WINDOW-DEPENDENT BASES FOR EFFICIENT REPRESENTATIONS OF THE STOCKWELL TRANSFORM

Abstract. Since its appearing in 1996, the Stockwell transform (S-transform) has been used as a tool in medical imaging, geophysics and signal processing in general. In this paper, we prove that the system of functions (so-called DOST basis) is indeed an orthonormal basis of \( L^2_{0,1} \), which is a time-frequency localized basis, in the sense of Donoh-Stark Theorem [11]. Our approach provides a unified setting in which to study the Stockwell transform and its orthogonal decomposition. Finally, we introduce a fast – \( O(N \log N) \) – algorithm to compute the Stockwell coefficients for general windows. This algorithm includes the one proposed in [33] as a special case.

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1. Introduction

Let \( f \) be a signal with finite energy, that is \( f \in L^2(\mathbb{R}) \), and let \( \varphi \) be a window in \( L^2(\mathbb{R}) \). Then, following M. W. Wong and H. Zhu [34], we define the Stockwell transform (S-transform) \( S_\varphi f \) as

\[
(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) \varphi(\xi (t - b)) \, dt, \quad b, \xi \in \mathbb{R}.
\]

(1.1)

It is possible to rewrite the S-transform with respect to the Fourier transform of the analyzed signal:

\[
(S_\varphi f)(b, \xi) = \int_{\mathbb{R}} e^{2\pi i b \xi} \hat{f}(\xi + \xi) \hat{\varphi}(\xi) \, d\xi, \quad b, \xi \in \mathbb{R}, \quad \xi \neq 0,
\]

(1.2)

where \( \hat{f} \) is the Fourier transform of the signal \( f \), given by

\[
\hat{f}(\xi) = (F f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) dt, \quad \xi \in \mathbb{R}.
\]

In the following, we denote with \( \hat{f} \) or \( F^{-1} f \) the inverse Fourier transform of a signal \( f \). The S-transform was initially defined by R. G. Stockwell, L. Mansinha and R. P. Lowe in [29] using a Gaussian window

\[
g(t) = e^{-t^2/2}, \quad t \in \mathbb{R}.
\]
In this case,

$$\mathbf{S}_g f \left( b, \xi \right) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i t \xi} f(t) |\xi| e^{-\left(t-b\right)^{2}/2} dt, \quad b, \xi \in \mathbb{R},$$

which in the alternative formulation becomes

$$\mathbf{S}_g f \left( b, \xi \right) = \int_{\mathbb{R}} e^{2\pi i \xi b} \hat{f}(\xi + \xi) e^{-2\pi^{2} \xi^{2} / \xi^{2}} d\xi, \quad b, \xi \in \mathbb{R}, \quad \xi \neq 0.$$  

The natural discretization of (1.4), introduced in [29], is given by

$$\mathbf{S}_g f \left( j, n \right) = \sum_{m=0}^{N-1} e^{2\pi i m j / N} \hat{f}(n + m) e^{-2\pi^{2} m^{2} / n^{2}},$$

where \( j = 0, \ldots, N - 1 \) and \( n = 1, \ldots, N - 1 \). For \( n = 0 \), it is set

$$\mathbf{S}_g f \left( j, 0 \right) = \frac{1}{N} \sum_{k=0}^{N-1} f(k), \quad j = 0, \ldots, N - 1.$$  

In the literature, (1.5) is called redundant discrete Stockwell transform. Unfortunately, the redundant Stockwell transform has a high computational cost – \( \mathcal{O} \left( N^{2} \log N \right) \). To overcome this problem, R. G. Stockwell introduced in [27], without a mathematical proof, a basis for periodic signals with finite energy, \( \text{i.e.} \ L^2 \left( [0, 1] \right) \), given by

$$\bigcup_{p \in \mathbb{Z}} D_p = \bigcup_{p \in \mathbb{Z}} \left( D_{p, \tau} \right)_{\tau = 0}^{\beta \left(p\right) - 1}.$$  

This basis, precisely defined in Section 3, is adapted to octave samples in the frequency domain. The decomposition of a periodic signal \( f \) in this basis is called in the literature the discrete orthonormal Stockwell transform (DOST transform). The related coefficients

$$f_{p, \tau} = \left( f, D_{p, \tau} \right)_{L^2\left([0, 1]\right)},$$

are called DOST coefficients.

In this paper we prove that this basis is not suited to the standard S-transform with Gaussian window (1.1), rather to a S-transform associated with a characteristic function (boxcar window). This fact was already pointed out by R. G. Stockwell himself in [27]. The computational complexity of the algorithm suggested by R. G. Stockwell is still high: \( \mathcal{O}(N^{2}) \). In 2009, Y. Wang and J. Orchard [33] proposed a fast algorithm which reduces drastically the complexity to \( \mathcal{O}(N \log N) \); the same complexity of the FFT. This achievement allowed the application of the S-transform to image analysis.

We provide an adapted basis of \( L^2 \left( [0, 1] \right) \) on which to decompose the Stockwell transform with a general window \( \varphi \), under an admissibility assumption on the window \( \varphi \). Assume that we can find such a basis \( E_{j}^{\varphi} \) of \( L^2 \left( [0, 1] \right) \), depending on the choice of \( \varphi \). Then, by linearity, we can write

$$\left( S_{\varphi} f \right) \left( b, \xi \right) = \sum_{j} \left( f, E_{j}^{\varphi} \right)_{L^2\left([0, 1]\right)} \left( S_{\varphi} E_{j}^{\varphi} \right) \left( b, \xi \right) = \sum_{j} f_{j}^{\varphi} \left( S_{\varphi} E_{j}^{\varphi} \right) \left( b, \xi \right),$$

where

$$f_{j}^{\varphi} = \left( f, E_{j}^{\varphi} \right)_{L^2\left([0, 1]\right)}$$

are the coefficients of the analyzed signal \( f \) with respect to the basis \( E_{j}^{\varphi} \).

An idealized basis would satisfy the following properties:

(i) \( E_{j}^{\varphi} \) is an orthonormal basis of \( L^2 \left( [0, 1] \right) \);

(ii) \( \left( S_{\varphi} E_{j}^{\varphi} \right) \left( b, f \right) \) is local in time;

(iii) \( \left( S_{\varphi} E_{j}^{\varphi} \right) \left( b, f \right) \) is local in frequency;
(iv) we can find a fast algorithm – $O(N \log N)$ – to compute the coefficients $f_{p,\tau}^\varphi$.

We prove that (1.6) is indeed an orthonormal basis of $L^2([0, 1])$ satisfying conditions (i), (ii) and (iii), (iv), associated to a particular window $\varphi$.

Moreover, for general windows $\varphi$,

$$(1.7)\quad \bigcup_{p \in \mathbb{Z}} E_{p,\tau}^\varphi = \bigcup_{p \in \mathbb{Z}} \{ E_{p,\tau}^\varphi \}_{\tau=0}^{\beta(p)-1}$$

is such that

$$(1.8)\quad (S_\varphi E_{p,\tau}^\varphi)(b, \nu(p)) = D_{p,\tau}(b),$$

where $\nu(p)$ is the center of the $p$-frequency band where the basis $D_p$ in (1.6) is supported and $\beta(p)$ is its width. In section 6 we introduce a fast – $O(N \log N)$ – algorithm to compute the coefficients

$$f_{p,\tau}^\varphi = (E_{p,\tau}^\varphi, f)_{L^2([0, 1])}.$$

Unfortunately, for a general window $\varphi$, the basis (1.7) fails to be orthogonal. Nevertheless, using a mild condition on $\varphi$, we prove that it forms a frame, which in general is not tight.

We are able to build a basis of $L^2([0, 1])$ satisfying conditions (ii) and (iii) and (iv) for general $\varphi$ and (i) for a particular window. In fact, taking

$$\tilde{\chi}_{[-\frac{1}{3}, \frac{1}{3}]}(x) = \left( F_{\xi=x} \chi_{[-\frac{1}{3}, \frac{1}{3}]}(\xi) \right)(x),$$

where

$$\chi_{[-\frac{1}{3}, \frac{1}{3}]}(\xi) = \begin{cases} 1, & \xi \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ 0, & \text{otherwise} \end{cases},$$

we can recover the DOST basis (1.6). Indeed,

$$E_{p,\tau}^\tilde{\chi} = D_{p,\tau}.$$

Moreover, we have

$$(1.9)\quad (S_{\tilde{\chi}} f) \left( \frac{\tau}{\beta(p)}, \nu(p) \right) = (-1)^\tau f_{p,\tau}, \quad \tau = 0, \ldots, \beta(p) - 1,$$

where

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0, 1])}.$$

Equation (1.9) is the representation of the DOST coefficients as skeleton of the redundant Stockwell transform given in [27], [28], [31]. Equation (1.9) can be checked directly, once the basis functions $D_{p,\tau}$ are given.

This paper is organized as follows: in Section 2 we provide a brief survey on the S-transform in the context of time-frequency analysis. In particular, we point out the similarities and the differences between Fourier transform, short-time Fourier transform and wavelet transform. In Section 3 we prove that (1.6) is a basis of $L^2([0, 1])$ and we highlight its time-frequency local properties. In Section 4 we decompose the Stockwell transform with a general window using (1.6). Moreover, we determine the explicit expression of $(S_\varphi D_{p,\tau})(b, \xi)$. In Section 5 we give a discretization of the S-transform. In Section 6 we determine the basis (1.7) adapted to a general window $\varphi$. We propose an algorithm which evaluates the coefficients related to the basis (1.7) with computational complexity $O(N \log N)$. Moreover, we show some numerical examples, considering different windows.
2. A Brief Survey on the S-transform

In many practical applications it is important to analyze signals, i.e. extracting the time-frequency content of the signal. Given a signal \( f \) in \( L^2(\mathbb{R}) \), we can precisely extract its frequency content using the Fourier transform \( \hat{f}(\xi) = (Ff)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi it\xi} f(t) \, dt \), \( \xi \in \mathbb{R} \).

Unfortunately, due to uncertainty principle, it is impossible to retain at the same time precise time-frequency information. In the past years, many techniques arose trying to deal with the uncertainty principle in order to obtain a sufficiently good time-frequency representation of a signal. One of the standard tools is the short-time Fourier transform \( \text{STFT}_\varphi f \) \( \hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi it\xi} f(t) \varphi(t-b) \, dt \), \( b, \xi \in \mathbb{R} \).

Loosely speaking, taking the short-time Fourier transform of a signal \( f \) at a certain time \( b \) is like taking the Fourier transform of the signal \( f \) cut by a window function \( \varphi \) centered in \( b \), see for example [13], [17]. It is possible to invert the short-time Fourier transform using the following theorem.

**Theorem 1.** Let \( f \) be a signal in \( L^2(\mathbb{R}) \). Then

\[
\hat{f}(\xi) = \int_{\mathbb{R}} (\text{STFT}_\varphi f) (b, \xi) \, db, \quad \xi \in \mathbb{R}.
\]

Notice that the width of the analyzing window remains fixed. Due to the Nyquist sampling theorem, it would be natural to consider a window whom width depends on the analyzed frequency. To accomplish this task, in [29], the S-transform \( S_g \) was introduced as

\[
(\text{STFT}_\varphi f) (b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi it\xi} f(t) \varphi(t-b) \, dt, \quad b, \xi \in \mathbb{R}.
\]

Notice that the width of the Gaussian window \( e^{-\frac{(t-b)^2}{2\xi^2}} \) shrinks as the analyzed frequency increases, providing a better time-localization for high frequencies. It is possible to rewrite the Stockwell transform with respect to the Fourier transform of the signal \( f \) as

\[
(\text{STFT}_\varphi f) (b, \xi) = \int_{\mathbb{R}} e^{2\pi i \zeta b} \hat{f}(\zeta + \xi) e^{-\frac{2\pi^2 \xi^2}{\zeta^2}} \, d\zeta, \quad b, \xi \in \mathbb{R}, \xi \neq 0.
\]

In [29] it has been stated an inversion formula similar to Theorem 1.

**Theorem 2.** Let \( f \) be a signal in \( L^2(\mathbb{R}) \). Then

\[
\hat{f}(\xi) = \int_{\mathbb{R}} (S_g f) (b, \xi) \, db, \quad \xi \in \mathbb{R}.
\]

Many extensions of this transform have been suggested in the last years. See for example [10], [18], [19], [34], [35]. We here recall the one introduced in [34].

**Definition 2.1.** Let \( f \) be a signal in \( L^2(\mathbb{R}) \) and let \( \varphi \) be a window function in \( L^2(\mathbb{R}) \). Then, we call \( S_\varphi \)-transform \( S_\varphi f \) of the signal \( f \) with respect to the window \( \varphi \)

\[
(\text{STFT}_\varphi f) (b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi it\xi} f(t) \varphi(t-b) \, dt, \quad b, \xi \in \mathbb{R}.
\]
It is possible to recover the original definition (2.1) taking \( \varphi \) to be the Gaussian window function
\[
\varphi(t) = e^{-t^2/2}.
\] The S-transform has been recently extended to the multi-dimensional case by the second author [26]. Theorem 2 still holds for the S-transform (2.3).

See [4, 7, 12, 16, 20, 22, 24, 36] for some applications of the S-transform to signal processing.

Heuristically, we can think at the S-transform as a short-time Fourier transform in which the width of the analyzing window varies with respect to the analyzed frequency. Therefore, the S-transform can also be interpreted as a particular non stationary Gabor transform, see [1].

We can give an equivalent definition of the S-transform using the following proposition.

**Proposition 3.** Let \( f \) be a signal in \( L^2(\mathbb{R}) \) and let \( \varphi \) be a window in \( L^2(\mathbb{R}) \). Then
\[
(S_\varphi f)(b, \xi) = e^{-2\pi i b \xi} \left( \mathcal{F}_{\xi=b}^{-1} f_\xi \right)(b), \quad b, \xi \in \mathbb{R}, \quad \xi \neq 0,
\]
where
\[
f_\xi(\zeta) = \hat{f}(\zeta) \overline{\varphi\left(\frac{\zeta - \xi}{\xi}\right)}, \quad \zeta \in \mathbb{R}.
\]

The following inversion formula has been proven in [34].

**Theorem 4.** Let \( \varphi \) be a function in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) such that
\[
c_\varphi = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty.
\]
We say that \( \varphi \) is an admissible window for the S-transform and we call \( c_\varphi \) the admissibility constant. Then,
\[
c_\varphi \left( f, f' \right)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} (S_\varphi f)(b, \xi) \overline{(S_\varphi f')(b, \xi)} db \frac{d\xi}{|\xi|},
\]
for all \( f \) and \( f' \) in \( L^2(\mathbb{R}) \).

At this point, it is useful to recall the wavelet transform \( W_\varphi f \) of a signal \( f \) in \( L^2(\mathbb{R}) \) with respect to the window \( \varphi \)
\[
(W_\varphi f)(b, a) = \int_{\mathbb{R}} f(t) |a|^{-1/2} \varphi(\cdot - b/a) dt, \quad \forall b, a \in \mathbb{R}.
\]
See for example [3, 9, 23] for details on wavelet analysis and filter banks.

**Theorem 5.** Let \( \varphi \) be a window in \( L^2(\mathbb{R}) \) such that
\[
c_\varphi = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty.
\]
We say that \( \varphi \) is an admissible wavelet and we call \( c_\varphi \) the admissibility constant. Then,
\[
c_\varphi \left( f, f' \right)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(b, a) \overline{(W_\varphi f')(b, a)} db \frac{da}{a^2},
\]
for all \( f \) and \( f' \) in \( L^2(\mathbb{R}) \).

Notice the similarities between Theorem 4 and Theorem 5. This follow from a deep connection among Stockwell transform, short-time Fourier transform and wavelet transform. In fact, these transforms are related to the affine Weyl-Heisenberg group studied in [21]. This connection has been highlighted in the multi-dimensional case by the second author in [25]. In [15, 30], the connections between Stockwell transform and wavelet transforms are pointed out. The affine Weyl-Heisenberg group is
Figure 1. DOST basis functions in increasing frequency \(p\)-bands. Black line = real, red line = imaginary. See Figure 2 in [27] for a comparison.

also connected to the definition of \(\alpha\)-modulation spaces, see [2 8 14], which represents, at the level of coorbit theory, a sort of interpolation between Modulation spaces and Besov spaces. A different group approach to the Stockwell transform has been studied in [6].

3. A Time-Frequency Localized Basis

In this section, we prove that the system of functions (1.6), proposed by R.G. Stockwell in [27], is indeed an orthonormal basis of \(L^2([0,1])\).

For \(p = 0\), we set
\[
\nu(0) = 0, \quad \beta(0) = 1, \quad \tau(0) = 0,
\]
for \(p = 1\)
\[
\nu(1) = 1, \quad \beta(1) = 1, \quad \tau(1) = 0,
\]
for all \(p \geq 2\)
\[
\nu(p) = 2^{p-1} + 2^{p-2}, \quad \beta(p) = 2^{p-1}, \quad \tau(p) = 0, \ldots, \beta(p) - 1.
\]
Setting, for each \(p\), the \(p\)-frequency band
\[
[\beta(p), 2\beta(p) - 1] = \left[\nu(p) - \frac{\beta(p)}{2}, \nu(p) + \frac{\beta(p)}{2} - 1\right],
\]
we obtain a partition of \(\mathbb{N}\); notice that \(\nu(p)\) is the center of each \(p\)-frequency band.

We recall here the definition of the so-called DOST functions, introduced in [27]
\[
D_0(t) = 1, \quad t \in \mathbb{R},
\]
\[
D_1(t) = e^{2\pi i t}, \quad t \in \mathbb{R},
\]
and
\[
D_p(t) = \{D_{p,\tau}(t)\}_{\tau=0,\ldots,\beta(p)-1}, \quad t \in \mathbb{R},
\]
Figure 2. DOST basis functions in the same $p$-band ($p = 5$). Black line = real, red line = imaginary. See Figure 1 in [27] for a comparison.

where

$$D_{p, \tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{2 \pi i ft} e^{-2 \pi i ft/\beta(p)}, \quad t \in \mathbb{R}. $$

We set for all negative integers $p$

$$D_{p, \tau}(t) = D_{-p, \tau}(t), \quad \tau = 0, \ldots, \beta(|p|) - 1. $$

For each $p \in \mathbb{N}$, $\nu(-p) = -\nu(p)$ and $\beta(-p) = -\beta(p)$. In the following we will call

$$(3.1) \bigcup_{p \in \mathbb{Z}} D_p$$

Stockwell basis.

Notice that, in the original paper of R. G. Stockwell [27], each $D_{p, \tau}$ had a multiplicative factor $e^{\tau \pi i}$. Since this factor is not crucial in proving that (3.1) is a basis of $L^2([0, 1])$, we have decided to drop it. In (5.14), we clarify the role of this multiplicative factor. In Figure 1 and Figure 2 we draw the DOST basis functions without this multiplicative factor. In Figure 2, notice that, with our choice, these functions are self-similar in each $p$-band, in contrast to the ones defined in [27]. Moreover, we have slightly changed the notation in the frequency domain. The $k$-element of the Fourier basis is $e^{2 \pi i kt}$, while, in the original paper, the $k$-element is $e^{-2 \pi i kt}$. The convention we adopt seems more transparent when compared to the standard Fourier analysis.

Theorem 6. $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal basis of $L^2([0, 1])$.

Proof. In the following we will consider positive $p$. For negative $p$, all results hold true using the adjoint property. We recall that $\{e^{2 \pi i kt}\}_{k \in \mathbb{Z}}$ is an orthonormal basis
of $L^2([0,1])$ and we notice that $D_{p,\tau}(t)$ is a finite linear combination of $e^{2\pi i kt}$ with $k$ in the $p$-frequency band

$$I_p = \left[ \nu(p) - \frac{\beta(p)}{2}, \nu(p) + \frac{\beta(p)}{2} - 1 \right].$$

Hence, we can conclude that

$$(D_{p,\tau}, D_{p',\tau'})_{L^2([0,1])} = 0, \quad \forall p, \tau',$$

since the $p$-band and the $p'$-band are disjoint. So, we can focus on the case $p = p'$. The proof is divided into three steps.

**Step I** - $\|D_{p,\tau}\|_{L^2([0,1])} = 1$.

Consider the inner product

$$\|D_{p,\tau}\|_{L^2([0,1])}^2 = (D_{p,\tau}, D_{p,\tau})_{L^2([0,1])}$$

$$= \frac{1}{\beta(p)} \left( \sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{2\pi i ft} e^{-2\pi ft/\beta(p)} \right) \left( \sum_{f'=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{-2\pi f't} e^{2\pi f't/\beta(p)} \right) dt.$$}

Since $\{e^{2\pi i kt}\}_{k \in \mathbb{Z}}$ is an orthonormal basis,

$$\|D_{p,\tau}\|_{L^2([0,1])}^2 = \frac{1}{\beta(p)} \sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} \int_{-1}^{1} dt = 1.$$}

**Step II** - $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal set.

If $p \neq p'$ the $L^2$-scalar product vanishes, so we can suppose $p = p'$. It is convenient to consider $j = f - \beta(p)$.

$$D_{p,\tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} e^{-2\pi i (\beta(p)+j)\tau/\beta(p)}$$

$$= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} e^{-2\pi i \tau j/\beta(p)}$$

$$= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} e^{-2\pi i \tau j/\beta(p)}.$$}

Using the orthonormality of the Fourier basis, we obtain

$$(D_{p,\tau}, D_{p,\tau'})_{L^2([0,1])} = \frac{1}{\beta(p)} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\tau' - \tau)j/\beta(p)}.$$}

Now, we need the following lemma.

**Lemma 6.1.** Let $k \in \mathbb{N}\setminus\{0\}$. Then

$$\sum_{j=0}^{2^k-1} e^{2\pi i jm/2^k} = 0, \quad m \in \mathbb{N}\setminus\{0\}.$$}

**Proof.** We notice that for $m = 1$

$$e^{2\pi i j/2^k} = -1 e^{2\pi i (j+2^k-1)/2^k} = -e^{2\pi i (j+2^{k-1})/2^k}, \quad j = 0, \ldots, 2^{k-1} - 1.$$}

So, (3.4) is zero. Notice that the $(2^k)$-roots of the unity represent the vertex of a $2^k$-polygon which is symmetric with respect to the origin, hence their sum vanishes.
If $m$ is odd we can repeat the same argument, since (3.5) implies
\[
(e^{2\pi i j/2^k})^m = -(e^{(2\pi i j + 2^{k-1})/2^k})^m.
\]
Let $m = 2^h h$ (h odd). We suppose now $h = 1$, we have
\[
e^{2\pi i j/2^{k-\tau}} = e^{2\pi i (j+2^{k-\tau})/2^k}.
\]
Hence, (3.4) turns into
\[
2^k \sum_{j=0}^{2^k-1} e^{2\pi i j/2^k} = 2^k \sum_{j=0}^{2^k-1} e^{2\pi i j/(2^{k-\tau})}.
\]
Thanks to (3.5), we have
\[
\sum_{j=0}^{2^k-1} e^{2\pi i j/2^{k-\tau}} = 0,
\]
and (3.4) follows immediately. If $h > 1$, in analogy with the case $m$ odd, we can prove the assertion.

Let $(\tau' - \tau) = m$ in (3.4), then Lemma 6.1 implies that
\[
(D_{0,\tau}, D_{0',\tau'})_{L^2([0,1])} = \delta_0(p - p')\delta_0(\tau - \tau'),
\]
i.e. $\bigcup_{p \in \mathbb{Z}} D_p$ is an orthonormal set.

**Step III -** $\bigcup_{p \in \mathbb{Z}} D_p$ is a basis of $L^2([0,1])$.
We use that
\[
D_p \subseteq \text{span}\{e^{2\pi i kt}\}_{k \in \{\beta(p), 2\beta(p) - 1\}}.
\]
Hence, to prove the assertion it is sufficient to show that the elements of the set $\{D_{p,\tau}\}_{\tau = 0, \ldots, \beta(p) - 1}$ are a basis of $\text{span}\{e^{2\pi i kt}\}_{k \in \{\beta(p), 2\beta(p) - 1\}}$. Since we deal with finite dimensional vector spaces it is enough to prove that they are linearly independent. We show that
\[
\sum_{\tau = 0}^{\beta(p) - 1} c_\tau D_{p,\tau} = 0 \Longrightarrow c_\tau = 0, \quad \forall \tau = 0, \ldots, \beta(p) - 1.
\]
Since $\{e^{2\pi i (\beta(p) + j)}\}_{j = 0, \ldots, \beta(p) - 1}$ is a basis, we can consider the projection of (3.6) on each term $\{e^{2\pi i (\beta(p) + j)}\}_{j = 0, \ldots, \beta(p) - 1}$ of the Fourier basis. We obtain the system
\[
\sum_{\tau = 0}^{\beta(p) - 1} c_\tau e^{-2\pi i j/\beta(p)} = 0, \quad j = 0, \ldots, \beta(p) - 1.
\]
We prove that, for all $k = 0, \ldots, p - 1$,
\[
\sum_{q = 0}^{\beta(p)/2^k - 1} c_{2^k q + s} = 0, \quad s = 0, \ldots, 2^k - 1.
\]
If we set $k = p - 1$ in equation (3.5), recalling that $\beta(p) = 2^{p-1}$ (for $p > 2$), we obtain
\[
\sum_{q = 0}^{2^{p-1} - 2^{p-1} - 1} c_{2^{p-1} q + s} = c_s = 0, \quad s = 0, \ldots, 2^{p-1} - 1 = \beta(p) - 1,
\]
that is (3.6). We prove (3.8) by induction. For $k = 0, \ldots, p - 2$, we show that

$$
\beta(p)/2^{k - 1} \sum_{q=0}^{2^k - 1} c_{2^k q + s} = 0, \quad s = 0, \ldots, 2^k - 1
$$

implies

$$
\beta(p)/2^{k+1 - 1} \sum_{q=0}^{2^{k+1} - 1} c_{2^{k+1} q + s} = 0, \quad s = 0, \ldots, 2^{k+1} - 1.
$$

To prove the inductive base, we set $j = 0$ in (3.7), so we have

$$
\beta(p) - 1 \sum_{\tau=0}^{\beta(p) - 1} c_{\tau} = 0,
$$

that is (3.8) with $k = 0$. It is simpler to show how the inductive step works in this case. For $j = \beta(p)/2$ we have

$$
e^{-\frac{2\pi i \beta(p)}{2\pi}} = e^{-\tau \pi} = (-1)^\tau.
$$

Hence, (3.7) with $j = \beta(p)/2$, becomes

$$
\beta(p) - 1 \sum_{\tau=0}^{\beta(p) - 1} (-1)^\tau c_{\tau} = 0.
$$

If we sum (3.11) and (3.12) and then we subtract (3.11) with (3.12) we obtain

$$
\beta(p)/2 - 1 \sum_{q=0}^{\beta(p)/2 - 1} c_{2q} = 0
$$

(3.13)

$$
\beta(p)/2 - 1 \sum_{q=0}^{\beta(p)/2 - 1} c_{2q + 1} = 0.
$$

(3.14)

Relations (3.13), (3.14) are precisely (3.10) for $k = 0$.

For a general $k$ we suppose that (3.9) holds true. Then, equation (3.7), for $j = (2n + 1)\beta(p)/2^{k+1}$, $n = 0, \ldots, 2^k - 1$, becomes

$$
\beta(p) - 1 \sum_{\tau=0}^{\beta(p) - 1} e^{-2\pi i (2n+1)/2^{k+1}} c_{\tau} = 0.
$$

We can write the above equation as

$$
\sum_{s=0}^{2^{k+1} - 1} \sum_{q=0}^{\beta(p)/2^{k+1} - 1} e^{-2\pi i (2n+1)(q2^{k+1} + s)/2^{k+1}} c_{q2^{k+1} + s} = 0.
$$

Notice that

$$
e^{-2\pi i (2n+1)(q2^{k+1} + s)/2^{k+1}} = e^{-2\pi i (2n+1)\frac{q2^{k+1} + s}{2^{k+1}}} e^{-2\pi i (2n+1)\frac{2^{k+1}}{2^{k+1}}}
$$

$$
= e^{-2\pi i (2n+1)s/2^{k+1}}, \; q = 0, \ldots, \beta(p)/2^{k+1} - 1,
$$

and

$$
e^{-2\pi i (2n+1)(s+2^{k})/2^{k+1}} = e^{-2\pi i (2n+1)s/2^{k+1}} e^{-2\pi i (2n+1)/2}
$$

$$
= e^{-2\pi i (2n+1)s/2^{k+1}} e^{-(2n+1)\pi i}
$$

$$
= e^{-2\pi i (2n+1)s/2^{k+1}}.
$$
We obtain
\[
\sum_{\tau=0}^{\beta(p)-1} e^{-2\pi \tau(2n+1)/2^{k+1}}
\]
\[
= \sum_{s=0}^{2^k-1} e^{-2\pi (2n+1)s/2^{k+1}} \sum_{q=0}^{\beta(p)/2^{k+1}-1} \left(c_{2q^{k+1}+s} - c_{2q^{k+1}+2s+2^k}\right)
\]
\[
= \sum_{s=0}^{2^k-1} e^{-2\pi (2n+1)s/2^{k+1}} \sum_{q=0}^{\beta(p)/2^{k+1}-1} \left(c_{2q^{k+1}+s} - c_{(2q+1)2^k+s}\right)
\]
\[
= \sum_{s=0}^{2^k-1} e^{-2\pi (2n+1)s/2^{k+1}} \sum_{q=0}^{\beta(p)/2^{k-1}-1} \left(-1\right)^q c_{2^k q+s}
\]
\[(3.15)\]
\[
\]
We can now sum with respect to \( n \) getting
\[
\sum_{n=0}^{2^k-1} \sum_{s=0}^{2^k-1} e^{-2\pi i ns/2^k} e^{-2\pi i s/2^{k+1}} \sum_{q=0}^{\beta(p)/2^k-1} \left(-1\right)^q c_{2^k q+s}
\]
\[
= \sum_{n=0}^{2^k-1} e^{-2\pi i ns/2^k} \sum_{q=0}^{\beta(p)/2^k-1} \left(-1\right)^q c_{2^k q+s}
\]
\[(3.16)\]

Lemma [6.1] implies, for \( s \neq 0 \),
\[
\sum_{n=0}^{2^k-1} e^{-2\pi i ns/2^k} = 0.
\]

Hence, \[(3.16)\] simplifies into
\[
\sum_{q=0}^{\beta(p)/2^k-1} \left(-1\right)^q c_{2^k q} = 0.
\]

Then, multiplying \[(3.15)\] for \( e^{2\pi i n/2^k} \) and summing as in \[(3.16)\], via the same argument as before, we obtain
\[
\sum_{q=0}^{2^k-1} \left(-1\right)^q c_{2^k q+1} = 0.
\]

By induction, multiplying at each step \[(3.16)\] for \( e^{2\pi i m/2^k} \) \( (m = 2, \ldots, 2^k - 1) \), we get
\[
\sum_{q=0}^{\beta(p)/2^k-1} \left(-1\right)^q c_{2^k q+s} = 0, \quad s = 0, \ldots, 2^k - 1.
\]

Now, as in the model case of \( k = 0 \), we sum and then subtract \[(3.9)\] and \[(3.17)\] for each \( s \) and we obtain equation \[(3.10)\]. The inductive step is proven. So, \( \{D_{\nu(p),\beta(p),\tau}(t)\}_{\tau=0,\ldots,\beta(p)-1} \) is a basis for the finite dimensional vector space generated by \( \{e^{2\pi i k t}\}_{k=0}^{\beta(p)-1} \) for each \( p \).

Lemma [6.1] implies the following corollary.

**Corollary 6.1.** For each \( p \in \mathbb{Z} \) and each \( \tau, \tau' = 0, \ldots, |\beta(p)| - 1 \) we have
\[
D_{p,\tau} \left( \frac{\tau'}{\beta(p)} \right) = \delta_0(\tau' - \tau).
\]
Proof. We check that

\[ D_{p,\tau} \left( \frac{\tau}{\beta(p)} \right) = 1. \]

Let us suppose \( p \) positive. Then,

\[ D_{p,\tau} \left( \frac{\tau'}{\beta(p)} \right) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j) \left( \frac{\tau'-\tau}{\beta(p)} \right)} \]

\[ = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i j \frac{(\tau'-\tau)}{\beta(p)}}. \]

Letting \((\tau'-\tau) = m\) we apply Lemma 6.1 and we obtain the assertion. For negative \( p \) we use the adjoint property. \( \square \)

It is clear that the DOST function are not dilations nor translations of a single functions. Nevertheless, for each \( p \),

\[ D_p \left( t \right) = \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)(t-\tau/\beta(p))} \right\}_{\tau=0,\ldots,\beta(p)-1} \]

is formed by translation of \( \tau/\beta(p) \) of the same function. Roughly speaking, we can state that the DOST basis is non self similar globally, but it is self similar in each band, see Figure 2. Hence, the S-transform in this setting appears different from the wavelet transform because the mother wavelet changes as the frequencies increases, in contrast to the usual formulation.

R. G. Stockwell proposed this basis because it is an efficient compromise between frequency localization in low frequencies and time localization for high frequencies. The price to pay is that, on one hand, for high frequencies, we do not have a precise frequency localization, but just a localization in a certain band, which is wider as the frequency increases and, on the other hand, in low frequencies, we lose time localization. The key of the time localization for high frequencies is that the basis \( D_{p,\tau} \) are, in large sense, local at \( t = \tau/\beta(p) \). This sentence must be understood in density terms, because it is not true that the function has compact support in time, but the mass is concentrated near the point \( t = \tau/\beta(p) \). We prove that basis \( D_{p,\tau} \) are 0.85-concentrated in the neighborhood

\[ I_{p,\tau} = \left[ \frac{\tau}{\beta(p)} - \frac{1}{2\beta(p)}, \frac{\tau}{\beta(p)} + \frac{1}{2\beta(p)} \right], \]

in the sense of the Donoh-Stark Theorem [11, 5].

**Proposition 7.** For each \( D_{\tau,\tau}(t) \) we have

\[ |D_{\tau,\tau}|_{L^2(I_{\tau,\tau})} = \left( \int_{\frac{\tau}{2\beta(p)}-\frac{1}{2\beta(p)}}^{\frac{\tau}{2\beta(p)}+\frac{1}{2\beta(p)}} |D_{\tau,\tau}(t)|^2 dt \right)^{1/2} > 0.85, \]

i.e. the \( L^2 \)-norm is concentrated in the interval

\[ I_{\tau,\tau} = \left[ \frac{\tau}{\beta(p)} - \frac{1}{2\beta(p)}, \frac{\tau}{\beta(p)} + \frac{1}{2\beta(p)} \right]. \]

Since \( \|D_{\tau}\| = 1 \), we can also state that the \( L^2 \)-norm of \( D_{\tau,\tau} \) is less that 0.15 out of \( I_{\tau,\tau} \). For \( \tau = 0 \), \( I_{\tau,0} \) must be considered as an interval in circle, that is

\[ I_{\tau,0} = \left[ 0, \frac{1}{2\beta(p)} \right) \cup \left( \frac{2\beta(p)-1}{2\beta(p)}, 1 \right]. \]
Proof. Since in each $p$-band the basis functions are a translation of $\tau/\beta(p)$ of the same function, we can prove the property for a fixed $\tau$. For simplicity, we consider $\tau = 0$. In order to take in account just one integral we extend by periodicity the function for negative $t$ and we evaluate

\begin{equation}
\int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} |D_{p,\tau}(t)|^2 \, dt.
\end{equation}

Notice that

\begin{equation}
|D_{p,0}|^2 = D_{p,0}(t) \cdot \overline{D_{p,0}(t)}
= \frac{1}{\beta(p)} \left( \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)t} \right) \cdot \left( \sum_{k=0}^{\beta(p)-1} e^{-2\pi i (\beta(p)+k)t} \right)
= \frac{1}{\beta(p)} \sum_{m=-\beta(p)+1}^{\beta(p)-1} (\beta(p) - |m|) e^{2\pi i mt}.
\end{equation}

Equation 3.19 can be proven by induction on the size of the band. Writing 3.19 in terms of cosine and sine we obtain

\begin{align*}
|D_{p,0}|^2
&= \frac{1}{\beta(p)} \sum_{m=-\beta(p)+1}^{\beta(p)-1} (\beta(p) - |m|)(\cos(2\pi mt) + i \sin(2\pi mt))
&= 1 + \frac{1}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) (\cos(2\pi mt) + \cos(-2\pi mt)) + (\sin(2\pi mt) + \sin(-2\pi mt))
&= 1 + \frac{2}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \cos(2\pi mt).
\end{align*}

So, we have

\begin{align*}
\int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} |D_{p,0}(t)|^2 \, dt
&= \int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} dt + \frac{2}{\beta(p)} \int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \cos(2\pi mt) \, dt
&= \frac{1}{\beta(p)} + \frac{2}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \frac{\sin(2\pi mt)}{2\pi m} \bigg|_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}}
&= \frac{1}{\beta(p)} + \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \frac{\sin(2\pi m)}{2\pi m}.
\end{align*}

By the Maclaurin expansion of $\sin(x)$

\begin{equation}
\int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} |D_{p,0}|^2 \, dt = \frac{1}{\beta(p)} + \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m) \left( \frac{2\pi m}{2\pi(p)} + R_m(q) \right).
\end{equation}

Where $R_m(q)$ is the Lagrange rest. Using Leibniz summation formula we get

\begin{equation}
\int_{-\frac{1}{2\pi(p)}}^{\frac{1}{2\pi(p)}} |D_{p,0}(t)|^2 \, dt \approx \frac{1}{\beta(p)} + \frac{2}{\beta^2(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m)
\end{equation}
we have

\[ \frac{1}{\beta(p)} + \frac{2}{\beta^2(p)} \left( \beta(p)(\beta(p) - 1) - \frac{1}{2} \beta(p)(\beta(p) - 1) \right) \]

\[ = \frac{1}{\beta(p)} + \frac{1}{\beta(p)}(\beta(p) - 1) = 1. \]

Hence, we obtain

\[ \text{We have to take into account the rests } R_m(\eta). \text{ Since} \]

\[ \sup \left| \frac{d^3}{dt^3} \left[ \sin(2\pi mt) \right] \right| = (2\pi m)^3, \]

we have

\[ \frac{4}{\beta(p)}\sum_{m=1}^{\beta(p)-1} \left| \frac{(\beta(p) - m) R_m(\eta)}{2\pi m} \right| \]

\[ \leq \frac{4}{\beta(p)} \sum_{m=1}^{\beta(p)-1} \frac{(\beta(p) - m)(2\pi m)^3}{2\pi m (2\beta(p))^3} \]

\[ \leq \frac{\pi^2}{3\beta^4(p)} \sum_{m=1}^{\beta(p)-1} (\beta(p) - m)m^2 \]

\[ \leq \frac{\pi^2}{3\beta^4(p)} \left( \frac{\beta(p)^2}{6}(\beta(p) - 1)(2\beta(p) - 1) - \frac{\beta(p)^2}{4}(\beta(p) - 1)^2 \right) \]

\[ \leq \frac{\pi^2}{3\beta^4(p)} \left( \frac{\beta(p) - 1}{3} - \frac{1}{6} \frac{\beta(p) - 1}{4} \right) \]

\[ \leq \frac{\pi^2}{36} \frac{\beta(p)^2 - 1}{\beta(p)^2} \]

\[ \leq \frac{\pi^2}{36} < 0, 275. \]

Hence, we obtain

\[ \left( \int \frac{1}{\pi\beta(p)} \left| D_{p,0}(t) \right|^2 dt \right)^{1/2} \geq (1 - 0, 275)^{1/2} \geq \sqrt{0.725} > 0, 85. \]

\[ \Box \]

4. **Diagonalization of the S-transform**

In this section, for the sake of clarity, we write $S_{\varphi}$-transform instead of S-transform to emphasize the window dependence. For each window $\varphi \in \mathcal{S}(\mathbb{R})$, the $S_{\varphi}$-transform is continuous from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R} \times \mathbb{R})$. By duality, the continuity of $S_{\varphi} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R} \times \mathbb{R})$ follows. In the following, we deal with periodic signals. Hence, we restrict our attention to $L^2([0, 1])$. Moreover, we consider windows with less regularity, namely $\hat{\varphi} \in L^2(\mathbb{R}) \cap C_{pwc}(\mathbb{R})$, where $C_{pwc}(\mathbb{R})$ is the set of piecewise continuous functions.

We consider the space $X$ as the extension by periodicity of $L^2([0, 1])$ in $\mathcal{S}(\mathbb{R})$. Using Fourier series, it is well known that if $f \in L^2([0, 1])$, then

\[ f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{2\pi i kt}, \quad t \in [0, 1], \quad a.e., \]

and

\[ \|f\|_{L^2([0, 1])} = \left( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \right)^{1/2}. \]
Therefore, we can define the Hilbert space \((X, (\cdot, \cdot)_X, \|x\|_X)\), where

\[
X = \left\{ \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t} \mid c_k \in \mathbb{C}, \sum_{k \in \mathbb{Z}} |c_k|^2 < \infty \right\} \subset \mathcal{S}'(\mathbb{R}),
\]

\[(f, f')_X = \int_0^1 f(t)f'(t)dt = \langle f|_{[0,1]}, f'|_{[0,1]} \rangle_{L^2([0,1])}, \quad f, f' \in X,
\]

\[
\|f\|_X = \sqrt{(f, f)_X} = \left( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}, \quad f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{2\pi i k t}, \quad \text{a.e.}.
\]

Clearly, if \(f \in X\) then \(f|_{[0,1]} \in L^2([0,1])\) and \(\|f\|_X = \|f|_{[0,1]}\|_{L^2([0,1])}\).

Analogously, we can define the Hilbert space \((Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)\)

\[
Y = \left\{ \sum_{k \in \mathbb{Z}} c_k(\xi)e^{2\pi i (k-\xi)b} \mid c_k(\xi) \in \mathbb{R}^2 \left( \mathbb{R}, \frac{1}{|\xi|} \right), \quad \text{and} \quad \sum_{k \in \mathbb{Z}} |c_k|^2_{L^2(\mathbb{R}, \frac{1}{|\xi|})} < \infty \right\},
\]

\[(g, g')_Y = \int_0^1 \int_{\mathbb{R}} g(b, \xi)\bar{g}'(b, \xi)\frac{d\xi}{|\xi|}db, \quad g, g' \in Y,
\]

\[
\|g\|_Y = \left( (g, g)_Y \right)^{\frac{1}{2}} = \left( \sum_{k \in \mathbb{Z}} |g_k|^2_{L^2(\mathbb{R}, \frac{1}{|\xi|})} \right)^{\frac{1}{2}}, \quad g(b, \xi) = \sum_{k \in \mathbb{Z}} g_k(\xi)e^{2\pi i (k-\xi)b}, \quad \text{a.e.}.
\]

**Theorem 8.** Let \(\varphi\) be an admissible window for the \(S_{\varphi}\)-transform. That is, following Theorem 4, \(\varphi\) is such that

\[
(4.1) \quad c_{\varphi} = \int |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|1 + \xi|} < \infty.
\]

Moreover, assume that \(\hat{\varphi} \in L^2(\mathbb{R}) \cap C_{pwc}(\mathbb{R})\). Then

\[
S_{\varphi} : X \longrightarrow Y
\]

\[
\sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t} \longmapsto \sum_{k \in \mathbb{Z}} c_k \left( S_{\varphi} \left( e^{2\pi i k t} \right) \right) (b, \xi)
\]

is continuous.

**Proof.** We start considering the \(S_{\varphi}\)-transform of \(e^{2\pi i k t}\). By Proposition 3 and the assumption \(\hat{\varphi} \in C_{pwc}(\mathbb{R})\), we have

\[
\left( S_{\varphi} e^{2\pi i k t} \right) (b, \xi) = e^{-2\pi i k b} \hat{\varphi} \left( \frac{\xi - \xi}{\xi} \right) \delta_k(\xi) (b)
\]

\[
= e^{-2\pi i k \xi} \hat{\varphi} \left( \frac{k - \xi}{\xi} \right) \delta_k(\xi) (b)
\]

\[
= e^{-2\pi i k b} \hat{\varphi} \left( \frac{k - \xi}{\xi} \right) e^{2\pi i k b}
\]

\[
= e^{2\pi i b(k - \xi)} \hat{\varphi} \left( \frac{k - \xi}{\xi} \right).
\]

Then

\[
\left\| \hat{\varphi} \left( \frac{k - \cdot}{\xi} \right) \right\|_{L^2([0,1], \frac{1}{|\xi|})}^2 = \int_{\mathbb{R}} \left| \hat{\varphi} \left( \frac{k - \xi}{\xi} \right) \right| \frac{1}{|\xi|} d\xi
\]

\[
= \int_{\mathbb{R}} \left| \hat{\varphi}(\omega - 1) \right|^2 \frac{1}{|\omega|} \frac{|k|}{|\omega|^2} d\omega.
\]
Using the definition of the $S_{\varphi}$-transform and equation (4.3), we have that
\[
\hat{S}_{\varphi} f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k t}, \quad \text{a.e.} \quad t \in \mathbb{R},
\]
then
\[
\| (S_{\varphi} f, \cdot ) \|_Y^2 = \sum_{k \in \mathbb{Z}} \| \hat{f}(k) \|_Y^2 \| S_{\varphi} e^{2\pi i k \cdot} \|_Y^2
\]
\[
= c_{\varphi} \| f \|_X^2.
\]
Therefore, the $S_{\varphi}$-transform is a continuous operator, and, apart from the multiplicative parameter $\sqrt{c_{\varphi}}$, it is an isometry. \hfill \square

**Remark 9.** Theorem 8 is the discrete counterpart of Theorem 5 in the case of periodic functions.

In Section 3, we proved that the DOST functions are a basis of $L^2([0, 1])$. Therefore, their periodic extensions are a basis of $X$. Let us assume that $\varphi$ satisfies the hypothesis of Theorem 5. Then, by Theorem 8, we have that $S_{\varphi} : X \to Y$ is continuous. So, we can write
\[
(S_{\varphi} f)(b, \xi) = \left( S_{\varphi} \sum_{j=0}^{\beta(p)} e^{2\pi i \beta(p)(t - \tau/\beta)} \right)(b, \xi)
\]
(4.4)
where
\[
f_{p, \tau} = (f, D_{p, \tau})_{L^2([0, 1])}
\]
and the sum in (4.4) is over all $D_{p, \tau}$ functions. Hence, in order to understand the $S_{\varphi}$-transform of a general function $f \in L^2([0, 1])$, it is enough to evaluate the coefficients $f_{p, \tau}$ and determine once for all the $S_{\varphi}$-transform of $D_{p, \tau}$.

Notice that
\[
D_{p, \tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i \left( \frac{\beta(p)+j}{\beta(p)} \right)(t-\tau/\beta)}
\]
\[
= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} T_{-\tau/\beta(p)} M_{\beta(p)+j} 1(t),
\]
To keep the notation simple, we set, for each fixed window \( c \)

\[
(F D_{p,\tau}) (\xi) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (F T_{-\tau/\beta(p)} M_{(\beta(p)+j)} \mathbb{1}) (\xi) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (M_{-\tau/\beta(p)} T_{-\beta(p)-j} \mathbb{1}) (\xi) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} (M_{-\tau/\beta(p)} T_{-\beta(p)-j} \delta_0) (\xi) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{\beta(p)+j}{\beta(p)} \delta_0} (\xi - \beta(p) - j) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{\beta(p)+j}{\beta(p)} \delta_0} \delta_{\beta(p)+j}(\xi).
\]

(4.5)

Hence, we can write

\[
Hence, (4.6) simplifies into
\]

\[
(\phi)\frac{(4.6)}{2} \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{\beta(p)+j}{\beta(p)} \delta_0} \delta_{\beta(p)+j}(\xi).
\]

(4.6)

Let us compute the \( S_\varphi \)-transform of a basis function \( D_{p,\tau} \) with a general window \( \varphi \). Using Proposition \( \frac{3}{2} \) by (4.5) we obtain

\[
e^{2\pi i \frac{b}{\beta(p)}} (S_\varphi D_{p,\tau}) (b, \xi)
\]

\[
= F_{\xi \rightarrow b}^{-1} \left( \hat{\varphi} \left( \frac{\xi - \beta(p)}{\xi} \right) (F D_{p,\tau}) (\xi) \right) (b)
\]

\[
= F_{\xi \rightarrow b}^{-1} \left( \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{(\beta(p)+j) \xi}{\beta(p)}} \hat{\varphi} \left( \frac{\xi - \beta(p)}{\xi} \right) \delta_{\beta(p)+j}(\xi) \right) (b)
\]

\[
= F_{\xi \rightarrow b}^{-1} \left( \sum_{j=0}^{\beta(p)-1} e^{-2\pi i \frac{(\beta(p)+j) \xi}{\beta(p)}} \hat{\varphi} \left( \frac{\beta(p) + j - \xi}{\xi} \right) \right) (b)
\]

(4.6)

To keep the notation simple, we set, for each fixed window \( \varphi \), the following functions

\[
c_{p,j}^\varphi : \mathbb{R} \rightarrow \mathbb{R}, \text{ given by}
\]

\[
c_{p,j}^\varphi (\xi) = \hat{\varphi} \left( \frac{\beta(p) + j - \xi}{\xi} \right).
\]

Hence, (4.6) simplifies into

\[
(4.7) \quad (S_\varphi D_{p,\tau}) (b, \xi) = e^{-2\pi i \frac{b}{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i \frac{(\beta(p)+j) b - \tau}{\beta(p)}} c_{p,j}^\varphi (\xi).
\]

Notice that, using (4.8), via (4.4), we get an explicit expression of the \( S_\varphi \)-transform of a periodic signal \( f \) in terms of its Stockwell coefficients \( f_{p,\tau} \).

**Remark 10.** Let \( \varphi \) be in \( \mathcal{S}(\mathbb{R}) \), which in general does not satisfy the admissibility condition (for example the Gaussian window). Then, \( X \rightarrow \mathcal{S}'(\mathbb{R}) \) is continuous. Hence, using the continuity of \( S_\varphi : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R} \times \mathbb{R}) \), we can repeat the same
argument as before in order to decompose the $S_{ϕ}$-transform of a periodic signal $f ∈ L^2 ([0, 1])$ into the DOST coefficients.

5. Discretization of the $S_{ϕ}$-transform

Considering a dyadic decomposition of the frequency domain (similar to the usual DOST), we evaluate the $S_{ϕ}$-transform of the periodic signal $f$ at $ξ = ν(p) = \frac{2}{3} β(p)$. Using equation (4.4), it is sufficient to evaluate $S_{ϕ}$ using the adjoint.

\begin{equation}
(5.1) \quad ⟨S_{ϕ}, D_{p', τ}⟩ (b, ν(p)) = e^{-2πi b ν(p)} \left( \sum_{j=0}^{β(p')-1} e^{2πi (β(p') + j)(t-τ β(p'))} c_{p', j}^x(ν(p)) \right).
\end{equation}

Clearly, it is crucial to understand the values $c_{p', j}^x(ν(p))$, which depend on the window $ϕ$ only. We consider the set of windows $ϕ$ such that

\begin{equation}
(5.2) \quad 2 \hat{ϕ} (ξ) = 0, \quad ξ ∈ \mathbb{R} \setminus \left[ -\frac{1}{3}, \frac{1}{3} \right].
\end{equation}

**Proposition 11.** Let $ϕ$ satisfy condition (5.2). Then

\begin{equation}
(5.3) \quad c_{p', j}^x(ν(p)) = 0, \quad ∀ j = 0, \ldots, β(p') - 1 \quad \text{if} \quad p' ≠ p.
\end{equation}

**Proof.** We restrict ourselves to positive $p'$. For $p' < 0$, it suffices to consider the adjoint.

Recalling the definition of $β(p')$ and $ν(p')$ and (5.2), we get

\begin{equation}
(5.4) \quad \frac{β(p') + j - ν(p)}{ν(p)} = \frac{β(p') + j - ν(p)}{ν(p)} - 1 = \frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right).
\end{equation}

We check that, if $p ≠ p'$, then

\begin{equation}
(5.5) \quad \frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right) - 1 ≠ \left[ -\frac{1}{3}, \frac{1}{3} \right], \quad ∀ j = 0, \ldots, β(p') - 1.
\end{equation}

Notice that, if $p$ is negative then

\begin{equation}
(5.6) \quad \frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right) < 0
\end{equation}

so

\begin{equation}
(5.7) \quad \frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right) - 1 < -1,
\end{equation}

hence (5.5) is fulfilled. We now consider $p > 0$. If $p ≠ p'$, then we have to consider two cases.

**Case I.** $p' < p$. By the definition of $β(p')$, we get $β(p') ≤ β(p)/2$ hence

\begin{equation}
\frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right) - 1 ≥ \frac{2}{3} \left( \frac{1}{2} + \frac{j}{β(p)} \right) - 1
\end{equation}

\begin{equation}
≤ -\frac{2}{3} + \frac{j}{3} β(p) ≤ -\frac{2}{3} + \frac{2 β(p') - 1}{3} β(p) ≤ -\frac{2}{3} + \frac{1}{3} - \frac{2}{3} β(p) ≤ -\frac{1}{3}.
\end{equation}

**Case II.** $p' > p$. We have $β(p) ≤ β(p')/2$, so we can write

\begin{equation}
\frac{2}{3} \left( \frac{β(p')}{β(p)} + \frac{j}{β(p)} \right) - 1 ≥ \frac{2}{3} \left( 2 + \frac{j}{β(p)} \right) - 1 ≥ \frac{1}{3} + \frac{j}{β(p)} ≥ \frac{1}{3}.
\end{equation}

Therefore, (5.5) is fulfilled in both cases. □
Let $\varphi$ satisfy condition (5.2). Then, using Proposition 11, the expression (5.1) assumes a simplified form since it vanishes for all $p' \neq p$. When $p = p'$ we have

$$
(5.8) \quad (S_{\varphi} D_{p, \tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} \left( \sum_{j=0}^{\beta(p)-1} \frac{c_{p,j}^{\nu}(\nu(p))}{\sqrt{\beta(p)}} \right).
$$

Assume that $c_{p,j}^{\nu}(\nu(p)) = 1$ for all $j = 0, \ldots, \beta(p) - 1$, then, via (5.8)

$$
(5.9) \quad (S_{\varphi} D_{p, \tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p, \tau}(b).
$$

In order to extend (5.9) to all $D_{p, \tau}$ we give the following proposition.

**Proposition 12.** Set $\varphi$ such that

$$
\bar{\varphi}(\xi) = \chi[-\frac{1}{2}, \frac{1}{2}](\xi) = \begin{cases} 1, & \xi \in [-\frac{1}{2}, \frac{1}{2}) \\ 0, & \text{otherwise} \end{cases}.
$$

Then

$$
(S_{\varphi} D_{p, \tau})(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p, \tau}(b), \quad \forall p \in \mathbb{Z}.
$$

**Proof.** It follows from the definition of $c_{p,j}^{\nu}$. \hfill \Box

For notation simplicity, we set

$$
(5.10) \quad \tilde{\chi}(x) = \left( F_{\xi \rightarrow x} \chi[-\frac{1}{2}, \frac{1}{2}] \right)(x).
$$

Using Proposition 12 and (4.4), we obtain

$$
(5.11) \quad (S_{\tilde{\chi}} f)(b, \nu(p)) = e^{2\pi i \nu(p)b} \sum_{\tau = 0}^{\beta(p)-1} f_{p, \tau} D_{p, \tau}(b).
$$

**Proposition 13.** Let $f$ be a periodic signal. Then

$$
(5.12) \quad (S_{\tilde{\chi}} f) \left( \frac{\tau}{\beta(p)}, \nu(p) \right) = (-1)^{\tau} f_{p, \tau}, \quad \tau = 0, \ldots, \beta(p) - 1.
$$

**Proof.** Equation (5.11) gives a precise evaluation of the $S_{\tilde{\chi}}$-transform of $f$ at $(\tau/\beta(p), \nu(p))$. We have

$$
(5.13) \quad (S_{\tilde{\chi}} f) \left( \frac{\tau}{\beta(p)}, \nu(p) \right) = \sum_{\tau' = 0}^{\beta(p)-1} (-1)^{\tau'} f_{p, \tau'} D_{p, \tau'} \left( \frac{\tau}{\beta(p)} \right).
$$

In Corollary 6.1 we proved that

$$
D_{p, \tau} \left( \frac{\tau'}{\beta(p)} \right) = \delta_{0}(\tau - \tau').
$$

Hence, it is sufficient to consider $\tau = \tau'$ in (5.13), that is

$$
(5.14) \quad (S_{\tilde{\chi}} f) \left( \frac{\tau}{\beta(p)}, \nu(p) \right) = e^{-2\pi i \nu(p) \tau / \beta(p)} f_{p, \tau} D_{p, \tau} \left( \frac{\tau}{\beta(p)} \right).
$$

Notice that $\nu(p) = 3/2 \beta(p)$, we have

$$
(-1)^{\tau} e^{-2\pi i \nu(p) \tau / \beta(p)} = e^{-3\pi i \tau} = (-1)^{\tau}.
$$

Finally,

$$
(5.14) \quad (S_{\tilde{\chi}} f) \left( \frac{\tau}{\beta(p)}, \nu(p) \right) = (-1)^{\tau} f_{p, \tau}.
$$

\hfill \Box
Figure 3. $E^{\varphi}_{p,\tau}$ basis functions in increasing frequency $p$-bands. Black line = real, red line = imaginary. $\hat{\varphi}$ is a Gaussian window with $\mu = 0$ and $\sigma = 1$. Notice the similarities with Figure 1. Indeed, in this case the ratio $(\delta/M)^2$ (see Theorem 19) is approximately 0.8836.

Equation (5.14) clarifies the representation of the S-transform of a periodic signal $f$ via the Stockwell coefficients $f_{p,\tau}$. Moreover, it explains the role of the multiplicative factor $(-1)^\tau$ in front of the basis functions $D_{p,\tau}$ used by R. G. Stockwell in [27].

Remark 14. In the paper we have always considered a symmetric partition of the frequency from the positive and negative side. Actually, the algorithm is slightly different: see [31, 32, 33] for details.

6. Window Adapted Basis Construction

In this section we determine a basis of $L^2([0,1])$ adapted to an admissible window $\varphi$. As explained in the introduction, we want to find a basis $E_j^\varphi$ such that $S\varphi E_j^\varphi$ is local both in time and in frequency and such that the evaluations of all coefficients $f_j^\varphi = (f, E_j^\varphi)_{L^2([0,1])}$ is fast - $O(N \log N)$. In Section 3, we have proven that $D_{p,\tau}$ is a basis of $L^2([0,1])$ which is local both in time and frequency. Moreover, in Section 5, we have shown that the natural discretization of the time-frequency domain in this setting is given by the dyadic decomposition in the frequency domain and the $\tau/\beta(p)$ grid in the time domain. So, it is natural to change our task in finding a basis $E_j^\varphi$ such that

$$
(S\varphi E_j^\varphi)(b,\nu(p)) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b).
$$

In order to obtain (6.1), we have to assume some conditions on the window $\varphi$. In view of Proposition 11, we require that $\varphi$ satisfies condition (5.2). Moreover, we ask that for all $p \in \mathbb{Z}$

$$
\varphi_p^{\varphi}(\nu(p)) \neq 0, \quad \forall j = 0, \ldots, \beta(p) - 1.
$$
Theorem 15. Let $\varphi$ be a window satisfying conditions (5.2) and (6.2). Then, setting

$$E_{p,\tau}^{\varphi}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}(\nu(p))]^{-1} e^{2\pi i (\beta(p)+j)(t-\frac{\tau}{\pi p})},$$

we get

$$(S_{\varphi} E_{p,\tau}^{\varphi})(b, \nu(p)) = e^{-2\pi i b \nu(p)} D_{p,\tau}(b).$$

Moreover,

$$\bigcup_{p \in \mathbb{Z}} E_{p}^{\varphi},$$

where

$$E_{p}^{\varphi} = \{ E_{p,\tau}^{\varphi} \}_{\tau=0,\ldots,\beta(|p|)-1}$$

is a basis of $L^2([0,1])$.

Remark 16. Take $\tilde{\chi}$ as in (5.10), then we have

$$c_{p,j}(\nu(p)) = 1,$$

for all $p$ and $j$. So, by (3.2),

$$E_{p,\tau}^{\tilde{\chi}}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}(\nu(p))]^{-1} e^{2\pi i (\beta(p)+j)(t-\frac{\tau}{\pi p})}$$

$$= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi i (\beta(p)+j)(t-\frac{\tau}{\pi p})}$$

$$= D_{p,\tau}(t).$$
Hence $E_p^{\varphi}$ is a proper generalization of the DOST functions.

**Proof.** By equation (4.8), it follows that the functions $E_p^{\varphi}$ do satisfy (6.3). So, we only need to prove that

$$
\bigcup_{p \in \mathbb{Z}} E_p^{\varphi} \text{ is a basis of } L^2 ([0,1]).
$$

Notice that

$$
E_p^{\varphi} \subseteq \text{span } \{ e^{2\pi i k t} \}_{k \in [\beta(p), 2\beta(p) - 1]} = \text{span } \{ D_p, \tau \}_{\tau = 0, \ldots, \beta(p) - 1}.
$$

It is enough to check that $E_p^{\varphi}$ is a linear independent set. Let us assume that there exist $\{ \alpha_{\tau} \}_{\tau = 0}^{\beta(p) - 1}$ such that

$$
\sum_{\tau = 0}^{\beta(p) - 1} \alpha_\tau E_p^{\varphi}_{p, \tau} (t) = 0.
$$

Then, by (6.3) we obtain

$$
0 = \left( S_p \sum_{\tau = 0}^{\beta(p) - 1} \alpha_\tau E_p^{\varphi}_{p, \tau} \right) (b, \nu(p))
= \sum_{\tau = 0}^{\beta(p) - 1} \alpha_\tau \left( S_p E_p^{\varphi}_{p, \tau} \right) (b, \nu(p))
= \alpha_\tau e^{-2\pi i b \nu(p)} \sum_{\tau = 0}^{\beta(p) - 1} D_p, \tau (b).
$$

Hence,

$$
\alpha_\tau \sum_{\tau = 0}^{\beta(p) - 1} D_p, \tau (b) = 0.
$$

(6.4)
Using the explicit expression of the basis $O$
By Plancharel's Theorem we can write

**Proof.** By Plancharel's Theorem we can write

$$f_{p, \tau} = (f, E_{p, \tau})_{L^2([0,1])}$$

has computational complexity $O(N \log N)$, where $N$ is the length of $f$.

**Proposition 17.** Let $E_{p, \tau}$ as in Theorem 15 and let $f$ be a finite signal. Then the evaluation of the coefficients

$$f_{p, \tau} = (f, E_{p, \tau})_{L^2([0,1])} = \left( \hat{f}, E_{p, \tau}^\varphi \right)_{L^2(\mathbb{Z})}.$$ 

Using the explicit expression of the basis $E_{p, \tau}$ we obtain

$$f_{p, \tau} = \left( \hat{f}, \sum_{j=0}^{\beta(p)-1} \left[ c_{p,j}^\varphi (\nu(p)) \right]^{-1} e^{-2\pi i (\beta(p) + j)(\tau/\beta(p))} \delta_{\beta(p) + j}(\cdot) \right)_{L^2(\mathbb{Z})}$$

where $R^\varphi$ is a sequence in $\mathbb{Z}$ such that

$$R^\varphi(\beta(p) + j) = \left[ c_{p,j}^\varphi (\nu(p)) \right]^{-1}$$

for all $p$ and $j$. Hence,

$$f_{p, \tau} = (f, E_{p, \tau})_{L^2([0,1])} = \left( \hat{f}, D_{p, \tau} \right)_{L^2([0,1])}$$

where $\hat{f} = F^{-1} R^\varphi \hat{\varphi}$. Given $\hat{f}$, computing (6.6) using the FDOST-algorithm introduced in [33] has complexity $O(N \log N)$ and computing $\hat{f}$ via FFT has complexity $O(N \log N)$. So, the computation complexity remains $O(N \log N)$.

**Remark 18.** It is worth checking explicitly the computational complexity of the algorithm. To perform this task, we start evaluating the column vector $f_p^\varphi$ given by

$$f_p^\varphi = \{ f_{p, \tau} \}_{\tau=0}^{\beta(p)-1}$$

$$= \left\{ (f, E_{p, \tau})_{L^2([0,1])} \right\}_{\tau=0}^{\beta(p)-1}$$

$$= \left\{ \left( \hat{f}, E_{p, \tau}^\varphi \right)_{L^2(\mathbb{Z})} \right\}_{\tau=0}^{\beta(p)-1}$$

$$= \left\{ \left( \hat{f}, \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} [c_{p,j}^\varphi (\nu(p))]^{-1} e^{-2\pi i (\beta(p) + j)(\tau/\beta(p))} \delta_{\beta(p) + j}(\cdot) \right)_{L^2(\mathbb{Z})} \right\}_{\tau=0}^{\beta(p)-1}$$

$$= \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p) + j) [c_{p,j}^\varphi (\nu(p))]^{-1} e^{2\pi i (\beta(p) + j)(\tau/\beta(p))} \right\}_{\tau=0}^{\beta(p)-1}$$

Since $D_{p, \tau}$ is a basis, [6.4] implies that $\alpha_{\tau}$ are all zeros. That is, $E_{p, \tau}$ are linear independent. □
Figure 6. Decompositions of a given test signal on different windowed basis.

\[
\begin{align*}
\mathcal{F} \{ f(\tau) \} &= \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p) + j) \left[ e^{\beta \pi j (\tau/\beta(p))} \right]^{-1} e^{2\pi i j (\tau/\beta(p))} \\
&= \left( F_{\beta \to \tau}^{-1} \left( \left( R_{\beta} \hat{f} \right) \left( |\beta(p)\ldots, 2\beta(p)-1(j)| \right) \right) \right)(\tau)
\end{align*}
\]

where \( R_{\beta} \) is defined as in \( \text{[6.5]} \). Therefore, first we have to perform the FFT of the signal \( f(\tau) \mathcal{O}(N \log N) \), and the multiplication by \( R_{\beta} \mathcal{O}(N) \), then at each \( p \) band we need to use the FFT to perform the anti Fourier transform with computational complexity \( \mathcal{O}(\beta(p) \log \beta(p)) \). Summing up the contribution for each \( p \)-band we get the computational complexity of \( \mathcal{O}(N \log N) \).

The basis \( \{ \mathcal{E}^{\varphi}_{p,\tau} \}_{p,\tau} \) is in general not orthogonal nor normal. Nevertheless, we can normalize it setting

\[
\mathcal{E}^{\varphi}_{p,\tau}(t) = \frac{E^{\varphi}_{p,\tau}(t)}{\| E^{\varphi}_{p,\tau} \|_{L^2([0,1])}},
\]

so that

\[
\| E^{\varphi}_{p,\tau} \|_{L^2([0,1])} = 1.
\]

Notice that

\[
\| E^{\varphi}_{p,\tau} \|_{L^2([0,1])} = \| E^{\varphi}_{p,\tau} \|_{L^2([0,1])} = N_p^{\varphi}
\]

depends just on the \( p \)-band, not on \( \tau \).

\( \{ E^{\varphi}_{p,\tau}(t) \}_{p,\tau} \) fails in general to be orthogonal. Assuming a mild condition on the window \( \varphi \), we can prove that it is a frame.

**Theorem 19.** Let \( \varphi \) a window function satisfying condition \( \text{[5.3]} \) such that

\[
\inf_{\xi \in [-1/3,1/3]} |\hat{\varphi}(\xi)| \geq \delta > 0
\]
where

\[ \sup_{\xi \in \mathbb{Z}} | \tilde{\varphi}(\xi) | \leq M < \infty \]

then the basis \( \bigcup_{p \in \mathbb{Z}} \tilde{\mathcal{E}}^p \) is a frame of \( L^2([0,1]) \), where

\[ \tilde{\mathcal{E}}^p = \{ \tilde{E}^p_{\tau, \tau'} \}_{\tau=0, \ldots, \beta(p)} . \]

In particular,

\[ \left( \frac{\delta}{M} \right)^2 \| f \|_{L^2([0,1])}^2 \leq \sum_{p, \tau} \left\| \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])} \right\|^2 \leq \| f \|_{L^2([0,1])}^2 . \]

**Proof.** Theorem [15] proves that \( \bigcup_{p \in \mathbb{Z}} \tilde{\mathcal{E}}^p \) is a basis of \( L^2([0,1]) \), the same holds true for \( \bigcup_{p \in \mathbb{Z}} \tilde{\mathcal{E}}^p \). So, for all \( f \in L^2([0,1]) \), we can write

\[
\begin{align*}
\| f \|_{L^2([0,1])}^2 &= (f, f)_{L^2([0,1])} \\
&= \left( \sum_{p, \tau} \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])} \tilde{E}^p_{\tau, \tau'}, \sum_{p', \tau'} \left( f, \tilde{E}^{p'}_{\tau', \tau''} \right)_{L^2([0,1])} \tilde{E}^{p'}_{\tau', \tau''} \right)_{L^2([0,1])} \\
&\geq \sum_{p, \tau} \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])}^2 \left( \tilde{E}^p_{\tau, \tau'}, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])} \\
&= \sum_{p, \tau} \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])}^2 .
\end{align*}
\]

(6.11)

Notice that under the hypothesis (6.9), (6.10), by (6.8)

\[ \frac{1}{M} \leq N^\varphi_p \leq \frac{\delta}{\tilde{N}} \quad \forall p \in \mathbb{Z}. \]

Observe, by a slight variation of (6.6), that

\[ \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])} = \left( F^{-1} \tilde{R}^\varphi \hat{f}, D_{\tau, \tau'} \right)_{L^2([0,1])} \]

where \( \tilde{R}^\varphi \) is a sequence such that

\[ \tilde{R}^\varphi(\beta(p) + j) = \frac{R^\varphi(\beta(p) + j)}{N^\varphi_p} = \frac{[\varphi, \mu(p)]^{-1}}{N^\varphi_p} , \]

where \( N^\varphi_p \) is as in (6.8). If the window \( \varphi \) satisfies condition (6.10), by (6.12), we have

\[ \inf_{k \in \mathbb{Z}} \left\{ \left| \tilde{R}^\varphi(k) \right| \right\} \geq \frac{\delta}{M} > 0. \]

Hence, since \( \bigcup_{p \in \mathbb{Z}} D_p \) is an orthonormal basis and since \( F \) is a unitary operator from \( L^2([0,1]) \) to \( l^2(\mathbb{Z}) \), we obtain

\[
\begin{align*}
\sum_{p, \tau} \left| \left( f, \tilde{E}^p_{\tau, \tau'} \right)_{L^2([0,1])} \right|^2 &= \sum_{p, \tau} \left| \left( F^{-1} \tilde{R}^\varphi \hat{f}, D_{\tau, \tau'} \right)_{L^2([0,1])} \right|^2 = \| F^{-1} \tilde{R}^\varphi \hat{f} \|_{L^2([0,1])}^2 \\
\leq \| \tilde{R}^\varphi \hat{f} \|_{l^2(\mathbb{Z})}^2 &= \left( \inf_{k \in \mathbb{Z}} \left| \tilde{R}^\varphi(k) \right| \right)^2 \| \hat{f} \|_{l^2(\mathbb{Z})}^2 \leq \left( \frac{\delta}{M} \right)^2 \| f \|_{L^2([0,1])}^2 .
\end{align*}
\]

\[ \square \]
Remark 20. Notice that, when in equations (6.9) and (6.10) $\delta = M$, we a have a tight-frame. In the case of the DOST basis, i.e. $E_{p,\tau}^{\chi}$, it is clear that $\delta = M = 1$. So $D_{p,\tau}$ is a tight-frame. Actually, we have proven more: $D_{p,\tau}$ is an orthonormal basis.

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