VIBRATIONS OF AN ELASTIC BAR, ISOSPECTRAL DEFORMATIONS, AND MODIFIED CAMASSA-HOLM EQUATIONS

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ABSTRACT. Vibrations of an elastic rod are described by a Sturm-Liouville system. We present a general discussion of isospectral (spectrum preserving) deformations of such a system. We interpret one family of such deformations in terms of a two-component modified Camassa-Holm equation (2-mCH) and solve completely its dynamics for the case of discrete measures (multipeakons). We show that the underlying system is Hamiltonian and prove its Liouville integrability. The present paper generalizes our previous work on interlacing multipeakons of the 2-mCH and multipeakons of the 1-mCH. We give a unified approach to both equations, emphasizing certain natural family of interpolation problems germane to the solution of the inverse problem for 2-mCH as well as to this type of a Sturm-Liouville system with singular coefficients.

CONTENTS

1. Introduction 2
2. The Lax formalism: the boundary value problem 4
3. Liouville integrability 7
  3.1. Bi-Hamiltonian structure 7
  3.2. Hamiltonian vector field 7
  3.3. Liouville integrability 8
4. Forward map: spectrum and spectral data 9
5. Inverse problem 11
  5.1. The first inverse problem and the interpolation problem 11
  5.2. The second inverse problem 17
6. Multipeakons for \( N = 2K \) 18
  6.1. Closed formulae for \( N = 2K \) 18
  6.2. Global existence for \( N = 2K \) 19
  6.3. Large time peakon asymptotics for \( n = 2K \) 20
7. Multipeakons for \( N = 2K + 1 \) 21
  7.1. Closed formulae for \( N = 2K + 1 \) 21
  7.2. Global existence for \( N = 2K + 1 \) 23
  7.3. Large time peakon asymptotics for \( N = 2K + 1 \) 24
8. Reductions of multipeakons 25
  8.1. From odd case to even case 25
  8.2. From the even case to the interlacing case 26
  8.3. From 2-mCH to 1-mCH 27
Acknowledgments 27
Appendix A. Lax pair for the 2-mCH peakon ODEs 28

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1. INTRODUCTION

One of the most important applications of the Sturm-Liouville systems is provided by the longitudinal vibrations of an elastic bar of stiffness \( p \) and density \( \rho \) [2, Chapter 10.3]. The longitudinal displacement \( v \) satisfies the wave equation

\[
\rho(x) \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left[ p(x) \frac{\partial v}{\partial x} \right],
\]

which after the separation of variables \( v = u(x) \cos \omega t \) leads to

\[
D_x [p(x)D_x u] + \omega^2 \rho(x) u = 0,
\]

where \( D_x = \frac{d}{dx} \). In a different area of applications, in geophysics, the Love waves were proposed by an early 20 century British geophysicist Augustus Edward Hugh Love who predicted the existence of horizontal surface waves causing Earth shifting during an earthquake. The wave amplitudes of these waves satisfy

\[
D_x (\mu D_x u) + (\omega^2 \rho - k^2 \mu) u = 0, \quad 0 < x < \infty,
\]

where \( \mu \) is the shear modulus, \( x \) is the depth below the Earth surface and the boundary conditions are \( D_x u(0) = u(\infty) = 0 \) which can be interpreted as the Neumann condition on one end and the Dirichlet condition on the other. In applications to geophysics the frequency \( \omega \) is fixed and the phase velocity is \( \omega/k \). In particular, in the infinity speed limit \( (k = 0) \), we obtain the same Sturm-Liouville system as in (1.2). In either case the problem can conveniently be written as the first order system:

\[
D_x \Phi = \begin{bmatrix} 0 & n \\ -\omega^2 \rho & 0 \end{bmatrix} \Phi,
\]

where \( n = \frac{1}{\mu} \), \( \Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \), \( \phi_1 = \phi, \phi_2 = \mu D_x \phi \). In the present paper we study (1.4) on the whole real axis and impose the boundary conditions \( \phi_1(\infty) = \phi_2(\infty) = 0 \) which can be interpreted as the Dirichlet condition at \( -\infty \) and the Neumann condition at \( +\infty \). Our motivation to study this problem comes from yet another area of applied mathematics dealing with integrable nonlinear partial differential equations. To focus our discussion we begin by considering the nonlinear partial differential equation

\[
m_t + (u^2 - u_x^2) m = 0, \quad m = u - u_{xx},
\]

which is one of many variants of the famous Camassa-Holm equation (CH) [3]:

\[
m_t + u m_x + 2 u_x m = 0, \quad m = u - u_{xx},
\]

for the shallow water waves. We will call (1.5) the \( m \)CH equation for short. The history of the \( m \)CH equation is long and convoluted: (1.5) appeared in the papers of Fokas [7], Fuchssteiner [8], Olver and Rosenau [15] and was, later, rediscovered by Qiao [16, 17].
Subsequently, in [19], Song, Qu and Qiao proposed a natural two-component general-
ization of (1.5)

\[
\begin{align*}
\frac{\partial m}{\partial t} + [(u - u_x)(\nu + \nu_x)m]_x &= 0, \\
\frac{\partial n}{\partial t} + [(u - u_x)(\nu + \nu_x)n]_x &= 0, \\
m &= u - u_{xx}, \\
n &= \nu - \nu_{xx},
\end{align*}
\]

which, for simplicity, we shall call the 2-mCH. Formally, the 2-mCH reduces to the mCH
when \( \nu = u \).

We are interested in the class of non-smooth solutions of (1.7) given by the peakon
ansatz [3, 4, 11]:

\[
\begin{align*}
u &= \sum_{j=1}^{N} n_j(t) e^{-|x-x_j(t)|}, \\
\nu &= \sum_{j=1}^{N} n_j(t) e^{-|x-x_j(t)|},
\end{align*}
\]

where all smooth coefficients \( m_j(t) \), \( n_j(t) \) are taken to be positive, and hence

\[
\begin{align*}
m &= u - u_{xx} = 2 \sum_{j=1}^{N} m_j \delta_{x_j}, \\
n &= \nu - \nu_{xx} = 2 \sum_{j=1}^{N} n_j \delta_{x_j}
\end{align*}
\]

are positive discrete measures.

For the above ansatz, (1.7) can be viewed as a distribution equation, requiring in
particular that we define the products \( Qm \) and \( Qn \), where

\[
Q = (u - u_x)(\nu + \nu_x).
\]

It is shown in Appendix A that the choice consistent with the Lax integrability discussed
in Section 2 is to take \( Qm, Qn \) to mean \( \langle Q \rangle m, \langle Q \rangle n \) respectively, where \( \langle f \rangle \) denotes the
average function (the arithmetic average of the right and left limits). Substituting (1.8)
into (1.7) and using the multiplication rule mentioned above leads to the system of ODEs:

\[
\begin{align*}
\dot{m}_j &= 0, \\
\dot{n}_j &= 0, \\
\dot{x}_j &= \langle Q \rangle(x_j).
\end{align*}
\]

In the present paper, we shall develop an inverse spectral approach to solve the peakon
ODEs (1.10) and hence (1.7) under the following assumptions:

1. all \( m_k, n_k \) are positive,
2. the initial positions are assumed to be ordered as \( x_1(0) < x_2(0) < \cdots < x_n(0) \).

We emphasize that the second condition is not restrictive since it can be realized by
relabeling positions as long as positions \( x_j(0) \) are distinct.

The present paper generalizes our previous work on interlacing multipeakons of the
2-mCH in [4] and multipeakons of the 1-mCH in [6]. It is worth mentioning, however,
that the technique of the present paper is a modification of the one employed in [6]
and is distinct from the inhomogeneous string approach adapted in [4]. As a result we
give a unified approach to both equations; this is accomplished by putting common
interpolation problems front and center of the solution to the inverse problem for (1.4).
Moreover, by solving (1.10), we furnish a family of isospectral flows for the Sturm-Liouville
system (1.4). The full explanation of the connection between (1.4) and (1.7) is reviewed
in the following sections.
2. THE LAX FORMALISM: THE BOUNDARY VALUE PROBLEM

The Lax pair for (1.7) can be written:

$$\frac{\partial}{\partial x} \Psi_x = \frac{1}{2} U \Psi, \quad \Psi_t = \frac{1}{2} V \Psi, \quad \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix},$$

where

$$U = \begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 4\lambda^{-2} + Q & -2\lambda^{-1}(u - u_x) - \lambda mQ \\ 2\lambda^{-1}(v + \nu) + \lambda nQ & -Q \end{pmatrix},$$

with $Q = (u - u_x)(v + \nu)$. This form of the Lax pair is a slight modification (in particular, $V$ have slightly different diagonal terms) of the original Lax pair in [19]. The modification is needed for consistency with the boundary value problem to be discussed below in Remark 2.3.

We recall that for smooth solutions the role of the Lax pair is to provide the Zero Curvature representation $\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + \frac{1}{2}(U, V) = 0$ of the original non-linear partial differential equation, in our case (1.7). In the non-smooth case the situation is more subtle as explained in Appendix A.

Following [4] we perform the gauge transformation $\Phi = \text{diag}(\frac{z^x}{x}, e^{-\frac{z}{x}}) \Psi$ which leads to a simpler $x$-equation

$$\Phi_x = \begin{bmatrix} 0 & h \\ -zg & 0 \end{bmatrix} \Phi, \quad g = \sum_{j=1}^{N} g_j \delta_{x_j}, \quad h = \sum_{j=1}^{N} h_j \delta_{x_j},$$

where $g_j = nj e^{-\lambda j}, h_j = mj e^{-\lambda j}, z = \lambda^2$, and thus $g_j h_j = mj n_j$. We note that (2.3), that is the $x$ member of the Lax pair, has the form of the Sturm-Liouville problem given by (1.4), provided we specify the boundary conditions. Our initial goal is to solve (2.3) subject to boundary conditions $\Phi_1(-\infty) = 0, \Phi_2(+\infty) = 0$ which are chosen in such a way as to remain invariant under the flow in $t$ whose infinitesimal change is generated by the matrix $V$ in the Lax equation.

Since the coefficients in the boundary value problem are distributions (measures), to make

$$\Phi_x = \begin{bmatrix} 0 & h \\ -zg & 0 \end{bmatrix} \Phi, \quad \Phi_1(-\infty) = \Phi_2(+\infty) = 0,$$

well posed, we need to define the multiplication of the measures $h$ and $g$ by $\Phi$. As suggested by the results in Appendix A we require that $\Phi$ be left continuous and we subsequently define the terms $\Phi_a \delta_{x_j} = \Phi_a(x_j) \delta_{x_j}, a = 1, 2$. This choice makes the Lax pair well defined as a distributional Lax pair and, as it is shown in the Appendix A, the compatibility condition of the $x$ and $t$ members of the Lax pair indeed implies (1.10). The latter result is more subtle than a routine check of compatibility for smooth Lax pairs.

Since the right-hand side of (2.4) is zero on the complement of the support of $g$ and $h$, which in our case consists of points $\{x_1, \ldots, x_N\}$, the solution $\Phi$ is a piecewise constant function, which solves a finite difference equation.

Lemma 2.1. Let $q_k = \Phi_1(x_k+), \ p_k = \Phi_2(x_k+)$, then the difference form of the boundary value problem (2.4) reads:

$$\begin{bmatrix} q_k \\ p_k \end{bmatrix} = T_k \begin{bmatrix} q_{k-1} \\ p_{k-1} \end{bmatrix}, \quad T_k = \begin{bmatrix} 1 & h_k \\ -zg & 1 \end{bmatrix}, \quad 1 \leq k \leq N,$$

where $q_0 = 0, p_0 = 1$, and the boundary condition on the right end (see (2.4)) is satisfied whenever $p_N(z) = 0$. 


By inspection we obtain the following corollary.

**Corollary 2.2.** \( q_k(z) \) is a polynomial of degree \( \lfloor k/2 \rfloor \) in \( z \), and \( p_k(z) \) is a polynomial of degree \( \lfloor k/2 \rfloor \), respectively.

**Remark 2.3.** Note that \( \det(T_k) = 1 + z h_k g_k = 1 + z m_k n_k \neq 1 \). In other words the setup we are developing goes beyond an \( SL_2 \) theory. In order to understand the origin of this difference we go back to the original Lax pair (2.2). If we assumed the matrix \( V \) to be traceless, with some coefficient \( \alpha(\lambda) \) on the diagonal, then in the asymptotic region \( x \gg 0 \) the second equation in (2.1) would read

\[
\frac{\dot{a} e^{-x/2}}{b e^{x/2}} = \frac{1}{2} \begin{bmatrix}
\alpha(\lambda) & -4 \lambda^{-1} u e^{-x} \\
0 & -\alpha(\lambda)
\end{bmatrix} \begin{bmatrix}
a e^{-x/2} \\
b e^{x/2}
\end{bmatrix},
\]

where \( u(x) = u_+ e^{-x} \) in the asymptotic region. The simplest way to implement the isospectrality is to require that \( \dot{b} = 0 \), which requires gauging away \(-\alpha(\lambda)\). This is justified on general grounds by observing that for any Lax equation \( \dot{L} = [B, L] \), \( B \) is not uniquely defined. In particular, any term commuting with \( L \) can be added to \( B \) without changing Lax equations. In our case, we are adding a multiple of the identity to the original formulation in [19]. Furthermore, the gauge transformation leading up to (2.3) is not unimodular which takes us outside of the \( SL_2 \) theory.

The polynomials \( p_k, q_k \) can be constructed integrating directly the initial value problem

\[
\Phi_x = \begin{bmatrix}
0 & h \\
-zg & 0
\end{bmatrix} \Phi, \quad \Phi_1(-\infty) = 0, \quad \Phi_2(-\infty) = 1,
\] (2.6)

with the same rule regarding the multiplication of discrete measures \( g, h \) by piecewise smooth left-continuous functions as specified for (2.4).

With this convention in place we obtain the following characterization of \( \Phi_1(x) \) and \( \Phi_2(x) \), proven in its entirety in [6, Lemma 2.4].

**Lemma 2.4.** Let us set

\[
\Phi_1(x) = \sum_{0 \leq k} \Phi_1^{(k)}(x) z^k, \quad \Phi_2(x) = \sum_{0 \leq k} \Phi_2^{(k)}(x) z^k.
\]

Then

\[
\Phi_1^{(0)}(x) = \int_{\eta_0 < x} h(\eta_0) \, d\eta_0, \quad \Phi_2^{(0)}(x) = 1
\] for \( k = 0 \), otherwise

\[
\Phi_1^{(k)}(x) = (-1)^k \int_{\eta_0 < \xi_1 < \eta_1 < \cdots < \xi_k < \eta_k < x} \left[ \prod_{p=1}^k h(\eta_p) g(\xi_p) \right] h(\eta_0) \, d\eta_0 \, d\xi_1 \ldots d\eta_k, \quad (2.7a)
\]

\[
\Phi_2^{(k)}(x) = (-1)^k \int_{\xi_1 < \eta_1 < \cdots < \xi_k < \eta_k < x} \left[ \prod_{p=1}^k g(\eta_p) h(\xi_p) \right] \, d\xi_1 \ldots d\eta_k. \quad (2.7b)
\]
If the points of the support of the discrete measure $g$ (and $h$) are ordered $x_1 < x_2 < \cdots < x_N$ then

\[
\Phi_1^{(k)}(x) = (-1)^k \sum_{\substack{I \subseteq [k] \cap [x_1, x_2] \ni |I| = k \\text{ and } I < J}} \left[ \prod_{p=1}^{k} h_{j_p} g_{i_p} \right] h_{j_k}, \tag{2.8a}
\]

\[
\Phi_2^{(k)}(x) = (-1)^k \sum_{\substack{I \subseteq [k] \cap [x_1, x_2] \ni |I| = k \\text{ and } I < J}} \left[ \prod_{p=1}^{k} g_{j_p} h_{i_p} \right]. \tag{2.8b}
\]

To simplify the formulas in Lemma 2.4 we introduce the following notation. Our basic set of indices is $\{1, 2, \ldots, N\}$ which we denote by $[N]$ and if $k \leq N$ we set $[k] = \{1, 2, \ldots, k\}$. We will denote by capital letters $I$ and $J$ any subsets of these sets and use the notation $\binom{[k]}{J}$ for the set of all $J$-element subsets of $[k]$, listed in increasing order; for example $I \in \binom{[k]}{J}$ means that $I = \{i_1, i_2, \ldots, i_J\}$ for some increasing sequence $i_1 < i_2 < \cdots < i_J \leq k$. Furthermore, given a multi-index $I = \{i_1, i_2, \ldots, i_J\}$ and a set of numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_J}$ indexed by $I$, we will abbreviate $a_I = a_{i_1} a_{i_2} \cdots a_{i_J}$ etc.

**Definition 2.5.** Let $I, J \in \binom{[k]}{1}$, or $I \in \binom{[k]}{I+1}, J \in \binom{[k]}{J}$.

Then $I, J$ are said to be *interlacing*, denoted $I < J$, if

\[ i_1 < j_1 < i_2 < j_2 < \cdots < i_J < j_J \]

or,

\[ i_1 < j_1 < i_J < j_J < \cdots < i_J < j_{J+1}, \]

in the latter case. The same notation is used in the degenerate case $I \in \binom{[k]}{0}, J \in \binom{[k]}{0}$.

Using this notation we can now express the results of Lemma 2.4 in a compact form.

**Corollary 2.6.** The unique solutions $q_k$ and $p_k$ to the recurrence equations (2.5) with initial conditions $q_0 = 0, p_0 = 1$ are given by

\[
q_k(z) = \sum_{\substack{I \subseteq [k] \cap [x_1, x_2] \ni |I| = k \\text{ and } I \subseteq J}} \left[ \prod_{l=0}^{k} h_{j_l} g_{i_l} \right] (-z)^I, \tag{2.9a}
\]

\[
p_k(z) = 1 + \sum_{\substack{I \subseteq [k] \cap [x_1, x_2] \ni |I| = k \\text{ and } I \subseteq J}} \left[ \prod_{l=1}^{k} h_{j_l} g_{i_l} \right] (-z)^I. \tag{2.9b}
\]

We can now make a brief comment about the spectrum of the boundary value problem (2.4). We observe that a complex number $z$ is an *eigenvalue* of the boundary value problem (2.4) if there exists a solution $(q_k(z), p_k(z))$ to (2.5) for which $p_N(z) = 0$. The set of all eigenvalues comprises the *spectrum* of the boundary value problem (2.4). Our choice of boundary conditions was picked to ensure the invariance of the spectrum under the time evolution. To verify that the flow is isospectral (spectrum preserving) we examine the $t$ part of the Lax pair (2.1) in the region $x > N$, as indicated in Remark 2.3 and perform the gauge transformation to determine the flow of $\Phi$.

**Lemma 2.7.** Let $(q_k, p_k)$ satisfy the system of difference equations (2.5). Then the Lax equations (2.1) imply

\[
\dot{q}_N = \frac{2}{z} q_N - \frac{2u_+}{z} p_N, \quad \dot{p}_N = 0, \tag{2.10}
\]
where \( u_+ = \sum_{j=1}^N h_j \).

This lemma implies that the polynomial \( p_N(z) \) is independent of time and, in particular, its zeros, i.e. the spectrum, are time invariant. Furthermore, Corollary 2.6 allows one to write the coefficients of \( p_N(z) \) in terms of the variables \( g_j, h_j \) (or equivalently \( m_j, n_j, x_j \)) and thus identify \( \left\lfloor \frac{N}{2} \right\rfloor \) constants of motion of the system (1.10):

\[
M_j = \sum_{I, J \in \binom{\{N\}}{2}} \delta_{IJ} \sum_{i < j} h_ig_j, \quad 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor. 
\]  

(2.11)

In the next section we will investigate the role of these constants in the integrability of (1.10).

3. LIOUVILLE INTEGRABILITY

3.1. Bi-Hamiltonian structure. The results of the previous section, especially the existence of \( \left\lfloor \frac{N}{2} \right\rfloor \) constants, suggests that the system (1.10) might be integrable in a classical Liouville sense which is proven below. For smooth solutions \( u(x, t), v(x, t) \) of (1.7) the Hamiltonian structure, in fact a bi-Hamiltonian one, of the 2-mCH equation (1.7) was given by Tian and Liu in [21]. By employing two compatible Hamiltonian operators

\[
\mathcal{L}_1 = \begin{pmatrix} D_xmD_x^{-1}mD_x & D_xmD_x^{-1}nD_x \\ D_xnD_x^{-1}mD_x & D_xnD_x^{-1}nD_x \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} 0 & -D_x^2 - D_x \\ D_x^2 - D_x & 0 \end{pmatrix}
\]

and the Hamiltonians

\[
H_1 = \int n(u - u_+ dx), \quad H_2 = \frac{1}{2} \int n(v + v_+)(u - u_+)^2 dx,
\]  

(3.1)

the 2-mCH equation (1.7) can be written as

\[
\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} = \mathcal{L}_1 \begin{pmatrix} \delta H_1/\delta m \\ \delta H_1/\delta n \end{pmatrix} = \mathcal{L}_2 \begin{pmatrix} \delta H_2/\delta m \\ \delta H_2/\delta n \end{pmatrix}. 
\]

(3.2)

We note that the word compatible mentioned above means that an arbitrary linear combination of the two Hamiltonian operators is also Hamiltonian. Since we work in the non-smooth context the results obtained for smooth functions will not hold in the non-smooth region, and one either has to formulate a limiting procedure leading to the non-smooth sector or study the non-smooth sector independently. At present, we prefer the second approach mainly because it is technically simpler, and also because it is not clear at this point which Hamiltonian structures have meaningful limits.

3.2. Hamiltonian vector field. We focus on the peakon sector of (1.7) described by the system of equations (1.10).

**Theorem 3.1.** The equations (1.10) for the motion of \( N \) peakons of the original PDE (1.7) are given by Hamilton’s equations of motion:

\[
\dot{x}_j = \{x_j, H\}, \quad \dot{m}_j = \{m_j, H\}, \quad \dot{n}_j = \{n_j, H\},
\]  

(3.3)

for the Hamiltonian

\[
H = -\frac{1}{2} \int n(\xi)(u(\xi) - u(\xi))d\xi = 2M_1 + \sum_{k=1}^N m_k n_k.
\]
Here $M_1$ is a constant of motion appearing in (2.11), the Poisson bracket $[,]$ is given by
\begin{align}
\{x_i, x_k\} &= \text{sgn}(x_i - x_k), \\
\{m_i, m_k\} &= \{m_i, x_k\} = \{n_i, n_k\} = \{n_i, x_k\} = \{n_i, m_k\} = 0,
\end{align}
and the ordering condition $x_1 < x_2 < \cdots < x_N$ is in place.

\textbf{Proof.} Clearly,
\[\{m_j, h\} = \{n_j, h\} = 0\]
under the above Poisson bracket, hence
\[\dot{m}_j = \dot{n}_j = 0.\]
We proceed with the computation of $\{x_j, H\}$:
\begin{align*}
\{x_j, H\} &= \left\{x_j, 2 \sum_{1 \leq k \leq N} m_i n_k e^{x_i - x_k} + \sum_{k=1}^{N} m_k n_k \right\} \\
&= 2 \sum_{1 \leq k \leq N} m_i n_k \{x_j, e^{x_i - x_k}\} \\
&= 2 \sum_{1 \leq k \leq N} m_i n_k e^{x_i - x_k} \left(\text{sgn}(x_j - x_i) - \text{sgn}(x_j - x_k)\right) \\
&= 2 \sum_{k=1}^{j-1} m_k n_j e^{x_i - x_j} + 2 \sum_{k=j+1}^{N} m_j n_k e^{x_j - x_k} + 4 \sum_{1 \leq i < j \leq N} m_i n_k e^{x_i - x_k} \\
&= (1.9) \quad \{Q_i\} \{x_j\},
\end{align*}
thus proving the results. \hfill \Box

3.3. Liouville integrability. We will introduce a natural Poisson manifold $(M, \pi)$ defined by the Poisson bracket (3.4). Since $m_j, n_j$ are constant we can restrict our considerations to the non-trivial part of the Poisson structure involving only $x_j$. Let us denote
\[M = \{x_1 < x_2 < \cdots < x_N\}\]
and define
\[\pi(f, g) = \{f, g\} = \sum_{1 \leq i < j \leq N} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\]
for all differentiable functions $f, g$ on $M$. Then $M$ has a structure of a Poisson manifold $M$ to be denoted $(M, \pi)$.

One can check directly from (3.4) that regardless whether $N = 2K$ or $N = 2K + 1$

\textbf{Lemma 3.2.} $\text{rank}(\pi) = 2K$.

Our objective now is to identify an appropriate number of Poisson commuting quantities. We will break down our analysis according to whether $N$ is even or odd.

1. Case $N = 2K$. It follows from Lemma 2.7 and (2.11) that the quantities
\[M_j = \sum_{l, j \in \{0, 2K\}} h_l g_j, \quad 1 \leq j \leq K,
\]
with $h_i = m_i e^{x_i}$, $g_l = n_l e^{-x_l}$, form a set of $K$ constants of motion for the system (1.10). We claim that these constants of motion Poisson commute. Short of giving a detailed proof, we would like to outline the argument which goes back to J. Moser in [14]. Since $M_j$ commute with the Hamiltonian $H$ (see Theorem 3.1) their Poisson bracket $\{M_j, M_k\}$ commutes with $H$ and thus $\{M_j, M_k\}$ is a constant
of motion for every pair of indices \( j, k \). For cases for which the inverse spectral methods allow one to express \( M_j \) in terms of leading asymptotic positions, in particular exploiting the asymptotic result that particles corresponding to adjacent positions \( x_j, x_{j+1} \) pair up, while distinct pairs do not interact, leads to a suppression of the majority of terms in \( M_j \). The precise argument is presented in [5, Theorem 3.8] while needed asymptotic results can be found in Theorem 6.6.

**Theorem 3.3.** The Hamiltonians \( M_1, \cdots, M_K \) Poisson commute.

2. **Case** \( N = 2K + 1 \). Again, following Lemma 2.7 and (2.11) we see

\[
M_j = \sum_{I,J \in [2K+1]} h_{IJ} g_{IJ}, \quad 1 \leq j \leq K,
\]

with \( h_i = m_i e^{x_i}, \quad g_i = n_i e^{-x_i} \), are constants of motion for the system (1.10) in the odd case.

In the odd case, there is an extra constant of motion, which can be computed from the value of the Weyl function \( W(z) \) at \( z = \infty \) (see Section 2 and Section 4 for details regarding the Weyl function). We point out that this constant is 0 in the even case. The computation is routine and produces

\[
c = \frac{\sum_{I \in [2K+1]} h_{II} g_{II}}{M_K}, \quad \text{with} \quad M_K = \sum_{I,J \in [2K+1]} h_{IJ} g_{IJ}.
\]

which, in turn, gives an extra constant of motion

\[
M_c = \sum_{I \in [2K+1]} h_{II} g_{II} = \prod_{j=1}^{K+1} m_{2j-1} e^{x_j} \prod_{j=1}^{K} n_{2j} e^{-x_j},
\]

so that \( \{M_1, M_2, \cdots, M_K, M_c\} \) form a set of \( K+1 \) constants of motion for the system (1.10) in this case.

It is not hard to see by using the same argument as in [5, Theorem 3.9] and the asymptotic results in Theorem 7.7, that the following theorem holds.

**Theorem 3.4.** The Hamiltonians \( M_1, \cdots, M_K, M_c \) Poisson commute.

Combining now both theorems above we conclude (the proof is similar to the argument in [5, Theorem 3.10]).

**Theorem 3.5.** The conservative peakon system given by (1.10) is Liouville integrable.

4. **Forward map: spectrum and spectral data**

The spectrum of the boundary value problem (2.4) (or equivalently, (2.5)) is given by the zeros of the polynomial \( p_N(z) \). However, one cannot recover the measures \( g \) and \( h \) from the spectrum alone. One needs extra data and the right object to turn to is the Weyl function

\[
W(z) = \frac{q_N(z)}{p_N(z)},
\]

which in our case is a rational function with poles located at the spectrum of the boundary value problem. Another compelling reason for using the Weyl function is that, as we will
show below, the residues of $W$ evolve linearly in time, while the value of $W$ at $z = \infty$ is a constant of motion. The investigation of the analytic properties of $W$ can be greatly simplified by observing that $W$ is built out of solutions to the recurrence (2.5). This suggests forming a recurrence of Weyl functions whose solution at step $N$ is $W(z)$. This leads to the following result which is an immediate consequence of (2.5).

**Lemma 4.1.** Let $(q_k, p_k)$ be the solution to (2.5) and let $w_{2k} = \frac{q_k}{p_k}$, $w_{2k-1} = \frac{q_{k-1}}{p_{k-1}}$. Then

$$w_1 = 0, \quad w_{2k} = (1 + zm_k n_k)w_{2k-1} + h_k, \quad 1 \leq k \leq N, \quad (4.2a)$$

$$\frac{1}{w_{2k}} = \frac{1}{w_{2k+1}} + zg_k, \quad 1 \leq k \leq N - 1. \quad (4.2b)$$

We will now show that all these Weyl functions, including the original $W(z)$, have the following properties in common:

1. they all have simple poles located on $\mathbb{R}_+$;
2. all the residues are positive;
3. the values at $z = \infty$ are non-negative.

The rational functions of this type have been studied, as a special case, in the famous memoir by T. Stieltjes [20]. The most relevant for our studies is the following theorem which is a special case of a more general theorem proved by Stieltjes.

**Theorem 4.2** (T. Stieltjes). Any rational function $F(z)$ admitting the integral representation

$$F(z) = c + \int \frac{d\nu(x)}{x - z}, \quad (4.3)$$

where $d\nu(x)$ is the (Stieltjes) measure corresponding to the piecewise constant non-decreasing function $\nu(x)$ with finitely many jumps in $\mathbb{R}_+$ has a finite (terminating) continued fraction expansion

$$F(z) = c + \frac{1}{a_1(-z) + \frac{1}{a_2 + \frac{1}{a_3(-z) + \ddots}}}, \quad (4.4)$$

where all $a_j > 0$ and, conversely, any rational function with this type of a continued fraction expansion has the integral representation (4.3).

We now apply Stieltjes’ result to our case.

**Lemma 4.3.** Given $h_j, g_j > 0, h_j g_j = m_j n_j > 0, 1 \leq j \leq N$, let $w_j$’s satisfy the recurrence relations of Lemma 4.1. Then $w_j$’s are shifted Stieltjes transforms of finite, discrete Stieltjes measures supported on $\mathbb{R}_+$, with nonnegative shifts. More precisely:

$$w_{2k-1}(z) = \int \frac{d\mu^{(2k-1)}(x)}{x - z},$$

$$w_{2k}(z) = c_{2k} + \int \frac{d\mu^{(2k)}(x)}{x - z},$$

where $c_{2k} > 0$ when $k$ is odd, otherwise, $c_{2k} = 0$. Furthermore, the number of points in the support $d\mu^{(2k)}(x)$ and $d\mu^{(2k-1)}$ is $\lfloor \frac{k}{2} \rfloor$. 

Proof. We only sketch the proof, for further details we refer to [6, Lemma 3.6]. The proof goes by induction on \( k \). The base case \( k = 1 \) is elementary. Assuming the induction hypothesis to hold up to \( 2k \), we invert (4.2b) to get:

\[
    w_{2k+1}(z) = \frac{1}{-zg_{k+1} + \frac{1}{w_k}}
\]

which, by induction hypothesis, implies that \( w_{2k+1} \) has the required continued fraction expansion covered by Stieltjes’ theorem, and thus has the required integral representation. We subsequently feed this integral representation into (4.2a) to obtain the Stieltjes integral representation for \( w_{2k+2} \). The analysis of the signs of the values of the Weyl functions at \( z = \infty \) is carried out in [6, Lemma 3.6].

\[\square\]

Remark 4.4. The recurrence in Lemma 4.1 can be viewed as the recurrence on the Weyl functions corresponding to shorter bars (keeping in mind the interpretation in terms of the longitudinal vibrations of an elastic bar) obtained by truncating at the index \( k \). Then \( W_k \) is precisely the Weyl function corresponding to the measures \( \sum_{j=1}^k h_j \delta_{x_j} \) and \( \sum_{j=1}^k g_j \delta_{x_j} \), while \( W_{2k-1} \) corresponds to the measures \( \sum_{j=1}^{k-1} h_j \delta_{x_j} \) and \( \sum_{j=1}^k g_j \delta_{x_j} \) respectively.

Now, in particular, we note that by Lemma 4.3

\[
    W(z) = \frac{q_N(z)}{p_N(z)} = c_{2N} + \int \frac{d\mu^{(2N)}(x)}{x-z}, \quad d\mu^{(2N)} = \sum_{j=1}^{[\frac{N}{2}]} b_j^{(2N)} \delta_{\zeta_j}
\]

and thus the following theorem holds.

Theorem 4.5. \( W(z) \) is a (shifted) Stieltjes transform of a positive, discrete measure \( d\mu \) with support inside \( \mathbb{R}_+ \). More precisely:

\[
    W(z) = c + \int \frac{d\mu(x)}{x-z}, \quad \mu = \sum_{j=1}^{[\frac{N}{2}]} b_j \delta_{\zeta_j}, \quad 0 < \zeta_1 < \cdots < \zeta_{[\frac{N}{2}]}, \quad 0 < b_j, \quad 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor,
\]

where \( c > 0 \) when \( N \) is odd and \( c = 0 \) when \( N \) is even.

The next corollary summarizes the properties of the spectrum of the boundary value problem (2.4), or equivalently (2.5).

Corollary 4.6.

1. The spectrum of the boundary value problem (2.3) is positive and simple.
2. \( W(z) = c + \sum_{j=1}^{[\frac{N}{2}]} \frac{b_j}{\zeta_j-z} \), where all residues satisfy \( b_j > 0 \) and \( c \geq 0 \).

5. INVERSE PROBLEM

5.1. The first inverse problem and the interpolation problem. The initial inverse problem we are interested in solving can be stated as follows:

Definition 5.1. Given a rational function (see Theorem 4.5)

\[
    W(z) = c + \int \frac{d\mu(x)}{x-z}, \quad \mu = \sum_{j=1}^{[\frac{N}{2}]} b_j \delta_{\zeta_j}, \quad 0 < \zeta_1 < \cdots < \zeta_{[\frac{N}{2}]}, \quad 0 < b_j, \quad 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor,
\]

where \( c > 0 \) when \( N \) is odd and \( c = 0 \) when \( N \) even, as well as positive constants \( m_1, m_2, \ldots, m_N, n_1, n_2, \ldots, n_N \), such that the products \( m_j n_j \) are distinct, find positive constants \( g_j, h_j, 1 \leq j \leq N \), for which
(1) \( g_j h_j = m_j n_j \),

(2) the unique solution of the initial value problem:

\[
\begin{bmatrix}
q_k \\
p_k
\end{bmatrix} = \begin{bmatrix}
1 & h_k \\
-zg_k & 1
\end{bmatrix} \begin{bmatrix}
q_{k-1} \\
p_{k-1}
\end{bmatrix}, \quad 1 \leq k \leq N,
\]

\[
\begin{bmatrix}
q_0 \\
p_0
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix},
\]

satisfies

\[
\frac{q_N(z)}{p_N(z)} = W(z).
\]

**Remark 5.2.** The restriction that the products \( m_j n_j \) be distinct has been made to facilitate the argument and will be eventually relaxed by taking appropriate limits of the generic case (see the comments below Theorem 5.13).

The basic idea of the solution of the above inverse problem is to associate to it a certain interpolation problem. We now proceed to explain how such an interpolation problem appears already in the solution of the forward problem, that is, in the solution of the difference boundary value problem (2.5), or equivalently (5.2) above. First, let us denote by

\[
T_j(z) = \begin{bmatrix}
1 & h_j \\
-zg_j & 1
\end{bmatrix}, \quad (5.3)
\]

Clearly,

\[
\begin{bmatrix}
q_N(z) \\
p_N(z)
\end{bmatrix} = T_N(z) T_{N-1}(z) \cdots T_{N-k+1}(z) \begin{bmatrix}
q_{N-k}(z) \\
p_{N-k}(z)
\end{bmatrix}. \quad (5.4)
\]

Let us introduce a different indexing \( j' = N - i + 1 \) which is a bijection of the set \([1, N]\) and represents counting points of the beam from right to left rather than left to right. Moreover, let us denote by \( \hat{T}_j(z) \) the classical adjoint of \( T_j(z) \). Thus

\[
\hat{T}_j(z) = \begin{bmatrix}
1 & -h_{N-j+1} \\
zg_{N-j+1} & 1
\end{bmatrix} = \begin{bmatrix}
1 & -h_{j'} \\
zg_{j'} & 1
\end{bmatrix}. \quad (5.5)
\]

Then (5.4) implies

\[
\hat{T}_k(z) \cdots \hat{T}_2 \hat{T}_1(z) \begin{bmatrix}
W(z) \\
1
\end{bmatrix} = \prod_{j'=1}^{k} \left( 1 + zm_{j'} n_{j'} \right) \begin{bmatrix}
q_{(k+1)(z)} \\
p_{(k+1)(z)}
\end{bmatrix}, \quad (5.6)
\]

where we used that \( \det(T_j(z)) = 1 + zg_j h_j \) and, subsequently, \( g_j h_j = m_j n_j \). We clearly have

**Lemma 5.3.** Let us fix \( 1 \leq k \leq N \) and denote \( \hat{S}_k(z) = \hat{T}_k(z) \cdots \hat{T}_2 \hat{T}_1(z) \). Then for every \( 1 \leq j \leq k \) the vector \( \begin{bmatrix}
W(z_j) \\
1
\end{bmatrix} \), where \( z_j = -\frac{1}{m_j n_j} \), is in the \( \ker(\hat{S}_k(z_j)) \).

We proceed by explicitly writing the conditions that the vectors \( \begin{bmatrix}
W(z_i) \\
1
\end{bmatrix} \) be null vectors of \( \hat{S}_k(z_i) \). To this end we write

\[
\hat{S}_k(z) = \begin{bmatrix}
\hat{q}_k(z) & \hat{Q}_k(z) \\
\hat{p}_k(z) & \hat{P}_k(z)
\end{bmatrix}, \quad (5.7)
\]
from which two sets of interpolation conditions

\[ \hat{q}_k(z_j) W(z_j) + \hat{Q}_k(z_j) = 0, \quad 1 \leq j \leq k, \]  
\[ \hat{p}_k(z_j) W(z_j) + \hat{P}_k(z_j) = 0, \quad 1 \leq j \leq k, \]  

(5.8a)  
(5.8b)

emerge. Whether these conditions, given \( W(z) \), determine the polynomials \( \hat{q}_k, \hat{Q}_k, \hat{p}_k, \hat{P}_k \) will depend on their degrees and this is the subject of the next result whose proof is an easy exercise in induction.

**Lemma 5.4.** For any \( k, 1 \leq k \leq N, \)

1. \( \deg \hat{q}_k(z) = \lfloor \frac{k}{2} \rfloor \), \( \deg \hat{Q}_k(z) = \lfloor \frac{k+1}{2} \rfloor \), \( \deg \hat{p}_k(z) = \lfloor \frac{k+1}{2} \rfloor \), \( \deg \hat{P}_k(z) = \lfloor \frac{k}{2} \rfloor \),
2. \( \hat{q}_k(0) = 1, \hat{p}_k(0) = 0, \hat{P}_k(0) = 1 \).

Now it is elementary to check that the number of interpolation conditions in equations (5.8) is the same as the number of unknown coefficients in \( \hat{q}_k, \hat{Q}_k, \hat{p}_k, \hat{P}_k \), so, in principle, the solution exists. Before we state our next lemma we revisit the notation introduced in Definition 2.5. For any multi-index \( I \in \{1, \ldots, k\} \), where we recall \( I = (i_1, i_2, \ldots, i_j) \) is an ordered set associated to an increasing sequence \( 1 \leq i_1 < i_2 < \cdots < i_j \leq k \), we assign its ordered image \( I' \) obtained by applying the bijection \( i \rightarrow N+1-i \) to \( I \) and reordering. The following result can be demonstrated by using induction on \( k \) and the definition of \( \hat{S}_k(z) \) (5.7).

**Lemma 5.5.** For any \( k, 1 \leq k \leq N, \)

\[ \hat{q}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left( \sum_{I \in \binom{\{1, \ldots, k\}}{j}} g_{I'} h_{I'} \right)(-z)^j, \] 
\[ \hat{Q}_k(z) = -\sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor} \left( \sum_{I \in \binom{\{1, \ldots, k\}}{j}} h_{I'} g_{I'} \right)(-z)^j, \] 
\[ \hat{p}_k(z) = -\sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \left( \sum_{I \in \binom{\{1, \ldots, k\}}{j}} g_{I'} h_{I'} \right)(-z)^j, \] 
\[ \hat{P}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left( \sum_{I \in \binom{\{1, \ldots, k\}}{j}} h_{I'} g_{I'} \right)(-z)^j, \]  

(5.9a)  
(5.9b)  
(5.9c)  
(5.9d)

with the convention that \( \hat{q}_1(z) = 1, \hat{Q}_1(z) = -h_N, \hat{p}_1(z) = zg_N, \hat{P}_1(z) = 1 \).

**Example 5.6.** It is instructive to display \( \hat{S}_k(z) \) for small \( k \). The notation is that of (5.7).

a) \[ \hat{S}_1(z) = \begin{bmatrix} \hat{q}_1(z) & \hat{Q}_1(z) \\ \hat{p}_1(z) & \hat{P}_1(z) \end{bmatrix} = \begin{bmatrix} 1 & -h_1 \\ g_1 z & 1 \end{bmatrix}. \]

b) \[ \hat{S}_2(z) = \begin{bmatrix} \hat{q}_2(z) & \hat{Q}_2(z) \\ \hat{p}_2(z) & \hat{P}_2(z) \end{bmatrix} = \begin{bmatrix} 1 - (g_1 h_2 + h_2 z) & -(h_1 + h_2) \\ (g_1 z + g_2 z) & 1 - (h_1 z + g_2 z) \end{bmatrix}. \]
Theorem 5.7. For any $g_3(z)$, we have

$$\begin{bmatrix}
\hat{q}_3(z) \\
\hat{Q}_3(z)
\end{bmatrix}
= \begin{bmatrix}
1 - (g_{1'} h_{1'} + g_{1'} h_{2'} + g_{2'} h_{2'}) z & -(h_{1'} + h_{2'} + h_{3'}) z + (h_{1'} g_{2'} h_{3'}) z \\
(g_{1'} + g_{2'}) z - (g_{1'} h_{2'} g_{3'}) z^2 & 1 - (h_{1'} g_{2'} + h_{1'} g_{3'} + h_{2'} g_{3'}) z
\end{bmatrix}.
$$

For a polynomial $f(z)$ let us denote by $f^+$ the coefficient at the highest power in $z$ and use the convention $q_0 = 1$. Then, by inspection, we obtain

$$g_1' = \frac{\hat{p}_1^+}{q_0'}, \quad g_2' = \frac{\hat{p}_2^+}{Q_1'}, \quad g_3' = \frac{\hat{p}_3^+}{q_2'},$$

and continuing with the help of induction we are led to

**Theorem 5.7.** For any $1 \leq k \leq N$,

$$g_{k'} = \frac{\hat{p}_k^+}{q_{k-1}'}, \quad \text{if } k \text{ is odd}, \quad (5.10a)$$

$$g_{k'} = \frac{\hat{p}_k^+}{Q_{k-1}'}, \quad \text{if } k \text{ is even}. \quad (5.10b)$$

**Remark 5.8.** Since, $g_{k'} h_{k'} = m_{k'} n_{k'}$, knowing $g_{k'}$ determines uniquely $h_{k'}$.

Now we can state our strategy for solving the original inverse problem stated in Definition 5.1:

1. given $W(z)$ we solve the interpolation problems (5.8) for all $1 \leq k \leq N$;
2. from the solution to the interpolation problem at stage $1 \leq k \leq N$, we use Theorem 5.7 to recover $g_{k'}, h_{k'}$ and thus $\hat{T}_k = \begin{bmatrix} 1 & -h_{k'} \\ z g_{k'} & 1 \end{bmatrix}$, finally all transitions matrices $T_k$ (see (5.3));
3. we then define

$$\begin{bmatrix} q_k(z) \\
p_k(z)
\end{bmatrix} = T_k(z) T_{k-1}(z) \cdots T_1(z) \begin{bmatrix} 0 \\
1
\end{bmatrix}; \quad (5.11)$$

4. we define $\hat{W}(z) = \frac{q_N(z)}{p_N(z)}$. The fact that $W(z) = \hat{W}(z)$ follows from (5.4) and (5.6) with $k = N$.

The interpolation problem is linear so the solution will be expressed in terms of determinants. Before, however, we present the final formulae it is helpful to introduce a family of generalized Cauchy-Vandermonde matrices \cite{9, 12, 13} attached to a Stieltjes transform of a positive measure. Matrices of this type arise in some interpolation problems, including the current one. We refer the reader to [6] for more details but to ease the presentation we provide in the paragraph below a simplified version of the interpolation problem specified by (5.8).

Given three positive integers $k, l, m$ such that $k = l + m$, a function $f(z)$, and a collection of distinct points $\{z_j\}_{1}^{k}$ we are seeking two polynomials $a(z) = 1 + \sum_{n=1}^{l} a_n z^n$ and $b(z) = \sum_{n=0}^{m} b_n z^n$ such that

$$a(z_j) f(z_j) + b(z_j) = 0, \quad 1 \leq j \leq k.$$
This interpolation problem is equivalent to the matrix problem

\[
\begin{bmatrix}
  z_1^1 f(z_1) & z_1^2 f(z_1) & \ldots & z_1^l f(z_1) & 1 & z_1 & \ldots & z_1^m \\
  z_2^1 f(z_2) & z_2^2 f(z_2) & \ldots & z_2^l f(z_2) & 1 & z_2 & \ldots & z_2^m \\
  \vdots & \vdots & \ddots & \vdots & 1 & \vdots & \ddots & \vdots \\
  z_k^1 f(z_k) & z_k^2 f(z_k) & \ldots & z_k^l f(z_k) & 1 & z_k & \ldots & z_k^m \\
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  \vdots \\
  a_l \\
  b_0 \\
  \vdots \\
  b_m 
\end{bmatrix} =
\begin{bmatrix}
  f(z_1) \\
  \vdots \\
  \vdots \\
  f(z_k) \\
\end{bmatrix}
\]

Denoting the determinant of the matrix on the left by \(D_k\) and assuming for now that \(D_k \neq 0\) we can succinctly write the solution

\[
a(z) + z^{l+1} b(z) = \frac{1}{D_k} \det
\begin{bmatrix}
  1 & z & z^2 & \ldots & z^l & z^{l+1} & z^{l+2} & \ldots & z^k \\
  f(z_1) & z_1^1 f(z_1) & z_1^2 f(z_1) & \ldots & z_1^l f(z_1) & 1 & z_1 & \ldots & z_1^m \\
  f(z_2) & z_2^1 f(z_2) & z_2^2 f(z_2) & \ldots & z_2^l f(z_2) & 1 & z_2 & \ldots & z_2^m \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  f(z_k) & z_k^1 f(z_k) & z_k^2 f(z_k) & \ldots & z_k^l f(z_k) & 1 & z_k & \ldots & z_k^m \\
\end{bmatrix}
\]

In the case of the function \(f\) being the Stieltjes transform of a measure the determinants in question are computable. Now we turn to spelling out the most important points in the solution of the inverse problem. We start with a definition.

**Definition 5.9.** Given a (strictly) positive vector \(e \in \mathbb{R}^k\), a non-negative number \(c\), an index \(l\) such that \(0 \leq l \leq k\), another index \(p\) such that \(0 \leq p\), \(p + l - 1 \leq k - l\), and a positive measure \(\nu\) with support in \(\mathbb{R}_+\), a **Cauchy-Stieltjes-Vandermonde (CSV) matrix** is that of the form

\[
CS^{(l,p)}_k(e, \nu, c) =
\begin{bmatrix}
  e_1^p \hat{\nu}_c(e_1) & e_1^{p+1} \hat{\nu}_c(e_1) & \ldots & e_1^{p+l-1} \hat{\nu}_c(e_1) & 1 & e_1 & \ldots & e_1^{k-l-1} \\
  e_2^p \hat{\nu}_c(e_2) & e_2^{p+1} \hat{\nu}_c(e_2) & \ldots & e_2^{p+l-1} \hat{\nu}_c(e_2) & 1 & e_2 & \ldots & e_2^{k-l-1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  e_k^p \hat{\nu}_c(e_k) & e_k^{p+1} \hat{\nu}_c(e_k) & \ldots & e_k^{p+l-1} \hat{\nu}_c(e_k) & 1 & e_k & \ldots & e_k^{k-l-1} \\
\end{bmatrix}
\]

where \(\hat{\nu}_c(y) = c + \int \frac{d\nu(x)}{y+x}\) is the (shifted) classical Stieltjes transform of the measure \(\nu\).

**Remark 5.10.** In this section we use a slightly different definition of the Stieltjes transform then the one in the context of the Weyl function (see Section 4). Thus, in this section, it is \(W(-z)\) which is the Stieltjes transform of the spectral measure; this notation being in fact more in line with Stieltjes’ notation in [20].

The explicit formulas for the determinant of the CSV matrix can be readily obtained. To this end we need some notations to facilitate the presentation. Recall that the multi-index
Theorem 5.11. Let $I$ be a positive measure with support in $\mathbb{R}$, and let $x$ denote the vector $[x_1, x_2, \ldots, x_l] \in \mathbb{R}^l$ and $d\nu^p(y) = y^p d\nu(y)$, respectively. Then

1. if either $c = 0$ or $p + l - 1 < k - 1$ then
\[
\det CSV_k^{(l,p)}(e, \nu, c) = (-1)^{l + [l-1]} \int_{0 < x_1 < x_2 < \cdots < x_l} \frac{\Delta_{[1,l]}(x)^2}{\Gamma_{[1,l],[1,l]}(e;x)} d\nu^p(x_1)d\nu^p(x_2)\cdots d\nu^p(x_l); \tag{5.12}
\]

2. if $c > 0$ and $p + l - 1 = k - 1$ then
\[
\det CSV_k^{(l,p)}(e, \nu, c) = (-1)^{p + [l-1]} \Delta_{[1,k]}(e) \cdot \left[ \int_{0 < x_1 < x_2 < \cdots < x_l} \frac{\Delta_{[1,l]}(x)^2}{\Gamma_{[1,k],[1,l]}(e;x)} d\nu^p(x_1)d\nu^p(x_2)\cdots d\nu^p(x_l) \right. \\
+ c \int_{0 < y_1 < y_2 < \cdots < y_{l-1}} \frac{\Delta_{[1,l-1]}(y)^2}{\Gamma_{[1,k],[1,l-1]}(e;y)} d\nu^p(y_1)d\nu^p(y_2)\cdots d\nu^p(y_{l-1}) \right]. \tag{5.13}
\]

Our next step is to give a complete solution to the inverse problem as stated in Definition 5.1 in terms of the determinants of the CSV matrices. To this end we set (see (5.1)):
\[
e_{j} = \frac{1}{m_j n_j}, \quad e_{[1,j]} = e_1 e_2 \cdots e_j, \quad \nu = \mu, \quad 1 \leq j \leq N, \tag{5.14}
\]
and
\[
\mathcal{D}_k^{(l,p)} = \left| \det CSV_k^{(l,p)}(e, \mu, c) \right|. \tag{5.15}
\]

Now, with Theorem 5.11 in hand, the main theorem of this section follows from the solution of the interpolation problem (5.8) with normalization conditions of Lemma 5.4, and Theorem 5.7. For details of computations we refer to [6].

Theorem 5.12. Suppose the Weyl function $W(z)$ is given by (5.1) along with positive constants (masses) $m_1, m_2, \ldots, m_N, n_1, n_2, \ldots, n_N$ such that the products $m_j n_j$ are distinct.
Then, there exists a unique solution to the inverse problem specified in Definition 5.1:

\[
g_k' = \frac{\mathcal{D}_k^{(\frac{1}{2}, 1)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 0)}}, \quad \text{if } k \text{ is odd,} \tag{5.16a}
\]

\[
g_k' = \frac{\mathcal{D}_k^{(\frac{1}{2}, 1)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 0)}}, \quad \text{if } k \text{ is even.} \tag{5.16b}
\]

Likewise,

\[
h_k' = \frac{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 0)}}{\mathcal{D}_k^{(1, 1)} \mathcal{D}_{k-1}^{(1, 1)}}, \quad \text{if } k \text{ is odd,} \tag{5.17a}
\]

\[
h_k' = \frac{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 0)}}{\mathcal{D}_k^{(1, 1)} \mathcal{D}_{k-1}^{(1, 1)}}, \quad \text{if } k \text{ is even.} \tag{5.17b}
\]

5.2. The second inverse problem. To proceed to the next step we recall that the original peakon problem (1.10) was formulated in the x space. To go back to the x space we use the relation \( h_j = m_j e^{x_j} \) (see equation (2.3)) to arrive at the inverse formulae relating the spectral data and the positions of peakons given by \( x_j \).

**Theorem 5.13.** Given positive constants \( m_j, n_j \) with distinct products \( m_j n_j \), let \( \Phi \) be the solution to the boundary value problem 2.4 with associated spectral data \( (d \mu, c) \). Then the positions \( x_j \) (of peakons) in the discrete measures \( m = 2 \sum_{j=1}^{N} m_j \delta_{x_j} \) and \( n = 2 \sum_{j=1}^{N} n_j \delta_{x_j} \) can be expressed in terms of the spectral data as:

\[
x_k' = \ln \frac{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}{m_k \mathcal{D}_k^{(\frac{1}{2}, 1)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}, \quad \text{if } k \text{ is odd,} \tag{5.18a}
\]

\[
x_k' = \ln \frac{\mathcal{D}_k^{(\frac{1}{2}, 0)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}{m_k \mathcal{D}_k^{(\frac{1}{2}, 1)} \mathcal{D}_{k-1}^{(\frac{1}{2}, 1)}}, \quad \text{if } k \text{ is even,} \tag{5.18b}
\]

with \( \mathcal{D}_k^{(l, p)} \) defined in (5.15), \( k' = N - k + 1, 1 \leq k \leq N \) and the convention that \( \mathcal{D}_0^{(l, p)} = 1 \).

Finally, we can relax the condition that the products of masses \( m_j, n_j \) be distinct. Indeed, it suffices to observe that the Vandermonde determinants \( \Delta_{[1, r]}(e), r = k, k-1 \), cancel out in all expressions of the type

\[
\mathcal{D}_k^{(l_1, p_1)} \mathcal{D}_{k-1}^{(l_2, p_2)} \mathcal{D}_k^{(l_3, p_3)} \mathcal{D}_{k-1}^{(l_4, p_4)} \]

as a result of Theorem 5.11 (see (5.12) and (5.13)).

In summary, we completed the full circle of starting with initial positions of peakons, mapping them to the spectral data \( (d \mu, c) \), while in the last chapter we solved explicitly the inverse problem of mapping back the spectral data to the positions \( x_j \) of peakons. In all this the time was fixed. In the following sections, we concentrate on the time evolution of the multipeakons of the 2-mCH. In light of the difference of the value of \( c \)
in the Stieltjes transform according to whether \( N \) is even or odd, the discussions will be presented separately for \( N \) even, \( N \) odd, respectively.

6. MULTIEAKONS FOR \( N = 2K \)

For even \( N, \) \( c = 0 \) (see Theorem 4.5). This impacts the asymptotic behaviour of solutions. Further comments on differences between solutions for odd and even \( N \) are in Section 8.1.

6.1. Closed formulae for \( N = 2K \). If we assume that \( x_1(0) < x_2(0) < \cdots < x_{2K}(0) \) then by continuity, this condition will hold at least in a small interval containing \( t = 0. \) At \( t = 0 \) we solve the forward problem (see Section 4) and obtain the Weyl function \( W(z) \) (see Theorem 4.5) hence the spectral measure \( d\mu(0) = \sum_{j=1}^{K} b_j(0)\delta_{\zeta_j} \) supported on the set of ordered eigenvalues \( 0 < \zeta_1 < \cdots < \zeta_K. \)

With the help of Theorem 5.13 we obtain the following result.

**Theorem 6.1.** The 2-mCH (1.7) with the regularization of the singular term \( Q_m \) given by \( \langle Q \rangle m \) admits the multipeakon solution

\[
\begin{align*}
  u(x, t) &= \sum_{k=1}^{2K} m_{k'} \exp(-|x - x_{k'}(t)|), \\
  v(x, t) &= \sum_{k=1}^{2K} n_{k'} \exp(-|x - x_{k'}(t)|),
\end{align*}
\]

(6.1)

where \( m_{k'}, n_{k'} \) are arbitrary positive constants, while \( x_{k'}(t) \) are given by equations (5.18a) and (5.18b) corresponding to the peakon spectral measure

\[
d\mu(t) = \sum_{j=1}^{K} b_j(t)\delta_{\zeta_j},
\]

(6.2)

with \( b_j(t) = b_j(0)e^{\alpha t}, \) \( 0 < b_j(0), \) and \( c = 0 \) in (5.15).

**Proof.** The only outstanding issue is the time evolution of \( b_j \) or, more generally, the time evolution of the spectral measure. Recall the Weyl function \( W(z) \) defined in (4.1). By employing the time evolution (2.10), one easily obtains

\[
\dot{W} = \frac{2}{z} W - \frac{2u_x}{z},
\]

which, in turn, implies \( \dot{b}_j = \frac{2}{z_j} b_j, 1 \leq j \leq K, \) by use of Corollary 4.6. \( \Box \)

In the following, we provide examples of multipeakons in the case of even \( N. \) Before this is done, it is useful to examine the explicit formulas for the CSV determinants following Theorem 5.11 (see equation (5.15) for notation). We remind the reader that the eigenvalues \( \zeta_j \) are positive and ordered \( 0 < \zeta_1 < \cdots < \zeta_K. \)

**Theorem 6.2.** Let \( N = 2K, \) \( 0 \leq l \leq K, \) \( 0 \leq p, p + l - 1 \leq k - l, 1 \leq k \leq 2K \) and let the peakon spectral measure be given by (6.2). Then

\[
\mathcal{D}_k^{(l,p)} = \left| \Delta_{[1,k]}(e) \right| \sum_{l \in \mathcal{I}_{[1,l]}^e} \frac{\Delta_{[1,l]}(e)\Delta_{[1,l]}^p}{\Gamma_{[1,k],l}(e; \xi)};
\]

(6.3)

(1) in the asymptotic region \( t \to +\infty \)

\[
\mathcal{D}_k^{(l,p)} = \left| \Delta_{[1,k]}(e) \right| \frac{\Delta_{[1,l]}^2(\xi)b_{[1,l]}(e; \xi)}{\Gamma_{[1,k],l}(e; \xi)} \left[ 1 + \mathcal{O}(e^{-\alpha t}) \right], \quad \text{for some } \alpha > 0;
\]

(6.4)
Theorem 6.5.

Example 6.4

tm question to understand the nature of collisions but we leave this topic for future work.

Example 6.3

··· <

Now we are ready to present examples of expressions for positions \(x_1, \ldots, x_{2K}\) of multipeakons based on formulas \(5.18a\), \(5.18b\), using \(6.3\) and \(e_j = \frac{1}{m_j n_j} \), \(j' = 2K - j + 1\).

Example 6.4 (2-peakon solution; \(K=1\)).

\[
x_1 = \ln \left( \frac{b_1}{\zeta_1 m_1(1 + \zeta_1 m_2 n_2)} \right), \quad x_2 = \ln \left( \frac{b_1 n_2}{1 + \zeta_1 m_2 n_2} \right).
\]

Example 6.4 (4-peakon solution; \(K=2\)).

\[
x_1 = \ln \left( \frac{b_1 b_2 (\zeta_2 - \zeta_1)^2}{\zeta_1 \zeta_2 (b_1 + b_2 m_1 n_1)(1 + \zeta_1 m_2 n_2)(1 + \zeta_2 m_3 n_3)(1 + \zeta_2 m_4 n_4)} \right)
\]

\[
x_2 = -\ln \left( \frac{b_1 b_2 (\zeta_2 - \zeta_1)^2}{\zeta_1 \zeta_2 (b_1 + b_2 m_1 n_1)(1 + \zeta_1 m_2 n_2)(1 + \zeta_2 m_3 n_3)(1 + \zeta_2 m_4 n_4)} \right)
\]

\[
x_3 = \ln \left( \frac{b_1 b_2 (\zeta_3 - \zeta_1)^2}{\zeta_1 \zeta_3 (b_1 + b_2 m_1 n_1)(1 + \zeta_1 m_2 n_2)(1 + \zeta_2 m_3 n_3)(1 + \zeta_3 m_4 n_4)} \right)
\]

\[
x_4 = -\ln \left( \frac{b_1 b_2 (\zeta_3 - \zeta_1)^2}{\zeta_1 \zeta_3 (b_1 + b_2 m_1 n_1)(1 + \zeta_1 m_2 n_2)(1 + \zeta_2 m_3 n_3)(1 + \zeta_3 m_4 n_4)} \right)
\]

6.2. **Global existence for \(N = 2K\).** Recall that our solution in Theorem 6.1 was obtained under the assumption that \(x_1 < x_2 < \cdots < x_{2K}\). However, even if we start with the initial positions satisfying \(x_1(0) < x_2(0) < \cdots < x_{2K}(0)\), the order might cease to hold for sufficiently large times, in other words some of the peakons might collide. It is an interesting question to understand the nature of collisions but we leave this topic for future work.

In this subsection we give sufficient conditions in terms of the spectrum and constant masses \(m_j, n_j\) which ensure that no collisions occur and thus the peakon solutions are global in \(t\). The readers might want to consult Appendix B for a detailed proof. Granted global existence, one can talk sensibly about the large time behaviour of peakons.

**Theorem 6.5.** Given arbitrary spectral data

\[
\{b_j > 0, 0 < \zeta_1 < \zeta_2 < \cdots < \zeta_K : 1 \leq j \leq K\},
\]

suppose the masses \(m_k, n_k\) satisfy

\[
\frac{k^{-1}}{\zeta_k} < m_k^{(k+1)/2} n_k^{(k-1)/2}, \quad \text{for all } k, \quad 1 \leq k \leq 2K - 1,
\]

\[
\frac{m_k^{(k+1)/2} n_k^{(k-1)/2}}{(1 + m_k^{(k+1)/2} n_k^{(k+1)/2}) (1 + m_k^{(k+1)/2} n_k^{(k+1)/2})} < \frac{2 \min_j (\zeta_j + 1)}{(k+1)(\zeta_K - \zeta_1)^{k-1} - 1},
\]

\[
\text{for all } k, \quad 1 \leq k \leq 2K - 3.
\]

Then the positions obtained from inverse formulas \(5.18a\), \(5.18b\) are ordered \(x_1 < x_2 < \cdots < x_{2K}\) and the multipeakon solutions \(7.1\) exist for arbitrary \(t \in R\).
Large time peakon asymptotics for $n = 2K$. Once the global existence of solutions is guaranteed, for example by imposing sufficient conditions of Theorem 6.5, one can study the asymptotic behaviour of multipeakon solutions for large (positive and negative) time by employing Theorem 6.1 and 6.2. More precisely, by using the formulae for positions (5.18a), (5.18b), as well as asymptotic evaluations of determinants (6.4) and (6.5), one arrives at

**Theorem 6.6.** Suppose the masses $m_j, n_j$ satisfy the conditions of Theorem 6.5. Then the asymptotic position of a $k$-th (counting from the right) peakon as $t \to +\infty$ is given by

$$x_k' = \frac{2t}{\zeta_k^{1/2}} + \ln \frac{b_{k+1}^{(0)}(0)e_{1,k-1}\Delta^2_{[1,k-1],[1,k-1]}(\zeta)}{m_k\Gamma_{1,k-1}(\zeta; e_{1,k-1}^{22q+y+1})} + O(e^{-\alpha_k t}),$$

for some positive $\alpha_k$ and odd $k$,

$$x_k' = \frac{2t}{\zeta_k^{1/2}} + \ln \frac{b_k^{(0)}(0)e_{1,k-1}\Delta^2_{[1,k-1],[1,k-1]}(\zeta)}{m_k\Gamma_{1,k-1}(\zeta; e_{1,k-1}^{22q+y+1})} + O(e^{-\alpha_k t}),$$

for some positive $\alpha_k$ and even $k$,

$$x_k' - x_{(k+1)'} = \ln m_{(k+1)'} n_k \zeta_k^{1/2} + O(e^{-\alpha_k t}),$$

for some positive $\alpha_k$ and odd $k$.

Likewise, as $t \to -\infty$, using the notation of Theorem 6.2, the asymptotic position of the $k$-th peakon is given by

$$x_k' = \frac{2t}{\zeta_k^{(1/2)^*}} + \ln \frac{b_{k+1}^{(0)}(0)e_{1,k-1}\Delta^2_{[1,k-1]^*,[1,k-1]^*}^{(1/2)^*}(\zeta)}{m_k\Gamma_{1,k-1}(\zeta; e_{1,k-1}^{22q+y+1})^{(1/2)^*}((\zeta; e_{1,k-1}^{22q+y+1})^{(1/2)^*}) + O(e^{\beta_k t}),$$

for some positive $\beta_k$ and odd $k$,

$$x_k' = \frac{2t}{\zeta_k^{(1/2)^*}} + \ln \frac{b_k^{(0)}(0)e_{1,k-1}\Delta^2_{[1,k-1]^*,[1,k-1]^*}^{(1/2)^*}(\zeta)}{m_k\Gamma_{1,k-1}(\zeta; e_{1,k-1}^{22q+y+1})^{(1/2)^*}((\zeta; e_{1,k-1}^{22q+y+1})^{(1/2)^*}) + O(e^{\beta_k t}),$$

for some positive $\beta_k$ and even $k$,

$$x_k' - x_{(k+1)'} = \ln m_{(k+1)'} n_k \zeta_k^{(1/2)^*} + O(e^{\beta_k t}),$$

for some positive $\beta_k$ and odd $k$.

**Remark 6.7.** It follows from the above theorem that multipeakons of the 2-mCH equation exhibit *Toda-like sorting properties* of asymptotic speeds and *an asymptotic pairing*. The latter can be partially explained by the fact that there are $K$ available eigenvalues to match $2K$ asymptotic speeds. Similar features were also observed in the mCH equation [6], as well as the interlacing cases of the 2-mCH equation [4]. It is clear now that these two features extend to the non-interlacing cases as well.

We end this section by providing graphs of a concrete 4-peakon solution. Let $K = 2$, and $b_1(0) = 10$, $b_2(0) = 1$, $\zeta_1 = 0.3$, $\zeta_2 = 3$, $m_1 = 8$, $m_2 = 24$, $m_3 = 5$, $m_4 = 10$, $n_1 = 12$, $n_2 = 10$, $n_3 = 24$, $n_4 = 16$. It is easy to check that the condition in Theorem 6.5 is satisfied. Hence, the order of $\{x_k, k = 1,2,3,4\}$ will be preserved and one can use the explicit formulae for the 4-peakon solution, resulting in the following sequence of graphs (Figure 1), illustrating the asymptotic pairing of peakons.
7. Multipeakons for $N = 2K + 1$

This section is devoted to the corresponding result for $N = 2K + 1$, which is presented in a way parallel to the previous section on the even case. The main source of difference between the two cases is of course the presence of the positive shift $c$ which impacts the evaluations of the CSV determinants as illustrated by Theorem 5.11, in particular formula (5.13). Nevertheless, we will present a comparison in the form of a correspondence between the odd case and the even case in Section 8.1.

7.1. Closed formulae for $N = 2K + 1$. As before we assume that $x_1(0) < x_2(0) < \cdots < x_{2K+1}(0)$. Then this condition will hold at least in a small interval containing $t = 0$. The following local existence result follows from Theorem 5.13.

**Theorem 7.1.** The 2-mCH equation (1.7) with the regularization of the singular term $Q_m$ given by $\langle Q \rangle_m$ admits the multipeakon solution

\[ u(x, t) = \sum_{k=1}^{2K+1} m_k \exp(-|x - x_k(t)|), \quad v(x, t) = \sum_{k=1}^{2K+1} n_k \exp(-|x - x_k(t)|), \quad (7.1) \]

where $m_k, n_k$ are arbitrary positive constants, while $x_k(t)$ are given by equations (5.18a) and (5.18b) corresponding to the peakon spectral measure

\[ d\mu = \sum_{j=1}^{K} b_j(t) \delta_{\zeta_j}, \quad (7.2) \]

\[ b_j(t) = b_j(0) e^{2t}, \quad 0 < b_j(0), \text{ with ordered eigenvalues } 0 < \zeta_1 < \cdots < \zeta_K \text{ and } c(t) = c(0) > 0 \text{ in } (5.15). \]
Proof. Similar to the even case, it is clear that the Weyl function $W(z)$ is defined in (4.1), undergoes the time evolution obtained earlier in the proof of Theorem 6.1, namely,

$$W = \frac{2}{z} W - \frac{2u_+}{z},$$

which, in turn, implies $\dot{b}_j = \frac{2}{\zeta_j} b_j$, $1 \leq j \leq K$ as well as $\dot{c} = 0$ by virtue of Corollary 4.6. The rest of the argument is the same as for the even case. \hfill $\square$

By using the above theorem, it is not hard to work out two simplest examples of solutions. Before we do that, however, we will examine the evaluation of CSV determinants presented in Theorem 5.11 (see equation (5.15) for notation), with due care to two facts: $N = 2K + 1$ and $c > 0$. The proof follows from the same steps as in Theorem 6.2 and we omit it.

Theorem 7.2. Let $N = 2K + 1$, $1 \leq k \leq 2K + 1$, $0 \leq l \leq K + 1$, $0 \leq p, p + l - 1 \leq k - l$, and let the peakon spectral measure be given by (7.2) and a shift $c > 0$. Then

(1)

$$\mathcal{D}^{(l,p)}_k = \left| \Delta_{[1,k]}(e) \right| \sum_{I \in \{1,K\}} \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)},$$

if $p + l - 1 < k - l$, \quad $k \leq 2K + 1$;

(7.3a)

$$\mathcal{D}^{(l,p)}_k = \left| \Delta_{[1,k]}(e) \right| \left( \sum_{I \in \{1,K\}} \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)} + c \sum_{I \in \{1,K\}} \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)} \right),$$

if $p + l - 1 = k - l$, \quad $k \leq 2K + 1$;

(7.3b)

with the proviso that the first term inside the bracket is set to zero if $l = K + 1$.

which only happens when $k = 2K + 1$, $p = 0$.

(2) In the asymptotic region $t \rightarrow +\infty$

$$\mathcal{D}^{(l,p)}_k = \left| \Delta_{[1,k]}(e) \right| \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)} \left[ 1 + \mathcal{O}(e^{-\alpha t}) \right], \quad 0 < \alpha,$$  

if $0 \leq l \leq K$;  

(7.4a)

$$\mathcal{D}^{(K+1,0)}_k = c \left| \Delta_{[1,2K+1]}(e) \right| \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,2K+1],I}(e;\zeta)},$$

if $k = 2K + 1, l = K + 1, p = 0$.

(4.1)

(3) In the asymptotic region $t \rightarrow -\infty$

$$\mathcal{D}^{(l,p)}_k = \left| \Delta_{[1,k]}(e) \right| \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)} \left[ 1 + \mathcal{O}(e^{-\beta t}) \right], \quad 0 < \beta,$$

if $p + l - 1 < k - l$, \quad $k \leq 2K + 1$;

(7.5a)

$$\mathcal{D}^{(l,p)}_k = c \left| \Delta_{[1,k]}(e) \right| \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,k],I}(e;\zeta)} \left[ 1 + \mathcal{O}(e^{-\beta t}) \right], \quad 0 < \beta,$$

if $p + l - 1 = k - l$, \quad $k < 2K + 1$;

(7.5b)

$$\mathcal{D}^{(K+1,0)}_k = c \left| \Delta_{[1,2K+1]}(e) \right| \frac{\Delta_k^2(e_2,\zeta^p_1)}{\Gamma_{[1,2K+1],I}(e;\zeta)},$$

if $k = 2K + 1, l = K + 1, p = 0$.

(7.5c)
where, as before, \([1, l]^* = [1^* = K - l + 1, 1^* = K]\).

There exists a relation between formulae with \(c > 0\) and \(c = 0\). Indeed, by comparing formulas (7.3a) and (7.3b) with (6.3), we arrive at the detailed dependence on \(c\).

**Corollary 7.3.** Let \(N = 2K + 1, 1 \leq k \leq 2K + 1, 0 \leq l \leq K + 1, 0 \leq p, p + l - 1 \leq k - l\), and let the peakon spectral measure be given by (7.2) and a shift \(c > 0\). Then

\[
\varphi_{k}^{(l,p)}(c) = \varphi_{k}^{(l,p)}(0), \quad \text{if } p + l - 1 < k - l, \quad k \leq 2K + 1; \tag{7.6a}
\]

\[
\varphi_{k}^{(l,p)}(c) = \varphi_{k}^{(l,p)}(0) + c\varphi_{k}^{(l-1,p)}(0), \quad \text{if } p + l - 1 = k - l, \quad k \leq 2K + 1; \tag{7.6b}
\]

with the convention that the first term in (7.6b) is set to zero if \(l = K + 1, k = 2K + 1, p = 0\).

Below the reader will find two examples of explicit peakon solutions for odd \(N = 2K + 1\).

**Example 7.4** (1-peakon solution; \(K = 0\)).

\[
x_1 = \ln \left( \frac{c}{m_1} \right).
\]

**Example 7.5** (3-peakon solution; \(K = 1\)).

\[
x_1 = \ln \left( \frac{b_1c}{\zeta_{1} m_1 (b_1 \zeta_{1} m_2 n_2 m_3 n_3 + c(1 + \zeta_{1} m_2 n_2)(1 + \zeta_{1} m_3 n_3))} \right),
\]

\[
x_2 = \ln \left( \frac{b_1 n_2}{b_1 \zeta_{1} m_2 n_2 m_3 n_3 + c(1 + \zeta_{1} m_2 n_2)(1 + \zeta_{1} m_3 n_3)} \left( \frac{b_1 m_3 n_3}{1 + \zeta_{1} m_3 n_3} + c \right) \right),
\]

\[
x_3 = \ln \left( \frac{1}{m_3} \left( \frac{b_1 m_3 n_3}{1 + \zeta_{1} m_3 n_3} + c \right) \right).
\]

**7.2. Global existence for \(N = 2K + 1\).** Similar to the even case, we can also provide a sufficient condition to ensure the global existence of peakon solutions when \(N = 2K + 1\). The main result is stated below while its proof is relegated to Appendix C.

**Theorem 7.6.** Given arbitrary spectral data

\[
\{b_{j} > 0, 0 < \xi_{1} < \xi_{2} < \cdots < \xi_{K}, c > 0: 1 \leq j \leq K\},
\]

suppose the masses \(m_{k}, n_{k}\) satisfy

\[
\frac{1}{m_{(k+1)^2} n_{(k+1)^2}} < \frac{\xi_{k}^{2}}{\xi_{k}^2} \min \left[ \frac{1}{k}, \hat{\beta} \right], \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1,
\]

\[
\frac{1}{m_{(k+2)^2} n_{(k+2)^2}} < \frac{\xi_{k}^{2}}{\xi_{k}^2} \min \left[ \frac{1}{k}, \hat{\beta}_1 \right], \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1,
\]

where

\[
\hat{\beta} = \begin{cases} 
2 \xi_{k} \min \{\xi_{j+1} - \xi_{j}, (1 + m_{k^2} n_{k} \xi_{1})^{1/2} \}^{1/3} & \text{for all odd } k, \\
3 \xi_{k} \min \{\xi_{j+1} - \xi_{j} \}^{1/3} & \text{for all odd } k, \\
+\infty & \text{for } k = 1,
\end{cases}
\]

\[
\hat{\beta}_1 = \frac{2 \min \{\xi_{j+1} - \xi_{j}, (1 + m_{(k+2)^2} n_{(k+2)^2})^{1/2} \}^{1/3}}{(k + 1)(\xi_{k} - \xi_{1})^{1/3}} \frac{m_{(k+2)^2} n_{(k+2)^2}}{m_{(k+1)^2} n_{(k+1)^2}} \text{ for all odd } k, 3 \leq k \leq 2K - 1,
\]

\[
\text{for } k = 1,
\]

Then the positions obtained from inverse formulas (5.18a), (5.18b) are ordered \(x_1 < x_2 < \cdots < x_{2K+1}\) and the multipeakon solutions (7.1) exist for arbitrary \(t \in \mathbb{R}\).
Theorem 7.7. Suppose the masses $m_j, n_j$ satisfy the conditions of Theorem 7.6. Then the asymptotic position of a $k$-th (counting from the right) peakon as $t \to +\infty$ is given by

$$x_k' = \frac{2t}{\zeta_{\frac{k}{2}, 1}} + \ln \frac{b_{\frac{k}{2}, 0} e_{[1, 0]} (0) \Delta^2_{[1, 1], [\frac{k}{2}, 1]} (\zeta)}{m_k' \Gamma_{[1, 1], [\frac{k}{2}, 1]} (e; \zeta)^2_{[1, 1]}} + \mathcal{O}(e^{-a_k t}), \quad a_k > 0$$

and odd $k \leq 2K - 1$;

$$x_{(2K+1)'} = \ln \frac{\mathcal{C} e_{[1, 2K]}(0)}{m_{(2K+1)'} \zeta_{[1, 2K]}^2} + \mathcal{O}(e^{-a_k t}), \quad a > 0;$$

$$x_k' = \frac{2t}{\zeta_{\frac{k}{2}, 1}} + \ln \frac{b_{\frac{k}{2}, 0} e_{[1, 0]} (0) \Delta^2_{[1, 1], [\frac{k}{2}, 1]} (\zeta)}{m_k' \Gamma_{[1, 1], [\frac{k}{2}, 1]} (e; \zeta)^2_{[1, 1]}} + \mathcal{O}(e^{-a_k t}), \quad a_k > 0$$

and even $k \leq 2K$;

$$x_k' - x_{k+1}' = \ln m_{(k+1)'} n_k \zeta_{\frac{k}{2}, 1} + \mathcal{O}(e^{-a_k t}), \quad a_k > 0$$

and odd $k \leq 2K - 1$.

Likewise, as $t \to -\infty$, using the notation of Theorem 6.2, the asymptotic position of the $k$-th peakon is given by

$$x_k' = \frac{2t}{\zeta_{\frac{k}{2}, 1}^*} + \ln \frac{b_{\frac{k}{2}, 0} e_{[1, 0]} (0) \Delta^2_{[1, 1], [\frac{k}{2}, 1]^*} (\zeta)}{m_k' \Gamma_{[1, 1], [\frac{k}{2}, 1]^*} (e; \zeta)^2_{[1, 1]}} + \mathcal{O}(e^{\beta_k t}), \quad \beta_k > 0$$

and odd $1 < k \leq 2K + 1$;

$$x_{1'} = \ln \frac{c}{m_{1'}} + \mathcal{O}(e^{\beta_k t}), \quad \beta_k > 0;$$

$$x_k' = \frac{2t}{\zeta_{\frac{k}{2}, 1}^*} + \ln \frac{b_{\frac{k}{2}, 0} e_{[1, 0]} (0) \Delta^2_{[1, 1], [\frac{k}{2}, 1]^*} (\zeta)}{m_k' \Gamma_{[1, 1], [\frac{k}{2}, 1]^*} (e; \zeta)^2_{[1, 1]}} + \mathcal{O}(e^{\beta_k t}), \quad \beta_k > 0$$

and even $k \leq 2K$;

$$x_k' - x_{k+1}' = \ln m_{(k+1)'} n_k \zeta_{\frac{k}{2}, 1}^* + \mathcal{O}(e^{\beta_k t}), \quad \beta_k > 0$$

and even $k \leq 2K$.

Remark 7.8. Similar to the even case, the Toda-like sorting property can also be observed in this case. It is perhaps interesting to examine the role of the constant $c$. The constant $c$ is playing the role of an additional eigenvalue $\zeta_{k+1} = \infty$ resulting in the formal asymptotic speed $0$. We observe that for large positive times the first particle counting from the left comes to a stop, while the remaining $2K$ peakons form pairs of bound states akin to what is occurring for even $N$, effectively sharing in pairs the remaining $K$ speeds. By contrast, for large negative times, the first particle counting from the right slows to a halt, while the remaining peakons form pairs. Note that a similar phenomenon also occurs in the mCH.

At the end of this section, we present graphs for a concrete 3-peakon solution, which confirm our theoretical predictions. Let $K = 1$, and $b_1 (0) = 1$, $c = 3$, $\zeta_1 = 5$, $m_1 = 3$, $m_2 =$
2, \ m_3 = 1.6, \ n_1 = 1.8, \ n_2 = 3, \ n_3 = 2.2. Then the sufficient conditions in Theorem 7.6 are satisfied. Hence the order of \ \{x_k, \ k = 1, 2, 3\} will be preserved and one can use the explicit formulae for the 3-peakon solution, resulting in the following sequence of graphs (Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{Graphs of 3-peakon at time \( t = -12, 2, 10, 30 \) in the case of \( b_1(0) = 1, \ c = 3, \ \zeta_1 = 5, \ m_1 = 3, \ m_2 = 2, \ m_3 = 1.6, \ n_1 = 1.8, \ n_2 = 3, \ n_3 = 2.2. \)}
\end{figure}

8. Reductions of multipeakons

We recall that the 2-mCH (1.7) is a two-component integrable generalization of the mCH (1.5). In the present contribution we constructed explicit formulae for generic multipeakons of the 2-mCH by using the inverse spectral method. Yet, in our past work we constructed the so-called interlacing multipeakons of the 2-mCH [4], while in [6] we gave explicit formulae for multipeakons of the mCH. In this section, as if to close the circle, we would like to show how the formulae obtained in previous sections reduce to those special cases.

8.1. From odd case to even case. In Section 6 and 7, we presented the multipeakon formulae according to the parity of the number \( N \) of masses. In this subsection, we will show how the multipeakon formula in the odd case can be used to derive the mutipeakon formula for the even case. Consider the multipeakons

\[ u = \sum_{j=1}^{2K+1} m_j(t)e^{-|x-x_j(t)|}, \quad v = \sum_{j=1}^{2K+1} n_j(t)e^{-|x-x_j(t)|}. \]

Suppose \( m_1 \to 0, \ n_1 \to 0 \), then \( h_1 \to 0, \ g_1 \to 0 \), since \( h_j = m_j e^{x_j}, \ g_j = n_j e^{-x_j} \). Moreover, by continuity, it is not hard to see that \( c \to 0 \) in view of (3.9) and results from Section 4.

Now, with the help of multipeakon formulae (5.18a)-(5.18b), we also see that when \( c \to 0 \), then \( \lim_{c \to 0} m_1 e^{-|x-x_1|} = 0. \) Moreover, we can rewrite the multipeakon formulae...
With the notation which was considered in [4].

Comparing the formulae (5.18a)-(5.18b) in the even case, these are nothing but the matrix, we obtain

$$
\lim_{c \to 0} x_{2k-2k+3} = \ln \frac{e^{[1,2k-2]2,[k,0]}_{2k-1)}{m_{2k-2k+3}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K + 1, \quad (8.1a)
$$

$$
\lim_{c \to 0} x_{2k-2k+2} = \ln \frac{e^{[1,2k-1]2,[k,0]}_{2k-1)}{m_{2k-2k+2}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K, \quad (8.1b)
$$

where $e_j = m_{2k-2-2j}$. Note, however, that neither $m_1$ nor $n_1$ appear in the expressions for $x_2, x_3, \ldots, x_{2K+1}$. Thus, we have

$$
\lim_{c \to 0} x_{2k-2k+3} = \ln \frac{e^{[1,2k-2]2,[k,0]}_{2k-1)}{m_{2k-2k+3}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K, \quad (8.2a)
$$

$$
\lim_{c \to 0} x_{2k-2k+2} = \ln \frac{e^{[1,2k-1]2,[k,0]}_{2k-1)}{m_{2k-2k+2}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K. \quad (8.2b)
$$

With the notation

$$
\tilde{x}_j = \lim_{c \to 0} x_{j+1}, \quad \tilde{m}_j = m_{j+1}, \quad \tilde{n}_j = n_{j+1},
$$

as well as

$$
\tilde{e}_j = \frac{1}{m_{2k-1-j}} \tilde{m}_{2k-1-j}, \quad \tilde{e}_{[1,j]} = \tilde{e}_1 \cdots \tilde{e}_j, \quad \tilde{\mathcal{G}}^{[1,0]}_{k} = \tilde{D}^{[1,0]}_{k}
$$

where $\tilde{\mathcal{G}}^{[1,0]}_{k}$ is obtained by replacing $m_{j+1}, n_{j+1}$ by $\tilde{m}_j, \tilde{n}_j$ in the corresponding CSV matrix, we obtain

$$
\tilde{x}_{2k-2k+3} = \ln \frac{\tilde{e}^{[1,2k-2]2,[k,0]}_{2k-1)}{\tilde{m}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K, \quad (8.3a)
$$

$$
\tilde{x}_{2k-2k+2} = \ln \frac{\tilde{e}^{[1,2k-1]2,[k,0]}_{2k-1)}{\tilde{m}^{[k-1,1]}_{2k-1)}}, \quad 1 \leq k \leq K. \quad (8.3b)
$$

Comparing the formulae (5.18a)-(5.18b) in the even case, these are nothing but the formulae for the positions $\tilde{x}_j$ corresponding the multipeakon ansatz:

$$
u = \sum_{j=1}^{2K+1} \tilde{n}_j(t)e^{-|x-\tilde{x}_j(t)|}, \quad v = \sum_{j=1}^{2K+1} \tilde{n}_j(t)e^{-|x-\tilde{x}_j(t)|}.
$$

So far, we have shown how the formulae for the odd case reduce to those for the even case. It is not hard to see that a comparison between Examples 6.3 and 7.5 supports our claim above.

8.2. From the even case to the interlacing case. Let us consider the multipeakon ansatz with the even number of masses:

$$
u = \sum_{j=1}^{2K} m_j(t)e^{-|x-x_j(t)|}, \quad v = \sum_{j=1}^{2K} n_j(t)e^{-|x-x_j(t)|}.
$$

Then the reduction $m_{2j} \to 0$, $n_{2j-1} \to 0$ gives the interlacing ansatz

$$
u = \sum_{j=1}^{K} m_{2j-1}(t)e^{-|x-x_{2j-1}(t)|}, \quad v = \sum_{j=1}^{K} n_{2j}(t)e^{-|x-x_{2j}(t)|},
$$

which was considered in [4].
Note that this reduction does not change the form of the Weyl function, that is, the Weyl function still possesses the form:

\[ W(z) = \int \frac{d\mu(x)}{x - z}, \quad d\mu = \sum_{j=1}^{K} b_j \delta_{\zeta_j}, \quad 0 < \zeta_1 < \cdots < \zeta_K, \quad 0 < b_j, \quad 1 \leq j \leq K. \]

Let us consider the multipeakon formulae (5.18a)-(5.18b) for the case of the even number of masses. It is not hard to verify that

\[\left( e_{[1,k]} \right)^{\phi_k^{(l,p)}} \left| \Delta_{[1,k]}(e) \right| \to \sum_{f \in \left[ 1,K \right]} b_j \zeta_j^p \Delta_j^2 \pm H_j^p, \quad \text{as} \quad m_{2j}, n_{2j-1} \to 0,\]

which leads to

\[ x_{2K+1-2k} \to \ln \left( \frac{1}{m_{2K+1-2k}} \frac{(H_k^0)^2}{H_k^1 H_{k-1}^1} \right), \quad 1 \leq k \leq K, \]

\[ x_{2K+2-2k} \to \ln \left( \frac{n_{2K+2-2k}}{m_{2K+2-2k}} \frac{H_k^0 H_{k-1}^0}{(H_k^1 H_{k-1}^1)} \right), \quad 1 \leq k \leq K. \]

These formulae are identical to those obtained in [4] for the interlacing multipeakons of 2-mCH equation. It is useful to compare examples 6.3 and 6.4 in the present paper and examples 5.4, 5.5 in [4] to see a concrete manifestation of the transition from a general configuration to an interlacing one. It is also helpful to observe that the inverse problem for the interlacing case was solved using diagonal Padé approximations at \( \infty \) while in the present paper we are formulating our interpolation problem at \( e_j \) (see (5.14) all of which get moved to \( \infty \) in the limit \( m_{2j} \to 0, n_{2j-1} \to 0 \).

8.3. From 2-mCH to 1-mCH. As is known, when \( v = u \), the 2-mCH (1.7) reduces to the mCH (1.5). As for the formulae (5.18a)-(5.18b) in Theorem 5.13, the identical mass setting

\[ n_j = m_j, \]

leads to

\[ v = u = \sum_{j=1}^{N} m_j(t) e^{-|x - x_j(t)|}, \quad n = m = 2 \sum_{j=1}^{N} m_j \delta_{x_j}, \]

In this case we have

\[ e_j = \frac{1}{m^2_j}, \]

and the formulae (5.18a)-(5.18b) in Theorem 5.13 reduce to the multipeakon formulae for the mCH equation (see [6, Theorem 4.21]). Moreover, it is not hard to see that the global existence condition and the long time asymptotics in Section 6 and 7 also cover those for the mCH equation in [6].

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APPENDIX A. LAX PAIR FOR THE 2-MCH PEAKON ODEs

The purpose of this appendix is to give a rigorous interpretation for the distributional Lax pair of the 2-mCH equation in the generic case, i.e. the Lax pair of the 2-mCH peakon ODEs (1.10). We note that the transition from the smooth sector to the distribution sector is not canonical. The argument presented below is closer to that in Appendix A of [6] than to the one used for the interlacing case of the 2-mCH [4]. Let us begin by reviewing some notation needed to present the argument.

Let \( \Omega_k \) denote the region \( x_k(t) < x < x_{k+1}(t) \), where \( x_k \) are smooth functions such that \( -\infty = x_0(t) < x_1(t) < \cdots < x_N(t) < x_{N+1}(t) = +\infty \).

Let the function space \( PC^\infty \) consist of all piecewise smooth functions \( f(x,t) \) such that the restriction of \( f \) to each region \( \Omega_k \) is a smooth function \( f_k(x,t) \) defined on \( \Omega_k \). Then, for each fixed \( t \), \( f(x,t) \) defines a regular distribution \( T_f(t) \) [18] in the class \( \mathcal{D}'(R) \) (for simplicity we will write \( f \) instead of \( T_f \)). Since distributions do not in general have values at individual points, we do not require \( f(x_k) \) to be defined. However, the left and right limits are defined. Let us denote them \( f_k(x_k-,t) \) and \( f_k(x_k+,t) \) and set

\[
\langle f \rangle(x_k) = \frac{f(x_k+,t) - f(x_k-,t)}{2},
\]

to denote the jump and the average, respectively. Denote by \( \partial_x \) (or \( \partial_t \)) the ordinary (classical) partial derivative with respect to \( x \) (or \( t \)), and by \( \frac{\partial f_k}{\partial x} \) (or \( \frac{\partial f_k}{\partial t} \)) their restrictions to \( \Omega_k \). Let \( D_x f \) denote the distributional derivative with respect to \( x \). Then we have a well-known identity

\[
D_x f = f_x + \sum_{k=1}^{N} \langle [f](x_k) \rangle \delta_{x_k}.
\]

Likewise, we can define \( D_t f \) as a distributional limit

\[
D_t f(t) = \lim_{a \to 0} \frac{f(t+a) - f(t)}{a},
\]

which can be computed explicitly to be

\[
D_t f = f_t - \sum_{k=1}^{N} \hat{x}_k \langle f(x_k) \rangle \delta_{x_k},
\]

where \( \hat{x}_k = dx_k/dt \).

The following formulas will be useful in further analysis:

\[
\langle fg \rangle = \langle f \rangle \langle g \rangle + \langle f \rangle \langle g \rangle, \\
\langle fg \rangle = \langle f \rangle \langle g \rangle + \frac{1}{4} \langle f \rangle \langle g \rangle (A.1)
\]

\[
\frac{d}{dt} \langle f \rangle(x_k) = \langle f_x \rangle(x_k) \hat{x}_k + \langle f_t \rangle(x_k), (A.2)
\]

\[
\frac{d}{dt} \langle f \rangle(x_k) = \langle f_x \rangle(x_k) \hat{x}_k + \langle f_t \rangle(x_k), (A.3)
\]

for any \( f, g \in PC^\infty \).

It is easy to see that the peakon solution \( u(x,t), v(x,t) \) and the corresponding functions \( \Psi_1, \Psi_2 \) in the Lax pair (2.1) are piecewise smooth functions of class \( PC^\infty \). Moreover, \( u, v \) are continuous throughout \( R \) but \( u_k, v_k, \Psi_1, \Psi_2 \) have a jump at each \( x_k \).

Let us now set \( \Psi = (\Psi_1, \Psi_2)^T \), and consider an overdetermined system (see (2.1))

\[
D_x \Psi = \frac{1}{2} \hat{L} \Psi, \quad D_t \Psi = \frac{1}{2} \hat{A} \Psi, (A.4)
\]
where

\[ \hat{L} = L + 2\lambda \left( \sum_{k=1}^{N} M_k \delta_{x_k} \right), \quad (A.5) \]

\[ \hat{A} = A - 2\lambda \left( \sum_{k=1}^{N} M_k Q(x_k) \delta_{x_k} \right) \quad (A.6) \]

with

\[ L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_k = \begin{pmatrix} 0 & m_k \\ -n_k & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 4\lambda^{-2} + Q \\ 2\lambda^{-1}(u - u_x) \end{pmatrix} \]

and \( Q = (u - u_x)(v + v_x) \). We observe that the above splitting of the Lax pair corresponds to the distributional splitting into a regular distribution and a distribution with a singular support. In particular the singular part of (A.5) requires that we multiply \( M_k \Psi(x_k) \delta_{x_k} \) by \( \delta_{x_k} \). This is not defined, and to define this multiplication we need to assign some values to \( \Psi_1, \Psi_2 \) at \( x_k \). This, in principle, is an arbitrary procedure but we want it to reflect the local behaviour of \( \Psi \) around \( x_k \) and postulate that our choice of values of \( \Psi \) depends linearly on the right and left hand limits.

Likewise, for the \( \tau \)-Lax equation (A.6) to be defined as a distribution equation, \( QM_k \Psi(x_k) \delta_{x_k} \) needs to be a multiplier of \( \delta_{x_k} \). Thus, in the same sense as above, the values of \( Q(x_k) \) need to be assigned as well.

Henceforth, we will refer to these assignments as regularizations, even though this name in the theory of distributions refers usually to re-defining divergent integrals.

We now specify more concretely a family of regularizations we consider. We will refer to them as invariant regularizations. The reason for that name is explained in [6, Appendix A].

**Definition A.1.** An invariant regularization of the Lax pair (A.4) consists of specifying the values of \( \alpha, \beta \in \mathbb{R} \) and \( Q(x_k) = (u - u_x)(v + v_x)(x_k) \) in the formulas

\[ \Psi(x) \delta_{x_k} = \Psi(x_k) \delta_{x_k}, \]

\[ \Psi'(x_k) = \alpha \Psi(x_k) + \beta \langle \Psi \rangle(x_k), \]

\[ Q(x) \delta_{x_k} = Q(x_k) \delta_{x_k}. \]

**Theorem A.2.** Let \( m \) and \( n \) be the discrete measures associated to \( u \) and \( v \) defined by (1.8). Given an invariant regularization in the sense of Definition A.1 the distributional Lax pair (A.4) is compatible, i.e. \( D_t D_x \Psi = D_x D_t \Psi \), if and only if the following conditions hold:

\[ \beta = 1, \quad \alpha = \pm \frac{1}{2}. \]

Then

\[ \dot{m}_k = -m_k(Q(x_k) - \langle Q \rangle(x_k)), \]

\[ \dot{n}_k = n_k(Q(x_k) - \langle Q \rangle(x_k)), \]

\[ \dot{x}_k = Q(x_k). \]

**Proof.** The proof follows almost verbatim [6, Theorem A.2] (also see [4, 10]). We omit the details. \( \Box \)

Now we can spell out the connection of the regularization problem to the content of the present paper.
Corollary A.3. Let \( m \) and \( n \) be the discrete measures associated to \( u \) and \( v \) defined by (1.8). Suppose that \( Q(x_k), \Psi(x_k) \) are assigned values

\[
Q(x_k) = \langle Q \rangle(x_k), \quad \Psi(x_k) = \Psi(x_k^+),
\]
or

\[
Q(x_k) = \langle Q \rangle(x_k), \quad \Psi(x_k) = \Psi(x_k^-).
\]

For either case, the compatibility condition \( D_1 D_x \Psi = D_x D_1 \Psi \) for the distributional Lax pair (A.4) reads

\[
\dot{m}_k = 0, \quad \dot{n}_k = 0, \quad \dot{x}_k = \langle Q \rangle(x_k).
\]

Appendix B. Proof of Theorem 6.5

Proof. It suffices to ensure that the ordering conditions \( x_1 < x_2 < \cdots < x_{2K} \) hold for all time \( t \). We write these conditions as:

\[
x_{(k+1)'} < x_k', \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1, \quad (B.1a)
\]

\[
x_{(k+2)'} < x_{(k+1)'}, \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 3, \quad (B.1b)
\]

and use equations (5.18a), (5.18b) to obtain equivalent conditions

\[
\frac{1}{m_{(k+1)'} n_{k'}} < \frac{\mathcal{D}_{(k+1)}^{(k+1)} \mathcal{D}_{k-1}^{(k),0}}{\mathcal{D}_{k+1}^{(k+1),0} \mathcal{D}_{k-1}^{(k),1}}, \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1, \quad (B.2a)
\]

\[
\frac{1}{m_{(k+2)'} n_{(k+1)'}} < \frac{\mathcal{D}_{k+2}^{(k+2),0} \mathcal{D}_{k+1}^{(k+1)} \mathcal{D}_{k-1}^{(k),1}}{\mathcal{D}_{k+2}^{(k+2),0} \mathcal{D}_{k+1}^{(k+1),0} \mathcal{D}_{k-1}^{(k),1}}, \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 3. \quad (B.2b)
\]

However, (6.3) implies that the inequality

\[
\frac{\mathcal{D}_{k+1}^{(k+1),0} \mathcal{D}_{k-1}^{(k),1}}{\mathcal{D}_{k+1}^{(k+1),0} \mathcal{D}_{k-1}^{(k),1}} > \frac{\zeta_k^{(k+1)}}{\zeta_k^{(k-1)}} \quad (B.3)
\]

holds uniformly in \( t \) and if we impose

\[
\frac{1}{m_{(k+1)'} n_{k'}} < \frac{\zeta_k^{(k+1)}}{\zeta_k^{(k-1)}}, \quad \text{for all odd } k, \quad k \leq 2K - 1,
\]

then (B.1a) holds.

Now we focus on the second inequality, namely (B.2b), which is valid whenever \( K \geq 2 \).

It is convenient to consider a slightly more general expression, namely,

\[
\frac{\mathcal{D}_{k+2}^{(l+1),0} \mathcal{D}_{k}^{(l),0}}{\mathcal{D}_{k+2}^{(l+1)},0} \mathcal{D}_{k}^{(l-1),1}} \quad 1 \leq l \leq K - 1,
\]

for which, using similar steps to those to in [6], we can prove the bound

\[
\frac{\mathcal{D}_{k+2}^{(l+1),0} \mathcal{D}_{k}^{(l),0}}{\mathcal{D}_{k+2}^{(l+1),0} \mathcal{D}_{k}^{(l-1),1}} > \frac{\zeta_k^{(l+1)}}{\zeta_k^{(l-1)}} \frac{\min \{ \zeta_j, \zeta_j^{(l+1)} \}}{\zeta_j^{(l-1)}} (e_{k+1} + e_{k+2} + \zeta_k) \cdot (1 + m_{(k+1)'}) \cdot (1 + m_{(k+2)'}) \cdot (1 + m_{(k+2)'})
\]

\[
= \frac{\zeta_k^{(l+1)}}{\zeta_k^{(l-1)}} \frac{\min \{ \zeta_j, \zeta_j^{(l+1)} \}}{\zeta_j^{(l-1)}} \frac{(1 + m_{(k+1)'}) n_{(k+1)'} (1 + m_{(k+2)'}) n_{(k+2)'} (1 + m_{(k+2)'})}{m_{(k+1)'}} \cdot \frac{(1 + m_{(k+2)'}) n_{(k+2)'}}{m_{(k+2)'}}.
\]
Hence, specializing to \( l = \frac{k+1}{2} \) and assuming

\[
\frac{1}{m_{(k+2)'} n_{(k+1)''}} < \frac{\xi_{\frac{k+1}{2}}}{2} \left( 1 + m_{(k+1)'n_{(k+1)''}} \xi_k \right),
\]

we obtain that (B.2b) and thus (B.1b) hold.

Finally, after rewriting the last condition as:

\[
\frac{n_{(k+2)'} m_{(k+1)'}}{(1 + m_{(k+1)'n_{(k+1)''}} \xi_k)} < \frac{\xi_{\frac{k+1}{2}}}{2} \left( 1 + m_{(k+1)'n_{(k+1)''}} \xi_k \right),
\]

we obtain the second condition (6.6b).

\[
\text{APPENDIX C. PROOF OF THEOREM 7.6}
\]

\textbf{Proof.} Again, it suffices to ensure that the ordering conditions \( x_1 < x_2 < \cdots < x_{2K+1} \) hold for all time \( t \). We write these conditions in an equivalent form:

\[
x_{(k+1)'} < x_{k'}, \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1, \quad \text{(C.1a)}
\]

\[
x_{(k+2)'} < x_{(k+1)'}, \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1, \quad \text{(C.1b)}
\]

and use equations (5.18a), (5.18b) to obtain

\[
\frac{1}{m_{(k+2)'} n_{k'}} < \frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (c) \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (c)}{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 0)} (c) \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 1)} (c)}, \quad \text{for all odd } k, 1 \leq k \leq 2K - 1, \quad \text{(C.2a)}
\]

\[
\frac{1}{m_{(k+2)'} n_{k'}} < \frac{\mathcal{D}_{k+2}^{(\frac{k+1}{2}, 1)} (c) \mathcal{D}_{k}^{(\frac{k+1}{2}, 0)} (c)}{\mathcal{D}_{k+2}^{(\frac{k+1}{2}, 0)} (c) \mathcal{D}_{k}^{(\frac{k+1}{2}, 1)} (c)}, \quad \text{for all odd } k, 1 \leq k \leq 2K - 1. \quad \text{(C.2b)}
\]

From (7.6a) and (7.6b) we obtain

\[
\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (c) \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (c) = \left( \mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (0) + c \mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (0) \right) \left( \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (0) + c \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (0) \right) \defeq \mathcal{A}_{1} + \mathcal{B}_{1},
\]

\[
\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 0)} (0) \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 1)} (c) = \left( \mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (0) + c \mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)} (0) \right) \left( \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (0) + c \mathcal{D}_{k-1}^{(\frac{k+1}{2}, 0)} (0) \right) \defeq \mathcal{A}_{2} + \mathcal{B}_{2},
\]
where $\mathcal{B}_2 = 0$ for $k = 1$. Hence the ratios $\mathcal{A}_1 / \mathcal{A}_2$, $\mathcal{B}_1 / \mathcal{B}_2$ satisfy (uniform in $t$) bounds

$$\frac{\mathcal{A}_1}{\mathcal{A}_2} > \frac{\zeta_1^{k+1}}{\zeta_k^2} \equiv \alpha,$$

$$\frac{\mathcal{B}_1}{\mathcal{B}_2} > \frac{\zeta_1^{k+1}}{\zeta_k^2} \frac{2 \min_j (\zeta_{j+1} - \zeta_j)^{k-1}}{(1 + m_k n_k \zeta_1)(1 + m_{(k+1)'} n_{(k+1)'} \zeta_1)} \frac{(1 + m_k n_k (k-1) (\zeta_K - \zeta_1)^{k-1})}{m_k n_k m_{(k+1)'} n_{(k+1)'}},$$

by equations (B.3) and (B.4), respectively, with the convention that $\beta = \infty$ for the special case $k = 1$. Thus

$$\min(\alpha, \beta) < \frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)}}{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 0)}} \frac{\mathcal{D}_{k-1}^{(\frac{k-1}{2}, 0)}}{\mathcal{D}_{k-1}^{(\frac{k-1}{2}, 1)}}(c)$$

holds uniformly in $t$ and if we impose

$$\frac{1}{m_{(k+1)'} n_{k'}} < \min(\alpha, \beta_1) \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1,$$

then equations (C.1a) will hold automatically.

Now we turn to the second inequality (C.2b). From Corollary 7.3 we obtain

$$\frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)}}{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 0)}}(c) \frac{\mathcal{D}_{k}^{(\frac{k-1}{2}, 1)}}{\mathcal{D}_{k}^{(\frac{k-1}{2}, 0)}}(c) > \frac{\mathcal{A}_1 + \mathcal{B}_1}{\mathcal{A}_2 + \mathcal{B}_2}.$$

Since

$$\frac{\mathcal{A}_1}{\mathcal{A}_2} > \frac{\zeta_1^{k+1}}{\zeta_k^2} \equiv \alpha,$$

$$\frac{\mathcal{B}_1}{\mathcal{B}_2} > \frac{\zeta_1^{k+1}}{\zeta_k^2} \frac{2 \min_j (\zeta_{j+1} - \zeta_j)^{k-1}}{(1 + m_k n_k \zeta_1)(1 + m_{(k+1)'} n_{(k+1)'} \zeta_1)} \frac{(1 + m_k n_k (k+1) (\zeta_K - \zeta_1)^{k+1})}{m_k n_k m_{(k+1)'} n_{(k+1)'}},$$

then

$$\min(\alpha, \beta_1) < \frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 1)}}{\mathcal{D}_{k+1}^{(\frac{k+1}{2}, 0)}} \frac{\mathcal{D}_{k}^{(\frac{k-1}{2}, 1)}}{\mathcal{D}_{k}^{(\frac{k-1}{2}, 0)}}(c)$$

is satisfied. Thus inequality

$$\frac{1}{m_{(k+2)'} n_{(k+1)'}} < \min(\alpha, \beta_1), \quad \text{for all odd } k, \quad 1 \leq k \leq 2K - 1,$$

implies (C.2b) and, consequently, (C.1b), thereby proving the claim.

\[\Box\]


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