THE STABILIZER BITORSORS OF THE MODULE AND ALGEBRA HARMONIC COPRODUCTS ARE EQUAL

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ABSTRACT. In earlier work, we constructed a pair of “Betti” and “de Rham” Hopf algebras and a pair of module-coalgebras over this pair, as well as the bitorsors related to both structures (which will be called the “module” and “algebra” stabilizer bitorsors). We showed that Racinet’s torsor constructed out of the double shuffle and regularization relations between multiple zeta values is essentially equal to the “module” stabilizer bitorsor, and that the latter is contained in the “algebra” stabilizer bitorsor. In this paper, we show the equality of the “algebra” and “module” stabilizer bitorsors. We reduce the proof to showing the equality of the associated “algebra” and “module” graded Lie algebras. The argument for showing this equality involves the relation of the “algebra” Lie algebra with the kernel of a linear map, the expression of this linear map as a composition of three linear maps, the relation of one of them with the “module” Lie algebra and the computation of the kernel of the other one by discrete topology arguments.

1. Introduction
2. The stabilizer Lie algebra \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \)
   2.1. The ambient Lie algebras \( \mathfrak{lie}(e_0, e_1), \mathfrak{lie}(v_0, v) \) and \( \mathfrak{lie}(v_0, v) \)
   2.2. Lie algebra action of \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \) on \( \mathfrak{Hom}_{\mathfrak{lie}}(W, W^{\otimes 2}) \)
   2.3. Corestriction of \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \)
   2.4. The stabilizer Lie algebras \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \) and \( \mathfrak{stab}(\mathfrak{lie}(e_0, e_1))(\Delta^W) \)
   2.5. Relation with the Lie algebra \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \) from \( \mathfrak{EF}^2 \)
   2.6. The stabilizer Lie algebra \( \mathfrak{stab}(\hat{\Delta}^{M, DR}) \)
   2.7. Lie algebra action of \( \mathfrak{stab}(\mathfrak{lie}(e_0, e_1))(\Delta^M) \)
   2.8. The stabilizer Lie algebra \( \mathfrak{stab}(\mathfrak{lie}(e_0, e_1))(\Delta^M) \) from \( \mathfrak{EF}^2 \)
   2.9. Relation with the Lie algebra \( \mathfrak{stab}(\mathfrak{lie}(e_0, e_1))(\Delta^M) \) from \( \mathfrak{EF}^2 \)
   2.10. Relation of \( \mathfrak{stab}(\mathfrak{lie}(e_0, e_1))(\Delta^M) \) with \( \mathcal{P}(\mathcal{M}) \) (Theorem 3.10 from \( \mathfrak{EF}^0 \))
   3. Decomposition of the map \( \mathfrak{Y}_0 \rightarrow \text{Der}_{\Delta^W}(W, W^{\otimes 2}) \)
   4. The map \( \mathfrak{H} : \mathfrak{Y}_0 \rightarrow (\oplus_{k\geq 0} W^{\otimes 2})^{\otimes 2} \oplus (\oplus_{k\geq 0} W) \)
   5. The map \( \mathfrak{h} : (\oplus_{k\geq 0} W^{\otimes 2})^{\otimes 2} \oplus (\oplus_{k\geq 0} W) \rightarrow \text{Map}(Z_{>0}, W^{\otimes 2}) \)
   6. The isomorphism \( \mathfrak{i} : \text{Map}(Z_{>0}, W^{\otimes 2}) \rightarrow \text{Der}_{\Delta^W}(W, W^{\otimes 2}) \)
   7. Decomposition of the map \( \mathfrak{Y}_0 \rightarrow \text{Der}_{\Delta^W}(W, W^{\otimes 2}) \)
   8. Study of the constituents \( \mathfrak{h} \) and \( \mathfrak{H} \) of the map \( \mathfrak{Y}_0 \rightarrow \text{Der}_{\Delta^W}(W, W^{\otimes 2}) \)
   9. The endomorphisms \( \partial_\alpha \) and degrees \( \deg^{(1)} \) and \( \deg^{(2)} \)
   10. Convergence results in the discrete topology
   11. Opposite versions of the results of \S\S\ 5.1 and 5.2
   12. Computation of \( \text{Ker}(\mathfrak{h}) \)
   13. A commutative square relating \( \mathfrak{H} \) and \( \Delta^M \) and a commutative triangle
   14. Proof of equality results
      14.1. Proof of the equality of the stabilizer Lie algebras \( \mathfrak{stab}(\hat{\Delta}^{W, DR}) \) and \( \mathfrak{stab}(\hat{\Delta}^{M, DR}) \)
      14.2. Proof of the equality of the stabilizer bitorsors \( \text{Stab}(\hat{\Delta}^{M, DR/B})(k) \) and \( \text{Stab}(\hat{\Delta}^{W, DR/B})(k) \)
   15. References
1. Introduction

The multiple zeta values (MZVs) are the real numbers defined by

$$\zeta(n_1, \ldots, n_s) := \sum_{k_1 > \cdots > k_s > 0} 1/(k_1^{n_1} \cdots k_s^{n_s})$$

for $s \geq 1$ and $n_1 \geq 2$, $n_2, \ldots, n_s \geq 1$ ([Z]). Two sets of algebraic relations between these numbers are particularly known: the double shuffle and regularization relations ([R] [IKZ], see also [EC] where these relations are given without proof) and the associator relations which follow from a combination of [DR] and [LM]. Both sets of relations give rise to $\mathbb{Q}$-schemes denoted $DMR$ and $M$, both of which are equipped with torsor structures. These sets of relations are conjectured to be equivalent in Racinet’s PhD thesis, this being equivalent to the equality of $Q$-we attach to each $\mathbb{Q}$-algebra $k$ (see [EF3], §3.9). Denote by $k$-alg (resp. $k$-Hopf, $k$-alg-mod, $k$-HAMC), the category of $k$-algebras (resp. $k$-Hopf algebras, pairs $(A, M)$ of a $k$-algebra $A$ and a $k$-module $M$ over $A$, pairs $((A, \Delta_A), (M, \Delta_M))$ of a $k$-Hopf algebra $(A, \Delta_A)$ and a $k$-module coalgebra $(M, \Delta_M)$ over it). For $\omega \in \{B, DR\}$, set $(\hat{\mathcal{W}}, \hat{\Delta^W}, \omega)$ and $((\hat{\mathcal{W}}, \hat{\Delta^W}), (\hat{\mathcal{M}}, \hat{\Delta^M}), \omega)$.

In the series of papers [EF0] [EF1] [EF2] [EF3], we give a proof of the inclusion $M \subset DMR$ based on the ideas of [DeT]. The fibered product bitorsors relies the geometric interpretations of $(\hat{\mathcal{W}}, \hat{\Delta^W})$ over it, by which one understand that the action map $\hat{\mathcal{W}} \times (\hat{\mathcal{W}}, \hat{\Delta^W}, \omega)$ and $\hat{\mathcal{M}} \times (\hat{\mathcal{M}}, \hat{\Delta^M}, \omega)$, which are based on the ideas of [DeT]. The fibered product bitorsors $G^{DR,B}(k) \times_{ISO_k-\text{alg-mod}}(\hat{\mathcal{W}}, \hat{\Delta^W}) \to ISO_k-\text{alg}(\hat{\mathcal{W}})$ and $G^{DR,B}(k) \times_{ISO_k-\text{alg-mod}}(\hat{\mathcal{W}}, \hat{\Delta^W}) \to ISO_k-\text{alg}(\hat{\mathcal{W}})$.

For each category $C$ and map $\{B, DR\} \to \text{Ob}(C)$ we denote $\omega \mapsto X_{\omega}$, we denote by $ISO_C(X_{DR/B})$ the bitorsors $ISO_C(X_{B}, X_{DR})$; the construction of the morphism $(\ast)$ relies the geometric interpretations of $(\hat{\mathcal{W}}, \hat{\Delta^W}, \omega)$ and $(\hat{\mathcal{M}}, \hat{\Delta^M}, \omega)$, which are based on the ideas of [DeT]. The fibered product bitorsors

$$G^{DR,B}(k) \times_{ISO_k-\text{alg-mod}}(\hat{\mathcal{W}}, \hat{\Delta^W}) \to ISO_k-\text{alg}(\hat{\mathcal{W}})$$

and

$$G^{DR,B}(k) \times_{ISO_k-\text{alg-mod}}(\hat{\mathcal{W}}, \hat{\Delta^W}) \to ISO_k-\text{alg}(\hat{\mathcal{W}})$$
are shown to be equal to bitorsors denoted respectively $\text{Stab}(\hat{\Delta}^{W,\text{DR}})(k)$ and $\text{Stab}(\hat{\Delta}^{M,\text{DR}})(k)$ (see [EF3], §3.9). The vertical maps being injective, the above diagram leads to a sequence of inclusions of bitorsors

$$M(k) \hookrightarrow \text{Stab}(\hat{\Delta}^{M,\text{DR}})(k) \hookrightarrow \text{Stab}(\hat{\Delta}^{W,\text{DR}})(k) \hookrightarrow G^{\text{DR},B}(k).$$

The inclusion $M(k) \subset DMR(k)$ then follows from the identification $DMR(k) = G_{\text{quad}}^B(k) \cap \text{Stab}(\hat{\Delta}^{M,\text{DR}})(k)$ (based on [EF0]) and the easy inclusion $M(k) \subset G_{\text{quad}}^B(k)$, where $G_{\text{quad}}^B(k) \subset G^{\text{DR},B}(k)$ is a subbitorsor defined by quadratic conditions (see [EF3], §3.1).

The main result of the present paper is:

**Theorem 1.1.** (see Theorem 6.7) The inclusion $\text{Stab}(\hat{\Delta}^{M,\text{DR}})(k) \hookrightarrow \text{Stab}(\hat{\Delta}^{W,\text{DR}})(k)$ is an equality of bitorsors.

Here is an outline of the proof. One reduces the proof of the equality of these bitorsors to that of the underlying $\mathbb{Q}$-group schemes $\text{Stab}(\hat{\Delta}^{M,\text{DR}})(k)$ and $\text{Stab}(\hat{\Delta}^{W,\text{DR}})(k)$, and then to that of their Lie algebra $\text{stab}(\hat{\Delta}^{M,\text{DR}})$ and $\text{stab}(\hat{\Delta}^{W,\text{DR}})$ (see proof of Theorem 6.7). Both Lie algebras are degree completions of semidirect products by a one-dimensional Lie algebra of graded Lie algebras $\text{stab}_{\text{Hic}(e_0,e_1)}(\Delta^M)$ and $\text{stab}_{\text{Hic}(e_0,e_1)}(\Delta^W)$, which are stabilizer Lie algebras of linear maps $\Delta^M$ and $\Delta^W$ (see §§2.2, 3.2 and Propositions 2.12, 3.3). This reduces the proof to that of the equality of these graded Lie algebras (see proof of Theorem 6.3).

This equality is proved as follows. One first expresses $\text{stab}_{\text{Hic}(e_0,e_1)}(\Delta^W)$ as the preimage by a Lie algebra morphism $\theta : \text{lie}(e_0,e_1) \to \mathcal{V}_0$ of the kernel of a linear map $- \cdot \Delta^W : \mathcal{V}_0 \to \text{Der}_{\Delta^W}(\mathcal{W},\mathcal{W}^{\otimes 2})$ (Lemma 2.5). One then decomposes $- \cdot \Delta^W$ as a composition $i \circ h \circ H$, where $i$, $h$ and $H$ are linear maps (§4.3). One shows that $i$ is injective (§4.3), computes the kernel of $h$ by discrete topology methods (§5.3), and relates $H$ with $\Delta^M$ (§5.4). This leads to a proof of the inclusion of the kernel of $- \cdot \Delta^W$ in a space $(- \cdot 1_M)^{-1}(\mathcal{P}(\mathcal{M}))$ defined in terms of $\Delta^M$ (Proposition 6.1). Using the main result of [EF0], one relates the preimage by $\theta$ of $(- \cdot 1_M)^{-1}(\mathcal{P}(\mathcal{M}))$ with $\text{stab}_{\text{Hic}(e_0,e_1)}(\Delta^M)$ (Proposition 5.7), which combined with the other results leads to the announced statement (Theorem 6.6).

**Notation.** For $A$ a $\mathbb{Q}$-algebra, we denote by $\text{Der}(A)$ the Lie algebra of its derivations. If $V,W$ are $\mathbb{Q}$-vector spaces, we denote by $\text{Hom}_{\mathbb{Q}-\text{vec}}(V,W)$ the $\mathbb{Q}$-vector space of linear maps from $V$ to $W$. We set $\text{End}_{\mathbb{Q}-\text{vec}}(V) := \text{Hom}_{\mathbb{Q}-\text{vec}}(V,V)$.

If $V$ is a $\mathbb{Z}$-graded vector space and $d \in \mathbb{Z}$, we denote by $V[d]$ the degree $d$ component of $V$, so $V = \bigoplus_{d \in \mathbb{Z}} V[d]$. For $v \in V$, we denote by $v[d]$ the component of $v$ of degree $d$, so $v[d] \in V[d]$ and $v = \sum_{d \in \mathbb{Z}} v[d]$.

2. **The stabilizer Lie algebra $\text{stab}(\hat{\Delta}^{W,\text{DR}})$**

This section deals with the Lie algebra $\text{stab}(\hat{\Delta}^{W,\text{DR}})$ from [EF2]. More precisely, in §2.1 we introduce Lie algebras $(\text{lie}(e_0,e_1),\{,\})$ and $(\mathcal{V}_0,\{,\})$ and a Lie algebra morphism $\theta : \text{lie}(e_0,e_1) \to \text{lie}(e_0,e_1)$.
$\mathcal{V}_0$. In §2.2 we recall from [EF1] the algebra $\mathcal{W}^{\text{DR}}$ (here denoted Lie) and the action of $\mathcal{V}_0$ on the vector space $\text{Hom}_{\mathbb{Q}-\text{vec}}(\mathcal{W}, \mathcal{W}^\otimes 2)$; this leads to a stabilizer Lie subalgebra $\text{stab}_{\mathcal{V}_0}(h)$ of $\mathcal{V}_0$ for any $h$ in this vector space. In §2.3 we show that if $h \in \text{Hom}_{\mathbb{Q}-\text{alg}}(\mathcal{W}, \mathcal{W}^\otimes 2)$, then $\text{stab}_{\mathcal{V}_0}(h)$ may be identified with the kernel of a linear map $\mathcal{V}_0 \to \text{Der}_h(\mathcal{W}, \mathcal{W}^\otimes 2)$ where the target is the set of $h$-derivations of the algebra $\mathcal{W}$ with values in $\mathcal{W}^\otimes 2$, viewed as a $\mathcal{W}$-bimodule via $h$. In §2.4 we show that when $h$ is equal to the algebra harmonic coproduct $\Delta^{\text{W,DR}}$ from [EF1] (here denoted $\Delta^W$), then the preimage under $\theta$ of $\text{stab}_{\mathcal{V}_0}(h)$ is a graded Lie subalgebra of $(\text{Lie}(e_0, e_1), (\cdot, \cdot))$, which after undergoing degree completion and semidirect product with the grading action of $\mathbb{Q}1$, gives the Lie algebra $\text{stab}(\Delta^{\text{W,DR}})$ from [EF2] (see Proposition 2.12).

2.1. **The ambient Lie algebras** $(\text{Lie}(e_0, e_1), (\cdot, \cdot))$ and $(\mathcal{V}_0, (\cdot, \cdot))$. Denote by $\mathcal{V}$ the free associative $\mathbb{Q}$-algebra with generators $e_0, e_1$. Let $\text{Lie}(e_0, e_1) \subset \mathcal{V}$ be the Lie subalgebra generated by $e_0, e_1$ (these objects are respectively denoted $\mathcal{V}^{\text{DR}}$ and $f_2$ in [EF1], §1.1 when $k = \mathbb{Q}$). Then $\mathcal{V}$ and $\text{Lie}(e_0, e_1)$ are equipped with compatible associative and Lie algebra $\mathbb{Z}_{\geq 0}$-gradings, where $e_0$ and $e_1$ have degree 1. Let $\mathcal{V}_0$ be the direct sum of components of $\mathcal{V}$ of positive degree.

For $v \in \mathcal{V}_0$, let $\text{der}^{\mathcal{V}}_{v}(1)$ be the algebra derivation of $\mathcal{V}$ such that

$$\text{der}^{\mathcal{V}}_{v}(1) : e_0 \mapsto [v, e_0], e_1 \mapsto 0.$$ 

For $v, v' \in \mathcal{V}_0$, set

$$(v, v') := \text{der}^{\mathcal{V}}_{v}(1)(v') - \text{der}^{\mathcal{V}}_{v}(1)(v) + [v', v].$$

**Lemma 2.1.**

(a) $(\mathcal{V}_0, (\cdot, \cdot))$ is a $\mathbb{Z}_{\geq 0}$-graded Lie algebra, of which $\text{Lie}(e_0, e_1)$ is a graded Lie subalgebra.

(b) The map $\text{der}^{\mathcal{V}}_{v}(1) : (\mathcal{V}_0, (\cdot, \cdot)) \to \text{Der}(\mathcal{V})$ is a Lie algebra morphism.

**Proof.** (a) Let $\hat{\mathcal{V}}_0$ be the degree completion of $\mathcal{V}_0$ (it is denoted $(\hat{\mathcal{V}}_0^{\text{DR}})_{\mathbb{Q}}$ in [EF2], §3.5). In [EF2], Lemma 3.8 (b), the space $\text{em}^{\text{DR}} := \mathbb{Q} \oplus \hat{\mathcal{V}}_0$ is equipped with a Lie bracket $(\cdot, \cdot)$. One checks that $\hat{\mathcal{V}}_0$ and $\mathcal{V}_0$ are preserved by this bracket, and are therefore Lie subalgebras of $\text{em}^{\text{DR}}$. One also checks the bracket to be graded.

Let also $\text{Lie}(e_0, e_1)^{\wedge}$ be the degree completion of $\text{Lie}(e_0, e_1)$ (these spaces are denoted $(f_2)_{\mathbb{Q}}^{\wedge}$ and $(f_2)_{\mathbb{Q}}$ in [EF2], §3.5). In loc. cit., it is proved that the space $\mathfrak{g}^{\text{DR}} := \mathbb{Q} \oplus \text{Lie}(e_0, e_1)^{\wedge}$ is a Lie subalgebra of $\text{em}^{\text{DR}}$. One checks that $\text{Lie}(e_0, e_1)^{\wedge}$ and $\text{Lie}(e_0, e_1)$ are Lie subalgebras of $\mathfrak{g}^{\text{DR}}$.

(b) In [EF2], Lemma 3.9, (c), a Lie algebra morphism denoted $\text{em}^{\text{DR}} \to \text{Der}(\hat{\mathcal{V}}_0, \hat{\mathcal{V}}_0)$, $(\nu, x) \mapsto (\text{der}^{\mathcal{V}}_{(\nu, x)}(1), \text{der}^{\mathcal{V}}_{(\nu, x)}(10))$ is constructed. By [EF2], Lemma 3.9 (a), $\text{Der}(\hat{\mathcal{V}}_0, \hat{\mathcal{V}}_0)$ is a Lie subalgebra of $\text{Der}(\hat{\mathcal{V}}_0) \times \text{End}_{\mathbb{Q}-\text{vec}}(\hat{\mathcal{V}}_0)$, so that the projection on the first factor of this product is a Lie algebra morphism. Post-composing the Lie algebra morphism $\text{em}^{\text{DR}} \to \text{Der}(\hat{\mathcal{V}}_0, \hat{\mathcal{V}}_0)$ with this projection and pre-composing it with the inclusion of $\mathcal{V}_0$, one obtains a Lie algebra morphism $\mathcal{V}_0 \to \text{Der}(\hat{\mathcal{V}})$, whose image can be shown to be contained in the Lie subalgebra $\text{Der}(\mathcal{V})$ of $\text{Der}$. The resulting map $\mathcal{V}_0 \to \text{Der}(\mathcal{V})$ is therefore a Lie algebra morphism, and one checks it to be given by $x \mapsto \text{der}^{\mathcal{V}}_{x}(1).$
Lemma 2.2. The map \( \theta : \mathfrak{lie}(e_0, e_1) \to \mathcal{V}_0 \) be the map defined by \( \theta(x) := x - (x|e_0)e_0 + \sum_{n \geq 1}(1/n)(x|e_0|_{n-1}e_1)e_{n1} \), where \((-|w)w \text{ word in } e_0, e_1) \) is the collection of maps \( \mathcal{V} \to \mathbb{Q} \) such that \( x = \sum_{w \text{ word in } e_0, e_1}(x|w|w)w \).

Proof. It follows from [EF2], Lemma 3.8 (c), that the map \( \theta \) is a subalgebra (it is denoted \( \mathcal{W} \)).

Lemma 2.3. (a) For \( \mathcal{V}_0, (,) \) is a subalgebra (it is denoted \( \mathcal{W} \)). From [EF1], Lemma 3.10, (b).

Proof. (a) follows from the statement preceding [EF2], Lemma 3.10; it is an immediate consequence of \( \mathcal{v} \).

(b) The map \( \mathcal{V}_0, (,) \) is a Lie algebra morphism.

Lemma 2.4. The \( \mathcal{Q} \)-vector space \( \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \) is equipped with a \( (\mathcal{V}_0, (,)) \)-module structure, the action of \( \mathcal{V}_0 \) on \( \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \) being given by

\[
(2.2.1) \quad v \cdot h := (\mathcal{der}_v^{\mathcal{W}, (1)} \otimes \mathcal{id} + \mathcal{id} \otimes \mathcal{der}_v^{\mathcal{W}, (1)}) \circ h - h \circ \mathcal{der}_v^{\mathcal{W}, (1)}.
\]

For \( \mathcal{h} \in \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \), we denote by \( - \cdot : \mathcal{V}_0 \to \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \) the map \( v \mapsto v \cdot h \).

Proof. Composing the Lie algebra morphism from Lemma 2.3 (b), the Lie algebra inclusion \( \mathcal{Der}(\mathcal{W}) \subset \mathcal{End}_{\mathcal{Q}}(\mathcal{W}) \), and the Lie algebra morphism \( \mathcal{End}_{\mathcal{Q}}(\mathcal{W}) \to \mathcal{End}_{\mathcal{Q}}(\mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2)) \) given by \( f \mapsto (f \otimes \mathcal{id}_{\mathcal{W}^\otimes 2} + \mathcal{id}_{\mathcal{W}^\otimes 2} \otimes f)h - h \circ f) \), one obtains a Lie algebra morphism \( (\mathcal{V}_0, (,)) \to \mathcal{End}_{\mathcal{Q}}(\mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2)) \), i.e. a \( (\mathcal{V}_0, (,)) \)-module structure over \( \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \), which is given by the announced formula.

Recall that if \( \varphi : \mathfrak{h} \to \mathfrak{g} \) is a Lie algebra morphism and if \( \mathcal{V} \) is a \( \mathfrak{g} \)-module, then \( \mathcal{V} \) is equipped with a \( \mathfrak{h} \)-module structure by \( y \cdot v := \varphi(y) \cdot v \), called the pull-back module of the latter structure by \( \varphi \). In particular, the pull-back of the \( (\mathcal{V}_0, (,)) \)-module structure from Lemma 2.4 on \( \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \) by \( \theta \) is a \( \mathcal{lie}(e_0, e_1) \)-module structure on the same vector space.

Recall that if \( \mathfrak{g} \) is a Lie algebra, the data of a pair \( (\mathcal{V}, v) \) of a \( \mathfrak{g} \)-module \( \mathcal{V} \) and an element \( v \in \mathcal{V} \) gives rise to a Lie subalgebra \( \mathfrak{stab}_{\theta}(v) \) of \( \mathfrak{g} \), defined as \( \{ x \in \mathfrak{g} | x \cdot v = 0 \} \).

Lemma 2.5. Let \( \mathcal{h} \in \mathcal{Hom}_{\mathcal{Q}}(\mathcal{V}_0, \mathcal{W}^\otimes 2) \).

(a) The stabilizer Lie subalgebra of \( (\mathcal{V}_0, (,)) \) of \( \mathcal{h} \) is \( \mathfrak{stab}_{\mathcal{V}_0}(\mathcal{h}) = \{ v \in \mathcal{V}_0 | v \cdot h = 0 \} \).

(b) One has \( \mathfrak{stab}_{\mathcal{lie}(e_0, e_1)}(\mathcal{h}) = \theta^{-1}(\mathfrak{stab}_{\mathcal{V}_0}(\mathcal{h})) \).

Proof. (a) is a specialization of the definition of a stabilizer Lie subalgebra. (b) is a consequence of the fact that if \( \varphi : \mathfrak{h} \to \mathfrak{g} \) is a Lie algebra morphism, if \( \mathcal{V} \) is a \( \mathfrak{g} \)-module and if \( v \in \mathcal{V} \), then \( \mathfrak{stab}_g(v) = \varphi^{-1}(\mathfrak{stab}_g(v)) \).
2.3. Corestriction of $- \cdot h$ to $\Der_h(W, W^{\otimes 2})$. Assume that $h \in \Hom_{\Qalg}(W, W^{\otimes 2})$.

**Definition 2.6.** $\Der_h(W, W^{\otimes 2})$ is the set of derivations of $W$ with values in $W^{\otimes 2}$, viewed as a $W$-bimodule using $h$; explicitly, this is the set of $\Q$-linear maps $\delta : W \to W^{\otimes 2}$ such that $\delta(ww') = \delta(w)h(w') + h(w)\delta(w')$.

**Lemma 2.7.** The map $D \mapsto (D \otimes \text{id} + \text{id} \otimes D) \circ h - h \circ D$ defines a linear map $\Der(W) \to \Der_h(W, W^{\otimes 2})$.

**Proof.** This follows from the facts that if $D \in \Der(W)$, then $h \circ D \in \Der_h(W, W^{\otimes 2})$ and $D \otimes \text{id} + \text{id} \otimes D \in \Der(W^{\otimes 2})$, and if $D' \in \Der(W^{\otimes 2})$, then $D' \circ h \in \Der_h(W, W^{\otimes 2})$. □

**Corollary 2.8.** For $v \in \mathcal{V}_0$, one has $v \cdot h \in \Der_h(W, W^{\otimes 2})$, so the map $\mathcal{V}_0 \to \Hom_{\Qvec}(W, W^{\otimes 2})$, $v \mapsto v : h$ admits a factorization $\mathcal{V}_0 \to \Der_h(W, W^{\otimes 2}) \subset \Hom_{\Qvec}(W, W^{\otimes 2})$.

**Proof.** Follows by application of Lemma 2.7 to $D := \text{der}_v^{\mathcal{V}((1))}$. □

**Lemma 2.9.** If $h \in \Hom_{\Qalg}(W, W^{\otimes 2})$, then $\text{stab}_{\mathcal{V}_0}(h) = \{v \in \mathcal{V}_0 | v \cdot h = 0 \text{ (equality in } \Der_h(W, W^{\otimes 2}))\}$.

**Proof.** Follows from Lemma 2.5 (a) and Corollary 2.8. □

2.4. The stabilizer Lie algebras $\text{stab}_{\mathcal{V}_0}(\Delta^W)$ and $\text{stab}_{\text{lie}\{e_0, e_1\}}(\Delta^W)$. Define the family of elements $(\tilde{y}_a)_{a \in \Z}$ of $W$ by

$$
\tilde{y}_a := e_0^{-a-1}e_1 \text{ for } a > 0, \quad \tilde{y}_0 := -1, \quad \tilde{y}_a := 0 \text{ for } a < 0.
$$

**Lemma 2.10.** ([EF1], §1 (2)) There is an element $\Delta^W \in \Hom_{\Qalg}(W, W^{\otimes 2})$ (denoted $\Delta^{W, \text{DR}}$ in [EF1]), uniquely determined by

$$
\forall n > 0, \quad \Delta^W(\tilde{y}_n) := -\sum_{i=0}^n \tilde{y}_i \otimes \tilde{y}_{n-i}.
$$

**Proof.** This follows from the fact that the algebra $W$ is freely generated by $(\tilde{y}_a)_{a > 0}$. □

Note that (2.4.1) is also valid for $n = 0$.

**Lemma 2.11.** (a) The stabilizer Lie subalgebra of $(\mathcal{V}_0, \langle , \rangle)$ of $\Delta^W$ is $\text{stab}_{\mathcal{V}_0}(\Delta^W) = \{v \in \mathcal{V}_0 | v \cdot \Delta^W = 0 \text{ (equality in } \Der_{\Delta^W}(W, W^{\otimes 2}))\}$.

(b) One has $\text{stab}_{\text{lie}\{e_0, e_1\}}(\Delta^W) = \theta^{-1}(\text{stab}_{\mathcal{V}_0}(\Delta^W))$.

**Proof.** (a) follows by specialization from Lemma 2.9 (b) follows by specialization from Lemma 2.5 (b). □
2.5. Relation with the Lie algebra $\text{stab}(\hat{\Delta}^{\text{W,DR}})$ from [EF2]. Recall that the Lie algebra $(\text{Lie}(e_0, e_1), \langle \cdot, \cdot \rangle)$ is $\mathbb{Z}_+$-graded. Denote by $(\text{Lie}(e_0, e_1)^\wedge, \langle \cdot, \cdot \rangle)$ its graded completion. Both Lie algebras are equipped with an action of the abelian Lie $\mathbb{Q}1$, the element $1 \in \mathbb{Q}$ acting by the grading action (multiplying each graded element by its degree). By [EF2], Lemma 3.8 (b), the Lie algebra $g^{\text{DR}}$ defined in [EF2], §3.5 is equal to the corresponding semidirect product, so that

\begin{equation}
    g^{\text{DR}} = \mathbb{Q}1 \ltimes (\text{Lie}(e_0, e_1)^\wedge, \langle \cdot, \cdot \rangle).
\end{equation}

**Proposition 2.12.** (a) The Lie subalgebra $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^\text{W})$ of $(\text{Lie}(e_0, e_1), \langle \cdot, \cdot \rangle)$ is graded. Denote by $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^\text{W})^\wedge$ its graded completion.

(b) The Lie subalgebra $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^\text{W})^\wedge$ of $(\text{Lie}(e_0, e_1)^\wedge, \langle \cdot, \cdot \rangle)$ is stable under the action of the abelian Lie algebra $\mathbb{Q}1$.

(c) The Lie subalgebra $\text{stab}(\hat{\Delta}^{\text{W,DR}})$ of $g^{\text{DR}}$ from [EF2], §3.5 is equal to the corresponding semidirect product, i.e.

\begin{equation}
    \text{stab}(\hat{\Delta}^{\text{W,DR}}) = \mathbb{Q}1 \ltimes \text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^\text{W})^\wedge.
\end{equation}

**Proof.** (a) follows from the fact that $\text{Hom}_{\mathbb{Q}-\text{vec}}(\mathcal{W}, \mathcal{W}^{\otimes 2})$ is equipped with a $\mathbb{Z}$-grading, compatible with the grading of the Lie algebra $\text{Lie}(e_0, e_1)$ and with its action on it, and that the element $\Delta^\text{W} \in \text{Hom}_{\mathbb{Q}-\text{vec}}(\mathcal{W}, \mathcal{W}^{\otimes 2})$ is homogeneous (of degree 0). (b) then follows from (a). Lemma 3.12 (d) in [EF2] implies that $\text{stab}(\hat{\Delta}^{\text{W,DR}})$ is the semidirect product of $\mathbb{Q}1$ with the degree completion of the Lie subalgebra \{ $x \in \text{Lie}(e_0, e_1)| (\Gamma \text{der}_{0, x}^\text{W}(1) \otimes \text{id}_\mathcal{W} + \text{id}_\mathcal{W} \otimes \Gamma \text{der}_{0, x}^\text{W}(1) \otimes \text{id}_\mathcal{W}) \circ \Delta^\text{W} = \Delta^\text{W} \circ \Gamma \text{der}_{0, x}^\text{W}(1) \otimes \text{id}_\mathcal{W}$ \} of $\text{Lie}(e_0, e_1)$. By the first equation of (3.5.4) in [EF2], $\Gamma \text{der}_{0, x}^\text{W}(1) = \text{der}_{0, x}^\text{W}(1)$, so the latter Lie algebra is \{ $x \in \text{Lie}(e_0, e_1)| (\text{der}_{0, x}^\text{W}(1) \otimes \text{id}_\mathcal{W} + \text{id}_\mathcal{W} \otimes \text{der}_{0, x}^\text{W}(1)) \circ \Delta^\text{W} - \Delta^\text{W} \circ \text{der}_{0, x}^\text{W}(1) = 0$ \}, which by Lemma 2.11 (a) is equal to $\theta^{-1}(\text{stab}_{\text{V}_{0}}(\Delta^\text{W}))$. Therefore, by Lemma 2.11 (b) equal to $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^\text{W})$. This implies (c). \[\square\]

3. The stabilizer Lie algebra $\text{stab}(\hat{\Delta}^{\text{M,DR}})$

This section deals with the Lie algebra $\text{stab}(\hat{\Delta}^{\text{M,DR}})$ from [EF2]. In §3.1 we recall from [EF1] the $\mathcal{W}$-module $\mathcal{M}^{\text{DR}}$ (here denoted $\mathcal{M}$) and we construct a Lie algebra action of $\mathcal{V}_{0}$ on $\text{Hom}_{\mathbb{Q}-\text{vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2})$. In §3.2 we recall from loc. cit. the definition of the element $\Delta^{\text{M,DR}}$ (henceforth denoted $\Delta^{\text{M}}$) of this vector space, called the module harmonic coproduct. This leads to the construction of the stabilizer Lie algebra $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^{\text{M}})$, a graded Lie subalgebra of $\text{Lie}(e_0, e_1)$. In §3.3 we show that the Lie algebra $\text{stab}(\hat{\Delta}^{\text{M,DR}})$ from [EF2] can be obtained from $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^{\text{M}})$ via degree completion and semidirect product with $\mathbb{Q}1$ for the grading action (Proposition 3.5). In §3.4 we recall the relation between $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^{\text{M}})$ and the set of primitive elements of $\mathcal{M}$ for $\Delta^{\text{M}}$ (Theorem 3.1 in [EF0]).

3.1. Lie algebra action of $(\mathcal{V}_{0}, \langle \cdot, \cdot \rangle)$ on $\text{Hom}_{\mathbb{Q}-\text{vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2})$. When equipped with the left regular action, $\mathcal{V}$ may be viewed as a graded module over the graded algebra $\mathcal{V}$. Then $\mathcal{V}e_0$ is a
graded submodule of this \( \mathcal{V} \)-module. We denote by
\[
\mathcal{M} := \mathcal{V} / \mathcal{V} \mathcal{e}_0
\]
the corresponding quotient graded module. We denote by \( 1_\mathcal{M} \in \mathcal{M} \) the image of \( 1 \in \mathcal{V} \). Then the canonical projection \( \mathcal{V} \to \mathcal{M} \) is the map \( v \mapsto v \cdot 1_\mathcal{M} \), which we denote by \(- \cdot 1_\mathcal{M} \) (the module \( \mathcal{M} \) and its element \( 1_\mathcal{M} \) are denoted \( \mathcal{M}^{\mathit{DR}}, 1_{\mathcal{M}^{\mathit{DR}}} \) in [\text{EF1}], §1.1).

For \( v \in \mathcal{V}_0 \), define \( \text{der}_v^{\mathcal{V},(10)} \) to be the \( \mathbb{Q} \)-vector space endomorphism of \( \mathcal{V} \) such that
\[
\forall a \in \mathcal{V}, \quad \text{der}_v^{\mathcal{V},(10)}(a) = \text{der}_v^{\mathcal{V},(1)}(a) + av.
\]
For any \( v \in \mathcal{V}_0 \), \( \text{der}_v^{\mathcal{V},(10)} \) preserves the subspace \( \mathcal{V} \mathcal{e}_0 \), therefore induces an endomorphism \( \text{der}_v^{\mathcal{M},(10)} \) of the \( \mathbb{Q} \)-vector space \( \mathcal{M} \).

**Lemma 3.1.** The maps \( (\mathcal{V}_0, \langle , \rangle) \to \text{End}_{\mathbb{Q}\text{-vec}}(\mathcal{V}), v \mapsto \text{der}_v^{\mathcal{V},(10)} \) and \( (\mathcal{V}_0, \langle , \rangle) \to \text{End}_{\mathbb{Q}\text{-vec}}(\mathcal{M}), v \mapsto \text{der}_v^{\mathcal{M},(10)} \) are Lie algebra morphisms.

**Proof.** By [\text{EF2}], Lemma 3.9 (c), the map \( \mathfrak{e} \mathfrak{m}^{\mathit{DR}} \to \text{Der}(\hat{\mathcal{V}}^{\mathit{DR}}, \hat{\mathcal{V}}^{\mathit{DR}}), (\nu, x) \mapsto (\text{der}^{\mathcal{V},(1)}_{(\nu, v)}, \text{der}^{\mathcal{V},(10)}_{(\nu, v)}) \) is a Lie algebra morphism. Precomposing it with the Lie algebra injection \( (\mathcal{V}_0, \langle , \rangle) \subset \mathfrak{e} \mathfrak{m}^{\mathit{DR}} \), \( v \mapsto (0, v) \) and post-composing it with the sequence of Lie algebra morphisms \( \text{Der}(\hat{\mathcal{V}}^{\mathit{DR}}, \hat{\mathcal{V}}^{\mathit{DR}}) \subset \text{Der}(\hat{\mathcal{V}}^{\mathit{DR}}) \times \text{End}_{\mathbb{Q}\text{-vec}}(\hat{\mathcal{V}}^{\mathit{DR}}) \to \text{End}_{\mathbb{Q}\text{-vec}}(\hat{\mathcal{V}}^{\mathit{DR}}) \), where the second map is the projection on the second factor, one sees that the map \( (\mathcal{V}_0, \langle , \rangle) \to \text{End}_{\mathbb{Q}\text{-vec}}(\mathcal{V}), v \mapsto \text{der}_v^{\mathcal{V},(10)} \) is a Lie algebra morphism. There is a diagram of Lie algebras \( \text{End}_{\mathbb{Q}\text{-vec}}(\hat{\mathcal{V}}^{\mathit{DR}}) \supset \bigoplus_{n \in \mathbb{Z}} \text{End}(\mathcal{V})[n] \subset \text{End}(\mathcal{V}) \), where \([-n]\) denotes the part of degree \( n \). One checks that \( \text{der}_v^{\mathcal{V},(10)} \) belongs to the Lie subalgebra \( \bigoplus_{n \in \mathbb{Z}} \text{End}(\mathcal{V})[n] \), and that the composition of the resulting morphism \( (\mathcal{V}_0, \langle , \rangle) \to \bigoplus_{n \in \mathbb{Z}} \text{End}(\mathcal{V})[n] \) with the inclusion \( \bigoplus_{n \in \mathbb{Z}} \text{End}(\mathcal{V})[n] \subset \text{End}(\mathcal{V}) \) is the map \( v \mapsto \text{der}_v^{\mathcal{V},(10)} \), which is therefore a Lie algebra morphism. The same argument applies with \( \mathcal{M} \) replacing \( \mathcal{V} \), using [\text{EF2}], Lemma 3.10, (b) instead of [\text{EF2}], Lemma 3.9 (c).

**Lemma 3.2.** The \( \mathbb{Q} \)-vector space \( \text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2}) \) is equipped with a \( (\text{Lie}(e_0, e_1), \langle , , \rangle) \)-module structure, the action of \( v \in \text{Lie}(e_0, e_1) \) on \( h \in \text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2}) \) being given by
\[
(3.1.1) \quad v \ast h := (\text{der}_{\theta(v)}^{\mathcal{M},(10)} \otimes \text{id} + \text{id} \otimes \text{der}_{\theta(v)}^{\mathcal{M},(10)}) \circ h - h \circ \text{der}_{\theta(v)}^{\mathcal{M},(10)}.
\]

**Proof.** Composing the Lie algebra morphisms \( \theta \) from Lemma 2.2 \( (\mathcal{V}_0, \langle , \rangle) \to \text{End}_{\mathbb{Q}\text{-vec}}(\mathcal{M}) \) from Lemma 3.1 and \( \text{End}_{\mathbb{Q}\text{-vec}}(\mathcal{M}) \to \text{End}_{\mathbb{Q}\text{-vec}}(\text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2})) \) given by \( f \mapsto (h \mapsto (f \otimes \text{id}_{\mathcal{M}} + \text{id}_{\mathcal{M}} \otimes f) \circ h - h \circ f) \), one obtains a Lie algebra morphism \( (\text{Lie}(e_0, e_1), \langle , , \rangle) \to \text{End}_{\mathbb{Q}\text{-vec}}(\text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2})) \), i.e. a \( (\text{Lie}(e_0, e_1), \langle , , \rangle) \)-module structure over \( \text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2}) \), which is given by the announced formula.

3.2. The stabilizer Lie algebra \( \text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^{\mathcal{M}}) \). The \( \mathbb{Q} \)-vector space \( \mathcal{M} \) is a free \( \mathcal{W} \)-module of rank 1 generated by \( 1_\mathcal{M} \) (see [\text{EF1}], §1.1). It follows that there is an element \( \Delta^{\mathcal{M}} \) in \( \text{Hom}_{\mathbb{Q}\text{-vec}}(\mathcal{M}, \mathcal{M}^{\otimes 2}) \), uniquely determined by
\[
(3.2.1) \quad \forall w \in \mathcal{W}, \quad \Delta^{\mathcal{M}}(w \cdot 1_\mathcal{M}) = \Delta^{\mathcal{W}}(w) \cdot 1^{\otimes 2}_{\mathcal{M}}
\]
Lemma 3.3. (a) The stabilizer Lie algebra of $\Delta^M \in \text{Hom}_{\mathbb{Q}-\text{vec}}(M, M \otimes 2)$ for the action \( \text{Lie}(e_0, e_1) \) is

$$\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M) = \{ v \in \text{lie}(e_0, e_1) | v \ast \Delta^M = 0 \}.$$  

(b) $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M)$ is a graded Lie subalgebra of $(\text{lie}(e_0, e_1), \langle , \rangle)$.

Proof. (a) follows from definitions. (b) follows from the facts that $M$, and therefore $\text{Hom}_{\mathbb{Q}-\text{vec}}(M, M \otimes 2)$, is a graded module over $(\text{lie}(e_0, e_1), \langle , \rangle)$, and that the element $\Delta^M$ of the latter module is homogeneous (of degree 0).}

Remark 3.4. As explained in the proof of Lemma 3.12, the map $(v, h) \mapsto (\text{der}_v^{M, (10)} \otimes \text{id} + \text{id} \otimes \text{der}_v^{M, (10)}) \circ h - h \circ \text{det}_v^{M, (10)}$ defines an action of the Lie algebra $(\mathcal{V}_0, \langle , \rangle)$ on the vector space $\text{Hom}_{\mathbb{Q}-\text{vec}}(M, M \otimes 2)$ so that as in Lemma 2.11 one may define the stabilizer Lie subalgebra $\text{stab}_{\mathcal{V}_0}(\Delta^M)$ of $(\mathcal{V}_0, \langle , \rangle)$ and show that $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M)$ is its preimage under $\theta$. Oppositely to $\text{stab}_{\mathcal{V}_0}(\Delta^M)$, the Lie algebra $\text{stab}_{\mathcal{V}_0}(\Delta^M)$ does not play a role in the sequel of the paper.

3.3. Relation with the Lie algebra $\text{stab}(\Delta^M_{\text{DR}})$ from [EF2]. Recall that $(\text{lie}(e_0, e_1)^\wedge, \langle , \rangle)$ denotes the degree completion of the $\mathbb{Z}_+$-graded Lie algebra $(\text{lie}(e_0, e_1), \langle , \rangle)$. Denote by $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)^\wedge$ the degree completion of its $\mathbb{Z}_+$-graded Lie subalgebra $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)$ (see Lemma 3.3 (b)).

Recall that $(\text{lie}(e_0, e_1)^\wedge, \langle , \rangle)$ is equipped with the grading action of the abelian Lie algebra $\mathbb{Q}^1$, and the identification of the Lie algebra $\mathcal{g}^{\text{DR}}$ from [EF2], §3.5 with the corresponding semidirect product (see \( \text{Lie}(e_0, e_1) \)). The following statement is an analogue of Proposition 2.12.

Proposition 3.5. (a) The Lie subalgebra $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M^\wedge)$ of $(\text{lie}(e_0, e_1)^\wedge, \langle , \rangle)$ is stable under the action of the abelian Lie algebra $\mathbb{Q}^1$.

(b) The Lie algebra isomorphism \( \text{Lie}(e_0, e_1) \) restricts to an isomorphism of the Lie subalgebra $\text{stab}(\Delta^M_{\text{DR}})$ of $\mathcal{g}^{\text{DR}}$ from [EF2], §3.5 with the corresponding semidirect product, i.e.

$$\text{stab}(\Delta^M_{\text{DR}}) = \mathbb{Q}^1 \ltimes \text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)^\wedge.$$  

Proof. (a) follows from the fact that $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)^\wedge$ is graded complete. Lemma 3.12 (e) in [EF2] implies that $\text{stab}(\Delta^M_{\text{DR}})$ is the semidirect product of $\mathbb{Q}^1$ with the degree completion of the Lie subalgebra \{ $x \in \text{lie}(e_0, e_1) | (\Gamma \text{det}_v^{M, (10)} \otimes \text{id}_M + \text{id}_M \otimes \Gamma \text{der}_v^{M, (10)} \otimes \text{id}_M) \circ \Delta^M = \Delta^M \circ \Gamma \text{det}_v^{M, (10)} \otimes \text{id}_M \}$ of $\text{lie}(e_0, e_1)$. By the second equation of (3.5.4) in [EF2], $\Gamma \text{der}_v^{M, (10)} = \text{der}_v^{M, (10)}$, which combined with Lemma 3.2 implies that this Lie algebra is equal to $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)^\wedge$. This implies (b).}

3.4. Relation of $\text{stab}_{\text{lie}(e_0, e_1)}(\Delta^M)$ with $\mathcal{P}(M)$ (Theorem 3.10 from [EF0]).

Lemma 3.6. The set $\mathcal{P}(M) := \{ m \in M | \Delta^M(m) = m \otimes 1_M + 1_M \otimes m \}$ is a graded subspace of $M$. 

Proof. Follows from the fact that $\Delta^M$ is graded. \hfill \Box

Then $\mathcal{P}(M)$ is the set of primitive elements of $(M, \Delta^M)$, which is a cocommutative coalgebra. One has

\begin{equation}
\mathcal{P}(M) = \ker(\Delta^M - (\text{id} \otimes 1_M + 1_M \otimes \text{id})): M \to M^{\otimes 2},
\end{equation}

where the map $\text{id} \otimes 1_M + 1_M \otimes \text{id}: M \to M^{\otimes 2}$ is defined by $m \mapsto m \otimes 1_M + 1_M \otimes m$.

For $d \geq 1$, we define $\text{Lie}(e_0, e_1)[d] := \{x \in \text{Lie}(e_0, e_1) | x[d] = 0\}$, where we recall that $x[d]$ is the degree $d$ component of an element $x \in \text{Lie}(e_0, e_1)$. Then $\text{Lie}(e_0, e_1)[d]$ is the direct sum of all the homogeneous components of $\text{Lie}(e_0, e_1)$ of degree $\neq d$.

Proposition 3.7. $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M)$ is the intersection with $\text{Lie}(e_0, e_1)[2]$ of the preimage of $\mathcal{P}(M) \subset M$ by the composed map $\text{Lie}(e_0, e_1) \xrightarrow{\theta} \mathcal{P}(M) \xrightarrow{\text{Lie}(e_0, e_1)[2]} M$, i.e.

\begin{equation}
\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M) = \theta^{-1}\left((-1)_1 \mathcal{P}(M) \cap \text{Lie}(e_0, e_1)[2]\right).
\end{equation}

In other terms, $\text{stab}_{\text{Lie}(e_0, e_1)}(\Delta^M) = \{x \in \text{Lie}(e_0, e_1)[2] \text{ and } \theta(x) \cdot 1_M \in \mathcal{P}(M)\}$.

Proof. Combining Definitions 3.1 and 3.2 and Theorem 3.10 from [EF0] in the case $\Gamma = \{1\}$, one obtains the equality

\begin{equation}
\text{stab}(\Delta_\ast) = \{\psi \in \text{Lib}(X) \langle \psi \rangle x_0 = (\psi \rangle x_1 = 0, (\psi \rangle y_{2,1}) = 0, \Delta_\ast(\psi_\ast) = \psi_\ast \otimes 1 + 1 \otimes \psi_\ast \} \oplus Qx_0 \oplus Qx_1,
\end{equation}

where $\text{stab}(\Delta_\ast)$ and $\text{Lib}(X)$ are the graded Lie algebras defined in [EF0], §2.5 and §2.1.1, the maps $\psi \mapsto (\psi \rangle x_0), \psi \mapsto (\psi \rangle x_1)$ and $\tilde{\psi} \mapsto (\tilde{\psi} \rangle y_{21})$ are defined in [R], §1.6, the map $\text{Lib}(X) \to Q(Y), \psi \mapsto \psi_\ast$ is defined in [EF0], (2.5), where $Q(Y)$ is the graded algebra defined in [EF0], §2.2, and $\Delta_\ast : Q(Y) \to Q(Y)^{\otimes 2}$ is the graded coproduct defined in [EF0], §2.2.

Both $\text{stab}(\Delta_\ast)$ and $\{\psi \in \text{Lib}(X) \langle \psi \rangle x_0 = \psi \otimes 1 + 1 \otimes \psi \}$ are graded subspaces of $\text{Lib}(X)$ and (3.3.3) implies that their components of any degree $d \neq 1, 2$ are equal. (3.3.3) also implies that $\text{stab}(\Delta_\ast)[1] = Qx_0 \oplus Qx_1$, while the fact that any element of $Q(Y)$ of degree one is primitive for $\Delta_\ast$, together with the fact that $\psi \mapsto \psi_\ast$ is graded, implies $\{\psi \in \text{Lib}(X) \langle \psi \rangle x_0 = \psi \otimes 1 + 1 \otimes \psi \} [1] = Qx_0 \oplus Qx_1$, therefore the degree 1 components of $\text{stab}(\Delta_\ast)$ and $\{\psi \in \text{Lie}(e_0, e_1) \langle \psi \rangle x_0 = \psi \otimes 1 + 1 \otimes \psi \}$ are equal. Finally, $\text{Lib}(X)[2]$ is one-dimensional, spanned by $[x_0, x_1]$. One computes $[x_0, x_1] = y_{2,1} - (1/2)(y_{1,1})^2$, which is primitive for $\Delta_\ast$, therefore $\{\psi \in \text{Lib}(X) \langle \psi \rangle x_0 = \psi \otimes 1 + 1 \otimes \psi \} [2] = Q \cdot [x_0, x_1]$. On the other hand, $[x_0, x_1] = y_{2,1} - (1/2)(y_{1,1})^2$ also implies $([x_0, x_1], [y_{2,1}]) = 1$, which together with (3.3.3) implies $\text{stab}(\Delta_\ast)[2] = 0$.

Putting together these results, one obtains

\begin{equation}
\text{stab}(\Delta_\ast) = \{\psi \in \text{Lib}(X) \langle \psi \rangle [2] = 0 \text{ and } \Delta_\ast(\psi_\ast) = \psi_\ast \otimes 1 + 1 \otimes \psi_\ast \}.
\end{equation}

The map $x_0 \mapsto e_0, x_1 \mapsto -e_1$ defines a Lie algebra isomorphism $\text{Lib}(X) \to \text{Lie}(e_0, e_1)$. The map taking, for any $r \geq 0$ and $(n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 1}$, the element $y_{n_1,1} \cdots y_{n_r,1} \in Q(Y)$ to $(-1)^r e_0^{n_1-1} e_1 \cdots e_0^{n_r-1} e_1 \cdot 1_M \in M$ defines a vector space isomorphism $Q(Y) \to M$ with $1 \mapsto
4. Decomposition of the map $V_0 \to \text{Der}_{\Delta W}(W,W^\otimes 2)$

In this section, we make explicit a decomposition of the map $V_0 \to \text{Der}_{\Delta W}(W,W^\otimes 2)$. Its constituents $H$, $h$ and $i$ are defined respectively in §§4.1, 4.2 and 4.3. The identification of the map $V_0 \to \text{Der}_{\Delta W}(W,W^\otimes 2)$ with $i \circ h \circ H$ is obtained in §4.4 (Proposition 4.10).

4.1. The map $H : V_0 \to (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{i \geq 0} W)$. Since $V$ is freely generated by $e_0,e_1$, a basis of $V_0$ is

\begin{equation}
\{ e_0^{a_0-1}e_1 \cdots e_l e_0^{a_l-1} \text{, where } l \geq 0, \ a_0, \ldots, a_l>0 \text{ and } a_0>1 \text{ if } l = 0. \}
\end{equation}

**Definition 4.1.** (a) For $k > 0$, $L_k, R_k : V_0 \to W^\otimes 2$ are the linear maps such that

\begin{equation}
L_k(e_0^{a_0-1}e_1 \cdots e_l e_0^{a_l-1}) = \delta_{a_l,k}(\Delta W - i \otimes 1 \otimes 1)(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}}),
\end{equation}

\begin{equation}
R_k(e_0^{a_0-1}e_1 \cdots e_l e_0^{a_l-1}) = \delta_{a_0,k}(\Delta W - i \otimes 1 \otimes 1)(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l}),
\end{equation}

for any $l \geq 0$ and $a_0, \ldots, a_l > 0$ such that $a_0 > 1$ if $l = 0$.

(b) For $i > 0$, $M_i : V_0 \to W$ is the linear map such that

\begin{equation}
M_i(e_0^{a_0-1}e_1 \cdots e_l e_0^{a_l-1}) = \tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \tilde{y}_{a_l-i} - \tilde{y}_{a_0-i} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l}
\end{equation}

for any $l \geq 0$ and $a_0, \ldots, a_l > 0$ such that $a_0 > 1$ if $l = 0$; recall that $\tilde{y}_0 = -1$ and $\tilde{y}_a = 0$ for $a < 0$.

**Definition 4.2.** $H : V_0 \to (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{i \geq 0} W)$ is the linear map

\[ H := (\oplus_{k \geq 0} L_{k+1}) \oplus (\oplus_{k \geq 0} R_{k+1}) \oplus (\oplus_{i \geq 0} M_{i+1}). \]

Explicitly, if $v \in V_0$, then

\[ H(v) = ((L_{k+1}(v))_{k \geq 0}, (R_{k+1}(v))_{k \geq 0}, (M_{i+1}(v))_{i \geq 0}) \in (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{i \geq 0} W). \]

4.2. The map $h : (\oplus_{k \geq 0} W^\otimes 2) \oplus (\oplus_{i \geq 0} W) \to \text{Map}(Z_{>0}, W^\otimes 2)$. The set $\text{Map}(Z_{>0}, W^\otimes 2)$ of maps from $Z_{>0}$ to $W^\otimes 2$ is equipped with the $\mathbb{Q}$-vector space structure inherited from the vector space structure of $W^\otimes 2$. Let $\Delta$ be the element of $\text{Map}(Z_{>0}, W^\otimes 2)$ defined by

\begin{equation}
\forall n > 0, \ \Delta(n) := -\Delta W(\tilde{y}_n) = \sum_{i=0}^{n} \tilde{y}_i \otimes \tilde{y}_{n-i} \in W^\otimes 2.
\end{equation}
Lemma 4.3. (a) For any $k \geq 0$, there is a unique pair of linear maps $\ell_k, \hat{r}_k : W \to \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$, such that

$$(4.2.2) \forall v \in W, \quad \forall n > 0, \quad \ell_k(v)(n) := w\Delta(n + k), \quad \hat{r}_k(v)(n) := \Delta(n + k)w \quad (\text{equality in } W^{\otimes 2}).$$

(b) For any $i \geq 0$, there is a unique linear map $m_i : \mathcal{V} \to \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$, such that

$$(4.2.3) \forall v \in \mathcal{V}, \quad \forall n > 0, \quad m_i(v)(n) := v \otimes \tilde{y}_{n+i} + \tilde{y}_{n+i} \otimes v \quad (\text{equality in } W^{\otimes 2}).$$

Proof. The follows from the fact that the right-hand sides of the equalities of (4.2.2) depend linearly on $w \in W$, and the right-hand side of the equality in (4.2.3) depends linearly on $v \in \mathcal{V}$. \hfill \square

Definition 4.4. $h : (\oplus_{k \geq 0}W^{\otimes 2}) \oplus (\oplus_{k \geq 0}W^{\otimes 2}) \oplus (\oplus_{i \geq 0}W) \to \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$ is the linear map

$$h := (\oplus_{k \geq 0}\ell_k) \oplus (\oplus_{k \geq 0} - \hat{r}_k) \oplus (\oplus_{i \geq 0}m_i).$$

Explicitly, if $(a, b, z) \in (\oplus_{k \geq 0}W^{\otimes 2}) \oplus (\oplus_{k \geq 0}W^{\otimes 2}) \oplus (\oplus_{i \geq 0}W)$, with $a = (a_k)_{k \geq 0}$, $b = (b_k)_{k \geq 0}$, $z = (z_i)_{i \geq 0}$, then $h(a, b, z)$ is the element of $\text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$ such that

$$(4.2.4) \forall n > 0, \quad h(a, b, z)(n) = \sum_{k \geq 0} (a_k\Delta(n + k) - \Delta(n + k)b_k) + \sum_{i \geq 0} (z_i \otimes \tilde{y}_{n+i} + \tilde{y}_{n+i} \otimes z_i),$$

where for $n > 0$, $\Delta(n)$ is given by (4.2.1).

4.3. The isomorphism $i : \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2}) \simi \text{Der}_{\Delta W}(W, W^{\otimes 2})$.

Definition 4.5. $\text{Der}_{\Delta W}(W, W^{\otimes 2})$ is the set of derivations of $W$ with values in $W^{\otimes 2}$, viewed as a $W$-bimodule using $\Delta W$; explicitly, this is the set of $\mathbb{Q}$-linear maps $\delta : W \to W^{\otimes 2}$ such that $\delta(ww') = \delta(w)\Delta W(w') + \Delta W(w)\delta(w')$.

Lemma 4.6. (a) For $(\delta_n)_{n \geq 0} \in \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$, there is a unique element $i((\delta_n)_{n \geq 0}) \in \text{Der}_{\Delta W}(W, W^{\otimes 2})$ such that for $l \geq 0$, $n_1, \ldots, n_l > 0$,

$$i((\delta_n)_{n \geq 0})(\tilde{y}_{n_1} \cdots \tilde{y}_{n_l}) = \sum_{i=1}^{l} \Delta W(\tilde{y}_{n_1} \cdots \tilde{y}_{n_{i-1}})\delta_{n_i} \Delta W(\tilde{y}_{n_{i+1}} \cdots \tilde{y}_{n_l}).$$

(b) The map $i : \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2}) \to \text{Der}_{\Delta W}(W, W^{\otimes 2})$ is a vector space isomorphism, inverse to the map $\text{Der}_{\Delta W}(W, W^{\otimes 2}) \to \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$ given by $\delta \mapsto (\delta(\tilde{y}_n))_{n \geq 0}$.

Proof. (a) Since $W$ is freely generated, as an algebra, by the family $(\tilde{y}_n)_{n > 0}$, a vector space basis of $W$ is the family $(\tilde{y}_{n_1} \cdots \tilde{y}_{n_l})_{0 \geq n_1, \ldots, n_l > 0}$ (the empty product, corresponding to $l = 0$, being equal to 1). It follows that $i((\delta_n)_{n \geq 0})$ is well-defined as an element of $\text{Hom}_{\mathbb{Q}\text{-vec}}(W, W^{\otimes 2})$. One checks that $i((\delta_n)_{n \geq 0})$ belongs to $\text{Der}_{\Delta W}(W, W^{\otimes 2})$.

(b) One checks that the map $i$ is linear. Let $ev$ be the map $\delta \mapsto (\delta(\tilde{y}_n))_{n \geq 0}$. Then for $(\delta_n)_{n \geq 0} \in \text{Map}(\mathbb{Z}_{>0}, W^{\otimes 2})$, the element $ev \circ i((\delta_n)_{n \geq 0})$ is the map taking $n > 0$ to $i((\delta_n)_{n \geq 0})(\delta_n) = \delta_n$ so $ev \circ i = id$. For $\delta \in \text{Der}_{\Delta W}(W, W^{\otimes 2})$, $i \circ ev(\delta)$ and $\delta$ are elements
of $\text{Der}_{\Delta^W}(W, W^\otimes 2)$ whose restrictions to the set $\{y_n | n > 0\}$, which is a generating set of $W$ coincide, therefore they are equal. It follows that $i \circ \text{ev} = \text{id}$. 

4.4. Decomposition of the map $\nu_0 \to \text{Der}_{\Delta^W}(W, W^\otimes 2)$.

**Lemma 4.7.** If $l \geq 0$, $a_1, \ldots, a_l > 0$ with $a_0 > 1$ if $l = 0$ and $v := e_0^{a_0-1}e_1 \cdots e_0^{a_l-1} \in \nu_0$, then the derivation $\text{de}^W_v(1)$ of $W$ is such that

$$\forall n > 0, \quad \text{de}^W_v(1)(\tilde{y}_n) = \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1+n} - \tilde{y}_{a_0-1+n} \tilde{y}_{1} \cdots \tilde{y}_{a_l}.$$  

**Proof.** In the following computation in $\mathcal{V}$:

$$\text{de}^W_v(1)(\tilde{y}_n) = \text{de}^W_v(1)(\tilde{y}_n) = \text{de}^W_v(1)(e_0^{n-1}e_1) = [v, e_0^{n-1}]e_1 = ve_0^{n-1}e_1 - e_0^{n-1}ve_1 = e_0^{a_0-1}e_1 \cdots e_0^{a_l+n-2}e_1 - e_0^{a_0+n-2}e_1 \cdots e_0^{a_l-1}e_1 = \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l+n-1} - \tilde{y}_{a_0+n} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l},$$

the first equality follows from the fact that $\text{de}^W_v(1)$ is the restriction of $\text{de}^W_v(1)$ to $W$, the second and last equalities follow from the definition of $\tilde{y}_n$ (see (2.3)), and the third equality follows from the definition of $\text{de}^W_v(1)$ (2.4). As $W \subset \mathcal{V}$, the resulting identity holds in $W$, as claimed. 

**Lemma 4.8.** For $l \geq 0$, $a_1, \ldots, a_l > 0$ with $a_0 > 1$ if $l = 0$, $v := e_0^{a_0-1}e_1 \cdots e_0^{a_l-1} \in \nu_0$ and for $n > 0$, there holds (equality in $W^\otimes 2$)

$$(\text{de}^W_v(1) \otimes \text{id} + \text{id} \otimes \text{de}^W_v(1)) \circ \Delta^W(\tilde{y}_n) = \sum_{j \geq 0} m_j(\tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1-j} - \tilde{y}_{a_0-1-j} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l})\tilde{y}_n + (\ell_{a_l-1}(\tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \otimes 1 + 1 \otimes \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1}) + r_{a_0-1}(\tilde{y}_{a_1} \cdots 1 + 1 \otimes \tilde{y}_{a_1} \cdots \tilde{y}_{a_l}))\tilde{y}_n.$$  

**Proof.** One computes

$$\begin{align*}
(\text{de}^W_v(1) \otimes \text{id} + \text{id} \otimes \text{de}^W_v(1)) \circ \Delta^W(\tilde{y}_n) &= -(\text{de}^W_v(1) \otimes \text{id})(\sum_{i=0}^{n} \tilde{y}_i \otimes \tilde{y}_{n-i}) \\
&= -\sum_{i=0}^{n} (\tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1+i} - \tilde{y}_{a_0-1+i} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l}) \otimes \tilde{y}_{n-i} \\
&= -\sum_{i=1-a_l}^{n} \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1+i} \otimes \tilde{y}_{n-i} + \sum_{i=1-a_l}^{0} \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1+i} \otimes \tilde{y}_{n-i} \\
&+ \sum_{i=1-a_0}^{n} \tilde{y}_{a_0-1+i} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes \tilde{y}_{n-i} - \sum_{i=1-a_0}^{0} \tilde{y}_{a_0-1+i} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes \tilde{y}_{n-i} \\
&= \sum_{j=0}^{a_l-1} \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_{a_l-1-j} \otimes \tilde{y}_{n+j} - \sum_{i=0}^{a_l-1} \tilde{y}_{a_0-1-j} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes \tilde{y}_{n+i} \\
&= \sum_{s=0}^{n+a_l-1} \tilde{y}_{a_0} \cdots \tilde{y}_{a_l-1} \tilde{y}_s \otimes \tilde{y}_{n+a_l-1-s} + \sum_{t=0}^{n+a_0-1} \tilde{y}_t \tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes \tilde{y}_{n+a_0-1-t}
\end{align*}$$
where the first equality follows from \((2.4.1)\), the second equality follows from Lemma \([4.7]\), the third equality follows from the addition of cancelling terms, and the last equality follows from the changes of variables \(j := -i, s := a_l - 1 + i\) and \(t := a_0 - 1 + i\). Then

\[
(\text{der}_v^W,1) \otimes \text{id} + \text{id} \otimes \text{der}_v^W,1) \circ \Delta^W (\tilde{y}_n) = -(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \otimes 1 + 1 \otimes \tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}}) \Delta(n + a_l - 1)
+ \Delta(n + a_0 - 1)(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes 1 + 1 \otimes \tilde{y}_{a_1} \cdots \tilde{y}_{a_l})
+ \sum_{j=0}^{a_l - 1} (\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \tilde{y}_{a_1 - 1 - j} \otimes \tilde{y}_{n+j} + \tilde{y}_{n+j} \otimes \tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \tilde{y}_{a_1 - 1 - j})
- \sum_{i=0}^{a_0 - 1} (\tilde{y}_{a_0 - 1 - j} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes \tilde{y}_{n+j} + \tilde{y}_{n+j} \otimes \tilde{y}_{a_0 - 1 - j} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l})

where the first equality follows from the symmetrization of \((4.2.1)\) with respect to the exchange of factors of \(W^{\otimes 2}\), together with the identification of the last line of this equation with 

\[-(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \otimes 1) \Delta(n + a_l - 1) + \Delta(n + a_0 - 1)(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l} \otimes 1),\]

and the second equality follows from \((4.2.2)\) and \((4.2.3)\). The statement then follows from \(\tilde{y}_a = 0\) for \(a < 0\) and the linearity of \(m_j\) for \(j \geq 0\). \(\square\)

**Lemma 4.9.** For \(l \geq 0, a_0, \ldots, a_l \geq 0\) with \(a_0 > 1\) if \(l = 0, v := e_0^{a_0-1}e_1 \cdots e_l e_0^{a_l-1} \in V_0\) and for \(n > 0\), there holds

\[
\Delta^W \circ \text{der}_v^W,1 (\tilde{y}_n) = \left( -\ell_{a_l - 1}(\Delta^W(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}})) + r_{a_0 - 1}(\Delta^W(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l})) \right) (\tilde{y}_n) \in W^{\otimes 2}.
\]

**Proof.** One computes

\[
\Delta^W \circ \text{der}_v^W,1 (\tilde{y}_n) = \Delta^W(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}} \tilde{y}_{a_1 - 1 + n} - \tilde{y}_{a_0 - 1 + n} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l})
= -\Delta^W(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}}) \Delta(a_l - 1 + n) + \Delta(a_0 - 1 + n) \Delta^W(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l})
= \left( -\ell_{a_l - 1}(\Delta^W(\tilde{y}_{a_0} \cdots \tilde{y}_{a_{l-1}})) + r_{a_0 - 1}(\Delta^W(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l})) \right) (\tilde{y}_n).
\]

where the first equality follows from Lemma \([4.7]\), the second equality follows from the algebra morphism property of \(\Delta^W\) and \((4.2.1)\), and the last equality follows from \((4.2.2)\). \(\square\)

**Proposition 4.10.** The map \(- : \Delta^W : V_0 \to \text{Der}_{\Delta^W}(W, W^{\otimes 2})\) (see \((2.2.1)\)) is equal to the composed map

\[
V_0 \xrightarrow{H} (\oplus_{k \geq 0} W^{\otimes 2})^{\otimes 2} \oplus (\oplus_{i \geq 0} W) \xrightarrow{h} \text{Map}(Z_{>0}, W^{\otimes 2}) \xrightarrow{i} \text{Der}_{\Delta^W}(W, W^{\otimes 2})
\]
Proof. Let $n > 0$ and let $l \geq 0$, $a_0, \ldots, a_l > 0$ with $a_0 > 1$ if $l = 0$, and $\nu := e_0^{a_0 - 1}e_1 \cdots e_l^{a_l - 1} \in \mathcal{V}_0$. Then

\begin{equation}
(4.4.2) \quad (v \cdot \Delta^W)(\tilde{y}_n) = (\text{der}_v^W(1) \otimes \text{id}) + \text{id} \otimes \text{der}_v^W(1) \circ \Delta^W(\tilde{y}_n) - \Delta^W \circ \text{der}_v^W(1)(\tilde{y}_n)
\end{equation}

\begin{align*}
&= \sum_{j \geq 0} m_j(\tilde{y}_a_0 \cdots \tilde{y}_{a_1-1} \tilde{y}_{a_1-1-j} - \tilde{y}_{a_1-1-j} \tilde{y}_{a_1} \cdots \tilde{y}_{a_l})(\tilde{y}_n) \\
&\quad + \left(\ell_{a_0-1}((\Delta^W - \text{id} \otimes 1 - 1 \otimes \text{id})(\tilde{y}_a_0 \cdots \tilde{y}_{a_1-1})) - r_{a_0-1}((\Delta^W - \text{id} \otimes 1 - 1 \otimes \text{id})(\tilde{y}_{a_1} \cdots \tilde{y}_{a_l}))\right)(\tilde{y}_n)
\end{align*}

where the first equality follows from (2.2.1) and the second equality from Lemmas 4.8 and 4.9 and from the linearity of $\ell_{a_0-1}$ and $r_{a_0-1}$. Using (4.1.2), (4.1.3), (4.1.4) and $L_k(v) = \delta_k a_0 a_{a_1} \cdots a_{a_l}$, this implies

\begin{equation}
(4.4.3) \quad (v \cdot \Delta^W)(\tilde{y}_n) = \sum_{k \geq 0} \ell_k \circ L_{k+1}(v)(\tilde{y}_n) - \sum_{k \geq 0} r_k \circ R_{k+1}(v)(\tilde{y}_n) + \sum_{i \geq 0} m_i \circ M_{i+1}(v)(\tilde{y}_n) \in \mathcal{W}^2.
\end{equation}

For each $n > 0$, both sides of this identity depend linearly on $v \in \mathcal{V}_0$, and since this identity is fulfilled for any $v$ in the family (4.1.1), which is a basis of $\mathcal{V}_0$, it holds for any $v \in \mathcal{V}_0$. Then for $v \in \mathcal{V}_0$, $v \cdot \Delta^W$ and $\sum_{k \geq 0} \ell_k \circ L_{k+1}(v) - \sum_{k \geq 0} r_k \circ R_{k+1}(v) + \sum_{i \geq 0} m_i \circ M_{i+1}(v)$ are elements of $\text{Der}_{\Delta^W}(\mathcal{W}, \mathcal{W}^2)$, and their images by the map $i : \text{Der}_{\Delta^W}(\mathcal{W}, \mathcal{W}^2) \to \text{Map}(\mathbb{Z}_{\geq 0}, \mathcal{W}^2)$ are the maps taking $n > 0$ to respectively the left and right hand sides of (4.4.3). By (4.4.3), these images are equal, and the injectivity of $i$ (see Lemma 4.10) then implies the equality

\begin{equation}
(4.4.4) \quad \forall v \in \mathcal{V}, \quad v \cdot \Delta^W = \sum_{k \geq 0} \ell_k \circ L_{k+1}(v) - \sum_{k \geq 0} r_k \circ R_{k+1}(v) + \sum_{i \geq 0} m_i \circ M_{i+1}(v) \in \text{Der}_{\Delta^W}(\mathcal{W}, \mathcal{W}^2),
\end{equation}

which by Definitions 4.2 and 4.4 implies the statement. \qed

5. Study of the constituents $\mathbf{h}$ and $\mathbf{H}$ of the map $\mathcal{V}_0 \to \text{Der}_{\Delta^W}(\mathcal{W}, \mathcal{W}^2)$

In this section, we study the maps $\mathbf{h}$ and $\mathbf{H}$. The first main result is the computation of $\text{Ker}(\mathbf{h})$. To obtain it, one first defines endomorphisms and degrees on the algebras $\mathcal{W}$ and $\mathcal{W}^2$ (5.1). These are used to construct sequences which are shown to be convergent in the discrete topology (i.e. eventually constant) in 5.2. The opposite analogues of the results of §§5.1 and 5.2 are obtained in 5.3 and the results of §§5.1, 5.3 are put together in 5.4 to obtain the computation of $\text{Ker}(\mathbf{h})$ (Proposition 5.15). The second main result is a commutative square relating $\mathbf{H}$ with $\Delta^M$ (5.5, Proposition 5.16).

5.1. The endomorphisms $\partial_n$ and degrees $\deg_n$, $\deg_n^{(1)}$ and $\deg_n^{(2)}$.

Lemma 5.1. (a) For each $n > 0$, there is a linear endomorphism $\partial_n$ of $\mathcal{W}$, uniquely determined by the conditions $\partial_n(a \tilde{y}_m) = \delta_{nm} a$ for any $a \in \mathcal{W}$ and $m > 0$, and $\partial_n(1) = 0$.

(b) One has

\begin{equation}
\forall a, b \in \mathcal{W}, \quad \partial_n(ab) = a \partial_n(b) + \partial_n(a) \epsilon(b).
\end{equation}

where $\epsilon : \mathcal{W} \to \mathbb{Q}$ is the projection of $\mathcal{W}$ on its degree 0 part.
Lemma 5.6. \(\text{(a)}\) As the algebra \(W\) is freely generated by the family \((\tilde{y}_n)_{n>0}\), a basis is given by the set of all the words in these elements. For each \(n > 0\), the conditions uniquely determine the image of this basis, which determines \(\partial_n\) uniquely. \(\text{(b)}\) can be checked for \(a, b\) elements of this basis, which by linearity implies its validity for any \(a, b\).

Lemma-Definition 5.2. \(\text{(a)}\) One has \(W = \mathbb{Q} \oplus (\oplus_{k>0} W\tilde{y}_k)\).

\(\text{(b)}\) For \(a \in W\), one defines \(\deg(a) := \min\{d \geq 0 | a \in \mathbb{Q} \oplus (\oplus_{k=1}^d W\tilde{y}_k)\} \in \mathbb{Z}_{\geq 0}\).

Proof. \(\text{(a)}\) follows from the fact that \(W\) is freely generated by the family \((\tilde{y}_n)_{n>0}\). It implies that the family \((\mathbb{Q} \oplus (\oplus_{k=1}^d W\tilde{y}_k))_{d \geq 0}\) is an increasing sequence of subsets of \(W\) whose union is \(W\), which justifies the definition \(\text{(b)}\).

Lemma 5.3. For \(a \in W\), one has \(\partial_n(a) = 0\) for any \(n > \deg(a)\).

Proof. One has \(a \in \mathbb{Q} \oplus (\oplus_{k=1}^{\deg(a)} W\tilde{y}_k)\), and the restriction of \(\partial_n\) to this space is 0 for any \(n > \deg(a)\).

Lemma-Definition 5.4. \(\text{(a)}\) One has \(|W|^2 = \mathbb{Q} \otimes W \oplus (\oplus_{k>0} W\tilde{y}_k \otimes W) = W \otimes \mathbb{Q} \oplus (\oplus_{k>0} W \otimes W\tilde{y}_k)\).

\(\text{(b)}\) For \(a \in |W|^2\), one defines \(\deg^{(1)}(a) := \min\{d \geq 0 | a \in \mathbb{Q} \otimes W \oplus (\oplus_{k=1}^d W\tilde{y}_k \otimes W)\} \in \mathbb{Z}_{\geq 0}\) and \(\deg^{(2)}(a) := \min\{d \geq 0 | a \in W \otimes \mathbb{Q} \oplus (\oplus_{k=1}^d W \otimes W\tilde{y}_k)\} \in \mathbb{Z}_{\geq 0}\).

Proof. \(\text{(a)}\) follows from Lemma-Definition 5.2 \(\text{(a)}\). It implies that \((\mathbb{Q} \otimes W \oplus (\oplus_{k=1}^d W\tilde{y}_k \otimes W))_{d \geq 0}\) and \((W \otimes \mathbb{Q} \oplus (\oplus_{k=1}^d W \otimes W\tilde{y}_k))_{d \geq 0}\) both are increasing sequences of subsets of \(|W|^2\) with union \(|W|^2\), which justifies the definition in \(\text{(b)}\).

Lemma 5.5. For any \(a \in |W|^2\), one has

\((\partial_n \otimes \text{id})(a) = 0\) for \(n > \deg^{(1)}(a)\) and \((\text{id} \otimes \partial_n)(a) = 0\) for \(n > \deg^{(2)}(a)\).

Proof. One has \(a \in (\mathbb{Q} \oplus (\oplus_{k=1}^{\deg^{(1)}(a)} W\tilde{y}_k)) \otimes W\), and the restriction of \(\partial_n \otimes \text{id}\) to this space is 0 for \(n > \deg^{(1)}(a)\). Similarly, \(a \in W \otimes (\mathbb{Q} \oplus (\oplus_{k=1}^{\deg^{(2)}(a)} W\tilde{y}_k))\), and the restriction of \(\text{id} \otimes \partial_n\) to this space is 0 for \(n > \deg^{(2)}(a)\).

5.2. Convergence results in the discrete topology. Recall that any set \(S\) can be equipped with its discrete topology. A sequence \((s_n)_{n>0}\) with values in \(S\) is convergent in this topology iff there exists \(s \in \overline{S}\), such that \(s_n = s\) for all but finitely many values of \(n\). If \((s_n)_{n>0}\) is convergent, an element \(s \in S\) with this property is necessarily unique and called the limit of \((s_n)_{n>0}\).

Lemma 5.6. If \((\mathfrak{a}, \mathfrak{b}, \mathfrak{z}) \in (\oplus_{i \geq 0} W^{|i|^2})_{\oplus^2} \oplus (\oplus_{i \geq 0} W)\), then for any \(k \geq 0\), the sequence \(Z_{>0} \ni n \mapsto (\partial_{k} \otimes \partial_{n+k})(h_{\mathfrak{a}}(\mathfrak{b}, \mathfrak{z}))(2n) \in W^{\otimes 2}\) is convergent in the discrete topology of \(W^{\otimes 2}\), with limit \(a_k - c^{\otimes 2}(b_k)1^{\otimes 2}\), where \(a = (a_i)_{i \geq 0}, \ b = (b_i)_{i \geq 0}\).
Proof. Define $z_i \in W$ for $i \geq 0$ by $z = (z_i)_{i \geq 0} \in \oplus_{i \geq 0} W$. Since the sequence $(b_l)_{l \geq 0}$ takes values zero for all but a finite number of indices, the same is true of the sequences $(\deg^{(1)}(b_l))_{l \geq 0}$, $(\deg^{(2)}(b_l))_{l \geq 0}$, so these sequences are bounded. Since the sequence $(z_i)_{i \geq 0}$ takes values zero for all but a finite number of indices, the same is true of the sequence $(\deg(z_i))_{i \geq 0}$, so this sequence is bounded. Set

$$N(b, \tilde{z}) := \max\left\{ \max\{\deg^{(1)}(b)\mid l \geq 0\}, \max\{\deg^{(2)}(b)\mid l \geq 0\}, \max\{\deg(z_i)\mid i \geq 0\} \right\}.$$  

Let $n > N(b, \tilde{z})$ and let $k \geq 0$.

For any $l \geq 0$, one has $(\partial_n \otimes \partial_{n+k})(a_l \Delta(2n + l)) = \sum_{i=0}^{2n+l} a_l \delta_{i,n} \delta_{2n+l-i,n+k}$ therefore

(5.2.1) \quad \forall l \geq 0, \quad (\partial_n \otimes \partial_{n+k})(a_l \Delta(2n + l)) = \delta_{kl} a_k.

Let $l \geq 0$. One has $n > \deg^{(1)}(b_l)$ and $n + k > \deg^{(2)}(b_l)$, which by Lemma 5.5 implies $(\partial_n \otimes \id)(b_l) = (\id \otimes \partial_{n+k})(b_l) = 0$ and therefore

(5.2.2) \quad (\partial_n \otimes \partial_{n+k})(b_l) = (\epsilon \otimes \partial_{n+k})(b_l) = (\partial_n \otimes \epsilon)(b_l) = 0.

Then

$$\begin{align*}
(\partial_n \otimes \partial_{n+k})(\Delta(2n+l)b_l) &= \Delta(2n+l)(\partial_n \otimes \partial_{n+k})(b_l) + (\partial_n \otimes \id)(\Delta(2n+l))(\epsilon \otimes \partial_{n+k})(b_l) \\
&+ (\id \otimes \partial_{n+k})(\Delta(2n+l))(\partial_n \otimes \epsilon)(b_l) + (\partial_n \otimes \partial_{n+k})(\Delta(2n+l))\epsilon^{\otimes 2}(b_l) \\
&= (\partial_n \otimes \partial_{n+k})(\Delta(2n+l))\epsilon^{\otimes 2}(b_l) = \left( \sum_{i=0}^{2n+l} \delta_{i,n} \delta_{2n+l-i,n+k} \right) \epsilon^{\otimes 2}(b_l) 1^{\otimes 2} = \delta_{kl} \epsilon^{\otimes 2}(b_l) 1^{\otimes 2},
\end{align*}$$

where the first equality follows from Lemma 5.1 (b), the second equality from (5.2.2) and the third equality from the definition of $\Delta(2n + l)$. Therefore

(5.2.3) \quad \forall l \geq 0, \quad (\partial_n \otimes \partial_{n+k})(\Delta(2n+l)b_l) = \delta_{kl} \epsilon^{\otimes 2}(b_l) 1^{\otimes 2}.

For any $i \geq 0$, one has $\partial_n(z_i) = \partial_{n+k}(z_i) = 0$ by Lemma 5.3 therefore

(5.2.4) \quad \forall i \geq 0, \quad (\partial_n \otimes \partial_{n+k})(z_i) = \partial_n(z_i) = \partial_{n+k}(z_i) = 0.

Subtracting the sum for all the values of $l \geq 0$ of (5.2.3) from the similar sum for (5.2.1), adding up the sum for all the values of $i \geq 0$ of (5.2.4), and applying $\partial_n \otimes \partial_{n+k}$ to the expression (4.2.3) of $h(a, b, z)(2n)$, one derives for any $k \geq 0$:

$$\forall n > N(b, \tilde{z}), \quad (\partial_n \otimes \partial_{n+k})(h(a, b, \tilde{z})(2n)) = a_k - \epsilon^{\otimes 2}(b_k) 1^{\otimes 2},$$

which implies the result. \hfill \square

Lemma 5.7. If $\tilde{z} = (z_i)_{i \geq 0} \in \oplus_{i \geq 0} W$, then for any $i \geq 0$, the sequence $Z_{i \geq 0} \ni n \mapsto ((\epsilon \circ \partial_{n+i}) \otimes \id)(h(0, 0, \tilde{z})(n)) \in W$ is convergent in the discrete topology of $W$, with limit $z_i$.

Proof. If $\tilde{z} \in \oplus_{i \geq 0} W$, then the sequence $j \mapsto \deg(z_j)$ is bounded. Set $N(\tilde{z}) := \max\{\deg(z_j)\mid j \geq 0\}$. Then if $n > N(\tilde{z})$ and $i \geq 0$, then $n + i > N(\tilde{z}) \geq \deg(z_i)$, which by Lemma 5.3 implies
\( \partial_{n+i}(z_i) = 0 \). Therefore
\[
\text{(5.2.5)} \quad \text{if } n > N(z) \text{ and } i \geq 0, \text{ then } \partial_{n+i}(z_i) = 0.
\]

Then if \( n > N(z) \) and \( i \geq 0, \)
\[
((\epsilon \circ \partial_{n+i}) \otimes \text{id})(h(0,0,z)(n)) = ((\epsilon \circ \partial_{n+i}) \otimes \text{id})(z_i \otimes \tilde{y}_{n+i} + \tilde{y}_{n+i} \otimes z_i) = 0 + z_i = z_i,
\]
where the first equality follows from \((1.2.4)\), and the second equation from \( \partial_{n+i}(\tilde{y}_{n+i}) = 1 \) and \((5.2.5)\). This implies the result. \( \square \)

5.3. Opposite versions of the results of §§5.1 and 5.2

Lemma 5.8. (a) For each \( n > 0 \), there is a linear endomorphism \( \partial'_n \) of \( W \), uniquely determined by the conditions \( \partial'_n(\tilde{y}_m) = \delta_{nm} a \) for any \( a \in W \) and \( m > 0 \), and \( \partial'_n(1) = 0 \).

(b) One has \( \partial'_n(ab) = \partial'_n(a)b + \epsilon(a)\partial'_n(b) \) for any \( a, b \in W \).

Proof. There is a unique automorphism \( \text{inv} \) of the vector space \( W \), which is the identity on 1 and the \( \tilde{y}_m \), \( m \geq 1 \) and satisfies \( \text{inv}(ab) = \text{inv}(b)\text{inv}(a) \). One checks that for any \( n \geq 1 \), \( \partial'_n = \text{inv} \circ \partial_n \circ \text{inv} \). The results then follow from Lemma 5.1. \( \square \)

Lemma-Definition 5.9. (a) One has \( W = \mathbb{Q} \oplus (\oplus_{k \geq 0} \tilde{y}_k W) \).

(b) For \( a \in W \), one defines \( \text{deg}'(a) := \text{min}\{d \geq 0|a \in \mathbb{Q} \oplus (\oplus_{k=1}^d \tilde{y}_k W)\} \in \mathbb{Z}_{\geq 0} \).

Proof. (a) follows from Lemma-Definition 5.2 by applying \( \text{inv} \) (see proof of Lemma 5.8). Then \( \text{deg}' = \text{deg} \circ \text{inv} \). \( \square \)

Lemma 5.10. For \( a \in W \), one has \( \partial'_n(a) = 0 \) for any \( n > \text{deg}'(a) \).

Proof. Follows from Lemma 5.3 and \( \text{deg}' = \text{deg} \circ \text{inv} \). \( \square \)

Lemma-Definition 5.11. (a) One has \( W^\otimes 2 = \mathbb{Q} \otimes W \oplus (\oplus_{k \geq 0} \tilde{y}_k W \otimes W) = W \otimes \mathbb{Q} \oplus (\oplus_{k \geq 0} \tilde{y}_k W \otimes \tilde{y}_k W) \).

(b) For \( a \in W^\otimes 2 \), one defines \( \text{deg}^{(1)}(a) := \text{min}\{d \geq 0|a \in \mathbb{Q} \otimes W \oplus (\oplus_{k=1}^d \tilde{y}_k W \otimes W)\} \in \mathbb{Z}_{\geq 0} \)
and \( \text{deg}^{(2)}(a) := \text{min}\{d \geq 0|a \in W \otimes \mathbb{Q} \oplus (\oplus_{k=1}^d W \otimes \tilde{y}_k W)\} \in \mathbb{Z}_{\geq 0} \).

Proof. (a) follows from Lemma-Definition 5.4 by applying \( \text{inv} \otimes 2 \). Then \( \text{deg}^{(1)} = \text{deg}^{(1)} \circ \text{inv} \otimes 2 \) and \( \text{deg}^{(2)} = \text{deg}^{(2)} \circ \text{inv} \otimes 2 \). \( \square \)

Lemma 5.12. For any \( a \in W^\otimes 2 \), one has
\[ (\partial'_n \otimes \text{id})(a) = 0 \quad \text{for} \quad n > \text{deg}^{(1)}(a) \quad \text{for} \quad (\text{id} \otimes \partial'_n)(a) = 0 \quad \text{for} \quad n > \text{deg}^{(2)}(a). \]

Proof. Follows Lemma 5.11 by applied to \( \text{inv} \otimes 2 \). \( \square \)

Lemma 5.13. If \( (a, b, z) \in (\oplus_{l \geq 0} W^\otimes 2) \oplus (\oplus_{i \geq 0} W) \), then for any \( k \geq 0 \), the sequence \( Z_{\geq 0} \ni n \mapsto (\partial'_n \otimes \partial'_n + k)(h(a, b, z)(2n)) \in W^\otimes 2 \) is convergent in the discrete topology of \( W^\otimes 2 \), with limit \( -b + \epsilon(2)(a)|^2 \), where \( a = (a_l)_{l \geq 0}, b = (b_l)_{l \geq 0} \).
Proof. Set $\tilde{N}(a, z) := \max\{\max\{\deg^{(1)}(a_i)|l \geq 0\}, \max\{\deg^{(2)}(a_i)|l \geq 0\}, \max\{\deg'(z_i)|i \geq 0\}\}$. Similarly to the proof of Lemma 5.6 and using Lemmas 5.10 and 5.13 one proves
\[ \forall k \geq 0, \forall n > \tilde{N}(a, z), (\partial_n^r \otimes \partial_{n+k}^s)(h(a, b, z)_n) = -b_k + \epsilon^{(2)}(a_k)1^{\otimes 2} \]
which implies the result. \hfill \square

5.4. Computation of $\text{Ker}(h)$.

Definition 5.14. $j : \oplus_{k \geq 0} \mathbb{Q} \to (\oplus_{k \geq 0} \mathcal{W}^{\otimes 2})^{\oplus 2} \oplus (\oplus_{i \geq 0} \mathcal{W})$ is the linear map given by $c \mapsto (c_1^{\otimes 2}, c_1^{\otimes 2}, 0)$.

Proposition 5.15. The sequence $\oplus_{k \geq 0} \mathbb{Q} \xrightarrow{j} (\oplus_{k \geq 0} \mathcal{W}^{\otimes 2})^{\oplus 2} \oplus (\oplus_{i \geq 0} \mathcal{W}) \xrightarrow{h} \text{Map}(\mathbb{Z}_{>0}, \mathcal{W}^{\otimes 2})$ is exact.

Proof. Using (4.2.4), one shows $h(c_1^{\otimes 2}, c_1^{\otimes 2}, 0) = 0$ for any $c \in \oplus_{i \geq 0} \mathbb{Q}$, which implies $h \circ j = 0$. It follows that $\text{Im}(j) \subset \text{Ker}(h)$. Let us prove the opposite inclusion. If $(a, b, z) \in (\oplus_{i \geq 0} \mathcal{W}^{\otimes 2})^{\oplus 2} \oplus (\oplus_{i \geq 0} \mathcal{W})$ belongs to $\text{Ker}(h)$, then $h(a, b, z) = 0$, which implies that for any $k \geq 0$, the sequence $((\partial_n^r \otimes \partial_{n+k}^s) \circ h(a, b, z))(2n)_{n \geq 0}$ is zero. Lemma 5.6 and the uniqueness of the limit of a convergent series in the discrete topology then implies
\[ (5.4.1) \quad \forall k \geq 0, \quad a_k = \epsilon^{(2)}(b_k)1^{\otimes 2}. \]

Similarly, for any $k \geq 0$, the sequence $((\partial_n^r \otimes \partial_{n+k}^s) \circ h(a, b, z))(2n)_{n \geq 0}$ is zero. Lemma 5.13 and the same uniqueness principle then implies
\[ (5.4.2) \quad \forall k \geq 0, \quad b_k = \epsilon^{(2)}(a_k)1^{\otimes 2}. \]

Equations (5.4.1) and (5.4.2) imply that for any $k \geq 0$, $\epsilon^{(2)}(a_k) = \epsilon^{(2)}(b_k)$, and if one sets $c_k := \epsilon^{(2)}(a_k) = \epsilon^{(2)}(b_k) \in \mathbb{Q}$, then $a_k = c_k1^{\otimes 2} = b_k$. Set now $c := (c_k)_{k \geq 0} \in \oplus_{k \geq 0} \mathbb{Q}$, then $a = c_1^{\otimes 2} = b$. Then $0 = h(a, b, z) = h(c_1^{\otimes 2}, c_1^{\otimes 2}, 0) + h(0, 0, z) = h(0, 0, z)$, where the first (resp. second, third) equality follows from $(a, b, z) \in \text{Ker}(h)$ (resp. $a = c_1^{\otimes 2} = b$, $(c_1^{\otimes 2}, c_1^{\otimes 2}, 0) \in \text{Ker}(h)$). Therefore $h(0, 0, z) = 0$. It follows that for any $i \geq 0$, the sequence $(((\iota \circ \partial_{n+i}^r) \otimes \text{id}) \circ h(0, 0, z))(n)_{n \geq 0}$ is zero. Then Lemma 5.7 together with the uniqueness of a limit in the discrete topology implies $z_i = 0$ for any $i \geq 0$, hence $z = 0$. So $(a, b, z) = (c_1^{\otimes 2}, c_1^{\otimes 2}, 0) = j(c) \in \text{Im}(j)$. \hfill \square

5.5. A commutative square relating $H$ and $\Delta^M$ and a commutative triangle.

Proposition 5.16. The following diagram is commutative
\[
\begin{array}{ccc}
\mathcal{V}_0 & \xrightarrow{H} & (\oplus_{k \geq 0} \mathcal{W}^{\otimes 2}) \oplus (\oplus_{k \geq 0} \mathcal{W}^{\otimes 2}) \oplus (\oplus_{i \geq 0} \mathcal{W}) \\
\downarrow^{-1 \cdot \text{id}} & & \downarrow^{-1 \cdot \text{id}} \\
\mathcal{M} & \xrightarrow{\Delta^M - \text{id} \otimes \text{id} - \text{id} \otimes \text{id}} & \mathcal{M}^{\otimes 2}
\end{array}
\]
where $p_0 : \oplus_{k \geq 0} \mathcal{W}^{\otimes 2} \to \mathcal{W}^{\otimes 2}$ is the projection on the summand $k = 0$. 

Proof. One checks that the restriction of $L_1 : \mathcal{V}_0 \to \mathcal{W}^{\otimes 2}$ to $\mathcal{V}e_0$ is zero, and that its restriction to $\mathcal{V}e_1 \subset \mathcal{W}$ coincides with $\Delta^W - \text{id} \otimes 1 - 1 \otimes \text{id}$. This implies

$$(5.5.1) \quad L_1 = (\Delta^W - \text{id} \otimes 1 - 1 \otimes \text{id}) \circ \pi,$$

where $\pi : \mathcal{V}_0 \to \mathcal{W}$ is the map whose restriction to $\mathcal{V}e_0$ is the injection $\mathcal{V}e_0 \subset \mathcal{W}$ and with kernel $\mathcal{V}e_0$. Let $v \in \mathcal{V}$. Then

$$(- \cdot 1_M)^{\otimes 2} \circ (p_0 \oplus 0 \oplus 0)(\mathbf{H}(v)) = L_1(v) \cdot 1^{\otimes 2}_M = (\Delta^W(\pi(v)) - \pi(v) \otimes 1 - 1 \otimes \pi(v)) \cdot 1^{\otimes 2}_M$$

$$= \Delta^M(\pi(v) \cdot 1_M) - \pi(v) \cdot 1_M \otimes 1_M - 1_M \otimes \pi(v) \cdot 1_M = \Delta^M(v \cdot 1_M) - v \cdot 1_M \otimes 1_M - 1_M \otimes v \cdot 1_M$$

where the first equality follows from $(p_0 \oplus 0 \oplus 0)(\mathbf{H}(v)) = L_1(v)$ (see Definition 4.2), the second equality follows from (5.5.1), the third equality from (2.4.1), and the last equality from $\pi(v) \cdot 1_M = v \cdot 1_M$ for any $v \in \mathcal{V}_0$. This implies the commutativity of the square. \qed

**Proposition 5.17.** The following triangle is commutative

$$\begin{array}{ccc}
\oplus_{k \geq 0}\mathbb{Q} & \xrightarrow{\rho_0 \cdot 1^\otimes_2_M} & \mathcal{M}^{\otimes 2} \\
\downarrow & & \downarrow \\
\oplus_{k \geq 0}\mathcal{W}^{\otimes 2} \oplus (\oplus_{i \geq 0}\mathcal{W}) & \xrightarrow{(- \cdot 1_M)^{\otimes 2} \circ (p_0 \oplus 0 \oplus 0)} & \mathcal{M}^{\otimes 2}
\end{array}$$

where $\rho_0$ is as in Proposition 6.16 and $\tilde{\rho}_0 : \oplus_{k \geq 0}\mathbb{Q} \to \mathbb{Q}$ is the projection on the component $k = 0$.

**Proof.** Let $\xi = (c_k)_{k \geq 0} \in \oplus_{k \geq 0}\mathbb{Q}$. Then

$$(- \cdot 1_M)^{\otimes 2} \circ (p_0 \oplus 0 \oplus 0) \circ \mathbf{j}(\xi) = (- \cdot 1_M)^{\otimes 2} \circ (p_0 \oplus 0 \oplus 0)(\xi^{\otimes 2}, \xi^{\otimes 2}, 0)$$

$$= (- \cdot 1_M)^{\otimes 2}(c_0^{\otimes 2}) = c_0^{\otimes 2} = \tilde{\rho}_0(\xi) = (\rho_0 \circ 1^{\otimes}_M)^{\otimes 2} - \rho_0(\xi).$$

where the first equality follows from $\mathbf{j}(\xi) = (\xi^{\otimes 2}, \xi^{\otimes 2}, 0)$, the second equality follows from $(p_0 \oplus 0 \oplus 0)(\xi^{\otimes 2}, \xi^{\otimes 2}, 0) = c_0^{\otimes 2}$, and the last equality follows from $\tilde{\rho}_0(\xi) = c_0$. \qed

6. Proof of equality results

In §6.1 we put together the results of §§4 and 5 to prove the equality between the Lie algebra $\text{stab}(\Delta^{W,\text{DR}})$ and $\text{stab}(\Delta^{M,\text{DR}})$ (Theorem 6.6). In §6.2 we derive from this the equality of the stabilizer bitorors $\text{Stab}(\Delta^{W,\text{DR}}/B)$ and $\text{Stab}(\Delta^{M,\text{DR}}/B)$ (Theorem 6.7).

6.1. Proof of the equality of the stabilizer Lie algebras $\text{stab}(\Delta^{W,\text{DR}})$ and $\text{stab}(\Delta^{M,\text{DR}})$. The subspace $\mathcal{P}(\mathcal{M}) \subset \mathcal{M}$ is graded. It follows that the same holds for the subspace $(- \cdot 1_M)^{-1}(\mathcal{P}(\mathcal{M})) \subset \mathcal{V}_0$.

**Proposition 6.1.** One has $\text{stab}_{\mathcal{V}_0}(\Delta^W) \subset (- \cdot 1_M)^{-1}(\mathcal{P}(\mathcal{M}))$ (inclusion of graded subspaces of $\mathcal{V}_0$).
Proof. Both sides of this inclusion are graded, so it suffices to show the inclusion of homogeneous components for each degree. Let \( a \in \text{stab}_{V_0}(\Delta^W) \subset V_0 \) be homogeneous. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{-1_M} & V_0 \\
\Delta^M - id \otimes _1 1_M - 1_M \otimes id & \downarrow H & \xrightarrow{-1} \text{Der}_{\Delta^W}(W, W \otimes 2) \\
M \otimes \mathbf{1}_{\mathbb{Q}}(1) & \xrightarrow{j} & \text{Map}(\mathbb{Z}_{>0}, W \otimes 2) \\
\oplus_{k \geq 0} W \otimes 2 & \xrightarrow{\iota} & \oplus (\oplus_{i \geq 0} W) \\
\end{array}
\]

where the left square commutes by Proposition \ref{11}, the right square commutes by Proposition \ref{11}, the triangle commutes by Proposition \ref{11}, and the lower sequence of maps \((h, j)\) is exact by Proposition \ref{11}.

Since \( \text{stab}_{V_0}(\Delta^W) = \text{Ker}(- \cdot \Delta^W) \) (see Lemma \ref{11} (a)) and by the commutativity of the right square, \( i \circ h \circ H(a) = 0 \), which by the injectivity of \( i \) (see Lemma \ref{11}) implies that \( H(a) \) belongs to \( \text{Ker}(h) \). The exactness of the lower sequence of maps then implies the existence of \( \mathcal{L} \in \oplus_{i \geq 0} \mathbb{Q} \), such that

\[ (6.1.1) \quad H(a) = j(\mathcal{L}). \]

Then

\[ (\Delta^M - id \otimes 1_M - 1_M \otimes id)(a \cdot 1_M) = (-1_M) \otimes 2 \circ (p_0 \otimes 0 \otimes 0) \circ H(a) = (-1_M) \otimes 2 \circ (p_0 \otimes 0 \otimes 0)(j(\mathcal{L})) = \tilde{p}_0(\mathcal{L})1^\otimes 2_M, \]

where the first equality follows from the commutativity of the left square, the second equality follows from \( (6.1.1) \), and the third equality follows from the commutativity of the triangle. It follows that \((\Delta^M - id \otimes 1_M - 1_M \otimes id)(a \cdot 1_M) \in \mathcal{Q}1^\otimes 2_{\mathcal{M}} \). The map \( a \mapsto (\Delta^M - id \otimes 1_M - 1_M \otimes id)(a \cdot 1_M) \) is graded and \( a \) has positive degree, which implies that \((\Delta^M - id \otimes 1_M - 1_M \otimes id)(a \cdot 1_M) = 0 \), which by \( (3.4.1) \) implies \( a \cdot 1_M \in \mathcal{P}(\mathcal{M}) \).

\[ \Box \]

Lemma 6.2. The derivation \( \Gamma_{\mathcal{W}}^{V(1)}_{[e_0, e_1]} \) of \( \mathcal{W} \) is such that for any \( n \geq 1 \),

\[ \tilde{y}_n \mapsto (\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2)\tilde{y}_n + \tilde{y}_n(\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) - \tilde{y}_1 \tilde{y}_{n+1} - \tilde{y}_{n+1} \tilde{y}_1. \]

Proof. One computes \( 0(\theta_0, e_1) = [e_0, e_1] + (1/2)(e_0, e_1)e_0 e_1^2 = [e_0, e_1] + (1/2)e_1^2 \), therefore \( \Gamma_{\mathcal{W}}^{V(1)}_{[e_0, e_1]} = \Gamma_{\mathcal{W}}^{V(1)}_{[e_0, e_1]} \) is given by \( e_0 \mapsto [e_0, e_1] + e_1^2/2, e_0 \) and \( e_1 \mapsto 0 \). Then for \( n \geq 1 \),

\[ \Gamma_{\mathcal{W}}^{V(1)}_{[e_0, e_1]}(\tilde{y}_n) = \Gamma_{\mathcal{W}}^{V(1)}_{[e_0, e_1]}(e_0^{n-1}e_1) = [(e_0, e_1) + e_1^2/2, e_0^{n-1}]e_1 \]

\[ = (\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2)\tilde{y}_n + \tilde{y}_n(\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) - \tilde{y}_1 \tilde{y}_{n+1} - \tilde{y}_{n+1} \tilde{y}_1. \]

\[ \Box \]
Lemma 6.3. The derivation $\theta([e_0, e_1]) \cdot \Delta^W = (\Gamma \text{der}_{[e_0, e_1]}^{W, (1)} \otimes \text{id} + \text{id} \otimes \Gamma \text{der}_{[e_0, e_1]}^{W, (1)}) \circ \Delta^W - \Delta^W \circ \Gamma \text{der}_{[e_0, e_1]}^{W, (1)}$ of $W$ is the element of $\text{Der}_{\Delta^W}(W, W^\otimes 2)$ such that for any $n \geq 1$,

$$\tilde{y}_n \mapsto 2(\tilde{y}_1 \otimes \tilde{y}_{n+1} + \tilde{y}_{n+1} \otimes \tilde{y}_1) - 2((\tilde{y}_2 + \tilde{y}_1^2) \otimes \tilde{y}_n + \tilde{y}_n \otimes (\tilde{y}_2 + \tilde{y}_1^2)) + 2(\tilde{y}_n \otimes 1 + 1 \otimes \tilde{y}_n - \sum_{k=1}^{n-1} \tilde{y}_k \otimes \tilde{y}_{n-k})(\tilde{y}_1 \otimes \tilde{y}_1)$$

Proof. Let $\sigma$ be the permutation of a tensor factor of $W^\otimes 2$. Then using Lemma 6.2 one computes, for $n \geq 1$

$$(\Gamma \text{der}_{[e_0, e_1]}^{W, (1)} \otimes \text{id} + \text{id} \otimes \Gamma \text{der}_{[e_0, e_1]}^{W, (1)}) \circ \Delta^W(\tilde{y}_n) = (\text{id} + \sigma)((\Gamma \text{der}_{[e_0, e_1]}^{W, (1)}(\tilde{y}_n) \otimes 1 - \sum_{k=1}^{n-1} \tilde{y}_k \otimes \tilde{y}_{n-k})$$

where the equality follows from (2.4.1) and from the $\sigma$-invariance of $\Delta^W(\tilde{y}_n)$, and

$$\Delta^W \circ \Gamma \text{der}_{[e_0, e_1]}^{W, (1)}(\tilde{y}_n) = \Delta^W((\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2) \tilde{y}_n + \tilde{y}_n (\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) - \tilde{y}_1 \tilde{y}_{n+1} - \tilde{y}_{n+1} \tilde{y}_1)$$

$$= (\text{id} + \sigma)((\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2) \otimes 1)((\tilde{y}_n \otimes 1 + 1 \otimes \tilde{y}_n) - \sum_{k=1}^{n-1} \tilde{y}_k \otimes \tilde{y}_{n-k})$$

$$+ (\tilde{y}_n \otimes 1 + 1 \otimes \tilde{y}_n - \sum_{k=1}^{n-1} \tilde{y}_k \otimes \tilde{y}_{n-k})((\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) \otimes 1) - \tilde{y}_1 \otimes \tilde{y}_1)$$

$$- (\tilde{y}_1 \otimes 1)(\tilde{y}_{n+1} \otimes 1 + 1 \otimes \tilde{y}_{n+1} - \tilde{y}_1 \otimes \tilde{y}_n - \sum_{k=1}^{n-1} \tilde{y}_{k+1} \otimes \tilde{y}_{n-k})$$

$$- (\tilde{y}_{n+1} \otimes 1 + 1 \otimes \tilde{y}_{n+1} - \tilde{y}_n \otimes \tilde{y}_n - \sum_{k=1}^{n-1} \tilde{y}_{k+1} \otimes \tilde{y}_{n-k})(\tilde{y}_1 \otimes 1)$$

where the first equality follows from Lemma 6.2 and the second equality from the identities $(\text{id} + \sigma)(a b) = (\text{id} + \sigma)(a b)$ and $b(\text{id} + \sigma)(a) = (\text{id} + \sigma)(b a)$ for $a, b \in W^\otimes 2$ such that $b$ is $\sigma$-invariant, the $\sigma$-invariance and the explicit expressions of $\Delta^W(\tilde{y}_n)$ and $\Delta^W(\tilde{y}_{n+1})$, and the equalities $\Delta^W((\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2) = (\text{id} + \sigma)((\tilde{y}_2 + \frac{1}{2} \tilde{y}_1^2) \otimes 1)$, $\Delta^W(\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) = (\text{id} + \sigma)((\tilde{y}_2 - \frac{1}{2} \tilde{y}_1^2) \otimes 1 - \tilde{y}_1 \otimes \tilde{y}_1)$ and $\Delta^W(\tilde{y}_1) = (\text{id} + \sigma)(\tilde{y}_1 \otimes 1)$. The result follows from the computation of the difference of the two expressions. □

Remark 6.4. It follows that $\theta([e_0, e_1]) \cdot \Delta^W = i \cdot H(\tilde{z}_0, \tilde{z}_0, \tilde{z}_0)$ where $\tilde{z}_0 = 0$, $\tilde{b}_0^0 = (b_0^0)_{i \geq 0}$ where $b_0^0 = 2\tilde{y}_1 \otimes \tilde{y}_1$ and $b_i^0 = 0$ for $i > 0$, $\tilde{z}_0^0 = (z_0^0)_{i \geq 0}$ where $z_0^0 = -2(\tilde{y}_2 + \tilde{y}_1^2)$, $z_1^0 = 2\tilde{y}_1$, and $z_i^0 = 0$ for $i > 1$. One can check that $H([e_0, e_1]) = (\tilde{z}_0^0, \tilde{b}_0^0, \tilde{z}_0^0)$. □

Recall that $\text{stab}_{\text{lie}}([e_0, e_1])(\Delta^W)$ is a graded Lie subalgebra of $\text{lie}([e_0, e_1], \langle, \rangle)$.

Lemma 6.5. The degree 2 component of $\text{stab}_{\text{lie}}([e_0, e_1])(\Delta^W)$ is zero.

Proof. Since the degree 2 part of $\text{lie}([e_0, e_1])$ is one-dimensional, spanned by $[e_0, e_1]$, the statement is equivalent to $[e_0, e_1] \notin \text{stab}_{\text{lie}}([e_0, e_1])(\Delta^W)$, which follows from Lemma 6.3. □
Theorem 6.6. The Lie subalgebras \( \text{stab}(\hat{\Delta}^W,\text{DR}) \) and \( \text{stab}(\hat{\Delta}^M,\text{DR}) \) of \( g^{\text{DR}} \) (see [EF2], §3.5) are equal.

Proof. The inclusion \( \text{stab}(\hat{\Delta}^M) \subset \text{stab}(\hat{\Delta}^W) \) follows from [EF2], Corollary 3.14 (c). By Propositions 2.12 and 3.5 it implies the inclusion

\[
(6.1.2) \quad \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^M) \subset \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W)
\]

of graded Lie subalgebras of \( \text{lie}(e_0,e_1) \).

On the other hand, one has

\[
\text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W) \cap \text{lie}(e_0,e_1)[2] = \theta^{-1}(\text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W)) \cap \text{lie}(e_0,e_1)[2]
\]

\[
\subset \theta^{-1}(Q1 \oplus P(\mathcal{M})) \cap \text{lie}(e_0,e_1)[2] = \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^M),
\]

where the first equality follows from Lemma 2.11 (c), the inclusion follows from Proposition 6.1, and the second equality follows from (3.4.2). Together with (6.1.2), this implies

\[
(6.1.3) \quad \forall d \neq 2, \quad \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W)[d] = \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^M)[d].
\]

Then \( \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W)[2] = 0 = \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^M)[2] \), where the first equality follows from Lemma 6.3 and the second equality follows from (3.4.2). Together with (6.1.3), this implies the equality of the graded components of the Lie algebras \( \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^W) \) and \( \text{stab}_{\text{lie}(e_0,e_1)}(\Delta^M) \) in any degree, hence the equality of these Lie algebras. The announced equality then follows from Propositions 2.12 and 3.5.

\[ \square \]

6.2. Proof of the equality of the stabilizer bitorsors \( \text{Stab}(\hat{\Delta}^M,\text{DR}/B)(k) \) and \( \text{Stab}(\hat{\Delta}^W,\text{DR}/B)(k) \).

6.2.1. Review of known results. In [EF2], Definitions 2.16 and 2.20 and [EF3], Definitions 3.5 and 3.8, we introduced \( Q \)-group schemes \( \text{Stab}(\hat{\Delta}^{?,?}) \), where \( ? \) (resp. \( ?? \)) is one of the symbols \( M, W \) (resp. DR,B); these are extensions of \( G_m \) by pronipotent group schemes and give rise to functors \( k \rightarrow \text{Stab}(\hat{\Delta}^{?,?})(k) \) from the category of \( Q \)-algebras to that of groups; the Lie algebras \( \text{Stab}(\hat{\Delta}^B,\text{DR}) \) are \( \text{stab}(\hat{\Delta}^1,\text{DR}) \) (see [EF2], §3.5), and the Lie algebras \( \text{stab}(\hat{\Delta}^B) \) of \( \text{Stab}(\hat{\Delta}^B) \) were defined in [EF3], §3.8.

In [EF2], Definitions 2.17 and 2.21, we introduced the \( Q \)-schemes \( \text{Stab}(\hat{\Delta}^{?,\text{DR}/B}) \), where \( ? \) is one of the symbols \( M, W \), giving rise to functors \( k \rightarrow \text{Stab}(\hat{\Delta}^{?,\text{DR}/B})(k) \) from the category of \( Q \)-algebras to that of sets. For any \( k \), the set \( \text{Stab}(\hat{\Delta}^{?,\text{DR}/B})(k) \) is equipped with commuting left and right actions of the groups \( \text{Stab}(\hat{\Delta}^{?,\text{DR}})(k) \) and \( \text{Stab}(\hat{\Delta}^{?,B})(k) \). By [EF2], Theorem 3.1 (b) and (c) and [EF3], Lemmas 3.6 and 3.9, each of these actions is free and transitive, so that \( \text{Stab}(\hat{\Delta}^{?,\text{DR}/B})(k) \) is a bitorsor (see [EF3], Definition 1.1 (a), and [Q], Chap. III, Definition 1.5.3).

In [EF2], Theorem 3.1 (a) and [EF3], Theorem 3.14 (b) we constructed for any \( k \), group inclusions \( \text{Stab}(\hat{\Delta}^M,\text{DR})(k) \subset \text{Stab}(\hat{\Delta}^W,\text{DR})(k) \) and \( \text{Stab}(\hat{\Delta}^M,\text{B})(k) \subset \text{Stab}(\hat{\Delta}^W,\text{B})(k) \) and set inclusions \( \text{Stab}(\hat{\Delta}^M,\text{DR}/B)(k) \subset \text{Stab}(\hat{\Delta}^W,\text{DR}/B)(k) \), which are compatible with the actions.
6.2.2. Equality results.

**Theorem 6.7.** Let $k$ be a $\mathbb{Q}$-algebra. Then the inclusion of the bitorsors attached to $\text{Stab}(\hat{\Delta}^{M,DR/B})(k)$ and $\text{Stab}(\hat{\Delta}^{W,DR/B})(k)$ is an equality, i.e.:

(a) the group inclusion $\text{Stab}(\hat{\Delta}^{M,DR})(k) \subset \text{Stab}(\hat{\Delta}^{W,DR})(k)$ is an equality;
(b) the inclusion $\text{Stab}(\hat{\Delta}^{M,DR/B})(k) \subset \text{Stab}(\hat{\Delta}^{W,DR/B})(k)$ is an equality;
(c) the group inclusion $\text{Stab}(\hat{\Delta}^{M,B})(k) \subset \text{Stab}(\hat{\Delta}^{W,B})(k)$ is an equality.

**Proof.** (a) The sets $\text{Stab}(\hat{\Delta}^{2,DR})(k)$ for $? \in \{M, W\}$ are subgroups of the group $G^{DR}(k)$, which is $k^\times \times G(\hat{V})$ equipped with the structure of semidirect product of the group $(G(\hat{V}), \circ)$ by the action of $k^\times$ (see [EF2], Definitions 2.16 and 2.20 and §1.6.3, where $\hat{V}$ is denoted $\check{V}^{DR}$). By [EF2], Lemmas 2.18 and 2.22, $\text{Stab}(\hat{\Delta}^{2,DR})(k)$ contain $k^\times$, which implies the equalities $\text{Stab}(\hat{\Delta}^{2,DR})(k) = k^\times \times \text{Stab}_1(\hat{\Delta}^{2,DR})(k)$ of subsets of $k^\times \times G(\hat{V})$, where $\text{Stab}_1(\hat{\Delta}^{2,DR})(k)$ are the images of the intersections of $\text{Stab}(\hat{\Delta}^{2,DR})(k)$ with $\{1\} \times G(\hat{V})$ by the canonical isomorphism $\{1\} \times G(\hat{V}) \rightarrow G(\hat{V})$ (see [EF2], Remark 2.23). The statement is therefore equivalent to the equality of the subgroups $\text{Stab}_1(\hat{\Delta}^{M,DR})(k)$ and $\text{Stab}_1(\hat{\Delta}^{W,DR})(k)$ of $(G(\hat{V}), \circ)$.

Equip the completion $\text{Lie}(e_0, e_1)^\hat{\otimes}k$ with the product $\text{cbh}_{(\cdot)}(\cdot, \cdot)$ taking $(x, y)$ to the image by the Lie algebra morphism from the topologically free Lie algebra with generators $a, b$ to $(\text{Lie}(e_0, e_1)^\hat{\otimes}k, (\cdot))$ given by $a \mapsto x, b \mapsto y$ of the element $\log(e^x e^y)$. Then $(\text{Lie}(e_0, e_1)^\hat{\otimes}k, \text{cbh}_{(\cdot)}(\cdot, \cdot))$ is a group. The map $\exp_{\hat{\circ}} : \text{Lie}(e_0, e_1)^\hat{\otimes}k \rightarrow G(\hat{V})$ defined in [R], (3.1.10.1) sets up a group isomorphism $(\text{Lie}(e_0, e_1)^\hat{\otimes}k, \text{cbh}_{(\cdot)}(\cdot, \cdot)) \simeq (G(\hat{V}), \hat{\circ})$ ([R], Corollary 3.1.10, see also [EF2], §3.5.2). It follows from [EF2], Lemma 3.14 (based on [EF0], Lemma 5.1) that $\exp_{\hat{\circ}}$ restricts to bijections $\text{stab}_0(\hat{\Delta}^{M,DR})\hat{\otimes}k \rightarrow \text{Stab}_1(\hat{\Delta}^{M,DR})(k)$ and $\text{stab}_0(\hat{\Delta}^{W,DR})\hat{\otimes}k \rightarrow \text{Stab}_1(\hat{\Delta}^{W,DR})(k)$, the index 0 meaning the intersection of a Lie subalgebra of $g^{DR}$ with $\text{Lie}(e_0, e_1)^\hat{\otimes}k$. As Theorem 6.6 implies the equality $\text{stab}_0(\hat{\Delta}^{M,DR})\hat{\otimes}k = \text{stab}_0(\hat{\Delta}^{W,DR})\hat{\otimes}k$, one obtains the equality $\text{Stab}_1(\hat{\Delta}^{M,DR})(k) = \text{Stab}_1(\hat{\Delta}^{W,DR})(k)$.

(b) follows from the fact that $\text{Stab}(\Delta^{M,DR/B})(k)$ and $\text{Stab}(\Delta^{W,DR/B})(k)$ are subtorsors of $G^{DR,B}(k)$ with respective groups $\text{Stab}(\Delta^{M,DR})(k)$ and $\text{Stab}(\Delta^{W,DR})(k)$ (see [EF2], Theorem 3.1, (b) and (c)), from the torsor inclusion $\text{Stab}(\Delta^{M,DR/B})(k) \subset \text{Stab}(\Delta^{W,DR/B})(k)$ (see [EF2], Theorem 3.1, (c)), from (a) and from [EF2], Lemma 2.7, (b).

(c) (a) and (b) imply that the subtorsors $\text{Stab}(\Delta^{M,DR/B})(k)$ and $\text{Stab}(\Delta^{W,DR/B})(k)$ of $G^{DR,B}(k)$ are equal. If $G \cdot X_H$ is a bitorsor and if $G \cdot X_H'$ and $G \cdot X_H''$ are subbitorsors such that $G \cdot X'$ and $G \cdot X''$ are equal, then the subgroups $H'$ and $H''$ of $H$ are equal; indeed, [EF3], Lemma 1.12 implies that $H' \subset H''$, which by symmetry implies the equality. Applying this to $G \cdot X_H'$ and $G \cdot X_H''$ being the subbitorsors of $G^{DR,B}(k)$ corresponding to $\text{Stab}(\Delta^{M,DR/B})(k)$ and $\text{Stab}(\Delta^{W,DR/B})(k)$ (see [EF3], Lemmas 3.6 and 3.9) using the equality of the underlying torsors yields the claimed equality.

**Corollary 6.8.** The Lie algebra inclusion $\text{stab}(\Delta^{M,B}) \subset \text{stab}(\Delta^{W,B})$ is an equality.
Proof. The group schemes $\text{Stab}(\hat{\Delta}^{M,B})$ and $\text{Stab}(\hat{\Delta}^{W,B})$ are extensions of $\mathbb{G}_m$ by pronipotent group schemes. Theorem 6.7 (c), then implies the equality of these group schemes, and therefore of their Lie algebras, which implies the statement. \hfill \Box

References

[Dr] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\mathbb{Q}/\mathbb{Q})$*, Leningrad Math. J. 2 (1991), no. 4, 829–860.
[DeT] P. Deligne, T. Terasoma, *Harmonic shuffle relation for associators*, preprint (2005)
[Ec] J. Écalle, *ARI/GARI, la dimorphie et l’arithmétique des multizetas: un premier bilan*. Journal de Théorie des Nombres de Bordeaux, 15 (2003), p 411–478.
[EF0] B. Enríquez, H. Furusho, *A stabilizer interpretation of double shuffle Lie algebras*. Int. Math. Res. Not. IMRN 2018, no. 22, 6870–6907.
[EF1] B. Enríquez, H. Furusho, *The Betti side of the double shuffle theory. I. The harmonic coproduct*. Selecta Math. (N.S.) 27 (2021), no. 5, Paper No. 79, 106 pp.
[EF2] B. Enríquez, H. Furusho, *The Betti side of the double shuffle theory. II. Double shuffle relations for associators*. Preprint [arXiv:1807.07780] v5.
[EF3] B. Enríquez, H. Furusho, *The Betti side of the double shuffle theory. III. Bitorsor structures*. Preprint [arXiv:1908.00444] v5.
[G] J. Giraud, *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.
[IKZ] K. Ihara, M. Kaneko, D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. 142 (2006), no. 2, 307–338.
[LM] T.T.Q. Le, J. Murakami, *Kontsevich’s integral for the Kauffman polynomial*. Nagoya Math. J. 142 (1996), 39–65.
[R] G. Racinet, *Doubles mélanges des polylogarithmes multiples aux racines de l’unité*, Publ. Math. Inst. Hautes Études Sci. 95 (2002), 185–231.
[Z] D. Zagier, *Values of zeta functions and their applications*. First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, Progr. Math., 120, Birkhäuser, Basel, 1994.

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