CHARACTERISTIC CLASSES FOR $G$-STRUCTURES

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Abstract. Let $G \subset GL(V)$ be a linear Lie group with Lie algebra $\mathfrak{g}$ and let $A(\mathfrak{g})^G$ be the subalgebra of $G$-invariant elements of the associative supercommutative algebra $A(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(V^*)$. To any $G$-structure $\pi : P \to M$ with a connection $\omega$ we associate a homomorphism $\mu_\omega : A(\mathfrak{g})^G \to \Omega(M)$. The differential forms $\mu_\omega(f)$ for $f \in A(\mathfrak{g})^G$ which are associated to the $G$-structure $\pi$ can be used to construct Lagrangians. If $\omega$ has no torsion the differential forms $\mu_\omega(f)$ are closed and define characteristic classes of a $G$-structure. The induced homomorphism $\mu'_\omega : A(\mathfrak{g})^G \to H^*(M)$ does not depend on the choice of the torsionfree connection $\omega$ and it is the natural generalization of the Chern Weil homomorphism.

1. $G$-STRUCTURES

1.1. $G$-structures. By a $G$-structure on a smooth finite dimensional manifold $M$ we mean a principal fiber bundle $\pi : P \to M$ together with a representation $\rho : G \to GL(V)$ of the structure group in a real vector space $V$ of dimension $\dim M$ and a 1-form $\sigma$ (called the soldering form) on $M$ with values in the associated bundle $P[V, \rho] = P \times_G V$ which is fiber wise an isomorphism and identifies $T_x M$ with $P[V](x)$ for each $x \in M$. Then $\sigma$ corresponds uniquely to a $G$-equivariant 1-form $\theta \in \Omega^1_{\text{hor}}(P; V)^G$ which is strongly horizontal in the sense that its kernel is exactly the vertical bundle $VP$. The form $\theta$ is called the displacement form of the $G$-structure. A $G$-structure is called 1-integrable if it admits torsionfree connections, see 1.4 below.

We fix this setting $((P, p, M, G), (V, \rho), \theta)$ from now on.

1.2. Invariant forms. We consider a multilinear form $f \in \bigotimes^k V^* = L^k(V)$ which is invariant in the sense that $f \circ (\bigotimes^k \rho(g)) = f$ for each $g \in G$. Let us denote by
\( L^k(V)^G \) the space of all these invariant forms. For each \( f \in L^k(V)^G \) we have for any \( X \in g \), the Lie algebra of \( G \),

\[
0 = \frac{d}{dt}|_0 f(\rho(\exp(tX))v_1, \ldots, \rho(\exp(tX))v_k),
\]

\[
= \sum_{i=1}^k f(v_1, \ldots, \rho'(X)v_i, \ldots, v_k),
\]

where \( \rho' = T_{e\rho} : g \to \mathfrak{gl}(V) \) is the differential of the representation \( \rho \).

1.3 Products of differential forms. For \( \varphi \in \Omega^p(P; g) \) and \( \Psi \in \Omega^q(P; V) \) let us define the form \( \rho'_\wedge(\varphi)\Psi \in \Omega^{p+q}(P; V) \) by

\[
(\rho'_\wedge(\varphi)\Psi)(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum \text{sign}(\sigma)\rho'(\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}))\Psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\]

Then \( \rho'_\wedge(\varphi) : \Omega^*(P; V) \to \Omega^{*+p}(P; V) \) is a graded \( \Omega(P) \)-module homomorphism of degree \( p \). Recall also that \( \Omega(P; g) \) is a graded Lie algebra with the bracket

\[
[\varphi, \psi]_\wedge(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum \text{sign}\sigma[\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}), \psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})]_g.
\]

One may easily check that for the graded commutator in \( \text{End}(\Omega(P; V)) \) we have

\[
\rho'_\wedge([\varphi, \psi]_\wedge) = [\rho'_\wedge(\varphi), \rho'_\wedge(\psi)] = \rho'_\wedge(\varphi) \circ \rho'_\wedge(\psi) - (-1)^{pq} \rho'_\wedge(\psi) \circ \rho'_\wedge(\varphi)
\]

so that \( \rho'_\wedge : \Omega^*(P; g) \to \text{End}^*(\Omega(P; V)) \) is a homomorphism of graded Lie algebras.

Let \( \otimes V \) be the tensoralgebra generated by \( V \). For \( \Phi, \Psi \in \Omega(P; \otimes V) \) we will use the associative bigraded product

\[
(\Phi \otimes_\wedge \Psi)(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum \text{sign}(\sigma)\Phi(X_{\sigma_1}, \ldots, X_{\sigma_p}) \otimes \Psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\]

1.4. The covariant exterior derivative. Let \( \omega \in \Omega^1(P; g)^G \) be a principal connection on the principal bundle \( (P, p, M, G) \). Let \( \chi : TP \to HP \) denote the corresponding projection onto the horizontal bundle \( HP := \ker \omega \). The covariant exterior derivative \( d_\omega : \Omega^k(P; V) \to \Omega^{k+1}(P; V) \) is then given as usual by \( d_\omega \Psi = \chi^*d\Psi = (d\Psi) \circ \Lambda^{k+1}(\chi) \).

**Lemma.** For \( \Psi \in \Omega_{\text{hor}}(P; V)^G \) the covariant exterior derivative is given by \( d_\omega \Psi = d\Psi + \rho'_\wedge(\omega)\Psi \).

**Proof.** If we insert one vertical vector field, say the fundamental vector field \( \zeta_\chi \) for \( X \in g \), into \( d_\omega \Psi \), we get 0 by definition. For the right hand side we use \( i_{\zeta_\chi} \Psi = 0 \) and
1.5. Definition. If $\theta \in \Omega^1_{\text{hor}}(P; V)^G$ is the displacement form of a $G$-structure then the torsion of the connection $\omega$ with respect to the $G$-structure is $\tau := d\omega \theta = d\theta + \rho^\omega_\theta(\omega)\theta$.

Recall that a $G$-structure is called 1-integrable if it admits a connection without torsion. This notion has also been investigated in [Kolář, Vadovičová] where it was called prolongable.

1.6. Chern-Weil forms. For differential forms $\psi_i \in \Omega^{p_i}(P; V)$ and $f \in L^k(V) = (\bigotimes^k V)^*$ we can construct the following differential forms:

$$\psi_1 \otimes \cdots \otimes \psi_k \in \Omega^{p_1+\cdots+p_k}(P; V \otimes \cdots \otimes V),$$

$$f^{\psi_1,\ldots,\psi_k} := f \circ (\psi_1 \otimes \cdots \otimes \psi_k) \in \Omega^{p_1+\cdots+p_k}(P).$$

The exterior derivative of the latter one is clearly given by

$$d(f \circ (\psi_1 \otimes \cdots \otimes \psi_k)) = f \circ d(\psi_1 \otimes \cdots \otimes \psi_k) =$$

$$= f \circ \left( \sum_{i=1}^k (-1)^{p_1+\cdots+p_{i-1}} \psi_1 \otimes \cdots \otimes d\psi_i \otimes \cdots \otimes \psi_k \right).$$

We also set $f^\psi := f^{\psi_1,\ldots,\psi}$ for $\psi \in \Omega^p(P; V)$. Note that the form $f^{\psi_1,\ldots,\psi_k}$ is $G$-invariant and horizontal if all $\psi_i \in \Omega^p_{\text{hor}}(P; V)^G$ and $f \in L^k(V)^G$. It is then the pullback of a form on $M$.

1.7. Lemma. Let $0 \neq \psi \in \Omega^{p}(P; V)$ and $f \in L^k(V)$. Then we have:

$$f^\psi \neq 0 \iff \begin{cases} \text{alt } f \neq 0, & \text{if } p \text{ is odd}, \\
\text{sym } f \neq 0, & \text{if } p \text{ is even}, \end{cases}$$

where alt and sym are the natural projections onto $\Lambda(V^*)$ and $S(V^*)$, respectively. □

1.8. Lemma. If $f \in L^k(V)^G$ is invariant then we have

$$f \circ \left( \sum_{i=1}^k (-1)^{p_1+\cdots+p_{i-1}} \psi_1 \otimes \cdots \otimes \rho^\omega_\psi(\omega) \psi_i \otimes \cdots \otimes \psi_k \right) = 0.$$  

Proof. This follows from the infinitesimal condition of invariance for $f$ given in 1.2 by applying the alternator. □
2. Obstructions to 1-integrability of $G$-structures

2.1. Proposition. Let $\pi : P \to M$ be a $G$-structure and let $f \in L^k(V)^G$ be an invariant tensor. For arbitrary $G$-equivariant horizontal $V$-valued forms $\psi_i \in \Omega^p_{\text{hor}}(P; V)^G$ we consider the $(p_1 + \cdots + p_k)$-form $f^{\psi_1, \ldots, \psi_k}$ on $M$ as above. If there is a connection $\omega$ for the $G$-structure $\pi$ such that $d_\omega \psi_i = 0$ for all $i$, then the form $f^{\psi_1, \ldots, \psi_k}$ is closed.

Proof. We use $d_\omega \psi_i = d \psi_i + \rho^i_\omega(\omega) \psi_i$ from lemma 1.4, and lemma 1.8, to obtain

$$df^{\psi_1, \ldots, \psi_k} = f \circ \left( \sum_{i=1}^{k} (-1)^{p_1 + \cdots + p_{i-1}} \psi_1 \otimes \cdots \otimes \rho^i_\omega d_\omega \psi_i \otimes \cdots \otimes \psi_k \right) = 0. \quad \Box$$

2.2. Corollary. 1. For a $G$-structure $\pi : P \to M$ with displacement form $\theta$ we have a natural homomorphism of associative algebras

$$\nu : \Lambda(V^*)^G \to \Omega(M),$$

$$f \mapsto f^\theta = f(\theta, \ldots, \theta).$$

2. If the $G$-structure is 1-integrable then the image of $\nu$ consists of closed forms and we get an induced homomorphism

$$\nu^* : \Lambda(V^*)^G \to H^*(M).$$

If $M$ and $G$ are compact then $\nu^*$ is injective.

Proof. If the $G$-structure is 1-integrable then there is a connection $\omega$ with vanishing torsion $\tau = d_\omega \theta = 0$. Then the result follows from proposition 2.1.

If $G$ is compact, any torsionfree connection $\omega$ for $\pi : P \to M$ is the Levi-Civita connection for some Riemannian metric. Any form $f^\theta$, which is parallel with respect to $\omega$, is harmonic and can thus not be exact for compact $M$. So $\nu^*$ is injective. \quad \Box

Problem. Is the homomorphism $\nu^*$ injective for compact $M$ but noncompact $G$?

2.3. Remark. Given a principal connection $\omega$ on $P$ there is the induced covariant exterior derivative $\nabla : \Omega^p(M; P[V]) \to \Omega^{p+1}(M; P[V])$ on the associated vector bundle $P[V]$. The soldering form (see 1.1) $\sigma : TM \to P[V]$ is an isomorphism of vector bundles and we may consider the pull back covariant derivative $\sigma^* \nabla$ on $TM$. Next we consider the ‘combined’ covariant derivative $D^{\sigma^* \nabla}$ on the vector bundle $L(TM, P[V])$ given by $D_X^{\sigma^* \nabla} A = \nabla_X \circ A - A \circ (\sigma^* \nabla)_X$. Obviously we have $D^{\sigma^* \nabla} \sigma = 0$. Consequently for any $f \in L^k(V)^G$ we have that $f^\theta \in \Omega^k(M)$ is parallel for the connection induced on $\Lambda^k T^* M$ from $\sigma^* \nabla$ on $TM$.

3. The generalized Chern-Weil homomorphism for $G$-structures

3.1. The Chern-Weil homomorphism. Let $\omega$ be a connection for a $G$-structure $\pi : P \to M$ with curvature form $\Omega \in \Omega^2_{\text{hor}}(P, g)$. Then the Bianchi identity $d_\omega \Omega = 0$ holds. If we apply proposition 2.1 to $\psi_i = \Omega$ we obtain a homomorphism

$$\gamma : S(g^*)^G \to \Omega(M),$$

given by $\gamma(f) = f^\Omega$. Since the image of $\gamma$ consists of closed forms we have an induced homomorphism

$$\gamma' : S(g^*)^G \to H^*(M).$$

This is the well known Chern-Weil homomorphism.
3.2. The algebra $A(g, V)$. In order to generalize the Chern Weil homomorphism we associate to a Lie algebra $g$ and a vector space $V$ the associative graded commutative algebra

$$A(g, V) := S(g^*) \otimes \Lambda(V^*),$$

where the generators of the symmetric algebra $S(g^*)$ have degree 2. We may also consider $A(g, V)$ as a graded Lie algebra with the bracket

$$[a \otimes \varphi, b \otimes \psi] := \{a, b\} \otimes \varphi \wedge \psi, \quad a, b \in S(g^*), \varphi, \psi \in \Lambda(V^*),$$

where $\{a, b\}$ is the usual Poisson-Lie bracket in $S(g^*)$.

Let now $\mathfrak{g}$ be the Lie algebra of the Lie group $G$ and let $\rho : G \to GL(V)$ be a representation. Then $G$ acts naturally on $A(g, V)$, and we denote $A(g, V)^G$ the subalgebra of $G$-invariant elements in $A(g, V)$.

3.3. Remark. The associative algebra $A(g, V)^G$ contains the subalgebra $S(g^*)^G \otimes \Lambda(V^*)^G$, in general as a proper subalgebra. Actually, let $G \subset GL(V)$ be the isotropy group of an irreducible Riemannian symmetric space $M$. Then the curvature tensor of $M$ defines an element of $(g^* \otimes \Lambda^2 V^*)^G \subset A(g, V)^G$ that does not belong to $(g^*)^G \otimes (\Lambda^2 V^*)^G = 0$.

3.4. The generalized Chern-Weil homomorphism. Now we are in a position to combine the constructions 2.2 and 3.1.

Theorem. Let $\pi : P \to M$ be a $G$-structure on $M$ with displacement form $\theta$. Any connection $\omega$ in $\pi$ defines a homomorphism of associative algebras

$$\mu : A(g, V)^G \to \Omega(M)$$

$$(S^p(g^*) \otimes \Lambda^q V^*)^G \ni f \mapsto f^{\Omega, \theta} = f_{\Omega, \ldots, \Omega, \theta, \ldots, \theta}$$

If the connection $\omega$ has no torsion then the image of $\mu$ consists of closed forms and $\mu$ induces a homomorphism

$$\mu' : A(g, V)^G \to H^*(M),$$

which is independent of the choice of the torsionfree connection.

In other words, any $G$-invariant tensor $f \in S^p(g^*) \otimes \Lambda^q(V^*)$ defines a cohomology class $[f^{\Omega, \theta}] \in H^{2p+q}(M)$ which is an invariant of the 1-integrable $G$-structure. We call it a characteristic class of the 1-integrable $G$-structure $\pi$.

Proof. It just remains to show that the cohomology class $[f^{\Omega, \theta}]$ does not depend on the choice of the torsionfree connection for the $G$-structure $\pi : P \to M$.

So let $\omega_0, \omega_1$ be two torsionfree connections for the $G$-structure, let $\varphi = \omega_1 - \omega_0$, and denote by $\Omega_t = d_{\omega_t} \Omega_t$ the curvature form of the torsionfree connection $\omega_t = \omega_0 + t \varphi = (1 - t) \omega_0 + t \omega_1$. We claim that for $f \in (S^p(g^*) \otimes \Lambda^q(V^*))^G$ we have

$$f^{\Omega_1, \theta} - f^{\Omega_0, \theta} = d(Tf),$$

where

$$Tf = p \int_0^1 f(\varphi, \Omega_t, \ldots, \Omega_t, \theta, \ldots, \theta) \, dt$$

(1)
is the transgression form of $f$ on $P$. The assertion is immediate from (1). To prove it we compute $\partial_t f^{\Omega_t, \theta}$ using the identities $\partial_t \Omega_t = d_{\omega_t} \varphi$ (see [Kobayashi, Nomizu II, p. 296]), $d_{\omega_t} \Omega_t = 0$, and $d_{\omega_t} \theta = 0$.

\[
\partial_t f^{\Omega_t, \theta} = p f(\partial_t \Omega_t, \Omega_t, \theta, \theta, \theta) = p f(d_{\omega_t} \varphi, \Omega_t, \theta, \theta) = p d_{\omega_t} f(\varphi, \Omega_t, \theta, \theta) = p d f(\varphi, \Omega_t, \theta, \theta). \quad \square
\]

3.5. Remarks about secondary characteristic classes. If the characteristic forms $f^{\Omega_t, \theta}$ and $f^{\Omega_0, \theta}$ associated with two torsionfree connections $\omega_1$ and $\omega_0$ vanish we obtain a secondary characteristic class $[Tf]$. It is a natural generalization of the classical Chern-Simons characteristic class, see [Chern, Simons], [Kobayashi, Ochiai].

Problem: study conditions when the secondary characteristic class $[Tf]$ does not depend on the choice of the torsionfree connections $\omega_1$ and $\omega_0$.

3.6. Examples of characteristic classes. Assume that a linear group $G \subset GL(V)$ preserves some pseudo Euclidean metric in $V = \mathbb{R}^n$. Then we may identify the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ with a subspace $\mathfrak{g} \subset \Lambda^2 V$. Suppose that there exists a $G$-invariant supplement $\mathfrak{d}$ to $\mathfrak{g}$ in $\Lambda^2 V$. Then the $G$-equivariant projection $\Lambda^2 V \rightarrow \mathfrak{g}$ along $\mathfrak{d}$ determines a $G$-invariant element $q \in \mathfrak{g} \otimes \Lambda^2 V^* \cong \mathfrak{g}^* \otimes \Lambda^2 V^*$. The element $q$ defines a 4-form $q^{\Omega, \theta}$ on the base of any $G$-structure $\pi : P \rightarrow M$ with a connection $\omega$ and curvature $\Omega$. It may be written as

\[
q^{\Omega, \theta} = q(\Omega, \theta, \theta) = q_{abcd} R_{aef}^b \theta^c \wedge \theta^d \wedge \theta^e \wedge \theta^f,
\]

where $(q_{abcd})$ is the coordinate expression of $q$ in the standard basis $(e_a)$ of $V = \mathbb{R}^n$, $\theta = e_a \otimes \theta^a$, and $\Omega = R_{bej}^a \theta^e \wedge \theta^f$.

If $\omega$ is torsionfree the 4-form $q^{\Omega, \theta}$ is closed and it defines a cohomology class $[q^{\Omega, \theta}] \in H^4(M)$ independently of the choice of $\omega$.

3.7. Remarks about the classification of characteristic classes. The classification of characteristic classes for $G$-structures with a given Lie group $G$ reduces to the construction of generators of the associative algebra $A(\mathfrak{g}, V)^G = (S(\mathfrak{g}^*) \otimes \Lambda(V^*))^G$. We may also use the bracket to multiply characteristic classes. It suffices to solve this problem for those Lie groups $G$ which appear as holonomy groups of torsionfree connection. Only for such groups $G$ there exist 1-integrable non-flat $G$-structures. Under the additional hypothesis of irreducibility, all such groups were classified by [Berger], up to some gaps which were filled by [Bryant] and [Alekseevsky, Graev].

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