CURVATURE OF THE TOTAL SPACE OF A GRIFFITHS NEGATIVE VECTOR BUNDLE AND QUASI-FUCHSIAN SPACE

INKANG KIM, XUEYUAN WAN, AND GENKAI ZHANG

Abstract. For a holomorphic vector bundle $E$ over a Hermitian manifold $M$ there are two important notions of curvature positivity, the Griffiths positivity and Nakano positivity. We study the consequence of these positivities and the relevant estimates. If $E$ is Griffiths negative over Kähler manifold, then there is a Kähler metric on its total space $E$, and we calculate the curvature and prove the non-positivity of the curvature along the tautological direction. The Nakano positivity can be formulated as a positivity for the Nakano curvature operator and we give estimate the Nakano curvature operator associated with a Nakano positive direct image bundle. As applications we construct a mapping class group invariant Kähler metric on the quasi-Fuchsian space $QF(S)$, which extends the Weil-Petersson metric on the Teichmüller space $T(S) \subset QF(S)$, and we obtain estimates for the Nakano curvature operator for the dual Weil-Petersson metric on the holomorphic cotangent bundle of Teichmüller space.

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INTRODUCTION

Let $E$ be a holomorphic Hermitian vector bundle over a Kähler manifold $M$. There are two important notions of curvature positivity, the Griffiths positivity and Nakano positivity. The aim of the present paper is to study the consequences of these positivities and apply the results to various cases related to the Teichmüller space.

The curvature on a holomorphic bundle $E$ over $M$ can be viewed as an operator on $T^{(1,0)}_m M \otimes E_m$, $m \in M$, and it is self-adjoint. It is called Griffiths positive if $R$ is positive on simple tensors $u \otimes v \in T^{(1,0)}_m \otimes E_m$, and Nakano positive if it is positive on the total tensor space $T^{(1,0)}_m \otimes E_m$, $m \in M$. We prove first that if $E$ is Griffith negative then there is a Kähler metric on its total space $E$, and we calculate the curvature and prove the non-positivity of the curvature along the fiber direction. We then give estimates of the Nakano curvature operator associated with a Nakano positive direct image bundle.

Let $p: \mathcal{X} \to M$ be a holomorphic fibration with compact fibers. Suppose the relative canonical line bundle $K_{\mathcal{X}/M}$ is positive over $\mathcal{X}$, and consider the following direct image bundle

$$E = p_*(K_{\mathcal{X}/M}^\otimes 2).$$

Following Berndtsson [3, 4], there exists a natural $L^2$-metric (a Hermitian metric) on the holomorphic vector bundle $E$, and the curvature of the $L^2$-metric is Nakano positive. It is natural to consider the extension of Nakano’s positivity. This suggests we estimate the Nakano curvature operator $Q$, which is defined as the quadratic form on $TM \otimes E$

$$Q(A, B) = \sum_{\alpha, \beta} a^{\alpha \bar{\beta} j} \left\langle R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) e_i, e_j \right\rangle$$

for any $A = a^{\alpha i} \frac{\partial}{\partial z^\alpha} \otimes e_i$ and $B = b^{\bar{\beta} j} \frac{\partial}{\partial \bar{z}^\beta} \otimes e_j$ in $TM \otimes E$. We identify it with an operator on $TM \otimes E$ denoted also by $Q$,

$$\langle Q(A), B \rangle_{TM \otimes E} = Q(A, B).$$

Let $\iota: T_t M \otimes E_t \to A^{n-1,1}(X_t)$, $n = \dim X_t$, be the diagonal map

$$\iota: T_t M \otimes E_t \to A^{0,1}(X_t, TX_t) \otimes H^0(X_t, L_t \otimes K_t) \to A^{n-1,1}(X_t, L_t),$$

(0.1)
where the first map is given by the Kodaira-Spencer tensor and the second one is the evaluation, see (6.6).

We obtain the following estimates on the Nakano curvature operator.

**Theorem 0.1.** For any fixed $t \in M$, let $\sigma$ be the maximum of the eigenvalue of $\Box' = \nabla' \nabla'^* + \nabla'^* \nabla'$ on the finite-dimensional subspace $\iota(T_t M \otimes E_t)$. Then we have

$$Q(A, A) \geq \left( \frac{1}{1+2n} + (1 + \sigma)^{-1} \right) \|\iota(A)\|^2.$$  

In particular, if the map $\iota$ is injective, then the Nakano curvature operator $Q$ satisfies

$$Q \geq \left( \frac{1}{1+2n} + (1 + \sigma)^{-1} \right) \lambda_{\text{min}}$$

where $\lambda_{\text{min}}$ is the lowest eigenvalue of $\iota^* \iota$ with respect to the Hilbert space norms in $T_t M \otimes E_t$ and $A^{n-1,1}(X_t, L_t)$.

We consider then the more concrete case related to these notions.

One of the most studied cases of the above notions is the case when $M$ is the Teichmüller space $T(S)$ equipped with the Weil-Petterson metric of a surface $S$ and $E$ the tangent bundle or relevant bundles. There have been active studies on the properties of this metric since its birth. More recently, some new Kähler metrics with more desirable properties such as Kähler hyperbolicity have been found where the Kähler hyperbolicity means that the Kähler metric is complete with bounded curvatures and it has a bounded Kähler primitive. Such Kähler hyperbolic metrics are studied by McMullen [17] and Liu-Sun-Yau [15].

In Kleinian group theory, the quasi-Fuchsian space $QF(S)$ is a quasi-conformal deformation space of the Fuchsian space $F(S)$ which can be identified with $T(S)$. By Bers’ simultaneous uniformization theorem, $QF(S)$ can be naturally identified with $T(S) \times T(S)$ where $\tilde{S}$ is $S$ with reversed orientation. With this identification, the mapping class group acts diagonally on $QF(S)$ and $F(S) = T(S)$ sits diagonally on $T(S) \times T(\tilde{S})$. But this diagonal embedding is totally real. Hence if one gives a product Kähler metric on $QF(S)$, this metric is not an extension of a Kähler metric on $F(S)$. There have been several attempts to extend a Kähler metric of $T(S)$ to $QF(S)$. Bridgeman and Taylor [5] described a quasi-metric which extends the Kähler metric of $T(S)$, but it vanishes along the pure bending deformation vectors [6].

As an application of our result we give a completely new mapping class group invariant Kähler metric on $QF(S)$ which extends any Kähler metric on $T(S)$. Indeed such a metric is already defined in the paper [13] a few years ago. The metric is defined by a Kähler potential, which is a combination of $L^2$-norm of a fiber and a Kähler potential on the base $T(S)$. We will see that $QF(S)$ can be embedded, via Bers embedding using the complex projective structures, in the holomorphic cotangent bundle $Q(S)$ over $T(S)$ with fibers being quadratic holomorphic differentials, as a bounded open neighborhood of the zero section.
On the holomorphic vector bundle $Q(S)$, one can give a mapping class group invariant Kähler metric as follows. By a theorem of Berndtsson [3], one can show that $Q(S)$ is Griffiths positive with respect to the $L^2$-metric; see [13] for a proof. Hence its dual bundle $B(S) = Q^*(S)$, which is a tangent bundle of Teichmüller space whose fiber is the set of Beltrami differentials, is Griffiths negative. We fix in the rest of the paper this realization of $QF(S)$ as a subset in $B(S)$. We recall that the $L^2$-norm of a Beltrami differential $w = \mu(v)\frac{dv}{\bar{dv}}$ is given by

$$||w||^2 = (w, w) = \int_Y |\mu(v)|^2 \rho(v) |dv|^2$$

where $v$ is a local holomorphic coordinate on $Y$ and $\rho(v) |dv|$ is a hyperbolic metric on $Y$. Here $(,)$ denotes the $L^2$ inner product over each fiber and $|| \cdot ||$ denotes its associated norm.

The Kähler metric depending on a constant $k > 0$ and a Kähler metric on $T(S)$, is constructed on $B(S)$ via Kähler potential

$$\Phi(w) = ||w||^2 + k\pi^* \psi(w),$$

where $w$ is an element in the fiber, $\psi$ is a Kähler potential on $T(S)$ and $\pi : B(S) \to T(S)$ is a projection.

In local holomorphic coordinates $(z, x)$ around $w_0$, where $z = (z_1, \cdots, z_{3g-3})$ is local holomorphic coordinates around $\pi(w_0) = z_0$, and $w = \sum x^a e_a(z)$ with respect to local holomorphic sections $e_a$, for a holomorphic tangent vector at $w$ $T = u + v$ with a canonical decomposition into $T(S)$ direction $u$ and vertical fiber direction $v$, the norm of $T$ with respect to the Kähler metric defined by the Kähler potential $\Phi$ is given by

$$||T||^2_\Phi = \bar{\partial}_T \partial_T \Phi(w) = -(R(u, \bar{u})w, w) + (\partial_w u + \bar{v}, \partial_w u + \bar{v}) + k\bar{\partial}_u \partial_u \psi > 0,$$

where $R$ is a curvature of the Chern connection $\nabla$ on $B(S)$ and $\nabla = D + \bar{\partial}$ is a decomposition into $(1,0)$ and $(0,1)$ part of the connection. See [13] for details.

The Kähler metric we construct on $QF(S)$ is the restriction to this open neighborhood. When we choose the Weil-Petersson metric on $T(S)$, we show that the new Kähler metric on $B(S)$ has similar properties, such as its Kähler form has a bounded primitive and the curvature has non-positivity in some directions.

**Theorem 0.2.** There exists a mapping class group invariant Kähler metric on $B(S)$, which extends the Weil-Petersson metric on $T(S) \subset B(S)$. Consequently, the quasi-Fuchsian space $QF(S)$ has such a Kähler metric as an invariant open set of $B(S)$ under the mapping class group. Furthermore, the curvature of the metric is non-positive when evaluated on the tautological section (and vanishes along vertical directions); its Ricci curvature is bounded from above by $-\frac{1}{\pi (g-1)}$ when restricted to Teichmüller space, and its Kähler form has a bounded primitive.
We specify Theorem 0.1 to the case of Teichmüller space $M = \mathcal{T}$. Let $\mathcal{X}$ be Teichmüller curve over Teichmüller space and consider the bundle

$$E = p_*(K_{\mathcal{X}/\mathcal{T}}^\otimes 2),$$

which is exactly the cotangent bundle $T^*\mathcal{T}$ of $\mathcal{T}$. In this case, the map $\iota$ is an isometric embedding and $n = 1$. Hence

**Corollary 0.3.** We have the following lower estimate for the Nakano curvature operator,

$$Q(A, A) \geq \left(\frac{1}{3} + \frac{1}{1 + \sigma}\right)\|A\|^2.$$

As an operator on the tensor product $T_{m}^{(1,0)} \otimes E_m$, we have $Q \geq \frac{1}{3} + \frac{1}{1 + \sigma}$. In particular, if $A$ has the form $A = \sum_i \lambda_i \mu_i \otimes q_i$ with $\lambda_i \geq 0$, then

$$Q(A, A) \geq \frac{2}{3}\|A\|^2,$$

where $q_i$ is a holomorphic quadratic differential and $\mu_i$ denotes the associated harmonic Beltrami differential of $q_i$.

This article is organized as follows. In Section 1, we will review the horizontal and vertical decomposition associated with a Hermitian vector bundle, the definition of a Griffiths negative vector bundle, and the Kähler metric on the total space. In Section 2, we will calculate the curvature of the Kähler metric and obtain some curvature properties. In Section 3, we will give some estimates on the Nakano curvature operator associated with a direct image bundle and prove Theorem 0.1 and Corollary 0.3. In Section 4, we will recall the definitions of quasi-Fuchsian space and complex projective structure, and we will embed the quasi-Fuchsian space into the space of complex projective structures. Then we will define a mapping class group invariant Kähler metric on the quasi-Fuchsian space and prove Theorem 0.2. In Section 5, we discuss some Kähler metrics on other geometric structures. In the appendix, we will derive the curvature formula of the Weil-Petersson metric on Teichmüller space by using Berndtsson’s method [4, Section 4.2].

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1. Kähler metric and Griffiths negative vector bundle

In this section, we will review the horizontal and vertical decomposition associated with a Hermitian vector bundle, the definition of a Griffiths negative vector bundle, and the Kähler metric on the total space.
1.1. Horizontal-vertical decomposition. Let $M$ be a complex manifold of complex dimension $n$. Let $\pi : E \to M$ be a holomorphic vector bundle of rank $r$ and the induced map $\pi_* : TE \to TM$. Let $\{e_1, \ldots, e_r\}$ be a local holomorphic frame for $E$ over a local holomorphic coordinate system $(U; z = (z^1, \ldots, z^n))$. Each element of $E|_U$ has the form $v = \sum_{i=1}^{r} v^i e_i$, and so $\{\partial/\partial z^1, \ldots, \partial/\partial z^n; \partial/\partial v^1, \ldots, \partial/\partial v^r\}$ is a local holomorphic frame of $TE|_{\pi^{-1}(U)}$. The vertical subbundle of $TE$ is defined as the kernel of $\pi_*$, i.e.

$$\mathcal{V} := \text{Ker} \pi_* \subset TE,$$

which is a holomorphic subbundle of rank $r$. In terms of local coordinates, the vertical subbundle is given by

$$\mathcal{V} = \text{Span}_C \left\{ \frac{\partial}{\partial v^i}, i = 1, \ldots, r \right\}.$$

Let $G = (G_{ij})$ be a Hermitian metric on $E$. It induces a Hermitian metric on $\mathcal{V}$,

$$\langle V, W \rangle = \sum_{i=1}^{r} V^i \overline{W}^j G_{ij},$$

where $V = \sum_{i=1}^{r} V^i \partial/\partial v^i$, $W = \sum_{i=1}^{r} W^i \partial/\partial v^i \in \mathcal{V}$. Let $\nabla^V$ denote the Chern connection of the Hermitian metric on the holomorphic vertical subbundle $\mathcal{V}$, and denote by

$$P = \sum_{i=1}^{r} v^i \frac{\partial}{\partial v^i}$$

the Euler vector field, which is a holomorphic section of $\mathcal{V}$. It is also called the tautological section. Then the horizontal subbundle $\mathcal{H}$ is defined as

$$\mathcal{H} = \{ X \in TE : \nabla^V_X P = 0 \}.$$

We adopt now the Einstein summation convention in the subsequent text. We denote $G := G_{ij} v^i \overline{v}^j$, and the differentiation of $G$ with respect to $v^i, \overline{v}^j, z^\alpha, \overline{z}^\beta, 1 \leq i, j \leq r, 1 \leq \alpha, \beta \leq n$, as

$$G_i = \partial G/\partial v^i, \quad G_j = \partial G/\partial \overline{v}^j, \quad G_{ij} = \partial^2 G/\partial v^i \partial \overline{v}^j,$$

$$G_{i\alpha} = \partial^2 G/\partial v^i \partial z^\alpha, \quad G_{i\overline{j}} = \partial^2 G/\partial v^i \partial \overline{v}^j \partial \overline{z}^\beta, \quad \text{etc.}.$$

We set

$$\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - G_{\alpha \overline{d}} \frac{\partial}{\partial \overline{v}^d}.$$

Then the horizontal subbundle $\mathcal{H}$ can be described as

$$\mathcal{H} = \text{Span}_C \left\{ \frac{\delta}{\delta z^\alpha}, \alpha = 1, \ldots, n \right\}.$$
The differential $\pi^* : \mathcal{H} \to TM$ is an isomorphism, and we have the following horizontal and vertical decomposition

$$TE = \mathcal{H} \oplus \mathcal{V}.$$ 

Denote

$$\delta v^i = dv^i + G_{\alpha l} G^i_{\bar{l} \alpha} dz^\alpha.$$ 

The dual bundle $T^*E$ has now a horizontal and vertical decomposition as follows

$$T^*E = \mathcal{H}^* \oplus \mathcal{V}^*,$$

where

$$\mathcal{H}^* = \text{Span}_\mathbb{C}\{dz^\alpha, \alpha = 1, \ldots, n\}, \quad \mathcal{V}^* = \text{Span}_\mathbb{C}\{\delta v^i, i = 1, \ldots, r\}.$$

1.2. Griffiths negative vector bundle. Let $(E, G)$ be a holomorphic Hermitian vector bundle over a complex manifold $M$. The Chern curvature tensor $R$ is given by

$$R = (R_{ij\alpha\beta} G^{ik} dz^\alpha \wedge d\bar{z}^\beta) e^i \otimes e_k \in A^{1,1}(M, \text{End}(E)),$$

where

$$R_{ij\alpha\beta} := -\partial_\alpha \partial_{\bar{\beta}} G_{ij} + G^{kl} \partial_\alpha G_{il} \partial_{\bar{\beta}} G_{kj}.$$

**Definition 1.1.** The Hermitian vector bundle $(E, G)$ is called Griffiths positive if

$$R_{ij\alpha\beta} v^i \bar{v}^j \xi^\alpha \bar{\xi}^\beta > 0$$

for any nonzero $v = v^i e_i \in E$ and $\xi = \xi^\alpha \partial/\partial z^\alpha \in TM$. It is called Nakano positive if

$$R_{ij\alpha\beta} u^i \bar{u}^j \xi^\alpha \bar{\xi}^\beta > 0$$

for any nonzero $u = u^i e_i \otimes \partial/\partial z^\alpha \in E \otimes TM$. Similarly we define the corresponding semi-positivity, negativity and semi-negativity.

Note that the metric $G$ defines a smooth function on the total space $E$, so $\partial \bar{\partial} G$ is a $(1,1)$-form on $E$. Moreover,

**Proposition 1.2.** We have

$$\partial \bar{\partial} G = -R_{ij\alpha\beta} v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta + G_{ij} \delta v^i \wedge d\bar{v}^j.$$

**Proof.** This follows by direct computations,

$$-R_{ij\alpha\beta} v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta + G_{ij} \delta v^i \wedge d\bar{v}^j$$

$$= -(-\partial_\alpha \partial_{\bar{\beta}} G_{ij} + G^{kl} \partial_\alpha G_{il} \partial_{\bar{\beta}} G_{kj}) v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta$$

$$+ G_{ij} (dv^i + G_{\alpha l} G^i_{\bar{l} \alpha}) \wedge (d\bar{v}^j + G_{\bar{\beta} k} G^{\bar{k} \bar{\beta} }d\bar{z}^\beta)$$

$$= G_{\alpha \bar{\beta}} d\bar{z}^\alpha \wedge d\bar{z}^\beta + G_{\alpha j} dz^\alpha \wedge d\bar{v}^j + G_{i \bar{\beta}} dv^i \wedge d\bar{z}^\beta + G_{ij} dv^i \wedge d\bar{v}^j$$

$$= \partial \bar{\partial} G.$$
Corollary 1.3. If \((E, G)\) is a Griffiths negative vector bundle, then \(\sqrt{-1}\partial \bar{\partial} G\) is a semi-positive \((1, 1)\)-form on \(E\), and is strictly positive on \(E^o := E - \{0\}\).

Proof. For any nonzero \(X = w^i \partial/\partial v^i + \xi^\alpha \partial/\partial z^\alpha \in TE\), by Proposition 1.2, one has
\[
(\partial \bar{\partial} G)(X, X) = -R_{i\bar{j} \alpha \bar{\beta}} v^i \bar{v}^j \xi^\alpha \bar{\xi}^\beta + G_{ij} w^i \bar{w}^j \geq 0,
\]
where the equality if and only if \(R_{i\bar{j} \alpha \bar{\beta}} v^i \bar{v}^j \xi^\alpha \bar{\xi}^\beta = 0\), \(G_{ij} w^i \bar{w}^j = 0\). Hence \(w = (w^1, \ldots, w^r) = 0\) and \(v = (v^1, \ldots, v^r) = 0\), and \(\xi = (\xi^1, \ldots, \xi^n) \neq 0\). Thus, \(\sqrt{-1}\partial \bar{\partial} G \geq 0\) on \(E\), and \(\sqrt{-1}\partial \bar{\partial} G > 0\) on \(E^o\). \(\square\)

1.3. Kähler metric on \(E\). Now we assume that \((E, G)\) is a Griffiths negative vector bundle over a Kähler manifold \((M, \omega)\), where
\[\omega = \sqrt{-1} g_{\alpha \bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta\]
denotes the Kähler form. Denote

\[
\Omega := \pi^* \omega + \sqrt{-1}\partial \bar{\partial} G. \tag{1.2}
\]

Proposition 1.4. \(\Omega\) is a Kähler metric on the total space \(E\).

Proof. In terms of the above local coordinates \(\Omega\) is

\[
\Omega = \sqrt{-1} \Omega_{\alpha \bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \sqrt{-1} G_{ij} \delta v^i \wedge \delta \bar{v}^j, \tag{1.3}
\]
where

\[
\Omega_{\alpha \bar{\beta}} := -R_{i\bar{j} \alpha \bar{\beta}} v^i \bar{v}^j + g_{\alpha \bar{\beta}} \tag{1.4}
\]
is a positive definite matrix due to Proposition 1.2 and \(g_{\alpha \bar{\beta}}\) being a Kähler metric. \(\square\)

Since \(G\) is a smooth function on \(E\), so the differential \(\partial G\) of \(G\) is a globally defined one-form on \(E\).

Proposition 1.5. The norm of the one-form \(\partial G\) with respect to \(\Omega\) is given by
\[
\|\partial G\|^2_{\Omega} = G
\]
for any metric \(\omega\) on \(M\). In particular,
\[
\|\partial G\|^2 < R
\]
on the disk bundle \(S_R = \{(z, v) \in E | G(z, v) < R\}\).
Proof. By a direct calculation and the (1.1), one has
\[
\partial G = G_\alpha dz^\alpha + G_i dv^i = G_i(dv^i + G_\alpha G^{\bar{\alpha}}_i dz^\alpha) = G_i \delta v^i.
\]
Its norm square with respect to the metric \(\Omega\) is
\[
\|\partial G\|^2 = G_i G_j \tilde{G}^{ji}.
\]
Since \(G = G_{ij} v^i \bar{v}^j\), so \(G_i = G_i \bar{v}^j\) and
\[
G_i G_j \tilde{G}^{ji} = G_i \bar{v}^j G_{kj} v^k \tilde{G}^{ji} = G_{kl} v^k \bar{v}^l = G,
\]
which yields that
\[
\|\partial G\|^2 = G_i G_j \tilde{G}^{ji} = G,
\]
which is independent of the metric \(\omega\).
□

**Definition 1.6.** A two form \(\omega\) is called \(d\)-bounded if \(\omega = d \beta\) for some (locally defined) bounded one-form \(\beta\).

As a result, we obtain

**Corollary 1.7.** If \(\omega\) is \(d\)-bounded, then \(\Omega\) is also \(d\)-bounded on any bounded domain of \(E\) with \(\Omega = d(\partial G + \pi^* \beta)\) and bounded one-form \(\partial G + \pi^* \beta\).

2. Curvature of the Kähler metric on \(E\)

2.1. General formulas. In the section, we will calculate the Chern curvature of the Kähler metric \(\Omega\) defined in (1.2).

Let \(\nabla = \nabla' + \bar{\partial}\) denote the Chern connection of \(\Omega\) and
\[
R^\Omega = \nabla^2 = \nabla' \circ \bar{\partial} + \bar{\partial} \circ \nabla' \in A^{1,1}(E, \text{End}(TE))
\]
denote the Chern curvature of \(\nabla\). Then
\[
\nabla' \left( \frac{\delta}{\delta z^\alpha} \right) = \left\langle \nabla' \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right\rangle \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma} + \left\langle \nabla' \left( \frac{\delta}{\delta v^i} \right), \frac{\partial}{\partial v^j} \right\rangle G^{ji} \frac{\partial}{\partial v^i}
\]
(2.1)
\[
= \partial \Omega_{\alpha \beta} - \left\langle \frac{\delta}{\delta z^\alpha}, \bar{\partial} \left( \frac{\delta}{\delta z^\beta} \right) \right\rangle \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma},
\]
where the last equality holds since \(\bar{\partial} \left( \frac{\delta}{\delta z^\beta} \right)\) is vertical, and
\[
\nabla' \left( \frac{\partial}{\partial v^i} \right) = \left\langle \nabla' \left( \frac{\partial}{\partial v^i} \right), \frac{\delta}{\delta z^\beta} \right\rangle \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma} + \left\langle \nabla' \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle G^{ji} \frac{\partial}{\partial v^k}
\]
(2.2)
\[
= - \left\langle \frac{\partial}{\partial v^i}, \bar{\partial} \left( \frac{\delta}{\delta z^\beta} \right) \right\rangle \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma} + \partial G^{ji} \frac{\partial}{\partial v^k}
\]
\[
= G_{ij} \partial \left( G_{k\bar{\beta}} G^{\bar{j}k} \right) \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma} + \partial G_{ij} G^{ji} \frac{\partial}{\partial v^k}
\]
\[
= G_{ij} \partial \left( G_{k\bar{\beta}} G^{\bar{j}k} \right) \Omega^{\beta \gamma} \frac{\delta}{\delta z^\gamma} + \partial_\alpha G_{ij} G^{ji} \frac{\delta}{\delta z^\alpha} \otimes \frac{\partial}{\partial v^k},
\]
where the last equality follows from the fact $G_{ij} = 0$ since $G_{ij}$ is independent of fiber $v$, and $\partial_j(G_{jk}) = \partial_j(G_{jl}v^l) = \partial_j\partial_j(G_{kl})v^l = 0$. From (2.1) and (2.2), the curvature $R^\Omega$ is

\begin{equation}
R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right) = (\nabla' \circ \bar{\partial} + \bar{\partial} \circ \nabla') \left( \frac{\delta}{\delta z^\alpha} \right)
= \nabla' \left( -\bar{\partial}(G_{\alpha l}G^{li}) \frac{\partial}{\partial v^l} \right) + \bar{\partial} \left( \partial\Omega_{\alpha\beta}^\gamma \Omega_{\beta\gamma}^\delta \frac{\delta}{\delta z^\gamma} \right)
= \left( -\bar{\partial}(G_{\alpha l}G^{li}) - \partial \bar{G}_{ij} G^{jk} \wedge \bar{\partial}(G_{\alpha l}G^{li}) + \partial \Omega_{\alpha\beta}^\gamma \Omega_{\beta\gamma}^\delta \wedge \bar{\partial}(G_{\gamma l}G^{lk}) \right) \frac{\partial}{\partial v^k}
+ \left( \bar{\partial}(\partial\Omega_{\alpha\beta}^\gamma) \wedge \bar{\partial}(G_{k\beta}G^{jk}) \wedge \bar{\partial}(G_{\gamma l}G^{lk}) \right) \frac{\delta}{\delta z^\gamma},
\end{equation}

and

\begin{equation}
R^\Omega \left( \frac{\partial}{\partial v^l} \right) = \bar{\partial} \circ \nabla' \left( \frac{\partial}{\partial v^l} \right)
= \bar{\partial} \left( G_{ij} \partial(G_{k\beta}G^{jk}) \Omega_{\beta\gamma}^\delta \frac{\delta}{\delta z^\gamma} + \bar{\partial}(G_{ij}G^{jk}) \frac{\partial}{\partial v^k} \right)
= \bar{\partial}(G_{ij} \partial(G_{k\beta}G^{jk}) \Omega_{\beta\gamma}^\delta \frac{\delta}{\delta z^\gamma}
+ \left( G_{ij} \partial(G_{k\beta}G^{jk}) \Omega_{\beta\gamma}^\delta \wedge \bar{\partial}(G_{\gamma l}G^{lk}) + \bar{\partial}(G_{ij}G^{jk}) \right) \frac{\partial}{\partial v^k}.
\end{equation}

**Proposition 2.1.** The Chern curvature $R^\Omega$ satisfies

(i) $\left< R^\Omega \left( \frac{\partial}{\partial v^l} \right), \frac{\partial}{\partial v^m} \right> = (R_{\alpha\beta\gamma\delta}R_{ij\gamma\delta}v^k v^l \Omega_{\beta\gamma}^\delta \wedge R_{ij\alpha\beta}dz^\alpha \wedge dz^\beta;$

(ii) $\left< R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right> = \bar{\partial}(\partial\Omega_{\alpha\beta}^\gamma)\Omega_{\beta\gamma}^\delta \wedge \bar{\partial}(G_{ij}G^{jk})G_{k\beta}G^{jk} \wedge \bar{\partial}(G_{ij}G^{jk})G_{k\beta}G^{jk}.$

**Proof.** (i) We compute the inner product according to (2.4),

$$\left< R^\Omega \left( \frac{\partial}{\partial v^l} \right), \frac{\partial}{\partial v^j} \right> = G_{ij} \partial(G_{k\beta}G^{jk}) \Omega_{\beta\gamma}^\delta \wedge \bar{\partial}(G_{\gamma l}G^{lk})G_{k\beta}G^{jk} + \bar{\partial}(G_{ij}G^{jk})G_{k\beta}G^{jk}.$$ 

Note that $G_{ij}G^{jk} = \delta^k_i$, hence

$$\partial\alpha(G^{k\beta}G_{ij}) = 0 = \partial\alpha(G^{k\beta}G_{ij}) + G^{k\beta}G_{ij}.$$ 

Then

$$G_{ij} \partial(G_{\beta k}G^{k\beta}) = \left[ G_{ij} (\partial\alpha(G_{\beta k}G^{k\beta}) + G_{ij}G_{\beta k}\partial\alpha(G^{k\beta})) \right] dz^\alpha
= (G_{ij}G_{\alpha\beta k}G^{k\beta} - G_{\beta k}G^{k\beta}G_{ij}) dz^\alpha
= (G_{\alpha\beta i} - G_{\beta k}G^{k\beta}G_{ij}) dz^\alpha.$$ 

But $R_{\alpha\beta i} = -\partial\alpha \partial\beta G_{ij} + G^{k\beta} \partial\alpha G_{ij} \partial\beta G_{k\beta}$ and $G = G(z, v) = G_{ij}(z)v^i v^j$, hence

$$G_{ij} = G_{ij} v_j, G_{\alpha\beta i} = G_{\alpha\beta i} v_i.$$
and

\[ R_{i\alpha\beta} v^j = -G_{\alpha\beta i} v^j + G_{\alpha j} G_{\beta k} v^k v^l \]

\[ = -G_{\alpha\beta i} + G_{\alpha j} G_{\beta k} \]

\[ = -G_{\alpha\beta i} + G_{\alpha q} G_{\beta k}. \]

Finally, we get

\[ G_{i\bar{q}}(G_{k\bar{eta}} G_{\bar{a}k}) = (G_{\alpha\bar{q}} - G_{k\bar{eta}} G_{i\bar{q}a} G_{\bar{a}k}) dz^\alpha = -R_{i\alpha\beta} v^i dz^\alpha. \]

Similar calculations give

\[ \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial z^\alpha} \right\rangle = (R_{i\alpha\beta} R_{k\bar{j}\gamma} v^k v^l \Omega^{\beta\gamma} + R_{ij\alpha\beta}) dz^\alpha \wedge dz^\sigma. \]

(ii) Using (2.5) we compute

\[ \left\langle R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right\rangle = \left\langle \delta(\partial(\partial_{\Omega^\beta\gamma}) - \partial(G_{k\bar{\beta}} G^{ar{a}k}) G_{ij} \Omega^{\beta\gamma} \wedge \bar{\partial}(G_{\mu i} G_{\bar{a}l})) \frac{\delta}{\delta z^\gamma}, \frac{\delta}{\delta z^\beta} \right\rangle \]

\[ = \delta(\partial_{\Omega^\beta\gamma}) \Omega_{\gamma\bar{\beta}} - R_{p\bar{q}\gamma\bar{\beta}} R_{k\bar{q}a} v^k v^i G_{i\bar{a}l} dz^\gamma \wedge dz^\sigma. \]

\[ \square \]

**Remark 2.2.** From (i), when evaluated on a vertical vector, \( \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle \) vanishes, that is \( \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle = 0. \)

There exists a canonical holomorphic section of \( \mathcal{V} \), that is

\[ P = v^i \frac{\partial}{\partial v^i} \in \mathcal{O}_E(\mathcal{V}) \]

which is called the tautological section (see e.g., [1, Section 3]). Denote

\[ \Psi_{\alpha\beta} = -R_{i\alpha\beta} v^i v^j. \]

From Proposition 2.1 the (1,1)-form \( \left\langle R^\Omega(P), P \right\rangle \) is

\[ \left\langle R^\Omega(P), P \right\rangle = \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle v^i v^j \]

\[ = (R_{i\alpha\beta} R_{k\bar{j}\gamma} v^k v^l \Omega^{\beta\gamma} + R_{ij\alpha\beta}) v^i v^j dz^\alpha \wedge dz^\sigma \]

\[ = \left( \Psi_{\alpha\beta} \Omega_{\gamma\bar{\beta}} - \Psi_{\gamma\bar{\alpha}} \right) dz^\alpha \wedge dz^\sigma. \]

For any point \((z, v)\) outside the zero section, i.e., in the set

\[ E^\circ := \{(z, v) \in E; v \neq 0\}, \]

the vector \(P(z, v) \neq 0.\) So \( (\Psi_{\alpha\beta}) \) is a positive definite matrix on \( E^\circ \) since \((E, G)\) is Griffiths negative. Since

\[ \Omega_{\alpha\beta} - \Psi_{\alpha\beta} = g_{\alpha\beta} \]
is positive definite, so
\[
\langle \sqrt{-1}R^\Omega(P), P \rangle = \sqrt{-1} \left( \Psi_{\alpha\beta} \Psi_{\gamma\delta} \Omega^{\beta\gamma} - \Psi_{\alpha\delta} \right) dz^\alpha \wedge dz^\sigma
\]
(2.7)
\[
\leq \sqrt{-1} \left( \Psi_{\alpha\beta} \Psi_{\gamma\delta} \Psi^{\beta\gamma} - \Psi_{\alpha\delta} \right) dz^\alpha \wedge dz^\sigma = 0.
\]
Thus, we obtain

**Proposition 2.3.** \( \langle \sqrt{-1}R^\Omega(P), P \rangle \) is a non-positive (1,1)-form on \( E \).

**Remark 2.4.** Moreover, \( \langle \sqrt{-1}R^\Omega(P), P \rangle \) is a strictly negative (1,1)-form on \( E^o \) along the horizontal directions, that is
\[
\langle R^\Omega(\xi, \xi)(P), P \rangle < 0
\]
for any nonzero vector \( \xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in \mathcal{H}_{(z,v)}, (z,v) \in E^o \). In fact, from (2.7), \( \langle R^\Omega(\xi, \xi)(P), P \rangle = 0 \) if and only if
\[
\left( \Psi^{\beta\gamma} - \Omega^{\beta\gamma} \right) (\Psi_{\alpha\beta} \xi^\alpha)(\Psi_{\gamma\delta} \tilde{\xi}^\delta) = 0.
\]
(2.8)
Since \( \left( \Psi^{\beta\gamma} - \Omega^{\beta\gamma} \right) \) is positive definite on \( E^o \), (2.8) is equivalent to
\[
\Psi_{\alpha\beta} \xi^\alpha = 0.
\]
On the other hand, \( (\Psi_{\alpha\beta}) \) is positive definite on \( E^o \) by Griffiths negativity of \( (E,G) \), which implies that \( \xi = 0 \).

The Ricci curvature of the Kähler metric is
\[
\text{Ric}^\Omega := \text{Tr}(R^\Omega) = G^{ji} \left( R^\Omega \left( \frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j} \right) + \Omega^{\beta\alpha} \left( \frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta z^\beta} \right) \right)
\]
(2.9)
\[
= G^{ji}(R_{i\alpha\beta} R_{k\gamma\sigma} v^k v^\gamma \Omega^{\beta\gamma} + R_{i\alpha\sigma} dz^\alpha \wedge dz^\sigma
\]
\[
+ \Omega^{\beta\alpha}(\overline{\partial}(\partial \Omega_{\alpha\sigma} \Omega^{\sigma\gamma}) \Omega_{\gamma\beta} - R_{p \gamma \beta} R_{k \sigma \alpha} v^k v^l G^{\rho\delta} dz^\gamma \wedge dz^\delta)
\]
\[
= \overline{\partial} \partial \log \det(G_{ij}) + \overline{\partial} \partial \log \det(\Omega_{\alpha\beta})
\]
\[
= \overline{\partial} \partial \log \left( \det(G_{ij}) \cdot \det(\Omega_{\alpha\beta}) \right).
\]

Denote by \( \iota : M \to E \) the natural embedding (as the zero section of \( E \)), then
\[
\iota^*(\text{Ric}^\Omega) = \iota^*(\overline{\partial} \partial \log \left( \det(G_{ij}) \cdot \det(\Omega_{\alpha\beta}) \right)) = \overline{\partial} \partial \log \left( \det(G_{ij}) \cdot \det(g_{\alpha\beta}) \right)
\]
is the (1,1)-form on \( M \).

2.2. **The case of Teichmüller space.** We specify our formulas above for \( E \to M \) to the case of Teichmüller space \( M \) with \( E \) its tangent and cotangent bundle, and derive some known results; see e. g. [?].

Let \( S \) be a closed surface. The holomorphic tangent bundle of Teichmüller space \( \mathcal{T}(S) \) is a holomorphic vector bundle \( B(S) \) over Teichmüller space whose fiber over \( X \) is the set of harmonic Beltrami differentials \( B(X) \). The cotangent bundle of \( \mathcal{T}(S) \) is \( Q(S) \) whose fiber over \( Y \in \mathcal{T}(S) \) is \( Q(Y) \), the set of holomorphic quadratic differentials over \( Y \).
Definition 2.5. For a harmonic Beltrami differential $\mu = \mu(z)\frac{dz}{dz}$ over $X$ with a hyperbolic metric $g = \rho(z)|dz|$, the $L^2$-norm is defined by
\[ ||\mu||^2 = \int_X |\mu(z)|^2 \rho(z)^2 |dz|^2. \]
The $L^2$-norm of a quadratic differential $\phi = \phi(z)dz^2$ over $X$ is
\[ ||\phi||^2 = \int_X \frac{|\phi(z)|^2}{\rho(z)^2} |dz|^2. \]
The $L^\infty$-norm of a quadratic differential $\phi = \phi(z)dz^2$ over $X$ is defined by
\[ ||\phi||_\infty = \sup_X \rho^{-2} |\phi(z)|. \]
This $L^2$-norm defines the Weil-Petersson metric
\[ ||\mu||_{WP}^2 = \int_X |\mu(z)|^2 \rho(z)^2 |dz|^2 \]
on the tangent space of $T(S)$.
Now let $E = TM$ with the Weil-Petersson metric $(\Omega_{ij}) = (g_{\alpha\beta})$. For any unit vector $\xi \in E = TM$, i.e. $||\xi||^2 = 1$, $\iota^*(\text{Ric}^\Omega)$ above becomes
\[ \iota^*(\text{Ric}^\Omega)(\xi, \bar{\xi}) = \bar{\partial}\partial \log \left( \det(g_{\alpha\beta}) \cdot \det(g_{\alpha\beta}) \right)(\xi, \bar{\xi}) \]
\[ = 2\text{Ric}(\xi, \bar{\xi}), \]
where $\text{Ric} := \bar{\partial}\partial \log \det(g_{\alpha\beta})$ denotes the Ricci curvature of Weil-Petersson metric. From [?], Lemma 4.6 (i), the Ricci curvature of the Weil-Petersson metric satisfies
\[ \text{Ric}(\xi, \bar{\xi}) \leq -\frac{1}{2\pi(g-1)}. \]
where $g$ denotes the genus of Riemann surfaces. Substituting (2.11) into (2.10), we obtains
\[ \iota^*(\text{Ric}^\Omega)(\xi, \bar{\xi}) \leq -\frac{1}{\pi(g-1)} \]
Thus
Proposition 2.6. Let $(M, \omega)$ be Teichmüller space with the Weil-Petersson metric, and let $E = TM$ be the holomorphic tangent bundle. When restricts to $M$, the Ricci curvature of $\Omega$ is bounded from above by $-\frac{1}{\pi(g-1)}$.

3. Estimates of Nakano curvature operator

In this section we shall obtain estimates of on the Nakano curvature operator.
3.1. Nakano curvature operator and its lowest eigenvalue.

**Definition 3.1.** Let \( \pi : (E, h) \to M \) be a Hermitian holomorphic bundle over a complex manifold \( M \) with a Hermitian metric \( g \) and \( R \) be the curvature tensor. The Nakano curvature operator \( Q \) is defined as the quadratic form on \( TM \otimes E \),

\[
Q(A, B) = \sum_{\alpha, \beta} a^{\alpha \beta} \overline{b^{\beta \alpha}} \langle R(\partial / \partial z^{\alpha}, \partial / \partial \bar{z}^{\beta}) e_i, e_j \rangle
\]

for any \( A = a^{\alpha \beta} \partial / \partial z^{\alpha} \otimes e_i \) and \( B = b^{\beta \alpha} \partial / \partial \bar{z}^{\beta} \otimes e_j \) in \( TM \otimes E \). We identify \( Q \) also with the operator on \( TM \otimes E \)

\[
\langle Q(A), B \rangle_{TM \otimes E} = Q(A, B).
\]

In other words \( E \) is Nakano positive if the Hermitian form \( Q \) is positive.

**Remark 3.2.** If the base manifold \( M \) is not equipped with a Hermitian metric, then the notion of Nakano positivity is well-defined but the curvature operator \( Q : T \otimes E \to T \otimes E \) is not defined.

Now we assume that \( p : \mathcal{X} \to M \) is a holomorphic fibration with an ample line bundle \( L \) over \( \mathcal{X} \), and

\[
E = p_* (K_{\mathcal{X}/M} \otimes L)
\]

is the direct image bundle. It follows immediately from Theorem 6.2 in Appendix below that Nakano curvature \( Q \) as an operator on \( TM \otimes E \) is positive definite. Indeed let \( \{\mu_{\alpha}\} \) be an orthonormal basis of \( T_t M \) and let \( c_0 := \min_{x \in \mathcal{X}_t} \lambda_{\min}(c(\phi)_{\alpha \beta}), \ X_t := p^{-1}(t), t \in M \), where \( \lambda_{\min} \) is the lowest eigenvalue of \( \iota^{*} \) with respect to the Hilbert space norms in \( T_t M \otimes E_t \) and \( A^{n-1,1}(X_t, L_t) \) and \( \iota \) is defined by (3.1). Then \( c_0 > 0 \) by the ampleness of \( L \). From Theorem 6.2, the Nakano curvature operator \( Q \) satisfies the following estimate

\[
Q \geq c_0
\]

as an operator on \( TM \otimes E \) with respect to a generalized Weil-Petersson metric on \( TM \) and the \( L^2 \)-metric on \( E \). We shall find a more accessible and geometric lower bound for \( Q \).

Let \( \iota : T_t M \otimes E_t \to A^{n-1,1}(X_t), n = \dim X_t \), be the diagonal map

(3.1) \( \iota : T_t M \otimes E_t \to A^{0,1}(X_t, TX_t) \otimes H^0(X_t, L_t \otimes K_t) \to A^{n-1,1}(X_t, L_t) \),

where the first map is given by the Kodaira-Spencer tensor and the second one is the evaluation, denoted alternatively as

\[
\iota(\mu \otimes u)(x) := (\iota \mu u)(x), \quad x \in X_t
\]

for any \( \mu \in T_t M \) and \( u \in E_t \); see (6.6).
Theorem 3.3. Fix \( t \in M \). Suppose \( L = K_{X/M} \) is the relative canonical bundle, and \( L_t := L|_{X_t} \) is a positive line bundle with curvature \( R^{1,1} \), which gives a Kähler metric on \( X_t \). Let \( \sigma \) be the maximum of the eigenvalue of \( \square = \nabla^*\nabla + \nabla^*\nabla \) on the finite-dimensional subspace \( i(T_t M \otimes E_t) \). Then we have

\[
Q(A, A) \geq (\frac{1}{1+2n} + (1+\sigma)^{-1})\|i(A)\|^2 \geq \frac{1}{1+2n}\|i(A)\|^2.
\]

In particular, if the map \( i \) is injective, then the Nakano curvature operator \( Q \) satisfies

\[
Q \geq (\frac{1}{1+2n} + (1+\sigma)^{-1})\lambda_{\text{min}}
\]

where \( \lambda_{\text{min}} \) is the lowest eigenvalue of \( i^*i \) with respect to the Hilbert space norms in \( T_t M \otimes E_t \) and \( A^{n-1,1}(X_t, L_t) \).

To avoid confusion, we write sometimes the point-wise metric square norm of a section \( u \) as \( \|u\|^2_x, x \in X_t \), i.e., \( \|u\|^2_x = |u(x)|^2e^{-\phi(x)} \), see the appendix Section 6 for the definition of \( |u(x)|^2e^{-\phi(x)} \), and the \( L^2 \)-norm as \( \|u\| \) and inner product \( \langle u, v \rangle \). We fix a point \( t \in M \) and write \( X = p^{-1}(t) \) for notational simplicity.

Lemma 3.4. Let \( (L, X) \) be a positive line bundle with \( X \) being equipped with the corresponding Kähler metric. Let \( \square = d^*d \) be the Laplace-Beltrami operator on scalar functions on \( X \). We have

1. \( (1 + \square)^{-1} \) preserves the positivity in the sense that if \( f \in C^\infty(X), f \geq 0 \) then \( (1 + \square)^{-1}f \geq 0 \).
2. For any \( u \in H^0(X, L) \),

\[
(\|u\|^2_x) \geq \frac{1}{1+n}\|u\|^2_x, \quad x \in X.
\]

Proof. Let \( f \geq 0 \) be a smooth function. Denote \( g = (1+\square)^{-1}f \). Then \((1+\square)g = f \) and \( \square g = f - g \). Let \( x_0 \) be the minimum point of \( g \), \( g(x_0) = \min_{x \in X} g(x) \). At this point \( x_0, f(x_0) - g(x_0) = \square g(x_0) \leq 0 \), and \( f(x_0) \leq g(x_0) \). But \( f \) is nonnegative we have then \( 0 \leq g(x_0) \). This proves that \( g \geq 0 \) on \( X \).

Let \( u \in H^0(X, L) \). We prove first

\[
\square\|u\|^2 \leq k\|u\|^2,
\]

where \( k \) denotes the scalar curvature of \( L \). We choose \( \{e_i\}_{1 \leq i \leq n} \) a local holomorphic frame of \( TX \) and orthonormal at any fixed point, and compute \( \square\|u\|^2 \) at this point,

\[
\square\|u\|^2 = -\sum_i \nabla_{e_i} \nabla_{e_i} \langle u, u \rangle = -\sum_i \nabla_{e_i} \langle \nabla_{e_i} u, u \rangle
\]

\[
= \left\langle \sum_i R(e_i, e_i) u, u \right\rangle - \|\nabla' u\|^2 \leq k\|u\|^2 = n\|u\|^2,
\]

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where the last equality holds since the Kähler metric on $X$ is given by the curvature of $L$. Hence
\[(1 + \Box)\|u\|^2 \leq (1 + n)\|u\|^2\]
or \[\|u\|^2 \geq \frac{1}{1+n}(1 + \Box)\|u\|^2.\] Write \[v = (1 + \Box)^{-1}\|u\|^2.\] Then \[(1 + \Box)v = \|u\|^2,\]
and the above becomes
\[(1 + \Box)v \geq \frac{1}{1+n}(1 + \Box)\|u\|^2,\]
which can be rewritten as
\[-\Box(v - \frac{1}{1+n}\|u\|^2) \leq v - \frac{1}{1+n}\|u\|^2.\]
The same proof above implies that \[v - \frac{1}{1+n}\|u\|^2 \geq 0.\] Indeed, let \(x_0\) be the minimum point of \(f = v - \frac{1}{1+n}\|u\|^2\). Thus \(-\Box f(x_0) \geq 0\) and thus \(f(x_0) \geq 0\). Consequently \(v - \frac{1}{1+n}\|u\|^2 \geq 0\) on $X$. \(\Box\)

We now prove Theorem 3.3.

Proof. From Theorem 6.2, the curvature of $E = p_*(L \otimes K_{X/M}) = p_*(K^{\otimes 2}_{X/M})$ has the following form
\[
\left< R\left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) u, u \right> = \int_X c(\phi)_{\alpha\beta}|u|^2 e^{-\phi} + \langle(1 + \Box')^{-1}i_{\bar{\partial}V} \frac{1}{24} u, i_{\bar{\partial}V} \frac{1}{24} u \rangle.
\]
Since $L|_t = K_{X/M}|_t = K_X$, is a positive line bundle, we can choose the metric $\phi$ on $L$ such that
\[e^\phi = \det \phi.\]

From Lemma 6.3, one has
\[
\left< R\left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) u, u \right> = \int_X (1 + \Box)^{-1}(\mu_\alpha, \mu_\beta)|u|^2 \frac{\omega^n}{n!} + \langle(1 + \Box')^{-1}i_{\mu_\alpha} u, i_{\mu_\beta} u \rangle.
\]
Here \((\cdot, \cdot)\) denotes the point-wise inner product on the holomorphic bundle $T^*X_t \otimes TX_t$, while \((\cdot, \cdot)\) denotes the global inner product for the sections in $A^{n-1,1}(X_t, L_t)$.

For any $A = a^{\alpha i} \frac{\partial}{\partial z^\alpha} \otimes e_i \in T_t M \otimes E_t$, then the Nakano curvature is given by
\[
Q(A, A) = \sum_{\alpha, \beta} a^{\alpha i} a^{\beta \bar{j}} \left< R\left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) e_i, e_j \right>
\]
\[
= \sum_{\alpha, \beta} a^{\alpha i} a^{\beta \bar{j}} \int_X (1 + \Box)^{-1}(\mu_\alpha, \mu_\beta)(e_i, e_j) \frac{\omega^n}{n!} + \langle(1 + \Box')^{-1}i_{\mu_\alpha} e_i, i_{\mu_\beta} e_j \rangle
\]
\[
= \sum_{\alpha, \beta} a^{\alpha i} a^{\beta \bar{j}} \int_X (\mu_\alpha, \mu_\beta)(1 + \Box)^{-1}(e_i, e_j) \frac{\omega^n}{n!} + \langle(1 + \Box')^{-1}i_{\mu_\alpha} e_i, i_{\mu_\beta} e_j \rangle,
\]
where the last equality holds since
\[
\int_X (1 + \Box)^{-1}(\mu_\alpha, \mu_\beta)(e_i, e_j) \frac{\omega^n_i}{n!} = \int_X ((1 + \Box)^{-1}(\mu_\alpha, \mu_\beta))(1 + \Box)^{-1}(e_i, e_j) \frac{\omega^n_i}{n!} = \int_X (1 + \Box)((1 + \Box)^{-1}(\mu_\alpha, \mu_\beta))(1 + \Box)^{-1}(e_i, e_j) \frac{\omega^n_i}{n!} = \int_X (\mu_\alpha, \mu_\beta)(1 + \Box)^{-1}(e_i, e_j) \frac{\omega^n_i}{n!}.
\]

From Lemma 3.4, the following matrix
\[
M_{ij} := (1 + \Box)^{-1}(e_i, e_j) - \frac{1}{1 + k_1}(e_i, e_j),
\]
is semi-positive definite, where
\[
k_1 = \sum_i R^{L_2}(e_i, e_i) = 2 \sum_i R^{L_1}(e_i, e_i) = 2n.
\]

On the other hand, \(a^{\alpha i}a^{\beta j}(\mu_\alpha, \mu_\beta)\) is also a semi-positive matrix, so
\[
Q(A, A) \geq \sum_{\alpha, \beta, i, j} a^{\alpha i}a^{\beta j}(\int_X \frac{1}{1 + 2n}(\mu_\alpha, \mu_\beta)(e_i, e_j) \frac{\omega^n}{n!} + \langle (1 + \Box)^{-1}i_{\mu_\alpha}e_i, i_{\mu_\beta}e_j \rangle)
\]
\[
= \sum_{\alpha, \beta, i, j} a^{\alpha i}a^{\beta j}(\int_X \frac{1}{1 + 2n}(i_{\mu_\alpha}e_i, i_{\mu_\beta}e_j) \frac{\omega^n}{n!} + \langle (1 + \Box)^{-1}i_{\mu_\alpha}e_i, i_{\mu_\beta}e_j \rangle)
\]
\[
\geq \left(\frac{1}{1 + 2n} + (1 + \sigma)^{-1}\right) \| \sum_{\alpha, i} a^{\alpha i}i_{\mu_\alpha}e_i \|^2,
\]
completing the proof. \(\square\)

3.2. **Nagano curvature for Teichmüller space.** Let \(\mathcal{X}\) be Teichmüller curve over Teichmüller space \(M = \mathcal{T}\), and \(L = K_{\mathcal{X}/\mathcal{T}}\), then
\[
E = p_*(K_{\mathcal{X}/\mathcal{T}}^\otimes 2),
\]
which is exactly the cotangent bundle of \(\mathcal{T}\). We fix a point \(t_0\) in \(\mathcal{T}\) and denote \(X = X_{t_0}\). The tangent space \(T_{t_0}(\mathcal{T})\) is the space \(H^0,1(X, TX) = H^0,1(X, K_X^{-1})\) of harmonic Beltrami differentials, \(\mu = \mu(v)\frac{\partial}{\partial v}dv\). They can be further identified with \(H^0(X, K_X^{-1})\) of holomorphic quadratic forms \(q = q(v)dv^2\) by the metric \(e^{-\phi}\) on \(K_X\),
\[
\mu = \overline{q(v)}e^{-\phi}\frac{\partial}{\partial v} \otimes dv = \overline{q}g, \quad g = e^{\phi(v)}dv \otimes \overline{dv}.
\]
We shall hereafter fix this realization. In the case of Riemann surfaces here, we have

**Lemma 3.5.** The paring \(\iota : H^0,1(X, K_X^{-1}) \otimes H^0(X, K_X^{-1}) \rightarrow A^0,1(X, K_X)\) is an isometric embedding, i.e. \(\iota\) is injective and preserves the natural global inner products of \(H^0,1(X, K_X^{-1}) \otimes H^0(X, K_X^{-1})\) and \(A^0,1(X, K_X)\),
Proof. For any
\[
A = \sum_{i,j} a_{ij} \mu_i \otimes q_j = \sum_{i,j} a_{ij} \frac{q_i}{g} \otimes q_j \in H^0,1(X, K_X^{-1}) \otimes H^0(X, K_X^2)
\]
then
\[
\iota(A) = \sum_{i,j} a_{ij} \frac{q_i}{g} \otimes q_j \in A^{0,1}(X, K_X)
\]
which implies that \(\iota\) is injective. Moreover, one has
\[
\|\iota(A)\|_2^2 = \|A\|_2^2,
\]
which completes the proof. \(\square\)

From Theorem 3.3 and using \(\dim X_t = 1\), one has

Corollary 3.6. We have the following lower estimate for the Nakano curvature operator,
\[
Q(A, A) \geq \left( \frac{1}{3} + \frac{1}{1+\sigma} \right)\|A\|^2.
\]
As an operator, we have \(Q \geq \frac{1}{3} + \frac{1}{1+\sigma}\). In particular, if \(A\) has the form \(A = \sum_{i} \lambda_i \frac{\mu_i}{g} \otimes q_i\) with \(\lambda_i \geq 0\), then
\[
Q(A, A) \geq \frac{2}{3}\|A\|^2.
\]

Proof. We need to prove the last part. If we consider
\[
A = \sum_{i} \lambda_i \frac{q_i}{g} \otimes q_i
\]
with \(\lambda_i \geq 0\), then
\[
\frac{\iota(A)}{g} = \sum_{i} \lambda_i \frac{|q_i|^2}{g^2} = \sum_{i} \lambda_i e^{-2\phi}|q_i(v)|^2,
\]
which follows that \(\iota(A)/g\) is real. Using the above argument as in the proof of Theorem 3.3 (see e.g. \cite[Lemma 5.1]{[19]}), we have
\[
(1 + \Box)^{-1}(e^{-2\phi}|q_i(v)|^2) \geq \frac{1}{3} e^{-2\phi}|q_i(v)|^2.
\]
Hence
\[
\int_X (1 + \Box)^{-1}\left(\frac{\iota(A)}{g}\right) \cdot \overline{\frac{\iota(A)}{g}} \omega_0 \geq \frac{1}{3} \int_X \left|\frac{\iota(A)}{g}\right|^2 \omega_0 = \frac{1}{3} \|\iota(A)\|^2 = \frac{1}{3} \|A\|^2.
\]
By (6.9), we obtain
\[
\langle (1 + \Box')^{-1}\iota(A), \iota(A) \rangle = \int_X (1 + \Box)^{-1}\left(\frac{\iota(A)}{g}\right) \cdot \overline{\frac{\iota(A)}{g}} \omega_0 \geq \frac{1}{3} \|A\|^2.
\]
Thus
\[
Q(A, A) \geq \frac{2}{3} \|A\|^2.
\]
\(\square\)
There have been some recent studies on the refined properties of the Weil-Petterson curvature at specific points on the Teichmüller space; see [7] and references therein.

4. Kähler metric on quasi-Fuchsian space

In this section, we will recall the definitions of quasi-Fuchsian space and complex projective structures on surfaces, and we will embed the quasi-Fuchsian space into the space of complex projective structures. Then we will define a mapping class group invariant Kähler metric on the quasi-Fuchsian space.

4.1. Quasi-Fuchsian space. Recall that the isometry group of the hyperbolic 3-space $H^3$ can be identified with $\text{PSL}(2,\mathbb{C})$. We use the unit ball in $\mathbb{R}^3$ as a realization of $H^3$. The ideal boundary is then $S^2$ and is further identified with $\mathbb{CP}^1$ such that the action of $\text{PSL}(2,\mathbb{C})$ on $S^2$ is the natural extension of its isometric action on $H^3$.

The Teichmüller space $T(S)$ is realized as the space of Fuchsian representations, i.e., discrete and faithful representations $\rho : \pi_1(S) \rightarrow \text{PSL}(2,\mathbb{R})$ up to conjugacy. Let $\Gamma_\rho$ be the image of $\rho$, whence $\Gamma_\rho$ acts on $S^2$ by Möbius map preserving the equator. Then any quasi-conformal map $f$ from $S^2$ into itself induces a quasi-conformal deformation $\rho_f$ defined by

$$\rho_f(\gamma) = f \circ \rho(\gamma) \circ f^{-1}.$$ 

If furthermore $\rho_f(\gamma)$ is an element of $\text{PSL}(2,\mathbb{C})$ for any $\gamma \in \pi_1(S)$ then it defines a representation of $\pi_1(S)$ in $\text{PSL}(2,\mathbb{C})$. Collection of such quasi-conformal deformations of Fuchsian representations is denoted $\text{QF}(S)$ and is identified with an open set of a character variety $\chi(\pi_1(S),\text{PSL}(2,\mathbb{C}))$. Hence it has a natural induced complex structure from $\chi(\pi_1(S),\text{PSL}(2,\mathbb{C}))$.

If $\phi : \pi_1(S) \rightarrow \text{PSL}(2,\mathbb{C})$ is a quasi-Fuchsian representation, then $M_\phi = H^3/\phi(\pi_1(S))$ is a quasi-Fuchsian hyperbolic 3-manifold which is homeomorphic to $S \times \mathbb{R}$. Then two ideal boundaries of $M_\phi$ define a pairs of points $(X,Y) \in T(S) \times T(S)$. This is known as Bers’ simultaneous uniformization of $\text{QF}(S)$; see [2]. In this case, we denote $M_\phi$ by $\text{QF}(X,Y)$. According to Bers’ uniformization, a Fuchsian representation $\rho : \pi_1(S) \rightarrow \text{PSL}(2,\mathbb{R})$ whose quotient $X = H^2/\rho(\pi_1(S))$ is a point in $T(S)$ gets identified with $(X,X)$.

The mapping class group $\text{Mod}(S)$ acts on the space of representations $\rho : \pi_1(S) \rightarrow \text{PSL}(2,\mathbb{C})$ by pre-composition $\phi \rho = \rho \circ \phi_*$ where $\phi \in \text{Mod}(S)$ and $\phi_*$ is the induced homomorphism on $\pi_1(S)$. Then $\text{Mod}(S)$ acts on $\text{QF}(S) = T(S) \times T(S)$ diagonally

$$\phi \rho = \phi(X,Y) = (\phi X, \phi Y).$$

4.2. Complex projective structure. In this subsection, we will recall the definition of complex projective structure.
A complex projective structure on $S$ is a maximal atlas $\{(\phi_i, U_i)|\phi_i : U_i \to S^2\}$ whose transition maps $\phi_i \circ \phi_j^{-1}$ are restrictions of complex Möbius maps. Then the developing map $\text{dev} : S \to S^2$ gives rise to a holonomy representation $\rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C})$. We denote the space of marked complex projective structures on $S$ by $\mathcal{P}(S)$. Since Möbius transformations are holomorphic, a projective structure determines a complex structure on $S$. In this way, we obtain a forgetful map

$$
\pi : \mathcal{P}(S) \to \mathcal{T}(S).
$$

Obviously a Fuchsian representation $\rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ preserving the equator of $S^2$ gives rise to an obvious projective structure by identifying $\mathbb{H}^2$ with the upper and lower hemisphere of $S^2$. This gives an embedding

$$
\sigma_0 : \mathcal{T}(S) \to \mathcal{P}(S).
$$

More generally, for $X \in \mathcal{T}(S)$ and $Z \in \pi^{-1}(X) := P(X)$, by conformally identifying $\bar{X} = \mathbb{H}^2$, we obtain a developing map $\text{dev} : \mathbb{H}^2 \to S^2 = \mathbb{CP}^1$ for $Z$. Hence the developing map can be regarded as a meromorphic function $f = \text{dev}$ on $\mathbb{H}^2$. Then the Schwarzian derivative

$$
S(f) = \left[ \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] dz^2
$$

descends to $X$ as a holomorphic quadratic differential. It is known that for any element in holomorphic quadratic differentials $Q(X)$ on $X$, one can show that there exists a complex projective structure over $X$ by solving the Schwarzian linear ODE.

In this way, $\mathcal{P}(S)$ can be identified with a holomorphic vector bundle $\mathcal{Q}(S)$ over $\mathcal{T}(S)$ whose fiber over $X$ is $Q(X)$. In particular this identifies $P(X)$ with $Q(X)$ as affine spaces [9], and the choice of a base point $Z_0$ in $P(X)$ gives an isomorphism $Z \mapsto Z - Z_0$. Hence we will choose $Z_0 = \sigma_0(X)$, and $\mathcal{T}(S)$ will be identified with zero section on $\mathcal{Q}(S)$.

### 4.3. Kähler metric on quasi-Fuchsian space. In this subsection, we will embed the quasi-Fuchsian space into the space of complex projective structures, and we will define a mapping class group invariant Kähler metric on the quasi-Fuchsian space.

#### 4.3.1. Embedding of quas-Fuchsian space. Recall that given $X \in \mathcal{T}(S), Y \in \mathcal{T}(S)$ the Bers’ uniformization determines the quasi-Fuchsian manifold $\text{QF}(X, Y)$. Then $\text{QF}(X, Y)$ has domain of discontinuity $\Omega_+ \cup \Omega_-$ in $S^2$ with $\Omega_+ / \text{QF}(X, Y) = X$, and $\Omega_- / \text{QF}(X, Y) = Y$ where $\text{QF}(X, Y)$ is viewed as a quasi-Fuchsian representation into $\text{PSL}(2, \mathbb{C})$.

As a quotient of a domain in $\mathbb{CP}^1$ by a discrete group in $\text{PSL}(2, \mathbb{C})$, the surface $\Omega_- / \text{QF}(X, Y)$ is a marked projective surface $\Sigma_Y(X)$. Then for a fixed $Y$, we
obtain a quasi-Fuchsian section, called a Bers’ embedding

\[ \beta_Y : T(S) \to P(Y) \subset P(\overline{S}). \]

It is known that this map

\[ \text{QF}(X, Y) \to \Omega_\infty / \text{QF}(X, Y) \]

is a homeomorphism onto its image in \( P(\overline{S}) \); see e.g. [9]. Under the identification of \( P(\overline{S}) \) with \( Q(\overline{S}) \) such that \( \sigma_0(T(\overline{S})) \) is a zero section,

\[ \text{QF}(X, Y) \to \Omega_\infty / \text{QF}(X, Y) - \sigma_0(Y), \]

this embedding includes a zero section, which is the image of \( T(S) \).

Then by Nehari’s bound [17] we get

**Theorem 4.1.** The above embedding of \( \text{QF}(X, Y) \) into \( Q(Y) \) is contained in a ball of radius \( \frac{3}{2} \) in \( Q(Y) \) where the norm is the \( L^\infty \)-norm on quadratic differentials.

**Corollary 4.2.** The quasi-Fuchsian space \( \text{QF}(S) \) embeds into a neighborhood of a zero section in \( Q(\overline{S}) \) which is contained in a ball of radius \( 9\pi(g - 1) \) in \( L^2 \)-norm on each fiber \( Q(Y) \).

**Proof.** The \( L^2 \)-norm of a quadratic differential \( \phi(z)dz^2 \) is given by

\[ \int_Y |\phi(z)|^2 \rho(z)^{-2} |dz|^2 \leq ||\phi||^2_\infty \cdot 2\pi(2g - 2) \leq 9\pi(g - 1). \]

By Corollary 4.2, we get

**Corollary 4.3.** Under this isomorphism between the cotangent bundle \( Q(S) \) and the holomorphic tangent bundle \( B(S) \) of \( T(S) \), the quasi-Fuchsian space \( \text{QF}(S) \) embeds into a neighborhood of a zero section in \( B(S) \) which is contained in a ball of radius \( 9\pi(g - 1) \) in \( L^2 \)-norm on each fiber \( B(X) \).

Now we prove Theorem 0.2.
Proof. Denote \( \pi : \mathcal{B}(S) \to \mathcal{T}(S) \). Since the tangent bundle \( \mathcal{B}(S) \) of \( \mathcal{T}(S) \) with the Weil-Petersson metric \( \omega_{WP} \) is Griffiths negative, then the following \((1,1)\)-form
\[
\Omega = \pi^* \omega_{WP} + \sqrt{-1} \partial \bar{\partial} G
\]
defines a mapping class group invariant Kähler metric on \( \mathcal{B}(S) \) by Proposition 1.4. From (1.3), one sees that \( \Omega \) is an extension of the Weil-Petersson metric \( \omega_{WP} \). From [17, Theorem 1.5], the Weil-Petersson metric \( \omega_{WP} \) has a bounded primitive with respect to the Weil-Petersson metric. By Corollary 1.7, the Kähler metric \( \Omega \) also has a bounded primitive with respect to \( \Omega \). By Remark 2.2, the curvature vanishes along vertical direction. And by Propositions 2.3, 2.6, the Chern curvature \( R^\Omega \) of \( \Omega \) is non-positive when evaluated on the tautological section \( P \), and its Ricci curvature is bounded from above by \(-\frac{1}{\pi(g-1)}\) when restricted to Teichmüller space.

From Corollary 4.3, the quasi-Fuchsian space \( \text{QF}(S) \) embeds into a neighborhood of a zero section in the holomorphic tangent bundle of \( \mathcal{T}(S) \) which is contained in a ball of radius \( 9\pi(g-1) \) in \( L^2 \)-norm on each fiber \( B(X) \). Hence as an open set invariant by the mapping class group, \( \text{QF}(S) \) inherits such a Kähler metric.

\[\square\]

5. Kähler metrics on other geometric structures

Finally, to put our results in perspective we remark that the space \( \mathcal{P}(S) \) of marked complex projective structures is identified with the cotangent bundle of \( \mathcal{T}(S) \) and the natural holonomy map \( \mathcal{P}(S) \to \chi = \chi(\pi_1(S), \text{PSL}(2, \mathbb{C})) \) to the character variety is a local biholomorphic map by the results of Earle-Hejhal-Hubbard [10, 11, 12] (see also [9, Theorem 5.1]). Thus our constructions and results are also valid for \( \mathcal{P}(S) \) and its image in \( \chi \). The space \( \text{QF}(S) \) of quasi-Fuchsian representations is also an open subset of \( \chi \), \( \mathcal{T}(S) \subset \text{QF}(S) \subset \chi \), and it might be interesting to understand the geometry of character variety \( \chi \) using our metric on these open subsets.

The above remark also applies to the Hitchin component for any real split simple Lie group \( G \) of real rank two, namely \( G = \text{SL}(3, \mathbb{R}), \text{Sp}(2, \mathbb{R}), G_2 \). Indeed Labourie [14] generalized the construction in [13] of Kähler metric for \( \text{SL}(3, \mathbb{R}) \) to the above \( G \). In this case, the Hitchin component is proved to be a bundle over Teichmüller space with fiber being a space of holomorphic differentials of degree 3, 4, 6, respectively.

In general, if we consider the bundle \( \mathcal{W} \) over the Teichmüller space whose fiber over \( X \) equal to \( \sum_{j \geq 2} H^0(X, K_X^j) \) for some integer \( N \geq 2 \), where \( K_X \) is the canonical line bundle of \( X \), then it is Griffiths positive, and hence our method applies to its dual space \( \mathcal{W}^* \).

Hence we obtain

\textbf{Corollary 5.1.} The curvature of the Kähler metric on \( \mathcal{W} \) vanishes along vertical directions and is non-positive along tautological sections. Such examples include
the Hitchin component for real split simple Lie groups of real rank two and the space of complex projective structures over $S$.

6. APPENDIX: CURVATURE FORMULA OF WEIL-PETERSSON METRIC

In this Appendix we recall the curvature formula of the Weil-Petersson metric on Teichmüller space using our setup and notation; see [?, 4, 16].

Let $p : \mathcal{X} \to M$ be a holomorphic fibration with compact fibers. Let $L$ be a relatively ample line bundle over $\mathcal{X}$ with the metric $e^{-\phi}$. We denote by $(z;v) = (z^1, \ldots, z^n; v^1, \ldots, v^r)$ a local admissible holomorphic coordinate system of $\mathcal{X}$ with $p(z;v) = z$. Denote

$$
\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - \phi_{\alpha \beta} \phi^{\beta i} \frac{\partial}{\partial v^i}.
$$

(6.1)

By a routine computation, one can show that $\{ \frac{\delta}{\delta z^\alpha} \}_{1 \leq \alpha \leq n}$ spans a well-defined horizontal subbundle of $T\mathcal{X}$. Let $\{ dz^\alpha; \delta v^i \}$ denote the dual frame of $\{ \frac{\delta}{\delta z^\alpha}; \frac{\partial}{\partial v^i} \}$.

One has

$$
\delta v^i = dv^i + \phi^{ij} \phi_{j\alpha} dz^\alpha.
$$

For any metric $\phi$ on $L$ with positive curvature on each fiber, the geodesic curvature $c(\phi)$ of $\phi$ is defined by

$$
c(\phi) = \sqrt{-1} e(\phi)_\alpha^\beta dz^\alpha \wedge \bar{d}z^\beta = \left( \phi_{\alpha \beta} - \phi_{\alpha \delta} \phi^{\delta j} \phi_{j\beta} \right) \sqrt{-1} dz^\alpha \wedge \bar{d}z^\beta,
$$

(6.2)

which is clearly a horizontal real $(1,1)$-form on $\mathcal{X}$.

**Lemma 6.1.** The following decomposition holds,

$$
\sqrt{-1} \partial \bar{\partial} \phi = c(\phi) + \sqrt{-1} \phi^{ij} \delta v^i \wedge \delta \bar{v}^j.
$$

(6.3)

Following Berndtsson (cf. [3, 4]), we define the following $L^2$-metric on the direct image bundle $E := p_* (K_{\mathcal{X} / M} \otimes L)$: for any $u \in E_z \equiv H^0(\mathcal{X}_z; (L \otimes K_{\mathcal{X} / M})_z)$, $z \in M$, then we define

$$
\|u\|^2 = \int_{\mathcal{X}_z} |u|^2 e^{-\phi}.
$$

(6.4)

Note that $u$ can be written locally as $u = f dv \wedge e$, where $e$ is a local holomorphic frame and locally

$$
|u|^2 e^{-\phi} = (\sqrt{-1})^n |f|^2 |e|^2 dv \wedge \bar{dv} = (\sqrt{-1})^n |f|^2 e^{-\phi} dv \wedge \bar{dv},
$$

where $dv := dv^1 \wedge \cdots \wedge dv^n$.

**Theorem 6.2** ([4, Theorem 1.2]). For any $z \in M$ and let $u \in E_z$, one has

$$
\langle \sqrt{-1} \Theta^E u, u \rangle = \int_{p^{-1}(z)} c(\phi) |u|^2 e^{-\phi} + \langle (1 + \square')^{-1} i_{\bar{\partial} v} \frac{\partial}{\partial z^\alpha} u, i_{\bar{\partial} v} \frac{\partial}{\partial z^\beta} u \rangle \sqrt{-1} dz^\alpha \wedge \bar{dz}^\beta
$$

$$
= \int_{p^{-1}(z)} c(\phi) |u|^2 e^{-\phi} + \langle (1 + \square')^{-1} \iota(\frac{\partial}{\partial z^\alpha} \otimes u), \iota(\frac{\partial}{\partial z^\beta} \otimes u) \rangle \sqrt{-1} dz^\alpha \wedge \bar{dz}^\beta.
$$

(6.5)
where $\Theta^E$ denotes the curvature of the Chern connection on $E$ with the $L^2$-metric defined above, here $\Box' = \nabla'\nabla'^{\ast} + \nabla'^{\ast}\nabla'$ is the Laplacian on $L|_{p^{-1}(z)}$-valued forms on $p^{-1}(z)$ defined by the $(1,0)$-part of the Chern connection on $L|_{p^{-1}(z)}$, and

$$\iota\left(\frac{\partial}{\partial z^\alpha} \otimes u\right) = i\bar{\partial}V \delta \delta z^\alpha u.$$  

(6.6)

Now we will derive the curvature formula of the Weil-Petersson metric by using Berndtsson’s curvature formula (see [4, Section 4.2]) or [16].

**Lemma 6.3** (Schumacher [18, Proposition 1]). If $e^\phi = \det \phi$, then

$$(\Box + 1)c(\phi)_{\alpha\bar{\beta}} = (\mu_\alpha, \mu_\beta),$$

where $\Box := -\phi^{ij} \frac{\partial^2}{\partial v^i \partial \bar{v}^j}$, $\mu_\alpha = \delta^{V} \frac{\delta}{\delta z^\alpha}$, $(\cdot, \cdot)$ denotes the point-wise inner product.

**Proof.** By direct computation, one has

$$\phi^{\bar{\beta}} \frac{\partial^2}{\partial \bar{v}^i \partial \bar{v}^j} c(\phi)_{\alpha\bar{\beta}} = (\partial\bar{\partial}\log \det \phi)\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta z^\beta}\right) - (\mu_\alpha)'^i (\mu_\beta)'^j \phi^i_\alpha \phi^j_\beta,$$

(6.7)

where $(\mu_\alpha)'^j = -\partial_j (\phi_{\alpha k} \phi^{k\bar{l}})$. By condition $e^\phi = \det \phi$, one has

$$-\Box c(\phi)_{\alpha\bar{\beta}} = c(\phi)_{\alpha\bar{\beta}} - (\mu_\alpha, \mu_\beta),$$

(6.8)

which completes the proof. □

Now we denote by $X$ the Teichmüller curve over Teichmüller space $M = \mathcal{T}$, $L = K_{\mathcal{X}/M}$, then $E = p_* (K_{\mathcal{X}/\mathcal{T}}^\otimes)$, which is the dual bundle of $T\mathcal{T}$, and the dual metric of $L^2$-metric (6.4) is exactly the Weil-Petersson metric. In fact,

$$\|u\|^2 = \int_{X_z} |u|^2 e^{-\phi} = \int_{X_z} |f|^2 e^{-\phi} \sqrt{-1} dv \wedge d\bar{v}$$

$$= \int_{X_z} |f|^2 \phi^{-2} (\phi e\sqrt{-1} dv \wedge d\bar{v}) = \int_{X_z} |u|^2 \omega_z,$$

where $\omega = \partial \bar{\partial} \phi$, $\omega_z = \omega|_{p^{-1}(z)}$.

**Lemma 6.4.** For any $\alpha \in A^0^1(X_z, K_{X_z})$, then

$$\Box \left(\frac{\alpha}{\omega'}\right) = \frac{1}{\omega'} \Box' \alpha.$$

Here $\omega' = \phi e \bar{dv} \otimes dv \in A^0^1(X_z, K_{X_z})$. 

Proof. We assume that \( \alpha = f d\bar{v} \otimes dv \), by noting that \(-\sqrt{-1}\nabla^v = [\Lambda, \bar{\partial}]\), then

\[
\frac{1}{\omega'} \square' \alpha = \frac{1}{\omega'} \nabla^v \nabla' \alpha \\
= \frac{1}{\omega'} \nabla^v ( (\partial_v f - f \partial_v \log \phi_{\bar{v}v}) dv \wedge d\bar{v} \otimes dv) \\
= -\frac{1}{\omega'} \bar{\partial}( \frac{1}{\phi_{\bar{v}v}} (\partial_v f - f \partial_v \log \phi_{\bar{v}v})) dv \\
= -\frac{1}{\phi_{\bar{v}v}} \partial_v ( \frac{1}{\phi_{\bar{v}v}} (\partial_v f - f \partial_v \log \phi_{\bar{v}v})) \\
= -\frac{1}{\phi_{\bar{v}v}} \partial_v \partial_v ( \frac{f}{\phi_{\bar{v}v}} ) = \square (\frac{\alpha}{\omega'}). \\
\]

Thus

\[
\langle (1 + \square')^{-1} i_{\mu_\alpha} u, i_{\mu_\beta} u \rangle = \langle \omega'(1 + \square)^{-1}(\omega' - 1) i_{\mu_\alpha} u, i_{\mu_\beta} u \rangle \\
= \int_{X_z} (1 + \square)^{-1}(\omega' - 1) i_{\mu_\alpha} u \cdot (\omega' - 1) i_{\mu_\beta} u \omega_z. \\
\tag{6.9}
\]

From Theorem 6.2, one has

\[
\langle \Theta^E_{\alpha \beta} u, u \rangle = \int_{X_z} c(\phi)_{\alpha \beta} |u|^2 e^{-\phi} + \langle (1 + \square')^{-1} i_{\beta'} \frac{1}{\omega'} u, i_{\beta'} \frac{1}{\omega'} u \rangle \\
= \int_{X_z} (1 + \square)^{-1}(\mu_\alpha \cdot \mu_\beta) |u|^2 + (1 + \square)^{-1}(\omega' - 1) i_{\mu_\alpha} u \cdot (\omega' - 1) i_{\mu_\beta} u \omega_z.
\]

Note that \( \{\mu_\alpha \}_{1 \leq \alpha \leq 3g-3} \in \mathbb{H}^{0,1}(X_z, K_{X_z}^{-1}) \) are harmonic, then

Lemma 6.5. \( \{u^\alpha := h^{\alpha \beta} i_{\beta'} \} \) is a basis of \( E = T^* \mathcal{T} \). Here

\[
h^{\alpha \bar{\beta}} = \int_{X_z} \mu_\alpha \cdot \mu_{\bar{\beta}} \omega_z = \int_{X_z} (i_{\mu_\alpha} \omega', i_{\mu_{\bar{\beta}}} \omega') \omega_z = \omega_z,
\]

and \( (h^{\alpha \bar{\beta}}) \) is the inverse matrix of \( (h_{\alpha \beta}) \).

Proof. We need to prove \( \bar{\partial} u^\alpha = 0 \) along each fiber. Note that

\[
u^\alpha = -h^{\alpha \bar{\beta}} \partial_v (\phi_{\bar{\beta}'v} \phi^{-1}_{\bar{v}v}) \phi_{\bar{v}v} dv^2.
\]

By taking \( \bar{\partial} \), one has

\[
\partial_v (\partial_v (\phi_{\alpha v} \phi^{-1}_{\bar{v}v}) \phi_{\bar{v}v}) = \partial_v \partial_v (\phi_{\alpha v}) - \partial_v (\phi_{\alpha v} \phi_{\bar{v}v}) \\
= \partial_v \partial_v e^{\phi} - e^{\phi} \phi_{\alpha v} \phi_{\bar{v}v} - e^{\phi} \phi_{\bar{v}v} = 0,
\]

which completes the proof. \( \square \)

Note that

\[
i_{\mu_\alpha} i_{\mu_\beta} \omega' = (\mu_\alpha \cdot \mu_\beta) \omega'.
\]
Thus
\[ R^\gamma_\alpha^\delta_\beta := \langle \Theta^{E \beta}_{\alpha \delta} u^\gamma, u^\delta \rangle \]
\[ = \int_{X_z} \left( (1 + \Box)^{-1}(\mu_\alpha \cdot \mu_\beta)(u^\gamma, u^\delta) \omega_z + (1 + \Box)^{-1}(\omega^{i \mu_\alpha} u^\gamma) \cdot (\omega^{i \mu_\beta} u^\delta) \right) \omega_z \]
\[ = h^\gamma_\alpha^\delta_\beta h^\delta_\tau^\sigma \int_{X_z} \left( (1 + \Box)^{-1}(\mu_\alpha \cdot \mu_\beta)(\mu_\tau \cdot \mu_\sigma) + (1 + \Box)^{-1}(\mu_\alpha \cdot \mu_\sigma) \cdot (\mu_\tau \cdot \mu_\beta) \right) \omega_z. \]

On the other hand, \( \langle u^\alpha, u^\delta \rangle = h^\alpha_\delta \), so the curvature of Weil-Petersson metric is
\[ R^\tau_\sigma^\alpha_\beta = -R^\gamma_\alpha^\delta_\beta h^\gamma_\tau^\sigma h^\delta_\beta \]
\[ = -\int_{X_z} \left( (1 + \Box)^{-1}(\mu_\alpha \cdot \mu_\beta)(\mu_\tau \cdot \mu_\sigma) + (1 + \Box)^{-1}(\mu_\alpha \cdot \mu_\sigma) \cdot (\mu_\tau \cdot \mu_\beta) \right) \omega_z. \]

**References**

[1] T. Aikou, *Finsler geometry on complex vector bundles*, A sampler of Riemann-Finsler geometry, 83-105, Math. Sci. Res. Inst. Publ., 50, Cambridge Univ. Press, Cambridge, 2004.

[2] L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), No. 2, 94-97.

[3] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) **169** (2009), 531-560.

[4] B. Berndtsson, *Strict and non strict positivity of direct image bundles*, Math. Z. **269** (3-4), (2011), 1201-1218.

[5] M. Bridgeman and E. Taylor, *An extension of the Weil-Petersson metric to quasi-Fuchsian space*, Math. Ann. **341** (2008), 927-943.

[6] M. Bridgeman, *Hausdorff dimension and the Weil-Petersson extension to quasi-Fuchsian space*, Geometry and Topology **14** (2010), 799-831.

[7] M. Bridgeman and Y. Wu, *Uniform bounds on harmonic Beltrami differentials and Weil-Petersson curvatures*, J. Reine Angew. Math. **770** (2021), 159-181.

[8] J. Cao, P. Wong, *Finsler Geometry of Projectivized Vector Bundles*, Journal of Mathematics of Kyoto University **43** (2003), no. 2, 369-410.

[9] D. Dumas, Complex projective structures, In Handbook of Teichmüller Theory, Volume II, Ed. Athanase Papadopoulos, EMS, 2009.

[10] C. Earle. On variation of projective structures. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Volume 97 of Ann. of Math. Stud., pages 87-99, Princeton, N.J., 1981. Princeton Univ. Press.

[11] D. Hejhal, *Monodromy groups and linearly polymorphic functions*, Acta Math., **135** (1) (1975), 1-55.

[12] J. Hubbard. The monodromy of projective structures. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Volume 97 of Ann. of Math. Stud., pages 257-275, Princeton, N.J., 1981. Princeton Univ. Press.

[13] I. Kim and G. Zhang, *Kähler metric on the space of convex real projective structures on surface*, J. Differential Geometry, **106** (2017), 127-137.

[14] F. Labourie, *Cyclic surfaces and Hitchin components in rank 2*, Ann. of Math. (2) **185** (2017), no. 1, 1-58.

[15] K. Liu, X. Sun, and S.-T. Yau, *Canonical metrics on the moduli space of Riemann surfaces*, I. II., J. Differential Geometry **68** (3) (2004), 571-637, **69** (1) (2005), 163-216.
Inkang Kim: School of Mathematics, KIAS, HEOGIRO 85, DONGDAEMUN-GU SEOUL, 02455, Republic of Korea
Email address: inkang@kias.re.kr

Xueyuan Wan: Mathematical Science Research Center, Chongqing University of Technology, Chongqing 400054, China
Email address: xwan@cqut.edu.cn

Genkai Zhang: Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-41296 Göteborg, Sweden
Email address: genkai@chalmers.se

[16] K. Liu, X. Sun, X. Yang, and S.-T. Yau, *Curvatures of moduli space of curves and applications*, Asian Journal of Mathematics 21, no. 5 (October 2017): 841-54.

[17] C. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, Ann. of Math. (2) 151 (2000), no. 1, 327-357.

[18] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), no. 1, 1-56.

[19] M. Wolf, *The Weil-Petersson Hessian of length on Teichmüller space*, J. Differential Geom. 91 (2012), no. 1, 129-169.