Matrix factorization identity for almost semi-continuous processes on a Markov chain

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In this article almost semi-continuous processes with stationary independent increments on a finite irreducible Markov chain are considered. For these processes the components of matrix factorization identity are concretely defined. On the basis of this concrete definition the relations for the distributions of extrema and distributions of their complements for the almost upper semi-continuous processes are established.

The processes with stationary independent increments on a Markov chain are considered in [1] - [4]. These processes also were considered in [7], where they were called as in [1,2] the risk processes in a Markovian environment or Markov additive processes.

We’ll consider the two-dimensional Markov process:

\[ Z(t) = \{ \xi(t), x(t) \} \quad (t \geq 0, \xi(0) = 0), \]

where \( x(t) \) is a finite ergodic Markov chain with state space \( E' = \{1, \ldots, m\} \) and transition matrix \( \mathbf{P}(t) = e^{t\mathbf{Q}}, \ t \geq 0, \ \mathbf{Q} = \mathbf{N}(\mathbf{P} - \mathbf{I}) \), where \( \mathbf{N} = \{ \| \delta_{kr} \nu_k \|_{k,r=1}^m \}, \nu_k \) are parameters of the exponentially distributed random variables \( \xi_k \) (the sojourn time of \( x(t) \) in state \( k \)), \( \mathbf{P} = \{ p_{kr} \} \) is the transition matrix of the embedded Markov chain.

Let \( \sigma_n = \sum_{k \leq n} \xi_k \), \( y_n = x(\sigma_n) \). We suppose that under conditions \( x(\sigma_n - 0) = k, x(t) = r, t \in [\sigma_n, \sigma_{n+1}) \) \( \xi(t) \) is determined by the processes with stationary independent increments \( \xi_r(t) (\xi_r(0) = 0) \) and by the independent jumps \( \chi_{kr} (k \neq r) \) at the moments \( \sigma_n, \xi_r(t) \) have the cumulant functions:

\[ \psi_r(\alpha) = \alpha a_r - \frac{1}{2} b_r^2 \alpha^2 + \int_{-\infty}^{\infty} \left[ e^{i \alpha x} - 1 - i \alpha x \delta(\{x\} = 1) \right] \Pi_r(dx), \]

\[ |\alpha| < \infty, b_r^2 < \infty, \Pi_r(\cdot) \] are spectral measures of \( \xi_r(t) \). And we denote \( \mathbf{F}(x) = \| \mathbf{P} \{ \chi_{kr} < x; y_1 = r/y_0 = k \} \| \).

The evolution of the process \( Z(t) \) is determined by the matrix characteristic function(ch.f.):

\[ \Phi_t(\alpha) = \| \mathbb{E} [e^{i \alpha (\xi(t + u) - \xi(u))}, x(t + u) = r/x(u) = k] \| \quad u \geq 0, \]

this ch.f. we can represent in the next form

\[ \Phi_t(\alpha) = \mathbb{E} e^{i \alpha \xi(t)} = e^{i \Psi_t(\alpha)} \quad \Psi(0) = \mathbf{Q}, \]

\[ \Psi(\alpha) = \| \psi_{kr} \| + \mathbb{N} \left[ \int_{-\infty}^{\infty} e^{i \alpha x} d\mathbf{F}(x) - \mathbf{I} \right], \Psi(0) = \mathbf{Q}. \]

Let \( \theta_s \) denote an exponentially distributed random variable with the parameter \( s > 0 \) \( \{ \theta_s > t \} = e^{-st}, t \geq 0 \). We assume that \( \theta_s \) is independent on \( Z(t) \), then

\[ \Phi(s, \alpha) = \mathbb{E} e^{i \alpha \xi(\theta_s)} = s \int_0^\infty e^{-st} \Phi_t(\alpha) dt = s (s \mathbf{I} - \Psi(\alpha))^{-1}. \]

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\[ \mathbf{P}_s = s \int_0^\infty e^{-st} \mathbf{P}(t)dt = s(sI - \mathbf{Q})^{-1}. \]

Let us denote the next functionals for \( \xi(t) \):

\[
\xi^+(t) = \sup_{0 \leq u \leq t} \xi(u), \quad \tilde{\xi}(t) = \xi(t) - \xi^+(t), \quad \xi^-(t) = \inf_{0 \leq u \leq t} \xi(u), \quad \tilde{\xi}(t) = \xi(t) - \xi^-(t),
\]

\[
\tau^+(x) = \inf\{ t : \xi(t) > x \}, \quad \gamma^+(x) = \xi(\tau^+(x)) - x,
\]

\[
\gamma_+(x) = x - \xi(\tau^+(x) - 0), \quad \gamma_x^+ = \gamma^+(x) + \gamma_+(x), \quad x > 0,
\]

\[
\tau^-(x) = \inf\{ t : \xi(t) < x \}, \quad x < 0.
\]

and the distributions

\[
\mathbf{P}_+(s, x) = \mathbf{P}\{ \xi^+(\theta_s) > x \}, x > 0, \quad \mathbf{P}_+(s) = \mathbf{P}\{ \xi^+(\theta_s) = 0 \}, Q_+(s) = \mathbf{P}_s - \mathbf{P}_+(s);
\]

\[
\mathbf{P}_-(s, x) = \mathbf{P}\{ \xi^-(\theta_s) < x \}, x < 0, \quad \mathbf{P}_-(s) = \mathbf{P}\{ \xi^-(\theta_s) = 0 \}, \quad \mathbf{P}_-(s, x) = \mathbf{P}\{ \xi^-(\theta_s) < x \}, x < 0.
\]

**Lemma 1.** [3] For the two-dimensional Markov process \( Z(t) = \{ \xi(t), x(t) \} \) the basic factorization identity is valid

\[
\Phi(s, \alpha) = \mathbb{E} e^{i\alpha \xi(\theta_s)} = \begin{pmatrix} \Phi_+(s, \alpha) \mathbf{P}_-^{-1} \Phi_-(s, \alpha), \\ \Phi_-(s, \alpha) \mathbf{P}_+^{-1} \Phi_+(s, \alpha), \end{pmatrix}, \quad \text{where}
\]

\[
\Phi_+(s, \alpha) = \mathbb{E} e^{i\alpha \xi^+(\theta_s)}, \quad \Phi_-(s, \alpha) = \mathbb{E} e^{i\alpha \xi^-(\theta_s)}, \quad \Phi_+(s, \alpha) = \mathbb{E} e^{i\alpha \xi(\theta_s)},
\]

\[
\Phi_+(s, \alpha) = \mathbb{E} e^{i\alpha \xi^-(\theta_s)}.
\]

In papers [3, 4] the concrete definition of components of (2) for the semi-continuous processes was obtained. We want to derive analogical concrete definition for the almost semi-continuous processes, that were investigated for the scalar case \( m = 1 \) in paper [5]. Before this let us consider some auxiliary definitions and statements.

Let \( B_m(\alpha) \) is the Banach algebra of matrices with dimension \( m \times m \), the elements of these matrices are the Fourier-Stieltjes transforms of the functions \( f_{kr}(x) \), \( k, r = 1..m \), with bounded variation.

If \( \Phi = \Phi(\alpha) \in B_m(\alpha), \Phi(\alpha) = \| \int_{-\infty}^{\infty} e^{i\alpha x} df_{kr}(x) \| \), then we determine projection operations by the next relations

\[
\| \Phi(\alpha) \|_{\pm} = \| \pm \int_{\pm\infty}^{\pm\infty} e^{i\alpha x} df_{kr}(x) \|,
\]

\[
\| \Phi(\alpha) \|^0_{\pm} = \| e_{kr}^{\pm} \int_{\pm\infty}^{\pm\infty} e^{i\alpha x} df_{kr}(x) \|.
\]

The relation between the distribution of \( \xi^+(\theta_s) \) and generating function of \( \tau^+(z) \) is valid:

\[
\mathbf{P}_+(s, z) = \mathbb{E} e^{-s\tau^+(z)} \tau^+(z) < \infty \mathbf{P}_s.
\]

In the sequel we will denote \( \mathbb{E} e^{-s\tau^+(z)}, \tau^+(z) < \infty \) = \( \mathbf{T}_+(s, x) \), taking into account that the generating function of \( \tau^+(z) \) is considered on the chain \( x(\tau^+(z)) \). Note that the state space of the chain \( x(\tau^+(z)) \) can be narrow due to nonattainability the level \( z > 0 (z < 0) \) by the process \( \xi(t) \). Therefore we impose the next conditions

\[
\{ k : P\{ x(\tau^+(z)) = k \} > 0 \} = E', \quad \{ k : P\{ x(\tau^+(z)) = k \} = 0 \} = \emptyset.
\]

\[
\| \mathbb{E} \|_H \xi(t) \|_t = r/x(0) = k \| < \infty.
\]
We’ll use the next notation

\[
dK_0(z) = NdF(z) + Π(dz), \quad K_0(x) = \int_x^\infty dK_0(z),
\]

\[
w_+(\alpha, u, v, μ) = \int_0^\infty e^{iαx} W(x, u, v, μ)dx = \int_0^\infty e^{iαx} e^{(u-ν)x-(u+μ)z} dK_0(z)dx,
\]

\[
v^+(s, α, u, v, μ) = \int_0^\infty e^{iαx} V_+(s, x, u, v, μ)dx
\]

\[
C_+(s) = \left\{s^{-1} \left[ \frac{1}{2} \frac{d}{dx} P_-(s, x) \right]_{x=0} B^2 + \left[ Φ^{-}(s, α) \right]_+^0 A_0^+ \right\}, \quad B \geq 0;
\]

\[
A_0^+ = \|δ_κ, δ(b_k = 0, a_k > 0)a_k\|, \quad A^+ = \|δ_κ, δ(a_k > 0)a_k\|, \quad B = \|κ, b^2\|.
\]

**Lemma 2.** [3] If condition (3) is satisfied, then

\[
v^+(s, α, u, v, μ) = Φ_+(s, α) P_+^{-1} \left( C_+(s) + s^{-1} \left[ Φ^{-}(s, α) w_+(α, u, v, μ) \right]_+^0 \right)
\]

(5)

If we denote

\[
K(s, x) = \int_{-∞}^0 dP_-(s, y) K_0(x - y), \quad k(s, α) = \int_0^∞ e^{iαx} K(s, x)dx,
\]

then from lemma 2 the next statement follows.

**Theorem 1.** If condition (4) is satisfied, then

\[
Φ_+(s, α) = (I - iα (C_+(s) + s^{-1} k(s, α)))^{-1} P_+.
\]

(6)

**Proof.** Let’s substitute \( u = v = μ = 0 \) in formula (5), then

\[
v^+(s, α, 0, 0, 0) = Φ_+(s, α) P_+^{-1} \left( C_+(s) + s^{-1} \left[ Φ^{-}(s, α) w_+(α, 0, 0, 0) \right]_+^0 \right),
\]

(7)

where

\[
[Φ^{-}(s, α) w_+(α, 0, 0, 0)]_+^0 = \left[ \int_{-∞}^0 e^{iαx} dP_-(s, x) \int_0^∞ e^{iαx} K_0(x)dx \right]_+^0
\]

\[
= \int_0^∞ e^{iαx} \int_{-∞}^0 dP_-(s, y) K_0(x - y)dx = k(s, α).
\]

Using formula (3) and definition of \( v^+ \), we have

\[
v^+(s, α, 0, 0, 0) = \int_0^∞ e^{iαx} E \left[ e^{-sτ^+(x)} , τ^+(x) < ∞ \right] dx = \int_0^∞ e^{iαx} P \{ ξ^+(θ_s) > x \} dx P_+^{-1}
\]

\[
= \frac{1}{iα} (Φ_+(s, α) - P_+) P_+^{-1}.
\]

Let’s substitute received relations in formula (7):

\[
\frac{1}{iα} (Φ_+(s, α) - P_+) P_+^{-1} = Φ_+(s, α) P_+^{-1} \left( C_+(s) + s^{-1} k(s, α) \right)
\]

(3)
proving formula (10) let us invert (5) with respect to $\alpha$

Let us consider the analogy of the almost upper semi-continuous scalar process analyzed in paper [5].

For the almost upper semi-continuous process $Z(t)$ we have the next concrete definition of components of the first part of (2).

**Theorem 2.** For the almost upper semi-continuous processes the distribution of $\xi^+(\theta_s)$ is determined by the next relations

\[
\Phi_+(s, \alpha) = (\mathbf{C} - \mathbf{\alpha} I) (\mathbf{p}_+(s) \mathbf{p}_+^{-1} \mathbf{C} - \mathbf{\alpha} I)^{-1} \mathbf{p}_+(s), \quad (8)
\]

\[
\mathbf{p}_+(s) = s (\mathbf{s} I + \mathbf{E} e^{\mathbf{\theta}(\theta_s) \mathbf{C}} \mathbf{A} \mathbf{F}_0(0))^{-1} \mathbf{P}_s; \quad (9)
\]

\[
\mathbf{P}_+(s, x) = q_+(s) e^{-\mathbf{p}_+ \mathbf{C} \mathbf{p}_+(s) x}, \quad x > 0, \quad (10)
\]

for $\mathbf{\xi}(\theta_s) = \xi(\theta_s) - \xi^+(\theta_s)$ we have

\[
\Phi^-(s, \alpha) = \mathbf{P}_s \mathbf{p}_+^{-1}(s) \left[ \mathbf{\Phi}(s, \alpha) \right]_- - \mathbf{q}_+(s) \mathbf{p}_+^{-1}(s) \mathbf{C} \left[ (\mathbf{C} - \mathbf{\alpha} I)^{-1} \mathbf{\Phi}(s, \alpha) \right]_-; \quad (11)
\]

\[
\mathbf{P}^-(s, x) = \mathbf{P}_s \mathbf{p}_+^{-1}(s) \mathbf{P}(s, x) - \mathbf{q}_+(s) \mathbf{p}_+^{-1}(s) \mathbf{C} \int_0^x e^{-\mathbf{C} y} \mathbf{P}(s, x - y) dy, \quad x < 0. \quad (12)
\]

Proof. Substituting $d\mathbf{K}_0(z) = \mathbf{A} \mathbf{F}_0(0) \mathbf{C} e^{-\mathbf{C}^2 z} dz, z > 0$ in formula (6), we obtain

\[
\Phi_+(s, \alpha) = \left( I - \mathbf{\alpha} s^{-1} \int_{-\infty}^0 d\mathbf{P}^-(s, y) e^{\mathbf{C} y} \mathbf{A} \mathbf{F}_0(0) (\mathbf{C} - \mathbf{\alpha} I)^{-1} \right)^{-1} \mathbf{P}_s. \quad (13)
\]

Under conditions of the theorem: $\mathbf{C}_s(s) \equiv \mathbf{0}$, then

\[
\Phi_+(s, \alpha) = \left( I - \mathbf{\alpha} s^{-1} \int_{-\infty}^0 d\mathbf{P}^-(s, y) e^{\mathbf{C} y} \mathbf{A} \mathbf{F}_0(0) (\mathbf{C} - \mathbf{\alpha} I)^{-1} \right)^{-1} \mathbf{P}_s. \quad (14)
\]

Hence

\[
\mathbf{p}_+(s) = \lim_{\alpha \to -\infty} \Phi_+(s, \alpha) = \left( I + s^{-1} \int_{-\infty}^0 d\mathbf{P}^-(s, y) e^{\mathbf{C} y} \mathbf{A} \mathbf{F}_0(0) \right)^{-1} \mathbf{P}_s. \quad (15)
\]

Substituting (15) in (14) and taking into account that $\int_{-\infty}^0 d\mathbf{P}^-(s, y) e^{\mathbf{C} y} = \mathbf{E} e^{\mathbf{\xi}(\theta_s) \mathbf{C}}$ we obtain (8). To prove formula (10) let us invert (5) with respect to $\alpha$:

\[
s \mathbf{V}_+(s, x, u, v, \mu) = \int_0^x d\mathbf{p}_+(s, z) \mathbf{p}_+^{-1} \int_{-\infty}^0 d\mathbf{P}^-(s, y) \mathbf{W}(x - y - z, u, v, \mu). \quad (16)
\]
Letting \( u = v = \mu = 0 \) we have
\[
sV_+(s,x,0,0,0) = \int_0^x \int_{-\infty}^0 \int_0^z dP_+(s,z)P_s^{-1} dP^-(s,y)K_0(x-y-z).
\]

Taking into account (3) and the condition of the almost semi-continuity we obtain the equation
\[
sT^+(s,x) = -\int_0^x \int_{-\infty}^0 dP^-(s,y) A\bar{F}_0(0)e^{C(y+z)} e^{-Cy}.
\]

Let’s differentiate with respect to \( x > 0 \) then we have the next equation for \( T^+(s,x) \):
\[
\frac{\partial}{\partial x} T^+(s,x) = -T^+(s,x)C p_+(s)P_s^{-1}, \quad x > 0,
\]
\[
T(s,0) = q_+(s)P_s^{-1}.
\]

The solution of equation (18) is expressed by the next formula
\[
T^+(s,x) = T(s,0)e^{-Cp_+(s)P_s^{-1}x}.
\]

Taking into account formulas (19) and (3), we have
\[
\bar{P}_+(s,x) = T^+(s,x)P_s
\]
\[
= q_+(s)P_s^{-1}e^{-Cp_+(s)P_s^{-1}x}P_s
\]
\[
= q_+(s)e^{-P_s^{-1}Cp_+(s)x}.
\]

The representations of \( \Phi_-(s,\alpha) \) in (11) we can obtain from the first part of factorization identity (2), substituting instead of \( \Phi_+(s,\alpha) \) its representation from formula (8). Formula (12) is received by the inversion of (11) with respect to \( \alpha \) and by the integration with respect to \( z \).

Further we consider the concrete definition of components of the second part of (2).

**Theorem 3.** For the almost upper semi-continuous processes the distribution of \( \hat{\xi}(\theta_s) = \xi(\theta_s) - \xi^- (\theta_s) \) is determined by the next relations:
\[
\Phi^+(s,\alpha) = \bar{p}_+(s) \left( CP_s^{-1}\bar{p}_+(s) - i\alpha\right)^{-1} (C - i\alpha),
\]
\[
\bar{p}_+(s) = sP_s \left( sI + \Lambda\bar{F}_0(0)\bar{E} e^{C\xi^- (\theta_s)} \right)^{-1},
\]
\[
\bar{P}^+(s,x) = e^{-P_+(s)CP_s^{-1}x}\bar{q}_+(s), \quad x > 0;
\]
for the minimum \( \xi^- (\theta_s) \) we have:
\[
\Phi_-(s,\alpha) = \left[ \Phi(s,\alpha) \right] \bar{p}_+^{-1}(s)P_s - \left[ \Phi(s,\alpha) (C - i\alpha) \right] \bar{C}p_+^{-1}(s)q_+(s),
\]
\[
P_-(s,x) = P(s,x)\bar{p}_+^{-1}(s)P_s - \int_0^\infty P(s,x-y)e^{-Cy}dy\bar{C}p_+^{-1}(s)q_+(s), \quad x < 0.
\]

**Proof.** Note that we’ll consider such trajectories of the process for which \( \{r^- (x) < \infty \} \). Then the stochastic relations for \( \tau^-_r(x) \) \( (k, r = 1, m) \) if \( x < 0 \), where lower indices denote the initial state and the state of \( x(t) \) at the moment of achievement the level \( x \), correspondingly \( (x(0) = k, x(\tau^- (x)) = r) \), have the next form
Combining formula (1) with the second row of (2) we obtain from (31) that

\[
\tau_{kr}^-(x) = \begin{cases} 
  x/a_k & \zeta'_k > x/a_k, \quad \zeta_k > x/a_k; \\
  \zeta_k & \zeta_k + x/a_k - \zeta_k' < x, \quad \zeta'_k < \zeta_k; \\
  \zeta_k' + \tau_{kr}^+ (x - a_k \zeta'_k - \zeta_k) & \zeta_k + a_k \zeta_k' < x, \quad \zeta'_k > \zeta_k; \\
  \zeta_k + \tau_{kr}^+ (x - a_k \zeta_k - \chi_{kj}) & \chi_{kj} + a_k \zeta_k > x, \quad \zeta'_k > \zeta_k;
\end{cases}
\tag{25}
\]

On the basis of (25), we derive the equation

\[
T_{kr}^-(s, x) = E \left[ e^{-st^-(x)}, t^-(x) < \infty, \ x(t^-(x)) = r/x(0) = k \right]
= \delta_{kr}e^{-(\lambda_k + \nu_k)x/a_k} + \frac{1}{a_k} \int_{-\infty}^{x} e^{-(\lambda_k + \nu_k)z} \tau_{kr}^-(z) \int_{-\infty}^{y} dK_{kr}^0(z) \, dy + \frac{1}{a_k} \sum_{j=1}^{m} \int_{-\infty}^{x} e^{-(\lambda_k + \nu_k)z} \tau_{kr}^-(z) \int_{-\infty}^{y} dK_{kr}^0(z) \, dy.
\tag{26}
\]

Let us differentiate left and right side of formula (26) with respect to \(x < 0\):

\[
a_k \frac{\partial}{\partial x} T_{kr}^-(s, x) = -(\lambda_k + \nu_k) T_{kr}^-(s, x) + \int_{-\infty}^{x} dK_{kr}^0(z) + \sum_{j=1}^{m} \int_{-\infty}^{x} dK_{kr}^0(z) T_{jr}^-(s, x - z).
\tag{27}
\]

Integro-differential equations (27) can be represent in the matrix form:

\[
A \frac{\partial}{\partial x} \mathbf{T}^-(s, x) = -(sI + N + \Lambda) \mathbf{T}^-(s, x) + \int_{-\infty}^{x} dK_0(z) + \int_{-\infty}^{x} dK_0(z) \mathbf{T}^-(s, x - z), \quad x \leq 0.
\tag{28}
\]

Analogically to (3) we have the formula of the relation between the distribution of \(\xi^-(\theta_s)\) and generating function of \(t^-(z)\):

\[
\mathbf{T}^-(s, x) = \mathbf{I} - \mathbf{F}^- (s, x) \mathbf{P}^{-1} \mathbf{P}, \quad \mathbf{F}^- (s, x) = \mathbf{P} \{ \xi^-(\theta_s) > x \}.
\tag{29}
\]

Substituting (28) in (29), we obtain the next equation

\[
-(sI + N + \Lambda) \mathbf{F}^- (s, x) = \mathbf{I} + \sum_{j=1}^{m} \mathbf{F}^- (s, x - z) \mathbf{K}_0(z) \mathbf{F}^- (s, x - z) - sI, \quad x \leq 0.
\tag{30}
\]

Let’s consider the Laplace transform of equation (30) with respect to \(x < 0\):

\[
(sI - \Psi(-iu)) \Phi^- (s, -iu) = sI + \int_{0}^{\infty} dK_0(z) \int_{-\infty}^{0} e^{iu(z-x)} \mathbf{F}^- (s, x - z) - e^{iu(z-x)} \mathbf{F}^- (s, x) \, dx.
\]

Taking into account that \(dK_0(z) = \Lambda \mathbf{F}^{-1}(0) C e^{-Cz} \, dz, \ z > 0\), we have:

\[
(sI - \Psi(-iu)) \Phi^- (s, -iu) = sI - u \Lambda \mathbf{F}^{-1}(0) (C - uI)^{-1} e^{C\xi^-(\theta_s)}.
\tag{31}
\]

Combining formula (1) with the second row of (2) we obtain from (31) that

\[
\Phi^+(s, -iu) = \mathbf{P} \left( I - us^{-1} \Lambda \mathbf{F}^{-1}(0) (C - uI)^{-1} e^{C\xi^-(\theta_s)} \right)^{-1}.
\tag{32}
\]
After the limit passage \( u \to \infty \) in (32) formula (21) follows. Substituting (21) in (32) we obtain (20).

After inversion of (20) with respect to \( \alpha \) formula (22) follows. Substituting (20) in the second row of (2) we obtain (23). Formula (24) is received by the inversion of (23) with respect to \( \alpha \) and by the integration with respect to \( z \).

**Remark 1.** Let \( Z(t) = \{ \xi(t), x(t) \} \) be the almost upper semi-continuous process, then the process \( Z_1(t) = \{-\xi(t), x(t)\} \) is the almost lower semi-continuous process. Taking into account the next relations between the c.h.f. of extrema for \( Z(t) \) and \( Z_1(t) \):

\[
\Phi_{\pm}^{(1)}(s, \alpha) = \Phi_{\pm}(s, -\alpha), \quad \Phi_{\pm}^{(1)}(s, \alpha) = \Phi_{\pm}(s, -\alpha),
\]

we can obtain the statements about the concrete definition of components of (2) for the almost lower semi-continuous processes.

**Remark 2.** In theorems 2 and 3 we have considered the case \( A < 0 \). Analogical results take place if we assume that \( A = 0 \). In this case we should take into account that \( p_-(s) = \mathbb{P} \{ \xi^{-}(\theta_s) = 0 \} \neq 0 \) and \( \Phi_-(s) = \mathbb{P} \{ \xi^{-}(\theta_s) = 0 \} \neq 0 \).

We’ll need some further notation. Let \( (\pi_1, \ldots, \pi_m) \) be the stationary distribution of \( x(t) \),

\[
m_1^0 = \sum_{k=1}^{m} \pi_k \sum_{r=1}^{m} m_{kr}, \quad m_{kr} = \delta_{kr} \left( c_k + \int_{R} x \Pi_k(dx) \right) + \int_{R} x \nu_k dF_{kr}(x),
\]

\[
K(r) = \Psi(-ir).
\]

By results of [6] it follows that

\[
\lim_{t \to \infty} \mathbb{P}(t) = \lim_{s \to 0^+} \mathbb{P}_s = \lim_{s \to 0^+} s(sI - Q)^{-1} = \mathbb{P}_0,
\]

if \( |m_1^0| > 0 \) then

\[
\lim_{r \to 0^+} r \mathbb{K}^{-1}(r) = \lim_{r \to 0^+} r (Q + r M_1)^{-1} = \frac{1}{m_1^0} \mathbb{P}_0, \quad M_1 = A + \int_{R} x d\mathbb{K}_0(x).
\]

**Theorem 4.** If \( m_1^0 > 0 \) then for the almost upper semi-continuous process \( Z(t) \) the distribution of strictly negative values of \( \xi^{-} = \inf_{0 \leq u \leq \infty} \xi(u) \) is determined by the generating function

\[
\mathbb{E} \left[ e^{r \xi^{-}}, \xi^{-} < 0 \right] = \left[ r \mathbb{K}^{-1}(r) (C - rI)^{-1} \right] \mathbb{R}_+.
\]

If \( A = 0 \) then

\[
\mathbb{P}_- = \mathbb{P} \{ \xi^{-} = 0 \} = (A - N(f(0) - I)) \mathbb{R}_+,
\]

where \( \mathbb{R}_+ = \lim_{s \to 0^+} s \mathbb{R}_+^{-1}(s) = \lim_{s \to 0^+} s \mathbb{P}_+^{-1}(s) \mathbb{P}_s, \) \( f(0) = \| \mathbb{P} \{ \chi_{kr} = 0, y_1 = r/y_0 = k \} \| \).

**Proof.** From formula (20) and factorization identity (2) it follows that

\[
\Phi_-(s, \alpha) = \Phi(s, \alpha) (C - i\alpha I)^{-1} (CP_s^{-1} \mathbb{P}_+(s) - i\alpha I) \mathbb{P}_+^{-1}(s) \mathbb{P}_s,
\]

or

\[
\mathbb{E} e^{r \xi^{-}(\theta_s)} = s (sI - K(r))^{-1} (C - rI)^{-1} (C - r \mathbb{R}_+^{-1}(s)).
\]

From relation (35) it follows that

\[
\lim_{r \to 0} \lim_{s \to 0} \mathbb{E} e^{r \xi^{-}(\theta_s)} = \frac{1}{m_1^0} \mathbb{P}_0 C^{-1} \mathbb{R}_+.
\]
Hence the condition \( m_1^0 > 0 \) provides the existence of \( \lim_{s \to 0} sR_+^{-1}(s) = R_+ \). Then from (35) after the limit passage \((s \to 0)\) formula (33) follows. If \( A = 0 \) let us consider the probability \( P^0_{kr}(t) = P \{ \xi(t) = 0, x(t) = r| x(0) = k \} \) which satisfies the next equation

\[
P^0_{kr}(t) = \delta_{kr}e^{-(\nu_k + \lambda_k)t} + \int_0^t \nu_k e^{-(\nu_k + \lambda_k)y} \sum_{j=1}^m P \{ \chi_{kj} = 0, y_1 = j| y_0 = k \} P^0_{kr}(t-y)dy. \tag{36}
\]

Applying Laplace-Karson transform to equation (36) we obtain the next equation

\[
\tilde{P}^0(s) = (sI + N + \Lambda)^{-1} (sI + Nf(0)) \tilde{P}^0(s)
\]

or

\[
\tilde{P}^0(s) = s(sI + \Lambda - N(f(0) - I))^{-1}. \tag{37}
\]

Applying to the first part of (2) operation \([\ ]^0\) and taking into account formula (37) we obtain relation

\[
p_-(s)P^0_s^{-1} \tilde{p}_+(s) = p_-(s)R_+(s) = s(sI + \Lambda - N(f(0) - I))^{-1}
\]

or

\[
p_-(s) = (sI + \Lambda - N(f(0) - I))^{-1} sR_+^{-1}(s). \tag{38}
\]

After the limit passage \((s \to 0)\) from (38) formula (34) follows.

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