THE ELEVEN DIMENSIONAL SUPERGRAVITY EQUATIONS
ON EDGE MANIFOLDS

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Abstract. We study the eleven dimensional supergravity equations which describe a low energy approximation to string theories and are related to M-theory under the AdS/CFT correspondence. These equations take the form of a non-linear differential system, on $\mathbb{B}^7 \times S^4$ with the characteristic degeneracy at the boundary of an edge system, associated to the fibration with fiber $S^4$. We compute the indicial roots of the linearized system from the Hodge decomposition of the 4-sphere following the work of Kantor, then using the edge calculus and scattering theory we prove that the moduli space of solutions, near the Freund–Rubin states, is parametrized by three pairs of data on the bounding 6-sphere.

1. Introduction

Supergravity is a theory of local supersymmetry, which arises in the representations of super Lie algebras $[VN81]$. A supergravity system is a low energy approximation to string theories $[Nas11]$, and can be viewed as a generalization of Einstein’s equation. Nahm $[Nah78]$ showed that the dimension of the system is at most eleven in order for the system to be physical, and in this dimension if the system exists it is unique. The existence of such systems was later shown $[CS77]$ by constructing a specific solution. More special solutions were constructed by physicists later $[CDF+84, VN85, BST87]$. Witten $[Wit97]$ showed that under the AdS/CFT correspondence M-theory is related to 11-dimensional supergravity, and there are more recent results $[BFOP02]$. Under dimensional reduction the fields break into many subfields and there are many such lower dimensional systems $[Nah78]$. The full eleven dimensional case, with only two fields, is in many ways the simplest to consider.

We are specifically interested in the bosonic sectors in the supergravity theory, which is a system of equations on the 11-dimensional product manifold $M = \mathbb{B}^7 \times S^4$, the product of a 7-dimensional ball and a 4-dimensional sphere. The fields are a metric, $g$, and a 4-form, $F$. Derived as the variational equations from a Lagrangian, the supergravity equations are

$$R_{\alpha\beta} = \frac{1}{12} (F_{\gamma_1\gamma_2\gamma_3} F_{\beta}^{\gamma_1\gamma_2\gamma_3} - \frac{1}{12} F_{\gamma_1\gamma_2\gamma_3\gamma_4} F_{\gamma_1\gamma_2\gamma_3\gamma_4} g_{\alpha\beta})$$

$$d*F = -\frac{1}{2} F \wedge F$$

$$dF = 0$$

(1)

The nonlinear supergravity operator has an edge structure in the sense of Mazzeo $[Maz91]$, which is a natural generalization in this context of the product of a conformally compact manifold and a compact manifold. We consider solutions as sections of the edge bundles, which are rescalings of the usual form bundles. The Fredholm
property of certain elliptic edge operators is related to the invertibility of the corresponding normal operator $N(L)$, which is the lift of the operator to the front face of the double stretched space $X^2$ which appears in the resolution of these operators. The invertibility of the normal operator is in turn related to its action on appropriate polyhomogeneous functions at the left boundary of $X^2$, the form of the expansion of the solution is determined by the indicial operator $I_s(L)$. The inverse of the indicial operator $I_s(L)^{-1}$ exists and is meromorphic on the complement of a discrete set $\{s \in \text{spec}_2 L\}$, the set of indicial roots of $L$ which are the exponents in the expansion. In this way the indicial operator as a model on the boundary determines the form of the leading order expansion of the solution.

One solution for this system is given by the metric which is the product of the round sphere with a Poincaré–Einstein metric on $\mathbb{B}^7$ with a volume form on the 4-sphere as the 4-form, in particular the Freund–Rubin solution [FR80] is contained in this class. Recall that a Poincaré–Einstein manifold is one that satisfies the vacuum Einstein equation and has a conformal degeneracy at the boundary. In the paper by Graham and Lee [GL91], solutions are constructed which are $C^{n-1,\gamma}$ close to the hyperbolic metric on the ball $\mathbb{B}^n$ near the boundary. They showed that every such perturbation is determined by the conformal data on the boundary sphere. We will follow a similar idea here for the equation (1), replacing the Ricci curvature operator by the nonlinear supergravity operator, considering its linearization around one of the product solutions, and using a perturbation argument to show that all the solutions nearby are determined by three pairs of data on the boundary.

Kantor studied this problem in his thesis [Kan09], where he computed the indicial roots of the system and produced one family of solutions by varying along a specific direction of the 4-form. In this paper we use Hodge decomposition to get the same set of indicial roots and show that all the solutions nearby are prescribed by boundary data for the linearized operator, more specifically, the indicial kernels corresponding to three pairs of special indicial roots.

1.1. Equations derived from the Lagrangian. The 11-dimension supergravity theory contains the following information on an 11-dimensional manifold $M$: a metric $g \in \text{Sym}^2(M)$ and a 4-form $F \in \bigwedge^4(M)$. The Lagrangian $L$ is

$$L(g, A) = \int_M R \text{Vol}_g - \frac{1}{2} \left( \int_M F \wedge *F + \int_M \frac{1}{3} A \wedge F \wedge F \right).$$

Here $R$ is the scalar curvature of the metric $g$, $A$ is a 3-form such that $F = dA$. The first term is the classical Einstein–Hilbert action, and the second and the third one are respectively of Yang–Mills and Maxwell type. Note here we are only interested in the equations derived from the variation of Lagrangian and the variation only depends on $F = dA$, therefore we only need $F$ to be globally defined but with only locally defined $A_i$ on coordinate patches $U_i$. Then

$$\delta A_i \left( \int_{U_i} A_i \wedge F \wedge F \right) = 3 \int_{U_i} \delta A_i \wedge F \wedge F$$

shows that the variation of this term is $F \wedge F$.

The supergravity equations (1) are derived from the Lagrangian above (see section 2). Since the Ricci operator is not elliptic, we follow [GL91] and add a gauge
breaking term

\[ \phi(g, t) = \delta^*_g g \Delta g \Id \]

to the first equation. Then we apply \( d^* \) to the 2nd equation, and combine this with the third equation to obtain the system:

\[
Q : S^2(T^*M) \oplus \bigwedge^4(M) \to S^2(T^*M) \oplus \bigwedge^4(M)
\]

\[
\left( \begin{array}{c} g \\ F \end{array} \right) \mapsto \left( \begin{array}{c} \text{Ric}(g) - \phi(g, t) - F \circ F \\ d \ast (d \ast F + \frac{1}{2} F \wedge F) \end{array} \right)
\]

which is the nonlinear system we will be studying.

1.2. Edge metrics and edge Sobolev spaces. Edge differential and pseudo-differential operators were formally introduced by Mazzeo [Maz91]. The general setting is a compact manifold with boundary, \( M \), where the boundary has in addition a fibration

\[ \pi : \partial M \to B. \]

In the setting considered here, \( M = \mathbb{B}^7 \times S^4 \) is the product of a seven dimensional closed ball identified as hyperbolic space and a four-dimensional sphere. The relevant fibration has the four-sphere as fibre:

\[
\begin{array}{ccc}
\mathbb{S}^4 & \to \partial M = \mathbb{S}^6 \times S^4 \\
\downarrow \pi & & \downarrow \mathbb{S}^6.
\end{array}
\]

The space of edge vector fields \( \mathcal{V}_e(M) \) is the Lie algebra consisting of those smooth vector fields on \( M \) which are tangent to the boundary and such that the induced vector field on the boundary is tangent to the fibre of \( \pi \). Another Lie algebra of vector fields we will be using is \( \mathcal{V}_b(M) \) which is the space of all smooth vector fields tangent to the boundary [Mel93]. As a consequence,

\[
\mathcal{V}_e \subset \mathcal{V}_b, \quad [\mathcal{V}_e, \mathcal{V}_b] \subset \mathcal{V}_b.
\]

In terms of local coordinates, let \( (x, y_1, y_2, \ldots, y_6) \) be the coordinates on the upper half space model for hyperbolic space \( \mathbb{H}^7 \), and \( (z_1, \ldots, z_4) \) be the coordinates on the sphere \( S^4 \). Then locally \( \mathcal{V}_b \) is spanned by \( \{x \partial_x, \partial_{y_i}, \partial_{z_j}\} \), while \( \mathcal{V}_e \) is spanned by \( \{x \partial_x, x \partial_{y_i}, \partial_{z_j}\} \). The edge forms are the wedge product of the dual form to the edge vector fields \( \mathcal{V}_e \), with a basis: \( \{dx, \partial x, dz^i\} \), and the edge 2-tensor bundle is formed by the tensor product of the basis. We will work on the following edge vector bundle:

**Definition 1.1.** Let \( K \) be the edge bundles, the sections of which are symmetric 2-tensors and 4-forms:

\[
K := \mathfrak{c} \text{Sym}^2(M) \oplus \mathfrak{c} \bigwedge^4 M.
\]

Edge differential operators form the linear span of products of edge vector fields over smooth functions. We denote the set of \( m \)-th order edge operator as \( \text{Diff}_e^m(M) \) and we will see that the supergravity operator \( Q \) is a nonlinear edge differential operator, so a nonlinear combination of elements of \( \text{Diff}_e^1(M) \).

The edge-Sobolev spaces are given by

\[
H^s_e(M) = \{ u \in L^2(M) | V_1 \ldots V_k u \in L^2(M), 0 \leq k \leq s, V_i \in \mathcal{V}_e(M) \}.
\]
However, for purpose of regularity we are also interested in hybrid spaces with additional tangential regularity, as the existence of solutions with infinitely smooth b-regularity gives polyhomogeneous expansions. Therefore we define the hybrid Sobolev space with both boundary and edge regularity:

\[ H^{s,k}_{e,b}(M) = \{ u \in H^s_{e}(M) | V_1 \ldots V_i u \in H^s_{e}(M), 0 \leq i \leq k, V_j \in \mathcal{V}_b(M) \}. \]

By the commutation relation (6), \( H^{s,k}_{e,b}(M) \) is well defined, that is, independent of the order in which edge and b-vector fields are applied, and the proof is given in proposition A.1. For the vector bundle \( K \) over \( M \), the Sobolev spaces \( H^{s,k}_{e,b}(M; K) \) can be similarly defined by choosing an orthonormal basis and are independent of choices.

These Sobolev spaces are defined so that edge operators maps between them, i.e., for any \( m \)-th order edge operator \( P \in \text{Diff}_m^e(M) \),

\[ P : H^{s,k}_{e,b}(M) \rightarrow H^{s-m,k}_{e,b}(M), m \leq s. \]

which is proved in proposition A.2.

1.3. Poincaré–Einstein metrics on \( \mathbb{B}^7 \). The product of an arbitrary Poincaré–Einstein metric with the spherical metric provides a large family of solutions to this system. For any Poincaré–Einstein metric \( h \) with curvature \( -6c^2 \) with \( c > 0 \), the following metric and 4-form gives a solution to equations (1):

\[ u = (h \times 9 \epsilon^2 g_{S^4}, c \text{Vol}_{S^4}). \]

According to [GL91], the Poincaré–Einstein metrics near the hyperbolic metric can be obtained by perturbation of the conformal boundary data. More specifically, there is the following result:

**Proposition 1.2 ([CS77]).** Let \( M = \mathbb{B}^{n+1} \) be the unit ball and \( \hat{h} \) the standard metric on \( S^n \). For any smooth Riemannian metric \( \hat{g} \) on \( S^n \) which is sufficiently close to \( \hat{h} \) in \( C^{2,\alpha} \) norm if \( n > 4 \) or \( C^{3,\alpha} \) norm if \( n = 3 \), for some \( 0 < \alpha < 1 \), there exists a smooth metric \( g \) on the interior of \( M \), with a \( C^0 \) conformal compactification satisfying

\[ \text{Ric}(g) = -ng, \quad g \text{ has conformal infinity} [\hat{g}]. \]

We are mainly interested in the solutions that are perturbations of such product solutions, in particular, we will focus on the solutions with \( c = 6 \) in (11) and \( h \) being the hyperbolic metric on the ball, i.e. on \( X = \mathbb{H}^7 \times S^4 \):

\[ u_0 = (t, W) = (g_{\mathbb{H}^7} \times \frac{1}{4} g_{S^4}, 6 \text{Vol}_{S^4}), \]

which is also known as the Freund–Rubin solution.

1.4. Main theorem. Besides the boundary conformal data that prescribes the Poincaré–Einstein metric, we will show that there are additionally three pairs of data on \( S^6 \) that together parametrize the solution to (1).

First we define three bundles on \( S^6 \) that correspond to the incoming and outgoing boundary data for the linearized supergravity operator.

**Definition 1.3.** Let \( V_1^\pm \) to be the space of 3-forms with \( *_g \) eigenvalue \( \pm i \):

\[ V_1^\pm := \{ v_1^\pm \in C^\infty(S^6, \bigwedge^3 T^*S^6) : *_{g_v} v_1 = \pm iv_1 \}. \]
Let \( V_2^+ \) and \( V_3^\pm \) be the smooth functions on the 6-sphere tensored with eigenforms on 4-sphere:

\[
V_2^+ = V_2^- := \{ v_2 = f_2 \otimes \xi_{16} : f_2 \in C^\infty(S^6), \xi_{16} \in E_{16}^1(S^4) \},
\]
\[
V_3^+ = V_3^- := \{ v_3 = f_3 \otimes \xi_{40} : f_3 \in C^\infty(S^6), \xi_{40} \in E_{40}^3(S^4) \},
\]
where \( E^i_k(S^4) \) is the space of closed 1-forms with eigenvalue \( \lambda \) on \( S^4 \). We also set \( V = \oplus_{i=1}^3 V_i^\pm \).

**Remark 1.4.** Note the dimension of the closed 1-forms with the first and second eigenvalues are determined by the degree 2 and 3 spherical harmonics in 4 variables, which, respectively, are 5 and 14 dimensional.

We also require three numbers that define the leading term in the expansion of the solution, which come from indicial roots computation listed in Appendix B:

\[
\theta_1^\pm = 3 \pm 6i, \quad \theta_2^\pm = 3 \pm i\sqrt{21116145}/1655, \quad \theta_3^\pm = 3 \pm i3\sqrt{582842}/20098.
\]

From the indicial calculation, we also see that the real parts of all other indicial roots are at least distance 1 away from 3. And from here we fix a number \( \delta \in (0, 1) \).

Then we define three scattering operators to relate the incoming and outgoing data:

**Definition 1.5.** Let \( S_i, i = 1, 2, 3 \), be the scattering operators defined as

\[
S_i : V_i^+ \to V_i^-
\]

for equation (3.17), and equation (3.18) with eigenvalues 16 and 40, such that the linearized operator \( dQ \) acting on the leading part of (14) is contained in \( O(x^{3+\delta}) \).

Now we can use the boundary data to parametrize the solution:

**Definition 1.6** (Leading expansion for the linear operator). When we say the leading expansion is given by the outgoing data \((v_1^+, v_2^+, v_3^+) \in V \), this means that \((k, H)\) has an expansion with the following leading terms:

\[
H_{(4,0)} = \frac{dx}{x} \wedge (v_1^+ x^{\theta_1^+} + S_1(v_1^+) x^{\delta_1^-}) + O(x^{3+\delta})
\]
\[
\text{Tr}_{\mathbb{R}^7} k = 7\delta S_3 (v_2^+ x^{\theta_2^+} + S_2(v_2^+) x^{\delta_2^-} + v_3^+ x^{\theta_3^+} + S_3(v_3^+) x^{\delta_3^-}) + O(x^{3+\delta})
\]
\[
\text{Tr}_{\mathbb{R}^7} k = 4\delta S_3 (v_2^+ x^{\theta_2^+} + S_2(v_2^+) x^{\delta_2^-} + v_3^+ x^{\theta_3^+} + S_3(v_3^+) x^{\delta_3^-}) + O(x^{3+\delta})
\]

\[
k_{(1,1)} = \delta S_3 (v_2^+ x^{\theta_2^+} + S_2(v_2^+) x^{\delta_2^-} + v_3^+ x^{\theta_3^+} + S_3(v_3^+) x^{\delta_3^-}) + O(x^{3+\delta})
\]
\[
H_{(1,3)} = \delta S_3 \ast S_3 (v_2^+ x^{\theta_2^+} + S_2(v_2^+) x^{\delta_2^-} + v_3^+ x^{\theta_3^+} + S_3(v_3^+) x^{\delta_3^-}) + O(x^{3+\delta})
\]
\[
H_{(0,4)} = \delta S_3 \ast S_3 (v_2^+ x^{\theta_2^+} + S_2(v_2^+) x^{\delta_2^-} + v_3^+ x^{\theta_3^+} + S_3(v_3^+) x^{\delta_3^-}) + O(x^{3+\delta})
\]

and the other components in \((k, H)\) are all in \( O(x^{3+\delta}) \).

Now the main result is the characterization of the solution:

**Theorem.** For \( k \gg 0, \delta \in (0, 1) \), there exists \( \rho > 0 \) and \( \epsilon > 0 \), such that, for a Poincaré–Einstein metric \( h \) that is sufficiently close to the base metric \( g_{117} \) with \( \|h - g_{117}\|_{H^1(S^7)} < \epsilon \), and any boundary value perturbation \( v \in V \) with \( \|v\|_{H^2(S^7; \pi)} < \rho \), there is a unique solution \( u = (g, H) \in D_{\nu, h} \subset x^{-\delta} H^{1,\delta}_{\nu, h}(M; K) \) satisfying the supergravity equations (1), with the leading expansion of \((g - h \times \frac{1}{4} g_{117}, H - 6 \text{Vol}_{S^7})\) given by (14).
Our approach is based on the implicit function theorem. From the boundary data $v$ we construct a perturbation term using the Poisson operator $O$, then consider a translation of the gauged operator:

$$Q_{h,v}(\cdot) = Q(\cdot + P v).$$

A right inverse of the linearization, denoted $(dQ)^{-1}$, is constructed, and we show that $Q_{h,v} \circ (dQ)^{-1}$ is an isomorphism on the Sobolev space $x^\delta H^{s,k}_{e,b}(M; K)$ which is the range space of $dQ$. From here we deduce there is a unique solution to $Q$ for each boundary parameter set $v$.

To get the isomorphism result on $Q_{h,v} \circ (dQ)^{-1}$, we note that the model operator on the boundary is $SO(5)$-invariant, and therefore utilize the Hodge decomposition of functions and forms on $S^4$ to decompose the equations into blocks that allows us to compute the indicial roots for each block. Indicial roots are those $s$ that the indicial operator has a nontrivial kernel, and as mentioned above these roots are related to the leading order of the solution expansions near the boundary.

Once the indicial roots are computed, we construct the right inverse $(dQ)^{-1}$. The operator exhibits different properties for large and small eigenvalues. For large ones, the projected operator is already invertible by constructing a parametrix in the small edge calculus. For small eigenvalues, two resolvents $R_{\pm} = \lim_{\epsilon \to 0} (dQ \pm i\epsilon)^{-1}$ are constructed and we combine them to get a real-valued right inverse. We show that those elements corresponding to indicial roots with real part equal to 3 are the boundary perturbations needed in the theorem.

The paper is organized as follows. In section 2 we show the derivation of the equations from the Lagrangian and discuss the gauge breaking condition. In section 3 we compute the linearization of the operator and its indicial roots. In section 4 we analyze the linearized operator, prove it is Fredholm and construct the boundary data. In section 5 we construct the solutions for the nonlinear equations using the implicit function theorem.

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2. Gauged operator construction

2.1. Equations derived from Lagrangian. As mentioned in Section 1.1, the supergravity system arises as the variational equations for the Lagrangian

$$L(g, A) = \int_M R \text{Vol}_g - \frac{1}{2} \left( \int_M F \wedge *F + \int_M \frac{1}{3} A \wedge F \wedge F \right).$$

Now we compute its variation along two directions, namely, the metric and the form direction. The first term is the Einstein-Hilbert action, for which the variation in $g$ is

$$\delta_g \left( \int M R \text{Vol}_g \right) = \int_M \left( R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \text{Vol}_g.$$

The variation of the second term $F \wedge *F$ in the metric direction is

$$\delta_g \left( \frac{1}{2} \int M F \wedge *F \right)$$

$$= \frac{2}{4!} \int F_{\eta_1 \ldots \eta_4} g^{\eta_2 \xi_2} g^{\eta_3 \xi_3} g^{\eta_4 \xi_4} \delta g^{\xi_2 \xi_3 \xi_4} \text{Vol}_g - \frac{1}{4} \int F \wedge *F g_{\alpha\beta} \delta g^{\alpha\beta} \text{Vol}_g.$$
Combining these we get the first equation on the metric

\[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{1}{12} F_{\alpha \eta_1 \eta_2 \eta_3 \beta} F^{\eta_1 \eta_2 \eta_3} - \frac{1}{4} (F, F) g_{\alpha\beta}. \]

Here \( (\cdot, \cdot) \) is the inner product on forms:

\[ \langle F, F \rangle = \frac{1}{4!} F_{\eta_1 \ldots \eta_4} F^{\eta_1 \ldots \eta_4}. \]

Taking the trace of the equation (2.3), we get

\[ R = \frac{1}{6} \langle F, F \rangle. \]

Finally, substituting \( R \) into (2.3), we get

\[ R_{\alpha\beta} = \frac{1}{12} \left( F_{\alpha \gamma_1 \gamma_2 \gamma_3 \beta} F^{\gamma_1 \gamma_2 \gamma_3} - \frac{1}{12} F_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} F^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} g_{\alpha\beta} \right), \]

which gives the first equation in (1).

The variation with respect to the 3-form \( A \) is

\[ \delta A L = \int \delta F \wedge \ast F - \frac{1}{6} \delta A \wedge F \wedge F - \frac{1}{3} A \wedge \delta F \wedge F = - \int \delta A \wedge (d \ast F + \frac{1}{2} F \wedge F), \]

which gives the second supergravity equation:

\[ d \ast F + \frac{1}{2} F \wedge F = 0. \]

Since \( F \) is locally exact, we have the third equation

\[ dF = 0. \]

2.2. Poincaré–Einstein metric. Product solutions to the supergravity equations are obtained as follows: let \( X \) be a 7-dimensional Einstein manifold with negative scalar curvature \( \alpha < 0 \) and \( K \) be a 4-dimensional Einstein manifold with positive scalar curvature \( \beta > 0 \). Consider \( X \times K \) with the product metric; then we have

\[ \text{Ric} = \begin{pmatrix} 6\alpha g_{AB}^X & 0 \\ 0 & 3\beta g_{ab}^K \end{pmatrix}. \]

Let \( F = c \text{Vol}_K \). A straightforward computation shows

\[ (F \circ F)_{\alpha\beta} = \frac{c^2}{12} \begin{pmatrix} -2g_{AB}^X & 0 \\ 0 & 4g_{ab}^K \end{pmatrix}. \]

Therefore any triple \((c, \alpha, \beta)\) satisfying

\[ -\frac{c^2}{6} = 6\alpha, \frac{c^2}{3} = 3\beta \]

corresponds to a solution to the supergravity equation.

2.3. Edge bundles. Such product metrics fit into the setting of edge bundles. As introduced in [Maz91], edge tangent bundles \( ^e TM \) are defined by declaring \( \mathcal{V}_e \) to be its smooth sections and its dual bundle, \( ^e T^* M \), is the edge cotangent bundle. We denote the edge form bundle,

\[ ^e \wedge^m (T^* M) =: ^e \wedge^m M, \]

of which the local sections can be written as the \( C^\infty (M) \) combinations of

\[ \frac{dx}{x} \wedge \frac{dy^1}{x} \ldots \wedge \frac{dy^k}{x} \wedge dz^j \ldots \wedge dz^l, 1 + k + l = m \]
and
\[
\frac{dy^1}{x} \cdots \wedge \frac{dy^k}{x} \wedge dz^1 \cdots \wedge dz^l, k + l = m.
\]
Similarly the edge symmetric 2-tensor bundle \(e \text{ Sym}^2(M) := \text{Sym}^2(\ast T^* M)\) is spanned by 2-tensors with local forms of
\[
\left( \frac{dx}{x} \frac{dy}{x} \frac{dz}{x} \right)
\begin{pmatrix}
k_{00} & k_{0j} & k_{0j} \\
k_{10} & k_{1j} & k_{1j} \\
k_{0j} & k_{1j} & k_{1j}
\end{pmatrix}
\begin{pmatrix} \frac{dx}{x} \\ \frac{dy}{x} \\ \frac{dz}{x} \end{pmatrix}
\]
with smooth coefficients \(k_{**}\).

It is easy to check that the supergravity operator \(S\) is an edge operator (2.14)
\[
S : e \text{ Sym}^2(M) \oplus e \wedge^4 M \rightarrow e \text{ Sym}^2(M) \oplus e \wedge^8 M \oplus e \wedge^5 M
\]
\[
\begin{pmatrix} g \\ F \end{pmatrix} \mapsto \begin{pmatrix} \text{Ric} g - F \circ F \\ d \ast F + \frac{1}{2} F \wedge F \end{pmatrix}.
\]

2.4. A square system. To get a square system, we apply \(d^*\) to the second equation. Because of the closed condition \(dF = 0\), \(d^* d \ast F\) is the same as \(\Delta F\). This leads to the following square system (here \(e \wedge^4 M\) denotes the bundle of closed edge 4-forms on \(M\)):
\[
\tilde{S} : e \text{ Sym}^2(M) \oplus e \wedge^4 M \rightarrow e \text{ Sym}^2(M) \oplus e \wedge^4 M
\]
\[
\begin{pmatrix} g \\ F \end{pmatrix} \mapsto \begin{pmatrix} \text{Ric} g - F \circ F \\ \Delta F + \frac{1}{2} d \ast (F \wedge F) \end{pmatrix}.
\]

**Proposition 2.1.** The kernel of the square supergravity operator \(\tilde{S}\) (2.15) is the same as the original supergravity operator \(S\) (2.14):
\[
\text{Nul}(S) \cap x^{-\delta} H^2_{e,b}(M; K) = \text{Nul}(\tilde{S}) \cap x^{-\delta} H^2_{e,b}(M; K)
\]

**Proof.** We only need to show that, in \(x^{-\delta} H^2_{e,b}(M; K)\), the null space of \(\tilde{S}\) does not have extra elements that are not in \(\text{Nul}(S)\). We show this by proving that if

\[
d \ast \left( d \ast F + \frac{1}{2} (F \wedge F) \right) = 0
\]
\[
dF = 0
\]
then

\[
\omega := d \ast F + \frac{1}{2} (F \wedge F) = 0.
\]

Note that (2.16) implies that \(\omega\) is a harmonic form on \(\mathbb{H}^7 \times S^4\).

Consider the Hodge decomposition of forms on \(S^4\) by taking \(\alpha_i\) and \(\beta_i\) to be the basis of coclosed and closed forms for the \(i\)-th eigenvalue
\[
dS \alpha_i = \lambda_i \beta_i, \quad \delta S \alpha_i = 0
\]
\[
\delta S \beta_i = \lambda_i \alpha_i, \quad dS \beta_i = 0
\]
This implies \(\Delta S \alpha_i = \lambda_i^2 \alpha_i, \Delta S \beta_i = \lambda_i^2 \beta_i\). Write \(\omega = \sum_{i=1}^{\infty} u_i \alpha_i + v_i \beta_i\), where \(u_i, v_i\) are forms on \(\mathbb{H}^7\). The decay condition on \(F\) implies that \(u_i, v_i\) are \(L^2\) forms on \(\mathbb{H}^7\).
Using the eigenspace decomposition to rewrite the closed and coclosed condition for $\omega$ and combining with (2.18), we get

\begin{align}
\tag{2.19} d_H u_i &= 0, \quad \lambda_i u_i + d_H v_i = 0 \\
\tag{2.20} \delta_H v_i &= 0, \quad \delta_H u_i + \lambda_i v_i = 0.
\end{align}

Then we get

\begin{align}
\Delta_H u_i &= \lambda_i^2 u_i, \quad \Delta_H v_i = \lambda_i^2 v_i.
\end{align}

Since there are no $L^2$ eigenforms on $H_7$ [Maz88], we get $u_i = 0, v_i = 0$, which proves (2.17).

\section{Gauge condition.}

Following [GL91] in the setting of a Poincaré–Einstein metric, we add a gauge operator to the curvature term where $g$ is the background metric:

$$
\phi(t, g) = \delta_t^* t g^{-1} \delta_t G_t g.
$$

Here

$$
[G_t g]_{ij} = g_{ij} - \frac{1}{2} g^k t_{ij}, \quad [\delta_t g]_i = -g^j_{ij},
$$

$\delta_g^*$ is the formal adjoint of $\delta_g$, which can be written as

$$
[\delta_g^* w]_{ij} = \frac{1}{2} (w_{i,j} + w_{j,i}),
$$

and

$$
[t g^{-1} w]_i = t_{ij} (g^{-1})^{jk} w_k.
$$

By adding the gauge term to the first equation of $\tilde{S}$ we get an operator $Q$, which is a map from the space of symmetric 2-tensors and closed 4-forms to itself:

$$
Q : \epsilon \text{Sym}^2 \oplus \epsilon \wedge^4 M \to \epsilon \text{Sym}^2 \oplus \epsilon \wedge^4 M
$$

\begin{equation}
\left( \begin{array}{c}
    g \\
    F
\end{array} \right) \mapsto \left( \begin{array}{c}
    \text{Ric}(g) - \phi(t, g) - F \circ F \\
    \Delta F + \frac{1}{2} d^* (F \wedge F)
\end{array} \right)
\end{equation}

which is the main object of study below.

As discussed in Lemma 2.2 in [GL91], $\text{Ric}(g) + n g - \phi(t, g) = 0$ holds if and only if $id : (M, g) \to (M, t)$ is harmonic and $\text{Ric}(g) + n g = 0$ when $(t, g)$ satisfies certain regularity restrictions. We will show that the gauged equations here yield the solution to the supergravity equations in a similar manner.

We first prove a gauge elimination lemma for the linearized operator $dQ$ which is computed in Proposition 3.1. As can be seen from (2.21), only the first part (the map on 2-tensors) involves the gauge term, therefore we restrict the discussion to the first part of $dQ$. We use $dQ_g(k, H)$ to denote the linearization of the tensor part of $Q$ along the metric direction at the point $(g, F)$, which acts on $(k, H) \in \Gamma(K)$.

First we give the following gauge-breaking lemma for the linearized operator, which is adapted from Theorem 4.1 and Theorem 4.2 in [Kan09].

\textbf{Proposition 2.2.} For fixed $\delta \in (-1, 0)$, there exists $\epsilon > 0$, such that for any $\|g - t\|_{x^2 H^2(M, \epsilon \text{Sym}^2(M))} < \epsilon$ and $k \in x^3 H^2(M, \epsilon \text{Sym}^2(M))$ satisfying $dQ_g(k, H) = 0$, there exists a 1-form $v$ and $\tilde{k} = k + L_v g$ such that $dS_g(\tilde{k}, H) = 0$.

To prove the proposition, we first determine the equation to solve for such a 1-form $v$, which appeared in Theorem 4.1 in [Kan09].
Lemma 2.3. Given \( k \in \mathfrak{e} \operatorname{Sym}^2(M) \), if a 1-form \( v \) satisfies

\[
(\Delta_{\text{rough}} - \operatorname{Ric})v = \frac{1}{2}(2\nabla^\alpha k_{\alpha \lambda} - \nabla_\lambda \operatorname{Tr}_g(k))
\]

then the 2-tensor \( \tilde{k} = k + L_v g \) satisfies the gauge condition

\[
d\phi_t(t, g)(\tilde{k}) = 0.
\]

Here \( d\phi_t(t, g) \) is the linearization of \( \phi \) with respect to the first variable.

Proof. Let \( \Psi(v, g) \) be the map

\[
\Psi(v, g)^k = (\phi_v^* g)^{\alpha \beta}(\Gamma^k_{\alpha \beta}(\phi_v^* g) - \Gamma^k_{\alpha \beta}(t)).
\]

This satisfies

\[
d_g D \Psi(0, t)(v) = d\phi_t(L_v, t, g) D_g \Psi(0, t)(k) = d\phi_t(k).
\]

Therefore in order to get \( d\phi_t(\tilde{k}) = 0 \), we only need

\[
-D_v \Psi(0, t)(v) = D_g \Psi(0, t)(k).
\]

The left hand side can be reduced to

\[
- g^{\alpha \beta} \nabla_\alpha \nabla_\beta v^k - R^k_{\mu \nu} v^\mu = (\Delta_{\text{rough}} - \operatorname{Ric})v^k
\]

and right hand side is

\[
\frac{1}{2} g^{\alpha \beta} g^{\lambda \gamma}(\nabla_\alpha k_{\beta \gamma} + \nabla_\beta k_{\alpha \gamma} - \nabla_\lambda k_{\alpha \beta}).
\]

Lowering the index on both side, we get

\[
((\Delta_{\text{rough}} - \operatorname{Ric})v)_\lambda = \frac{1}{2}(2\nabla^\alpha k_{\alpha \lambda} - \nabla_\lambda \operatorname{Tr}_g(k)).
\]

\[\square\]

Next we discuss the solvability of the operator defined in the left hand side of (2.22).

Lemma 2.4. If \( |\delta| < 1 \), then at the point \( t = g_{\operatorname{S}^7} \times \frac{1}{4} g_{\operatorname{S}^4} \) the operator

\[
\Delta_{\text{rough}} - \operatorname{Ric} : x^\delta H^2_c(\mathfrak{e}^* M) \to x^\delta L^2_c(\mathfrak{e}^* M)
\]

is an isomorphism.

Proof. Using the splitting

\[
\mathfrak{e}^* M \cong \pi_{\mathbb{H}}^* T^* \mathbb{H}^7 \oplus \pi_{\mathbb{S}}^* T^* \mathbb{S}^4
\]

and the product structure of the metric, we write the operator as

\[
\Delta_{\text{rough}} - \operatorname{Ric} = \Delta_{\mathbb{H}}^{\text{rough}} + \Delta_{\mathbb{S}}^{\text{rough}} - \operatorname{diag}(-6, 12).
\]

It decomposes into two parts: trace and trace-free 2-tensors. Decomposing into eigenfunctions on the 4-sphere, consider the following two operators:

\[
L_{tr} = \Delta_{\mathbb{H}} + \lambda - 24 : C^\infty(\mathbb{H}^7) \to C^\infty(\mathbb{H}^7),
\]

(2.23) \[
L_{tr} = \Delta_{\mathbb{H}}^{\text{rough}} + \lambda' + 6 : \mathfrak{e}^* \wedge^\ast \mathbb{H}^7 \to \mathfrak{e}^* \wedge^\ast \mathbb{H}^7
\]

Consider the smallest eigenvalue in each case: \( \lambda_{tr} = 16, \lambda_{tf} = 0 \). The indicial radius for \( L_{tr} \) is 1 and for \( L_{tf} \) is 4. By the same argument as in proposition 4.14, \( L_{tr} : x^\delta H^2_c(\mathbb{H}^7) \to x^\delta L^2_c(\mathbb{H}^7) \) is an isomorphism. The same argument holds for \( L_{tf} \),
which is also Fredholm and an isomorphism on $x^\delta H^2_e(\ast\ast H^7) \to x^\delta L^2_e(\ast\ast H^7)$ for $|\delta| < 4$.

Combining the statements for $L_{t_1}$ and $L_{t_f}$, we conclude that $\Delta^{\text{rough}} - \text{Ric}$ is an isomorphism between $x^\delta H^2_e(\ast T^*M) \to x^\delta L^2_e(\ast T^*M)$ for $|\delta| < 1$.

The isomorphism holds true for metrics nearby, by a simple perturbation argument.

**Corollary 2.5.** There exists $\epsilon > 0$, such that for any metric $g$ with $\|g-t\|_{x^\delta H^2_e(M;\ast \text{Sym}^2(M))} < \epsilon$, for $|\delta| < 1$, $\Delta^{\text{rough}} - \text{Ric}$ is an isomorphism as a map

$$\Delta^{\text{rough}} - \text{Ric} : x^\delta H^2_e(\ast T^*M) \to x^\delta L^2_e(\ast T^*M).$$

**Proof.** Let $A_g = \Delta^{\text{rough}} - \text{Ric}$, by writing the coefficients out, we have that for any $u \in x^\delta H^2_e(\ast T^*M)$,

$$\|A_g - A_t)u\|_{x^\delta L^2_e(\ast T^*M)} \leq C\|g - t\|_{x^\delta H^2_e(M;\ast \text{Sym}^2(M))}\|u\|_{x^\delta H^2_e(\ast T^*M)},$$

which shows that $A_g$ is also an isomorphism for $g$ sufficiently close to $t$. \qed

With the lemmas above, we can prove the proposition.

**Proof of Proposition 2.2.** From Corollary 2.5 we know $\Delta^{\text{rough}} - \text{Ric} : x^\delta H^2_e(\ast T^*M) \to x^\delta L^2_e(\ast T^*M)$ is an isomorphism for $g$ close to $t$, therefore there exists a one-form $v$ satisfying 2.3. Then from Lemma 2.3, $\tilde{k} = k + L_{\ast \gamma} g$ satisfies $d\phi_g(\tilde{k}) = 0$. Putting it back to the linearized equation, we get $dS_g(\tilde{k}, H) = 0$. \qed

Next we prove the nonlinear version of gauge elimination by using.

**Proposition 2.6.** If a metric and a closed 4-form $(g, V)$ satisfies the gauged equations $Q(g, V) = 0$, then there is a diffeomorphism $g \mapsto \tilde{g}$ such that $\phi(t, \tilde{g}) = 0$ and $(\tilde{g}, V)$ is a solution to equation (1) i.e. $S(\tilde{g}, V) = 0$.

**Proof.** Consider the vector field in the affine Sobolev space $\{t\} + x^\delta H^2_e(M;\ast \text{Sym}^2(M))$ with $\delta \in (-1, 0)$, defined by

$$X_g = \tilde{k}_g = k_g + L_{\ast \gamma} g,$$

(2.25)

It is easy to check that for any $g \in x^\delta H^2_e(M;\ast \text{Sym}^2(M))$, $\tilde{k}_g \in x^\delta H^2_e(M;\ast \text{Sym}^2(M))$. Now consider the integral curve $g(s)$ starting from $g(0) = g$. From the integral curve theory on an infinite dimensional manifold (see for example Proposition 1 in Chapter 4 of [Lan85]), since the map $g \mapsto \tilde{k}_g$ is Lipchitz, the integral curve exists and we take $\tilde{g} = g(1)$. From the construction of $\tilde{k}_g$ we know that $\tilde{g}$ satisfies $S(\tilde{g}, V) = 0$. \qed

3. The linearized and the indicial operator

We now consider the linearization of the gauged supergravity operator near the base solution $(t, W) = (g_{H^7} \times 4, g_{S^4}, 6 \text{Vol}_{S^4})$. The first step in proving the invertibility of $Q$ as an edge operator is to compute the indicial roots and indicial kernels of this linearized operator, which is done with respect to the Hodge decomposition on the 4-sphere. There is a pair of indicial roots associated to each eigenvalue and as the eigenvalues becomes larger, the pairs move apart. The distribution of indicial roots are illustrated in Figure 1.
With respect to the volume form on $\mathbb{H}^7 \times S^4$, there is an inclusion of weighted functions and forms. For $\text{Re}(s) > 3$,

$$x^* C^\infty(\mathbb{C} \wedge^p M) \subset L^2_{c, \mathbb{C}}(\mathbb{C} \wedge^p M),$$

and therefore $\text{Re}(s) = 3$ line is the $L^2$ cutoff line. There are only three pairs of exceptional indicial roots corresponding to the lowest three eigenvalues that lie on the $L^2$ line, in the sense that they have real parts equal to 3 and the remainders are pure imaginary and symmetric around this line.

### 3.1. Linearization of the operator $Q$

The nonlinear supergravity operator contains two parts: the gauged curvature operator $\text{Ric}(\cdot) - \phi(t, \cdot)$ with a nonlinear part $F \circ F$, and the first order differential operator $d * F$ with a nonlinear part $F \wedge F$. Note that since the Hodge operator $*$ depends on the metric, the linearized operator couples the metric with the 4-form in both equations.

**Proposition 3.1.** The operator $Q : K \to K$ has linearization at $(t, W)$:

$$dQ_{t, W} : \Gamma(K) \to \Gamma(K)$$

$$\left( \begin{array}{c} k \\ H \end{array} \right) \mapsto \left( \begin{array}{c} \Delta k + L \\ d * (d * H + 6 \text{Vol}_3 \wedge H + 6d *_{\mathbb{H}} k_{1,1} + 3d(\text{tr}_3(k) - \text{tr}_3(k)) \wedge \text{Vol}_3) \end{array} \right)$$

where the lower order term $L$ is:

$$L = \left( \begin{array}{cc} -k_{1,1} - 6 \text{tr}_3(k)t_{1,1} + \text{tr}_3(k)t_{1,1} + 2 *_{\mathbb{H}} H_{0,4} t_{1,1} + 6k_{1,1} - 3 *_{\mathbb{H}} H_{1,3} \\ 6k_{1,1} - 3 *_{\mathbb{H}} H_{1,3} \\ 4k_{ij} + 8 \text{tr}_3(k)t_{ij} - *_{\mathbb{H}} H_{0,4} t_{ij} \end{array} \right).$$

The computation involves a curvature part and a form part.

**Lemma 3.2.** For $k \in \mathfrak{c} \text{ Sym}^2(M)$, the linearization of the gauged Ricci operator at the base metric $t$ is

$$d(\text{Ric}(\cdot) - \phi(t, \cdot))_t(k) = \frac{1}{2} \Delta_{t, \text{rough}} k + R(k),$$

where

$$R(k) = \left( \begin{array}{cc} -7k_{1,1} + \text{Tr}_{\mathbb{H}}(k)g_{1,1} \\ 0 \end{array} \begin{array}{c} 0 \\ 16k_{ij} - \text{Tr}_{\mathbb{H}}(g_{ij}) \end{array} \right).$$

**Proof.** Following the result in [GL91], the linearization of the gauged operator at the base metric $t$ is

$$d(\text{Ric} - \phi(t, \cdot))_t(k) = \frac{1}{2} \Delta_{t, \text{rough}} k + k^{\alpha\beta} R_{\beta\gamma\delta\alpha} + \frac{1}{2}(R_{\gamma}^\beta k_{\beta\delta} + R_{\alpha}^\beta k_{\beta\gamma}).$$

Specifically, if the metric has constant sectional curvature near the boundary of $M$ which is the case for the conformal compact metric considered here (with sectional curvature $-1$), the curvature term is diagonal and can be written as

$$R_{\alpha\beta\gamma\delta} = -(g_{\alpha\delta} g_{\gamma\beta} - g_{\alpha\beta} g_{\gamma\delta}),$$

so the linearization of this total operator is as above. \qed
Lemma 3.3. The linearization of the term $F \circ F$ acting on a 2-tensor $k \in \mathfrak{e} \text{Sym}^2(M)$ and a 4-form $H \in \mathfrak{e} \wedge^4 M$ are respectively:

$$
\text{(3.5)} \quad d(F \circ F)_{t,W}(k) = \left( \begin{array}{c}
\frac{1}{36} Tr_S(k) t_{IJ} - \frac{1}{36} k + 2(W,H)t \\
\frac{1}{144} (W,W) k_{IJ}
\end{array} \right) + \frac{1}{12} \left( -3 W_{ab_1 b_2 b_3} W^{b_2 b_3} t_{i_1 i_2} k_{i_1 i_2} t_{j_1 j_2} \\
+ \frac{4}{12} F_{a_1 b_1 c_1} F^{a_1 b_1 c_1} t_{i_1 i_2} k_{i_1 i_2} t_{i_2 i_3} t_{i_3 i_4} + \frac{4}{12} F_{i_1 i_2 i_3 i_4} t_{i_1 i_2} k_{i_1 i_2} t_{i_2 i_3} t_{i_3 i_4} t_{i_4 i_5} - \frac{1}{12} (W,W) k_{ab} \right)
$$

and

$$
\text{(3.6)} \quad d(F \circ F)_{t,W}(H) = \left( \frac{1}{72} c_2^{*} S H_{(0,4)} t_{AB} + \frac{1}{3} (S H_{(1,3)})_{AB} - \frac{1}{72} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2} k_{i_1 i_2} t_{i_2 i_3} t_{i_3 i_4} t_{i_4 i_5} \right).
$$

Proof. The proof is by direct computation. Note that

$$
D_{t,W}(F \circ F)(H) = \left( HH \quad HS \quad SS \right),
$$

where

$$
\text{(3.7)} \quad HH_{AB} = -\frac{1}{72} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{AB} = -\frac{1}{72} c_2 h_{t_{AB}} = \frac{1}{72} c_2^{*} S H_{(0,4)} t_{AB};
$$

$$
\text{(3.8)} \quad HS_{ab} = \frac{1}{12} H_{a_1 b_1} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} = 3 (S H_{(1,3)})_{ab},
$$

$$
\text{(3.9)} \quad SS_{ab} = \frac{1}{6} H_{a_1 b_1} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} = \frac{1}{72} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{i_2 i_3 i_4} t_{i_3 i_4} t_{i_4 i_5}.
$$

Then for metric variation $k \in \text{Sym}^2(T^*M)$

$$
D_{t,W}(F \circ F)(k) = \left( HH \quad HS \quad SS \right),
$$

$$
\text{(3.10)} \quad HH = \frac{1}{144} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} k_{IJ},
$$

$$
\text{(3.11)} \quad HS = \frac{1}{144} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} k_{IJ},
$$

$$
\text{(3.12)} \quad SS_{ab} = \frac{1}{12} \left( -3 W_{a_1 b_1} W_{b_1 b_2 b_3} t_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} k_{i_1 i_2} t_{i_2 i_3} t_{i_3 i_4} t_{i_4 i_5} - \frac{1}{12} W_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} t_{i_1 i_2 i_3 i_4} k_{IJ} \right),
$$

which using inner product $W_{i_1 i_2 i_3 i_4} W_{i_1 i_2 i_3 i_4} = \langle W,W \rangle$ will give the expressions above.

Next we compute the linearization for the second part:
Lemma 3.4. The linearization of the equation
\[ d * F + \frac{1}{2} F \wedge F = 0, \]
in the form and tensor directions respectively are:
\begin{equation}
(3.13) \quad d(d * F + \frac{1}{2} F \wedge F)_{\ell,W}(H) = d * H + H \wedge F,
\end{equation}
\begin{equation}
(3.14) \quad d(d * F + \frac{1}{2} F \wedge F)_{\ell,W}(k) = 6d *_{\mathbb{H}} k_{(1,1)} + 3d(tr_{\mathbb{H}}(k) - tr_{\mathbb{H}}(k)) \text{Vol}_{\mathbb{H}}.
\end{equation}

Proof. The linearization along the form direction is straightforward, as the terms are linear and quadratic in \( F \). Along the metric direction, the linearization comes from the Hodge star:
\begin{equation}
(3.15) \quad D(*F)_{\beta_1, \beta_2, \beta_3}(k) = D(\frac{1}{4} V^{\alpha_1, \alpha_4} W_{\alpha_1, \alpha_4} \delta g)_{\gamma_1, \gamma_4} V^{\alpha_2, \alpha_4} W_{\alpha_2, \alpha_4} + \frac{1}{4} V^{\alpha_2, \alpha_4} \delta g_{\gamma_1, \gamma_4} V^{\alpha_1, \alpha_4} W_{\alpha_1, \alpha_4} = 6d *_{\mathbb{H}} k_{(1,1)} + 3d(tr_{\mathbb{H}}(k) - tr_{\mathbb{H}}(k)) \text{Vol}_{\mathbb{H}}
\end{equation}
which gives the expressions above. \( \square \)

Proof of Proposition 3.1. Combining the components above, the linearized equations are
\begin{equation}
(3.16) \quad \frac{1}{2} \Delta^{\text{rough}} k + k^{\alpha \beta} R_{\beta \gamma \delta \alpha} + \frac{1}{2} (R_{\beta \gamma}^3 k_{\beta \delta} + R_{\beta \gamma}^4 k_{\beta \gamma}) + L = 0
\end{equation}
\[ d * (6d *_{\mathbb{H}} k_{(1,1)} + 3d(tr_{\mathbb{H}}(k) - tr_{\mathbb{H}}(k)) \text{Vol}_{\mathbb{H}} + d * H + H \wedge \text{Vol}_{\mathbb{H}} bH) = 0 \]
which after rearrangement gives equation (3.1). \( \square \)



3.2. Indicial roots computation. Having obtained the linearized operator \( dQ \), we next compute its indicial roots on the boundary of \( \mathbb{H}^7 \), which together with the indicial kernels give the parametrization of the kernel of this linear operator. Using the Hodge decomposition on the 4-sphere, the operator \( dQ \) acts on sections of \( \wedge^*(\mathbb{H}^7) \) tensored with the finite dimensional eigenspace of \( \wedge^*(\mathbb{S}^4) \).

Lemma 3.5. Sections of the bundle \( K \) decompose with respect to the Hodge decomposition of \( \mathbb{S}^4 \).

Proof. We identify the symmetric edge 2-tensor bundle with
\[ (\text{Sym}^2(T^*\mathbb{H}^7)) \oplus (T^*\mathbb{H}^7 \otimes T^*\mathbb{S}^4) \oplus \text{Sym}^2(T^*\mathbb{S}^4), \]
and decompose the 4-form bundle according to its degree on \( \mathbb{H}^7 \) and \( \mathbb{S}^4 \), i.e.
\[ e \wedge^4 T^* M = \oplus_{i+j=4} e \wedge^i T^*\mathbb{H}^7 \otimes \wedge^j T^*\mathbb{S}^4. \]
For each element of the form $u \otimes v$ with 
$$u \in \Gamma(e \wedge T^*H^7), \quad v \in \Gamma(\wedge^*T^*S^4)$$
the projection operator $\pi_\lambda$ maps it to $u \otimes \pi_\lambda v$, which by linearity extends to the whole bundle $K$. \hfill \square

We denote the projection by $\pi_\lambda$ on the sections of the bundle $K$ as the linear extension of the eigenvalue projection on $S^4$. Note here we have a collection of eigenvalues on both functions and forms, specifically we have:

- on functions: $4k(k + 3), \quad k \geq 0$;
- on closed 1-forms: $4k(k + 3), \quad k \geq 1$;
- on co-closed 1-forms: $4(k + 1)(k + 2), \quad k \geq 0$.

It follows from Lemma 3.5 that the operator decomposes to a sum of infinitely many operators, each acting on a subbundle.

**Lemma 3.6.** The operator $dQ$ preserves the eigenspaces of $S^4$, and decomposes as

$$dQ = \sum_{\lambda \geq 0} dQ^\lambda := \sum_{\lambda} \pi_\lambda \circ dQ \circ \pi_\lambda$$

**Proof.** We only need to show that Hodge laplacian $\Delta$ commutes with the linearized operator. Since the linear operator is composed from $\Delta^{\text{hodge}}, \Delta^{\text{rough}}$ (which are related by Bochner formula), Hodge $\ast$ operator, differential, and scalar operator, all of which commute with $\Delta$, $dQ$ therefore commutes with the eigenvalue projections. \hfill \square

Next we compute the indicial roots and kernels for the linearized operator as an edge differential operator. Recall that $\partial M$ is the total space of fibration over $Y = \partial B^7$.

**Definition 3.7 (Indicial operator).** Let $L : \Gamma(E_1) \to \Gamma(E_2)$ be an edge operator between two vector bundles over $M$. For any boundary point $p \in Y$, and $s \in \mathbb{C}$, the indicial operator of $L$ at point $p$ is defined as

$$I_p[L](s) : \Gamma(E_1|_{\pi^{-1}(p)}) \to \Gamma(E_2|_{\pi^{-1}(p)})$$

$$(I_p[L](s))v = x^{-s}L(x^{-s}\hat{v})|_{\pi^{-1}(p)}$$

where $\hat{v}$ is an extension of $v$ to a neighborhood of $\pi^{-1}(p)$. The indicial roots of $L$ at point $p$ are those $s \in \mathbb{C}$ such that $I_p[L](s)$ has a nontrivial kernel, and the corresponding kernels are called indicial kernels.

In the conformally compact case, the indicial operator is a bundle map from $E_1|_p$ to $E_2|_p$ (which is simpler than a partial differential operator as in the general edge case). Moreover, since we have an $SO(7)$ symmetry for the operator, the indicial roots are invariant on $S^6$.

**Proposition 3.8.** The indicial roots of operator $dQ$ are symmetric around $\text{Re} \ z = 3$, with three special pairs of roots

$$\theta_1^\pm = 3 \pm 6i, \quad \theta_2^\pm = 3 \pm i\sqrt{21116145/1655}, \quad \theta_3 = 3 \pm i\sqrt{582842/20098}.$$ 

and all other roots lying in $\{ \| \text{Re} \ z - 3 \| \geq 1 \}$.
Figure 1. Indicial roots of the linearized supergravity operator on $\mathbb{C}$

Proof. With the harmonic decomposition on sphere $S^4$, the linearized operator $dQ$ is block-diagonalized and we compute the indicial roots for the linear system $dQ$ in Appendix B. We summarize the results below and Figure 1 is an illustration of the indicial roots distribution. The indicial roots fall into the following three categories:

1. The roots corresponding to harmonic forms:
   (a) The equation for trace-free 2-tensors on $H^7$ arising from the first component of (3.1) is
   \[(\Delta_S + \Delta_H - 2)\hat{k}_{IJ} = 0,\]
   and the corresponding indicial equation is
   \[(-s^2 + 6s)\hat{k}_{IJ} = 0.\]
   We have indicial roots
   \[S_1^+ = 0, S_1^- = 6.\]
   This corresponds to the perturbation of the hyperbolic metric to a Poincaré–Einstein metric.
   (b) The equation for trace-free 2-tensors on $S^4$ is
   \[\Delta_S^{\text{rough}}\hat{k}_{ij} + \Delta_H\hat{k}_{ij} + 8\hat{k}_{ij} = 0\]
   where indicial equation is
   \[(-s^2 + 6s + 8)\hat{k}_{ij} = 0,\]
   and the indicial roots are
   \[S_2^+ = 3 \pm \sqrt{17}.\]
   (c) Equations for $H_{(4,0)}$:
   \[
   \begin{align*}
   d_H \ast H_{(4,0)} + W \wedge H_{(4,0)} &= 0 \\
   d_H H_{(4,0)} &= 0
   \end{align*}
   \]
   where the indicial equation is
   \[-(s - 3)(\ast_6 N) \wedge dx/x - 6dx/x \wedge N = 0,\]
   with indicial roots
   \[\theta_1^\pm = 3 \pm 6i.\]
This corresponds to a perturbation of the 4-form on hyperbolic space.

(2) The roots corresponding to functions / closed 1-forms / coclosed 3-forms / closed 4-forms

(a) The equations for $7\sigma = Tr_H(k), 4\tau = Tr_S(k), k_{(1,1)}, H_{(1,3)}, H_{(0,4)}$ are

$$
6d_H*h_{(1,1)}^{cl} + d_S(3Tr_H(k) = 3Tr_S(k)) \land V + d_S * H_{(0,4)}^{cl} + d_H * H_{(1,3)}^{cc} = 0
$$

$$
d_H H_{(0,4)}^{cl} + d_S H_{(1,3)}^{cc} = 0
$$

$$
\triangle_s k_{(1,1)}^{cl} + \triangle_H k_{(1,1)}^{cl} + 12k_{(1,1)}^{cl} - 6 * S H_{(1,3)}^{cc} = 0
$$

$$
\Delta_s \tau + \Delta_H \tau + 72\tau - 8* S H_{0,4}^{cl} = 0
$$

$$
\Delta_s \sigma + \Delta_H \sigma + 12\sigma + 4* S H_{0,4}^{cl} - 48\tau = 0
$$

The indicial equations are

$$
(3.19) \quad \lambda^4 - 4S^2\lambda^3 + 24S * \lambda^2 - 90\lambda^3 + 6S^4\lambda^2 - 72S^3\lambda^2 + 342S^2\lambda^2 - 756S * \lambda^2 + 1152\lambda^2 - 456\lambda + 72S^5\lambda - 414S^4 \lambda + 648S^3 \lambda + 1152S^2 \lambda - 3024S * \lambda + 10368 \lambda
$$

$$+ S^8 - 24S^7 + 162S^6 + 108S^5 - 6192S^4 + 31536S^3 - 33696S^2 - 155520S = 0
$$

When $\lambda = 16$ there is a pair of roots with real part 3

$$
(3.20) \quad s = \theta_3^\pm = 3 \pm i\sqrt{21116145}/1655
$$

and when $\lambda = 40$ there is a pair of roots with real part 3

$$
\theta_3^\pm = 3 \pm i3\sqrt{582842}/20098
$$

And here the five variables are related by

$$
H_{(0,4)}^{cl} = d_s * d_s \xi, H_{(1,3)}^{cc} = -d_H * d_s \xi, k_{(1,1)}^{cl} = -d_s \delta_H \xi, 4\sigma = 7\tau = \xi
$$

where

$$
\xi \in \delta_s \wedge_{10}^\perp S^4
$$

similarly we have another indicial kernel corresponding to $\theta_3^\pm$ with

$$
\xi \in \delta_s \wedge_{10}^\perp S^4.
$$

(b) The equations for $H_{(3,1)}, H_{(4,0)}$ are

$$
d_S * H_{(3,1)}^{cl} + d_H * H_{(4,0)}^{cc} + 6^4V \land H_{(4,0)}^{cc} = 0
$$

$$
d_H H_{(3,1)}^{cl} + d_S H_{(4,0)}^{cc} = 0
$$

where the indicial equations are

$$
(s - 3)^2 \pm 6i(s - 3) - 16 = 0
$$

with indicial roots

$$
S_3^\pm = 3 \pm \sqrt{7} \pm 3i.
$$

(3) The roots corresponding to coclosed 1-forms / closed 2-forms / coclosed 2-forms / closed 3-forms

(a) The equations for $k_{(1,1)}, H_{(1,3)}, H_{(2,2)}$ are

$$
6d_H * h_{(1,1)}^{cc} + d_H * H_{(1,3)}^{cl} = 0
$$

$$
d_S * H_{(1,3)}^{cl} + d_H * H_{(2,2)}^{cc} + 6d_S * H_{(1,1)}^{cc} = 0
$$

$$
d_H H_{(1,3)}^{cl} + d_S H_{(2,2)}^{cc} = 0
$$

$$
\frac{1}{2} \triangle_s k_{(1,1)}^{cc} + \frac{1}{2} \triangle_H k_{(1,1)}^{cc} + 6k_{(1,1)}^{cc} - \frac{1}{2} * S H_{(1,3)}^{cl} = 0
$$
The indicial equation is
\[ \lambda^2 - (36 + (s - 1)(s - 5) + s^2 - 6s - 1)\lambda - (s - 1)(s - 5)(-s^2 + 6s + 1) = 0. \]

With the smallest eigenvalue for coclosed 1-forms being \( \lambda = 24 \), the indicial roots are
\[ S_4^\pm = 3 \pm \sqrt{97} + 31. \]

(b) The equations for \( H_{(2,2)}, H_{(3,1)} \) are
\[
\begin{align*}
    d_S * H^{cl}_{(2,2)} + d_H * H^{cc}_{(3,1)} &= 0 \\
    d_H H^{cl}_{(2,2)} + d_S H^{cc}_{(3,1)} &= 0
\end{align*}
\]

The indicial equations are
\[ (\Delta^{H^{edge}}_S - (2 - s)(4 - s))H_{(3,1)} = 0, \]
and for \( \lambda = 24 \) we have
\[ S_5^\pm = 3 \pm \sqrt{17}. \]

\[
\square
\]

4. Fredholm property of the linearized operator

Once we identify these indicial roots, we proceed using various strategies according to whether the indicial roots land on the \( L^2 \) line or not. We show that, for all of the indicial roots bigger than a certain number, the linearized operator after projection is Fredholm on suitable edge Sobolev spaces. This is done by using small edge calculus and \( SO(5) \) invariance with respect to the boundary. For the rest of finite indicial roots, we discuss them individually. For the three exceptional pairs we use the scattering theory to construct two generalized inverses, which encode the boundary data that parametrize the kernel of the linearized operator.

We then describe the kernel of this linearized operator in terms of the two generalized inverses, and a scattering matrix construction that gives the Poisson operator. Near any Poincaré–Einstein metric product that is close to base metric \( t \), a perturbation argument shows that the space given by the difference of the two generalized inverses is transversal to the range space of the linearized operator and therefore this space gives the kernel of the linearized operator, which later will provide the kernel parametrization for the nonlinear operator.

First of all, we define the domain for the linearized operator:

**Definition 4.1.** Fix \( \delta \in (0,1) \), and take any small \( \epsilon > 0 \), define the domain as
\[ D_k(\delta) = \{ u \in x^{-\delta}H^{2,k}_{e,b}(M;K) : (dQ \pm i\epsilon)u \in x^{\delta}H^{0,k}_{e,b}(M;K) \}. \]

**Remark 4.2.** The domain is well defined independent of any sufficiently small \( \epsilon \). This is proved by Proposition 4.25.

Using the projection operator \( \pi_\lambda \) defined above, the domain can be decomposed:
\[ D_k(\delta) = \oplus_{\lambda \in \Lambda} D_k(\lambda, \delta), \]
where \( \Lambda = \{ 4k(k+3), k \geq 0 \} \cup \{ 4(k+1)(k+2), k \geq 0 \} \) is the set of eigenvalues on the 4-sphere. And we denote the operator acting on each subbundle as
\[ dQ^\lambda := \pi_\lambda \circ dQ \circ \pi_\lambda, \quad dQ^{\lambda > M} = \sum_{\lambda > M} dQ^\lambda. \]
Moreover the eigenvalue decomposition extends to the hybrid Sobolev spaces. Consider the bundle $K$ over $M = \mathbb{B}^7 \times S^4$ which carries a unitary linear action of $\text{SO}(5)$ covering the action on $S^4$. There is an induced action of $\text{SO}(5)$ on $C^\infty(\mathbb{B}^7 \times S^4; K)$, which extends to all the weighted hybrid Sobolev spaces $x^s H^{k,j}_{e,b}(\mathbb{B}^7 \times S^4; W)$ since the group acts through diffeomorphisms. The linearized operator $dQ \in \text{Diff}_c^2(\mathbb{B}^7 \times S^4; K)$ is an elliptic edge operator for the product edge structure and we have shown that $dQ$ commutes with the induced action of $\text{SO}(5)$ on $C^\infty(\mathbb{B}^7 \times S^4; K)$.

The Sobolev spaces of sections of $K$ decompose according to the irreducible representations of $\text{SO}(5)$, all finite dimensional and forming a discrete set. In particular these may be labelled by the eigenvalues, $\lambda$, of the Casimir operator for $\text{SO}(5)$ with a finite dimensional span when $\lambda$ is bounded above. The $\text{SO}(7,1)$ action on $\mathbb{H}^7$ commutes with the $\text{SO}(5)$ action on $K$ and acts transitively on $\mathbb{H}^7$, so the multiplicity of the $\text{SO}(5)$ representation does not vary over $\mathbb{H}^7$. The individual representations of $\text{SO}(5)$ in the decomposition of $K$ therefore form bundles over $\mathbb{H}^7$. Therefore we have

**Lemma 4.3.** The group $\text{SO}(5)$ acts on $x^s H^{s,k}_{e,b}(M; K)$ transitively, and the bundle decomposes to subbundles on $\mathbb{H}^7$.

**Proof.** By the transitivity of $\text{SO}(7,1)$ and $\text{SO}(5)$ actions discussed above. □

We will separately discuss three parts. One part is the infinite dimensional subspace formed by large eigenvalues

$$\oplus_{\lambda > \lambda_0} D_k(\lambda, \delta),$$

on which the operators $dQ^{\lambda > \lambda_0} \pm i\epsilon$ are isomorphisms, and their inverses approach two limits $R_{\epsilon}^{\lambda > \lambda_0}$ uniformly as $\epsilon$ goes to zero. This is shown by using ellipticity and a parametrix construction.

### 4.1. Large eigenvalues

We are going to show the following proposition for the operator with projection off finitely many small eigenvalues.

**Proposition 4.4.** There is a sufficiently large $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ and any small $\epsilon > 0$ the two operators

$$dQ^{\lambda > \lambda_0} \pm i\epsilon : \oplus_{\lambda > \lambda_0} D_k(\lambda, \delta) \to \oplus_{\lambda > \lambda_0} \pi_{\lambda} x^s H^{0,k}_{e,b}(M; K)$$

are both isomorphisms. And their inverses approach two operators uniformly as $\epsilon \downarrow 0$.

To prove this proposition, we will bundle all the large eigenvalues together.

**Definition 4.5.** For $\lambda \in [0, \infty)$, let $\pi_{\geq \lambda} : K \to K$ be defined as the projection off the span of the eigenspaces of the Casimir operator for $\text{SO}(5)$ with eigenvalues smaller than $\lambda$, i.e. $\pi_{\geq \lambda} := \text{Id} - \sum_{\lambda' < \lambda} \pi_{\lambda'}$.

**Proposition 4.6.** For any weight $s \in \mathbb{R}$ and any orders $p, k$, the bounded operator defined as

$$dQ : x^s H^{p+2,k}_{e,b}(M; K) \to x^s H^{p,k}_{e,b}(M; K)$$

is such that $\pi_{\geq \lambda_0} dQ$ is an isomorphism onto the range of $\pi_{\geq \lambda_0}$ for some $\lambda_0 \in [0, \infty)$ (depending on $s$ but not on $p$ and $k$). Moreover, the range of $\text{Id} - \pi_{\geq \lambda_0}$ on $C^\infty(M; K)$ is the space $C^\infty(M; \oplus_{\lambda' < \lambda_0} \pi_{\lambda'} K)$ of sections of a smooth vector bundle over $M$ and $dQ$ restricts to it as an elliptic element of $\text{Diff}_0^2(M; \oplus_{\lambda' < \lambda_0} \pi_{\lambda'} K)$. 
The second part of the proposition is from the definitions of 0-operator and edge operator and the fact that ellipticity in edge symbols implies ellipticity in zero symbols once fiber directions are removed. To prove the first part of the proposition, we first construct an $SO(5)$-invariant parametrix in the small edge calculus by finding an appropriate kernel on the edge stretched product space $M_2^e$ which is defined from $M^2$ by blowing up the fiber diagonal over the boundary of $M$ [Maz91].

**Definition 4.7.** The edge stretched product $M_2^e$ for an edge manifold $M$ is defined as the blow up $[M^2; S]$ where $S$ consists of all fibers of the product fibration $\pi^2 : (\partial M)^2 \to Y^2$ which intersect the diagonal of $(\partial M)^2$.

Notice that from the definition of fiber diagonal, the blow up actually preserves the product structure of $\mathbb{H}^7 \times S^4$, i.e. the fiber diagonal contained in $M^2$ is just the product $\Delta \times S^4 \times S^4$, and the manifold after the blow up is $[(\mathbb{H}^7)^2; \partial \Delta_0] \times (S^4)^2$.

**Lemma 4.8.** For $M = \mathbb{H}^7 \times S^4$, the edge stretched product is actually a product: $M_2^e = [(\mathbb{H}^7)^2, \partial \Delta_0] \times (S^4)^2$.

The elliptic element $dQ \in \text{Diff}^2_e(M; K)$ lift to be transversely elliptic to the fiber diagonal down to the front face. Therefore we have a parametrix construction in the small edge calculus.

**Lemma 4.9.** The $SO(5)$-invariant elliptic operator $dQ \in \text{Diff}^2_e(M; K)$ has an $SO(5)$-invariant parametrix $\tilde{E}$ in $\Psi^{-\infty}_e(M; K)$ such that $\text{Id} - dQ \circ \tilde{E}, \text{Id} - \tilde{E} \circ dQ \in \Psi^{-\infty}_e(M; K)$ are also $SO(5)$-invariant.

**Proof.** Any elliptic edge differential operator has a parametrix in the small edge calculus, following Theorem 3.8 in [Maz91]. The construction gives the kernel of $E$ as a classical conormal distribution with respect to the ‘lifted diagonal’ of the stretched edge produce $M_2^e$. Because of the product structure of $M_2^e$, in fact the action of $SO(5)$ on the kernel $E$, through the product action on $M^2$, lifts smoothly to $M_2^e$ and preserves the lifted diagonal (which is the closure of the diagonal in the interior). So we may average under the product action and define

$$\tilde{E} = \int_{g \in SO(5)} g \cdot E.$$ 

Since $dQ$ is $SO(5)$ invariant, $\tilde{E}$ is also a parametrix,

$$dQ \circ \tilde{E} = \text{Id} + \tilde{R},$$

and the average remainder $\tilde{R}$ is also $SO(5)$ invariant. \qed

As a consequence, now $\tilde{E}$ and $\tilde{R}$ both commute with the spherical eigenvalue projection $\pi_{\geq \lambda}$. The remainder $\tilde{R}$ can be characterized as:

**Lemma 4.10.** The Schwartz kernel of $\tilde{R}$ is in $C^\infty((S^4)^2, \Psi^{-\infty}_0(\mathbb{H}^7) \otimes \text{Hom}(K)) \subset C^\infty(M_2^e; K)$. In consequence it is a smooth map from $(S^4)^2$ to bounded operators on $x^sH^p_0(\mathbb{H}^7; K)$ for any $s, p$, with a norm depending on some $C^k$ norm for any bounded range of $s$. 

Proof. As an element in $\Psi^{-\infty}(M; K)$, the Schwartz kernel $\tilde{R}$ is smooth on the double edge space $M_2$, with values in the bundle $\text{Hom}(K) \otimes K$ where $K$ is the kernel density bundle. From the properties of the small calculus, $\tilde{R}$ vanishes to infinite order at the left and right boundary faces. Because $M_2$ has the product structure $(\mathbb{H}^7)^0 \times (S^4)^2$, the Schwartz kernel is in $C^\infty((S^4)^2, C^\infty((\mathbb{H}^7)^0, \text{Hom}(K) \otimes K))$ where $C^\infty((\mathbb{H}^7)^0, \text{Hom}(K) \otimes K)$ is the $\Psi^{-\infty}(\mathbb{H}^7; K)$ operator acting on $K$.

Consider the map from $\Psi^{-\infty}(\mathbb{H}^7; K)$ to bounded operators on $x^sH_0^2(S^4; K)$: since it is a continuous map from a Fréchet space to a normed space, the norm is bounded by some norm on $\Psi^{-\infty}(\mathbb{H}^7; K)$, i.e. the operator norm of $\tilde{R}$ on $x^sH_0^2(\mathbb{H}^7; K)$ is bounded by $C(s)||\tilde{R}||_{C^\infty((\mathbb{H}^7; K))}$ where $C(s)$ is a constant only depending on $s$. For any bounded interval $s \in [-S, S]$, the bound of $||\tilde{R}||_{x^sH_0^2(\mathbb{H}^7; K)}$ is uniform. □

We can use the following interpolation result to show that $\pi_\lambda \tilde{R}$ rapidly decays as $\lambda$ tends to infinity.

**Lemma 4.11.** $x^sH_{e,b}^{p,k}(M; K) = L^2(S^4; x^s H_0^p(S^4; K)) \cap H^{p+k}(S^4; x^s L^2(S^4; K))$.  

**Proof.** We only prove the case $s = 0$ since the weight on the boundary defining function $x$ transfer to the 0-Sobolev spaces on $\mathbb{H}^7$ directly. The direction of “$\subset$” is obvious. On the other hand, if a function $f$ is in $L^2(S^4; H_0^p(S^4; K)) \cap H^{p+k}(S^4; x^s L^2(S^4; K))$, that is $V_1 \ldots V_p(f) \in L^2(M; K)$ for any $V_i \in V_e$. Then applying an elliptic k-th order differential b-operator to $f$ we obtain an element in $H^p(S^4; L^2(H^7; K))$, therefore by elliptic regularity $f \in H_{e,b}^{p,k}(M; K)$. □

**Lemma 4.12.** As $\lambda$ tends to infinity, the bounded operators $\pi_{\geq \lambda} \tilde{R}$ decay in operator norm on any Sobolev space $x^sH_{e,b}^{p,k}(M; K)$, i.e.

$$\lim_{\lambda \to \infty} ||\pi_{\geq \lambda} \tilde{R}||_{x^sH_{e,b}^{p,k}(\mathbb{H}^7; K) \to x^sH_{e,b}^{p,k}(\mathbb{H}^7; K)} = 0.$$  

**Proof.** Using Plancherel it follows that the Schwartz kernel of $\pi_{\geq \lambda} \tilde{R}$ rapidly converges to 0 in $C^\infty((\mathbb{H}^7)^2, B(x^s L^2(S^4; K)))$ and $L^2((\mathbb{H}^7)^2, B(x^s H_0^p(S^4; K)))$. Then we obtain $||\pi_{\geq \lambda} \tilde{R}|| \to 0$ as bounded operators on $x^sH_{e,b}^{p,k}(M; K)$ by the above lemma. □

As a consequence, for any fixed $s, k, l$, there is a $\lambda_0$ such that $||\pi_{\geq \lambda_0} \tilde{R}||_{x^sH_{e,b}^{k,l}(M; K)} \leq \frac{1}{2}$, and this $\lambda_0$ only depends on some $C^k$ norm. In the case that $\pi_{\geq \lambda_0} \tilde{R}$ is small, we get that $\pi_{\geq \lambda_0} dQ \pi_{\geq \lambda_0} \tilde{E}$ is a perturbation of the identity, which is still an isomorphism, that is,

**Lemma 4.13.** For any $s, k, l$, there is a $\lambda_0$ depending only on $s$, such that

$$\pi_{\geq \lambda_0} dQ \pi_{\geq \lambda_0} \tilde{E} = 1d_{\geq \lambda_0} \kappa + \pi_{\geq \lambda_0} \tilde{R}$$

where the right hand side is an isomorphism from $x^sH_{e,b}^{k,l}(M; K)$ to itself.

**Proof.** The norm of the operator on the right hand side acting on $x^sH_{e,b}^{k,l}(M; K)$ is bounded away from 0. □

The same argument applies to $\tilde{E} \circ \pi_{\geq \lambda_0} dQ \pi_{\geq \lambda_0}$. Then from the above lemma, we get that $\pi_{\geq \lambda_0} dQ$ is an isomorphism mapping from $\pi_{\geq \lambda_0} x^sH_{e,b}^{k+2,l}(M; K)$ to $\pi_{\geq \lambda_0} x^sH_{e,b}^{k,l}(M; K)$, proving the first part of Proposition 4.6. And this implies Proposition 4.4.
Proof of Proposition 4.4. From Proposition 4.6, there is a constant $C$ such that we have
\[ C^{-1} \| u \|_{x^+H^2_{e,b}((M;K))} \leq \| dQ^{λ > λ_0}(u) \|_{x^+H^0_{e,b}((M;K))} \leq C \| u \|_{x^+H^2_{e,b}((M;K))} \]
With any sufficiently small $\epsilon$, with the triangle inequality we have, for another constant $\bar{C}$ that
\[ \bar{C}^{-1} \| u \|_{x^+H^2_{e,b}((M;K))} \leq \| (dQ^{λ > λ_0} + i\epsilon)u \|_{x^+H^0_{e,b}((M;K))} \leq \bar{C} \| u \|_{x^+H^2_{e,b}((M;K))} \]
Therefore $(dQ^{λ > λ_0} + i\epsilon)^{-1}$ approaches two operators uniformly in the operator norm of $x^δ H^0_{e,b}(M;K) \to x^δ H^2_{e,b}(M;K)$ as $\epsilon \to 0^+$.

4.2. Individual eigenvalues with $λ \neq 0,16,40$. Now we consider those eigenvalues smaller than $λ_0$. Consider the projected operator
\[ dQ^λ : D_k(λ, δ) \to x^δ H^0_{e,b}(M;π_λ K), \]
which is viewed as a 0-problem on $\mathbb{H}^7$ (each tensored with fixed eigenforms on $S^4$).

For the purpose of simplicity, we denote the set of special indicial roots as
\[ \Lambda = \{0,16,40\}. \]
From Proposition 3.8, except for $λ \in \Lambda$, the indicial roots of $dQ^λ$ are contained in the range $(-∞, 3 - δ] \cup [3 + δ, ∞)$ with $δ = 1$. Moreover each pair of indicial roots is separated further when $λ$ is bigger. With this information, we show that,

Proposition 4.14. For $λ \in \Lambda$, $dQ^λ : π_λ x^δ H^s_{e,b}(M;K) \to π_λ x^δ H^{s,2}_{e,b}(M;K)$ is Fredholm for any $|δ| < δ$. Moreover, when $δ < 0$, this map is injective; when $δ > 0$, the map is surjective.

The idea of the proof essentially follows the proof of the proposition below:

Proposition 4.15 (Theorem 6.1 from [Maz91]). Suppose $L ∈ \text{Diff}^m_e(M)$ is elliptic and satisfies

1. constant indicial roots over the boundary;
2. unique continuation property of $N(L)$;
and the weight $δ$ satisfies $|δ| < δ$, then $L : x^δ H^{1+m}_e(M) \to x^δ H^1_e(M)$ is an isomorphism.

Remark 4.16. Here $dQ^λ$ is a uniformly degenerate edge operators, i.e. a 0-operator in the sense of [MM87], which satisfies both conditions (1) and (2).

We first introduce two operators related to $dQ^λ$: the normal operator and reduced normal operator.

Definition 4.17 (Normal operator). For $L ∈ \text{Diff}^m_e(M)$ the normal operator $N(L)$ is defined to be the restriction to the front face $B_{11}$ of the lift of $L$ to $M^2_e$. In terms of the local coordinate, if

\[ L = \sum_{j+|α|+|β| \leq m} a_{j,α,β}(x,y,z)(x\partial_x)^j(x\partial_y)^α\partial_z^β \]
then
\[ N(L) = \sum_{j+|α|+|β| \leq m} a_{j,α,β}(0,\tilde{y},z)(s\partial_s)^j(s\partial_u)^α\partial_z^β, \]
where \( s, u, \tilde{x}, \tilde{y}, z, \tilde{z} \) is the lifted coordinate system on \( M^2_\epsilon \) covering \( B_{11} \) such that
\[
s = \frac{x}{\tilde{x}}, \quad u = \frac{y - \tilde{y}}{\tilde{x}}.
\]

Since \( dQ^\lambda \) does not have \( \partial_z \) component, \( N(dQ^\lambda) \) acts on the half space \( \mathbb{R}^+ \times \mathbb{R}^6 \ni (s,u) \) which is the tangent space for each fixed boundary point \( \tilde{y} \in Y \). This may be further reduced to be the reduced normal operator which is a family of differential \( b \)-operators.

**Definition 4.18** (Reduced normal operator). The reduced normal operator \( N_0(L) \) is defined by applying the Fourier transform in the \( \mathbb{R}^6 \) direction to \( N(L) \) then doing a rescaling. Specifically, if we denote \( \eta \) as the dual variable to \( u \), and let \( t = s|\eta|, \tilde{\eta} = \frac{\eta}{|\eta|} \), then
\[
N_0(L) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(0, \tilde{y}, z)(t\partial_t)\hat{j}(it\tilde{\eta})^\alpha \partial_z^\beta, \quad t \in \mathbb{R}^+, \quad \tilde{\eta} \in S^*_\eta Y.
\]

Since the reduced normal operator of \( dQ^\lambda \) is independent of the fiber variables \( z \), then for fixed \( (\tilde{y}, \tilde{\eta}) \), \( N_0(dQ^\lambda) \) is an ordinary \( b \)-differential operator on \( \mathbb{R}^+_t \) and has the following mapping property:

**Lemma 4.19.** For \( \lambda \notin \Lambda \), given any fixed \( (\tilde{y}, \tilde{\eta}) \in S^6 \times S^6_{\bar{\eta}} \) and \( |\delta| < \bar{\delta} \), the reduced normal operator
\[
N_0(dQ^\lambda) : t^\delta \mathcal{H}^2(\mathbb{R}^+_t) \to t^\delta L^2(\mathbb{R}^+_t)
\]
is an isomorphism.

**Proof.** For each \( \lambda \notin \Lambda \), \( N_0(dQ^\lambda) \) has a pair of indicial roots \( 3 \pm \delta \lambda \) with \( |\delta \lambda| \geq \bar{\delta} \). Near \( t = 0 \) the operator is an ordinary \( b \)-operator controlled by the pair of indicia roots; near \( t = +\infty \), the operator is of the form \( t^2Q + O(t) \) where \( Q \) is elliptic. Therefore two parametrices can be constructed near two ends and patched together to give \( H_0 \) that gives the Fredholm property:
\[
(4.1) \quad H_0 \circ N_0 = I - P_{01}, \quad N_0 \circ H_0 = I - P_{02}
\]
such that \( P_{01} \) and \( P_{02} \) are compact. And from basic ODE theory, for weights between the two indicial roots, the operator is bijective. \( \square \)

**Lemma 4.20.** For \( \lambda \notin \Lambda \) and \( |\delta| < \bar{\delta} \), the normal operator \( N(dQ^\lambda) \) is Fredholm on \( t^\delta \mathcal{H}^0_c(M; \pi_\lambda K) \).

**Proof.** The reduced normal operator is obtained by Fourier transform and normalization of the operator \( N(dQ^\lambda) \), so we may do an inverse Fourier transform and rescaling to get the parametrix. Specifically,
\[
\hat{H}(s, \tilde{s}, \eta) = H_0(s|\eta|; \tilde{s}|\eta|; \eta)|\eta|
\]
is a bounded operator from \( s^\delta \hat{H}_c^0(M; K) \) to \( s^\delta \hat{H}_c^2(M; K) \) where in the definition of \( \hat{H}_c^* \) the differentiation \( s\partial_u \) is replaced by multiplication of \( s\eta \). And this gives \( \hat{P}_{01} \) and \( \hat{P}_{02} \) with the correct bounds. Then by doing a inverse Fourier transform
\[
N(H)(s, \tilde{s}, u, \tilde{u}) = \int e^{i(u-\tilde{u})\eta} \hat{H}(s, \tilde{s}, \eta) d\eta
\]
we obtain the normal operator for the generalized inverse
\[
N(H) : s^\delta \mathcal{H}_c^0(M; \pi_\lambda K) \to s^\delta \mathcal{H}_c^2(M; \pi_\lambda K)
\]
with corresponding compact errors $N(P_{0i})$, $i = 1, 2$. □

Then we use representation theory to show this operator is injective on any space contained in $L^2$:

**Lemma 4.21.** The kernel of the normal operator $N(dQ^\lambda)$ on $x^\delta H^2_{e,b}(M; \pi_\lambda K)$ is zero for $\delta > 0$.

**Proof.** This follows from the fact that there are no finite dimensional $L^2$ eigenspaces for functions and tensors on $\mathbb{H}^7$. Indeed, consider the representation of $SO(7, 1)$ on tensor bundles on $\mathbb{H}^7$. There are no finite dimensional $L^2$ invariant subspace of forms on $\mathbb{H}^7$ [Maz88]; and there are no $L^2$ eigentensors, from Delay’s result [Del02]. □

As a result we have:

**Lemma 4.22.** For any $\bar{\delta} > \delta > 0$, the normal operator $N(dQ^\lambda)$ is injective on $x^{\bar{\delta}} H^2_{e,b}(M; \pi_\lambda K)$ and surjective on $x^{-\delta} H^2_{e,b}(M; \pi_\lambda K)$.

**Proof.** The kernel of this map on $x^\delta H^2_{e,b}(M; \pi_\lambda K)$ is contained in $L^2$ eigenspace of forms and tensors on $\mathbb{H}^7$, which from the lemma above does not have any nontrivial elements. Therefore it is injective. By considering $N(dQ^\lambda)^*$ and duality with respect to the $L^2$ space, we can see that the operator $N(dQ^\lambda)$ is surjective on the bigger space $x^{-\delta} H^2_{e,b}(M; \pi_\lambda K)$. □

We now return to the original operator $dQ^\lambda$ and show that it is Fredholm.

**Proof of Proposition 4.14.** From $N(G)$ one can extend from the front face $B_{11}$ to $M^2_e$ and solve off the errors on the other two boundary faces using the indicial operator. This gives a correct generalized inverse (where the $b$-regularity comes from the conjugation with the operator)

$$G : x^\delta H^0_{e,b}(M; \pi_\lambda K) \rightarrow x^\delta H^2_{e,b}(M; \pi_\lambda K),$$

which shows that $dQ^\lambda$ is Fredholm.

For a general kernel element of $dQ^\lambda$, we decompose it using $SO(7, 1)$ action, so it falls into the kernel space of the normal operator $N(dQ^\lambda)$ for which there is not any. Therefore the kernel is also trivial for the operator $dQ^\lambda$. So it is injective on the smaller space, and by duality surjective on the bigger space. □

4.3. **Individual eigenvalues with $\lambda \in \Lambda$.** For those eigenvalues corresponding to indicial roots with real part equal to 3, we consider each subspace $\pi_\lambda x^{-\delta} H^2_{e,b}(M; K)$ separately. Restricted to these subspaces, the linearized operator is a 0-operator on hyperbolic space, of which the main part is the hyperbolic Laplacian $\Delta_{\mathbb{H}}$. From Melrose-Mazzeo [MM87] and Guillarmou [Gui05], the resolvent of $\Delta_{\mathbb{H}} - \lambda$, denoted as $R(\lambda)$, extends to a meromorphic family with finite degree poles. Similarly, we want to show that $dQ$ has two generalized inverses $R_\pm$, which is the limit of the resolvent $(dQ^\pm i\epsilon)^{-1}$ when $\epsilon \downarrow 0$ that extends to the spectrum, and this contains the boundary data for the kernel of $dQ$.

First we show that the indicial roots become separated from the $L^2$ line by adding an imaginary perturbation.

**Lemma 4.23.** For $\lambda \in \Lambda$ and any $\epsilon > 0$, the two indicial roots of operator $dQ^\lambda \pm i\epsilon$ lie off the $\text{Re}(s) = 3$ line.
Proof. Suppose \( s \in \mathbb{C} \) is an indicial root for an operator \( P \) on a point \( p \) at the boundary, then we have \( P(x^s) = O(x^{s+1}) \) by definition. For \( \epsilon \neq 0 \), the following computation shows that \( s \) is no longer an indicial root: \((P + i\epsilon)(x^s) = \epsilon x^s + O(x^{s+1}) \neq O(x^{s+1})\). Instead, take the harmonic 4-form part which has indicial roots \( 3 \pm 6i \) which in the indicial root computation is \( dQ^\lambda(x^{3+\theta}) = (\theta^2 + 36)x^{3+\theta} + O(x^4) \), after perturbation it becomes

\[
(dQ^\lambda + i\epsilon)(x^{3+\theta}) = (\theta^2 + i\epsilon + 36)x^{3+\theta} + O(x^4)
\]

so the indicial equation becomes \( \theta^2 = -i\epsilon - 36 \) which moves the two roots \( 3 \pm \theta_1 \) off the line of \( \text{Re}(s) = 3 \). A similar argument applies to other two pairs of roots \( 3 \pm \theta_i, \ i = 2, 3 \).

Lemma 4.24. For \( \lambda \in \Lambda \) and any sufficiently small \( \epsilon > 0 \), the inverse \((dQ^\lambda \pm i\epsilon)^{-1}\) : \( x^\delta H^{2,k}_{e,b}(M; \pi \lambda K) \to D_k(\lambda, \delta) \subset x^{-\delta} H^{0,k}_{e,b}(M; \pi \lambda K) \) exists as a bounded operator.

Proof. Using the indicial roots separation and same argument as in Proposition 4.14, the operator \( dQ^\lambda \pm i\epsilon \) is Fredholm on \( x^\delta H^{2,k}_{e,b}(M; \pi \lambda K) \), injective on the smaller space and surjective on the larger space.

Moreover with the limiting absorption principle below we show that the bound is uniform with respect to \( \epsilon \downarrow 0 \). Consider the reduced normal operator of \( dQ^\lambda + i\epsilon \), which is a differential operator (parametrized by \( y \) and \( \epsilon \)), is injective from \( x^\delta H^2(\mathbb{R}^+) \to x^\delta L^2(\mathbb{R}^+) \) for any fixed \( \delta > 0 \). This ODE operator may be extended holomorphically as \( \epsilon \) approaches zero from above, and the solution of the ODE extends holomorphically as well. However after extending \( \epsilon \) past zero, the smaller indicial root becomes the larger one, which is excluded from the solution. This is reflected in the resolvent \( R^\lambda_c := \lim_{\epsilon \downarrow 0} (dQ^\lambda + i\epsilon)^{-1} \) as the expansion of \( R^\lambda_c u \) for \( u \in x^\delta H^{0,k}_{e,b}(M; K) \) has only the indicial roots. And similarly for the other direction, the expansion of \( R^\lambda u \) only has the other half of the indicial roots.

Proposition 4.25 (Limiting absorption principle). For \( 0 < \delta < \delta_0, \lambda \in \Lambda, \) and \( \epsilon > 0 \) the operators \((dQ^\lambda \pm i\epsilon)^{-1}\) converges uniformly to bounded operators on weighted Sobolev spaces,

\[
\lim_{\epsilon \downarrow 0} \|(dQ^\lambda \pm i\epsilon)^{-1} - R^\lambda_c\|_{x^\delta H^{0,k}_{e,b}(M; \pi \lambda K) \to x^{-\delta} H^{2,k}_{e,b}(M; \pi \lambda K)} = 0.
\]

Proof. Since we are considering the operator after projection acting on sections of edge spaces, which is the Beltrami-Laplace operator acting on 0-Sobolev spaces on hyperbolic space \( \mathbb{H}^7 \). The result can be directly obtained by using commutator with b-vector field if we know

\[
\lim_{\epsilon \downarrow 0} \|(\Delta_{\mathbb{H}^7} - s(6 - s) \pm i\epsilon)^{-1} - R(s^{\pm})\|_{x^\delta L^2(\mathbb{H}^7) \to x^{-\delta} H^2(\mathbb{H}^7)} = 0.
\]

The proof is essentially contained in [MM87] with the description of the kernel of the resolvent \( R(s) = (\Delta_{\mathbb{H}^7} - s(6 - s))^{-1} \) for \( s \) approaching \( \text{Re}s = 3 \) which corresponds to the continuous spectrum. Here we use proposition 6.2 in [MM87] that the kernel of \( R(s) \) decays as \( x^{\text{Re}s(\text{Re}s)} \) \( s(6 - s) \) on the boundary of the double space where \( q(x, y, x', y') \) is a smooth function on \( [x, y) - (x', y') \). To show the bounds, we use the fact that \( x^3 L^2(\mathbb{H}^7; x^{-1}dx dy) = L^2(\mathbb{H}^7; x^{-7}dx dy) = L^2(\mathbb{H}^7) \).

Therefore if we consider the kernel \( \bar{R}(s) \) on \( L^2 BB^7; x^{-3}dx dy \), then \( \bar{R}(s, x, x') = \)
\[ x^{-3}R(s, x, x')(x')^{-3}. \] For fixed \( \delta > 0 \) and any \( |\text{Re } s - 3| << 1 \),
\[
\sup_{s' \in \mathbb{S}^7} \int_{x, y} x^\delta \hat{R}(s, x, x', y') x^{-1} dx \, dy < C,
\]
\[
\sup_{s' \in \mathbb{S}^7} \int_{x, y} x^\delta \hat{R}(s, x, x', y') x^{-1} dx \, dy' < C,
\]
where \( C \) does not depend on \( \epsilon \). Then by Schur’s lemma \( \hat{R}(s) : x^\delta L^2(\mathbb{S}^7; x^{-1} \, dx \, dy) \rightarrow x^{-3} L^2(\mathbb{S}^7; x^{-3} \, dx \, dy) \) is bounded in the operator norm for \( s \) approaching \( s^k \). Transform back to the hyperbolic space, \( R(s) : x^\delta L^2_0(\mathbb{H}^7) \rightarrow x^{-3} L^2_0(\mathbb{H}^7) \) is uniformly bounded. The bound on \( x^\delta L^2_0(\mathbb{H}^7) \rightarrow x^{-3} L^2_0(\mathbb{H}^7) \) follows from ellipticity by commuting with the 0-elliptic operator \( \Delta_{\mathbb{H}^7} \).

\[ \square \]

### 4.4. Boundary data parametrization

Combining the analysis for \( \lambda \) off the \( L^2 \) line and on the \( L^2 \) line, we conclude:

#### Proposition 4.26

For \( \delta \in (0, \delta_0) \) and a product metric \( h \times \frac{1}{4} g_{g_{4}} \) on \( M \) with \( h \) being sufficiently close to hyperbolic metric, there are two generalized inverses \( R_{\pm} : x^\delta H^{0, k}_{e,b}(M; K) \rightarrow x^{-3} H^{2,k}_{e,b}(M; K) \) for operator \( dQ \), such that

\[
dQ \circ R_+ = Id, dQ \circ R_- = Id : x^\delta H^{4,k}_{0,b}(M; K) \rightarrow x^\delta H^{0,k}_{e,b}(M; K).
\]

**Proof.** When \( h \) is the hyperbolic metric, we just need to combine result from Proposition 4.4, 4.14, and 4.25. When \( \| h - g^H \|_{H^1(\mathbb{H}^7)} << 1 \), one can choose a boundary defining function on \( \mathbb{S}^7 \) such that the normal operator \( N(dQ) \) is the same as before, therefore is invertible. This implies that \( dQ \) is Fredholm. The injectivity of \( dQ \) is obtained by considering the finite dimensional \( L^2 \) subspaces which by perturbation from \( \mathbb{S}^7 \) there is not any.

As a consequence, we define the following right inverse

\[(dQ)^{-1} := \frac{1}{2}(R_+ + R_-),\]

with the property that \( dQ \circ (dQ)^{-1} = Id_{x^{-3} H^{0,k}_{0,b}(M; K)} \) and \( (dQ)^{-1} \) being a real-valued operator.

To get the main theorem, we will parametrize the domain by the boundary data, which amounts to show that there is a Poisson operator that maps boundary data into the kernel space. From the analysis in previous sections, we know that the only nontrivial kernel comes from the three pairs of special indicial roots, and therefore it is a geometric scattering problem. For the hyperbolic space, the scattering operator for Laplacian operators on functions and forms have been studied in various settings [MM87, Maz88, MP90, Mel95, GZ03, Gui05, Gui06, Lee06]. In the context of \( dQ^\lambda \), as we see from the computations in appendix B the three special cases all turn into problems of \( (\Delta_{\mathbb{H}^7} - s_\lambda) \) on functions and forms within the continuous spectrum, therefore we will use scattering operators to parametrize the kernels. We start with the base case with the metric \( g^H \times \frac{1}{4} g_{g_{4}} \).

#### Lemma 4.27

The real null space \( \text{Null}(dQ) \subset D_k(\delta) \) is parametrized by \( \oplus V_i \) defined in Definition 1.3. There is a Poisson operator \( P : H^k(\mathbb{S}_i^6; V) \rightarrow x^{-3} H^{2,k}_{e,b}(M; K) \) such that \( dQ \circ P = 0 \) and for any \( v \in V \), \( P(v) \) has the expansion as in (14).

**Proof.** For the construction of the Poisson operator, we follow Graham–Zworski [GZ03] and Guillarmou [Gui05] and first construct a formal solution operator \( \hat{P} \) by the standard asymptotic method. From the indicial root computation, given the terms
consisting of \( v = (v^+_1, v^+_2, v^+_3) \) in (14), if we write \((k_0, H_0)\) by adding to the scattering part and extend it from the boundary by a cut-off function, then
\[
dQ(k_0, H_0) = I(dQ)(k_0, H_0) + O(x^{3+\delta}) \in x^\delta H_{c,b}^0(M; K)
\]
then one can solve the subsequent terms by iteratively constructing the expansion. By Borel’s lemma, we arrive at a formal solution \( \tilde{P}(v) = (\tilde{k}, \tilde{H}) \) with \( dQ(\tilde{k}, \tilde{H}) = O(x^{\infty}) \) where \((\tilde{k}, \tilde{H})\) has the same leading order expansion. Then following proposition 3.4 in [GZ03] there is a unique Poisson operator defined as
\[
(4.3) \quad P = (I - R_+ \circ dQ) \circ \tilde{P}
\]
with the correct mapping property.

To show that the resulting kernel \( P(v) \) is real, we use the description of scattering matrix in hyperbolic space in Guillarmou–Naud [GN06]:
\[
S(s) = 2^{n-2s} \frac{\Gamma(n/2-s) \Gamma\left(\sqrt{s+\frac{n-1}{2}}^2 + \frac{1-n}{2} + s\right)}{\Gamma(s-\frac{n}{2}) \Gamma\left(\sqrt{s+\frac{n+3}{2}}^2 + \frac{n+1}{2} - s\right)}
\]
where if we put in \( s = \theta_2^+ \) denoted as \( 3 + i\alpha \), we get the scattering operator
\[
S(3 + i\alpha) = 2^{-2n\alpha} \frac{\Gamma(-i\alpha) \Gamma\left(\sqrt{\frac{2\theta_2 + 1}{4} + \frac{1}{2} + i\alpha}\right)}{\Gamma(i\alpha) \Gamma\left(\sqrt{\frac{2\theta_2 + 1}{4} + \frac{1}{2} - i\alpha}\right)}
\]
Since the scattering matrix is a function of the Laplacian on the boundary \( S^6 \) we can take the eigenvalue expansion on 6-sphere with real eigenform \( f_\lambda \), we would consider the following expression, which is real and forms the leading order of the actual solution:
\[
x^{3+i\alpha} f_\lambda + x^{3-i\alpha} S(3 + i\alpha) f_\lambda = x^{3+i\alpha} f_\lambda + x^{3-i\alpha} (2^{-2n\alpha} e^{i\theta} \lambda^{i\alpha}) f_\lambda.
\]
Here \( \lambda \) is a real number determined by
\[
\gamma^{2\theta(\lambda)} = \frac{\Gamma(-i\alpha) \Gamma\left(\sqrt{\lambda + \frac{2\theta_2}{4} + \frac{1}{2} + i\alpha}\right)}{\Gamma(i\alpha) \Gamma\left(\sqrt{\lambda + \frac{2\theta_2}{4} + \frac{1}{2} - i\alpha}\right)}
\]
by using the relation of
\[
\Gamma(\bar{z}) = \overline{\Gamma(z)}
\]
so that the right hand side of (4.4) is a complex number with norm 1 and \( \theta \) is a real number determined by \( \lambda \).

Rearranging the expression, the solution in the eigenvalue \( \lambda \) component is
\[
(4.5) \quad \pi_\lambda u = x^{3+i\alpha} f_\lambda + x^{3-i\alpha} 2^{-2n\alpha} e^{2\theta} f_\lambda
\]
\[
(4.6) \quad = x^{3} 2^{-i\alpha} e^{i\theta} \left( (2x)^{i\alpha} e^{i\theta} + (2x)^{-i\alpha} e^{i\theta} \right) f_\lambda
\]
\[
(4.7) \quad = x^{3} 2^{1-i\alpha} e^{i\theta} \text{Re} \left( (2x)^{i\alpha} e^{i\theta(\lambda)} \right) f_\lambda
\]
which is a product of a real 3-form with complex constant \( x^{3} 2^{1-i\alpha} e^{i\theta} \). Therefore in this case,
\[
f = |f| e^{it}, t \in \mathbb{R},
\]
and \( f \) is a real function. The same argument applies to \( \theta_3^+ \). For \( \theta_4^+ \) we use the scattering matrix for forms on hyperbolic space and get the same conclusion. \( \square \)
And in the exact case with hyperbolic space, we can characterize the range of the Poisson operator using the two resolvents.

**Lemma 4.28.** The range of the Poisson operator \( P \) acting on \( H^k(S^6; V) \) is the same as the range of \( i(R_+ - R_-) \) acting on \( x^d H^{0,k}_{e,b}(M; K) \).

Proof. By Stone’s theorem, see for example (4.4) in [Gui06], the difference of \( R_+ \) and \( R_- \) is the spectral projector of \( dQ \). For any element \( u \in x^d H^{0,k}_{e,b}(M; K) \), \( dQ \circ (R_+ - R_-)u = 0 \). Since \( R_+ - R_- \) is imaginary, the range of \( i(R_+ - R_-) \) is the real null space in \( x^{-d} H^{2,k}_{e,b}(M; K) \), therefore is the same as the range of \( P \). □

Now we consider the perturbation from the base hyperbolic metric to a nearby Poincaré-Einstein metric \( h \). As discussed before, we still have the two generalized inverses.

**Lemma 4.29.** For a Poincaré–Einstein metric \( h \) that is closed to the background metric \( g_0 \) with \( \| h - g_0 \|_{H^1(B^7)} << 1 \), the range space of the sum of two generalized inverses \( R_+ \) is transversal to the range of their difference: \( \text{Range}(R_+ + R_-) \) is transversal to \( \text{Range}(R_+ - R_-) \).

Proof. For the hyperbolic case, the range of \( (R_+ - R_-) \) is the kernel of \( dQ \). However, the range of \( R_+ + R_- \) doesn’t contain any element of the kernel. Otherwise there will be \( 0 \neq u \in x^d H^{0,k}_{e,b}(M; K) \) with \( dQ \circ (R_+ + R_-)u = 0 \), then since \( dQ \circ (R_+ - R_-)u = 0 \), this contradicts with \( dQ \circ R_+ = Id, \quad dQ \circ R_- = Id \). And for any \( f \in D_k(\delta) \), consider \( \hat{f} \) the projection of \( f \) off the kernel \( f \) of \( dQ \), then \( \frac{1}{2}(R_+ + R_-)dQ(\hat{f}) = \hat{f} \). Therefore the two range spaces are transversal in \( D_k(\delta) \).

Since transversality is stable under small perturbations, the result follows for nearby Poincaré-Einstein metrics.

The Poisson operator defined in (4.3) (with respect to the hyperbolic metric) exists for nearby Poincaré-Einstein metric as well. However, \( P \) maps to a real element in the domain which is not necessarily an element in the null space, but a perturbation of the kernel. That is, the range of \( P \) is a perturbation of the range of \( i(R_+ - R_-) \) such that the leading term expansion is the same. With this we conclude:

**Proposition 4.30.** The range of the Poisson operator \( P \) is transversal to the range of \( (dQ)^{-1} = (R_+ + R_-) \) acting on \( x^d H^{2,k}_{e,b}(M; K) \).

Proof. Since the range of \( i(R_+ - R_-) \) is transversal to the range of \( (R_+ + R_-) \) by lemma 4.29, the transversality in the statement follows from that the range of \( P \) is close to the range of \( i(R_+ - R_-) \). □

5. **Solvability of the nonlinear operator**

From the discussion of the linear operator \( dQ \) above, we now can apply the implicit function theorem to get results for the nonlinear operator. To do this we first need to show that the nonlinear terms are controlled. Then we will use a perturbation argument to show that for each solution with Poincaré–Einstein metric close to hyperbolic metric, the nearby solutions are parametrized by the three parameters on \( S^6 \) as in the linear case.

To deal with the fact that the domain changes with the base metric and the boundary parameters, we will use an implicit function theorem for a map from
range space to itself, and show this map is a perturbation of identity, therefore an isomorphism.

First of all we define the domain that depends on the choice of the base Poincaré–Einstein metric $h$ and the boundary parameter $v = (v_1, v_2, v_3) \in V$. From proposition 4.30, we know that the image of $(dQ)^{-1} = \frac{1}{2}(R_+ + R_-)$ is transversal to the image of the Poisson operator $P$, which for a nearby Poincaré–Einstein metric is close to (but not equal to) the kernel of the linearized operator. For each fixed parameter $v$, we define the domain as an affine section of $(dQ)^{-1}(x^\delta H_{e,b}^{0,k}(M; K))$ translated by $Pv$.

**Definition 5.1.** (Domain of nonlinear operator) For a Poincaré–Einstein metric $h$ with $\|h - g_{\mathbb{R}^7}\|_{H^1(\mathbb{R}^7)} << 1$ and a set of parameters $v = (v_1, v_2, v_3) \in V$, the domain $D_{h,v}$ for the nonlinear operator $Q$ is defined as

$$D_{h,v} := \left\{ (dQ)^{-1}f + Pv \mid f \in x^\delta H_{e,b}^{0,k}(M; K) \right\}.$$ 

Note that the domain depends on the choice of $h$ and $v$, where the dependence of $h$ comes from the construction $\frac{1}{2}(R_+ + R_-) = (dQ)^{-1}$. Because of the transversality from 4.30, $D_{h,v}$ can be viewed as a slice in $D_k(\delta)$:

$$\cup_{v \in V} D_{h,v} = D_k(\delta).$$ 

The domain has the property that, if $h = g_{\mathbb{R}^7}$ is the hyperbolic metric, then each slice $D_{h,v}$ is mapped by $dQ$ isomorphically back to the range space $x^\delta H_{e,b}^{0,k}(M; K)$ where the kernel in each slice is exactly $Pv$.

For nearby metric $h$, one important property of this domain is that $D_{h,v}$ is mapped surjectively to the range space $x^\delta H_{e,b}^{0,k}(M; K)$ by the linear operator $dQ_h$.

**Lemma 5.2.** Acting on the domain defined in 5.1, the linear operator

$$dQ_h : D_{h,v} \to x^\delta H_{e,b}^{0,k}(M; K)$$

is a surjective map.

**Proof.** By direct computation, for any $f \in x^\delta H_{e,b}^{0,k}(M; K)$ and $v \in V$,

$$dQ_h \left( \frac{1}{2}(R_+ + R_-)f + Pv \right) = dQ_h(dQ_h)^{-1}f + dQ_h(Pv) = f + dQ_h(Pv).$$

Here we used the fact that $R_+$ and $R_-$ are both right generalized inverses for $dQ_h$. And by definition $dQ_h(Pv) \in x^\delta H_{e,b}^{0,k}(M; K)$ for any $v$. Since $f$ can be any element in the vector space, it follows the range of $dQ_h$ acting on $D_{h,v}$ is the whole space. \hfill $\square$

On this domain we define a nonlinear operator $Q_{h,v}$ that can be viewed as a translation of the original operator $Q_{h,0}$.

**Definition 5.3.** We define the parametrized nonlinear operator $Q_{h,v}$ as:

$$Q_{h,v} : D_k(\delta) \to x^\delta H_{e,b}^{0,k}(M; K), \ u \mapsto Q_{h,0}(u + Pv).$$

Next we show that the nonlinear terms are well controlled, the nonlinear operator $Q$ maps $D_{h,v}$ to $x^\delta H_{e,b}^{2,k}(M; K)$. This is proved by showing that the difference of $Q$ and $dQ$ is small. And we only need to consider the action on the $x^\delta H_{e,b}^{2,k}(M; K)$ since the only part that has a worse decay is eliminated by $dQ$. 


Lemma 5.4. For sufficiently large $k$, the product type nonlinear terms: $F \circ F - d(F \circ F)$, and $F \wedge F - d(F \wedge F)$ are both contained in $x^\delta H^{2,k}_{e,b}(M;K)$.

Proof. The nonlinear parts are $F \wedge F$ and $F \circ F$ which are products of two elements in the range space $x^\delta H^{2,k}_{e,b}(M;K)$. Take a basis of the edge bundles, these can be considered locally as functions in $x^\delta H^{2,k}_{e,b}(M)$. Using proposition A.3, we know that for $r > -3$, and sufficiently large $k$, and any $f, g \in x^\delta H^{s,k}_{e,b}(M)$, the product $fg$ is also in $x^\delta H^{s,k}_{e,b}(M)$. Since in our case $\delta > 0$, the result follows. \hfill \Box

The other nonlinear term is the remainder from the linearization of Ric operator, for which we show below that it is also contained in the range space.

Lemma 5.5. The nonlinear remainder of Ric, $\text{Ric} - d(\text{Ric})$ acting on $k \in x^\delta H^{0,k}_{e,b}(M;K)$ is contained in $x^\delta H^{0,k}_{e,b}(M;K)$.

Proof. We compute the linearization $d(\text{Ric})$, which acting on a 2-tensor $h$ can be written as

$$d(\text{Ric})[h] = -\frac{1}{2}g^{ml}(\nabla_m \nabla_l h_{jk} - \nabla_m \nabla_k h_{jl} - \nabla_l \nabla_j h_{mk} - \nabla_j \nabla_k h_{ml}).$$

Comparing Ric and $d(\text{Ric})$, the difference is a $3rd$ order polynomial of $g, g^{-1}$ and first order derivatives of these with smooth coefficients. Since the metric component $g$ and $g^{-1}$ are smooth, hence in $x^\delta H^{s,k}_{e,b}(M;K)$, it follows again by the algebra property that their product is contained in $x^\delta H^{s,k}_{e,b}(M;K)$. \hfill \Box

As a translation of the original operator, the linearization of $Q_{h,v}$ is closely related to the original linearized operator $dQ$:  

Lemma 5.6. The linearization of $Q_{h,v}$ at $(h \times \frac{1}{6} g_{33}, 6 \text{Vol}_{3M})$ is the same as $dQ_{h,0}$:

$$dQ_{h,v}(\cdot) = dQ_{h,0}(\cdot).$$

Proof. Since $Q_{h,v}$ is defined as a translation of $Q_{h,0}$ by $Pv$, and using the fact that the nonlinear terms are all quadratic,

$$dQ_{h,v}(u) = dQ_{h,0}(\cdot + Pv)(u) = dQ_{h,0}(u) = dQ_{h,0}(u).$$

The composed operator $Q_{h,v} \circ (dQ)^{-1}$ is this well-defined operator as a map on the following space:

$$Q_{h,v} \circ (dQ)^{-1} : x^\delta H^{0,k}_{e,b}(M;K) \rightarrow x^\delta H^{0,k}_{e,b}(M;K).$$

$$f \mapsto Q_{h,0}\left(\frac{1}{2}(R_+ + R_-)f + Pv\right)$$

We now discuss the properties of this operator using the following implicit function theorem, which can be found for example in [Lan99].

Lemma 5.7 (Implicit function theorem). Consider the following smooth map $f : V \times M \rightarrow M$ near a point $(v_0, m_0) \in V \times M$ with $f(v_0, m_0) = c$, if the linearization of the map with respect to the second variable $df_2(v_0, m_0) : M \rightarrow M$ is an isomorphism, then there is neighborhood $v_0 \in U \subset V$ and a smooth map $g : V \rightarrow M$, such that $f(v, g(v)) = c, \forall v \in U$. 

Theorem 1. For $k \gg 0$, $\delta \in (0, \delta)$, there exists $\rho > 0$ and $\epsilon > 0$, such that, for a Poincaré–Einstein metric $h$ that is sufficiently close to the base metric $g_{37}$ with $\|h - g_{37}\|_{H^1(S^7)} < \epsilon$, and any boundary value perturbation $v \in V$ with $\|v\|_{H^1(S^7; V)} < \rho$, there is a unique solution $u = (g, H) \in D_{e,h} \subset x^{-\delta}H^{s,k}_{e,b}(M; K)$ satisfying the gauged supergravity equations $Q(u) = 0$ with the leading expansion of $(g - h \times \frac{1}{2}g_{37}, H - 6\text{Vol}_{37})$ given by (14).

To prove the theorem, we will apply the implicit function theorem to the following operator:

$$Q_h \circ (dQ)^{-1} : H^k(S^6; V) \times x^\delta H^{0,k}_{e,b}(M; K) \to x^\delta H^{0,k}_{e,b}(M; K)$$

$$(v, f) \mapsto Q_h \circ (dQ)^{-1} f$$

From the previous discussion, this map is well defined. The following is a consequence of Lemma 5.6.

Lemma 5.8. The linearization of $Q_h \circ (dQ)^{-1}$ at point $(v, f) = (0, 0) \in V \times x^\delta H^{0,k}_{e,b}(M; K)$ is an isomorphism.

Proof. From Lemma 5.6 we know that at the point $(v, f) = (0, 0) \in V \times x^\delta H^{0,k}_{e,b}(M; K)$ the linearization, which is the composition of linearizations, is

$$d(Q_h \circ (dQ)_{h,0})^{-1}(0, 0) = \text{Id} : x^\delta H^{0,k}_{e,b}(M; K) \to x^\delta H^{0,k}_{e,b}(M; K).$$

Lemma 5.9. For a given metric $h$, the map $Q_h \circ (dQ)^{-1}$ as an edge operator varies smoothly with the parameter $v \in V$.

Proof. From the construction of $(dQ)^{-1}$ we know it is an edge operator. And from the discussion for $Q_{h,0}$, this nonlinear operator is also edge. Now we we only need to show that when the nonlinear operator $Q$ applies to elements of type $f + P_v$, it varies smoothly with the parameter $v$. This follows from the algebra property and the fact that a second order elliptic edge operator maps from $H^s_x(M)$ to $H^{s-2}_x(M)$ smoothly as shown in proposition A.2.

Now as a direct result of the implicit function theorem, we now prove the main theorem:

Proof of Theorem 1. Using the implicit function theorem, we can find neighborhoods of $0 \in V$ and $f \in x^\delta H^{0,k}_{e,b}(M; K)$, in this case, $U_1 = \{v \in V : \|v\|_{H^k(S^6; V)} < \rho\}$ and $U_2 = \{f \in x^\delta H^{0,k}_{e,b}(M; K) : \|f\|_{x^\delta H^{0,k}_{e,b}(M; K)} < \rho_2\}$, such that the nonlinear map $Q_{h,v} \circ (dQ_{h,0})^{-1}$ is a bijective smooth map on $U_2$ for any $v \in U_1$. And this gives us the parametrized map $g$ from $U_1$ to $U_2$ such that

$$Q_{h,v}(dQ_{h,0})^{-1}(g(v)) = 0.$$  

And we can rewrite it as

$$Q_{h,0}(dQ_{h,0})^{-1}(g(v)) + P_v = 0.$$  

That is, for each parameter set $v$, $u = (dQ_{h,0})^{-1}(g(v)) + P_v$ is the unique solution in the space $D_{e,h} \subset x^{-\delta}H^{s,k}_{e,b}(M; K)$. When the base metric is $g_{37}$, since the nonlinearity is quadratic, the leading order expansion of the solution is given by $P_v$. By continuity, $(dQ_{h,0})^{-1}(g(v))$ is contained in $O(x^{3+\delta})$ when $\|g - h\|$ and $\|v\|$ is
sufficiently small. Therefore the leading order behavior is again given by $Pv$ which is (14).

Next we show that the solution obtained above is smooth if the boundary data is smooth.

**Proposition 5.10.** If the boundary data $v \in C^\infty(\mathbb{S}^6; V)$, then the solution $u$ is in $C^\infty(M; K)$.

**Proof.** This is done by elliptic regularity. For any $k$,

$$
\|u - \left(h \times \frac{1}{4} g_{\mathbb{S}^4}, 6 \text{Vol}_{\mathbb{S}^4}\right)\|_{x^{-k} H^{2, k}_{e,b}(M; K)} \leq C(\|v\|_{H^k(\mathbb{S}^6; V)} + \|Q(u)\|_{x^k H^{0, k}_{e,b}(M; K)}).
$$

For the linearized operator we use the elliptic estimate for edge operators, and the difference with nonlinear operator is lower order therefore can be controlled by $\|u\|_{x^k H^{0, k}_{e,b}(M; K)}$ so it is absorbed to the left hand side. Since the estimate holds for every $k$ we get the smoothness of $u$. \qed

We can also obtain the polyhomogeneous expansion of the solution.

**Proposition 5.11.** When the boundary data $v \in C^\infty(\mathbb{S}^6; V)$, the solution $u$ has a classical polyhomogeneous expansion in the sense of [Mel93], with leading terms given by (14) and the exponent of the logarithmic terms grows linearly with the order.

**Proof.** We solve the problem iteratively to obtain a formal expansion. For the first order problem, from the linearization and its inverse construction, we have $u_1$ as in (14) and

$$
Q(u_1) = x^{3+\delta} e_1, \quad e_1 \in C^\infty(M; K).
$$

Then we solve away the $x^{3+\delta} e_1$ term and one log term appear because the appearance of indicial roots. Then iteratively we obtain the terms

$$
u_j = x^{s_j} \left(\sum_{i=0}^{\infty} x^i u_{i,j}\right), \quad u_{i,j} \in (\log x)^i C^\infty(M; K),$$

with $s_j$ being the indicial roots bigger than 3 and each time the power of log increases by at most one with the power of $x$. \qed

Finally we prove the main theorem for the original supergravity operator.

**Proof of Theorem.** From proposition 2.6, there is a diffeomorphism $g \rightarrow \tilde{g}$ such that we obtain the solution to the original supergravity equations $S(\tilde{g}, H) = 0$. The parametrization of solutions $(g, H)$ to the gauged equation is given in Theorem 1. The regularity of $(\tilde{g}, H)$ is the same as $(g, H)$ because of the diffeomorphism. Since $\tilde{g}$ and $g$ differ by a lower order term $O(x^{3+\delta})$, they have the same leading order expansion. \qed

**Appendix A. Edge operators**

**Proposition A.1.** $H^{s,k}_{e,b}(M)$ is defined independent of the order of applying edge- and b-vectors.
Proof. We prove it by induction. Take \( s = k = 1 \), using the commutator relation 
\( [V_c, V_b] \subset V_b \), we have 
\[ V_c V_b u = V_b V_c u + V'_b u, \quad V_c \in V_c, \quad V_b, V'_b \in V_b. \]
Therefore \( V_c V_b u \in L^2(M) \) if and only if \( V_b V_c u \in L^2(M) \) (since \( u \in H^1_b(M) \) is implied by both sides.) For \( s, k > 1 \), for an arbitrary order of vector fields applied to \( u \), we use the commutator to reduce to the sum \( \sum_{i=0}^{s} V'^i_b V^i_c u \) and use the induction that \( H_e^{s,k} \subset H_e^{1,k} \) for any \( i < s \).

\[ \square \]

Proposition A.2. Any \( m \)-th order edge operator \( P \) maps \( H_{e,b}^{s,k}(M) \) to \( H_{e,b}^{s-m,k}(M) \), for \( m \leq s \).

Proof. Locally, any \( m \)-th order edge operator \( P \) can be written as 
\[ P = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x,y,z)(x\partial_x)^\alpha(y\partial_y)^\beta(z\partial_z)^\beta \]
If we can prove for \( m = 1 \), \( P \) maps \( H_{e,b}^{s,1}(M) \) to \( H_{e,b}^{s-1,1}(M) \), then by induction we can prove for any \( m \). Therefore we restrict to the case \( m = 1 \).

We just need to check that, for a function \( u \in H_{e,b}^{s,1}(M) \), \( Pu \) satisfies 
\[ V^i_b Pu \in H^1_b(M), \quad 0 \leq i \leq s - 1. \]

The we prove the proposition by induction on \( k \). For \( k=1 \) case, since a boundary vector field \( V \in V_b(M) \) satisfies the commutator relation \( VP = PV + [V,P] \) where the Lie bracket \( [V,P] \in V_b \), then 
\[ VP(u) = PV(u) + V_b(u) \]
by definition of \( u \in H_{e,b}^{s,1} \), both \( V(u) \) and \( V_b(u) \) are in \( H^s_b(M) \), therefore \( PV(u) \in H^{s-1}_e(M) \).

If it holds for \( k-1 \), then by the relation 
\[ V^i_b P(u) = V^{i-1}_b PV_b(u) + V^i_b(u), \]
since \( V_b(u) \in H_{e,b}^{s,k-1} \) and from induction assumption \( PV_b(u) \in H_{e,b}^{s-1,k-1} \), therefore the first term \( V^{i-1}_b PV_b(u) \in H^{s-1}_e(M) \), and the second term is in \( H^s_e(M) \) by definition. Therefore \( Pu \in H_{e,b}^{s-1,k-1} \), which completes the induction.

\[ \square \]

Proposition A.3. For sufficiently large \( k, r \geq -3 \), and any \( s \in \mathbb{R} \), \( x^r H_{r,b}^{s,k}(M) \) is an algebra.

Proof. We first prove that, for the case \( r = -3 \), the \( b \)-Sobolev space \( x^{-3} H^k_b(M) \) is an algebra for sufficiently large \( k \). Working in the upper half plane model with coordinates \((x, y_1, \ldots, y_n, z_1, \ldots, z_{n'})\). For any element \( f \in x^{-3} H^k_b(M) \), by definition, its Sobolev norm is 
\[ \int_{\mathbb{R}^+ \times \mathbb{R}^{n+n'}} |V^k_b(x^3 f)|^2 x^{-7} dx dy dz \]
Since the commutator relation satisfies \([V_b, x^3] f = x^3 V_b f + 3 x^3 f\), the definition of the Sobolev norm above is the same as 
\[ \int |x^3 (V^k_b f)|^2 x^{-7} dx dy dz \]
We do a coordinate transformation to change the problem back to \( \mathbb{R}^m \) with \( m = 1 + n + n' \); let \( \rho = \ln(x) \), then \( x\partial_x = \partial_\rho \). Therefore under the new coordinates, the
b-vector fields are spanned by \((\partial_\rho, \partial_\gamma, \partial_\zeta)\). Let \(F\) be the function after coordinate transformation

\[ F(\rho, \gamma, \zeta) = f(e^\rho, \gamma, \zeta) \]

then from the discussion above we can see the norm for \(x^{-3}H_b^k(M)\) is characterized by

\[ \|f\|_{x^{-3}H_b^k}^2 = \int_{\mathbb{R}^+ \times \mathbb{R}^{n+3}} |x^3(V_b f)|^2 x^{-7} dxdydz = \int_{\mathbb{R}^m} |V_b^3 F|^2 dxdydz < \infty \]

which means \(F \in H^k(\mathbb{R}^m)\). From [Tay11], the usual Sobolev space \(H^k(\mathbb{R}^m)\) is closed under multiplication if and only if \(k > \frac{n}{2}\). Therefore, take two elements \(f, g \in x^{-3}H_b^k(M)\), then the corresponding functions in \(\mathbb{R}^m\) satisfy \(FG \in H^k(\mathbb{R}^m)\). It follows that \(fg \in x^{-3}H_b^k(M)\) by taking the inverse coordinate transformation.

Then it is easy to see that \(x^r H_b^k(M)\) is an algebra for \(r > -3\) from the result above:

\[ (x^r H_b^k) \cdot (x^s H_b^k) = x^{3+r}(x^{-3}H_b^k) \cdot x^{3+r}(x^{-3}H_b^k) \]
\[ \subset x^{6+2r}(x^{-3}H_b^k) \subset x^r H_b^k(M). \]

Now that we proved \(x^r H_b^k(M)\) is closed under multiplication, then we want to prove \(x^r H_{e,b}^{s,k}(M)\) is also an algebra for any \(s\). For any functions \(f, g \in x^r H_{e,b}^{s,k}(M)\), by Leibniz rule,

\[ V_e^j(fg) = \sum_{i=0}^{j} V_e^j(f)V_e^{j-i}(g) \]

where by assumption, both \(V_e^j(f)\) and \(V_e^{j-i}(g)\) are in \(x^r H_b^k(M)\), therefore their product is also in \(x^r H_b^k(M)\) from the above result. Hence we proved \(V_e^j(fg) \in x^r H_b^k(M)\) for \(0 \leq j \leq s\), which shows \(fg \in x^r H_{e,b}^{s,k}(M)\). \(\square\)

APPENDIX B. COMPUTATION OF THE INDICIAL ROOTS

B.1. Hodge decomposition. The system contains the following equations, where the \((i, j)\) notations mean the splitting of forms with respect to the product structure of \(\mathbb{H}^7 \times \mathbb{S}^4\), i.e. \(H(i, j)\) has the form \(\sum_k f_k \alpha_k \wedge \beta_k, \alpha_k \in e \wedge i \mathbb{H}^7, \beta_k \in \wedge j \mathbb{S}^4\).

- From the first order equation

\[ (7, 1) : \quad 6d_H \ast \gamma k(1, 1) + 3d_S(Tr_H^7 k) - Tr_{S^4} k) \wedge 7V + d_S * H(0, 4) + d_H * H(1, 3) = 0 \]
\[ (6, 2) : \quad d_S * H(1, 3) + d_H * H(2, 2) + 6d_S \ast \gamma k(1, 1) = 0 \]
\[ (5, 3) : \quad d_S * H(2, 2) + d_H * H(3, 1) = 0 \]
\[ (4, 4) : \quad d_S * H(3, 1) + d_H * H(4, 0) + W \wedge H(4, 0) = 0 \]

- From \(dH = 0\)

\[ (B.5) \quad d_H H(0, 4) + d_S H(1, 3) = 0 \]
\[ (B.6) \quad d_H H(1, 3) + d_S H(2, 2) = 0 \]
\[ (B.7) \quad d_H H(2, 2) + d_S H(3, 1) = 0 \]
\[ (B.8) \quad d_H H(3, 1) + d_S H(4, 0) = 0 \]
\[ (B.9) \quad d_H H(4, 0) = 0 \]
• From the laplacian:

\[
\frac{1}{2} \Delta_s k_{IJ} + \frac{1}{2} \Delta_H k_{IJ} + 6k_{IJ} - 3 \ast_s H_{(1,3)} = 0 
\]  
(B.10)

\[
\frac{1}{2} (\Delta_s + \Delta_H) k_{IJ} - k_{IJ} - 6Tr_S(k)t_{IJ} + Tr_H(k)t_{IJ} + 2H_{(0,4)}t_{IJ} = 0 
\]  
(B.11)

\[
\frac{1}{2} (\Delta_s + \Delta_H) k_{ij} + 4k_{ij} + 8Tr_S(k)t_{ij} - H_{(0,4)}t_{ij} = 0 
\]  
(B.12)

B.2. **Indicial roots.** Then we decompose further with respect to Hodge theory on the sphere, and compute the indicial roots for each part.

1. **\( \hat{k}_{IJ} \): trace-free 2-tensor on \( \mathbb{H}^7 \)**

   The equation is

   \[
   (\Delta_S + \Delta_H - 2) \hat{k}_{IJ} = 0, 
   \]
   
   and the indicial equation is

   \[
   (\lambda - s^2 + 6s) \hat{k}_{IJ} = 0. 
   \]
   
   we have indicial roots

   \[
   s = 3 \pm \sqrt{9 + \lambda} 
   \]
   
   The first pair of indicial roots, when \( \lambda = 0 \), correspond to the perturbation of hyperbolic metric to Poincaré–Einstein metric.

2. **\( \hat{k}_{ij} \): trace-free 2-tensor on \( S^4 \)**

   The equation is

   \[
   \Delta^{\text{rough}}_S \hat{k}_{ij} + \Delta_H \hat{k}_{ij} + 8\hat{k}_{ij} = 0 
   \]
   
   where indicial equation is

   \[
   (\lambda - s^2 + 6s + 8) \hat{k}_{ij} = 0, 
   \]
   
   indicial roots

   \[
   s = 3 \pm \sqrt{17 + \lambda}. 
   \]
   
3. **\( H_{(4,0)} \) with harmonic functions**

   We have

   \[
   d_H \ast H_{(4,0)} + W \wedge H_{(4,0)} = 0 
   \]  
(B.13)

   \[
   d_H H_{(4,0)} = 0 
   \]  
(B.14)

   The second equation can be deduced from the first one. Since the indicial operator for \( d_H \) on a k-form is

   \[
   I[d](s)w = (-1)^k(s - k)w \wedge dx/x 
   \]
   
   Let

   \[
   H_{(4,0)} = T + dx/x \wedge N 
   \]
   
   be the decomposition with respect to tangential and normal decomposition, then the indicial equations are

   \[
   -(s - 3)(\ast_6 N) \wedge dx/x - 6dx/x \wedge N = 0 
   \]
   
   \[
   (s - 4)T \wedge dx/x = 0 
   \]
   
   where the first equation gives

   \[
   (s - 3) \ast_6 N - 6N = 0 
   \]
i.e. $N$ is an eigenform of $*_6$ and the corresponding indicial roots are

\[ s_3^- = 3 - 6i, N \in \bigwedge^3(S^6); *_6 N = iN; \]
\[ s_3^+ = 3 + 6i : N \in \bigwedge^3(S^6); *_6 N = -iN. \]

And plugging into the second equation, we have the vanishing of tangential form

\[ T = 0. \]

Therefore the kernel in this case is

\[ H_{(4,0)} = dx/x \wedge N, N \in \{ \bigwedge^3(S^6), *_6 N = \pm iN \}. \]

(4) $\tau = \frac{1}{4} Tr_S(k), \sigma = \frac{1}{7} Tr_H(k), k_{(1,1)}, H_{(0,4)}, H_{(1,3)}$ on eigenfunctions / exact 1-form / coexact 3-form / exact 4-form

We have the following equations:

\[ 6d_H *_H k_{(1,1)}^{cl} + d_S(3Tr_H(k) - 3Tr_S(k)) \bigwedge^7 V \]
\[ + d_s * H_{(0,4)}^c + d_H * H_{(1,3)}^{cc} = 0 \]

(B.15)

\[ d_H H_{(0,4)}^c + d_S H_{(1,3)}^{cc} = 0 \]

(B.16)

\[ d_H H_{(1,3)}^{cc} = 0 \]

(B.17)

\[ \Delta_s k_{(1,1)}^{cl} + \Delta_H k_{(1,1)}^{cl} + 12k_{(1,1)}^{cl} - 6 * S H_{(1,3)}^{cc} = 0 \]

(B.18)

\[ \Delta_S \sigma + \Delta_H \sigma + 12 \sigma + 4 * S H_{(0,4)}^{cl} - 48 \tau = 0 \]

(B.19)

First note that B.17 can be derived from B.16. Let $H_{(0,4)}^{cc} = d_S \eta$, here $\eta$ is a (0,3)-form. Then $H_{(1,3)}^{cc} = -d_H \eta$ by B.17. Let $f = * S d_S \eta$. Let $k_{(1,1)}^{cl} = d_S w$, $w$ is (1,0)-form. Put it back to B.15 we get

(B.21) \[ 6d_H *_H d_S w + d_S (*_H (21 \sigma - 12 \tau) + *_H d_S *_S d_S \eta - *_S d_H *_H d_H \eta = 0 \]

Apply $*_H (s_H^2 = 1)$, we get

\[ 6 *_H d_H *_H d_S w + d_S (21 \sigma - 12 \tau) + d_S *_S d_S \eta - *_S *_H d_H *_H d_H \eta = 0 \]

Then let $\eta = *_S d_S \xi, \xi$ be a function, and pull out $d_S$

(B.22) \[ -6d_H w + (21 \sigma - 12 \tau) - \Delta_S \xi - \Delta_H \xi = 0 \]

and put the expression to B.18,

\[ \Delta_S d_S w + \Delta_H d_S w + 12d_S w + 6 *_S d_H *_S d_S \xi = 0 \]

Apply $\delta_H$ and pull out $d_S$

(B.23) \[ \Delta_S \delta_H w + \Delta_H \delta_H w + 12 \delta_H w + 6 \Delta_H \xi = 0 \]

Now B.19 becomes

(B.24) \[ \Delta_S \tau + \Delta_H \tau + 72 \tau + 8 \Delta_S \xi = 0 \]
And B.20 is

\[ \Delta_S \sigma + \Delta_H \sigma + 12\sigma - 4\Delta_S \xi - 48\tau = 0 \]

Putting the above four equations together, and suppose the eigenvalue of \( \Delta_S \) is \( \lambda \), we get

\[
\begin{pmatrix}
12 + \lambda + \Delta_H & -48 & -4\lambda & 0 \\
0 & 72 + \lambda + \Delta_H & 8\lambda & 0 \\
21 & -12 & -\lambda - \Delta_H & -6 \\
0 & 0 & 6\Delta_H & 12 + \lambda + \Delta_H
\end{pmatrix}
\begin{pmatrix}
\sigma \\
\tau \\
\xi \\
\delta_H w
\end{pmatrix} = 0
\]

The determinant, after putting in the indicial operator of \( \Delta_H \), is

\[ \lambda^4 - 4S^2\lambda^3 + 24S \ast \lambda^3 - 90\lambda^3 + 6S^4\lambda^2 - 72S^3\lambda^2 \\
+ 342S^2\lambda^2 - 756S \ast \lambda^2 + 1152\lambda^2 - 4S^6\lambda + 72S^5\lambda - 414S^4\lambda \\
+ 648S^3\lambda + 1152S^2\lambda - 3024S \ast \lambda + 10368\lambda \\
+ S^8 - 24S^7 + 162S^6 + 108S^5 - 6192S^4 \\
+ 31536S^3 - 33696S^2 - 155520S = 0
\]

Putting the lowest two eigenvalues for closed 1-form, we get the following two pairs of roots: for \( \lambda = 16 \) the indicial roots are \( \theta_2 = 3\pm i\sqrt{21116145/1655} \), with kernel

\[ \xi_{16} \in \bigwedge^c_{\lambda=16} (S) \]

which is the closed 1-form on 4-sphere with eigenvalue 16. and the other pair is for \( \lambda = 40 \) then

\[ \theta_3 = 3 \pm i3\sqrt{58284220098} \],

with kernel

\[ \xi_{40} \in \bigwedge^c_{\lambda=40} (S). \]

(5) \( H_{(3,1)}, H_{(4,0)} \) with closed 1-form / eigenfunctions

We have

\[ d_S \ast H_{(3,1)}^c + d_H \ast H_{(4,0)}^c + 6^4 V \wedge H_{(4,0)}^c = 0, \]

\[ d_H H_{(3,1)}^c + d_S H_{(4,0)}^c = 0. \]

Let

\[ H_{(3,1)}^c = d_S \eta \]

where \( \eta \) is (3,0), put into second equation to get

\[ H_{(4,0)}^c = -d_H \eta \]

Put everything back to first equation, we get

\[ d_S \ast d_S \eta - d_H \ast d_H \eta - 6^4 V \wedge d_H \eta = 0. \]

Apply \( \ast_S \), and note \( s^2 = (-1)^{k(4-k)} = 1 \), \( \delta_S = (-1)^{4(k+1)+1} \ast_S d_S \ast_S = - *_S d_S \ast_S \), \( \Delta_S = d\delta + \delta d \),

\[ \ast_H(-\delta_S) d_S \eta - d_H \ast_H d_H \eta + d_H \eta = 0 \]
Then apply $*_H$, note $(*_H)^2 = 1$, get
$$-\Delta_S \eta - *_H d_H *_H d_H \eta + 6 *_H d_H \eta = 0.$$ Let $\Delta_S \eta = \lambda \eta$
$$-\lambda \eta - \Delta_H \eta + 6 *_H d_H \eta = 0$$
The indicial equation: using $I(d)(s)w = (-1)^k(s - k)w \wedge \frac{dx}{s}$, 
$$-\lambda \eta + (s - 3)^2 \eta + 6(s - 3) *_6 \eta = 0$$
that is
$$(s - 3)^2 \pm 6i(s - 3) - 16 = 0$$
with roots
$$s = 3 \pm \sqrt{7} \pm 3i.$$ (6) $k_{(1,1)}, H_{(1,3)}, H_{(2,2)}$ with coclosed 1-form / closed 3-form / coclosed 2-form
We have

\begin{align*}
(B.29) & \quad 6d_H *_H k^{cc}_{(1,1)} + d_H *_H H^{cl}_{(1,3)} = 0 \\
(B.30) & \quad d_S *_H H^{cl}_{(1,3)} + d_H *_H H^{cc}_{(2,2)} + 6d_S *_H k^{cc}_{(1,1)} = 0 \\
(B.31) & \quad d_H H^{cl}_{(1,3)} + d_S H^{cc}_{(2,2)} = 0 \\
(B.32) & \quad \frac{1}{2} \Delta_S k^{cc}_{(1,1)} + \frac{1}{2} \Delta_H k^{cc}_{(1,1)} + 6k^{cc}_{(1,1)} - \frac{1}{2} *_S H^{cl}_{(1,3)} = 0
\end{align*}

First note that (B.29) can be derived from (B.30) Let $H^{cl}_{(1,3)} = d_s \eta$, where $\eta$ is (1,2)-form. Then $H^{cl}_{(2,2)} = -d_H \eta$ from (B.31). Put it to (B.30), $d_S *_S d_S \eta - d_H *_H d_H \eta + 6d_s *_H k^{cc}_{(1,1)} = 0$. Apply $*_S, *_H$, get $-\Delta_S \eta - \Delta_H \eta + 6 *_S d_S k^{cc}_{(1,1)} = 0$. Apply $*_S d_S$ again, get $-\Delta_S(*_S d_S \eta) - \Delta_H(*_S d_S \eta) - 6 \Delta_S k^{cc}_{(1,1)} = 0$. Combining with (B.32), and let $\lambda$ be the eigenvalue for $\Delta_S$ on coclosed 1-form, we get
$$\begin{pmatrix}
-\lambda - \Delta_H & -6\lambda \\
-1 & \lambda + \Delta_H + 12
\end{pmatrix}
\begin{pmatrix}
*_S d_S \eta \\
_k^{cc}_{(1,1)}
\end{pmatrix}
= 0$$
The indicial equation is
$$\lambda^2 - (36 + (s - 1)(s - 5) + s^2 - 6s - 1)\lambda - (s - 1)(s - 5)(-s^2 + 6s + 1) = 0.$$ With smallest eigenvalue for coclosed 1-form to be $\lambda = 24$, indicial roots are
$$s = 3 \pm \sqrt{3\sqrt{97} + 31}$$ (7) $H_{(2,2)}, H_{(3,1)}$ with closed 2-form / coclosed 1-form

\begin{align*}
(B.33) & \quad d_S *_H H^{cl}_{(2,2)} + d_H *_H H^{cc}_{(3,1)} = 0 \\
(B.34) & \quad d_H H^{cl}_{(2,2)} + d_S H^{cc}_{(3,1)} = 0 \\
\end{align*}
Apply $d_H$ and $d_S$ to the equations, we have

\begin{align*}
(B.35) & \quad d_H d_S *_H H^{cl}_{(2,2)} = 0, d_S d_H H^{cl}_{(2,2)} = 0
\end{align*}

let $H^{cl}_{(2,2)} = d_S \eta$ where $\eta$ is a coclosed (2,1)-form. Putting it back, and using $d_S$ is an isomorphism, $d_H \eta = -H^{cc}_{(3,1)}$. Then from first equation,
\[ d_S * d_S \eta - d_H * d_H \eta = 0, \text{ which is } - *_{H} *_{S} \Delta_{S} \eta - *_{S} *_{H} \Delta_{H} \eta = 0 \text{ then it requires } \Delta_{H} \eta = -\lambda \eta. \text{ Putting } \lambda = 4(k + 2)(k + 3), \text{ the result is} \]
\[ s = 3 \pm \sqrt{17}. \]

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