FORCED GRADINGS IN INTEGRAL QUASI-HEREDITARY
ALGEBRAS
WITH APPLICATIONS TO QUANTUM GROUPS

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Abstract. Let \( \mathcal{O} \) be a discrete valuation ring with fraction field \( K \) and residue field \( k \). A quasi-hereditary algebra \( \tilde{A} \) over \( \mathcal{O} \) provides a bridge between the representation theory of the quasi-hereditary algebra \( \tilde{A}_K := K \otimes \mathcal{O} \tilde{A} \) over the field \( K \) and the quasi-hereditary algebra \( A_k := k \otimes_{\mathcal{O}} \tilde{A} \) over \( k \). In one important example, \( \tilde{A}_K \)-mod is a full subcategory of the category of modules for a quantum enveloping algebra while \( \tilde{A}_k \)-mod is a full subcategory of the category of modules for a reductive group in positive characteristic. This paper considers first the question of when the positively graded algebra \( \text{gr} \tilde{A} := \bigoplus_{n \geq 0} (\tilde{A} \cap \text{rad}^n \tilde{A}_K)/(\tilde{A} \cap \text{rad}^{n+1} \tilde{A}_K) \) is quasi-hereditary. A main result gives sufficient conditions that \( \text{gr} \tilde{A} \) be quasi-hereditary. The main requirement is that each graded module \( \text{gr} \tilde{A}_\lambda \) arise from a \( \tilde{A} \)-standard (Weyl) module \( \tilde{A}_\lambda \) have an irreducible head. An additional hypothesis requires that the graded algebra \( \text{gr} \tilde{A}_K \) be quasi-hereditary, a property recently proved [16] to hold in some important cases involving quantum enveloping algebras. In the case where \( \tilde{A} \) arises from regular dominant weights for a quantum enveloping algebra at a primitive \( p \)th root of unity for a prime \( p > 2h - 2 \) (where \( h \) is the Coxeter number), a second main result shows that \( \text{gr} \tilde{A} \) is quasi-hereditary. The proof of this difficult result involves interesting new methods involving integral quasi-hereditary algebras. It also depends on previous work [16] of the authors, including a continuation of the methods there involving tightly graded subalgebras, and a development of a quantum deformation theory over \( \mathcal{O} = \mathbb{Z}_{(p)}(\zeta) \), where \( \zeta \) is a \( p \)th root of unity. This (new) deformation theory, given in Appendix II (§7), extends work of Andersen-Jantzen-Soergel [2], and is worthy of attention in its own right. As we point out, the present paper provides an essential step in our work on \( p \)-filtrations of Weyl modules for reductive algebraic groups over fields of positive characteristic.

1. Introduction

Quasi-hereditary algebras are certain finite dimensional algebras over a field which arise naturally in Lie-theoretic representation theory, in both geometric and algebraic contexts. For example, if \( G \) is a reductive algebraic group in positive characteristic, the category of finite dimensional rational
G-modules generated by the irreducible modules having highest weights in a finite saturated set of dominant weights is equivalent to the module category $A$–mod for a quasi-hereditary algebra $A$; see [3]. In a geometric vein, certain categories of perverse sheaves are similarly equivalent to $A$–mod for a quasi-hereditary algebra $A$; see [14]. On the other hand, quasi-hereditary algebras are defined abstractly, so they are objects of interest to finite dimensional algebraists; see [19], for example.

In [16], the authors studied naturally occurring hypotheses which, given a quasi-hereditary algebra $A$, guarantee a quasi-hereditary structure on the positively graded algebra

$$\text{gr} A := \bigoplus \text{rad}^i A / \text{rad}^{i+1} A.$$  

For general quasi-hereditary algebras $A$, it is quite unexpected that $\text{gr} A$ should also be quasi-hereditary, though [16] demonstrated that this does occur in several situations of interest in the theory of quantum and algebraic groups.

In [3], the notion of an integral quasi-hereditary $\tilde{A}$ over a commutative ring $O$ was introduced. See §2 below for a summary. In this paper, $O$ is taken to be a DVR with fraction field $K$ and residue field $k = O/(\pi)$. We take up the question of when, for the integral quasi-hereditary algebra $\tilde{A}$ over $O$, the positively graded algebra

$$\text{gr} \tilde{A} := \bigoplus_{i \geq 0} \frac{\tilde{A} \cap \text{rad}^i \tilde{A}_K}{\tilde{A} \cap \text{rad}^{i+1} \tilde{A}_K}$$

is an integral quasi-hereditary algebra over $O$. The (integral) quasi-heredity of $\tilde{A}$ implies, for example, that the $K$-algebra $\tilde{A}_K := K \otimes_O \tilde{A}$ and the $k$-algebra $A = \tilde{A}_k := \tilde{A} \otimes_O k$ are both quasi-hereditary. We will always assume that $\tilde{A}$ is associated with a fixed weight poset $\Lambda$ indexing its irreducible modules, over either $K$ or $k$, and is “split” (see §2). The algebra $\tilde{A}$ provides a link between the representation theory of the $K$-algebra $\tilde{A}_K$ and the $k$-algebra $A$.

Of course, if $\text{gr} \tilde{A}$ is quasi-hereditary over $O$, then $\text{gr} \tilde{A}_K$ is also quasi-hereditary (with the same weight poset $\Lambda$ as $\tilde{A}$, and with standard modules $\text{gr} \tilde{\Delta}(\lambda)_K$)—already a difficult property to prove. We simply assume, as a starting point, that $\text{gr} \tilde{A}_K$ is quasi-hereditary. In fact, in the applications we have in mind, this assumption will be valid. The first main theorem, Theorem 4.17, establishes that $\text{gr} \tilde{A}$ is integral quasi-hereditary (with the same poset as $\tilde{A}$ and standard modules $\text{gr} \tilde{\Delta}(\lambda)$) if and only if each $\text{gr} \tilde{\Delta}(\lambda)$,
\[ \lambda \in \Lambda, \text{ has a simple head. Here} \]

\[ \operatorname{gr} \tilde{\Delta}(\lambda) := \bigoplus_{i \geq 0} \frac{\tilde{\Delta}(\lambda) \cap \text{rad}^i \tilde{\Delta}(\lambda)_K}{\Delta(\lambda) \cap \text{rad}^{i+1} \Delta(\lambda)_K}. \]

Corollary 4.15 gives conditions under which \( \tilde{\mathcal{A}} \)-modules \( \tilde{\mathcal{M}} \) with standard filtrations give rise to \( \operatorname{gr} \tilde{\mathcal{A}} \)-modules \( \operatorname{gr} \tilde{\mathcal{M}} \) with standard filtrations.

In §4, we show that the simple head hypothesis required in Theorem 4.17 holds, provided certain conditions involving an integral subalgebra \( \tilde{a} \) of \( \tilde{A} \) are satisfied\(^2\). The conditions are stated in Conditions 5.1 and the hypotheses of Theorem 5.3. The latter theorem is based on the (new) notion of a tight lattice (see Definition 2.2 just below). All this is pulled together in Corollary 5.4 and Theorem 5.5.

In §5, we consider the case in which the integral quasi-hereditary algebra \( \tilde{A} \) arises as a suitable “regular weight” homomorphic image of a quantum enveloping algebra at a root of unity. (The regular blocks of the famous \( q \)-Schur algebras provide examples of such algebras.) The first main result here is Theorem 6.1 which verifies that Conditions 5.1 hold in case \( p > h \) (the Coxeter number) and \( \tilde{A} \) is associated to a finite ideal of regular weights. Then the main result in this section, given in Theorem 6.3, shows that, when \( p \geq 2h - 2 \), \( \tilde{A} \) is a quasi-hereditary algebra with standard modules \( \operatorname{gr} \tilde{\Delta}(\Lambda) \). Its proof completes the verification process begun in Theorem 6.1 by checking that the hypotheses of Theorem 5.3 hold, so that all the results of §4 are available.

A key step in the proof of Theorem 6.1, namely, the verification of Condition 5.1(5), is postponed to Appendix II (§6), entitled “Quantum deformation theory over \( \mathcal{O} \).” The appendix is inspired by, and heavily uses, the deformation theory in the paper [2], in which Andersen-Jantzen-Sörgel showed that the (known to be true) Lusztig conjecture for quantum groups at a \( p \)th root of unity implies the characteristic \( p \) version of the conjecture for primes \( p \gg 0 \), depending on the root system. The paper [2] presents two parallel deformation theories, the first in positive characteristic for restricted Lie algebra representations, and the second for the small quantum group at a root of unity \( \zeta \). In our appendix, we extend the results of [2] to include a third deformation theory, for a natural integral form of the small quantum group over an appropriate DVR \( \mathcal{O} \subseteq \mathbb{Q}(\zeta) \) with \( \zeta \) a \( p \)th root of unity. We require that \( p \) be a prime \( > h \). The main consequence of this theory is the demonstration of a positive grading on the \( \mathcal{O} \)-integral form which base changes to the grading on the small (“regular block”) quantum group over \( \mathbb{Q}(\zeta) \) studied in [2], §17, §18. See Theorem 8.1 which can also be regarded as a main result of this paper. This is just what is needed to complete the

\(^1\)These \( \operatorname{gr} \)-constructions make sense for any integral domain and lattices for algebras over them, and, as far as we know, have not been previously studied.

\(^2\) The use of such a special subalgebra has been previously explored by us in [16], and this section is a natural extension of those results.
verification of Condition 5.1(5), as required in the proof of Theorem 6.1(c). Observe that [2, §18] is quoted earlier in checking Condition 5.1(1) (in “case 2” of [2]).

One area where the results of this paper are very useful is the study of filtrations of standard modules both for \( \tilde{A} \) and for \( A = \tilde{A}_k \). For example, an important question concerns the existence of \( p \)-filtrations of Weyl modules \( \Delta(\lambda) \) (even in the non-regular case) for a reductive group \( G \), i.e., filtrations with sections of the form \( L(\mu_0) \otimes \Delta(\mu_1) \) with \( \mu_0 \) a \( p \)-restricted dominant weight and \( \mu_1 \) an arbitrary dominant weight. See [1], though the terminology goes back earlier to an 1990 MSRI lecture of Donkin, and the concept already appears in the work of Jantzen [9]. Of course, anything that can be said about filtrations of standard modules for \( \text{gr} \, \tilde{A} \) and \( \text{gr} \, A \) can be phrased as statements about filtrations of standard modules for \( \tilde{A} \) and \( A \). The additional structure afforded by the gradings makes such filtration assertions even more interesting. Indeed, [17], which uses our results here to present new results on \( p \)-filtrations of Weyl modules for semisimple groups from this perspective, using tools unavailable in the ungraded (or non-integral) setting.

We also expect to use [17] and the Lusztig modular character conjecture to improve the Koszulity theory given in [16]. One unexpected feature of our results in the present paper is that the Lusztig modular conjecture is not required. We do use the root of unity quantum analog of the conjecture, but that is known to be a theorem for \( p > h \) [21].

2. Some preliminary notation/results

Let \((K, O, k)\) is a \( p \)-local system, i.e., \( O \) is a discrete valuation ring with maximal ideal \( m = (\pi) \), fraction field \( K \), and residue field \( k = O/m \). The field \( k \) is allowed to have arbitrary characteristic; in our applications it will have positive characteristic \( p \).

Let \( A_O \) be an \( O \)-algebra, which will always be assumed to be an \( O \)-free module of finite rank. Base change to \( K \) and \( k \) defines algebras \( A_K := K \otimes_O A_O \) and \( A_k := k \otimes_O A_O = A_O/\pi A_O \) over \( K \) and \( k \), respectively. It will be often convenient to denote \( A_O \) by \( \tilde{A} \) and denote \( A_k \) simply by \( A \). More generally, if \( \tilde{M} \) is an \( \tilde{A} \)-module, write \( \tilde{M}_K \) for \( K \otimes_O \tilde{M} \) and \( \tilde{M}_k \) for \( k \otimes_O \tilde{M} \). The \( \tilde{A} \)-module \( \tilde{M} \) is said to be a \( \tilde{A} \)-lattice if it is \( O \)-finite and torsion free. (This definition applies when \( O \) is any integral domain; see Reiner [18, pp. 44, 129].) Any \( O \)-finite \( \tilde{A} \)-submodule \( \tilde{M} \) of a finite dimensional \( \tilde{A}_K \)-module \( N \) is a \( \tilde{A} \)-lattice, and is said to be a full lattice in \( N \) if also \( K \tilde{M} = N \), in which case \( K \otimes_O \tilde{M} \cong N \). Let \( \tilde{A}\text{-mod} \) be the category of \( O \)-finite \( \tilde{A} \)-modules. The

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\[3\text{In the text, given a ring or algebra } R, \text{ an } R\text{-module } M \text{ is finite (or } R\text{-finite) if it is finitely generated as an } R\text{-module.} \]
category $\tilde{A}$-mod contains all $\tilde{A}$-lattices, and also contains $A$-mod as a full subcategory.

It will often be very convenient, for a non-negative integer $n$, to denote the ideal $\tilde{A} \cap \text{rad}^n \tilde{A}_K$ of $\tilde{A}$ by $\tilde{\text{rad}}^n \tilde{A}$. Here $\text{rad}^n \tilde{A}_K = (\text{rad} \tilde{A}_K)^n$, the $n$th power of the radical (= maximal nilpotent ideal) of the finite dimensional algebra $\tilde{A}_K$. Of course, if $n \geq 1$, $\tilde{\text{rad}}^n \tilde{A}$ is a nilpotent ideal in $\tilde{A}_K$ and $(\text{rad}^n \tilde{A})_K = \text{rad}^n \tilde{A}_K$. Define

\begin{equation}
\text{gr} \tilde{A} := \bigoplus_{n \geq 0} \frac{\tilde{A} \cap \text{rad}^n \tilde{A}_K}{\tilde{A} \cap \text{rad}^{n+1} \tilde{A}_K} = \bigoplus_{n \geq 0} \tilde{\text{rad}}^n \tilde{A} / \tilde{\text{rad}}^{n+1} \tilde{A}.
\end{equation}

Necessarily the graded algebra $\text{gr} \tilde{A}$ is a finite $\mathcal{O}$-free module and $(\text{gr} \tilde{A})_K \cong \text{gr} \tilde{A}_K = \bigoplus_{n \geq 0} \text{rad}^n \tilde{A}_K / \text{rad}^{n+1} \tilde{A}_K$.

A similar notation will be used for modules and lattices. For any $\tilde{A}$-lattice $\tilde{M}$ and non-negative integer $n$, let $\tilde{M} = \tilde{M} \cap \text{rad}^n \tilde{M}_K$ (where $\text{rad}^n \tilde{M}_K = (\text{rad}^n \tilde{A}_K)\tilde{M}_K$), and set

\begin{equation}
\text{gr} \tilde{M} := \bigoplus_{n \geq 0} \frac{\tilde{M} \cap \text{rad}^n \tilde{M}_K}{\tilde{M} \cap \text{rad}^{n+1} \tilde{M}_K} = \bigoplus_{n \geq 0} \tilde{\text{rad}}^n \tilde{M} / \tilde{\text{rad}}^{n+1} \tilde{M}.
\end{equation}

Then $\text{gr} \tilde{M}$ is a graded $\text{gr} \tilde{A}$-lattice, and

$$(\text{gr} \tilde{M})_K \cong \text{gr} \tilde{M}_K = \bigoplus_{n \geq 0} \text{rad}^n \tilde{M}_K / \text{rad}^{n+1} \tilde{M}_K.$$

There are one-to-one correspondences of isomorphism classes of irreducible modules:

\begin{equation}
\text{Irr}((\text{gr} \tilde{A})_K) \leftrightarrow \text{Irr}(\text{gr} \tilde{A}_K) \leftrightarrow \text{Irr}(\tilde{A}_K).
\end{equation}

Also, $\tilde{A} \cap \text{rad} \tilde{A}_K$ is an $\mathcal{O}$-pure nilpotent ideal in $\tilde{A}$. Hence,

\begin{equation}
\text{Irr}((\text{gr} \tilde{A})_k) \leftrightarrow \text{Irr}(\text{gr} \tilde{A}) \leftrightarrow \text{Irr}((\tilde{A}/(\text{rad} \tilde{A})_k)) \leftrightarrow \text{Irr}(\tilde{A}) \leftrightarrow \text{Irr}(A).
\end{equation}

The relation between the sets $\text{Irr}(A_K)$ and $\text{Irr}(A)$ may be subtle (or non-existent); however, in the next section, we consider a family of $\mathcal{O}$-algebras $\tilde{A}$ in which the irreducible $A$-modules correspond bijectively, up to isomorphism, to the irreducible $A_K$-modules. Further, there is an evident common indexing set $\Lambda$. If $\lambda \in \Lambda$, we will let $L_K(\lambda)$ (resp., $L(\lambda)$) be the corresponding irreducible $A_K$-(resp., $A$-) module.

The following general lemma which will be needed in §5.

**Lemma 2.1.** Suppose $a_K \subseteq b_K \subseteq A_K$ are finite dimensional $K$-algebras. Assume that every irreducible $A_K$-module is absolutely irreducible and a direct sum of irreducible $b_K$-modules, each of which is absolutely irreducible over $a_K$. Then

(a) $b_K \cap \text{rad} A_K = \text{rad} b_K$, and $a_K \cap \text{rad} A_K = a_K \cap \text{rad} b_K = \text{rad} a_K$. 

(b) The algebras $a_K$ and $b_K$ (as well as $A_K$) are split semisimple. Any Wedderburn complement $a_{K,0}$ of $a_K$ is contained in a Wedderburn complement $b_{K,0}$ of $b_K$, and every Wedderburn complement $b_{K,0}$ of $b_K$ is contained in a Wedderburn complement $A_{K,0}$ of $A_K$.

Proof. We first prove (a). Since $b_K \cap \text{rad} A_K$ is a nilpotent ideal in $b_K$, we have $b_K \cap \text{rad} A_K \subseteq \text{rad} b_K$. Since $\text{rad} b_K$ kills every irreducible $A_K$-module, we also have $\text{rad} b_K \subseteq b_K \cap \text{rad} A_K$. Thus, $b_K \cap \text{rad} A_K = \text{rad} b_K$. A similar argument shows that $a_K \cap \text{rad} A_K = \text{rad} a_K$.

Since $a_K$ and $b_K$ are subalgebras of $A_K$, all their irreducible modules can be found as composition factors of irreducible $A_K$-modules. Thus, all irreducible $a_K$-modules are absolutely irreducible, and all irreducible $b_K$-modules are absolutely irreducible. In particular, $a_K/\text{rad} a_K$ and $b_K/\text{rad} b_K$ are split semisimple algebras, as is $A_K/\text{rad} A_K$. By part (a) above, we have $a_K/\text{rad} a_K \subseteq b_K/\text{rad} b_K \subseteq A_K/\text{rad} A_K$. Now (b) follows from standard separability arguments (vanishing of Hochschild 1- and 2-cohomology).

The following general definition makes sense when $O$ is any integral domain, though we will apply it only in our DVR context.

**Definition 2.2.** Let $\tilde{b}$ be an $O$-finite and torsion free $O$-algebra, and let $\tilde{M}$ be a $\tilde{b}$-lattice. Call $\tilde{M}$ is tight if

$$(2.5) \quad \text{rad}^r \tilde{M} = (\text{rad}^r \tilde{b}) \tilde{M}, \quad \forall r \in \mathbb{N}.\quad (2.5)$$

Observe that, for any $\tilde{b}$-lattice $\tilde{M}$, the left-hand side always contains the right-hand side in (2.5), and that equality holds for $\tilde{M} = \tilde{b}$ or any finite projective $\tilde{b}$-module, which are, thus, tight lattices for $\tilde{b}$.

Finally, this section concludes with the following well-known elementary lemma which will often be used without comment. Part (a) is [18, Thm. 4.0]. Part (b) gives one of the many characterizations of purity in the case of a DVR. We leave the easy proof to the reader.

**Lemma 2.3.** Let $R$ be an integral domain, and let $\tilde{N}$ be a submodule of an $R$-finite, torsion free module $\tilde{M}$. (That is, $\tilde{M}$ is an $R$-lattice in the sense of [18] p. 44) Then

(a) $\tilde{N}$ is pure in $\tilde{M}$ (meaning that if $\tilde{M}/\tilde{N}$ is torsion free) if and only if $\tilde{N} = \tilde{M} \cap \tilde{N}_K$, where $K$ is the fraction field of $R$;

(b) if $R = O$ is a DVR with fraction field $K$, then $\tilde{N}$ is pure in $\tilde{M}$ if and only if the natural map $\tilde{N}_k \to \tilde{M}_k$ obtained by base-change is injective.

3. INTEGRAL QUASI-HEREDITARY ALGEBRAS

We maintain the general notation from above, unless otherwise noted. Assume that $\tilde{A}$ is a “split” quasi-hereditary algebra (QHA) over $O$ in the sense of [4] Defn. (3.2)]. This definition requires that there exists a sequence
0 = \bar{J}_0 \subset \bar{J}_1 \subset \cdots \subset \bar{J}_t = \tilde{A} \text{ of ideals in } \tilde{A} \text{ such that } \bar{J}_i/\bar{J}_{i-1} \text{ is an heredity ideal in } \bar{A}/\bar{J}_{i-1} \text{ for } 0 < i \leq t.\footnote{An ideal } \bar{J} \text{ is heredity provided that (i) } \bar{A}/\bar{J} \text{ is } \mathcal{O}\text{-free; (ii) } \bar{J}^2 = \bar{J}; (iii) } \bar{J} \text{ is a projective left } \bar{A}\text{-module; and (iv) } E = \text{End}_A(\bar{J}) \text{ is semisimple over } \mathcal{O}. \text{ The split case, used here, requires (over DVRs, such as } \mathcal{O}\text{) the stronger, but easier to state, condition that } E \text{ is a direct sum of full matrix algebras } M_n(\mathcal{O}) \text{ over } \mathcal{O}.

It follows that } \bar{A}_K\text{-mod and } A\text{-mod are highest weight categories (HWCs)}\footnote{See also [7] Prop. 2.1.2, Rem./Defn. 2.1.3, which explains the essential equivalence of the } \mathcal{O}\text{-finite highest weight category notion (in the finite weight poset case) with that of a split QHA (over a commutative Noetherian ring } \mathcal{O}, \text{ which is a DVR here). If } \bar{A}\text{-mod is a HWC with poset } \Lambda, \text{ we often informally say that } \bar{A} \text{ is a QHA with poset } \Lambda.\text{ The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.\text{\footnote{The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.}

\text{with the same poset } \Lambda \text{ indexing the (isomorphism classes of) irreducible modules } \bar{L}_K(\lambda) \text{ (resp., } L(\lambda)) \text{ for } \bar{A}_K \text{ (resp., } A). \text{ Indeed, by [7] Prop. 2.1.1], the category } \bar{A}\text{-mod is an } \mathcal{O}\text{-finite highest weight category in the sense of [7] Defn. 2.1}\footnote{The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.\text{\footnote{The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.} \text{. The assumption that } \bar{A} \text{ is “split” implies that the irreducible modules for } \bar{A}_K \text{ and } A \text{ are absolutely irreducible. Also, the category } \bar{A}\text{-mod has standard modules } \tilde{\Delta}(\lambda), \lambda \in \Lambda, \text{ which are } \bar{A}\text{-lattices with the property that the HWCs } \bar{A}_K\text{-mod (resp., } A\text{-mod) has standard objects } \{\Delta_K(\lambda) := \tilde{\Delta}(\lambda)k\}_{\lambda \in \Lambda} \text{ (resp., } \{\Delta(\lambda) = \Delta_k(\lambda) := \tilde{\Delta}(\lambda)k\}_{\lambda \in \Lambda}). \text{ Similarly, } \bar{A}\text{-mod has costandard objects } \nabla(\lambda), \lambda \in \Lambda, \text{ etc. Each irreducible } A\text{-module } L(\lambda) \text{ has a (unique, up to isomorphism) projective cover } \bar{P}(\lambda) \text{ in } \bar{A}\text{-mod; see [7] Prop. 2.3 or [4] p. 157}\footnote{The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.\text{\footnote{The arguments in [7] and [4] do not require that } \mathcal{O} \text{ be complete, or make any use of a completion of } \mathcal{O}.} \text{ and } \bar{P}(\lambda) \text{ has a } \tilde{\Delta}\text{-filtration (i.e., a filtration with sections isomorphic to } \tilde{\Delta}(\mu), \mu \in \Lambda). \text{ In this filtration, } \tilde{\Delta}(\lambda) \text{ appears as the top section, and all other sections } \tilde{\Delta}(\mu) \text{ have } \mu > \lambda. \text{ Clearly, } \bar{P}(\lambda) \text{ is a } \bar{A}\text{-lattice, and a direct summand of } \bar{A} \text{ (viewed as a module over itself). Moreover, the (left) } \bar{A}\text{-module } \tilde{\Delta}(\lambda) \text{ decomposes a direct sum of various copies of } \bar{P}(\lambda) \text{ (each appearing dim } L(\lambda) \text{ times).}

\text{Observe that } \text{gr } \bar{P}(\lambda) \text{ is the projective cover of } L(\lambda) \text{ in either gr } \bar{A}\text{-mod (ungraded module category) or in gr } \bar{A}\text{-grmod (graded module category). This claim follows from dimension considerations, since (} \text{gr } \bar{P}(\lambda))_0, \text{ as a nonzero quotient of } \bar{P}(\lambda), \text{ has } L(\lambda) \text{ as a homomorphic image. (Note head } \bar{A} \cong \text{ head (} \text{gr } \bar{A})_0 \cong \text{ head } \bar{A}.\text{)}}

\text{While } \bar{P}(\lambda)_K \text{ is the projective cover } P(\lambda) \text{ in } A\text{-mod of } L(\lambda), \text{ it is not generally true that } P(\lambda)_K = \bar{P}(\lambda)_K \text{ is the projective cover } P_K(\lambda) \text{ of } \bar{L}_K(\lambda) \text{ in } \bar{A}_K\text{-mod. However, we do have}

(3.1) \quad \bar{P}(\lambda)_K \cong P_K(\lambda) \oplus \bigoplus_{\mu > \lambda} P_K(\mu)^{\oplus m_{\lambda, \mu}},
for non-negative integers $m_{\lambda,\mu}$. This follows since $\Delta_K(\lambda)$ is a quotient of $\tilde{P}(\lambda)_K$, and since all other standard modules $\Delta_K(\mu)$ appearing in any $\Delta_K$-filtration of $\tilde{P}(\lambda)_K$ have weights $\mu > \lambda$. Applying the functor $\text{gr}$ to (3.1) gives a similar decomposition of $\text{gr}(\tilde{P}(\lambda)_K) = (\text{gr}\tilde{P}(\lambda))_K$, although we, as yet, know little about $\text{gr}\Delta_K$-filtrations of $\text{gr}(\tilde{P}(\lambda))_K$ or of $P_K(\lambda)$. This issue will be addressed in Hypothesis 4.7 below.

4. A MAIN RESULT

This section approaches the question of determining conditions on an integral quasi-hereditary algebra $\tilde{A}$ over $\mathcal{O}$ which guarantee that the graded algebra $\text{gr } \tilde{A}$ as defined in (2.1) is quasi-hereditary. The first step is to find some properties $\tilde{A}$ and $\text{gr } \tilde{A}$ might “obviously” share, at least in common situations. To this end we introduce a general framework of “weight algebras”. In this framework we can replace the DVR $\mathcal{O}$ by a general commutative ring $R$. If $\mathfrak{p} \in \text{Spec } R$ and if $M$ is an $R$-module, let $M_\mathfrak{p}$ denote the $R_\mathfrak{p}$-module obtained by localizing $M$ at $\mathfrak{p}$. Finally set $M(\mathfrak{p}) := M_\mathfrak{p}/pM_\mathfrak{p}$, a vector space over the residue field $R(\mathfrak{p}):= R_\mathfrak{p}/pR_\mathfrak{p}$.

Let $B$ be an $R$-finite algebra. We call $B$ a weight algebra, if there is a set $\{e_\nu | \nu \in X\}$ of orthogonal idempotents in $B$, summing to 1, such that, for each $\mathfrak{p} \in \text{Spec } R$, some subset $X_\mathfrak{p}$ of $X$ bijectively indexes the irreducible $B_\mathfrak{p}$-modules in such a way that, if $\nu \in X_\mathfrak{p}$ corresponds to the irreducible $B_\mathfrak{p}$-module $L_\mathfrak{p}(\nu)$, then $e_\nu L_\mathfrak{p}(\nu) \neq 0$. Somewhat loosely, we refer to $X$ as the set of “weights” of $B$. If $M$ is a $B$-module and $\nu \in X$, we write $M_\nu := e_\nu M$, the $\nu$-weight space of $M$.

In this paper, we only consider weight algebras $B$ as above for which the sets $X_\mathfrak{p}$ can be uniformly chosen to be the same subset $\Lambda$ of $X$. In this case, we say $B$ is $\Lambda$-uniform. The weights in $X \setminus \Lambda$ then becomes largely irrelevant. In particular, we note the following.

**Proposition 4.1.** Suppose that $B$ is a $\Lambda$-uniform weight algebra with weight set $X$. Then $B$ is Morita equivalent to a $\Lambda$-uniform weight algebra with weight set $\Lambda$.

**Proof.** Put $P = \bigoplus_{\lambda \in \Lambda} Be_\lambda$. Then $P$ is a projective generator because of the non-vanishing conditions $e_\lambda L_\mathfrak{p}(\lambda) \neq 0$ above. Thus, $B' := \bigoplus_{\lambda, \mu \in \Lambda} e_\lambda B_\mu \cong (\text{End}_B P)^{\text{op}}$ is $R$-finite as a module and is Morita equivalent to $B$. The idempotents $e_\lambda, \lambda \in \Lambda$, all belong to $B'$, and sum to the identity in $B'$. It is easily checked that these idempotents, indexed by $\Lambda$, give $B'$ the structure of a $\Lambda$-uniform weight algebra with weight set $\Lambda$. \qed

Some useful concepts can be defined in the generality of $\Lambda$-uniform weight algebras $B$. Let $\Gamma$ be a nonempty subset of $\Lambda$. Let $N$ be a finite $B$-module. Let $N_\Gamma$ be the largest quotient module for which, given $\nu \in \Lambda$, $e_\nu N_\Gamma \neq 0$ implies $\nu \in \Gamma$. Explicitly, $N_\Gamma = N/\sum_{\nu \in \Lambda \setminus \Gamma} Be_\nu N$. Considering $B$ as a left module over itself, it follows that $B_\Gamma$ is an $R$-algebra, and $N_\Gamma$ is a $B_\Gamma$-module. The notation agrees with the terminology at the end of Example
2.1. In case $B = \tilde{A}$ is a split QHA over $\mathcal{O}$ with weight poset $\Lambda$, then, in case $\Gamma$ is an ideal in $\Lambda$, $\tilde{A}_\Gamma$ is an integral QHA with poset $\Gamma$ and standard modules $\tilde{A}(\gamma)$, $\gamma \in \Gamma$ (the standard modules of $\tilde{A}$ indexed by elements in $\Gamma$).

The following definition introduces an even stronger property for weight algebras which will hold in our applications. The definition is somewhat redundant, in that conditions (1) and (2) on the set $\Lambda$ of weights for algebras which will hold in our applications. The definition is somewhat "primitive" used there. Notice, if $v$ be maximal among $\mu$ such that, for all $p \in \text{Spec} R$, and all $\lambda, \mu \in \Lambda$, if $L_p(\lambda) \neq 0$, then $\mu \leq \lambda$.

When $R = \mathcal{O}$, then $\text{Spec} \mathcal{O} = \{m, 0\}$, with $\mathcal{O}(0) = K$ and $\mathcal{O}(m) = k$.

The following lemma, which applies for any Noetherian integral domain $R$, is one of the main points of the “weight algebra” development of this section. The proof is essentially obvious, noting that the ideal $B \cap \text{rad} B_K$ and its $gr \ B$ counterpart are both nilpotent ideals. The Noetherian hypotheses on $R$ insures that $gr \ B$ is finite $B$-module.

**Lemma 4.3.** Assume that $B$ is a weight algebra over a Noetherian integral domain $R$ with fraction field $K$, and put $gr \ B = \bigoplus_{n \geq 0} \frac{-\text{rad}^n B}{\text{rad}^{n+1} B}$. If $\{e_\mu | \mu \in X\}$ is a set of idempotents giving $B$ the structure of a weight algebra over $R$, the “same” set $\{e_\mu | u \in X\} \subseteq (gr \ B)_0 \subseteq gr \ B$ gives $gr \ B$ the structure of a weight algebra. If $\Lambda$ is a subset of $X$, then $B$ is $\Lambda$-standard if and only if $gr \ B$ is $\Lambda$-standard.

Now return to the situation in which $R = \mathcal{O}$ is a DVR as above. Let $\tilde{A}$ be a $\Lambda$-standard weight algebra over $\mathcal{O}$, and suppose that $\tilde{N}$ is an $\tilde{A}$-lattice. If $\lambda \in \Lambda$, let $\tilde{N}_\lambda^{\prime} (\lambda)$ denote the $\tilde{A}_\lambda$-submodule of $\tilde{N}_\lambda$ generated by all $\mu$-weight vectors in $\tilde{N}_\lambda$ with $\mu \in \Lambda$ and $\mu > \lambda$. An element $v \in \tilde{N}_\lambda$ is called $\lambda$-primitive\(^7\) in $\tilde{N}$ if the image of $\mathcal{O} v$ in $\tilde{N}/\tilde{N} \cap \tilde{N}_\lambda^{\prime} (\lambda)$ is pure and nonzero. (Equivalently,\(^8\) $v$ has nonzero image in $(\tilde{N}/\tilde{N} \cap \tilde{N}_\lambda^{\prime} (\lambda))_k$)

We say that $v \in \tilde{N}_\lambda$ is strongly $\lambda$-primitive if $v$ represents a homogeneous element $[v]$ in $\text{gr} \tilde{N}$ such that the image of $\mathcal{O} [v]$ in $\text{gr} \tilde{N}/\tilde{N} \cap \tilde{N}_\lambda^{\prime} (\lambda)$ is pure.
and nonzero. (Equivalently, \([v]\) has nonzero image in \((\text{gr } \overline{N}/\overline{N} \cap \tilde{N}'_K(\lambda))_k\).

If \(v\) is strongly \(\lambda\)-primitive, then it is \(\lambda\)-primitive; see the proposition below. Necessarily, \(v\) and, of course, \([v]\) are non-zero. If \(v\) is strongly \(\lambda\)-primitive, we also say that \([v]\) is strongly \(\lambda\)-primitive.

An element \(v \in \overline{N}\) is said to be primitive (resp., strongly primitive) if it is \(\lambda\)-primitive (resp., strongly \(\lambda\)-primitive) for some \(\lambda\). The astute reader will observe that “primitive” can be defined for \(\text{gr } \overline{N}\) or any \(\text{gr } \tilde{A}\)-lattice, and that every strongly primitive element of \(\text{gr } \overline{N}\) is both primitive and homogeneous.

For some well-behaved lattices, the notions coincide; that is, the primitive homogeneous elements of \(\text{gr } \overline{N}\) are strongly primitive. See Remark \[1.18(c)\]. It is useful to use the technical translation of the “strongly primitive” notion in (b) of the result below.

**Proposition 4.4.** Let \(\tilde{A}\) be a \(\Lambda\)-standard weight algebra over \(\mathcal{O}\) and let \(\overline{N}\) be a \(\tilde{A}\)-lattice. Let \(\lambda \in \Lambda\) and \(v \in \tilde{N}_\lambda\).

(a) The element \(v\) is \(\lambda\)-primitive if and only if \(v \not\in \overline{N} \cap \tilde{N}'_K(\lambda)\) and \(\mathcal{O}v + (\overline{N} \cap \tilde{N}'_K(\lambda))\) is \(\mathcal{O}\)-pure.

(b) The element \(v\) is strongly \(\lambda\)-primitive if and only if there is a non-negative integer \(i\) such that \(v \in \overline{\text{rad}}^i \overline{N}, v \not\in \overline{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{N}'_K(\lambda))\), and \(\mathcal{O}v + \overline{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{N}'_K(\lambda))\) is \(\mathcal{O}\)-pure in \(\overline{N}\).

(c) If \(v\) is strongly \(\lambda\)-primitive, it is \(\lambda\)-primitive.

**Proof.** The element \(v \in \tilde{N}_\lambda\) is \(\lambda\)-primitive if and only if the image of \(\mathcal{O}v\) in \(\overline{N}/\overline{N} \cap \tilde{N}'_K(\lambda)\) is nonzero and pure, i. e., if and only if \(v \not\in \overline{N} \cap \tilde{N}'_K(\lambda)\), and the quotient

\[
\frac{\overline{N}}{\overline{N} \cap \tilde{N}'_K(\lambda)}/\frac{\mathcal{O}v + (\overline{N} \cap \tilde{N}'_K(\lambda))}{\overline{N} \cap \tilde{N}'_K(\lambda)}
\]

is torsion free. Equivalently,

\[
\overline{N}/(\mathcal{O}v + (\overline{N} \cap \tilde{N}'_K(\lambda))
\]

is torsion free, which just means that \(\mathcal{O}v + (\overline{N} \cap \tilde{N}'_K(\lambda))\) is pure in \(\overline{N}\). This proves (a).

We now prove (b). The element \(v \in \tilde{N}_\lambda\) represents a homogeneous element \([v]\) of grade \(i\) in \(\text{gr } \overline{N}\) with nonzero image in \(\text{gr } (\overline{N}/\overline{N} \cap \tilde{N}'_K(\lambda))\) if and only if \(v \in \overline{N} \cap \text{rad}^i \tilde{N}_K\) and \(v \not\in \text{rad}^{i+1} \tilde{N}_K + \tilde{N}'_K(\lambda)\) for some (uniquely determined) \(i\). Purity of the image of \(\mathcal{O}[v]\) in \((\text{gr } (\overline{N}/\overline{N} \cap \tilde{N}'_K(\lambda)))\) is equivalent to the purity of \(\mathcal{O}[v]\) in \((\text{gr } (\overline{N}/(\overline{N} \cap \tilde{N}'_K(\lambda))))\) for this integer \(i\). The is equivalent to the purity of the sum \(\mathcal{O}v + (\overline{N} \cap (\tilde{N}'_K(\lambda) + \text{rad}^{i+1} \tilde{N}_K))\) in \(\overline{N}\). This proves (b).

Finally, to see (c), assume that \(v\) is strongly \(\lambda\)-primitive, and choose the index \(i\) as in the proof of (b) above. Thus, \(v \not\in \tilde{F} := (\overline{N} \cap (\tilde{N}'_K(\lambda) + \text{rad}^{i+1} \tilde{N}_K)), and \(\mathcal{O}v + \tilde{F}\) is pure in \(\overline{N}\). Put \(\tilde{E} := (\overline{N} \cap (\tilde{N}'_K(\lambda)) \subseteq \tilde{F}\). Then
\(Ov \cap \tilde{F} = 0\), so that the quotient
\[
(Ov + \tilde{F})/(Ov + \tilde{E}) \cong \tilde{F}/(\tilde{F} \cap (Ov + \tilde{E})) = \tilde{F}/\tilde{E}
\]
is torsion free. It follows that \(\tilde{N}/(Ov + \tilde{E})\) is torsion free, which implies, by (a), that \(v\) is \(\lambda\)-primitive. This proves (c). \(\square\)

The gr \(\tilde{A}\)-submodule of gr \(\tilde{N}\) generated by all of its strongly primitive elements is denoted gr \(\grave{\lambda}\tilde{N}\) (or by gr \(\grave{\lambda}\tilde{N}\) to emphasize the dependence on \(\tilde{A}\)). Since strongly primitive elements in gr \(\tilde{N}\) are, by definition, homogeneous, the gr \(\tilde{A}\)-module gr \(\grave{\lambda}\tilde{N}\) is a graded submodule of gr \(\tilde{N}\).

Now suppose that \(\tilde{A}\) is a split integral QHA over \(O\) with poset \(\Lambda\). We call \(\tilde{A}\) a toral quasi-hereditary (TQHA) if it is a \(\Lambda\)-standard weight algebra over \(O\), using the given poset structure on \(\Lambda\). Many integral QHAs come naturally equipped with such a structure. In any case, it can usually be assumed, by passing to a Morita equivalent algebra. In particular, we note the following.

**Proposition 4.5.** Suppose that \(\tilde{A}\) is a split integral QHA over \(O\) with poset \(\Lambda\). Then \(\tilde{A}\) is Morita equivalent to a TQHA \(\tilde{B}\) over \(O\). In addition, the algebras gr \(\tilde{A}\) and gr \(\tilde{A}_K\) are Morita equivalent to the algebras gr \(\tilde{B}\) and gr gr \(\tilde{B}_K\), respectively.

**Proof.** Let \(\tilde{P}(\lambda)\) be the PIM (projective indecomposable module) associated to \(\lambda \in \Lambda\), and put \(\tilde{P} := \bigoplus_{\lambda \in \Lambda} \tilde{P}(\lambda)\). Clearly, \(\tilde{P}\) is a finite projective generator for \(\tilde{A}\), so that \(\tilde{E} := (\text{End}_{\tilde{A}}(\tilde{P}))^{\text{op}}\) is Morita equivalent to \(\tilde{A}\). If we take \(\tilde{P}(\lambda) = A e \lambda\) for orthogonal idempotents \(\{e_\lambda\}_{\lambda \in \Lambda}\), then \(\tilde{E}\) identifies with the algebra \(\bigoplus_{\lambda, \mu \in \Lambda} e_\lambda e_\mu\). With this identification, \(\{e_\lambda\}_{\lambda \in \Lambda} \subseteq \tilde{E}\) is a set of orthogonal idempotents summing to the identity \(e := \sum e_\lambda\) of \(\tilde{E}\). It is easily checked that these idempotents give \(\tilde{E} = e\tilde{A}e\) the structure of a \(\Lambda\)-standard weight algebra. (Use the equivalence \(M \mapsto eM\) from \(\tilde{A}\)-mod to \(\tilde{E}\)-mod.)

Since \(\tilde{A}e = \tilde{P}\) is a projective generator for \(\tilde{A}\), we have \(\tilde{A}\tilde{e}\tilde{A} = \tilde{A}\). This implies that
\[
(e \text{ rad} \tilde{A}_K e)^n = e \text{ rad}^n \tilde{A}_K e, \quad n \in \mathbb{N},
\]
from which the remaining Morita equivalences can be deduced. \(\square\)

**Lemma 4.6.** Let \(\tilde{A}\) be a TQHA over \(O\). For \(\lambda \in \Lambda\), the standard \(\tilde{A}\)-module \(\tilde{\Delta}(\lambda)\) is generated by its \(\lambda\)-weight space \(\tilde{\Delta}(\lambda)_\lambda := e_\lambda \tilde{\Delta}(\lambda) = \tilde{\Delta}(\lambda) \cap \tilde{\Delta}_K(\lambda)_\lambda\).

**Proof.** The head of \(\tilde{\Delta}(\lambda)\) is isomorphic to \(L(\lambda)\). Also, \(\tilde{\Delta}(\lambda)\) is \(O\)-free of rank 1. The lemma follows from Nakayama’s lemma. \(\square\)

\(^9\)The set of strongly primitive elements of gr \(\tilde{N}\) does not depend on \(\tilde{A}\), so long as \(\tilde{N}\) is an \(\tilde{A}\)-lattice. (So, for example, we could use instead a quotient of \(\tilde{A}\) by an \(O\)-pure ideal acting trivially on \(\tilde{N}\).) However, the gr \(\tilde{A}\)-submodule generated by these strong primitive elements might depend on gr \(\tilde{A}\), and, thus, on \(\tilde{A}\). (The dependence may sometimes be eliminated with strong hypotheses on \(\tilde{N}\); see Remark 4.18(b).)
Hypotheses 4.7. (1) \( \tilde{A} \) is a TQHA over \( \mathcal{O} \) with poset \( \Lambda \).

(2) The graded \( K \)-algebra \( \text{gr} \tilde{A}_K \) is a QHA with the same weight poset \( \Lambda \) as \( \tilde{A}_K \). For each \( \lambda \in \Lambda \), the standard modules in the HWC \( \text{gr} \tilde{A}_K \)-mod of not necessarily graded \( \text{gr} \tilde{A}_K \)-modules are the modules \( \text{gr} \Delta_K(\lambda) \).

Lemma 4.8. Assume that Hypothesis 4.7(1) holds. Let \( \tilde{P} \) be projective in \( \tilde{A} \)-mod. Then \( \text{gr}^b \tilde{P} = \text{gr} \tilde{P} \).

Proof. First, it is useful to note that every finite projective \( \tilde{A} \)-module \( \tilde{P} \) has the property that the positive grade terms of \( \text{gr} \tilde{P} \) are contained in \( \text{rad}(\text{gr} \tilde{P}) \), the intersection of all maximal submodules. This follows from the case of \( \tilde{P} = \tilde{A} \) and the nilpotence of the ideal of positive grade terms. In particular, \( \tilde{P} \) and \( \tilde{P} \) have the same head. See also the discussion above [24].

Now to prove the lemma, it suffices to treat the case \( \tilde{P} = \tilde{P}(\lambda), \lambda \in \Lambda \). The map \( \tilde{P}(\lambda)_\lambda \to \tilde{\Delta}(\lambda)_\lambda \) is surjective. So Lemma 4.6 implies there exists an element \( v \in \tilde{P}(\lambda)_\lambda \) whose image in \( \tilde{\Delta}(\lambda)_\lambda \) is a generator for \( \tilde{\Delta}(\lambda) \).

The image of \( v \) in \( L(\lambda) = \text{head} \tilde{\Delta}(\lambda) \) is nonzero, as is the image of \( [v] \) in \( L(\lambda) = \text{head} \text{gr} \tilde{P}(\lambda) \). Thus, \( [v] \) generates \( \text{gr} \tilde{P}(\lambda) \) by Nakayama. It is, thus, sufficient to show that \( v \) is strongly \( \lambda \)-primitive. However, the image of \( v \) in \( \tilde{P}(\lambda)/\tilde{P}(\lambda) \cap (\text{rad} \tilde{P}(\lambda)_K + \tilde{P}(\lambda)_K^\#(\lambda)) \) is an \( \tilde{A} \)-generator of the latter nonzero module, hence generates an \( \mathcal{O} \)-pure submodule.

Assuming Hypothesis 4.7(1), \( \tilde{\Delta}(\lambda) \) is generated by a strongly \( \lambda \)-primitive element \( v_\lambda \in \tilde{\Delta}(\lambda)_\lambda \) and there are no \( \mu \)-primitive elements in \( \tilde{\Delta}(\lambda) \) for \( \mu \neq \lambda \). Thus, \( \text{gr}^b \tilde{\Delta}(\lambda) \) is generated by \( [v_\lambda] \in \text{gr} \tilde{\Delta}(\lambda) \), but we do not, in general, know that \( \text{gr} \Delta(\lambda) = \text{gr}^b \Delta(\lambda) \), although this does hold if \( \lambda \) is maximal in \( \Lambda \), by Lemma 4.8.\(^\text{10}\) In favorable cases—see Lemma 4.10—if \( \tilde{R} \) is a pure \( \tilde{A} \)-submodule of \( N \), the map \( \text{gr} \tilde{N} \to \text{gr} \tilde{N}/\tilde{R} \) induces a surjection \( \text{gr}^b \tilde{N} \to \text{gr}^b \tilde{N}/\tilde{R} \), though we do not have independent conditions which guarantee surjectivity for \( \text{gr} \tilde{N} \to \text{gr} \tilde{N}/\tilde{R} \). (The best we have is Corollary 4.16 whose proof uses the surjection \( \text{gr}^b \tilde{N} \to \text{gr}^b \tilde{N}/\tilde{R} \) of Lemma 4.11 and requires additional hypotheses.

An immediate consequence of the discussion above is the following result.

Lemma 4.9. Assume Hypothesis 4.7(1). For \( \lambda \in \Lambda \), \( \text{gr} \tilde{\Delta}(\lambda) \) has a simple head if and only if \( \text{gr}^b \tilde{\Delta}(\lambda) = \text{gr} \Delta(\lambda) \).

The next two results are technical results needed in Lemma 4.12.

\(^{10}\) We cannot just reduce to the maximal case without changing the algebra \( \text{gr} \tilde{A} \) that is acting on \( \text{gr} \tilde{\Delta}(\lambda) \). This is a subtle point, since the physical graded \( \mathcal{O} \)-module structure of the latter \( \text{gr} \tilde{A} \)-module remains the same, if we attempt such a reduction. However, the submodule generated by a given vector might change, as \( \text{gr} \tilde{A} \) changes.
Lemma 4.10. Assume Hypothesis (4.7(1)). Suppose \( \tilde{N} \) is an \( \tilde{A} \)-lattice, and \( \tilde{R} \) is an \( O \)-pure \( \tilde{A} \)-submodule. \( \lambda \in \Lambda, \) maximal with respect to \( \tilde{N}_\lambda \neq 0, \) such that the following conditions hold.

1. \( \tilde{R} \subseteq \tilde{A}_K \tilde{N}_\lambda. \)
2. \( \tilde{R}_\lambda + (\tilde{N} \cap \text{rad}^i \tilde{N}_K)_\lambda \) is pure in \( \tilde{N} \) (or, equivalently, in \( \tilde{N}_\lambda \)), for all \( i \in \mathbb{N}. \)
3. Let \( j_\lambda \) be the largest index \( j \) such that \( \tilde{N}_\lambda \subseteq \text{rad}^j \tilde{N}_K. \) Then \( \tilde{N}_\lambda = \tilde{R}_\lambda \bigoplus (\text{rad}^{j_\lambda} \tilde{N}_\lambda) \)

Then the map \( \text{gr} \tilde{N} \to \text{gr} \tilde{N} / \tilde{R} \) induces a surjection \( \text{gr}^b \tilde{N} \to \text{gr}^b (\tilde{N} / \tilde{R}). \)

Proof. We first prove that each strongly primitive element in \( \text{gr}^b \tilde{N} / \tilde{R} \) is the image of a strongly primitive element in \( \text{gr} \tilde{N}. \)

Let \( [v] \in (\text{gr} \tilde{N} / \tilde{R})_i \) be such a strongly primitive element, represented (through some abuse of notation) by \( v \in \tilde{N}. \) We take \([v] \) to be \( \mu \)-strongly primitive for some \( \mu \in \Lambda. \) Thus, for some grade \( i \), \( v \in \tilde{N} \cap (\text{rad}^i \tilde{N}_K + \tilde{R}_K), \) the \( O \)-module \( Ov + \tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{R}_K + \tilde{N}'(\mu)_K) \) is pure, and does not collapse to the right hand summand. Without loss, \( v \in \tilde{N}_\mu \neq 0. \) Since \( \lambda \) is maximal with \( \tilde{N}_\lambda \neq 0, \) there are three cases to consider.

Case 1: \( \lambda > \mu \)

Using condition (1), we have that \( \tilde{R}_K \subseteq \tilde{A}_K \tilde{N}_K \subseteq \tilde{N}'(\mu)_K. \)

Thus, \( \tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{R}_K + \tilde{N}'(\mu)_K) = \tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{N}'(\mu)_K). \)

Thus, the image of \( v \in (\text{gr} \tilde{N})_i \) is \( \mu \)-strongly primitive, and maps to \([v]\) in \((\text{gr} \tilde{N} / \tilde{R})_i. \)

Case 2: \( \mu \neq \lambda \) and \( \mu \neq \lambda. \)

Then \( (\tilde{N}'(\mu)_K)_\lambda = 0, \) so that, using condition (1), we get

\[
\tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{R}_K + \tilde{N}'(\mu)_K)_\mu = (\tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{N}'(\mu)_K)_\mu
\]

so that, again, \( Ov + \tilde{N} \cap (\text{rad}^{i+1} \tilde{N}_K + \tilde{N}'(\mu)_K) \) is pure and does not collapse to the right-hand summand. Again, the image of \( v \) in \((\text{gr} \tilde{N})_i \) is strongly primitive, and maps to \([v]\) in \((\text{gr} \tilde{N} / \tilde{R})_i. \)

Case 3: \( \mu = \lambda \)

Here \( \tilde{N}'(\mu) = \tilde{N}'(\lambda) = 0 \) by the maximality of \( \lambda. \) For each \( j \in \mathbb{N}, \) \( \tilde{R}_\lambda + (\text{rad}^j \tilde{N})_\lambda \) is a lattice in \( (\tilde{R}_K + \text{rad}^j \tilde{N}_K)_\lambda, \) and is pure in \( \tilde{N} \) by condition (2). Hence,

\[
(\tilde{N} \cap (\text{rad}^j \tilde{N}_K + \tilde{R}_K))_\lambda = \tilde{R}_\lambda + (\text{rad}^j \tilde{N})_\lambda, \quad j \in \mathbb{N}.
\]

We claim that \( i > j_\lambda. \) If not, then \( \tilde{N}_\lambda \subseteq \tilde{R}_\lambda + (\tilde{N} \cap \text{rad}^{i+1} \tilde{N}_K) \) by condition (3). Thus,

\[
Ov + (\tilde{R}_\lambda + \tilde{N} \cap \text{rad}^{i+1} \tilde{N}_K) = \tilde{R}_\lambda + \tilde{N} \cap \text{rad}^{i+1} \tilde{N}_K,
\]

contradicting the strong primitivity of \([v]\) in \( \text{gr} \tilde{M} / \tilde{R}, \) taking \( j = i \) in the previous display.

\footnote{It can be shown that (3) \( \implies \) (2).}
Thus, we may assume that $i > j$. Notice $[v] = [v']$ if $v' \in v + \tilde{R}_\lambda$ and, of course, the expression $Ov + (\tilde{R} + \text{rad}^{i+1} \cap \tilde{N}_K)$ remains unchanged if $v'$ replaces $v$. Using condition (3), we may now assume that $v \in \tilde{N}_\lambda \cap \text{rad}^{j+1} \tilde{N}_K$.

$$
(\tilde{N}_\lambda \cap \text{rad}^{j+1} \tilde{N}_K) \cap (Ov + \tilde{R} + \tilde{N} \cap \text{rad}^{i+1} \tilde{N}_K)_{\lambda} \\
= Ov + \tilde{R}_\lambda \cap \text{rad}^{j+1} \tilde{N}_K + \tilde{N}_\lambda \cap \text{rad}^{i+1} \tilde{N}_K \\
= Ov + (\tilde{N} \cap \text{rad}^{i+1} \tilde{N}_K).
$$

So the image of $v$ in $(\text{gr} \tilde{N})_i$ is strongly $\lambda$-primitive, and maps to to $[v] \in (\text{gr} \tilde{N}/\tilde{R})_i$. This completes the proof in all cases that $[v]$ is the image of a strongly primitive element of $\text{gr} \tilde{N}$.

It remains to prove that the image of $\text{gr}^b \tilde{N}$ in $\text{gr} \tilde{N}/\tilde{R}$ is contained in $\text{gr}^b \tilde{N}/\tilde{R}$. For this it is sufficient to show that the image in $\text{gr} \tilde{N}/\tilde{R}$ of a strongly primitive element in $\text{gr} \tilde{N}$ is either strongly primitive or zero.

Suppose that $u \in \tilde{N}_\mu \cap \text{rad}^i \tilde{N}_K$ represents a strongly primitive element $[u]$ in $(\text{gr} \tilde{N})_i$, and suppose that the image of $[u]$ in $(\text{gr} \tilde{N}/\tilde{R})_i$ is not zero. There are again three cases, as above, depending on the relationship of $\mu$ to $\lambda$.

If $\mu \neq \lambda$, we may use the formulas developed in Cases 1 and 2 above to prove the image of $[u]$ in $\text{gr} (\tilde{N}/\tilde{R})$ is strongly primitive. If $\mu = \lambda$ and $i > j$, condition (3) implies the image of $[u]$ in $\text{gr} \tilde{N}/\tilde{R}$ is zero, as is, essentially argued in the discussion of Case 3. This completes the proof of the lemma.

A prestandard module of weight $\lambda \in \Delta$ is a graded $\tilde{A}$-module $\tilde{D}(\lambda)$ which is submodule of $\text{gr} \tilde{D}(\lambda)$, and which satisfies the condition that $(\text{gr} \tilde{D}(\lambda))_{\lambda} = \tilde{D}(\lambda)_{\lambda}$. This implies that $\tilde{D}(\lambda)$ is a full $\tilde{A}$-lattice in $\text{gr} \tilde{D}(\lambda)$. Equivalently, $\tilde{D}(\lambda)$ is a graded $\tilde{A}$-module satisfying $\text{gr}^b \tilde{D}(\lambda) \subseteq \tilde{D}(\lambda) \subseteq \text{gr} \tilde{D}(\lambda)$. For example, both $\text{gr}^b \tilde{A}(\lambda)$ and $\text{gr} \tilde{A}(\lambda)$ are prestandard modules of weight $\lambda$. The notation $\text{gr}^b \tilde{A}(\lambda)$ depends on the algebra $\tilde{A}$. Thus, if $\tilde{A}' = \tilde{A}/\tilde{I}$ is a natural quasi-hereditary quotients associated with the poset ideal $\Gamma$ containing $\lambda$, then $\text{gr} \tilde{D}(\lambda)$ is a module for $\text{gr} \tilde{A}'$, but the analogue $\text{gr}^b \tilde{A}'(\lambda)$ of $\text{gr}^b \tilde{D}(\lambda)$ constructed for $\tilde{A}'$, rather than $\tilde{A}$, may be larger (because the map $\tilde{A} \rightarrow \text{gr} \tilde{A}'$ may not be surjective). However, if $\tilde{D}(\lambda) \subseteq \tilde{D}(\lambda)_{\lambda}$ is prestandard with respect to $\text{gr} \tilde{A}$, and is also a $\tilde{A}'$-module, then it remains prestandard and still satisfies the required sandwich property. We have

$$
\text{gr}^b \tilde{A}(\lambda) \subseteq \text{gr}^b \tilde{A}'(\lambda) \subseteq \tilde{D}(\lambda) \subseteq \text{gr} \tilde{A}'(\lambda) = \text{gr} \tilde{D}(\lambda).
$$

This is because $\text{gr}^b \tilde{A}'(\lambda)$ is generated over $\text{gr} \tilde{A}'$ by $(\text{gr} \tilde{D}(\lambda))_{\lambda}$. In particular, $\text{gr}^b \tilde{A}'(\lambda)$ is itself prestandard. Later, we introduce much stronger
hypotheses which guarantee that gr$^b\Delta(\lambda) = \text{gr}\Delta(\lambda)$, so that all of this structure collapses. But, for now, we will keep track of it.

**Lemma 4.11.** Assume Hypothesis 4.7(1). Suppose that $\tilde{E}$ is a graded gr$\bar{A}$-module satisfying

$$\bigoplus_{i=1}^{n} \text{gr}^b\tilde{\Delta}(\lambda_i)(s_i) \subseteq \tilde{E} \subseteq \bigoplus_{i=1}^{n} \text{gr}\Delta(\lambda_i)(s_i)$$

for $\lambda_1, \cdots, \lambda_n \in \Lambda, s_1, \cdots, s_n \in \mathbb{Z}$. Then there are prestandard modules $\tilde{D}(\lambda_i)$ such that $\tilde{E}$ has a filtration $0 = \tilde{F}_0 \subseteq \tilde{F}_1 \subseteq \cdots \subseteq \tilde{F}_n = \tilde{E}$ with sections $\tilde{F}_i/\tilde{F}_{i-1} \cong \tilde{D}(\lambda_i)(s_i)$ with $\lambda_i$ and $s_i$ as above, $i = 1, \cdots, n$. (It may be that $\tilde{D}(\lambda_i)$ is distinct from $\tilde{D}(\lambda_j)$, even if $\lambda_i = \lambda_j$.)

**Proof.** The proof is an easy induction, using projections, on the number $n$ of summands. \hfill \qed

Now we are ready to prove the main lemma. For convenience, we assume both parts of Hypotheses 4.7.

**Lemma 4.12.** Assume Hypothesis 4.7. Let $\tilde{N}$ be an $\bar{A}$-lattice which has a $\tilde{\Delta}$-filtration. Assume also that $\text{gr}\tilde{N}_K = (\text{gr}\tilde{N})_K$ has a gr$\Delta_K$-filtration as a graded gr$\bar{A}_K$-module. Then gr$^b\tilde{N}$ has a graded filtration with sections graded gr$\bar{A}$-modules which have the form $\tilde{D}(\lambda)(s)$, $\lambda \in \Lambda$ and $s \in \mathbb{N}$, for prestandard modules $\tilde{D}(\lambda)$. (Different sections can be associated to non-isomorphic prestandard modules of the same weight $\lambda$.) Applying $K \otimes_{\mathcal{O}} -$ gives a graded gr$\Delta_K$-filtration of gr$\tilde{N}_K$. (In particular, gr$^b\tilde{N}$ is a full lattice in gr$\tilde{N}_K$.) Moreover, we may choose the filtration

$$0 = \tilde{F}_0 \subseteq \tilde{F}_1 \subseteq \cdots \subseteq \tilde{F}_r = \text{gr}^b\tilde{N}$$

so that, if $\lambda < \lambda'$ belong to $\Lambda$ and $\tilde{F}_i/\tilde{F}_{i-1} \cong \tilde{D}(\lambda)(s), \tilde{F}_j/\tilde{F}_{j-1} \cong \tilde{D}(\lambda')(s')$, for some $i, j, s, s'$, then $i > j$.

**Proof.** Since $\tilde{N}$ has a $\tilde{\Delta}$-filtration, it contains a pure submodule $\tilde{M}$ such that $\tilde{M}$ is a direct sum of copies of $\tilde{\Delta}(\lambda)$, for some $\lambda \in \Lambda$ such that all weights of irreducible sections of $\tilde{N}/\tilde{M}$ are smaller than $\lambda$ or not comparable to it. The module $\tilde{N}/\tilde{M}$ also has a $\tilde{\Delta}$-filtration.

Put

$$\text{gr}^\#\tilde{M} = \bigoplus_{n \geq 0} \frac{\tilde{M} \cap (\text{rad} \bar{A}_K)^n \tilde{N}_K}{\tilde{M} \cap (\text{rad} \bar{A}_K)^{n+1} \tilde{N}_K}.$$ 

There is an obvious inclusion map $\iota : \text{gr}^\#\tilde{M} \rightarrow \text{gr}\tilde{N}$ of graded gr$\bar{A}$-modules, and the image $\text{Im} \ i$ is the kernel of the natural map $\text{gr}\tilde{N} \rightarrow \text{gr}\tilde{N}/\tilde{M}$. (We do not claim this latter map is surjective.) The module gr$^\#\tilde{M}$ is a different ordering of the weights $\lambda_1, \cdots, \lambda_n$ could conceivably result in different prestandard modules $\tilde{D}(\lambda_i)$.\footnote{\textcopyright 2022, Springer Nature Switzerland AG, part of Springer Nature.}
full lattice in an analogously defined graded gr $\tilde{A}_K$-module gr $\# M_K$, as discussed in [16] (without the $K$ subscript), and there is an analogous inclusion $\iota_K : \text{gr} \# M_K \to \text{gr} \tilde{N}_K$ of graded $A_K$-modules, with Im $\iota_K$ the kernel of the natural map gr $\tilde{N}_K \to \text{gr} (\tilde{N}_K/\tilde{M}_K)$.

Let $d = d(\lambda) = \dim N_{K,\lambda}$. Thus, $\tilde{M}$ is a direct sum of $d$ copies of $\tilde{A}(\lambda)$ by the construction of $\tilde{M}_K$. Also, gr $\tilde{N}_K$ must contain a direct sum of $d$ copies of the modules gr $\tilde{\Delta}_K(\lambda)$, viewed, for the moment, as ungraded gr $\tilde{A}_K$-modules. ($\tilde{N}_K$ will also contain a graded version of the direct sum; see below.) Since $\tilde{N}_K/\tilde{M}_K$ has zero $\lambda$-weight space, this direct sum must be contained in Im $\iota_K$. By dimension considerations, it must equal Im $\iota_K$.

Taking gradings into account, we have a decomposition

$$\text{gr} \# \tilde{M}_K \cong \bigoplus_{i=1}^d \text{gr} \tilde{\Delta}_K(\lambda)(m_i)$$

for some non-negative integers $m_1 \leq m_2 \leq \cdots \leq m_d$. As usual, the notation $\text{gr} \tilde{\Delta}_K(\lambda)(m)$ indicates that the usual grade 0 head of gr $\tilde{\Delta}_K(\lambda)$ (viz., gr $\tilde{\Delta}_K(\lambda)(0)$) is given grade $m$ ($m \in \mathbb{Z}$).

Write $\tilde{M}_\lambda$ as a direct sum

$$\tilde{M}_\lambda = \bigoplus_{n \geq 0} \tilde{M}_{\lambda,n}$$

where $\tilde{M}_{\lambda,n}$ is an $O$-module chosen as a complement to $\tilde{M}_\lambda \cap (\text{rd} \tilde{A}_K)^{n+1} \tilde{N}_K$ in $\tilde{M}_\lambda \cap (\text{rd} \tilde{A}_K)^n \tilde{N}_K$. The latter $O$-module is pure in $\tilde{M}$ and is the full $\lambda$-weight space in $\tilde{M} \cap (\text{rd} \tilde{A}_K)^n \tilde{N}_K$. The image $\text{gr} \# \tilde{M}_{n,\lambda}$ of $\tilde{M}_{\lambda,n}$ in (gr $\# \tilde{M}$)$_n$ is the full $\lambda$-weight space (gr $\# \tilde{M}$)$_{n,\lambda}$. The image is 0 unless $n = m_i$ for some $i$. In fact, the number of $m_i$ equal to $n$ is the rank of $\tilde{M}_{\lambda,n}$. We have

$$\begin{cases} 
\tilde{M} = \tilde{A} \tilde{M}_\lambda = \bigoplus_{n \geq 0} \tilde{A} \tilde{M}_{\lambda,n} \\
\tilde{A} \tilde{M}_{\lambda,n} \cong \tilde{\Delta}(\lambda) \otimes_O \tilde{M}_{\lambda,n}.
\end{cases}$$

Here, in the tensor product, $\tilde{M}_{\lambda,n}$ is regarded as an $O$-module only.

Recall that $m_1$ is the smallest integer $n$ with $\tilde{M}_{\lambda,n} \neq 0$. Thus,

$$\begin{cases} 
\tilde{M} \subseteq (\text{rd} \tilde{A}_K)^{m_1} \tilde{N}_K \\
\tilde{M} \nsubseteq (\text{rd} \tilde{A}_K)^{m_1+1} \tilde{N}_K.
\end{cases}$$

The two displays above show that the two naturally isomorphic $O$-submodules $\tilde{M}_{\lambda,m_1} \subseteq \tilde{N}$ and gr $\# \tilde{M}_{\lambda,m_1} \subseteq (\text{gr} \tilde{N})_{m_1} \subseteq \text{gr} \tilde{N}$ are generated as $O$-modules by strongly primitive elements of $\tilde{N}$ and gr $\tilde{N}$, respectively. Put $\tilde{R} = \tilde{A} \tilde{M}_{\lambda,m_1} \cong \tilde{\Delta}(\lambda) \otimes_O \tilde{M}_{\lambda,m_1}$, and $R_K = K \tilde{R}$. Form the graded module gr $\# \tilde{R} \hookrightarrow \text{gr} \tilde{N}$ as constructed for $\tilde{M}$, and note (gr $\# \tilde{R})_{\lambda} \subseteq \text{gr}^2 \tilde{N}$ identifies naturally with $\tilde{M}_{\lambda,m_1}$. 
We may identify \( \text{gr} \# \tilde{R} \) with a submodule of \( \text{gr} \# \tilde{M} \). As such,

\[
\text{gr} \# \tilde{R}_{\lambda,m_1} = \text{gr} \# \tilde{M}_{\lambda,m_1} \cong \text{gr} \tilde{N}_{\lambda,m_1}
\]

has rank \( \dim M_{K,\lambda,m_1} \). Consequently, since \( \text{gr} \tilde{N}_K \) has a graded submodule consisting of a direct sum of dim \( M_{K,\lambda,m_1} \)-copies of \( \text{gr} \Delta_K(\lambda)(m_1) \), the image of \( \text{gr} \# \tilde{R}_K \) must be that submodule. Thus, there is a graded \( \tilde{A}_K \)-isomorphism \( \text{gr} \# \tilde{R}_K \cong \text{gr} \Delta_K(\lambda)(m_1) \otimes_K \tilde{M}_{K,\lambda,m_1} \).

The isomorphism (4.1) has implications for the filtration terms \( \tilde{R}_K \cap (\text{rad} \tilde{A}_K)^n \tilde{N}_K, n \geq 0 \), whose successive quotients define the grades of \( \text{gr} \# \tilde{R} \). In fact, it is easy to see

\[
(\text{rad} \tilde{A}_K)^s \tilde{R}_K = \tilde{R}_K \cap (\text{rad} \tilde{A}_K)^{m_1+s} \tilde{N}_K, \quad s \geq 0.
\]

The proof that each \( \tilde{R}_K \cap (\text{rad} \tilde{A}_K)^n \tilde{N}_K \) must be some \( (\text{rad} \tilde{A}_K)^s \tilde{R}_K \) is an easy downward induction on \( n \), starting the largest \( n \) for which \( \tilde{R}_K \cap (\text{rad} \tilde{A}_K)^n \tilde{N}_K \neq 0 \). The precise indexing can then be made from the fact that \( \tilde{R}_K \cap (\text{rad} \tilde{A}_K)^{m_1+1} \tilde{N}_K \) is the largest intersection not equal to \( \tilde{R}_K \).

Since the grade \( n \) terms of \( \text{gr} \# \tilde{R} \) are defined by successive quotients of the filtration terms \( \tilde{R} \cap (\text{rad} A_K)^n \tilde{N}_K \), (4.2) gives

\[
\text{gr} \# \tilde{R} \cong (\text{gr} \tilde{R})(m_1) \cong (\text{gr} \Delta(\lambda))(m_1) \otimes_{\tilde{O}} \tilde{M}_{\lambda,m_1}
\]

in \( \text{gr} \tilde{A}\text{-grmod} \).

To complete the proof by induction on \( \dim \tilde{N}_K \), we next check that \( \tilde{N}/\tilde{R} \) satisfies the hypotheses of the lemma. By construction, \( \tilde{N}/\tilde{M} \) has a filtration with sections \( \Delta(\nu), \nu \in \Lambda \setminus \{\lambda\} \), while \( \tilde{M}/\tilde{R} \) is visibly an ungraded direct sum of copies of \( \Delta(\lambda) \). Similarly, \( \text{gr} (\tilde{N}_K/\tilde{M}_K) \) must, by construction, have a graded filtration with sections modules \( \text{gr} \Delta_K(\mu)(s), \mu \in \Lambda \setminus \{\lambda\}, s \geq 0 \). The kernel of the map \( \text{gr} (\tilde{N}_K/\tilde{R}_K) \to \text{gr} (\tilde{N}_K/\tilde{M}_K) \) is the cokernel of the map \( \text{gr} \# \tilde{R}_K \to \text{gr} \# \tilde{M}_K \), which is, visibly, a direct sum of graded \( \tilde{A}_K \)-modules \( \text{gr} \Delta_K(\lambda)(t), t \geq 0 \). Thus, \( \tilde{N}/\tilde{R} \) satisfies the hypotheses of the lemma.

By induction, \( \text{gr}^s(\tilde{N}/\tilde{R}) \) has a graded filtration with sections various modules \( \text{gr} \Delta(\nu)(s), \nu \in \Lambda, s \in \mathbb{N} \), here, and in the statement of the lemma. Applying \( K \otimes_{\tilde{O}} - \), these become sections of a graded \( \Delta_K \)-filtration of \( \text{gr} \tilde{N}_K/\tilde{R}_K \). It is easily checked that the hypotheses of Lemma 4.11 apply to \( \tilde{N} \) and \( \tilde{R} \), so that the map \( \text{gr} \tilde{N} \to \text{gr} \tilde{N}/\tilde{R} \) is surjective. The latter map has kernel \( \text{gr}^s \tilde{N} \cap \text{gr} \# \tilde{R} \), which contains \( \text{gr} \tilde{M}_{\lambda,m_1} \). Thus,

\[
\text{gr}^s \Delta(\lambda)(m_1) \otimes_{\tilde{O}} \tilde{M}_{\lambda,m_1} \subseteq \text{gr}^s \tilde{N} \cap \text{gr} \# \tilde{R} \subseteq \text{gr} \# \tilde{R} \cong \text{gr} \Delta(\lambda)(m_1) \otimes_{\tilde{O}} \tilde{M}_{\lambda,m_1},
\]

noting the isomorphism \( \text{gr}^s \Delta(\lambda)(m_1) \otimes_{\tilde{O}} \tilde{M}_{\lambda,m_1} \cong (\text{gr} \tilde{A}) \text{gr} \tilde{M}_{\lambda,m_1} \). The lemma now follows by Lemma 4.11 applied to \( \tilde{E} = \text{gr}^s \tilde{N} \cap \text{gr} \# \tilde{R} \), and induction.

\( \square \)

Remark 4.13. (a) Character arguments in the (common) Grothendieck group for \( \tilde{A}_K\text{-mod} \) and \( \text{gr} \tilde{A}_K\text{-mod} \) show that \([\tilde{N}_K : \Delta_K(\lambda)] = [\text{gr} \tilde{N}_K : \text{gr} \Delta_K(\lambda)]\)
for \( \tilde{N} \) as in Lemma 4.12. Also, \( [\tilde{N}_K : \Delta_K(\lambda)] \) is obviously the same as \( [\tilde{N} : \tilde{\Delta}(\lambda)] \) and \( [\text{gr} \tilde{N}_K : \text{gr} \Delta_K(\lambda)] \) counts the number of (collective occurrences) of prestandard modules \( \tilde{D}(\lambda) \) in any prestandard filtration of \( \text{gr}^b \tilde{N} \), viewed as an ungraded \( \tilde{A} \)-module (a full lattice in \( \text{gr} \tilde{N}_K \)).

(b) Assuming Hypothesis 4.7 any PIM \( \tilde{P}(\lambda), \lambda \in \Lambda \), may be used for \( \tilde{N} \) in Lemma 4.12. Here \( \text{gr}^b \tilde{P}(\lambda) = \text{gr} \tilde{P}(\lambda) \) by Lemma 4.8. It has a prestandard filtration by Lemma 4.12 and the proof of Lemma 4.12 shows that the top term of the filtration may be taken to be \( \text{gr}^b \tilde{\Delta}(\lambda) \). Also, the kernel of the map from \( \tilde{P}(\lambda) = \tilde{P}(\lambda) \) onto \( \text{gr}^b \tilde{\Delta}(\lambda) \) is filtered by prestandard modules \( \tilde{D}(\mu), \mu > \lambda \). This is also guaranteed by part (a) above of this remark.

The following proposition is, in some sense, a corollary of the proof of Lemma 4.12, though additional argument is required.

**Proposition 4.14.** Let \( \tilde{N} \) be an \( \tilde{A} \)-lattice satisfying the hypothesis of Lemma 4.12 (which includes Hypotheses 4.7). Assume also that \( \text{gr} \tilde{\Delta}(\nu) \) has a simple head, for each \( \nu \in \Lambda \) satisfying \( [\tilde{N}_K : \tilde{\Delta}_K(\nu)] \neq 0 \). Then \( \text{gr}^b \tilde{N} = \text{gr} \tilde{N} \) and all the prestandard modules in the filtration of Lemma 4.12 are “standard” (i.e., if a shifted copy of \( \tilde{D}(\nu) \) occurs in the filtration, then \( \tilde{D}(\nu) \cong \text{gr} \tilde{\Delta}(\nu) \)).

Before giving the proof, we record the following immediate corollary (of the proposition and Lemma 4.12 using the Morita equivalence of Proposition 4.5).

**Corollary 4.15.** Suppose that \( \tilde{A} \) is a split QHA over \( \mathcal{O} \) with poset \( \Lambda \), and that \( \text{gr} \tilde{A}_K \) is a QHA algebra with the same poset \( \Lambda \). Let \( \tilde{N} \) be a \( \tilde{A} \)-lattice with a \( \tilde{\Delta} \)-filtration, and suppose \( \text{gr} \tilde{\Delta}(\nu) \) has a simple head for each \( \nu \in \Lambda \) satisfying \( [\tilde{N}_K : \tilde{\Delta}_K(\nu)] \neq 0 \). Then \( \text{gr} \tilde{N} \) has a filtration with sections \( \text{gr} \tilde{\Delta}(\lambda), \lambda \in \Lambda \).

**Proof of Proposition 4.14.** According to Lemma 4.8 the head of a given \( \text{gr} \tilde{\Delta}(\lambda) \) guarantees that \( \text{gr}^b \tilde{\Delta}(\lambda) = \text{gr} \tilde{\Delta}(\lambda) \). Since any module \( \tilde{D}(\lambda) \) is sandwiched between \( \text{gr}^b \tilde{\Delta}(\lambda) \) and \( \text{gr} \tilde{\Delta}(\lambda) \), we must have \( \tilde{D}(\lambda) = \text{gr} \tilde{\Delta}(\lambda) \).

We now continue with the notation and proof of Lemma 4.12. The displayed inclusions in the last paragraph are now equalities, and so \( \text{gr}^b \tilde{N} \cap \text{gr}^b \tilde{R} = \text{gr}^b \tilde{R} \). That is, \( \text{gr}^b \tilde{R} \subseteq \text{gr}^b \tilde{N} \).

We recall that \( \text{gr}^b \tilde{R} \) is the kernel of the map \( \text{gr} \tilde{N} \to \text{gr} \tilde{N}/\tilde{R} \). This map sends \( \text{gr}^b \tilde{N} \) onto \( \text{gr}^b \tilde{N}/\tilde{R} \), according to the paragraph quoted above. The paragraph before that, in the proof of Lemma 4.12, notes that \( \tilde{N}/\tilde{R} \) satisfies the hypotheses of the lemma required on \( \tilde{N} \). In particular, \( \tilde{N}/\tilde{R} \) has a \( \tilde{\Delta} \)-filtration. This is part of a \( \tilde{\Delta} \)-filtration of \( \tilde{N} \), since \( \tilde{R} \) is a direct sum of standard modules by construction (see its introduction earlier in the proof of the lemma).

It follows that \( \tilde{N}/\tilde{R} \) satisfies the hypothesis of the current proposition, and we can assume inductively (using induction on the rank of \( \tilde{N} \)) that
\[ \text{gr}^b \tilde{N}/\tilde{R} = \text{gr} \tilde{N}/\tilde{R}. \] But now \( \text{gr}^b \tilde{N} \) and \( \text{gr} \tilde{N} \) have the same image and kernel, so must be equal. This proves the proposition.

As a further corollary of Lemma 4.12, Proposition 4.14, and their proofs, we have

**Corollary 4.16.** Let \( \tilde{A}, \tilde{N}, \Lambda \) satisfy the hypotheses of the preceding corollary, and let \( \Gamma \subseteq \Lambda \) be a poset ideal. Then \( \text{gr} \tilde{N}_\Gamma \cong (\text{gr} \tilde{N})_\Gamma \) as \( \tilde{A} \)-lattices. In particular, the natural map \( \tilde{N} \to \text{gr} \tilde{N}_\Gamma \) is surjective.

**Proof.** Again, we may assume the hypothesis of Proposition 4.14, using Morita equivalence. Now, we simply continue the discussion, in the proof of the latter result, choosing \( \lambda \) to belong to \( \Gamma \). (If \( \Gamma = \emptyset \), there is nothing to prove.) Proceeding by induction on the rank of \( \tilde{N} \), it can be assumed that

\[ \text{gr} (\tilde{N}/\tilde{R})_\Gamma \cong (\text{gr} \tilde{N}/\tilde{R})_\Gamma. \]

Since \( \tilde{R} \) is a direct sum of copies of \( \tilde{A}(\lambda) \), it is contained in the kernel of the surjection \( \tilde{N} \to \tilde{N}_\Gamma \), and so \( \text{gr} (\tilde{N}/\tilde{R})_\Gamma \cong \text{gr} \tilde{N}_\Gamma \). Also, since \( \text{gr} \# \tilde{R} \) is a direct sum of shifted copies of \( \text{gr} \tilde{A}(\lambda) \), each of which has head which is a shift of \( L(\lambda) \), \( \text{gr} \# \tilde{R} \) is contained in the kernel of the surjection \( \text{gr} \tilde{N} \to (\text{gr} \tilde{N})_\Gamma \). Of course, \( \text{gr} \# \tilde{R} \) is the kernel of the map \( \text{gr} \tilde{N} \to \text{gr} \tilde{N}/\tilde{R} \). The latter map is surjective, as Lemma 4.12 and Proposition 4.14 (and their proofs) show. Thus, \( (\text{gr} \tilde{N})_\Gamma \cong (\text{gr} \tilde{N}/\tilde{R})_\Gamma. \) The corollary now follows.

An analogue of Corollary 4.16 holds for QHAs over fields (without any integral hypothesis or conclusion). See Appendix I.

We are now ready to establish a main theorem of this paper.

**Theorem 4.17.** Suppose that \( \tilde{A} \) is a split QHA over \( \mathcal{O} \) with poset \( \Lambda \), and \( \text{gr} \tilde{A}_K \) is a QHA over \( K \) with the same poset \( \Lambda \). Further, suppose that \( \text{gr} \tilde{A}(\lambda) \) has a simple head, for all \( \lambda \in \Lambda \). Then \( \text{gr} \tilde{A} \) is a split QHA over \( \mathcal{O} \) with poset \( \Lambda \) and standard objects \( \text{gr} \tilde{A}(\lambda), \lambda \in \Lambda \).

**Proof.** Using Proposition 4.5 to pass to a Morita equivalent algebra, we can assume that Hypothesis 4.7 holds. Let \( \tilde{P} \) be any \( \mathcal{O} \)-finite projective \( \tilde{A} \)-module. Then \( \tilde{N} := \tilde{P} \) satisfies the hypotheses of Proposition 4.14. Now apply Remark 4.13(b) to \( \tilde{P} = \tilde{P}(\lambda) \), for any given \( \lambda \in \Lambda \). This provides an exact sequence \( 0 \to \Omega \to \text{gr} \tilde{P}(\lambda) \to \text{gr} \tilde{A}(\lambda) \to 0 \) in which \( \Omega \) has a filtration with sections \( \text{gr} \tilde{A}(\nu), \nu > \lambda \). In particular,

\[
\begin{align*}
\text{Hom}_{\text{gr} \tilde{A}}(\Omega, L(\mu)) &= 0, \\
\text{Hom}_{\text{gr} \tilde{A}}(\Omega, \text{gr} \tilde{A}(\mu)) &= 0,
\end{align*}
\]

unless \( \mu > \lambda \). Thus

\[
\begin{align*}
\text{Ext}^1_{\text{gr} \tilde{A}}(\text{gr} \tilde{A}(\lambda), L(\mu)) &= 0, \\
\text{Ext}^1_{\text{gr} \tilde{A}}(\text{gr} \tilde{A}(\lambda), \text{gr} \tilde{A}(\mu)) &= 0,
\end{align*}
\]
unless \( \lambda < \mu \) (\( \lambda, \mu \in \Lambda \)). In particular, if \( \tilde{M} \) is a gr \( \tilde{A} \)-module such that the composition factors \( L(\mu) \) of \( \tilde{M} \) satisfy \( \nu \not\succ \lambda \), then \( \operatorname{Ext}_{\tilde{A}}^1(\operatorname{gr} \tilde{A}(\lambda), \tilde{M}) = 0 \).

Next, take \( \tilde{P} = \tilde{A} \), viewed as a left module over itself in Proposition 4.14. We can rearrange the (ungraded version of the) filtration guaranteed by Proposition 4.14 so that all copies of a given \( \operatorname{gr} \tilde{A}(\lambda) \) occur contiguously. Removing all redundant filtration indices \( j \) and those for which \( \tilde{F}_{j+1}/\tilde{F}_j \) and \( \tilde{F}_j/\tilde{F}_{j-1} \) are isomorphic to the same module \( \operatorname{gr} \tilde{A}(\lambda) \), gives a new filtration

\[
0 \subseteq \tilde{F}_1 \subseteq \tilde{F}_2 \subseteq \cdots \subseteq \tilde{F}_s = \operatorname{gr} \tilde{A},
\]

with \( 1 = j_1 < j_2 < \cdots < j_s = r \) and \( \tilde{F}_{j_i+1}/\tilde{F}_{j_i} \cong \tilde{A}(\lambda_i)^{\oplus m_i}, \; i = 1, \ldots, s \), for suitable elements \( \lambda_i \in \Lambda \) and positive integers \( m_i \). We may assume that \( \lambda_i \neq \lambda_j \) if \( i \neq j \), and also that \( \lambda_i \neq \lambda_j \) if \( i < j \).

Now take \( \tilde{J}_i = \tilde{F}_{j_i} \). By definition, \( \tilde{J}_i \) is a left ideal in \( \operatorname{gr} \tilde{A} \). On the other hand, let \( x \in \operatorname{gr} \tilde{A} \). We claim that \( \tilde{J}_i x \subseteq \tilde{J}_i \). Right multiplication by \( x \) defines an endomorphism \( \operatorname{gr} \tilde{A} \to \operatorname{gr} \tilde{A} \), and it suffices to show that the induced map \( \tilde{J}_i \xrightarrow{\sim} \operatorname{gr} \tilde{A} \to (\operatorname{gr} \tilde{A})/\tilde{J}_i \) is the zero map. If this map is nonzero, tensoring with \( K \) defines a nonzero map \( \tilde{J}_i \to (\operatorname{gr} \tilde{A})/\tilde{J}_i \). However, \( \tilde{J}_i \) is filtered by standard modules \( \operatorname{gr} \tilde{A}(\lambda_j) \), \( j = 1, \ldots, s' \) for some non-negative integer \( s' \), while \((\operatorname{gr} \tilde{A})/\tilde{J}_i \) is filtered by modules \( \operatorname{gr} \tilde{A}(\mu) \) with \( \mu \geq \lambda_i \) than, or equal to, any \( \lambda_j \). Hence, \( \operatorname{Hom}_{\tilde{A}}(\tilde{J}_i, (\operatorname{gr} \tilde{A})/\tilde{J}_i) = 0 \), a contradiction. Therefore, \( \tilde{J}_i x \subseteq \tilde{J}_i \) as claimed, and \( \tilde{J}_i \) is an ideal in \( \operatorname{gr} \tilde{A} \).

Next, for \( \lambda \in \Lambda \), \( \operatorname{End}_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda)) \) is a finite \( O \)-subalgebra of \( \operatorname{End}_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda)) = \operatorname{End}_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda)) = K \), so \( \operatorname{End}_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda)) = K \). Thus,

\[
\operatorname{End}_{\tilde{A}}(\tilde{J}_i/\tilde{J}_{i-1}) \cong \operatorname{End}_{\tilde{A}}(\tilde{A}(\lambda_i)^{\oplus m_i}) \cong M_{m_i}(O).
\]

Fix \( i \), and put \( \Gamma_i = \{ \gamma \in \Lambda \mid \gamma \leq \lambda_{i'}, \text{for some } i' > i \} \). Then \( \tilde{J}_i \subseteq B := \operatorname{gr} \tilde{A} \) and \( B/\tilde{J}_i \) is the largest quotient algebra of \( B \) whose composition factors \( L(\nu) \) satisfy \( \nu \in \Gamma_i \). Since \( \tilde{J}_i/\tilde{J}_i^2 \) is a \( B/\tilde{J}_i \)-module, its composition factors \( L(\gamma) \) satisfy \( \gamma \in \Gamma_i \). On the other hand, \( \tilde{J}_i/\tilde{J}_i^2 \) is a homomorphic

\[13\] There are many ways to prove this. First, note the hypothesis on \( \tilde{M} = \tilde{M}/\pi \tilde{M} \) holds for any of its homomorphic images \( \pi^{a-1} \tilde{M}/\pi^a \tilde{M} \), with \( a > 0 \), and, thus, on \( \tilde{M}/\pi^a \tilde{M} \). The required vanishing holds if \( \tilde{M} \) is replaced by \( \tilde{M}/\pi^a \tilde{M} \), so it suffices to show it holds when \( \tilde{M} \) is replaced by \( \pi^a \tilde{M} \). For large enough \( a \), the latter module is torsion free. So we may assume to start that \( \tilde{M} \) is torsion free. All composition factors \( \tilde{L}_K(\mu) \) of \( \tilde{M}_K \) satisfy \( \mu \not\succ \lambda \) since the analogous property over \( k \) holds. Thus, \( \operatorname{Ext}^1_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda)(\lambda), \tilde{M}_K) = 0 \), since \( \operatorname{gr} \tilde{A}_K \) is a QHA with poset \( \Lambda \). Thus, \( \operatorname{Ext}^1_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda), \tilde{M}) = 0 \). However, multiplication by \( \pi \) on this \( \operatorname{Ext}^1 \)-group is surjective, since \( \operatorname{Ext}^1_{\tilde{A}}(\operatorname{gr} \tilde{A}(\lambda), \tilde{M}) = 0 \). So the \( \operatorname{Ext}^1 \)-group itself vanishes.
image of \( \tilde{J}_{ik} \), which is filtered with sections \( \text{gr} \tilde{\Delta}(\lambda), \lambda \not\in \Gamma_i \), each of which has head \( L(\lambda) \). So, if \( \tilde{J}_{ik}/\tilde{J}_{ik}^2 \neq 0 \), it must have a composition factor \( L(\lambda), \lambda \not\in \Gamma_i \). Therefore, \( \tilde{J}_{ik}^2 = \tilde{J}_{ik} \), so \( \tilde{J}_i = \tilde{J}_i^2 \) by Nakayama’s lemma.

Finally, to show \( \tilde{J}_i/\tilde{J}_{i-1} \) is \( (\text{gr} \tilde{A})/\tilde{J}_{i-1} \)-projective, it suffices to show that \( \text{gr} \tilde{\Delta}(\lambda_i) \) is \( (\text{gr} \tilde{A})/\tilde{J}_{i-1} \)-projective. But this follows from the sentence immediately following (4.3).

In conclusion, \( \text{gr} \tilde{A} \) is a split QHA over \( \mathcal{O} \), as claimed. This construction also shows that \( \text{gr} \tilde{\Delta}(\lambda) \) is standard. \( \square \)

**Remark 4.18.** (a) We could give an alternative version of part of the above proof by appealing to Corollary 4.16. In particular, we note, for \( \tilde{A} \) and \( \Lambda \) satisfying the hypothesis of Theorem 4.17, and any poset ideal \( \Gamma \subseteq \Lambda \), that

\[
\text{gr} (\tilde{A}_\Gamma) \cong (\text{gr} \tilde{A})_\Gamma.
\]

(b) Consequently, the map \( \text{gr} \tilde{A} \to \text{gr} \tilde{A}_\Gamma \) is surjective. So, if \( \tilde{N} \) is an \( \tilde{A}_\Gamma \)-lattice, with \( \tilde{A} \) and \( \Gamma \) as above, there is a physical equality of \( \mathcal{O} \)-modules

\[
\text{gr}^{\tilde{A}_\Gamma} \tilde{N} = \text{gr}^{\tilde{A}_\Gamma} \tilde{N}.
\]

(c) In a similar spirit, another consequence of Corollary 4.16. If \( \tilde{A}, \Lambda, \tilde{N} \) satisfy the hypothesis of the corollary, then the strongly primitive elements of \( \text{gr} \tilde{N} \) are the same as the primitive elements of \( \text{gr} \tilde{N} \). This follows by taking \( \Gamma = \{ \gamma \mid \gamma \not\in \lambda \} \) for any given \( \lambda \in \Lambda \), and using the isomorphism

\[
\text{gr} \tilde{N}/\tilde{N} \cap \tilde{N}_K(\lambda) = (\text{gr} \tilde{N})_\Gamma \cong (\text{gr} \tilde{N})_\Gamma \cong \text{gr} \tilde{N}/\text{gr} \tilde{N} \cap (\text{gr} \tilde{N})_K(\lambda).
\]

5. A SPECIAL CASE

In this section, unless otherwise noted, we continue the notation and assume both the conditions of Hypotheses 4.7 of the previous section. In particular, the algebra \( \tilde{A} \) is a split QHA with standard objects \( \tilde{\Delta}(\lambda), \lambda \in \Lambda \). The irreducible modules and PIMS of the QHA \( \tilde{A}_K \) are denoted \( L_K(\lambda) \) and \( P_{\lambda} \), respectively. We sometimes write \( \Delta_K(\lambda) \) (which is isomorphic to \( \tilde{\Delta}(\lambda)_{\tilde{K}} \)) for its standard modules. The algebra \( \text{gr} \tilde{A}_K \) is also a QHA under Hypothesis 4.7 specifically item (2). It has standard objects \( \text{gr} \tilde{\Delta}(\lambda)_{\tilde{K}} \cong \text{gr} \Delta_K(\lambda), \lambda \in \Lambda \).

In addition, we assume, for the rest of this section, that \( \tilde{A} \) has a pure subalgebra \( \tilde{a} \) and a Wedderburn complement \( \tilde{A}_{K,0} \) of \( \tilde{A}_K \). For use in the results below, we record the following conditions.

**Conditions 5.1.**

1. \( \tilde{a}_K \) has a tight grading \( \tilde{a}_K = \tilde{a}_{K,0} \oplus \tilde{a}_{K,1} \oplus \cdots \).
2. \( \text{rad} \tilde{A}_K = (\text{rad} \tilde{a}_K) \tilde{A}_K = \tilde{A}_K (\text{rad} \tilde{a}_K) \).
3. For \( \lambda \in \Lambda \), \( \Delta_K(\lambda) \) has a graded \( \tilde{a}_K \)-structure, and is generated as an \( \tilde{a}_K \)-module by \( \Delta_K(\lambda)_0 \).

\[\text{This means that } \tilde{a}_K \text{ is positively graded, } \tilde{a}_K = \bigoplus_{n \geq 0} \tilde{a}_{K,n} \text{ with } \tilde{a}_{K,0} \text{ semisimple and with } \tilde{a}_K \text{ generated by } \tilde{a}_{K,1}; \text{ see } [4] \S 4. \text{ Equivalently, } \tilde{a}_K \cong \text{gr} \tilde{a}_K.\]
(4) In (3), $\Delta_K(\lambda)_0$ is $\tilde{A}_{K,0}$-stable. Also, $\tilde{A}_{K,0}$ contains $\tilde{a}_{K,0}$ (defined in (1)) and all idempotents $e_{\lambda}, \lambda \in \Lambda$.

(5) $\tilde{a}$ has a positive grading $\tilde{a} = \oplus_{r \geq 0} \tilde{a}_r$ such that $K\tilde{a}_r = \tilde{a}_{K,r}$, the $r$th grade of $\tilde{a}_K$ in (1), for each $r \in \mathbb{N}$.

Observe that Conditions (5.1)(1) & (5) can be made independently of the QHA algebra $\tilde{A}$. In this spirit, we have the following result.

Proposition 5.2. Let $\tilde{a}$ be an algebra which is free and finite over the DVR $O$ and satisfies Conditions (5.1)(1) & (5). Then the following statements hold:

(a) For each $r \in \mathbb{N}$, we have $\tilde{a}_r = \tilde{a} \cap \tilde{a}_{K,r}$ and $\sum_{i \geq r} \tilde{a}_i = \tilde{a} \cap \text{rad}^r \tilde{a}_K$.

There is an isomorphism $\tilde{a} \sim \text{gr} \tilde{a}$ of graded $O$-algebras sending $x \in \tilde{a}_r$ to its image $[x]$ in $\text{rad}^r \tilde{a}/\text{rad}^{r+1} \tilde{a} = (\text{gr} \tilde{a})_r$.

(b) If $\tilde{M}$ is an $\tilde{a}$-lattice, the following statements are equivalent:

(i) $\tilde{M}$ is tight. (See Definition 2.2.)

(ii) $\sum_{i \geq r} \tilde{a}_i \tilde{M} = \text{rad}^r \tilde{M}$ for each $r \in \mathbb{N}$.

(iii) The $\text{gr} \tilde{a}$-lattice $\text{gr} \tilde{M}$ is generated by $(\text{gr} \tilde{M})_0$.

Proof. We begin with (a). Since $\tilde{a}_r$ is an $O$-direct summand of $\tilde{a}$, we have that $\tilde{a}_r = \tilde{a} \cap \tilde{a}_{K,r}$ and $\sum_{i \geq r} \tilde{a}_i = \tilde{a} \cap \text{rad}^r \tilde{a}_K$, since $\text{rad}^r \tilde{a}_K = (\text{rad} \tilde{a}_K)^r = (\sum_{i \geq 1} \tilde{a}_{K,i})^r = \sum_{i \geq r} \tilde{a}_{K,i} = (\sum_{i \geq r} \tilde{a}_i)_K$.

In particular, there is a well-defined $O$-linear graded map sending $x \in \tilde{a}_r$ to its image $[x] \in \tilde{a} \cap \text{rad}^r \tilde{a}_K/\tilde{a} \cap \text{rad}^{r+1} \tilde{a}_K$, and the map is clearly multiplicative. Thus, it is a graded homomorphism of $O$-algebras. The equations $\sum_{i \geq r} \tilde{a}_i = \tilde{a} \cap \text{rad}^r \tilde{a}_K$ and $\sum_{i \geq r+1} \tilde{a}_i = \tilde{a} \cap \text{rad}^{r+1} \tilde{a}_K$ show that the map is an isomorphism. Part (a) is now proved.

We now prove (b). The equivalence (i) $\iff$ (ii) is immediate from the sum formula for $\tilde{a} \cap \text{rad}^r \tilde{a}_K$ in part (a).

Now suppose that (ii) holds. Then, using the isomorphism of (a),

$$(\text{gr} \tilde{a})_0, (\text{gr} \tilde{M})_0 = [\tilde{a}_r, \tilde{M}]/(\tilde{M} \cap \text{rad}^r \tilde{M}_K) = (\tilde{a}_r \tilde{M} + \tilde{M} \cap \text{rad}^{r+1} \tilde{M}_K)/\tilde{M} \cap \text{rad}^{r+1} \tilde{M}_K,$$

the second equality holding by definition of the action of $\text{gr} \tilde{a}$ on $\text{gr} \tilde{M}$. But (ii) gives that

$$\tilde{a}_r \tilde{M} + \tilde{M} \cap \text{rad}^{r+1} \tilde{M}_K = \tilde{a}_r \tilde{M} + \sum_{i \geq r+1} \tilde{a}_i \tilde{M} = \sum_{i \geq r} \tilde{a}_i \tilde{M} = \tilde{M} \cap \text{rad}^r \tilde{M}_K.$$

Thus, $(\text{gr} \tilde{a})_r, (\text{gr} \tilde{M})_0 = \text{gr} \tilde{M}_r$, and (iii) holds.

If (iii) holds, then $(\text{gr} \tilde{a})_r, (\text{gr} \tilde{M})_0 = \text{gr} \tilde{M}_r$ for all $r \in \mathbb{N}$. This gives that

$$\tilde{a}_r \tilde{M} + \tilde{M} \cap \text{rad}^{r+1} \tilde{M}_K = \tilde{M} \cap \text{rad}^r \tilde{M}_K$$
For $\lambda \in \Lambda$, define the $\tilde{A}_K$-submodule $E_K(\lambda)$ of $P_K(\lambda)$ by the short exact sequence

\begin{equation}
0 \to E_K(\lambda) \hookrightarrow P_K(\lambda) \to \Delta_K(\lambda) \to 0.
\end{equation}

**Theorem 5.3.** Let $\tilde{A}$ be a QHA over $\mathcal{O}$ with weight poset $\Lambda$, which satisfies Hypothesis 4.7 and Conditions 5.1(1)–(5). Let $\lambda \in \Lambda$ be such that the PIM $P_K(\lambda)$ in $\tilde{A}_K$-mod contains a full $\tilde{A}$-lattice $\tilde{P}(\lambda)\uparrow$ with the following properties:

(i) $\tilde{P}(\lambda)\uparrow = \tilde{A}v$, where $v \in \tilde{P}(\lambda)\uparrow$;

(ii) $\tilde{P}(\lambda)\uparrow = \tilde{P}(\lambda)_0\uparrow \oplus \tilde{P}(\lambda)\uparrow \cap \text{rad} P_K(\lambda)$, where $\tilde{P}(\lambda)_0\uparrow$ is an $\mathcal{O}$-direct summand of $\tilde{P}(\lambda)\uparrow$ such that $K\tilde{P}(\lambda)_0\uparrow + E_K(\lambda)$ is an $\tilde{A}_{K,0}$-submodule of $K\tilde{P}(\lambda)\uparrow = P_K(\lambda)$;

(iii) $\tilde{P}(\lambda)\uparrow_{\tilde{A}}$ is tight.

Then $\tilde{A}(\lambda)_{\tilde{A}}$ is tight.

**Proof.** Since $L_K(\lambda) \cong P_K(\lambda)/\text{rad} P_K(\lambda)$, (ii) implies $K\tilde{P}(\lambda)_0\uparrow + E_K(\lambda)/E_K(\lambda) \cong L_K(\lambda)_{\tilde{A}_{K,0}}$ as an $A_{K,0}$-module. Now let $\varphi : P_K(\lambda) \to \Delta_K(\lambda)$ be a surjection. Then $\varphi(\text{rad} P_K(\lambda)) = \text{rad} \Delta_K(\lambda)$, so that $\varphi(K\tilde{P}(\lambda)_0\uparrow)$ is nonzero and isomorphic to $L_K(\lambda)_{\tilde{A}_{K,0}}$. Since the latter module appears with multiplicity one as an $\tilde{A}_{K,0}$-composition factor of $\Delta_K(\lambda)$, we must have $\varphi(K\tilde{P}(\lambda)\uparrow) = \Delta_K(\lambda)_0$, the $\tilde{A}_{K,0}$-submodule of $\Delta_K(\lambda)$ in Conditions 5.1(3) & (4). (Observe that $\Delta_K(\lambda)_0 \cap \text{rad} \Delta_K(\lambda) = \Delta_K(\lambda)_0 \cap \sum_{i \geq 1} \tilde{a}_i \Delta_K(\lambda)_0 = 0$ by Conditions 5.1(1), (2), (3).)

Since $\tilde{P}(\lambda)_0\uparrow$ is a complement in $\tilde{P}(\lambda)\uparrow$ to $\tilde{P}(\lambda)\uparrow \cap \text{rad} P_K(\lambda) = \tilde{P}(\lambda)_0\uparrow \cap \text{rad} \tilde{P}(\lambda)_K$, which is $\sum_{i \geq 1} \tilde{a}_i \tilde{P}(\lambda)_i\uparrow$, by the assumed tightness (iii) and Proposition 5.2 we have

\[\tilde{P}(\lambda)\uparrow = \tilde{P}(\lambda)_0 + \sum_{i \geq 1} \tilde{a}_i \tilde{P}(\lambda)_i\uparrow = \tilde{P}(\lambda)_0\uparrow + \sum_{i \geq 1} \tilde{a}_i \tilde{P}(\lambda)_0 + \sum_{i \geq 1} \tilde{a}_i \tilde{P}(\lambda)_i\uparrow.\]

Further iterating this equation (or just applying Nakayama’s lemma), we get that

\[\tilde{P}(\lambda)\uparrow = \tilde{P}(\lambda)_0\uparrow + \sum_{i \geq 1} \tilde{a}_i \tilde{P}(\lambda)_0\uparrow.\]

Thus,

\[\varphi(\tilde{P}(\lambda)\uparrow) = \varphi(\tilde{P}(\lambda)_0\uparrow) + \sum_{i \geq 1} \tilde{a}_i \varphi(\tilde{P}(\lambda)_0\uparrow).\]
Since \( K \phi(\tilde{P}(\lambda)_0^\dagger) = \phi(K\tilde{P}(\lambda)_0^\dagger) = \Delta_K(\lambda)_0 \), the \( a_K \)-graded structure of Condition (3) shows the above sum is direct, even inside the right-hand side sum. That is,

\[
\sum_{i \geq 1} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger) = \bigoplus_{i \geq 1} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger).
\]

We claim that the \( \tilde{a} \)-lattice \( \phi(\tilde{P}(\lambda)_0^\dagger)|_{\tilde{a}} \) is tight. By Proposition 5.2 it is sufficient to check the equality

\[
\sum_{i \geq r} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger) = \phi(\tilde{P}(\lambda)_0^\dagger) \cap \text{rad}^r \phi(\tilde{P}(\lambda)_0^\dagger), \quad \forall r \in \mathbb{N}.
\]

For \( r = 0 \), this is trivial, since \( \sum_{i \geq 0} \tilde{a}_i = \tilde{a} \), and \( \tilde{a} \phi(\tilde{P}(\lambda)_0^\dagger) = \phi(\tilde{P}(\lambda)_0^\dagger) \).

For \( r \geq 1 \), we may argue as above that \( \sum_{i \geq r} \tilde{a}_i \tilde{P}(\lambda)_0 = \sum_{i \geq r} \tilde{a}_i \tilde{P}(\lambda)_0 \).

Consequently,

\[
\sum_{i \geq r} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger) = \sum_{i \geq r} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger).
\]

Multiplying by \( K \), the left-hand side gives \( (\text{rad} \tilde{a}_K)^r \phi(K\tilde{P}(\lambda)_0^\dagger) = \text{rad}^r \phi(\tilde{P}(\lambda)_0^\dagger) \).

The right-hand side, however, is an \( O \)-direct summand of \( \phi(\tilde{P}(\lambda)_0^\dagger) \), as shown by the direct sum decomposition above. Hence, it is equal to the intersection with \( \phi(\tilde{P}(\lambda)_0^\dagger) \) of its product with \( K \), i.e., it is equal to

\[
\phi(\tilde{P}(\lambda)_0^\dagger) \cap \text{rad}^r \phi(\tilde{P}(\lambda)_0^\dagger),
\]

and so \( \phi(\tilde{P}(\lambda)_0^\dagger) \) is tight as an \( \tilde{a} \)-lattice. Since \( \phi(\tilde{P}(\lambda)_0^\dagger) = \tilde{A} \phi(v) \), with \( v \)
and \( \phi(v) \) being \( \lambda \)-weight vectors, we have \( \phi(\tilde{P}(\lambda)_0^\dagger) \cong \tilde{\Delta}(\lambda) \), proving the theorem. \( \square \)

**Corollary 5.4.** Let \( \tilde{A} \) be a QHA over \( O \) with weight poset \( \Lambda \). If \( \lambda \in \Lambda \) satisfies the hypothesis of Theorem 5.3, then \( \text{gr} \tilde{\Delta}(\lambda) \) has a simple head. Explicitly, \( \text{head}(\text{gr} \tilde{\Delta}(\lambda)) \cong L(\lambda) \), the head of \( \tilde{\Delta}(\lambda) \). The assertion still holds if \( \lambda \) belongs to an ideal \( \Gamma \), and \( \text{gr} \tilde{\Delta}(\lambda) \) is regarded as a \( \text{gr} \tilde{A}_\Gamma \)-module.

**Proof.** By Theorem 5.2, \( \Delta(\lambda) \) is tight as a \( \tilde{a} \)-lattice. Next, note the action of \( \text{gr} \tilde{a} \) on \( \text{gr} \tilde{\Delta}(\lambda) \) factors naturally through the graded map \( \text{gr} \tilde{a} \to \text{gr} \tilde{A} \). By Proposition 5.2(c), \( \text{gr} \tilde{\Delta}(\lambda)_0 \) generates \( \text{gr} \tilde{\Delta}(\lambda) \) as a \( \text{gr} \tilde{a} \)-module, and thus as a \( \text{gr} \tilde{A} \)-module. But \( \text{gr} \tilde{\Delta}(\lambda)_0 \) has a simple \( \text{gr} \tilde{A} \)-head, since it has a simple \( \tilde{A} \)-head (which, of course, is also the head \( L(\lambda) \) of \( \tilde{\Delta}(\lambda) \)). \( \square \)

Now we can use Theorem 4.17 applied to ideals in \( \Lambda \), to obtain some integral QHAs of the form \( \text{gr} \tilde{A}_\Gamma \). The assertion below regarding \( \tilde{N} \) follows from Corollary 4.15.

**Theorem 5.5.** Let \( \tilde{A} \) be a QHA over \( O \) with weight poset \( \Lambda \). If \( \Gamma \) is an ideal in \( \Lambda \) such that the hypothesis of Theorem 5.3 holds for all \( \lambda \in \Gamma \), then \( \text{gr} \tilde{A}_\Gamma \) is a QHA (over \( O \)) with weight poset \( \Gamma \), and standard modules \( \text{gr} \tilde{\Delta}(\lambda), \lambda \in \Gamma \). In addition, if \( \tilde{N} \) is a \( \tilde{A}_\Gamma \)-lattice with a \( \tilde{\Delta} \)-filtration, then

\[
\sum_{i \geq 1} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger) = \bigoplus_{i \geq 1} \tilde{a}_i \phi(\tilde{P}(\lambda)_0^\dagger).
\]
the gr $\tilde{A}$-lattice gr $\tilde{N}$ has a gr $\Delta$-filtration (a filtration with sections gr $\Delta(\mu)$, $\mu \in \Gamma$).

6. Quantum enveloping algebras

For the rest of this paper, let $\Phi$ be a (classical) irreducible root system. Let $C = (c_{i,j})$ be the $r \times r$ Cartan matrix of $\Phi$; thus, $c_{i,j} = (\alpha_i^\vee, \alpha_j)$ if $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a fixed simple set of roots of $\Phi$. Let $\Phi^+$ be the corresponding set of positive roots, and let $\alpha_0 \in \Phi^+$ be the maximal short root. For $\alpha \in \Phi$, let

$$d_\alpha := (\alpha, \alpha)/(\alpha_0, \alpha_0) \in \{1, 2, 3\}. \tag{6.1}$$

Next, let $p > 2$ be a prime integer. If $\Phi$ has type $G_2$, then assume $p > 3$. (Later additional assumptions will be made on $p$.) Thus, $p$ does not divide any of the integers $d_\alpha$ defined above in (6.1). In the polynomial ring $\mathbb{Z}[v]$, let $m = (p, v - 1)$ and put $A = \mathbb{Z}[v]_m$. Setting $O = A/(\phi_p(v))$, with $\phi_p(v) = v^{p-1} + \cdots + 1$, the $p$th cyclotomic polynomial, $(K, O, \mathbb{F}_p)$ is a $p$-modular system, where $K = \mathbb{Q}(\zeta)$. Here $\zeta := v + (\phi_p(v))$ is a primitive $p$th root of unity. Under the quotient map $\pi : O \rightarrow \mathbb{F}_p$, we have $\pi(\zeta) = 1$. Later it will be important to observe that, given $\alpha \in \Phi$,

$$\zeta^{d_\alpha - 1} = u_\alpha(\zeta - 1), \quad \text{for some unit } u_\alpha \in O. \tag{6.2}$$

In fact, we can take $u_\alpha = \zeta^{d_\alpha - 1} + \cdots + \zeta + 1 \in O$, which is a unit since $\pi(u_\alpha) = d_\alpha \cdot 1 \in \mathbb{F}_p$ by comments just above.

Let $U'$ be the quantum enveloping algebra over the function field $\mathbb{Q}(v)$, with generators $E_\alpha, F_\alpha, K_\alpha, K_{\alpha}^{-1}, \alpha \in \Pi$, satisfying the usual relations. For $\beta \in \Phi$, write $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ (for integers $n_\alpha$) and set

$$K_\beta := \prod_{\alpha \in \Pi} K_\alpha^{n_\alpha} \in U'. \tag{6.3}$$

Let $U'_A$ be the $A$-subalgebra generated by the divided powers $E_\alpha^{(i)} = E_\alpha^i/[i]_\alpha^i$, $F_\alpha^{(i)} = F_\alpha^i/[i]_\alpha^i$ and the elements $K_\alpha, K_{\alpha}^{-1}, \alpha \in \Pi, i \in \mathbb{N}$.

Let $U'_\zeta$ be the $\zeta$-mod of finite dimensional integrable, type 1 $U'_\zeta$-modules has irreducible modules $L_\zeta(\lambda)$ indexed by the poset $X^+$ of dominant weights $[\Pi]$, §5.1. In fact, it is a highest weight category (in the sense of $[3]$) with standard (resp., costandard) modules denoted $\Delta_\zeta(\lambda)$ (resp., $\nabla_\zeta(\lambda)$), $\lambda \in X^+$. Given a finite ideal $\Lambda$ in the poset $X^+$, let $U'_\zeta$-$\text{mod}[\Lambda]$ be the full subcategory of $U'_\zeta$-$\text{mod}$ consisting of objects whose composition factors have the form $L_\zeta(\lambda)$, $\lambda \in \Lambda$. Let $\tilde{I}$ be the annihilator in $U'_\zeta$ of $U'_\zeta$-$\text{mod}[\Lambda]$, and set $\tilde{A} = U'_\zeta \otimes \tilde{I}$. Then $\tilde{A}$ is an integral QHA in the sense above. See $[5], [7]$ for more details.

\footnote{Here $[i]_\alpha^i$ means that the polynomial $[i]_\alpha^i$ is to be evaluated at $v^{d_\alpha}$.}
We can repeat the previous paragraph, replacing \( \Lambda \) by a finite non-empty ideal of the poset of \( p \)-regular dominant weights. We wish to verify that the Hypothesis 4.7 holds for the algebra \( \tilde{A} = A_\Lambda \) for \( \Lambda \) a suitably large ideal of \( p \)-regular weights. Recall from [16, §8] that \( \Lambda \) is fat provided, given any \( p \)-restricted, \( p \)-regular weight \( \lambda \), we have \( 2(p-1)\rho + w_0\lambda \in \Lambda \). This implies, in particular, that \( \Lambda \) contains all \( p \)-restricted, \( p \)-regular dominant weights.

For \( \alpha \in \Pi \), \( K^p_\alpha - 1 \) is central in \( U'_\zeta \). Consider the \( \mathcal{O} \)-Hopf algebra

\[
\tilde{U}_\zeta := \tilde{U}'_{\zeta,K}/(K^p_\alpha - 1 | \alpha \in \Pi).
\]

Let \( U_\zeta := \tilde{U}_{\zeta,K} \). (Of course, the \( K^p_\alpha - 1 \in \tilde{I} \).) Following Lusztig [13, 6.5(b)] and [12, Thm. 8.3.4], let \( \tilde{u}_\zeta \) and \( u_\zeta = \tilde{u}_\zeta,K \) be the small quantum enveloping algebras. Thus, \( \tilde{u}_\zeta \) (resp., \( u_\zeta \)) is a (normal) subalgebra of \( \tilde{U}_\zeta \) (resp., \( U_\zeta \)). In addition, \( \tilde{u}_\zeta \) is a free \( \mathcal{O} \)-module of rank \( p^{\dim \tilde{\zeta}} \). Also, \( u_\zeta \) admits a triangular decomposition

\[
\tilde{u}_\zeta^- \otimes u_\zeta^0 \otimes \tilde{u}_\zeta^+ \xrightarrow{\text{mult}} u_\zeta,
\]

where \( u_\zeta^+, u_\zeta^- \), and \( u_\zeta^0 \) are certain subalgebras of the algebras \( U_\zeta^+, U_\zeta^-, U_\zeta^0 \) defined in [12], [13].

Finally, let \( u'_\zeta \) is defined to be the product of the \( p \)-regular blocks in the algebra \( u_\zeta \). Define \( \tilde{u}'_\zeta = \tilde{u}_\zeta \cap u'_\zeta \). It is easy to see that \( \tilde{u}'_\zeta \) is a direct factor of \( \tilde{u}_\zeta \), and \( k \otimes \tilde{u}'_\zeta \) is the product of all regular blocks of \( k \otimes \tilde{u}_\zeta \).

**Theorem 6.1.** Let \( \tilde{A} = \tilde{U}_{\zeta,A} \) for a finite, non-empty ideal \( \Lambda \) of \( p \)-regular dominant weights.

(a) Hypothesis 4.7 holds provided \( p > h \).

(b) If, in addition, \( \Lambda \) is a fat ideal, then the image \( \tilde{a} \) of \( \tilde{u}_\zeta \) in \( \tilde{A} \) is a pure subalgebra such that Conditions 5.1(1)–(4) hold.

(c) Also, for \( p > h \) and \( \Lambda \) fat, Condition 5.1(5) holds for the grading in Condition 5.1(1).

**Proof.** We begin by showing that Hypothesis 4.7(1) holds. We have already remarked that \( \tilde{A} \) is an integral QHA. We need to check that \( \tilde{A} \) contains an appropriate set of idempotents. We recall from [7, Cor. 3.3] that the left \( \tilde{U}_\zeta \)-module \( \tilde{A} \) is a direct sum of weight spaces, i. e., spaces upon which \( \tilde{U}_\zeta \) acts in a uniform way determined by \( \mu \). Let \( X = X_\Lambda \) be the (integral) weights \( \mu \) for which \( \tilde{U}_\zeta \)-module \( \tilde{A} \) has a nonzero \( \mu \)-weight space. All weights

\[\lambda \in X \text{ is } p \text{-regular if } (\lambda + \rho, \alpha^\vee) \neq 0 \text{ mod } p \text{ for all roots } \alpha. \text{ (Here } \rho \text{ is the Weyl weight.) So we assume that (1) } \Lambda \subseteq X^+ \text{ consists of } p \text{-regular weights, and (2) if } \mu \in X^+ \text{ is } p \text{-regular and } \mu \leq \lambda, \text{ then } \mu \in \Lambda.\]

Write \( u_\zeta = u_\zeta^+ \oplus u_\zeta^- \), where \( u_\zeta^\pm \) is the product of all singular blocks of \( u_\zeta \). Then all composition factors of any “reduction mod p” of (a full \( \tilde{u}_\zeta \)-lattice in) a \( u_\zeta^\pm \)-module, such as \( k \otimes (\tilde{u}_\zeta \cap u_\zeta^\pm) \), have \( p \)-regular weights. A similar statement holds for \( u_\zeta^0 \), using \( p \)-singular weights. Now \( \tilde{u}_\zeta/(\tilde{u}_\zeta \cap u_\zeta^+ \oplus \tilde{u}_\zeta \cap u_\zeta^-) \) is a homomorphic image of the lattices \( \tilde{u}_\zeta \cap u_\zeta^\pm \) and \( \tilde{u}_\zeta \cap u_\zeta^0 \) in \( u_\zeta^\pm \) and \( u_\zeta^0 \), respectively. This common quotient must be zero, giving the claimed properties of \( \tilde{u}_\zeta \).
in $X$ are $p$-regular, and $\Lambda \subseteq X$. Let $\bar{C}$ be the image of $\bar{A}$ of $\bar{U}_C^0$. Thus, $\bar{C}$ is commutative. Since $\bar{A}$ is a direct sum of rank 1 $\bar{C}$-modules (on which $\bar{U}_C^0$ acts with some weight $\lambda \in X$), the algebra $\bar{C}_K$ has a faithful completely reducible module, namely, $\bar{A}_K$. Hence, $\bar{C}_K$ is (split) semisimple and a direct sum of copies of $K$, one copy for each weight. Similarly, the image $C$ of $\bar{C}$ in $\bar{A}_K = A$ is split semisimple, and its irreducible (1-dimensional) modules are also indexed by the set $X$. If we base change $\bar{A}$ to $\bar{A}_\bar{O}$, where $\bar{O}$ is the completion of $\mathcal{O}$, idempotents in $\bar{C} \subseteq \bar{A}$ lift to idempotents in $\bar{O}\bar{C} \subseteq \bar{A}_\bar{O}$, preserving orthogonality. In this way, we get a set of $|X|$ orthogonal primitive idempotents in $\bar{O}\bar{C} \subseteq K\bar{O}\bar{C}$. But $K\bar{C}$ already has a complete set of $|X|$ orthogonal primitive idempotents in the (commutative) algebra $K\bar{C} \subseteq K\bar{O}\bar{C}$. By the uniqueness of complete sets of central primitive idempotents, the two sets of $|X|$ idempotents we have constructed are the same. Hence, the idempotents $\{e_\mu \mid \mu \in X\}$ constructed for $K\bar{C}$ actually lie in $K\bar{C} \cap \bar{O}\bar{C} = \bar{C} \subseteq \bar{A}$. This set of idempotents, together with its subset indexed by $\Lambda$, gives $\bar{A}$ the structure of a TQHA. Thus, Hypothesis 4.7(1) holds.

Property (2) holds for $p > h$ by [16, Thm. 8.4] (with $p$ playing the role of $e$). This completes the proof of (a).

Now we prove (b). Since $\Lambda$ is fat, $\bar{u}_\zeta^{\prime} = u_\zeta^{\prime}$ maps injectively onto its image $\bar{a}_K^{\prime}$ in $\bar{A}_K$. The same holds for $\bar{u}_\zeta^{\prime} = u_\zeta^{\prime}$ since $\bar{u}_\zeta$ is contained in $u_\zeta^{\prime}$, so, in particular, $\bar{u}_\zeta^{\prime} \cong \bar{a}$. Similarly, using fatness, $k \otimes \bar{u}_\zeta^{\prime}$ maps isomorphically onto the image of $k \otimes \bar{a}^{\prime}$ in $k \otimes \bar{a}$. The latter image is not $a$ priori isomorphic to $k \otimes \bar{a}$, but follows here, because $\bar{u}_\zeta^{\prime} \cong \bar{a}$. Now Lemma 2.3 implies that $\bar{a}$ is pure in $\bar{A}$.

In [2, §§18.17–18.21], it is proved that $u_\zeta^{\prime}$ is a Koszul algebra, a property which implies that $u_\zeta^{\prime}$ has a tight grading (i. e., a grading making it isomorphic to $\text{gr} u_\zeta^{\prime}$), provided that $p > h$ and the Lusztig character conjecture holds in the quantum enveloping algebra case (which is true [21]). Thus, Condition 5.1(1) holds. Condition 5.1(2) follows from left-right symmetry and [16, Lem. 8.3], and Condition 5.1(3) follows from [16, Thm. 6.4]. In fact, choose a larger ideal $\Lambda'$ containing $\Lambda$ and consisting of $p$-regular weights, such that the projective cover $P(\lambda)$ of $L(\lambda)$ in $U_{\Lambda'}$ is projective for $\bar{u}_K^{\prime}$. The final condition of (4) follows from Lemma 2.1. This proves (b).

The proof of (c) requires further results from [2], and it is given in §7, Appendix II. \[\Box\]

Remark 6.2. As remarked in the proof, the graded algebra $u_\zeta^{\prime}$ is isomorphic to $\text{gr} u_\zeta^{\prime}$ as graded algebras. Also, Conditions 5.1(2) show that the $\text{gr} \bar{a}_K \subseteq \text{gr} \bar{A}_K$ (whether $\Lambda$ is fat or not). Thus, the natural map $\bar{u}_\zeta^{\prime} \rightarrow \bar{a}$ induces a graded map $\bar{u}_\zeta^{\prime} \rightarrow \text{gr} \bar{A}$, under the hypotheses of the theorem (using the
grading on $\mathfrak{u}^*_\zeta$ in the proof of part (c)). The map is an injection when $\Lambda$ is fat.

**Theorem 6.3.** Assume that $p > 2h - 2$ is an odd prime and consider the algebra $A = \widehat{U}_\zeta \Gamma$, where $\Gamma$ is an non-empty ideal of $p$-regular dominant weights. Then $\text{gr } A$ is a QHA over $\mathcal{O}$, with standard modules $\text{gr } A(\lambda), \lambda \in \Gamma$.

**Proof.** Our goal is to verify the hypotheses of Theorem 6.1, leaving checking hypotheses (i)—(iii) in the statement of Theorem 5.3. Conditions 5.1 have already been dealt with in Theorem 6.3. Fixing $\lambda \in \Gamma$. In the category of $U_\zeta$-modules, we have

$$P_K(\lambda) \cong Q_K(\lambda_0) \otimes L_K(\lambda_1)^[1],$$

(6.4) Enlarging $\Gamma$ to a larger poset $\Lambda$ of weights (with $\Gamma$ an ideal in $\Lambda$), we can insure that $P_K(\lambda)$ in the display makes sense as an $\tilde{A}_K$-module. We assume this has been done.

Assuming $p \geq 2h - 2$, there is an $\tilde{A}$-module $\tilde{Q}(\lambda_0)$, a PIM in a certain highest weight category, lifting a corresponding truncated projective module for $\tilde{A}/p\tilde{A}$, with $\tilde{Q}(\lambda_0)_K \cong Q_K(\lambda_0)$. (See [7].) Also, $Q(\lambda_0) := \tilde{Q}(\lambda_0)/\pi \tilde{Q}(\lambda_0)$ is $\tilde{a}_K$-projective, and so $\tilde{Q}(\lambda_0)$, which is $\mathcal{O}$-free, is $\tilde{a}$-projective. Thus, $\tilde{Q}(\lambda_0)$ can be given an $\tilde{a}$-grading, as

$$\tilde{Q}(\lambda_0) \cong \tilde{a} \otimes_{\tilde{a}_0} \tilde{L}(\lambda_0) \subseteq a_K \otimes a_{K,0} L_K(\lambda_0),$$

where $\tilde{L}(\lambda_0)$ is a full $\tilde{a}$-lattice in $L_K(\lambda_0)$ isomorphic to the projection $\tilde{Q}(\lambda_0)/\tilde{Q}(\lambda_0) = \tilde{Q}(\lambda_0)_K \cong \tilde{Q}(\lambda_0)/\text{rad } \tilde{Q}(\lambda_0)_K \cong \tilde{P}_K(\lambda_0)/\text{rad } \tilde{P}_K(\lambda_0)$. Fix a nonzero $\lambda$-weight vector $v = v_0 \otimes v_1 \in Q_K(\lambda_0)_0 \otimes L_K(\lambda_1)^[1]$, with $v_0 \in Q_K(\lambda_0)_0 = (A \otimes a_0 L_K(\lambda_0))_0 \cong a_0 \otimes a_0 L_K(\lambda_0) \cong L_K(\lambda_0)$. Put

$$\tilde{P}(\lambda)^\dagger = \tilde{Q}(\lambda_0) \otimes \tilde{L}^{\text{min}}(\lambda_1)^[1],$$

(6.5) where $\tilde{L}^{\text{min}}(\lambda_1)^[1]$ is the full $\tilde{U}_\zeta$-lattice in $L_K(\lambda_1)^[1]$ generated by $v_1$. (Of course, this lattice, as well as $\tilde{Q}(\lambda_0)$, must be viewed as $\tilde{U}_\zeta$-modules to make sense of the tensor product.) In general, $\tilde{P}(\lambda)^\dagger$ is not a projective $\tilde{A}$-module, though it is projective for $\tilde{a}$. Its scalar extension $\tilde{P}(\lambda)_K \cong L_K(\lambda)$ is $A_K$-projective. The projectivity of $\tilde{P}(\lambda)^\dagger|_{\tilde{a}}$ implies that it is tight. (See the observation following Definition 2.2.) This establishes part (iii) of the hypothesis of Theorem 5.3.

---

18 If $M$ is a $U(\mathfrak{g}_K)$-module, then $M^{[1]}$ denotes the $U_\zeta$-module obtained by “pulling $M$ back” through the Frobenius morphism $U_\zeta \rightarrow U(\mathfrak{g}_K)$. If $M = L_K(\lambda)$ is the irreducible $\mathfrak{g}_K$-module of highest weight $\lambda$, then $M^{[1]} \cong L_K(p\lambda)$.

19 Reference [7] discusses integral lifting of truncated projective modules. $Q(\lambda_0)$ is a PIM in the category of $G$-modules whose composition factors $L(\nu)$ all satisfy $\nu \leq 2(p-1)\rho + w_0 \lambda_0$. This assumes $p \geq 2h - 2$; see [10].
Also, $\tilde{P}(\lambda)^\dagger = \tilde{A}v$, since adding $\tilde{P}(\lambda)^\dagger_{\geq 1} = \tilde{Q}(\lambda_0)_{\geq 1} \otimes \tilde{L}(\lambda_1)[1]$ to both sides of $\tilde{A}v$ gives $\tilde{P}(\lambda)^\dagger$, as may be checked in

$$(\tilde{P}(\lambda)^\dagger + \text{rad } P_K(\lambda))/\text{rad } P_K(\lambda) \cong \tilde{L}(\lambda_0) \otimes \tilde{L}^{\text{min}}(\lambda_1)[1].$$

(Notice that the latter module is a homomorphic image of $\tilde{\Delta}(\lambda)$. This follows from a result of Z. Lin, c.f., \cite{5, Prop. 1.9}.) This establishes hypothesis (i) of Theorem 5.3.

To establish hypothesis (ii) of Theorem 5.3, observe that

$$K\tilde{P}(\lambda) + E_K(\lambda)/E_K(\lambda) \text{ and } \tilde{A}_{K,0}v + E_K(\lambda)/E_K(\lambda) \text{ have the same dimension and are thus equal (in view of the above containment). This establishes hypothesis (ii).}$$

\section{Appendix I}

We prove the following result.

\textbf{Lemma 7.1.} Suppose that $\Gamma$ is a poset ideal in a finite poset $\Lambda$, and that $B$ is a QHA over a field $K$ with weight poset $\Lambda$. Assume that gr $B$ is also a QHA with weight poset $\Lambda$, where we identify irreducible modules for gr $B$ and $B$ through $B/\text{rad } B$. Then gr $(P(\gamma)_{\Gamma})$ is a PIM for $(\text{gr } B)_{\Gamma}$, for each $\gamma \in \Gamma$. Here $P(\gamma)$ is the PIM for $B$ associated to $\gamma$. In particular, for $\lambda \in \Lambda$, gr $\Delta(\lambda)$ is the standard module for $\text{gr } B$ associated to $\lambda$.

\textit{Proof.} If $\Gamma = \Lambda$, there is nothing to prove. Otherwise, let $\mu \in \Lambda \setminus \Gamma$ be maximal in $\Lambda$, and put $m = |P(\gamma) : L(\mu)|$. Thus, $P(\gamma)$ contains a submodule $D \cong (\Delta(\mu)^{\oplus m})|_{\Lambda'}$, and $P(\gamma)/D \cong P(\gamma)_{\Lambda'}$, where $\Lambda' = \Lambda \setminus \{\mu\}$.

We have $D/\text{rad } D \cong L(\mu)^{\oplus m}$, and $\text{gr } D/\text{gr } \text{rad } D$ is a graded version of $D/\text{rad } D$, as may be checked by counting composition factors. Write

$$\text{gr } D/\text{gr } \text{rad } D = \bigoplus_{i \in \mathbb{N}} L(\mu)(i)^{\oplus m_i},$$

with $m = \sum m_i$, $m_i \geq 0$.

Since $\mu$ is maximal, gr $\Delta(\mu)$ is projective both as a graded or ungraded gr $B$-module, and its shifts $(\text{gr } \Delta(\mu))(i)$ are also projective. Also, the head of $(\text{gr } \Delta(\mu))(i)$ is $L(\mu)(i)$. Consequently, there is a graded module map

$$\phi : D := \bigoplus_{i \in \mathbb{N}} \Delta(\mu)(i)^{\oplus m_i} \rightarrow \text{gr } D,$$

giving a surjection when composed with $\text{gr } D \rightarrow \text{gr } D/\text{gr } \text{rad } D$. By the assumption that gr $B$ is a QHA, the ungraded projective gr $B$-module gr $P(\gamma)$ has a filtration by standard modules, which may be assumed to have bottom term $D'$, a direct sum of standard modules. Consequently,
if \((\text{gr } \# D)'\) denotes the ungraded \(\text{gr } B\)-modules obtained by forgetting the grading on \(\text{gr } \# D\), we have

\[
D' \subseteq \text{Image } \phi' \subseteq (\text{gr } \# D)',
\]

where \(\phi'\) is the ungraded version of the map \(\phi\). (Of course, the underlying image is the same as for the graded version. A similar remark applies for \((\text{gr } \# D)\).)

Since 
\[
\dim D' = m \dim \Delta(\mu) = \dim D = \dim \text{gr } \# D,
\]

it follows that equality holds in the inclusions

\[
(\text{gr } P(\gamma))_{\Lambda'} = \text{gr } P(\gamma)/D' \cong \text{gr } (P(\gamma)/D) = \text{gr } (P(\gamma)_{\Lambda'})
\]
as ungraded \(\text{gr } A\)-modules. (Recall the graded isomorphism \(\text{gr } P(\gamma)/\text{gr } \# D \cong (\text{gr } P(\gamma)/D)\).

The lemma now follows by induction on \(\Lambda \setminus \Gamma\). \(\square\)

As a corollary of the proof, we have the following result. In the statement, we maintain the notation and hypotheses of the above lemma. The corollary is parallel to Corollary 4.16.

**Corollary 7.2.** Let \(M\) be a (finite dimensional) \(B\)-module such that \(M\) and \(\text{gr } M\) have standard filtrations. Then \(\text{gr } M_\Gamma = \text{gr } (M_\Gamma)\).

**Proof.** This is obtained by essentially the same argument as above, using induction on \(\Lambda \setminus \Gamma\). \(\square\)

8. **Appendix II: Quantum deformation theory over \(O\)**

The goal of this appendix is provide a proof of Theorem 6.1(c). The proof will require some preparation and relies on results in [2]. In the process, we extend (when \(p > h\) is a prime) the quantum deformation theory of [2] to a version over \(O = \mathbb{Z}(p)[\zeta]\), where \(\zeta\) is a \(p\)th root of unity.

We let \(\Phi\) be an irreducible root system as in §5, and let \(\zeta\) be a primitive \(p\)th root of unity. Assume that \(p\) is an odd prime, \(p \neq 3\) in case \(\Phi\) has type \(G_2\). We also use other notation of §5 above, e. g., \(d_\alpha = (\alpha, \alpha)/(\alpha_0, \alpha_0)\). Notations of [2] will be used as they arise. For instance, \(U\) denotes the quantum algebra defined in [2; §1.3] over \(\mathbb{Q}(\zeta)\), a homomorphic image of the DeConcini-Kac quantum algebra \(U_2\). Thus, \(U = U^+ U^- U^0\) is the quantum enveloping algebra defined on [2; p. 15]. Here \(U^+, U^-\) are finite dimensional, while \(U^0\) is the algebra of Laurent polynomials in \(K_\alpha, \alpha \in \Pi\).

Let

\[
S' : = O \left[ \frac{K_\alpha - 1}{\zeta^{d_\alpha}_\alpha - 1} \alpha \in \Pi \right].
\]

This is a polynomial algebra over \(O\), an \(O\)-form of the polynomial algebra \(K [K_\alpha, \alpha \in \Pi]\). Observe that, for any \(\alpha \in \Phi^+\) (not just in \(\Pi\)), the expression

\[
H'_\alpha := \frac{K_\alpha - 1}{\zeta^{d_\alpha}_\alpha - 1}
\]
makes sense, using the definition of \( K_\alpha \) as a product given on [2] p. 47 (see also [3]). A recursive argument, applying the identity \( xy - 1 = (x - 1)y + (y - 1) \) to the numerator of \( H'_\alpha \), shows that \( H'_\alpha \) belongs to \( S' \). (Note that the denominator may be adjusted to have the correct \( d_\alpha \) by multiplying by a unit in \( \mathcal{O} \), since all the numbers \( d_\alpha \) are relatively prime to \( p \).)

We now make the following additional observations for \( \alpha \in \Phi^+ \), using the notation \([K_\alpha; j], 0 \leq j < p\) on [2] p. 48, and given implicitly below:

1. \( K_\alpha \in S' \). (This is obvious, since \( (\zeta^{d_\alpha} - 1)\frac{K_\alpha - 1}{K_\alpha} = K_\alpha - 1 \in S' \), for \( \alpha \in \Phi^+ \).

2. Let \( \hat{S}' \) be the completion (or just localization) with respect to the augmentation ideal of \( S' \) (i.e., the kernel of the augmentation map \( S' \to \mathcal{O} \)). Then \( K_\alpha^{-1} \in \hat{S}' \).

3. \( [K_\alpha; 0] \in K_\alpha^{-1}S' \), since

\[
[K_\alpha; 0] = \frac{K_\alpha^{-1}}{\zeta^{-d_\alpha}} \cdot \frac{K_\alpha + 1}{\zeta^{d_\alpha} + 1} \cdot \frac{K_\alpha - 1}{\zeta^{d_\alpha} - 1},
\]

and \( p \) is odd (which implies \( \zeta^{d_\alpha} + 1 \) is a unit in \( \mathcal{O} \)).

4. \( [K_\alpha; j] \in K_\alpha^{-1}S' \), \( 1 \leq j < p \), since \( [K_\alpha; j] = \frac{K_\alpha \zeta^{d_\alpha j} - \zeta^{-d_\alpha j} K_\alpha^{-1}}{\zeta^{d_\alpha} - \zeta^{d_\alpha}} \)

\[
= \frac{(K_\alpha - 1)\zeta^{d_\alpha j} + \zeta^{-d_\alpha j} (1 - K_\alpha^{-1})}{\zeta^{d_\alpha} - \zeta^{d_\alpha}} + \frac{\zeta^{d_\alpha j} - \zeta^{-d_\alpha j}}{\zeta^{d_\alpha} - \zeta^{-d_\alpha}}
\]

\[
= \left( \frac{K_\alpha - 1}{\zeta^{d_\alpha} - 1} \right) \cdot \frac{1}{\zeta^{d_\alpha} + 1} \zeta^{-d_\alpha} \left( \zeta^{d_\alpha j} + \zeta^{-d_\alpha j} K_\alpha^{-1} \right) + \frac{\zeta^{d_\alpha j} - \zeta^{-d_\alpha j}}{\zeta^{d_\alpha} - \zeta^{-d_\alpha}}.
\]

5. \( [K_\alpha; j]^{-1} \in \hat{S}' \) (or the corresponding localization of \( S' \)), as follows from (2) and (4), \( 1 \leq j < p \).

6. \( \log K_\alpha \in \hat{S}' \), since \( \frac{(\zeta^{d_\alpha - 1})^r}{r} \in \mathcal{O} \) for all integers \( r \geq 1 \).

As mentioned above, the proofs in this section will require the results and methods of [2]. In particular, we follow the notation of that paper closely. Consider the algebras \( S \) and \( B_S \) [2] p. 180] (expanded on [2] p. 179, bottom) to allow direct sums of compositions of wall-crossing functors; indeed, we should take \( B_S := \text{End}_{\overline{\mathcal{C}}_K(W_{a,S})}(\mathcal{Q}_I(S)) \)—see [2] p. 219]—for sufficiently large \( I \). The algebra \( S \) is a symmetric algebra on the integral root lattice associated to \( \Phi \) (the latter denoted \( R \) in [2]). The algebra \( B_S \) is \( \mathbb{Z} \)-graded, compatibly with a grading on \( S \) in which every root has grade 2. It gives rise, by base change, to various algebras \( B_A \), for any commutative \( S \)-algebra \( A \). Taking \( A = \mathbb{Q}(\zeta) \) yields an algebra isomorphic to the ungraded endomorphism algebra of the “\( Y \)-projective generator” of a certain category \( \mathcal{C}_K(\Omega) \), \( K = \mathbb{Q}(\zeta) \), with \( \Omega \) any regular orbit of the affine Weyl group \( W_a \cong W_p \) on integral weights. This result requires restrictions on \( p \): it must be a positive
odd integer $> h$, not divisible by 3 if $\Phi$ is of type $G_2$. (In [2], $\Phi$ is allowed to be decomposable.) Examination of this ungraded endomorphism algebra shows that $B^\text{op}_K$ is Morita equivalent to the block algebra of the small quantum group $u_\zeta$ associated to the orbit $\Omega$. See [2, Thm. 16.18], which is essentially already proved in this “Case 2” by the Claim [2, §14.13(4)].

When $p > h$ is a prime, we can define the above Morita equivalence over a DVR $\mathcal{O}$ with fraction field $K$, namely, $\mathcal{O} := \mathbb{Z}(p)[\zeta] = \mathbb{Z}[\zeta](p,\zeta - 1)$. This is not done in [2], but, with $S'$ and $\hat{S}'$ above in hand, it is possible to modify the arguments there to establish this Morita equivalence. Here is an outline: We wish to transport many of the results stated in [2] for the algebra $U$ over $K := \mathbb{Q}(\zeta)$ in “Case 2” to an $O$-form $U_O$ of $U$. Like $U$, the form $U_O$ has a triangular decomposition $U_O = U_O^- U_O^0 U_O^+$ with $U_O^-, U_O^+$ generated over $\mathcal{O}$ by the $F_\alpha, E_\alpha, \alpha \in \Phi^+$, and $U_O^0 := \mathcal{O} \left[ K_\alpha^{\pm 1}, H'_\alpha, \alpha \in \Pi \right]$, the localization of $S'$ above with respect to the multiplicatively closed set generated by the $K_\alpha$'s. Using (3) and (4) above, and standard identities (e.g., Kac’s identity [6, Lem. 5.27]), we find that $U_O$ is an $\mathcal{O}$-algebra, an $\mathcal{O}$-form of $U$. It also inherits the comultiplication on $U$, defined in [2, §7.1].

Observe

$$\Delta(\frac{K_\alpha - 1}{\xi^{d_\alpha} - 1}) = \frac{K_\alpha - 1}{\xi^{d_\alpha} - 1} \otimes K_\alpha + 1 \otimes \frac{K_\alpha - 1}{\xi^{d_\alpha} - 1}.$$ 

Define $B_\mathcal{O}$ to be the $\mathcal{O}$-algebra obtained by localizing $U_O^0$ at the multiplicatively closed set generated by all $[K_\alpha; j], \alpha \in \Phi^+, 0 \leq j < p$. Then $B_\mathcal{O}$ is an $\mathcal{O}$-form of the $K$-algebra $B$ defined in [2, p. 48, bottom]. Put $H_\alpha = [K_\alpha; 0]$ as at the beginning of [2, §8]. The results of [2, §8.6] hold (with similar proofs) if $B$ is replaced by $B_\mathcal{O}$. In particular, they hold for the $B_\mathcal{O}$-algebra $A = \hat{S}'$, which has other properties we need, such as those required on [2, p. 103, top]. Thus, the functor $\mathcal{V}$, given in [2, p. 86], may be defined on the category $\mathcal{C}_A(\Omega)$, using for the $e_\beta(\lambda)$, on [2, p. 86], generators of the Ext$^1$-groups described in terms of $A$ in [2, §§8.6, 8.7]. The orbit $\Omega$ on [2, p. 86] need not be regular, and this flexibility is important for discussing translation to, from, and through the walls. The target category of $\mathcal{V}$ is a certain “combinatorial category” $\mathcal{K}(\Omega, A)$ on which “combinatorial translation functors” are defined. The “combinatorial” category $\mathcal{K}(\Omega, A) = \mathcal{K}(\zeta)$ is defined by analogy with [2, §9.4, §14.4].

(If $\Omega$ is regular, $\mathcal{K}(\Omega, A)$ is isomorphic to $\mathcal{K}(W_\alpha, A)$ used above.) These translation functors are shown in [2, Prop., §10.11] to commute with the module theoretic translation functors. (Commutativity there implies commutativity in our case.)

Since $S'$ is a unique factorization domain, so are $U_O^0$ and $B_\mathcal{O}$ also unique factorization domains. In particular, the critical intersection property

$$A = \bigcap_{\beta \in \Phi^+} A^\beta$$
holds for $A$ above, since it is flat over $B_O$. See [2, Lem. 9.1] and its proof. The algebra $A^\beta$ is obtained by inverting all $H_\alpha = \frac{K_\alpha - K_{-1}}{\zeta \alpha - \zeta^{-1} \alpha} \in B_O$ with $\beta \neq \alpha \in \Phi^+$, and setting $A^\beta = A \otimes_{B_O} B_O^\beta$. Now [2, Prop. 9.4] essentially holds as stated and with the same proof, for any $A$ flat over $B_O$, and, in particular, for $A = \hat{S}'$. This gives a fully faithful functor

$$\mathcal{V}_\Omega : \mathcal{FC}_A(\Omega) \rightarrow \mathcal{K}(\Omega),$$

with $\mathcal{FC}_A(\Omega)$ denoting the category of $A$-flat $U_O$-modules, which are a direct sum of weight spaces ("$X$-graded" in the sense of [4]) with all weights in $\Omega$, and satisfying the conditions of [2, §2.3] with $U$ replaced by $U_O$.

The symmetric algebra $S = S(\mathbb{Z}\Phi)$ is written in [2, §14.3] using the symbol $h_\alpha$ to denote a root $\alpha \in \Phi$. It is then given two different interpretations, as $\log K_\alpha$ and as $d_\alpha H_\alpha$ in “Case 1” and “Case 2,” respectively. We can essentially handle both interpretations in our set-up at the same time: Fix $h_\alpha := \log K_\alpha \in \hat{S}'$ as in (6) above. Then $d_\alpha H_\alpha$ differ from $h_\alpha$ only by multiplication by a unit in $\hat{S}'$, and a similar comparison may be made to the generators $H'_\alpha$ of $S'$. In particular, $\hat{S}'$ is isomorphic to the completion of $S \otimes_{\mathbb{Z}} O$ with respect to its augmentation ideal. So $A = \hat{S}'$ is flat over $S$, giving the nice base-change property of [2, Lem. 14.8] for passing from objects and morphisms of $\mathcal{K}(\Omega, S)$ to $\mathcal{K}(\Omega, A)$. Together with the isomorphism $\mathcal{V}_\Omega$ above, this implies that $B_S \otimes_S A$ is the endomorphism algebra of a projective generator of $\mathcal{C}_A(\Omega)$. Base changing this generator from $A$ to $O$ shows, as in [2, 14.13(4)], that $B_S \otimes_S O$ is the endomorphism algebra of a projective generator for a regular block of $\tilde{u}_\zeta$-mod. See [2, §16.9] for a more complete treatment over $K$.

The algebra $B_S \otimes_S O$ retains the $\mathbb{Z}$-grading of $B_S$ and is compatible with the grading of $B \otimes_S K$. The latter grading is shown to be positive and even Koszul in [2, §§18.17–8.21]. (The hypothesis there on Lusztig conjecture holds for the quantum case when $p > h$.) The positive grading on $B_S \otimes_S O$ transports to a positive grading on $\tilde{u}_\zeta$, with the required properties in (c). (The algebra $\tilde{u}_\zeta$ is Morita equivalent to a product of copies of the algebra $(B_S \otimes_S O)^{op}$, so it has the form $eMe$, where $M$ is a full matrix algebra over a finite product of copies of the algebra $(B_S \otimes_S O)^{op}$, and $e \in M$ is an idempotent such that $Me$ is a projective generator of $M$-mod. Then $gr eMe \cong e(gr M)e \cong eMe$, the latter isomorphism coming from a similar one for $(B_S \otimes_S O)^{op}$, completing the proof.\[\square\]

\footnote{In checking the analogues of [2, §9.4] as well as the previous results in §8 mentioned above, a good strategy is to read through these results and, when references are made to previous sections in [2], to look back at those references and check that they hold in our context.}
As a corollary of the proof, we have the following new result, independent of Theorem 6.1 and worthy of attention in its own right. The notations $u'_\zeta$ and $\tilde{u}'_\zeta$ may be found above Theorem 6.1. They refer to the “regular” part of the small quantum group $u_\zeta$ and of its integral form, respectively.

**Theorem 8.1.** Assume that $p > h$ is a prime. Then the algebra $\tilde{u}'_\zeta$ over $O$ has a positive grading which base changes to the Koszul grading on $u'_\zeta$ obtained in [2 §§17-18 and p. 231].

**References**

1. H. Andersen, $p$-Filtrations and the Steinberg module, *J. Algebra* **244** (2001), 664–683.
2. H. Andersen, J. Jantzen, W. Soergel, *Representations of quantum groups at a $p$th root of unity and of semisimple groups in characteristic $p$*, Astérisque **220** (1994).
3. E. Cline, B. Parshall, and L. Scott, Finite dimensional algebras and highest weight categories, *J. reine angew. Math.* **391** (1988), 85–99.
4. E. Cline, B. Parshall and L. Scott, Integral and graded quasi-hereditary algebras, I, *J. Algebra* **131** (1990), 126–160.
5. E. Cline, B. Parshall, and L. Scott, Reduced standard modules and cohomology, *Trans. Amer. Math. Soc.* **361** (2009), 5223-5261.
6. B. Deng, J. Du, B. Parshall, and J.-P. Wang, *Finite Dimensional Algebras and Quantum Groups*, Math. Surveys and Monographs **150**, Amer. Math. Soc. (2008).
7. J. Du and L. Scott, Lusztig conjectures, old and new, I, *J. reine Angew. math.* **455** (1994), 141–182.
8. J. Du, B. Parshall, and L. Scott, Quantum Weyl reciprocity and tilting modules, *Comm. Math. Physics* **195** (1998), 321–352.
9. J. C. Jantzen, Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne, *J. Reine und Angew. Math.* **317** (1980), 157-199.
10. J.C. Jantzen, *Representations of algebraic groups, 2nd ed.* American Mathematical Society (2003).
11. J. Jantzen, *Lectures on Quantum Groups*, Grad. Studies in Math., 6, Amer. Math. Soc., 1996.
12. G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, *J. A.M.S.* **3**, 257–296.
13. G. Lusztig, Quantum groups at roots of 1, *Geo. Dedicata* **35** (1990), 89–114.
14. B. Parshall and L. Scott, Derived categories, quasi-hereditary algebras, and algebraic groups, Lecture Notes in Mathematics Series, Carleton University, 3 (1988), 1–105.
15. B. Parshall and L. Scott, Quantum Weyl reciprocity for cohomology, *Proc. London Math. Soc.* **90** (2005), 655-688.
16. B. Parshall and L. Scott, A new approach to the Koszul property in representation theory using graded subalgebras, *J. Inst. Math. Jussieu*, doi:10.1017/S1474748012000679 (2012) (see also arXiv:0910.0633).
17. B. Parshall and L. Scott, On $p$-filtrations of Weyl modules, to appear.
18. I Reiner, *Maximal Orders*, Academic Press (1975).
19. C. Ringel, The category of good modules over a quasi-hereditary algebra has almost-split sequences, *Math. Zeit.* **208** (1991), 209–225.
20. L. Scott, Semistandard filtrations and highest weight categories, *Mich. Math. J.* **58** (2009), 339–360.
21. T. Tanisaki, Character formulas of Kazhdan-Lusztig type, *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, Fields Inst. Commun. **40**, Amer. Math. Soc., Providence, RI (2004) 261–270.
22. N. Xi, Maximal and primitive elements in Weyl modules for type $A_2$, *J. Algebra* 215 (1999), 735–756.

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