Some Results on cubic and higher order extensions of the Poincaré algebra

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Abstract. In these lectures we study some possible higher order (of degree greater than two) extensions of the Poincaré algebra. We first give some general properties of Lie superalgebras with some emphasis on the supersymmetric extension of the Poincaré algebra or Supersymmetry. Some general features on the so-called Wess-Zumino model (the simplest field theory invariant under Supersymmetry) are then given. We further introduce an additional algebraic structure called Lie algebras of order F, which naturally comprise the concepts of ordinary Lie algebras and superalgebras. This structure enables us to define various non-trivial extensions of the Poincaré algebra. These extensions are studied more precisely in two different contexts. The first algebra we are considering is shown to be an (infinite dimensional) higher order extension of the Poincaré algebra in (1 + 2)−dimensions and turns out to induce a symmetry which connects relativistic anyons. The second extension we are studying is related to a specific finite dimensional Lie algebra of order three, which is a cubic extension of the Poincaré algebra in D−space-time dimensions. Invariant Lagrangians are constructed.

1. Introduction

Describing the laws of physics in terms of underlying symmetries has always been a powerful tool. For instance the Casimir operators of the Poincaré algebra are related to the mass and the spin of elementary particles. Moreover, it has been understood that all the fundamental interactions (electromagnetic, weak and strong) are related to the Lie algebra $u(1)_Y \times su(2)_L \times su(3)_c$, in the so-called Standard Model. The Standard Model is then described by the Lie algebra $\mathfrak{iso}(1,3) \times u(1)_Y \times su(2)_L \times su(3)_c$, where $\mathfrak{iso}(1,3)$ is related to space-time symmetries and $u(1)_Y \times su(2)_L \times su(3)_c$ to internal symmetries. The elementary particles, namely, the quarks, the leptons and the Higgs boson are in appropriate representations of $u(1)_Y \times su(2)_L \times su(3)_c$. Even if the Standard Model is the physical theory were the confrontation between experimental results and theoretical predictions is in an extremely good agreement, there are strong arguments (which cannot be summarised here) that it is not the final theory.

Thus, to understand the properties of elementary particles, it is then interesting to study the kind of symmetries which are allowed in space-time. Within the framework of Quantum Field Theory (unitarity of the $S$ matrix etc.), S. Coleman and J. Mandula have shown [1] that if the symmetries are described in terms of Lie algebras, only trivial extensions of the Poincaré algebra can be obtained. Namely, the fundamental symmetries are based on $\mathfrak{iso}(1,3) \times \mathfrak{g}$ with $\mathfrak{g} \supseteq u(1)_Y \times su(2)_L \times su(3)_c$ a compact Lie algebra describing the fundamental interactions and $[\mathfrak{iso}(1,3), \mathfrak{g}] = 0$. Several algebras, in relation to phenomenology, have been investigated (see e.g. [2]) such as $su(5), so(10), e_6$ etc. Such theories are usually refer to “Grand-Unified-Theories”.
or GUT, \textit{i.e.} theories which unify all the fundamental interactions. The fact that elements of $g$ and $Iso(1,3)$ commute means that we have a trivial extension of the Poincaré algebra.

Then, R. Haag, J. T. Lopuszanski and M. F. Sohnius \cite{3} understood that is was possible to extend in a non-trivial way, \textit{i.e.} to introduce new generators which are not Lorentz scalars but spinors, the symmetries of space-time within the framework of Lie superalgebras (see Definition 2.1). This extension is unique (up to the number of supercharges) and called supersymmetry. For a discussion of the Coleman & Mandula theorem see also the Appendix B of \cite{4}.

In the context of the two above mentioned theorems, two mathematical structures emerge naturally in the description of the symmetries in physics: the Lie algebras and the Lie superalgebras. Since, the properties of elementary particles are dictated by its symmetries, it should be interesting to know, whether or not other types of symmetries might be possible. The main feature of the above descriptions of physical symmetries are based upon algebras, \textit{i.e.} vector spaces equipped with a binary multiplication, the commutator or the anticommutator. This observation leads immediately to the simple question: would it be possible to obtain higher order algebras in Quantum Field Theory? Of course these new types of symmetries should not be in conflict with the basic principle of Quantum Field Theory. Furthermore, since we do not want to leave aside the basic principles of Relativity this new structure, if acceptable, should in some sense reproduce the Poincaré symmetries. Ternary algebras and higher order algebras have been considered in physics only occasionally (see for instance \cite{5, 6, 7, 8, 9, 10, 11, 12} and references therein). For some mathematical references one can see \cite{13, 14, 15, 16}. Recently they was some revival of interest in ternary algebras when it has been realised that a ternary algebra defined by a fully antisymmetric product appears in the description of multiple M2-branes \cite{17} (see also \cite{18}).

In a series of papers \cite{19, 20, 21} a specific $F-$ary algebra, called Lie algebra of order $F$ was introduced. This algebra presents the interesting feature to have a Lie algebra as a subalgebra. It is thus a kind of hybrid mathematical structure with two products: one binary and the second of order $F$. Furthermore, Lie algebras of order $F$ can be seen as a possible generalisation of Lie (super)algebras. A Lie algebra of order $F$ admits a $\mathbb{Z}_F-$grading, the zero-graded part being a Lie algebra. An $F-$fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part. In the same manner than the Lie (super)algebras lead to description of (extended) space-time symmetries, Lie algebras of order $F$ allow to construct higher order extensions of the Poincaré algebras. See also \cite{22} for different extensions of the Poincaré algebra.

These lectures are devoted to give some examples of higher order extensions of the Poincaré algebra. These extensions will be studied with many details, and a collection of useful technical points will be given in some appendices. We also give some emphasise on the way we by-pass the no-go theorems. Since these types of structures can be seen as a possible "generalisations" of supersymmetry, and in order to stress on the analogy on both structures, we give in Section 2 of this lecture some well-known results on supersymmetry. Some results upon the Wess-Zumino model are given. Section three is the main subject of these lectures. We firstly give the precise definition of Lie algebra of order $F$ that we apply in two different contexts to extend the Poincaré symmetries. It is known that small dimensional space-times present exceptional properties. In particular, in three space-time dimensions there exists states which are neither bosons nor fermions but anyons. We construct a higher order extension of the Poincaré algebra in three space-time dimensions \cite{23}. Studying its representations explicitly shows that this extension induces a symmetry between relativistic anyons in a straight analogy with supersymmetry, which is a Fermi-Bose symmetry. In the next subsection, we construct a cubic extension of the Poincaré algebra in any space-time dimensions. In this case, the cubic extension considered turns out to induce a symmetry on generalised gauge fields or $p-$forms. The transformation laws have a geometrical interpretation in terms of the natural operations.
on $p$–forms. Then invariant Lagrangians are constructed [24]. In a series of appendices useful technical details are given, in order to illustrate some points, or to give all the needed definitions and identities to prove the results. In Appendix A we give our conventions for spinors in four space-times dimensions. In Appendix B, some emphasis on relativistic wave equations for anyons in three space-times dimensions is given. The main point of our algebras is that they are partially $F$–ary. This means that studying their representations goes along the same lines of studying the representations of supersymmetric algebras, but in the context of Clifford algebras of polynomial’s instead of Clifford algebras. In Appendix C, we give some results on Clifford algebras of polynomial’s. In Appendix D, infinite dimensional representations of Lie algebras are studied with some emphasis on Verma modules or indecomposable representations. Appendix E is a collection of useful identities on spinors and $p$–forms in $4n$–dimensional space-time.

2. Lie superalgebras and supersymmetry

The purpose of this section is to give some general features of Lie superalgebras and to show its implementation in Quantum Field Theory. The basic idea is very simple. It is a consequence of the Noether and spin-statistics theorems. The Noether theorem establishes a deep relation between symmetries and conservation laws. The spin-statistics theorem is related to the quantisation of fields. It is known that, when the space-time dimension is higher than three, there are two types of fields, the bosons of integer spin and the fermions of half-integer spin. The former are quantised using commutation relations whilst the latter are quantised with anticommutation relations. Thus, if we assume that we have some conserved charges of spin, the only allowed possibility, is given by an algebraic structure involving commutation relations and anticommutation relations. Furthermore, since both sides of the equality have to behave in the same manner with respect to the Poincaré algebra, the only possibility is given by

\[ [X_i, X_j] = f_{ij}^k X_k, \quad [X_i, Y_a] = R_{ia}^b Y_b, \quad \{Y_a, Y_b\} = C_{abc} X_c. \quad \tag{1} \]

The mathematical structure hidden behind relations (1) is called a Lie superalgebra. In fact it is the discovery of supersymmetry in Quantum Field Theory [25, 26] which gave rise to the concept of Lie superalgebra and its subsequent classification [27].

**Definition 2.1** A Lie (complex or real) superalgebra is a $\mathbb{Z}_2$–graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with the following structure

(i) $\mathfrak{g}_0$ is a Lie algebra, we denote by $[\cdot , \cdot ]$ the bracket on $\mathfrak{g}_0 ([\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0);$  
(ii) $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0 ([\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1);$  
(iii) there exists a $\mathfrak{g}_0$–equivariant mapping $\{\cdot , \cdot \} : S^2 (\mathfrak{g}_1) \to \mathfrak{g}_0$ where $S^2 (\mathfrak{g}_1)$ denotes the two-fold symmetric product of $\mathfrak{g}_1$ ($\{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \mathfrak{g}_0);$  
(iv) The following Jacobi identities hold ($\forall b_1, b_2, b_3 \in \mathfrak{g}_0, \forall f_1, f_2, f_3 \in \mathfrak{g}_1$)

\[
\begin{align*}
[b_1, [b_2, b_3]] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2] &= 0 \\
[[b_1, b_2], f_3] + [[b_2, b_3], f_1] + [[f_3, b_1], b_2] &= 0 \\
[b_1, \{f_2, f_3\}] - \{[b_1, f_2], f_3\} - \{f_2, [b_1, f_3]\} &= 0 \\
\{f_1, \{f_2, f_3\}\} + [f_2, \{f_3, f_1\}] + [f_3, \{f_1, f_2\}] &= 0.
\end{align*}
\]

In the definition above, the generators of zero (resp. one) gradation are called the bosonic (resp. fermionic) generators or $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$), is called the bosonic (resp. fermionic) part of the Lie superalgebra. The first Jacobi identity is the usual Jacobi identity for Lie algebras, the
second says that \( g_1 \) is a representation of \( g_0 \), the third identity is the equivariance of \( \{ , \} \). These identities are just consequences of 1., 2. and 3., respectively. However, the fourth Jacobi identity, which is an extra constraint, is just the \( \mathbb{Z}_2 \)-graded Leibniz rule. This means that in Definition 2.1 we could have avoided the three first Jacobi identities. Let us mention however that, for some authors the Jacobi identities are given in the definition of Lie superalgebras, and as a consequence they deduce that \( g_0 \) is a Lie algebra and \( g_1 \) is a representation of \( g_0 \). Finally all the Jacobi identities unify. Indeed, if for homogeneous elements \( X,Y,Z \) of \( g \) we denote by \( |X| \) etc their \( \mathbb{Z}_2 \)-grade and define the graded-commutator by \( [X,Y]|_\pm = (-1)^{|X||Y|}[Y,X]|_\pm \), we have

\[
(-1)^{|Z|}|X|[X,[Y,Z]|_\pm \pm (-1)^{|X||Y|}[Y,[Z,X]|_\pm \pm (-1)^{|Y||Z|}[Z,[X,Y]|_\pm = 0.
\]

2.1. The super-Poincaré algebra

The supersymmetric extension of the Poincaré algebra is constructed, in the framework of Lie superalgebras, by adjoining to the Poincaré generators anticommuting elements, called supercharges (we denote \( Q \)), which belong to the spinor representation of the Poincaré algebra. Thus the supersymmetric algebra is a Lie superalgebra \( g = Iso(1,d-1) \oplus S \) with brackets

\[
[L,L] = L, \quad [L,P] = P, \quad [L,Q] = Q, \quad [P,Q] = 0, \quad \{Q,Q\} = P,
\]

with \( (L,P) \) the generators of the Poincaré algebra that belong to the bosonic part of the algebra and \( Q \) the fermionic part of the algebra which is called the supercharges. This extension is non-trivial, because the supercharges \( Q \) are spinors, and thus do not commute with the generators of the Lorentz algebra. In these lectures we are considering the simplest supersymmetric extension of the Poincaré algebra, i.e. where there is only one spinor supercharge. For \( N \)-extended supersymmetry \( (N \leq 8) \), i.e. where \( N \) spinor-supercharges are introduced, the reader is referred to the literature. There are many good text books on the subject. For historical references one can see [28, 29, 30, 31]. For a more modern presentation see e.g. [32], and for the construction of supersymmetric models in particle physics see for instance [33, 34, 35].

We now set up our conventions. We recall that the left- (right-)handed spinors are respectively in the \( 2 \) (resp. \( \bar{2} \)) representation of \( SL(2,\mathbb{C}) \). We denote \( 2 = \langle \psi_\alpha, \alpha = 1,2 \rangle \) and \( \bar{2} = \langle \bar{\psi}^\dot{\alpha}, \dot{\alpha} = 1,2 \rangle \) the two-dimensional representation of \( SL(2,\mathbb{C}) \) and its complex conjugate representation. A Majorana spinor is given by

\[
\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^\dot{\alpha} \end{pmatrix},
\]

with \( (\psi_\alpha)^* = \bar{\psi}^\dot{\alpha} \) (where * denotes the complex conjugation). See Appendix Appendix A for our conventions and useful identities. With the notations above the supersymmetric algebra is given by \( g = Iso(1,3) \oplus 2 \oplus \bar{2} \), with \( Iso(1,3) = \langle L_\mu\nu, P_\mu, \mu, \nu = 0, \cdot \cdots , 3 \rangle \) the Poincaré algebra and \( 2 = \langle Q_\alpha, \alpha = 1,2 \rangle, \bar{2} = \langle \bar{Q}^\dot{\alpha}, \dot{\alpha} = 1,2 \rangle \) the supercharges that we take Majorana \( (Q_\alpha)^\dagger = Q_\dot{\alpha} \). We now determine the various brackets of the supersymmetric algebra:

(i) The even-even part of the algebra is just the Lie algebra structure of the Poincaré algebra

\[
[L_\alpha\beta, L_\mu\nu] = \eta_{\beta\nu}L_{\mu\alpha} - \eta_{\alpha\nu}L_{\mu\beta} + \eta_{\beta\mu}L_{\alpha\nu} - \eta_{\alpha\mu}L_{\beta\nu},
\]

\[
[L_\alpha\beta, P_\mu] = \eta_{\beta\mu}P_\alpha - \eta_{\alpha\mu}P_\beta,
\]

\[
[P_\alpha, P_\beta] = 0.
\]

with \( \eta = \text{diag}(1,-1,-1,-1) \) the Minkowski metric. We have to emphasise that with our conventions, since our structure constants are real, for a unitary representation,
the operators $L_{\mu\nu}$ and $P_\mu$ are antihermitian. The usual quadri-momentum and angular momentum of physical applications are given by $-iP_\mu, -iL_{\mu\nu}$, and are thus hermitian for unitary representations.

(ii) The odd-even part of the algebra is given by the action of the Poincaré algebra on the spinors $Q_\alpha$ and $\bar{Q}^\alpha$. The action of the Lorentz generator is known. For the action of $P$ on $Q$ we have 

$$[P_\mu, Q_\alpha] = p\sigma_{\mu\alpha}Q_\beta, \quad [P_\mu, \bar{Q}^\alpha] = p'\bar{\sigma}_\mu^{\alpha\beta}Q_\beta.$$ 

To write the R.H.S. of the equation above we have identified the only tensors with the appropriate index structure. But the Jacobi identity with $(P_\mu, P_\nu, Q_\alpha)$ gives $pp' = 0$, and since $Q$ is the complex conjugate of $\bar{Q}$, $p = 0$ automatically implies that $p' = 0$ and conversely. Thus we have the non-vanishing brackets

$$[L_{\mu\nu}, Q_\alpha] = \sigma_{\mu\nu}^{\beta\gamma}Q_\beta,$$

$$[L_{\mu\nu}, \bar{Q}^\alpha] = \bar{\sigma}_\mu^{\alpha\beta}\beta.$$ 

We have to pose for a while with this part of the algebra. Indeed, the relations (5) seem to be incompatible with the fact that we are dealing with a real Lie superalgebra since the structure constants are complex. However, it is known that when the space-time dimension is four one can find a representation where the Dirac $\Gamma-$matrices are purely imaginary ($\Gamma^M_0 = \sigma_2 \otimes \sigma_1, \Gamma^M_1 = i\sigma_3 \otimes \sigma_0, \Gamma^M_2 = -i\sigma_3 \otimes \sigma_3, \Gamma^M_3 = -i\sigma_1 \otimes \sigma_0$), the Majorana representation. This means that in this representation the matrices $\Gamma^M_{\mu\nu}$ are real. Thus we clearly see, in this representation, that the structure constants become real.

(iii) The odd-odd part of the algebra has to close on the even part of the algebra. Since $2 \otimes 2 = 1 \oplus 3_+, \bar{2} \otimes \bar{2} = 1 \oplus 3_-$ and $2 \otimes \bar{2} = 4$ with 1 the scalar representation, 3$_\pm$ the (anti-)self-dual two-forms and 4 the vector representation of $SL(2, \mathbb{C})$, we have \textit{a priori}

$$\{Q_\alpha, Q_\beta\} = a\sigma^{\mu\nu}_{\alpha\beta}L_{\mu\nu},$$

$$\{\bar{Q}^\alpha, \bar{Q}^\beta\} = b\bar{\sigma}_\mu^{\alpha\beta}\beta L_{\mu\nu},$$

$$\{Q_\alpha, \bar{Q}^\beta\} = -i\sigma^{\mu}_{\alpha\beta}P_\mu.$$ 

The right handed part of (6) is obtained by means of the natural tensors acting on the spinor space. Now using the Jacobi identity with $(P, Q, Q)$ gives $a = 0$, since $P$ commute with $Q$ and do not commute with $L$. Similarly we have $b = 0$. We identify now the constant $c$. The relation above gives $\{Q_1, Q_1\} + \{Q_2, Q_2\} = -2icP_0$. Now if we assume that the supercharges act on a Hilbert space, since $\bar{Q}_\alpha = Q_\beta$, for any element $|\psi\rangle$ we have $\sum_\alpha \langle \psi|\{Q_\alpha, Q_\beta\}|\psi\rangle = -2ic<\psi|P_0|\psi\rangle$. Thus $-2ic<\psi|P_0|\psi\rangle$ is positive. Assuming $-iP_0$ is a positive operator (the energy with our convention) gives $c > 0$. The conventional normalisation is to take $c = 2$. The odd-odd part of the algebra is then

$$\{Q_\alpha, Q_\beta\} = -2i\sigma^{\mu}_{\alpha\beta}P_\mu.$$ 

The relation above seems to be in conflict with the reality of the Lie superalgebra. To solve this discrepancy, write the relations (7) in the four dimensional representation.
\[ \{ (Q^\alpha, Q^\beta), (Q^\delta, Q^\epsilon) \} = -2i \begin{pmatrix} 0 & \sigma_{\mu\alpha\gamma} \\ \bar{\sigma}_{\mu}^{\alpha\gamma} & 0 \end{pmatrix} \begin{pmatrix} -\epsilon_{\gamma\beta} & 0 \\ 0 & -\bar{\epsilon}_{\gamma\beta} \end{pmatrix} P^\mu, \] (8)

introduce

\[ C = -i\Gamma_0 \Gamma_2 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\bar{\varepsilon} \end{pmatrix}, \]

the charge conjugation matrix. Denoting \( Q_a \) the components of the four dimensional Majorana spinor, the relations (8) become

\[ \{ Q_a, Q_b \} = -2i (\Gamma^\mu C)_{ab} P^\mu. \]

Now, we write the equation above in the Majorana representation for which the Dirac \( \Gamma \)-matrices and \( C \) are purely imaginary. If we denote \( Q^M_a \) the supercharge in the Majorana representation and substitute \( Q^M_a \rightarrow S_a = e^{i\frac{\pi}{4}} Q^M_a \), then we obtain

\[ \{ S_a, S_b \} = -2(\Gamma^M C^M)_{ab} P^\mu, \]

and the structure constant becomes real. We however prefer to write the odd-odd part of the algebra in the form (7).

The algebra given by equation (4), (5) and (7) defines the simplest supersymmetric extension of the Poincaré algebra. Extended supersymmetric algebras with a number of spinor charges \( N \leq 8 \) [3, 26, 36, 37] or supersymmetric algebras in any space-times dimensions \( D \leq 11 \) [38] can however be defined. As we are concerned, we just focus on the simplest extension of the Poincaré symmetry (4), (5) and (7). We will however not follow the standard approach where irreducible representations of the Poincaré superalgebra were systematically investigated [37] leading to the implementation of supersymmetry in Quantum Field Theory and then to construction of supersymmetric extensions of the Standard Model of Particle Physics [33, 34, 35]. We simply quote some results that can be found in the text book given in references: (i) we clearly see that \( P^\mu P_\mu \) is a Casimir operator, this means that in any representation all states have the same mass; (ii) for any representation of the supersymmetric algebra there is an equal number of bosonic and fermionic degrees of freedom. Following the method of Wigner induced representations we identify to types of relevant representations the massive representations \( P^\mu P_\mu < 0 \) (recall that with our conventions the mass operator is given by \( M^2 = -P^\mu P_\mu \)) and the massless representations \( P^\mu P_\mu = 0 \). When supersymmetry is implemented in Quantum Field Theory, it turns out that it is a new symmetry that mixes bosons and fermions. Beyond its purely mathematical interest this has also some phenomenological interesting consequences. This was probably why supersymmetry becomes so popular in the description of physical laws in particle physics (see for instance [33, 34]). Finally, observing that the symmetric product of two supersymmetric transformations gives rise to a space-time translation, the construction of local supersymmetric theory is necessarily a theory of gravity called supergravity. Several authors introduced independently supergravity [39].

2.2. Finite dimensional matrix representation of the supersymmetric algebra

We now give a finite dimensional matrix representation of the algebra above. At first, we introduce a five dimensional representation of the Poincaré algebra

\[ L_{\mu\nu} \rightarrow L_{5\mu\nu} = \begin{pmatrix} J_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_\mu \rightarrow P_{5\mu} = \begin{pmatrix} 0 & \delta_\mu \\ 0 & \delta_\mu \end{pmatrix}, \quad \begin{pmatrix} 4 \times 4 & 4 \times 1 \\ 1 \times 4 & 1 \times 1 \end{pmatrix} \] (9)
where $\delta_\mu$ is a column vector with components $\delta_\mu^\nu$. In this representation, the group element’s read

$$\langle \Lambda, T \rangle = e^{t J_{0\alpha\beta} J_{0\alpha\beta}} e^{t P_{0\alpha}} = \left( \begin{array}{c} \Lambda \\ 0 \\
1 \end{array} \right),$$

and the quadrivector $x$ is embedded in a five-dimensional rep. $X_5 = \left( x \atop 1 \right)$ such that in the Poincaré transformation above we have

$$X' = \left( x' \atop 1 \right) = \left( \begin{array}{c} \Lambda \\ 0 \\
1 \end{array} \right) \left( x \atop 1 \right) = \left( \Lambda x + t \atop 1 \right).$$

In a similar manner, we can define a nine-dimensional matrix representation of the supersymmetric algebra. We define a nine-dimensional vector with indices $I = (\mu, 4, \alpha, \dot{\alpha})$ given by

$$X_9^I = \begin{pmatrix} x_\mu \\ \theta^\alpha \\ \bar{\theta}^\dot{\alpha} \end{pmatrix}.$$

Since we consider $\theta^\alpha$ instead of $\theta_\alpha$ we have for its transformation law under a Lorentz transformation

$$\delta_\omega \psi = \frac{1}{2} \omega^{\mu\nu} \sigma_{\mu\alpha}^\beta \psi_\beta \Rightarrow \delta_\omega \psi^\alpha = -\frac{1}{2} \omega^{\mu\nu} \sigma_{\mu\alpha}^\beta \psi_\beta$$

but we have $\sigma_{\mu\nu}^\alpha = \sigma_{\mu\nu}^{\alpha\beta}.\sigma_\beta.$

We define now the $9 \times 9$ matrices (to simplify notations we keep the same symbols)

$$L_{\mu\nu} = \begin{pmatrix} J_{\mu\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_{\mu\nu} & 0 \\ 0 & 0 & 0 & \sigma_{\mu\nu} \end{pmatrix}, \quad P_\mu = \begin{pmatrix} 0 & \delta_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix} = \begin{pmatrix} 5 \times 5 & 5 \times 4 \\ 4 \times 5 & 4 \times 4 \end{pmatrix}$$

with $J_{\mu\nu}$ the Lorentz generators in the defining vector representation with matrix elements $(J_{\mu\nu})_{\alpha\beta} = \delta_\mu^\alpha \eta_{\beta\gamma} - \delta_\nu^\gamma \eta_{\alpha\mu}$. We introduce $Q_{01}^M, Q_{10}^A$ with $M = 0, 1, 2, 3, 4$ and $A = 1, 2, 1, 2$ the components of the supercharges. If we note $I = (\mu\beta)$ and $J = (\nu\gamma)$ the indices of lines and columns of the following matrices, we get

$$\begin{align*}
(Q_{01}^\alpha)^M_A &= \begin{pmatrix} 0 & i \sigma_\mu^\alpha \gamma_\alpha \\ 0 & 0 \end{pmatrix}, & (Q_{01}^\dot{\alpha})^M_A &= \begin{pmatrix} -i \sigma_\mu^\dot{\alpha} \gamma_\dot{\alpha} \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 4 \times 2 & 4 \times 2 \\ 1 \times 2 & 1 \times 2 \end{pmatrix}, \\
(Q_{10}^\alpha)^A_M &= \begin{pmatrix} 0 & \delta_\mu^\alpha \\ 0 & 0 \end{pmatrix}, & (Q_{10}^\dot{\alpha})^A_M &= \begin{pmatrix} 0 & 0 \\ 0 & -\delta_\mu^{\dot{\alpha}} \end{pmatrix}, & \begin{pmatrix} 2 \times 4 & 2 \times 1 \\ 2 \times 4 & 2 \times 1 \end{pmatrix}. \quad (10)
\end{align*}$$

A direct calculation shows that the matrices (10) satisfy the algebra (4), (5) and (7). For the even-odd part we use the relation

$$(J_{\alpha\beta})^\mu_{\nu} \Gamma^\nu = -\left( \sigma_{\alpha\beta} \sigma_\mu - \sigma_\mu \sigma_{\alpha\beta} \right).$$
A matrix representation of the supersymmetric algebra was also given in [40], but with different notations. The transformations are now parameterised by

$$(\Lambda, t, e) = e^{\frac{1}{2} \omega_{\alpha\beta} L_{\alpha\beta}} e^{\epsilon^{\alpha} P_\alpha} e^{\alpha Q_\alpha + Q_\alpha \epsilon^\alpha},$$

with $\epsilon^\alpha, \bar{\epsilon}^\alpha$ Majorana spinors. And since $(\epsilon^\alpha Q_\alpha + Q_\alpha \epsilon^\alpha)^2 = 0$, we have for a supersymmetric transformation

$$X' = \left( \begin{array}{c} x'^\mu \\ \theta'^\alpha \\ \bar{\theta}'\bar{\alpha} \end{array} \right) = e^{\epsilon^\alpha Q_\alpha + Q_\alpha \epsilon^\alpha} \left( \begin{array}{c} x\mu \\ \theta^\alpha \\ \bar{\theta}\bar{\alpha} \end{array} \right) = \left( \begin{array}{c} x\mu + \bar{\epsilon}^\alpha \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta - \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\epsilon}^\beta \\ 1 \\ \theta^\alpha + \bar{\epsilon}^\alpha \\ \bar{\theta}^\beta - \bar{\epsilon}^\beta \end{array} \right).$$

Several remarks are in order here. The minus sign in the transformation of the variable $\theta$ seems to be rather surprising. In addition, since the matrices (10) act on the nine-dimensional ($Z_2$-graded) vector space parametrised by $X$, at that level, there is no need to have anticommuting variables. As a simple consequence the variable $x$ cannot be real. All this will be corrected by the introduction of Grassmann variables in the next subsection.

### 2.3. Superspace

Since the supersymmetric algebra contains fermionic supercharges its seems natural to promote the space-time to a superspace by adjoining to the space-time coordinates fermionic coordinates that we take Majorana as it was done in [41]. We then consider a point in a superspace as given by $(x^\mu, \theta^\alpha, \bar{\theta}^\bar{\alpha})$, together with the conjugate momenta $(\partial_{x^\mu}, \partial_{\theta^\alpha}, \partial_{\bar{\theta}^\bar{\alpha}})$ which satisfy (we give the only non-vanishing graded-commutators)

$$[\partial_{x^\mu}, x^\nu] = \delta_{\mu}^{\nu}, \quad \left\{ \frac{\partial}{\partial \theta^\alpha}, \theta^\beta \right\} = \delta^\alpha_\beta, \quad \left\{ \frac{\partial}{\partial \bar{\theta}^{\bar{\alpha}}}, \bar{\theta}^{\bar{\beta}} \right\} = \delta^{\alpha}_{\bar{\beta}}. \quad (11)$$

Differently from the previous subsection we now assume from the very beginning that the variables $\theta$ are of Grassmann type. It should be emphasise that from the properties of spinor indices we have the relation $\{\partial_{\theta^\alpha}, \theta^\beta\} = -\delta^\alpha_\beta$. To make contact with the matrices given in (10) and the standard approach of superspace we also introduce $z, \bar{z}$ such that $[\partial_z, z] = 1$. Now, to construct a realisation of the supersymmetric algebra, we use several properties. Since these properties are not specific to our superalgebra, we state them to $g$ a Lie superalgebra which admits a finite dimensional representation of dimension $m + n$ ($m$ is the dimension of the even part and $n$ of the odd part). Denote

$$X_i = \left( \begin{array}{c} X_i^{00} \\ 0 \\ X_i^{11} \end{array} \right), \quad Y_a = \left( \begin{array}{c} 0 \\ Y_a^{10} \\ Y_a^{01} \end{array} \right),$$

the matrix representation of $g$ ($X$ are bosonic operators and $Y$ fermionic operators), satisfying

$$[X_i, X_j] = f_{ij}^k X_k, \quad [X_i, Y_a] = R_{ia}^b Y_b, \quad \{Y_a, Y_b\} = C_{ab}^i X_i. \quad (12)$$

Introduce $m$ bosonic ($n$ fermionic) oscillators $(x^A, \partial_A)$ $(\theta^I, \partial_{\theta^I})$ satisfying the relation (11) and set

$$X = \left( \begin{array}{c} x^1 \\ \vdots \\ x^m \end{array} \right), \quad \Theta = \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^n \end{array} \right), \quad \partial_X = (\partial_1, \cdots, \partial_m), \quad \partial_{\Theta} = (\partial_{\theta^1}, \cdots, \partial_{\theta^n}).$$
Assume further that the vectors $Z = \begin{pmatrix} X \\ \theta \end{pmatrix}$ and $\partial Z = (\partial X \ \partial \theta)$ transform contravariantly (resp. covariantly) under the action of the matrices $X,Y$. Finally observing that the matrices $\bar{X}_i = -X^i_\ell$ and $\bar{Y}_a = iY^a_\ell$, with $M^\ell$ the transpose of the matrix $M$ satisfy the relations (12), one can show that the operators

\begin{equation*}
X_i = X^\ell_i \partial^\ell_X + \Theta^\ell_i \partial^\ell_\Theta, \quad Y_a = X^\ell_a \bar{Y}^0_{\ell} \partial^\ell_X,
\end{equation*}

are a differential realisation of $g$, i.e. satisfy (12).

In our special case, we define (we normalise slightly differently our operators in order that $\mathcal{P}_\mu$ leads to $\partial_\mu$)

\begin{equation*}
\mathcal{L}_{\mu\nu} = X^\ell \tilde{\mathcal{L}}_{\mu\nu}^\ell \partial^\ell_X + \Theta^\ell \tilde{\mathcal{L}}_{\mu\nu}^{11} \partial^\ell_\Theta = x_\mu \partial_\nu - x_\nu \partial_\mu + \theta^\alpha \sigma_{\mu\nu} \beta \partial_{\theta^\beta} - \bar{\theta}^\alpha \bar{\sigma}_{\mu\nu} \beta \bar{\partial}_{\beta},
\end{equation*}

\begin{equation*}
\mathcal{P}_\mu = -X^\ell \tilde{\mathcal{P}}_{\mu}^\ell \partial^\ell_X - \Theta^\ell \tilde{\mathcal{P}}^{11}_{\mu} \partial^\ell_\Theta = z \partial_\mu,
\end{equation*}

\begin{equation*}
Q_\alpha = -iX^\ell \tilde{Q}_\alpha^0 \partial^\ell_\Theta - i\Theta^\ell \tilde{Q}_\alpha^{10} \partial^\ell_X = z \partial_{\theta^\alpha} + i\sigma^\mu_{\alpha\beta} \bar{\theta}^\beta \partial_\mu,
\end{equation*}

\begin{equation*}
\bar{Q}_\dot{\alpha} = -iX^\ell \tilde{Q}_{\dot{\alpha}}^0 \partial^\ell_\Theta - i\Theta^\ell \tilde{Q}_{\dot{\alpha}}^{10} \partial^\ell_X = -z \partial_{\theta^\dot{\alpha}} - i\theta^\beta \sigma^\mu_{\dot{\alpha}\beta} \partial_\mu,
\end{equation*}

which reduces to a differential realisation of the algebra (4), (5) and (7) for $z = 1$. From now on in order to simplify the notations $Q$ and $\bar{Q}$ will be denoted $\bar{Q}$ and $\bar{Q}$ respectively.

\begin{equation*}
Q_\alpha = \partial_{\theta^\alpha} + i\sigma^\mu_{\alpha\beta} \bar{\theta}^\beta \partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = -\partial_{\theta^\dot{\alpha}} - i\theta^\beta \sigma^\mu_{\dot{\alpha}\beta} \partial_\mu \quad (14)
\end{equation*}

since no confusion would be possible. For further use we are looking to some operators $D_\alpha$ and $\bar{D}_{\dot{\alpha}}$ which anticommute with $Q_\beta$ and $\bar{Q}_{\dot{\beta}}$:

\begin{equation*}
\{Q_\alpha, D_\beta\} = 0, \quad \{\bar{Q}\dot{\alpha}, D_\beta\} = 0, \quad \{\bar{Q}\dot{\alpha}, \bar{D}_{\dot{\beta}}\} = 0. \quad (15)
\end{equation*}

A simple calculation gives

\begin{equation*}
D_\alpha = \partial_{\theta^\alpha} - i\sigma^\mu_{\alpha\beta} \bar{\theta}^\beta \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = \partial_{\theta^\dot{\alpha}} - i\theta^\beta \sigma^\mu_{\dot{\alpha}\beta} \partial_\mu. \quad (16)
\end{equation*}

It is easy to see that they also satisfy the supersymmetric algebra, and in particular we have

\begin{equation*}
\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu.
\end{equation*}

All these operators might be obtained in the more formal way of superspace. The concept of superspace was first introduced in [41]. Formally, in the same manner as we can see the Minkowski space-time as a coset of the Poincaré group by the Lorentz group, the superspace can be seen as a coset of the super-Poincaré group by the Lorentz group (see also [28]). This leads naturally to the structure of supermanifold (i.e. structures generalising the concept of manifold and containing Grassmann variables together with the usual “bosonic” coordinates). One can see [42] for a more formal approach (convergence problem, differentiability etc). Supermanifolds are intimately related to Lie supergroups as manifolds are related to Lie groups (see for instance [40]). A point in the superspace has coordinates
\[ X = (x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}). \]

In the coset approach a point in the superspace is parametrised by

\[ G(x, \theta, \bar{\theta}) = e^{x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}^\dot{\alpha} \bar{Q}_{\dot{\alpha}}}, \]

and the action of a supersymmetric transformation is given by

\[ G(x, \theta, \bar{\theta}) G(0, \varepsilon, \bar{\varepsilon}), \]

we consider the right action in order to have the correct sign in the supersymmetric algebra. Since we are dealing with anticommuting variables we assume further \( \{\theta, \varepsilon\} = \{\bar{\theta}, \bar{\varepsilon}\} = 0 \) etc. We recall that \( [\varepsilon^\alpha Q_\alpha, \bar{\varepsilon}^\dot{\alpha} \bar{Q}_{\dot{\alpha}}] = \varepsilon^\alpha \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \bar{\theta}^\dot{\alpha} \) etc. Using the Baker-Campbell-Hausdorff formulæ which reduces to

\[ e^A e^B = e^{A + B + \frac{1}{2}[A, B]}, \]

since \([A, B], A = [A, B], B = 0\), we finally obtain

\[ G(x, \theta, \bar{\theta}) G(0, \varepsilon, \bar{\varepsilon}) = e^{\left[ x^\mu + i(\theta \sigma^\mu \varepsilon - \varepsilon \sigma^\mu \bar{\theta}) \right] P_\mu + [\theta^\alpha + \varepsilon^\alpha] Q_\alpha + [\bar{\theta}^\dot{\alpha} + \bar{\varepsilon}^\dot{\alpha}] \bar{Q}_{\dot{\alpha}}}. \]

This means that under a supersymmetry transformation we have

\[
\begin{align*}
\delta_\varepsilon x^\mu &= i(\theta \sigma^\mu \varepsilon - \varepsilon \sigma^\mu \bar{\theta}), \\
\delta_\varepsilon \theta^\alpha &= \varepsilon^\alpha, \\
\delta_\varepsilon \bar{\theta}^\dot{\alpha} &= \bar{\varepsilon}^\dot{\alpha}. \\
\end{align*}
\]

Assume now that the transformations (17) are given by some supercharges

\[ \delta_\varepsilon x^\mu = [\varepsilon, Q + \bar{Q}, \varepsilon, x^\mu], \quad \delta_\varepsilon \theta^\alpha = [\varepsilon, Q + \bar{Q}, \varepsilon, \theta^\alpha], \quad \delta_\varepsilon \bar{\theta}^\dot{\alpha} = [\varepsilon, Q + \bar{Q}, \varepsilon, \bar{\theta}^\dot{\alpha}], \]

gives (14) for the supercharges. We see that the benefit of introducing Grassmann variables is two-fold. Firstly the unwanted minus sign in the \( \bar{\theta} \) transformation disappear. Secondly, \( \delta x \) is real because of (A.9).

2.4. The superfield formalism: the Wess-Zumino model

The natural objects which live in the superspace are the superfields. Superfields where firstly introduced in [43]. A superfield \( \Phi \) is a field which depends on the superspace coordinates. Since the Grassmann variables are nilpotent, the superfield has a finite number of components

\[ \Phi(x, \theta, \bar{\theta}) = z(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^\dot{\alpha} \bar{\chi}^\dot{\alpha}(x) + \theta^\alpha \theta_\alpha \eta(x) + \bar{\theta}^\dot{\alpha} \bar{\eta}_{\dot{\alpha}}(x) + \theta^\alpha \sigma_{\mu\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} v^\mu(x) + \theta^\alpha \theta_\alpha \bar{\theta}^\dot{\alpha} \lambda^\dot{\alpha}(x), \]

with \( z, m, n, d \) complex scalar field’s, \( v_\mu \) a complex vector field, \( \psi, \lambda \) left-handed spinors and \( \bar{\chi}, \bar{\omega} \) right-handed spinors. Since in all supersymmetric calculus one has to be careful with the signs, and since for Grassmann variables we have \( \theta^\alpha \psi_\alpha = -\theta_\alpha \psi^\alpha \), when the summation on the spinor indices will be omitted \( \theta, \psi \) means \( \theta^\alpha \psi_\alpha \) and \( \bar{\theta}, \bar{\psi} \) means \( \bar{\theta}^\dot{\alpha} \bar{\psi}^\dot{\alpha} \). In the sequel we will intensively follow this convention. See the reference of Wess and Bagger [29] for more details and useful identities (see also Appendix Appendix A).

A scalar superfield is a superfield which transforms under a supersymmetric transformation
\[ \Phi'(X') = \Phi(X). \]

At the infinitesimal level

\[ \delta \Phi(X) = \Phi'(X) - \Phi(X) = (\varepsilon \cdot Q + \bar{Q} \cdot \bar{\varepsilon}) \Phi(X). \quad (19) \]

It turns out that a scalar superfield is a reducible representation of the supersymmetric algebra. Several types of superfields constructed from \( \Phi \) may be defined. The chiral superfield is defined by

\[ \bar{D}_\alpha \Phi = 0. \quad (20) \]

We can have also defined an antichiral superfield \( \bar{\Phi} \) satisfying \( D_\alpha \bar{\Phi} = 0 \). The chiral superfield leads to the first supersymmetric (where supersymmetry is realised in a linear way) model in field theory in four space-time dimensions. It is known as the Zess-Zumino model [26]. In fact this model was historically constructed in components and not in the superfield language.

Observing that

\[ \bar{D}_\alpha y^\mu = 0, \quad y^\mu = x^\mu - i \theta \sigma^\mu \bar{\theta}, \]

means that the chiral superfield \( \Phi \) depends only on the variables \( y \) and \( \theta \). Developing \( \Phi(y, \theta) \) using (A.12) and (A.13) we get

\[ \Phi(y, \theta) = z(y) + \sqrt{2} \theta \psi(y) - \theta \cdot F(y) \]

\[ = z(x) + \sqrt{2} \theta \psi(x) - i \theta \sigma^\mu \bar{\theta} \partial_\mu z + \frac{i}{\sqrt{2}} \theta \theta \theta \sigma^\mu \bar{\theta} \bar{\psi}(x) - \frac{1}{4} \theta \theta \theta \theta \Box z. \quad (21) \]

The minus sign and \( \sqrt{2} \) factor are introduced such that the kinetic part of the Lagrangian is correctly normalised. A chiral superfield contains two complex scalars \( z \) and \( F \) and one left-handed Weyl spinors \( \psi \). The adjoint of \( \Phi \) is given by

\[ \Phi^\dagger(y, \theta) = z^\dagger(x) + \sqrt{2} \bar{\theta} \bar{\psi} - \bar{\theta} \cdot F^\dagger(x) + i \theta \sigma^\mu \bar{\theta} \partial_\mu z^\dagger(x) + \frac{i}{\sqrt{2}} \bar{\theta} \bar{\theta} \theta \sigma^\mu \partial_\mu \bar{\psi}(x) - \frac{1}{4} \theta \theta \theta \theta \Box z^\dagger(x). \quad (22) \]

and contains two complex scalars and one right-handed Weyl spinor. It turns out that \( \Phi^\dagger \) is an anti-chiral superfield. To determine the transformation of the chiral superfield under a supersymmetric transformation, we first observe that

\[ Q_\alpha y^\mu = 0, \quad \bar{Q}_{\dot{\alpha}} y^\mu = -2i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}}. \quad (23) \]

Since \( \Phi \) only depends on \( y \) and \( \theta \), we have

\[ Q_\alpha \Phi(y, \theta) = Q_\alpha y^\mu \frac{\partial \Phi}{\partial y^\mu} + Q_\alpha \theta^\beta \frac{\partial \Phi}{\partial \theta^\beta} = \frac{\partial \Phi}{\partial \theta^\alpha}, \]

\[ \bar{Q}_{\dot{\alpha}} \Phi(y, \theta) = \bar{Q}_{\dot{\alpha}} y^\mu \frac{\partial \Phi}{\partial y^\mu} + \bar{Q}_{\dot{\alpha}} \theta^\beta \frac{\partial \Phi}{\partial \theta^\beta} = -2i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \frac{\partial \Phi}{\partial x^\mu}. \quad (24) \]
The Wess-Zumino model is then given by

\[ W = \psi \bar{Q} \bar{z} \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \]

For renormalisation arguments (see below) the most general invariant Lagrangian is given by the superpotential

\[ L = \psi \bar{Q}\bar{z} \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \]

Expressed in components this gives

\[ \begin{align*}
\delta z &= \sqrt{2}\bar{\psi} \theta = \sqrt{2}\bar{\psi} \theta, \\
\delta \phi &= -\sqrt{2}\bar{\psi} \theta = -\sqrt{2}\bar{\psi} \theta, \\
\delta \bar{\phi} &= -i\sqrt{2}\theta \partial \phi \sigma_{\alpha \dot{\alpha}} \bar{z}. \end{align*} \]

Identity (A.13) has been used to simplify \(-2i\sqrt{2}\theta \sigma_{\alpha \dot{\alpha}} \bar{z} \phi\). These transformation laws show two points. First, they show that supersymmetry is a symmetry which maps bosons into fermions and fermions into bosons. Secondly they show that the highest component of the superfield \(\Phi\), namely \(F\), transforms as a total derivative. This is the key point to construct invariant Lagrangians. Indeed, by definition the product of superfields is a superfield. However, the highest component of a superfield transforms as a total derivative. This means that it is a good candidate to construct invariant Lagrangians. Two types of terms may be constructed

\[ \Phi^i \Phi_{\alpha \dot{\alpha}} = \partial \phi \bar{z} \phi \theta \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \]

where we have used (A.12) to simplify the \(z\) parts and (A.13) and (A.14) for the \(\psi\) part. These terms contribute to the kinetic part of the Lagrangian. The interacting part is given by the superpotential \(W(\phi)\), which is a polynomial in \(\Phi\). This means that \(W\) is a holomorphic function. For renormalisation arguments (see below) \(W\) is of degree three. Using

\[ \begin{align*}
\Phi_i \Phi_j &= z_i z_j + \sqrt{2}\theta \left(z_i \psi_j + z_j \psi_i\right) - \theta \theta \left(z_i F_j + z_j F_i + \psi_i \psi_j\right), \\
\Phi_i \Phi_j \Phi_k &= z_i z_j z_k + \sqrt{2}\theta \left(z_i z_j z_k \psi_j + z_i z_k \psi_j + z_j z_k \psi_j\right) \quad (28)
\end{align*} \]

the most general invariant Lagrangian is given by the superpotential

\[ W(\Phi) = \alpha^i \Phi_i + \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{6} \lambda^{ijk} \Phi_i \Phi_j \Phi_k. \quad (29) \]

The Wess-Zumino model is then given by

\[ \mathcal{L}_{W.Z.} = (\Phi^i \Phi_i)_{\alpha \dot{\alpha}} + W(\Phi)_{\theta \theta} + W(\Phi)_{\bar{\theta} \bar{\theta}} + \partial \phi \bar{z} \phi \theta \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \]

\[ \begin{align*}
& \mathcal{L}_{W.Z.} = \partial \phi \bar{z} \phi \theta \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \\
& = \partial \phi \bar{z} \phi \theta \phi + \sqrt{2}\bar{\psi} \theta \phi + \sqrt{2} \phi \bar{z}. \\
& - \alpha^i F_i - \frac{1}{2} m^{ij} (z_i F_j + z_j F_i + \psi_i \psi_j) + c.c. \\
& - \frac{1}{6} \lambda^{ijk} (z_i z_j F_k + z_i z_k F_j + z_j z_k F_i + z_i \psi_j \psi_k + z_j \psi_i \psi_k + z_k \psi_i \psi_j) + c.c. \quad (30) \end{align*} \]
The equations of motion for the auxiliary fields $F$ give

$$F^{\dagger i} = \alpha^i + m^{ij} z_j + \frac{1}{2} \lambda^{ijk} z_j z_k = \frac{\partial W(z)}{\partial z_i}. \quad (31)$$

If we eliminate $F$ from the Lagrangian and use $W(z)$, the Lagrangian reduces to

$$\mathcal{L}_{W.Z.} = \partial_\mu z^i \partial^\mu \bar{z}_i + \frac{i}{2} \left( \bar{\psi}_i \sigma^\mu \partial_\mu \bar{\psi}_i - \partial_\mu \psi_i \sigma^\mu \psi_i \right) - \frac{1}{2} \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \bar{\psi}_i \psi_i - \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{z}_i \partial \bar{z}_j} \bar{\psi}_i \bar{\psi}_i, \quad (32)$$

$$V_F = \frac{1}{2} \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2,$$

the so-called $F$-terms, contributes to a scalar potential of degree four (since $W$ is of degree three) and $\frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \bar{\psi}_i \psi_i$ to Yukawa interactions between fermions and scalars. It has to be emphasised that all the interaction terms of the Wess-Zumino model are encoded into the superpotential. Furthermore, the auxiliary fields gives an equal number of degrees of freedom (d.o.f) on-shell and off-shell:

|           | on-shell | off-shell |
|-----------|----------|----------|
| $z$       | 2 d.o.f. |          |
| $\psi$    | 2 d.o.f. |          |
| $F$       | 0 d.o.f. |          |
| $\bar{F}$ | 2 d.o.f. |          |

To proceed further in a construction of supersymmetric models, one uses another type of superfields: the real superfield $V$ such that $V^\dagger = V$. This superfield enables us to construct supersymmetric Yang-Mills theories [44], and to couple chiral superfields to vector superfields in an invariant way. This means that if the various fields are in some representation of the gauge group, the superpotential has to be invariant. Finally, it should be observed that models constructed along these lines are not acceptable physically, since in their spectrum they contain bosons and fermions of the same masses. Since scalar particles with the quantum numbers of known fermions (electron, etc) have not been observed supersymmetry has to be broken.

3. Lie algebras of order $F$

As it has been mentioned previously, Lie (super)algebras are binary algebras. This means that one can always multiply two elements. Indeed, the multiplication law is given by the commutators or the anticommutators. In this section we introduce higher order algebras \textit{i.e.} defined by higher order products and try to implement them for constructing new symmetries in physics. Of course the new structures considered would have to respect the principle of Relativistic and Quantum Physics. Thus we will by-pass the no-go theorems which restrict drastically the possible extensions of the symmetry of the space-time. In this lecture, we construct some $F$-ary algebras which can be seen as a possible extension of the Lie (super)algebras. We then show that these new structures can be applied in physics in two different ways, leading to higher order non-trivial extensions of the Poincaré algebra. It is well known that $(1 + 2)$-dimensional space-time is exceptional. In such dimensional space-time, one can consider representations which are neither bosons nor fermions but anyons. An anyon is a particle of arbitrary spin. We take advantage of this situation to define an $F$-order non-trivial extension of the Poincaré algebra, which maps a relativistic anyon of spin $s \in \mathbb{R}$ to a relativistic anyons of spin $s \pm 1/F$. In the second application, we play with the way the Noether theorem is implemented in Quantum Field Theory, and construct a non-trivial cubic-extension of the Poincaré algebra for arbitrary dimensional space-time.
### 3.1. Lie algebras of order $F$: definition

The basic idea to construct higher order extensions of the Poincaré algebra is to define higher order extensions of Lie superalgebras. In supersymmetric theories, the extensions of the Poincaré algebra are obtained from a “square root” of the translations, “$Q Q \sim P$”. It is tempting to consider other alternatives where the new algebra is obtained from yet higher order roots. Basically, in such extensions, the generators of the Poincaré algebra are obtained as $F-$fold symmetric products of more fundamental generators, leading to the “$F^{th}$-root” of translation: “$Q^F \sim P^r$” with $F$ a positive integer. It is important to stress that such structures are not Lie (super)algebras (even though they contain a Lie subalgebra), and as such avoid a priori the Coleman-Mandula [1] as well as the Haag-Lopuszanski-Sohnius no-go theorems [3]. Furthermore, as far as we know, no no-go theorem associated with such types of extensions has been considered in the literature. This can open interesting possibilities to search for a field theoretic realization of a non-trivial extension of the Poincaré algebra which is not the supersymmetric one. If successful, this might throw a new light on how to construct physical models. In small space-time dimensions $D \leq 2$, several authors were able to define along these lines an extension of supersymmetry called Fractional Supersymmetry [45]. In the same manner than Lie superalgebras are the underlying mathematical structure of supersymmetry, Lie algebras of order $F$ are the mathematical structure associated to Fractional Supersymmetry. These algebras were introduced in [19, 20]. Subsequently Lie algebras of order three were studied on some formal ground. The basis of the theory of contractions and deformations in the context of Lie algebras of order three has been studied in [21, 46] and a group together with the parameters of the transformation for Lie algebras of order three were defined [47].

We give now the abstract mathematical structure which generalises the theory of Lie superalgebras and their representations. Let $F$ be a positive integer and $q = \exp\left(\frac{2\pi i}{F}\right)$. We consider a complex vector space $\mathfrak{g}$ together with a linear map $\varepsilon$ from $\mathfrak{g}$ into itself satisfying $\varepsilon^F = 1$. We set $\mathfrak{g}_k$ ($k = 0, \ldots, F - 1$) the eigenspace corresponding to the eigenvalue $q^k$ of $\varepsilon$, so that $\mathfrak{g} = \bigoplus_{k=0}^{F-1} \mathfrak{g}_k$. The map $\varepsilon$ is called the grading. If $\mathfrak{g}$ is endowed with the following structures we will say that $\mathfrak{g}$ is Lie algebra of order $F$ [19].

**Definition 3.1** Let $F \in \mathbb{N}^*$. A $\mathbb{Z}_F$-graded $\mathbb{C}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex Lie algebra of order $F$ if

(i) $\mathfrak{g}_0$ is a complex Lie algebra.

(ii) For all $i = 1, \ldots, F - 1$, $\mathfrak{g}_i$ is a representation of $\mathfrak{g}_0$. If $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_i$ then $[X, Y]$ denotes the action of $X$ on $Y$ for any $i = 1, \ldots, F - 1$.

(iii) For all $i = 1, \ldots, F - 1$, there exists an $F-$linear, $\mathfrak{g}_0-$equivariant map

$$
\{ \cdots \} : \mathcal{S}^F(\mathfrak{g}_i) \to \mathfrak{g}_0,
$$

where $\mathcal{S}^F(\mathfrak{g}_i)$ denotes the $F-$fold symmetric product of $\mathfrak{g}_i$, satisfying the following (Jacobi) identity

$$
\sum_{j=1}^{F+1} [Y_j, \{Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{F+1}\}] = 0, \quad (33)
$$

for all $Y_j \in \mathfrak{g}_i$, $j = 1, \ldots, F + 1$.

We would like to make now several observations. We clearly see that Definition 3.1 reduces to Lie algebras when $F = 1$ and to Lie superalgebras when $F = 2$. Thus Lie algebras of order
$F$ constitute a possible generalisation of Lie (super)algebras. Furthermore, we have given the definition of Lie superalgebras (Definition 2.1) and of Lie algebra of order $F$ (Definition 3.1) in a same manner in order to stress on there similitude. It should also be observed that for any $i = 1, \ldots, F - 1$, the $\mathbb{Z}_F-$graded vector spaces $g_0 \oplus \hat{g}_i$ is a Lie algebra of order $F$. We call these type of algebras elementary Lie algebras of order $F$. Finally, as for Lie superalgebras one can deduce more Jacobi identities from the definition above. Indeed, we have

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0, \quad J_1$$

$$[[X_1, X_2], Y_1] + [[X_2, Y_1], X_1] + [[Y_1, X_1], X_2] = 0, \quad J_2$$

$$[X_1, \{Y_1, \ldots, Y_F\}] = \{[X_1, Y_1], \ldots, Y_F\} + \cdots + \{Y_1, \ldots, [X, Y_F]\}. \quad J_3$$

for any $X_1, X_2, X_3 \in g_0$, $Y_1, \cdots, Y_F \in g_1$. The identity $J_1$ is the consequence of the fact that $g_0$ is a Lie algebra, $J_2$ is equivalent to assume that $g_1$ is a representation of $g_0$ and $J_3$ is related to the $g_0-$equivariance of the mapping $S^F(g_1) \to g_0$. Only (33) is an extra constraint.

A representation of an elementary Lie algebra of order $F$ is a linear map $\rho: g = g_0 \oplus g_1 \to \text{End}(V)$, such that (for all $X_i \in g_0, Y_j \in g_1$)

$$\rho([[X_1, X_2]]) = \rho(X_1)\rho(X_2) - \rho(X_2)\rho(X_1)$$

$$\rho([[X_1, Y_2]]) = \rho(X_1)\rho(Y_2) - \rho(Y_2)\rho(X_1)$$

$$\rho(\{Y_1, \cdots, Y_F\}) = \sum_{\sigma \in S_F} \rho(Y_{\sigma(1)}) \cdots \rho(Y_{\sigma(F)}) \quad (34)$$

($S_F$ being the group of permutations of $F$ elements). If $V = V_0 \oplus \cdots \oplus V_{F-1}$ then for all $a \in \{0, \cdots, F - 1\}$, $V_a$ is a $g_0-$module and we have

$$\rho(g_1)(V_a) \subseteq V_{a+1}. \quad (35)$$

There is no need to impose further conditions, since the Jacobi identity (33) is just a consequence of the associativity of the product in $\text{End}(V)$.

In Definition 3.1 we have defined complex Lie algebras of order $F$. Since we want to implement these new structures to extend non-trivially the Poincaré algebra and since this last algebra is a real Lie algebra, real forms of Lie algebras of order $F$ should be defined. A real elementary Lie algebra of order $F$ is given by a real Lie algebra $g_0$ and $g_1$ a real representation of $g_0$ which satisfy the axioms of complex elementary Lie algebras of order $F$. Of course when considering a real form of a complex $g$ some structures are lost such that the grading map $\varepsilon$. Many examples of Lie algebras of order $F$ (complex and real) were constructed. In these lectures, since we study non-trivial higher order extensions of the Poincaré algebra we only give Lie algebras of order $F$ which are related to the Poincaré algebra. There is two types of such extensions infinite-dimensional and finite-dimensional that will be studied below.

### 3.2. Lie algebras of order $F$ in $(1+2)-dimensions$

In this subsection we construct explicitly an infinite dimensional extension of the Poincaré algebra in $(1 + 2)-dimensions$ [23]. Studying the representations of this extension gives that this higher-order symmetry is a symmetry between relativistic anyons. We recall firstly some results upon the Lorentz and the Poincaré groups in three dimensions, and then construct a higher order extension of the Poincaré symmetry.
3.2.1. Discrete series of \( \text{SO}(1,2) \) Since \( \text{SO}(1,2) \) is a non-compact Lie group its unitary representations are infinite dimensional. There are two types of unitary representations. The discrete and the continuous series. The former are either bounded from below or bounded from above and the latter are unbounded representations. These representations were studied by Bargmann [48] (see also [49]). The generators of the Poincaré algebra in \((1+2)\)-dimensions are \((P_0, P_1, P_3)\) and \((J_0, J_1, J_2)\) and the algebra is given by

\[
\begin{align*}
[J_\mu, J_\nu] &= -\epsilon_{\mu\nu\rho}\eta^{\rho\sigma}J_\sigma, \\
[J_\mu, P_\nu] &= -\epsilon_{\mu\nu\rho}\eta^{\rho\sigma}P_\sigma, \\
[P_\mu, P_\nu] &= 0,
\end{align*}
\]

with \(\eta_{\mu\nu} = \text{diag}(1, -1, -1)\) the three-dimensional Minkowski metric and \(\epsilon_{\mu\nu\rho}\) the Levi-Civita tensor normalised as follows \(\epsilon_{012} = \epsilon^{012} = 1\). If we consider the complexified of the Poincaré algebra and introduce \(L_0 = -iJ_0, L_\pm = -iJ_1 \mp iJ_2\) and \(\Pi_0 = -iP_0, \Pi_\pm = -iP_1 \mp iP_2\) (these unconventional notations come from our normalisation and from the fact that our structure constants are real for Lie algebras) the commutation relations reduce to

\[
\begin{align*}
[L_0, L_\pm] &= \pm L_\pm \\
[L_0, \Pi_\pm] &= \Pi_\mp \\
[L_\pm, \Pi_\mp] &= -2\Pi_0 \\
[L_\pm, L_-] &= -2L_0
\end{align*}
\]

\[ (36) \]

Since \(\pi_1(\text{SO}(1,2)) = \mathbb{Z}\) there exists, in some universal covering group noted \(\text{SO}(1,2)\), representations with arbitrary spin \(s \in \mathbb{R}\). This means that the eigenvalues of \(L_0\) are not any more integer (bosonic) or half-integer (fermionic). There are two types of unitary representations for \(\text{SO}(1,2)\) the discrete or the continuous series [48, 49]. The only representations that are relevant in the sequel are the discrete series which are either bounded from below or bounded from above. We have

\[
\begin{align*}
\mathcal{D}_s^+: L_{s,0}|s_+,n\rangle &= (s+n)|s_+,n\rangle \\
L_{s,0}|s_-,n\rangle &= \sqrt{(2s+n)(n+1)}|s_-,n+1\rangle \\
L_{s,-}|s_+,n\rangle &= \sqrt{(2s+n-1)n}|s_+,n-1\rangle
\end{align*}
\]

\[ (37) \]

for representations bounded from below and

\[
\begin{align*}
\mathcal{D}_s^-: L_{s,0}|s_-,n\rangle &= -(s+n)|s_-,n\rangle \\
L_{s,0}|s_+,n\rangle &= -\sqrt{(2s+n+1)n}|s_+,n+1\rangle \\
L_{s,-}|s_-,n\rangle &= -\sqrt{(2s+n)(n+1)}|s_-,n-1\rangle
\end{align*}
\]

\[ (38) \]

for representations bounded from above. For the first representation we have \(L_{s,-}|s_+,0\rangle = 0\) and for the second \(L_{s,+}|s_-,0\rangle = 0\). These representations are of dimensions \(2|s|+1\) if \(2s\) is a negative integer but in the general case we have an infinite number of states. In both cases the quadratic Casimir operator \(Q = L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+)\) has eigenvalues \(s(s-1)\). Furthermore, when \(s < 0\) the representations are not unitary. There is a well-known way to characterise these representations, using functions of complex variables (see the first reference of [49]).

To define higher order extensions of the Poincaré algebra, taking the supersymmetric extension as a guideline, we choose representations for which \(s = -1/F\) to build a non-trivial

\[ ^1 \text{In the mathematical literature one prefers to take } K_0 = J_1 \text{ for the Cartan subalgebra with } K_\pm = J_0 \mp J_3. \text{ We have made of different choice – in the complexified of } \mathfrak{so}(1,2) – \text{ in such way that the eigenvalues of } L_0 \text{ correspond to the eigenvalues of the } \mathfrak{SO}(2) \text{ subgroup of } \mathfrak{so}(1,2) \text{ and can be identified with the spin of the state.} \]
extension of the Poincaré algebra. If we observe the relations (38) and (39) with \( s = -1/F \), we see an ambiguity in the square root of \(-2/F\). So a priori we have four different representations for \( s = -1/F \), (two bounded from below/above) with the two choices \( \sqrt{-1} = \pm i \). We note \( \mathcal{D}_{1/F; \pm}^\pm \) (with obvious notations) these representations. Next, we can make the following identifications:

- the dual representation of \( \mathcal{D}_{1/F; +}^- \) is obtained through the substitution \( J_\mu \rightarrow - (J_\mu)^\dagger \) and is given by \( \left[ \mathcal{D}_{-1/F; +}^+ \right]^* \cong \mathcal{D}_{-1/F; +}^- \);
- the complex conjugate representation of \( \mathcal{D}_{1/F; +}^- \) is defined by \( J_\mu \rightarrow J_\mu \) (we have to be careful when we do such a transformation because we have by definition \( L_\pm = -i J_1 \mp J_2 \) for any representation) is given by \( \mathcal{D}_{-1/F; +}^+ \cong \mathcal{D}_{-1/F; -}^+ \);
- the dual of the complex conjugate representation of \( \mathcal{D}_{1/F; -}^+ \) is given by \( \left[ \mathcal{D}_{-1/F; +}^- \right]^* \cong \mathcal{D}_{-1/F; -}^- \).

If we note \( \psi_a \in \mathcal{D}_{1/F; +}^+ \), \( \psi^a \in \mathcal{D}_{-1/F; +}^-, \bar{\psi}_a \in \mathcal{D}_{-1/F; -}^- \) and \( \bar{\psi}^a \in \mathcal{D}_{-1/F; -}^+ \) then we have the following transformation laws:

\[
\begin{align*}
\psi_a' &= S_a^b \psi_b \\
\psi^a' &= (S^{-1})_b^a \psi^b \\
\bar{\psi}_a' &= \bar{(S)}_a^b \bar{\psi}_b \\
\bar{\psi}^a' &= ((S^{-1})^b_a) \bar{\psi}^b.
\end{align*}
\]

Furthermore, if we define

\[
\psi^a = g^{a\alpha} \bar{\psi}_\alpha,
\]

we can write the following scalar product

\[
\varphi^a \psi_a = -\bar{\varphi}_0 \psi_0 + \sum_{a > 0} \bar{\varphi}_a \psi_a,
\]

where the infinite matrix \( g^{a\alpha} \) and its inverse \( g_{a\alpha} \) are given by \( \text{diag}(-1,1,\cdots,1,\cdots) \). The reason why we have a pseudo-hermitian scalar product is because we are dealing with a non-unitary representation of a non-compact Lie group. The invariant scalar product gives an explicit isomorphism between the two representations bounded from below (or above) \( ((S^{-1})_b^a = g^{a\alpha} \bar{(S)}_\alpha^b \gamma_{\alpha\beta}) \). From now on, we choose \( \sqrt{-2/F} = i \sqrt{2/F} \) for representations bounded from below and \( \sqrt{-2/F} = -i \sqrt{2/F} \) for those bounded from above.

### 3.2.2. Representations of the three-dimensional Poincaré algebra

Particles are classified according to the values of the Casimir operators of the Poincaré algebra. More precisely, for a mass \( m \) particle of positive/negative energy, the unitary irreducible representations are obtained by studying the little group leaving the rest-frame momentum \( \Pi^\alpha = -i P^\alpha = (m,0,0) \) invariant. (In the same manner we consider \( L_\mu = -i J_\mu \) the angular momentum.) This stability group in \( SO(1,2) \) is simply the universal covering group \( \mathbb{R} \) of the abelian subgroup of rotations \( SO(2) \) (generated by \( L^0 \)). As is well-known, such a group is not quantised. This means that the substitution \( L^0 \rightarrow L^0 + s \) leaves the \( SO(2) \) part invariant. But the remarkable property of \( SO(1,2) \) is that the concomitant transformation on the Lorentz boosts \( L^i \rightarrow L^i + s \frac{\Pi^i}{\Pi_{\alpha\beta} \gamma_{\alpha\beta}} \) leaves the algebraic structure (36) unchanged. Indeed, on account of the mass-shell condition
where 

\[ \Pi_0^2 - \Pi_1^2 - \Pi_2^2 = m^2, \]

which proves that \((L_0 + s, L_1 + s \frac{H_m}{\Pi_2})\) satisfy the same algebra as \((L_0, L_1)\). Anyway, following the method of induced representations of groups expressible as a semi-direct product, we find that unitary irreducible representations for a massive particles are one dimensional.

The main difference between \(SO(1,2)\), or more precisely the proper orthochronous Lorentz group, and \(SO(3)\) is that \(\Pi_0 + m\) never vanishes with \(SO(1,2)\) and \(s\) does not need to be quantised. In Ref.\([51, 52]\), a relativistic wave equation for massive anyons was given. First, notice that the two Casimir operators are the two scalars \(P.P\) and \(P.J\) and their eigenvalues for a spin--s unitary irreducible representation are \(-m^2\) and \(-ms\), respectively. The equations of motion are then

\[
(P^2 + m^2)\Psi = 0 \quad (43)
\]

\[
(P.J + s m)\Psi = 0.
\]

However, to obtain manifestly covariant equations one has to go beyond the mass-shell conditions \((43)\) given by the induced representation. Therefore, we can start with a field which belongs to the appropriate spin--s representation of the full Lorentz group instead of the little group. When \(s\) is a negative integer, or a negative half-integer, this representation is not unitary and is \(2|s| + 1\) dimensional, and the solution of the relativistic wave equations reduces to the appropriate induced representation (see \([50, 51]\) for an explicit calculation in the case \(|s| = 1, 1/2\)). When \(s\) is an arbitrary number, the representation is infinite dimensional and belongs to the discrete series of \(SO(1,2)\). A relativistic wave equation for an anyon in the continuous series was also considered in \([52, 53]\). For completeness, we recall in Appendix B, how M. Plyushchay obtained a very nice relativistic wave equation in \([53]\).

3.2.3. Non-trivial extension of the Poincaré algebra in \((1 + 2)-\)dimensions

Using the representations \((39-38)\), and with the sign ambiguity resolved, we can define two series of operators, belonging to a non-trivial representation of the Poincaré algebra. We denote now \(\sqrt{-1} = i\). Note \(A^+_1/F_{-n}\) those built from the representation bounded from below \(D^+_{-1/F_{1+}}\) and \(A^-_{1/F_{1+}}\) the charges of the representation bounded from above \(D^-_{-1/F_{1-}}\). Recall that \((D^+_{-1/F_{1+}}) \cong D^-_{-1/F_{1-}}\). Using \((39, 38)\) we get

\[
\begin{align*}
L_0, A^+_{1/F_{-n}} &= (n - 1/F) A^+_{1/F_{-n}} \\
L_+, A^+_{1/F_{-n}} &= \sqrt{(-2/F + n)(n + 1)} A^+_{1/F_{n+1}} \\
L_-, A^+_{1/F_{-n}} &= \sqrt{(-2/F + n - 1)n} A^+_{1/F_{n-1}} \\
L_0, A^-_{1/F_{n+1}} &= -(n - 1/F) A^-_{1/F_{n+1}} \\
L_+, A^-_{1/F_{n+1}} &= -\left(\sqrt{(-2/F + n - 1)n}\right)^* A^-_{1/F_{n+1}} \\
L_-, A^-_{1/F_{n+1}} &= -\left(\sqrt{(-2/F + n)(n + 1)}\right)^* A^-_{1/F_{n+1}}.
\end{align*}
\]

We want to combine this algebra \((44)\) in a non-trivial way with the Poincaré algebra \((36)\). With such a choice, \(A^+_1\) (resp. \(A^-_1\)) has helicity \(h = -\frac{1}{F}\) (\(\frac{1}{F}\) resp.). With the above choices
for the square roots of the negative numbers, we know that the representations are conjugate to each other, i.e. $(A^+_{-1/F+n})^\dagger = A^-_{-1/F+n}$.

By Definition 3.1, to associate a Lie algebra of order $F$ with $\mathcal{D}^\pm_{-1/F,\pm}$, we have to relate $S^F(\mathcal{D}^\pm_{-1/F,\pm})$ to $\mathcal{D}_{-1}$. But here some care has to be taken, since $\mathcal{D}^\pm_{-1/F,\pm}$ are infinite-dimensional representations of $\mathfrak{so}(1,2)$, although $\mathcal{D}_{-1}$ is finite-dimensional. Indeed, using (44) we have

$$[L_\pm, (A^\pm_{-1/F,\pm})^F] = 0, \quad [L_0, (A^\pm_{-1/F,\pm})^F] = \mp(A^\pm_{-1/F,\pm})^F.$$ 

This means that $(A^\pm_{-1/F,\pm})^F$ might be identified with a primitive vector for the vector representation. Consider hence $\langle (A^\pm_{-1/F,\pm})^F \rangle \subset S^F(\mathcal{D}^\pm_{-1/F,\pm})$, the representation built from the primitive vector $(A^\pm_{-1/F,\pm})^F$. For this representation, a priori we have

$$[L_\pm, [L_\pm, (A^\pm_{-1/F,\pm})^F]] \neq 0, \quad (45)$$

but using (37) we easily see that

$$[L_\pm, [L_\pm, [L_\pm, (A^\pm_{-1/F,\pm})^F]]] = 0. \quad (46)$$

This can be represented by means of a diagram. We denote now $[X, Y] = \text{ad}(X)Y, (A^\pm_{-1/F,\pm})^F = V_\pm, \text{ad}(L_\pm)^n(A^\pm_{-1/F,\pm})^F = V_{\pm(n-1)}$

$$\begin{array}{c}
V_\mp \xrightarrow{L_\pm} V_\pm \xrightarrow{L_\pm} V_\mp \xrightarrow{L_\pm} V_{\pm2} \xrightarrow{L_\pm} V_{\pm3} \xrightarrow{L_\pm} \cdots \xrightarrow{L_\pm} V_{\pm n} \xrightarrow{L_\pm} \cdots \\
0 \xrightarrow{L_\mp} 0 \xrightarrow{L_\mp}
\end{array} \quad (47)$$

This means that we can safely impose $V_n = 0$ for $n = 1, 2, 3, \ldots$. This can be presented on a more formal ground. See Appendix Appendix C for more details. To construct a non-trivial extension of the Poincaré algebra we then proceed as follows. We identify $(A^\pm_{-1/F,\pm})^F$ with $\Pi_\mp$ and we impose the relation $\text{ad}^3(L_\pm)(A^\pm_{-1/F,\pm})^F = 0$. Thus we have the following isomorphism

$$\mathcal{D}_{-1} = \langle \Pi_0, \Pi_+, \Pi_- \rangle = \left\{ (A^\pm_{-1/F,\pm})^F, \left[ L_\pm, (A^\pm_{-1/F,\pm})^F \right], \left[ L_\pm, \left[ L_\pm, (A^\pm_{-1/F,\pm})^F \right] \right] \right\}$$

which leads to the following brackets
\[
\frac{1}{F!} \left\{ A_{-\frac{1}{F}}, \ldots, A_{-\frac{1}{F}} \right\} = \Pi_+ \\
\frac{1}{F!} \left\{ A_{-\frac{1}{F}}, A_{-\frac{1}{F}}, A_{-\frac{1}{F}} \right\} = \pm i \sqrt{2} \Pi_0 \\
-\left(\frac{F-1}{F!}\right) \left\{ A_{-\frac{1}{F}}, \ldots, A_{-\frac{1}{F}}, A_{-\frac{1}{F}}, A_{-\frac{1}{F}} \right\} = \Pi_+ \\
\left[ L_\pm, L_\pm, \left[ L_\pm, \left( A_{-\frac{1}{F}} \right)^F \right] \right] = 0
\]

How can we address the question of the remaining brackets? In fact, it is impossible to find a decomposition

\[
S^F \left( D_{-1/F}^\pm \right) = D_{-1} \oplus D_{-1}^\perp,
\]

where \( D_{-1} \) is stable under \( SO(1, 2) \). Indeed, if there were such a decomposition, there would exist a \( SO(1, 2) \) equivariant projection

\[
\pi : S^F \left( D_{-1/F}^\pm \right) \rightarrow D_{-1}.
\]

But then \( X^\pm = \pi \left( S^F \left( A_{-1/F}, \ldots, A_{-1/F}, A_{-3/F} \right) \right) \in D_{-1} \) would satisfy (see 48)

\[
\left[ L_\mp, \left[ L_\mp, \left( X^\pm \right) \right] \right] = i F! \sqrt{2} \sqrt{2(1-2/F)} \sqrt{3(2-2/F)} \Pi_\mp \neq 0,
\]

and this is impossible because in the vector representation \( D_{-1} \), \( \text{ad}^3(L_-) \) acts as zero. This means that the algebra given by (48) is not a Lie algebra of order three. There are in fact two ways to define a Lie algebra of order three. Either to embed \( D_{-1} \) into an infinite dimensional (but indecomposable) representation of \( \mathfrak{so}(1, 2) \) or to embed the algebra \( \mathfrak{so}(1, 2) \) into an infinite dimensional algebra, which here turns out to be the De Witt algebra (the centerless Virasoro algebra) [19, 54]. Since this part is technical and not relevant for the sequel, it will be developed in Appendix Appendix C.

3.2.4. Representations Before studying the representations of the algebra given by the quadratic relations (36) and (44) and the higher order brackets (48), let us draw some general feature. First, since \( P^2 \) commutes with all the generators, all states in an irreducible representation have the same mass. Second, define an anyonic-number operator (this is in fact the grading map of the definition of Lie algebras of order \( F \)) \( \exp(2i\pi N A) \) which gives the phase \( e^{2ixs} \) on a spin \(-s\) anyon and assume we have a finite dimensional representation of our algebra. We have \( \text{tr} \exp(2i\pi N A) = 0 \) showing that in each irreducible representation there are \( F \) possible states with helicity \((s, s \pm \frac{1}{F}, \ldots, s \pm \frac{F-1}{F})\), where \( s \) will be specified later and the dimension of the space with a given helicity is always the same. This can be checked proving by that \( (\exp(2i\pi N A))A_s = e^{2i\pi s A_s} \exp(2i\pi N A) \) and using cyclicity of the trace.
\[
\text{tr}\left(\exp(2i\pi NA) \left\{ A_{\frac{1}{F}}^+, \ldots, A_{-\frac{1}{F}}^+, A_{1-\frac{1}{F}}^+ \right\}_F \right) = (F-1)! \times \text{tr}\left( \sum_{a=0}^{F-1} e^{2i\pi N A} \left( A_{\frac{1}{F}}^+ \right)^a \left( A_{1-\frac{1}{F}}^+ \right) \right) = (F-1)! \times \left( \sum_{a=0}^{F-1} e^{-2i\pi N A} \right) \left( \left( A_{1-\frac{1}{F}}^+ \right)^{F-1} e^{2i\pi N A} \left( A_{1-\frac{1}{F}}^+ \right) \right) = 0.
\]

To study the representations of the higher order extension of the Poincaré algebra, we follow the method of induced representations of Wigner. We restrict ourselves to massive representations. The massive representations are characterised by \( P^\mu P_\mu = -m^2 \) (recall that \( P_\mu \) are antihermitian). Going in the rest-frame where \( P^\mu = (im, 0, 0) \) gives that the only non-vanishing bracket are the ones involving the generator \( A_{-\frac{1}{F}+1/F}^\pm, A_{1-\frac{1}{F}}^\pm \). Thus we assume that \( A_{s/2}^\pm = 0 \) with \( s \neq -1/F, 1 - 1/F \) i.e. the “active” charges are \( A_{-\frac{1}{F}}^\pm, A_{1-\frac{1}{F}}^\pm \) and the little algebra is generated by \( 1, L_0 \) and \( A_{-\frac{1}{F}}^\pm, A_{1-\frac{1}{F}}^\pm \). After appropriate rescaling of the generators, the higher order brackets are given by

\[
(F-1)! = \left\{ A_{-\frac{1}{F}}^\pm, \cdots, A_{-\frac{1}{F}}^\pm, A_{1-\frac{1}{F}}^\pm \right\} \quad \quad \quad (51)
\]

This algebra, called the Clifford algebra of polynomial’s is studied in Appendix Appendix D. A relevant representation is given by

\[
A_{\pm\frac{1}{F}}^+ = \begin{pmatrix}
\frac{1}{\sqrt{F-1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{\sqrt{2(F-2)}} & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{3(F-3)}} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \sqrt{(F-1)!} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad A_{1-\frac{1}{F}}^+ = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \quad \quad \quad (52)
\]

In [23] another equivalent representation was exhibited. From the basic conjugation we obtain

\[
A_{-\frac{1}{F}}^- = (A_{-\frac{1}{F}}^+)^\dagger, \quad A_{1-\frac{1}{F}}^- = (A_{1-\frac{1}{F}}^+)^\dagger.
\]

As a direct consequence, the generators satisfy the additional relations

\[
A_{1-\frac{1}{F}}^+ = \frac{1}{(F-1)!} (A_{-\frac{1}{F}}^-)^{F-1},
\]

together with the quadratic relations

\[
\begin{align*}
\left[ A_{-\frac{1}{F}}^-, A_{1-\frac{1}{F}}^+ \right] &= 2N = 2\text{diag}\left( \frac{F-1}{2}, \frac{F-3}{2}, \cdots, \frac{1-F}{2} \right) \\
\left[ N, A_{1-\frac{1}{F}}^\pm \right] &= \mp A_{-\frac{1}{F}}^\pm.
\end{align*}
\]

This means that \( iN, \frac{1}{2}(A_{-\frac{1}{F}}^+ + A_{-\frac{1}{F}}^-), \frac{1}{2}(A_{1-\frac{1}{F}}^+ - A_{1-\frac{1}{F}}^-) \) are antihermitian and generate the finite dimensional unitary irreducible representation of \( su(2) \). Thus the representation
of the little algebra build with $A_{\pm 1/F}$ will be unitary. In fact, we obtain an $F$–dimensional representation of $su(2)$. To identify the helicity content of the representation, we assume that we are starting from a vacuum state $\Omega^+_{\lambda} = \Omega_{\lambda}$ in the spin–$\lambda$ representation of $SO(1, 2)$ which is annihilated by $A^{-1/F}_-$. Thus acting on the vacuum state with the operator $A^+_{-1/F}$ we identify the helicity content of the representation. Indeed the states $(A^+_{-1/F})^n \Omega_\lambda$, $n = 0, \ldots, F - 1$ are of helicity $h = \lambda - n/F$. We assume further that these states are of positive energy. Since the representation has to be CPT invariant, we consider also a conjugated vacuum $\Omega^-_{-\lambda} = \Omega_{\lambda}$ of helicity $h = -\lambda$, negative energy and annihilated by $A^+_{-1/F}$. Acting with $A^{-1/F}_-$ on the conjugated vacuum, we obtain the states of helicity $n/F - \lambda$. This can be summarised in the following table:

| states | helicity | states | helicity |
|--------|----------|--------|----------|
| $\Omega^+_{\lambda}$ | $\lambda$ | $\Omega^-_{-\lambda}$ | $-\lambda$ |
| $A^+_{-1/F} \Omega^+_{\lambda}$ | $\lambda - 1/F$ | $A^{-1/F}_- \Omega^-_{-\lambda}$ | $-\lambda + 1/F$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $(A^+_{-1/F})^n \Omega^+_{\lambda}$ | $\lambda - a/F$ | $(A^{-1/F}_-)^n \Omega^-_{-\lambda}$ | $-\lambda + a/F$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $(A^+_{-1/F})^{F-1} \Omega^+_{\lambda}$ | $\lambda - (F - 1)/F$ | $(A^{-1/F}_-)^{F-1} \Omega^-_{-\lambda}$ | $-\lambda + (F - 1)/F$ |

The states of positive energy and helicity $(\lambda, \lambda - 1/F, \ldots, \lambda - F^{-1})$ are CP- conjugated to the states of negative negative energy and helicity $(-\lambda, -\lambda + 1/F, \ldots, -\lambda + F^{-1})$. Having the states on shell, the states of positive energy are boosted and belong to the representations bounded from below $D^+_{\lambda - n/F}$ while those of negative energy correspond to the representations bounded from above $D^-_{\lambda - n/F}$. It should be interesting to have a explicit Lagrangian where the higher order extensions of the Poincaré algebra are realised, but up to now there are no results in these direction. As a final remark let us mention that if we had considered only one series of operator $A$, say $A_{n-1/F}$, the representation would not have been unitary.

3.3. Cubic extensions of the Poincaré algebra in any dimensions

Now, we would like to extend the ideas developed previously in any space-time dimensions, namely to construct higher order extensions of the Poincaré algebra in $D$–dimensions. We have shown that considering any semi-simple Lie algebra $\mathfrak{g}$ and any representation $\mathcal{D}$, along the lines of Appendix Appendix C, one is able to define a Lie algebra of order $F$ starting from $\mathfrak{g} \oplus \mathcal{D}$ [19]. In particular this applies to $\mathfrak{so}(1, D - 1) \oplus \mathcal{D}_\nu$, with $\mathcal{D}_\nu$ the vector representation of $\mathfrak{so}(1, D - 1)$. However, this construction leads to infinite dimensional representations of the Lorentz group which are not exponentiable, and as a consequence is unacceptable for realistic physical models. This obstruction is valid as soon as the space-time dimension is greater than four. Indeed, even in four dimensions, where the situation is somewhat exceptional, we cannot construct cubic extensions along the lines of infinite dimensional algebras. In this case, the little group for massless particles is $E(2)$, the group of affine transformations in the plane. Representing the translational part by zero gives rise to $SO(2)$, which is not quantised. So we could have expected to consider, in this case, some massless states with fractional helicity. In [55] we defined a relativistic wave equation along the lines of Appendix Appendix B, considering infinite-dimensional representations corresponding formally to the massless states with fractional (real)
helicity. But the solutions of the relativistic equation, however, break down “spontaneously” to the $(1 + 3)D$ Poincaré invariance to the $(1 + 2)D$ Poincaré invariance and induce a compactification on a circle that produces a consistent theory for massive anyons in $D = 1 + 2$.

This seems to be a serious obstacle to define higher order extensions of the Poincaré algebra in the framework of Lie algebras of order $F$ when the space-time dimension is greater than four. Therefore, if we expect to apply higher order extensions of the Poincaré algebra, we have to follow new lines. Subsequently, it was realised that finite-dimensional Lie algebras of order $F$ could be defined. Indeed induction theorems enable us to define Lie algebras of order $F$ from Lie algebras or Lie superalgebras [20]. Thus, this opens the way to obtain higher order extensions of the Poincaré algebra. However, we should understand in that context, how these structures would not be in conflict with the Noether and spin-statistics theorems. Since we continue to assume that bosonic fields are quantised by commutation relations although fermionic fields are quantised by anticommutation relations, a fresh look to the Noether theorem should be given in order to implement Lie algebras of order $F$.

### 3.3.1. Noether theorem in higher order algebras

To understand in which way the Noether theorem can be implemented in higher order symmetries, we first recall some well-known results and give some general features in the conventional case i.e. when the symmetries correspond to a Lie (super)algebra. According to Noether theorem, to all the symmetries there correspond conserved currents. The symmetries are then generated by charges which are expressed in terms of the fields. We actually start from the classical field theory case, then go to the quantum case through the usual canonical quantisation procedure. Starting from a general Lagrangian $\mathcal{L}$ at the classical level, invariant under some Lie (super)algebra $g$, one constructs through the standard procedure the Noether charges $\hat{T}_a$

$$\varepsilon^a \hat{T}_a = \int d^3x \frac{\partial L}{\partial \partial_0 \Psi} \delta_\varepsilon \Psi$$  \hspace{1cm} (54)

associated with the transformation

$$\delta_\varepsilon \Psi = \varepsilon^a T_a \Psi$$  \hspace{1cm} (55)

where $T_a$ generate the Lie (super)algebra in some appropriate (matrix) representation. Upon use of eq.(55) in eq.(54) one gets

$$\hat{T}_a = \int d^3x \Pi(x) T_a \Psi(x)$$  \hspace{1cm} (56)

where $\Pi(x) = \frac{\partial L}{\partial \partial_0 \Psi}$ is the conjugate momentum. Equation (56) is the general relation between $\hat{T}_a$ and $T_a$. At the quantum level, $\hat{T}_a$ and $\Psi$ are operators acting in some Hilbert space and we have

$$[\varepsilon^a \hat{T}_a, \Psi(x)] = \delta_\varepsilon \Psi.$$  \hspace{1cm} (57)

Thus the conserved charges $\hat{T}_a$ from Noether’s currents, realise the algebra in the sense that

$$[\delta_a, \delta_b] \Psi = \delta_a(\delta_b \Psi) - \delta_b(\delta_a \Psi) = f_{abc} \delta_c \Psi$$  \hspace{1cm} (58)

where $\Psi$ is a field operator and $\delta$ is by definition given by

$$\delta_a \Psi = [\hat{T}_a, \Psi].$$  \hspace{1cm} (59)

Equation (58) reads then equivalently

$$[\hat{T}_a, [\hat{T}_b, \Psi]] - [\hat{T}_b, [\hat{T}_a, \Psi]] = f_{ab}^c [\hat{T}_c, \Psi].$$  \hspace{1cm} (60)
Now due to Jacobi identity \([[A, [B, C]] + \text{cyclic} = 0\)], one can recast the left-hand side of Eq. (60) in the form \([[\hat{T}_a, \hat{T}_b], \Psi]] to get

\[
[[\hat{T}_a, \hat{T}_b], \Psi] = f_{ab}^c [\hat{T}_c, \Psi] \quad \text{(for any } \Psi) \tag{61}
\]

meaning that

\[
[\hat{T}_a, \hat{T}_b] = f_{ab}^c \hat{T}_c \tag{62}
\]
at least on some (sub-)space of field operators \(\Psi\).

Now consider a Lie algebra of order three where the cubic brackets are given by

\[
\{Y_i, Y_j, Y_k\} = Q_{ijk}^a X_a, \tag{63}
\]

all the results above remain true (Eqs. (54) and (59)) but the analogy with the steps described above stops at Eq. (60). Now the algebra is realised (with \(\hat{Y}_i, \hat{X}_a\) the conserved charges associated to (63))

\[
(\delta_i, \delta_j, \delta_k + \text{perm}) \Psi = [\hat{Y}_i, [\hat{Y}_j, [\hat{Y}_k, \Psi]]] + \text{perm} = Q_{ijk}^a [\hat{X}_a, \Psi], \tag{64}
\]

but, the (generalised) Jacobi identities (see Eq. (33)), do not allow to obtain

\[
[[\hat{Y}_i, \hat{Y}_j, \hat{Y}_k], \Psi] = Q_{ijk}^a [\hat{X}_a, \Psi] \tag{65}
\]

which would have been the analog of Eq. (61) above. In other words, due to the lack of some appropriate Jacobi identities, the quantised version of the Noether charges algebra is just (64) and cannot be cast simply in a \(\Psi\) independent form. We should stress at this level that the difference with the conventional algebras and algebras of the type (63) we are pointing out, does not mean the absence of a realisation of this algebra in terms of Noether charges, as we will see latter on.

We thus define the adjoint representation where the generators of the Lie algebra of order three act on some given operator \(\Phi\) which belongs to some endomorphism space \(\text{End}(V)\) of some vector space \(V\). In Quantum Mechanics, \(V\) reduces to some Hilbert space. The adjoint representation of an elementary Lie algebra of order three is a linear map

\[
\text{ad} : \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \rightarrow \mathcal{L}(\text{End}(V), \text{End}(V)) \quad \text{ad}(g) \tag{66}
\]

such that for all \(\Phi \in \text{End}(V)\) we have \(\text{ad}(g)\Phi = [\text{ad}(g), \Phi]\). In the adjoint representation, the algebra is realised through multiple-commutators. Furthermore, it is a matter of calculation to check that the Jacobi identities (33) are satisfied.

Another difference with usual quadratic algebras lies on the tensor product of representations (this has implications on \(N\)–particles states at the quantum level). Assume that we have two representations \(\rho_1\) of and \(\rho_2\) of (63), then \(\rho_1 \otimes 1 + 1 \otimes \rho_2\) is a representation if \(\otimes\) is the twisted tensor product, such that for homogeneous elements \(G_1, G_2, G_3, G_4\) of \(\mathfrak{g}\) we have \((G_1 \otimes G_2)(G_3 \otimes G_4) = q^{[G_2][G_3]}G_1G_3 \otimes G_2G_4\), where \([G]\) denotes the \(\mathbb{Z}_3\)–grading of \(G\) and \(q\) is a cubic primitive root of the unity. Indeed, if we calculate

\[
\{\rho_1(Y_i) \otimes 1 + 1 \otimes \rho_2(Y_i), \rho_1(Y_j) \otimes 1 + 1 \otimes \rho_2(Y_j), \rho_1(Y_k) \otimes 1 + 1 \otimes \rho_2(Y_k)\} \tag{68}
\]
The algebra (69) enables us to define a cubic extension of the Poincaré algebra in any space-time structure given by a basis of $g$. It is also possible to construct Lie algebras of order three. A major progress, in this direction, was done when it was realised that it was possible to construct Lie algebras of order three have been identified, and turn out to be related to the three-exterior algebra (see Eq. (D.10), Appendix D). Secondly, a group associated to a Lie algebra of order three was constructed. This latter being simply some matrix group where the matrix elements belong to the three-exterior algebra, in a straight analogy with Lie supergroups associated to Lie superalgebras [47].

### 3.3.2. Lie algebras of order three in any space-time dimensions

As mentioned previously, it seems rather difficult to define higher order extensions of the Poincaré algebra in the framework of infinite dimensional Lie algebras of order $F$ when the space-time dimension is bigger than three. A major progress, in this direction, was done when it was realised that it was possible to construct finite-dimensional Lie algebras of order $F$ from the usual Lie algebras and Lie superalgebras [20], by an induction theorem.

**Theorem 3.2** (M. Rausch de Traubenberg, M. J. Slupinski, [20]). Let $g_0$ be a Lie algebra and $g_1$ be a $g_0$-module such that:

1. $g = g_0 \oplus g_1$ is a Lie algebra of order $F_1 > 1$;
2. $g_1$ admits a $g_0$-equivariant symmetric form of order $F_2 \geq 2$.

Then $g = g_0 \oplus g_1$ inherits the structure of a Lie algebra of order $F_1 + F_2$.

The proof of this theorem is not very difficult, the only delicate point is to prove the Jacobi identities (33). Indeed one can show that these last identities result on some factorisation property [20]. This theorem can be extended to include the $F_1 = 1$ cases, and as a simple consequence, one can define elementary Lie algebras of order three from any Lie algebras $^2$. Let $g_0$ be any Lie algebra and let $g_1$ be its adjoint representation. Introduce $\{J_a, a = 1, \cdots, \dim g_0\}$ a basis of $g_0$, $\{A_a, a = 1, \cdots, \dim g_0\}$ the corresponding basis of $g_1$. The invariant forms are given by the Casimir operators. In particular, for the Killing form, set $g_{ab} = Tr(A_aA_b)$ and denote $f_{abc}$ the structure constants. Then one can endow $g = g_0 \oplus g_1$ with a Lie algebra of order 3 structure given by

\[
\begin{align*}
[J_a, J_b] &= f_{abc}J_c, \\
[J_a, A_b] &= f_{abc}A_c, \\
\{A_a, A_b, A_c\} &= g_{ab}J_c + g_{ac}J_b + g_{bc}J_a.
\end{align*}
\]

The algebra (69) enables us to define a cubic extension of the Poincaré algebra in any space-time dimension. Consider the real Lie algebra $g_0 = so(1, D)$ (we could have equally chosen $g_0 = g_0^\mathbb{C}$).
\( \mathfrak{so}(2, D-1) \) and define \( \mathfrak{g} = \mathfrak{so}(1, D) \oplus \text{ad}(\mathfrak{so}(1, D)) = (L_{MN} = -L_{NM}, 0 \leq M < N \leq D + 1) \oplus (A_{MN} = -A_{NM}, 0 \leq M < N \leq D + 1) \). The algebra is given by

\[
\begin{align*}
\{L_{MN}, L_{PQ}\} &= \eta_{NQ}L_{PM} - \eta_{MQ}L_{PN} + \eta_{NP}L_{MQ} - \eta_{MP}L_{NQ}, \\
\{L_{MN}, A_{PQ}\} &= \eta_{NQ}A_{PM} - \eta_{MQ}A_{PN} + \eta_{NP}A_{MQ} - \eta_{MP}A_{NQ}, \\
\{A_{MN}, A_{PQ}, A_{RS}\} &= (\eta_{MP}N_Q - \eta_{MQ}P_N)L_{RS} + (\eta_{MR}N_{QS} - \eta_{MS}P_{NR})L_{PQ} + (\eta_{PR}Q_N - \eta_{PS}R_Q)L_{MN}
\end{align*}
\]

with \( \eta_{MN} = \text{diag}(1, -1, \ldots, -1) \). Using vector indices of \( \mathfrak{so}(1, D-1) \) coming from the inclusion \( \mathfrak{so}(1, D-1) \subset \mathfrak{so}(1, D) \), \( g_0 \) is generated by \( L_{\mu\nu}, L_{\mu D} \), with \( \mu, \nu = 0, \ldots, D - 1 \) and the graded part by \( A_{\mu\nu}, A_{\mu D} \). Letting \( R \to \infty \) after the Inönü-Wigner contraction [20],

\[
J_{\mu\nu} = L_{\mu\nu}, \quad P_\mu = \tfrac{1}{R}L_{\mu D}, \quad V_\mu = \tfrac{1}{R}A_{\mu D},
\]

one sees that \( L_{\mu\nu} \) and \( P_\mu \) generate the Poincaré algebra in \( D \)--dimensions and that \( T_{\mu\nu}, V_\mu \) are in respectively the adjoint and vector representations of \( \mathfrak{so}(1, D - 1) \). This Lie algebra of order three is therefore a non-trivial extension of the Poincaré algebra where translations are cubes of more fundamental generators. The subspace generated by \( L_{\mu\nu}, P_\mu, V_\mu \) is also a Lie algebra of order three extending the Poincaré algebra. The trilinear symmetric brackets have the simple form

\[
\{V_\mu, V_\nu, V_\rho\} = \eta_{\mu\nu}P_\rho + \eta_{\mu\rho}P_\nu + \eta_{\nu\rho}P_\mu.
\]

where \( \eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1) \) is the Minkowski metric.

This algebra has been firstly studied in four-space time dimensions [56, 57, 58] and then in any dimensions [24], and it has been realised that it induces naturally a symmetry on generalised gauge fields or \( p \)--forms. An analysis of the implementation of the Noether theorem in relation with the spin-statistics theorem was given in [58] in four dimensions. Furthermore, the cubic algebra (72) is not the only non-trivial extension of the Poincaré algebra one can construct. In fact in [21] a classification of all cubic extensions of the Poincaré algebra in four space-time dimensions was given. Furthermore in [46], along the lines of the classification of Bacry and Lévy-Leblond (where a classification of kinematical algebras was undertaken [59]), we have classified all kinematical Lie algebras of order three in four dimensions and showed that they are related through generalised Inönü-Wigner contractions of the algebra (70) [46].

3.3.3. Representations In this section we construct some multiplets of the cubic extension of the Poincaré algebra (72). Since, the generators \( V_\mu \) are in the vector representation of the Lorentz algebra, a multiplet contains states of the same statistics. This is rather different from the supersymmetric extension of the Poincaré algebra. To proceed further, from the matrix representations given in (D.8) or (D.9), one can construct representations of the algebra (72). The representations are specified by the representation of the vacuum. If the vacuum is in the trivial representation of the Lorentz group the multiplet consists of three spinors

\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix},
\]

transforming like \( \delta_\mu \Psi = \varepsilon^\mu V_\mu \Psi \). An invariant Lagrangian with three spinors was constructed in [56, 57]. However, there is more interesting multiplets obtained when the vacuum is in the
spinor representation of the Lorentz group \[24\]. In this case, a multiplet contains \(p\)-forms. Generalised gauge fields or \(p\)-forms which are fully antisymmetric tensors are generalisations of the usual electromagnetic gauge field. The revival of interest for the \(p\)-forms is mainly due to there appearance in supergravity or string theory. Furthermore, the \(p\)th antisymmetric gauge fields naturally couple to \((p-1)\)-dimensional extended objects. However, the \(p\)-forms are known to be relatively rigid in the sense that there is a few number of consistent interactions for them \[60\]. Despite these restrictions \(p\)-forms may present interesting symmetry properties, as it is the case, for instance, with the duality transformations between types IIA and IIB string theories \[61\]. In this section we investigate a new possible symmetry among \(p\)-forms.

Since the properties of spinors and \(p\)-forms depends on the space-time dimension, from now on, we restrict to the case where \(D = 4n\). The other cases are analogous. We thus concentrate on the case where the vacua are in the spinor representations of the Lorentz algebra. We take two copies \(\Psi_\pm, \Lambda_\pm\) transforming with \(V_\pm\), and two copies of the vacuum in the spinor representation \(\Omega_\pm, \omega_\pm\). Since \(\Psi_\pm\) transforms with \((D.9)\) we have

\[
\Psi_\pm = \begin{pmatrix} \Psi_1 \pm \\ \Psi_2 \pm \\ \Psi_3 \pm \end{pmatrix},
\]

with \(\Psi_{++}\) (resp. \(\Psi_{--}\)) left-handed (resp. right-handed) spinors. Thus for instance we have

\[
\Psi_+ \otimes \Omega_+ = \begin{pmatrix} \Psi_1 \otimes \Omega_+ \\ \Psi_2 \otimes \Omega_+ \\ \Psi_3 \otimes \Omega_+ \end{pmatrix}.
\]

Consequently, from the decomposition of the product of spinors \((E.7)\) one gets the four possible multiplets (See Appendix for notations)

\[
\begin{align*}
\Xi_{++} &= \Psi_+ \otimes \Omega_+ = \begin{pmatrix} \xi_{1++} \\ \xi_{2++} \\ \xi_{3++} \end{pmatrix} = \begin{pmatrix} A_{0}^1 & A_{0}^2 & \cdots & A_{0}^{2n} \\ A_{1}^1 & A_{1}^2 & \cdots & A_{1}^{2n} \\ A_{2}^1 & A_{2}^2 & \cdots & A_{2}^{2n} \end{pmatrix}, \\
\Xi_{--} &= \Psi_- \otimes \Omega_- = \begin{pmatrix} \xi_{1--} \\ \xi_{2--} \\ \xi_{3--} \end{pmatrix} = \begin{pmatrix} A'_{0}^1 & A'_{0}^2 & \cdots & A'_{0}^{2n} \\ A'_{1}^1 & A'_{1}^2 & \cdots & A'_{1}^{2n} \\ A'_{2}^1 & A'_{2}^2 & \cdots & A'_{2}^{2n} \end{pmatrix}, \\
\Xi_{-+} &= \Lambda_- \otimes \omega_+ = \begin{pmatrix} \xi_{1-+} \\ \xi_{2-+} \\ \xi_{3-+} \end{pmatrix} = \begin{pmatrix} A_{1}^1 & A_{1}^2 & \cdots & A_{1}^{2n} \\ A_{2}^1 & A_{2}^2 & \cdots & A_{2}^{2n} \\ A_{3}^1 & A_{3}^2 & \cdots & A_{3}^{2n} \end{pmatrix}, \\
\Xi_{+-} &= \Lambda_+ \otimes \omega_- = \begin{pmatrix} \xi_{1+-} \\ \xi_{2+-} \\ \xi_{3+-} \end{pmatrix} = \begin{pmatrix} A'_{1}^1 & A'_{1}^2 & \cdots & A'_{1}^{2n} \\ A'_{2}^1 & A'_{2}^2 & \cdots & A'_{2}^{2n} \\ A'_{3}^1 & A'_{3}^2 & \cdots & A'_{3}^{2n} \end{pmatrix}.
\end{align*}
\]

From the general property of representations of Lie algebras of order three \((35)\), one can assume that, for instance, for the multiplet \(\Xi_{++}\), the fields \(A\) are in the \((-1)\)-graded sector, \(\tilde{A}\) in the 0-graded sector and \(\check{A}\) in the 1-graded sector. The same classification also holds for the other multiplets. Since anti-self dual \(2n\)-forms are complex (see \((E.11)\)), we take all the \(p\)-forms above as being complex. But in \(4n\)-dimensional space-time a left-handed spinor is complex conjugate to a right-handed spinor, this gives us the opportunity to take the multiplets above as being conjugated \(\Xi_{++}^* = \Xi_{--}\) and \(\Xi_{+-}^* = \Xi_{-+}\). This gives for the multiplets \(\Xi_{++}\) and \(\Xi_{--}\).
We now calculate the transformation law of the different fields. Since the results are similar for all the multiplets (73), we only give the results for the multiplet $\Xi_{++}$. From the transformation law $\delta_{\varepsilon} \Xi_{++} = (\varepsilon^\mu V^\mu_+ \Psi_+) \otimes \Omega_+$, we obtain

$$
\begin{align*}
\delta_{\varepsilon} \Xi_{1++} &= \varepsilon^\mu \Sigma_\mu \Xi_{2-+}, \\
\delta_{\varepsilon} \Xi_{2--} &= \varepsilon^\mu \Sigma_\mu \Xi_{3++}, \\
\delta_{\varepsilon} \Xi_{3++} &= \varepsilon^\mu \partial_\mu \Xi_{1++}.
\end{align*}
$$

To calculate (75), we use the identity (E.5). We proceed differently for $\Xi_{1++}$ and $\Xi_{2-+}$, in order to avoid the presence of $\Sigma^{(\ell)}$ or $\Sigma^{(\ell)}$ with $\ell > 2n$. Starting from $\delta_{\varepsilon} \Xi_{2-+} = \varepsilon^\mu \Sigma_\mu \Xi_{3++}$, and using (E.8) we first have $\delta_{\varepsilon} \tilde{A}_{[2p+1]} = \varepsilon^\mu 2^{-n} \text{Tr} \left( \Sigma_\mu \Xi_{3++} \eta_{\mu2p+1} \Sigma^{(2p+1)} \right) = \varepsilon^\mu 2^{-n} \text{Tr} \left( \Sigma^{(2p+1)} \Sigma_\mu \Xi_{3++} \eta_{\mu2p+1} \right).$

Using (E.5), we calculate $\Sigma^{(2p+1)} \Sigma_\mu$. Then, from the trace identities for the $\Sigma$ matrices (E.4), one obtains the transformations laws for $\tilde{A}_{[2p+1]}$. In order to calculate $\delta_{\varepsilon} \Xi_{1++} = \varepsilon^\mu \Sigma_\mu \Xi_{2-+}$, we proceed in the reverse order. Firstly, using (E.7) and the identity (E.5), we compute the product $\Sigma_\mu \Xi_{3++}$. Then, using the trace formulae, we get the transformation laws of $\tilde{A}_{[2p]}$. The last case $\delta_{\varepsilon} \Xi_{3++}$ is not difficult to handle. For instance, this procedure gives for $\delta_{\varepsilon} \Xi_{1++}$ (when $k < n$)

\[
\begin{align*}
\delta_{\varepsilon} A_{[2k]}^{\mu_1 \cdots \mu_{2k}} &= \frac{1}{2^n} \text{Tr} \left( \sum_{p=0}^{n} \frac{1}{(2p+1)!} \left[ \Sigma_\mu^{\mu_1 \cdots \mu_{2k}} \Sigma_{\mu_2 \mu_{2p+1} \cdots \mu_1} + (2p+1) \eta_{\mu_2 \mu_{2p+1}} \Sigma_\mu^{\mu_1 \cdots \mu_{2k}} \Sigma_{\mu_2 \cdots \mu_1} \right] \right) \\
&\quad \times \varepsilon^\mu A_{[2p+1]}^{\mu_1 \cdots \mu_{2p+1}}.
\end{align*}
\]

Using the trace formulae (E.4), the first term gives $\tilde{A}_{[2k-1]} \wedge \varepsilon$ and the second $i_{\varepsilon} \tilde{A}_{[2k+1]}$. Finally we get [24]

\[
\begin{align*}
\delta_{\varepsilon} A_{[0]} &= i_{\varepsilon} \tilde{A}_{[1]}, \\
\vdots \\
\delta_{\varepsilon} A_{[2p]} &= i_{\varepsilon} \tilde{A}_{[2p+1]} + \tilde{A}_{[2p-1]} \wedge \varepsilon, \\
\vdots \\
\delta_{\varepsilon} A_{[2n-1]} &= i_{\varepsilon} \tilde{A}_{[2n-1]} + \tilde{A}_{[2n-2]} \wedge \varepsilon, \\
\delta_{\varepsilon} \tilde{A}_{[2n]} &= i_{\varepsilon} \tilde{A}_{[2n]} + \tilde{A}_{[2n]} \wedge \varepsilon.
\end{align*}
\]

(76)

the second term in $\delta_{\varepsilon} A_{[2n]+}$ ensure its self-duality. For the definition and conventions see (E.13) and (E.15). It is interesting to observe that the transformation (76) have a geometrical interpretation in terms of the natural operations on $p$--forms.
3.3.4. Invariant Lagrangian The transformations (76) suggest that an invariant Lagrangian should be obtained coupling the field $A$ with the fields $\tilde{A}$ and the field $\tilde{\tilde{A}}$ with themselves. In other words to couple $(-1)\text{–graded sector with 1–graded sector, and grade 0–graded sector with itself. Furthermore, if we consider for example the } \Xi_{++} \text{ multiplet, in order to have a real Lagrangian, one has also to take into consideration the conjugate multiplet } \Xi_{--} \text{ (see (74)). For the } \Xi_{++} \text{ and } \Xi_{--} \text{ multiplets, the Lagrangian writes [24]}

$$
\mathcal{L} = \mathcal{L}(\Xi_{++}) + \mathcal{L}(\Xi_{--}) = \mathcal{L}_{[0]} + \ldots + \mathcal{L}_{[2n]} + \mathcal{L}'_{[0]} + \ldots + \mathcal{L}'_{[2n]} = 
$$

$$
= dA_{[0]}d\tilde{A}_{[0]} + \ldots + 
- \frac{1}{2} (2p + 2)! dA_{[2p+1]} d\tilde{A}_{[2p+1]} - \frac{1}{2} (2p)! d^l \tilde{A}_{[2p+1]} d^l \tilde{\tilde{A}}_{[2p+1]} 
+ \frac{1}{(2p + 3)!} dA_{[2p+2]} d\tilde{A}_{[2p+2]} + \frac{1}{(2p + 1)!} d^l A_{[2p+2]} d^l \tilde{A}_{[2p+2]} 
+ \ldots + 
+ \frac{1}{2} (2n + 1)! dA_{[2n]_+} d\tilde{A}_{[2n]_+} + \frac{1}{2} (2n - 1)! d^l A_{[2n]_+} d^l \tilde{A}_{[2n]_+} 
+ dA_{[0]} d\tilde{A}_{[0]} + \ldots + 
- \frac{1}{2} (2p + 2)! dA'_{[2p+1]} d\tilde{A}'_{[2p+1]} - \frac{1}{2} (2p)! d^l \tilde{A}'_{[2p+1]} d^l \tilde{\tilde{A}}'_{[2p+1]} 
+ \frac{1}{(2p + 3)!} dA'_{[2p+2]} d\tilde{A}'_{[2p+2]} + \frac{1}{(2p + 1)!} d^l A'_{[2p+2]} d^l \tilde{A}'_{[2p+2]} 
+ \ldots + 
+ \frac{1}{2} (2n + 1)! dA'_{[2n]_-} d\tilde{A}'_{[2n]_-} + \frac{1}{2} (2n - 1)! d^l A'_{[2n]_-} d^l \tilde{A}'_{[2n]_-} 
$$

Here $\omega_{[p]}^\prime \omega_{[-p]}'$ stands for $\omega_{[p]}^\mu_1 \ldots \mu_p, \omega_{[-p]}'^\mu_1 \ldots \mu_p$, where $\omega_{[p]}^\prime$ and $\omega_{[-p]}'$ are two $p$–forms. For the definition of the exterior derivative $d$ and its adjoint $d^l$ see (E.17) and (E.18). To prove that (77) is invariant under (76), we firstly note that $\delta_{\varepsilon} \mathcal{L}(\Xi_{++})$ and $\delta_{\varepsilon} \mathcal{L}(\Xi_{--})$ do not mix. It is thus sufficient to check separately their invariance, which we do here only for $\mathcal{L}(\Xi_{++})$ as an illustration. Starting from a specific normalisation for $\mathcal{L}_{[0]}$, its variation fixes the normalisation for $\mathcal{L}_{[1]}$. By a step-by-step process, the normalisations for $\mathcal{L}_{[p]}$, $0 \leq p \leq 2n$ are also fixed. At the very end, all the terms of $\delta_{\varepsilon} \mathcal{L}$ compensate each other, up to a total derivative. The key observations in this compensation process, is to remark that we have

$$(i_{\varepsilon} A_{[p+1]}) A_{[p]} = A_{[p+1]} (A_{[p]} \wedge \varepsilon),$$

together with the fact that the Lagrangian can be rewritten in a Fermi-like form (see below (84)). This shows, the Lagrangian (77) is invariant.

If one considers the terms involving the (anti–)self–dual $2n$–form one can have further simplifications. Indeed, for the self–dual $2n$–forms we have

$$
\mathcal{L}_{[2n]} = \frac{1}{2} \frac{1}{(2n + 1)!} dA_{[2n]_+} d\tilde{A}_{[2n]_+} + \frac{1}{2} \frac{1}{(2n - 1)!} d^l A_{[2n]_+} d^l \tilde{A}_{[2n]_+} = 
$$

$$
= \frac{1}{(2n + 1)!} dA_{[2n]_+} d\tilde{A}_{[2n]_+},
$$

because of the self–duality condition $^* A_{[2n]_-} = i A_{[2n]_+}$. A more interesting way of regrouping the terms involving the self–dual and the anti–self–dual $2n$–forms is to introduce the real $2n$–forms
A_{1[2n]} = \frac{1}{\sqrt{2}} \left(A_{2n} + A'_{2n} \right), \quad \tilde{A}_{1[2n]} = \frac{1}{\sqrt{2}} \left(\tilde{A}_{2n} + \tilde{A}'_{2n} \right),

(78)

such that

\mathcal{L}_{2n} + \mathcal{L}'_{2n} = \frac{1}{(2n+1)!} dA_{1[2n]} d\tilde{A}_{1[2n]} + \frac{1}{(2n-1)!} d^\dagger A_{1[2n]} d^\dagger \tilde{A}_{1[2n]}

(79)

since \ A_{2n+1} A'_{2n-} = 0 \text{ when } D = 1 + (4n - 1), \text{ for a self–dual and an anti–self–dual } 2n–\text{form.}

The final real \ 2n–\text{forms} are neither self–dual nor anti–self–dual, which is in agreement with the representation theory of the Poincaré algebra. In the same manner, we introduce the real fields

\begin{align*}
A_{1[2p]} &= \frac{1}{\sqrt{2}} \left(A_{2p} + A'_{2p} \right), \quad A_{2[2p]} = \frac{i}{\sqrt{2}} \left(A_{2p} - A'_{2p} \right), \\
\tilde{A}_{1[2p]} &= \frac{1}{\sqrt{2}} \left(\tilde{A}_{2p} + \tilde{A}'_{2p} \right), \quad \tilde{A}_{2[2p]} = \frac{i}{\sqrt{2}} \left(\tilde{A}_{2p} - \tilde{A}'_{2p} \right),
\end{align*}

(80)

With this new fields the Lagrangian is not diagonal, introducing the fields

\begin{align*}
\hat{A}_{1[2p]} &= \frac{1}{\sqrt{2}} \left(\hat{A}_{1[2p]} + \hat{\tilde{A}}_{1[2p]} \right), \quad \hat{\tilde{A}}_{1[2p]} = \frac{1}{\sqrt{2}} \left(\hat{A}_{1[2p]} - \hat{\tilde{A}}_{1[2p]} \right), \quad p = 0, \ldots, n, \\
\hat{A}_{2[2p]} &= \frac{1}{\sqrt{2}} \left(\hat{A}_{2[2p]} + \hat{\tilde{A}}_{2[2p]} \right), \quad \hat{\tilde{A}}_{2[2p]} = \frac{1}{\sqrt{2}} \left(\hat{A}_{2[2p]} - \hat{\tilde{A}}_{2[2p]} \right), \quad p = 0, \ldots, n - 1,
\end{align*}

(81)

we have \ (2p \neq 2n \text{ since for } \mathcal{L}_{2n} \text{ we have only one term})

\begin{align*}
\mathcal{L}_{2p+1} + \mathcal{L}'_{2p+1} &= - \frac{1}{2} \frac{1}{(2p+2)!} d\hat{A}_{1[2p+1]} d\hat{\tilde{A}}_{1[2p+1]} - \frac{1}{2} \frac{1}{(2p)!} d^\dagger \hat{A}_{1[2p+1]} d^\dagger \hat{\tilde{A}}_{1[2p+1]} \\
&+ \frac{1}{2} \frac{1}{(2p+2)!} d\hat{A}_{2[2p+1]} d\hat{\tilde{A}}_{2[2p+1]} + \frac{1}{2} \frac{1}{(2p)!} d^\dagger \hat{A}_{2[2p+1]} d^\dagger \hat{\tilde{A}}_{2[2p+1]}
\end{align*}

\begin{align*}
\mathcal{L}_{2p} + \mathcal{L}'_{2p} &= - \frac{1}{2} \frac{1}{(2p+1)!} d\hat{A}_{1[2p]} d\hat{\tilde{A}}_{1[2p]} - \frac{1}{2} \frac{1}{(2p-1)!} d^\dagger \hat{A}_{1[2p]} d^\dagger \hat{\tilde{A}}_{1[2p]} \\
&- \frac{1}{2} \frac{1}{(2p+1)!} d\hat{A}_{2[2p]} d\hat{\tilde{A}}_{2[2p]} - \frac{1}{2} \frac{1}{(2p-1)!} d^\dagger \hat{A}_{2[2p]} d^\dagger \hat{\tilde{A}}_{2[2p]}
\end{align*}

(82)

Usually, the kinetic term for a \ p–\text{form } \omega_{[p]} \text{ writes, with our conventions for the metric, } (-1)^p \frac{1}{2(2p+1)!} d\omega_{[p]} d\omega_{[p]}. \text{ However, in the Lagrangian written as sum of terms above (79) and (82) some of the kinetic terms have the wrong sign. This implies that these fields have an energy density not bounded from below. We propose a possible way to construct a Lagrangian with correct signs for the various kinetic terms, based on a special choice for the physical fields. The main idea is related to Hodge duality. However, the duality transformation will act here...
on the $p$–forms with respect to the Lorentz group $SO(1, D – 1)$. This should be contrasted with the case of the usual duality transformations (generalising the electric-magnetic duality) which act on the field strengths with respect to $SO(1, D – 1)$, or equivalently on the potentials themselves but with respect to the little group $SO(D – 2)$. Firstly, the decomposition (E.6) is purely conventional, and we could also have decomposed this product on the set of $p$–forms with $p \geq 2n$. Moreover, looking at the Lagrangians (79) and (82) there is a kind of duality between the kinetic term and the gauge fixing term. This means that for the field with the wrong signs, we define the Hodge dual

$$\begin{align*}
\hat{A}_1^{[2]} & \rightarrow \hat{B}_1^{[4n-2]} = *\hat{A}_1^{[2]}, \\
\hat{A}_2^{[2]} & \rightarrow \hat{B}_2^{[4n-2]} = *\hat{A}_2^{[2]}, \\
\hat{A}_2^{[2p+1]} & \rightarrow \hat{B}_2^{[4n-2p-1]} = *\hat{A}_2^{[2p+1]}, \\
\hat{A}_1^{[2n]} & \rightarrow \hat{B}_1^{[D-2n]} = *\hat{A}_1^{[2n]},
\end{align*}$$

(83)

with $p = 0, \ldots, n – 1$. Thus, starting from the Lagrangian (77) and performing the field redefinitions (80), (81) and (83) we get a new Lagrangian with the correct signs (that we do not write but is easy to obtain), because of relation (E.19). This Lagrangian is given in [24]. In this transformation the kinetic term of $A$ becomes the gauge fixing term (see below Section 3.3.5) of $B$ and vice versa. Using (E.16), one can obtain the transformations law of the new fields. The field content is then one one–form, one three–form, \ldots, one $(4n – 1)$–form and two zero–forms, two two–forms, \ldots and two $4n$–form, all the $p$–forms have a kinetic term and a gauge fixing term; the only exceptions are the zero–forms, which have only kinetic terms, and the $4n$–forms, which have only gauge fixing terms. Let us emphasise that the substitutions (83) are done with respect to the gauge fields. This is quite different from the usual duality transformations (generalising the electric–magnetic duality) where the duality transformations are done with respect to the field strengths.

With the “duality” transformations (83) the number of degree of freedom is not the same for $\omega^{[p]}$ and $\rho^{[D-\rho]} = *\omega^{[p]}$, which is not the case for the usual duality transformations. Hence, the two Lagrangians describe inequivalent theories.

Note finally that the sum of the kinetic terms and the gauge fixing terms gives rises to Fermi–like terms. For instance

$$\frac{1}{2} \frac{1}{(p+1)!} dA^{[p]} dA^{[p]} + \frac{1}{2} \frac{1}{(p-1)!} d^* A^{[p]} d^* A^{[p]} = \frac{1}{2} \frac{1}{p!} \partial_{\mu} A^{[\rho]} \partial^{\mu} A^{[\rho]},$$

(84)

and note, in particular that it is more easy to check the invariance of the Lagrangian (77) when it is written as a sum of Fermi-like terms.

### 3.3.5. Gauge invariance

Let us analyse gauge invariance. Gauge transformations for $p$–forms, are given by

$$A_{[p]} \rightarrow A_{[p]} + d\chi_{[p-1]}, \quad p > 0,$$

(85)

where $\chi_{[p-1]}$ is a $(p – 1)$–form. We observe that (77) (or the new Lagrangian that we do not have given) is not invariant. Indeed, the terms $d^* A_{[p]} d^* A_{[\rho]}$ in the Lagrangian (77) fixes partially the gauge, and the Lagrangian is invariant provided $\chi_{[p-1]}$ satisfies an additional constraint.
\[ d^1 d\chi_{[p-1]} = 0. \] (86)

This means that this last term is a gauge fixing term, analogous to the Feynman gauge fixing term for the electromagnetic field. The peculiar form of our Lagrangian leads also to another gauge invariance. Since it contains a kinetic term and a gauge fixing term, there is some kind of duality between these two terms. It was in fact this duality which enabled us to make the field redefinition (83). At the level of gauge invariance, this duality translates in the invariance of the Lagrangian under the transformation

\[ A_{[p]} \to A_{[p]} + d^1 \chi_{[p+1]}, \quad p < 4n, \] (87)
such that \( dd^1 \chi_{[p+1]} = 0 \). However, from the Poincaré theorem (since we are in \( \mathbb{R}^{4n} \) there is no topological obstruction), \( dd^1 \chi_{[p+1]} = 0 \) implies that there exists a \( (p-1) \)-form \( \lambda_{[p-1]} \) such that \( d^1 \chi_{[p+1]} = d\lambda_{[p-1]} \). Thus (87) is not a new symmetry. While in the case of gauge theories the gauge invariance guarantees that the physical (on-shell) quantities are gauge fixing independent, in our case \( (d^1A_{[p]})^2 \) cannot be traded for any other gauge-fixing function since it is imposed by the invariance under (76), and is thus expected to affect the physical degrees of freedom. This shows that the effective degrees of freedom of \( A_{[p]} \) are dictated only by the gauge freedom eq.(85), supplemented by \( d^1 d\chi_{[p-1]} = 0 \). An immediate consequence of the latter constraint is that the usual Lorentz condition \( d^1A_{[p]} = 0 \) cannot be imposed in general to eliminate the unphysical components. This means that the way one should eliminate the unphysical components cannot be handled in a usual manner [62]. On top of that, such a condition is not stable under our transformation laws (76). For instance, if we put \( \partial_\mu A^\mu_{[1]} = 0 \), then \( \partial_\mu \partial_\nu A^\mu_{[1]} = 0 \) gives \( \varepsilon^\mu \partial_\mu A_{[0]} = 0 \), which is obviously too strong.

The gauge invariance (85) and the field equations imply (for a \( p \)-form \( A_{[p]} \), with \( p \leq D-2 \)) \( P^\mu A_{[p]\mu_1\cdots\mu_p} = 0 \) and \( P^2 = 0 \) (with \( P^\mu \) the momentum), thus \( A_{[p]} \) gives rise to a massless state in the \( p \)-order antisymmetric representation of the little group \( SO(4n-2) \). But, in our decomposition, there are also appearing \( p \)-forms with \( p = 4n-1, 4n \). Of course these \( p \)-forms do not propagate. It is interesting to note that similar phenomena are well-known in the context of type IIA, IIB string theory [63] in 10 space-time dimensions where 9- and 10- forms appear. Actually, subsequent to the early works on two-forms in [64, 65], several authors studied the classical and quantum properties of the non-propagating 3- and 4-forms [66, 67] in four dimensions. In particular, it was pointed out in [67] that the gauge fixing term for a 4-form takes the form of a kinetic term for a scalar field, in exact analogy with our results. The gauge condition for \( p \)-forms is crucial in order to eliminate the unphysical degrees of freedom such that massless \( p \)-form \( p < D-1 \) has \( \binom{p}{D-1} \) degrees of freedom off-shell and \( \binom{p}{D-2} \) on-shell. In our case, relation (86) is not enough to eliminate the unphysical degrees of freedom. However, here the situation is not as simple, since in our case there is some mixing between gauge invariance and (76). This means that (76) should itself have some role in the elimination of the superfluous degree of freedom. For a discussion of the compatibility of the transformation (76) and gauge transformation see [57].

The Lagrangian constructed so far is a free theory. It should be interesting to construct interacting theories invariant under (76). It has been proved by a brute force method, that when \( D = 4 \) and we consider only multiplets of the types (73), no interacting terms are allowed [57]. In the general case, it is still an open question to know whether or not interacting terms...
compatible with (76) exist.

Let us mention to conclude this section that invariant Lagrangians including mass terms were also constructed [24].

4. Conclusion
There are many mathematical structures generalising Lie superalgebras which can be defined. In particular one of this structure called Lie algebras of order $F$ can be used to implement higher order extensions of the Poincaré algebra. In these lectures we have studied explicitly two examples of non-trivial higher order extensions of the Poincaré algebra based on Lie algebras of order $F$, which are not the supersymmetric ones. In the former, we have constructed higher order symmetries in three space-time dimensions which act on relativistic anyons. In the latter, cubic extensions of the Poincaré algebra in any space-time dimensions are obtained. These new symmetries have a natural geometrical interpretation on generalised gauge fields or $p$–forms. Invariant Lagrangians in this last cases have revealed some problems to be solved: construction of interacting Lagrangian, relation between the cubic symmetries and gauge invariance.

In order to understand deeper these structures a program to study Lie algebras of order $F$ on a formal way has been investigated. The basis of the theory of contractions and deformations in the Gerstenhaber sense together with a classification of Lie algebras associated to $\mathfrak{sl}(2)$ and the four dimensional-Poincaré algebra have been initiated [21]. A classification of kinematical algebras of order three have been investigated in [46]. Since Lie algebras of order three correspond to transformations at the infinitesimal level, group associated to ternary algebras has been defined in the context of Hopf algebras, and the parameters of the transformation have been identified [47]. However some points remains to be studied. Simple Lie algebras of order three have been defined in [20], however there is no general classification of simple complex Lie algebras of order $F$ (and even of order three) analogous to the classification of the Lie (super)algebras. It should be also interesting to have a analogous of a Coleman-Mandula theorem in that context. Finally, in order to construct interacting Lagrangians some adapted superspace will certainly be relevant.

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Appendix A. Convention and useful identities for spinor calculus in four dimensions
In this appendix, we collect useful relations and conventions for the four dimensional spinor calculus. The metric is taken to be

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$  \hspace{1cm} (A.1)

In the $\mathfrak{so}(1, 2) = \mathfrak{sl}(2, \mathbb{C})$ notations of dotted and undotted indices for two-dimensional spinors, the spinor conventions to raise/lower indices are as follows (we have minor differences as compared to the notations of Wess and Bagger [29]): $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$, $\bar{\psi}_\dot{\alpha} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^\dot{\beta}$, $\bar{\psi}_\dot{\alpha} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^\dot{\beta}$ with $(\psi_\alpha)^* = \bar{\psi}_\dot{\alpha}$, $\varepsilon_{12} = \varepsilon_{1\dot{2}} = -1$, $\varepsilon^{12} = \varepsilon^{1\dot{2}} = 1$.

The 4D Dirac matrices, in the Weyl representation, are

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix}$$  \hspace{1cm} (A.2)
with

\[ \sigma_\mu = (1, \sigma_i), \bar{\sigma}_\mu = (1, -\sigma_i), \]  

(A.3)

the \( \sigma_i \) \((i = 1, 2, 3)\) denoting the Pauli matrices. The index structure of the \( \sigma_\mu \)-matrices is as follows: \( \sigma_\mu \rightarrow \sigma_{\mu\alpha\dot{\alpha}}, \bar{\sigma}_\mu \rightarrow \bar{\sigma}_{\mu\dot{\alpha}} \). The following relation holds,

\[ \bar{\sigma}_\mu \dot{\alpha} = \sigma_{\mu\beta}\dot{\alpha} \varepsilon^{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}}, \]  

(A.4)

and (A.3) leads to

\[ \text{Tr}(\sigma_\mu \bar{\sigma}_\nu) = 2\eta^{\mu\nu}. \]  

(A.5)

Furthermore, the Lorentz generators for the spinor representations are given by

\[
\begin{align*}
\sigma_{\mu\nu\alpha\beta} &= \frac{1}{4} \left( \sigma_{\mu\alpha\dot{\alpha}} \sigma_{\nu\dot{\beta}} - \sigma_{\nu\alpha\dot{\alpha}} \sigma_{\mu\dot{\beta}} \right), \\
\bar{\sigma}_{\mu\nu\dot{\alpha}\dot{\beta}} &= \frac{1}{4} \left( \bar{\sigma}_{\mu\dot{\alpha}\alpha} \sigma_{\nu\beta} - \bar{\sigma}_{\nu\dot{\alpha}\alpha} \sigma_{\mu\beta} \right)
\end{align*}
\]  

(A.6)

We adopt the usual spinor summation convention

\[
\psi.\lambda = \psi^\alpha \lambda_\alpha = -\psi_\alpha \lambda^\alpha, \quad \bar{\psi}.\bar{\lambda} = \bar{\psi}^\dot{\alpha} \bar{\lambda}_\dot{\alpha}.
\]  

(A.7)

Since \( \psi^\dagger_\alpha = \bar{\psi}_{\dot{\alpha}} \) and \( \sigma^{\dagger\mu} = \sigma^\mu \), we have

\[
\begin{align*}
(\psi.\lambda)^\dagger &= \bar{\lambda}.\bar{\psi}, \\
(\psi \sigma^\mu \bar{\lambda})^\dagger &= \lambda \sigma^\mu \bar{\psi}.
\end{align*}
\]  

(A.8)  

(A.9)

For anticommuting Grassmann spinors we have

\[
\begin{align*}
\theta^\alpha \bar{\theta}^\dot{\beta} &= -\frac{1}{2} \varepsilon^{\alpha\beta} \bar{\theta}.\bar{\theta}, \\
\bar{\theta}^\dot{\alpha} \bar{\theta}^\dot{\beta} &= \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}.\bar{\theta},
\end{align*}
\]  

(A.10)  

(A.11)

\[
\begin{align*}
(\theta \sigma^\mu \bar{\theta}) \ (\theta \sigma^\nu \bar{\theta}) &= \frac{1}{2} \theta.\theta \bar{\theta}.\bar{\theta} \eta^{\mu\nu}, \\
(\theta \sigma^\mu \bar{\theta}) \ \theta.\lambda &= -\frac{1}{2} \theta.\theta \lambda \sigma^\mu \bar{\theta}, \\
(\theta \sigma^\mu \bar{\theta}) \ \bar{\theta}.\bar{\lambda} &= -\frac{1}{2} \bar{\theta}.\bar{\theta} \theta \sigma^\mu \bar{\lambda}.
\end{align*}
\]  

(A.12)  

(A.13)  

(A.14)
Appendix B. Relativistic wave equations for anyons

We give now a relativistic wave equation for anyons. Following M. Plyushchay it is based on the $R$-deformed Heisenberg algebra with reflection. This algebra is defined by generators and relations. Consider the generators $a^\pm, R$ that satisfy

$$[a^-, a^+] = 1 + \nu R, \quad \{ R, a^\pm \} = 0, \quad R^2 = 1,$$

with $\nu \in \mathbb{R}$ the deformation parameter. The operators $a^\pm$ together with the quadratic operators generate the superalgebra $\mathfrak{osp}(1|2)$ whose bosonic part is $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(1, 2)$ and fermionic part the two-dimensional (Majorana) spinor representation

$$\{ a^\pm, a^\mp \} = 4L_\pm, \quad \{ a^+, a^- \} = 4L_0,$$

$$[L_\pm, a^\mp] = \mp a^\mp, \quad [L_0, a^\pm] = \pm \frac{1}{2} a^\pm,$$

This is the generalisation of the well-known realisation of the algebra $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(1, 2)$ by the usual harmonic oscillator. When $\nu > -1$, the algebra (B.1) admits an infinite dimensional unitary representation (when $\nu = -(2k+1)$ the algebra admits finite dimensional representations) [53]

$$a^+ |n\rangle = \sqrt{n + 1 + \frac{\nu}{2}(1 - (-1)^{n+1})} \ |n + 1\rangle,$$

$$a^- |n\rangle = \sqrt{n + \frac{\nu}{2}(1 - (-1)^n)} \ |n - 1\rangle,$$

this representation is bounded from below since we have $a^- |0\rangle = 0$, unitary $(a^-)^\dagger = a^+$ and the vectors $|n\rangle$ are orthonormal $< n|m > = \delta_{nm}$. We denote $\mathcal{R}_\nu = \{|n\rangle, \ n \in \mathbb{N}\}$ this representation. We also have $\mathcal{R} |n\rangle = (-1)^n |n\rangle$. Furthermore, because of relations (B.2) the representation of (B.3) is an irreducible representation of $\mathfrak{osp}(1|2)$ but a reducible representation of $\mathfrak{sl}(2, \mathbb{R})$. Indeed, it decomposes on the direct sum of the two irreducible representations

$$\mathcal{R}_\nu = \mathcal{D}^+_\nu \oplus \mathcal{D}^-_\nu$$

with

$$\mathcal{D}^+_\nu = \frac{1}{2}(1 + \mathcal{R})\mathcal{R}_\nu = \left\{ |2n\rangle, \ n \in \mathbb{N}\right\},$$

$$\mathcal{D}^-_\nu = \frac{1}{2}(1 - \mathcal{R})\mathcal{R}_\nu = \left\{ |2n + 1\rangle, \ n \in \mathbb{N}\right\}.$$

In order to obtain a relativistic wave equation for relativistic anyons, and to make contact with the literature [53], in this appendix we are working with physical quantities. This means that differently from the rest of the text we are directly working with the tri-momentum and the angular momentum. In particular, this means that they are given by hermitian quantities (although in the rest of the text they are antihermitian). In order to avoid confusion, we denote here $\Pi_\mu = -iP_\mu, L_\mu = -iJ_\mu$, the tri-momentum and angular momentum. In this
basis the Lie algebra \(\mathfrak{so}(1,2)\) writes \([L_\mu, L_\nu] = i\varepsilon_{\mu\nu\rho} \eta^\rho L_\sigma\) and \(L_\pm = L_1 \mp iL_2\). We also define \(\Pi_\pm = \Pi_1 \mp i\Pi_2\). The Dirac \(\Gamma\)–matrices in the Majorana representation are taken to be \(\gamma^0 = -i\sigma^2, \gamma^1 = i\sigma^2, \gamma^2 = -i\sigma^3\) with spinor matrix elements \(\gamma_\mu \alpha^\beta\), and the spinor indices can be raised and lowered as in Appendix Appendix A. We also define the hermitian spinor operators \(L_1 = \frac{1}{\sqrt{2}} (a^+ + a^-), L_2 = \frac{i}{\sqrt{2}} (a^+ - a^-)\).

A direct calculation using (B.1) and (B.2) gives
\[
[L_\alpha, L_\beta] = -i\varepsilon_{\alpha\beta}(1 + \nu)R, \quad (B.6)
\]
\[
\{L_\alpha, L_\beta\} = -4i(\Pi L_\alpha + \varepsilon m L_\beta).
\]

Finally we introduce the spinor operator
\[
D_\alpha = (\Pi \gamma)_\alpha^\beta L_\beta + \varepsilon m L_\alpha,
\]
with \(\varepsilon = \pm 1\), and a little algebra gives
\[
L^\alpha L_\alpha = -i(1 + \nu)R,
\]
\[
D^\alpha D_\alpha = -i(\Pi^2 - m^2)(1 + \nu)R,
\]
\[
L^\alpha D_\alpha = 4i(\Pi L - m\varepsilon \frac{1}{4}(1 + \nu)R).
\]

Now we have all the material and identities to define the spinor set of equations. Taking \(|\psi\rangle \in \mathbb{R}_\mu\) we have
\[
|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle = \sum_{n=0}^{+\infty} \psi_{2n}(x) |2n\rangle + \sum_{n=0}^{+\infty} \psi_{2n+1}(x) |2n + 1\rangle
\]
and we assume the relativistic wave equations
\[
D_\alpha |\psi\rangle = 0. \quad (B.9)
\]

The covariance of (B.9) under the \((1 + 2)\)–dimensional Poincaré group is checked in [53]. Using the identities (B.7) a direct calculation gives
\[
(\Pi^2 - m^2)|\psi_+\rangle = 0, \quad (\Pi L - m\varepsilon \frac{1}{4} + \nu)|\psi_+\rangle = 0,
\]
\[
(\Pi^2 - m^2)|\psi_-\rangle = 0, \quad (\Pi L - m\varepsilon \frac{1}{4} - \nu)|\psi_-\rangle = 0. \quad (B.10)
\]

This means that \(|\psi_\pm\rangle\) describe a particle of mass \(m\) and spin \(\frac{1+\nu}{4}\). Now if we solve equation (B.9) in the rest frame we obtain that \(|\psi_+\rangle = \psi_0 |0\rangle\), which describes a anyons of mass \(m\), energy \(\varepsilon m\) and helicity \(\frac{1+\nu}{4}\) although we have \(|\psi_-\rangle = 0\). Solving the equation in any frame gives
\[
|\psi_+\rangle = \sum_{n=0}^{+\infty} \sqrt{\frac{\Gamma(2s + n)}{\Gamma(2s)\Gamma(m + 1)}} \left(\frac{\Pi_1 + i\Pi_2}{\Pi_0 + \varepsilon m}\right)^n \psi_0 |2n\rangle. \quad (B.11)
\]
This is proven by induction. Assuming
\[
\psi_{2n} = \left( \frac{\Gamma(2s+n)}{\Gamma(2s)\Gamma(m+1)} \frac{\Pi_1 + i\Pi_2}{\Pi_0 + \varepsilon m} \right)^n \psi_0,
\]
using
\[
\frac{\Pi_0 - \varepsilon m}{\Pi_1 - i\Pi_2} = \frac{\Pi_1 + i\Pi_2}{\Pi_0 + \varepsilon m},
\]
gives the correct value for \(\psi_{2n+2}\). Thus equations (B.9) describe a relativistic anyon of mass \(m\), helicity \(\frac{1+\nu}{4}\) and energy of sign \(\varepsilon\). It has one degree of freedom as it should.

**Appendix C. Lie algebras of order \(F\) associated with anyons**

In this appendix we give an abstract and an explicit construction to define a Lie algebra of order \(F\) for relativistic anyons. Recall that the vector representation may be obtained in the Verma module language formalism [69]. Consider \(\mathfrak{sl}(1,2)\) the universal enveloping algebra. The Poincaré-Birkhoff-Witt theorem gives
\[
\mathfrak{u}(\mathfrak{sl}(1,2)) = \{ L^m_n L^n_0, m, n \in \mathbb{N} \}.
\]
Consider now the two-sided ideal generated by \(L_-, L_0 + I\) (where \(I\) denotes the identity of \(\mathfrak{u}(\mathfrak{sl}(1,2))\)) and set
\[
\mathcal{V}_{-1} = \mathfrak{u}(\mathfrak{sl}(1,2))/I.
\]
We have, in \(\mathcal{V}_{-1}\)
\[
L_0 \cdot I = -I, \quad L_- \cdot I = 0.
\]
Thus \(I\) is the highest weight representation of \(\mathcal{V}_{-1}\), and we have
\[
\mathcal{V}_{-1} = \{ L^m_n, n \in \mathbb{N} \}.
\]
But as seen in (46) \((L_- L^3_- = 0)\) and (47), \(M_{-1} = \langle L^+_{-n}, n > 2 \rangle\) is an invariant subspace of \(\mathcal{V}_{-1}\), and the standard finite dimensional vector representation is given by
\[
\mathcal{D}_{-1} = \mathcal{V}_{-1}/M_{-1}.
\]
In the same vain, we define the Verma module associated with the representation of spin \(-1/F\). Recall that among the four representations of Section 3.2.1, there are two inequivalent representations, one bounded from below and one bounded from above. Since the construction works equally well on both representations, in the discussion, we only consider the representation bounded from below. In this language, we have
\[
\mathcal{D}^+_{-1/F} = \mathcal{V}^+_{-1/F} = \mathfrak{u}(\mathfrak{sl}(1,2))/ < L_-, L_0 + \frac{1}{F} I >.
\]
In \(\mathcal{D}^+_{-1/F}\) we have \(L_0 \cdot I = -1/F I, L_- \cdot I = 0\), this means that in \(\mathcal{S}^F(\mathcal{D}^+_{-1/F})\), we have \(I \otimes \cdots \otimes I\) \((F\text{--times})\) is such that \(L_-, I \otimes \cdots \otimes I = 0, L_0 \otimes \cdots \otimes I = -I \otimes \cdots \otimes I\). Thus \(I \otimes \cdots \otimes I\) is a primitive vector of \(\mathcal{V}_{-1}\) meaning that there is an \(\mathfrak{so}(1,2)\)--equivariant map
\[
i : \mathcal{V}_{-1} \hookrightarrow \mathcal{S}^F(\mathcal{D}^+_{-1/F}).
\]
Taking the coset by \(M_{-1}\) we obtain an \(\mathfrak{so}(1,2)\)--equivariant inclusion.
The construction above, we obtain the same result in an explicit way. It is well known that
so \( (1,2) \) are representations of \( D \). Restricting to \( S^F(D_{-1/F})_{\text{red}} \) (indecomposability), and to clarify the abstract construction above, we obtain the same result in an explicit way. It is well known that \( \mathfrak{so}(1,2) \) can be realised by differential operators

\[
L_- = x \partial_y, L_+ = -y \partial_x, L_0 = \frac{1}{2} (y \partial_y - x \partial_x).
\]

Consider now \( \mathcal{F} \) the vector space of functions on \( \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, x, y > 0\} \). The following subspaces of \( \mathcal{F} \)

\[
\mathcal{D}_{-n} = \left\{ x^{2n}, x^{2n-1}y, \ldots, xy^{2n-1}, y^{2n} \right\}, \quad n \in \mathbb{N}/2,
\]

\[
\mathcal{D}^+_{-\lambda} = \left\{ x^{2\lambda} (\frac{y}{x})^m, m \in \mathbb{N} \right\}, \quad \lambda \in \mathbb{R} \setminus \mathbb{N}/2,
\]

\[
\mathcal{D}^-_{-\lambda} = \left\{ y^{2\lambda} (\frac{x}{y})^m, m \in \mathbb{N} \right\}, \quad \lambda \in \mathbb{R} \setminus \mathbb{N}/2,
\]

are representations of \( \mathfrak{so}(1,2) \). The representation \( \mathcal{D}_{-n} \) is the \((2n+1)\)-dimensional irreducible representation and the representations \( \mathcal{D}^\pm_{-\lambda} \) are infinite dimensional representations, bounded from below and above respectively. It is important to emphasise that the representations given in (C.1) do not have the normalisations conventionally taken.

To define \( S^F(D_{-1/F})_{\text{red}} \), we consider the multiplication map \( m_F : \mathcal{F} \times \cdots \times \mathcal{F} \to \mathcal{F} \) given by

\[
m_F(f_1, \cdots, f_F) = f_1 \cdots f_F
\]

which is multilinear and totally symmetric and hence induces a map \( \mu_F \) from \( S^F(\mathcal{F}) \) into \( \mathcal{F} \). Restricting to \( S^F(D_{-1/F})_{\text{red}} \) one sees that

\[
S^F(D^+_{-1/F})_{\text{red}} \overset{\text{def}}{=} \mu_F \left( S^F(D^+_{-1/F}) \right) = \left\{ x^2 (\frac{y}{x})^m, m \in \mathbb{N} \right\} \supset \mathcal{D}_{-1} \quad \text{(C.3)}
\]

\[
S^F(D^-_{-1/F})_{\text{red}} \overset{\text{def}}{=} \mu_F \left( S^F(D^-_{-1/F}) \right) = \left\{ y^2 (\frac{x}{y})^m, m \in \mathbb{N} \right\} \supset \mathcal{D}_{-1}.
\]
If we construct a diagram analogous to (47) (for $D^+_{-1/F}$), we have

\[
\begin{align*}
x^2 & \xrightarrow{y\partial_x} xy \xrightarrow{y\partial_y} y^2 \xrightarrow{x\partial_y} x^2 (\frac{y}{x})^3 \xrightarrow{x\partial_y} x^2 (\frac{y}{x})^4 \xrightarrow{y\partial_x} \cdots \xrightarrow{y\partial_x} x^2 (\frac{y}{x})^n \xrightarrow{y\partial_x} \cdots \\
\end{align*}
\]

Looking at the representations defined in (C.3) \textit{i.e.} $S^F(D^\pm_{-1/F})_{\text{red}}$, one sees that, even though $D_{-1}$ is a subspace stable under $\mathfrak{so}(1,2)$ there is no complement stable under $\mathfrak{so}(1,2)$ [23]. Indeed, these representations cannot be built from a primitive vector. This is due to the fact that $L^3_n(x^2)=0$ and consequently we cannot reach $x^{-1}y^3$ from $x^2$ but conversely $L^3_n(x^{-1}y^3)=6x^2$. This is the reason why there is no $F$--Lie algebra structure on $\mathfrak{so}(1,2) \oplus D_{-1}$. With the normalisations (C.1) and (C.3), denoting $D^+_{-1/F} = \{A_{-1/F+n}, n \in \mathbb{N}\}$ and $S^F_{\text{red}}(D^+_{-1/F}) = \{P_{-1+n}, n \in \mathbb{N}\}$, the trilinear brackets are given by

\[
\left\{A^+_{-1/F+n_1}, \cdots, A^+_{-1/F+n_p}\right\} = P_{-1+n_1+\cdots+n_p}.
\]

Appendix D. Clifford algebras of polynomial

Clifford algebras of a polynomial's is a direct generalisation of usual Clifford algebras for higher degree polynomials. Consider $p$ a polynomial of degree $n$ with $k$ variables. Since degree $n$ polynomials are isomorphic to symmetric tensors of order $n$, we can write

\[
p(x^1, \cdots, x^k) = x^{i_1} \cdots x^{i_n} g_{i_1 \cdots i_n}.
\]

The Clifford algebra of the polynomial $p$ denoted $\mathcal{C}_p$ is the algebra generated by $k$ primitive elements $g_1, \cdots, g_k$ such that

\[
\{g_{i_1}, \cdots, g_{i_n}\} = n! g_{i_1 \cdots i_n}.
\]

The algebra defined by relations (D.2) can be real if the tensor $g$ is real, or complex. It appears as an example of a Lie algebras of order $n$ where $\mathfrak{g}_0 = \mathbb{R}$ or $\mathbb{C}$. The relations (D.2) means that if we consider $P$ in $\mathcal{C}_p$ defined by $P = x^k g_k$, we have

\[
(x^1 g_1 + \cdots + x^k g_k)^n = p(x_1, \cdots, x_k).1,
\]

where 1 denotes the unity of $\mathcal{C}_p$ that we omit from now on. This algebra has been introduced in a formal way by N. Roby [70]. It is a natural generalisation of the usual Clifford algebra, but this algebra is very different from the usual Clifford algebra. Since it is defined through $n$-th order relations, the number of independent monomials increases with polynomial's degree (for instance, $(g_1 g_2)^k, k \geq 0$ are all independent). This means that we do not have enough constraints among the generators to order them in some fixed way and, as a consequence, $\mathcal{C}_p$ turns out to be an infinite-dimensional algebra. Thus any finite dimensional representation are non-faithful. Several properties on representations was then established such that the dimension...
of a representation of Clifford algebras is a multiple of the degree of the polynomial [71], but
the first systematic way to obtain a matrix representation was given in [72]. Subsequently, an
extensive study of the representations of Clifford algebras of cubic polynomials was undertaken
by Revoy [73] and a family of inequivalent representations can be obtained. See also Ref.[12] for
many references on Clifford algebras of polynomials.

For a self contained presentation, we give an algorithm to construct a matrix representation
of the Clifford algebra of a given polynomial, for more details and comments see [72, 12]. In this
process, two basic polynomials will be considered. The sum polynomial \( S(x) = (x^1)^n + \cdots + (x^k)^n \)
and the product polynomial \( \pi(x) = x^1 \cdots x^n \).

For the linearisation of the sum polynomial, consider the \( 2r + 1 \) following matrices (with
\( r = [k/2] \))

\[
\begin{align*}
\Pi_1 &= \pi_1 \otimes I^{\otimes (r-1)} \\
\Pi_2 &= \pi_2 \otimes I^{\otimes (r-1)} \\
&\vdots \\
\Pi_{2\ell-1} &= \pi_3^{\otimes (\ell-1)} \otimes \pi_1 \otimes I^{\otimes (r-\ell-1)} \\
&\vdots \\
\Pi_{2r-1} &= \pi_3^{\otimes (r-1)} \otimes \pi_1 \\
\Pi_{2r+1} &= \pi_3^{\otimes r}.
\end{align*}
\]

with the matrices \( \pi_1, \pi_2, \pi_3 \) being defined by

\[
\pi_1 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\pi_3 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & q & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & q^{n-1}
\end{pmatrix},
\pi_2 = (\sqrt{q})\pi_3\pi_1,
\]

\( \sqrt{q} \) being there only when \( n \) is even and \( q = e^{2i\pi/n} \). Many authors have considered this set of
matrices, see [12] for references. It is not difficult to see that the \( \Pi \)-matrices satisfy the relation

\[
\Pi_i \Pi_j = q \Pi_j \Pi_i, \quad \Pi_i^n = 1, \quad i < j
\]

and as a consequence of the relation root-coefficients that we have \( (x^1\Pi_1 + \cdots x^k\Pi_k)^n = (x^1)^n + \cdots (x^k)^n \). The algebra generated by elements satisfying (D.6) generate the generalised
Clifford algebra. This algebra together with its representations have been classified by Morris
[74].

The product polynomial is linearised by the matrices \( H_{i+1} \) (with \( H_{ij} \) the canonical matrices
with a one at the intersection of the \( i \)-th line and \( j \)-th column and zero elsewhere). Indeed an
easy calculation gives \( (x^1H_{12} + \cdots x^nH_{1n})^n = x^1 \cdots x^n \).

The matrices \( \Pi \) and \( H \) allow to linearise any polynomial, since an arbitrary polynomial is a
sum of monomials

\[
p(x) = \sum_{\ell=1}^q m_\ell(x).
\]

Each monomial is a particular case of the polynomial \( (x_1)^{a_1}(x_2)^{a_2} \cdots (x_p)^{a_p} \) which can be
linearised by the \( n \times n \) \( H \)-matrices
Indeed, (51) implies that the first power of $A$ are not equal to one (see Section 3.2.4). The matrix i.e., which would have been obtained by the algorithm above (for physical reasons the normalisation of these matrices is not the same of the normalisation of $A$). Thus we take the matrices as in (D.7).

Thus any monomial of the polynomial $p$ can be linearised as above and we have $p(x) = \sum_{\ell=1}^{q} M_{\ell}^{n}$, with $M_{\ell}$, $q$ $n \times n$ matrices. Introducing the commuting matrices $\tilde{M}_{\ell}$ we have using the linearisation of the sum polynomial by the $\Pi$--matrices

$$p(x) = \sum_{\ell=1}^{q} M_{\ell}^{n} = \sum_{\ell=1}^{q} (\Pi_{\ell} \otimes \tilde{M}_{\ell})^{n}.$$  

Since $\Pi_{\ell} \otimes \tilde{M}_{\ell}$ are linear in $x^{k}$ we have $\sum_{\ell=1}^{q} \Pi_{\ell} \otimes \tilde{M}_{\ell} = x_{1}G_{1} + \cdots x_{k}G_{k}$. This ends end the process of linearisation. Of course this process is far from being unique and many different matrices of different size can be obtained. See [72, 12] for examples and comments.

As an illustration of this process consider the little algebra (51) of Section 3.2.4. Equation (51) say that the generators $A_{-1/F}$ and $A_{1-1/F}$ generate the Clifford algebra of the polynomial $x^{F-1}y$. The process above allows to find a linearisation of the polynomial by the $n \times n$ matrices $A_{-1/F}^+$, $A_{1-F}^+$

$$A_{-1/F}^+ = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
a_{1} & 0 & 0 & \ldots & 0 & 0 \\
0 & a_{2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & a_{F-1} & 0
\end{pmatrix}, \quad A_{1-F}^+ = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & \frac{1}{a_{1} \cdots a_{F-1}} \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}. \quad (D.7)$$

(For physical reasons the normalisation of these matrices is not the same of the normalisation which would have been obtained by the algorithm above i.e. that the non-zero matrix elements are not equal to one (see Section 3.2.4.).) The matrix $A_{-1/F}^+$ can be obtained in a different way. Indeed, (51) implies that the first power of $A_{-1/F}^+$ which is equal to zero is $F$ (in other words the rank of $A_{-1/F}^+$ is $F - 1$). Writing $A_{-1/F}^+$ in its Jordan form using the relations (51) gives a solution for $A_{1-F}^+$ of the type above. However, it is known for $F = 3$ that there exists other solutions for the matrix $A_{1-F}^+$ (see [70, 12]). But these matrices would not respect the grading, see (35). Indeed, if some of the matrix elements which are equal to zero in (D.7) are different from zero, the matrices would not be consistent with the Poincaré algebra i.e. we obtain equations where both sides do not have the same helicity. Thus we take the matrices as in (D.7).

In the same way, the representation of the cubic extension of the Poincaré algebra in any space-time dimensions (see Section 3.3) are related to Clifford algebras of polynomial’s. Indeed, writing the R.H.S. of (72) as $g_{\mu \rho} = \frac{1}{5}(\eta_{\mu \nu} \delta_{\rho}^{\sigma} + \eta_{\nu \rho} \delta_{\mu}^{\sigma} + \eta_{\mu \rho} \delta_{\nu}^{\sigma})P_{\sigma}$, we define $p(x) = g_{\mu \rho} x^{\mu} x^{\nu} x^{\rho} = \frac{1}{5}(x.x)(x.P)$. In [56, 57], along the algorithm above, we have found two different representations of the Clifford algebra of the polynomial $\frac{1}{5}(x.x)(x.P)$, but only one was compatible with the Poincaré algebra (see [56]). For the second ones, writing
\[ \frac{1}{2} (x, x) (x, p) = \frac{1}{2} (x^\mu \Gamma_\mu)^2 (x, p) = \frac{1}{2} \begin{pmatrix}
0 & \Lambda^{\frac{1}{2}} x^\mu \Gamma_\mu & 0 \\
0 & 0 & \Lambda^{\frac{1}{2}} x^\mu \Gamma_\mu \\
\Lambda^{-\frac{1}{2}} x^\mu P_\mu & 0 & 0
\end{pmatrix}^3 \]

we obtain

\[ V_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \Lambda^{\frac{1}{2}} \Gamma_\mu & 0 \\
0 & 0 & \Lambda^{\frac{1}{2}} \Gamma_\mu \\
\Lambda^{-\frac{1}{2}} P_\mu & 0 & 0
\end{pmatrix}, \quad (D.8) \]

where \( \Lambda \) is a parameter with dimension of mass (that we take normalised to one) and \( \Gamma_\mu \) are the Dirac \( \Gamma \)-matrices in \( D \)-space-time dimensions. We also renormalise the matrices \( V \) in such a way that the factor \( \sqrt{2} \) cancels. If \( D \) is even, the representation (D.8) is reducible. Taking the Dirac \( \Gamma \)-matrices in the chiral representation

\[ \Gamma_\mu = \begin{pmatrix}
0 & \Sigma_\mu \\
\Sigma_\mu & 0
\end{pmatrix}, \]

(\( \Sigma_0 = \bar{\Sigma}_0 = 1 \) and \( \Sigma_i = -\Sigma_i \)) are the generators of the Clifford algebra of \( SO(D - 1) \)) we obtain

\[ V^+_\mu = \begin{pmatrix}
0 & \Sigma_\mu & 0 \\
0 & 0 & \Sigma_\mu \\
P_\mu & 0 & 0
\end{pmatrix}, \quad V^-_\mu = \begin{pmatrix}
0 & \bar{\Sigma}_\mu & 0 \\
0 & 0 & \Sigma_\mu \\
P_\mu & 0 & 0
\end{pmatrix}, \quad (D.9) \]

two inequivalent representations.

A peculiar Clifford algebra of polynomial is given when \( p = 0 \). In this case, the algebra (D.2) reduces to

\[ \{ e_{i_1}, \cdots, e_{i_n} \} = 0. \quad (D.10) \]

This algebra has been defined in [75] and called the \( n \)-exterior algebra. An explicit basis of this infinite dimensional algebra was exhibited. This is clearly a generalisation of the Grassmann algebra.

Appendix E. Some properties on spinors and \( p \)-forms in \((1 + (4n - 1))\)-dimensions

In this section, we give a collection of useful identities on \( p \)-forms and spinors. The Minkowski metric is taken to be \( \eta_{\mu\nu} = \text{diag}(1, -1, \cdots, -1) \), and the Levi-Civita tensor is normalised as follows \( \varepsilon_{01 \cdots D-1} = -\epsilon_{01' \cdots D-1} = 1 \). For latter convenience, we denote

\[ \delta^{(\mu)}_{(\nu)\ell} = \begin{vmatrix}
\delta_{\nu_1}^{\mu_1} & \cdots & \delta_{\nu_1}^{\mu_\ell} \\
\vdots & \ddots & \vdots \\
\delta_{\nu_\ell}^{\mu_1} & \cdots & \delta_{\nu_\ell}^{\mu_\ell}
\end{vmatrix}, \]

\[ \varepsilon^{(\mu)(\nu)} = \varepsilon_{\mu_1 \cdots \nu_2n}, \varepsilon_{\nu_1 \cdots \nu_2n}, \]

\[ \eta(\mu)(\nu) = \eta_{\mu_1 \nu_1} \cdots \eta_{\nu_2n \mu_2n}. \]

All the properties below come, for instance, from an explicit matrix realisation of the Dirac matrices and from the properties of spinors (see e.g. [76]).
Appendix E.1. Dirac matrices

The Dirac $\Gamma$–matrices in the chiral representation are given by

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \bar{\Sigma}_\mu & 0 \end{pmatrix},$$

(where $\Sigma_0 = \bar{\Sigma}_0 = 1$ and $\bar{\Sigma}_i = -\Sigma_i$ are the generators of the Clifford algebra of $SO(D - 1)$). We introduce further the fully antisymmetric matrices:

$$\Gamma^{(\ell)}: \Gamma_{\mu_1 \cdots \mu_\ell} = \frac{1}{\ell!} \sum_\sigma \Gamma_{\mu_\sigma(1)} \cdots \Gamma_{\mu_\sigma(\ell)}$$

which write

$$\Gamma_{\mu_1 \cdots \mu_{2\ell}} = \frac{1}{(2\ell)!} \begin{pmatrix} \Sigma_{\mu_1} \bar{\Sigma}_{\mu_2} \cdots \bar{\Sigma}_{\mu_{2\ell}} + \text{perm} & 0 \\ 0 & \bar{\Sigma}_{\mu_1} \Sigma_{\mu_2} \cdots \Sigma_{\mu_{2\ell}} + \text{perm} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{\mu_1 \cdots \mu_{2\ell}} & 0 \\ 0 & \bar{\Sigma}_{\mu_1 \cdots \mu_{2\ell}} \end{pmatrix}$$

(E.3)

$$\Gamma_{\mu_1 \cdots \mu_{2\ell+1}} = \frac{1}{(2\ell + 1)!} \begin{pmatrix} 0 & \bar{\Sigma}_{\mu_1} \Sigma_{\mu_2} \cdots \Sigma_{\mu_{2\ell+1}} + \text{perm} \\ \Sigma_{\mu_1} \bar{\Sigma}_{\mu_2} \cdots \bar{\Sigma}_{\mu_{2\ell+1}} + \text{perm} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \Sigma_{\mu_1 \cdots \mu_{2\ell+1}} + \text{perm} \\ \Sigma_{\mu_1 \cdots \mu_{2\ell+1}} & 0 \end{pmatrix}$$

These matrices are generically denoted by

$$\Gamma^{(\nu)}= \Gamma^{\mu_1 \cdots \mu_\ell}, \quad \Gamma^{(\ell)} = \Gamma_{\mu_1 \cdots \mu_\ell}$$

and we have the trace formulæ

$$\frac{1}{2^n} \text{Tr} \left( \Sigma^{(\mu)}_{2a} \Sigma_{(\nu)2b} \right) = \delta_{ab} \left( \delta^{(\mu)2a}_{(\nu)2a} - i \delta_{an} \varepsilon^{(\mu)(\mu')}(\nu) \eta(\mu')(\nu) \right)$$

$$\frac{1}{2^n} \text{Tr} \left( \bar{\Sigma}^{(\mu)}_{2a} \bar{\Sigma}_{(\nu)2b} \right) = \delta_{ab} \left( \delta^{(\mu)2a}_{(\nu)2a} + i \delta_{an} \varepsilon^{(\mu)(\mu')}(\nu) \eta(\mu')(\nu) \right)$$

$$= \delta_{ab} \delta^{(\mu)2a+1}_{(\nu)2a+1}$$

(E.4)

We have assumed here that $\Sigma^{(\mu)2n}$ projects onto the self-dual $2n$–forms. Correspondingly we assume that $\bar{\Sigma}^{(\mu)2n}$ projects onto the anti-self-dual $2n$–forms. This convention fixes the second term in the first equation in (E.4). Moreover, we have the following properties

$$\Gamma_{\mu_1 \cdots \mu_\ell} \Gamma_\mu = \Gamma_{\mu_1 \cdots \mu_{\ell+1}} + \eta_{\mu\nu} \Gamma_{\mu_1 \cdots \mu_{\ell-1}} + \cdots + (-1)^{\ell-1} \eta_{\mu\nu} \Gamma_{\mu_2 \cdots \mu_{\ell}}. \quad (E.5)$$
Appendix E.2. Spinors and \( p \)-forms

The Dirac matrices act naturally on spinors. When the dimension is even, the spinor space decomposes into left-handed and right-handed spinors \( \mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_- \). Furthermore, when \( D = 4n \), left-handed spinors are in the complex conjugate representation of right-handed spinors. In a straight analogy with Appendix Appendix A, we denote \( \Psi_L \in \mathcal{S}_+ \) and \( \Psi_R \in \mathcal{S}_- \) by there components \( \Psi_L \rightarrow \Psi_A \) and \( \Psi_R \rightarrow \bar{\Psi}^A \). This leads to the following index structure for the \( \Gamma \)-matrices: \( \Sigma_\mu \rightarrow \Sigma_{\mu AB}, \bar{\Sigma}_\mu \rightarrow \bar{\Sigma}_{\mu AB} \). The charge conjugation matrix is given by

\[
\bar{\epsilon} = \begin{pmatrix} C_{+AB} & 0 \\ 0 & C_{-\bar{A}\bar{B}} \end{pmatrix}
\]

and allows to raise and lower the indices \( \Psi^A = \Psi_B C_{+AB}, \bar{\Psi}_A = \Psi^B C_{-\bar{A}\bar{B}} \) etc.

The tensor product of two spinors decomposes on the set of \( p \)-forms (\([p]\) denotes \( p \)-forms and \([2n]_\pm\) (anti-)self-dual \( 2n \)-form see (E.11))

\[
\begin{align*}
\mathcal{S} \otimes \mathcal{S} &= [0] \oplus [1] \oplus \cdots [4n], \\
\mathcal{S}_+ \otimes \mathcal{S}_+ &= [0] \oplus [2] \oplus \cdots [2n]_+ \\
\mathcal{S}_- \otimes \mathcal{S}_- &= [0] \oplus [2] \oplus \cdots [2n]_- \\
\mathcal{S}_+ \otimes \mathcal{S}_- &= [1] \oplus [3] \oplus \cdots [2n-1].
\end{align*}
\]

(E.6)

Introducing, \( A_{[p]} \in [p], 0 \leq p \leq 2n-1 \) and \( A_{[2n]_{\pm}} \in [2n]_{\pm} \) (E.6) give (in the sequel \( A_{[p]} \Gamma^{(p)} \) alway means \( A_{[p]\mu_1\cdots\mu_p} \Gamma^{\mu_1\cdots\mu_p} \)):

\[
\begin{align*}
\Xi_{++} &= \Psi_+ \otimes \Psi_+ = \sum_{p=0}^{n-1} \frac{1}{(2p)!} A_{[2p]} \Sigma^{(2p)} \bar{\epsilon}^{-1} + \frac{1}{2} \frac{1}{(2n)!} A_{[2n]} \Sigma^{(2n)} \bar{\epsilon}^{-1} \\
\Xi_{--} &= \Psi_- \otimes \Psi_- = \sum_{p=0}^{n-1} \frac{1}{(2p)!} A'_{[2p]} \bar{\Sigma}^{(2p)} \bar{\epsilon}^{-1} + \frac{1}{2} \frac{1}{(2n)!} A'_{[2n]} \bar{\Sigma}^{(2n)} \bar{\epsilon}^{-1} \\
\Xi_{+-} &= \Psi_+ \otimes \Psi_- = \sum_{p=0}^{n-1} \frac{1}{(2p+1)!} A_{[2p+1]} \Sigma^{(2p+1)} \bar{\epsilon}^{-1} \\
\Xi_{-+} &= \Psi_- \otimes \Psi_+ = \sum_{p=0}^{n-1} \frac{1}{(2p+1)!} A'_{[2p+1]} \bar{\Sigma}^{(2p+1)} \bar{\epsilon}^{-1}.
\end{align*}
\]

(E.7)

In the first relation in (E.7) the \( 2n \)-form is self-dual, and in the second the \( 2n \)-form is anti-self-dual. Using the trace relations (E.4), one gets

\[
\begin{align*}
A_{[2p]} &= \frac{1}{2^n} \text{Tr} \left( \Xi_{++} \bar{\epsilon} \Sigma^{(2p)} \right), \quad A_{[2n]}_+ = \frac{1}{2^n} \text{Tr} \left( \Xi_{++} \bar{\epsilon} \Sigma^{(2n)} \right), \\
A'_{[2p]} &= \frac{1}{2^n} \text{Tr} \left( \Xi_{--} \bar{\epsilon} \bar{\Sigma}^{(2p)} \right), \quad A'_{[2n]}_- = \frac{1}{2^n} \text{Tr} \left( \Xi_{--} \bar{\epsilon} \bar{\Sigma}^{(2n)} \right), \\
A_{[2p+1]} &= \frac{1}{2^n} \text{Tr} \left( \Xi_{+-} \bar{\epsilon} \bar{\Sigma}^{(2p+1)} \right), \\
A'_{[2p+1]} &= \frac{1}{2^n} \text{Tr} \left( \Xi_{-+} \bar{\epsilon} \Sigma^{(2p+1)} \right),
\end{align*}
\]

(E.8)

where \( \text{Tr} \left( \Xi_{--} \bar{\epsilon} \bar{\Sigma}^{(2p+1)} \right) = \text{Tr} \left( \Xi_{--} \bar{\epsilon} \bar{\Sigma}_{\mu_1\cdots\mu_{2p+1}} \right) \) to simplify notations.
Finally, we introduce some operations on \( p \)-forms. The Hodge duality is a linear map \( \star : [p] \rightarrow [D-p] \). If \( A_{[p]} \in [p] \), \( \star A_{[p]} = B_{[D-p]} \in [D-p] \) is given by

\[
B_{[D-p]}{}^{\mu_1 \cdots \mu_{D-p}} = \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_{D-p} \nu_1 \cdots \nu_p} A_{[p]}^{\nu_1 \cdots \nu_p},
\]

(E.9)

and it is easy to prove that

\[
\star \star A_{[p]} = (-1)^{(D-1)(p-1)} A_{[p]},
\]

(E.10)

When the dimension is even, one can define a (anti-)self-dual \((D/2)\)-form:

\[
\star A_{[D/2]} = \begin{cases} 
\pm A_{[D/2]} & \text{when } D/2 \text{ is an odd number } (\star^2 = 1) \\
\pm i A_{[D/2]} & \text{when } D/2 \text{ is an even number } (\star^2 = -1)
\end{cases}
\]

(E.11)

This means that (anti-)self-dual \(2n\)-forms are complex representations of the Lorentz group when \(D = 4n\).

Next, introducing \( \varepsilon \in [1] \), one defines

- The inner product

\[
i_\varepsilon : \quad [p] \rightarrow [p-1]
A_{[p]} \mapsto i_\varepsilon A_{[p]},
\]

(E.12)

in components we have

\[
(i_\varepsilon A_{[p]}){}^{\mu_1 \cdots \mu_{p-1}} = A_{[p]}{}^{\mu_1 \cdots \mu_p} \varepsilon^{\mu_p}.
\]

(E.13)

Notice the difference of convention, useful for our purpose: the summation is done on the last index instead of the first one.

- The exterior product

\[
\wedge : \quad [p] \rightarrow [p+1]
A_{[p]} \mapsto A_{[p]} \wedge \varepsilon,
\]

(E.14)

in components reads

\[
(A_{[p]} \wedge \varepsilon){}^{\mu_1 \cdots \mu_{p+1}} = \frac{1}{p!} \varepsilon^{\nu_1 \cdots \nu_{p+1}} A_{[p]}{}^{\nu_1 \cdots \nu_p} \varepsilon_{\nu_p+1}.
\]

(E.15)

- Now, with \( A_{[p]} \in [p] \) and \( \star A_{[p]} = B_{[D-p]} \in [D-p] \), we have that

\[
\star (A_{[p]} \wedge \varepsilon) = (-1)^p i_\varepsilon B_{[D-p]},
\]

(E.16)

\[
\star (i_\varepsilon A_{[p]}) = (-1)^{D(p-D)} B_{[D-p]} \wedge \varepsilon.
\]
Then, one defines the exterior derivative \( d \) which maps \([p] \rightarrow [p+1]\) by

\[
(dA_{[p]})_{\mu_1 \cdots \mu_{p+1}} = \frac{1}{p!} \delta_{\mu_1 \cdots \mu_{p+1}}^\nu \partial_{\nu_1} A_{[p]}^{\nu_2 \cdots \nu_{p+1}}
\]

(E.17)

and its adjoint \( d^\dagger \) which maps \([p] \rightarrow [p-1]\) is defined by \( d^\dagger = (-1)^{pD+D^*} d^* \), and gives

\[
(d^\dagger A_{[p]})_{\mu_2 \cdots \mu_p} = \partial^{\mu_1} A_{[p]}^{\mu_1 \nu_2 \cdots \nu_p}.
\]

(E.18)

It can the be shown that we have the following relation

\[
\frac{1}{(p+1)!} dA_{[p]} dA_{[p]} + \frac{1}{(p-1)!} d^\dagger A_{[p]} d^\dagger A_{[p]}
= (-1)^{D-1} \left( \frac{1}{(D-p-1)!} d^\dagger B_{[D-p]} d^\dagger B_{[D-p]} + \frac{1}{(D-p+1)!} dB_{[D-p]} dB_{[D-p]} \right)
\]

with \( B_{[D-p]} = * A_{[p]} \).

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