SOME EXAMPLES OF CALABI-YAU PAIRS WITH MAXIMAL INTERSECTION AND NO TORIC MODEL

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Abstract. It is known that a maximal intersection log canonical Calabi-Yau surface pair is crepant birational to a toric pair. This does not hold in higher dimension: this article presents some examples of maximal intersection Calabi-Yau pairs that admit no toric model.

1. Introduction and motivation

A Calabi–Yau (CY) pair \((X, D_X)\) consists of a normal projective variety \(X\) and a reduced sum of integral Weil divisors \(D_X\) such that \(K_X + D_X \sim_{\mathbb{Q}} 0\).

The class of CY pairs arises naturally in a number of problems and comprises examples with very different birational geometry. Indeed, on the one hand, a Gorenstein Calabi–Yau variety \(X\) can be identified with the CY pair \((X, 0)\). On the other hand, if \(X\) is a Fano variety, and if \(D_X\) is an effective reduced anticanonical divisor, then \((X, D_X)\) is also a CY pair.

Definition 1.1. (a) A pair \((X, D_X)\) is (t,dlt) (resp. (t,lc)) if \(X\) is \(\mathbb{Q}\)-factorial, terminal and \((X, D_X)\) divisorially log terminal (resp. log canonical).

(b) A birational map \((X, D_X) \xrightarrow{\varphi} (Y, D_Y)\) is volume preserving if \(a_E(K_X + D_X) = a_E(K_Y + D_Y)\) for every geometric valuation \(E\) with centre on \(X\) and on \(Y\).

The dual complex of a dlt pair \((Z, D_Z = \sum D_i)\) is the regular cell complex obtained by attaching an \((|I| - 1)\)-dimensional cell for every irreducible component of a non-empty intersection \(\bigcap_{i \in I} D_i\).

The dual complex encodes the combinatorics of the lc centres of a dlt pair and [4] show that its PL homeomorphism class is a volume preserving birational invariant.

By [3, Theorem 1.9], a (t,lc) CY pair \((X, D_X)\) has a volume preserving (t,dlt) modification \((X, D_X) \to (X, D_X)\), and the birational map between such modifications is volume preserving.

Abusing notation, I call dual complex the following volume preserving birational invariant of a (t,lc) CY pair \((X, D_X)\).

Definition 1.2. \(\mathcal{D}(X, D_X)\) is the PL homeomorphism class of the dual complex of a volume preserving (t,dlt) modification of \((X, D_X)\).

As the underlying varieties of CY pairs range from CY to Fano varieties, they can have very different birational properties. However, \(X\) being Fano is
not a volume preserving birational invariant of the pair \((X, D_X)\). Following [13], I consider the following volume preserving birational invariant notion:

**Definition 1.3.** A \((t,lc)\) CY pair \((X, D_X)\) has maximal intersection if 
\[
\dim D(X, D_X) = \dim X - 1.
\]

In other words, \((X, D_X)\) has maximal intersection if there is a volume preserving \((t,dlt)\) modification of \((X, D_X)\) with a 0-dimensional log canonical centre. Maximal intersection CY pairs have some Fano-type properties; Kollár and Xu show the following:

**Theorem 1.4.** Let \((X, D_X)\) be a dlt maximal intersection CY pair, then:
1. [13, Proposition 19] \(X\) is rationally connected,
2. [13, Theorem 21] there is a volume preserving map \((X, D_X) \rightarrow (Z, D_Z)\) such that \(D_Z\) fully supports a big and semiample divisor.

**Remark 1.5.** The expression “Fano-type” should be understood with a pinch of salt. Having maximal intersection is a degenerate condition: a general \((t,lc)\) CY pair \((X, D_X)\) with \(X\) Fano and \(D_X\) a reduced anticanonical section needs not have maximal intersection.

**Definition 1.6.** A toric pair \((X, D_X)\) is a \((t,lc)\) CY pair formed by a toric variety and the reduced sum of toric invariant divisors.

A toric model is a volume preserving birational map to a toric pair.

**Example 1.7.** A CY pair with a toric model has maximal intersection.

**Remark 1.8.** In dimension 2, the converse holds: maximal intersection CY surface pairs are precisely those with a toric model [6].

The characterisation of CY pairs with a toric model is an open and difficult problem. A characterisation of toric pairs was conjectured by Shokurov and is proved in [1], but it is not clear how to refine it to get information on the existence of a toric model. A motivation to better understand the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry.

The mirror conjecture extends from a duality between Calabi-Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models, i.e. non-compact Kähler manifolds endowed with a superpotential. Most known constructions of mirror partners rely on toric features such as the existence of a toric model or of a toric degeneration. In an exciting development, Gross, Hacking and Keel conjecture the following construction for mirrors of maximal intersection CY pairs.

**Conjecture 1.9.** [6] Let \((Y, D_Y)\) be a simple normal crossings maximal intersection CY pair. Assume that \(D_Y\) supports an ample divisor, let \(R\) be the ring \(k[\text{Pic}(Y)^\times]\), \(\Omega\) the canonical volume form on \(U\) and
\[
U^{\text{trop}}(\mathbb{Z}) = \left\{ \text{divisorial valuations } v: k(U) \setminus \{0\} \to \mathbb{Z} \text{ with } v(\Omega) < 0 \right\} \cup \{0\}.
\]
Then, the free $R$-module $V$ with basis $U^\text{trop}(\mathbb{Z})$ has a natural finitely generated $R$-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on $U$.

Denote by $K$ the torus $\text{Ker}\{\text{Pic}(Y) \to \text{Pic}(U)\}$. The fibration

$$p: \text{Spec}(V) \to \text{Spec}(R) = T_{\text{Pic}(Y)}$$

is a $T_K$-equivariant flat family of affine maximal intersection log CY varieties. The quotient

$$\text{Spec}(V)/T_K \to T_{\text{Pic}(U)}$$

only depends on $U$ and is the mirror family of $U$.

Versions of Conjecture 1.9 are proved for cluster varieties in [7], but relatively few examples are known.

The goal of this note is to present examples of maximal intersection CY pairs that do not admit a toric model and for which one can hope to construct the mirror partner proposed in Conjecture 1.9 (see Section 2 for a precise statement).

2. Auxiliary results on 3-fold CY pairs

The examples in Section 3 are 3-fold maximal intersection CY pairs whose underlying varieties are birationally rigid. In particular, such pairs admit no toric model; this shows that [6]'s results on maximal intersection surface CY pairs do not extend to higher dimensions. In this section, I first recall some results on birational rigidity of Fano 3-folds. Then, I introduce the $(t,\text{dlt})$ modifications suited to the construction outlined in Conjecture 1.9 and discuss the singularities of the boundary $D_X$.

2.1. Birational rigidity. Let $X$ be a terminal $\mathbb{Q}$-factorial Fano 3-fold. When $X$ has Picard rank 1, $X$ is a Mori fibre space, i.e. an end product of the classical MMP.

**Definition 2.1.** A birational map $Y/S \xrightarrow{\varphi} Y'/S'$ between Mori fibre spaces $Y/S$ and $Y'/S'$ is square if it fits into a commutative square

$$
\begin{array}{c}
Y \xrightarrow{\varphi} Y' \\
\downarrow \\
S \xrightarrow{g} S'
\end{array}
$$

where $g$ is birational and the restriction $Y_\eta \xrightarrow{\varphi_\eta} Y'_\eta$ is biregular, where $\eta$ is the function field of the base $k(S)$.

A Mori fibre space $Y/S$ is (birationally) rigid if for every birational map $Y/S \xrightarrow{\varphi} Y'/S'$ to another Mori fibre space, there is a birational self map $Y/S \xrightarrow{\alpha} X/S$ such that $\varphi \circ \alpha$ is square.
In particular, if $X$ is a rigid Mori fibre space, then $X$ is non-rational and no (t,lc) CY pair $(X,D_X)$ admits a toric model.

Non-singular quartic hypersurfaces $X_4 \subset \mathbb{P}^4$ are probably the most famous examples of birationally rigid 3-folds \[9\]. Some mildly singular quartic hypersurfaces are also known to be birationally rigid, in particular, we have:

**Proposition 2.2.** \[2, 16\] Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface with no worse than ordinary double points. If $|\text{Sing}(X)| \leq 8$, then $X$ is $\mathbb{Q}$-factorial (in particular, $X$ is a Mori fibre space) and is birationally rigid.

### 2.2. Singularities of the boundary.

I now state some results on the singularities of the boundary of a 3-fold (t,lc) CY pair. Let $(X, D_X)$ be a 3-fold (t,lc) CY pair and $(\tilde{X}, D_{\tilde{X}})$ a (t,dlt) modification. A stratum of $(\tilde{X}, D_{\tilde{X}})$ is an irreducible component of a non-empty intersection of components of $D_{\tilde{X}}$. Given a stratum $W$, there is a divisor $\text{Diff}_W D_{\tilde{X}}$ on $W$ such that $(W, \text{Diff}_W D_{\tilde{X}})$ is a lc CY pair and $K_W + \text{Diff}_W D_{\tilde{X}} \sim_{\mathbb{Q}} (K_{\tilde{X}} + D_{\tilde{X}})|_W$.

When $K_{\tilde{X}} + D_{\tilde{X}}$ is Cartier and $D_{\tilde{X}}$ reduced, $\text{Diff}_W D_{\tilde{X}}$ is the sum of the restrictions of the components of $D_{\tilde{X}}$ that do not contain $W$.

In particular, for any irreducible component $S$ of $D_{\tilde{X}}$, the link of $[S]$ in $\mathcal{D}(X,D_X)$ is the dual complex $\mathcal{D}(S, \text{Diff}_S D_{\tilde{X}})$. Therefore, if $(X,D_X)$ has maximal intersection, so does $(S,\text{Diff}_S D_{\tilde{X}})$. By the results of \[9\], $(S,\text{Diff}_S D_X)$ then has a toric model.

As $X$ has terminal singularities, $X$ is normal and Cohen-Macaulay. Any Cartier component $S$ of the boundary $D_X$ is Cohen-Macaulay and satisfies Serre’s condition $S_2$. By \[12\] Proposition 16.9, $(S,\text{Diff}_S D_X)$ is semi log canonical (slc). In particular, if $X$ is Gorenstein and $D_X$ irreducible, $D_X$ has slc singularities.

I am particularly interested in producing examples of (t,lc) CY pairs for which the mirror partners proposed in Conjecture 1.9 (see also \[8\]) can be constructed; this motivates the following definition:

**Definition 2.3.** A (t,dlt) modification $(\tilde{X}, D_{\tilde{X}}) \to (X,D_X)$ is called good if $(\tilde{X}, D_{\tilde{X}})$ is log smooth in the sense of log geometry, that is if the components of $D_{\tilde{X}}$ are non-singular and if $\tilde{X}$ has only cyclic quotient singularities.

An immediate consequence of the definition is that if $(\tilde{X}, D_{\tilde{X}}) \xrightarrow{f} (X,D_X)$ is a good (t,dlt) modification and $D_X = \sum_i D_i$, then

$$D_{\tilde{X}} = \sum_i f_*^{-1}D_i + E,$$

where $E$ is reduced and $f$-exceptional, and the restriction of $f$ to $f_*^{-1}D_i$ is a resolution for all $i$.

**Normal singularities** Let $p \in \text{Sing}(D_i)$ be an isolated singularity lying on a single component of the boundary. The restriction $f_i : D_i \to D_i$ is a
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Resolution and we have:

\[ K_{\tilde{D}_i} = (K_{\tilde{X}} + D_i)_{|\tilde{D}_i} = (f_{|\tilde{D}_i})^*K_{D_i} - (E)_{|\tilde{D}_i} \]

where \( E \) is defined by \( K_{\tilde{X}} + f^{-1}D_{\tilde{X}} + E = f^*(K_X + D_X) \).

We now assume that \( D_i \) is Cartier, as is the case when \( X \) is Gorenstein and \( D_X \) irreducible. Without loss of generality, assume that \( \text{Sing}(D_i) = p \).

Then, \( p \) is canonical if \( E \cap \tilde{D}_i = \emptyset \), and elliptic otherwise. Indeed, let

\[ f_i; \tilde{D}_i \rightarrow \overline{D}_i \rightarrow D_i \]

be the factorisation through the minimal resolution of \((p \in D_i)\). Then, \( q \) is either an isomorphism or an isomorphism at the generic point of each component of \( E_{|\tilde{D}_i} \) because \( f \) is volume preserving. We have: \( K_{\overline{D}_i} = \mu^*K_{D_i} - Z \), where the effective cycle \( Z = q_*(E_{D_i}) \) is either empty (and \( p \) is canonical) or a reduced sum of \( \mu \)-exceptional curves (and \( p \) is elliptic). In the second case, \( Z \sim -K_{\overline{D}_i} \) is the fundamental cycle of \((p \in D_i)\). If \( Z \) is irreducible, it is reduced and has genus 1; if not, every irreducible component of \( Z \) is a smooth rational curve of self-intersection \(-2\).

When \( p \) is elliptic, \( Z \) is reduced and \( p \) is a Kodaira singularity \([10, Theorem 2.9]\), i.e. a resolution is obtained by blowing up points of the singular fibre in a degeneration of elliptic curves; further, in Arnold’s terminology, the singularity \( p \) is uni or bimodal.

Further, \( p \in D_i \) is a hypersurface singularity (resp. a codimension 2 complete intersection, resp. not a complete intersection) when \(-3 \leq Z^2 \leq -1\) (resp. \( Z^2 = -4\), resp. \( Z^2 \leq -5\) \([14]\)). When \( -1 \leq Z^2 \leq -4\), normal forms are known for \( p \in D_i \); Table 1 lists normal forms of slc hypersurface singularities, while normal forms of codimension 2 complete intersections elliptic singularities are given in \([19]\).

3. Examples of Rigid Maximal Intersection 3-Fold CY Pairs

All the examples below are \((t, lc)\) CY pairs \((X, D_X)\) which admit no toric model. Except for Example \([3.4]\), all underlying varieties \( X \) are birationally rigid quartic hypersurfaces by Proposition \([2.2]\), the underlying variety in Example \([3.4]\) is a smooth cubic 3-fold, and therefore non-rational.

3.1. Examples with Normal Boundary.

**Example 3.1.** Consider the CY pair \((X, D_X)\) where \( X \) is the nonsingular quartic hypersurface

\[ X = \{x_1^4 + x_2^4 + x_3^4 + x_0x_1x_2x_3 + x_4(x_0^3 + x_4^3) = 0\} \]

and \( D_X \) is its hyperplane section \( X \cap \{x_4 = 0\} \).

The quartic surface \( D_X \) has a unique singular point \( p = (1:0:0:0:0) \), and using the notation of Table 1, \( p \) is locally analytically equivalent to a \( T_{4,4,4} \) cusp

\[ \{0\} \in \{x^4 + y^4 + z^4 + xyz = 0\}. \]
\[ \text{Table 1. Dimension 2 slc hypersurface singularities} \]

\[ D_X \text{ is easily seen to be rational: the projection from the triple point } p \text{ is} \]
\[ D_X \to \mathbb{P}^2_{x_1,x_2,x_3}; \]

this map is the blowup of the 12 points \( \{x_1^4 + x_2^4 + x_3^4 = x_1x_2x_3 = 0\} \), of which 4 lie on each coordinate line \( L_i = \{x_i = 0\} \), for \( i = 1, 2, 3 \).

I treat this example in detail and construct explicitly a good \((t,\delta t)\) modification of the pair \((X,D_X)\).

Let \( f : X_p \to X \) be the blowup of \( p \), then \( X_p \) is non-singular, the exceptional divisor \( E \) satisfies \( (E,\mathcal{O}_E(E)) = (\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-1)) \), and if \( D \) denotes the proper transform of \( D_X \), we have:
\[ K_{X_p} + D + E = f^*(K_X + D). \]

Explicitly, the blowup \( F \to \mathbb{P}^4 \) of \( \mathbb{P}^4 \) at \( p \) is the rank 2 toric variety \( \text{TV}(I,A) \), where \( I = (u,x_0) \cap (x_1,\ldots,x_4) \) is the irrelevant ideal of \( \mathbb{C}[u,x_0,\ldots,x_4] \) and \( A \) is the action of \( \mathbb{C}^* \times \mathbb{C}^* \) with weights:

\[
\begin{pmatrix}
  u & x_0 & s_1 & s_2 & s_3 & s_4 \\
  1 & 0 & -1 & -1 & -1 & -1 \\
  0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

The equation of \( X_p \) is
\[ X_p = \{u^2(u(s_1^4 + s_2^4 + s_3^4 + (x_0s_1s_2s_3)) + s_4(x_0^3 + u^3s_3^3) = 0\}, \]
while $E = \{ u = 0 \}$ and $D = \{ u(s_1^4 + s_2^4 + s_3^4) + x_0s_1s_2s_3 = 0 \}$. By construction, $E$ is the projective plane with coordinates $s_1, s_2, s_3$. Note that $(X_p, D + E)$ is not dlt because $D \cap E = \{ x_0s_1s_2s_3 = 0 \}$ consists of 3 concurrent lines $C_1, C_2, C_3$.

Consider $g_1: X_1 \to X_p$ the blowup of the nonsingular curve

$C_1 = \{ u = s_1 = s_4 = 0 \} \subset X_p$.

The exceptional divisor of $g_1$ is a surface $E_1 \simeq \mathbb{P}(\mathcal{N}_{C_1/X_p})$, and since $C_1 \simeq \mathbb{P}^1$, the restriction sequence of normal bundles gives

\[
\mathcal{N}_{C_1/X_p} \simeq \mathcal{N}_{C_1/E} \oplus (\mathcal{N}_E/X_p)|_{C_1} \simeq \mathcal{O}_{C_1}(1) \oplus \mathcal{O}_E(-1)|_{C_1},
\]

so that $E_1 = \mathbb{F}_2$. Further,

\[
K_{X_1} + D + E + E_1 = g_1^*(K_{X_p} + D + E)
\]

where, abusing notation, I denote by $D$ and $E$ the proper transforms of the divisors $D$ and $E$. The “restricted pair” on $E_1$ is a surface CY pair $(E_1, (D + E)|_{E_1})$ by adjunction. By construction, $E \cap E_1$ is the negative section $\sigma$. The curve $\Gamma = D \cap E_1$ is irreducible, and since $(D + E)|_{E_1}$ is anticanonical, we have

\[
\Gamma \sim \sigma + 4f \text{ where } f \text{ is a fibre of } \mathbb{F}_2 \to \mathbb{P}^1, \text{ and } \Gamma^2 = 6, \Gamma \cdot E_{E_1} = 2.
\]

The divisors $D, E, E_1$ meet in two points, the dual complex $\mathcal{D}(X_1, D + E + E_1)$ is not simplicial it is a sphere $S^2$ whose triangulation is given by 3 vertices on an equator. While not strictly necessary, we consider a further blowup to obtain a (t,dlt) pair with simplicial dual complex.

Denote by $C_2$ the proper transform of the curve

$\{ u = s_2 = s_4 = 0 \}$.

Then $C_2 \subset E \cap D$ is rational, and as above

\[
\mathcal{N}_{C_2/X_1} \simeq \mathcal{N}_{C_2/E} \oplus (\mathcal{N}_E/X_2)|_{C_2} = \mathcal{O}_{C_2}(1) \oplus \mathcal{O}_{C_2}(-2).
\]

Let $g_2: X_2 \to X_1$ be the blowup of $C_2$, then the exceptional divisor of $g_2$ is a Hirzebruch surface

\[
E_2 \simeq \mathbb{P}^1(\mathcal{N}_{C_2/X_1}) \simeq \mathbb{F}_3.
\]

Still denoting by $D, E, E_1$ the strict transforms of $D, E, E_1$, we have:

\[
K_{X_2} + D + E + E_1 + E_2 = g_2^*(K_{X_1} + D + E + E_1).
\]

The pair $(X_2, D + E + E_1 + E_2)$ is dlt; the composition

\[
g_2 \circ g_1 \circ f: (\tilde{X}, D_{\tilde{X}}) = (X_2, D + E + E_1 + E_2) \to (X, D_X)
\]

is a good (t,dlt) modification.

The “restrictions” of $(\tilde{X}, D_{\tilde{X}})$ to the component of the boundary are the following surface anticanonical pairs:

- On $D$: $(E + E_1 + E_2)|_D$ is a cycle of $(-3)$-curves, the morphism $D \to D_X$ is the familiar resolution of the $T_{4,4,4}$ cusp singularity;
- On $E$: $(D + E_1 + E_2)_E$ is the triangle of coordinate lines with self-intersections $(1,1,1)$;
- On $E_1$: $(D + E + E_2)_{E_1}$ is an anticanonical cycle with self-intersections $(5,-3,-1)$;
- On $E_2$: $(D + E + E_1)_{E_2}$ is an anticanonical cycle with self-intersections $(5,-3,0)$ (as above, $E_{1|E_2} \sim \sigma$ is a negative section, $E_{1|E_2} \sim f$ a fibre of $\mathbb{F}_3 \to \mathbb{P}^1$, and $D_{|E_2} \sim 4f + \sigma$).

It follows that the dual complex $D(X,D_X)$ is PL homeomorphic to a tetrahedron and $(X,D_X)$ has maximal intersection. Note that $(0 \in D_X)$ is a maximal intersection lc point, and since $D_X$ is a rational surface, it has a toric model.

**Example 3.2.** Let $X$ be the hypersurface $X = \{x_3(x_0^3 + x_1^3) + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\}$, and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic $X$ has 3 ordinary double points at the intersection points $L \cap \{x_0^3 + x_1^3 = 0\}$, where $L$ is the line $\{x_2 = x_3 = x_4 = 0\}$. The singular locus of $D_X$ is $\text{Sing}(X) \cup \{p\}$, where $p = (0:0:0:1:0)$ is a $T_{3,3,4}$ cusp, i.e. locally analytically equivalent to $\{0\} \in \{x^3 + y^3 + z^4 + xyz = 0\}$.

The quartic surface $D_X$ is rational; the projection of $D_X$ from $p$ is $D_X \dashrightarrow \mathbb{P}^2_{x_0,x_1,x_2}$; this map is defined outside of the 12 points (counted with multiplicity) defined by $\{x_2^4 = x_0^3 + x_1^3 + x_0x_1x_2 = 0\}$.

If $\tilde{X} \to X$ is the composition of the blowups at the ordinary double points and at $p$, $\tilde{X}$ is smooth and $D_{\tilde{X}}$ is non-singular, so that $f$ is a good (t,dlt) modification.

The minimal resolution of $p \in D_X$ is a rational curve with self intersection $C^2 = -3$. Explicitly, taking the blowup of $X$ at $p$, the proper transform is a rational surface $D$. The exceptional curve is the preimage of a nodal cubic in $\mathbb{P}^2$ blown up at 12 points counted with multiplicities. Note that $(\tilde{X},D + E)$ is not dlt, but in order to obtain a (t,dlt) modification, we just need to blowup the node of $D \cap E$ which is a nonsingular point of $\tilde{X}, D$ and $E$. The (t,dlt) modification of $(X,D_X)$ in a neighbourhood of $p$ is good and the associated dual complex is 2-dimensional.

The pair $(X,D_X)$ has maximal intersection; but as in the previous examples, $X$ is rigid, so that $(X,D_X)$ can have no toric model.

**Example 3.3.** Let $X$ be the nonsingular quartic hypersurface $X = \{x_0^3x_3 + x_4^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \subset \mathbb{P}^4$ and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$. 
The surface \(D_X\) has a unique singular point \(p = (0:0:0:1:0)\) of \(D_X\), which is a cusp \(T_{3,4,4}\), i.e. is locally analytically equivalent to
\[
\{0\} \in \{x^3 + y^4 + z^4 + xyz = 0\}.
\]

As in Example 3.1, \(X\) is non-singular, and finding a good \((t,dlt)\) modification of \((X,D_X)\) will amount to taking a minimal resolution of the singular point of \(D_X\). Let \(X_p \rightarrow X\) be the blowup of \(X\) at \(p\); \(X_p\) is non-singular and if \(D\) denotes the proper transform of \(D_X\), and \(E\) the exceptional divisor, \(D \cap E\) consists of 2 rational curves of self intersection \(-3\) and \(-4\). These curves are the proper transforms of \(\{x_0 = 0\}\) and of \(\{x_0^2 + x_1x_2\}\) under the blow up of \(\mathbb{P}^2_{x_0,x_1,x_2}\) at the points
\[
\{x_1^2 + x_2^4 = x_0(x_1x_2 + x_0^2) = 0\}.
\]

The dual complex consists of 3 vertices that are joined by edges and span 2 distinct faces: \(\mathcal{D}(X,D_X)\) is PL homeomorphic to a sphere \(S^2\) whose triangulation is given by 3 vertices on an equator. The CY pair \((X,D_X)\) has maximal intersection but no toric model.

3.2. Examples with non-normal boundary.

**Example 3.4.** This example is due to R. Svaldi. Consider the cubic 3-fold
\[
X = \{x_0x_1x_2 + x_1^2 + x_2^3 + x_3q + x_4q' = 0\} \subset \mathbb{P}^4
\]
where \(q,q'\) are homogeneous polynomials of degree 2 in \(x_0,\ldots, x_4\). If the quadrics \(q\) and \(q'\) are general and if
\[
(q(1,0,0,0,0),q'(1,0,0,0,0)) \neq (0,0),
\]
then \(X\) and \(S = \{x_3 = 0\} \cap X\) and \(T = \{x_4 = 0\} \cap X\) are nonsingular.

Let \(D_X\) be the anticanonical divisor \(S + T\). The curve \(C = S \cap T = \Pi \cap X\) for \(\Pi = \{x_3 = x_4 = 0\}\) is a nodal cubic. It follows that both \((S,C)\) and \((T,C)\) are log canonical, and therefore so is \((X,D_X)\).

Since \(S\) and \(T\) are smooth, \(\text{Sing}(D_X) = S \cap T = C\), and if \(p\) is the node of \(C\), we have:
\[
(p \in D_X) \sim \{0\} \in \{(xy + x^3 + y^3 + z)(xy + x^3 + y^3 + t) = 0\}
\]
\[
\sim \{0\} \in \{(xy + z)(xy + t) = 0\} \sim \{0\} \in \{(xy + z)(xy - z) = 0\}.
\]

Thus, \(p \in D_X\) is a double pinch point, i.e. \(p\) is locally analytically equivalent to \(\{0\} \subset \{x^2y^2 - z^2 = 0\}\).

We now construct a good \((t,dlt)\) modification of \((X,D_X)\). Let \(f : X_C \rightarrow X\) be the blowup of \(X\) along \(C\); \(\text{Sing}(X_C)\) is an ordinary double point.

Indeed, let \(\Pi = \{x_3 = x_4 = 0\}\), then \(f\) is the restriction of \(X\) to the blowup \(\mathcal{F} \rightarrow \mathbb{P}^4\), where \(\mathcal{F}\) is the rank 2 toric variety \(TV(I,A)\), where \(I = (u,x_0,x_1,x_2)\cap (x_3,x_4)\) is the irrelevant ideal of \(\mathbb{C}[u,x_0,\ldots, x_4]\) and \(A\) is the action of \(\mathbb{C}^* \times \mathbb{C}^*\) with weights:
\[
\begin{pmatrix}
u & x_0 & x_1 & x_2 & x_3 & x_4 \\
1 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
The equation of $X_C$ is
\[ \{x_0x_1x_2 + x_1^3 + x_2^2 + u(x_3q + x_4q') = 0\}, \]
so that $X_C$ has a unique singular point at
\[ x_0 - 1 = u = x_1 = x_2 = x_3q(1,0,0,0,0) + x_4q'(1,0,0,0,0) = 0, \]
and this is a 3-fold ordinary double point. In addition, denoting by $E_f = \{u = 0\} \cap X_C$ the exceptional divisor, we have
\[ K_{X_C} + \tilde{S} + \tilde{T} + E_f = K_X + S + T, \]
so that the pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is a (t,lc) CY pair.

The pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is not dlt as the boundary has multiplicity 3 along the fibre $F$ over the node of $S \cap T$. The blowup of $F$ is not $\mathbb{Q}$-factorial, therefore in order to obtain a good (t,dlt) modification, we consider the divisorial contraction $g: \tilde{X} \to X_C$ centred along $F$. This is obtained by (a) blowing up the node, (b) then blowing up the proper transform of $F$, (c) flopping a pair of lines with normal bundle $(-1, -1)$ and (d) contracting the proper transform of the $\mathbb{P}^1 \times \mathbb{P}^1$ above the node to a point $\frac{1}{2}(1,1,1)$. The exceptional divisor of $g$ is denoted by $E_g$.

The pair $(\tilde{X}, \tilde{S} + \tilde{T} + E_f + E_g)$ is the desired (t,dlt) modification of $(X, D_X)$, and it has maximal intersection. The dual complex is PL homeomorphic to a tetrahedron.

**Example 3.5.** Let $X$ be the quartic hypersurface
\[ X = \{x_1^2x_2^2 + x_1x_2x_3l + x_2^2q + x_4f_3 = 0\} \subset \mathbb{P}^4, \]
where $l$ (resp. $q$) is a general linear (resp. quadratic) form in $x_0, \ldots, x_3$, and $f_3$ a general homogeneous form of degree 3 in $x_0, \ldots, x_4$. Let $D_X$ be the hyperplane section $X \cap \{x_4 = 0\}$.

As $l, q$ and $f_3$ are general, $X$ has 6 ordinary double points. Indeed, denote by $L = \{x_1 = x_3 = x = 4 = 0\}$ and $L' = \{x_2 = x_3 = x = 4 = 0\}$, then
\[ \text{Sing}(X) = \{L \cap \{f_3 = 0\}\} \cup \{L' \cap \{f_3 = 0\}\} = \{q_1, q_2, q_3\} \cup \{q'_1, q'_2, q'_3\} \]
which consists of 3 points on each of the lines. In the neighbourhood of each point $q_i$ (resp. $q'_i$) for $i = 1, 2, 3$, the equation of $X$ is of the form
\[ \{0\} \in \{xy + zt = 0\} \]
(and $D_X = \{t = 0\}$) so that all singular points of $X$ are ordinary double points. The quartic hypersurface $X$ is birationally rigid by Proposition 2.22.

The surface $D_X$ is non-normal as it has multiplicity 2 along $L$ and $L'$. The point $p = L \cap L'$ is locally analytically equivalent to
\[ \{0\} \in \{x^2y^2 + z^2 = 0\}, \]
so that $p \in D_X$ is a double pinch point. We conclude that the surface $D_X$ has slc singularities, and hence $(X, D_X)$ is a (t,lc) CY pair.

We construct a good (t,dlt) modification as follows.
First, since $\text{Sing}(X) \cap L$ (resp. $\text{Sing}(X) \cap L'$) is non-empty, the blowup of $X$ along $L$ (resp. along $L'$) is not $\mathbb{Q}$-factorial. In order to remain in the $(t,\text{dlt})$ category, we consider the divisorial extraction $f: X_L \rightarrow X$ centered on $L$ (resp. along $L'$). This is obtained by (a) blowing up the 3 nodes lying on $L$, (b) blowing up the proper transform of $L$, (c) flopping 3 pairs of lines with normal bundle $(-1,-1)$ and (d) contracting the proper transforms of the three exceptional divisors $\mathbb{P}^1 \times \mathbb{P}^1$ lying above the nodes to points $\frac{1}{2}(1,1,1)$. The exceptional divisor of $f$ is denoted by $E$. Let $p: \tilde{X} \rightarrow X$ denote the morphism obtained by composing the divisorial extraction centered on $L$ with that centered on $L'$ (in any order), and let $E, E'$ denote the exceptional divisors of the divisorial extractions. Then

$$K_{\tilde{X}} + \tilde{D} + E + E' = p^*(K_X + D)$$

is a $(t,\text{dlt})$ modification of $(X, D_X)$ and it has maximal intersection. The dual complex $\mathcal{D}(X, D_X)$ is PL homeomorphic to a sphere $S^2$ whose triangulation is given by 3 vertices on an equator.

4. Further results on quartic 3-fold CY pairs: beyond maximal intersection

This section concentrates on $(t,\text{lc})$ CY pairs $(X, D_X)$, where $X$ is a factorial quartic hypersurface in $\mathbb{P}^4$ and $D$ is an irreducible hyperplane section of $X$. I give some more detail on the possible dual complexes of such pairs.

As explained in Section 2.2, $D_X$ is slc because $(X, D_X)$ is lc. In order to study completely the dual complexes of such $(t,\text{lc})$ CY pairs, one needs a good understanding of the normal forms of slc singularities that can lie on $D$. In the case of a general Fano $X$, this step would require additional work, but here, $D_X$ is a quartic surface in $\mathbb{P}^3$ and the study of singularities of such surfaces has a rich history. I recall some results directly relevant to the construction of degenerate CY pairs $(X, D_X)$. The classification of singular quartic surfaces in $\mathbb{P}^3$ can be broken in three independent cases.

(a) Quartic surfaces with no worse than rational double points: the minimal resolution is a $K3$ surface. Possible configurations of canonical singularities were studied by several authors using the moduli theory of $K3$ surfaces; there are several thousands possible configurations. The pair $(X, D_X)$ is $(t,\text{dlt})$ and the dual complex of $(X, D_X)$ is reduced to a point.

(b) Non-normal quartic surfaces were classified by Urabe [17]; there are a handful of cases recalled in Theorem 4.1.

(c) Non-canonical quartic surfaces with isolated singularities. These are studied by Wall [20] and Degtyarev [5] among others; their results are recalled in Theorem 4.5.

**Theorem 4.1.** [17] A non-normal quartic surface $D \subset \mathbb{P}^3$ is one of:

1. the cone over an irreducible plane quartic curve with a singular point of type $A_1$ or $A_2$.
2. a ruled surface over a smooth elliptic curve $G$, $D = \varphi_L(Z)$, where:
(a) $L = \mathcal{O}_Z(C_1) \otimes \pi^*M$, and $Z = \mathbb{P}_G(\mathcal{O}_G \oplus N)$, for
- $M$ a line bundle of degree 2 and
- $N$ a non-trivial line bundle of degree 0.

Denoting by $L_i$ the images by $\varphi_L$ of the sections of $Z$ associated to $\mathcal{O}_G \oplus N \to \mathcal{O}_G$ and $\mathcal{O}_G \oplus N \to N$, $\text{Sing}(D) = L_1 \cup L_2$.

(b) $L = \mathcal{O}_Z(C) \otimes \pi^*M$, and $Z = \mathbb{P}_G(E)$ for a rank 2 vector bundle $E$ that fits in a non-splitting

$$0 \to \mathcal{O}_G \to E \to \mathcal{O}_G \to 0.$$ 

Denoting by $L$ the image by $\varphi_L$ of a section $G \to Z$, $\text{Sing} D = L$.

3. a rational surface $D \subset \mathbb{P}^3$ which is
(a) the image of a smooth $S \subset \mathbb{P}^3$ under the projection from a line disjoint from $S$; $D$ has no isolated singular point and
- $S = v_2(\mathbb{P}^2)$, where $v_2$ is the Veronese embedding; $D$ is the Steiner Roman surface and is homeomorphic to $\mathbb{R}^2$;
- $S = \varphi(\mathbb{P}^1 \times \mathbb{P}^1)$, where $\varphi$ is the embedding defined by $|l_1 + 2l_2|$ for $l_{1,2}$ the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$;
- $S = \varphi(\mathbb{P}_2)$, where $\varphi$ is the embedding defined by $|\sigma + f|$ for $\sigma$ the negative section and $f$ the fibre of $\mathbb{P}_2$.

(b) the image of a surface $\tilde{D} \subset \mathbb{P}^4$ with canonical singularities under the projection from a point not lying on it; $\tilde{D}$ is a degenerate dP4 surface which is the blowup of $\mathbb{P}^2$ in 5 points in almost general position.

(c) a rational surface embedded by a complete linear system on its normalisation $\tilde{D}$; the non-normal locus of $D$ is a line $L$ and $D$ may have isolated singularities outside $L$. The minimal resolution of the normalisation of $D$ is a blowup of $\mathbb{P}^2$ in 9 points. The normalisation of $D$ has at most two rational triple points lying on the inverse image of the non-normal locus; their images on $D$ are also triple points.

**Remark 4.2.** $D$ is not slc in case 1.

**Corollary 4.3.** Let $(X, D_X)$ be a $(t,lc)$ quartic CY pair with non-normal boundary. Then, $(X, D_X)$ has maximal intersection except in the cases described in 2.(a) and (b) of Theorem 4.1.

**Example 4.4.** Consider the pair $(X, D_X)$ where:

$$X = \{ x_0^2x_3^2 + x_1x_2x_3 + x_2^2q(x_0, x_1) + x_4f_3 = 0 \}, D_X = X \cap \{ x_4 = 0 \},$$

where $q$ is a general quadratic form in $(x_1, x_2)$ and $f_3$ a general cubic in $x_0, \ldots, x_4$.

When $q$ and $f_3$ are general, the quartic hypersurface $X$ has 3 ordinary double points. Indeed, denote by $L = \{ x_0 = x_1 = x_4 = 0 \}$, then $\text{Sing}(X)$ consists of points of intersection of $L$ with $\{ f_3 = 0 \}$; there are 3 such points $\{ q_1, q_2, q_3 \}$ when $f_3$ is general. In the neighbourhood of each point $q_i$ for $i = 1, 2, 3$, the equation of $X$ is of the form

$$\{ 0 \} \in \{ xy + zt = 0 \}$$
(and $D_X = \{ t = 0 \}$) so that all singular points of $X$ are ordinary double points. The nodal quartic $X$ is terminal and $\mathbb{Q}$-factorial because it has less than 9 ordinary double points; $X$ is birationally rigid by $[2, 16]$.

Taking the divisorial extraction of the line $L$ is enough to produce a dlt modification $(\tilde{X}, D_{\tilde{X}} + E)$ of $(X, D_X)$; this shows that $(X, D_X)$ does not have maximal intersection. The dual complex has a single 1-stratum, the elliptic curve $D_{\tilde{X}} \cap E$, which is a $(2, 2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. The quartic surface $D_X$ is a ruled surface over an elliptic curve isomorphic to $D_{\tilde{X}} \cap E$; it is an example of case 2(b) in Theorem 4.1.

Theorem 4.5. $[20]$ A normal quartic surface $D \subset \mathbb{P}^3$ with at least one non-canonical singular point is one of:

1. $D$ has a single elliptic singularity and $D$ is rational, or
2. $D$ is a cone, or
3. $D$ is elliptically ruled and
   (a) $D$ has a double point $p$ with tangent cone $z^2$, the projection away from $p$ is the double cover of $\mathbb{P}^2$ branched over a sextic curve $\Gamma$. The curve $\Gamma$ is the union of 3 conics in a pencil that also contains a double line. When this line is a common chord, $D$ has two $T_{2,3,6}$ singularities, when this line is a common tangent, $D$ has one singularity of type $E_{4,0}$. In the first case, $D$ may have an additional $A_1$ singular point.
   (b) $D$ is $\{(x_0x_3 + q(x_1, x_2))^2 + f_4(x_1, x_2, x_3) = 0\}$ and $\{f_4 = 0\}$ is four concurrent lines. Depending on whether $L = \{x_3 = 0\}$ is one of these lines or not and on whether the point of concurrence lies on $L$, $D$ has either two $T_{2,4,4}$ singular points or one trimodal elliptic singularity. The surface may have additional canonical points $A_n$ for $n = 1, 2, 3$ or $2A_1$.

Example 4.6. Let $X$ be the nonsingular quartic hypersurface

$$X = \{ x_0^2x_2^2 + x_0x_1^3 + x_3x_2^3 + x_0x_1x_2x_3 + x_4(x_0^3 + x_3^3 + x_4^3) = 0 \}$$

and $D_X$ its hyperplane section $X \cap \{x_4 = 0\}$. The surface $D_X$ is normal,

$$\text{Sing}(D_X) = \{ p, p' \} = \{ (1:0:0:0:0), (0:0:0:1:0) \},$$

and each singular point is simple elliptic $J_{2,0} = T_{2,3,6}$, i.e. is locally analytically equivalent to $\{ 0 \} \in \{ x^2 + y^3 + z^6 + xyz = 0 \}$.

Here $X$ is nonsingular and $D_X$ is irreducible and normal, and as I explain below, finding a good $(t, \text{dlt})$ modification amounts to constructing a minimal resolution of $D_X$. Let $\tilde{X} \rightarrow X$ be the composition of the weighted blowups at $p = (1:0:0:0:0)$ with weights $(0, 2, 1, 3, 1)$ and at $p' = (0:0:1:0)$ with weights $(3, 1, 2, 0, 1)$, and denote by $E$ and $E'$ the corresponding exceptional divisors. Note that $\tilde{X}$ is terminal and $\mathbb{Q}$-factorial by $[11]$ Theorem 3.5 and has no worse than cyclic quotient singularities. The morphism

$$(\tilde{X}, D + E + E') \xrightarrow{f} (X, D)$$
is volume preserving and the intersection of $D$ with each exceptional divisor is a smooth elliptic curve $C_6 \subset \mathbb{P}(1, 1, 2, 3)$ not passing through the singular points of $E$ and $E'$; $f$ is a good (t,dlt) modification.

The dual complex $\mathcal{D}(X,D_X)$ is 1-dimensional, it has 3 vertices and 2 edges; $(X,D_X)$ does not have maximal intersection. The quartic surface $D_X$ is an example of case 3.(a) in Theorem 4.5.

**Corollary 4.7.** Let $(X,D_X)$ be a $(t,lc)$ quartic CY pair. Assume that $D_X$ is normal, has non-canonical singularities but is not a cone. Then $(X,D_X)$ has maximal intersection except in cases 3.(a) and (b) of Theorem 4.5.

**Remark 4.8.** When $\dim \mathcal{D}(X,D_X) = 1$, $D_X$ either has two $T_{2,3,6}$ or two $T_{2,4,4}$ singularities. Indeed, as is explained in Section 2.2 singular points $p \in D$ are Kodaira singularities, and in particular are at worst bimodal. The description of cases 3.(a) and (b) of Theorem 4.5 immediately implies the result, because a surface singularity of type $E_{4,0}$ is trimodal.

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