BLOW-UP AND GLOBAL SOLUTIONS TO $L^p$ NORM PRESERVING NON-LOCAL FLOWS

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Abstract. In this paper, we study global existence and blow up properties to $L^p$ norm preserving non-local heat flows. We first study two kinds of $L^p$ norm preserving non-local flows and prove that these flows have the global solutions. Finally, we give an example to show that one kind of this heat flow may blow up in $L^\infty$ norm though its $L^p$ norm is preserved.

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1. Introduction

In this paper, we study global existence and blow up properties of positive solutions to $L^p$ norm preserving non-local heat flows

$$\partial_t u^r = \Delta u + \lambda(t) u^s, \quad M \times (0, T)$$

on the Riemannian manifold $(M, g)$ with the Cauchy data, where $T > 0$, $r > 0$, $s > 0$, and $\lambda(t)$ is chosen to make the $L^p$ norm of the solution $u$ be constant. We shall show that when $r = s = p - 1 > 0$, the global smooth solution exists. Assume for example, $M = \Omega$ is a bounded convex domain in $\mathbb{R}^n$. When $r = 1$, $1 < s = p < \frac{n+2}{n-2}$, the global existence of positive is also true, however, when $r = 1$ and $s = p \geq \frac{n+2}{n-2}$, we have blowup result. Our work is motivated by the recent excellent work of C.Caffarelli and F.Lin [5], where they have studied the global existence and regularity of $L^2$ norm preserving heat flow on bounded domains $\Omega \subset \mathbb{R}^n$ such as

$$\partial_t u = \Delta u + \lambda(t) u,$$

with

$$\lambda(t) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$
They also extend the method to study a family of singularly perturbed systems of non-local parabolic equations and study the partition problem for eigenvalues. After that the authors studied the global existence, asymptotic behavior, stability and gradient estimates for two kinds of non-homogenous $L^2$ norm preserving heat flows in [19].

We remark that the non-local heat flow also naturally arises in geometry such that the flow preserves some $L^p$ norm in the sense that some the geometrical quantity (such as length, area and so on) is preserved in the geometric heat flows. For more references on geometric flows such as harmonic map heat flows and non-local curve shortening flows, one may see [1], [17], [18] and [24].

We first study the following Yamabe type heat flow on a closed smooth Riemannian manifold $M^n$

\begin{equation}
\begin{aligned}
\left\{ \begin{array}{ll}
  u^{p-2} \partial_t u &= \Delta u + \lambda(t) u^{p-1} & \text{in } M \times \mathbb{R}_+, \\
  u(x, 0) &= g(x) & \text{in } M,
\end{array} \right.
\end{aligned}
\end{equation}

where $p > 1$, which has the positive solution and preserves the $L^p$ the norm. We call equation (1) Yamabe type heat flow since it relates to following Yamabe flow on closed manifolds $M^n$ which introduced by Hamilton

\begin{equation}
\frac{\partial g}{\partial t} = (s - R) g,
\end{equation}

where $R$ denotes the scalar curvature of metric $g$ and $s$ denotes the average scalar curvature. Write $g = u^{-n-2} g_0$, $n \geq 3$, with $u$ is a positive function and change time by a constant scale. Then (2) is equivalent to the following heat equation

\[ \frac{\partial u^p}{\partial t} = L_{g_0} u + c(n) u^p, \]

where \( p = \frac{n+2}{n-2} \), \( c(n) = \frac{n-2}{4(n-1)} \) and \( L_{g_0} u = \Delta_{g_0} u - c(n) R_{g_0} u \). For more references about Yamabe problem and Yamabe flow, one may see [2], [3], [13], [14], and [30]. Now direct computation to (1) shows that

\[ \frac{1}{p} \frac{d}{dt} \int_M u^p \, dx = \int_M u^{p-1} u_t \, dx = - \int_M |\nabla u|^2 \, dx + \lambda(t) \int_M u^p \, dx. \]

Thus, one must have \( \lambda(t) = \frac{\int_M |\nabla u|^2 \, dx}{\int_M g^p \, dx} \) to preserve the $L^p$ norm. Without loss of generality we assume $\int_M g^p \, dx = 1$. Then we consider the following problem on closed smooth Riemannian manifold $M^n$

\begin{equation}
\begin{aligned}
\left\{ \begin{array}{ll}
  u^{p-2} \partial_t u &= \Delta u + \lambda(t) u^{p-1} & \text{in } M \times \mathbb{R}_+, \\
  u(x, 0) &= g(x) & \text{in } M,
\end{array} \right.
\end{aligned}
\end{equation}
Thus, one must have \( \lambda \in g \) where \( \lambda \in g \)

\[ \text{Theorem 2.} \] Problem (3) has a positive global smooth solution \( u(t) \in L^\infty(\mathbb{R}_+, H^1(M)) \cap L^\infty(\mathbb{R}_+, L^\infty(M)) \). Furthermore, \( \lambda(t) \) is non-increasing function such that \( \lambda(t) \to 0 \) at exponential rate as \( t \to \infty \) and \( u(t) \) converges (passing by a subsequence) smoothly to a positive constant.

We next study the non-local heat flow on bounded smooth domain in \( \mathbb{R}^n \) which relates to the semilinear heat equations,

\[
\begin{cases}
\partial_t u = \Delta u + \lambda(t)u^p & \text{in } \Omega \times \mathbb{R}_+, \\
u(x, 0) = g(x) & \text{in } \Omega, \\
u(x, t) = 0 & \text{on } \partial \Omega \times \mathbb{R}_+,
\end{cases}
\]

where \( 1 < p < \frac{n+2}{n-2} \), which has the positive solution and preserves the \( L^2 \) the norm. Likewise,

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx = \int_\Omega uu_t = -\int_\Omega |\nabla u|^2 dx + \lambda(t) \int_\Omega u^{p+1} dx.
\]

Thus, one must have \( \lambda(t) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^{p+1} dx} \) to preserve the \( L^2 \) norm. Then we consider the following problem on bounded smooth domain in \( \mathbb{R}^n \)

\[
\begin{cases}
\partial_t u = \Delta u + \lambda(t)u^p & \text{in } \Omega \times \mathbb{R}_+, \\
u(x, 0) = g(x) & \text{in } \Omega, \\
u(x, t) = 0 & \text{on } \partial \Omega \times \mathbb{R}_+,
\end{cases}
\]

where \( 1 < p < \frac{n+2}{n-2} \), \( \lambda(t) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^{p+1} dx} \), \( g(x) \geq 0 \) in \( \Omega \), \( \int_\Omega g^2 dx = 1 \) and \( g \in C^1(\Omega) \). Similar to theorem 1 we also have the global solution to problem (4).

\[ \text{Theorem 2.} \] Problem (4) has a global positive smooth solution

\[ u(t) \in L^\infty(\mathbb{R}_+, H^1(\Omega)) \cap L^\infty(\mathbb{R}_+, L^{\frac{2n}{n-2}}(\Omega)). \]

Moreover, one can take \( t_i \to \infty \) such that \( \lambda(t_i) \to \lambda_\infty > 0 \), \( u(x, t_i) \to u_\infty(x) \) in \( L^2(\Omega) \), \( u(x, t_i) \to u_\infty(x) \) in \( H^1(\Omega) \) and \( u_\infty \) solves the equation \( \Delta u_\infty + \lambda_\infty u_\infty^p = 0 \) in \( \Omega \) and \( u_\infty = 0 \) on \( \partial \Omega \) with \( \int_\Omega |u_\infty|^2 dx = 1 \).

Finally, we find an interesting phenomenon that not all the \( L^p \) norm preserving non-local flow has such good properties as problem (3). And we shall give an example following to show that some \( L^p \) norm preserving non-local flow must blow up in \( L^\infty \) norm. We study the following nonlinear heat flow on bounded smooth domain in \( \mathbb{R}^n \),

\[
\begin{cases}
\partial_t u = \Delta u + \lambda(t)u^p & \text{in } \Omega \times \mathbb{R}_+, \\
u(x, 0) = g(x) & \text{in } \Omega, \\
u(x, t) = 0 & \text{on } \partial \Omega \times \mathbb{R}_+,
\end{cases}
\]
where $p \geq \frac{n+2}{n-2}$, which has the positive solution and preserves the $L^{p+1}$ norm. Likewise,

$$
\frac{1}{p+1} \frac{d}{dt} \int_{\Omega} u^{p+1}dx = \int_{\Omega} u^p u_t = -p \int_{\Omega} u^{p-1} |\nabla u|^2 dx + \lambda(t) \int_{\Omega} u^p dx.
$$

Thus, one must have $\lambda(t) = \frac{p \int_{\Omega} u^{p-1} |\nabla u|^2 dx}{\int_{\Omega} u^p dx}$ to preserve the $L^{p+1}$ norm. Then we consider the following problem on bounded smooth domain in $\mathbb{R}^n$

$$
\left\{ \begin{array}{ll}
\partial_t u = \Delta u + \lambda(t) u^p & \text{in } \Omega \times \mathbb{R}_+,
\end{array} \right.
\begin{array}{ll}
\, u(x, 0) = g(x) & \text{in } \Omega,
\end{array}
\begin{array}{ll}
\, u(x, t) = 0 & \text{on } \partial \Omega \times \mathbb{R}_+,
\end{array}
\right.
$$

where $p \geq \frac{n+2}{n-2}$, $\lambda(t) = \frac{p \int_{\Omega} u^{p-1} |\nabla u|^2 dx}{\int_{\Omega} u^p dx}$, $g(x) \geq 0$ in $\Omega$ and $g \in C^1(\Omega)$. We have the following blow up property for problem (5).

**Theorem 3.** Suppose that $\Omega$ is a bounded smooth star-shaped domain in $\mathbb{R}^n$. Then the $L^{p+1}$ norm preserving flow (5) must blow up with $L^\infty$ norm in time interval $[0, \infty)$.

This paper is organized as follows. In section 2 we prove Theorem 1 and Theorem 2. In section 3 we prove the blowup result, Theorem 3.

### 2. Global Solutions

In this section we study the global existence property for the $L^p$ energy preserving non-local flows. First we give the proof of theorem 1.

**Proof of theorem 1** Firstly by the maximum principle, we know that $u(t) > 0$. Since

$$
\frac{1}{2} \frac{d}{dt} \int_M |\nabla u|^2 dx = -\int_M u_t \Delta u dx
$$

$$
= -\int_M u_t (u^{p-2} u_t - \lambda(t) u^{p-1}) dx
$$

$$
= -\int_M u^{p-2} (u_t)^2 dx + \frac{\lambda(t)}{p} \frac{d}{dt} \int_M u^p dx
$$

$$
= -\int_M u^{p-2} (u_t)^2 dx \leq 0,
$$

where $p \geq \frac{n+2}{n-2}$, which has the positive solution and preserves the $L^{p+1}$ norm. Likewise,
we know that \( \lambda(t) \) is non-increasing and uniformly bounded. We also have

\[
\frac{1}{2} \frac{d}{dt} \int_M |\nabla u|^2 dx = - \int_M u_t \Delta u dx
\]

\[
= - \int_M (u^{2-p} \Delta u + \lambda(t) u) \Delta u dx
\]

\[
= - \int_M u^{2-p} (\Delta u)^2 dx + \lambda(t) \int_M |\nabla u|^2 dx
\]

Note that at the maximum point of \( u \), by setting \( u_{\max}(t) = \max_{x \in M} (x, t) \), we have

\[
(u_{\max})_t \leq \lambda(t) u_{\max}(t).
\]

Hence

\[
\log \frac{u_{\max}(t)}{u_{\max}(0)} \leq \int_0^t \lambda(t) dt.
\]

Likewise, setting \( u_{\min}(t) = \min_{x \in M} u(x, t) \), we have

\[
(u_{\min})_t \geq \lambda(t) u_{\min}(t).
\]

Hence

\[
\log \frac{u_{\min}(t)}{u_{\min}(0)} \geq \int_0^t \lambda(t) dt.
\]

Combining with (8) and (9), we conclude the Harnack inequality

\[
u_{\max}(t) \leq C u_{\min}(t).
\]

Since \( \int_M u^p(t) dx \equiv 1 \), we get

\[
0 < C' \leq u(x, t) \leq C.
\]

Now we have

\[
\int_0^\infty \lambda(t) dt < C,
\]

by (9) and (11). Note that the solution \( u(x, t) \) is smooth for \( t > 0 \) by standard bootstrap argument. Hence \( \lambda(t) \) of course is continuous and problem (3) has a global solution. Now one can take a sequence \( \lambda(t_i) \) such that \( \lambda(t_i) \to 0 \) as \( t_i \to \infty \). Moreover, since \( \lambda(t) \) is non-increasing, we know that \( \lambda(t) \to 0 \) as \( t \to \infty \). Furthermore, by (7), (11) and the Poincare inequality, we conclude that

\[
\lambda(t) \leq \lambda(0) \exp(-Ct).
\]
Now we integrate (6) with $t$, we get
\[
\int_0^\infty \int_M u^{p-2}(u_t)^2 dx \leq C.
\]
Hence we can take a subsequence $\{t_i\}$ with $t_i \to \infty$ such that $u_i(x) = u(x, t_i)$ and we have
\[
\begin{cases}
  u_i \to u_\infty & \text{in } L^p(M), \\
  u_i \rightharpoonup u_\infty & \text{in } H^1_0(M), \\
  \partial_t u_i \to 0 & \text{in } L^2(M).
\end{cases}
\]
Note that $u_\infty \in H^1_0(M)$ solves the equation $\Delta u_\infty = 0$ in $M$ and satisfies $\int_M |u_\infty|^p dx = 1$. Hence $u_\infty$ must be a positive constant. Combine with (10), (11) and (13), one can use the same argument in [30] theorem 1 to prove $u(t_i)$ converges to $u_\infty$ in $C^\infty$ sense. We omit the details here. □

Next we give the proof of theorem 2.

**Proof of theorem 2.** Firstly by the maximum principle, we know that $u(t) > 0$. Note that
\[
\begin{align*}
  (14) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx &= - \int_\Omega u_t \Delta u dx \\
  &= - \int_\Omega u_t (u_t - \lambda(t)u^p) dx \\
  &= - \int_\Omega (u_t)^2 dx + \frac{\lambda(t)}{p + 1} \frac{d}{dt} \int_\Omega u^{p+1} dx.
\end{align*}
\]
we also have
\[
\begin{align*}
  (15) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx &= - \int_\Omega u_t \Delta u dx \\
  &= - \int_\Omega (\Delta u + \lambda(t)u^p) \Delta u dx \\
  &= - \int_\Omega (\Delta u)^2 + p\lambda(t) \int_\Omega u^{p-1} |\nabla u|^2 dx.
\end{align*}
\]
We denote that $B = \int_\Omega u^{p+1} dx$. Then we have
\[
\int_\Omega (u_t)^2 dx + \frac{1}{2} \frac{d}{dt} (\lambda B) = \frac{\lambda}{p + 1} \frac{d}{dt} B,
\]
hence
\[
\frac{2 \int_\Omega (u_t)^2 dx}{\lambda B} + \frac{d}{dt} (\log(\lambda B^{\frac{p-1}{p+1}})) = 0.
\]
This implies that $\lambda B^{\frac{p}{p+1}}$ is non-increasing and hence

$$\lambda B^{\frac{p}{p+1}}(t) \leq C.$$  \hfill (16)

By Hölder inequality, we have

$$B(t) \geq c_0(\int_{\Omega} u^2 dx)^{\frac{p+1}{2}} = c_0.$$  \hfill (17)

By (16), we have

$$\lambda(t) \leq C.$$  \hfill (18)

Furthermore, by (16), we conclude that

$$||\nabla u||_2 \leq C||u||_{p+1}.$$  \hfill (19)

Note that $1 < p < \frac{n+2}{n-2}$. Hence by Sobolev inequality, we get

$$||u||_{\frac{n}{n-2}} \leq C||\nabla u||_2 \leq C||u||_{p+1} \leq C||u||^{\theta}||u||^{1-\theta} = C||u||^{1-\theta},$$

where $\theta = \frac{n}{(p+1)(n-1)}$. So we have

$$||u||_{p+1} \leq C||u||_{\frac{n}{n-2}} \leq C.$$  \hfill (20)

We integrate (14) with $t$, we get

$$\int_0^\infty \int_M (u_t)^2 dx \leq C.$$  \hfill (21)

Note that the solution $u(x, t)$ is smooth for $t > 0$ by standard bootstrap argument. Hence $\lambda(t)$ of course is continuous and problem (4) has a global solution. Now we can take a subsequence $\{t_i\}$ with $t_i \to \infty$ such that $u_i(x) = u(x, t_i)$ and we have

$$\begin{align*}
  u_i \to u_\infty & \quad \text{in } L^2(\Omega), \\
  u_i \to u_\infty & \quad \text{in } H^1_0(\Omega), \\
  \partial_t u_i \to 0 & \quad \text{in } L^2(\Omega),
\end{align*}$$

by (13), (19) and (21). Hence one can take $t_i \to \infty$ such that $\lambda(t_i) \to \lambda_\infty$, $u(x, t_i) \to u_\infty(x)$ in $L^2(\Omega)$, $u(x, t_i) \to u_\infty(x)$ in $H^1_0(\Omega)$ and $u_\infty$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty^p = 0$ in $\Omega$ and $u_\infty = 0$ on $\partial \Omega$ with $\int_{\Omega} |u_\infty|^2 dx = 1$. Note that $\lambda_\infty \neq 0$ since $\Delta u_\infty = 0$ only has zero solution in this case which contradict to $\int_{\Omega} |u_\infty|^2 dx = 1$. \hfill $\square$
3. BLOW UP

This section is devoted to the proof of theorem 3. Our proof is based on the observation that the flow (5) would converge to elliptic equation
\[ \Delta u + \lambda u^p = 0 \] in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), \( p \geq \frac{n+2}{n-2} \), with \( \int_{\Omega} |u|^{p+1} dx = 1 \) if we assume the \( L^\infty \) norm is uniformly bounded. But this equation only has the vanishing solution if \( \Omega \) is bounded smooth star-shaped domain.

**Proof of theorem 3.** Firstly by the maximum principle, we know that \( u(t) > 0 \). Note that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} u_t \Delta u dx = - \int_{\Omega} u_t (u - \lambda(t) u^p) dx = - \int_{\Omega} (u_t)^2 dx + \frac{\lambda(t)}{p+1} \frac{d}{dt} \int_{\Omega} u^{p+1} dx \]
\[ = - \int_{\Omega} (u_t)^2 dx \leq 0. \]
Hence, we have
\[ ||u||_{H^1} \leq C. \] (22)

Now we argue by contradiction, supposing that \( ||u||_\infty \) is uniformly bounded on time interval \([0, +\infty)\). Since \( \int_{\Omega} u^{2p} dx \geq C \int_{\Omega} u^{p+1} dx = C \), we have
\[ \lambda(t) \leq C p \frac{\int_{\Omega} \nabla u^2 dx}{\int_{\Omega} u^{2p} dx}. \] (23)
Integrate (22) with \( t \), we get
\[ \int_{0}^{t} \int_{\Omega} (u_t)^2 dx = \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla g|^2 dx. \] (24)
Hence
\[ \int_{0}^{\infty} \int_{\Omega} u_t^2 dx \leq C. \] (25)
Note that the solution \( u(x, t) \) is smooth for \( t > 0 \) by standard bootstrap argument. Hence \( \lambda(t) \) of course is continuous and problem (5) has a global solution. Then we can take a subsequence \( \{t_i\} \) with \( t_i \to \infty \) such that \( u_i(x) = u(x, t_i), \lambda(t_i) \to \lambda_\infty \). By (22), (25) and the assumption
$|u|_\infty$ is uniformly bounded on time interval $[0, +\infty)$, we have

$$
\begin{align*}
&u_i \rightarrow u_\infty \quad \text{in } L^{p+1}(\Omega), \\
u_i \rightarrow u_\infty \quad \text{in } H^1_0(\Omega), \\
\partial_t u_i \rightarrow 0 \quad \text{in } L^2(\Omega).
\end{align*}
$$

Hence $u_\infty \in H^1_0(\Omega)$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty^p = 0$ in $\Omega$ and satisfies

$$
\int_\Omega |u_\infty|^{p+1} \, dx = 1.
$$

This contradicts to the fact the equation

$$
\begin{align*}
\Delta u + \lambda u^p &= 0 \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\lambda \geq 0$ and $p > \frac{n+2}{n-2}$, only has the solution $u \equiv 0$ in bounded smooth star-shaped domain (see [24]). □

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