On Semigroups Generated by Two Consecutive Integers and Improved Hermitian Codes

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Abstract

Analysis of the Berlekamp-Massey-Sakata algorithm for decoding one-point codes leads to two methods for improving code rate. One method, due to Feng and Rao, removes parity checks that may be recovered by their majority voting algorithm. The second method is to design the code to correct only those error vectors of a given weight that are also geometrically generic. In this work, formulae are given for the redundancies of Hermitian codes optimized with respect to these criteria as well as the formula for the order bound on the minimum distance. The results proceed from an analysis of numerical semigroups generated by two consecutive integers. The formula for the redundancy of optimal Hermitian codes correcting a given number of errors answers an open question stated by Pellikaan and Torres in 1999.

Keywords: Numerical semigroup, Hermitian curve, Feng-Rao improved code.

Introduction

Numerical semigroups have proven to be very useful in the study of one-point algebraic-geometry codes. On one hand the arithmetic of the numerical semigroup associated to the one-point yields a good bound—called the order bound—on minimum distance [1, 2, 3]. On the other hand, a close analysis of the numerical semigroup and the decoding algorithm commonly used for one-point codes shows that significant improvements in rate may be achieved while maintaining a given error correction capability [4]. In this article we discuss the order bound and improvements to the rate for codes constructed from Hermitian curves.

Let us briefly recall the definition of one-point algebraic geometry codes and state the notation we will use. Suppose \( \mathbb{F} \) is a finite field, \( \mathbb{F}/\mathbb{F} \) a function

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field and $P$ a rational point of $F/\mathbb{F}$. For $m \in \mathbb{N}_0$ let $\mathcal{L}(mP)$ be the ring of functions in $F$ having poles only at $P$ and of order at most $m$. Let $v_P$ be the valuation of $F$ associated with $P$ and let $\Lambda = \{-v_P(f) : f \in \bigcup_m \mathcal{L}(mP)\}$. $\Lambda$ is a numerical semigroup. That is, a subset of $\mathbb{N}_0$, closed under summation, containing 0 and with finite complement in $\mathbb{N}_0$. It is called the Weierstrass semigroup associated to $P$. Let $P_1, \ldots, P_n$ be pairwise distinct rational points of $F/\mathbb{F}$ which are different from $P$ and let $\varphi$ be the map $\bigcup_m \mathcal{L}(mP) \to \mathbb{F}^n$ such that $f \mapsto (f(P_1), \ldots, f(P_n))$. Suppose that $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 < \ldots\}$. The $i$-th one-point algebraic-geometry code associated with $P$ and $P_1, \ldots, P_n$ is $[\varphi(\mathcal{L}(\lambda_i P))]_{i}$. Naturally, the semigroup which will give us information about the one-point codes on $P$ will be the Weierstrass semigroup associated to $P$.

The Hermitian curve over $\mathbb{F}_{q^2}$, where $q$ is a prime power, is defined by its affine equation $x^{q+1} = y^q + y$. It has a single point $P_\infty$ at infinity and $q^3$ proper rational points $P_1, \ldots, P_{q^3}$. The ring of functions on the curve with poles only at $P_\infty$ is generated, as a vector space over $\mathbb{F}_q$, by the set $\{x^j y^k : j < q\}$. Moreover, $v_{P_\infty}(x) = -q$ and $v_{P_\infty}(y) = -q - 1$. Thus, the Weierstrass semigroup at $P_\infty$ is generated by $q$ and $q + 1$. Hermitian codes are the one-point codes defined on the Hermitian curve associated with $P_\infty$ and $P_1, \ldots, P_{q^3}$. For details on the Hermitian curve and the Hermitian codes we refer to [5, 2, 6].

The scope of this work is to analyze some aspects of Hermitian codes based on the Weierstrass semigroup at $P_\infty$. Since the only thing we will be using about the Hermitian codes is that the associated numerical semigroup is generated by two consecutive integers, all the results can be stated more generally for all those one-point codes for which the associated semigroup is generated by two consecutive integers. In Section 1 we analyze the enumeration of semigroups generated by two consecutive integers. Then we mention the known results on the sequence $\nu_i$ and the order bound. In Section 2 we give formulas for the number of checks of optimal codes correcting all errors of a given weight, whenever the associated numerical semigroup is generated by two consecutive integers. In the case of Hermitian codes this is the answer of an open question stated in [7]. In Section 3 we give formulas for the number of checks of optimal codes correcting all generic errors of a given weight.

1 On the enumeration and the $\nu$-sequence of semigroups generated by two consecutive integers

We start this section with a small survey of the nomenclator and notations we will use on numerical semigroups and, more specifically, those numerical semigroups generated by two consecutive integers. Then we will analyze the enumeration of the latter semigroups and give the tools we will use in Section 2 and Section 3.
1.1 Semigroups Generated by Two Consecutive Integers

By a numerical semigroup we mean a subset of \( \mathbb{N}_0 \), whose complement in \( \mathbb{N}_0 \) is finite and which contains any sum of its elements. Given a numerical semigroup \( \Lambda \) we denote the elements in its complement in \( \mathbb{N}_0 \). The genus \( g \) of \( \Lambda \) is the number of gaps while its conductor \( c \) is equal to the largest gap plus one. The enumeration \( \lambda \) of \( \Lambda \) is the unique increasing bijective map \( \lambda : \mathbb{N}_0 \longrightarrow \Lambda \). We say \( \lambda_i \) to denote \( \lambda(i) \). Notice that if \( \lambda_i \) is larger than or equal to the conductor or, equivalently, \( i \geq c - g \), then \( \lambda_i = i + g \).

In this work we just deal with numerical semigroups generated by two consecutive integers. If the consecutive integers are \( a, a + 1 \) then the numerical semigroup consists of any element \( ia + j(a + 1) \) with \( i, j \in \mathbb{N}_0 \). By properties of semigroups generated by two integers \([2]\), we know that the genus of this semigroup is \( g = (a - 1)a^2 \) and its conductor is \( c = (a - 1)a \). Furthermore, the semigroup generated by \( a, a + 1 \) admits two alternative descriptions. The first one is given by the disjoint union \( 0 \sqcup \{a, a + 1\} \sqcup \{2a, 2a + 1, 2a + 2\} \sqcup \cdots \sqcup \{(a - 2)a, (a - 2)a + 1, \ldots, (a - 2)a + a - 2\} \sqcup \{i : i \geq (a - 1)a\} \). The second one was proved in \([8]\) and it is given in the next lemma.

**Lemma 1.1** The numerical semigroup generated by \( a, a + 1 \) is the set with all nonnegative integers whose remainder when dividing by \( a \) is at most the quotient.

1.2 Enumeration

As one can see from Lemma 1.1, numerical semigroups generated by two consecutive integers are highly related to the set of pairs \( \mathcal{P} = \{(x, y) : x, y \in \mathbb{N}_0, y \leq x\} \). In fact, the numerical semigroup generated by \( a, a + 1 \) is the image of the map \( \alpha_a : \mathcal{P} \rightarrow \mathbb{N}_0 \)

\[
(x, y) \mapsto ax + y
\]

It turns out that this map is one-to-one whenever \( \alpha_a(x, y) \) is strictly less than \( a(a + 1) \). Indeed, if \( l < a(a + 1) \) and \( (x, y) \in \alpha_a^{-1}(l) \) then \( x \) must be less than or equal to \( a \) and \( y \) must be strictly less than \( a \). So \( x \) and \( y \) are the quotient and the remainder of the Euclidean division of \( l \) by \( a \), which are unique. In particular, \( \alpha_a \) is one-to-one whenever \( \alpha_a(x, y) \) is less than or equal to the conductor of the semigroup, which is \( c = a(a - 1) \).

Furthermore, the total order

\[
(x, y) < (x', y') \text{ if } \begin{cases} x < x', \\ x = x' \text{ and } y < y', \end{cases}
\]

is compatible with the natural order of the semigroup for all those values in the semigroup which are less than \( a(a + 1) \). That is, for any \( l, l' \in \Lambda \) with \( l, l' < a(a + 1) \), then \( l < l' \) if and only if \( \alpha_a^{-1}(l) < \alpha_a^{-1}(l') \).

Now, since \( \sum_{j=0}^k j = \frac{k(k + 1)}{2} \), the sequence \( a_k = \frac{k(k + 1)}{2} \) is increasing and \( a_{k+1} - a_k = k + 1 \). So any integer \( i \in \mathbb{N}_0 \) can be written uniquely as \( i = \frac{x(x + 1)}{2} + y \).
for some $x \in \mathbb{N}_0$ and some $0 \leq y \leq x$. Thus, the map

$$\beta : \mathcal{P} \rightarrow \mathbb{N}_0 \quad (x, y) \mapsto \frac{x(x+1)}{2} + y$$

is one-to-one everywhere and it is also compatible with the former total order.

As a conclusion, and taking into consideration that the genus and the conductor of the numerical semigroup generated by $a, a+1$ are, respectively, $\frac{(a-1)a}{2}$ and $(a-1)a$, one can see that the map $\lambda : \mathbb{N}_0 \rightarrow \Lambda$ with

$$\lambda(i) = \begin{cases} \alpha o \beta^{-1}(i) & \text{if } i \leq \frac{(a-1)a}{2}, \\ i + \frac{(a-1)a}{2} & \text{otherwise}, \end{cases}$$

is increasing and one-to-one. Hence, it is exactly the enumeration of the semigroup generated by $a, a+1$.

### 1.3 The $\nu$-Sequence and the Order Bound

Given a numerical semigroup $\Lambda$ with enumeration $\lambda$ define the sequence $\nu_i$ by

$$\nu_i = |\{ j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda \}|.$$

The sequence $\nu_i$ is used to define the order bound on the minimum distance of one-point algebraic-geometry codes:

$$\delta_i = \min \{ \nu_j : j > i \}.$$

The order bound, also known as Feng-Rao bound, is a lower bound on the minimum distance of the $i$-th one-point code on $P$. In this case the numerical semigroup is the Weierstrass semigroup associated to $P$. Details can be found in [1, 2, 3].

The Feng-Rao improved codes [4] are defined by means of the sequence $\nu_i$ as well. First a set of functions on the curve $\{ z_i : i \in \mathbb{N}_0 \}$ having only poles at $P$ is considered such that the valuation of $z_i$ at $P$ is $-\lambda_i$. Now, the Feng-Rao code designed to correct $t$ errors has as parity checks the evaluation in certain points of the curve of functions $z_i$ for all $i$ with $\nu_i < 2t + 1$.

In this subsection we derive the sequence $\nu_i$ as well as the order bound for numerical semigroups generated by two consecutive integers. For Hermitian codes this information has appeared previously (see [7, 2, 9]). We choose to include our own proofs since our methods are new and will be needed later in the analysis of improved codes.

From now on, let $\Lambda$ be the semigroup generated by $a$ and $a+1$ and let $g$ and $c$ be respectively its genus and its conductor, and let $\lambda$ be its enumeration. In order to compute the values in the sequence $\nu_i$ we need to distinguish between those elements $\lambda_i \in \Lambda$ for which $\lambda_i = ax + y$ for unique nonnegative integers $x, y$ with $y \leq x$ from those for which $x, y$ are not unique.

Let us denote by $\Lambda^x$ the subset of $\Lambda$ containing the elements $l = ax + y$ with $0 \leq y \leq x$. Then $l$ is uniquely expressible as $l = ax + y$ for nonnegative integers
Lemma 1.2 Let \( \lambda_i \in \Lambda \) and suppose that the Euclidean division of \( \lambda_i \) by \( a \) has quotient \( x \) and remainder \( y \). If \( x - a \leq y \leq a - 1 \), then \( \nu_i = (x - y + 1)(y + 1) = xy - y^2 + x + 1 \).

Proof: Suppose \( \lambda_i = \lambda_j + \lambda_k \). It is easy to check that if \( \lambda_i \in \Lambda^x \setminus (\cup_{x' \neq x} \Lambda^{x'}) \) for some \( x \), then \( \lambda_j \in \Lambda^{x'} \setminus (\cup_{x' \neq x'} \Lambda^{x'}) \) and \( \lambda_k \in \Lambda^{x''} \setminus (\cup_{x'' \neq x''} \Lambda^{x''}) \) for some \( x', x'' \).

So,

\[
\nu_i = \left| \{(x', y') \in \mathcal{P} : \lambda_i - ax' - y' \in \Lambda \} \right|
= \left| \{(x', y') \in \mathcal{P} : (x - x', y - y') \in \mathcal{P} \} \right|
= \left| \{(x', y') \in \mathbb{N}_0 \times \mathbb{N}_0 : x' \leq x, y' \leq y, y' \leq x', y' \geq x' - x + y \} \right|
= \sum_{0 \leq x' \leq x} \{|y' : \max\{0, y + x' - x\} \leq y' \leq \min\{y, x'\}|\}.
\]

This last number is the number of integer points inside a parallelogram with base \( x - y + 1 \) and height \( y + 1 \) (see Figure 1). Hence it is equal to \( (x - y + 1)(y + 1) \).

![Figure 1: Parallelogram in proof of Lemma 1.2](image)

To approach the case in which \( \lambda_i = ax + y = ax' + y' \) with \( x \neq x', y \neq y' \), we need a result from [10]. It says that if a numerical semigroup \( \Lambda \) is such that its conductor \( c \) is two times its genus, then for all \( \lambda_i \in \Lambda \) such that \( \lambda_i - c + 1 \in \Lambda \),
we have $\nu_i = \lambda_i - c + 1$. We already know that for the numerical semigroup generated by $a, a+1$ the conductor is two times the genus. Let us check that if $\lambda_i \in \Lambda^x \cap \Lambda^{x+1}$ then $\lambda_i - c + 1 \in \Lambda$. Indeed, suppose $\lambda_i \in \Lambda^x \cap \Lambda^{x+1}$. Since $\lambda_i \in \Lambda^{x+1}$, $\lambda_i = (x+1)a + y$ with $y \leq x+1$. Now, since $\lambda_i \in \Lambda^x$ and $\lambda_i = xa + (a+y)$, we have $a+y \leq x$. Thus, $\lambda_i - c + 1 = (x+1)a + y - a(a-1) + 1 = a(x-a+2) + y + 1$ with $y + 1 \leq x - a + 2$ and so $\lambda_i - c + 1 \in \Lambda$. Consequently, if $\lambda_i = ax + y = ax' + y'$ with $x \neq x'$, $y \neq y'$, then $\nu_i = \lambda_i - c + 1$.

The next theorem is a consequence of the former arguments.

Theorem 1.3 Let $\lambda_i \in \Lambda$ and suppose that the Euclidean division of $\lambda_i$ by $a$ has quotient $x$ and remainder $y$. Then,

$$
\nu_i = \begin{cases} 
(x - y + 1)(y + 1) & \text{if } -a + x \leq y \leq a - 1, \\
\lambda_i - c + 1 & \text{otherwise.}
\end{cases}
$$

Once we have found a formula for the values in the sequence $\nu_i$, the next step is to find a formula for the values of the order bound defined as $\delta_i = \min\{\nu_j : j > i\}$. Notice that this definition has a lot to do with the increasingness of the sequence $\nu_i$.

From Theorem 1.3 we deduce that $\nu_i$ is quadratic in $y$ for the integers $i$ corresponding to the values $\lambda_i = ax + y$ inside $\Lambda^x$ with $-a + x \leq y \leq a - 1$, while it is increasing elsewhere. See Figure 4, Figure 5, Figure 6. By analyzing the parabola we see that $\nu_i$ is increasing for $y \leq \frac{x}{2}$ and decreasing for $y \geq \frac{x}{2}$, being symmetric with respect to $y = \frac{x}{2}$. In the case when $x < a$ all values $ax + y \in \Lambda^x$ satisfy $-a + x \leq y \leq a - 1$. Then the first and last elements in $\Lambda^x$ (i.e. $y = 0, y = x$) have the same value for $\nu_i$, which is $x + 1$ and which is minimal. In the case when $x \geq a$, the first element (i.e. $y = -a + x$) attains the minimal value for $\nu_i$, which is $ax - a^2 + x + 1$; the second and last elements (i.e. $y = -a + x + 1, y = a - 1$) have the same value for $\nu_i$, which is $a(x - a + 2)$ and which is minimal if we take the first element away. Thus,

- If $x < a$ then
  - $\Lambda^x \cap \Lambda^{x'} = \emptyset$ for any $x' \neq x$ and
    $$\min\{\nu_i : \lambda_i \in \Lambda^x\} = x + 1, \quad (1)$$
  - if $\lambda_i \in \Lambda^x$ and $\lambda_x \neq ax + x$ then
    $$\min\{\nu_j : j > i \text{ and } \lambda_j \in \Lambda^x\} = x + 1.$$
- If $a \leq x < 2a$ then
  - $\Lambda^x \cap \Lambda^{x'} \neq \emptyset$, $\Lambda^x \setminus (\cup_{x' \neq x} \Lambda^{x'}) \neq \emptyset$, and
    $$\min\{\nu_i : \lambda_i = ax + y \in \Lambda^x, -a + x \leq y \leq a - 1\} = (a + 1)x - a^2 + 1, \quad (2)$$
\[ \lambda_i = ax + y \in \Lambda^x \quad \text{and} \quad -a + x \leq y < a - 1 \]

\[
\min \{ \nu_j : j > i, \lambda_j = ax + y \in \Lambda^x, 
- a + x \leq y \leq a - 1 \} = a(x - a + 2), \tag{3}
\]

\[ \min \{ \nu_i : \lambda_i \in \Lambda^x \cap \Lambda^{x+1} \} = \min \{ \nu_i : a(x + 1) \leq \lambda_i \leq ax + x \} = \nu_{x+1} = a(x + 1) - a(a - 1) + 1, \]

if \( \lambda_i \in \Lambda^x \cap \Lambda^{x+1} \) and \( \lambda_i \neq ax + x \), then

\[ \min \{ \nu_j : j > i \text{ and } \lambda_j \in \Lambda^x \cap \Lambda^{x+1} \} = \lambda_{i+1} - c + 1 = \lambda_i - c + 2. \]

\[
\delta_i = \begin{cases} 
  x + 1 & \text{if } x < a \text{ and } y \neq x, \\
  x + 2 & \text{if } x < a \text{ and } y = x, \\
  a(x - a + 2) & \text{if } x \geq a \text{ and } -a + x \leq y < a - 1, \\
  \lambda_i - c + 2 & \text{otherwise.}
\end{cases}
\]

The graphics in Figure 2, Figure 3, and Figure 4 show the first values of \( \nu_i \) and \( \delta_i \) for the Hermitian codes over \( \mathbb{F}_{4^2} \), \( \mathbb{F}_{8^2} \), and \( \mathbb{F}_{16^2} \), respectively.

In fact, it is proven \([11, 2]\) that for Hermitian codes the order bound on the minimum distance is exactly the real minimum distance of the codes.

## 2 Minimizing redundancy

The decoding algorithm commonly used for one-point codes is an adaptation of the Berlekamp-Massey-Sakata algorithm \([12]\) together with the majority voting algorithm of Feng-Rao-Duursma \([13, 14, 2]\). By analyzing majority voting, one realizes that only some of the parity checks are really necessary to perform correction of a given number of errors. New codes can be defined with just these few checks, yielding larger dimensions while keeping the same correction capability as standard codes \([4, 2]\). These codes are often called Feng-Rao...
improved codes. The redundancy of standard one-point codes correcting a given number $t$ of errors is

$$r(t) = \lambda^{-1}(\max\{i \in \mathbb{N}_0 : \nu_i < 2t + 1\}) + 1,$$

where the enumeration $\lambda$ and the sequence $\nu$ are derived from the Weierstrass semigroup of the distinguished point. The redundancy of the Feng-Rao improved codes correcting the same number of errors is

$$\tilde{r}(t) = |\{i \in \mathbb{N}_0 : \nu_i < 2t + 1\}|.$$

This section is devoted to finding explicit formulae for these redundancies in the case when the associated Weierstrass semigroup is generated by two consecutive integers $a, a + 1$. Recall that this is the case of Hermitian codes.

**Theorem 2.1** Let $a > 1$. Then,

$$r(t) = \begin{cases} 
\frac{t(2t + 1)}{(a^2 - a)/2 + (a + 1) \frac{a}{2} + 1} & \text{if } t \leq a/2, \\
(a^2 - a)/2 + 2t & \text{if } a/2 < t < a(\frac{a}{2} + 1)/2, \\
(a^2 - a)/2 + 2t - \sum_{x'=[\sqrt{ax'^2 + 4ax' - 8t}] \atop x' < a} \lfloor |\sqrt{ax'^2 + 4ax' - 8t} + \delta_{x',t}| \rfloor & \text{if } t \geq a/2.
\end{cases}$$

$$\tilde{r}(t) = \begin{cases} 
\frac{t(2t + 1)}{(a^2 - a)/2 + (a + 1) \frac{a}{2} + 1} & \text{if } t \leq a/2, \\
(a^2 - a)/2 + 2t - \sum_{x'=[\sqrt{ax'^2 + 4ax' - 8t}] \atop x' < a} \lfloor |\sqrt{ax'^2 + 4ax' - 8t} + \delta_{x',t}| \rfloor & \text{if } a/2 < t \leq a(\frac{a}{2} + 1)/2, \\
(a^2 - a)/2 + 2t - \sum_{x'=[\sqrt{ax'^2 + 4ax' - 8t}] \atop x' < a} \lfloor |\sqrt{ax'^2 + 4ax' - 8t} + \delta_{x',t}| \rfloor & \text{if } t \geq a(\frac{a}{2} + 1)/2.
\end{cases}$$

where

$$\delta_{x,t} = \begin{cases} 
1 & \text{if } x = [\sqrt{ax'^2 + 4ax' - 8t}] \text{ mod } 2 \\
0 & \text{if } x \neq [\sqrt{ax'^2 + 4ax' - 8t}] \text{ mod } 2.
\end{cases}$$

Proof: By the arguments in the previous section, the maximum non-gap whose $\nu$ is bounded by a certain constant must be 1) the last element in a parabola, that is, $ax + x$ for some $x < a$ or $ax + a - 1$ for some $x \geq a$; 2) the first element in a parabola for some $x \geq a$, that is, $ax + x - a$; 3) some value in $\Lambda_x' \cap \Lambda_x'+1$ for some $x'$. In case 1) and 2), $x$ is the largest integer such that $\Lambda_x \setminus (\cup_{x' \neq x} \Lambda_x') \neq \emptyset$ and such that the minimum $\nu$ value in $\Lambda_x \setminus (\cup_{x' \neq x} \Lambda_x')$ is at most $2t$. That is, the corresponding parabola is not empty and its minimum value is at most $2t$. In case 3), if the largest integer $x$ such that $\Lambda_x \setminus (\cup_{x' \neq x} \Lambda_x') \neq \emptyset$ and such that the minimum $\nu$ value in $\Lambda_x \setminus (\cup_{x' \neq x} \Lambda_x')$ is at most $2t$, satisfies $x < 2a - 1$, then $x' = x$. Otherwise, $x' \geq x$.

By formulas 1 and 2] the set of all minimum $\nu$ values among all non-empty parabolas is

$$M = \{x' + 1 : 0 \leq x' \leq a - 1\} \cup \{(a + 1)x' - a^2 + 1 : a \leq x' < 2a\}$$

$$= \{z : 1 \leq z \leq a\} \cup \{z(a + 1) : 1 \leq z \leq a\}.$$
Now, the maximum among these values which is at most 2t is

\[
\max\{m \in M : m \leq 2t\} = \begin{cases} 
2t & \text{if } 2t \leq a, \\
\frac{|2t|}{a+1}(a+1) & \text{if } a+1 \leq 2t \leq a(a+1), \\
a(a+1) & \text{if } 2t > a(a+1).
\end{cases}
\]

Therefore,

\[
x = \begin{cases} 
2t - 1 & \text{if } 2t \leq a, \\
\frac{|2t|}{a+1} + a - 1 & \text{if } a + 1 \leq 2t \leq a(a+1), \\
2a - 1 & \text{if } 2t > a(a+1).
\end{cases}
\]

If 2t \leq a then \(\Lambda^x \cap \Lambda^{x+1} = \emptyset\) and we are in case 1). Otherwise, if 2t > a then \(\Lambda^x \cap \Lambda^{x+1} \neq \emptyset\). If 2t < a(x - a + 2), by formulas (2) and (3), then we are in case 2). Otherwise, we will be either in case 1) or 3). Consequently,

\[
r(t) = \begin{cases} 
\lambda^{-1}(ax + x) + 1 & \text{if } 2t \leq a, \\
\lambda^{-1}(ax + x - a) + 1 & \text{if } a < 2t < a(x - a + 2), \\
\lambda^{-1}(ax + a - 1) + 1 + \{\lambda_i \in \bigcup_{x' > x} \Lambda^{x'} : \nu_i \leq 2t\} & \text{if } 2t \geq a(x - a + 2).
\end{cases}
\]

Replacing x by its value and taking into consideration that the value \(\nu_i\) increases constantly by one within \(\{\lambda_i \in \bigcup_{x' > x} \Lambda^{x'} : \nu_i \leq 2t\}\), we obtain

\[
r(t) = \begin{cases} 
t(2t + 1) & \text{if } t \leq a/2, \\
(a^2 - a)/2 + (a + 1)\left\lfloor \frac{2t}{a+1} \right\rfloor & \text{if } a/2 < t < a(\left\lfloor \frac{2t}{a+1} \right\rfloor + 1)/2, \\
(a^2 - a)/2 + 2t & \text{if } t \geq a(\left\lfloor \frac{2t}{a+1} \right\rfloor + 1)/2.
\end{cases}
\]

For the result on \(\hat{r}(t)\) recall that the parabola \((x - y + 1)(y + 1)\) gives the values of \(\nu_i\) for the non-gaps \(\lambda_i = ax + y\) with \(x - a \leq y \leq a - 1\). Fixed \(x\), the maximum on \(y\) of \((x - y + 1)(y + 1)\) is attained at \(y = x/2\) and it is equal to \(x^2/4 + x + 1\). From the values \(\lambda_i\) with \(i < r(t)\) we want to take away all those values whose corresponding \(\nu_i\) is larger than 2t. Our first aim is to identify which parabolas have nonempty intersection with the line at height \(2t + 1\). That is, \(x^2/4 + x + 1 \geq 2t + 1\). Those are exactly the parabolas for which \(x \geq [2\sqrt{2t + 1} - 1]\).

Now, from each parabola we need to know which is the number of integers \(y\) for which the \(\nu_i\) corresponding to \(\lambda_i = ax + y\) is at least \(2t + 1\). Since the parabola \((x - y + 1)(y + 1)\) is symmetric with respect to \(y = x/2\), there will be an odd number of such integers if \(x\) is even and an even number if \(x\) is odd.
The real values $y$ where the parabola equals $2t + 1$ are given by the equation 
$-y^2 + xy + x + 1 = 2t + 1$, and are exactly \( \frac{x \pm \sqrt{x^2 + 4x - 8t}}{2} \). Thus, the length of 
the real interval where the parabola is at least $2t + 1$ is \( \sqrt{x^2 + 4x - 8t} \). Now, 
from this interval we only want its integer values. It is easy to check that the 
number of such integers is \( \lfloor \sqrt{x^2 + 4x - 8t} \rfloor + \delta_{xt} \).

\[\Box\]

3 Minimizing redundancy for correcting generic errors

In [15] another improvement on one-point codes is described. Under the Berlekamp-
Massey-Sakata algorithm with majority voting, an error vector whose weight is 
larger than half the minimum distance of the code is often correctable. In particular 
this occurs for generic errors (also called independent errors in [16, 17]), 
whose technical algebraic definition can be found in [18]. Generic errors of 
weight $t$ can be a very large proportion of all possible errors of weight $t$, as 
in the case of the examples worked out in [15]. This suggests that a code be 
designed to correct only generic errors of weight $t$ rather than all error words 
of weight $t$. Using this restriction, one obtains new codes with much larger 
dimension than that of standard one-point codes correcting the same number 
of errors. In [18], the redundancy of standard one-point codes correcting all 
generic errors of weight up to $t$ is shown to be 
\[
r^*(t) = \lambda^{-1}(\max(\Lambda \setminus \{\lambda_i + \lambda_j : i, j \geq t\})) + 1.
\]

However, taking full advantage of the Feng and Rao improvements due to the 
majority voting step [4], one can get optimal codes correcting all generic errors 
of weight up to $t$ with redundancy 
\[
\tilde{r}^*(t) = |\Lambda \setminus \{\lambda_i + \lambda_j : i, j \geq t\}|.
\]

This section is devoted to finding explicit formulae for these redundancies.

It is easy to check that if $t$ is such that $\lambda_i$ is larger than or equal to the 
conductor then both $r^*(t)$ and $\tilde{r}^*(t)$ are equal to $\lambda_i + t$. If $c$ is the conductor 
and $g$ is the genus, $\lambda_i \geq c$ is equivalent to $t \geq c - g$. More specifically, for the 
semigroup generated by $a, a + 1$ this is equivalent to $t \in \Lambda^x$ for $x \geq a - 1$. In the 
next theorem we deal with the case when $t$ is strictly less than the conductor, 
that is, when $t \in \Lambda^x$ with $x < a - 1$.

**Theorem 3.1** Suppose $t = \frac{x(x+1)}{2} + y$ with $0 \leq y \leq x < a - 1$. That is,
\[ \lambda_t = xa + y \text{ with } 0 \leq y \leq x < a - 1. \]  

Then,  

\[
  r^*(t) = \begin{cases} 
    2x^2 + x & \text{if } 2x < a, \ y = 0, \\
    2x^2 + 3x + y + 1 & \text{if } 2x < a, \ y > 0, \\
    2xa + y - \frac{a^2 - 3a}{2} & \text{if } 2x \geq a, \ y > 2x - a + 1, \\
    2xa + 2y - \frac{a^2 - a}{2} & \text{if } 2x \geq a, \ y \leq 2x - a + 1.
  \end{cases}
\]

\[
  \tilde{r}^*(t) = \begin{cases} 
    2x^2 + x + 3y & \text{if } 2x < a, \\
    2xa + 3y - 2x - \frac{a^2 - 3a}{2} - 1 & \text{if } 2x \geq a, \ y > 2x - a + 1, \\
    2xa + 2y - \frac{a^2 - a}{2} & \text{if } 2x \geq a, \ y \leq 2x - a + 1.
  \end{cases}
\]

**Proof:** We have \( \{ \lambda_i + \lambda_j : i, j \geq t \} = \{ l \in \Lambda^{2x} : l \geq 2xa + 2y \} \cup \{ l \in \Lambda^{2x+1} : l \geq (2x + 1)a + y \} \cup (\cup_{x' \geq 2x+2} \Lambda^{x'}) \). Notice that \( \{ l \in \Lambda^{2x+1} : l < (2x+1)a + y \} \cap \Lambda^{2x+2} = \emptyset \) because \( y < a \). So,  

\[ \Lambda \setminus \{ \lambda_i + \lambda_j : i, j \geq t \} = \{ l \in \Lambda : l < 2xa + 2y \} \cup (\{ l \in \Lambda^{2x+1} : l < (2x+1)a + y \} \setminus \Lambda^{2x}). \]  

Let  

\[
  A = \{ l \in \Lambda : l < 2xa + 2y \}, \\
  B = \{ l \in \Lambda^{2x+1} : l < (2x+1)a + y \} \setminus \Lambda^{2x}.
\]

If \( 2x < a \) then \( |A| = \frac{2x(2x+1)}{2} + 2y \) and \( |B| = y \) because \( \Lambda^{2x} \cap \Lambda^{2x+1} = \emptyset \). So,  

\[
  \tilde{r}^*(t) = |A| + |B| = 2x^2 + x + 3y,
\]

\[
  r^*(t) = \begin{cases} 
    \frac{2x(2x+1)}{2} = 2x^2 + x & \text{if } y = 0, \\
    \frac{(2x+1)(2x+2)}{2} + y = 2x^2 + 3x + y + 1 & \text{if } y > 0.
  \end{cases}
\]

elements in \( \Lambda^{2x} \) are larger than the conductor and \( |A| = 2xa + 2y - g = 2xa + 2y - \frac{a^2 - a}{2} \). In order to compute \( |B| \), notice that \( |\{ l \in \Lambda^{2x+1} : l < (2x+1)a + y \}| = y \), while \( |\Lambda^{2x} \cap \Lambda^{2x+1}| = 2x - a + 1 \). Now, if \( y > 2x - a + 1 \), then \( \Lambda^{2x} \cap \Lambda^{2x+1} \subseteq \{ l \in \Lambda^{2x+1} : l < (2x+1)a + y \} \), so \( |B| = y - 2x + a - 1 \) and  

\[
  \tilde{r}^*(t) = |A| + |B| = 2xa + 3y - 2x - \frac{a^2 - 3a}{2} - 1,
\]

\[
  r^*(t) = 2xa + y - \frac{a^2 - 3a}{2}.
\]
Otherwise, if \( y \leq 2x - a + 1 \), then \( \Lambda^{2x} \cap \Lambda^{2x+1} \supseteq \{ l \in \Lambda^{2x+1} : l < (2x + 1)a + y \} \), so \(|B| = 0\) and

\[
\hat{r}^*(t) = |A| = 2xa + 2y - \frac{a^2 - a}{2},
\]

\[
r^*(t) = |A| = 2xa + 2y - \frac{a^2 - a}{2}.
\]

\(\square\)

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Figure 3: Graph of \( \nu_i \) and \( \delta_i \) for the Hermitian code over \( \mathbb{F}_{8^2} \).

Figure 4: Graph of \( \nu_i \) and \( \delta_i \) for the Hermitian code over \( \mathbb{F}_{16^2} \).
Figure 5: Graph of $\nu_i$ and $\delta_i$ for the Hermitian code over $F_{32^2}$.

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