the model (a)). Therefore, the model (b) stems the absence of the SG-effect for the free electrons, protons and neutrons (the later already being well-known from [18]), for the bare atomic nuclei as well as for the hydrogen atom. The effect can be considered to be rather weak for the small atoms ($Z < 10$) and more effective for larger atoms. The precise form of $\hat{H}_{CM+R}$ for the atoms with “incomplete shells” can be easily obtained by adapting eq. (23)—e.g. by adding the new terms defined by the states of the electrons not (classically speaking) belonging to the “closed shells”. This, however, does not seem to change anything in our qualitative estimation of the $Z$-dependence of $\hat{H}_{CM+R}$.

The physical basis of a possible experimental test of the model (b) can be now directly borrowed from Ref. [5]. While we do not guarantee that the *screen really captures the atomic-nucleus-, instead of the atomic-center-of-mass*, the experimentally observed absence of the effect e.g. for the massive, bare atomic nucleus might be a significant support for the model considered.

Needless to say, if the screen captures the center-of-mass of the *atom as a whole*, one seems to be obliged to reinterpret the SG experiment. Actually, (as emphasized in Introduction) then the screen itself would be responsible for the decoherence-like effect by transforming the pure state eq. (2) into incoherent mixture eq. (1).

4. Discussion

The choice of the collective degrees of freedom of atomic nucleus naturally—and, probably, unavoidably—calls for the adiabatic approximation for the atomic electrons system. The nuclear physics theory seems silent in this regard [11]. The estimates of the mass-ratios (cf. Appendix 2) justify the adiabatic cut of the electrons from both $CM$ and $R$ systems, not yet jus-
tifying the adiabatic cut of $R$ from $CM$–the later giving possibly rise to the occurrence of decoherence that is induced by the bipartite interaction $\hat{H}_{CM+R}$.

The interaction model eq. (23) justifies (cf. Section 3.2) the wave-packets as the approximate pointer basis thus recovering the standard interpretation of SG experiment. As a benefit, we obtain that for the larger atoms, the decoherence-inducing interaction is stronger, while for the free electrons (likewise the protons, and particularly the neutrons [18,19]), as well as for the hydrogen atom, one can not expect decoherence due to nonexistence of the subsystem $R$. Of course, we have not proved the occurrence of decoherence–this requires the different methods (e.g. in estimating the decoherence- and the recurrence-time [15]). Nevertheless, we have prepared all the necessary data in the case the decoherence may be considered to be effective, yet bearing some subtlety.

Actually, one may pose the following question. The straightforward extrapolation of our model may seem to imply the occurrence of decoherence for the larger systems–such as the large molecules, and even more for the macroscopic systems–due to the internal environment. Bearing this in mind, one may ask if our model contradicts the experimentally verified coherence (interference) of the large-molecules’ center-of-mass states [20]. Interestingly enough, our model directly answers this question: the $R$ system can not be considered to induce decoherence of $CM$ system for large quantum systems. Namely, the mass ratio (Appendix 2) $\kappa_3 = \mu/M$ decreases with the increase of the number of particles in the system, thus in effect allowing the adiabatic separation between $CM$ and $R$; the larger the system, the larger the total mass $M$, while $\mu$ remains virtually unchanged. In the formal terms: the large system’s state is presented approximately (the larger the system the better the approximation) by the first term in eq. (13). Therefore, we conclude that the occurrence of decoherence for the large systems is not induced by the internal environment–as supported by the macromolecules decoherence experiments [21].

Finally, we should clarify the relation between the two different models of “atom” as defined by (a) and (b) in Section 2.1. Actually, one may ask, after all, if “atom” is an isolated or a decoherent (open) system. And this is another subtle issue, indeed.

The two models are the legitimate physical models of the one and the same physical system–the “atom”. They represent the different faces (facets) of “atom”, thus not necessarily providing the same or even equivalent informa-
tions about “atom”. To this end, a particular approach has recently been proposed in the context of the problem “what is system” [22-24]. Actually, the different models of a composite system may still give rise to the mutually non-equivalent physical and information-theoretic descriptions of the composite system itself. On the intuitive level, answering what is system may bear some arbitrariness, thus promoting relativity of the very fundamental physical concept of “system” [22, 23]. In a sense, the two models (a) and (b) are paradigmatic for the system-relativity issue: both models describe the one and the same composite system—the “atom”—while the different separations of the system into subsystems may still give rise to the different physical descriptions of “atom”. Now, the observation of the atomic center-of-mass would describe the “atom” as an isolated quantum system. On the other side—if we trust the model (b)—the observation of the nucleus center-of-mass (as it might be the case in SG experiment) might give rise to the opposite answer—the “atom” may appear as a decoherent, and certainly as an open system. And the system relativity stems [23, 24]: the two answers do not raise a contradiction; they just represent the two different descriptions of “atom” still depending on the “point of view”, i.e. on the observables targeted in the course of observation. So, there is not a unique answer to the above question regarding “atom” as isolated/open system. Observation of “atom” as isolated quantum system might be equally physically realistic as the observation of “atom” as open quantum system.

Needless to say, we always observe (or perceive) only a “fraction” (the comparatively small subsystems) of a composite system, and still try to generate the proper description of the composite system as a whole. This, however is a usual conjecture not necessarily supported by the system-relativity [22-24]. To this end, now appears the following question: whether or not the models (a) and (b) may be reduced to each other, or to another (not yet known), more fundamental model of “atom”? In this regard, we just want to emphasize: mutual irreducibility of the two models seems to stem as a corollary of the standard interpretation of the decoherence process [6]. Conversely, the mutual reducibility of the models (as well as existence of the alternative, more fundamental model) might hide a clue for giving the answer to the problem of the “transition from quantum to classical” [6]—e.g. to the transition from the isolated to the open “atom”. In this sense, our study may sharpen this truly fundamental issue of modern quantum mechanics.

So, one may say, that we offer a consistent physical model of “atom” that might support the standard interpretation of the Stern-Gerlach exper-
iment. Needless to say, in the case of e.g. experimental disproving of this (the decoherence–based model), there is the alternative (cf. Introduction) interpretation of the Stern-Gerlach experiment that still requires elaboration in the context of the non-repeatable quantum measurement which is out of the scope of the present study.

5. Conclusion

If the screen in the Stern-Gerlach experiment capturing the atoms brings the information about the atomic-nucleus center-of-mass ($CM$), then there exists a decoherence-based model of the experiment that fits essentially with the standard interpretation of the experiment. The $CM$ system is assumed to be monitored by the “relative particles” ($R$) subsystem of the atomic nucleus. The possible occurrence of decoherence is due to the (electrons-system-mediated) interaction between $CM$ and $R$. The interaction scales approximately as $Z^2$ ($Z$ being the number of protons) for larger atoms providing the $CM$-wave-packets as the approximate pointer basis. In the case of e.g. experimental disproving of this model, there seems only one alternative to remain: the screen is responsible for “reduction” (“collapse”) of the entangled state of $CM + S$ system ($S$ denoting the atomic spin) thus mimicking existence (as assumed in the standard interpretation of the experiment) of the definite trajectories of $CM$ system in front of the screen.

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Appendix 1
The canonical transformations eqs. (3), (4) define the center-of-mass system $\text{CM}$ with the total mass $M$, and the set of the “relative particles” defined by the relative coordinate $\hat{\vec{\rho}}_{R\alpha}$, with the reduced masses $\mu_\alpha$ which define the kinetic terms $\hat{T}_{R\alpha} = \frac{\hat{\vec{P}}_{R\alpha}^2}{2\mu_\alpha}$, that appear in eq. (9).

Due to the distance-dependence of all of the Coulomb- and the nuclear-interaction-terms, all these interactions obtain the form $V(\hat{\vec{\rho}}_{R\alpha})$. I.e. the interactions become the (effective) external potentials for $R$. E.g.,

$$V_{\text{Coul}}^{(R)} = k \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{1}{|\hat{\vec{\rho}}_i^R|},$$

(26)

where $\hat{\vec{\rho}}_i^R = \hat{\vec{r}}_{Ei} - \hat{\vec{r}}_{pj}$, $|k| = \frac{e^2}{4\pi\varepsilon_0}$ and, according to eq. (7)

$$V_{\text{nucl}} = -\gamma^2 \frac{\exp(-\mu|\hat{\vec{\rho}}_R^{nn'}|)}{|\hat{\vec{\rho}}_R^{nn'}|}.$$

(27)

It can be easily shown that, regarding eqs. (3), (4), the Hamiltonian obtains the form (8), where (cf. eq. (9)) appears the internal interaction in the $R$ system [25]:

$$\hat{M}_{\eta\nu}^{(R)} = \sum_{n=1}^{A+Z-1} \sum_{\nu=1}^{A+Z-1} \frac{m_{\eta+1}m_{\nu+1}}{M} \hat{\vec{\rho}}_{\eta} \cdot \hat{\vec{\rho}}_{\nu}.$$  

(28)

Considering the atom as a whole, the CM mass $M = Z(m_e + m_p) + (A-Z)m_n$, where $A$ is the mass number of the atom. The relative masses $\mu_\alpha$ take the general form

$$\mu_\alpha = \frac{m_{\alpha+1}(M - m_{\alpha+1})}{M},$$

(29)

where $m_\alpha$ are the constituent-particles masses.

On the other side, for the nucleus center-of-mass, $M = Z m_p + (A-Z)m_n \approx Am$, while eq. (29) gives:

$$\mu = (1 - \frac{1}{A})m,$$

(30)

where we simplify $m_n = m_p \approx m$. 

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Regarding the CM and R as the nucleus subsystems, the original Coulomb interaction (cf. eq. (6)) that reads:

$$\hat{V}_{\text{Coul}}^{ep} = k \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{1}{|\vec{r}_{Ei} - \vec{r}_{pj}|}.$$  \hspace{1cm} (31)

obtains the following form due to eq. (4):

$$\hat{V}_{\text{Coul}}^{ep} \equiv \hat{H}_{E+CM+R} = k \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{1}{|\vec{r}_{Ei} - \hat{R}_{CM} - \sum_{a=1}^{A-1} \omega_{(j)} \hat{\rho}_{R_{(j)}}|}. \hspace{1cm} (32)$$

The rhs of eq. (32) is apparently a tripartite interaction $\hat{H}_{E+CM+R}$, coupling $E$, CM and R, as it appears in eq. (12).

**Appendix 2**

The kinetic terms $\hat{T}_{CM}$ and $\hat{T}_{R_{\alpha}}$, as well as $\hat{T}_{E}$ are proportional to $M^{-1}$, $\mu_{\alpha}^{-1}$, $m_{e}^{-1}$ respectively, where (cf. Appendix 1) $M$ is the total mass of the nucleus, $\mu$ is the reduced mass for the nucleus–system and $m_{e}$ is the mass of an electron.

Then there appear the three parameters,

$$\kappa_{1} \equiv \frac{m_{e}}{M}, \quad \kappa_{\alpha} \equiv \frac{m_{e}}{\mu_{\alpha}}, \quad \kappa_{3} \equiv \frac{\mu}{M}, \hspace{1cm} (33)$$

where $\mu = \min\{\mu_{\alpha}\}$; in eq. (30), we approximated $\mu_{\alpha} \approx \mu$, \forall $\alpha$. So, for the realistic atoms, $Z \lesssim 10^{2}$, one may state the following estimates:

$$\kappa_{1} \lesssim 10^{-4}, \quad \kappa_{2} \lesssim 10^{-3}, \quad \kappa_{3} \lesssim 10^{-2}, \hspace{1cm} (34)$$

and $\kappa_{2} = \max\{\kappa_{\alpha}^{2}\}$.

The values eq. (34) justify the applicability of the adiabatic approximation [2, 13, 14] for $E + CM + R$ as follows: the small values of $\kappa_{1,2}$ justify the adiabatic cut of the electronic system ($E$) from both $CM$ and $R$ systems, while $CM$ and $R$ can not be properly mutually separated.

Now, the standard adiabatic approximation stems [2, 13, 14]: (a) the exact state of $E + CM + R + S$ system reads:

$$|\chi_{E}(\Phi)_{CM+R+S} + |O(\kappa))_{E+CM+R+S}, \hspace{1cm} (35)$$
where $\kappa = \max\{\kappa_1, \kappa_2\}$, while (b) the “slow” system $CM + R + S$ is described by the following effective Hamiltonian:

$$\hat{H}_{CM + R + S} \cong_E \langle \chi | \hat{H} | \chi \rangle_E.$$  \hspace{1cm} (36)

For the larger systems (e.g. macromolecules, or the macroscopic systems), due to the large number of the particles, the total mass $M$ increases, while the relative mass $\mu' = \max\{\mu_\alpha\}$ does not significantly change. Then the parameter $\kappa'_3 \equiv \mu'/M$ may satisfy the applicability of the adiabatic approximation also for $CM$ and $R$ systems.

**Appendix 3**

The calculation of expression (22) can be performed on the basis of the “addition theorem” for the spherical harmonics [16]:

$$\frac{1}{|\vec{\xi} - \vec{\tau}|} = \sum_{s=0}^{\infty} \frac{\rho_{<s+1}}{|\rho_>^{s+1}|} \left[ \frac{4\pi}{2s + 1} \sum_{m_s = -s}^{s} Y^{m_s*}_s(\vartheta_{\tau}, \varphi_{\tau})Y^m_s(\vartheta_{\xi}, \varphi_{\xi}) \right]$$  \hspace{1cm} (37)

where $\vec{\tau} = (\tau, \vartheta_{\tau}, \varphi_{\tau})$ and $\vec{\xi} = (\xi, \vartheta_{\xi}, \varphi_{\xi})$, while $\rho_\leq = \xi, \rho_\geq = \tau$ if $\xi < \tau$, and $\rho_\leq = \tau, \rho_\geq = \xi$ if $\tau < \xi$. By “$Y$” we denote the spherical harmonics.

Taking for the one-electron states the standard stationary states for the hydrogen-like atom:

$$\phi_i(\vec{\xi}) \equiv \phi_{n\ell m}(\vec{\xi}) = R_n(\xi)Y^\ell_m(\vartheta_\xi, \varphi_\xi),$$  \hspace{1cm} (38)

while bearing in mind $\vec{\tau} \equiv \vec{\Omega}$, the rhs of eq. (22) takes the following form:

$$k_Z \sum_n \sum_{\ell=0}^{n-1} \sum_{\ell'=-\ell}^{\ell} \sum_{s=0}^{\infty} \frac{4\pi}{2s + 1} \sum_{m_s = -s}^{s} Y^{m_s*}_s(\vartheta_{\tau}, \varphi_{\tau}) \times$$

$$\times \int \frac{\rho_<}{|\rho_>^{s+1}|} R^{2}_n(\xi)\xi^2d\xi \times$$

$$\times \int Y^m_{\ell'}(\vartheta_\xi, \varphi_\xi)Y^{m'*}_{\ell'}(\vartheta_\xi, \varphi_\xi)Y^{m_s*}_{s}(\vartheta_{\xi}, \varphi_{\xi}) \sin \vartheta_{\xi}d\varphi_{\xi}d\vartheta_{\xi}. \hspace{1cm} (39)$$

In eq. (39) we first calculate the sum $\sum_{m_{\ell} = -\ell}^{\ell} I^{(m_{\ell})}_{\vartheta\varphi}$, where $I_{\vartheta\varphi}$ is the integral over $\vartheta_{\xi}$ and $\varphi_{\xi}$. Following the result given in [26] (cf. eqs. (1.2-26), (1.2-27) therein), we obtain for the atoms with the “closed shells”:

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\[
\sum_{m=\ell}^{\ell} I_{\delta_0}^{(m)} = \frac{2\ell + 1}{\sqrt{4\pi}} \delta_0 \delta_{m,0},
\]  
(40)

where appear the two Kronecker-deltas.

The integration over \( \xi \) in eq. (39) is performed in a few steps. First, the integral over \( \xi \) can be written as [16]:

\[
I_R = \int_0^\tau \frac{\xi^s}{\tau^s+1} R_{n\ell}^2(\xi)\xi^2 d\xi + \int_\tau^\infty \frac{\tau^s}{\xi^s+1} R_{n\ell}^2(\xi)\xi^2 d\xi.
\]  
(41)

Then the substitution of eqs. (40), (41) in eq. (39) gives the following expression for eq. (39):

\[
kZ \sum_{n=1}^{n-1} (2\ell + 1) \left\{ \tau^{-1} \int_0^\tau R_{n\ell}^2(\xi)\xi^2 d\xi + \int_\tau^\infty R_{n\ell}^2(\xi)\xi d\xi \right\}.
\]  
(42)

The integrals appearing in eq. (42) can be managed by the use of the well-known “incomplete” gamma functions as defined by the following expressions [27]:

\[
\gamma(1 + n, x) \equiv \int_0^x e^{-t} t^n dt = n! \left[ 1 - e^{-x} \left( \sum_{m=0}^{n} \frac{x^m}{m!} \right) \right]
\]  
(43)

and

\[
\Gamma(1 + n, x) \equiv \int_x^\infty e^{-t} t^n dt = n! e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}
\]  
(44)

Now, the integrations over \( \xi \) give finally the following form for \( \hat{H}_{CM+R} \) (where we return \( \hat{\Omega} \) instead of \( \tau \)):

\[
\hat{H}_{CM+R} = kZ \sum_{n=1}^{n-1} \sum_{\ell=0}^{n-\ell-1} \sum_{g=0}^{2g} \frac{(2\ell + 1)}{2n2^{2(\ell-1)-1}} \left( \frac{2(n - \ell - 1) - 2g}{n - \ell - 1 - g} \right) \times
\]
\[
\times \frac{(2g)!}{g!(2\ell + 1 + g)!} \left( \frac{2g + 2(2\ell + 1)}{2g - t} \right) \frac{(-2)^t}{t!} \left\{ (2\ell + t + 2)! \left( 1 - \exp \left( -\frac{2Z \hat{\Omega}}{na_\mu} \right) \sum_{f=0}^{2\ell+t+2} \left( \frac{2Z \hat{\Omega}}{na_\mu} \right)^f \frac{f!}{f!} \right) \right\} \hat{\Omega}^{-1}
\]
\[ + \frac{2Z}{na_{\mu}} (2\ell + t + 1)! \exp \left( - \frac{2Z \Omega}{na_{\mu}} \right) \sum_{f=0}^{2\ell+t+1} \frac{(2Z \Omega)^f}{f!} \], \quad (45) \]

as given in eq. (23) in the body text.