Frequency Characteristics of Dissipative and Generative Fractional RLC Circuits

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Abstract
Equations governing the transient- and steady-state regimes of the fractional series RLC circuits containing dissipative and/or generative capacitor and inductor are posed by considering the electric current as a response to electromotive force. Further, fractional RLC circuits are analyzed in the steady-state regime and their energy consumption/production properties are established depending on the angular frequency of electromotive force. Frequency characteristics of the modulus and argument of transfer function, i.e., of circuit’s equivalent admittance, are analyzed through the Bode diagrams for the whole frequency range, as well as for low and high frequencies employing the asymptotic expansions of transfer function modulus and argument.

Keywords Dissipative and generative capacitor and inductor · Fractional series RLC circuits · Frequency characteristics of transfer function modulus and argument
1 Introduction

Electric elements made of materials having history-dependent polarization and magnetization processes rather than only instantaneous ones can be constitutively modeled by adding a hereditary type term in a classical constitutive equation, as proposed in [16], so that in the case of capacitor one may express either charge $q$ at the time instant $t > 0$ in terms of history of capacitor voltage $u_C$, or vice versa, as

$$q(t) = C u_C(t) + C_\alpha 0^1_{t} t^{\alpha - 1} u_C(t), \quad \alpha \in (0, 1), \quad (1)$$

$$u_C(t) = \frac{1}{C} q(t) + \frac{1}{C_\mu} 0^\mu_{t} t^{\mu - 1} q(t), \quad \mu \in (0, 1), \quad (2)$$

where $0^\xi_t f(t)$ denotes the Riemann–Liouville fractional integral of order $\xi > 0$, defined as in [28] by

$$0^\xi_t f(t) = \frac{t^{\xi - 1}}{\Gamma(\xi)} \ast f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(t')}{(t - t')^{1 - \xi}} dt',$$

where $\Gamma$ is the Euler gamma function, with $\ast$ denoting the convolution and $C[F]$, $C_\alpha[F s^{1 - \alpha}]$, and $C_\mu[F s^{\mu}]$ being classical and fractional capacitances, while for the inductor, the same type of equations expresses either magnetic flux $\phi$ in terms of history of inductor current $i_L$, or vice versa, reading

$$\phi(t) = L i_L(t) + L_\beta 0^1_{t} t^{\beta - 1} i_L(t), \quad \beta \in (0, 1), \quad (3)$$

$$i_L(t) = \frac{1}{L} \phi(t) + \frac{1}{L_\nu} 0^\nu_{t} \phi(t), \quad \nu \in (0, 1), \quad (4)$$

where $L[H]$, $L_\beta[H s^{1 - \beta}]$, and $L_\nu[H s^{\nu}]$ are classical and fractional inductances. Although constitutive models (1)–(4) share the same mathematical form, they describe different elements regarding the energy consumption and production properties, since models (1) and (3) correspond to the dissipative capacitor and inductor, while models (2) and (4) represent elements’ generative counterparts, according to the thermodynamic analysis of elements in the steady-state regime, conducted in [16]. Fractional electric elements modeled by (2) and (3) can be viewed as a series connection of either classical and generative fractional capacitor, or classical and dissipative fractional inductor, while models (1) and (4) represent a parallel connection of either classical and dissipative fractional capacitor, or classical and generative fractional inductor, that is emphasized in circuits’ schemes, shown in Figs. 1, 2, 3 and 4.

Aiming to analyze the thermodynamic properties in the steady-state regime, i.e., energy consumption/production properties, as well as the frequency characteristics of the fractional $RLC$ circuits formed by the series connection of the resistor with the dissipative/generative capacitor and dissipative/generative inductor, four fractional $RLC$ circuits are considered.

The first fractional $RLC$ circuit is the dissipative–dissipative $RLC$ circuit, already considered in [17] in transient and steady-state regimes, formed by the series connec-
Fig. 1 Scheme of the dissipative–dissipative RLC circuit

![Dissipative–Dissipative RLC Circuit](image)

Fig. 2 Scheme of the generative–generative RLC circuit

![Generative–Generative RLC Circuit](image)

The scheme of the resistor with the dissipative capacitor and dissipative inductor and having the scheme depicted in Fig. 1. The equation governing dissipative–dissipative RLC circuit’s response to the electromotive force $E$ is obtained as

$$R \left( \tau_L \tau_C \frac{d^2}{dt^2} + \tau_L \tau_\alpha 0D_i^{1+\alpha} + \tau_C \tau_\beta 0D_i^{1+\beta} \right. \left. + \tau_\alpha \tau_\beta 0D_i^{\alpha+\beta} + \tau_C \frac{d}{dt} + \tau_\alpha 0D_i^{\alpha} + 1 \right) i(t) = \left( \tau_C \frac{d}{dt} + \tau_\alpha 0D_i^{\alpha} \right) E(t), \quad (5)$$

by coupling the constitutive Eqs. (1) and (3), corresponding to the dissipative capacitor and inductor, with the second Kirchhoff’s and Ohm’s laws. The derivation of governing Eq. (5), as well of the all other governing Eqs. (6)–(8), along with the description of time constants appearing in the equations is given in Sect. 2.

The second fractional RLC circuit is the generative–generative RLC circuit, having the scheme depicted in Fig. 2, that is formed by the series connection of the resistor with the generative capacitor and generative inductor and whose governing equation

$$R \left( \tau_\nu \frac{d^2}{dt^2} + \tau_\nu \tau_\mu 0D_i^{1+\nu} + \tau_\nu \frac{d}{dt} + \tau_\nu 0D_i^{\nu} + 1 \right) i(t) = \left( \tau_\nu \frac{d}{dt} + \tau_\nu 0D_i^{\nu} + 1 \right) E(t), \quad (6)$$

is obtained by coupling the constitutive Eqs. (2) and (4), corresponding to the generative capacitor and inductor, with the second Kirchhoff’s and Ohm’s laws.

The third and fourth RLC circuits are the dissipative–generative and generative–dissipative RLC circuits, with the corresponding schemes shown in Figs. 3 and 4, that are formed by connecting the resistor in series either with the dissipative capacitor and generative inductor, or with the generative capacitor and dissipative inductor, having the response to electromotive force governed either by

$$R \left( \tau_\nu \frac{d^2}{dt^2} + \tau_\nu \tau_\alpha 0D_i^{1+\nu} + \tau_\nu \tau_\alpha 0D_i^{1+\nu} + \tau_C \tau_\nu 0D_i^{1+\nu} \right. \left. + \tau_\nu \tau_\alpha \tau_\nu 0D_i^{1+\nu} + \tau_C \tau_\nu 0D_i^{1+\nu} \right) i(t) = \left( \tau_\nu \frac{d}{dt} + \tau_\nu 0D_i^{\nu} + 1 \right) E(t), \quad (7)$$
Motivated by the need for phenomenological models of electric elements that would suit for the behavior description of various newly developed electrical devices, the constitutive models (1) and (2), describing the capacitor, as well as (3) and (4), describing the inductor, that account for instantaneous and memory contributions and that are already developed and analyzed for the thermodynamical properties and for the transient- and steady-state responses of generalized \( RC \) and \( RL \) circuits in [16], are used in order to model four generalized \( RLC \) circuits: dissipative–dissipative, generative–generative, dissipative–generative, and generative–dissipative, having the corresponding schemes depicted in Figs. 1, 2, 3 and 4. Equations governing the response of these four \( RLC \) circuits are derived in complex and time domain in Sect. 2, and the circuits are further analyzed in Sect. 3 in the steady-state regime for

\[
\begin{align*}
+ & \tau_C \frac{d}{dt} + \tau_\alpha \tau_\nu \frac{0D_t^{\alpha+v}}{\tau_L} + \tau_\alpha 0D_t^\alpha + \frac{\tau_\nu}{\tau_L} 0D_t^\nu + 1) i(t) \\
& = \left( \begin{array}{c}
\tau_C \tau_\nu \frac{0D_t^{1+v}}{\tau_L} + \tau_C \frac{d}{dt} + \frac{\tau_\alpha \tau_\nu}{\tau_L} 0D_t^{\alpha+v} + \frac{\tau_\alpha}{\tau_\mu} 0I_t^{1+\mu} \end{array} \right) \mathcal{E}(t), \quad (7)
\end{align*}
\]

or by

\[
R \left( \tau_L \frac{d}{dt} + \tau_\beta \frac{0D_t^\beta}{1} + 1 + \frac{1}{\tau_C} 0I_t + \frac{1}{\tau_\mu} 0I_t^{1+\mu} \right) i(t), = \mathcal{E}(t), \quad (8)
\]

obtained either by employing the constitutive equations of dissipative capacitor and generative inductor (1) and (4), or by the use of generative capacitor and dissipative inductor models (2) and (3) in the second Kirchhoff’s law coupled with Ohm’s law.

Governing Eqs. (5)–(8) share the same notation: \( 0D_t^{n+\xi} \), with \( n \in \mathbb{N}_0 \) and \( \xi \in (0, 1) \), denotes the operator of Riemann–Liouville fractional differentiation, defined as in [28] by

\[
0D_t^{n+\xi} f (t) = \frac{d^{n+1}}{dt^{n+1}} 0^{1-\xi}I_t f (t) = \frac{d^{n+1}}{dt^{n+1}} \left( \frac{t^{-\xi}}{\Gamma(1-\xi)} * f (t) \right),
\]

\( R \) denotes resistor’s resistance, and \( \tau_C [s] \) and \( \tau_L [s] \) are classical time constants, while \( \tau_\alpha [s^\alpha], \tau_\mu [s^{1+\mu}], \tau_\beta [s^\beta], \) and \( \tau_\nu [s^{1+v}] \) are fractional time constants.

Motivated by the need for phenomenological models of electric elements that would suit for the behavior description of various newly developed electrical devices, the constitutive models (1) and (2), describing the capacitor, as well as (3) and (4), describing the inductor, that account for instantaneous and memory contributions and that are already developed and analyzed for the thermodynamical properties and for the transient- and steady-state responses of generalized \( RC \) and \( RL \) circuits in [16], are used in order to model four generalized \( RLC \) circuits: dissipative–dissipative, generative–generative, dissipative–generative, and generative–dissipative, having the corresponding schemes depicted in Figs. 1, 2, 3 and 4. Equations governing the response of these four \( RLC \) circuits are derived in complex and time domain in Sect. 2, and the circuits are further analyzed in Sect. 3 in the steady-state regime for
the thermodynamical properties, i.e., their energy consumption/production properties are established depending on the angular frequency of the electromotive force, having the significance in models’ further use in describing more complex electric devices. Moreover, whether the analyzed \( RLC \) circuit, for the given angular frequency, has predominantly capacitive or inductive properties is established as well. Mentioned newly derived properties of such circuits are illustrated in several numerical examples revealing their characteristics. Note the transient regime of the dissipative–dissipative \( RLC \) circuit is already discussed in [17]. Among the original contributions, the frequency characteristics of transfer function modulus and argument, derived in Sect. 4 along with their asymptotic expansions, stand out, since model parameters can be easily estimated from the Bode plots according to the high- and low-frequency asymptotics of transfer function modulus and argument, representing the important result useful in interpretation and fitting experimental data, that can be one of the possible further implementations of the obtained results.

Modeling electrical devices utilized for storing energy, like supercapacitors, ultracapacitors, and electrochemical double-layer capacitors (EDLC), require the use of non-classical constitutive models and the application of fractional calculus proved to be useful in formulating constitutive equations for such devices. In particular, the soundness of fractional-order models of electric elements is discussed in [59] regarding their physical properties. The review of supercapacitor’s models involving fractional calculus along with their application is presented in [2], while [51] reviews the characteristics of electric elements of fractional order, as well as the possibilities of their modeling, realizations, and applications. In addition, the review of energy storage systems, such as supercapacitors and ion-lithium batteries, along with their applications is presented in [18]. Mathematically, the behavior of supercapacitors and ultracapacitors is described either by linear constitutive models, see [9, 40, 41], or by the nonlinear constitutive equations, see [11]. Moreover, the constitutive equation of fractional capacitor may also involve fractional differentiation orders higher than one, as in [24]. The experimental work includes testing of supercapacitors at various frequencies and comparison of obtained results with the theoretical models, see [1, 26], or even manufacturing electric elements of fractional order, see [25, 29, 33], as well as their realizations by the use of constant phase element, as demonstrated in [5, 6]. The modeling of electrochemical double-layer capacitors also includes the fractional calculus, that is investigated in [23] through the frequency characteristics, in [31, 42] through the time domain analysis, and in [32] by the analysis of capacitor’s quality properties. The experiments conducted in [3] aimed to test the presence of hereditariness effects in electric double-layer capacitors, while discharge properties of electrolytic computer-grade capacitors (ECGCs) are investigated in [8]. Fractional-order models of the memory effects in inductor are discussed in [30, 49, 56], while [48, 61] investigate the complex electric networks containing electric elements of fractional order. Derivation of elements’ constitutive models from Maxwell’s equations and models of material can be found in [36, 54, 55].

The equations describing time domain behavior of \( RLC \) and \( RC \) circuits are generalized in [13, 15] by the simple replacement of the integer-order derivatives by the fractional ones, while the transient response investigations of the series \( RC_\alpha \) circuit as well of the series and parallel \( RL_\beta C_\alpha \) are conducted in [20–22], where the frac-
tional models of capacitor and inductor are taken into account. The analytical tools in obtaining the time domain response of electrical circuits containing fractional capacitor and inductor are applied in [4], while [7, 52, 53] use numerical tools for the same purpose. Different types of fractional derivatives, including Caputo, Caputo–Fabrizio, Atangana–Baleanu, and conformable derivative, are used in [14, 19, 27, 34, 35, 50] in order to replace the ordinary derivatives appearing in the governing equations of RC, RL, LC, and RLC electrical circuits. Further, such governing equations are solved using various analytical and numerical methods in order to analyze properties of such circuits as well as to compare the obtained responses with experimental data. The analysis of $RL\beta C\alpha$ and fractional RC, RL, and LC circuits in the frequency domain is performed in [44–46], while [10, 47] investigate the Wien bridge oscillators. Fractional-order filters, like the Kalman filter, and filter realizations are studied in [37, 57], while [43, 58] deal with the resonance phenomena in fractional electric circuits.

Comprehensive material regarding the modeling of classical and fractional systems, signal propagation, and fractional-order circuits is contained in [38, 39, 60].

2 Model Formulation

In order to derive the Eqs. (5)–(8), governing the response of fractional RLC circuits to the electromotive force, the second Kirchhoff’s law, combined with Ohm’s law,

$$E(t) = R i(t) + u_L(t) + u_C(t)$$

is coupled with the constitutive models of dissipative/generative capacitor and inductor, see also Figs. 1, 2, 3 and 4.

Rewriting the constitutive models of dissipative capacitor and inductor in terms of current and voltage by differentiation of (1) and (3) and subsequent use of either defining relation of current $i(t) = \frac{d}{dt} q(t)$, or Faraday’s law of electromagnetic induction $u_L(t) = \frac{d}{dt} \phi(t)$, one obtains

$$i(t) = C \frac{d}{dt} u_C(t) + C \alpha_0 D^{\alpha}_t u_C(t) = \frac{1}{R} \left( \tau_C \frac{d}{dt} u_C(t) + \tau_\alpha 0D^{\alpha}_t u_C(t) \right), \quad (10)$$

$$u_L(t) = L \frac{d}{dt} i(t) + L \beta_0 D^{\beta}_t i(t) = R \left( \tau_L \frac{d}{dt} i(t) + \tau_\beta 0D^{\beta}_t i(t) \right), \quad (11)$$

where $\tau_C = RC [s]$ and $\tau_L = L[R] [s]$ are classical time constants, while $\tau_\alpha = RC_\alpha [s^{\alpha}]$ and $\tau_\beta = L_\beta R [s^{\beta}]$ are fractional time constants. Since the dissipative capacitor, according to (10), can be viewed as the parallel connection of the classical and fractional capacitors, see also Figs. 1 and 3, the combination of classical and fractional time constants $\tau_C$ and $\tau_\alpha$ can be considered as the timescale of the charging/discharging processes of the equivalent capacitor through the resistor of resistance $R$. Similarly, the dissipative inductor, according to (11), can be viewed as the series connection of the classical and fractional inductors, see also Figs. 1 and 4, so that the combination
of classical and fractional time constants $\tau_L$ and $\tau_\beta$ can be considered as the timescale of the charging/discharging processes of the equivalent inductor through the resistor of resistance $R$.

On the other hand, the constitutive models of generative capacitor and inductor (2) and (4), rewritten in terms of current and voltage by the use of $q(t) = \int_0^t i(t') dt' = 0_1^t I^1_i(t)$ and $\phi(t) = \int_0^t u_L(t') dt' = 0_1^t I^1_u(t)$, provided that $q(0) = 0$ and $\phi(0) = 0$, as well as by using the semigroup property for fractional integrals $0_1^\xi 0_1^\eta t \Rightarrow 0_1^\xi t 0_1^\eta t$, take the respective forms

$$u_C(t) = \frac{1}{C} 0_1^1 I^1_i(t) + 0_1^\xi 0_1^{1+\mu} i(t) = R \left( \frac{1}{\tau_C} 0_1^1 I^1_i(t) + \frac{1}{\tau_\mu} 0_1^{1+\mu} i(t) \right),$$

(12)

$$i(t) = \frac{1}{L} 0_1^1 I^1_u(t) + 0_1^\nu 0_1^{1+v} u_L(t) = \frac{1}{R} \left( \frac{1}{\tau_L} 0_1^1 I^1_u(t) + \frac{1}{\tau_\nu} 0_1^{1+v} u_L(t) \right),$$

(13)

where $\tau_C = RC [s]$ and $\tau_L = L_R [s]$ are classical time constants, while $\tau_\mu = RC_\mu [s^{1+\mu}]$ and $\tau_\nu = L_\nu [s^{1+v}]$ are fractional time constants. The generative capacitor, according to (12), can be viewed as the series connection of the classical and fractional capacitors, see also Figs. 2 and 4, hence the combination of classical and fractional time constants $\tau_C$ and $\tau_\mu$ can be considered as the timescale of the charging/discharging processes of the equivalent capacitor through the resistor of resistance $R$. Similarly, the generative inductor, according to (13), can be viewed as the parallel connection of the classical and fractional inductors, see also Figs. 2 and 3, so that the combination of classical and fractional time constants $\tau_L$ and $\tau_\nu$ can be considered as the timescale of the charging/discharging processes of the equivalent inductor through the resistor of resistance $R$.

Aiming to derive the governing equation of the dissipative–dissipative $RLC$ circuit (5), the Laplace transform is applied to the second Kirchhoff’s law (9) and constitutive models of dissipative capacitor (10) and dissipative inductor (11), yielding

$$\hat{E}(s) = R \hat{i}(s) + \hat{u}_L(s) + \hat{u}_C(s)$$

(14)

and

$$\hat{i}(s) = \left( C s + C_\alpha s^{\alpha} \right) \hat{u}_C(s) = \frac{1}{R} \left( \tau_C s + \tau_\alpha s^{\alpha} \right) \hat{u}_C(s),$$

(15)

$$\hat{u}_L(s) = \left( L s + L_\beta s^{\beta} \right) \hat{i}(s) = R \left( \tau_L s + \tau_\beta s^{\beta} \right) \hat{i}(s),$$

(16)

respectively, where the Laplace transform is defined by

$$\hat{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t) e^{-st} dt, \quad \text{for } \text{Re} s > 0,$$
and where the Laplace transforms of Riemann–Liouville fractional derivative and fractional integral

\[
L\left[0D_t^\xi f(t)\right](s) = s^\xi \hat{f}(s) - \left[0I_t^{1-\xi} f(t)\right]_{t=0} = s^\xi \hat{f}(s) \quad \text{and}
L\left[0I_t^\xi f(t)\right](s) = \frac{1}{s^{\xi}} \hat{f}(s),
\]

holding for \(f\) bounded at zero, are employed. Reducing the system of Eqs. (14)–(16) to the single equation written in terms of the circuit current and electromotive force in complex domain, one obtains the governing equation of the dissipative–dissipative \(RLC\) circuit in the complex domain in the form

\[
R(\tau_L \tau_C s^2 + \tau_L \tau_{\alpha}s^{1+\alpha} + \tau_C \tau_{\beta}s^{1+\beta} + \tau_{\alpha} \tau_{\beta}s^{\alpha+\beta} + \tau_C s + \tau_{\alpha}s^{\alpha} + 1)\hat{i}(s) = (\tau_C s + \tau_{\alpha}s^{\alpha})\hat{E}(s),
\]

implying the transfer function

\[
\hat{g}^{(dd)}_{i}(s) = \frac{\hat{i}(s)}{\hat{E}(s)} = \frac{1}{R} \frac{\tau_C s + \tau_{\alpha}s^{\alpha}}{\Phi_{dd}(s)}, \quad \text{with}
\Phi_{dd}(s) = \tau_L \tau_C s^2 + \tau_L \tau_{\alpha}s^{1+\alpha} + \tau_C \tau_{\beta}s^{1+\beta} + \tau_{\alpha} \tau_{\beta}s^{\alpha+\beta} + \tau_C s + \tau_{\alpha}s^{\alpha} + 1,
\]

so that the governing Eq. (5) follows from (17) after the inverse Laplace transform is applied.

Analogously, the governing equation of the generative–generative \(RLC\) circuit (6) is derived by coupling the second Kirchhoff’s law in complex domain (14) with the Laplace transforms of constitutive Eq. (12) for generative capacitor and (13) for generative inductor, being of the form

\[
\hat{u}_C(s) = \left(\frac{1}{C s} + \frac{1}{C_{\mu} s^{1+\mu}}\right)\hat{i}(s) = R \left(\frac{1}{\tau_C s} + \frac{1}{\tau_{\mu} s^{1+\mu}}\right)\hat{i}(s),
\]

\[
\hat{i}(s) = \left(\frac{1}{L s} + \frac{1}{L_{\nu} s^{1+\nu}}\right)\hat{u}_L(s) = \frac{1}{R} \left(\frac{1}{\tau_L s} + \frac{1}{\tau_{\nu} s^{1+\nu}}\right)\hat{u}_L(s),
\]

so that the system of equations reduced to a single equation, written in terms of the circuit current and electromotive force in complex domain, yield the governing equation of generative–generative \(RLC\) circuit in complex domain as

\[
R \left(\tau_{\nu}s^{1+\nu} + \frac{\tau_{\nu}}{\tau_L} s^{1+\nu} + 1 + \frac{1}{\tau_C s} + \frac{\tau_{\nu}}{\tau_L \tau_{\mu} s^{1+\mu-\nu}} + \frac{1}{\tau_{\mu} s^{1+\mu}}\right)\hat{i}(s) = \left(\frac{\tau_{\nu}s^{\nu} + 1}{\tau_L}\right)\hat{E}(s),
\]
implying the governing Eq. (6) after the inversion of Laplace transform. The transfer function

$$\hat{g}_i^{(gg)}(s) = \frac{\hat{i}(s)}{\mathcal{E}(s)} = \frac{1}{R} \frac{\tau_C \tau_v s^{1+\mu} \tau_v s^{1+\nu} + \tau_L \Phi_{gg}(s)}{\Phi_{gg}(s)}, \quad \text{with}$$

$$\Phi_{gg}(s) = \frac{\tau_L \tau_C \tau_v s^{2+\mu+\nu} + \tau_C \tau_v s^{1+\mu+\nu} + \tau_L \tau_C \tau_v s^{1+\mu} + \tau_v s^{\mu+\nu} + \tau_L \tau_v s^{\mu} + \tau_C \tau_0 s^{\nu} + \tau_L \tau_C, \quad (22)$$

corresponding to the generative–generative $RLC$ circuit, follows from the governing equation in complex domain (21).

Governing Eqs. (7) and (8), corresponding to the dissipative–generative and generative–dissipative $RLC$ circuits, are obtained by the application of inverse Laplace transform to the equations in complex domain

$$\begin{align*}
R \left( \tau_C \tau_v s^{2+\nu} + \tau_\alpha s^{1+\nu} + \frac{\tau_C \tau_v s^{1+\nu}}{\tau_L} \\
+ \tau_C s + \frac{\tau_\alpha \tau_v s^{\alpha+\nu}}{\tau_L} + \tau_\alpha s^{\alpha} + \frac{\tau_v s^{\nu} + 1}{\tau_L} \right) \hat{i}(s) \\
= \left( \frac{\tau_C \tau_v s^{1+\nu}}{\tau_L} + \tau_C s + \frac{\tau_\alpha \tau_v s^{\alpha+\nu}}{\tau_L} + \tau_\alpha s^{\alpha} \right) \mathcal{E}(s), \quad (23)
\end{align*}$$

corresponding to the dissipative–generative $RLC$ circuit and

$$\begin{align*}
R \left( \tau_L s + \tau_\beta s^\beta + 1 + \frac{1}{\tau_C s} + \frac{1}{\tau_\mu s^{1+\mu}} \right) \hat{i}(s) = \mathcal{E}(s), \quad (24)
\end{align*}$$

corresponding to the generative–dissipative $RLC$ circuit, that are obtained as a consequence of the reduction to the single equation, expressed in terms of the circuit current and electromotive force in complex domain, of the second Kirchhoff’s law (14), combined either with the constitutive equations of dissipative capacitor (15) and generative inductor (20), or with the models of generative capacitor (19) and dissipative inductor (16). The transfer function

$$\begin{align*}
\hat{g}_i^{(dg)}(s) = \frac{\hat{i}(s)}{\mathcal{E}(s)} = \frac{1}{R} \frac{\tau_C \tau_v s^{1+\nu} + \tau_L \tau_C s + \tau_\alpha \tau_v s^{\alpha+\nu} + \tau_L \tau_\alpha s^{\alpha}}{\Phi_{dg}(s)}, \quad \text{with}
\end{align*}$$

$$\begin{align*}
\Phi_{dg}(s) = \tau_L \tau_C \tau_v s^{2+\nu} + \tau_L \tau_\alpha \tau_v s^{1+\alpha+\nu} + \tau_C \tau_0 s^{1+\nu} + \tau_L \tau_C s \\
+ \tau_\alpha \tau_v s^{\alpha+\nu} + \tau_L \tau_\alpha s^{\alpha} + \tau_v s^{\nu} + \tau_L 
\end{align*} \quad (25)$$

corresponding to the dissipative–generative $RLC$ circuit and

$$\begin{align*}
\hat{g}_i^{(gd)}(s) = \frac{\hat{i}(s)}{\mathcal{E}(s)} = \frac{1}{R} \frac{\tau_C \tau_\mu s^{1+\mu}}{\Phi_{gd}(s)}, \quad \text{with}
\end{align*}$$

$$\Phi_{gd}(s) = \tau_L \tau_C \tau_\mu s^{2+\mu} + \tau_C \tau_\beta \tau_\mu s^{1+\beta+\mu}.$$
corresponding to the generative–dissipative RLC circuit, respectively, follows from the governing equations in complex domain (23) and (24).

3 Thermodynamical Considerations in the Steady-State Regime

In order to analyze energy consumption/production properties of fractional RLC circuits containing dissipative and/or generative elements, the steady-state regime of circuits is assumed, i.e., the electromotive force, circuit current, and voltages of dissipative/generative capacitor and inductor are assumed as harmonic functions of angular frequency \( \omega \) as

\[
\begin{align*}
E(t) &= E_0 e^{j \omega t}, \\
i(t) &= i_0 e^{j(\omega t + \phi_i)}, \\
\mathcal{U}_C(t) &= u_{C0} e^{j(\omega t + \phi_C)}, \quad \text{and} \quad \mathcal{U}_L(t) = u_{L0} e^{j(\omega t + \phi_L)},
\end{align*}
\]

(27)

where \( E_0, i_0, u_{C0}, \) and \( u_{L0} \) are amplitudes and \( \phi_i, \phi_C, \) and \( \phi_L \) are phase angles. Electromotive force and current, assumed as (27)1 and (27)2, when plugged into governing Eqs. (5)–(8) of fractional RLC circuits, lead to the sine and cosine of current’s phase angle \( \phi_i \), expressed in terms of ratio of current and electromotive force amplitudes \( \frac{i_0}{E_0} \), due to the linearity of governing equations, properties of integer-order derivatives, and large time asymptotics of Riemann–Liouville fractional derivative and fractional integral, being given by

\[
\begin{align*}
0D_t^\xi e^{j(\omega t + \phi)} &= (j\omega)^\xi e^{j(\omega t + \phi)} = \omega^\xi e^{j(\omega t + \phi + \frac{\xi \pi}{2})} \quad \text{as} \quad t \to \infty, \\
0I_t^\xi e^{j(\omega t + \phi)} &= \frac{1}{(j\omega)^\xi} e^{j(\omega t + \phi)} = \frac{1}{\omega^\xi} e^{j(\omega t + \phi - \frac{\xi \pi}{2})} \quad \text{as} \quad t \to \infty,
\end{align*}
\]

(28)

(29)

see [12]. The same result would be achieved by substituting \( s = j\omega \) into the transfer functions (18), (22), (25), and (26), followed by the separation of real and imaginary parts in such obtained expressions, assuming \( \hat{i}(j\omega) = i_0 e^{j(\omega t + \phi_i)} \) and \( \hat{\mathcal{E}}(j\omega) = \mathcal{E}_0 e^{j\omega t} \).

The sign of current’s phase angle cosine determines whether circuit dissipates or generates energy, since the energy consumed/produced by the fractional RLC circuit during one period \( T \) of harmonic functions (27) is determined by

\[
W = \int_{nT}^{(n+1)T} \mathcal{E}(t) i(t) \, dt \\
W = \mathcal{E}_0 i_0 \int_{nT}^{(n+1)T} \cos(\omega t) \cos(\omega t + \phi_i) \, dt = \frac{1}{2} \mathcal{E}_0 i_0 T \cos \phi_i,
\]

(30)

where

\[
\mathcal{E} = \text{Re} \hat{\mathcal{E}} \quad \text{and} \quad i = \text{Re} \hat{i},
\]
while the sign of phase angle sine determines whether the circuit has capacitive or inductive character, so that if \( \cos \phi_i > 0 \) (\( \cos \phi_i < 0 \)), then circuit dissipates (generates) energy and if \( \sin \phi_i > 0 \) (\( \sin \phi_i < 0 \)), then circuit has capacitive (inductive) character, since current leads (lags) the electromotive force.

Rather than plugging harmonic electromotive force and current into governing equations, harmonic current and appropriate harmonic voltage are inserted into constitutive equations for dissipative elements (10) and (11) yielding dissipative capacitor’s admittance and dissipative inductor’s impedance as

\[
Y_C^{(d)}(\omega) = \frac{i(t)}{u_C(t)} = C\omega e^{i\alpha\pi/2} + C\alpha\omega e^{i\alpha\pi/2},
\]

\[
Z_L^{(d)}(\omega) = \frac{u_L(t)}{i(t)} = L\omega e^{i\beta\pi/2} + L\beta\omega e^{i\beta\pi/2},
\]

by the large time asymptotics of Riemann–Liouville fractional derivative (28), so that

\[
\frac{1}{Y_C^{(d)}(\omega)} = \frac{C\alpha\omega^\alpha \cos \frac{\alpha\pi}{2} - j \left( C\omega + C\alpha\omega^\alpha \sin \frac{\alpha\pi}{2} \right)}{C^2\omega^2 + 2C\alpha\omega^1+\alpha \sin \frac{\alpha\pi}{2} + C^2\alpha\omega^{2\alpha}},
\]

(31)

\[
Z_L^{(d)}(\omega) = L\beta\omega^\beta \cos \frac{\beta\pi}{2} + j \left( L\omega + L\beta\omega^\beta \sin \frac{\beta\pi}{2} \right),
\]

(32)

while generative capacitor’s impedance and generative inductor’s admittance

\[
Z_C^{(g)}(\omega) = \frac{u_C(t)}{i(t)} = \frac{1}{C\omega e^{i\alpha\pi/2}} + \frac{1}{C\mu \omega^{1+\mu} e^{i(1+\mu)\pi/2}},
\]

\[
Y_L^{(g)}(\omega) = \frac{i(t)}{u_L(t)} = \frac{1}{L\omega e^{i\beta\pi/2}} + \frac{1}{L\nu \omega^{1+\nu} e^{i(1+\nu)\pi/2}},
\]

are obtained using large time asymptotics of the fractional integral (29) in generative capacitor’s and inductor’s models (12) and (13), so that

\[
Z_C^{(g)}(\omega) = -\frac{\sin \frac{\mu\pi}{2}}{C\mu \omega^{1+\mu}} - j \left( \frac{1}{C\omega} + \frac{\cos \frac{\mu\pi}{2}}{C\mu \omega^{1+\mu}} \right),
\]

(33)

\[
\frac{1}{Y_L^{(g)}(\omega)} = -LL\nu \omega^{1+\nu} L \sin \frac{\nu\pi}{2} - j \left( L\nu \omega^{\nu} + L \cos \frac{\nu\pi}{2} \right) \frac{L^2\omega^{2\nu} + 2LL\nu \omega^{\nu} \cos \frac{\nu\pi}{2} + L^2}.
\]

(34)

The second Kirchhoff’s law (9) in the steady-state regime

\[
\mathcal{E}(t) = R i(t) + u_L(t) + u_C(t), \quad \text{i.e.,} \quad \mathcal{E}(t) = Z_e i(t),
\]

(35)

with \( Z_e = R + Z_C + Z_L \) being the equivalent impedance of the fractional RLC circuit, using (27) yields

\[
Z_e = \frac{\mathcal{E}_0}{i_0} e^{-j\phi_i} \implies \cos \phi_i = \frac{\Re Z_e}{|Z_e|} \quad \text{and} \quad \sin \phi_i = -\frac{\Im Z_e}{|Z_e|},
\]

(36)
so that in the case of dissipative–dissipative $RLC$ circuit the equivalent impedance

$$Z_e^{(dd)}(\omega) = R + \frac{1}{Y_C^{(d)}(\omega)} + Z_L^{(d)}(\omega) \quad (37)$$

gives

$$\cos \phi_i^{(dd)}(\omega) = \frac{1}{|Z_e^{(dd)}(\omega)|} \times \left( R + \frac{C_\omega \omega^\alpha \cos \frac{\alpha \pi}{2} + C_\omega^2 \omega^{2\alpha} + L_\omega \omega^\beta \cos \frac{\beta \pi}{2} \right), \quad (38)$$

$$\sin \phi_i^{(dd)}(\omega) = \frac{1}{|Z_e^{(dd)}(\omega)|} \times \left( \frac{C_\omega + C_\omega \omega^\alpha \sin \frac{\alpha \pi}{2}}{C^2 \omega^2 + 2CC_\omega \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + C_\omega^2 \omega^{2\alpha}} - L_\omega - L_\omega \omega^\beta \sin \frac{\beta \pi}{2} \right), \quad (39)$$

using (36) along with capacitor’s and inductor’s impedances (31) and (32). Clearly, the circuit is dissipative for all frequencies since $\cos \phi_i > 0$, see (38), having the asymptotics

$$\cos \phi_i^{(dd)}(\omega) \sim \cos \frac{\alpha \pi}{2} \quad \text{as} \quad \omega \to 0 \quad \text{and}$$

$$\cos \phi_i^{(dd)}(\omega) \sim \frac{L_\omega}{L} \frac{1}{\omega^{1-\beta}} \cos \frac{\beta \pi}{2} \to 0^+ \quad \text{as} \quad \omega \to \infty, \quad (40)$$

while its character changes from capacitive for low frequencies to inductive for high frequencies, due to the low- and high-frequency asymptotics of phase angle sine

$$\sin \phi_i^{(dd)}(\omega) \sim \sin \frac{\alpha \pi}{2} > 0 \quad \text{as} \quad \omega \to 0 \quad \text{and}$$

$$\sin \phi_i^{(dd)}(\omega) \sim -1 \quad \text{as} \quad \omega \to \infty. \quad (41)$$

The phase angle cosine and sine as functions of the angular frequency, obtained by (38) and (39), are depicted in Fig. 5, illustrating that the dissipative–dissipative $RLC$ circuit consumes energy for all frequencies, with the possibility that the phase angle cosine may even have an additional minimum, compare Fig. 5a, b, nevertheless slowly tending to zero from above, as predicted by the asymptotics (40), while the phase angle sine either monotonically or non-monotonically changes from positive to negative values inferring the change of circuit’s character from capacitive to inductive, as predicted by the asymptotics (41).
The equivalent impedance of the generative–generative fractional RLC circuit

\[
Z_e^{(gg)}(\omega) = R + Z_C^{(g)}(\omega) + \frac{1}{Y_L^{(g)}(\omega)}
\]

using (36) together with capacitor’s and inductor’s impedances (33) and (34) yields

\[
\cos \phi_i^{(gg)}(\omega) = \frac{1}{|Z_e^{(gg)}(\omega)|} \times \left( R - \frac{\sin \frac{\mu \pi}{2}}{C_\mu \omega^{1+\mu}} - LL_v \omega^{1+v} \frac{L \sin \frac{\nu \pi}{2}}{L_v^2 \omega^{2\nu} + 2LL_v \omega^{\nu} \cos \frac{\nu \pi}{2} + L^2} \right),
\]

\[
\sin \phi_i^{(gg)}(\omega) = \frac{1}{|Z_e^{(gg)}(\omega)|} \times \left( \frac{1}{C\omega} + \frac{\cos \frac{\mu \pi}{2}}{C_\mu \omega^{1+\mu}} - LL_v \omega^{1+v} \frac{L_v \omega^{\nu} \cos \frac{\nu \pi}{2}}{L_v^2 \omega^{2\nu} + 2LL_v \omega^{\nu} \cos \frac{\nu \pi}{2} + L^2} \right),
\]

inferring that circuit is generative for both low and high frequencies and may be dissipative for mid-range frequencies, due to

\[
\cos \phi_i^{(gg)}(\omega) \sim - \sin \frac{\mu \pi}{2} < 0 \quad \text{as} \quad \omega \to 0 \quad \text{and}
\]

\[
\cos \phi_i^{(gg)}(\omega) \sim - \frac{L}{L_v \omega^{\nu}} \sin \frac{\nu \pi}{2} \to 0^- \quad \text{as} \quad \omega \to \infty,
\]
Fig. 6 Energy consumption/production properties and capacitive/inductive character of the generative–generative $RLC$ circuit: $\cos \phi_{i}^{(gg)}$ and $\sin \phi_{i}^{(gg)}$ as functions of angular frequency $\omega$, obtained for model parameters: $\mu = 0.7$, $\nu = 0.9$, $\tau_{C} = 0.75$, $\tau_{L} = 0.75$, and $\tau_{\nu} = 0.025$

and, as for the dissipative–dissipative $RLC$ circuit, capacitive properties prevail for low frequencies, while the circuit is inductive for high frequencies, since

$$
\sin \phi_{i}^{(gg)}(\omega) \sim \cos \frac{\mu \tau_{\nu}}{2} > 0 \quad \text{as} \quad \omega \to 0 \quad \text{and}
$$

$$
\sin \phi_{i}^{(gg)}(\omega) \sim -1 \quad \text{as} \quad \omega \to \infty.
$$

Figure 6 depicts the phase angle cosine and sine versus the angular frequency for the generative–generative $RLC$ circuit, obtained according to (43) and (44), where the energy is produced for all frequencies in the case of model parameters used to obtain the plot from Fig. 6c, while in the other two cases, the energy is produced for both low and high frequencies, see Fig. 6a, b, that is in accordance with the asymptotics (45), while the circuit consumes energy for the mid-range frequencies, with the possibility of the abrupt change in energy consumption/production properties, as illustrated in Fig. 6b. The change of circuit’s predominant character from capacitive to inductive, see also the asymptotics (46), is non-monotonic for all depicted cases of model parameters, again with the possibility of the abrupt character change.
The equivalent impedances corresponding to the dissipative–generative and generative–dissipative RLC circuits are

\[ Z^{(dg)}_e(\omega) = R + \frac{1}{Y_C^{(d)}(\omega)} + \frac{1}{Y_L^{(g)}(\omega)} \quad \text{and} \quad Z^{(gd)}_e(\omega) = R + Z_C^{(g)}(\omega) + Z_L^{(d)}(\omega), \]

so that for the dissipative–generative RLC circuit the equivalent impedance \( Z^{(dg)}_e(\omega) \), given by (47), with capacitor’s and inductor’s impedances (31) and (34), according to (36) gives

\[
\cos \phi^{(dg)}_i(\omega) = \frac{1}{|Z^{(dg)}_e(\omega)|} \left( R + \frac{C\alpha\omega^\alpha \cos \frac{\alpha\pi}{2}}{C^2\omega^2 + 2CC\omega^{1+\alpha} \sin \frac{\alpha\pi}{2} + C^2\omega^{2\alpha}} - LLv\omega^{1+v} \frac{L \sin \frac{\nu\pi}{2}}{L^2\omega^{2\nu} + 2LLv\omega^\nu \cos \frac{\nu\pi}{2} + L^2} \right),
\]

\[
\sin \phi^{(dg)}_i(\omega) = \frac{1}{|Z^{(dg)}_e(\omega)|} \left( C\omega + C\alpha\omega^\alpha \sin \frac{\alpha\pi}{2} \right. \\
\left. - LLv\omega^{1+v} \frac{L\omega^\nu + L \cos \frac{\nu\pi}{2}}{L^2\omega^{2\nu} + 2LLv\omega^\nu \cos \frac{\nu\pi}{2} + L^2} \right),
\]

along with their asymptotics

\[
\cos \phi^{(dg)}_i(\omega) \sim \cos \frac{\alpha\pi}{2} > 0 \quad \text{as} \quad \omega \to 0 \quad \text{and} \quad \cos \phi^{(dg)}_i(\omega) \sim -\frac{L}{Lv\omega^\nu \sin \frac{\nu\pi}{2}} \to 0^- \quad \text{as} \quad \omega \to \infty,
\]

\[
\sin \phi^{(dg)}_i(\omega) \sim \sin \frac{\alpha\pi}{2} > 0 \quad \text{as} \quad \omega \to 0 \quad \text{and} \quad \sin \phi^{(dg)}_i(\omega) \sim -1 \quad \text{as} \quad \omega \to \infty,
\]

while for the generative–dissipative RLC circuit, the equivalent impedance \( Z^{(gd)}_e \), see (47), employing capacitor’s and inductor’s impedances (32) and (33), by (36) yields

\[
\cos \phi^{(gd)}_i(\omega) = \frac{1}{|Z^{(gd)}_e(\omega)|} \left( R - \frac{\sin \frac{\mu\pi}{2}}{C\mu\omega^{1+\mu}} + L\beta\omega^\beta \cos \frac{\beta\pi}{2} \right),
\]

\[
\sin \phi^{(gd)}_i(\omega) = \frac{1}{|Z^{(gd)}_e(\omega)|} \left( \frac{1}{C\omega} + \frac{\cos \frac{\mu\pi}{2}}{C\mu\omega^{1+\mu}} - L\omega - L\beta\omega^\beta \sin \frac{\beta\pi}{2} \right),
\]
Fig. 7 Energy consumption/production properties and capacitive/inductive character of the dissipative–generative RLC circuit: \( \cos \phi_i^{(dg)} \) and \( \sin \phi_i^{(dg)} \) as functions of angular frequency \( \omega \), obtained for model parameters: \( \alpha = 0.25, \nu = 0.85, \tau_C = 0.25, \tau_\alpha = 0.005, \) and \( \tau_L = 0.75 \)

having the asymptotics

\[
\cos \phi_i^{(gd)} (\omega) \sim - \sin \frac{\mu \pi}{2} < 0 \quad \text{as} \quad \omega \to 0 \quad \text{and} \\
\cos \phi_i^{(gd)} (\omega) \sim \frac{L^\beta}{L} \frac{1}{\omega^{1-\beta}} \cos \frac{\beta \pi}{2} \to 0^+ \quad \text{as} \quad \omega \to \infty, \\
\sin \phi_i^{(gd)} (\omega) \sim \cos \frac{\mu \pi}{2} > 0 \quad \text{as} \quad \omega \to 0 \quad \text{and} \\
\sin \phi_i^{(gd)} (\omega) \sim -1 \quad \text{as} \quad \omega \to \infty.
\]  

(54) 

Regarding the energy consumption/production properties, one concludes that the dissipative–generative \( RLC \) circuit dissipates energy for low frequencies and generates it for high frequencies, see (50), while generative–dissipative circuit behaves exactly in the opposite way, see (54). Capacitive and inductive character of both circuits remains same as in all previous cases, compare (51) and (55) with (41) and (46).

Plots of the cosine and sine of phase angle as functions of the angular frequency, shown in Figs. 7 and 8, are obtained according to (48) and (49) for dissipative–generative and according to (52) and (53) for generative–dissipative \( RLC \) circuits,
Fig. 8 Energy consumption/production properties and capacitive/inductive character of the generative–dissipative RLC circuit: \( \cos \phi_i^{(gd)} \) and \( \sin \phi_i^{(gd)} \) as functions of angular frequency \( \omega \), obtained for model parameters: \( \mu = 0.2 \), \( \beta = 0.6 \), \( \tau_C = 0.025 \), \( \tau_\mu = 0.01 \), and \( \tau_L = 0.95 \) respectively. Figure 7a and 8c illustrate that the energy is consumed, respectively, produced, for quite a large frequency range, changing the energy consumption/production properties for high frequencies in accordance with the high-frequency asymptotics (50) and (54), respectively. One notices from Fig. 7a that the circuit changed its character from capacitive (capacitor is the dissipative element) to inductive (inductor is the generative element) and still being energy consuming, contrary to the case depicted in Fig. 7c, where the circuit started to produce energy while still being of predominantly capacitive (dissipative) character. Figure 8a, c illustrates the similar behavior of the generative–dissipative RLC circuit as well, since the circuit became energy consuming while still being of predominantly capacitive character (capacitor is the generative element), see Fig. 8a, contrary to the case depicted in Fig. 8c, where the circuit changed its character to predominantly inductive (inductor is the dissipative element), while still producing energy. The possibility that both types of RLC circuits suddenly change their energy consumption/production properties and predominant character is illustrated in Figs. 7b and 8b.
representing the ideal energy consuming element, see Fig. 6c. In all other cases, there is an interplay of dissipative and generative properties of electric elements, implying the non-monotonic character of the energy consumption/production properties of the fractional \( RLC \) circuit with respect to the angular frequency change, see Figs. 6a, b, 7, and 8. Nevertheless, the energy consumption/production characteristics for low and high frequencies are fully governed by the capacitor’s character for low frequencies and inductor’s character for high frequencies, since the predominant character of the fractional \( RLC \) circuit is capacitive for low and inductive for high frequencies. Neither of the fractional \( RLC \) circuits consumes/produces energy for zero and infinite angular frequency, since the current amplitude is zero although the phase angle cosine is different from zero in the former case and in addition to the zero value of the current amplitude, the cosine is also zero in the latter case. The change in the energy consumption/production properties of the fractional \( RLC \) circuits can also be abrupt and occurring for a specified angular frequency determined by the model parameters, see Figs. 6b, 7b, and 8b.

4 Frequency Characteristics and their Asymptotics

Traditionally, frequency characteristics of the transfer function modulus and argument are obtained by substituting \( s = j\omega \) into the transfer function and subsequently by determining its modulus and argument, where, in this particular case, the transfer function \( \hat{g} \) takes the forms, respectively, given by (18), (22), (25), and (26) for the dissipative–dissipative, generative–generative, dissipative–generative, and generative–dissipative fractional \( RLC \) circuits, respectively.

However, the approach of considering the second Kirchhoff’s law (35), corresponding to the fractional \( RLC \) circuit in the steady-state regime, is adopted, so that the transfer function is equivalently defined by

\[
\hat{g} = R \frac{i(t)}{E(t)} \quad \text{yielding} \quad \hat{g} = \frac{R}{Z_e} \left( \text{Re} Z_e - j \text{Im} Z_e \right)
\]

and implying that the transfer function \( \hat{g} \) is physically the equivalent admittance of the \( RLC \) circuit multiplied by resistor’s resistance \( R \). Hence, the transfer function modulus and argument, along with their asymptotics, are determined using (56) as

\[
|\hat{g}(\omega)|_{\text{dB}} = 20 \log |\hat{g}(\omega)| = -10 \log \left( \frac{\text{Re}^2 Z_e(\omega)}{R} + \frac{\text{Im}^2 Z_e(\omega)}{R} \right) \quad \text{and} \quad \arg \hat{g}(\omega) = \arccot \frac{\text{Re} Z_e(\omega)}{-\text{Im} Z_e(\omega)},
\]

with the equivalent impedance \( Z_e \) given by (37) in the case of dissipative–dissipative fractional \( RLC \) circuit, expression (42) for generative–generative circuit, and by (47) for dissipative–generative and generative–dissipative circuits. Note that the definition

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of transfer function (56), using $i$ and $E$ given by (27), also implies

$$
\hat{g} = R \frac{i_0}{E_0} e^{i \phi_i} \text{ yielding } |\hat{g}|_{\text{dB}} = 20 \log \frac{|\hat{g}|}{E_0} \text{ and } \arg \hat{g} = \phi_i.
$$

4.1 Dissipative–Dissipative Fractional RLC Circuit

Considering the dissipative–dissipative fractional RLC circuit and by rewriting the real and imaginary parts of circuit’s equivalent impedance (37) in terms of the classical and fractional time constants, one has

$$
\text{Re } Z_e^{(dd)}(\omega) = R \left( 1 + \frac{\tau_\alpha \omega^\alpha \cos \frac{\alpha \pi}{2} \tau_\alpha + 2 \tau_c \tau_\alpha \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_\beta \omega^\beta \cos \frac{\beta \pi}{2} }{\tau_c \omega^2 + 2 \tau_c \tau_\alpha \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_\alpha \omega^{2\alpha}} \right),
$$

(58)

$$
\text{Im } Z_e^{(dd)}(\omega) = -R \left( \frac{\tau_\alpha \omega^\alpha \sin \frac{\alpha \pi}{2} \tau_\alpha + 2 \tau_c \tau_\alpha \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_\beta \omega^\beta \sin \frac{\beta \pi}{2} }{\tau_c \omega^2 + 2 \tau_c \tau_\alpha \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_\alpha \omega^{2\alpha}} - \tau_L \omega - \tau_\beta \omega^\beta \sin \frac{\beta \pi}{2} \right),
$$

(59)

yielding the transfer function modulus and argument according to (57).

Figure 9 presents frequency characteristics of the transfer function modulus and argument, i.e., Bode diagrams, for same two sets of model parameters as in the case of plots from Fig. 5. The transfer function modulus, shown in Fig. 9a, is a non-monotonic function of the angular frequency, tending to the negative infinity for low frequencies, implying that it has a zero of non-integer order at the origin, as obvious from the form (18) of the transfer function $\hat{g}^{(dd)}$. Further, the transfer function modulus attains a maximum and tends to the negative infinity for high frequencies, suggesting that the transfer function $\hat{g}^{(dd)}$ has a pair of complex conjugated poles, that is exactly the case for the model parameters used to produce dashed-line plot, while the parameters used for the solid-line plot yield no poles of the transfer function. The shape of the transfer function modulus suggests that the dissipative–dissipative RLC circuit behaves as the band-pass filter with different bandwidths.

The transfer function argument, so as its sine, either monotonically or non-monotonically changes from positive to negative values as the angular frequency increases, as obvious from Fig. 9b, implying the change of circuit’s behavior from predominantly capacitive to predominantly inductive, with argument’s span between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ implying dissipativity of the circuit for the whole range of frequencies, since argument’s cosine is positive.

As already noted in Sect. 3, the capacitive character of the fractional RLC circuit prevails for low frequencies and therefore the corresponding asymptotics of $\text{Re } Z_e^{(dd)}$ and $\text{Im } Z_e^{(dd)}$, given by (58) and (59), is

$$
\text{Re } Z_e^{(dd)}(\omega) = R \frac{\cos \frac{\alpha \pi}{2}}{\tau_\alpha \omega^\alpha} \left\{ \begin{array}{ll}
1 + \frac{\tau_\alpha \omega^\alpha}{\cos \frac{\alpha \pi}{2}} + O(\omega^{1-\alpha}), & \text{if } \alpha \in \left(0, \frac{1}{2}\right), \\
1 + O(\omega^{1-\alpha}), & \text{if } \alpha \in \left[\frac{1}{2}, 1\right),
\end{array} \right.
$$

(60)
Fig. 9 Frequency characteristics of transfer function modulus and argument for dissipative–dissipative RLC circuit, obtained for model parameters: $\alpha = 0.4$, $\beta = 0.8$, $\tau_C = 0.5$, $\tau_L = 0.5$, $\tau_\beta = 1.5$, and $\tau_\alpha = 3$—solid line and $\tau_\alpha = 0.2$—dashed line.

\[
\text{Im } Z_e^{(dd)}(\omega) = -R \frac{\sin \frac{\alpha \pi}{2}}{\tau_\alpha \omega^\alpha} \left( 1 + O(\omega^{1-\alpha}) \right), \quad (61)
\]
as $\omega \to 0$, where only the leading terms of real and imaginary parts of dissipative capacitor’s impedance $\frac{1}{Y_C^{(dd)}}$, given by (31), are taken into account, since they are of the order $-\alpha$, which is certainly smaller than the order $\beta > 0$ of the leading terms of dissipative inductor’s impedance $\text{Re } Z_L^{(dd)}$ and $\text{Im } Z_L^{(dd)}$, see (32), so that, by (57)1, (60), and (61), for the transfer function modulus one has

\[
\left| \hat{g}^{(dd)}(\omega) \right|_{\text{dB}} = 20 \log \left( \tau_\alpha \omega^\alpha \right) - 10 \log \begin{cases} 
1 + 2 \tau_\alpha \omega^\alpha \cos \frac{\alpha \pi}{2} + O(\omega^{1-\alpha}), & \text{if } \alpha \in (0, \frac{1}{3}), \\
1 + 2 \tau_\alpha \omega^\alpha \cos \frac{\alpha \pi}{2} + O(\omega^{1-\alpha}), & \text{if } \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right), \\
1 + O(\omega^{1-\alpha}), & \text{if } \alpha \in \left[\frac{1}{2}, 1\right],
\end{cases} \quad (62)
\]
as $\omega \to 0$, while the transfer function argument, by (57)2, (60), and (61), is

\[
\cot \arg \hat{g}^{(dd)}(\omega) = \cot \frac{\alpha \pi}{2} \begin{cases} 
1 + \frac{\tau_\alpha \omega^\alpha}{\cos \frac{\alpha \pi}{2}} + O(\omega^{1-\alpha}), & \text{if } \alpha \in (0, \frac{1}{3}), \\
1 + O(\omega^{1-\alpha}), & \text{if } \alpha \in \left[\frac{1}{3}, 1\right],
\end{cases} \quad (63)
\]
as $\omega \to 0$.

On the other hand, the inductive properties of the fractional RLC circuits are more prominent for the high frequencies and therefore $\text{Re } Z_e^{(dd)}$ and $\text{Im } Z_e^{(dd)}$, given by (58) and (59), are of the form

\[
\text{Re } Z_e^{(dd)}(\omega) = R \tau_\beta \omega^\beta \frac{\beta \pi}{2} \left( 1 + \frac{\omega^{-\beta}}{\tau_\beta \cos \frac{\beta \pi}{2}} + O(\omega^{1-\beta}) \right), \quad (64)
\]
\[
\text{Im } Z_e^{(dd)}(\omega) = R \tau_L \omega \left( 1 + \frac{\tau_\beta}{\tau_L} \omega^{-1+\beta} \sin \frac{\beta \pi}{2} + O(\omega^{-2}) \right), \quad (65)
\]
\[ \omega \to \infty, \text{ due to the leading terms of dissipative capacitor's impedance (31), that are} \]
\[
\begin{align*}
\text{Re} \frac{1}{Y_C^{(d)}(\omega)} &\sim R \frac{\tau_d}{\tau_C} \omega^{-2+\alpha} \cos \frac{\alpha \pi}{2} \quad \text{and} \\
\text{Im} \frac{1}{Y_C^{(d)}(\omega)} &\sim -R \frac{1}{\tau_C} \omega^{-1}, \quad \text{as } \omega \to \infty,
\end{align*}
\]

so that (64) and (65), by (57), yield the high-frequency asymptotics of transfer function modulus and argument in the respective forms

\[
\begin{align*}
\left| \hat{g}^{(dd)}(\omega) \right|_{\text{dB}} &= -20 \log (\tau_L \omega) - 10 \log \left( 1 + 2 \frac{\tau_{\beta}}{\tau_L} \omega^{-1+\beta} \sin \frac{\beta \pi}{2} \right) \\
&+ \frac{\tau_{\beta}^2}{\tau_L} \omega^{-2+2\beta} + 2 \frac{\tau_{\beta}}{\tau_L} \omega^{-2+\beta} \cos \frac{\beta \pi}{2} + O(\omega^{-2}), \\
\cot \arg \hat{g}^{(dd)}(\omega) &= -\frac{\tau_{\beta}}{\tau_L} \omega^{-1+\beta} \cos \frac{\beta \pi}{2} \left( 1 - \frac{\tau_{\beta}}{\tau_L} \omega^{-1+\beta} \sin \frac{\beta \pi}{2} \right) \\
&+ \begin{cases}
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} - \frac{1}{\tau_L} \omega^{-1} \tan \frac{\beta \pi}{2} + O(\omega^{-1-\beta}), & \text{if } \beta \in \left( 0, \frac{1}{3} \right], \\
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} - \frac{1}{\tau_L} \omega^{-1} \tan \frac{\beta \pi}{2}, & \text{if } \beta \in \left( \frac{1}{3}, \frac{1}{2} \right], \\
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} + \frac{\tau_{\beta}^2}{\tau_L^2} \omega^{-2+2\beta} \sin^2 \frac{\beta \pi}{2}, & \text{if } \beta \in \left[ \frac{1}{2}, \frac{2}{3} \right), \\
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} + \frac{\tau_{\beta}^2}{\tau_L^2} \omega^{-2+2\beta} \sin^2 \frac{\beta \pi}{2}, & \text{if } \beta \in \left[ \frac{2}{3}, \frac{3}{4} \right), \\
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} + O(\omega^{-3+3\beta}), & \text{if } \beta \in \left[ \frac{3}{4}, 1 \right), \\
\frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} + O(\omega^{-3+3\beta}), & \text{if } \beta \in \left[ \frac{3}{4}, 1 \right),
\end{cases}
\end{align*}
\]

as \( \omega \to \infty, \) since (64) and (65) used in (57)\textsubscript{2} yield

\[
\cot \arg \hat{g}^{(dd)}(\omega) = -\frac{\tau_{\beta}}{\tau_L} \omega^{-1+\beta} \cos \frac{\beta \pi}{2} \left( 1 + \frac{\omega^{-\beta}}{\tau_L \cos \frac{\beta \pi}{2}} + O(\omega^{-1-\beta}) \right)
\times \left( 1 - \frac{\tau_{\beta}}{\tau_L} \omega^{-1+\beta} \sin \frac{\beta \pi}{2} + \frac{\tau_{\beta}^2}{\tau_L^2} \omega^{-2+2\beta} \sin^2 \frac{\beta \pi}{2} \right)
+ \begin{cases}
O(\omega^{-2}), & \text{if } \beta \in \left( 0, \frac{1}{3} \right], \\
O(\omega^{-3+3\beta}), & \text{if } \beta \in \left( \frac{1}{3}, 1 \right),
\end{cases}
\]

as \( \omega \to \infty, \) according to the series expansion \( \frac{1}{1+x} = 1 - x + x^2 + O(x^3) \) as \( x \to 0. \)
Fig. 10 Comparison of frequency characteristics of transfer function modulus and argument (solid line) with their asymptotics (dashed line) for dissipative–dissipative RLC circuit, obtained for model parameters: $\alpha = 0.2$, $\tau_C = 0.1$, $\tau_\alpha = 2.5$, $\tau_L = 0.75$, and $\tau_\beta = 0.25$.

Transfer function modulus and argument versus the angular frequency for different values of fractional order of dissipative capacitor are depicted in Fig. 10, along with their asymptotics. The transfer function modulus, see Fig. 10a, is a linear function of log $\omega$, for both low and high frequencies, with the slope either determined by the fractional-order $\alpha$ for low frequencies, or with the slope equal to one for high frequencies, as predicted by the asymptotics formulae (62) and (67). The transfer function argument, depicted in Fig. 10b, agrees very well with the low- and high-frequency asymptotics (63) and (68).

Note that the leading term in the low-frequency asymptotics of the transfer function modulus (62) provides the possibility of determining the model parameters $\alpha$ and $\tau_\alpha$, since they, respectively, represent the slope and intercept of the function linear in log $\omega$, while the leading term of the high-frequency asymptotics (67) yields $\tau_L$ as the intercept of the linear function in log $\omega$. Further, by considering the logarithm of the absolute value of transfer function argument’s cotangent, see the first term in high-frequency asymptotics (68), one determines the remaining model parameters $\beta$ and $\tau_\beta$, since

$$\log|\cot \arg \hat{g}^{(dd)}(\omega)| \sim (-1 + \beta) \log \omega + \log \left( \frac{\tau_\beta}{\tau_L} \frac{\beta \pi}{2} \cos \frac{\beta \pi}{2} \right)$$

as $\omega \to \infty$.

4.2 Generative–Generative Fractional RLC Circuit

The real and imaginary parts of the equivalent impedance $Z_e^{(gg)}(\omega)$, given by (42) and corresponding to the generative–generative fractional RLC circuit, rewritten in terms of classical and fractional time constants read

$$\text{Re } Z_e^{(gg)}(\omega) = R \left( 1 - \frac{\sin \frac{\mu \pi}{2}}{\tau_{\mu} \omega^{1+\mu}} - \tau_L \tau_v \omega^{1+v} \frac{\tau_L \sin \frac{\nu \pi}{2}}{\tau_v \omega^{2v} + 2 \tau_L \tau_v \omega^{v} \cos \frac{\nu \pi}{2} + \tau_L^2} \right),$$

(69)
\[
\text{Im } Z_e^{(gg)}(\omega) = -R \left( \frac{1}{\tau_C \omega} + \frac{\cos \frac{\mu \pi}{2}}{\tau_\mu \omega^{1+\mu}} - \tau_L \frac{\tau_\nu \omega^{1+\nu}}{\tau_C \omega^{2\nu} + 2 \tau_L \tau_\nu \omega^{\nu} \cos \frac{\nu \pi}{2} + \tau_L^2} \right), \tag{70}
\]
yielding the transfer function modulus and argument according to (57).

The frequency characteristics of transfer function modulus and argument, presented in Fig. 11 and obtained for the same model parameters as in the case of plots from Fig. 6, illustrate the possibility that the transfer function modulus has a vertical asymptote, while the argument abruptly changes by \(\pi\), see the solid-line plots from Fig. 11a, b, which is the property of transfer function having purely imaginary poles, that is exactly the case for the selected set of model parameters. In addition to non-integer zeros of the transfer function, its modulus tends to negative infinity as the frequency tends to zero, see also the form of transfer function \(\hat{g}^{(gg)}\) given by (22)—the occurrence of a maximum followed by modulus’ tendency to negative infinity as the frequency tends to infinity, see dashed- and dot-dashed-line plots from Fig. 11a, is a consequence of the existence of complex conjugated poles of the transfer function, as it is the case for the selected sets of model parameters.

The transfer function argument in the case of dashed-line plot from Fig. 11b decreases, which underlines the similarity with the integer-order case, since the complex conjugated poles for selected model parameters have negative real part, contrary to the case of transfer function argument represented by dot-dashed line, that actually increases over \(\pi\), although in Fig. 11b drops by \(2\pi\) due to the codomain of arcus tangent function being \((-\pi, \pi]\), again underlining the similarity with the integer-order case, since the complex conjugated poles for this set of model parameters have positive real part.

Again, the transfer function argument illustrates the transition from the capacitive to inductive character of the circuit with the increase in the angular frequency, as well as its energy consumption/production properties, that in the case of solid- and dashed-line plots change with frequency from generative to dissipative, since the argument drops below \(\pi\), and again to generative when the argument further drops below \(-\frac{\pi}{2}\), while for the dot-dashed-line plot circuit’s character is generative for the whole frequency range, since the argument has values either higher than \(\pi\), or lower than \(-\frac{\pi}{2}\) in the whole frequency range.

As for the dissipative–dissipative fractional \(RLC\) circuit, the capacitive character is dominant for low frequencies, so that the asymptotics of expressions (69) and (70), corresponding to \(\text{Re } Z_e^{(gg)}\) and \(\text{Im } Z_e^{(gg)}\), takes the following forms

\[
\text{Re } Z_e^{(gg)}(\omega) = -R \frac{\sin \frac{\mu \pi}{2}}{\tau_\mu \omega^{1+\mu}} \left( 1 - \frac{\tau_\mu \omega^{1+\mu}}{\sin \frac{\mu \pi}{2}} + O(\omega^{2+\mu}) \right), \tag{71}
\]

\[
\text{Im } Z_e^{(gg)}(\omega) = -R \frac{\cos \frac{\mu \pi}{2}}{\tau_\mu \omega^{1+\mu}} \left( 1 + \frac{\tau_\mu \omega^{\mu}}{\tau_C \cos \frac{\mu \pi}{2}} + O(\omega^{2+\mu}) \right). \tag{72}
\]

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as \( \omega \to 0 \), since the leading terms of real and imaginary parts of generative inductor’s impedance \( \frac{1}{Y_L^{(g)}} \), given by (34), are

\[
\begin{align*}
\text{Re} \left( \frac{1}{Y_L^{(g)}}(\omega) \right) & \sim -R \tau_v \omega^{1+\nu} \sin \frac{\nu \pi}{2} \\
\text{Im} \left( \frac{1}{Y_L^{(g)}}(\omega) \right) & \sim R \tau_v \omega^{1+\nu} \cos \frac{\nu \pi}{2}
\end{align*}
\]

as \( \omega \to 0 \), while the transfer function argument, by (57)2, (71), and (72), is

\[
\cot \arg \hat{g}^{(gg)}(\omega) = -\tan \left( \frac{\mu \pi}{2} \right) \left( 1 - \frac{\tau_\mu \omega^{\mu}}{\tau_C \cos \frac{\mu \pi}{2}} + \frac{\tau_\mu^2 \omega^{2\mu}}{\tau_C^2 \cos^2 \frac{\mu \pi}{2}} - \frac{\tau_\mu^3 \omega^{3\mu}}{\tau_C^3 \cos^3 \frac{\mu \pi}{2}} \right)
\]
\[
\begin{align*}
\frac{O(\omega^4\mu)}{1+\frac{\tau_4\omega^{1+\mu}}{\sin \frac{\mu \pi}{2}}} + O(\omega^4\mu), & \quad \text{if } \mu \in \left(0, \frac{1}{3}\right], \\
\frac{\tau_4\omega^{1+\mu}}{\sin \frac{\mu \pi}{2}} + O(\omega^4\mu), & \quad \text{if } \mu \in \left(\frac{1}{3}, \frac{1}{2}\right], \\
+ \frac{\tau_4^2\omega^{1+2\mu}}{\tau_4 \sin \frac{\mu \pi}{2} \cos \frac{\mu \pi}{2}} + O(\omega^4\mu), & \quad \text{if } \mu \in \left(\frac{1}{2}, \frac{2}{3}\right], \\
- \tau_4\omega^{1+\mu} \sin \frac{\mu \pi}{2} + O(\omega^4\mu), & \quad \text{if } \mu \in \left(\frac{2}{3}, \frac{1}{2}\right].
\end{align*}
\]

(75)

as \(\omega \to 0\), since

\[\cot \arg \hat{g}^{(gg)}(\omega) = -\tan \left(\frac{\mu \pi}{2} \left(1 - \frac{\tau_4 \omega^{1+\mu}}{\sin \frac{\mu \pi}{2}} + O(\omega^2+\mu)\right)\right)\]

as \(\omega \to 0\), since for the generative inductor, according to (34), the high-frequency asymptotics of \(\frac{1}{Y^{(g)}_L} \) is

\[
\begin{align*}
\text{Re } Z^{(gg)}_e(\omega) &= -R \frac{\tau_4^2}{\tau_v} \omega^{1-\nu} \sin \frac{\nu \pi}{2} \left(1 - 2 \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2}\right) \\
&+ \begin{cases} 
O(\omega^{-2\nu}), & \text{if } \nu \in \left(0, \frac{1}{3}\right], \\
- \frac{\tau_v \omega^{-1+\nu}}{\tau_L^2 \sin \frac{\nu \pi}{2}} + O(\omega^{-2\nu}), & \text{if } \nu \in \left(\frac{1}{3}, 1\right],
\end{cases} \\
\text{Im } Z^{(gg)}_e(\omega) &= R \tau_L \omega \left(1 - \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2}\right) \\
&- \frac{\tau_v^2}{\tau_L} \omega^{-2\nu} \left(1 - 2 \cos^2 \frac{\nu \pi}{2}\right) + O(\omega^{-3\nu}).
\end{align*}
\]

(76, 77)

as \(\omega \to \infty\), since for the generative inductor, according to (34), the high-frequency asymptotics of \(\frac{1}{Y^{(g)}_L} \) is

\[
\text{Re } \frac{1}{Y^{(g)}_L}(\omega) = -R \frac{\tau_4^2}{\tau_v} \omega^{1-\nu} \sin \frac{\nu \pi}{2} \left(1 + 2 \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2} + \frac{\tau_v^2}{\tau_L} \omega^{-2\nu}\right) \\
= -R \frac{\tau_4^2}{\tau_v} \omega^{1-\nu} \sin \frac{\nu \pi}{2} \left(1 - 2 \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2} + O(\omega^{-2\nu})\right). 
\]

(78)
due to the series expansion \( \frac{1}{1+x} = 1 - x + O(x^2) \) as \( x \to 0 \), while according to (34), the high-frequency asymptotics of \( \text{Im} \left( \frac{1}{Y_{L}(\omega)} \right) \) is

\[
\text{Im} \left( \frac{1}{Y_{L}(\omega)} \right) = R \tau_L \omega \left( 1 + \frac{\tau_L \omega^{-v} \cos \frac{\nu \pi}{2}}{1 + 2 \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} + \frac{\tau_L^2}{\tau_v^2} \omega^{-2v}} \right)
\]

\[
= R \tau_L \omega \left( 1 + \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} \right) \times \left( 1 - 2 \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} - \frac{\tau_L^2}{\tau_v^2} \omega^{-2v} \left( 1 - 4 \cos^2 \frac{\nu \pi}{2} \right) + O(\omega^{-3v}) \right)
\]

\[
= R \tau_L \omega \times \left( 1 - \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} - \frac{\tau_L^2}{\tau_v^2} \omega^{-2v} \left( 1 - 2 \cos^2 \frac{\nu \pi}{2} \right) + O(\omega^{-3v}) \right),
\]

(79)

due to the series expansion \( \frac{1}{1+x} = 1 - x + x^2 + O(x^3) \) as \( x \to 0 \), taking into account that the leading terms of generative capacitor’s impedance (33) are

\[
\text{Re} Z_{C}(\omega) \sim -R \frac{1}{\tau_{\mu}} \omega^{-1-\mu} \sin \frac{\mu \pi}{2} \quad \text{and}
\]

\[
\text{Im} Z_{C}(\omega) \sim -R \frac{1}{\tau_C} \omega^{-1} \quad \text{as} \quad \omega \to \infty,
\]

(80)

so that (76) and (77), by (57), yield the high-frequency asymptotics of transfer function modulus and argument in the respective forms

\[
\begin{align*}
\left| \hat{g}^{(gg)}(\omega) \right|_{\text{dB}} &= -20 \log (\tau_L \omega) \\
&- 10 \log \left( 1 - 2 \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} - \frac{\tau_L^2}{\tau_v^2} \omega^{-2v} \left( 1 - 4 \cos^2 \frac{\nu \pi}{2} \right) \right) \\
&+ \begin{cases} 
O(\omega^{-3v}), & \text{if } \nu \in \left( 0, \frac{1}{3} \right], \\
-2 \frac{1}{\tau_{\mu}} \omega^{-1-v} \sin \frac{\nu \pi}{2} + O(\omega^{-3v}), & \text{if } \nu \in \left( \frac{1}{2}, \frac{2}{3} \right], \\
-2 \frac{1}{\tau_C} \omega^{-1} \sin \frac{\nu \pi}{2} + \frac{\tau_L}{\tau_v} \omega^{-2} + O(\omega^{-3v}), & \text{if } \nu \in \left( \frac{2}{3}, 1 \right), 
\end{cases}
\end{align*}
\]

(81)

\[
\cot \arg \hat{g}^{(gg)}(\omega) = \frac{\tau_L}{\tau_v} \omega^{-v} \sin \frac{\nu \pi}{2} \left( 1 - \frac{\tau_L}{\tau_v} \omega^{-v} \cos \frac{\nu \pi}{2} \right) \\
+ \begin{cases} 
O(\omega^{-2v}), & \text{if } \nu \in \left( 0, \frac{1}{3} \right], \\
-\frac{\omega^{-1+v}}{\tau_v \sin \frac{\nu \pi}{2}} + O(\omega^{-2v}), & \text{if } \nu \in \left( \frac{1}{2}, \frac{1}{3} \right], \\
-\frac{\omega^{-1+v}}{\tau_C \sin \frac{\nu \pi}{2}} - \frac{1}{\tau_L} \omega^{-1} \cos \frac{\nu \pi}{2} + O(\omega^{-2v}), & \text{if } \nu \in \left( \frac{1}{2}, 1 \right), 
\end{cases}
\]

(82)
Fig. 12  Comparison of frequency characteristics of transfer function modulus and argument (solid line) with their asymptotics (dashed line) for generative–generative RLC circuit, obtained for model parameters: \( \nu = 0.9, \tau_C = 0.75, \tau_{\mu} = 0.15, \tau_L = 0.75, \) and \( \tau_{v} = 0.025 \)
as \( \omega \to \infty \), since (76) and (77) used in (57) yield

\[
\cot \arg \hat{g}^{(gg)}(\omega) = \frac{\tau_L}{\tau_v} \omega^{-\nu} \sin \frac{\nu \pi}{2} \left( 1 - 2 \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2} + O(\omega^{-2\nu}) \right) + \begin{cases} O(\omega^{-2\nu}), & \text{if } \nu \in \left(0, \frac{1}{3}\right], \\ -\frac{\nu}{2} \omega^{-1+\nu} + O(\omega^{-2\nu}), & \text{if } \nu \in \left(\frac{1}{3}, 1\right), \end{cases}
\]

\times \left( 1 + \frac{\tau_L}{\tau_v} \omega^{-\nu} \cos \frac{\nu \pi}{2} + \frac{\tau_L^2}{\tau_v^2} \omega^{-2\nu} \sin^2 \frac{\nu \pi}{2} + O(\omega^{-3\nu}) \right),
\]
as \( \omega \to \infty \), according to the series expansion \( \frac{1}{1+x} = 1 - x + x^2 + O(x^3) \) as \( x \to 0 \).

Figure 12 presents the plots of transfer function modulus and argument versus the angular frequency for different values of fractional order of generative capacitor, together with their asymptotics. According to the low-frequency asymptotics (74), the transfer function modulus is a linear function of \( \log \omega \), with the slope and intercept, respectively, determined by the model parameters \( \mu \) and \( \tau_{\mu} \), see also Fig. 12a, while the low-frequency asymptotics of transfer function argument (75) implies the dependence on the parameter \( \mu \), as obvious from Fig. 12b. On the other hand, the high-frequency asymptotics of transfer function modulus (81) yields \( \tau_{L} \) as the intercept of the linear function in \( \log \omega \), see Fig. 12a, while the remaining model parameters \( \nu \) and \( \tau_{v} \) are determined by the logarithm of the absolute value of transfer function argument’s cotangent for high frequencies, see the asymptotics (82) and agreement between the asymptotic curve and the frequency characteristics in Fig. 12b.

### 4.3 Dissipative–Generative Fractional RLC Circuit

Rewriting the real and imaginary parts of the equivalent impedance \( Z_{eq}^{(dg)} \), given by (47) of the dissipative–generative fractional RLC circuit using classical and frac-
tional time constants yields

\[
\text{Re } Z_e^{(dg)}(\omega) = R \left( 1 + \frac{\tau_C \omega \alpha \cos \frac{\alpha \pi}{2}}{\tau_C^2 \omega^2 + 2 \tau_C \tau_a \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_a^2 \omega^{2\alpha}} \right. \\
- \left. \tau_L \tau_v \omega^{1+v} \left( \frac{\tau_L \sin \frac{\frac{\alpha \pi}{2}}{2}}{\tau_C^2 \omega^2 + 2 \tau_C \tau_a \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_a^2 \omega^{2\alpha}} \right) \right),
\]

\[
\text{Im } Z_e^{(dg)}(\omega) = -R \left( \frac{\tau_C \omega + \tau_a \omega \alpha \sin \frac{\alpha \pi}{2}}{\tau_C^2 \omega^2 + 2 \tau_C \tau_a \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_a^2 \omega^{2\alpha}} \right. \\
- \left. \tau_L \tau_v \omega^{1+v} \left( \frac{\tau_v \omega^{1+v} + \tau_L \omega \cos \frac{\alpha \pi}{2}}{\tau_C^2 \omega^{2v} + 2 \tau_L \tau_v \omega^{1+v} \cos \frac{\alpha \pi}{2} + \tau_L^2} \right) \right)
\]

giving the transfer function modulus and argument according to \((57)\).

Figure 13 presents the Bode diagrams of transfer function modulus and argument, obtained for the same set of model parameters as the plots in Fig. 7, corresponding to the band-pass filter. The transfer function \(\hat{g}^{(dg)}\), given by \((25)\), has a non-integer zero at the origin that can be recognized from the plots of transfer function modulus from Fig. 13a, as well as complex conjugated poles, obtained as a consequence of the chosen sets of model parameters, either with nonzero real part, that correspond to the plots of transfer function modulus depicted by dot-dashed and dashed lines, or purely imaginary ones, corresponding to the solid-line plot. Plots of the transfer function argument versus the angular frequency support the statement regarding the type of poles and their real part, since if poles have nonzero real part, the argument decreases if pole’s real part is obtained to be negative and increases over \(\pi\) if pole’s real part is obtained to be positive, while in the case of purely imaginary poles, the value of argument drops by \(\pi\).

Clearly, the fractional \(RLC\) circuit is dissipative for relatively low frequencies, since arguments’ values belong to interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) and eventually becomes generative: abruptly for the model parameters corresponding to the solid-line plot when the circuit simultaneously becomes of inductive type as well, since the argument drops below \(-\frac{\pi}{2}\), and gradually either for mid-range frequencies in the case of dashed-line plots, or for high frequencies in the case of dot-dashed-line plots.

Although the dissipative–generative \(RLC\) circuit contains dissipative capacitor so as the dissipative–dissipative circuit, due to the different orders of the leading terms corresponding to the generative and dissipative inductor, compare \((73)\) with \((32)\), the low-frequency asymptotics of real and imaginary parts of dissipative capacitor’s impedance \(\text{Re } \frac{1}{Y_C^{(d)}}\) and \(\text{Im } \frac{1}{Y_C^{(d)}}\), according to \((31)\), is calculated as

\[
\text{Re } \frac{1}{Y_C^{(d)}(\omega)} = R \frac{\cos \frac{\alpha \pi}{2}}{\tau_a \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \frac{\tau_a^2 \omega^{2-2\alpha}}{\tau_C^2 \omega^2}} \left( 1 - 2 \frac{\tau_C}{\tau_a} \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + O(\omega^{2-2\alpha}) \right),
\]

\(\text{Im } \frac{1}{Y_C^{(d)}(\omega)}\)
(a) Modulus of the transfer function $\hat{g}^{(dg)}$. (b) Argument of the transfer function $\hat{g}^{(dg)}$.

**Fig. 13** Frequency characteristics of transfer function modulus and argument for dissipative–generative \(RLC\) circuit, obtained for model parameters: \(\alpha = 0.25, \nu = 0.85, \tau_C = 0.25, \tau_\alpha = 0.005, \tau_L = 0.75, \) and \(\tau_\nu = 5\)—dot-dashed line, \(\tau_\nu = 0.23329\ldots\)—solid line, and \(\tau_\nu = 0.09\)—dashed line

\[
\text{Im } \frac{1}{Y_C^{(d)}(\omega)} = -R \frac{\sin \alpha \pi}{\tau_\alpha \omega^\alpha} \frac{1 + \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} - \frac{1}{\sin \frac{\alpha \pi}{2}}}{1 + 2 \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} + \frac{\tau_C^2}{\tau_\alpha^2} \omega^{2-2\alpha}}
\]

\[
= -R \frac{\sin \alpha \pi}{\tau_\alpha \omega^\alpha} \left( 1 - 2 \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} \left( 1 - \frac{1}{2 \sin^2 \frac{\alpha \pi}{2}} \right) \right) + O(\omega^{2-2\alpha})
\]

as \(\omega \to 0\), by the series expansion \(\frac{1}{1-x} = 1 - x + O(x^2)\), as \(x \to 0\), transforming the real and imaginary parts of circuit’s equivalent impedance \(\text{Re } Z_e^{(dg)}\) and \(\text{Im } Z_e^{(dg)}\), given by (83) and (84), into

\[
\text{Re } Z_e^{(dg)}(\omega) = R \frac{\cos \alpha \pi}{\tau_\alpha \omega^\alpha}
\]

\[
\times \left( 1 - 2 \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} + \left\{ \begin{array}{ll}
\frac{\tau_\alpha \omega^\alpha}{\cos \frac{\alpha \pi}{2}} & \text{if } \alpha \in (0, \frac{2}{3}), \\
O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[ \frac{2}{3}, 1 \right),
\end{array} \right. \right)
\]

\[
\text{Im } Z_e^{(dg)}(\omega) = -R \frac{\sin \alpha \pi}{\tau_\alpha \omega^\alpha}
\]

\[
\times \left( 1 - 2 \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} \left( 1 - \frac{1}{2 \sin^2 \frac{\alpha \pi}{2}} \right) + O(\omega^{2-2\alpha}) \right)
\]

as \(\omega \to 0\), so that, by (57)\(_1\), (85), and (86), for the transfer function modulus one has

\[
\left| \hat{g}^{(dg)}(\omega) \right|_{\text{dB}} = 20 \log \left( \tau_\alpha \omega^\alpha \right) - 10 \log \left( 1 - 2 \frac{\tau_C}{\tau_\alpha} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} \right)
\]
\[
\begin{align*}
&+ \begin{cases}
2\tau C \omega^\alpha \cos \frac{\alpha \pi}{2} + \frac{\tau C^2 \omega^2}{2}, & \text{if } \alpha \in \left(0, \frac{1}{2}\right), \\
-4\tau C \omega \sin \frac{\alpha \pi}{2} \cos \frac{\alpha \pi}{2} + O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[\frac{1}{2}, \frac{2}{3}\right), \\
2\tau C \omega^\alpha \cos \frac{\alpha \pi}{2} + O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[\frac{2}{3}, 1\right), \\
O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[\frac{2}{3}, 1\right].
\end{cases}
\end{align*}
\]  

(87)

as \(\omega \to 0\), while the transfer function argument, by (57), (85), and (86), is

\[
\cot \arg \hat{g}^{(dg)}(\omega) = \cot \frac{\alpha \pi}{2} \left(1 - \frac{\tau C}{\tau C} \omega^{1-\alpha} \sin \frac{\alpha \pi}{2} \right) + \begin{cases}
\frac{\tau C^2 \omega^2}{2}, & \text{if } \alpha \in \left(0, \frac{1}{2}\right), \\
O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[\frac{1}{2}, \frac{2}{3}\right), \\
O(\omega^{2-2\alpha}), & \text{if } \alpha \in \left[\frac{2}{3}, 1\right].
\end{cases}
\]

(88)

as \(\omega \to 0\) according to \(1 - x = 1 - x + O(x^2)\), as \(x \to 0\).

In the case of dissipative–generative fractional \(RLC\) circuit, as for all previously considered circuits, the inductive properties are dominant for high frequencies and therefore one uses already calculated asymptotics (78) and (79) of the real and imaginary parts of generative inductor’s impedance \(Y_{L}^{(g)}\), given by (34), since the upper limits of orders of leading terms of dissipative and generative capacitors’ impedances are equal, compare (66) with (80), which implies that the high-frequency asymptotics of the real and imaginary parts of circuit’s equivalent impedance \(\frac{1}{Y_{L}^{(g)}}\) is the same as the real and imaginary parts of equivalent impedance of the generative–generative circuit \(\frac{1}{Y_{L}^{(gg)}}\) as \(\omega \to \infty\), thus being given by (76) and (77), yielding the transfer function modulus and argument in the same forms as for the generative–generative circuit, i.e., given by the expression (81) for the transfer function modulus and (82) for the transfer function argument.

Frequency characteristics of transfer function modulus and argument, together with their asymptotics, are presented in Fig. 14. The discussion about the low-frequency asymptotic behavior of the dissipative–generative \(RLC\) circuit is the same as for
Fig. 14 Comparison of frequency characteristics of transfer function modulus and argument (solid line) with their asymptotics (dashed line) for dissipative–generative $RLC$ circuit, obtained for model parameters: $\nu = 0.75$, $\tau_C = 0.2$, $\tau_\alpha = 0.5$, $\tau_L = 0.75$, and $\tau_v = 0.5$

The dissipative–dissipative $RLC$ circuit, since the capacitive character of circuit prevails, compare also the leading terms in the low-frequency asymptotics of transfer function modulus and argument (62) with (87) and (63) with (88), respectively. On the other hand, the discussion about the high-frequency asymptotic behavior of the dissipative–generative circuit is the same as for the generative–generative circuit, since the inductive character of circuit prevails, see the leading terms in the high-frequency asymptotics of transfer function modulus and argument (81) and (82).

4.4 Generative–Dissipative Fractional $RLC$ Circuit

Equivalent impedance $Z_e^{(gd)}$ of the generative–dissipative fractional $RLC$ circuit, given by (47)$_2$, implies

\[
\text{Re } Z_e^{(gd)}(\omega) = R \left( 1 - \frac{\sin \frac{\mu \pi}{2}}{\tau_\mu \omega^{1+\mu}} + \tau_\beta \omega^\beta \cos \frac{\beta \pi}{2} \right),
\]

(89)

\[
\text{Im } Z_e^{(gd)}(\omega) = -R \left( \frac{1}{\tau_C \omega} + \frac{\cos \frac{\mu \pi}{2}}{\tau_\mu \omega^{1+\mu}} - \tau_L \omega - \tau_\beta \omega^\beta \sin \frac{\beta \pi}{2} \right),
\]

(90)

when its real and imaginary parts are rewritten in terms of classical and fractional time constants, yielding the transfer function modulus and argument according to (57).

Frequency characteristics of transfer function modulus and argument corresponding to the generative–dissipative $RLC$ circuit, presented in Fig. 15 and obtained for the same set of model parameters as the plots in Fig. 8, are of the same shape as the frequency characteristics corresponding to the dissipative–generative circuit, see Fig. 13, since the transfer function $g^{(gd)}$, given by (26), for selected sets of model parameters also has either complex conjugated poles with positive or negative real parts, or purely imaginary poles.

Although the plots of transfer function argument from Fig. 15b are qualitatively the same as the ones from Fig. 13b, they quantitatively correspond to the generative circuit for relatively low frequencies, since arguments’ values are above $\frac{\pi}{2}$, while the circuit
eventually becomes dissipative: abruptly for the model parameters corresponding to the solid-line plot when the circuit simultaneously becomes of inductive type as well, since the argument drops to interval \((-\frac{\pi}{2}, 0)\), and gradually either for mid-range frequencies in the case of dashed-line plots, or for high frequencies in the case of dot-dashed-line plots.

In the low-frequency limit, the impedance \(Z_C^{gd}(\omega)\) of dissipative capacitor, given by (33), dominates the equivalent impedance and therefore expressions (89) and (90) become

\[
\begin{align*}
\text{Re} \ Z_C^{gd}(\omega) &= -R \frac{\sin \frac{\mu \pi}{2}}{\tau_C \omega^{1+\mu}} \left( 1 - \frac{\tau_C \omega^{1+\mu}}{\sin \frac{\mu \pi}{2}} + O(\omega^{1+\mu+\delta}) \right), \\
\text{Im} \ Z_C^{gd}(\omega) &= -R \frac{\cos \frac{\mu \pi}{2}}{\tau_C \omega^{1+\mu}} \left( 1 + \frac{\tau_C \omega^{1+\mu}}{\tau_C \cos \frac{\mu \pi}{2}} + O(\omega^{1+\mu+\delta}) \right),
\end{align*}
\]

as \(\omega \to 0\), where \(\delta\) is chosen to be smaller than \(\beta\), i.e., smaller than the order of the leading terms of the real and imaginary parts of dissipative inductor’s impedance \(\text{Re} \ Z_L^{d}(\omega)\) and \(\text{Im} \ Z_L^{d}(\omega)\), see (32), implying by (57), (91), and (92) the low-frequency asymptotics of the transfer function modulus as

\[
\left| \hat{g}^{(gd)}(\omega) \right|_{\text{dB}} = 20 \log \left( \tau_C \omega^{1+\mu} \right) - 10 \log \left( 1 + \frac{\tau_C \omega^{1+\mu}}{\tau_C \cos \frac{\mu \pi}{2}} + \frac{\tau_C^2 \omega^{2\mu}}{\tau_C^2 \cos^2 \frac{\mu \pi}{2}} \right),
\]

as \(\omega \to 0\), while the transfer function argument, by (57), (91), and (92), is

\[
\cot \arg \hat{g}^{(gd)}(\omega) = -\tan \frac{\mu \pi}{2} \left( 1 - \frac{\tau_C \omega^{1+\mu}}{\tau_C \cos \frac{\mu \pi}{2}} + \frac{\tau_C^2 \omega^{2\mu}}{\tau_C^2 \cos^2 \frac{\mu \pi}{2}} \right)
\]
as \( \omega \to 0 \), since

\[
\cot \arg \hat{g}^{(gd)}(\omega) = -\tan \frac{\mu \pi}{2} \left( 1 - \frac{\tau_\mu \omega^{1+\mu}}{\sin \frac{\mu \pi}{2}} + O(\omega^{1+\mu+\delta}) \right)
\times \left( 1 - \frac{\tau_\mu \omega^\mu}{\tau_C \cos \frac{\mu \pi}{2}} + \frac{\tau^2_\mu \omega^{2\mu}}{\tau^2_C \cos^2 \frac{\mu \pi}{2}} \right.

\left. + O(\omega^{3\mu}), \quad \text{if } \mu \in (0, \frac{1}{2}], \right)

\begin{align*}
\left\{ O(\omega^{3\mu}), \quad \text{if } \mu \in (0, \frac{1}{2}], \right) \quad \text{and } \quad \left\{ O(\omega^{1+\mu+\delta}), \quad \text{if } \mu \in (\frac{1+\delta}{2}, 1) \right. \}
\end{align*}

In the case of generative–dissipative fractional RLC circuit, as for all previously considered circuits, the inductive properties are dominant for high frequencies and since the upper limits of orders of leading terms of generative and dissipative capacitors’ impedances are equal, compare (80) with (66), the high-frequency asymptotics of the real and imaginary parts of circuit’s equivalent impedance \( \text{Re } Z_e^{(gd)}(\omega) \) and \( \text{Im } Z_e^{(gd)}(\omega) \) are the same as the real and imaginary parts of equivalent impedance of the dissipative–dissipative circuit \( \text{Re } Z_e^{(dd)}(\omega) \) and \( \text{Im } Z_e^{(dd)}(\omega) \), i.e., \( \text{Re } Z_e^{(gd)} = \text{Re } Z_e^{(dd)} \) and \( \text{Im } Z_e^{(gd)} = \text{Im } Z_e^{(dd)} \) as \( \omega \to \infty \), and therefore given by (64) and (65), yielding the transfer function modulus and argument in the same forms as for the dissipative–dissipative circuit, i.e., given by the expression (67) for the transfer function modulus and (68) for the transfer function argument.

Frequency characteristics of transfer function modulus and argument, together with their asymptotics, are presented in Fig. 16. The discussion about the low-frequency asymptotic behavior of the generative–dissipative RLC circuit is the same as for the generative–generative RLC circuit, since the capacitive character of circuit pre-
vails, compare also the leading terms in the low-frequency asymptotics of transfer function modulus and argument (74) with (93) and (75) with (94), respectively. On the other hand, the discussion about the high frequency asymptotic behavior of the generative–dissipative circuit is the same as for the dissipative–dissipative circuit, since the inductive character of circuit prevails, see the leading terms in the high-frequency asymptotics of transfer function modulus and argument (67) and (68).

5 Conclusion

Constitutive equations of dissipative and generative capacitor (1) and (2) along with the constitutive models (3) and (4) corresponding to the dissipative and generative inductor are employed to model: dissipative–dissipative fractional series RLC circuit, schematically depicted in Fig. 1 and consisting of dissipative electric elements, by the governing Eq. (5); generative–generative circuit, schematically depicted in Fig. 2 and consisting of generative elements, using the governing Eq. (6); as well as to model dissipative–generative and generative–dissipative circuits, schematically depicted in Figs. 3 and 4 and consisting of dissipative capacitor and generative inductor for the former and generative capacitor and dissipative inductor for the latter, by the governing Eqs. (7) and (8). Governing equations in time domain (5)–(8), as well as the governing equations in complex domain, along with the corresponding transfer functions, are derived in Sect. 2 using the Laplace transform method. The physical meaning of classical and fractional time constants appearing in (5)–(8) is also explained and emphasized.

Constitutive models of dissipative and generative electric elements, expressed in terms of current and voltage, are used in the steady-state regime to define the corresponding impedances and admittances, further to be used in equivalent impedances of the aforementioned RLC circuits. Through the phase angle, the equivalent impedance determines the energy consumption/production properties of the circuit as well as its predominant behavior, see (30) and (36). It is concluded that for each of the circuits capacitive properties prevail for low frequencies, while for the high frequencies inductive properties are dominant, see (41), (46), (51), and (55). As far as the energy consumption/production properties of the fractional RLC circuits are concerned, the dissipative–dissipative circuit consumes energy for all frequencies, see expression (38) and Fig. 5, since its constituents are dissipative elements, while all other RLC circuits may both consume and generate energy, depending on the frequency range, see Figs. 6, 7, and 8. The energy can be produced in the generative–generative RLC circuit for all frequencies in spite of the presence of a resistor representing the ideal energy consuming element, see Fig. 6c. In the case of the dissipative–generative and generative–dissipative RLC circuit, there is an interplay of dissipative and generative properties of electric elements, implying the non-monotonic character of the energy consumption/production properties of the fractional RLC circuit with respect to the angular frequency change, see Figs. 6a, b, 7, and 8. Moreover, the change in the energy consumption/production properties can also be abrupt, occurring for a specified angular frequency determined by the model parameters, see Figs. 6b, 7b, and 8b. Since capacitive (inductive) properties are dominant for low (high) frequencies,
whether the capacitor (inductor) is dissipative or generative determines the energy consumption/production properties of the circuit itself for low (high) frequencies.

The equivalent impedance of the fractional $RLC$ circuit also determines the explicit form of transfer function modulus and argument, see (57), governing the frequency characteristics. The Bode diagrams, presented in Figs. 9, 11, 13, and 15, underline the similarities of the transfer functions corresponding to the fractional $RLC$ circuits with the integer-order transfer functions. Namely, aforementioned figures illustrate that if the transfer function has complex conjugated poles, then the corresponding frequency characteristics of its modulus attain a maximum and then tend to the negative infinity as the frequency tends to infinity, while the occurrence of purely imaginary poles implies that modulus’ characteristics has a vertical asymptote. Also, modulus’ frequency characteristics illustrate the fact that transfer function has a zero of non-integer order at the origin and indicate that all considered $RLC$ circuits behave as the band-pass filter. Whether the complex conjugated poles of the transfer function have negative or positive real part, determine whether the frequency characteristics of the transfer function argument decrease or increase with the increase in frequency, while if poles are purely imaginary, then argument’s characteristics display a sudden drop by $\pi$. Predominant character of the fractional $RLC$ circuit as well as its energy consumption/production properties is discussed by the use of argument’s frequency characteristics.

The leading terms in the low- and high-frequency asymptotics of transfer function modulus and argument are useful in determining model parameters, since for low frequencies, the transfer function modulus is a linear function in $\log \omega$, with the slope $\alpha$ and intercept $\tau_\alpha$ in cases of dissipative–dissipative and dissipative–generative $RLC$ circuits, see (62) and (87), while the slope is determined by $1+\mu$ and intercept by $\tau_\mu$ in cases of generative–generative and generative–dissipative $RLC$ circuits, see (74) and (93). For high frequencies, the transfer function modulus is a linear function in $\log \omega$, with the intercept being $\tau_L$ for all fractional $RLC$ circuits, while $\log |\cot \arg \hat{g}(\omega)|$ proves to be a linear function in $\log \omega$, with the slopes $-1+\beta$ and $-\nu$ and intercepts proportional to $\tau_\beta$ and $\tau_\nu$, see (68) and (82), in cases of dissipative/generative–dissipative $RLC$ circuit and generative/dissipative–generative $RLC$ circuit, respectively. Figures 10, 12, 14, and 16 vividly illustrate these statements.

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References

1. A. Allagui, A.S. Elwakil, M.E. Fouda, A.G. Radwan, Capacitive behavior and stored energy in supercapacitors at power line frequencies. J. Power Sources 390, 142–147 (2018)
2. A. Allagui, T.J. Freeborn, A.S. Elwakil, M.E. Fouda, B.J. Maundy, A.G. Radwan, Z. Said, M.A. Abdelkareem, Review of fractional-order electrical characterization of supercapacitors. J. Power Sources 400, 457–467 (2018)
3. A. Allagui, D. Zhang, A.S. Elwakil, Short-term memory in electric double-layer capacitors. Appl. Phys. Lett. 113, 253901-1–5 (2018)
4. M.C. Bošković, T.B. Šekara, B. Lutovac, M. Daković, P.D. Mandić, M.P. Lazarević. Analysis of electrical circuits including fractional order elements, in 6th Mediterranean Conference on Embedded Computing (MECO), Bar, Montenegro (2017)
5. A. Buscarino, R. Caponetto, S. Graziani, E. Murgano, Realization of fractional order circuits by a constant phase element. Eur. J. Control 54, 64–72 (2020)
6. R. Caponetto, S. Graziani, E. Murgano, Realization of a fractional-order RLC circuit via constant phase element. Int. J. Dyn. Control 9, 1589–1599 (2021)
7. X. Chen, Y. Chen, B. Zhang, D. Qiu, A modeling and analysis method for fractional-order DC-DC converters. IEEE Trans. Power Electron. 32, 7034–7044 (2017)
8. J.M. Cruz-Duarte, M. Guía-Calderón, J.J. Rosales-García, R. Correa, Determination of a physically correct fractional-order model for electrolytic computer-grade capacitors. Math. Methods Appl. Sci. 44, 4366–4380 (2021)
9. A. Dzieliński, G. Sarwas, D. Sierociuk, Comparison and validation of integer and fractional order ultracapacitor models. Adv. Differ. Equ. 2011(11), 1–15 (2011)
10. O. Elwy, L.A. Said, A.H. Madian, A.G. Radwan, All possible topologies of the fractional-order Wien oscillator family using different approximation techniques. Circuits Syst. Signal Process. 38, 3931–3951 (2019)
11. M.E. Fouda, A. Allagui, A.S. Elwakil, S. Das, C. Psychalinos, A.G. Radwan, Nonlinear charge-voltage relationship in constant phase element. Int. J. Electron. Commun. AEÜ 117, 153104-1–4 (2020)
12. R. Garrappa, E. Kaslik, M. Popolizio, Evaluation of fractional integrals and derivatives of elementary functions: overview and tutorial. Mathematics 7, 407-1–21 (2019)
13. F. Gómez, J. Rosales, M. Guía, RLC electrical circuit of non-integer order. Cent. Eur. J. Phys. 11, 1361–1365 (2013)
14. J.F. Gómez-Aguilar, R. Razo-Hernández, D. Granados-Lieberman, A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response. Revista Mexicana de Física 60, 32–38 (2014)
15. M. Guía, J. Rosales, F. Gómez, Analysis on the time and frequency domain for the RC electric circuit of fractional order. Cent. Eur. J. Phys. 11, 1366–1371 (2013)
16. K. Haška, S.M. Cvetičanin, D. Zorica, Dissipative and generative fractional electric elements in modeling RC and RL circuits. Nonlinear Dyn. 105, 3451–3474 (2021)
17. K. Haška, D. Zorica, S.M. Cvetičanin, Fractional RLC circuit in transient and steady state regimes. Commun. Nonlinear Sci. Numer. Simul. 96, 105670-1–17 (2021)
18. J.I. Hidalgo-Reyes, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, V.M. Alvarado-Martínez, M.G. López-López, Classical and fractional-order modeling of equivalent electrical circuits for supercapacitors and batteries, energy management strategies for hybrid systems and methods for the state of charge estimation: a state of the art review. Microelectron. J. 85, 109–128 (2019)
19. J.I. Hidalgo-Reyes, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, V.M. Alvarado-Martínez, M.G. López-López, Determination of supercapacitor parameters based on fractional differential equations. Int. J. Circuit Theory Appl. 47, 1225–1253 (2019)
20. A. Jakubowska, J. Walczak, Analysis of the transient state in a circuit with supercapacitor. Poznan Univ. Technol. Acad. J. Electr. Eng. 81, 71–77 (2015)
21. A. Jakubowska, J. Walczak, Analysis of the transient state in a series circuit of the class $RL\beta C\alpha$. Circuits Syst. Signal Process. 35, 1831–1853 (2016)
22. A. Jakubowska-Ciszek, J. Walczak, Analysis of the transient state in a parallel circuit of the class $RL\beta C\alpha$. Appl. Math. Comput. 319, 287–300 (2018)
23. I.S. Jesus, J.A.T. Machado, Development of fractional order capacitors based on electrolyte processes. Nonlinear Dyn. 56, 45–55 (2009)
24. Y. Jiang, B. Zhang, X. Shu, Z. Wei, Fractional-order autonomous circuits with order larger than one. J. Adv. Res. 25, 217–225 (2020)
25. D.A. John, K. Biswas, Electrical equivalent circuit modelling of solid state fractional capacitor. Int. J. Electron. Commun. AEÜ 78, 258–264 (2017)
26. A. Kartci, A. Agambayev, N. Herencsar, K.N. Salama, Series-, parallel-, and inter-connection of solid-state arbitrary fractional-order capacitors: theoretical study and experimental verification. IEEE Access 6, 10933–10943 (2018)
27. M.M. Khader, J.F. Gómez-Aguilar, M. Adel, Numerical study for the fractional RL, RC, and RLC electrical circuits using Legendre pseudo-spectral method. Int. J. Circuit Theory Appl. 49, 3266–3285 (2021)

28. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier, Amsterdam, 2006)

29. M.S. Krishna, S. Das, K. Biswas, B. Goswami, Fabrication of a fractional order capacitor with desired specifications: a study on process identification and characterization. IEEE Trans. Electron Devices 58, 4067–4073 (2011)

30. J.A.T. Machado, A.M.S.F. Galhano, Fractional order inductive phenomena based on the skin effect. Nonlinear Dyn. 68, 107–115 (2012)

31. V. Martynyuk, M. Ortigueira, Fractional model of an electrochemical capacitor. Signal Process. 107, 355–360 (2015)

32. V. Martynyuk, M. Ortigueira, M. Fedula, O. Savenko, Methodology of electrochemical capacitor quality control with fractional order model. Int. J. Electron. Commun. AEÜ 91, 118–124 (2018)

33. D. Mondal, K. Biswas, Packaging of single-component fractional order element. IEEE Trans. Device Mater. Reliab. 13, 73–80 (2013)

34. V.F. Morales-Delgado, J.F. Gómez-Aguilar, M.A. Taneco-Hernández, Analytical solutions of electrical circuits described by fractional conformable derivatives in Liouville-Caputo sense. Int. J. Electron. Commun. AEÜ 85, 108–117 (2018)

35. V.F. Morales-Delgado, J.F. Gómez-Aguilar, M.A. Taneco-Hernández, R.F. Escobar-Jiménez, Fractional operator without singular kernel: applications to linear electrical circuits. Int. J. Circuit Theory Appl. 46, 2394–2419 (2018)

36. M.A. Moreles, R. Lainez, Mathematical modelling of fractional order circuit elements and bioimpedance applications. Commun. Nonlinear Sci. Numer. Simul. 46, 81–88 (2017)

37. K. Nosrati, M. Shafiee, On the convergence and stability of fractional singular Kalman filter and Riccati equation. J. Frankl. Inst. Eng. Appl. Math. 357, 7188–7210 (2020)

38. A.V. Oppenheim, A.S. Willsky, S.H. Nawab, Signals and Systems. Prentice-Hall Signal Processing Series (Prentice-Hall, Hoboken, 1997)

39. M.D. Ortigueira, D. Valério, Fractional Signals and Systems, volume 7 of Fractional Calculus in Applied Sciences and Engineering (de Gruyter, Berlin, 2020)

40. R. Prasad, K. Kothari, U. Mehta, Flexible fractional supercapacitor model analyzed in time domain. IEEE Access 7, 122626–122633 (2019)

41. R. Prasad, U. Mehta, K. Kothari, Various analytical models for supercapacitors: a mathematical study. Resour. Effici. Technol. 1, 1–15 (2020)

42. J.J. Quintana, A. Ramos, I. Nuez, Modeling of an EDLC with fractional transfer functions using Mittag-Leffler equations. Math. Probl. Eng. 2013, 807037–1–7 (2013)

43. A.G. Radwan, Resonance and quality factor of the $RLC^n$ fractional circuit. IEEE J. Emerg. Sel. Top. Circuits Syst. 3, 377–385 (2013)

44. A.G. Radwan, M.E. Fouda, Optimization of fractional-order $RLC$ filters. Circuits Syst. Signal Process. 32, 2097–2118 (2013)

45. A.G. Radwan, K.N. Salama, Passive and active elements using fractional $L^n C$ circuit. IEEE Trans. Circuits Syst. I Regul. Pap. 58, 2388–2397 (2011)

46. A.G. Radwan, K.N. Salama, Fractional-order $RC$ and $RL$ circuits. Circuits Syst. Signal Process. 31, 1901–1915 (2012)

47. A.G. Radwan, A.M. Soliman, A.S. Elwakil, Design equations for fractional-order sinusoidal oscillators: four practical circuit examples. Int. J. Circuit Theory Appl. 36, 473–492 (2008)

48. M.S. Sarafraz, M.S. Tavazoei, Realizability of fractional-order impedances by passive electrical networks composed of a fractional capacitor and $RLC$ components. IEEE Trans. Circuits Syst. I Regul. Pap. 62, 2829–2835 (2015)

49. I. Schäfer, K. Krüger, Modelling of coils using fractional derivatives. J. Magn. Magn. Mater. 307, 91–98 (2006)

50. N. Sene, J.F. Gómez-Aguilar, Analytical solutions of electrical circuits considering certain generalized fractional derivatives. Eur. Phys. J. Plus 134, 260–1–14 (2019)

51. Z.M. Shah, M.Y. Kathjo, F.A. Khanday, K. Biswas, C. Psychalinos, A survey of single and multi-component fractional-order elements (FOEs) and their applications. Microelectron. J. 84, 9–25 (2019)

52. M. Sowa, A subinterval-based method for circuits with fractional order elements. Bull. Pol. Acad. Sci. Tech. Sci. 62, 449–454 (2014)
53. M. Sowa, “gcdAlpha”—a semi-analytical method for solving fractional state equations. Poznan Univ. Technol. Acad. J. Electr. Eng. 96, 231–242 (2018)
54. T.P. Stefański, J. Gulgowski, Electromagnetic-based derivation of fractional-order circuit theory. Commun. Nonlinear Sci. Numer. Simul. 79, 104897-1–13 (2019)
55. T.P. Stefański, J. Gulgowski, Signal propagation in electromagnetic media described by fractional-order models. Commun. Nonlinear Sci. Numer. Simul. 82, 105029-1–16 (2020)
56. R. Süße, A. Domhardt, M. Reinhard, Calculation of electrical circuits with fractional characteristics of construction elements. Forsch. Ingenieurwes. 69, 230–235 (2005)
57. M.S. Tavazoei, Passively realizable approximations of non-realizable fractional order impedance functions. J. Frankl. Inst. Eng. Appl. Math. 357, 7037–7053 (2020)
58. J. Walczak, A. Jakubowska, Resonance in series fractional order $RL_{\beta}C_{\alpha}$ circuit. Przegląd Elektrotechniczny 90, 210–213 (2014)
59. S. Westerlund, L. Ekstam, Capacitor theory. IEEE Trans. Dielectr. Electr. Insul. 1, 826–839 (1994)
60. B. Zhang, X. Shu, Fractional-Order Electrical Circuit Theory. CPSS Power Electronics Series (Springer, Singapore, 2022)
61. L. Zhou, Z. Tan, Q. Zhang, A fractional-order multifunctional $n$-step honeycomb $RLC$ circuit network. Front. Inf. Technol. Electron. Eng. 18, 1186–1196 (2017)

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