Canonical variational completion of differential equations

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Abstract
Given a non-variational system of differential equations, the simplest way of turning it into a variational one is by adding a correction term. In the paper, we propose a way of obtaining such a correction term, based on the so-called Vainberg-Tonti Lagrangian, and present several applications in general relativity and classical mechanics.

Keywords: jet bundle, source form, variationality conditions, Einstein field equations, canonical energy-momentum tensor

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1 Introduction

For a given non-variational system of differential equations, there are multiple ways of transforming it into a variational one - among these, variational multipliers (or variational integrating factors), [1], are maybe the most well known. Another possibility is to simply add a correction term.

In the paper, we consider systems of ordinary or partial differential equations - represented by source forms, similar to Euler-Lagrange systems for extremals of integral variational functionals in the calculus of variations. We propose a way of obtaining such a correction term - which we call a variational completion, as follows. Any ordinary or partial differential system can be expressed as the vanishing of some source form $\varepsilon$ on sections of an appropriate jet bundle. Further, to this source form, one can naturally attach a Lagrangian $\lambda_\varepsilon$, called the Vainberg-Tonti Lagrangian of $\varepsilon$, [5]; this Lagrangian has the property that the difference

$$\tau := E(\lambda_\varepsilon) - \varepsilon$$

(1)

between its Euler-Lagrange form $E(\lambda_\varepsilon)$ and $\varepsilon$ offers a measure of the non-variationality of $\varepsilon$. Using $\tau$ in (1) as the correction term, the system $\varepsilon + \tau = 0$ becomes variational.
The method appears to have several interesting applications. We present here three of them.

1) **Einstein tensor obtained from variational completion of the Ricci tensor.** Historically, the first variant of gravitational field equations proposed by Einstein was:

\[ R_{ij} = \frac{8\pi\kappa}{c^4} T_{ij}, \tag{2} \]

where: \( R_{ij} \) is the Ricci tensor of a 4-dimensional Lorentzian manifold \((X,g)\), \( T_{ij} \) is the energy-momentum tensor, while \( \kappa \) and \( c \) are constants, \[8\]. This variant correctly predicted some physical facts, but failed to fulfil another request: local energy-momentum conservation. This led Einstein to adding in the left hand side the "correction term" \(-\frac{1}{2}R_{gij}\) (by a reasoning based on Bianchi identities), thus leading to the nowadays famous:

\[ R_{ij} - \frac{1}{2}R_{gij} = \frac{8\pi\kappa}{c^4} T_{ij}. \tag{3} \]

The variational deduction of (3), due to Hilbert, relies on a heuristic argument - simplicity. Hilbert chose to construct the action for the left hand side using the "simplest scalar" which can be constructed from the metric tensor and its derivatives alone. Happily, the Euler-Lagrange expression ensuing from this simplest scalar (which is the scalar curvature \( R \)) coincides with the left hand side of (3).

There is, still, another way of finding this correction term. Equation (2) is not variational. Actually, the term which fails to be variational is \( R_{ij} \); in the paper, we prove that Hilbert's "simplest scalar" Lagrangian is (up to multiplication by a non-essential constant), nothing else than the Vainberg-Tonti Lagrangian corresponding to \( R_{ij} \). Accordingly, the correction term \(-\frac{1}{2}R_{gij}\) can be determined as the canonical variational completion of \( R_{ij} \).

2) **Symmetrization of canonical energy-momentum tensors.** In special relativity, energy-momentum tensors are obtained by adding to the Noether current corresponding to the invariance of the field Lagrangian to space-time translations a symmetrization term. The way of obtaining the symmetrization term is subject to an old and complicated debate, \[6\]. We show that the canonical variational completion method offers the possibility of recovering the expression of a full, symmetrized energy-momentum tensor from just one of its terms - e.g., from a non-symmetrized Noether current. In particular, the energy-momentum tensor of the electromagnetic field can be obtained this way.

3) In classical mechanics, equations of **damped small oscillations** are known to be non-variational. Without aiming to give a general physical interpretation of the obtained correction term, we determine the canonical variational completion of these equations.

In Sections 2 and 3, we briefly present some known notions and results to be used in the following.
2 Differential forms on jet bundles

The mathematical background for a modern formulation of both field theory and mechanics are fibered manifolds and their jet bundles.

Consider a fibered manifold $Y$ of dimension $m + n$, with $n$-dimensional base $X$ and projection $\pi : Y \to X$. Fibered charts $(V, \psi)$, $\psi = (x^i, y^\sigma)$ on $Y$ induce the fibered charts $(V^r, \psi^r)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1...j_r})$ on the $r$-jet prolongation $J^rY$ of $Y$ and $(U, \phi)$, $\phi = (x^i)$ on $X$. The manifold $J^rY$ can be regarded as a fibered manifold in multiple ways, by means of the projections:

$$\pi^r : J^rY \to J^sY, \quad (x^i, y^\sigma, y^\sigma_{j_1...j_r}) \mapsto (x^i, y^\sigma, y^\sigma_{j_1...j_r})$$

where $r > s$, $J^0Y := Y$ and:

$$\pi^r : J^rY \to X.$$

The set of $C^\infty$-smooth sections $\gamma : X \to Y$, locally expressed by some functions $(x^i) \mapsto \gamma(x^i) = (x^i, y^\sigma(x^i))$ is denoted by $\Gamma(Y)$. Given a section $\gamma \in \Gamma(Y)$, its prolongation to $J^rY$ is: $J^r\gamma : (x^i) \mapsto J^r\gamma(x^i) = (x^i, y^\sigma, y^\sigma_{j}(x)), \ldots, y^\sigma_{j_1...j_r}(x))$, where the symbol $\cdot_j$ denotes partial differentiation with respect to $x^j$.

In field theoretical applications, the coordinates $x^i$ play the role of spacetime coordinates, while $y^\sigma$ are "field" coordinates (to be accurate, real fields are encoded in sections $y^\sigma = y^\sigma(x^i)$). The case of mechanics is characterized by $\dim X = 1$; in this case, the coordinates on $J^rY$ are usually denoted by $(t, q^\sigma, \dot{q}^\sigma, \ldots, q^{(r)})$ and are interpreted as: time, generalized coordinates, generalized velocity etc.

By $\Omega^r_W$, we denote the set of $k$-forms of order $r$ over an open set $W \subset Y$, i.e., the set of $k$-forms over the $r$-th prolongation $J^rW \subset J^rY$. In particular, $\mathcal{F}(W) := \Omega^0_W$ is the set of real-valued smooth functions over $J^rW$.

The subset of $\Omega^r_W$ consisting of $k$-forms:

$$\rho = \frac{1}{k!} a_{i_1i_2...i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k},$$

(4)

(where $a_{i_1i_2...i_k}, \ k \leq n$, are smooth functions of the coordinates $x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1...j_r}$) is called the set of $(\pi^r)$-horizontal $k$-forms of order $r$; similarly, one can speak about $\pi^{r,s}$-horizontal forms of order $r$ as forms generated by exterior products of the differentials $dx^i, dy^\sigma, \ldots, dy^\sigma_{j_1...j_r}$.

Examples of $\pi^r$-horizontal forms are volume forms and Lagrangians.

For $X = \mathbb{R}^n$, the Euclidean volume form is:

$$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n.$$

(5)

On Riemannian manifolds $(X, g)$, a coordinate-invariant volume form is locally given by: $dV = \sqrt{|g|} \omega_0$. 

3
A Lagrangian of order \( r \) is defined as a \( \pi^r \)-horizontal \( n \)-form of order \( r \):
\[
\lambda = \mathcal{L} \omega_0, \quad \mathcal{L} = \mathcal{L}(x^i, y^\sigma, ..., y^\sigma_{i_1...i_r}).
\] (6)

A form \( \theta \in \Omega^k_Y \) is a contact form if it is annihilated by all jets \( J^r \gamma \) of sections \( \gamma \in \Gamma(Y) \). Important examples are contact 1-forms on \( J^r Y \) defined on a coordinate neighborhood by:
\[
\omega^\sigma = dy^\sigma - y^\sigma_j dx^j, \quad \omega^\sigma_{i_1} = dy^\sigma_{i_1} - y^\sigma_{i_1 j} dx^j, ...
\] (7)
A differential form is called \( p \)-contact if it is generated by \( p \)-th exterior powers of contact forms.

3 Source forms and variationality conditions

A source form of order \( r \) on a fibered manifold \( Y \), \( \mathfrak{M} \), is a \( \pi^{r,0} \)-horizontal, 1-contact \((n+1)\)-form on \( J^r Y \). In local coordinates, any source form is expressed as:
\[
\varepsilon = \varepsilon_\sigma \omega^\sigma \land \omega_0, \quad \varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, ..., y^\sigma_{j_1...j_r}).
\] (8)

The set of source forms of order at most \( r \) over \( Y \) is closed under addition and under multiplication with functions \( f \in \mathcal{F}(J^r Y) \).

The most notable example of a source form is the Euler Lagrange form \( \mathcal{E}(\lambda) \) of a Lagrangian \( \lambda = \mathcal{L}(x^i, y^\sigma, ..., y^\sigma_{i_1...i_r}) \omega_0 \in \Omega^r_n(Y) \):
\[
\mathcal{E}_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_{k_1} \frac{\partial \mathcal{L}}{\partial y^\sigma_{k_1}} + ... + (-1)^r d_{k_1...k_r} \frac{\partial \mathcal{L}}{\partial y^\sigma_{k_1...k_r}}.
\]

A section \( \gamma : X \to Y \) is critical for the Lagrangian \( \lambda \) if and only if the \( \mathcal{E}(\lambda) \) is annihilated by the \( r \)-jet of \( \gamma \), i.e., \( \mathcal{E}_\sigma(\lambda) \circ J^r \gamma = 0, \sigma = 1, ..., m \).

A source form \( \varepsilon \) is called:

a) locally variationai if around any point of the fibered manifold \( Y \), there exists a local fibered chart \((V, \psi)\) and a Lagrangian \( \lambda \) on some jet prolongation \( V^r \) \((r \in \mathbb{N})\) of \( V \), such that, on \( V^r \), \( \varepsilon = \mathcal{E}(\lambda) \);

b) globally variationai if there exists a Lagrangian \( \lambda \) on the whole manifold \( Y \) such that \( \varepsilon = \mathcal{E}(\lambda) \).

Local variationality of a source form \( \varepsilon = \varepsilon_\sigma \omega^\sigma \land \omega_0 \) of order \( r \) is equivalent to a generalization of classical Helmholtz conditions, \( \mathfrak{I} \):
\[
H_{\sigma_{j_1...j_k}}(\varepsilon) = 0, \quad k = 0, ..., r,
\] (9)
where:
\[
H_{\sigma}^{j_1...j_k}(\varepsilon) = \frac{\partial \varepsilon_\sigma}{\partial y_{j_1...j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y_{j_1...j_k}} - \sum_{l=k+1}^{r} (-1)^l \binom{l}{k} d_{i_{k+1}} d_{i_{k+2}} ... d_{i_l} \frac{\partial \varepsilon_\nu}{\partial y_{j_1...j_k l+1...i_l}}. \]

4 Canonical variational completion

By variational completion of a given source form \( \varepsilon \) on \( Y \), we will mean any source form \( \tau \) on \( Y \) with the property that \( \varepsilon + \tau \) is variational. Of course, one can speak about local and about global variational completions.

In the following, we will only study local variational completions. Clearly, every source form has infinitely many variational completions: indeed, any Lagrangian \( \lambda \) induces the completion \( \tau := E(\lambda) - \varepsilon \). Thus, the question is how to choose the Lagrangian \( \lambda \) in a meaningful way. In the following, we will try to give an answer to this question.

Given an arbitrary source form \( \varepsilon = \varepsilon_\sigma \omega^\sigma \land \omega_0 \in \Omega_{\varepsilon + 1}(Y) \) of order \( r \), a local Lagrangian attached to \( \varepsilon \) is the Vainberg-Tonti Lagrangian \( \lambda_\varepsilon = L_\varepsilon \omega_0 \), \( \ref{5} \), \ref{7} \), defined by:

\[
L_\varepsilon(x^i, y^\sigma, ..., y^\sigma_{j_1...j_s}) = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, uy^\sigma, ..., uy^\sigma_{j_1...j_s}) du. \]

The Euler-Lagrange form \( E(\lambda_\varepsilon) = E_\nu \omega^\nu \land \omega_0 \) of the Vainberg-Tonti Lagrangian \( \lambda_\varepsilon \) is given, \( \ref{5} \), by:

\[
E_\nu = \varepsilon_\nu - \int_0^1 u \{ y^\sigma (H_{\nu \sigma} \circ \chi_u) + y^\sigma_{j_1...j_s} (H_{\nu \sigma} \circ \chi_u) + ... + y^\sigma_{j_1...j_r} (H_{\nu \sigma}^{j_1...j_r} \circ \chi_u) \} du, \]

where \( \chi_u : J^r Y \rightarrow J^r Y \) denotes the homothety \( (x^i, y^\sigma, y^\sigma_{j_1...j_s}) \mapsto (x^i, uy^\sigma, uy^\sigma_{j_1...j_s}) \) and the coefficients \( H_{\sigma \nu}^{j_1...j_k} \) are as in \( \ref{9} \).

From \( \ref{6} \), it follows that the coefficients \( H_{\sigma \nu}^{j_1...j_k} \) in \( \ref{12} \) have the meaning of ”obstructions from variationality” of the source form \( \varepsilon \). In particular, if the source form \( \varepsilon \) is variational, then \( E(\lambda_\varepsilon) = \varepsilon \).

It thus appears as natural

**Definition 1** The canonical variational completion of a source form \( \varepsilon \in \Omega_{\varepsilon + 1}(Y) \), is the source form \( \tau(\varepsilon) \) given by the difference between the Euler-Lagrange form of the Vainberg-Tonti Lagrangian of \( \varepsilon \) and \( \varepsilon \) itself:

\[
\tau(\varepsilon) = E(\lambda_\varepsilon) - \varepsilon. \]
The local coefficients $\tau_{\nu}$ of the canonical variational completion $\tau(\varepsilon) = \tau_{\nu}\omega^\nu \wedge \omega_0$ can be directly expressed in terms of the coefficients $H_{\nu\sigma}^{j_1\cdots j_r}$:

$$\tau_{\nu} = -\int_0^1 u\{y^\sigma(H_{\nu\sigma} \circ \chi_u) + y^j_j(H_{\nu\sigma}^j \circ \chi_u) + \ldots + y^\sigma_{j_1\ldots j_r}(H_{\nu\sigma}^{j_1\cdots j_r} \circ \chi_u)\} du.$$ 

**Remark.** Generally, the Vainberg-Tonti Lagrangian and, accordingly, the canonical variational completion of a source form of order $r$, are of order $2r$. Still, under certain conditions, [7] (which are fulfilled by a large number of equations in physics), the Vainberg-Tonti Lagrangian is actually equivalent to a Lagrangian of order $r$.

## 5 Source forms in general relativity

Consider a Lorentzian manifold $(X, g_{ij})$ of dimension 4, with local charts $(U, \phi)$, $\phi = (x^i)_{i=0,3}$ and Levi-Civita connection $\nabla$. We denote by $R_{ij}$ the Ricci tensor of $\nabla$ and by $R = g^{ij}R_{ij}$, the scalar curvature; semicolons $;j$ will mean $\nabla$-covariant derivatives with respect to $\partial_j := \partial/\partial x^j$. We assume in the following that measurement units are chosen in such a way that $c = 1$.

Einstein field equations (3) arise by varying with respect to the metric tensor the Lagrangian $\lambda = \lambda_g + \lambda_m$, where:

i) $\lambda_g = -\frac{1}{16\pi\kappa}R \sqrt{|g|}\omega_0$ is the Hilbert Lagrangian;

ii) the matter Lagrangian $\lambda_m = L_m(\sqrt{|g|}\omega_0)$, is given by a differential invariant $L_m = L_m(g_{ij},g_{ij},\ldots;y^\sigma,y^j_j,\ldots,y^\sigma_{j_1\ldots j_r})$ depending on the metric tensor components and their derivatives up to a certain order $s \in N$ and on the $r$-jet of a field $g^\sigma$. Typically, in classical general relativity, $s = 0$.

In the case of vacuum Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0,$$ (14)

the “field components” to be varied are the inverse metric tensor components $g^{ij}$, hence the fibered manifold $Y$ is the bundle of metrics $\text{Met}(X)$, defined as the set of symmetric nondegenerate tensors of type $(0,2)$ on $X$. Since both $R_{ij}$ and $R$ are of second order in $g^{ij}$, the space we have to work on is the second order jet bundle $J^2\text{Met}(X)$.

We denote the local charts on $\text{Met}(X)$ by $(V, \psi)$, with $\psi = (x^i, g^{jk})$ and the induced fibered chart on $J^2\text{Met}(X)$, by $(V^2, \psi^2)$, with $\psi^2 = (x^i, g^{jk}, g^{jk}, g^{jk})$. We will also use the following notations:

- $\omega_0 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ ("non-invariant" volume form on a chart domain);

- $\omega^{jk} = dg^{jk} - g^{jk} dx^i$;

- $\omega^{jk} = dg^{jk} - g^{jk} dx^i$ (basic contact forms on $J^2\text{Met}(X)$).
The Riemann tensor, the Ricci tensor and the Ricci scalar thus become objects on $J^2 \text{Met}(X)$.

In the case of Einstein equations with matter (3), we will have to work on a fibered product $Y \times_X \text{Met}(X)$ over $X$ (where $Y$ is a fibered manifold with base $X$) with coordinate charts $(V, \psi)$, $\psi = (x^i, y^\sigma, g^{ik})$; in this case, we have on $J^l(Y \times_X \text{Met}(X))$, $l = \max(r, s)$, the induced fibered charts $(V^l, \psi^l)$, $\psi^l = (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_l}, g^{ik}, \ldots, g^{ik}_{i_1 \ldots i_l})$. We can speak separately about variations with respect to the "non-gravitational field" $y^\sigma$ and with respect to the metric $g^{ij}$ of a Lagrangian and, accordingly, about $Y$- and $\text{Met}(X)$- variationality, $Y$- and $\text{Met}(X)$- variational completions of a source form.

5.1 Canonical variational completion of the Ricci tensor

We will prove in the following that vacuum Einstein equations (14) can be obtained by means of the canonical variational completion of the source form with components $R_{ij}$.

Take the following source form on $J^2 \text{Met}(X)$:

$$\varepsilon_R = -\frac{1}{16\pi\kappa} R_{ij} \sqrt{|g|} \omega^{ij} \wedge \omega_0,$$

with components $\varepsilon_{ij} = \varepsilon_{ij}(g^{kl}, g^{kl}_{i}, g^{kl}_{ij})$ given by $\varepsilon_{ij} = -\frac{1}{16\pi\kappa} R_{ij} \sqrt{|g|}$.

The Vainberg-Tonti Lagrangian of $\varepsilon_R$ is given by

$$L_\varepsilon = g^{ij} \int_0^1 \varepsilon_{ij}(ug^{kl}, ug^{kl}_{i}, ug^{kl}_{ij}) du.$$

Let us study the behavior of the integrand with respect to homotheties $\chi_u: (g^{kl}, g^{kl}_{i}, g^{kl}_{ij}) \mapsto (ug^{kl}, ug^{kl}_{i}, ug^{kl}_{ij})$. These homotheties induce the transformation $(g^{kl}, g^{kl}_{i}, g^{kl}_{ij}) \mapsto (u^{-1}g^{kl}, u^{-1}g^{kl}_{i}, u^{-1}g^{kl}_{ij}, u^{-1}g^{kl}_{ij})$ of the covariant metric tensor components and of their derivatives. The Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2}g^{ih}(g_{hj,k} + g_{hk,j} - g_{jk,h})$$

are invariant to $\chi_u$ and hence the curvature tensor components $R_{ij}^{kl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^h_{jkl} - \Gamma^h_{jkl} \Gamma^i_{hk}$ are also invariant. The Ricci tensor $R_{jk} = R_{ij}^{ij} k_i$ is obtained just by a summation process from $R_{ij}^{kl}$, which means that it is also insensitive to $\chi_u$.

It remains to compute the contribution of $\chi_u$ to the factor $\sqrt{|g|}$. Each line of the matrix $(g^{jk})$ is multiplied by $u^{-1}$, that is, $g = \det(g_{ij})$ will acquire a factor of $u^{-4}$ and finally,

$$\sqrt{|g \circ \chi_u|} = u^{-2} \sqrt{|g|}.$$

We get this way,
\[
\mathcal{L}_\varepsilon = -\frac{1}{16\pi\kappa}g^{ij}\int_0^1 u^{-2}R_{ij}\sqrt{|g|}du = -\frac{1}{16\pi\kappa}g^{ij}R_{ij}\sqrt{|g|}\int_0^1 u^{-2}du = \frac{1}{16\pi\kappa}R\sqrt{|g|},
\]
that is, the Vainberg-Tonti Lagrangian \(\lambda_\varepsilon = \mathcal{L}_\varepsilon\omega_0\) is minus the Hilbert Lagrangian:
\[
\lambda_\varepsilon = -\lambda_g. \tag{15}
\]

But, the Euler-Lagrange form of \(\lambda_g\) is well-known, as:
\[
E(\lambda_g) = -\sqrt{|g|}(R_{ij} - \frac{1}{2}Rg_{ij})\omega^i_0\wedge\omega_0, \quad \text{hence, the variational completion of } \tau = E(\lambda_\varepsilon) - \varepsilon R
\]
has the local components
\[
\tau_{ij} = \frac{1}{16\pi\kappa}(R_{ij} - \frac{1}{2}Rg_{ij})\sqrt{|g|} + \frac{1}{16\pi\kappa}R_{ij}\sqrt{|g|} = \frac{\sqrt{|g|}}{16\pi\kappa}(2R_{ij} - \frac{1}{2}Rg_{ij}).
\]

**Remark.** The minus sign in (15) does not affect the outcome - the resulting Euler-Lagrange equation is the correct vacuum Einstein equation. Still, if we want the variational completion \(\tau_{ij}\) not to contain any \(R_{ij}\) term, we can use \(\mathcal{L}' = -\mathcal{L}\) as a Lagrangian. More generally, if for a source form \(\varepsilon = \varepsilon_\sigma\omega^\sigma\wedge\omega_0\) (on an arbitrary fibered manifold \(\mathcal{Y}\)) the Euler-Lagrange form \(E(\mathcal{L}_\varepsilon)\) already contains a multiple of \(\varepsilon\) as a term, i.e., if there exists a constant \(\alpha \neq 0\) such that:
\[
E_\sigma(\mathcal{L}_\varepsilon) = \alpha\varepsilon_\sigma + \phi_\sigma
\]
(where \(\phi_\sigma\) is an independent term), we can use the ”adjusted” Lagrangian
\[
\mathcal{L}'_\varepsilon := \alpha^{-1}\mathcal{L}_\varepsilon,
\]
thus getting \(E_\sigma(\mathcal{L}'_\varepsilon) = \varepsilon_\sigma + \alpha^{-1}\phi_\sigma\) and the correction term \(\tau_\sigma = \alpha^{-1}\phi_\sigma\).

## 5.2 Canonical energy-momentum tensors

In relativistic theories, there are two major ways of defining energy-momentum tensors, corresponding to two different contexts:

1) **The canonical energy-momentum tensor**, corresponding to special relativity (where \(X = \mathbb{R}^4\) and the metric tensor is fixed as \(\eta_{ij} = \text{diag}(1, -1, -1, -1)\)). A Lagrangian \(\Lambda_m = L_m\omega_0\), which is invariant to the group of space-time translations \(\tilde{x}^i = x^i + a^i\), \(a^i = \text{const.}\), gives rise to a system of conserved Noether currents, called the canonical energy-momentum tensor (**Note**: Invariance to space-time translations amounts to the fact that \(L_m\) does not explicitly depend on \(x^i\)). For a first order Lagrangian, \(L_m = L_m(y^\sigma, y_\sigma^i)\), these Noether currents are given by, [8]:
\[
\tilde{T}_{ik} = \eta_{kh}(y^\sigma_0 \frac{\partial L_m}{\partial y^\sigma_k}) - \delta_i^h L_m. \tag{16}
\]
The canonical energy-momentum tensor $\tilde{T}_{ik}$ is, generally, not symmetric - which is a problem, since symmetry is required on physical grounds (angular momentum conservation). This is usually solved by adding a divergence-free term, thus obtaining a tensor $\tilde{T}_{ik}$ which is symmetric and still conserved, i.e., $\tilde{T}_{ik,k} = 0$. There are multiple possibilities of choosing the symmetrization term, [6].

2) In general relativity (where $(X, g)$ is an arbitrary Lorentzian manifold), energy-momentum tensors (Hilbert, or metric energy-momentum tensors) $\tilde{T}_{ij}$ are defined by means of functional derivatives of the matter Lagrangian $\lambda_m = \mathcal{L}_m \omega_0$, $\mathcal{L}_m = \mathcal{L}_m \sqrt{|g|}$, with respect to $g^{ij}$:

$$\frac{1}{2} \tilde{T}_{ij} := \frac{\delta \mathcal{L}_m}{\delta g^{ij}}; \quad \tilde{T}_{ij} := \frac{1}{\sqrt{|g|}} \tilde{T}_{ij}. \quad (17)$$

Here, $\mathcal{L}_m = \mathcal{L}_m(y^\sigma, y^\sigma_j, ..., y^\sigma_{j_1...j_r}, g^{ij})$ is a differential invariant (a "scalar"), hence the Lagrangian $\lambda_m$ is invariant to (transformations on $J^r Y$ induced by) arbitrary diffeomorphisms on $X$. As a result, $\tilde{T}_{ij}$ obeys the covariant conservation law $\tilde{T}_{ij;k} = 0$. Moreover, $\tilde{T}_{ij}$ is, by construction, symmetric.

The two procedures of defining the energy-momentum tensor are fundamentally different and obviously require a thorough geometric analysis. They generally don’t even make sense at the same time: in special relativity, where the metric is fixed, it makes no sense to speak about variations of a Lagrangian with respect to the metric. On the other hand, in general relativity, where $X$ is an arbitrary manifold, space-time translations $\tilde{x}^i = x^i + a^i$, $a^i = \text{const.}$, cannot be defined geometrically. However, there is a realm (see, e.g., [10]) where both procedures can be applied, namely, when:

$$X = \mathbb{R}^4, \quad g_{ij} \text{ - arbitrary} \quad (18)$$

(actually, in [10], it is pointed out the particular case of weak metrics – in which the author studies the equivalence between the two definitions. Still, for our purposes, we do not need the assumption that the metric is weak).

We will prove that, for a first order, special-relativistic Lagrangian

$$\lambda_m = \mathcal{L}_m \omega_0, \quad \mathcal{L}_m = \mathcal{L}_m(y^\sigma, y^\sigma_i, g^{ij} = \eta^{ij}),$$

the canonical variational completion offers a recipe of symmetrization of the Noether current $\tilde{T}_{ij}$. We will do this in three steps:

**Step 1.** We leave for the moment the special relativistic context and allow $g^{ij}$ to vary, as in (18). Abiding by the principle of general covariance, [8], a straightforward curved-space generalization of (16) is given by the tensor density:

$$\tilde{T}_{ik} = g_{kh}(y^\sigma_i; \partial \mathcal{L}_m / \partial y^\sigma_h - \delta^h_i \mathcal{L}_m), \quad \mathcal{L}_m = \mathcal{L}_m \sqrt{|g|}. \quad (19)$$
Step 2. Consider the source form \( \varepsilon = \bar{T}_{ik} \omega^i \wedge \omega^0 \) on \( J^1(Y \times_X \text{Met}(X)) \), with \( \bar{T}_{ik} = \bar{T}_{ik}(y^\sigma, y^\sigma_j, g^{ih}) \). Its \( \text{Met}(X) \)-Vainberg-Tonti Lagrangian \( \lambda_\varepsilon := \mathcal{L}_\varepsilon \omega_0 \) is:

\[
\mathcal{L}_\varepsilon = g^{ik} \int_0^1 (\bar{T}_{ik} \circ \chi_u) du,
\]

where \( \chi_u(y^\sigma, y^\sigma_i, g^{ih}) := (y^\sigma, y^\sigma_i, u g^{ih}) \) only affects the metric components. Substituting \( \bar{T}_{ik} \) from (19) and taking into account that \( \chi_u \) leaves Christoffel symbols invariant and that \( \delta^i_i = \dim(X) = 4 \), we have:

\[
\mathcal{L}_\varepsilon = \int_0^1 u^{-1}(y^\sigma_i \frac{\partial (L_m \circ \chi_u)}{\partial y^\sigma_i} - 4L_m \circ \chi_u) du. \tag{20}
\]

Further, we calculate the Hilbert energy-momentum tensor of \( \lambda_\varepsilon \) as:

\[
\frac{1}{2} \mathcal{T}_{ij} := \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_\varepsilon}{\delta g^{ij}}. \tag{21}
\]

Step 3. Finally, particularize in (21) \( g_{ij} \) as \( \eta_{ij} \) and define

\[
\text{met} \ T_{ij}|_{g_{ij} = \eta_{ij}} = T_{ij}.
\]

The covariant conservation law of \( \text{met} \ T^{ij} \) (obeyed by \( \text{met} \ T^{ij} \)) by virtue of the fact that it is a Hilbert energy-momentum tensor) now transforms into the usual conservation law: \( T^{ij}_{;j} = 0 \).

Thus, the obtained energy-momentum tensor \( T_{ij} \) is both symmetric and conserved. The needed symmetrization term is defined by:

\[
\tau = \tau_{ij} \omega^i \wedge \omega_0, \quad \tau_{ij} := T_{ij} - \bar{T}_{ij}. \tag{22}
\]

Particular case: energy-momentum tensor of the electromagnetic field.

The electromagnetic field is described by the potential 1-form \( A = A_i dx^i \) on \( X \) and by the 2-form \( F := dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j \).

In the special relativistic case \( g_{ij} = \eta_{ij} \), we have \( F_{ij} = A_{j,i} - A_{i,j} \), or, in terms of the contravariant components \( A^i \); \( F_{ij} = \eta_{jk} A^k_{;i} - \eta_{ik} A^k_{;j} \). The Lagrangian of the electromagnetic field is \( \lambda_f = L_f \omega_0 \) with:

\[
L_f = - \frac{1}{16\pi} F_{ij} F^{ij}; \tag{23}
\]
Translational invariance of $\lambda_f$ leads to the Noether current, \cite{8}:

$$\tilde{T}_{ij} = - \frac{1}{4\pi} \frac{\partial A^l}{\partial x^j} F_{jl} + \frac{1}{16\pi} \eta_{ij} F_{kl} F^{kl}. \quad (24)$$

The curved space generalization of $\tilde{T}_{ij}$ in (24) is the tensor density:

$$\tilde{T}_{ij} = \left(- \frac{1}{4\pi} A^l;^i F_{jl} + \frac{1}{16\pi} g_{ij} F_{kl} F^{kl} \right) \sqrt{|g|}. \quad (25)$$

where

$$F_{ij} = g_{jk} A^k;^i - g_{ik} A^k;^j. \quad (26)$$

Further, we calculate the Vainberg-Tonti Lagrangian of the source form $\varepsilon = \tilde{T}_{ij} \omega^i \wedge \omega^0$. We prefer to use $A^k$, rather than $A_k$ as our variables for a reason which will be explained below.

Expressing $F_{ij}$ as in (26), we can now determine the effect of $\chi_u$ on each term of $\tilde{T}_{ij}$:

$$A^l;^i \circ \chi_u = A^l;^i; \quad F_{jl} \circ \chi_u = u^{-1} F_{jl}; \quad F^{kl} \circ \chi_u = u F^{kl}, \quad \sqrt{|g|} \circ \chi_u = u^{-2} \sqrt{|g|}. \quad \text{All in all, we have:}$$

$$\tilde{T}_{ij}(u g^{kl}) = u^{-3} \tilde{T}_{ij}(g^{kl})$$

and hence, the Vainberg-Tonti Lagrangian $\lambda_{\varepsilon} = \mathcal{L}_\varepsilon \omega_0$ is given by:

$$\mathcal{L}_\varepsilon = g^{ij} \int_0^1 \tilde{T}_{ij}(g^{kl}) u^{-3} du = \frac{1}{2} g^{ij} \tilde{T}_{ij}, \quad \text{that is,}$$

$$\mathcal{L}_\varepsilon = \left(\frac{1}{8\pi} A^{ik} F_{kl} - \frac{1}{8\pi} F_{kl} F^{kl} \right) \sqrt{|g|}. \quad (27)$$

Taking into account that $F_{kl} = - F_{lk}$, the first term in the above can be re-expressed as: $\frac{1}{8\pi} A^{ik} F_{kl} = \frac{1}{16\pi} (A^{ik} - A^{ki}) F_{kl} = \frac{1}{16\pi} F^{kl} F_{kl}$; substituting into (27), we finally obtain the $\text{Met}(X)$-Vainberg-Tonti Lagrangian of (25) as:

$$\lambda_{\varepsilon} = - \frac{1}{16\pi} F^{ij} F_{ij} \sqrt{|g|} \omega_0,$$

which is nothing else than the curved space variant of the Lagrangian $\lambda_f$ of the electromagnetic field.

But, variation of $\lambda_f$ with respect to $g^{ij}$ is well-known, \cite{8}, to lead to the Hilbert energy-momentum tensor:

$$\text{met} \quad T_{ij} = \left(- \frac{1}{4\pi} F^{ij} F_{jl} + \frac{1}{16\pi} g_{ij} F_{kl} F^{kl} \right).$$

Particularizing now $g_{ij} = \eta_{ij}$, we get $\text{met} \quad T_{ij} = T^{\text{met}}_{ij}$ and

$$\tau_{ij} = \frac{1}{4\pi} A_{i,^l} F_{j}^{^l}.$$
as the canonical variational completion. This is the classical symmetrization term, yet, obtained here by a completely different reasoning.

Remarks.
1) The $\text{Met}(X)$-Vainberg-Tonti Lagrangian $\lambda_\varepsilon$ of (25) coincides with the Lagrangian $\lambda_f$ of the free electromagnetic field, used in deducing the Noether currents (24). Still, for a general matter Lagrangian $\lambda_m$, we cannot state that $\lambda_m = \lambda_\varepsilon$.

2) If we had worked with the potential 1-form components $A_i$ (instead of the vector field components $A^i$) as the field variables, we would have had $F_{ij} = A_{ij} - A_{ij}$ - invariant to $\chi_u$ and by a similar reasoning to the above, we would have got $\tilde{T}_{ij}(u^k) = u^{-1}\tilde{T}_{ij}(g^k)$ and, consequently, to $L_\varepsilon = (g^{ij}\tilde{T}_{ij})\int_0^u u^{-1}du$. But, since the latter integral does not have a finite value, we could not have calculated $L_\varepsilon$ this way. Hence, it appears that, at least in this case, the 4-potential vector field components $y^\alpha := A^i$ are a more natural choice as the dynamical variables.

6 An example in first order mechanics

Take $Y = \mathbb{R} \times \mathbb{R}^n$, with local coordinates $(t, q^\alpha)$; on the second jet prolongation $J^2Y$, we denote the induced local coordinates by $(t, q^\alpha, \dot{q}^\alpha, \ddot{q}^\alpha)$.

Consider the second order source form

$$\varepsilon = \varepsilon_\sigma \omega^\tau \wedge dt,$$

$$\varepsilon_\sigma = m_{\sigma\nu} \ddot{q}^\nu + k_{\sigma\nu} q^\nu + \frac{\partial F}{\partial \dot{q}^\sigma},$$

(28)

where:
- $m_{\sigma\nu}$, $k_{\sigma\nu}$ are constant and symmetric;
- $F = F(\dot{q}^\sigma)$ is homogeneous of some degree $p \in \mathbb{R} \setminus \{0\}$ in $\dot{q}^\sigma$.

The ODE system $\varepsilon_\sigma = 0$ is generally non-variational. Let us determine its canonical variational completion. The Vainberg-Tonti Lagrangian attached to $\varepsilon$ is $\lambda_\varepsilon = L_\varepsilon dt$, with

$$L_\varepsilon = q^\sigma \int_0^1 \varepsilon_\sigma(t, uq^\nu, u\dot{q}^\nu, u\ddot{q}^\nu)du = q^\sigma \int_0^1 (m_{\sigma\nu} u\ddot{q}^\nu + k_{\sigma\nu} uq^\nu + \frac{\partial F}{\partial \dot{q}^\sigma}(u\dot{q}))du.$$

Taking into account the homogeneity degree of $F$, this is:

$$L_\varepsilon = q^\sigma \int_0^1 u(m_{\sigma\nu} \ddot{q}^\nu + k_{\sigma\nu} q^\nu) + u^{p-1} \frac{\partial F}{\partial \dot{q}^\sigma}(u\dot{q})]du =$$

$$= \frac{1}{2} (m_{\sigma\nu} \ddot{q}^\nu + k_{\sigma\nu} q^\nu) + \frac{1}{p} q^\sigma \frac{\partial F}{\partial q^\sigma}.$$
The term $\frac{1}{2}m_{\sigma\nu}\ddot{q}^\nu q^\sigma$ differs by a total derivative $d_t(\frac{1}{2}m_{\sigma\nu}\ddot{q}^\nu q^\sigma)$ from $-\frac{1}{2}m_{\sigma\nu}\dot{q}^\nu \dot{q}^\sigma$, hence the two expressions are dynamically equivalent. We will thus prefer to take the latter, which is of lower order and thus, we obtain the following Lagrangian function, which is equivalent to the Vainberg-Tonti one:

$$L = \frac{1}{2}(-\frac{1}{2}m_{\sigma\nu}\dot{q}^\nu \dot{q}^\sigma + k_{\sigma\nu} q^\sigma q^\nu) + \frac{1}{p} \dot{q}^\rho \frac{\partial F}{\partial \dot{q}^\rho}. \quad (29)$$

Let us determine the Euler-Lagrange form of $L$. We have, on one hand:

$$\frac{\partial L}{\partial q^\rho} = k_{\sigma\rho} q^\sigma + \frac{1}{p} \frac{\partial F}{\partial \dot{q}^\rho}$$

and, on the other hand,

$$\frac{\partial L}{\partial \dot{q}^\rho} = -m_{\sigma\rho} \ddot{q}^\sigma + \frac{1}{p} \frac{\partial^2 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma} \dot{q}^\sigma,$$

$$d_t(\frac{\partial L}{\partial \dot{q}^\rho}) = -m_{\sigma\rho} \ddot{q}^\sigma + \frac{1}{p} \frac{\partial^3 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma \partial \dot{q}^\nu} \ddot{q}^\nu q^\sigma + \frac{1}{p} \frac{\partial^2 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma} \dot{q}^\sigma;$$

taking again into account that $\frac{\partial F}{\partial \dot{q}^\rho}$ is homogeneous of degree $p - 1$, the latter term is:

$$\frac{1}{p} \frac{\partial^2 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma} \ddot{q}^\sigma = \frac{p - 1}{p} \frac{\partial F}{\partial \dot{q}^\rho}$$

and, finally,

$$E_{\rho}(L) = (m_{\sigma\rho} \ddot{q}^\sigma + k_{\sigma\rho} q^\sigma) + \frac{2 - p}{p} \frac{\partial F}{\partial \dot{q}^\rho} - \frac{1}{p} \frac{\partial^3 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma \partial \dot{q}^\nu} \ddot{q}^\nu q^\sigma.$$

We find the variational completion $\tau = \tau_{\rho}(t, q, \dot{q}) \omega^\rho \wedge dt$ as:

$$\tau_{\rho} = 2(\frac{1}{p} - 1) \frac{\partial F}{\partial \dot{q}^\rho} - \frac{1}{p} \frac{\partial^3 F}{\partial \dot{q}^\rho \partial \dot{q}^\sigma \partial \dot{q}^\nu} \ddot{q}^\nu q^\sigma. \quad (30)$$

**Particular cases:**

1) If $F = 0$, the system $\varepsilon = 0$ is equivalent to

$$m_{\sigma\nu} \ddot{q}^\nu + k_{\sigma\nu} q^\nu = 0.$$  

These equations characterize free small oscillations with multiple degrees of freedom, [8]. They are known to be variational; their Lagrangian function $L = \frac{1}{2}(-m_{\sigma\nu} \dot{q}^\nu \dot{q}^\sigma + k_{\sigma\nu} q^\nu q^\sigma)$ coincides (as expected), with (29).

2) If $p = 2$, i.e., $F$ is quadratic in $\dot{q}$:

$$F = \frac{1}{2} \alpha_{\sigma\nu} \ddot{q}^\nu \ddot{q}^\sigma.$$
(where $\alpha_{\sigma\nu} = \alpha_{\nu\sigma} \in \mathbb{R}$) the ODE system $\varepsilon_{\sigma} = 0$ characterizes, [9], Section 25, linearly damped oscillations with multiple degrees of freedom. In this case, the function $F$ is called the Rayleigh dissipation function and is interpreted as the rate of energy dissipation in the system. In (28), the last term (with a minus in front) $-\frac{\partial F}{\partial \dot{q}^\rho} = -a_{\sigma\nu}\dot{q}^\nu$ is interpreted as a friction force. In this case, the canonical variational completion (30) is given by

$$\tau^\rho = -\frac{\partial F}{\partial \dot{q}^\rho}$$

and the variationally completed equations are:

$$m_{\rho\nu}\ddot{q}^\nu + k_{\rho\nu}q^\nu = 0,$$  \hspace{1cm} (31)

which are precisely the equations of "undamped" oscillations. That is, the friction force $\frac{\partial F}{\partial \dot{q}^\rho}$ has, in this case, the meaning of obstruction from variationality of the equations.

**Remark.** In other cases (e.g., when $-\frac{\partial F}{\partial \dot{q}^\rho}$ is quadratic or cubic in $\dot{q}^\sigma$), the variationally completed equations will not coincide anymore with the equations (31) of undamped oscillations.

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