Optimization of Virtual Networks

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Abstract

We introduce a general and comprehensive model for the design and optimization of Virtual Networks, and the related concept of Network Slicing. The model is flexible, so that by adjusting some of its elements, it can accommodate many different specific cases of importance. Yet, surprisingly, it still allows efficient optimization, in the sense that the global optimum can be well approximated by efficient algorithms.

I. Introduction

The general concept of virtualization refers to the creation of a logical structure that is distinct from the underlying physical infrastructure. In the networking field virtualization plays an important role: virtual networks already have a long history. One of the first precursors was when companies could connect various sites through leased lines over the public switched telephone network, thus forming an early, still primitive, embodiment of a Virtual Private Network (VPN). The early data networks, such as X.25 in the 1970s, Frame Relay in the 1980s, and Asynchronous Transfer Mode (ATM) in the 1980-90s, already supported VPN-like configurations. Virtual Local Area Networks (VLANs) also emerged in the 1980s; the IEEE later standardized VLANs over Ethernet in the IEEE 802.1Q standard. In the Internet, some efforts were present from the very beginning to allow the logical separation of some part of the traffic, or to provide dedicated treatment to a traffic class. This intent was first represented by the 8-bit Type of Service field of the IPv4 header, which later became Differentiated Services Code Point (DSCP), to support Differentiated Services. These could also be viewed as primordial early precursors of Network Slicing (see below). Virtual Networks could be configured via Multiprotocol Label Switching (MPLS), as well, which came into view in the 1990s. In the modern Internet, initially the possibility of flexible experimentation with new protocols and architectures motivated the concept of Overlay Networks, which are a type of Virtual Networks, see Anderson, Peterson, Shenker, and Turner [1]. By now, the Virtual Network concept became ubiquitous.

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A related, but not fully identical, concept is *Network Slicing*, which emerged in connection with fifth generation (5G) wireless networks, and created quite a bit of excitement in the cellular network research community. This is based on perceiving the concept as a revolutionary new way to extending network capabilities, in combination with the evolutionary improvement of efficiency. To illustrate it, let us quote from the survey article of Foukas et al. [11]:

“What 5G systems are going to be has yet to be determined. However, it is conceivable that the eventual 5G system will be a convergence of two complementary views that are currently driving the research and industrial activity on 5G. One is an *evolutionary* view focusing on significantly scaling up and improving the efficiency of mobile networks (e.g., 1000x traffic volume, 100x devices, and 100x throughput). Much of the research focused around this view is on the radio access side looking at novel technologies and spectrum bands (e.g., massive multiple-input multiple-output, MIMO; millimeter-wave). The other *service-oriented* view envisions 5G systems catering to a wide range of services differing in their requirements and types of devices, and going beyond the traditional human-type communications to include various types of machine-type communications. This requires the network to take different forms depending on the service in question, leading naturally to the notion of *slicing* the network on a per-service basis (…) Realizing this service-oriented view requires a radical rethink of the mobile network architecture to turn it into a more flexible and programmable fabric, leveraging technologies like software defined networking (SDN) and network functions virtualization (NFV), which can be used to simultaneously provide a multitude of diverse services over a common underlying physical infrastructure.”

It is natural to ask at this point: *how do Network Slices and Virtual Networks differ?* There is clearly significant overlap between the two concepts, since they both consist of a *logically* separated subset of network resources, dedicated to serve some demand. On the other hand, in case of Virtual Networks this aims mostly at serving a customer, such as setting up a VPN for a company. Therefore, it is typically horizontal logical separation. In a Network Slice, the key goal is to logically separate resources for *services*, at different levels, including cloud, fog, edge, processing power and storage capacity, switches, base stations, radio access network etc, so that they together can best provide the resources for the service. Thus, in this sense, the Network Slice has a more vertical focus.

It is not difficult to believe that the need for designing, configuring, dimensioning, as well as physically embedding Network Slices and Virtual Networks leads to hard problems. We review some issues in the next section.

**A. Previous Work**

The most investigated task in connection with Virtual Networks is how to *embed* them into a *substrate network*. The substrate is most often an actual physical network, but it may also be another virtual network.
This embedding entails a number of sub-tasks, that may be logically separated, or can also be handled in a more complex joint optimization model. The most important sub-tasks are:

- Assigning the virtual nodes into physical ones, while taking into account processing demands and physical node capabilities, as well as various preferences, such as geographic distance.
- Setting up physical routes to implement the logical links. This may encompass disjointness constraints, and various cost and reliability considerations.
- Dimensioning the logical links, i.e., assigning transmission capacities to them along the physical routes, taking traffic related issues, such as blocking probabilities into account.

These tasks have been intensely researched for more than a decade. Many variants have been investigated. For example, the underlying physical network may be static, but may also be dynamically reconfigurable. The latter happens, e.g., in data center networks, see Curran et al. [5]. The demand for Virtual Networks, and their parameters, may also be static, or stochastic. The 2013 survey paper by Fischer et al. [10] already reviews as many as 80 published algorithms, and since the time the survey has been published, many more have been proposed. (More than 1,100 papers refer to this survey!) Some of the approaches solve the various sub-tasks separately, while others set up a larger joint optimization problem, and search for a global optimum. However, a common feature in all cases that one has to solve NP-hard optimization tasks, due to the involvement of NP-complete problems, which are known to be notoriously hard. As a result, the proposed solutions either lead to prohibitively long computation time, or are forced to give up optimality.

The Network Slicing literature is somewhat smaller, since it is a more recent concept. There are attempts to formalize it in various ways: as a complex optimization problem (Han et al. [12]), to approach it with the tools of machine learning (Le et al. [18]), using game theory based auction models (Jiang et al. [13]), big data analytics (Raza et al. [19]), and a number of others. Again, a common feature is that hardness unavoidably forces the algorithms to sacrifice either speed or optimality. Furthermore, there is not even clear agreement on what is the “right” model that can best capture the problem. With some sarcastic exaggeration, we could describe the status of research with this statement: we do not know how to solve the problem, but we are not even sure what exactly to solve.

Thus, by and large, we can characterize the current situation in Virtual Networks and Network Slicing with the following features:

- There is no clear consensus on how to capture the problems, which model is best for which situation.
- The algorithmic solutions are either too slow, or else they sacrifice optimality. In the latter case the authors resort to various heuristics, typically without any performance guarantee. Nevertheless, this is usually accepted with the argument “what else can you do?”

B. Our Goal

The above outlined situation motivates us to propose a new approach, with the following main features:
• We intend to make our model general and comprehensive enough, so that it can capture many special cases. In this sense, we do not want to propose just another model, after hundreds of previous ones. Rather, we aim at creating a common generalization of many special cases, applicable both to Virtual Networks and Network Slicing. The specific problem we focus on is this: how much capacity should be allocated to each virtual/logical entity, such that the total carried traffic (or its weighted version, the network revenue) is maximized, under the constraint that the logical capacities together fit in the available physical capacities.

• We intend to make the model flexible, so that by adjusting some of its elements, it can accommodate many different specific cases of importance.

• The hardest part is this: what to do with the evergreen desire for optimality? We cannot reasonably aim at optimally solving NP-hard optimization problems efficiently. On the other hand, we are also not satisfied by resorting to mere heuristics without any performance guarantee, that is, finding ad-hoc solutions that can be arbitrarily far from the optimum. But is there anything else that one can reasonably do? We believe the answer is yes, but in a quite nontrivial way. We elaborate on it in the subsequent sections.

II. The Proposed Model

A. Elements of the Model

• Physical Entities. They represent the physical parts of the network, such as nodes, links, computers, routers, storage units, base stations, sensors, etc., i.e., anything that is physically present and takes part in the operation of the network. We denote the physical entities by \( P_1, \ldots, P_n \).

• Physical Capacities. Each physical entity is represented by a numerical parameter, called its capacity. The unit in which the capacity is measured depends on the nature of the physical entity. For example, if it is a link, then its capacity means its maximum transmission rate, which can be measured, say, in Mb/s. If it is a processor, then its capacity is the processing speed in some unit, such as megaflops. If it is a storage device, then its capacity is its storage capacity, expressed, say, in gigabytes. If it is a radio link, then its capacity is bandwidth, which can be measured in MHz.

• Capacity Types. The physical nature of the capacity parameter (whether it is transmission rate, bandwidth, processing speed, storage size, etc) is referred to as the type of the capacity.

• Vector of Physical Capacities. We collect the physical capacity values in a vector, denoted by \( C_{\text{phys}} \). That is, the \( i^{th} \) coordinate of \( C_{\text{phys}} \) is the capacity of physical entity \( P_i, i = 1, \ldots, n \).

• Physical Entities with Multiple Parameters. A network can naturally contain entities that cannot be represented by a single parameter. For example, a computer can have a certain processing speed and a certain storage capacity. To capture this, we could certainly represent the capacity by a vector. That, however, would make the notation messy, as these vectors could have a different number of components, but we still want to collect them in \( C_{\text{phys}} \). Therefore, we rather reduce it to the single
parameter case, by representing the physical entity as several entities, one for each parameter. In this example, the computer would be represented as two physical entities: a processor with some processing speed, and a storage unit with some storage capacity.

- **Logical Entities.** Any subset of physical entities with the same capacity type can form a logical entity. For example, if a logical link \( L \) is embedded in a network by a physical route, containing the physical links \( L_1, \ldots, L_k \), each characterized by its transmission rate, then they together form a logical entity. Note that different logical entities, as subsets, are allowed to overlap.

It is also allowed that the set contains only a single physical entity. For example, a logical node, such as a Virtual Machine, may be mapped into a single physical node. Such Virtual Machines can carry out various service functions for packets, such as firewall, proxy server, they can perform deep packet inspection, access control, network address translation, traffic compression, QoS policy enforcement, traffic optimization, etc. This allows implementing the functionality of *Service Function Chaining*.

We denote the logical entities by \( L_1, \ldots, L_m \).

- **Incidence of Logical and Physical Entities.** This is represented by an \( m \times n \) matrix \( S = [s_{ij}] \) with 0-1 entries. It expresses the incidence of logical and physical entities such that the \( r \)th row indicates which physical entities are contained in (i.e., used by) logical entity \( L_r \). That is,

\[
s_{ij} = \begin{cases} 
1 & \text{if } P_j \in L_i \\
0 & \text{if } P_j \notin L_i.
\end{cases}
\]

- **Logical Capacities.** Each logical entity, just like the physical ones, is characterized by a parameter called logical capacity. This has the same type as the physical capacities of the physical entities that make up the logical entity (these physical entities are required to have the same type of capacity).

- **Vector of Logical Capacities.** Let us denote the capacity of logical entity \( L_i \) by \( C_i \). We collect them in a vector, denoted by \( C = (C_1, \ldots, C_m) \).

- **Capacity Constraints.** The summed capacities of the logical entities that use a given physical entity cannot be more than the capacity of the physical entity (note that they all must have the same type, due to the shared physical entity). This holds for every physical entity, which can be concisely expressed in matrix-vector notation as

\[
SC \preceq C_{\text{phys}}. \tag{1}
\]

- **Offered Load to a Logical Entity.** The offered load represents the demand for a logical entity. For example, if it is a logical link, then the demand is the offered traffic. For processors it may be the requested processing speed. For storage units, the offered load is the data amount to be stored. The demand can be random, represented by a stochastic process, which is typically the case, e.g., for the offered traffic load. We assume, for (initial) simplicity, that this stochastic process has a time-invariant expected value, denoted by \( \rho_i \), which is the offered load to logical entity \( L_i \).

- **Loss Functions.** If a logical entity \( L_i \) receives an offered load of \( \rho_i \), then it may not be able to fully satisfy this demand. We say that loss occurs, in the sense that the carried load, the part of
the demand that is actually satisfied, is less than the offered load. The carried load is also captured (for initial simplicity) by a time-invariant expected value. If \( \rho_i > C_i \), i.e., the demand exceeds the capacity, then loss must clearly occur. But even if \( \rho_i \leq C_i \), loss can still occur, due to the possible random fluctuation of the offered load around its average.

We capture the loss by a *loss function*, denoted by \( F_i(\rho_i, C_i) \). The meaning of the loss function is that if logical entity \( L_i \) receives an offered load \( \rho_i \), and has capacity \( C_i \), then an \( F_i(\rho_i, C_i) \) fraction of the offered load is lost. We can also view it as loss probability or blocking probability. As a result, the carried load will be \( \rho_i(1 - F_i(\rho_i, C_i)) \).

These loss functions may be different for different logical entities, that is why they are indexed by \( i \). Note that even though the arrival process (offered load) is characterized by its expected value \( \rho_i \), the choice of the loss function can express much more information about the detailed stochastic behavior. After all, \( F_i(\rho_i, C_i) \) can represent the probability that the *actual* random load exceeds a threshold \( C_i \). In this sense, the values of \( F_i(\rho_i, C_i) \) with different \( \rho_i, C_i \) values can describe the entire probability distribution of the load (assuming it is stationary, which we assume for initial simplicity). To provide maximum model flexibility, we allow arbitrary real-valued functions for \( F_i \), except that they are required to satisfy the following basic conditions, which are needed for making the mathematical analysis feasible:

(i) \( 0 \leq F_i(\rho_i, C_i) \leq 1 \), so that we can view it as loss probability.

(ii) \( F_i(\rho_i, C_i) \) is a continuous function in both variables. 

*Note*: the stronger property of being differentiable is not required. For example, the function may have breakpoints where the derivative does not exist.

(iii) \( F_i(\rho_i, C_i) \) is increasing in \( \rho_i \), for any fixed \( C_i \), and decreasing in \( C_i \), for any fixed \( \rho_i \). (the increase/decrease does not have to be strict, the function may remain constant). In words, these express the natural expectation that putting more load on the same capacity cannot result in smaller loss, and adding more capacity to carry the same load cannot increase the loss.

- **Flows.** The definition of a flow has two parts: (1) a set of logical entities; we say that the flow *traverses* these entities; and (2) it is characterized by the following parameters:
  - **Offered flow:** A flow amount that we would like to push through.
  - **Capacity demand:** A capacity demand per unit offered flow for each traversed logical entity.

To understand the meaning of flow amount, let us bring a classical analogy. Imagine that we want to use a route for broadband calls in the telephone network. Then the offered flow is the number of such calls we would like carry by the route. The capacity demand tells how many circuits are needed on each link for one call.

We generalize this classical scenario in several ways. First, the flow can be served by an arbitrary set of entities, not only by a route. Second, these entities can have different types of capacities. For example, transmission rates of links, processing capacities of nodes, storage capacity of storage units, bandwidth of radio links, etc. Three, the amount of capacity needed for one unit of flow may be
different on different entities. Four, these are logical entities. For example, logical links that may be implemented by physical routes. (Keep also in mind, however, that a logical link/node may contain a single physical entity; so we can also represent the case when the components are physical.)

- **Virtual Networks or Network Slices.** A Virtual Network or Network Slice is defined as a system of flows. (For initial simplicity, we first consider a static set.) Observe that we have a nested system of abstractions here. For example, a Network Slice is described by a set of flows, each flow incorporates a subset of logical entities, and each logical entity is made up by a subset of physical entities. Of course, multiple network slices may exist in the network simultaneously, and share the underlying physical capacities.

- **Goal.** The key goal of the optimization is to tell how much capacity should be allocated to each logical entity, such that the total carried traffic (or a weighted version of thereof) is maximized, under the constraint that the logical capacities together fit in the available physical capacities.

  **Important note:** the loss functions can reflect very different requirements. Allowing this is a key feature of the model. For example, one loss function can express classical blocking probability. Another one can express, e.g., that a service, such as emergency notification, needs resources which provide extremely low delay and high reliability, or else its loss will be unacceptably high. Yet another service, such as video conferencing, may need resources that guarantee low jitter, small packet loss, and high bandwidth, or else its quality will be unacceptable, leading to high loss. It is a main feature of the model that we can treat these very different requirements in a unified way, without losing the ability of efficient optimization.

- **Notational convention:** To make the notation easier to follow, whenever it does not cause confusion, we denote an entity simply by its index. In this vein, let us number the flows that exist in the network by $1, \ldots, R$, and the logical entities by $1, \ldots, m$. For every $j, r$, let $A_{j r}$ denote the demand (capacity units) that is requested on logical entity $j$ by flow $r$, per unit offered flow. We assume that $A_{j r}$ is integer valued, which can always be achieved by an appropriate scaling of units. If the flow does not traverse $j$, then $A_{j r} = 0$. Further, let us denote the offered load of flow $r$ by $\nu_r$.

**B. Fundamental Equations**

For (initial) simplicity we introduce the assumptions below, because they significantly help the analysis. These assumptions lead to asymptotically exact results, i.e., only cause vanishing errors, when the capacities grow large, as was analyzed under classical scenarios, see Kelly [14], [16], Labourdette [17].

**Independence Assumptions:**

1) The losses on different logical entities are considered independent random events.

2) When multiple units of capacity are needed for a unit of flow on a logical entity, it is modeled as grabbing each capacity unit independently, if available.
Let us now compute the carried load of flow \( r \). It has an offered load \( v_r \). On each logical entity \( j \) that the flow traverses it suffers a (relative) loss of \( F_j(\rho_j, C_j) \), where \( F_j, \rho_j, C_j \) are the loss function, offered load, and capacity of logical entity \( j \), respectively. That is, the probability that the flow can successfully grab a unit of available capacity on \( j \) is \( 1 - F_j(\rho_j, C_j) \). Since each unit of flow needs \( A_{jr} \) units of capacity, therefore, by the second independence assumption, its success probability on \( j \) (the probability that the unit of flow gets through) will be \( (1 - F_j(\rho_j, C_j))^{A_{jr}} \). Then we can obtain the success probability of a unit of the entire flow, as the product of these probabilities (by the first independence assumption) over the set of all logical entities \( \mathcal{F}_r \) that flow \( r \) uses:

\[
\prod_{j \in \mathcal{F}_r} \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}.
\]

Observe now that whenever \( j \notin \mathcal{F}_r \), we have \( A_{jr} = 0 \), leading to \( (1 - F_j(\rho_j, C_j))^{A_{jr}} = 1 \). These factors of 1 do not change the product value, so we can take the product over all \( j \), rather than just \( j \in \mathcal{F}_r \). Multiplying it with the offered load of flow \( r \) we get the carried load of flow \( r \) as

\[
v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}.
\]

Consider now a logical entity \( i \). Recall that if its offered load is \( \rho_i \), then its carried load will be \( \rho_i (1 - F_i(\rho_i, C_i)) \), where \( F_i \) is its loss function, and \( C_i \) is its capacity. But all the carried load on \( i \) must come from the flows that use \( i \). Therefore, if we sum up the carried loads of all flows, taking into account that flow \( r \) uses \( A_{ir} \) capacity on \( i \) per unit load, then we get the following equation:

\[
\rho_i (1 - F_i(\rho_i, C_i)) = \sum_r A_{ir} v_r \left[ \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}} \right].
\]

Note that whenever \( r \) does not use \( i \), we have \( A_{ir} = 0 \), so it is safe to do the summation for all flows \( r \). The above equation holds for every logical entity \( i \). Therefore, after rearranging, we get the following system of equations for the offered loads \( \rho_i, i = 1, \ldots, m \), of the \( m \) logical entity:

\[
\rho_i = (1 - F_i(\rho_i, C_i))^{-1} \sum_r A_{ir} v_r \left[ \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}} \right].
\]

If the logical capacities \( C_1, \ldots, C_m \) are known, then the \( \rho_i \) values can be computed from this system by iterated substitution. Once \( \rho_1, \ldots, \rho_m \) are computed, we get the total carried load, by summing up \( (2) \) for all flows. Let us denote the total carried load by \( T \), then we have

\[
T = \sum_r \left( v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}} \right).
\]

However, the logical capacities \( C_j \) are not known! They are precisely what we want to optimize, so that we can tell what the optimal allocation is of logical capacities, within the physical constraints. The latter are expressed by the linear system of equations \( (1) \). Note that adding more logical capacity to a logical entity can only happen at the expense of others, since the physical limits are given.

\[1\]This system is a generalization of what is known as Erlang Fixed Point Approximation, see Kelly \[15\].
Thus, we face the following, rather complicated looking, optimization problem:

\[
\text{Maximize} \quad T(C) = \sum_r v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}} \quad (3)
\]

Subject to

\[
\rho_i = \left(1 - F_i(\rho_i, C_i)\right)^{-1} \sum_r A_{ir} v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}, \quad i = 1, \ldots, m \quad (4)
\]

\[
\text{SC} \leq C_{\text{phys}} \quad (5)
\]

\[
C \geq 0 \quad (6)
\]

III. Optimization

Clearly, the optimization task described above by (3), (4), (5), and (6), is quite complex and heavily nonlinear. At first, it appears hopeless to find (or at least closely approximate) a globally optimal solution. Surprisingly, however, the objective function can be well approximated by a concave function, which is much easier to maximize. Specifically, we can prove a theorem presented below. To explain it, let us first introduce a useful concept:

Asymptotically concave function. A function \( f \) is called asymptotically concave if for each \( x \geq 1 \) the point-wise limit

\[
\tilde{f}(x) = \lim_{n \to \infty} \frac{1}{n} f(nx)
\]

exists and \( \tilde{f} \) is a concave function in the ordinary sense.

In other words, an asymptotically concave function is almost concave for large variable values, which is the case we consider (large capacities).

Theorem 1: If the independence assumptions\(^2\) hold, then there exist a correction function \( \epsilon(C) \) with the following properties:

(i) \( \tilde{T}(C) = T(C) + \epsilon(C) \) is an asymptotically concave function of \( C \).

(ii) The correction function \( \epsilon(C) \) is small in the following sense: \( 0 \leq \epsilon(C) \leq \sum_i \rho_i B_i \), where \( B_i = F(\rho_i, C_i) \) is the blocking probability (loss) of logical entity \( i \).

Note: Observe that \( \rho_i B_i \) is the blocked load on logical entity \( i \). Since under normal operation the overall blocked load is expected to be small, therefore, we can expect \( \epsilon(C) \ll T(C) \), resulting in a small

\(^2\)See at the beginning of Section II-B.
difference between the original objective function $T(C)$ and its modified version $\tilde{T}(C)$. Also note that all this is valid for any system of loss functions, as long as they satisfy the mild mathematical requirements outlined in Section II-A.

**Proof of Theorem 1:** See in Appendix A.

Theorem 1 gives hope to find an approximation of the global optimum, since globally maximizing a concave function over a convex domain is a well solved problem. In our case, the modified objective function $\tilde{T}(C)$ is indeed concave. Regarding the constraints, the linear inequalities (5), (6) alone would indeed define a convex domain. Unfortunately, however, this is badly messed up by the heavily nonlinear system (4) of equations. Nevertheless, surprisingly again, we can prove the following:

**Theorem 2:** If the independence assumptions hold, then there exist a function $\phi(C)$ with the following properties:

(i) $\phi(C)$ is an asymptotically concave function of $C$.

(ii) The value of $\phi(C)$ can be computed by a polynomial time algorithm.

(iii) If $C^*$ is an optimal solution of the new optimization problem

$$\text{maximize } \phi(C)$$

subject to

$$SC \leq C_{phys}, \quad C \geq 0$$

then $C^*$ is also an optimal solution to the modified version of the original optimization problem

$$\text{maximize } \tilde{T}(C)$$

subject to (4), (5), (6).

**Proof:** See in Appendix A.

Thus, we may say that the new objective function $\phi(C)$ can “swallow” the badly nonlinear system (4) of constraints, and leaves only the linear part to be considered. Yet (and this is the surprising part!) $\phi(C)$ still can be chosen such that it remains an efficiently computable asymptotically concave function. Once we have it, we see that for large capacities the new optimization task (7) requires only the maximization of an (efficiently computable) nearly concave function over a convex domain, given by linear inequalities. This optimization task can already be solved globally and efficiently by standard methods of convex optimization, for which off-the-shelf commercial software is also available.

A. Interlude: A Bold Conjecture About Optimization

The above theorems suggest that even a very complicated-looking optimization problem may be approximated by one that is efficiently (i.e., polynomial-time) solvable. (Convex optimization is known to
be solvable in polynomial time, see, e.g., [2], [3].) One may wonder: is it only good luck in the considered case that complicated optimization can be approximated efficiently, or is it perhaps the manifestation of a more general phenomenon?

Our recent papers [6], [9] strongly suggest that it is indeed a more general phenomenon. We do not have space here to elaborate the rather complex details (they are detailed in the referenced papers), so let us just briefly state the essence. We were able to formally prove that for a large class of decision problems (i.e., questions with a yes/no answer) the following holds: every problem in the class can be approximated by a polynomial-time solvable one, in the sense that they differ only on an asymptotically vanishing subset of instances. One might view it as an analogy to the classical Weierstrass Theorem in real analysis:

> Every continuous function on a bounded interval can be arbitrarily well approximated by a polynomial. In other words, the polynomials constitute a dense subset of all continuous functions over the interval.

Our result can be formulated analogously this way: with an appropriate, more sophisticated definition of density, the polynomial-time solvable decision problems are dense in a much larger class of decision problems. What is very surprising, all known natural NP-complete problems are in the considered class! *Natural* means here that it has been studied on its own right, rather than having been constructed artificially (e.g., by diagonalization), just for the sake of an example or counterexample.

The above result of ours provides strong motivation to conjecture that this phenomenon may carry over to optimization tasks from decision problems. Let us informally state the conjecture:

**Conjecture 1:** There is a (reasonable) definition of density, such that in the set of natural optimization problems the ones that are solvable in polynomial time constitute a dense subset. In other words, all natural optimization tasks can be well approximated by efficiently solvable ones.

While the above conjecture may sound very bold and surprising, our results in [6], [9] suggest that it has a quite reasonable chance to hold. Note that if the conjecture is indeed true, and one can find a constructive proof, then it would have a huge impact. It would mean that all the notoriously hard natural optimization problems can be well approximated with efficient algorithms! But this takes quite a bit of more work, we do not intend to engage into it in the present paper.

IV. **TREATING RECONFIGURABLE PHYSICAL NETWORKS**

In some cases the physical network is not fixed, it is reconfigurable. An example is when in a data center network the racks of computers communicate via free space laser links, which can be quickly reconfigured, when needed, see Curran et al. [5]. Interestingly, this case can also fit in our model.

Consider the case, when the $n$ physical entities of our model are not fixed in advance. Rather, they can be chosen arbitrarily from a set of $N$ potential physical entities. Let us look at the simplest case,
when all potential physical entities have unit capacity, measured in relative units. The logical entities, and the flows on top of them, can now use all the potential physical entities. Of course, eventually only those among them can actually operate, which only use the chosen physical entities, i.e., the ones that are selected from the potential set to come into existence. Can we still fit this in the optimization? The answer is yes! Let $\mathbf{e}_N$ denote the $N$-dimensional vector in which all components are 1. We treat now the physical capacity vector $\mathbf{C}_{\text{phys}}$ as a variable, not constant. Its value may be any vector in which each coordinate is in the interval $[0, 1]$, expressing the values in relative units, with a maximum of 1. Then we can modify our optimization task to

$$\text{Maximize } T(\mathbf{C}) = \sum_r v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}$$

Subject to

$$\rho_i = (1 - F_i(\rho_i, C_i))^{-1} \sum_r A_{ir} v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}, \quad i = 1, \ldots, m$$

$$\mathbf{SC} \leq \mathbf{C}_{\text{phys}}$$

$$\mathbf{C} \geq 0, \mathbf{C}_{\text{phys}} \geq 0$$

$$\mathbf{e}_N \geq \mathbf{C}_{\text{phys}} \geq 0$$

$$\mathbf{e}_N^T \mathbf{C}_{\text{phys}} \leq n$$

It turns out, as in Section III, that this task can also be (approximately) converted to the maximization of a concave function over a convex domain. This convex setting yields that in the optimal solution the $\mathbf{C}_{\text{phys}}$ vector will be at a vertex of the polyhedral feasible domain. This forces it to be a 0-1 vector, thus identifying (via its 1-coordinates) the optimal choice of the $n$ physical entities, out of the potential $N$, i.e., the optimal configuration of the reconfigurable physical network.

V. CONCLUSION

We have introduced a general and comprehensive model for the design and optimization of Network Slicing and Virtual Networks. The model is flexible, so that by adjusting some of its elements, it can accommodate many different specific cases of importance. Yet, surprisingly, it still allows efficient optimization, in the sense that the global optimum can be well approximated by efficient algorithms. In the present paper our goal was to describe the conceptual model, elaborate its fundamental equations, as well as issues related to its efficient optimization. Numerical investigations and simulation will be the subject of future papers.
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APPENDIX

PROOFS OF THEOREMS 1 AND 2 (SKETCH)

The modification we use in the objective function lies in adding a term that is characteristic to *link utilization*. Adopting ideas from Kelly [16] and from our previous work Faragó et al. [7], [8] (where we used it in a more restricted context), let us define the utilization function $U(y, C_j)$ on a logical entity $j$ of capacity $C_j$ by the *implicite* relation

$$U(-\log(1 - F_j(\rho_j,C_j)),C_j) = \rho_j(1 - F(\rho_j,C_j)).$$
$U(y, C_j)$ is exactly the mean amount of capacity in use, i.e. the average occupancy, when the loss probability is $1 - \exp(-y)$. In other words, $U(y, C_j)$ is the average utilization of the logical entity with respect to a logarithmically scaled loss probability.

Consider a logical entity of capacity $C_j$ and blocking probability $B_j$. Let us define the utilization measure as

$$\tilde{U}(B_j, C_j) = \int_0^{-\log(1-B_j)} U(z, C_j) \, dz.$$  

Note that the value $y = -\log(1 - B_j)$ is a logarithmic measure of loss. If $B_j = 0$ then $y = 0$, if $B_j$ approaches 1 then $y$ tends to infinity and the mapping between $B_j$ and $y$ is strictly increasing.

Now we state our modified optimization problem as follows.

Maximize

$$\sum_r v_r \prod_j (1 - B_j)^{A_{jr}} + \sum_j \tilde{U}(B_j, C_j)$$

Subject to

$$\rho_i = (1 - F_i(\rho_i, C_i))^{-1} \sum_r A_{ir} v_r \prod_j \left(1 - F_j(\rho_j, C_j)\right)^{A_{jr}}, \quad i = 1, \ldots, m,$$

$$\text{SC} \leq C_{phys}$$

$$C \geq 0$$

Remark: The difference between the original and the modified problem is the additional utilization term in the objective function. Note, however, that this difference is quite negligible for realistic values. This follows from the following argument.

Let $T$ be the total carried traffic. Further, let $T'$ be a weighted version of the total carried traffic such that the carried traffic on each logical route is multiplied by the length of the route (number of logical entities on the route). For example, if a route $r$ carries 5 units of traffic and $r$ consists of 3 logical entities, then $r$ contributes 5 to $T$ and 15 to $T'$.

Now we can observe that $T'$ equals to the sum of logical entity utilizations, summed over all logical entities, since the total number of circuits in use (on the average) is exactly the average carried traffic if we take into account that a route occupies capacity on a number of logical entities, equal to the length of the route.

Let $L$ be the length of the longest route. Then $T' \leq LT$. Now, if we approximate $-\log(1 - B_j)$ by $B$, which is good for small values of $B$, and use the fact that $U(y, C)$ is increasing in $y$, then we can bound the link utilization term in the objective function from above by $BLT$, where $B$ is the largest blocking probability. In other words, instead of optimizing the total carried traffic $T$, we optimize a quantity $Q$ that satisfies

$$T < Q < (1 + BL)T.$$
If $B$ is small and $L$ is not too large, then $1 + BL$ can be quite close to 1. For example, if $B < 0.5\%$ and $L = 4$, then we have

$$T < Q < 1.02T.$$  

Thus, we are optimizing an objective function that is a good approximation of the total carried traffic. Extending this informal argument to the general case, we obtain a proof of (ii).

To prove (i), let us define an auxiliary function $\phi(C)$ as follows.

$$\phi(C) = \min_{y \geq 0} \left\{ \sum_r y_r e^{-\Sigma_j y_j A_{jr}} + \sum_j \int_0^{y_j} U(z,C) \, dz \right\},$$  

where $y = (y_1, \ldots, y_J)$. Now, using the methods of Kelly [16], we can prove the following:

**Lemma 1.** $\phi(C)$ is an asymptotically concave function of $C$.

**Lemma 2:** The logical capacity vector $C$ is an optimal solution to problem (14) if and only if it is an optimal solution to the problem

$$\text{Maximize} \quad \phi(C) \quad \text{Subject to} \quad SC \leq C_{phys} \quad \text{and} \quad C \geq 0.$$  

Observe that by Lemma 1, problem (16) means the maximization of an asymptotically concave function over a convex domain. Thus, in the asymptotic sense, that is, when the capacities are large, the task tends to an ordinary convex programming task that can be solved by standard techniques. The result yields a solution to problem (14), as well, by Lemma 2.

Of course, to carry out the optimization, we need to be able to compute the value of $\phi(C)$ efficiently for any given $C$. This can be done again by convex optimization, since the computation of $\phi(C)$ for a given $C$ requires the minimization of a convex function over a convex domain. This is so, because one can directly check that the argument of minimization in (15) is a convex function of $y$. 

\diamond