Another proof to Kotschick-Morita’s Theorem of Kontsevich homomorphism

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Abstract

In [4], Kotschick and Morita showed that the Gel’fand-Kalinin-Fuks class in $H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8$ is decomposed as a product $\eta \wedge \omega$ of some leaf cohomology class $\eta$ and a transverse symplectic class $\omega$. In other words, the Kontsevich homomorphism $\omega \wedge : H^5_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2, \mathbb{R}))_{10} \to H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8$ is isomorphic.

In this paper, we give proof for the Kotschick and Morita’s theorem by using the Gröbner Basis theory and computer symbol calculations.

1 Introduction

On the symplectic space $(\mathbb{R}^{2n}, \omega)$, let $\mathfrak{ham}_{2n}$ be the Lie algebra of the formal Hamiltonian vector fields, and let $H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$ be the relative Gel’fand-Fuks cohomology group with the weight $w$. When $n = 1$, Gel’fand-Kalinin-Fuks ([2]) showed that $H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2, \mathbb{R}))_w = 0$ for the weight $w = 2, 4, 6$ and the $H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8 \cong \mathbb{R}$ whose generator is called the Gel’fand-Kalinin-Fuks class. The next non-trivial result in this context is $H^9_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_{14} \cong \mathbb{R}$, which is proved by S. Metoki ([5]) in 1999.

D. Kotschick and S. Morita ([4]) studied $H^*_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2n, \mathbb{R}))_w$ and determined the whole space for $n = 1$ and $w \leq 10$, where $\mathfrak{ham}_0^0$ is the Lie subalgebra of the formal Hamiltonian vector fields which vanish at the origin of $\mathbb{R}^{2n}$.

There is a natural homomorphism due to Kontsevich ([3])

$$\omega^n : H^*_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2n, \mathbb{R}))_w \to H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{w-2n}$$

D. Kotschick and S. Morita show in [4] the next theorem.

**Theorem 1.1 ([4]).** There is a unique element $\eta \in H^5_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2n, \mathbb{R}))_w$ such that

$$\text{Gel’fand-Kalinin-Fuks class } = \eta \wedge \omega \in H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8$$

where $\omega$ is the cochain associated with the linear symplectic form of $\mathbb{R}^2$.

About mathematical background, we refer to [4] or a draft “An affirmative answer to a conjecture for Metoki class” by K. Mikami([6]). For more precise notations or notions in this paper, we refer to [4], [7] or [6].

Our aim of this draft is to give another proof of the theorem above by using Gröbner basis theory (cf. [1] or [6]).

We use Maple Groebner Package for computing Groebner Basis and the normal form. There are several symbol calculus softwares beside Maple, Mathematica, Risa/Asir and so on. Risa/Asir is popular among Japanese mathematicians because it is bundled in Math Libre Disk which is distributed at annual meetings of the Mathematical Society of Japan. So, the author presents the source code and the output about Risa/Asir concerning to the Theorem by D. Kotschick and S. Morita in Appendix XYZ.

You can compare the results by Maple and Risa/Asir and you will see that the both are the same, up to non-zero scalar multiples.

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2 Preliminaries

Let \( x, y \) be the standard basis of \( \mathbb{R}^2 \) with the Poisson bracket is \( \{x, y\} = 1 \). We denote the standard basis of \( A \)-homogeneous polynomials of \( x \) and \( y \) as \( x^a \ y^{A-a} \ \frac{a!}{(A-a)!} \) and the dual basis is written by \( z_A^a \).

3 About \( C^*_G \( \mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}) \))\(_{\mathfrak{w}} \)

Using the method in [7], we understand the structures of \( C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{\mathfrak{w}} \) concretely. We denote \( C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{\mathfrak{w}} \) by \( C^* \). We choose our concrete bases as \( \{q_i\}_{i=1}^9 \) of \( C^4 \), \( \{w_i\}_{i=1}^{12} \) of \( C^5 \), and \( \{r_i\}_{i=1}^6 \) of \( C^6 \).

3.1 Basis of \( C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{\mathfrak{w}} \)

\[
q_1 = \frac{1}{8} z_1^3 z_2^4 + \frac{3}{4} z_1^2 z_2^2 z_4^2 + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_2^2 z_4 + \frac{1}{2} z_1^2 z_2 z_3^2 - \frac{1}{4} z_2^2 z_3^2 + \frac{1}{2} z_1^2 z_2 z_4 + \frac{1}{2} z_1^2 z_2 z_3^2 \]
\[
q_2 = \frac{3}{16} z_1^3 z_2^4 + \frac{3}{4} z_1^2 z_2^2 z_4^2 + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_2^2 z_4 + \frac{1}{2} z_1^2 z_2 z_3^2 - \frac{1}{4} z_2^2 z_3^2 + \frac{1}{2} z_1^2 z_2 z_4 + \frac{1}{2} z_1^2 z_2 z_3^2 \]
\[
q_3 = -\frac{3}{8} z_1^3 z_2^4 - \frac{3}{4} z_1^2 z_2^2 z_4^2 - \frac{1}{2} z_1 z_2 - \frac{1}{4} z_2^2 z_4 - \frac{1}{2} z_1^2 z_2 z_3^2 + \frac{1}{4} z_2^2 z_3^2 - \frac{1}{2} z_1^2 z_2 z_4 + \frac{1}{2} z_1^2 z_2 z_3^2 \]
\[
q_4 = -\frac{1}{16} z_1^3 z_2^4 - \frac{1}{8} z_1^2 z_2^2 z_4^2 - \frac{1}{4} z_1 z_2 - \frac{1}{8} z_2^2 z_4 - \frac{1}{4} z_1^2 z_2 z_3^2 + \frac{1}{8} z_2^2 z_3^2 - \frac{1}{4} z_1^2 z_2 z_4 + \frac{1}{4} z_1^2 z_2 z_3^2 \]
\[
q_5 = -\frac{3}{16} z_1^3 z_2^4 - \frac{3}{8} z_1^2 z_2^2 z_4^2 - \frac{1}{4} z_1 z_2 - \frac{1}{8} z_2^2 z_4 - \frac{1}{4} z_1^2 z_2 z_3^2 + \frac{1}{8} z_2^2 z_3^2 - \frac{1}{4} z_1^2 z_2 z_4 + \frac{1}{4} z_1^2 z_2 z_3^2 \]
\[-2s_3s_2s_1s_0 + 4s_3s_2s_1s_0 + \frac{7}{6}s_3s_2s_1s_0 + \frac{1}{12}s_3s_2s_1s_0 + \frac{5}{2}s_3s_2s_1s_0 + \frac{2}{3}s_3s_2s_1s_0 + \frac{5}{3}s_3s_2s_1s_0 + \frac{8}{3}s_3s_2s_1s_0 - s_1^2s_2s_0 s_1s_0 \]
\[-3s_3s_2s_1s_0 + 2s_3s_2s_1s_0 + \frac{11}{2}s_3s_2s_1s_0 + \frac{5}{2}s_3s_2s_1s_0 - \frac{11}{6}s_3s_2s_1s_0 + \frac{1}{6}s_3s_2s_1s_0 + \frac{5}{2}s_3s_2s_1s_0 - \frac{1}{6}s_3s_2s_1s_0 - s_1^2s_2s_0 s_1s_0 \]
\[-\frac{7}{2}s_3s_2s_1s_0 - 5s_3s_2s_1s_0 - \frac{11}{4}s_3s_2s_1s_0 + \frac{3}{2}s_3s_2s_1s_0 - \frac{8}{3}s_3s_2s_1s_0 + \frac{1}{4}s_3s_2s_1s_0 + \frac{3}{2}s_3s_2s_1s_0 + s_1^2s_2s_0 s_1s_0 - \frac{4}{3}s_1^2s_2s_0 s_1s_0 \]
\[+ s_3^2s_2s_1s_0 - \frac{9}{2}s_3^2s_1s_0 s_0 s_1 \]

\[q_6 = -s_1^2s_2s_0 s_1s_0 - \frac{13}{6}s_1^2s_2s_0 s_1s_0 + \frac{35}{12}s_1^2s_2s_0 s_1s_0 + \frac{35}{12}s_1^2s_2s_0 s_1s_0 - \frac{25}{6}s_1^2s_2s_0 s_1s_0 + \frac{15}{4}s_1^2s_2s_0 s_1s_0 - \frac{27}{20}s_1^2s_2s_0 s_1s_0 - \frac{25}{6}s_1^2s_2s_0 s_1s_0 - \frac{5}{2}s_1^2s_2s_0 s_1s_0 + \frac{5}{2}s_1^2s_2s_0 s_1s_0 - \frac{3}{4}s_1^2s_2s_0 s_1s_0 \]
\[q_7 = -s_1^2s_2s_0 s_1s_0 - \frac{3}{4}s_1^2s_2s_0 s_1s_0 + \frac{3}{4}s_1^2s_2s_0 s_1s_0 + \frac{3}{4}s_1^2s_2s_0 s_1s_0 - \frac{3}{4}s_1^2s_2s_0 s_1s_0 - \frac{3}{4}s_1^2s_2s_0 s_1s_0 + \frac{7}{4}s_1^2s_2s_0 s_1s_0 \]
\[q_8 = \frac{3}{2}s_1^2s_2s_0 s_1s_0 - \frac{3}{5}s_1^2s_2s_0 s_1s_0 + \frac{4}{5}s_1^2s_2s_0 s_1s_0 + \frac{4}{5}s_1^2s_2s_0 s_1s_0 - \frac{2}{5}s_1^2s_2s_0 s_1s_0 + \frac{2}{5}s_1^2s_2s_0 s_1s_0 - \frac{3}{5}s_1^2s_2s_0 s_1s_0 + \frac{3}{5}s_1^2s_2s_0 s_1s_0 \]

3.2 Basis of $C_5^{0}(\text{ham}^0_2,\text{sp}(2,\mathbb{R}))(0)$

\[w_1 = -\frac{1}{6}s_1^2s_2s_1s_0 + \frac{2}{3}s_1^2s_2s_1s_0 - \frac{2}{3}s_1^2s_2s_1s_0 - \frac{1}{6}s_1^2s_2s_1s_0 + \frac{1}{6}s_1^2s_2s_1s_0 + \frac{2}{3}s_1^2s_2s_1s_0 + \frac{2}{3}s_1^2s_2s_1s_0 \]
\[+ \frac{2}{3}s_1^2s_2s_1s_0 - \frac{2}{3}s_1^2s_2s_1s_0 + \frac{1}{6}s_1^2s_2s_1s_0 + \frac{1}{6}s_1^2s_2s_1s_0 + \frac{2}{3}s_1^2s_2s_1s_0 + \frac{2}{3}s_1^2s_2s_1s_0 \]
\[w_2 = \frac{3}{2}s_1^2s_2s_1s_0 - 6s_1^2s_2s_1s_0 + 9s_1^2s_2s_1s_0 - 6s_1^2s_2s_1s_0 + \frac{3}{2}s_1^2s_2s_1s_0 - \frac{3}{2}s_1^2s_2s_1s_0 + \frac{3}{2}s_1^2s_2s_1s_0 + \frac{3}{2}s_1^2s_2s_1s_0 \]
\[+ \frac{3}{2}s_1^2s_2s_1s_0 - 4s_1^2s_2s_1s_0 + \frac{7}{2}s_1^2s_2s_1s_0 - \frac{7}{2}s_1^2s_2s_1s_0 + \frac{3}{2}s_1^2s_2s_1s_0 + \frac{3}{2}s_1^2s_2s_1s_0 \]

3
\[-\frac{1}{2} z_1^2 z_2 z_3 z_4 + \frac{1}{2} z_1 z_2^2 z_3 z_4 + \frac{3}{2} z_1 z_2 z_3^2 z_4 - \frac{3}{2} z_1 z_2 z_3 z_4^2 - 9 z_1 z_2 z_3 z_4 - 8 z_1 z_2 z_3 z_4^2 + 6 z_1 z_2 z_3 z_4^2\]

\[w_3 = \frac{1}{24} - \frac{1}{8} z_1^2 z_2 z_3 z_4 + \frac{1}{8} z_1 z_2 z_3^2 z_4 - \frac{1}{24} z_1 z_2 z_3 z_4^2 - \frac{1}{3} z_1^2 z_2 z_3 z_4^2 + \frac{1}{3} z_1 z_2 z_3^2 z_4^2 - \frac{1}{8} z_1 z_2 z_3 z_4^2 + \frac{1}{8} z_1 z_2 z_3 z_4^2\]

\[w_4 = \frac{1}{12} z_1^2 z_2 z_3 z_4 - 2 z_1 z_2 z_3^2 z_4 + \frac{5}{12} z_1 z_2 z_3 z_4^2 - 2 z_1^2 z_2 z_3 z_4 + 3 z_1 z_2 z_3^2 z_4^2 - 2 z_1 z_2 z_3 z_4^2 + \frac{3}{2} z_1 z_2 z_3 z_4^2 - \frac{1}{2} z_1^2 z_2 z_3 z_4 - 2 z_1 z_2 z_3^2 z_4 + \frac{3}{2} z_1 z_2 z_3 z_4^2 - 2 z_1 z_2 z_3 z_4^2 + \frac{3}{2} z_1 z_2 z_3 z_4^2\]

\[w_5 = \frac{1}{6} z_1^2 z_2 z_3 z_4 - \frac{1}{6} z_1 z_2^2 z_3 z_4 + \frac{1}{6} z_1 z_2 z_3^2 z_4 - \frac{1}{3} z_1^2 z_2 z_3 z_4 + \frac{1}{3} z_1 z_2 z_3 z_4^2 + \frac{1}{6} z_1 z_2 z_3 z_4^2 - \frac{1}{6} z_1 z_2^2 z_3 z_4 + \frac{1}{6} z_1^2 z_2 z_3 z_4\]

\[w_6 = \frac{1}{3} z_1^2 z_2 z_3 z_4 - \frac{1}{3} z_1 z_2^2 z_3 z_4 + \frac{1}{3} z_1 z_2 z_3^2 z_4 - \frac{1}{3} z_1 z_2 z_3 z_4^2 + \frac{1}{3} z_1 z_2 z_3 z_4^2 - \frac{1}{3} z_1 z_2 z_3 z_4^2 + \frac{1}{3} z_1^2 z_2 z_3 z_4^2 - \frac{1}{3} z_1 z_2^2 z_3 z_4^2 + \frac{1}{3} z_1^2 z_2 z_3 z_4^2\]

\[w_7 = \frac{1}{4} z_1^2 z_2 z_3 z_4 - \frac{1}{2} z_1 z_2^2 z_3 z_4 + \frac{1}{2} z_1 z_2 z_3^2 z_4 - \frac{1}{4} z_1 z_2 z_3 z_4^2 - \frac{1}{2} z_1^2 z_2 z_3 z_4^2 + \frac{1}{2} z_1 z_2 z_3 z_4^2 - \frac{1}{4} z_1 z_2 z_3 z_4^2 - \frac{1}{2} z_1 z_2^2 z_3 z_4 + \frac{1}{2} z_1^2 z_2 z_3 z_4^2 - \frac{1}{4} z_1^2 z_2 z_3 z_4\]

\[w_8 = \frac{1}{8} z_1^2 z_2 z_3 z_4 - \frac{1}{4} z_1 z_2^2 z_3 z_4 + \frac{1}{4} z_1 z_2 z_3^2 z_4 - \frac{1}{8} z_1 z_2 z_3 z_4^2 - \frac{1}{2} z_1^2 z_2 z_3 z_4^2 + \frac{1}{2} z_1 z_2 z_3 z_4^2 - \frac{1}{4} z_1 z_2 z_3 z_4^2 - \frac{1}{2} z_1 z_2^2 z_3 z_4 + \frac{1}{2} z_1^2 z_2 z_3 z_4^2 - \frac{1}{4} z_1^2 z_2 z_3 z_4\]
\[ -6z^1 z^2 z^3 z^4 + 2 z^1 z^2 z^4 z^6 - 2z^1 z^4 z^1 z^4 - 4z^1 z^3 z^1 z^3 - \frac{1}{6} z^3 z^2 z^1 z^3 + 3 z^2 z^1 z^1 z^3 - 2z^2 z^2 z^4 z^5 \]

\[ w_9 = \frac{4}{5} z^1 z^2 z^4 z^5 - 2 z^1 z^2 z^4 z^6 - \frac{10}{3} z^1 z^4 z^1 z^4 - \frac{1}{6} z^3 z^2 z^1 z^3 + 3 z^2 z^1 z^1 z^3 - 2z^2 z^2 z^4 z^5 + \frac{1}{4} z^1 z^3 z^1 z^3 \]

\[ w_{10} = \frac{2}{3} z^1 z^2 z^4 z^5 - 4 z^1 z^2 z^4 z^6 - \frac{3}{2} z^1 z^2 z^4 z^6 + z^1 z^2 z^4 z^6 + 4z^1 z^2 z^4 z^6 + \frac{1}{3} z^3 z^2 z^1 z^3 - \frac{1}{3} z^3 z^2 z^1 z^3 \]

\[ w_{11} = \frac{4}{3} z^1 z^2 z^4 z^5 - z^1 z^2 z^4 z^6 - \frac{3}{2} z^1 z^2 z^4 z^6 + \frac{1}{3} z^3 z^2 z^1 z^3 + \frac{2}{3} z^3 z^2 z^1 z^3 - z^1 z^2 z^4 z^6 \]

\[ w_{12} = \frac{1}{3} z^1 z^2 z^4 z^5 + z^1 z^2 z^4 z^6 - \frac{2}{3} z^1 z^2 z^4 z^6 + \frac{4}{3} z^1 z^2 z^4 z^6 - \frac{1}{3} z^3 z^2 z^1 z^3 - \frac{1}{3} z^3 z^2 z^1 z^3 \]

3.3 Basis of $C^6_{GF}(\text{ham}_0^0, sp(2, \mathbb{R}))_{10}$

\[ r_1 = \frac{1}{5} z^1 z^2 z^3 z^4 z^6 + \frac{2}{3} z^1 z^2 z^3 z^6 z^8 + 2 z^2 z^3 z^4 z^6 \]

\[ r_2 = -3 z^1 z^2 z^3 z^4 z^6 + \frac{1}{5} z^1 z^2 z^4 z^6 z^8 - \frac{2}{3} z^1 z^2 z^4 z^6 z^8 + 4 z^2 z^3 z^4 z^6 - z^2 z^3 z^4 z^6 + 6 z^1 z^2 z^3 z^6 + 6 z^1 z^2 z^3 z^6 \]

\[ r_3 = z^1 z^2 z^3 z^4 z^6 - \frac{1}{5} z^1 z^2 z^4 z^6 z^8 + \frac{4}{3} z^1 z^2 z^4 z^6 z^8 + z^2 z^3 z^4 z^6 + \frac{2}{3} z^2 z^3 z^4 z^6 \]

\[ r_4 = \frac{1}{3} z^1 z^2 z^3 z^4 z^6 - \frac{3}{2} z^1 z^2 z^3 z^6 z^8 + 3 z^1 z^2 z^3 z^6 z^8 - 2 z^2 z^3 z^4 z^6 + \frac{2}{3} z^2 z^3 z^4 z^6 \]

3.4 Matrix representation of $d_0$

We denote $C^6_{GF}(\text{ham}_0^0, sp(2, \mathbb{R}))_{10}$ by $C^*$. We choose our concrete bases as \{q_i\}_{i=1}^9 of $C^4$, \{w_i\}_{i=1}^{12} of $C^5$, and \{r_i\}_{i=1}^4 of $C^6$. Then the matrix representations of linear maps $d_0 : C^4 \to C^5$ and
$d_0 : C^5 \rightarrow C^6$ are given as
\[ [d_0 (q_1), \ldots, d_0 (q_9)] = [w_1, \ldots , w_{12}]M \]
and
\[ [d_0 (w_1), \ldots, d_0 (w_{12})] = [r_1, r_2, r_3, r_4]N \]
where
\[ tM = \begin{bmatrix} -\frac{135}{4} & 0 & -60 & \frac{15}{2} & -45 & -15 & \frac{5}{4} & -\frac{45}{4} & \frac{75}{2} & 0 & 0 & 0 \\ \frac{108}{11} & 18 & 0 & 0 & 0 & 60 & \frac{46}{11} & \frac{90}{11} & \frac{156}{11} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 12 & -\frac{9}{2} & -9 & 0 & 0 & 0 & 0 & \frac{27}{4} & 18 & 0 \\ 0 & 0 & -10 & -\frac{29}{3} & -2 & 2 & 1 & 0 & 6 & 4 & -1 & 0 \\ 0 & \frac{5}{2} & 29 & \frac{47}{3} & -23 & 43 & \frac{13}{2} & 9 & 25 & 16 & -\frac{71}{2} & 0 \\ 0 & 5 & 45 & \frac{155}{6} & -40 & 65 & 10 & 0 & 50 & 20 & -\frac{115}{2} & 0 \\ 0 & \frac{3}{2} & 18 & \frac{23}{2} & -3 & 30 & \frac{11}{2} & \frac{9}{2} & 9 & 6 & -33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 7 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 70 \end{bmatrix} \]
\[ N = \begin{bmatrix} 0 & 140 & 0 & 0 & 0 & -15 & 15 & 30 & \frac{5}{2} & 0 & 0 & 0 \\ -5 & -4 & 1 & \frac{3}{4} & -\frac{11}{2} & 31 & \frac{31}{2} & -3 & -2 & \frac{5}{3} & -1 & 2 & 0 \\ -16 & 32 & -2 & -12 & \frac{22}{3} & 58 & \frac{3}{3} & -18 & -12 & -\frac{5}{3} & 0 & 8 & 0 \\ 0 & 0 & 0 & 42 & 7 & 0 & 0 & 0 & 0 & 0 & 14 & 3 \end{bmatrix} \]

Since rank $M = 7$ and rank $N = 4$, we see the dimensions of $d_0 (C^4)$ and ker$(d_0 : C^5 \rightarrow C^6)$, and so on. The precise data of the structures of $C^\bullet_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_1$ and $H^\bullet_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_1$ is in the table below, where dim and rank mean the dimension of $C^p$ and the rank of $d_0 : C^p \rightarrow C^{p+1}$, and Betti num is the Betti number, which is the dimension of the cohomology group $H^\bullet_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_1$.

| $\text{ham}_2^0, w=10$ | $0 \rightarrow C^2 \rightarrow C^3 \rightarrow C^4 \rightarrow C^5 \rightarrow C^6 \rightarrow 0$ |
|------------------------|----------------------------------|
| dim                    | 1 | 3 | 9 | 12 | 4 |
| rank                   | 0 | 1 | 2 | 7 | 4 | 0 |
| Betti num              | 0 | 0 | 0 | 1 | 0 |

### 4 About $C^\bullet_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_w$

We also know the structures of $C^\bullet_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_8$ well.

#### 4.1 Basis of $C^6_{\text{GF}}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_8$

\[ q_1 = \frac{1}{8} z_1 z_2 z_3 z_4 z_5 + \frac{3}{4} z_1 z_2 z_3 z_4 z_7 + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{1}{8} z_1 z_2 z_3 z_4 z_9 - \frac{1}{4} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 \\
\] 
\[ + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{3}{4} z_1 z_2 z_3 z_4 z_9 + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{1}{8} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 \\
\] 
\[ + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{3}{4} z_1 z_2 z_3 z_4 z_9 + \frac{3}{4} z_1 z_2 z_3 z_4 z_9 - \frac{1}{8} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 + \frac{1}{2} z_1 z_2 z_3 z_4 z_9 \\
\]
\[ q_9 = -\frac{3}{8} z_2^2 z_4^2 + \frac{3}{2} z_1 z_2 z_3 z_4 - 3 z_1^2 z_2 z_3 z_4 + 3 z_1 z_2 z_3^2 z_4^2 - \frac{2}{3} z_1 z_2 z_3 z_4 - 4 z_1 z_3 z_2^2 z_4^2 \]
\[ q_{10} = -\frac{2}{3} z_1^2 z_2^2 z_3 z_4^2 + \frac{2}{3} z_1 z_2 z_3 z_4^2 - z_1 z_2 z_3 z_4^2 - \frac{1}{4} z_1 z_2 z_3^2 z_4^2 - \frac{1}{3} z_1 z_2 z_3 z_4^2 \]
\[ q_{11} = -\frac{1}{3} z_1^2 z_2 z_3 z_4^2 + \frac{1}{3} z_1 z_2 z_3 z_4^2 + z_1 z_2 z_3 z_4^2 - \frac{4}{3} z_1^2 z_2 z_3^2 z_4^2 - \frac{1}{3} z_1^2 z_2 z_3 z_4^2 \]
\[ q_{12} = -\frac{1}{2} z_1^2 z_2 z_3 z_4^2 - \frac{2}{3} z_1 z_2 z_3 z_4^2 + \frac{3}{2} z_1^2 z_2 z_3^2 z_4^2 - 2 z_1^2 z_2 z_3 z_4^2 - \frac{1}{2} z_1^2 z_2 z_3^2 z_4^2 + 4 z_1^2 z_2 z_3 z_4^2 \]
\[ q_{13} = -\frac{1}{2} z_1^2 z_2 z_3 z_4^2 - 2 z_1 z_2 z_3 z_4^2 + 3 z_1^2 z_2 z_3^2 z_4^2 - 2 z_1^2 z_2 z_3 z_4^2 - \frac{1}{2} z_1^2 z_2 z_3 z_4^2 + 2 z_1^2 z_2 z_3^2 z_4^2 \]
\[ q_{14} = -\frac{1}{6} z_1^2 z_2 z_3 z_4^2 + \frac{2}{3} z_1 z_2 z_3 z_4^2 - \frac{2}{3} z_1^2 z_2 z_3^2 z_4^2 + z_1^2 z_2 z_3 z_4^2 - \frac{2}{3} z_1^2 z_2 z_3^2 z_4^2 - \frac{1}{3} z_1^2 z_2 z_3 z_4^2 + \frac{1}{3} z_1^2 z_2 z_3^2 z_4^2
\[ q_{15} = -\frac{7}{6} z^6 + \frac{10}{3} z^5 - \frac{1}{3} z^4 + \frac{1}{2} z^3 - z^2 - z - 1 \]

\[ q_{16} = -\frac{1}{4} z^8 + \frac{1}{2} z^7 - \frac{3}{2} z^6 + \frac{3}{2} z^5 - \frac{3}{2} z^4 + \frac{3}{2} z^3 - \frac{3}{2} z^2 + \frac{3}{2} z + 1 \]

\[ q_{17} = -\frac{1}{3} z^8 + \frac{1}{6} z^7 - \frac{1}{2} z^6 + \frac{1}{2} z^5 - \frac{1}{2} z^4 + \frac{1}{2} z^3 - \frac{1}{2} z^2 + \frac{1}{2} z + 1 \]
\[ + z_1^3 z_3 z_4 z_6^2 + \frac{1}{4} z_1^2 z_3 z_4 z_6^2 - \frac{3}{16} z_1^4 z_3 z_4 z_6^2 - \frac{1}{8} z_1^3 z_3 z_4 z_6^2 - \frac{1}{3} z_1^2 z_3 z_4 z_6^2 + \frac{1}{6} z_1 z_3 z_4 z_6^2 \]
\[ + 2 z_1^5 z_3 z_4 z_6^2 - \frac{5}{3} z_1^4 z_3 z_4 z_6^2 + z_1^3 z_3 z_4 z_6^2 + \frac{2}{3} z_1^2 z_3 z_4 z_6^2 - z_1 z_3 z_4 z_6^2 + \frac{2}{3} z_1 z_3 z_4 z_6^2 \]
\[ - \frac{1}{2} z_1^4 z_3 z_4 z_6^2 + \frac{1}{6} z_1^3 z_3 z_4 z_6^2 + \frac{1}{6} z_1^2 z_3 z_4 z_6^2 + \frac{1}{4} z_1 z_3 z_4 z_6^2 \]
\[ - \frac{1}{2} z_1^3 z_3 z_4 z_6^2 + \frac{1}{6} z_1^2 z_3 z_4 z_6^2 - z_1 z_3 z_4 z_6^2 + \frac{2}{3} z_1 z_3 z_4 z_6^2 + \frac{5}{3} z_1 z_3 z_4 z_6^2 \]
\[ + z_1^2 z_3 z_4 z_6^2 - \frac{5}{3} z_1^1 z_3 z_4 z_6^2 + 2 z_1^0 z_3 z_4 z_6^2 - \frac{1}{3} z_1^{-1} z_3 z_4 z_6^2 + \frac{1}{21} z_1^{-2} z_3 z_4 z_6^2 - \frac{13}{38} z_1^{-3} z_3 z_4 z_6^2 \]
\[ + \frac{1}{8} z_1^{-4} z_3 z_4 z_6^2 + \frac{5}{16} z_1^{-3} z_3 z_4 z_6^2 + \frac{1}{4} z_1^{-2} z_3 z_4 z_6^2 - \frac{3}{16} z_1^{-1} z_3 z_4 z_6^2 + \frac{1}{2} z_1^{0} z_3 z_4 z_6^2 - \frac{1}{8} z_1^{1} z_3 z_4 z_6^2 \]
\[ + 9 z_1^{2} z_3 z_4 z_6^2 + \frac{9}{16} z_1^{3} z_3 z_4 z_6^2 - \frac{1}{2} z_1^{4} z_3 z_4 z_6^2 + \frac{3}{4} z_1^{5} z_3 z_4 z_6^2 + z_1^{6} z_3 z_4 z_6^2 - \frac{1}{8} z_1^{7} z_3 z_4 z_6^2 \]
\[ + z_1^{8} z_3 z_4 z_6^2 - \frac{3}{8} z_1^{9} z_3 z_4 z_6^2 - \frac{3}{2} z_1^{10} z_3 z_4 z_6^2 - \frac{3}{8} z_1^{11} z_3 z_4 z_6^2 - \frac{3}{8} z_1^{12} z_3 z_4 z_6^2 \]

4.2 Basis of $C_{\text{GP}}(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$
+ \frac{21}{2}z_1^2z_3z_4z_5z_6z_7 - \frac{1}{2}z_1^2z_2z_3z_4z_5z_6z_7 + \frac{1}{7}z_1^2z_2z_3z_4z_5z_6z_7 + 3z_1^2z_2z_3z_4z_5z_6z_7 - 2z_1^2z_2z_3z_4z_5z_6z_7 - \frac{1}{7}z_1^2z_2z_3z_4z_5z_6z_7 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6z_7

\omega_{10} = \frac{1}{9}z_1^3z_2z_3z_4z_5z_6 - \frac{1}{9}z_1^3z_2z_3z_4z_5z_6 + \frac{2}{9}z_1^3z_2z_3z_4z_5z_6 + \frac{4}{9}z_1^3z_2z_3z_4z_5z_6 - \frac{2}{9}z_1^3z_2z_3z_4z_5z_6

\omega_{11} = \frac{1}{9}z_1^3z_2z_3z_4z_5z_6 - \frac{4}{9}z_1^3z_2z_3z_4z_5z_6 + \frac{4}{9}z_1^3z_2z_3z_4z_5z_6 - \frac{16}{9}z_1^3z_2z_3z_4z_5z_6 - \frac{10}{9}z_1^3z_2z_3z_4z_5z_6

\omega_{12} = z_1^3z_2z_3z_4z_5z_6

\omega_{13} = z_1^3z_2z_3z_4z_5z_6 + z_1^3z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^3z_2z_3z_4z_5z_6 - \frac{1}{2}z_1^3z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^3z_2z_3z_4z_5z_6 - \frac{1}{2}z_1^3z_2z_3z_4z_5z_6

\omega_{14} = z_1^3z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^3z_2z_3z_4z_5z_6 - \frac{1}{2}z_1^3z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^3z_2z_3z_4z_5z_6 + \frac{3}{2}z_1^3z_2z_3z_4z_5z_6

4.3 Basis of $C^8_{G_P}(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$

$\rho_1 = -\frac{1}{5}z_1^2z_2z_3z_4z_5z_6 + \frac{z_1^2z_2z_3z_4z_5z_6 - 2z_1^2z_2z_3z_4z_5z_6 - 3z_1^2z_2z_3z_4z_5z_6 - 4z_1^2z_2z_3z_4z_5z_6 + 3z_1^2z_2z_3z_4z_5z_6 + 3z_1^2z_2z_3z_4z_5z_6

\rho_2 = \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{3}{2}z_1^2z_2z_3z_4z_5z_6 + 3z_1^2z_2z_3z_4z_5z_6 - 4z_1^2z_2z_3z_4z_5z_6 + 3z_1^2z_2z_3z_4z_5z_6 + 3z_1^2z_2z_3z_4z_5z_6

\rho_3 = -\frac{3}{2}z_1^2z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{3}{2}z_1^2z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^2z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^2z_2z_3z_4z_5z_6 - \frac{3}{2}z_1^2z_2z_3z_4z_5z_6

\rho_4 = \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6

\rho_5 = \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6 + \frac{1}{2}z_1^2z_2z_3z_4z_5z_6
\[-\frac{3}{4}z_1^3 z_1 z_3 z_4 + \frac{3}{4}z_1 z_1 z_3 z_4 z_4 + \frac{9}{2}z_1 z_1 z_3 z_4 z_4 - \frac{9}{2}z_1 z_3 z_3 z_4 - \frac{3}{4}z_1 z_1 z_3 z_4 z_4 z_4 + 3 z_1 z_1 z_3 z_4 - z_1 z_3 z_3 z_4\]

For simplicity, we denote $C^*_G(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ by $C^*_G$.

For instance, the matrix representations of $d : C^6 \rightarrow C^7$ and $d : C^7 \rightarrow C^8$ are given as follows:

\[
\begin{bmatrix}
135/4 & 0 & -20 & -60 & 15 & 5 & -135/4 & -15 & -15 & 0 & 0 & 0 & 0 \\
-18 & -12 & 0 & 0 & 0 & 0 & 0 & -45 & 21 & 10 & 0 & 0 & 0 \\
27/4 & 0 & -20 & 12 & 3 & 0 & 0 & 0 & 0 & 0 & 63/8 & -18 & 0 & 0 \\
0 & 0 & 2 & -10 & 2 & 0 & 1 & 2 & 1 & 2 & -7 & 1 & 0 & 0 \\
0 & -11/2 & -17 & 9 & 12 & 7 & 3 & 51 & 233 & 10 & 149 & 4 & 0 & 0 \\
0 & -1/2 & -9 & 5 & 12 & 4 & 9 & 7 & 55 & 4 & 28 & 1 & 0 & 0 \\
0 & -1/2 & -20 & 0 & 4 & 12 & 15 & 35 & 3 & 35 & 70 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -28 \\
1/2 & 2 & 2 & 14 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\
-1/2 & 2 & 0 & 16 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 & 0 & -112 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & -35 & 0 & 0 & 0 & -30 & 0 & -15 & 25/2 & 0 & 0 & 0 & -5/2 & 0 \\
-11 & -9 & -39/8 & -31/8 & -61/2 & -75 & -10 & -10 & 85/2 & -2 & 20 & -20 & 3 & 0 & -1 & -3 \\
-16 & 8 & 9 & 5 & 52 & 20 & 8 & 3 & -2 & 16 & 8 & 0 & 1 & 4 \\
0 & 0 & 63/2 & 21/2 & 84 & 0 & 0 & 0 & 0 & 0 & 0 & -14 & 3 & 0 & 3
\end{bmatrix}
\]

Since rank $\tilde{M} = 9$ and rank $\tilde{N} = 4$, we see the dimensions of $d : C^6 \rightarrow C^7$ and ker($d : C^7 \rightarrow C^8$), and so on.

The precise data of the structures of $C^*_G(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ and $H^*_G(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ are in the table below.

| $\text{hammer}_w$, w=8 | $0 \rightarrow C^3 \rightarrow C^4 \rightarrow C^5 \rightarrow C^6 \rightarrow C^7 \rightarrow C^8 \rightarrow 0$ |
|------------------------|-------------------------------------------------|
| dim        | 5 | 13 | 17 | 18 | 14 | 14 |
| rank       | 0 | 5 | 8 | 9 | 9 | 4 |
| Betti num  | 0 | 0 | 0 | 0 | 1 | 0 |
5 Another proof by Gröbner bases

Since $H_{GF}^1(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_{10}$ and $H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ are both 1-dimensional, if

$$\omega^\wedge : H_{GF}^5(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_{10} \longrightarrow H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$$

is non-zero map, then it is an isomorphism.

We need to check if $\omega \wedge \ker(d_0) \subset d(C_{6}^0(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8)$ or not. For that purpose, choose a basis $k_1, \ldots, k_8$ of $\ker(d_0)$ and linear independent cochains $b_1, \ldots, b_9$ in $C_{6}^0(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ such that $d(b_1), \ldots, d(b_9)$ is a basis of $d(C_{6}^0(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8)$.

By taking matrix representation, we see that

$$\text{rank}(\omega \wedge k_1, \ldots, \omega \wedge k_8, d(b_1), \ldots, d(b_9)) = 10 > 9$$

Thus, for an element, say $h$, which represents the non-trivial cohomology class, we have to check if $\omega \wedge h$ is absorbed in $d(C_{6}^0(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8)$ or not, namely, if $\omega \wedge h$ realizes the non-trivial cohomology class in $H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ or not. This is our strategy to complete the proof of Theorem.

Since the both methodologies of using Gröbner bases in order to investigate the cohomology groups $H_{GF}^1(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_{10}$ or $H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ are the same, we discuss in the case of $H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_{10}$ in detail and write down only the result for $H_{GF}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$. In particular, we discuss the key issue where $\omega^\wedge$ is involved, carefully. Let $\{w_1, \ldots, w_{12}\}$ be the basis of $C^5$ and $\{q_1, \ldots, b_9\}$ be the basis of $C^4$ as before. From the matrix representation (1) of the coboundary operator $d_0$ of $C^1 \rightarrow C^5$, we define the linear functions

$$g_j(y) = \sum_{k=1}^{12} \lambda_{k,j} y_k \quad (j = 1, \ldots, 9)$$

where $(\lambda_{k,j}) = M$ and $\{y_1, \ldots, y_{12}\}$ are the auxiliary variables.

Fixing a monomial order of polynomials induced, say $y_1 \succ \cdots \succ y_{12}$, we get the Gröbner basis $GB_e$ of the ideal generated by $\{g_j(y) \mid j = 1, \ldots, 9\}$. This corresponds to the non-zero rows of the elementary matrix of $M$ obtained by the elementary row operations for $M$. Thus, the cardinality of $GB_e$ is equal to the rank of $M$, namely, to dim($d_0(C^4)$) and $\{\hat{g}(w) \mid \hat{g} \in GB_e\}$ gives a basis of $d_0(C^4)$ (cf. Proposition 3.1 in [1]). In our case,

$$GB_e = [21y_7 - 9y_8 - 18y_9 - 15y_{10} + 30y_{11} - 140y_{12},
18y_6 + 9y_8 + 15y_{10} - 30y_{11} + 140y_{12},
1512y_5 + 75y_8 - 900y_9 - 666y_{10} - 1461y_{11} + 3290y_{12},
36y_4 - 3y_8 + 36y_9 - 18y_{10} + 57y_{11} - 770y_{12},
72y_3 + 3y_8 - 36y_9 - 18y_{10} + 15y_{11} - 70y_{12},
63y_2 - 327y_8 + 396y_9 - 258y_{10} - 660y_{11} + 3080y_{12},
189y_1 - 12y_8 + 144y_9 + 99y_{10} + 390y_{11} - 1820y_{12}]$$

In general, the normal form of a given polynomial $g$ with respect to the Gröbner basis is the “smallest” remainder of $g$ modulo by the Gröbner basis. For a linear function $L(y)$ of $y_1, \ldots, y_{12}$, that $L(w)$ belongs to $d_0(C^4)$ is equivalent to the normal form of $L(y)$ with respect to $GB_e$ is zero.

Let $\{r_1, r_2, r_3, r_4\}$ be the basis of $C^6$ as before. The kernel space of $d_0 : C^5 \longrightarrow C^6$, whose element is given by $\sum_{j=1}^{12} c_j w_j$ satisfying $\sum_{j=1}^{12} c_j d_0(w_j) = 0$, is characterized by 4 linear functions, say $f_1(c), f_2(c), f_3(c), f_4(c)$ of $c_1, \ldots, c_{12}$ given by

$$[f_1(c), f_2(c), f_3(c), f_4(c)] = [c_1, \ldots, c_{12}]^T N$$
where $N$ is the matrix representing the operator $d_0 : C^5 \to C^6$ (This means we deal with the dual map $d_0^* : (C^6)^* \leftarrow (C^6)^*$). In our case,
\begin{align*}
f_1 &= 140c_2 - 15c_6 + 15c_7 + 30c_8 + \frac{5}{2} c_9 \\
f_2 &= -5c_1 - 4c_2 + \frac{1}{4} c_3 - \frac{11}{2} c_4 + \frac{31}{12} c_5 + \frac{31}{6} c_6 - 3c_7 - 2c_8 + \frac{5}{3} c_9 - c_{10} + 2c_{11} \\
f_3 &= -16c_1 + 32c_2 - 2c_3 - 12c_4 + \frac{22}{3} c_5 + \frac{58}{3} c_6 - 18c_7 - 12c_8 - \frac{5}{3} c_9 + 8c_{11} \\
f_4 &= 42c_4 + 7c_5 + 14c_{11} + 3c_{12}
\end{align*}

By taking a monomial order, say $c_1 \succ \cdots \succ c_{12}$, we get the Gröbner basis $GB$ of the ideal $(f_1(c), f_2(c), f_3(c), f_4(c))$. In our case,
\[
GB = [42c_4 + 7c_5 + 14c_{11} + 3c_{12}, 42c_3 + 28c_5 - 114c_6 + 198c_7 + 228c_8 + 117c_9 - 48c_{10} + 4c_{11} + 6c_{12}, 56c_2 - 6c_6 + 6c_7 + 12c_8 + c_9, 168c_1 - 112c_5 - 182c_6 + 126c_7 + 84c_8 - 35c_9 + 24c_{10} - 128c_{11} - 12c_{12}]
\]

The $GB$ gives a basis of the subspace $(d_0^* : (C^5)^* \leftarrow (C^6)^*) ((C^6)^*)$.

Consider the polynomial $h = \sum_{j=1}^{12} c_j y_j$ where $\{y_1, \ldots, y_{12}\}$ are the other auxiliary variables.

Proposition 3.3 in [1] says that the normal form of $h$ with respect to the Gröbner basis $GB$ is written as $\sum_j c_j \tilde{f_j}(y)$ where $J$ is a subset of $\{1, 2, \ldots, 12\}$, $\tilde{f_j}(y)$ is linear in $\{y_1, \ldots, y_{12}\}$, the cardinality of $J$ is dim $\ker(d_0)$, and $\{\tilde{f_j}(w) \mid j \in J\}$ gives a basis of $\ker(d_0)$. We continue the discussion in our case, then we have
\[
\tilde{f_1} = 0, \quad \tilde{f_2} = 0, \quad \tilde{f_3} = 0, \quad \tilde{f_4} = 0, \\
\tilde{f_5} = \frac{2}{3} y_1 - \frac{2}{3} y_3 - \frac{1}{6} y_4 + y_5, \quad \tilde{f_6} = \frac{13}{12} y_1 + \frac{3}{28} y_2 + \frac{19}{7} y_3 + y_6, \\
\tilde{f_7} = -\frac{3}{4} y_1 - \frac{3}{28} y_2 - \frac{33}{7} y_3 + y_7, \quad \tilde{f_8} = \frac{1}{14} y_1 + \frac{3}{28} y_2 - \frac{38}{7} y_3 + y_8, \\
\tilde{f_9} = \frac{5}{24} y_1 - \frac{1}{56} y_2 - \frac{39}{14} y_3 + y_9, \quad \tilde{f_9} = \frac{1}{14} y_1 + \frac{8}{7} y_3 + y_{10}, \\
\tilde{f_{11}} = \frac{16}{21} y_1 - \frac{2}{21} y_2 - \frac{1}{14} y_4 + y_{11}, \quad \tilde{f_{12}} = \frac{1}{14} y_1 - \frac{1}{7} y_3 - \frac{1}{14} y_4 + y_{12}
\]

Again, fixing the monomial order of $\{y_j\}$, we get the Gröbner basis $GB_k$ of the ideal generated by $\tilde{f_j}$ ($j \in J$) as
\[
GB_k = [3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}, \quad 3y_7 - 18y_9 - 33y_{10} + 3y_{11} - 14y_{12}, \\
18y_6 + 108y_9 - 231y_{10} - 21y_{11} + 98y_{12}, \quad 36y_5 + 27y_{10} - 33y_{11} + 70y_{12}, \\
2y_4 - 5y_{10} + 3y_{11} - 42y_{12}, \quad 12y_3 + 9y_{10} + 3y_{11} - 14y_{12}, \\
9y_2 - 504y_9 - 1158y_{10} - 141y_{11} + 658y_{12}, \quad 3y_1 - 3y_{10} + 6y_{11} - 28y_{12}]
\]
in our case.

The cohomology $H^n_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{\tilde{g}}$ corresponds to the Gröbner basis $GB_k/e$ of the ideal generated by the normal form of $\tilde{g} \in GB_k$ with respect to $GB_e$. In our case, this is given by
\[
GB_{k/e} = [3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}]
\]
$H_{\text{GF}}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ case: In the case of $H_{\text{GF}}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$, we use the notations $\overline{GB}_k$, $\overline{GB}_e$ and $\overline{GB}_{k/e}$ for the Gröbner bases corresponding to the kernel, $d$-image and $H_{\text{GF}}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ respectively. The space $d(\mathfrak{e}^6)$ is characterized by the following Gröbner basis:

$$\overline{GB}_e = \left[ 3y_{10} - 3y_{11} - 20y_{12} + 6y_{14},
100y_8 + 36y_9 - 15y_{11} - 420y_{12} - 420y_{13} + 350y_{14},
300y_7 + 84y_9 - 135y_{11} - 980y_{12} + 420y_{13} + 350y_{14},
100y_6 + 204y_9 - 135y_{11} - 1380y_{12} - 180y_{13} + 750y_{14},
40y_5 - 12y_9 + 15y_{11} - 460y_{12} - 60y_{13} - 590y_{14},
4800y_4 + 84y_9 + 2565y_{11} + 6020y_{12} + 420y_{13} - 10850y_{14},
1600y_3 - 84y_9 + 1035y_{11} - 620y_{12} - 420y_{13} - 5950y_{14},
400y_2 - 12y_9 - 195y_{11} - 1860y_{12} - 5660y_{13} + 950y_{14},
450y_1 + 24y_9 + 315y_{11} + 220y_{12} + 120y_{13} + 1250y_{14} \right]$$

The kernel space of $d : \mathfrak{e}^7 \to \mathfrak{e}^8$ is generated by

$$f_1 = -35c_2 - 30c_6 - 15c_8 + \frac{25}{2}c_9 - \frac{5}{2}c_{13},$$
$$f_2 = 11c_1 - 9c_2 - \frac{39}{8}c_3 - \frac{31}{8}c_4 - \frac{61}{2}c_5 - 75c_6 - 10c_7 - 10c_8 + \frac{85}{2}c_9 - \frac{2}{3}c_{10} - \frac{20}{3}c_{11} - c_{13} - 3c_{14},$$
$$f_3 = -16c_1 + 8c_2 + 9c_3 + 5c_4 + 52c_5 + 20c_6 + \frac{8}{3}c_7 - 2c_8 - \frac{55}{3}c_9 + 8c_{11} + c_{13} + 4c_{14},$$
$$f_4 = \frac{63}{2}c_3 + \frac{21}{2}c_4 + 8c_5 - 14c_{11} + 3c_{12} + 3c_{14}$$

and the kernel space of $d : \mathfrak{e}^7 \to \mathfrak{e}^8$ is characterized by the following Gröbner basis.

$$\overline{GB}_k = \left[ 3y_{10} - 3y_{11} - 20y_{12} + 6y_{14},
y_8 - 15y_{11} - 102y_{12} - 6y_{13} + 32y_{14},
y_7 - 36y_{11} - 238y_{12} + 70y_{14},
y_6 - 171y_{11} - 1136y_{12} - 24y_{13} + 338y_{14},
y_5 + 51y_{11} + 280y_{12} - 154y_{14},
y_4 - 3y_{11} - 56y_{12} - 14y_{14},
y_3 + 45y_{11} + 168y_{12} - 126y_{14},
y_2 + 3y_{11} + 14y_{12} - 56y_{13},
y_1 - 3y_{11} - 28y_{12} + 14y_{14} \right]$$

thus, $H_{\text{GF}}^7(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$ is characterized by

$$\overline{GB}_{k/e} = \left[ 12y_9 + 495y_{11} + 3260y_{12} + 60y_{13} - 950y_{14} \right]$$

Take $h(y) = 3y_8 - 36y_9 - 72y_{10} - 3y_{11} + 14y_{12}$ from $GB_{k/e}$ of (5). Now $h(w)$ is in $\ker(d_0 : C^5 \to C^6) \setminus d_0(C^4)$. We express the following element

$$\omega \wedge h(w) = z_1^0 \wedge z_1^1 \wedge h(w)$$

by the basis of $\mathfrak{e}^7$. We see that

$$\omega \wedge h(w) = z_1^0 \wedge z_1^1 \wedge h(w) = -9\overline{w}_7 + 105\overline{w}_{10} + 3\overline{w}_{11} + 14\overline{w}_{12} = \overline{h}(w)$$

where $\overline{h} = -9y_7 + 105y_{10} + 3y_{11} + 14y_{12}$. The normal form of $\overline{h}$ with respect to $\overline{GB}_e$ is

$$\overline{h} = \frac{63}{25}y_9 + \frac{2079}{20}y_{11} + \frac{3423}{5}y_{12} + \frac{63}{5}y_{13} - \frac{399}{2}y_{14}$$

and is not zero. This finishes the proof of the Theorem.
Remark 5.1. We emphasize that everything starts from the concrete bases of cochain complexes $C_4^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{10}$, $C_5^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{10}$, $C_6^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{10}$, $C_7^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{8}$.

Even though we make use of Gröbner Base theory or use of classical linear algebra argument, we are based on some concrete matrix representations.

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Appendix A

In this Appendix, we make use of Risa/Asir, which is another Symbol Calculus Software, and show the results we got by Maple and Risa/Asir are the same up to non-zero scalar multiples. We remark that we added some line breaks so that we get better look.

Basis of $d_0 (C_4^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{10}) \subset C_5^{GF}(\text{ham}^0_2, \text{sp}(2, \mathbb{R}))_{10}$:

Our source file for Risa/Asir is this:

```plaintext
/* ##### On C^4 -> C^5 #### */ 
G1 = -135/4*y1-60*y3+15/2*y4-45*y5-15*y6+5/4*y7-45/4*y8+75/2*y9$
G2 = 108/11*y1+18/11*y2+60/11*y6+46/11*y7-90/11*y8+156/11*y9$
G3 = 27/4*y1+12*y3-9/2*y4-9*y5+27/4*y10+18*y11$
G4 = -10*y3+2/3*y4-2*y5+2*y6+y7+6*y9+4*y10-y11$
G5 = 5/2*y3+29*y4+47/3*y4+23*y5+43*y6+13/2*y7+9/2*y8+25*y9+16*y10-71/2*y11$
G6 = 5*y2+45*y3+155/6*y4+40*y5+65*y6+10*y7+50*y9+20*y10-115/2 *y11$
G7 = 3/2*y2+18*y3+23/2*y4-3*y5+30*y6+11/2*y7+9/2*y8+9*y9+6* y10-33*y11$
G8 = 6*y6+7*y7-6*y9$
G9 = -6*y3-3*y4+3*y10-6*y11+70*y12$
GBe = gr([G1,G2,G3,G4,G5,G6,G7,G8,G9], [y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12], 1) ;
*/
```

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The output of Groebner Basis is the next:

\[-140*y_{12}+30*y_{11}-15*y_{10}-18*y_9-9*y_8+21*y_7,\]
\[140*y_{12}-30*y_{11}+15*y_{10}+9*y_8-18*y_6,\]
\[-3290*y_{12}+1461*y_{11}+666*y_{10}+900*y_9-75*y_8-1512*y_5,\]
\[-770*y_{12}+57*y_{11}-18*y_{10}+36*y_9-3*y_8+36*y_4,\]
\[70*y_{12}-15*y_{11}+18*y_{10}+36*y_9-3*y_8-72*y_3,\]
\[3080*y_{12}-660*y_{11}-258*y_{10}+396*y_9-327*y_8+63*y_2,\]
\[-1820*y_{12}+390*y_{11}+99*y_{10}+144*y_9-12*y_8+189*y_1\]

Kernel space of \(d_0\) : \(\mathbb{C}_{5}^{6}\) \(\text{GF}(\text{ham}^0, \text{sp}(2,\mathbb{R}))_{10} \rightarrow \mathbb{C}_{6}^{6}\) \(\text{GF}(\text{ham}^0, \text{sp}(2,\mathbb{R}))_{10}\):

Our source file for Risa/Asir is this:

```plaintext
/* ##### On C^5 \rightarrow C^6 #### */
F1 = -5*w2-16*w3$
F2 = 140*w1-4*w2+32*w3$
F3 = 1/4*w2-2*w3$
F4 = -15*w1+31/6*w2+58/3*w3$
F5 = 15*w1-3*w2-18*w3$
F6 = 30*w1-2*w2-12*w3$
F7 = 5/2*w1+5/3*w2-5/3*w3$
F8 = -w2$
F9 = 2*w2+8*w3+14*w4$
F10 = 168*y_{12}+28*y_{11}+112*y_{10}+128*y_{9}+182*y_{8}+189*y_{1}

FList = [F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,F11,F12]$
WList = [w1,w2,w3,w4]$

/* ########################################################## */
NagawaW = length(WList)$
NagasaF = length(FList)$
CC = [c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12]$
YY = [y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12]$
for ( Uke = [], J=1; J <= NagawaW; J++ ) { MyA = WList[J-1]; Atai = 0;
for (K=1 ; K <= NagasaF; K++ ){ MyB = FList[K-1];
Atai += diff( MyB, MyA)* CC[K-1];}; Uke = cons(Atai, Uke );
}
print("mark A")$
Uke = reverse( Uke );
print("mark B")$
GBadj = gr( Uke, CC, 0); /* Groebner Basis */
for (H=0, I=1; I <= NagasaF; I++) { H += CC[I-1]* YList[I-1]; }
Hnf = p_nf(H, GBadj, CC, 0)$ /* Normal Form */
for( MyUkez = [], T=CC; T != []; T = cdr(T)){
    MyA = car(T); MyV = diff( Hnf, MyA); MyUkez = cons( MyV, MyUkez );
}
print("mark C")$
MyUkez = reverse(MyUkez);$
print("mark D")$
GBk = gr( MyUkez, YY, 0); /* Groebner Basis */
end$
```

The outputs are the follows:

**mark A**

\([-140*y_{12}+30*y_{11}-15*y_{10}-18*y_9-9*y_8+21*y_7,\]
\[140*y_{12}-30*y_{11}+15*y_{10}+9*y_8-18*y_6,\]
\[-3290*y_{12}+1461*y_{11}+666*y_{10}+900*y_9-75*y_8-1512*y_5,\]
\[-770*y_{12}+57*y_{11}-18*y_{10}+36*y_9-3*y_8+36*y_4,\]
\[70*y_{12}-15*y_{11}+18*y_{10}+36*y_9-3*y_8-72*y_3,\]
\[3080*y_{12}-660*y_{11}-258*y_{10}+396*y_9-327*y_8+63*y_2,\]
\[-1820*y_{12}+390*y_{11}+99*y_{10}+144*y_9-12*y_8+189*y_1]\n
**mark B**

\[-42*x_{12}-28*x_{11}+198*x_{10}+228*x_9-117*x_8+48*x_7-14*x_6-128*x_5+56*x_4+6*x_3+68*x_2+8*x_1,\]
\[168*y_{12}-112*y_{11}+182*y_{10}+126*y_{9}+784*y_{8}-35*c9+24*c10-128*c11-12*c12]\n
**mark C**

\[0, 0, 0, 0,\]
\[-168*y_{5}+28*y_{4}+112*y_{3}-112*y_{1},\]
\[-168*y_{6}-456*y_{3}-18*y_{2}-182*y_{1},\]
Basis of $H_{GF}^5(ham^0_2, sp(2, \mathbb{R}))_{10}$

The next is a source file for Risa/Asir. GBe and GBk are data gotten above.

GBe = [-140*y12-30*y11+15*y10-18*y9-9*y8+21*y7, 140*y12-30*y11+15*y10+9*y8+18*y6, -3290*y12+1461*y11+666*y10+900*y9-75*y8-1512*y5, -770*y12+1461*y11+15*y10+36*y9+36*y7+90*y6, 70*y12-1461*y11+18*y10+36*y9-3*y8+36*y4, 3080*y12-660*y11-258*y10+396*y9-327*y8+63*y2, -1820*y12+390*y11+99*y10+144*y9-12*y8+189*y1]

GBk = [ 14*y12-3*y11-72*y10+36*y9-3*y8, 14*y12-3*y11+33*y10+18*y9-3*y7, 98*y12-21*y11+231*y10+108*y9+18*y6, 70*y12-33*y11+27*y10+36*y5, 42*y12-3*y11+5*y10-2*y4, 14*y12-3*y11-9*y10-12*y3, -658*y12+141*y11+1158*y10+504*y9-9*y2, 28*y12-6*y11+3*y10-3*y1]

YY = [y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12]

for(Uke=[], T= GBk; T != [] ; T = cdr(T)) {
    MyA = car(T); Atai = p_nf( MyA, GBe, YY , 0) ; /* NormalForm */
    Uke = cons(Atai, Uke); }

Uke = reverse(Uke)$

GBh = gr( Uke, YY , 0); /* Groebner Basis */

A basis of $H_{GF}^5(ham^0_2, sp(2, \mathbb{R}))_{10}$ is given by the output

[-14*y12+3*y11+72*y10+36*y9-3*y8]

Check $d_0 \circ d_0 : C_{GF}^4(ham^0_2, sp(2, \mathbb{R}))_{10} \rightarrow C_{GF}^0(ham^0_2, sp(2, \mathbb{R}))_{10}$ is zero identically:

The next is a source file for Risa/Asir.

/* Feb 07, 2014 n=1, type 1, weight 10, C^4 -- C^5 -- C^6 */
G1 = -135/4*y1-60*y3+15/2*y4-45*y5-15*y6+5/4*y7-45/4*y8+75/2*y9$
G2 = 108/11*y1+18/11*y2+60/11*y6+46/11*y7-90/11*y8+156/11*y9$
G3 = 27/4*y1+12*y3-9/2*y4-9*y5+27/4*y10+18*y11$
G4 = -10*y3+2/3*y4-2*y5+2*y6+y7+6*y9+4*y10-11*y11$
G5 = 5/2*y2+29*y3+47/3*y4-23*y5+43*y6+13/2*y7+9/2*y8+25*y9+16 *y10-71/2*y11$
G6 = 5*y2+45*y3+155/6*y4-40*y5+65*y6+10*y7+50*y9+20*y10-115/2 *y11$
G7 = 3/2*y2+18*y3+23/2*y4-3*y5+30*y6+11/2*y7+9/2*y8+9*y9+6* y10-33*y11$
G8 = 6*y6+7*y7-6*y9$
G9 = -6*y3-3*y4+3*y10-6*y11+70*y12$

/* The next data are gotten by replacing y to F and G to GG in the above. */
GG1 = -135/4*F1-60*F3+15/2*F4-45*F5-15*F6+5/4*F7-45/4*F8+75/2*F9$
GG2 = 108/11*F1+18/11*F2+60/11*F6+46/11*F7-90/11*F8+156/11*F9$
GG3 = 27/4*F1+12*F3-9/2*F4-9*F5+27/4*F10+18*F11$
GG4 = -10*F3+2/3*F4-2*F5+2*F6+F7+6*F9+4*F10-F11$
GG5 = 5/2*F2+29*F3+47/3*F4-23*F5+43*F6+13/2*F7+9/2*F8+25*F9+16*F10-71/2*F11$
GG6 = 5*F2+45*F3+155/6*F4-40*F5+65*F6+10*F7+50*F9+20*F10-115/2*F11$
GG7 = 3/2*F2+18*F3+23/2*F4-3*F5+30*F6+11/2*F7+9/2*F8+9*F9+6*F10-33*F11$
GG8 = 6*F6+7*F7-6*F9$
GG9 = -6*F3-3*F4+3*F10-6*F11+70*F12$

/* ### On $C^5 \rightarrow C^6$: */
F1 = -5*w2-16*w3
F2 = 140*w1-4*w2+32*w3
F3 = 1/4*w2-2*w3
F4 = -11/2*w2-12*w3+42*w4
F5 = 31/12*w2+22/3*w3+7*w4
F6 = -15*w1-31/6*w2-12*w3
F7 = 5/2*w1+5/3*w2-5/3*w3
F8 = 30*w1-2*w2-12*w3
F9 = 5/2*w1+5/3*w2-5/3*w3
F10 = -w2
F11 = 2*w2+8*w3+14*w4
F12 = 3*w4

/* ########################################## */
L = [GG1,GG2,GG3,GG4,GG5,GG6,GG7,GG8,GG9]
print(L)$ end$

The output is as expected.

[0,0,0,0,0,0,0,0,0]

Appendix B

In section 5, we have shown a basis of $H^7_{GF}(\text{ham}_2, sp(2, \mathbb{R}))$ concretely by Groebner Basis Package of Maple.

In this Appendix, we do the same job by using Risa/Asir, which is another Symbol Calculus Software, and show that the results we got by Maple and Risa/Asir are the same up to non-zero scalar multiples. We remark that in this note we added some line breaks so that we get better look and we use nd_gr() instead of gr().

We stock two matrix representations of $d$ in the two files:

Mat_w8_6and7_type0.rr
Mat_w8_7and8_type0.rr

Basis of $d$ $(C^6_{GF}(\text{ham}_2, sp(2, \mathbb{R}))) \subset C^7_{GF}(\text{ham}_2, sp(2, \mathbb{R}))$:
Our source file for Risa/Asir is this:

load("./Mat_w8_6and7_type0.rr")$ /* GB1 = gr( [ ], [ ], 0) */
ord( YList )$
GB1 = reverse(
nd_gr(GList , YList ,0,0))
print(["GBe",GB1])$
end$

The output of Groebner Basis is the next:

[GBe, [3*y10-3*y11-20*y12+6*y14, 100*y8+36*y9-15*y11-420*y12-420*y13+350*y14, 
-300*y7-84*y9+135*y11+980*y12-420*y13-350*y14, 
100*y6+204*y9-135*y11-1380*y12-180*y13+750*y14, 
40*y5-12*y9+15*y11-460*y12-60*y13-590*y14, 
4800*y4+84*y9+2565*y11+6020*y12+420*y13-10850*y14, 
1600*y3-84*y9+1035*y11-6020*y12-420*y13-5950*y14, 
400*y2-12*y9-95*y11-1860*y12-5660*y13+950*y14, 
-450*y1-24*y9-315*y11-220*y12-120*y13-1250*y14]]
Kernel space of $d : C^7_{GF}(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8 \to C^8_{GF}(\text{ham}_2, \text{sp}(2, \mathbb{R}))_8$:

Our source file for Risa/Asir is this:

```plaintext
load("Mat_w8_7and8_type0.rr")$ NagasaW = length(WList)$ NagasaF = length(FList)$
for ( Uke = [], J=1; J <= NagasaW; J++ ) { MyA = WList[J-1]; Atai = 0;
    for (K=1 ; K <= NagasaF; K++ ){ MyB = FList[K-1];
        Atai += diff( MyB, MyA) * CList[K-1];}
    Uke = cons(Atai, Uke ); }
print("mark A")$ Uke = reverse( Uke );
print("mark B")$ GBadj = nd_gr( Uke, CList, 0, 0 );
for (H=0, I=1; I <= NagasaF; I++){ H += CList[I-1]* YList[I-1]; }
Hnf = p_nf(H, GBadj, CList, 0)$
for( MyUkez = [], T=CList; T != []; T = cdr(T)){
    MyA = car(T); MyV = diff( Hnf, MyA);
    MyUkez = cons( MyV, MyUkez);}
print("mark C")$ MyUkez = reverse(MyUkez);
ord(YList)$
print("mark D")$ GBk = reverse( nd_gr( MyUkez, YList, 0, 0 ));
end$
```

The outputs are the follows:

```
mark A
[-35*c2-30*c6+15*c8+25/2*c9-5/2*c13, 
-11*c1-9*c2-39/8*c3-31/8*c4-61/2*c5-75*c6-10*c7
-14*c8+85/2*c9-2/3*c10+20/3*c11-13-3*c14, 
-16*c1+8*c2+9*c3+65/2*c4+20*c6+6/3*c7-2*c8-55/3*c9+8*c11+c13+4*c14, 
63/2*c3+21/2*c4+84*c5+14*c11+3*c12+3*c14]
mark B
[-168*c1+336*c5-1260*c6-168*c7-294*c8+525*c9-16*c10+112*c11-12*c12+3*c13+24*c14, 
-14*c2-12*c6-6*c8+5*c9+c13, 
-126*c3-420*c5+2796*c6+392*c7+474*c8-1375*c9+32*c10+84*c11-6*c12+3*c13+6*c14, 
-42*c4+84*c5-2796*c6-392*c7-474*c8+1375*c9-32*c10-28*c11-6*c12-3*c13-18*c14]
mark C
[0, 0, 0, 0, 
1008*y1-1680*y3+1008*y4+504*y5, 
-3780*y1-432*y2+11184*y3-33552*y4+504*y6, 
-504*y1+1568*y3-4704*y4+504*y7, 
-882*y1-216*y2+1896*y3-5688*y4+504*y8, 
1575*y1+180*y2-5500*y3+16500*y4+504*y9, 
-48*y1+128*y3-384*y4+504*y10, 
336*y1+336*y3-336*y4+504*y11, 
-36*y1-24*y3-72*y4+504*y12, 
9*y1-36*y2+12*y3-36*y4+504*y13, 
72*y1+24*y3-216*y4+504*y14]
mark D
[-3*y10+3*y11+20*y12-6*y14, 
-12*y9-495*y11+3260*y12-60*y13+950*y14, 
y8-15*y11-102*y12-6*y13+32*y14, 
3*y7-36*y11-238*y12+70*y14, 
-2*y6+171*y11+1136*y12+24*y13-338*y14, 
4*y5+51*y11+280*y12-154*y14, 
16*y4-3*y11-56*y12-14*y14, 
```
Basis of $H^7_{\text{GF}}(\text{Ham}_2, \text{sp}(2, \mathbb{R}))_8$

The next is a source file for Risa/Asir. $G\text{Be}$ and $G\text{Bk}$ are data gotten above.

\begin{verbatim}
GBe = [ 3*y10-3*y11-20*y12+6*y14,
         100*y8+36*y9-15*y11-420*y12-420*y13+350*y14,
         40*y5-12*y9+15*y11-460*y12-60*y13-590*y14,
        4800*y4+84*y9+2565*y11+6020*y12+420*y13-10850*y14,
       1600*y3-84*y9+1035*y11-6020*y12-420*y13-5950*y14,
       400*y2-12*y9-195*y11-1860*y12-5660*y13+950*y14,
      -450*y1-24*y9-315*y11-220*y12-120*y13-1250*y14 ]$

GBk = [ -3*y10+3*y11+20*y12-6*y14,
         -12*y9-495*y11-3260*y12-60*y13+950*y14,
        3*y7-36*y11-238*y12+70*y14,
         -2*y6+171*y11+1136*y12+24*y13-338*y14,
         4*y5+51*y11+280*y12-154*y14,
        16*y4-3*y11-56*y12-14*y14,
        16*y3+45*y11+168*y12-126*y14,
        4*y2+3*y11+14*y12-56*y13,
         -2*y1+3*y11+28*y12-14*y14 ]$

YList = [y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14]$
\end{verbatim}

for(Uke=[], T = GBk; T != []; T = cdr(T)) {
    MyA = car(T); /* print(MyA); */
    Atai = p_nf( MyA, GBe, YList , 0) ; Uke = cons(Atai, Uke); }  
Uke = reverse(Uke)$
ord(YList)$
GBe = reverse( nd_gr( Uke, YList , 0, 0) ); end$

A basis of $H^7_{\text{GF}}(\text{Ham}_2, \text{sp}(2, \mathbb{R}))_8$ is given by the output

$[-12*y9-495*y11-3260*y12-60*y13+950*y14]$

We may omit the job of $\text{Check } d \circ d : C^6_{\text{GF}}(\text{Ham}_2, \text{sp}(2, \mathbb{R}))_8 \rightarrow C^6_{\text{GF}}(\text{Ham}_2, \text{sp}(2, \mathbb{R}))_8$ is identically zero.

Appendix C Final stage by Risa/Asir

We have already studied of $H^5_{\text{GF}}(\text{Ham}_0, \text{sp}(2, \mathbb{R}))_{10}$, $G\text{Be}_e$, $G\text{Bk}$ and $G\text{Bk}/e$, also of $H^7_{\text{GF}}(\text{Ham}_2, \text{sp}(2, \mathbb{R}))_8$, $G\text{Be}_e$, $G\text{Bk}$ and $G\text{Bk}/e$.

We calculate $\omega \wedge h(w_j)$ and we have $\widetilde{h} = -9y_7 + 105y_{10} + 3y_{11} + 14y_{12}$.

We will check NormalForm($\widetilde{h}$, $G\text{Be}_e$, Ord$_p$) does not vanish. Then a proof to a Theorem in The Gel’fand-Kalinin-Fuks class and characteristic classes of transversely symplectic foliations, arXiv:0910.3414, October 2009 by D. Kotschick and S. Morita will be done.

We remark that in this note we added some line breaks so that we get better look and we use $\text{nd}\_\text{gr}()$ instead of $\text{gr}()$.  

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Final stage:
Our source file for Risa/Asir is this:

$$YList = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}\}$$

ord(YList)

$$GBe = [3y_{10} - 3y_{11} - 20y_{12} + 6y_{14},
100y_8 + 36y_9 - 15y_{11} - 420y_{12} - 420y_{13} + 350y_{14},
-300y_7 - 84y_9 + 135y_{11} + 980y_{12} - 420y_{13} - 350y_{14},
100y_6 + 204y_9 - 135y_{11} - 1380y_{12} - 180y_{13} + 750y_{14},
40y_5 - 15y_{11} + 460y_{12} - 60y_{13} - 590y_{14},
4800y_4 + 84y_9 + 2565y_{11} + 6020y_{12} + 420y_{13} - 10850y_{14},
1600y_3 - 84y_9 + 2565y_{11} - 6020y_{12} - 420y_{13} - 5950y_{14},
400y_2 - 12y_{11} + 195y_{12} - 180y_{13} + 5660y_{14} + 950y_{14},
-450y_1 + 24y_9 - 315y_{11} - 220y_{12} - 120y_{13} - 1250y_{14}]$$

$$pnf(H, GBe, YList, 0);$$
end$

The output of Groebner Basis is the next:

$$-252y_9 - 10395y_{11} - 68460y_{12} + 19950y_{14}$$