Encoding graphs into quantum states: an axiomatic approach

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A fundamental problem in quantum information is to describe efficiently multipartite quantum states. An efficient representation in terms of graphs exists for several families of quantum states (graph, cluster, stabilizer states), motivating us to extend this construction to other classes. We introduce an axiomatic framework for mapping graphs to quantum states of a suitable physical system. Starting from three general axioms we derived a rich structure which includes and generalizes several classes of multipartite entangled state, like graph/stabilizer states, Gaussian cluster states, quantum random networks and projected entangled pair states (PEPS). Due to its flexibility we can extend the present formalism to include directed and weighted graphs.

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I. INTRODUCTION

A key feature of quantum systems is the exponentially large Hilbert space required to describe them. This property is a mixed blessing. On the one hand, quantum systems can efficiently simulate quantum dynamics with polynomial resources. On the other, characterization of an arbitrary state of the Hilbert space needs an exponentially large number of parameters, a well-known problem in quantum tomography. Thus it becomes crucial to characterize efficiently quantum states. One such class of parameters, a well-known problem in quantum tomography.

A second motivation comes from graph theory. There are several hard problems in graph theory (graph isomorphism, 3-colorability) for which an efficient (polynomial) solution would be highly desirable. Several authors attempted to solve problems like graph isomorphism using a physically motivated approach\textsuperscript{11,12,14}. The intuition is to encode a graph into the quantum state of a system (bosons hopping on a graph\textsuperscript{14}, quantum walks\textsuperscript{15,17}) and to derive properties of the underlying graph $G$ from the associated quantum state $|G\rangle$.

These considerations bring us to the problem we investigate in this article (Fig.\textsuperscript{1}): Given a graph $G$, how do we associate to it a state $|G\rangle$ of a suitable quantum system? Our approach to address this question is axiomatic. Starting from a few general principles we construct an axiomatic framework for mapping (encoding) graphs to states of a quantum system.

The structure of the article is the following. After a quick overview of graph theory and the main notations, in Section \textsuperscript{II} we develop the framework starting from three intuitive axioms. In Section \textsuperscript{III} we show that a number of well known quantum states, like graph states, Gaussian and continuous-variable cluster states, quantum random networks, projected entangled pair states (PEPS), share the structure introduced here and emerge naturally from our construction. In Section \textsuperscript{IV} we extend this approach to directed and weighted graphs and finally we conclude in Section \textsuperscript{V}.

Notations and background. A graph is a pair $G = (V,E)$, where $V(E)$ is the set of vertices (edges). The order $|G|$ of the graph is the number of its vertices. The empty graph $E_n = (V,\emptyset)$ has $n$ vertices and no edges. $K_n$ is the complete graph with $n$ vertices, e.g., $K_2$ is an edge and $K_3$ a triangle. In a regular graph each vertex has the same degree, i.e., number of incident edges. In a directed graph the edges have an orientation, hence $(x,y) \in E$ is an ordered pair. The edges in a weighted graph have an associated label (weight).

A graph $G$ is described by its adjacency matrix $A(G)$, defined as $A(G)_{ij} = 1$, if the edge $(i,j) \in E$ and $A(G)_{ij} = 0$ otherwise. For simple graphs (undirected graphs without loops and at most one edge between any pair of vertices) $A$ is symmetric with zero diagonal. Two graphs of order $n$ are isomorphic $G \simeq G'$, iff there is a vertex permutation $P \in S_n$ (the symmetric group of $n$ objects) preserving edge incidence: $A(G') = PA(G)P^\dagger$. A graph automorphism is an isomorphism of $G$ into itself, $[A(G), P] = 0$, i.e., a graph symmetry.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{(color online). Given a graph $G = (V,E)$, how do we map it to a quantum state $|G\rangle \in \mathcal{H}$, for a suitable quantum system described by $\mathcal{H}$?}
\end{figure}
II. AXIOMS

In this section we develop the main framework for mapping graphs to quantum states. Starting from three intuitive principles (or axioms), we derive a structure which is general enough to include several well-known families of multipartite quantum states. In Section IV we will see that this structure is also flexible enough to construct novel states starting from directed and weighted graphs.

We now formulate the problem more precisely. Given a graph \( G \), we want to: (i) find a suitable Hilbert space \( \mathcal{H} \), and (ii) associate to \( G \) a state \( |G\rangle \in \mathcal{H} \). For simplicity we consider only pure states \( \rho(G) = |G\rangle\langle G| \) (a mixed state can be obtained from the pure state of a larger system by a suitable partial trace).

For point (i), a preliminary question is the following: given a graph \( G \), how large should the associated Hilbert space be? An answer requires an estimate of the size of the graph space. For a simple graph of order \( n \) the number of edges satisfies \( 0 \leq L \leq n(n-1)/2 \). Thus the number of simple graphs of order \( n \) is \( N(n) \leq 2^{n(n-1)/2} \); this is an upper bound and does not take into account graph isomorphisms. Using a similar argument, the number of graphs with \( n \) vertices and \( L \) edges is bounded by \( N(n, L) \leq (n-1)^{L} \). Thus, if we want to capture enough of the structure of the graph space, the Hilbert space of the quantum system should have a similar dimension.

The second point (ii) is more difficult. \textit{A priori} there are several possible ways to tackle the problem \([2, 4, 11, 12, 18, 22]\). As it is not clear which method is best suited (mapping vertices to qubits? edges to qubits? others?), we reframe the problem by asking a different question: \textit{What properties should the mapping \( G \mapsto |G\rangle \) have?} This gives a new perspective and provides us with a starting point. In the following we discuss three desirable properties for our mapping, quantified as a set of axioms.

Suppose we have two disjoint graphs \( G_1 \) and \( G_2 \). What is a natural way to map the disjoint sum of two graphs \( G_1 \oplus G_2 \) to a quantum state \( |G_1 \oplus G_2\rangle \)? As physicists we are used to think in terms of subsystems and the fourth postulate of quantum mechanics \([23]\) gives us a hint in the right direction. Thus we choose as the first axiom the following:

\textbf{A1: Separability (tensor product).} For a disjoint sum of two graphs \( G_1, G_2 \) we have:

\[ |G_1 \oplus G_2\rangle = |G_1\rangle \otimes |G_2\rangle \]

This property immediately implies the following corollary:

\textbf{Corollary 1} If \( E_n = (V, \emptyset) \) is the empty graph on \( n \) vertices, then

\[ |E_n\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle \]

Corollary 2 \textbf{Given a graph} \( G = (V, E) \) \textbf{of order} \( n \), \textbf{we associate to each vertex} \( i \in V \) \textbf{a Hilbert space} \( \mathcal{H}_i \). The total Hilbert space is

\[ \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i \]

It is worth stressing the power of the separability axiom \textbf{A1} – this axiom alone automatically implies the tensor product structure of the Hilbert space.

While the first axiom is inspired by quantum theory, the second one has its roots in graph theory. Graph isomorphism is a central concept and we would like to preserve it under the mapping. The second axiom naturally captures this property:

\textbf{A2: Graph isomorphism.} If \( G_1 \simeq G_2 \) are isomorphic, the corresponding density operators \( \rho_{1,2} = |G_{1,2}\rangle\langle G_{1,2}| \) satisfy:

\[ \rho_2 = D(P)\rho_1 D(P)^{-1} \]

where \( D(P) \) is a matrix representation of the permutation \( P \in S_n \) mapping \( G_1 \) to \( G_2 \), i.e.,

\[ A(G_2) = P A(G_1) P^{-1} \]

Notice that following Corollary 2 the permutation \( P \) is well defined, as it interchanges the Hilbert spaces of the corresponding vertices. Also, expressing \textbf{A2} in terms of the density matrix \( \rho \) naturally avoids possible extra phases related to the statistics of identical particles.

Axiom \textbf{A2} implies straightforwardly the following corollary for graph automorphisms:

\textbf{Corollary 3} If \( P \in S_n \) is a graph automorphism of \( G \), then we have

\[ [\rho, D(P)] = 0 \]

Graph symmetries are thus captured naturally – as expected, they commute with \( \rho \).

Corollary 2 alone does not impose any restriction on the Hilbert spaces \( \mathcal{H}_i \) (they are completely arbitrary). However, together with Corollary 3 (graph automorphism) it implies that all the Hilbert spaces are identical, \( \mathcal{H}_i = \mathcal{H}_1, \forall i \). This follows from the automorphism group of the empty graph \( E_n \), which is the whole symmetric group \( S_n \): swapping any two Hilbert spaces \( \mathcal{H}_i, \mathcal{H}_j \) is a symmetry of the mapping \( E_n \rightarrow |E_n\rangle \). It also implies that the empty graph is mapped to \( |E_n\rangle = |\psi\rangle^{\otimes n} \), with \( |\psi\rangle \in \mathcal{H}_1 \) a free parameter of the theory. Hence we have:

\textbf{Proposition 1} Given a graph \( G = (V, E) \) of order \( n \), the corresponding quantum state \( |G\rangle \in \mathcal{H} \) belongs to a Hilbert space of \( n \) identical quantum systems

\[ \mathcal{H} = \mathcal{H}_1^{\otimes n} \]

where \( \mathcal{H}_1 \) is the Hilbert space associated to a single vertex. Moreover, the empty graph is mapped to \( E_n \rightarrow |E_n\rangle = |\psi\rangle^{\otimes n} \).
The dimension of $\mathcal{H}_1$ is arbitrary and is a free parameter of the theory. This gives us the freedom to consider various families of mappings with finite or infinite Hilbert spaces. Specific examples are $\mathcal{H}_1 = \mathbb{C}^d$ (qudit), $\mathcal{H}_2 = \text{span}\{|k\rangle\}_{k=0}^{\infty}$ (Fock space of a harmonic oscillator) or the (uncountably) infinite dimensional Hilbert space of a continuous variable.

Axioms A1 and A2 determine the structure of the Hilbert space $\mathcal{H}$ and can intuitively be thought of as giving the “kinematics” of the model. However, we also need the equivalent of “dynamics”: given a graph $G$, how do we construct $|G\rangle$? Proposition 1 implies that all graphs of order $n$ are mapped to the same Hilbert space $\mathcal{H} = \mathcal{H}_1^\otimes n$. Therefore, given two graphs of order $n$, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, there exists a linear operator $U \in \mathcal{L}(\mathcal{H})$ such that

$$|G_2\rangle = U(G_1, G_2)|G_1\rangle$$  

(6)

In particular, for any graph $G$ we have

$$|G\rangle = U(G)|E_n\rangle = U(G)|\psi\rangle^\otimes n$$  

(7)

with the obvious notation $U(G) := U(E_n, G)$. Of course, this only shifts the problem to one of finding $U(G)$. Since the last equation still does not tell us how to find the operator $U(G)$, we supplement our set of axioms with a final one:

A3: Universal edge operator. If the graphs $G = (V, E)$ and $G' = (V', E')$ differ by a single edge, i.e., $V' = V$ and $E' = E \cup \{(x, y)\}$, then

$$|G'\rangle = U(x, y)|G\rangle$$  

(8)

The edge operator $U(x, y)$ is independent of both $G, G'$ and depends only on the edge $(x, y)$.

Axiom A3 is particularly strong since it requires the edge operator $U$ to be independent of all graphs. From an axiomatic perspective, it is worth noting that replacing axiom A3 with a different one (while keeping A1 and A2 the same) is analogous to changing the “dynamics” of the model, i.e., akin to modifying the “Hamiltonian” of the system.  

Given a graph $G$, axiom A3 together with eq. 4 provides a constructive way to obtain the corresponding quantum state $|G\rangle$: starting from the empty graph $|E_n\rangle$ we apply successively the edge operator corresponding to all edges:

$$|G\rangle = \prod_{(x, y) \in E} U(x, y)|\psi\rangle^\otimes n$$  

(9)

Obviously, this construction is consistent only if the edge operator satisfies certain constraints. First, since the graph is undirected, $U(x, y)$ has to be symmetric in its inputs (swapping two vertices is a symmetry of the edge graph $K_2$). Second, the order in which we apply the edge operators in eq. 9 should be irrelevant. Finally, $U$ has a natural local action, as can be seen from the following argument. Consider the graph of order $n$ having a single edge $(x, y)$, $G = (V, \{(x, y)\})$. From A3 we have $|G\rangle = U(x, y)|E_n\rangle = U(x, y)|\psi\rangle^\otimes n$. On the other hand, from the separability axiom A1 we know that $|G\rangle = |K_2 \otimes E_{n-2}\rangle = |K_2\rangle \otimes |\psi\rangle^\otimes n - 2$, with $K_2$ an edge.  

A natural way of satisfying this for all $n$ is to require that the edge operator is local, i.e., acts only on the Hilbert spaces of the corresponding vertices $\mathcal{H}_x \otimes \mathcal{H}_y$. The following three properties summarize the (sufficient) consistency conditions required for the edge operator:

C1: Locality. The edge operator $U(x, y)$ acts nontrivially only on the Hilbert spaces $\mathcal{H}_x \otimes \mathcal{H}_y$ associated with vertices $x, y$

$$U(x, y) = U \otimes I^\otimes n - 2$$  

(10)

with $I$ the identity acting on the rest. Without the risk of confusion we denote the edge operator by either $U(x, y)$ or $U$.

C2: Symmetry. For undirected graphs $G$, the edge operator is symmetric in the inputs $U(x, y) = U(y, x)$. Let $S(x, y) = \sum_{i,j} |i\rangle\langle j|/|j\rangle\langle i|$ be the swap operator acting on the Hilbert spaces of two vertices $\mathcal{H}_x \otimes \mathcal{H}_y$. Since $U(x, y) := S(x, y)U(y, x)S(x, y)$, the symmetry condition is:

$$[U(x, y), S(x, y)] = 0$$  

(11)

C3: Edge commutativity. Consider two edges sharing a common vertex. Then the corresponding $U$’s should commute:

$$[U(x, y), U(y, z)] = 0$$  

(12)

or, equivalently, $[U \otimes I, I \otimes U] = 0$.

To summarize, the axiomatic framework presented here defines a class of theories characterized by a triplet

$$\mathcal{G} = (\mathcal{H}_1, |\psi\rangle, U)$$  

(13)

with $\mathcal{H}_1$ the Hilbert space associated to a vertex, $|\psi\rangle \in \mathcal{H}_1$ the initial state and $U \in \mathcal{L}(\mathcal{H}_1^\otimes 2)$ the (local) edge operator. The graph state $|G\rangle$ is constructed from the initial state $|\psi\rangle^\otimes n$ by applying the edge operator $U(x, y)$ for each edge $(x, y) \in E$,

$$|G\rangle = \prod_{(x, y) \in E} U(x, y)|\psi\rangle^\otimes n$$  

This construction is consistent if the edge operator satisfies the conditions C1–C3. Physically important cases correspond to $U$ a unitary operator, a projector, or a combination of both, as shown in the next section.

III. EXAMPLES

After developing the general framework in the previous section, we now discuss several important classes of entangled states emerging from the present formalism.

(i) Graph states. $\mathcal{G} = (\mathbb{C}^2, |\rangle\langle +|, C(Z))$. This is the simplest case and has been studied extensively [2,4]. At each vertex there is a qubit ($\mathcal{H}_1 = \mathbb{C}^2$) initialized in $|\psi\rangle = |\rangle + 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The edge operator is $U = C(Z) = \text{diag}(1, 1, 1, -1)$. The graph state $|G\rangle$ is constructed by applying a $C(Z)$ operator for each graph edge. An important example is cluster states, an essential resource for the one-way quantum computing model [5].

(ii) Qudit graph states. $\mathcal{G} = (\mathbb{C}^d, |\rangle\langle +|_d, C(Z_d))$, Fig. 2a). These are a straightforward generalization of graph states [24]: each vertex is a qudit $\mathcal{H}_1 = \mathbb{C}^d$ initialized in $|\rangle +_d := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$. The edge operator is the controlled-$Z_d$, $U = \text{diag}(1, 1, \ldots, 1, -1)$.
tors after all the unitary operators. The next two examples fit invertible, a further generalization is to include projector operators. In this case the edge operator can take the form of a unitary V followed by a projector, \( U = |\Psi\rangle\langle\Psi|V \). However, since the projector does not commute with the unitary part, we can ensure the constraints C2–C3 by applying the projector after all the unitary operators. The next two examples fit neatly in this class.

(iv) Projected entangled pair states (PEPS). PEPS are higher-dimensional generalizations of valence bond states/matrix product states, Fig. 2(b). They play an important role in solid-state as ground states of local Hamiltonians – the AKLT model is a particular example [26–28]. Each vertex shares with its nearest neighbors a \( d \)-dimensional maximally entangled state \( |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle = V|i\rangle^{\otimes d}_d \). We construct this state by acting on the initial state \( |\psiangle = |i\rangle^{\otimes d}_d \) with the symmetric 2-qudit operator \( V = d^{-3/2} \sum_{i,j,k,l=0}^{d-1} \omega^{(i-j)(k-l)} |ij\rangle\langle kl| \). The Hilbert space at each vertex is \( \mathcal{H}_1 = \left( \mathbb{C}^d \right)^{\otimes g} \), where \( g \) is the vertex degree; \( g = 4 \) for the regular 2D lattice in Fig. 2(b). PEPS are obtained by projecting the state at each vertex (the \( g \) singlets) onto a subspace of dimension \( k \) (the “physical” subspace). The edge operator \( U \) is given by \( V \) followed by this projector.

(v) Quantum random networks (QRN). These states [29] are the quantum analog of the classic Erdős-Rényi random networks. Each graph node shares with the other \( n \) – 1 nodes an entangled state \( |\Omega\rangle = \sqrt{1 - \frac{2}{p}} |00\rangle + \sqrt{\frac{2}{p}} |11\rangle \), with \( 0 \leq p \leq 1 \). The Hilbert space at each vertex is \( \mathcal{H}_1 = \left( \mathbb{C}^2 \right)^{\otimes n-1} \), see Fig. 2(c). To construct a QRN each pair of nodes tries to convert, using only (stochastic) local operations and classical communication, their shared link into the maximally entangled state \( |\Phi^+\rangle \) success with probability \( p \). The edge operator is \( U = |\Phi^+\rangle\langle \Phi^+|V \); we choose the symmetric 2-qubit unitary \( V \) to satisfy \( V|00\rangle = |\Omega\rangle \), giving \( V = \sqrt{1 - \frac{2}{p}} |00\rangle\langle 00| + \frac{\sqrt{2}}{p} (\sigma_x - \sigma_y) |00\rangle\langle 00| \). The initial state is \( |\psi\rangle = |00\rangle^{\otimes n-1} \).

So far we discussed well-known examples which fit in the above axiomatic framework. Are there any novel quantum states which emerge from this framework? In the following we present the general solution for qubits, \( d = 2 \), which contains (and generalizes) graph states. Imposing the constraints C1–C3 we obtain two families of solutions for the edge operator \( U \). The first is diagonal:

\[
U_I = \text{diag}(a, b, b, c)
\]

(14)

Two important examples belong to this family:

(a) Graphs states. Taking \( a = b = c = 1 \) we recover the construction for graphs states given above. More generally, the edge operator is a product of single-qubit phase shifts and a controlled phase shift, \( U = \text{diag}(1, 1, 1, e^{i\varphi}) P(\alpha)^{\otimes 2} \), with \( P(\alpha) := \text{diag}(1, e^{i\alpha}) \).

(b) Parity projectors. The two parity operators \( P_0 = \text{diag}(1, 0, 0, 1) \) and \( P_1 = \text{diag}(0, 1, 1, 0) \) project, respectively, on the even parity, \( \text{span}\{|00\rangle, |11\rangle\} \), and odd parity, \( \text{span}\{|01\rangle, |10\rangle\} \) subspaces of two qubits. Parity gates, together with single-qubit gates, are universal for quantum computation [30, 31].

The second solution is parametrized by \( a, b, c \) and \( T \): 

\[
U_{II} = a I^{\otimes 2} + b (T \otimes I + I \otimes T) + c T \otimes T
\]

with \( T = \begin{bmatrix} 0 & 1 \\ \gamma & -\alpha \end{bmatrix} \) or \( T = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix} \).
There are some intriguing similarities between the present approach and quantum lattice gas models [32, 33] which can prove insightful for future research.

IV. EXTENSION

The modular structure of our framework (given by the axioms A1–A3 and consistency conditions C1–C3) enables us to generalize it to non-simple graphs. In this section we extend the formalism to directed graphs and weighted graphs. For directed graphs, we replace the consistency conditions C2–C3 while keeping the others unchanged. For weighted graphs, the edge operator will be different for different edges.

**Directed graphs.** Since in this case the edge operator \( V \) is no longer symmetric, \( V(y, x) \neq V(x, y) \), we need to change the consistency conditions C2–C3 (locality C1 still holds). Let \( V'(x, y) \) be the swapped version of the edge operator

\[
V'(x, y) := V(y, x) = S(x, y)V(x, y)S(x, y)
\]  
(16)

We represent the two operators graphically as quantum networks:

\[
\begin{align*}
V(x, y) := & \quad \begin{array}{c}
\text{x} \\
\text{y}
\end{array} \\
V'(x, y) := & \quad \begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\end{align*}
\]

The previous commutation relations on two and three vertices now become:

\[
D2 : \quad [V(x, y), V'(x, y)] = 0, \quad \begin{array}{c}
\text{x} \\
\text{y}
\end{array} = \begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\]  
(17)

and

\[
D3a : \quad [V(x, y), V(y, z)] = 0, \quad \begin{array}{c}
\text{x} \\
\text{y} \\
\text{z}
\end{array} = \begin{array}{c}
\text{z} \\
\text{y} \\
\text{x}
\end{array}
\]  
(18)

\[
D3b : \quad [V(x, y), V'(y, z)] = 0, \quad \begin{array}{c}
\text{x} \\
\text{y} \\
\text{z}
\end{array} = \begin{array}{c}
\text{z} \\
\text{y} \\
\text{x}
\end{array}
\]  
(19)

\[
D3c : \quad [V'(x, y), V(y, z)] = 0, \quad \begin{array}{c}
\text{x} \\
\text{y} \\
\text{z}
\end{array} = \begin{array}{c}
\text{z} \\
\text{y} \\
\text{x}
\end{array}
\]  
(20)

The fourth condition \( [V'(x, y), V'(y, z)] = 0 \) is equivalent to\( D3a: \quad 0 = S(x, z)V(x, y)S(x, z)S(y, z)V(y, z) = [V'(z, y), V'(y, z)], \)

Any solution of the edge operator for directed graphs automatically provides a solution for undirected graphs. Let \( V(x, y) \) be a directed edge operator satisfying the constraints D2–D3. Define

\[
U(x, y) := V(x, y)V'(x, y)
\]

Using the graphical notations introduced above, it is straightforward to prove that \( U(x, y) \) satisfies the constraints C2–C3 and thus \( U \) is an edge operator for undirected graphs. This solution for undirected graphs captures intuitively the well-known fact from graph theory that an undirected edge is equivalent to two oppositely-directed edges (as can be seen from the entries of the adjacency matrix of the graph).

An example of a unitary solution for directed edge operator for qubits \((d = 2)\) is given pictorially by

\[
\begin{array}{c}
\text{M} \\
\alpha \\
\phi
\end{array}
\]

with \( M \) an arbitrary unitary, \( \alpha, \beta \) and \( \varphi \) free parameters; equivalently

\[
V(x, y) = M^{\dagger} \text{diag}(1, 1, 1, e^{i\varphi})P(\alpha) \otimes P(\beta)M \otimes M \quad (21)
\]

In this case the edge asymmetry condition is enforced by choosing \( \alpha > \beta \).

**Weighted graphs.** Several important problems in graph theory involve weighted graphs, in which the edges are labeled by one (or more) parameters (the weight). Notable examples are the traveling salesmen problem (the edge weight is the distance between the nodes) or problems in network optimization (weights are the network capacity of the link). In general a weight can have several numbers, each characterizing a different parameter of the link.

The solutions for the edge operator (in both undirected and directed cases) contain free parameters, see eqs. (14), (15), (21). This suggests a straightforward generalization of the present formalism to weighted graphs. We construct a quantum state associated to a weighted graph by choosing the parameters for the edge operator to be different for different edges (provided they satisfy the constraints C2–C3 or D2–D3).

Since the solution (14) is diagonal, the parameters \( a, b, c \) can be taken different for all edges. Modulo an overall phase, this gives two free parameters to label an edge. For the solutions (15) and (21), the matrices \( T \), and respectively \( M \), are fixed for all edges in order to ensure edge commutativity (C3 or D3). In this case only the parameters \( a, b, c \), and respectively \( \alpha, \beta, \varphi \), are free to label different edges.

For undirected graphs, eq. (14) corresponds to weighted graph states which arise naturally from an Ising-type interaction between spins located on a graph [34, 35]. The solution (21) is a straightforward generalization to directed graphs, where the asymmetry of the edge operator appears from an extra local phase.

V. CONCLUSION

Graphs play an important role in characterizing efficiently several families of multipartite quantum states. A notable example are cluster states which are an essential resource for the one-way quantum computing paradigm. Due to the visual impact of a graph, it is easier to understand the entanglement content of the associated graph state. For instance,
includes and generalizes several classes of multipartite entanglement from three general axioms we derived a rich structure which works and PEPS. Due to its modular structure (in terms of axioms and consistency conditions), the axiomatic approach developed here is remarkably flexible. By changing some of the consistency conditions while keeping the others the same, we can incorporate in the model non-simple graphs, namely directed and weighted graphs. Specifically, directed graphs can be included by modifying the consistency conditions C2–C3 in order to take into account the edge asymmetry.

There are several directions in which the present research can be developed in the future. First, one would like to find the general solution of the edge operator (for both undirected and directed graphs) in the case of a qudit, $\mathcal{H}_1 = 4^d$. Second, given a graph $G$ and an edge operator $U$, we would like to characterize the entanglement of the resulting state $|G\rangle$ as a function of the entangling power of $U$. Third, one can envisage a more general extension by changing the “dynamics” of the system by modifying Axiom A3. The tensor product structure of $\mathcal{H}$ remains the same (following from A1–A2), but the edge operator and the consistency conditions will be different. Last but not least, it will be valuable to apply methods based on this approach to graph-theoretical problems (e.g., finding novel graph invariants).

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