GRADIENT ESTIMATES AND LOWER BOUND FOR THE BLOW-UP TIME OF STAR-SHAPED MEAN CURVATURE FLOW

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ABSTRACT. In this paper we consider a star-shaped hypersurface flow by mean curvature. Without any assumption on the convexity, we give a new proof of gradient estimate for a short time. As an application, we also give a lower bound for the blowing up time.

1. INTRODUCTION

Let $F_0 : M_n \to \mathbb{R}^{n+1}$ be a smooth immersion of an $n$-dimensional hypersurface in Euclidean space. $M_0 = F_0(M_n)$ is a star-shaped hypersurface and $F : M_n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t} (p,t) = -H(p,t)\nu(p,t)$$
$$F(\cdot,0) = M_0.$$ (1.1)

Here $\nu$ is the outer normal vector, $H$ is the mean curvature of $M_t$.

For closed hypersurfaces, it is well known that the solution of (1.1) exists on a finite time interval $[0, T_c)$, $0 < T_c < \infty$, and the curvature of the hypersurfaces becomes unbounded as $t \to T_c$. The study of a detailed description of the singular behavior as $t \to T_c$ has drawn lots of attentions for the past decades (see [1], [4], [5], [6], [7], and [9]). We can see that, the growth rate of the second fundamental form $A$ as $t$ approaches $T_c$ plays an important role in determining the shape of singularities (see [2], [3], [8], and [10]).

Inspired by [11], we give a new proof for the short time gradient estimate for $(X, \nu)^{-1}$ of $M_t$, where $M_t$ is a family of smooth closed $n$-dimensional hypersurfaces immersed in $\mathbb{R}^{n+1}$ evolving by mean curvature, and $M_0$ is star-shaped. As an application, we give a lower bound on the blow-up time under the assumption that $H$ remains bounded through the flow. Our main theories are stated as following:
Theorem 1.1. Let $M_t$, $t \in [0, T_c)$ be a solution of equation (1.1). Then for every $0 < T_0 < T_c$, there exists $0 < T \leq T_0$, such that when $t \in [0, T]$ we have

$$\langle X, \nu \rangle^{-1} \leq 2c_3f_0,$$

here $T = \min\{\frac{1}{Gf_0}, T_0\}$, $f_0 = \max_{M_0} \langle X, \nu \rangle^{-1}$, $G$ depends on $M_0$, $T_0$, and $c_3$ is a constant depending on $M_0$.

Theorem 1.2. Let $M_t$, $t \in [0, T_c)$ be a solution of (1.1). If $\sup_{M_n \times [0, T_c)} H \leq C_H < \infty$, then $T_c > \frac{1}{24c_2c_3f_0}$, where $c_2$ and $c_3$ depends on $M_0$.

2. GRADIENT ESTIMATES

Proof of Theorem 1.1. We define

$$\rho = \rho_{(X_0, t_0)}(X, t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp \left\{ -\frac{|X - X_0|^2}{4(t_0 - t)} \right\}.$$

Then, for any $u \in C^2$ we have

$$\frac{d}{dt} \int_{M_t} u\rho d\mathcal{H}^n = \int_{M_t} u_t \rho + u \rho_t - H^2 u \rho d\mu_t$$

$$= \int_{M_t} \left\{ u \left( \left( \frac{d}{dt} + M_t \right) \rho - H^2 \rho \right) + \rho \left( \frac{d}{dt} - M_t \right) u \right\} d\mu_t$$

$$= -\int_{M_t} u \rho \left| \vec{H} + \frac{1}{2\tau} \vec{F} \right|^2 d\mu_t + \int_{M_t} u \rho \left( \frac{d}{dt} - M_t \right) u d\mu_t,$$

where $\tau = t_0 - t$.

Now let $f = \langle X, \nu \rangle^{-1}$, then it is easy to see that $f$ satisfies

$$f_t - \Delta f = \frac{-2}{f} |\nabla f|^2 + 2f^2 H - |A|^2 f.$$
Plugging equation (2.3) into (2.2) we obtain
\[\frac{d}{dt} \int_{M_t} f \rho d\mu_t = - \int_{M_t} f \rho \left| \vec{H} + \frac{1}{2\tau} \vec{F} \right|^2 d\mu_t - \int_{M_t} \frac{2\rho}{f} |\nabla f|^2 - 2f^2H\rho + |A|^2 f \rho d\mu_t \leq 2 \int_{M_t} f^2 H \rho d\mu_t.\]

Denote \(f_\infty = \sup_{M_n \times [0,T]} f = f(Y,s), \tilde{\rho} = \rho(Y,s)(X,t), |X_0| = \text{diam}(M_0), c_1 = 2 \max_{M_n \times [0,T]} H\), where \(T \leq T_0 < T_c\), and \(T_c\) is the blow-up time.

Since \(M_0\) is a compact star-shaped hypersurface, we know that \(c_1 > 0\). Therefore, we have
\[\frac{d}{dt} \int_{M_t} \tilde{\rho} d\mu_t \leq c_1 f_\infty \int_{M_t} \tilde{\rho} d\mu_t.\]

Next, we are going to estimate \(\int_{M_t} \tilde{\rho} d\mu_t\). When \(M_t\) is star-shaped, we can represent \(M_t\) as a radial graph over \(S_n\), i.e. for \(X(t) \in M_t\), we have\[X(t) = e^{\nu(z,t)}z = r(z,t)z, z \in S_n.\]

It is easy to see that \(g_{ij} = e^{2\nu} (\delta_{ij} + v_i v_j)\), and \(\det g = e^{2nu} (1 + |\nabla v|^2)\).

Therefore,
\[\int_{M_t} \tilde{\rho} d\mu_t = \int_{S^n} \tilde{\rho} e^{nu} \sqrt{1 + |\nabla v|^2} d\mu = \int_{S^n} \tilde{\rho} \sqrt{1 + |\nabla v|^2} r^n d\mu.\]

Since \(f = \langle X, \nu \rangle^{-1} = \frac{\sqrt{1 + |\nabla v|^2}}{e^v}\), we get
\[\int_{M_t} \tilde{\rho} d\mu_t = \int_{S^n} \tilde{\rho} r^{n+1} d\mu = \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{|X(t)-Y|^2}{4\tau}} f r^{n+1} d\mu \leq f_\infty \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{|X(t)-Y|^2}{4\tau}} r^{n+1} d\mu.\]
We only need to show
\[ Q = \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{|rz-r_0z_0|^2}{4\tau}} r^{n+1} dz \]
is bounded.

Case 1: When \( Y \) is the origin, we have
\[ Q = \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{r^2}{4\tau}} r^{n+1} dz \]
\[ \leq \frac{\omega_n}{\pi^{n/2}} |X_0| \max_{\phi \in \mathbb{R}_+} e^{-\phi} \phi^{n/2} \]
\[ \leq \tilde{c}_1 |X_0|. \]

Case 2: When \( Y \) is not the origin, we have
\( \frac{r}{r_0} \geq 2 \), we have
\[ |rz-r_0z_0|^2 \geq (r-r_0)^2 \geq \frac{r^2}{4}. \]
Thus,
\[ Q \leq \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{r^2}{4\tau}} r^{n+1} dz \]
\[ \leq \frac{\omega_n}{\pi^{n/2}} |X_0| \max_{\phi \in \mathbb{R}_+} e^{-\phi} \phi^{n/2} \]
\[ \leq \tilde{c}_2 |X_0|. \]
\( \frac{r}{r_0} < 2 \), we divide \( S_n \) into two parts as shown in the graph, and denoted by \( S_1, S_2 \) respectively.

Then, we have
\[ Q = \int_{S^n} \frac{1}{(4\pi \tau)^{n/2}} e^{-\frac{r^2}{4\tau} \frac{(|r/r_0|z-z_0|^2)}{4\tau}} r^{n+1} dz \]
\[ = \int_{S_1} + \int_{S_2} = I + II. \]
In \( S_1 \), it’s easy to see that
\[ |(r/r_0)z - z_0|^2 \geq |z - z_0|^2 \cos^2 \frac{\pi}{6} = \frac{3}{4} |z - z_0|^2, \]
so we have
\begin{equation}
I \leq \int_{S_1} \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{3}{4}\tau_0^2 |z-z_0|^2} \rho^{n+1} d\rho.
\end{equation}

Since \( S_1 \) can be represented as a vertical graph over the tangent plane at \( z_0 \), we assume \( z = (x,u(x)) \in S_1 \), where \( x \in \text{proj}(S_1) \). Let \( \tilde{c}_0 = \max_{\text{proj}(S_1)} \sqrt{1 + |\nabla u|^2} \), then
\begin{equation}
I \leq \tilde{c}_0 \int_{\text{proj}(S_1)} \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{3}{4}\tau_0^2 |x|^2} \rho^{n+1} dx.
\end{equation}

In \( S_2 \) we have
\[ |(r/r_0)z - z_0|^2 \geq 1/2. \]

Therefore,
\begin{equation}
II \leq \int_{S_2} \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{3}{4}\tau_0^2 |x|^2} \rho^{n+1} d\rho.
\end{equation}

We conclude that
\[ \int_{M_t} \tilde{p} d\mu_t \leq c_{f\infty} |X_0|. \]

Now we have
\begin{equation}
\frac{d}{dt} \int_{M_t} f \tilde{p} d\mu_t \leq cc_1 |X_0| f_\infty^3 = c_2 f_\infty^3.
\end{equation}

Let \( f_0 = \max_{M_0}(X, \nu)^{-1} \), then we have
\[ \int_{M_t} f \tilde{p} d\mu_t |_{t=0} \leq c |X_0| f_0^2 \leq c_3 f_0^2, \]
\( c_3 \) is chosen such that \( c_3 f_0 \geq 1 \). Since
\[ \lim_{t\to s} \int_{M_t} f \tilde{p} d\mu_t = f(Y, s) = f_\infty, \]
we get
\begin{equation}
f_\infty - c_3 f_0^2 \leq \int_0^s \frac{d}{dt} \int_{M_t} f \tilde{p} d\mu_t dt \leq sc_2 f_\infty^3.
\end{equation}
Now consider function \( h(r) = sc^2r^3 - r + c_3f_0^2 \), we have
\[
h(0) = c_3f_0^2 > 0, \quad \text{and} \quad h(f_0) > 0.
\]
Furthermore,
\[
h'(r) = 3c_2sr^2 - 1 < -1/2 \quad \text{for} \quad r \in \left(0, \frac{1}{\sqrt{24c_2s}}\right).
\]
We can see if \( 2c_3f_0^2 < \frac{1}{\sqrt{24c_2s}} \), then \( h'(r) < -1/2 \) for any \( r \in (0, 2c_3f_0^2) \). Therefore, there exists \( \alpha \in (f_0, 2c_3f_0^2) \) such that \( h(r) < 0 \). Let \( T = \min\left\{ \frac{1}{24c_2c_3f_0^2}, T_0 \right\} \) and \( s \leq T \), we assume \( f_\infty > 2c_3f_0^2 \). Then, there exists \( s' \in (0, s) \) such that
\[
v_1 := \max_{M_n \times [0, s']} \langle X, \nu \rangle^{-1} = \alpha \quad \text{and} \quad h(v_1) < 0,
\]
leads to a contradiction. So, we have when \( t \leq \min\left\{ \frac{1}{24c_2c_3f_0^2}, T_0 \right\} \), \( f_\infty \leq 2c_3f_0^2 \). □

3. Lower Bound for Blow-up Time

Proof of Theorem 1.2. Now we assume
\[
\sup_{M_n \times [0, T_0]} H \leq c_H < \infty,
\]
then \( c_2 = c_H|X_0| \) is always bounded. Let \( T = \frac{1}{24c_2c_3f_0^2} \) and \( T_0 \leq T \), consider
\[
\Phi = \log |A|^2 + 2 \log f - Bt \quad \text{in} \quad M_n \times [0, T_0],
\]
where \( B > 0 \) will be determined later. Denote \( g = |A|^2 \), then we have
\[
\Phi_t - \Delta \Phi = \frac{1}{g}(g_t - \Delta g) + \frac{2}{f}(f_t - \Delta f) + \frac{|
abla g|^2}{g^2} + 2\frac{|
abla f|^2}{f^2} - B.
\]
Since
\[
\frac{\partial}{\partial t} |A|^2 - \Delta |A|^2 = -2|\nabla A|^2 + 2|A|^4,
\]
we have
\[
\Phi_t - \Delta \Phi = \frac{1}{|A|^2} (-2|\nabla A|^2 + 2|A|^4)
\]
\[
+ \frac{2}{f} \left( -\frac{2}{f} |\nabla f|^2 + 2f^2H - |A|^2f \right)
\]
\[
+ \frac{|
abla g|^2}{g^2} + 2\frac{|
abla f|^2}{f^2} - B
\]
\[
= -2\frac{|
abla A|^2}{|A|^2} + 4fH + \frac{|
abla g|^2}{g^2} - 2\frac{|
abla f|^2}{f^2} - B.
\]

(3.1)
Choose $B > 4f_\infty c_H$,

\begin{equation}
\Phi_t - \Delta \Phi < -2 \frac{|\nabla A|^2}{|A|^2} + \frac{|\nabla g|^2}{g^2} - 2 \frac{|\nabla f|^2}{f^2}.
\end{equation}

At an interior point where $\Phi$ achieves its local maximum, we have

$$\frac{\nabla g}{g} = -2 \frac{\nabla f}{f}.$$ 

Moreover,

$$\frac{|\nabla |A|^2|^2}{2|A|^4} \leq 2 \frac{|\nabla A|^2}{|A|^2}.$$ 

So, we have $\Phi_t - \Delta \Phi < 0$ at the point, leads to a contradiction. We conclude that

$$\Phi \leq C(M_0)$$

and

$$T_c > \frac{1}{24c_2c_3^2f_0^4} = \frac{1}{24c_2c_3^2|X_0|^3f_0^4}.$$ 

\[\square\]

**Remark 3.1.** From the proof of Theorem 1.2, we can also conclude that during a mean curvature flow, $|A|^2$ is staying bounded as long as $\langle X, \nu \rangle^{-1}$ is bounded.

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