Phase Transitions of Quintessential AdS Black Holes in M-theory/Superstring Inspired Models

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\textbf{ABSTRACT:} We study $d$-dimensional $AdS$ black holes surrounded by Dark Energy (DE), embedded in $D$-dimensional M-theory/superstring inspired models having $AdS_d \times S^{d+k}$ space-time with $D = 2d + k$. We focus on the thermodynamic Hawking-Page phase transitions of quintessential DE black hole solutions, whose microscopical origin is linked to $N$ coincident $(d-2)$-branes supposed to live in such $(2d+k)$-dimensional models. Interpreting the cosmological constant as the number of colors $\propto N^{d-1}$, we calculate various thermodynamical quantities in terms of brane number, entropy and DE contributions. Computing the chemical potential conjugated to the number of colors in the absence of DE, we show that a generic black hole is more stable for a larger number of branes for lower dimensions $d$. In the presence of DE, we find that the DE state parameter $\omega_q$ should take particular values, for $(D, d, k)$ models, providing a non trivial phase transition structure.

\textbf{KEYWORDS:} $AdS$ black holes, M-theory, String theory, Spherical compactification, Dark energy.
1 Introduction

Recent observations, consistent with expectations for the shadow of a Kerr black hole (BH) as predicted by general relativity, have been, for the first time, presented [1, 2]. This result emphasise the need to address the long-standing puzzles presented by its BH solutions. A classical stationary BH solution is characterised by its mass $M$, angular momentum $J$ and charge $Q$ alone. In particular, its horizon area is a simple function of these three quantities. Identifying the horizon area as an entropy the BHs obeys a set of laws being directly analogous to those of thermodynamics [3–7]. According to Hawkings prediction [6, 7], BHs emit thermal radiation at the semiclassical level fixing the Bekenstein-Hawking area/entropy relation to be (where $c = \hbar = G = 1$),

$$S_{BH} \sim \frac{A}{4}. \quad (1.1)$$

This suggests that the thermodynamic interpretation of BH mechanics is more than a mere analogy and points to the existence of sublying degrees of freedom[8]. Any complete theory
of quantum gravity should address this challenge in some way or at least advance in this direction. More detailed view of black holes could be found in [9–21].

The thermodynamic properties of $AdS$ black holes have been extensively studied, the existence of a minimal Hawking temperature and the Hawking-Page phase transition [22–25]. The Hawking-Page phase transition between large stable black holes and thermal gas in the $AdS$ space has been approached using different methods [26, 27].

For example, an analogy between phase structures of various $AdS$ black holes and statistical modes associated with Van der Waals like phase transitions has been suggested [28–30]. Interpreting the cosmological constant as a kind of thermodynamic pressure, and its conjugate variable as the thermodynamic volume, several non trivial results have been presented [31–41]. Thermodynamics of $AdS$ black holes, in supergravity theories, have been also investigated by exploiting the $AdS$/CFT correspondence which provides an interplay between gravitational models in $d$-dimensional $AdS$ geometries and $(d - 1)$-dimensional conformal field theories living in the boundary of such $AdS$ spaces. Using the physics of solitonic branes, different models in type IIB superstrings and M-theory have been studied by considering the cosmological constant in the bulk closely related to the number of colors associated with branes in question. The thermodynamic stability behavior of such $AdS$ black holes in higher dimensional known supergravity theories has been examined in this context [36–41].

Dark Energy (DE) is needed to explain the, well established, observation of the existence of an accelerated expansion of the universe. On lack of a deeper microscopical explanation, the ratio of a pressure and energy density $\omega_q = \frac{p_{dark}}{\rho_{dark}}$, interpreted as a DE fluid equation of state appearing in the Einstein stress tensor, is usually used to model DE. Distinguished DE models have been discussed in terms of such ratio covering the range $]-1,0[ [43]$. Among others, “quintessence models”, associated with ratio values in the range $\left[-1, -\frac{1}{3}\right]$, interpreted as a dynamical field with a negative pressure has been proposed in order to explain the universe acceleration [44–46]. In higher dimensional theories, like M-theory supergravity, a massless pseudoscalar axion like field, obtained from the compactification to lower dimensions, has been considered as a candidate to explain DE contributions [47].

Black holes could carry information about the nature of the elusive dark energy (DE), et vice versa. For example several BH solutions surrounded by a static spherically symmetric quintessence DE have been considered. Typically, the presence of DE acts as a cooling fluid agent, largely modifying several thermodynamical quantities [48–50].

The aim of this work is to advance in the investigation of the thermodynamical phase transitions of $d$-dimensional $AdS$ black holes ($d \geq 4$) surrounded by quintessential DE described by the ratio $\omega_q$. These quintessential black holes solutions are embedded in $D$-dimensional superstring/M-theory inspired models having $AdS_d \times S^{d+k}$ space-time, where $D = 2d + k$. These solutions, which could be associated with $N$ coincident $(d - 2)$-branes assumed to move in such higher dimensional models, are labeled by a triplet $(D, d, k)$ where $k$ is associated with the internal space, the $S^{d+k}$ sphere. By interpreting the cosmological
constant as the number of colors (in fact proportional to $N^{d-1}$), we compute various thermodynamical quantities in terms of the brane number $N$, the entropy $S$ and DE contributions. By computing the chemical potential conjugated to the number of colors in the absence of DE, we find that the black hole is more stable for for configuration with a larger number of branes, for small dimensions $d$. In the presence of DE, we observe that the state parameter $\omega_q$ takes specific values, for $(D,d,k)$ models. Non trivial properties of the Hawking-Page phase transition in each case are obtained. In this work, we use dimensionless units in which one has $\hbar = c = 1$.

The organization of this paper is as follows. In section 2, we provide detailed formulas for $d$-dimensional AdS black holes embedded in $D$-dimensional superstring/M-theory inspired models and compute several thermodynamical quantities. In sections 3 and 4, we study in detail a model indexed by the triplet $(D,d,k) = (11, 7, -3)$, associated with the compactification of M-theory on the sphere $S^4$ with the M5-branes without and with DE respectively. Similar results are presented in full detail in appendices. In sections 5, we present further discussions, conclusions and open questions. The last sections are devoted to appendices. In Appendices A, B, C and D, we present detailed results for $AdS_4 \times S^7$ and $AdS_5 \times S^5$ models with and without DE.

2 AdS black holes in M-theory/superstring inspired models

In this work, we focus on the investigation of $d$-dimensional AdS black holes embedded in $D$-dimensional superstring/M-theory inspired models (where $d \geq 4$).

We assume they are obtained from the compactification on $(D - d)$-dimensional real spheres denoted by $S^{D-d}$. In the presence of brane solitonic objects, the associated $D$-dimensional geometry can be factorized as follows

$$AdS_d \times S^{D-d}. \quad (2.1)$$

This spacetime geometry could be interpreted as the near horizon geometry of $(d - 2)$ branes in such superstring/M-theory inspired models [51]. An examination on the sphere compactification shows that such higher dimensional models should, at least, involve a $(D - d)$ strength gauge field $F_{D-d}$ contributing with the term $\int d^d x F_{D-d}^2$ in the associated lower dimensional black hole action. The presence of such a term is supported by a $(D - d - 1)$ gauge form coupled to a $(D - d - 2)$-brane supposed to live in such higher dimensional inspired models. After a close inspection, it has been remarked that one has two possible distinct brane objects in the proposed models. They could be classified as

- $(d - 2)$-branes associated with the $AdS_d$ geometry of the black hole,
- $(D - d - 2)$-branes corresponding to the $S^{D-d}$ sphere compactification.

In the study of such higher dimensional inspired models, two cases could arise

1. $D - d - 2 = d - 2$ leading to $D = 2d$,
2. $D - d - 2 \neq d - 2$ giving $D \neq 2d$.

It is worth noting that particular cases appear in known theories associated with $D = 10$ and $D = 11$. For instance, the first case arises in type IIB superstring in the presence of D3-branes [52]. However, the second situation takes place in M-theory involving M2 and M5-branes [53, 54]. Roughly, the two previous conditions can be put in one relation given by

$$D = 2d + k,$$  \hspace{1cm} (2.2)

where now $k$ is an arbitrary integer which will be used to specify the internal space of $(D - d)$ dimensions. In this notation, the $d$-dimensional AdS black holes are obtained by the compactification of the $D$-dimensional theory on the real spheres $S^{d+k}$. The resulting models will be classified by a triplet $(D, d, k)$ subject to the relation (2.2). Using this notation, the electric-magnetic duality is assured by the transformation

$$(D, d, k) \leftrightarrow (D, d + k, -k).$$  \hspace{1cm} (2.3)

In terms of the AdS geometry, this duality can be rewritten as

$$AdS_d \times S^{d+k} \leftrightarrow AdS_{d+k} \times S^d.$$  \hspace{1cm} (2.4)

Well known theories correspond to special choices of the triplet $(D, d, k)$. For instance, the space-time of type IIB superstring theory associated with the triplet $(10, 5, 0)$ is given by $(AdS_5)_L \times (S^5)_L$ linked to D3-brane physics. For M-theory, one has the triplet $(11, 4, 3)$ described by $(AdS_4)_{L/2} \times (S^7)_L$ based on M2-branes being dual to the triplet $(11, 7, -3)$ associated with the $(AdS_7)_{2L} \times (S^4)_L$ geometry relying on M5-branes.

The near horizon geometry of the $(d - 2)$-branes spacetime manifold becomes now

$$AdS_d \times S^{d+k},$$  \hspace{1cm} (2.5)

with the associated line element given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx^i dx^j + L^2 d\Omega_{d+k}^2.$$  \hspace{1cm} (2.6)

It is noted that $h_{ij} dx^i dx^j$ is the line element of a $(d - 2)$-dimensional Einstein space $(\Xi_{d-2}) [55]$. The quantity $d\Omega_{d+k}^2$ is the metric of the $S^{d+k}$ real sphere with radius $L$.

In an AdS/CFT context, this radius is linked to the brane number [56]. The function $f(r)$ depends on physical parameters including the possible existence of non trivial backgrounds, i.e. quintessence. Such a situation will be dealt with in the present study.

Using the well established procedures (see [48–50],[57–59]), we can check that a line element metric with the $f(r)$ function of the form

$$f(r) = 1 - \frac{M}{r^{d-3}} - \sum n \left( \frac{r_n}{r} \right)^{(d-1)\omega_n + (d-3)},$$  \hspace{1cm} (2.7)
where $\omega_n$ are free parameters and $r_n$ are dimensional normalization constants, is a solution of the Einstein Equations for a suitable energy-momentum tensor.

Some well known situations are particular cases of (2.7). For instance, one can consider the case

$$f(r) = 1 - \frac{m}{r^{d-3}} - \frac{c}{r^{(d-1)\omega_q+(d-3)}},$$

(2.8)

where $m$ is an integration constant and $c$ represents the DE contributions [60, 61]. The associated quintessence energy density $\rho_q$ is given by

$$\rho_q = -\frac{c\omega_q(d-1)(d-2)}{4r^{(d-1)\omega_q+1}}.$$  

(2.9)

For $\omega_0 = -1$, we obtain

$$f(r) = 1 - \frac{m}{r^{d-3}} + \left(\frac{r}{L}\right)^2,$$

(2.10)

producing an AdS-Schwarzschild black hole solution in $d$-dimensions. However, the spherical Schwarzschild solution is obtained by taking the large limit of AdS radius ($L^2 \rightarrow \infty$).

Another case associated with the $d$-dimensional Reissner-Nordstrom black hole is obtained by taking $\omega_1 = \frac{d-3}{d-1}$

$$f(r) = 1 - \frac{m}{r^{d-3}} + \left(\frac{r}{L}\right)^2 + \frac{Q^2}{r^{2(d-3)}}$$

(2.11)

where $Q$ denotes the associated charge. A close inspection shows that a $d$-dimensional AdS-Schwarzschild black hole, in the presence of quintessence, can be described by the following metric function

$$f(r) = 1 - \frac{m}{r^{d-3}} + r^2 \frac{c}{L^2} - \frac{c}{r^{(d-1)\omega_q+d-3}},$$

(2.12)

where $c$ is a positive normalization factor associated with DE intensity given by $r_q^{(d-1)\omega_q+(d-3)}$.

This last case will treated in full detail in the next sections.

2.1 Thermodynamics of $d$-dimensional AdS-Schwarzschild black hole in the presence of quintessence

A $d$-dimensional AdS-Schwarzschild black hole, in the presence of quintessence with parameter $\omega_q$, can be described by the metric function given in Eq.(2.12). The event horizon $r_h$ in this case, is determined by setting Eq.(2.12) equal to zero ($f(r) = 0$). This gives as solution for the coefficient $m$ the value

$$m = r_h^{d-3} + r_h^{d-1} \frac{c}{L^2} - c r_h^{\omega_q(1-d)}.$$  

(2.13)

It turns out that the general form of the black hole mass reads as

$$M_d = \frac{(d-2) \omega_{d-2}}{16\pi G_d} m,$$

(2.14)
where the factor \( \varpi_{d-2} \) is given by

\[
\varpi_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\left(\frac{d-3}{2}\right)!},
\]
(2.15)

identified with the volume of the Einstein space \( \Xi_{d-2} \) [62]. It is recalled that \( G_d \) is the gravitational constant [62, 63]. Using Eq.(2.13), we get the mass expression

\[
M_d = \frac{(d-2) \varpi_{d-2}}{16\pi G_d} \left( r_h^{d-3} + \frac{r_h^{d-1}}{L^2} - c r_h \omega_d^{(1-d)} \right).
\]
(2.16)

For \( d \)-dimensional AdS black holes [64], the general formula of Bekenstein-Hawking entropy takes the following form

\[
S_d = \frac{\varpi_{d-2} r_h^{d-2}}{4G_d}.
\]
(2.17)

In terms of such an entropy, \( r_h \) is given by

\[
r_h = \left( \frac{4G_d}{\varpi_{d-2}} \right)^{\frac{1}{d-2}} S_d^{\frac{1}{d-2}}.
\]
(2.18)

The gravitational constant \( G_d \) in such a \( d \)-dimensional AdS black hole should be related to the one corresponding to \((2d+k)\)-dimensional inspired models. It is noted that \( d \)-dimensional AdS black hole theory can be obtained by the compactification of the \((2d+k)\)-dimensional theory on the \( S^{d+k} \) sphere of radius \( L \) [34]. From such dimensional spherical reductions, we get

\[
G_d = \frac{G_{2d+k}}{\text{Vol}(S^{d+k})} = \frac{G_{2d+k}}{\omega_{d+k} L^{d+k}},
\]
(2.19)

where \( \omega_{d+k} \) is given by

\[
\omega_{d+k} = \frac{2\pi^{(d+k+1)/2}}{\Gamma\left(\frac{d+k+1}{2}\right)}.
\]
(2.20)

After a close inspection, we show that the radius \( L \) verifies

\[
L^{2(d-1)+k} = 2^{-\left(\frac{d(4-d)+3}{2}\right)} \pi^{7(k+2(d-5))-4} N^{\frac{d-1}{2}} \ell_p^{2(d-1)+k}
\]
(2.21)

where \( \ell_p \) and \( N \) are the Planck length and the brane number, respectively.

By putting Eqs.(2.15),(2.18) and (2.19) in Eq.(2.16), the general mass form reads as

\[
M_d^{(k)}(S, N, c) = \frac{(d-2) \pi^{\frac{d-3}{2}} \ell_p^{d+k-2} \omega_{d+k} B^{\frac{d+k-2}{2}}}{8 \left(\frac{d-3}{2}\right)!} G_{2d+k}
\times \left\{ \ell_p^2 B \left[ A^{\frac{d-3}{2}} - c A^{\frac{-\omega_d^{(d-1)}}{d-k}} \right] + A^{\frac{d-1}{d-k}} \right\},
\]
(2.22)

where \( A \) and \( B \) are given by

\[
B(N) = \frac{\ell_p^{2(d-4)+3}}{\pi} \frac{2(7(k+2(d-5))-4)}{k+2(d-1)} N^{\frac{d-1}{k+2(d-1)}},
\]
(2.23)

\[
A(S, N) = \frac{(d-3)!}{\pi^{\frac{d-1}{2}}} \omega_{d+k} \ell_p^{d+k} B^{\frac{d+k}{2}}
\]
(2.24)
Exploiting the first law of black hole thermodynamics
\[ dM = TdS + \mu dN^{\frac{d-1}{2}} \] (2.25)
and using Eq. (2.22), we find the associated thermodynamic temperature \( T \)
\[
T^{(k)}_d(S, N, c) = \left. \frac{\partial M^{(k)}_d}{\partial S} \right|_N = \frac{1}{8S} \frac{\pi \left( \frac{d-3}{2} \right) \ell_p^{d+k-2} \omega_{d+k} B^{\frac{d+k-2}{2}}}{G_{2d+k}} \times \left\{ \ell_p^2 B \left[ (d - 3) A^{\frac{d-3}{2}} - c \omega_q (d - 1) A^{-\omega_q(d-1)} \right] + (d - 1) A^{\frac{d-1}{2}} \right\}.
\] (2.26)
and the chemical potential \( \mu \), being conjugate variable associated with the number of colors \( N^{(d-1)/2} \), as
\[
\mu^{(k)}_d(S, N, c) = \left. \frac{\partial M^{(k)}_d}{\partial N^{\frac{d-1}{2}}} \right|_S = \frac{2}{(d - 1) \cdot N^{\frac{d-3}{2}}} \left( \frac{\partial M^{(k)}_d}{\partial N} \right) \bigg|_S.
\] (2.27)
A simple calculation gives
\[
\mu^{(k)}_d(S, N, c) = \frac{1}{8N^{\frac{d-1}{2}} k + 2(d - 1)} \left( \frac{d-3}{2} \right)! \frac{\pi \left( \frac{d-3}{2} \right) \ell_p^{d+k-2} \omega_{d+k} B^{\frac{d+k-2}{2}}}{G_{2d+k}} \times \left\{ \ell_p^2 (d + k) B \left[ A^{\frac{d-3}{2}} - c ((d - 2) + (d - 1) \omega_q) A^{-\omega_q(d-1)} \right] + (3d + k - 4) A^{\frac{d-1}{2}} \right\}.
\] (2.28)
The Gibbs (or Helmholtz) free energy can be also computed by applying the following relation
\[ G^{(k)}_d(S, N, c) = M^{(k)}_d(S, N, c) - T^{(k)}_d(S, N, c) \cdot S. \] (2.29)
Using Eqs (2.22) and (2.26), we get
\[
G^{(k)}_d(S, N, c) = \frac{\pi \left( \frac{d-3}{2} \right) \ell_p^{d+k-2} \omega_{d+k} B^{\frac{d+k-2}{2}}}{8 G_{2d+k}} \times \left\{ \ell_p^2 B \left[ A^{\frac{d-3}{2}} - c [ (\omega_q + 1) (d - 1) - 1] A^{-\omega_q(d-1)} - A^{\frac{d-1}{2}} \right] \right\}.
\] (2.30)

3 The baseline case: Hawking-Page phase transitions without dark energy

We consider first the Hawking-Page phase transition in the absence of DE (the case \( c = 0 \) in the previous section) in different models. These models will be denoted by the triplet \((D, d, k)\). Fixing the space-time dimension \( D \), the thermodynamical quantities will be indicated by two numbers \( d \) and \( k \). The thermodynamical quantities will be referred to as \( X^{(k)}_d \). In absence of DE, \( X^{(k)}_d \) depend only on the brane number \( N \) and the entropy \( S \).

Here, we deal with a model associated with the compactification of 11-dimensional M-theory in the presence of \( N \) coincident M5-branes, indexed by the triplet
\[ (D, d, k) = (11, 7, -3). \] (3.1)
This corresponds to the AdS$_7 \times S^4$ space-time, where the gravitational constant in eleven dimensions is $G_{11} = 2^4 \pi^7 \ell_p^6$. In this case, the internal space is the four-dimensional real sphere $S^4$ with the volume factor

$$\omega_4 = \frac{8 \pi^2}{3},$$

(3.2)

obtained from Eq.(2.20). This compactification produces a seven-dimensional AdS black hole. Using Eq.(3.2), we compute the corresponding thermodynamical quantities $X_d^{(k)}$ denoted by the doublet $(7, -3)$. They are listed as follows

- **mass**

$$M_7^{(-3)}(S, N) = \frac{5 \left[ 8 \times 3^{3/5} \pi^{2/5} S^{2/5} N^{4/5} + 3 \times 2^{3/5} S^{6/5} N^{14/5} \right]}{16 \times 6^{4/5} 3^{1/5} \pi^{23/15} \ell_p},$$

(3.3)

- **Hawking temperature**

$$T_7^{(-3)}(S, N) = \frac{16 \times 3^{3/5} \pi^{2/5} S^{-1/5} N^{4/5} + 9 \times 2^{3/5} S^{6/5} N^{14/5}}{8 \times 6^{4/5} \pi^{23/15} \ell_p},$$

(3.4)

- **chemical potential**

$$\mu_7^{(-3)}(S, N) = \frac{16 \times 3^{3/5} \pi^{2/5} S^{2/5} N^{-11/5} - 21 \times 2^{3/5} S^{6/5} N^{29/5}}{24 \times 6^{4/5} \pi^{23/15} \ell_p},$$

(3.5)

- **Gibbs free energy**

$$G_7^{(-3)}(S, N) = \frac{8 \times 3^{3/5} \pi^{2/5} S^{2/5} N^{4/5} - 3 \times 2^{3/5} S^{6/5} N^{14/5}}{16 \times 6^{4/5} \pi^{23/15} \ell_p}.$$

(3.6)

In order to investigate the behaviour of the system in presence of the Hawking-Page transitions, we first consider the dependence of the Hawking temperature with respect to the entropy. This is illustrated in figure 1. For a generic number $N$ of M5-branes, the Hawking temperature has a minimum for

$$S_7^{\text{min}} = \frac{2^{17/2}}{3^{7/2}} \pi N^3,$$

(3.7)

corresponding to the minimal temperature

$$T_7^{\text{min}} = \frac{2^{1/2}}{2^{1/2} \pi^{4/3} N^{1/3} \ell_p}.$$

(3.8)

Below such a minimal temperature, no black hole solution can exist. However, it follows from figure 1 that above this temperature one can distinguish two branches. One corresponding to a small entropy is associated with a thermodynamically unstable black hole. The second branch, with a large entropy, describes a thermodynamically stable black hole.
The sign of the Gibbs free energy changes at the Hawking-Page temperature $T^H_P$ associated with $S^H_P$. Taking into account the Gibbs free energy given in Eq. (3.6), this phase transition occurs at ($G(T^H_P, N) = 0, S^H_P = S_{G=0}$)

$$S^H_P = \frac{96}{3} \pi N^3,$$

(3.9)
corresponding to the Hawking-Page temperature

$$T^H_P = \frac{5}{4 \pi^{4/3} N^{1/3} \ell_p},$$

(3.10)
which is always greater than $T^{min}_I$. The dependence of the brane number $N$ on the Hawking-Page phase transition of the $AdS$ black hole is represented in figure 2. In particular, we plot the Gibbs free energy as a function of the Hawking temperature $T^I$ for different values of $N$. In this plot, it becomes apparent that below $T^{min}_I$ (the higher black dot) no thermodynamically stable black hole can survive. The Gibbs free energy reaches a maximum at $T^{min}_I$. For lower Gibbs free energy values, one finds two branches. The upper branch describes a small (unstable) black hole with a negative specific heat. The lower branch corresponds to (large) stable black hole solutions with positive specific heat values. The crossing point in this branch for which $G = 0$ corresponds to $T^H_P$. At this point, the (first order) Hawking-Page phase transition occur between large (stable) black holes and a thermal radiation state [26]. In figure 2, we plot the chemical potential as a function of the entropy $S$ for a fixed number of $M5$-branes. The chemical potential is positive for smaller entropy $S$, and negative for large values. The sign change in the chemical potential occurs at the entropy value

$$S^\mu=0 = \frac{2^{17/2}}{3 \cdot 7^{5/2}} \pi N^3,$$

(3.11)
which corresponds the temperature

$$T^\mu=0 = \frac{5}{\sqrt{14} \pi^{4/3} \ell_p N^{1/3}}.$$

(3.12)
We have $S^\mu=0 < S^{min}_I < S^H_P$. The entropy where the chemical potential changes the sign is less than the one at which no black hole can exist. The chemical potential dependence as a function of $N$ with a fixed entropy $S$ is showed in 2. It follows from figure 2 that the maximum of the chemical potential corresponds to the entropy

$$S^{max}_I = \frac{2^{17/2}}{3 \cdot 7^{5/2}} \left(\frac{41}{59}\right)^{5/2} \pi (N^{max}_I)^3.$$

In figure 2, we plot the chemical potential as a function of the temperature $T$ for a fixed number of $M5$-branes $N$. It is noted that the dot, appearing in figure 2, indicates the minimum of the temperature $T^{min}_I$. However, bellow this point one can find the Hawking-Page temperature $T^H_P$, separating the stable lower branch (very low values of the chemical potential) and the upper unstable branch where $T^{min}_I$ resides. From Eq.(3.11), the chemical
potential is positive for the brane number condition $N^3 > \frac{3.75/2}{2^{1/2}} \frac{S}{\pi}$. This limit can be saturated for the temperature $T_{\mu=0}^d \simeq 1.1 T_{\mu=0}^{d+1}$. This can be seen clearly in the transition points listed in table 1, figures 1, 6 and 11. This can simply be understood from the extensive character of the Bekenstein Hawking entropy. However, although increasing with the dimension, from the table 1, we remark the universal character of the normalized ratio $\frac{S_{\mu=0}^d}{S_{\mu=0}^{d+1}}$, in fact in all the cases, at different space-time dimensions,

$$\frac{S_{\mu=0}^d}{S_{\mu=0}^{d+1}} \simeq \frac{1}{3}.$$  (3.14)

3.1 The Darkness case: further models

Similar results are obtained for other cases. In particular, the results for $(D, d, k) = (11, 4, 3)$ and $(D, d, k) = (10, 5, 0)$ will be presented in appendices A and B, respectively. A summary of different quantities associated to each model is shown in table 1.

From this table, one can observe some systematic features among the models. First, we notice an increasing behavior for the entropy as increases the dimension $d$, the dimension in which the AdS black holes live. This can be seen clearly in the transition points listed in table 1, figures 1, 6 and 11. This can simply be understood from the extensive character of the Bekenstein Hawking entropy. However, although increasing with the dimension, from the table 1, we remark the universal character of the normalized ratio $\frac{S_{\mu=0}^d}{S_{\mu=0}^{d+1}}$, in fact in all the cases, at different space-time dimensions,
Independently of the dimension $d$, the entropy where the large black holes transit to thermal AdS is approximately three times the entropy for which there is no black hole solution. This relation seems to be a universal one.

Second, one can notice an obvious decrease in the number of branes ($N_{\text{max}}$, the value for which the chemical potential is maximum for fixed entropy) as the dimension $d$ of the AdS space increases: the four-dimensional AdS space “needs” a larger number of $M2$-branes in comparison to the seven-dimensional AdS space with $M5$-branes in the context of the M-theory compactification.

Third, the Hawking temperature of five-dimensional AdS black hole in type IIB superstring is bigger than the one associated with other AdS black holes appearing in M-theory. Taking a particular case of the ratio $T_5^{\text{min}}/T_7^{\text{min}}$ which is equal to

$$\frac{2^{7/8}\pi^{5/6}N^{1/12}}{3^{1/2}},$$

we find $T_5^{\text{min}} > T_7^{\text{min}}$. Since, below $T_d^{\text{min}}$ no black hole can exist, we see that this phase
is larger for $AdS_5$ followed by $AdS_7$ and $AdS_4$. However, the radiation phase of $AdS_5$ is larger than the radiation phase of $AdS_4$. The smaller radiation phase correspond to $AdS_7$ case. Taking the ratio $T_4^{HP}/T_7^{HP}$ for the particular case of $N = 1$, we find the opposite behavior.

4 Hawking-Page phase transitions in presence of dark energy

In this section, we investigate the effect of DE surrounding AdS black holes. The DE state parameters are the free parameter $c$ (see Eq.2.12) and $\omega_q = \frac{\rho}{p}$ defined in terms of a ratio of the pressure and the density of DE in the universe [65–67]. Many forms of such energy have been proposed depending on particular values of $\omega_q$. Among others, quintessence associated with a dynamic field has been extensively investigated including string theory and brane physics [68, 69]. Here, such DE contributions will be approached in the context of $AdS$ black holes in M-theory/IIB superstring inspired models.

We first compute the AdS black hole thermodynamical quantities like the mass, the temperature, the Gibbs free energy and the chemical potential. This will be made using the general relations obtained in section 2. Then, we discuss the effect of the quintessence on the associated phase transitions. The previous thermodynamical quantities $X_d^{(k)}(N,S)$ will be replaced by $X_d^{DE (k)}(N,S,c)$ where $c$ represents the DE contributions. The parameter $w_q$ will be fixed by dimensional considerations for any of the models.

We discuss in detail a model denoted by the triplet $(D,d,k) = (11,7,−3)$, i.e. a compactification of M-theory on the sphere $S^4$ in the presence of $M5$-branes: $AdS_7 \times S^4$. Further results for $AdS_4 \times S^7$ and $AdS_5 \times S^5$ spacetimes will be presented in the appendices C and D, respectively.

Using Eq.(3.2), we get the following expressions for thermodynamic variables of interest

- mass
  
  \[ M_7^{(-3) \ DE}(S, N, c) = M_7^{(-3)}(S, N) - \alpha_7^{\ DE}(S, N, c), \]  
  \[ (4.1) \]

- temperature
  
  \[ T_7^{(-3) \ DE}(S, N, c) = T_7^{(-3)}(S, N) + \frac{6\omega_q}{5} \cdot \frac{\alpha_7^{\ DE}(S, N, c)}{S}, \]  
  \[ (4.2) \]

- chemical potential
  
  \[ \mu_7^{(-3) \ DE}(S, N, c) = \mu_7^{(-3)}(S, N) - \frac{4(6\omega_q + 5)}{45} \cdot \frac{\alpha_7^{\ DE}(S, N, c)}{N^2}, \]  
  \[ (4.3) \]

- Gibbs free energy
  
  \[ G_7^{(-3) \ DE}(S, N, c) = G_7^{(-3)}(S, N) - \left(\frac{6\omega_q + 5}{5}\right) \cdot \alpha_7^{\ DE}(S, N, c). \]  
  \[ (4.4) \]
For all these quantities, $\alpha^{DE}_7(S,N,c)$ denotes the DE contribution which is given by

$$\alpha^{DE}_7(S,N,c) = \frac{5c \, 2^{6\omega_q - 5} \, N^{4(6\omega_q + 5)} \, 15}{3 \, \pi \, \frac{6\omega_q + 5}{5} \, \frac{6\omega_q + 25}{5} \, S^{6\omega_q + 5} \, \ell_p^{3}}. \quad (4.5)$$

It is noted that $c$ is a free parameter (in a certain range), the quantity $w_q$ is restricted by dimensional analysis. In associated black hole physics, the thermodynamical quantities should be proportional to the Planck mass $m_p$. Indeed, an examination shows that such quantities are all proportional to $\ell_p^{-1}$. In the $AdS_7 \times S^4$ for instance, we have

$$[M_7^{(-3)}] = [M_7^{(-3)}] = [\alpha_7^{(-3)}]. \quad (4.6)$$

Since $M_7^{(-3)}$ should be proportional to $\ell_p^{-1}$, $\alpha_7^{(-3)}$ should have the same physical dimension. Using the expression of $\alpha_7^{(-3)}$ given in Eq.(4.1), we find that this quantity is proportional to $\ell_p^{-(6\omega_q + 5)}$. A simple calculation gives

$$\omega_q = -\frac{2}{3}. \quad (4.7)$$

Other values $\omega_q$ will be obtained for $S^7$ and $S^5$ compactifications (see appendices C and D, respectively). In this way, (4.5) reduces to

$$\alpha_7^{DE} (S,N,c) = \frac{5c \, 2^{-9/5} \, N^{4/15}}{3^{4/5} \pi^{17/15} \, S^{-4/5} \, \ell_p}. \quad (4.8)$$

To investigate the associated phase transitions behaviours, we first plot the Hawking temperature as a function of the entropy as illustrated in figure 3 for different values of the intensity $c$.

For a general number of $M5$-branes and a general intensity of DE dynamical field, we can show that the Hawking temperature has a minimum for

$$S_7^{DE-min} = \frac{2^{17/2}}{3^{1/2}} \, (1 - c)^{5/2} \, \pi N^3. \quad (4.9)$$

This corresponds to the following minimal temperature

$$T_7^{DE-min} = \frac{3^{1/2} \sqrt{1 - c}}{2^{1/2} \, \pi^{1/3} \, N^{1/3} \, \ell_p^3}. \quad (4.10)$$

From figure 3, we see that the temperature is decreasing when the DE intensity $c$ increases. This shows that DE behaves like a cooling system surrounding the black hole. This confirms the results obtained in [50]. The minimum of the temperature and the corresponding entropy noted by the black dotes in such a figure are also affected in the same way.

For more investigations in the phase transitions of the seven-dimensional $AdS$ black hole, we illustrate the Gibbs free energy as a function of the Hawking temperature $T_7^{(-3)}$ for different values $N$ of $M5$-branes and a fixed value of $c$. This is given in figure 3.

It is remarked, from this figure, that the phases discussed in figure 3 are getting smaller. The Gibbs free energy decreases also, yielding a smaller unstable black hole phase. From
Figure 3. (Left) Temperature as function of the entropy using $N = 3$ and $\ell_p = 1$ and different values of $c$. The dots denote the minimum of the temperature for each case. (Right) The Gibbs free energy as function of temperature for different values of $N$, $\ell_p = 1$ and $c = 0.2$. The sign of Gibbs free energy changes at the Hawking-Page temperature $T_{HP}^\ell$.

the Gibbs free energy given in Eq.(4.4), one can find the Hawking-Page phase transition corresponding to

$$S_{DE}^{DE-HP} = \frac{2^6}{3} (1 - c)^{5/2} \pi N^3. \quad (4.11)$$

Thus, the Hawking-Page temperature is

$$T_{DE}^{DE-HP} = \frac{5\sqrt{1-c}}{4 \pi^{3/2} N^{1/3} \ell_p}, \quad (4.12)$$

which is smaller than $T_{DE}^{DE-min}$. To identify the difference in the behavior of the Gibbs free energy in the presence of quintessence, we plot this function for $c = 0$ (absence of DE) and for $c = 0.2$ for several values $N$ of M5-branes in figure 4.

We see that the decrease in the radiation phase and the phase where no black hole can exist comes from the temperature diminution. However, the stable and the unstable phases are directly affected by the diminution of the Gibbs free energy.

It has been observed that the quintessence field affects also the chemical potential. To inspect the corresponding modifications, we plot the chemical potential as a function of entropy $S$ for $N = 3$ in figure 5.

It is remarked that the chemical potential is positive for small entropy $S$, and negative for large $S$. Moreover, the figure 5 shows a diminution of both the chemical potential and the entropy when DE is present. Besides, the gap between the curves seems to increase when the entropy increases. The chemical potential changes the sign at the following entropy

$$S_{DE-\mu=0}^7 = \frac{2^{17/2}}{3 \cdot 7^{5/2}} (1 - c)^{5/2} \pi N^3. \quad (4.13)$$

As mentioned above, we can show that $S_{DE-\mu=0}^7 < S_{DE-min}^7 < S_{DE-HP}^7$.

Furthermore, we can also study the behavior of the chemical potential as a function of the number $N$ of M5-branes in such a M-theory compactification. This can be illustrated...
Figure 4. The Gibbs free energy as function of the temperature in the absence of DE and in its presence for different values of $N$, $\ell_p = 1$.

In figure 5. Here, the entropy $S$ has a fixed value. The maximum of the chemical potential corresponds to the point

$$S_{T}^{DE-max} = \frac{2^{17/2}}{3 \cdot 75/2} \left( \frac{41}{59} \right)^{5/2} (1 - c)^{5/2} \pi \left( N_{T}^{DE-max} \right)^{3},$$

namely, $N_{T}^{DE-max} \simeq 1.81$ for $S_{T}^{max} = 4$ and $c = 0.2$. From $N_{T}^{DE-max}$, we see that the number of M5-branes grows in the presence of DE in the M-theory compactification on $S^4$.

Besides, we can also reveal that DE stabilises the AdS black hole. In figure 5, we plot the chemical potential as a function of temperature $T_{T}^{(-3)DE}$ for a fixed $N$, where the dots denotes $T_{T}^{DE-min}$. Lower from this point, we have $T_{T}^{DE-HP}$ which separates the lower stable branch and the upper unstable branch where $T_{T}^{DE-min}$ resides.

We notice that $T_{T}^{DE-min}$ is getting higher in the curves when the intensity of DE $c$ is bigger, which is also true for $T_{T}^{DE-HP}$. In the presence of DE, then, the unstable branch is getting smaller. However, the stable one becomes relevant.
Figure 5. (Top) The chemical potential as a function of entropy for $N = 3, \ell_p = 1$. The sign change of the chemical potential happens at $S_{DE}^{\mu=0}$. (Center) The chemical potential as a function of the number of M5-branes $N$. Here we take $\ell_p = 1$ and $S_{7}^{\text{max}} = 4$. (Bottom) The chemical potential as a function of temperature $T$. We take $\ell_p = 1$ and $N = 3$.

4.1 DE effect on AdS black holes

Let us have a closer look on the effects of DE energy presence on $d$-dimensional AdS black holes embedded in M-theory/superstring inspired models. A close examination shows that the $d$-dimensional AdS black hole entropy can be put in a compact formula given by

$$S_{d}^{DE-i} (N, c) = (1 - c)^{\frac{d+2}{2}} \cdot S_{d}^{i} (N),$$

(4.15)

where $i$ stands for the set $\{\text{min}, \text{HP}\}$. It is worth noting that $(1 - c)$ is interpreted as a DE scaling factor depending on the dimension $d$ of the considered AdS black hole. We observe
a decrease in the entropy values of such $AdS$ black holes. This means that DE induces the reduction of the associated number of microstates.

Putting this entropy in the equation of the Hawking temperature given in (2.26), the associated $T_{d}^{DE-i}$ temperature can be obtained. Indeed, it satisfies the following general formula

$$T_{d}^{DE-i} (N, c) = (1 - c)^{\frac{1}{2}} \cdot T_{d}^{i} (N).$$

(4.16)

For the temperature, however, DE scaling factor does not depend on the dimension of the $AdS$ black holes. This can be understood that the temperature does not depend on the size parameter of the theory in question. It follows also that Eq.(4.16) provides a colder black hole, being a more stable one. Moreover, we observe an increasing behavior regarding the number $N$ of $(d - 2)$-branes. In $d$ dimensions, such a number takes the following general form

$$N_{d}^{DE-max} (S, c) = \frac{N_{d}^{max} (S)}{(1 - c)^{\frac{d+2}{d-1}}}. \quad (4.17)$$

In M-theory/superstring inspired models, it follows that $N_{d}^{DE-max} (S, c)$ grows with the DE contributions. Thus, DE enhances the number of branes by generating non trivial extra branes which we refer to as "Dark-branes."

5 Conclusion and open questions

In this work, we have investigated the thermodynamical phase transitions of $d$-dimensional $AdS$ black holes surrounded by DE. These black hole solutions have been embedded in $D$-dimensional superstring/M-theory inspired models with the $AdS_d \times S^{d+k}$ space-time, where $D = 2d + k$ is their Minkowski dimension. These models, which could be associated with $N$ coincident $(d - 2)$-branes supposed to live in such higher dimensional inspired theories, have been labeled by a triplet $(D, d, k)$ where $k$ carries data on the internal space $S^{d+k}$. By interpreting the cosmological constant as the number of colors $N_{d-1}^{\frac{d+1}{2}}$, we have computed various thermodynamical quantities denoted by $X^{(k)DE}_{d}$ in terms of the brane number $N$, the entropy $S$ and DE contributions via a dynamical quintessence scalar field.

By calculating the chemical potential conjugated to the number of colors in the absence of DE, we have found that the black hole is more stable, enjoying a large number of branes for lower dimensions $d$. In the presence of DE, we have realised that the state parameter $\omega_q$ should have specific values, for $(D, d, k)$ models, providing non trivial phase transition results.

Among others, we have obtained a smaller no black hole phase, a more stable and colder black hole. Furthermore, we have found an enhancement regarding the number of branes which we refer to as Dark-branes. We believe that such suggestions need deeper investigations. We hope to come back to this non trivial remark in connection with cosmology in future works.

Inspired by sphere compactifications in higher dimensional theories, various models could be examined for quintessential $AdS$ black holes. A possible situation is associated
with trivial sphere fibrations with the same dimension $n$. In this way, the internal space $X^{d+k}$ can be factorized as

$$X^{d+k} = S^n \times S^n \times \cdots \times S^n.$$  \hfill (5.1)

Borrowing ideas from intersecting attractors [70], these models could provide non trivial phase transitions corresponding to such geometric fibrations. In this context, lower dimensional cases could be approached using group theory techniques. Another road is to think about orbifolding spheres generating twisted sectors in the resulting compactified theories. This could bring new features to $AdS$ black holes surrounded by non trivial contributions including dark matter. Once these sphere geometries become accessible, Calabi-Yau manifolds could find places in the building of $AdS$ black holes from such M-theory/superstring inspired models.

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**A Eleven-dimensional compactification: The $AdS_4 \times S^7$ case without DE**

We consider here eleven-dimensional M-theory, with $AdS_4 \times S^7$ space-time in the presence of $N$ coincident $M2$-branes, i.e. a model with characterised by the triplet

$$(D, d, k) = (11, 4, 3).$$  \hfill (A.1)

Using the same procedure as in previous sections, we compute the thermodynamical quantities $X_4^{(3)}$ for such four-dimensional $AdS$ black holes obtained from the compactification of M-theory on seven dimensional real sphere $S^7$. In this case the volume factor is

$$\omega_7 = \frac{\pi^4}{3}.$$  \hfill (A.2)
We list the final expressions of some of these quantities:

- the mass parameters
  \[ M_4^{(3)}(S,N) = \frac{\pi^{3/2} S^{1/2} N^{3/2} + 96\sqrt{2} \times S^{3/2} N^{-11/12}}{8 \times 2^{7/12} \sqrt{3} \pi^{25/12} \ell_p}, \]  

- the temperature
  \[ T_4^{(3)}(S,N) = \frac{\pi^{3/2} S^{-1/2} N^{7/12} + 3 \times 96\sqrt{2} \times S^{3/2} N^{-11/12}}{16 \times 2^{7/12} \sqrt{3} \pi^{25/12} \ell_p}, \]  

- chemical potential
  \[ \mu_4^{(3)}(S,N) = \frac{7 \times \pi^{3/2} S^{1/2} N^{-7/12} - 11 \times 96\sqrt{2} \times S^{3/2} N^{-11/12}}{96 \times 2^{7/12} \sqrt{3} \pi^{25/12} \ell_p}, \]  

- Gibbs free energy
  \[ G_4^{(3)}(S,N) = \frac{\pi^{3/2} S^{1/2} N^{7/12} - 96\sqrt{2} \times S^{3/2} N^{-11/12}}{16 \times 2^{7/12} \sqrt{3} \pi^{25/12} \ell_p}. \]  

To investigate the existence of phase transitions, we first consider the behavior of the Hawking temperature with respect to entropy. This is illustrated in figure 6.

The temperature reaches its minimum at \( S_4^{\text{min}} \approx 0.07 \). For a general number of M2-branes \( N \), one can reveal that the Hawking temperature has a minimum at

\[ S_4^{\text{min}} = \frac{\pi^{3/2}}{3^2 \cdot 2^{11/2}} N^{3/2}, \]  

corresponding to the minimal temperature

\[ T_4^{\text{min}} = \frac{3^{1/2}}{2^{5/6} \pi^{4/3} N^{1/6} \ell_p}. \]
Figure 7. The Gibbs free energy as function of temperature for different values of $N$ and $\ell_p = 1$.

Under the minimal temperature no black hole solution can survive. However, we observe from figure 6 that above such a temperature two branches arise. The first one is associated with small entropy corresponding to a thermodynamically unstable black hole. The second one, with large entropy, indicates a thermodynamically stable black hole.

To discuss the brane number $N$ effect on the phase transition, we use the computed Gibbs free energy. In figure 7, we plot the Gibbs free energy as a function of the Hawking temperature $T_{4}^{(3)}$ for different values $N$ of $M2$-branes and such M-theory backgrounds.

As before the sign of Gibbs free energy changes at the Hawking-Page temperature $T_{4}^{HP}$ associated with $S_{4}^{HP}$, corresponding to the Hawking-Page phase transition and given by the lower dot in figure 7. From the Gibbs free energy given in Eq.(A.6), such a phase transition appears at

$$S_{4}^{HP} = \frac{\pi^{3/2} N^{3/2}}{3 \cdot 2^{11/2}}.$$  \hfill (A.9)

The corresponding Hawking-Page temperature is

$$T_{4}^{HP} = \frac{2^{1/6}}{\pi^{3/3} N^{1/6} \ell_p},$$  \hfill (A.10)

being smaller than $T_{4}^{min}$.

In figure 7, a first order Hawking-Page phase transition arises at $T_{4}^{HP}$ between large (stable) black holes and the thermal radiations. Besides, one can see that below the higher dot representing $T_{4}^{min}$ no black hole can survive. For lower Gibbs free energy, we have two branches. The upper branch describes a small (unstable) black hole with a negative specific heat. However, the lower one indicates a (large) stable black hole solution with a positive specific heat.

The chemical potential can be also exploited to examine the phase transitions of the studied four-dimensional AdS black hole embedded in the M-theory compactification on
Figure 8. The chemical potential as a function of entropy for $N = 3, \ell_p = 1$. The sign change of the chemical potential happens at $S_4^{\mu=0} \simeq 0.13$.

Figure 9. The chemical potential as a function of the number of $M2$-branes $N$. Here we take $\ell_p = 1$ and $S = 4$.

$S^7$. In figure 8, we plot the chemical potential as a function of the entropy $S$ for a fixed number of $M2$-branes. For simplicity reasons, we consider the case associated with $N = 3$.

From such a figure, we remark that the chemical potential is positive for small entropy $S$, and negative for large entropy $S$. It has been found that the sign change in the chemical potential occurs at the following entropy

$$S_4^{\mu=0} = \frac{7 \pi^{3/2}}{11 \times 3 \cdot 2^{11/2}} N^{3/2}. \quad \text{(A.11)}$$

In this case, the four-dimensional $AdS$ black hole verifies $S_4^{\min} < S_4^{\mu=0} < S_4^{HP}$. We remark that $S_4^{\min}$ is the lowest value of the entropy.

In terms of the temperature, the chemical potential changes its sign at the following thermal point

$$T_4^{\mu=0} = \frac{8^{1/6}}{\sqrt{77 \pi^{1/3}} \ell_p N^{1/6}}. \quad \text{(A.12)}$$

In figure 9, the chemical potential is plotted as a function of $N$ with a fixed entropy $S$.

From figure 9, it is observed that the maximum of the chemical potential is associated with the value $S_4^{\max} = \frac{7 \times \pi^{3/2}}{3 \times 29 \cdot 2^{11/2}} (N_4^{\max})^{3/2}$, namely, $N_4^{\max} \simeq 54$ for $S_4^{\max} = 4.$
Figure 10. The chemical potential as a function of temperature $T$. We take $\ell_p = 1$ and $N = 9$.

In figure 10, we plot the chemical potential as a function of the temperature $T_4^{(3)}$ for a fixed number $N$ of $M2$-branes.

It is noted that the upper dot in figure 10 indicates the minimum of temperature $T_4^{\text{min}}$. Below this point, however, one can find the Hawking-Page temperature that separates the lower stable branch and the upper unstable branch where $T_4^{\text{min}}$ resides. Here, the chemical potential changes the sign at unphysical regions. From Eq.(A.11), the chemical potential is positive when the number of $M2$-branes is constrained by $N^{3/2} > \frac{11 \times 2^{11/2}}{4} \frac{S}{\pi^{7/2}}$. This limit can be saturated for

$$T_4^{\mu=0} \approx 0.9 T_4^{\text{HP}}.$$  \hfill (A.13)

We remark that $T_4^{\mu=0} < T_4^{\text{HP}}$ implying that the pure $AdS$ geometry is preferred over the black hole.

B Ten-dimensional compactification: The $AdS_5 \times S^5$ case without DE

In this part, we reconsider the case of ten-dimensional compactification associated with the odd-dimensional spheres. In our formulation, the model that we deal with is indexed by

$$(D, d, k) = (10, 5, 0).$$  \hfill (B.1)

This case is nothing but the ten-dimensional type IIB superstring theory with the $AdS_5 \times S^5$ space-time in the presence of $D3$-branes. The corresponding five-dimensional $AdS$ black hole is obtained from the compactification on the odd dimensional real sphere $S^5$ with the following volume factor

$$\omega_5 = \pi^3.$$  \hfill (B.2)

In this way, the gravitational constant is $G_{10} = \ell_p^2$. Taking into account Eq.(B.2), we get
Figure 11. Temperature as function of the entropy using $N = 3$ and $\ell_p = 1$.

- mass

$$M_5^{(0)}(S, N) = \frac{3 \left( \pi^{2/3} S^{\frac{2}{3}} N^{\frac{2}{11}} + S^{\frac{4}{3}} N^{-\frac{11}{12}} \right)}{4 \times 2^{1/8} \pi^{5/6} \ell_p}, \quad (B.3)$$

- temperature

$$T_5^{(0)}(S, N) = \frac{\pi^{2/3} S^{-\frac{1}{3}} N^{\frac{1}{12}} + 2 \times S^{\frac{1}{3}} N^{-\frac{11}{12}}}{2 \times 2^{1/8} \pi^{5/6} \ell_p}, \quad (B.4)$$

- chemical potential

$$\mu_5^{(0)}(S, N) = \frac{5 \times \pi^{2/3} S^{\frac{2}{3}} N^{-\frac{19}{12}} - 11 \times S^{\frac{4}{3}} N^{-\frac{35}{12}}}{32 \times 2^{1/8} \pi^{5/6} \ell_p}, \quad (B.5)$$

- Gibbs free energy

$$G_5^{(0)}(S, N) = \frac{\pi^{2/3} S^{\frac{2}{3}} N^{\frac{5}{12}} - S^{\frac{4}{3}} N^{-\frac{11}{12}}}{4 \times 2^{1/8} \pi^{5/6} \ell_p}. \quad (B.6)$$

The obtained results match perfectly with the ones given in [55]. However, we reconsider and refine the associated results by presenting detailed discussions. This will be needed later on when we turn on the DE effects.

To discuss the associated phase transitions, we first plot the behavior of the Hawking temperature with respect to the entropy. This is illustrated in figure 11.

From this figure, the temperature riches its minimum at $S_5^{\text{min}} \approx 9.99$. As in the eleven Minkowski space-time associated with M-theory in the presence of $M5$ and $M2$-branes, one can show that the Hawking temperature has a minimum for

$$S_5^{\text{min}} = \frac{\pi N^2}{2^{3/2}}, \quad (B.7)$$

which corresponds to the minimal temperature

$$T_5^{\text{min}} = \frac{2^{3/8}}{\pi^{1/2} N^{1/4} \ell_p}. \quad (B.8)$$
Under the minimal temperature, no black hole solution can exist. Above such a temperature, one can distinguish two branches. The first branch is associated with a thermodynamical unstable black hole. The second one with large entropy describes a thermodynamic stable black hole.

To investigate the D3-brane number effect on the phase transition, we exploit the computed Gibbs free energy. In figure 12, we plot the Gibbs free energy as a function of the Hawking temperature $T^{(0)}$ for different values of $N$.

The sign of the Gibbs free energy changes at the Hawking-Page temperature $T^{HP}_5$ associated with $S^{HP}_5$. From the Gibbs free energy given in Eq.(B.6), the Hawking-Page phase transition happens at

$$S^{HP}_5 = \pi N^2,$$

(B.9)

giving the following Hawking-Page temperature

$$T^{HP}_5 = \frac{3}{2 \times 2^{1/8} \pi^{1/2} N^{1/4} \ell_p},$$

(B.10)

being smaller than $T^{min}_5$. In figure 12, a first order Hawking-Page phase transition can occur at $T^{HP}_5$ between large (stable) black holes and the thermal radiation. Besides, we see that below the higher point representing $T^{min}_5$ no black hole can survive. For lower Gibbs free energy values, we find two branches. The upper branch describes a small (unstable) black hole with negative specific heat values. The lower branch, however, represents (large) a stable black hole solution with positive specific heat values.

The chemical potential can be also used to discuss phase transitions of the five-dimensional AdS black hole in type IIB superstring. In figure 13, we plot the chemical
Figure 13. The chemical potential as a function of entropy for $N = 3, \ell_p = 1$. The sign change of the chemical potential happens at $S_5^{\mu=0} \simeq 8.6$.

Figure 14. The chemical potential as a function of the number of D3-branes $N$. Here we take $\ell_p = 1$ and $S = 4$.

potential as a function of the entropy $S$ for a fixed value of the D3-brane number. For simplicity reasons, we take $N = 3$.

We notice that the chemical potential is positive for a small entropy $S$, and negative for a large entropy $S$. The sign change, in the chemical potential, occurs at the entropy

$$S_5^{\mu=0} = \left( \frac{5}{11} \right)^{3/2} \pi N^2.$$  \hspace{1cm} (B.11)

Using this equation, one has $S_5^{\mu=0} < S_5^{\text{min}} < S_5^{\text{HP}}$ which is the same as the seven-dimensional $AdS$ black hole case appearing in M-theory in the presence of $M5$-branes.

We observe, then, that for the odd values of the dimension $d$ we have $S_d^{\mu=0} < S_d^{\text{min}} < S_d^{\text{HP}}$. The entropy of Eq.(B.11) corresponds to the temperature

$$T_5^{\mu=0} = \frac{21}{2 \times 2^{1/8} \sqrt{55} \pi^{1/2} \ell_p N^{1/4}}.$$  \hspace{1cm} (B.12)

In figure 14, the chemical potential is plotted as a function of $N$ in the case of a fixed entropy $S$.

From figure 14, the maximum of the chemical potential corresponds to the value $S_5^{\text{max}} = \left( \frac{19}{77} \right)^{3/2} \pi (N_5^{\text{max}})^2$, namely, $N_5^{\text{max}} \simeq 3.2$ for $S_5^{\text{max}} = 4$. 

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Figure 15. The chemical potential as a function of the temperature $T_5^{(0)}$. We take $\ell_p = 1$ and $N = 3$.

In figure 15, we plot the chemical potential as a function of the temperature $T_5^{(0)}$ for a fixed value of $N$.

In figure 15, the lower dot denotes $T_5^{HP}$ and the upper one indicates $T_5^{\text{min}}$. The Hawking-Page temperature separates the lower stable branch and the upper unstable branch where $T_5^{\text{min}}$ resides. In this case, the chemical potential changes the sign at an unphysical region. From Eq. (B.11), the chemical potential is positive when the number of D3-branes, $N$, is constrained by the condition $N^2 > \frac{11^{1/2}}{3^{1/2}} \frac{2}{\pi}$. This limit can be saturated for

$$T_5^{\mu=0} \simeq 0.9 T_5^{HP}.$$  \hspace{1cm} (B.13)

We remark that $T_5^{\mu=0} < T_5^{HP}$ which implies that the pure AdS$_5$ geometry is preferred over the black hole backgrounds.

C The $AdS_4 \times S^7$ case in presence of DE

We consider now the model labeled by the triplet $(11, 4, 3)$ associated with the compactification of M-theory on $S^7$ in the presence of M2-branes. Using (A.2), we obtain

- the mass

$$M_4^{(3) \text{DE}}(S, N, c) = M_4^{(3)}(S, N) - \alpha_4^{\text{DE}}(S, N, c),$$ \hspace{1cm} (C.1)

- the temperature

$$T_4^{(3) \text{DE}}(S, N, c) = T_4^{(3)}(S, N) + \frac{3 \omega_q}{2} S \frac{\alpha_4^{\text{DE}}(S, N, c)}{S},$$ \hspace{1cm} (C.2)

- the chemical potential

$$\mu_4^{(3) \text{DE}}(S, N, c) = \mu_4^{(3)}(S, N) - \frac{7 (3 \omega_q + 2)}{18} S \frac{\alpha_4^{\text{DE}}(S, N, c)}{N^{3/2}},$$ \hspace{1cm} (C.3)
the Gibbs free energy

\[ G_4^{(3)DE}(S, N, c) = G_4^{(3)}(S, N) - \left( \frac{3\omega q + 2}{2} \right) \cdot \alpha_4^{DE}(S, N, c). \tag{C.4} \]

In all these quantities associated with M2-branes, \( \alpha_4^{DE}(S, N, c) \) which denotes the DE contribution reads as

\[ \alpha_4^{DE}(S, N, c) = \frac{c\pi 5\omega q - \frac{1}{6} N^7\omega q + \frac{7}{9}}{2^{3\omega q + \frac{37}{6}} S^{\frac{2\omega q}{2}} \ell_p^{3\omega q + 2}}. \tag{C.5} \]

For such a quintessential AdS black hole solution, it has been remarked that the parameter \( \omega q \) has a fixed value which is \(-1/3\). As in the previous M-theory model, this can be obtained from the dimensional analysis of the thermodynamical quantities given in Eqs. (C.1), (C.2), (C.3) and (C.4) since they are all proportional to \( \ell_p^{-1} \).

To study the phase transitions behavior in the presence of quintessence, we first plot the behavior of the Hawking temperature with respect to the entropy in figure 16 for different values of the intensity \( c \) of the quintessence field providing DE contributions.

From figure 16, we observe that the temperature is decreasing when the intensity of DE \( c \) increases. For a general number \( N \) of M2-branes and a generic intensity \( c \), the Hawking temperature has a minimum at

\[ S_4^{DE-min} = \frac{\pi^{3/2}}{3^2 211/2} (1 - c) N^{3/2}. \tag{C.6} \]

This corresponds to the following minimal temperature

\[ T_4^{DE-min} = \frac{3^{1/2} \sqrt{1 - c}}{2^{5/6} \pi^{4/3} N^{1/6} \ell_p}. \tag{C.7} \]

To get more information about the phase transitions of the four-dimensional AdS black hole in M-theory, the Gibbs free energy as a function of the Hawking temperature \( T_4^{(3)DE} \) for different values of \( N \) and a fixed value of \( c \) is illustrated in figure 17.
In figure 17, we observe that the phases are smaller than the ones discussed in figure 7. The Gibbs free energy decreases producing a smaller unstable black hole phase. Using the Gibbs free energy, given in Eq.(C.4), one can obtain the Hawking-Page phase transition. This occurs at the following entropy

\[
S_{DE-HP}^4 = \frac{(1 - c) \pi^{3/2} N^{3/2}}{3 \cdot 2^{11/2}},
\]

which corresponds to the Hawking-Page temperature

\[
T_{DE-HP}^4 = \frac{2^{1/6} \sqrt{1 - c}}{\pi^{1/3} N^{1/6} \ell_p^{1/6}}.
\]

An examination shows that \( T_{DE-HP}^4 < T_{DE-min}^4 \).

To see the difference in the Gibbs free energy behavior in the presence of quintessence, we plot this function for \( c = 0 \) (absence of DE) and for \( c = 0.2 \) for certain values of \( N \) in figure 18.

From figure 18, we see that the decrease in the radiation phase and the phase where no black hole can survive comes from the temperature diminution. However, the stable and the unstable phase are directly affected by the diminution of the Gibbs free energy.

As in the seven-dimensional AdS black hole case, the quintessence field affects also the chemical potential. To examine the corresponding effects, we plot the chemical potential as a function of the entropy \( S \) for \( N = 3 \) in figure 19. It shows a diminution of the chemical potential and the entropy when DE is present.

Moreover, we observe that the chemical potential is positive for small entropy \( S \), and negative for large \( S \) values. The sign change of the chemical potential happens at the entropy

\[
S_{DE-\mu=0}^4 = \frac{7 (1 - c) \pi^{3/2}}{11 \times 3 \cdot 2^{11/2}} N^{3/2}.
\]

The four-dimensional AdS black hole verifies \( S_{DE-\mu=0}^4 < S_{DE-min}^4 < S_{DE-HP}^4 \).

Furthermore, we can also study the modification of the chemical potential when it is plotted
Figure 18. The Gibbs free energy as function of temperature for different in the absence of DE and in its presence for different values of $N$, $\ell_p = 1$.

Figure 19. The chemical potential as a function of entropy for $N = 3$, $\ell_p = 1$.

as a function of the number $N$ of $M2$-branes in M-theory. This modification is shown in figure 20. Here, the entropy $S$ has a fixed value.

The maximum of the chemical potential corresponds to the point

$$S_4^{DE-max} = \frac{7(1-c)}{3 \times 29 \cdot 2^{11/2}} \left( N_4^{DE-max} \right)^{3/2},$$

namely, $N_4^{DE-max} \simeq 63,42$ for $S_4^{max} = 4$ and $c = 0.2$. From $N_4^{DE-max}$, we see that the
number of $M2$-branes grows in the presence of DE.

Finally, DE could be used to stabilise the AdS black hole. To reveal this, we plot the chemical potential as a function of the temperature $T_{4}^{(3)DE}$ for a fixed number of $M2$-branes $N$ in figure 21.

The higher dots denotes $T_{4}^{DE-min}$. Lower from such points, we have the $T_{4}^{DE-HP}$ dots which separates the lower stable branch and the upper unstable one where $T_{4}^{DE-min}$ resides. It is observed that, for large $c$ values, the $T_{4}^{DE-min}$ is getting higher, which is also true for $T_{4}^{DE-HP}$. Thus, it is clear that the unstable branch is getting smaller and the stable one is bigger.

D The $AdS_{5} \times S^{5}$ case in presence of DE

Now, we consider a model of ten-dimensional theory associated with $D3$-brane physics. This model has been labeled by the triplet $(10,5,0)$ concerning the compactification of
In this way, the DE manifestly term $\alpha_{5}^{DE}(S, N, c)$ reads

$$\alpha_{5}^{DE}(S, N, c) = \frac{3c\pi^{\frac{10\omega_{q}}{3} + \frac{3}{2}} N^{\frac{5\omega_{q}}{3} + \frac{1}{2}}}{2^{\frac{19}{4}} + 19 S^{\frac{5\omega_{q}}{3} + \frac{1}{2}} \ell_{p}^{3\omega_{q} + 3}}.$$  \hspace{1cm} (D.5)

For such a black hole solution in type IIB superstring compactification on $S^5$, it has been remarked that the parameter $\omega_{q}$ has a fixed value being $-1/2$.

To investigate the phase transition behaviours in type IIB superstring, we first plot the Hawking temperature as a function of the entropy in figure 22 for different values of the intensity $c$. 

\hspace{1cm} Figure 22. Temperature as function of the entropy using $N = 3$ and $\ell_{p} = 1$ and different values of $c$. 

In particular, we list the following computations

• mass

$$M_{5}^{(0)DE}(S, N, c) = M_{5}^{(0)}(S, N) - \alpha_{5}^{DE}(S, N, c), \hspace{1cm} (D.1)$$

• temperature

$$T_{5}^{(0)DE}(S, N, c) = T_{5}^{(0)}(S, N) + \frac{4\omega_{q}}{3} \cdot \frac{\alpha_{5}^{DE}(S, N, c)}{S}, \hspace{1cm} (D.2)$$

• chemical potential

$$\mu_{5}^{(0)DE}(S, N, c) = \mu_{5}^{(0)}(S, N) - \frac{5 (4\omega_{q} + 3)}{24} \cdot \frac{\alpha_{5}^{DE}(S, N, c)}{N^2}, \hspace{1cm} (D.3)$$

• Gibbs free energy

$$G_{5}^{(0)DE}(S, N, c) = G_{5}^{(0)}(S, N) - \left(\frac{4\omega_{q} + 3}{3}\right) \cdot \alpha_{5}^{DE}(S, N, c). \hspace{1cm} (D.4)$$
The dots in figure 22 denote the minimum of the temperature. For a general number of $D3$-branes and a generic intensity of DE field, we can show that the Hawking temperature has a minimum at the following entropy

$$S_{DE}^{\text{min}} = \frac{\pi (1 - c)^{3/2}}{2^{3/2} N^2},$$

(D.6)
corresponding to the minimal temperature

$$T_{DE}^{\text{min}} = \frac{2^{3/8} \sqrt{1 - c}}{\pi^{1/2} N^{1/4} \ell_p}.$$  

(D.7)

From figure 22, we see that the temperature is decreasing when the intensity of DE $c$ increases. This confirms that DE can be considered as a cooling system surrounding the black hole [50]. The minimum of the temperature and the corresponding entropy denoted by the black dotes in figure 22 are also affected in the same way.

For more investigations about the phase transitions of the five-dimensional $AdS$ black hole in type IIB superstring, we illustrate the Gibbs free energy as a function of the Hawking temperature $T_{DE}^{(0)}$ for certain values of the $D3$-brane number $N$ and fixed values of $c$ in figure 23.

We notice that the phases discussed previously are reduced. Indeed, the Gibbs free energy decreases generating smaller unstable black hole phase. To see the difference in the behavior of the Gibbs free energy when the quintessence is present, we plot this function for $c = 0$ (absence of DE) and $c = 0.2$ for several values of $N$ associated with $D3$-branes.

It follows from figure 24 that the decrease in the radiation phase and the phase where no black hole can exist can be understood from the diminution of temperature. However, the stable and the unstable phase are directly affected by the diminution of the Gibbs free energy.
energy. From such an energy given in Eq.(D.4), we find the Hawking-Page phase transition which occurs at

\[ S_{DE-HP}^5 = (1 - c)^{3/2} \pi N^2, \]

(D.8)
corresponding to the Hawking-Page temperature

\[ T_{DE-HP}^5 = \frac{3 \sqrt{1 - c}}{2 \times 2^{1/8} \pi^{1/2} N^{1/4} \ell_p}. \]

(D.9)

This temperature is smaller than \( T_{DE-min}^5 \).

As in the M-theory compactification on \( S^4 \) and \( S^7 \), the quintessence field affects also the chemical potential. To examine the corresponding modifications, we plot the chemical potential as a function of entropy \( S \) for \( N = 3 \) in figure 25.

This figure shows a diminution of both the chemical potential and the entropy when DE is present. The sign change in the chemical potential occurs at the following entropy

\[ S_{DE-\mu=0}^5 = \left( \frac{5}{\Pi} \right)^{3/2} (1 - c)^{3/2} \pi N^2. \]

(D.10)
Figure 25. The chemical potential as a function of entropy for $N = 3, \ell_p = 1$. The sign change of the chemical potential happens at $S_5^{DE-\mu=0}$.

Figure 26. The chemical potential as a function of the number of M5-branes $N$. Here we take $\ell_p = 1$ and $S_5^{DE-max} = 4$.

Using this equation, one can show that $S_5^{DE-\mu=0} < S_5^{DE-min} < S_5^{DE-HP}$.

Furthermore, we can also discuss the chemical potential as a function of the number $N$ of D3-branes in type IIB superstring which is illustrated in figure 26.

The maximum of the chemical potential corresponds to the point

$$S_5^{DE-max} = \left(\frac{19}{41}\right)^{3/2} (1 - c)^{3/2} \pi \left(N_5^{DE-max}\right)^2$$

(D.11)

namely, $N_5^{DE-max} \approx 3.8$ for $S_5^{DE-max} = 4$ and $c = 0.2$. From $N_5^{DE-max}$, we see that the number $N$ of D3-branes grows in the presence of DE. In figure 27, we plot the chemical potential as a function of the temperature $T_5^{(0)DE}$ for a fixed number $N$ of D3-brane in the compactification of type IIB superstring on $S^5$.

The higher dots are associated with $T_5^{DE-min}$. Lower from these points, we have the $T_5^{DE-HP}$ point that separates the lower stable branch and the upper unstable branch where $T_5^{DE-min}$ resides. We notice that $T_5^{DE-min}$ is getting higher in the curves when the intensity of DE $c$ is bigger, which is also true for $T_5^{DE-HP}$. Thus, we observe that DE can be considered as a establishing factor for the five-dimensional quintessential AdS black hole embedded in type IIB superstring theory.
Figure 27. The chemical potential as a function of temperature $T$. We take $\ell_p = 1$ and $N = 3$.

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