Coends of Higher Arity

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Received: 30 November 2020 / Accepted: 25 June 2021 / Published online: 30 August 2021
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Abstract

We specialise a recently introduced notion of generalised dinaturality for functors $T : (\text{C}^{\text{op}})^p \times C^q \to \mathcal{D}$ to the case where the domain (resp., codomain) is constant, obtaining notions of ends (resp., coends) of higher arity, dubbed herein $(p, q)$-ends (resp., $(p, q)$-coends). While higher arity co/ends are particular instances of ‘totally symmetrised’ (ordinary) co/ends, they serve an important technical role in the study of a number of new categorical phenomena, which may be broadly classified as two new variants of category theory. The first of these, \textit{weighted category theory}, consists of the study of weighted variants of the classical notions and construction found in ordinary category theory, besides that of \textit{a limit}. This leads to a host of varied and rich notions, such as \textit{weighted Kan extensions}, \textit{weighted adjunctions}, and \textit{weighted ends}. The second, \textit{diagonal category theory}, proceeds in a different (albeit related) direction, in which one replaces universality with respect to natural transformations with universality with respect to \textit{dinatural transformations}, mimicking the passage from limits to ends. In doing so, one again encounters a number of new interesting notions, among which one similarly finds \textit{diagonal Kan extensions}, \textit{diagonal adjunctions}, and \textit{diagonal ends}.

Keywords Coend · Cowedge · Dinatural transformation

Communicated by Nicola Gambino.

The first author was supported by the ESF funded Estonian IT Academy research measure (Project 2014-2020.4.05.19-0001). The first author would like to thank A. Santamaria and his delightful talk at ItaCa [18], without which this paper would probably not exist, the entire Italian community of category theorists that has made ItaCa possible, and sensei Inoue Takehiko. The second author is greatly indebted to many people for their immense generosity and kindness, including their parents, friends, as well as Jonathan Beardsley, Lennart Meier, Igor Mencattini, Oziride Manzoli Neto, and Eric Peterson, among so many others who I’ve had the pleasure of knowing. The second author was supported by grant #2020/02861-7, São Paulo Research Foundation (FAPESP).

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1 Introduction

A functor \( T : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \) can be thought as a generalised form of pairing defined on objects of \( \mathcal{C} \). Such a \( T \) depends on two variables \( C, C' \) running over a ‘generalised space’ \( \mathcal{C} \), and given its action on morphisms, covariant in one component and contravariant in the other, the ‘generalised quantity’ \( T(C, C') \) can be ‘integrated’, yielding two objects with dual universal properties:

(C1) A coend, resulting by the symmetrisation along the diagonal of \( T \), i.e. modding out the coproduct \( \bigsqcup_{C \in \mathcal{C}} T(C, C) \) by the equivalence relation generated by the arrow functions \( T(-, C') : \mathcal{C}^{\text{op}}(X, Y) \to \mathcal{D}(TX, TY) \) and \( T(C, -) : \mathcal{C}(X, Y) \to \mathcal{D}(TX, TY) \);

(C2) An end, i.e. an object \( \int_{C} T(C, C) \) of \( \mathcal{C} \) arising as an ‘object of invariants’ of ‘fixed points’ for the same action of \( T \) on arrows. By dualisation, if a coend is a quotient of \( \bigsqcup_{C \in \mathcal{C}} T(C, C) \), an end is a subobject of the product \( \prod_{C \in \mathcal{C}} T(C, C) \).

It is the fact that, given \( T : \mathcal{B}^{\text{op}} \times \mathcal{B} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \), the coend operation satisfies the commutativity rule

\[
\int_{A} \int_{B} T(B, B; C, C) \cong \int_{B} \int_{A} T(B, B; C, C),
\]

that motivated N. Yoneda [20] to adopt for them an integral-like notation and terminology. For obvious reasons, this is called the Fubini rule for coends. Obviously, an analogous result holds for ends.

Since, given a functor \( T \) as above, the (co)end of \( T \) can be computed as a certain (co)equaliser, (co)ends can be regarded as just particular (co)limits, associated to functors of a particular type of variance. Central to this reduction rule of (co)ends to (co)limits is the twisted arrow category construction for \( \mathcal{C} \), i.e. the category of elements of the hom functor \( \text{hom}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set} \): it is the case that

\[
\int_{A} T(A, A) \cong \text{lim} \left( \text{Tw}(\mathcal{C}) \xrightarrow{T} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\text{Tw}} \mathcal{D} \right)
\]

and coends are colimits over a similar diagram with domain \( \text{Tw}(\mathcal{C}^{\text{op}})^{\text{op}} \).

Throughout the years, coend calculus, i.e. the set of rules allowing one to formally manipulate integrals of the above kind in order to prove statements in category theory, has found applications in many different fields of mathematics intersecting category theory, and of computer science; see [13] for a comprehensive account of the topic.

Question The particular variance \( T \) is forced to have begs the question of whether there is an analogue of the above picture (a universal property, a Fubini rule, a mass of examples and applications), for ends and coends associated to more general functors

\[
T : \mathcal{C}^{\text{op}} \times \cdots \times \mathcal{C}^{\text{op}} \times \mathcal{C} \times \cdots \times \mathcal{C} \to \mathcal{D}
\]
taking \( p \geq 1 \) contravariant arguments, and \( q \geq 1 \) covariant arguments, admitting the possibility that \( p \neq q \).

The present paper aims to answer this question in the positive.

Such generalised (or \(' (p, q)\)-)co/ends exist, they can be given a universal property that is in all respect analogous to the one for ‘classical’ co/ends (obtained when \( p = q = 1 \)).

A fundamental ‘structure theorem’ of our theory is that each \((p, q)\)-co/ends exist, they can be given a universal property that is in all respect analogous to the one for ‘classical’ co/ends (obtained when \( p = q = 1 \)).

The possibility to define a higher arity analogue of the Day convolution monoidal structure [3,4,8] on the presheaf category \( \text{Cat}(\mathcal{C}^{\text{op}}, \text{Set}) \) over a monoidal category \((\mathcal{C}, \otimes)\) (\(\mathcal{C}\) can be either a small category, or we can consider restriction to the category of small presheaves to avoid size issues). Classically, the Day convolution of two presheaves \( F, G : \mathcal{C}^{\text{op}} \to \text{Set} \) is defined by

\[
\mathcal{F} \otimes \mathcal{G} : \int_{X,Y \in \mathcal{C}} \mathcal{F}(X) \times \mathcal{G}(Y) \times \mathcal{C}(-, \mathcal{C}(X \otimes Y)).
\]

We generalise this notion in Definition 4.5: given \( n \) presheaves \( \mathcal{F}_1, \ldots, \mathcal{F}_n \), their \( n \)-ary Day convolution is the \((n, n)\)-coend

\[
(\mathcal{F}_1 \otimes_n \cdots \otimes_n \mathcal{F}_n)(A) \overset{\text{def}}{=} \int_{A \in \mathcal{C}} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times \mathcal{C}(-, A^{\otimes n}).
\]

The sets of various \( n \)-ary Day convolutions, together with the convolution of order \( k \leq n \), organise into an operad that we dub the Day operad in Example 4.16.

The object of higher arity dinatural transformations from a functor \( F : (\mathcal{C}^{\text{op}})^p \times \mathcal{C}^q \to \mathcal{D} \) to a functor \( G : (\mathcal{C}^{\text{op}})^q \times \mathcal{C}^p \to \mathcal{D} \) (note that the arity of \( F \) and the arity of \( G \) are opposite to each other) turns out to be a natural example of a \((p, q)\)-end. Firstly noticed by Street and Dubuc in [5, Theorem 1] for (ordinary, i.e. \((1, 1)\)-) dinatural transformations, this result holds in the higher arity context and provides us with an important tool for the study of higher arity co/ends.

Extending Street and Dubuc’s work, we provide an canonical way to appropriately resolve a functor \( F \) of type \( \left[ \begin{array}{c} \alpha \vspace{1em} \beta \end{array} \right] \) into a functor \( J^{p,q}(F) \) of type \( \left[ \begin{array}{c} \beta \vspace{1em} \alpha \end{array} \right] \) satisfying

\[ \text{Nat}(J^{p,q}(F), G) \cong \text{DiNat}^{(p,q)}(F, G). \]

That is, \( J^{p,q}(F) \) is characterised by the universal property that every dinatural transformation from \( F \) to \( G \) amounts exactly to a natural transformation from \( J^{p,q}(F) \) to \( G \). Dually, one can ‘resolve’ \( G \) into a functor \( \Gamma^{q,p}(G) \) satisfying

\[ \text{Nat}(F, \Gamma^{q,p}(G)) \cong \text{DiNat}^{(p,q)}(F, G). \]

We study these functors in detail in Sect. 5, where, as an application, we construct an analogue of the twisted arrow category \( \text{Tw}(\mathcal{C}) \) of \( \mathcal{C} \) in the higher arity setting, dubbed the higher arity twisted arrow category of \( \mathcal{C} \) in Sect. 5.3.

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1 In [11], the author says that a functor \( T : \mathcal{C}^n \to \mathcal{D} \) has ‘type \( n \)’; mimicking this nomenclature, we shortly refer to our functors as having ‘type \( \left[ \begin{array}{c} p \\
q \end{array} \right] \)’.
Higher arity co/ends also pave the way to a number of future research directions:

1. **Weighted category theory (besides co/limits).** The notion of weighted co/limit arises as the solution to a representability problem. Computing the ‘conical’ colimit of a diagram \( D : C \to D \), we aim to find an object \( \text{colim}(D) \) that represents the functor sending \( X \in D \) to the set of cocones for \( D \), i.e. natural transformations between the ‘constant at the point’ functor \( * \) and the functor \( D(D\text{-}, X) \). In a similar fashion, when we want to compute the weighted colimit of \( D \) we try to represent the functor \( X \mapsto \text{Cat}(C^{\text{op}}, \text{Set})(W, D(D\text{-}, X)) \).

Now, what if we try to do the same for the other usual categorical notions apart from that of co/limits, such as adjunctions, Kan extensions, monads, or co/ends? For example, what if, instead of trying to represent the functor that sends \( T : C^{\text{op}} \times C \to D \) to the set of its co/wedges, we try to represent the functor that sends \( X \) to \( \text{DiNat}(W, D(T, X)) \)? We dub **weighted category theory** the piece of technology that addresses this problem.

2. **Diagonal category theory.** In what we call ‘diagonal’ category theory, a different (albeit related) path is taken. In analogy to the passage from limits (universal cones) to ends (universal wedges), we seek a general framework for other categorical constructions, in which the passage from cones to wedges is meaningful. Thus, we aim to replace naturality requests with dinaturality in categorical concepts besides that of a limit, obtaining, as a result, a very rich theory with notions such as diagonal Kan extensions, adjunctions and monads, all standing to their classical counterparts as co/ends stand to co/limits. The theory so obtained is non-trivial and sheds light on a number of aspects of classical category theory, such as the disparity between limits and colimits assorting themselves into a triple adjunction

\[
(\text{colim} \dashv \Delta_{(-)} \dashv \text{lim}) : \text{Cat}(C, D) \xleftrightarrow{\text{Diag}} D
\]

with no such result existing for ends and coends: rather, we have **diagonally adjoint functors**, of which ends and coends are a fundamental example, assembling into a ‘triple diagonal adjunction’ \( \int_A \Delta_{(-)} \dashv \delta \Delta_{(-)} \dashv \delta \int_A \).

We find the fact that an end (universal dinatural transformation from a constant) and a limit (universal natural transformation from a constant) an interesting perspective.

### 1.1 Structure of the Paper

In Sect. 1.2, we motivate higher arity dinaturality, showing how such a notion arises geometrically as a ‘diagonal’ version of natural transformations between functors with domain a product category. All the material there contained is very well-known, and this is only meant to fix notation at the outset. We borrow from [2, pp. 48–50] an intuitive explanation of how dinaturality arises from elementary considerations,\(^2\) first recalling it in Remark 1.1 and then generalising that classical argument to the higher arity case in Remark 1.2.

In Sect. 2, we review and specialise the notion of dinaturality introduced in [17,19] to the appropriate setting for considering ‘universal higher arity dinaturality’. In detail, we first recall the notion of a \((p, q)\)-**dinatural transformation** and study its properties, generalising results of Street and Dubuc [5] to functors of arbitrary arity. We then proceed to discuss \((p, q)\)-dinatural transformations from constant functors, which we dub \((p, q)\)-**wedges**, in analogy with the classical case.

\(^2\) See also [7] for a similar presentation.
In Sect. 3, we formulate the notion of a higher arity co/end. Just as ends are universal wedges, higher arity ends are universal \((p, q)\)-wedges. After introducing them in Definition 3.1, we discuss some of their basic properties (Proposition 3.1). The intuition behind the definition is that a \((p, q)\)-end of a functor \(T\) is an end of a version of \(T\) that has been ‘completely symmetrised’. Then, in Sect. 3.2, we state and prove a Fubini rule for higher arity co/ends, generalising the classical Fubini rule for co/ends.

In Sect. 4, we illustrate the theory developed so far by working out a large number of examples. We study naturally-appearing instances of higher arity co/ends in category theory as well as in related areas. The machinery employed here is elementary but provides insightful examples when applied to laying down the rules of a ‘calculus of weighted ends’, and to ‘diagonal category theory’ (see Sects. 4.2.1, 4.2.3 for reference).

In Sect. 5, we introduce the notion of co/kusarigama. These are fundamental constructions in higher arity co/end calculus, allowing us to reduce the study of \((p, q)\)-dinaturality to that of (ordinary) naturality. Co/kusarigama also provide us with a way to express higher arity co/ends as weighted co/limits, as well as with a higher arity version of the twisted arrow category (Sect. 5.3).

1.2 Geometric Motivation for Higher Arity Dinaturality

Notation 1.1 \(((p, q)\text{-products, tensor calculus notation})\) The entire paper deals with categories that are the product of \(q\) copies of a (small) category \(C\), and \(p\) copies of the opposite category \(C^{\text{op}}\), for \(p, q\) two non-negative integers: throughout the entire discussion we will adopt the notation

\[
C^{(p,q)} \overset{\text{def}}{=} C^{\text{op}} \times \cdots \times C^{\text{op}} \times C \times \cdots \times C.
\]

An alternative, short notation for the category \((C^{\text{op}})^p = (C^p)^{\text{op}}\) is \(C^{-p} \times C^q\), but this has to be used cum grano salis, as some of the usual sign conventions do not apply (for example, exponents of discordant signs do not add: \(C^{-1} \times C^1 = C^{\text{op}} \times C = C^{(1,1)}\) is ‘irreducible’).

A functor having domain \(C^{(p,q)}\) will be called a functor of type \([p\ q]\).

To denote the action of a functor of type \([p\ q]\) on objects we write

\[
F_{A^{\prime \prime}}^{A^{\prime}} \overset{\text{def}}{=} F_{A^{\prime \prime}_1,\ldots,A^{\prime \prime}_p}^{A^{\prime}_1,\ldots,A^{\prime}_q} = F \left( A^{\prime \prime}_1,\ldots,A^{\prime \prime}_p, A^{\prime}_1,\ldots,A^{\prime}_q \right)
\]

for (tuples of) objects \(A^{\prime} \overset{\text{def}}{=} (A^{\prime}_1,\ldots,A^{\prime}_q) \in C^q\) and \(A^{\prime \prime} \overset{\text{def}}{=} (A^{\prime \prime}_1,\ldots,A^{\prime \prime}_p) \in C^{-p}\). (see Sect. 1.3 below for variations and specialisations of this notation.)

This is somewhat reminding of the way in which one writes the coordinates of a tensor of type \([p\ q]\): contravariant indices are superscripts, covariant are subscripts.

With this notation in place, we start by recalling the notion of dinatural transformation. We borrow an argument from [2, pp. 48–50] that shows how dinaturality is, if not unavoidable, at least motivated by elementary geometric considerations on commutative \(n\)-dimensional cubes.

Informally, we may summarise the discussion below as follows: to arrive at the notion of dinaturality, one performs a symmetrisation process, first considering a ‘naturality cube’ and then removing its ‘non-diagonal’ pieces.

Baez–Stay’s argument can be adapted to motivate our notion of \((p, q)\)-dinaturality (Definition 2.1) as similarly motivated (see Remark 1.2 below).
Remark 1.1 (From naturality cubes to dinaturality hexagons) Let $C$ be a category, and consider the usual product category $\mathcal{C}^{\text{op}} \times \mathcal{C}$. Each morphism $f : A \to B$ of $\mathcal{C}$ gives rise to a commutative square in $\mathcal{C}^{\text{op}} \times \mathcal{C}$ of the form

$$
\begin{array}{c}
(B, A) 
\xrightarrow{(B, f)} (B, B) \\
\downarrow_{(f^{\text{op}}, A)} 
\xRightarrow{\sim} 
\downarrow_{(f^{\text{op}}, B)} \\
(A, A) 
\xrightarrow{(A, f)} (A, B),
\end{array}
$$

to which we can apply parallel functors $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \Rightarrow \mathcal{D}$, obtaining two naturality squares

$$
\begin{array}{c}
F^B_A F^B_B 
\xrightarrow{F^f_A} F^B_B 
\downarrow_{F^f_A} 
\xRightarrow{\alpha^B_A} 
G^B_A G^B_B \\
G^B_A G^B_B 
\downarrow_{G^f_A} 
\xRightarrow{\alpha^B_A} 
G^A_A G^A_B,
\end{array}
$$

from which we derive that $F^f_B \circ F^B_A = F^A_f \circ F^f_A$, and similarly for $G$.

A natural transformation from $F$ to $G$ is then a collection of morphisms $\alpha^B_A : F^A_A \to G^B_A$ making the diagram

$$
\begin{array}{c}
F^A_B 
\xrightarrow{F^f_A} F^{A'}_{B'} 
\downarrow_{\alpha^B_A} 
\xRightarrow{\alpha^{A'}_{B'}} 
G^B_A G^{A'}_{B'} \\
G^B_A G^{A'}_{B'} 
\downarrow_{G^f_A} 
\xRightarrow{\alpha^{A'}_{B'}} 
G^A_A G^A_B,
\end{array}
$$

commute for every $\begin{bmatrix} A' \\ g \end{bmatrix}$ and $\begin{bmatrix} B' \\ f \end{bmatrix}$. When $g = f : A \to B$, this reduces to

$$
\begin{array}{c}
F^B_A F^B_B 
\xrightarrow{F^f_A} F^A_B 
\downarrow_{\alpha^B_A} 
\xRightarrow{\alpha^B_A} 
G^B_A G^A_B \\
G^B_A G^A_B 
\downarrow_{G^f_A} 
\xRightarrow{\alpha^A_B} 
G^A_A G^A_B,
\end{array}
$$

which we may rewrite as the following commutative cube, using that $(f, f) = (\text{id}_B, f) \circ (f, \text{id}_A)$ in $\mathcal{C}^{\text{op}} \times \mathcal{C}$:

$$
\begin{array}{c}
F^B_A F^B_B 
\xrightarrow{F^f_A} F^A_B 
\downarrow_{\alpha^B_A} 
\xRightarrow{\alpha^A_B} 
G^B_A G^A_B \\
G^B_A G^A_B 
\downarrow_{G^f_A} 
\xRightarrow{\alpha^A_B} 
G^A_A G^A_B,
\end{array}
$$

(in all arrows, the action of $F$ on its covariant or contravariant component is taken into account).
A notion of ‘diagonal natural transformation’ between $F$ and $G$ is then a collection of morphisms of $D$ from $F_A$ to $G_A$. To obtain it, we should remove the ‘non-diagonal pieces’ $\alpha_{A}^{B}$ and $\alpha_{B}^{A}$ from this cube, arriving at the diagram.

Writing $\alpha$ (resp. $\alpha_B$) for $\alpha_{A}^{A}$ (resp. $\alpha_{B}^{B}$) and ‘flattening’ the resulting diagram, we get the dinaturality hexagon

$$
\begin{array}{ccc}
F_A & \xrightarrow{\alpha} & G_A \\
F_B & \xleftarrow{\alpha_B} & G_B \\
\end{array}
$$

for $\alpha : F \Rightarrow G$.

Given functors $F : C^{(p, q)} \to D$ and $G : C^{(q, p)} \to D$, we seek a similar intuitive explanation for an analogous notion of $(p, q)$-dinaturality. Of course, as the sum $p + q$ grows bigger, it is more difficult to visualise the underlying geometry, since we have to work in dimension $p + q + 1 \geq 4$. For this reason, we content ourselves with the case $(p, q) = (2, 1)$.

Recall that we write $C^{(2,1)}$ for the category $C^{\text{op}} \times C^{\text{op}} \times C$.

**Remark 1.2** (From Naturality Hypercubes to $(2, 1)$-Dinaturality) In a similar fashion to Remark 1.1, a morphism $f : A \to B$ induces a commutative cube in $C^{(2,1)}$ which, under the action of $F$ and $G$, yields two commutative cubes

$$
\begin{array}{ccc}
F_{A}^{BB} & \xrightarrow{\alpha_{A}^{B}} & F_{A}^{BB} \\
& & \downarrow \\
F_{A}^{BA} & \xleftarrow{\alpha_{B}^{A}} & F_{A}^{BA} \\
& & \downarrow \\
F_{A}^{AA} & \xrightarrow{\alpha_{C}^{AB}} & F_{A}^{AA} \\
\end{array}
$$

in $D$. Now, a natural transformation $\alpha$ from $F$ to $G$ is a collection

$$
\left\{ \alpha_{C}^{AB} : F_{C}^{AB} \to G_{C}^{AB} \mid (A, B, C) \in C^{(2,1)} \right\}
$$

of morphisms of $D$ making the hypercube diagram below-left commute:
Remove “non-diagonal” pieces

( vermillion: the $F$ cube; blue: the $G$ cube). Again deleting the nodes $F^Y_Z, G^Y_Z$ for which $X, Y, Z$ are not all equal, we get the above-right diagram. Flattening the result, this gives us the octagonal diagram below-left, which becomes the ‘$(2, 1)$-dinaturality hexagon’ below-right upon using that $F_A^f = F_A^f \circ F_A^B$ and $G_B^f = G_B^f \circ G_B^B$:

In order not to clutter the page with too many unwanted apices and pedices, it is vital that we establish appropriate notation to represent how functors act on tuples. This is the scope of the following subsection.
1.3 Notation and Preliminaries

All the basic notation for categories and functors used in this paper follows standard practice. Apart from this, and apart from what we already introduced in Notation 1.1, we need notation for:

(N1) A generic tuple of objects,
\[ A \overset{\text{def}}{=} (A_1, \ldots, A_n) \]
often split as the juxtaposition \( A' \); \( A'' \) of two subtuples of length \( q, p \) respectively:
\[ A' \overset{\text{def}}{=} (A_1, \ldots, A_q), \quad A'' \overset{\text{def}}{=} (A_{p+1}, \ldots, A_{p+q}) \]

(N2) As already said, the image of a split \((p + q)\)-tuple \( A' \); \( A'' \) under a functor of type \([q]\), \( F : C^{(p, q)} \rightarrow D \) is denoted \( F_{A'}^{A''} \); the contravariant components come first/top, and the covariant component come second/bottom. So: contravariant components are always \textit{left} in the typing
\[ F : C^{(p, q)} \rightarrow D \]
of a functor, and \textit{up} in its action on objects.

(N3) Denoting a functor \( F \) of type \([q]\) evaluated at a diagonal tuple \( A \overset{\text{def}}{=} (A, \ldots, A) \) with \( A \in C_o \), we write
\[ F_A^A \overset{\text{def}}{=} F_{A,\ldots,A} \]
where the superscript has \( p \) elements, and the subscript has \( q \) elements.

In the following definition, for every set \( I \) we consider the functor \( \Delta^C_I : C \rightarrow C^I \), the constant functor sending an object \( C \in C \) into the constant family \( \{C \mid i \in I\} \); in particular, if \( I = \{1, \ldots, p\} \) we write \( \Delta^C \) to denote \( \Delta^C_I \). Now,

**Definition 1.1** The \((p, q)\)-diagonal functor is the functor \( \Delta_{p,q} : C^{op} \times C \rightarrow C^{(p,q)} \) defined by \( \Delta_{p,q} \overset{\text{def}}{=} \Delta^C_p \times \Delta^C_q \). If there is no risk of ambiguity, we just write \( \Delta_{p,q} \) for \( \Delta_{p,q}^C \).

More explicitly, \( \Delta_{p,q} : C^{op} \times C \rightarrow C^{(p,q)} \) is the functor sending
\[ (UD1) \quad \text{An object } (A, B) \text{ of } C^{op} \times C \text{ to the object } (A, B) \overset{\text{def}}{=} (A, \ldots, A, B, \ldots, B) \text{ of } C^{(p,q)}, \]
and
\[ (UD2) \quad \text{A morphism } (f^{op}, g) : (A, B) \rightarrow (A', B') \text{ of } C^{op} \times C \text{ to the morphism } (f^{op}, g) \overset{\text{def}}{=} (f^{op}, \ldots, f^{op}, g, \ldots, g) \text{ of } C^{(p,q)}, \]
where in the expression \((A, B)\) we have \( p \) repeated copies of \( A \) and \( q \) repeated copies of \( B \), and similarly for \((f^{op}, g)\).

The notational choices and lemmas below will be used only from Sect. 3.2 on, so we advise the reader to skip them at this point, referring to them later as needed.

**Notation 1.2** (Mixed Products and Coproduts) Let \( C \) be a category with finite products and coproducts. Given tuples \((A_1, \ldots, A_p)\) and \((B_1, \ldots, B_q)\), we write
\[ W_{p,q}(A, B) \overset{\text{def}}{=} \left( \prod_{i=1}^{p} A_i, \prod_{j=1}^{q} B_j \right) \]
\[ M_{p,q}(A, B) \overset{\text{def}}{=} \left( \prod_{i=1}^{p} A_i, \prod_{j=1}^{q} B_j \right) \]
and regard them as functors of type \([q]\).
Lemma 1.1 (Adjoints to the \((p, q)\)-Diagonal Functor) If \(C\) has products and coproducts, then we have a triple adjunction

\[
(W_{p,q} \dashv \Delta_{p,q} \dashv M_{p,q}) : \mathcal{C}^{(p,q)} \leftrightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}.
\]

Proof The lemma follows from the universal property of the co/product, as a standard manipulation of natural isomorphisms yields an isomorphism

\[
\text{hom}_{\mathcal{C}^{(p,q)}} ((A, B), \Delta_{p,q}(C, D)) \cong \text{hom}_{\mathcal{C}^{\text{op}} \times \mathcal{C}} (W_{p,q}(A_i, B_j), (C, D))
\]
natural in \(A_1, \ldots, A_p, B_1, \ldots, B_q, C, D \in \mathcal{C}_0\). The proof that \(\Delta_{p,q}\) admits a right adjoint is dual. \(\square\)

We also collect a couple of standard results on generating strings of adjunctions by left/right Kan extending a given adjunction:

Lemma 1.2 (Applying Kan Extensions to an Adjunction) Every adjunction

\[
L : \mathcal{C} \leftrightarrows \mathcal{D} : K
\]
induces a quadruple adjunction \(\text{Lan}_K \dashv K^* \dashv L^* \dashv \text{Ran}_L\) such that

\[
\text{Lan}_K \cong L^*, \quad \text{Ran}_L \cong K^*.
\]

Proof Both \(L\) and \(K\) induce triple adjunctions between \(\text{Cat}(\mathcal{C}, \mathcal{E})\) and \(\text{Cat}(\mathcal{D}, \mathcal{E})\). Proving that \(\text{Lan}_K \cong L^*\) and \(\text{Ran}_L \cong K^*\) would show that these are actually parts of a single quadruple adjunction, which is the stated one. This follows from the string of isomorphisms

\[
L^*(F) \overset{\text{def}}{=} F \circ L \cong \int_X \mathcal{C}(L(-), X) \circ F(X) \cong \int_X \mathcal{D}(-, K(X)) \circ F(X) \cong \text{Ran}_L F
\]
natural in \(F\). Hence \(K^* \cong \text{Lan}_L\). By a similar argument, \(K^* \cong \text{Ran}_L\), finishing the proof. \(\square\)

Combining two applications of Lemma 1.1, as well as uniqueness of adjoint functors up to natural isomorphism, with Lemma 1.2, we get the following corollary:

Corollary 1.1 We have a string of five adjoint functors

\[
\left(\text{Lan}_{M_{p,q}} \dashv M^*_{p,q} \dashv \Delta^*_{p,q} \dashv W^*_{p,q} \dashv \text{Ran}_{W_{p,q}}\right) : \text{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D}) \leftrightarrow \text{Cat}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{D}),
\]
with natural isomorphisms

\[
\text{Lan}_{\Delta_{p,q}} \cong M^*_{p,q}, \quad \text{Ran}_{\Delta_{p,q}} \cong W^*_{p,q}.
\]
2 Higher Arity Wedges

This section formalises completely the notion that we dubbed ‘(2,1)-dinaturality’ in Remark 1.1 above and presents it for general \( p, q \geq 0 \).

The definition of dinaturality given below is not new: it was recently introduced in Santamaria’s PhD thesis [17,19], building on previous work by M. Kelly [10,11] in far more generality than needed for our purposes.

In [17,19], however, an ‘unbiased’ arrangement of the factors in \( \mathcal{C}^{(p,q)} \) is considered, in the sense that [17, Definition 2.4] takes into account functors \( \mathcal{C}^\alpha \to \mathcal{B} \), where \( \alpha \) is a ‘binary multi-index’, i.e. an element in the free monoid over the set \( \{ \oplus, \ominus \} \), and the convention is that \( \mathcal{C}^\ominus \overset{\text{def}}{=} \mathcal{C} \), \( \mathcal{C}^\oplus \overset{\text{def}}{=} \mathcal{C}^{op} \), and \( \mathcal{C}^{\ominus \oplus \ominus} \overset{\text{def}}{=} \mathcal{C}^\alpha \times \mathcal{C}^\alpha \).

Here instead, we adopt a different convention: a generic power \( \mathcal{C}^\alpha \) is always ‘reshuffled’ in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right. The categories \( \mathcal{C}^\alpha \) and \( \mathcal{C}^{(p,q)} \) so obtained are, of course, canonically isomorphic, and the tuple \( \alpha \) is equivalent to the reshuffled tuple \( (\ominus, \ldots, \ominus_p, \ominus_1, \ldots, \ominus_q) \).

2.1 Higher Arity Dinaturality

Let \( p, q \in \mathbb{N} \) and \( \mathcal{C} \) be a category. The definition of a \((p, q)\)-dinatural transformation from a functor \( F : \mathcal{C}^{(p,q)} \to \mathcal{D} \) of type \( \left[ \begin{array}{c} q \\ p \end{array} \right] \) to a functor \( G : \mathcal{C}^{(q,p)} \to \mathcal{D} \) of type \( \left[ \begin{array}{c} p \\ q \end{array} \right] \) can be shortly stated as the condition that a dinaturality hexagon commutes, when filled with the conjoint action of \( F \) (resp. \( G \)) in all its contravariant and covariant components separately.

The choice of joining with a transformation just functors of opposite types deserves a bit of explanation.

In Definition 2.1 we dub the condition we are interested in ‘\((p, q)\)-to-(q, p) dinaturality’. It sits between a rigid one (a ‘\((p, q)\)-to-(p, q)’ dinaturality with \( F \) and \( G \) of the same type) and a loose one (a ‘\((p, q)\)-to-(r, s)’ dinaturality, where \( F \) and \( G \) can have completely different types: this second is the path chosen by [19], recalled in Definition 2.6 below).

We could have employed the tighter notion, but some of the characterisations we give would not hold: for example, the set of \((p, q)\)-dinatural transformations between \( F \) and \( G \) is a \((p, q)\)-end only with our convention (see Example 4.7).

We could have employed the looser one; but the definition of co/wedge given in Definition 2.3 wouldn’t have changed (a constant functor can be ‘-muted’, in the sense of Notation 3.1, to have whatever type is needed).

Both the loose and rigid notions of generalised dinaturality mentioned above allow for component-wise composition, but, as is well-known in the classical case for ordinary dinaturality, the resulting family may fail to be dinatural. This does not present a problem for developing the theory of higher arity co/ends, as one just needs generalised dinaturality to be pre or post-composable with natural transformations, and this is indeed the case (Proposition 2.1) for the notion of higher arity dinaturality introduced below.

**Definition 2.1** Let \( F : \mathcal{C}^{(p,q)} \to \mathcal{D} \) and \( G : \mathcal{C}^{(q,p)} \to \mathcal{D} \) be functors. A \((p, q)\)-dinatural transformation \( \alpha : F \Rightarrow G \) is a collection

\[
\{ \alpha_A : F_{A,\ldots, A}^{p,\ldots, p} \to G_{A,\ldots, A}^{q,\ldots, q} \mid A \in \mathcal{C}_o \}
\]

of morphisms of \( \mathcal{D} \) indexed by the objects of \( \mathcal{C} \) such that, for each morphism \( f : A \to B \) of \( \mathcal{C} \), the diagram
Example 2.1 For \((p, q) = (2, 1)\), a \((2, 1)\)-dinarl natural transformation is a collection

\[
\{ \alpha_A : F_A^A \rightarrow G_A^A \mid A \in C_o \}
\]

of morphisms of \(\mathcal{D}\) such that, for each morphism \(f : A \rightarrow B\) of \(\mathcal{C}\), the following hexagonal diagram commutes:

\[
\begin{array}{ccc}
F_A^A & \xrightarrow{\alpha_A} & G_A^A \\
\downarrow & & \downarrow \\
F_B^B & \xrightarrow{\alpha_B} & G_B^B
\end{array}
\]

Notation 2.1 We write \(\operatorname{DiNat}^{(p, q)}(F, G)\) for the set of \((p, q)\)-dinarl natural transformations from \(F\) to \(G\).

Remark 2.1 The same convention of Sect. 1.3 applies to morphisms as well as to objects: \(F_B^B \overset{\text{def}}{=} F_{B,\ldots,B}^B\) is the morphism \(F_{A,\ldots,A}^B \rightarrow F_{A,\ldots,A}^B\) induced by the conjoint action of \(f\) in all the covariant components of \(F\), and similarly for \(F_A^F, G_A^F\), etc.

Dinarl natural transformations can always be composed with natural ones of the appropriate arity, on the left and on the right.

Definition 2.2 (Composing dinatural with naturals) Let \(F\) and \(G\) be a functors of type \([\quad p \quad q \quad]\), let \(H\) and \(K\) be functors of type \([\quad q \quad p \quad]\), let \(\alpha : F \Longrightarrow G\) and \(\beta : H \Longrightarrow K\) be natural transformations, and let \(\theta : G \Longrightarrow H\) be a \((p, q)\)-dinatural transformation.

(DC1) The \textit{vertical composition of} \(\theta\) \textit{with} \(\alpha\) \textit{is the} \((p, q)\)-dinatural transformation \\

\[
\theta \circ \alpha : F \Longrightarrow H
\]

\[
\left\{ (\theta \circ \alpha)_A : F_A^A \rightarrow H_A^A \mid A \in C_o \right\},
\]

where \((\theta \circ \alpha)_A = \theta_A \circ \alpha_A^A;\)

(DC2) The \textit{vertical composition of} \(\beta\) \textit{with} \(\theta\) \textit{is the} \((p, q)\)-dinatural transformation \\

\[
\beta \circ \theta : G \Longrightarrow K
\]

\[
\left\{ (\beta \circ \theta)_A : G_A^A \rightarrow K_A^A \mid A \in C_o \right\},
\]

where \((\beta \circ \theta)_A = \beta_A^A \circ \theta_A.\)
Remark 2.2 Note that the definition of \((p, q)\)-dinaturality in Definition 2.1 does not allow one to construct a dinatural transformation from a natural transformation \(\alpha^A_B \mapsto \alpha^A_A\), in fact, this request does not even make sense, as natural transformations are defined only between functors of the same type. For instance, given \(F\) and \(G\) of variance \((1, 2)\), a natural transformation \(\alpha : F \Rightarrow G\) is a collection of morphisms of the form
\[
\alpha^A_{A,A} : F^A_{A,A} \to G^A_{A,A},
\]
rather than of the form
\[
\alpha^A_{A,A} : F^A_{A,A} \to G^A_{A,A},
\]
as required in Definition 2.1. More formally, the non-existence of 'associated \((p, q)\)-differential transformations' for natural transformations between functors of different variance boils down to the absence of identity dinatural transformations for \(p \neq q\). We note, however, that these two aspects of higher arity dinaturality are completely irrelevant for developing the theory of higher arity co/ends.

Proposition 2.1 \(\theta \circ \alpha\) and \(\beta \circ \theta\) are \((p, q)\)-dinatural transformations.

Proof The \((p, q)\)-dinaturality condition for \(\theta \circ \alpha\) is the requirement that the boundary of the diagram

\[
\begin{array}{c}
F^A_A \xrightarrow{\alpha^A_A} G^A_A \xrightarrow{\theta_A} H^A_A \\
F^B_A \xrightarrow{\beta^B_A} G^B_A \xrightarrow{\theta_B} H^B_A
\end{array}
\]

commutes. Since

1. Sub-diagrams (1) and (2) commute by the naturality of \(\alpha\), and
2. Sub-diagram (3) commutes by the \((p, q)\)-dinaturality of \(\theta\),

so does the boundary diagram: \(\theta \circ \alpha\) is indeed a \((p, q)\)-dinatural transformation.

Similarly, for \(\beta \circ \theta\): one considers instead the corresponding diagram having the form

\[
\begin{array}{c}
\bullet \xrightarrow{} \bullet \xrightarrow{} \bullet \\
\bullet \xrightarrow{\bullet} \bullet \xrightarrow{} \bullet
\end{array}
\]

where again each sub-diagram commutes by either the dinaturality of \(\theta\) or the naturality of \(\beta\).

For the next proposition, recall the definition of the \((p, q)\)-diagonal functor \(\Delta_{p,q} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}^{(p,q)}\) of \(\mathcal{C}\) introduced in Definition 1.1.
Proposition 2.2 (Higher arity dinaturality via ordinary dinaturality) Let \( F : \mathcal{C}^{(p,q)} \to \mathcal{D} \) and \( G : \mathcal{C}^{(q,p)} \to \mathcal{D} \) be functors. We have a natural bijection
\[
\text{DiNat}^{(p,q)}(F, G) \cong \text{DiNat}^{(1,1)}(\Delta_{p,q}^* (F), \Delta_{q,p}^* (G)).
\] (2)

**Proof** This is simply a matter of unwinding the definitions: since \((F \circ \Delta_{p,q})_B \overset{\text{def}}{=} F_B^A\) (and similarly for morphisms and for \(G\)), it follows that a \((p,q)\)-dinaratural transformation \( F \Longrightarrow G \) is precisely a dinatural transformation \( \Delta_{p,q}^* (F) \Longrightarrow \Delta_{q,p}^* (G) \).

\[\square\]

2.2 Higher Arity Wedges

The notion of wedge (resp., cowedge) for a diagram \( D : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \) arises when assuming that the domain (resp., codomain) of a dinatural transformation to/from \( D \) is constant; similarly, a \((p,q)\)-wedge (resp., \((p,q)\)-cowedge) for a diagram \( D : \mathcal{C}^{(p,q)} \to \mathcal{D} \) consists of a \((p,q)\)-dinatural transformation whose domain (resp., codomain) is a constant functor \( \Delta_X : \mathcal{C}^{(q,p)} \to \mathcal{D} \) of type \([q/p] \) with value \( X \in \mathcal{D}_o \).

**Definition 2.3** Let \( D : \mathcal{C}^{(p,q)} \to \mathcal{D} \) be a functor and let \( X \in \mathcal{D}_o \).

(CW1) A \((p,q)\)-wedge for \( D \) under \( X \) is a \((p,q)\)-dinatural transformation \( \theta : \Delta_X \Longrightarrow D \) from the constant functor of type \([q/p] \) with value \( X \) to \( D \);

(CW2) A \((p,q)\)-cowedge for \( D \) over \( X \) is a \((p,q)\)-dinatural transformation \( \zeta : D \Longrightarrow \Delta_X \) from \( D \) to the constant functor of type \([q/p] \) with value \( X \).

**Remark 2.3** (Unwinding Definition 2.3)

(CWU1) A \((p,q)\)-wedge \( \theta : \Delta_X \Longrightarrow D \) is a collection
\[
\{ \theta_A : X \to D_A^A : A \in \mathcal{C}_o \}
\]
of morphisms of \( \mathcal{D} \) such that, for each morphism \( f : A \to B \) of \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\theta_B} & D_B^B \\
\downarrow{\theta_A} & & \downarrow{\rho_B^f} \\
D_A^A & \xrightarrow{\rho_B^f} & D_B^A
\end{array}
\]

commutes.

(CWU2) A \((p,q)\)-cowedge \( \zeta : D \Longrightarrow \Delta_X \) is a collection
\[
\{ \xi_A : D_A^A \to X : A \in \mathcal{C}_o \}
\]
of morphisms of \( \mathcal{C} \) such that, for each morphism \( f : A \to B \) of \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\xi_B} & D_B^B \\
\downarrow{\xi_A} & & \downarrow{\rho_B^f} \\
D_A^A & \xleftarrow{\rho_B^f} & D_B^A
\end{array}
\]

commutes.
Remark 2.4 For the sake of clarity, we remind the reader that in our notation, the commutativity of the diagram in item (CWU1) of Remark 2.3 above means that, for every \( f \in \text{Mor}(C) \),
\[
D_{f_{1}, \ldots, f_{p}} \circ \theta_{B} = \theta_{A} \circ D_{f_{1}, \ldots, f_{q}}^{A_{1}, \ldots, A_{p}}
\]
where \( f_{i} \equiv f \), \( A_{i} \equiv \text{src} f \), \( B_{i} \equiv \text{trg} f \) are the domain and codomain of \( f \), for every index in the relevant range, and \( \theta_{A} : X \rightarrow D_{A_{1}, \ldots, A_{p}}^{A_{1}, \ldots, A_{p}} \) is a morphism in \( D \).

Notation 2.2 We write \( Wd_{X}^{(p,q)}(D) \) for the set of \((p, q)\)-wedges of \( X \) under \( D \), and similarly, \( CWd_{X}^{(p,q)}(D) \) for \((p, q)\)-cowedges.

Proposition 2.3 Let \( D : C^{(p,q)} \rightarrow \mathcal{D} \) be a functor.

(WDF1) The assignment \( X \mapsto Wd_{X}^{(p,q)}(D) \) defines a presheaf
\[
Wd_{X}^{(p,q)}(D) : C^{op} \rightarrow \text{Set}.
\]

(WDF2) The assignment \( X \mapsto CWd_{X}^{(p,q)}(D) \) defines a functor
\[
CWd_{X}^{(p,q)}(D) : C \rightarrow \text{Set}.
\]

Proof item (WDF1): Let \( f : X \rightarrow Y \) be a morphism of \( C \). We have a map
\[
Wd_{X}^{(p,q)}(D) : Wd_{Y}^{(p,q)}(D) \longrightarrow Wd_{X}^{(p,q)}(D)
\]
\[
(Y \longmapsto D) \longmapsto \left( X \overset{f}{\longrightarrow} Y \right)
\]
where we have used Proposition 2.1. As it is clear that this construction preserves composition and identities, we get our desired presheaf.

item (WDF1): This is dual to item (WDF1).

\[\square\]

Proposition 2.4 Let \( D : C^{(p,q)} \rightarrow \mathcal{D} \) be a functor.$^{3}$

(WDF’1) The assignment \( D \mapsto Wd_{X}^{(p,q)}(D) \) defines a functor
\[
Wd_{X}^{(p,q)} : \text{Cat}(C^{(p,q)}, \mathcal{D}) \rightarrow \mathcal{D}.
\]

(WDF’2) The assignment \( D \mapsto CWd_{X}^{(p,q)}(D) \) defines a functor
\[
CWd_{X}^{(p,q)} : \text{Cat}(C^{(p,q)}, \mathcal{D}) \rightarrow \mathcal{D}.
\]

Proof Let \( \alpha : D \rightarrow D’ \) be a natural transformation. We have a map
\[
Wd_{X}^{(p,q)}(\alpha) : Wd_{X}^{(p,q)}(D) \longrightarrow Wd_{X}^{(p,q)}(D’)
\]
\[
(X \longmapsto D) \longmapsto \left( X \longmapsto D \overset{\alpha}{\longrightarrow} D’ \right)
\]

$^{3}$ More generally, the assignments \( F, G \mapsto \text{DiNat}^{(p,q)}(F, G) \) define functors
\[
\text{DiNat}^{(p,q)}(-, -) : \text{Cat}(C^{(p,q)}, \mathcal{D})^{op} \times \text{Cat}(C^{(p,q)}, \mathcal{D}) \rightarrow \text{Set},
\]
\[
\text{DiNat}^{(p,q)}(F, -) : \text{Cat}(C^{(p,q)}, \mathcal{D}) \rightarrow \text{Set},
\]
\[
\text{DiNat}^{(p,q)}(-, G) : \text{Cat}(C^{(p,q)}, \mathcal{D})^{op} \rightarrow \text{Set}.
\]
where we have used Proposition 2.1. As it is clear that this construction preserves composition and identities, we get our desired functor. □

**Definition 2.4** Let \( \theta : X \longrightarrow D \) be a \((p, q)\)-wedge, and \( \zeta : D \longrightarrow Y \) be a \((p, q)\)-cowedge;

- (PC1) The \((p, q)\)-wedge post-composition natural transformation associated to a \((p, q)\)-wedge \( \theta : X \longrightarrow D \) is the natural transformation

  \[ \theta_\ast : h_X \Longrightarrow \mathcal{W}d^{(p, q)}_A(D) \]

  consisting of the collection

  \[ \{ \theta_\ast, A : h_X(A) \rightarrow \mathcal{W}d^{(p, q)}_A(D) : A \in \mathcal{C}_o \} , \]

  where \( \theta_\ast, A \) is the map

  \[
  \begin{pmatrix}
  A \\
  \xi
  \end{pmatrix} \longmapsto \left( \Delta_A \overset{\Delta f}{\longrightarrow} \Delta_X \longrightarrow D \right).
  \]

- (PC2) The \((p, q)\)-cowedge pre-composition natural transformation associated to a \((p, q)\)-cowedge \( \zeta : D \longrightarrow Y \) is the natural transformation

  \[ \zeta_\ast : h^Y \longrightarrow \mathcal{C}Wd^{(p, q)}_A(D) \]

  consisting of the collection

  \[ \{ \zeta_\ast, A : h^Y(A) \rightarrow \mathcal{C}Wd^{(p, q)}_A(D) : A \in \mathcal{C}_o \} , \]

  where \( \zeta_\ast, A \) is the map

  \[
  \begin{pmatrix}
  Y \\
  \xi
  \end{pmatrix} \longmapsto \left( D \longrightarrow \Delta_Y \overset{\Delta f}{\longrightarrow} \Delta_A \right).
  \]

The notion of dinaturality introduced in [17] is in fact more general, as [17, Definition 2.4] introduces what would be called here a \((p, q)\)-to-\((r, s)\)-dinatural transformation. Recall that in the setting of [17], the tuple of powers of \( \mathcal{C} \) is unbiased. Their definition is as follows:

**Definition 2.5** Let \( \alpha, \beta \) be two multi-indices, and let \( F : \mathcal{C}^\alpha \rightarrow \mathcal{D}, G : \mathcal{C}^\beta \rightarrow \mathcal{D} \) be functors. A transformation \( \phi : F \rightarrow G \) of type \( \ell\alpha \xrightarrow{\sigma} n \xleftarrow{\tau} \ell\beta \) (with \( n = \ell A \) a positive integer) is a family of morphisms in \( \mathcal{D} \)

\[ \phi_A : F(A\sigma) \rightarrow G(A\tau) \] \( A \in \mathcal{C}^n \).

This translates into a family \( \phi_{A_1, \ldots, A_n} : F(A_{\sigma_1}, \ldots, A_{\ell\alpha}) \rightarrow G(A_{\tau_1}, \ldots, A_{\ell\beta}) \).

Notice that \( \alpha \) and \( \beta \) are different multi-indices in this definition, and \( \sigma, \tau \) need not be injective or surjective, so we may have repeated or unused variables.

**Notation 2.3** Before proceeding, we need two pieces of notation:

1. Substitution of an object at a prescribed index

\[ A[X/i] \overset{\text{def}}{=} (A_1, \ldots, A_{i-1}, X, A_{i+1}, \ldots, A_n). \]
2. Substitution of a tuple at a prescribed tuple of indices

\[ A[X_1, \ldots, X_r/i_1, \ldots, i_r] \overset{\text{def}}{=} (A[X_1/i_1])(X_2/i_2)\cdots[X_r/i_r]. \]

**Definition 2.6** Let \( \phi = (\phi_{A_1, \ldots, A_n}) : F \to G \) be a transformation. For \( i \in \{1, \ldots, n\} \), we say that \( \phi \) is dinatural in \( A_i \) (or, more precisely, dinatural in its \( i \)-th variable) if and only if for all \( A_1, A_{i-1}, A_{i+1}, \ldots, A_n \) objects of \( C \) and for all \( f : A \to B \) in \( C \) the following hexagon commutes:

\[
\begin{array}{ccc}
F(A[A/i]_\sigma) & \xrightarrow{\phi_{A/A}} & G(A[A/i]_\tau) \\
F(A[f,A/i]_\sigma) & & G(A[A,f/i]_\tau) \\
F(A[B,A/i]_\sigma) & \xrightarrow{\phi_{B/A}} & G(A[A,B/i]_\tau) \\
F(A[B,f/i]_\sigma) & & G(A[f,B/i]_\tau) \\
F(A[B/i]_\sigma) & & G(A[B/i]_\tau)
\end{array}
\]

where \( A \) is the \( n \)-tuple \((A_1, \ldots, A_n)\) of the objects above with an additional (unused in this definition) object \( A_i \) of \( C \).

As far as higher arity co/wedges (i.e. higher arity dinatural transformations from/to a constant functor) are concerned, however, the notions of \((p, q)\)-dinaturality and \((p, q)\)-to-(\(r, s\))-dinaturality agree and yield the same theory of higher arity co/ends.

**3 Higher Arity Ends**

In the present section, we start a systematic study of higher arity co/ends; our main purpose is to establish to what extent the classical theory (for \( p = q = 1 \)) can be reproduced in the more general setting of \((p, q)\)-dinatural transformations. The technology employed in the first half of the section is quite basic, and no result is particularly surprising. On a different note, establishing the Fubini rule for higher arity co/ends, allowing to exchange the order of integration in an iterated higher co/end, required a deeper understanding of adjoints to a functor \( \text{hom}_{\Pi,p,q} : C^{(p,q)} \to \text{Set} \) sending a pair of tuples \((A, B)\) to the product \( \prod_{i,j=1}^{p,q} \text{hom}_C(A_i, B_j) \); this is the content of Sect. 3.2.

**3.1 Basic Definitions**

**Definition 3.1** Let \( D : C^{(p,q)} \to \mathcal{D} \) be a functor.

\((PQ1)\) The \((p, q)\)-end of \( D \) is, if it exists, the pair \( \left(\int_{A \in C} D_A^A, \omega\right) \) formed by an object

\[ \int_{A \in C} D_A^A \]

of \( \mathcal{D} \), and a \((p, q)\)-wedge

\[ \omega : \int_{A \in C} D_A^A \Rightarrow \circ D \]

\( \circ \) Springer
for \((p, q)\int_{A \in C} D^A_A\) over \(D\), such that the \((p, q)\)-wedge post-composition natural transformation

\[
\omega_\ast : h\left(-, (p, q)\int_{A \in C} D^A\right) \Longrightarrow Wd^{(p, q)}_{(-)}(D)
\]

is a natural isomorphism.

(PQ2) The \((p, q)\)-coend of \(D\) is, if it exists, the pair \(\left((p, q)\int_{A \in C} D^A, \xi\right)\) formed by an object

\[
(p, q)\int_{A \in C} D^A
\]

of \(D\), and a \((p, q)\)-cowedge

\[
\xi : D \Longrightarrow (p, q)\int_{A \in C} D^A
\]

for \((p, q)\int_{A \in C} D^A_A\) under \(D\), such that the \((p, q)\)-cowedge post-composition natural transformation

\[
\xi^\ast : h\left((p, q)\int_{A \in C} D^A, -\right) \Longrightarrow CWd^{(p, q)}_{(-)}(D)
\]

is a natural isomorphism.

We follow the customary abuse of notation of denoting the \((p, q)\)-end of \(D\) as just the tip \((p, q)\int_{A \in C} D^A\) of the terminal \((p, q)\)-wedge \(\omega\). The object \((p, q)\int_{A \in C} D^A\) can also be shortly denoted as \((p, q)\int_{A \in C} D\), or \((p, q)\int D\).

**Remark 3.1** The co/representability conditions of Definition 3.1 unwind as the following universal properties:

**(UPQ1)** The \((p, q)\)-end of \(D\) consists of a pair \((p, q)\int_{A \in C} D^A, \omega\) with

1. \((p, q)\int_{A \in C} D^A\) an object of \(D\), and
2. \(\omega\) a natural isomorphism with components

\[
\omega_E : D\left(E, (p, q)\int_{A \in C} D^A\right) \cong Wd_E^{(p, q)}(D).
\]

The family of such morphisms of \(D\) is such that evaluating the isomorphism \(\omega_E\) at the identity of \(E = (p, q)\int_{A \in C} D^A\) gives a \((p, q)\)-wedge

\[
\left\{\omega_A : (p, q)\int_{A \in C} D^A \to D^A_A : A \in C\right\}
\]

indexed by the objects of \(C\). This \((p, q)\)-wedge has the following universal property:

**(*)** Given another such pair \((E, \theta)\), there exists a unique morphism \(E \xrightarrow{\exists!} (p, q)\int_A D^A\) filling the diagram.
(UPQ2) The \((p, q)\)-coend of \(D\) consists of a pair \(((p, q)\int_{A \in C} D_A^A, \xi)\) with

1. \(((p, q)\int_{A \in C} D_A^A\) an object of \(D\), and
2. \(\xi\) a natural isomorphism with components

\[
\xi_E : D \left( (p, q)\int_{A \in C} D_A^A, E \right) \cong CWd^{(p, q)}_E(D).
\]

The family of such morphisms of \(D\) is such that evaluating the isomorphism \(\xi_D\) at the identity of \(C = (p, q)\int_{A \in C} D_A^A\) gives a \((p, q)\)-cowedge

\[
\left\{ \xi_A : D_A^A \to (p, q)\int_{A \in C} D_A^A : A \in C_o \right\}
\]

indexed by the objects of \(C\). This \((p, q)\)-cowedge has the following universal property:

(*) Given another such pair \((C, \zeta)\), there exists a unique morphism \((p, q)\int_{A} D_A^A \xrightarrow{\exists!} C\) filling the diagram

**Remark 3.2** This means that the \((p, q)\)-end of \(D\) is the terminal object of the category of wedges of \(D\), whose morphisms \(h : (\alpha : \Delta X \Longrightarrow D) \to (\beta : \Delta Y \Longrightarrow D)\) are defined as the morphisms \(h : X \to Y\) of \(D\) such that for every \(A \in C_o\) one has \(\beta_A \circ h = \alpha_A\):

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{\alpha_A} & & \downarrow{\beta_A} \\
D_A^A & & \\
\end{array}
\]
Remark 3.2 can be dualised to define \((p, q)\)-coends as initial \((p, q)\)-cowedges. This is straightforward, and we leave it to the reader to spell out.

In the following proposition, we will make use of the \((p, q)\)-diagonal functor \(\Delta_{p,q}\) introduced in Definition 1.1, and duplicated in the following

**Notation 3.1** We say that

- A functor \(F : C^{(p+r,q+s)} \rightarrow D\) is \((r, s)\)-mute if it factors through the canonical projection \(\pi_{r,s} : C^{(p+r,q+s)} \rightarrow C^{(p,q)}\) that cancels the last \(r\) contravariant components, and the last \(s\) covariant components.
- Given a functor \(F : C^{(p,q)} \rightarrow D\), we write \(\varnothing^F_F : C^{(p+r,q+s)} \longrightarrow D\) for the composition \(C^{(p+r,q+s)} \xrightarrow{\pi_{r,s}} C^{(p,q)} \xrightarrow{F} D\); this promotes every functor of type \([\varnothing]\) to an \((r, s)\)-mute one.

Obviously, every functor that is mute in some of its variables can be made into an \((r, s)\)-mute one by suitably reshuffling its arguments.

**Proposition 3.1** (Properties of \((p, q)\)-ends and \((p, q)\)-coends) Let \(D : C^{(p,q)} \rightarrow D\) be a functor.

\((PE1)\) **Functoriality.** Let \(D : C^{(p,q)} \rightarrow D\) be a functor. The assignments \(D \mapsto (p,q) \int_A D^A_{\Delta^{(p,q)}} \rightarrow D\) define functors

\[
\int_{(p,q)} : \text{Cat}(C^{(p,q)}, D) \rightarrow D,
\]

\[
\int_{(p,q)}^{(p,q)} : \text{Cat}(C^{(p,q)}, D) \rightarrow D,
\]

with domain the category of functors from \(C\) of type \([\varnothing]\) to \(D\) and natural transformations between them.

\((PE2)\) (\(p, q\))-Wedges and (\(p, q\))-diagonals For each \(X \in C\) we have natural isomorphisms

\[
\text{Wd}_{(-)}(D) \cong \text{Wd}_{(-)}(\Delta^*_{p,q}(D)),
\]

\[
\text{Cwd}_{(-)}(D) \cong \text{Cwd}_{(-)}(\Delta^*_{p,q}(D)).
\]

where \(\Delta_{p,q}\) is the functor introduced in Definition 1.1.

\((PE3)\) (\(p, q\))-Ends as ordinary ends We have natural isomorphisms

\[
\int_{(p,q)} D_{\Delta_{p,q}} \cong \int_{A \in C} \Delta^*_{p,q}(D)_A,
\]

\[
\int_{(p,q)} D_{\Delta_{p,q}} \cong \int_{A \in C} \Delta^*_{p,q}(D)_A.
\]

where \(\Delta_{p,q}\) is the functor introduced in Definition 1.1. In other words, the \((p, q)\)-end functor factors as a composition

\[
\text{Cat}(C^{(p,q)}, D) \xrightarrow{\Delta^*_{p,q}} \text{Cat}(\text{op} \times C, D) \xrightarrow{f_A} D,
\]

and similarly so do \((p, q)\)-coends.
(PE4) \((p, q)\)-Ends as limits. The \((p, q)\)-end and \((p, q)\)-coend of \(D\) fit respectively into an equaliser and into a coequaliser diagram

\[
\begin{align*}
\int_{A \in \mathcal{C}} D_A^A &\xrightarrow{} \prod_{A \in \mathcal{C}_o} D_A^A \\
\prod_{u: A \to B} D_A^B &\xrightarrow{} \prod_{A \in \mathcal{C}_o} D_A^A
\end{align*}
\]

for suitable maps \(\lambda, \rho, \lambda', \rho'\), induced by the morphisms \(D_A^u, D_B^u\).

(PE5) \((p, q)\)-Ends as limits, again. We have natural isomorphisms

\[
\begin{align*}
\int_{A \in \mathcal{C}} D_A^A &\cong \lim \left( \text{Tw}(\mathcal{C}) \xrightarrow{\Sigma_{p,q}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right), \\
\int_{A \in \mathcal{C}} D_A^A &\cong \operatorname{colim} \left( \text{Tw}(\mathcal{C}) \xrightarrow{\Sigma_{p,q}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right),
\end{align*}
\]

where \(\Sigma_{p,q}: \text{Tw}(\mathcal{C}) \to \mathcal{C}^{(p,q)}\) is the composition \(\Delta^{(p,q)} \circ \Sigma\), with \(\Sigma\) the usual forgetful functor from \(\text{Tw}(\mathcal{C})\) to \(\mathcal{C}^{\text{op}} \times \mathcal{C}\). Explicitly, \(\Sigma_{p,q}\) is the functor

\[
\begin{align*}
\Sigma_{p,q}: \text{Tw}(\mathcal{C}) &\to \mathcal{C}^{(p,q)}, \\
\begin{bmatrix} f & A \hline B \end{bmatrix} &\mapsto (A, B), \\
\begin{bmatrix} \phi & A \hline C \psi & B \hline g \end{bmatrix} &\mapsto (\phi, \psi).
\end{align*}
\]

(PE6) \((p, q)\)-Ends as limits, yet again. There exists a category \(\text{Tw}^{(p,q)}(\mathcal{C})\) together with a universal fibration

\[
\Sigma: \text{Tw}^{(p,q)}(\mathcal{C}) \to \mathcal{C}^{(p,q)}
\]

inducing natural isomorphisms

\[
\begin{align*}
\int_{A \in \mathcal{C}} D_A^A &\cong \lim \left( \text{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right), \\
\int_{A \in \mathcal{C}} D_A^A &\cong \operatorname{colim} \left( \text{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right).
\end{align*}
\]

(PE7) \((p, q)\)-Ends as \((p + r, q + s)\)-ends. We have

\[
\begin{align*}
\int_{A \in \mathcal{C}} D_A^A &\cong \lim \left( \int_{A \in \mathcal{C}} D_A^{(p+r,q+s)} \right), \\
\int_{A \in \mathcal{C}} D_A^A &\cong \operatorname{colim} \left( \int_{A \in \mathcal{C}} D_A^{(p+r,q+s)} \right),
\end{align*}
\]

where \(\Omega\) is the functor introduced in Definition 3.1.

(PE8) Commutativity of \((p, q)\)-ends with homs. We have natural isomorphisms

\[
\mathcal{D} \left( -, \int_{A \in \mathcal{C}} D_A^A \right) \cong \int_{A \in \mathcal{C}} \mathcal{D} \left( -, D_A^A \right)
\]
\[ \mathcal{D} \left( \int_{A \in \mathcal{C}} (p, q) A^l \right) \cong \mathcal{D} \left( \int_{A \in \mathcal{C}} (q, p) A^r \right). \]

**Proof** We provide proofs of the above statements for \((p, q)\)-ends only; the case of \((p, q)\)-coends is dual.

item (PE1): Let \( \alpha : D \to D' \) be a natural transformation and consider the composition

\[
(\omega \circ \alpha)_A \xrightarrow{\omega(p, q)} \text{We}_{\mathcal{C}}^{(p, q)}(D) \xrightarrow{\text{We}_{\mathcal{C}}^{(p, q)}(q)} \text{We}_{\mathcal{C}}^{(p, q)}(D') \xrightarrow{(\omega \circ \alpha)_A} (\omega \circ \alpha)_A,
\]

where we have used Proposition 2.4. This gives us a morphism between the representable functors associated to the \((p, q)\)-ends of \( D \) and \( D' \). The Yoneda lemma now yields a morphism

\[
\int_A (p, q) A^l \to \int_A (q, p) A^r.
\]

between the \((p, q)\)-ends. Since all constructions involved are functorial, it follows that \((p, q)\)-ends preserve composition and identities, and hence define a functor.

item (PE2): This is the special case of Proposition 2.2 where \( F = \Delta \).

item (PE3): We have

\[
\int_A (p, q) A^l \cong \text{We}_{\mathcal{C}}^{(p, q)}(D) \cong \text{We}_{\mathcal{C}}^{(p, q)}(\Delta^{(p, q)}(D)) \cong h \left( - , \int_{A \in \mathcal{C}} (p, q) A^r \right),
\]

from which the result follows from the Yoneda lemma.

item (PE4): This is again a combination of item (PE3) with the ‘products-and-equalisers’ formula for ends.

item (PE5): This is just a combination of item (PE3) with the usual formula computing ends as limits of diagrams from the twisted arrow category.

item (PE6): This problem is studied in Sect. 5.3, with the statement of item (PE6) being proved in Proposition 5.2.

item (PE7): We have

\[
\int_A (p, q) A^l \cong \text{We}_{\mathcal{C}}^{(p, q)}(D) \cong \text{We}_{\mathcal{C}}^{(p + r, q + s)}(\Delta^{(p + r, q + s)}(D)) \cong h \left( - , \int_{A \in \mathcal{C}} \Delta^{(p + r, q + s)} A^r \right),
\]

from which the result follows from the Yoneda lemma.

item (PE8): This follows from item (PE3) and the fact that co/ends commute with homs.

\(\square\)

### 3.2 Adjoints and the Fubini Rule

The scope of this section is to prove that \((p, q)\)-ends are right adjoints (and, dually, that \((p, q)\)-coends are left adjoints), and from this to derive a Fubini rule. Although the proofs are elementary, these properties of \((p, q)\)-ends are more subtle to assess and require a certain amount of new terminology. Thus we separate them from the above list. Let’s start with a simple definition:

**Definition 3.2** (The \( \text{hom}_{\Pi} \) functor) Let \( p, q \geq 1 \) be natural numbers; define a functor

\[
\text{hom}_{\Pi, p, q} : \mathcal{C}^{(p, q)} \to \text{Set}
\]

by sending a pair of tuples \((A, B)\) to the product \( \prod_{i, j=1}^{p, q} \text{hom}_{\mathcal{C}}(A_i, B_j) \), namely to the iterated product \( \prod_{i=1}^{p} \prod_{j=1}^{q} (A_i, B_j) \).
Remark 3.3 If $C$ has finite products and finite coproducts, then we have a canonical factorisation

$$
\begin{array}{ccc}
C^{(p,q)} & \xrightarrow{M_{p,q}} & C^{\text{op}} \times C \\
\downarrow \text{hom}_{\Pi,p,q} & & \downarrow \text{Set.}
\end{array}
$$

where $M_{p,q}$ is the functor of Notation 1.2.

Proposition 3.2 Let $G : C^{(p,q)} \to \text{Set}$ be a functor. There is an isomorphism, natural in $G$,

$$
\text{DiNat}^{(p,q)}(\text{pt}, G) \cong \text{Nat}(\text{hom}_{\Pi,p,q}, G)
$$

The proof of Proposition 3.2 requires several lemmas (Lemmas 3.1, 3.2, 3.3), which we now discuss.

Lemma 3.1 Let $F, G : C^{\text{op}} \times C \to \text{Set}$ be functors. We have a natural isomorphism

$$
\text{DiNat}(F, G) \cong \text{Nat}(h, [F, G]).
$$

Proof The proof is a formal derivation and mimics Example 4.6:

$$
\text{DiNat}(F, G) \cong \int_{A \in C} \text{hom}_{D}(F^A, G^A) \\
\cong \int_{A, B \in C} \left[ \text{hom}_{C}(A, B), \text{hom}_{D}(F^B, G^A) \right] \\
\cong \text{Nat} \left( \text{hom}_{C}(-1, -2), \text{hom}_{D}(F_{-1}, G_{-2}) \right).
$$

Lemma 3.2 Let $G : C^{(q,p)} \to D$ be a functor. If $D$ is cocomplete, then

$$
\text{DiNat}^{(p,q)}(\Delta_{\text{pt}}, G) \cong \text{Nat}(\text{Lan}_{\Delta_{q,p}} h, G).
$$

Proof We have

$$
\text{DiNat}^{(p,q)}(\Delta_{\text{pt}}, G) \cong \text{DiNat} \left( \Delta^*_p \Delta_{\text{pt}}, \Delta^*_q \Delta_{q,p} G \right), \\
\cong \text{Nat} \left( h, \left[ \Delta^*_p \Delta_{\text{pt}}, \Delta^*_q \Delta_{q,p} G \right] \right), \\
\cong \text{Nat} \left( h, \left[ \Delta^*_p \Delta_{q,p} G \right] \right), \\
\cong \text{Nat} \left( h, \Delta^*_q \Delta_{q,p} G \right), \\
\cong \text{Nat} \left( \text{Lan}_{\Delta_{q,p}} h, G \right).
$$

Remark 3.4 (Computing $\text{Lan}_{\Delta_{q,p}} h$) We have

$$
\text{Lan}_{\Delta_{q,p}} h \cong \int_{A, B \in C} \text{hom}_{C^{(q,p)}}(\Delta_{q,p}(A, B); (-, -)) \odot h^A_B \\
\cong \int_{A, B \in C} \text{hom}_{C^{(q,p)}}((A, B); (-, -)) \odot h^A_B
$$
\[ \cong \int_{A \in \mathcal{C}} \hom_{\mathcal{C}(q,p)} ((A, A); (-, -)) \]
\[ \overset{\text{def}}{=} \int_{A \in \mathcal{C}} h^{-1}_A \times \cdots \times h^{-q}_A \times h^A_{-1} \times \cdots \times h^A_{-p}, \quad (2) \]
meaning the end of
\[ (A, B) \mapsto h^{-1}_B \times \cdots \times h^{-q}_B \times h^A_{-1} \times \cdots \times h^A_{-p}. \quad (3) \]

**Lemma 3.3** There is an isomorphism of functors
\[ \text{Lan}_{\Delta, q, p} h \cong \hom_{\Pi, p, q}. \quad (4) \]

**Proof** We shall prove the case \((p, q) = (2, 1)\), as the general case is analogous. Namely, we claim that
\[ \int_{X \in \mathcal{C}} h^{-1}_X \times h^{X}_{-2} \times h^{X}_{-3} \cong \text{Lan}_{\Delta, 2, 1} h \]
(\text{Remark 3.4})
\[ \cong \hom_{\Pi, 2, 1} \]
\[ \overset{\text{def}}{=} h^{-1}_- \times h^{-1}_-. \]
Fix \(A, B, C \in \mathcal{C}_0\). A standard inspection of the universal property shows that the diagram
\[ \bigsqcup_{u \colon X \rightarrow Y} h^A_X \times h^Y_B \times h^Y_C \xrightarrow{\lambda} \prod_{X \in \mathcal{C}} h^A_X \times h^X_B \times h^X_C \xrightarrow{\sigma} h^A_B \times h^A_C, \]
where \(\lambda\) and \(\rho\) are induced by the maps given by
\[ \lambda \left( \left[ \begin{array}{c} X \\ Y \end{array} \right] ; \left[ \begin{array}{c} A \\ X \end{array} \right] , \left[ \begin{array}{c} Y \\ B \end{array} \right] , \left[ \begin{array}{c} Y \\ C \end{array} \right] \right) = (f, g \circ u, h \circ u), \]
\[ \rho \left( \left[ \begin{array}{c} X \\ Y \end{array} \right] ; \left[ \begin{array}{c} A \\ X \end{array} \right] , \left[ \begin{array}{c} Y \\ B \end{array} \right] , \left[ \begin{array}{c} Y \\ C \end{array} \right] \right) = (u \circ f, g, h), \]
and \(\sigma\) is induced by the maps
\[ \sigma_{X,X,X} \left( \left[ \begin{array}{c} A \\ X \end{array} \right] , \left[ \begin{array}{c} X \\ B \end{array} \right] , \left[ \begin{array}{c} X \\ C \end{array} \right] \right) \overset{\text{def}}{=} (g \circ f, h \circ f), \]
is a coequaliser. \(\square\)

Taken all together, Lemmas 3.1, 3.2, and 3.3 yield Proposition 3.2.

**Corollary 3.1** ((\(p, q\))-co/ends as weighted co/limits) Let \(G : \mathcal{C}^{(p,q)} \to \text{Set}\) be a functor. We have functorial isomorphisms
\[ \int_{(p,q)} \int_{A \in \mathcal{C}} G_A \cong \lim_{\hom_{\Pi, p,q}} G, \quad \int_{(p,q)} \int_{A \in \mathcal{C}} G_A \cong \colim_{\hom_{\Pi, p,q}} G, \]
where \(\hom_{\Pi, p,q}\) is the functor of Notation 3.2.

\[{}^4\text{For \((p, q) = (1, 1)\), this amounts to the well-known statement that the co/end of} T : \mathcal{C}^{\text{op}} \times \mathcal{C}\text{is the weighted co/limit of} T \text{by the hom functor} \hom_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}; \text{see [12, Section 3.10].}\]

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Proof We have isomorphisms, natural in \( A \in \mathcal{D} \)

\[
\mathcal{D} \left( A, \int_{(p, q)} G^A \right) \overset{\text{def}}{=} \text{DiNat}^{(p, q)} (\text{pt}, \mathcal{D}(A, G)) \cong \text{Nat} \left( \text{hom}_{\mathcal{P}, p, q}, \mathcal{D}(A, G) \right) \overset{\text{def}}{=} \mathcal{D}(A, \lim \text{hom}_{\mathcal{P}, p, q} G).
\]

The result then follows from the Yoneda lemma. A dual argument yields the second identity.

\[ \square \]

From Corollary 3.1, a general fact about weighted limits [13, Lemma 4.3.1] yields

**Corollary 3.2** There is an adjunction

\[
\text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D}) \xleftarrow{(p, q) \circ -} \mathcal{D},
\]

where the left adjoint \( \text{hom}_{\mathcal{P}, p, q} \circ - \) is defined by \( D \mapsto (A, \mathcal{B}) \mapsto \text{hom}_{\mathcal{P}, p, q} (A, \mathcal{B}) \odot D) \).

Dually, there is an adjunction

\[
\text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D}) \xrightarrow{(p, q) \cdot -} \mathcal{D},
\]

where the left adjoint \( \text{hom}_{\mathcal{P}, p, q} \cdot - \) is defined by \( D \mapsto (A, \mathcal{B}) \mapsto \text{hom}_{\mathcal{P}, p, q} (A, \mathcal{B}) \bullet D) \), and the right adjoint \( \text{hom}_{\mathcal{P}, p, q} \cdot - \) is defined by \( D \mapsto (A, \mathcal{B}) \mapsto \text{hom}_{\mathcal{P}, p, q} (A, \mathcal{B}) \bullet D) \).

**Lemma 3.4** (Shishido identity, first form) The product-hom functor of Definition 3.2 satisfies the Shishido identity:\(^5\)

\[
\text{hom}_{\mathcal{P}, p, q} \times \text{hom}_{\mathcal{P}, r, s} \cong \text{hom}_{\mathcal{P}, r, s} \times \text{hom}_{\mathcal{P}, p, q} \cong \prod_{i=1}^{p+q+s} h_{(-i, -i)}^{(-1, -1)}.
\]

**Proof** The first isomorphism is clear. For the second one, let \((A, B) \in \mathcal{C}^{(p, q)}\) and \(A', B' \in \mathcal{C}^{(r, s)}\); we then have natural isomorphisms

\[
\text{hom}_{\mathcal{P}, p, q} \times \text{hom}_{\mathcal{P}, r, s} ((A; A'), (B; B')) = \prod_{i=1}^{p, q} \text{hom}_{\mathcal{C}} (A_i, B_j) \times \prod_{h, k=1}^{r, s} \text{hom}_{\mathcal{C}} (A'_{h}, B'_{k})
\]

\[
\cong \prod_{i, j=1}^{p, q} \text{hom}_{\mathcal{C}} (A_i, B_j) \times \text{hom}_{\mathcal{C}} (A'_{i}, B'_{j}),
\]

\[
\cong \prod_{i, j=1}^{p+q+s} \text{hom}_{\mathcal{C}} ((A; A'), (B; B')).
\]

where the tuple \(A; A'\) is the juxtaposition of \(A\) and \(A'\). \[ \square \]

\(^5\) Shishido Baiken is the name of a Japanese swordsman (his existence is attested in the Nitenki written in 1776, but the reliability of the text is currently an object of debate). Baiken was a skilled master of kusarigama-jutsu and, according to the legend, lost a duel (and his life) with Miyamoto Musashi.
Theorem 3.1 (The Fubini Rule) Let $D : A^{(p,q)} \times B^{(r,s)} \to \mathcal{D}$ be a functor. Then

$$
(p + r, q + s) \int_{(A, B)} D_{A, B}^{A, B} \cong (p, q) \int_{(A, r, s)} D_{A, B}^{A, B} \cong (r, s) \int_{(B, p, q)} D_{A, B}^{A, B},
$$

as objects of $\mathcal{D}$, meaning that any of these expressions exist if and only if the others do, and, if so, they are all canonically isomorphic.

Proof To prove that three expressions in Eq. 5 are isomorphic, it suffices to show that their adjoints (Corollary 3.2)

$$
\left(\text{hom}_{\Pi, p, q} \cap \text{hom}_{\Pi, r, s}\right) \cap (-)
$$

$$
\left(\text{hom}_{\Pi, r, s} \cap \text{hom}_{\Pi, p, q}\right) \cap (-)
$$

$$
\left(\prod_{i=1}^{p+r} \prod_{j=1}^{q+s} h_{(-i,-j)}\right) \cap (-)
$$

are isomorphic, since adjoints are unique. As $(A \cap B) \cap C \cong (A \times B) \cap C$, this follows from Lemma 3.4.

A suitably dualised argument yields the result for higher arity coends. $\square$

Remark 3.5 (Fubini does not reduce arity) Note that in an iterated higher arity end like

$$(p, q) \int_{(A, r, s)} \int_{(B)} D_{A, B}^{A, B}$$

above, each end cannot be simplified further: given a functor $G$ of type $\left[\begin{array}{c} q \end{array}\right]$, its $(p, q)$-end isn’t in general expressible in terms of $(p - j, q - i)$-ends for suitable $i, j \geq 1$. This confirms the fact that iterated ends are not higher arity ends. Instead, as we have already seen, higher arity ends are particular ends.

More explicitly, what just said means that Theorem 3.1 does not allow us to reduce the arity of a higher arity co/end when $A = B$:

$$(p, q) \int_{(A, r, s)} \int_{(B)} D_{A, B}^{A, B} \cong (p + r, q + s) \int_{(A, B) \in A \times A} D_{A, B}^{A, B} \ncong (p + r, q + s) \int_{A \in C} D_{A}^{A, A}.$$

This is already apparent from the classical Fubini rule, where, given a functor $T : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{E}^{\text{op}} \times \mathcal{E} \to \mathcal{D}$ with $\mathcal{C} = \mathcal{E}$, we have once again

$$
\int_{(A, B) \in \mathcal{C} \times \mathcal{C}} T((A, B), (A, B)) \ncong \int_{A \in \mathcal{C}} T(A, A, A).
$$

The main point in both cases is that we are ‘integrating’ over a pair $(A, B)$, and not over a single variable $A$.

From the point of view of adjoints, we have in (e.g.) the $(p, q) = (1, 1)$ case

$$(-) \circ \left(h_{-1}^{-1} \times h_{-4}^{-1} \times h_{-3}^{-2} \times h_{-4}^{-2}\right) \n\int_{A \in \mathcal{C}} D_{A, A}^{A, A} \n(2.2)$$
and of course
\[ h_{-1}^{-1} \times h_{-1}^{-1} \times h_{-1}^{-2} \times h_{-1}^{-2} \neq h_{(-1,-2)}^{-1} = h_{-1}^{-1} \times h_{-1}^{-2}, \]
so \( \int_{A \in C} D_{A,A}^{A} \) and \( \int_{(A,B) \in C \times C} D_{(A,B)}^{A,B} \) are different as well.

### 4 Examples: A Session of Callisthenics

This section is devoted to examples of higher arity ends arising ‘in nature’; the main result of the first subsection, starting with low dimensional examples of \((0, 0)\), \((0, n)\) and \((2, 1)\)-ends, generalises a well-known theorem by Street and Dubuc (Example 4.6) to the \((p, q)\)-case (see Example 4.7).

The heart of the following subsection consists of a glance at weighted co/ends (where we look for representatives of a certain functor \(X \mapsto \text{DiNat}(W, \text{hom}(X, D))\) depending not on a presheaf \(W : C \to \text{Set} \) but on an endo-distributor \(W : C^{\text{op}} \times C \to \text{Set} \), and at diagonal category theory (see Sect. 4.2.3), where we replace instances of naturality in many categorical constructions with dinaturality; both settings naturally give rise to examples of higher arity co/ends.

#### 4.1 Examples Arranged by Dimension

**Example 4.1** ((0, 0)-co/ends) For the case where \(p = q = 0\), we look at functors of the form \(D : \text{pt} \to \mathcal{D}\), where \(\text{pt}\) is the terminal category. It is evident that such a functor corresponds precisely to an object of \(\mathcal{D}\), a \((0, 0)\)-wedge corresponds to the identity on that object, and the \((0, 0)\)-end of \(D\) is precisely that object.

**Example 4.2** ((1, 0)- and (0, 1)-co/ends) When \((p, q) = (1, 0)\), we consider functors of the form \(D : C^{\text{op}} \to \mathcal{D}\), and we see from the universal property of \((p, q)\)-ends that the \((1, 0)\)-end of \(D\) is the limit of \(D\). Similarly, the \((0, 1)\)-end of a functor \(D : C \to \mathcal{D}\) is again the limit of \(D\).

In particular, starting with a functor \(D : C \to \mathcal{D}\) and passing to the opposite functor \(D^{\text{op}} : C^{\text{op}} \to \mathcal{D}^{\text{op}}\), we get isomorphisms
\[
\int_{(0, 1)} A \in C \quad D = \text{lim}(D),
\]
\[
\int_{(1, 0)} A \in C \quad D^{\text{op}} = \text{colim}(D).
\]

**Example 4.3** ((1, 1)-co/ends) Let \((p, q) = (1, 1)\) and consider a diagram in \(\mathcal{D}\) of the form \(D : C^{\text{op}} \times C \to \mathcal{D}\). Again from the universal property of \((p, q)\)-ends, we see that \((1, 1)\)-ends are nothing but ordinary ends. That is:
\[
\int_{(1, 1)} A \in C \quad D_{A}^{A} = \int_{A \in C} D_{A}^{A}.
\]
Furthermore, \((n, 0)\)-co/ends and \((0, n)\)-co/ends are just suitable co/limits:
Example 4.4 ((2, 0)-, (0, 2)-co/ends; (n, 0)- and (0, n)-co/ends) Given a diagram \( D : C^2 \to \mathcal{D} \), we have

\[
\int_{A \in C} D^{A,A} = \lim (D \circ \Delta_C), \quad \int_{A \in C} D^{A,A} = \operatorname{colim} (D \circ \Delta_C),
\]

where \( \Delta_C : C \to C \times C \) is the diagonal functor of \( C \) in the Cartesian monoidal structure of \( \mathbf{Cat} \).

A similar argument yields, for a diagram \( D : C^n \to \mathcal{D} \):

\[
\int_{A \in C} D^{A} = \lim (D \circ \Delta^n_C), \quad \int_{A \in C} D^{A} = \operatorname{colim} (D \circ \Delta^n_C).
\]

We consider next the first nontrivial example:

Example 4.5 ((2, 1)- and (1, 2)-co/ends) Given a functor \( T : C^{-2} \times C \to \mathcal{D} \) let’s flesh out what a (2, 1)-wedge is: it consists of an object \( X \) endowed with maps

\[
\omega_A : X \to T(AA; A)
\]

with the property that, for every \( f : A \to B \) in \( C \), the square

\[
\begin{array}{ccc}
X & \xrightarrow{\omega} & T(AA; A) \\
\downarrow{\omega} & & \downarrow{T(11f)} \\
T(BB; B) & \xrightarrow{T(1f)} & T(AA; B)
\end{array}
\]

The (2, 1)-end of \( T \) is the terminal object in the category \( \mathbf{Wd}_{2,1}(T) \) of wedges for \( T \).

As a particular example, let \( C \) be a Cartesian category. Let us consider the functor

\[
T = \text{hom}_{(-1 \times -2, -3)} : C^{\text{op}} \times C^{\text{op}} \times C \to \mathbf{Set} \quad (A, B; C) \mapsto C(A \times B, C)
\]

What is a (2, 1)-wedge for \( T \)? It consists of a set \( X \), and a family of functions \( \omega_A : X \to C(A \times A, A) \) with the property that for each \( f : A \to B \), the square

\[
\begin{array}{ccc}
X & \to & C(A \times A, A) \\
\downarrow & & \downarrow f_* \\
C(B \times B, B) & \xrightarrow{(f \times f)^*} & C(A \times A, B)
\end{array}
\]

commutes. In other words, each \( \omega_A(x) \) is a morphism \( A \times A \to A \) in \( C \) with the property that each \( f : A \to B \) is a ‘homomorphism’ with respect to \( \omega_A(x), \omega_B(x) \):

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\omega_A(x)} & A \\
\downarrow{f \times f} & & \downarrow{f} \\
B \times B & \xrightarrow{\omega_B(x)} & B
\end{array}
\]
This structure is easy to determine: let \( b : 1 \to B \) be a point of \( A \) (e.g., let \( C = \text{Set} \)). Then the commutativity of
\[
\begin{array}{ccc}
1 & \xrightarrow{\omega_A(x)} & 1 \\
\downarrow{f \times f} & & \downarrow{f} \\
B \times B & \xrightarrow{\omega_B(x)} & B
\end{array}
\]
tells that \( \omega_B(x) : B \times B \to B \) is a section of the diagonal \( \Delta_B \) (this means: \( \omega_B(x)(b, b) = b \) for every \( b \in B \)). Moreover, the family \( \omega_A : A \times A \to A \) is natural in \( A \), i.e. it is a natural transformation
\[
\times \circ \Delta \xrightarrow{\omega} \text{id}
\]
that is a section of the natural transformation in the opposite direction, unit of the adjunction constant-product.

There are few such transformations. First, observe that the functor \( \times \circ \Delta \) coincides with the functor \( X \mapsto \mathbb{2} \), so corresponds to the cotensoring with \( \mathbb{2} = \{0, 1\} \) in an abstract category, and it is just the corepresentable presheaf on \( \mathbb{2} \) in the category of sets. Similarly, the identity is the corepresentable over the point in the category of sets. In the category of sets
\[
\int_{(2, 1)} \text{hom}(A \times A, A) \cong [\text{Set}, \text{Set}](\text{Set}(2, \_), \text{Set}(1, \_)) \\
\cong \text{Set}(1, 2) \cong 2
\]
by the Yoneda lemma.

Similarly, in a category \( C \) with \( \text{Set} \)-cotensors,
\[
\int_{(2, 1)} C(A \times A, A) \cong [C, \text{Set}](\text{Set}(n \uplus \_), \text{Set}(1 \uplus \_))
\]
where the functor \( (n \uplus \_) \) coincides with the \( n \)-fold iterated product \( X \mapsto X^n = X \times \cdots \times X \). A similar argument shows that \( \int_{(n, 1)} C(A^n, A) \cong [C, \text{Set}](\text{Set}(n \uplus \_), \text{Set}(1 \uplus \_)) \).

**Example 4.6** (Dinatural transformations via \((2, 2)\)-ends) This example was first discovered by Street and Dubuc in [5, Theorem 1]. We give an account of it in our language.

Let \( F, G : C^{\text{op}} \times C \to D \) be functors. Then
\[
\text{DiNat}(F, G) \cong \int_{A \in C} \text{hom}_D(F^A_A, G^A_A), \\
\cong \int_{(2, 2)} \text{hom}_D(F^A_A, G^A_A).
\]

**Proof** The proof of the first isomorphism is divided in two steps:

(DEP1) First, consider the functor
\[
\text{hom}_D \left( F^{-2}_1, G^{-1}_2 \right) : C^{\text{op}} \times C \to \text{Set}
\]
sending
(a) An object \((A, X)\) of \( C^{\text{op}} \times C\) to the set \( \text{hom}_D \left( F^X_A, G^X_A \right) \).
(b) A morphism \( (A \xrightarrow{f} B, X \xrightarrow{g} Y) \) of \( \mathcal{C}^{\text{op}} \times \mathcal{C} \) to the map

\[
\mathcal{D}
\left(
F^g_f, G^f_g
\right): \mathcal{D}
\left(
F^X_A, G^X_X
\right) \to \mathcal{D}
\left(
F^Y_B, G^Y_Y
\right)
\]

defined as the composition

\[
\mathcal{D}(F^g_f, G^f_g)
\]

\[
\mathcal{D}(F^X_A, G^X_X) \to \mathcal{D}(F^Y_B, G^Y_Y)
\]

By functoriality of homs, the assignment \( (A, X) \mapsto \mathcal{D}(F^X_A, G^X_X) \) preserves identities and composition, defining therefore a functor.

(DEP2) Second, we compute the end \( \int_{A \in \mathcal{C}} \text{hom}_{\mathcal{D}} (F^A_A, G^A_A) \); this is given by the equaliser of the pair of maps

\[
\prod_{A \in \mathcal{C}} \mathcal{D}(F^A_A, G^A_A) \xrightarrow{\lambda} \prod_{f: A \to B} \text{Set}(\mathcal{C}(A, B), \mathcal{D}(F^B_B, G^B_B))
\]

where \( \lambda \) and \( \rho \) are the morphisms induced by the universal property of the product by the morphisms

\[
\lambda_{A,B}: \prod_A \mathcal{D}(F^A_A, G^A_A) \to \text{Set}((\mathcal{C}(A, B), \mathcal{D}(F^B_B, G^B_B)));
\]

\[
\rho_{A,B}: \prod_A \mathcal{D}(F^A_A, G^A_A) \to \text{Set}((\mathcal{C}(A, B), \mathcal{D}(F^B_B, G^B_B)));
\]

acting on elements as

\[
\left( \alpha_A: F^A_A \to G^A_A \right) \mapsto \left( \left[ \int^A_A \right] \mapsto \left( G^{\text{id}_A}_f \circ \alpha_A \circ F^f_f \right) \right),
\]

\[
\left( \alpha_B: F^B_B \to G^B_B \right) \mapsto \left( \left[ \int^A_A \right] \mapsto \left( G^{\text{id}_B}_f \circ \alpha_B \circ F^{\text{id}_B}_f \right) \right).
\]

As for the second isomorphism, we define a functor

\[
\mathcal{D}(F^f_f, G^f_f): \mathcal{C}^{(2,2)} \to \text{Set}
\]

in a similar manner as we did above and then invoke item (PE3) of Proposition 3.1. The universal property of the \((2,2)\)-end of \( \mathcal{D}(F^f_f, G^f_f) \) is the same as the universal property of the equaliser defining \( \text{DiNat}(F, G) \).

Generalising Example 4.6, we have the following.

**Example 4.7** \(((p, q)-\text{Dinatural transformations via } (q, p)-\text{ends})\) Let \( F \) and \( G \) be functors of type \([p^q]\) and \([q^p]\), respectively. Then

\[
\text{DiNat}^{(p, q)}(F, G) \cong \int_{(A,B) \in \mathcal{C}} \text{hom}_{\mathcal{D}}(F^A_A, G^A_A),
\]

where the ‘integrand’ is the functor equation

\[
\mathcal{C}^{(q,p)} \xrightarrow{\square} \mathcal{D}
\]

\[
(A, B) \mapsto \text{hom}_{\mathcal{D}}(F^A_A, G^A_A).
\]

**Proof** This is a combination of Proposition 2.2, Example 4.6, and item (PE3) of Proposition 3.1. \(\square\)
4.2 Classes of Higher Arity Coends

4.2.1 A Glance at Weighted Co/ends

We now introduce a natural factory of examples for higher arity co/ends. In a nutshell, weighted co/ends are to co/ends as weighted co/limits are to co/limits.

**Definition 4.1 (Weighted co/ends)** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-enriched categories and $W : \mathcal{C}^{\text{op}} \otimes \mathcal{V} \to \mathcal{V}$ a $\mathcal{V}$-functor, and $\mathcal{D}$ a $\mathcal{V}$-category.

(WE1) The *end* of $\mathcal{D}$ weighted by $W$ is, if it exists, the object $\int^W A \in \mathcal{C} \mathcal{D} A$ of $\mathcal{D}$ with the property that
\[
\text{hom}_{\mathcal{D}} \left( -, \int^W A \in \mathcal{C} \mathcal{D} A \right) \cong \text{DiNat}_\mathcal{V}(W, \mathcal{D}(-)) .
\]

(WE2) The *coend* of $\mathcal{D}$ weighted by $W$ is, if it exists, the object $\int^A \in \mathcal{C} W A$ of $\mathcal{D}$ with the property that
\[
\text{hom}_{\mathcal{D}} \left( \int^A \in \mathcal{C} W A, - \right) \cong \text{DiNat}_\mathcal{V}(W, \mathcal{D}(D, -)) .
\]

**Example 4.8 (Weighted co/ends are $(2, 2)$-co/ends)** For $\mathcal{V}$ a co/tensored monoidal category, there are $(2, 2)$-co/end formulas for weighted co/ends:
\[
\int^W A \in \mathcal{C} \mathcal{D} A \cong \int^A \in \mathcal{C} W A \otimes D A.
\]

**Example 4.9 (Weights increase arity)** Let $F, G : \mathcal{C} \to \mathcal{D}$ and $W : \mathcal{C}^{\text{op}} \otimes \mathcal{V} \to \mathcal{V}$ be $\mathcal{V}$-functors. In analogy with
\[
\text{Nat}_\mathcal{V}(F, G) \overset{\text{def}}{=} \int_{A \in \mathcal{C}} \text{hom}_{\mathcal{D}}(F A, G A),
\]
we define the *object* $\text{Nat}^W(F, G)$ of natural transformations from $F$ to $G$ weighted by $W$ by
\[
\text{Nat}^W(F, G) \overset{\text{def}}{=} \int_{A \in \mathcal{C}} \text{hom}_{\mathcal{D}}(F A, G A) .
\]

Taking $W$ to be mute in its contravariant variable, we can give a reformulation of the universal property of weighted limits:
\[
h \left( -, \lim^W (D) \right) \cong \text{Nat}^W(\Delta(-), D) .
\]

Defining $\text{DiNat}_\mathcal{V}^W(F, G)$ by a similar formula, we also obtain the following isomorphism in the case of weighted ends:
\[
h \left( -, \int^W A \in \mathcal{C} D A \right) \cong \text{DiNat}_\mathcal{V}^W(\Delta(-), D) .
\]
This naturally suggests a definition of ‘doubly-weighted ends’:

\[ h(-, \int_{A \in C}^{[W_1, W_2]} D^A_A) \cong \text{DiNat}_{\mathcal{V}}^{[W_1]}(W_2, D). \]

Proceeding inductively leads to the notion of an end weighted by a collection of functors \([W_1, \ldots, W_n]\). These ‘\(n\)-weighted ends’ however, can actually be computed as \((n+1, n+1)\)-ends:

\[ \int_{A \in C}^{[W_1, \ldots, W_n]} D^A_A \cong \int_{A \in C}^{(n+1, n+1)} (W_1)^A_A \times \cdots \times (W_n)^A_A \circ D^A_A. \]

As such, we see that weighting an end increases its arity by \((1, 1)\).

### 4.2.2 Weighted Kan Extensions

Another source of examples comes from ‘weighing’ left and right Kan extensions. While the most general such weight is a profunctor, having type \([\mathbb{1}], \mathbb{1}\] or \([\mathbb{0}, \mathbb{1}]\) are specially interesting, as they give a more direct parallel to the classical theory of weighted co/limits (see Example 4.10).

For Definitions 4.2, 4.3 below, recall from Eq. 1 the definition of the object \(\text{Nat}_{\mathcal{V}}^{[W]}(F, G)\) of weighted natural transformations.

**Definition 4.2** The left Kan extension of \(F\) along \(K\) weighted by \(W\) is, if it exists, the \(\mathcal{V}\)-functor

\[ \left( \text{Lan}_{K}^{[W]} F : \mathcal{D} \to \mathcal{E} \right) : \]

\[ \xymatrix{ \mathcal{D} \ar[dr]^{\text{Lan}_{K}^{[W]} F} & \mathcal{C} \ar[l]_{W} \ar[r]^{F} & \mathcal{E}, } \]

for which we have a \(\mathcal{V}\)-natural isomorphism

\[ \text{Nat}_{\mathcal{V}} \left( \text{Lan}_{K}^{[W]} F, G \right) \cong \text{Nat}_{\mathcal{V}}^{[W]} (F, G \circ K), \quad (2) \]

natural in \(G\).

One defines weighted right Kan extensions in a dual manner:

**Definition 4.3** The right Kan extension of \(F\) along \(K\) weighted by \(W\) is, if it exists, the \(\mathcal{V}\)-functor

\[ \left( \text{Ran}_{K}^{[W]} F : \mathcal{D} \to \mathcal{E} \right) : \]

\[ \xymatrix{ \mathcal{D} \ar[dr]^{\text{Ran}_{K}^{[W]} F} & \mathcal{C} \ar[l]_{W} \ar[r]^{F} & \mathcal{E}, } \]

for which we have a \(\mathcal{V}\)-natural isomorphism

\[ \text{Nat}_{\mathcal{V}} \left( G, \text{Ran}_{K}^{[W]} F \right) \cong \text{Nat}_{\mathcal{V}}^{[W]} (G \circ K, F), \quad (3) \]

natural in \(G\).
Example 4.10 (Weighted co/limits as weighted Kan extensions) Let $D: C \to D$ be a diagram on a category $D$. Then we may canonically identify the left Kan extension of $D$ along the terminal functor with its colimit:

$$\text{Lan}_! D \cong \left[ \text{colim}(D) \right] \overset{pt}{\leftarrow} \left[ \text{colim}(D) \right] / D.$$  

Similarly, given a weight $W: C^{\text{op}} \to \text{Set}$, we have

$$\text{Lan}_!^{[W]} D \cong \left[ \text{colim}^W(D) \right] \overset{pt}{\leftarrow} \left[ \text{colim}^W(D) \right] / D.$$  

Weighted Kan extensions may also be expressed as $(p, q)$-co/ends:

$$\text{Lan}_{[W]}^K F \cong \int_{[W]} \overset{(2, 2)}{\hom}_C(K_A, -) \otimes F_A \cong \left( W_A \times \hom_C(K_A, -) \right) \otimes F_A, \quad (4)$$

$$\text{Ran}_{[W]}^K F \cong \int_{[W]} \overset{(2, 2)}{\hom}_C(-, K_A) \cap F_A \cong \left( W_A \times \hom_C(-, K_A) \right) \cap F_A, \quad (5)$$

Equipped with this description, we now proceed to compute a few weighted Kan extensions.

Example 4.11 Consider the functor $[0]^{\text{op}}: \text{pt}^{\text{op}} \to \Delta^{\text{op}}$. The left and right Kan extensions of a set $X_\bullet: \text{pt} \to \text{Set}$ along $[0]^{\text{op}}$ are given by

$$\text{Lan}_{[0]^{\text{op}}} (X) \cong X_\bullet$$
$$\text{Ran}_{[0]^{\text{op}}} (X) \cong \check{C}_\bullet(X).$$

Passing from ordinary Kan extensions to weighted ones and hence picking a weight $W: \text{pt}^{\text{op}} \times \text{pt} \to \text{Set}$ (whose image in $\text{Set}$ we also denote by $W$), we obtain

$$\text{Lan}_{[0]^{\text{op}}}^W (X) \cong W \times X_\bullet,$$
$$\text{Ran}_{[0]^{\text{op}}}^W (X) \cong \check{C}_\bullet(W \times X),$$

Example 4.12 The above example has a more interesting counterpart, in which we consider the left adjoint

$$r^{\text{op}}: \Delta^{\text{op}} \longrightarrow \text{pt}^{\text{op}}$$

of $[0]^{\text{op}}$. The left and right Kan extensions of a simplicial set $X_\bullet: \Delta^{\text{op}} \to \text{Set}$ along $r^{\text{op}}$ are given by

$$\text{Lan}_{r^{\text{op}}} (X) \cong \pi_0(X_\bullet)$$
$$\text{Ran}_{r^{\text{op}}} (X) \cong \text{ev}_0(X_\bullet) \overset{\text{def}}{=} X_0.$$
We now have a much wider range of choices for the weight \( W \): we may choose it to be any cosimplicial space \( W \): \( \Delta^{\text{op}} \times \Delta \to \text{Set} \):

\[
\begin{array}{ccc}
\text{pt}^{\text{op}} & \xrightarrow{r^{\text{op}}} & \text{Lan}_{\Delta}^{[W]} X_* \\
W & \xleftarrow{} & \Delta^{\text{op}} \xrightarrow{\gamma} \text{Set}.
\end{array}
\]

For instance, taking \( W = \Delta^* \) almost gives the geometric realisation of \( X_* \):

\[
\text{Lan}_{\Delta}^{[\Delta^*]}(X_*) \cong \int_{[n] \in \Delta} \Delta^n \times X_n,
\]

with the caveat that the geometric realisation involves \(|\Delta^n|\), rather than \( \Delta^n \) itself. Dually, taking again \( W = \Delta^* \) but now for a cosimplicial object \( X^* : \Delta \to \text{Set} \), we have

\[
\text{Ran}_{\Delta}^{[\Delta^*]}(X^*) = \text{Tot}(X_*).
\]

**Example 4.13** (Stalks of a sheaf ([1, Paragraph 6.8 and Section 7.1])) Let \( i_p : \{p\} \to X \) be the inclusion of a point into a topological space \( X \). We get an induced functor

\[
\text{Op}(i_p) : \text{Op}(X) \longrightarrow \text{Op}(\{p\})
\]

\[
U \longmapsto i_p^{-1}(U).
\]

Considering now left Kan extensions along the opposite of \( \text{Op}(i_p) \),

\[
\begin{array}{ccc}
\text{Op}(i_p)^{\text{op}} & \xrightarrow{\text{Lan}_{\text{Op}(i_p)^{\text{op}}}} & \text{Op}(\{p\})^{\text{op}} \\
\text{Op}(X)^{\text{op}} & \xrightarrow{} & \text{Set},
\end{array}
\]

we obtain a functor \( \text{Lan}_{\text{Op}(i_p)^{\text{op}}} : \text{PSh}(X) \to \text{PSh}(\{p\}) \), whose image at \( \mathcal{F} \) we write \( [\mathcal{F}_p] \) for simplicity. The restriction of this functor to \( \text{Shv}(X) \) can be identified with the stalk functor \( (-)_p : \text{Shv}(X) \to \text{Set} \): we have \( \text{Op}(\{p\}) = \{\emptyset \to \{p\}\} \) and computing the images of \( \emptyset \) and \( \{p\} \) under \( [\mathcal{F}_p] \) via the usual colimit formula for left Kan extensions yields

\[
[\mathcal{F}_p](\{p\}) \cong \mathcal{F}_p \quad [\mathcal{F}_p](\emptyset) \cong \mathcal{F}(\emptyset)
\]

(in case \( \mathcal{F} \) is a sheaf, \( \mathcal{F}(\emptyset) \) is the singleton set.) Consider the same situation, but now with a weight \( W : \text{Op}(X) \times \text{Op}(X)^{\text{op}} \to \text{Set} \):

\[
\begin{array}{ccc}
\text{Op}(i_p)^{\text{op}} & \xrightarrow{\text{Lan}_{\text{Op}(i_p)^{\text{op}}}} & \text{Op}(\{p\})^{\text{op}} \\
\text{Op}(X)^{\text{op}} & \xrightarrow{} & \text{Set}.
\end{array}
\]

Using Eq. 4, we may compute \( \text{Lan}_{\text{Op}(i_p)^{\text{op}}^{\mathcal{F}}} \) as the weighted coend

\[
[\mathcal{F}_p^{[W]}] \overset{\text{def}}{=} \int_{U \in \text{Op}(X)} \text{hom}_{\text{Op}(X)^{\text{op}}}(\text{Op}(i_p)^{\text{op}}(U), -) \times \mathcal{F}(U)
\]

\[
\overset{\text{def}}{=} \int_{U \in \text{Op}(X)} W_U^U \times \text{hom}_{\text{Op}(X)}(\chi_p(U), -) \times \mathcal{F}(U),
\]

\(\Box\) Springer
where

\[ \chi_p(U) = \begin{cases} \emptyset & \text{if } p \notin U, \\ U & \text{otherwise}. \end{cases} \]

For instance, taking \( W \) to be a sheaf \( G \) on \( X \) gives

\[ \mathcal{F}_p[G] \overset{\text{def}}{=} [\mathcal{F}_p[G]](\{p\}) \cong (\mathcal{F} \times G)_p. \]

4.2.3 A Glance at Diagonality

‘Diagonal’ category theory arises when, instead of considering a natural transformation filling a higher-dimensional cell, we consider a dinatural one. Transformations that are more general than natural ones notoriously do not compose (see [10, 11], and mostly [19] for a modern account); yet, the category theory arising from this generalisation is interesting when approached with higher arity co/ends.

For the purposes of our exposition here, left/right Kan extensions are the most interesting categorical gadget to ‘diagonalise’; when this is done, they provide examples of higher arity co/ends.

**Definition 4.4** The diagonal left Kan extension of a functor \( F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \) along a functor \( K : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \) is, if it exists, the functor \( \text{DiLan}_K F : \mathcal{D} \to \mathcal{E} \) such that we have an isomorphism

\[ \text{Nat}(\text{DiLan}_K F, G) \cong \text{DiNat}(F, G \circ K), \]

natural in \( G \).

**Example 4.14** (Ends as diagonal left Kan extensions) Classically, the left Kan extension of a functor \( D : \mathcal{C} \to \mathcal{D} \) along the terminal functor \( t : \mathcal{C} \to \text{pt} \) may be canonically identified with its colimit. Passing to diagonal category theory, one obtains a similar result: the diagonal left Kan extension of a functor \( D : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \) along \( t : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{pt} \) may similarly be canonically identified with its coend.

\[ \text{DiLan}_K F \cong \int^{A \in \mathcal{C}} \mathcal{D}(\text{colim}(D)_A, F^A), \]

**Remark 4.1** While ordinary Kan extensions may be computed as co/ends, diagonal Kan extensions admit \((2, 2)\)-co/end formulas:

\[ \text{DiLan}_K F \cong \int^{A \in \mathcal{C}} \mathcal{D}(K^A \cdot -, F^A). \quad (6) \]

6 Upon inspection, one observes that there is a striking similarity between Eqs. 6, 7 and 4, 5. This is not a coincidence, as it is possible to show that diagonal Kan extensions are precisely hom-weighted Kan extensions:}

\[ \text{DiLan}_K F \cong \int^{A, B \in \mathcal{C}}_{\text{[hom}(\mathcal{C}(\cdot, -))]} \mathcal{D}(K^B_A \cdot -, F^A_B). \]
\[
\text{DiRan}_K F \cong \int_{A \in \mathcal{C}} \mathcal{D} \left( -, K_A^A \right) \succeq F_A^A,
\]
where the pairing in Eq. 6 is such that \(\text{DiLan}_K F\) is the coend of
\[
(A, B) \mapsto \mathcal{D} \left( K_A^B, - \right) \odot F_B^A.
\]

**Example 4.15** (Application to \((p, p)\)-dinaturality) As diagonal Kan extensions along identity functors satisfy
\[
\text{Nat} \left( \text{DiLan}_{\text{id}_C} F, G \right) \cong \text{DiNat}^{(p, p)}(F, G),
\]
\[
\text{Nat} \left( F, \text{DiLan}_{\text{id}_C} G \right) \cong \text{DiNat}^{(p, p)}(F, G),
\]
they provide us with a tool to study \((p, p)\)-dinaturality in terms of (ordinary) naturality. A variant of this construction allowing also the case \(p \neq q\) will be studied in Sect. 5.

### 4.2.4 Weighted Diagonal Kan Extensions

In the same spirit, we may combine the two perspectives found in Definitions 4.1, 4.4, thus obtaining notions of *weighted diagonal Kan extensions*. While this topic is outside the scope of the present paper, we note that these constructions turn out to be examples of \((4, 4)\)-co/ends:
\[
\text{DiLan}_{\text{id}_C}^{[W]} F \cong \int_{A \in \mathcal{C}} (W_A^A \times \text{hom}_C(\mathcal{K}^A_A, -)) \odot F_A^A,
\]
\[
\text{DiRan}_{\text{id}_C}^{[W]} F \cong \int_{(A, B) \in \mathcal{C} \times \mathcal{C} \text{op}} (\text{hom}_C(\mathcal{K}^B_A, -)) \succeq F_B^A.
\]

At this point, it is evident that the list of examples is virtually endless. We plan to dedicate separate works [14,15] to a thorough investigation of the topic.

### 4.2.5 The Day Operad

Day convolution was introduced by B. Day in [3,4], in order to classify monoidal structures on the category \(\text{PSh}(\mathcal{C})\) of presheaves on \(\mathcal{C}\). Day proved that \(\text{PSh}(\mathcal{C})\) can be turned into a monoidal category in as many ways as \(\mathcal{C}\) can be turned into a pseudomonoid in the bicategory \(\text{Prof}\) of profunctors.

We now propose an analogue of this framework based on higher arity coends: let \((\mathcal{C}, \otimes, 1)\) be a monoidal category, and let \(\mathcal{K} \overset{\text{def}}{=} \text{PSh}(\mathcal{C})\). Higher arity Day convolution is defined as a certain family of functors \(\odot_n : \mathcal{K}^n \to \mathcal{K}\).

\[
\text{DiRan}_K F \cong \int_{A, B \in \mathcal{C}} [\text{hom}_C(-, -)] \mathcal{D} \left( -, K_A^B \right) \succeq F_B^A.
\]

This is both an analogy as well as a generalisation (Example 4.14) of the fact that co/ends are exactly hom-weighted co/limits.

---

7 More formally, let \(S : \text{Cat} \to \text{Cat}\) be the 2-monad of pseudomonoids; let \(\tilde{S} : \text{Prof} \to \text{Prof}\) be the lifting of \(S\) to the bicategory of profunctors (i.e. to the Kleisli bicategory of the presheaf construction \(\text{PSh}\)); then, given an object \(\mathcal{C}\) of \(\text{Cat}\), there is a bijection between pseudo-\(S\)-algebra structures on \(\text{PSh}(\mathcal{C})\) and pseudo-\(\tilde{S}\)-algebras on \(\mathcal{C}\), as an object of \(\text{Prof}\).
Definition 4.5 The Day \((n, n)\)-convolution of an \(n\)-tuple of presheaves \(\mathcal{F}_1, \ldots, \mathcal{F}_n\) is the presheaf

\[
\oplus_n(\mathcal{F}_1, \ldots, \mathcal{F}_n) : \mathcal{C}^{\text{op}} \to \text{Set}
\]

defined at \(A \in \mathcal{C}\) as the \((n, n)\)-coend

\[
\oplus_n(\mathcal{F}_1, \ldots, \mathcal{F}_n) \overset{\text{def}}{=} A \mapsto \int^{A \in \mathcal{C}} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times \mathcal{C}(-, A^\otimes n),
\]

where \(A^\otimes n\) is shorthand for the \(n\)-fold tensor product of \(A\) with itself.

Example 4.16 (Day convolution operad) The Day convolution operad associated to \((\mathcal{C}, \otimes, 1)\) is the symmetric operad \(\text{Day}\) whose set of generating operations (see [6, Section 1.2.5]) is given by \([\text{id}, \oplus_2, \oplus_3, \ldots, \oplus_n, \ldots]\).

Remark 4.2 (Unwinding Example 4.16) The set \(\text{Day}_n\) can be succinctly described by the formula

\[
\text{Day}_n = \{\oplus_n\} \cup \sum_{p_1 + \cdots + p_k = n} \oplus_k \circ (\text{Day}_{p_1} \times \cdots \times \text{Day}_{p_k}),
\]

while the operadic composition of \(\text{Day}\) is defined via ‘grafting’ in the usual way: equation\(\@\@unnumberedDiagramCounter\)

\[
\text{Day}_n \times \text{Day}_{k_1} \times \cdots \times \text{Day}_{k_n} \longrightarrow \text{Day}_{\sum_{i} k_i}
\]

\[
(\theta; \theta_1, \ldots, \theta_k) \longmapsto \theta(\theta_1(-1, \ldots, -k_1), \ldots, \theta_k(-1, \ldots, -k_n)).
\]

So, for example

\[
\begin{align*}
\text{Day}_1 &= \{\text{id}\} \\
\text{Day}_2 &= \{\oplus_2(-, -)\} \\
\text{Day}_3 &= \{\oplus_3(-, -,-), \oplus_2(\oplus_2(-, -,-), -), \oplus_2(-, \oplus_2(-, -,-))\}.
\end{align*}
\]

We refrain from going further in the analysis of the properties of the family of \(n\)-ary Day convolutions; it seems to the authors this constitutes a mildly interesting object, especially because as already said, convolution monoidal structures on a presheaf category \([\mathcal{A}^{\text{op}}, \text{Set}]\) correspond to promonoidal structures on \(\mathcal{A}\); similarly here, the full assignment of the \(n\)-ary Day convolutions gives rise to a family of profunctors

\[
p_n : A^n \to \mathcal{A}.
\]

What are the properties of this family of profunctors? We leave this as an open question. Conjecturally, this family is akin to an ‘unbiased’ version of promonoidal structure on \(\mathcal{A}\), and should determine some sort of correspondence à la Day; yet, the Day \((2, 2)\)-convolution evidently is not the usual Day convolution on \(\text{PSh}(\mathcal{A})\), due to the difference between a \((2, 2)\)-coend and an iterated coend.
5 Kusarigamas and Twisted Arrow Categories

The aim of this section is to introduce and study a fundamental computational tool that will endow higher arity coends with a fairly rich calculus, serving as the next best substitute to a ‘higher arity Yoneda lemma’, whose most naïve formulation turns out to be false. Generalising a construction of Street–Dubuc introduced in [5, Theorem 2], we introduce in Definition 5.1 functors

\[ \Gamma_{p,q} : \text{Cat}(\mathcal{C}(p,q), \mathcal{D}) \to \text{Cat}(\mathcal{C}(q,p), \mathcal{D}), \]

\[ \Gamma_{q,p} : \text{Cat}(\mathcal{C}(q,p), \mathcal{D}) \to \text{Cat}(\mathcal{C}(p,q), \mathcal{D}), \]

which we dub co/kusarigama. These constructions also generalise the product-hom functor of Definition 3.2, in the sense that

\[ \text{hom}_{\Pi,p,q}(A, B) \cong \Gamma_{p,q}(\text{pt})^A_B, \]

where pt is the terminal functor. Co/kusarigama allow us to pass from dinaturality to naturality, underpinning a number of results in the theory of higher arity co/ends, such as the construction of higher arity twisted arrow categories.

Overall, the entire structure of this section concentrates on studying the properties of the functors \( \Gamma_{p,q} \) and \( \Gamma_{q,p} \), which may be regarded as

1. Universal objects among \((p, q)\)-dinatural transformations, through which all other \((p, q)\)-dinaturals factor (Definition 5.1, Remark 5.2):

\[ \text{Nat}(\mathcal{J}_{p,q}(F), G) \cong \text{DiNat}^{(p,q)}(F, G) \cong \text{Nat}(F, \Gamma_{p,q}(G)); \]

2. Functors that can be inductively defined through suitable Kan extensions (item (PK5)), starting from the case [ ]:

\[ \mathcal{J}_{p,q}(F) \cong \text{Lan}_{\Delta_{p,q}}(\mathcal{J}(\Delta_{p,q}^*(F))), \quad \Gamma_{p,q}(G) \cong \text{Ran}_{\Delta_{p,q}}(\Gamma(\Delta_{p,q}^*(G))). \]

3. ‘Twisted versions’ of \( F \) and \( G \), which may be computed by use of a similar formula as the one computing co/ends as co/limits via the twisted arrow category (Sect. 5.4).

Finally, the paramount property of the co/kusarigama functors is that given a category \( \mathcal{C} \), the category of elements of \( \Gamma_{p,q}(\text{pt}) \), where \( \text{pt} : \mathcal{C}(p,q) \to \text{Set} \) is the terminal presheaf, is the universal fibration needed to build a higher-arity version of the twisted arrow category (i.e., the category of elements of \( \text{hom}_C \)): we study this construction in Sect. 5.3. This makes it possible to express the \((p, q)\)-co/end of a diagram \( G : \mathcal{C}(p,q) \to \mathcal{D} \) as a co/limit over the \((p, q)\)-twisted arrow category of \( \mathcal{C} \).

5.1 Co/kusarigama: Basic Definitions

Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories.

**Definition 5.1** Let \( F \) and \( G \) be functor from \( \mathcal{C} \) to \( \mathcal{D} \) of types \([p \quad q]\) and \([q \quad p]\).

**CK1** The kusarigama\(^8\) of \( G \) is, if it exists, the object

\[ \Gamma_{q,p}(G) : \mathcal{C}(p,q) \to \mathcal{D} \]

\(^8\) A kusarigama (鎌鎖) is a Japanese compound weapon made of a sickle (kama) and a blunt weight (fundo) attached to the opposite ends of a chain (kusari). The weight was used to disarm the opponent by entangling their sword in the chain or as a single weapon; disarmed or damaged the opponent, the sickle was then used
of \( \mathbf{Cat}(\mathcal{C}(p, q), \mathcal{D}) \) representing the functor

\[ \text{DiNat}^{(p, q)}(-, G) : \mathbf{Cat}(\mathcal{C}(p, q), \mathcal{D}) \to \mathbf{Set}. \]

(CK2) The cokusarigama of \( F \) is, if it exists, the object

\[ \mathcal{J}^{p, q}(F) : \mathcal{C}(q, p) \to \mathcal{D} \]

of \( \mathbf{Cat}(\mathcal{C}(p, q), \mathcal{D}) \) corepresenting the functor

\[ \text{DiNat}^{(p, q)}(F, -) : \mathbf{Cat}(\mathcal{C}(q, p), \mathcal{D}) \to \mathbf{Set}. \]

**Remark 5.1** Thus, co/kusarigama are defined by the following relations:

\[ \text{Nat}(\mathcal{J}^{p, q}(F), -) \equiv \text{DiNat}^{(p, q)}(F, -), \]
\[ \text{Nat}(-, \Gamma^{p, q}(G)) \equiv \text{DiNat}^{(p, q)}(-, G). \]

It is crucial to focus on the exact way in which the types of \( F, G \), and of \( \Gamma^{p, q}(F) \), \( \Gamma^{p, q}(G) \) interchange: asking that \( F, G \) be of type \( [p q] \) and \( [q p] \) is the only possible choice for the three objects \( \text{Nat} (\mathcal{J}^{p, q}(F), G) \), \( \text{Nat}(F, \Gamma^{p, q}(G)) \) and \( \text{DiNat}^{(p, q)}(F, G) \) to exist, according to our Definition 2.1.

This means that \( \Gamma^{p, q}, J^{p, q} \) are candidates to be functors

\[ J^{p, q} : \mathbf{Cat}(\mathcal{C}(p, q), \mathcal{D}) \to \mathbf{Cat}(\mathcal{C}(q, p), \mathcal{D}) \]
\[ \Gamma^{p, q} : \mathbf{Cat}(\mathcal{C}(q, p), \mathcal{D}) \to \mathbf{Cat}(\mathcal{C}(p, q), \mathcal{D}) \]

Among many other properties, we prove in Proposition 5.1 that these correspondences are indeed functors.

**Remark 5.2** (Unwinding Definition 5.1) The co/representability conditions defining co/kusarigama unwind as the following universal properties:

(UK1) The cokusarigama of a functor \( F : \mathcal{C}(p, q) \to \mathcal{D} \) is, if it exists, the pair \((\mathcal{J}^{p, q}(F), \eta)\) with

\[ \mathcal{J}^{q, p}(F) : \mathcal{C}(q, p) \to \mathcal{D} \]

a functor of type \([q p]\), and

\[ \eta : F \Rightarrow \mathcal{J}^{p, q}(F) \]

a \((p, q)\)-dinatural transformation satisfying the following universal property:

...to deliver the final, fatal strike. Kusarigama was probably adapted from an old farming tool, first adopted by Koga ninjas as a fast, compact weapon; its use then spread to tactic-oriented esoteric weaponry schools like Shinkage-ryū and Suiō-ryū. See [9] for more information.

The reason for this terminological choice is the following: as proved in Construction 5.1, \( J^{p, q}(F) \) is computed via the \((p, q)\)-coend formula

\[ J^{p, q}(F) \equiv \int \limits_{\mathcal{A} \in \mathcal{C}} (h_{A_1}^{-} \times \cdots \times h_{A_q}^{-} \times h_{A_1}^{A_1} \times \cdots \times h_{A_p}^{A_p}) \otimes F_A^A. \]

In this equation, we have a sickle \((p, q)\)-connected to the weight \( F \) by the chain \( h_{A_1}^{-} \times \cdots \times h_{A_q}^{-} \times h_{A_1}^{A_1} \times \cdots \times h_{A_p}^{A_p} \) of hom-functors.
Given a \((p, q)\)-dinaratural transformation \(\theta : F \Rightarrow G\), there exists a unique natural transformation \(\Gamma_{p, q}(F) \Rightarrow G\) making the diagram

\[
\begin{array}{ccc}
\Gamma_{p, q}(F) \\
\downarrow \exists! \\
F \\
\downarrow \theta \\
G
\end{array}
\]

commute.

The kusarigama of a functor \(G : C^{(q, p)} \rightarrow D\) is, if it exists, the pair \((\Gamma^{q,p}(G), \epsilon)\) with

\[
\Gamma^{q,p}(G) : C^{(p, q)} \rightarrow D
\]

a functor of type \([_{q}^{p}]\), and

\[
\epsilon : \Gamma^{p,q}(G) \Rightarrow G
\]

a \((p, q)\)-dinaratural transformation satisfying the following universal property:

Given a \((p, q)\)-dinaratural transformation \(\theta : F \Rightarrow G\), there exists a unique natural transformation \(F \Rightarrow \Gamma^{q,p}(G)\) making the diagram

\[
\begin{array}{ccc}
\Gamma^{q,p}(G) \\
\downarrow \exists! \\
F \\
\downarrow \theta \\
G
\end{array}
\]

commute.

**Notation 5.1** Given tuples \(A, C \in C^{-p} = (C^{p})^{op}\), \(B, D \in C^{q}\) we make use of the notation

\[
\text{hom}_{C^{(p, q)}}(A, B), (C, D)) \overset{\text{def}}{=} h_{(C, D)}^{(A, B)},
\]

as well as of the equalities

\[
h_{(A, B)}^{(C, D)} \overset{\text{def}}{=} h_{A}^{C} \times h_{A}^{B} = h_{A_{1}}^{C_{1}} \times \cdots \times h_{A_{p}}^{C_{p}} \times h_{B_{1}}^{D_{1}} \times \cdots \times h_{B_{q}}^{D_{q}}.
\]

**Construction 5.1** (Constructing Cokusarigama) Suppose that \(D\) is cocomplete. Then

\[
\int_{C^{(p, q)}}^{(p, q)} A \in C \quad (h_{A}^{-} \times h_{A}^{A} \cup F_{A}^{A})
\]

(meaning the \((p, q)\)-coend of

\[
C^{(p, q)} \rightarrow \text{Cat}(C^{(q, p)}, D)
\]

\[(A, B) \mapsto \text{hom}_{C^{(q, p)}}((B, A), (-, -)) \cup F_{B}^{A},\]

satisfies the universal property in Definition 5.1.
Proof The proof is merely a formal manipulation:

\[
\text{DiNat}^{(p, q)}(F, G) \equiv \int_{(q, p)} \int_{X \in \mathcal{C}} \hom_{\mathcal{D}} \left( F^X_X, G^X_X \right)
\]

\[
\equiv \int_{(q, p)} \int_{X \in \mathcal{C}} \hom_{\mathcal{D}} \left( F^X_X, \int_{A, B \in \mathcal{C}} \left( h^A_X \times h^B_X \right) \cap G^A_B \right)
\]

\[
\equiv \int_{(q, p)} \int_{X \in \mathcal{C}} \hom_{\mathcal{D}} \left( \int_{A, B \in \mathcal{C}} \left( h^A_X \times h^B_X \right) \cap F^X_X, G^A_B \right)
\]

\[
\equiv \int_{A, B \in \mathcal{C}} \hom_{\mathcal{D}} \left( (p, q) \int_{X \in \mathcal{C}} \left( h^A_X \times h^B_X \right) \cap F^X_X, G^A_B \right)
\]

\[
\equiv \int_{A, B \in \mathcal{C}} \hom_{\mathcal{D}} \left( J^{p, q}(F)^A_B, G^A_B \right)
\]

\[
\equiv \text{Nat} \left( J^{p, q}(F), G \right).
\]

\[\square\]

Construction 5.2 (Constructing Kusarigamas) Suppose that \(\mathcal{D}\) is complete. Then

\[
\int_{(q, p)} \int_{A \in \mathcal{C}} \left( h^A_+ \times h^-_A \right) \cap G^A_B,
\]

meaning the \((q, p)\)-end of equation \ref{equation:universal_property}

\[
\mathcal{C}^{(q, p)} \longrightarrow \text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D})
\]

\[
(A, B) \longmapsto \hom_{\mathcal{C}^{(q, p)}} ((A, B); (-, -)) \cap G^A_B,
\]

satisfies the universal property in Definition 5.1.

Proof The derivation is dual to Construction 5.1, so the proof is straightforward. \[\square\]

Notation 5.2 ((1, 1)-kusarigama) For the sake of brevity, we often write \(\Gamma(D)\) and \(J(D)\) for \(\Gamma^{1,1}(D)\) and \(J^{1,1}(D)\), respectively.

Proposition 5.1 (Properties of co/kusarigama) Let \(D, F, G : \mathcal{C}^{(p, q)} \rightrightarrows \mathcal{D}\) be functors, where \(\mathcal{D}\) is a bicomplete category.

(PK1) Functoriality. The assignments \(D \mapsto \Gamma(D), J(D)\) define functors

\[
J^{p, q} : \text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D}) \to \text{Cat}(\mathcal{C}^{(q, p)}, \mathcal{D}),
\]

\[
\Gamma^{p, q} : \text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D}) \to \text{Cat}(\mathcal{C}^{(q, p)}, \mathcal{D}).
\]

(PK2) Adjointness. We have an adjunction

\[
\text{Cat}(\mathcal{C}^{(p, q)}, \mathcal{D}) \xrightarrow{\Gamma^{p, q}} \text{Cat}(\mathcal{C}^{(q, p)}, \mathcal{D}).
\]
(PK3) Commutativity with homs. Let \( F : C^{(p,q)} \to \mathcal{D} \) be a functor, and let us consider the functors
\[
\mathcal{D}(F,1) : \mathcal{D} \to \mathcal{Cat}(C^{(q,p)}, \text{Set}), D \mapsto ((A, B) \mapsto \mathcal{D} \left( F^A_B, D \right)),
\]
\[
\mathcal{D}(1, F) : \mathcal{D}^{op} \to \mathcal{Cat}(C^{(p,q)}, \text{Set}), D \mapsto ((A, B) \mapsto \mathcal{D} \left( D, F^A_B \right)).
\]
then the diagrams
\[
\begin{array}{ccc}
\mathcal{D}(\Gamma^{p,q}(F), 1) & \xrightarrow{\mathcal{D}} & \mathcal{D}(F, 1) \\
\mathcal{Cat}(C^{(q,p)}, \text{Set}) & \xrightarrow{\mathcal{Cat}(C^{(p,q)}, \text{Set})} & \mathcal{Cat}(C^{(p,q)}, \text{Set})
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{D}(1, \Gamma^{p,q}(F)) & \xrightarrow{\mathcal{D}} & \mathcal{D}(1, F) \\
\mathcal{Cat}(C^{(p,q)}, \text{Set}) & \xrightarrow{\mathcal{Cat}(C^{(p,q)}, \text{Set})} & \mathcal{Cat}(C^{(p,q)}, \text{Set})
\end{array}
\]
commute:
\[
\mathcal{D}(\Gamma^{p,q}(F), 1) \cong \Gamma^{q,p}(\mathcal{D}(F, 1)), \quad \mathcal{D}(1, \Gamma^{p,q}(F)) \cong \Gamma^{p,q}(\mathcal{D}(1, F)).
\]

(PK4) Limits of kusarigama. Let \( F : C^{(p,q)} \to \mathcal{D} \) be a functor; we have functorial isomorphisms
\[
\int_{(p,q)} \int_{A \in C} F^A_p \cong \lim \left( \Gamma^{p,q}(F) \right), \quad \int_{(p,q)} \int_{A \in C} F^A_p \cong \colim \left( \Gamma^{q,p}(F) \right).
\]

(PK5) Higher arity co/kusarigama from. \((1, 1)\)-co/kusarigama. The kusarigama
\[
\Gamma^{p,q}(F) : C^{(q,p)} \to \mathcal{D}
\]
of a functor \( F : C^{(p,q)} \to \mathcal{D} \) is the left Kan extension of the \((1, 1)\)-kusarigama of \( \Delta^*_{p,q}(F) \) along \( \Delta_{q,p} \):
\[
\begin{array}{ccc}
\Delta_{q,p} & \xrightarrow{\mathcal{D}} & \mathcal{C}^{(q,p)} \\
\int_{\Gamma^{p,q}(F)} & \xrightarrow{\mathcal{D}} & \int_{\Gamma^{p,q}(F)} \\
\mathcal{C}^{op} \times \mathcal{C} & \xrightarrow{\mathcal{D}} & \mathcal{D}.
\end{array}
\]
Moreover, if \( \mathcal{C} \) has finite products and finite coproducts, then \( \Gamma^{p,q}(-) \) factors as
\[
\mathcal{Cat}(C^{(p,q)}, \mathcal{D}) \xrightarrow{\mathcal{Cat}(C^{op} \times \mathcal{C}, \mathcal{D}) \circ \mathcal{Cat}(C^{op} \times \mathcal{C}, \mathcal{D}) \circ \mathcal{Cat}(C^{(p,q)}, \mathcal{D})}.
\]
Dually, the kusarigama
\[
\Gamma^{q,p}(G) : C^{(p,q)} \to \mathcal{D}
\]
of \( G : C^{(q,p)} \to \mathcal{D} \) is the right Kan extension of the \((1, 1)\)-kusarigama of \( \Delta^*_{q,p}(G) \) along \( \Delta_{p,q} \):
\[
\begin{array}{ccc}
\Delta_{p,q} & \xrightarrow{\mathcal{D}} & \mathcal{C}^{(p,q)} \\
\int_{\Gamma^{q,p}(G)} & \xrightarrow{\mathcal{D}} & \int_{\Gamma^{q,p}(G)} \\
\mathcal{C}^{op} \times \mathcal{C} & \xrightarrow{\mathcal{D}} & \mathcal{D}.
\end{array}
\]
Moreover, if $C$ has finite products and finite coproducts, then $\Gamma^{p,q} (\cdot)$ factors as
\[
\text{Cat} \left( C^{(p,q)} ; D \right) \xrightarrow{\Delta^*_{p,q}} \text{Cat} \left( C^{op} \times C ; D \right) \xrightarrow{\Gamma} \text{Cat} \left( C^{op} \times C ; D \right) \xrightarrow{(M_{p,q})^*} \text{Cat} \left( C^{(p,q)} ; D \right).
\]

In fact, the adjunction yielding $\mathcal{J}^{p,q} \dashv \Gamma^{p,q}$ can be extended as in the following diagram of adjunctions:

\[
\begin{array}{ccc}
\text{Cat} \left( C^{(p,q)} ; D \right) & \xrightarrow{\Delta^*_{p,q}} & \text{Cat} \left( C^{op} \times C ; D \right) \\
\downarrow^{\mathcal{J}} & & \downarrow^{\Gamma} \\
\text{Cat} \left( C^{(p,q)} ; D \right) & \xrightarrow{(M_{p,q})^*} & \text{Cat} \left( C^{op} \times C ; D \right)
\end{array}
\]

\[\xrightarrow{\text{Lan}_{\mathcal{W}_{p,q}}} \]

see Corollary 1.1.

**Proof** We often prove the statements for kusarigama only, as the ones for kusarigama follow by an easy dualisation.

item (PK1): This follows from [16, Theorems IX.7.2 and IX.7.3]. item (PK2): This follows straight from the definition of co/kusarigama.

item (PK3): For the first statement, we have
\[
\mathcal{D} \left( 1, \int_{A \in C} h^{(A,\ldots,A)} \cap F_A^A \right) \cong \int_{(q,p)} \int_{A \in C} \mathcal{D} \left( (q,p) \int_{A \in C} h^{(A,\ldots,A)} \cap F_A^A \right) \\
\cong \int_{(q,p)} \int_{A \in C} h^{(A,\ldots,A)} \cap \mathcal{D} \left( (q,p) \int_{A \in C} F_A^A \right) \\
\cong \Gamma^{p,q} (\mathcal{D}(1, F)).
\]

item (PK4): We just prove the first statement, the other being a straightforward dualisation. We have
\[
\mathcal{D} \left( (p,q) \int_{A \in C} D_A^A \right) \cong \text{DiNat}^{(p,q)} (\Delta_{p,q}, h_D) \\
\cong \text{Nat} (\Delta_{p,q}, \Gamma^{p,q} (h_D)) \quad \text{by Remark 5.1}, \\
\cong \text{Nat} (\Delta_{p,q}, h_{\Gamma^{p,q}(D)}) \quad \text{by item (PK3),} \\
\cong \text{Nat} (\Gamma^{p,q}(D)).
\]

The result then follows from the Yoneda lemma.

item (PK5): We have
\[
\text{Nat} \left( \text{Lan}_{\Delta_{q,p}} \mathcal{J} \left( \Delta_{p,q}^* (F) \right), G \right) \cong \text{Nat} \left( \mathcal{J} \left( \Delta_{p,q}^* (F) \right), \Delta_{q,p}^* (G) \right) \\
\cong \text{DiNat} \left( \Delta_{p,q}^* (F), \Delta_{q,p}^* (G) \right) \quad \text{by Remark 5.1}, \\
\cong \text{DiNat}^{(p,q)} (F, G) \quad \text{by Proposition 2.2}, \\
\cong \text{Nat} \left( \mathcal{J}^{p,q} (F), G \right) \quad \text{by Remark 5.1 again}.
\]

The stated factorisation follows from the isomorphism $\text{Lan}_{\Delta_{q,p}} \cong (W^{q,p})^*$ of Corollary 1.1.
5.2 Examples of Co/kusarigama

**Example 5.1** (Cokusarigama of hom functors) The computation given in the proof of Lemma 3.3 generalises to show that, given \((A, B) \in C^{(p,q)}\), the cokusarigama of the functor \(h_{(A, B)}: C^{(q,p)} \to \text{Set}\), which may be written as

\[
\text{hom}_{C^{(p,q)}}((-,-); (A, B)) \overset{\text{def}}{=} \text{hom}_{C^{p}}(A, -) \times \text{hom}_{C^{q}}(-, B)
\]

is given by

\[
J^{q,p}(h_{(A, B)}) \overset{\text{def}}{=} \int \left( X \in C \right) h_{p+1}^{-1} \times \cdots \times h_{p+q}^{-1} \times h_{-1}^{-1} \times \cdots \times h_{-p+1}^{-1} \times \cdots \times h_{-p+q}^{-1} \\
\times \left( h_{B_1}^{A_1} \times \cdots \times h_{B_q}^{A_1} \times \cdots \times h_{B_1}^{A_p} \times \cdots \times h_{B_q}^{A_p} \right) \\
\times \left( h_{B_1}^{-1} \times \cdots \times h_{B_q}^{-1} \times \cdots \times h_{B_1}^{-p} \times \cdots \times h_{B_q}^{-p} \right) \\
\times \left( h_{B_1}^{-1} \times \cdots \times h_{B_q}^{-1} \times \cdots \times h_{B_1}^{-p+1} \times \cdots \times h_{B_q}^{-p+1} \right)
\]

**Example 5.2** (Cokusarigama of constant functors) Let \(E\) be a set and let’s equally denote \(E : C^{(p,q)} \to \text{Set}\) the constant functor on \(E\); assume \(C\) has finite products and coproducts; then, we can compute the kusarigama of \(E\) as the integral

\[
\Gamma(E) \overset{\text{def}}{=} \int_{A \in C} (C[Y|A] \times C[A|X]) \cap E
\]

\[
\overset{\text{def}}{=} \int_{A \in C} \text{Set}(C(A, \coprod X_i), \text{Set}(C(\coprod Y_j, A), E))
\]

\[
\overset{\text{def}}{=} \text{Cat}(C^{(p,q)}, \text{Set}(C(-, \coprod X_i), \text{Set}(C(\coprod Y_j, -), E))
\]

\[
\overset{\text{def}}{=} \text{Set}(C(Y, X), E),
\]

where \(X \overset{\text{def}}{=} \coprod X_i\), \(Y \overset{\text{def}}{=} \coprod Y_j\).

In particular, when \(D = \text{Set}\):

\[
\Gamma(\text{pt}) = \int_{A \in C} [h_A \times h_A^{A}, \text{pt}] \overset{\text{def}}{=} \text{pt}. \]

This is in accordance with the fact that dinatural transformations to \(\Delta_{\text{pt}}\) coincide with natural transformations to \(\Delta_{\text{pt}}\).

Dually,

\[
J^{Y}(E) \overset{\text{def}}{=} \int^{A} (h^A)^p \times (h^A)^q \times E
\]

\[
\overset{\text{def}}{=} \int^{A} C[Y|A] \times C[A|X] \times E
\]

\[
\overset{\text{def}}{=} \left( \int^{A} C(Y, A) \times C(A, X) \right) \times E
\]
\[ \cong \mathcal{C}(Y, X) \times E, \]  
(7)

where \( X \overset{\text{def}}{=} \prod X_i \) and \( Y \overset{\text{def}}{=} \bigsqcup Y_j \).

**Example 5.3** *(The co/kusarigama of the identity functor)* Let \( \mathcal{C} \) be a complete and cocomplete category (so that the co/ends in Constructions 5.1 and 5.2 exist).

We want to compute the co/kusarigama of the identity functor \( \text{id}_{\mathcal{C}(p,q)} : \mathcal{C}(p,q) \rightarrow \mathcal{C}(p,q) \).

By virtue of the universal property of the product category \( \mathcal{C}(p,q) \), it is then enough to determine the functor

\[ \mathcal{C}(p,q)^{\ast} \Gamma(p,q)(\text{id}) \overset{\pi_j}{\longrightarrow} \mathcal{C} \]

where the functor \( \pi_j \) projects to the factor \( \mathcal{C} \) for \( 1 \leq j \leq q \), and to \( \mathcal{C}^{\text{op}} \) for \( p+1 \leq j \leq q+p \).

In case \( (p,q) = (2,1) \) one sees that for objects \( (X_1, X_2, Y) \) the diagram

\[
\begin{array}{ccc}
\prod_{f : B \rightarrow A} (h_B^{X_1} \times h_B^{X_2} \times h_Y) \circ B & \overset{\beta}{\rightarrow} & \prod_{A \in \mathcal{C}} (h_A^{X_1} \times h_A^{X_2} \times h_Y) \circ A \\
\downarrow^\alpha & & \downarrow^\zeta \\
(h_Y^{X_1} \times h_Y^{X_2}) \circ Y & & (w \circ u, w \circ v, w \circ a)
\end{array}
\]

commutes and in fact that it is a coequaliser: every other \( \zeta : \prod_{A \in \mathcal{C}} \mathcal{C}(X_1, A) \times \mathcal{C}(X_2, A) \times \mathcal{C}(A, Y) \circ A \rightarrow E \) coequalising the pair \((\alpha, \beta)\) must factor through \( \mathcal{C}(X_1, Y) \times \mathcal{C}(X_2, Y) \circ Y \) with a uniquely determined map.

A standard argument, carried over the general case, to find the coequaliser defining the end and coend in Constructions 5.1 and 5.2 now yields

\[ J_{p,q}^\ast(\text{id})(X, Y) = \text{hom}_{\mathcal{C},p,q}(Y, X) \circ (Y, X), \]
\[ \Gamma_{p,q}^\ast(\text{id})(X, Y) = \text{hom}_{\mathcal{C},p,q}(X, Y) \cap (Y, X). \]

**Remark 5.3** The previous argument hides a technical point. It holds by virtue of the following fact: if two categories \( \mathcal{A}, \mathcal{B} \) are co/tensored over Set, then so is their product \( \mathcal{A} \times \mathcal{B} \), with the component-wise action of a functor \( \circ \) : \( \text{Set} \times \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B} \).

A similar result does *not* hold for a generic base of enrichment.

### 5.3 Higher Arity Twisted Arrow Categories

Classically, it is possible to compute the co/end of a diagram \( \mathcal{D} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D} \) as the co/limit of \( \mathcal{D} \) over the twisted arrow category \( \text{Tw}(\mathcal{C}) \) of \( \mathcal{C} \), i.e. over the category of elements of the hom functor of \( \mathcal{C} \). The purpose of this section is to formulate and prove an analogous description for higher arity co/ends.

In this section, we abbreviate \( J_{p,q}^\ast(\Delta_{\text{pt}}) \) as \( J_{p,q}^\ast(\text{pt}) \).

**Definition 5.2** The \((p, q)\)-twisted arrow category of \( \mathcal{C} \) is the category \( \text{Tw}^{(p, q)}(\mathcal{C}) \) defined as the category of elements of \( J_{p,q}^\ast(\text{pt}) \):
**Remark 5.4 (Unwinding Definition 5.2)** By the calculation in the proof of Lemma 3.3, we have $J^{p,q}(pt) \cong \text{hom}_{\Pi,p,q}$. As a result, we see that $\text{Tw}^{(p,q)}(C)$ may be described as the category whose 

(KCC1) Objects are collections $\{f_{ij} : A_i \rightarrow B_j\}$ of morphisms of $\mathcal{D}$ with $0 \leq i \leq p$ and $0 \leq j \leq q$;

(KCC2) Morphisms are collections of factorisations of the codomain through the domain, of the form

$$A_i \xrightarrow{f} B_j \xrightarrow{\psi_j} B'_{j'} \xrightarrow{g} A'_i,$$

one for each $0 \leq i \leq p$ and each $0 \leq j \leq q$.

**Lemma 5.1** Let $D : \mathcal{C}^{(p,q)} \rightarrow \mathcal{D}$ be a diagram. We have natural isomorphisms

$$\int^{A \in \mathcal{C}} D^A \cong \text{colim}^{J^{p,q}(pt)}(D),$$

$$\int^{A \in \mathcal{C}} D^A \cong \text{lim}^{J^{p,q}(pt)}(D),$$

generalising the well-known isomorphisms

$$\int^{A \in \mathcal{C}} D^A \cong \text{colim}^{\text{hom}_C}(D),$$

$$\int^{A \in \mathcal{C}} D^A \cong \text{lim}^{\text{hom}_C}(D),$$

valid for $(p, q) = (1, 1)$.

**Proof** We have

$$h\left(\int^{(p,q)}_{A \in \mathcal{C}} D^A\right) \cong \text{DiNat}(\Delta_{pt}, h_D)$$

$$\cong \text{Nat}(J^{p,q}(pt), h_D)$$

$$\cong h_{\text{lim}^{J^{p,q}(pt)}}(D).$$

The proof of the remaining isomorphism is formally dual to the above one. $\Box$

**Proposition 5.2 ((p, q)-Ends as limits, yet again)** Let $D : \mathcal{C}^{(p,q)} \rightarrow \mathcal{D}$ be a functor. We have isomorphisms

$$\int^{(p,q)}_{A \in \mathcal{C}} D^A \cong \text{lim} \left( \text{Tw}^{(p,q)}(C) \xrightarrow{\Sigma^{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right),$$

$$\int^{(p,q)}_{A \in \mathcal{C}} D^A \cong \text{colim} \left( \text{Tw}^{(p,q)}(C)^{op} \xrightarrow{\Sigma^{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D^{op}} \mathcal{D} \right).$$

**Proof** This result follows from Lemma 5.1 and the classical description of weighted colimits as conical ones [12, Section 3.4, Equation 3.33]. $\Box$
5.4 Twisted Arrow Categories Associated to Cokusarigama

In this short section, we give a co/comma category formula for computing co/kusarigama. These generalise the construction in Sect. 5.3 and work for arbitrary \((p, q)\). However, these turn out to be too complicated for \(p, q \geq 2\) as to be practically useful\(^9\), so we restrict our attention to the case \((p, q) = (1, 1)\) below. Let \(F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}\) be a functor and fix \(A, B \in \mathcal{C}_o\).

**Definition 5.3** The twisted arrow category of \(\mathcal{C}\) for \((1, 1)\)-cokusarigama at \((A, B)\) is the category \(\text{Tw}_{A,B}^{1,1}(\mathcal{C})\) defined as the category of elements of \(h_B^A \times h_{-2}^A \times h_B^{-1} \times h_{-2}^{-1}\).

**Remark 5.5** Concretely, \(\text{Tw}_{A,B}^{1,1}(\mathcal{C})\) may be described as the category whose

(KCC1) Objects are squares of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & B \\
\downarrow{f} & & \downarrow{g} \\
Y & \longleftarrow{\psi} & A
\end{array}
\]

with \(X, Y \in \mathcal{C}_o\) and \(f, g, \phi, \psi \in \text{Mor}(\mathcal{C})\);

(kcc2) Morphisms are twisted commutative cubes

![Twisted commutative cube](image)

\(\text{(8)}\)

**Remark 5.6** \((\text{Tw}_{A,B}^{1,1}(\mathcal{C})\) as a generalisation of the twisted arrow category\) The twisted arrow category of \(\mathcal{C}\) naturally fits inside \(\text{Tw}_{A,B}^{1,1}(\mathcal{C})\):

This comes from the identity \(J(\text{pt}) \cong \text{hom}\).

---

\(^9\) Similarly to how a morphism of \(\text{Tw}^{(p,q)}(\mathcal{C})\) turned out to involve \(pq\) arrows of \(\mathcal{C}\), unravelling the construction given in this section for arbitrary \((p, q)\) gives a category \(\text{Tw}^{(p,q)}(\mathcal{C})\) whose morphisms now consist of \(4pq\) morphisms of \(\mathcal{C}\). Additionally, each of these points now in a different direction (i.e. they cannot anymore be arranged as morphisms in product categories). Together, these two points make \(\text{Tw}^{(p,q)}(\mathcal{C})\) unusable in practice when \(p\) and \(q\) are too large. As a compromise, we work out the case \((p, q) = (1, 1)\), which is both the simplest case as well as the most useful one.
Proposition 5.3 (Co/kusarigama as limits) Given a functor $D : C^{op} \times C \to D$, we have isomorphisms

$$J(D)^A_B \cong \text{colim} \left( TW^A_B(C) \xrightarrow{pr} C^{op} \times C \xrightarrow{D} D \right),$$

$$\Gamma(D)^A_B \cong \text{lim} \left( TW^A_B(C) \xrightarrow{pr} C^{op} \times C \xrightarrow{D} D \right).$$

Proof Firstly, observe that we may compute $J(D)$ as the following weighted coend:

$$\int_{X \in C} \left[ h^{-2}_A \times h^{-1}_B \right] D^X_X \cong \int_{X \in C} \left( h^A_X \times h^X_B \right) \odot D^X_X$$

$$\cong J(D)^A_B.$$ 

Now, weighted coends corepresent functors of the form $DiNat(W, h^D)$, but since

$DiNat(W, h^D) \cong \text{Nat}(J(W), h^D),$

we see that the above weighted coend is the weighted colimit of $D$ by $J(h^{-2}_A \times h^{-1}_B)$. From the computation in the proof of Lemma 3.3, we have $J(h^{-2}_A \times h^{-1}_B) \cong h^A_B \times h^2_A \times h^1_B \times h^2_A$. The result then follows from the classical description of weighted colimits as conical ones [12, Section 3.4, Equation 3.33].

The second formula is proved in a dual fashion. ☐

data availability

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