Chaotic behaviour of countable products of homeomorphism groups

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\section*{ABSTRACT}
Relationships between a chaotic behaviour and closely related properties of topological transitivity, sensitivity to initial conditions, density of closed orbits of homeomorphism groups and their countable products are investigated. We provide a large number of new examples of chaotic groups of homeomorphisms of countable products of various metrizable topological spaces, including infinite-dimensional topological manifolds, whose factors can be as noncompact surfaces, so triangulable closed manifolds of an arbitrary dimension.

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\section*{1. Introduction}
In theory of chaos, the chaotic behaviour of infinite products of transformation groups has not been investigated. The reason is that, by definition, the chaotic behaviour of a group of homeomorphisms assumes the density of the set of finite orbits (see, e.g. \cite{8, 9, 19}). This requirement may not be fulfilled when moving to the infinite product of spaces. Following \cite{6}, we give a more general definition of the chaotic behaviour of groups of homeomorphisms by weakening the above condition by requiring the density of the set of closed orbits (Definition 1.3). Here by a closed orbit we mean an orbit which is a closed subset of the respective topological space. This allows us to investigate the chaotic behaviour of arbitrary infinite products of homeomorphism groups.

\subsection*{1.1. Devaney chaos}
Let $T : X \to X$ be a continuous map of metric space. The family $\{T^n\}_{n \in \mathbb{N}}$ denoted by the pair $(X, T)$ is called a dynamical system. Devaney \cite{13} proposed the following notion of chaos, which is usually called Devaney chaos.
**Definition 1.1:** A dynamical system \((X, T)\) is called Devaney chaotic if it satisfies the following three properties:

1. \((X, T)\) is transitive;
2. the set of periodic points of \((X, T)\) is dense in \(X\);
3. \((X, T)\) has sensitive dependence on initial conditions.

Sensitive dependence on initial conditions is widely understood as being the central idea of chaos. In [4], it was shown that in Devaney’s definition of chaos, the sensitive dependence follows from transitivity and density of periodic orbits. It was found in [3] that neither transitivity nor density of periodic trajectories are deducible from the remaining two conditions.

In [8], Cairns et al. introduced the following notion of a chaotic group action as a generalization of chaotic dynamical systems (Definition 1.2). They showed that, if a group \(G\) acts chaotically on a compact Hausdorff space, then \(G\) is residually finite. Moreover, the reverse is also true, i.e. for every residually finite group \(G\) there exists a Hausdorff space on which \(G\) acts chaotically. As in [8], we don’t assume any topology on the group \(G\), but we assume that each element of \(G\) acts on a topological space \(X\) as a homeomorphism of \(X\), and the set \(X\) is infinite. All group actions are assumed to be faithful, i.e. the only element of a group \(G\) which acts as identity homeomorphism is the neutral element in \(G\).

**Definition 1.2:** A group of homeomorphisms \(G\) of a Hausdorff topological space \(X\) is called chaotic if the following two conditions are met:

1. **Topological transitivity:** For every pair of nonempty open subsets \(U\) and \(V\) in \(X\), there exists an element \(g \in G\) such that \(g(U) \cap V \neq \emptyset\).
2. **Finite orbits dense:** The union of finite orbits is dense in \(X\).

Following [6], we give and use in this work a different definition of the chaotic behaviour of an arbitrary group \(G\).

**Definition 1.3:** A group of homeomorphisms \(G\) of a topological space \(X\) is called chaotic (or \(G\) has a chaotic behaviour) on \(X\) if the following two conditions are met:

1. there exists a dense non-closed orbit of the group \(G\) in \(X\) (the existence of a dense orbit);
2. the union of closed orbits is dense in \(X\) (the density of closed orbits).

Recall that a separable space homeomorphic to a complete metric space is referred to as a Polish space. Note that Definition 1.3 is more general than Definition 1.2 in the wide class of Polish spaces. Emphasize that in the case when \(G\) is a countable homeomorphism group of a metrizable compact space \(X\), Definitions 1.2 and 1.3 are equivalent (Proposition 3.4 and Lemma 7.2).

If \((X, d)\) is a metric space, then we define the notion of a sensitive dependence of a homeomorphism group \(G\) on initial conditions (in Section 6).
1.2. The organization of this work. Main results

For the convenience of the reader, we provide the basic notions in Sections 2 and 5.

Let $A$ be any set and let $X_{\alpha}$, $\alpha \in A$, be any topological spaces. We prove that the direct product of groups $G = \prod_{\alpha \in A} G_{\alpha}$ is topologically transitive on the Tychonoff product of topological spaces $X = \prod_{\alpha \in A} X_{\alpha}$ if and only if every homeomorphism group $G_{\alpha}$, $\alpha \in A$, is topologically transitive on the respective factor $X_{\alpha}$ (Theorem 3.2). The analogous statement is proved for the existence of dense orbits (Theorem 3.3). We get also an analog of the Birkhoff theorem (Proposition 3.4).

We investigated density of closed orbits in Section 4 and show that the direct product of groups $G = \prod_{\alpha \in A} G_{\alpha}$ has a dense subset of closed orbits in $X = \prod_{\alpha \in A} X_{\alpha}$ if and only if for every $\alpha \in A$, the group $G_{\alpha}$ has a dense subset of closed orbits in $X_{\alpha}$ (Theorem 4.1).

In Section 5, we recall the definition of the product of a countable family of metric spaces.

Section 6 is devoted to the sensitive dependence of group actions on initial conditions. Recall that a topological space $X$ is a Baire space if every countable intersection of open dense subsets of $X$ is dense in $X$ [18, Def. 8.2]. A topological space $X$ is referred to a completely metrizable space, if it admits an agreed complete metric [18, Def. 3.1]. According to the Baire category theorem, every completely metrizable space is a Baire space. Emphasize that every Polish space, and, in particular, every compact metric space are completely metrizable spaces. We prove the following theorem which is a generalization of the main result of [4] in the case of invertible dynamical systems.

**Theorem 1.4:** Let $(X, d)$ be a metric Baire space. Assume that every closed orbit is a nowhere dense set in $X$. If a homeomorphism group $G$ acts chaotically on $X$, then the group $G$ is sensitive to initial conditions.

Let $G_{i}$, $i \in J \subset \mathbb{N}$, be a homeomorphism group of a metric space $X_{i}$, and on the Tychonoff product $X = \prod_{i \in J} X_{i}$ the canonical action of the direct product of groups $G = \prod_{i \in J} G_{i}$ is given. We prove that, in contrast to the transitivity and density of closed orbits, in order for the canonical action of the group $G$ on the product $X$ to be sensitive to initial conditions, it is sufficient to have one group $G_{n}$, $n \in J$, which is sensitive to initial conditions on $X_{n}$ (Theorem 6.7). In the case when the index set $J$ is finite, this condition is also necessary (Theorem 6.8).

In Section 7, we prove the following theorem.

**Theorem 1.5:** For every set $A$ of indexes, let $G_{\alpha}$, $\alpha \in A$, be a homeomorphism group of a topological space $X_{\alpha}$, and on the Tychonoff product $X = \prod_{\alpha \in A} X_{\alpha}$ the canonical action of the product of groups $G = \prod_{\alpha \in A} G_{\alpha}$ is given. Then the group $G$ acts chaotically on $X$ if and only if every group $G_{\alpha}$, $\alpha \in A$, acts chaotically on $X_{\alpha}$.

The application to Polish spaces is considered (Section 7.2). In particular, we prove the following theorem.

**Theorem 1.6:** Let $G_{i}$, $i \in \mathbb{N}$, be a countable group of homeomorphisms of a metrizable compact space $X_{i}$. Assume that every $G_{i}$ acts chaotically on $X_{i}$. Then:
Figure 1. The Loch Ness monster.

(1) the canonical action of the product of groups $G = \prod_{i \in \mathbb{N}} G_i$ is chaotic on the Tychonoff product $X = \prod_{i \in \mathbb{N}} X_i$;
(2) exists a dense subset $F \subset X$ which is the union of continual compact orbits, and every such orbit is a perfect subset of $X$;
(3) exists a dense continuum orbit of the group $G$ in $X$;
(4) all groups $G_i$, $i \in \mathbb{N}$, and $G$ are residually finite;
(5) all groups $G_i$, $i \in \mathbb{N}$, and $G$ are sensitive to initial conditions;
(6) if each group $G_i$ has a fixed point, then the union of the finite orbits of group $G$ is dense in $X$, and $G$ has a fixed point.

Since every topological manifold is a Polish space, then all results of our work for Polish spaces are applicable to topological manifolds.

Sections 8 and 9 contain the construction of families of homeomorphism groups of various topological spaces.

In Section 8, we check the chaoticity of the group generated by the full $N$-shift of the space of bi-infinite sequences $\Sigma^N$ of $N$ symbols. This allows us to get series of new chaotic groups of homeomorphisms of different finite and infinite products of spaces $\Sigma^{N_i}$, $N_i \in \mathbb{N}$. Emphasize, that the space $\Sigma^N$ is homeomorphic to the $(2N - 1)$-ary Cantor set.

In Section 9, we construct numerous examples of chaotic homeomorphism groups of topological manifolds including noncompact manifolds. Using the method from [8], we construct a countable series of examples of chaotic groups of homeomorphisms, isomorphic to the group $\mathbb{Z}$, on every closed surface as well as on various noncompact surfaces, examples of which are shown in Figures 1 and 2.

Emphasize that all examples of chaotic group of homeomorphisms on noncompact topological manifolds are new and they are represented for the first time.

We use the obtained chaotic actions as building blocks for constructions of chaotic actions of homeomorphism groups on Tychonoff products of topological manifolds. Due to the result of G. Cairns and A. Kolganova [9] and Theorem 1.5, every triangulable closed manifold of arbitrary dimension can be taken as a factor on which arbitrary countably generated free groups act chaotically. In particular, we get a continuum set of examples of chaotic actions of homeomorphism groups on infinite dimensional topological manifolds.

Notations. If a group $G$ acts on a set $X$, we denote by $g.x$ the action of an element $g \in G$ on a point $x \in X$. By $G.x$ we denote the orbit of $x$ respectively $G$. 

Figure 2. The surface homeomorphic to the plane without the Cantor set.

We use the notations $D_r(x) = \{ y \in X \mid d(x, y) < r \}$ for the open ball of a radius $r$ with the centre at $x$ and $D_\varepsilon(B) = \{ y \in X \mid d(y, B) < \varepsilon \} = \bigcup_{b \in B} D_\varepsilon(b)$ for the $\varepsilon$-neighbourhood of a subset $B$ in a metric space $(X, d)$.

Assumptions. Inclusions do not exclude equality. All neighbourhoods are assumed to be open. By a countable set we mean an infinite countable set as well as a finite set.

2. The canonical action of the direct product of groups

2.1. The Tychonoff product of topological spaces

Let $A$ be an arbitrary set, let $\{X_\alpha \mid \alpha \in A\}$ be a family of any sets. The direct (Cartesian) product $X = \prod_{\alpha \in A} X_\alpha$ is the set of all maps of $x : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for any $\alpha \in A$. If $x \in X$, then the point $x(\alpha) \in X_\alpha$ we will denote by the symbol $x_\alpha$ and call the $\alpha$-coordinate of the element $x$. The symbol $\{x_\alpha\}$ will denote the point of the product $X$, $\alpha$-coordinate of which is the point $x_\alpha \in X_\alpha$.

Let $\{ (X_\alpha, \tau_\alpha) \mid \alpha \in A \}$ be a family of topological spaces. Assume that $X = \prod_{\alpha \in A} X_\alpha$ is provided by the weakest topology $\tau$ such that all projections $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$, $\pi_\alpha(\{x_\alpha\}) := x_\alpha$, are continuous. This topology $\tau$ is called the Tychonoff topology. Note that

$$\zeta = \{ \pi_\beta^{-1}(U) \subset X \mid U \in \tau_\beta, \beta \in A \}$$

is a subbase of $\tau$. The base of the Tychonoff topology formed by all possible finite intersections of subsets of $\zeta$, is called the canonical base.
The topological space \((X, \tau)\) is called the Tychonoff product of topological spaces \((X_\alpha, \tau_\alpha)\) and is denoted by \((X, \tau) = \prod_{\alpha \in A} (X_\alpha, \tau_\alpha)\).

### 2.2. A direct product of groups and its canonical action

Let \(G_\alpha, \alpha \in A\), be a family of groups. On the direct product of sets \(G = \prod_{\alpha \in A} G_\alpha\), the group operation is introduced as follows:

\[
\Psi : G \times G \rightarrow G, \quad \Psi((g_\alpha), (h_\alpha)) = \{g_\alpha \cdot h_\alpha\} \quad \forall ((g_\alpha), (h_\alpha)) \in G \times G,
\]

where \(g_\alpha \cdot h_\alpha\) is the product of elements \(g_\alpha\) and \(h_\alpha\) in \(G_\alpha\). The group \(G = \prod_{\alpha \in A} G_\alpha\) is referred to as a direct product of groups \(G_\alpha, \alpha \in A\).

Let \(X = \prod_{\alpha \in A} X_\alpha\) be a direct product of sets. Assume that for every \(\alpha \in A\) a group \(G_\alpha\) acts on \(X_\alpha\). Consider the direct product of groups \(G = \prod_{\alpha \in A} G_\alpha\). Then the following action of the group \(G\) on \(X\) is defined

\[
\Phi : G \times X \rightarrow G, \quad \Phi((g_\alpha), (x_\alpha)) = \{g_\alpha \cdot x_\alpha\} \quad \forall ((g_\alpha), (x_\alpha)) \in G \times X.
\]

We call this action the canonical action of the direct product of groups \(G = \prod_{\alpha \in A} G_\alpha\) on the direct product of sets \(X = \prod_{\alpha \in A} X_\alpha\).

Further in this work, we assume that the direct product of groups \(G = \prod_{\alpha \in A} G_\alpha\) acts on \(X = \prod_{\alpha \in A} X_\alpha\) canonically.

### 3. Transitivity of the canonical product of homeomorphism groups

**Definition 3.1:** A homeomorphism group \(G\) of a topological space \(X\) is called topological transitive on \(X\) if for every nonempty open subsets \(U\) and \(V\) in \(X\) there exists such an element \(g \in G\) that

\[
g(U) \cap V \neq \emptyset.
\]

**Theorem 3.2:** Let \(A\) be an arbitrary set of indexes. For every \(\alpha \in A\), a homeomorphism group \(G_\alpha\) of the topological space \(X_\alpha\) is topologically transitive on \(X_\alpha\) if and only if the direct product of groups \(G = \prod_{\alpha \in A} G_\alpha\) topologically transitive on the Tychonoff product of topological spaces \(X = \prod_{\alpha \in A} X_\alpha\).

**Proof:** Suppose that for every \(\alpha \in A\), the group of homeomorphisms \(G_\alpha\) of the topological space \(X_\alpha\) is topologically transitive on \(X_\alpha\). Let \(X = \prod_{\alpha \in A} X_\alpha\) be the Tychonoff product of topological spaces \(X_\alpha\). Show that for every nonempty open subsets \(U\) and \(V\) in \(X\) there exists an element \(g = \{g_\alpha\} \in G\) such that \(g(U) \cap V \neq \emptyset\). It is enough to prove this fact for every sets \(U\) and \(V\) from the canonical base of the Tychonoff topology on \(X\). Let \(U = \prod_{\alpha \in A} U_\alpha, \quad V = \prod_{\alpha \in A} V_\alpha\) where \(U_\alpha\) and \(V_\alpha\) are nonempty open subsets in \(X_\alpha\), and for some finite subsets \(A_1, A_2 \subset A\) the equalities are fulfilled

\[
U_\alpha = X_\alpha \quad \forall \alpha \in A \setminus A_1, \quad V_\beta = X_\beta \quad \forall \beta \in A \setminus A_2.
\]

The topological transitivity of \(G_\alpha\) on \(X_\alpha\) implies the existence of elements \(g_\alpha \in G_\alpha\) satisfying \(g_\alpha(U_\alpha) \cap V_\alpha \neq \emptyset\) for all \(\alpha \in A\). Put \(g = \{g_\alpha\} \in G\), then \(g(U) \cap V = \prod_{\alpha \in A} (g_\alpha(U_\alpha) \cap V_\alpha) \neq \emptyset\). Thus the group \(G\) is topologically transitive on \(X\).
The opposite. Suppose that the homeomorphism group $G = \prod_{\alpha \in A} G_\alpha$ is topologically transitive on the Tychonoff product $X = \prod_{\alpha \in A} X_\alpha$. Fix an arbitrary element $\delta \in A$. Let $U_\delta$ and $V_\delta$ be any nonempty open subsets in $X_\delta$. Let $U = \prod_{\alpha \in A} U_\alpha$, $V = \prod_{\alpha \in A} V_\alpha$ where $U_\delta = \tilde{U}_\delta$, $V_\delta = \tilde{V}_\delta$, $U_\alpha = V_\alpha = X_\alpha$ for every $\alpha \in A \setminus \{\delta\}$. Note that $U$ and $V$ are nonempty open subsets in $X$. Since the group $G$ acts topologically transitive on $X$, then there is an element $g = \{g_\alpha\} \in G$ such that $\emptyset \neq g(U) \cap V = \prod_{\alpha \in A} (g_\alpha(U_\alpha) \cap V_\alpha)$, therefore $g_\delta(U_\delta) \cap V_\delta \neq \emptyset$. This implies topological transitivity of the group $G_\delta$ on $X_\delta$ for every $\delta \in A$.

Theorem 3.3: Let $A$ be an arbitrary index set and let $X = \prod_{\alpha \in A} X_\alpha$ be the Tychonoff product of topological spaces $X_\alpha$. Assume that $G_\alpha$ is a homeomorphism group of $X_\alpha$. Then the direct product of groups $G = \prod_{\alpha \in A} G_\alpha$ has a dense orbit in $X$ if and only if the group $G_\alpha$ has a dense orbit in $X_\alpha$ for every $\alpha \in A$.

Proof: As known [15, Prop. 2.3.3], for every family of subsets $B_\alpha \subset X_\alpha$ in the product $X = \prod_{\alpha \in A} X_\alpha$ the closures satisfy the following relation:

$$\prod_{\alpha \in A} B_\alpha = \prod_{\alpha \in A} \overline{B_\alpha}. \quad (1)$$

Since the orbit $G.x$ of $x = \{x_\alpha\} \in X$ is equal to the product of orbits $G_\alpha.x_\alpha$, i.e. $G.x = \prod_{\alpha \in A} G_\alpha.x_\alpha$, then taking into account (1) we get a chain of equalities $G.x = \prod_{\alpha \in A} G_\alpha.x_\alpha = \prod_{\alpha \in A} \overline{G_\alpha.x_\alpha}$. Therefore

$$G.x = \prod_{\alpha \in A} G_\alpha.x_\alpha \quad \forall x = \{x_\alpha\} \in X = \prod_{\alpha \in A} X_\alpha. \quad (2)$$

Using (2), it is easy to obtain a statement of the theorem being proved.

An analog of the Birkhoff transitivity theorem

If the action of the group $G$ on the space $X$ has a dense orbit, then it is topologically transitive. Indeed, let $U$ and $V$ are any open nonempty subsets of $X$ and let $G.x = X$ for some $x \in X$. Then there are elements $g_1, g_2 \in G$ such that $g_1.x \in U$ and $g_2.x \in V$. It follows that $g.U \cap V \neq \emptyset$ where $g = g_2g_1^{-1} \in G$.

According to [10, Prop. 1], if the group $G$ is topologically transitive on a Baire space $X$ with a countable base, then there exists a point $x \in X$ with a dense orbit. Therefore we get the following analog of the Birkhoff theorem for homeomorphism groups of Baire spaces with a countable base.

Proposition 3.4: If $G$ is a homeomorphism group of a topological space $X$, then the existence of a dense orbit of $G$ implies topological transitivity of $G$.

When $X$ is a Baire space with a countable base, the converse is also true.

In particular, for Polish spaces $X$, the existence of a dense orbit of $G$ on $X$ is equivalent to topological transitivity of $G$. 

Example 3.5: Let \( \{e_k\}_{k=1}^{\infty} \) be a basis of the vector space \( \mathbb{R}^n \) for \( n \geq 1 \). Define homeomorphisms of the Euclidean space \( \mathbb{R}^n \) by the following equalities: \( g_k(x) = x + e_k \), for \( k = 1, n \), and \( g_{n+1}(x) = \lambda x \) where \( \lambda > 1 \) for all \( x \in \mathbb{R}^n \). Consider the homeomorphism group \( G = \langle g_k \mid k = 1, n+1 \rangle \). According to [24, Prop. 16], the homeomorphism group \( G \) is topologically transitive, and every its orbit is dense in \( \mathbb{R}^n \). In other words, \( \mathbb{R}^n \) is a minimal set of the group \( G \). Note that there is no transitive subgroup of \( G \) with the number of generators less than \( n + 1 \).

4. Density of closed orbits

Theorem 4.1: Let \( A \) be an arbitrary index set and let \( X = \prod_{\alpha \in A} X_\alpha \) be the Tychonoff product of topological spaces \( X_\alpha \). Assume that \( G_\alpha \) is a homeomorphism group of \( X_\alpha \). Then the direct product of groups \( G = \prod_{\alpha \in A} G_\alpha \) has a dense union of closed orbits in \( X \) if and only if for every \( \alpha \in A \), the group \( G_\alpha \) has a dense union of closed orbits in the topological space \( X_\alpha \).

Proof: Let \( x = \{x_\alpha\} \in X \), then \( G.x = \prod_{\alpha \in A} G_\alpha.x_\alpha \). According to (2), we get

\[
\overline{G.x} = G.x \iff \overline{G_\alpha.x_\alpha} = G_\alpha.x_\alpha \quad \forall \alpha \in A.
\] (3)

This means that an orbit \( G.x \) is closed in \( X \) if and only if the orbit \( G_\alpha.x_\alpha \) is closed in \( X_\alpha \) for every \( \alpha \in A \). Let \( B \) be the union of all closed orbits of \( G \) in \( X \). Denote by \( B_\alpha \) the union of all closed orbits of \( G_\alpha \) in \( X_\alpha \). Therefore \( B = \prod_{\alpha \in A} B_\alpha \). Suppose that \( B \) is dense in \( X \), hence applying the equality (1), we get the following chain of equalities \( X = B = \prod_{\alpha \in A} B_\alpha = \prod_{\alpha \in A} \overline{B_\alpha} \). Consequently \( X_\alpha = \overline{B_\alpha} \), i.e. \( B_\alpha \) is dense in \( X_\alpha \).

Conversely, let for every \( \alpha \in A \) the subset of \( B_\alpha \) be dense in \( X_\alpha \). As \( B = \prod_{\alpha \in A} B_\alpha \), applying the equality (1), we have \( B = \prod_{\alpha \in A} \overline{B_\alpha} = \prod_{\alpha \in A} X_\alpha = X \). This means that the union of all closed orbits of \( G \) is dense in \( X \).

5. Countable products of metric spaces

5.1. The direct product of a countable family of metric spaces

The direct product of two metric spaces

Let \( (X_1, d_1) \) and \( (X_2, d_2) \) be metric spaces. A metric \( d \) on the product of two metric spaces \( X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\} \) may be introduced in the following ways ([14, Sec. 4.2]):

1. \( d(x, y) = \sqrt[p]{d_1^{p}(x_1, y_1) + d_2^{p}(x_2, y_2)} \) where \( p \geq 1 \).
2. \( d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \).
3. \( d(x, y) = \sum_{i=1}^{2} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \).

All these methods can be easily extended to the case of the product of any finite number of factors.
The direct product of a countable family of metric spaces

Based on any metric $d$ on the set $X$, you can get a metric bounded by the number 1, by the formula

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \forall x, y \in X,$$

(4)

and metric topologies defined by the metrics $d$ and $\tilde{d}$ are coincided.

Let $(X_i, d_i), i \in \mathbb{N}$, be a countable family of metric spaces. According to the above, without loss of generality, we will assume that all metrics $d_i$ are limited by the number 1. Metric on the product $X = \prod_{i \in \mathbb{N}} X_i$ can be given by the equality ([15, Th. 4.2.2]):

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \tilde{d}_i(x_i, y_i),$$

(5)

where $\tilde{d}_i$ is defined by the formula (4) if the metric $d_i$ not limited by a number 1, otherwise $\tilde{d}_i = d_i$.

Emphasize that the topology on $X$ generated by the metric $d$, defined by (5), coincides with the Tychonoff topology of the product $X = \prod_{i \in \mathbb{N}} X_i$ of topological spaces $X_i$.

Definition 5.1: The metric $d$ on $X = \prod_{i \in \mathbb{N}} X_i$ given by the formula (5) is called the direct product of metrics $\tilde{d}_i$. The metric space $(X, d)$ is called the direct product of countable family of metric spaces $(X_i, \tilde{d}_i)$ and it is denoted by $(X, d) = \prod_{i \in \mathbb{N}} (X_i, \tilde{d}_i)$.

Remark 5.1: As it is known [14], the metric (5) is a special case of the metric $d(x, y) = \sum_{i=1}^{\infty} A_i \tilde{d}_i(x_i, y_i)$, where the series $\sum_{i=1}^{\infty} A_i$ converges and all of its members are positive.

5.2. Nonmetrizability of the product of an uncountable family of topological spaces

Let $(M, d)$ be a metric space and $D_r(x) = \{z \in M \mid d(x, z) < r\}$ be the ball of radius $r > 0$ centred at $x$. Recall that a topological space satisfies the first axiom of countability if it has a countable base of topology at each point. Every metric space $(M, d)$ has a countable base $\Sigma_x = \{D_{1/n}(x) \mid n \in \mathbb{N}\}$ at each point $x \in M$. According to [15, Cor. 4.2.4], an uncountable product of metrizable spaces each of which contains at least two points does not satisfy the first axiom of countability. Thus an uncountable product of such spaces is not metrizable.

Therefore, we will consider further only products of a countable family of metric spaces in investigations of sensitivity of homeomorphism groups.

6. Sensitivity to initial conditions

6.1. Sufficient conditions for sensitivity to initial conditions

Definition 6.1: A homeomorphism group $G$ of a metric space $(X, d)$ is called sensitive to initial conditions at a point (or, for short, sensitive at a point) $x \in X$, if there exists a number
\[ \delta(x) > 0 \text{ such that for every } \varepsilon > 0 \text{ there exists an element } g \in G \text{ such that:} \]
\[ \text{diam}(g.D_{\varepsilon}(x)) > \delta. \]  
\[ (6) \]

A group \( G \) is called \textit{pointwise sensitive to initial conditions} (or, for short, \textit{pointwise sensitive}), if it is sensitive at every point \( x \in X \).

Usually, group actions are investigated on compact topological spaces, and the following definition of sensitivity to initial conditions is used.

**Definition 6.2:** A homeomorphism group \( G \) of a metric space \((X, d)\) is called \textit{sensitive to initial conditions} (or, for short, \textit{sensitive}) if there exists a number \( \delta > 0 \) such that for each \( x \in X \) and for every \( \varepsilon > 0 \) there exists an element \( g \in G \) such that:
\[ \text{diam}(g.D_{\varepsilon}(x)) > \delta. \]
\[ (7) \]

The number \( \delta \) is referred to as \textit{the sensitivity constant} for \( G \).

Emphasize that every group \( G \) satisfying Definition 6.2, satisfies also Definition 6.1.

**Example 6.3:** Recall that a homeomorphism group \( G \) of a metric space \((X, d)\) is said to be \textit{expansive} on \( X \), if there exists a constant \( c > 0 \) such that for every \( x \neq y \) in \( X \), there is \( g \in G \) satisfying \( d(g.x, g.y) > c \). Such a constant \( c \) is called an \textit{expansivity constant} of this group [5]. Every expansive group \( G \) of homeomorphisms of a metric space \((X, d)\) is sensitive to initial conditions, and the role of \( \delta \) in Definition 6.2 plays the expansivity constant \( c \).

**Proposition 6.4:** Let \( G \) be a homeomorphism group of a metric Baire space \( X \) having a dense non-closed orbit. Then \( G \) is pointwise sensitive if and only if it has sensitive dependence on initial conditions.

**Proof:** As the sensitivity implies pointwise sensitivity, proves the inverse. Let \( X \) be a metric Baire space. Assume that a group \( G \) is pointwise sensitive and has a dense non-closed orbit. For each \( n \in \mathbb{N} \) consider the following subset:
\[ V_n = \{ x \in X | \exists \varepsilon > 0 : \text{diam}(g.D_{\varepsilon}(x)) \leq 1/n \quad \forall g \in G \}. \]
\[ (8) \]

Note that \( V_n \) is an open \( G \)-invariant subset in \( X \), and \( V_n \supset V_{n+1} \supset V_{n+2} \supset \cdots \). Since every open set contains a point with dense orbit, then if \( V_n \neq \emptyset \) for some \( n \), due to \( G \)-invariance of \( V_n \), the set \( V_n \) is dense in \( X \). Emphasize that each \( x \in \bigcap_{n \in \mathbb{N}} V_n \) is not a sensitive point respectively \( G \), that contradicts the assumption. Therefore \( \bigcap_{n \in \mathbb{N}} V_n = \emptyset \), and there exists \( m \in \mathbb{N} \) for which \( V_m = \emptyset \), because otherwise \( \bigcap_{n \in \mathbb{N}} V_n \) is dense in the Baire space \( X \). Therefore, \( V_n = \emptyset \) for every \( n \geq m \). According to (8), this means that there exists \( \delta = 1/m \) such that for every \( \varepsilon > 0 \) there exists an element \( g \in G \) satisfying \( \text{diam}(g.D_{\varepsilon}(x)) > \delta \).
By Definition 6.2, the group \( G \) is sensitive to initial conditions on \( X \). This completes the proof.

**Remark 6.1:** For a continuous action of a topological group \( G \) on a compact metric space \((X, d)\), as indicated by F. Polo [20, Prop. 1.3] one can prove Proposition 6.4 using ideas...
from [1]. In fact, we have implemented such a possibility under a more general assumption, replacing the compactness condition of the metric space X with the assumption that X is a Baire space.

**Remark 6.2:** As it is known, the sensitivity is not invariant with respect to topological conjugacy. However, if dynamical systems are defined on a compact topological space, in [4] it is shown that the sensitivity is invariant respectively topological conjugacy.

The following statement contains conditions that guarantee that sensitivity to initial conditions of the group of homeomorphisms of the topological space X does not depend on the choice of the metric that metrizes X.

**Proposition 6.5:** Let X be a Baire space. Let (X, d) and (X, ρ) be metric spaces such that the metric topologies given by the metrics d and ρ coincide with the topology of the space X. Then the homeomorphism group G of X, having a dense non-closed orbit, is sensitive to initial conditions in (X, d) if and only if G is sensitive to initial conditions in (X, ρ).

**Proof:** According to Proposition 6.4, it is sufficient to prove that pointwise sensitivity of G in the spaces (X, d) and (X, ρ) are equivalent.

Let τ_d and τ_ρ be metric topologies defined by metrics d and ρ on X respectively, and τ_d = τ_ρ. Recall that τ_d is the topology with a base σ = {D_τ(x) | x ∈ X, r > 0}. Analogously, σ = {D_τ′(x) | x ∈ X, r > 0} is a base of τ_ρ.

Assume that a homeomorphism group G of X is pointwise sensitive to initial conditions in (X, d). According to Definition 6.1, for every x ∈ X there is a number δ = δ(x) > 0 such that for each ε > 0 there exists an element g ∈ G satisfying the inequality diam_d(g.D_δ(x)) > δ, where diam_d(g.D_δ(x)) denotes the diameter of the set g.D_δ(x) in (X, d). Consider an arbitrary ε > 0, then there will be ε′ > 0 such that D_ε′(x) ⊂ D_δ(x). Hence there is g ∈ G for which diam_d(g.D_ε′(x)) > δ. Therefore there exists y ∈ D_ε′(x) such that d(g.x, y) > δ/2. This means that

\[ g.y \notin D_{\delta/2}(g.x). \]  

(9)

According to the equality τ_d = τ_ρ, there will be a number \( \tilde{\delta} = \tilde{\delta}(x) > 0 \) satisfying the following inclusion:

\[ \tilde{D}_{\tilde{\delta}}(g.x) \subset D_{\delta/2}(g.x). \]  

(10)

The relations (9)–(10) imply that \( g.y \notin \tilde{D}_{\tilde{\delta}}(g.x) \), i.e. \( \rho(g.x, g.y) > \tilde{\delta} \). This means that the group G is pointwise sensitive to initial conditions in the metric space (X, ρ).

Thus, for every x ∈ X there is a number \( \tilde{\delta} = \tilde{\delta}(x) > 0 \) such that for every ε > 0 there exists an element g ∈ G satisfying the inequality diam_ρ(g.D_\tilde{\delta}(x)) > \tilde{\delta}. This means that the group G is pointwise sensitive to initial conditions in the metric space (X, ρ).

Similarly, the sensitivity of the group G to initial conditions in the metric space (X, ρ) implies the sensitivity of G to initial conditions in (X, d).

**6.2. Proof of Theorem 1.4**

Suppose the opposite, let the group of homeomorphisms G of the metric space (X, d) acts chaotically, but G is not sensitive to initial conditions. According to Proposition 6.4, there exists a point x ∈ X which is not sensitive to initial conditions.
Case 1: The orbit $Gx$ is not dense in $X$, i.e. the $Gx \neq X$. Then $U = X \setminus Gx$ is nonempty open subset in $X$. Therefore there exists $a \in U$, and $d(a, Gx) > 0$. Put $\delta = \frac{1}{2}d(a, Gx)$. According to the condition (1) of Definition 1.3, there exists a dense orbit $Gz$ in $X$. Hence for every $\varepsilon > 0$, there is $y \in Gz \cap D_\delta(x)$ and there exists $y' \in Gz \cap D_\delta(a)$. Hence there exists an element $g \in G$ for which $y' = gy$. Show that $\text{diam}(g.D_\varepsilon(x)) > \delta$. Suppose the opposite, i.e. $\text{diam}(g.D_\varepsilon(x)) \leq \delta$. Using the triangle inequality in the metric space $(X, d)$, we get

$$d(a, g.x) \leq d(a, y') + d(g.y, g.x) < \delta + \delta = 2\delta = d(a, Gx)$$

which contradicts the definition of distance $d(a, Gx)$. The contradiction proves inequality $\text{diam}(g.D_\varepsilon(x)) > \delta$. Therefore, $x$ is sensitive to initial conditions.

Case 2: The orbit $Gx$ is dense in $X$, i.e. $Gx = X$, and at point $x \in X$ the group $G$ is not sensitive to initial conditions. Hence for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that $\text{diam}(g.D_{\varepsilon_0}(x)) \leq \delta$ for each $g \in G$. Therefore for every $y \in D_{\varepsilon_0}(x)$ and each $x' \in Gx$ there exists $y' \in G.y$ satisfying the inequality

$$d(x', y') \leq \delta. \quad (11)$$

As the union of closed orbits of $G$ is dense in $X$, there is a point $y \in D_{\varepsilon_0}(x)$, through which the closed orbit $G.y$ passes. Consider $z \in U = X \setminus G.y$. Since the orbit $Gx$ is dense in $X$, for any $\varepsilon > 0$ there exists $x' \in Gx \cap D_\delta(z)$. Due to the arbitrariness of $\varepsilon > 0$, given (11) and applying the triangle inequality, we get

$$d(z, y') \leq d(z, x') + d(x', y') < \varepsilon + \delta \Rightarrow z \in D_\delta(G.y). \quad (12)$$

Consequently,

$$X \supset D_\delta(G.y) \Rightarrow X = D_\delta(G.y). \quad (13)$$

Take a sequence of positive numbers $\delta_n \to 0$. Hence for every $\delta_n$ there exists $\varepsilon_n > 0$ and $y_n \in D_{\varepsilon_n}(x)$ such that the orbit $G.y_n$ is closed, and $d(g.x, g.y_n) \leq \delta_n$ for each $g \in G$. Hence

$$X = \overline{D_{\delta_n}(G.y_n)} \quad \forall n \in \mathbb{N} \Rightarrow X = \bigcap_{n \in \mathbb{N}} \overline{D_{\delta_n}(G.y_n)}. \quad (14)$$

Consequently, for every point $v \in X$ we get $v \in \overline{D_{\delta_n}(G.y_n)}$ for every $n \in \mathbb{N}$. This means that for every $\varepsilon > 0$ there exists $z_n \in D_{\delta_n}(G.y_n)$ such that $d(v, z_n) < \varepsilon$. Therefore for every $n \in \mathbb{N}$, there exists $y_n \in G.y_n$, satisfying the inequality $d(z_n, y_n) < \delta_n$. Applying the triangle inequality, we get

$$d(v, y_n) \leq d(v, z_n) + d(z_n, y_n) < \varepsilon + \delta_n \quad \forall n \in \mathbb{N}.$$

As $\varepsilon$ and $\delta_n$ are arbitrarily small quantities, it is necessary that

$$v \in \bigcup_{n \in \mathbb{N}} G.y_n \Rightarrow X = \bigcup_{n \in \mathbb{N}} G.y_n. \quad (15)$$

By the condition of theorem being proved, every closed orbit $G.y_n$ is a nowhere dense subset in $X$, hence a countable union $\bigcup_{n \in \mathbb{N}} G.y_n$ is also nowhere dense in the Baire space $X$. Therefore the equality in (15) is not possible. The contraction completes the proof. □
As in each Hausdorff space a finite orbit is closed, Theorem 1.4 implies the following statement.

**Corollary 6.6:** Let X be a metric Baire space with a countable base. If a homeomorphism group G of X is chaotic in the sense of Definition 1.2, then the group G is sensitive to initial conditions.

### 6.3. Sensitivity to initial conditions of the direct product of homeomorphism groups

Emphasize that all metric spaces are considered with the metric topology, unless otherwise specified. We use notations from Section 5. In the case when J is a finite subset of \( \mathbb{N} \), the metric \( d \) of the product \( \prod_{i \in J} (X_i, \tilde{d}_i) \) is defined by the formula (5), in which \( i \in J \).

The sensitivity of the group \( G = \prod_{i \in J} G_i \) is not changed if the metric \( d \) will be defined by the formula

\[
d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \quad \forall x = \{x_i\}, y = \{y_i\},
\]

where boundedness of metrics \( d_i \) is not assumed.

**Theorem 6.7:** Let \( (X, d) = \prod_{i \in J} (X_i, \tilde{d}_i) \) where \( (X_i, \tilde{d}_i) \) is a metric space and let \( J \subset \mathbb{N} \). Let \( G_i, i \in J \), be a homeomorphism group of \( X_i \). If there exists \( n \in J \) such that \( G_n \) is sensitive to initial conditions on \( (X_n, \tilde{d}_n) \), then the direct product of groups \( G = \prod_{i \in J} G_i \) is also sensitive to initial conditions on \( (X, d) \).

**Proof:** Let \( x = \{x_i\} \) be any point in \( X \). Assume that a group \( G_n \) is sensitive to initial conditions on \( (X_n, \tilde{d}_n) \). This means that there exists a number \( \sigma > 0 \) such that for the point \( x_n \in X_n \) and for any \( \eta > 0 \) there is an element \( \tilde{g} \in G_n \) such that \( \text{diam}(\tilde{g} \cdot D_\eta(x_n)) > \sigma \).

Therefore there exists a point \( \tilde{y} \in D_\eta(x_n) \) satisfying the inequality \( \tilde{d}_n(\tilde{g} \cdot x_n, \tilde{g} \cdot \tilde{y}) > \frac{\sigma}{2} \).

Consider \( y = \{y_i\} \in X \) where \( y_n = \tilde{y} \) and \( y_i = x_i \) for \( i \in J \setminus \{n\} \). Pick an element \( g = \{g_i\} \in G \) where \( g_i \) is an arbitrary element of \( G_i \) where \( i \in J \setminus \{n\} \) and \( g_n = \tilde{g} \). Then

\[
d(x, y) = \frac{\tilde{d}_n(x_n, y_n)}{2^n} < \frac{\eta}{2^n} = \epsilon \quad \text{and} \quad d(g \cdot x, g \cdot y) = \frac{\tilde{d}_n(g_n \cdot x_n, g_n \cdot y_n)}{2^n} > \frac{\sigma}{2^n} = \delta.
\]

Hence \( \text{diam}(g \cdot D_\epsilon(x)) > \delta \). Since \( \eta \) is any arbitrarily small positive number, then \( \epsilon = \frac{\eta}{2^n} \) is also any arbitrarily small positive number. As \( x \) is any point from \( X \), the proven means the sensitivity of the group \( G = \prod_{i \in J} G_i \) to initial conditions.

**Theorem 6.8:** Let \( G_i, i = 1, \ldots, m, \) be a homeomorphism group of a metric space \((X_i, d_i)\). Assume that \( (X, d) = \prod_{i=1}^{m} (X_i, d_i) \). Then the direct product of groups \( G = \prod_{i=1}^{m} G_i \) is sensitive to initial conditions on \((X, d)\) if and only if there exists \( n, 1 \leq n \leq m \), such that the group \( G_n \)  is sensitive to initial conditions.

**Proof:** Sufficiency is proved similarly to Theorem 6.7. Let’s prove the necessity. To do this, assume that the product of groups \( G = \prod_{i=1}^{m} G_i \) is sensitive to initial conditions and at the same time each group \( G_i \) is not sensitive to initial conditions. Therefore for every \( \delta_i > 0 \)
there exist point $x_i \in X_i$ and $\varepsilon_i > 0$ such that for every $g_i \in G_i$ the following inequality
\[ \text{diam}(g_i.D_{\varepsilon_i}(x_i)) \leq \delta_i \] is satisfied. Put $\varepsilon = \min\{\varepsilon_i \mid i = 1, m\} \Rightarrow \varepsilon > 0,$ $\delta_i = \frac{\delta}{2m},$ where $\delta$ is the sensitivity constant of the action of the group $G.$ Let $x = \{x_i\}.$ Then for every $y = \{y_i\} \in D_\varepsilon(x)$ we get
\[
d(g.x, g.y) = \sum_{i=1}^{m} d_i(g_i.x_i,g_i.y_i) \leq \sum_{i=1}^{m} \frac{\delta}{2m} = \frac{\delta}{2},
\]
consequently $\text{diam}(g.D_{\varepsilon}(x)) \leq \delta$ that contradicts the sensitivity of the group $G$ to initial conditions. Thus there exists $n, 1 \leq n \leq m,$ such that the group $G_n$ is sensitive to initial conditions. 

**Corollary 6.9:** Let $(X, d) = \prod_{i=1}^{m} (X_i, d_i),$ where $(X_i, d_i)$ are metric spaces. Let $G_i$ be a homeomorphism group of $X_i.$ Then the expansiveness of one of the groups $G_i$ entails the sensitivity of the product group $G = \prod_{i=1}^{m} G_i$ to initial conditions.

**Example 6.10:** If the group of homeomorphisms $G_1$ is expansive on $(X_1, d_1),$ and for $i = 2, m$ the group $G_i$ is an isometry group of a metric space $(X_i, d_i),$ then, according to Corollary 6.9, the group $G = \prod_{i=1}^{m} G_i$ is sensitive to initial conditions on the product $(X, d) = \prod_{i=1}^{m} (X_i, d_i).$

Here is a concrete example. The straight line $\mathbb{R}^1$ and the plane $\mathbb{R}^2$ are considered as metric spaces with standard metrics $d_1$ and $d,$ respectively, and $(\mathbb{R}^2, d) = (\mathbb{R}^1, d_1) \times (\mathbb{R}^1, d_1).$ Consider two groups of homeomorphisms $G_1 = \langle f \rangle,$ where $f(x) = 2x,$ $x \in \mathbb{R}^1,$ and $G_2 = \langle g \rangle,$ where $g(y) = y + 1,$ $y \in \mathbb{R}^1.$ Since the group $G_2$ is expansive, then, according to Corollary 6.9, the group $G = G_1 \times G_2$ of plane homeomorphisms is sensitive to initial conditions (see Figure 3). Note that the group $G$ is not expansive.
7. Chaotic actions of groups

7.1. Proof of Theorem 1.5

Let $A$ be an arbitrary set of indexes. Suppose that the group $G = \prod_{\alpha \in A} G_\alpha$ acts chaotically on the Tychonoff product $X = \prod_{\alpha \in A} X_\alpha$ of topological spaces $X_\alpha$. By Definition 1.3, $G$ has a dense non-closed orbit and a dense union of closed orbits. According to Theorem 3.3, the group $G$ has a dense non-closed orbit in $X$ if and only if for every $\alpha \in A$ the group $G_\alpha$ has a dense orbit in $X_\alpha$. By Theorem 4.1, $G$ has a dense union of closed orbits if and only if the union of closed orbits of the group $G_\alpha$ is dense on $X_\alpha$ for every $\alpha \in A$. Thus a chaotic behaviour of the group $G$ on $X$ is equivalent to a chaotic behaviour of each group $G_\alpha$ on $X_\alpha$ for every $\alpha \in A$.

Further we use notations introduced in Section 5.

Theorem 7.1: For every subset $J \subset \mathbb{N}$, let $G_i, i \in J$, be a homeomorphism group of metric space $(X_i, \tilde{d}_i)$ and on the product of metric spaces $(X, d) = \prod_{i \in J} (X_i, \tilde{d}_i)$ the direct product of groups $G = \prod_{i \in J} G_i$ is given. Assume that $(X, d)$ is a Baire space. If $G$ acts on $X$ chaotically, then:

(1) the group $G$ is sensitive to initial conditions;
(2) for every $i \in J$ the group $G_i$ is chaotic and sensitive.

Proof: First note that by Theorem 1.5, every group $G_i, i \in J$, is chaotic. Since $(X, d)$ is a Baire space, then every factor $(X_i, \tilde{d}_i), i \in J,$ is also a Baire space. Hence, according to Theorem 1.4, chaoticity of $G$ implies its sensitivity to initial conditions on $X$ and chaoticity of $G_i$ implies the sensitivity of $G_i$ on $(X_i, \tilde{d}_i)$.

7.2. Chaotic products of countable homeomorphism groups of Polish spaces

Lemma 7.2: Let $H$ be a countable group of homeomorphisms of a Polish space $X$. Then:

(1) every closed orbit of $H$ is discrete;
(2) if $X$ is compact, then every closed orbit of $H$ is finite.

Proof: Assume that $H.x$ is a closed orbit of $x \in X$. Then $H.x$ is a Polish space as a closed subset of a Polish space $X$. It is shown that the induced topology on the orbit $H.x$ is discrete. Assume that it is not true, hence $H.x$ has a non-isolated point $x_0$. Since $H$ is countable, then $H$ is represented in the form $H = \{h_i | i \in \mathbb{N}\}$. As $x_0$ is non-isolated, there exists a sequence $z_n = g_n.x, g_n \in H$, such that $z_n \to x_0$ as $n \to +\infty$. Now check that every point of the orbit $H.x$ is non-isolated. Pick $y \in H.x$, then there is an element $g \in H$ for which $y = g.x_0$. Hence $y_n = g.z_n \to y$ as $n \to +\infty$, i.e. $y$ is non-isolated.

As $h_i.x$ is not isolated in $H.x$, the set $U_i = H.x \setminus h_i.x$ is dense and open in $H.x$. Since $H.x$ is a Polish space, then $H.x$ is also a Baire space. According to the definition of Baire space, the intersection $\bigcap_{i \in \mathbb{N}} U_i$ is dense in $H.x$. As $\bigcap_{i \in \mathbb{N}} U_i = \emptyset$, we get a contradiction. Hence every point of $H.x$ is isolated. Thus the statement (1) is proved.

Suppose now that a Polish space $X$ is compact. In this case, every closed orbit $H.x, x \in X$, of $H$ is finite as a discrete compact subspace of the compact $X$. Therefore, the statement (2) is also true.
Recall that a group $G$ is referred to as \textit{residually finite} or \textit{finitely approximable}, if for every non-neutral element $g \in G$ there exists a normal subgroup, not containing $g$, of finite index in $G$.

\textbf{Theorem 7.3:} If a countable group of homeomorphisms $G$ is chaotic (in the sense of Definition 1.3) on a metrizable compact space $X$, then $G$ is residually finite and sensitive to initial conditions.

\textbf{Proof:} Lemma 7.2 and Proposition 3.4 imply that definitions of chaos 1.2 and 1.3 are equivalent. Hence the homeomorphism group $G$ is chaotic in the sense of Definition 1.2. It follows from [8, Th. 1] that every chaotic group of homeomorphisms is residually finite. According to Theorem 1.4, chaoticity of the group $G$ in the sense of Definition 1.3 implies sensitivity of $G$ to initial conditions. ■

As it is known, the following groups are residually finite:

1. matrix groups $SL(n, \mathbb{Z})$ for all $n \geq 2$;
2. finitely generated linear groups;
3. (finite or infinite) direct products of residually finite groups;
4. countable generated free groups;
5. finitely generated nilpotent groups;
6. quotients of residually finite groups by finite normal subgroups;
7. fundamental groups of compact 3-manifolds

and some others.

Moreover, groups having infinite simple subgroups and, in particular, simple groups are not residually finite [8].

\textbf{Remark 7.1:} In [8], the sensitivity to initial conditions is not investigated.

\textbf{Proof of Theorem 1.6}

Note that every compact metrizable space is Polish. As it is well known, unlike the Baire space, any Polish space is either countable or has the continuum cardinality.

Assume that every group $G_i, i \in \mathbb{N}$, is chaotic. According to Theorem 1.5, the canonical action of the direct product of groups $G = \prod_{i \in \mathbb{N}} G_i$ is chaotic, i.e. the statement (1) is true.

As the group $G_i$ is chaotic, the union of closed orbits of $G_i$ is dense in $X_i$. Due to countability of $G_i$ and compactness of $X_i$, by Lemma 7.2, every closed orbit of $G_i$ is finite. The union $F_i \subset X_i$ of closed orbits of the group $G_i$, containing greater than one point, is dense in $X_i$. Otherwise there exists an open subset $U \subset X_i$ such that $U \cap F_i = \emptyset$. Since $G_i$ is chaotic, then the union of its closed orbits is dense in $X_i$. Therefore the union of one-point orbits of $G_i$ is dense in $U$. Continuation of $G_i$ implies $G_i|_U = \text{id}_U$ that contradicts the chaoticity of $G_i$. Thus $F_i$ is dense in $X_i$. Analogously to the proof Theorem 4.1, we get that the subset $F = \prod_{i \in \mathbb{N}} F_i$ of $X = \prod_{i \in \mathbb{N}} X_i$ is a union of closed orbits of the group $G = \prod_{i \in \mathbb{N}} G_i$, and $F$ is dense in $X$. Every orbit $G.x$ where $x = \{x_i\} \in X, x_i \in F_i$, has continuum cardinality as a product of countable set of finite orbits $G_i.x_i, i \in \mathbb{N}$. By the Tychonoff theorem, the orbit $G.x$ is compact as the product of compacts orbits $G_i.x_i$. 
Assume that there exists a point \( x_0 \in G.x \) which is isolated in \( G.x \). Now check that every point of the orbit \( G.x \) is isolated. Otherwise there is non-isolated \( y = g.x_0 \) where \( g \in G \). Hence there exists a sequence \( y_n \to y \) as \( n \to +\infty \). Then \( z_n = g^{-1}.y_n \to x_0 \) as \( n \to +\infty \), that contradicts to our assumption. Therefore orbit \( G.x \) has no isolated points, i.e. \( G.x \) is perfect subset of \( X \). Consequently, the statement (2) is proved.

According Definition 1.3, every chaotic group \( G_i \) has a dense orbit \( G_i . v_i, v_i \in X_i \). Note that \( G_i . v_i \) is a countable subset of \( X_i \), hence the orbit \( G_i . v = \prod_{i \in \mathbb{N}} G_i . v_i \) has continuum cardinality. Since \( G_i . v = \prod_{i \in \mathbb{N}} G_i . v_i = \prod_{i \in \mathbb{N}} X_i = X \), the orbit \( G_i . v \) is dense in \( X \). Hence the statement (3) is proved.

The statements (4)–(5) are corollaries of Theorem 7.3.

Let for every \( i \in \mathbb{N} \), the group \( G_i \) has a fixed point \( x_i^0 \). Let \( Y \) be the union of all finite orbits of the group \( G = \prod_{i \in \mathbb{N}} G_i \). Note that \( x_0 = \{ x_i^0 \} \) is a fixed point of \( G \), hence \( x_0 \in Y \). Observe that \( z = \{ z_i \} \in Y \) if and only if there exists a finite subset \( A \subset \mathbb{N} \) such that \( z_i = x_i^0 \) for \( i \in \mathbb{N} \setminus A \) and \( z_i \) has a finite orbit \( G_i . z_i \) for \( i \in A \). As the intersection of \( Y \) with every set from the canonical base of the Tychonoff topology of \( X \) is a nonempty set, \( Y \) is dense in \( X \). Thus the statement (6) is proved. □

8. Products of groups generated by generalized horseshoe maps

8.1. Bi-infinite sequence of symbols

Let \( S = \{ 1, 2, \ldots, N \} \) with \( N \geq 2 \). Equip the set \( S \) with discrete metric \( d \). Note that the metric topology coincides with the discrete topology on \( S \). Let \( \{ S_i \mid i \in \mathbb{Z} \} \) be a family of topological spaces, and \( S_i = S, d_i = d \) for every \( i \in \mathbb{Z} \). Let \( \Sigma^N \) be a Tychonoff product of this family: \( \Sigma^N = \prod_{i \in \mathbb{Z}} S_i \). Every point of \( \Sigma^N \) can be represented as a bi-infinite sequence of \( N \) symbols:

\[
\sigma \in \Sigma^N \iff \sigma = (\ldots, \sigma_{-k}, \ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_k, \ldots),
\]

where \( \sigma_i \in \{ 1, 2, \ldots, N \} \) for every \( i \in \mathbb{Z} \). The topological space \( \Sigma^N \) is compact (as product of compact spaces), totally disconnected (as product of totally disconnected spaces), perfect and uncountable. Also \( \Sigma^N \) is a metrizable space. The metric on \( \Sigma^N \) can be defined by the following way:

\[
\rho(\sigma, \tau) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \frac{d_i(\sigma_i, \tau_i)}{1 + d_i(\sigma_i, \tau_i)},
\]

where \( d_i \) is the metric on the space \( S_i \). The topology induced by the metric \( \rho \) on \( \Sigma^N \) is the same as the Tychonoff topology on the product \( \prod_{i \in \mathbb{Z}} S_i \).

As is known, the space \( \Sigma^2 \) is homeomorphic to the standard ternary Cantor set [15, Example 3.1.28].

Now we define a map \( g: \Sigma^N \to \Sigma^N \) as follows: \( (g(\sigma))_i = \sigma_{i+1} \). The map \( g \) is referred to as the full \( N \)-shift. Consider a homeomorphism group \( G = \langle g \rangle \) generated by \( g \).

**Proposition 8.1:** The homeomorphism group \( G = \langle g \rangle \) is chaotic on the space \( \Sigma^N \), and the union of finite orbits is infinite countable and dense in \( \Sigma^N \). Besides, the group \( G \) is sensitive to initial conditions.
Figure 4. Three steps of construction of the 5-ary Cantor set.

Proof: According to [22, Prop. 3.9.4], the group $G$ has a dense orbit.

Let $\sigma = (\sigma_i) \in \Sigma^N$. It is sufficiently to show that for every $\varepsilon > 0$ there is periodic point $\tau \in D_\varepsilon(\sigma)$ of the group $G$. Let $m > \log_2 \frac{1}{\varepsilon}$ and $m \in \mathbb{N}$. The point $\tau = [\tau_i]$, where $\tau_{k(2m+1)+j} = \sigma_j \forall k \in \mathbb{Z}$, $\forall j \in [-m, m] \cap \mathbb{Z}$, is periodic with the minimal period $2m+1$. Besides, $\tau \in D_\varepsilon(\sigma)$ because $\rho(\sigma, \tau) \leq \sum_{i=m+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^m} < \varepsilon$. Thus the set of periodic orbits is dense in $\Sigma^N$.

By analogy with [17, Corollary 2.5.1], one can prove that the set of periodic orbits is infinite countable.

Moreover, the topological space $\Sigma^N$ has a countable base, then it is a Polish space, hence Theorem 1.4 implies that the chaotic group of homeomorphisms $G$ of $\Sigma^N$ is sensitive to initial conditions.

Remark 8.1: Let $J$ be any subset of $\mathbb{N}$ and $\Sigma^N_i = \Sigma^N$ for all $i \in J$. Applying [15, Prop. 2.3.7], we get that the Tychonoff product $\Sigma = \prod_{i\in J} \Sigma^N_i$ is homeomorphic to the space $\Sigma^N$.

8.2. N-ary Cantor sets

Here we recall the known generalization of the standard ternary Cantor set and the relation between this generalization and the space $\Sigma^N$.

Let $N \in \mathbb{N}$, $N > 2$ and $N$ is odd. The $N$-ary Cantor set can be constructed in the similar way as the standard ternary Cantor set. Let $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{N}] \cup [\frac{2}{N}, \frac{3}{N}] \cup \cdots \cup [\frac{N-1}{N}, \frac{N}{N}]$. Further we subdivide every closed interval of the set $C_1$ into $N$ equal parts and delete open intervals with even numbers. The remaining set is denoted $C_2$. By repeating this process, we get sets $C_i$ for every $i \geq 2$. The set $C = \bigcap_{i\in \mathbb{N}} C_i$ is the $N$-ary Cantor set [11, section 2.3]. In Figure 4, this definition for case $N = 5$ is illustrated.

Another equivalent approach to the definition of the $N$-ary Cantor set is known. Let $N \in \mathbb{N}$, $N > 2$ and $N$ is odd. The set $\left\{ \sum_{i\in \mathbb{N}} \frac{a_i}{N^i} \mid a_i \in \mathbb{Z}, 0 \leq a_i < N, a_i \text{ is even} \right\}$ is called the $N$-ary Cantor set [2, section 1.9.1].

The $N$-ary Cantor set is homeomorphic to the space $\Sigma^{\frac{N+1}{2}}$. The proof of this fact is fully similar the proof that the standard ternary Cantor set is homeomorphic to the space $\Sigma^2$ [15, Example 3.1.28].
8.3. Generalized horseshoe maps

Original notion of the horseshoe map belongs to S. Smale. As is known, the invariant set \( \Lambda \) of the horseshoe map is homeomorphic to the standard ternary Cantor set and the restriction of the horseshoe map to the set \( \Lambda \) is conjugate to the 2-shift map on the space \( \Sigma^2 \) [17, Section 2.5].

The definition of generalized horseshoe map of length \( N \) can be found in [16, Definition 2.7]. In Figure 5, the definition for case \( N = 3 \) is illustrated.

Let \( D = S_0 \cup R \cup S_1 \). The generalized horseshoe map \( f \) of length 3 has the set \( D \) as its domain. The image of the set \( D \) respectively \( f \) is shown in Figure 5(a).

The invariant set \( \Lambda \) of the generalized horseshoe map of length \( N \) is homeomorphic to the \((2N - 1)\)-ary Cantor set and, consequently, to the space \( \Sigma^N \). The full \( N \)-shift is conjugate to the restriction of the generalized horseshoe map of length \( N \) to the invariant set \( \Lambda \) [22, Section 4.2.3], [16, Remark 2.9].

8.4. Chaotic groups of homeomorphisms on countable products of Cantor sets

Let \( A \) be any index set. Consider a family of chaotic groups \( \{ G_\alpha = \langle g_\alpha \rangle \}_{\alpha \in A} \), where \( g_\alpha \) is the full \( N_\alpha \)-shift of \( \Sigma^{N_\alpha} \). According to Theorem 1.5, the product of groups \( G = \prod_{\alpha \in A} G_\alpha \) of homeomorphisms of the space \( \Sigma = \prod_{\alpha \in A} \Sigma^{N_\alpha} \) is chaotic.

Let the set \( A \) be countable. As we have seen above, every \( \Sigma^{N_\alpha} \) is a compact metric space, hence \( \Sigma^{N_\alpha} \) is a metric Baire space. Consequently, \( \Sigma \) is a metric Baire space, so according to Theorem 1.4, \( G \) is sensitive to initial conditions on \( \Sigma \).
In the case when \( N_\alpha = N \) for every \( \alpha \in A \subset \mathbb{N} \), we get \( \Sigma_\alpha^N := \Sigma^{N_\alpha} = \Sigma^N \) and \( \Sigma = \prod_{\alpha \in A} \Sigma_\alpha^N \). According to Remark 8.1, \( \Sigma \) is homeomorphic to \( \Sigma^N \). Thus we get new chaotic actions of free Abelian groups \( G \) with countable set of generators on the space \( \Sigma \) homeomorphic to \((2N - 1)\)-ary Cantor set.

9. Construction of chaotic actions of groups on topological manifolds

9.1. Chaotic actions of the group \( \mathbb{Z} \) on every closed surface

By a closed surface we mean a connected compact topological two-dimensional manifold \( M \) without boundary.

When proving Theorem 9.1, we use the concept of a two-dimensional orbifold. A simple exposition of the theory of compact two-dimensional orbifolds can be found in [23]. The necessary information about orbifolds used by us is contained in [7].

**Theorem 9.1:** For every closed surface \( M \) there exists a countable family of chaotic groups of homeomorphisms, isomorphic to the group \( \mathbb{Z} \), such that the union of finite orbits of every such group is dense in \( M \).

**Proof:** Show that for each \( k, m \in \mathbb{N}, k, m \geq 3 \), the following matrix

\[
A = A(k, m) = \begin{pmatrix}
1 & k \\
m & 1 + km
\end{pmatrix}
\]

induces a chaotic homeomorphism \( g_A \) of the closed two-dimensional disk \( \mathbb{B}^2 \). Let \( f_A \) be the Anosov torus automorphism defined by \( A \).

We represent the torus \( \mathbb{T}^2 \) as the square \([-1/2, 1/2] \times [-1/2, 1/2]\) in a Cartesian coordinate system \( Oxy \) on a plane \( \mathbb{R}^2 \) with identified opposite sides. In other words \( \mathbb{T}^2 \) has coordinates \((x, y)\) where \( x \) and \( y \) are periodic of period one. For each fixed pair of numbers \( k, m \in \mathbb{N}, k, m \geq 3 \), consider subsets \( P = [-1/2, 1/2] \times [-1/k, 1/k] \) and \( Q = [-1/m, 1/m] \times [-1/2, 1/2] \) of \( \mathbb{T}^2 \). Note that \( P \) and \( Q \) are two overlapping annuli on the torus \( \mathbb{T}^2 \). Denote the union of the annuli by \( R = P \cup Q \) and the intersection by \( S = P \cap Q \), see Figure 6. Define maps \( f : R \to R \) and \( h : R \to R \) by the following equations:

\[
f(x, y, k) = \begin{cases} 
(x + ky, y) & \text{if } (x, y) \in P, \\
(x, y) & \text{if } (x, y) \in R \setminus P,
\end{cases}
\]

\[
h(x, y, m) = \begin{cases} 
(x, y + mx) & \text{if } (x, y) \in Q, \\
(x, y) & \text{if } (x, y) \in R \setminus Q.
\end{cases}
\]

The map \( g = h \circ f : R \to R \) is called the toral linked twist map defined by composing \( f \) and \( h \). According to [12, Theorem A], the constructed linked twist map \( g : R \to R \) is topologically mixing and the periodic points of \( g \) are dense in \( R \). Therefore, the homeomorphism group \( G = \langle g \rangle \) generated by \( g \) is topologically transitive and chaotic on the topological space \( R \). Note that \( g = g(k, m) \) and \( A = A(k, m) \) are related by the equality \( g(x, y) = A(\frac{x}{y}) \) in some neighbourhood \( U \) of a point \((0, 0), U \subset S \subset \mathbb{T}^2 \).

Let \( E_2 \) be the unit two-dimensional matrix. Let us identify the quotient space \( \mathcal{N} = \mathbb{T}^2 / \pm E_2 \) with the rectangle \([-1/2, 1/2] \times [0, 1/2] \) the sides of which are identified in the
Figure 6. Representation of the torus on the unit square.

way indicated by the arrows in Figure 7. Denote by $p : \mathbb{T}^2 \to \mathcal{N}$ the quotient map. Note that $\mathcal{N}$ is the orbifold ‘Pillow’, which is homeomorphic to the standard sphere $\mathbb{S}^2$, and both subsets $p(R)$, coloured green, and $p(\mathbb{T}^2 \setminus R)$, coloured yellow, are homeomorphic to a closed disk $\mathbb{B}^2$ (see Figure 7 b). Identify the topological space of $\mathcal{N}$ with the sphere $\mathbb{S}^2$.

It is easy to check that $g|_{\partial R} = \text{id}|_{\partial R}$. Therefore we can continue $g$ for the entire torus $\mathbb{T}^2$ such that $g|_{\mathbb{T}^2 \setminus R} = \text{id}|_{\mathbb{T}^2 \setminus R}$. The resulting torus homeomorphism will continue to denote by $g$. Emphasize that the points coloured yellow in Figure 6 are fixed relative to the homeomorphism $g$. Since the map $g$ satisfies the equality $g(-x, -y) = -g(x, y)$ for every $(x, y) \in \mathbb{T}^2$, it induces a homeomorphism $g_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{S}^2$ satisfying the following equality $p \circ g = g_{\mathbb{S}^2} \circ p$. Therefore the restriction $g_{\mathbb{B}^2}$ of $g_{\mathbb{S}^2}$ to the closed disk $\mathbb{B}^2 = p(R)$ (coloured green in Figure 7b) satisfies the commutative diagram

$$
\begin{array}{ccc}
R & \stackrel{g}{\longrightarrow} & R \\
p \downarrow & & \downarrow p \\
\mathbb{B}^2 & \stackrel{g_{\mathbb{B}^2}}{\longrightarrow} & \mathbb{B}^2,
\end{array}
$$

and the projection $p : R \to \mathbb{B}^2$ is a surjective continuous open map. As a finite orbit of the group $G$ maps onto a finite orbit of the group $\Gamma = \langle g_{\mathbb{B}^2} \rangle$ with respect to $p : R \to \mathbb{B}^2$, the union of finite orbits of $\Gamma$ is dense in $\mathbb{B}^2$. A dense orbit of $G$ maps onto a dense orbit of
Figure 7. Orbifold ‘Pillow’: (a) representation of the orbifold ‘Pillow’ and (b) the orbifold ‘Pillow’ is homeomorphic to the standard sphere.

Thus, the group $\Gamma$ is chaotic in $\mathbb{B}^2$. Since $g_{\mathbb{B}^2}$ is equal to identity on another closed disc coloured yellow, complementary to $\mathbb{B}^2$, the group $\Gamma$ fixes every point of the boundary $\partial \mathbb{B}^2$. Therefore it is possible to glue the boundary $\partial \mathbb{B}^2$ in an arbitrary way and to obtain a new surface $M$, and as a result, we can get every closed surface $M$. Denote the corresponding quotient map by $k : \mathbb{B}^2 \to M$. Since $\Gamma|_{\partial \mathbb{B}^2} = \text{id}_{\partial \mathbb{B}^2}$, then $\Gamma$ induces an isomorphic group of homeomorphisms $\tilde{\Gamma}$ of the surface $M$, and the mapping $k : \mathbb{B}^2 \to M$ is a topological semi-conjugation of the groups $\Gamma$ and $\tilde{\Gamma}$. Emphasize that the interior $\text{Int}(\mathbb{B}^2)$ is invariant respectively $\Gamma$, and the restriction $k|_{\text{Int}(\mathbb{B}^2)}$ is a homeomorphism conjugating $\Gamma|_{\text{Int}(\mathbb{B}^2)}$ with $\tilde{\Gamma}|_{k(\text{Int}(\mathbb{B}^2))}$. Therefore, the group $\tilde{\Gamma}$ is chaotic on $M$. Further we write $\tilde{\Gamma} = \Gamma(k, m)$, since this group is defined by the matrix $A = A(k, m)$ of the form (16).

Thus we get a countable family of chaotic groups $\{\tilde{\Gamma}(k, m) \mid k, m \in \mathbb{N}, k, m \geq 3\}$ on every closed surface $M$. For short, further we denote such families by $\{\tilde{\Gamma}(k, m)\}$.

More specifically, we consider the following ways of gluing the disk boundary $\partial \mathbb{B}^2$.

**Case I.** Consider the boundary $\partial \mathbb{B}^2 \cong S^1$ as a $4q$-polygon whose sides are glued together according to the scheme $a_1 b_1^{-1} b_1^{-1} \cdots a_q b_q a_q^{-1} b_q^{-1}$, $q \in \mathbb{N}$. The scheme is obtained as follows: the direction of movement along the sides of the polygon is selected, the sides are written out in a row, the glued sides are marked with one letter, the degree $-1$ means the direction of gluing opposite to the direction of movement along the side. As a result we get a closed surface homeomorphic to the sphere with $q$ handles. Denote this surface by $S^2_q$. Thus we get a countable family of chaotic groups $\{\tilde{\Gamma}(k, m)\}$ on $S^2_q$ for every $q \in \mathbb{N}$.

**Case II.** Consider the boundary $\partial \mathbb{B}^2 \cong S^1$ as a $2q$-polygon whose sides are glued together according to the scheme $a_1 a_1 \cdots a_q a_q$, $q \in \mathbb{N}$. In this case, we get a closed non-orientable surface homeomorphic to the sphere with $q$ Mobius bands which will be denoted by $NS^2_q$. Thus we get an infinite series of chaotic groups $\{\tilde{\Gamma}(k, m)\}$ on each surface $NS^2_q$, $q \in \mathbb{N}$.

**Case III.** Let us glue the boundary $\partial \mathbb{B}^2 \cong S^1$ into an arbitrary segment denoted by $L_0$ such that as the result we get a topological sphere $S^2$. In the same way as above, we get an infinite series of chaotic groups $\{\tilde{\Gamma}(k, m)\}$ on $S^2$. 
Since every closed surface $M$ is homeomorphic to one of the canonical surfaces $S^2$, $S^2_q$ or $NS^2_q$ where $q \geq 1$, then in the above way we obtain a countable family of chaotic homeomorphism groups $(\tilde{\Gamma}(k, m) \cong \mathbb{Z})$ on $M$.

It is well known that finite orbits of every Anosov automorphism of the torus $T^2$ form a countable dense subset in $T^2$. However, this is not true for the constructed homeomorphism $g$. In the next section, we will show that $g$ has a continuum of fixed points. Hence, the set of all finite orbits of both groups $\Gamma$ and $\tilde{\Gamma}$ has the cardinality of the continuum. ■

**Remark 9.1:** In contrast to [8], we have constructed an infinite countable family of chaotic actions of the group $\mathbb{Z}$ on each closed surface.

**9.2. Chaotic actions of the group $\mathbb{Z}$ on noncompact two-dimensional manifolds**

Consider the toral linked twist map $g = g(k, m) : R \to R$ constructed in the previous section. Recall that $R$ is the union of two annuli $P \cup Q$ on the torus $T^2$. Let $p : R \to \mathbb{B}^2$ be the projection satisfying the commutative diagram (19), hence $p(\partial R) = \partial \mathbb{B}^2 \cong S^1$.

At first we will pick out some point $z \in \partial \mathbb{B}^2$. Identify $\partial \mathbb{B}^2 \setminus \{z\}$ with the real line $\mathbb{R}^1 = \bigcup_{n=\infty}^{+\infty} [n, n + 1]$. Represent $\mathbb{B}^2 \setminus \{z\}$ as an polygon without a point $z$, the boundary of which is divided into a countable set of pairwise glued sides. Consider the following gluing rules.

**Case IV.** $[n, n + 1] \sim a_n b_n a_n^{-1} b_n^{-1} \forall n \in \mathbb{Z}$. As a result of gluing, we will get a noncompact two-dimensional manifold $M$ without boundary, homeomorphic to a plane with a countable family of handles which we denote by $\mathbb{R}^2_{\infty}$. Emphasize that $M$ is homeomorphic to the Loch Ness monster (see Figure 1).

**Case V.** $[n, n + 1] \sim a_n a_n \forall n \in \mathbb{Z}$. As a result of gluing, we will get a noncompact two-dimensional manifold $M$ without boundary, homeomorphic to a plane with a countable family of Mobius bands which we denote by $\mathbb{N} \mathbb{R}^2_{\infty}$.

**Case VI.** Let us glue $p(\partial R) = \partial \mathbb{B}^2 \cong S^1$ to a segment $L_0$ as in Case III. As a result, we get a topological sphere $S^2$. As above, let $k : \mathbb{B}^2 \to S^2$ be the respective quotient map. The image $k(\partial \mathbb{B}^2)$ of $\partial \mathbb{B}^2$ we denote by $L_0 \subset S^2$. Consider an arbitrary nonempty compact subset $K_0$ of $L_0$ and a point $z_0 \in L_0 \setminus K_0$. We get a noncompact two-dimensional manifold denoted by $\mathbb{R}^2(K_0) := S^2 \setminus (K_0 \cup \{z_0\})$ homeomorphic to a plane without $K_0$. In Figure 2, you can see the noncompact surface $\mathbb{R}^2(K_0)$ where $K_0$ is the standard Cantor set.

Now we consider the restriction of $g : R \to R$, constructed in the previous section, to a circle $C = \{(x, 0) \mid -1/2 \leq x \leq 1/2\}$. The image $g(C)$ is a closed curve on the torus $T^2$. Observe that

$$g(x, 0) = (x, 0) \quad \forall x \in [-1/2, -1/m] \cup [1/m, 1/2].$$

Emphasize that $[-1/2, -1/m] \cup [1/m, 1/2]$ is a connected segment in $R \subset T^2$. The set $L = p([-1/2, -1/m] \cup [1/m, 1/2])$ is a connected topological segment in the inside of the disk $\mathbb{B}^2$ highlighted in red in Figure 7(b), and $g|_L = \text{id}_L$. Assume that the boundary $\partial \mathbb{B}^2$ of $\mathbb{B}^2$ is glued in one of the ways specified above in Cases I–VI. The result is a manifold $M$ where $M$ is either one of arbitrary closed surfaces $S^2$, $S^2_q, NS^2_q$ where $q \geq 1$, or one of noncompact surfaces $\mathbb{R}^2_{\infty}, N \mathbb{R}^2_{\infty}, \mathbb{R}^2(K_0)$. Let $k : \mathbb{B}^2 \to M$ be the corresponding quotient map. As no points from $L$ were not glued together, we identify the image $k(L)$ of $L$ in $M$ with $L$ and consider $L$ as subset of $M$. This fact allows us to remove an arbitrary
closed subset \( K \) of \( L \) from \( L \) and to obtain a noncompact two-dimensional topological manifold \( \hat{M} = \widehat{M}(K) := M \setminus K \). Emphasize that we do not exclude the possibility \( K = \emptyset \). The homeomorphism \( \hat{g} = \hat{g}(k, m) \) induces a chaotic homeomorphism \( \hat{g} = \hat{g}(k, m) \) of \( \hat{M} \), and the group \( \hat{\Gamma} = \langle \hat{g} \rangle \) has a dense set periodic orbits in \( \hat{M} \).

Indicate some important classes of obtained manifolds \( \hat{M} = \hat{M}(K) \).

If \( K \) is a finite subset of \( L \), then \( \hat{M} \) is a noncompact manifold with a finite number of ends.

If \( K \) is an infinite subset of \( L \) with one limit point \( z' \), then \( z' \in K \) and we get a noncompact manifold \( \hat{M} = M \setminus K \) with a discrete countable set of ends.

When \( K \) is the standard Cantor set on \( L \), we get a noncompact manifold \( \hat{M} \) containing the Cantor set of ends. It is possible that \( \hat{M} \) has other ends obtained by removing a compact subset \( K_0 \subset L_0 \) where \( L_0 \subset k(\partial B^2) \) similarly to Case VI.

Emphasize that for every closed subset \( K \), the set of handles or Mobius bands on \( \hat{M} \) may be arbitrary finite as well as infinite.

In Case VI, for \( M = \mathbb{R}^2(K_0) \) we get \( \hat{M} = \widehat{M}(K_0, K) \). When \( K_0 = \emptyset \) and \( K \) is the standard Cantor set, the surface \( \hat{M}(K_0, K) \) is homeomorphic to the surface in Figure 2.

On every such constructing noncompact topological two-dimensional manifold \( \hat{M} \) the homeomorphism \( \hat{g} = \hat{g}(k, m) \) induces a chaotic homeomorphism \( \hat{g} = \hat{g}(k, m) \), and the group \( \hat{\Gamma} = \langle \hat{g} \rangle \) has a dense set periodic orbits in \( \hat{M} \).

Thus we get an infinite countable family of chaotic groups \( \{ \hat{\Gamma}(k, m) = \langle \hat{g}(k, m) \rangle \} \) on each surface \( \hat{M} \) constructed above.

### 9.3. Chaotic groups acting on closed \( n \)-dimensional manifold

Let \( G \) be any countably generated free group. In [9], Cairns et al. showed that every compact triangulable manifold \( M \) of an arbitrary dimension \( n \) greater than 1 admits a faithful chaotic action of the group \( G \).

Emphasize that the free group \( \Gamma \) of two generators may be implemented as a chaotic group of homeomorphisms of every such manifold \( M \) [9].

### 9.4. Chaotic groups of homeomorphisms on products of manifolds

Since every open subset of a Polish space is also Polish, then all topological manifolds defined above in Sections 9 satisfies conditions of Theorems 1.4–1.5. For an arbitrary index set \( A \), we use the constructed above chaotic actions of homeomorphism groups \( G_\alpha, \alpha \in A \), on topological manifolds \( M_\alpha \) as building blocks for constructions of chaotic canonical action of the product of groups \( G = \prod_{\alpha \in A} G_\alpha \) on \( M = \prod_{\alpha \in A} M_\alpha \).

According to Theorem 1.6, if \( G_i, i \in J \subset \mathbb{N} \), is a countable chaotic group of homeomorphisms of \( n_i \)-dimensional closed triangulable manifold \( M_i, n_i \geq 2 \), then the canonical action of the direct product of groups \( G = \prod_{i \in I} G_i \) is chaotic on the product \( M = \prod_{i \in I} M_i \).

Here \( M \) is an infinite-dimensional topological manifold, if the set \( J \) is infinite countable, otherwise \( M \) is a finite-dimensional topological manifold. In both cases, by Theorem 1.4, every groups \( G \) and \( G_i \) have sensitivity to initial conditions. Emphasize that the dimension of \( M_i \) may be an arbitrary greater than 1.
Moreover, in the case \( J = \mathbb{N} \), according to Theorem 1.6, there exists a dense subset \( F \subset M \) which is the union of compact continuum orbits of the group \( G \), and every such orbit is a perfect subset of \( M \). Besides there exists a dense continuum orbit of \( G \) in \( M \).

If, moreover, every \( M_i \) is a two-dimensional manifold with a chaotic action of the group \( G_i \cong \mathbb{Z} \) constructed in Sections 9.1 and 9.2, then every group \( G_i \) has finite orbits and, in particular, a fixed point. Therefore, in this case, the group \( G = \prod_{i \in \mathbb{N}} G_i \) on the infinite-dimensional topological manifold \( M = \prod_{i \in J} M_i \) has a continuum set of finite orbits, and the union of finite orbits is dense in \( M \).

**Example 9.2:** Consider the standard two-dimensional torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). Anosov torus automorphism \( T^2 \), given by a matrix \( A \in SL(2, \mathbb{Z}) \), is denoted by \( g_A \). As is well known [21], every matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

where \( ad - bc = 1 \) and \( a + d > 2 \) define an Anosov automorphism \( g_A \) of the torus \( T^2 \) preserving its orientation. The group \( G = \langle g_A \rangle \cong \mathbb{Z} \) generated by \( g_A \) acts chaotically on \( T^2 \). It is well known that there exists a countable set of finite orbits of \( G \), and this set is dense in \( T^2 \). There exists a countable family \( \mathcal{A} = \{ A_k \mid k \in \mathbb{N} \} \) such matrices defining Anosov automorphisms.

Consider the infinite-dimensional torus \( T^\infty = \prod_{i \in \mathbb{N}} T^2_i \) where \( T^2_i = T^2 \) for each \( i \in \mathbb{N} \). Let \( g_{A_i}, A_i \in \mathcal{A} \), be an Anosov automorphism on \( T^2_i \). Let \( G_i = \langle g_{A_i} \rangle \) and \( G = \prod_{i \in \mathbb{N}} G_i \). According to Theorem 1.6, we get a chaotic group of homeomorphisms \( G \) of the torus \( T^\infty \). Taking into account that \( A_i \) may be an arbitrary matrix belonging to \( \mathcal{A} \), we see that different groups \( G \) form a continuum set. Emphasize that \( G \) also has continuum cardinality. As every group \( G_i \) has a fixed point, by Theorem 1.6, the union of finite orbits of group \( G \) is dense in \( T^\infty \), and \( G \) has a fixed point. Moreover, the union of its compact orbits of continuum cardinality is also dense in \( T^\infty \), and every such orbit is a perfect subset of \( T^\infty \). Note that every dense orbit has continuum cardinality. Besides, by Theorem 1.6, the group \( G \) is sensitive to initial conditions.

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