Localization for a one-dimensional split-step quantum walk with bound states robust against perturbations

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Abstract

For given two unitary and self-adjoint operators on a Hilbert space, a spectral mapping theorem was proved in [14]. In this paper, as an application of the spectral mapping theorem, we investigate the spectrum of a one-dimensional split-step quantum walk. We give a criterion for when there is no eigenvalues around ±1 in terms of a discriminant operator. We also provide a criterion for when eigenvalues ±1 exist in terms of birth eigenspaces. Moreover, we prove that eigenvectors from the birth eigenspaces decay exponentially at spatial infinity and that the birth eigenspaces are robust against perturbations.

1 Introduction

During the last two decades, increasing attention has been paid to discrete-time quantum walks (see [1, 17, 21, 30, 31, 28] and references therein), which are quantum counterparts of classical random walks. Motivated by Grover’s search algorithm [9], Szegedy [38] quantized a random walk on a finite bipartite graph, define an evolution operator as a product of two unitary and self-adjoint operators, and compute its spectrum from the transition probabilities of the random walk. The bipartite walk was updated in [26, 27] and then reformulated in [34, 11] as a quantum walk on a digraph (without assuming bipartiteness). Nowadays, such a generalization is called the (twisted) Szegedy walk [10, 11, 12], which as a special case includes the Grover walk [34, 41]. The Szegedy walk on a symmetric digraph $G = (V, D)$ is described by the evolution operator $U = SC$, which is a product of two unitary self-adjoint operators $S$ and $C$ on the Hilbert space $l^2(D)$ of square summable functions on directed edges $D$. Here $S$ and $C$ is called the shift and coin operators. Moreover, $C$ can be expressed as $2d^*d - 1$, where $d : l^2(D) \rightarrow l^2(V)$ is coisometry, i.e., $dd^* = 1$, and is called a boundary operator. A remarkable feature of the Szegedy walk is that the spectrum $\sigma(U)$ of $U$ can be expressed in terms of the discriminant operator $T = dSd^*$ and the birth eigenspaces $B_{\pm} = \ker d \cap \ker (S \pm 1)$ as

$$\sigma(U) = \varphi^{-1}(\sigma(T)) \cup \{+1\}^{M_+} \cup \{-1\}^{M_-},$$

(1.1)

where $\varphi(z) = (z + z^{-1})/2$ and $M_{\pm} = \dim B_{\pm}$ denotes the cardinality of the set $\{\pm 1\}$ with the convention $\{\pm 1\}^{M_+} = \emptyset$ when $M_+ = 0$. This statement is called the spectral mapping theorem of quantum walks [34, 11] and $\varphi^{-1}(\sigma(T))$ is called the inherited part [29, 13]. In the case of the Grover walk, the discriminant operator $T$ is unitarily equivalent to the transition probability operator $P$ of the symmetric random walk on the graph where the Grover walk itself is defined.

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Hence, the quantized evolution $U$ inherits the spectrum form the transition probability operator $P$ of the classical random walk. In [11], the multiplicities $M_\pm$ were characterized in terms of graph structure.

The spectral mapping theorem was extended to a more general setting in [14, 35]. Let $d$ be coisometry from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ and $S$ be a unitary and self-adjoint operator on $\mathcal{H}$. Then $U := S(2d^2d - 1)$ and $T := dSd^*$ satisfy (1.1). The spectral mapping theorem of this form can be applied to the spectral analysis of various types of quantum walks. Actually, in a previous paper [6], the authors of the current paper used it for analyzing a $d$-dimensional split-step quantum walk, which was a unified model including Kitagawa’s split-step quantum walks [18] and $d$-dimensional quantum walks [25, 33, 15]. In particular, the authors performed the spectral analysis of the inherited part from the discriminant operator $T$ and provided a criterion for $T$ and hence $U$ to have eigenvalues.

In this paper, we perform the analysis of the birth eigenspaces $\mathcal{B}_\pm$ of the one-dimensional split-step quantum walk. We provide a criterion for when $\mathcal{B}_\pm$ is nontrivial. Moreover, we prove that the norm of vectors in $\mathcal{B}_\pm$ (if exists) decay exponentially at spatial infinity and show the robustness of $\mathcal{B}_\pm$ against perturbations. Here we note that the criterion for the nontriviality of $\mathcal{B}_\pm$ is given in terms of the asymptotic behavior of local coins $C(x)$ as $x$ tends to $\pm\infty$. This adapts to two phase quantum walks [4, 5] and anisotropic quantum walks [32, 33] and leads us to define a topological index such as those introduced in [13, 2]. In a forthcoming paper [37], the third author studies such an index in terms of supersymmetric quantum mechanics.

The spectral analysis of the quantum walk is of particular interest, because the asymptotic behavior is governed by the spectral properties of the evolution operator $U$. The presence of an eigenvalue ensures that localization occurs if and only if the initial state has an overlap with its eigenvector [23, 35]. Hence, if $\mathcal{B}_\pm$ is nontrivial, the localization can occur. A weak limit theorem originated from Konno [19, 20] (see also [8]) says that at large time $t$, the position $X_t$ of the quantum walker behaves like $X_t \sim tV$, where $V$ is interpreted as the asymptotic velocity.

As put into evidence in [36, 33], if $U$ is asymptotically homogeneous, then the distribution $\mu_V$ of $V$ is given by

$$\mu_V(dv) = \|\Pi_p(U)\Psi_0\|^2\delta_0(dv) + \|E_\hat{v}(dv)\Pi_{ac}(U)\Psi_0\|^2.$$  

Here $\Pi_p(U)$ and $\Pi_{ac}(U)$ are orthogonal projections onto the eigenspaces and the subspace of absolute continuity for $U$, $E_\hat{v}$ is the spectral measure of the velocity operator $\hat{v}$, and $\Psi_0$ is the initial state. This statement is based on the fact that $U$ has no singular continuous spectrum [31] (see also [32]). A weak limit theorem for the one-dimensional split-step quantum walk will be reported in a subsequent paper [7].

This paper is organized as follows. In Section 2, we define a shift operator $S$ and a coin operator $C$ so that both are unitary and self-adjoint on $l^2(\mathbb{Z}; \mathbb{C}^2)$. The coin operator $C$ is also assumed to be the multiplication operator by unitary and self-adjoint matrices $C(x) \in M(2; \mathbb{C})$ ($x \in \mathbb{Z}$). The evolution operator of the split-step quantum walk is defined as $U = SC$ and the state of a walker at time $t$ is given by $\Psi_t = U^t\Psi_0$, where $\Psi_0$ is the initial state of the walker. Then the state evolution is governed by

$$\Psi_{t+1}(x) = P(x+1)\Psi_t(x+1) + Q(x-1)\Psi_t(x-1) + R(x)\Psi_t(x), \quad x \in \mathbb{Z}, \ t = 0, 1, 2, \ldots,$$

where $P(x)$, $Q(x)$, and $R(x) \in M(2; \mathbb{C})$ are determined by $S$ and $C$. In Examples 2.1 and 2.2 we see that $U$ becomes the standard one-dimensional quantum walk and Kitagawa’s split-step quantum walk [18] as special cases.

In Section 3, we see that the spectral mapping theorem (1.1) is applicable to the split-step quantum walk and we give an explicit expression of the discriminant operator $T$ in terms of eigenvectors of $C(x)$ (Lemma 3.3). Here we also provide a criterion for when $T$ has no eigenvalues around $\pm 1$ (Theorem 3.5).
In Section 4, we introduce positive constants $B_\pm$ and $b_\pm$ and prove that: if $B_\pm < 1$, then \( \dim \mathcal{B}_\pm = 1 \); if $b_\pm > 1$, then $\mathcal{B}_\pm$ is a trivial subspace (Theorem 4.2). Here, the constants $B_\pm$ and $b_\pm$ are defined in terms of the asymptotic behavior of local coins $C(x)$ as $x$ tends to $\pm\infty$.

In Section 5, we prove two characteristic properties of vectors in $\mathcal{B}_\pm$. In Subsection 5.1, we show that if $B_\pm < 1$, then $\Psi \in \mathcal{B}_\pm$ exhibits an exponential decay, i.e., there exist positive constants $c_\pm$, $c'_\pm$, $\kappa_\pm$, and $\kappa'_\pm$, such that

$$ \kappa'_\pm e^{-c'_\pm |x|} \leq \| \Psi(x) \|_{\mathcal{C}^2} \leq \kappa_\pm e^{-c_\pm |x|}, \quad |x| \geq R_\pm $$

with some $R_\pm$ sufficiently large. Let $X_t$ be the random variable denoting the position of the quantum walker at time $t$. Then the probability distribution of $X_t$ is given by

$$ P(X_t = x) = \| \Psi_t(x) \|_{\mathcal{C}^2}^2, \quad x \in \mathbb{Z}, \ t = 0, 1, 2, \ldots, $$

where $\Psi_0$ is the initial state. Combining (1.2) and (1.3) yields the fact that $P(X_t = x)$ decays exponentially for the initial state $\Psi_0 \in \mathcal{B}_\pm$. In Subsection 5.2, we show that $\mathcal{B}_\pm$ is robust against local perturbations of $C(x)$. To this end, we consider two local coins $C(x)$ and $C'(x)$ that satisfy

$$ \lim_{x \to \pm\infty} C'(x) = \lim_{x \to \pm\infty} C(x) =: C_{\pm\infty}, $$

i.e., the difference between $C'(x)$ and $C(x)$ vanish at spatial infinity. Hence, we can regard $C$ and $C'$ as an unperturbed coin and a perturbed coin. We use $\mathcal{B}_\pm(C)$ for $\mathcal{B}_\pm$ to make the dependence on $C$ explicit. We introduce constants $\beta_\pm$ determined only by $C_\pm$ and prove that if $\beta_\pm < 1$, then $\dim \mathcal{B}_\pm(C) = \dim \mathcal{B}_\pm(C') = 1$ (Theorem 5.2). This implies that $\mathcal{B}_\pm$ are robust against perturbations that vanish at spatial infinity.

In Section 6, we give two examples. The first one is an anisotropic quantum walk. The second one is Kitagawa’s split-step quantum walk. In these examples, we see that the following three cases are possible: (i) $\dim \mathcal{B}_+ = \dim \mathcal{B}_- = 1$; (ii) $\mathcal{B}_+ = \mathcal{B}_- = \{0\}$; (iii) $\dim \mathcal{B}_+ = 1$ and $\mathcal{B}_- = \{0\}$.

## 2 Definition of the model

Let

$$ \mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C}^2) = \{ \Psi : \mathbb{Z} \to \mathbb{C}^2 \mid \sum_{x \in \mathbb{Z}} \| \Psi(x) \|_{\mathcal{C}^2}^2 < \infty \} $$

be the Hilbert space of states and define a shift operator $S$ and a coin operator $C$ on $\mathcal{H}$ as follows. For a vector $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathcal{H}$ and $x \in \mathbb{Z},$

$$ (S\Psi)(x) = \begin{pmatrix} p\Psi_1(x) + q\Psi_2(x + 1) \\ q\Psi_1(x - 1) - p\Psi_2(x) \end{pmatrix}, $$

where $(p, q) \in \mathbb{R} \times \mathbb{C}$ satisfies $p^2 + |q|^2 = 1$. Then, $S$ is unitary and self-adjoint. Let \( \{ C(x) \}_{x \in \mathbb{Z}} \subset U(2) \) be a family of unitary and self-adjoint matrices such that

$$ C(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & -a(x) \end{pmatrix}, $$

(2.2)

where $a(x) \in \mathbb{R}$ and $a(x)^2 + |b(x)|^2 = 1$. Since, by (2.2), $\text{tr} C(x) = 0$ and $\det C(x) = -1$, we have $\ker(C(x) \pm 1) = 1$. For $\Psi \in \mathcal{H}$, $C\Psi$ is given by

$$ (C\Psi)(x) = C(x)\Psi(x), \quad x \in \mathbb{Z}. $$
Then, \( C(x) \) is unitary and self-adjoint and so is \( C \). We now define an evolution operator as

\[
U = SC.
\]

Let \( \Psi_0 \in \mathcal{H} (\| \Psi_0 \| = 1) \) be the initial state of a quantum walker. We define the state of the walker at time \( t \in \mathbb{N} \) as \( \Psi_t = U^t \Psi_0 \) and we obtain the state evolution

\[
\Psi_{t+1}(x) = P(x+1)\Psi_t(x+1) + Q(x-1)\Psi_t(x-1) + R(x)\Psi_t(x), \quad x \in \mathbb{Z},
\]

(2.3)

where

\[
P(x) = q \begin{pmatrix} b(x) & -a(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \bar{q} \begin{pmatrix} 0 & 0 \\ a(x) & b(x) \end{pmatrix}, \quad R(x) = p \begin{pmatrix} a(x) & b(x) \\ -b(x) & a(x) \end{pmatrix}.
\]

From (2.3), this walk is interpreted as a lazy quantum walk. We emphasize that our walk is defined as a two-state quantum walk on \( \ell^2(\mathbb{Z}; \mathbb{C}^2) \), whereas standard lazy quantum walks \([16], [24]\) are defined as a three-state quantum walk on \( \ell^2(\mathbb{Z}; \mathbb{C}^3) \).

Our evolution \( U \) partially covers several examples of one-dimensional two-state quantum walks as seen below.

**Example 2.1** (Ambainis-type QW). In the one-dimensional quantum walk defined by Ambainis \([1]\), the shift operator \( S_A \) is defined as

\[
(S_A \Psi)(x) = \begin{pmatrix} \Psi_1(x+1) \\ \Psi_2(x-1) \end{pmatrix}, \quad x \in \mathbb{Z}, \ \Psi \in \mathcal{H}.
\]

Let \( C(x) \) be of the form (2.2) and set \( \tilde{C}(x) = \sigma_1 C(x) \), where \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Define an evolution operator \( U_A \) as \( U_A = S_A \tilde{C} \). Then

\[
(S_A \sigma_1 \Psi)(x) = \begin{pmatrix} \Psi_2(x+1) \\ \Psi_1(x-1) \end{pmatrix}, \quad \Psi \in \mathcal{H}.
\]

Let \( S \) satisfy (2.1) with \( p = 0 \) and \( q = 1 \). Then \( U \) becomes \( U_A \). Indeed, \( S = S_A \sigma_1 \) and

\[
U = SC = (S_A \sigma_1)(\sigma_1 \tilde{C}) = U_A.
\]

We emphasize that the evolution \( U_A \) is unitarily equivalent to standard quantum walks (see [30] for more information).

**Example 2.2** (Split-step QW). Let \( S_+ \) and \( S_- \) be shift operators defined as

\[
(S_+ \Psi)(x) = \begin{pmatrix} \Psi_1(x-1) \\ \Psi_2(x) \end{pmatrix}, \quad (S_- \Psi)(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x+1) \end{pmatrix}, \quad x \in \mathbb{Z}, \ \Psi \in \mathcal{H}.
\]

The evolution \( U_{ss}(\theta_1, \theta_2) \) of the split-step quantum walk introduced by Kitagawa et al \([18]\) is defined as

\[
U_{ss}(\theta_1, \theta_2) = S_- R(\theta_2) S_+ R(\theta_1),
\]

where \( \theta_1, \theta_2 \in [0, 2\pi) \) and

\[
R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
\]
By direct calculation,

\[
(\sigma_1 S \cdot R(\theta) S_+ \Psi)(x) = \left( p_\theta \Psi_1(x) + q_\theta \Psi_2(x + 1) \right) \quad \text{with } p_\theta = \sin(\theta/2) \text{ and } q_\theta = \cos(\theta/2)
\]

and

\[
R(\theta) \sigma_1 = \begin{pmatrix} a_\theta & b_\theta \\ b_\theta & -a_\theta \end{pmatrix} \quad \text{with } a_\theta = -\sin(\theta/2) \text{ and } b_\theta = \cos(\theta/2).
\]

When \( p = p_{\theta_2} \) and \( q = q_{\theta_2} \), \( S = \sigma_1 S \cdot R(\theta) S_+ \). Let \( C(x) = R(\theta_1) \sigma_1 \). Then,

\[
U = SC = \sigma_1 U_{ss}(\theta_1, \theta_2) \sigma_1.
\]

Hence, \( U \) and \( U_{ss}(\theta_1, \theta_2) \) are unitarily equivalent.

We call the quantum walk with the evolution \( U \) a split-step quantum walk, because it is a generalization of Kitagawa’s split-step quantum walk in the sense of Example 2.2. Throughout this paper, we consider the shift operator \( S \) and coin operator \( C \) defined by (2.1) and (2.2) unless otherwise stated.

### 3 Spectral mapping theorem

In this section, we apply spectral mapping techniques obtained in [14, 35] for the product of two unitary self-adjoint operators. Since the shift \( S \) and coin \( C \) are unitary and self-adjoint, these techniques can be applied to the evolution \( U \) of the split step quantum walk. To applying these techniques, we need to define a coisometry operator

\[
d : \mathcal{H} \rightarrow \ell^2(\mathbb{Z}) =: \mathcal{K}
\]

so that \( C = 2d^*d - 1 \). Since \( \dim \ker (C(x) - 1) = 1 \), we can choose a unique normalized vector \( \chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix} \in \ker (C(x) - 1) \) up to a constant factor. We now define an operator \( d : \mathcal{H} \rightarrow \mathcal{K} \) as

\[
(d \Psi)(x) = 1 < \chi(x), \Psi(x) >_{\mathbb{C}^2}, \quad x \in \mathbb{Z} \quad \text{for } \Psi \in \mathcal{H}.
\]

We use \( \text{id}_{\mathcal{K}} \) to denote the identity on \( \mathcal{K} \).

**Lemma 3.1.** Let \( d \) be as above.

1. \( d \) is bounded and its adjoint \( d^* : \mathcal{K} \rightarrow \mathcal{H} \) is given by

\[
d^* \psi = \psi \chi \quad \text{for } \psi \in \mathcal{K}.
\]

2. \( d \) is coisometry, i.e., \( dd^* = \text{id}_{\mathcal{K}} \).

3. \( C = 2d^*d - 1 \).

**Proof.** Because \( \chi(x) \) is a normalized vector in \( \mathbb{C}^2 \),

\[
\|d \Psi\|_{\mathcal{K}}^2 = \sum_{x \in \mathbb{Z}} |\langle \chi(x), \Psi(x) \rangle_{\mathbb{C}^2}|^2 \leq \|\Psi\|^2_{\mathcal{K}} \quad \text{for } \Psi \in \mathcal{H}.
\]
Hence, \( d \) is bounded. For \( \psi \in \mathcal{K} \),

\[
\langle \psi, d\Psi \rangle_{\mathcal{X}} = \sum_{x \in \mathbb{Z}} \bar{\psi}(x) \langle \chi(x), \Psi(x) \rangle_{\mathcal{C}^2} = \langle \psi \chi, \Psi \rangle_{\mathcal{X}},
\]

which completes (1).

By direct calculation,

\[
(dd^* \psi)(x) = \langle \chi(x), (d^* \psi)(x) \rangle_{\mathcal{C}^2} = \psi(x), \quad x \in \mathbb{Z},
\]

which implies that \( d \) is isometry. Hence, (2) is proved.

Because \( d^* d = \sum_{x \in \mathbb{Z}} |\chi(x)|^2 \langle \chi(x) \rangle \) and \( C(x) = 2|\chi(x)|\langle \chi(x) \rangle - 1 \), we obtain (3).

With the terminology of [38, 12], we call \( T = dSd^* \) the **discriminant** of \( U \) and

\[
\mathcal{B}_\pm = \ker d \cap \ker (S \pm 1)
\]

the **birth eigenspaces**. We set \( M_\pm = \dim \mathcal{B}_\pm \). Let \( \varphi \) be the Joukowsky transformation:

\[
\varphi(z) = \frac{z + z^{-1}}{2},
\]

which maps \( S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \} \) onto \([-1, 1]\). We use \( \sigma(A) \) to denote the spectrum of an operator \( A \) and \( \sigma_p(A) \) the point spectrum of \( A \). The following proposition is a direct consequence of [13].

**Proposition 3.2** (Spectral mapping theorem [14]). Let \( U, T, d, M_\pm \), and \( \varphi \) as above.

1. \( T \) is bounded and self-adjoint on \( \mathcal{K} \) with \( \|T\| \leq 1 \).
2. \( \sigma(U) = \varphi^{-1}(\sigma(T)) \cup \{1\}^{M_+} \cup \{-1\}^{M_-} \).
3. \( \sigma_p(U) = \varphi^{-1}(\sigma_p(T)) \cup \{1\}^{M_+} \cup \{-1\}^{M_-} \).
4. \( \dim \ker(U \mp 1) = M_\pm + m_\pm \), where \( m_\pm = \dim \ker(T \mp 1) \).

We use \( L \) to denote the left shift operator on \( \mathcal{K} \):

\[
(L\psi)(x) = \psi(x + 1), \quad x \in \mathbb{Z} \quad \text{for} \quad \psi \in \mathcal{K}.
\]

**Lemma 3.3.** Let \( V \) denote the multiplication operator by

\[
V(x) = p \left( |\chi_1(x)|^2 - |\chi_2(x)|^2 \right)
\]

and \( D = q\bar{\chi}_1 L\chi_2 \). Then

\[
T = D + D^* + V.
\]

**Proof.** The proof proceeds along the same lines as the proof of [9][Lemma 3.2]. Identifying \( \mathcal{K} \) with \( \mathcal{K} \oplus \mathcal{K} \), we observe that \( S = \left( \begin{array}{cc} p & qL \\ \bar{q}L^* & -p \end{array} \right) \). Hence, for \( \psi \in \mathcal{K} \), \( Sd^* \psi = S\psi \chi = \left( \begin{array}{c} p\chi_1 + q\chi_2 \\ \bar{q}L^*\chi_1 - p\chi_2 \end{array} \right) \)

and

\[
T\psi = \langle \chi(\cdot), (Sd^* \psi)(\cdot) \rangle_{\mathcal{C}^2}
\]

\[
= \bar{\chi}_1 (p\chi_1 + q\chi_2)\psi + \bar{\chi}_2 (\bar{q}L^*\chi_1 - p\chi_2)\psi
\]

\[
= q\bar{\chi}_1 L\chi_2 \psi + q\bar{\chi}_2 L^*\chi_1 \psi + p(|\chi_1|^2 - |\chi_2|^2)\psi.
\]

This completes the proof. \( \square \)
In what follows, we provide a criterion for when $T$ has no eigenvalues $\pm E$ with $E > |V|_{\infty} := \sup_{x \in \mathbb{Z}} |V(x)|$. Because, for such an $E$, $E \not\equiv V(x) > 0 \ (x \in \mathbb{Z})$, we can define an operator $K_E$ as

$$K_E^\pm = \frac{1}{\sqrt{E \mp V}}(D + D^*) \frac{1}{\sqrt{E \mp V}}.$$  

**Lemma 3.4.** Let $E$ be as above. The following are equivalent.

(i) $\pm E \in \sigma_p(T)$.

(ii) $\pm 1 \in \sigma_p(K_E^\pm)$.

In this case, $\dim \ker(T \mp E) = \dim \ker(K_E \mp 1)$.

**Proof.** The assertion of the lemma follows from $T \mp E = \frac{1}{\sqrt{E \mp V}}(K_E \mp 1) \frac{1}{\sqrt{E \mp V}}$, $E > |V|_{\infty}$.

**Theorem 3.5.** If $E > |q| + |V|_{\infty}$, then $\pm E \not\in \sigma_p(T)$.

**Proof.** By Lemma 3.4 it suffices to prove that $\|K_E^\pm\| < 1$. To this end, we consider the range of $\langle \psi, K_E^\pm \psi \rangle$ for $\psi \in \mathcal{K}$. Let $\psi_i = \frac{\chi_i}{\sqrt{E \mp V}} \psi$ $(i = 1, 2)$. Then

$$|\langle \psi, K_E^\pm \psi \rangle| = 2|q| \|\psi_1\| \|\psi_2\| \leq |q| (\|\psi_1\|^2 + \|\psi_2\|^2).$$

Because $|\chi_1(x)|^2 + |\chi_2(x)|^2 = 1$,

$$\|\psi_1\|^2 + \|\psi_2\|^2 = \sum_{x \in \mathbb{Z}} \frac{|\psi(x)|^2}{E \pm V(x)}.$$ 

Hence, $\|K_E^\pm\| \leq |q|/(E - |V|_{\infty})$ and the proof of the lemma is complete.

**4 Nontriviality of the birth eigenspaces**

In this section, we address the problem when the birth eigenspaces $B_{\pm}$ defined in (3.2) becomes nontrivial. To this end, we characterize $B_{\pm}$.

In the case of $|p| = 1$, $S$ becomes a constant matrix and $U$ becomes a multiplication operator. In this case, the quantum walker never moves and hence the quantum walk becomes trivial (see also (2.3) with $q = 0$). To avoid this trivial case, we suppose the following.

**Hypothesis 1.** $|p| \neq 1$.

**Lemma 4.1.** Assume Hypothesis 1. Then

$$\ker(S \pm 1) = \left\{ \left(\begin{array}{c}
\frac{-q}{p \pm 1} L \psi \\
\psi
\end{array}\right) \mid \psi \in \mathcal{K}\right\}.$$
**Proof.** Let \( \Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \in \mathcal{H} \). Because \( S \pm 1 = \left( \begin{array}{cc} p \pm 1 & qL \\ qL^* & -p \pm 1 \end{array} \right) \), we observe that \( \Psi \in \ker(S \pm 1) \) if and only if \( \Psi_1 \) and \( \Psi_2 \) belong to \( \mathcal{H} \) and satisfy

\[
\begin{cases}
(p + 1)\Psi_1(x) + q(L\Psi_2)(x) = 0, \\
q(L^*\Psi_1)(x) + (-p \pm 1)\Psi_2(x) = 0
\end{cases}
\] for all \( x \in \mathbb{Z} \). (4.1)

By Hypothesis \( \Psi \) is equivalent to

\[
\begin{cases}
\Psi_1(x) = -\frac{q}{p \pm 1} (L\Psi_2)(x), \\
\left( -\frac{|q|^2}{p \pm 1} - p \pm 1 \right) \Psi_2(x) = 0
\end{cases}
\] for all \( x \in \mathbb{Z} \).

Because \( p^2 + |q|^2 = 1 \) implies that

\[
-\frac{|q|^2}{p \pm 1} - p \pm 1 = 0,
\]

we obtain the desired result. \( \square \)

Combining Lemma 4.1 with (3.1) yields the following,

\[
\mathcal{B}_\pm = \left\{ \Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \left| \Psi \in \mathcal{H}, \ -q_{\chi_1}L\psi + (p \pm 1)\chi_2 \psi = 0 \right. \right\}. \quad (4.2)
\]

In order to provide a criterion for \( \mathcal{B}_\pm \) to be nontrivial, we suppose the following.

**Hypothesis 2.** \( \chi_1(x)\chi_2(x) \neq 0 \) for all \( x \in \mathbb{Z} \).

We define four constants \( B_\pm \) and \( b_\pm \) as

\[
B_\pm = \max \{ B_\pm(-\infty), B_\pm(+\infty) \}, \quad b_\pm = \min \{ b_\pm(-\infty), b_\pm(+\infty) \},
\]

where

\[
B_\pm(-\infty) = \limsup_{x \to -\infty} \left| \frac{q\chi_1(x)}{(p \pm 1)\chi_2(x)} \right|^2, \quad B_\pm(+\infty) = \limsup_{x \to +\infty} \left| \frac{(p \pm 1)\chi_2(x)}{q\chi_1(x)} \right|^2,
\]

\[
b_\pm(-\infty) = \liminf_{x \to -\infty} \left| \frac{q\chi_1(x)}{(p \pm 1)\chi_2(x)} \right|^2, \quad b_\pm(+\infty) = \liminf_{x \to +\infty} \left| \frac{(p \pm 1)\chi_2(x)}{q\chi_1(x)} \right|^2.
\]

We are now in a position to state our main result.

**Theorem 4.2.** Assume Hypotheses \( \Psi \) and \( \Psi \)

(1) If \( B_\pm < 1 \), then \( \dim \mathcal{B}_\pm = 1 \).

(2) If \( b_\pm > 1 \), then \( \mathcal{B}_\pm = \{0\} \).

In order to prove Theorem 4.2, we use the following lemma.

**Lemma 4.3.** Assume Hypotheses \( \Psi \) and \( \Psi \). Let \( \psi : \mathbb{Z} \to \mathbb{C} \) be a nonzero solution to

\[
L\psi = \left( \frac{p \pm 1}{q\chi_1} \right) \psi.
\] (4.3)

Then

\[
B_\pm(-\infty) = \limsup_{x \to -\infty} \left| \frac{\psi(x - 1)}{\psi(x)} \right|^2, \quad B_\pm(+\infty) = \limsup_{x \to +\infty} \left| \frac{\psi(x + 1)}{\psi(x)} \right|^2,
\]

\[
b_\pm(-\infty) = \liminf_{x \to -\infty} \left| \frac{\psi(x - 1)}{\psi(x)} \right|^2, \quad b_\pm(+\infty) = \liminf_{x \to +\infty} \left| \frac{\psi(x + 1)}{\psi(x)} \right|^2.
\] (4.4) (4.5)
Proof. Because Hypotheses 1 and 2 imply that \( p, q, \chi_1(x), \) and \( \chi_2(x) \) are not zero, (4.3) is equivalent to

\[
\begin{cases}
\psi(x + 1) = \frac{(p \pm 1) \chi_2(x)}{q \chi_1(x)} \psi(x), & x \geq 0, \\
\psi(x - 1) = \frac{q \chi_1(x)}{(p \pm 1) \chi_2(x)} \psi(x), & x \leq 0.
\end{cases}
\]  

(4.6)

Since \( \psi \neq 0 \), (4.6) implies that \( \psi(x) \neq 0 \) for all \( x \in \mathbb{Z} \). Hence,

\[
\left| \frac{q \chi_1(x)}{(p \pm 1) \chi_2(x)} \right|^2 = \frac{\psi(x - 1)^2}{\psi(x)^2}, \quad x \leq 0,
\]

\[
\left| \frac{(p \pm 1) \chi_2(x)}{q \chi_1(x)} \right|^2 = \frac{\psi(x + 1)^2}{\psi(x)^2}, \quad x > 0.
\]

Taking the limits of both sides, we obtain the desired result. \( \square \)

Proof of Theorem 4.2. By (1.2), \( \Psi \in B_\pm \) if and only if there exists a vector \( \psi \in \mathcal{K} \) such that \( \Psi = \left( \frac{-q}{p+1} L \psi \right) \) and \( \psi \) satisfies (4.3). Now we suppose that \( B_\pm < 1 \). We define a function \( \psi_0 : \mathbb{Z} \to \mathbb{C} \) inductively as follows. Let \( \psi_0(0) = 1 \) and define \( \psi_0(x) (x \neq 0) \) by (4.6). Then, from the above argument, \( \psi_0 \) satisfies (4.3). By definition, \( \psi_0 \neq 0 \). Hence, we have \( \psi_0 \in \mathcal{K} \) by combining Lemma 4.3 with the ratio test. Thus, defining \( \Psi_0 = \left( \frac{-q}{p+1} L \psi_0 \right) \), we observe that \( \Psi_0 \) is nonzero and belongs to \( B_\pm \). Hence, \( B_\pm \) is nontrivial. If there is another nonzero vector \( \Psi = \left( \frac{-q}{p+1} L \psi \right) \in B_\pm \), then \( \psi \) also satisfies (4.6). Taking a constant \( \alpha = \psi(0)/\psi_0(0) \), we observe from (4.6) that \( \psi = \alpha \psi_0 \). Hence, \( \dim B_\pm = 1 \). Thus, (1) is proved.

We next suppose that \( b_+ > 1 \) and \( \Psi = \left( \frac{-q}{p+1} L \psi \right) \in B_\pm \) is nonzero. Similarly to the above, the ratio test implies that \( \psi \not\in \mathcal{K} \). This contradicts \( \Psi \in B_\pm \). Hence, \( B_\pm = \{0\} \). Thus, (2) is proved. \( \square \)

5 Properties of vectors in the birth eigenspaces

Throughout this section, we suppose that Hypotheses 1 and 2 are satisfied. Summarizing the arguments in Sec. 4, we observe that

\[
B_\pm = \left\{ \Psi = \left( \frac{-q}{p+1} L \psi \right) \mid \psi \in \mathcal{K} \text{ satisfies (4.6)} \right\}.
\]  

(5.1)

5.1 Exponential decay

We prove that the birth eigenvector \( \Psi \in B_\pm \) decays exponentially at spatial infinity.

Theorem 5.1. Suppose that \( B_\pm < 1 \) and \( \Psi \in B_\pm \). Then, there exist positive constants \( c_\pm, c'_\pm, \kappa_\pm, \kappa'_\pm, \) and \( R_\pm > 0 \) such that

\[
k'_\pm e^{-c'_\pm |x|} \leq \|\Psi(x)\|_{C^2} \leq k_\pm e^{-c_\pm |x|}, \quad |x| \geq R_\pm.
\]  

(5.2)
Proof. Let $\Psi \in \mathcal{B}_\pm$. From (5.1), there exists a $\psi \in \mathcal{K}$ such that $\Psi = \left( \frac{-q}{p|x|} L \psi \right) \in \mathcal{B}_\pm$ and $\psi$ satisfy (1.6).

We first prove the right-hand side of (5.2). Because $\|\Psi(x)\|_{L^2}^2 = (|q|^2/(p \pm 1)^2)|\psi(x + 1)|^2 + |\psi(x)|^2$, it suffices to prove that

$$|\psi(x)|^2 \leq \kappa_+ e^{-c_-|x|}, \quad |x| \geq R_+$$

with some $\kappa_+, c_+$, and $R_+ > 0$. Let $\epsilon$ satisfy $0 < \epsilon < 1 - B_+(\infty)$. By the definition of $B_+(\infty)$, there exists $x_0 \in \mathbb{N}$ such that if $x \geq x_0$,

$$0 \leq \sup_{y \geq x} \left| \frac{(p \pm 1) \chi_2(x - 1)}{q \chi_1(y)} \right|^2 - B_+(\infty) \leq \epsilon.$$

Suppose that $x \geq x_0$. By (4.6),

$$|\psi(x)| = \left| \frac{(p \pm 1) \chi_2(x - 1)}{q \chi_1(x - 1)} \right| |\psi(x - 1)|$$

$$= \left| \frac{(p \pm 1) \chi_2(x - 1)}{q \chi_1(x - 1)} \right| \left| \frac{(p \pm 1) \chi_2(x - 2)}{q \chi_1(x - 2)} \right| \cdots \left| \frac{(p \pm 1) \chi_2(x)}{q \chi_1(x)} \right| |\psi(x)|$$

$$\leq (B_+(\infty) + \epsilon)^{(x-x_0)/2} |\psi(x_0)|$$

Since $B_+(\infty) + \epsilon < 1$,

$$c_+(\infty) := -\log(B_+(\infty) + \epsilon) > 0.$$

Hence, if $x \geq x_0$,

$$|\psi(x)|^2 \leq \kappa_+(\infty) e^{-c_+(\infty)x}$$

with $\kappa_+(\infty) := |\psi(x_0)|^2 e^{c_+(\infty)x_0}$. Similarly, taking $0 < \epsilon < 1 - B_+(-\infty)$ and $c_+(-\infty) = -\log(B_+(-\infty) + \epsilon)$, we can prove that there exists $x_1 \in \mathbb{N}$ such that if $x \leq -x_1$,

$$|\psi(x)|^2 \leq \kappa_+(-\infty) e^{-c_+(-\infty)|x|}$$

with $\kappa_+(-\infty) := |\psi(x_1)|^2 e^{c_+(-\infty)x_1}$. Taking $c_+ = \min\{c_+(-\infty), c_+(\infty)\}$, $\kappa_+ = \max\{\kappa_+(-\infty), \kappa_+(-\infty)\}$, $R_+ := \max\{x_0, x_1\}$ yields the right-hand side of (5.2).

Next we prove the left-hand side of (5.2). Taking $\epsilon$ as $0 < \epsilon < b_+(\infty)$, we have

$$0 \leq b_+(\infty) - \inf_{y \geq x} \left| \frac{(p \pm 1) \chi_2(y)}{q \chi_1(y)} \right|^2 \leq \epsilon$$

and

$$|\psi(x)| \geq (b_+(\infty) - \epsilon)^{(x-x_0)/2} |\psi(x_0)|$$

for $x$ greater than some $x_0 > 0$. Because $0 < b_+(\infty) - \epsilon < 1$,

$$c_+(\infty) := -\log(b_+(\infty) - \epsilon) > 0,$$

we observe from the same argument as above that

$$|\psi(x)|^2 \geq \kappa_+(\infty) e^{-c_+(\infty)x}, \quad x \geq x_0$$

with some $\kappa_+(\infty) > 0$. Thus, the right-hand side of (5.2) is proved for $x > 0$ sufficiently large. For $x \leq 0$ sufficiently small, the same argument works. Therefore, we complete the proof. $\square$
5.2 Robustness against perturbations

In this subsection, we illustrate the robustness of the birth eigenspaces against perturbations. For simplicity, we focus here on the case in which the limits \( \lim_{x \to \pm \infty} C(x) \) exist. This is a slight generalization of the anisotropic quantum walk introduced in [32, 33], where the authors addressed the case of \( p = 0 \). We address the case in which \( p \) can be nonzero.

Let \( C_{\pm \infty} \in U(2) \) be self-adjoint unitary matrices with \( \det C_{\pm \infty} = -1 \) and choose normalized vectors \( \chi_{\pm \infty} = (\chi_{\pm \infty, 1}, \chi_{\pm \infty, 2}) \) such that
\[
C_{\pm \infty} = 2 |\chi_{\pm \infty} \rangle \langle \chi_{\pm \infty}| - 1.
\]

We define two constants \( \beta_+ \) and \( \beta_- \) as
\[
\beta_+ = \max \{ \beta_+(-\infty), \beta_+(+\infty) \}, \quad \beta_- = \min \{ \beta_-(\infty), \beta_-(+\infty) \},
\]
where
\[
\beta_+(-\infty) = \left| \frac{q \chi_{-\infty, 1}}{(p \pm 1) \chi_{-\infty, 2}} \right|^2, \quad \beta_+(+\infty) = \left| \frac{(p \pm 1) \chi_{+\infty, 2}}{q \chi_{+\infty, 1}} \right|^2.
\]

**Theorem 5.2.** Let \( C(x) \) be defined in (2.2) and satisfy in addition \( \lim_{x \to \pm \infty} C(x) = C_{\pm \infty} \). Then \( B_\pm = b_\pm = \beta_\pm \). In particular, the following hold.

1. If \( \beta_\pm < 1 \), then \( \dim B_\pm = 1 \).
2. If \( \beta_\pm > 1 \), then \( B_\pm = \{0\} \).

**Proof.** By assumption, \( \lim_{x \to \pm \infty} |\chi_j(x)| = |\chi_{\pm \infty, j}| \) and hence \( B_\pm = b_\pm = \beta_\pm \). Therefore, by Theorem 4.2 we obtain the desired result.

We write \( B_\pm(C) \) for the birth eigenspaces \( B_\pm \) to make the dependence on the coin \( C \) explicit. The following is a direct consequence of Theorem 5.2 and reveals the robustness of the birth eigenspaces against perturbations. See Examples in the next section for more details.

**Corollary 5.3.** Let \( C(x) \) be as in Theorem 5.2 and \( C'(x) \) satisfy the same condition as \( C(x) \), so that \( \lim_{x \to \pm \infty} (C'(x) - C(x)) = 0 \).

1. If \( \beta_\pm < 1 \), then \( \dim B_\pm(C') = \dim B_\pm(C) = 1 \).
2. If \( \beta_\pm > 1 \), then \( B_\pm(C') = B_\pm(C) = \{0\} \).

**Proof.** Because the assertions of Theorem 5.2 depend only on the limit of the coin operator, and both \( C'(x) \) and \( C(x) \) have the same limits, we obtain the desired result.

6 Examples

In this section, we provide examples.
6.1 An anisotropic coin model

Let \( \varepsilon > 0 \) and define
\[
C_{+\infty} := \begin{pmatrix} 1 - 2\varepsilon^2 - 2\varepsilon\sqrt{1 - \varepsilon^2} & 2\varepsilon\sqrt{1 - \varepsilon^2} \\ 2\varepsilon\sqrt{1 - \varepsilon^2} & 2\varepsilon^2 - 1 \end{pmatrix}, \quad C_{-\infty} := \begin{pmatrix} 2\varepsilon^2 - 1 & 2\varepsilon\sqrt{1 - \varepsilon^2} \\ 2\varepsilon\sqrt{1 - \varepsilon^2} & 1 - 2\varepsilon^2 \end{pmatrix}.
\]

We can choose \( \chi_{\pm\infty} \in \ker(C_{\pm\infty} - 1) \) as follows.
\[
\chi_{+\infty} = \begin{pmatrix} \sqrt{1 - \varepsilon^2} \\ \varepsilon \end{pmatrix}, \quad \chi_{-\infty} = \begin{pmatrix} \varepsilon \sqrt{1 - \varepsilon^2} \\ \sqrt{1 - \varepsilon^2} \end{pmatrix}
\]

are eigenvectors of \( C_{\pm\infty} \) corresponding to the eigenvalues 1. Let \( \{\chi(x)\} \subset \mathbb{C}^2 \) be a family of normalized vectors and satisfy \( \chi_1(x)\chi_2(x) \neq 0 \) and \( \lim_{x \to \pm\infty} \chi(x) = \chi_{\pm\infty} \). Then \( C(x) := 2|\chi(x)\rangle\langle\chi(x)| - 1 \) satisfies
\[
\lim_{x \to \pm\infty} C(x) = C_{\pm\infty}.
\] (6.1)

By direct calculation,
\[
\beta_{\pm}(-\infty) = g(\varepsilon)\frac{1 + p}{1 \mp p}, \quad \beta_{\pm}(+\infty) = g(\varepsilon)\frac{1 \mp p}{1 + p},
\] (6.2)
where \( g(\varepsilon) := \varepsilon^2/(1 - \varepsilon^2) \).

**Theorem 6.1.** Let \( \varepsilon_0 \in (0, 1) \) be a unique solution to
\[
g(\varepsilon) = \min \left\{ \frac{1 - p}{1 + p}, \frac{1 + p}{1 - p} \right\}.
\] (6.3)

1. If \( \varepsilon < \varepsilon_0 \), then \( \dim \mathcal{B}_+ = \dim \mathcal{B}_- = 1 \).
2. If \( \varepsilon > \varepsilon_0 \), then \( \mathcal{B}_+ = \mathcal{B}_- = \{0\} \).

**Remark 6.2.** Combining Theorem 6.1 with Proposition 3.2, we observe that
\[
\dim \ker(U \mp 1) \geq \dim \mathcal{B}_\pm = 1 \quad \text{for} \quad \varepsilon < \varepsilon_0.
\]
Moreover, the above statement is independent of the choice of \( C(x) = 2|\chi(x)\rangle\langle\chi(x)| - 1 \) that satisfies (6.1). See Corollary 5.3. In the case of the two phase quantum walk with
\[
C(x) = \begin{cases} C_{+\infty}, & x > 0, \\ C_{-\infty}, & x \leq 0 \end{cases}
\]
and \( \varepsilon < 1/\sqrt{2} \), Theorem 6.1 implies that if \(|q| < 2\varepsilon^2\), then \( \pm 1 \notin \sigma_p(T) \). Hence,
\[
\dim \ker(U \mp 1) = 1 \quad \text{for} \quad \varepsilon < \min\{\varepsilon_0, 1/\sqrt{2}\}.
\]

**Proof.** Because \( g \) is strictly increasing in \((0, 1)\) with \( \lim_{\varepsilon \downarrow 0} g(\varepsilon) = 0 \) and \( \lim_{\varepsilon \uparrow 1} g(\varepsilon) = +\infty \), there is a unique solution \( \varepsilon_0 \) to (6.3) in \((0, 1)\). By (6.2),
\[
\beta_+ = g(\varepsilon) \max \left\{ \frac{1 - p}{1 + p}, \frac{1 + p}{1 - p} \right\} = \beta_-.
\]
Hence, \( \beta_{\pm} < 1 \) if and only if
\[
g(\varepsilon) < \min \left\{ \frac{1 - p}{1 + p}, \frac{1 + p}{1 - p} \right\} = g(\varepsilon_0).
\]
Therefore, Theorem 5.2 provides the desired results. \( \square \)
6.2 Kitagawa’s split-step quantum walk

Here we slightly generalize the one discussed in Example 2.2. Let \( p = \sin(\theta_2/2) \) and \( q = \cos(\theta_2/2) \) with \( \theta_2 \in [-2\pi, 2\pi] \). Let \( \theta_1 : \mathbb{Z} \to [0, 2\pi) \) be a function and define \( C(x) \) as in (2.2) with \( a(x) = -\sin(\theta_1(x)/2) \) and \( b(x) = \cos(\theta_1(x)/2) \). Similarly to the argument in Example 2.2 \( U \) is unitarily equivalent to \( U_{ss}(\theta_1(\cdot), \theta_2) \). In this case, we can take a normalized vector \( \chi(x) \in \ker(C(x) - 1) \) as

\[
\chi(x) = \frac{1}{\sqrt{2(1 + \sin(\theta_1(x)/2))}} \left( \frac{\cos(\theta_1(x)/2)}{1 + \sin(\theta_1(x)/2)} \right).
\]

(6.4)

By definition, if \( \theta_2, \theta_1(x) \neq \pi \), then Hypotheses 1 and 2 are satisfied. Suppose that the limits \( \theta_{\pm \infty} := \lim_{x \to \pm \infty} \theta_1(x) \in [0, 2\pi) \setminus \{ \pi \} \) exist. We define \( \chi_{\pm \infty} \) as in (6.4) with \( \theta_1(x) \) replaced by \( \theta_{\pm \infty} \). By direct calculation,

\[
\beta_{\pm}(\infty) = \frac{1 - \sin(\theta_{-\infty}/2)}{1 + \sin(\theta_{-\infty}/2)} \frac{1 \mp \sin(\theta_{2}/2)}{1 \pm \sin(\theta_{2}/2)}, \quad \beta_{\pm}(\infty) = \frac{1 + \sin(\theta_{+\infty}/2)}{1 - \sin(\theta_{+\infty}/2)} \frac{1 \pm \sin(\theta_{2}/2)}{1 \mp \sin(\theta_{2}/2)}.
\]

(6.5)

**Theorem 6.3.**

1. If \( \sin(\theta_{-\infty}/2) < \sin(\theta_{+\infty}/2) \), then \( \mathcal{B}_{\pm} = \{0\} \).

2. If \( \sin(\theta_{-\infty}/2) > \sin(\theta_{+\infty}/2) \), then the following hold:
   - If \( \mp \sin(\theta_{2}/2) \in (\sin(\theta_{+\infty}/2), \sin(\theta_{-\infty}/2)) \), then \( \dim \mathcal{B}_{\pm} = 1 \);
   - If \( \mp \sin(\theta_{2}/2) < \sin(\theta_{+\infty}/2) \) or \( \sin(\theta_{-\infty}/2) < \mp \sin(\theta_{2}/2) \), then \( \mathcal{B}_{\pm} = \{0\} \).

**Proof.** From (6.5), we obtain the following assertions.

(a) \( \beta_{\pm}(\infty) < 1 \) if and only if \( \mp \sin(\theta_{2}/2) < \sin(\theta_{-\infty}/2) \).

(b) \( \beta_{\pm}(\infty) < 1 \) if and only if \( \sin(\theta_{+\infty}/2) < \mp \sin(\theta_{2}/2) \).

Combining (a) and (b) with Theorem 5.2 we obtain the desired results. \( \square \)

**Remark 6.4.** In the case of (2) in Theorem 6.3 we observe the following.

- If both \( -\sin(\theta_{2}/2) \) and \( +\sin(\theta_{2}/2) \) are in \( (\sin(\theta_{+\infty}/2), \sin(\theta_{-\infty}/2)) \), then

\[
\dim \mathcal{B}_{+} = \dim \mathcal{B}_{-} = 1.
\]

- If \( \mp \sin(\theta_{2}/2) \in (\sin(\theta_{+\infty}/2), \sin(\theta_{-\infty}/2)) \) and \( \pm \sin(\theta_{2}/2) \notin [\sin(\theta_{+\infty}/2), \sin(\theta_{-\infty}/2)] \), then

\[
\dim \mathcal{B}_{\pm} = 1, \quad \mathcal{B}_{+} = \{0\}.
\]

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