All static spherically symmetric anisotropic solutions for general relativistic polytropes

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An algorithm presented by K. Lake to obtain all static spherically symmetric perfect fluid solutions was recently extended by L. Herrera to the interesting case of locally anisotropic fluids (principal stresses unequal). In this work we develop an algorithm to construct all static spherically symmetric anisotropic solutions for general relativistic polytropes. Again the formalism requires the knowledge of only one function (instead of two) to generate all possible solutions. To illustrate the method some known cases are recovered.

I. INTRODUCTION

The general formalism to study polytropes for anisotropic matter has been presented in recent papers [1,2]. The motivations to undertake such a task are the fact that the polytropic equations of state allow us to deal with a variety of fundamental astrophysical problems (see Refs. [3,21] and references therein). Also, the local anisotropy of pressure may be caused by a large variety of physical phenomena of the kind we expect to find in compact objects [22,52]. Among all possible sources of anisotropy let us mention two which might be particularly related to our primary interest: (i) the intense magnetic field observed in compact objects such as white dwarfs, neutron stars, or magnetized strange quark stars [36–41] (in some way, the magnetic field can be addressed as a fluid anisotropy) and, (ii): the viscosity, which is another source of anisotropy expected to be present in neutron stars and, in general, in highly dense matter (see Refs. [17–51]). We are not concerned by how small the resulting anisotropy produced in this last case might be, since the occurrence of an interesting phenomena such as cracking [53] may happen even for slight deviations from isotropy. On the other hand, in the context of Newtonian gravity, polytropic equations of state are particularly useful to describe a great variety of situations (see Refs. [4,11]), their great success stemming mainly from the simplicity of the equation of state and the ensuing main equation (Lane-Emden). Polytropes in the context of general relativity have been considered in Refs. [3,12–15].

The theory of polytropes is based on the polytropic equation of state, which in the Newtonian case reads

\[ P = k\rho^n = k\rho^{1+\frac{n}{k}}, \]

where \( P \) and \( \rho \) denote the radial pressure and the energy density, respectively. The constants \( k, \gamma, \) and \( n \) are usually called the polytropic constant, polytropic exponent, and polytropic index, respectively. In the general relativistic anisotropic case, two possible extensions of the above equation of state are possible, namely

\[ P_r = K\rho^n = K\rho^{1+\frac{k}{n}}, \]

(2)

\[ P_r = K\rho^n = K\rho^{1+\frac{1}{n}}, \]

(3)

where \( P_r \) and \( \rho \) denote the radial pressure and the energy density, respectively. It is important to emphasize that the assumption of either (2) or (3) is not enough to integrate completely the field equations, since the appearance of two principal stresses (instead of one) leads to a system of two equations for three unknown functions. Thus, in order to integrate the obtained system of equations, we need to provide further information about the anisotropy, inherent to the problem under consideration. Also we can proceed in a different way in order to integrate the set of equations. We impose certain conditions on the metric variables that can have physical relevance. An example of this procedure is adopting the vanishing of the Weyl tensor, usually referred to as the conformally flat condition [54].

Some years ago, Lake [57] developed an algorithm which produced all static spherically symmetric and isotropic perfect fluid solutions based on a single generating function, thus constructing an infinite number of previously unknown solutions physically relevant. This work was subsequently extended by Lake himself [58] and by Herrera et al. [59] to the case of locally anisotropic fluids, which describes very reasonably the matter distribution of a great variety of situations of interest (see, for example, Ref. [60] and references therein). In this case, the protocol to obtain all the static and anisotropic solutions of Einstein’s equations can be summarized as follows. Given a line element parametrized as

\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2, \]

(4)

from where, considering an anisotropic fluid, the Einstein
field equations read

\[ \frac{e^{-\lambda}}{r^2} - \frac{1}{r^2} e^{-\lambda} = -8\pi \rho \]
\[ \frac{e^{-\lambda}}{r^2} - \frac{1}{r^2} e^{-\lambda} = 8\pi P_r \]
\[ \frac{e^{-\lambda}}{4r} (2r\nu' - (r\nu' + 2) (\lambda' - \nu')) = 8\pi P_\perp, \]

we introduce the variables

\[ e^\nu = e^{\int(2z(r) - 2/r)dr} \]
\[ e^{-\lambda} = y(r), \]

where \( z \) and \( y \) are the so-called generating functions. In terms of these functions, we obtain

\[ e^\lambda = \frac{z^2 e^{\int(4/r^2 z + 2z)dr}}{r^6 (-2 \int z (1 + r^2 z + 4z)dr + C)}, \]

where \( C \) is a constant of integration and \( \Pi = 8\pi (P_r - P_\perp) \). Then, replacing (8) in (5), (6) and (7), we arrive to

\[ 4\pi \rho = \frac{m'}{r^2} \]
\[ 4\pi P_r = \frac{z(r - 2m) + m/r - 1}{r^2} \]
\[ 4\pi P_\perp = \left( 1 - \frac{2m}{r} \right) F(z) + z \left( \frac{m}{r^2} - \frac{m'}{r} \right), \]

where

\[ F(z) = z' + z^2 - \frac{z}{r} + \frac{1}{r^2} \]

and the mass function \( m(r) \) is defined by

\[ e^{-\lambda} = 1 - \frac{2m}{r}. \]

Note that, in contrast with the isotropic case reported in Ref. 53, three physical variables (principal stresses unequal) came into play. Therefore, in order for this protocol to work, two generating functions are needed; namely, \( z(r) \) and \( y(r) \). Interestingly, if a less general technique of generation is required, only one input function is necessary, as shown in 61, to convert isotropic Newtonian static fluid spheres into general relativistic anisotropic static fluid spheres. A different possibility is to provide one generating function and an additional ansatz such as the conformally flat condition, certain energy density distribution or a non local equation of state 59, for example.

In this work we shall develop a protocol to generate all static and spherically symmetric anisotropic solutions for general relativistic polytropes. We note that the constraint introduced by the polytropic equation of state gives place to only one generating function. Given the interest in polytropic equations of state for the relativistic community, the protocol here developed could be of interest in several contexts (for applications of polytropes in astrophysics and related fields, see, for example, Ref. 62).

In the next section we shall present the general equations and the algorithm which permits to construct all the solutions previously mentioned and then we shall obtain the generating function for a specific solution previously considered in the literature.

II. THE ALGORITHM

In this section we develop an algorithm to obtain all static and spherically symmetric anisotropic solutions for general relativistic polytropes. For this purpose we shall consider a line element parametrized in Schwarzschild-like coordinates as Eq. (1).

Let us introduce the generating function, \( y(r) \), by means of

\[ e^{-\lambda} = y(r). \]

Considering the polytropic equation of state

\[ P_r = k\rho^{1+\frac{1}{\epsilon}}, \]

Eqs. (5), (6) and (7) can be written as

\[ \frac{y}{r^2} - \frac{1}{r^2} + \frac{y'}{r} = -8\pi \rho \]
\[ \frac{y}{r^2} - \frac{1}{r^2} + \frac{y'}{r} = 8\pi k\rho^{\frac{1+\epsilon}{\epsilon + 1}} \]
\[ \frac{(r\nu' + 2)(y' + y\nu')}{4r} + \frac{2r y\nu''}{4r} = 8\pi P_\perp. \]

Using (18) and (19) we obtain

\[ \nu' = k(8\pi)^{-1/n}\rho^{-2/n} \left( -r y' - y + 1 \right)^{\frac{1}{\epsilon} + 1} - \frac{y + 1}{r y}. \]

Integrating the above equation, the metric function \( \nu(r) \) turns to be

\[ \nu = \int r^{-\frac{n+2}{n}} \left( k \left( -r y' - y + 1 \right)^{\frac{1}{\epsilon} + 1} - (8\pi r^2)^{\frac{1}{\epsilon} - 1} (y - 1) \right) dr. \]

Then, replacing Eqs. (10) and (22) in Eq. (11), the line element takes the form
\[ ds^2 = -e^{\gamma}(\int_{r}^{r'} \frac{r^{2n+2}}{8\pi} (k(r^y - y + 1)^{\frac{n+1}{n}}) dr) \ dt^2 + y dr^2 + r^2 d\Omega^2. \] (23)

Finally, after a long but straightforward calculation, the tangential pressure can be expressed in terms of the generating function as

\[ 8\pi P_\perp = \frac{r^{-2(n+2)}}{4\gamma} \left( \frac{64^{-1/n\pi-2/n}}{n} (k^2 (r^y - y + 1)^{\frac{n+2}{n}} + k(8\pi)^{1/n} r^{2/n} (r^y - y + 1)^{1/n} \times \right. \]
\[ \left. \times (y (-2(n + 1)(r^y + 2) - n r^y) + n r^y (r^y - 3) + 2(n + 2) y^2 + 2n) + \right. \]
\[ \left. + 64^{1/n} \pi^2/n^4 r^{4/n} (r(y - 1) y' + (y - 2) y') + r^{4/n} \right). \] (24)

Therefore, the system is completely determined once the generating function, \( y(r) \), is provided.

At this point, a couple of comments are in order. First, note that the mass function, \( m(r) \), is related to the generating function by means of

\[ y = 1 - \frac{2m}{r}. \] (25)

Therefore, using the fact that Einstein’s equations imply \( 4\pi \rho = \frac{\dot{m}}{\dot{r}} \), we have

\[ \rho = \frac{1 - y - ry'}{8\pi r^2}. \] (26)

Then, Eqs. (17), (24) and (26) express the physical variables in terms of the generating function. Even more, using the polytropic equation of state (see Eq. (17)), and Eqs. (25) and (8), it can be seen that the anisotropic system described by (11), (12) and (13) is reduced to the polytropic case under study. In this sense, the protocol for the polytropic fluid developed here is consistent with the most general case reported in [26]. Second, physically meaningful solutions should satisfy the weak and null energy conditions (WEC and NEC, respectively) which read \( \rho > 0 \) and \( \rho + P_r > 0 \) and \( \rho + P_\perp > 0 \). Considering the fluid satisfies the WEC and \( k > 0 \), which, although discards some cosmological fluids such as the cosmological constant, quintessence and k-essence and some phantom models [69], is a reasonable assumption, the NEC is automatically satisfied for the radial pressure. Therefore, only the NEC for the perpendicular component of the pressure remains to be considered. However, due to the intricate expression obtained for \( P_\perp(r) \) in terms of the generating function, this last condition has to be studied in detail for any particular case in which we are interested.

\[ \text{III. A PARTICULAR CASE} \]

In this section we obtain the generating function for a particular solution reported in Ref. [1] based on the generalized Tolman-Oppenheimer-Volkoff equation in the form

\[ P_r' = -\frac{\nu'}{2}(\rho + P_r) + \frac{2(P_\perp - P_r)}{r}, \] (27)

together with a specific form for the anisotropy \( \Delta = P_\perp - P_r = C f(P_r, r)(\rho + P_r)r^N \). In this previous equation, \( C \) is a parameter encoding the anisotropy, and \( f \) and \( N \) are certain function and number specific for each model studied (see [64] for details). In particular, if \( f(P_r, r)r^{N-1} = \frac{\nu'}{2} \) is assumed, then Eq. (27) reads

\[ \frac{dP_r}{dr} = -h(\rho + P_r)\frac{\nu'}{2}, \] (28)

where \( h = 1 - 2C \). As noted in [1], after defining the usual variable entering the Lane-Emden equation, \( \psi^\alpha \), as

\[ \rho(r) = \rho_c \psi^\alpha(r) \] (29)

where \( \rho_c \) is the energy density at the center of the object, Eq. (28) can be integrated, which produces

\[ e^{\nu(r)} = \left( 1 - \frac{2M}{r\Sigma} \right)^{(1 + \alpha)} \psi^{\frac{2(n+1)}{n}}, \] (30)

where \( M \) and \( r\Sigma \) are the mass and radius of the object and \( \alpha = P_c/\rho_c \) with \( P_c \) the radial pressure at the center.

Now, in order to obtain the generating function for this case, let us replace Eq. (29) in Eq. (18) and let us solve for \( y'(r) \) to obtain

\[ y' = \frac{-8\pi \rho_c r^2 \psi^\alpha - y + 1}{r}. \] (31)
Then, combining (31) and (16) we arrive to
\[ \nu' = \frac{8\pi kr^2 \rho c^\frac{1}{2} \psi^{n+1} + y + 1}{ry}. \] (32)

Finally, after deriving Eq. (30) to obtain
\[ \nu' = -\frac{2\alpha k(m + 1)\psi'}{\alpha h\psi + h}, \] (33)

and using (32) and (33), the generating function takes the form
\[ y = \frac{h(\alpha \psi + 1)(8\pi kr^2 \rho c^\frac{1}{2} \psi^{m+1} + 1)}{\alpha h\psi + h - 2\alpha k(m + 1)\psi'}, \] (34)

which concludes the protocol. At this point, it is worth mentioning that although once the generating function is obtained the Einstein’s equations are formally solved, as previously mentioned, the constraint given by Eq. (31) must be fulfilled in order to obtain \( \nu(r) \).

In the above result we have implemented some kind of inverse protocol to demonstrate that, once a solution of the Einstein’s equations is given, the generating function can be constructed. However, we can go a step further by implementing an extra condition in order to obtain a differential equation whose solution is the generating function. This can be done for example, by implementing a geometric constraint as the conformally flat condition \[ \frac{2}{3}. \]

This assumption is based on the role of the Weyl tensor in the structure and evolution of self-gravitating systems. Indeed, for spherically symmetric distributions of fluid, the Weyl tensor may be defined exclusively in terms of the density contrast and the local anisotropy of the pressure that can affect the fate of gravitational collapse (see for instance [62]). So, in order to find the generating function, we may provide an additional ansatz. Thus, for example, in the spherically symmetric case we know that there is only one independent component of the Weyl tensor, namely
\[ W = \frac{\rho c^\frac{1}{2} \psi^{n+1}}{6} \left( \frac{e^\lambda}{r^2} + \frac{\nu'\lambda'}{4} - \frac{1}{r^2} - \frac{\nu'^2}{4} - \frac{\nu''2 - \lambda' - \nu'}{2r} \right) \] (35)

Now, using (16), the conformally flat condition, namely \( W = 0 \), reduces to
\[ \frac{r(\nu r' - 2) y' + y (2 r \nu'' + \nu' (r v' - 2)) + 4}{ry} = 0, \] (36)

from where we obtain
\[ \nu = \int \frac{2}{r} \left( \sqrt{y(u) + 1} \right) \frac{du}{u \sqrt{y(u)}} + c_1. \] (37)

Replacing the above equation in Eqs. (18) and (19) we get
\[ 8\pi \rho + y' + \frac{y - 1}{r} = 0 \] (38)
\[ 3y + 2\sqrt{y} - 1 = \frac{8\pi kr^2}{r^2} \] (39)

from where, after eliminating \( \rho \) using the polytropic equation of state, we arrive to
\[ 3y + 2\sqrt{y} - 1 = \frac{k(8\pi)^{-1/2} (y' y - y + 1)}{r^2} \] (40)

Although the above result is a differential equation for the generating function \( y(r) \) which does not admit analytical solutions in general, it can be solved algebraically for \( y \) for the \( n = -1 \) case, giving
\[ y = \frac{5}{9} + \frac{8}{3} \frac{\pi kr^2}{9} \pm \frac{4}{9} \sqrt{6\pi kr^2 + 1}. \] (41)

Therefore, for this particular case, the conformally flat condition results in a simple form for the generating function. However, in order to apply the protocol here presented by using this condition to find the generating function, numerical solutions to Eq. (40) have to be explored.

IV. REMARKS AND CONCLUSIONS

In this work we have developed an algorithm to construct all static spherically symmetric anisotropic solutions for general relativistic polytropes in terms of one generating function, illustrating the method by recovering some known cases. In this sense, on one hand, once certain solution to the field equations is given, the generating function can be obtained. On the other hand, the direct computation of this generating function requires the knowledge of an extra constraint which, in order to give an example, here we have chosen to be the conformally flat condition. The protocol here presented could be useful for dealing with several phenomena in which anisotropic polytropes appear, with emphasis in the physics of compact objects.

Finally, it should be stressed that the models are presented with the only purpose to illustrate the method. The natural way to obtain models is by providing the specific information about the kind of anisotropy present in the problem under consideration or, in other case, presenting some relations of the metric functions as in the conformally flat case. It has to be stressed that the conformally flat condition implies energy density homogeneity for the perfect fluid sphere. This in turn implies that there are not bounded isotropic conformally flat polytropes. Therefore, our conformally flat models are necessarily anisotropic and are not continuously linked to the isotropic sphere.
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[1] L. Herrera and W. Barreto, Phys. Rev. D 87, 087303 (2013).
[2] L. Herrera and W. Barreto, Phys. Rev. D 88, 084022 (2013).
[3] L. Herrera, A. Di Prisco, W. Barreto, and J. Ospino, Gen. Relativ. Gravit. 46, 1827 (2014).
[4] S. Chandrasekhar, An Introduction to the Study of Stellar Structure (University of Chicago, Chicago, 1939).
[5] M. Schwarzschild, Structure and Evolution of the Stars (Dover, New York, 1958).
[6] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars (John Wiley and Sons, New York, 1983).
[7] R. Kippenhahn and A. Weigert, Stellar Structure and Evolution (Springer Verlag, Berlin, 1990).
[8] C. Hansen and S. Kawaler, Stellar Interiors: Physical Principles, Structure and Evolution (Springer Verlag, Berlin, 1994).
[9] A. Kovetz, Astrophys. J. 154, 999 (1968).
[10] P. Goldreich and S. Weber, Astrophys. J. 238, 991 (1980).
[11] M. A. Abramowicz, Acta Astronaut. 33, 313 (1983).
[12] R. Tooper, Astrophys. J. 140, 434 (1964).
[13] R. Tooper, Astrophys. J. 142, 1541 (1965).
[14] D. Reimers, S. Jordan, D. Koester, N. Bade, Th. Kohler, and L. Wisotzki, Astron. Astrophys. 311, 572 (1996).
[15] A. Perez Martinez, R. G. Felipe, and D. M. Paret, Int. J. Mod. Phys. D 19, 1511 (2010).
[16] D. Alvear Terrero, P. Bargueno, E. Contreras, A. P. Martinez and G. Quintero Angulo, Int. J. Mod. Phys. D (on-line ready, https://doi.org/10.1142/S0218271819500901, 2019).
[17] R. Tooper, Astrophys. J. 143, 465 (1966).
[18] S. Bludman, Astrophys. J. 183, 637 (1973).
[19] L. Herrera and W. Barreto, Phys. Rev. D 87, 087303 (2013).
[20] F. Shojai, M. R. Fazel, A. Estepenian, and M. Kohandel, Phys. Rev. D 93, 024047 (2016).
[21] S. Thirukkanesh and F. C. Ragel, Pramana J. Phys. 78, 67 (2012).
[22] X. Y. Lai and R. X. Xu, Astropart. Phys. 31, 128 (2009).
[23] J. Krisch and E. N. Glass, J. Math. Phys. 54, 082501 (2013).
[24] L. Herrera, J. Martin, and J. Ospino, J. Math. Phys. 43, 4889 (2002).
[25] L. Herrera, A. Di Prisco, J. Ospino, N. O. Santos, and O. Troconis, Phys. Rev. D 69, 084026 (2004).
[26] L. Herrera, J. Ospino and A. Di Prisco, Phys. Rev. D 77, 027502 (2008).
[27] L. Herrera, N. O. Santos, and A. Wang, Phys. Rev. D 78, 064026 (2008).
[28] P. H. Nguyen and J. F. Pedraza, Phys. Rev. D 88, 064020 (2013).
[29] P. H. Nguyen and M. Lingam, Mon. Not. R. Astron. Soc. 436, 2014 (2013).
[30] J. Krisch and E. N. Glass, J. Math. Phys. 54, 082501 (2013).
[31] R. Sharma and B. Ratanpal, Int. J. Mod. Phys. D 22, 1350074 (2013).
[32] E. N. Glass, Gen. Relativ. Gravit. 45, 2661 (2013).
[33] K. P. Reddy, M. Govender, and S. D. Maharaj, Gen. Relativ. Gravit. 47, 35 (2015).
[34] J. Ovalle, Phys. Rev. D 95, 104019 (2017).
[35] J. Ovalle, R. Casadio, R. da Rocha, and A. Sotomayor, Eur. Phys. J. C 78, 122 (2018).
[36] E. Contreras, A. Rincón, and P. Bargueño, Eur. Phys. J. C 19, 216 (2019).
[37] J. C. Kemp, J. B. Swedlund, J. D. Landstreet, and J. R. P. Angel, Astrophys. J. 161, L77 (1970).
[38] G. D. Schmidt and P. S. Schmidt, Astrophys. J. 448, 305 (1995).
[39] A. Putney, Astrophys. J. 451, L67 (1995).
[40] D. Reimers, S. Jordan, D. Koester, N. Bade, Th. Kohler, and L. Wisotzki, Astron. Astrophys. 311, 572 (1996).
[41] A. Perez Martinez, R. G. Felipe, and D. M. Paret, Int. J. Mod. Phys. D 19, 1511 (2010).
[42] D. Alvear Terrero, P. Bargueno, E. Contreras, A. P. Martinez and G. Quintero Angulo, Int. J. Mod. Phys. D (online ready, https://doi.org/10.1142/S0218271819500901, 2019).
[43] M. Chaihian, S. S. Masood, C. Montonen, A. Perez Martinez, and H. Perez Rojas, Phys. Rev. Lett. 84, 5261 (2000).
[44] A. Perez Martinez, H. Perez Rojas, and H.J. Mosquera Cuesta, Eur. Phys. J. C 29, 111 (2003).
[45] A. Perez Martinez, H. Perez Rojas, and H.J. Mosquera Cuesta, Int. J. Mod. Phys. D 17, 2107 (2008).
[46] E. J. Ferrer, V. de la Incera, J. P. Keith, I. Portillo, and P. L. Springsteen, Phys. Rev. C 82, 065802 (2010).
[47] R.D. Blandford and L. Hernquist, J. Phys. C 15, 6233 (1982).
[48] N. Andersson, G. Comer, and K. Glampedakis, Nucl. Phys. A763, 212 (2005).
[49] B. Sad, I. Shovkovy, and D. Rischke, Phys. Rev. D 75, 125004 (2007).
[50] M. Alford and A. Schmitt, AIP Conf. Proc. 964, 256 (2007).
[51] D. B. Blaschke and J. Berdermann, AIP Conf. Proc. 8, 290 (2008).
[52] A. Drago, A. Lavagnino, and G. Pagliara, Phys. Rev. D 71, 103004 (2005).
[53] P. B. Jones, Phys. Rev. D 64, 084003 (2001).
[54] E. N. van Dalen and A. E. L. Dieperink, Phys. Rev. C 69, 025802 (2004).
[55] H. Dong, N. Su, and O. Wang, J. Phys. G 34, S643 (2007).
[56] L. Herrera, E. Fuenmayor and P. León, Phys. Rev. D93, 024047, (2016).
[57] L. Herrera, A. Di Prisco, J. Ospino, and E. Fuenmayor, J. Math. Phys. (N.Y.) 42, 2129 (2001).
[58] K. Lake, Phys. Rev. D 67, 104015 (2003).
[59] K. Lake, Phys. Rev. Lett. 92, 051101 (2004).
[60] L. Herrera, J. Ospino and A. Di Prisco, Phys. Rev. D 77, 027502 (2008).
[61] L. Herrera and N. O. Santos, Phys. Rep. 286, 53 (1997).
[62] K. Lake, Phys. Rev. D 80, 064039 (2009).
[62] G. P. Horedt, *Politropes: Applications in Astrophysics and Related Fields*, Kluwer academic publishers (2004).

[63] L. Rezzolla and O. Zanotti, Relativistic Hydrodynamics, Oxford University Press (2013).

[64] M. Cosenza, L. Herrera, M. Esculpi and L. Witten, J. Math. Phys. (N. Y.) 22, 118 (1982).

[65] Herrera, L., Di Prisco, A., Hernandez-Pastora, J., Santos, N.O. Phys. Lett. A 237, 113 (1998).