Connections Between Real Polynomial Solutions Of Hypergeometric-type Differential Equations With Rodrigues Formula

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Abstract

Starting from the Rodrigues representation of polynomial solutions of the general hypergeometric-type differential equation complementary polynomials are constructed using a natural method. Among the key results is a generating function in closed form leading to short and transparent derivations of recursion relations and addition theorem. The complementary polynomials satisfy a hypergeometric-type differential equation themselves, have a three-term recursion among others and obey Rodrigues formulas. Applications to the classical polynomials are given.

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Running title: Connections between polynomials with Rodrigues
1 Introduction

Real polynomial solutions $P_l(x)$ of the hypergeometric-type differential equation (ODE)

$$\sigma(x)\frac{d^2 P_l}{dx^2} + \tau(x)\frac{dP_l}{dx} + \Lambda_l P_l(x) = 0, \quad \Lambda_l = -l\tau' - \frac{l}{2}(l-1)\sigma''$$

(1)

with $l = 0, 1, \ldots$ and real, first and second-order coefficient polynomials

$$\sigma(x) = ex^2 + 2fx + g, \quad \tau = a_l + b_l x$$

(2)

are analyzed in ref. [1], [2]. The (unnormalized) polynomials are generated from the Rodrigues formula

$$P_l(x) = \frac{1}{w(x)} \frac{d^l}{dx^l} (\sigma^l(x)w(x)), \quad l = 0, 1, \ldots,$$

(3)

where $w(x)$ is the possibly $l$ dependent weight function on the fundamental interval $(a, b)$ that satisfies Pearson’s ODE

$$\sigma(x)w'(x) = [\tau(x) - \sigma'(x)]w(x)$$

(4)

to assure the self-adjointness of the differential operator of the hypergeometric ODE. Polynomial solutions of ODEs with $l$ dependent coefficients are studied in ref. [3] along with their orthogonality properties and zero distributions, which we therefore do not address here.

Here our first goal is to construct complementary polynomials for them by reworking their Rodrigues representation, Eq. (3), in a simple and natural way. The generating function of these complementary polynomials is obtained in closed form allowing for short and transparent derivations of general properties shared by the complementary polynomials.

The paper is organized as follows. In the next section we introduce and construct the complementary polynomials. In Section 3 we establish their generating function, the key result from which recursion relations and an addition theorem are derived in Section 4. The Sturm-Liouville ODE is derived in Section 5. Classical polynomial examples are given in Section 6.
2 Complementary Polynomials

Definition. We now introduce the complementary polynomials \( P_\nu(x; l) \) defining them in terms of the generalized Rodrigues representation

\[
P_l(x) = \frac{1}{w(x)} \frac{d^{l-\nu}}{dx^{l-\nu}} \left( \sigma(x)^{l-\nu} w(x) P_\nu(x; l) \right),
\]

where \( \nu = 0, 1, \ldots, l; \ l = 0, 1, \ldots \).

Theorem 1. \( P_\nu(x; l) \) is a polynomial of degree \( \nu \) that satisfies the recursive differential equation:

\[
P_{\nu+1}(x; l) = \sigma(x) \frac{dP_\nu(x; l)}{dx} + [\tau(x) + (l - \nu - 1)\sigma'(x)]P_\nu(x; l).
\]

By the Rodrigues formula (3), \( P_0(x; l) \equiv 1 \).

Proof. Equations (5), and (6) follow by induction. The first step, \( \nu = 1 \), is derived by carrying out explicitly the innermost differentiation in Eq. (3), which is a natural way of working with the Rodrigues formula (3) that yields

\[
P_l(x) = \frac{1}{w(x)} \frac{d^{l-1}}{dx^{l-1}} \left( l\sigma^{l-1}(x) w(x) \sigma'(x) + \sigma'(x) w'(x) \right)
\]

showing, upon substituting Pearson’s ODE (4), that

\[
P_1(x; l) = (l - 1)\sigma'(x) + \tau(x).
\]

Assuming the validity of the Rodrigues formula (5) for \( \nu \) we carry out another differentiation in Eq. (5) obtaining

\[
P_l(x) = \frac{1}{w(x)} \frac{d^{l-\nu-1}}{dx^{l-\nu-1}} \left\{ (l - \nu)\sigma(x)^{l-\nu-1} \sigma'(x) w(x) P_\nu(x; l) \right. \\
+ \left. \sigma^{l-\nu}(x) w'(x) P_\nu(x; l) + \sigma(x)^{l-\nu} w(x) P'_{\nu}(x; l) \right\}
\]

\[
= \frac{1}{w(x)} \frac{d^{l-\nu-1}}{dx^{l-\nu-1}} \left( \sigma(x)^{l-\nu-1} w(x) [l - \nu] \sigma'(x) P_\nu \\
+ (\tau - \sigma'(x)) P_\nu(x; l) + \sigma P'_{\nu}(x; l) \right)
\]

\[
= \frac{1}{w(x)} \frac{d^{l-\nu-1}}{dx^{l-\nu-1}} \left( \sigma(x)^{l-\nu-1} w(x) P_{\nu+1}(x; l) \right).
\]
Comparing the rhs of Eq. (9) proves Eq. (5) by induction along with the recursive ODE (6) which allows constructing systematically the complementary polynomials starting from $P_0(x; l) \equiv 1$. For example, $\nu = 0$ of the recursive ODE (6) confirms Eq. (8).

In terms of a generalized Rodrigues representation we have

**Theorem 2.** The polynomials $P_\nu(x; l)$ satisfy the Rodrigues formulas

$$P_\nu(x; l) = w^{-1}(x) \sigma^{\nu-l}(x) \frac{d^\nu}{dx^\nu} [w(x) \sigma^l(x)];$$  \hspace{1cm} (10)

$$P_\nu(x; l) = w^{-1}(x) \sigma^{\nu-l}(x) \frac{d^{\nu-\mu}}{dx^{\nu-\mu}} \left( \sigma^{l-\mu}(x) w(x) P_\mu(x; l) \right).$$  \hspace{1cm} (11)

**Proof.** We prove the Rodrigues formulas for the $P_\nu(x; l)$ polynomials by integrating first the homogeneous ODE (6) while dropping the inhomogeneous term $P_{\nu+1}(x; l)$. This yields

$$\ln P_\nu(x; l) - \ln c_\nu = (-l + \nu + 1) \ln \sigma(x) - a_l \int \frac{dx}{\sigma(x)} - b_l \int \frac{xdx}{\sigma(x)},$$  \hspace{1cm} (12)

where $c_\nu$ is an integration constant and $\int \frac{dx}{\sigma(x)}$, $\int \frac{xdx}{\sigma(x)}$ are indefinite integrals. Exponentiating Eq. (12) we obtain

$$P_\nu(x; l) = c_\nu \sigma^{l+\nu+1} e^{-a_l \int \frac{dx}{\sigma(x)} - b_l \int \frac{xdx}{\sigma(x)}}.$$  \hspace{1cm} (13)

Note that, if the zeros of $\sigma(x)$ are real, they lie outside the fundamental interval $(a, b)$ of $w(x)$ and the hypergeometric Eq. (1) by definition, while $x$ lies within it. So, these zeros pose no problem for the indefinite integrals.

Now we allow for the $x$ dependence of $c_\nu$ and vary it to include the inhomogeneous term $P_{\nu+1}(x; l)$. Differentiating Eq. (13) and substituting the recursive ODE (6) yields

$$P_{\nu+1}(x; l) = c_{\nu}'(x) \sigma^{-l+\nu+2}(x) e^{-a_l \int \frac{dx}{\sigma(x)} - b_l \int \frac{xdx}{\sigma(x)}},$$  \hspace{1cm} (14)

or

$$P_{\nu+1}(x; l) \sigma^{-2-\nu}(x) e^{a_l \int \frac{dx}{\sigma(x)} + b_l \int \frac{xdx}{\sigma(x)}} = \frac{d}{dx} [\sigma^{-\nu-1}(x) e^{a_l \int \frac{dx}{\sigma(x)} + b_l \int \frac{xdx}{\sigma(x)} P_\nu(x; l)]$$

$$= c_{\nu}'(x).$$  \hspace{1cm} (15)
Noting that the expression in brackets on the rhs of Eq. (15) differs from the coefficient of \( P_{\nu+1}(x;l) \) on the lhs only by one unit in the exponent of \( \sigma(x) \) suggests iterating the differentiation and then replacing \( \nu+1 \to \nu \). This leads to the formula

\[
P_{\nu}(x;l) = \sigma^{-l+1+\nu}e^{-a_l \int \frac{dx}{\sigma}} \int \frac{d\nu}{dx} \left[ \sigma^{l-1+\nu} e^{a_l \int \frac{dx}{\sigma}} \right].
\] (16)

Integrating Pearson’s ODE (4),

\[
\ln w(x) = \int \left( \tau - \sigma' \right) dx = -\ln \sigma(x) + a_l \int \frac{dx}{\sigma} + b_l \int \frac{xdx}{\sigma(x)}
\] (17)

and exponentiating this, gives

\[
w(x) = \sigma^{-1} e^{a_l \int \frac{dx}{\sigma} + b_l \int \frac{xdx}{\sigma(x)}}.
\] (18)

Substituting this result into Eq. (16) allows casting it in the general form of Eq. (10).

When we carry out the innermost differentiation in Eq. (10) we obtain the first step (\( \mu = 1 \)) of the inductive proof of the generalized Rodrigues representation of Eq. (11). Equation (11) yields trivially \( P_{\nu}(x;l) \) for \( \mu = \nu \), while for \( \mu = \nu - 1 \) it reproduces Eq. (6) and the case \( \mu = 1 \) is Eq. (10). The inductive step from \( \mu \) to \( \mu + 1 \) is similar to that leading to Eqs. (5) and (6).

### 3 Generating Function

**Definition.** The generating function for the polynomials \( P_{\nu}(x;l) \) is

\[
P(y, x; l) = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} P_{\nu}(x; l).
\] (19)

The series converges for \( |y| < \epsilon \) for some \( \epsilon > 0 \) and can be summed in closed form if the generating function is regular at the point \( x \).

**Theorem 3.** The generating function for the polynomials \( P_{\nu}(x;l) \) is given in closed form by

\[
P(y, x; l) = \frac{w(x+y\sigma(x))}{w(x)} \left( \frac{\sigma(x+y\sigma(x))}{\sigma(x)} \right)^l
\] (20)
\[
\frac{\partial \mu}{\partial y} \mathcal{P}(y, x; l) = \frac{w(x + y\sigma(x))}{w(x)} \left( \frac{\sigma(x + y\sigma(x))}{\sigma(x)} \right)^{\mu - \mu} \mathcal{P}_\mu(x + y\sigma(x); l). \tag{21}
\]

**Proof.** Equation (20) follows by substituting the Rodrigues representation, Eq. (10) in Eq. (19) which yields, with \( z \equiv x + y\sigma(x) \),

\[
\mathcal{P}(y, x; l) = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} \mathcal{P}_\nu(x; l) = \left( w(x)\sigma'(x) \right)^{-1} \sum_{\nu=0}^{\infty} \frac{(y\sigma(x))^\nu}{\nu!} \frac{d^\nu}{dz^\nu} \left( \sigma'(z)w(z) \right) \bigg|_{z=x}, \tag{22}
\]

converging for \( |y\sigma(x)| < \epsilon \) for a suitable \( \epsilon > 0 \) if \( x \in (a, b) \) is a regular point of the generating function, i.e. \( w \) is regular at \( x \) and \( x + y\sigma(x) \). The series can be summed exactly because the expression inside the derivatives is independent of the summation index \( \nu \) and we deal with the Taylor expansion of the function \( \sigma'(z)w(z) \) at the point \( x \) with increment \( y\sigma(x) \).

Differentiating Eq. (19) and substituting the generalized Rodrigues formula (11) in this yields Eq. (21) similarly.

In preparation for recursion relations we translate the case \( \mu = 1 \) of Eq. (21) into partial differential equations (PDEs).

**Theorem 4.** The generating function satisfies the partial differential equations (PDEs)

\[
(1 + y\sigma'(x) + \frac{1}{2} y^2 \sigma''(x)) \frac{\partial \mathcal{P}(y, x; l)}{\partial y} = [\mathcal{P}_1(x; l) + y\sigma(x)\mathcal{P}'_1(x; l)] \cdot \mathcal{P}(y, x; l), \tag{23}
\]

\[
\frac{\partial \mathcal{P}(y, x; l)}{\partial y} = [(l - 1)\sigma'(x + y\sigma(x)) + \tau(x + y\sigma(x))]\mathcal{P}(y, x; l - 1), \tag{24}
\]

\[
\left( 1 + y\sigma'(x) + \frac{1}{2} y^2 \sigma''(x) \right) \frac{\partial \mathcal{P}(y, x; l)}{\partial x} = \mathcal{P}(y, x; l)y \left\{ (1 + y\sigma'(x))\mathcal{P}'_1(x; l) - \frac{1}{2} y\sigma''\mathcal{P}_1(x; l) \right\}, \tag{25}
\]

\[
\sigma(x) \frac{\partial \mathcal{P}(y, x; l)}{\partial x} = (1 + y\sigma'(x))[\tau(x) + (l - 1)\sigma'(x) + y\sigma(x)(\sigma' + (l - 1)\sigma'')] \cdot \mathcal{P}(y, x; l - 1) - [\tau(x) + (l - 1)\sigma'(x)]\mathcal{P}(y, x; l). \tag{26}
\]
Proof. From Eq. (21) for \( \mu = 1 \) in conjunction with Eq. (20) we obtain
\[
\sigma(x + y\sigma(x)) \frac{\partial \mathcal{P}(y, x; l)}{\partial y} = \sigma(x)[\tau(x + y\sigma(x))]
\]
\[+ (l - 1)\sigma'(x + y\sigma(x))\mathcal{P}(y, x; l). \tag{27} \]
Substituting in Eq. (27) the Taylor series-type expansions
\[
\sigma(x + y\sigma(x)) = \sigma(x)(1 + y\sigma'(x) + \frac{1}{2}y^2\sigma''(x)),
\]
\[
\sigma'(x + y\sigma(x)) = \sigma'(x) + y\sigma''(x),
\]
\[
\tau(x + y\sigma(x)) = \tau(x) + y\tau'(x)\sigma(x) \tag{28} \]
following from Eq. (2), we verify Eq. (23). Using the exponent \( l - 1 \) instead of \( l \) of the generating function we can similarly derive Eq. (24).

By differentiation of the generating function, Eq. (22), with respect to \( (x, l, l; l) = (\tau) = (\mathcal{P}_1); (29) \) we find Eq. (26). Using the exponent \( l \) instead of \( l - 1 \) of the generating function in conjunction with Eq. (26) leads to Eq. (25).

4 Recursion and Other Relations

Our next goal is to rewrite various PDEs for the generating function in terms of recursions for the complementary polynomials.

**Theorem 5.** The polynomials \( \mathcal{P}_\nu(x; l) \) satisfy the recursion relations
\[
\mathcal{P}_{\nu+1}(x; l) = [\tau(x) + (l - 1 - \nu)\sigma'(x)]\mathcal{P}_\nu(x; l)
\]
\[+ \nu\sigma(x)[\tau' + (l - 1 - \frac{1}{2}(\nu - 1))\sigma'']\mathcal{P}_{\nu-1}(x; l); \tag{29} \]
\[
\mathcal{P}_{\nu+1}(x; l) = [\tau(x) + (l - 1)\sigma'(x)]\mathcal{P}_\nu(x; l - 1)
\]
\[+ \nu\sigma(x)[\tau' + (l - 1)\sigma'']\mathcal{P}_{\nu-1}(x; l - 1)
\]
\[= \mathcal{P}_1(x; l)\mathcal{P}_\nu(x; l - 1) + \mathcal{P}_1'(x; l - 1)\mathcal{P}_{\nu-1}(x; l - 1); \tag{30} \]
\[
\nu (\nu - 1)\frac{1}{2}\sigma''(x)\frac{d\mathcal{P}_{\nu-2}(x; l)}{dx} + \nu\sigma'(x)\frac{d\mathcal{P}_{\nu-1}(x; l)}{dx} + \frac{d\mathcal{P}_\nu(x; l)}{dx}
\]
\[= \nu[\tau' + (l - 1)\sigma'']\mathcal{P}_{\nu-1}(x; l) + \nu(\nu - 1)\mathcal{P}_{\nu-2}(x; l)
\]
\[\cdot \left\{ \sigma'(x)[\tau' + (l - 1)\sigma''] - \frac{1}{2}\sigma''[\tau(x) + (l - 1)\sigma'(x)] \right\} \]
\[= \nu\mathcal{P}_1'(x; l)\mathcal{P}_{\nu-1}(x; l) + \nu(\nu - 1)\mathcal{P}_{\nu-2}(x; l)
\]
\[\cdot \left\{ \sigma'(x)\mathcal{P}_1'(x; l) - \frac{1}{2}\sigma''\mathcal{P}_1(x; l) \right\}. \tag{31} \]
Proof. Substituting Eq. (19) defining the generating function in Eq. (23) we rewrite the PDE as Eq. (29). The recursion (30) is derived similarly from Eq. (24). The same way Eq. (25) translates into the differential recursion relation (31).

Corollary. Comparing the recursion (29) with the recursive ODE (6) we establish the basic recursive ODE

$$
\frac{d}{dx} P_\nu(x; l) = \nu [\nu' + (l - 1 - \frac{1}{2} (\nu - 1) \sigma'')] P_{\nu - 1}(x; l)
$$

(32)

with a coefficient that is independent of the variable $x$.

Parameter Addition Theorem.

$$
P(y, x; l_1 + l_2) P(y, x; 0) = P(y, x; l_1) P(y, x; l_2).
$$

(33)

$$
\sum_{\mu=0}^\nu \binom{\nu}{\mu} [P_\mu(x; l_1 + l_2) P_{\nu-\mu}(x; 0) - P_\mu(x; l_1) P_{\nu-\mu}(x; l_2)] = 0.
$$

(34)

Proof. The multiplicative structure of the generating function of Eq. (20) involving the parameter $l$ in the exponent implies the identity (33). Substituting Eq. (19) into this identity leads to Eq. (34).

We can also separate the $l$ dependence in the polynomials using Eq. (28) in the generating function, Eq. (21). If $\sigma(x) = \text{constant}$ (as is the case for Hermite polynomials), the generating function only depends on the weight function, and the Taylor expansion of $w(x + y \sigma(x))$ for $|y \sigma(x)| < 1$ is equivalent to the Rodrigues formula (10).

Corollary 1.

$$
P(y, x; l) = \frac{w(x + y \sigma(x))}{w(x)} (1 + y \sigma'(x))^l
$$

$$
= \sum_{N=0}^{\infty} y^N \sum_{N-l \leq m \leq N} \binom{l}{N-m} \sigma(x)^m \sigma'(x)^{N-m} \frac{w(m)(x)}{m! w(x)}.
$$

(35)

$$
P_N(x; l) = \sum_{N-l \leq m \leq N} \binom{l}{N-m} \frac{N!}{m!} P_m(x; 0) \sigma'(x)^{N-m}
$$

(36)

Proof. When $\sigma'(x) \neq 0$, the Taylor expansion of the weight function in conjunction with a binomial expansion of the $l$th power of Eq. (28) yields
Eq. (35). Using Eq. (19) this translates into the polynomial expansion (36) that separates the complementary polynomials $P_N(x;l)$ into the simpler polynomials $P_m(x;0)$ and the remaining $\sigma'(x)$ and $l$ dependence. Pearson’s ODE (4) guarantees the polynomial character of the $P_m(x;0)$ that are defined in Eq. (10).

Let us also mention the following symmetry relations.

**Corollary 2.** If $\sigma(-x) = (-1)^m \sigma(x), w(-x) = (-1)^n w(x)$ hold with integers $m, n$ then $P_l(-x) = (-1)^{(m+1)} P_l(x)$ and

$$P_l(-x;l) = P_l(x;l), \text{ } m \text{ odd},$$

$$P_l(-x;l) = (-1)^\nu P_l(x;l), \text{ } m \text{ even}. \quad (37)$$

**Proof.** The parity relation for $P_l(x)$ follows from substituting $-x$ in the Rodrigues formula (3). The other polynomial parity relations follow from the identities

$$P(y,-x;l) = P(y,x;l), \text{ } m \text{ odd} \quad (39)$$

$$P(-y,-x;l) = P(y,x;l), \text{ } m \text{ even} \quad (40)$$

which, in turn, result from substituting $-x$ into the first formula of Theorem 3. Expanding the generating functions according to their definition yields the relations (37), (38).

## 5 Sturm–Liouville ODE

**Theorem 6.** The polynomials $P_\nu(x;l)$ satisfy the Sturm-Liouville differential equation

$$\frac{d}{dx} \left( \sigma(x)^{l-\nu+1} w(x) \frac{dP_\nu(x;l)}{dx} \right) = -\lambda_\nu \sigma(x)^{l-\nu} w(x) P_\nu(x;l), \quad (41)$$

which is equivalent to

$$\sigma(x) \frac{d^2 P_\nu(x;l)}{dx^2} + [(l-\nu)\sigma'(x) + \tau(x)] \frac{dP_\nu(x;l)}{dx} = -\lambda_\nu P_\nu(x;l), \quad (42)$$

and the eigenvalues are given by

$$\lambda_\nu = -\nu[(l - \nu + 1/2)\sigma'' + \tau'], \text{ } \nu = 0, 1, \ldots \quad (43)$$
Proof. This is derived by natural induction again. The first step for \( \nu = 1 \) is straightforward to verify.

The step from \( \nu \) to \( \nu + 1 \) proceeds from the lhs of Eq. (41) for \( \nu + 1 \), where we replace \( P_{\nu+1} \) by \( P_{\nu} \) using the recursive ODE (6) so that, after some elementary manipulations, we end up with

\[
\frac{d}{dx} \left( \sigma(x)^{l-\nu} w(x) \frac{dP_{\nu+1}(x;l)}{dx} \right) = \sigma(x)^{l-\nu-1} w(x)
\]

\[
\cdot \left\{ [(l - \nu - 1)\sigma'' + \tau'] [\sigma(x) \frac{dP_{\nu}(x;l)}{dx} + \tau(x) + (l - \nu - 1)\sigma'(x)]P_{\nu}(x;l) \right\}
\]

\[
+ \left\{ [(l - \nu - 1)\sigma'(x) + \tau(x)] [\sigma(x) \frac{d^2P_{\nu}(x;l)}{dx^2} + (l - \nu)\sigma'(x) + \tau(x)] \frac{dP_{\nu}(x;l)}{dx} \right\}
\]

\[
+ \sigma(x) \frac{d}{dx} \left[ \sigma(x) \frac{d^2P_{\nu}(x;l)}{dx^2} + [(l - \nu)\sigma'(x) + \tau(x)] \frac{dP_{\nu}(x;l)}{dx} \right] \}
\]

\[
= \sigma(x)^{l-\nu-1} w(x) \left\{ [(l - \nu - 1)\sigma'' + \tau']P_{\nu+1}(x;l) - \lambda_{\nu} P_{\nu+1}(x;l) \right\}
\]

\[
= -\lambda_{\nu+1} P_{\nu+1}(x;l),
\]

where we have used the recursive ODE (6) and the ODE (42) for the index \( \nu \) repeatedly. Eq. (6) introduces a third derivative of \( P_{\nu}(x;l) \), a term which shows up as the next to last term on the rhs of the first equality sign in Eq. (44). This completes the proof by induction and establishes the recursion

\[
\lambda_{\nu+1} = \lambda_{\nu} - [(l - \nu - 1)\sigma'' + \tau']
\]

for the eigenvalues, whose solution is Eq. (43).

6 Classical Polynomial Examples

In the case of Hermite polynomials \([4],[5],[6],[7],[8]\) \( \sigma \) has no roots, so \( \sigma(x) = \text{constant} = 1 \), without loss of generality, and \( \sigma' = 0 \); moreover, we may take \( a_l = 0, b_l = -2 \) so \( \tau(x) = -2x \). Hence Pearson’s ODE yields the weight function \( w(x) = e^{-x^2} \) on \(( -\infty, \infty) \) that is characteristic of Hermite polynomials. The Rodrigues formula (3) then identifies the polynomials \( P_l(x) = (-1)^l H_l(x) \) as Hermite’s, while the Rodrigues formula (10) for the complementary polynomials implies \( P_{\nu}(x;l) = P_{\nu}(x) \), so they are independent of the index \( l \) and also Hermite polynomials. The recursive ODE (6) becomes the well known differential recursion

\[
H_{n+1}(x) = 2x H_n(x) - H'_n(x).
\]
The Sturm-Liouville ODE becomes the usual ODE of the Hermite polynomials. The recursion (29) is the basic \( H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \). Eq. (32) gives the differential recursion \( H_n'(x) = 2nH_{n-1}(x) \). The parity relation is also the well known one. The generating function is the standard one. Equation (35) reproduces the usual expansion of Hermite polynomials in powers of the variable \( x \).

For Laguerre polynomials, \( \sigma(x) \) has one real root, so \( \sigma(x) = x \) and \( \tau(x) = 1 - x \) without loss of generality. Pearson’s ODE gives the familiar weight function \( w(x) = e^{-x} \) on \([0, \infty)\). Rodrigues formula (3) identifies \( P_l(x) = \frac{l!}{l!} L_{l + \nu}^{l + \nu}(x) \).

The Sturm-Liouville ODE (42)
\[
x \frac{d^2 P_\nu(x; l)}{dx^2} + (l + 1 - \nu - x) \frac{dP_\nu(x; l)}{dx} = -\lambda_\nu P_\nu(x; l), \quad \lambda_{\nu + 1} = \lambda_\nu + 1
\] (46)

allows identifying \( P_\nu(x; l) = \nu! P_{\nu}(x; l) \) as an associated Laguerre polynomial. So, in the following we shift \( l \rightarrow l + \nu \), as a rule. The recursive ODE (6) yields the differential recursion

\[
(\nu + 1) L_{\nu+1}^{l-1}(x) = x \frac{dL_\nu(x)}{dx} + (l - x) L_\nu(x)
\] (47)

which, in conjunction with

\[
L_{\nu+1}^{l-1}(x) = L_{\nu+1}^l(x) - L_\nu(x),
\] (48)

leads to the standard three-term recursion

\[
(\nu + 1) L_{\nu+1}^l(x) = (l + \nu + 1 - x) L_\nu(x) + x \frac{dL_\nu(x)}{dx}.
\] (49)

The formula (10) of Theorem 2 is the usual Rodrigues formula for associated Laguerre polynomials, while the generalized Rodrigues formula (11)

\[
L_\nu(x) = \frac{\mu!}{\nu!} e^x x^{-\nu-\mu} \frac{d^{\nu-\mu}}{dx^{\nu-\mu}} \left( x^{l+\nu-\mu} e^{-x} L_{l+\nu-\mu}(x) \right)
\] (50)

is not part of the standard lore.

The generating function (20) for this case becomes

\[
L(y, x; l) = \sum_{\nu=0}^{\infty} y^\nu L_{\nu}^{l-\nu}(x) = e^{-xy}(1 + y)^l
\] (51)
and is simpler than the usual one for associated Laguerre polynomials, which is the reason why our method is more elementary and faster than the standard approaches. The recursion (29) becomes

\[(\nu + 1)L_{\nu+1}^{l-1}(x) = (l - x)L_{\nu}^{l}(x) - xL_{\nu+1}^{l+1}(x), \quad (52)\]

while the recursion (30) becomes

\[(\nu + 1)L_{\nu+1}^{l}(x) = (l + \nu + 1 - x)L_{\nu}^{l}(x) - xL_{\nu-1}^{l+1}(x), \quad (53)\]

and Eq. (31) translates into

\[
\frac{dL_{\nu+1}^{l+1}(x)}{dx} + \frac{dL_{\nu}^{l}(x)}{dx} = -L_{\nu-1}^{l+1}(x) - L_{\nu+2}^{l+2}(x), \quad (54)\]

a sum of the known recursion \(\frac{dL_{\nu}^{l}(x)}{dx} = -L_{\nu-1}^{l+1}(x)\) which is the basic recursive ODE (32). Equation (35) gives the standard expansion

\[
L_{N}^{l}(x) = \sum_{n=0}^{N} \binom{l + n}{N - n} \frac{(-x)^{n}}{n!}. \quad (55)\]

The simplest addition theorem originates from the elegant identity

\[
L(y, x_1; n_1)L(y, x_2; n_2) = L(y, x_1 + x_2; n_1 + n_2) \quad (56)\]

which translates into the polynomial addition theorem

\[
\mathcal{P}_{\nu}(x_1 + x_2; n_1 + n_2) = \sum_{k=0}^{\nu} \binom{\nu}{k} \mathcal{P}_{\nu-k}(x_1; n_1)\mathcal{P}_{k}(x_2; n_2) \quad (57)\]

and

\[
L_{\nu}^{n_1+n_2}(x_1 + x_2) = \sum_{k=0}^{n_1} L_{k}^{n_1-k}(x_1)L_{\nu-k}^{n_2+k}(x_2) \quad (58)\]

for associated Laguerre polynomials which is not listed in the standard ref. [8] or elsewhere.

In the case of Jacobi polynomials, \(\sigma(x)\) has two real roots at \(\pm 1\), without loss of generality; so

\[
\sigma(x) = (1 - x)(1 + x), \quad \tau(x) = b - a - (2 + a + b)x, \quad (59)\]
in a notation that will allow us to use the standard parameters. Pearson’s ODE (41) leads to
\[ w(x) = (1 - x)^a(1 + x)^b, \] (60)
and Rodrigues formula (3) and (11) identify the polynomials
\[ P_l(x) = \frac{2^l(l!)^2}{l!(a + l)(b + l)} P_{a,b}^{(a + l, b + l)}(x). \] (61)
Thus, we shift \( l \to l + \nu \) in translating our general results to Jacobi polynomials, as a rule. We may also set \( l = 0 \) because this index merely shifts the parameters \( a, b \).

The recursive ODE (6) translates into
\[ -2(\nu + 1)P^{(a,b)}_{\nu+1}(x) = [b - a - (a + b)x]P^{(a,b)}_{\nu}(x) \]
\[ + (1 - x^2)\frac{dP^{(a,b)}_{\nu}(x)}{dx}. \] (62)

The Sturm-Liouville ODE (42) reproduces the usual ODE of Jacobi polynomials. The generating function, Eq. (20),
\[ P(y, x; l) = \frac{[1 - x - y(1 - x^2)]^a[1 + x + y(1 - x^2)]^b}{(1 - x)^a(1 + x)^b} \]
\[ \left\{ 1 - \frac{(x + y(1 - x^2))^2}{1 - x^2} \right\}^l \]
\[ = [1 - y(1 + x)]^a[1 + y(1 - x)]^b[1 - 2xy - y^2(1 - x^2)]^l \] (63)
is much simpler than the standard one [4], especially when we set \( l = 0 \), allowing for the transparent derivation of many recursion relations. For example, Eq. (30) becomes
\[ -4(\nu + 1)P^{(a-1,b-1)}_{\nu+1}(x) = 2[b - a - x(a + b)]P^{(a,b)}_{\nu}(x) \]
\[ - (1 - x^2)[\nu + 1 + a + b]P^{(a+1,b+1)}_{\nu-1}(x), \] (64)
Eq. (30) translates into
\[ -4(\nu + 1)P^{(a-1,b-1)}_{\nu+1}(x) = 2[b - a - x(a + b + 2\nu)]P^{(a-1,b-1)}_{\nu}(x) \]
\[ + (1 - x^2)[a + b + 2\nu]P^{(a,b)}_{\nu}(x), \] (65)
and Eq. (31) takes the form
\[ (x^2 - 1)\frac{dP^{(a+2,b+2)}_{\nu-2}(x)}{dx} + 4x \frac{dP^{(a+1,b+1)}_{\nu-1}(x)}{dx} + 4 \frac{dP^{(a,b)}_{\nu}(x)}{dx} \]
\[ = 2[a + b + 2\nu]P^{(a+1,b+1)}_{\nu-2}(x) + [b - a + x(a + b + 2\nu)] \]
\[ \cdot P^{(a+2,b+2)}_{\nu-2}(x). \] (66)
Equation (35) gives
\[
P_N^{(a,b)}(x) = (-2)^N N!(1-x)^{-a}(1+x)^{-b} \sum_{n=0}^{N} \binom{N}{n} \cdot \frac{(-2x)^n(1-x^2)^{N-n}}{(N-n)!} \frac{d^n}{dx^n}[(1-x)^{a}(1+x)^{b}]. \tag{67}
\]

A product formula for Jacobi polynomials is obtained from an addition theorem in the variable \(y\) for our generating function for \(l = 0\) (where we display the upper parameters now for clarity)
\[
\mathcal{P}^{(a,b)}(y_1, x; 0) \mathcal{P}^{(a,b)}(y_2, x; 0) = \left[ (1 + y_1(1-x))(1 + y_2(1-x)) \right]^b
\cdot
(1 - y_1(1+x))(1 - y_2(1+x))^a
\cdot
[1 + (y_1 + y_2)(1-x)]^b
\cdot
[1 - (y_1 + y_2)(1+x)]^a
\mathcal{P}(y_1 + y_2, x; 0) \sum_{j,k=0}^{\infty} \binom{a}{k} \binom{b}{j} \left( \frac{y_1y_2(1-x)^2}{1 + (y_1 + y_2)(1-x)} \right)^j
\cdot
\left( \frac{y_1y_2(1+x)^2}{1 - (y_1 + y_2)(1+x)} \right)^k
= \mathcal{P}^{(a,b)}(y_1 + y_2, x; 0) \sum_{j,k=0}^{\infty} \binom{a}{k} \binom{b}{j} \left( \frac{y_1y_2(1-x)^2}{1 + (y_1 + y_2)(1-x)} \right)^j
\cdot
\left( \frac{y_1y_2(1+x)^2}{1 - (y_1 + y_2)(1+x)} \right)^k
\cdot
\mathcal{P}^{(a-k,b-j)}(y_1 + y_2, x; 0) y_1^{j+k} y_2^{j+k} (1-x)^{2j} (1+x)^{2k}. \tag{68}
\]

Expanding into Jacobi polynomials according to Eq. (19), comparing like powers of \(y_1y_2\), converting to Jacobi polynomials and shifting \(a \rightarrow a + \nu_1, b \rightarrow b + \nu_2\) yields the product formula
\[
P_{\nu_1}^{(a,b)}(x) P_{\nu_2}^{(a+\nu_1-\nu_2,b+\nu_1-\nu_2)}(x) = \sum_{0\leq \nu \leq (\nu_1 + \nu_2)/2} 2^{-2\nu} \binom{\nu_1 + \nu_2 - 2\nu}{\nu_1 - \nu}
\cdot
\sum_{k=0}^{\nu} \binom{a + \nu_1}{k} \binom{b + \nu_1}{\nu - k} (1+x)^{2k} (1-x)^{2(\nu-k)}
\cdot
P_{\nu_1+\nu_2-2\nu}^{(a+2\nu-\nu_2-k,b+\nu-\nu_2+k)}(x). \tag{69}
\]

7 Conclusions

We have used a natural way of working with the Rodrigues formula of a given set of orthogonal polynomials which leads to a set of closely related comple-
complementary polynomials that obey their own Rodrigues formulas, always have a generating function that can be summed in closed form leading to a transparent derivation of numerous recursion relations and addition theorems. These complementary polynomials satisfy a homogeneous second-order differential equation similar to that of the original polynomials.

Our method generates all the basics of the Hermite polynomials. It generates the associated Laguerre polynomials and many of their known properties and new ones from the Laguerre polynomials in an elementary way. It also simplifies the derivations of various results for Jacobi polynomials.

Our method is not restricted to the classical polynomials; when it is applied to the polynomials that are part of the wave functions of the Schrödinger equation with the Rosen-Morse and the Scarf potentials, it links these polynomials to the Romanovski polynomials which will be shown elsewhere.

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