Constrain on the mass of Fuzzy Dark Matter from the rotation curve of Milky Way

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ABSTRACT

The fuzzy Dark Matter (FDM) is one of the recent models for dark matter. According to this model, dark matter is made of very light scalar particles with considerable quantum mechanical effects on the galactic scales which solves many problems of the Cold Dark Matter (CDM). Here we use the observed data from the rotation curve of the Milky Way (MW) galaxy to compare the results from FDM and CDM models. We show that FDM adds a local peak on the rotation curve close to the center of the bulge where its position and amplitude depend on the mass of FDM particles. From fitting the observed rotation curve with our expectation from FDM, we find the mass of FDM to be $m = 2.5^{+3.6}_{-2.0} \times 10^{-21}$ eV. We note that the local peak of the rotation curve in MW can also be explained in the CDM model with an extra inner bulge model for the MW galaxy. We conclude that the FDM model explains this peak without need to extra structure for the bulge.

Key words: Fuzzy dark matter, Milky way rotational curve

1 INTRODUCTION

The rotation curve of the galaxies has been a useful tool for studying the kinematics and the mass distribution of the galaxies (Oort et al. 1927; Lindblad 1927; Babcock 1939; De Swart et al. 2017). Based on these studies, it has been observed that the rotation curve outside the galaxies, reaches a constant value, which is in contrast to our expectation of having a decrease in velocity according to the Keplerian law (Freeman 1970; Rubin & Ford Jr 1970). This flat behavior of rotational velocity gave very strong evidence in favor of existing an inconsistency between the theory and observation of the galactic dynamics (Zwicky 1933; Kahn & Woltjer 1959; Page 1959). It is supposed that there must be a component for the structure of the galaxy, which is called Dark Matter halo in literature, with a mass density that could provide a flat rotation curve for the galaxies. Dark matter is also needed in the cosmological scales to have the consistency of dynamics and the structure formation with the observation (Bertone et al. 2005; Peebles 1993). Although, there are modified gravity theories trying to explain the dynamics of galaxies without need to the mysterious dark matter fluid (Milgrom 1983; Moffat 2006; Nolli et al. 2017; Moffat & Rahvar 2013, 2014). In the context of dark matter theories, understanding the properties and the nature of dark matter is one of the challenging and active fields of study in physics and cosmology. Based on the standard cosmological constant–Cold Dark Matter ($\Lambda$CDM) model, and latest data acquired by the Planck satellite, the Universe consists of 68% dark energy and 28% dark matter with 4% baryonic matter (Aghanim et al. 2018). The $\Lambda$CDM model considers non-relativistic, collision-less particles with very weak interaction with the baryonic matter (Bernabei et al. 2003; Markevitch et al. 2004). Simulations based on $\Lambda$CDM model showed that it is very successful in explaining the Universe in the large scales (Springel et al. 2005). However, it seems that the interpretation of the observations in the small scales faces some serious problems (Weinberg et al. 2015). The important problems are as the "core-cusp" problem (Alvarado-Flores & Primack 1993; Moore et al. 1999), "missing satellites" problem (Klypin et al. 1999), "too big to fail" problem (Boylan-Kolchin et al. 2011). We also should note that all the efforts for the detection of the dark matter particles were unsuccessful (Aprile et al. 2017).

There had been many proposals to solve the small scale problems of CDM (Weinberg et al. 2015; Hu et al. 2000), such as Warm Dark Matter (WDM) (Avila-Reese et al. 2001), Self-Interacting Dark Matter (SIDM) (Rocha et al. 2013), taking into account the feedback of the effects of the baryonic matter on the profile of halo (Governato et al. 2010; Garrison-Kimmel et al. 2013), or modified initial condition (Nakama et al. 2017; Kameli & Baghram 2019) Another in-
teresting approach to solving these problems is to consider dark matter made of very light particles with the mass of \((m \simeq 10^{-23} - 10^{-21} \text{eV})\) with a large de Broglie wavelength where their quantum properties in the small scales (kpc) play an important role and enable this model to solve the small scale problems of CDM (Hu et al. 2000). These particles form a Bose-Einstein Condensation (BEC) in the halo's central region and result in a condensate core which due to their solitonic properties often called a soliton (Böhmer & Harko 2007; Schive et al. 2014b; Li et al. 2020). In this model, the quantum pressure stabilizes the gravitational collapse and prevents the formation of cusp by suppressing the small scale structures (Hu et al. 2000; Woo & Chiueh 2009; Lee &Lim 2010). Simulations based on this model also showed that its prediction on the large scale is the same as the CDM but in the small scales, it has different predictions which is interestingly consistent with the observational data (Schive et al. 2014a). The observational tests such as weak lensing of the colliding FDM cores and X-ray emission from the colliding area to test the hypothesis of FDM is studied in Maleki et al. (2019). Also, in Kendall & Easther (2019) using the rotation curve of galaxies in SPARC data a mass for the FDM is obtained as \(m \geq 10^{-23} \text{eV}\). Bar et al. (2018) also studied some tensions of the FDM model with observational data form rotation curves of some galaxies and suggested that the study of the MW galaxy could probe the FDM model particle mass of \(m < 10^{-19} \text{eV}\).

In this model, the dark matter particles are assumed to be non-self-interacting scalar fields. One important candidate can be Axion-like particles (Schive et al. 2014a). These particles are predicted in high energy physics theories. For example, in the string theory, all models have at least several bosonic Axion-like fields (Hui et al. 2017). There are many names for this model of Dark matter in the literature, such as 'Wave Dark Matter', 'Ultra-light Axionic Particles (ULP)', 'BEC DM', 'fuzzy Dark Matter (FDM)' (Lee 2018). Here we use the term "FDM" for this model. One main concern about the FDM model is to use observational data to investigate the existence of these types of particles as a candidate for dark matter. Our attempt is to introduce the observational features of FDM in the rotation curve of the MW galaxy.

In Section (2) we discuss about the FDM properties. In section (3) we introduce a model for the Milky Way galaxy mass components and the contribution of each component in the rotation curve of MW. We also calculate the rotation curve based on the FDM halo model for the MW and find the best value for the mass of the scalar field. In Section (4) we compare the difference between the CDM and FDM model. Section (5) is the summary and conclusion of this work.

2 FDM HALO CORE (SOLITON)

In this section, we will investigate the properties of a FDM halo core. We use Schrodinger-Poisson (SP) equations for the dynamics of a quantum system under its self-gravity (Paredes & Michinel 2018; Navarrete et al. 2017) as

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + mU\psi, \quad (1)
\]

\[
\nabla^2 U = 4\pi G\rho. \quad (2)
\]

where \(m\) is the mass of FDM particles. The mass density defines as \(\rho = |\psi|^2\) and \(U\) is the gravitational potential. Simulations based on the fuzzy dark matter with SP equations in a co-moving frame shows that the system will form a halo with a core. The profile of FDM core \(\rho_c(r)\) which is called soliton has a profile as below (Schive et al. 2014a):

\[
\rho_c(r) = \rho_0[1 + 0.091\left(\frac{r}{r_c}\right)^2]^{-8}, \quad (3)
\]

in which \(r_c\) is the core radius where the density reduces to half of its peak value. The central mass density of the core is given by

\[
\rho_0 = 1.9 \times \left(\frac{10^{-23}\text{eV}}{m}\right)^2(\frac{\text{kpc}}{r_c})^4 M_\odot\text{pc}^{-3}. \quad (4)
\]

The relation between the halo mass \(M_h\) and the core radius is (Robles et al. 2018):

\[
r_c = 1.6\text{kpc} \times \left(\frac{10^{-23}\text{eV}}{m}\right)(\frac{M_h}{M_\odot \times 10^9})^{-1/3}. \quad (5)
\]

The mass enclosed in this radius called the core mass \(M_c\). The outer region of FDM halo behaves like an ordinary cold dark matter which is well approximated by the Navarro-Frenk-White (NFW) profile (Schive et al. 2014a; Du et al. 2016). So generally the full mass density of halo can be written as

\[
\rho = \rho_c(r_1 - r) + \rho_{NFW}\theta(r - r_1),
\]

where \(\theta\) is a step function and \(r_1\) is scale that the transition from BEC to ordinary phase happens. This specific scale is proportional to the core size (i.e. \(r_1 = \alpha r_c\)) where \(\alpha \sim 2-4\) (Robles et al. 2018). Figure (1) shows the mass density profile of the FDM halo for the Milky Way-like galaxy. For the outer regions, we used Sofia (2017) data for the rotation curve and the best value for the NFW parameters. Also, if we adapt the mass of FDM as \(m = 8.1 \times 10^{-23}\text{eV} \) (Schive et al. 2014a), from equation (5), the core size would be \(r_c \sim 200\text{pc}\).

We also investigate the intersection of core and NFW halo in the FDM model, using the continuity condition. With the adapted values for the mass of FDM, the transition radius for the MW Galaxy is \(r_1 \sim 500\text{pc}\) (as shown in Figure 1) which happens for \(\alpha \simeq 2.5\). By decreasing the mass of FDM the core size increases and according to the Equations (4) and (5), the central density decreases. To satisfy the continuity condition between the density of NFW and solitonic core, the mass of the FDM has to be larger than \(m > 10^{-25} \text{eV}\). In the next section, we will investigate the best mass for FDM particles to fit with the rotation curve of the Milky Way galaxy.

3 THE MASS MODEL FOR MILKY WAY

In this section, we introduce the density profile of MW. In this direction, we model the mass distribution in the MW galaxy by considering a bulge, a disk and the dark matter halo with a supermassive black hole at the center of MW
with an additional solitonic structure at the center of the bulge. The circular rotation velocity of a test particle around a mass density is given by
\[ v^2(r) = G\rho I_0(r) \int \frac{\rho(r')d^3r'}{|r-r'|^2}. \]

Accordingly, the total rotational velocity is a sum of all the components of Galaxy as follows
\[ v_{rot}^2 = v_{bh}^2 + v_{ob}^2 + v_{ib}^2 + v_{dm}^2, \quad (7) \]

where subscript “bh”, “ob”, “ib”, “dm” indicate to the central black-hole, bulge, disk and dark matter halo correspondingly.

In what follows, we explain the model that we use for each component of the Galaxy.

### 3.1 The Bulge

The mass profile for the bulge of galaxies is parameterized by de Vaucouleur’s suggestion (de Vaucouleurs 1958; Sofue et al. 2009). It has been investigated that this profile cannot explain the MW rotation curve in the central region. The suggested profile is made of two components of the inner bulge and the outer bulge where the overall density profile can be written as
\[ \rho_b(r) = \rho_{b0} e^{-r/a_b} + \rho_{ob0} e^{-r/a_{ob}}, \quad (8) \]

where \( \rho_{b0} \) and \( \rho_{ob0} \) are the central density of the inner and outer bulge profiles and \( a_b \) and \( a_{ob} \) are the corresponding characteristic core scales. Using this profile the rotational velocity is:
\[ v_b^2(r) = \frac{GM_b F(x)}{a_b} + \frac{GM_{ob} F(y)}{a_{ob}}, \quad (9) \]

where we define \( x = r/a_b \), \( y = r/a_{ob} \), \( M_b = 8\pi a_b^2 \rho_{b0} \) and \( M_{ob} = 8\pi a_{ob}^2 \rho_{ob0} \) are the total mass for the inner and the outer bulges. The function \( F \) defines as:
\[ F(z) = 1 - e^{-z} (1 + z + \frac{z^2}{2}). \quad (10) \]

Figure (2) shows the rotation curve of the MW (Sofue 2013, 2017) where form one hand, we will use the two bulge model to fit the observational data and from the other hand we will apply the FDM to explain the extra peak in the rotation curve of the galaxy without need to use the second bulge model.

### 3.2 The Disk

The MW galaxy has a baryonic matter disk which can be approximated by an exponential function. The surface mass density of disk (Mo et al. 2010) in the cylindrical coordinate is given by:
\[ \Sigma_d(R) = \Sigma_0 e^{-\frac{R}{r_d}}, \quad (11) \]

in which the \( \Sigma_0 \) is the central value of the profile and \( r_d \) is the characteristic radius of the disk. The rotational velocity corresponding to the mass profile is
\[ v_d(R) = \sqrt{\frac{GM_d}{a_d}} D(y), \quad (12) \]

where \( y = R/a_d \), and \( M_d = 2\pi \Sigma_0 a_d^2 \) is the total mass of the disk and the function \( D(y) \) is defined as:
\[ D(y) = \left( \frac{y}{\sqrt{2}} \right) \left[ I_0\left(\frac{y}{2}\right) k_0\left(\frac{y}{2}\right) - I_1\left(\frac{y}{2}\right) k_1\left(\frac{y}{2}\right) \right]. \quad (13) \]

The functions \( I_0, I_1, k_0, k_1 \) are the first and second kinds of modified Bessel functions, respectively (Binney & Tremaine 2008).

### 3.3 Central Supermassive Black hole

It is believed that there is a supermassive black hole at the center of the MW known as the Sagittarius A* with the mass of \( M \sim 4 \times 10^6 M_\odot \) (Ghez et al. 1998; Narayan et al. 1995; Broderick et al. 2011). This black hole dominates the dynamics of the rotation curve in inner regions and causes an increase of the rotation curve in the center of the MW galaxy.

### 3.4 Dark Matter

#### 3.4.1 Cold Dark Matter

Dark matter is the dominant mass component in the universe. Even though in the MW, the central region is composed of baryonic matter mainly, in the outer region the rotation curve is mostly obtained from the dark matter mass density. For the cold dark matter, the well-known mass density profile is NFW profile which is (Navarro et al. 1995):
\[ \rho(r) = \frac{\rho_H}{R(1+R)}, \quad (14) \]

where \( \rho_H \) is the characteristic density and \( R \) is defined as \( R = r/R_H \), in which \( R_H \) is the characteristic radius of the halo. So the enclosed mass within the radius \( r \) is:
\[ M_b(r) = 4\pi \rho_H R_H^2 \ln(1+R) - \frac{R}{1+R}, \quad (15) \]

where the rotational velocity obtain from \( v_b^2(r) = GM_b(r)/r \).

#### 3.4.2 Fuzzy Dark Matter

The FDM mass density profile consists of a core and a transition to the NFW profile, where using equation (3) the rotational velocity of the core can be obtained. Inside the core, using relation (3) the rotational velocity is
\[ v_{rot}^2 = \frac{4\pi G \rho_0}{r} I_1, \quad (16) \]
In what follows we let the mass of FDM be a variable parameter whereby increasing the mass of FDM, the rotation curve in Figure (2) shifts to the left side of the diagram. Due to a natural bump in the rotation curve of FDM, we examine the possibility of replacing the contribution of the second bulge with the contribution of FDM in the rotation curve. We again let the parameters of the CDM and FDM to be free parameters and find the best values for these two models. The results are given in Table (2). Figure (3) shows the rotation curve for the best values of FDM. Figure (4) represents the dependence of the $\chi^2$ as a function of mass of FDM particle from fitting to the rotation curve of Galaxy. The best value of this mass is $m = 2.5^{+3.6}_{-2.0} \times 10^{-21} \text{eV}$. The inner bump of the MW rotation curve can be explained with the FDM without need the second bulge as used in the NFW model. The best values of the parameters of the FDM model are given in Table (2).

**Table 1.** The best values of the NFW model from fitting with the rotation curve of the Milky Way galaxy. The dashed line in Figure (2) represents the best fit to the rotation curve with normalized $\chi^2_N$ is 0.0209. The last row is not entered in the fitting process to the rotation curve. We use this parameter to calculate the rotation curve from the solitonic part of the FDM structure in Figure (2).

| Mass component                  | Mass ($M_\odot$) | Characteristic length scale (kpc) |
|---------------------------------|------------------|----------------------------------|
| Super massive black hole        | $3.6^{+2.8}_{-2.0} \times 10^6$ |                                  |
| Inner bulge                     | $5.4^{+1.0}_{-1.3} \times 10^7$ | $0.0040^{+0.0024}_{-0.0018}$     |
| Main bulge                      | $9.4^{+2.0}_{-1.8} \times 10^8$ | $0.134^{+0.028}_{-0.029}$        |
| Disk                            | $4.1^{+0.5}_{-0.5} \times 10^{10}$ | $2.830^{+0.190}_{-0.190}$        |
| Drak matter (NFW)               | $8\times2.10^{-7}(r < 1Mpc)$   | $12.0 \pm 2$                     |
| Solitonic core                  | $M_c = 4.1 \times 10^9$          | $0.2$                            |

where

$$I_1 = \Sigma_{n=0}^{\infty} a_n r_n^2 r (\frac{r_c}{r})^{2n} \left(1 + 0.991(\frac{r_c}{r})^2\right)^{-\frac{1}{2}} + a r_n^3 \arctan(a r_n^3),$$  \hspace{1cm} (17)$$

in which the parameters are $a_0 = -0.1771$, $a_1 = 0.2259$, $a_2 = 0.3907$, $a_3 = 0.0030$, $a_4 = 0.0002$, $a_5 = 7.3664 \times 10^{-6}$, $a_6 = 1.0656 \times 10^{-7}$, $a_7 = 0.5870$ and $a_8 = 0.3017$. For the outer region, We use the relation (6) which considers the ordinary phase of FDM.

### 4 COMPARISON BETWEEN THE FDM AND NFW COLD DARK MATTER MODELS

In this section, we compare these two models of standard NFW Dark Matter and FDM for the rotation curve of MW. Figure (2) represents the rotation curves resulting from the cold dark matter halo and that of FDM. Here we adapt the mass of FDM particles to be $m = 8.1 \times 10^{-23} \text{eV}$ (Schive et al. 2014a). The rotation curve is not consistent with the CDM model at the inner bulge of the MW and we need to add a second bulge with the mass of $5.4 \times 10^7 M_\odot$ with the characteristic size of 4.1 pc. This structure results in an additional contribution to the rotation curve of the central part of the Galaxy as indicated by the dashed blue line in Figure (2). The dashed black line in this figure represents the rotation curve from the NFW model. In order to fit the observational data, we let the parameter of CDM to be free and taking into account the second bulge structure, we find the best parameters as indicated in Table (1). Our results for NFW parameters are almost consistent with the values in (Sohn et al., 2013, 2017).

In what follows we let the mass of FDM be a variable parameter whereby increasing the mass of FDM, the rotation curve in Figure (2) shifts to the left side of the diagram. Due to a natural bump in the rotation curve of FDM, we examine the possibility of replacing the contribution of the second bulge with the contribution of FDM in the rotation curve. We again let the parameters of the CDM and FDM to be free parameters and find the best values for these two models. The results are given in Table (2). Figure (3) shows the rotation curve for the best values of FDM. Figure (4) represents the dependence of the $\chi^2$ to the mass of FDM. Here we find the best value for the mass of FDM to be $m = 2.5^{+3.6}_{-2.0} \times 10^{-21} \text{eV}$ with corresponding one sigma error. With this mass for the FDM, we can fit the rotation curve of Galaxy close to the Galactic core without need to take into account the extra component to the bulge structure.
Table 2. The best values of the FDM model from fitting with the rotation curve of the Milky Way galaxy. The dashed line in Figure (3) represents the best fit to the rotation curve with normalized $\chi^2_N = 0.0214$.

| Mass component     | Mass ($M_\odot$) | characteristic length scale (kpc) |
|--------------------|------------------|---------------------------------|
| Super massive black hole | $3.8^{+2.7}_{-2.1} \times 10^9$ |                          |
| Main bulge         | $9.4^{+0.6}_{-0.6} \times 10^9$ | $0.13^{+0.04}_{-0.06}$ |
| Disk               | $4.1 \pm 0.5 \times 10^{10}$   | $2.83^{+0.06}_{-0.10}$ |
| Dark matter (NFW)  | $8\pm2 \times 10^{14} (r < 1 \text{Mpc})$ | $12.0 \pm 2$ |
| Solitonic core (2.5$^{+3.6}_{-2.0} \times 10^{-21} \text{eV}$) | $M_c = 1.2^{+1.7}_{-1.0} \times 10^{10}$ | $0.006^{+0.002}_{-0.004}$ |

5 SUMMARY AND CONCLUSION

In this work, we used the Milky Way rotation curve to examine the FDM with the observational data. Also, we compare it with the conventional Galactic model with the structures of one disk, two bulges (inner and outer), central black hole and a halo with the NFW profile. The inner bulge is needed in this model to generate the central bump in the rotation curve of the MW. We have shown that using the FDM as the dark halo structure not only produces the correct rotation curve at the outer part of the Galaxy, also it can play the role of the central bulge to produce the same bump profile in the rotation curve of the Galaxy.

From the best fit to the rotation curve of MW, we find the parameters of luminous components and halo in both NFW and FDM halo models. From the best fit, we obtain the mass of the FDM to be $m = 2.5^{+2.6}_{-2.0} \times 10^{-21} \text{eV}$ with the core mass of $m_c = 1.2 \times 10^{10} M_\odot$. Using the continuity condition for the mass profile of the core and the outer halo where dark matter transits from the BEC at the central part of halo to the NFW state puts a lower bound of $m > 10^{-23} \text{eV}$ for the mass of FDM particles.

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