Three-Point Vortex Dynamics as a Lie-Poisson Reduced Space

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Abstract
This paper studies the reduced dynamics of the three-vortex problem from the point of view of Lie-Poisson reduction on the dual of the Lie algebra of $U(2)$. The algebraic study leading to this point of view has been given by Borisov and Lebedev [1, 2] (see also [3]). The main contribution of this paper is to properly describe the dynamics as a Lie-Poisson reduced system on $(u(2)^*,\{\ ,\}_{\text{LP}})$, giving a systematic construction of a one-parameter family of covectors $\{\sigma_1, \sigma_2, \sigma_3\}$ satisfying Pauli-commutation relations.

Keywords: N-vortex problem, Lie-Poisson reduction, Pauli matrices.

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1 Introduction

The \(N\)-point-vortex problem arises as a model on incompressible, homogeneous, inviscid fluid flows, governed by Euler’s equation on \(\mathbb{R}^2\), where the vorticity is assumed to be concentrated at \(N\) discrete points. The equations of motion are (cf. [5])

\[
\dot{z}_\alpha = \frac{i}{2\pi} \sum_{\beta \neq \alpha}^N \Gamma_\beta \frac{z\alpha - z\beta}{|z\alpha - z\beta|^2}
\]

where the \(\Gamma_\beta\)’s are the vortex strengths. These equations are equivalent to Hamilton’s equations \(i_{X_h} \Omega = dh\) with Hamiltonian

\[
h = -\frac{1}{2\pi} \sum_{\alpha < \beta} \Gamma_\alpha \Gamma_\beta \ln|z_\alpha - z_\beta|
\]

and symplectic form

\[
\Omega_0(z, w) = -\text{Im} \sum_{\alpha=1}^n \Gamma_\alpha z_\alpha \bar{w}_\alpha.
\]

The Hamiltonian and symplectic form are invariant with respect to the diagonal action of \(SE(2)\) on the phase space of the system identified with \(\mathbb{C}^N\) minus collision points:

\[
z_i \mapsto e^{i\theta} z_i + a
\]

\((\theta, a) \in SE(2) \cong S^1 \times \mathbb{C}\)

The model admits various conserved quantities related to \(SE(2)\) invariance, time-translation and rescaling symmetries:

\[
Z_0 = \Gamma_{\text{tot}}^{-1} \sum_k \Gamma_k z_k, \quad \Theta_0 = \sum_k \Gamma_k |z_k|^2, \quad (2)
\]

\[
\Psi_0 = -\sum_{n < k} \Gamma_n \Gamma_k \ln|z_n - z_k|
\]

\[
V_0 = \frac{1}{2i} \sum_k \Gamma_k (\bar{z}_k \dot{z}_k - z_k \dot{\bar{z}}_k) = \sum_{n < k} \Gamma_n \Gamma_k.
\]

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It should be noted that the expression

\[ M = \sum_{n<k} \Gamma_n \Gamma_k |z_n - z_k|^2, \]

also a conserved quantity, is not independent. Indeed, it is expressed in terms of \( \Theta_0 \) and \( Z_0 \):

\[ M = \Gamma_{\text{tot}} \Theta_0 - \Gamma_{\text{tot}}^2 |Z_0|^2, \]

where \( \Gamma_{\text{tot}} := \sum_k \Gamma_k. \)

The symplectic form (1) induces a Poisson structure on \( \mathbb{C}^N \). This Poisson structure can be written in terms of the following group invariants: the square of the distances between each pair of vortices and the oriented areas of each triad of vortices. By regarding these quantities as independent, Bolsinov, Borisov and Mamaev [3] are able to describe the reduced vortex dynamics as a subsystem of a hamiltonian system on the Lie algebra \( \mathfrak{u}(n-1) \). More concretely, Borisov and Lebedev [1, 2] study the 3-vortex compact and non-compact vortex dynamics using this point of view.

In this paper we concentrate on the 3-point vortex problem. One of the objectives is to clarify that, properly understood, the point of view of Borisov and Lebedev lead to a description of the reduced dynamics as a Lie-Poisson reduced system in \( \mathfrak{u}^*(2) \). The Poisson bracket for the reduced dynamics turns out to be the standard Lie-Poisson bracket on the dual of a Lie algebra (cf. [4, chap. 13]).

Moreover, this paper contributes by giving a systematic construction of a one-parameter family of covectors \( \{ \sigma_1, \sigma_2, \sigma_3 \} \) satisfying Pauli-commutation relations and relates such construction with canonical transformations involving Jacobi-Bertrand-Haretu coordinates for the original three point vortex system. The origin of the Pauli symbols is explained in a broader context and simple expressions for Casimirs are given in terms of the coordinates \( (a_0, a_1, a_2, a_3) \) induced by the dual basis of the Pauli symbols. This allows for the foliation of \( \mathfrak{u}(2)^* \) by level sets of Casimirs to be made explicit.

As a side note, it should be noticed that our preferred choice of Pauli symbols differs from the corresponding one given in [1]. With our choice, the positive direction of the \( a_1 \)-axis intersects the coadjoint orbit at a binary collision and the expression for the hamiltonian takes a simpler form.

## 2 Reduced space as coadjoint orbit in \( \mathfrak{u}(2)^* \)

The purpose of this section is to realize the symplectic reduced space in the 3-vortex problem as a coadjoint orbit \( O_\mu \) in \( \mathfrak{u}(2)^* \).
2.1 The extended vortex configuration space

Consider the Poisson structure on $C^3$ induced by the symplectic form $\Omega_0$:

$$\{f, g\}_{C^3} = \Omega_0(X_f, X_g)$$

with $X_h$ defined by Hamilton’s equation $i_{X_h} \Omega_0 = df$. Let $b_i = |z_j - z_k|^2$ and let $\Delta = \text{Im}[(z_3 - z_1)(z_1 - z_2)]/2$, the oriented area of the triangle with vertices at $z_1, z_2, z_3$. Let $C_3$ denote the cyclic permutations of indices $(1, 2, 3)$.

It is verified that, with $(i,j,k) \in C_3$,

$$\{b_i, \Delta\}_{C^3} = \frac{1}{2} \left[ \left( \frac{1}{\Gamma_j} - \frac{1}{\Gamma_k} \right) b_i + \left( \frac{1}{\Gamma_j} + \frac{1}{\Gamma_k} \right) (b_j - b_k) \right],$$

$$\{b_i, b_j\}_{C^3} = -8 \frac{\Delta}{\Gamma_k}.$$  (4)

Let $\mathcal{V} := \mathbb{R}^4$ and let $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{\Delta})$ be the dual to the standard basis in $\mathcal{V}$. (We emphasize that in this notation the over-bar does not denote complex conjugation.) Give $\mathcal{V}$ a Poisson structure by defining $\{\bar{b}_i, \Delta\}_{\mathcal{V}}$ and $\{\bar{b}_i, b_j\}_{\mathcal{V}}$ as the right-hand sides of (4), putting bars on top of the $b_i$’s and $\Delta$. Then $\psi : C^3 \rightarrow \mathcal{V}$ given by $(z_1, z_2, z_3) \mapsto (b_1, b_2, b_3, \Delta)$ is an $SE(2)$-invariant Poisson map. Moreover, restriction of $\psi$ to $P_0 \subset C^3$, where $P_0 \subset C^3$ is the set of vortex configurations with center of vorticity at the origin, gives an $SO(2)$-invariant Poisson map.

We call $\mathcal{V}$ the extended vortex configuration space. It is to be regarded as the space of “triangle configurations” with sides of length $\sqrt{b_i}$ and oriented area $\Delta$. Of course, only those points in $\mathcal{V}$ for which $b_i \geq 0$ and satisfy Heron’s condition relating the area and the sides of a triangle,

$$(4\Delta)^2 + b_1^2 + b_2^2 + b_3^2 - 2(b_1b_2 + b_2b_3 + b_3b_1) = 0,$$  (5)

have geometric and physical meaning.

Let $H(b_1, b_2, b_3, \Delta)$ be defined as the left-hand side of (5). We will refer to $H$ as Heron’s function. It is verified that both $I_2$ and $H$ are Casimirs of the Poisson structure $\{ , \}_{\mathcal{V}}$.

Observe that $\{ , \}_{\mathcal{V}}$ is closed in $\mathcal{V}^* \subset \mathcal{F}(\mathcal{V})$; that is to say, the bracket of two linear functionals is again a linear functional. It follows that $\{ , \}_{\mathcal{V}}$ makes $\mathcal{V}^*$ a four-dimensional real Lie algebra. It’s center $Z(\mathcal{V}^*)$ is

$$Z(\mathcal{V}^*) = \text{span}(\sigma_0), \quad \text{where} \quad \sigma_0 := \frac{1}{2\Gamma_{\text{tot}}} \sum_{(i,j,k) \in C_3} \Gamma_j \Gamma_k \bar{b}_i.$$
Moreover, let
\[
\sigma_1 := \frac{1}{2\Gamma_{\text{tot}}} \left( \Gamma_2 \Gamma_3 \tilde{b}_1 + \Gamma_3 \Gamma_1 \tilde{b}_2 - \Gamma_1 \Gamma_2 \frac{\Gamma_{\text{tot}} + \Gamma_3}{\Gamma_{\text{tot}} - \Gamma_3} \tilde{b}_3 \right),
\]
\[
\sigma_2 := \frac{1}{2} \sqrt{\frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_{\text{tot}}}} \left( -\tilde{b}_1 + \tilde{b}_2 + \frac{\Gamma_1 - \Gamma_2}{\Gamma_1 + \Gamma_2} \tilde{b}_3 \right),
\]
\[
\sigma_3 := 2 \sqrt{\frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_{\text{tot}}}} \Delta.
\]

(6)

**Proposition 2.1.** The \(\sigma_1, \sigma_2, \sigma_3\) satisfy Pauli's commutation relations:

\[
\{\sigma_i, \sigma_j\}_V = -2\sigma_k, \quad (i, j, k) \in C_3.
\]

(7)

It follows that \((\mathcal{V}^*, \{\ , \}_V)\) is isomorphic, as a Lie algebra, to \(u(2)\). Hence \((\mathcal{V}, \{\ , \}_\mathcal{V})\) is isomorphic, as a Poisson vector space, to \((u(2)^*, \{\ , \}_{LP})\), where \(\{\ , \}_{LP}\) denotes the Lie-Poisson braquet\footnote{See \cite[chap. 10]{[4]} for background on the Lie-Poisson braquet defined on the dual of a Lie algebra.}

\[
\{F, G\}_{LP}(\nu) \equiv \left\langle \nu, \left[ \frac{\delta F}{\delta \nu}, \frac{\delta G}{\delta \nu} \right] \right \rangle.
\]

**Definition 2.2.** We will refer to the \(\sigma_i\)'s as the Pauli symbols in \(\mathcal{V}^*\).

**Remark 2.3.** Commutation relations (7) follow from a direct computation. Nevertheless, we will leave the proof of theorem 2.1 to section 3 where we also discuss the origin of the definitions of the \(\sigma_i\)'s in a broader context.

### 2.2 The shape sphere

We start by considering the compact case \(W_0 > 0\), where

\[
W_0 \overset{\text{def}}{=} \frac{1}{\Gamma_1 \Gamma_2} + \frac{1}{\Gamma_2 \Gamma_3} + \frac{1}{\Gamma_3 \Gamma_1}.
\]

Given \(\nu \in \mathcal{V} \cong u(2)^*\), let \(\nu = \sum_{i=0}^{3} a_i \sigma^i\), where \(\{\sigma^0, \ldots, \sigma^3\}\) is the dual bases of \(\{\sigma_0, \ldots, \sigma_3\}\). Then the Casimirs \(I_2\) and \(H\) take the form

\[
I_2 = a_0, \quad H = 4 \frac{\Gamma_{\text{tot}}}{\Gamma_1 \Gamma_2 \Gamma_3} \left( a_1^2 + a_2^2 + a_3^2 - a_0^2 \right).
\]
Hence, the extended vortex configuration space $\mathcal{V}$ is foliated by the three-dimensional hyperboloids $a_1^2 + a_2^2 + a_3^2 - a_0^2 = h = \text{constant}$, corresponding to level sets of Heron’s function. The manifold $\psi(P_0) \subset \mathcal{V}$ (i.e., the physically meaningful portion of $\mathcal{V}$) corresponds to the half-cone $\tilde{h} = 0, a_0 > 0$. This half-cone is further foliated by the two-dimensional spheres of radius $\mu$

$$
S_\mu^2 = \left\{ \sum_{i=0}^{3} a_i \sigma^i \mid a_1^2 + a_2^2 + a_3^2 = \mu^2, a_0 = \mu \right\},
$$

each sphere lying on the three-dimensional hyperplane $a_0 = \mu > 0$. (Note that the vertex of the cone at $\mu = 0$ corresponds to triple collision.)

### 2.3 Jacobi-Bertrand-Haretu coordinates

We want to relate the above construction to Jacobi-Bertrand-Haretu coordinates. To accomplish this we will consider the composition of the following three canonical transformations.

Define $T_1 : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$(z_1, z_2, z_3) \mapsto (Z_0, r, s)$$

with

$$Z_0 = \frac{1}{\Gamma_{\text{tot}}} \sum_{j=1}^{3} \Gamma_j z_j \quad \text{(center of vorticity)},
$$

$$r = z_2 - z_1,
$$

$$s = z_3 - \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}.
$$

One computes that $\det T_1 = 1$, hence $T_1$ is invertible. Vectors $r$ and $s$ are the Jacobi-Bertrand-Haretu coordinates on the original system. The symplectic form after $T_1$ becomes

$$\Omega_1 \equiv T_1^* \Omega_0 = \Gamma_{\text{tot}} dZ_{0x} \wedge dZ_{0y} + A \, dr_x \wedge dr_y + B \, ds_x \wedge ds_y,$$

where

$$A := \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}, \quad B := \frac{\Gamma_1 + \Gamma_2) \Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3},$$

and the subindices $x$ and $y$ indicate real and imaginary parts.

Define $T_2 : \mathbb{C}^3 \to \mathbb{R}^2 \times \mathbb{R}^2 \times T^2$ by

$$(Z_0, r, s) \mapsto (K_x, K_y; j_1, j_2; \theta_1, \theta_2)$$

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with
\[ Z_0 = \frac{1}{\sqrt{\Gamma_{\text{tot}}}} (K_x + iK_y) \]
and
\[ r = \frac{\sqrt{2j_1} e^{i\theta_1}}{\sqrt{A}}, \quad s = \frac{\sqrt{2j_2} e^{i\theta_2}}{\sqrt{B}} \] (10)
with \( A \) and \( B \) as in (9). Then,
\[ \Omega_2 \overset{\text{def}}{=} T_2^* \Omega_1 = dK_x \wedge dK_y + dj_1 \wedge d\theta_1 + dj_2 \wedge d\theta_2. \]
Note that reduction by translational symmetry is achieved by setting the center of vorticity at the origin: \( K_x = K_y = 0 \).

Define \( T_3 : \mathbb{R}^2 \times \mathbb{R}^2 \times T^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^2 \times T^2 \) by
\[ (K_x, K_y; j_1, j_2; \theta_1, \theta_2) \mapsto (\tilde{K}_x, \tilde{K}_y; I_1, I_2; \varphi_1, \varphi_2) \]
through the (type II) generating function
\[ G_2 = K_x \tilde{K}_y + j_1(\varphi_2 - \varphi_1) + j_2(\varphi_1 + \varphi_2) \]
with the relations
\[ \theta_k = \frac{\partial G_2}{\partial j_k}, \quad I_k = \frac{\partial G_2}{\partial \varphi_k}, \quad k = 1, 2 \]
and \( K_y = \partial G_2/\partial K_x, \tilde{K}_x = \partial G_2/\partial K_y \). That is to say,
\[ (\tilde{K}_x, \tilde{K}_y) = (K_x, K_y), \]
and
\[ I_1 = j_2 - j_1, \quad I_2 = j_1 + j_2, \quad \varphi_1 = \frac{\theta_2 - \theta_1}{2}, \quad \varphi_2 = \frac{\theta_1 + \theta_2}{2}. \] (11) (12)
Note that, being half the angle between vectors \( r \) and \( s \), \( \varphi_1 \in [0, \pi] \). Also note that \( \Omega_3 := T_3^* \Omega_2 \) is again the standard symplectic form
\[ \Omega_3 = d\tilde{K}_x \wedge d\tilde{K}_y + dI_1 \wedge d\varphi_1 + dI_2 \wedge d\varphi_2. \]
A computation shows that:
\[ a_0 = I_2 \]
\[ a_1 = I_1 \]
\[ a_2 = \sqrt{I_2^2 - I_1^2} \cos(2\varphi_1) \]
\[ a_3 = \sqrt{I_2^2 - I_1^2} \sin(2\varphi_1) \] (13)
Figure 1: Phase portrait on $\mathcal{O}_\mu$ and on its cylindrical coordinates chart for $\Gamma_1 = 0.08904$, $\Gamma_2 = 0.28196$, and $\Gamma_3 = 0.629$. Solid dots on the cylindrical chart indicate binary collisions.

The symplectic leaves of $\mathcal{V}$ are its coadjoint orbits (the group orbits of the coadjoint action of $U(2)$ on the dual of its Lie algebra). Coadjoint orbits in $u(2)^*$ are 2-dimensional and connected. Therefore:

**Proposition 2.4.** For each $\mu \in \mathbb{R}^+$,

$$\mathcal{O}_\mu = S^2_\mu.$$

That is to say, each symplectic reduced space is identified with a two-dimensional sphere of radius $\mu$ in the hyperplane $a_0 = \mu > 0$, with center on the $a_0$-axis.

**Remark 2.5.** Since each point in the reduced space $\mathcal{O}_\mu$ represents an equivalence class of vortex configurations with the same “shape” (its orientation disregarded), it is natural to call $\mathcal{O}_\mu$ the **shape sphere**.

**Remark 2.6.** The case $\mu = 0$ corresponds to triple collision.

With this picture in mind, we see that $(I_1, 2\varphi_1) \in (-\mu, \mu) \times S^1$ are cylindrical coordinates of the shape sphere $\mathcal{O}_\mu$, with the cylindrical axis in the direction of $\sigma^1$. Figure 1-(a) shows, as an example, the phase portrait on $\mathcal{O}_\mu$ for particular choices of the vortex strengths and $\mu = 1$. 
2.4 The $\sigma^3$-axis

By definition of dual basis, each Pauli symbol $\sigma_i$ is the projection functional giving the $\sigma_i$-component of a vortex configuration $\nu$, i.e. $\sigma_i(\nu) = a_i$. This gives $a_3$ a simple interpretation in terms of the oriented area of the vortex triangle. Indeed, from (6):

$$a_3 = 2 \sqrt{\frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_{tot}}} \Delta.$$  \hspace{1cm} (14)

Thus, the $\sigma^3$-component of a point on the shape sphere $O_\mu$ is proportional to the oriented area of the vortex triangle.

Remark 2.7. Relation (14) can be recovered directly from (13) and the geometric interpretation of the canonical transformation $T_3 \circ T_2 \circ T_1$ constructed in section 2.3. Indeed:

$$\Delta = \frac{1}{2} |r'||s| \sin(\theta_2 - \theta_1) = \sqrt{\frac{\Gamma_{tot}}{\Gamma_1 \Gamma_2 \Gamma_3}} \sqrt{j_1 j_2} \sin(2\varphi_1) = \frac{1}{2} \sqrt{\frac{\Gamma_{tot}}{\Gamma_1 \Gamma_2 \Gamma_3}} a_3,$$

which agrees with (14).

2.5 Hamiltonian flow on $O_\mu$

The Hamiltonian

$$h = -\frac{1}{4\pi} \sum_{i=1}^{3} \Gamma_j \Gamma_k \ln b_i , \quad (i,j,k) \in C_3,$$  \hspace{1cm} (15)

induces a dynamic flow on the extended vortex configuration space $V$ which restricts to the reduced flow on the coadjoint orbits $O_\mu, \mu > 0$. Since $h$ does not depend on $\Delta$, it follows that, for a fixed $a_0 = \mu$, $h$ is a function of $a_1$ and $a_2$ only. Thus, the level sets of $h_\mu \equiv h|_{O_\mu}$ on the shape sphere $O_\mu$ are obtained by intersecting the cylinders

$$C_{\mu,E} = \left\{ \sum_i a_i \sigma^i \in V \mid h(a_0,a_1,a_2) = E, \quad a_0 = \mu \right\}$$

with the sphere $O_\mu$. Now, since the cylindrical axis of $C_{\mu,E}$ is along the $\sigma^3$-direction, the phase portrait of the Hamiltonian flow will be most symmetrical when represented using the cylindrical coordinates $(a_3, \alpha)$ with respect
to the $\sigma^3$-axis, that is to say,

$$a_1 = \sqrt{\mu^2 - a_3^2} \cos \alpha, \quad a_2 = \sqrt{\mu^2 - a_3^2} \sin \alpha.$$  

The coordinate transformation giving the cylindrical coordinates $(a_3, \alpha)$ of $O_\mu$ in terms of $(a_1, 2\varphi_1)$ is given by the equations

$$a_3 = \sqrt{\mu^2 - a_1^2} \sin(2\varphi_1),$$

$$\tan \alpha = \frac{\sqrt{\mu^2 - a_1^2} a_1}{a_1} \cos(2\varphi_1).$$

As an example, figure (b) shows the phase portrait on a cylindrical chart using coordinates $(a_3, \alpha)$, for particular choices of the vorticities.

### 3 Pauli symbols for the 3-vortex problem

The objective of this section is to justify the definitions of the Pauli symbols $\sigma_i, i = 1, 2, 3,$ that appear in (6) and the commutation relations (7).

Recall that $V \cong \mathbb{R}^4$ represents the extended vortex configuration space. Let $(b_1, b_2, b_3, \Delta)$ be the standard basis in $V$ and $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{\Delta})$ its dual basis in $V^*$. We have equipped $V$ with the Poisson structure

$$\{b_i, \Delta\} = \frac{1}{2} \left[ \left( \frac{1}{\Gamma_j} - \frac{1}{\Gamma_k} \right) b_i + \left( \frac{1}{\Gamma_j} + \frac{1}{\Gamma_k} \right) (b_j - b_k) \right],$$

$$\{\bar{b}_i, \bar{b}_j\} = -\frac{8 \Delta}{\Gamma_k},$$

which also defines a Lie bracket in $V^*$. Let $W^* = \text{span}\{\bar{b}_1, \bar{b}_2, \bar{b}_3\} \subset V^*$ and let $A : W^* \rightarrow W^*$ defined by $A(x) = \{x, \Delta\}$. The matrix of $A$ with respect to the basis $\beta = \{\bar{b}_1/\Gamma_1, \bar{b}_2/\Gamma_2, \bar{b}_3/\Gamma_3\}$ is

$$[A]_\beta = \frac{1}{2 \Gamma_1 \Gamma_2 \Gamma_3} \begin{pmatrix} \Gamma_1 (\Gamma_3 - \Gamma_2) & -\Gamma_1 (\Gamma_1 + \Gamma_3) & \Gamma_1 (\Gamma_1 + \Gamma_2) \\ \Gamma_2 (\Gamma_2 + \Gamma_3) & \Gamma_2 (\Gamma_1 - \Gamma_3) & -\Gamma_2 (\Gamma_1 + \Gamma_2) \\ -\Gamma_3 (\Gamma_2 + \Gamma_3) & \Gamma_3 (\Gamma_1 + \Gamma_3) & \Gamma_3 (\Gamma_2 - \Gamma_1) \end{pmatrix}.$$  

The kernel of both $[A]_\beta$ and $[A]_\beta^T$ is spanned by $(1, 1, 1)$. Moreover, it is easy to check that $\text{span}(u)$ is the center of $(V^*, \{\ , \})$, where $u \in W^*$ is such that its coordinate vector relative to $\beta$ is $(1, 1, 1)$, i.e.

$$u := \frac{\bar{b}_1}{\Gamma_1} + \frac{\bar{b}_2}{\Gamma_2} + \frac{\bar{b}_3}{\Gamma_3}.$$
Let $\langle \cdot , \cdot \rangle_\Gamma$ be the inner product on $W^*$ whose matrix representation in the basis $\beta$ is
\[
\begin{pmatrix}
\Gamma_2 + \Gamma_3 \\
\Gamma_3 + \Gamma_1 \\
\Gamma_1 + \Gamma_2
\end{pmatrix}.
\]
Let $S \overset{\text{def}}{=} \{ x \in W^* \mid \langle u, x \rangle_\Gamma = 0 \}$. Then

**Proposition 3.1.** It is verified that $S = \text{range}(A)$.

*Proof:* Since $u \in \ker A^*$, $\langle u, Ax \rangle = \langle A^*u, x \rangle = 0$ for all $x \in W^*$. □

The next two propositions (3.2 and 3.3) are easily proved by verifying the claim on a basis of $S$.

**Proposition 3.2.** For all $x \in S$, $\{ \{ x, \Delta \}, \Delta \} = -\Gamma_\text{tot}/(\Gamma_1 \Gamma_2 \Gamma_3) x$. In other words:
\[
A^2|_S = -\frac{\Gamma_\text{tot}}{\Gamma_1 \Gamma_2 \Gamma_3} \text{Id}|_S.
\]

*Proof:* It suffices to verify the claim for a basis of $S$, e.g. $\{ v_1, v_2 \}$ with
\[
[v_1]_\beta = (-\Gamma_3 - \Gamma_1, \Gamma_2 + \Gamma_3, 0), \quad [v_2]_\beta = (-\Gamma_1 - \Gamma_2, 0, \Gamma_2 + \Gamma_3), \quad (16)
\]
which is easily done. □

**Proposition 3.3.** Let $Q_1 : W^* \rightarrow \mathbb{R}$ be the quadratic form defined by $Q_1(x) \Delta = \{ x, \{ x, \Delta \} \}$. Then $Q_1|_S = Q_2|_S$, where $Q_2 : W^* \rightarrow \mathbb{R}$ is the quadratic form whose matrix representation in $\beta$ is:
\[
[Q_2]_\beta = -\frac{8}{\Gamma_1^2 \Gamma_2 \Gamma_3} Q,
\]
with
\[
Q \overset{\text{def}}{=} \begin{pmatrix}
(\Gamma_2 + \Gamma_3)^2 & \Gamma_1 \Gamma_2 & \Gamma_1 \Gamma_3 \\
\Gamma_1 \Gamma_2 & (\Gamma_3 + \Gamma_1)^2 & \Gamma_2 \Gamma_3 \\
\Gamma_1 \Gamma_3 & \Gamma_2 \Gamma_3 & (\Gamma_1 + \Gamma_2)^2
\end{pmatrix}.
\]

*Proof:* It suffices to verify that the bilinear forms associated with $Q_1$ and $Q_2$ coincide when evaluated at $(v_i, v_j)$, with $\{ v_1, v_2 \}$ as in (16). □

**Corollary 3.4.** Let $Q : S \rightarrow \mathbb{R}$ be defined by $Q(x) \Delta = \{ x, \{ x, \Delta \} \}$. Then $Q$ is negative definite.
Proof: It is easily verified that
\[
Q = T^t \cdot T + 2 \begin{pmatrix}
\Gamma_2 \Gamma_3 & \Gamma_3 \Gamma_1 \\
\Gamma_1 \Gamma_2 & \Gamma_1 \Gamma_2
\end{pmatrix}
\]
with \( T = \begin{pmatrix}
0 & \Gamma_3 & \Gamma_2 \\
\Gamma_3 & 0 & \Gamma_1 \\
\Gamma_2 & \Gamma_1 & 0
\end{pmatrix} \). Therefore \( Q \) is positive definite. Hence \([Q], \beta\), and thus \( Q \), are negative definite. \( \square \)

3.1 Pauli symbols

We will use the following notation: if \( E \) is a real vector space, its projective space is
\[
P(E) := (E \setminus \{0\}) / \sim
\]
where \( \sim \) is the equivalence relation on \( E \) defined by \( v \sim w \) if \( v = \lambda w \) for some \( \lambda \in \mathbb{R}, \lambda \neq 0 \).

Given \([x] \in P(S)\), let:
\[
\sigma_3 = \gamma \Delta, \quad \sigma_1 = \alpha x, \quad \text{and} \quad \sigma_2 = -\frac{1}{2} \{\sigma_3, \sigma_1\}.
\]

Then, from propositions 3.2, 3.3 and corollary 3.4,
\[
\begin{align*}
\{\sigma_1, \sigma_2\} &= \frac{\alpha^2 \gamma}{2} \{x, \{x, \Delta\}\} = \frac{\alpha^2 \gamma}{2} Q(x) \Delta \\
\{\sigma_2, \sigma_3\} &= \frac{\alpha \gamma}{2} \{\{x, \Delta\}, \Delta\} = -\frac{\alpha \gamma}{2} \frac{\Gamma_{\text{tot}}}{\Gamma_1 \Gamma_2 \Gamma_3} x \\
\{\sigma_3, \sigma_1\} &= -2 \sigma_2
\end{align*}
\]

Therefore, the Pauli commutation relations \( \{\sigma_i, \sigma_j\} = -2 \sigma_k, \ (i, j, k) \in \text{cyclic}(1, 2, 3) \), are equivalent to
\[
-\alpha^2 Q(x) = 4, \quad \gamma^2 \Gamma_{\text{tot}} = 4 \Gamma_1 \Gamma_2 \Gamma_3.
\]

Thus we arrive at the next proposition. (Note that the superindex \( \perp \) denotes the perpendicular subspace with respect to the usual dot product in \( \mathbb{R}^n \).)

Proposition 3.5. Let \( S_\Gamma := (\Gamma_2 + \Gamma_3, \Gamma_3 + \Gamma_1, \Gamma_1 + \Gamma_2)^\perp \subset \mathbb{R}^3 \). For any \([x] \in P(S_\Gamma)\), let
\[
\begin{align*}
\sigma_3 &= 2 \sqrt{\frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_{\text{tot}}}} \Delta, \quad \sigma_1 = \frac{\Gamma_1 \Gamma_2 \Gamma_2}{\sqrt{2 \sqrt{x^T \cdot Q \cdot x}}} \hat{x}, \quad \sigma_2 = -\frac{1}{2} \{\sigma_3, \sigma_1\} \quad (17)
\end{align*}
\]
where
\[
\wedge : \mathbb{R}^3 \to \mathcal{W}^* , \quad (x_1, x_2, x_3)^\wedge = x_1 \vec{b}_1/\Gamma_1 + x_2 \vec{b}_2/\Gamma_2 + x_3 \vec{b}_3/\Gamma_3 ,
\]
and \( \mathcal{Q} \) is the matrix defined in proposition 3.3. Then \( \sigma_1, \sigma_2, \sigma_3 \) satisfy the Pauli commutation relations
\[
\{ \sigma_1, \sigma_2 \} = -2 \sigma_3, \quad \{ \sigma_2, \sigma_3 \} = -2 \sigma_1, \quad \{ \sigma_3, \sigma_1 \} = -2 \sigma_2.
\]

We are now ready to establish that the Lie-algebra \((\mathcal{V}^*, \{ \; , \; \})\) is isomorphic to \(u(2)\). Let
\[
\tilde{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
(Note that \( \tilde{\sigma}_0 \) spans the center of \( u(2) \) and that \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \) are the Pauli spin matrices.) Let
\[
\sigma_0 = \frac{\Gamma_1 \Gamma_2 \Gamma_3}{2 \Gamma_{\text{tot}}} \left( \frac{\vec{b}_1}{\Gamma_1} + \frac{\vec{b}_2}{\Gamma_2} + \frac{\vec{b}_3}{\Gamma_3} \right) = \frac{1}{2 \Gamma_{\text{tot}}} \sum_{(i,j,k) \text{ cyclic}} \Gamma_i \Gamma_j \vec{b}_k . \tag{18}
\]

**Proposition 3.6.** For any \([x] \in \mathcal{P}(\mathcal{S}_\Gamma)\), let \( \sigma_1, \sigma_2, \sigma_3 \) be as in (17). Then the linear isomorphism \( \psi : \mathcal{V}^* \to u(2) \) given by
\[
\sigma_k \mapsto i \tilde{\sigma}_k , \quad k = 0, 1, 2, 3,
\]
defines a Lie algebra isomorphism.

**Proof** Both \( \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3 \} \), basis of \( \mathcal{V}^* \), and \( \{ i\tilde{\sigma}_0, i\tilde{\sigma}_1, i\tilde{\sigma}_2, i\tilde{\sigma}_3 \} \), basis of \( u(2) \), satisfy the same commutation relations. \( \square \)

**Proposition 3.7.** Let \((V, \{ \; , \; \})\) be a Poisson vector space such that \( \mathcal{V}^* \subset \mathcal{F}(V) \) is closed under \( \{ \; , \; \} \); hence \((\mathcal{V}^*, \{ \; , \; \})\) is a Lie algebra. Suppose that \( \psi : \mathcal{V}^* \to g \) is a Lie algebra isomorphism, i.e.
\[
[\psi(\alpha), \psi(\beta)] = \psi(\{ \alpha, \beta \})
\]
for all \( \alpha, \beta \in \mathcal{V}^* \). (Here \( \{ \; , \; \} \) denotes the Lie bracket on \( \mathfrak{g} \).) Let \( \varphi : g^* \to V \) be the adjoint operator to \( \psi \), i.e. defined by
\[
\langle \alpha, \varphi(\mu) \rangle = \langle \mu, \psi(\alpha) \rangle . \tag{19}
\]
Then \( \varphi \) is a Poisson transformation; that is to say,
\[
\{ f \circ \varphi, h \circ \varphi \}_{\text{LP}} = \{ f, h \} \circ \varphi
\]
for all \( f, h \in \mathcal{F}(V) \). (Here \( \{ \; , \; \}_{\text{LP}} \) denotes the Lie-Poisson bracket on \( g^* \); for its definition see [4, chap. 10].)
Proof: It suffices to consider the case $f = \alpha, g = \beta$, with $\alpha, \beta \in V^*$. Observe that

$$(f \circ \varphi)(\mu) = \langle f \circ \varphi, \mu \rangle = \langle \alpha, \varphi(\mu) \rangle = \langle \mu, \psi(\alpha) \rangle.$$ 

Thus, under the identification $g^{**} = g, f \circ \varphi = \psi(\alpha)$. Analogously, $g \circ \varphi = \psi(\beta)$. Hence

$$\{f \circ \varphi, g \circ \varphi\}_{LP}(\mu) = \langle \mu, \left[ \frac{\delta(f \circ \varphi)}{\delta \mu}, \frac{\delta(g \circ \varphi)}{\delta \mu} \right] \rangle$$

$$= \langle \mu, [\psi(\alpha), \psi(\beta)] \rangle = \langle \mu, \psi(\{\alpha, \beta\}) \rangle$$

$$= \langle \{\alpha, \beta\}, \varphi(\mu) \rangle = \langle \{f, g\} \circ \varphi, \mu \rangle$$

$$= (\{f, g\} \circ \varphi)(\mu)$$

as claimed. □

Propositions 3.6 and 3.7 allow us to identify the extended vortex configuration space $V$ with the Poisson vector space $u(2)^*$:

Corollary 3.8. For any $[x] \in P(S_\Gamma)$, let $\psi : V^* \to u(2)$ be the Lie algebra isomorphism defined in proposition 3.6. Let $\varphi : u(2)^* \to V$ be its adjoint operator (defined by \(19\)). Then $\varphi$ is a Poisson transformation.

The symplectic leaves of $V$ are its coadjoint orbits. These are the common level sets of Casimirs $I_2$ and $H$, the semi-moment of circulation and Heron’s function defined in section 2. These Casimirs take very simple forms when expressed in terms of the dual to the Pauli symbols basis. Indeed, a computation shows that:

Proposition 3.9. For all $[x] \in P(S_\Gamma)$,

$$I_2 = \Gamma_{\text{tot}} a_0 \quad \text{and} \quad H = \frac{4\Gamma_{\text{tot}}}{\Gamma_1 \Gamma_2 \Gamma_3} \left( a_1^2 + a_2^2 + a_3^2 - a_0^2 \right).$$

Here $(a_0, a_1, a_2, a_3)$ denotes the coordinates in $V$ with respect to the basis $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$, defined as the dual to the basis of Pauli symbols $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, which in turn were defined in (17) and (18).

We conclude that the symplectic leaves of $V$ are the submanifolds

$$\mathcal{O}_{(\mu, \tilde{H})} \overset{\text{def}}{=} \left\{ \sum_i a_i \sigma^i \mid a_0 = \mu, a_1^2 + a_2^2 + a_3^2 - \mu^2 = \tilde{H} \right\}.$$
Physically (and geometrically) meaningful dynamics occur only when $H = 0$ and $\mu \geq 0$. Therefore, the symplectic reduced spaces for the three-vortex problem are the coadjoint orbits

$$O_\mu \overset{\text{def}}{=} \{ \sum_i a_i \sigma^i \mid a_0 = \mu, a_1^2 + a_2^2 + a_3^2 = \mu^2 \}, \quad \mu \in \mathbb{R}^+.$$  

Note that $\mu = I_2/\Gamma_{\text{tot}}$ and that $\mu = 0$ corresponds to triple collision.

In this way, given $\mu \in \mathbb{R}^+$, the symplectic reduced space $O_\mu$ is a two-dimensional sphere of radius $\mu$, embedded in the hyperplane $a_0 = \mu$. The “vertical” $a_3$-axis represents the oriented area of the triangle formed by the vortices. Hence, the equator $a_3 = 0$ corresponds to collinear configurations and the “north” and “south” hemispheres correspond to the two possible orientations of the triangle, with the poles representing equilateral triangles with opposite orientation.

### 3.2 Choice of orientation for the $a_1$-$a_2$ plane

The three possible binary collision appear as points on the equator of the sphere $O_\mu$. It is convenient to choose an orientation of the $a_1, a_2$ axes (within the plane containing the equator of the sphere) so that the binary collision $z_1 - z_2 = 0$, which in coordinates $(I_1, I_2, \varphi_1, \varphi_2)$ corresponds to $I_1 = I_2$, lies on the positive direction of the $a_1$-axis. This collision is represented by a vector in $V$ of the form $B_{12} = \lambda(\mathbf{b}_1 + \mathbf{b}_2) = (\lambda, \lambda, 0, 0)$. The condition $\sigma_2(B_{12}) = 0$ yields

$$-\frac{1}{2} \{\sigma_3, \sigma_1\}(\mathbf{b}_1 + \mathbf{b}_2) = 0$$

hence

$$\{\mathbf{\Delta}, \hat{x}\}(\mathbf{b}_1 + \mathbf{b}_2) = 0.$$  

Therefore,

$$\left( (\Gamma_3 - \Gamma_2)x_1 - (\Gamma_1 + \Gamma_3)x_2 + (\Gamma_1 + \Gamma_2)x_3 \right) \mathbf{b}_1 + \left( (\Gamma_2 + \Gamma_3)x_1 + (\Gamma_1 - \Gamma_3)x_2 - (\Gamma_1 + \Gamma_2)x_3 \right) \mathbf{b}_2 \mathbf{b}_1 + \mathbf{b}_2 = 2\Gamma_3(x_1 - x_2) = 0.$$  

Thus $x_1 = x_2$. Now, from the condition $\hat{x} \in \mathcal{S}_\Gamma$, i.e.

$$x_1(\Gamma_2 + \Gamma_3 + \Gamma_1) + x_3(\Gamma_1 + \Gamma_2) = 0,$$  

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we obtain $x_3 = -(\Gamma_{\text{tot}} + \Gamma_3)/(\Gamma_{\text{tot}} - \Gamma_3)x_1$. Therefore, setting

$$\hat{x} = x\left(1, 1, -\frac{\Gamma_{\text{tot}} + \Gamma_3}{\Gamma_{\text{tot}} - \Gamma_3}\right), \quad x \in \mathbb{R} \setminus \{0\}$$

is equivalent to positioning the binary collision $B_{12}$ at the point where the positive direction of the $q_3$-axis intersects the sphere $O_\mu$. It is straightforward to check that, with this choice of $\hat{x}$, the Pauli symbols defined in (17) and (18) take the form (6).

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