A FAMILY OF TETRAVALENT ONE-REGULAR GRAPHS

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Abstract. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper, 4-valent one-regular graphs of order $5p^2$, where $p$ is a prime, are classified.

1. Introduction

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set, arc set and its full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to $u$ and $v$ in $X$, and $N(u)$ is the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$. A graph $X$ is said to be vertex-transitive and arc-transitive (or symmetric) if $\text{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. In particular, if $\text{Aut}(X)$ acts regularly on $A(X)$, then $X$ is said to be one-regular.

Clearly, a one-regular graph is connected, and it is of valency 2 if and only if it is a cycle. In this sense the first non-trivial case is that of cubic graphs. The first example of a cubic one-regular graph was constructed by Frucht [14] and later on lot of works have been done along this line (as part of the more general investigation of cubic arc-transitive graphs) see [9, 10, 11, 12]. 4-valent one-regular graphs have also received considerable attention. In [1], 4-valent one-regular graphs of prime order were constructed. In [24], an infinite family of 4-valent one-regular Cayley graphs on alternating groups is given. 4-valent one-regular circulant graphs were classified in [34] and 4-valent one-regular Cayley graphs on abelian groups were classified in [35]. Next, one may deduce a classification of 4-valent one-regular Cayley graphs on dihedral groups from [23, 29, 31]. Let $p$ and $q$ be primes. Then, clearly every 4-valent one-regular graph of order $p$ is a circulant graph. Also, by [31, 27, 28, 30, 31, 33, 35] every 4-valent one-regular graph of order $pq$ or...
$p^2$ is a circulant graph. Furthermore, the classification of 4-valent one-
regular graphs of order $3p^2$, $4p^2$, $6p^2$ and $2pq$ are given in [8, 15, 17, 37].
Along this line the aim of this paper is to classify 4-valent one-regular
graphs of order $5p^2$, see Theorem 3.3.

2. Preliminaries

In this section, we introduce some notations and definitions as well
as some preliminary results which will be used later in the paper.

For a regular graph $X$, use $d(X)$ to represent the valency of $X$, and
for any subset $B$ of $V(X)$, the subgraph of $X$ induced by $B$ will be
denoted by $X[B]$. Let $X$ be a connected vertex-transitive graph, and
let $G \leq \text{Aut}(X)$ be vertex-transitive on $X$. For a $G$-invariant partition
$B$ of $V(X)$, the quotient graph $X_B$ is defined as the graph with vertex
set $B$ such that, for any two vertices $B, C \in B$, $B$ is adjacent to $C$ if
and only if there exist $u \in B$ and $v \in C$ which are adjacent in $X$. Let
$N$ be a normal subgroup of $G$. Then the set $B$ of orbits of $N$ in $V(X)$
is a $G$-invariant partition of $V(X)$. In this case, the symbol $X_B$ will be
replaced by $X_N$. For a positive integer $n$, denote by $\mathbb{Z}_n$ the cyclic group of order $n$ as
well as the ring of integers modulo $n$, by $\mathbb{Z}_n^*$ the multiplicative group
of $\mathbb{Z}_n$ consisting of numbers coprime to $n$, by $D_{2n}$ the dihedral group
of order $2n$, and by $C_n$ and $K_n$ the cycle and the complete graph of
order $n$, respectively. We call $C_n$ an $n$-cycle.

For a finite group $G$ and a subset $S$ of $G$ such that $1 \notin S$ and
$S = S^{-1}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined
to have vertex set $G$ and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given a
g \in G, define the permutation $R(g)$ on $G$ by $x \mapsto xg$, $x \in G$. The
permutation group $R(G) = \{R(g) \mid g \in G\}$ on $G$ is called the right
regular representation of $G$. It is easy to see that $R(G)$ is isomor-
phic to $G$, and it is a regular subgroup of the automorphism group
$\text{Aut}(\text{Cay}(G, S))$. Also it is easy to see that $X$ is connected if and only
if $G = \langle S \rangle$, that is, $S$ is a connection set. Furthermore, the group
$\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\text{Aut}(\text{Cay}(G, S))$.
Actually, $\text{Aut}(G, S)$ is a subgroup of $\text{Aut}(\text{Cay}(G, S))_1$, the stabilizer of
the vertex 1 in $\text{Aut}(\text{Cay}(G, S))$. A Cayley graph $\text{Cay}(G, S)$ is said to be
normal if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$. Xu [36], proved that
$\text{Cay}(G, S)$ is normal if and only if $\text{Aut}(\text{Cay}(G, S))_1 = \text{Aut}(G, S)$. Sup-
pose that $\alpha \in \text{Aut}(G)$. One may easily prove that $\text{Cay}(G, S)$ is normal
if and only if $\text{Cay}(G, S^\alpha)$ is normal. Also later much subsequent work
was done along this line (see [11, 13, 18, 29, 31]).
For \( \mathbf{v} \in V(X) \), denote by \( N_X(\mathbf{u}) \) the \textit{neighbourhood} of \( \mathbf{u} \) in \( X \), that is, the set of vertices adjacent to \( \mathbf{u} \) in \( X \). A graph \( \tilde{X} \) is called a \textit{covering} of a graph \( X \) with projection \( p : \tilde{X} \to X \) if there is a surjection \( p : V(\tilde{X}) \to V(X) \) such that \( p|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \to N_X(v) \) is a bijection for any vertex \( v \in V(X) \) and \( \tilde{v} \in p^{-1}(v) \). A covering \( \tilde{X} \) of \( X \) with a projection \( p \) is said to be \textit{regular} (or \textit{K-covering}) if there is a semiregular subgroup \( K \) of the automorphism group \( \text{Aut}(\tilde{X}) \) such that graph \( X \) is isomorphic to the quotient graph \( \tilde{X}/K \), say by \( h \), and the quotient map \( \tilde{X} \to \tilde{X}/K \) is the composition \( ph \) of \( p \) and \( h \) (for the purpose of this paper, all functions are composed from left to right). If \( K \) is cyclic or elementary abelian then \( \tilde{X} \) is called a \textit{cyclic} or an \textit{elementary abelian covering} of \( X \), and if \( \tilde{X} \) is connected \( K \) becomes the covering transformation group. In this case we also say \( p \) is a \textit{regular covering projection}. The fibre of an edge or a vertex is its preimage under \( p \). An automorphism of \( \tilde{X} \) is said to be \textit{fibre-preserving} if it maps a fibre to a fibre, while every covering transformation maps a fibre on to itself. All of fibre-preserving automorphisms form a group called the \textit{fibre-preserving group}.

Let \( \tilde{X} \) be a \( K \)-covering of \( X \) with a projection \( p \). If \( \alpha \in \text{Aut}(X) \) and \( \tilde{\alpha} \in \text{Aut}(\tilde{X}) \) satisfy \( \tilde{\alpha}p = p\alpha \), we call \( \tilde{\alpha} \) a lift of \( \alpha \), and \( \alpha \) the \textit{projection} of \( \tilde{\alpha} \). Concepts such as a lift of a subgroup of \( \text{Aut}(X) \) and the projection of a subgroup of \( \text{Aut}(\tilde{X}) \) are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in \( \text{Aut}(\tilde{X}) \) and \( \text{Aut}(X) \) respectively.

For two groups \( M \) and \( N \), \( N \rtimes M \) denotes a semidirect product of \( N \) by \( M \). For a subgroup \( H \) of a group \( G \), denote by \( C_G(H) \) the centralizer of \( H \) in \( G \) and by \( N_G(H) \) the normalizer of \( H \) in \( G \). Then \( C_G(H) \) is normal in \( N_G(H) \).

**Proposition 2.1.** [21, Chapter I, Theorem 4.5] The quotient group \( N_G(H)/C_G(H) \) is isomorphic to a subgroup of the automorphism group \( \text{Aut}(H) \) of \( H \).

Let \( G \) be a permutation group on a set \( \Omega \) and \( \alpha \in \Omega \). Denote by \( G_\alpha \) the stabilizer of \( \alpha \) in \( G \), that is, the subgroup of \( G \) fixing the point \( \alpha \). We say that \( G \) is \textit{semiregular} on \( \Omega \) if \( G_\alpha = 1 \) for every \( \alpha \in \Omega \) and \textit{regular} if \( G \) is transitive and semiregular. For any \( g \in G \), \( g \) is said to be \textit{semiregular} if \( \langle g \rangle \) is semiregular.

**Proposition 2.2.** [33, Chapter I, Theorem 4.5] Every transitive abelian group \( G \) on a set \( \Omega \) is regular.
The following proposition is due to Praeger et al, refer to [19, Theorem 1.1].

**Proposition 2.3.** Let $X$ be a connected 4-valent $(G, 1)$-arc-transitive graph. For each normal subgroup $N$ of $G$, one of the following holds:

1. $N$ is transitive on $V(X)$;
2. $X$ is bipartite and $N$ acts transitively on each part of the bipartition;
3. $N$ has $r \geq 3$ orbits on $V(X)$, the quotient graph $X_N$ is a cycle of length $r$, and $G$ induces the full automorphism group $D_{2r}$ on $X_N$;
4. $N$ has $r \geq 5$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected 4-valent $G/N$-symmetric graph, and $X$ is a $G$-normal cover of $X_N$.

Moreover, if $X$ is also $(G, 2)$-arc-transitive, then case (3) can not happen.

The following classical result is due to Wielandt [33, Theorem 3.4]

**Proposition 2.4.** Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a permutation group $G$ acting on a set $\Omega$. Let $\omega \in \Omega$. If $p^m$ divides the length of the $G$-orbit containing $\omega$, then $p^m$ also divides the length of the $P$-orbit containing $\omega$.

To state the next result we need to introduce a family of 4-valent graphs that were first defined in [20]. The graph $C_{\pm 1}(p; 5p, 1)$ is defined to have the vertex set $Z_p \times Z_{5p}$ and edge set $\{(i, j)(i \pm 1, j + 1)| i \in Z_p, j \in Z_{5p}\}$. Also from [20, Definition 2.2], the graphs $C_{\pm 1}(p; 5p, 1)$ are Cayley graphs over $Z_p \times Z_{5p}$ with connection set $\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$. In the proof of Theorem 3.3 we will need $C_{\pm 1}(p; 5p, 1)$ with $p > 11$. It can be readily checked from [20, Definition 2.2] that for $p > 11$ these graphs are actually normal Cayley graphs over $Z_p \times Z_{5p}$.

**Proposition 2.5.** [20, Theorem 1.1] Let $X$ be a connected, $G$-symmetric, 4-valent graph of order $5p^2$, and let $N = Z_p$ be a minimal normal subgroup of $G$ with orbits of size $p$, where $p$ is an odd prime. Let $K$ denote the kernel of the action of $G$ on $V(X_N)$. If $X_N = C_{5p}$ and $K_v \cong Z_2$ then $X$ is isomorphic to $C_{\pm 1}(p; 5p, 1)$.

The graphs defined in [20, Lemma 8.4] are all one-regular (see [20, Section 8]) and therefore we refer to [20] for an intrinsic description of these families.

**Proposition 2.6.** [20, Theorem 1.2] Let $X$ be a connected, $G$-symmetric, 4-valent graph of order $5p^2$, and let $N = Z_p \times Z_p$ be a minimal normal subgroup of $G$ with orbits of size $p^2$, where $p$ is an odd prime. Let
Let $K$ denote the kernel of the action of $G$ on $V(X_N)$. If $X_N = C_5$ and $K_v \cong \mathbb{Z}_2$ then $X$ is isomorphic to one of the graphs in [20] Lemma 8.4.

Finally in the following example we introduce $G(5p; 2, 2, u)$, which first was defined in [27].

**Example 2.7.** Let $2$ be a divisor of $p - 1$. Let $H(5, 2) = \langle a \rangle$, let $t \in \mathbb{Z}_p^*$ be such that $t \in -H(p, 2)$, and let $u$ be the least common multiple of $2$ and the order of $t$ in $\mathbb{Z}_p^*$. Then $X = G(5p; 2, 2, u)$ is defined as the graph with vertex set

$$V(X) = \mathbb{Z}_5 \times \mathbb{Z}_p = \{(i, x)|i \in \mathbb{Z}_5, x \in \mathbb{Z}_p\}$$

such that vertices $(i, x)$ and $(j, y)$ are adjacent if and only if there is an integer $l$ such that $j - i = a^l$ and $y - x \in t^lH(p, 2)$. Also $X$ as defined above is independent of the choice of generator $a$ of $H(5, 2)$ up to isomorphism, and $X$ is also independent of the choice of $t$, such that $\text{lcm}\{o(t), 2\} = u$, up to isomorphism. Moreover, the above graph is circulant, that is, admits a cyclic group of automorphisms of order $5p$ acting regularly on vertices.

We may extract the following results from [3] pp. 76-80].

**Proposition 2.8.** Let $p$ be a prime and $p > 5$. Also let $G$ be a non-abelian group of order $5p^2$.

(i) If $G$ has a normal subgroup of order $p$, say $N$, such that $G/N$ is cyclic, then $G$ is isomorphic to $\langle x, y, z|x^p = y^5 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i\rangle$, where $i^5 \equiv 1 \pmod{p}$ and $(i, p) = 1$;

(ii) If $G$ has a normal subgroup of order $p^2$, say $N$, such that $G/N$ is cyclic, then $G$ is isomorphic to $\langle x, y|xy^2 = y^5 = 1, y^{-1}xy = x^i\rangle$, where $i^5 \equiv 1 \pmod{p^2}$.

3. One-regular graphs of order $5p^2$

For proving the main theorem we need the following two lemmas.

**Lemma 3.1.** Let $p$ be a prime, $p > 5$ and $G = \langle x, y, z|x^p = y^5 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i\rangle$, where $i^5 \equiv 1 \pmod{p}$ and $(i, p) = 1$. Then there is no 4-valent one-regular normal Cayley graph $X$ of order $5p^2$ on $G$.

**Proof.** Suppose to the contrary that $X$ is a 4-valent one-regular normal Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to the generating set $S$. Since $X$ is one-regular and normal, the stabilizer $A_1 = \text{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on $S$ and so that elements in $S$ are all of the same order. The elements of $G$ of order 5 lie in $\langle x, y \rangle$ and the elements of $G$ of order $p$ lie in $\langle x, z \rangle$. Since $X$ is connected, $G = \langle S \rangle$ and
hence $S$ consists of elements of order $5p$. Denote by $S_{5p}$ the elements of $G$ of order $5p$. Therefore

$$S \subseteq S_{5p} = \{x^sy^t \mid s \in \mathbb{Z}_p, t \in \mathbb{Z}_5, j \in \mathbb{Z}_p^*\}.$$  

Clearly $\sigma : x \mapsto x^s, y \mapsto y, z \mapsto z^i (s,j \neq 0)$ is an automorphism of $G$, we may suppose that

$$S = \{xy^iz^{-1}z^{-1}, x^my^n, y^n, x^{-m}z^{-k}\}.$$  

Since $\text{Aut}(G, S)$ acts transitively on $S$, it implies that there is $\alpha \in \text{Aut}(G, S)$ such that $(xy^iz^{-1}z^{-1})^\alpha = y^{-t}x^{-1}z^{-1}$. Since $[x, z] = 1$, and $[y, z] = 1$, the element $x^\alpha$ needs to commute with $x^\alpha$ and $y^\alpha$. Thus $(x^\alpha(y^i)^\alpha = y^{-t}x^{-1}z^{-1} = x^{-i4}y^{-t}z^{-1}$. We may assume that $(y^i)^\alpha = x^t$ where $t_1 \in \mathbb{Z}_p$, and $t_2 \in \mathbb{Z}_5^*$. Thus $(x^\alpha)_1 \alpha t_2 = x^{-i4}y^{-t}z^{-1}$ and so $(xz)^\alpha = x^{-i4}x^{-t_1}x^{k(-1-t_2)}y^{-t-t_2}z^{-1}$. Since $o(xz) = p$, we have $t = -t_2$. Therefore $(xz)^\alpha = x^{-i4}z^{-t_1}z^{-1}$. Also let $z^\alpha = x^{s_1}z^{s_2}$ where $s_1, s_2 \in \mathbb{Z}_p$. So $(x)^\alpha = x^{-i4}x^{-s_1}x^{t_1}z^{1-s_2}$. Since $z^\alpha$ commutes with $(y^i)^\alpha$, it follows that $s_1 = 0$ or $i4t_2 = 1$.

Since $t_2 \in \mathbb{Z}_5^*$ and $i5 \equiv 1 \pmod{p}$, it follows that $i4t_2 \neq 1$. Thus we may suppose that $s_1 = 0$. Therefore $x^\alpha = x^{-i4}z^{-1-s_2}$, $(y^i)^\alpha = x^t$, $z^{1-t_1}z^{-1}z^{s_2}$. Since $x^t = x^i$, we have $(x^\alpha)^{(y^i)^\alpha} = (x^\alpha)^t$ and so $s_2 = -1$ and $(-i4-t_1)(i4t_2-i) = 0$. Since $t = -t_2$ and $t_2 \in \mathbb{Z}_5^*$, we have $(i4t_2-i) \neq 0$. Thus we may suppose that $(i4t_1 = 0$. Therefore $x^\alpha = z^{-1-s_2} = z^0 = 1$, a contradiction.

Lemma 3.2. Let $p$ be a prime, $p > 5$ and $G = \langle x, y \mid x^{p^2} = y^5 = 1, y^{-1}xy = x^i \rangle$, where $i5 \equiv 1 \pmod{p^2}$. Then there is no 4-valent one-regular normal Cayley graph $X$ of order $5p^2$ on $G$.

Proof. Suppose to the contrary that $X$ is a 4-valent one-regular normal Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to the generating set $S$. Since $X$ is one-regular and normal, the stabilizer $A_1 = \text{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on $S$ and so that elements in $S$ are all of the same order. Clearly $x^p$ is the only element of order $p$. Also $x^r$ where $r \in \mathbb{Z}_p^*$ are the only elements of order $p^2$. The elements of $G$ of order 5 lie in $\langle x, y \rangle$. Since $X$ is connected, $G = \langle S \rangle$ and hence $S$ consists of elements of order 5. Denote by $S_5$ the elements of $G$ of order 5. Therefore

$$S \subseteq S_5 = \{x^sy^t \mid r \in \mathbb{Z}_p^2, s \in \mathbb{Z}_5^*\}.$$  

Clearly $\sigma : x \mapsto x^r, y \mapsto y (r \neq 0)$ is an automorphism of $G$, we may suppose that $S = \{xy^i, y^{-x}x^{-1}, x^uy^v, y^{-v}x^{-u}\}$. Since $\text{Aut}(G, S)$ acts transitively on $S$, it implies that there is $\alpha \in \text{Aut}(G, S)$ such
that \((xy^s)^\alpha = y^{-s}x^{-1}\). We may assume that \(y^\alpha = x^m y^n\), where \(m \in \mathbb{Z}_{p^2}\), \(n \in \mathbb{Z}_{p^2}^*\). Also let \(x^\alpha = x^r\), where \(r \in \mathbb{Z}_{p^2}^*\). Therefore \(x^r(x^m y^n)^s = y^{-s}x^{-1}\), and so \(ns = -s\). Thus \(s = 0\) or \(n = -1\). Clearly, \(s \neq 0\), and so \(n = -1\). Now \(y^\alpha = x^m y^{-1}\). Since \(x^y = x^r\), we have \((x^\alpha)^y^\alpha = (x^\alpha)^i\) and so \(ri^4 - ri = 0\). Thus \(i^3 = 1\), a contradiction.

Let \(X\) be a tetravalent one-regular graph of order \(5p^2\). If \(p \leq 11\), then \(|V(X)| = 20, 45, 125, 245,\) or 605. Now, a complete census of the tetravalent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret \[25, 26\]. Therefore, a quick inspection through this list (with the invaluable help of \texttt{magma}\(\)) gives the number of tetravalent one-regular graphs in the case that \(p \leq 11\). Thus we may suppose that \(p > 11\).

The following result is the main result of this paper.

**Theorem 3.3.** Let \(p\) be a prime. A 4-valent graph \(X\) of order \(5p^2\) is 1-regular if and only if one of the following holds:

- (i): \(X\) is a Cayley graph over \(\langle x, y | x^p = y^{5p} = [x, y] = 1 \rangle\), with connection sets \(\{y, y^{-1}, xy, x^{-1}y^{-1}\}\) and \(\{y^2, xy, x^{-2}y^{-2}\}\).
- (ii): \(X\) is connected arc-transitive circulant graph with respect to every connection set \(S\).
- (iii): \(X\) is one of the graphs described in [20, Lemma 8.4].

**Proof.** Let \(X\) be a 4-valent one-regular graph of order \(5p^2\). If \(p \leq 11\), then \(|V(X)| = 20, 45, 125, 245,\) or 605. Now, a complete census of the 4-valent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret \[25, 26\]. Therefore, a quick inspection through this list (with the invaluable help of \texttt{magma}\(\)) gives the proof of the theorem in the case that \(p \leq 11\).

Now, suppose that \(p > 11\). Let \(A = \text{Aut}(X)\) and let \(A_v\) be the stabilizer in \(A\) of the vertex \(v \in V(X)\). Let \(P\) be a Sylow \(p\)-subgroup of \(A\). Since \(A\) is one-regular, it follows that \(|A| = 20p^2\). We show that \(P\) is normal in \(A\). Since \(|A| = 20p^2\), the Sylow’s theorems show that the number of Sylow \(p\)-subgroups of \(A\) is equal to \(|A : N_A(P)| = 1 + kp\), for some \(k \geq 0\). If \(k = 0\), then \(P\) is normal in \(A\) and thus we may assume that \(k \geq 1\). Now, \(1 + kp\) divides 20 and this is possible if and only if \(k = 1\) and \(p = 19\). Now \(|A : N_A(P)| = 20\). So \(N_A(P) = P\) and \(C_A(P) = N_A(P)\). Therefore, by the Burnside’s \(p\)-complement theorem \[32, page 76\], we see that \(A\) has a normal subgroup \(N\) of order 20. In particular, \(P\) acts by conjugation as a group of automorphisms on \(N\). As a group of order 20 does not admit non-trivial automorphisms of order 19, we see that \(P\) centralizes \(A\). Thus \(A \cong N \times P\) and \(P\) is
normal in $A$.

Assume first that $P$ is cyclic. Let $X_P$ be the quotient graph of $X$ relative to the orbits of $P$ and let $K$ be the kernel of $A$ acting on $V(X_P)$. By Proposition 2.4, the orbits of $P$ are of length $p^2$. Thus $|V(X_P)| = 5$, $P \leq K$ and $A/K$ acts arc-transitively on $X_P$. By Proposition 2.3 either $X_P \cong C_5$ and hence $A/K \cong D_{10}$ forcing that $|K| = 2p^2$, or $P$ acts semiregularly on $V(X)$, the quotient graph $X_P$ is a tetravalent connected $A/P$-arc-transitive graph and $X$ is a regular cover of $X_P$. First assume that $X_P \cong C_5$. If $A/P$ is an abelian then, since $A/K$ is a quotient group of $A/P$, also $A/K$ is an abelian. But since $A/K$ is vertex-transitive on $X_P$, Proposition 2.2 implies that it is regular on $X_P$, contradicting arc-transitivity of $A/K$ on $X_P$. Thus $A/P$ is non-abelian group. Clearly $K$ is not semiregular on $V(X)$. Then $K_v \cong \mathbb{Z}_2$, where $v \in V(X)$. By Proposition 2.1 $A/C \leq \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since $A/P$ is not abelian we have that $P$ is a proper subgroup of $C$. If $C \cap K \neq P$, then $C \cap K = K$ ($|K| = 2p^2$). Since $K_v$ is a Sylow 2-subgroup of $K$, $K_v$ is characteristic in $K$ and so normal in $A$, implying that $K_v = 1$, a contradiction. Thus $C \cap K = P$ and $1 \neq C/P = C/C \cap K \cong CK/K \leq A/K \cong D_{10}$. If $C/P \cong \mathbb{Z}_2$, then $C/P$ is in the center of $A/P$ and since $A/P/C/P \cong A/C$ is cyclic, $A/P$ is abelian, a contradiction. It follows that $|C/P| \in \{5, 10\}$, and hence $C/P$ has a characteristic subgroup of order 5, say $H/P$. Thus $|H| = 5p^2$ and $H/P \leq A/P$, implies that $H \leq A$. In addition since $H \leq C = C_A(P)$, we have that $H$ is abelian. Clearly $|H_v| \in \{1, 5\}$. If $|H_v| = 5$, then $H_v$ is a Sylow 5-subgroup of $H$, implying that $H_v$ is characteristic in $H$. The normality of $H$ in $A$ implies that $H_v \leq A$, forcing $H_v = 1$, a contradiction. If $H_v = 1$, then since $|H| = 5p^2$, $H$ is regular on $V(X)$. It follows that $X$ is a Cayley graph on an abelian group with a cyclic Sylow $p$-subgroup $P$. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{5p^2}$, where $p > 11$, is the only abelian group with a cyclic Sylow $p$-subgroup. Also by [34, Theorem 7], $X$ is one-regular.

Now assume that $X_P$ is a tetravalent connected $A/P$-symmetric graph. Clearly, $X_P \cong K_5$ and by [1, Theorem 2.2], $X_P$ is non-normal Cayley graph on $\mathbb{Z}_5$. On the other hand $A/P$ is isomorphic to a subgroup of index 6 in $\text{Aut}(K_5) \cong S_5$. Thus $A/P$ is isomorphic to affine group $\text{AGL}(1, 5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Therefore $A/P$ has a normal subgroup of order 5, say $PM/P$. Thus $PM \leq A$ and $PM$ is transitive on $V(X)$. Since $|PM| = 5p^2$, $PM$ is also regular on $V(X)$, implying that $X$ is a normal Cayley graph on $PM$. If $PM$ is an abelian group, then $PM$
is isomorphic to $\mathbb{Z}_{5p^2}$. Also if $PM$ is not abelian, then by Proposition 2.8 part (ii), $PM$ is isomorphic to $\langle x, y | x^{p^2} = y^5 = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod{p^2}$. If $PM \cong \mathbb{Z}_{5p^2}$, then by [31, Theorem 7] $X$ is one-regular. In the latter case, $X$ is not one-regular, by Lemma 3.2.

Now assume that $P$ is an elementary abelian. Suppose first that $P$ is a minimal normal subgroup of $A$, and consider the quotient graph $X_P$ of $X$ relative to the orbits of $P$. Let $K$ be the kernel of $A$ acting on $V(X_P)$. By Proposition 2.3 either $X_P \cong C_5$ and hence $A/K \cong D_{10}$ forcing that $|K| = 2p^2$, or $P$ acts semiregularly on $V(X)$, the quotient graph $X_P$ is a tetravalent connected $A/K$-arc-transitive graph and $X$ is a regular cover of $X_P$. First assume that $X_P \cong C_5$. Thus $K_v = \mathbb{Z}_2$. Proposition 2.6 implies that $X$ is isomorphic to one of the graphs described in [20, Lemma 8.4].

Now assume that $X_P$ is a tetravalent connected $A/P$-symmetric graph. So $X$ is a $\mathbb{Z}_p \times \mathbb{Z}_p$-regular cover of $K_5$. By [22, Table 1], $AGL(1, 5)$, lifts along $p$. Now we use the fact that the lift of an $s$-regular group that lifts along a regular covering projection is $s$-regular (see [6, 7]). We recall that $AGL(1, 5)$ is a one-regular subgroup of $Aut(K_5)$. Now by [22, Theorem 2.1, Propositions 3.4, 3.5], $X$ is not one-regular.

Suppose now that $P$ is not a minimal normal subgroup of $A$. Then a minimal normal subgroup $N$ of $A$ is isomorphic to $\mathbb{Z}_p$. Let $X_N$ be the quotient graph of $X$ relative to the orbits of $N$ and let $K$ be the kernel of $A$ acting on $V(X_N)$. Then $N \leq K$ and $A/K$ is transitive on $V(X_N)$, moreover we have that $|V(X_N)| = 5p$. By Proposition 2.3 $X_N$ is a cycle of length $5p$, or $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is 4-valent connected $A/N$-arc-transitive graph and $X$ is a regular cover of $X_N$. If $X_N \cong C_{5p}$, and hence $A/K \cong D_{10p}$, then $|K| = 2p$ and thus $K_v \cong \mathbb{Z}_2$. Applying Proposition 2.5 we get that $X$ is isomorphic to $C^{\pm 1}(p; 5p, 1)$. If, however $X_N$ is a 4-valent connected $A/N$-symmetric graph, then, by Proposition 2.3 $X$ is a covering graph of a symmetric graph of order $5p$. By [27], $G(5p; 2, 2, u)$ is the just 4-valent symmetric graph of order $5p$ (see Example 2.7). Observe that in this case a one-regular subgroup of automorphism contains a normal regular subgroup isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_p$. Let $H$ be a one-regular subgroup of automorphism of $X_N$. Since $X$ is one-regular graph, $A$ is the lift of $H$. Since $H$ contains a normal regular subgroup isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_p$ also $A$ contains a normal regular subgroup. Therefore $X$ is a normal Cayley graph of order $5p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_5 \times \mathbb{Z}_p \leq H$, there exists a normal subgroup $G$ of $A$ such that $G/\mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_5$. If $G$ is an abelian group, then $G$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{5p}$, or $\mathbb{Z}_{5p^2}$. Also
if \( G \) is not abelian, then by Proposition 2.8 part (i), \( G \) is isomorphic to \( \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i \rangle \), where \( i^5 \equiv 1 \pmod{p} \) and \( (i, p) = 1 \). If \( G \cong \mathbb{Z}_{5^2} \), then by [34, Theorem 7] \( X \) is one-regular. Also if \( G \cong \mathbb{Z}_p \times \mathbb{Z}_{5^2} \) then by [35, Proposition 3.3, Example 3.2] \( X \) is isomorphic to either \( \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{5^2}, \{a, a^{-1}, ab, a^{-1}b^{-1}\}) \) or \( \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{5^2}, \{a, a^{-2}, ab, a^{-2}b^{-2}\}) \) which are 1-regular. These graphs are in Theorem 3.3 part (ii). Finally, in the latter case, \( X \) is not one-regular, by Lemma 3.1. This complete the proof.

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