In this work we study a modified version of vacuum $f(R)$ gravity with a kinetic term which consists of the first derivatives of the Ricci scalar. We develop the general formalism of this kinetic Ricci modified $f(R)$ gravity and we emphasize on cosmological applications for a spatially flat cosmological background. By using the formalism of this theory, we investigate how it is possible to realize various cosmological scenarios. Also we demonstrate that this theoretical framework can be treated as a reconstruction method, in the context of which it is possible to realize various exotic cosmologies for ordinary Einstein-Hilbert action. Finally, we derive the scalar-tensor counterpart theory of this kinetic Ricci modified $f(R)$ gravity, and we show the mathematical equivalence of the two theories.

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I. INTRODUCTION

Modified gravity in its most general forms, serves as a formal theoretical framework which can potentially harbor in a consistent way both cosmological and astrophysical phenomena. For example, in the context of the most sound and simple modified gravity, $f(R)$ gravity, it is possible to provide a unified description of the late-time acceleration and of inflation as it was demonstrated firstly in Ref. [9]. Also generalized modified gravities, like for example Gauss-Bonnet modified gravities [10–17], teleparallel gravity [18–26] and extensions of these [27–29], can also describe a plethora of cosmological and astrophysical phenomena. Now the question is, which theory provides the correct description of our Universe, and this question can be answered only by confronting each modified gravity with the observational data. It is possible that the answer is simple, however most of the theories can be compatible with the observations, so in principle, all possible modified gravities should be scrutinized in order to reveal the phenomenology these suggest.

In this line of research, recently the $f(R)$ gravity framework was extended to include first and higher derivatives.
of the Ricci scalar in the \( f(R) \) gravity action \[30\]. It was shown that the resulting theory is free from ghost, under an appropriate choice of the functional form of the Lagrangian. Later on several studies in this framework were performed, see for example \[31, 37, 38, 48\].

In the present work, we shall investigate an extended \( f(R) \) gravity model which contains first derivatives of the Ricci scalar in the \( f(R) \) gravity action. We shall derive the gravitational field equations and we shall emphasize to cosmological applications of the model at hand. Particularly, we firstly demonstrate in a formal way how to obtain the gravitational field equations, and we introduce appropriate variables in order to cast the field equations in a convenient and compact way. We specialize the field equations for a spatially flat metric, and we study how it is possible to realize various cosmological evolutions. As we demonstrate, the realization of specific cosmological evolutions results to a system of ordinary linear coupled differential equations, and the general case can be quite tedious, so we focus our study on specific limiting cases of the theory, which have some physical significance. Also we demonstrate that the field equations can be used as a reconstruction method, in which by providing the cosmological evolutions results to a system of ordinary linear coupled differential equations, and the general case can be quite tedious, so we focus our study on specific limiting cases of the theory, which have some physical significance.

This paper is organized as follows: In section II we present the theoretical framework of kinetic scalar curvature-corrected \( f(R) \) gravity, and we derive the gravitational equations for a general metric. In section III, we focus our study on cosmological applications, by using a spatially flat metric. We investigate various cosmological evolutions of physical interest and we find the approximate form of the kinetic scalar curvature-corrected \( f(R) \) gravity which may realize such an evolution. We also show how to treat the theory at hand as a reconstruction method for realizing various cosmological evolutions, by specifying the \( f(R) \) gravity and the cosmological scale factor, and then finding the function \( X(R) \) of the kinetic term that realizes the given cosmological scenario. Finally, in section IV we demonstrate that the kinetic Ricci modified \( f(R) \) gravity is equivalent with a multi-tensorial scalar-tensor theory of gravity.

II. \( f(R) \) GRAVITY WITH KINETIC SCALAR CURVATURE

In this section we shall present the general formalism of kinetic scalar curvature extended \( f(R) \) gravity, and we shall derive the field equations of this modified gravity. The action of the \( f(R) \) gravity with kinetic scalar curvature term is the following,

\[
S = \int d^4x \sqrt{-g} \left( X(R)R + f(R) \right) + S_{\text{matter}}.
\]

where \( X(R) \) and \( f(R) \) are differentiable functions of the Ricci scalar, and \( S_{\text{matter}} \) stands for the action of the matter fluids present. The deviation from the standard \( f(R) \) gravity action is obvious, and it is quantified by the term \( \sim X(R)R + f(R) \), which justifies the terminology kinetic, since this is simply a kinetic term for the scalar curvature \( R \). A similar action to the above, in the context of the Lagrange multipliers formalism \[40, 41\], was given in Ref. \[43\]. Upon variation of the action \[1\] with respect to the metric tensor \( g^\mu\nu \), we obtain the following,

\[
\delta S = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[ -\frac{1}{2} g_{\mu\nu} (R, R) X(R) + f(R) \right] + X'(R) R_{\mu\nu} R^{\mu\nu} R^{\alpha\beta} +
\]

\[
+ X(R) R_{\mu\nu} R_{\mu^\nu} + f'(R) R_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) (R, R) X' \right] \]

\[
- 2 \left[ \nabla_\alpha X \nabla_\sigma (R R_{\mu\nu} - 2 \Box RR_{\mu\nu} - 2 (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) (\Box X) \nabla_\sigma R + X \Box R) \right]
\]

\[
+ \int d^4x \sqrt{-g} \nabla_\mu M_\alpha - \int d^4x \sqrt{-g} \nabla_\mu L^{\mu} + 2 \int d^4x \sqrt{-g} \nabla_\alpha N_\sigma
\]

\[
+ 2 \int d^4x \sqrt{-g} \nabla_\sigma P_\sigma - 2 \int d^4x \sqrt{-g} \nabla_\sigma V_\alpha - 2 \int d^4x \sqrt{-g} \nabla_\sigma W_\sigma + 2 \int d^4x \sqrt{-g} \nabla_\mu Z^{\mu},
\]

where we introduced the tensorial quantities \( M_\alpha, L^{\mu}, N^\alpha, P_\sigma, V_\alpha, W_\sigma, Z^{\mu} \), which are defined as follows,

\[
M_\alpha = (R, R) X' + f') \nabla_\beta \delta g^{\beta\gamma} - \nabla_\alpha (R, R) X' + f') \delta g^{\beta\gamma},
\]

\[
L^{\mu} = (R, R) X' + f') \nabla_\beta \delta g^{\mu\beta} - \nabla_\beta (R, R) X' + f') \delta g^{\mu\beta},
\]

\[
N_\sigma = (R, R) X' + f') \nabla_\sigma \delta g^{\mu\nu} - \nabla_\mu (R, R) X' + f') \delta g^{\mu\nu},
\]

\[
P_\sigma = (R, R) X' + f') \nabla_\sigma \delta g^{\mu\nu} - \nabla_\mu (R, R) X' + f') \delta g^{\mu\nu},
\]

\[
V_\alpha = (R, R) X' + f') \nabla_\alpha \delta g^{\mu\nu} - \nabla_\mu (R, R) X' + f') \delta g^{\mu\nu},
\]

\[
W_\sigma = (R, R) X' + f') \nabla_\sigma \delta g^{\mu\nu} - \nabla_\mu (R, R) X' + f') \delta g^{\mu\nu},
\]

\[
Z^{\mu} = (R, R) X' + f') \nabla_\mu \delta g^{\beta\gamma} - \nabla_\beta (R, R) X' + f') \delta g^{\beta\gamma},
\]
\[ N_\sigma = X \nabla_\sigma R R_{\mu \nu} \delta g^{\mu \nu}, \]  
\[ P_\sigma = X R,_{\sigma} g_{\alpha \beta} \Box \delta g^{\alpha \beta}, \]  
\[ V_\alpha = (\nabla^\gamma X \nabla_\gamma R + X \Box R) g_{\beta \gamma} \nabla_\alpha \delta g^{\beta \gamma} - \nabla_\alpha (\nabla^\gamma X \nabla_\gamma R + X \Box R) g_{\beta \gamma} \delta g^{\beta \gamma}, \]  
\[ W_\sigma = X \nabla_\sigma R \nabla_\mu \nabla_\beta \delta g^{\mu \beta}, \]  
\[ Z^\mu = (\nabla^\sigma X \nabla_\sigma R + X \Box R) \nabla_\beta \delta g^{\mu \beta} - \nabla_\beta (\nabla^\sigma X \nabla_\sigma R + X \Box R) \delta g^{\mu \beta}. \]

By taking into account the fact that the 4-divergence terms lead to three dimensional hypersurface integrals, that according to Gauss-Stokes theorem tend to zero as surface terms, the vacuum gravitational field equations are the following,

\[- \frac{1}{2} g_{\mu \nu} f + R_{\mu \nu} f' + D_{\mu \nu} f' + (- \frac{1}{2} g_{\mu \nu} R,_{\sigma} R^{\sigma} + R,_{\nu} R,_{\mu} - 2 R_{\mu \nu} \Box R) X +
\]  
\[+ ( - R,_{\sigma} R^{\sigma} R_{\mu \nu} ) X' - 2 D_{\mu \nu}( \Box R X ) - D_{\mu \nu}( R,_{\sigma} R^{\sigma} X') = 0, \]  
where \( D_{\mu \nu} = g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu \). After some algebra, the field equations (10) become,

\[ A^\mu_\nu X + B^\mu_\nu X' + C^\mu_\nu X'' + F^\mu_\nu X''' - \frac{1}{2} \delta^\mu_\nu f + f' R^\nu_\mu + D^\mu_\nu f' = 0, \]  
where we introduced for notational convenience the tensorial quantities \( A^\mu_\nu, B^\mu_\nu, C^\mu_\nu, F^\mu_\nu \), which are defined as follows,

\[ A^\mu_\nu = R,_{\nu} R^{\mu} - \frac{1}{2} \delta^\mu_\nu R,_{\sigma} R^{\sigma} - 2 \Box R R^\nu_\mu - 2 D^\mu_\nu \Box R, \]  
\[ B^\mu_\nu = - R^\mu_\nu R,_{\sigma} R^{\sigma} - D^\mu_\nu (R,_{\sigma} R^{\sigma}) - 2 \Box R D^\mu_\nu R, \]  
\[ C^\mu_\nu = - R,_{\sigma} R^{\sigma} D^\mu_\nu R - 2 \Box R (\delta^\mu_\nu R,_{\sigma} R^{\sigma} - R,_{\mu} R^{\sigma}), \]  
\[ F^\mu_\nu = - R,_{\sigma} R^{\sigma} (\delta^\mu_\nu R,_{\sigma} R^{\sigma} - R,_{\mu} R^{\sigma}). \]

Obviously, by setting \( X(R) = 0 \), we recover the standard vacuum \( f(R) \) gravity field equations,

\[- \frac{1}{2} g_{\mu \nu} f(R) + f'(R) R_{\mu \nu} + (g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu) f' = 0, \]  
where the prime in all the above equations denotes differentiation with respect to the Ricci scalar. Thus we derived the gravitational equation of the \( f(R) \) gravity with kinetic scalar curvature terms, for the vacuum case of the action (1), which are Eqs. (11) along with the definitions of the tensorial quantities \( A^\mu_\nu, B^\mu_\nu, C^\mu_\nu, F^\mu_\nu \), namely Eqs. (12)-(15).

It is useful to present the trace of the gravitational field equations (11). Performing a contraction of the indices, we obtain the following equation,

\[ AX + BX' + CX'' + FX''' - 2 f + R f' + 3 \Box f' = 0 \]  
where the parameters \( A, B, C, F \), are defined as follows,

\[ A = - \dot{R} - 2 R \Box R - 6 \Box^2 R, \quad \dot{R} = R,_{\sigma} R^{\sigma}, \]  
\[ B = - \dot{R} R - 3 (\dot{R}) - 6 (\Box R)^2, \]  
\[ C = - 9 \dot{R} \Box R, \]  
\[ (17) \]
For cosmological applications in this paper we shall consider a spatially flat Friedmann-Robertson-Walker (FRW) Universe with metric,

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + r^2 \left( d\theta^2 + \sin^2 \theta \right) d\phi^2 \right), \]  

(22)

where \( a \) is the scale factor. For later purposes, it is worth finding the functional form of the quantities \( A, B, C, F \) appearing in Eqs. (13)-(21) for a FRW Universe, which are,

\[ A = 1728H^4\dot{H} - 288H^3\ddot{H} + (-576(\dot{H})^2 - 1044\dddot{H})H^2 + (-3420\dddot{H}\dddot{H} - 360\dddot{H})H - 576(\dot{H})^3 - 720\dot{H}\dddot{H} - 504(\dot{H})^2 - 36\dddot{H} \]  

(23)

\[ B = -3456(\dot{H})^4 - 6912H^2(\dddot{H})^3 + (6912H^4 - 18144\dot{H}\dddot{H} - 1728\dddot{H})(\dot{H})^2 + (-6912H^3\dot{H} - 6048H^2\dddot{H} - 864H\dddot{H} - 2376(\dot{H})^2\dot{H} - 5616H^2(\dddot{H})^2 - 3240H\dot{H}\dddot{H} - 216\dot{H}^2 - 216(\dddot{H})^2 \]  

(24)

\[ C = -1944(4H\dot{H} + \dddot{H})^2(12H^2\dot{H} + 7H\dddot{H} + 4(\dot{H})^2 + \dddot{H}) \]  

(25)

\[ F = -3888(4H\dot{H} + \dddot{H})^4 \]  

(26)

where \( H \) stands for the Hubble rate of the Universe \( H = \frac{\dot{a}}{a} \). The above Eqs. (24)-(26) will be useful later on, when we discuss cosmological solutions.

Also it will be convenient to divide the general equation (11) in two parts,

\[ Q^\mu_\nu + L^\mu_\nu = 0 \]  

(27)

where \( Q^\mu_\nu \) is the part containing the function \( X(R) \) and its derivatives,

\[ Q^\mu_\nu = A^\mu_\nu X + B^\mu_\nu X' + C^\mu_\nu X'' + F^\mu_\nu X''', \]  

(28)

the trace of which is equal to,

\[ Q = AX + BX' + CX'' + FX''', \]  

(29)

and \( L^\mu_k \) is the part containing the function \( f(R) \) and its derivatives,

\[ L^\mu_\nu = -\frac{1}{2} \delta^\mu_\nu f + f' R^\mu_\nu + D^\mu_\nu (f'), \]  

(30)

with the corresponding trace being in this case,

\[ L = L^\mu_\mu = -2f + Rf' + 3\Box f'. \]  

(31)

Having described the general formalism of the kinetic Ricci modified \( f(R) \) gravity, in the next sections we shall consider several cosmological realizations of this theory, focusing on inflationary evolutions mainly. Also we demonstrate the equivalence of the theory with a scalar-tensor theory at some later section. Before we continue, we need to briefly discuss a somewhat critical issue regarding the kinetic Ricci modified \( f(R) \) gravity, having to do with the stability of the theory. All the cosmological realizations of this formalism, which we present in the next section, should be examined towards their stability. According to Ref. [30], ghost instabilities are absent, since the kinetic terms are appropriately chosen, however a potential source of instability in this theory can be traced in Eq. (2.17) of Ref. [30], namely in the potential, which is unbounded from below since it is linear in the scalar field. This issue in turn could make the theory less self-consistent, due to the severe instabilities caused. This issue should be appropriately examined in a future study, which is beyond the scopes of this demonstrational version of our work.
III. COSMOLOGY WITH $f(R, (\nabla R)^2)$ GRAVITY

Let us now focus on cosmological applications of the kinetic Ricci modified $f(R)$ gravity, and we discuss the general framework of the theory in the context of a flat FRW background. The non-zero components of the Ricci tensor for the FRW metric are,

\[ R^0_0 = \frac{\ddot{a}}{a}, \]
\[ R^1_1 = R^2_2 = R^3_3 = \frac{\ddot{a}}{a} + 2 \frac{a'^2}{a^2}, \]

and the corresponding Ricci scalar is,

\[ R = 6 \left( \frac{a'^2}{a^2} + \frac{\ddot{a}}{a} \right), \quad (32) \]

Using the equation (11),

\[ A^\mu_X + B^\mu_X' + C^\mu_X'' + F^\mu_X''' - \frac{1}{2} \delta^\mu_n f + f' R^\mu_n + D^\mu_n (f') = 0 \]

we may calculate each component of it, for the metric (22) in terms of Hubble rate $H$, and the resulting set of the components of $A^\mu_0$, $B^\mu_1$, $C^\mu_n$ and $F^\mu_n$ are the following,

\[ A^0_0 = 18 \left( 24H^4\dot{H} - 10H^3\dddot{H} - 32\dddot{H}H^2 - 12\dddot{H}\dot{H}^2 \right) - 24\dot{H}\dot{H}\dddot{H} + 8\dddot{H}^3 - 2H\dddot{H} + 2\dddot{H} - \dddot{H}^2 \right), \quad (33) \]

\[ A^1_1 = A^2_2 = A^3_3 = 6 \left( 72H^4\dot{H} - 6H^3\dddot{H} - 46\dddot{H}\dot{H}^2 - 166\dddot{H}\dot{H} - 40\dddot{H} - 18H^3\dddot{H} - 42\dddot{H}H - 27\dddot{H}^2 - 2H^5 \right), \quad (34) \]

\[ B^0_0 = -1728 \left( \dot{H}\dddot{H} + \frac{1}{4} \left( 17H^2 - \dddot{H} \right) \dddot{H} + H\dddot{H} \left( H^2 + 3\dddot{H} \right) \right) \left( \dot{H} \dddot{H} + \frac{1}{4} \dddot{H} \right) \quad (35) \]
\[ B^1_1 = B^2_2 = B^3_3 = -576 \left( H\dddot{H} + \frac{1}{4} \dddot{H} \right) \left( \frac{1}{2} \dddot{H} + \frac{1}{2} \dddot{H} + \frac{57}{4} H^2 + \frac{23}{4} \dddot{H} \right) \dddot{H} + H\dddot{H} \left( H^2 + 13\dddot{H} \right) \right), \quad (36) \]

\[ C^0_0 = -648H \left( 4\dddot{H} + \dddot{H} \right)^3 \quad (37) \]
\[ C^1_1 = C^2_2 = C^3_3 = -216 \left( 4\dddot{H} + \dddot{H} \right)^2 \left( 32\dddot{H}H^2 + 20\dddot{H} + 12\dddot{H}^2 + 3\dddot{H} \right), \quad (38) \]
\[ F^0_0 = 0 \quad (39) \]
\[ F^1_1 = F^2_2 = F^3_3 = -1296 \left( 4\dddot{H} + \dddot{H} \right)^4. \quad (40) \]

The above components, will be useful for the study of specific cosmological solutions, as we will shortly show. The kinetic Ricci modified $f(R)$ gravity is a geometric theory, as the $f(R)$ gravity theory is too, so it is useful to rewrite the gravitational field equations in a way so that the geometric effects can be formed as a perfect fluid and the resulting Einstein equations can be written in the standard way as in the Einstein-Hilbert case. To this end, consider the $f(R)$ gravity part tensor $L^\mu_\nu$ of Eq. (40),

\[ L^\mu_\nu = -\frac{1}{2} \delta^\mu_n f + f' R^\mu_n + D^\mu_n (f'). \quad (41) \]

the non-zero components of which are,

\[ L^0_0 = -3H^2 f' + \frac{R f' - f}{2} - 3H\dddot{H} f' \quad (42) \]
\[ L^1_1 = L^2_2 = L^3_3 = -\frac{1}{2} f + \left( \frac{R}{2} + 2\dot{H} + 3H^2 \right) f' + \left( \dddot{H} + 2\dot{H} \dddot{H} \right) f'' + \left( \dddot{H} \right)^2 f'''. \quad (43) \]
In effect, we can rewrite Eq. (27) in a perfect fluid form, by separating the terms $H^2$ and $2H + 3H^2$, as follows,

$$H^2 = \frac{1}{3f'} \left( \kappa \rho + \frac{Rf' - f}{2} - 3H \dot{R}f'' + A_0^0 X + B_0^0 X' + C_0^0 X'' \right),$$ \hspace{1cm} (44)

$$2\dot{H} + 3H^2 = \frac{1}{f'} \left( -\kappa \rho + \frac{f - Rf'}{2} - \left( \ddot{R} + 2H \dot{R} \right) f'' - \left( \dot{R} \right)^2 f''' - A_1^1 X - B_1^1 X' - C_1^1 X'' - F_1^1 X'''' \right).$$ \hspace{1cm} (45)

We can bring the above equations to the standard Friedmann equations form of Einstein-Hilbert gravity, if we assume that a perfect fluid originating from geometry has a stress tensor with energy and pressure components defined as follows,

$$\rho_{eff} = \frac{1}{f'} \left( \frac{Rf' - f}{2} - 3H \dot{R}f'' + A_0^0 X + B_0^0 X' + C_0^0 X'' \right)$$ \hspace{1cm} (46)

$$p_{eff} = \frac{1}{f'} \left( \frac{Rf' - f}{2} + \left( \ddot{R} + 2H \dot{R} \right) f'' + \left( \dot{R} \right)^2 f''' + A_1^1 X + B_1^1 X' + C_1^1 X'' + F_1^1 X'''' \right).$$ \hspace{1cm} (47)

Hence, in the absence of standard matter fluids, in which case $\rho = p = 0$ in Eqs. (44) and (45), we can rewrite these as follows,

$$H^2 = \frac{\rho_{eff}}{3},$$ \hspace{1cm} (48)

$$2\dot{H} + 3H^2 = -\rho_{eff}.$$ \hspace{1cm} (49)

The same equations can be easily rewritten in form of Friedmann equations of standard Einstein-Hilbert gravity,

$$H^2 = \frac{\rho_{eff}}{3},$$ \hspace{1cm} (50)

$$\frac{\ddot{a}}{a} = -\frac{1}{6} \left( \rho_{eff} + 3p_{eff} \right),$$ \hspace{1cm} (51)

and the contribution in $\rho_{eff}$ and $p_{eff}$ is purely geometrically originating. Having the Friedmann equations at hand, we can proceeding in considering several cosmological solutions in the context of kinetic Ricci $f(R)$ gravity.

A. Specific Cosmological Solutions

Recall that the part of the field equations which contain the $X(R)$ contribution in Eq. (28), is,

$$Q^\mu_{\nu} = A^\mu_{\nu} X + B^\mu_{\nu} X' + C^\mu_{\nu} X'' + F^\mu_{\nu} X''''$$

while the part containing the $f(R)$ contribution is the one appearing in Eq. (30), which we quote here for reading convenience,

$$L^\mu_{\nu} = -\frac{1}{2} \delta^\mu_{\nu} f + f' R^\mu_{\nu} + D^\mu_{\nu} (f').$$

Let us now consider some simple cosmological solutions, and we start off our analysis with a toy model of a static Universe, in which case $a(t) = C$, so the field equations (28) and (30), reduce both to,

$$-\frac{f}{2} = 0$$

Thus in this case we have $f(R) = 0$, and $X(R)$ is an arbitrary function. Consider now the case of a de Sitter cosmological evolution, in which case $a(t) = e^{kt}$, where $k > 0$. In this case, the field equations take the following form,

$$-\frac{f}{2} + \frac{3}{4} k^2 f' = 0,$$

The solution of the above equation is $f = \mathcal{C} \exp(\frac{2R}{3k^2})$, where $\mathcal{C}$ is an arbitrary integration constant. For the purely de Sitter evolution, all the coefficients $A^i_1$, $B^i_1$, $C^i_{si}$, $F^i_1$, $i = 1, 2$ are equal to zero, so the field equations are satisfied for
an arbitrary function $X(R)$. Phenomenologically, this might be interesting for this particular cosmological solution, due to the fact that the effects of the $X(R)$ gravity are trivial, and therefore have no direct effect on the slow-roll indices of inflation. However, the situation might change if the evolution is a quasi de Sitter evolution, as we show later on.

Let us now consider a power law evolution, with scale factor $a(t) = t^m$. We shall make use of the following formulas,

$$X' = \frac{X}{R}, \quad (52)$$

$$X'' = \frac{\ddot{X}(t)}{R(t)^2} - \frac{\dot{X}(t)R''(t)}{R(t)^3}, \quad (53)$$

$$X''' = \frac{X^{(3)}(t)}{R'(t)^2} - \frac{2\dot{X}(t)R''(t)}{R'(t)^3} - \frac{R^{(3)}(t)\dot{X}(t)}{R'(t)^3} + \frac{2\ddot{X}(t)R''(t)^2}{R'(t)^4}, \quad (54)$$

which relate the derivatives of the function $X(R)$ with respect to the Ricci scalar, denoted with a “prime”, with the derivatives of the function $X(t)$ with respect to the cosmic time, denoted with a “dot”. From (11) we have,

$$Q_0^2 = \frac{5832m^2}{t^6} \left( -m^2 \left( m - \frac{1}{2} \right)^2 t^2 \ddot{X} + \right.
+ m \left( \frac{11}{9} + m^2 t + \left( -\frac{t}{2} - \frac{1}{9} \right) m \right) \left( m - \frac{1}{2} \right) t \dot{X} + \frac{2}{27} \left( m^2 + \frac{8}{3} m - \frac{10}{3} \right) X \right), \quad (55)$$

$$Q_1^2 = Q_2^2 = Q_3^2 = \frac{17496m (m - \frac{1}{2})}{t^6} \left( -m^3 \left( m - \frac{1}{2} \right)^3 t^2 \dddot{X} + \right. 
+ m^2 \left( m - \frac{1}{2} \right)^2 \left( 1 + m^2 t + \left( -\frac{t}{2} - \frac{8}{9} \right) m \right) t^2 \ddot{X} + 
+ \frac{8}{9} m \left( m - \frac{1}{2} \right) \left[ -\frac{1}{3} + m^3 t + \left( -\frac{13t}{8} - \frac{1}{72} \right) m^2 + \left( \frac{9t}{16} + \frac{41}{72} \right) m \right] t \dot{X} + 
+ \frac{2}{81} (m - 2) \left( m^2 + \frac{8}{3} m - \frac{10}{3} \right) X \right), \quad (56)$$

From a first observation, it seems that for certain values of the parameter $m$, significant simplifications occur in the field equations, which we consider now. For example, the case $m = 0$ is the static Universe toy model we discussed earlier. More significantly, the value $m = \frac{1}{2}$ simplifies considerably the field equations, and in this case we have,

$$\frac{f}{2} - \frac{3}{4} f' = 0$$

$$\frac{f}{2} + \frac{4}{5} f' = 0$$

In this case too, all the coefficients $A_i^i, B_i^i, C_{i,ni}^i, F_{i,si}^i, i = 1, 2$ are equal to zero as it can be checked, so the function $X(R)$ is arbitrary, and also the solution of the differential equations is, $f(R) = Ce^{2R}$, where $C$ is an integration constant.

Let us now assume that the function $X(R)$ has a simple form, $X(R) =$const. In this case, all derivatives of $X(R)$ with respect to the Ricci scalar are equal to zero. Also for $m = -\frac{4}{3} + \frac{\sqrt{46}}{3}$, the $X$-dependent part of the field equations vanish. For general $f(R)$ dependent part,

$$L_0^0 = -\frac{f}{2} + \left( \frac{-22\sqrt{46} + 148}{8 (-136 + 19\sqrt{46})} t \right) \frac{3 (1696\sqrt{46} - 11344) t^2}{32 (-136 + 19\sqrt{46})^2} f + \frac{3 (1696\sqrt{46} - 11344) t^2}{32 (-136 + 19\sqrt{46})^2} \ddot{f} = 0, \quad (57)$$

$$L_1^1 = L_2^2 = L_3^3 = -\frac{f}{2} + \left( \frac{-54\sqrt{46} + 396}{24 (-136 + 19\sqrt{46})} t \right) \frac{4760\sqrt{46} - 32480) t^2}{32 (-136 + 19\sqrt{46})^2} f + \left( \frac{4760\sqrt{46} - 32480) t^2}{32 (-136 + 19\sqrt{46})^2} + \frac{-69768\sqrt{46} + 473877) t^3}{18 (-136 + 19\sqrt{46})^2} \right) f + \frac{69768\sqrt{46} - 473877) t^3}{18 (-136 + 19\sqrt{46})^2} \ddot{f} = 0, \quad (58)$$
which can be solved with respect to $f(t)$ if the Hubble rate of the cosmological evolution is given. This technique is the core of a general reconstruction method for kinetic Ricci $f(R)$ gravity, which we shall present in the next section.

As a final example, let us consider a quasi-de Sitter evolution, with scale factor $a(t) = e^{eta t - \epsilon t^2}$, where $\epsilon$ is assumed to take small values. Note that this case of quasi-de Sitter evolution is different from a phenomenological point of view from an inflationary quasi-de Sitter evolution $a(t) = e^{H_0 t - H_1 t^2}$, in which case the parameters $H_0$ and $H_1$ are constrained by observations for the ordinary Starobinsky model [45], see for example Ref. [46] for a concrete analysis of the parameter space. Coming to the problem at hand, by assuming that terms $\sim \epsilon^2$ and higher orders tend to zero, obtain

$$H = H_\ast - 2\epsilon t; \quad \dot{H} = -2\epsilon; \quad \dot{H}^2 \approx 0; \quad \frac{\ddot{a}}{a} = -2\epsilon + (H_\ast - 2\epsilon t)^2; \quad R \approx 12 \left( H_\ast^2 - (4H_\ast t + 1) \epsilon \right). \quad (59)$$

Upon substitution (59) into (11) the coefficients (11) become,

$$A_\mu'' \approx -864\epsilon H_\ast^4; \quad B_\mu'' \approx 0; \quad C_\mu'' \approx 0; \quad F_\mu'' \approx 0. \quad (60)$$

As a result we obtain the following field equations,

$$288\epsilon H_\ast^2 f'' + \left( \frac{R}{2} - 6\epsilon \right) f' - 1728\epsilon H_\ast^4 X - f = 0, \quad (61)$$

$$192\epsilon H_\ast^2 f'' + \left( \frac{R}{2} + 2\epsilon \right) f' - 1728\epsilon H_\ast^4 X - f = 0. \quad (62)$$

From (61) and (62), we can find the approximate forms of $f(R)$ and of $X(R)$, which are,

$$f(R) = 1728\epsilon H_\ast^4 \left( 24C_2 H_\ast^2 \exp \left( \frac{R}{12 H_\ast^2} \right) - C_1 \right), \quad (63)$$

$$X(R) = C_1 + C_2 \exp \left( \frac{R}{12 H_\ast^2} \right) (R - 24 H_\ast^2 + 36\epsilon). \quad (64)$$

The solutions (63) and (64) correspond to the model with quasi-de Sitter evolution. In the next section, we also discuss how the model at hand actually constitutes a reconstruction method, in the context of which, several cosmological evolutions can be realized.

B. A General Reconstruction Technique for Kinetic Ricci Extended $f(R)$ Gravity

Essentially, the field equations (49), (51) with the definitions (47), can be used as a reconstruction method in which, given the Hubble rate and one of the functions $f(R)$ and $X(R)$, it is possible to determine the other function, and hence determine the kinetic Ricci modified $f(R)$ gravity which realizes the given cosmological evolution. Apparently, the most interesting case is by choosing $f(R) = R$, so the theory is standard Einstein-Hilbert gravity with scalar curvature kinetic corrections. In the following we shall investigate this case, however other choices for the function $f(R)$ are possible.

Let us demonstrate explicitly how the reconstruction method works, by using some well known cosmological evolutions, starting with the symmetric bounce case, in which case the scale factor and the Hubble rate are,

$$a(t) = e^{\beta t^2}, \quad H(t) = 2\beta t, \quad (65)$$

and since $a(0) \neq 0$ this is a non-singular bounce. Clearly the bouncing behavior is acquired from the change in the sign of the Hubble rate before and after the bouncing point $t = 0$. In the literature, there are standard works that study this type of cosmological evolution, in the context of $f(R)$ gravity [46]. To start with, assume that $f(R) = R$, so the standard general relativity is assumed in the $f(R)$ part, and the corresponding vacuum action is,

$$S = \int d^4x \sqrt{-g} \left( R + X(R) R_{\sigma R^\sigma} \right). \quad (66)$$
In this case, the non-zero components of the tensorial quantities $A^\mu_n$, $B^\mu_n$, $C^\mu_n$ and $F^\mu_n$ given in Eqs. (63)-(70) are,

$$A_0^1 = 18 \left( 64 \beta^3 + 768 \beta^3 t^4 - 512 \beta^3 t^2 \right),$$

$$A_1^1 = A_2^1 = A_3^1 = 6 \left( 2304 \beta^5 t^4 - 320 \beta^3 \right),$$

$$B_0^0 = -27648 \beta^4 t^2 (6 \beta + 4 \beta^2 t^2),$$

$$B_1^1 = B_2^2 = B_3^3 = -36864 \beta^6 t^4 - 239616 \beta^5 t^2,$$

$$C_0^0 = -5308416 \beta^7 t^4,$$

$$C_1^1 = C_2^2 = C_3^3 = -331776 \beta^5 t^3 (48 \beta^2 + 256 \beta^3 t^2),$$

$$F_0^0 = 0,$$

$$F_1^1 = F_2^2 = F_3^3 = 331776 \beta^4 t^2.$$

It is conceivable that even in the case $f(R) = R$, the resulting differential equation for the function $X(t)$ is impossible to solve analytically. So we shall focus our analysis for cosmic times near the bouncing point, so for $t \sim 0$. By making this assumption, we shall keep only the lowest powers of the cosmic time dependent terms, and the resulting approximate differential equation for $X(t)$ [44] is,

$$-576 \beta^3 t^2 \ddot{X}(t) - 1152 \beta^4 t \dot{X}(t) + 1152 \beta^3 X(t) - 4 \beta^2 t^2 = 0,$$

which can be solved analytically, and the resulting solution is,

$$X(t) = -\frac{t^2}{576 \beta} + \frac{\Lambda_a}{t^2} + \Lambda_b t,$$

where $\Lambda_a$ and $\Lambda_b$ are integration constants. Also the Ricci scalar for the symmetric bounce [65] is equal to $R(t) = 12 \beta + 48 \beta^2 t^2$, so by inverting the function $R(t)$ we obtain two solutions $t(R) = \pm \frac{\sqrt{R - 12 \beta}}{4 \sqrt{\beta}}$. Finally, substituting $t(R)$ in the solution (70), we obtain the following two approximate solutions near the bouncing point,

$$X(R) \simeq \frac{R - 12 \beta}{27648 \beta^3} + \frac{48 \beta^2 A_b}{R - 12 \beta} + \frac{\Lambda_a \sqrt{R - 12 \beta}}{4 \sqrt{3} \beta},$$

$$X(R) \simeq -\frac{R - 12 \beta}{27648 \beta^3} + \frac{48 \beta^2 A_b}{R - 12 \beta} - \frac{\Lambda_a \sqrt{R - 12 \beta}}{4 \sqrt{3} \beta}.$$
where \( \Lambda_a \) and \( \Lambda_b \) are integration constants. By substituting the function \( t(R) = \pm \frac{\sqrt{R - 12\beta}}{4\sqrt{3}\beta} \) in Eq. (80), we find the following two solutions for \( X(R) \),

\[
X(R) \simeq -\frac{1}{576\beta} + \frac{48\beta^2\Lambda_b}{R - 12\beta} + \frac{\Lambda_a\sqrt{R - 12\beta}}{4\sqrt{3}\beta},
\]

(81)

so in this case too, the condition \( R > 12\beta \) is required for the consistency of the solution.

As a final example, let us consider the quasi-de Sitter evolution with scale factor and Hubble rate, \( a(t) = e^{H_0-t-H_1t}, \ H(t) = H_0 - 2H_1t \).

(82)

The quasi-de Sitter evolution (82) is known to be realized by the Starobinsky \( R^2 \) model, and the parameters \( H_0 \) and \( H_1 \) are constrained from the Planck data, see Ref. [46] for a thorough analysis of the parameter space. Instead of using the \( R^2 \) model, we shall assume that \( f(R) = R \), and we shall find which kinetic Ricci \( X(R) \) gravity can realize this inflationary evolution, emphasizing to small cosmic times. The action is again given by Eq. (54), and in this case, the non-zero components of the tensorial quantities \( A^\mu_0, B^{\mu
u}_0, C^\mu_0 \) and \( F^{\mu
u}_0 \) given in Eqs. (33)-(40), at leading order in the cosmic time \( t \), are,

\[
A^0_0 = -864H_0^3H_1 + 6912H_0^3H_1^2t - 2304H_0^3H_1^2t + 9216H_0H_1^2t^2 - 1152H_1^3,
\]

\[
A^1_1 = A^2_2 = A^3_3 = -864H_0^3H_1 + 6912H_0^3H_1^2t + 1920H_1^3,
\]

\[
B^0_0 = -6912H_0^3H_1^2 + 55296H_0^3H_1^3t + 41472H_0^3H_1^3t - 165888H_0H_1^4t,
\]

\[
B^1_1 = B^2_2 = B^3_3 = -2304H_0^3H_1^2 + 18432H_0^3H_1^3t + 59904H_0^3H_1^3t - 239616H_0H_1^4t,
\]

\[
C^0_0 = 331776H_0^3H_1^3 - 2654208H_0^3H_1^4t,
\]

\[
C^1_1 = C^2_2 = C^3_3 = 2654208H_0^3H_1^4t - 2654208H_0^3H_1^4t - 1900656H_0^3H_1^4t + 11943936H_0^3H_1^4t,
\]

\[
F^0_0 = 0,
\]

\[
F^1_1 = F^2_2 = F^3_3 = 82944H_0^3H_1^2 - 331776H_0H_1^3t.
\]

Then, it is possible to obtain the differential equation which will yield the function \( X(t) \) at leading order, by using Eq. (44), which becomes in this case,

\[
144H_0^2H_1\dot{X}(t) + (144H_0^3H_1 - 864H_0H_1^2t)\ddot{X}(t) - (864H_0^3H_1 + 2304H_0^3H_1^2t + 1152H_1^3)X(t) - H_0^2 = 0,
\]

(84)

which can be solved analytically, and the solution is in this case,

\[
X(t) = C_a e^{\mu t} + C_b e^{-\nu t} + \Lambda,
\]

(85)

where \( \Lambda, C_a \) and \( C_b \) are integration constants, and the parameters \( \mu \) and \( \nu \) are equal to,

\[
\mu = -H_0^2 + \sqrt{25H_0^2 + 52H_0^2H_1 + 6H_0^2H_1^2 + 6H_0H_1}, \quad \nu = \frac{H_0^2 + \sqrt{25H_0^2 + 52H_0^2H_1 + 6H_0^2H_1^2 - 6H_0H_1}}{2H_0^2}.
\]

(86)

The Ricci scalar for the quasi-de Sitter evolution at hand is at leading order \( R(t) = 12H_0^2 - 48H_0H_1t - 12H_1 \), so by inverting this function, we obtain \( t(R) = \frac{12H_0^2 - 12H_1 - r}{48H_0H_1} \). So substituting \( t(R) \) in \( X(t) \) (85), we obtain the solution,

\[
X(R) \simeq C_a e^{\frac{\rho(12H_0^2 - 12H_1 - r)}{48H_0H_1}} + C_b e^{-\frac{\rho(12H_0^2 - 12H_1 - r)}{48H_0H_1}} + \Lambda.
\]

(87)

Hence, the inflationary quasi-de Sitter evolution (82), which is realized by the vacuum \( R^2 \) model, can be realized by standard Einstein-Hilbert gravity with exponential kinetic corrections of the scalar curvature, of the form (87).

IV. SCALAR-MULTI-TENSORIAL EQUIVALENCE

The kinetic Ricci \( f(R) \) gravity we considered in the previous sections, belongs to a general class of models of the form \( f(R, \nabla_{\mu}R, ..., \nabla^\nu R) \), studied in Ref. [47]. In this section we shall demonstrate the equivalence of the theory at hand with a scalar-tensor theory. To this end we can define the functional dependence

\[
\Phi(R, R_{\mu}) = f(R) + X(R)R_{\mu}R^{\nu},
\]

(88)
and the action (11) is written in the following form,

\[ S = \int d^4x \sqrt{-g} \Phi(R, R_\mu) + S_{\text{matter}}. \] (89)

Following the formalism firstly proposed in Ref. [47], we make the following substitutions, we can use the substitution

\[ \xi = R \]
(90)
\[ \xi_\mu = R_\mu, \]
(91)

Applying the above, the function (88) transforms as follows,

\[ \Phi(\xi, \xi_\mu) = f(\xi) + X(\xi)\xi_\mu \xi^\mu. \] (92)

The first order derivatives are equal to,

\[ \frac{\partial \Phi}{\partial \xi} = \frac{\partial f(\xi)}{\partial \xi} + \frac{\partial X(\xi)}{\partial \xi} \xi_\mu \xi^\mu, \quad \frac{\partial \Phi}{\partial \xi_\mu} = 2Xg^{\mu\nu} \xi_\nu. \]

For the sake of simplicity, we will not use the arguments of functions, and the “prime” hereafter denotes differentiation with respect to \( \xi \). To obtain the correct transformation from the initial model (1), we must impose the following constraint,

\[ \det \left( \frac{\partial^2 \Phi}{\partial \xi \partial \xi^\nu} \frac{\partial^2 \Phi}{\partial \xi \partial \xi_\mu} \right) = \det \left( f'' + X'' \xi_\mu \xi^\mu \frac{2Xg^{\mu\nu} \xi_\nu}{2Xg^{\mu\nu}} \right) \neq 0, \]
(93)

or equivalently,

\[ 2g^{\mu\nu} (X f'' + X'' \xi_\alpha \xi^\alpha) - 4(X')^2 \xi^\mu \xi^\nu \neq 0 \]

In effect, the trace of (93) is,

\[ 2X f'' + \xi_\alpha \xi^\alpha (2X'' X - X'^2) \neq 0. \]
(94)

Let us check the constraint for the Starobinsky-Podolsky action

\[ S = \int d^4x \sqrt{-g} \left[ R + \frac{c_0}{2} R^2 + \frac{c_1}{2} R_\mu R^\mu \right] + S_{\text{matter}}. \] (95)

The condition (93) in the case at hand is,

\[ \det \left( \begin{array}{cc} c_0 & 0 \\ 0 & c_1 g^{\mu\nu} \end{array} \right) \neq 0, \]
(96)

while the corresponding trace gives,

\[ 4c_0 c_1 \neq 0. \]

Now we introduce the scalar field \( \phi \) and the vector fields \( \phi^\mu \), defined as follows,

\[ \phi = \frac{\partial \Phi}{\partial \xi} = f' + X' \xi_\mu \xi^\mu \]
(97)
\[ \phi^\mu = \frac{\partial \Phi}{\partial \xi_\mu} = 2X \xi^\mu. \]
(98)

Note that the condition (93) ensures that \( \xi = \xi(\phi, \phi^\mu) \) and \( \xi^\nu = \xi^\nu(\phi, \phi^\mu) \). The scalar potential \( U(\phi, \phi_\mu, \phi^\mu) \) is reduced to,

\[ U(\phi, \phi_\mu) = \phi \xi + \phi^\mu \xi_\mu - \Phi(\xi, \xi_\mu) = \phi \xi - f + \frac{\phi^\mu \phi_\mu}{4X}. \]
(99)
The action integral $S' = S'(\phi, \phi^\mu, R, R, \mu)$ is cast as follows,

$$
S' = \int d^4x \sqrt{-g} \left[ \phi R + \phi^\mu R,_{\mu} - \phi \xi + f - \frac{\phi^\mu \phi_{\mu}}{4X} \right] + S_{\text{matter}},
$$

(100)

where the scalar and the vector fields $\xi$ and $\xi_{\mu}$ are the fundamental fields, and $R$ along with its derivative are considered as variables. The theory with action (100), is the scalar-vector-tensor equivalent theory to the theory with action (1). By using the relation $\sqrt{-g} \phi^\mu R_{\mu} = \partial_{\mu} (\sqrt{-g} \phi^\mu R) - \sqrt{-g} \nabla_{\mu} \phi^\mu R$ and by introducing the new field $\psi = (\phi - \nabla_{\mu} \phi^\mu)$, after some algebra, the action (100) can be cast as follows,

$$
S' = \int d^4x \sqrt{-g} \left[ \phi R - \xi (\psi + \nabla_{\mu} \phi^\mu) + f(\psi, \phi_{\mu}, \nabla_{\mu} \phi^\mu) - \frac{\phi^\mu \phi_{\mu}}{4X} \right] + S_{\text{matter}}.
$$

(101)

Thus the scalar-tensor equivalent theory of (1) is the following,

$$
S' = \int d^4x \sqrt{-g} [\psi R - U(\psi, \phi_{\mu}, \nabla_{\mu} \phi^\mu)] + S_{\text{matter}},
$$

(102)

where the scalar potential $U$ is,

$$
U(\psi, \phi_{\mu}, \nabla_{\mu} \phi^\mu) = \xi (\psi + \nabla_{\mu} \phi^\mu) - f + \frac{\phi^\mu \phi_{\mu}}{4X},
$$

(103)

and $\xi$, $f$ are functions of the following arguments,

$$
\xi = \xi(\psi, \phi_{\mu}, \nabla_{\mu} \phi^\mu)
$$

(104)

$$
f = f(\psi, \phi_{\mu}, \nabla_{\mu} \phi^\mu)
$$

(105)

It should be noted here that we obtained a Brans-Dicke-like theory without a kinetic term for the corresponding Brans-Dicke scalar. By variation of the action (102), one gets the field equations [47],

$$
\psi G_{\mu\nu} - (\nabla_{\mu} \nabla_{\nu} \psi - g_{\mu\nu} \Box \psi) + \frac{1}{2} g_{\mu\nu} U = kT_{\mu\nu},
$$

(106)

$$
R = \frac{\partial U}{\partial \psi}
$$

(107)

$$
\frac{\partial U}{\partial \phi^\rho} - \nabla_{\mu} \frac{\partial U}{\partial (\nabla_{\mu} \phi^\mu)} = 0,
$$

(108)

$$
\frac{\delta L_M}{\delta \phi} = 0.
$$

(109)

Substituting the potential from Eq. (103) into Eq. (107) we obtain,

$$
\psi G_{\mu\nu} - (\nabla_{\mu} \nabla_{\nu} \psi - g_{\mu\nu} \Box \psi) + \frac{1}{2} g_{\mu\nu} \left( \xi (\psi + \nabla_{\mu} \phi^\mu) - f + \frac{\phi^\mu \phi_{\mu}}{4X} \right) = kT_{\mu\nu}.
$$

(110)

By contracting the above, we obtain the following relation,

$$
- \psi \xi + 3 \Box \psi + 2 \left( \xi (\psi + \nabla_{\mu} \phi^\mu) - f + \frac{\phi^\mu \phi_{\mu}}{4X} \right) - kT = 0.
$$

(111)

After some algebra, we finally obtain,

$$
\psi \xi + 3 \Box \psi + 2 \xi \nabla_{\mu} \phi^\mu - 2f + \frac{\phi^\mu \phi_{\mu}}{2X} - kT = 0.
$$

(112)

For the Starobinsky-Podolsky action (95), Eq. (112) transforms to,

$$
- 3 \Box \psi - kT - 2 \left( \frac{1}{2c_0} (\psi + \nabla_{\mu} \phi^\mu - 1)^2 + \frac{\phi^\mu \phi_{\mu}}{2c_1} \right) + \psi \frac{1}{c_0} (\psi + \nabla_{\mu} \phi^\mu - 1) = 0.
$$

(113)
Now let us derive the equation (109) using (103).

\[
\frac{\partial \xi}{\partial \phi^\rho} (\psi + \nabla_\mu \phi^\mu) - \frac{\partial f}{\partial \phi^\rho} + \frac{2\phi^\mu \phi^\rho}{4X^2} - \nabla_\mu \left( \frac{\partial \xi}{\partial (\nabla_\rho \phi^\mu)} (\psi + \nabla_\mu \phi^\nu) + \xi - \frac{\partial f}{\partial (\nabla_\rho \phi^\mu)} - \frac{\phi^\mu \phi^\rho}{4X^2} \frac{\partial X}{\partial (\nabla_\rho \phi^\mu)} \right).
\]

(114)

For the Starobinsky-Podolsky action (114), the result is the same as in Ref. [47], that is,

\[
\frac{\phi^\mu}{c_1} - \nabla_\mu \left( \frac{1}{c_0} (\psi + \nabla_\mu \phi^\nu - 1) \right) = 0.
\]

(115)

An issue we did not address is the occurrence of ghost instabilities, as we mentioned briefly some sections earlier. Although ghosts are in general absent, due to the appropriate choice of the kinetic terms, as was also shown in Ref. [30], however these instabilities could make there presence visible via the fluctuations, and particularly in the sound speed of these fluctuations. It is possible that the sound speed becomes super-luminal. In addition, one relevant issue would be to check for gradient instabilities, as for example in the multi-Galileon theories [48], see also [49]. In addition we should note that in the Einstein frame there are 4 degrees of freedom, to a perturbative level, due to the presence of a propagating degree of freedom related to the higher derivatives of the Ricci scalar. The viability of the whole theoretical framework crucially depends on this study, so this issue should be carefully addressed in a focused future work, since it lies beyond the scopes of this introductory to the subject paper.

V. CONCLUDING REMARKS

In this paper we investigated the cosmological implications of a kinetic scalar curvature-corrected $f(R)$ gravity. Particularly, we included first derivatives of the Ricci scalar in the $f(R)$ gravity action, and we derived the resulting gravitational equations of motion. We realized various cosmological evolutions corresponding to a flat FRW Universe, and we demonstrated how the gravitational equations can be used as a reconstruction technique for realizing various cosmological scenarios. The interesting feature of the kinetic Ricci $f(R)$ gravity is that cosmological evolutions such as bouncing cosmologies and quasi-de Sitter evolutions, can be realized by a theory which is an Einstein-Hilbert theory in the $f(R)$ gravity part.

What we did not address in this paper is the study on the evolutions of cosmological perturbations in the Jordan frame. Such a study is demanding but compelling in order to find explicit forms of the spectral index of the primordial curvature perturbations and of the scalar-to-tensor ratio, and work is in progress along this research line.

Furthermore, another important issue we need to discuss, is related to phenomenological aspects of the kinetic Ricci theory, and particular the question is how to distinguish this theory from other modified gravity theories. Perhaps the effective field theory approach may shed some light on this question, as for example in Refs. [50, 51], where the effective field theory approach was found to be able to tell the difference between one particular class of modified gravity theories from general relativity and other theories. In general, this study is compelling in many theoretical cosmology contexts, and along with the effective field theory approach, the gravitational wave generation of each theory, may also provide useful hints towards distinguishing each theory. Finally, another way to distinguish various modified gravities is to study the growth factor of matter perturbations during the matter domination era, see for example [52].

Finally, in this paper only inflationary cosmological solutions were considered and reconstructed. However, a major alternative theoretical framework able to produce a nearly scale invariant framework is that of bouncing cosmology [53, 57]. The bounce cosmology paradigm has the appealing feature of producing a cosmological evolution without the unappealing feature of having an initial singularity. This class of solutions should also be considered and we hope to address this issue in a future work.

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