O(N) colour-flavour transformations and characteristic polynomials of real random matrices

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Abstract

The fermionic, bosonic and supersymmetric variants of the colour-flavour transformation are derived for the orthogonal group. These transformations are then used to calculate the ensemble averages of characteristic polynomials of real random matrices.

1 Statement of main results

Since the pioneering work of Wigner, many physical systems have been successfully studied with the help of random matrix models. Among these asymmetric real random matrices arising in applications in neural networks \cite{1}, quantum chaos \cite{2} and QCD \cite{3} are known to be the most difficult. In fact, until the very recent breakthrough \cite{4, 5, 6, 7, 8}, the eigenvalue correlation functions of real and complex eigenvalues were not accessible even for Gaussian matrices and calculating the ensemble averages of eigenvalue statistics in the complex plane for a sufficiently general class of real random matrices remains a challenging problem. The mathematical difficulties in calculating the eigenvalue correlation functions of real random matrices are mainly due to the fact that their eigenvalues are either real or pairwise complex conjugate and the mathematical tools available to study real random matrices are very limited, especially when compared to those available for complex matrices. In this paper we derive several integral transformations dealing with integrations over real orthogonal matrices which we believe might be useful in the above context. These integral transformations are known under the name of the Colour-Flavour Transformations.

The Colour-Flavor Transformations (CFT) are certain types of integral transformations based on Howe’s dual pair theory. They were was first derived by Zirnbauer \cite{9} in 1996 and since then have became a standard tool in mesoscopic physics, random matrix theory, lattice QCD as well as other fields. The CFTs were originally derived for U(N) and Sp(2N), and later generalized to other classical groups \cite{10, 11, 12, 13}. However, both the fermionic and bosonic variants of the O(N) CFT appearing in the literature \cite{12, 13} do not seem to be reproduced in the correct form, possibly suffering from typos. In this paper, we first correct...
the bosonic and fermionic versions of the O(N) CFT and then derive the supersymmetric version, which is a new result.

Fermionic O(N) CFT

$$\int_{O(N)} dO \exp(\overline{\psi}_a^i O_{ij} \psi_a^j) = C_0^F \int_{Z=-Z^\dagger} d\mu(Z, Z^\dagger) \exp \left( \frac{1}{2} \left( \overline{\psi}_a^i Z_{ab} \overline{\psi}_b^i + \psi_a^i Z_{ab}^\dagger \psi_b^i \right) \right),$$

(1.1)

where $C_0^F$ is the normalization constant dependent on $N$ and $n$, see Eq. (2.20) and

$$d\mu(Z, Z^\dagger) = \frac{dZdZ^\dagger}{\det \frac{N}{2} + n-1 (1 + ZZ^\dagger)}. \quad (1.2)$$

Integral on the left-hand side of Eq. (1.1) is over the real orthogonal group O(N). Here, $\psi_a^i$ and $\overline{\psi}_a^i$, $i = 1, \ldots, N$ and $a = 1, \ldots, n$ are Grassmann variables. Note that the $\overline{\psi}$'s are not necessarily related to $\psi$'s by complex conjugation. That is, one can replace $\overline{\psi}_a^i$ with an arbitrary set of independent Grassmann variables which are not related to $\psi$ by any operation and the identity will still hold. The integral on right-hand side of Eq. (1.1) is over complex skew-symmetric matrices of dimension $n \times n$.

Bosonic O(N) CFT

$$\int_{O(N)} dO \exp(\phi_a^i O_{ij} \phi_b^j) = C_0^B \int_{1-ZZ^\dagger>0} d\mu(Z, Z^\dagger) \exp \left( \frac{1}{2} \left( \phi_a^i Z_{ab} \phi_b^i + \phi_a^i Z_{ab}^\dagger \phi_b^i \right) \right),$$

(1.3)

where

$$d\mu(Z, Z^\dagger) = \det \frac{N}{2} - n-1 (1 - ZZ^\dagger)dZdZ^\dagger. \quad (1.4)$$

Integral on the left-hand side of Eq. (1.3) is again over the real orthogonal group O(N). Here, $\phi_a^i$ and $\overline{\phi}_a^i$, $i = 1, \ldots, N$ and $a = 1, \ldots, n$ are complex variables. The normalization constant $c_0^B$ is defined in Eq. (2.20). Similar to the fermionic case, one can also replace $\overline{\phi}_a^i$ with an arbitrary set of independent complex variables and the identity will still hold. The integral on right-hand side of Eq. (1.3) is over complex symmetric matrices of dimension $n \times n$ such that $1 - ZZ^\dagger$ is positive definite.

Supersymmetric O(N) CFT

$$\int_{O(N)} dO \exp(\overline{\psi}_a^i O_{ij} \psi_b^j) = \int_{M_B \times M_F} d\mu(Z, \tilde{Z}) \exp \left( \frac{1}{2} \left( \overline{\psi}_a^i Z_{ab} \overline{\psi}_b^i + \psi_a^i Z_{ab}^\dagger \psi_b^i \right) \right),$$

(1.5)

As before, the integration on the left-hand side is over O(N) and $i, j = 1, \ldots, N$. Here $\psi$ and $\overline{\psi}$ are graded vectors whose elements $\psi_a^i$ and $\overline{\psi}_a^i$ are bosonic when $a = (\alpha, B)$ and fermionic when $a = (\alpha, F)$, where $\alpha = 1, \ldots, n$. On the right-hand side, $Z$ and $\tilde{Z}$ are $2n \times 2n$ dimensional supermatrices subject to the following condition

$$Z = Z^T \sigma, \quad \tilde{Z} = \sigma \tilde{Z}^T,$$

(1.6)

where $\sigma$ is the superparity defined as $\sigma = \left( \begin{array}{cc} I_n & 0 \\ 0 & -I_n \end{array} \right)$. Here, $Z$ and $\tilde{Z}$ parameterize the Riemannian symmetric superspace CI|DIII defined as OSp(2n|2n)/GL(n|n), such that

$$M_B = \text{Sp}(2n, \mathbb{R})/U(n), \quad M_F = \text{SO}(2n)/U(n). \quad (1.7)$$

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We defined $d\mu(Z, \tilde{Z}) = dZd\tilde{Z} \text{SDet}(1 - \tilde{Z}Z)$, where $dZd\tilde{Z}$ is the flat Berezin measure on the space of supermatrices $Z$ and $\tilde{Z}$. As usual, SDet denotes the superdeterminant. The integration domain $M_B \times M_F$ is fixed by

$$Z_{BB} = \tilde{Z}_{BB}^\dagger, \quad Z_{FF} = -\tilde{Z}_{FF}^\dagger. \quad (1.8)$$

The symbol $|a|$ denotes the parity of index $a$, $|a| = 0$ when $a$ is bosonic and $|a| = 1$ when $a$ is fermionic.

With the fermionic CFT, we calculated the ensemble average of spectral determinants of certain real random matrices. The first one is the positive integer moments of modulus of the characteristic polynomial of matrices of the form $GO(N)$, where $G$ is diagonal. Its average over the orthogonal group $O(N)$ can be written as an integral of products of Pfaffians,

$$\langle |z - GO|^{2m} \rangle_{O(N)} = \int_{O(N)} dO \det^m [(z - GO)(z - GO)^\dagger]$$

$$= \text{const.} \int_{Z = -Z^T} d\mu(Z, Z^\dagger) \prod_{i=1}^{N} \text{pf} \begin{bmatrix} g_i^2 Z \otimes I_m & \mathcal{F} \otimes I_m \\ -\mathcal{F} \otimes I_m & Z^\dagger \end{bmatrix}. \quad (1.9)$$

Here we introduced $2 \times 2$ matrix $\mathcal{F} = \text{diag}(z, \bar{z})$ and the integration measure is defined in Eq.(1.2). In the second example, we calculated the ensemble average of characteristic polynomials of real random matrices $A$ from the Jacobi ensemble,

$$I_W(\lambda, \gamma) = \int dAdA^T (AAT)^a(1 - AA^T) \det(\lambda - A) \det(\gamma - A^T), \quad (1.10)$$

where $a$ and $b$ are non-negative integers. This average is again written in terms of Pfaffians,

$$I_W(\lambda, \gamma) = \text{const.} \int_0^1 dr \frac{1}{(1 + r)^{N+2}} \text{pf} [\alpha_{i,j}|i,j=0,...,2s-1], \quad \text{when } N = 2s. \quad (1.11)$$

and

$$I_W(\lambda, \gamma) = \text{const.} \int_0^1 dr \frac{1}{(1 + r)^{N+2}} \text{pf} \begin{bmatrix} \alpha_{i,j} & k_i(a, b; 1) \\ -k_j(a, b; 1) & 0 \end{bmatrix}_{i,j=0,...,2s}, \quad \text{when } N = 2s + 1. \quad (1.12)$$

Here $\alpha_{ij}$ is defined in Eq.(3.43) and $k_i$ is defined in Eq.(3.42).

This paper is organized as follows. In section 2 we give a detailed proof of the fermionic CFT and then outline the derivations of the bosonic and supersymmetric CFT. In section 3, we calculate the ensemble average of characteristic polynomials as applications of the fermionic CFT. A summary is given in section 4.

## 2 Proof of colour-flavor transformation

We derive the $O(N)$ colour-flavor transformations in this section. A detailed derivation is given for the fermionic case. Derivations for the bosonic and supersymmetric cases are given in more brief way with only important issues addressed, whereas details are referred to either the fermionic case or literatures.

In this paper, all the three types of colour-flavor transformations are established by Howe’s dual pair theory, see [9, 15] and references therein. It is worth mentioning that in certain cases symmetric polynomials can also be used to derive these transformations or even transformations with different forms [11, 16]. However, it turns out that here the derivations based on dual pair theory are more convenient.
2.1 Fermionic $O(N)$ CFT

To establish the fermionic colour-flavor transformation over $O(N)$, we first derive the transformation over the special orthogonal group $SO(N)$,

$$
\int_{SO(N)} d\mu(O) \exp(\bar{\psi}_a^i O_{ij} \psi^i_a) = C_0 \int_{Z=-Z^T} d\mu(Z, Z^T) \exp \left( \frac{1}{2} \left( \bar{\psi}_a^i Z_{ab} \bar{\psi}_b^i + \psi^i_a Z_{ab} \psi^i_b \right) \right) (1 + K \det M) ,
$$

(2.13)

where $C_0^F$ and $K = \frac{C_0^F}{c_{0N}^F}$ are constants defined later and $M$ is an $N \times N$ matrix defined as $M_{ij} = \left[ \bar{\psi}_a^i (1 + ZZ^T) \right]_{ab} \psi^j_b$.

**Remark:** To get Eq. (1.1), we exploit the fact that $O(N) = O_+(N) \oplus O_-(N)$, where $O_+ \cong SO(N)$ and $O_-$ is a rotation followed by a ‘reflection’ $R$, which can be chosen as $R = \text{diag}(I_{N-1}, -1)$. Note that $\bar{\psi}^i R_{ij}$ flips the sign of all $\bar{\psi}_i^N$s therefore inverts the sign of $\det M$. Hence, the parts containing $\det M$ are cancelled when we combine the contributions from the normal rotation $SO(N)$ and the improper rotation $R \cdot SO(N)$. Which proofs Eq. (1.1).

In fact, it turns out that one can ‘naively’ use a similar mapping as defined above Eq. (2.14) for $O(N)$ and get Eq. (1.1) with less effort. However, due to the fact that $\det O = \pm 1$ for $O \in O(N)$ this kind of mapping is mathematically better defined for $SO(N)$.

In the following paragraphs, we will mainly follow the method of paper [10]. Introduce fermionic creation and annihilation operator $\hat{f}_a^i$ and $\hat{f}_a^i$, where $i = 1, \ldots, N$ and $a = 1, \ldots, n$. As usual, we borrow the terminology from lattice gauge theory where the upper indices are referred to as ‘colour’ and the lower indices are referred to as ‘flavor’. This set of operators satisfies the canonical fermion anticommutation relations $\{ f^i_a, f^j_b \} = \delta^{ij} \delta_{ab}$ and $\{ \bar{f}^i_a, \bar{f}^j_b \} = 0$. These operators construct a Fock space $\mathcal{F}_F$ for a fermionic system. Let $\ket{0}$ be the vacuum state, then we have $f_a^i \ket{0} = 0$. The quadratic operators $\bar{f}^i_a \bar{f}^j_b, f_a^i f_b^j$ and $\bar{f}_a^i \bar{f}_b^j - f_a^i f_b^j$ therefore define a representation of the Lie algebra $\text{so}(2n, \mathbb{C})$. This algebra has two commuting subalgebras. Which are $\text{so}(2n, \mathbb{C})$ generated by $\bar{f}_a^i \bar{f}_b^j, f_a^i f_b^j$ and $\bar{f}_a^i \bar{f}_b^j - f_a^i f_b^j$ and $\text{so}(N, \mathbb{C})$ generated by $\bar{f}_a^i f_d^j + f_a^i \bar{f}_d^j$.

To derive colour-flavor transformation we need to construct two types of projection operators $\hat{P}_C$ and $\hat{P}_F$ which project states to a subspace of $\mathcal{F}_F$, named colour-single space, which is invariant under the $SO(N)$ group action $O \mapsto TO := \exp(\hat{f}_a^i (\ln O)_{ij} f_a^j)$. As usual, we define

$$
\hat{P}_C = \int_{SO(N)} dO \ T_O .
$$

(2.14)

Note that for $SO(N)$, the colour-single space has two disconnected components, each of which carries an irreducible representation of $SO(N)$. One of them contains the vacuum state $\ket{\psi_0} = \ket{0}$ and is in addition $O(N)$ invariant, and the other one contains the state $\ket{\psi_1} := \hat{f}_1^1 \hat{f}_2^2 \cdots \hat{f}_N^N \ket{0}$. Acting on the vacuum and $\ket{\psi_1}$ by the operators $\hat{f}^i \hat{f}^j$ span each colour-single subspace. Therefore, we have $\hat{P}_F = \hat{P}_{F_0} + \hat{P}_{F_1}$.

It is convenient to define the notation

$$
\bar{c}_A^i = \left\{ \begin{array}{ll} f_A^i & A = 1, \ldots, n \\ f_{A-n}^i & A = n + 1, \ldots, 2n \end{array} \right. \quad \text{and} \quad \bar{c}_A^i = \left\{ \begin{array}{ll} f_A^i & A = 1, \ldots, n \\ f_{A-n}^i & A = n + 1, \ldots, 2n \end{array} \right. .
$$

(2.15)

For $g \in \text{SO}(2n)$, it is direct to check the mapping $g \mapsto T_g := \exp(1/2 \bar{c}_A^i (\ln g)_{AB} \bar{c}_{B}^i)$ constructs a representation of $\text{SO}(2n)$, such that $T_g \bar{c}_A^i T_g^{-1} = \bar{c}_{B}^i g_{BA}$. An element of $\text{SO}(2n)$ can be parameterized as

$$
g = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}, \quad \text{where} \quad UU^T + VV^T = I, \quad \text{and} \quad UV^T + VU^T = 0 .
$$

(2.16)
Note that this $G$ is a subgroup of $U(2n)$ that is isomorphic to $SO(2n)$. It is clear that $G$ has a $U(n)$ subgroup $G \supset H = \text{diag}(U, U) \cong U(n)$. We parametrize the coset space $G/H$ as

$$G/H = \left( \begin{array}{cc} (1 + ZZ^\dagger)^{-\frac{1}{2}} & Z(1 + Z^\dagger Z)^{-\frac{1}{2}} \\ Z^\dagger(1 + ZZ^\dagger)^{-\frac{1}{2}} & (1 + Z^\dagger Z)^{-\frac{1}{2}} \end{array} \right),$$

(2.17)

where $Z = U^{-1}V$ are complex skew-symmetric matrices of dimension $n \times n$. Combining the above formula, we complete the proof of Eq. (2.13) with the following few changes. We construct a bosonic Fock space $F_B$ with bosonic operators $A$. The bosonic colour-flavor transformation is derived with almost the same method \cite{13} but with the following few changes. We construct a bosonic Fock space $F_B$ with bosonic operators $A$. The two commuting algebras are $so(N, \mathbb{C})$ and $sp(2n, \mathbb{C})$ \cite{13}. Define

$$c_A^i = \begin{cases} b_A^i & A = 1, \ldots, n \\ b_{A-n}^i & A = n + 1, \ldots, 2n \end{cases} \quad \text{and} \quad c_A^i = \begin{cases} b_A^i & A = 1, \ldots, n \\ -b_{A-n}^i & A = n + 1, \ldots, 2n \end{cases}$$

(2.24)
For \( g \in \text{Sp}(2n, \mathbb{R}) \), it is straightforward to check that the mapping \( g \mapsto T_g := \exp\left(\frac{1}{2} \bar{c}_A (\ln g)_{AB} c^B_B\right) \) constructs a representation of \( \text{Sp}(2n, \mathbb{R}) \), such that \( T_g \bar{c}_A T_g^{-1} = \bar{c}_B g_{BA} \). An element of \( \text{Sp}(2n, \mathbb{R}) \) can be parameterized as

\[
g = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}, \quad \text{where} \quad UU^\dagger - V^T \bar{V} = I, \quad \text{and} \quad U^\dagger V = V^T \bar{U}.
\]

(2.25)

Here \( G \) is a subgroup of \( U(n, n) \) that is isomorphic to \( \text{Sp}(2n, \mathbb{R}) \). Note that \( G \) has a \( U(n) \) subgroup. The coherent states are similarly defined as before, \( |Z\rangle = \exp\left(\frac{1}{2} \bar{b}_a a^Z_{ab} \bar{b}_b\right) |0\rangle \) and \( |\phi\rangle = \exp(\bar{b}_a \phi^a) |0\rangle \). Following the same procedure as before and [13], we get Eq. (1.3).

In order to fix the normalization factor, choosing the group volume of \( O(N) \) to be 1 and using Eq. of [17], we get

\[
C_0^B = \pi^{-\frac{n(n+1)}{4}} \frac{N^3 - 2n}{2} \prod_{i=1}^{n-1} \frac{(N - i)\Gamma(N - 1 - i)}{\Gamma(N - 1 - 2i)}.
\]

(2.26)

### 2.3 Supersymmetric \( O(N) \) CFT

When \( \psi \) and \( \bar{\psi} \) are graded vectors, we need to extend the flavor group to incorporate this 'supersymmetry'. The Lie superalgebra we need satisfies the following condition

\[
Q = -\gamma Q^T \gamma.
\]

(2.27)

Let \( \sigma \)'s being the Pauli matrices, \( \gamma \) can be defined as

\[
\gamma = \begin{pmatrix} i\sigma_y & 0 \\ 0 & i\sigma_x \end{pmatrix} \otimes I_n.
\]

(2.28)

Therefore, \( Q \) defines the Lie superalgebra \( \text{osp}(2n|2n) \) whose boson-boson block is the Lie algebra \( \text{sp}(2n) \) and fermi-fermi block is \( \text{so}(2n) \). The remaining derivation closely follows [9].

To do a simple check, setting the bosonic (or fermionic) degree of freedom on both sides of Eq.(1.5) to zero and choose corresponding integral measure, we can recover the fermionic (or bosonic) CFT Eq.(1.1).

### 3 Applications in real random matrices

In this section, we calculate the averaged characteristic polynomials of real random matrices from certain ensembles. These calculations are simple applications of the fermionic \( O(N) \) CFT.

#### 3.1 Modulus square of characteristic polynomials of certain type of real random matrices averaged over \( O(N) \)

Let \( z \) be a complex number and \( G \) be a real diagonal matrix. We calculate the following quantity,

\[
F_G(z) = \langle |z - GO|^{2m} \rangle_{O(N)} = \int_{O(N)} dO \det^m \left[(z - GO)(z - GO)^\dagger\right].
\]

(3.29)

Note that \( \langle |z - GU|^{2m} \rangle_{U(N)} \), where the average is over unitary group, has been calculated with the method of symmetric functions in [21]. And it is not hard to show that the method we will use in this section can be applied to the unitary case as well, in which case one will use the \( U(N) \) fermionic CFT.
Introducing Grassmann vectors, \( \eta^i_a \) and \( \xi^i_a \), \( i = 1, \ldots, N \) and \( a = 1, \ldots, m \), we can re-write Eq. (3.31) as
\[
F_G(z) = \int d\bar{\eta} d\bar{\xi} e^{-z\bar{\eta}^i_a \eta^i_a - z\bar{\xi}^i_a \xi^i_a} dO \int d\bar{\eta} d\bar{\xi} e^{-z\bar{\eta}^i_a \eta^i_a - z\bar{\xi}^i_a \xi^i_a} dO \exp\left(\bar{\psi}^i_a O_{ij} \psi^j_a\right),
\]
where we introduced the composite notation \((\bar{\psi}^i_{1a}, \bar{\psi}^i_{2a}) = (\bar{\eta}^i_g, -\xi^i_g)\) (no summation) and \((\psi^i_{1a}, \psi^i_{2a}) = (\eta^i_a, \xi^i_a)\) and changed the order of integration. Defining the \(2m \times 2m\) complex skew-symmetric matrix \(Z\) and applying the fermionic CFT Eq. (1.1), we get
\[
F_G(z) = C^O_0 \int d\bar{\eta} d\bar{\xi} e^{-z\bar{\eta}^i_a \eta^i_a - z\bar{\xi}^i_a \xi^i_a} \int_{Z = -Z^T} d\mu(Z, Z^T) \exp\left(-\frac{1}{2} \left(\bar{\psi}^i_a Z_{ab} \psi^j_b + \psi^i_a Z^i_{ab} \psi^j_b\right)\right).
\]

Next, we change the order of integration and use the standard Gaussian integral formula for Grassmann vectors. The final result is written in terms of Pfaffians and given in Eq. (1.9).

When \( m = 1 \), we can integrate over \( Z \) explicitly since
\[
Z = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.
\]

By Eq. (3.31), we get
\[
F_G(z) = \langle |z - GO|^2 \rangle_{O(N)} = \text{const.} \int da d\bar{a} \frac{1}{(1 + a\bar{a})^{N+2}} \prod_{i=1}^{N} (|z|^2 + a\bar{a}g^2_i)
\]
\[
= \text{const.} \sum_{l=0}^{N} \frac{1}{C_N^l} |z|^{2(N-l)} S^l(G^2),
\]
where \( C_N^l = N! / l!(N-l)! \) is the binomial function and \( S^l(G^2) \) denotes the \( l \)-th order elementary symmetric polynomials of \( g^2_i \)'s, i.e. \( S^0(G^2) = 1 \), \( S^1(G^2) = \sum_i g^2_i \), \( S^2(G^2) = \sum_{i<j} g^2_i g^2_j \), etc.

### 3.2 Characteristic polynomials averaged over Jacobi ensemble

As another simple application of the transformations we derived in the previous section, we calculate the following average,
\[
\int dAdA^T W(AA^T) \text{det}(\lambda - A) \text{det}(\gamma - A^T).
\]

Where \( W(AA^T) \) is an arbitrary invariant ensemble which depends on singular values of \( A \) only, i.e. \( W(A) = W(\text{OAO'}) = W(G) \), where \( O \) and \( O' \) are arbitrary elements in \( O(N) \) and \( G \) are the singular values of \( A \) defined by \( A = O_1 GO_2 \). In particular, we are interested in ensembles whose potentials are also separable functions of \( G \), i.e.
\[
W(AA^T) = \prod_i W(g^2_i).
\]

First, we make singular decomposition \( A = O_1 GO_2 \), where \( G = \text{diag}(g_1, \ldots, g_N) \geq 0 \).
\[
\int dAdA^T W(AA^T) = \int_{O(N)} dO_1 dO_2 \int W(g^2_i) dg_i \prod_{i<j} |g^2_i - g^2_j|.
\]
By invariance of Haar measure, we have \( \det(\lambda - O_1 G O_2) = \det(\lambda - O_2 O_1 G) = \det(\lambda - G) \).

Introducing Grassmann vectors as before, we can re-write Eq. (3.34) as

\[
\int_{O(N)} dO \prod_i W(g_i^2) \prod_{i < j} |g_i^2 - g_j^2| d\eta d\xi e^{-\lambda \eta^2} d\xi e^{\gamma \eta^2} \cdot
\]

\[
\int d\bar{\eta} d\bar{\xi} e^{-\bar{\lambda} \eta^2} e^{-\gamma \xi^2} e^{\gamma \eta^2} e^{-\lambda \xi^2} e^{\gamma \eta^2} e^{\gamma \xi^2} \cdot
\]

(3.37)

Use the same method as in Eq. (3.30)-Eq. (3.31), we get for separable invariant potentials Eq. (3.35),

\[
I_W(\lambda, \gamma) = \int dA dA^T W(A A^T) \det(\lambda - A) \det(\gamma - A^T)
\]

\[
= \text{const.} \int_0^1 dr \frac{1}{(1 + r)^{N+2}} \int \prod_i |g_i^2 - g_j^2| \prod_i (\lambda \gamma + r g_i^2) W(g_i^2) dg_i .
\]

(3.38)

One benefit of using singular values instead of eigenvalues is that all integral variables are real non-negative. From the above formula it is clear that when \( \gamma = \bar{\lambda} \) the integral on the left-hand side depends only on \( |\lambda|^2 \).

Note that Eq. (3.38) applies to all ensembles of real random matrices satisfying Eq. (3.35). These ensembles include \( W(A A^T) = e^{-\text{Tr}(V(A A^T))} \), which becomes the Ginibre ensemble when \( V(x) = \frac{1}{2} x \), which has been studied intensively. In the remaining part of this section, we consider the Jacobi ensemble,

\[
W(x) = x^a(1 - x)^b .
\]

(3.39)

To proceed, we use the method in Chapter 5 of [19].

\[
\int_0^1 \prod_i |g_i^2 - g_{j}\rangle \prod_i (1 + r g_i^2) W(g_i^2) dg_i = N! \int_{0 < g_1 < g_2 < \cdots < g_N \leq 1} \prod_i \det [W(g_i^2) R_{ij} - 1 (g_i^2)(1 + r g_{i}^2)] .
\]

(3.40)

where \( R_{ij}(x) \) can be arbitrary monic \( j \) order polynomials of \( x \). Here we choose \( R_j(x) = x^j \).

Define the following functions

\[
h(a, b; x) = \int_0^x dg g^{2a} (1 - g^2)^b = \frac{1}{2} \Gamma(b + 1) \Gamma(a + \frac{1}{2}) \sum_{i=0}^b \frac{x^{2(a+i)+1}(1 - x^2)^{b-i}}{\Gamma(b - i + 1) \Gamma(a + i + \frac{1}{2})} ,
\]

(3.41)

and

\[
k_i(a, b; x) = \int_0^x dg (1 + r g^2) g^{2(a+i)} (1 - g^2)^b = h(a + i, b; x) + \frac{r}{\lambda \gamma} h(a + i + 1, b; x) ,
\]

(3.42)

Let \( B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \) be the Beta function. When \( N = 2s \), introduce the skew-symmetric
matrix \( \alpha \),

\[
\alpha_{ij} = \int_{0}^{1} dg \left( 1 + rg^2 \right) g^{2a} \left( 1 - g^2 \right)^b \left( g^{2j} k_j(g) - g^{2j} k_i(g) \right)
\]

\[
= \frac{1}{4} \left\{ B(b + 1, a + j + \frac{1}{2}) \sum_{l=0}^{b} \frac{B(2b - l + 1, 2a + 1 + i + j + l)}{B(l + 1, a + j + l + \frac{3}{2})} + r \frac{B(b + 1, a + j + \frac{1}{2})}{\lambda \gamma} \sum_{l=0}^{b} \frac{B(2b - l + 1, 2a + 2 + i + j + l)}{B(l + 1, a + j + l + \frac{3}{2})} + \right.
\]

\[
\left. r \frac{B(b + 1, a + j + \frac{3}{2})}{\lambda \gamma} \sum_{l=0}^{b} \frac{B(2b - l + 1, 2a + 3 + i + j + l)}{B(l + 1, a + j + l + \frac{5}{2})} - i \leftrightarrow j \right\} .
\]

By Eq.(5.5.8) and Eq.(5.5.9) of [19], we get Eq.(1.11) and Eq.(1.12).

Similarly, for \( W(x) = \exp(-\frac{1}{2}x) \), we can check that Eq.(3.38) and Eq.(3.40) gives the same results in [20].

\[
\int d\lambda dA^T e^{-\frac{1}{2}TrAA^T} \det(\lambda - A) \det(\gamma - A^T) = \text{const.} \sum_{n=0}^{N} \frac{1}{n!} (\lambda \gamma)^n .
\]

### 4 Conclusions

In this paper, we derived the \( O(N) \) colour-flavor transformations. We believe that these transformations will be useful in the study of real random matrices, just as the \( U(N) \) CFT are for complex random matrices. As a simple application, we calculated averaged characteristic polynomials for two types of ensembles, which, to the best of our knowledge, have not been calculated before.

We have only showed examples with the fermionic CFT whereas applications of the bosonic and supersymmetric CFTs will be postponed to future works. In which, as usual, we need to pay attention to analytical issues, especially when we need to change the order of integrations, [21, 22]. Also, by its nature the supersymmetric CFT often requires more work. However, when \( n = 1 \) Eq.(1.5) enjoys the simplicity that

\[
Z = \begin{pmatrix} a & \sigma \\ -\sigma & 0 \end{pmatrix} \quad \text{and} \quad \tilde{Z} = \begin{pmatrix} \bar{a} & \rho \\ \bar{\rho} & 0 \end{pmatrix} .
\]

This is because \( O(2)/U(1) \) consists of only single points.

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