Direction dependence of the extraordinary refraction index in uniaxial nematic liquid crystals

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Abstract
This paper presents a straightforward experiment that directly and illustratively demonstrates double refraction. For this purpose, two liquid crystalline cells were designed to enable qualitative and quantitative measurements of the extraordinary refractive index direction dependence in a uniaxial nematic liquid crystal.

(Some figures may appear in colour only in the online journal)

1. Teaching the concept of anisotropy

When a physical property in a material varies with direction, the material is said to be anisotropic. Anisotropy of physical properties is crucial in materials that are extremely important in contemporary science and technology. The liquid crystals that enable production of several devices used in everyday life (flat-screen TVs, notebooks, iPads, calculators, etc) are an example of such materials. Even though students encounter such devices on a daily basis, they are not aware of anisotropy as a key property of the materials that enable the high-tech products they use so eagerly. This ignorance might be due to the fact that the concept of anisotropy and in particular its consequences is quite difficult to comprehend [1, 2]. It is thus important to introduce the concept as soon as possible, taking into account students’ knowledge of physics and mathematics [3–5].

In this paper we focus on birefringence as one of the most widely used properties of anisotropic materials [6]. In a birefringent material the index of refraction depends on polarization of light and the direction of light propagation. The most easily observed and striking optical property of transparent birefringent materials is double refraction: the light incident on the interface between the isotropic and anisotropic material refracts into the anisotropic material in such a way that there are in general two rays of light propagating through the material in different directions; the light in both rays is polarized and the polarization of light in one ray is perpendicular to that in the other.
Double refraction is usually demonstrated by observing the doubling of a text observed through the calcite [7]. When a polarizer is placed behind the calcite (or in front of it), one of the figures disappears if the polarizer’s transmission direction coincides with the polarization of the transmitted light. Although the explanation for this phenomenon is quite straightforward for a trained physicist, it is usually not so easily comprehended by students. A more straightforward experiment demonstrates the splitting of the unpolarized incident light ray into two rays of linearly polarized light. One needs a large birefringent crystal that is thick enough so that two transmitted rays can be observed as two light spots on a distant screen. By changing the incident angle of light by rotating the crystal, one can observe the direction dependence of the extraordinary wave refractive index in uniaxial crystals, as well as the direction dependence of both indices in biaxial crystals.

In this paper we offer an alternative way to demonstrate the direction dependence of refractive indices. We present the use of liquid crystals to prepare a wedge shaped cell of a uniaxial material, which can then be used for quantitative measurements of the refractive index for light propagating at angles ranging from 0° to 90° with respect to the optic axis. Besides the fact that such liquid crystal cells are cheap and easy to prepare, the anisotropy of liquid crystals is much larger than in typical birefringent solid crystals, so the measurements can be performed with high accuracy already on cells with a thickness of a few hundred micrometres. In addition, liquid crystals can be synthesized in a school laboratory, not only at university level, but also at high school level [4, 8–10].

The concept of anisotropy can be qualitatively introduced even at elementary school level using liquid crystals. Of course a birefringent crystal would do just as well, but it is very important to connect physics lessons to modern topics and to materials that students use in everyday life. Liquid crystals offer an excellent opportunity for the introduction of modern interdisciplinary topics into education [11]. At high school level, anisotropic properties can be introduced through a set of carefully designed experiments. The concept can be efficiently reinforced at university level, with an interdisciplinary teaching module consisting of lectures and laboratory work [4, 12]. The module is appropriate for both social and natural science students. While such a teaching module has proven to be very efficient in achieving a conceptual understanding of optical anisotropy, physics students also require experiments that allow quantitative measurements. Measuring the angular dependence of the refractive index presented in this paper is a candidate for this.

The paper is structured as follows. In section 2 we revise the basic properties of light propagation in anisotropic media and focus on double refraction. This section will be welcomed by readers who are not specialists in the field; readers familiar with the field can skip it. In section 3 we present the preparation of liquid crystal cells that enable quantitative measurement of the refractive index through the whole span of light propagation directions. Two cells are prepared: one presenting the anisotropic material with the optic axis perpendicular to the cell surface (homeotropic cell), the other with the optic axis along the surface (planar cell). We show that from the measurements with the homeotropic cell one can obtain the refractive index values at angles from 0° to 30° with respect to the optic axis, while with the planar cell one obtains the refractive index at angles close to 45°. We show how an old experimenters’ trick, phase matching by the use of another material, can be used to obtain the refractive index at angles close to 45°. Finally, in section 4 we sum up the results.

2. Propagation of light in anisotropic media

In an anisotropic dielectric material, the polarization ($\vec{P}$) in the material depends on the direction of the external electric field ($\vec{E}$), and it is, in general, not in the direction of the external field.
This property is described by the electric susceptibility ($\chi$) being a tensor quantity (not scalar, as in isotropic materials), and the relation between the material polarization and the electric field is:

$$\vec{P} = \varepsilon_0 \chi \vec{E}. \quad (1)$$

Light is the propagation of an electromagnetic wave. The electric field in an electromagnetic wave interacts with the material and induces polarization, which varies over time with the same frequency as light. Interaction of light with matter reduces its speed. In materials where the field-induced polarization is larger (materials with greater electric susceptibility), the speed of light is lower. The ratio between the speed of light in a vacuum and in a particular material is given by the index of refraction. Since in anisotropic materials polarization in the material depends on the direction of the external electric field, in such materials the index of refraction varies with the direction of light propagation, and it depends on the direction of the electric field (i.e. the direction of light polarization). We point out that polarization of light (direction of $\vec{E}$ in light) should not be confused with material polarization ($\vec{P}$), which is a result of $\vec{E}$ in light. The reader should once again carefully examine equation (1) to fully grasp the difference.

Propagation of light is described by the wave equation. We shall assume a nonmagnetic material that contains no volume charge and no conducting current. Assuming a monochromatic plane wave with angular frequency $\omega$ propagating in any direction given by the direction of the wave vector $\vec{k}$, the electric ($\vec{E}$) and magnetic ($\vec{H}$) field vectors may be described by the harmonic representation, e.g. $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$. The general plane wave equation in the anisotropic media is

$$\vec{k} \times (\vec{k} \times \vec{E}) + k_0^2 \varepsilon \vec{E} = 0, \quad (2)$$

where $k_0$ is the magnitude of the wave vector in a vacuum, and $\varepsilon = I + \chi$ is the dielectric tensor; it is the sum of an identity tensor $I$ and the electric susceptibility. For ordinary nonabsorbing materials the dielectric tensor is symmetric, so there is always a coordinate system with a set of axes, called the principal axes, in which the dielectric tensor is diagonal. In optically uniaxial materials with the optic axis in the $z$-direction, the $x$- and $y$-components of the dielectric tensor are equal:

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}. \quad (3)$$

where $\varepsilon_1$ and $\varepsilon_3$ are the eigenvalues of the dielectric tensor. We shall solve the wave equation only for a special case of the wave vector direction, assuming that the wave vector is in the $xz$-plane: $\vec{k} = (k_x, 0, k_z)$. For the electric field, we assume a general direction $\vec{E} = (E_x, E_y, E_z)$. One cannot assume that the electric field is perpendicular to the wave vector, since such orthogonality is valid only in isotropic materials. Using the ansatz for $\vec{k}$ and $\vec{E}$ and the expression (3) for the dielectric tensor, a set of three equations is obtained from the wave equation (2):

$$-k_x^2 E_x + k_x k_z E_z + k_0^2 \varepsilon_1 E_x = 0, \quad (4)$$

$$-k_x^2 E_y - k_z^2 E_y + k_0^2 \varepsilon_1 E_y = 0, \quad (5)$$

$$-k_z^2 E_z + k_x k_z E_z + k_0^2 \varepsilon_3 E_z = 0. \quad (6)$$

4 For a quick overview, we suggest the concise treatment in [13], but, of course, any book on optics or modern optics (such as [14–17]) will also have a thorough treatment of light propagation in anisotropic media.
The allowed wave vector magnitudes in the $xz$-plane for an anisotropic material with the optic axis along the $z$-axis lie on the circle (blue) for the ordinary wave and on the ellipse (red) for the extraordinary wave. $\theta$ is the angle between the wave vector $\vec{k}$ and the optic axis, $n_o$ and $n_e$ are the ordinary and the extraordinary refractive indices, respectively, and $n(\theta)$ is the refractive index of the extraordinary wave, which propagates with $\vec{k}$ at an angle $\theta$ with respect to the optic axis. At a given direction of $\vec{k}$, the direction of the electric field $\vec{E}$ in the extraordinary wave is tangential to the ellipse (green dashed line). The direction of energy propagation, given by the direction of the Poynting vector $\vec{S}$, is in the direction perpendicular to $\vec{E}$. $k_x$ and $k_z$ are the $x$- and $z$-components of the wave vector in the anisotropic material and $k_0$ is the wave vector magnitude in vacuum.

The nontrivial solutions to the set of equations (4)–(6) is obtained if the determinant of the coefficients in front of the electric field components is zero. The determinant is zero if one of the conditions

$$k_x^2 + k_z^2 = k_0^2 \varepsilon_1$$

or

$$\frac{k_x^2}{\varepsilon_3} + \frac{k_z^2}{\varepsilon_1} = k_0^2$$

is satisfied.

The condition (7) gives the magnitude of the wave vector in the anisotropic material, which is the same for all directions of propagation in the $xz$-plane. This solution presents the ordinary wave. The refractive index for the ordinary wave is isotropic and equal to the ordinary index of refraction $n_o = \sqrt{\varepsilon_1}$. Equation (8) presents the constraint on the magnitude of the wave vector of the extraordinary wave and on a circle for the ordinary wave (figure 1). When rotated around the optic axis, the circle draws the surface of a sphere, and the ellipse the surface of an ellipsoid; we thus obtain the wave vector surfaces, giving us the magnitude of the wave vector for light propagation in a general direction. The reader is referred to any book on modern optics to prove this, or he/she can check it on his/her own by choosing different planes of wave vector direction (for example, in addition to the already studied $xz$-plane, consider the $xy$- and $yz$-planes).

Expressing the magnitude of the wave vector of the extraordinary wave as $k = k_0 n(\theta)$, where $n(\theta)$ is the index of refraction of the extraordinary wave propagating at an angle $\theta$ with respect to the optic axis, the wave vector can be expressed as $\vec{k} = k(\sin \theta, 0, \cos \theta)$. Using this expression, the angular dependence of the index of refraction can be derived from equation (8):

$$n^2(\theta) = \frac{n_o^2 n_e^2}{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}.$$  

(9)
The ordinary and extraordinary waves are linearly polarized. By using the condition (7) in equations (4)–(6), we find that $E_x = E_z = 0$ and $E_y \neq 0$. The ordinary wave is thus linearly polarized in the $y$-direction, i.e. in the direction perpendicular to the plane defined by the optic axis and the wave vector. When the condition (8) is used in equations (4)–(6), we find that $E_y = 0$ while $E_x$ and $E_z$ can be different from zero. The polarization of the extraordinary wave is in the plane defined by the optic axis and the wave vector and is thus perpendicular to the polarization of the ordinary wave. From equations (4) and (6), it is straightforward to show that the polarization of the extraordinary wave is along the $z$-axis if $\vec{k}$ is along the $x$-axis and vice versa. To find the direction of $\vec{E}$ in the extraordinary wave with $\vec{k}$ at a general angle with respect to the optic axis, is more elaborate. So we state without a proof that $\vec{E}$ is always in the direction tangential to the ellipse. The direction of energy propagation is given by that of the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$, so energy propagates in the direction perpendicular to $\vec{E}$. The energy (ray) propagating in a different direction to the momentum (wave) is a crucial property of an anisotropic material. In order to avoid confusion one should strictly state whether the direction of ray or wave propagation is being considered. The direction of ray at a given direction of $\vec{k}$ is shown in figure 1. The difference in the two directions is obvious. However, one must bear in mind that even in materials with large anisotropy, the ordinary and the extraordinary indices differ by approximately 10%, so the actual difference in the direction of propagation of $\vec{k}$ and $\vec{S}$ is much smaller. In calcite $n_e/n_o = 1.486/1.658 = 0.896$, in quartz $n_e/n_o = 1.553/1.544 = 1.006$ and in liquid crystals, where the anisotropy is very large, $n_e/n_o = 1.75/1.52 = 1.15$ for the liquid crystal used in the experiment presented in this paper.

2.1. Double refraction

When a beam of unpolarized light is incident on the anisotropic uniaxial transparent medium, it refracts into two beams. The direction of the wave vectors for the two beams is determined by the boundary conditions at the interface; these conditions follow from applying the Maxwell equations to suitable regions containing the interface. The boundary conditions give us the amplitude matching condition, which determines the reflectivity and the transmissivity at the interface, and the phase matching condition, which leads to the law of reflection and law of refraction [13–17]. The phase matching condition requires the components of the wave vector along the interface to be equal in all the waves: the incident, the reflected and the refracted. Figure 2 shows the refraction from an isotropic to an anisotropic medium. In the anisotropic medium the wave vector surfaces are shown for the ordinary and the extraordinary waves for the case of the optic axis being perpendicular (figure 2(a)) and parallel (figure 2(b)) to the interface. The light is incident at an angle $\alpha$, and $\beta_o$ and $\beta_e$ are the refraction angles for the wave vector direction of the ordinary and the extraordinary waves, respectively. Assuming that the isotropic medium is air with the refractive index approximately 1, the phase matching boundary condition requires:

$$k_0 \sin \alpha = k_o \sin \beta_o \quad \text{and} \quad k_0 \sin \alpha = k_e \sin \beta_e,$$

where $k_o = k_0 n_o$ and $k_e = k_0 n(\theta)$ are the wave vector magnitudes of the ordinary and the extraordinary wave, respectively. Equations (10) reduce to

$$\sin \alpha = n_o \sin \beta_o \quad \text{and} \quad \sin \alpha = n(\theta) \sin \beta_e.$$

We note that Snell’s law is valid for the direction of the wave vector propagation. However, since it is the direction of energy propagation (the ray direction) that we actually observe, Snell’s law for the extraordinary ray (as opposed to the extraordinary wave) is not obeyed (see figure 1 and draw the directions of energy propagation in figure 2). The refraction can be
Figure 2. Double refraction at the interface between the isotropic (air) and anisotropic (AM) materials if the optic axis lies in the plane of incidence and is (a) perpendicular and (b) parallel to the interface. In the anisotropic material the allowed magnitudes of the wave vector lie on the circle (blue) for the ordinary wave and on the ellipse (red) for the extraordinary wave. $\alpha$ is the incident angle, $\beta_o$ and $\beta_e$ are the refraction angles of the ordinary and extraordinary waves, respectively, and $\theta$ is the angle between the wave vector of the extraordinary wave and the optic axis. The phase matching boundary condition requires that the component of the wave vector parallel to the interface be conserved. The magnitude of this component is presented by a vertical solid line parallel to the interface normal (dashed line). Polarization of the ordinary wave is in the direction perpendicular to the incident plane (denoted by a circle with a dot), polarization of the extraordinary wave is in the direction tangential to the ellipse and it is not perpendicular to $\vec{k}$.

especially striking if the optic axis is at an angle with respect to the interface, since one can observe the ray which refracts to the same side of the normal (not only towards the normal). For the demonstration of this fascinating property of anisotropic materials using liquid crystals, the reader is referred to [18].

2.2. Optical anisotropy in liquid crystals

Nematic liquid crystals are composed of elongated molecules with orientationally ordered long molecular axes. The average direction of the long molecular axes is called the director. Since all directions of molecular motion in the direction perpendicular to the long molecular axis are equally probable, the system is optically uniaxial with the optic axis along the director. In bulky samples, clusters of oriented molecules are formed, and the director varies in space. In nematic liquid crystals, the orientational correlation length over which one can expect the same orientation of molecules extends to 500 $\mu$m; therefore, the well-ordered samples have to be thinner than that [19]. Within the range of optical frequencies, elongated molecules have greater polarizability along the long molecular axis than perpendicular to it. Thus, when the external electric field is along the director, the polarization induced in the liquid crystalline material will be larger than in the case where the electric field is perpendicular to the director.
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Figure 3. The homeotropic and planar wedge cells. (a) In the homeotropic cell molecular long axes are oriented perpendicular to the cell surface and the optic axis is thus perpendicular to the surface. The wedge angle $\delta$ is defined by the length of the cell $h$ and the thickness of the foil $d$. (b) In the planar cell molecular long axes are oriented parallel to the longer side of the cell. The optic axis is parallel to the surface. In both cells the allowed magnitudes of the wave vectors of the ordinary (blue circle) and the extraordinary (red ellipse) waves are shown (compare also to figure 2). The wave vector directions of the ordinary ($o$) and the extraordinary ($e$) waves are shown. $\alpha$ is the incident angle and $\theta$ is the angle between the wave vector of the extraordinary wave and the optic axis.

Because of that, the light being polarized perpendicularly to the director ($\vec{E}$ in the light is perpendicular to the director) will be faster than the light with polarization parallel to the director.

3. Experiment

To provide the refraction situations shown in figure 2, two wedge cells were designed with different surface treatments in order to achieve different orientations of the director in the cell and thus different orientations of the optic axis with respect to the surface (figure 3). The cells were approximately 1 cm long ($h$) and half a centimetre wide (figure 4(a)). Typical laser beams are too wide to enable the study of double refraction in cells having parallel surfaces. In thin samples, which guarantee homogeneity of orientation of the long molecular axis (the director gives the direction of the optical axis), the spatial separation of the ordinary and extraordinary beam can be obtained by the prismatic effect [1, 20]. To prepare a wedge cell, a foil with a thickness $d = 360 \mu m$ was inserted and glued between two pieces of microscope glass in one of the narrower sides, while the other narrow side of the cell was glued together directly, thus forming a wedge. By rubbing the surfaces, the planar cell in which molecules are aligned in the surface plane along the long side of the surface (figure 3(b)) was designed. The elongated molecules align with their long axes along the scratches, and the director is parallel to the rubbing direction. Therefore, the optic axis also coincides with the rubbing direction. To align
Figure 4. (a) Experimental setup with a close-up of a liquid crystalline cell. (b) Schematic presentation. $l$ is the distance between the cell and the screen. $x_o$ and $x_e$ define the position of the ordinary and extraordinary rays, respectively. For example, if one takes a planar cell with a wedge angle $\delta = 3.2^\circ$, then at the incident angle $\alpha = 10^\circ$ and $l = 5.07$ m, the ordinary and the extraordinary rays hit the screen at $x_o = 15.5$ cm and $x_e = 22.0$ cm, respectively.

the molecules perpendicular to the surface (homeotropic cell, figure 3(a)), a polymer coating was applied to the glass. Professional cells normally use carefully engineered coatings, but for simple experiments a satisfactory effect is provided by simply dipping the glass into a detergent or lecithin solution and allowing it to dry. The capillary effect was used to fill the cells with the liquid crystal E18 heated above the transition temperature from the nematic to the isotropic phase. For E18 at room temperature, the values of the ordinary and extraordinary indices are $n_o = 1.52$ and $n_e = 1.75$.

The experimental setup is shown in figure 4(a). The wedge cell is fixed into a holder and placed on a rotatable table with the longer side parallel to the table surface. A helium–neon (He–Ne) laser is used as the source for the unpolarized light. The direction of the incident light is always in the plane perpendicular to the wedge. When the laser beam of unpolarized light is incident on the wedge cell, two bright spots are observed on a distant screen; this is

5 Nematic liquid crystals with appropriate properties are commercially available from several companies like Merck, Sigma Aldrich or Nematel. Different commercial mixtures are available which are in the liquid crystalline state around the room temperature. The measurements can be done with any nematic liquid crystals that have the nematic phase at the room temperature and high birefringence. Instead, we can also use the liquid crystal MBBA which is quite easy to synthesize in the school lab. We have used E18, which was available in the lab, however, this mixture cannot be bought any more.
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Because of the birefringence of the liquid crystal in the wedge cell. The refractive indices are determined by measurements of the relative position of the spots with respect to the position of the direct beam spot, when light does not pass through the cell (figure 4(b)). The position of the spots changes as the incident angle of light alters when the table rotates; this enables measurement of the angular dependence of both indices.

Figure 5 shows the geometry of light propagation through the wedge cell. The angle \( \alpha \) is the controlled incident angle, and \( \beta \) is the refraction angle giving the direction of the wave vector of either the ordinary (\( \beta_o \)) or the extraordinary (\( \beta_e \)) wave. When exiting the cell the direction of light propagation deviates from the direction of the incident beam by an angle of \( \gamma \), which differs for the ordinary and the extraordinary waves and can be calculated from the beam position on the screen and the distance between the cell and the screen as \( \gamma \approx x/l \) (see figure 4(b)) where, again, one should use \( x \) for either the ordinary (\( x_o \)) or the extraordinary (\( x_e \)) wave.

The refractive indices of the ordinary and the extraordinary waves are obtained by measuring \( \gamma \) as a function of \( \alpha \) and knowing the wedge angle of the cell \( \delta \approx d/h \) (figure 3). Applying Snell’s law at the two interfaces of the wedge cell, we find (see figure 5):

\[
\frac{\sin \alpha}{\sin \beta} = n \quad \text{and} \quad \frac{\sin (\alpha \pm \gamma \pm \delta)}{\sin (\beta \pm \delta)} = n,
\]

where \( n \) is the refractive index of either the ordinary or the extraordinary wave. In equation (11) the upper sign in \( \pm \) stands for the wave propagation given in figure 5 by the red solid line (we shall call this incident angle positive \( \alpha \)), and the lower sign for the wave propagation denoted by the red dashed line (negative \( \alpha \)). Equation (11) is an approximation, since the direction of the optic axis in the cell varies slightly because of the wedge. However, since \( \delta \) is small, one can assume that the refractive indices of the extraordinary wave at angles \( \beta \) and \( \beta \pm \delta \) are the same. To confirm this, we measured \( x_e \) at incidence angles \( \pm \alpha \) and found them the same within the experimental error and the width of the laser beam spots on the screen.

Because the energy and momentum in the extraordinary wave do not propagate in the same direction, the position at which the extraordinary ray exits the cell will be displaced from the position shown in figure 5, where the wave vector directions are drawn. The displacement is of the order of the cell thickness multiplied by the angle between the ray and wave direction and is thus much smaller than the cell thickness. Since in air energy and momentum again propagate in the same direction, and the laser beam is of a width much larger than the cell.
Since the wedge angle $\delta$ and the angle $\gamma$ are very small, $\sin \delta \approx \delta$, $\cos \delta \approx 1$ and $\sin \gamma \approx \gamma$, $\cos \gamma \approx 1$. Using the sine addition formulae and equating the left parts of equations (11), we find (up to the first order in $\delta$ and $\gamma$):

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin \alpha + \gamma \cos \alpha + \delta \cos \alpha}{\sin \beta + \delta \cos \beta}$$

from which the refraction angle $\beta$ follows:

$$\tan \beta = \frac{\delta}{\delta + \gamma} \tan \alpha.$$  

With $\beta$ obtained from equation (13), the value of the refractive index is found from equation (11). Equations (11) and (13) are general expressions that can be used to obtain the refractive index when light passes through a thin wedge sample of any, not necessarily birefringent, material and they allow evaluation of the refractive indices of both the ordinary and the extraordinary waves exiting a cell filled by birefringent material.

In the homeotropic cell (figure 3(a)), the optic axis is perpendicular to the glass plate, and the refraction angle of the extraordinary wave ($\beta_e$) is equal to the angle $\theta$ between the optic axis and the wave vector, so the refractive index of the extraordinary wave is:

$$n(\theta) = \frac{\sin \alpha}{\sin \beta_e},$$

where $\beta_e$ is calculated from equation (13) using the measured value of $\gamma$ for the extraordinary wave.

From the measurements performed on the homeotropic cell, the refractive indices of the ordinary and extraordinary waves at angles ranging from $\theta = 0^\circ$ to approximately $40^\circ$ can be obtained.

In the planar cell the optic axis is parallel to the glass, and the refraction angle of the extraordinary wave is related to the angle between the optic axis and the wave vector direction as $\beta_e = \frac{\pi}{2} - \theta$. The refractive index of the extraordinary wave is therefore given by:

$$\frac{\sin \alpha}{\sin \beta_e} = \frac{\sin \alpha}{\sin \left(\frac{\pi}{2} - \theta\right)} = n(\theta).$$

From the measurements in the planar cell, the values are obtained for the refractive index at angles $\theta$ ranging from approximately $60^\circ$ to $90^\circ$. The planar cell can thus be used to study the direction dependence of the refractive index of the extraordinary wave in the region where the difference in the indices of the ordinary and the extraordinary waves is close to its largest value.

There are a few limitations in the experiment that must be considered. Although the ordinary and extraordinary waves can propagate in any direction, experimentally we are limited by the refraction of the incident light, since the light source is outside the birefringent material. In the situation presented, the wave vector direction of the ordinary wave was theoretically limited (at the incident angle $\alpha = 90^\circ$) to $\theta = 41^\circ$ at $n_o = 1.52$ and for the extraordinary wave to $\theta = 35^\circ$ at $n_e = 1.75$. The cell size and the cell holder additionally limit the incident angle $\alpha$. Figure 6 shows the combined results of the angular dependence of the refractive indices of the ordinary and the extraordinary wave measured in the planar and the homeotropic cell. The theoretical dependence was calculated from equation (9) using the known values of $n_o$ and $n_e$.

From figure 6 it can be seen clearly that with such a simple experiment we were not able to measure the values of the refractive indices for directions of propagation close to $\theta = 45^\circ$. To measure the refractive indices for these directions as well, one should use an old experimental
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Figure 6. The refractive indices of the ordinary \((n_o)\) and the extraordinary \((n(\theta))\) wave as a function of the angle \(\theta\) between the optic axis and the wave vector direction in the nematic liquid crystal E18. Squares: values obtained from the measurements in the planar cell. Circles: values obtained from the measurements in the homeotropic cell. Red: \(n(\theta)\), blue: \(n_o\). Green: measurements of \(n(\theta)\) in the sandwiched cell (see figure 8). The theoretical dependence \(n(\theta)\) given by equation (9) with \(n_e = 1.75\) and \(n_o = 1.52\) is given for comparison.

Figure 7. The sandwich of prisms and the cell.

trick: phase matching with the help of an additional material [22]. In order to enter the liquid crystal under a general angle, one should prevent refraction between two materials that have a large difference in refractive indices. Therefore, the light should pass a surface where refractive indices of materials on both sides of the surface are similar.

To achieve such conditions, the wedge cell is sandwiched between two glass prisms (refractive index \(n_g = 1.50\)) with an apex angle of \(45^\circ\) (figure 7). To prevent any possibility of a tiny air interface between the prism and the wedge cell, the contact areas are covered by glycerol, which has a refractive index similar to the refractive indices of glass and liquid crystal\(^6\). If the incident angle on the prism surface is zero, the beam does not refract and the incident angle to the liquid crystal in the wedge is \(45^\circ\) (figure 8). Since the refractive indices

\(^6\) Refractive index of glycerol (http://refractiveindex.info/).
of glass and liquid crystal are similar, the angle of refraction does not differ much from the incident angle, and in the liquid crystal light propagates in a direction close to $\theta = 45^\circ$. The prism at the other side of the cell provides a change in light direction that is opposite to the first one. Without the wedge cell, the light beam should be straight. Therefore measuring the positions of the two light spots again allows simultaneous measurement of both refractive indices.

Let us follow the light beam through the sandwich and calculate the refractive indices (figure 8). If the glycerol film is of uniform thickness, it does not influence the analysis of the refractive indices. The first relevant interface is thus the glass–liquid crystal interface. The incident angle is $\alpha = 45^\circ$, and the beam refracts only slightly ($\beta$ is close to $45^\circ$) because the refractive indices of liquid crystal are close to 1.5. At the liquid crystal–glass interface, the incident angle changes, owing to the wedge angle $\delta$, to $\beta + \delta$, as in figure 8, or to $\beta - \delta$ if the sandwich is rotated $90^\circ$ clockwise. Therefore, Snell’s law gives:

$$\frac{\sin(\beta \pm \delta)}{\sin \beta'} = \frac{n_g}{n},$$

(16)

where $\beta'$ is the refraction angle from the liquid crystal to the glass (see figure 8). The last refraction occurs at the glass–air interface. The incident angle is equal to $\beta' - \pi/4$, as in figure 8, or $\beta' + \pi/4$ if the sandwich is rotated $90^\circ$ degrees clockwise, as in figure 7. One can write Snell’s law as

$$\sin(\beta' \mp \pi/4) = \frac{\sin(\gamma \pm \delta)}{n_g}.$$  

(17)

The angle $\theta$ between the optic axis and the wave vector direction of the extraordinary/ordinary wave is equal to $\beta$ in the homeotropic cell and $\pi/2 - \beta$ in the planar cell (see figure 3).

As explained above, the sandwiched cells provide extra measurements of refractive indices at angles $\theta$ close to $45^\circ$. Since we have two cells with different alignments of molecules and two different orientations of sandwich as in figures 7 and 8, the refractive indices in four different directions of light propagation in the cell with respect to the optic axis can be measured. These measurements are shown as green diamonds in figure 6. One can clearly see that the accuracy of the refractive index measurement when the wedge cell is fixed between the prisms is much lower than when the prisms are absent. The reason is the glycerol, because it forms a slight wedge as well, which could not be controlled precisely. In most cases, the wedge of the glycerol has the same orientation as the wedge of the liquid crystalline cell, a fact that results in larger values for refractive indices. Less frequently, the glycerol wedge is in the opposite

![Figure 8. The geometry of the upgraded experiment (sandwich). The angles are defined in the text.](image)
direction, which leads to lower values for the refractive index. The effect was verified in the absence of the liquid crystal. Nevertheless, the experiment demonstrates nicely that values of the refractive indices at $\theta$ between 40° and 50° are consistent with the calculated direction dependence of the refractive index of the extraordinary wave in the uniaxial liquid crystal.

4. Conclusion

Students are confronted by the difficult concept of birefringence during physics lessons at university. We have shown that the concept can be fully demonstrated using nematic liquid crystalline wedge cells with different orientation of molecules. Most importantly, the experiment, the setup of which consists of a laser, rotatable table, two different wedge cells, two prisms with an apex angle of 45° and a drop of glycerol, enables quantitative measurements of the angular dependence of the refractive index of the extraordinary wave for the whole range of propagation directions. The homeotropic cell permits the demonstration of the direction dependence of the refractive index of the extraordinary wave, as well as its measurement, when its value is close to the value of the ordinary refractive index. The planar cell can be used for quantitative measurement of the refractive index direction dependence close to its largest value. By using additional glass prisms to form a sandwich of prisms and the wedge cell, one can study the angular dependence of the refractive indices in a range of directions that cannot be attained with a simple setup.

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