Quantum theory does not admit local or noncontextual hidden variable (NCHV) models. This is manifest in the nonlocality of contextuality and the theory. Contextuality precludes the possibility that quantum measurements reveal pre-existing outcomes. It arises from the non-existence of a global joint probability distribution over measurement outcomes that can reproduce the measurement statistics predicted by quantum theory. Nonlocality is a special case of contextuality, applicable when measurements are spacelike separated. Traditionally, contextuality has been shown with respect to NCHV models of projective measurements for Hilbert spaces of dimension three or greater.

For projective measurements, noncontextuality is the assumption that the outcome of a measurement \( A \) is independent of whether it is performed together with a measurement \( B \), where \([A, B] = 0\), or with measurement \( C \), where \([A, C] = 0\) and \( B \) and \( C \) are not compatible, i.e., \([B, C] \neq 0\). B and C provide contexts for measurement of A. A qubit cannot yield a proof of traditional contextuality because it does not allow projective measurements \( A, B, C \) such that \([A, B] = 0, [A, C] = 0\), and \([B, C] \neq 0\). The existence of a triple of such projective measurements is necessary for any proof of traditional contextuality. While a state-independent proof of contextuality holds for any state-preparation, a state-dependent proof requires a special choice of the prepared state. The minimal state-independent proof of traditional contextuality requires a qutrit and 13 projectors. The minimal state-dependent proof, first given by Klyachko et al. [9], requires a qutrit and five projectors (Fig. 1). Thus a qutrit is the simplest quantum system that allows a proof of traditional contextuality, both state-independent and state-dependent. However, we note that contextuality for a qubit has been considered earlier in a manner that is conceptually distinct from the approach we adopt here. Our approach builds upon the work of Spekkens [16] and Liang et al. [17].

A contextuality scenario is a collection of subsets, called ‘contexts’, of the set of all measurements. A context refers to measurements that are jointly implementable. Conceptually, the simplest possible contextuality scenario, first considered by Specker [3] (Fig. 2), requires three two-valued measurements, \( \{M_1, M_2, M_3\} \), to allow for three non-trivial contexts: \( \{\{M_1, M_2\}, \{M_1, M_3\}, \{M_2, M_3\}\} \). Any other choice of contexts will be trivially non-contextual, e.g., \( \{\{M_1, M_2\}, \{M_1, M_3\}\} \) is non-contextual because the joint probability distribution \( p(M_1, M_2, M_3) \equiv p(M_1, M_2)p(M_1, M_3)/p(M_1) \) reproduces the marginal statistics. In Specker’s scenario, measurement statistics that always shows perfect anti-correlation between any two measurements sharing a context is necessarily contextual. On assigning outcomes \( \{+, -\} \) noncontextually to the three measurements \( \{M_1, M_2, M_3\} \), it becomes obvious that the maximum number of anti-correlated contexts possible in a single assignment is two, e.g., for the assignment \( \{M_1 \rightarrow +, M_2 \rightarrow -, M_3 \rightarrow +\} \), \( \{M_1, M_2\} \) and \( \{M_2, M_3\} \) are anti-correlated but \( \{M_1, M_3\} \) is not. This puts an upper bound of \( \frac{2}{3} \) on the probability of anti-correlation when a context is chosen uniformly at random. Specker’s scenario precludes projective measurements because a set of three pairwise commuting projective measurements is trivially jointly measurable and cannot show contextuality. One may surmise that it represents a kind of contextuality that is not seen in quantum theory. However, as Liang et al. [17] showed, this contextuality scenario can be realized using noisy spin-1/2 observables. They showed that if one does not assume outcome determinism for unsharp measurements and models them stochastically but noncontextually, then this generalized-noncontextual model [16] for noisy spin-1/2 observables will obey a bound of \( 1 - \frac{2}{3} \eta \), where \( \eta \in [0, 1] \) is
That is, $M$ Specker’s inequality so the noncontextual bound is larger for noisy spin-1/2 measurements. We will refer to this noncontextual inequality as Specker’s inequality. After giving examples of orthogonal and trine spin-axes that did not seem to show a violation of this inequality, Liang et al. 17 left open the question of whether such a violation exists. They conjectured that all such triples of POVMs will admit a generalized-noncontextual model 16, i.e., Specker’s inequality will not be violated.

Our main result is a proof that a state-dependent violation of Specker’s inequality is possible. To show this, we set up Specker’s inequality for three unsharp qubit POVMs, obtain constraints on $\eta$ from joint measurability, and construct the joint measurement POVMs. We prove that noisy spin-1/2 observables do not allow a state-independent violation of Specker’s inequality, followed by our main result: a state-dependent violation of Specker’s inequality and the optimal choice of state and measurements for it.

**Specker’s inequality.**—The three POVMs considered, $M_k = \{E^{k\pm}_k, E^k_\pm\}$, $k \in \{1, 2, 3\}$, are noisy spin-1/2 observables of the form

$$E^k_\pm = \frac{1}{2} I \pm \eta \frac{1}{2} \vec{\sigma} \cdot \hat{k}_k, \quad 0 \leq \eta \leq 1.$$  \(2\)

That is,

$$E^k_\pm = \frac{1 - \eta}{2} I + \eta \Pi_{\pm}^k, \quad (3)$$

where $\Pi_{\pm}^k$ are the corresponding projectors. So $E^k_\pm$ are noisy versions of the projectors $\Pi_{\pm}^k$, and the observable $\{E^k_+, E^k_-\}$ is therefore a noisy (or unsharp) version of the projective measurement $P_k = \{\Pi_{\pm}^k, \Pi_{\mp}^k\}$ (for $k \in \{1, 2, 3\}$).

Specker’s inequality concerns the following quantity:

$$R_3 \equiv \frac{1}{3} \sum_{(ij) \in \{(12),(23),(13)\}} \Pr(M_i \neq M_j) \leq 1 - \frac{\eta}{3}, \quad (1)$$

where $\Pr(M_i \neq M_j)$ is the probability of anticorrelation between measurements $M_i$ and $M_j$. Note that $1 - \frac{\eta}{3} \geq \frac{2}{3}$, so the noncontextual bound is larger for noisy spin-1/2 measurements. We will refer to this noncontextual inequality as Specker’s inequality. After giving examples of orthogonal and trine spin-axes that did not seem to show a violation of this inequality, Liang et al. 17 left open the question of whether such a violation exists. They conjectured that all such triples of POVMs will admit a generalized-noncontextual model 16, i.e., Specker’s inequality will not be violated.

The question is: Does there exist a triple of noisy spin-1/2 observables that will violate this inequality, perhaps for some specific state-preparation?

**Joint measurability constraints on $\eta$.**—Testing Specker’s inequality for a quantum mechanical violation requires a special kind of joint measurability, denoted by jointly measurable contexts $\{\{M_1, M_2\}, \{M_2, M_3\}, \{M_1, M_3\}\}$, i.e., pairwise joint measurability but no triplewise joint measurability. For a given choice of measurement directions $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ in eq. 2, the necessary and sufficient condition for this kind of joint measurability is

$$\eta < \eta \leq \eta_u$$  \(6\)

where

$$\eta = \frac{8}{\sum_{X_1, X_2, X_3 \in \{\pm 1\}} \sqrt{3 + 2 \sum_{k \neq l} E_k X_k \hat{n}_k \cdot \hat{n}_l}} \quad (7)$$

and

$$\eta_u = \frac{\eta}{\min_{(ij) \in \{(12),(23),(13)\}} \frac{\sqrt{2}}{\sqrt{1 + \hat{n}_i \cdot \hat{n}_j}}} \quad (8)$$

These are obtained as special cases of the more general joint measurability conditions obtained in Appendix F of 17. Explicit bounds on $\eta$ are computed in Appendix A for reference.

**Joint measurement effects.**—We construct the joint measurement POVM, $G_{ij} = \{G_{ij}^{+}, G_{ij}^{-}, G_{ij}^{0}, G_{ij}^{0'}\}$, such that the given POVMs, $M_i = \{E^{k\pm}_i, E^k_\pm\}$ and $M_j = \{E^{l\pm}_j, E^l_\pm\}$, are recovered as marginals, i.e., $\sum_{X_i} G_{ij}^{0} = E^i_{X_i}$, $\sum_{X_j} G_{ij}^{0} = E^j_{X_j}$, $0 \leq G_{ij}^{0} \leq 1$, and $\sum_{X_i, X_j} G_{ij}^{0} = I$, where $X_i, X_j \in \{\pm, -\}$. The joint measurement POVM has the following general form:

$$G_{ij}^{+} = \frac{1}{2} \frac{\alpha_{ij}}{2} \left[ I + \vec{\sigma} \cdot \frac{1}{2} \left( \eta (\hat{n}_i + \hat{n}_j) - \hat{a}_{ij} \right) \right] \quad (9)$$

$$G_{ij}^{-} = \frac{1}{2} \frac{\alpha_{ij}}{2} \left[ I - \vec{\sigma} \cdot \frac{1}{2} \left( \eta (\hat{n}_i - \hat{n}_j) + \hat{a}_{ij} \right) \right] \quad (10)$$

$$G_{ij}^{0} = \frac{1}{2} \left[ (1 - \alpha_{ij}) \frac{1}{2} I + \vec{\sigma} \cdot \frac{1}{2} \left( \eta (\hat{n}_i + \hat{n}_j) + \hat{a}_{ij} \right) \right] \quad (11)$$

$$G_{ij}^{0'} = \frac{1}{2} \left[ (1 - \alpha_{ij}) \frac{1}{2} I - \vec{\sigma} \cdot \frac{1}{2} \left( \eta (\hat{n}_i - \hat{n}_j) - \hat{a}_{ij} \right) \right] \quad (12)$$

FIG. 2. Specker’s [3] contextuality scenario.
where $I$ is the $2 \times 2$ identity matrix and $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the $2 \times 2$ Pauli matrices. The necessary and sufficient conditions for these to be valid qubit effects, $0 \leq G_{X_i,X_j}^i \leq I$, $\forall X_i, X_j \in \{+, -\}$, are equivalent to the following inequalities [13].

$$\sqrt{2\eta^2(1 + \hat{n}_i \hat{n}_j) + |\bar{a}_{ij}|^2 + 2\eta(|\hat{n}_i + \hat{n}_j| \bar{a}_{ij})} \leq \alpha_{ij}$$  \hspace{1cm} (13)

$$\alpha_{ij} \leq 2 - \sqrt{2\eta^2(1 - \hat{n}_i \hat{n}_j) + |\bar{a}_{ij}|^2 + 2\eta(|\hat{n}_i - \hat{n}_j| \bar{a}_{ij})},$$  \hspace{1cm} (14)

where $\eta_i < \eta \leq \eta_a$. The construction of the joint measurement POVM and derivation of the necessary and sufficient condition for its validity, (13)-(14), are provided in Appendix B. The joint measurement effects corresponding to anti-correlation sum to

$$G_{i+}^{ij} + G_{i-}^{ij} = (1 - \frac{\alpha_{ij}}{2})I + \frac{1}{2} \bar{\sigma}.\bar{a}_{ij}. $$  \hspace{1cm} (15)

No state-independent violation. — We will now show that no state-independent violation of Specker’s inequality is possible.

**Theorem 1** There exists no state-independent violation of the generalized-noncontextual inequality $R_{\alpha} \leq 1 - \frac{3}{4}$ using a triple of qubit POVMs, $\{M_k\} = \{E_{\pm k}\}_{k \in \{1, 2, 3\}}$, that are pairwise jointly measurable but not triplywise jointly measurable.

Proof.— In quantum theory, the probability $R_{\alpha}^Q$ for anti-correlation of measurement outcomes for pairwise joint measurements of $M_k = \{E_{\pm k}\}$ (where $k \in \{1, 2, 3\}$) has the following form:

$$R_{\alpha}^Q \equiv \frac{1}{3} \sum_{(ij) \in \{(12), (23), (13)\}} \text{Tr} (\rho (G_{i-}^{ij} + G_{i+}^{ij})).$$  \hspace{1cm} (16)

The condition for violation of noncontextual inequality [5] is $R_{\alpha}^Q > 1 - \frac{3}{4}$. Using (15), this reduces to

$$\text{Tr} (\rho \sum_{(ij)} (\alpha_{ij} I - \bar{\sigma}.\bar{a}_{ij})) < 2\eta $$  \hspace{1cm} (17)

Using the standard $2 \times 2$ Pauli matrices and $\rho$ parameterized by $0 \leq q \leq 1$ and $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$:

$$\rho = q|\psi\rangle \langle\psi| + (1 - q)(I - |\psi\rangle \langle\psi|)$$  \hspace{1cm} (18)

$$|\psi\rangle = \left(\begin{array}{c}
\cos \frac{\theta}{2} \\
e^{i\phi} \sin \frac{\theta}{2}
\end{array}\right) = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$  \hspace{1cm} (19)

the condition for violation becomes

$$\sum_{(ij)} \alpha_{ij} + \lambda_\rho < 2\eta $$  \hspace{1cm} (20)

where

$$\lambda_\rho \equiv (1 - 2q)\bar{a} \cdot \hat{n} \in [\bar{a}^2, |\bar{a}|] $$  \hspace{1cm} (21)

denotes the state-dependent term in the condition and $\bar{a} = (a_x, a_y, a_z)$ is given by

$$a_x = \sum_{(ij)} (\bar{a}_{ij})_x, a_y = \sum_{(ij)} (\bar{a}_{ij})_y, a_z = \sum_{(ij)} (\bar{a}_{ij})_z.$$  \hspace{1cm} (22)

For a state-independent violation, either the state-dependent term in (20), $\lambda_\rho$, must vanish for all qubit states $\rho$, or $\sum_{(ij)} \alpha_{ij} + \bar{a} \cdot \hat{n} < 2\eta$ should hold. The first case, $\lambda_\rho = 0 \forall \rho$, requires $\bar{a} = 0$, since $\bar{a}$ is the only term in $\lambda_\rho$ that depends on the joint measurement POVM. This means $a_x = a_y = a_z = 0$, so that $\lambda_\rho = 0$ for all $\rho$. The second case requires $\sum_{(ij)} \alpha_{ij} + \bar{a} < 2\eta$. In both cases, we have the following lower bound on $\alpha_{ij}$, from inequality (13):

$$\alpha_{ij} > \sqrt{2\eta^2} + \hat{n}_i \hat{n}_j $$  \hspace{1cm} (23)

Taking the sum of $\alpha_{ij}, (ij) \in \{(12), (23), (13)\}$, we have

$$\sum_{(ij)} \alpha_{ij} > \sqrt{2\eta^2} \sum_{(ij)} \sqrt{1 + \hat{n}_i \hat{n}_j} $$  \hspace{1cm} (24)

For the first case, the condition for state-independent violation is $\sum_{(ij)} \alpha_{ij} < 2\eta$, while for the second case the condition for such a violation is $\sum_{(ij)} \alpha_{ij} + |\bar{a}| < 2\eta$. Given the lower bound on $\sum_{(ij)} \alpha_{ij}$, it follows that a necessary condition for state-independent violation of Specker’s inequality is:

$$\sum_{(ij)} \sqrt{1 + \hat{n}_i \hat{n}_j} < \sqrt{2} $$  \hspace{1cm} (25)

We will show that there exists no choice of measurement directions that will satisfy this necessary condition, thereby ruling out a state-independent violation of Specker’s inequality. The particular cases of orthogonal axes ($\hat{n}_i, \hat{n}_j = -1/2$) used in [17], are clearly ruled out by this necessary condition. Denoting $\hat{n}_i, \hat{n}_j \equiv \cos \theta_{ij}$, the necessary condition for violation is

$$|\cos \frac{\theta_{12}}{2}| + |\cos \frac{\theta_{13}}{2}| + |\cos \frac{\theta_{23}}{2}| < 1 $$  \hspace{1cm} (26)

Without loss of generality, the three directions can be parameterized as:

$$\hat{n}_1 \equiv (0, 0, 1),$$  \hspace{1cm} (27)

$$\hat{n}_2 \equiv (\sin \theta_{12}, 0, \cos \theta_{12}),$$  \hspace{1cm} (28)

$$\hat{n}_3 \equiv (\sin \theta_{13} \cos \phi_3, \sin \theta_{13} \sin \phi_3, \cos \theta_{13}).$$  \hspace{1cm} (29)

where

$$0 < \frac{\theta_{ij}}{2} < \frac{\pi}{2}, \forall (ij) \in \{(12), (13), (23)\}, \quad 0 \leq \phi_3 < 2\pi,$$

and $\cos \theta_{23} = \sin \theta_{12} \sin \theta_{13} \cos \phi_3 + \cos \theta_{12} \cos \theta_{13}$. This implies:

$$\cos(\theta_{12} + \theta_{13}) \leq \cos(\theta_{23}) \leq \cos(\theta_{12} - \theta_{13}).$$  \hspace{1cm} (30)
Then
\[
\min_{\theta_{12}, \theta_{13}, \theta_{23}} \left\{ |\cos \theta_{12}/2 + |\cos \theta_{13}/2| + |\cos \theta_{23}/2| \right\} \geq \frac{1}{2}.
\]
\[
\min_{\theta_{12}, \theta_{13}} \left\{ |\cos \theta_{12}/2 + |\cos \theta_{13}/2| + \sqrt{1 + \cos(\theta_{12} + \theta_{13})} \right\} > 1.
\]

This contradicts the necessary condition \( \sum \). Hence, there is no state-independent violation of Specker’s inequality \( \{ \alpha, \beta \} \) allowed by noisy spin-1/2 observables. \( \Box \)

State-dependent violation of Specker’s inequality. — Our main result is that Specker’s inequality can be violated in a state-dependent manner. From the condition for violation \( (20) \), it follows that a necessary condition for state-dependent violation is \( \sum_{ij} \alpha_{ij} - |\vec{a}| < 2\eta \). An optimal choice of \( \rho \) that yields \( \lambda_{\rho} = -|\vec{a}| \) corresponds to \( q = 1 \) and \( \vec{a}, \vec{n} = |\vec{a}| \), i.e.,
\[
\cos \theta = \frac{a_z}{|\vec{a}|}, \quad \tan \phi = \frac{a_y}{a_x}.
\]

With this choice of \( \rho \) the question becomes: Does there exist a choice of \( \{ n_1, n_2, n_3 \} \), \( \{ \alpha_{ij}, \vec{a}_{ij} \} \) such that \( \sum_{ij} \alpha_{ij} - |\vec{a}| < 2\eta \)? We show that this is indeed the case. We define
\[
C \equiv 2\eta - \left( \sum_{\langle i \rangle} \alpha_{ij} - |\vec{a}| \right), \quad (31)
\]
so that \( C > 0 \) indicates a state-dependent violation. Note that violation of Specker’s inequality \( R_{ij}^Q \leq 1 - \frac{4}{3} \) is characterized by
\[
S \equiv R_{ij}^Q - (1 - \frac{\eta}{3}) = \frac{C}{6} \quad (32)
\]
where \( S > 0 \) for a state-dependent violation. The optimal value of \( C \),
\[
C_{\text{max}} = \max C, \quad (33)
\]
denotes the maximum possible violation, and \( S_{\text{max}} = C_{\text{max}}/6 \). It turns out that there is a range of choices that one could make for these parameters that will allow a state-dependent violation. Our main result is:

Theorem 2 The optimal violation of Specker’s inequality corresponds to coplanar measurements along \( \{\hat{n}_1, \hat{n}_2, \hat{n}_3\} \) such that \( \hat{n}_1, \hat{n}_2, \hat{n}_3 \rightarrow -1 \), \( \hat{n}_2, \hat{n}_3 = 2(\hat{n}_1, \hat{n}_2)^2 - 1 \), and \( |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) if the plane of measurements is the Z-X plane. Under this choice of state and measurement directions, the lower and upper bounds on \( \eta \) are given by \( \eta_l \rightarrow \frac{4}{9} \approx 0.6667 \) and \( \eta_u \rightarrow 1 \), and the optimal violation \( C_{\text{max}} \rightarrow 2\eta(1 - \eta) = \frac{4}{9} \approx 0.4444 \), \( S_{\text{max}} = C_{\text{max}}/6 \rightarrow \eta(1 - \eta) = \frac{4}{9} \approx 0.0741 \) or 7.41% for \( \eta \rightarrow \eta_l \).

Thus the quantum probability of anti-correlation can exceed the classical bound by an amount arbitrarily close to \( \frac{4}{9} \approx 0.0741 \) or about 7.41%. The quantum degree of anti-correlation is \( R_{ij}^Q = S_{\text{max}} + (1 - \frac{4}{9}) \rightarrow \frac{23}{27} \approx 0.8519 \) and the classical bound is \( (1 - \frac{4}{9}) \rightarrow \frac{7}{9} \approx 0.7778 \) for the optimal scenario. Geometrically, the optimal choice corresponds to choosing measurement directions \( \{ \hat{n}_1, \hat{n}_2, \hat{n}_3 \} \) in a plane of the Bloch sphere passing through its centre and preparing the qubit in a pure state perpendicular to this plane (Fig. 3). A detailed proof of Theorem 2 is provided in Appendix C.

Discussion. — Specker’s scenario (Fig. 2) is the simplest contextuality scenario since at least three two-valued measurements are required for contextuality to make sense. A qubit is the simplest quantum system and our proof shows that Specker’s scenario can be realized on a qubit and stronger-than-classical anti-correlations obtained. This provides a novel no-go theorem for generalized noncontextuality \( 16 \), of which traditional noncontextuality is a special case.

It is interesting to note that the classical bound \( 1 - \frac{4}{9} \geq \frac{2}{3} \), so adding noise to the noncontextual model seems to make it more difficult to obtain stronger-than-classical anti-correlation. This was one of the reasons that Liang et al. \( 17 \) did not expect a violation of this inequality, since the quantum probability of anti-correlations in their examples did not exceed \( \frac{2}{3} \). In view of our result, it is clear that quantum theory catches up with the classical bound and does better if one makes a careful choice of qubit state and measurements. An interesting open question is whether such a violation is possible in higher dimensional systems and whether the amount of violation could be higher for these than for a qubit. Besides, our result hints at the fact that perhaps all contextuality scenarios may be realizable and contextuality demonstrated if we consider the possibilities that general quantum measurements allow. In particular, scenarios that involve pairwise compatibility between all measurements but no global compatibility may be realizable within quantum theory. Specker’s scenario is the simplest such example we have considered.

Conclusion.— The joint measurability allowed in a theory restricts the kind of contextuality scenarios that can arise in it. Quantum theory admits Specker’s contextuality scenario if one uses unsharp measurements \( 17 \). Further, as we have shown, quantum theory allows violations of the noncontextual bound for anti-correlations in this scenario. Thus, quantum theory is contextual even in the simplest contextuality scenario. It may be interesting to verify these violations in experiments.
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Appendix A: Bounds on $\eta$ from joint measurability

We obtain bounds on $\eta$ from the more general constraints proven in Appendix F of [17]. Let us rewrite the expressions [7,8] as

$$\eta = \frac{4}{\sum_{i=1}^{4} \cos \theta_{ij} \cos \theta_{kz} \cos \theta_{lk} - \hat{p}_i} \quad (A1)$$

where $\cos \theta_{ij} = \hat{n}_i \cdot \hat{n}_j$, $\hat{p}_i = (1, 1, 1)$, $\hat{p}_2 = (-1, 1, -1)$, $\hat{p}_3 = (-1, 1, 1)$, $\hat{p}_4 = (1, -1, 1)$, $(\cos \theta_{ij}, \cos \theta_{kz}, \cos \theta_{lk}) = (\cos \theta_{ij} + \cos \theta_{kz} + \cos \theta_{lk})$, etc., and

$$\eta_u = \min_{(ij) \in \{(12), (23), (13)\}} \frac{1}{\cos \theta_{ij} \frac{1}{2} + \sin \theta_{ij} \frac{1}{2}}. \quad (A2)$$

Bounds on $\eta_u$: One may check that

$$6 \leq \sum_{i=1}^{4} \sqrt{3} + 2(\cos \theta_{ij} \cos \theta_{kz} \cos \theta_{lk}) \frac{1}{\hat{p}_i} \leq 4\sqrt{3} \approx 6.9282 \quad (A3)$$

where $0 < \theta_{ij}, \theta_{kz}, \theta_{lk} < \pi$, so that

$$\frac{1}{\sqrt{3}} \leq \eta_u \leq \frac{2}{3}. \quad (A4)$$

or

$$0.577 \leq \eta_u \leq 0.667.$$ 

Clearly, $\eta_u = \frac{1}{\sqrt{3}}$ for $\theta_{ij} = \theta_{kz} = \theta_{lk} = 0$, and $\eta_u = \frac{2}{3}$ for $\theta_{ij} = \theta_{kz} = \theta_{lk} = \frac{2\pi}{3}$.

Bounds on $\eta_u$:

$$1 < |\cos \frac{\theta_{ij}}{2}| + |\sin \frac{\theta_{ij}}{2}| \leq \sqrt{2} \quad (A5)$$

where $0 < \theta_{ij} < \pi$ and the maxima occurs at $\theta_{ij} = \frac{\pi}{2}$. Therefore,

$$\frac{1}{\sqrt{2}} \leq \eta_u < 1, \quad (A6)$$

or

$$0.707 \leq \eta_u < 1.$$ 

The necessary condition for the joint measurability required is $\frac{1}{\sqrt{3}} < \eta < 1$ for any choice of $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$. Also, the sufficient condition for this joint measurability for all choices of $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ is $\frac{2}{3} < \eta \leq \frac{1}{\sqrt{2}}$. Combining these, the necessary and sufficient condition that ensures pairwise joint measurability but no triplewise joint measurability for all choices of $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ is

$$\frac{2}{3} < \eta \leq \frac{1}{\sqrt{2}} \quad (A7)$$

That is, given $\frac{2}{3} < \eta \leq \frac{1}{\sqrt{2}}$, all triples of noisy spin-1/2 observables allow for pairwise joint measurability but no triplewise joint measurability.

Orthogonal spin axes: $\hat{n}_i \cdot \hat{n}_j = 0 \ \forall (ij) \in \{(12), (13), (23)\}$. From $[A1, A2]$, the necessary and sufficient joint measurability condition is

$$\frac{1}{\sqrt{3}} \leq \eta \leq \frac{1}{\sqrt{2}} \quad (A8)$$

Trine spin axes: $\hat{n}_i \cdot \hat{n}_j = -\frac{1}{2} \ \forall (ij) \in \{(12), (13), (23)\}$. From $[A1, A2]$, the necessary and sufficient joint measurability condition is

$$\frac{2}{3} < \eta \leq \sqrt{3} - 1 \quad (A9)$$
Appendix B: Constructing the joint measurement

The joint measurement $G_{ij}$ for $\{M_i, M_j\}$ should satisfy the marginal condition:

\begin{align}
G_{++}^{ij} + G_{+-}^{ij} &= E_+^i, \\
G_{-+}^{ij} + G_{--}^{ij} &= E_+^j, \\
G_{++}^{ij} + G_{-+}^{ij} &= E_+^i, \\
G_{--}^{ij} + G_{+-}^{ij} &= E_+^j.
\end{align}

(B1)

Also, the joint measurement should consist of valid effects:

\[ 0 \leq G_{ij}^{++}, G_{ij}^{+-}, G_{ij}^{-+}, G_{ij}^{--} \leq 1, \]

(B3)

where $I$ is the $2 \times 2$ identity matrix. The general form of the joint measurement effects is:

\begin{align}
G_{++}^{ij} &= \frac{1}{2} \alpha_{ij} I + \sigma \bar{a}_{ij}^+, \\
G_{+-}^{ij} &= \frac{1}{2} \left[ (1 - \alpha_{ij}) I + \sigma \bar{a}_{ij}^+ \right], \\
G_{-+}^{ij} &= \frac{1}{2} \left[ (1 - \alpha_{ij}) I + \sigma \bar{a}_{ij}^- \right], \\
G_{--}^{ij} &= \frac{1}{2} \alpha_{ij} I + \sigma \bar{a}_{ij}^-.
\end{align}

(B4)

where each effect is parameterized by four real numbers. From the marginal condition (B1–B2) it follows that:

\begin{align}
\bar{a}_{ij}^+ + \bar{a}_{ij}^- &= \eta_{ij}, \\
\bar{a}_{ij}^+ + \bar{a}_{ij}^- &= -\eta_{ij}.
\end{align}

(B5)

These can be rewritten as:

\begin{align}
2\bar{a}_{ij}^+ + \bar{a}_{ij}^- &= \eta_{ij}, \\
2\bar{a}_{ij}^- + \bar{a}_{ij}^+ &= \eta_{ij}, \\
2\bar{a}_{ij}^+ + \bar{a}_{ij}^- &= \eta_{ij}, \\
2\bar{a}_{ij}^- + \bar{a}_{ij}^+ &= \eta_{ij}.
\end{align}

(B6)

From (B1–B2) it follows that:

\[ (\bar{a}_{ij}^+ + \bar{a}_{ij}^-) + (\bar{a}_{ij}^+ + \bar{a}_{ij}^-) = 0. \]

So one can define:

\[ \bar{a}_{ij} \equiv \bar{a}_{ij}^+ + \bar{a}_{ij}^- \Rightarrow \bar{a}_{ij}^+ + \bar{a}_{ij}^- = -\bar{a}_{ij}. \]

Now, from equations (B10–B13) the following are obvious:

\begin{align}
\bar{a}_{ij}^+ &= \frac{1}{2} \left[ \eta (\hat{n}_i + \hat{n}_j) - \bar{a}_{ij} \right], \\
\bar{a}_{ij}^- &= \frac{1}{2} \left[ \eta (\hat{n}_i - \hat{n}_j) + \bar{a}_{ij} \right], \\
\bar{a}_{ij}^+ &= \frac{1}{2} \left[ \eta (\hat{n}_i + \hat{n}_j) + \bar{a}_{ij} \right], \\
\bar{a}_{ij}^- &= \frac{1}{2} \left[ \eta (\hat{n}_i - \hat{n}_j) - \bar{a}_{ij} \right].
\end{align}

(B7)

This gives the general form of the joint measurement POVMs in (9–12). For qubit POVMs, $G_{X_i,X_j}$ (cf. Eqs. (9–12)), where $X_i, X_j \in \{+, -\}$, the valid effect condition (B3) is equivalent to the following:

\begin{align}
|\bar{a}_{ij}^+| &\leq \frac{\alpha_{ij}}{2} \leq 2 - |\bar{a}_{ij}^+|, \\
|\bar{a}_{ij}^-| &\leq 1 - \frac{\alpha_{ij}}{2} \leq 2 - |\bar{a}_{ij}^-|, \\
|\bar{a}_{ij}^+| &\leq 1 - \frac{\alpha_{ij}}{2} \leq 2 - |\bar{a}_{ij}^-|, \\
|\bar{a}_{ij}^-| &\leq \frac{\alpha_{ij}}{2} \leq 2 - |\bar{a}_{ij}^+|.
\end{align}

(B18)

(B19)

(B20)

(B21)

These inequalities can be combined and rewritten as:

\[ 2 \max \{|\bar{a}_{ij}^+|, |\bar{a}_{ij}^-|\} \leq \alpha_{ij} \leq 2 - 2 \max \{|\bar{a}_{ij}^+|, |\bar{a}_{ij}^-|\} \]

(B22)

where

\[ \max \{|\bar{a}_{ij}^+|, |\bar{a}_{ij}^-|\} = \sqrt{\frac{\eta^2}{2} (1 + \hat{n}_i \hat{n}_j) + \frac{|\bar{a}_{ij}|^2}{4} + \frac{\eta}{2} (\hat{n}_i + \hat{n}_j) \bar{a}_{ij}} \]

and

\[ \max \{|\bar{a}_{ij}^-|, |\bar{a}_{ij}^+|\} = \sqrt{\frac{\eta^2}{2} (1 - \hat{n}_i \hat{n}_j) + \frac{|\bar{a}_{ij}|^2}{4} + \frac{\eta}{2} (\hat{n}_i - \hat{n}_j) \bar{a}_{ij}}. \]

This is the condition for a valid joint measurement used in inequalities (13–14).

Appendix C: Optimal state-dependent violation

We need to maximize $C \equiv 2 \eta - (\sum_{ij} \alpha_{ij} - |\bar{a}|)$ (Eq. (1)) to obtain the optimal violation of Specker’s inequality [5]. Subject to satisfaction of the joint measurability constraints (13–14) we have

\[ C_{\max} = \max_{\{\hat{n}_i, \hat{n}_j, \bar{a}_{ij}, \eta\}} \left\{ 2 \eta - \sum_{ij} \alpha_{ij} \right\} \]

\[ \leq \max_{\{\hat{n}_i, \bar{a}_{ij}, \eta\}} \left\{ 2 \eta - \sum_{ij} |\bar{a}_{ij}| \right\} \]

\[ - \sum_{ij} \sqrt{2\eta^2 (1 + \hat{n}_i \hat{n}_j) + |\bar{a}_{ij}|^2} \]

(C1)

The inequality above follows from the fact that

\[ |\bar{a}| = \sqrt{\sum_{ij} |\bar{a}_{ij}|^2 + 2(\bar{a}_{12} \bar{a}_{13} + \bar{a}_{12} \bar{a}_{23} + \bar{a}_{13} \bar{a}_{23})}, \]

(C2)

so that $|\bar{a}| \leq \sum_{ij} |\bar{a}_{ij}|$, and

\[ \sum_{ij} \sqrt{2\eta^2 (1 + \hat{n}_i \hat{n}_j) + |\bar{a}_{ij}|^2} \leq \sum_{ij} \sqrt{2\eta^2 (1 + \hat{n}_i \hat{n}_j) + |\bar{a}_{ij}|^2} \]

\[ \leq \sum_{ij} \alpha_{ij}. \]
Also, we have
\[
\sum_{(ij)} \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2 \\
\geq \sum_{(ij)} \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2 |_{\text{coplanar}, \phi_3 = \pi}
\]
That is, for a fixed $|\vec{a}_{ij}|$, $\sum_{(ij)} \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2$ is smallest when the measurement directions $\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}$ are coplanar and $\phi_3 = \pi$. From Eqs. (27,29), $\hat{n}_1,\hat{n}_2,\hat{n}_3 = cos \theta_2 = sin \theta_12 sin \theta_13 cos \phi_3 + cos \theta_12 cos \theta_13$. When $\phi_3 = 0$ or $\pi$, the three measurements are coplanar and there are only two free angles, $\hat{n}_1,\hat{n}_2 = cos \theta_12$ and $\hat{n}_1,\hat{n}_3 = cos \theta_13$, while the third angle is fixed by these two: $\hat{n}_2,\hat{n}_3 = cos(\theta_12 - \theta_13)$ or $cos(\theta_12 + \theta_13)$. Since $cos(\theta_12 + \theta_13) \leq cos(\theta_12 - \theta_13)$, for any given $\theta_12$ and $\theta_13 \in (0,\pi)$, $cos \theta_2$ is smallest when $\phi_3 = \pi$. Hence, we choose the three measurements to be coplanar such that $\phi_3 = \pi$ and $cos \theta_2 = cos(\theta_12 + \theta_13)$. Any other choice of $\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}$ will give a larger value of $cos \theta_2$, hence also $\sum_{(ij)} \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2$. So,
\[
C_{\text{max}} \leq \max_{\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}} \eta \left\{ 2\eta + \sum_{(ij)} |\vec{a}_{ij}| \right\} - \sum_{(ij)} \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2 |_{\text{coplanar}, \phi_3 = \pi}
\]
We will now argue that this inequality for $C_{\text{max}}$ can be replaced by an equality. Let us take coplanar measurement directions $\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}$ such that $\phi_3 = \pi$. We also take all the $\vec{a}_{ij}$ parallel to each other, i.e., $\vec{a}_{ij} = \vec{a}_{12} = |\vec{a}_{12}| |\vec{a}_{13}|$, $\vec{a}_{12} \cdot \vec{a}_{13} = |\vec{a}_{12}| |\vec{a}_{13}|$, and $\vec{a}_{13} \cdot \vec{a}_{23} = |\vec{a}_{13}| |\vec{a}_{23}|$, so that $|\vec{a}| = |\vec{a}_{12}| + |\vec{a}_{13}| + |\vec{a}_{23}|$. Besides, $|\langle \hat{n}_1, \hat{n}_j \rangle| \cdot \eta = 0$ $\forall (ij) \in \{12,13,23\}$. From these conditions it follows that each $\vec{a}_{ij}$ is perpendicular to the plane and $\forall (ij) \in \{12,13,23\}$, $\vec{a}_{ij} \cdot \hat{n}_i = \vec{a}_{ij} \cdot \hat{n}_j = 0$. This allows us to choose $\alpha_{ij} = \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2$. So, in our optimal configuration, the measurement directions are coplanar while the $\vec{a}_{ij}$’s are parallel to each other and perpendicular to the plane of measurements. Note that this also means $\vec{a}$ will be parallel to $\vec{a}_{ij}$ and therefore perpendicular to the plane of measurements, and so will be the optimal state (which is parallel to $\vec{a}$). With these optimality conditions satisfied, the optimal violation can now be written as
\[
C_{\text{max}} = \max_{\{\hat{n}_1,\hat{n}_2,\hat{n}_3,\}|\vec{a}_{ij}|,\eta} \left\{ 2\eta + \sum_{(ij)} |\vec{a}_{ij}| \right\} - \sqrt{2\eta^2(1 + \hat{n}_i \cdot \hat{n}_j)} + |\vec{a}_{ij}|^2 \right\}.
\]
The constraints from joint measurability \{13,14\} become
\[
\sqrt{1 - 2\eta^2} \leq |\vec{a}_{ij}|
\]
\[
\leq \sqrt{1 + \eta^4(\hat{n}_1,\hat{n}_j)^2 - 2\eta^2} - 1
\]
Of course, $\frac{1}{\sqrt{3}} \leq \eta < \eta < \eta < 1$ (cf. Appendix A). Now,
\[
C_{\text{max}} \leq \max_{\{\hat{n}_1,\hat{n}_2,\hat{n}_3\},|\vec{a}_{ij}|,\eta} \left\{ 2\eta + \sum_{(ij)} \left( \sqrt{1 + \eta^4(\hat{n}_1,\hat{n}_j)^2 - 2\eta^2} - (1 + \eta^2 \hat{n}_i,\hat{n}_j) \right) \right\}
\]
The upper bound follows from the fact that $f(x, y) = x - \sqrt{x^2 + 2\eta^2(1 + y)}$, where $\sqrt{1 - 2\eta^2} \leq x \leq \sqrt{1 + \eta^4y^2 - 2\eta^2}$ and $-1 < y < 1$, is an increasing function of $x$ for a fixed $y$, i.e., $(\frac{df}{dy})_y > 0$. Here $x \equiv |\vec{a}_{ij}|$ and $y \equiv \hat{n}_1,\hat{n}_j$. So, taking $|\vec{a}_{ij}| = \sqrt{1 + \eta^4(\hat{n}_1,\hat{n}_j)^2 - 2\eta^2}$, we have
\[
C_{\text{max}} \approx 2\eta + \sum_{(ij)} \left( \sqrt{1 + \eta^4(\hat{n}_1,\hat{n}_j)^2 - 2\eta^2} - (1 + \eta^2 \hat{n}_i,\hat{n}_j) \right)
\]
(C8)
Note that $\alpha_{ij} = 1 + \eta^2 \hat{n}_i,\hat{n}_j$ for $|\vec{a}_{ij}| = \sqrt{1 + \eta^4(\hat{n}_1,\hat{n}_j)^2 - 2\eta^2}$. $C_{\text{max}}(\eta,\eta)$ is the maximum value of $C$ for a given choice of measurement directions, $\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}$, and sharpness parameter, $\eta$. To simplify notation, let us define $x' \equiv \hat{n}_1,\hat{n}_2$ and $y' \equiv \hat{n}_1,\hat{n}_3$. Then, $\hat{n}_2,\hat{n}_3 = x'y' - \sqrt{1 - x'^2} \sqrt{1 - y'^2}$. On maximizing $C_{\text{max}}(\eta,\eta)$ as a function of $(x', y')$, where $-1 < x' < 1$ and $-1 < y' < 1$, the maximum occurs as $(x', y')$ approach the boundary values, $(x', y') \rightarrow (-1, 1), (x', y') \rightarrow (-1, -1), (x', y') \rightarrow (1, 1)$, and is given by
\[
C_{\text{max}}(\eta,\eta) \rightarrow 2\eta (1 - \eta).
\]
(C9)
That is, for a given choice of $\eta$, the optimal violation, $C_{\text{max}}(\eta)$, can approach a maximum value of $C_{\text{ub}} \equiv 2\eta (1 - \eta)$. This upper bound on $C_{\text{max}}(\eta)$, $C_{\text{ub}}(\eta) = 2\eta (1 - \eta)$, is a decreasing function of $\eta$, i.e., $\frac{\partial C_{\text{ub}}(\eta)}{\partial \eta} = 2 - 4\eta < 0$ since $\eta > \frac{1}{2}$ from the bounds on $\eta$ (cf. Appendix A). This means for an optimal violation, $\eta \rightarrow \eta$ and $C_{\text{max}} \rightarrow \max_{\eta} C_{\text{max}}(\eta) \rightarrow 2\eta (1 - \eta)$. Since $\eta \rightarrow \frac{1}{2}$ for $\hat{n}_1,\hat{n}_2 = \hat{n}_1,\hat{n}_3 \rightarrow -1$, this gives an upper bound on the optimal quantum violation is given by $C_{\text{ub}} = \frac{2}{27}$, so that $S_{\text{max}} = \frac{C_{\text{ub}}}{\eta}$ is given by
\[
S_{\text{max}} \rightarrow \frac{2}{27} \approx 0.0741. \text{ or } 74.1\%.
\]
(C10)
Note that all these calculations correspond to the optimal scenario $\hat{n}_1,\hat{n}_2 = \hat{n}_1,\hat{n}_3 \rightarrow -1$, and $\hat{n}_2,\hat{n}_3 = 2(\hat{n}_1,\hat{n}_2)^2 - 1$. For a given choice of $\{\hat{n}_1,\hat{n}_2,\hat{n}_3\}$ in a plane, the optimal violation is given by $\max_{\eta} C_{\text{max}}(\eta)$ for $\eta \rightarrow \eta$. In the case of trine spin axes, where $\hat{n}_1,\hat{n}_j = -1/2$, this gives $C_{\text{max}} \rightarrow \frac{23}{37} \approx 0.20185$, and $S_{\text{max}} \rightarrow 0.03364$, i.e., a violation of about 3.36%.
As an example, consider the following choice: $\hat{n}_1, \hat{n}_2 = \hat{n}_1, \hat{n}_3 = -0.9$, and $\hat{n}_2, \hat{n}_3 = 0.62$, where

$$\vec{a}_{23} = (0, 0.3351, 0),$$

the violation is given by

$$C = 2\eta - (\sum_{ij} \alpha_{ij} - |\vec{a}|) = 0.2395,$$

and

$$S = 0.0399 \text{ or } 3.99\%.$$  

The state that yields this violation is given by $\cos \theta = a_z/|\vec{a}| = 0$ and $\tan \phi = a_y/a_x = \infty$, so that $\theta = \pi/2$ and $\phi = \pi/2$. That is, the qubit is prepared in a spin-up state along the y-axis:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).$$