Exact results for ‘bouncing’ Gaussian wave packets

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Abstract

We consider time-dependent Gaussian wave packet solutions of the Schrödinger equation (with arbitrary initial central position, \(x_0\), and momentum, \(p_0\), for an otherwise free-particle, but with an infinite wall at \(x = 0\), so-called bouncing wave packets. We show how difference or mirror solutions of the form \(\psi(x,t) - \psi(-x,t)\) can, in this case, be normalized exactly, allowing for the evaluation of a number of time-dependent expectation values and other quantities in closed form. For example, we calculate \(\langle p^2 \rangle\), explicitly which illustrates how the free-particle kinetic (and hence total) energy is affected by the presence of the distant boundary. We also discuss the time dependence of the expectation values of position, \(\langle x \rangle_t\), and momentum, \(\langle p \rangle_t\), and their relation to the impulsive force during the ‘collision’ with the wall. Finally, the \(x_0, p_0 \to 0\) limit is shown to reduce to a special case of a non-standard free-particle Gaussian solution. The addition of this example to the literature then expands on the relatively small number of Gaussian solutions to quantum mechanical problems with familiar classical analogs (free particle, uniform acceleration, harmonic oscillator, unstable oscillator, and uniform magnetic field) available in closed form.

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I. INTRODUCTION

Closed-form wave packet solutions of the time-dependent Schrödinger equation are excellent exemplary models to study the time-evolution of quantum systems for comparison to their classical counterparts. Because of their special properties, Gaussian solutions are possible for a few of the most familiar classical systems. Free-particle Gaussian wave packet solutions are standard fare in introductory textbooks in quantum mechanics, and this example goes back at least to Darwin [1]. The problem of a particle acting under a uniform force has similar Gaussian solutions which were first derived by Kennard [2] and occasionally appear in undergraduate-level presentations [3]. Wave packet solutions for the harmonic oscillator are discussed in some textbooks [4] (most often using propagator techniques) and, with a simple change of variables, can also be used to describe particles in unstable equilibrium [5], giving rise to solutions which exhibit the expected exponential runaway behavior. Gaussian solutions corresponding to classical helical motion in a uniform magnetic field [6] have also been constructed. In all of these cases, the special form of the Gaussian solutions allows for the explicit evaluation of time-dependent expectation values for position (\(\langle x \rangle_t, \langle x^2 \rangle_t, \Delta x_t\)), momentum (\(\langle p \rangle_t, \langle p^2 \rangle_t, \Delta p_t\)), and other quantities, for easy comparison to classical expectations.

Another simple system, also with a clear classical analog, is given by an otherwise free particle, but restricted to the half-line by an infinite wall at \(x = 0\), namely the problem defined by the 1D potential

\[
V(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\infty & \text{for } x \geq 0
\end{cases}.
\]

In the classical case, a point particle would move freely, with constant kinetic (and hence total) energy, exhibiting an impulsive collision at the wall, resulting in a discontinuous change in momentum.

Localized time-dependent solutions for this problem, dubbed \textit{bouncing wave packets}, can be constructed in a very straightforward way from solutions of the free-particle problem. Andrews [7] has noted that simple difference solutions of the form \(\psi(x, t) - \psi(-x, t)\) not only satisfy the free-particle Schrödinger equation for all \(x\) values (if \(\psi(x, t)\) does), but also accommodate the new boundary condition at the wall, namely that \(\psi(0, t) = 0\). This construction is very similar to image methods in electrostatics and has been used in numerical
evaluations and visualizations of such systems [8], as well as for discussions of wave packet propagation in the infinite square well [9], [10] (with two infinite walls producing an infinite series of image wave functions.) Such solutions have also been used to visualize many aspects of the collision with the wall [11] and we show in Fig. 1 an example of the time-dependent $|\psi(x, t)|^2$ using just such a construction, illustrating the obvious interference effects between the $\psi(x, t)$ and $\psi(-x, t)$ terms, near the time of the ‘bounce’ with the wall.

The study of general relationships between expectation values, such as

$$m \frac{d\langle x \rangle}{dt} = \langle p \rangle, \quad \text{and} \quad \frac{d\langle p \rangle}{dt} = -\left\langle \frac{dV(x)}{dx} \right\rangle_t,$$

is straightforward for many of the familiarly treated quantum mechanical systems [12], especially for well-behaved potential energy functions. On the other hand, for infinite well type potentials of the type considered here, the relationship to the classical force can be more subtle and so having exact or approximate closed-form solutions for quantities such as $\langle x \rangle_t$ and $\langle p \rangle_t$ to probe such relationships is very useful. For example, the force exerted on the walls in an infinite square well potential has been recently examined in a somewhat similar context, using more numerical methods, focusing on the time-dependent relationships in Eqn. (2) as related to wave packet revivals and fractional revivals [13].

In this note, we will show that if one uses standard Gaussian solutions in this mirror or difference solution approach for the potential in Eqn. (1), one can also perform many (but not all) of the standard calculations including exact normalization of the wave packet solution, evaluation of many expectation values, and the calculation of the autocorrelation function, $A(t)$, to obtain exact closed-form results, which is the main thrust of Sec. II. This case then adds another example to the otherwise rather small pantheon of closed-form quantum mechanical Gaussian solutions to one-dimensional problems with classical analogs. Using these solutions, we will also be able to discuss the nature of the collision with the wall, deriving approximate expressions for the impulsive force exerted on the particle during the collision. We will also be able to explicitly evaluate the change in kinetic (and hence total) energy induced by the addition of the infinite barrier, making connections to earlier work on the effects of distant boundaries [14] on the energy spectrum of quantum mechanical systems; this example extends those results to an exact time-dependent wave packet solution and is also discussed in Sec. II. Finally, the mirror solutions discussed here can be connected to special, non-standard Gaussian free-particle wave packets in the special
limit when \(x_0, p_0 \to 0\), as shown in Sec. III.

II. “BOUNCING” GAUSSIAN WAVE PACKET SOLUTIONS

We first review well-known textbook results for the standard time-dependent Gaussian free-particle momentum- and position-space wave packet solutions, for arbitrary initial \(x_0\) and \(p_0\). These can be written in the forms

\[
\phi(G)(p,t) = \sqrt{\frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 (p-p_0)^2/2}} e^{-ipx_0/\hbar} e^{-ip^2 t/2m\hbar}
\]

\[
\psi(G)(x,t) = \frac{1}{\sqrt{\sqrt{\pi\alpha\hbar}(1+i\ell/t_0)}} e^{ip_0(x-x_0)/\hbar} e^{-ip^2 t/2m\hbar} e^{-(x-x_0-p_0 t/m)^2/2(\alpha\hbar)^2(1+i\ell/t_0)}.
\]

These solutions are characterized by

\[
\langle p \rangle_t = p_0, \quad \Delta p_t = \frac{1}{\alpha\sqrt{2}}
\]

\[
\langle x \rangle_t = x_0 + (p_0/m)t \equiv X(t), \quad \Delta x_t = \frac{\beta}{\sqrt{2}} \sqrt{1 + (t/t_0)^2} \equiv \frac{\beta t}{\sqrt{2}}
\]

where \(t_0 \equiv m\hbar\alpha^2\) is the spreading time and \(\beta \equiv \alpha\hbar\). This gives the familiar uncertainty principle product

\[
\Delta x_t \cdot \Delta p_t = \frac{\hbar}{2} \sqrt{1 + (t/t_0)^2} \longrightarrow \left(\frac{\hbar}{2}\right) \left(\frac{t}{t_0}\right)
\]

for \(t \gg t_0\).

For Gaussian wave packet solutions of this type, the corresponding difference or mirror solution of Andrews \[7\] is written as

\[
\tilde{\psi}(G)(x,t) = \begin{cases} 
N \left[ \psi(G)(x,t) - \psi(G)(-x,t) \right] & \text{for } x < 0 \\
0 & \text{for } x \geq 0
\end{cases}
\]

where \(N\) is a normalization constant. For initial free-particle wave packets, \(\psi(G)(x,t)\), which are already normalized correctly (over all space), and which are sufficiently far apart in phase space \((x_0, p_0\) not too small), one would expect that \(N\) would be very close (exponentially so) to unity, a result which we will confirm by explicit calculation below. Once normalized, of course, any solution of the form Eqn. [8] will remain normalized for later times.

In order to evaluate the various required integrals involved in the normalization of \(\tilde{\psi}(G)(x,t)\), we make use of the fact that the difference or mirror solution is antisymmetric in \(x\), so that \(|\tilde{\psi}(G)(x,t)|^2\) is automatically symmetric in \(x\). This allows us, for example,
to determine the normalization constant \( N \) using the requirement

\[
1 = \int_{-\infty}^{\infty} |\tilde{\psi}(G)(x,t)|^2 dx
\]

\[
= N^2 \int_{-\infty}^{0} |\psi(G)(x,t) - \psi(G)(-x,t)|^2 dx
\]

\[
= \frac{N^2}{2} \int_{-\infty}^{+\infty} |\psi(G)(x,t) - \psi(G)(-x,t)|^2 dx
\]

and because the integration region can be extended (by symmetry) over all space, one can do the resulting integrals in closed form to obtain

\[
N = \frac{1}{\sqrt{1 - \exp[-(x_0/\beta)^2 - (p_0\beta/\hbar)^2]}} = \frac{1}{\sqrt{1 - e^{-z_0}}} \tag{10}
\]

where

\[
z_0 \equiv \left( \frac{x_0}{\beta} \right)^2 + \left( \frac{p_0\beta}{\hbar} \right)^2 = \frac{1}{2} \left( \frac{x_0}{\Delta x_0} \right)^2 + \left( \frac{p_0}{\Delta p_0} \right)^2 \tag{11}
\]

The integrals are, of course, done most simply for \( t = 0 \), but it is then easy to confirm that one obtains the same answer for \( t > 0 \), so that the wave function can be explicitly shown to remain normalized at all later times. While this expression is consistent with expectations for large values of \( x_0, p_0 \), we stress that it is exact for arbitrary initial parameter values, even in the limit where \( x_0, p_0 \to 0 \), which is discussed in Sec. III.

The same method can be used to evaluate various expectation values involving even powers of variables, such as \( \langle x^2 \rangle_t \) and \( \langle p \rangle_t^2 \). For example, one can show that

\[
\langle x^2 \rangle_t \equiv \int_{-\infty}^{0} x^2 |\tilde{\psi}(G)(x,t)|^2 dx
\]

\[
= \frac{1}{2} \int_{-\infty}^{+\infty} x^2 |\tilde{\psi}(G)(x,t)|^2 dx
\]

\[
= \left[ \left( x_0 + \frac{p_0 t}{m} \right)^2 + \frac{\beta^2}{2} \right] + \beta^2 F(z_0) \tag{12}
\]

where

\[
F(z_0) \equiv \frac{z_0 e^{-z_0}}{1 - e^{-z_0}} \tag{13}
\]

and \( z_0 \) is defined in Eqn. (11). The corresponding result for momentum is

\[
\langle p^2 \rangle_t = \langle p^2 \rangle_0 = \left[ p_0^2 + \frac{\hbar^2}{2\beta^2} \right] + \frac{\hbar^2}{\beta^2} F(z_0) \tag{14}
\]

We note the similarity in form between the expressions in Eqns. (12) and (14), so that the effect is the largest (not surprisingly) in the limit when \( z_0 = 0 \), namely when \( x_0, p_0 \to 0 \).
This special case also turns out to be a particular limit of a less-familiar free-particle solution which we discuss further in Sec. III.

The result in Eqn. (14) is also useful as it implies that the change in kinetic (and hence total) energy caused by the addition of the distant boundary is given by

$$\frac{\Delta E}{E} \approx \frac{2F'(z_0)}{1 + 2(p_0\beta/\hbar)^2},$$

that is, an exponential suppression, which is consistent with more general arguments [14].

For expectation values of odd powers of $x, p$, the resulting integrals can be done in terms of error functions ($\text{erf}(\zeta)$), but simple expressions for special cases of interest are perhaps more useful. For example, the classical prediction for the position is $x_{CL}(t) = -|X(t)|$, where $X(t) \equiv (x_0 + p_0t/m)$; the time of the classical collision with the wall, $t_c$, is determined by the condition $X(t_c) = 0$ and we can expand the integrals required for an evaluation of $\langle x \rangle_t$ in terms of $X(t \approx t_c)$, treated as a small parameter, to obtain the approximation

$$\langle x \rangle_{t \approx t_c} = \int_{-\infty}^{0} x|\psi_G(x, t)|^2 dx \approx \left( -\frac{\beta t}{\sqrt{\pi}} - \frac{[X(t)]^2}{\beta t \sqrt{\pi}} + \cdots \right)_{t \approx t_c}. \quad (16)$$

The first term in this expansion was derived in Ref. [11]. We show, in Fig. 11, the numerically calculated value of $\langle x \rangle_t$ versus $t$ for a sample solution and note how the expression in Eqn. (16) reproduces the softened ‘parabolic’ shape of the curve near the collision time, to be compared to the purely classical result, namely $x_{CL}(t) = -|X(t)|$, shown as the dashed lines.

The result in Eqn. (16) can then be differentiated and evaluated at the collision time $t_c$, to give

$$\langle p \rangle_{t \approx t_c} = m \frac{d\langle x \rangle_t}{dt} \bigg|_{t \approx t_c} = -\frac{\hbar}{\beta \sqrt{\pi}} \left[ \frac{t_c/t_0}{\sqrt{1 + (t_c/t_0)^2}} \right] \approx -\frac{\hbar}{\beta \sqrt{\pi}} = -\frac{1}{\sqrt{\pi} \alpha} \quad (17)$$

for $t_c \gg t_0$. This implies that the expectation value of momentum at the classical collision time does not vanish, for reasons which are most easily visualized using the behavior of the time-dependent momentum-space probability densities, as in Fig. 3 of Ref. [11]. For times long before and long after the ‘bounce’, the momentum distribution is peaked at $+p_0$ (within $\pm \Delta p_0$) and $-p_0$ (within $\pm \Delta p_0$) respectively. Roughly speaking, at the classical collision time, the momentum components corresponding to the fastest speeds (in the $+p_0 + \Delta p_0$ half of the distribution) have already been reflected from the wall and are already flipped in sign to
have values $-(p_0 + \Delta p_0)$, while the slower components are still predominantly in the lower $+p_0 - \Delta p_0$ half of the distribution, but still positive. There is a small resulting asymmetry in the momentum-distribution, giving an expectation value of order $-\Delta p_0$, as in Eqn. (16).

The result in Eqn. (16) is also useful in that it gives similarly valid approximations for higher derivatives, such as

$$m \frac{d^2 \langle x \rangle_{t \approx t_c}}{dt^2} \approx - \frac{2}{\sqrt{\pi}} \left( \frac{p_0^2}{m \beta_t} \right)$$

which can then be used to describe the effective force exerted on the particle by the wall near the classical collision time. It is not clear how one would obtain this result more directly from the Ehrenfest theorem approach in Eqn. (2) using the potential energy function in Eqn. (1).

The dimensional dependence of this result on $p_0, m, \beta_t$ can be easily understood from simply assuming that the change in momentum during the collision is of order $\Delta p = p_f - p_i \approx -2p_0$, while the collision time, $\Delta t$, is determined by

$$\frac{p_0}{m} \sim \frac{\Delta x}{\Delta t} \quad \text{or} \quad \Delta t \sim \frac{\beta_t m}{p_0} \quad \text{giving} \quad F \sim \frac{\Delta p}{\Delta t} \sim - \frac{2p_0^2}{m \beta_t},$$

assuming we identify $\Delta x \sim \beta_t$ near the time of the collision.

Finally, another useful quantity which can be evaluated in closed form for the solution in Eqn. (8) is the autocorrelation function, which measures the overlap between the initial quantum state, $\psi(x, 0)$, and its time-evolved value at later times, $\psi(x, t)$. This is defined most generally by

$$A(t) \equiv \int_{-\infty}^{\infty} [\psi(x, 0)]^* \psi(x, t) \, dx = \int_{-\infty}^{\infty} [\phi(p, 0)]^* \phi(p, t) \, dp.$$

For the free-particle Gaussian solutions in Eqns. (3) and (4) this can be written in the equivalent forms [15]

$$A_{(G)}(t) = \frac{1}{\sqrt{1 - it/2t_0}} e^{ip_0^2/(2m\hbar)} \exp \left[- \frac{(X(t) - x_0)^2}{4\beta^2(1 - it/2t_0)} \right]$$

$$= \frac{1}{\sqrt{1 - it/2t_0}} \exp \left[ \frac{i \alpha^2 p_0^2 t}{2t_0(1 - it/2t_0)} \right].$$

Both of these forms give

$$|A_{(G)}(t)|^2 = \frac{1}{\sqrt{1 + (t/2t_0)^2}} \exp \left[-2\alpha^2 p_0^2 \frac{(t/2t_0)^2}{(1 + (t/2t_0)^2)} \right].$$
One sees that the free-particle autocorrelation function decreases monotonically with time, due to both the *dynamic* exponential dependence on $p_0$, as well as to the *dispersive* pre-factor which can be attributed to wave packet spreading.

For the bouncing wavepacket solution, we must evaluate the autocorrelation using the closed form expression for the position-space solution in Eqn. (8), giving

$$\tilde{A}_G(t) = \int_{-\infty}^{0} [\tilde{\psi}_G(x,0)]^* \tilde{\psi}_G(x,t) \, dx$$

and the same trick of extending the integral over all space (because the integrand is still odd under $x \to -x$) can be used to find

$$\tilde{A}_G(t) = A_G(t) \left( \frac{1 - \exp \left[ - \{ (x_0/\beta)^2 + (p_0 \beta / \hbar)^2 \} / (1 + it / 2t_0) \} \right]}{1 - \exp \left[ - x_0/\beta)^2 + (p_0 \beta / \hbar)^2 \} \right]} \right) \right. \right.$$}

(24)

Once again, there is a monotonic decrease in $|A(t)|$ with no distinction between the smoother time-evolution before and after the collision and the time during the impulsive *splash* at the wall.

III. OTHER FREE-PARTICLE GAUSSIAN SOLUTIONS RELATED TO THE BOUNCING WAVE PACKET

While the standard Gaussian solutions for the free-particle case given in Eqns. (3) and (4) are the most familiar examples found in textbooks, it is straightforward to construct other localized Gaussian-like wave packet solutions, using the fact that a wide variety of Gaussian integrals can be performed in closed form. Some of these can then be easily used as special case solutions for the ‘bouncing’ wavepacket case as well.

For example, a modified free-particle momentum-space solution of the form

$$\phi_G(p, t) = \sqrt{\frac{2\alpha^3}{\sqrt{\pi}}} \left( p - p_0 \right) e^{-a^2(p-p_0)^2/2} \frac{e^{-ipx_0/\hbar} e^{-ip^2t/2m\hbar}}{1 - \exp[-z_0/\beta] (1 + it/2t_0)}$$

(26)

gives the expectation values

$$\langle p \rangle_t = p_0, \quad \langle p^2 \rangle_t = p_0^2 + \frac{3}{2\alpha^2}, \quad \text{and} \quad \Delta p_t = \sqrt{\frac{3}{2\alpha^2}}$$

(27)
and can be Fourier transformed to yield the position-space wavefunction

\[ \psi_{(G')}(x, t) = i \sqrt{\frac{2}{\beta^3(1 + it/t_0)^3}} e^{ip_0(x-x_0)/\hbar} e^{-ip_0^2t/2m\hbar} (x - X(t)) e^{-(x-x(t))^2/2\beta^2(1+it/t_0)} \] (28)

with

\[ \langle x \rangle_t = X(t) \equiv x_0 + \frac{p_0t}{m} \quad \text{and} \quad \Delta x_t = \sqrt{\frac{3}{2}} \beta_t. \] (29)

The uncertainty principle product in this case is given by

\[ \Delta x_t \cdot \Delta p_t = \frac{3\hbar}{2} \sqrt{1 + (t/t_0)^2} \] (30)

which is similar to that for the first excited state of the harmonic oscillator, at least for \( t = 0 \). In the same way, initial momentum-distributions with higher powers of \((p - p_0)\) can be used to exhibit localized position-space wave packets.

For the case described by Eqns. (26) and (28), if we consider the special case of \( x_0, p_0 \to 0 \), we find the solution

\[ \psi_{(0)}(x, t) = i \sqrt{\frac{2}{\sqrt{\pi} \beta^3(1 + it/t_0)^3}} x e^{-x^2/2\beta^2(1+it/t_0)} \] (31)

which is valid for all space. This clearly satisfies \( \psi(0, t) = 0 \) for all \( t \) and so can be used as a solution for the ‘bouncing’ packet case corresponding to the potential in Eqn. (1); the solution must then be ‘renormalized’ by multiplying by a factor of \( \sqrt{2} \) to account for the different range of definition. This then gives a ‘bouncing’ packet solution

\[ \tilde{\psi}_{(0)}(x, t) = \begin{cases} \sqrt{2} \psi_{(0)}(x, t) & \text{for } x \leq 0 \\ 0 & \text{for } x \geq 0 \end{cases}. \] (32)

The ‘bouncing’ wave packet solution of Eqn. (10) which is valid for arbitrary values of \( x_0 \) and \( p_0 \) can be considered in the limit when \( x_0 = 0 \) and \( p_0 \to 0 \). In that limit, we recover the form in Eqn. (32) (except for trivial multiplicative factors of \( i \)).

For this special solution, almost all of the relevant expectation values can be obtained in closed form, and we find, for example, that

\[ \langle x \rangle_t = -\frac{2\beta_t}{\sqrt{\pi}} \quad \text{and} \quad \langle x^2 \rangle_t = \frac{3\beta_t^2}{2} \] (33)

which combine to give

\[ \Delta x_t = \beta_t \sqrt{\frac{3\pi - 8}{\pi}} \] (34)
which increases with time in the same way as for the standard free-particle solution in Eqn. (13).

Using the explicit form for \( \tilde{\psi}(0)(x,t) \) in Eqn. (32), and the operator form for \( \hat{p} = (\hbar/i)(\partial/\partial x) \), we also find that

\[
\langle p \rangle_t = -\frac{2\hbar}{\beta \sqrt{\pi}} \left[ \frac{t/t_0}{\sqrt{1 + (t/t_0)^2}} \right] \quad \text{and} \quad \langle p^2 \rangle_t = \frac{3\hbar^2}{2\beta^2}.
\]

The first of these expressions is consistent with differentiation of the result in Eqn. (33), while the second is related to the free-particle result in Eqn. (27). The momentum uncertainty is then given by

\[
\Delta p_t = \frac{\hbar}{\beta \sqrt{\frac{3}{2} - \frac{4}{\pi} \frac{(t/t_0)^2}{1 + (t/t_0)^2}}} \quad (36)
\]

which actually decreases in time. This effect can be understood crudely as being due to the fact that the \( t = 0 \) solution has both positive and negative momentum components in the range \((-\Delta p_0, +\Delta p_0)\), and over a time interval of order \( t_0 \), the positive momentum components are reflected from the infinite wall so that the momentum distribution is then more localized to the range \((-p_0, 0)\).

The effective force on the particle due to its interaction with the wall, can be associated with

\[
\frac{d\langle p \rangle_t}{dt} = -\left( \frac{2}{\alpha \sqrt{\pi} t_0} \right) \frac{1}{(1 + (t/t_0)^2)^{3/2}}
\]

which decreases monotonically with time, with an initial value which scales as \( \Delta p_0/t_0 \) which is dimensionally correct.

These results can be combined to give the uncertainty principle product as

\[
\Delta x_t \cdot \Delta p_t = \frac{\hbar}{2} \sqrt{\frac{3(3\pi - 8)}{\pi}} \sqrt{1 + (1 - 8/3\pi)(t/t_0)^2}
\]

\[
\approx (0.58\hbar) \sqrt{1 + (0.15)(t/t_0)^2}.
\]

This localized solution has an initial uncertainty principle product which is only slightly larger than the minimum value but for long times is substantially smaller (about half) than that of the standard Gaussian free-particle solution in Eqn. (7), since for \( t/t_0 \gg 1 \), Eqn. (38) reduces to

\[
\Delta x_t \cdot \Delta p_t \longrightarrow \frac{\hbar}{2} \left( \frac{3\pi - 8}{\pi} \right) \left( \frac{t}{t_0} \right) = (0.45) \frac{\hbar}{2} \left( \frac{t}{t_0} \right).
\]
Variations on the momentum-space distribution in Eqn. (26) which are also odd in $p - p_0$ (higher power solutions of the simple harmonic oscillator, for example) can be used to evaluate other generalized solutions for the free-particle case which can also satisfy the appropriate boundary conditions (at $x = 0$) for the ‘bouncing’ particle case and provide additional examples.

IV. CONCLUSIONS AND DISCUSSIONS

Using the familiar method-of-images or mirror wavefunction technique, we have shown how to construct normalizable Gaussian solutions with arbitrary initial $x_0, p_0$ for the bouncing particle problem, for which many expectation values and related quantities are calculable in closed form. This example adds another case to the limited number of time-dependent wave packet solutions of one-dimensional quantum mechanical problems with familiar classical analogs. It also provides an explicitly calculable example of the effect of the introduction of a distant boundary on an explicitly time-dependent solution for a quantum system for comparison to more general discussions. Because of the methods used, not all of the expectation values (or related quantities such as the momentum-space wave function or the Wigner quasi-probability distribution) can be evaluated as easily, but all of the even expectation values are readily calculable and many of the others, such as $\langle x \rangle_t$ can be usefully approximated, especially near the time of the classical collision with the wall. A class of non-standard free-particle Gaussian solutions can also be used to provide special $p_0 = 0$ which vanish at $x = 0$ and are therefore also solutions for the bouncing well case, with interesting long-time behavior.

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FIG. 1: Plots of the position-space probability density, $|\tilde{\psi}(G)(x,t)|^2$ versus $x$, for the Gaussian bouncing packet of Eqn. [8], for various times before, during, and after the ‘collision’ with the wall are shown on the left. On the right, we show numerical calculations of the quantum mechanical expectation value of position, $\langle x \rangle_t$ versus $t$ (solid curve), over the same time range. The softening of the classical trajectory result, $x_{CL}(t) = -|X(t)|$ (dashed lines), near the collision in the quantum case is well described by Eqn. [16].