Solutions of several coupled discrete models in terms of Lamé polynomials of order one and two

AVINASH KHARE$^{1,3,}$*, and AVADH SAXENA$^2$

$^1$Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India
$^2$Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
$^3$Current address: Indian Institute of Science Education and Research, Sai Trinity Building, Pashan, Pune 411 021, India

*Corresponding author. E-mail: khare@iiserpune.ac.in

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Abstract. Coupled discrete models abound in several areas of physics. Here we provide an extensive set of exact quasiperiodic solutions of a number of coupled discrete models in terms of Lamé polynomials of order one and two. Some of the models discussed are: (i) coupled Salerno model, (ii) coupled Ablowitz–Ladik model, (iii) coupled saturated discrete nonlinear Schrödinger equation, (iv) coupled $\phi^4$ model and (v) coupled $\phi^6$ model. Furthermore, we show that most of these coupled models in fact also possess an even broader class of exact solutions.

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1. Introduction

In a recent paper [1] we have obtained solutions of a coupled $\phi^4$ and a coupled $\phi^6$ model in terms of Lamé polynomials of order one and two even though most of these solutions are not the solutions of the corresponding uncoupled problem. The purpose of the present paper is to carry out a similar study for a number of coupled discrete field theory models. In particular, we obtain exact solutions of (i) coupled Ablowitz–Ladik (AL) model, (ii) coupled Salerno model, (iii) coupled saturated discrete nonlinear Schrödinger equation (DNLSE), (iv) coupled $\phi^6$ model and (v) coupled $\phi^4$ model. We also show that unlike the continuum field theory models, many of the discrete coupled field theory models possess an even broader class of exact solutions. Moreover, we show that as in the uncoupled case [2], even the coupled AL, coupled Salerno and coupled DNLSE models follow from the same Hamiltonian but with a different Poisson bracket (PB) structure.
The motivation for this work comes from the fact that there are many physical situations where a discrete field theory is appropriate to model the phenomena of interest with a specific coupling between the two fields. One such phenomenon of current intense interest is the coexistence of magnetism and ferroelectricity (i.e. magnetoelectricity) in a given material. This is a highly desired functionality in technological applications involving cross-field response, switching and actuation. In general, this phenomenon is referred to as multiferroic behaviour [3]. Recently, two different classes of (single-phase) multiferroics, namely the orthorhombically distorted perovskites [4] and rare-earth hexagonal structures [5], have emerged. The latter shows magnetic domain walls in the basal planes which can be modelled by a coupled $\phi^4$ model [6] in the presence of a magnetic field. Coupled $\phi^4$ models [7–9] also arise in the context of many ferroelectric and other second-order phase transitions. The coupled $\phi^4$ model for multiferroics [6] has a biquadratic coupling whereas the coupled $\phi^4$ model for a surface phase transition with hydration forces [9], relevant in biophysics context, has a bilinear coupling. Other types of couplings are also known for structural phase transitions with strain [10].

Examples of coupled discrete AL, coupled discrete Salerno and coupled saturated DNLS models are also known [11–13]. Similarly, there are analogous coupled models in field theory [14,15]. Several related models have been discussed in the literature and their soliton solutions have been found [16–23] including periodic ones [24–26].

The paper is organized as follows. In §2 we first show that the coupled AL, coupled Salerno and coupled DNLS models can all be obtained from the same Hamiltonian but with different PB structure. We also obtain additional conserved quantities in these models. In §3 we provide the solutions for the coupled Salerno model in terms of Lamé polynomials of order one and two as well as a broader class of solutions. In §4 we provide similar solutions for the coupled AL model. In §5 we show that unlike the coupled Salerno and coupled AL models, the coupled saturated DNLS model only admits Lamé polynomial solutions of order one but not of order two. Besides, we have not been able to obtain a broader class of solutions in this case. Section 6 is devoted to solutions of a coupled discrete $\phi^6$ model [1,28] in terms of Lamé polynomials of order one and two and also a broader class of solutions. In §7 we discuss solutions of a coupled $\phi^4$ model introduced by us recently [29] in terms of Lamé polynomials of order two as well as a broader class of solutions. Note that the solutions in terms of Lamé polynomials of order one have already been obtained by us in [29]. Section 8 contains the summary of main results and possible future directions.

2. The model for coupled saturated DNLS, coupled AL and coupled Salerno equations

We have previously shown [2] that the uncoupled Salerno model [30], the uncoupled AL model [31] and the uncoupled saturated DNLS model can all be deduced from the same Hamiltonian

$$
H = \sum_{n=1}^N \left[ |u_n - u_{n+1}|^2 - \frac{\nu}{\mu} |u_n|^2 + \frac{\nu}{\mu^2} \ln(1 + \mu |u_n|^2) \right],
$$

(1)
but with different PB structures. We now show that even the coupled Salerno, coupled AL and the coupled saturated DNLSE models can all be derived from the same Hamiltonian given by

\[ H = N \sum_{n=1}^{N} \left[ |u_n - u_{n+1}|^2 + |v_n - v_{n+1}|^2 - \frac{v_1}{\mu_1} |u_n|^2 \right. \]

\[ - \frac{v_2}{\mu_2} |v_n|^2 + \frac{v_1}{\mu_1^2} \ln(1 + \mu_1 |u_n|^2 + \mu_2 |v_n|^2) \] \hspace{1cm} (2)

with the equations of motion in the two field variables \( u_n \) and \( v_n \) in all three cases being

\[ i \dot{u}_n = [u_n, H], \quad i \dot{v}_n = [v_n, H]. \] \hspace{1cm} (3)

The difference in the equations of motion comes from a different definition of the PB and consequently a different definition of the time derivative. The PB structure in all three cases can be compactly written as

\[ [U, V] = N \sum_{n=1}^{N} \left[ \frac{\partial U}{\partial u_n} \frac{\partial V}{\partial u^*_n} - \frac{\partial U}{\partial u^*_n} \frac{\partial V}{\partial u_n} + \frac{\partial U}{\partial v_n} \frac{\partial V}{\partial v^*_n} - \frac{\partial U}{\partial v^*_n} \frac{\partial V}{\partial v_n} \right] \]

\[ \times \left[ 1 + \lambda_1 |u_n|^2 + \lambda_2 |v_n|^2 \right]. \] \hspace{1cm} (4)

2.1 Coupled saturated DNLSE

On using eqs (2)–(4) with \( \lambda_1 = \lambda_2 = 0 \) yields the coupled saturated DNLS equations

\[ i \frac{du_n}{dt} + [u_{n+1} + u_{n-1} - 2u_n] + \frac{v_1 (\mu_1 |u_n|^2 + \mu_2 |v_n|^2) u_n}{\mu_1 (1 + \mu_1 |u_n|^2 + \mu_2 |v_n|^2)} = 0, \] \hspace{1cm} (5)

\[ i \frac{dv_n}{dt} + [v_{n+1} + v_{n-1} - 2v_n] + \frac{(v_2 - \frac{v_1 \mu_2}{\mu_1}) v_n + v_2 (\mu_1 |u_n|^2 + \mu_2 |v_n|^2) v_n}{\mu_2 (1 + \mu_1 |u_n|^2 + \mu_2 |v_n|^2)} = 0. \] \hspace{1cm} (6)

It is easily checked that in this case, apart from the Hamiltonian (1), the two other conserved quantities are power \( P_u \) and \( P_v \) defined by

\[ P_u = \sum_{n=1}^{N} |u_n|^2, \quad P_v = \sum_{n=1}^{N} |v_n|^2. \] \hspace{1cm} (7)

2.2 Coupled Salerno model

If instead, we use eqs (2)–(4) with \( \lambda_1 = \mu_1 \) and \( \lambda_2 = \mu_2 \), then we obtain the coupled Salerno model with field equations

\[ i \frac{du_n}{dt} + [u_{n+1} + u_{n-1} - 2u_n] + (\mu_1 |u_n|^2 + \mu_2 |v_n|^2) \]

\[ \times \left[ u_{n+1} + u_{n-1} + \frac{v_1 - 2\mu_1}{\mu_1} u_n \right] = 0, \] \hspace{1cm} (8)
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\[\text{id}v_n/\text{d}t + \left[ v_{n+1} + v_{n-1} - \left( 2 + \frac{v_1\mu_2}{\mu_1^2} - \frac{v_2}{\mu_2} \right) v_n \right] + \left( \mu_1|u_n|^2 + \mu_2|v_n|^2 \right) \left[ v_{n+1} + v_{n-1} + \frac{v_2 - 2\mu_2}{\mu_2} v_n \right] = 0. \quad (9)\]

It is easily checked that in this case, apart from the Hamiltonian (1), the other conserved quantity is power \(P\) given by

\[P = \sum_{n=1}^{N} \ln[1 + \mu_1|u_n|^2 + \mu_2|v_n|^2]. \quad (10)\]

2.3 Coupled AL model

In the special case when \(v_1 = 2\mu_1\) and \(v_2 = 2\mu_2\), the coupled Salerno model reduces to the coupled AL model with the field equations

\[\text{id}u_n/\text{d}t + [u_{n+1} + u_{n-1} - 2u_n] + (\mu_1|u_n|^2 + \mu_2|v_n|^2)[u_{n+1} + u_{n-1}] = 0, \quad (11)\]

\[\text{id}v_n/\text{d}t + \left[ v_{n+1} + v_{n-1} - \frac{2\mu_2}{\mu_1} v_n \right] + (\mu_1|u_n|^2 + \mu_2|v_n|^2)[v_{n+1} + v_{n-1}] = 0. \quad (12)\]

It is interesting to note that in this case, apart from the Hamiltonian (1) and power \(P\) as given by eq. (10), generalized momentum \(P_m\) given by

\[P_m = \sum_{n=1}^{N} i[\mu_1(u_n u_{n+1}^* - u_{n+1}^* u_n) + \mu_2(v_n v_{n+1}^* - v_{n+1}^* v_n)], \quad (13)\]

is also conserved.

One remark is in order here. Just as the uncoupled Salerno model interpolates between AL and DNLSE, it is easy to see that the coupled Salerno model given by eqs (8) and (9) also interpolates between coupled AL model (as given by eqs (11) and (12)) and coupled DNLSE. In particular, in the limit \(v_1 = 2\mu_1\) and \(v_2 = 2\mu_2\), the coupled Salerno model eqs (8) and (9) go over to the coupled AL model eqs (11) and (12). On the other hand, in the limit \(\mu_1 = \mu_2 = 0\) but with \(\mu_2/\mu_1 = c\) and \(v_2\mu_1^2 = v_1\mu_2^2\) the coupled Salerno model eqs (8) and (9) go over to the coupled DNLSE equations

\[\text{id}u_n/\text{d}t + [u_{n+1} + u_{n-1} - 2u_n] + v_1(|u_n|^2 + c|v_n|^2)u_n = 0, \quad (14)\]

\[\text{id}v_n/\text{d}t + [v_{n+1} + v_{n-1} - 2v_n] + v_1 c(|u_n|^2 + c|v_n|^2)v_n = 0. \quad (15)\]

3. Solutions of the coupled Salerno model

We now show that the coupled Salerno model given by eqs (8) and (9) has Lamé polynomial solutions of order one and two. In fact, it turns out that the coupled model has an even broader class of exact solutions of which Lamé polynomial solutions of order one...
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and two are the special cases. We remind that as far as we know, the uncoupled Salerno model has no known exact solutions.

We start with the ansatz

\[
    u_n = f_n \exp[-i(\omega_1 t + \delta_1)], \quad v_n = g_n \exp[-i(\omega_2 t + \delta_2)],
\]

with \( f_n \) and \( g_n \) satisfying

\[
    f_n^2 + ag_n^2 = b, \quad a, b > 0.
\]

Here \( a, b > 0 \) are two positive numbers while \( \delta_1, \delta_2 \) are two arbitrary parameters. On substituting this ansatz in eqs (8) and (9) we find that this is a consistent ansatz provided

\[
    a = \frac{\mu_2}{\mu_1}, \quad b = -\frac{1}{\mu_1}, \quad \omega_1 = \frac{\nu_1}{\mu_1}, \quad \omega_2 = \frac{\nu_1 \mu_2}{\mu_1^2}.
\]

This implies that such solutions are possible only if \( \mu_1, \mu_2 < 0 \). Further, since

\[
    \frac{\omega_1}{\omega_2} = \frac{\mu_1}{\mu_2},
\]

which is a real number, the solutions obtained from here are in general only quasiperiodic. Only if \( \mu_1/\mu_2 \) is a rational number, will the solutions be periodic.

Clearly this is a very general ansatz which admits a broad class of solutions including Lamé polynomials of order one and two.

3.1 Lamé polynomial solutions of order one

(i) One solution is

\[
    f_n = A \text{dn}[\beta(n + c_2), m], \quad g_n = B \sqrt{m} \text{sn}[\beta(n + c_2), m],
\]

provided eq. (18) is satisfied and further

\[
    b = A^2, \quad \mu_1 A^2 = \mu_2 B^2.
\]

Note that \( \beta \) is completely arbitrary. Using the fact that \( \text{dn}(x, m) \) has period \( 2K(m) \) while \( \text{cn}(x, m) \) and \( \text{sn}(x, m) \) are periodic functions with period \( 4K(m) \), it follows that for the solution (20), \( u_n, v_n \) satisfy the boundary condition

\[
    u_{n + \frac{2K(m)}{\beta}} = u_n, \quad v_{n + \frac{4K(m)}{\beta}} = v_n.
\]

Here \( K(m) \) is the complete elliptic integral of the first kind.

(ii) Another solution is

\[
    f_n = A \sqrt{m} \text{cn}[\beta(n + c_2), m], \quad g_n = B \sqrt{m} \text{sn}[\beta(n + c_2), m],
\]

provided eq. (18) is satisfied and further

\[
    b = mA^2, \quad \mu_1 A^2 = \mu_2 B^2.
\]

For the solution (23), \( u_n, v_n \) satisfy the boundary condition

\[
    u_{n + \frac{4K(m)}{\beta}} = u_n, \quad v_{n + \frac{4K(m)}{\beta}} = v_n.
\]
In the limit \( m = 1 \), both these solutions go over to the hyperbolic solution

\[
  f_n = A \text{sech}[\beta(n + c_2)], \quad g_n = B \tanh[\beta(n + c_2)].
\]  

(26)

3.2 Lamé polynomial solutions of order two

(iii) One solution is

\[
  f_n = A \text{dn}^2[\beta(n+c_2), m] + B, \quad g_n = F \sqrt{m} \text{sn}[\beta(n+c_2), m] \text{dn}[\beta(n+c_2), m],
\]  

(27)

provided eq. (18) is satisfied and further

\[
  b = \frac{A^2}{4}, \quad \mu_1 A^2 = \mu_2 F^2, \quad A = -2B.
\]  

(28)

For the solution (27), \( u_n, v_n \) satisfy the boundary condition (22).

(iv) Another solution is

\[
  f_n = A \text{dn}^2[\beta(n+c_2), m] + B, \quad g_n = F m \text{sn}[\beta(n+c_2), m] \text{cn}[\beta(n+c_2), m],
\]  

(29)

provided eq. (18) is satisfied and further

\[
  b = \frac{m^2 A^2}{4}, \quad \mu_1 A^2 = \mu_2 F^2, \quad (2 - m) A = -2B.
\]  

(30)

For the solution (29), \( u_n, v_n \) satisfy the boundary condition

\[
  u_{n+\frac{2\pi \omega}{\beta}} = u_n, \quad v_{n+\frac{2\pi \omega}{\beta}} = v_n.
\]  

(31)

In the limit \( m = 1 \), both solutions (27) and (29) go over to the hyperbolic solution

\[
  f_n = A \text{sech}^2[\beta(n+c_2)] + B, \quad g_n = F \tanh[\beta(n+c_2)] \text{sech}[\beta(n+c_2)].
\]  

(32)

(v) Apart from these, several other solutions are possible. For example, one can have nonperiodic solutions like

\[
  f_n = A \frac{\sqrt{1 + n^2}}{\sqrt{1 + n^2}}, \quad g_n = Bn \frac{\sqrt{1 + n^2}}{\sqrt{1 + n^2}}.
\]  

(33)

provided eq. (18) is satisfied and further

\[
  b = A^2, \quad \mu_1 A^2 = \mu_2 B^2.
\]  

(34)

One can obviously write down a wider class of such solutions. For example

\[
  f_n = A \frac{\sqrt{1 + n^2}}{\sqrt{1 + n^2 + n^4}}, \quad g_n = Bn^2 \frac{\sqrt{1 + n^2}}{\sqrt{1 + n^2 + n^4}}.
\]  

(35)

provided eqs (18) and (34) are satisfied.
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(vi) Yet another possible periodic solution is
\[ f_n = A \cos[\beta(n + c_2)], \quad g_n = B \sin[\beta(n + c_2)], \]
provided eq. (18) is satisfied and further
\[ b = A^2, \quad \mu_1 A^2 = \mu_2 B^2. \]  
(37)

In this case both \( u_n, v_n \) satisfy the periodicity condition
\[ u_{n + \frac{c_2}{\beta}} = u_n, \quad v_{n + \frac{c_2}{\beta}} = v_n. \]  
(38)

It turns out that apart from the general solution given by eqs (16) and (17), there is another possible general solution given by
\[ u_n = f_n \exp[-i(\omega_1 t + \delta_1)], \quad v_n = g_n \exp[-i(\omega_2 t + \delta_2)], \]
but now \( f_n \) and \( g_n \) satisfy
\[ f_n^2 - ag_n^2 = b, \quad a, b > 0. \]  
(40)

On substituting this ansatz in eqs (8) and (9) we find that this is a consistent ansatz provided
\[ a = -\frac{\mu_2}{\mu_1}, \quad b = -\frac{1}{\mu_1}, \quad \omega_1 = \frac{v_1}{\mu_1}, \quad \omega_2 = \frac{v_1 \mu_2}{\mu_1^2}. \]  
(41)

This implies that such solutions are possible only if \( \mu_1, \mu_2 \) have opposite signs. Clearly this is a very general ansatz which admits a broad class of solutions. As an illustration we discuss a few such solutions.

(vii) One solution is
\[ f_n = A \operatorname{dn}^2[\beta(n + c_2), m], \quad g_n = B \sqrt{m} \operatorname{sn}[\beta(n + c_2), m], \]
provided eq. (41) is satisfied and further
\[ b = A^2, \quad \mu_1 < 0, \quad \mu_2 > 0, \quad |\mu_1| A^2 = \mu_2 B^2. \]  
(43)

In this case both \( u_n, v_n \) satisfy the periodicity condition (22). Note that if we interchange \( f_n \) and \( g_n \), then \( \mu_1 > 0, \mu_2 < 0 \). In the limit \( m = 1 \), this solution goes over to the hyperbolic solution
\[ f_n = A \cosh[\beta(n + c_2)], \quad g_n = B \sinh[\beta(n + c_2)]. \]  
(44)

(viii) Another solution is
\[ f_n = A \operatorname{dn}[\beta(n + c_2), m] + B, \quad g_n = B \sqrt{m} \operatorname{sn}[\beta(n + c_2), m], \]
provided eq. (41) is satisfied and further
\[ b = A^2, \quad \mu_1 < 0, \quad \mu_2 > 0, \quad |\mu_1| A^2 = \mu_2 F^2, \quad A = -2B. \]  
(46)

In this case both \( u_n, v_n \) satisfy the periodicity condition (22). In the limit \( m = 1 \), this solution goes over to the hyperbolic solution
\[ f_n = A \cosh^2[\beta(n + c_2)] + B, \quad g_n = F \sinh[\beta(n + c_2)] \cosh[\beta(n + c_2)]. \]  
(47)
(ix) Apart from these, several other solutions are possible. For example, one can have the nonperiodic solution

\[ f_n = A \sqrt{\frac{2 + n^2}{1 + n^2}}, \quad g_n = B \frac{1}{\sqrt{1 + n^2}}, \] (48)

provided eq. (41) is satisfied and further

\[ b = A^2, \quad \mu_1 < 0, \quad \mu_2 > 0, \quad |\mu_1|A^2 = \mu_2 F^2. \] (49)

One can, easily write down a wider class of such solutions.

4. Solutions of the coupled AL model

We show that for the coupled AL model characterized by eqs (11) and (12), one not only has solutions similar to those in the previous section (for the coupled Salerno case), but just like the uncoupled AL case [32], even coupled AL equations have moving periodic solutions in terms of Lamé polynomials of order one.

As in the previous section, if we start with the ansatz given by eqs (16) and (17) or eqs (39) and (40), then it is easy to show that the entire discussion of the previous section goes through except that since in the coupled AL model \( v_1 = 2\mu_1, \ v_2 = 2\mu_2, \) in the coupled AL model with the above two ansatz, \( \omega_1 = 2, \ \omega_2 = 2\mu_2/\mu_1. \) But for this minor change, all the nine solutions given in the previous section are also solutions of the coupled AL model under identical conditions (except \( \omega_1 = 2, \ \omega_2 = 2\mu_2/\mu_1. \))

In addition to these nine solutions, we now show that as in the uncoupled case [32], the coupled AL model also admits moving periodic solutions in terms of Lamé polynomials of order one.

(i) For example, it admits mixed moving periodic kink-pulse solution

\[ u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)]dn(\beta(n - vt + \delta_2), m), \]
\[ v_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)]\sqrt{m}\ sn(\beta(n - vt + \delta_2), m), \] (50)

provided

\[ \omega_1 = 2 \left[ 1 - (1 + \mu_2 B^2)\frac{\cos(k_1)dn(\beta, m)}{\cn(\beta, m)} \right], \]
\[ \omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - (1 + \mu_2 B^2)\frac{\cos(k_2)dn(\beta, m)}{\cn(\beta, m)} \right], \]
\[ 1 = \mu_1 A^2 \cs^2(\beta, m) - \mu_2 B^2 \ns^2(\beta, m), \]
\[ \beta v = \frac{2 \sin(k_1)(1 + \mu_2 B^2)}{\cs(\beta, m)}, \quad \cn(\beta, m) = \frac{\sin(k_2)}{\sin(k_1)}, \] (51)
where \(cs(\beta, m) = cn(\beta, m)/sn(\beta, m)\) and \(ns(\beta, m) = 1/sn(\beta, m)\). Note that since it is a moving periodic kink-pulse solution, it must satisfy not only the periodicity condition (22) but also the periodicity condition
\[
u_{n+\frac{2\pi}{\beta}} = v_n.
\] (52)

The periodicity conditions (22) and (52) imply that \(u_n, v_n\) are periodic solutions provided there exist integers \(n_1, n_2, n_3, n_4\) such that
\[
n_1 \frac{2K(m)}{\beta} = n_2 \frac{2\pi}{k_1}, \quad n_3 \frac{4K(m)}{\beta} = n_4 \frac{2\pi}{k_2}.
\] (53)

(ii) Another mixed moving periodic kink-pulse solution that it admits is
\[
u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \sqrt{m cm[\beta(n - vt + \delta_2), m]},
\]
\[
u_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \sqrt{m sn[\beta(n - vt + \delta_2), m]},
\] (54)

provided
\[
\omega_1 = 2 \left[ 1 - (1 + m \mu_2 B^2) \frac{\cos(k_1) cn(\beta, m)}{dn(\beta, m)} \right],
\]
\[
\omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - (1 + m \mu_2 B^2) \frac{\cos(k_2) cn(\beta, m)}{dn(\beta, m)} \right],
\]
\[
1 = \mu_1 A^2 ds^2(\beta, m) - \mu_2 B^2 ns^2(\beta, m),
\]
\[
\beta v = \frac{2 \sin(k_1)}{ds(\beta, m)} [1 + m \mu_2 B^2],\quad dn(\beta, m) = \frac{\sin(k_2)}{\sin(k_1)},
\] (55)

where \(ds(\beta, m) = dn(\beta, m)/sn(\beta, m)\). Note that since it is a moving periodic kink-pulse solution, it must satisfy not only the periodicity condition (25) but also the periodicity condition (52). The periodicity conditions (25) and (52) imply that \(u_n, v_n\) are periodic solutions provided there exist integers \(n_1, n_2, n_3, n_4\) such that
\[
n_1 \frac{4K(m)}{\beta} = n_2 \frac{2\pi}{k_1}, \quad n_3 \frac{4K(m)}{\beta} = n_4 \frac{2\pi}{k_2}.
\] (56)

In the limit \(m = 1\), both these solutions reduce to the moving pulse-kink solution
\[
u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \text{sech}[\beta(n - vt + \delta_2)],
\]
\[
u_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \text{tanh}[\beta(n - vt + \delta_2)],
\] (57)

provided
\[
\sinh^2(\beta) = \mu_1 A^2 - \mu_2 B^2 \cosh^2(\beta),
\]
\[
\omega_1 = 2[1 - (1 + \mu_2 B^2) \cos(k_1) \cosh(\beta)],
\]
\[
\omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - (1 + \mu_2 B^2) \cos(k_2) \right],
\]
\[
\nu \beta = 2(1 + \mu_2 B^2) \sin(k_1) \sinh(\beta),\quad \frac{\sin(k_2)}{\sin(k_1)} = \text{sech}(\beta).
\] (58)
Notice that in case \( k_1 = k_2 = v = 0 \), the solutions (50), (54) and (57) become stationary coupled, periodic pulse-kink solutions provided relations (51), (55) and (58) with \( k_1 = k_2 = v = 0 \) are satisfied. However, we have already shown (and it can also be verified from relations (51), (55) and (58)) that the solutions (50), (54) and (57) with \( k_1 = k_2 = v = 0 \) also hold good under stronger conditions given by eqs (18), (21) and (24) with \( \nu_1 = 2\mu_1, \nu_2 = 2\mu_2 \).

(iii) It also admits a coupled moving periodic pulse solution

\[
\begin{align*}
  u_n &= A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \text{dn}[\beta(n - vt + \delta_2), m], \\
  v_n &= B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \text{dn}[\beta(n - vt + \delta_2), m],
\end{align*}
\]

provided

\[
\begin{align*}
  k_1 &= k_2, \quad \omega_1 = 2 \left[ 1 - \frac{\cos(k_1)\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right], \\
  \omega_2 &= 2 \left[ \frac{\mu_2}{\mu_1} - \frac{\cos(k_1)\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right], \\
  1 &= (\mu_1 A^2 + \mu_2 B^2)\text{cs}^2(\beta, m), \quad \beta v = \frac{2\sin(k_1)}{\text{cs}(\beta, m)}. \quad (60)
\end{align*}
\]

Note that since it is a moving periodic pulse solution, it must not only satisfy the periodicity condition (25) but also the periodicity condition (52). The periodicity conditions (31) and (52) imply that \( u_n, v_n \) are periodic solutions provided there exist integers \( n_1, n_2, n_3, n_4 \) such that

\[
\begin{align*}
  n_1 \frac{2K(m)}{\beta} &= n_2 \frac{2\pi}{k_1}, \quad n_3 \frac{2K(m)}{\beta} &= n_4 \frac{2\pi}{k_2}. \quad (61)
\end{align*}
\]

(iv) Another coupled periodic moving pulse solution is

\[
\begin{align*}
  u_n &= A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \sqrt{m} \text{cn}[\beta(n - vt + \delta_2), m], \\
  v_n &= B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \sqrt{m} \text{cn}[\beta(n - vt + \delta_2), m],
\end{align*}
\]

provided

\[
\begin{align*}
  k_1 &= k_2, \quad \omega_1 = 2 \left[ 1 - \frac{\cos(k_1)\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right], \\
  \omega_2 &= 2 \left[ \frac{\mu_2}{\mu_1} - \frac{\cos(k_1)\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right], \\
  1 &= (\mu_1 A^2 + \mu_2 B^2)\text{ds}^2(\beta, m), \quad \beta v = \frac{2\sin(k_1)}{\text{ds}(\beta, m)}. \quad (63)
\end{align*}
\]

Note that since it is a moving periodic pulse solution, it must not only satisfy the periodicity condition (25) but also the periodicity condition (52). The periodicity
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conditions (25) and (52) imply that $u_n, v_n$ are periodic solutions provided there exist integers $n_1, n_2, n_3, n_4$ such that the condition (56) is satisfied.

(v) Finally, it also admits a mixed coupled moving periodic pulse solution

\[ u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \text{dn} [\beta(n - vt + \delta_2), m] , \]
\[ v_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \sqrt{m} \text{cn} [\beta(n - vt + \delta_2), m] , \] (64)

provided

\[ \omega_1 - 2 = -[1 - (1 - m)\mu_2 B^2] \frac{2 \cos(k_1) \text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} , \]
\[ \omega_2 - 2 \frac{\mu_2}{\mu_1} = -[1 - (1 - m)\mu_2 B^2] \frac{2 \cos(k_2)}{\text{cn}(\beta, m)} , \]
\[ 1 = \mu_1 A^2 \text{cs}^2(\beta, m) + \mu_2 B^2 \text{ds}^2(\beta, m) , \]
\[ \beta v = \frac{2 \sin(k_1)}{\text{cs}(\beta, m)} [1 - (1 - m)\mu_2 B^2] . \]
\[ \sin(k_1) \text{cn}(\beta, m) = \sin(k_2) \text{dn}(\beta, m) . \] (65)

Note that since it is a moving periodic pulse solution, it must satisfy not only the periodicity condition (22) but also the periodicity condition (52). The periodicity conditions (22) and (52) imply that $u_n, v_n$ are periodic solutions provided there exist integers $n_1, n_2, n_3, n_4$ such that the condition (53) is satisfied.

In the limit $m = 1$, these three solutions (iii), (iv) and (v) reduce to

\[ u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \text{sech}[\beta(n - vt + \delta_2)] , \]
\[ v_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \sqrt{m} \text{sn}[\beta(n - vt + \delta_2)] , \] (66)

provided

\[ k_1 = k_2 , \quad \sinh^2(\beta) = (\mu_1 A^2 + \mu_2 B^2) , \]
\[ \omega_1 = 2[1 - \cos(k_1) \cosh(\beta)] , \quad \omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - \cos(k_1) \cosh(\beta) \right] , \]
\[ v \beta = 2 \sin(k_1) \sinh(\beta) . \] (67)

Thus the periodic pulse solutions exist provided at least one out of $\mu_1, \mu_2$ is positive.

For an entirely different coupled AL model, solution (66) has also been obtained in [33].

(vi) Finally, it also admits a coupled periodic kink solution

\[ u_n = A \exp[-i(\omega_1 t - k_1 n + \delta_1)] \sqrt{m} \text{sn}[\beta(n - vt + \delta_2), m] , \]
\[ v_n = B \exp[-i(\omega_2 t - k_2 n + \delta_3)] \sqrt{m} \text{sn}[\beta(n - vt + \delta_2), m] , \] (68)
provided
\[
k_1 = k_2, \quad \omega_1 = 2[1 - \cos(k_1)\text{cn}(\beta, m)\text{dn}(\beta, m)],
\]
\[
\omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - \cos(k_1)\text{cn}(\beta, m)\text{dn}(\beta, m) \right],
\]
\[
\mu_1 A^2 + \mu_2 B^2 = -\text{sn}^2(\beta, m), \quad \beta v = 2\sin(k_1)\text{sn}(\beta, m).
\] (69)

Thus this solution only exists if at least one out of \(\mu_1, \mu_2\) is negative. Note that since it is a moving periodic kink solution, it must satisfy not only the periodicity condition (25) but also the periodicity condition (52). The periodicity conditions (25) and (52) imply that \(u_n, v_n\) are periodic solutions provided there exist integers \(n_1, n_2, n_3, n_4\) such that the condition (56) is satisfied.

In the limit \(m = 1\), this solution reduces to
\[
u_n = A \exp\left[-i(\omega_1 t - k_1 n + \delta_1)\right]\text{dn}\left[\beta(n + \delta_2), m\right],
\]
\[
u_n = B \exp\left[-i(\omega_2 t - k_2 n + \delta_3)\right]\sqrt{m}\text{sn}\left[\beta(n + \delta_2), m\right],
\] (70)

provided
\[
k_1 = k_2, \quad \omega_1 = 2[1 - \cos(k_1)\text{sech}^2(\beta)],
\]
\[
\omega_2 = 2 \left[ \frac{\mu_2}{\mu_1} - \cos(k_1)\text{sech}^2(\beta) \right],
\]
\[
\mu_1 A^2 + \mu_2 B^2 = -\text{tanh}^2(\beta), \quad \beta v = 2\sin(k_1)\text{tanh}(\beta).
\] (71)

While obtaining these solutions, we have made use of several identities for the Jacobi elliptic functions \([34]\).

5. Solutions of the coupled saturated DNLS equations

We show that unlike the coupled Salerno and the coupled AL models, the coupled saturated DNLS eqs (5) and (6) while they admit Lamé polynomial solutions of order one, they do not admit general solutions characterized by eqs (16) and (17) or eqs (39) and (40). In particular, this model does not admit Lamé polynomial solutions of order two. It is worth noting here that the uncoupled saturated DNLSE model does admit Lamé polynomial solutions of order one \([35]\).

It is easy to check that the coupled eqs (5) and (6) have the following exact solutions in terms of Lamé polynomials of order one:

(i) It admits a coupled mixed pulse-kink solution
\[
u_n = A \exp\left[-i(\omega_1 t + \delta_1)\right]\text{dn}\left[\beta(n + \delta_2), m\right],
\]
\[
u_n = B \exp\left[-i(\omega_2 t + \delta_3)\right]\sqrt{m}\text{sn}\left[\beta(n + \delta_2), m\right],
\] (72)
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provided

\[ \omega_1 = 2 - \frac{v_1}{\mu_1}, \ \omega_2 = 2 - \frac{v_2}{\mu_2}, \ \mu_2 = \mu_1 \text{cn}(\beta, m), \]

\[ \mu_1 A^2 = \frac{v_1}{2\mu_1 \text{dn}(\beta, m)} - 1, \ \mu_2 B^2 = \frac{v_1 \text{cn}(\beta, m)}{2\mu_1 \text{dn}(\beta, m)} - 1. \]

(73)

For the solution (72), \( u_n, v_n \) satisfy the boundary condition (22).

(ii) Another coupled mixed pulse-kink solution is

\[ u_n = A \exp[-i(\omega_1 t + \delta_1)]\sqrt{m} \text{cn}[(n + \delta_2), m], \]

\[ v_n = B \exp[-i(\omega_2 t + \delta_3)]\sqrt{m} \text{sn}[(n + \delta_2), m], \]

(74)

provided

\[ \omega_1 = 2 - \frac{v_1}{\mu_1}, \ \omega_2 = 2 - \frac{v_2}{\mu_2}, \ \mu_2 = \mu_1 \text{dn}(\beta, m), \]

\[ m\mu_1 A^2 = \frac{v_1}{2\mu_1 \text{cn}(\beta, m)} - 1, \ m\mu_2 B^2 = \frac{v_1 \text{dn}(\beta, m)}{2\mu_1 \text{cn}(\beta, m)} - 1. \]

(75)

For the solution (74), \( u_n, v_n \) satisfy the boundary condition (25).

In the limit \( m = 1 \), these two solutions (72) and (74) go over to the mixed hyperbolic pulse-kink solution

\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \text{sech}[(n + \delta_2)], \]

\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \text{tanh}[(n + \delta_2)], \]

(76)

provided

\[ \omega_1 = 2 - \frac{v_1}{\mu_1}, \ \omega_2 = 2 - \frac{v_2}{\mu_2}, \ \mu_2 = \mu_1 \text{sech}(\beta), \]

\[ \mu_1 A^2 = \frac{v_1 \cosh(\beta)}{2\mu_1} - 1, \ \mu_2 B^2 = \frac{v_1}{2\mu_1 \cosh(\beta)} - 1. \]

(77)

(iii) This model also admits two coupled pulse solutions. One solution is

\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \text{dn}[(n + \delta_2), m], \]

\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \text{dn}[(n + \delta_2), m], \]

(78)

provided

\[ \omega_1 = \omega_2 = 2 \left[ 1 - \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right], \ v_1 = v_2, \ \mu_1 = \mu_2, \]

\[ \mu_1 (A^2 + B^2) = \frac{\text{sn}^2(\beta, m)}{\text{cn}^2(\beta, m)}. \]

(79)

For the solution (78), \( u_n, v_n \) satisfy the boundary condition (31).
Another pulse solution is
\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \sqrt{\mu} \cn[(n + \delta_2), m], \]
\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \sqrt{\mu} \cn[(n + \delta_2), m], \]
\[ \text{(80)} \]
provided
\[ \omega_1 = \omega_2 = 2 \left[1 - \frac{\cn(\beta, m)}{\dn^2(\beta, m)}\right], \; \nu_1 = \nu_2, \; \mu_1 = \mu_2, \]
\[ \mu_1(A^2 + B^2) = \frac{\sn^2(\beta, m)}{\dn^2(\beta, m)}. \]
\[ \text{(81)} \]

For the solution (80), \(u_n, v_n\) satisfy the boundary condition (25).

In the limit \(m = 1\), these two solutions (78), (80) reduce to the coupled hyperbolic pulse solution
\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \sech[\beta(n + \delta_2)], \]
\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \sech[\beta(n + \delta_2)], \]
\[ \text{(82)} \]
provided
\[ \omega_1 = \omega_2 = -2[\cosh(\beta) - 1], \; \nu_1 = \nu_2, \; \mu_1 = \mu_2, \]
\[ \mu_1(A^2 + B^2) = \sinh^2(\beta). \]
\[ \text{(83)} \]

Notice that the coupled pulse solutions are admitted only if \(\mu_1 = \mu_2 > 0\).

Finally, these coupled equations also admit a mixed kink solution
\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \sqrt{\mu} \sn[(n + \delta_2), m], \]
\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \sqrt{\mu} \sn[(n + \delta_2), m], \]
\[ \text{(84)} \]
provided
\[ \omega_1 = \omega_2 = 2[1 - \cn(\beta, m) \dn(\beta, m)], \]
\[ \nu_1 = \nu_2, \; \mu_1 = \mu_2, \; \mu_1(A^2 + B^2) = -\sn^2(\beta, m). \]
\[ \text{(85)} \]

Thus unlike the coupled pulse solution, the coupled kink solution is only valid if \(\mu_1 = \mu_2 < 0\). For the solution (84), \(u_n, v_n\) satisfy the boundary condition (25).

In the limit \(m = 1\), this solution reduces to the hyperbolic kink solution
\[ u_n = A \exp[-i(\omega_1 t + \delta_1)] \tanh[\beta(n + \delta_2)], \]
\[ v_n = B \exp[-i(\omega_2 t + \delta_3)] \tanh[\beta(n + \delta_2)], \]
\[ \text{(86)} \]
provided
\[ \omega_1 = \omega_2 = 2 \tanh^2(\beta), \; \nu_1 = \nu_2, \; \mu_1(A^2 + B^2) = -\tanh^2(\beta). \]
\[ \text{(87)} \]
Before ending this section, it might be worthwhile explaining why this model (unlike coupled Salerno or coupled AL models) does not admit Lamé polynomial solutions of order two. If we look at the field equations which follow by using the general PB structure given by eq. (4), then it is easily seen that the model admits Lamé polynomials of order two as solution provided
\[ 1 + \lambda_1 |u_n|^2 + \lambda_2 |v_n|^2 = 0. \tag{88} \]
It is easily checked that while this condition can be easily satisfied in both coupled Salerno and coupled AL models (where \( \lambda_1 = \mu_1, \lambda_2 = \mu_2 \)), this condition can never be satisfied in the coupled DNLS model because in that case \( \lambda_1 = \lambda_2 = 0 \). Note however that in view of the nontrivial identities for Jacobi elliptic functions [34] the coupled DNLS model still admits Lamé polynomial solutions of order one.

6. Solutions for a coupled discrete \( \phi^6 \) model

We start from the same continuum coupled \( \phi^6 \) model for which recently we have obtained Lamé polynomial solutions of order two [1]. We now show that if we consider the following discrete variant of the same model, then it has solutions not only in terms of Lamé polynomials of order one but also in terms of Lamé polynomials of order two, even though the Lamé polynomials of order two are not the solutions of the uncoupled discrete \( \phi^6 \) model.

The field equations for the static coupled continuum model, which we had considered recently [1], are given by (modulo a factor of 2 in the definitions of \( c_1, c_2, e, f \))
\[
\frac{d^2 \phi}{dx^2} = a_1 \phi - b_1 \phi^3 + c_1 \phi^4 + d \phi \psi^2 + e \phi^3 \psi^2 + 2 f \phi \psi^4, \tag{89}
\]
\[
\frac{d^2 \psi}{dx^2} = a_2 \psi - b_2 \psi^3 + c_2 \psi^4 + d \phi^2 \psi + e \psi^4 + 4 f \phi^2 \psi^3. \tag{90}
\]
Let us consider the following coupled discrete model:
\[
\frac{1}{h^2} (\phi_{n+1} + \phi_{n-1} - 2 \phi_n) = a_1 \phi_n - b_1 \phi_n^3 + d \psi_n^2 \phi_n + [c_1 \phi_n^4 + e \phi_n^2 \psi_n^2 + f \psi_n^4][\phi_{n+1} + \phi_{n-1}], \tag{91}
\]
\[
\frac{1}{h^2} (\psi_{n+1} + \psi_{n-1} - 2 \psi_n) = a_2 \psi_n - b_2 \psi_n^3 + d \phi_n^2 \psi_n + [c_2 \psi_n^4 + \frac{e}{2} \phi_n^4 + 2 f \phi_n^2 \psi_n^2][\psi_{n+1} + \psi_{n-1}], \tag{92}
\]
which in the continuum limit go over to eqs (89) and (90). Here \( h \) denotes the discreteness parameter.

6.1 Solutions of the uncoupled model

Let us first note that the uncoupled field eq. (91) for field \( \phi \) (similar conclusion is also valid for the field \( \psi \)) given by
\[
\frac{1}{h^2} (\phi_{n+1} + \phi_{n-1} - 2 \phi_n) = a_1 \phi_n - b_1 \phi_n^3 + c_1 \phi_n^4[\phi_{n+1} + \phi_{n-1}]. \tag{93}
\]
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has three solutions in terms of Lamé polynomials of order one. However, Lamé polynomials of order two do not satisfy the uncoupled eq. (93). In particular, it is easily shown that

\[ \phi_n = A \text{dn}[\beta(n + x_0), m], \tag{94} \]

is an exact solution to the field eq. (93) provided

\[ A^4 h^2 c_1 \text{cs}^4(\beta, m) = 1, \quad a_1 = \frac{2}{h^2} \left[ \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} - 1 \right], \tag{95} \]

\[ \frac{b_1^2}{2a_1 c_1} = \frac{\text{dn}^2(\beta, m)}{\text{cn}^2(\beta, m)[\text{dn}(\beta, m) - \text{cn}^2(\beta, m)]}. \]

For the solution (94), \( \phi_n \) satisfies the boundary condition

\[ \phi_{n + \frac{2\kappa}{\beta}} = \phi_n. \tag{96} \]

Yet another solution to the field eq. (93) is given by

\[ \phi_n = A \sqrt{m} \text{cn}[\beta(n + x_0), m], \tag{97} \]

provided

\[ A^4 h^2 c_1 \text{ds}^4(\beta, m) = 1, \quad a_1 = \frac{2}{h^2} \left[ \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} - 1 \right], \tag{98} \]

\[ \frac{b_1^2}{2a_1 c_1} = \frac{m^2 \text{cn}^2(\beta, m)}{\text{dn}^2(\beta, m)[\text{cn}(\beta, m) - \text{dn}^2(\beta, m)]}. \]

For the solution (97), \( \phi_n \) satisfies the boundary condition

\[ \phi_{n + \frac{4\kappa}{\beta}} = \phi_n. \tag{99} \]

In the limit \( m = 1 \), both these solutions go over to the pulse solution

\[ \phi_n = A \text{sech}[\beta(n + x_0)], \tag{100} \]

provided

\[ h^2 A^4 c_1 = \text{sinh}^4(\beta), \quad a_1 = \frac{2}{h^2}[\cosh(\beta) - 1] > 0, \quad \frac{b_1^2}{2a_1 c_1} = \frac{\cosh^2(\beta)}{\cosh(\beta) - 1}. \tag{101} \]

The third periodic solution to the field eq. (93) is given by

\[ \phi_n = A \sqrt{m} \text{sn}[\beta(n + x_0), m], \tag{102} \]

provided

\[ A^4 h^2 c_1 \text{ns}^4(\beta, m) = 1, \quad a_1 = \frac{2}{h^2}[\text{cn}(\beta, m) \text{dn}(\beta, m) - 1] < 0, \tag{103} \]

\[ \frac{b_1^2}{2|a_1| c_1} = \frac{m^2 \text{cn}^2(\beta, m) \text{dn}^2(\beta, m)}{1 - \text{cn}(\beta, m) \text{dn}(\beta, m)}. \]

For the solution (102), \( \phi_n \) satisfies the boundary condition (99).
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In the limit \( m = 1 \), this solution goes over to the kink solution
\[
\phi_n = A \tanh[\beta(n + x_0)],
\]
provided
\[
h^2 A^4 c_1 = \tanh^4(\beta), \quad a_1 = -\frac{2}{h^2} \tanh^2(\beta) < 0, \quad \frac{b_1^2}{2|a_1| c_1} = \frac{4}{\sinh^2(2\beta)}. \tag{105}
\]

Let us now discuss the solutions of the coupled eqs (91) and (92). It turns out that as in the coupled AL model, the \( \phi^6 \) coupled equations have solutions satisfying the ansatz similar to (17) (but no solutions satisfying the ansatz similar to (40)), and also solutions in terms of Lamé polynomials of order one (by making use of the identities for the Jacobi elliptic functions [34]).

6.2 Solutions of the coupled model satisfying ansatz similar to (17)

On substituting the ansatz
\[
\phi_n^2 + a \psi_n^2 = b, \quad a, b > 0, \tag{106}
\]
(which is similar to the ansatz (17)) in the coupled field eqs (91) and (92), we find that such solutions exist provided
\[
c_1 = c_2 = f = \frac{e}{2}, \quad b_1 = b_2 = -d, \quad a_1 = a_2, \quad c_1 h^2 b^2 = 1, \tag{107}
\]
\[
a = 1, \quad a_1 + \frac{2}{h^2} = \frac{b_1}{\sqrt{h^2 c_1}}.
\]
This is a rather general ansatz and there are several solutions of this type which exist for this model.

6.3 Lamé polynomial solutions of order one

(i) One solution is
\[
\phi_n = A \operatorname{dn}[\beta(n + c_2), m], \quad \psi_n = B \sqrt{m} \operatorname{sn}[\beta(n + c_2), m], \tag{108}
\]
provided eq. (107) is satisfied and further
\[
b = A^2, \quad A^2 = B^2, \quad a_1 - b_1 A^2 + 2 c_1 A^4 = 0. \tag{109}
\]
Note that the width \( \beta \) is completely arbitrary. For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (22) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

(ii) Another solution is
\[
\phi_n = A \sqrt{m} \operatorname{cn}[\beta(n + c_2), m], \quad \psi_n = B \sqrt{m} \operatorname{sn}[\beta(n + c_2), m], \tag{110}
\]
provided eq. (107) is satisfied and further
\[
b = mA^2, \quad A^2 = B^2, \quad a_1 - m b_1 A^2 + 2 m^2 c_1 A^4 = 0. \tag{111}
\]
For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (25) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

In the limit \( m = 1 \), both these solutions go over to the hyperbolic solution
\[
f_n = A \operatorname{sech}[\beta(n + c_2)], \quad g_n = B \tanh[\beta(n + c_2)]. \tag{112}
\]
6.4 Lamé polynomial solutions of order two

(iii) One solution is given by
\[ \phi_n = A \text{dn}^2[\beta(n + c_2), m] + B, \quad \psi_n = F\sqrt{m} \text{sn}[\beta(n + c_2), m] \text{dn}[\beta(n + c_2), m], \]
(113)
provided eq. (107) is satisfied and further
\[ b = \frac{A^2}{4}, \quad A^2 = F^2, \quad A = -2B, \quad a_1 - b_1 B^2 + 2c_1 B^4 = 0. \]
(114)
For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (22) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

(iv) Another solution is
\[ \phi_n = A \text{dn}^2[\beta(n + c_2), m] + B, \quad \psi_n = F m \text{sn}[\beta(n + c_2), m] \text{cn}[\beta(n + c_2), m], \]
(115)
provided eq. (107) is satisfied and further
\[ b = \frac{m^2 A^2}{4}, \quad A^2 = F^2, \quad (2 - m) A = -2B, \quad 8a_1 - 2b_1 m^2 F^2 + c_1 m^4 F^4 = 0. \]
(116)
For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (31) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

In the limit \( m = 1 \), both the solutions (113) and (115), go over to the hyperbolic solution
\[ \phi_n = A \text{sech}^2[\beta(n + c_2)] + B, \quad \psi_n = F \tanh[\beta(n + c_2)] \text{sech}[\beta(n + c_2)]. \]
(117)

(v) Apart from these, several other solutions are possible. For example, one can have the following nonperiodic solution:
\[ \phi_n = \frac{A}{\sqrt{1 + n^2}}, \quad \psi_n = \frac{Bn}{\sqrt{1 + n^2}}, \]
(118)
provided eq. (107) is satisfied and further
\[ b = A^2, \quad A^2 = B^2. \]
(119)

(vi) Yet another solution is
\[ \phi_n = A \cos[\beta(n + c_2)], \quad \psi_n = B \sin[\beta(n + c_2)], \]
(120)
provided eq. (107) is satisfied and further
\[ b = A^2, \quad A^2 = B^2. \]
(121)
For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (38) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.
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6.5 Solutions following from identities for Jacobi elliptic functions

Using the identities for the Jacobi elliptic functions [34], we now show that there are six Lamé polynomial solutions of order one to the coupled field eqs (91) and (92).

**Solution 1:** It is not difficult to show that

\[
\phi_n = A \text{dn}[\beta(n + x_0), m], \quad \psi_n = B \sqrt{m} \text{sn}[\beta(n + x_0), m],
\]

is an exact solution to the coupled field eqs (91) and (92) provided

\[
b_1 A^2 + d B^2 = 2(c_1 A^4 + f B^4 - e A^2 B^2) \text{ds}(\beta, m) \text{ns}(\beta, m),
\]

\[
a_1 + \frac{2}{h^2} + d B^2 + 2(e A^2 B^2 - 2 f B^4) \text{ds}(\beta, m) \text{ns}(\beta, m)
\]

\[
= 2(c_1 A^4 + f B^4 - e A^2 B^2) \text{ds}(\beta, m) \text{ns}(\beta, m) \text{cs}^2(\beta, m),
\]

\[
\frac{1}{h^2} - f B^4 + (e A^2 B^2 - 2 f B^4) \text{cs}^2(\beta, m) = (c_1 A^4 + f B^4 - e A^2 B^2) \text{cs}^4(\beta, m),
\]

\[
b_2 B^2 + d A^2 = -2 \left( c_2 B^4 + \frac{e}{2} A^4 - 2 f A^2 B^2 \right) \text{ds}(\beta, m) \text{cs}(\beta, m),
\]

\[
a_2 + \frac{2}{h^2} + d A^2 - 2(2 f A^2 B^2 - e A^4) \text{ds}(\beta, m) \text{cs}(\beta, m)
\]

\[
= 2 \left( c_2 B^4 + \frac{e}{2} A^4 - 2 f A^2 B^2 \right) \text{ds}(\beta, m) \text{cs}(\beta, m) \text{ns}^2(\beta, m),
\]

\[
\frac{1}{h^2} - \frac{e}{2} A^4 - (2 f A^2 B^2 - e A^4) \text{ns}^2(\beta, m) = \left( c_2 B^4 + \frac{e}{2} A^4 - 2 f A^2 B^2 \right) \text{ns}^4(\beta, m).
\]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (22) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

These equations are of course trivially satisfied if relations (107) and (109) are satisfied. It is worth pointing out that while relations (107) and (109) are sufficient so that (122) constitutes an exact solution to the coupled eqs (91) and (92), it is not obvious if relations (107) and (109) are also necessary. The necessary relations are given by eqs (123)–(128).

**Solution 2:** Another solution is given by

\[
\phi_n = A \sqrt{m} \text{cn}[\beta(n + x_0), m], \quad \psi_n = B \sqrt{m} \text{sn}[\beta(n + x_0), m],
\]

which is an exact solution to the coupled field eqs (91) and (92) provided

\[
b_1 A^2 + d B^2 = 2(c_1 A^4 + f B^4 - e A^2 B^2) \text{cs}(\beta, m) \text{ns}(\beta, m),
\]
For this solution, $\phi_n, \psi_n$ satisfy the boundary condition given by eq. (25) with $\phi_n, \psi_n$ replacing $f_n, g_n$ respectively. These equations are trivially satisfied if relations (107) and (111) are satisfied.

In the limit $m = 1$, both the solutions (122) and (129) go over to the hyperbolic soliton solution

\[
\phi_n = A \sech[\beta(n + x_0)], \quad \psi_n = B \tanh[\beta(n + x_0)],
\]

provided relations (123)–(128) with $m = 1$ are satisfied.

**Solution 3:** It is not difficult to show that

\[
\phi_n = A \dn[\beta(n + x_0), m], \quad \psi_n = B \sqrt{m} \cn[\beta(n + x_0), m],
\]

is an exact solution to the coupled field eqs (91) and (92) provided

\[
b_1 A^2 - dB^2 = 2(c_1 A^4 + f B^4 + e A^2 B^2) \dn(\beta, m) \cn(\beta, m),
\]

\[
a_1 + \frac{2}{h^2} - (1 - m)dB^2 - 2(1 - m)(e A^2 B^2 + 2 f B^4) \dn(\beta, m) \cn(\beta, m)
\]

\[
= 2(c_1 A^4 + f B^4 + e A^2 B^2) \dn(\beta, m) \cn(\beta, m) \cs^2(\beta, m),
\]

\[
\frac{1}{h^2} - (1 - m)^2 f B^4 - (1 - m)(e A^2 B^2 + 2 f B^4) \cs^2(\beta, m)
\]

\[
= (c_1 A^4 + f B^4 + e A^2 B^2) \cs^4(\beta, m),
\]
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\[ b_2 B^2 - dA^2 = -2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ns}(\beta, m) \text{cs}(\beta, m), \quad (141) \]

\[ a_2 + \frac{2}{h^2} + (1 - m)dA^2 + 2(1 - m)(2 f A^2 B^2 + eA^4) \text{ns}(\beta, m) \text{cs}(\beta, m) = 2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ns}(\beta, m) \text{cs}(\beta, m) \text{ds}^2(\beta, m), \quad (142) \]

\[ \frac{1}{h^2} - \frac{e}{2}(1 - m)^2 A^4 + (1 - m)(2 f A^2 B^2 + eA^4) \text{ds}^2(\beta, m) = \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ds}^4(\beta, m). \quad (143) \]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (22) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

**Solution 4:** Another solution to the coupled eqs (91) and (92) is given by

\[ \phi_n = A \text{dn}[\beta(n + x_0), m], \quad \psi_n = B \text{dn}[\beta(n + x_0), m], \quad (144) \]

provided

\[ b_1 A^2 - dB^2 = 2(c_1 A^4 + f B^4 + eA^2 B^2) \text{ds}(\beta, m) \text{ns}(\beta, m), \quad (145) \]

\[ a_1 + \frac{2}{h^2} = 2(c_1 A^4 + f B^4 + eA^2 B^2) \text{ds}(D, m) \text{ns}(\beta, m) \text{cs}^2(\beta, m), \quad (146) \]

\[ \frac{1}{h^2} = (c_1 A^4 + f B^4 + eA^2 B^2) \text{cs}^4(\beta, m), \quad (147) \]

\[ b_2 B^2 - dA^2 = 2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ds}(\beta, m) \text{ns}(\beta, m), \quad (148) \]

\[ a_2 + \frac{2}{h^2} = 2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ns}(\beta, m) \text{ds}(\beta, m) \text{cs}^2(\beta, m), \quad (149) \]

\[ \frac{1}{h^2} = \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{cs}^4(\beta, m). \quad (150) \]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (31) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

**Solution 5:** Yet another solution to the coupled eqs (91) and (92) is given by

\[ \phi_n = A \sqrt{m} \text{cn}[\beta(n + x_0), m], \quad \psi_n = B \sqrt{m} \text{cn}[\beta(n + x_0), m], \quad (151) \]
provided
\[ b_1 A^2 - d B^2 = 2(c_1 A^4 + f B^4 + e A^2 B^2) \cs(\beta, m) \ns(\beta, m), \]  
(152)
\[ a_1 + \frac{2}{h^2} = 2(c_1 A^4 + f B^4 + e A^2 B^2) \cs(\beta, m) \ns(\beta, m) \ds^2(\beta, m), \]  
(153)
\[ \frac{1}{h^2} = (c_1 A^4 + f B^4 + e A^2 B^2) \ds^4(\beta, m), \]  
(154)
\[ b_2 B^2 - dA^2 = 2\left(c_2 B^4 + \frac{e}{2} A^4 + 2f A^2 B^2\right) \cs(\beta, m) \ns(\beta, m), \]  
(155)
\[ a_2 + \frac{2}{h^2} = 2\left(c_2 B^4 + \frac{e}{2} A^4 + 2f A^2 B^2\right) \ns(\beta, m) \cs(\beta, m) \ds^2(\beta, m), \]  
(156)
\[ \frac{1}{h^2} = \left(c_2 B^4 + \frac{e}{2} A^4 + 2f A^2 B^2\right) \ds^4(\beta, m). \]  
(157)

For this solution, \(\phi_n, \psi_n\) satisfy the boundary condition given by eq. (25) with \(\phi_n, \psi_n\) replacing \(f_n, g_n\) respectively.

In the limit \(m = 1\), all three solutions given by (137), (144) and (151) go over to the hyperbolic soliton solution
\[ \phi_n = A \sech[\beta(n + x_0)], \quad \psi_n = B \sech[\beta(n + x_0)], \]  
(158)
provided
\[ a_1 = a_2 = \frac{2}{h^2}[\cosh(\beta) - 1] > 0, \]  
\[ b_1 A^2 - d B^2 = b_2 B^2 - dA^2 = \frac{2}{h^2} \sinh^2(\beta) \cosh(\beta), \]  
(159)
\[ c_1 A^4 + e A^2 B^2 + f B^4 = c_2 B^4 + 2f A^2 B^2 + \frac{e}{2} A^4 = \frac{\sinh^4(\beta)}{h^2}. \]  
(160)

**Solution 6:** Finally, another solution to the coupled eqs (91) and (92) is given by
\[ \phi_n = A \sqrt{m} \sn[\beta(n + x_0), m], \quad \psi_n = B \sqrt{m} \sn[\beta(n + x_0), m], \]  
(161)
provided
\[ b_1 A^2 - d B^2 = 2(c_1 A^4 + f B^4 + e A^2 B^2) \cs(\beta, m) \ds(\beta, m), \]  
(162)
\[ a_1 + \frac{2}{h^2} = 2\left(c_1 A^4 + f B^4 + e A^2 B^2\right) \cs(\beta, m) \ds(\beta, m) \ns^2(\beta, m), \]  
(163)
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\[
\frac{1}{h^2} = (c_1 A^4 + f B^4 + e A^2 B^2) \text{ ns}^4 (\beta, m), \quad (164)
\]

\[
b_2 B^2 - dA^2 = 2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ cs}(\beta, m) \text{ ds}(\beta, m), \quad (165)
\]

\[
a_2 + \frac{2}{h^2} = 2 \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ ds}(\beta, m) \text{ cs}(\beta, m) \text{ ns}^2 (\beta, m), \quad (166)
\]

\[
\frac{1}{h^2} = \left( c_2 B^4 + \frac{e}{2} A^4 + 2 f A^2 B^2 \right) \text{ ns}^4 (\beta, m). \quad (167)
\]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (25) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

In the limit \( m = 1 \), this solution goes over to the hyperbolic soliton solution

\[
\phi_n = A \tanh[\beta(n + x_0)], \quad \psi_n = B \tanh[\beta(n + x_0)], \quad (168)
\]

provided

\[
a_1 = a_2 = \frac{2}{h^2} [\tanh^2 (\beta)] < 0, \quad b_1 A^2 - dB^2 = b_2 B^2 - dA^2 = \frac{2}{h^2} \frac{\sinh^2 (\beta)}{\cosh^4 (\beta)}, \quad (169)
\]

\[
c_1 A^4 + e A^2 B^2 + f B^4 = c_2 B^4 + 2 f A^2 B^2 + \frac{e}{2} A^4 = \frac{\tanh^4 (\beta)}{h^2}. \quad (170)
\]

7. Solutions for a coupled discrete \( \phi^4 \) model

We start from the same coupled static discrete field equations as in our recent paper [29] for which we had obtained six solutions in terms of Lamé polynomials of order one. We now show that the same model also admits two Lamé polynomial solutions of order two, even though they are not the solutions of the corresponding uncoupled problem. Let us start from the field equations considered in [29]:

\[
\frac{1}{h^2} (\phi_{n+1} + \phi_{n-1} - 2\phi_n) - 2\alpha_1 \phi_n - [2\beta_1 \phi_n^2 + \gamma \psi_n^2][\phi_{n+1} + \phi_{n-1}] = 0, \quad (171)
\]

\[
\frac{1}{h^2} (\psi_{n+1} + \psi_{n-1} - 2\psi_n) - 2\alpha_2 \psi_n - [2\beta_2 \psi_n^2 + \gamma \phi_n^2][\psi_{n+1} + \psi_{n-1}] = 0. \quad (172)
\]

Let us now discuss the solutions of the coupled eqs (171) and (172). It turns out that as in the coupled \( \phi^6 \) case, the \( \phi^4 \) coupled equations have solutions satisfying ansatz similar to the one given by eq. (17) (but no solutions satisfying ansatz similar to (40)). Further, they also have solutions in terms of Lamé polynomials of order one which we have already discussed in [29]. Note that these solutions were obtained by making use of the identities for the Jacobi elliptic functions [34].
7.1 Solutions satisfying ansatz similar to (17)

On substituting the ansatz given by eq. (106) (which is similar to the ansatz given by eq. (17)) in the coupled field eqs (171) and (172), we find that such solutions exist provided

\[ 2\beta_1 = 2\beta_2 = \gamma, \; \alpha_1 = \alpha_2 = -\frac{1}{h^2}, \; a = 1. \tag{173} \]

This is a rather general ansatz and there are several solutions of this type which exist in this model.

7.2 Lamé polynomial solutions of order one

(i) One solution is

\[ \phi_n = A \text{dn}[\beta(n + c_2), m], \; \psi_n = B\sqrt{m} \text{sn}[\beta(n + c_2), m], \tag{174} \]

provided eq. (173) is satisfied and further

\[ b = A^2 = \frac{1}{2\beta_1 h^2}, \; A^2 = B^2. \tag{175} \]

Note that the width \( \beta \) is completely arbitrary. For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (22) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

(ii) Another solution is

\[ \phi_n = A\sqrt{m} \text{cn}[\beta(n + c_2), m], \; \psi_n = B\sqrt{m} \text{sn}[\beta(n + c_2), m], \tag{176} \]

provided eq. (173) is satisfied and further

\[ b = mA^2 = \frac{1}{2\beta_1 h^2}, \; A^2 = B^2. \tag{177} \]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (25) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

In the limit \( m = 1 \), both these solutions go over to the hyperbolic solution

\[ \phi_n = A \text{sech}[\beta(n + c_2)], \; \psi_n = B \text{tanh}[\beta(n + c_2)]. \tag{178} \]

7.3 Lamé polynomial solutions of order two

(iii) One solution is given by

\[ \phi_n = A \text{dn}^2[\beta(n+c_2), m]+B, \; \psi_n = F\sqrt{m} \text{sn}[\beta(n+c_2), m] \text{dn}[\beta(n+c_2), m], \tag{179} \]

provided eq. (173) is satisfied and further

\[ b = \frac{A^2}{4} = \frac{1}{2\beta_1 h^2}, \; A^2 = F^2, \; A = -2B. \tag{180} \]

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (25) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.
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(iv) Another solution is

\[ \phi_n = A \operatorname{dn}^2[\beta(n+c_2), m] + B, \quad \psi_n = F m \operatorname{sn}[\beta(n+c_2), m] \operatorname{cn}[\beta(n+c_2), m], \]

(181)

provided eq. (173) is satisfied and further

\[ b = \frac{m^2 A^2}{4}, \quad A^2 = F^2, \quad (2 - m) A = -2 B. \]

(182)

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (31) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

In the limit \( m = 1 \), both solutions (179) and (181) go over to the hyperbolic solution

\[ \phi_n = A \operatorname{sech}^2[\beta(n+c_2)] + B, \quad \psi_n = F \tanh[\beta(n+c_2)] \operatorname{sech}[\beta(n+c_2)]. \]

(183)

(v) Apart from these, several other solutions are possible. For example, one can have a nonperiodic solution

\[ \phi_n = \frac{A}{\sqrt{1 + n^2}}, \quad \psi_n = \frac{B n}{\sqrt{1 + n^2}}, \]

(184)

provided eq. (173) is satisfied and further

\[ b = A^2, \quad A^2 = B^2. \]

(185)

(vi) Yet another solution is

\[ \phi_n = A \cos[\beta(n+c_2)], \quad \psi_n = B \sin[\beta(n+c_2)]. \]

(186)

provided eq. (173) is satisfied and further

\[ b = A^2, \quad A^2 = B^2. \]

(187)

For this solution, \( \phi_n, \psi_n \) satisfy the boundary condition given by eq. (31) with \( \phi_n, \psi_n \) replacing \( f_n, g_n \) respectively.

8. Summary

In this paper we have shown that for a number of coupled discrete models, e.g. coupled Salerno, coupled Ablowitz–Ladik, coupled saturated nonlinear Schrödinger equation, coupled \( \phi^6 \), coupled \( \phi^4 \), while the uncoupled equations do not admit solutions in terms of Lamé polynomials of order two, the coupled models do admit such solutions. These solutions have relevance in physical contexts ranging from ferroelectric [7–9] to multiferroic [4–6] materials to the models in field theory [10,14] as well as for various discrete contexts [11–13].

The stability of various solutions found here remains an open issue to be explored numerically, particularly some solutions have an arbitrary soliton width. In addition, the
scattering of solitons of various discrete models is an important issue with these static solutions boosted with a certain velocity. Similarly, the Peierls–Nabarro (discreteness) barrier for the solutions remains to be explored. Given the solutions in terms of Lamé functions of order one and two, it is then worth enquiring if one considers coupling of three discrete fields, would they admit solutions in terms of Lamé polynomials of order three? And if true, can one generalize it to the case of $N$ coupled fields? We hope to address these issues in the near future.

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