INVERSE-SYST.LIB
Singular library for computing
Macaulay’s inverse systems *

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For J.L. Sánchez Palacios
who loved algebra with passion.

Abstract
In this note we review the Singular library INVERSE-SYST.LIB that implements
Macaulay’s correspondence and other related constructions for local rings.

1 Introduction
Let $k$ be an arbitrary field. Let $R = k[[x_1, \ldots, x_n]]$ be the ring of the formal series with
maximal ideal $m = (x_1, \ldots, x_n)$ and let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, we denote
by $m = (x_1, \ldots, x_n)$ the homogeneous maximal ideal of $S$.

In 1916 Macaulay stated a one-to-one correspondence between Artin Gorenstein ideals
of $R$ and polynomials of $S$, [13]. This correspondence can be extended to Artin ideals of $R$
and finitely sub-$R$-modules of $S$. Recall that Macaulay’s correspondence is a particular case
of Matlis duality, see Theorem 2.3 and Proposition 2.4.

Classically Macaulay’s correspondence has been mainly used for studying homogenous
ideals, [11], [12]. Recently Macaulay’s correspondence has been applied to the classification
of local Artin Gorenstein algebras, see [6], [1], [7], [8]. Most of the examples appearing in
these papers have been computed by using Singular, [3].

In this note we review the main commands of the Singular library INVERSE-SYST.LIB
that we used for these computations, [5]. The main purpose of this library is to implement

\footnotesize *

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Macaulay’s correspondence if the action of \( R \) in \( E_R(k) = S \) is defined by differentiation or contraction, Theorem[2,2] We also implement some useful operations of \( S \) as \( R \)-module. See Section[4] for a listing of all commands of INVERSE-SYST.LIB. In Section[5] we give a new proof of the classification of Artin Gorenstein local rings with Hilbert function \( \{1, 3, 3, 1\} \) by using INVERSE-SYST.LIB and the Weierstrass equation of an elliptic curve instead of Legendre equation as it was done in [6].

## 2 Macaulay’s correspondence

Let \( A = R/I \) be an Artin ring with maximal ideal \( n = m/I \). The Hilbert function of \( A \) is the numerical function \( HF_A : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( HF_A(i) = \dim_k(n^i/n^{i+1}), i \geq 0. \) The socle degree of \( A \) is the last integer \( s \) such that \( HF_A(s) \neq 0. \) The socle of \( A \) is the \( k \)-vector subspace of \( \mathcal{A} soc(A) = (0 :_A n) \), and the Cohen-Macaulay type of \( A \) is \( \tau(A) = \dim_k(soc(A)). \)

**Definition 2.1.** An Artin ring \( A \) with socle degree \( s \) is level if \( soc(A) = n^s. \) In particular, \( A \) is Gorenstein iff \( \tau(A) = 1. \)

The function \texttt{isAG(I)} returns \(-2\) if the quotient \( A = R/I \) is not Artin, returns \(-1\) if \( A \) is Artin but not Gorenstein and returns the socle degree of \( A \) if the ring \( A \) is an Artin Gorenstein ring. A similar function is implemented for checking if \( A \) is level, see Section[4].

The functions \texttt{socle} and \texttt{cmType} compute the socle and the Cohen-Macaulay type of \( A. \)

```plaintext
// we define the ring r
>ring r=0,(x(1..3)),ds;
// loading the library
>LIB "inverse-syst-v.4.lib";
// the quotient r/i is not Artin:
>ideal i=x(1)^2+x(2)^3, x(2)^4;
>isAG(i);
-2
// the quotient r/i is Artin but Gorenstein:
>ideal i=x(1)^2+x(2)^3, x(2)^4+x(1)^2, x(3)^2+x(1)*x(2),
   x(1)*x(2)^2*x(3);
>isAG(i);
-1
// the quotient r/i is Artin Gorenstein and we get
// the socle degree
> ideal i=x(1)^2+x(2)^3, x(2)^4+x(1)^2, x(3)^2+x(1)*x(2);
>isAG(i);
4
// we define an Artin no Gorenstein ideal:
> ideal i=x(1)^2+x(2)^3, x(2)^4+x(1)^2, x(3)^2+x(1)*x(2),
   x(1)*x(2)^2*x(3);
>isAG(i);
-1
// we compute the socle ideal of r/i
```

2
The polynomial ring $S$ can be considered as $R$-module with two linear structures: by derivation and by contraction. If $\text{char}(k) = 0$, the $R$-module structure of $S$ by derivation is defined by

$$ R \times S \rightarrow S $$

$$(x^\alpha, x^\beta) \mapsto x^\alpha \circ x^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} & \beta \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

where for all $\alpha, \beta \in \mathbb{N}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$

If $\text{char}(k) \geq 0$, the $R$-module structure of $S$ by contraction is defined by:

$$ R \times S \rightarrow S $$

$$(x^\alpha, x^\beta) \mapsto x^\alpha \circ x^\beta = \begin{cases} x^{\beta-\alpha} & \beta \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

In Singular we can use the above products as follows:

```plaintext
> ideal F=x(1)^2*x(3)^4+ x(2)^3*x(1)*x(3)+x(2)^5;
> diff(x(1)^2,F);  
_1[1]=2*x(3)^4 

> contract(x(1)^2,F); 
_1[1]=x(3)^4
```

It is easy to prove that for any field $k$ there is a $R$-module homomorphism

$$ \sigma : (S, \text{der}) \rightarrow (S, \text{cont}) $$

$$ x^\alpha \mapsto \alpha! x^\alpha $$

If $\text{char}(k) = 0$ then $\sigma$ is an isomorphism of $R$-modules. The $R$-module $S$ is the injective hull $E_R(k)$ of the $R$-module $k$:

**Theorem 2.2.** ([10]) If $k$ is of characteristic zero then

$$ E_R(k) \cong (S, \text{der}) \cong (S, \text{cont}). $$

If $k$ is of positive characteristic then

$$ E_R(k) \cong (S, \text{cont}). $$
Given a commutative ring $R$ we denote by $R\text{-mod}$, resp. $R\text{-mod.Noeth}$, $R\text{-mod.Artin}$, the category of $R$-modules, resp. category of Noetherian $R$-modules, Artinian $R$-modules. The Matlis dual of an $R$-module $M$ is $M^\vee = \text{Hom}_R(M, E_R(k))$. We write $(-)^\vee = \text{Hom}_R(-, E_R(k))$, which is an additive contravariant exact functor from the category $R\text{-mod}$ into itself.

**Theorem 2.3** (Matlis duality). The functor $(-)^\vee$ is contravariant, additive and exact, and defines anti-equivalence between $R\text{-mod.Noeth}$ and $R\text{-mod.Artin}$ (resp. between $R\text{-mod.Artin}$ and $R\text{-mod.Noeth}$). The composition $(-)^\vee \circ (−)^\vee$ is the identity functor of $R\text{-mod.Noeth}$ (resp. $R\text{-mod.Artin}$). Furthermore, if $M$ is a $R$-module of finite length then $\ell_R(M^\vee) = \ell_R(M)$.

From the previous result we can recover the classical result of Macaulay, [13], for the power series ring, see [9], [12]. If $I \subset R$ is an ideal, then $(R/I)^\vee$ is the sub-$R$-module of $S$

$$I^\perp = \{ g \in S \mid I \circ g = 0 \},$$

this is the Macaulay’s inverse system of $I$. Given a sub-$R$-module $M$ of $S$ then dual $M^\vee$ is an ideal of $R$

$$M^\perp = \{ f \in R \mid f \circ g = 0 \text{ for all } g \in M \}.$$

**Proposition 2.4** (Macaulay’s duality). Let $R = k[x_1, \ldots, x_n]$ be the $n$-dimensional power series ring over a field $k$. There is a order-reversing bijection $\perp$ between the set of finitely generated sub-$R$-submodules of $S = k[y_1, \ldots, y_n]$ and the set of $m$-primary ideals of $R$ given by: if $M$ is a submodule of $S$ then $M^\perp = (0 : M)$, and $I^\perp = (0 : S I)$ for an ideal $I \subset R$.

Given a polynomial $H \in S$ of degree $l$ we denote by $\text{top}(H)$ the degree $l$ homogeneous form of $H$.

**Proposition 2.5** (Proposition 3.7 and Corollary 3.8 [11], [2]). Let $A = R/I$ be an Artin ring of socle degree $s$ and Cohen-Macaulay $t$. The following conditions are equivalent:

(i) $A$ is level,

(ii) $I^\perp$ is generated by $t$ polynomials $H_1, \ldots, H_t \in S$ such that $\deg(H_i) = s$, for $i = 1, \ldots, t$, and the homogeneous forms $\text{top}(H_1), \ldots, \text{top}(H_t)$ are $k$–linear independent.

In particular, $A = R/I$ is Gorenstein of socle degree $s$ if and only if $I^\perp$ is a cyclic $R$-module generated by a polynomial of degree $s$.

Given a collection of polynomials $H_1, \ldots, H_t \in S$ we denote by $\langle H_1, \ldots, H_t \rangle$ the sub-$R$-module of $S$ generated by $H_1, \ldots, H_t$. Notice that $\langle H_1, \ldots, H_t \rangle$ is not an ideal of $S$, is the $k$-vector space generated by the collection $H_1, \ldots, H_t$ and their derivatives of any order. Notice that in Singular ideals are handled by the list of a given system of generators. In
the library \textsc{inverse-syst.lib} the sub-$R$-modules of $S$ are handled by using the Singular's structure of ideal, i.e. by the list of a given system of generators.

We denote by $S_{\leq i}$ (resp. $S_{< i}$, resp. $S_i$), $i \in \mathbb{N}$, the $k$-vector space of polynomials of $S$ of degree less or equal (resp. less, resp. equal to) to $i$, and we consider the following $k$-vector space

$$(I^\perp)_i := \frac{I^\perp \cap S_{\leq i} + S_{< i}}{S_{< i}}.$$

\textbf{Proposition 2.6.} \((\mathbb{P})\) For all $i \geq 0$ it holds

$$\text{HF}_A(i) = \dim_k (I^\perp)_i.$$ 

In the library \textsc{inverse-syst.lib} Macaulay’s correspondence has been programmed with respect the two $R$-module structures of $S$. i.e. with respect to the differentiation and with respect to the contraction. Here we will show how works Macaulay’s correspondence with respect the differentiation. Recall that for technical reasons the sub-$R$-modules of $S$ are handled in \textsc{inverse-syst.lib} by using the Singular’s structure of ideal.

The command \texttt{invSyst} computes the inverse system $I^\perp \subset S$ of an ideal $I$ of $R$, the command \texttt{idealAnn} computes the annihilator $M^\perp \subset R$ of a finitely sub-$R$-module $M$ of $S$.

In the next example we will show that the composition \texttt{idealAnn} $\circ$ \texttt{invSyst} is the identity map on the set of Artin ideals as Proposition 2.4 predicts.

```plaintext
> ring r=0,(x(1..3)),ds;
> LIB "inverse-syst-v.4.lib";
// we define an ideal i of r, notice that the first generator is
// a random polynomial with monomials of degree between 2 and 3
// and random coefficients between -2 and 2. The second generator
// is a random homogenous form of degree 3 and coefficients
// between -1 and 1.
> ideal i=genPole(2,3,2), genPole(3,3,1), x(2)^3+x(1)*x(3)^4,
  x(1)^2+x(2)^2*x(3);
> i;
```

```plaintext
i[1]=2*x(1)^2+2*x(2)^2-2*x(1)*x(3)+2*x(2)*x(3)-x(3)^2-2*x(1)^3
   +x(1)^2*x(2)+2*x(1)*x(2)^2-2*x(2)^3-2*x(1)^2*x(3)
   +2*x(1)*x(2)*x(3)+2*x(2)^2*x(3)-2*x(2)*x(3)^3
i[2]=-x(1)^2*x(2)^2-2*x(2)^3+3*x(1)*x(2)*x(3)+x(2)^2*x(3)+x(1)*x(3)^2
   +x(3)^3
i[3]=x(2)^3+x(1)*x(3)^4
i[4]=x(1)^2+x(2)^2*x(3)
```

// we compute the inverse system of i:
```plaintext
> ideal iv=invSyst(i);
> iv;
```

```plaintext
iv[1]=3*x(1)^2+69*x(2)^2-42*x(1)*x(2)*x(3)-3*x(2)^2*x(3)
   -42*x(1)*x(3)^2+15*x(2)*x(3)^2+22*x(3)^3
iv[2]=24*x(1)*x(3)+3*x(1)*x(2)^2+6*x(1)*x(3)^2-2*x(3)^3
```

```
```
// Notice that iv is not cyclic so i is not Gorenstein
// we compute the annihilator of iv. Should be the ideal i.
> ideal j=idealAnn(iv);
> j;
j[1]=7*x(1)^2+x(1)*x(2)*x(3)
j[2]=14*x(2)^2-7*x(1)*x(3)+14*x(2)*x(3)-7*x(3)^2+28*x(1)*x(2)^2-52*x(1)*x(2)*x(3)+196*x(2)^2*x(3)^2-98*x(2)*x(3)^2
nj[3]=343*x(1)*x(2)*x(3)-686*x(2)^2*x(3)+343*x(2)*x(3)^2
nj[4]=5*x(1)*x(3)^2-8*x(2)*x(3)^2+5*x(3)^3
nj[5]=x(2)*x(3)^3
nj[6]=x(3)^4
// we check that i and j are the same ideal:
> eqIdeal(i,j);
1

In the next example we will show that the composition \text{invSyst} \circ \text{idealAnn} is the identity map on the set of finitely sub-$R$-modules of $S$ as Proposition 2.4 predicts.

// We start with a random polynomial
> ideal q=genPol(2,3,2);
> q;
q[1]=2*x(1)^2-2*x(1)*x(2)+2*x(2)^2+2*x(1)*x(3)-2*x(2)*x(3)
-x(3)^2+x(1)^2+2*x(2)+2*x(1)*x(2)^2-2*x(2)^3
-2*x(1)*x(2)*x(3)+x(3)^2
// we compute the annihilator qa of q.
// The ideal qa is a Gorenstein ideal.
> ideal qa=idealAnn(q);
> qa;
qa[1]=4*x(1)^2-17*x(1)*x(2)-5*x(2)^2+12*x(2)*x(3)+x(1)^3
qa[2]=2*x(1)*x(2)+2*x(2)^2-8*x(1)*x(3)-3*x(1)^2+2*x(2)
qa[3]=x(1)*x(3)-x(3)^2+2*x(1)*x(2)*x(3)
qa[4]=x(2)^2+6*x(1)*x(2)*x(3)
qa[5]=x(2)^2+2*x(2)*x(3)^2
qa[6]=x(3)^3
// We get that qa is a Gorenstein ideal and the socle degree
// of r/i is three that coincides with the degree of q:
> isAG(qa);
3
// we compute the inverse system of qa
> ideal q2=invSyst(qa);
> q2;
q2[1]=6*x(1)*x(2)-24*x(1)*x(3)+22*x(3)^2+17*x(1)^3
+34*x(1)^2-34*x(1)*x(2)^2+34*x(2)^3
+34*x(1)*x(2)*x(3)+34*x(2)*x(3)^2+17*x(2)*x(3)^2
// from Macaulay’s correspondence q and q2 should coincide:
> eqModIH(q,q2);
1
3 A case study: Artin Gorenstein rings with Hilbert function \{1, 3, 3, 1\}

As a corollary of the main result of [6] we got the classification of Artin Gorenstein local rings with Hilbert function \(\{1, 3, 3, 1\}\) by using the Legendre equation of an elliptic curve. In this section we recover this classification by using the Weierstrass equation of an elliptic curve and the library \textsc{Inverse-syst.lib}.

**Theorem 3.1 ([6]).** The classification of Artin Gorenstein local \(k\)-algebras with Hilbert function \(HF_A = \{1, n, n, 1\}\) is equivalent to the projective classification of the hypersurfaces \(V(F) \subset \mathbb{P}^{n-1}_k\) where \(F\) is a degree three non degenerate form in \(n\) variables.

See [6] for the classification of Artin Gorenstein local \(k\)-algebras with Hilbert function \(HF_A = \{1, n, n, 1\}\) for \(n = 1, 2\). Assume \(n = 3\). Any plane elliptic cubic curve \(C \subset \mathbb{P}^2_k\) is defined, in a suitable system of coordinates, by a Weierstrass’ equation, \cite{[14]} proof of Proposition 1.4,

\[
W(a, b) = x_2^2x_3 - x_1^3 + ax_1x_3^2 + bx_3^3
\]

with \(a, b \in k\) such that \(4a^3 + 27b^2 \neq 0\). The \(j\) invariant of \(C\) is

\[
j(a, b) = 1728 \frac{4a^3}{4a^3 + 27b^2}
\]

It is well known that two plane elliptic cubic curves \(C_i = V(W_{a_i, b_i}) \subset \mathbb{P}^2_k, i = 1, 2\), are projectively isomorphic if and only if \(j(a_1, b_1) = j(a_2, b_2)\).

For elliptic curves the inverse moduli problem can be done as follows. We denote by \(W(j)\) the following elliptic curves with \(j\) as moduli: \(W(0) = x_2^2x_3 + x_2x_3^2 - x_1^3\), \(W(1728) = x_2^2x_3 - x_1x_3^2 - x_1^3\), and for \(j \neq 0, 1728\)

\[
W(j) = (j - 1728)(x_2^2x_3 + x_2x_3^2 - x_1^3) + 36x_1x_3^2 + x_3^3.
\]

We will show by using the library \textsc{Inverse-syst.lib} that:

**Proposition 3.2.** Let \(A\) be an Artin Gorenstein local \(k\)-algebra with Hilbert function \(HF_A = \{1, 3, 3, 1\}\). Then \(A\) is isomorphic to one and only one of the following quotients of \(R = k[[x_1, x_2, x_3]]\):

| Model for \(A = R/I\) | Inverse system \(F\) | Geometry of \(C = V(F) \subset \mathbb{P}^2_k\) |
|------------------------|----------------------|----------------------------------|
| \((x_1^2, x_2^2, x_3^2)\) | \(x_1x_2x_3\) | Three independent lines |
| \((x_1^2, x_1x_3, x_1x_2^2, x_2^2, x_3^2 + x_1x_2)\) | \(x_2(x_1x_2 - x_3^2)\) | Conic and a tangent line |
| \((x_1^2, x_2^2 + 6x_1x_2)\) | \(x_3(x_1x_2 - x_3^2)\) | Conic and a non-tangent line |
| \((x_1^2, x_1x_2, x_1x_3, x_2^2, x_3^2 + 3x_1x_3)\) | \(x_2^2x_3 - x_1^3(x_1 + x_3)\) | Irreducible nodal cubic |
| \((x_1^2, x_1x_2 + 3x_2^2x_3, x_1x_3 + x_2^2 - x_2x_3 + x_3^2, x_1x_2)\) | \(x_2^2x_3 - x_1^3\) | Irreducible cuspidal cubic |
| \((x_1^2, x_1^2 + 3x_2^2x_3, x_1x_3 + x_2^2 - x_2x_3 + x_3^2, x_1x_2)\) | \(W(0)\) | Elliptic curve \(j = 0\) |
| \((x_1^2 + x_1x_3, x_1x_2, x_1^2 - 3x_2^2)\) | \(W(1728)\) | Elliptic curve \(j = 1728\) |
| \(I(j) = (x_2(x_2 - 2x_1), H_j, G_j)\) | \(W(j)\) | Elliptic curve with \(j \neq 0, 1728\) |
with:

\[ H_j = 6j x_1 x_2 - 144(j - 1728)x_1 x_3 + 72(j - 1728)x_2 x_3 - (j - 1728)^2 x_3^2, \]
\[ G_j = j x_1^2 - 12(j - 1728)x_1 x_3 + 6(j - 1728)x_2 x_3 + 144(j - 1728)x_3^2; \]
\[ I(j_1) \cong I(j_2) \text{ if and only if } j_1 = j_2. \]

The first 7 models can be obtained from the corresponding inverse system \( F \) by using the command \texttt{idealAnn}. Assume that \( j \neq 0, 1728 \). Let \( J(j) \) be the ideal \( \langle W(j) \rangle \); a simple computation shows that \( HF_{R/J(j)} = \{1, 3, 3, 1\}, \) Proposition 2.6.

// we define a ring of characteristic zero, three variables and ground field
// a field of functions with indeterminate c(1)
> def r=workringc(0,1,3);
> setring r;
> r;
// characteristic : 0
// 1 parameter : c(1)
// minpoly : 0
// number of vars : 3
// block 1 : ordering ds
// : names x(1) x(2) x(3)
// block 2 : ordering C
// the ideal p defines a elliptic curve with j=c(1)
> ideal p=weierstrassp();
> p;
p[1]=(-c(1)+1728)*x(1)^3+(c(1)-1728)*x(1)*x(2)*x(3)+
      (c(1)-1728)*x(2)^2*x(3)+36*x(1)*x(3)^2+x(3)^3
// we define the ideal q. We will prove that the inverse system of q
// is p.
> ideal q=idealwp();
> q;
q[1]=(6*c(1))*x(1)*x(2)+(-144*c(1)+248832)*x(1)*x(2)*x(3)+
      (72*c(1)-124416)*x(2)*x(3)+(-c(1)^2+3456*c(1)-2985984)*x(3)^2
q[2]=(c(1))*x(1)^2+(-12*c(1)+20736)*x(1)*x(3)+(6*c(1)-10368)*x(2)*x(3)+
      (144*c(1)-248832)*x(3)^2
q[3]=-2*x(1)*x(2)+x(2)^2
// we check that q is contained in p^\perp
> diff(q,p);
_\[1,1\]=0
_\[1,2\]=0
_\[1,3\]=0
// If we perform de division of the \$4\$-th power of the maximal ideal
// by q we get three matrices Q, R, U such that (see Singular’s manual)
// generators(maxideal(4))*U=generators(q)*Q + R
// U is the 15x15 identity matrix, R is the 15x1 zero matrix
// and Q is a 6x15 matrix with coefficients in the ground field (see below
// for more details). The command is:
> division(maxideal(4),q);

From the last computation we get that the denominators of the coefficients of the matrix Q
are constant polynomials or polynomials with roots in \( \{0, 1728\} \). Hence for all \( j = c(1) \neq \)
we get that $m^4 \subset q$, so $q$ is an Artin ideal. Notice that for all $j = c(1) \in k$, $q = I(j)$ and $p = \langle W(j) \rangle$. Since $I(j)$ is generated by three homogeneous elements, $I(j)$ is a homogeneous complete intersection ideal. In particular $I(j)$ is a homogeneous Artin Gorenstein ideal, so $HF_{R/I(j)}$ is symmetric. Notice that the generators of $I(j)$ are three homogeneous forms of degree two, so the Hilbert function of $A = R/I(j)$ is $\{1, 3, 3, 1\}$. We know that $I(j)$ is contained $J(j) = \langle W(j) \rangle^\perp$. Since $HF_{R/I(j)} = HF_{R/J(j)} = \{1, 3, 3, 1\}$, we get that $I(j) = J(j) = \langle W(j) \rangle^\perp$, i.e. $I(j) = \langle W(j) \rangle^\perp$.

## 4 Commands

Next, we list the most important commands of INVERSE-SYST.LIB.

**Ideal Theory**

- **genPol(i,j,a)**: Returns a generic polynomial sum of forms of degrees between $i$ and $j$, with integer coefficients in $[-a,a]$.
- **eqIdeal(J,I); I, J ideals.** Returns 1 if $I=J$, 0 otherwise.
- **socle(J); I ideal** Returns $-1$ if $J$ is not Artin, returns the ideal of $J$ if $J$ is Artin.
- **cmType(J); J ideal.** Returns $-1$ if $J$ is not Artin, returns the Cohen-Macaulay type of $J$ otherwise.
- **isAG(I); I ideal.** Returns $-2$ if $J$ is not Artin, returns $-1$ if $J$ is Artin but not Gorenstein, and returns the socle degree if $J$ is Artin Gorenstein.
- **isLevel(I); I ideal.** Returns $-2$ if $J$ is not Artin, returns $-1$ if $J$ is Artin but not Level, and returns the socle degree if $J$ is Artin Level.

**Macaulay Inverse System Correspondence with Coefficients**

- **invSystG(ideal J)** returns the inverse system of $J$; $J$ is Artin Gorenstein
- **invSyst(J)** returns the inverse system of $J$; $J$ is Artin
- **idealAnnG(poly f)** returns the Artin Gorenstein ideal with inverse system $f$
- **idealAnn(I)** returns the Artin ideal with inverse system $I$

**Structure of Injective Hull with Coefficients**

- **memberIH(g,I); I=f1,... list of polynomials, g polynomial.** returns 1 if $g$ belongs to the $R$-submodule of $S$ generated by $f1,...fs$ in $S$, 


returns 0, otherwise
subModIH(I,J); I=f1,... list of polynomials, J=g1,... list of polynomials.
   Returns 1 if I is a sub-R-submodule of J, both sub-R-modules of S;
   0 otherwise
eqModIH(I,J); I=f1,... list of polynomials, J=g1,... list of polynomials.
   Returns 1 if I=J, both sub-R-modules of S;
   0 otherwise
minGensIH(I); I=f1,...,fs list of polynomials.
   Returns a minimal system of generators of <f1,...,fs>, sub-R-module of S
colonInvSyst(f,g); f, g polynomials.
   Returns an element g of R such that Aof=g if exists,
   0 otherwise.

MACAULAY INVERSE SYSTEM CORRESPONDENCE WITH NO COEFFICIENTS

invSystGNC(ideal J)
   returns the inverse system of J; J is Artin Gorenstein
invSystNC(J)
   returns the inverse system of J; J is Artin
idealAnnGNC(poly f)
   returns the Artin Gorenstein ideal with inverse system f
idealAnnNC(I)
   returns the Artin ideal with inverse system I

STRUCTURE OF INJECTIVE HULL WITH NO COEFFICIENTS

memberIHNC(g,I); I=f1,... list of polynomials, g polynomial.
   Returns 1 if g belongs to the R-submodule of S generated by f1,...fs in S,
   returns 0 otherwise
subModIHNC(I,J); I=f1,... list of polynomials, J=g1,... list of polynomials.
   Returns 1 if I is a sub-R-submodule of J, both sub-R-modules of S;
   returns 0 otherwise
eqModIHNC(I,J); I=f1,... list of polynomials, J=g1,... list of polynomials.
   Returns 1 if I=J, both sub-R-modules of S;
   returns 0, otherwise
minGensIHNC(I); I=f1,...,fs list of polynomials.
   Returns a minimal system of generators of <f1,...,fs>, sub-R-module of S
colonInvSystNC(f,g); f, g polynomials.
   Returns an element g of R such that Aof=g if exists,
   returns 0 otherwise.

RINGS WITH PARAMETERS

workringc(p, t, n)
   returns the def of a ring r with
t coefficients c (1),...,c(t),
n vars x(1),...,x(n), char is p, and order ds

ELLIPTIC CURVES

weiertrassj(t)
returns the ideal generated by Weierstrass equation of the elliptic curve with $j$ invariant

\text{idealwj}(t)

returns the ideal with inverse system \text{weierstrassj}(j)

\text{weierstrassp()}

returns the ideal generated by Weierstrass equation of the elliptic curve with moduli $j=c(1)$

\text{idealwp}(t)

returns the ideal with inverse system \text{weierstrassp()}

with moduli $j=c(1)$

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