Convergence analysis of a numerical scheme for a tumour growth model

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Abstract

We consider a one-spatial dimensional tumour growth model that consists of three variables; volume fraction and velocity of tumour cells, and nutrient concentration. The model variables form a coupled system of semi-linear advection equation (hyperbolic), simplified stationary Stokes equation (elliptic), and linear diffusion equation (parabolic) with appropriate conditions on the time-dependent boundary which is governed by an ordinary differential equation. An equivalent formulation in a fixed domain is used to overcome the difficulty associated with the time-dependent boundary in the original model. Though this reduces complexity of the model, the tight coupling between the component equations and their non-linear nature offer challenges in proving suitable a priori estimates. A numerical scheme that employs a finite volume method for the hyperbolic equation, a finite element method for the elliptic equation and a mass-lumped finite element method for the parabolic equation is developed. We establish the existence of a time interval (0, T∗) over which the convergence of the scheme using compactness techniques is proved, thus proving the existence of a solution. Numerical tests and justifications that confirm the theoretical findings conclude the paper.

1 Introduction

One spatial dimensional tumour growth models are generally obtained by assuming that a higher spatial dimensional tumour is growing radially [1, 5, 15, 23]. Though such models are much simpler than their intricate higher dimensional versions [11, 12, 16, 19], they still offer severe theoretical and computational difficulties. The time-dependent boundary, non-linearity in the equations, non-coercive coefficient functions, and entangled coupling between the equations are a few challenges worth
mentioning. In this article, we consider a modified version of a tumour growth model proposed by C. J. W. Breward et al. [2, 3]. The model assumes that the tumour cells (cell phase) are embedded in a fluid medium (fluid phase). The mechanical interactions between these two phases along with the differential distribution of the limiting nutrient (in this case, oxygen) cause the growth or depletion of the tumour. The relative volume of the cell phase is called the cell volume fraction, the velocity by which the cells are moving is called the cell velocity, and the concentration of the limiting nutrient is quantified by the oxygen tension, and these three space-time dependent variables are denoted by $\hat{\alpha}$, $\hat{u}$, and $\hat{c}$, respectively. The model seeks variables $(\hat{\alpha}, \hat{u}, \hat{c}, \hat{\ell})$ such that for every $t \in (0, T)$ and $x \in \hat{\Omega}(t) := (0, \hat{\ell}(t))$,

\[
\frac{\partial \hat{\alpha}}{\partial t} + \frac{\partial}{\partial x}(\hat{u}\hat{\alpha}) = \hat{\alpha}f(\hat{\alpha}, \hat{c}), \quad (1.1a)
\]
\[
k\hat{u}\hat{\alpha} - \mu \frac{\partial}{\partial x}\left(\hat{\alpha}\frac{\partial \hat{u}}{\partial x}\right) = -\frac{\partial}{\partial x}(\mathcal{H}(\hat{\alpha})), \quad (1.1b)
\]
\[
\frac{\partial \hat{c}}{\partial t} - \lambda \frac{\partial^2 \hat{c}}{\partial x^2} = -Q\hat{\alpha}\hat{c}, \quad (1.1c)
\]
\[
\hat{\ell}(t) = \hat{u}(t, \hat{\ell}(t)).
\]

The initial and boundary conditions are

\[
\hat{\alpha}(0, x) = \alpha_0(x) \text{ and } \hat{c}(0, x) = c_0(x) \quad \forall x \in \hat{\Omega}(0);
\]
\[
\hat{u}(t, 0) = 0, \quad \mu \frac{\partial \hat{u}}{\partial x}(t, \hat{\ell}(t)) = \frac{(\hat{\alpha}(t, \hat{\ell}(t)) - \alpha^*)^+}{(1 - \hat{\alpha}(t, \hat{\ell}(t)))^2}, \quad (1.1d)
\]
\[
\frac{\partial \hat{c}}{\partial x}(t, 0) = 0, \quad \hat{c}(t, \hat{\ell}(t)) = 1 \quad \forall t \in (0, T); \quad \text{and} \quad (1.1e)
\]
\[
\hat{\ell}(0) = \ell_0.
\]

Here, $f(\hat{\alpha}, \hat{c}) := (1 + s_1)(1 - \hat{\alpha})\hat{c} / (1 + s_3) - (s_2 + s_3\hat{c}) / (1 + s_4\hat{c})$, $\mathcal{H}(\hat{\alpha}) := \hat{\alpha}(\hat{\alpha} - \alpha^*)^+ / (1 - \hat{\alpha})^2$, and $\alpha^+$ and $\alpha^-$ used in the sequel are defined by $a^+ := \max(a, 0)$ and $a^- := -\min(a, 0)$. The positive constants $s_1$, $s_2$, $s_3$, and $s_4$ control the cumulative production rate of the tumour cells provided by $\hat{\alpha}f(\hat{\alpha}, \hat{c})$. The constant $\alpha^+$ regulates repulsive and attractive interactions between the tumour cells. The positive constant $k$ controls traction between the cell and fluid phases, while $\mu$ is the viscosity coefficient in the cell phase. The fluid phase is assumed to be inviscid. The diffusivity coefficient of oxygen is denoted by $\lambda$. The constant $Q$ is non-negative, and controls the oxygen consumption rate by the tumour cells. For more details on physical constants, refer to the reviews [4, 18] and the references therein. It holds $0 < m_{01} \leq a_0 \leq m_{02} < 1$, where $m_{01}$ and $m_{02}$ are constants, and $0 \leq c_0(x) \leq 1$ for every $x \in (0, \ell_0)$. Define $\mathcal{D}_T := \cup_{0 \leq t < T} \{t\} \times \hat{\Omega}(t)$, and $\mathcal{D}_T := (0, T) \times (0, \ell_m)$, where $\ell_m > \hat{\ell}(t)$ for $t \in (0, T)$.

**Remark 1.1 (Variation on the model).** The function $Q\hat{\alpha}\hat{c} / (1 + \hat{Q}_1\hat{c})$ is a more generic source term [2] for $Q_1f$, where $\hat{Q}_1 \geq 0$. For technical simplicity we take $\hat{Q}_1 = 0$, and the analysis can easily be extended to the case when $\hat{Q}_1 > 0$.

The original model in [2] is presented with the terms, $(\hat{\alpha} - \alpha^*)^+ \in \mathcal{H}$ in (1.1b) and $(\hat{\alpha} - \alpha^*)^+ \in \mathcal{H}$ in (1.1d), replaced by $(\hat{\alpha} - \alpha^*)\chi_{\alpha > \alpha_{\min}}$, where $\alpha_{\min}$ is a constant and $\chi_X$ denotes the characteristic function of the set $X$ (that is, $\chi_X(x) = 1$ if $x \in X$ and $0$ otherwise).
zero otherwise). In the case \( \alpha_{\text{min}} \neq \alpha^* \), this non-linear term is discontinuous with respect to \( \alpha \), which makes any proof of existence of a solution to (1.1) difficult – and even questions the well-posedness of the model. The continuity of \( (\tilde{\alpha} - \alpha^*)^+ \) is essential to obtain a priori estimates (see in particular the proof of Proposition 5.10), and for passing to the limit in the numerical scheme.

It is shown in [17] that the model (1.1) can be recast into an extended model, where (1.1a) is set in \( \mathcal{D}_T \) with \( \tilde{\alpha} \) being extended by 0 outside \( \mathcal{D}_T \), the variable \( \tilde{\ell} \) is eliminated, and the variables \( \tilde{u} \) and \( \tilde{c} \) are extended to \( \mathcal{D}_T \backslash \mathcal{D}_T \) by 0 and 1, respectively. In terms of mathematical analysis, the defect of this model is that it does not allow any uniform lower bounds on \( \tilde{\alpha} \) inside the computational domain \( \mathcal{D}_T \), which means that the velocity equation (1.1b) can lose its coercivity properties. In the present work, we therefore consider a modification of this extended model, hereafter called the threshold model, in which we introduce a (small) threshold which determines the computational domain used for \( \tilde{u} \) and \( \tilde{c} \) (see Figure 1). The formulation of a numerical scheme for the threshold model with a suitable notion of the solution, and analysis of the same to obtain the convergence of the iterates, are the primary objectives of this article. This approach has the added benefit of establishing the existence of a solution. The threshold model and the extension to a fixed domain help to reduce the computational cost of re-meshing \( \Omega(t) \) satisfying a Courant–Friedrich–Levy condition (C.F.L.) at each time step.

Despite the fact that tumour growth models have been popular since the seventies [4, 18], the theoretical literature available on this field are very few. Recently, J. Zheng and S. Cui [22] considered existence of solutions for a tumour growth model with volume fraction and pressure in the tumour region as the unknown variables. The model equations in [22] are fully linear, while the boundary conditions are non-linear, and a local well-posedness result is proved. A similar linear model is considered by C. Calzada et al. [6], and equivalence to an extended problem in a larger domain is proved. A more advanced model is considered by N. Zhang and Y. Tao [21], where the nutrient concentration is also considered as a variable and the existence of solutions is obtained by transforming the fixed domain to a unit ball in \( \mathbb{R} \). Studies from the numerical analysis point of view are scarce. J. A. Mackenzie and A. Madzvamuse [14] have shown the convergence of a finite difference scheme for a single variable tumour growth model with a non-linear source term on a time dependent boundary.

The contributions of the current work can be summarised as follows.

- A numerical scheme based on finite volume and Lagrange \( P^1 \)-finite element methods is designed such that the physical properties of the system (1.1); for example, conservation of mass is preserved.
- Bounded variation estimates for the volume fraction, \( H^1 \)-norm and \( L^\infty \) estimates for the cell velocity, and spatial and temporal estimates for the derivatives of oxygen tension are derived.
- The convergence analysis of numerical solutions for a tumour growth model that caters the variables volume fraction, cell velocity and nutrient concentration is studied; and to the best of our knowledge, is first of its kind.
• It is established that the limit of the numerical solutions is indeed a solution to the threshold model thus proving existence of a solution for this model.
• Results of numerical experiments that justify the theory developed are presented.

This paper is organised in the following way. In Section 2, we define the weak solution to the threshold model and in Section 3, a numerical scheme is formulated. In Section 4, the main theorems are stated. The compactness and convergence properties of the numerical solutions are derived in Section 5. In Section 6, we show that the limit of numerical solutions obtained in Section 5 is a solution to the threshold model in an appropriate sense. In Section 7, we present numerical results of examples, and discuss the optimal time below which a solution exists. In Section 8, possible extensions of the current work to other models in single and several spatial dimensions are discussed. We provide the expansions used for indexing abbreviations and a series of classical results used in this article in Appendix A and B, respectively.

This article is set in such a way that an overall reading of Sections 1–4, steps (IS.1)–(IS.4) of Section 5.1, steps (CR.1)–(CR.7) of Section 5.2 and steps (CA.1)–(CA.4) of Section 6 helps to understand the gist of the paper. Proofs of the steps mentioned above in their respective sections provide the details.

We conclude this section by introducing a few important notations. The notation \( \nabla_{t,x} \) stands for \((\partial_t, \partial_x)\). The notation \((\cdot, \cdot)_X\) is the standard \(L^2\) inner product in \(X \subset \mathbb{R}^d\) with \(d \geq 1\). We define the norms \(||u||_{0,X} := (u, u)_X^{1/2}\) and \(||u||_{k,X} := \sum_{j, |j| \leq k} |\partial^j_x u|_0, X\), where \(j\) is a multi-index. The vector space \(\mathbb{P}_1(X)\) is the collection of all first degree polynomials on \(X\).

## 2 Threshold model and well-posedness

We first introduce the notion of a threshold solution. A constant and positive parameter, \(\alpha_{\text{thr}}\), characterises each threshold solution. The source term \(\hat{\alpha} f(\hat{\alpha}, \hat{c})\) in (1.1a) is modified to \((\hat{\alpha} - \alpha_{\text{thr}})^+ f(\hat{\alpha}, \hat{c})\), and tumour radius at time \(t\), \(\ell(t)\), is the smallest number above which the cell volume fraction \(\hat{\alpha}(t, x)\) is below \(\alpha_{\text{thr}}\). In the limiting case \(\alpha_{\text{thr}}\) approaches zero, the continuous function \((\hat{\alpha} - \alpha_{\text{thr}})^+ f(\hat{\alpha}, \hat{c})\) approaches \(\hat{\alpha} f(\hat{\alpha}, \hat{c})\), and \(\ell(t)\) is the smallest number above which there are no tumour cells present. Theorem 3 in [17] proves that the threshold solution with \(\alpha_{\text{thr}} = 0\) and the weak solution of the model (1.1) are equivalent. The introduction of the threshold into the definition of the domain and in the source term helps to obtain boundedness and bounded variation estimates for the numerical solution of (1.1a), and thus enables the numerical scheme to converge to the weak form (2.1a). The source term modification is also a way to account for the fact that, in the absence of sufficient amount of cells, the reaction term that drives their growth remains dormant. The details presented in Subsection 3.1 complement this discussion.

**Definition 2.1 (Threshold solution).** A threshold solution (with threshold \(\alpha_{\text{thr}} \in (0, 1)\)) and domain \(D_{\text{thr}}^T\) of the threshold model in \(\mathcal{D}_T\) is a 4-tuple \((\alpha, u, c, \Omega)\) such that \(0 < m_{11} \leq \alpha_{\Omega(t)} \leq m_{12} < 1\) for every \(t \in [0, T]\), where \(m_{11} \leq m_{01}\) and \(m_{12} \geq m_{02}\) are constant numbers, \(c \geq 0\), and the following conditions hold.
The volume fraction $\alpha \in L^\infty(\mathcal{D}_T)$ is such that $\forall \varphi \in C^\infty_c([0,T) \times (0,\ell_m))$,
\[
\int_{\mathcal{D}_T} (\alpha, u\alpha) \cdot \nabla_t x \varphi \, dt \, dx + \int_{\Omega(0)} \varphi(0, x) \alpha_0 \, dx \\
+ \int_{\mathcal{D}_T} (\alpha - \alpha_{thr})^+ f(\alpha, c) \, dx = 0. \tag{2.1a}
\]

The set $D_T^{thr}$ is of the form $D_T^{thr} = \cup_{0<t<T} \{t\} \times \Omega(t)$, where $\Omega(t) = (0, \ell(t))$, and we have $\alpha \leq \alpha_{thr}$ on $\mathcal{D}_T \setminus D_T^{thr}$.

Define $H^{1,c}_{\partial x}(D_T^{thr}) := \{v \in L^2(D_T^{thr}) : \partial_x v \in L^2(D_T^{thr}) \text{ and } v(t,0) = 0 \forall t \in (0,T)\}$. The velocity $u$ is such that $u \in H^{1,u}_{\partial x}(D_T^{thr})$ and $\forall v \in H^{1,c}_{\partial x}(D_T^{thr})$,
\[
\int_0^T a'(u(t,\cdot), v(t,\cdot)) \, dt = \int_0^T \mathcal{L}'(v(t,\cdot)) \, dt, \tag{2.1b}
\]
where $a' : H^1(\Omega(t)) \times H^1(\Omega(t)) \to \mathbb{R}$ is a bilinear form and $\mathcal{L} : H^1(\Omega(t)) \to \mathbb{R}$ is a linear form as follows:
\[
a'(u, v) = k \left( \frac{\alpha}{1 - \alpha} u, v \right)_{\Omega(t)} + \mu (\alpha \partial_x u, \partial_x v)_{\Omega(t)} \quad \text{and} \\
\mathcal{L}'(v) = (\mathcal{H}(\alpha), \partial_x v)_{\Omega(t)}.
\]

Extend $u$ to $\mathcal{D}_T$ by setting $u|_{\mathcal{D}_T \setminus D_T^{thr}} := 0$.

Define $H^{1,c}_{\partial x}(D_T^{thr}) := \{v \in L^2(D_T^{thr}) : \partial_x v \in L^2(D_T^{thr}) \text{ and } v(t,\ell(t)) = 0 \forall t \in (0,T)\}$. The oxygen tension $c$ is such that $c - 1 \in H^{1,c}_{\partial x}(D_T^{thr})$ and satisfies
\[
- \int_{D_T^{thr}} c \partial_t v \, dx \, dt + \lambda \int_{D_T^{thr}} \partial_x c \partial_x v \, dx \, dt - \int_{\Omega(0)} c_0(x) v(0, x) \, dx \\
- Q \int_{D_T^{thr}} \alpha c v \, dx \, dt = 0, \tag{2.1c}
\]
$\forall v \in H^{1,c}_{\partial x}(D_T^{thr})$ such that $\partial_t v \in L^2(D_T^{thr})$. Extend $c$ to $\mathcal{D}_T$ by setting $c|_{\mathcal{D}_T \setminus D_T^{thr}} := 1$.

Remark 2.2. In the paragraph before Definition 2.1, we described the tumour radius as the smallest number above which the cell volume fraction $\alpha(t, x)$ is below $\alpha_{thr}$. However, this description is subtly different from the definition of $\ell(t)$ in Definition 2.1. In TS.2, we only demand that the volume fraction of the tumour cells outside the domain must be less than or equal to $\alpha_{thr}$. The convergence analysis in this article assures the existence of such a domain. It remains open whether such a domain is unique, and if at all unique, it coincide with $\cup_{0<t<T} \{t\} \times (0,\tilde{\ell}(t))$, where $\tilde{\ell}(t) := \min\{x : \alpha(t, x) < \alpha_{thr}\}$. 

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Given the bounds of $\alpha$ in Definition 2.1, it can easily be checked that $a'$ is uniformly continuous and coercive on $H^1(\Omega(t))$, and $\mathcal{L}'$ is uniformly continuous on $H^1(\Omega(t))$. To prove existence of a solution for (2.1a), we need uniform supremum norm bounds on $u$, $\partial_x u$ [10, p. 153] and $c$. However, Definition 2.1 does not assume these bounds a priori. We overcome this difficulty by proving that $u$ and $\partial_x u$ satisfies uniform supremum norm bounds at the discrete level, which leads to the existence of a discrete solution for (2.1a) with uniform bounded variation, and limit of which is a solution of (2.1a). The boundedness of $\alpha$ helps to obtain existence of solutions to (2.1c). However, strong convergence of discrete solutions of (2.1a) is needed to obtain convergence of (2.1b) and (2.1c). It is readily noted that the bounds on $\alpha$, $u$ and $c$ are interdependent, and we address this issue also.

### 3 Discretisation

We discretise (1.1a) using a finite volume method, (1.1b) using a Lagrange $P^1$-finite element method, and (1.1c) using backward Euler in time and $P^1$- mass lumped finite element method in space. The space and time variables are discretised as follows. Let $0 = x_0 < \cdots < x_J = \ell_m$ be a uniform spatial discretisation with $h = x_{j+1} - x_j$, and $0 = t_0 < \cdots < T_N = T$ be a uniform temporal discretisation with $\delta = t_{n+1} - t_n$. Define the intervals $X_j := (x_j, x_{j+1})$ and $T_n := [t_n, t_{n+1})$. The node-centred intervals are defined by $\tilde{X}_j := (x_j - h/2, x_j + h/2)$ for $0 \leq j \leq J - 1$, $\tilde{X}_0 := [x_0, x_0 + h/2]$, and $\tilde{X}_J := [x_J - h/2, x_J]$. For any real valued function $f$ on $\mathbb{R}$, define the pointwise average $\{f\}_{X_j} = (f(x_j) + f(x_{j+1}))/2$.

**Definition 3.1 (Discrete scheme).** Define

- $\alpha_h^0$ by $\alpha_h^0 := \alpha_j^0$ on $X_j$ for $0 \leq j \leq J - 1$, where $\alpha_j^0 = \frac{1}{\pi} \int_{X_j} \alpha_0(x) \, dx$,
- $c_h^0$ by $c_h^0 \in P^1(X_j)$ for $0 \leq j \leq J - 1$ and $c_h^0(x_j) := c_0(x_j)$ for $0 \leq j \leq J$, and
- $\Omega_h^n := (0, \ell_h^n)$, where $\ell_h^n = 1$.

Fix a threshold $\alpha_{\text{thr}} \in (0, 1)$ and $\ell_m > \ell_0$, to ensure that the extended domain contains the initial domain. Obtain $u_h^0$ from [DS.c] by taking $n = 0$. Then, construct a finite sequence of 3-tuple of functions $(\alpha_h^n, u_h^n, c_h^n)_{0 \leq n \leq N-1}$ on $(0, \ell_m)$ as in [DS.a] [DS.d]
Set $\alpha_h^n := \alpha_h^n$ on $\mathcal{X}_j$ for $0 \leq j \leq J - 1$, where
\[
\frac{1}{\delta} (\alpha_j^n - \alpha_j^{n-1}) + \frac{1}{h} \left[ u_{j+1}^{(n-1)} + u_{j+1}^{n-1} - u_j^{(n-1)} - u_j^{n-1} + \ell_j^{(n-1)} + \ell_j^{n-1} - \alpha_j^{n-1} \right] = \left( \alpha_j^n - \alpha_j^{n-1} \right)^+ (1 - \alpha_j^{n-1}) b_j^{n-1} - (\alpha_j^n - \alpha_{\text{thr}}) + d_j^{n-1} \tag{3.1}
\]
with $u_j^n = u_h^n(x_j)$, $b_j^n = \mathbb{I}((1 + s_1) \tilde{c}_h^n/(1 + s_1 c_h^n))^\chi_j$, and $\alpha_j^n = \mathbb{I}(s_2 + s_3 c_h^n)/(1 + s_4 c_h^n)^\chi_j$. Note that, when $j = 0$, $u_0^{(n-1)} = 0$ and thus the value of $\alpha_{n-1}^n$ can be arbitrarily fixed, say for example $\alpha_{n-1}^n = m_{11}$.

Set $\Omega_h^n := (0, \ell_h^n)$, where the recovered radius at step $n$, $\ell_h^n$, is provided by $\ell_h^n = \min\{x_j : \alpha_j^{n} < \alpha_{\text{thr}} \text{ on } (x_j, \ell_m)\}$.

Set the conforming $\mathbb{P}^1$ finite element space on $\Omega_h^n$, and its subspace with homogeneous boundary condition at $x = 0$, by
\[
S_h^n := \{ v_h^n \in \mathcal{C}^0(\Omega_h^n) : v_h^n|_{\mathcal{X}_j} \in \mathbb{P}^1(\mathcal{X}_j) \text{ for } 0 \leq j \leq J_n := \ell_h^n/h \} \quad \text{and} \quad S_{0,h}^n := \{ v_h^n \in S_h^n : v_h^n(0) = 0 \}.
\]

Then,
\[
u_h^n := \begin{cases} 
\bar{u}_h^n & \text{on } \Omega_h^n, \\
0 & \text{on } (0, L) \setminus \Omega_h^n,
\end{cases}
\]
where $\bar{u}_h^n \in S_{0,h}^n$ satisfies
\[
\alpha_h^n(\bar{u}_h^n, v_h^n) = \mathcal{L}_h^n(v_h^n) \quad \forall v_h^n \in S_{0,h}^n, \tag{3.2}
\]
with $\alpha_h^n : S_h^n \times S_h^n \to \mathbb{R}$ and $\mathcal{L}_h^n : S_h^n \to \mathbb{R}$ defined by
\[
\alpha_h^n(w, v) = k \left( \frac{\alpha_h^n}{1 - \alpha_h^n} w, v \right)_{\Omega_h^n} + \mu \left( \alpha_h^n \partial_x w, \partial_x v \right)_{\Omega_h^n} \quad \text{and} \quad \mathcal{L}_h^n(v) = (\mathcal{H}(\alpha_h^n), \partial_x v)_{\Omega_h^n}. \tag{3.3}
\]

Define the finite dimensional vector spaces
\[
S_{h,0}^n := \{ v_h^n \in S_h^n : v_h^n(\ell_h^n) = 0 \} \quad \text{and} \quad S_{h,ML} := \left\{ w_h : w_h = \sum_{j=0}^{J} w_j \chi_{\mathcal{X}_j}, \ w_j \in \mathbb{R}, \ 0 \leq j \leq J \right\},
\]
and the mass lumping operator $\Pi_h : \mathcal{C}^0([0, L]) \to S_{h,ML}$ such that $\Pi_h w = \sum_{j=0}^{J} w(x_j) \chi_{\mathcal{X}_j}$. Then,
\[
\bar{c}_h^n := \begin{cases} 
\bar{c}_h^n & \text{on } \Omega_h^n, \\
1 & \text{on } (0, L) \setminus \Omega_h^n,
\end{cases}
\]
where $\bar{c}_h^n \in S_h^n$ satisfies $\bar{c}_h^n(\ell_h^n) = 1$, and with $\Pi_h \bar{c}_h^n := (\Pi_h \bar{c}_h^n)_{\Omega_h^n}$,
\[
(\Pi_h \bar{c}_h^n, \Pi_h v_h^n)_{\Omega_h^n} - (\Pi_h c_h^{n-1}, \Pi_h v_h^n)_{\Omega_h^n} + \delta \lambda (\partial_x \bar{c}_h^n, \partial_x v_h^n)_{\Omega_h^n} = -Q \delta (\alpha_h^n \Pi_h \bar{c}_h^n, \Pi_h v_h^n)_{\Omega_h^n} \quad \forall v_h^n \in S_{h,0}^n. \tag{3.5}
\]
Definition 3.2 (Time-reconstruct). For a family of functions \((f^n_h)_{0 \leq n < N}\) on a set \(X\), define the time-reconstruct \(f_{h,\delta} : (0, T) \times X \rightarrow \mathbb{R}\) as \(f_{h,\delta} := f^n_h\) on \(T_n\) for \(0 \leq n < N\).

Definition 3.3 (Discrete solution). The 4-tuple \((\alpha_{h,\delta}, u_{h,\delta}, c_{h,\delta}, \ell_{h,\delta})\), where \(\alpha_{h,\delta}\), \(u_{h,\delta}\), \(c_{h,\delta}\), and \(\ell_{h,\delta}\) are the respective time-reconstructs corresponding to the families \((\alpha^n_h)_n\), \((u^n_h)_n\), \((c^n_h)_n\), and \((\ell^n_h)_n\) obtained from \((\text{DS.a})\) \((\text{DS.d})\) is called the discrete solution of the problem \((1.1)\).

3.1 Comments on the numerical method

This subsection substantiates the particular choices of numerical methods used to compute the discrete solution in Definition 3.3.

3.1.1 Volume fraction equation

The volume fraction equation \((1.1a)\) is a continuity equation with the source term \(\alpha f(\tilde{\alpha}, c)\), and the conserved variable \(\tilde{\alpha}\) is transported with a velocity \(\tilde{u}\). Finite volume methods are the natural choice of numerical methods that preserve conservation property at the discrete level \([13]\). An upwinding finite volume scheme is used in \([3.1]\). This means that the flux at the boundary \(x_j\) between any two intervals \(X_{j-1}\) and \(X_j\) is approximated by: for any \(t \in (0, T)\)

\[
(u, \alpha)(t, \cdot)_{|x_j} \approx u_{h,\delta}(t, x_j)^+ \alpha_{h,\delta}(t, \cdot)_{|X_{j-1}} - u_{h,\delta}(t, x_j)^- \alpha_{h,\delta}(t, \cdot)_{|X_j}. \tag{3.6}
\]

The upwinding flux \((3.6)\) is one of the simplest numerical fluxes and leads to a stable scheme.

The upwind method \([3.1]\) introduces significant numerical diffusion in the discrete solution \(\alpha_{h,\delta}\). Hence if we locate the time-dependent boundary \(\ell^n_h\) as \(\min\{x_j : \alpha^n_h = 0\}\) on \((x_j, \ell_m]\), then \(\ell_{h,\delta}\) will have notable deviation from the exact solution, which will further tamper the quality of the solutions \(u_{h,\delta}\) and \(c_{h,\delta}\). To eliminate this propagating error, the boundary point \(\ell^n_h\) is located by \(\min\{x_j : \alpha^n_h < \alpha_{\text{thr}}\}\) on \((x_j, \ell_m]\) (see Figure \([1]\)). However, the residual volume fraction of \(\alpha_{\text{thr}}\) on \([\ell^n_h, \ell_m]\) might cause the reaction term \(\alpha f(\tilde{\alpha}, c)\) to contribute a spurious growth; the modification \((\tilde{\alpha} - \alpha_{\text{thr}})^+ f(\tilde{\alpha}, c)\) overcomes this problem. More importantly, \(\alpha_{\text{thr}}\) acts as a lower bound on the value of \(\alpha_{h,\delta}\) on \(X_{j-1}\) (the right most control volume in \((0, \ell^n_h]\)) at each time \(t_n\). A detailed study of the dependence of the discrete solution on \(\alpha_{\text{thr}}\) and the optimal choice of \(\alpha_{\text{thr}}\) that minimises the error incurred in \(\ell_{h,\delta}\) is done in \([17]\). Naturally, we anticipate that the numerical solutions converge to a threshold solution.

3.1.2 Velocity equation

The velocity equation \((1.1b)\) is elliptic with Dirichlet boundary condition at \(x = 0\) and Neumann boundary condition at \(x = \ell^n_h\) for each \(t_n\), and hence the Lagrange \(P^1\) finite element method is used to discretise \((1.1b)\). Since the unknowns in the Lagrange \(P^1\) finite element method are located at the boundary nodes of \(X_j\) for \(j = 0, \ldots, J − 1\), the values of the discrete solution \(u_{h,\delta}\) at these nodes are useful to compute the upwinding flux in \((3.6)\).
3.1.3 Oxygen tension equation

The choice of time-implicit mass lumped finite element method for the oxygen tension equation (1.1c) is substantiated mainly by two reasons. Firstly, the choice of mass lumping as opposed to a more natural Lagrange $P^1$ finite element method is important to obtain a discrete maximum principle for $c_{h,\delta}$. Secondly, the backward time procedure ensures $L^2(0,T;H^1(0,\ell_m))$ - stability of the mass lumped solutions. This is essential to prove Proposition 5.18 and Theorem 5.19 that lead to the compactness and convergence of the iterates.

4 Main theorems

Define the function $\hat{u}_{h,\delta}(t,\cdot)$ on $D_T$ such that for every $t \in [0,T]$,

$$
\hat{u}_{h,\delta}(t,\cdot) = \begin{cases} u_{h,\delta}(t,\cdot) & \text{in } (0,\ell_{h,\delta}(t)), \\
 u_{h,\delta}(t,\ell_{h,\delta}(t)) & \text{in } (\ell_{h,\delta}(t),\ell_m).
\end{cases}
$$

Note that $\hat{u}_{h,\delta}$ is continuous on the contrary to $u_{h,\delta}$ (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The left-hand side plot illustrates the discontinuous function $u_{h,\delta}$ and the right-hand side plot illustrates the continuous modification $\hat{u}_{h,\delta}$.}
\end{figure}

Define the function $\Pi_{h,\delta}c_{h,\delta}$ by $(\Pi_{h,\delta}c_{h,\delta})(t,\cdot) := \Pi_h(c_{h,\delta}(t,\cdot))$ for every $t \in [0,T]$. Now, we state the main results of this article concerning the compactness and convergence of the iterates from the Discrete scheme 3.1. The results are presented as two separate theorems, Theorem 4.1 and Theorem 4.2. Define the spaces:

$$
L^2_c(0,T;H^1(0,\ell_m)) := \{ f \in L^2(0,T;H^1(0,\ell_m)) : f(t,\ell(t)) = 0 \text{ for a.e. } t \in [0,T] \},
$$

$$
L^2_u(0,T;H^1(0,\ell_m)) := \{ f \in L^2(0,T;H^1(0,\ell_m)) : f(t,0) = 0 \text{ for a.e. } t \in [0,T] \}.
$$

**Theorem 4.1** (compactness). Let the properties stated below be true.

- The initial volume fraction $\alpha_0$ belongs to $BV(0,\ell_m)$ and $0 < m_{01} \leq \alpha_0 \leq m_{02} < 1$, where $m_{01}$ and $m_{02}$ are constants.
- The discretisation parameters $h$ and $\delta$ satisfy the following conditions:

$$
\rho \mathcal{C}_{CFL} \leq \frac{\delta}{h} \leq \mathcal{C}_{CFL} := \frac{\sqrt{\nu s \mu}}{2\ell_m} \frac{|1-a^*|^2}{|a^* - \alpha^*|} \quad \text{and} \quad \delta < \min \left( \frac{1-\rho}{s_2}, \frac{2(1-\rho)}{1+s_2} \right), \quad (4.1)
$$

Define the spaces:
where \( \rho, a_\ast \) and \( a^\ast \) are constants chosen such that \( \rho < 1, 0 < a_\ast < m_{01}, \) and \( 0 < m_{02} < a^\ast. \)

Then, there exists a finite time \( T_* \) depending on the choice of \( \rho, a_\ast, \) and \( a^\ast, \) a subsequence of the family of functions \( \{ (\alpha_{h, \delta}, \hat{u}_{h, \delta}, c_{h, \delta}, \ell_{h, \delta}) \}_{h, \delta} \) and a 4-tuple of functions \( (\alpha, \hat{u}, c, \ell) \) such that \( \alpha \in BV(D_{T_*}) \) with \( D_{T_*} = (0, T_*) \times (0, \ell_m), c \in L^2_c((0, T_*) ; H^1(0, \ell_m)), \hat{u} \in L^2_h((0, T_*) ; H^1(0, \ell_m)), \ell \in BV(0, T_*) \), and as \( h, \delta \to 0, \)

- \( \alpha_{h, \delta} \to \alpha \) almost everywhere and in \( L^\infty \) weak* on \( D_{T_*}, \)
- \( \Pi_{h, \delta} c_{h, \delta} \to c \) strongly in \( L^2(D_{T_*}) \) and \( \partial_x c_{h, \delta} \to \partial_x c \) weakly in \( L^2(D_{T_*}), \)
- \( \hat{u}_{h, \delta} \to \hat{u} \) and \( \partial_x \hat{u}_{h, \delta} \to \partial_x \hat{u} \) weakly in \( L^2(D_{T_*}), \) and
- \( \ell_{h, \delta} \to \ell \) almost everywhere in \( (0, T_*). \)

**Theorem 4.2** (convergence). Let \( (\alpha, \hat{u}, c, \ell) \) be the limit provided by Theorem 4.1. Define \( \Omega(t) := (0, \ell(t)) \) and the threshold domain \( D^{br}_{T_*} := \{ (t, x) : x < \ell(t), t \in (0, T_*), \}, \) and let \( u := \hat{u} \) on \( D^{br}_{T_*} \) and \( u := 0 \) on \( D_{T_*} \setminus D^{br}_{T_*}. \) Then, \( (\alpha, u, c, \Omega) \) is a threshold solution in the sense of Definition 2.1 with \( T = T_*. \)

## 5 Proof of Theorem 4.1

The proof of Theorem 4.1 involves several steps which are described here. In Subsection 5.1 we prove: existence and uniqueness of the discrete solutions \( \alpha_{h, \delta}, u_{h, \delta}, \) and \( c_{h, \delta}; \) boundedness of \( u_{h, \delta}; \) positivity, boundedness and bounded variation property of \( \alpha_{h, \delta}; \) and positivity and boundedness of \( c_{h, \delta}. \) In Subsection 5.2, we show that the families of functions \( \{ \alpha_{h, \delta} \}_{h, \delta}, \{ u_{h, \delta} \}_{h, \delta}, \{ c_{h, \delta} \}_{h, \delta} \) and \( \{ \ell_{h, \delta} \}_{h, \delta} \) are relatively compact in appropriate spaces.

### 5.1 Existence and uniqueness of the iterates

The proof of existence and uniqueness of the discrete solutions \( \alpha_{h, \delta}, u_{h, \delta}, \) and \( c_{h, \delta} \) involves many interrelated results. For clarity, we provide a sketch of the steps involved.

Fix two constants \( a^\ast \in (\max(a^\ast, m_{02}), 1) \) and \( a_\ast \in (0, \min(m_{01}, a_{br})). \) We establish the existence of a time \( T_* \) (explicitly determined in the analysis), depending in particular on \( a_\ast \) and \( a^\ast, \) such that the following theorem holds.

**Theorem 5.1.** For all \( n \in \mathbb{N} \) such that \( t_n \leq T_* \), \( \alpha_{h, \delta}(t_n, \cdot) \) and \( c_{h, \delta}(t_n, \cdot) \) are well defined, and it holds \( a_\ast < \alpha_{h, \delta}(t_n, \cdot)|_{\Omega_h} < a^\ast \) and \( 0 \leq c_{h, \delta}(t_n, \cdot)|_{(0, \ell_m)} \leq 1. \)

The proof of Theorem 5.1 is done in several steps by strong induction on \( n \in \mathbb{N}. \) The base case obviously holds, for any choice of \( a_\ast \) and \( a^\ast \) as above. Let \( n \in \mathbb{N} \) be such that \( t_n \leq T_*, \) and assume that the statement of Theorem 5.1 holds for the indices \( 0, \ldots, n. \) The steps (IS.1) (IS.4) below show that the same holds for the index \( n + 1. \)

**Inductive steps:**

(IS.1) **Well-posedness of (3.2):** We establish that there exists a unique solution \( \tilde{u}_h^n \) for the variational problem (3.2) and derive energy estimates.

10
(IS.2) Bounded variation and $L^\infty$ estimates on $\alpha_{h,\delta} u_{h,\delta}$: We show that
\begin{align}
(a) \quad \|\mu \alpha_{h,\delta}(t_n, \cdot) \partial_x u_{h,\delta}(t_n, \cdot) - \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot))\|_{BV(0, \ell_m)} &\leq \mathcal{C} \quad \text{and} \\
(b) \quad \|\mu \alpha_{h,\delta}(t_n, \cdot) \partial_x u_{h,\delta}(t_n, \cdot)\|_{L^\infty(0, \ell_m)} &\leq \mathcal{C},
\end{align}
where $\mathcal{C}$ is a generic constant that depends on $T$, $\ell_m$, $\ell$, $\alpha_*$, $a_*$, $a^*$ and the model parameters as explicitly defined in (5.3a)–(5.3c), and $\mathcal{H}(\alpha) = \alpha(\alpha - a^*)^+/(1 - \alpha)^2$.

(IS.3) $L^\infty$ estimates on $\alpha_{h,\delta}$: It holds $a_* < \alpha_{h,\delta}(t_{n+1, \cdot})|_{\Omega_{n+1}} < a^*$.

(IS.4) Well-posedness of (3.5): We show that there exists a unique solution $\tilde{\alpha}_{h,\delta}(t_{n+1, \cdot})$ to (3.5) and that $0 \leq \tilde{\alpha}_{h,\delta}(t_{n+1, \cdot})(0, \ell_m) \leq 1$.

5.1.1 Proofs of (IS.1)–(IS.4)

In this subsection we verify the steps (IS.1)–(IS.4) in Lemmas 5.2, 5.4, 5.7 and Proposition 5.5. The time $T_\alpha$ is explicitly obtained in the proof of Proposition 5.5.

Lemma 5.2 (Step IS.1). There exists a unique solution $\tilde{u}_h^n$ to (3.2) and it satisfies the following estimates:
\begin{align}
\left\| \sqrt{\alpha_{h,\delta}(t_n, \cdot)} \partial_x \tilde{u}_h^n \right\|_{0, \Omega_h^n} &\leq \frac{\sqrt{\ell_m}[a^* - a^*]}{\mu |1 - a^*|^2} \quad \text{and} \\
\left\| \sqrt{\alpha_{h,\delta}(t_n, \cdot)} \tilde{u}_h^n \right\|_{0, \Omega_h^n} &\leq \frac{\ell_m |a^* - a^*|}{\sqrt{\kappa \mu} |1 - a^*|^2}.
\end{align}

Proof. Coercivity and continuity of the bilinear form $\alpha_h^n$ and continuity of the linear form $L_h^n$ are clear from $0 < a_* \leq \alpha_{h,\delta}(t_n, \cdot) \leq a^* < 1$. An application of the Lax-Milgram lemma [8, p. 297] establishes the existence of a unique discrete solution to (3.2). A choice of $\tilde{v}_h^n = \tilde{u}_h^n$ in (3.2), the fact that $0 < \alpha_{h,\delta}(t_n, \cdot) < 1$ and a use of Cauchy-Schwarz inequality on (3.4) yield
\begin{align}
\mu \left\| \sqrt{\alpha_{h,\delta}(t_n, \cdot)} \partial_x \tilde{u}_h^n \right\|_{0, \Omega_h^n}^2 + k \left\| \sqrt{\alpha_{h,\delta}(t_n, \cdot)} \tilde{u}_h^n \right\|_{0, \Omega_h^n}^2 &\leq \sqrt{\ell_m} \frac{|a^* - a^*|}{|1 - a^*|^2} \left\| \sqrt{\alpha_{h,\delta}(t_n, \cdot)} \partial_x \tilde{u}_h^n \right\|_{0, \Omega_h^n},
\end{align}
which proves (5.1a) and (5.1b).

\hfill \Box

Remark 5.3 ($L^\infty$ estimate on velocity). Since $\alpha_{h,\delta}(t_n, \cdot) \geq a_*$, the estimate (5.1a) yields an upper bound on $\|\partial_x \tilde{u}_h^n\|_{0, \Omega_h^n}$, which after an application of the boundary condition $\tilde{u}_h^n(0) = 0$ and a use of Cauchy-Schwarz inequality, yields
\begin{align}
\|u_{h,\delta}(t_n, \cdot)\|_{L^\infty(0, \ell_m)} \leq \frac{\ell_m}{\sqrt{a_* \mu}} \frac{|a^* - a^*|}{|1 - a^*|^2}. \quad (5.2)
\end{align}
Lemma 5.4 (Step [IS2]). It holds true that

\[ ||\mu_{\alpha,\delta}(t_n, \cdot)\partial_x u_{h,\delta}(t_n, \cdot) - \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot))||_{BV(0,\ell_m)} \leq \ell_m \sqrt{\frac{k}{\mu}} \frac{|a^* - \alpha^*|}{|1 - \alpha^*|^{5/2}}, \]  

(5.3a)

\[ ||(\mu_{\alpha,\delta}(t_n, \cdot)\partial_x u_{h,\delta}(t_n, \cdot)) - (\mathcal{H}(\alpha_{h,\delta}(t_n, \cdot)) - \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot)))||_{L^\infty(0,\ell_m)} \leq \ell_m \sqrt{\frac{k}{\mu}} \frac{|a^* - \alpha^*|}{|1 - \alpha^*|^{5/2}}, \]  

(5.3b)

\[ ||\mu_{\alpha,\delta}(t_n, \cdot)\partial_x u_{h,\delta}(t_n, \cdot)||_{L^\infty(0,\ell_m)} \leq \ell_m \sqrt{\frac{k}{\mu}} \frac{|a^* - \alpha^*|}{|1 - \alpha^*|^{5/2}} + \frac{a^*(a^* - \alpha^*)}{(1 - a^*)^2}. \]  

(5.3c)

Proof. Consider the Lagrange \(P^1\) basis functions \(\{\varphi_h^n\}_{1 \leq j \leq J_n}\) of \(S^n_{0,h}\), and choose \(v_n = \varphi_h^n\) in (3.2) for \(j \in \{1, \ldots, J_n - 1\}\), where \(J_n = \ell_n^* / h\) to obtain

\[ \mu \left( \alpha_{j-1} \partial_x \tilde{u}_h^n, x_{j-1} - \alpha_j \partial_x \tilde{u}_h^n, x_j \right) - (\mathcal{H}(\alpha_j^n) - \mathcal{H}(\alpha_{j-1}^n)) = -k \int_{x_{j-1}}^{x_{j+1}} \alpha_{h,\delta}(t_n, \cdot) \tilde{u}_h^n \varphi_{h,j} \, dx. \]  

(5.4a)

Choose \(v_h^n = \varphi_h^n\) in (3.2) to obtain

\[ \mu \partial_x \tilde{u}_h^n, x_{j-1} - \mathcal{H}(\alpha_j^n) = -k \int_{x_{j-1}}^{x_{j+1}} \alpha_{h,\delta}(t_n, \cdot) \tilde{u}_h^n \varphi_{h,j} \, dx. \]  

(5.4b)

Recall that \(u_h^n = \tilde{u}_h^n\) on \((0, \ell_n^*)\), and that \(u_h^n = 0 = \mathcal{H}(\alpha_j^n)\) outside this interval. Then, for any \(j \in \{1, \ldots, J_n - 1\}\), (5.4a) and (5.4b) show that

\[ \mu \left( \alpha_{j-1} \partial_x u_h^n, x_{j-1} - \alpha_j \partial_x u_h^n, x_j \right) - (\mathcal{H}(\alpha_j^n) - \mathcal{H}(\alpha_{j-1}^n)) \]

\[ = -k \int_{x_{j-1}}^{x_{j+1}} \alpha_{h,\delta}(t_n, \cdot) u_h^n \varphi_{h,j} \, dx, \]

where \(\varphi_h^n = 0\) if \(j \geq J_n + 1\). Then, a use of the triangle inequality, a summation over \(j = 1, \ldots, J_n - 1\), Cauchy-Schwarz inequality, (5.1b), and an observation that \(0 \leq \varphi_{h,j-1} + \varphi_{h,j} \leq 1\) everywhere leads to (5.3a). As a consequence, since \(\mu \alpha_{h,\delta}(t_n, \cdot)\partial_x u_{h,\delta}(t_n, \cdot) - \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot))\) vanishes at \(x = \ell_m\),

\[ ||\mu_{\alpha,\delta}(t_n, \cdot)\partial_x u_{h,\delta}(t_n, \cdot) - \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot))||_{L^\infty(0,\ell_m)} \leq \ell_m \sqrt{\frac{k}{\mu}} \frac{|a^* - \alpha^*|}{|1 - \alpha^*|^{5/2}}. \]

Since \(0 \leq \mathcal{H}(\alpha_{h,\delta}(t_n, \cdot)) \leq a^*(a^* - \alpha^*)/(1 - a^*)^2\), the bounds (5.3b) and (5.3c) follow. \(\Box\)

The positivity and boundedness of \(\alpha_{h,\delta}(t_{n+1}, \cdot)\) are shown next. The next proposition establishes the existence of a finite time \(T_*\) such that the strong induction assumption holds in \([0, T_*]\).

Proposition 5.5 (Step [IS3]). There exists \(T_* > 0\) such that if \(n+1 \leq T_* := T_*/\delta\), then

\[ a_* \leq \min_{j : x_j \in \Omega_{n+1}} \alpha_{j+1} \leq \max_{0 \leq j \leq J-1} \alpha_{j+1} \leq a^*. \]
Proof. Substitute $u_{n+1}^n = u_{n+1}^n + u_{n+1}^{n-}$ and $u_{n+1}^n = u_{n+1}^n - u_{n}^n$ in (3.1) written for $n+1$ instead of $n$ to obtain

\[
\alpha_{j}^{n+1} + \delta (\alpha_{j}^{n+1} - \alpha_{\text{thr}}) + d_{j}^{n} = \alpha_{j}^{n} + \delta (\alpha_{j}^{n} - \alpha_{\text{thr}})(1 - \alpha_{j}^{n})b_{j}^{n} - \delta \alpha_{j}^{n} (u_{j+1}^{n} - u_{j}^{n}) + \delta \left( u_{j+1}^{n}(\alpha_{j+1}^{n} - \alpha_{j}^{n}) + u_{j}^{n}(\alpha_{j-1}^{n} - \alpha_{j}^{n}) \right). \tag{5.5}
\]

Define the linear combination

\[
\mathcal{L}(\alpha_{j-1}^{n}, \alpha_{j}^{n}, \alpha_{j+1}^{n}) := \frac{\delta}{h}u_{j}^{n+1} + \frac{\delta}{h} u_{j-1}^{n} + \left( 1 - \frac{\delta}{h} u_{j+1}^{n} - \frac{\delta}{h} u_{j}^{n} \right) \alpha_{j}^{n} + \frac{\delta}{h} u_{j+1}^{n}\alpha_{j+1}^{n}. \tag{5.6}
\]

The condition (4.1) and (5.2) show that all the coefficients in (5.6) are positive, and thus this linear combination is convex. Moreover, (5.5) can be recast as

\[
\alpha_{j}^{n+1} + \delta (\alpha_{j}^{n+1} - \alpha_{\text{thr}}) + d_{j}^{n} = \mathcal{L}(\alpha_{j-1}^{n}, \alpha_{j}^{n}, \alpha_{j+1}^{n}) + \delta (\alpha_{j}^{n} - \alpha_{\text{thr}})(1 - \alpha_{j}^{n})b_{j}^{n} - \delta \alpha_{j}^{n} \partial_{x} u_{n}^{n}|_{x_{j}}. \tag{5.7}
\]

Since $0 \leq c_{h}^{n} \leq 1$ (this is the induction hypothesis (IS.4) at step $n$), we have $0 \leq d_{j}^{n} \leq s_{2}$ and $b_{j}^{n} \geq 0$. Then, a use of (5.3c) and the positivity of $1 - \alpha_{j}^{n}$ in (5.7) yield

\[
\alpha_{j}^{n+1}(1 + \delta s_{2}) \geq \min(\alpha_{j-1}^{n}, \alpha_{j}^{n}, \alpha_{j+1}^{n}) - \delta \mathcal{F}_{\min}, \tag{5.8}
\]

where

\[
\mathcal{F}_{\min} = \mu \sqrt{k} \left| a^{+} - a^{-} \right| \left[ \frac{1}{1 - a^{-}} \right]^{3/2} + \frac{1}{\mu} \left( 1 - a^{-} \right)^{2}.
\]

Step (DS.b) in the Discrete scheme 3.1 implies that $\alpha_{j-1}^{n}, \alpha_{j}^{n}, \alpha_{j+1}^{n} < \alpha_{\text{thr}}$ for $j \geq J_{n+1}$. This fact along with an observation that $u_{n}^{n} = 0$ in $(0, \ell_{m}) \setminus \Omega_{h}^{n}$ ensures that the right hand side of (5.7) is strictly bounded above by $\alpha_{\text{thr}}$ (the linear combination remains, and the other terms vanish); hence $\alpha_{j}^{n+1} < \alpha_{\text{thr}}$ for all $j \geq J_{n+1}$. Thus the domain $\Omega_{h}^{n+1}$ is either a subset of $\Omega_{h}^{n}$, or equal to $\Omega_{h}^{n} \cup \mathcal{X}_{j_{n}}$. These two cases are considered separately.

Case 1 ($\Omega_{h}^{n+1} \subseteq \Omega_{h}^{n}$). If $\Omega_{h}^{n+1} = \Omega_{h}^{n}$, the last value $\alpha_{j_{n+1}}^{n+1}$ depends on $\alpha_{j_{n-2}}^{n}$, $\alpha_{j_{n-1}}^{n}$, and $\alpha_{j_{n}}^{n}$ (see Figure 3(a)). The domain selection procedure (DS.b) in the Discrete scheme 3.1 shows $\alpha_{j_{n+1}}^{n+1} \geq \alpha_{\text{thr}}$. All other values $\alpha_{j}^{n+1}$ depend on $\alpha_{k}^{n}$ with $k \leq J_{n-1}$, which are values inside $\Omega_{h}^{n}$. Therefore, for all $j \leq J_{n+1} - 1$, by (5.8)

\[
\alpha_{j}^{n+1}(1 + \delta s_{2}) \geq \min \left( \min_{k : x_{k} \in \Omega_{h}^{n}} \alpha_{j_{k}}^{n}, \alpha_{\text{thr}} \right) - \delta \mathcal{F}_{\min}. \tag{5.9}
\]

The same argument follows in the case $\Omega_{h}^{n+1} \subset \Omega_{h}^{n}$ (see Figure 3(b)).

Case 2 ($\Omega_{h}^{n+1} = \Omega_{h}^{n} \cup \mathcal{X}_{j_{n}}$). By the domain selecting procedure (DS.b) in the Discrete scheme 3.1, we have $\alpha_{j_{n+1}}^{n+1} \geq \alpha_{\text{thr}}$ (see Figure 3(c)). This with the facts $\alpha_{j_{n}}^{n} < \alpha_{\text{thr}}$ and $u_{j}^{n} = 0$ for $j > J_{n}$, implies that some volume fraction must flow from $\Omega_{h}^{n}$ to $\mathcal{X}_{j_{n}}$. This implies that $u_{j_{n+1}} > 0$. We note here that our usage of $(\alpha - \alpha_{\text{thr}})^{+}$ in the source term is essential to ensure this property (the reaction term
cannot yield the growth above \( \alpha_{\text{thr}} \) in \( X_{J_n} \). Therefore, since \( J_{n+1} - 2 = J_n - 1 \) in this case, choosing \( j = J_n - 1 \) in (5.7), the term involving \( \alpha_{j+1}^{n+1} \) vanishes from \( \mathcal{L}(\alpha_{j-1}^{n}, \alpha_{j}^{n}, \alpha_{j+1}^{n}) \) (since it is multiplied by \( u_{J_n}^{n} \)) and we obtain

\[
\alpha_{J_{n+1}-2}^{n+1}(1 + \delta s_2) \geq \min(\alpha_{J_{n-2}}^{n}, \alpha_{J_{n-1}}^{n}) - \delta F_{\min}. \tag{5.10}
\]

The values \( \alpha_{j+1}^{n+1} \) with \( j \leq J_{n+1} - 3 \) can be dealt as in (5.9).

Combine (5.9) and (5.10) to obtain, for \( j \leq J_{n+1} - 1 \)

\[
\alpha_{j+1}^{n+1}(1 + \delta s_2) \geq \min\left( \min_{k : x_k \in \Omega_h^k} \alpha_{j}^{n}; \alpha_{\text{thr}} \right) - \delta F_{\min}.
\]

A use of \((1 + \delta s_2)^{-1} \geq \exp(-\delta s_2)\) yields

\[
\min_{j : x_j \in \Omega_h^{n+1}} \alpha_{j+1}^{n} \geq \exp(-\delta s_2) \min\left( \min_{j : x_j \in \Omega_h^{n}} \alpha_{j}^{n}; \alpha_{\text{thr}} \right) - \delta \exp(-\delta s_2) F_{\min}.
\]

This relation is obviously also true if the left-hand side is replaced by \( \alpha_{\text{thr}} \), and therefore,

\[
\min\left( \min_{j : x_j \in \Omega_h^{n+1}} \alpha_{j+1}^{n}; \alpha_{\text{thr}} \right) \geq \exp(-\delta s_2) \min\left( \min_{j : x_j \in \Omega_h^{n}} \alpha_{j}^{n}; \alpha_{\text{thr}} \right) - \delta \exp(-\delta s_2) F_{\min}. \tag{5.11}
\]

Define

\[
y_n = \exp(s_2 n \delta) \min\left( \min_{j : x_j \in \Omega_h^{n}} \alpha_{j}^{n}; \alpha_{\text{thr}} \right).
\]
The estimate (5.11) shows that
\[ y_{n+1} \geq y_n - \delta \exp(s_2 n \delta) F_{\min}. \]
Write this relation for a generic \( k \leq n \), and sum over \( k = 0, \ldots, n \) to obtain
\[ y_{n+1} \geq y_0 - \sum_{n=0}^{n} \delta \exp(s_2 n \delta) F_{\min}. \] (5.12)

The fact that the sum in (5.12) is the lower Riemann sum for the function \( \exp(s_2 \tau) \) from \( \tau = 0 \) to \( \tau = (n + 1) \delta \) yields
\[ y_{n+1} \geq y_0 - \sum_{n=0}^{n} \delta \exp(s_2 n \delta) F_{\min}. \] (5.13)

To obtain an upper bound, note that (5.7) yields \( \alpha_{n+1} \leq \max_{0 \leq j \leq J-1} \alpha_j + \delta (1 - \alpha_{\text{thr}}) + \frac{\delta}{\mu} \| (\mu a_h(\cdot) \partial_x u_h^n)^- \|_{L^\infty(0, \ell_m)}. \) (5.14)
Define the function
\[ F_{\max} = 1 - \alpha_{\text{thr}} + \ell_m \sqrt{k} \frac{|a^* - a^*|}{a^* \mu^{3/2} |1 - a^*|^{5/2}}. \] (5.15)
Then, (5.14) and (5.3b) imply
\[ \max_{0 \leq j \leq J-1} \alpha_j^{n+1} \leq \max_{0 \leq j \leq J-1} \alpha_j^n + \delta F_{\max}. \]
Write this relation for a generic \( k \leq n \) and sum over \( k = 0, \ldots, n \) to obtain
\[ \max_{0 \leq j \leq J-1} \alpha_j^{n+1} \leq \max_{0 \leq j \leq J-1} \alpha_j^0 + (n + 1) \delta F_{\max} \leq m_{02} + t_{n+1} F_{\max}. \]
Selection of time \( t_{n+1} \) such that
\[ t_{n+1} \leq \frac{a^* - m_{02}}{F_{\max}} := T_M \] (5.16)
implies \( \max_{0 \leq j \leq J-1} \alpha_j^{n+1} \leq a^* \). Finally to ensure that the extended domain \((0, \ell_m)\) contains the time-dependent domains \((0, \ell(t))\) for every \( t \in [0, T^*) \) we impose a restriction on \( T^* \). Since the domain increases at most by \( h \) at each time step, and there are \( T^* / \delta \) such time steps, we set \( T^* < \ell_t := \rho \mathcal{C} \ell_{m_0} \) such that
\[ T^* = \min(T_m, T_M, T^\epsilon) \] to conclude the proof. \( \square \)
Then, the same reasoning used to obtain the positivity implies

\[ \frac{1}{\sqrt{3}} \| \Pi_h w \|_{0, \Omega_h^N} \leq \| w \|_{0, \Omega_h^N} \leq \| \Pi_h w \|_{0, \Omega_h^N}. \]  

(5.17)

This is an easy consequence of estimating \( \| w \|_{0, \Omega_h^N} \) by Simpson’s quadrature rule, which is exact for second degree polynomials.

The existence, uniqueness, positivity and boundedness of the iterates \( c_h^{n+1} \) defined by (3.5) are proved next. The proof of the first part of next lemma is a direct consequence of the bounds on \( \alpha_{h,\delta}(t_{n+1}, \cdot) \) and Lax-Milgram lemma.

**Lemma 5.7 (Step [IS.4]).** The equation (3.5) has a unique solution \( c_h^{n+1} \), and it holds \( 0 \leq c_h^{n+1} \leq 1 \).

**Proof.** For \( r = n, n + 1 \), define the vector

\[ c_h^r := [c_h^r(x_0), c_h^r(x_1), \ldots, c_h^r(x_{J_{n+1}}-1)]. \]

Note that we do not compute \( c_h^{n+1}(x_{J_{n+1}}) \) at the discrete level since \( x_{J_{n+1}} \) is a Dirichlet boundary point at every \( t_{n+1} \). The matrix equation corresponding to (3.5) is

\[ (M + \delta \lambda D + Q \delta S) c_h^{n+1} = M c_h^n - \delta b_h, \]

where \( b_h \) is \( J_{n+1} \times 1 \) vector with entries \( b_{h,i} = 0 \) for \( 0 \leq i \leq J_{n+1} - 2 \) and \( b_{h,J_{n+1}-1} = -\lambda/h \). Here, \( M \) is the \( J_{n+1} \times J_{n+1} \) positive, diagonal, lumped mass matrix. The matrix \( D \) is the stiffness matrix with all off diagonal entries negative.

The entries of the positive, diagonal and lumped mass matrix \( S \) are as follows:

\[ S_{ij} = \sum_{X_j \subseteq \text{supp}(\varphi_{i,h})} h \alpha_j^*(\langle \Pi_h \varphi_{i,h} \rangle^2) \chi_j/2, \quad 0 \leq i \leq J_{n+1} - 1, \]

where \( \{ \varphi_{i,h} \}_{0 \leq i \leq J_{n+1} - 1} \) is the nodal basis of \( S_{h,0}^{n+1} \). The symbol \( \langle f \rangle_{X_j} \) denotes the average of \( f \) over the cell \( X_j \). An application of [20] Theorem 3.1, 3.2] shows that the discrete operator \( \epsilon_{h,\delta} := (1_{J_{n+1}} + \delta M^{-1}(\lambda D + QS))^{-1} \) is positive. A use of the facts \( \alpha_{h,\delta}(t_{n+1}, \cdot) > 0 \), \( c_h^n \geq 0 \) and \( b_h \leq 0 \) yields \( c_h^{n+1} \geq 0 \). Next, we obtain the upper bound. For \( r = n, n + 1 \), define

\[ \tilde{c}_h^r := [c_h^r(x_0) - 1, c_h^r(x_1) - 1, \ldots, c_h^r(x_{J_{n+1}}-1) - 1]. \]

It is easy to observe that

\[ (M + \delta \lambda D + Q \delta S) \tilde{c}_h^{n+1} = M \tilde{c}_h^n - \delta \hat{b}_h, \]

where \( \hat{b}_h \) has non-negative entries

\[ \hat{b}_{h,i} = \sum_{X_j \subseteq \text{supp}(\varphi_{i,h})} Q h \alpha_j^*(\Pi_h \varphi_{i,h}) \chi_j/2, \quad 0 \leq j \leq J_{n+1} - 1. \]

Then, the same reasoning used to obtain the positivity implies \( c_h^{n+1} - 1 \leq 0 \).
5.2 Compactness results

The next goal is to establish necessary compactness properties for the iterates, which enables us to extract a convergent subsequence of discrete solutions, whose limit is a threshold solution. We list the main steps involved in this section. Establish

(CR.1) a uniform $L^2(0, T_\ast; H^1(0, \ell_m))$ estimate for the family $\{ c_{h, \delta} \}_{h, \delta}$.  
(CR.2) a uniform spatial BV estimate for the family $\{ \alpha_{h, \delta} \}_{h, \delta}$.  
(CR.3) a uniform temporal BV estimate for the family $\{ \alpha_{h, \delta} \}_{h, \delta}$.  
(CR.4) a uniform BV estimate for the family $\{ \ell_{h, \delta} \}_{h, \delta}$.  
(CR.5) that the family $\{ \Pi_{h, \delta} c_{h, \delta} \}_{h, \delta}$ is relatively compact in $L^2(\mathcal{D}_T)$.  
(CR.6) a uniform $L^2(0, T_\ast; H^1(0, \ell_m))$ estimate for the family $\{ \hat{\alpha}_{h, \delta} \}_{h, \delta}$.  
(CR.7) Theorem 4.1 with the help of (CR.1) (CR.6)

In this subsection, $C_G$ denotes a generic constant that depends $\alpha_0$, $c_0$, $a_\ast$, $a^\ast$, $\ell_m$, $T_\ast$ and the model parameters. Let us start with a preliminary lemma, the proof of which is an easy consequence of local Taylor expansions.

**Lemma 5.8.** [7, Section 8.4] For any $w \in H^1(0, \ell_m)$, the following estimates hold:

\[
|w - \Pi_h w_h|_{0, (0, \ell_m)} \leq \frac{h}{2} |\partial_x w|_{0, (0, \ell_m)} \text{ and } \tag{5.18}
\]

\[
|\Pi_h w_h|_{0, (0, \ell_m)} \leq \frac{h}{2} |\partial_x w|_{0, (0, \ell_m)} + ||w||_{0, (0, \ell_m)}. \tag{5.19}
\]

We now prove an $L^2(0, T_\ast; H^1(0, \ell_m))$ stability estimate for $c_{h, \delta}$.

**Proposition 5.9** (Step (CR.1). It holds $||c_{h, \delta}||_{L^2(0, T_\ast; H^1(0, \ell_m))} \leq \mathcal{C}_G$.

**Proof.** Define the continuous function $\tilde{c}_h^n$ on $(0, \ell_m)$ by $\tilde{c}_h^n := c_h^n - 1$ in $\Omega_h^n$, and $\tilde{c}_h^n := 0$ on $(0, \ell_m) \setminus \Omega_h^n$. Note that

\[
2(\Pi_h \tilde{c}_h^{n-1}, \Pi_h \tilde{c}_h^n)_{\Omega_h^n} \leq ||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^n}^2 + ||\Pi_h \tilde{c}_h^n||_{0, \Omega_h^n}^2. \tag{5.20}
\]

If $\ell_h^n \leq \ell_h^{n-1}$, then $||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^n} \leq ||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^{n-1}}$ since $\Omega_h^0 \subseteq \Omega_h^{n-1}$. If $\ell_h^n = \ell_h^{n-1} + h$, then $\Pi_h \tilde{c}_h^{n-1} = 0$ on $\Omega_h^n \setminus \Omega_h^{n-1}$, and $||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^n} = ||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^{n-1}}$. Hence by (5.20) in any case

\[
2(\Pi_h \tilde{c}_h^{n-1}, \Pi_h \tilde{c}_h^n)_{\Omega_h^n} \leq ||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^{n-1}}^2 + ||\Pi_h \tilde{c}_h^n||_{0, \Omega_h^{n}}^2. \tag{5.21}
\]

Take $v_h^n = \tilde{c}_h^n \in S_{h, 0}^{n}$ as the test function in (3.5) with a Dirichlet lift of $-1$, and use (5.21) to obtain

\[
\frac{1}{2} ||\Pi_h \tilde{c}_h^n||_{0, \Omega_h^n}^2 - \frac{1}{2} ||\Pi_h \tilde{c}_h^{n-1}||_{0, \Omega_h^{n-1}}^2 + \delta \lambda ||\partial_x \tilde{c}_h^n||_{0, \Omega_h^n}^2 \leq -Q\delta (\alpha_h^n, \Pi_h \tilde{c}_h^n)_{\Omega_h^n}. \tag{5.21}
\]
A use of Young’s and Poincaré’s inequalities (together with (5.19)) and a summation on the index \( n \) yield
\[
\frac{1}{2} \| \Pi_h \partial_t c_h^n \|_{0, \Omega_h}^2 + \frac{\lambda \delta}{2} \sum_{r=0}^{n} \| \partial_r c_h^n \|_{0, \Omega_h}^2 \leq \mathcal{E}_G^2.
\] (5.22)

Since \( \partial_t c_h^n = \partial_\ell c_h^n \) on \( \Omega_h^n \) and \( \partial_\ell c_h^n = 0 \) outside this set, (5.22) yields a bound on \( \partial_\ell c_{h, \ell} \) in \( L^2(\mathcal{D}_T) \). We obtain the desired conclusion from the fact \( c_{h, \ell}(t, \ell_m) = 1 \) for all \( t \in (0, T) \) and a Poincaré inequality. \( \square \)

Proposition 5.9 is crucial in obtaining a bounded variation estimate for the piecewise constant function \( \alpha_{h, \ell} \). The idea is then to use Helly’s selection theorem (see Theorem B.11) to extract an almost everywhere convergent subsequence of functions out of the family of functions \( \{ \alpha_{h, \ell} \}_{h, \ell} \). A spatial BV estimate and a temporal BV estimate are derived separately.

Proposition 5.10 (Step [CR.2]). For \( t \in (0, T_*) \) it holds
\[
\| \alpha_{h, \ell}(t, \cdot) \|_{BV(a, t_m)} \leq \mathcal{E}_G.
\] (5.23)

Proof. Let \( j \in \{1, \ldots, J - 1\} \) and subtract (5.7) for \( \alpha_{j-1} \) from (5.7) for \( \alpha_j \). This yields \( T_0 = T_1 + \delta T_2 - \delta T_3 \), where
\[
T_0 = (\alpha_j^n + 1 - \alpha_{j-1}^{n+1}) + \delta((\alpha_j^n + 1 - \alpha_{\text{thr}}) + d^n_j - (\alpha_{j-1}^{n+1} - \alpha_{\text{thr}}) + d^n_{j-1}),
\]
\[
T_1 = \mathcal{L} (\alpha_j^n - \alpha_{j-1}^{n+1}, \alpha_{j+1}^n, \alpha_{j+1}^{n+1}) - \mathcal{L} (\alpha_{j-2}^n, \alpha_{j-1}^n, \alpha_j^n),
\]
\[
T_2 = (\alpha_j^n - \alpha_{\text{thr}}) + (1 - \alpha_j^n) b^n_j - (\alpha_{j-1} - \alpha_{\text{thr}}) + (1 - \alpha_{j-1}^n) b^n_{j-1},
\]
\[
T_3 = \alpha_j^n \partial_\ell u_h^n \partial_\ell \chi_j - \alpha_{j-1}^n \partial_\ell u_h^n \partial_\ell \chi_{j-1}.
\]

The terms in \( T_1 \) can be grouped in the following way:
\[
T_1 = (\alpha_j^n - \alpha_{j-1}^n) \left( 1 - \frac{\delta}{\lambda} \alpha_{j-1}^n - \frac{\delta}{\lambda} \alpha_j^n \right) + \frac{\delta}{\lambda} \alpha_{j-1}^n - \alpha_j^n.
\] (5.24a)

Split the terms in \( T_0 \) and \( T_2 \) using (B.11) in Appendix B to obtain
\[
T_0 = (\alpha_j^n + 1 - \alpha_{j-1}^{n+1}) + \delta((\alpha_j^n + 1 - \alpha_{\text{thr}}) + (\alpha_{j-1}^{n+1} - \alpha_{\text{thr}}) + d^n_j + d^n_{j-1})\frac{d^n_j}{2} - d^n_{j-1},
\] (5.24b)
\[
T_2 = ((\alpha_j^n - \alpha_{\text{thr}}) + (1 - \alpha_j^n) b^n_j - (\alpha_{j-1}^n - \alpha_{\text{thr}}) + (1 - \alpha_{j-1}^n) b^n_{j-1}) + ((\alpha_j^n - \alpha_{\text{thr}}) + (1 - \alpha_j^n) b^n_j + (\alpha_{j-1}^n - \alpha_{\text{thr}}) + (1 - \alpha_{j-1}^n) b^n_{j-1})\frac{b^n_j}{4} + b^n_{j-1}.
\] (5.24c)

Substitute (5.24a), (5.24b) and (5.24c) in \( T_0 = T_1 + \delta T_2 - \delta T_3 \), use the facts that \( d^n_j \geq 0, 0 \leq \alpha_{j-1} \leq 1, 0 \leq b^n_j \leq 1 \), the CFL condition (4.1) together with the bound

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The CFL condition (4.1) yields 1 − δ that, for all admissible
and since d
\[ (5.27) \]
and (5.27) imply
Further note that

\[ (1 - \delta s_2) \alpha_j^{n+1} - \alpha_j^{n+1} \leq \sum_{j=1}^{J} |\alpha_j^n - \alpha_j^{n-1}| \left( 1 - \frac{\delta}{h} u_j^n - \frac{\delta}{h} u_j^{n+1} \right) + \frac{\delta}{h} u_{j+1}^n \alpha_j^{n+1} - \alpha_j^n] \]
\[ + \frac{\delta}{h} u_{j-1}^n |\alpha_j^n - \alpha_j^{n-1}| + \delta |d_j^n - d_j^{n-1}| + \delta |b_j^n - b_j^{n-1}| \]
\[ + 2\delta |\alpha_j^n - \alpha_j^{n-1}| + \delta \left| \alpha_j^n \partial_x u_h^n|_{x_j} - \alpha_j^{n-1} \partial_x u_h^{n-1}|_{x_j} \right|. \quad (5.25) \]
Sum the expression \[ (5.25) \] from \( j = 1 \) to \( j = J \), and apply the facts \( u_0^n = 0 \), \( u_j^n = 0 \), \( u_{j+1}^n = 0 \) and \( 0 \leq (\delta/h) \alpha_i^n - \alpha_i^0 |u_i^0| \) to obtain

\[ (1 - \delta s_2) \sum_{j=1}^{J} |\alpha_j^{n+1} - \alpha_j^{n+1}| \leq (1 + 2\delta) \sum_{j=1}^{J} |\alpha_j^n - \alpha_j^{n-1}| + \sum_{j=1}^{J} |d_j^n - d_j^{n-1}| \]
\[ + \delta \sum_{j=1}^{J} |b_j^n - b_j^{n-1}| + \delta \sum_{j=1}^{J} |\alpha_j^n \partial_x u_h^n|_{x_j} - \alpha_j^{n-1} \partial_x u_h^{n-1}|_{x_j}|. \quad (5.26) \]
Further note that

\[ ||\mu \alpha_{h,\delta}(t_n,\cdot) \partial_x u_{h,\delta}(t_n,\cdot)||_{BV(0,\ell_m)} \leq ||\mu \alpha_{h,\delta}(t_n,\cdot) \partial_x u_{h,\delta}(t_n,\cdot) - \mathcal{H}(\alpha_{h,\delta}(t_n,\cdot))||_{BV(0,\ell_m)} \]
\[ + ||\mathcal{H}(\alpha_{h,\delta}(t_n,\cdot))||_{BV(0,\ell_m)}. \]
A use of \[ (5.3a) \] and the fact that \( \mathcal{H} \) is continuous and piecewise differentiable (see definition of \( \mathcal{H} \) from (IS.2)) yield

\[ ||\mu \alpha_{h,\delta}(t_n,\cdot) \partial_x u_{h,\delta}(t_n,\cdot)||_{BV(0,\ell_m)} \leq \mathcal{C}_G + \mathcal{C}_G ||\alpha_{h,\delta}(t_n,\cdot)||_{BV(0,\ell_m)}. \quad (5.27) \]
The CFL condition \[ (4.1) \] yields \( 1 - \delta s_2 \geq \rho \). Moreover, there exists a \( c_g > 0 \) such that, for all admissible \( \delta \), \( (1 + 2\delta)/(1 - s_2 \delta) \leq 1 + c_g \delta \). Hence the estimates \[ (5.26) \] and \[ (5.27) \] imply

\[ ||\alpha_{h,\delta}(t_{n+1},\cdot)||_{BV(0,\ell_m)} \leq (1 + c_g \delta)||\alpha_{h,\delta}(t_{n},\cdot)||_{BV(0,\ell_m)} + \delta \mathcal{C}_G (\rho \mu)^{-1} \]
\[ + \rho^{-1} \delta ||d_{h,\delta}(t_{n},\cdot)||_{BV(0,\ell_m)} + \rho^{-1} \delta ||b_{h,\delta}(t_{n},\cdot)||_{BV(0,\ell_m)}. \]
Induction on the right hand side of the above expression yields

\[ ||\alpha_{h,\delta}(t_{n+1},\cdot)||_{BV(0,\ell_m)} \leq ||\alpha_{h,\delta}(0,\cdot)||_{BV(0,\ell_m)} \exp( T_s c_g ) + \mathcal{C}_G (\rho \mu)^{-1} T_s \exp( T_s c_g ) \]
\[ + \rho^{-1} \exp( T_s c_g ) \int_0^{T_s} ||d_{h,\delta}(t,\cdot)||_{BV(0,\ell_m)} \, dt \]
\[ + \rho^{-1} \exp( T_s c_g ) \int_0^{T_s} ||b_{h,\delta}(t,\cdot)||_{BV(0,\ell_m)} \, dt, \]
and since \( d_{h,\delta} \) and \( b_{h,\delta} \) are smooth functions of \( c_{h,\delta} \), the estimates from Proposition \[ 5.9 \] conclude the proof. \( \square \)

**Proposition 5.11** (Step \[ CR.3 \]). The function \( \alpha_{h,\delta} \) satisfies the upper bound

\[ \int_0^{\ell_m} ||\alpha_{h,\delta}(\cdot, x)||_{BV(0,T_s)} \, dx \leq \mathcal{C}_G. \]
Proof. Start with (5.5) and apply (B.1) to obtain
\[
\alpha_{j+1}^n - \alpha_j^n = \delta (\alpha_j^n - \alpha_{\text{thr}}^n) + (1 - \alpha_j^n) b_j^n - \delta (\alpha_j^{n+1} - \alpha_{\text{thr}}^n) d_j^n + \frac{\delta}{h} u_{j+1}^n (\alpha_{j+1}^n - \alpha_j^n) \\
+ \frac{\delta}{h} u_j^{n+1} (\alpha_j^{n+1} - \alpha_j^n) - \frac{\delta}{h} \alpha_j^n (u_j^{n+1} - u_j^n) \\
= \delta ((\alpha_j^n - \alpha_{\text{thr}}^n) + (\alpha_j^{n+1} - \alpha_{\text{thr}}^n) ) \frac{(1 - \alpha_j^n) b_j^n - d_j^n}{2} \\
+ \delta ((\alpha_j^n - \alpha_{\text{thr}}^n) + (\alpha_j^{n+1} - \alpha_{\text{thr}}^n) ) \frac{(1 - \alpha_j^n) b_j^n + d_j^n}{2} \\
+ \frac{\delta}{h} u_j^{n+1} (\alpha_j^{n+1} - \alpha_j^n) - \frac{\delta}{h} \alpha_j^n (u_j^{n+1} - u_j^n).
\]

Use the facts that \(0 \leq b_j^n \leq 1, 0 \leq d_j^n \leq s_2, 0 \leq \alpha_j^n \leq 1, g(x) = (x - \alpha_{\text{thr}})^+\) is a Lipschitz function with Lipschitz constant one, and group the terms appropriately to obtain, for \(j = 1, \ldots, J-1\)
\[
|\alpha_{j+1}^n - \alpha_j^n| \leq \delta \left(1 + s_2 + |\alpha_j^n - \alpha_j^{n+1}| \frac{1 + s_2}{2}\right) + \frac{\delta}{h} ||u_{h,\delta}||_{L^\infty(\mathbb{R}_T)} |\alpha_j^{n+1} - \alpha_j^n| \\
+ \frac{\delta}{h} ||u_{h,\delta}||_{L^\infty(\mathbb{R}_T)} |\alpha_j^{n+1} - \alpha_j^n| + \delta ||\alpha_{h,\delta} \partial_x u_{h,\delta}||_{L^\infty(\mathbb{R}_T)}.
\]

(5.28)

Since \(u_0^n = 0\), for \(j = 0\) the same estimate holds with \(\alpha_{-1}^n := \alpha_0^n\). Multiply (5.28) by \(h\) and sum over \(j = 0, \ldots, J-1\) and \(n = 0, \ldots, N_* - 1\) with \(N_* = T_*/\delta\) to obtain
\[
\left(1 - \delta \frac{(1 + s_2)}{2}\right) \sum_{j=0}^{J-1} h \sum_{n=0}^{N_*-1} |\alpha_{j+1}^n - \alpha_j^n| \leq T_* \ell_m (1 + s_2 + ||\alpha_{h,\delta} \partial_x u_{h,\delta}||_{L^\infty(\mathbb{R}_T)}) \\
+ 2 ||u_{h,\delta}||_{L^\infty(\mathbb{R}_T)} \sum_{n=0}^{N_*-1} \frac{\delta}{h} \sum_{j=0}^{J-1} |\alpha_{j+1}^n - \alpha_j^n|.
\]

A use of the estimates (5.2), (5.3c), (5.23) and (4.1) concludes the proof. \(\square\)

Next, we need to obtain an estimate on the total variation of \(\ell_{h,\delta}\). From Proposition 5.5 it is evident that at each time step, \(\ell_{h,\delta}\) can either increase by \(h\) or decrease by any value. We show that \(\ell_{h,\delta}\) can be expressed as sum of a decreasing function and a bounded variation function as discussed in the next proposition.

Proposition 5.12 (Step CR.4)). The piecewise constant function \(\ell_{h,\delta} : [0, T_*) \to \mathbb{R}\) is of the form \(\ell_{h,\delta} = \ell_{h,\delta,BV} + \ell_{h,\delta,D}\), where \(\ell_{h,\delta,BV}\) is a function with uniform bounded variation in \((0, T_*)\) and \(\ell_{h,\delta,D}\) is a monotonically decreasing function. Consequently,
\[
\sum_{n=1}^{N_*} |\ell^n_h - \ell^{n-1}_h| \leq \mathcal{E}_G.
\]

(5.29)

Proof. Define \(\ell_{h,\delta,D}(t) = \ell_{h,\delta}(t) - (\rho \mathcal{E}_{CFL} t)^{-1} t\) and \(\ell_{h,\delta,BV}(t) = (\rho \mathcal{E}_{CFL} t)^{-1} t\), where \(\rho\) and \(\mathcal{E}_{CFL}\) are defined in (4.1). In this case \(\ell_{h,\delta}(t) = \ell_{h,\delta,BV}(t) + \ell_{h,\delta,D}(t)\). Clearly, the function \(\ell_{h,\delta,BV}\) is of uniform bounded variation. For the function \(\ell_{h,\delta,D}\) note that
\[
\ell_{h,\delta,D}(t_{n+1}) - \ell_{h,\delta,D}(t_n) = \ell_h^{n+1} - \ell_h^n - (\rho \mathcal{E}_{CFL})^{-1} \delta.
\]
If $\ell^{n+1}_h - \ell^n_h = h$ then, by [4.1], $\ell^{n+1}_h - \ell^n_h \leq (\rho E^{CF})^{-1} \delta$ and thus $\ell_{h,\delta,D}(t_{n+1}) \leq \ell_{h,\delta,D}(t_n)$. If $\ell^{n+1}_h \leq \ell^n_h$, then $\ell_{h,\delta,D}(t_{n+1}) \leq \ell_{h,\delta,D}(t_n)$, trivially. Since $\ell_{h,\delta,D}$ is decreasing and uniformly bounded, the bounded variation estimate [5.29] follows.

The compactness results for the function $c_{h,\delta}$ are proved next. Note that Proposition 5.9 already guarantees that $c_{h,\delta} \in L^2(0, T; H^1(0, \ell_m))$, and the Hilbert space structure of this space allows us to extract a weakly convergent subsequence. However, the right hand side of (3.5) involves product of two discrete functions $a_{h,\delta}$ and $\Pi_{h,\delta,c_{h,\delta}}$. Therefore, the weak convergence of $\Pi_{h,\delta,c_{h,\delta}}$ is not sufficient to prove that the limit of $\Pi_{h,\delta,c_{h,\delta}}$ is a weak solution. Similarly, (3.1) has non linear rational terms $b_{h,\delta}$ and $d_{h,\delta}$ that involve $\Pi_{h,\delta,c_{h,\delta}}$. Therefore, we require strong $L^2(\Theta_T)$ convergence for $\Pi_{h,\delta,c_{h,\delta}}$. A standard method to achieve this is to use a discrete Aubin–Simon theorem (see Theorem B.1).

We state the definition of a compactly and continuously embedded sequence of Banach spaces next.

**Definition 5.13 (Compactly-continuous embedding).** [7, p. 450, 451] Let $B$ be a Banach space. The families of Banach spaces $\{X_h, ||\cdot||_{X_h}\}_h$ and $\{Y_h, ||\cdot||_{Y_h}\}_h$ are such that $Y_h \subset X_h \subset B$. We say that the family $\{(X_h, Y_h)\}_h$ is compactly embedded in $B$ if the following conditions hold.

(C.C.1) Any sequence $\{u_h\}_h$ such that $u_h \in X_h$ and $\{||u_h||_{X_h}\}_h$ uniformly bounded is relatively compact in $B$.

(C.C.2) Any sequence $\{u_h\}_h$ such that $u_h \in X_h$, $\{||u_h||_{X_h}\}_h$ uniformly bounded, $\{u_h\}_h$ converges in $B$, and $||u_h||_{Y_h} \to 0$, converges to zero in $B$.

Define $B := L^2(0, \ell_m)$ and $X_h := \Pi_h(H^1(0, \ell_m))$ with norm

$$||u||_{X_h} := \inf \left\{ ||w||_{1,0,\ell_m} : w \in H^1(0, \ell_m), u = \Pi_h w \right\}. \quad (5.30a)$$

Set $Y_h := X_h$ with the discrete dual norm $||\cdot||_{Y_h}$ defined by: $\forall u \in Y_h$,

$$||u||_{Y_h} := \sup \left\{ \int_0^{\ell_m} u \Pi_h v \, dx : v \in H^1(0, \ell_m), ||v||_{1,0,\ell_m} \leq 1 \right\}. \quad (5.30b)$$

**Lemma 5.14.** The family of Banach spaces $\{(X_h, Y_h)\}$ with $X_h = \Pi_h(H^1(0, \ell_m)) = Y_h$ and $||\cdot||_{X_h}$ and $||\cdot||_{Y_h}$ as defined in (5.30a) and (5.30b), respectively, is compactly-continuously embedded in $B = L^2(0, \ell_m)$.

**Proof.** Firstly, we establish (CC.1). Let $\{u_h\}_h \subset B$ be a sequence of functions such that $u_h \in X_h$ and $\{||u_h||_{X_h}\}_h$ is bounded. Consider the corresponding sequence $\{w_h\}_h \subset H^1(0, \ell_m)$ such that $u_h = \Pi_h w_h$ and $||u_h||_{X_h} = ||w_h||_{1,0,\ell_m}$. The boundedness of $\{||u_h||_{X_h}\}_h$ shows that $\{||w_h||_{1,0,\ell_m}\}_h$ is also bounded. Since $H^1(0, \ell_m)$ is compactly embedded in $L^2(0, \ell_m)$, there exists a subsequence $\{w_h\}_h$ up to re-indexing such that $w_h \to w$ weakly in $H^1(0, \ell_m)$ and $w_h \to w$ strongly in $L^2(0, \ell_m)$. We claim that $u_h \to w$ strongly in $L^2(0, \ell_m)$. To prove this, use the triangle inequality and then apply (5.18) and (5.19) to obtain

\[
||u_h - w||_{0,0,\ell_m} \leq ||u_h - \Pi_h w||_{0,0,\ell_m} + ||\Pi_h w - w||_{0,0,\ell_m} \\
\leq ||\Pi_h(w_h - w)||_{0,0,\ell_m} + ||\Pi_h w - w||_{0,0,\ell_m} \\
\leq ||w_h - w||_{0,0,\ell_m} + \rho \partial_x(w_h - w)||_{0,0,\ell_m}. \quad (5.31)
\]
Since \( w_h \to w \) in \( L^2(0, \ell_m) \) while being bounded in \( H^1(0, \ell_m) \), (5.31) shows that 
\[
||u_h - w||_{0,(0,\ell_m)} \to 0 \quad \text{as} \quad h \to 0.
\]
This proves (CC.1)

Next we show (CC.2). Let \( \{u_h\} \subset B \) be such that \( u_h \in X_h \), \( ||u_h||_{X_h} \) is bounded, \( ||u_h||_{Y_h} \to 0 \) as \( h \to 0 \), and \( u_h \) converges in \( B \). Let \( w_h \in X_h \) be such that 
\[
\Pi_h u_h = u_h \quad \text{and} \quad ||w_h||_{1,(0,\ell_m)} = ||u_h||_{X_h}.
\]
Then, note that 
\[
\|u_h\|^2_{0,(0,\ell_m)} = \int_0^{\ell_m} u_h \Pi_h w_h \, dx \leq ||u_h||_{Y_h} ||w_h||_{1,(0,\ell_m)} \leq ||u_h||_{Y_h} ||u_h||_{X_h}.
\]
The assumed properties on \( \{u_h\}_h \) then show that \( u_h \to 0 \) in \( L^2(0, \ell_m) \), which concludes the proof.

To obtain the relative compactness of \( \{\Pi_{h,\delta} \hat{c}_{h,\delta}\}_h,\delta \) in \( L^2(\mathcal{D}_{T_n}) \), we start with the following definition.

**Definition 5.15.** Define \( \hat{c}_{h,\delta} := c_{h,\delta} - 1 \). For a fixed \( \epsilon > 0 \), define the auxiliary function \( \varphi_{h,\epsilon}^{n} : [0, \ell_m] \to [0, 1] \) (see Figure 4) by 
\[
\varphi_{h,\epsilon}^{n}(x) = \begin{cases} 
1 & 0 \leq x \leq \ell_h^n - \epsilon, \\
\frac{(\ell_h^n - x)}{\epsilon} & \ell_h^n - \epsilon < x \leq \ell_h^n, \\
0 & \ell_h^n < x \leq \ell_m.
\end{cases}
\]

![Figure 4: The auxiliary function \( \varphi_{h,\epsilon}^{n} \).](image)

The mass lumped function can be split into 
\[
\Pi_{h,\delta} \hat{c}_{h,\delta} = \Pi_{h,\delta}(\hat{c}_{h,\delta} \varphi_{h,\epsilon}) + \Pi_{h,\delta}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon})),
\]
where \( \varphi_{h,\epsilon} = \varphi_{h,\epsilon}^{n} \) on \( T_n \) for \( 0 \leq n \leq N_s - 1 \). Consider the second term \( \Pi_{h,\delta}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon})) \), which is equal to \( \Pi_{h}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon})) \) on \( T_n \). A use of the facts \( 1 - \varphi_{h,\epsilon} = 0 \) on \( [0, \ell_h^n - \epsilon] \), \( \Pi_{h} \hat{c}_{h}^{n} = 0 \) (see Figure 4) on \( (\ell_h^n, \ell_m) \) and the property \( \Pi_{h}(fg) = (\Pi_{h}f)(\Pi_{h}g) \) yield 
\[
||\Pi_{h}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon}))||_{0,(0,\ell_m)}^2 = \int_{\ell_h^n - \epsilon}^{\ell_h^n} ||\Pi_{h}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon}))||_{L^\infty(0,\ell_m)}^2 \, dx 
\leq \epsilon \ ||\Pi_{h}(\hat{c}_{h,\delta}(1 - \varphi_{h,\epsilon}))||_{L^\infty(0,\ell_m)}^2.
\]

(5.32)
Multiply (5.32) by \( \delta \), sum over \( n = 0, \ldots, N_* - 1 \), and use the bounds \( ||\Pi_h(1 - \varphi_{h,\epsilon})||_{L^\infty(0, \ell_m)} \leq 1 \) and \( ||\Pi_h \tilde{c}_h^n||_{L^\infty(0, \ell_m)} \leq 1 \) to obtain

\[
||\Pi_{h,\delta}(\tilde{c}_{h,\delta}(1 - \varphi_{h,\epsilon}))||_{L^2(\mathcal{D}_{T_*})} \leq \sqrt{T_* \epsilon}.
\]  

Proposition 5.18 establishes that the family of functions \( \{\Pi_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})\}_{h,\delta} \) is relatively compact in \( L^2(\mathcal{D}_{T_*}) \). Then, Proposition 5.18 and (5.33) are used to prove Theorem 5.19.

**Definition 5.16 (Discrete time derivative).** The discrete time derivative of a function \( f \) on \( \mathcal{D}_{T_*} \) is defined as follows: on \( \mathcal{T}_n \),

\[
D_{h,\delta}^n f := \frac{\Pi_h f(t_{n+1}\cdot) - \Pi_h f(t_n\cdot)}{\delta}.
\]

**Definition 5.17 (Piecewise linear interpolant).** The piecewise linear interpolator \( I_h : H^1(0, \ell_m) \to S_h \) is defined by

\[
I_h f(x) = f(x_j) \frac{x_{j+1} - x}{h} + f(x_{j+1}) \frac{x - x_j}{h} \quad \forall x \in X_j, \ j = 0, \ldots, J - 1.
\]  

We are now in a position to prove the relative compactness of \( \{\Pi_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})\}_{h,\delta} \) in \( L^2(\mathcal{D}_{T_*}) \), which is required to prove Step (CR.5).

**Proposition 5.18.** The family of functions \( \{\Pi_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})\}_{h,\delta} \) is relatively compact in \( L^2(\mathcal{D}_{T_*}) \).

**Proof.** The desired result follows from the discrete Aubin–Simon theorem (see Theorem B.IV), for which we need to verify the conditions (AS.1)–(AS.3) with \( B = L^2(0, \ell_m) \) and \( Y_h = X_h = \Pi_h(H^1(0, \ell_m)) \). The family

- (AS.1) \( \{\Pi_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})\}_{h,\delta} \) is bounded in \( L^2(0, T_*; B) \).

- (AS.2) \( \{ ||\Pi_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})||_{L^2(0, T_*; X_h)} \}_{h,\delta} \) is bounded.

- (AS.3) \( \{ ||D_{h,\delta}(\varphi_{h,\epsilon} \tilde{c}_{h,\delta})||_{L^1(0, T_*; Y_h)} \}_{h,\delta} \) is bounded.

**Step 1** (verification of (AS.1)). Proposition 5.9 and the bound \( |\varphi_{h,\epsilon}| \leq 1 \) yield the result.

**Step 2** (verification of (AS.2)). We have \( |\varphi_{h,\epsilon}| \leq 1 \) and \( |\partial_x \varphi_{h,\epsilon}| \leq 1/\epsilon \), so for all \( t \in (0, T_*) \),

\[
|\varphi_{h,\epsilon}(t, \cdot) \tilde{c}_{h,\delta}(t, \cdot)|_{1,0, \ell_m} \leq |\tilde{c}_{h,\delta}(t, \cdot)|_{1,0, \ell_m} + \epsilon^{-1} |\tilde{c}_{h,\delta}(t, \cdot)|_{0,0, \ell_m},
\]

and (AS.2) follows from (5.30) \( a\), (5.17) and Proposition 5.9 (the bound depending on \( \epsilon \)). The facts \( ||\varphi_{h,\epsilon}||_{L^\infty(0, \ell_m)} \leq 1 \), \( ||\tilde{c}_h^n||_{L^\infty(0, \ell_m)} \leq 1 \), \( |\partial_x \varphi_{h,\epsilon}| \leq 1/\epsilon \) and \( \partial_x \varphi_{h,\epsilon} = 0 \) on \( [0, \ell_h^n - \epsilon - h] \) and \( (\ell_h^n + h, \ell_m) \) yield

\[
|\varphi_{h,\epsilon} \tilde{c}_h^n|_{1,0, \ell_m}^2 \leq 2 \int_0^{\ell_m} |\partial_x \tilde{c}_h^n|^2 \, dx + 2 \int_{\ell_h^n + h}^{\ell_h^n + h - \epsilon} \frac{1}{\epsilon^2} |\tilde{c}_h^n|^2 \, dx
\]

\[
\leq 2|\tilde{c}_h^n|_{1,0, \ell_m}^2 + \frac{2(\epsilon + 2h)}{\epsilon^2},
\]

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Therefore, (3.5) with a Dirichlet lift of 
the rest of the proof, 

\[ \| \Pi_{h,\delta}(\varphi_{h,\epsilon}\hat{c}_{h,\delta}) \|_{L^2(0,T;X_h)} \leq \| \varphi_{h,\epsilon}\hat{c}_{h,\delta} \|_{L^2(0,T;H^1(0,\ell_m))} \leq c_G + \frac{2T_*(\epsilon + 2h)}{\epsilon}, \]

which verifies \([AS.2]\).

**Step 3** (verification of \([AS.3]\)). We first estimate \( ||D^{n-1}_{h,\delta}(\varphi_{h,\epsilon}\hat{c}_{h,\delta})||_{Y_h} \). Let \( v_h \in H^1(0,\ell_m) \) with \( ||v_h||_{1,(0,\ell_m)} \leq 1 \). Note that (5.34) along with the identity (B.2) yields

\[ D^{n-1}_{h,\delta}(\varphi_{h,\epsilon}\hat{c}_{h,\delta}) = (D^{n-1}_{h,\delta}\hat{c}_{h,\delta})\Pi_h\varphi^n_{h,\epsilon} + (D^{n-1}_{h,\delta}\varphi_{h,\epsilon})\Pi_h\hat{c}^{n-1}_{h}, \]

and hence

\[ \int_0^{\ell_m} D^{n-1}_{h,\delta}(\varphi_{h,\epsilon}\hat{c}_{h,\delta})\Pi_h v_h dx = \int_0^{\ell_m} (D^{n-1}_{h,\delta}\hat{c}_{h,\delta})\Pi_h\varphi^n_{h,\epsilon} v_h dx \\
+ \int_0^{\ell_m} (D^{n-1}_{h,\delta}\varphi_{h,\epsilon})\Pi_h\hat{c}^{n-1}_{h} v_h dx \]

\( =: T_1 + T_2. \)

To estimate \( T_1 \), observe that \( \varphi^n_{h,\epsilon} \) is zero on \( [\ell^n_h, \ell_m] \). Use the result \((\Pi_h f)(\Pi_h g) = \Pi_h(fg)\) to obtain

\[ T_1 = \int_0^{\ell^n_h} (D_{h,\delta}\hat{c}^{n-1}_{h})\Pi_h(\varphi^n_{h,\epsilon} v_h) dx. \]

Now observe that \( \Pi_h(\varphi^n_{h,\epsilon} v_h) = \Pi_h(\mathcal{I}_h(\varphi^n_{h,\epsilon} v_h)) \), where \( \mathcal{I}_h \) is defined by (5.35). Therefore, (3.5) with a Dirichlet lift of \(-1\) tested against \( \mathcal{I}_h(\varphi^n_{h,\epsilon} v_h) \in S^n_{h,0} \) yields

\[ T_1 = -\lambda \int_0^{\ell^n_h} \partial_x \hat{c}^{n-1}_{h}\partial_x (\mathcal{I}_h(v_h\varphi^n_{h,\epsilon})) dx - Q \int_0^{\ell^n_h} \alpha_{h,\delta}(t_n,\cdot)\Pi_h\hat{c}^{n}_{h}(v_h\varphi^n_{h,\epsilon}) dx \\
- Q \int_0^{\ell^n_h} \alpha_{h,\delta}(t_n,\cdot)\Pi_h(v_h\varphi^n_{h,\epsilon}) dx. \]

We have \( ||\mathcal{I}_h w||_{1,(0,\epsilon^n_h)} \leq ||w||_{1,(0,\epsilon^n_h)} \) and \( ||\varphi^n_{h,\epsilon} v_h||_{1,(0,\epsilon^n_h)} \leq c'(\epsilon) \), where, here and in the rest of the proof, \( c'(\epsilon) \) is a generic constant that depends on \( \epsilon \). Hence

\[ T_1 \leq c'(\epsilon)\|\partial_x \hat{c}^{n-1}_{h}\|_{0,(0,\epsilon^n_h)} + 2Q||\Pi_h\hat{c}^{n}_{h}||_{0,(0,\epsilon^n_h)} + 2Q\sqrt{\ell_m}. \]

(5.36)

Next, we estimate the term \( T_2 \). The function \( \varphi_{h,\epsilon} \) has the property \( \varphi^{n-1}_{h,\epsilon}(x) = \varphi^{n-1}_{h,\epsilon}(x - \ell^{n-1}_h + \ell^n_h) \) by definition. This with the fact that \( \varphi^n_{h,\epsilon} \) is 1/\epsilon–Lipschitz, implies \( |D_{h,\delta}^{n-1}\varphi_{h,\epsilon}| \leq |\ell^n_h - \ell^{n-1}_h|/(\delta\epsilon) \). Consequently,

\[ |T_2| \leq \frac{\ell_m}{\delta\epsilon} |\ell^n_h - \ell^{n-1}_h|. \]

(5.37)
Now let us conclude the argument. The estimates \((5.36)\) and \((5.37)\) yield
\[
\int_0^{T_\ast} \|D_h,\delta(\varphi_h,\widehat{c}_h,\delta)\|_{L^2}\,dt \leq \mathcal{C}(\epsilon) + \mathcal{C}(\epsilon) \sum_{n=1}^{N,} |\ell_h^n - \ell_h^{n-1}|^2 + \mathcal{C}(\epsilon) \sum_{n=1}^{N,} \delta(\|\Pi_h \widehat{c}_{h,n}\|_{L^2} + \|\partial_x \widehat{c}_{h,n}\|_{L^2}).
\]
Therefore, taking the supremum over the considered \(v_h\), multiplying \((5.38)\) by \(\delta\) and summing over \(n = 1, \ldots, N\) yields
\[
\int_0^{T_\ast} \|D_h,\delta(\varphi_h,\widehat{c}_h,\delta)\|_{L^2}\,dt \leq \mathcal{C}(\epsilon) + \mathcal{C}(\epsilon) \sum_{n=1}^{N,} |\ell_h^n - \ell_h^{n-1}|^2 + \mathcal{C}(\epsilon) \sum_{n=1}^{N,} \delta(\|\Pi_h \widehat{c}_{h,n}\|_{L^2} + \|\partial_x \widehat{c}_{h,n}\|_{L^2}).
\]
\((AS.3)\) then follows from an application of discrete Cauchy-Schwarz inequality, \((5.29)\) and Proposition \(5.9\).

The estimates in Steps 1, 2, and 3 yield the desired conclusion. \(\Box\)

**Theorem 5.19** (Step [CR.5]). The family of functions \(\{\Pi_h,\delta c_{h,\delta}\}_{h,\delta}\) is relatively compact in \(L^2(\mathcal{D}_T)\).

**Proof.** Since \((5.33)\) holds true, for any \(\epsilon > 0\),
\[
\{\Pi_h,\delta c_{h,\delta}\}_{h,\delta} \subset \{\Pi_h,\delta(\varphi_h,\widehat{c}_h,\delta)\}_{h,\delta} + B_{L^2(\mathcal{D}_T)} \left(0; \sqrt{T_\ast} \epsilon\right),
\]
where \(B_{L^2(\mathcal{D}_T)} \left(0; \sqrt{T_\ast} \epsilon\right)\) is the ball in \(L^2(\mathcal{D}_T)\) centered at the zero function with radius \(\sqrt{T_\ast} \epsilon\). The relative compactness of the set \(\{\Pi_h,\delta(\varphi_h,\widehat{c}_h,\delta)\}_{h,\delta}\) and \((5.39)\) show that \(\{\Pi_h,\widehat{c}_{h,\delta}\}_{h,\delta}\) can be covered by finite number of \(L^2(\mathcal{D}_T)\) balls with radius \(\eta\) for any \(\eta > 0\), hence is totally bounded in \(L^2(\mathcal{D}_T)\), and thus relatively compact. Then, the relation \(c_{h,\delta} = \widehat{c}_{h,\delta} + 1\) yields the desired result. \(\Box\)

The next result is a direct consequence of Lemma \(5.2\), Proposition \(5.5\) and \((5.2)\).

**Proposition 5.20** (Step [CR.6]). The family of functions \(\{\widehat{u}_{h,\delta}\}_{h,\delta}\) is uniformly bounded in \(L^2(0, T_\ast; H^1(0, \ell_m))\).

Now we have all compactness results to prove Theorem 4.1. In particular, we use Helly’s selection theorem for \(\{\alpha_{h,\delta}\}\) and \(\{\ell_{h,\delta}\}\), weak compactness of \(\{\widehat{u}_{h,\delta}\}\) in \(L^2(0, T_\ast; H^1(0, \ell_m))\), and relative compactness of \(\{\Pi_h,\delta c_{h,\delta}\}\) in \(L^2(\mathcal{D}_T)\).

**Proof of Theorem 4.1** (Step [CR.7] convergence of the iterates). Proposition \(5.5\) establishes the existence of a time \(T_\ast\) such that \(\alpha_{h,\delta} \in L^\infty(\mathcal{D}_T)\). Propositions \(5.10\) and \(5.11\) show that \(\alpha_{h,\delta} \in BV(\mathcal{D}_T)\). Therefore, Helly’s selection theorem guarantees the existence of a subsequence \(\{\alpha_{h,\delta}\}\) up to re-indexing and a function \(\alpha \in BV(\mathcal{D}_T) \cap L^\infty(\mathcal{D}_T)\) such that \(\alpha_{h,\delta} \rightharpoonup \alpha\) in \(L^1(\mathcal{D}_T)\) and almost everywhere in \(\mathcal{D}_T\). Proposition \(5.9\) yields a subsequence \(\{c_{h,\delta}\}_{h,\delta}\) up to re-indexing, and a function \(c \in L^2(0, T_\ast; H^1(0, \ell_m))\) such that \(c_{h,\delta} \rightharpoonup c\) and \(\partial_x c_{h,\delta} \rightharpoonup \partial_x c\) weakly in
$L^2(\mathcal{D}_{T_*})$. Theorem 5.19 establishes the strong convergence of $\Pi_{h,\delta}c_{h,\delta}$ in $L^2(\mathcal{D}_{T_*})$ and, by \cite{5.18}, $c_{h,\delta} - \Pi_{h,\delta}c_{h,\delta} \to 0$ in this space; hence, the strong limit of $\Pi_{h,\delta}c_{h,\delta}$ is $c$. An application of Proposition \cite{5.20} shows that there exist a subsequence $\{\tilde{u}_{h,\delta}\}_{h,\delta}$ and a function $\tilde{u} \in L^2(0, T_*; H^1(0, \ell_m))$ such that $\tilde{u}_{h,\delta} \to \tilde{u}$ weakly and $\partial_x\tilde{u}_{h,\delta} \to \partial_x\tilde{u}$ weakly in $L^2(\mathcal{D}_{T_*})$. Finally, Proposition \cite{5.12} shows that the family $\{\ell_{h,\delta}\}_{h,\delta}$ is bounded in $BV(0, T_*)$. Therefore, Helly’s selection theorem guarantees the existence of a function $\ell \in BV(0, T_*) \cap L^\infty(0, T_*)$ such that $\ell_{h,\delta} \to \ell$ strongly in $L^1(0, T_*)$ and almost everywhere in $(0, T_*)$.

\section{Proof of Theorem 4.2}

The proof of Theorem 4.2 involves four main steps which are listed below.

\begin{enumerate}[(CA.1)]
    \item The domains $A_{h,\delta} := \{(t, x) : x < \ell_{h,\delta}(t), t \in (0, T_*)\}$ converge to $D_{T_*}^{thr} := \{(t, x) : x < \ell(t), t \in (0, T_*)\}$ as defined in Theorem 4.2.
    \item The limit function $\alpha$ satisfies (2.1a) with $T = T_*$. \label{CA.2}
    \item The restricted limit function $\hat{u}_{D_{T_*}^{thr}}$ satisfies (2.1b) with $T = T_*$. \label{CA.3}
    \item The limit function $c_{D_{T_*}^{thr}}$ satisfies (2.1c) with $T = T_*$. \label{CA.4}
\end{enumerate}

\begin{proposition}[Step \ref{CA.1}] The characteristic functions $\chi_{A_{h,\delta}}$ converge (up to a subsequence) almost everywhere to $\chi_{D_{T_*}^{thr}}$. \label{Proposition 6.1}
\end{proposition}

\begin{proof} Theorem 4.1 yields a subsequence $\{\ell_{h,\delta}\}$ (up to re-indexing) such that $\ell_{h,\delta} \to \ell$ almost everywhere, where $\ell \in BV(0, T_*)$. Define the set $E = \{t \in (0, T_*) : \ell_{h,\delta}(t) \neq \ell(t)\}$. Let $\mu_d$ denotes the $d$-dimensional Lebesgue measure. The almost everywhere convergence of $\ell_{h,\delta}(t)$ to $\ell(t)$ implies that $\mu_1(E) = 0$. Tonelli’s theorem applied to $\chi_{E \times (0, \ell_m)}$ yields $\mu_2(E \times (0, \ell_m)) = 0$. Define the graph of $\ell$ as $F_\ell = \{(t, x) \in \mathcal{D}_{T_*} : x = \ell(t), t \in (0, T_*)\}$ (see Figure \ref{figure}). Again an application of the Tonelli’s theorem shows $\mu_{2d}(F_\ell) = 0$. Let $(t, x) \notin (E \times (0, \ell_m)) \cup F_\ell$. Then, either $\ell(t) > x$ or $\ell(t) < x$. When $\ell(t) < x$, $\chi_{A}(t, x) = 0$. Since $(t, x) \notin E \times (0, \ell_m)$, $\ell_{h,\delta}(t) \to \ell(t)$. Therefore, for $h$ and $\delta$ small enough $\ell_{h,\delta}(t) < x$. That is, $\chi_{A_{h,\delta}}(t, x) = 0$, and hence $\chi_{A_{h,\delta}}(t, x) \to \chi_A(t, x)$. A similar argument yields the convergence for the case $\ell(t) > x$. Hence we have the almost everywhere convergence $\chi_{A_{h,\delta}} \to \chi_A$. \qedhere \end{proof}

\begin{proposition}[Step \ref{CA.2}] Let $\alpha : \mathcal{D}_{T_*} \to \mathbb{R}$ be a limit provided by Theorem 4.1 such that $\alpha_{h,\delta} \to \alpha$ almost everywhere in $\mathcal{D}_{T_*}$. Then, $\alpha$ satisfies (2.1a) with $T = T_*$ for every $\varphi \in \mathcal{C}_c^\infty([0, T_*) \times (0, \ell_m))$. \label{Proposition 6.2}
\end{proposition}

\begin{proof} Let $\varphi \in \mathcal{C}_c^\infty([0, T_*) \times (0, \ell_m))$. Multiply $\ell_{n+1} - \ell_n$ by $\varphi^n :=$
\( \langle \varphi(n\delta, \cdot) \rangle \chi_j \) and sum over the indices to obtain \( T_1 + T_2 = T_3 \), where

\[
T_1 := h \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \left( \alpha_{n,j}^n + \alpha_{n,j}^n - \alpha_{n,j+1}^n + \alpha_{n,j-1}^n \right) \varphi_j^n,
\]

\[
T_2 := \delta \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \left( u_{n,j+1}^n \alpha_{n,j+1}^n - u_{n,j+1}^n \alpha_{n,j+1}^n + u_{j,j}^n \alpha_{n,j}^n - u_{j,j}^n \alpha_{n,j}^n \right) \varphi_j^n, \text{ and}
\]

\[
T_3 := h\delta \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \left( \alpha_j^n - \alpha_{\text{thr}}^n + (1 - \alpha_i^n) b_j^n - (\alpha_j^n - \alpha_{\text{thr}}^n) d_j^n \right) \varphi_j^n,
\]

with \( N_s = T_s / \delta \). The fact \( \varphi_j^{N_s} = 0 \) for all \( j \) and a use of (B.3) yield

\[
T_1 = -h \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \left( \alpha_j^{n+1} - \alpha_j^n \right) \alpha_j^{n+1} - \int_0^{\ell_0} \alpha_h^0(x) \varphi(0, x) \, dx =: T_{11} + T_{12},
\]

where \( \alpha_h^0 \) is a piecewise constant function defined by \( \alpha_h^0 \chi_j = \langle \alpha_0 \rangle \chi_j \) for \( j = 0, \ldots, J - 1 \) (see Discrete scheme [3.1]). A direct calculation yields

\[
T_{11} = -\sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \alpha_j^{n+1} \int_{\chi_j} \int_{n \delta}^{(n+1) \delta} \partial_t \varphi(t, x) \, dt
\]

\[
= -\int_0^{\ell_m} \int_0^{T_s} \alpha_{h, \delta}(t, x) \partial_t \varphi(t - \delta, x) \, dt \, dx.
\]

Since \( \alpha_{h, \delta} \to \alpha \) almost everywhere (see Theorem 4.1) as \( h, \delta \to 0 \), a use of Lebesgue’s dominated convergence theorem leads to

\[
T_{11} \to -\int_0^{\ell_m} \int_0^{T_s} \alpha(t, x) \partial_t \varphi(t, x) \, dt \, dx.
\]

Since \( \alpha_h^0 \to \alpha_0 \) in \( L^2(0, \ell_0) \) (see Discrete scheme [3.1]), \( T_{12} \to -\int_0^{\ell_0} \alpha_0(x) \varphi(0, x) \, dx \).

An application of (B.4) on \( T_2 \) yields

\[
T_2 = \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \varphi_j^n \left( u_{n,j+1}^n \alpha_{n,j+1}^n - u_{n,j+1}^n \alpha_{n,j+1}^n + u_{n,j}^n \alpha_{n,j}^n - u_{n,j}^n \alpha_{n,j}^n \right)
\]

\[
+ \delta \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-1} \varphi_j^n \left( |u_{n,j+1}^n| \alpha_{n,j+1}^n - |u_{n,j+1}^n| \alpha_{n,j+1}^n - |u_{n,j}^n| \alpha_{n,j}^n - |u_{n,j}^n| \alpha_{n,j}^n \right) =: T_{21} + T_{22}.
\]

A use of the facts \( u_0^n = 0 \) and \( u_j^n = 0 \) leads to

\[
T_{22} = \delta \sum_{n=0}^{N_s-1} \sum_{j=0}^{J-2} (\varphi_j^n - \varphi_{j+1}^n) |u_{n,j+1}^n| \frac{\alpha_j^n - \alpha_{j+1}^n}{2}.
\]

Therefore,

\[
|T_{22}| \leq \frac{h}{2} ||u_{h, \delta}||_{L^\infty(\chi_j)} ||\partial_x \varphi(t, x)||_{L^\infty(\chi_j)} \sum_{n=0}^{N_s-1} \delta \sum_{j=0}^{J-2} |\alpha_j^n - \alpha_{j+1}^n|,
\]

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and hence (5.2) and (5.23) yield $|T_{22}| \to 0$ as $h \to 0$. Use (B.3) and the facts $u^n_0 = 0$ and $\varphi^n_j = 0$ to obtain

$$T_{21} = -\delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} (\varphi^n_{j+1} - \varphi^n_j) u^n_{j+1} \frac{\alpha^n_j + \alpha^n_{j+1}}{2}. \quad (6.1)$$

Add and subtract $\delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} (\varphi^n_{j+1} - \varphi^n_j) u^n_{j+1} / 2 \alpha^n_j$ to (6.1) to obtain

$$T_{21} = \delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} \frac{u^n_{j+1} \alpha^n_j}{2} (\varphi^n_{j+1} - \varphi^n_{j-1} + \varphi^n_{j+1} - \varphi^n_{j+2})$$

$$- \delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} (\varphi^n_{j+1} - \varphi^n_j) \left( \frac{u^n_{j+1} + u^n_j}{2} \alpha^n_j \right) =: T_{211} + T_{212}.$$

We show that $T_{211}$ converges to zero. A use of the definition of $\varphi^n_j$, mean value theorem and the C.F.L. condition (4.1) yields,

$$T_{211} = \delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} \frac{u^n_{j+1} \alpha^n_j}{2} (\varphi^n_{j+1} - \varphi^n_{j-1} + \varphi^n_{j+1} - \varphi^n_{j+2})$$

$$\leq \mathcal{C}_g \delta \| u_{h,\delta} \alpha h, \delta \|_{L^\infty(\mathcal{P}_T)} \| \partial_x \varphi \|_{L^\infty(\mathcal{P}_T)} \sum_{n=0}^{N_z-1} \sum_{j=0}^J h_{n,\delta} \to 0$$

as $\delta \to 0$, where $\mathcal{C}_g$ is a constant independent of $h$ and $\delta$. Define $\partial_{h,\delta} \varphi : \mathcal{P}_T \to \mathbb{R}$ by $\partial_{h,\delta} \varphi := (\varphi^n_{j+1} - \varphi^n_j)/h$ on $T_n \times X_j$. Use the fact $u_{h,\delta} = \chi_{A_{h,\delta} \delta h, \delta}$ and the trapezoidal quadrature rule on the piecewise linear function $u_{h,\delta}$ to express sum $T_{212}$ as

$$T_{212} = -\int_0^T \int_0^\ell_m u_{h,\delta} \alpha_{h,\delta} \partial h, \delta \varphi \, dt \, dx = -\int_0^T \int_0^\ell_m \chi_{A_{h,\delta} \delta h, \delta} \partial h, \delta \varphi \, dt \, dx.$$

An application of Lemmas [BV][a] and [BV][b] yields

$$T_{212} \to -\int_0^T \int_0^\ell_m \chi_A \partial \varphi \, dt \, dx = -\int_0^T \int_0^\ell_m u \partial \varphi \, dt \, dx.$$

Let $T_3 := T_{31} - T_{32}$, where

$$T_{31} := h \delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} (\alpha^n_j - \alpha_{\text{thr}}) \alpha^n_j b^n_j \varphi^n_j$$

and

$$T_{32} := h \delta \sum_{n=0}^{N_z-1} \sum_{j=0}^{J-1} (\alpha^{n+1}_j - \alpha_{\text{thr}}) d^n_j \varphi^n_j.$$

Lemmas [BV][a] and [BV][b], and the definitions of $b^n_j$, $d^n_j$, and $\varphi^n_j$ show that

$$T_{31} = \int_0^T \int_0^\ell_m (\alpha h, \delta (t, x) - \alpha_{\text{thr}}) \alpha^n_j b^n_j \varphi^n_j \frac{(1 + s_1) \Pi_{h, \delta} c_{h, \delta} (t, x) - (1 + s_1) c \varphi (t, x) \, dx \, dt}{1 + s_1}$$

$$\to \int_0^T \int_0^\ell_m (\alpha - \alpha_{\text{thr}}) + (1 - \alpha) \frac{(1 + s_1) c}{1 + s_1} \varphi \, dx \, dt.$$
A similar argument shows that

\[
T_{32} = \int_0^T \int_0^{\ell_m} (\alpha_{h,\delta} - \alpha_{thr})^+(t + \delta, x) \frac{s_2 + s_3 \Pi_{h,\delta} c_{h,\delta}(t, x)}{1 + s_4 \Pi_{h,\delta} c_{h,\delta}(t, x)} \varphi(t_n, x) \, dx \, dt
\]

\[
\rightarrow \int_0^T \int_0^{\ell_m} (\alpha - \alpha_{thr})^+ \frac{s_2 + s_3 c}{1 + s_4 c} \varphi \, dx \, dt.
\]

Plugging the above convergences in \( T_1 + T_2 = T_3 \) concludes the proof. \( \square \)

**Proposition 6.3** (Step (CA.3)). Let \( \hat{u} : \mathcal{D}_T \to \mathbb{R} \) be the limit provided by Theorem [1.1] such that \( \hat{u}_{h,\delta} \to \hat{u} \) weakly in \( L^2(\mathcal{D}_T) \) and \( \partial_x \hat{u}_{h,\delta} \to \partial_x \hat{u} \) weakly in \( L^2(\mathcal{D}_T) \). Then, for every \( v \in H_{\partial x}^{1, u}(D_{1T}^{thr}) \) such that \( v(\cdot, 0) = 0 \), \( \hat{u}_{D_{1T}^{thr}} \) satisfies (2.1b).

**Proof.** Let \( v \in \mathcal{C}^\infty(D_{1T}^{thr}) \) with \( v(\cdot, 0) = 0 \). Redefine \( v \) to be a smooth extension to \( \mathcal{D}_T \) for ease of notation. Define \( v_{h,\delta}(t, x) = T_n v(t_n, x) \) on \( T_n \times \mathcal{X}_j \) for \( n, j \geq 0 \).

The piecewise linear in space and piecewise constant in time function \( v_{h,\delta} \) satisfies \( v_{h,\delta} \to v \) and \( \partial_x v_{h,\delta} \to \partial_x v \) strongly in \( L^2(\mathcal{D}_T) \). Take the test function as \( \varphi \delta x A_{h,\delta}(t_n, \cdot) \) in (\ref{2.1.1}) and multiply with \( \delta x A_{h,\delta}(t_n, \cdot) \), use the fact that \( u_{h,\delta} = \chi A_{h,\delta} u_{h,\delta} \) and sum over \( n = 1, \ldots, N_v - 1 \) to obtain \( T_1 + T_2 = T_3 \), where

\[
T_1 := \int_0^T \int_0^{\ell_m} \chi A_{h,\delta} \frac{k_{\alpha_{h,\delta}}}{1 - \alpha_{h,\delta}} \hat{u}_{h,\delta} v_{h,\delta} \, dx \, dt,
\]

\[
T_2 := \int_0^T \int_0^{\ell_m} \chi A_{h,\delta} \mu_{\alpha_{h,\delta}} \partial_x \hat{u}_{h,\delta} \partial_x v_{h,\delta} \, dx \, dt, \text{ and}
\]

\[
T_3 := \int_0^T \int_0^{\ell_m} \chi A_{h,\delta} \mathcal{H}(\alpha_{h,\delta}) \partial_x v_{h,\delta} \, dx \, dt.
\]

We have \( \chi A_{h,\delta} \to \chi D_{1T}^{thr} \) almost everywhere and \( \alpha_{h,\delta} \to \alpha \) in \( L^2(\mathcal{D}_T) \). Therefore, Lemmas [4V(a)] and [4V(b)] show that

\[
T_1 \to \int_0^T \int_0^{\ell_m} \chi D_{1T}^{thr} \frac{k_{\alpha}}{1 - \alpha} \hat{u} v \, dx \, dt = \iint \chi D_{1T}^{thr} \frac{k_{\alpha}}{1 - \alpha} u v \, dx \, dt.
\]

A similar argument for \( T_2 \) shows that

\[
T_2 \to \int_0^T \int_0^{\ell_m} \chi D_{1T}^{thr} \mu_{\alpha} \partial_x \hat{u} \partial_x v \, dx \, dt = \iint \chi D_{1T}^{thr} \mu_{\alpha} \partial_x u \partial_x v \, dx \, dt.
\]

Since \( \mathcal{H} \) is continuous, \( \mathcal{H}(\alpha_{h,\delta}) \to \mathcal{H}(\alpha) \) almost everywhere in \( \mathcal{D}_T \). Therefore,

\[
T_3 \to \int_0^T \int_0^{\ell_m} \chi D_{1T}^{thr} \mathcal{H}(\alpha) \partial_x v \, dx \, dt = \iint \chi D_{1T}^{thr} \mathcal{H}(\alpha) \partial_x v \, dx \, dt.
\]

These convergences, the relation \( T_1 + T_2 = T_3 \), and the density of \( \mathcal{C}^\infty(D_{1T}^{thr}) \) in \( H_{\partial x}^{1, u}(D_{1T}^{thr}) \) yield the desired result. \( \square \)

To establish (2.1c) we start with a definition and a covering lemma.
**Definition 6.4** (Right leaning type parallelogram). A right leaning type parallelogram is of the form, for some $x_0 < x_1$ and $t_0 < t_1$ (see Figure 5).

\[
P = \bigcup_{t_0 \leq t \leq t_1} \{t\} \times [x_0 - (\rho \mathcal{C}_{\text{CFL}})^{-1}(t_1 - t), x_1 - (\rho \mathcal{C}_{\text{CFL}})^{-1}(t_1 - t)] \tag{6.2}
\]

![Figure 5: The domain $A$ and $A^-$ are the geometries described in Lemma 6.5 and $P$ is a right leaning parallelogram, and $d = (\rho \mathcal{C}_{\text{CFL}})^{-1}(t_1 - t_0)$.

**Lemma 6.5** (Covering lemma). If $P$ is a right-leaning parallelogram contained in $A^- := D_{T_\nu}^{\text{thr}} \cup ([0] \times [0, \ell(0)]) \cup ([0, T] \times \mathbb{R}^\text{−})$, then there exists an $h_P > 0$ and a $\delta_P > 0$ such that, for every $h \leq h_P$ and $\delta \leq \delta_P$, $P \subseteq A^-_{h, \delta} := A_{h, \delta} \cup ([0] \times [0, \ell(0)]) \cup ([0, T] \times \mathbb{R}^\text{−})$.

**Proof.** From (6.2) and $P \subseteq A^-$, we have $\ell(t_1) > x_1 + \epsilon$ for some $\epsilon > 0$. Without loss of generality, assume that $\ell_{h, \delta}(t_1) \to \ell(t_1)$ or consider a $\ell_1$ arbitrarily close to $t_1$ such that $\ell_{h, \delta}(\ell_1) \to \ell(\ell_1)$. The existence of $\ell_1$ is guaranteed by the fact that $\ell_{h, \delta} \to \ell$ almost everywhere. In this case, there exists an $h_P$ and a $\delta_P$ such that $\ell_{h, \delta}(t_1) > x_1$ for every $h \leq h_P$ and $\delta \leq \delta_P$, which means that $\ell_{h, \delta, D}(t_1) > x_1 - \ell_{h, \delta, BV}(t_1)$, where $\ell_{h, \delta, D}$ and $\ell_{h, \delta, BV}$ are obtained from the proof of Proposition 5.12. Since $\ell_{h, \delta, D}$ is decreasing, for $t \in [t_0, t_1]$ we have $\ell_{h, \delta, D}(t) > x_1 - \ell_{h, \delta, BV}(t_1)$ and

\[
\ell_{h, \delta, D}(t) + \ell_{h, \delta, BV}(t) > x_1 - \ell_{h, \delta, BV}(t_1) + \ell_{h, \delta, BV}(t) \\
\geq x_1 - (\rho \mathcal{C}_{\text{CFL}})^{-1}(t_1 - t).
\]

Therefore, for $t \in [t_0, t_1]$, $\ell_{h, \delta}(t) > x_1 - (\rho \mathcal{C}_{\text{CFL}})^{-1}(t_1 - t)$, which yields $P \subseteq A^-_{h, \delta}$.

**Remark 6.6.** Let $v \in \mathcal{C}_c^\infty(A^-)$. Then, $\text{supp}(v)$ is compact in $A^-$ and can be covered by a finite number of right leaning type parallelograms $\{P_i\}_i$. Since there exists a $C^\infty$ partition of unity $\{\zeta_i\}_i$ subordinate to $\{P_i\}_i$, we can write $v = \sum_i v_i \zeta_i$ and $\text{supp}(v_i) \subset P_i$. Then, for any $h < h_0$ and $\delta < \delta_0$, where $h_0 = \min_i h_{P_i}$, $\delta_0 = \min_i \delta_{P_i}$, the support of $v$ is contained in $A^-_{h, \delta}$, and $v \in \mathcal{C}_c^\infty(A^-_{h, \delta})$. 

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Remark 6.7. The fact that oxygen tension satisfies the Neumann boundary condition \((1.1c)\) forces a test function in \((2.1c)\) not to vanish at the boundary \((0, T_*) \times \{0\}\) of \(D^\text{thr}_{T_*}\). This requirement forces us to consider \(A^-\) instead of \(D^\text{thr}_{T_*}\) in Lemma 6.5. Since we can extend any function \(v \in \mathcal{C}^\infty(D^\text{thr}_{T_*})\) with \(v(t, \ell(t)) = 0\) smoothly to \(A^-\), the proof of Proposition 6.8 is not affected by this consideration of \(A^-\).

Next, we show that oxygen tension \(c\) satisfies \((2.1c)\).

Proposition 6.8 (Step \((\text{CA.4})\)). Let \(c : \mathcal{D}_{T_*} \to \mathbb{R}\) be the limit provided by Theorem 4.1. Then, for every \(v \in H^1_{\text{loc}}(D^\text{thr}_{T_*})\) such that \(\partial_t v \in L^2(D^\text{thr}_{T_*})\), \(c|_{D^\text{thr}_{T_*}}\) satisfies \((2.1c)\).

Proof. Since \(v \in H^1_{\text{loc}}(D^\text{thr}_{T_*})\) can be approximated by functions in \(\mathcal{C}^\infty(D^\text{thr}_{T_*})\) with \(v(t, \ell(t)) = 0\) for all \(t \in (0, T_*)\), by Remarks 6.6 and 6.7 it is sufficient to consider functions \(v \in \mathcal{C}^\infty(P)\), where \(P \subset A^-\) is a right-leaning parallelogram.

Choose \(v \in \mathcal{C}^\infty(P)\). There exists an \(h\) and a \(\delta\) small enough such that \(v \in \mathcal{C}^\infty(A^\text{thr}_{h, \delta})\) by Remark 6.6. Define \(v_{h, \delta}(t, x) = I_h v(t_n, x)\) for \((t, x) \in T_n \times X_j\) for \(n, j \geq 0\). The piecewise linear in space and piecewise constant in time function \(v_{h, \delta}\) satisfies the following properties: (a) \(v_{h, \delta} \in L^2(0, T_*; H^1(0, \ell_m))\), (b) for \(n \geq 0\), \(v_{h, \delta}(t_n, \ell^*_m) = 0\), (c) \(v_{h, \delta} = 0\) on \(\mathcal{D}_{T_*} \setminus A^\text{thr}_{h, \delta}\), and (d) \(v_{h, \delta}(T_*, \cdot) = 0\).

In \((3.5)\), take the test function as \(v_{h, \delta}(t_n, \cdot)\) and sum over \(n = 1, \ldots, N, T_1 + T_2 = T_3\), where

\[
T_1 = \sum_{n=1}^{N_*} \int_0^{t_m} (\Pi c_{h, \delta}(t_n, x) - \Pi c_{h, \delta}(t_{n-1}, x)) \Pi v_{h, \delta}(t_n, x) \, dx,
\]

\[
T_2 := \sum_{n=1}^{N_*} \lambda \delta \int_0^{t_m} \partial_x c_{h, \delta}(t_n, x) \partial_x v_{h, \delta}(t_n, x) \, dx, \quad \text{and}
\]

\[
T_3 := -Q \sum_{n=1}^{N_*} \delta \int_0^{t_m} \alpha_{h, \delta}(t_n, x) \Pi c_{h, \delta}(t_n, x) \Pi v_{h, \delta}(t_n, x) \, dx.
\]

Note that the space integrals in \(T_1\), \(T_2\), and \(T_3\) are on \((0, \ell^*_m)\) for each \(t_n\) by the property (c). A use of \((B.3)\) leads to

\[
T_1 = -\sum_{n=1}^{N_*} \int_0^{t_m} (\Pi h v_{h, \delta}(t_n, x) - \Pi h v_{h, \delta}(t_{n-1}, x)) \Pi h c_{h, \delta}(t_n, x) \, dx
+ \int_0^{t_m} \Pi h v_{h, \delta}(T_* x) \Pi h c_{h, \delta}(T_* x) \, dx - \int_0^{t_m} \Pi h v_{h, \delta}(0, x) \Pi h c_{h, \delta}(0, x) \, dx.
\]

Using the property (c) and the strong convergences \(\Pi h c_{h, \delta}(0, \cdot) \to c_0(\cdot)\), \(\Pi h v_{h, \delta}(0, \cdot) \to v(0, \cdot)\), \(\partial_t v_{h, \delta} \to \partial_t v\), \(\Pi h c_{h, \delta} \to c\) in \(L^2(\mathcal{D}_{T_*})\), we deduce

\[
T_1 \to - \int_0^{T_*} \int_0^{t_m} c \partial_t v \, dx \, dt - \int_0^{t_m} c_0(x) v(0, x) \, dx
= - \int_{D^\text{thr}_{T_*}} c \partial_t v \, dx \, dt - \int_0^{t_m} c_0(x) v(0, x) \, dx.
\]
The weak convergence $\partial_x c_{h,\delta} \to c$, the strong convergence $\partial_x v_{h,\delta} \to \partial_x v$ in $L^2(\mathcal{D}_T)$, and an application of Lemma B.V[a] yield

$$T_2 = -\lambda \int_0^{\ell_m} \int_0^{T_*} \partial_x c_{h,\delta} \partial_x v_{h,\delta} \, dx \, dt \to -\lambda \int_0^{\ell_m} \int_0^{T_*} \partial_x c \partial_x v \, dx \, dt = -\lambda \int_{D_{h,\delta}^{\text{br}}} \partial_x c \partial_x v \, dx \, dt.$$ 

A use of Lemma B.V[b] shows that $\alpha_{h,\delta} \Pi_{h,\delta} \to \alpha c$ in $L^2(\mathcal{D}_T)$. Since $\Pi_{h,\delta} v_{h,\delta} \to v$ in $L^2(\mathcal{D}_T)$,

$$T_3 \to -Q \int_0^{T_*} \int_0^{\ell_m} \alpha c v \, dx \, dt = -Q \int_{D_{h,\delta}^{\text{br}}} \alpha c v \, dx \, dt.$$

Plugging the above convergences in $T_1 + T_2 = T_3$ yields the desired result. \hfill \Box

This concludes the proof of Theorem 4.2 and thereby convergence of the Discrete scheme 3.1 to a threshold solution (see Definition 2.1).

7 Numerical results

In Subsection 7.1 we study the dependency of $T_*$, the time below which a threshold solution exists, on the parameters $a_*$, $a^*$, $m_{02}$ and $\alpha^*$. In Subsection 7.2 we present the solution of the Discrete scheme 3.1 for a fixed set of parameters and discretisation factors, and discuss some important physical and numerical features of it.

7.1 Optimal time of existence

The time $T_*$ below which a threshold solution exists that is obtained in Proposition 5.5 depends on the parameters $a_*$, $a^*$, $m_{02}$, and $\alpha^*$. We can always fix $\ell_m$ large enough so that $\rho \mathcal{C} F_L (\ell_m - \ell_0)$ is greater than $T_m$ and $T_M$. Hence $T_*$ can be taken as the minimum of $T_m$ and $T_M$ in the proof of Proposition 5.5. The time $T_m$ provided by (5.13) is a decreasing function of $\mathcal{F}_{\text{min}}$. The fact that $\mathcal{F}_{\text{min}} \geq 0$ yields $T_m \leq \log(\alpha_{\text{thr}}/a_*)/s_2$, which precisely occurs when $a^* = \alpha^*$ (iff $\mathcal{F}_{\text{min}} = 0$). The time $T_M$ provided by (5.16) requires a more careful analysis. The domain of $T_M$ as a function of $a^*$ is $(m_{02}, 1)$. However, $T_M$ is zero at both $a^* = m_{02}$ and $a^* = 1$ (since $\lim_{a^* \to 1} \mathcal{F}_{\text{max}} = \infty$). Therefore, $T_M$ has the maximum between $a^* = m_{02}$ and $a^* = 1$. Here, we need to consider three cases. If $m_{02} > \alpha^*$, then $T_*$ attains the maximum at an $a^*$ between $m_{02}$ and 1 (see Figure 6).

If $m_{02} = \alpha^*$, then $T_M$ attains the maximum between $a^* = \alpha^*$ and $a^* = 1$. Since $T_m$ is decreasing on $[a^*, 1]$, $T_*$ attains the maximum at an $a_*$ in $(a^*, 1)$ (see Figure 7(a)). However, if $m_{02} < \alpha^*$, then $T_*$ attains maximum exactly at $a^*$ since $\mathcal{F}_{\text{max}}$ is minimal at $a^*$ and $a^* - m_{02}$ is increasing on $(m_{02}, 1)$ (see Figure 7(b)).

The time $T_M$ depends also on the lower bound $a_*$. The range of $a_*$ is $(0, \alpha_{\text{thr}})$. From (5.15) it is easy to observe that $\mathcal{F}_{\text{max}}$ is a decreasing function of $a_*$. Hence $T_*$ increases as $a_*$ approaches $\alpha_{\text{thr}}$ which is evident from Figures 6, 7 and 8.
Figure 6: Variation of $T_*$ with respect to $a^*$ and $a_*$ when $m_{02} > \alpha^* = 0.8$.

Figure 7: Variation of $T_*$ with respect to $a^*$ and $a_*$ when $m_{02} \leq \alpha^* = 0.8$.

7.2 Numerical example

The parameters are chosen as in [2]: $k = 1$, $\mu = 1$, $Q = 0.5$, $s_1 = 10 = s_4$, $s_2 = 0.5 = s_3$ and $\alpha^* = 0.8$. The bounds of the cell volume fraction are set to be $a_* = 0.4$ and $a^* = 0.82$. The extended domain length $\ell_m$ is set as 10. The threshold value is taken as $\alpha_{thr} = 0.1$. With these choices the constant $C_{CFL}$ is 0.0361. Set $\rho = 0.1$ and choose $\delta = 1E - 3$ and $h = 5E - 2$, so that the condition (4.1) is satisfied.

The final time is set to be $T_* = 50$. We plot the variation of $\alpha_{h,\delta}(t, \cdot)$, $u_{h,\delta}(t, \cdot)$ and $c_{h,\delta}(t, \cdot)$ for the times $t \in \{5, 10, \ldots, 50\}$ on the corresponding domains $(0, \ell_{h,\delta}(t))$ in Figures 9(a), 9(b), and 9(c), respectively. The variation of $\ell_{h,\delta}(t)$ with respect to time is depicted in 9(d). We observe from Figures 9(a) and 9(c) that the volume fraction and oxygen tension decrease towards $x = 0$ due to the slower diffusion of oxygen towards $x = 0$ and the accelerated cell death owing to nutrient starvation. This effect is more noticeable in larger tumours than smaller ones. The positive
Figure 8: The dependence of optimal $T_\ast$ on $a_\ast$.

Figure 9: Numerical solution of the Discrete scheme 3.3 with $\delta = 1E - 3$ and $h = 5E - 2$ is depicted. A curve in each of the Figures 9(a), 9(b), and 9(c) represents the spatial variation of cell volume fraction, cell velocity, and oxygen tension, respectively on the tumour domain $(0, \ell_{h,\delta}(t))$ at a time $t$ as colour-coded in the legends. Figure 9(d) represents the evolution of the tumour radius $\ell(t)$ with respect to the time.

The value of cell velocity towards the tumour boundary and negative value towards the interior suggests that the outermost cells flow outwards and the internal cells flow
inwards. Note that $c_{h,\delta}$ is unity at $\ell_{h,\delta}(t)$, and this unlimited supply of nutrient results in the steady increase of tumour size as illustrated in Figure 9(d).

**Remark 7.1** (Sufficiency of Theorem 4.1). The optimal value of $T_\ast$ from Subsection 7.1 is of the order of $1E - 7$ to $1E - 5$; except when $m_{02} < \alpha^\ast$, in this case $T_\ast \approx 0.12$. However, in practice, we can observe that the Discrete scheme 3.1 is convergent up to at least a time of the order of $1E + 2$ as shown in Section 7.2. In other words, the time $T_\ast$ derived in the proof of Proposition 5.5 is not restrictive, and provides a sufficient condition for the convergence.

8 Discussion

The flexible design of the tools in Sections 4 and 6 allows us to extend these results to similar models described in Subsection 8.1. Subsection 8.2 describes the challenges in extension to higher dimensional setting.

8.1 Extension to similar models

8.1.1 Cut-off model

The supremum norm bound on the cell volume fraction is required to ensure uniform coercivity and continuity of the bilinear form (3.3), continuity of the linear form (3.4), and well-posedness of (3.5). In this article, we achieved this by introducing the threshold value $\alpha_{th}$ and a careful selection of the domain $D_{th}^T$. A mathematical trick to bypass the threshold value is to consider the following modification of (1.1):

$$ku\tilde{\alpha} \left( \frac{\partial}{\partial x} \tilde{\alpha} \frac{\partial u}{\partial x} \right) = -\frac{\partial}{\partial x} (\mathcal{H}(\tilde{\alpha})),\quad \frac{\partial c}{\partial t} - \lambda \frac{\partial^2 c}{\partial x^2} = -\frac{Q\tilde{\alpha}c}{1 + Q_1c},$$

where $\tilde{\alpha} := \min(\max(\alpha, \alpha_m), \alpha_M)$ and $0 < \alpha_m < \alpha_M < 1$ are fixed positive numbers. Though this modification helps us to obtain coercivity and continuity of the bilinear forms and continuity of the linear forms (thus avoiding selection of boundary using a threshold value), it must be noted that there is no biological motivation for this modification, and the selection of the parameters $\alpha_m$ and $\alpha_M$ is heuristic. Nevertheless, the analysis in this article applies to the cut-off model also.

8.1.2 Growth in an external medium

System (1.1) can be interpreted as an ideal model of symmetric tumour growth in a free suspension (a medium surrounding the tumour is absent). The absence of a surrounding medium helps oxygen to diffuse uniformly in the vicinity of the tumour. This makes the oxygen tension equal to one on the boundary of the tumour. However, in the study of in vitro or in vivo tumour growth, where the tumour is surrounded by a medium, oxygen may not diffuse as quickly as it does in the free
Figure 10: Figure 10(a) illustrates oxygen diffusion through the medium in which tumour is embedded, and Figure 10(b) illustrates oxygen diffusion from nearby capillaries through the tissue surrounding the tumour.

Figure 10: Figure 10(a) illustrates oxygen diffusion through the medium in which tumour is embedded, and Figure 10(b) illustrates oxygen diffusion from nearby capillaries through the tissue surrounding the tumour.

8.2 Challenges in higher dimensional setting

- We frequently use the embedding result that every function in $H^1(0, \ell_m)$ is continuous and bounded. However, this result is not valid in $\mathbb{R}^2$ or $\mathbb{R}^3$. Consequently, we cannot use the energy norm estimates to obtain the boundedness of $u$ in supremum norm, which in turn is essential to obtain boundedness and bounded variation of $\alpha$.

- Secondly, to control the bounds on $\alpha$, we need an additional supremum norm estimate on $\partial_x u$ or divergence of the cell velocity field in a higher-dimensional setting. In the one-dimensional case, it is not hard to obtain these bounds. It is crucial to note that in higher dimensions, we need to consider the variations in fluid phase pressure also. The cell velocity and pressure form a
coupled system of a visco-elastic equation and a Laplace equation. We need to design a stable finite element scheme for the cell velocity-pressure system that guarantees a uniform supreme norm bound on the cell velocity field and its divergence in two and three dimensions, and this is an open problem to investigate.

- In higher dimensions, the challenges offered by the moving boundary are many fold. For instance, the moving boundary can make loops or knots, and these situations demand careful theoretical investigations.

9 Conclusion

In this paper, we achieve the following objectives: (a) design a convergent numerical scheme for the threshold model and (b) establish the existence of a threshold solution up to finite time. It is possible to extend the results derived in this article to similar models without additional work. A few embedding results used in here apply only to the one-dimensional case, and hence a direct extension to higher dimensional models is challenging. However, the article provides a proper framework to approach similar coupled problems of elliptic, hyperbolic, and parabolic equations in single or several spatial dimensions. It remains mostly open to develop a general theory for problems with degenerate equations; for instance, \( \text{(1.1b)} \) which is only non-uniformly elliptic, defined in time-dependent domains, which includes the study of well-posedness, design and analysis of numerical schemes.

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Appendix

A  Expansions of abbreviations
For \( x = 1, 2, \ldots \) expansions of the abbreviations are as follows.

| Abbreviation | Definition                     | Abbreviation | Definition                     |
|--------------|--------------------------------|--------------|--------------------------------|
| TS.x         | Threshold Solution.x           | CR.x         | Compactness Results.x          |
| DS.x         | Discrete Solution.x            | AS.x         | Aubin - Simon.x                |
| CC.x         | Compactly - Continuously.x     | CA.x         | Convergence Analysis.x         |

B  Identities and results

I. If \( a, b, c, d \in \mathbb{R} \), then the following identities hold:

\[
ab - cd = \frac{(a + c)(b - d)}{2} + \frac{(a - c)(b + d)}{2}, \quad (B.1)
\]

\[
ac - bd = (a - b)c + (c - d)b, \quad \text{and} \quad (B.2)
\]

\[
a = a^+ - a^-, |a| = a^+ + a^-,
\]

where \( a^+ = \max(a, 0) \) and \( a^- = -\min(a, 0) \).

II. Discrete integration by parts formula.\footnote{\textbf{[7] p. 464}} If \( (a_n)_{n=0, \ldots, N} \) and \( (b_n)_{n=0, \ldots, N} \) are two families of real numbers, then

\[
\sum_{n=0}^{N-1} (a_{n+1} - a_n)b_n = -\sum_{n=0}^{N-1} a_{n+1}(b_{n+1} - b_n) + a_N b_N - a_0 b_0. \quad (B.3)
\]
III. Theorem (Helly’s selection theorem). [9, Theorem 4, p. 176]. Let \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)) be an open and bounded set with a Lipschitz boundary \( \partial \Omega \), and \((f_n)_{n \in \mathbb{N}} \) be a sequence in \( BV(\Omega) \) such that \( (||f_n||_{BV(\Omega)})_n \) is uniformly bounded. Then, there exists a subsequence \((f_n)_n \) up to re-indexing and a function \( f \in BV(\Omega) \) such that as \( n \to \infty \), \( f_n \to f \) in \( L^1(U) \) and almost everywhere in \( \Omega \).

IV. Theorem (discrete Aubin-Simon). [7, Theorem C.8]. Let \( p \in [1, \infty) \), \((X_m,Y_m)_{m \in \mathbb{N}} \) be a compactly-continuously embedded sequence in a Banach space \( B \), and \((f_m)_{m \in \mathbb{N}} \) be a sequence in \( L^p(0,T;B) \), where \( T > 0 \) such that the properties \( (a), (b), \) and \( (c) \) are satisfied.

(a) Corresponding to each \( m \in \mathbb{N} \), there exists an \( N \in \mathbb{N} \), a partition \( 0 = t_0 < \cdots < t_N = T \), and a finite sequence \((g_n)_{n=0,\ldots,N} \) in \( X_m \) such that \( \forall n \in \{0,\ldots,N-1\} \) and almost every \( t \in (t_n,t_{n+1}) \), \( f_m(t) = g_n \). Then, the discrete derivative \( \delta_m f_m \) is defined almost everywhere by \( \delta_m f_m(t) := (g_{n+1} - g_n)/(t_{n+1} - t_n) \) on \( (t_n,t_{n+1}) \) for all \( n \in \{0,\ldots,N-1\} \).

(b) The sequence \((f_m)_{m \in \mathbb{N}} \) is bounded in \( L^p(0,T;B) \).

(c) The sequences \((||f_m||_{L^p(0,T;X_m)})_m \) and \((||\delta_m f_m||_{L^1(0,T;Y_m)})_m \) are bounded.

Then, \((f_m)_{m \in \mathbb{N}} \) is relatively compact in \( L^p(0,T;B) \).

V. (a) Lemma (weak-strong convergence). [7, Lemma D.8]. If \( p \in [0,\infty) \) and \( q := p/(1-p) \) are conjugate exponents, \( f_n \to f \) strongly in \( L^p(X) \), and \( g_n \to g \) weakly in \( L^q(X) \), where \((X,\mu) \) is a measured space, then

\[
\int_X f_n g_n \, d\mu \to \int_X f g \, d\mu.
\]

The next result follows from Lebesgue’s dominated convergence theorem.

(b) Lemma (bounded-strong convergence). If \( f_n \to f \) in \( L^2(X) \), \( g_n \to g \) almost everywhere on \( X \), \( ||g_n||_{L^\infty(X)} \) is uniformly bounded, then \( f_n g_n \) converges to \( fg \) in \( L^2(X) \).