A simple PDE-constrained Optimization Problem to Evaluate the Strategy for Fishery Resource Transportation

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Abstract

We present a new impulse control problem of degenerate parabolic Fokker-Planck equations that govern stochastic growth dynamics of biological resource from the viewpoint of probability density functions of the body weight. The problem we focus on is impulsively transporting a resource population from a habitat to other habitat(s), which stems from a recent engineering problem in a river in Japan. A formula of the optimal control is theoretically derived with the help of an adjoint equation method. A remarkable point is simplicity of the problem such that finding the optimal control can be achieved by only solving an adjoint equation. We present a demonstrative computational example of Ayu sweetfish \textit{Plecoglossus altivelis altivelis} utilizing a very recent Weighted Essentially Non-Oscillatory method.

1 Introduction

Partial differential equations (PDEs) are versatile mathematical tools for describing a variety of physical, biological, chemical, and many other phenomena occurring in the world [1]. Especially, degenerate parabolic PDEs frequently arise in dynamic optimization problems of stochastic systems in which the dynamics to be controlled follow some stochastic differential equations (SDEs) [2]. Dynamic optimization problems arising in biological resources management are no exception where degenerate PDEs play central roles.

In this paper, we consider a new PDE-constrained optimization problem where the decision-maker, the controller, can intervene the target system dynamics only impulsively. More specifically, we consider a transportation problem of a fishery resource population from a habitat to other habitat(s) artificially. Such a problem has arisen in H River in Japan where the first author has been studying as a field survey site. This river has a weir that Ayu sweetfish \textit{Plecoglossus altivelis altivelis} (abbreviated as \textit{P. altivelis}), which is a major inland fishery resource in Japan [3], has a difficulty to ascend. It has been implied, although deeper investigation will be necessary, that the habitat downstream of the weir has a lower quality (food resource may be less available) than that upstream of the weir. Currently, a local fishery cooperative is planning to artificially transport the fish from the downstream habitat to the upstream habitat. The transportation will be carried out within a short duration (several hours) that is significantly shorter than the life history of the fish (one year). Then, when, and how much of the fish should be transported?

The previous study indicated that the growth dynamics of the fish are stochastic and can be described with an SDE of a logistic type [4]. The stochastic logistic model has been applied to a problem of statistically estimating the intrinsic growth rates [5]. The logistic model is an SDE having non-Lipschitz coefficients, but is well-posed and solvable in a path-wise sense [6]. Therefore, the classical theory of SDEs requiring the Lipschitz continuity of the coefficients does not apply to this unique and sound model.

Rather than the SDEs, we utilize the Fokker-Planck equation that governs their probability density functions (PDFs) [7]. Assuming an exponential population decay that is reasonable when the resource competition is weak, the total population in a habitat is a multiple of a PDF. The optimization problem is then formulated as an impulse control problem of a system of Fokker-Planck equations governing the upstream and downstream stochastic growth dynamics of the fish. In our case, the Fokker-Planck equation has degenerate coefficients due to the logistic nature.

Impulse control problems of PDEs have not been well-studied compared with the regular control problems of PDEs [8]. Impulse control of the heat equations, the simplest PDEs, has been mathematically and numerically studied only recently by Yu et al. [9]. Mbogne and Thron [10] analyzed an anthracnose treatment problem as a mixed regular-impulse control problem based on a non-degenerate parabolic PDE. To the best of the authors’ knowledge, impulse control of degenerate parabolic PDEs...
has not been studied so far.

The objective as well as contribution of this paper are formulation and brief numerical analysis of an impulse control problem of transporting the population of *P. alivelis*. This is a PDE-constrained problem based on degenerate parabolic PDEs. For the sake of simplicity, we assume that the possible intervention times are prescribed *a priori* as in the existing models [9-10]. We show that the adjoint method [8-10] formally applies to our problem and that finding the optimal control efficiently reduces to solving only the adjoint equation. This is a difference between the existing models [9-10] and ours because the former usually requires iteratively solving the primal and adjoint problems. Our model avoids this difficulty.

We present a detailed derivation procedure of the adjoint equation, especially the internal boundary conditions to be satisfied at the intervention times, because the form of the equation itself is non-trivial. We finally present a computational example using the very recent high-resolution finite difference scheme based on a Weighted Essentially Non-Oscillatory (WENO) method [11].

## 2 Mathematical Model

### 2.1 Logistic Model

We briefly review the stochastic logistic model as it is a foundation of the proposed mathematical model. The time is denoted as \( t \geq 0 \) and the body weight of an individual fish in a habitat at time \( t \) is considered as a non-negative continuous-time process \( X = (X_t)_{t \geq 0} \). Its stochasticity is assumed to be driven by a Gaussian white noise in a multiplicative form using a 1-D standard Brownian motion process \( B = (B_t)_{t \geq 0} \) so that \( X \) is non-negative as required biologically. The stochastic logistic model is given by

\[
dX = rX \left( 1 - \frac{X}{K} \right) (r dt + \sigma dB_t), \quad t > 0, \tag{1}
\]

where \( r > 0 \) is the intrinsic (deterministic) growth rate, \( \sigma^2 > 0 \) with \( \sigma > 0 \) represents the stochastic growth rate, and \( K > 0 \) is the capacity serving as the maximum body weight. The SDE (1) subject to an initial condition \( X_0 \in (0, K) \) gives a path of the biological growth of the individual fish. A more realistic model can consider individual differences of the growth and maximum weight [12]. The presented logistic model is therefore a simplified model such that all the individuals in the habitat share the generic form of the growth dynamics.

The SDE (1) has the quadratic drift and diffusion coefficients, meaning that the standard methodology for SDEs having Lipschitz continuous coefficients do not apply [13]. Nevertheless, it has been show using a contradiction argument that the SDE (1) is uniquely solvable in the path-wise sense and that each path satisfies the constraint \( X_t \in (0, K) \) almost surely for \( t > 0 \) if the initial condition satisfies \( X_0 \in (0, K) \) [6]. Especially, the former result means that we can consider feedback control problems of the stochastic logistic model.

### 2.2 System of Fokker-Planck Equations

Assume that there exist the two habitats, which are referred to as the (downstream) habitat 1 and (upstream) habitat 2. We assume, as briefly discussed in the previous section, that the habitat 1 has a lower quality than the habitat 2, and that the individuals cannot migrate from 2 to 1. There exists a possibility that the individuals migrate or are flushed out from the habitats 2 to 1, but this is not considered in this paper for the sake of simplicity of the mathematical modeling. The lower letters 1 and 2 represent the parameters of the habitats 1 and 2. In addition, assume \( 0 < K_1 < K_2 \) to consider a situation that the individuals possibly become larger in the habitat 2. The initial time \( t = 0 \) is assumed to be taken in a season where the fish is at the juvenile stage. We want to find a transport plan of the population from the habitat 1 to habitat 2.

The population density of the resource population in the habitat \( i = 1, 2 \) is denoted as \( y^{(i)}(t, x) \), where \( x \) represents the body weight. The total population at time \( t \) is then given as the total integral

\[
M_{i,t} = \int_0^t y^{(i)}(t, x) \, dx, \tag{2}
\]

where the range of integration is the compact interval \([0, K_i]\). Set \( \Omega = (0, K_i) \). Based on the SDE (1) without interventions, the Fokker-Planck equation governing the PDF \( y^{(i)} \) is given as the degenerate parabolic PDE [7]

\[
\frac{\partial y^{(i)}}{\partial t} + \frac{\partial F^{(i)}}{\partial x} = -R_i y^{(i)}, \quad t > 0, \quad x \in \Omega \tag{3}
\]

with the flux

\[
F^{(i)} = rX \left( 1 - \frac{x}{K_i} \right) y^{(i)} - \frac{\sigma^2}{2} x^2 \left( 1 - \frac{x}{K_i} \right)^2 y^{(i)}, \tag{4}
\]

where \( R_i < 0 \) is the mortality rate in the habitat \( i \).

The Fokker-Planck equation (3) is defined over the open set \( \Omega \). At the domain boundaries \( x = 0, K_i \), we implicitly prescribe the no-flux condition \( F^{(i)} = 0 \). In addition, we impose an initial condition \( y^{(i)}(0, x) \) such that \( y^{(i)}(0, 0) = y^{(i)}(0, K_i) = 0 \). This problem setting assumes that initially all the individuals have the body weights such that \( X_0 \in \Omega \). This assumption is not restrictive from an engineering viewpoint because the body weight 0 is meaningless and the individuals having the maximum body weight would not appear at the juvenile stage.

For the later use, we present an alternative operator rep-
representation of the Fokker-Planck equation (3):

\[
\frac{\partial y^{(i)}}{\partial t} + A^{(i)} y^{(i)} = -R_i y^{(i)} \quad \text{with} \quad A^{(i)} y^{(i)} = \frac{\partial E^{(i)}}{\partial x}.
\]  

(5)

We assume that \( y^{(i)} \) is non-negative, uniformly bounded, integrable over \( \Omega_i \), and satisfies (3) in a distributional sense at least locally in time. Sharper assumptions on the Fokker-Planck equations maybe necessary, but the topic is beyond the scope of this paper.

### 2.3 Optimization Problem

Now, we consider an optimization problem of the Fokker-Planck equation (3) by impulse controls. We assume that the impulse controls can be activated at prescribed times \( 0 < \tau_1 < \tau_2 < \ldots < \tau_j < T \) with a fixed \( J \in \mathbb{N} \) and a fixed terminal time \( T > 0 \). The time just after \( \tau_j \) is denoted as \( \tau_{j+} \).

We consider proportional transportation strategies of the following form: at the time \( \tau_j \) transport the population from the habitat 1 to the habitat 2 as an internal boundary condition in the temporal direction

\[
\begin{pmatrix}
    y^{(1)}(\tau_j) \\
    y^{(2)}(\tau_j)
\end{pmatrix} = \begin{pmatrix}
    1-u_j & 0 \\
    u_j & 1
\end{pmatrix} \begin{pmatrix}
    y^{(1)}(\tau_j) \\
    y^{(2)}(\tau_j)
\end{pmatrix}
\]

(6)

with \( u_j \in L^\infty(\Omega_j) \), \( 0 \leq u_j \leq \bar{u} \) with a given \( 0 < \bar{u} < 1 \).

This impulsive intervention is additionally imposed to the Fokker-Planck equations explained above. More rigorously, the admissible set \( \mathcal{U} \) of the controls is set as

\[
\mathcal{U} = \left\{ u = \{ u_j \}_{1 \leq j \leq J} \mid u_j \in L^\infty(\Omega_j), 0 \leq u_j \leq \bar{u}, 1 \leq j \leq J \right\}.
\]

(7)

In (6), we are suppressing the dependence of \( y^{(i)} \) on \( x \) to simplify the notations. Similar representations are used in the sequel. Note that any element \( \nu \in L^\infty(\Omega_j) \) can be trivially extended to an element in \( L^\infty(\Omega_j) \) by setting \( \nu = 0 \) in \( (K_1, K_2) \) by \( K_1 \leq K_2 \).

The series of functions \( u = \{ u_j \}_{1 \leq j \leq J} \) represents the control in our problem. In this context, \( u_j y^{(i)} \) represents the transportation density of the population in the habitat 1 at time \( \tau_j \), and the total population transported at this time is simply given by its integration over \( \Omega_i \).

We consider a problem that the control variable \( u \) is optimized so that the objective function \( \phi \) of the following form is minimized:

\[
\phi(y, u) = \sum_{j=1}^{J} \int_{\Omega_j} v_j y^{(i)}(\tau_j) - \int_{\mathcal{T}} z^{(2)}(T).
\]

(8)

The first term represents the total transportation cost with a proportional coefficient \( c > 0 \), and the second term represents the opposite of the utility by increasing the population in some target range \( \omega \subseteq \Omega_2 \) of the body weight in the habitat 2 at the terminal time \( T \).

A minimizer \( u \in \mathcal{U} \) is called an optimal control and is denoted as \( u = u^* \). The goal of the problem is to find \( u^* \).

### 2.4 System of Adjoint Equations

Since directly minimizing \( \phi \) seems to be impossible due to its PDE-based nature, we employ the adjoint method to optimize it [8]. We demonstrate that solving an adjoint equation is sufficient to find \( u^* \) in our case.

We explain the derivation procedure of the adjoint equation in the rest of this section. Choose some \( u \in \mathcal{U} \). Set the sensitivity \( z^{(i)}(t, x; u) \) as the directional derivative at \( u \in \mathcal{U} \) in the direction of \( \nu \) such that \( \nu \in \mathcal{U} \). The sensitivity equation serving as its governing equation of the sensitivity \( z^{(i)}(t, x; u) \) is

\[
\frac{\partial z^{(i)}}{\partial t} + A^{(i)} z^{(i)} = -R^{(i)} z^{(i)}, \quad t \neq \tau = \{ \tau_j \}_{1 \leq j \leq J}
\]

subject to the zero-flux terminal condition as in the Fokker-Planck equation and the initial condition \( z^{(i)}(0) = 0 \).

By (6), the internal boundary condition is

\[
\begin{pmatrix}
    z^{(1)}(\tau_j) \\
    z^{(2)}(\tau_j) \\
    y^{(1)}(\tau_j) \\
    y^{(2)}(\tau_j)
\end{pmatrix} = \begin{pmatrix}
    1-u & 0 \\
    u & 1 \\
    -v & 0 \\
    v & 0
\end{pmatrix} \begin{pmatrix}
    z^{(1)}(\tau_j) \\
    z^{(2)}(\tau_j) \\
    y^{(1)}(\tau_j) \\
    y^{(2)}(\tau_j)
\end{pmatrix}.
\]

(10)

By utilizing \( z^{(i)} \), the directional derivative of \( \phi \) at \( u \in \mathcal{U} \) in the direction of \( \nu \) such that \( \nu \in \mathcal{U} \), expressed as \( \phi' \), satisfies

\[
\phi'(y, u; \nu) = \sum_{j=1}^{J} \int_{\Omega_j} v_j y^{(i)}(\tau_j) + \sum_{j=1}^{J} \int_{\Omega_j} u z^{(i)}(\tau_j) - \int_{\mathcal{T}} z^{(2)}(T).
\]

(11)

We rewrite the right-hand side of (11) so that we can clearly identify a necessary optimality condition. For this purpose, we set the adjoint equation governing adjoint variables \( p^{(i)} \) as follows:
(12) \[
\frac{\partial p^{(i)}}{\partial t} + A^{(i)^*} p^{(i)} = -R^{(i)} p^{(i)}
\]

with a formal adjoint operator \( A^{(i)^*} \) of \( A^{(i)} \), and
\[
A^{(i)^*} p^{(i)} = -\frac{\sigma^2}{2} x^2 \left( 1 - \frac{x}{k_i} \right) \frac{\partial^2 p^{(i)}}{\partial x^2} - \frac{\sigma^2}{2} \left( 1 - \frac{x}{k_i} \right) \frac{\partial p^{(i)}}{\partial x} - \sum_{j=1}^{M} \frac{\sigma^2}{2} p^{(j)}.
\] (13)

The boundary condition consistent with the formal adjoint operator is given as
\[
\frac{\sigma^2}{2} x^2 \left( 1 - \frac{x}{k_i} \right) \frac{\partial^2 p^{(i)}}{\partial x^2} = 0 \quad \text{at} \quad x = 0, k_i.
\] (14)

and
\[
p^{(i)}(T) = 0, \quad p^{(2)}(T) = \begin{cases} -1 & (x \in \omega) \\ 0 & (x \notin \omega) \end{cases} = -\delta_{i,2,\omega}.
\] (15)

The adjoint equations should also associate internal boundary conditions, but their functional forms are not prescribed here since they seem not to be trivial. Instead, these conditions are found through the discussion below.

Set \( K_1 = K_2 \) (\( \Omega_1 = \Omega_2 \)) for the sake of simplicity, but the case \( K_1 \neq K_2 \) can be handled similarly at the cost of the increase of the formal complexity. By utilizing the adjoint equation, we get the following primal-adjoint relationship as a key equality to rewrite (11) as follows:
\[
\int_{0}^{T} p^{(i)} \cdot A^{(i)^*} z^{(i)} \, dx = \int_{0}^{T} z^{(i)} \cdot A^{(i)^*} p^{(i)} \, dx, \quad t \notin \tau.
\] (16)

Now, we rewrite (11). For \( i = 1, 2 \), we have
\[
\int_{0}^{T} p^{(i)} \cdot z^{(i)} \, dx = \int_{0}^{T} p^{(1)} \cdot z^{(1)} \, dx + \sum_{j=1}^{M} \int_{f_{z^{(j)}}}^{i} p^{(j)} \cdot z^{(j)} \, dx - \int_{0}^{T} z^{(i)} \cdot \frac{\partial p^{(i)}}{\partial t} \, dx
\] (17)

and thus
\[
\int_{0}^{T} p^{(i)} \cdot z^{(i)} \, dx = \int_{0}^{T} \left[ p^{(i)} z^{(i)} \right]_0 + \sum_{j=1}^{M} \left[ p^{(j)} z^{(j)} \right]_{f_{z^{(j)}}}^{i} + \left[ p^{(i)} z^{(i)} \right]_{T} - \int_{0}^{T} z^{(i)} \cdot \frac{\partial p^{(i)}}{\partial t} \, dx
\]

by the definition of the adjoint equation and the relationship (16). Combining (17) and (19) yields the equality connecting the variables at the interfaces:
\[
\int_{0}^{T} \left[ p^{(i)} z^{(i)} \right]_0 + \sum_{j=1}^{M} \left[ p^{(j)} z^{(j)} \right]_{f_{z^{(j)}}}^{i} + \left[ p^{(i)} z^{(i)} \right]_{T} - \int_{0}^{T} z^{(i)} \cdot \frac{\partial p^{(i)}}{\partial t} \, dx = 0.
\] (20)

The integrands of (20) are evaluated as follows:
\[
\left[ p^{(i)} z^{(i)} \right]_0 = p^{(i)}(\tau_i) z^{(i)}(\tau_i) - p^{(i)}(0) z^{(i)}(0),
\]
\[
\left[ p^{(j)} z^{(j)} \right]_{f_{z^{(j)}}}^{i} = p^{(j)}(\tau_i) z^{(j)}(\tau_i),
\]
and
\[
\left[ p^{(i)} z^{(i)} \right]_{T} = p^{(i)}(T) z^{(i)}(T) - p^{(i)}(\tau_i) z^{(i)}(\tau_i),
\]
\[
\int_{0}^{T} z^{(i)} \cdot \frac{\partial p^{(i)}}{\partial t} \, dx = \delta_{i,2,\omega} z^{(i)}(T) - p^{(i)}(\tau_i) z^{(i)}(\tau_i).
\] (23)

Consequently, we get
\[
\phi'(y,u) \cdot v = \sum_{j=1}^{J} v_j \phi_j'(y,u) \cdot v_j + \sum_{j=1}^{J} u_j \phi_j''(y,u) \cdot v_j + \sum_{j=1}^{J} z_j \phi_j'''(y,u) \cdot v_j \\
- \int_{\Omega} z^{(2)}(T) \, dx \\
= \sum_{j=1}^{J} \left( \sum_{i=1}^{2} \phi_j'(y,u) \cdot v_j \phi_j''(y,u) \cdot v_j + \phi_j''(y,u) \cdot v_j \phi_j'''(y,u) \cdot v_j \right) \\
= \sum_{j=1}^{J} I_j \\
where \\
I_j = \int_{\Omega} \phi_j'(y,u) \cdot v_j \, dx + \int_{\Omega} u_j \phi_j''(y,u) \cdot v_j \, dx + \int_{\Omega} z_j \phi_j'''(y,u) \cdot v_j \, dx.
\]

By (10), we have

\[
\phi'(y,u) \cdot (v-u) \\
= \sum_{j=1}^{J} \int_{\Omega} (v_j - u_j) \phi_j(y,u) \left[ c - p^{(1)}(\tau_j) + p^{(2)}(\tau_j) \right] \, dx \\
\geq 0
\]
for all \( v \) such that \( v \in \mathfrak{U} \) (For example, see [9,10]). Assume that \( y^{(0)} \geq 0 \). We get, at each state \( x \), if
\[ c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) > 0 \text{ then } u_{j}^{*} = 0. \] In fact, we have 
\[ v_{j} - u_{j}^{*} = v_{j} \geq 0 \] in this case by the requirement 
\[ v_{j} \in [0, u] \] and (35) is certainly satisfied. On the other hand, if 
\[ c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) < 0, \] then 
\[ u_{j}^{*} = \bar{u}. \] In fact, we get 
\[ v_{j} - u_{j}^{*} = v_{j} - \bar{u} \leq 0 \] in this case by the requirement 
\[ v_{j} = 0 \] and (35) is again satisfied. Consequently, we state that the optimal control, which is feasible, has the form

\[ u_{j}^{*} = u_{j}^{*}(x) = \begin{cases} 0 & \text{if } c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) \geq 0, \\ \bar{u} & \text{otherwise}. \end{cases} \] (36)

assuming that the set where 
\[ c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) = 0 \] has the Lebesgue measure zero. Namely, (36) is valid if the set of points such that 
\[ c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) = 0 \] has the 1-D length 0.

Firstly, we have the continuity 
\[ p^{(2)}(\tau_{j}) = p^{(2)}(\tau_{j}) , \] meaning that no specific internal boundary condition is necessary for 
\[ p^{(2)} \]. On the other hand, for 
\[ p^{(0)} \], we get

\[
p^{(0)}(\tau_{j}) = cu_{j} + (1 - u_{j})p^{(0)}(\tau_{j}) + u_{j}p^{(2)}(\tau_{j}) \]

\[ = p^{(0)}(\tau_{j}) 
+ u_{j} \left( c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) \right) \] \hspace{1cm} (37)

For the optimal control, by (36), we can effectively replace (37) by

\[
p^{(0)}(\tau_{j}) = p^{(0)}(\tau_{j}) + u_{j} \left( c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) \right) \]

\[ = p^{(0)}(\tau_{j}) + u_{j} \min \left\{ c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}), 0 \right\} \] \hspace{1cm} (38)

In summary, to compute the optimal control \( u^{*} \), what we should do is to solve the adjoint equation(s) (12) backward in time starting from the terminal condition (15) with the internal boundary conditions (32) and (38). The optimal control is given by (36) as a quantity determined from the adjoint variables \( p^{(0)} \).

**2.6 Computational Example**

The above-presented solution procedure of the optimization problem is implemented by numerical computation of the adjoint equation(s) (12). We employ a finite difference scheme based on the WENO reconstruction of a local Lax-Friedrichs spatial discretization in space [11] and a classical explicit Euler scheme in time. The internal boundary conditions are local updating equations and thus do not require any special treatment. The WENO reconstruction has already been verified against test cases, demonstrating its excellent performance to accurately solve hyperbolic (completely degenerate parabolic) problems [11].

We consider a case of \( P. \text{altivelis} \), and employ a set of hypothetical parameter values because the detailed parameter values are still collecting in H River. The proposed mathematical framework can be applied to individuals transport problems between arbitrary physically disconconnected habitats such as problems between two different rivers. We consider the parameter values of the fish \( P. \text{altivelis} \) in G River (habitat 1) and H River (habitat 2) in San-in area, Japan. Basically, the growth rate are larger in H river (\( r_{1} < r_{2} \)) based on field survey results in 2018-2020. The magnitudes of the stochasticity \( \sigma \) has been evaluated in H River, while it is not in G River. As a demonstrative example, we assume \( \sigma_{1} = \sigma_{2} \). However, the mathematical and numerical approaches in this paper can also be applied to the cases \( \sigma_{1} > \sigma_{2} \) and \( \sigma_{1} < \sigma_{2} \) as well without theoretical difficulties. Furthermore, set \( K_{1} = K_{2} \) for the sake of simplicity of numerical computation because we want to discuss a simplest case.

We set the following parameter values based on the available biological data: \( r_{1} = 0.03 > 0.02 = r_{1} \) (1/day), \( \sigma_{1} = \sigma_{2} = 0.062 \) (1/day), \( u = 0.1 \), \( c = 0.2 \), \( R_{1} = R_{2} = 0.015 \) (1/day), \( r_{1} = 10j \) (\( j = 1, 2, 3, ..., 6 \)).

The terminal time is set as \( T = 70 \) (day) and the time increment for the temporal integration as 0.01 (day). The spatial domain is normalized to be the unit interval \( \Omega_{i} = [0, 1] \), and is uniformly discretized into 200 sub-intervals having the common length of 0.005. This normalization is justified because we are assuming \( K_{1} = K_{2} \) and we normalize (1) as \( X_{i} \rightarrow XK \). The normalized \( X \) is valued in \([0, 1]\). Set \( \omega = (0.2, 0.8) \).

The initial populations are \( y^{(i)} = 0 \) (\( x < 0.1, x > 0.2 \)), \( y^{(i)} = 10 \) (\( 0.1 \leq x \leq 0.2 \)), and \( y^{(i)} = 0 \).

In the computation below, the Fokker-Planck equation (3) under the optimal control is also numerically computed to show behavior of the controlled population dynamics; however, we again stress that the equation does not have to be solved for finding the optimal control. The WENO method for conservative hyperbolic problems [14] combined with the central differencing to the diffusion term is used for discretizing the Fokker-Planck equation.

Figs. 1 and 2 show the computed \( p^{(0)} \) and \( p^{(2)} \), respectively. Similarly, Figs. 3 and 4 show the controlled population \( y^{(i)} \) and \( y^{(i)} \). The computed optimal control \( u^{*} \) at each \( \tau_{j} \) is also plotted in each figure. The computational results demonstrate that the PDEs are computed without spurious oscillation. The obtained optimal control is the bang-bang type as expected, and the assumption that the set where \( c - p^{(0)}(\tau_{j}) + p^{(2)}(\tau_{j}) = 0 \) has the Lebesgue
measure zero seems to be satisfied. The controlled population $y^{(1)}$ is high in and around the prescribed target set $\omega = (0.2, 0.8)$ as preferred by the controller.

Fig. 1. The computed adjoint variable $p = p^{(1)}$ (Colored surface). The computed optimal control $u^*$ at each $\tau_j$ is also plotted (black circles).

Fig. 2. The computed adjoint variable $p = p^{(2)}$ (Colored surface). The computed optimal control $u^*$ is also plotted as in Fig. 1.

Fig. 3. The computed population $y = y^{(1)}$ (Colored surface). The computed optimal control $u^*$ is also plotted as in Fig. 1.

Fig. 4. The computed population $y = y^{(2)}$ (Colored surface). The computed optimal control $u^*$ is also plotted as in Fig. 1.

3 Conclusions

We formulated and analyzed a new impulse control problem as a PDE-constrained optimization problem. The optimal control was theoretically obtained using an adjoint equation method and was numerically computed focusing on a realistic case. Especially, the finite difference scheme based on the recent WENO reconstruction successfully handled the degenerate parabolic adjoint equation. The established framework is applicable to problems with more realistic and complicated growth dynamics where the environmental capacity is life history-dependent [12]. In this example, the dimension of the PDEs increases and the admissible set of controls must be modified since not all the variables are directly observable. Furthermore, we have fixed the chances to execute the impulsive interventions, while they can also become additional variables to be optimized. Theoretically, the adaptive timing approach that has been employed for impulse control problems of ordinary differential equations arising in cancer therapy [15] can be extended to our case. However, its computational cost needs to be carefully evaluated. Sufficiency of the optimality condition should also be considered in future for both the proposed and extended models.

The proposed model can be straightforwardly extended to models with multiple habitats if the internal boundary conditions are properly specified. Numerical computation of such an extended model, of course, requires higher computational costs. From an engineering viewpoint, data collection for parameter identification becomes harder as the total number of habitats increases. In addition, real implementability of theoretically optimal controls should also be considered. We believe that this statement is true as the set of admissible controls is bounded, closed, and convex, and the objective function is bounded provided that solutions to the Fokker-Planck equations are integrable.

The PDE-constrained problem discussed in this paper has a limitation that the biological growth of the resource is stochastic but gradual. In the real world, flood and draught would serve as catastrophes for the resource population dynamics. Problems considering such events are doubly
stochastic in the sense that the stochastic biological dynamics are affected by stochastic environmental shocks. Lévy processes are mathematically rigorous and sufficiently general candidates for modeling the environmental shocks [16]. Numerical techniques to discretize stochastic PDEs having degenerate coefficients [17] can be effectively utilized in such cases and would be satisfactory from an engineering viewpoint. However, convergence of optimal controls and associated resource dynamics would be a highly non-trivial issue. Specifically, there is no guarantee that the WENO-based finite difference scheme can handle stochastic PDEs. These issues will be analyzed from both engineering and scientific sides in our future research.

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