Matrix-Scaled Consensus

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Abstract—This paper proposes matrix-scaled consensus algorithm, which generalizes the scaled consensus algorithm in [1]. In (scalar) scaled consensus algorithms, the agents’ states do not converge to a common value, but to different points along a straight line in the state space, which depends on the scaling factors and the initial states of the agents. In the matrix-scaled consensus algorithm, a positive/negative definite matrix weight is assigned to each agent. Each agent updates its state based on the product of the sum of relative matrix scaled states and the sign of the matrix weight. Under the proposed algorithm, each agent asymptotically converges to a final point differing with a common consensus point by the inverse of its own scaling matrix. Thus, the final states of the agents are not restricted to a straight line but are extended to an open subspace of the state-space. Convergence analysis of matrix-scaled consensus for single and double-integrator agents are studied in detail. Simulation results are given to support the analysis.

I. INTRODUCTION

Consensus algorithm and its variations [1]–[6] have been the main model for studying networked systems. Though simple, consensus algorithms can describe intricate phenomena such as bird flocking, synchronization behaviors, or how a group of people eventually reaches an agreement after discussions [7]–[10]. The consensus algorithm is also used to coordinate large-scale systems such as formation of vehicles, electrical, sensor, and traffic networks [11].

Consider a network in which the interactions between subsystems, or agents, is modeled by a graph. In the consensus algorithm, each agent updates its state based on the sum of the relative states with its nearby agents. If the interaction graph is connected, the agents’ states asymptotically converge to a common point in the space, and we say that the system asymptotically reaches a consensus.

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The author in [1] proposed a scaled consensus model, in which each agent has a scaling gain $s_i$ and updates its state variable $x_i$ based on the consensus law

$$\dot{x}_i = \text{sign}(s_i) \sum_{j \in N_i} (s_j x_j - s_i x_i), \quad i = 1, \ldots, n. \quad (1)$$

The system (1) achieves a scaled-consensus globally asymptotically, that is, $s_i x_i(t) \to s_j x_j(t)$, as $t \to \infty$ and agents with the same $s_i$ will converge to the same point (or cluster). The system (1) can describe a cooperative network, where agents have different levels of consensus on a single topic. Further extensions of the scaled consensus algorithm with consideration to switching graphs, time delays, disturbance attenuation, or different agents’ models can be found in the literature, for examples, see [12]–[17].

This paper generalizes the consensus model (1) by assuming that each agent has a state vector and a positive or negative definite scaling matrix. The proposed model has some interesting features. First, thanks to the matrix weights, the system still achieves clustering behavior, but the final states are not restricted to be distributed along a straight line. Under the matrix-scaled consensus algorithm, a virtual consensus point is jointly determined by the initial states and the scaling matrices of all agents. The state vector of each agent converges to a point differently from the virtual consensus point by the inverse of its scaling matrix. As a result, clustering behaviors usually happen, and agents with the same scaling matrix converge to a common cluster in the space. Second, although the proposed consensus law has similarities with the biased consensus [18] and orientation estimation algorithms [19], [20], in the proposed model, the scaling matrices are not limited to the rotation matrices. Under the assumption that the interaction graph is undirected and connected, it is shown that if the scaling matrices are positive definite or negative definite (possibly asymmetric), then the system achieves a matrix-scaled consensus globally asymptotically. Finally, the proposed algorithm can be used as a multi-dimensional model for studying clustering behaviors in a social network. Unlike the matrix-weighted consensus algorithm [6], in which a positive definite/semidefinite matrix weight associated with an edge in the graph characterizes the degree of cooperation between the agents in the network, in the matrix-scaled consensus algorithm, a positive/negative definite matrix weight corresponds to a vertex in the

1Notations will be defined in detail in Section II.
graph and represents a local coordinate system of each agent. Each local coordinate system can be interpreted as the private belief system of an individual on $d$ logically dependent topics, and this belief system is usually not perfectly aligned with a social norm (a global coordinate system). If each individual updates his/her opinion based on his/her own belief system, due to the existence of negative definite scaling matrices, the system will be unstable. By self-realizing the negative/positive of his/her own belief’s system relative to a social norm, each individual adjusts the opinion along a direction that is not contrary to the social norm. The sign of the scaling matrix in the proposed algorithm, thus, realizes the readiness of each individual to compromise in order to prevent the society from divergence. Moreover, we propose a matrix-scaled consensus algorithm for a network of double integrator agents. A corresponding convergence condition related to the damping gain and the eigenvalues of the scaled Laplacian is also given.

The remainder of this paper is organized as follows. Section II provides notations and the theoretical framework that will be used throughout the paper. The matrix-scaled consensus algorithms for single and double integrator agents are proposed and examined in Sections III and IV, respectively. Simulation results are given in Section V and Section VI concludes the paper.

II. Preliminaries

A. Notations

The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Scalars are denoted by lowercase letters, while bold font normal and capital letters are used for vectors and matrices, respectively. Let $A \in \mathbb{R}^{m \times n}$, its transpose is given by $A^\top$. The kernel, image, rank, and determinant of $A$ are denoted by $\ker(A)$, $\mathrm{im}(A)$, $\mathrm{rank}(A)$, and $\det(A)$, respectively. For a vector $x = [x_1, \ldots, x_d]^\top$, its 2-norm is $\|x\| = \sqrt{\sum_{i=1}^d x_i^2}$. Let $x_1, \ldots, x_n \in \mathbb{R}^d$, the vectorization operator is defined as vec$(x_1, \ldots, x_n) = [x_1^\top, \ldots, x_n^\top]^\top \in \mathbb{R}^{dn}$. A matrix $A \in \mathbb{R}^{d \times d}$ is positive definite (negative definite) if and only if $\forall x \in \mathbb{R}^d$, $x \neq 0_d$, then $x^\top Ax > 0$ (resp., $x^\top Ax < 0$).

B. Useful lemma

Lemma 1: [8] The complex-coefficient polynomial $p(s) = a^2 + (a + bj)s + c + jd$, where $j^2 = -1$, is Hurwitz if and only if $abd + a^2c - d^2b > 0$.

C. Algebraic graph theory

An undirected graph is given by $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the vertex set and $\mathcal{E} \subset \mathcal{V}^2$ is the set of $|\mathcal{E}| = m$ edges. If there is an edge $(i, j) \in \mathcal{E}$ connecting vertices $i, j \in \mathcal{V}$, then $i$ and $j$ are adjacent to each other. The neighbor set of a vertex $i$, denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$, contains all adjacent vertices of $i$. A path in $G$ is a sequence of edges connecting adjacent vertices in the graph. A graph is connected if and only if there is a path between any pair of vertices in $\mathcal{V}$. Let the edges be indexed as $\mathcal{E} = \{e_1, \ldots, e_m\}$, and oriented such that for each edge $e_k = (i, j)$, $i$ is the starting vertex and $j$ is the end vertex of $e_k$. The incidence matrix $H = [h_{kl}] \in \mathbb{R}^{m \times n}$ has $h_{kl} = -1$ if $l = i$, $h_{kl} = 1$ if $l = j$, and $h_{kl} = 0$, otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ of $G$ is defined as follows:

$$l_{ij} = \begin{cases} -1, & i \neq j, (i, j) \in \mathcal{E}, \\ 0, & i \neq j, (i, j) \notin \mathcal{E}, \\ -\sum_{i=1,j \neq l}^n l_{ij}, & i = j. \end{cases}$$

As $G$ is undirected, $L$ is symmetric positive semidefinite with eigenvalues given by $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. We can write $L = H^\top H$. For a connected graph, $\lambda_2 > 0$ and $\ker(L) = \ker(H) = \mathrm{im}(1_n)$.

D. Matrix-scaled consensus

Consider a multi-agent system consisting of $n$ agents. Each agent $i \in \{1, \ldots, n\}$ has a state vector $x_i \in \mathbb{R}^d$ ($d \geq 2$) and a scaling matrix $S_i \in \mathbb{R}^{d \times d}$, which is either positive definite or negative definite. Define the signum function

$$\mathrm{sign}(S_i) = \begin{cases} 1, & S_i \text{ is positive definite}, \\ -1, & S_i \text{ is negative definite}. \end{cases}$$

Then $|S_i| \triangleq \mathrm{sign}(S_i)S_i$ is a positive definite matrix. It is worth noting that $\mathrm{sign}(S_i) = \mathrm{sign}(S_i^\top) = \mathrm{sign}(S_i^{-1})$ and $|S_i^{-1}| = |S_i|^{-1}$. Let $x = \mathrm{vec}(x_1, \ldots, x_n)$, the following definition will be used in this paper:

**Definition 1**: The $n$-agent system achieves a matrix scaled consensus (MSC) in a state $x$ if and only if $x \in \mathcal{A}$, where

$$\mathcal{A} = \{x \in \mathbb{R}^{dn} | S_1x_1 = S_2x_2 = \ldots = S_nx_n = x^n \},$$

and $x^n \in \mathbb{R}^d$ is called the virtual consensus point of the system.

Equivalently, the $n$-agent system achieves a matrix-scaled consensus if and only if

$$x_i = S_i^{-1}x^n = S_i^{-1}S_jx_j, \ \forall i, j \in \mathcal{V}.$$  

III. Matrix-scaled consensus of single-integrator agents

A. The proposed consensus law

Consider a system of $n$ single-integrator modeled agents in $\mathbb{R}^d$ ($d \geq 2$):

$$x_i = u_i, \ i = 1, \ldots, n,$$

where $x_i, u_i \in \mathbb{R}^d$ are respectively the state variable and the input of agent $i$. The matrix-scaled consensus is proposed as follows:

$$u_i = \mathrm{sign}(S_i) \sum_{j \in \mathcal{N}_i} (S_jx_j - S_jx_i),$$

$$= |S_i| \sum_{j \in \mathcal{N}_i} (S_j^{-1}S_jx_j - x_i), \ i = 1, \ldots, n.$$  

2Note that $S_i$ is not required to be symmetric.
In the consensus algorithm (4), each agent $i$ measures the relative state vector $S^{-1}_i x_j$ from its neighboring agents, sums up the relative vector $r_i = \sum_{j \in N_i} (S^{-1}_i x_j - x_i)$, and updates its state variable along the direction of $|S^{-1}_i| r_i$. The $n$-agent system can be written in the compact form as follows:

$$\dot{x} = - (\text{diag}(\text{sign}(S_i)) L \otimes I_d) \text{blkdiag}(S_i) x,$$  

(5)

where $x = \text{vec}(x_1, \ldots, x_n)$, $\otimes$ stands for the Kronecker product and $S \triangleq \text{blkdiag}(S_i)$ is the block diagonal matrix with the matrices $S_1, \ldots, S_n$ in the main diagonal. Introduce the variable transformation $x_c = S x$ and let $\Theta = |S| L$, where $L = I \otimes I_d$ and $|S| = \text{blkdiag}(|S_i|)$. We can reexpress the $x_c$-dynamics in the matrix form as follows:

$$\dot{x}_c = - \Theta \dot{x}_c.$$  

(6)

Note that if the signum term is omitted, the system (6) becomes $\dot{x}_c = -S L x_c$. Since $S L$ is the product of a nondefinite matrix $S$ with a positive semidefinite matrix $L$, it will be likely that $S L$ contains eigenvalues with positive real parts, and the system is unstable.

**B. Stability analysis**

We will study the system (6) in this subsection. Since $|S_i|$, $i \in \mathcal{V}$, are positive definite, $|S| = \text{blkdiag}(|S_i|)$ is positive definite. Thus, rank($\Theta$) = rank($L$) = $dn - d$ and $\ker(\Theta) = \ker(L) = \text{im}(I_n \otimes I_d)$. The following lemma characterizes the spectrum of $\Theta$:

**Lemma 2:** Suppose that $G$ is undirected and connected. The matrix $\Theta$ has $d$ zero eigenvalues and $dn - d$ eigenvalues with positive real parts.

**Proof:** Following the proof of [21][Lem. 8.2.4], we define the following matrices $H = H \otimes I_d$, $X = \begin{bmatrix} I_{dn} & s^{-1}|S| H^\top \\ H \otimes I_{dn} \end{bmatrix}$, and $Y = \begin{bmatrix} I_{dn} \\ -H \otimes I_{dn} \end{bmatrix}$.

Then,

$$XY = \begin{bmatrix} I_{dn} & s^{-1}|S| H^\top \\ 0_{dn \times dn} & I_{dn} \end{bmatrix},$$

$$YX = \begin{bmatrix} I_{dn} \\ 0_{dn \times dn} \end{bmatrix} (I_{dn} - s^{-1}|S| H^\top H).$$

From the fact that $\det(XY) = \det(YX)$, one has

$$\det(I_{dn} - s^{-1}|S| H^\top H) = \det(s^{-1}|S| H^\top H) \text{ sdd}.$$  

(7)

Thus, the nonzero eigenvalues of two matrices $\Theta = |S| L$ and $N = H |S| H^\top$ are the same. Since $|S|$ is positive definite, $N + N^\top = H |S| + |S| H^\top$ is symmetric and positive semidefinite, $\text{rank}(N + N^\top) = \text{rank}(H) = dn - d$. Therefore, $\Theta$ has $dn - d$ eigenvalues with positive real parts.

Let $\Theta$ be expressed in the Jordan canonical form $\Theta = W J W^{-1}$, where $W = [w_1, \ldots, w_{dn}] \in \mathbb{C}^{dn \times dn}$. For brevity, the notation $W_{[j:k]} = [w_j, \ldots, w_k]$ is adopted to denote the columns from $j$ to $k$ ($j < k$) of the matrix $W$.

By selecting $W_{[1:d]} = 1_n \otimes I_d$, we have $\left(\left(W^{-1}\right)^\top \right)^{[1:d]} = \left(|S^{-1}| \otimes I_d\right) P^\top$, where $|S^{-1}| = \text{blkdiag}(|S^{-1}_i|)$ and $P \in \mathbb{R}^{d \times d}$ is included so that the normalization condition $\left(\left(W^{-1}\right)^\top \right)^{[1:d]} W_{[1:d]} = I_d$ (8) is satisfied. The equation (8) is equivalent to

$$P(1_n \otimes I_d)|S^{-1}|(1_n \otimes I_d) = I_d,$$  

(9)

and it follows that $P = \left(\sum_{i=1}^n |S^{-1}_i|\right)^{-1}$ . Since all nonzero eigenvalues of $-\Theta$ have negative real parts, there holds

$$\lim_{t \to +\infty} x_c(t) = \lim_{t \to +\infty} \exp(-\Theta t) x_c(0) = (1_n \otimes I_d) P(1_n \otimes I_d)|S^{-1}|x_c(0) = (1_n \otimes I_d) P \sum_{i=1}^n |S^{-1}_i| x_i(0) = 1_n \otimes \left(P \sum_{i=1}^n \text{sign}(S_i) x_i(0)\right).$$  

(10)

Thus, $\lim_{t \to +\infty} S x(t) = 1_n \otimes x^a$, where

$$x^a = \left(\sum_{i=1}^n |S^{-1}_i|\right)^{-1} \sum_{i=1}^n \text{sign}(S_i) x_i(0).$$  

(11)

Thus, the system (5) asymptotically achieves a matrix-scaled consensus. Because $\dot{x}_c(t) = (\sum_{i=1}^n |S^{-1}_i|)^{-1}(1_n \otimes I_d)L x_c(t) = 0_d$, and which shows that $x_c(t)$ is time-invariant. We can now state the main theorem of this section.

**Theorem 1:** Suppose that $G$ is undirected and connected. Under the matrix-scaled consensus algorithm (4), $x(t) \to S^{-1}(1_n \otimes x^a)$ as $t \to +\infty$.

Below, another proof of Theorem 1 will be given based on Barbalat’s lemma.

**Proof:** Consider the function $V(x_c) = x_c^\top \bar{L} x_c$ which is positive definite with regard to $L x_c$ and continuously differentiable. Moreover,

$$\dot{V} = -x_c^\top \bar{L} (|S| + |S|^\top) \bar{L} x_c.$$  

Since $|S| + |S|^\top$ is symmetric positive definite, it follows that $\dot{V} \leq 0$. It follows that $\lim_{t \to +\infty} V$ exists and is finite and $L x_c$ is bounded. Thus, $V = 2x_c^\top \bar{L} (|S| + |S|^\top) \bar{L} x_c$ is also bounded. It follows from Barbalat’s lemma [22] that $\lim_{t \to +\infty} \dot{V} = 0$. Thus, $x_c(t) \to \text{im}(1_n \otimes I_d)$ as $t \to +\infty$. Since $x_c = S x$, there holds

$$(1_n \otimes I_d)|S^{-1}|x_c = -(1_n \otimes I_d) L x_c = 0_d.$$  

This means $(1_n \otimes I_d)|S^{-1}|x_c(t) = (1_n \otimes I_d)|S^{-1}|x_c(0) = \sum_{i=1}^n |S^{-1}_i| x_i(0) = \sum_{i=1}^n \text{sign}(S_i) x_i(0), \quad \forall t \geq 0.$

From $(1_n \otimes I_d)|S^{-1}|(1_n \otimes x^a) = \sum_{i=1}^n |S^{-1}_i| x_i(0)$, it follows that $x_c^a = P \sum_{i=1}^n \text{sign}(S_i) x_i(0) = x^a$. Thus, $x_c(t) \to x^a,$
$x_i(t) \to S^{-1}_i x^a$ as $t \to +\infty$, or i.e., the system (5) globally asymptotically achieves a matrix-scaled consensus.

IV. MATRIX-SCALED CONSENSUS OF DOUBLE-INTEGRATOR AGENTS

A. Proposed consensus laws

This section studies the matrix-scaled consensus algorithm for a system of double integrators modeled by

$$
\begin{align}
\dot{x}_i^1 &= x_i^2, \\
\dot{x}_i^2 &= u_i, \quad i = 1, \ldots, n,
\end{align}
$$

where $x_i^1, x_i^2 \in \mathbb{R}^d$ are states of agent $i$, and $u_i \in \mathbb{R}^d$ is its control input. Let $x_i = \text{vec}(x_i^1, x_i^2)$, $x_i^1 = \text{vec}(x_i^1, \ldots, x_i^1)$, and $x_i^2 = \text{vec}(x_i^2, \ldots, x_i^2)$. The objective is to make the agents’ states $x_i^1$ asymptotically achieve a matrix-scaled consensus, i.e., to make $x = \text{vec}(x^1, x^2)$ asymptotically converge to the set

$$
A' = \{ x \in \mathbb{R}^{2dn} | S_i x_i^1 = S_2 x_2^1 = \ldots = S_n x_n^1, \quad x^2 = 0_{dn}\}
$$

The following consensus law is proposed to achieve the matrix-scaled consensus:

$$
u_i = -\text{sign}(S_i) \sum_{j \in N_i} (S_i x_j^1 - S_j x_i^1) - \alpha x_i^2,
$$

where $\alpha > 0$ is a control gain. The $n$-agent system under (13) is given as follows

$$
\begin{align}
x_i^1 &= x_i^2, \\
x_i^2 &= -(\text{sign}(S_i)) \otimes I_d) \bar{L} S x^1 - \alpha x_i^2.
\end{align}
$$

B. Stability analysis

The behavior of the system (14) is given in the following theorem.

**Theorem 2**: Suppose that $G$ is undirected and connected. Under the consensus law (13), $x(t)$ asymptotically converges to a point in $A'$.

**Proof**: Let $x_i^1 = S x_i^1$ and $x_i^2 = S x_i^2$, we can rewrite the system (14) as follows

$$
\begin{bmatrix}
\dot{x}_i^1 \\
\dot{x}_i^2
\end{bmatrix} = \begin{bmatrix}
0_{dn} & I_{dn} \\
-SL & -\alpha I_{dn}
\end{bmatrix} \begin{bmatrix}
x_i^1 \\
x_i^2
\end{bmatrix} = N_1 \begin{bmatrix}
x_i^1 \\
x_i^2
\end{bmatrix}.
$$

Substituting $\Theta = |S|L = WJW^{-1}$ into the characteristic equation $\det(s^2J - N_1) = 0$, one has

$$
\det(s^2I_{dn} + \alpha s I_{dn} + WJW^{-1}) = 0,
$$

or, equivalently

$$
\prod_{k=1}^{dn} (s^2 + \alpha s + \mu_k) = 0,
$$

where $\mu_1 = \ldots = \mu_d = 0$ and $\mathbb{C} \ni \mu_k = \alpha_k + j b_k \neq 0, \ \forall k = d+1, \ldots, dn$. Based on Lemma 1, each polynomial $s^2 + \alpha s + \mu_k$ is Hurwitz if and only if $\alpha_k > b_k^2/\alpha_2$. Thus, by choosing $\alpha$ so that

$$
\min_{k=d+1, \ldots, dn} \text{Re}(\mu_k) > \frac{(\max_{k=d+1, \ldots, dn} \text{Im}(\mu_k))^2}{\alpha^2},
$$

the matrix $N_1$ has $d$ zero eigenvalues and $2dn - d$ eigenvalues with negative real parts. The right and left eigenvectors of $N_1$, corresponding to the zero eigenvalues, are columns and rows of the matrices $\begin{bmatrix} I_n \otimes I_d \\ 0_n \otimes I_d \end{bmatrix}$ and $P [(1_n^T \otimes I_d) |S|^{-1} \alpha^{-1}(1_n^T \otimes I_d) |S|^{-1}]$. Thus,

$$
\lim_{t \to +\infty} \begin{bmatrix} x_i^1(t) \\
2dn \end{bmatrix} = \lim_{t \to +\infty} \text{exp}((N_1 t) \begin{bmatrix} x_i^1(0) \\
2dn \end{bmatrix}
$$

Fig. 1: The 16-vertex graph used in the simulations.

V. Simulation results

Consider a system of 16 agents having the interaction graph as depicted in Fig. 1. We will provide some simulations to support the results in the previous sections.

A. Simulation 1: MSC of single integrators

Let the SO(2) rotation matrix of angle $\theta$ (rad) be denoted by $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Let the scaling matrices be chosen as $S_1 = \ldots = S_6 = R(0) = I_2$ (positive definite), $S_7 = \ldots = S_{11} = R(\frac{\pi}{4})$, and $S_{12} = \ldots = S_{16} = R(\frac{3\pi}{4})$ (negative definite). The initial condition $x(0)$ is randomly selected. The simulation results of (4) depicted in Fig. 2 show that the agents converge to three clusters, which are three vertices of an equilateral triangle. Agents with the same matrix $S_i$ converge to the same cluster. Notice that $\theta$ should be not equal to $\frac{\pi}{2}$ so that the analysis holds. See [23] for further simulation of this algorithm.

B. Simulation 2: MSC of double-integrators

Next, let the agents be modeled by double-integrators. The scaling matrices are $S_i = R(\frac{\pi}{4}), i = 1, \ldots, 4$, $S_i = R(\frac{3\pi}{4}), i = 5, \ldots, 8$, $S_i = R(-\frac{3\pi}{4}), i = 9, \ldots, 12$, and $S_i = R(-\frac{\pi}{4}), i = 13, \ldots, 16$. We conduct three simulations of the MSC algorithm (13) with $\alpha = 1.8,
1.9724 and 3, respectively. The trajectories of the agents, corresponding to these parameters are depicted in Fig. 3. For $\alpha = 1.8$, the system is unstable. Correspondingly, Figs. 3 (a), (d)–(e) show the state variables grow unbounded. For $\alpha = 1.9724$, $\Theta$ has pairs of imaginary eigenvalues with the corresponding independent eigenvectors, $x^l_i$, $l = 1, 2$, are asymptotic to sinusoidal functions (see Figs. 3 (b), (h)–(k)). Finally, for $\alpha = 3$, the condition (17) is satisfied. The agents converge to 4 clusters as shown in Figs. 3 (c), (l)–(o).

VI. Conclusions

A novel matrix-scaled consensus model, which can describe a multi-dimensional opinion dynamics system with heterogeneous individuals’ private belief systems was proposed. The dissimilarities on individuals’ belief systems cause clustering phenomenon to happen frequently. The matrix scaling gains allow agents keeping their own biased states (in both amplitude and direction) with regard to a virtual consensus point. Extension of the model to double-integrator agents was also proposed. For further studies, it will be of interests to combine the matrix-scaled consensus with the Altafini model, and study other applications such as scaled synchronization and formation control.

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Fig. 2: The agents converge to 3 clusters in the plane under the matrix-scaled consensus algorithm (4).
Fig. 3: Simulations of double-integrator agents: (a)–(c): Trajectories of agents with \( \alpha = 1.8, 1.9724, \) and 3, respectively; (d)–(g): \( x_{i1}^k \) and \( x_{i2}^k, k = 1, 2, \) vs time \( t \) [s] corresponding to \( \alpha = 1.8; \) (h)–(k): \( x_{i1}^k \) and \( x_{i2}^k, k = 1, 2, \) vs time \( t \) [s] corresponding to \( \alpha = 1.9724; \) (l)–(o): \( x_{i1}^k \) and \( x_{i2}^k, k = 1, 2, \) vs time \( t \) [s] corresponding to \( \alpha = 3. \)