FINITE DIFFERENCE QUANTUM TODA LATTICE VIA EQUIVARIANT K-THEORY

ALEXANDER BRAVERMAN AND MICHAEL FINKELBERG

To Vladimir Drinfeld with admiration

Abstract. We construct the action of the quantum group $U_v(sl_n)$ by the natural correspondences in the equivariant localized $K$-theory of the Laumon based Quasiflags' moduli spaces. The resulting module is the universal Verma module. We construct geometrically the Shapovalov scalar product and the Whittaker vectors. It follows that a certain generating function of the characters of the global sections of the structure sheaves of the Laumon moduli spaces satisfies a $v$-difference analogue of the quantum Toda lattice system, reproving the main theorem of Givental-Lee (cf. [7]). Similar constructions are performed for the affine Lie algebra $\hat{sl}_n$.

1. Introduction

1.1. This work arose from an attempt to understand the results of the paper [7] of A. Givental and Y.-P. Lee where the authors perform some computations related to "quantum $K$-theory" of flag varieties (as well as some results from [14] related to 5d $SU(n)$-gauge theory compactified on a circle) in the framework of representation theory. Similar approach to quantum cohomology of flag varieties (and to partition functions of 4d gauge theory) is discussed in [1] and [2].

In [7] the authors consider the moduli spaces $Q_d$ introduced by G. Laumon in [10], [11]. These are certain closures of the moduli spaces of based maps of degree $d$ from $\mathbb{P}^1$ to the flag variety $B$ of $sl_n$.

A Cartan torus $T$ of $SL_n$ acts on $Q_d$. The multiplicative group $\mathbb{C}^*$ of dilations of $\mathbb{P}^1$ (loop rotations) also acts on $Q_d$. The formal character of the (infinite dimensional) $T \times \mathbb{C}^*$-module $RT(Q_d, 0_d)$ turns out to be a rational function on $T \times \mathbb{C}^*$. One may form a certain generating function $J$ of these rational functions for all degrees $d$. Computing the function $J$ presumably should give rise to a computation of the $SL_n$-equivariant quantum $K$-theory ring of $B$ (which to the best of the authors' knowledge has not yet been defined in the literature).

A. Givental and Y.-P. Lee prove that $J$ satisfies a certain $v$-difference version of the quantum Toda lattice equations (here $v$ stands for the tautological character of $\mathbb{C}^*$). Moreover, they suggest another way to construct solutions of the $v$-difference Toda system: as the Shapovalov scalar product of the Whittaker vectors in the universal Verma module for the quantum group $U_v(sl_n)$. The latter construction was worked out independently in [4], [17].

1.2. The principal goal of the present paper is to identify these two constructions of solutions of the $v$-difference Toda system. Namely, we prove that the natural correspondences between the moduli spaces $Q_d$ (for the degrees differing by a simple root) give rise to the action of the standard generators of $U_v(sl_n)$ on the localized equivariant $K$-theory $\mathbb{P}^1 K^{T \times \mathbb{C}^*}(Q_d)$. Here the localization is taken with respect to the $K^{T \times \mathbb{C}^*}(\cdot) = \mathbb{C}[T \times \mathbb{C}^*]$, that is, we tensor everything
with the fraction field of $\mathbb{C}[T \times \mathbb{C}^*]$. This is needed since the above correspondences are not proper, but the subspaces of their $T \times \mathbb{C}^*$-fixed points are proper (in fact, they are finite), so their action is well defined only in the localized equivariant $K$-theory. This way we get a $U_v(sl_n)$-module, and we identify it with the universal Verma module $M$. We also compute in geometric terms the Shapovalov scalar product on $M$, and the Whittaker vectors. It turns out that the generating function for the Shapovalov scalar product of the Whittaker vectors is a simple modification of the Givental-Lee generating function $J$. Thus we reprove the Main Theorem of Givental-Lee.

1.3. There is a similar generating function $J$ for equivariant integrals of the unit cohomology classes of $\Omega_d$ which controls the $T$-equivariant quantum cohomology of $B$. It satisfies the quantum Toda lattice differential system, as proved originally by A. Givental and B. Kim.

For the simple Lie algebras $\mathfrak{g}$ other than $sl_n$ there is no analogue of the Laumon moduli spaces $\Omega_d$ but there is Drinfeld’s moduli space of Quasimaps $Z_d(\mathfrak{g})$. It also exists for the case of affine Lie algebras, under the name of Uhlenbeck compactification. In the $sl_n$ case $\Omega_d$ is a small resolution of $Z_d(sl_n)$. In the affine $\mathfrak{sl}_n$ case $Z_d(sl_n)$ possesses a semismall resolution of singularities: the moduli space $P_d$ of torsion free parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ endowed with some additional structures. Thus in the affine case we can define an analog of the function $J$ which we denote by $J_{aff}$ (this is discussed in [1]).

The generating function $J$ (for any simple $G$) is known to satisfy the quantum (differential) Toda equations (cf. [8] and [9]).

In the work [1], the generating function $J$ for equivariant integrals of the unit cohomology classes of $Z_d(\mathfrak{g})$ was proved to satisfy the quantum Toda lattice by constructing the action of the Langlands dual Lie algebra $\hat{\mathfrak{g}}$ in the equivariant Intersection Cohomology of the Drinfeld compactifications. Also in the affine case the function $J_{aff}$ was shown to satisfy some non-stationary analog of “the most basic” (quadratic) Toda equation. Thus [1] offered a representation theoretic explanation of the Givental-Kim results as well as generalized them to the affine case. And the present work is a multiplicative analogue of [1] in the simplest case of $sl_n$.

1.4. It would be extremely interesting to extend our work to other simple and affine Lie algebras. It would require something like an equivariant “IC $K$-theory” of $Z_d(\mathfrak{g})$ which is not defined at the moment. In case of $sl_n$ the IC cohomology of $Z_d(sl_n)$ coincides with the cohomology of the small resolution $\Omega_d$, while in the affine case of $\mathfrak{sl}_n$ the IC cohomology of $Z_d(\hat{sl}_n)$ is a direct summand in the cohomology of the semismall resolution $P_d$. Accordingly, one might look for the correct “IC $K$-theory” of $Z_d(\hat{sl}_n)$ as an appropriate direct summand of the usual $K$-theory of $P_d$. This is sketched in the Section 8. Namely, similarly to the case of Laumon spaces, the quantum affine group $U_v(sl_n)$ acts by the natural correspondences on the direct sum of localized equivariant $K$-groups $\oplus \Omega_d K^{\mathbb{T} \times \mathbb{C}^* \times \mathbb{C}^*}(P_d)$. However, this module looks more like the universal Verma module for $U_v(sl_n)$, and we have to specify a certain submodule isomorphic to the universal Verma module for $U_v(sl_n)$. Then we construct geometrically the Shapovalov scalar product, and the Whittaker vectors. It turns out that the Shapovalov scalar product of the Whittaker vectors can be expressed via the formal characters of the global sections $RT(P_d, O_d)$ as in the case of $sl_n$. However, we were unable to derive any $\nu$-difference equation for the affine version of the generating function $J$.

1.5. **Acknowledgments.** M.F. is obliged to V. Schechtman, A. Stoyanovsky, B. Feigin, E. Vasserot, and R. Bezrukavnikov who, ever since the appearance of [1], urged him to consider its equivariant $K$-theory analogue. While trying to guess the correct formulae in the
low ranks, we profited strongly from the computational help of V. Dotsenko, V. Golyshev, A. Kuznetsov. We are also grateful to P. Etingof and A. Joseph for very useful explanations; to M. Kashiwara for bringing the reference [13] to our attention, and to the referee for the valuable comments. Last but not least, our thanks go to A. Tsymbaliuk for the careful reading of our note and spotting several mistakes. We would like to thank the Weizmann Institute and RIMS, Kyoto, as well as the University of Chicago, for the hospitality and support.

M.F. was partially supported by the CRDF award RM1-2545-MO-03. A.B. was partially supported by the NSF grant DMS-0300271.

2. LAUMON SPACES AND QUANTUM GROUPS

2.1. We recall the setup of [6]. Let $\mathbb{C}$ be a smooth projective curve of genus zero. We fix a coordinate $z$ on $\mathbb{C}$, and consider the action of $\mathbb{C}^\ast$ on $\mathbb{C}$ such that $v(z) = v^{-2}z$. We have $\mathbb{C}^\ast = \{0, \infty\}$.

We consider an $n$-dimensional vector space $W$ with a basis $w_1, \ldots, w_n$. This defines a Cartan torus $T \subset G = SL_n \subset Aut(W)$. We also consider its $2^{n-1}$-fold cover, the bigger torus $\tilde{T}$, acting on $W$ as follows: for $\tilde{T} \ni t = (t_1, \ldots, t_n)$ we have $\tilde{t}(w_i) = t_i^2 w_i$. We denote by $\mathbb{B}$ the flag variety of $G$.

2.2. Given an $(n-1)$-tuple of nonnegative integers $d = (d_1, \ldots, d_{n-1})$, we consider the Laumon’s quasiflags’ space $\Omega_d$, see [11], 4.2. It is the moduli space of flags of locally free subsheaves of $\mathbb{C}$-supported by the NSF grant DMS-0300271.

2.3. We consider the following locally closed subvariety $\Omega_d \subset \Omega_{\mathbb{B}}$ (quasiflags based at $\infty \in \mathbb{C}$) formed by the flags

$$0 \subset W_1 \subset \ldots \subset W_{n-1} \subset W = W \otimes \mathcal{O}_\mathbb{C}$$

such that $\text{rank}(W_k) = k$, and $\text{deg}(W_k) = -d_k$.

It is known to be a smooth projective variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim \mathbb{B}$, see [11], 2.10.

2.4. The group $G \times \mathbb{C}^\ast$ acts naturally on $\Omega_d$, and the group $\tilde{T} \times \mathbb{C}^\ast$ acts naturally on $\Omega_{\mathbb{B}}$. The set of fixed points of $\tilde{T} \times \mathbb{C}^\ast$ on $\Omega_d$ is finite; we recall its description from [6], 2.11.

Let $d$ be a collection of nonnegative integers $(d_{ij})$, $i \geq j$, such that $d_i = \sum_{j=1}^i d_{ij}$, and for $i \geq k \geq j$ we have $d_{kj} \geq d_{ij}$. Abusing notation we denote by $d$ the corresponding $\tilde{T} \times \mathbb{C}^\ast$-fixed point in $\Omega_d$.

$$W_1 = \mathcal{O}_\mathbb{C}(-d_{11} \cdot 0) w_1, \quad W_2 = \mathcal{O}_\mathbb{C}(-d_{21} \cdot 0) w_1 \oplus \mathcal{O}_\mathbb{C}(-d_{22} \cdot 0) w_2, \quad \ldots$$

$$W_{n-1} = \mathcal{O}_\mathbb{C}(-d_{n-1,1} \cdot 0) w_1 \oplus \mathcal{O}_\mathbb{C}(-d_{n-1,2} \cdot 0) w_2 \oplus \ldots \oplus \mathcal{O}_\mathbb{C}(-d_{n-1,n-1} \cdot 0) w_{n-1}.$$

2.5. For $i \in \{1, \ldots, n-1\}$, and $d = (d_1, \ldots, d_{n-1})$, we set $d + i := (d_1, \ldots, d_i + 1, \ldots, d_{n-1})$. We have a correspondence $E_{d, i} \subset \Omega_d \times \Omega_{d+i}$ formed by the pairs $(W_i, W'_i)$ such that for $j \neq i$ we have $W_j = W'_j$, and $W'_i \subset W_i$, see [6], 3.1. In other words, $E_{d, i}$ is the moduli space of flags of locally free sheaves

$$0 \subset W_1 \subset \ldots W_{i-1} \subset W'_i \subset W_i \subset W_{i+1} \ldots \subset W_{n-1} \subset W$$

such that $\text{rank}(W_k) = k$, and $\text{deg}(W_k) = -d_k$, while $\text{rank}(W'_i) = i$, and $\text{deg}(W'_i) = -d_i - 1$. 
According to [10], 2.10, $E_{d,i}$ is a smooth projective algebraic variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim \mathcal{B} + 1$.

We denote by $p$ (resp. $q$) the natural projection $E_{d,i} \to \Omega_{d,i}$ (resp. $E_{d,i} \to \Omega_{d,i+1}$). We also have a map $r : E_{d,i} \to C$,

\[(0 \subset W_1 \subset \ldots \subset W_i \subset W'_i \subset W_i \subset W_{i+1} \subset \ldots \subset W_{n-1} \subset W) \mapsto \text{supp}(W_i/W'_i).
\]

The correspondence $E_{d,i}$ comes equipped with a natural line bundle $L_i$ whose fiber at a point

\[(0 \subset W_1 \subset \ldots \subset W_i \subset W'_i \subset W_i \subset W_{i+1} \subset \ldots \subset W_{n-1} \subset W)
\]

equals $\Gamma(C, W_i/W'_i)$.

Finally, we have a transposed correspondence $\tau E_{d,i} \subset \Omega_{d,i+1} \times \Omega_{d,i}$.

2.6. Restricting to $\Omega_{d,i} \subset \Omega_{d}$ we obtain the correspondence $E_{d,i} \subset \Omega_{d,i} \times \Omega_{d,i+1}$ together with line bundle $L_i$ and the natural maps $p : E_{d,i} \to \Omega_{d,i}$, $q : E_{d,i} \to \Omega_{d,i+1}$, $r : E_{d,i} \to C = \infty$.

We also have a transposed correspondence $\tau E_{d,i} \subset \Omega_{d,i+1} \times \Omega_{d,i}$. It is a smooth quasiprojective variety of dimension $2d_1 + \ldots + 2d_{n-1} + 1$.

2.7. We denote by $'M$ the direct sum of equivariant (complexified) $K$-groups: $'M = \oplus_2 K^{T \times C^*}(\Omega_{d,i})$. It is a module over $K^{T \times C^*}(pt) = \mathbb{C}[T \times C^*] = \mathbb{C}[t_1, \ldots, t_n, v : t_1 \cdots t_n = 1]$. We define $M = 'M \otimes K^{T \times C^*}(pt) \text{ Frac}(K^{T \times C^*}(pt))$.

We have an evident grading $M = \oplus_2 M_d$, $M_d = K^{T \times C^*}(\Omega_{d,i}) \otimes K^{T \times C^*}(pt) \text{ Frac}(K^{T \times C^*}(pt))$.

2.8. The grading and the correspondences $\tau E_{d,i}$, $E_{d,i}$ give rise to the following operators on $M$ (note that though $p$ is not proper, $p_*$ is well defined on the localized equivariant $K$-theory due to the finiteness of the fixed point sets):

\[K_i = t_{i+1}^{-1}v^{2d_i - d_i - d_i+1+1} : M_d \to M_d;\]
\[L_i = t_i^{-1} \cdots t_i^{-1}v^{d_i+1} : M_d \to M_d;\]
\[f_i = p_*, q^* : M_d \to M_{d-i};\]
\[F_i = t_i^{-1}v^{2d_i - d_i - d_i+1+1}p_*, q^* : M_d \to M_{d-i};\]
\[e_i = t_i^{-1}v^{d_i+1}q_{e_i}(L_i \otimes p^*) : M_d \to M_{d+i},\]
\[E_i = t_i^{-1}v^{d_i+1}q_{e_i}(L_i \otimes p^*) : M_d \to M_{d+i}.\]

2.9. We recall the notations and results of [10] in the special case of quantum group of $SL_n$ type.

$U$ is the $\mathbb{C}[v, v^{-1}]$-algebra with generators $E_i, L_i^{\pm 1}, K_i^{\pm 1}, F_i$, $1 \leq i \leq n - 1$, subject to the following relations:

(1) $L_i L_j = L_j L_i$, $K_i L_i = L_i K_i$, $K_i L_i = L_i K_i$, $K_{n-1} L_{n-1} = L_{n-1} K_{n-1}$

(2) $L_i E_j L_i^{-1} = v^{\delta_{i,j}} E_j$, $L_i F_j L_i^{-1} = v^{-\delta_{i,j}} F_j$

(3) $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}$

(4) $|i - j| > 1 \implies E_i E_j - E_j E_i = 0 = F_i F_j - F_j F_i$

(5) $|i - j| = 1 \implies E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 = F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2$
Sevostyanov considers elements $e_i, f_i \in U$ depending on a choice of $(n-1) \times (n-1)$-matrices $n_{ij}, c_{ij}$. We make the following choice:

\begin{align}
(6) & \quad n_{i,i} = -2i; \quad n_{i,i+1} = n_{i,i-1} = i, \\
(7) & \quad i < n - 1 \implies c_{i,i+1} = -1, \quad c_{i+1,i} = 1, \\
\end{align}

otherwise $n_{ij} = 0$.

\begin{align}
(8) & \quad f_i := L_i^{-1}L_i^{-2i}L_iF_i = K_i^{-1}F_i, \quad e_i := E_iL_i^{-1}L_i^{2i}L_i^{-1} = E_iK_i^{-1}.
\end{align}

Clearly, the algebra $U$ is generated by $e_i, L_i^{\pm 1}, K_i^{\pm 1}, f_i$, $1 \leq i \leq n - 1$, and the relations (6–5) above are equivalent to the relations (9)–(12) below.

\begin{align}
(9) & \quad L_i e_j L_i^{-1} = v^{\delta_{i,j}} e_j, \quad L_i f_j L_i^{-1} = v^{-\delta_{i,j}} f_j \\
(10) & \quad e_i f_j - v^{e_{ij}} f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}} \\
(11) & \quad |i - j| > 1 \implies e_i e_j - e_j e_i = 0 = f_i f_j - f_j f_i \\
(12) & \quad |i - j| = 1 \implies e_i^2 e_j - v^{e_{ij}} (v + v^{-1}) e_i e_j e_i + v^{2e_{ij}} e_j e_i^2 = 0 = f_i^2 f_j - v^{e_{ij}} (v + v^{-1}) f_i f_j f_i + v^{2e_{ij}} f_j f_i^2
\end{align}

2.10. **Remark.** The elements $f_i$ of the subalgebra $U_{\leq 0}$ generated by $F_1, \ldots, F_{n-1}, K_1, \ldots, K_{n-1}$ were introduced by C. M. Ringel in [15]. They are the natural generators of the Hall algebra of the $A_n$-quiver with the set of vertices $1, \ldots, n - 1$, and orientation $i \to i + 1$. More generally, Ringel’s construction works for an arbitrary orientation of an $ADE$ quiver, and produces Sevostyanov’s generators $f_i$ (in the simply laced case). It can be seen easily that the set of Sevostyanov’s matrices $c_{ij}$ (parametrizing the choices of his “Coxeter realizations”) is in a natural bijection with the set of orientations of the corresponding quiver.

2.11. We are finally able to formulate our main theorem. Recall the operators $E_i, e_i, L_i^{\pm 1}, K_i^{\pm 1}, F_i, f_i$ on $M$ defined in (2.8)

**Theorem 2.12.** The operators $E_i, L_i^{\pm 1}, K_i^{\pm 1}, F_i, 1 \leq i \leq n - 1$, on $M$ satisfy the relations (1)–(5). Equivalently, the operators $e_i, L_i^{\pm 1}, K_i^{\pm 1}, f_i, 1 \leq i \leq n - 1$, on $M$ satisfy the relations (6), (7)–(12).

The relations (1) and (2) are evident. The relation (3) for $i \neq j$ follows from a transversality property formulated in the next subsection.
2.13. We consider the subvarieties \( p_{12}^{-1}(\mathcal{E}_{d,i}) \) and \( p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) in \( \Omega_d \times \Omega_{d+i} \times \Omega_{d+i-j} \). Similarly, we consider the subvarieties \( p_{12}^{-1}(\tau \mathcal{E}_{d-j,i}) \) and \( p_{23}^{-1}(\mathcal{E}_{d-j,i}) \) in \( \Omega_d \times \Omega_{d-j} \times \Omega_{d+i-j} \).

**Lemma 2.14.** For \( i \neq j \) the intersection (a) \( p_{12}^{-1}(\mathcal{E}_{d,i}) \cap p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) in \( \Omega_d \times \Omega_{d+i} \times \Omega_{d+i-j} \) (resp. (b) \( p_{12}^{-1}(\tau \mathcal{E}_{d-j,i}) \cap p_{23}^{-1}(\mathcal{E}_{d-j,i}) \) in \( \Omega_d \times \Omega_{d-j} \times \Omega_{d+i-j} \)) is transversal.

(c) \( p_{12}^{-1}(\mathcal{E}_{d,i}) \cap p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \approx p_{12}^{-1}(\tau \mathcal{E}_{d-j,i}) \cap p_{23}^{-1}(\mathcal{E}_{d-j,i}) \).

**Proof.** We prove (a). By definition, \( p_{12}^{-1}(\mathcal{E}_{d,i}) \) is the moduli space of pairs of flags of prescribed ranks and degrees, while \( p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) is the moduli space of pairs of flags of prescribed ranks and degrees.

Their intersection is the moduli space of flags (say, \( i < j \))

\[
0 \subset W_1 = W_1'' \subset \ldots \subset W_i'' \subset W_i \subset \ldots \subset W_j = W_j'' \subset \ldots \subset W_j'''' \subset W_{n-1} = W_{n-1}'' \subset W
\]

of prescribed ranks and degrees which is smooth according to [10], 2.10. This implies that at any closed point of the scheme-theoretic intersection \( p_{12}^{-1}(\mathcal{E}_{d,i}) \cap p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) the Zariski tangent space to \( p_{12}^{-1}(\mathcal{E}_{d,i}) \cap p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) is the intersection of tangent spaces to \( p_{12}^{-1}(\mathcal{E}_{d,i}) \) and \( p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \).

Comparing the dimensions we conclude that the sum of tangent spaces to \( p_{12}^{-1}(\mathcal{E}_{d,i}) \) and \( p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) must coincide with the tangent space to \( \Omega_d \times \Omega_{d+i} \times \Omega_{d+i-j} \). Hence the intersection is transversal. This completes the proof of (a).

In (b) we prove similarly that \( p_{12}^{-1}(\tau \mathcal{E}_{d-j,i}) \cap p_{23}^{-1}(\mathcal{E}_{d-j,i}) \) is the moduli space of flags (say, \( i < j \))

\[
0 \subset W_1 = W_1''' \subset \ldots \subset W_i''' \subset W_i \subset \ldots \subset W_j = W_j''' \subset \ldots \subset W_j'''' \subset W_{n-1} = W_{n-1}''' \subset W
\]

of prescribed ranks and degrees which is smooth according to [10], 2.10. Hence the intersection is transversal by the same argument as in the proof of (a). This completes the proof of (b).

Part (c) was proved in [6], 3.6. We just recall that the mutually inverse isomorphisms send a triple \((W_*, W', W'')\) to \((W_*, W', W'')\) where \( W'' := W_0 + W''_0 \), and a triple \((W_*, W', W'')\) to \((W_*, W', W'')\) where \( W_0 := W_0 \cap W''_0 \).

2.15. We return to the proof of relation (3) for \( i \neq j \). The composition \( F_j E_i \) is given by the action of correspondence

\[
f(\hat{t})g(v)p_{13*}(p_{12}^* \mathcal{L}_i \otimes_{\mathcal{O}_{\Omega_d} \times \mathcal{O}_{\Omega_{d+i}} \times \mathcal{O}_{\Omega_{d+i-j}}} p_{23}^* \mathcal{O}_{\mathcal{E}_{d+i-j,j}})\]

where \( f \) (resp. \( g \)) is a certain monomial in \( \hat{t} \) (resp. \( v \)).

Because of the transversality in 2.14(a), \( p_{12}^* \mathcal{L}_i \otimes_{\mathcal{O}_{\Omega_d} \times \mathcal{O}_{\Omega_{d+i}} \times \mathcal{O}_{\Omega_{d+i-j}}} p_{23}^* \mathcal{O}_{\mathcal{E}_{d+i-j,j}} \) is a line bundle \( \mathcal{L}_{i,j} \) on \( p_{12}^{-1}(\mathcal{E}_{d,i}) \cap p_{23}^{-1}(\tau \mathcal{E}_{d+i-j,j}) \) whose fiber at a point \((W_*, W', W'')\) is equal to \( \Gamma(C, W_0/W''_0) \).

Similarly, due to the transversality in 2.14(b), the composition \( E_i F_j \) is given by the action of correspondence

\[
f'(\hat{t})g'(v)p_{13*}(\mathcal{L}_{i,j})\]
where $f'$ (resp. $g'$) is a certain monomial in $t$ (resp. $v$), and $E_{i,j}$ is a line bundle on $p_{23}^{-1}([1] \in \mathcal{P}_{23})$ whose fiber at a point $(W_i, W_j, W_k)$ is equal to $\Gamma(C, W_i/W_k)$.  

Now the isomorphism in (2.14) clearly takes $E_{i,j}$ to $E_{i',j'}$, and a routine check shows that $f'(t)g(v) = f'(t')g'(v)$.  This completes the proof of the relations (3) for $i \neq j$.

2.16. To prove the relation (2) for $i = j$ we use the localization to the fixed points.

According to the Thomason localization theorem (see e.g. [3]), restriction to the $T \times C^*$-fixed point set induces an isomorphism

$$K_{T \times C^*}(\mathcal{O}_d) \otimes K_{T \times C^*}(pt) \xrightarrow{\text{Frac}(K_{T \times C^*}(pt))} K_{T \times C^*}(\mathcal{O}_d) \otimes K_{T \times C^*}(pt) \xrightarrow{\text{Frac}(K_{T \times C^*}(pt))}$$

(resp.

$$K_{T \times C^*}(\mathcal{E}_d) \otimes K_{T \times C^*}(pt) \xrightarrow{\text{Frac}(K_{T \times C^*}(pt))} K_{T \times C^*}(\mathcal{E}_d) \otimes K_{T \times C^*}(pt) \xrightarrow{\text{Frac}(K_{T \times C^*}(pt))}$$

The classes of the structure sheaves $[\mathcal{O}_d]$ of the $T \times C^*$-fixed points $\mathcal{O}_d$ (see [2.14]) form a basis in $\oplus_2 K_{T \times C^*}([1] \in \mathcal{P}_{23}) \otimes K_{T \times C^*}(pt) \xrightarrow{\text{Frac}(K_{T \times C^*}(pt))}$.  In order to compute the matrix coefficients of $E_i, F_i$ in this basis, we have to know the character of the $T \times C^*$-action in the tangent spaces $T_{\mathcal{O}_d} \mathcal{O}_d$ and also in the tangent spaces to the fixed points in the correspondences.  This is the subject of the following Proposition.

2.17. Note that a point $(\mathbf{d}, \mathbf{d}')$ lies in the correspondence $\mathcal{O}_d$ if and only if $d_{k,j} = d_{k,j}'$ with a single exception $d_{i,j}' = d_{i,j} + 1$ for certain $j \leq i$.

**Proposition 2.18.** a) The character $\chi_{\mathcal{O}_d}$ of $T \times C^*$ in the tangent space $T_{\mathcal{O}_d}$ equals

$$\sum_{1 \leq j < k \leq n} t_k^2 t_j^2 - \sum_{i=1}^{n-1} t_k^2 t_i^2 \sum_{i=1, k \leq n-1}^{n-1} v^{2i} \sum_{i=1, k \leq n-1}^{n-1} v^{2i}$$

where we set $d_{n,k} = 0$.

b) The character $\chi_{\mathcal{E}_d}$ of $T \times C^*$ in the tangent space $T_{\mathcal{E}_d}$ equals

$$\chi_{\mathcal{E}_d} = \sum_{1 \leq k \leq i \leq n} t_k^2 t_j^2 - v^{2d_{i,k}-2d_{i,j}} - \sum_{k \leq i-1} t_k^2 t_j^2 v^{2d_{i,k}-2d_{i,j}}$$

if $d_{i,j}' = d_{i,j} + 1$ for certain $j \leq i$.

c) The character $\chi_{\mathcal{E}_d}$ of $T \times C^*$ in the fiber of $\mathcal{E}_d$ at the point $(\mathbf{d}, \mathbf{d}')$ equals $t_j^2 v^{-2d_{i,j}}$ if $d_{i,j}' = d_{i,j} + 1$.

**Proof.** Let $\mathcal{O}$ be the moduli space of flags of locally free subsheaves

$$\emptyset \subset W_1 \subset W_2 \subset \ldots \subset W_r \subset W$$

of fixed ranks.  Then the tangent space $T_{\mathcal{O}_{W_1}} \mathcal{O}$ equals the kernel of

$$\sum_{1 \leq l < r} p_{l-1} \otimes \text{Id} - \text{Id} \otimes q_l : \oplus_1 \text{Hom}(W_l, W/W_l) \rightarrow \oplus_1 \text{Hom}(W_l, W/W_{l+1})$$

where $p_l : W_l \rightarrow W_{l+1}$; $q_l : W/W_l \rightarrow W/W_{l+1}$ (see e.g. [2], 3.2).

Now the parts a), b) follow easily from the obvious equalities $\text{ch}(\text{Hom}(\mathcal{O}_C(-a), \mathcal{O}_C)) = \sum_{a=0}^a v^{2c}$ and $\text{ch}(\text{Hom}(\mathcal{O}_C(-a), \mathcal{O}_C(-b_1)/\mathcal{O}_C(-b_2))) = \sum_{a-b_1}^{a-b_2} v^{2c}$.  The part c) is obvious.  

$\square$
2.19. Let us denote by $S_{\chi_{\tilde{d}}} = \Lambda^{-1} \chi_{\tilde{d}}$ (resp. $S_{\chi_{\tilde{d}}'} = \Lambda^{-1} \chi_{\tilde{d}}'$) the character of $\tilde{T} \times \mathbb{C}^*$ in the symmetric algebra $\text{Sym} \tilde{T} \mathbb{C}^* \tilde{d}$ (resp. $\text{Sym}^* \tilde{T} \mathbb{C}^* \tilde{d}$). It is the inverse of the character of the corresponding exterior algebra, thus it lies in the fraction field $\text{Frac}(K^\tilde{T} \times \mathbb{C}^*(pt))$.

According to the Bott-Lefschetz fixed point formula, the matrix coefficient $p_\star q_\star$ of $p_\star q_\star^*$ : $M_d \to M_d'$ with respect to the basis elements $[\tilde{d}] \in K^\tilde{T} \times \mathbb{C}^*(\mathbb{Q}_d)$, $[\tilde{d}'] \in K^\tilde{T} \times \mathbb{C}^*(\mathbb{Q}_d)$ (see 2.10) equals $S_{\chi_{\tilde{d}}} / \chi_{\tilde{d}}$. Similarly, the matrix coefficient $q_\star (\mathfrak{L} \otimes p^*)_{\tilde{T} \mathbb{C}^* \tilde{d}}$ of $q_\star (\mathfrak{L} \otimes p^*)$ : $M_d \to M_d'$ equals $\lambda_{\tilde{T} \mathbb{C}^* \tilde{d}} S_{\chi_{\tilde{d}}} / \chi_{\tilde{d}}$.

Hence, the matrix coefficient $E_{ij}$ of $E_i : M_d \to M_d'$ equals
\[ -t_{i+1}^{-1} t_i^{-1} v_{(i-1)d_i-1+(i+1)d_i+1-2d_i-i} \sum_{j \neq k \leq i} \left( 1 - t_j^2 v_{2d_i+k-2d_i-j} \right)^{-1} \prod_{k \leq i-1} \left( 1 - t_j^2 v_{2d_i+k-2d_i-j} \right) \]
if $d_{i,j}' = d_{i,j} + 1$ for certain $j \leq i$;

\[ t_i^{-1} t_{i+1}^{-1} t_i^{-1} v^{d_i-1-2d_i-i+1} \left( 1 - t_j^2 v_{2d_i,j-2d_i,i+1} \right)^{-1} \prod_{j \neq k \leq i} \left( 1 - t_j^2 v_{2d_i,j-2d_i,k} \right)^{-1} \prod_{k \leq i+1} \left( 1 - t_j^2 v_{2d_i,j-2d_i,k} \right) \]
if $d_{i,j}' = d_{i,j} - 1$ for certain $j \leq i$;

All the other matrix coefficients of $E_i, E_i$ vanish.

Now the relation $E_i$ boils down to the following identity.

**Proposition 2.21.**
\[ t_i^{-1} t_{i+1}^{-1} v^{d_i-1-2d_i,d_i+1} \frac{1}{v-1} (1 - v^2)^{2} v^{d_i-1-2d_i,i+1} t_i t_{i+1} = \]
\[ \sum_{j \leq i} t_j^2 v^{-2d_i,j} \left( 1 - t_j^2 v_{2d_i,j-2d_i,i+1} \right) \left( 1 - t_j^2 v_{2d_i,j-2d_i,i+1} \right) \times \]
\[ \prod_{k \leq j} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right)^{-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right)^{-1} \times \]
\[ \prod_{k \leq i-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right)^{-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right)^{-1} \times \]
\[ \sum_{j \leq i} t_j^2 v^{-2d_i,j} \left( 1 - t_j^2 v_{2d_i,j-2d_i,i+1} + 2 \right) \left( 1 - t_j^2 v_{2d_i,j-2d_i,i+1} + 2 \right) \times \]
\[ \prod_{k \leq j} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} + 2 \right)^{-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} + 2 \right)^{-1} \times \]
\[ \prod_{k \leq i-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} + 2 \right)^{-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} + 2 \right)^{-1} \times \]
\[ \prod_{k \leq i-1} \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right) \left( 1 - t_k^2 v_{2d_i,j-2d_i,k} \right). \]
Proof. We introduce the new variables \( q := v^2; \) \( s_j := t_j^2 v^{-2d_{ij}}, 1 \leq j \leq i; \) \( r_k := t_k^2 v^{-2d_{ik}+1}, 1 \leq k \leq i+1; \) \( p_k := t_k^2 v^{-2d_{ik}-1}, 1 \leq k \leq i-1. \) Then the LHS of (2.21) equals

\[
(1 - q) \left( q \prod_{k=1}^{i+1} r_k \prod_{j=1}^{i} s_j^{-1} - \prod_{j=1}^{i} s_j \prod_{k=1}^{i} p_k^{-1} \right)
\]

while the RHS of (2.21) equals

\[
\prod_{j=1}^{i} s_j \prod_{k=1}^{i} p_k^{-1} \left( q \sum_{j=1}^{i} s_{j-1} \prod_{k=1}^{i} (p_k - q s_j) \prod_{k \leq i} (s_k - q s_j)^{-1} \right)
\]

Dividing both the LHS and the RHS by \( \prod_{j=1}^{i} s_j \prod_{k=1}^{i} p_k^{-1} \) we arrive at

\[
(1 - q) \left( q \prod_{j=1}^{i} s_{j-1} \prod_{k=1}^{i} p_k \prod_{j=1}^{i+1} r_k - 1 \right) =
\]

\[
q \sum_{j=1}^{i} s_{j-1} \prod_{k=1}^{i} (p_k - q s_j) \prod_{k \leq i} (s_k - q s_j)^{-1} -
\]

\[
\sum_{j=1}^{i} s_{j-1} \prod_{k=1}^{i} (p_k - s_j) \prod_{k \leq i} (s_k - s_j)^{-1}.
\]

If we subtract the LHS from the RHS we obtain a rational expression in \( s_j \) of degree 0, that is, the degree of numerator is not bigger than the degree of denominator. We see easily that as \( s_j \) tends to \( \infty \), the difference of the RHS and the LHS tends to 0. The possible poles of the difference can occur at \( s_j = 0, s_j = s_k, s_j = q s_k, s_j = q^{-1} s_k. \) We see easily that the principal parts of the difference at these points vanish. We conclude that the difference is identically 0. This completes the proof of the Proposition. \( \Box \)

2.22. To finish the proof of relation (3) we note that the commutator correspondence \( E_1 F_i - F_i E_i \) is concentrated on the diagonal of \( \mathcal{Q}_d \times \mathcal{Q}_d. \) This is proved exactly as in Lemma 2.14.

In other words, \( E_1 F_i - F_i E_i \) is given by tensor product \( q \mapsto L \otimes X_i \) for certain \( X_i \in M_2. \) This means that in the basis \( \partial \) the operator \( E_1 F_i - F_i E_i \) is diagonal. Now the Proposition 2.21 computes the matrix coefficient \( (E_1 F_i - F_i E_i)_{\mathcal{Q}_d, \mathcal{Q}_d} \) and proves that it equals \( K_1 - K_0 \) \( |_{M_2}. \) This completes the proof of the relation (3).

2.23. Alternatively, the relation (3) follows from the next Conjecture. We consider a 2-dimensional vector space with a basis \( \mathfrak{w}_1, \mathfrak{w}_2. \) Let \( T \) be a torus acting on \( \mathfrak{w}_1 \) (resp. \( \mathfrak{w}_2 \)) via a character \( \tau^2_1 \) (resp. \( \tau^2_2 \)). Let \( \mathfrak{B}_1, \mathfrak{B}_2 \) be the moduli stack of flags of coherent sheaves \( \mathfrak{M}_1 \subset \mathfrak{M}_2 \) on \( \mathbb{C} \) locally free at \( \infty \in \mathbb{C}, \) equipped with a trivialization \( \mathfrak{M}_1|_\infty = (\mathfrak{w}_1), \mathfrak{M}_2|_\infty = (\mathfrak{w}_1, \mathfrak{w}_2), \) and such that deg \( \mathfrak{M}_1 = -\partial_1, \) deg \( \mathfrak{M}_2/\mathfrak{M}_1 = -\partial_2. \) We have a natural correspondence \( \mathfrak{E}_1 \subset \mathfrak{B}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_1, \mathfrak{B}_2 \) formed by the pairs \( (\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_1', \mathfrak{M}_2') \) such that \( \mathfrak{M}_1' \subset \mathfrak{M}_1 \subset \mathfrak{M}_2 = \mathfrak{M}_2'. \) The projection \( \mathfrak{E}_1 \rightarrow \mathfrak{B}_1 \) (resp. \( \mathfrak{E}_2 \rightarrow \mathfrak{B}_2 \)) is denoted by \( p \) (resp. \( q \)). Finally, \( \mathfrak{E}_1 \) is equipped with the line bundle \( \mathfrak{L}_0, \) whose fiber at the point \( (\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_1', \mathfrak{M}_2') \) equals \( \Gamma(C, \mathfrak{M}_1/\mathfrak{M}_1'). \)
The stack $\mathfrak{Z}_{\mathfrak{a}_{1}, \mathfrak{a}_{2}}$ is smooth, and acted upon by $\mathbb{T} \times \mathbb{C}^*$. So it makes sense to consider the operators

\[ f := p_\ast q^*: \]

\[ K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}, \mathfrak{a}_{2}}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)) \rightarrow K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}-1, \mathfrak{a}_{2}+1}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)), \]

\[ e := -\tau_{1}^{-1}\tau_{2}^{-1}v^{\mathfrak{a}_{1}}\mathfrak{p}_{\mathfrak{a}_{1}}(\mathfrak{Q}_{\mathfrak{a}_{1}} \otimes p^*): \]

\[ K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}, \mathfrak{a}_{2}}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)) \rightarrow K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}+1, \mathfrak{a}_{2}-1}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)), \]

\[ K = \tau_{1}^{-1}\tau_{2}v^{\mathfrak{a}_{1}}b_{2+1}^2: \]

\[ K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}, \mathfrak{a}_{2}}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)) \rightarrow K^{\mathbb{T} \times \mathbb{C}^*}(\mathfrak{Z}_{\mathfrak{a}_{1}, \mathfrak{a}_{2}}) \otimes K^{\mathbb{T} \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\mathbb{T} \times \mathbb{C}^*}(pt)) \]

**Conjecture 2.24.** $ef - fe = \frac{K-K^{-1}}{K^{-1}}$.

**2.25.** To derive the relation (12), or equivalently, (13) for $j = i$ from Conjecture 2.24 we consider the map

\[ \mathfrak{Z}_{\mathfrak{d}}: \Omega_{\mathfrak{d}} \rightarrow \mathfrak{Z}_{d_{i-1}d_{i+1}d_{i}} \text{, } \mathfrak{W}_{\mathfrak{e}} \mapsto (\mathfrak{W}_{i}/\mathfrak{W}_{i-1}, \mathfrak{W}_{i+1}/\mathfrak{W}_{i-1}). \]

Then we have

\[ (\Omega_{\mathfrak{d}} \times \mathfrak{Z}_{d_{i-1}d_{i+1}d_{i}}) \times \mathfrak{Z}_{d_{i-1}d_{i}d_{i+1}d_{i}} \times \mathfrak{Z}_{d_{i-1}d_{i+1}d_{i}} \mathfrak{E}_{d_{i}-d_{i-1}} = \mathfrak{E}_{d_{i}} \subset \Omega_{\mathfrak{d}} \times \Omega_{\mathfrak{d}+i}. \]

We also have the natural maps

\[ \epsilon_{d,i}: \mathfrak{E}_{d_{i}} \rightarrow \mathfrak{E}_{d_{i-1}}, \]

\[ \tau_{d,i}: \tau \mathfrak{E}_{d_{i}} \rightarrow \tau \mathfrak{E}_{d_{i-1}}, \]

\[ h_{d,i}: \mathfrak{E}_{d_{i}} \otimes \tau \mathfrak{E}_{d_{i}} \rightarrow \mathfrak{E}_{d_{i}-d_{i-1}} \otimes \tau \mathfrak{E}_{d_{i}-d_{i-1}}, \]

\[ \dot{h}_{d,i}: \tau \mathfrak{E}_{d_{i}} \otimes \mathfrak{E}_{d_{i}} \rightarrow \tau \mathfrak{E}_{d_{i}-d_{i-1}} \otimes \mathfrak{E}_{d_{i}-d_{i-1}}. \]

We may consider $e_{i}$ (resp. $f_{i}, e, f$) as an element of $K^{\tau \times \mathbb{C}^*}(\mathfrak{E}_{d_{i}})$ (resp. $K^{\tau \times \mathbb{C}^*}(\tau \mathfrak{E}_{d_{i}})$, $K^{\tau \times \mathbb{C}^*}(\mathfrak{E}_{d_{i}-d_{i-1}})$, $K^{\tau \times \mathbb{C}^*}(\tau \mathfrak{E}_{d_{i}-d_{i-1}})$). We evidently have

\[ e_{d,i}^\ast e = e_{i}, \tau_{d,i}^\ast f = f_{i}. \]

Moreover, according to [12], 8.2 (Restriction of the convolution to submanifolds), we have

\[ (13) \quad h_{d,i}^\ast (e * f) = e_{i} * f_{i}, \dot{h}_{d,i}^\ast (f * e) = f_{i} * e_{i}. \]

We already know from the argument in 2.22 that the correspondence $e_{i} * f_{i} - f_{i} * e_{i}$ acts as tensor multiplication with a certain class $X_{i} \in M_{\mathfrak{d}}$. Similarly, the correspondence $e * f - f * e$ acts in $K^{\tau \times \mathbb{C}^*}(\mathfrak{Z}_{d_{i-1}d_{i+1}d_{i}}) \otimes K^{\tau \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\tau \times \mathbb{C}^*}(pt))$ as tensor multiplication with a certain class $\mathfrak{X} \in K^{\tau \times \mathbb{C}^*}(\mathfrak{Z}_{d_{i-1}d_{i+1}d_{i}}) \otimes K^{\tau \times \mathbb{C}^*}(pt) \text{ Frac}(K^{\tau \times \mathbb{C}^*}(pt))$. By (13) we must have $X_{i} = \delta_{\mathfrak{d}}^{\ast} \mathfrak{X}$. Thus the relation (13) for $j = i$ follows from Conjecture 2.24.
2.26. To complete the proof of Theorem 2.12 it remains to check the relations (1), (3). To this end we consider the algebra $\tilde{U}$ given by the generators $E_i, L_i^{\pm 1}, K_i^{\pm 1}, F_i, 1 \leq i \leq n - 1$, and the relations (1)–(3). Thus, $U$ is the quotient of $\tilde{U}$ by the Serre relations.

We extend the scalars to $\text{Frac}(\mathbb{C}[L_i^{\pm 1}, \ldots, L_{n-1}^{\pm 1}])$; we set

$$U' = U \otimes \text{Frac}(\mathbb{C}[L_i^{\pm 1}, \ldots, L_{n-1}^{\pm 1}]), \quad \tilde{U}' = \tilde{U} \otimes \text{Frac}(\mathbb{C}[L_i^{\pm 1}, \ldots, L_{n-1}^{\pm 1}])$$

Note that $\tilde{U}'$ acts in $M$, so $U'$ acts in the quotient $\overline{M}$ of $M$ by the two-sided ideal $I$ in $\tilde{U}'$ generated by the Serre relations. So it suffices to check that $\overline{M} = M$, or equivalently, $3M = 0$.

Now $M$ has the size of the universal Verma module over $U'$ which is an irreducible $U'$- (and $\tilde{U}'$-) module. In effect, a bijection between the set $\{[\underline{a}]\}$, and the set of Kostant partitions for $\mathfrak{sl}_n$ is defined e.g. in [6, 2.1.1]. Hence we only have to check that $3M = 0$. But any element $x \in I$ of principal grading degree 0 annihilates the lowest weight vector $[(0, \ldots, 0)]$ of $M$ since we may shift the generators $e_i$ in the expression of $x$ to the right.

This completes the proof of the Serre relations in $M$ along with the proof of Theorem 2.12.

2.27. Remark. (A. Joseph) We have constructed a basis $\{[\underline{a}]\}$ in the universal Verma module $M$ over $U$. Though we can not identify it with any known type of basis, the parametrization of this basis coincides with the polyhedral realization of the crystal base of $U^+_p(\mathfrak{sl}_n)$ corresponding to the reduced expression in the Weyl group of $SL_n$:

$$w_0 = s_{n-1}s_{n-2} \cdots s_1s_{n-1}s_{n-2} \cdots s_{n-1}s_{n-2}s_{n-1}$$

(see [13]).

2.28. Recall that the universal Verma module $M$ over $U$ is equipped with the symmetric Shapovalov form $\langle , \rangle$ with values in $\text{Frac}(\mathbb{C}[T \times T^*])$. It is characterized by the properties

(a) $\langle [\underline{a}], [\underline{a}] \rangle = 1$ where $[\underline{a}] = [(0, \ldots, 0)]$ is the lowest weight vector;
(b) $\langle E_i x, y \rangle = (x, F_i y) \quad \forall x, y \in M$.

We will write down a geometric expression for the Shapovalov form. Evidently, the different weight spaces of $M$ are orthogonal with respect to the Shapovalov form. So it suffices to check that $\overline{M} = M$, or equivalently, $3M = 0$.

Now $M$ has the size of the universal Verma module over $U'$ which is an irreducible $U'$- (and $\tilde{U}'$-) module. In effect, a bijection between the set $\{[\underline{a}]\}$, and the set of Kostant partitions for $\mathfrak{sl}_n$ is defined e.g. in [6, 2.1.1]. Hence we only have to check that $3M = 0$. But any element $x \in I$ of principal grading degree 0 annihilates the lowest weight vector $[(0, \ldots, 0)]$ of $M$ since we may shift the generators $e_i$ in the expression of $x$ to the right.

This completes the proof of the Serre relations in $M$ along with the proof of Theorem 2.12.

2.29. Proposition. For $\mathfrak{g}_1, \mathfrak{g}_2 \in M_{\underline{d}}$ we have

$$\langle \mathfrak{g}_1, \mathfrak{g}_2 \rangle = (-1)^{\sum_{i=1}^{d-1} \sum_{i=1}^{n-1} 2i - \sum_{i=2}^{n-1} 2(i-1)d_i} \prod_{i=1}^{n} t_i^{2(2i-1)(d_i-1)-d_i} |R\Gamma(\Omega_{\underline{d}}, \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathcal{D}_{\underline{d}})|$$

Proof. Since $\text{det} R\Gamma$ is multiplicative in short exact sequences, we have an equality of line bundles on the correspondence $\mathfrak{e}_{\underline{d}}, \mathfrak{p}^* \mathcal{D}_{\underline{d}} = \mathfrak{q}^* \mathcal{D}_{\underline{d}} \otimes \mathfrak{L}_i$. Now the projection formula shows that the operators $\mathfrak{p}, \mathfrak{q}^*$ and $\mathfrak{q}_i (\mathfrak{L}_i \otimes \mathfrak{p}^*)$ are adjoint with respect to the pairing $\langle \mathfrak{g}_1, \mathfrak{g}_2 \rangle \mapsto R\Gamma(\Omega_{\underline{d}}, \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathcal{D}_{\underline{d}})$. Finally, it is easy to see that the $v, t$-factor takes care of the scaling coefficients of our $E_i, F_i$.

2.30. While the operators $E_i, F_i$ are conjugate to each other with respect to the Shapovalov form, the operators $e_i, f_i$ are not. In fact, obviously, $e_i = K_i f_i$. It is known that a completion of the universal Verma module $M$ contains a unique vector $\mathfrak{t} = \sum_{\underline{d}} \mathfrak{t}_{\underline{d}}$ (resp. $\mathfrak{w} = \sum_{\underline{d}} \mathfrak{w}_{\underline{d}}$) such that $\mathfrak{t}_{(0, \ldots, 0)} = \mathfrak{w}_{(0, \ldots, 0)} = [(0, \ldots, 0)]$, and $f_i \mathfrak{t} = (1 - v^2)^{-1} \mathfrak{t}$ (resp. $e_i^* \mathfrak{w} = (1 - v^2)^{-1} \mathfrak{w}$) for any $i$ (the Whittaker vectors).

The following proposition gives a geometric construction of the Whittaker vectors $\mathfrak{t}, \mathfrak{w} \in M$.:
Proposition 2.31. a) \( \mathfrak{t}_d = [\mathcal{O}_d] \) (the class of the structure sheaf of \( \Omega_d \)); 
b) \( \mathfrak{w}_d = v \sum_{i=1}^{n-1} (1-2i) d_i^2 - \sum_{i=2}^{n-1} (2-2i) d_{i-1} - \sum_{i=1}^{n-1} d_i \prod_{i=1}^{n} t_i (2-2i)(d_{i-1} - d_i)[\mathcal{D}_d^{-1}] \).

Proof. a) We have \( q^* \mathcal{O}_d = 0 \mathfrak{e}_d \). Furthermore, since \( p \times r : \mathfrak{e}_{d,i} \to \Omega_d \times (\mathbf{C} - \infty) \) is proper and birational, and both the source and the target are smooth, we have \( (p \times r)_*[\mathcal{O}_{d,i}] = [\mathcal{O}_d] \boxtimes [\mathcal{O}_{\mathbf{C} - \infty}] \). In effect, \( (p \times r)_*[\mathcal{O}_{d,i}] = \mathcal{O}_d \boxtimes \mathcal{O}_{\mathbf{C} - \infty} \), and the higher direct images \( R^\geq 0 (p \times r)_* \mathfrak{e}_{d,i} \) vanish. Finally, \( p_* [\mathcal{O}_d \boxtimes \mathcal{O}_{\mathbf{C} - \infty}] = (1 - v^2)^{-1} [\mathcal{O}_d] \) where \( p_* : \Omega_d \times (\mathbf{C} - \infty) \to \Omega_d \) is the projection to the first factor.

b) Recall that \( e_i^* = K_i^2 f_i \). Thus we have to check that \( f_i[\mathcal{D}_d^{-1}] = t_i^2 v^{2d_{i-1} - 2d_i}(1 - v^2)^{-1}[\mathcal{D}_d^{-1}] \). Furthermore, recall that on \( \mathfrak{e}_{d,i} \) we have a canonical isomorphism \( q^* \mathcal{D}_d^{-1} = \mathcal{E}_i \boxtimes p^* \mathcal{D}_d^{-1} \). By the projection formula we are reduced to

\[
(14) \quad p_*[\mathcal{E}_i] = t_i^2 v^{2d_{i-1} - 2d_i}(1 - v^2)^{-1}[\mathcal{O}_d]
\]

This can be calculated in the basis \([d]\) where we already know the matrix coefficients of our operators (see Corollary 2.20). More precisely, by the Bott-Lefschetz fixed point formula, we have to check

\[
\sum_{j \leq i} t_j^2 v^{-2d_{j,1}} (1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - t_j q_k^2 v^{2d_{k,1} - 2d_{j,1}})^{-1} \prod_{k \leq i - 1} (1 - t_k^2 q_k^2 v^{2d_{i-1,1} - 2d_{i,1}}) =
\]

\[
= t_i^2 v^{2d_{i-1} - 2d_i}(1 - v^2)^{-1}
\]

Recall the change of variables we used in the proof of Proposition 2.21 \( s_j := t_j^2 v^{-2d_{j,1}}, 1 \leq j \leq i; \ p_k := t_k^2 v^{2d_{i-1,1}}, 1 \leq k \leq i - 1. \) Then we have to prove

\[
\sum_{j \leq i} s_j \prod_{k \leq i} (1 - s_j s_k^{-1})^{-1} \prod_{k \leq i - 1} (1 - s_j p_k^{-1}) = s_1 \cdots s_i p_1^{-1} \cdots p_{i-1}^{-1}
\]

This follows immediately from the well known identity

\[
\sum_{j \leq i} \prod_{k \leq i - 1} (p_k - s_j) \prod_{k \leq i} (s_k - s_j)^{-1} = 1.
\]

This completes the proof of the Proposition. \( \square \)

Corollary 2.32. The Shapovalov scalar product of the Whittaker vectors equals \( (\mathfrak{t}_d, \mathfrak{w}_d) = (-1)^{\sum_{i=1}^{n-1} d_i \sum_{i=2}^{n-1} d_{i-1} - \sum_{i=1}^{n-1} d_i \prod_{i=1}^{n} t_i (2-2i)(d_{i-1} - d_i)} [RT(\mathfrak{t}_d, \mathfrak{w}_d)] \).

2.33. According to the works \([4, 17]\), the appropriate generating function of the Shapovalov scalar product of the Whittaker vectors satisfies a \( v \)-deformed (\( v \)-difference) version of the quantum Toda lattice equations. Let us recall the required notations and results.

We introduce the formal variables \( z_1, \ldots, z_n \), and we set \( Q_i = \exp(z_i - z_{i+1}), \ i = 1, \ldots, n - 1. \) We set \( \hbar = \log(v) \), so that \( v = \exp(\hbar) \). We introduce the shift operators \( T_i, \ i = 1, \ldots, n, \) acting on the space of functions of \( z_1, \ldots, z_n \) invariant with respect to the simultaneous translations \( f(z_1, \ldots, z_n) = f(z_1 + z, \ldots, z_n + z) \). Namely, we set \( T_i f(z_1, \ldots, z_n) = f(z_1, \ldots, z_i + \hbar, \ldots, z_n) \).

We define the following \( v \)-difference operators:

\[
(15) \quad \mathfrak{S} := \sum_{j=1}^{n} T_j^2 + v^{-2} \sum_{i=1}^{n-1} Q_i T_i T_{i+1}
\]

\[
(16) \quad \mathfrak{S} := T_1^2 + T_2^2 (1 - Q_1) + \ldots + T_n^2 (1 - Q_{n-1})
\]
We also consider the following generating functions:

\[ J := \prod_{i=1}^{n-1} Q_i^{\log(t_i - t_{i+1})} \sum_d (t_d, m_d) Q_1^{d_1} \cdots Q_{n-1}^{d_{n-1}} \]  
(17)  
\[ \mathcal{G} := \prod_{i=1}^{n-1} Q_i^{\log(t_i - t_{i+1})} \sum_d [\mathcal{R} \Gamma(\Omega_{d}, \mathcal{O}_{d})] Q_1^{d_1} \cdots Q_{n-1}^{d_{n-1}} \]  
(18)  

Then according to the last formula of (17) (or equivalently, the formula (5.7) of [4]), we have

\[ \mathcal{G} J = \left( \sum_{i=1}^{n} t_i^2 \right) J \]  
(19)  

In effect, the seeming discrepancy between the formula (15) above, and the formula (5.7) of [4] is explained by the fact that (a) our \( v \) corresponds to \( q \) of [4]; (b) our Whittaker vectors have eigenvalue \( (1 - v^2)^{-1} \), whereas the Whittaker vectors of [4] have eigenvalue 1, which takes care of the factor \( (q - q^{-1})^2 \) in the second summand of the formula (5.7) of [4].

Now the argument of [4], section 6 (see the formula (6.5)) together with Corollary 2.3.2 establishes

\[ \mathcal{G} J = \left( \sum_{i=1}^{n} t_i^2 \right) J \]  
(20)  

thus reproving the Main Theorem 2 of [7].

3. PARABOLIC SHEAVES AND AFFINE QUANTUM GROUPS

In this section we want to generalize the previous results to the affine setting.

3.1. Parabolic sheaves. We recall the setup of [5]. Let \( X \) be another smooth projective curve of genus zero. We fix a coordinate \( x \) on \( X \), and consider the action of \( \mathbb{C}^* \) on \( X \) such that \( u(x) = u^{-2}x \). We have \( X^{\mathbb{C}^*} = \{0, \infty_X \} \). Let \( S \) denote the product surface \( \mathbb{C} \times X \). Let \( D_{\infty} \) denote the divisor \( \mathbb{C} \times X \cup \{\infty\} \times X \). Let \( D_0 \) denote the divisor \( \mathbb{C} \times 0 \).

Given an \( n \)-tuple of nonnegative integers \( \underline{d} = (d_0, \ldots, d_{n-1}) \), we say that a parabolic sheaf \( \mathcal{F}_\bullet \) of degree \( \underline{d} \) is an infinite flag of torsion free coherent sheaves of rank \( n \) on \( S : \ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \) such that:

(a) \( \mathcal{F}_{k+n} = \mathcal{F}_k(D_0) \) for any \( k \);
(b) \( ch_1(\mathcal{F}_k) = k[D_0] \) for any \( k \): the first Chern classes are proportional to the fundamental class of \( D_0 \);
(c) \( ch_2(\mathcal{F}_k) = d_i \) for \( i \equiv k \) (mod \( n \));
(d) \( \mathcal{F}_0 \) is locally free at \( D_{\infty} \) and trivialized at \( D_{\infty} : \mathcal{F}_0|D_\infty = W \otimes \mathcal{O}_{D_\infty} \);
(e) For \( -n \leq k \leq 0 \) the sheaf \( \mathcal{F}_k \) is locally free at \( D_\infty \), and the quotient sheaves \( \mathcal{F}_k/\mathcal{F}_{-n} \), \( \mathcal{F}_0/\mathcal{F}_k \) (both supported at \( D_0 = \mathbb{C} \times 0 \subset S \)) are both locally free at the point \( \infty_X \); moreover, the local sections of \( \mathcal{F}_k|_{\mathbb{C} \times X} \) are those sections of \( \mathcal{F}_0|_{\mathbb{C} \times X} = W \otimes \mathcal{O}_X \) which take value in \( \langle w_1, \ldots, w_{n-k} \rangle \subset W \) at \( \infty_X \in X \).

According to [5], 3.5, the fine moduli space \( \mathcal{P}_{d} \) of degree \( d \) parabolic sheaves exists and is a smooth connected quasi-projective variety of dimension \( 2d_0 + \ldots + 2d_{n-1} \).

The group \( \tilde{T} \times \mathbb{C}^* \times \mathbb{C}^* \) acts naturally on \( \mathcal{P}_{d} \), and its fixed point set is finite.
3.2. Correspondences. If the collections $d$ and $d'$ differ at the only place $i \in I := \mathbb{Z}/n\mathbb{Z}$, and $d'_i = d_i + 1$, then we consider a correspondence $E_{d,i} \in \mathcal{P}_d \times \mathcal{P}_{d'}$ formed by the pairs $(\mathcal{F}_i, \mathcal{F}'_i)$ such that for $j \neq i \ (\text{mod} \ n)$ we have $\mathcal{F}_j = \mathcal{F}'_j$, and for $j \equiv i \ (\text{mod} \ n)$ we have $\mathcal{F}'_j \subset \mathcal{F}_j$.

It is a smooth quasiprojective algebraic variety of dimension $2\sum_{i \in I} d_i + 1$. In effect, the argument of [9], Lemma 3.3, reduces this statement to the corresponding fact about Laumon correspondences (see [10], 2.10).

We denote by $p$ (resp. $q$) the natural projection $E_{d,i} \to \mathcal{P}_d$ (resp. $E_{d,i} \to \mathcal{P}_{d'}$). For $j \equiv i \ (\text{mod} \ n)$ the correspondence $E_{d,i}$ is equipped with a natural line bundle $L_j$ whose fiber at $(\mathcal{F}_i, \mathcal{F}'_i)$ equals $\Gamma(C_i, \mathcal{F}_j - n/\mathcal{F}'_j - n)$. Finally, we have a transposed correspondence $\mathcal{F}'_{i} E_{d,i} \subset \mathcal{P}_{d'} \times \mathcal{P}_d$.

3.3. We denote by $\mathcal{M}$ the direct sum of equivariant (complexified) $K$-groups:

$$\mathcal{M} = \bigoplus d K^T \times C^* \times C^*(\mathcal{P}_d).$$

It is a module over $K^T \times C^* \times C^*(pt) = \mathbb{C}[T \times C^* \times C^*] = \mathbb{C}[t_1, \ldots, t_n, v, u : t_1 \cdots t_n = 1]$. We define $M = \mathcal{M} \otimes K^T \times C^* \times C^*(pt). \text{Frac}(K^T \times C^* \times C^*(pt))$.

We have an evident grading $M = \bigoplus d M_d$, $M_d = K^T \times C^* \times C^*(\mathcal{P}_d) \otimes K^T \times C^* \times C^*(pt)$ Frac$(K^T \times C^* \times C^*(pt))$.

3.4. The grading and the correspondences $\mathcal{F}'_{i} E_{d,i}, E_{d,i}$ give rise to the following operators on $M$ (note that though $p$ is not proper, $p_*$ is well defined on the localized equivariant $K$-theory due to the finiteness of the fixed point sets):

$$K_i = t_i + t_i^{-1} v^{d_0, i} v^{d_1, d_2, -d_3, -d_4, \ldots, -d_i, \ldots, i+1} : M_d \to M_d,$$

$$C = uv^b,$$

For $i = 0, \ldots, n - 1$ we define $L_i = t_i^{-1} \cdots t_i^{-1} v^{d_i, \ldots, d_0} \delta_i (n-i) : M_d \to M_d$ (that is, $L_0 = v^{d_0}$),

$$f_i = p_* q^* : M_d \to M_{d-i},$$

For $n > 2$ and $i = 0, \ldots, n - 1$ we define $F_i = t_i^{-1} v^{d_i, \ldots, d_0} \delta_i (n-i) : M_d \to M_{d-i},$

For $n > 2$ we define $F_i = f_i,$

$$e_i = -t_i^{-1} t_i^{-1} v^{d_i, \ldots, d_0} \delta_i (n-i) : M_d \to M_{d+i},$$

For $n > 2$ and $i = 0, \ldots, n - 1$ we define $E_i = -t_i^{-1} v^{d_i, \ldots, d_0} \delta_i (n-i) : M_d \to M_{d+i},$

For $n > 2$ we define $E_i = e_i.$

3.5. Sevostyanov’s form of affine quantum $SL_n$. Let $I$ denote the set $\mathbb{Z}/n\mathbb{Z}$ of residue classes modulo $n$.

$u$ is the $C[v, v^{-1}]$-algebra with generators $E_i, E_i^\pm, K_i^\pm, C^\pm, F_i$, $i \in \mathbb{Z}/n\mathbb{Z}$, subject to the following relations:

$$L_i L_j = L_j L_i, \quad K_i = L_i^2 L_i^{-1} L_i^{-1} C_{d_i, 0}$$

$$L_j E_i L_j^{-1} = v^{b_{i,j}} E_i, \quad L_j F_i L_j^{-1} = v^{-b_{i,j}} F_i, \quad C \quad \text{is central}$$

$$E_i F_j - F_j E_i = \delta_{i,j} K_i - K_i^{-1}$$

$$|i - j| > 1 \implies E_i E_j - E_j E_i = 0 = F_i F_j - F_j F_i$$

$$n > 2 \& |i - j| = 1 \implies E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 = F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2$$
(26) \( n = 2 \) \& \(|i-j|=1 \implies E_i^3E_j-(v^2+1+v^{-2})E_i^2E_j+(v^2+1+v^{-2})E_iE_jE_iE_j^2-E_jE_i^3 = 0 \)

(27) \( n = 2 \) \& \(|i-j|=1 \implies F_i^3F_j-(v^2+1+v^{-2})F_i^2F_j+(v^2+1+v^{-2})F_iF_jF_i^2-F_jF_i^3 = 0 \)

For \( n > 2 \) we also consider elements \( e_i, f_i \in \mathcal{U} \) depending on the following choice of \( n \times n \)-matrices \( n_{ij}, c_{ij} \) (cf. [16], Remark 3):

(28) \( n_{i,i} = 1, \ n_{i,i+1} = -1, \ n_{i+1,i} = 0, \) otherwise \( n_{ij} = 0. \)

(29) \( c_{i,i+1} = -1, \ c_{i+1,i} = 1, \) otherwise \( c_{ij} = 0. \)

Then we set

\[ f_i := L_iL_i^{-1}F_i, \ c_i := E_iL_i^{-1}L_{i+1}. \]

Clearly, the algebra \( \mathcal{U} \) is generated by \( e_i, L_i^\pm 1, K_i^\pm 1, C^\pm 1, f_i, i \in \mathbb{Z}/n\mathbb{Z}, \) and the relations (22)–(26) above are equivalent to the relations (31)–(34) below.

(31) \( L_j e_i L_j^{-1} = v^{\delta_{ij}} e_i, \ L_j f_i L_j^{-1} = v^{-\delta_{ij}} f_i, \) \( C \) is central

(32) \( e_i f_j - v^{c_{ij}} f_j e_i = \delta_{ij} \frac{K_i-K_i^{-1}}{v-v^{-1}} \)

(33) \(|i-j| > 1 \implies e_i e_j - e_j e_i = 0 = f_i f_j - f_j f_i \)

(34) \(|i-j| = 1 \implies e_i^2 e_j - v^{c_{ij}}(v+v^{-1})e_i e_j e_i + v^{2c_{ij}} e_i e_i^2 = 0 = f_i^2 f_j - v^{c_{ij}}(v+v^{-1})f_i f_j f_i + v^{2c_{ij}} f_j f_i^2 \)

3.6. The following is an affine analogue of Theorem 2.12. Recall the operators \( E_i, e_i, K_i^\pm 1, L_i^\pm 1, C^\pm 1, F_i, f_i, i \in I, \) on \( \mathcal{M} \) defined in 3.4.

**Conjecture 3.7.** The operators \( E_i, K_i^\pm 1, L_i^\pm 1, C^\pm 1, F_i, i \in I, \) on \( \mathcal{M} \) satisfy the relations (22)–(27). Equivalently, if \( n > 2, \) the operators \( e_i, K_i^\pm 1, L_i^\pm 1, C, f_i, i \in I, \) satisfy the relations (22), (27)–(34).

3.8. We can prove Conjecture 3.7 for \( n > 2. \) Let us sketch this proof. It is parallel to the proof of Theorem 2.12. In fact, the relation (32) for \( i \neq j \) follows from the transversality statement absolutely similar to Lemma 2.14. More precisely, the argument of [53] (Lemma 3.3), reduces the required smoothness to that proved in Lemma 2.14.

The relation (32) for \( j = i \) follows from Conjecture 3.7 by the argument of [2.25]. Since we cannot prove Conjecture 2.24 at the moment, we will derive the relation (32) for \( j = i \) from its weaker but accessible form.

To this end we consider the following closed substack \( 3_{a_1 a_2} \subset 3_{a_1 a_2}. \) Recall that a coherent sheaf \( \mathcal{M}_1 \) (resp. \( \mathcal{M}_2 \)) contains the maximal torsion subsheaf \( \mathcal{M}_1^\text{tors} \) (resp. \( \mathcal{M}_2^\text{tors} \)) with the locally free quotient sheaf \( \mathcal{M}_1^\text{free} \) (resp. \( \mathcal{M}_2^\text{free} \)). Moreover, we have \( \mathcal{M}_1 \simeq \mathcal{M}_1^\text{tors} \oplus \mathcal{M}_1^\text{free} \)
(resp. \( \mathcal{M}_1 \cong \mathcal{M}_1^{tor} \oplus \mathcal{M}_1^{free} \)). The closed substack \( \mathcal{Z}_{0,1,2} \subset \mathcal{Z}_{0,1,2} \) classifies the flags of coherent sheaves (with trivialization at \( \infty \in \mathbb{C} \)) \( \mathcal{M}_1 \subset \mathcal{M}_2 \) such that \( \deg \mathcal{M}_1^{free} \leq 0 \geq \deg \mathcal{M}_2^{free} \). We define \( K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \) as the \( K \)-group of \( T \times C^* \)-equivariant coherent sheaves on the smooth stack \( \mathcal{Z}_{0,1,2} \) supported on the closed substack \( \mathcal{Z}_{0,1,2} \). Note that for any \( d = (d_1, \ldots, d_n) \) the map \( \mathfrak{d} : \mathcal{M}_d \to \mathcal{Z}_{d_i-d_{i-1},d_{i+1}-d_i} \) factors through the same named map into the closed substack \( \mathcal{Z}_{d_i-d_{i-1},d_{i+1}-d_i} \). Similarly, for any \( d = (d_0, d_1, \ldots, d_{n-1}) \) the map

\[
\mathfrak{d}_d : \mathcal{M}_d \to \mathcal{Z}_{d_i-d_{i-1},d_{i+1}-d_i}, \quad \mathcal{F}_* \mapsto (\mathcal{F}_0/\mathcal{F}_1, \mathcal{F}_1/\mathcal{F}_2/\cdots/\mathcal{F}_n)
\]

factors through the same named map into the closed substack \( \mathcal{Z}_{d_i-d_{i-1},d_{i+1}-d_i} \).

Let \( (\mathcal{M}_1 \subset \mathcal{M}_2) \) be a \( T \times C^* \)-fixed point of \( \mathcal{Z}_{0,1,2} \). Let \( \iota : (\mathcal{M}_1 \subset \mathcal{M}_2) \) denote its locally closed embedding into \( \mathcal{Z}_{0,1,2} \). Let \( Aut(\mathcal{M}_1 \subset \mathcal{M}_2) \) stand for its automorphisms’ group. One can easily check the following

**Lemma 3.9.** There exists \( n, i, 1 \leq i \leq n-1, d = (d_1, \ldots, d_n) \), and a fixed point \( d \in \mathcal{Z}_{0,1,2} \) such that

(a) \( \mathfrak{d}_d (\mathcal{M}_1) = (\mathcal{M}_1 \subset \mathcal{M}_2) \);

(b) \( \mathfrak{d}_d (T \times C^*) \) is a maximal torus of \( Aut(\mathcal{M}_1 \subset \mathcal{M}_2) \).

3.10. One way to prove Conjecture 2.24 would be to reverse the argument of 2.25 and derive it from the relations (10) for all \( n, i \). In effect, we must compute (notations of 2.25) \( \mathcal{X} \in K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt)) \) while we know \( \mathcal{X} \) for all \( n, i, d \) such that \( d_i - d_{i-1} = \mathfrak{d}_1, d_{i+1} - d_i = \mathfrak{d}_2 \) (also, the homomorphism of tori \( \tilde{T}_n \to T \) acts on the characters as \( \tau_1 = \tau_i, \tau_2 = \tau_{i+1} \)).

Let us denote by \( \mathcal{Y} \in K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt)) \) the restriction of \( \mathcal{X} \) to \( \mathcal{Z}_{0,1,2} \).

The Lemma 3.9 implies that the kernel \( Ker_1 \) of the direct product of inverse images

\[
\prod_{n,i,d} \mathfrak{d}_d : K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt)) \to \prod_{n,i,d} M_d
\]

coincides with the kernel \( Ker_2 \) of the direct product of restrictions

\[
\prod_{(\mathcal{M}_1 \subset \mathcal{M}_2) \in (\mathcal{M}_1 \subset \mathcal{M}_2)} \iota_\ast : K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt)) \to
\]

\[
\prod_{(\mathcal{M}_1 \subset \mathcal{M}_2) \in (\mathcal{M}_1 \subset \mathcal{M}_2)} K^{T \times C^* \times Aut(\mathcal{M}_1 \subset \mathcal{M}_2)} (pt) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt))
\]

It follows that for any \( n, i, 0 \leq i \leq n-1, d = (d_0, \ldots, d_n), \) such that \( d_1 = d_i - d_{i-1}, d_2 = d_{i+1} - d_i \), the kernel \( Ker_1 = Ker_2 \) is contained in the kernel \( Ker_3 \) of the inverse image

\[
\mathfrak{d}_d : K^{T \times C^*} (\mathcal{Z}_{0,1,2}) \otimes_{K^{T \times C^*}(pt)} \text{Frac}(K^{T \times C^*}(pt)) \to \mathcal{M}_d
\]

By the argument of 2.25 we know that \( \mathcal{Y} = \mathcal{Y} \) modulo \( Ker_1 \), and hence the same holds modulo \( Ker_3 \). The argument of *loc. cit.* then shows that the relation (22) for \( j = i \) holds in \( \mathcal{M} \).

3.11. It remains to check the Serre relations. The relations for negative generators follow from the relations for positive generators because they are adjoint with respect to the nondegenerate Shapovalov form, see 3.13 below. So it suffices to consider the relations (24), (25) between \( E_i, E_j, i \neq j \). It is here that we need the assumption \( n > 2 \) for technical reasons. Namely, for \( n > 2 \) we can find \( k \in I \) such that \( i \neq k \neq j \).
We consider an \( n \)-dimensional vector space with a basis \( w_1, \ldots, w_n \), and a torus \( \mathbb{T} \) acting on \( w_i \) by the character \( \tau_i^2 \). Let \( \mathcal{Z}_n \) be the moduli stack of flags of coherent sheaves \( \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n \) on \( \mathbb{C} \) locally free at \( \infty \in \mathbb{C} \), equipped with compatible trivializations \( \mathcal{M}_i|_\infty = (w_1, \ldots, w_i) \). Note that \( \mathcal{Z}_n \) has connected components numbered by the degrees of \( \mathcal{M}_i \), which for \( n = 2 \) coincide with the stacks \( \mathcal{Z}_{3,0} \). Absolutely similarly to Section 2.24 we introduce the correspondences between various connected components, which give rise to the operators \( E_1^3, \ldots, E_{n-1}^3 \) on the localized equivariant \( K \)-theory of \( \mathcal{Z}_n \).

As in 3.8 above, we have a closed substack \( \mathcal{Z}'_n \subset \mathcal{Z}_n \) classifying the flags such that \( \deg \mathcal{M}_i^{free} \leq 0 \), \( 1 \leq i \leq n \).

We have a map
\[ \mathcal{Z}_n \to \mathcal{Z}_n, \quad (\mathcal{F}_i) \mapsto (\mathcal{F}_{i+1}/\mathcal{F}_i) \]

factoring through the same named map \( \mathcal{P}_d \to \mathcal{Z}_n \). For any \( N \geq n \), and \( m \) such that \( 0 \leq m \leq N - n \), and \( d = (d_1, \ldots, d_N) \), we also have a map
\[ \mathcal{Z}_n \to \mathcal{Z}_n, \quad (\mathcal{F}_i) \mapsto (\mathcal{F}_{i+m+1}/\mathcal{F}_{i+m}) \]

factoring through the same named map \( \mathcal{Q}_d \to \mathcal{Z}_n \).

Now the argument of Section 2.24 shows that the Serre relation between \( E_i, E_j \) would follow from the Serre relation between \( E_i^3, E_j^3 \) for certain \( i', j' \). Though we cannot establish the latter relations, the argument of Section 3.10 shows that they hold modulo the subspace \( \text{Ker} r_1 \) (because we already know the Serre relations for \( \mathfrak{sl}_N \) with arbitrary \( N \)), and also shows that this suffices to derive the former relations.

This completes the proof of the Serre relations for \( n > 2 \). Thus, Conjecture 3.7 is proved for \( n > 2 \).

3.12. Similarly to Section 2.24, we will write down a geometric expression for a Shapovalov form on \( \mathcal{M} \), that is a symmetric \( \text{Frac}(\mathbb{C}[T \times \mathbb{C}^* \times \mathbb{C}^*]) \)-valued bilinear form on \( \mathcal{M} \) such that \( (E_i m_1, m_2) = (m_1, F_im_2) \) for any \( i \in I \), and \( m_1, m_2 \in \mathcal{M} \). The different weight spaces of \( \mathcal{M} \) will be orthogonal with respect to this geometric Shapovalov form. For \( i = 0, \ldots, n - 1 \), we consider the line bundle \( \mathcal{D}_i \) on \( \mathcal{P}_d \) whose fiber at the point \( (\mathcal{F}_i) \) equals \( \text{det} \Gamma(\mathcal{F}_i) \). We also define the line bundle \( \mathcal{D}_d := \bigotimes_{i=0}^{n-1} \mathcal{D}_i \). For \( \mathcal{G}_1, \mathcal{G}_2 \in \mathcal{M}_d \), we set
\[ (\mathcal{G}_1, \mathcal{G}_2) := (-1)^{\sum_{i=0}^{n-1} d_i} \prod_{i=0}^{n-1} d_i! \sum_{i=0}^{n-1} d_i d_{i+1} + \sum_{i=0}^{n-1} (n-2i) d_i! \prod_{i=0}^{n-1} t_i^{d_i} \prod_{i=0}^{n-1} t_i^{d_i-d_i+1} \text{det} \Gamma(\mathcal{P}_d, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{D}_d) \]

Clearly, the form \( (, ) \) is nondegenerate, since the classes of the structure sheaves of the \( \mathbb{C}^* \)-fixed points form an orthogonal basis of \( \mathcal{M} \).

The following proposition is proved exactly as Proposition 2.29.

**Proposition 3.13.** For \( i \in I \), \( \mathcal{G}_1 \in \mathcal{M}_d \), \( \mathcal{G}_2 \in \mathcal{M}_{d+1} \), we have \( (E_i \mathcal{G}_1, \mathcal{G}_2) = (\mathcal{G}_1, F_i \mathcal{G}_2) \).

3.14. We define a formal sum in a completion of \( \mathcal{M} \) as follows: \( v = \sum_d v_d := \sum_d [\mathcal{O}_d] = \sum_d [\mathcal{O}_d] \). We also consider the following formal sum: \( u = \sum_d u_d \) where
\[ u_d = v^2 \sum_{i=0}^{n-2} d_i! d_{i+1} d_i^{-(n-2i-1)} d_i^{n-2i+1} \prod_{i=1}^{n-2} d_i^{2d_i-2} \prod_{i=1}^{n-2} d_i \]

**Proposition 3.15.**

a) \( v \) is a common eigenvector of the operators \( f_i \) with the eigenvalue \( (1 - v^2)^{-1} \).

b) \( u \) is a common eigenvector of the operators \( e_i^* \) with the eigenvalue \( (1 - v^2)^{-1} \).
Proof. a) is proved exactly as Proposition 2.31 (a).

To check b) we argue as in the proof of Proposition 2.31 (b), and reduce it to

\[ p_*[L_i] = t_i^2 u^{2d_i} (1 - u^2)^{-1} \mathbb{O}_d \]

To verify this we recall the setup of 2.23 and claim that in the notations of loc. cit. we have

\[ p_*[L_i] = \pi_t^2 v^{2d_i} (1 - v^2)^{-1} \mathbb{O}_{\delta_1, \delta_2} \]

In effect, (38) is deduced from (14) by the argument of 3.10. Finally, (37) is deduced from (38) by the argument of 2.25.

The Proposition is proved.

\[ \square \]

**Corollary 3.16.** The Shapovalov scalar product of the Whittaker vectors equals \( (n_d, u_d) = (-1) \sum_{i=0}^{\infty} \delta_i d_i v^{d_i} (1 - v^2)^{-1} \mathbb{O}_d \).

**3.17.** We define \( M' \subset M \) as a minimal \( \mathcal{U} \)-submodule containing the lowest weight vector \([0, \ldots, 0] \). The relations (32) show that \( M' \) is generated from \([0, \ldots, 0] \) by the action of operators \( e_i \), \( i \in I \). Clearly, \( M' \) is isomorphic to a universal Verma module over \( \mathcal{U} \).

**Conjecture 3.18.** The class of the structure sheaf \([0_d] \) lies in \( M' \).

In what follows we shall assume the validity of Conjecture 3.7 (as was explained above this is actually not an assumption for \( n > 2 \)).

**Proposition 3.19.** The class of \([\mathcal{D}^{-1}_d] \) lies in \( M' \).

**Proof.** We have \( M = M' \oplus M'' \) where \( M'' \) is the orthogonal complement of \( M' \) in \( M \) with respect to the Shapovalov form. We have to prove that \([\mathcal{D}^{-1}_d] \) is orthogonal to \( M'' \). Let \( A \in M'' \). Suppose \( A = e_i B \) for some \( i \in I \) and \( B \in M'_{d-i} \). Then \([A, [\mathcal{D}^{-1}_d]] = (B, e_i^*[\mathcal{D}^{-1}_d]) \). Thus up to (an invertible) monomial in \( t, u, v \) we have \([A, [\mathcal{D}^{-1}_d]] = (B, e_i^*[\mathcal{D}^{-1}_d]) \). Hence, arguing by induction in \( d \) we may assume that \( A \in M'' \) is orthogonal to the image of any \( e_i \). Then \( e_i^* A = 0 \) or, equivalently, \( f_i A = 0 \) for any \( i \in I \). Up to (an invertible) monomial in \( t, u, v \) we have \([A, [\mathcal{D}^{-1}_d]] = (B, e_i^*[\mathcal{D}^{-1}_d]) \). Thus we are reduced to the following claim for \( d \neq (0, \ldots, 0) \):

\[ f_i A = 0 \ \forall \ i \in I \implies \Gamma(\mathcal{P}_d, A) = 0. \]

We will derive (39) from the corresponding claim in the equivariant (complexified) Borel-Moore homology \( \hat{H}_{BM}^{\mathbb{T} \times C^*}(\mathcal{P}_d) \). Let \( \text{Td}_{\mathcal{P}_d} \) denote the equivariant Todd class in the completion of the equivariant cohomology. Let also \( \text{ch}_* \) denote the homological Chern character map from the equivariant K-theory to the completion of the equivariant Borel-Moore homology (see e.g. [3]). We define

\[ a := \text{Td}_{\mathcal{P}_d} \cup \text{ch}_* A \in \hat{H}_{BM}^{\mathbb{T} \times C^*}(\mathcal{P}_d) \]

By the bivariant Riemann-Roch Theorem (see e.g. [3], 5.11.11) we have \( \text{ch}_*(f_i A) = \text{ch}_*(p_* q_* A) = p_* q_* a \) where in the RHS \( p_* \) and \( q_* \) refer to the operations in the (localized and completed) equivariant Borel-Moore homology. We also have \( \Gamma(\mathcal{P}_d, A) = \int_{\mathcal{P}_d} a \). Since \( \text{ch}_* \) is injective, and the operation \( ? \mapsto \text{Td}_{\mathcal{P}_d} ? \) is invertible, the claim (39) follows from the corresponding claim in the equivariant Borel-Moore homology \( \hat{H}_{BM}^{\mathbb{T} \times C^*}(\mathcal{P}_d) \):

\[ f_i a = 0 \ \forall \ i \in I \implies \int_{\mathcal{P}_d} a = 0. \]
Here \( f_i = p_i q^* \) is a part of the action of the affine Lie algebra \( \widehat{\mathfrak{sl}}_n \) on \( \mathcal{M} := \bigoplus_{d} \tilde{H}^F_{BM} \big( \mathcal{P}_d \big) \) (localized and completed equivariant Borel-Moore homology). The positive generators act as \( c_i = -q_i p^* \). This can be checked along the lines of [3.8][3.11] but simpler.

Reversing the argument in the beginning of the proof, we see that [4.10] is equivalent to the statement that the fundamental cycle \([\mathcal{P}_d] \in \tilde{H}^F_{BM}(\mathcal{P}_d)\) is contained in the subspace \( \mathcal{M}' \) of \( \mathcal{M} \) generated by the action of \( c_i, \ i \in I \), from \([\mathcal{P}_{(0,...,0)}]\).

Recall the semismall resolution morphism \( \pi_d : \mathcal{P}_d \to \mathcal{P}_d \) to the Uhlenbeck flag space, see [5]. By the Decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber, the direct sum of (localized and completed) equivariant Intersection Homology \( '\mathcal{M} := \bigoplus_d \tilde{H}^F_{BM} C' \big( \mathcal{P}_d \big) \) is a direct summand of \( \mathcal{M} \).

Now [1] defines the action of \( \widehat{\mathfrak{sl}}_n \) on \( \mathcal{M} \), and one can check that the action of [1] is the restriction of the above \( \widehat{\mathfrak{sl}}_n \)-action on \( \mathcal{M} \). It follows that \( \mathcal{M} = \mathcal{M}' \). Finally, it is proved in [1] that \([\mathcal{P}_d] \in '\mathcal{M} \).

This completes the proof of the Proposition. \( \square \)

### 3.2.20. We conclude that \( u \) is the unique Whittaker vector in the completion of the Verma module \( \mathcal{M}' \) with the lowest weight component \( u_{(0,...,0)} = [(0,\ldots,0)] \) (the common eigenvector of \( e_i^* \), \( i \in I \), with the eigenvalue \( (1 - \nu^2)^{-1} \)).

Let \( n' \in \tilde{\mathcal{M}}' \) be the unique common eigenvector of \( f_i, \ i \in I \), with the eigenvalue \( (1 - \nu^2)^{-1} \) and with the lowest weight component \( n'_{(0,...,0)} = [(0,...,0)] \). Then \( n' \) is the orthogonal projection of \( n \) onto \( \tilde{\mathcal{M}}' \) along \( \tilde{\mathcal{M}}'' \). Hence the Corollary 3.16 yields the following

**Corollary 3.21.** One has

\[
(n'_d, u_d) = (-1)^{\sum_{i=0}^{n-1} d_i \nu + \sum_{i=0}^{n-1} d_i^2 - \sum_{i=0}^{n-1} d_i d_{i-1} - \sum_{i=0}^{n-1} d_i u d_0} \prod_{i=1}^{n} \frac{d_{i-1} - d_i}{d_i} \left[ \Gamma(\mathcal{P}_d, \mathcal{O}_d) \right].
\]

### 3.2.22. Some further remarks. The next natural step would be to study the generating function of all \( [\Gamma(\mathcal{P}_d, \mathcal{O}_d)] \)'s in a way similar to subsection 2.3.3. Let us denote this function by \( \mathcal{J}_{\text{eff}} \). The cohomology (as opposed to \( K \)-theory) analogue of this is performed in [1] and [2]. In particular, in [1] it is shown that such a function is an eigen-function of a certain linear differential operator of 2nd order (the "non-stationary analogue" of the quadratic affine Toda hamiltonian). This fact is used in [2] in order to show that certain asymptotic of this function is given by the Seiberg-Witten prepotential of the corresponding classical affine Toda system. This agrees well with the results of [14] about a similar asymptotic of the partition function of \( N=2 \) supersymmetric gauge theory in 4 dimensions.

Unfortunately, in the present (\( K \)-theoretic) case we can’t derive any good equation for the function \( \mathcal{J}_{\text{eff}} \). Thus we do not know how to generalize the results of [2] to this case. One can probably show that the results of [14] on 5d gauge theory imply that a similar asymptotic (when the classical affine Toda lattice is replaced by the classical affine relativistic Toda) is valid for the function \( \mathcal{J}_{\text{eff}} \), but we do not know how to derive it from Corollary 3.21.

### References

[1] A. Braverman, *Instanton counting via affine Lie algebras I. Equivariant J-functions of (affine) flag manifolds and Whittaker vectors*, CRM Proc. Lecture Notes 38, Amer. Math. Soc., Providence, RI (2004), 113–132.

[2] A. Braverman and P. Etingof, *Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential*, [math.AG/0409441](http://arxiv.org/abs/math.AG/0409441).

[3] N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser, Boston (1997).
P. Etingof, *Whittaker functions on quantum groups and q-deformed Toda operators*, Amer. Math. Soc. Transl. Ser. 2 194 (1999), 9–25.

M. Finkelberg, D. Gaitsgory, A. Kuznetsov, *Uhlenbeck spaces for $\mathfrak{h}^2$ and affine Lie algebra $\hat{\mathfrak{sl}}_n$*, Publ. RIMS, Kyoto Univ. 39 (2003), 721–766.

M. Finkelberg, A. Kuznetsov, *Global Intersection Cohomology of Quasimaps' spaces*, Intern. Math. Res. Notices 7 (1997), 301–328.

A. Givental, Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. math. 151 (2003), 193–219.

A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. 168 (1995), 609-641.

B. Kim, *Quantum cohomology of flag manifolds $G/P$ and quantum Toda lattices*, Annals of Math. 149 (1999), 129-148.

G. Laumon, *Un Analogue Global du Cône Nilpotent*, Duke Math. Journal 57 (1988), 647–671.

G. Laumon, *Faisceaux Automorphes Liés aux Séries d’Eisenstein*, Perspect. Math. 10 (1990), 227–281.

H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, Journal of the AMS 14, no. 1 (2001), 145–238.

T. Nakashima, A. Zelevinsky, *Polyhedral Realizations of Crystal Bases for Quantized Kac-Moody Algebras*, Advances in Mathematics, 131, No.1 (1997), 253–278.

N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, to appear in “Unity of Mathematics” (proceedings of a conference dedicated to I. M. Gelfand’s 90th birthday, Harvard University, 2003).

C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. 101 (1990), 583–592.

A. Sevostyanov, *Regular Nilpotent Elements and Quantum Groups*, Commun. Math. Phys. 204 (1999), 1–16.

A. Sevostyanov, *Quantum deformation of Whittaker modules and the Toda lattice*, Duke Math. J. 105 (2000), 211–238.

Address:

A.B.: DEPT. OF MATH., BROWN UNIV., PROVIDENCE, RI 02912, EINSTEIN INSTITUTE OF MATHEMATICS EDMOND J. SAFRA CAMPUS, GIVAT RAM THE HEBR cave UNIVERSITY OF JERUSALEM JERUSALEM, 91904, ISRAEL

M.F.: INDEPENDENT MOSCOW UNIV., 11 BOLSHOJ VLASJEVSKIJ PER., MOSCOW 119002, RUSSIA

E-mail address:
braval@math.brown.edu; fnklberg@mccme.ru