Nonlinear and memory boundary feedback stabilization for Schrödinger equations with variable coefficients

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Abstract

In this paper, the boundary stabilization of Schrödinger equations with variable coefficients by nonlinear and memory feedback is considered. The approach adopted uses Riemannian geometry methods and multipliers techniques.©2020 All rights reserved.

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1. Introduction

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \) with boundary \( \Gamma := \partial \Omega \). It is assumed that \( \Gamma \) consists of two parts \( \Gamma_0 \) and \( \Gamma_1 \) such that \( \Gamma_0, \Gamma_1 \neq \emptyset, \Gamma_0 \cap \Gamma_1 = \emptyset \). We consider the mixed problem for Schrödinger equation

\[
\begin{align*}
    y_t - iAy &= 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\
    y(0,x) &= y_0(x) \quad \text{in } \Omega, \\
    y &= 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \\
    \frac{\partial y}{\partial \nu_A} &= u \quad \text{on } \Gamma_1 \times \mathbb{R}_+, 
\end{align*}
\]

where

\[
A_y = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} (x) \frac{\partial y}{\partial x_j} \right).
\]

\( a_{ij} = a_{ji} \) are \( C^\infty \) functions in \( \mathbb{R}^n \),

\[
\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij} (x,t) \frac{\partial u}{\partial x_i} \nu_i, \quad \nu = (\nu_1, \nu_2, ..., \nu_n)
\]

is the unit normal of \( \Gamma \) pointing toward the exterior of \( \Omega \), \( \nu_A = \nu \nu \), and \( A = (a_{ij}) \) is a matrix function. We suppose that

\[
\sum_{i,j=1}^{n} a_{ij} (x) \xi_i \xi_j > \alpha_1 \sum_{i,j=1}^{n} \xi_i^2, \quad \forall x \in \Omega, \xi \in \mathbb{C}^n, \xi \neq 0.
\]

for some positive constant \( \alpha_1 \).

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Throughout the paper we assume
\[
    u = - \int_0^t k' (t - s) F \left( \frac{|y|}{|y|} \right) (y(s) - y_1(t)) \, ds - \frac{1}{2} ky. \tag{1.3}
\]
where \( k : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}_+ \in C^2 (\mathbb{R}_+^+, L^\infty (\Omega)) \), and \( F : \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous.

The goal of this work is to stabilize the system (1.1a - 1.1d) and (1.3); to find a suitable feedback \( u = F(x, y_1) \) such that the energy (1.3) decays to zero exponentially as \( t \to +\infty \) for every solution \( y \) of which \( E(0) < +\infty \). The approach adopted uses Riemannian geometry, this method was first introduced into the boundary-control problem by Yao [10] for the exactly controllability of wave equations.

The stabilization of partial differential equations has been considered by many authors.

On the other hand, the stabilization of the Schrödinger equation has been studied by Machtyngier Zuazua [9] in the Neumann boundary conditions, and by Cipolatti et al [3] with nonlinear feedbacks, this study has been considered by Lasiecka Triggani [7] with constant coefficients acting in the Dirichlet boundary conditions.

The objective of this work, we consider the boundary stabilization for system (1.1a - 1.1d) and (1.3) with variable coefficients and memory with nonlinear feedbacks by using multipliers techniques.

Our paper is organized as follows. In subsection 1.1, we introduce some notations and results on Riemannian geometry. Our main results are studied in section 2. Section 3 is devoted to the proof of the main results.

**Euclidean metric on \( \mathbb{R}^n \)**

Let \( (x_1, \ldots, x_n) \) be the natural coordinate system in \( \mathbb{R}^n \). For each \( x \in \mathbb{R}^n \), denote
\[
    X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i, \quad |X|_0^2 = X \cdot X, \quad \text{for} \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x \mathbb{R}^n.
\]

For \( f \in C^1(\overline{\Omega}) \) and \( X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \), we denote by
\[
    \nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad \text{div}_0 (X) = \sum_{i=1}^n \frac{\partial \alpha_i (x)}{\partial x_i}
\]

the gradient of \( f \) and the divergence of \( X \) in the Euclidean metric.

**Riemannian metric**

For each \( x \in \mathbb{R}^n \), define the inner product and the corresponding norm on the tangent space \( T_x \mathbb{R}^n \) by
\[
    g(X, Y) = \langle X, Y \rangle_g = X \cdot G(x)Y = \sum_{i,j=1}^n g_{ij}(x) \alpha_i \beta_j
\]
\[
    |X|_g^2 = \langle X, Y \rangle_g \quad \text{for} \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x \mathbb{R}^n
\]
Then \((\mathbb{R}^n, g)\) is a Riemannian manifold with a Riemannian metric \(g\). Denote the Levi-Cività connection in metric \(g\) by \(\nabla\). Let \(H\) be a vector field on \((\mathbb{R}^n, g)\). The covariant differential \(DH\) of \(H\) determines a bilinear form on \(T_x\mathbb{R}^n \times T_x\mathbb{R}^n\). For each \(x \in \mathbb{R}^n\), by

\[
DH(X, Y) = \langle D_X H, Y \rangle_g, \forall X, Y \in T_x\mathbb{R}^n
\]

where \(D_X H\) is the covariant derivative of \(H\) with respect to \(X\). The following lemma provides some useful equalities.

**Lemme 1.1** [Yao, 1999, lemma 2.1]. Let \(f, h \in C^1(\Omega)\) and let \(H, X\) be vector field on \(\mathbb{R}^n\). Then with reference to the above notation, we have

(i) \[
\langle H(x), A(x)X(x) \rangle_g = H(x)X(x), \quad \forall x \in \mathbb{R}^n
\]

(ii) The gradient \(\nabla_g f\) of \(f\) in the Riemannian metric \(g\) is given by

\[
\nabla_g f(x) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla_0 f.
\]

(iii) \[
\frac{\partial y}{\partial \nu} = (A(x) \nabla_0 y) \cdot \nu = \nabla_g y \cdot \nu.
\]

(iv) \[
\langle \nabla_g f, \nabla_g H \rangle_g = \nabla_g f(h) = \nabla_0 f \cdot A(x) \nabla_0 h.
\]

(v) \[
\langle \nabla_g f, \nabla_g H(f) \rangle = DH(\nabla_g f, \nabla_g f) + \frac{1}{2} \text{div}_0 \left( |\nabla_g f|^2 \right)(x) - \frac{1}{2} |\nabla_g f|^2 \text{div}_0(H) \quad x \in \mathbb{R}^n.
\]

2. **Statement of main result**

To obtain the boundary stabilization of problem (1.1a − 1.1d) and (1.2) , the following hyptheses are assumed.

There exists a vector field \(H\) on the Riemannian manifold \((\mathbb{R}^n, g)\) such that

\[
\forall X \in T_x\mathbb{R}^n, \ a > 0, \ (D_X H, X)_g \geq a |X|^2_g. \tag{2.1}
\]

\[
H(x) \cdot \nu < 0, \quad \text{on } \Gamma_0, \tag{2.2}
\]

and

\[
H(x) \cdot \nu \geq 0, \quad \text{on } \Gamma_1. \tag{2.3}
\]

We also assumed the feedback function \(F\) satisfies

\[
F(|y|) \leq |y|, \tag{2.4}
\]

and \(k \geq 0\) and \(k' \leq 0\), on \(\Gamma_0 \times \mathbb{R}_+\). Moreover,

\[
k(0) = 0 \tag{2.5}
\]

\[
\varphi = \inf_{(x, t) \in \Gamma_0 \times \mathbb{R}_+} (-k') \neq 0 \tag{2.6}
\]
We have the following result of existence and uniqueness of strong and weak solution to (1.1a – 1.1d) and (1.2).

**Theorem 2.1.** For all initial data \( y_0 \in V = H^1_0(\Omega) = \{ y \in H^1(\Omega); y = 0 \text{ on } \Gamma_0 \}, \) problem (1.1a – 1.1d) and (1.3) admits a unique global weak solution \( y \in C(\mathbb{R}^+, V). \) Furthermore, if \( y_0 \in H^3(\Omega) \cap H^1_0(\Omega) \) and \( \frac{\partial u_0}{\partial y_x} = -\frac{1}{2} k y(0) \) on \( \Gamma_1, \) then the solution has the regularity

\[
y \in C^1(\mathbb{R}^+, V).
\]

**Proof.** We can prove, using Galerkin method [6]. Suppose \( y \) is the unique global weak solution of problem (1.1a – 1.1d) and (1.2). Our main result is the following:

**Theorem 2.2**

Under the hypothesis (1.2)-(1.4) and (2.1),(2.7).

The problem

\[
y_t - iAy = 0 \quad \text{on } \Omega \\
y = 0 \quad \text{in } \Sigma_0 \\
\frac{\partial y}{\partial y_A} = 0 \quad \text{in } \Sigma
\]

and \( y = 0 \) is the unique solution of the problem.

Then for all given initial data \( y_0 \in H^1_0(\Omega), \) there exist two positive constants \( M > 0 \) and \( \omega > 0 \) such that

\[
E(t) \leq Me^{-\omega t}E(0), \quad t > 0.
\]

3. PROOF OF MAIN RESULT

For simplicity, we assume that \( y \) is a strong solution. By a classical density argument, Theorem 2.2 still holds for a weak solution.

**Lemma 3.1.** The energy defined by (1.4) is decreasing and satisfies

\[
E(T) - E(0) = -\int_0^T \int_{\Gamma_1} b |y_t|^2 \, d\Gamma_1 \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} k' |y|^2 \, d\Gamma_1 \, dt \\
- \frac{1}{2} \int_0^T \int_{\Gamma_1} k'' (t-s) |y(t)-y(s)|^2 \, d\Gamma_1 \, ds.
\]

Where \( b = -\int_0^1 k' / (t-s) \frac{f(y)}{|y|} \) \( ds, \) and whenever \( 0 < T < \infty. \)

**Proof.** Differentiating the energy \( E(\cdot) \) defined by (1.4) and using Green’s second theorem, we have

\[
E'(t) = -Re \int_{\Gamma_1} k y y_{\Gamma} d\Gamma_1 - Re \int_{\Gamma_1} \left( \int_0^t k' (t-s) \rho(y(s)) \right) y_{\Gamma} d\Gamma_1 \\
+ \int_{\Gamma_1} b |y_t|^2 \, d\Gamma_1 + 2Re \int_{\Gamma_1} y y_{\Gamma} d\Gamma_1 + \int_0^t \int_{\Gamma_1} k' (t-s) F(|y(s)|) |y(s)| \, d\Gamma_1 \, ds.
\]

By the boundary condition, and by the assumption (2.5) and several technics we obtain that

\[
E'(t) = -\int_{\Gamma_1} b |y_t|^2 \, d\Gamma_1 + \frac{1}{2} \int_{\Gamma_1} k' |y|^2 \, d\Gamma_1 \\
- \frac{1}{2} \int_0^t \int_{\Gamma_1} k'' (t-s) |y(t)-y(s)|^2 \, d\Gamma_1 \, ds.
\]
This completes the proof of lemma 3.1.

**Lemma 3.2.**
For all 
\[0 \leq S < T < \infty\] we have
\[
\int_{S}^{T} \int_{\Gamma} \left( \frac{\partial y}{\partial \nu_{A}} \right)^{2} \frac{H_{y}}{|y_{A}(x)|_{y_{g}}} \, d\Gamma \, ds - \int_{S}^{T} \int_{\Gamma} |\nabla g y|^{2} H \nu_{g} \, d\Gamma \, ds
+ 2 \text{Re} \int_{S}^{T} \int_{\Gamma} \left( \frac{\partial y}{\partial \nu_{A}} \right) H(y) \, d\Gamma \, ds + \text{Im} \int_{S}^{T} \int_{\Gamma} y y_{A} \, d\Gamma \, ds = 0
\]

**Proof.** We multiply (1.1a) by \(H \nabla y\) and integrate over \(S, T \times \Omega\) we obtain
\[
\int_{S}^{T} \int_{\Omega} y \nabla y \cdot H \, d\Omega \, ds = \int_{S}^{T} \int_{\Omega} y \nabla y \cdot H \, d\Omega \, ds
+ \int_{S}^{T} \int_{\Gamma} \partial y_{A} \cdot H y \, d\Gamma \, ds
- \int_{S}^{T} \int_{\Omega} D H(\nabla y, \nabla y) \, d\Omega \, ds
+ \text{Im} \int_{S}^{T} \int_{\Omega} \nabla y \cdot H \, d\Omega \, ds
\]

Substituting (3.1b) into (3.1a), we get
\[
\int_{S}^{T} \int_{\Omega} y \nabla y \cdot H \, d\Omega \, ds - \int_{S}^{T} \int_{\Gamma} \partial y_{A} \cdot H y \, d\Gamma \, ds - \text{Im} \int_{S}^{T} \int_{\Omega} \nabla y \cdot H \, d\Omega \, ds
= 0.
\]

Hence
\[
2 \text{Re} \int_{S}^{T} \int_{\Omega} Ay \nabla y \cdot H \, d\Omega \, ds = \text{Im} \int_{S}^{T} \int_{\Omega} y \nabla y \cdot H \, d\Omega \, ds
- \text{Im} \int_{S}^{T} \int_{\Gamma} \partial y_{A} \cdot H y \, d\Gamma \, ds
+ \text{Im} \int_{S}^{T} \int_{\Omega} \nabla y \cdot H \, d\Omega \, ds
\]

Using (1.13) of lemma 1.1, we rewrite the integral on the left-hand side of (3.1c) as
\[
\int_{S}^{T} \int_{\Omega} Ay \nabla y \cdot H \, d\Omega \, ds
= \int_{S}^{T} \int_{\Gamma} \frac{\partial y}{\partial \nu_{A}} H(y) \, d\Gamma \, ds - \int_{S}^{T} \int_{\Omega} D H(\nabla y, \nabla y) \, d\Omega \, ds
- \frac{1}{2} \int_{S}^{T} \int_{\Omega} \text{div} (\nabla y_{g}^{2} H) \, d\Omega \, ds
+ \frac{1}{2} \int_{S}^{T} \int_{\Omega} |\nabla y_{g}|^{2} \, d\Omega \, ds
\]
Recalling the boundary condition (1.1b), we have on $\Gamma$

\[ y = y_1 = 0; |\nabla_y y|_g^2 = \frac{1}{|\nabla A(y)|_g^2} \left( \frac{\partial y}{\partial \nu_A} \right)^2 \]; \quad \mathcal{H}(y) = \frac{\mathcal{H}_v}{|\nabla A(y)|_g^2} \left( \frac{\partial y}{\partial \nu_A} \right). \] \tag{3.2}

Thus using (1.1c) and (3.2), we find that this simplifies to the sought-after identity.

4. Completion of the proof of theorem 2.1.

Set $C_0 = \left( \alpha_1 \sup_{x \in \Omega} |\mathcal{H}| \right)^2$, $C_1 = \frac{\alpha_1}{2} \sup_{x \in \Omega} |\text{div} \mathcal{H}|$, $C_2 = \frac{\alpha_2}{2} \sup_{x \in \Omega} |\nabla g| |\text{div} \mathcal{H}| + \frac{1}{4} \sup_{x \in \Omega} |\text{div} \mathcal{H}| + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \Omega} |\mathcal{H}(x)|$

From assumptions (2.1), (2.2) and (2.3), we deduce that

\[ 2a \int_{\Omega} \int_{\Omega} |\nabla_y y|_g^2 \, d\Omega \, ds \leq - \int_{\Gamma} \int_{\Gamma} |\nabla_y y|_g^2 \mathcal{H} \cdot \nu_\Gamma ds + 2 \operatorname{Re} \int_{\Gamma} \left( \frac{\partial y}{\partial \nu_A} \right) \mathcal{H}(\overline{y}) \, d\Gamma_1 ds \]

\[ + \operatorname{Im} \int_{\Gamma} \int_{\Gamma} y \overline{y} \mathcal{H} \cdot \nu_\Gamma ds + \int_{\Omega} \int_{\Omega} |\nabla_y y|_g^2 \text{div} \mathcal{H} \, d\Omega \, ds \]

\[ - \operatorname{Im} \int_{\Omega} \int_{\Omega} y \mathcal{H} \, d\Omega \, ds \leq C_0 \int_{\Omega} \int_{\Gamma} \left( \frac{\partial y}{\partial \nu_A} \right)^2 \, d\Gamma_1 ds \] \tag{3.3b}

We also have the estimates, in (3.3c) and (3.3d) $\varepsilon$ is an arbitrary positive constant.

\[ \sup_{x \in \Omega} |\mathcal{H}| \int_{\Gamma} \int_{\Gamma} |y|^2 \, d\Gamma_1 ds + 2a_1 \sup_{x \in \Omega} |\mathcal{H}| \int_{\Gamma} \int_{\Gamma} |y|^2 \, d\Gamma_1 ds + \varepsilon \sup_{x \in \Omega} |\mathcal{H}(x)| \, E(S) \]

\[ + \left( \frac{\varepsilon \alpha_1}{2} \sup_{x \in \Omega} |\mathcal{H}(x)| \right) \left| y^2 \right|_{C([S,T], L^2(\Omega))} \] \tag{3.3c}

\[ \left| \int_{\Omega} \int_{\Omega} |\nabla_y y|_g^2 \text{div} \mathcal{H} \, d\Omega \, ds \right| - \operatorname{Im} \int_{\Omega} \int_{\Omega} \overline{y} \mathcal{H} \, d\Omega \, ds \leq C_1 \int_{\Gamma} \int_{\Gamma} \left( \frac{\partial y}{\partial \nu_A} \right)^2 \, d\Gamma_1 ds \]

\[ + \frac{\varepsilon}{2} \int_{\Omega} \int_{\Omega} |\nabla_y y|_g^2 \, d\Omega \, ds \]

\[ + \left( \alpha_1 \sup_{x \in \Omega} |\nabla g| |\text{div} \mathcal{H}| + \frac{1}{4} \sup_{x \in \Omega} |\text{div} \mathcal{H}| \right) \int_{\Gamma} \int_{\Gamma} |y|^2 \, d\Gamma_1 ds \] \tag{3.3d}
Choosing $\varepsilon$ now using a compactness-uniqueness argument in the style of Lasiecka and Triggiani [7], we deduce

$$\left| \int_{\Omega} \left( \frac{\partial y}{\partial \nu_A} \right)^2 \, d\Sigma \right| \leq 3 \left[ \int_{S}^{T} \int_{\Gamma_1} \left( - \int_{0}^{t} k' (t - s) y(s) \, ds \right)^2 \, d\Gamma_1 \, ds \right.
\left. + \int_{S}^{T} \int_{\Gamma_1} k(0) |y|^2 \, d\Gamma_1 \, ds + \int_{S}^{T} \int_{\Gamma_1} b^2 |y_t|^2 \, d\Gamma_1 \, ds \right]. \quad (3.3e)$$

Now using a compactness-uniqueness argument in the style of Lasiecka and Triggiani [7], we deduce

$$|y_{C([S,T],L^2(\Omega))}|^2 \leq \int_{S}^{T} \int_{\Gamma_1} |b y_t|^2 \, d\Gamma_1 \, ds \quad (3.3f)$$

Insertion (3.3b), (3.3c), (3.3d) and (3.3f) in (3.3a), leads to

$$2a \int_{S}^{T} \int_{\Omega} |\nabla g y|^2 \, d\Omega \, ds \leq 3 \left[ (c_0 + c_1) \left[ \int_{S}^{T} \int_{\Gamma_1} \left( - \int_{0}^{t} k' (t - s) y(s) \, ds \right)^2 \, d\Gamma_1 \, ds \right.
\left. + \int_{S}^{T} \int_{\Gamma_1} k(0) |y|^2 \, d\Gamma_1 \, ds + \int_{S}^{T} \int_{\Gamma_1} |b y_t|^2 \, d\Gamma_1 \, ds \right]
\right]$$

$$+ C_2 \int_{S}^{T} \int_{\Gamma_1} |y|^2 \, d\Gamma_1 \, ds + \frac{\varepsilon}{2} \int_{S}^{T} \int_{\Omega} |\nabla g y|^2 \, d\Omega \, ds$$

$$+ \varepsilon \sup_{x \in \Omega} |H(x)| E(S) + 2\alpha_1 \sup_{x \in \Omega} |H| \int_{S}^{T} \int_{\Gamma_1} |y_t|^2 \, d\Gamma_1 \, ds$$

$$+ \left( \frac{\varepsilon \alpha_1}{2} \sup_{x \in \Omega} |H(x)| \right) |y|_{C([S,T],L^2(\Omega))}^2 \quad (3.4)$$

Choosing $\varepsilon$ so that $4a - \varepsilon > 0$, we obtain

$$(4a - \varepsilon) \int_{S}^{T} E(t) \, dt \leq \left[ 3c_3 (c_0 + c_1) + \varepsilon \sup_{x \in \Omega} |H(x)| E(S) \right.$$

$$\left. + 2\alpha_1 \sup_{x \in \Omega} |H| \frac{1}{\sup_{x \in \Gamma_1} |b|} E(S) \right]$$

$$+ \left[ 3 (c_0 + c_1) |k(0)|_{L^\infty} + C_2 \right] \frac{1}{f} E(S)$$

$$+ \left[ 3 (c_0 + c_1) + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \Omega} |H(x)| \sup_{x \in \Gamma_1} |b| \right] E(S).$$

Where

$$\int_{S}^{T} E(t) \, dt \leq CE(S)$$

$$C = \frac{1}{4a - \varepsilon} \left[ 3c_3 (c_0 + c_1) + \varepsilon \sup_{x \in \Omega} |H(x)| + 2\alpha_1 \sup_{x \in \Omega} |H| \frac{1}{\sup_{x \in \Gamma_1} |b|} \right.$$

$$\left. + \left[ 3 (c_0 + c_1) + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \Omega} |H(x)| \sup_{x \in \Gamma_1} |b| \right] \right]$$

$$+ \left[ 3 (c_0 + c_1) |k(0)|_{L^\infty} + C_2 \right] \frac{1}{f}. \]
Letting $T \to +\infty$, we obtain for every $S \in \mathbb{R}_+$,

$$\int_{S}^{+\infty} E(t)\,dt \leq CE(S).$$

The desired conclusion follows now from Komornik [6].

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