Solution to the satisfiability problem using a complete Grover search with trapped ions

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Abstract
The main idea in the original Grover search (1997 \textit{Phys. Rev. Lett.} 79 325) is to single out a target state containing the solution to a search problem by amplifying the amplitude of the state, following the Oracle’s job, i.e., a black box giving us information about the target state. We design quantum circuits to accomplish a complete Grover search involving both the Oracle’s job and the amplification of the target state, which are employed to solve satisfiability (SAT) problems. We explore how to carry out the quantum circuits with currently available ion-trap quantum computing technology.

1. Introduction
Quantum algorithms, such as the well-known Shor’s factoring algorithm [1] and Grover’s search algorithm [2], have shown an increase in computational speed over their classical counterparts, due to the capabilities of exploiting the parallelism of quantum mechanics or interference effects. It is expected that future quantum computers should be able to solve some classically intractable problems [1, 3], e.g., nondeterministic polynomial-complete (NPC) problems [4, 5], among which the random Boolean $K$-satisfiability ($K$-SAT) problem [6] is a central issue in computer science. Many proposals have so far focused on the solution of hard instances of $K$-SAT problems by the method of the so-called DPLL algorithm [7], quantum adiabatic algorithm [8–11], Grover’s search algorithm [12], Hogg’s algorithm [13], local search algorithms [14–17], statistical mechanics approach [18], etc.

We will concentrate in the present paper on the solution to $K$-SAT problems using the Grover search algorithm. Our motivation is to show the possibility of the Grover search finding answers to some solvable $K$-SAT problems. Although most $K$-SAT problems, even if $K = 3$, are NP problems and the Grover search does not have an exponential speed-up capability in solution, we will consider some solvable cases of $K$-SAT as examples. As we know from Grover’s original papers and other subsequent work, the Grover search was only considered as a method to efficiently single out the answer states (or target state in the language of the Grover search), whereas the job to find the target state is assigned to a black box named Oracle. To work out a realistic problem, however, we should have to consider how to take account of the Oracle’s work. So there naturally arises a question: is it possible to design a scheme with consideration of both the job of Oracle and the job of amplitude amplification? We will give a positive answer to the question with some quantum circuits for solutions to some $K$-SAT problems, which enables us to work with trapped-ion QC.

The ion-trap system [19–21] favours QC owing to the long coherence time of qubits, high efficiency of detection and full controllability of operations. In the context of trapped-ion QC, there have been some schemes [22, 23] for implementation of the Grover search by constructing multi-qubit conditional operations based on the Cirac–Zoller (CZ) gate [24] or Mølmer–Sørensen gate [25]. Experimentally, a two-qubit Grover search has been proposed in $^{111}$Cd\textsuperscript{+} [26] and eight ions have been confined stably in entanglement in the trap [27], which means that most qubits are ready for QC among...
the currently available QC candidate systems. Most recently, the Grover search was carried out in a relatively simple way [28] making use of collective states of trapped ions [29].

We will explore below the experimental feasibility of solving K-SAT problems using a complete Grover search with trapped ions, where the Grover search will be implemented either by sequences of single-qubit and multi-qubit quantum gates based on light-shift (LS) gates [30] or directly by multi-qubit conditional phase flip (CPF) gates [31]. We will compare the two methods for implementation of our solutions. The outline of the paper is as follows. In section 2, the K-SAT problem and Grover search algorithm are briefly introduced. In section 3, we apply the complete Grover search to some satisfiable K-SAT problems by quantum circuits, which compared to [12], is much simplified with a reduced number of auxiliary qubits and gating. Then we will discuss the experimental feasibility with trapped ions for implementation in section 4. The last section is for our conclusion.

2. K-SAT problem and Grover search algorithm

Let us first of all review the K-SAT problem briefly. As a paradigmatic example of an NPC problem, the well-known K-SAT problem is actually a combinatorial search problem in theoretical computer science4. Generally speaking, the K-SAT problem could be expressed by a logical configuration involving K Boolean variables $b_i$ ($i = 1, 2, \ldots, K$) with the values $0 = \text{FALSE}$ and $1 = \text{TRUE}$. A K-SAT formula $F_K$, which consists of $m$ clauses $\{C_{\mu}\}_{\mu=1,2,\ldots,m}$, could be written as [10]

$$ F_K = C_1 \land C_2 \land C_3 \cdots \land C_m, \quad (1) $$

where $\land$ means the logical AND gate, and each clause $C_i$ contains a number of logical variables $b_i$ or its negation $\bar{b}_i$ (i.e., logical NOT gate on $b_i$). These variables ($b_i$ or $\bar{b}_i$) are connected with each other by $\lor$ (logical OR gate), and the maximal number of variables in each clause is $K$. For instance, $F_2 = (b_2 \lor b_3) \land (b_1 \lor b_3) \land (b_1 \lor b_2)$ is a 2-SAT formula with three clauses and four logical variables. A solution to $F_K$ is to find the values of the logical variables satisfying all clauses simultaneously in order to make $F_K$ be TRUE. In some cases, there is no solution to a SAT problem, which is called unsatisfiable, and in some other cases, there are possibilities to have multiple solutions to a SAT problem. In the present scheme, for simplicity, we will restrict our study exclusively to formulae with only one solution.

The primary idea of Grover’s algorithm in [2] is to boost the probability amplitude of the target state so that we can measure the target state with a considerably high probability. Generally speaking, provided that the initial state of the system has been prepared in an average superposition state [33], and implementation of OR consists of a single $C_{\text{NOT}}$ and some NOT operations. To accomplish each of them, an

Figure 1. The required basic logic gates, where $\cdot$ denotes the control qubit and $\oplus$ the target qubit. (a) Two-qubit controlled-NOT gate $C_{\text{NOT}}$, where qubits 1 and 2 denote control and target qubits, respectively; (b) multi-qubit controlled-NOT gate $C_{\text{NOT}}$ with $n-1$ control qubits ($q_{1}, \ldots, q_{n-1}$); (c) the $f-C_{\text{NOT}}$ (function-controlled-NOT) gate, where the Boolean expression $f(q_{1}, \ldots, q_{n-1})$ can be satisfied by one or more than one Boolean variable assignment; (d) AND gate; (e) OR gate.

|\psi_0\rangle = (1/\sqrt{N}) \sum_{i=0}^{N-1} |i\rangle \quad (N = 2^n \text{ with } n \text{ being the qubit number}), \text{ the Grover search can be depicted as the iterative operation } G = \hat{D}^{(n)} I_{\pi} \text{ (defined later) by at least } \pi \sqrt{N}/4 \text{ times for finding the marked state } |\tau\rangle \text{ with an optimal probability } [32], \text{ where the quantum phase gate } I_{\pi} = I - 2|\tau\rangle\langle\tau| \text{ (with } I \text{ being the identity matrix) plays an important role of selective inversion (SI) to invert the amplitude of the target state, and the diffusion transform } \hat{D}^{(n)} \text{ defined as } \hat{D}^{(n)} = 2/N - \delta_{ij} (i,j = 1, 2, 3, \ldots, N) \text{ is called inversion-about-average (IAA).}

3. Solution to K-SAT problems using a complete Grover search

In this section, we will show how the Grover search can be carried out to find a satisfiable solution to the K-SAT problem. For simplicity, we first consider a 2-SAT formula $F_2$ involving three clauses and two Boolean variables,

$$ F_2 = (a \lor \bar{b}) \land (a \lor b) \land a. \quad (2) $$

To solve $F_2$, we have to employ some basic quantum logical gates including controlled-NOT (CNOT), AND and OR, as depicted in figure 1, where the multi-qubit CNOT gate $C_{\text{NOT}}$ involves $(n-1)$ qubits as control and the $n$th qubit as target. The $f-C_{\text{NOT}}$ (called function-CNOT) gate inverts the target state on the condition that the control states satisfy a specific Boolean function. The $f-C_{\text{NOT}}$ could be used for the eigenvalue-kickback effect to induce $\pi$ phase shifts on some component states if the auxiliary state is initially prepared as $|0\rangle - |1\rangle)/\sqrt{2}$ (see the appendix).

The logic operation AND is carried out by a single $C_{\text{NOT}}$ [33], and implementation of OR consists of a single $C_{\text{NOT}}$ and some NOT operations. To accomplish each of them, an
auxiliary qubit, initially prepared in $|0\rangle$, is employed to store the output of each operation.

Using the above-mentioned logical gates, we depict the quantum circuits in figure 2 for a complete Grover search to solve the 2-SAT problem in equation (2). We can find that, besides two qubits $a$ and $b$ encoding the two variables, respectively, four auxiliary qubits are required, where $q_1$ and $q_2$ are to store the results of $\bar{a} \lor b$ and $a \lor b$, respectively, $q_3$ will record the result of $F_2$, and $q_4$ is the ancilla to do eigenvalue kickback. The first part of the quantum circuit $\hat{U}_0$ is for Oracle’s job, including solution to $F_2$ and the SI operation $I_{t} = I - 2|\tau\rangle\langle\tau|$ using the effect of eigenvalue kickback. It consists of the following five steps: (i) gates 1 and 2 perform the first clause $(\bar{a} \lor b)$ in equation (2); (ii) gates 3–5 implement the second clause $(a \lor b)$; (iii) gate 6 is to restore the states of the qubits $a$ and $b$; (iv) gate 7 executes AND for three clauses to obtain the value of $F_2$, which achieves
\[
(1/2\sqrt{2})(|00\rangle_{ab}|00\rangle_{q_1q_2q_3} + |01\rangle_{ab}|10\rangle_{q_1q_2q_3} - |10\rangle_{ab}|11\rangle_{q_1q_2q_3} + |11\rangle_{ab}|01\rangle_{q_1q_2q_3})\times(|0\rangle - |1\rangle)_{q_4}.
\]
(3)
Therefore, the target state is $|00\rangle_{ab}|11\rangle_{q_1q_2q_3}(|0\rangle - |1\rangle)_{q_4}$, which implies the SAT solution to be $a = 1$ and $b = 0$. To efficiently single out the target state, we have to discard the auxiliary qubits $q_{1,3}$, but keep the phase flipping in the qubit subspace. To this end, we employ the reverse operations in an additional circuit $\hat{U}_{add}$ to do $\hat{U}_0^{-1}$. After the gating $\hat{U}_0^{-1}$, we obtain $(1/2)(|00\rangle + |01\rangle - |10\rangle + |11\rangle)_{ab}|00\rangle_{q_1q_2q_3}(|0\rangle - |1\rangle)_{Q_4}$. Consequently, all the auxiliary qubits are decoupled from the qubits. We may discard $q_1, q_2$ and $q_3$, but keep $q_4$ for later use.

The function of IAA performed in the subsequent dotted-box of the quantum circuit is to single out the target state. As mentioned previously [23, 32], in the implementation of IAA, the diffusion transform $\hat{D}^{(a)}$ is always unchanged no matter which state is to be searched. As a result, we may design the IAA as a fixed module. Since the target state is labelled by SI operation, once IAA (i.e., gate 8) is carried out, we should be able to single out the target state. Specifically, the operation in the step of IAA could be represented by
\[
\hat{D}^{(\alpha)} = -H^{\otimes 2}\sigma_x.b\sigma_x.a C^{(2)}_{P} \sigma_x.a \sigma_x.b H^{\otimes 2},
\]
(4)
where $\hat{D}^{(\alpha)}$ is the diffusion transform $\hat{D}^{(a)}$ in the case of two qubits, $C^{(2)}_{P} = \text{diag}[1, 1, 1, -1]$ in the subspace spanned by $\{|0\rangle_a, |1\rangle_a, |0\rangle_b, |1\rangle_b\}$ inverts the state $|11\rangle$ to be $-|11\rangle$ and $C^{(2)}_{P}$ can be performed in terms of $C^{\otimes 2}_{NOT}$ gate and Hadamard gates. In equation (4),
\[
H^{\otimes 2} = \prod_{j=1}^{2} H_j = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},
\]
(5)
and $\sigma_{x,i}$ is the single-qubit NOT gate acting on the qubit $i$ ($i = a$ or $b$).

With more variables, the accomplishment of our scheme for solution to the $K$-SAT problem would be more complicated [10]. For example, we have designed a quantum circuit in figure 3 for a 2-SAT formula with three clauses and three variables, i.e.,
\[
F_2 = (a \lor b) \land (\bar{a} \lor c) \land \bar{b},
\]
(6)
where $a, b$ and $c$ denote different variables. Due to the similarity to operations in figure 2, we will not discuss it in
Figure 4. Quantum circuit for one iteration in a Grover search to solve the 3-SAT problem in equation (7), where the dots mean omitted operations SI and IAA.

Figure 5. Schematic setup for implementing a Grover search in a linear trap, where $n$ ions are individually addressed by $n$ lasers. The left inset shows the ionic-level configuration in the LB gate [30], where the bold lines mean the levels encoding qubits, and the transition excited by the laser pulse depends on the laser polarization. The right inset shows the ionic level configuration in [31], where the qubit encoding in the $n$th ion is different from in other ions, as labelled by the bold lines.

detail. In addition, we have also designed a quantum circuit in figure 4 for a 3-SAT problem with four clauses and three variables, which reads

$$F_3 = (a \lor b \lor c) \land (a \lor \bar{b} \lor c) \land b \land \bar{c}.$$  

(7)

It is evident that more operations are needed with more variables involved, and for more clauses, more auxiliary qubits are required. One thing we have to mention is that a Grover search for more than two qubits is carried out with probability. So to obtain the final result, we have to perform SI and IAA repeatedly. But for clarity, we have only plotted in figures 4 and 5 a single iteration.

4. Experimental feasibility with trapped ions by two methods

In the previous section, we have designed some quantum circuits for a complete Grover search to solve $K$-SAT problems, where the two-qubit $C_{\text{NOT}}^2$ gate and multi-qubit $C_{\text{NOT}}^n$ gate play crucial roles. To have efficient quantum operations, however, we need fewer operations and shorter implementing time. So in contrast with conventional methods [34, 35] using a number of single-qubit and two-qubit gates to compose a $C_{\text{NOT}}^n$ gate, a direct performance of a $C_{\text{NOT}}^n$ gate seems to be more attractive [31, 36]. In what follows, we will try to carry out those quantum circuits with ultracold trapped ions by two different methods for $C_{\text{NOT}}^n$ gating: the conventional way and the straightforward way.

Let us first consider the conventional way by taking LS gates [30] as an example. In ion trap QC [24], the qubits are encoded in each ion’s internal ground state $|\downarrow\rangle$ and excited state $|\uparrow\rangle$, and quantum gates are made via excitation of the common vibrational modes. We assume that the ions could be irradiated individually by lasers and we have a string of $n$ ions confined in a linear trap, with strong confinement along the $\vec{x}$- and $\vec{y}$-directions, but less strong confinement along the $\vec{z}$-axis. According to [30], conditional on the resonance condition $\Omega_0 = \omega_z/2$ with $\Omega_0$ being the Rabi frequency and $\omega_z$ the $\vec{z}$-axis vibrational frequency, the ac Stark shift induced by the laser in resonance with the carrier transition frequency allows the ionic internal state to couple ionic motional state in a way exactly analogous to the red detuning transition. As a result, the two-qubit $C_{\text{NOT}}^2$ gate could be achieved by sequences of laser pulses corresponding to following unitary operators on
In the straightforward way, the total time $T$ is the summation of the operational time for $B$, $B^*$ gate, specifically, $T = |N[B]| + 2N[B^*]T_y$, with $|N[B]|$, and $N[B^*]$ the number of the gates $B$, and $B^*$ in the circuits, respectively, and $T_y$ the time for $B$ gating. In the straightforward way, $C_{PF}^n$ gating time is irrelevant to the number of the qubits involved, and $N[C_{PF}^n]$ denotes the total number of $C_{PF}^n$ required in the circuits with $i = 2, 3, 4, 5$ being the number of the qubits.

| $\bar{\nu}_o/2\pi$ (MHz) | Conventional way | Straightforward way |
|--------------------------|------------------|-------------------|
| $N[B]$ | $N[B^*]$ | $T_B$ ($\mu$s) | $T$ (ms) | $N[C_{PF}^n]$ | $N[C_{PF}^n]$ | $N[C_{PF}^n]$ | $N[C_{PF}^n]$ | $T$ (ms) |
| Circuit I 2.92 | 118 | 59 | 8.562 | 2.021 | 1 | 5 | 2 | 0 | 1.370 |
| Circuit II 2.50 | 134 | 67 | 10.0 | 2.680 | 1 | 4 | 3 | 0 | 1.600 |
| Circuit III 1.94 | 246 | 123 | 12.887 | 6.340 | 1 | 8 | 1 | 2 | 3.093 |

$C_{PF}^n$ gating time is almost equal to the $C_{PF}^n$ gating time and the realization of a $C_{PF}^n$ gate by a direct way requires only $(n+2)$ individually addressing laser pulses.

In what follows, we compare the two above-mentioned methods working with trapped $^{40}$Ca$^+$ to carry out the quantum circuits in figures 2-4. To have a good confinement of the trapped ions, we have fixed the Lamb–Dicke parameter to be $\eta = 0.02$ throughout our calculation, where $\eta = (\bar{\nu}/\omega_o)/2\pi M\bar{\nu}$, with $\bar{\nu}$ being the wave vector of the laser, $\omega_o$ the angle of the laser radiation with respect to the $\bar{z}$-axis, $n$ the number of the ions and $M$ the mass of the ion. In our estimate, we have set the angle $\theta = 30^\circ$. As a result, we have different axial trapping frequencies for different numbers of ions. For example, in quantum circuits I, II and III, we have $\bar{\nu}_o/2\pi = 2.92$ MHz, 2.50 MHz and 1.94 MHz, respectively. Moreover, in both methods, we consider strong laser radiation, i.e., $\Omega_0 = \bar{\nu}_o/2$ in the conventional way, and $\Omega_{PF}^\max = \bar{\nu}_o/2 (i \neq n)$ in the direct way. For the scheme with LB gates [30], the $C_{PF}^n$ gating time is mainly determined by $B$ and $B^*$ gating time since other gates work much faster. So we may only consider the values of $T_B$ and $T_{B^*}$ to achieve $B$ and $B^*$ gates with $T_B = \pi/2\Omega_{PF}^\max = 50\pi/\omega_o$, and $T_{B^*} = 2T_B$. In contrast, for our CPF proposal [31], in each circuit, once $n$ is determined, the $C_{PF}^n$ gating time is irrelevant to the number of the qubits involved, but determined by the Rabi frequency regarding the last qubit, i.e., $\Omega_{PF}^\max$. Like in [31], we set $\Omega_{PF}^\max = \Omega_{PF}^\max/10$, which yields the $C_{PF}^n$ gating time to be $T = \pi/\Omega_{PF}^\max = 1000\pi/\omega_o$. We have listed our results in table 1, where both methods take a millisecond timescale for an implementation of our scheme.

In addition, as $C_{PF}^n$ in the direct way has intrinsic success probability regarding $m = \Omega_{PF}^\max/\Omega_{PF}^\max$, we have also shown by numerical simulation the implementation of the Grover search for solution to the K-SAT problem in figure 6, in which both the success probability of the $C_{PF}^n$ gate and the intrinsic probability of the Grover search itself are involved.

5. Discussion

To discuss the experimental feasibility of our scheme, we considered ultracold trapped calcium ions $^{40}$Ca$^+$ in a linear trap as an example. If we adopt the conventional way [30], the ground state $|\downarrow\rangle$ and excited state $|\uparrow\rangle$ could be encoded in $S_{1/2}(m_j = -1/2)$ and $S_{1/2}(m_j = 1/2)$, respectively,
and $D_{5/2}(m_j = -3/2)$ can be one of the candidates for the auxiliary state $|\text{aux}\rangle$. In contrast, once the direct way is utilized, as done in [31], for the first $(n - 1)$ ions, the qubits $|\downarrow\rangle$ and $|\uparrow\rangle$ are encoded in $S_{1/2}(m_j = 1/2)$ and $S_{1/2}(m_j = -1/2)$, respectively, namely, the Zeeman sublevels of the ground state $S_{1/2}$ [38], but for the last $n$th ion, the qubits $|\downarrow\rangle$ and $|\uparrow\rangle$ are encoded into $S_{1/2}(m_j = 1/2)$ and $D_{5/2}(m_j = -1/2)$, respectively. The prerequisite of the experiment includes the accurate tuning of the laser pulses to the desired frequencies and phases, and the initial preparation of the vibrational mode of the ions to the ground state.

Both the conventional way and the straightforward way include the following common features: (i) the COM motion is utilized as the data bus, namely, the quantum logic operations (except for the carrier transition) involving the degrees of freedom of the quantized motion. So the vibrational mode should be laser cooled to the ground state, and thereby heating becomes a dominant source of decoherence; (ii) individually addressing by lasers is required. As a result, the laser intensity fluctuation $\Delta \Omega_0$ and the phase fluctuation $\Delta \phi$ should be well controlled for achieving high fidelity of gating [22, 37].

Based on current experimental progress, we argue that 8–12 qubit operations with ions do not look unreasonable in the near future. As we know, heating time of the ground vibrational state of the ions in the linear trap is currently on a timescale of milliseconds [39, 40]. But the lifetime of the metastable level $D_{3/2}$ of $^{40}\text{Ca}^+$ is much longer, i.e., for about 1.16 s [41]. So our operational time should be restricted to within tens of milliseconds. Fortunately, from our calculation, the required total time $T$ regarding both methods is shorter than 10 ms. Nevertheless, to have a completely heating-free operation, we may accelerate our manipulation by some other ways. One of the ways is to optimize the quantum circuits [42], i.e., minimizing the number of gates required by using a geometric approach [43]. Alternatively, we may consider improving the efficiency of the quantum gates. For example, we may employ a tighter trap to enhance $\omega_z$, or consider exactly adjusting the magic numbers regarding the Lamb–Dicke parameter [45].

On the other hand, we can also find advantages of the straightforward way over the conventional way. First, the smaller number of laser pulses makes operation easier in practice. In the conventional way [30], the number of laser pulses required to accomplish $C_N^{\text{NOT}}$ gate is much larger than the counterpart required in the direct way. So the overhead could be much reduced in the direct way to generate a multi-qubit $C_N^{\text{NOT}}$. Second, the increase of the qubits in the direct way [31] could improve the fidelity and the success probability of the $C^\text{PF}_n$ gate, which are favourable for a scalable Grover search. Third, the reduction of the manipulation steps could diminish computational errors.

As shown in table 1, however, the straightforward way, although with much fewer operational steps, takes a comparable time to the conventional way. The reason is that the $C^\text{PF}_n$ gating rate in the straightforward way mainly depends on the value of the Rabi frequency $\Omega^\text{max}_n$ regarding the $n$th ion by a laser, namely, to meet the condition $\text{erf} \left[ n \Omega^\text{max}_n / \sqrt{\pi} \right] \rightarrow 1$ [31]. On the other hand, to obtain a $C^\text{PF}_n$ gating with high enough fidelity and success probability, we require $m = \Omega^\text{max}_n / \Omega^\text{max}_n \ll 1$, which greatly restricts the value of $\Omega^\text{max}_n$, and thereby limits the speed of $C^\text{PF}_n$ gate. As a result, in the few-qubit case, the direct way works only slightly faster than the conventional way, whereas it would be more and more efficient than the conventional way with more qubits involved. Furthermore, as operational overhead could be much reduced in the direct way, we prefer to use it for the implementation of our scheme even in the few-qubit case.

Another point we should mention is that the scheme in [31] is very sensitive to $m$. As shown in figure 6, with the value of $m$ larger than 0.04, the laser intensity fluctuation and phase fluctuation would lead to considerable affect on implementation. To avoid this detrimental influence, we have to keep the laser in high stability, or we take the case with $m < 0.04$, which, however, would yield a longer gating time. So a trade-off would be taken in realistic implementation with the straightforward way.

For solving K-SAT problems with multiple solutions, we will need more gates to exclude the target states that have been found previously. As the design of the circuit for this job is straightforward and strongly relevant to the specific solution [12], we will not go further along this line in the present work. But more gates will yield longer time and a higher requirement for implementation, which brings about more challenges.

The solution to K-SAT problems with more clauses needs more qubits. With more than two qubits involved, however, the Grover search works only probabilistically. So several iterations are needed to accomplish a solution. On the other hand, with more ions confined, the vibrational mode spectrum becomes more and more complicated and the ions’ spacing would be decreasing, which yields individually addressing of the ions to be more intricate. As a result, the extension of QC from a few qubits to a large number of qubits is quite technically challenging. Nevertheless, as ultracold trapped ions in entanglement have been experimentally available [27], our treatment in quantum circuits I, II and III, which involves six, seven and nine ions, respectively, looks very promising under currently achieved technology in the trapped ion system.
6. Conclusion

It is still hard to find a practical quantum algorithm. That is why we have had few working quantum algorithms so far. As one of the most frequently mentioned quantum algorithms, however, the Grover search had never been applied to any really practical problems, although it was regarded as useful for searching for some personal information from a phone book. Most relevant proposals and experiments so far have neglected the Oracle’s work. Actually when the Oracle’s job is involved in the implementation, much overhead would be needed, which makes the implementation of the Grover search more complicated, but more realistic and practical.

The contribution of the present work lies in two aspects. First, some quantum circuits for solutions to K-SAT problems are designed. We have shown the possibility to enable solutions to K-SAT problems by less than ten qubits, which is important in view of experimental realization. Second, we have explored the possibility to accomplish the operations required in the quantum circuits with trapped ions. Our discussion would be also helpful to apply our scheme to other QC candidate systems.

In summary, we have concentrated on the application of the Grover search algorithm on some solvable K-SAT problems with few ultracold trapped ions. Specifically, we have designed quantum circuits to solve some 2-SAT and 3-SAT problems, and presented the feasibility, challenge and possible efforts to accomplish the schemes with currently or near future available ion trap technology. We believe that our design could be further optimized and would be useful for exploring the application of QC.

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Appendix. Eigenvalue kickback effect

Provided that we prepare the state of the target qubit \( q_t \) as \(|0\rangle - |1\rangle)/\sqrt{2} \) and the quantum state of the control qubits as \( |\Psi\rangle = \sum_{\rho \in [A']} c_{\rho} |\rho\rangle + \sum_{\nu \in [B']} c_{\nu} |\nu\rangle \), where \([A'] = \{ i | f(i) = 1, i \in S = \{0, 1, \ldots, 2^n - 1\} \} \) is the set of satisfied assignments and \([B'] = S - [A'] \) is the complementary set, the \( f^{-}C_{\text{NOT}} \) gate (figure 1(c)) will result in \textit{eigenvalue kickback}, which could effectively flip the phase of some components (i.e., \(|\rho\rangle\)) of the state \(|\Psi\rangle_0\). This mechanism can be concretely explained as

\[
\begin{align*}
&f^{-}C_{\text{NOT}} \left( \left( \sum_{\rho \in [A']} c_{\rho} |\rho\rangle + \sum_{\nu \in [B']} c_{\nu} |\nu\rangle \right) \\
&\otimes (|0\rangle - |1\rangle)/\sqrt{2} \right) \\
&= \left( \sum_{\rho \in [A']} c_{\rho} |\rho\rangle \otimes \text{NOT}(|0\rangle - |1\rangle) \right) \\
&+ \sum_{\nu \in [B']} c_{\nu} |\nu\rangle \otimes (|0\rangle - |1\rangle)/\sqrt{2}.
\end{align*}
\]

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