We investigate exact solutions and the asymptotic dynamics for the Friedmann–Lemaître–Robertson–Walker universe with nonzero spatial curvature in the fourth-order modified teleparallel gravitational theory known as $f(T, B)$ theory. We show that the field equations admit a minisuperspace description, and they can reproduce any exact form of the scale factor. Moreover, we calculate the equilibrium points and analyze their stability. We show that Milne and Milne-like solutions are supported, and the de Sitter universe is provided. To complete our analysis, we use Poincaré variables to investigate the dynamics at infinity.

**KEYWORDS**
closed universe, modified gravity, open universe, teleparallel cosmology

**MSC CLASSIFICATION**
83F05, 83D05, 34C60

1 | INTRODUCTION

Cosmologists have widely studied alternative and modified theories of gravity in recent years\(^1\)\(^-\)\(^3\) because geometrodynamical degrees of freedom are introduced into the gravitational Action Integral to explain recent cosmological observations.\(^4\)\(^-\)\(^9\) In particular, geometric invariants are used to modify the Einstein–Hilbert Action. See for instance previous works\(^1\)\(^-\)\(^2\),\(^10\)\(^-\)\(^12\) and references therein.

The fundamental geometric invariant function of General Relativity is the Ricciscalar $R$. The Levi–Civita connection defines $R$. However, Einstein showed that if the fundamental connection curvature-less Weitzenböck connection\(^13\) and the torsion scalar $T$ are used for the definition of the gravitational theory, then the resulting theory is equivalent to General Relativity known as the teleparallel equivalence of General Relativity (TEGR).\(^14\)\(^,\)\(^15\) There is a plethora of modified theories inspired by teleparallelism with many interesting results in cosmology and astrophysics.\(^16\)\(^-\)\(^24\) For reviews in teleparallelism, we refer the reader to references.\(^25\)\(^,\)\(^26\)
We are interested in the fourth-order teleparallel theory of gravity known as the $f(T, B)$ theory. The gravitational Action Integral is defined by an arbitrary function $f$ of the torsion scalar $T$ and of the boundary term $B$, which is related to the torsion scalar and the Ricci scalar, that is, $B = R + T$. The theory was introduced in detail in Bahamonde et al.\textsuperscript{27} However, a similar fourth-order teleparallel theory was introduced before in Myrzakulov.\textsuperscript{26} There are various studies in the literature on $f(T, B)$ theory. The cosmological dynamics in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe were investigated in detail in a series of works\textsuperscript{29–32} where it was found that for a plethora of functions $f$ the modified teleparallel theory can describe the main eras of the cosmological history. Exact and analytic solutions were found in Paliathanasis and Karpathopoulos et al.\textsuperscript{33,34} while some bouncing solutions were determined in Caruana et al.\textsuperscript{35} Recently, a new inhomogeneous exact solution was derived in Najera et al.\textsuperscript{36} The quantization of $f(T, B)$ by using the minisuperspace description to write the Wheeler–DeWitt equation of quantum cosmology was studied in Paliathanasis.\textsuperscript{37} Cosmological constraints of $f(T, B)$ theory can be found in previous studies\textsuperscript{38–40} while some astrophysics applications are presented in previous works.\textsuperscript{41–46} Anisotropic spacetimes in $f(T, B)$ theory investigated before in previous studies.\textsuperscript{47–49} Specifically, the dynamical evolution for the physical parameters investigated in the case of Bianchi I,\textsuperscript{47} Kantowski-Sachs,\textsuperscript{48} and Bianchi III\textsuperscript{49} background geometries.

In the following, we consider in $f(T, B)$ theory in the background space of an FLRW universe with nonzero spatial curvature. It has been found that the inflationary scenario is not affected by the presence of negative curvature in the background space.\textsuperscript{50,51} Thus, such an analysis is important for studying teleparallelism in the very early stages of the universe. We investigate the existence of power-law and exponential scale factors in the $f(T, B)$ theory, which can describe inflation. An accelerated expansion of the universe described by the exponential scale factor solves various problems in cosmology, such as the “flatness,” “horizon” and monopole problems.\textsuperscript{52,53} A recent study on the effects of curvature in inflation. An accelerated expansion of the universe described by the exponential scale factor solves various problems in cosmology.\textsuperscript{54–56} The analysis of the asymptotic behavior for the theory is essential for a better understanding of the viability of the model.\textsuperscript{50,61}

The plan of the paper is as follows.

In Section 2, we present the basic properties of teleparallelism and we give the gravitational field equations for the $f(T, B)$ theory. Moreover, for the case of FLRW geometries, we derive the minisuperspace description for the field equations and write the point-like Lagrangian using a Lagrange multiplier. For the case of $f(T, B) = T + F(B)$ in Section 3, we prove that the gravitational theory supports exact solutions of interest, specifically those with power-law and exponential scale factor. The stability properties of these important solutions are investigated in Section 4. In particular, we investigate the global evolution of the field equations by investigating the equilibrium points and their stability properties. We discuss our results in Section 5.

## 2 | $F(T,B)$ GRAVITY

The fundamental geometric objects in teleparallelism are the vierbein fields $e^\mu(x^\nu)$. The vierbein fields introduce the dynamical variables of the theory, and they form an orthonormal basis for the tangent space at each point $P$ such that $g(e^\mu, e^\nu) = e^\mu \cdot e^\nu = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the line element of the Minkowski spacetime, $\eta_{\mu\nu} = \text{diag}(\ldots, +, +, +)$.

Furthermore, for the vierbein fields, it holds that

$$[e^\mu, e^\nu] = c^\mu_{\nu\rho} e^\rho \quad \text{where} \quad c^\mu_{\nu\rho} = 0. \quad (1)$$

In general, in the nonholonomic coordinates, the covariant derivative $\nabla_\nu$ is defined by the nonsymmetric connection

$$\hat{\nabla}_\nu = \{^{\mu}_{\nu\beta} \} + \frac{1}{2} \eta^{\alpha\sigma}(c_{\nu\sigma,\beta} + c_{\nu\beta,\sigma} - c_{\mu\beta,\sigma}), \quad (2)$$

in which $\{^{\mu}_{\nu\beta} \}$ is the symmetric Levi–Civita connection of Riemannian geometry.

When $e^\mu \cdot e^\nu = \eta_{\mu\nu}$, it follows that\textsuperscript{13}

$$\hat{\nabla}_{\nu\beta} = \frac{1}{2} \eta^{\alpha\sigma}(c_{\nu\sigma,\beta} + c_{\nu\beta,\sigma} - c_{\mu\beta,\sigma}), \quad (3)$$
where now $\tilde{\Gamma}^{\mu}_{\nu\rho}$ describes the Ricci rotation coefficients. $\tilde{\Gamma}^{\mu}_{\nu\rho}$ has the property that it is antisymmetric in the first two indices, that is, $\tilde{\Gamma}^{\mu}_{\nu\rho} = -\tilde{\Gamma}^{\mu}_{\rho\nu}$, with $\tilde{\Gamma}^{\mu}_{\nu\rho} = \eta_{\mu\nu} \tilde{\Gamma}^{\rho}_{\nu\rho}$.

Consequently, the non-null torsion tensor can be defined as $T^{\beta}_{\mu\nu} = \tilde{\Gamma}^{\beta}_{\nu\mu} - \tilde{\Gamma}^{\beta}_{\mu\nu}$, with the scalar $T = S^{\beta\mu\nu}T^{\beta}_{\mu\nu}$. The geometric object $S^{\beta\mu\nu}$ is given by the expression

$$S^{\beta\mu\nu} = \frac{1}{2}(K^{\mu\nu}_\beta + \delta^{\mu}_{\beta}T^{\theta\nu}_\theta - \delta^{\nu}_{\beta}T^{\theta\mu}_\theta),$$

(4)

with $K^{\mu\nu}_\beta = -\frac{1}{2}(T^{\mu\nu}_\beta - T^{\nu\mu}_\beta - T^{\theta\mu}_\theta)$.

In the teleparallel equivalent of General Relativity, the fundamental scalar is the torsion scalar $T$, instead of the Ricci-scalar and the dynamical variables are the vierbein fields, instead of the metric tensor, that is, the gravitational Action Integral is

$$S_T = \frac{1}{16\pi G} \int d^4x T,$$

(5)

In this study, we are interested in an extension of the teleparallel theory known as $f(T, B)$ gravity. Specifically, we consider the modified Action Integral

$$S_{f(T, B)} = \frac{1}{16\pi G} \int d^4x f(T, B),$$

(6)

where $B = 2e^{-1}\delta_i (eT^\nu_{\rho})$ corresponds to the boundary term which relates the torsion scalar $T$ with the Ricciscalar $R$, that is $R = -T + B$.

Function $f(T, B)$ is an arbitrary function which should be defined. In the case for which $f(T, B)$ is a linear function. Then, the field equations of the General Relativity are recovered, with or without the cosmological constant term. Moreover, for $f(T, B) = f(-T + B)$, another well-known theory of gravity is recovered, the so-called fourth-order modified $f(R)$ theory.

The gravitational field equations follow from the variation of the Action Integral (6) with respect to the vierbein fields. They are

$$4\pi G T^{(m)\lambda}_a = \frac{1}{2}e h^\mu_\lambda(f_B)^{\mu\nu}_\lambda g_{\mu\nu} - \frac{1}{2}e h^\mu_\lambda(f_B)^{\mu\sigma}_{,\sigma} + \frac{1}{4}e f_B S^\mu_{,\mu} f_T + e((f_B)_{,\mu} + (f_T)_{,\mu}) S^\mu_{,\mu} - e f_T T^{\sigma\mu} S^\mu_{,\sigma},$$

or equivalently

$$4\pi G T^{(m)\lambda}_a = e f_T G^\lambda_a + \left[\frac{1}{4} (T f_T - f) e h^\mu_\lambda + e(f_T)_{,\mu} S^\mu_{,\mu}\right]$$

$$+ \left[e((f_B)_{,\mu} S^\mu_{,\mu} - \frac{1}{2} e \left(h^\mu_\lambda(f_B)^{\mu\sigma}_{,\sigma} - h^\lambda_\mu(f_B)^{\mu\nu}_{,\nu}\right) g_{\mu\nu} + \frac{1}{4} eB h^\lambda_\mu(f_B)\right].$$

(7)

(8)

### 2.1 Minisuperspace description

For the FLRW universe with nonzero curvature, that is, for the line element

$$ds^2 = N^2 dt^2 - a^2(t) \left( dr^2 + \sin^2 (r) \left( d\theta^2 + \sin^2 (\theta) d\phi^2 \right) \right), K = 1,$$

(9)

or

$$ds^2 = N^2 dt^2 - a^2(t) \left( dr^2 + \sinh^2 (r) \left( d\theta^2 + \sin^2 (\theta) d\phi^2 \right) \right), K = -1,$$

(10)

we consider the vierbein fields

$$e_\mu = (N dt, a(t) E^t(K), a(t) E^\phi(K), a(t) E^\phi(K)),$$

where the components of $E(K)$ depend upon the spatial curvature $K$ and, for simplicity, in the following, we set the lapse function $N = 1$.

For the “closed” universe, i.e. $K = 1$, we have

$$E^t(K = 1) = -\cos (\theta) dr + \sin (r) \sin (\theta) (\cos (r) d\theta - \sin (r) \sin (\theta) d\phi),$$

(11)

$$E^\phi(K = 1) = \sin (\theta) \cos (\phi) dr - \sin (r) \sin (\phi) (\cos (r) \cos (\theta) \sin (\phi)) d\theta$$

$$- \sin (r) \sin (\theta) (\cos (r) \sin (\phi) + \sin (r) \cos (r) \cos (\theta) \sin (\phi)) d\phi,$$

(12)

$$E^\phi(K = 1) = -\sin (\theta) \sin (\phi) dr - \sin (r) \cos (\phi) (\cos (r) \sin (\theta) (\cos (\phi) + \cos (r) \cos (\theta) \sin (\phi)) d\theta$$

$$- \sin (r) \sin (\theta) (\cos (r) \cos (\phi) - \sin (r) \cos (\theta) \sin (r) \sin (\phi)) d\phi.$$
For the “open" universe, that is, \( K = -1 \), it holds that

\[
E'(K = -1) = \cos(\theta)dr + \sinh(r)\sin(\theta)(-\cosh(r)d\theta + i\sinh(r)\sin(\theta)d\phi),
\]

\[
E^\theta(K = -1) = -\sin(\theta)\cos(\phi)dr + \sinh(r)(i\sinh(r)\sin(\phi) - \cos(r)\cos(\theta)\cos(\phi))d\theta
+ \sinh(r)\sin(\phi)(\cosh(r)\sin(\phi) + i\sinh(r)\cos(\theta)\cos(\phi))d\phi,
\]

\[
E^\phi(K = -1) = \sin(\theta)\sin(\phi)dr + \sinh(r)(i\sinh(r)\cos(\phi) + \cosh(r)\cos(\theta)\sin(\phi))d\theta
+ \sinh(r)\sin(\theta)(\cosh(r)\cos(\phi) - \sinh(r)\cos(\theta)\sin(\phi))d\phi.
\]

Thus, for this frame, the torsion scalar \( T \) is calculated as

\[
T = 6\left(\frac{K}{a^2} - H^2\right),
\]

where \( H = \frac{\dot{a}}{a} \) is the Hubble function \( \dot{a} = \frac{da}{dt} \), while the boundary term is

\[
B = -6\left(\dot{H} + 3H^2\right).
\]

We introduce the Lagrange multipliers \( \lambda_1, \lambda_2 \). Thus, the Action Integral (6) for a the FLRW spacetime is written in the equivalent form

\[
S_{f(T,B)} = \frac{1}{16\pi G} \int dt \left( a^3 f(T,B) - \lambda_1 a^3 \left( T - 6\left(\frac{K}{a^2} - H^2\right)\right) - \lambda_2 a^3 \right( B + 6 \left(\dot{H} + 3H^2\right)\right).
\]

Variation with respect to the variables \( T \) and \( B \) of (19) constrain the Lagrange multiplier. Indeed, from the equations of motion \( \frac{\delta}{\delta T} \left( S_{f(T,B)} \right) = 0 \) and \( \frac{\delta}{\delta B} \left( S_{f(T,B)} \right) = 0 \), it follows that \( \lambda_1 = f,T \) and \( \lambda_2 = f,B \). Consequently, expression (19) becomes

\[
S_{f(T,B)} = \frac{1}{16\pi G} \int dt \left( a^3 f(T,B) - f,T \left( a^3 T - 6\left( aK - a\dot{a}^2\right)\right) - f,B \left( B + 12a^2 - 6f,Ba^2\dot{a}\right).\right.
\]

Integration by parts of the last term of (20) gives

\[
\int dt \left( 6f,Ba^2\dot{a}\right) = -\int dt \left( 12f,Ba^2\dot{a}^2 + 6a^2 f,BBa\dot{B}\right).
\]

Thus, we can write the point-like Lagrangian function

\[
L(a, \dot{a}, T, B, \dot{B}) = -6f,Ta\dot{a}^2 - 6a^2 f,BBa\dot{B} + a^3 \left( f - T f,T - B f,B\right) + 6aK f,T,
\]

which generates the gravitational field equations. We remark that \( L(a, \dot{a}, T, B, \dot{B}) \) provides field equations of second-order. However, \( B \) has been introduced by a Lagrange multiplier and includes the higher order derivatives, such that \( f(T,B) \) theory is to be of fourth order.

### 2.2 \( f(T,B) = T + F(B) \) theory

We proceed with our analysis by assuming the functional form \( f(T,B) \) to be linear in \( T \), that is, \( f(T,B) = T + F(B) \). A such function has been considered before in Paliathanasis.\(^29\) The main mathematical novelty of this approach is that the point-like Lagrangian (22) is regular, while for small values of function \( F(B) \), we are very close to the limit of General Relativity. In such consideration, we assume a modification of the Action Integral for the TEGR, which follows from the existence of the boundary function \( B \).

Now, we introduce the new field, and potential definition

\[
\phi = F_B, \text{ and } V(\phi) = (F - BF_B) / 6.
\]

(23)
Given an explicit form $V(\phi)$, we reconstruct $f(B)$ from the singular solution of Clairaut’s equation. Therefore, the Lagrangian of the field equations is

$$\mathcal{L} (a, a, \phi, \dot{\phi}) = \frac{1}{N} \left[-6a\dot{a}^2 - 6a^2\ddot{a}\phi\right] + N \left[6a^3V(\phi) + 6aK\right], \quad (24)$$

whereby convenience, we have reinserted the lapse function $N$.

Taking the variation of (24) with respect to $(a, \phi, N)$, we derive the field equations

$$2\dot{H} + 3H^2 + \dot{\phi} + 3V(\phi) + Ka^{-2} = 0, \quad (25)$$

with constraint equation

$$\dot{H} + 3H^2 + V'(\phi) = 0, \quad (26)$$

and

$$H^2 + H\dot{\phi} + V(\phi) + Ka^{-2} = 0, \quad (27)$$

where Equation (25) is provided by the Euler–Lagrange equations with respect to the scale factor $a$, $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} - \frac{\partial \mathcal{L}}{\partial a} = 0$. The scalar field Equation (26) arises from $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$. Finally, the equation $\partial \mathcal{L}/\partial N|_{N=1} = 0$ gives the Friedmann constrain (27). As usual, in all the above equations, one can set $N = 1$ after the derivations.

The field equations (25) and (27) can be written in the equivalent form:

$$3H^2 + 3Ka^{-2} = \rho_\phi, \quad (28)$$

$$2\dot{H} + 3H^2 + Ka^{-2} = -p_\phi, \quad (29)$$

where $\rho_\phi$ and $p_\phi$ are the cosmological fluid components that correspond to the geometrodynamical degrees of freedom given by the nonlinear $F(B)$ function. They are

$$\rho_\phi = -3 \left(H\dot{\phi} + V(\phi)\right), \quad (30)$$

$$p_\phi = \dot{\phi} + 3V(\phi). \quad (31)$$

Thus, the equation of state parameter for the geometric fluid source is defined as

$$w_\phi = -\frac{\frac{1}{3}\dot{\phi} + V(\phi)}{H\dot{\phi} + V(\phi)}. \quad (32)$$

We observe that, when $\dot{\phi} \approx 0$ and $H \approx 0$, it follows $w_\phi \approx -1$; that is, the limit of the cosmological constant is recovered. The dust matter domination is provided by $\dot{\phi} + 3V(\phi) \approx 0$. When $V(\phi) \approx 0$, we have $w_\phi \approx -\frac{1}{3}\dot{\phi}/(H\dot{\phi})$. Therefore, radiation-dominated epoch corresponds to $V(\phi) \approx 0$ together with $\dot{\phi}/(H\dot{\phi}) \approx -1$. That is, $V(\phi) \approx 0$ and $d \ln(\dot{\phi})/d \ln a \approx -1$; that is, $\dot{\phi} \approx a^{-1}$, $\phi \approx \int a^{-1} dt$ mimics a radiation-dominated universe. These models of $f(T, B)$ gravity offer a unified description of the universe evolution (i.e. the matter era and the late-time acceleration epoch), similarly to the analysis of Leon in scalar-torsion theory. This analysis has been presented before in Paliathanasis, where this theory can describe various eras of cosmological history. However, the present work focuses on the existence of spatial curvature and how $f(T, B)$ gravity solves the flatness problem.

In the following sections, we investigate the existence of exact solutions. Also, we study the cosmological dynamics for the field equations (25)–(27). Such an analysis provides important information that will help us understand spatial curvature’s effects on the application of teleparallelism in cosmology.

## 3 | EXACT SOLUTIONS

We proceed by investigating the existence of exact solutions in which the scale factor is a power-law function, that is, $a(t) = a_0 t^p$, the exponential function, that is, $a(t) = a_0 e^{t/t_0}$, and the Einstein-static universe, $a(t) = a_0$. 
3.1 Scaling solution

For the scaling solution \( a(t) = a_0 t^p \), with \( H(t) = pt^{-1} \), from the field equations (25)–(27), we find the linear second-order ordinary differential equation

\[
\ddot{\phi} - 3pt^{-1}\dot{\phi} + 2\left(pt^{-2} + Kt^{-2p}\right) = 0
\]

with analytic solution

\[
\phi(t) = \phi_0 + \frac{\phi_1}{1 + 3p}t^{1+3p} - \frac{2p}{1 + 3p} - \frac{K}{(1 - p)(1 - 5p)}t^{2(1-p)},\ p \not\in \left\{ 1, \frac{1}{5}, -\frac{1}{3} \right\}, \quad (33)
\]

\[
\phi(t) = \phi_0 + \frac{\phi_1}{4}t^4 - \frac{1 + K}{2} \ln t, \ p = 1, \quad (34)
\]

\[
\phi(t) = \phi_0 + \frac{5}{8} \left( \phi_1 - \frac{5}{4}K \right)t^5 - \frac{1}{4} \left( 1 - 5Kt^5 \right) \ln t, \ p = \frac{1}{5}, \quad (35)
\]

\[
\phi(t) = \phi_1 + \phi_1 \ln t - \frac{1}{3}(\ln t)^2 + \frac{9}{32}Kt^5, \ p = -\frac{1}{3}. \quad (36)
\]

Similarly, for the scalar field potential \( V(\phi(t)) \), we derive

\[
V(\phi(t)) = -\phi_1 pt^{-1+3p} + \frac{p^2(1-p)}{1 + 3p}t^{-2} - \frac{(1 - 3p)K}{(1 - 5p)}t^{-2p}, \ p \not\in \left\{ 1, \frac{1}{5}, -\frac{1}{3} \right\}, \quad (37)
\]

\[
V(\phi(t)) = -\phi_1 t^2 - \frac{K + 1}{2}t^{-2}, \ p = 1, \quad (38)
\]

\[
V(\phi(t)) = -\left( \phi_1 + \frac{5K}{5} \right)t^\frac{5}{2} + \frac{1}{100}t^{-2} - \frac{2}{5}t^{-2}\ln t, \ p = \frac{1}{5}, \quad (39)
\]

\[
V(\phi(t)) = \frac{3\phi_1 - 1}{9}t^{-2} - \frac{2}{3}Ke^\frac{6}{5} - \frac{2}{9}t^{-2}\ln t, \ p = -\frac{1}{3}. \quad (40)
\]

Let us focus now on the case where \( p = 1 \). The scale factor \( a(t) = a_0 t \), describes Milne (for \( K = -1 \)) and Milne-like (for \( K = 1 \)) universes. For \( K = -1 \), it follows \( \phi(t) = \phi_0 + \frac{\phi_1}{4}t^4 \) and \( V(\phi) = -\phi_1 t^2 \), for which we observe that \( \rho_\phi = 0 \). Thus, there is not any contribution to the cosmological fluid from the \( F(B) \) component. On the other hand, for the \( K = 1 \), and the Milne-like solution, we observe that \( \rho_\phi \neq 0 \). For large values of \( t \), \( \phi(t) \approx \phi_0 + \frac{\phi_1}{4}t^4 \) and \( V(\phi(t)) \approx -\phi_1 t^2 - t^{-2} \), that is,

\[
V(\phi) \approx -\left( 4\phi_1 (\phi - \phi_0) \right)^\frac{1}{2} - \left( \frac{4}{\phi_1} (\phi - \phi_0) \right)^{-\frac{1}{2}}. \quad (41)
\]

Furthermore, for small values of \( t \), that is, near to the initial singularity, it follows \( \phi(t) \approx -\ln t, \ V(\phi(t)) = -t^{-2} \), that is,

\[
V(\phi) \approx -e^{2\phi}, \quad (42)
\]

which leads to

\[
F \approx \frac{1}{2}B \left( \ln \left( \frac{B}{12} \right) - 1 \right). \quad (43)
\]

3.2 Exponential scale factor solution

Assume now the exponential scale factor \( a(t) = a_0 e^{H_0 t} \). Then the scalar field satisfies the second-order ordinary differential equation,

\[
\ddot{\phi} - 3H_0\dot{\phi} - 2Ke^{-2H_0 t} = 0, \quad (44)
\]

with analytic solution

\[
\phi(t) = \phi_0 + \frac{\phi_1}{3H_0}e^{3H_0 t} + \frac{K}{5H_0^2}e^{-2H_0 t}. \quad (45)
\]
For the potential function, we calculate

\[ V(\phi) = 3H_0^2 \left( -\frac{1}{3} + \frac{K}{5H_0^2} e^{-2H_0 t} + \frac{\phi_1}{3H_0} e^{3H_0 t} \right) = 3H_0^2 \left( -\frac{1}{3} + \phi - \phi_0 \right). \]

Thus, the scalar field potential is described by the linear function

\[ V(\phi) = -\alpha + \beta \phi, \quad \alpha = \beta \left( \frac{1}{3} + \phi_0 \right), \quad \beta = 3H_0^2, \]

which leads to

\[ F(B) = \left( \frac{1}{3} + \phi_0 \right) B. \quad (47) \]

### 3.3 Einstein-static universe

For a static universe, \( a(t) = a_0 \), we calculate

\[ \phi(t) = Kt^2 + \phi_0 + \phi_1 t, \quad (48) \]

\[ V(\phi) = -K, \quad (49) \]

so \( F(B) = F_1 B - 6K \), where \( F_1 \) is the integration constant.

Below, we continue our analysis by investigating the stability properties of the above solutions and the asymptotic behavior of the field equations. Specifically, we study the dynamics of the field equations by determining the equilibrium points and their stability.

### 4 DYNAMICAL ANALYSIS

To study the dynamics of the field equations (25)–(27) we define the new variables

\[ x = \frac{\phi}{\sqrt{H^2 + |K| a^{-2}}}, \quad y = \frac{V(\phi)}{H^2 + |K| a^{-2}}, \quad \eta = \frac{H}{\sqrt{H^2 + |K| a^{-2}}}, \quad \lambda = \frac{V_{\phi}(\phi)}{V(\phi)}, \quad (50) \]

which satisfies

\[ (1 - \text{sgn}(K)) \eta^2 + \text{sgn}(K) \eta x + y = 0. \quad (51) \]

Furthermore, we consider the new independent variable, \( d\tau = \sqrt{H^2 + |K| a^{-2}} dt \), leading to the dynamical system

\[ \frac{dx}{d\tau} = \text{sgn}(K) (\eta^2 - 1) + \eta (3\eta + 2\eta^2 x + x) + y(2\lambda + \lambda \eta x - 3), \quad (52) \]

\[ \frac{dy}{d\tau} = y \left( \lambda x + 2\eta (2\eta^2 + \lambda y + 1) \right), \quad (53) \]

\[ \frac{d\eta}{d\tau} = (\eta^2 - 1) \left( 2\eta^2 + \lambda y \right), \quad (54) \]

\[ \frac{d\lambda}{d\tau} = hx, \quad (55) \]

where

\[ h = \frac{V_{\phi\phi}}{V} - \frac{V_{\phi}^2}{V}. \quad (56) \]

In the new variables, the equation of state parameter for the effective fluid, \( w_{\text{tot}} = -1 - \frac{2}{3H^2} \). The deceleration parameter,

\[ q = -1 - \frac{\ddot{H}}{H^2}, \]

is

\[ w_{\text{tot}}(x, y, \eta, \lambda) = 1 + 2\frac{\lambda}{3\eta^2} y \quad (57) \]

and

\[ q(x, y, \eta, \lambda) = 1 + \frac{\lambda y}{\eta^2}. \quad (58) \]
It is useful to define the quantity $\Omega_K = \frac{|K|}{a H^2}$, which is a dimensionless measure of the spatial curvature in an FLRW universe. Hence, although the analysis is for $K = 1$ and $K = -1$ respectively, some points correspond to flat FRW, which is the situation where asymptotically $\Omega_K \to 0$. In other words, with the term flat, we mean asymptotically flat. Notice that $\eta^2 = 1/(1 + \Omega_K)$ so that $\Omega_K \to 0$ implies $\eta \to \pm 1$, which means asymptotically flat FRW universe.

Finally,

$$w_\phi = \frac{\text{sgn}(K) - (\text{sgn}(K) + 3) \eta^2 - 2 \lambda y}{3(\eta x + y)}. \quad (59)$$

We proceed with our analysis by considering $K = 1$ and $K = -1$. Moreover, in the following we assume that $\lambda = \text{const}$, that is, we consider the exponential potential $V(\phi) = -V_0 e^{i \phi}$, which leads to

$$\phi = \frac{1}{\lambda} \ln \left( \frac{B}{6\lambda V_0} \right), \quad F(B) = \frac{B}{\lambda} \left( \ln \left( \frac{B}{6\lambda V_0} \right) - 1 \right), \quad (60)$$

and to $h \equiv 0$.

### 4.1 Positive curvature

For an FLRW spacetime with positive spatial curvature, that is, $K = 1$, in the new variables, the field equations become

$$\frac{dx}{d\tau} = -1 + 4\eta^2 + (2\eta^3 + \eta) x + y(2\lambda + \lambda \eta) x - 3), \quad (61)$$

$$\frac{dy}{d\tau} = y(\lambda x + 2\eta(2\eta^2 + \lambda y + 1)), \quad (62)$$

$$\frac{d\eta}{d\tau} = (\eta^2 - 1)(2\eta^2 + \lambda y). \quad (63)$$

with constraint equation

$$1 + x\eta + y = 0. \quad (64)$$

With the use of (64), we can write the equivalent two-dimensional system

$$\frac{dx}{d\tau} = - (\eta x + 1)(2\lambda - 2\eta^2 + \lambda \eta x - 4), \quad (65)$$

$$\frac{d\eta}{d\tau} = (1 - \eta^2)(\lambda - 2\eta^2 + \lambda \eta x). \quad (66)$$

The observable quantities are reduced to

$$\{q, w_{tot}, w_\phi\} = \left\{ 2 - \frac{\lambda(\eta x + 1)}{\eta^2}, 1 - \frac{2\lambda(\eta x + 1)}{3\eta^2}, \frac{1}{3} \left( -2\lambda + 4\eta^3 - 2\lambda \eta x - 1 \right) \right\}. \quad (67)$$

The equilibrium points $P = (x(P), \eta(P))$ of the dynamical system (65) and (66) are given by the algebraic equations

$$(\eta x + 1)(2\lambda - 2\eta^2 + \lambda \eta x - 4) = 0, \quad (\eta^2 - 1)(\lambda - 2\eta^2 + \lambda \eta x) = 0, \quad (68)$$

that is

$$A_1 = (1, -1), A_2 = (-1, 1),$$

$$A_3 = \left( \frac{1}{\sqrt{\lambda}}, -\frac{\sqrt{\lambda}}{2} \right), A_4 = \left( -\frac{1}{\sqrt{\lambda}}, \frac{\sqrt{\lambda}}{2} \right),$$

$$A_5 = \left( 2 - \frac{6}{\lambda}, -1 \right), A_6 = \left( -2 + \frac{6}{\lambda}, 1 \right).$$

Points $A_1, A_2$ describe exact solutions for which the kinetic part of the scalar field dominates, that is, $y(A_1) = y(A_2) = 0$, with $w_{tot}(A_1) = w_{tot}(A_2) = 1$ and $q(A_1) = q(A_2) = 2$. The spacetime is described asymptotically by the spatially flat FLRW universe with scale factor $a(t) = a_0 e^{\tau}$. 

Moreover, points $A_3$ and $A_4$ exist only when $\lambda > 0$ and describe Milne-like solutions with $a(t) = a_0t$. At these two points, the physical parameters are derived $y(A_3) = y(A_4) = -\frac{1}{2}$, $w_{\text{tot}}(A_3) = w_{\text{tot}}(A_4) = -\frac{1}{3}$ and $q(A_3) = q(A_4) = 0$.

Finally, the family of points $A_5$ and $A_6$ describe scaling solutions with scale factor $a(t) = a_0 e^{Ht}$. We derive $w_{\text{tot}}(A_5) = w_{\text{tot}}(A_6) = -3 + \frac{2\lambda}{3}$ and $q(A_5) = q(A_6) = \lambda - 4$, from which we observe that the exact solution describes an accelerated universe when $\lambda < 4$. The spatial curvature for the background space is asymptotically zero. In the special case for which $\lambda = 3$, the exact solution at the equilibrium points is $a(t) = a_0 e^{Ht}$, which is the de Sitter solution. Evaluating at the equilibrium points $A_1$, we have $w_{\text{tot}} = w_{\phi}$.

To investigate the stability properties of the equilibrium points, we determine the eigenvalues of the matrix

$$A = \begin{pmatrix} \frac{\partial}{\partial x} (d^2_1 + d^2_2) & \frac{\partial}{\partial y} (d^2_1 + d^2_2) \\ \frac{\partial}{\partial x} (d^2_1 - d^2_2) & \frac{\partial}{\partial y} (d^2_1 - d^2_2) \end{pmatrix}_{(x,y) \rightarrow (x(P),y(P))}. \tag{69}$$

Let $e_1(P)$, $e_2(P)$ be the two eigenvalues of the matrix $A$. We say that the equilibrium point $P$ is an attractor and describes a stable asymptotic solution when the real parts of the two eigenvalues are negatives. When the real parts of the eigenvalues are positive, point $P$ is called a source, and the asymptotic solution is an attractor. Otherwise, the equilibrium point $P$ is characterized as a saddle point.

For each of the six equilibrium points, we derive the following set of eigenvalues

$$e_1(A_1) = -4, \quad e_2(A_1) = \lambda - 6, \tag{70}$$

$$e_1(A_2) = 4, \quad e_2(A_2) = -(\lambda - 6), \tag{71}$$

$$e_1(A_3) = \frac{1}{2} \sqrt{\lambda} \left( 1 + \sqrt{(\lambda - 8)\lambda + 17} \right), \quad e_2(A_3) = \frac{1}{2} \sqrt{\lambda} \left( 1 - \sqrt{(\lambda - 8)\lambda + 17} \right), \tag{72}$$

$$e_1(A_4) = -\frac{1}{2} \sqrt{\lambda} \left( 1 + \sqrt{(\lambda - 8)\lambda + 17} \right), \quad e_2(A_4) = -\frac{1}{2} \sqrt{\lambda} \left( 1 - \sqrt{(\lambda - 8)\lambda + 17} \right), \tag{73}$$

$$e_1(A_5) = 6 - \lambda, \quad e_2(A_5) = -2(\lambda - 4), \tag{74}$$

$$e_1(A_6) = -(6 - \lambda), \quad e_2(A_6) = 2(\lambda - 4). \tag{75}$$

Points $A_{1,2}$ are nonhyperbolic when $\lambda = 6$. Hence, point $A_1$ is a sink when $\lambda < 6$; otherwise, it is a saddle. Point $A_2$ is a source when $\lambda < 6$; otherwise, it is a saddle point. Points $A_3$ and $A_4$ exist for $\lambda > 0$, and they are nonhyperbolic for $\lambda = 4$, and a saddle otherwise. The equilibrium points $A_{5,6}$ are nohyperbolic for $\lambda \in (4, 6)$. We find that $A_5$ is a sink for $\lambda > 6$, a source when $\lambda < 4$ and a saddle for $4 < \lambda < 6$, while $A_6$ is a source for $\lambda > 6$, a sink when $\lambda < 4$ and a saddle for $4 < \lambda < 6$.

The results are summarized in Table 1. Figure 1 draws a phase plot of system (65) and (66) for $\lambda = 1, 2, 4, 6$.

### Table 1: Asymptotic solutions for the field equations with positive spatial curvature

| Point | Curvature of FLRW | $a(t)$ | $w_{\phi}$ | Attractor? |
|-------|------------------|-------|------------|------------|
| $A_1$ | Flat             | $t^1$ | $-1$       | $\lambda < 6$ |
| $A_2$ | Flat             | $t^1$ | $-1$       | No         |
| $A_3$ | $>0$             | $t$   | $-\frac{1}{3}$ | No         |
| $A_4$ | $>0$             | $t$   | $-\frac{2}{3}$ | No         |
| $A_5$ | Flat             | $t^{3,4}, \lambda \neq 3$ | $-3 + \frac{2\lambda}{3}$ | $\lambda > 6$ |
| $A_6$ | Flat             | $t^{3,4}, \lambda \neq 3$ | $-3 + \frac{2\lambda}{3}$ | $\lambda < 4$ |

#### 4.1.1 Poincaré variables

To perform a complete analysis of the dynamics, we should investigate if there exist equilibrium points when the dynamical variable, $x$, and $\eta$ take values at infinity.
Thus, we define the Poincaré variables
\[
x = \frac{X}{\sqrt{1 - X^2 - Z^2}}, \quad \eta = \frac{Z}{\sqrt{1 - X^2 - Z^2}}
\] (76)

and the new independent variable \(d\sigma = (1 - X^2 - Z^2)^{-1/2}d\tau\).

The two-dimensional dynamical system (65) and (66) becomes
\[
\frac{dX}{d\sigma} = -2(\lambda + (\lambda - 2)X^4 - 2(\lambda - 1)X^2Z + X^2(-2\lambda + (2\lambda - 1)Z^2 + 4) - XZ(-2\lambda + (\lambda + 2)Z^2 + 2) - (\lambda - 1)Z^2 - 2),
\] (77)
\[
\frac{dZ}{d\sigma} = \lambda - 2(\lambda - 2)X^2Z + X^2(4(\lambda - 1)Z^2 - \lambda) + XZ(3\lambda + (2 - 4\lambda)Z^2 - 4) + Z^2(-3\lambda + 2(\lambda + 2)Z^2 - 2).
\] (78)

The equilibrium points of the latter system at infinity that is, on the surface \(1 - X^2 - Z^2 = 0\), are
\[
P_1 = (1, 0), \quad P_2 = (-1, 0).
\]

Furthermore, in the new variables for the deceleration parameter, the effective equation of state parameter, and the effective equation of state parameter of \(\phi\), we derive
\[
q(X, Z) = \lambda + \frac{\lambda(X^2 - XZ - 1)}{Z^2} + 2,
\] (79)
\[
w_{tot}(X, Z) = \frac{2\lambda(X^2 - XZ + Z^2 - 1)}{3Z^2} + 1.
\] (80)
Negative curvature

For an FLRW spacetime with negative spatial curvature, the field equations in the dimensionless variables are

\[ P_{1} \quad (1,0) \quad a_{0} \quad \text{Sink for } \lambda < 0, \text{source for } \lambda > 0 \]

\[ P_{2} \quad (-1,0) \quad a_{0} \quad \text{Source for } \lambda < 0, \text{sink for } \lambda > 0 \]

and

\[ w_{\varphi}(X, Z) = \frac{1}{3} \left( -2\lambda + \frac{2(2X^{2} + \lambda XZ - 2)}{X^{2} + Z^{2} - 1} - 5 \right). \] (81)

Thus, points \( P_{1} \) and \( P_{2} \) describe static universes with \( a(t) = a_{0} \). We proceed with the study of the stability properties of the equilibrium points at infinity by using the parametrization

\[ X = (1 - \varphi) \cos(\varphi), \quad Z = (1 - \varphi) \sin(\varphi), \quad 0 \leq \varphi < 1, \] (82)

and a time re-scaling \( ds = d\sigma/(1 - \varphi) \), such that the region at infinity for \( (x, \eta) \), that is, \( X^{2} + Z^{2} = 1 \), is approached as \( \varphi \to 0^{+} \).

Taking the Taylor expansion centered in \( \varphi = 0 \), neglecting higher order terms, we have

\[ \frac{d\varphi}{ds} = \sin^{2}(\varphi)(\lambda \cos(\varphi) - 2 \sin(\varphi)), \quad \frac{d\varphi}{ds} = \frac{1}{2} \sin(\varphi)(\lambda + 2(\lambda - 3) \sin(2\varphi) + (\lambda + 2) \cos(2\varphi) - 2). \] (83)

To find the equilibrium points at infinity, we solve the algebraic equation

\[ \sin^{2}(\varphi)(\lambda \cos(\varphi) - 2 \sin(\varphi)) = 0, \quad \sin(\varphi)(\lambda + 2(\lambda - 3) \sin(2\varphi) + (\lambda + 2) \cos(2\varphi) - 2) = 0. \] (84)

Equation (83) does not depend on the radial coordinate. Therefore, the stability analysis considers the nature of the eigenvalues

\[ \Lambda_{1}(P) = 0, \quad \Lambda_{2}(P) = \frac{1}{4}(-2(\lambda - 3)(\sin(\varphi) - 3 \sin(3\varphi)) + \lambda - 6) \cos(\varphi) + 3(\lambda + 2) \cos(3\varphi)). \] (85)

of the Jacobian matrix evaluated at the values \( \varphi \) that satisfy (84).

The solutions of (84) are \( P_{1} \) := \( \varphi = 2\pi c_{1} \quad \text{if} \quad c_{1} \in \mathbb{Z} \), with \( \Lambda_{1}(P_{1}) = 0, \Lambda_{1}(P_{1}) = \lambda \). Sink for \( \lambda < 0 \), source for \( \lambda > 0 \).

\( P_{2} \) := \( \varphi = 2\pi c_{1} + \pi \quad \text{if} \quad c_{1} \in \mathbb{Z} \), with \( \Lambda_{1}(P_{2}) = 0, \Lambda_{1}(P_{2}) = -\lambda \). Source for \( \lambda < 0 \), sink for \( \lambda > 0 \).

The results are summarized in Table 2.

In Figure 2, a phase plot of system (77)–(78) is presented for \( \lambda = 1, 2, 4, \) and 6. In the figures, it is confirmed for \( \lambda > 0 \) that \( P_{1} \) is unstable and \( P_{2} \) is stable. Additionally, the information in Table 1 relative to the stability of the points at the finite region \( A_{i}, \ i = 1 \ldots 6 \) is confirmed.

### 4.2 Negative curvature

For an FLRW spacetime with negative spatial curvature, the field equations in the dimensionless variables are

\[ \frac{dx}{d\tau} = (2\eta^{2} + 1)(\eta x + 1) + y(2\lambda + \lambda \eta x - 3), \] (86)

\[ \frac{dy}{d\tau} = y \left( \lambda x + 2\eta \left( 2\eta^{2} + \lambda y + 1 \right) \right), \] (87)

\[ \frac{d\eta}{d\tau} = \left( \eta^{2} - 1 \right) \left( 2\eta^{2} + \lambda y \right), \] (88)

with constraint

\[ \eta(2\eta + x) + y - 1 = 0. \] (89)

Thus, with the use of the constraint equation (89), the dynamical system is reduced to the two-dimensional system

\[ \frac{dx}{d\tau} = (2\eta^{2} + 1)(\eta x + 1) - (\eta(2\eta + x) - 1)(2\lambda + \lambda \eta x - 3), \] (90)
\[ \frac{d\eta}{dt} = (\eta^2 - 1) (\lambda - \eta(2\lambda - 1)\eta + \lambda x)). \]  \hspace{1cm} (91)

The observable quantities are reduced to

\[ \{q, w_{\text{tot}}, w_{\phi}\} = \left\{ -2\lambda + \frac{\lambda - \lambda \eta x}{\eta^2} + 2, \frac{1}{3} \left( -4\lambda - \frac{2\lambda(\eta x - 1)}{\eta^2} + 3 \right), \frac{2\lambda + 2\eta(\eta - \lambda(2\eta + x)) + 1}{6\eta^2 - 3} \right\}. \]  \hspace{1cm} (92)

The equilibrium points for the dynamical system (90) and (91) are

\[ B_1 = (1, -1), B_2 = (-1, 1), \]
\[ B_3 = \left( \sqrt{\frac{2}{\lambda (\lambda - 2)}}, \sqrt{\frac{2}{\lambda (\lambda - 2)}} \right), B_4 = \left( -\sqrt{\frac{\lambda}{2(\lambda - 2)}}, \sqrt{\frac{\lambda}{2(\lambda - 2)}} \right), \]
\[ B_5 = \left( 2 - \frac{6}{\lambda}, -1 \right), B_6 = \left( -2 + \frac{6}{\lambda}, 1 \right). \]

The physical properties of the asymptotic solutions at the latter equilibrium points are similar to those of points \( A_i \). Indeed, points \( B_1 \) and \( B_2 \) describe spatially flat FLRW spacetimes with scale factor \( a(t) = a_0 t^\frac{1}{3} \); points \( B_3 \) and \( B_4 \) are real when \( \lambda (\lambda - 2) > 0 \) and describe Milne universes, while for the points \( B_5 \) and \( B_6 \), the asymptotic solution is that of spatially flat FLRW with scale factor \( a(t) = a_0 e^{t\frac{1}{2}}, \lambda = 3 \). Evaluating at the equilibrium points \( B_i \), we have \( w_{\text{tot}} = w_{\phi} \).

The eigenvalues of the linearized system (90) and (91) around the equilibrium points \( B_i \) are calculated

\[ e_1 (B_1) = -4, e_2 (B_1) = \lambda - 6, \]  \hspace{1cm} (93)
\[ e_1 (B_2) = 4, e_2 (B_2) = -\lambda + 6. \]  \hspace{1cm} (94)
TABLE 3  Asymptotic solutions for the field equations with negative spatial curvature

| Point | FLRW | \( \alpha(t) \) | \( w_\phi \) | Attractor? |
|-------|------|----------------|-------------|------------|
| \( B_1 \) | Flat | \( t^5 \) | \( \lambda < 6 \) | Yes |
| \( B_2 \) | Flat | \( t^5 \) | No | No |
| \( B_3 \) | \( <0 \) | \( t \) | No | No |
| \( B_4 \) | \( <0 \) | \( t \) | \( \lambda > 4 \) | No |
| \( B_5 \) | Flat | \( t^{1/5}, \lambda \neq 3 \) | \( \lambda > 6 \) | No |
| \( B_6 \) | Flat | \( t^{1/5}, \lambda \neq 3 \) | \( \lambda < 4 \) | Yes |

The stability properties of points \( B_1, B_2, B_3, \) and \( B_6 \) are the same as those of points \( A_1, A_2, A_3, \) and \( A_6, \) respectively. Point \( B_3 \) is a source for \( 4 < \lambda \leq \frac{17}{4} \) (unstable node) or \( \lambda > \frac{17}{4} \) (unstable spiral), or a saddle point for \( \lambda < 0 \) or \( 2 < \lambda < 4 \). On the other hand, point \( B_4 \) is an attractor for \( 4 < \lambda \leq \frac{17}{4} \) (stable node) or \( \lambda > \frac{17}{4} \) (stable spiral), or a saddle point for \( \lambda < 0 \) or \( 2 < \lambda < 4 \). The results are summarized in Table 3. Figure 3 draws a phase plot of system (90) and (91) for \( \lambda = 1, 3, 4, 6 \). [Colour figure can be viewed at wileyonlinelibrary.com]
4.2.1 Poincaré variables

We introduce the Poincaré variables (76), and we write the dynamical system (90) and (91) into the equivalent form

\[
\frac{dX}{d\sigma} = 2\left(\lambda + (\lambda - 1)X^4 - 2X^3Z + X^2(-2\lambda + (2\lambda - 5)Z^2 + 2) + XZ((2 - 3\lambda)Z^2 + 2) + (5 - 3\lambda)Z^2 - 1\right),
\]

\[
\frac{dZ}{d\sigma} = \lambda(2Z(X - Z) + 1)(X^2 + XZ + 3Z^2 - 1) - 2Z(X^3 + 2X^2Z + X(5Z^2 - 1) - 2Z^3 + Z).
\]

Furthermore, the physical variables in the new variables are

\[
q(X, Z) = -3\lambda + \frac{\lambda - \lambda X(X + Z)}{Z^2} + 2,
\]

\[
w_{\text{tot}}(X, Z) = \frac{2\lambda - 2\lambda X^2 - 2\lambda XZ + (3 - 6\lambda)Z^2}{3Z^2},
\]

and

\[
w_{\phi}(X, Z) = \frac{1}{9}\left(-6\lambda + \frac{4X^2 - 6\lambda XZ + 4}{X^2 + 3Z^2 - 1} + 1\right).
\]

The equilibrium points for the dynamical system (99) and (100) at infinity, that is, on the surface \(X^2 + Z^2 = 1\), are

\[
Q_1 = (1, 0), \quad Q_2 = (-1, 0).
\]

Points \(Q_1, Q_2\) describe static universe, \(a(t) = a_0\), similarly with points \(P_1, P_2\).

To investigate the stability of the equilibrium points at infinite, we use the parameterization

\[
X = (1 - \rho)\cos(\phi), \quad Z = (1 - \rho)\sin(\phi), \quad 0 \leq \rho < 1,
\]

and a time rescaling \(ds = d\sigma/(1 - \rho)\), such as the region at infinite for \((x, \eta)\), that is, \(X^2 + Z^2 = 1\), is approached as \(\rho\) tends to 0.

Taking the Taylor expansion centered in \(\rho = 0\) and neglecting error terms, we obtain that

\[
\frac{d\rho}{ds} = \sin^2(\phi)(2(\lambda - 1)\sin(\phi) + \lambda \cos(\phi)), \quad \frac{d\phi}{ds} = \frac{1}{2}\sin(\phi)(5\lambda + (4\lambda - 6)\sin(2\phi) + (8 - 3\lambda)\cos(2\phi) - 8).
\]

To find the equilibrium points at infinity, we solve the algebraic equation

\[
\sin^2(\phi)(2(\lambda - 1)\sin(\phi) + \lambda \cos(\phi)) = 0, \quad \sin(\phi)(5\lambda + (4\lambda - 6)\sin(2\phi) + (8 - 3\lambda)\cos(2\phi) - 8) = 0.
\]

In the first order, Equation (105) does not depend upon the radial coordinate. Therefore, the stability analysis considers the nature of the eigenvalues

\[
\Lambda_1(Q) = 0, \quad \Lambda_2(Q) = \frac{1}{4}(-2(2\lambda - 3)\sin(\phi) - 3\sin(3\phi)) + (13\lambda - 24)\cos(\phi) + (24 - 9\lambda)\cos(3\phi)
\]

of the Jacobian matrix valuated at the values, \(\varphi\), that satisfy (106).

The solutions of (106) are:

- \(Q_1 := \varphi = 2\pi c_1\) if \(c_1 \in \mathbb{Z}\), with \(\Lambda_1(Q_1) = 0, \Lambda_2(Q_1) = \lambda\). Sink for \(\lambda < 0\), source for \(\lambda > 0\).
- \(Q_2 := \varphi = 2\pi c_1 + \pi\) if \(c_1 \in \mathbb{Z}\), with \(\Lambda_1(Q_2) = 0, \Lambda_2(Q_2) = -\lambda\). Source for \(\lambda < 0\), sink for \(\lambda > 0\).

The dynamics at infinity (\(\rho = 0\)) are governed by the one-dimensional dynamical system (105). In Figure 4, a phase plot of system (99)–(100) is presented for \(\lambda = 1, 3, 4, 6\). In the figures, it is confirmed that for \(\lambda > 0\), \(Q_1\) is unstable and \(Q_2\) is stable. Moreover, the information in Table 3 relative to the stability of the points at the finite region \(B_i, i = 1 \ldots 6\) is confirmed. The stability of the rest of the points at the infinite region is as discussed in Table 4.
5 | CONCLUSIONS

In this piece of work, we studied the cosmological model of the fourth-order teleparallel theory gravity known as \( f(T, B) \) in the case of FLRW background space with nonzero spatial curvature. In particular, we assumed that \( f(T, B) \) is a linear function of \( T, f(T, B) = T + F(B) \), such that the modifications of the gravitational Action Integral of the TEGR to be introduced by the term \( F(B) \). In the case where the function \( F(B) \) is linear, the TEGR is recovered; thus, in this work, we consider \( F(B) \) to be a nonlinear function.

For the proper vierbein fields, we derived the field equations for the background space of our consideration. Using Lagrange multipliers, we introduced a scalar field that attributes the geometrodynamical degrees of freedom to writing the field equations in the form of second-order theory. The dynamical variables are the scale factor \( a(t) \) and the scalar field \( \phi(t) \). Moreover, we show that this cosmological model admits a minisuperspace description. Thus, there exists a point-like Lagrangian, the variation of which provides the field equations. That specific characteristic of the gravitational theory is essential because various techniques from analytic mechanics can be applied for the investigation of the differential equations, also, the Wheeler–DeWitt equation of quantum cosmology can be calculated straightforward.

We investigated the existence of exact solutions where the scale factor is a power-law function, scaling solution, or exponential function. These exact solutions are essential because they can describe specific eras of cosmological history. Moreover, we proved that both cases have a scalar field potential, so the field equations can be solved explicitly.

Assume now, an arbitrary scale factor \( a(t) \), then from Equations (25) and (27), it follows

\[
\ddot{\phi} - H\dot{\phi} - 2Ka^{-2} + 2\dot{H} = 0,
\]  

(108)

\[
\text{TABLE 4 Asymptotic solutions at the infinity for the field equations with negative spatial curvature}
\]

| Point | \((X, Z)\) | \(a(t)\) | Stability |
|-------|-----------|---------|-----------|
| \(Q_1\) | (1, 0) | \(a_0\) | Sink for \(\lambda < 0\), source for \(\lambda > 0\) |
| \(Q_2\) | (−1, 0) | \(a_0\) | Source for \(\lambda < 0\), sink for \(\lambda > 0\) |
which is a second-order equation of the form

$$\ddot{\phi} + \alpha(t)\dot{\phi} + \beta(t) = 0 \quad (109)$$

with $\alpha(t) = -H$ and $\beta(t) = -2Ka^{-2} + 2\dot{H}$. Equation (109) is a linear equation and it is maximally symmetric, which means that it always admits a solution for an arbitrary function $\alpha(t)$ and $\beta(t)$, that is, for arbitrary selection of the scale factor $a(t)$. Hence, it is easy to infer that the field equations in these cosmological scenarios are always integrable. Such analysis generalizes previous results on this theory in the case of a spatially flat FLRW geometry.

Finally, we studied the general evolution of the dynamical variables described by the field equations. In particular, we derived the equilibrium points and investigated their stability properties. We performed our analysis separately for the cases of positive and negative spatial curvature. The equilibrium points were studied in the finite and infinite regions using Poincaré variables. While, from a first read, it seems that there are similarities in the cosmological evolution for the two cases of positive and negative spatial curvature, from the detailed analysis, we found that the results differ.

The results of this work are essential to understanding the spatial curvature in teleparallelism. The $f(T,B)$ theory provides important asymptotic behaviors of particular interest. In future work, we plan to study further the applications of $f(T,B)$ theory in the preinflationary era.

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**CONFLICT OF INTEREST**

The authors declare to have no conflict of interest.

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