TWO COUNTEREXAMPLES IN RATIONAL AND INTERVAL DYNAMICS

NICOLAE MIHALACHE

Abstract. In rational dynamics, we prove the existence of a polynomial that satisfies the Topological Collet-Eckmann condition, but which has a recurrent critical orbit that is not Collet-Eckmann. This shows that the converse of the main theorem in [11] does not hold.

In interval dynamics, we show that the Collet-Eckmann property for recurrent critical orbits is not a topological invariant for real polynomials with negative Schwarzian derivative. This contradicts a conjecture of Świątek [22].

1. Introduction

In one-dimensional real and complex dynamics, there are several conditions which guarantee some form of non-uniform hyperbolicity, which in turn gives a reasonable understanding on statistical and geometric properties of the dynamics. Classical examples include the Misiurewicz condition [12], semi-hyperbolicity [5] and the Collet-Eckmann condition (CE) [6, 11, 2, 13, 15, 14, 17, 18, 9, 19]. More recent examples include the Topological Collet-Eckmann condition (TCE) [14, 19, 20], summability conditions [16, 8, 3] and the Collet-Eckmann condition for recurrent critical orbits (RCE) [22, 11].

In the setting of rational dynamics, CE implies TCE [9, 20], but there are TCE polynomials which do not verify the CE condition [20]. Graczyk asked whether RCE is equivalent to TCE. In a previous paper, the author showed that RCE implies TCE [11]. Here, we show that the converse is not true.

Theorem A. There exists a TCE rational map that is not RCE.

On the interval, TCE is equivalent to CE in the S-unimodal setting [15, 14], thus CE is a topological invariant. However, with more than one critical point this is no longer true. Świątek conjectured that RCE is topologically invariant for multimodal analytic maps with negative Schwarzian derivative [22]. We show this conjecture to be false.

Theorem B. In interval dynamics, the RCE condition for S-multimodal analytic maps is not topologically invariant.

1.1. History. In one-dimensional dynamics, the orbits of critical points (zeros of the derivative in the smooth case) play a special role. Conditions on critical orbits and their consequences for the dynamics were first studied in the context of interval dynamics.

Two important results in this direction were published in 1981. Misiurewicz showed that S-unimodal maps with non-recurrent critical orbit have an absolutely continuous invariant
probability measure [12], see Definitions 2.9 and 2.18. Jakobson proved that real quadratic maps which have such invariant measures are abundant, that is, the set of their parameters is of positive measure [10]. Collet and Eckmann introduced the CE condition and showed that S-unimodal CE maps have absolutely continuous invariant probability measures [6]. Benedicks and Carleson showed that CE maps are abundant in the quadratic family [2].

Nowicki and Sands showed that CE is equivalent to several non-uniform hyperbolicity conditions for S-unimodal maps [13, 15]. One of them is Uniform Hyperbolicity on repelling Periodic orbits (UHP), see Definition 2.5. Shortly after, it was noticed by Nowicki and Przytycki that CE is also equivalent to TCE in this setting [14], see Definition 2.7. Therefore CE becomes a topological invariant for S-unimodal maps. This is no longer true for S-multimodal maps [14, 22]. Świątek conjectured that RCE (see Definition 2.10) is topologically invariant for multimodal analytic maps with negative Schwarzian derivative, see Conjecture 1 in [22]. We provide a counterexample to this conjecture, Theorem B.

A generalization of the Misiurewicz condition, semi-hyperbolicity, was studied in complex polynomial dynamics by Carleson, Jones and Yoccoz [5], see Definition 2.2. They show that semi-hyperbolicity is equivalent to TCE with $P = 1$ (see Definition 2.7), but also to John regularity of Fatou components. They also prove that such polynomials satisfy the Exponential Shrinking of components condition (ExpShrink), using a telescopic construction, see Definition 2.4.

CE rational were initially studied by Przytycki [17, 18]. Later, he showed that TCE implies CE if the Julia set contains only one critical point (unicritical case) [19]. Graczyk and Smirnov show that CE implies the backward or second Collet-Eckmann condition ($CE2(z_0)$), which in turn implies the Hölder regularity of Fatou components [9]. The first implication is obtained using a telescopic construction along the backward orbit of $z_0$. Przytycki and Rohde proved that CE implies TCE [21]. Therefore, in the unicritical case they are equivalent and CE is topologically invariant. Recently, Aspenberg showed that CE rational maps are abundant [1].

Przytycki, Rivera-Letelier and Smirnov establish the equivalence of several non-uniform hyperbolicity conditions, as $TCE, CE2(z_0), UHP, ExpShrink$ and the existence of a positive lower bound for the Lyapunov exponent of invariant measures [20]. A semi-hyperbolic counterexample shows that TCE does not imply CE. Another counterexample, involving semi-hyperbolic maps, shows that CE is not a quasi-conformal invariant.

In an attempt to characterize TCE in terms of properties of critical orbits, the author studied RCE for rational maps (see Definition 2.3). In [11] it is shown that RCE implies TCE. We provide a counterexample to the converse, Theorem A.

1.2. A short overview. In the following section we present some definitions and basic results and lemmas necessary for our study.

In Section 3 we describe a technique of building real polynomials with prescribed topological and analytical properties by specifying their combinatorial properties. The critical orbits of the polynomials we shall construct will be on the interval $[0, 1]$. Therefore, we can restrict our attention to the dynamics on the unit interval. We shall make use of the theory of kneading sequences to construct our maps.
A kneading sequence is a sequence of symbols associated to the points of a critical orbit. The critical points of an interval map define a partition of the interval. To each element of the partition we associate a symbol. The orbit of a critical point will thus generate a symbol sequence which describes the itinerary of the point through the various partition elements. Knowledge of the kneading sequences is enough to fully describe the combinatorics of a map. Moreover, in the absence of homtervals (Definition 2.14), two maps with the same kneading sequences are topologically conjugate.

We shall consider one-parameter families of bimodal maps (i.e. with two critical points) of the interval, see Definition 2.9. While the theory of multimodal maps and kneading sequences is generally well understood [7], for the most part it is related to topological properties of the dynamics. We develop new tools to obtain a prescribed growth (or lack thereof) of the derivative on the critical orbits.

In Section 4 we prove Theorem A, constructing an \textit{ExpShrink} polynomial (thus TCE) which is not RCE. In the vicinity of critical points the diameter of a small domain decreases at most in the power rate, while the derivative can approach 0 as fast as one wants (in comparison with the diameter of the domain). This important difference in the behaviour of derivative and diameter is the main idea of the (rather technical) proof.

In Section 5 we prove Theorem B, a counterexample to the conjecture of Świątek. Using careful estimates of the derivative on the critical orbits we construct two polynomials with negative Schwarzian derivative with the same combinatorics, thus topologically conjugate on the interval, such that only one is RCE. This situation is in sharp contrast with the unimodal case, where the Collet-Eckmann condition is topologically invariant \cite{15,14}. An important feature of our counterexample is that the corresponding critical points of this two polynomials are of different degree. One should be aware that considered as maps of the complex plane they are not conjugate.

2. Preliminaries

Let \( R \) be a rational map, \( J \) its Julia set and \( \text{Crit} \) the set of critical points.

\textbf{Definition 2.1.} We say that \( c \in \text{Crit} \) satisfies the Collet-Eckmann condition \((c \in \text{CE})\) if \( |(R^n)'(R(c))| > C \lambda^n \) for all \( n > 0 \) and some constants \( C > 0, \lambda > 1 \). We say that \( R \) is Collet-Eckmann if all critical points in \( J \) are CE.

\textbf{Definition 2.2.} Given \( c \in \text{Crit} \) we say that it is non-recurrent \((c \in \text{NR})\) if \( c \notin \omega(c) \), where \( \omega(c) \) is the \( \omega \)-limit set, the set of accumulation points of the orbit \((R^n(c))_{n>0}\). We call \( R \) semi-hyperbolic if all critical points in \( J \) are non-recurrent and \( R \) has no parabolic periodic orbits.

Recurrent Collet-Eckmann condition is weaker than Collet-Eckmann or semi-hyperbolicity alone. Indeed, in \cite{9} it is shown that a Collet-Eckmann rational map cannot have parabolic cycles.

\textbf{Definition 2.3.} We say that \( R \) satisfies the Recurrent Collet-Eckmann \((RCE)\) condition if every recurrent critical point in the Julia set is Collet-Eckmann and \( R \) has no parabolic periodic orbits.
Let us remark that a RCE rational map may have critical points in $J$ that are Collet-Eckmann and non-recurrent in the same time. Moreover any critical orbit may accumulate on other critical points.

Several weak hyperbolicity standard conditions are shown to be equivalent in [20]. Among these conditions we recall Topological Collet-Eckmann condition $(TCE)$, Uniform Hyperbolicity on Periodic orbits $(UHP)$, Exponential Shrinking of components $(ExpShrink)$ and Backward Collet-Eckmann condition at some $z_0 \in \mathbb{C}$ $(CE2(z_0))$.

Let us define these conditions.

**Definition 2.4.** $R$ satisfies the Exponential Shrinking of components condition $(ExpShrink)$ if there are $\lambda > 1$, $r > 0$ such that for all $z \in J$, $n > 0$ and every connected component $W$ of $R^{-n}(B(z,r))$

$$\text{diam } W < \lambda^{-n}.$$  

Here we use the spherical distance as $\infty$ may be contained in the Julia set.

**Definition 2.5.** $R$ satisfies Uniform Hyperbolicity on Periodic orbits $(UHP)$ if there is $\lambda > 1$ such that for all periodic points $z \in J$ with $R^n(z) = z$ for some $n > 0$

$$|{(R^n)'}(z)| > \lambda^n.$$ 

**Definition 2.6.** $R$ satisfies the Backward Collet-Eckmann condition at some $z_0 \in \mathbb{C}$ $(CE2(z_0))$ if there are $\lambda > 1, C > 0$ and $z_0 \in \mathbb{C}$ such that for any preimage $z \in \mathbb{C}$ of $z_0$ with $R^n(z) = z_0$ for some $n > 0$

$$|{(R^n)'}(z)| > C\lambda^n.$$ 

The definition of $TCE$ is technical but it is stated exclusively in topological terms, therefore it is invariant under topological conjugacy.

**Definition 2.7.** $R$ satisfies the Topological Collet-Eckmann condition $(TCE)$ if there are $M \geq 0, P \geq 1$ and $r > 0$ such that for all $z \in J$ there exists a strictly increasing sequence of integers $(n_j)_{j \geq 1}$ such that for all $j \geq 1$, $n_j \leq P \cdot j$ and

$$\# \{i : 0 \leq i < n_j, \text{Comp}_y R^{-(n_j-i)} B(R^{n_j}(x), r) \cap \text{Crit} \neq \emptyset \} \leq M,$$

where $\text{Comp}_y$ means the connected component containing $y$.

Using the equivalence of these conditions, we may formulate the main result in [11] as follows.

**Theorem 2.8.** The RCE condition implies TCE for rational maps.

To produce our counterexamples we restrict to dynamics of real polynomials on the interval with all critical points real. This is a particular case of multimodal dynamics.

In the sequel all distances and derivatives are considered with respect to the Euclidean metric if not specified otherwise.

Let us define multimodal maps and state some classical results about their dynamics.
Definition 2.9. Let $I$ be the compact interval $[0,1]$ and $f: I \to I$ a piecewise strictly monotone continuous map. This means that $f$ has a finite number of turning points $0 < c_1 < \ldots < c_l < 1$, points where $f$ has a local extremum, and $f$ is strictly monotone on each of the $l+1$ intervals $I_1 = [0,c_1), I_2 = (c_1,c_2), \ldots, I_{l+1} = (c_l,1]$. Such a map is called $l$-modal if $f(\partial I) \subseteq \partial I$. If $l = 1$ then $f$ is called unimodal. If $f$ is $C^{1+r}$ with $r \geq 0$ it is called a smooth $l$-modal map if $f'$ has no zeros outside $\{c_1,\ldots,c_l\}$.

If $f$ is a $l$-modal map, let us denote by $\text{Crit}_f$ the set of turning points - or critical points

$$\text{Crit}_f = \{c_1, \ldots, c_l\}.$$ 

Let us define the Recurrent Collet-Eckmann condition (RCE) in the context of multimodal dynamics. Remark that it is similar to Definition 2.3.

Definition 2.10. We say that $f$ satisfies RCE if every recurrent critical point $c \in \text{Crit}_f$, $c \in \omega(c)$ is Collet-Eckmann, that is, there exist $C > 0, \lambda > 1$ such that for all $n \geq 0$

$$|(f^n)'(f(c))| > C\lambda^n.$$ 

Our counterexamples are polynomials which have all critical points in $I = [0,1]$ which is included in the Julia set, as they do not have attracting or neutral periodic orbits and $I$ is forward invariant. Therefore in this case the previous definition is equivalent to Definition 2.3. Analogously, semi-hyperbolicity, UHP, CE2($x$) and TCE admit very similar definitions to the rational case.

For all $x \in I$ we denote by $O(x)$ or $O^+(x)$ its forward orbit

$$O(x) = (f^n(x))_{n \geq 0}.$$ 

Analogously, let $O^-(x) = \{y \in f^{-n}(x) : n \geq 0\}$ and $O^\pm(x) = \{y \in f^n(x) : n \in \mathbb{Z}\}$. We also extend these notations to orbits of sets. For $S \subseteq I$ let $O^+(S) = \{f^n(x) : x \in S, n \geq 0\}$, $O^-(S) = \{y \in f^{-n}(x) : x \in S, n \geq 0\}$ and $O^\pm(S) = O^+(S) \cup O^-(S)$.

One of the most important questions in all areas of dynamics is when two systems have similar underlying dynamics. A natural equivalence relation for multimodal maps is topological conjugacy.

Definition 2.11. We say that two multimodal maps $f, g: I \to I$ are topologically conjugate or simply conjugate if there is a homeomorphism $h: I \to I$ such that

$$h \circ f = g \circ h.$$ 

One may remark that if $f$ and $g$ are conjugate by $h$ then $h(f^n(x)) = g^n(h(x))$ for all $x \in I$ and $n \geq 0$ so $h$ maps orbits of $f$ onto orbits of $g$. It is easy to check that $h$ is a monotone bijection form the critical set of $f$ to the critical set of $g$. We may also consider combinatorial properties of orbits and use the order of the points of critical orbits to define another equivalence relation between multimodal maps. Theorem II.3.1 in [1] shows that it is enough to consider only the forward orbit of the critical set.

Theorem 2.12. Let $f, g$ be two $l$-modal maps with turning points $c_1 < \ldots < c_l$ respectively $\tilde{c}_1 < \ldots < \tilde{c}_l$. The following properties are equivalent.
(1) There exists an order preserving bijection \( h \) from \( O^+(\text{Crit}_f) \) to \( O^+(\text{Crit}_g) \) such that
\[
h(f(x)) = g(h(x)) \quad \text{for all} \; x \in O^+(\text{Crit}_f).
\]
(2) There exists an order preserving bijection \( \tilde{h} \) from \( O^+(\text{Crit}_f) \) to \( O^+(\text{Crit}_g) \) such that
\[
\tilde{h}(f(x)) = g(\tilde{h}(x)) \quad \text{for all} \; x \in O^+(\text{Crit}_f).
\]

If \( f \) and \( g \) satisfy the properties of the previous theorem we say that they are combinatorially equivalent. Note that if \( f \) and \( g \) are conjugate by an order preserving homeomorphism \( h \) then the restriction of \( h \) to \( O^+(\text{Crit}_f) \) is an order preserving bijection onto \( O^+(\text{Crit}_g) \) so \( f \) and \( g \) are combinatorially equivalent. The converse is true only in the absence of homtervals. It is the case of all the examples in this chapter. There is a very convenient way to describe the combinatorial type of a multimodal map using symbolic dynamics. We associate to every point \( x \in I \) a sequence of symbols \( i(x) \) that we call the itinerary of \( x \). The itineraries \( k_1, \ldots, k_j \) of the critical values \( f(c_1), \ldots, f(c_l) \) are called the kneading sequences of \( f \) and the ordered set of kneading sequences the kneading invariant. Combinatorially equivalent multimodal maps have the same kneading invariants but the converse is true only in the absence of homtervals. We use the kneading invariant to describe the dynamics of multimodal maps in one-dimensional families. We build sequences \( (F_n)_{n \geq 0} \) of compact families of \( C^1 \) multimodal maps with \( \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \) for all \( n \geq 0 \) and obtain our examples as the intersection of such sequences.

When not specified otherwise, we assume \( f \) to be a multimodal map.

**Definition 2.13.** Let \( O(p) \) be a periodic orbit of \( f \). This orbit is called attracting if its basin
\[
B(p) = \{ x \in I : f^k(x) \to O(p) \text{ as } k \to \infty \}
\]
contains an open set. The immediate basin \( B_0(p) \) of \( O(p) \) is the union of connected components of \( B(p) \) which contain points from \( O(p) \). If \( B_0(p) \) is a neighborhood of \( O(p) \) then this orbit is called a two-sided attractor and otherwise a one-sided attractor. Suppose \( f \) is \( C^1 \) and let \( m(p) = |(f^n)'(p)| \) where \( n \) is the period of \( p \). If \( m(p) < 1 \) we say that \( O(p) \) is attracting respectively super-attracting if \( m(p) = 0 \). We call \( O(p) \) neutral if \( m(p) = 1 \) and we say it is repelling if \( m(p) > 1 \).

Let us denote by \( B(f) \) the union of the basins of periodic attracting orbits and by \( B_0(f) \) the union of immediate basins of periodic attractors. The basins of attracting periodic contain intervals on which all iterates of \( f \) are monotone. Such intervals do not intersect \( O^-(\text{Crit}_f) \) and they do not carry too much combinatorial information.

**Definition 2.14.** Let us define a homterval to be an interval on which \( f^n \) is monotone for all \( n \geq 0 \).

Homtervals are related to wandering intervals and they play an important role in the study of the relation between conjugacy and combinatorial equivalence.

**Definition 2.15.** An interval \( J \subseteq I \) is wandering if all its iterates \( J, f(J), f^2(J), \ldots \) are disjoint and if \( (f^n(J))_{n \geq 0} \) does not tend to a periodic orbit.
Homintervals have simple dynamics described by the following lemma, Lemma II.3.1 in [7].

**Lemma 2.16.** Let $J$ be a hominterval of $f$. Then there are two possibilities:

1. $J$ is a wandering interval;
2. $J \subseteq B(f)$ and some iterate of $J$ is mapped into an interval $L$ such that $f^p$ maps $L$ monotonically into itself for some $p \geq 0$.

Multimodal maps satisfying some regularity conditions have no wandering intervals. Let us say that $f$ is *non-flat* at a critical point $c$ if there exists a $C^2$ diffeomorphism $\phi : \mathbb{R} \to I$ with $\phi(0) = c$ such that $f \circ \phi$ is a polynomial near the origin.

The following theorem is Theorem II.6.2 in [7].

**Theorem 2.17.** Let $f$ be a $C^2$ map that is non-flat at each critical point. Then $f$ has no wandering intervals.

Guckenheimer proved this theorem in 1979 for unimodal maps with *negative Schwarzian derivative* with non-degenerate critical point, that is with $|f''(c)| \neq 0$. The Schwarzian derivative was first used by Singer to study the dynamics of quadratic unimodal maps $x \to ax(1-x)$ with $a \in [0, 4]$. He observed that this property is preserved under iteration and that it is has important consequences in unimodal and multimodal dynamics.

**Definition 2.18.** Let $f : I \to I$ be a $C^3$ $l$-modal map. The *Schwarzian derivative* of $f$ at $x$ is defined as

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f(x)} \right)^2,$$

for all $x \in I \setminus \{c_1, \ldots, c_l\}$.

We may compute the Schwarzian derivate of a composition

$$S(g \circ f)(x) = Sg(f(x)) \cdot |f'(x)|^2 + Sf(x),$$

therefore if $Sf < 0$ and $Sg < 0$ then $S(f \circ g) < 0$ so negative Schwarzian derivative is preserved under iteration. Let us state an important consequence of this property for $C^3$ maps of the interval proved by Singer (see Theorem II.6.1 in [7]).

**Theorem 2.19** (Singer). If $f : I \to I$ is a $C^3$ map with negative Schwarzian derivative then

1. the immediate basin of any attracting periodic orbit contains either a critical point of $f$ or a boundary point of the interval $I$;
2. each neutral periodic point is attracting;
3. there are no intervals of periodic points.

Combining this result with Theorem 2.17 and Lemma 2.16 we obtain the following

**Corollary 2.20.** If $f$ is $C^3$ multimodal map with negative Schwarzian derivative that is non-flat at each critical point and which has no attracting periodic orbits then it has no hominterval. Therefore $O^-(\text{Crit}_f)$ is dense in $I$. 
The following corollary is a particular case of the corollary of Theorem II.3.1 in [7].

**Corollary 2.21.** Let \( f, g \) and \( h \) be as in Theorem 2.12. If \( f \) and \( g \) have no homtervals then they are topologically conjugate.

All our examples of multimodal maps in this chapter are polynomials with negative Schwarzian derivative and without attracting periodic orbits. We prefer however to use slightly more general classes of multimodal maps, as suggested by the previous two corollaries. As combinatorially equivalent multimodal maps have the same monotonicity type we only use maps that are increasing on the leftmost lap \( I_1 \), that is exactly the multimodal maps \( f \) with \( f(0) = 0 \). Let us define some classes of multimodal maps

\[
S_l = \{ f : I \to I : f \text{ is a } C^l \text{-modal map with } f(0) = 0 \},
\]

\[
S'_l = \{ f \in S_l : f \text{ is } C^3 \text{ and } Sf < 0 \},
\]

\[
P_l = \{ f \in S'_l : f \text{ non-flat at each critical point} \}
\]

\[
P'_l = \{ f \in P_l : \text{all periodic points of } f \text{ are repelling} \}.
\]

We have seen that in the absence of homtervals combinatorially equivalent multimodal maps are topologically conjugate. Using symbolic dynamics is a more convenient way to describe the combinatorial properties of forward critical orbits. Let \( \mathcal{A}_l = \{I_1, \ldots, I_{l+1}\} \) and \( \mathcal{A}_c = \{c_1, \ldots, c_l\} \) be two alphabets and \( \mathcal{A} = \mathcal{A}_l \cup \mathcal{A}_c \). Let

\[
\Sigma = \mathcal{A}_l^\mathbb{N} \cup \bigcup_{n \geq 0} (\mathcal{A}_l^n \times \mathcal{A}_c)
\]

be the space of sequences of symbols of \( \mathcal{A} \) with the following property. If \( \vec{i} \in \Sigma \) and \( m = |\vec{i}| \in \mathbb{N} \) is its length then \( m = \infty \) if and only if \( \vec{i} \) consists only of symbols of \( \mathcal{A}_l \). Moreover, if \( m < \infty \) then \( \vec{i} \) contains exactly one symbol of \( \mathcal{A}_c \) on the rightmost position. Let \( \Sigma' = \Sigma \setminus \mathcal{A}_c \) be the space of sequences \( \vec{i} \in \Sigma \) with \( |\vec{i}| > 1 \). Let us define the shift transformation \( \sigma : \Sigma' \to \Sigma \) by

\[
\sigma(i_0 i_1 \ldots) = i_1 i_2 \ldots.
\]

If \( f \in S_l \) let \( \vec{i} : I \to \Sigma \) be defined by \( \vec{i}(x) = i_0(x)i_1(x)\ldots \) where \( i_n(x) = I_k \) if \( f^n(x) \in I_k \) and \( i_n(x) = c_k \) if \( f^n(x) = c_k \) for all \( n \geq 0 \). The map \( \vec{i} \) relates the dynamics of \( f \) on \( I \setminus \{c_1, \ldots, c_l\} \) with the shift transformation \( \sigma \) on \( \Sigma' \)

\[
\vec{i}(f(x)) = \sigma(\vec{i}(x)) \text{ for all } x \in I \setminus \{c_1, \ldots, c_l\}.
\]

Moreover, we may define a **signed lexicographic ordering** on \( \Sigma \) that makes \( \vec{i} \) increasing. It becomes strictly increasing in the absence of homtervals.

**Definition 2.22.** A signed lexicographic ordering \( \prec \) on \( \Sigma \) is defined as follows. Let us define a sign \( \epsilon : \mathcal{A} \to \{-1, 0, 1\} \) where \( \epsilon(I_j) = (-1)^{j+1} \) for all \( j = 1, \ldots, l+1 \) and \( \epsilon(c_j) = 0 \) for all \( j = 1, \ldots, l \). Using the natural ordering on \( \mathcal{A} \) we say that \( x \prec y \) if there exists \( n \geq 0 \) such that \( x_i = y_i \) for all \( i = 0, \ldots, n - 1 \) and

\[
x_n \cdot \prod_{i=0}^{n-1} \epsilon(x_i) < y_n \cdot \prod_{i=0}^{n-1} \epsilon(y_i).
\]
Let us observe that $<$ is a complete ordering and that $\epsilon \cdot f' > 0$ on $I \setminus \{c_1, \ldots, c_l\}$, that is $\epsilon$ represents the monotonicity of $f$. The product $\prod_{i=0}^{n-1} \epsilon(x_i)$ represents therefore the monotonicity of $f^n$. This is the main reason for the monotonicity of $\bar{i}$ with respect to $\prec$.

**Proposition 2.23.** Let $f \in S_l$ for some $l \geq 0$.

1. If $x < y$ then $\bar{i}(x) \leq \bar{i}(y)$.
2. If $\bar{i}(x) < \bar{i}(y)$ then $x < y$.
3. If $f \in P'_l$ then $x < y$ if and only if $\bar{i}(x) < \bar{i}(y)$.

**Proof.** The first two points are Lemma II.3.1 in [7]. If $f \in P'_l$ then by Corollary 2.20 $O^-(\text{Crit}_f)$ is dense in $I$. Let us note that

$$O^-(\text{Crit}_f) = \{x \in I \mid |\bar{g}(x)| < \infty\}.$$  
Moreover, $O^-(\text{Crit}_f)$ is countable as $f^{-1}(x)$ is finite for all $x \in I$, therefore $\bar{i}$ is strictly increasing.  

Let us define the kneading sequences of $f \in S_l$ by $k_i = \bar{i}(f(c_i))$ for $i = 1, \ldots, l$, the itineraries of the critical values. The kneading invariant of $f$ is $K(f) = (k_1, \ldots, k_l)$. The last point of the previous lemma shows that if $f, g \in P'_l$ and $K(f) = K(g)$ then there is an order preserving bijection $h : O^+(\text{Crit}_f) \rightarrow O^+(\text{Crit}_g)$. Therefore, by Corollaries 2.20 and 2.21 $f$ and $g$ are topologically conjugate.

Let us define one-dimensional smooth families of multimodal maps. They are the central object of this paper.

**Definition 2.24.** We say that $\mathcal{F} : [\alpha, \beta] \rightarrow S_l$ is a family of $l$-modal maps if $\mathcal{F}$ is continuous with respect to the $C^1$ topology of $S_l$.

Note that we do not assume the continuity of critical points in such a family - as in the general definition of a family of multimodal maps in [7] - as it is a direct consequence of the smoothness conditions we impose.

When not stated otherwise we suppose $\mathcal{F} : [\alpha, \beta] \rightarrow S_l$ is a family of $l$-modal maps and denote $f_\gamma = \mathcal{F}(\gamma)$.

**Lemma 2.25.** The critical points $c_i : [\alpha, \beta] \rightarrow I$ of $f_\gamma$ are continuous maps for all $i = 1, \ldots, l$.

**Proof.** Fix $\gamma_0 \in [\alpha, \beta]$ and

$$0 < \epsilon < \frac{1}{2} \min_{i \neq j} |c_i(\gamma_0) - c_j(\gamma_0)|.$$

Let $A = \{x \in [0, 1] \mid \epsilon \leq \min_i |x - c_i(\gamma_0)|\}$, a finite union of compact intervals and

$$\theta = \min_{x \in A} |f'_{\gamma_0}(x)| > 0$$
by Definition 2.9. Then the monotonicity of $f_{\gamma_0}$ alternates on the connected components of $A$. Let $\delta > 0$ be such that $||f_\gamma - f_{\gamma_0}||_{C^1} < \frac{\theta}{2}$ for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$. Therefore the critical points $c_i(\gamma)$ satisfy

$$|c_i(\gamma) - c_i(\gamma_0)| < \epsilon$$
for all $i = 1, \ldots, l$ and $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ as $f'_\gamma(x) \cdot f'_{\gamma_0}(x) > 0$ for all $x \in A$. \qed

Let us show that the $C^1$ continuity of families of multimodal maps is preserved under iteration.

**Lemma 2.26.** Let $G, H : [a, b] \to C^1(I, I)$ be continuous. Then the map

$$c \mapsto G(c) \circ H(c)$$

is continuous on $[a, b]$.

**Proof.** Fix $c_0 \in [a, b]$ and $\varepsilon > 0$. We show that there is $\delta > 0$ such that

$$\|G(c) \circ H(c) - G(c_0) \circ H(c_0)\|_{C^1} < \varepsilon$$

for all $c \in (c_0 - \delta, c_0 + \delta) \cap [a, b]$. For transparency we denote $g_c = G(c)$ and $h_c = H(c)$ for all $c \in [a, b]$. Let

$$M = \max \{\|g_c\|_{C^1}, \|h_c\|_{C^1}, 1 : c \in [a, b]\}.$$  

As $g'_{c_0}$ is uniformly continuous on $I$, there is $\delta' > 0$ such that

$$|g'_{c_0}(x) - g'_{c_0}(y)| < \frac{\varepsilon}{4M}$$

for all $x, y \in I$ with $|x - y| < \delta'$.

Let $\delta > 0$ such that

$$\sup \{\|g_c - g_{c_0}\|_{C^1}, \|h_c - h_{c_0}\|_{C^1} : c \in (c_0 - \delta, c_0 + \delta) \cap [a, b]\} < \min \left(\frac{\varepsilon}{4M}, \delta' \right).$$

We compute a bound for $\|g_c \circ h_c - g_{c_0} \circ h_{c_0}\|_{C^1}$ for all $c \in (c_0 - \delta, c_0 + \delta) \cap [a, b]$

$$\|g_c \circ h_c - g_{c_0} \circ h_{c_0}\|_{C^1} \leq \|g_c \circ h_c - g_c \circ h_{c_0}\|_{\infty} + \|g_c \circ h_{c_0} - g_{c_0} \circ h_{c_0}\|_{\infty} \leq \frac{M \varepsilon}{4M} + \frac{\varepsilon}{4M} < \varepsilon.$$  

Analogously

$$\|g'_c \circ h_c \cdot h'_c - g'_{c_0} \circ h_{c_0} \cdot h'_{c_0}\|_{\infty} \leq \\|(g'_c \circ h_c - g'_{c_0} \circ h_c) \cdot h'_c\|_{\infty} + \\|(g'_{c_0} \circ h_c - g'_{c_0} \circ h_{c_0}) \cdot h'_c\|_{\infty} \leq \frac{\varepsilon}{4M}M + \frac{\varepsilon}{4M}M + \frac{\varepsilon}{4M} < \varepsilon$$

as $\|h_c - h_{c_0}\|_{\infty} < \delta'$.

Remark that by iteration $\gamma \mapsto f^n_\gamma$ is continuous for all $n \geq 1$.

The following proposition shows that pullbacks of given combinatorial type of continuous maps are continuous in a family of multimodal maps.

**Proposition 2.27.** Let $\gamma : [\alpha, \beta] \to \mathcal{I}$ be continuous and $S \in \mathcal{A}_1^n$, where $\mathcal{I}$ denotes the interior of $I$. A maximal connected domain of definition $D$ of the map $\gamma \mapsto x_\gamma$ such that

$$f^n_\gamma(x_\gamma) = y(\gamma)$$

and $\gamma : (x_\gamma) \in S \times \Sigma$

is open in $[\alpha, \beta]$ and $\gamma \mapsto x_\gamma$ is unique and continuous on $D$. 

Proof. Suppose that for some $\gamma$ there are $x_1 < x_2 \in I$ such that $f^n_{\gamma}(x_1) = f^n_{\gamma}(x_2) = y(\gamma)$ and
and such that $\hat{z}(x_1) = \hat{z}(x_2) = S_i(y(\gamma))$ for some $\gamma \in [\alpha, \beta]$. But $S \in A^I$ so $f^n$ is strictly
monotone on $[x_1, x_2]$, which contradicts $f^n_{\gamma}(x_1) = f^n_{\gamma}(x_2)$ so $\gamma \to x_2$ is unique.

Let $x_{\gamma_0}$ be as in the hypothesis and $\varepsilon > 0$ such that $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon) \subseteq I$. We show that
there exists $\delta > 0$ such that $\gamma \to x_\gamma$ is defined on $(\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ and takes values
in $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$. Let
$$\theta = (f^n_{\gamma_0})'(x_{\gamma_0}) \neq 0$$
and by eventually diminishing $\varepsilon$ we may suppose that
$$|(f^n_{\gamma_0})'(x) - \theta| < \frac{\theta}{4} \text{ for all } x \in (x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon).$$

Let $\delta_1 > 0$ be such that
$$||f^n_{\gamma} - f^n_{\gamma_0}||_{C^1} < \frac{\theta \varepsilon}{4} < \frac{\theta}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_1, \gamma_0 + \delta_1) \cap [\alpha, \beta].$$

Let also $\delta_2 > 0$ be such that
$$|y(\gamma) - y(\gamma_0)| < \frac{\theta \varepsilon}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_2, \gamma_0 + \delta_2) \cap [\alpha, \beta].$$

We choose $\delta = \min(\delta_1, \delta_2)$ and show that
$$y(\gamma) \in f^n_{\gamma}((x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)) \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta].$$

Indeed, $f^n_{\gamma}$ is monotone on $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$ and

$$|f^n_{\gamma}(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| > \frac{\theta \varepsilon}{4}$$

for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ as $|f^n_{\gamma}(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| = |f^n_{\gamma}(x_{\gamma_0} \pm \varepsilon) - f^n_{\gamma_0}(x_{\gamma_0} \pm \varepsilon) + f^n_{\gamma_0}(x_{\gamma_0} \pm \varepsilon) - f^n_{\gamma_0}(x_{\gamma_0})| \text{ and } |f^n_{\gamma_0}(x_{\gamma_0} \pm \varepsilon) - f^n_{\gamma_0}(x_{\gamma_0})| > \frac{3}{4} \theta \varepsilon.$

As an immediate consequence of the previous proposition and Lemma 2.25 we obtain the following corollary.

Corollary 2.28. If $F$ realizes a finite itinerary sequence $i_{\alpha} \in \Sigma$, that is for all $\gamma \in [\alpha, \beta]$ there is $x(i_{\alpha})(\gamma) \in I$ such that
$$\hat{z}(x(i_{\alpha})(\gamma)) = i_{\alpha},$$
then $x(i_{\alpha}) : [\alpha, \beta] \to I$ is unique and continuous.

One may observe that if $x, y : [\alpha, \beta] \to I$ are continuous and for some $k \geq 0$

$$(f_k^y(x(\alpha)) - y(\alpha)) \cdot (f_k^y(x(\beta)) - y(\beta)) < 0$$

then there exists $\gamma \in [\alpha, \beta]$ such that
(2) \hspace{1cm} $$f_k^\gamma(x(\gamma)) = y(\gamma).$$

Therefore if $\hat{z}(x(\alpha)) \neq \hat{z}(x(\beta))$ then there exists $\gamma \in [\alpha, \beta]$ such that $\hat{z}(x(\gamma))$ is finite.
Let $m = \min\{k \geq 0 : \exists \gamma \in [\alpha, \beta] \text{ such that } \hat{z}(x(\alpha))(k) \neq \hat{z}(x(\gamma))(k)\}$ then the itinerary
$\sigma^m \hat{z}(x(\gamma)) = \hat{z}(f_{\gamma}^m(x(\gamma)))$ changes the first symbol on $[\alpha, \beta]$. Without loss of generality we
may assume that $\sigma^m(x(\alpha)) < \sigma^m(x(\beta))$. Therefore there exists $i \in \{1, \ldots, l\}$ such that $f^m_\gamma(x(\alpha)) \leq c_i(\alpha)$ and $f^m_\gamma(x(\beta)) \geq c_i(\alpha)$, which yields $\gamma$ using the previous remark.

A simplified version of the proof of Proposition 2.27 shows that if $F : [\alpha, \beta] \to C^1(I)$ is continuous, $r_0 \in I$ is a root of $F(\gamma_0)$ and $(F(\gamma_0))'(r_0) \neq 0$ then there are $J \subseteq [\alpha, \beta]$ a neighborhood of $\gamma_0$ and $r : J \to I$ continuous such that $F(\gamma)(r(\gamma)) = 0$ for all $\gamma \in J$. For $F(\gamma)(x) = f^m_\gamma(x) - x$ we obtain the following corollary.

**Corollary 2.29.** Let $r_0$ be a periodic point of $f_{\gamma_0}$ of period $n \geq 1$ that is not neutral. There exists a connected neighborhood $J \subseteq [\alpha, \beta]$ of $\gamma_0$ and $r : J \to I$ continuous such that $r(\gamma)$ is a non-neutral periodic point of $f_\gamma$ of period $n$. Moreover, provided $r(\gamma)$ is not super-attracting for any $\gamma \in J$, the itinerary $i(r(\gamma))$ is constant.

**Proof.** As a periodic point, $r(\gamma)$ exists and is continuous on a connected neighborhood $J_0$ of $\gamma_0$, using the previous remark. As $|(f^n_\gamma)'(r_0)| \neq 1$, there is a connected neighborhood $J_1$ of $\gamma_0$ such that

$$|(f^n_\gamma)'(r(\gamma))| \neq 1 \text{ for all } \gamma \in J_1.$$ 

Let $J = J_0 \cap J_1$ so $r(\gamma)$ is a non-neutral periodic point of period $n$ for all $\gamma \in J$. Suppose that its itinerary $i(r(\gamma))$ is not constant, then there is $\gamma_1 \in J$ such that $i(r(\gamma_1))$ is finite so the orbit of $r(\gamma_1)$ contains a critical point thus it is super-attracting. \qed

Let us define the **asymptotic kneading sequences** $k_j^-(\gamma)$ and $k_j^+(\gamma)$ for all $\gamma \in [\alpha, \beta]$ and $j = 1, \ldots, l$. When they exist, the asymptotic kneading sequences capture important information about the local variation of the kneading sequences.

**Definition 2.30.** Let $j \in \{1, \ldots, l\}$ and $\gamma \in [\alpha, \beta]$. If $\gamma > \alpha$ and for all $n \geq 0$ there exists $\delta > 0$ such that $k_j^-(\gamma - \theta) \in S_n \times \Sigma$ with $S_n \in A^l_\theta$ for all $\theta \in (0, \delta)$ then we set $k_j^-(\gamma)(k) = S_n(k)$ for all $0 \leq k < n$. Analogously, if $\gamma < \beta$ and for all $n \geq 0$ there exists $\delta > 0$ such that $k_j^+(\gamma + \theta) \in S'_n \times \Sigma$ with $S'_n \in A^l_\theta$ for all $\theta \in (0, \delta)$ then we set $k_j^+(\gamma)(k) = S'_n(k)$ for all $0 \leq k < n$.

Let us define a sufficient condition for the existence of the asymptotic kneading sequences for all $\gamma \in [\alpha, \beta]$.

**Definition 2.31.** We call a family $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ of $l$-modal maps natural if for all $j = 1, \ldots, l$ the set

$$k_j^{-1}(\hat{i}) = \{ \gamma \in [\alpha, \beta] : k_j^-(\gamma) = \hat{i} \}$$

is finite for all $\hat{i} \in \Sigma$ finite.

This property does not hold in general for $C^1$ families of multimodal maps, even polynomial, as such a family could be reparametrized to have intervals of constancy in the parameter space. It is however generally true for analytic families such as the quadratic family $a \mapsto ax(1-x)$ with $a \in [0, 4]$.

The following proposition shows that this property guarantees the existence of all asymptotic kneading sequences.
Proposition 2.32. Let \( F : [\alpha, \beta] \rightarrow S_l \) be a natural family of \( l \)-modal maps and \( j \in \{1, \ldots, l\} \). Then \( k_j^- (\gamma) \) exists for all \( \gamma \in (\alpha, \beta) \) and \( k_j^+ (\gamma) \) exists for all \( \gamma \in [\alpha, \beta] \). Moreover, if \( k_j^-(\gamma) \in \mathcal{A}_l^\gamma \) for some \( \gamma \in (\alpha, \beta) \) then \( k_j^- (\gamma) = k_j^-(\gamma) = k_j^+(\gamma) \). If \( k_j^-(\gamma) = S \in \mathcal{A}_l^\gamma \) for some \( n \geq 0 \) and \( i \in \{1, \ldots, l\} \) then \( k_j^- (\gamma) = Sl_1 l_2 \ldots \) and \( k_j^+ (\gamma) = Sr_1 r_2 \ldots \) with \( l_1, r_1 \in \{I_i, I_{i+1}\} \).

Proof. If \( F \) is natural then the set of all \( \gamma \in [\alpha, \beta] \) that have at least one kneading sequence of length at most \( n \) for some \( n > 0 \)

\[
K_n = \bigcup_{j=1}^{l} \{ \gamma \in [\alpha, \beta] : |k_j(\gamma)| \leq n \}
\]

is finite. This is sufficient for the existence of all asymptotic kneading sequences.

If \( k_j^- (\gamma_0) \in S \times \Sigma \) with \( S \in \mathcal{A}_l^\gamma \) and \( n \geq 0, j \in \{1, \ldots, l\} \) then by the continuity of \( \gamma \rightarrow f_\gamma^m(c_j) \) and of \( \gamma \rightarrow c_i \) for all \( m = 0, \ldots, n-1 \) and \( i = 1, \ldots, l \) there exists \( \delta > 0 \) such that

\[
k_j^- (\gamma) \in S \times \Sigma \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta].
\]

Therefore if \( k_j^- (\gamma) \in \mathcal{A}_l^\gamma \) then \( k_j^- (\gamma) = k_j^- (\gamma) = k_j^-(\gamma) \). If \( k_j^- (\gamma) = S \in \mathcal{A}_l^\gamma \) for some \( i \in \{1, \ldots, l\} \) then \( k_j^- (\gamma) = Sl_1 l_2 \ldots \) and \( k_j^+ (\gamma) = Sr_1 r_2 \ldots \). Again by the continuity of \( \gamma \rightarrow f_\gamma^m(c_j) \) and of \( \gamma \rightarrow c_k \) for all \( k = 1, \ldots, l \)

\[
l_1, r_1 \in \{I_i, I_{i+1}\}.
\]

\( \square \)

Note that we may omit the parameter \( \gamma \) whenever there is no danger of confusion but \( c_j, i \) and \( k_j \) for some \( j \in \{1, \ldots, l\} \) should always be understood in the context of some \( f_\gamma \).

However, the symbols of the itineraries of \( \Sigma \) are \( I_1, \ldots, I_{l+1}, c_1, \ldots, c_l \) and do not depend on \( \gamma \).

3. One-parameter families of bimodal maps

In this section we consider a natural family \( G : [\alpha, \beta] \rightarrow \mathcal{P}_2 \) of bimodal polynomials with negative Schwarzian derivative satisfying the following conditions

\[
(3) \quad 0, 1 \in \partial I \text{ are fixed and repelling for } g_\alpha,
\]

\[
(4) \quad g_\gamma (c_1) = 1 \text{ for all } \gamma \in [\alpha, \beta],
\]

\[
(5) \quad g_\gamma (c_2) = 0 \text{ if and only if } \gamma = \alpha.
\]

Let us denote by \( v_n = g_{\gamma_{n+1}}(c_2) \) for \( n \geq 0 \) the points of the second critical orbit and let \( k = k_2^- (\gamma) = k_0 k_1 \ldots \). If \( S \in \mathcal{A}_l^\gamma, k \geq 1 \) and \( n \geq 1 \) we write \( S^n \) for \( SS \ldots S \in \mathcal{A}_l^{kn} \) repeated \( n \) times and \( S^\infty \) for \( SS \ldots S \in \mathcal{A}_l^{\infty} \).

Proposition 2.32 shows the existence of \( k^+ (\alpha) = k (\alpha) = I_1^\infty \) therefore, there is \( \delta_0 > 0 \) such that

\[
k \in I_1^2 \times \Sigma
\]
for all $\gamma \in [\alpha, \alpha + \delta_0]$. Figure 1 represents the graph of a bimodal map with the second kneading sequence $I_1 c_1 > \gamma$ for all $\gamma \in [\alpha, \alpha + \delta_0]$.

Let us observe that $O^+(\text{Crit}_{g_\alpha}) = \{0, c_1, c_2, 1\}$ and that by Singer’s Theorem 2.19 $g_\alpha$ has no homtervals. Therefore by Corollary 2.21 if $\mathcal{H} : [\alpha', \beta'] \rightarrow \mathcal{P}_2$ is a natural family satisfying conditions (3) to (5) then $g_\alpha$ and $h_{\alpha'}$ are topologically conjugate. Moreover, $g_\alpha$ is conjugate to the second Chebyshev polynomial (on $[-2, 2]$) and topological properties of its dynamics are universal. Let us study this dynamics and extend by continuity some of its properties to some neighborhood of $\alpha$ in the parameter space.

We have seen that $g_\alpha$ has no homtervals and that all its periodic points are repelling. Proposition 2.23 shows that the map $\tilde{i}(g_\alpha) : I \rightarrow \Sigma$ is strictly increasing.

Let us denote by $\sigma^- (\tilde{i})$ the set of all preimages of $\tilde{i}$ by some shift

$$\sigma^- (\tilde{i}) = \{ \tilde{i}' \in \Sigma : \exists k \geq 0 \text{ such that } \sigma^k(\tilde{i}') = \tilde{i} \} .$$

As $(0, 1) = \tilde{i} \subseteq g_\alpha(I_j)$ for $j = 1, 2, 3$, $g_\alpha(c_1) = 1$, $g_\alpha(c_2) = 0$, $\tilde{i}(g_\alpha)(0) = I_1^\infty$ and $\tilde{i}(g_\alpha)(1) = I_3^\infty$, $\tilde{i}(g_\alpha)(I) = \Sigma \setminus (\sigma^-(I_1^\infty) \cup \sigma^-(I_3^\infty))$.

Let us denote by $\Sigma_0 = \tilde{i}(g_\alpha)(I) = \tilde{i}(g_\alpha)(\tilde{i}) \cup \{ I_1^\infty, I_3^\infty \}$. Then

(7) $\tilde{i}(g_\alpha) : I \rightarrow \Sigma_0$ is an order preserving bijection.

Remark also that $\Sigma_0$ is the space of all itinerary sequences of $I$ under a bimodal map.

As $g_\alpha$ is decreasing on $I_2$, $g_\alpha(c_1) > c_1$ and $g_\alpha(c_2) < c_2$ it has exactly one fixed point $r \in I_2$ and it is repelling. Moreover, $g_\alpha$ has no fixed points in $I_1$ or $I_3$ other than 0 and 1 as this would contradict the injectivity of $\tilde{i}(g_\alpha)$. As 0 and 1 are repelling fixed points
g_\alpha(x) > x for all x ∈ (0, c_1) and g_\alpha(x) < x for all x ∈ (c_2, 1). Then by the C^1 continuity of G and Corollary 2.29 we obtain the following lemma.

**Lemma 3.1.** There is δ_1 > 0 such that g_\gamma has exactly one fixed point r(\gamma) in (0, 1) and all its fixed points 0, 1 and r(\gamma) are repelling for all \gamma ∈ [α, α + δ_1]. Moreover, the map γ → r(\gamma) is continuous and i(r) = I_∞ 2.

Let p be a periodic point of period 2 of g_\alpha. Then i(p) is periodic of period 2 and infinite. So i(p) ∈ \{(I_j I_k)^\infty : j, k = 1, 2, 3\}. But i(g_\alpha) is injective, i(g_\alpha)(0) = I_1^\infty, i(g_\alpha)(r) = I_2^\infty and i(g_\alpha)(1) = I_3^\infty so

i(p) ∈ \{(I_j I_k)^\infty : j \neq k and j, k = 1, 2, 3\} ⊆ \Sigma_0.

Therefore g_\alpha has exactly 3 periodic orbits of period 2 with itinerary sequences (I_1 I_2)^∞, (I_1 I_3)^∞, (I_2 I_3)^∞ and their shifts. Figure 2 illustrates the periodic orbits of period 2 of g_\alpha. By continuity of γ → g_\gamma^2 and Corollary 2.29 we obtain the following lemma.

**Lemma 3.2.** There is δ_2 > 0 such that g_\gamma has exactly 3 periodic orbits of period 2 with itinerary sequences (I_1 I_2)^∞, (I_1 I_3)^∞, (I_2 I_3)^∞ for all \gamma ∈ [α, α + δ_2]. Moreover, the periodic orbits of period 2 are repelling and continuous with respect to \gamma on [α, α + δ_2].

Let us define

(8) \quad \beta' = α + \min\{δ_0, δ_1, δ_2\}

so that G satisfies equality (6), Lemma 3.1 and the previous lemma for all γ ∈ [α, β'].

Let us consider the dynamics of all maps g_\gamma with γ ∈ [α, β'] from the combinatorial point of view. We observe that if x ≥ v = g_\gamma(c_2) then g_\gamma^n(x) ≥ v for all n ≥ 0. This means that any itinerary of g_\gamma is of the form i_\gamma = I_1^k a ⋯ ∈ \Sigma_0 with k ≥ 0, a ≠ I_1 and such that
Lemma 3.3. Let \( \gamma_0 \in [\alpha, \beta'] \) and \( k = k_2(\gamma_0) \). Then every finite itinerary
\[
i_0 \in \{i \in \Sigma(k) : |i| < \infty\}
\]
is realized by a unique point \( x(i) \in I \) and \( \gamma \mapsto x(i) \) is continuous on a neighborhood of \( \gamma_0 \).

A kneading sequence \( k \in \Sigma(k) \) satisfies the following property.

Definition 3.4. We call \( m \in \Sigma_0 \) minimal if
\[
m \leq \sigma^k m \text{ for all } 0 \leq k < |m|.
\]

The following proposition shows that the minimality is an almost sufficient condition for an itinerary to be realized as the second kneading sequence in the family \( \mathcal{G} \). This is very similar to the realization of maximal kneading sequences in unimodal families but the proof involves some particularities of our family \( \mathcal{G} \). For the convenience of the reader, we include a complete proof.

Proposition 3.5. Let \( \alpha \leq \alpha_0 < \beta_0 \leq \beta' \) and \( m \) be a minimal itinerary such that
\[
k(\alpha_0) < m < k(\beta_0).
\]
Then there exists \( \gamma \in (\alpha_0, \beta_0) \) such that
\[
k(\gamma) = m.
\]

Proof. Suppose that \( k(\gamma) \neq m \) for all \( \gamma \in (\alpha_0, \beta_0) \). Let \( \gamma_0 = \sup \{ \gamma \in [\alpha_0, \beta_0] : k(\gamma) \leq m \} \) and \( n = \min \{ j \geq 0 : k(\gamma_0)(j) \neq m(j) \} < \infty \). Then, using the continuity of \( g_\gamma^n \), \( c_1 \) and \( c_2 \) one may check that
\[
k_n = k(\gamma_0)(n) \in A_c = \{ c_1, c_2 \},
\]
otherwise the maximality of \( \gamma_0 \) is contradicted as \( k(0), \ldots, k(n-1) \) and \( k(n) \) would be constant on an open interval that contains \( \gamma_0 \). There are two possibilities

1. \( k_n = c_1 \) so \( g_\gamma^n(c_2) = c_1 \) therefore \( c_2 \) is preperiodic.
2. \( k_n = c_2 \) so \( g_\gamma^n(c_2) = c_2 \) therefore \( c_2 \) is super-attracting.

Therefore \( \gamma_0 > \alpha \) and \( \gamma_0 \leq \beta' < \beta \). Let us recall that \( \mathcal{G} \) is a natural family so the asymptotic kneading sequences \( k^-(\gamma_0) \) and \( k^+(\gamma_0) \) do exist and are infinite. Then the definition of \( \gamma_0 \) shows that
\[
\min(k(\gamma_0), k^-(\gamma_0)) \leq m \leq k^+(\gamma_0).
\]
Let \( m = m_0 m_1 \ldots m_n \ldots \) and \( S = m_0 \ldots m_{n-1} \in A_i^n \) be the maximal common prefix of \( k(\gamma_0) \) and \( m \), so \( k(\gamma_0) = Sc_j \) with \( j \in \{1, 2\} \). Therefore, using Proposition 2.32 \( m_n \in \{I_j, I_{j+1}\} \).

Suppose \( k_n = c_1 \) so \( g_\gamma^n(c_2) = c_1 \). Lemma 3.3 and property (6) show that the sequences \( I_1 I_2 c_2 \) and \( I_2 I_3 c_2 \) are realized as itineraries by all \( g_\gamma \) with \( \gamma \in [\alpha, \beta'] \) for all \( k \geq 0 \). Moreover
Let us prove a complementary combinatorial property.
Lemma 3.6. Let $S \in \mathcal{A}^n_j$ with $k(\alpha) \preceq SI^\infty_2 \preceq k(\beta')$ and such that $SI^\infty_2$ is minimal. If $i_1 i_2 \ldots \in \Sigma$ and $i_1, i_2, \ldots \in \mathcal{A} \setminus \{I_1\}$ then

$$SI^k_{i_1 i_2} \ldots \in \Sigma \text{ is minimal for all } k \geq |S|.$$ 

Proof. Let $\hat{i} = SI^k_{i_1 i_2} \ldots \in \Sigma, n = |S|$ and $k \geq n$. Suppose there exists $j > 0$ such that

$$\sigma^j(\hat{i}) \prec \hat{i}.$$ 

As $SI^\infty_2 \preceq k(\beta') = I_1 \ldots$

$$\hat{i} \in I_1 \times \Sigma.$$ 

Then $j < n$ and we set $m = \min \{p \geq 0 : \sigma^j(\hat{i})(p) \neq \hat{i}(p)\}$. Therefore $m \leq n - 1$ so

$$\sigma^j(SI^\infty_2) \prec SI^\infty_2$$

as $\hat{i}$ coincides with $SI^\infty_2$ on the first $2n$ symbols, a contradiction. □

Using relation (6), $k(\gamma) = I_1 \ldots$ so $I^k_2 c_j \in \Sigma(k())$ for all $k \geq 0$, $j = 1, 2$ and $\gamma \in [\alpha, \beta']$. Then by Lemma 3.3 the maps

$$\gamma \rightarrow p_k(\gamma) = x(I^k_2 c_1)(\gamma) \text{ and } \gamma \rightarrow q_k(\gamma) = x(I^k_2 c_2)(\gamma)$$

are uniquely defined and continuous on $[\alpha, \beta']$ for all $k \geq 0$. Let us recall that $g_\gamma$ is decreasing on $I_2$ so

$$c_1 < I^2_2 c_2 < I^3_2 c_1 < I^3_2 c_2 < \ldots < I^\infty_2 < \ldots < I^3_2 c_1 < I^3_2 c_2 < I_2 c_1 < c_2,$$

therefore

$$c_1 = p_0 < q_1 < p_2 < q_3 < \ldots < r < \ldots < p_3 < q_2 < p_1 < q_0 = c_2$$

for all $\gamma \in [\alpha, \beta']$.

Let us show that $g_\gamma \rightarrow r$ and $g_k \rightarrow r$ as $k \rightarrow \infty$ for all $\gamma \in [\alpha, \beta']$. Let

$$r^- = \lim_{k \rightarrow \infty} p_k = \lim_{k \rightarrow \infty} q_k,$$

$$r^+ = \lim_{k \rightarrow \infty} q_k = \lim_{k \rightarrow \infty} p_{k+1}.$$ 

Suppose that $r^- < r^+$ then by continuity $g_\gamma(r^-) = r^+$ and $g_\gamma(r^+) = r^-$, as $g_\gamma(p_{k+1}) = p_k$ and $g_\gamma(q_k) = q_k$ for all $k \geq 0$. Then $r^- \text{ and } r^+$ are periodic points of period 2 and with itinerary sequence $I^\infty_2$, which contradicts Lemma 3.2. By compactness (10)

$$p_k, q_k \rightarrow r \text{ uniformly as } k \rightarrow \infty.$$ 

The following proposition shows that these convergences have a counterpart in the parameter space.

Proposition 3.7. Let $S \in \mathcal{A}^n_j$ for some $n \geq 0$ be such that $SI^\infty_2$ is minimal and $k^{-1}(SI^\infty_2)$ is finite. Let $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta'$ be such that $k(\alpha_0) < SI^\infty_2 < k(\beta_0)$ and $S' = SI^k_{i_1 i_2}$ with $k \geq 0$ and such that $e(S') = 1$. If $\hat{i}_1 = S' c_1$, $\hat{i}_2 = S' c_2$ and $k$ is sufficiently large then we may define

$$\gamma_1 = \max \left( k^{-1}(\hat{i}_1) \cap (\alpha_0, \beta_0) \right) \text{ and }$$

$$\gamma_2 = \min \left( k^{-1}(\hat{i}_2) \cap (\gamma_1, \beta_0) \right).$$
and then
\[ \lim_{k \to \infty} (\gamma_2 - \gamma_1) = 0. \]

Proof. First let us remark that the condition \( \epsilon(S') = 1 \) guarantees that
\[ i_1 < SI_{2}^{\infty} < i_2. \]

Using for example convergences \( \square \) and the bijective map \( \hat{g}_a \) defined by \( \Box \) there exists \( N_0 > 0 \) such that for all \( k \geq N_0, k(\alpha_0) < i_1 < i_2 < k(\beta_0) \). Moreover, if \( k \geq n \) then \( i_1 \) and \( i_2 \) are minimal, using Lemma \( \Box \).

Therefore for \( k \geq \max(N_0, n) \) we may apply Proposition \( \Box \) to show that there exist \( \gamma_1 \in k^{-1}(i_1) \cap (\alpha_0, \beta_0) \) and \( \gamma_2 \in k^{-1}(i_2) \cap (\gamma_1, \beta_0) \). As \( i_1 \) and \( i_2 \) are finite and the family \( G \) is natural, \( k^{-1}(i_1) \) and \( k^{-1}(i_2) \) are finite.

We may apply again Proposition \( \Box \) to see that \( \gamma_1 \) is increasing to a limit \( \gamma^- \) as \( k \to \infty \). Again by Proposition \( \Box \) and by the finiteness of \( k^{-1}(SI_{2}^{\infty}) \) there exists
\[ \gamma_{\infty} = \max (k^{-1}(SI_{2}^{\infty}) \cap (\alpha_0, \beta_0) < \beta_0 \) and \( \gamma^- \leq \gamma_{\infty} \).

For the same reasons there is \( N > 0 \) such that \( \gamma_2 > \gamma_{\infty} \) for all \( k \geq N \), therefore \( \gamma_2 \) becomes decreasing and converges to some \( \gamma^+ \geq \gamma_{\infty} \).

Suppose that the statement does not hold, that is
\[ \gamma^- < \gamma^+ . \]

The map \( \hat{g}_a : I \to \Sigma_0 \) is bijective and order preserving and \( p_i \to r, q_i \to r \) as \( i \to \infty \) therefore
\[ \{ i \in \Sigma_0 : i_1 \leq i \leq i_2 \text{ for all } k > 0 \} = \{ SI_{2}^{\infty} \}. \]

Then the definitions of \( \gamma^- \) and \( \gamma^+ \) imply that
\[ k(\gamma) = SI_{2}^{\infty} \text{ for all } \gamma \in [\gamma^-, \gamma^+], \]
which contradicts the hypothesis. \( \square \)

From the previous proof we may also retain the following Corollary.

Corollary 3.8. Assume the hypothesis of the previous proposition. Then
\[ \lim_{k \to \infty} \gamma_1 = \lim_{k \to \infty} \gamma_2 = \gamma_{\infty} \]
and \( k(\gamma_{\infty}) = SI_{2}^{\infty} \).

We may also control the growth of the derivative on the second critical orbit in the setting of the last proposition. In fact, letting \( k \to \infty \), the second critical orbit spends most of its time very close to the fixed repelling point \( r \). Therefore the growth of the derivative along this orbit is exponential.

Let us also compute some bounds for the derivative along two types of orbits.

Lemma 3.9. Let \( [\gamma_1, \gamma_2] \subseteq [a, \beta'], n \geq 0, S \in A^0_l \) and \( i_1, i_2 \in S \times \Sigma \) with \( i_1 < i_2 \) be finite or equal to \( I_{1}^{\infty} \), \( I_{2}^{\infty} \) or \( I_{3}^{\infty} \). If \( i_1, i_2 \) are realized on \( [\gamma_1, \gamma_2] \) then there exists \( \theta > 0 \) such that
\[ \theta < \left| \left( g^j_l \right)'(x) \right| < \theta^{-1} \]
for all \( \gamma \in [\gamma_1, \gamma_2] \), \( x \in [x(i_1), x(i_2)] \) and \( j = 1, \ldots, n. \)
Proof. Let us remark that \( \dot{i}(x) \in S \times \Sigma \) therefore \( (g^x_j)'(x) \neq 0 \) for all \( \gamma \in [\gamma_1, \gamma_2] \), \( x \in [x(i_1), x(i_2)] \) and \( j = 1, \ldots, n \). As \( x(i_1) \) and \( x(i_2) \) are continuous by Lemmas 3.1 and 3.3 the set
\[
\{ (\gamma, x) \in \mathbb{R}^2 : \gamma \in [\gamma_1, \gamma_2], x \in [x(i_1), x(i_2)] \}
\]
is compact. Therefore the continuity of \((\gamma, x) \rightarrow (g^x_j)'(x)\) for all \( j = 1, \ldots, n \) implies the existence of \( \theta \).

The previous lemma helps us estimate the derivative of \( g^x_j(x) \) on a compact interval of parameters if \( \dot{i}(x) \in I^n_j \times \Sigma \) and \( n \) is sufficiently large. Let us denote
\[
I_j(n)(\gamma) = \{ x \in I_j : g^x_j(x) \in I_j \text{ for all } k = 1, \ldots, n \}
\]
for \( j=1,2,3 \), the interval of points of \( I_j \) that stay in \( I_j \) under \( n \) iterations. Let also \( s_j \) be the unique fixed point in \( I_j \).

Lemma 3.10. Let \([\gamma_1, \gamma_2] \subseteq [\alpha, \beta'] \), \( j \in \{1, 2, 3\} \) and \( \varepsilon > 0 \). Let also
\[
\lambda_1 = \lambda_1(j) = \min_{\gamma \in [\gamma_1, \gamma_2]} |g^x_j(s_j)|,
\]
\[
\lambda_2 = \lambda_2(j) = \max_{\gamma \in [\gamma_1, \gamma_2]} |g^x_j(s_j)|.
\]
There exists \( N > 0 \) such that for all \( k > 0 \), \( \gamma \in [\gamma_1, \gamma_2] \) and \( x \in I_j(\max(k, N))(\gamma) \)
\[
\lambda_1^{k(1-\varepsilon)} < |(g^x_j)'(x)| < \lambda_2^{k(1+\varepsilon)}.
\]

Proof. Let us first observe that by the definition (8) of \( \beta' \)
\[
1 < \lambda_1 \leq \lambda_2.
\]
Lemma 3.3 shows that the itinerary sequences \( I^n_j c_1, I^n_j c_2 \) are realized on \([\alpha, \beta']\) for all \( n \geq 0 \). We may easily obtain analogous convergences to (10) if \( j \in \{1, 3\} \), therefore
\[
x(I^n_j c_1), x(I^n_j c_2) \rightarrow s_j \text{ uniformly as } n \rightarrow \infty.
\]
Moreover \( \partial I_j(n)(\gamma) \subseteq \{ x(I^n_j c_1), x(I^n_j c_2), s_j \} \) for all \( \gamma \in [\gamma_1, \gamma_2] \) and \( n \geq 0 \). By continuity of \( s_j \) and of \((\gamma, x) \rightarrow g^x_j(x)\) there exists \( N_0 > 0 \) such that
\[
\lambda_1^{1-\varepsilon} < |g^x_j(x)| < \lambda_2^{1+\varepsilon}
\]
for all \( \gamma \in [\gamma_1, \gamma_2] \) and \( x \in I_j(N_0)(\gamma) \).

Using Lemma 3.9 there exists \( \theta > 0 \) such that
\[
\theta < |(g^x_j)'(x)| < \theta^{-1}
\]
for all \( \gamma \in [\gamma_1, \gamma_2] \), \( x \in I_j(N_0)(\gamma) \) and \( 1 \leq i \leq N_0 \). Let \( N_1 > 0 \) be such that
\[
\lambda_1^{N_1/2} > \theta^{-1} \lambda_2^{N_0(1+\varepsilon)}
\]
and set \( N = N_0 + N_1 \). Let \( k > N_1 \) and \( n = \max(N_1, k - N_0) \) then
\[
\theta \lambda_1^{n(1-\varepsilon)} < |(g^x_j)'(x)| < \theta^{-1} \lambda_2^{n(1+\varepsilon)}
\]
for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)(\gamma)$, where $m = \max(k, N)$. As $n \geq N_1$ and $1 < \lambda_1 \leq \lambda_2$
\[ \lambda_1^{k(1-\varepsilon)} < \left| (g^n_\gamma)'(x) \right| < \lambda_2^{k(1+\varepsilon)} \]
for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)(\gamma)$. If $k \leq N_1$ then $g^n_\gamma(x) \in I_j(N_0)(\gamma)$ for all $n = 0, \ldots, k - 1$ so
\[ \lambda_1^{k(1-\varepsilon)} < \lambda_1^{k(1-\frac{1}{2})} < \left| (g^n_\gamma)'(x) \right| < \lambda_2^{k(1+\frac{1}{2})} < \lambda_2^{k(1+\varepsilon)} \]
for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)(\gamma)$. 

We may remark that if we assume the hypothesis of the previous lemma then $g^n_\gamma$ is monotone on $I_j(m)(\gamma)$ therefore for all $\gamma \in [\gamma_1, \gamma_2]$
\begin{equation}
\lambda_2(j)^{-k(1+\varepsilon)} < |I_j(m)(\gamma)| < \lambda_1(j)^{-k(1-\varepsilon)}.
\end{equation}

Let $d_n : [\alpha, \beta'] \to \mathbb{R}_+$ be defined by
\[ d_n(\gamma) = \left| (g^n_\gamma)'(v) \right|, \]
where $v = g_\gamma(c_2)$ the second critical value. As $\gamma \to v$ and $\gamma \to g^n_\gamma$ are continuous, $d_n$ is continuous. The family $\mathcal{G}$ is natural so $d_n$ has finitely many zeros for all $n \geq 0$.

**Corollary 3.11.** Assume the hypothesis of Proposition 3.7 and let $\lambda_0 = |g''_{\gamma_m}(v)| > 1$. For all $0 < \varepsilon < 1$ there exists $N > 0$ such that if $k \geq N$ then
\[ \lambda_0^{(n+k)(1-\varepsilon)} < d_{n+k}(\gamma) < \lambda_0^{(n+k)(1+\varepsilon)} \text{ for all } \gamma \in [\gamma_1, \gamma_2]. \]

**Proof.** Let us remark that $|f(\gamma)| > n$ for all $\gamma \in [\gamma_1, \gamma_2]$ therefore there exists $\theta > 0$ such that
\[ \theta < d_n(\gamma) < \theta^{-1} \text{ for all } \gamma \in [\gamma_1, \gamma_2]. \]

Using the previous argument and Corollary 3.8 there exists $N_0 > 0$ such that if $k \geq N_0$ then
\[ \lambda_0^k(1-\frac{1}{2}) < \left| (g^n_\gamma)'(v_n) \right| < \lambda_0^k(1+\frac{1}{2}) \text{ for all } \gamma \in [\gamma_1, \gamma_2]. \]

Therefore it is enough to choose $N \geq N_0$ such that
\[ \lambda_0^{N\frac{1}{2}} > \theta^{-1} \lambda_0^{n(1-\varepsilon)}. \]

\[ \square \]

4. TCE does not imply RCE

In this section we consider a family $\mathcal{G} : [\alpha, \beta] \to \mathcal{P}_2$ (see the definition of $\mathcal{P}_2$ at page 8) satisfying all properties (3) to (6) and Lemmas 3.1 and 3.2 for all $\gamma \in [\alpha, \beta]$. We build a decreasing sequence of families $\mathcal{G}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2$ with $\mathcal{G}_0 = \mathcal{G}$, $\alpha_n \nearrow \bar{\gamma}$ and $\beta_n \searrow \bar{\gamma}$ as $n \to \infty$. This means that $\mathcal{G}_n(\gamma) = \mathcal{G}(\gamma)$ for all $n \geq 0$ and $\gamma \in [\alpha_n, \beta_n]$. We obtain our counterexample as a limit $g_\gamma = \mathcal{G}(\bar{\gamma}) = \mathcal{G}_n(\bar{\gamma})$ for all $n \geq 0$. For all $n \geq 0$ we choose two finite minimal itinerary sequences $\underline{i}_n(n + 1)$ and $\underline{j}_n(n + 1)$ as in Proposition 3.7 such that
\[ k_{\underline{a}}(\alpha_n) < \underline{i}_n(n + 1) < \underline{j}_n(n + 1) < k_{\underline{a}}(\beta_n). \]
We set $\alpha_{n+1} = \gamma_1$ and $\beta_{n+1} = \gamma_2$. Choosing sufficient long sequences $\zeta_1(n+1)$ and $\zeta_2(n+1)$ we obtain the convergences $\alpha_n \rightarrow \gamma$ and $\beta_n \rightarrow \gamma$ as $n \rightarrow \infty$.

Let $T_2(x) = x^3 - 3x$ be the second Chebyshev polynomial. Observe that $-2, 0$ and $2$ are fixed and that the critical points $c_1 = -1$ and $c_2 = 1$ are sent to $2$ respectively $-2$. Its Schwarzian derivative $S(T_2)(x) = -\frac{4x^2+1}{(x^2-1)^2}$ is negative on $\mathbb{R} \setminus \{c_1, c_2\}$. Let $h > 0$ small and for each $\gamma \in [0, h]$ two order preserving linear maps $P_\gamma(x) = x(4 + \gamma) - 2 - \gamma$ and $Q_\gamma(y) = \frac{y - T_2(-2 - \gamma)}{2 - T_2(-2 - \gamma)}$ that map $[0, 1]$ onto $[-2 - \gamma, 2]$ respectively $[T_2(-2 - \gamma), T_2(2)]$ onto $[0, 1]$. Let then

\begin{equation}
\gamma \mapsto Q_\gamma \circ T_2 \circ P_\gamma
\end{equation}

be a bimodal degree $3$ polynomial. As $S(P_\gamma) = S(Q_\gamma) = 0$ for all $\gamma \in [0, h]$, using equality \([1]\), one may check that

$S(g_\gamma) < 0$ on $I \setminus \{c_1(\gamma), c_2(\gamma)\}$ for all $\gamma \in [0, h]$.

If we write

\begin{equation}
    g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma)x^k
\end{equation}

it is not hard to check that $\gamma \mapsto a_k(\gamma)$ is continuous on $[0, h]$ for $k = 0, \ldots, 3$ therefore $\gamma \mapsto g_\gamma$ is continuous with respect to the $C^1$ topology on $I$. By the definition of $P_2$ (see page \([8]\), as $g_\gamma(0) = 0$ for all $\gamma \in [0, h]$, $\mathcal{G} : [0, h] \rightarrow P_2$ with $\mathcal{G}(\gamma) = g_\gamma$ for all $\gamma \in [0, h]$ is a family of bimodal maps with negative Schwarzian derivative. Observe that $0$ and $1$ are fixed points for all $\gamma \in [0, h]$ and that they are repelling for $g_0$, with $g_0'(0) = g_0'(1) = 9$, which is condition \([3]\). Moreover, $g_\gamma(c_1) = 1$ for all $\gamma \in [0, h]$ thus $\mathcal{G}$ satisfies also \([4]\). Observe that if $\gamma \in [0, h]$ then $Q_\gamma(-2) = 0$ if and only if $\gamma = 0$, therefore condition \([5]\) is also satisfied by $\mathcal{G}$. We show that $\mathcal{G}$ is also natural and that any minimal sequence $SI_2^\infty$ with $S \in \mathcal{A}_2^\gamma$ and $n \geq 0$ equals the second kneading sequence $k(\gamma)$ for at most finitely many $\gamma \in [0, h]$. This allows us to use all the results of the previous section for the family $\mathcal{G}$.

Let $G : [0, h] \times [0, 1] \rightarrow \mathbb{R}$ be defined by

$G(\gamma, x) = g_\gamma(x)$ for all $\gamma \in [0, h]$ and $x \in [0, 1]$.

Then

$G(\gamma, x) = \frac{P_1(\gamma, x)}{P_2(\gamma)}$,

where $P_1$ and $P_2$ are polynomials. Using definition \([13]\), we may compute $P_2$ easily

$P_2(\gamma) = 2 - T_2(-2 - \gamma) = (\gamma + 1)^2(\gamma + 4)$.

We may therefore extend $G$ analytically on a neighborhood $\Omega \subseteq \mathbb{R}^2$ of $[0, h] \times [0, 1]$. The critical points $c_1$ and $c_2$ are continuously defined on $[0, h]$ by Lemma \([3.3]\). They are also analytic in $\gamma$ as a consequence of the Implicit Functions Theorem for real analytic maps.
applied to $\frac{\partial \gamma}{\partial x}$. Therefore for all $n \geq 0$ the map $g^n_\gamma(c_2)$ is analytic on a neighborhood of $[0, h]$ so
\[ c_j(\gamma) - g^n_\gamma(c_2) \text{ has finitely many zeros in } [0, h] \]
for all $j \in \{1, 2\}$ and $n \geq 0$ as $g^n_\gamma(c_2) = 0$ and $c_1(\gamma), c_2(\gamma) \in (0, 1)$ for all $\gamma \in [0, h]$. The family $G$ is therefore natural so by eventually shrinking $\gamma$ we may also suppose that $G$ satisfies property (6) and Lemmas 3.1 and 3.2 for all $\gamma \in [0, h]$. Then the repelling fixed point $r$ is continuously defined on $[0, h]$ and again by the Implicit Functions Theorem applied to $G(\gamma, x) - x$, it is analytic on a neighborhood of $[0, h]$. Then
\[ r(\gamma) - g^n_\gamma(c_2) \text{ has finitely many zeros in } [0, h] \]
for all $n \geq 0$ as $r(0) - g^n_\gamma(c_2) = \frac{1}{2}$.

Let then $G_0 = \mathcal{G}$ so $\alpha_0 = 0$ and $\beta_0 = h$. Our counterexample $g_\gamma$ should be TCE but not RCE (see Definitions 2.3 and 2.5). Its first critical point is non-recurrent as $g_\gamma(c_1) = 1$ and 1 is fixed for all $\gamma \in [\alpha_0, \beta_0]$. Therefore the second critical point $c_2$ should be recurrent and not Collet-Eckmann. We let $c_2$ accumulate on $c_1$ also to control the growth of the derivative along its orbit. We build $g_\gamma$ such that its second critical orbit spends most of the time near $r$ or 1 so its derivative accumulates sufficient expansion. We show that $g_\gamma$ is $ExpShrink$ (thus TCE) using a telescopic construction, in an analogous way to the proof of Theorem 2.8.

### 4.1. A construction.

The construction of the sequence $(\mathcal{G}_n)_{n \geq 0}$ is realized by imposing at the $n$-th step the behavior of the second critical orbit for a time span $t_{n-1} + 1, t_{n-1}, \ldots, t_n$. This is achieved specifying the second kneading sequence and using Proposition 3.7. We set $t_0 = 0$.

We have seen that $k^+(0) = I_1^\infty$ and that $g_\gamma(x) > x$ for all $x \in (0, c_1)$ and all $\gamma \in [0, h]$ as 0 is repelling and $g_\gamma$ has no fixed point in $(0, c_1)$. Therefore the backward orbit of $c_1$ in $I_1$ converges to 0 and by compactness the convergence is uniform. Then
\[ k^{-1}(I_1^k c_1) \rightarrow 0 \text{ as } k \rightarrow \infty, \]
using Proposition 3.5 for their existence. Then for any $\epsilon_0 > 0$ there is $k_0 > 0$ such that $I_1^{k_0} c_1 < k(\beta_0)$ and $|g_\gamma(x) - g_\gamma(c_1)| < \epsilon_0$ for all $\gamma \in [0, k^{-1}(I_1^{k_0} c_1)]$. In particular, if
\[ 1 < \lambda < \lambda' < |g_\gamma'(0)| = 3 < |g_\gamma'(0)| = |g_\gamma'(1)| = 9 \]
then for $\epsilon_0$ sufficiently small
\[ \lambda < |g_\gamma'(r)| \text{ and } \lambda < |g_\gamma'(0)| \text{ and } \lambda < |g_\gamma'(1)| \]
for all $\gamma \in [0, k^{-1}(I_1^{k_0} c_1)]$. Let $S_0 = I_1^{k_0+1} \in \mathcal{A}_1^{k_0+1}$ so $i < I_1^{k_0} c_1$ for all $i \in S_0 \times \Sigma$. Moreover, $S_0 I_2^\infty$ is minimal. Using Proposition 3.7 we find $\alpha_0 < \gamma_1 < \gamma_2 < \beta_0$ such that
\[ k(\alpha_0) < k(\gamma_1) < S_0 I_2^\infty < k(\gamma_2) < k(\beta_0) \]
with $k(\gamma_1), k(\gamma_2) \in S_0 I_2 \times \Sigma$ and
\[ |\gamma_2 - \gamma_1| < 2^{-1}. \]
We set $\alpha_1 = \gamma_1$ and $\beta_1 = \gamma_2$ and define $G_1 : [\alpha_1, \beta_1] \rightarrow \mathcal{P}_2$ by $G_1(\gamma) = G(\gamma) = g_2$ for all $\gamma \in [\alpha_1, \beta_1]$. Moreover, let $t_1 = k + |S_0|$ and $S_1 = S_0 I_2^k$, where $k$ is specified by Proposition 3.7, then

$$k(\gamma) \in S_1 I_2 \times \Sigma,$$

for all $\gamma \in [\alpha_1, \beta_1]$. Using Corollary 3.11 we may also assume that

$$d_m(\gamma) > \lambda^m,$$

for all $\gamma \in [\alpha_1, \beta_1]$, where $m = t_1 = |S_1|$. Let us recall that $d_n(\gamma) = \left| (g_n^\gamma)'(v) \right|$ and $v = g_\gamma(c_2)$.

Then we build inductively the decreasing sequence of families $(G_n)_{n \geq 0}$ such that for all $n \geq 1$, $G_n$ satisfies

$$k(\gamma) \in S_n I_2 \times \Sigma,$$

$$|k(\alpha_n)|, |k(\beta_n)| < \infty,$$

$$|\beta_n - \alpha_n| < 2^{-n},$$

$$k(\alpha_n) < S_n I_2^\infty < k(\beta_n),$$

and conditions (15) and (16) for all $\gamma \in [\alpha_n, \beta_n]$, for some $S_n \in \mathcal{A}_m^n$ with $S_n I_2^\infty$ minimal, where $m = t_n$. As the sequence $(G_n)_{n \geq 0}$ is decreasing, inequality (15) is satisfied by all $G_n$ with $n \geq 1$. For transparency we denote $v_n = g_n^\gamma(v)$ and

$$d_{n,p}(\gamma) = \left| (g_n^\gamma)'(v_n) \right|,$$

which also equals $d_{n+p}(\gamma) d_n^{-1}(\gamma)$, whenever $|k(\gamma)| > n$ so $d_n(\gamma) \neq 0$.

Let us describe two types of steps, one that takes the second critical orbit near $c_1$ to control the growth of the derivative and the other that takes it near $c_2$ to make the second critical point $c_2$ recurrent. We alternate the two types of steps in the construction of the sequence $(G_n)_{n \geq 0}$ to obtain our counterexample.

The following proposition describes the passage near $c_1$.

**Proposition 4.1.** Let the family $G_n$ with $n \geq 1$ satisfy conditions (15) to (20) and

$$0 < \lambda_1 < \lambda_2 < \lambda.$$

Then there exists a subfamily $G_{n+1}$ of $G_n$ satisfying the same conditions and such that there exists $2t_n < p < t_{n+1}$ with the following properties

1. $\max_{\gamma \in [\alpha_{n+1}, \beta_{n+1}]} \left| \log |g_\gamma'(r)| - \frac{1}{p-1} \log d_{p-1}(\gamma) \right| < \log \lambda_2 - \log \lambda_1$.
2. $\lambda_1^p < d_p(\gamma) < \lambda_2^p$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
3. $d_{t_n}(\gamma) > \lambda_1$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ and $l = 1, \ldots, p - 1 - t_n$.
4. $d_{p,l}(\gamma) > \lambda_1$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ and $l = 1, \ldots, t_{n+1} - p$.
5. $d_{t_n,t_n-t_n}(\gamma) > \lambda_{n+1}^{t_n-t_n}$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$. 


Proof. This proof follows a very simple idea, to define the family $\mathcal{G}_{n+1}$ with

$$S_{n+1} = S_n I_{k_1 + 1}^k I_{k_2}^k I_{k_3}^k,$$

as described by properties (17) and (20). For $k_1$ and $k_3$ sufficiently large there exist $k_2$ such that the conclusion is satisfied for $p = t_n + k_1 + 1$.

Let us apply Proposition 3.7 to $S_n$, $\alpha_n$ and $\beta_n$. Let $k_1 = k + 1$, $\lambda_0 = |g'\gamma_\infty(r)|$ and $\lambda_3 = |g'_\gamma_\infty(1)|$. By inequality (15)

$$0 < \lambda_1 < \lambda_2 < \lambda < \lambda_0,$$

therefore there exists $\varepsilon_0 \in (0, 1)$ such that

$$\frac{(1 + \varepsilon_0) \log \lambda_0 - \log \lambda_2}{(1 - \varepsilon_0) \log \lambda_3} < \frac{(1 - \varepsilon_0) \log \lambda_0 - \log \lambda_1}{(1 + \varepsilon_0) \log \lambda_3}.$$

We choose $0 < \varepsilon < \varepsilon_0$ such that

$$\varepsilon < \frac{\log \lambda_2 - \log \lambda_1}{8 \log \lambda_0}.$$

Let us recall that

$$\ell(\gamma_1) = S_n I_{k_1}^k c_1 < S_{n+1} \times \Sigma < \ell(\gamma_2) = S_n I_{k_1}^k c_2.$$

Using Lemma 3.10 and Corollaries 3.8 and 3.11 there exists $N_0$ such that if $k_1 > N_0$ then the first and the third conclusions are satisfied provided $[\alpha_n+1, \beta_n+1] \subseteq [\gamma_1, \gamma_2]$.

Let $y(\gamma) \in I$ with $\hat{y}(y) \in I_2 I_{k_2}^k I_2 \times \Sigma$ and $y' = g_\gamma(y)$. By Corollary 3.8 Lemma 3.10 and inequality (12) there exist $N_1, N_0_1 > 0$ such that if $k_1 > N_1$ and $k_2 > N_0_1$ then for all $\gamma \in [\gamma_1, \gamma_2]$

$$\lambda_3^{-k_2(1+\varepsilon)} < |1 - y'| < \lambda_3^{-(k_2-1)(1-\varepsilon)},$$

as $y \in I_3(k_2 - 1) \setminus I_3(k_2)$.

Let us recall that $g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma)x^k$ with $a_i$ continuous and $g'_\gamma(c_1) = 0$, $g''_\gamma(c_1) \neq 0$ for all $\gamma \in [\alpha, \beta]$ and $c_1$ is continuous. Therefore there exist constants $M > 1$, $\delta > 0$ and $N_2 > 0$ such that if $k_1 > N_2$ and $\gamma \in [\gamma_1, \gamma_2]$ then

$$M^{-1}(x - c_1)^2 < |1 - g_\gamma(x)| < M(x - c_1)^2$$

and

$$M^{-1}(x - c_1) < |g'_\gamma(x)| < M(x - c_1)$$

for all $x \in (c_1 - \delta, c_1 + \delta)$. Using inequality (22) there exists $N_1'$ such that if $k_2 > N_1'$ then $|1 - y'| = |1 - g_\gamma(y)| < M^{-1}\delta^2$ so $|y - c_1| < \delta$, therefore

$$M^{-\frac{1}{2}} \lambda_3^{\frac{k_2}{2}(1+\varepsilon)} < |g'_\gamma(y)| < M^{\frac{1}{2}} \lambda_3^{\frac{k_2-1}{2}(1-\varepsilon)}.$$

Let $k_1 > \max(t_n, N_0, N_1, N_2)$ and $k_2 > \max(N_0_1, N_1')$. Lemma 3.6 shows that $S_{n+1} I_{k_3}^\infty$ is minimal. We may therefore apply Proposition 3.7 with $S = S_n I_{k_1+1}^k I_{k_2}^k$, using inequality (21). Let $k_3 = k$ and $\alpha_{n+1}$ and $\beta_{n+1}$ be the new bounds for $\gamma$ provided by Proposition 3.7. Let us recall that $p = t_n + k_1 + 1$ and $v_n = g_{\gamma_\infty}^{n+1}(c_2)$ for all $n \geq 0$, therefore $\hat{y}(v_{n-1}) \in I_2 I_{k_3}^2 \times \Sigma$ so we may set $y = v_{p-1}$ and $y' = v_p$. Let us remark that

$$d_p(\gamma) = d_{p-1}(\gamma) \cdot |g'_\gamma(y)|$$

to all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$. 

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By Corollary 3.11 if \( k_1 \) is sufficiently large, then for all \( \gamma \in [\alpha_{n+1}, \beta_{n+1}] \)
\[
M^{\frac{3}{2}} \lambda_0^{(p-1)(1-\varepsilon)} \lambda_3^{\frac{k_2}{2}(1+\varepsilon)} < d_p(\gamma) < M^{\frac{3}{2}} \lambda_0^{(p-1)(1+\varepsilon)} \lambda_3^{\frac{k_2}{2}(1-\varepsilon)}.
\]
Therefore the second conclusion is satisfied if
\[
p \log \lambda_1 < -\frac{3}{2} \log M + (p - 1)(1 - \varepsilon) \log \lambda_0 - \frac{k_2}{2} (1 + \varepsilon) \log \lambda_3
\]
and
\[
p \log \lambda_2 > \frac{3}{2} \log M + (p - 1)(1 + \varepsilon) \log \lambda_0 - \frac{k_2 - 1}{2} (1 - \varepsilon) \log \lambda_3.
\]
We may let \( p \to \infty \) and \( \frac{k_2}{2p} \to 0 \) so it is enough to find \( \eta > 0 \) such that
\[
\log \lambda_1 < (1 - \varepsilon) \log \lambda_0 - \eta(1 + \varepsilon) \log \lambda_3
\]
and
\[
\log \lambda_2 > (1 + \varepsilon) \log \lambda_0 - \eta(1 - \varepsilon) \log \lambda_3.
\]

The existence of \( \eta \) is guaranteed by the choice of \( \varepsilon < \varepsilon_0 \).

Again by inequality (15), Lemma 3.10 and Corollary 3.11, if \( k_2 \) and \( k_3 \) are sufficiently large then the last two conclusions are satisfied. If \( k_3 \) is sufficiently large then by Corollary 3.8 inequality (19) is also satisfied. \( \square \)

The following proposition describes the passage of the second critical orbit near \( c_2 \).

**Proposition 4.2.** Let the family \( \mathcal{G}_n \) with \( n \geq 1 \) satisfy conditions (15) to (20) and \( \Delta > 0 \).

Then there exists a subfamily \( \mathcal{G}_{n+1} \) of \( \mathcal{G}_n \) satisfying the same conditions and such that there exists \( t_n < p < t_{n+1} \) with the following properties

1. \( |g^p_\gamma(c_2) - c_2| < \Delta \) for all \( \gamma \in [\alpha_{n+1}, \beta_{n+1}] \).
2. \( d_{\gamma,\lambda}(\gamma) > \lambda \) for all \( \gamma \in [\alpha_{n+1}, \beta_{n+1}] \) and \( l = 1, \ldots, t_{n+1} - t_n \).
3. \( d_{p-1,t_{n+1}-p+1}(\gamma) > \lambda^{t_{n+1}-p+1} \) for all \( \gamma \in [\alpha_{n+1}, \beta_{n+1}] \).

**Proof.** Once again, we build the family \( \mathcal{G}_{n+1} \) using the prefix of the kneading sequence
\[
S_{n+1} = S_n I_2^{k_1} S_n I_2^{k_2+1} I_3 I_2^{k_3},
\]
and show that we may choose \( k_2 \) such that if \( k_1 \) and \( k_3 \) are sufficiently large then the conclusion is satisfied for \( p = t_n + k_1 \).

We apply Proposition 3.7 to \( S_n, \alpha_n \) and \( \beta_n \). Let \( k_1 = k + 2 \), \( \lambda_0 = |g_{\gamma} r' | > \lambda \) and
\[
S' = S_n I_2^{k_2+1} I_3.
\]
In the sequel \( k_2 \) is chosen such that \( \epsilon(S') = 1 \) therefore \( \lambda(\gamma) = S_n I_2^{k_2} < S' \ldots, \) so
\[
S_{n+1} I_2^{\infty} \text{ is minimal if } k_1 - 1 > k_2 > t_n.
\]
Indeed, suppose that there exists \( j > 0 \) such that \( \sigma^j(S_{n+1} I_2^{\infty}) < S_{n+1} I_2^{\infty} \). Let us recall that \( t_n = |S_n| \) and \( S_n I_2^{\infty} \) is minimal, using property (20) of \( \mathcal{G}_n \). A similar reasoning to the proof of Lemma 3.6 shows that \( j \) can only be equal to \( t_n + k_1 \) so
\[
S' \ldots < S_n I_2^{k_1} \ldots
\]
which contradicts $S_n I_2^\infty \prec S' \ldots$, as $k_1 \geq k_2 + 2$. Moreover,

$$k(\gamma_1) = S_n I_2^{k_1-1} c_1 \prec S_{n+1} I_2^\infty \prec k(\gamma_2) = S_n I_2^{k_1-1} c_2,$$

and $I'_2 c_1 \prec I'_2 S' c_2 = I''_2 < c_2$ are realized for all $\gamma \in [\gamma_1, \gamma_2]$, using Lemma 3.3. Let us remark that $g_{\gamma_\infty}$ has no homoterval as $v_n = r$, using Singer's Theorem 2.19. Therefore

$$\lim_{k_2 \to \infty} g_{\gamma_\infty}(x(\hat{\gamma}'')) = c_2$$

as $g_{\gamma_\infty}(x(\hat{\gamma}'')) = x(\sigma(y'')) < c_2$ is increasing with respect to $k_2$ and

$$\{i \in \Sigma_0 : I_2 S' c_2 \prec i \prec c_2 \text{ for all } k_2 > 0\} = \emptyset.$$

Let $k_2$ be such that $|c_2(\gamma_\infty) - x(\sigma(y''))(\gamma_\infty)| < \Delta$. Using Corollary 3.8 and the continuity of $c_2$ and of $x(\sigma(y'')) < x(\sigma(y')) < c_2$ there exists $N_0 > 0$ such that if $k_1 > N_0$ then

$$|c_2 - x| < \Delta,$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in [x(\sigma(y''))$, $(x(\sigma(y'))]$. Lemma 3.9 applied to $\hat{\gamma}'$ and $\hat{\gamma}''$ yields $\theta > 0$ such that if $l = t_n + k_2 + 4$ then

$$\theta < \left| (g_{\gamma})' (x) \right| < \theta^{-1},$$

for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\hat{\gamma}')$, $(x(\hat{\gamma}'')]$ and $j = 1, \ldots, l$. Lemma 3.10 provides $N_1 > 0$ such that if $k_1 > N_1$ then

$$\lambda' \lambda < d_{n,j}(\gamma) \text{ for all } \gamma \in [\gamma_1, \gamma_2] \text{ for all } j = 1, \ldots, k_1 - 2.$$

As $\lambda > \lambda'$ there exists also $N_2 > 0$ such that

$$\theta^{-1} \lambda^{N_2 - 2 + l} < (\lambda')^{N_2 - 2}.$$

Let $k_1 > \max(k_2 + 1, N_0, N_1, N_2)$ and $S'' = S_n I_2^k S'$. Let us remark that $S'' I_2^\infty = S_{n+1} I_2^\infty$ thus we may apply Proposition 3.7 to $S''$, $\gamma_1$ and $\gamma_2$. Let $\alpha_{n+1}$ and $\beta_{n+1}$ be the new bounds for $\gamma$ provided by Proposition 3.7 and $k_3 = k$.

As $g_{\gamma}(c_2) = v_{p-1}$ and $\sigma^{p-1} (S'' I_2^{k_3} \ldots) = I_2 S' I_2 \ldots$

$$g_{\gamma}(c_2) \in [x(\sigma(y''))$, $(x(\sigma(y'))],$$

for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ thus, by inequality (24), the first conclusion is satisfied. Moreover, using inequalities (25), (26) and (27)

$$\lambda' \lambda < d_{n,j}(\gamma) \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}] \text{ and } j = 1, \ldots, |S''| = k_1 - 2 + l.$$

Using Lemma 3.10 and Corollary 3.11 for $k_3$ sufficiently large the last two conclusions are satisfied. If $k_3$ is sufficiently large then by Corollary 3.8 inequality (19) is also satisfied by $\alpha_{n+1}$ and $\beta_{n+1}$.

\[ \square \]
4.2. **Some properties of polynomial dynamics.** Let us introduce some notations. For any set $E \subseteq \overline{\mathbb{C}}$ and $\alpha > 0$, we define the $\alpha$-neighborhood of $E$ by

$$E_+ = B(E, \alpha) = \{ x \in \overline{\mathbb{C}} \mid \text{dist} (x, E) < \alpha \}.$$ 

One may easily check that if $f, g : \Omega \to \overline{\mathbb{C}}$ with $\Omega \subseteq \overline{\mathbb{C}}$ and $\delta > \| f - g \|_\infty$ then for all $B \subseteq \overline{\mathbb{C}}$

\begin{equation}
(28) \quad g^{-1}(B) \subseteq f^{-1}(B + \delta).
\end{equation}

Using this simple observation we show that in a neighborhood of an $\text{ExpShrink}$ polynomial some weaker version of Backward Stability is satisfied, see Proposition 4.5.

**Definition 4.3.** We say that a rational map $f$ has Backward Stability if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in J$, the Julia set of $f$, all $n \geq 0$ and every connected component $W$ of $f^{-n}(B(z, \delta))$

$$\text{diam } W < \varepsilon.$$

Let us first show that the Julia set is $\text{continuous}$ in the sense of Lemma 4.4. For transparency we introduce additional notations. We denote by $\mathbb{C}_d[z]$ the space of complex polynomials of degree $d$. If $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{C}_d[z]$ let us also denote

$$|f| = \max_{0 \leq i \leq d} |a_i|.$$ 

By convention, when $f \in \mathbb{C}_d[z]$ and we compare it to another polynomial $g$ writing $|f - g|$ we also assume that $g \in \mathbb{C}_d[z]$.

Let us observe that the coefficients of $f^n = f \circ f \circ \ldots \circ f$, the $n$-th iterate of $f$, are continuous with respect to $(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$ for all $n > 0$. Therefore given $f \in \mathbb{C}_d[z]$, $m > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$|f^i - g^i| < \varepsilon \quad \text{for all } i = 1, \ldots, m.$$

Given a compact $K \subseteq \mathbb{C}$, the map $\mathbb{R}^{d+1} \ni (a_0, a_1, \ldots, a_d) \mapsto f \in \mathbb{C}_d[z]$ is continuous with respect to the topology of $C(K, \mathbb{C})$. Therefore for all $f \in \mathbb{C}_d[z]$, $\varepsilon > 0$ and $m > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

\begin{equation}
(29) \quad \| f^i - g^i \|_{\infty, K} < \varepsilon \quad \text{for all } i = 1, \ldots, m.
\end{equation}

**Lemma 4.4.** Let $f \in \mathbb{C}_d[z]$ with $d \geq 2$ and such that its Fatou set is connected and let $J$ be its Julia set. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$J_g \subseteq J_{+\varepsilon}.$$

**Proof.** The Fatou set of $f$ is the basin of attraction of $\infty$ and $J$ is compact and invariant. Let $|J| = \max_{z \in J} |z|$, then for all $M \geq |J|

$$J = \{ z \in \mathbb{C} : |f^n(z)| \leq M \text{ for all } n \geq 0 \}.$$

Let $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{C}_d[z]$. There exists $R > 1$ such that if $|f - g| < \frac{1}{2} |a_d|$ then

$$|J_g| \leq R.$$
Indeed, it is enough to choose
\[ R > 4d + 2|a_d|^{-1}\left(1 + \sum_{i=0}^{d-1} |a_i|\right), \]
and check that if \(|z| > R\) then \(|g(z)| > |z| + 1\).

Let \( T = \{ z \in \mathbb{C} : \text{dist}(z, J) \geq \varepsilon \} \). As \( T \) is compact in \( \overline{\mathbb{C}} \) and contained in the basin of attraction of \( \infty \), there is \( m > 0 \) such that
\[ |f^m(z)| > R + 1 \text{ for all } z \in T. \]

Let \( K = \overline{B}(0, R + 1) \) a compact such that \( J_{+\varepsilon} \subseteq K \) if \( |f - g| < \frac{1}{2}|a_d| \). Inequality (29) yields \( 0 < \delta < \frac{1}{2}|a_d| \) such that if \( |f - g| < \delta \) then
\[ \left\| f^i - g^i \right\|_{\infty,K} < 1 \text{ for all } i = 1, \ldots, m. \]

Therefore by the definitions of \( R \) and \( m \), if \( |f - g| < \delta \) then
\[ |g^m(z)| > R \text{ for all } z \in T, \]
thus \( J_g \cap T = \emptyset. \)

**Remark.** The hypothesis \( f \) polynomial and its Fatou set connected are somewhat artificial, introduced for the elegance of the proof. It may be easily generalized to rational maps with attracting periodic orbits but without parabolic periodic orbits nor rotation domains.

**Proposition 4.5.** Let \( f \) be an \( \text{ExpShrink} \) polynomial satisfying the hypothesis of Lemma 4.4. There exists \( \delta > 0 \) such that for all \( 0 < r < \delta \) there exist \( N > 0 \) and \( d > 0 \) such that for all \( g \) with \( |f - g| < d \) and \( z \in J_g \)
\[ \text{diam Comp } g^{-N}(B(z,\delta)) < r. \]

We use the notation \( \text{Comp } A \) for connected components of the set \( A \). The previous statement means that the inequality holds for any such component.

**Proof.** Let us denote \( J \) the Julia set of \( f \). Let \( r_0 > 0 \) and \( \lambda_0 > 1 \) be provided by Definition 2.4 such that for all \( z \in J \)
\[ \text{diam Comp } f^{-n}(B(z,r_0)) < \lambda_0^{-n} \text{ for all } n \geq 0. \]

Let \( \delta = \frac{r_0}{4} \) and choose \( N \geq 1 \) such that
\[ \lambda_0^{-N} < r. \]

Inequality (29) provides \( d_0 \) such that if \( |f - g| < d_0 \) then
\[ |f^N(z) - g^N(z)| < \delta \text{ for all } z \in J_{+r_0}. \]

Lemma 4.4 yields \( d_1 > 0 \) such that if \( |f - g| < d_1 \) and \( z \in J_g \) then there exists \( z' \in J \) such that \( |z - z'| < 2\delta \) therefore
\[ B(z, 2\delta) \subseteq B(z', r_0). \]

We choose \( d = \min(d_0, d_1) \) and \( g \in \mathbb{C}_d[z] \) with \( |f - g| < d \). Using inequality (28)
\[ \text{diam Comp } g^{-N}(B(z,\delta)) < \lambda_0^{-N} < r \text{ for all } z \in J_g. \]
Corollary 4.6. Let \( f \) satisfy the hypothesis of Proposition 4.5 and \( \varepsilon > 0 \). There exist \( d, \delta > 0 \) such that if \( |f - g| < d \) then for all \( z \in J_g \) and \( n \geq 0 \)

\[
diam \text{Comp} g^{-n}(B(z, \delta)) < \varepsilon.
\]

Proof. Let us use the notations defined by the proof of Proposition 4.5. It is straightforward to check that \( f \) has Backward Stability and that, by eventually decreasing \( r_0 \), we may also suppose

\[
diam \text{Comp} f^{-n}(B(z, r_0)) < \varepsilon \text{ for all } z \in J \text{ and } n \geq 0.
\]

Let \( m \geq 1 \) such that

\[
\lambda_0^{-m} < \delta.
\]

Inequality (29) provides \( d_0 \) such that if \( |f - g| < d_0 \) then

\[
|f^i(z) - g^i(z)| < \delta \text{ for all } z \in J_{r_0} \text{ and } i = 1, \ldots, m.
\]

Let \( d_1, d \) and \( g \) be as in the proof of Proposition 4.5. By inequality (28), for all \( z \in J_g \)

\[
diam \text{Comp} g^{-m}(B(z, \delta)) < \delta
\]

and

\[
diam \text{Comp} g^{-i}(B(z, \delta)) < \varepsilon \text{ for all } i = 0, \ldots, m.
\]

For some \( z \in J_g \), let \( W \in \text{Comp} g^{-m}(B(z, \delta)) \) and \( z_1 \in W \cap J_g \). Then

\[
W \subseteq B(z_1, \delta)
\]

and the proof is completed by induction. \( \square \)

Let us show that the hypothesis of Lemma 4.4 is easy to check for polynomials in \( G_0 \).

Lemma 4.7. If \( g_\gamma \in G_0 \) and its second critical orbit \( (v_n)_{n \geq 0} \) accumulates on a repelling periodic orbit then \( g_\gamma \) satisfies the hypothesis of Lemma 4.4. Moreover, if \( (v_n)_{n \geq 0} \) is preperiodic then \( g_\gamma \) has \( \text{ExpShrink} \).

Proof. By Theorems III.2.2 and III.2.3 in [4], the immediate basin of attraction of an attracting or parabolic periodic point contains a critical point. But \( c_1 \) is strictly preperiodic and \( (v_n)_{n \geq 0} \) accumulates on a repelling periodic orbit thus it cannot converge to some attracting or parabolic periodic point. Using Theorem V.1.1 in [4] we rule out Siegel disks and Herman rings as their boundary should be contained in the closure of the critical orbits which is contained in \([0,1]\) for all \( g_\gamma \in G_0 \). Using Sullivan’s classification of Fatou components, Theorem IV.2.1 in [4], the Fatou set equals the basin of attraction of infinity which is connected for all polynomials by the maximum principle.

If \( (v_n)_{n \geq 0} \) is preperiodic then \( g_\gamma \) is semi-hyperbolic therefore by the main result in [5] or by Theorem 2.8 it has \( \text{ExpShrink} \). \( \square \)

Let us recall some general distortion properties of rational maps. The following result is a classical distortion estimate due to Koebe, see for example Lemma 2.5 in [8].
Lemma 4.8 (Koebe). Let $g : B \to \mathbb{C}$ be a univalent map from the unit disk into the complex plane. Then the image $g(B)$ contains the ball $B \left( g(0), \frac{1}{4} |g'(0)| \right)$. Moreover, for all $z \in B$ we have that
\[
\frac{(1 - |z|)}{(1 + |z|)^2} \leq \frac{|g'(z)|}{|g'(0)|} \leq \frac{(1 + |z|)}{(1 - |z|)^2},
\]
and
\[
|g(z) - g(0)| \leq |g'(z)| \frac{|z|(1 + |z|)}{1 - |z|}.
\]

For the remainder of this section, let $f$ be any rational map and $\text{Crit}$ the set of critical points of $f$.

**Distortion.** This is a reformulation of the previous lemma. For all $D > 1$ there exists $\varepsilon > 0$ such that if the open $W$ satisfies
\[
\text{diam } W \leq \varepsilon \text{ dist } (W, \text{Crit}),
\]
then the distortion of $f$ in $W$ is bounded by $D$.

**Pullback.** Once a sufficiently small $r > 0$ is fixed, there exists $M \geq 1$ such that for any open $U$ with $\text{diam } U \leq r$ and for all $W \in \text{Comp } f^{-1}(U)$ and all $z \in W$
\[
\text{diam } W \leq M |f'(z)|^{-1} \text{ diam } U.
\]

We shall use this estimate for $W$ close to Crit.

4.3. A counterexample. Using Propositions 4.1 and 4.2 we build a sequence of families $(G_n)_{n \geq 1}$ which converge to a bimodal polynomial $g$ that has $\text{ExpShrink}$. Its first critical point $c_1$ is non-recurrent as $g(c_1) = 1$ and $1$ is a repelling fixed point. The second critical point $c_2$ is recurrent and it does not satisfy the Collet-Eckmann condition. Therefore $g$ does not satisfy $\text{RCE}$.

We obtain the following theorem which states that the converse of Theorem 2.8 does not hold. We use the equivalence of $\text{TCE}$ and $\text{ExpShrink}$ [20].

**Theorem A.** There exists an $\text{ExpShrink}$ polynomial that is not $\text{RCE}$.

The proof that $g$ has $\text{ExpShrink}$ is analogous to that of Theorem 2.8. This paper contains a complete proof of Theorem A. However, remarks about the proof of Theorem 2.8 are present for the convenience of the reader. As $g$ is not $\text{RCE}$ we have to modify some of the tools like Propositions 9, 10 and 11 in [11]. The polynomial $g_0$ is Collet-Eckmann and semi-hyperbolic thus $\text{RCE}$. By the main result of [5] or by Theorem 2.8 $g_0$ has $\text{ExpShrink}$. Choosing the family $G_1$ in a sufficiently small neighborhood of $g_0$ we show two contraction results similar to Propositions 9 and 10 in [11] that hold on $G_1$, Corollary 4.9 and Proposition 4.13 below. As $g \in G_1$ we may choose constants $\mu, \theta, \varepsilon, R$ and $N_0$ - as described in the sequel - that do not depend on $g$.

The main idea of the proof of Theorem A is that in inequality (31) the right term may be much larger than the left term, see also Lemma 4.12. This means that when pulling back a ball $B$ to $B^{-1}$ near a degree two critical point, the diameter of $B^{-1}$ is comparable
to the square root of the radius of $B$ but $|f'(z)|^{-1}$ may be as large as one wants for some $z \in B^{-1}$. This is the main difference between growth conditions in terms of the derivative or in terms of the diameter of pullbacks.

Corollary [4.10] an immediate consequence of Corollary [4.6] replaces Proposition 11 in [11] in the proof of Theorem A.

**Corollary 4.9.** There exists $\delta > 0$ such that for all $0 < r < R \leq \delta$ there exist $\beta > \alpha_0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of $g_\gamma$

$$\text{diam } \text{Comp} g_\gamma^{-N}(B(z, R)) < r.$$

**Proof.** Using Lemma 4.7, $g_0$ satisfies the hypothesis of Proposition 4.5. Using the continuity of coefficients of $g_\gamma$ (14) there exists $\beta > \alpha_0$ such that $|g_0 - g_\gamma| < d$ for all $\gamma \in [\alpha_0, \beta]$. \hfill \Box

The following consequence of Corollary 4.6 is a weaker version of uniform Backward Stability. The proof is analogous to the proof of the previous proposition.

**Corollary 4.10.** For all $\varepsilon > 0$ there exist $\beta > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of $g_\gamma$

$$\text{diam } \text{Comp} g_\gamma^{-n}(B(z, \delta)) < \varepsilon \text{ for all } n \geq 0.$$

Let us compute an estimate of the diameter of a pullback far from critical points.

**Lemma 4.11.** Let $h : B(z, 2R) \to \mathbb{C}$ be an analytic univalent map and $U \ni z$ a connected open with $\text{diam } U \leq R$. If

$$\sup_{x,y \in B(z,2R)} \left| \frac{h'(x)}{h'(y)} \right| \leq D$$

then

$$\text{diam } U \leq D \left| h'(z) \right|^{-1} \text{diam } h(U).$$

**Proof.** Let $x, y \in \partial U$ such that $|x - y| = \text{diam } U$. Let $a = h(x)$, $b = h(y)$ and consider the pullback of the line segment $[a, b]$ that starts at $x$. Then there exists $t_0 \in (0, 1]$ such that $[a, t_0 a + (1 - t_0) b] \subseteq h(B(z, 2R))$ and such that the length of $h^{-1}([a, t_0 a + (1 - t_0) b])$ is at least $\text{diam } U$. We also notice that

$$\left| (h^{-1})'(ta + (1 - t)b) \right| \leq D |h'(z)|^{-1} \text{ for all } t \in [0, t_0],$$

which completes the proof as $|(t_0 - 1)a + (1 - t_0)b| \leq \text{diam } h(U)$. \hfill \Box

Proposition 10 in [11] relies on inequalities (30) and (31). We remark that they are satisfied uniformly on a neighborhood of $g_0$. By Koebe’s Lemma 4.8, the definition (30) of $\varepsilon$ does not depend on $f$. Let us prove the uniform version of inequality (31) in $G$. 
Lemma 4.12. There exist $M > 1$, $\beta_M > \alpha_0$ and $r_M > 0$ such that for all $\gamma \in [\alpha_0, \beta_M]$ if $W$ is a connected open with $\text{diam} \ W < r_M$, $W^{-1}$ a connected component of $g_\gamma^{-1}(W)$ and $x \in W^{-1}$ then

$$\text{diam} \ W^{-1} < M \left| g'_\gamma(x) \right|^{-1} \text{diam} \ W.$$  

Proof. Let $\gamma \in [\alpha_0, \beta_1]$, $x \in W^{-1}$ and suppose

$$3 \text{diam} \ W^{-1} \leq \text{dist} \left( W^{-1}, \text{Crit} \right),$$

where we denote by Crit the set of critical points $\{c_1, c_2\}$. Then by Koebe’s Lemma 4.8 the distortion is bounded by an universal constant $M_1 \geq 1$ on the ball $B(x, 2 \text{diam} \ W^{-1})$.

Using Lemma 4.11

$$\text{diam} \ W^{-1} \leq M_1 \left| g'_\gamma(x) \right|^{-1} \text{diam} \ W.$$  

Let us remark some properties of the map $f_b : \mathbb{C} \to \mathbb{C}$ defined by $f_b(z) = bz^2$ for all $z \in \mathbb{C}$ and $b > 0$. Let $U$ be a connected open and $V = f_b(U)$. If $3 \text{diam} \ U > \text{dist} \left( U, 0 \right)$ then there exist universal constants $M_2, M_3 > 1$ such that

$$bM_2^{-1} \text{diam} \ U < \sup_{z \in U} \left| f'_b(z) \right| < bM_2 \text{diam} \ U,$$

$$bM_3^{-1} (\text{diam} \ U)^2 < \text{diam} \ V < bM_3 (\text{diam} \ U)^2.$$  

Let us also remark that using equality (14) if $\gamma \in [\alpha_0, \beta_1]$ and $c \in \text{Crit}$ then

$$g_{\gamma}(x) = g_{\gamma}(c) + \frac{g''_{\gamma}(c)}{2} (x - c)^2 + \frac{g'''_{\gamma}(c)}{6} (x - c)^3.$$  

As $g''_{0}(c) \neq 0$ and $g_{\gamma}(c)$, $g''_{\gamma}(c)$ and $g'''_{\gamma}(c)$ are continuous there exist $r_M > 0$, $\beta_M > \alpha_0$ and $M_4 > 1$ such that if $\gamma \in [\alpha_0, \beta_M]$, $\text{diam} \ W < r_M$ and

$$3 \text{diam} \ W^{-1} > \text{dist} \left( W^{-1}, \text{Crit} \right),$$

then

$$M_4^{-1} \text{diam} \ W^{-1} < \sup_{x \in W^{-1}} \left| g'_{\gamma}(x) \right| < M_4 \text{diam} \ W^{-1},$$

$$M_4^{-1} (\text{diam} \ W^{-1})^2 < \text{diam} \ W < M_4 (\text{diam} \ W^{-1})^2.$$  

The previous inequality together with inequality (32) complete the proof. \qed

We may now prove a uniform contraction result on a neighborhood of $g_0$ in $\mathcal{G}$. It replaces Proposition 10 in [11] in the proof of Theorem A.

Proposition 4.13. For any $1 < \lambda_0 < \lambda$ and $\theta < 1$ there exist $\beta > \alpha_0$, $\delta > 0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$, $0 < R \leq \delta$, $n \geq N$ and $z \in J_\gamma$, the Julia set of $g_\gamma$, if $W \in \text{Comp} g_\gamma^{-n}(B(z, R))$ and there exists $x \in \overline{W}$ such that $\left| (g_\gamma^n)'(x) \right| > \lambda^n$ then

$$\text{diam} \ W < \theta R \lambda_0^{-n}.$$  

Proof. Let us fix $D \in (1, \lambda/\lambda_0)$. Let $\varepsilon \in (0, 1)$ be provided by inequality \(30\). Let also $r_M > 0$ be small and $M > 1$ provided by the Lemma 4.12. Let $l \geq 1$ such that 

(35) \[ 2M^{2l}D^j \lambda^{-j} \leq \theta \lambda_0^{-j} \text{ for all } j \geq l. \]

Let $N = 2l$. There exists $r_1 < r_M$ such that for all $i = 1, 2, k = 1, \ldots, N$ and any connected component $W$ of $g_0^{-k}(B(c_i, 4r_1))$

\[ \text{diam } W \leq 2\varepsilon \text{ dist } (W, \text{Crit}). \]

An argument similar to the proof of Proposition 4.5 and the continuity of the critical points and of the coefficients \(14\) of $g_2$ show that there exists $b_0 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_0]$, $i = 1, 2$ and $k = 1, \ldots, N$

\[ g_\gamma^{-k}(B(c_i, 2r_1)) \subseteq g_0^{-k}(B(c_i, 4r_1)). \]

There are only a finite number of connected components of $g_0^{-k}(B(c_i, 4r_1))$ for all $i = 1, 2$ and $k = 1, \ldots, N$. Therefore by the continuity of the critical points there exists $b_1 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_1]$, $i = 1, 2$ and $k = 1, \ldots, N$ all connected components of $g_\gamma^{-k}(B(c_i, 2r_1))$ satisfy inequality \(30\).

Corollary 4.10 provides $b_2 > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, b_2]$, $z \in J_\gamma$ and $k \geq 0$

\[ \text{diam } \text{Comp } g_\gamma^{-k}(B(z, \delta)) < \varepsilon r_1. \]

Let us define $\beta = \min (\beta_M, b_0, b_1, b_2)$ and fix $\gamma \in [\alpha_0, \beta]$, $z \in J_\gamma$ and $n > N$. Then

\[ \text{diam } \text{Comp } g_\gamma^{-k}(B(z, R)^{-k}) < \varepsilon r_1 < r_M \text{ for all } 0 \leq k \leq n. \]

Let us also fix $W$ and $x$ as in the hypothesis. Denote $x_k = g_\gamma^{-k}(x) \in W_k = g_\gamma^k(W)$ for all $k = 0, \ldots, n$. Let $0 < k_1 < \ldots < k_t \leq N$ be all the integers $0 \leq k \leq n$ such that $W_k$ does not satisfy the inequality \(30\). As $\varepsilon r_1 \geq \text{diam } W_{k_i}$

\[ r_1 > \text{dist } (W_{k_i}, \text{Crit}) \text{ for all } 1 \leq i \leq t. \]

Then for all $1 \leq i \leq t$ there exists $c \in \{c_1, c_2\}$ such that $W_{k_i} \subseteq B(c, 2r_1)$. By the definition of $r_1$

(36) \[ k_{i+1} - k_i > N \text{ for all } 1 \leq i < t. \]

We may begin estimates. For all $0 < j \leq n$ with $j \neq k_i$ for all $1 \leq i \leq t$, $W_j$ satisfies the inequality \(30\), so the distortion on $W_j$ is bounded by $D$. Thus by Lemma 4.11

(37) \[ \text{diam } W_j \leq D|g_\gamma'(x_j)|^{-1} \text{ diam } W_{j-1}. \]

If $j = k_i$ for some $1 \leq i \leq t$ we use Lemma 4.12 to obtain

(38) \[ \text{diam } W_j \leq M|g_\gamma'(x_j)|^{-1} \text{ diam } W_{j-1}. \]

Let us recall that $x_n = x$ with $|g_\gamma'(x)| > \lambda^n$ and that $W_0 = B(z, R)$ so diam $W_0 = 2R$. If $t \geq 2$ inequality \(36\) yields $lt \leq 2l(t - 1) = N(t - 1) < n$. Consequently, as $n > 2l = N$,

\[ t < \frac{n}{l}. \]
Multiplying all the relations (37) and (38) for all 0 < j ≤ n we obtain
\[
\text{diam } W_n \leq M^t D^{n-t} \left| (g_n^\gamma)'(x_n) \right|^{-1} \text{diam } W_0
\]
\[
< 2M^{n/t} D^n \lambda^{-n} R
\]
\[
\leq \theta R\lambda_0^{-n}.
\]
The last inequality is inequality (35). □

As a direct consequence of inequality (36) we obtain the following corollary.

**Corollary 4.14.** Assume the hypothesis of Proposition 4.13. If there exist
\[
-1 \leq k_1 < k_2 < n
\]
such that \( v \in \overline{g_{k_1}^{k+1}(W)} \) and \( g_{k_2}^{k+1}(W) \cap \{c_1,c_2\} \neq \emptyset \) then \( k_2 - k_1 > N \) therefore condition \( n \geq N \) is superfluous.

Let us compute a diameter estimate similar to (12).

**Lemma 4.15.** There exist \( \delta > 0 \) and \( N > 0 \) such that for all \( \gamma \in [\alpha_0,\beta_1] \), \( k \geq 1 \) and \( x \in I_3(N) \) with \( i(x) = I_x^I_x \ldots \) where \( I_x \in \{I_2,I_3\} \), the following statement holds. If \( x \in W \subseteq \mathbb{C} \) is a connected open such that \( \text{diam } g_i^\gamma(W) < \delta \) for all \( i = 0, \ldots, k - 1 \) then
\[
\text{diam } W < \lambda^{-k} \text{diam } g_k^\gamma(W).
\]

**Proof.** Let us denote \( x_i = g_i^\gamma(x) \) and \( W_i = \overline{g_i^\gamma(W)} \) for all \( i = 0, \ldots, k \). Using Lemma 3.10 inequalities (15) and Lemma 3.9 for \( i_1 = I_3c_1, i_2 = I_3c_2 \) if \( I_x = I_2 \) and \( i_1 = I_3c_2, i_2 = I_3 \) if \( I_x = I_3 \) there exists \( N_0 > 0 \) that does not depend on \( \gamma \) such that if \( N \geq N_0 \) then
\[
\left| (g_k^\gamma)'(x) \right| > (\lambda)^k.
\]

Let \( D \in (1,\frac{\lambda}{\delta}) \) and \( \varepsilon > 0 \) given by inequality (30). Using Lemma 4.11 it is enough to show that \( B(x_i,2\delta) \) satisfies inequality (30) for all \( i = 0, \ldots, k - 1 \).

Let us recall that \( g_\gamma(y) < y \) for all \( y \in (c_2,1) = I_3 \setminus \{1\} \). Therefore for all \( i = 0, \ldots, k - 1 \)
\[
\text{dist } (x_i,\{c_1,c_2\}) \geq \text{dist } (x_{k-1},\{c_1,c_2\}) \geq \text{dist } (x(I_3c_1),\{c_1,c_2\})
\]

Let
\[
d = \min_{\gamma \in [\alpha_0,\beta_1]} \text{dist } (x(I_3c_1),\{c_1,c_2\})
\]
and recall that \( \varepsilon \) does not depend on \( \gamma \). Therefore there exists
\[
\delta = \frac{d}{2(1 + 2\varepsilon^{-1})} > 0
\]
such that if \( \text{dist } (y,\{c_1,c_2\}) \geq d \) then \( B(y,2\delta) \) satisfies inequality (30). □

The following corollary admits a very similar proof.
Corollary 4.16. There exist \( \delta > 0 \) and \( N > 0 \) such that for all \( \gamma \in [\alpha_0, \beta_1], \) \( k \geq 1 \) and \( x \in I_2(\max(k, N) + 1), \) the following statement holds. If \( x \in W \subseteq \mathbb{C}, \) a connected open such that \( \text{diam} \ g_\gamma^i(W) < \delta \) for all \( i = 0, \ldots, k - 1, \) then

\[
\text{diam} \ W < \lambda^{-k} \text{diam} \ g_\gamma^k(W).
\]

Let us recall that all distances and diameters are considered with respect to the Euclidean metric, as we deal exclusively with polynomial dynamics. Let us state Lemma 3 in [11] in this setting.

Lemma 4.17. Let \( f \) be a polynomial, \( z \in \mathbb{C} \) and \( 0 < r < R. \) Let \( W \subseteq \text{Comp} f^{-1}(B(z, R)) \) and \( W' \subseteq \text{Comp} f^{-1}(B(z, r)) \) with \( W' \subseteq W. \) If \( \text{deg}_W(f) \leq \mu \) then

\[
\frac{\text{diam} \ W'}{\text{diam} \ W} < 32 \left( \frac{r}{R} \right)^{\frac{1}{r}}.
\]

Let us set some constants that define the telescope construction used in the proof of Theorem A. For convenience, we use the same notations as in the proof of Theorem 2.8 in [11]. Let \( \mu = 2 \) and \( \theta = \frac{1}{2} \frac{32}{R}. \) Let \( \delta_0 > 0 \) be provided by Corollary 4.9 and \( \beta'_0 > \alpha_0, \delta_1 > 0, N_1 > 0 \) be provided by Proposition 4.13 applied to \( \lambda^+ \). Let \( \delta' > 0, N_2 > 0 \) be provided by Lemma 4.15, \( \delta'' > 0, N_3 > 0 \) be provided by Corollary 4.16 and \( \beta_M > \alpha_0, r_M > 0 \) and \( M > 1 \) defined by Lemma 4.12.

Let us observe that

\[
I_1^\infty < I_1 c_2 < c_1 < I_2 c_2 < I_2^\infty < I_2 c_1 < c_2 < I_3 c_1 < I_3^\infty
\]

and that all these sequences are continuously realized on \( [\alpha_0, \beta_1]. \) Let us define

\[
\epsilon_0 = \min_{\gamma \in [\alpha_0, \beta_1]} (|x(I_1 c_2) - c_1|, |x(I_2 c_2) - c_1|, |x(I_2 c_1) - c_2|, |x(I_3 c_1) - c_2|)
\]

therefore \( \epsilon_0 > 0 \) is smaller than \( |c_1 - c_2|, |c_1| \) and \( |1 - c_2| \) for all \( \gamma \in [\alpha_0, \beta_1]. \) We set

\[
(39) \quad \epsilon = \min (\epsilon_0, \delta', \delta'', r_M).
\]

Corollary 4.10 provides \( \beta'_1 > \alpha_0 \) and \( \delta_2 > 0 \) such that for all \( \gamma \in [\alpha_0, \beta'_1] \) the diameter of any pullback of a ball of radius at most \( \delta_2 \) centered on \( J_\gamma \) is smaller than \( \epsilon. \) Let \( \beta_2 = \min (\beta_1, \beta'_0, \beta'_1, \beta_M) \) and \( \delta_3 = \min (\delta_0, \delta_1, \delta_2), \) such that Proposition 4.13 applies for balls centered on \( J_\gamma \) of radius at most \( \delta_3, \) for all \( \gamma \in [\alpha_0, \beta'_1]. \) Moreover, Lemma 4.15 and Corollary 4.16 apply and inequalities (32) and (33) hold on all pullbacks of such balls.

Corollary 4.13 provides \( \delta_4 \) such that for

\[
r = \theta R < R = \min (\delta_3, \delta_4),
\]

there exist \( \beta'_3 > \alpha_0 \) and \( N_0 > 0 \) the time span needed to contract the pullback of balls of radius \( R \) into components of diameter smaller than \( \theta R \) for all \( \gamma \in [\alpha_0, \beta'_3]. \) We define

\[
\beta = \min (\beta'_2, \beta'_3).
\]
Let \( f \) be a rational map and \( \text{Crit} \) its critical set. If \( W \subset \mathbb{C} \) is an open set and \( \overline{f^k(W)} \) contains at most one critical point for all \( 0 \leq k < n \), let us define

\[
\deg_W(f^n) = \prod_{c \in \overline{f^k(W) \cap \text{Crit}}} \deg(c),
\]

counted with multiplicities.

The following fact provides the hypothesis of Corollary 4.14.

**Corollary 4.18.** For all \( \gamma \in [\alpha_0, \beta] \), \( z \in J_\gamma \), \( 0 < r \leq R \) and \( (W_k)_{k \geq 0} \) a backward orbit of \( B(z, r) = W_0 \), if \( \deg_W g_\gamma^k > \mu \) then there exist \( 0 < k_1 < k_2 \leq n \) such that \( \overline{W_{k_1}} \cap \{c_1, c_2\} \neq \emptyset \) and \( c_1, c_2 \in \overline{W_{k_2}} \).

**Proof.** By the definition of \( R \), \( \text{diam} \ W_k < \varepsilon \leq \varepsilon_0 < |c_1 - c_2| \) therefore \( \overline{W_k} \) contains at most one critical point for all \( k \geq 0 \). As \( \mu = \mu_{c_1} = \mu_{c_2} \) there exist \( 0 < k_1 < k_2 \leq n \) such that \( \overline{W_{k_1}} \) and \( \overline{W_{k_2}} \) contain exactly one critical point each. Suppose \( c_1 \in \overline{W_{k_2}} \) therefore \( 1 \in \overline{W_k} \) for all \( 0 \leq k < k_2 \) which contradicts \( \text{diam} \ W_{k_1} < \varepsilon \leq \varepsilon_0 \).

Let us prove the main result of this section.

**Proof of Theorem A.** This proof has two parts. The first part describes the construction of a convergent sequence of families \( (\mathcal{G}_n)_{n \geq 0} \) of bimodal polynomials with negative Schwarzian derivative. Its limit \( g \) does not satisfy \( \text{RCE} \). The second part shows that \( g \) has \text{ExpShrink} and it is similar to the proof of Theorem 2.8.

Let us recall the construction of the family \( \mathcal{G}_1 \). It is described by the common prefix \( S_1 \) of its kneading sequences \( \mathcal{K}(\gamma) \) for all \( \gamma \in [\alpha_1, \beta_1] \). We defined \( S_1 = I_1^{k_0} I_2^{k_1} \) so \( \beta_1 < k^{-1} (I_1 c_1) \) which converges to \( c_0 = 0 \) as \( k_0 \to \infty \). Using this convergence, inequalities (15), Lemma 3.10 applied to \( v = g_\gamma(c_2) \) and Lemma 3.9 applied to \( \mathcal{I}_1 = I_1 c_1 \), \( \mathcal{I}_2 = I_1 c_2 \) to bound \( |g_\gamma'(v_{k_0})| \) there exists \( k_0 > 0 \) such that the following inequalities hold

\[
\beta_1 < \max k^{-1} (I_1^{k_0} c_1) < \beta, \\
d_\gamma(k) < \lambda^k \text{ for all } \gamma \in [\alpha_0, \beta_1] \text{ and } k = 1, \ldots, k_0 + 1.
\]

Again by Lemma 3.10 property (17) and inequalities (15), if \( k_1 \) is sufficiently large then

\[
d_\gamma(k) > \lambda^k \text{ for all } \gamma \in [\alpha_1, \beta_1] \text{ and } k = 1, \ldots, t_1,
\]

where \( t_1 = k_0 + 1 + k_1 = |S_1| \). Let us choose \( k_1 \) such that the previous inequality holds and such that \( t_1 > N_1 \) and

\[
\max (\varepsilon R^{-1}, 2M_4 (\theta R)^{-1}, \varepsilon^2 (\theta R)^{-2}, 2M_4) < \lambda^{t_1 - 1},
\]

where \( M_1 \) and \( M_4 \) are defined by inequalities (32) respectively (33). This achieves the construction of the family \( \mathcal{G}_1 \).

For all \( k \geq 1 \) we construct \( \mathcal{G}_{2k} \) using Proposition 4.1 with

\[
\lambda^{-1} < \lambda_1 < \lambda_2 < 1,
\]
and \( G_{2k+1} \) using Proposition 4.2 with
\[
\Delta_k = 2^{-k}.
\]
Using inequality (19) the sequence \( (G_n)_{n \geq 1} \) converges to a map \( g = g_\gamma \). Let us denote \( d(n) = d_n(\gamma) = |(g^n)'(v)| \) and \( d(n,p) = d_{n,p}(\gamma) = |(g^p)'(v_p)| \) for all \( n, p \geq 0 \), where \( v \) is the second critical value and \( v_n = g^n(v) \). For all \( n \geq 2 \) let \( p_n = p \) be provided by Proposition 4.1 or Proposition 4.2 used to construct \( G_n \). Therefore for all \( n \geq 1 \)
\[
t_n < p_{n+1} < t_{n+1},
\]
where \( t_n = |S_n| \) the length of the common prefix \( S_n \) of kneading sequences in \( G_n \). As \( \gamma \in [\alpha_n, \beta_n] \) for all \( n \geq 1 \),
\[
k = k(\gamma) \in S_n \times \Sigma \text{ for all } n \geq 1.
\]
Let us also recall that for all \( k \geq 1 \)
\[
S_{2k} = S_{2k-1} I_2^{k_1} I_3^{k_2} I_2^{k_3},
\]
and that we may choose \( k_1, k_2 \) and \( k_3 \) as large as we need. We impose therefore for all \( k \geq 1 \)
\[
(42) \quad k_1 > N_3, k_2 > N_2 \text{ and } k_3 > N_3.
\]
Let us remark that \( g(c_1) = 1, g(1) = 1 \) and \( |g'(1)| > 1 \) therefore \( c_1 \in J \) the Julia set of \( g \) and \( c_1 \) is non-recurrent and Collet-Eckmann. Let us remark that \( \Delta_k \rightarrow 0 \) as \( k \rightarrow \infty \) and \( \gamma \in [\alpha_{2k+1}, \beta_{2k+1}] \) for all \( k \geq 1 \) therefore the second critical orbit is recurrent. By construction and inequality (12) the second critical orbit accumulates on \( r \) and on 1. Therefore \( c_2 \in J \) using for example a similar argument to the proof of Lemma 4.7. Let us show that \( c_2 \) is not Collet-Eckmann. Indeed, by Proposition 4.1 for all \( k \geq 1 \)
\[
d(p_{2k}) < \lambda^2 < 1,
\]
and \( p_{2k} \rightarrow \infty \) as \( k \rightarrow \infty \). Therefore by Definition 2.3
\[
g \text{ is not RCE}.
\]
Combining inequalities (40) and (16), the third claim of Proposition 4.1 and the second claim of Proposition 4.2
\[
(43) \quad d(n) > \lambda^n \text{ for all } n \in \bigcup_{k \geq 0} \{t_{2k}, \ldots, p_{2k+2} - 1\}.
\]
Let us check that for all \( m > 0 \) such that \( |g^m(c_2) - c_2| < \varepsilon \)
\[
(44) \quad d(m) > \lambda^m.
\]
Let us recall that \( \varepsilon \leq \varepsilon_0 \) by its definition (39) so \( |g^m(c_2) - c_2| < \varepsilon \) implies that \( v_m = g^{m+1}(c_2) \in I_1 \) therefore \( k(m) = I_1 \) so there exists \( k \geq 1 \) such that
\[
t_{2k} < m < t_{2k+1}.
\]
as Proposition 4.1 extends $S_{2n-1}$ to $S_{2n}$ using only the symbols $I_2$ and $I_3$ for all $n \geq 1$. Therefore $m \in \{l_{2k}, \ldots, p_{2k+2} - 1\}$ thus inequality (44) is a direct consequence of inequality (13).

Let us show that $g$ has $ExpShrink$. We use a telescope that is very similar to the one used in the proof of Theorem 2.8. Loosely speaking, the strategy is the following. We consider pullbacks of a small ball centered on the Julia set $J$ of $g$. We show that after some time, the pullbacks are contracted. We include such a component in another ball centered on $J$. The construction is achieved inductively and contractions at each step are used to show uniform contraction, thus $ExpShrink$. We call blocks of the telescope the sequences of pullbacks associated to each step. In order to deal with various configurations of critical points inside the telescope, we need three types of blocks.

Let us introduce additional notation and rigorously define the telescope.

For $B \subseteq \mathbb{C}$ connected and $n \geq 0$, we write $B^{-n}$ or $g^{-n}(B)$ for some connected component of $g^{-n}(B)$. When $z \in B$ and some backward orbit $z_n \in g^{-n}(z)$ are fixed, $B^{-n}$ is the connected component of $g^{-n}(B)$ that contains $z_n$.

We consider a pullback of an arbitrary ball $B(z, R)$ with $z \in J$, of length $N > 0$. We show that there are constants $C_1 > 0$ and $\lambda_3 > 1$ that do not depend on $z$ nor on $N$ such that

$$\text{diam } B(z, R)^{-N} \leq C_1 \lambda_3^{-n}.$$ 

It is easy to check that the previous inequality for all $z \in J$ and $N > 0$ implies the $ExpShrink$ condition.

Let $(z_n)_{n \geq 0} \subseteq J$ be a backward orbit of $z$, that is $z_0 = z$ and $g(z_{n+1}) = z_n$ for all $n \geq 0$. We consider preimages of $B(z, R_0') := B(z, R)$ up to time $N$. We show that there is some moment $N_0'$ when the pullback $B(z, R_0')^{-N_0'}$ observes a strong contraction. Then $B(z, R_0')^{-N_0'}$ can be nested inside some ball $B(z_{N_0'}, R_1')$ where $R_1' \leq R$. This new ball is pulled back and the construction is achieved inductively. The pullbacks $B(z, R_0')$, $B(z, R_0')^{-1} \ldots B(z, R_0')^{-N_0'}$ form the first block of the telescope. The pullbacks $B(z_{N_0'}, R_1')$, $B(z_{N_0'}, R_1')^{-1} \ldots B(z_{N_0'}, R_1')^{-N_1'}$ form the second block and so on. Lemma 4.17 is essential to manage the passage between two such consecutive telescope blocks. We show contraction for every block using either Corollary 4.11 or Proposition 4.13. This leads to a classification of blocks depending on the presence and on the type of critical points inside them.

Let $R'$ be the radius of the initial ball of some block and $N'$ be its length. We introduce a new parameter $r' < R$ for each block, a lower bound for $R'$. It is an upper bound of the diameter of the last pullback of the previous block. This choice guarantees that consecutive blocks are nested. A block that starts at time $n$ is defined by the choice of $R'$ with $r' \leq R' \leq R$ and of $N'$ with $1 \leq N' \leq N - n$. It is the pullback of length $N'$ of $B(z_n, R_0')$.

For all $n, t \geq 0$ and $r > 0$ we denote

$$d(n, r, t) = \deg_{B(z_n, r)^{-t}} (g^t) \quad \text{and} \quad \overline{d}(n, r, t) = \deg_{B(z_n, r)^{-\overline{t}}} (g^t).$$
Fix $n \geq 0$ and $t \geq 1$ and consider the maps $d$ and $\overline{d}$ defined on $[r', R]$. They are increasing and $d \leq \overline{d}$. Moreover, for all $n \geq 0$, $r > 0$, $t \geq 0$ and $s > 0$,
$$\overline{d}(n, r, t) \leq d(n, r + s, t).$$

The set $\{ r \in [r', R] \mid d(n, r, t) < \overline{d}(n, r, t) \}$ is the common set of discontinuities of $d$ and $\overline{d}$. Note also that $d$ is lower semi-continuous and $\overline{d}$ is upper semi-continuous.

For transparency, we also denote
$$W_k = B(z_n, R')^{-k}.$$

Let us define the three types of blocks. For convenience, we keep the same notations as in the proof of Theorem 2.8 in [11].

**Type 1:** Blocks with $R' = r'$ and $N'$ such that $\overline{d}(n, R', N') > 1$ and $c_2 \in W_{N'+1}$.

**Type 2:** Blocks with $R' = R$, $N' = \min(N_0, N - n)$ and $d(n, R, N - n) \leq \mu$.

**Type 3:** Blocks with $\overline{d}(n, R', N') > 1$, $c_2 \in W_{N'+1}$ and $d(n, R', N - n) \leq \mu$.

Let us define $r'$. It is the diameter of the last pullback of the previous block of type 1 or 3. It is $r = \theta R$ if the previous block is of type 2 and $r' = R$ for the first block.

Let us first show that for all $z \in J$ and $N > 0$ we may define a telescope using the three types of blocks. The construction is inductive and we show that given $0 \leq n < N$ and $0 < r' \leq R$ we can find $R' \in [r', R]$ and $0 < N' \leq N$ that define a block of one of the three types. We also show contraction along every block so $r'$ defined as above is smaller than $R$, thus completing the proof of the existence of the telescope.

If $\overline{d}(n, r', N - n + 1) > \mu = 2$ then by Corollary 4.18 there is $1 \leq N' \leq N - n$ that defines a type 1 pullback for $R' = r'$. If $\overline{d}(n, R, N - n) \leq \mu$ then we define a type 2 block, as $d \leq \overline{d}$. Note that the first block of the telescope is already constructed as $r' = R$. In all other cases $r' < R$. If $\overline{d}(n, R, N - n + 1) > \mu$ there is a smallest $R'$, with $r' < R' \leq R$, such that $\overline{d}(n, R', N - n + 1) > \mu$. Thus $R'$ is a point of discontinuity of $\overline{d}$ so $d(n, R', N - n + 1) < \overline{d}(n, R', N - n + 1)$, therefore $d(n, R', N - n) \leq d(n, R', N - n + 1) \leq \mu$. Then by Corollary 4.18 there is $1 \leq N' \leq N - n$ that defines a type 3 pullback.

Let us be more precise with our notations. We denote by $n'_0$, $N'_0$, $r'_0$ and $R'_0$ the parameters $n$, $N'$, $r'$ and $R'$ of the $i$-th block. Let also $W_{i,k}$ be $W_k$ in the context of the $i$-th block with $i \in \{0, \ldots, b\}$, where $b + 1$ is the number of blocks of the telescope. So $n'_0 = 0$, $r'_0 = R$ and $n'_1 = N'_0$. In the general case $i > 0$, we have
$$n'_i = n'_{i-1} + N'_{i-1} \quad \text{and} \quad r'_i \geq \text{diam } W_{i-1,N'_{i-1}}.$$

Let us also denote by $T_i \in \{1, 2, 2', 3\}$ the type of the $i$-th block. The type 2' is a particular case of the second type, when $N' < N_0$. This could only happen for the last block, when $N - n'_b < N_0$. So $T_i \in \{1, 2, 3\}$ for all $0 \leq i < b$. We may code our telescope by the type of its blocks, from right to left
$$T_b \ldots T_2 T_1 T_0.$$

Our construction shows that
$$\text{diam } W_{i-1,N'_{i-1}} \leq r'_i < R \quad \text{for all } 0 < i < b.$$
is a sufficient condition for the existence of the telescope that contains the pullback of $B(z, R)$ of length $N$.

If $T_i = 2$ we apply Corollary 4.10 so
\[(46) \quad \text{diam } W_{i,N'_i} < r'_{i+1} = \theta R < R.\]

If $T_i \in \{1, 3\}$ we show that there exists $\lambda_0 > 1$ such that
\[(47) \quad \text{diam } W_{i,N'_i} < \theta R' \lambda_0^{-N'_i} < R,\]
as $\theta < \frac{1}{2}$, $R'_i \leq R$ and $\lambda_0^{-N'_i} < 1$. This inequality completes the proof of the existence of the telescope.

Let us fix $i \geq 0$ such that $T_i \in \{1, 3\}$. Suppose that $i > 0$ and $T_{i-1} \in \{1, 3\}$ also, therefore
\[c_2 \in W_{i-1,N'_{i-1}+1} \subseteq W_{i,1} = B(z_{n_i}, R'_i) = g^{N'_i}(W_{i,N'_i+1}) \subseteq B(z_{n_i}, R)^{-1}.\]

But $c_2 \in W_{i,N'_i+1}$ also and $\text{diam } B(z_{n_i}, R)^{-1} < \varepsilon$ by the definition of $R$. Therefore by inequality (44)
\[d(N'_i) > \lambda^{N'_i}\]
so by Corollary 4.14 we may apply Proposition 4.13 to obtain
\[\text{diam } W_{i,N'_i} < \theta R' \lambda^{-N'_i}.\]

We have proved that for all $i > 0$ with $T_{i-1} \in \{1, 3\}$ inequality (47) holds for all $\lambda_0 \leq \lambda^{\frac{1}{2}}$.

If $i = 0$ or $T_{i-1} = 2$ then $R'_i \in [\theta R, R]$. Therefore it is enough to show that there exists $\lambda_0 > 1$ such that for all $z \in J$, $\tau \in [\theta R, R]$, $n > 0$ and $W$ a connected component of $g^{-n}(B(z, \tau))$ the following statement holds. If $v \in W$ and there exist $0 \leq m < n$ such that
\[g^m(W) \cap \text{Crit} \neq \emptyset\] then
\[(48) \quad \text{diam } W < \theta \lambda^{-n}.\]

Again, if $d(n) > \lambda^n$ using Corollary 4.14 and Proposition 4.13 the previous inequality is satisfied for all $1 < \lambda_0 \leq \lambda^{\frac{1}{2}}$. Therefore using inequality (43) we may suppose that there exist $k' \geq 1$ such that
\[p_{2k'} \leq n < t_{2k'}.\]

Let us denote $p = p_{2k'}$, $t = t_{2k' - 1}$ and $W^k = g^k(W)$ for all $k = 0, \ldots, n$. By the definition of $p$ in Proposition 4.1
\[(49) \quad 2t < p.\]

Using Corollary 4.16 inequalities (42) and (41)
\[\text{diam } W^t < \lambda^{-(p-1-t)} \text{diam } W^{p-1} < \lambda^{-(p-1-t)} \varepsilon < R.\]

As $t_1 > N_1$, inequality (16) lets us apply Proposition 4.13 to $B(v_t, \text{diam } W^t)$ which combined to the previous inequality shows that
\[(50) \quad \text{diam } W < \theta \lambda^{-(p-1-t)} \text{diam } W^{p-1}.\]
By inequalities (42), using Lemma 4.15 and eventually Corollary 4.16 if \( v_n \in I_2 \)
\[
\text{diam } W^p < \lambda^{-(n-p)} \text{diam } W^n = 2\lambda^{-(n-p)} \tau.
\]

Therefore the only missing link is an estimate of \( \text{diam } W^{p-1} \) with respect to \( \text{diam } W^p \).
We distinguish the following two cases.

1. \( \text{dist } (W^{p-1}, c_1) < 3 \text{diam } W^{p-1} \).
2. \( \text{dist } (W^{p-1}, c_1) \geq 3 \text{diam } W^{p-1} \).

Suppose the first case. The by the definition (39) of \( \varepsilon \) we may use inequality (33) therefore
\[
\text{diam } W^{p-1} < (M_4 \text{diam } W^p)^{\frac{1}{2}} < (2M_4\tau)^{\frac{1}{2}} \lambda^{-\frac{n-p}{2}}
\]
using inequality (51). Recall that \( \tau \geq \theta R \) and \( t \geq t_1 \). Therefore inequalities (50), (49) and (41) imply that
\[
\text{diam } W < \theta \lambda^{-\frac{n}{2}} \tau (2M_4\tau^{-1})^{\frac{1}{2}} \lambda^{-\frac{n}{2}} < \theta \lambda^{-\frac{n}{2}} \tau.
\]

Therefore in the first case it is enough to choose \( \lambda_0 \leq \lambda^\frac{1}{2} \).

Suppose the second case. Using inequalities (50), (49) and (41) we may compute
\[
\text{diam } W < \theta \lambda^{-\frac{n}{2}} \tau \theta R \leq \theta \lambda^{-\frac{n}{2}} \tau = \theta \lambda^{-n} (\frac{\tau}{R}) \tau.
\]

This is not enough as \( \lambda_0 \) should depend only on \( g \). We may remark that we are in position to use inequality (32) for \( W^p \) therefore
\[
\text{diam } W^{p-1} < M_1 |g'(v_{p-1})|^{-1} \text{diam } W^p.
\]

Let us compute an upper bound for \( |g'(v_{p-1})|^{-1} = d(p - 1, 1)^{-1} \). Using the first two claims of Proposition 4.1
\[
d(p)^{-1} = d(p - 1)^{-1}d(p - 1, 1)^{-1} < \lambda_1^{-p} < \lambda^p,
\]
and
\[
d(p - 1) < \lambda_1^{p-1} \lambda^{p-1},
\]
where we denote \( \lambda_r = |g'(r(\tau))| \). Let \( \nu = \frac{\log \lambda_r}{\log \lambda} \). Combining the previous inequalities
\[
d(p - 1, 1)^{-1} < \lambda^{p(\nu + 2)},
\]
therefore using inequalities (50), (51), (49) and (41)
\[
\text{diam } W < 2M_1 \theta \lambda^{-(p-1-\frac{1}{2})} \lambda^{p(\nu + 2) - (n-p) \tau} < \theta(2M_1) \lambda^{p(\nu + 2)+p+1+\frac{1}{2}} \lambda^{-n} \tau < \theta \lambda^{-n+p(\nu+3)} \tau.
\]
If \( n > 2p(\nu + 3) \) then inequality (48) is satisfied for all \( \lambda_0 \leq \lambda_1^2 \). If \( n \leq 2p(\nu + 3) \) then using inequality (52), inequality (48) is satisfied for all

\[
\lambda_0 \leq \lambda_0^{-\nu + 3} \leq \lambda_0^{-\frac{p}{\lambda_0}},
\]

which completes the proof of inequality (47) and therefore of the existence of the telescope.

The remainder of the proof shows the global exponential contraction of the diameter of pullbacks. It is identical to the second part of the proof of Theorem 2.8 in [11]. We reproduce it here for convenience.

Note that if \( T_i = 1 \) then we may rewrite inequality (47) as follows

\[
(53) \quad r'_{i+1} < \theta r'_{i}^{-N'_i} < r'_{i}^{-N'_i}.
\]

Recall also that if there are \( \lambda_3 > 1 \) and \( C_1 > 0 \) such that

\[
(54) \quad \text{diam } B(z, R)^{-N} = \text{diam } W_{0,N} < C_1 \lambda_3^{-N},
\]

then the theorem holds. We may already set

\[
(55) \quad \lambda_3 = \min \left( 2^{\frac{1}{2}} 3^2, \lambda_0^2 \right).
\]

As inequality (53) provides an easy way to deal with the first type of block, we compute estimates only for sequences of blocks of types 1...1, 1...12 and 1...13, as the sequence \( T_i \) can be decomposed in such sequences. Sequences with only one block of type 2 or 3 are allowed as long as the following block is not of type 1. For a sequence of blocks \( T_{i+p} \ldots T_i \), let

\[
N'_{i,p} = N'_{i+p} + \ldots + N'_i
\]

be its length.

A sequence 1...1 may only occur as the first sequence of blocks, thus \( i = 0 \). As \( r'_0 = R \), iterating inequality (53) for such a sequence we obtain

\[
(56) \quad r'_{p+1} < \theta^{p+1} R \lambda_0^{-N'_0}_{0,p} < 2 \theta R' \lambda_3^{-\mu N'_{0,p}}.
\]

Combining inequalities (53), (46) and the definition (55) of \( \lambda_3 \), for a sequence 1...12

\[
(57) \quad r'_{i+p+1} < r'_{i+1} \lambda_0^{-N'_{i+1,p-1}} < 2 \theta R' \lambda_3^{-\mu N'_{i+1,p-1}} \leq 2 \theta R' \lambda_3^{-\mu N'_{i,p}},
\]

as \( N'_i = N_0, N'_{i,p} = N'_i + N'_{i+1,p-1} \) and \( R'_i = R \).

For a sequence 1...13, inequalities (53) and (47) yield

\[
(58) \quad r'_{i+p+1} < r'_{i+1} \lambda_0^{-N'_{i+1,p-1}} < \theta R' \lambda_0^{-N'_{i,p}} < 2 \theta R' \lambda_3^{-\mu N'_{i,p}}.
\]
We also find a bound for $r_{b+1}'$ in the case $T_b = 2'$. Using notations introduced in Section 4.2, we define $K = \|g^{'}\|_{\infty,J_{s+}}$. We compute

$$r_{b+1}' < R_b' \mu K N_b'$$

$$= \mu (K \lambda_3)^{N_b'} R_b' \lambda_3^{-N_b'} .$$

We decompose the telescope into $m + 1$ sequences $1 \ldots 1$, $1 \ldots 12$, $1 \ldots 13$ and eventually $2'$ on the leftmost position

$$S_m \ldots S_2 S_1 S_0 .$$

Consider a sequence of blocks

$$S_j = T_{i+p} \ldots T_i .$$

Denote $n_j'' = n_i$, $N_j'' = N_i$, $r_j'' = r_i'$ and $R_j'' = R_i'$. Let also

$$\Delta_j = \text{diam} W_i, N - n_i'$$

be the diameter of the pullback of the first block of the sequence up to time $-N$.

With the eventual exception of $S_m$, inequalities (56), (57) and (58) provide good contraction estimates for each sequence $S_j$

$$r_{j+1}'' < 2 \theta R_{j+1}'' \lambda_3^{-\mu N_j''} .$$

If $T_b = 2'$ then inequality (59) yields a constant $\mu (K \lambda_3)^{N_b'} < C_1 = \mu (K \lambda_3)^{N_0}$ such that

$$r_{m+1}'' < C_1 R_m'' \lambda_3^{-N_m''} ,$$

as $R_m'' = R_b'$ and $N_m'' = N_b'$. Note that the previous inequality also holds if $T_b \in \{1, 2, 3\}$. We cannot simply multiply these inequalities as $R_j'' > r_j''$ for all $0 < j \leq m$.

By the definitions of types 2 and 3, the degree $d(n_j'', R_j'', N - n_j'')$ is bounded by $\mu$ in all cases. So Lemma 4.17 provides a bound for the distortion of pullbacks up to time $-N$

$$\frac{\Delta_{j+1}}{\Delta_j} < \frac{32 \left( \frac{R_{j+1}''}{R_j''} \right)^{\frac{1}{\mu}}}{\lambda_3^{N_j''}} .$$

$$= \lambda_3^{-N_j''} \left( \frac{R_{j+1}''}{R_j''} \right)^{\frac{1}{\mu}} ,$$

for all $0 < j \leq m$. Therefore

$$\frac{\Delta_0}{\Delta_m} < \lambda_3^{-N + N_m''} \left( \frac{R_0''}{R_m''} \right)^{\frac{1}{\mu}} .$$

Recall that $R_j'' \leq R < 1$ for all $0 \leq j \leq m$ and $\Delta_m = r_{m+1}''$, so

$$\Delta_0 < \lambda_3^{-N + N_m''} C_1 R_m'' \lambda_3^{-N_m''} \left( \frac{R}{R_m''} \right)^{\frac{1}{\mu}}$$

$$< \lambda_3^{-N} C_1 \left( R_m'' \right)^{\frac{1}{\mu}}$$

$$< C_1 \lambda_3^{-N} .$$
By definition $\Delta_0 = \text{diam} W_{0,N}$, therefore the previous inequality combined with inequality (54) completes the proof of the theorem.

5. RCE is not a topological invariant for real polynomials with negative Schwarzian derivative

Let $H : [0, h] \to P_2$ (see the definition of $P_2$ at page 8) be equal to the family $G$ defined in the previous section. Let us define another family of bimodal maps $\tilde{H} : [0, h'] \to P_2$ in an analogous fashion. Let $T \in \mathbb{R}_7[x]$ be a degree 7 polynomial such that $T(0) = 0$ and such that $T'(x) = (x + 1)^3(x - 1)^3$. Therefore $T$ has two critical points $-1$ and $1$ of degree 4 and $T(-x) = -T(x)$ for all $x \in \mathbb{R}$. Let $y_0 = T(-1)$ and $x_0 > 1$ such that $T(x_0) = y_0$. Let $h' > 0$ be small and for each $\gamma \in [0, h']$ two order preserving linear maps $R_{\gamma'}(x) = x(2x_0 + \gamma') - x_0 - \gamma'$ and $S_{\gamma'}(y) = \frac{y - T(-x_0 - \gamma')}{y_0 - T(-x_0 - \gamma')}$ that map $[0, 1]$ onto $[-x_0 - \gamma', x_0]$ respectively $[T(-x_0 - \gamma'), T(x_0)]$ onto $[0, 1]$. One may show by direct computation that if a real polynomial $P$ is such that all the roots of $P'$ are real then $P$ has negative Schwarzian derivative. Therefore

$$\tilde{h}_{\gamma'} = S_{\gamma'} \circ T \circ R_{\gamma'} \in P_2$$

for all $\gamma' \in [0, h']$.

We define $\tilde{H}(\gamma') = \tilde{h}_{\gamma'}$ for all $\gamma' \in [0, h']$. Let us remark that $y_0 \in (0, 1)$ and $x_0 \in (\frac{3}{2}, 2)$ therefore all three fixed points of $\tilde{h}_0$ are repelling. Let $\tilde{r}(\gamma')$ be the only fixed point of $\tilde{h}_{\gamma'}$ in $(0, 1)$ and $\tilde{c}_1 < \tilde{c}_2$ its critical points. The proofs that for $h' > 0$ sufficiently small $H$ satisfies properties (3) to (6), Lemmas 3.1 and 3.2, that it is natural, that $\tilde{r}(\gamma') \to \tilde{c}_1$ as $\gamma' \to 0$ and $(\gamma') = 0$ and $|\tilde{h}_0'(r(0))| = \frac{\gamma'}{y_0}$ one may compute that

$$\frac{1}{2} \log |h_0'(1)| = 1 < \frac{3}{4} \log |h_0'(1)|$$

We may also suppose $h > 0$ and $h' > 0$ sufficiently small such that there exist $1 < \lambda < \lambda'$, $1 < \lambda < \lambda'$ and $\theta_1 < \theta_2$ such that for all $\gamma \in [0, h]$ and $\gamma' \in [0, h']$

$$\lambda' < \min \left( \left| \tilde{h}_{\gamma'}'(0) \right|, \left| \tilde{h}_{\gamma'}'(1) \right| \right)$$

and

$$\tilde{\lambda}' < \min \left( \left| \tilde{h}_{\gamma'}'(0) \right|, \left| \tilde{h}_{\gamma'}'(r(\gamma')) \right| \right)$$

and

$$\frac{1}{2} \log \left| \tilde{h}_{\gamma'}'(\gamma) \right| < \theta_1 < \theta_2 < \frac{3}{4} \log \left| \tilde{h}_{\gamma'}'(1) \right|$$

Let us denote $\tilde{k}(\gamma)$ the second kneading sequence of $h_{\gamma'}$ and $\tilde{k}(\gamma')$ the second kneading sequence of $\tilde{h}_{\gamma'}$. We construct two decreasing sequences of families of bimodal maps $(\mathcal{H}_n)_{n \geq 1}$
and \((\tilde{H})_{n \geq 1}\). Let \(\mathcal{H}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2\) with \(\mathcal{H}_n(\gamma) = \mathcal{H}(\gamma)\) for all \(\gamma \in [\alpha_n, \beta_n]\) and \(\tilde{\mathcal{H}}_n : [\alpha'_n, \beta'_n] \to \mathcal{P}_2\) with \(\tilde{\mathcal{H}}_n(\gamma) = \tilde{\mathcal{H}}(\gamma)\) for all \(\gamma' \in [\alpha'_n, \beta'_n]\). By construction we choose that for all \(n \geq 1\)

\[
\tilde{k}(\alpha_n) = \tilde{k}(\alpha'_n) \quad \text{and} \quad \tilde{k}(\beta_n) = \tilde{k}(\beta'_n).
\]

Let us denote \(v = h_\gamma(c_2)\), \(\tilde{v} = h'_{\gamma'}(c_2)\) and \(v_n = h_\gamma^n(v)\), \(\tilde{v}_n = h'_{\gamma'}^n(\tilde{v})\) for all \(n \geq 0\), \(\gamma \in [0, h]\) and \(\gamma' \in [0, h']\). Let also \(d_n(\gamma) = |(h_\gamma^n)'(v)|\), \(\tilde{d}_n(\gamma') = |(\tilde{h}_{\gamma'}^n)'(\tilde{v})|\), \(d_{n,p}(\gamma) = |(h_\gamma^n)'(v_n)|\) and \(\tilde{d}_{n,p}(\gamma') = |(\tilde{h}_{\gamma'}^n)'(\tilde{v}_n)|\) for all \(n, p \geq 0\), \(\gamma \in [0, h]\) and \(\gamma' \in [0, h']\). The basic construction tool is again Proposition 3.7 and we build the sequences \((\mathcal{H}_n)_{n \geq 1}\) and \((\tilde{\mathcal{H}})_{n \geq 1}\) by specifying the common prefix \(S_n\) of the kneading sequences in \(\mathcal{H}_n\) and \(\tilde{\mathcal{H}}_n\) for all \(n \geq 1\). We also reuse the notation \(t_n = |S_n|\) for all \(n \geq 1\). In an analogous way to the construction of the family \(G_1\), see inequality (40), we choose

\[
S_1 = I_1^{k_0+1}I_2^{k_1}
\]

such that

\[
d_k(\gamma) > \lambda^k \quad \text{and} \quad \tilde{d}_k(\gamma') > \tilde{\lambda}^k
\]

for all \(\gamma \in [\alpha_1, \beta_1]\), \(\gamma' \in [\alpha'_1, \beta'_1]\) and \(k = 1, \ldots, t_1\) and

\[
\beta_1 < h \quad \text{and} \quad \beta'_1 < h'.
\]

Let us describe the construction of the sequences \((\mathcal{H}_n)_{n \geq 1}\) and \((\tilde{\mathcal{H}})_{n \geq 1}\) which satisfy properties (16) to (20) and

\[
\tilde{d}_{n}(\gamma') > \tilde{\lambda}^n
\]

for all \(n \geq 1\).

Let us recall that Proposition 4.1 employs twice Proposition 3.7, to construct a subfamily \(G_{n+1}\) of \(G_n\) with

\[
S_{n+1} = S_n I_1^{k_1+1}I_2^{k_2}I_3^{k_3}
\]

Let \(\gamma_\infty\) and \(\gamma'_\infty\) be provided by Proposition 3.7 such that \(k(\gamma) = k(\gamma') = S_n I_2^\infty\). We use the same strategy as in the proof of Proposition 4.1 to define both \(\mathcal{H}_{n+1}\) and \(\tilde{\mathcal{H}}_{n+1}\) with the same combinatorics. Taking \(k_1, k_2, k_3\) sufficiently large we may control the growth of \(d_m(\gamma)\) and \(\tilde{d}_m(\gamma')\) uniformly for all \(t_n < m \leq t_{n+1}\). We let

\[
k_1 \to k_2 \to n > 0,
\]

\(p = t_n + k_1 + 1\) and compute some bounds for \(d_p(\gamma)\) and \(\tilde{d}_p(\gamma')\). For transparency, let us denote \(\lambda_0 = |h_{\gamma_\infty}'(r)|\), \(\tilde{\lambda}_0 = |\tilde{h}_{\gamma'_\infty}'(\tilde{r})|\), \(\lambda_3 = |h_{\gamma_\infty}'(1)|\) and \(\tilde{\lambda}_3 = |\tilde{h}_{\gamma'_\infty}'(1)|\). As in the proof of Proposition 4.1, we obtain

\[
\lim_{k_1 \to \infty} \frac{1}{k_1} \log d_p(\gamma) = \log \lambda_0 - \frac{1}{2n} \log \lambda_3 \quad \text{for all} \quad \gamma \in [\alpha_{n+1}, \beta_{n+1}].
\]
We may observe that inequalities \((23)\) hold exactly when \(c_1\) is a second degree critical point. We may however write similar bounds for \(\tilde{H}_{n+1}\). By the same arguments there exist constants \(\tilde{M} > 1\), \(\tilde{d} > 0\) and \(\tilde{N}_2 > 0\) such that if \(k_1 > \tilde{N}_2\) and \(\gamma' \in [\gamma_1', \gamma_2']\) then
\[
\tilde{M}^{-1}(x - \tilde{c}_1)^4 < |1 - \tilde{h}_{\gamma'}(x)| < \tilde{M}(x - \tilde{c}_1)^4 \quad \text{and} \quad \tilde{M}^{-1}(x - \tilde{c}_1)^3 < |\tilde{t}_{\gamma'}(x)| < \tilde{M}(x - \tilde{c}_1)^3
\]
for all \(x \in (\tilde{c}_1 - \tilde{d}, \tilde{c}_1 + \tilde{d})\), where \(\gamma_1', \gamma_2'\) are the bounds for \(\gamma'\) provided by Proposition \(3.7\) applied to \(S_n\) and \(\hat{H}_n\). Therefore we obtain
\[
\lim_{k_1 \to -\infty} \frac{1}{k_1} \log \tilde{d}_p(\gamma') = \log \tilde{\lambda}_0 - \frac{3}{4\eta} \log \tilde{\lambda}_3 \quad \text{for all} \quad \gamma' \in [\alpha'_{n+1}, \beta'_{n+1}].
\]
Using inequalities \((60)\) and the limits \((63)\) and \((64)\) it is enough to choose
\[
\theta_1 < \eta < \theta_2
\]
to obtain the following corollary of Proposition \(4.1\)

**Corollary 5.1.** There exist
\[
0 < \lambda_1 < 1 < \lambda_2 < \min \left(\lambda, \frac{\lambda}{\lambda'}\right)
\]
that depend only on \(H_1\) and \(\hat{H}_1\) such that if \(H_n\) is a subfamily of \(H_1\) and \(\hat{H}_n\) is a subfamily of \(\hat{H}_1\) both satisfying conditions \((16)\) to \((24)\) and \((62)\) then there exist \(H_{n+1}\) a subfamily of \(H_n\) and \(\hat{H}_{n+1}\) a subfamily of \(\hat{H}_n\) satisfying the same condition and \(2t_n < p < t_{n+1}\) with the following properties

1. \(d_p(\gamma) > \lambda'_2\) for all \(\gamma \in [\alpha_{n+1}, \beta_{n+1}]\).
2. \(\tilde{d}_p(\gamma') < \lambda'_2\) for all \(\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]\).
3. \(d_{t,n}(\gamma) > \lambda'\) for all \(\gamma \in [\alpha_{n+1}, \beta_{n+1}]\) and \(l = 1, \ldots, p - 1 - t_n\).
4. \(\tilde{d}_{t,n}(\gamma') > \tilde{\lambda}'\) for all \(\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]\) and \(l = 1, \ldots, p - 1 - t_n\).
5. \(d_{p,l}(\gamma) > \lambda'\) for all \(\gamma \in [\alpha_{n+1}, \beta_{n+1}]\) and \(l = 1, \ldots, t_{n+1} - p\).
6. \(\tilde{d}_{p,l}(\gamma') > \tilde{\lambda}'\) for all \(\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]\) and \(l = 1, \ldots, t_{n+1} - p\).

Proposition \(4.2\) has an immediate corollary for the families \(H\) and \(\hat{H}\).

**Corollary 5.2.** Let the subfamilies \(H_n\) and \(\hat{H}_n\) of \(H_1\) respectively \(\hat{H}_1\) with \(n \geq 1\) satisfy conditions \((16)\) to \((24)\) and \((62)\) and \(\Delta > 0\).

Then there exist subfamilies \(H_{n+1}\) of \(H_n\) and \(\hat{H}_{n+1}\) of \(\hat{H}_n\) satisfying the same conditions and such that there exists \(t_n < p < t_{n+1}\) with the following properties

1. \(|h_{\gamma}(c_2) - c_2| < \Delta\) for all \(\gamma \in [\alpha_{n+1}, \beta_{n+1}]\).
2. \(|\tilde{h}_{\gamma'}(\tilde{c}_2) - \tilde{c}_2| < \Delta\) for all \(\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]\).
3. \(d_{t,n}(\gamma) > \lambda'\) for all \(\gamma \in [\alpha_{n+1}, \beta_{n+1}]\) and \(l = 1, \ldots, t_{n+1} - t_n\).
4. \(\tilde{d}_{t,n}(\gamma') > \tilde{\lambda}'\) for all \(\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]\) and \(l = 1, \ldots, t_{n+1} - t_n\).
For all \( k \geq 1 \) we define \( H_{2k} \) and \( \tilde{H}_{2k} \) using Corollary 5.1 and \( H_{2k+1} \) and \( \tilde{H}_{2k+1} \) using Corollary 5.2 with \( \Delta = 2^{-k} \). Let \( h \) be the limit of \( (H_n)_{n \geq 1} \) and \( \tilde{h} \) be the limit of \( (\tilde{H}_n)_{n \geq 1} \). Then \( h \) is CE therefore RCE and the second critical point \( \tilde{c}_2 \) of \( \tilde{h} \) is recurrent but not CE therefore \( \tilde{h} \) is not RCE. Both \( h \) and \( \tilde{h} \) have negative Schwarzian derivative and their second critical orbits accumulate on \( r \) and 1 respectively on \( \tilde{r} \) and 1. Moreover, using Lemma 4.7, \( h \) and \( \tilde{h} \) do not have attracting or neutral periodic points on \([0, 1]\). We may therefore apply Corollaries 2.20 and 2.21 to obtain the following theorem that contradicts Conjecture 1 in [22].

**Theorem B.** The RCE condition for S-multimodal maps is not topologically invariant.

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KTH - Royal Institute of Technology
Department of Mathematics
100 44 Stockholm, Sweden

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E-mail address: nicolae@kth.se