DETERMINANTAL VARIETIES OVER TRUNCATED POLYNOMIAL RINGS

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Abstract. We study higher order determinantal varieties obtained by considering generic $m \times n$ ($m \leq n$) matrices over rings of the form $F[t]/(t^k)$, and for some fixed $r$, setting the coefficients of powers of $t$ of all $r \times r$ minors to zero. These varieties can be interpreted as generalized tangent bundles over the classical determinantal varieties; a special case of these varieties first appeared in a problem in commuting matrices. We show that when $r = m$, the varieties are irreducible, but when $r < m$, these varieties have at least $\lfloor k/2 \rfloor + 1$ components. In fact, when $r = 2$ (for any $k$), or when $k = 2$ (for any $r$), there are exactly $\lfloor k/2 \rfloor + 1$ components. We give formulas for the dimensions of these components in terms of $k$, $m$, and $n$. In the case of square matrices with $r = m$, we show that the ideals of our varieties are prime and that the coordinate rings are complete intersection rings, and we compute the degree of our varieties via the combinatorics of a suitable simplicial complex.

1. Introduction

Let $F$ be an algebraically closed field and $A^k_F$ the affine space of dimension $k$ over $F$. The varieties $Z_{r,n}^m \subset A^m_{F^n}$ consisting of $m \times n$ matrices ($m \leq n$) with entries in $F$ and of rank at most $r - 1$ are of course a natural and very well understood class of objects; their various geometric and algebraic properties and their connections to representation theory and combinatorics have been extensively studied (see [2] for instance). By contrast, very little is known about the following class of objects $Z_{r,k}^m, n$ that are very closely related to the classical varieties $Z_{r,n}^m$: Consider the truncated polynomial ring $R = F[t]/(t^k)$ ($k = 1, 2, 3, \ldots$), and let $X(t)$ be the generic $m \times n$ matrix over this ring; thus, each entry of $X$ is of the form $x_{i,j}(t) = x_{i,j}^{(0)} + x_{i,j}^{(1)} t + \cdots + x_{i,j}^{(k-1)} t^{k-1}$. Each $r \times r$ minor of this matrix is an element of $R = F[t]/(t^k)$. Let $I_{r,n}^{m,k}$ be the ideal of $F[x_{i,j}^{(l)}; 1 \leq i \leq m, 1 \leq j \leq n, 0 \leq l < k]$ generated by the coefficients of $t$ in each $r \times r$ minor of the generic matrix $X(t)$, and define $Z_{r,k}^m, n \subset A^m_{F^n}$ to be the zero set of $I_{r,k}^{m,n}$. These varieties $Z_{r,k}^m, n$ are therefore natural generalizations of the classical varieties $Z_{r,n}^m$, and when $k = 1$, of course, we simply recover the original $Z_{r,n}^m$. 

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Our interest in these varieties arises from previous work on commuting triples of matrices. In the paper [7], the second author and Neubauer determined the variety of commuting pairs in the centralizers of 2-regular matrices (a matrix is said to be $r$-regular if each eigenspace is at most $r$ dimensional). They observed there that when $C$ is a 2-regular $n \times n$ matrix, the variety of commuting pairs in the centralizer of $C$ is the product of $A_p^F$ (for suitable $p$) and the subvariety of $2 \times 3$ matrices over $F[t]/(t^k)$ where the coefficients of $t$ of all $2 \times 2$ minors vanish. This second factor is of course just the variety $Z_{2,k}^{2,3}$ introduced above. It was then natural to recognize $Z_{2,k}^{2,3}$ as belonging to the larger class of varieties $Z_{m,n}^{r,k}$, and to commence a program to study this larger family.

It is worth giving a geometric interpretation to these varieties $Z_{r,k}^{m,n}$ as suitable “bundles” over the classical objects $Z_{r}^{m,n}$. Suppose $f(x, y, \ldots)$ is an irreducible polynomial, and suppose we were to replace the variables $x$ by $\sum_{i=0}^{k-1} x(i)t^i$, $y$ by $\sum_{i=0}^{k-1} y(i)t^i$, and so on, and suppose we were to expand the polynomial in powers of $t$ and write it as $f_0 + f_1t + f_2t^2 + \cdots$. (Note that $f_0$ will just be $f$.) Setting each of $f_0, f_1, f_2, \ldots, f_{k-1}$ to zero is to ask for a point $(x(0), y(0), \ldots)$ on the hypersurface $f = 0$, and then to ask for those degree $k - 1$ curves $x = x(0) + \sum_{i=1}^{k-1} x(i)t^i$, $y = y(0) + \sum_{i=1}^{k-1} y(i)t^i$, and so on, parameterized by $t$, that vanish on the variety of $f$ at that point up to order $k$. Applying these considerations to the equations defining our varieties $Z_{r,k}^{m,n}$ (and recalling that the obvious defining equations for $Z_{r}^{m,n}$ also generate the ideal of $Z_{r}^{m,n}$), it is clear then that $Z_{r,k}^{m,n}$ consists of the classical varieties $Z_{r}^{m,n}$, and at each point of $Z_{r}^{m,n}$, those parameterized degree $k - 1$ curves vanishing on $Z_{r}^{m,n}$ at that point to order $k$. In particular, when $k = 2$, $Z_{r,2}^{m,n}$ may be considered as the “tangent bundle” to the classical determinantal variety $Z_{r}^{m,n}$. Of course, this is not really a bundle in the usual sense, since different fibers will have different dimensions, but we will use the words tangent bundle freely in the paper. (More accurately, $Z_{r,2}^{m,n}$ is the variety associated to the symmetric algebra on the module of derivations of the classical determinantal variety $Z_{r}^{m,n}$.)

In a different language, the varieties $Z_{r,k}^{m,n}$ appear as the $(k - 1)$-th jet scheme of the classical determinantal varieties $Z_{r}^{m,n}$ [I, §1]. From another point of view, the varieties $Z_{r,k}^{m,n}$ are the restriction (or direct image) from $R$ to $F$, in the sense of Weil ([4, I, §1, 6.6]) of the scalar extension of the classical varieties $Z_{r}^{m,n} \times_F R$.

Since in general the fibers over the base $Z_{r}^{m,n}$ will not all be of the same dimension, it is not a priori clear whether the assemblage of the base space and its fibers should be reducible or irreducible. We show
here that if we set all maximal minors to zero (that is, if we take \( r = m \)), then \( Z^{m,n}_{m,k} \) is indeed irreducible, but if we consider submaximal minors (that is, if we take \( r \leq m - 1 \)), then \( Z^{m,n}_{r,k} \) breaks up into several components, not all of the same dimension. In fact, we have a complete picture of what these various components look like in the \( r = 2 \), i.e., in the case of \( 2 \times 2 \) minors (for any \( k \)); we also have a complete picture of the components when \( k = 2 \) (for any \( r \)). In both these situations, there are exactly \( \lfloor k/2 \rfloor + 1 \) components (so in particular the tangent bundle has two components), and we have formulas for their dimensions in terms of \( k, m, \) and \( n \). In general, we show that when \( r < m \), the minimum number of components must be \( \lfloor k/2 \rfloor + 1 \). Also, from the fact that these components intersect nontrivially, we are able to show that in the submaximal minors case, our varieties are not normal, and that since the components (with one exception) are of different dimensions, that our varieties (with one possible exception) are not Cohen-Macaulay.

In the special case where \( m = n \), and where we consider maximal minors, we are able to say considerably more. We show in that case that the defining equations form a Groebner basis (with respect to a suitable ordering) for the ideal generated by these equations, and we are then able to show that the ideal is actually prime. It follows easily that the coordinate ring is a complete intersection ring. Moreover, the quotient of the polynomial ring in \( m^2 k \) variables by the ideal generated by the leading terms of these equations turns out to be the Stanley-Reisner ring of a particularly nice simplicial complex, and from the combinatorics of that complex, we are able to determine the Hilbert polynomial of our original ideal \( \mathcal{I}^{m,m}_{m,k} \).

We introduce here some alternative notation for the entries of the generic matrix that will be of much use: We will denote the \( i \)-th row of the matrix \( X(t) \) over \( \mathbb{F}[t]/(t^k) \) by \( u_i(t) \): this is an element of \( (\mathbb{F}[t]/t^k)^n \). We will write \( u_i(t) = \sum_{l=0}^{k-1} u_i^{(l)} t^l \), so the various \( u_i^{(l)} \) are row vectors from \( \mathbb{F}^n \). We will sometimes refer to \( u_i^{(l)} \) by itself as the “row \( u_i^{(l)} \).” We will also refer to a vector of the form \( u_i^{(l)} \) as being “of degree \( l \).” In a similar vein, we will talk of a variable of the form \( x_{i,j}^{(l)} \) as being of “of degree \( l \).” In particular, when we talk of a “degree zero” minor, we will mean a minor of the matrix \( X(0) = (x_{i,j}^{(0)}) \).

The methods we use in the paper are totally elementary. We also note that the paper \(^8\) discusses a related set of objects: the quantum Grassmannians, whose coordinate rings are the subalgebras of \( \mathbb{F}[x_{i,j}^{(l)}] \).
generated by the coefficients of \( t \) of the various \( m \times m \) minors of an \( m \times n \) matrix.

2. The Fundamental Reduction Process

We describe here a reduction process that exhibits our varieties \( Z_{m,n}^{r,k} \) to be a union of two subvarieties, one isomorphic to \( Z_{r-k}^{m,n} \times A_{mn}^{m(r-1)} \) (or to \( A_{mn}^{mn(k-1)} \) when \( k \leq r \)), and another whose components are in one-to-one correspondence with the components of \( Z_{r-k}^{m-1,n-1} \times A_{mn}^{(m+n-1)k} \). This reduction will be a key tool in understanding the components of \( Z_{r,k}^{m,n} \).

Lemma 2.1. The subvariety of \( Z_{r,k}^{m,n} \) where all \( x_{i,j}^{(0)} \) are zero is isomorphic to \( Z_{r-k}^{m,n} - r \times A_{mn}^{m(r-1)} \) when \( k > r \), and isomorphic to \( A_{mn}^{mn(k-1)} \) when \( k \leq r \).

Proof. This can be seen easily by writing the equations defining \( Z_{r,k}^{m,n} \) in terms of the rows \( u_i^{(l)} \). Our determinantal equations read

\[
\begin{align*}
& u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r} = 0 \\
& \sum_{d_1 + d_2 + \cdots + d_r = l} u_{i_1}^{(d_1)} \wedge u_{i_2}^{(d_2)} \wedge \cdots \wedge u_{i_r}^{(d_r)} = 0, \quad l = 0, \ldots, k - 1.
\end{align*}
\]

It is clear that if all \( u_{i,j}^{(0)} \) are zero, then all terms in the coefficients of \( t^l \), for \( l = 0, 1, \ldots, r - 1 \) become zero, since any \( r \)-fold product of degree at most \( r - 1 \) must contain at least one term of degree 0. If \( k \leq r \), this just means that there are no equations governing the remaining variables \( x_{i,j}^{(l)} \) for \( l \geq 1 \), so the subvariety is isomorphic to \( A_{mn}^{mn(k-1)} \). When \( k > r \), the equation for the coefficient of \( t^l \) for \( l = r, r + 1, \ldots, k - 1 \) now reads

\[
\sum_{d_1 + d_2 + \cdots + d_r = l} u_{i_1}^{(d_1)} \wedge u_{i_2}^{(d_2)} \wedge \cdots \wedge u_{i_r}^{(d_r)} = 0, \quad l = r, \ldots, k - 1.
\]

Observe that none of the rows \( u_i^{(k-1)}, u_i^{(k-2)}, \ldots, u_i^{(k-(r-1))} \) show up in these equations. For, every summand is an \( r \)-fold wedge product of degree \( l \), and if, for instance, \( u_i^{(k-(r-1))} \) were to appear in a summand, then the minimum degree of that summand would be \( k - (r - 1) + (r - 1) = k > k - 1 \). Thus, there are no equations governing the variables...
Let $a$ be an element of $I$ an ideal of $R$ (see [2], Prop. 2.4), for instance).

Proof. This is elementary. \hfill $\square$

But these are precisely the equations that one would obtain if one were to consider the generic matrix $m \times n$ matrix with rows $u_1^{(1)} + u_2^{(2)}t + \cdots + u_i^{(k-r)}t^{k-r-1} (1 \leq i \leq m)$ and set determinants of $r \times r$ minors to zero modulo $t^{k-r}$. This proves the lemma. \hfill $\square$

Our next theorem will be crucial to understanding the closure of the open set where at least one $x_{i,j}^{(l)}$ is nonzero. It is merely an extension to the case $k > 1$ of a result that is well known in the classical case (see [2], Prop. 2.4), for instance).

We first need the following:

**Lemma 2.2.** Let $R$ be a ring, and let $I$ be an ideal of $R[t]/(t^k)$. Let $I$ be generated by $a_1(t), \ldots, a_u(t)$, with each $a_i = \sum_{0}^{k-1} a_i^{(l)}t^l$. Let $J$ be the ideal of $R$ generated by the various $a_i^{(l)}$. Then $J = K$, where $K$ is the set of all $r \in R$ such that $r$ is the coefficient of $t^l$, for some $l$, in some element of $I$. In particular, if $I$ is also generated by $b_1(t), \ldots, b_v(t)$ with $b_i = \sum_{0}^{k-1} b_i^{(l)}t^l$, then the ideal of $R$ generated by the various $b_i^{(l)}$ also equals $J$.

**Proof.** This is elementary. \hfill $\square$

Write $S = F[x_{i,j}^{(l)}]$, and observe that in the ring $S[(x_{m,n}^{(0)})^{-1}][t]/(t^k)$, the element $x_{m,n}(t)$ is invertible. If we were to perform row reduction in $S[(x_{m,n}^{(0)})^{-1}][t]/(t^k)$ on the matrix $X$ to bring all the elements in the last column above $x_{m,n}$ to zero, we would subtract from row $u_i$ the row $u_m$ multiplied by $x_{m,n}^{-1}x_{i,n}$. Thus, we would replace $X$ by a matrix

$$Y = ((y_{i,j})),$$

where

$$y_{i,j} = \begin{cases} 
  x_{i,j} - x_{m,j}x_{i,n}x_{m,n}^{-1} & \text{for } 1 \leq i \leq m - 1, \ 1 \leq j \leq n - 1 \\
  0 & \text{for } j = n \text{ and } 1 \leq i \leq m - 1 \\
  x_{i,j} & \text{for } i = m \text{ and } 1 \leq j \leq n
\end{cases}$$
Since the inverse of \( x_{m,n}(t) \) can be written as a polynomial in the various entries \( x_{m,n}^{(l)} \) (for \( 1 \leq l \leq k-1 \)) and various negative powers of \( x_{m,n}^{(0)} \), we find that each \( y_{i,j}^{(l)} \), for \( 1 \leq i \leq m-1, 1 \leq j \leq n-1 \) can be written in terms of the \( x_{i,j}^{(l)} \) in the form

\[
y_{i,j}^{(l)} = x_{i,j}^{(l)} - q_{i,j}^{(l)}(x_{m,j}, x_{i,n}, x_{m,n}, (x_{m,n}^{(0)})^{-1}),
\]

for a suitable polynomial expression \( q_{i,j}^{(l)} \) in the indicated variables (\( 0 \leq p, r < k, 1 \leq s < k \)).

With the observations above about row reduction as our motivation, and continuing with the same notation, we have the following:

**Theorem 2.3.** (see [Prop. 2.4]) Assume \( r \geq 2 \). Let \( z_{i,j}^{(l)}, 1 \leq i \leq m-1, 1 \leq j \leq n-1, 0 \leq l < k \) be a new set of variables, and write \( T \) for the ring \( F[z_{i,j}^{(l)}] \), and \( T' \) for the ring \( F[z_{i,j}^{(l)}, x_{1,n}^{(l)}, \ldots, x_{m,n}^{(l)}, x_{m,1}^{(l)}, \ldots, x_{m,n-1}^{(l)}] \) \((0 \leq l < k)\). Also, write \( Z \) for the \( m-1 \times n-1 \) matrix \((z_{i,j}(t))\) over \( T[t]/(t^k) \), where \( z_{i,j}(t) = \sum_{l=0}^{k-1} z_{i,j}^{(l)} t^l \). We have an isomorphism

\[
S[(x_{m,n})^{-1}] \cong T'[(x_{m,n}^{(0)})^{-1}],
\]

given by

\[
f: \begin{align*}
x_{i,n}^{(l)} & \to x_{i,n}^{(l)}, 1 \leq i \leq m \\
x_{m,j}^{(l)} & \to x_{m,j}^{(l)}, 1 \leq j \leq n-1 \\
x_{i,j}^{(l)} & \to z_{i,j}^{(l)} + q_{i,j}^{(l)}(x_{m,j}, x_{i,n}, x_{m,n}, (x_{m,n}^{(0)})^{-1}), 0 \leq p, r < k, 1 \leq s < k; \\
& \text{for } 1 \leq i \leq m-1, 1 \leq j \leq n-1, 0 \leq l < k.
\end{align*}
\]

Under this isomorphism, the localization of \( I_{r,k}^{m,n} \) at \( (x_{m,n})^{-1} \) corresponds to the localization of the ideal \( I_{r-1,k}^{m-1,n-1}T' \) at \( (x_{m,n}^{(0)})^{-1} \), where \( I_{r-1,k}^{m-1,n-1} \) is the ideal of \( T \) determined by the coefficients of \( t \) of the various \((r-1) \times (r-1)\) minors of the matrix \( Z \). Moreover, this gives a one-to-one correspondence between the prime ideals \( P \) of \( S \) that are minimal over \( I_{r,k}^{m,n} \) and do not contain \( x_{m,n}^{(0)} \) and the prime ideals \( Q \) of \( T \) that are minimal over \( I_{r-1,k}^{m-1,n-1} \). In this correspondence, if \( P \) corresponds to \( Q \), then the codimension of \( P \) in \( S \) equals the codimension of \( Q \) in \( T \).
Proof. The fact that $f$ is an isomorphism is clear, since the map

$$
\tilde{f}: \begin{align*}
x^{(l)}_{i,n} &\rightarrow x^{(l)}_{i,n}, 1 \leq i \leq m \\
x^{(l)}_{m,j} &\rightarrow x^{(l)}_{m,j}, 1 \leq j \leq n - 1 \\
z^{(l)}_{i,j} &\rightarrow x^{(l)}_{i,j} - q^{(l)}_{i,j}(x^{(r)}_{m,j}, x^{(s)}_{i,n}, (x^{(0)}_{m,n})^{-1}), 0 \leq p, r < k, 1 \leq s < k, \\
&\text{for } 1 \leq i \leq m - 1, 1 \leq j \leq n - 1, 0 \leq l < k
\end{align*}
$$

provides the necessary inverse.

As for the second assertion, write $I$ for the localization of $\mathcal{I}_{r,k}^{m,n}$ at $(x^{(0)}_{m,n})^{-1}$ and $J$ for the localization of the $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$ at $(x^{(0)}_{m,n})^{-1}$. We wish to show that $f(I) = J$. Subtracting a multiple of the $m$-th row from the $i$-th row of a matrix preserves the ideal of $S[(x^{(0)}_{m,n})^{-1}]/(t^k)$ generated by $r \times r$ minors, so by Lemma 2.2, the coefficients of $t$ of the various $r \times r$ minors of the matrix $Y$ can be taken as the generators of $I$. Write $\tilde{Y}$ for the upper-left $m - 1 \times n - 1$ block of $Y$. Recall that $r \geq 2$. The $r \times r$ minors of $Y$ fall into two classes. The first class consists of minors that involve the last column of $Y$, so by Laplace expansion, these minors are either zero or of the form $x_{m,n}(t)$ times an $(r - 1) \times (r - 1)$ minor of $\tilde{Y}$. Since $x_{m,n}(t)$ is invertible, we find that up to multiplication by a unit, this class of generators is precisely the set of all $(r - 1) \times (r - 1)$ minors of $\tilde{Y}$. The second class of generators of $I$ consists of minors that do not involve the last column of $Y$. By Laplace expansion, these minors can be written as $S[(x^{(0)}_{m,n})^{-1}]$ linear combinations of suitable $(r - 1) \times (r - 1)$ minors of $\tilde{Y}$. It follows that $I$ is generated precisely by the set of all $(r - 1) \times (r - 1)$ minors of $\tilde{Y}$. On the other hand, $J$ is generated by all $(r - 1) \times (r - 1)$ minors of the matrix $Z$. Thus, under the map $f$, these generators of $I$ map precisely to generators of $J$, so $f(I) = J$.

As for the last assertion, we have a one-to-one correspondence between the minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that do not contain $x^{(0)}_{m,n}$ and the minimal primes of the localized ideal $I$ in $S[(x^{(0)}_{m,n})^{-1}]$. By the isomorphism described above, these are in one-to-one correspondence with the minimal primes of the ideal $J$ of $T'[(x^{(0)}_{m,n})^{-1}]$. These, in turn, are in one-to-one correspondence with the minimal primes of the ideal $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$ of $T'$ that do not contain $x^{(0)}_{m,n}$. But since $T'$ is just an extension of $T$ obtained by adding the indeterminates $x^{(l)}_{1,n}, \ldots, x^{(l)}_{m,n}$, $x^{(l)}_{m,1}, \ldots, x^{(l)}_{m,n-1}$, the minimal primes of $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$ are in one-to-one correspondence with the minimal primes of $\mathcal{I}_{r-1,k}^{m-1,n-1}$ in $T$; specifically, the minimal prime $Q$ of $\mathcal{I}_{r-1,k}^{m-1,n-1}$ corresponds to $Q[x^{(l)}_{1,n}, \ldots, x^{(l)}_{m,n}, x^{(l)}_{m,1}, \ldots, x^{(l)}_{m,n-1}]$. 

Moreover, tracing through this correspondence, since localization and adding indeterminates does not change the codimension of a prime ideal that avoids the localization set, we find that the correspondence preserves the codimension of the respective prime ideals in their respective rings. This gives us the assertion. □

Remark 2.4. The theorem above shows that there is a birational isomorphism between $\mathcal{Z}_{r,k}^{m,n}$ and $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbb{A}^{k(m+n-1)}$, with the domain of definition of this isomorphism being the open set of $\mathcal{Z}_{r,k}^{m,n}$ where $x_{m,n}^{(0)} \neq 0$, and the image being the open set of $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbb{A}^{k(m+n-1)}$ where the “free” variable $x_{m,n}^{(0)} \neq 0$.

Remark 2.5. Notice that there is nothing special in these considerations about the variable $x_{m,n}^{(0)}$. Essentially the same result holds for localization at any other variable $x_{i,j}^{(0)}$. The matrix $\bar{Y}$ in that case would arise from the removal of the $j$-th column and $i$-th row of the matrix $X$.

The ideas in the proof of the theorem above also lead to the following:

Proposition 2.6. Let $S = F[x_{i,j}^{(l)}]$ as before. Let $P$ be a minimal prime ideal of $\mathcal{T}_{r,k}^{m,n}$. Then for any two pairs of indices $(i,j)$ and $(i',j')$, $P$ contains $x_{i,j}^{(0)}$ iff it contains $x_{i',j'}^{(0)}$.

Proof. It is sufficient to prove this for the case where $(i',j') = (m,n)$. Let $P$ be a prime ideal minimal over $\mathcal{T}_{r,k}^{m,n}$ that does not contain $x_{m,n}^{(0)}$. Assume to the contrary that it contains $x_{m,n}^{(0)}$. Then the localization $\bar{P}$ of $P$ at $x_{m,n}^{(0)}$ will also contain $x_{i,j}^{(0)}$, and will be minimal over the localization of $f$ of $\mathcal{T}_{r,k}^{m,n}$. Hence, the ideal $f(\bar{P})$, where $f$ is as in the theorem above, will contain $f(x_{i,j}^{(0)}) = z_{i,j}^{(0)} + q_{i,j}^{(0)}$, and will be minimal over $J = f(I)$. But as is readily seen, $q_{i,j}^{(0)}$ is just $x_{i,n}^{(0)}x_{m,j}^{(0)}(x_{m,n})^{-1}$. Since $x_{m,n}^{(0)}$ is a unit, it follows that $f(P)$ will contain $x_{m,n}^{(0)}z_{i,j}^{(0)} + x_{i,n}^{(0)}x_{m,j}^{(0)}$. Under the localization map from $T'$ to $T'[(x_{m,n}^{(0)})^{-1}]$, $f(P)$ will correspond to a prime ideal $Q'$ of $T'$, that is minimal over the ideal $\mathcal{T}_{r-1,k}^{m-1,n-1}T'$. Moreover, $Q'$ will contain $x_{m,n}^{(0)}z_{i,j}^{(0)} + x_{i,n}^{(0)}x_{m,j}^{(0)}$. As we saw in the proof of the last assertion of Theorem 2.3 above, $Q'$ must be of the form $Q[x_{1,n}^{(l)}, \ldots, x_{m,n}^{(l)}, x_{m,1}^{(l)}, \ldots, x_{m,n-1}^{(l)}]$ for some minimal prime $Q$ of the ideal $\mathcal{T}_{r-1,k}^{m-1,n-1}$ of $T$. But this is impossible, since the $Q$ coefficient of $x_{i,n}^{(0)}x_{m,j}^{(0)}$ in the element $x_{m,n}^{(0)}z_{i,j}^{(0)} + x_{i,n}^{(0)}x_{m,j}^{(0)}$, namely, 1, is not in $Q$. Hence $P$ cannot contain $x_{i,j}^{(0)}$.

Conversely, if $P$ is a minimal prime ideal of $\mathcal{T}_{r,k}^{m,n}$ that does not contain $x_{i,j}^{(0)}$ but contains $x_{m,n}^{(0)}$, then the same argument as above, applied
to the corresponding isomorphism obtained on localizing at $x_{i,j}^{(0)}$ (see Remark (2.5) above), gives us a contradiction. This proves the corollary. □

We are now ready to decompose our variety $\mathcal{Z}_{r,k}^{m,n}$ as described at the beginning of this section. Let $Z_0$ represent the union of the zero sets of all those minimal prime ideals of $\mathcal{I}_{r,k}^{m,n}$ in $S = F[x_{i,j}^{(l)}]$ that do not contain some $x_{i,j}^{(0)}$ (and hence, by Proposition 2.6 above, do not contain any $x_{i,j}^{(0)}$ for $1 \leq i \leq m, 1 \leq j \leq n$). $Z_0$ is not empty: There are clearly points in our variety where $x_{i,j}^{(0)}$ is not zero, hence there exist minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that do not contain $x_{i,j}^{(0)}$ for any $(i,j)$. The following is elementary:

**Lemma 2.7.** For any pair $(i, j)$, let $U_{i,j}$ represent the open set of $\mathcal{Z}_{r,k}^{m,n}$ where $x_{i,j}^{(0)} \neq 0$, and let $\overline{U}$ represent the open set of $\mathcal{Z}_{r,k}^{m,n}$ where no $x_{i,j}^{(0)}$ (for $1 \leq i \leq m, 1 \leq j \leq n$) is zero. Then $Z_0 = \overline{U_{i,j}} = \overline{U}$, where the bar represents the closure of the respective sets.

**Proof.** It is clear that $U_{i,j} \subset Z_0$, from which it follows that $\overline{U_{i,j}} \subset Z_0$. For any minimal prime $P$ of $\mathcal{I}_{r,k}^{m,n}$ that does not contain $x_{i,j}^{(0)}$, let $Z(P)$ denote its zero set. Then $U_{i,j} \cap Z(P)$ is nonempty, since otherwise, $Z(P) \subset Z(< x_{i,j}^{(0)} >)$, which would force $x_{i,j}^{(0)} \in P$. Hence, $U_{i,j} \cap Z(P)$ is dense in $Z(P)$, so $\overline{U_{i,j}}$ must contain all of $Z(P)$. This shows that $\overline{U_{i,j}} = Z_0$. A similar argument shows that $\overline{U} = Z_0$. □

Now let $Z_1$ represent the subvariety of $\mathcal{Z}_{r,k}^{m,n}$ where all $x_{i,j}^{(0)}$, for $1 \leq i \leq m, 1 \leq j \leq n$, are zero. If there are minimal prime ideals of $\mathcal{I}_{r,k}^{m,n}$ in $S = F[x_{i,j}^{(l)}]$ that contain some $x_{i,j}^{(0)}$ (and hence, by Proposition 2.4 above, contain all $x_{i,j}^{(0)}$ for all $i, j$), then the zero sets of such prime ideals will clearly be components of $Z_1$. (These zero sets will also be components of $\mathcal{Z}_{r,k}^{m,n}$, of course. If no such minimal primes exist, then $Z_1$ will be contained in $Z_0$.)

Since every minimal prime ideal of $\mathcal{I}_{r,k}^{m,n}$ either contains some (hence all) $x_{i,j}^{(0)}$ or does not contain some (hence any) $x_{i,j}^{(0)}$, we have the following:

**Theorem 2.8.** The variety $\mathcal{Z}_{r,k}^{m,n}$ is the union of two subvarieties $Z_0$ and $Z_1$. The variety $Z_0$ is the closure of any of the open sets $U_{i,j}$ (1 $\leq i \leq m, 1 \leq j \leq n$) of $\mathcal{Z}_{r,k}^{m,n}$ where $x_{i,j}^{(0)}$ is nonzero (as also the closure of the open set $U$ where all $x_{i,j}^{(0)}$ are nonzero). $Z_0$ is also the union of the
components of \( Z_{r,k}^{m,n} \) that correspond to minimal primes of \( I_{r,k}^{m,n} \) that do not contain some (hence do not contain any) \( x_{i,j}^{(0)} \). Such components always exist, and are in one-to-one correspondence with the components of the variety \( Z_{r-1,n-1}^{m-1,k} \). The correspondence preserves the codimension (in \( A^{mnk} \) and \( A^{(m-1)(n-1)k} \) respectively) of the components. In fact, \( Z_0 \) is birational to \( Z_{r-1,k}^{m-1,n-1} \times A^{(m+n-1)k} \). The variety \( Z_1 \) is the subvariety of \( Z_{r,k}^{m,n} \) where all \( x_{i,j}^{(0)} \) are zero, and is isomorphic to \( Z_{r,k-r} \times A^{mn(r-1)} \) when \( k > r \), and isomorphic to \( A^{mn(k-1)} \) when \( k \leq r \).

The subvariety \( Z_1 \) will be wholly contained in \( Z_0 \) precisely when there are no minimal primes of \( I_{r,k}^{m,n} \) that contain some (hence all) \( x_{i,j}^{(0)} \). If there exist minimal primes of \( I_{r,k}^{m,n} \) that contain some (hence all) \( x_{i,j}^{(0)} \), then these will correspond to some (possibly all) components of \( Z_1 \).

Proof. This is just a summary of the discussions in this section. \( \square \)

3. The Case of Maximal Minors

When \( r = m \), i.e., when we consider the situation where we set all maximal minors to zero, we have the following easy result:

**Theorem 3.1.** The varieties \( Z_{m,k}^{m,n} \) are all irreducible, of codimension \( k(n-m+1) \).

Proof. We prove the irreducibility by induction on \( m \). If \( m = 1 \), then the varieties \( Z_{1,k}^{1,n} \) (for any \( n \) and \( k \)) are clearly irreducible, in fact, \( Z_{1,k}^{1,n} \) is just the origin in \( A^{nk} \). So assume that \( Z_{m-1,k}^{m-1,n-1} \) is irreducible. Then there is only one minimal prime ideal lying over the ideal \( I_{r-1,k}^{m-1,n-1} \) in the ring \( T = F[z_{i,j}^{(l)}] \) (see the statement of Theorem 2.3 for notation). Tracing through the isomorphism of Theorem 2.3 above, the localization of \( I_{r,k}^{m,n} \) at \( x_{m,n}^{(0)} \) has only one minimal prime ideal, so in particular, there is only one minimal prime ideal of \( I_{r,k}^{m,n} \) in \( S = F[x_{i,j}^{(l)}] \) that does not contain \( x_{m,n}^{(0)} \). As in the discussion in Section 2 (in particular, see Theorem 2.3), this means that subvariety \( Z_0 \) is irreducible. It is now sufficient to show that \( Z_1 \subset Z_0 \). We will do this by showing that each point in \( Z_1 \) is on a line, all but a finite number of points of which lie inside one of the open sets \( U_{i,j} \). Since \( Z_0 \) is the closure of any of the open sets \( U_{i,j} \), this will establish that \( Z_1 \subset Z_0 \).

Let \( Q \) be a point in \( Z_1 \). If \( Q \) is the origin in \( A^{mnk} \), then \( Q \) lies on the line \( \lambda P \) (\( \lambda \in F \)) for any \( P \) in any \( U_{i,j} \). (Recall that \( U_{i,j} \) is nonempty.) Since for \( \lambda \neq 0 \) the point \( \lambda P \) is in \( U_{i,j} \), our point \( Q \) must lie in the closure of \( U_{i,j} \).
Now assume $Q$ is not the origin. In the representation of $Q$ as rows $(u_1(t), \ldots, u_m(t))^T$, with $u_i(t) = \sum_l u_i^{(l)}t^l$ (see the notation introduced just before Lemma 2.1), $u_i^{(0)} = 0$ for $i = 1, \ldots, m$. Since $Q$ is not the origin, some $u_i^{(s)} \neq 0$ for some $i$ with $1 \leq i \leq m$, and some $s$ with $1 \leq s < k$ and with $s$ minimal for this $i$. Write $v(t)$ for the vector $u_i^{(s)} + u_i^{(s+1)}t + \cdots + u_i^{(k-1)}tk^{k-s-1}$ in $(F[t]/(t^k))^n$. Consider the point $P(\lambda) = (u_1(t), \ldots, u_i(t), u_{i+1}(t) + \lambda v(t), u_{i+2}(t), \ldots, u_m(t))^T$ (with the understanding that if $i = m$, then $P(\lambda) = (u_1(t), \ldots, u_{m-2}(t), u_{m-1}(t) + \lambda v(t), u_m(t))^T$). Then the $m$-fold wedge product of these vectors contains two summands: $u_i(t) \wedge \cdots \wedge u_i(t) \wedge u_{i+1}(t) \wedge \cdots \wedge u_m(t)$, and $\lambda u_i(t) \wedge \cdots \wedge u_i(t) \wedge v(t) \wedge \cdots \wedge u_m(t)$ (suitably modified if $i = m$).

The first is zero, since $Q$ is in $\mathcal{Z}_{m,k}$, and the second is zero since $u_i(t) = t^s v(t)$. Thus, the point $P(\lambda)$ is in $\mathcal{Z}_{m,k}$. When $\lambda \neq 0$, $P(\lambda)$ is actually in $U_{i,t}$ for some $l$ (corresponding to any one coordinate of $u_i$, that is nonzero). Hence, the point $Q = P(0)$ is in the closure of $U_{i,t}$, which is $Z_0$.

As for the codimension, the codimension of $\mathcal{L}_{m,k}$ is the codimension of its unique minimal prime ideal. Tracing through the localization correspondence of Theorem 2.3, this is the same as the codimension of $\mathcal{L}_{m-1,k}$. Continuing this localizing process, we find that the codimension of $\mathcal{L}_{m,n}$ is the same as the codimension of $\mathcal{L}_{1,k}^{n-m+1}$, which is clearly $k(n - m + 1)$.

\begin{remark}
Notice how this proof technique breaks down when considering $r$-fold wedge products with $r < m$: if one were to consider a wedge product that includes the $i$-th row but not the $(i+1)$-th row (or contains the $m$-th row but not the $(m-1)$-th row if $i = m$), then the second summand of the wedge product need not be zero, so the point $P(\lambda)$ need not be in our variety at all.
\end{remark}

3.1. **Square Matrices.** When $m = n$, i.e., when our matrices are square, and when we are still in the situation of maximal minors, we can say considerably more. Let us denote the $k$ coefficients of $t$ of the determinant of our square matrix $X(t)$ by $d_l$, $l = 0, \ldots, k-1$. It is easy to determine the structure of the polynomial expressions $d_l$. The first term $d_0$ is just the determinant of the matrix $X(0) = ((x_{i,j}^{(0)}))_{1 \leq i,j \leq m}$. The remaining terms can be obtained by the following process: Every monomial appearing in $d_0$ is the form

$$m_{\sigma} = x_{\sigma(1),1}^{(0)}x_{\sigma(2),2}^{(0)} \cdots x_{\sigma(m),m}^{(0)}$$
for some permutation $\sigma$ of $\{1, 2, \ldots, m\}$. Given such a monomial, we define
\[
\mu_s(m_\sigma) = \sum_{k_i \geq 0, \sum k_i = s} x_{\sigma(1)}^{(k_1)} x_{\sigma(2)}^{(k_2)} \cdots x_{\sigma(m)}^{(k_m)}
\]
We then find
\[
d_s = \sum_{\sigma \in S_n} \mu_s(m_\sigma) sgn(\sigma)
\]
We will prove that the ideals $I_{m,k}^m$ are prime and that the coordinate rings of the varieties $Z_{r,k}^r$ are complete intersection rings and hence Cohen-Macaulay. We will do so by showing that the $d_i$ form a Groebner basis for $I_{m,k}^m$ with respect to a suitable monomial ordering.

We consider the graded reverse lexicographic ordering (grevlex) on the monomials of $S = F[x_{1,j}, \ldots, x_{m,m}]$ given by the following scheme: $x_{1,1}^{(k_1)} > x_{1,2}^{(k_1)} > \cdots > x_{1,m}^{(k_1)} > x_{2,1}^{(k_2)} > \cdots > x_{m,1}^{(k_2)} > x_{1,2}^{(k_2)} > \cdots > x_{1,m}^{(k_2)} > x_{2,1}^{(k_2)} > \cdots > x_{m,1}^{(k_2)} > \cdots > x_{1,m}^{(0)} > x_{2,1}^{(0)} > \cdots > x_{m,1}^{(0)} > \cdots > x_{1,m}^{(0)} > x_{2,1}^{(0)} > \cdots > x_{m,1}^{(0)} > x_{1,m}^{(0)} > \cdots > x_{2,1}^{(0)} > \cdots > x_{m,1}^{(0)}$.

(Recall that in the graded reverse lexicographic ordering the monomials of $S$ are first ordered by the total degree, and for two monomials $\alpha$ and $\beta$ of the same degree, $\alpha$ is greater than $\beta$ if the rightmost nonzero element in $\alpha - \beta$ (with $\alpha$ and $\beta$ thought of as elements of $\mathbb{Z}^{km^2}$) is negative—see [3, Chapter 2, §2], for instance.)

**Theorem 3.3.** Under the grevlex ordering on $S$ described above, the generators $d_l$, $l = 0, 1, \ldots, k - 1$ of the ideal $I_{m,k}^m$ form a Groebner basis for $I_{m,k}^m$.

**Proof.** The grevlex order is designed so as to favor monomials in which the lower order variables do not appear. It is easy to see then that the leading monomials (LM) of the various $d_l$ are as follows:

\[
\begin{align*}
\text{LM}(d_0) &= x_{1,m}^{(0)} x_{2,m-1}^{(0)} \cdots x_{m,1}^{(0)} \\
\text{LM}(d_1) &= x_{1,m-1}^{(0)} x_{2,m-2}^{(0)} \cdots x_{m-1,1}^{(1)} x_{m,m}^{(1)} \\
\text{LM}(d_2) &= x_{1,m-2}^{(0)} x_{2,m-3}^{(0)} \cdots x_{m-2,1}^{(1)} x_{m-1,1}^{(1)} x_{m,m-1}^{(1)} \\
&\vdots \quad = \vdots \\
\text{LM}(d_{m-1}) &= x_{1,1}^{(1)} x_{2,m-1}^{(1)} \cdots x_{m,2}^{(1)} \\
\text{LM}(d_m) &= x_{1,m}^{(1)} x_{2,m-1}^{(1)} \cdots x_{m,1}^{(1)} \\
\text{LM}(d_{m+1}) &= x_{1,m-1}^{(1)} x_{2,m-2}^{(1)} \cdots x_{m-1,1}^{(1)} x_{m,m}^{(2)} \\
&\vdots \quad = \vdots
\end{align*}
\]
so in general, we have

\[ \text{LM}(d_{\lambda m+\mu}) = x_{1,m-\mu}^{(\lambda)} x_{2,m-\mu-1}^{(\lambda)} \cdots x_{m-\mu,1}^{(\lambda+1)} x_{m-\mu+1,m}^{(\lambda+1)} \cdots x_{m,m-\mu+1}^{(\lambda+1)}, \]

where \( 0 \leq \mu \leq m - 1. \)

It is clear that the leading monomials of \( d_i \) and \( d_j \), for distinct \( i \) and \( j \), are formed of sets of variables that are disjoint from one another. Hence, the leading terms of the various \( d_k \) are all pairwise relatively prime. It follows, e.g. from \([3, \text{Chapter 2, §9, Prop. 4 and Theorem 3}]\), that the \( d_l \) form a Groebner basis for \( I_{m,m}^{m,m,k} \). \( \square \)

We are now ready to prove our main result about the ideals \( I_{m,m}^{m,m,k} \).

**Theorem 3.4.** The ideals \( I_{m,k}^{m,m} \) are prime ideals, of codimension \( k \). The coordinate rings of the varieties \( Z_{m,k}^{m,m} \) are consequently complete intersection rings, and hence Cohen-Macaulay.

**Proof.** Since the polynomials \( d_l \) form a Groebner basis for \( I_{m,k}^{m,m} \), and since the lead terms of these \( d_l \) in (8) are obviously square free, it follows that the ideals \( I_{m,k}^{m,m} \) are radical. (This is well known and easy: if \( f^r \in I_{m,k}^{m,m} \), then \( (\text{LM}(f))^r \in< \text{LM}(d_0), \ldots, \text{LM}(d_{k-1}) \rangle > \), so \( \text{LM}(d_i) \) divides \( (\text{LM}(f))^r \) for some \( i \). Since \( \text{LM}(d_i) \) is square free, this means that \( \text{LM}(d_i) \) divides \( \text{LM}(f) \), so \( \text{LM}(d_i)e = \text{LM}(f) \) for some monomial \( e \). Then \( f - ed_i \) is also in the radical of \( I_{m,k}^{m,m} \), and has lower lead monomial. We proceed thus to find that \( f \) is in \( I_{m,k}^{m,m} \).) But we have already seen in Theorem 3.1 above that the radical of \( I_{m,k}^{m,m} \) must be prime. It follows that the ideals \( I_{m,k}^{m,m} \) are prime and that their codimension is equal to \( k \). Then, by the definition of a complete intersection ring, it follows that the coordinate ring \( S/I_{m,k}^{m,m} \) of the variety \( Z_{m,k}^{m,m} \) is a complete intersection ring. By e.g. \([3, \text{Chapter 18, Prop. 18.13}]\), it is Cohen-Macaulay. \( \square \)

We can obtain more insights into \( Z_{m,k}^{m,m} \) by considering the quotient ring of \( F[x_{i,j}^{(l)}] \) by the ideal \( \text{LM}(I_{m,k}^{m,m}) \) generated by the leading monomials of the elements of \( I_{m,k}^{m,m} \). By Theorem 3.3, the generators of \( \text{LM}(I_{m,k}^{m,m}) \) are given by (8) above. As is well known, the Hilbert function of \( I_{m,k}^{m,m} \) is the same as that of \( \text{LM}(I_{m,k}^{m,m}) \). In turn, because \( \text{LM}(I_{m,k}^{m,m}) \) is generated by the squarefree monomials in (8), we may consider the simplicial complex attached to \( F[x_{i,j}^{(l)}]/\text{LM}(I_{m,k}^{m,m}) \), for which this ring is the Stanley-Reisner ring. (The connection between simplicial complexes and its associated Stanley-Reisner ring may be found, for instance, in \([4, \text{Chap. 5}]\). Briefly, if \( \Delta \) is a simplicial complex on the vertex set \( \{v_1, \ldots, v_n\} \), then the Stanley-Reisner ring associated to \( \Delta \)
is the ring $F[x_1, \ldots, x_n]/I_{\Delta}$, where $I_{\Delta}$ is generated by all monomials $x_{i_1} \ldots x_{i_s}$, $s \leq n$, such that $x_{i_1} \ldots x_{i_s}$ is not a face of $\Delta$. This correspondence can be reversed: given an ideal $I$ of $F[x_1, \ldots, x_n]$ such that $I \subset (x_1, \ldots, x_n)^2$ and such that $I$ is generated by squarefree monomials, then we take $\Delta$ to be the simplicial complex on the vertex set $\{v_1, \ldots, v_n\}$ whose faces are all those subsets $\{x_{i_1}, \ldots, x_{i_s}\}$, $s \leq n$, such that $x_{i_1} \ldots x_{i_s}$ is not in $I$.

We first determine the simplicial complex attached to $\mathrm{LM}(\mathcal{I}_{m,k}^m)$. We start with the following trivial observation:

**Lemma 3.5.** The ring $F[x_{i,j}^{(l)}]/\mathrm{LM}(\mathcal{I}_{m,k}^m)$ is isomorphic to $L[x_{km+1}, \ldots, x_{km^2}]$, where $L$ is isomorphic to $F[y_1, \ldots, y_{km}]$ modded out by the ideal generated by $y_1y_2 \cdots y_m, y_{m+1}y_{m+2} \cdots y_{2m}, \ldots, y_{(k-1)m+1}y_{(k-1)m+2} \cdots y_{km}$. (Here, $y_1, \ldots, y_{km}$ are variables.)

**Proof.** Notice from Equality (3) that the leading monomials of the generators of $\mathcal{I}_{m,k}^m$ only involve $km$ variables. The form of the leading monomials now gives us the result above. \hfill $\square$

For any $l \geq 1$, write $C_l$ for the simplicial complex defined by the set of all subsets of the $l$ element set $\{x_1, \ldots, x_l\}$, and write $S_l$ for the simplicial complex defined by the set of all subsets of the $l$ element set $\{x_1, \ldots, x_l\}$ except the full set $\{x_1, \ldots, x_l\}$. (Thus, $S_l$ is the skeleton of the complex $C_l$.) Let $S_l^k$ be the join of $k$ disjoint copies of $S_l$. (The join of two simplicial complexes $\Delta_1$ and $\Delta_2$ on the disjoint vertex sets $V_1$ and $V_2$ is the complex on the vertex set $V_1 \cup V_2$ with faces $F_1 \cup F_2$, where $F_1$ is a face of $\Delta_1$ and $F_2$ is a face of $\Delta_2$.) We now have the following essentially trivial result:

**Lemma 3.6.** The simplicial complex whose Stanley-Reisner ring is the ring $F[x_{i,j}^{(l)}]/\mathrm{LM}(\mathcal{I}_{m,k}^m)$ is the complex $S_m^k \ast C_k^{(m_2-m)}$, where $\ast$ denotes the join.

**Proof.** This is clear from Lemma 3.5 above. \hfill $\square$

Recall that the face vector $f(\Delta)$ of a simplicial complex $\Delta$ of dimension $d-1$ is the $d$-tuple $(f_0, f_1, \ldots, f_{d-1})$, where $f_i$ is the number of $i$ dimensional faces of $\Delta$. Recall too that a face of dimension $s$ corresponds to a subset of cardinality $s+1$. Hence, for the simplicial complex $S_m$, $f_0 = m$, $f_1 = \binom{m}{2}$, $\ldots$, $f_{m-2} = \binom{m}{m-1}$. We have the following:

**Lemma 3.7.** The $f$-vector of the simplicial complex $S_m^k$ is determined by the coefficients of $x^l$, $l = 1, \ldots, k(m-1)$, in the $k$-th power of the polynomial $(1 + mx + \binom{m}{2}x^2 + \cdots + \binom{m}{m-1}x^{m-1})$. 

Proof. This is clear from the definition of the join of two simplicial complexes, and that the fact that the coefficients \( m, \binom{m}{2}, \) etc. in the polynomial above are just the components of the face vector of \( S_m \).
Note that an \((s - 1)\)-dimensional face of \( S^k_m \) (which corresponds to a subset of cardinality \( s \)) arises from a choice of an \((s_1 - 1)\) dimensional face from the first copy of \( S_m \) (corresponding to a subset of cardinality \( s_1 \)), an \((s_2 - 1)\) dimensional face from the second copy of \( S_m \) (corresponding to a subset of cardinality \( s_2 \)), and so on, to a choice of an \((s_k - 1)\) dimensional face from the \( k \)-th copy of \( S_m \) (corresponding to a subset of cardinality \( s_k \)), where \( s_1 + s_2 + \cdots + s_k = s \).

In the same vein, we have the following, which gives us the \( f \)-vector for the simplicial complex attached to \( F[x_{i,j}] / \text{LM}(I_{m,k}^{m,m}) \):

**Proposition 3.8.** Write \( b \) for \( k(m^2 - m) \). Then, the \( f \)-vector of the simplicial complex attached to \( F[x_{i,j}] / \text{LM}(I_{m,k}^{m,m}) \) is determined by the coefficients of \( x^l, l = 1, \ldots, k(m^2 - 1) \), in the polynomial \( (1 + mx + \binom{m}{2}x^2 + \cdots + \binom{m}{m-1}x^{m-1})^k \cdot (1 + bx + \binom{b}{2}x^2 + \cdots + \binom{b}{b-1}x^{b-1} + x^b) \).

Proof. The proof is similar to the proof of the lemma above. The simplicial complex attached to \( F[x_{i,j}] / \text{LM}(I_{m,k}^{m,m}) \) is the join of \( S^k_m \) and \( C_b \) by Lemma 3.6 above. The \( f \)-vectors for \( S^k_m \) are determined by Lemma 3.7 above. The \( f \)-vectors of \( C_b \) are easy to determine: the number of \( s - 1 \) dimensional faces is the number of subsets of a \( b \) element set of cardinality \( s \), so it is \( \binom{b}{s} \). The rest of the argument is the same.

Note immediately that the dimension of the simplicial complex attached to \( F[x_{i,j}] / \text{LM}(I_{m,k}^{m,m}) \) is \( k(m - 1) + b - 1 = k(m^2 - 1) - 1 \), and that \( f_{k(m^2 - 1) - 1} = m^k \). From the connection between the face vector of \( \Delta \) and the Hilbert polynomial of the associated Stanley Reisner ring (see [1], p. 204 for instance), and from the fact that the Hilbert function of \( F[x_{i,j}] / \text{LM}(I_{m,k}^{m,m}) \) and \( F[x_{i,j}] / I_{m,k}^{m,m} \) are the same, we have the following:

**Proposition 3.9.** The (projective) Hilbert polynomial of \( F[x_{i,j}] / I_{m,k}^{m,m} \) is given by \( H(n) = \sum_{i=0}^{k(m^2 - 1) - 1} f_i \binom{n-1}{i} \), where the \( f_i \) are the components of the face vector of the simplicial complex \( S^k_m \ast C_{k(m^2-m)} \) determined by Proposition 3.8. In particular, the degree of \( Z_{m,k}^{m,m} \) is \( m^k \).

Proof. The degree of \( Z_{m,k}^{m,m} \) is just the coefficient \( f_{k(m^2 - 1) - 1} \) attached to the highest degree term \( \binom{n-1}{2} \) in the (projective) Hilbert polynomial when it is expressed as a linear combination of the polynomials \( \binom{n-1}{i} \). □
Remark 3.10. The Hilbert polynomial above also shows the (projective) dimension of $Z_{m,k}^{m,m}$ to be $k(m^2 - 1) - 1$, which is consistent with Theorem 3.1.

4. Equations for Some Open Sets of $Z_{r,k}^{m,n}$

In this section, we will derive the key equations that will hold in certain open sets of our variety and will enable us to show that our varieties are reducible when $r < m$ (i.e., we set submaximal minors to zero).

We start with an elementary and well-known result:

Lemma 4.1. Let $R$ be a commutative ring, and $R^n$ the free module of rank $n$. Suppose $u_1, \ldots, u_{r-1} \in R^n$ are such that for some $w_1, \ldots, w_{n-r+1} \in R^n$, the product $u_1 \wedge \cdots \wedge u_{r-1} \wedge w_1 \wedge \cdots \wedge w_{n-r+1} \in R^s$ (i.e., the elements $u_1, \ldots, u_{r-1}$ can be extended to a basis of $R^n$). Here $R^s$ denotes the set of all invertible elements of $R$. If $u_1 \wedge \cdots \wedge u_{r-1} \wedge v = 0$ for some element $v \in R^n$, then $v = \sum_{i=1}^{r-1} \alpha_i u_i$ for some $\alpha_i \in R$.

Proof. Since $u_1 \wedge \cdots \wedge u_{r-1} \wedge w_1 \wedge \cdots \wedge w_{n-r+1}$ is the determinant of the matrix $[u_1, \ldots, u_{r-1}, w_1, \ldots, w_{n-r+1}]$, the hypothesis shows that this matrix is invertible in $R$. Hence we can find the unique solution $x = (\alpha_1, \ldots, \alpha_n)^T$ to the equation

$[u_1, \ldots, u_{r-1}, w_1, \ldots, w_{n-r+1}]x = v$

by Cramer’s rule. But the assumption $u_1 \wedge \cdots \wedge u_{r-1} \wedge v = 0$ shows that $\alpha_j$ must be zero for $j = r, \ldots, n$, since for such $j$, one of the $w_i$ will be replaced by $v$ during the solution. Hence $v = \sum_{i=1}^{r-1} \alpha_i u_i$. □

For the rest of this section we will write $R$ for the polynomial ring $F[t]$, and write $\overline{R}$ for the ring $F[t]/(t^k)$. Let $M$ be the free $R$ module of rank $n$, and $\overline{M} = M \otimes_R F[t]/(t^k)$ be the free $\overline{R}$ module of rank $n$. For any $v \in M$, write $\overline{v}$ for the image of $v$ under the map $M \mapsto \overline{M} = M/t^k M$.

Lemma 4.1 leads to the following:

Corollary 4.2. Suppose $u_1, \ldots, u_r \in \overline{M}$ are such that $u_1 \wedge \cdots \wedge u_r = 0$ and $u_1^{(0)} \wedge \cdots \wedge u_r^{(0)} \neq 0$ in $\bigwedge^{(r-1)} F^n$ (here, $u_j = \sum_{i=0}^{k-1} u_j^{(i)} t^i + t^k M$). Then, there are $\alpha_1, \ldots, \alpha_{r-1} \in \overline{R}$ such that $u_r = \sum_{i=1}^{r-1} \alpha_i u_i$.

Proof. Since $u_1^{(0)} \wedge \cdots \wedge u_r^{(0)} \neq 0$ in $\bigwedge^{(r-1)} F^n$, there are elements $w_1, \ldots, w_{n-r+1} \in F^n$ such that $u_1^{(0)}, \ldots, u_r^{(0)}, w_1, \ldots, w_{n-r+1}$ form a basis for the vector space $F^n$. In particular, $u_1^{(0)} \wedge \cdots \wedge u_r^{(0)} \wedge w_1 \wedge \cdots \wedge w_{n-r+1} \neq 0$ in $F$. Thus, the constant term in the wedge product
u_1 \wedge \cdots \wedge u_{r-1} \wedge w_1 \wedge \cdots \wedge w_{n-r+1} will be nonzero, so u_1 \wedge \cdots \wedge u_{r-1} \wedge w_1 \wedge \cdots \wedge w_{n-r+1} is in \overline{R^r}. The result now follows from Lemma 4.1. □

We now come to the main result that generates equations for certain open sets of our variety:

**Theorem 4.3.** Suppose u_1, \ldots, u_m \in R^n (with m ≤ n) are of degree at most k−1 (that is, each component of each u_j is a polynomial of degree at most k−1 in t), and suppose that \overline{u_{j_1}} \wedge \cdots \wedge \overline{u_{j_r}} = 0 for all sequences 1 ≤ j_1 < \cdots < j_r ≤ m. Further, assume that u_1^{(0)} \wedge \cdots \wedge u_r^{(0)} \neq 0 (here, u_j = \sum_{l=0}^{k} u_j^{(l)} t^l). Then

\[u_{j_1} \wedge \cdots \wedge u_{j_r} \wedge u_{j_{r+1}} \in t^{2k} \bigwedge M\]

for all sequences 1 ≤ j_1 < \cdots < j_r < j_{r+1} ≤ m.

**Proof.** Applying Corollary 4.2 to the elements \overline{u_1}, \ldots, \overline{u_{r-1}}, \overline{u_r} (r ≤ j ≤ m), we find

\[\overline{u_j} = \sum_{i=1}^{r-1} \alpha_{j,i} \overline{u_i}\]

for elements \alpha_{j,i} in \overline{R}. If \alpha_{j,i} = p_{j,i}(t) + t^k R for uniquely determined polynomials p_{j,i} ∈ R of degree at most k−1, define

\[v_j = \begin{cases} u_j & j = 1, \ldots, r-1, \\ \sum_{i=1}^{r-1} p_{j,i}u_i & j = r, r+1, \ldots, m. \end{cases}\]

Then, since the v_j depend linearly on the r−1 vectors v_1, \ldots, v_{r-1}, we find, this time in M, that

\[v_{j_1} \wedge \cdots \wedge v_{j_r} \wedge v_{j_{r+1}} = 0\]

for all sequences 1 ≤ j_1 < \cdots < j_r < j_{r+1} ≤ m.

Now, for any j, u_j and v_j are equal modulo t^k, so we may write u_j = v_j + t^k y_j for suitable y_j. Then,

\[u_{j_1} \wedge \cdots \wedge u_{j_r} \wedge u_{j_{r+1}} = (v_{j_1} + t^k y_{j_1}) \wedge \cdots \wedge (v_{j_{r+1}} + t^k y_{j_{r+1}}).\]

Expanding the right side, we get the following: a term v_{j_1} \wedge \cdots \wedge v_{j_r} \wedge v_{j_{r+1}} which is zero by Equality (9), a sum of terms of the form t^k(\pm y_{j,i}) \wedge v_{j_1} \wedge \cdots \wedge \hat{v}_{j,i} \wedge \cdots \wedge v_{j_{r+1}} (where the hat denotes the omission of the term under the hat), and then terms that are clearly in t^{2k} \bigwedge^{r+1} M or higher. But the product v_{j_1} \wedge \cdots \wedge \hat{v}_{j,i} \wedge \cdots \wedge v_{j_{r+1}} is already in t^k \bigwedge^r M, since on reduction modulo t^k, this is just the r-fold wedge product of the various \overline{u_j}. It follows that u_{j_1} \wedge \cdots \wedge u_{j_r} \wedge u_{j_{r+1}} is in t^{2k} \bigwedge^{r+1} M. This proves the theorem. □
Corollary 4.4. Assume the \( u_j \) \((1 \leq j \leq m)\) are as in the preceding theorem. Then,

\[
\sum_{l_1+\cdots+l_{r+1}=w \atop 0 \leq l_i < k} u^{(l_1)}_{j_1} \wedge \cdots \wedge u^{(l_r)}_{j_r} \wedge u^{(l_{r+1})}_{j_{r+1}} = 0
\]

for each \( w \) such that \( 0 \leq w < 2k \), and for all sequences \( 1 \leq j_1 < \cdots < j_r < j_{r+1} \leq m \). In particular, when \( r = 2 \),

\[
\sum_{l_1+l_2+l_3=w \atop 0 \leq l_1,l_2,l_3 < k} u^{(l_1)}_{j_1} \wedge u^{(l_2)}_{j_2} \wedge u^{(l_3)}_{j_3} = 0
\]

on the subvariety \( Z_0 \) of \( Z_{2,k}^{m,n} \), for each \( w \) such that \( 0 \leq w < 2k \), and for all sequences \( 1 \leq j_1 < j_2 < j_3 \leq m \).

Proof. The expressions on the left hand side of the equalities (10) and (11) are merely the coefficients of \( t^w \), \( 0 \leq w < 2k \), of \( u_{j_1} \wedge \cdots \wedge u_{j_r} \wedge u_{j_{r+1}} \), so by the previous theorem these are zero whenever \( u^{(0)}_{j_1} \wedge \cdots \wedge u^{(0)}_{j_{r+1}} \neq 0 \). When \( r = 2 \), the subvariety \( Z_0 \) is the closure of the open set where \( u^{(0)}_{j_1} \neq 0 \).

Remark 4.5. There is another set of equations that hold on the closure of the open set of \( Z_{r,k}^{m,n} \) where \( u^{(0)}_{j_1} \wedge \cdots \wedge u^{(0)}_{j_m} \neq 0 \). By Corollary 1.2, all other vectors \( u_i \) can be expressed as an \( R \)-linear combination of \( u_{j_1}, \ldots, u_{j_m} \). In particular, all other vectors \( u^{(l)}_i \) can be expressed as an \( F \)-linear combination of the \((r-1)\) vectors \( u^{(l)}_{j_1}, \ldots, u^{(l)}_{j_{r-1}} \), \( l = 0, \ldots, k-1 \). It follows that the \((k(r-1)+1)\)-fold wedge product of any of the vectors \( u^{(l)}_i \) must be zero on this closure, for \( i = 1, \ldots, m \) and for \( l = 0, \ldots, k-1 \). This will be trivially true if \( (k(r-1)+1) > n \).

5. The Case of \( 2 \times 2 \) Submaximal Minors

In this section, we will completely describe the components of the variety \( Z_{2,k}^{m,n} \) when \( n \geq m \geq 3 \) and \( k \geq 2 \).

For this section, we will write \( Y_0 \) for our variety \( Z_{2,k}^{m,n} \), and \( X_0 \) for the closure \( Z_0 \) of any of the open sets \( U_{i,j} \) described in Theorem 2.8.

We will write \( Y_1 \) for the subvariety \( Z_1 \) of Theorem 2.8 where all \( x^{(0)}_{i,j} \) are zero. We will also write \( \Sigma_0 \) for the set \( \{0, 1, \ldots, k-1\} \).

Now \( Y_1 \) is isomorphic to \( Z_{2,k-2}^{m,n} \times A^{mn} \) when \( k > 2 \), and isomorphic to \( A^{mn} \) when \( k = 2 \).

In particular, recall from the proof of Lemma 2.1 that in the case \( k > 2 \), the variety \( Y_1 \) is really determined by considering the generic \( m \times n \) matrix with rows \( u^{(1)}_i + u^{(2)}_i t + \cdots + u^{(k-2)}_i t^{k-3} \) \((1 \leq i \leq m) \) and
setting determinants of $2 \times 2$ minors to zero modulo $t^{k-2}$. We will write $\Sigma_1$ for the set $\{1, \ldots, k-2\}$ which indexes the powers of $t$ from each row of the original matrix which are now governed by determinantal equations modulo $t^{k-2}$, and we will write $W_1$ for the factor isomorphic to $Z_{2,k-2}^{m,n}$ determined by these rows.

Notice that if $k = 3$, the factor $W_1$ is the classical determinantal variety of $2 \times 2$ minors of the generic matrix $((x_{i,j}^{(1)}))$, and this variety is known to be irreducible (\cite{2}). Hence, when $k = 3$, $Y_1$ is just an irreducible variety cross an affine piece, and is hence irreducible.

If $k > 3$, we will write $T_1$ for the subvariety “$Z_0$” of $W_1$, i.e., the closure in $W_1$ of the open set where some $x_{i,j}^{(1)} \neq 0$. Also, we will write
X_1 for the closure of any of the open sets of Y_1 where some \( x_{i,j}^{(1)} \neq 0 \). It is clear that X_1 is just \( T_1 \times A^{mn} \).

Write \( k = 2L + 1 \) or \( k = 2L \) according to whether \( k \) is odd or even. Proceeding thus, we will have subvarieties \( Y_s, X_s \), for \( s = 0, 1, \ldots, L = [k/2] \). We have the following:

- \( Y_0, X_0 \) and \( \Sigma_0 = \{0, 1, \ldots, k - 1\} \) are as already described.
- (For \( 0 < s < L \)) On \( Y_s \), all row vectors \( u_i^{(0)}, \ldots, u_i^{(s-1)} \) are zero, \( 1 \leq i \leq m \). The various rows \( u_i^{(s)}, \ldots, u_i^{(k-1-s)} \), \( 1 \leq i \leq m \), are governed by the condition that all \( 2 \times 2 \) minors of the matrix with rows \( u_i^{(s)} + u_i^{(s+1)} t + \cdots + u_i^{(k-1-s)} t^{k-2s-1} \) are zero modulo \( t^{k-2s} \). We write \( \Sigma_s \) for the set \( \{s, \ldots, k - 1 - s\} \) which indexes these rows. The rows \( u_i^{(k-s)}, \ldots, u_i^{(k-1)} \), \( 1 \leq i \leq m \), are all free. Hence, \( Y_s \cong W_s \times A^{mn} \cong Z_{2,k-2s} \times A^{mn} \). We write \( T_s \) for the subvariety "\( Z_0 \)" of \( W_s \), this is the closure in \( W_s \) of the open set where some \( x_{i,j}^{(s)} \neq 0 \).
- (For \( 0 < s < L \)) \( X_s \) is the closure of any of the open sets of \( Y_s \) where some \( x_{i,j}^{(s)} \neq 0 \). It is clear that \( X_s \) is just \( T_s \times A^{mn} \).
- If \( k = 2L \), then \( Y_L \) is given by setting all row vectors \( u_i^{(0)}, \ldots, u_i^{(L-1)} \) to zero, \( 1 \leq i \leq m \), with all remaining row vectors \( u_i^{(L)}, \ldots, u_i^{(k-1)} \), \( 1 \leq i \leq m \), being free. Thus, \( Y_L \cong A^{mnL} \).
- If \( k = 2L + 1 \), then on \( Y_L \), all row vectors \( u_i^{(0)}, \ldots, u_i^{(L-1)} \) are zero, \( 1 \leq i \leq m \). The row vectors \( u_i^{(L)}, 1 \leq i \leq m \), satisfy the classical determinantal equations for the \( 2 \times 2 \) minors of the generic \( m \times n \) matrix \( ((x_{i,j}^{(L)}) \). \( W_L \) will denote the classical determinantal variety determined by the \( u_i^{(L)} \). The various row vectors \( u_i^{(L+1)}, \ldots, u_i^{(k-1)} \), \( 1 \leq i \leq m \), are all free. Thus, \( Y_L \cong W_L \times A^{mnL} \), and since \( W_L \) is a classical determinantal variety, \( Y_L \) is itself irreducible (3).

Our result is the following:

**Theorem 5.1.** The variety \( Z_{2,k}^{mn} \) \((n \geq m \geq 3, k \geq 2)\) is reducible. Its components are the subvarieties \( X_0, X_1, \ldots, X_{L-1} \), and \( Y_L \) described above. The components \( X_s, s = 0, 1, \ldots, L - 1 \), have codimension \((m - 1)(n - 1)(k - 2s) + mn\). If \( k = 2L \), \( Y_L \) has codimension \( mnL \), while if \( k = 2L + 1 \), \( Y_L \) has codimension \((m - 1)(n - 1) + mnL \).

**Proof.** The fact that the various \( X_s \) and \( Y_L \) are irreducible and have the stated codimension follows easily from the descriptions of the various \( X_s \) and \( Y_L \) above and from Theorem 2.3. We have seen that for \( 0 \leq s < L \), \( X_s \cong T_s \times A^{mn} \), and that \( X_s \) sits in the portion of \( A^{mn(k-2s)} \)
determined by setting all rows $u^{(l)}_i$ to zero, $l = 0, 1, \ldots, s - 1$ (when $s = 0$ this condition is vacuous). Recall, too, that $T_s$ is the closure of the open set of $Z_{2,k-2s}^{m,n}$ where some $x^{(s)}_{i,j} \neq 0$. By Theorem 2.3, we have a one-to-one correspondence between the components of $T_s$ and the components of $Z_{1,k-2s}^{m-1,n-1}$, a correspondence which preserves codimension in the respective spaces $A^{mn(k-2s)}$ and $A^{(m-1)(n-1)(k-2s)}$. Since $Z_{1,k-2s}^{m-1,n-1}$ is clearly irreducible of codimension $(m - 1)(n - 1)(k - 2s)$ (it is the origin in $A^{(m-1)(n-1)(k-2s)}$), we find that each $X_s$ is irreducible. It follows too that $X_s$ has codimension $(m - 1)(n - 1)(k - 2s) + mns$ in $A^{mnk}$, where the extra summand $mns$ accounts for the rows $u^{(0)}_i \ldots u^{(s-1)}_i$ set to zero.

As for $Y_L$, we have already observed in the discussion before this theorem that it is irreducible. In the case $k = 2L$, $Y_L$ is just $A^{mnL}$. It follows that $Y_L$ has codimension $mnL$ (corresponding to the rows $u^{(0)}_i \ldots u^{(L-1)}_i$ set to zero). If $k = 2L + 1$, then the codimension of $Y_L$ is $mnL$ plus the codimension of the variety $W_L$. But this is known to be $(m - 1)(n - 1)$ (see 2 for instance, this can also be derived from Theorem 2.3).

We will now prove that the components of $Z_{2,k}^{m,n}$ are as described. It is easily seen from the codimension formulas above that except when $(m, n) = (3, 3)$ or $(m, n) = (3, 4)$, the codimension decreases as a function of $s$. This shows that if $s' > s$, then $X_{s'}$ (or $Y_{s'}$) cannot be contained in $X_s$. Since the reverse containment is ruled out as $x^{(s)}_{i,j} \neq 0$ on $X_s$, we find that the components of $Z_{2,k}^{m,n}$ are indeed as described, except in the two special cases.

To take care of these two special cases, we will use results from Section 4. (The proof works for all $(m, n)$ pairs actually.) Using reverse induction on $s$, we will show that at the $s$-th stage, $s = L, L - 1, \ldots, 0$, the components of $Y_s$ are $X_s, X_{s+1}, \ldots, X_L$, and $Y_L$. We have already observed that $Y_L$ is irreducible, so assume that $s < L$. Note that $Y_s$ is the union of $X_s$ and $Y_{s+1}$, and by induction, $Y_{s+1}$ will have components $X_{s+1}, \ldots, X_L$, and $Y_L$. We will prove that none of these subvarieties $X_{s+1}, \ldots, X_L$, and $Y_L$ can be contained in $X_s$. The reverse containment is once again ruled out, and we will indeed find that $Y_s$ has components $X_s, X_{s+1}, \ldots, X_L$, and $Y_L$.

We will first show that no $X_{s+\alpha}$ ($\alpha = 1, \ldots, L - s - 1$) can be contained in $X_s$. For, assume to the contrary. Recall that $X_{s+\alpha}$ decomposes as $T_{s+\alpha} \times A^{mn(s+\alpha)}$, where the factor $T_{s+\alpha}$ comes from the various entries of the rows $u^{(l)}_i$, $1 \leq i \leq m$, $l = s + \alpha, \ldots, k - 1 - (s + \alpha)$, and the factor $A^{mn(s+\alpha)}$ comes from the various entries of the rows $u^{(l)}_i$, $1 \leq
$i \leq m, \ l = k - (s + \alpha), \ldots, k - 1$. Recall too that $T_{s+\alpha}$ is the closure in $W_{s+\alpha}$ where some $u_{i}^{(s+\alpha)} \neq 0$, where $W_{s+\alpha}$ has the description given earlier. The following point $P$ is therefore in $X_{s+\alpha}$: $u_{1}^{(s+\alpha)} = (1, 0, 0, \ldots)$, $u_{2}^{(k-(s+\alpha))} = (0, 1, 0, \ldots)$, $u_{3}^{(k-1-s)} = (0, 0, 1, 0, \ldots)$, and all other rows of all possible degrees zero. (The nonzero coordinates coming from the row $u_{1}^{(s+\alpha)}$ belong to $T_{s}$ while those coming from $u_{2}^{(k-(s+\alpha))}$ and $u_{3}^{(k-1-s)}$ belong to the other factor $A_{mn(s+\alpha)}$.)

Now $X_{s}$ decomposes as $T_{s} \times A_{mn}$, where the factor $T_{s}$ comes from the various entries of $u_{i}(l), 1 \leq i \leq m, \ l = s, \ldots, k - 1 - s$. Since $P \in X_{s}$ by assumption, an examination of its coordinates show that this point $P$ does not satisfy Equation (11) for $T_{s}$ but does not satisfy Equation (11) for $T_{s}$. This completes the proof.

By our choice of these rows, this wedge product is clearly nonzero. The coefficient of $t^{2(k-2s)-1}$, on specializing to the rows $u_{1}, u_{2},$ and $u_{3}$, reads

\begin{equation}
\sum_{a+b+c=2(k-2s)-1}^{0 \leq a, b, c < k-2s} u_{1}^{(s+\alpha)} \wedge u_{2}^{(s+b)} \wedge u_{3}^{(s+c)} = 0
\end{equation}

Examining the coordinates of $P$, and recognizing that $u_{2}^{(k-(s+\alpha))} = u_{2}^{(s+(k-2s-\alpha))}$ and $u_{3}^{(k-1-s)} = u_{3}^{(s+(k-1-2s))}$ we find that the only wedge product that is nonzero in this equation is $u_{1}^{(s+\alpha)} \wedge u_{2}^{(k-(s+\alpha))} \wedge u_{3}^{(k-1-s)}$. But by our choice of these rows, this wedge product is clearly nonzero. This shows that $X_{s+\alpha}$ is not contained in $X_{s}$.

To show that $Y_{L}$ is not contained in $X_{s}$, consider first the case $k = 2L+1$. Then $Y_{L} \cong Z_{2,1}^{m,n} \times A_{mnL}$, where the factor $Z_{2,1}^{m,n}$ comes from the entries of $u_{i}(L), 1 \leq i \leq m$, and the factor $A_{mnL}$ comes from the entries of $u_{i}(l), 1 \leq i \leq m, \ l = L+1, \ldots, 2L$. Choose $P$ to be the point with $u_{1}^{(L)} = (1, 0, 0, \ldots)$, $u_{2}^{(L+1)} = (0, 1, 0, \ldots)$, $u_{3}^{(k-1-s)} = (0, 0, 1, 0, \ldots)$, and all other rows of all possible degrees zero. (The nonzero coordinates coming from the row $u_{1}^{(L)}$ belong to $Z_{2,1}^{m,n}$ while those coming from $u_{2}^{(L+1)}$ and $u_{3}^{(k-1-s)}$ belong to the other factor $A_{mnL}$.) Exactly as before, this point $P$ is in $Y_{L}$, but we find that $P$ does not satisfy Equation (11) for $T_{s}$.

When $k = 2L$, we take $P$ to be the point with $u_{1}^{(L)} = (1, 0, 0, \ldots)$, $u_{2}^{(L)} = (0, 1, 0, \ldots)$, $u_{3}^{(k-1-s)} = (0, 0, 1, 0, \ldots)$, and all other rows of all possible degrees zero. Once again, this point $P$ is in $Y_{L}$ but does not satisfy Equation (11) for $T_{s}$. This completes the proof.
We get the following corollary from this:

**Corollary 5.2.** The variety \( Z_{m,n}^{2} \) has 1 + \( \lfloor k/2 \rfloor \) components. The codimension of \( Z_{2,k}^{m,n} \) (in \( A^{mn} \)) is \((m-1)(n-1) + mn\lfloor k/2 \rfloor \) if \( k \) is odd and the codimension is \( mn\lfloor k/2 \rfloor \) if \( k \) is even, except in the case where \((m,n) = (3,3) \) or \((m,n) = (3,4) \). In these two special cases, \( Z_{2,k}^{3,3} \) has codimension 4\( k \), while \( Z_{2,k}^{3,4} \) has codimension 6\( k \).

**Proof.** By Theorem 5.1, \( Z_{2,k}^{m,n} \) clearly has 1 + \( \lfloor k/2 \rfloor \) components. These have codimension \((m-1)(n-1)(k-2s) + mns\), for \( s = 0,1,\ldots,L = mn\lfloor k/2 \rfloor \) (the case \( s = L \) corresponds to the component \( Y_{L} \), but is also covered by this formula). This is linear in \( s \), and as already observed, is decreasing in \( s \) except when \((m,n) = (3,3) \) or \((m,n) = (3,4) \). It follows that except for these two cases, the component \( Y_{L} \), has the least codimension. This yields the formula for the codimension of \( Z_{2,k}^{m,n} \) in the general case. In the two special cases, the component \( X_{0} \) must have least codimension. Hence, the codimension of \( Z_{2,k}^{m,n} \) in these cases is given by the codimension of \( X_{0} \), which is \((m-1)(n-1)k \). \( \square \)

**Remark 5.3.** Note that when \((m,n) = (3,4) \), the codimension of the components is constant in \( s \). Hence, in this case, all components have the same dimension.

### 6. Submaximal Minors: The General Situation

In this section, we will use Theorem 5.1 as a building block to derive results about \( Z_{r,k}^{m,n} \) in general, when \( r < m \). The first one is easy:

**Theorem 6.1.** The variety \( Z_{r,k}^{m,n} \) in the submaximal case \((r < m) \) has at least 1 + \( \lfloor k/2 \rfloor \) components.

**Proof.** By Theorem 2.8, \( Z_{r,k}^{m,n} \) has at least as many components as its subvariety \( Z_{0} \), and the components of this subvariety are in one-to-one correspondence with those of \( Z_{r-1,k}^{m-1,n-1} \). Proceeding thus, \( Z_{r,k}^{m,n} \) has at least as many components as \( Z_{2,k}^{m-r+2,n-r+2} \), and this last variety has 1 + \( \lfloor k/2 \rfloor \) components. \( \square \)

It is quite clear that the components of \( Z_{2,k}^{m,n} \) intersect one another. For instance, the origin is in \( Y_{L} \) and is also in each of the \( X_{s} \). (For, as in the proof of Theorem 3.1, the line between the origin and any point \( P \in T_{s} \) with some \( x_{i,j}^{(s)} \neq 0 \) has to lie in \( X_{s} \).) But, as well, it is quite easy to find lots of other points of intersection between the
various components. For example, if \( k \geq 4 \), then there are at least three components, \( X_0, X_1 \) and \( Y_L \). The rows \( u_i^{(k-1)} \) are free on both \( X_1 \) and \( Y_L \), and it is then easy to see that the point with zeros in all rows except in the rows \( u_i^{(k-1)} \) is in both \( X_1 \) and \( Y_L \). If \( k = 2 \), then the components are \( X_0 \) and \( Y_1 \). For any nonzero \( \lambda \in F \) and for any point \( P \in X_0 \) with some \( u_i^{(0)} \neq 0 \), the point with coordinates \( \lambda u_i^{(0)}, u_i^{(1)}, \ i = 1, \ldots, m \), also satisfies the equations of \( Z_{2,2}^{m,n} \), so when \( \lambda \) equals zero, we get a point in \( X_0 \cap Y_1 \). Similarly, if \( k = 3 \), then the point with coordinates \( \lambda^2 u_i^{(0)}, \lambda u_i^{(1)}, u_i^{(2)}, i = 1, \ldots, m \), also satisfies the equations of \( Z_{2,3}^{m,n} \), so once again, if we start with a point in \( X_0 \) with some \( u_i^{(0)} \neq 0 \) and let \( \lambda \) equal zero, we get a point in \( X_0 \cap Y_1 \).

From this, we get the following trivially:

**Theorem 6.2.** The varieties \( Z_{r,k}^{m,n} \) when \( r < m \) are not normal. They are also not Cohen-Macaulay, except possibly in the case where \( (m, n) = (1 + r, 2 + r) \).

**Proof.** In the case of \( 2 \times 2 \) minors, the fact that the varieties are not normal follows from the fact that we have explicitly found components that intersect nontrivially, while the fact that they are not Cohen-Macaulay except possibly when \( (m, n) = (3, 4) \) follows from the fact that components with different dimensions intersect nontrivially (see Remark 5.3). In the general case, we repeatedly invoke the birational isomorphism of Theorem 2.3 between the subvariety \( Z_0 \) (which is a union of some of the components of \( Z_{r,k}^{m,n} \)) and the variety \( Z_{r-1,k}^{m-1,n-1} \times A^{k(m+n-1)} \), and then reduce to the case \( Z_{2,k}^{m-r+2,n-r+2} \). Note that the image of the birational isomorphism is the open subset of \( Z_{r-1,k}^{m-1,n-1} \times A^{k(m+n-1)} \) where the free variable \( x_{m,n}^{(0)} \neq 0 \) (see Remark 2.4), so any intersection between components of \( Z_{r-1,k}^{m-1,n-1} \) of dimensions \( d_1 \) and \( d_2 \) will indeed manifest itself as an intersection between components of \( Z_{r,k}^{m,n} \) (in fact of the subvariety \( Z_0 \) of \( Z_{r,k}^{m,n} \)) of dimensions \( d_1 + k(m+n-1) \) and \( d_2 + k(m+n-1) \). \( \square \)

For \( r \geq 3 \) (and \( r < m \)), it is somewhat more difficult (and cumbersome) to determine explicitly the components of \( Z_{r,k}^{m,n} \). Recall that our variety \( Z_{r,k}^{m,n} \) decomposes into \( Z_0 \) and \( Z_1 \) as in Theorem 2.8. The components of \( Z_0 \) are in one-to-one correspondence with those of \( Z_{r-1,k}^{m-1,n-1} \), while the components of \( Z_1 \) are determined by those of \( Z_{r,k-r}^{m,n} \) when \( k > r \), and of course, when \( k \leq r \), \( Z_1 \) is isomorphic to \( A^{mn/(k-1)} \) (by Lemma 2.1). Now, even if we are able to determine inductively the components of \( Z_{r-1,k}^{m-1,n-1} \) and those of \( Z_{r,k-r}^{m,n} \) (when \( k > r \)), it is quite
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It is quite difficult to determine if any of the components of $Z_1$ live inside any of the components of $Z_0$. (It is clear that no component of $Z_0$ can live inside any component of $Z_1$, since every component of $Z_0$ contains points with $x_{i,j}^{(0)} \neq 0$.) But there is one special situation where one can indeed answer this question explicitly, and this is in the case where $k < r$. We have the following:

**Proposition 6.3.** In the case where $k < r$, the subvariety $Z_1$ of $Z_{r,k}^{m,n}$ is contained in $Z_0$. The components of $Z_{r,k}^{m,n}$ and their codimensions in $A_{mnk}$ in this case are hence determined completely by the components of $Z_{r−1,k}^{m−1,n−1}$ and their codimensions in $A^{(m−1)(n−1)}$.

**Proof.** Consider the subvariety $V$ of $Z_{r,k}^{m,n}$ defined by setting all $2 \times 2$ minors of degree zero to zero, i.e., defined by setting all $u_i^{(0)} \wedge u_j^{(0)} = 0$ for all $1 \leq i < j \leq m$. The equations for $V$ are thus the standard equations for $Z_{r,k}^{m,n}$ along with all $2 \times 2$ minors of degree zero. It is clear that every point on $Z_1$ satisfies these equations, so $Z_1 \subset V$. Since $r \geq (k−1) + 2$, every $r$-fold wedge product of vectors $u_i^{(l)}$ of total degree at most $k−1$ must contain at least two factors of degree zero. It follows that $Z_{r,k}^{m,n}$ is already contained in the ideal generated by all $2 \times 2$ minors of degree zero, which is an ideal that is known classically to be prime. Hence, $V$ is an irreducible variety, isomorphic to $Z_{2,1}^{m,n}$. The components of $Z_{r,k}^{m,n}$ come from either $Z_1$ or $Z_0$. Since $V$ cannot be contained wholly in any component of $Z_1$ (as there are clearly points on $V$ where not all $x_{i,j}^{(0)}$ are zero), we find $V \subset Z_0$. It follows that $Z_1 \subset Z_0$. Thus the components of $Z_{r,k}^{m,n}$ all come from $Z_0$, and Theorem 2.8 now finishes the proof.

We use the result above to determine the components of the tangent bundle to the classical determinantal varieties in the case of submaximal minors:

**Corollary 6.4.** When $k = 2$ (i.e., when we consider the tangent bundle to $Z_{r,1}^{m,n}$), and when $r < m$, $Z_{r,2}^{m,n}$ has exactly two components. One of them is the closure of any of the open sets $U_{[i_1,\ldots,i_{r−1};j_1,\ldots,j_{r−1}]}$ of $Z_{r,k}^{m,n}$, where the $(r − 1) \times (r − 1)$ minor of degree zero determined by rows $i_1,\ldots,i_{r−1}$ and columns $j_1,\ldots,j_{r−1}$ is nonzero, and hence also of their union. This component has codimension $2(m − r + 1)(n − r + 1)$. The other is the subvariety defined by setting all $(r − 1) \times (r − 1)$ minors of degree zero to zero, and has codimension $(m − r + 2)(n − r + 2)$.

**Proof.** The number of components and their codimension comes from repeated applications of Proposition 6.3 above. By the proposition,
the components of $\mathcal{Z}_{r,2}^{m,n}$ are in one-to-one correspondence (via the birational isomorphism of Theorem 2.3) to the components of $\mathcal{Z}_{r-1,2}^{m-1,n-1}$. If $r - 1 = 2$, then the components of $\mathcal{Z}_{r-1,2}^{m-1,n-1}$ and their codimensions are described by Theorem 5.1. Otherwise, we repeat the process, until we come to $\mathcal{Z}_{2,2}^{m-r+2,n-r+2}$.

It remains to establish the description of the components. We proceed by induction on $r$. When $r = 2$, this is precisely the content of Theorem 5.1, as also of Lemma 2.7. Note that if $P_0$ is the prime corresponding to $Z_0$, then $P_0$ does not contain any $x_{i,j}^{(0)}$, while if $P_1$ is the prime corresponding to $Z_1$, then $P_1$ is the minimal prime of $\mathcal{I}_{2,2}^{m,n}$ that contains all $x_{i,j}^{(0)}$ (Lemma 2.4). Now assume that the theorem is true for the variety $\mathcal{Z}_{r-1,2}^{m-1,n-1}$. We will invoke the notation of Theorem 2.3. The assumption shows that there are exactly two minimal primes of $\mathcal{I}_{r-1,2}^{m-1,n-1}$ in $T$. One of them, call it $Q_0$, does not contain any $(r - 2) \times (r - 2)$ minors of the degree zero matrix $((z_{i,j}^{(0)}))$ (1 ≤ $i$ ≤ $m - 1$, 1 ≤ $j$ ≤ $n - 1$), since otherwise, the component $Z(Q_0)$ corresponding to $Q_0$ cannot contain the open set where this minor is nonzero. The other, call it $Q_1$, is the unique minimal prime over the ideal of $T$ generated by $\mathcal{I}_{r-1,2}^{m-1,n-1}$ and the various $(r - 2) \times (r - 2)$ minors of the degree zero matrix $((z_{i,j}^{(0)}))$. Let $\tilde{Q}_0$ and $\tilde{Q}_1$ in $S[[x_{m,n}^{(0)}]]$ be the images (respectively) of $Q_0$ and $Q_1$ in $S[[x_{m,n}^{(0)}]]$ under the inverse map $\tilde{f}$. Notice that the various $(r - 2) \times (r - 2)$ minors of the degree zero matrix $((z_{i,j}^{(0)}))$ are just the generators of the ideal $\mathcal{I}_{r-2,1}^{m,n-1}$. By the proof of Theorem 2.3, these generators go to the generators of $\mathcal{I}_{r-1,1}^{m,n}[[x_{m,n}^{(0)}]]$ under the inverse map $\tilde{f}$. Thus, $\tilde{Q}_1$ contains all generators of $\mathcal{I}_{r-1,1}^{m,n}[[x_{m,n}^{(0)}]]$. On the other hand, the standard generators for $\mathcal{I}_{r-1,1}^{m,n}[[x_{m,n}^{(0)}]]$ are those that come from $\mathcal{I}_{r,1}^{m,n}$: these are just the various $(r - 1) \times (r - 1)$ minors of our $m \times n$ degree zero matrix $X(0) = ((x_{i,j}^{(0)}))$. Thus, the pullback of $\tilde{Q}_1$, call it $P_1$, contains all $(r - 1) \times (r - 1)$ minors of degree zero of $X(t)$. Moreover, since $\tilde{Q}_1$ is the unique minimal ideal of $S[[x_{m,n}^{(0)}]]$ lying over the ideal generated by $\mathcal{I}_{r,1}^{m,n}$ and the $(r - 1) \times (r - 1)$ minors of degree zero of $X(t)$, $P_1$, being its pullback, is the unique minimal prime of $\mathcal{I}_{r,1}^{m,n}$ lying over the ideal generated by $\mathcal{I}_{r,1}^{m,n}$ and the $(r - 1) \times (r - 1)$ minors of degree zero of $X(t)$. Thus, the component corresponding to $P_1$ is the subvariety of $\mathcal{Z}_{r,2}^{m,n}$ obtained by setting all $(r - 1) \times (r - 1)$ minors of degree zero to zero.
As for the description of $P_0$, it is clear that $P_0$ cannot contain any $(r - 1) \times (r - 1)$ minor $[i_1, \ldots, i_{r-1} | j_1, \ldots, j_{r-1}]$ of degree zero, since this minor is already in $P_1$ and would hence be zero on all of $Z_{r,2}^{m,n}$ if it were also in $P_0$, a contradiction. Recasting this in the language of open sets, for any $(r - 1) \times (r - 1)$ minor of degree zero of $X(t)$ indexed by rows $i_1, \ldots, i_{r-1}$ and columns $j_1, \ldots, j_{r-1}$, we must have $U_{[i_1, \ldots, i_{r-1} | j_1, \ldots, j_{r-1}]} \subset Z(P_0)$, since $Z_{r,2}^{m,n} = Z(P_0) \cup Z(P_1)$ and since the minor $[i_1, \ldots, i_{r-1} | j_1, \ldots, j_{r-1}]$ is zero on $Z(P_1)$. Because $Z(P_0)$ is irreducible and $U_{[i_1, \ldots, i_{r-1} | j_1, \ldots, j_{r-1}]}$ is clearly nonempty, the closure of $U_{[i_1, \ldots, i_{r-1} | j_1, \ldots, j_{r-1}]}$ must be all of $Z(P_0)$. This must then trivially be true of the union of all these open sets indexed by these minors. (As well, this is true of their intersection, as the intersection is also nonempty). □

Remark 6.5. Except when $(m, n) = (1 + r, 1 + r)$ or $(m, n) = (1 + r, 2 + r)$, the corollary can be obtained very easily without recourse to the machinery of this paper. Write $U$ for the union of the open sets where some $(r - 1) \times (r - 1)$ degree zero minor of $X(0)$ is nonzero. Note that the portion of the classical degree zero variety $Z_{r,1}^{m,n}$ where some $(r - 1) \times (r - 1)$ minor is nonzero is precisely the set of smooth points of $Z_{r,1}^{m,n}$. The variety $Z_{r,2}^{m,n}$ is the union of two subvarieties: one, call it $X$, is the closure of $U$, and the other, call it $Y$ is the subvariety where all $(r - 1) \times (r - 1)$ degree zero minors of $X(0)$ are zero. It is easy to see that $U$ is irreducible of the stated codimension, since the fibers over any point of $Z_{r,1}^{m,n}$ where some $(r - 1) \times (r - 1)$ is nonzero are all linear spaces of the same dimension. Hence $X$ is irreducible of the stated codimension. It is also easy to see that the Jacobian matrix defining tangent spaces to classical variety $Z_{r,1}^{m,n}$ is zero when all $(r - 1) \times (r - 1)$ minors are zero, so indeed, the tangent spaces at such points are simply copies of $A^{m,n}$. Since the base space $Z_{r-1,1}^{m,n}$ is irreducible, $Y$ is irreducible as well, and it has the stated codimension. It is clear that $X$ cannot be contained in $Y$ as $U$ is nonempty. Except for the given exceptional values of $(m, n)$, the dimension of $Y$ is greater than that of $X$, so $Y$ cannot be contained in $X$ as well. It follows that $X$ and $Y$ are precisely the components of $Z_{r,2}^{m,n}$.

We end this section with an inductive scheme for computing the codimension of $Z_{r,k}^{m,n}$ in the case $r < m$. The induction is based on $r$, and we will assume that for all $r'$ with $2 \leq r' < r$ and for all $m$, $n$ with $r' < m \leq n$, and for all $k \geq 2$, we know the codimension of $Z_{r',k}^{m,n}$. (The starting point for the induction is Theorem 5.1, and the ideas here parallel the codimension computations of Theorem 5.1).
Write \( k = \lambda r + \mu \), for \( \lambda \geq 1 \), and \( 0 \leq \mu < k \). (When \( k < r \), we already know that the components, and their codimensions, are determined by those of \( Z_{r-1,k}^{m-1,n-1} \), thanks to Proposition 5.3 above.) We now have the following sequence of subvarieties:

- We will write \( Y_0 \) for our variety \( Z_{r,k}^{m,n} \), and \( X_0 \) for its subvariety \( Z_0 \). Thus, \( X_0 \) is birational to \( Z_{r-1,k}^{m-1,n-1} \times A^{k(m+n-1)} \). Write \( c_0 \) for the codimension of \( Z_{r-1,k}^{m-1,n-1} \) in \( A^{(m-1)(n-1)k} \). Then \( X_0 \) also has codimension \( c_0 \). We will assume that \( c_0 \) is known by induction.

- We will write \( Y_1 \) for the subvariety \( Z_1 \) of \( Z_{r,k}^{m,n} \)—this is obtained by setting all \( x_{i,j}^{(0)} \) to zero. \( Y_1 \) is isomorphic to \( Z_{r,k-r}^{m,n} \times A^{mn(r-1)} \). We will write \( X_1 \) for the subvariety “\( Z_0 \)” of \( Y_1 \). It is birational to \( Z_{r-1,k-r}^{m-1,n-1} \times A^{m+n-1} \). Write \( c_1 \) for the codimension of \( Z_{r-1,k-r}^{m-1,n-1} \) in \( A^{(m-1)(n-1)(k-r)} \). Then the codimension of \( X_1 \) is \( c_1 + mn \), where the extra codimension \( mn \) comes from the fact that \( X_1 \) sits in the portion of \( A^{mnk} \) where all \( x_{i,j}^{(0)} \) are zero. We will assume that \( c_1 \) is known.

- Proceeding thus, let \( Y_s \) \((s = 1, \ldots, \lambda - 1)\) be the subvariety of \( Y_{s-1} \) where all \( x_{i,j}^{(s-1)} \) are zero, and, let \( X_s \) be the subvariety “\( Z_0 \)” of \( Y_s \). Then \( X_s \) has codimension \( c_s + smn \), where \( c_s \) is the codimension of \( Z_{r-1,k-rs}^{m-1,n-1} \) in \( A^{(m-1)(n-1)(k-rs)} \). We will assume that \( c_s \) is known.

- If \( \mu = 0 \), i.e., if \( k = \lambda r \), then \( Y_\lambda \), the subvariety of \( Y_{\lambda-1} \) where all \( x_{i,j}^{(\lambda-1)} \) are zero, is already an affine space of codimension \( mn\lambda \) in \( A^{mnk} \). For convenience we will take \( c_\lambda = 0 \) in this case, so the codimension of \( Y_\lambda \) may be written for this case as \( c_\lambda + \lambda mn \).

- If \( \mu > 0 \), then \( Y_\lambda \) is isomorphic to \( Z_{r,\mu}^{m,n} \times A^{\lambda(r-1)mn} \). Since \( \mu < r \), we can reduce its codimension computations to that of \( Z_{r-1,\mu}^{m-1,n-1} \) by Proposition 5.3, so we will assume that \( c_\lambda \), the codimension of \( Z_{r,\mu}^{m,n} \) is known. It follows that the codimension of \( Y_\lambda \) is \( c_\lambda + \lambda mn \). (\( X_\lambda \) in this case will equal \( Y_\lambda \) by Proposition 5.3.)

Our result is the following:

**Theorem 6.6.** The codimension of \( Z_{r,k}^{m,n} \) in the case \( r < m \) is the minimum of the numbers \( c_s + smn, s = 0, 1, \ldots, \lambda \).

**Proof.** We will show that the codimension of the subvariety \( Y_s, s = 0, 1, \ldots, \lambda \) is the minimum of \( c_{s'} + s'mn, s' = s, s+1, \ldots, \lambda \). When \( s = \lambda \) the result is clear. For \( s < \lambda \), the codimension of \( Y_{s+1} \), by reverse induction, is the minimum of \( c_{s'} + s'mn, s' = s+1, \ldots, \lambda \). For such \( s \),
the codimension of $Y_s$ is the minimum of the codimension of $Y_{s+1}$ and $X_s$, since $Y_s$ is the union of $Y_{s+1}$ and $X_s$. Putting this together with our inductive result, the codimension of $Y_s$, is the minimum of $c_{s'} + s'mn$, $s' = s, s+1, \ldots, \lambda$. □

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