INFINITE-DIMENSIONAL ALGEBRAS IN
DIMENSIONALLY REDUCED STRING THEORY

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Abstract

We examine 4-dimensional string backgrounds compactified over a two torus. There exist two alternative effective Lagrangians containing each two $SL(2)/U(1)$ sigma-models. Two of these sigma-models are the complex and Kähler structures on the torus. The effective Lagrangians are invariant under two different $O(2,2)$ groups and by the successive applications of these groups the affine $\hat{O}(2,2)$ Kac-Moody algebra is emerged. The latter has also a non-zero central term which generates constant Weyl rescalings of the reduced 2-dimensional background. In addition, there exists a number of discrete symmetries relating the field content of the reduced effective Lagrangians.

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It is known that higher-dimensional gravitational theories exhibit unexpected new symmetries upon reduction \[1\]. Dimensional reduction of the string background equations \[2\] with dilaton and antisymmetric field also exhibit new symmetries as for example dualities \[3\]–\[5\]. However, the exact string symmetries will necessarily be subgroups or discrete versions of the full symmetry group of the string background equations and thus, a study of the latter would be useful. The empirical rule is that the rank of the symmetry group increases by one as the dimension of the space-time is decreased by one after dimensional reduction \[6\]. However, the appearance of non-local currents in two-dimensions in addition to the local ones, turns the symmetry group infinite dimensional. Let us recall the $O(8,24)$ group of the heterotic string after reduction to three dimensions \[7\] which turns out to be the affine $\hat{O}(8,24)$ algebra by further reduction to two dimensions \[8\] or the $\hat{O}(2,2)$ algebra after the reduction of 4-dimensional backgrounds \[9\]. The latter generalizes the Geroch group of Einstein gravity \[10\]–\[12\]. We will examine here the “affinization” of the symmetry group of the string background equations for 4-dimensional space-times with two commuting Killing vectors and we will show the emergence of a central term. Generalization to higher dimensions is straightforward.

The Geroch group is the symmetry group which acts on the space of solutions of the Einstein equations \[10\]. Its counterpart in string theory, the “string Geroch group”, acts, in full analogy, on the space of solutions of the one-loop beta functions equations \[9\]. The Geroch group, as well as its string counterpart, results by dimensional reducing four-dimensional backgrounds with zero cosmological constant over two commuting, orthogonal transitive, Killing vectors or, in other words, by compactifying $M_4$ to $M_2 \times T^2$. In dimensional reduced Einstein gravity, there exist two $SL(2,\mathbb{R})$ groups (the Ehlers’ and the Matzner-Misner groups \[13\]) acting on the space of solutions, the interplay of which produce the infinite dimensional Geroch group. In the string case, we will see that apart from the Ehlers and Matzner-Misner groups acting on the pure gravitational sector, there also exist two other $SL(2,\mathbb{R})$ groups, one of which generates the familiar S-duality, acting on the antisymmetric-dilaton fields sector.

The Geroch group was also studied in the Kaluza-Klein reduction of supergravity theories \[1\]. It was B. Julia who showed that the Lie algebra of the Geroch group in Einstein gravity is the affine Kac-Moody algebra $\hat{sl}(2)$ and he pointed out the existence of a central term \[13\]. We will show here that in the string case, after the reduction to $M_2 \times T^2$, there exist four $SL(2,\mathbb{R})$ groups, the interplay of which produce the infinite dimensional Geroch group. However, there is also a central term which rescales the metric of $M_2$ so that the Lie algebra of the string Geroch group turns out to be the $\hat{sl}(2) \times \hat{sl}(2) \simeq \hat{o}(2,2)$ affine
Kac-Moody algebra. The appearance of a non-zero central term already at the tree-level is rather surprising since usually such terms arise as a consequence of quantization [15]. Here however, the central term acts non-trivially even at the “classical level” by constant Weyl rescalings of the reduced two-dimensional space $M_2$.

String propagation in a critical background $\mathcal{M}$, parametrized with coordinates $(x^M)$ and metric $G_{MN}(x^M)$, is described by a two-dimensional sigma-model action

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left( G_{MN} + B_{MN} \right) \partial x^M \partial x^N - \frac{1}{8\pi} \int d^2z \phi R^{(2)},$$

where $B_{MN}, \phi$ are the antisymmetric and dilaton fields, respectively. The conditions for conformal invariance at the 1-loop level in the coupling constant $\alpha'$ are

$$R_{MN} - \frac{1}{4} H_{MKA} H_N^{KA} - \nabla_M \nabla_N \phi = 0$$
$$\nabla^M (e^\phi H_{MNK}) = 0$$
$$-R + \frac{1}{12} H_{MNK} H^{MNK} + 2 \nabla^2 \phi + (\partial_M \phi)^2 = 0,$$

and the above equations may be derived from the Lagrangian [16]

$$\mathcal{L} = \sqrt{-G} e^\phi (R - \frac{1}{12} H_{MNK} H^{MNK} + \partial_M \phi \partial^M \phi),$$

where $H_{MNA} = \partial_M B_{NA} + \text{cycl. perm.}$ is the field strength of the antisymmetric tensor field $B_{MN}$.

The right-hand side of the last equation in eq. (2) has been set to zero assuming that the central charge deficit $\delta c$ is of order $\alpha'^2$ (no cosmological constant). We will also assume that the string propagates in $M_4 \times K$ with $c(M_4) = 4 + \mathcal{O}(\alpha'^2)$ and that the dynamics is completely determined by $M_4$ while the dynamics of the internal space $K$ is irrelevant for our purposes. Thus, we will discuss below general 4-dimensional curve backgrounds in which $H_{\mu\nu\rho}$ can always be expressed as the dual of $H^M$

$$H_{MNA} = \frac{1}{2} \sqrt{-G} \eta_{MNAK} H^K,$$

with $\eta_{1234} = +1$ and $M, N, ... = 0, 1, 2, 3$. The Bianchi identity $\partial_K H_{MNA} = 0$ gives the constraint

$$\nabla_M H^M = 0,$$

which can be incorporated into (3) as $b \nabla_M H^M$ by employing the Lagrange multiplier $b$ so that (3) turns out to be

$$\mathcal{L} = \sqrt{-G} e^\phi (R - \frac{1}{2} s H_M H^M + e^{-\phi} b \nabla_M H^M + \partial_M \phi \partial^M \phi).$$
s = ±1 for spaces of Euclidean or Lorentzian signature, respectively and we will assume that s = −1 since the results may easily be generalized to include the s = +1 case as well. We may now eliminate $H_M$ by using its equation of motion

$$H_M = e^{-\phi} \partial_M b,$$

and the Lagrangian (8) turns out to be

$$\mathcal{L} = \sqrt{-G} e^\phi (R - \frac{1}{2} e^{-2\phi} \partial_M b \partial^M b + \partial_M \phi \partial^M \phi).$$ (8)

Let us now suppose that the space-time $M_4$ has an abelian space-like isometry generated by the Killing vector $\xi_1 = \partial/\partial\theta_1$ so that the metric can be written as

$$ds^2 = G_{11} d\theta_1^2 + 2 G_{1\mu} d\theta_1 dx^\mu + G_{\mu\nu} dx^\mu dx^\nu,$$ (9)

where $\mu, \nu, ... = 0, 2, 3$ and $G_{11}, G_{1\mu}, G_{\mu\nu}$ are functions of $x^\mu$. We may express the metric (9) as

$$ds^2 = G_{11} (d\theta_1 + 2 A_\mu dx^\mu)^2 + \gamma_{\mu\nu} dx^\mu dx^\nu,$$ (10)

where

$$\gamma_{\mu\nu} = G_{\mu\nu} - G_{1\mu} G_{1\nu}/G_{11},$$

$$A_\mu = G_{1\mu}/G_{11}.$$ (11)

The metric (10) indicates the $M_3 \times S^1$ topology of $M_4$ and $\gamma_{\mu\nu}$ may be considered as the metric of the 3-dimensional space $M_3$. Space-times of this form have extensively been studied in the Kaluza-Klein reduction where $A_\mu$ is considered as a U(1)–gauge field. The scalar curvature $R$ for the metric (10) turns out to be

$$R = R(\gamma) - \frac{1}{4} G_{11} F_{\mu\nu} F^{\mu\nu} - \frac{2}{G_{11}^{1/2}} \nabla^2 G_{11}^{1/2},$$ (12)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\nabla^2 = 1/\sqrt{-\gamma} \partial_\mu \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\nu$. By replacing (12) into (8) and integrating by parts we get the reduced Lagrangian

$$\mathcal{L} = \sqrt{-\gamma} G_{11}^{1/2} e^\phi \left( R(\gamma) - \frac{1}{4} G_{11} F_{\mu\nu} F^{\mu\nu} + \frac{1}{G_{11}} \partial_\mu G_{11} \partial^\mu \phi - \frac{1}{4} \frac{1}{G_{11}} H_{\mu\nu} H^{\mu\nu} + \partial_\mu \phi \partial^\mu \phi \right)$$ (13)

where $H_{\mu\nu} = H_{\mu\nu 1} = \partial_\mu B_{\nu 1} - \partial_\nu B_{\mu 1}$. (A general discussion on the dimensional reduction of various tensor fields can be found in [17]). We have taken $H_{\mu\nu\rho} = 0$ since in three dimensions
$B_{\mu \nu}$ has no physical degrees of freedom. Let us note that the Lagrangian (13) is invariant under the transformation

\[
\begin{align*}
G_{11} & \rightarrow \frac{1}{G_{11}}, \\
H_{\mu \nu} & \rightarrow F_{\mu \nu}, \\
\phi & \rightarrow \phi - \ln G_{11}, \\
\gamma_{\mu \nu} & \rightarrow \gamma_{\mu \nu},
\end{align*}
\]

which, in terms of $G_{11}$, $G_{1\mu}$, $G_{\mu \nu}$, $B_{1\mu}$ and $\phi$ may be written as

\[
\begin{align*}
G_{11} & \rightarrow \frac{1}{G_{11}}, \\
B_{\mu 1} & \rightarrow \frac{G_{1\mu}}{G_{11}}, \\
G_{1\mu} & \rightarrow \frac{B_{\mu 1}}{G_{11}}, \\
G_{\mu \nu} & \rightarrow G_{\mu \nu} - \frac{G_{1\mu}^2 - B_{\mu 1}^2}{G_{11}}, \\
\phi & \rightarrow \phi - \ln G_{11},
\end{align*}
\]

(14)

and it is easily be recognized as the abelian duality transformation.

Let us further assume that $M_3$ has also an abelian spece-like isometry generated by $\xi_2 = \frac{\partial}{\partial \theta_2}$ so that $M_3 = M_2 \times S^1$. We will further assume that the two Killings ($\xi_1, \xi_2$) of $M_4$ are orthogonal to the surface $M_2$. Thus, the metric (9) can be written as

\[
\begin{align*}
ds^2 = G_{11} d\theta_1^2 + 2G_{12} d\theta_1 d\theta_2 + G_{22} d\theta_2^2 + G_{ij} dx^i dx^j,
\end{align*}
\]

(16)

where $i, j, ... = 0, 3$ and $G_{11}, G_{12}, G_{22}, G_{ij}$ are functions of $x^i$ only. We may write the metric above as

\[
\begin{align*}
ds^2 = G_{11}(d\theta_1 + A d\theta_2)^2 + V d\theta_2^2 + G_{ij} dx^i dx^j,
\end{align*}
\]

(17)

where

\[
A = \frac{G_{12}}{G_{11}}, \quad V = \frac{G_{11}G_{22} - G_{12}^2}{G_{11}}.
\]

(18)

By further reducing (13) with respect to $\xi_2$ and using the fact that the only non-vanishing components of $F_{\mu \nu}$ and $H_{\mu \nu}$ are

\[
\begin{align*}
F_{i2} &= \partial_i A, \\
H_{i2} &= \partial_i B,
\end{align*}
\]

(19)

with $B = B_{21}$, we get

\[
\begin{align*}
\mathcal{L} &= \sqrt{-G^{(2)}G_{11}^{1/2}V^{1/2}} e^{\phi} \left( R(G^{(2)}) - \frac{1}{2} (\partial A)^2 \frac{G_{11}}{V} - \frac{1}{8} (\partial \ln \frac{G_{11}}{V})^2 \\
&\quad - \frac{1}{2} (\partial B)^2 \frac{1}{G_{11}V} - \frac{1}{8} (\partial \ln G_{11}V)^2 + (\partial \phi)^2 \right),
\end{align*}
\]

(20)
where $\tilde{\phi} = \phi + \frac{1}{2} \ln G_{11} V$ and $(\partial \phi)^2 = \partial_i \phi \partial^i \phi$. Let us now introduce the two complex coordinates $\tau, \rho$ \[18\] defined by

$$
\tau = \tau_1 + i \tau_2 = \frac{G_{12}}{G_{11}} + i \sqrt{G_{11}} G_{12}, \quad (21)
$$

$$
\rho = \rho_1 + i \rho_2 = B_{21} + i \sqrt{G}, \quad (22)
$$

where $G = G_{11} G_{22} - G_{12}^2$ is the determinant of the metric on the 2-torus $T^2 = S^1 \times S^1$, so that $\tau, \rho$ turn out to be the complex and Kähler structure on $T^2$. In terms of $\tau, \rho$, the Lagrangian \[20\] is written as

$$
\mathcal{L} = \sqrt{-G^{(2)}} e^{\tilde{\phi}} \left( R(G^{(2)}) + 2 \frac{\partial \tau \partial \bar{\tau}}{(\tau - \bar{\tau})^2} + 2 \frac{\partial \rho \partial \bar{\rho}}{(\rho - \bar{\rho})^2} + (\partial \bar{\phi})^2 \right), \quad (23)
$$

where $R(G^{(2)})$ is the curvature scalar of $M_2$. The Lagrangian above is clearly invariant under the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \simeq O(2, 2, \mathbb{R})$ transformation

$$
\begin{align*}
\tau &\rightarrow \tau' = \frac{a \tau + b}{c \tau + d}, \quad ad - bc = 1, \\
\rho &\rightarrow \rho' = \frac{\alpha \rho + \beta}{\gamma \rho + \delta}, \quad \alpha \delta - \gamma \beta = 1.
\end{align*} \quad (24)
$$

There also exist discrete symmetries acting on the $(\tau, \rho)$ space which leave $\tilde{\phi}$ invariant. One of these interchanges the complex and Kähler structures

$$
D : \quad \tau \leftrightarrow \rho, \quad \tilde{\phi} \rightarrow \tilde{\phi}. \quad (25)
$$

In terms of the fields $G_{11}, G_{12}, G_{22},$ and $B_{12}$ the above transformation is written as

$$
\begin{align*}
G_{11} &\rightarrow \frac{1}{G_{11}}, \quad G_{12} \rightarrow \frac{B_{21}}{G_{11}}, \\
B_{21} &\rightarrow \frac{G_{12}}{G_{11}}, \quad G_{22} \rightarrow \frac{G_{12}^2 - B_{21}^2}{G_{11}},
\end{align*} \quad (26)
$$

which may be recognized as the factorized duality.

Other discrete symmetries are \[4\]

$$
W : \quad (\tau, \rho) \leftrightarrow (\tau, -\bar{\rho}), \quad \tilde{\phi} \rightarrow \tilde{\phi}, \quad (27)
$$

as well as

$$
R : \quad (\tau, \rho) \leftrightarrow (-\bar{\tau}, \rho), \quad \tilde{\phi} \rightarrow \tilde{\phi}, \quad (28)
$$

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with $R = DWDW$. The $W, R$ discrete symmetries leave invariant the fields $G_{ij}, G_{11}, G_{22}$ and $\phi$ while

$$
G_{12} \xrightarrow{W} G_{12}, \quad B_{21} \xrightarrow{W} -B_{21},
$$
$$
G_{12} \xrightarrow{R} -G_{12}, \quad B_{21} \xrightarrow{R} -B_{21}.
$$

(29)

Let us note that there exists another Lagrangian which leads to the same equations as (23). It can be constructed by using the fact that in 3-dimensions, two-forms like $F_{\mu\nu}$ and $H_{\mu\nu}$ can be written as

$$
F_{\mu\nu} = \frac{1}{\sqrt{3}} \sqrt{-\gamma} \eta^{\mu\rho} F_{\rho},
$$
$$
H_{\mu\nu} = \frac{1}{\sqrt{3}} \sqrt{-\gamma} H_{\rho}.
$$

(30)

The Bianchi identities for $F_{\mu\nu}, H_{\mu\nu}$ are then imply

$$
\nabla_{\mu} F_{\mu} = 0, \quad \nabla_{\mu} H_{\mu} = 0.
$$

(31)

Thus, we may express (13) as

$$
L^* = \sqrt{-\gamma} G_{11}^{1/2} e^{\phi} \left( R(\gamma) + \frac{1}{2} G_{11} F_{\mu} F_{\mu} + G_{11}^{1/2} e^{-\phi} \nabla_{\mu} F_{\mu} + \frac{1}{2} H_{\mu} H_{\mu} + G_{11}^{1/2} e^{-\phi} b \nabla_{\mu} H_{\mu} + \partial_{\mu} \phi \partial_{\mu} \phi \right),
$$

(32)

where the constraints (31) have been taken into account by employing the auxiliary fields $(b, \psi)$. The equations of motions for the $H_{\mu}, F_{\mu}$ give

$$
F_{\mu} = G_{11}^{-3/2} e^{-\phi} \partial_{\mu} \psi,
$$
$$
H_{\mu} = G_{11}^{1/2} e^{-\phi} \partial_{\mu} b,
$$

(33)

so that $L^*$ is written as

$$
L^* = \sqrt{-\gamma} G_{11}^{1/2} e^{\phi} \left( R(\gamma) - \frac{1}{2} G_{11}^{1/2} e^{-2\phi} \partial_{\mu} \psi \partial_{\mu} \psi \\
-\frac{1}{2} e^{-2\phi} \partial_{\mu} b \partial_{\mu} b + \partial_{\mu} \phi \partial_{\mu} \phi \right).
$$

(34)

If we further reduce it with respect to $\xi_2$, we get

$$
L^* = \sqrt{-G^{(2)} G_{11}^{1/2} V^{1/2} e^{\phi}} \left( R(G^{(2)}) + \frac{1}{2} \frac{\partial V}{V} \frac{\partial G_{11}}{G_{11}} - \frac{1}{2} G_{11}^{1/2} e^{-2\phi} \partial \psi \right)^2 \\
+ \frac{1}{2} e^{-2\phi} \left( \partial b \right)^2 + \left( \partial \phi \right)^2.
$$

(35)
The two Lagrangians $\mathcal{L}, \mathcal{L}^*$ given by (20) (or (23)) and (35), respectively lead to the same equations of motions. $\mathcal{L}$ is invariant under $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ while the symmetries of $\mathcal{L}^*$ are less obvious. In order the invariance properties of both $\mathcal{L}, \mathcal{L}^*$ to become transparent, we adapt the following parametrization

$$G_{11} = e^{-\phi} \sigma \quad V = e^{-\phi} \mu^2 \sigma$$

$$G_{ij} = e^{-\phi} \lambda^2 \eta_{ij} \quad , \quad (36)$$

where $\eta_{ij} = (-1, 1)$. The metric (17) is then written as

$$ds^2 = e^{-\phi} \sigma (d\theta_1 + Ad\theta_2)^2 + e^{-\phi} \frac{1}{\sigma} (\mu^2 d\theta_2^2 + \lambda^2 \eta_{ij} dx^i dx^j). \quad (38)$$

As a result, $\mathcal{L}, \mathcal{L}^*$ turn out to be

$$\mathcal{L} = \mu \left( 2\partial \ln \mu \partial \left( \frac{e^{-\phi/2} \mu^{1/2}}{\sigma^{1/2}} \right) - \frac{1}{2} \frac{\sigma^2}{\mu^2} (\partial A)^2 - \frac{1}{2} (\partial \ln \frac{\sigma}{\mu})^2 \right)$$

$$- \frac{1}{2} \mu^2 (\partial B)^2 - \frac{1}{2} (\partial \ln e^{-\phi} \mu)^2 \right) , \quad (39)$$

and

$$\mathcal{L}^* = \mu \left( 2\partial \mu \partial \ln \lambda - \frac{1}{2} \frac{1}{\sigma^2} (\partial \sigma)^2 - \frac{1}{2} \frac{1}{\sigma^2} (\partial \psi)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{-2\phi} (\partial b)^2 \right) \quad . \quad (40)$$

Note that $(A, \psi)$ and $(B, b)$ are related through the relations

$$\partial_i A = - \frac{1}{\sqrt{3}} \varepsilon_{ij} \frac{\mu}{\sigma^2} \eta_{jk} \partial_k \psi , \quad (41)$$

$$\partial_i B = - \frac{1}{\sqrt{3}} \varepsilon_{ij} e^{-2\phi} \mu \eta_{jk} \partial_k b , \quad (42)$$

where $\varepsilon_{12} = 1$ is the antisymmetric symbol in two-dimensions.

Let us now define, in addition to the $(\tau, \rho)$ fields given in eqs. (21,22), the complex fields $(S, \Sigma)$

$$S = b + ie^\phi \quad , \quad \Sigma = \psi + i\sigma \quad . \quad (43)$$

Then $\mathcal{L}, \mathcal{L}^*$ may be expressed as

$$\mathcal{L} = \mu \left( 2\partial \ln \mu \partial \left( \frac{e^{-\phi/2} \mu^{1/2}}{\sigma^{1/2}} \right) + 2 \frac{\partial \tau \partial \bar{\tau}}{(\tau - \bar{\tau})^2} + 2 \frac{\partial \rho \partial \bar{\rho}}{(\rho - \bar{\rho})^2} \right)$$

$$\mathcal{L}^* = \mu \left( 2\partial \mu \partial \ln \lambda + 2 \frac{\partial S \partial \bar{S}}{(S - \bar{S})^2} + 2 \frac{\partial \Sigma \partial \bar{\Sigma}}{(\Sigma - \bar{\Sigma})^2} \right) \quad . \quad (44)$$

$$\mathcal{L}^* = \mu \left( 2\partial \mu \partial \ln \lambda + 2 \frac{\partial S \partial \bar{S}}{(S - \bar{S})^2} + 2 \frac{\partial \Sigma \partial \bar{\Sigma}}{(\Sigma - \bar{\Sigma})^2} \right) \quad . \quad (45)$$
Thus, there exist four $SL(2, \mathbb{R})/U(1)$-sigma models, $\mathcal{L}$ is invariant under the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ transformations (24) and $\mathcal{L}^*$ is invariant under

$$S \rightarrow \frac{kS + m}{nS + \ell}, \quad \Sigma \rightarrow \frac{\kappa \Sigma + \eta}{\nu \Sigma + \theta}. \quad (46)$$

These transformation do not affect $\mu$. There also exist discrete $Z_2$ transformations, besides those that have already been noticed in eqs. (25,27,28), namely

$$D' : (S, \Sigma) \leftrightarrow (\Sigma, S) \quad \text{(47)}$$
$$W' : (S, \Sigma) \leftrightarrow (S, -\Sigma) \quad \text{(48)}$$
$$R' : (S, \Sigma) \leftrightarrow (-\bar{S}, \Sigma). \quad \text{(49)}$$

Moreover, the transformations

$$N : (\tau, \rho) \leftrightarrow (S, \Sigma), \quad \lambda \leftrightarrow e^{-\phi/2} \frac{h_{1/2}}{\sigma^{1/2}} \lambda, \quad \text{(50)}$$
$$N' : (\tau, \rho) \leftrightarrow (\Sigma, S), \quad \lambda \leftrightarrow e^{-\phi/2} \frac{h_{1/2}}{\sigma^{1/2}} \lambda, \quad \text{(51)}$$

indentify the two Lagrangians and thus, may be considered as the string counterpart of the Kramer-Neugebauer symmetry [19]. Note that $\mathcal{L}, \mathcal{L}^*$ may also be written as

$$\mathcal{L} = \mu \left(2\partial \ln \mu \partial \left(e^{-\phi/2} \lambda \frac{h_{1/2}}{\sigma^{1/2}} - \frac{1}{4}Tr(h_{1}^{-1} \partial h_{1})^{2} - \frac{1}{4}Tr(h_{2}^{-1} \partial h_{2})^{2}\right)\right) \quad \text{(52)}$$
$$\mathcal{L}^* = \mu \left(2\partial \mu \partial \lambda - \frac{1}{4}Tr(g_{1}^{-1} \partial g_{1})^{2} - \frac{1}{4}Tr(g_{2}^{-1} \partial g_{2})^{2}\right). \quad \text{(53)}$$

where the $2 \times 2$ matrices $h_{1}, h_{2}, g_{1}$ and $g_{2}$ are

$$h_{1} = \begin{pmatrix} \frac{2}{\mu} A & \frac{2}{\mu} B \\ \frac{2}{\mu} A + \mu & \mu \end{pmatrix}, \quad h_{2} = \begin{pmatrix} \frac{2}{\mu} B & \frac{2}{\mu} B \\\n \frac{2}{\mu} B + \mu & \mu \end{pmatrix}, \quad \text{(54)}$$
$$g_{1} = \begin{pmatrix} \frac{1}{\sigma} \psi & \frac{1}{\sigma} \psi^{2} + \sigma \\ \frac{1}{\sigma} \psi & \sigma \end{pmatrix}, \quad g_{2} = \begin{pmatrix} e^{-\phi} & e^{-\phi} \\\n e^{\phi} b & e^{\phi} \psi b + e^{-\phi} \end{pmatrix}. \quad \text{(55)}$$

The Lagrangian $\mathcal{L}$ is invariant under the infinitesimal transformations

$$\delta \sigma = \sqrt{2} \frac{1}{\sigma} A e_{1}^{+} - 2e_{1}^{0}, \quad \delta A = -\frac{1}{\sqrt{2}} \left(\frac{\sigma^{2}}{\mu^{2}} - A^{2}\right)e_{1}^{+} + 2A e_{1}^{0} + \sqrt{2} e_{1}^{-}, \quad \text{(56)}$$

while $\mathcal{L}^*$ is invariant under

$$\delta \sigma = -\sqrt{2} \psi \sigma e_{3}^{+} + 2 \sigma e_{3}^{0}, \quad \delta \psi = -\frac{1}{\sqrt{2}} \left(\frac{1}{\sigma^{2}} - \psi^{2}\right)e_{3}^{+} - 2 \psi e_{3}^{0} + \sqrt{2} e_{3}^{-}, \quad \text{(57)}$$
$$\delta \phi = \sqrt{2} b e_{4}^{+} - 2 e_{4}^{0}, \quad \delta b = -\frac{1}{\sqrt{2}} \left(e^{2\phi} - b^{2}\right)e_{4}^{+} - 2 b e_{4}^{0} + \sqrt{2} e_{4}^{-}. \quad \text{(58)}$$

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The above infinitesimal transformations are generated by a set of four Killing vectors \((K^i_a, a = 1, 2, 3, i = 1, 2, 3, 4)\) which can easily be written down by recalling that the metric

\[ ds^2 = dx^2 + e^{2x}dy^2 \]  

(58)

has a three-parameter group of isometries generated by

\[
K_+ = -\sqrt{2}y\partial_x - \frac{1}{\sqrt{2}}(e^{-2x} - y^2)\partial_y, \\
K_0 = 2(\partial_x - y\partial_y), \\
K_- = \sqrt{2}\partial_y, \\
(59)
\]

which satisfy the \(SL(2)\) commutation relations

\[
[K_+, K_0] = 2K_+, [K_-, K_0] = -2K_-, [K_-, K_+] = -K_0. \\
(60)
\]

Among these Killing vectors, let us consider \(K^{(3)}_0\) which scales both \(\psi\) and \(\sigma\) as

\[
K^{(3)}_0 : (\psi, \sigma) \rightarrow (\alpha\psi, \alpha\sigma). \\
(61)
\]

In view of eq. (51), \(A\) is also scaled as

\[
A \rightarrow \frac{1}{\alpha}A, \\
(62)
\]

so that \((A, \sigma)\) is transformed into \((\frac{1}{\alpha}A, \alpha\sigma)\) which is generated by \(-K^{(1)}_0\). However, \(\mathcal{L}\) is not invariant unless we also scale the conformal factor \(\lambda\) as \(\sqrt{\alpha}\lambda\). Let us denote the generator of constant Weyl transformations by \(k\). Then we have the relation

\[
K^{(1)}_0 + K^{(3)}_0 = k. \\
(63)
\]

In the same way, one may see that \(K^{(2)}_0, K^{(4)}_0\) which transform \((B, \phi)\) and \((b, \phi)\) as \((e^{-\alpha}B, \phi + \alpha), (e^{\alpha}, \phi + \alpha)\) respectively satisfy

\[
K^{(2)}_0 + K^{(4)}_0 = k. \\
(64)
\]

As a result, the algebra turns out to be

\[
[K^{(1)}_0, K^{(1)}_0] = 2K^{(1)}_+, [K^{(1)}_0, K^{(1)}_0] = -2K^{(2)}_-, [K^{(1)}_0, K^{(1)}_0] = K^{(1)}_0, \\
[K^{(2)}_0, K^{(2)}_0] = 2K^{(2)}_+, [K^{(2)}_0, K^{(2)}_0] = -2K^{(2)}_-, [K^{(2)}_0, K^{(2)}_0] = K^{(2)}_0, \\
[K^{(3)}_0, k-K^{(1)}_0] = 2K^{(3)}_+, [K^{(3)}_0, k-K^{(1)}_0] = -2K^{(3)}_-, [K^{(3)}_0, K^{(3)}_0] = k-K^{(1)}_0, \\
[K^{(4)}_0, k-K^{(2)}_0] = 2K^{(4)}_+, [K^{(4)}_0, k-K^{(2)}_0] = -2K^{(4)}_-, [K^{(4)}_0, K^{(4)}_0] = k-K^{(2)}_0. \\
(65)
\]
If we define the generators \((h_i, k_i, f_i)\) by
\[
\begin{align*}
    h_i &= K_0^{(i)}, \
    f_i &= K_+^{(i)}, \
    e_i &= K_-^{(i)},
\end{align*}
\] (66)
then the algebra \([65]\) may be written as
\[
\begin{align*}
    [h_i, h_j] &= 0, \
    [h_i, e_j] &= A_{ij} e_j, \
    [h_i, f_j] &= -A_{ij}, \
    [e_i, f_j] &= \delta_{ij} h_j,
\end{align*}
\] (67)
where the Cartan matrix \(A_{ij}\) is
\[
A_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{pmatrix}, \quad a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\] (68)
In addition, one may verify the Serre relation
\[
(ade_i)^{1-A_{ij}}(e_j) = 0, \quad (adf_i)^{1-A_{ij}}(f_j) = 0.
\] (69)
As a result, the algebra generated by the successive applications of the transformations \((56,57)\) is the affine Kac-Moody algebra \(\hat{o}(2, 2)\) with a central term corresponding to constant Weyl rescalings of the 2-dimensional background metric. The central term survives in higher dimensions as well, since its emergence is related to the existence of two alternative effective Lagrangians after reducing the 3-dimensional theory down to two dimensions over an abelian isometry. It is the interplay of the symmetries of these Lagrangians which produce the Kac-Moody algebra.
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