Discrete sticky couplings of functional autoregressive processes

Alain Durmus\textsuperscript{1}, Andreas Eberle\textsuperscript{2}, Aurélien Enfroy\textsuperscript{3}, Arnaud Guillin\textsuperscript{4}, and Pierre Monmarché\textsuperscript{5}

\textsuperscript{1}Université Paris-Saclay, ENS Paris-Saclay, CNRS, Centre Borelli, F-91190 Gif-sur-Yvette, France.
\textsuperscript{2}Institute for Applied Mathematics - University of Bonn, Germany.
\textsuperscript{3}Samovar, Télécom SudParis, département CITI, TIPIC, Institut Polytechnique de Paris, Palaiseau.
\textsuperscript{4}Laboratoire de Mathématiques Blaise Pascal - Université Clermont-Auvergne, France.
\textsuperscript{5}LJLL - Sorbonne Université, France.

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Abstract

In this paper, we provide bounds in Wasserstein and total variation distances between the distributions of the successive iterates of two functional autoregressive processes with isotropic Gaussian noise of the form

\[ Y_{k+1} = T_\gamma(Y_k) + \sqrt{\gamma \sigma^2} Z_{k+1} \]

and

\[ \tilde{Y}_{k+1} = \tilde{T}_\gamma(\tilde{Y}_k) + \sqrt{\gamma \sigma^2} \tilde{Z}_{k+1} \].

More precisely, we give non-asymptotic bounds on \( \rho(\mathcal{L}(Y_k), \mathcal{L}(\tilde{Y}_k)) \), where \( \rho \) is an appropriate weighted Wasserstein distance or a \( V \)-distance, uniformly in the parameter \( \gamma \), and on \( \rho(\pi_\gamma, \tilde{\pi}_\gamma) \), where \( \pi_\gamma \) and \( \tilde{\pi}_\gamma \) are the respective stationary measures of the two processes. The class of considered processes encompasses the Euler-Maruyama discretization of Langevin diffusions and its variants. The bounds we derive are of order \( \gamma \) as \( \gamma \to 0 \). To obtain our results, we rely on the construction of a discrete sticky Markov chain \( (W_k(\gamma))_{k \in \mathbb{N}} \) which bounds the distance between an appropriate coupling of the two processes. We then establish stability and quantitative convergence results for this process uniformly on \( \gamma \). In addition, we show that it converges in distribution to the continuous sticky process studied in [20, 18]. Finally, we apply our result to Bayesian inference of ODE parameters and numerically illustrate them on two particular problems.

1 Introduction

We are interested in this paper in Markov chains \( (Y_k)_{k \in \mathbb{N}} \) starting from \( y \in \mathbb{R}^d \) and defined by recursions of the form

\[ Y_{k+1} = T_\gamma(Y_k) + \sigma \sqrt{\gamma} Z_{k+1} , \]  \hspace{1cm} (1)

where \( \sigma > 0, \gamma \in (0, \bar{\gamma}] \), for some \( \bar{\gamma} > 0 \), \( \{T_\gamma : \gamma \in (0, \bar{\gamma}]\} \) is a family of continuous functions from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) and \( (Z_k)_{k \geq 1} \) is a sequence of i.i.d. \( d \)-dimensional standard Gaussian random variables. Note that the Euler-Maruyama discretization of overdamped Langevin diffusions or of general Komolgorov processes and its variants belong to this class of processes and in

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\textsuperscript{1}Email: alain.durmus@ens-paris-saclay.fr
\textsuperscript{2}Email: eberle@uni-bonn.de
\textsuperscript{3}Email: aurelien.enfroy@ens-paris-saclay.fr
\textsuperscript{4}Email: arnaud.guillin@uca.fr
\textsuperscript{5}Email: pierre.monmarche@sorbonne-universite.fr
that setting $\gamma$ corresponds to the discretization step size. Indeed, the Euler scheme consists in taking for any $\gamma \in (0, \bar{\gamma}]$, $T_\gamma(y) = y + \gamma b(y)$ for some $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. When $b = -\nabla U$ for some potential $U$ and $\sigma^2 = 2$, these methods are now popular Markov Chain Monte Carlo algorithms to sample from the target density $x \mapsto e^{-U(x)}/\int_{\mathbb{R}^d} e^{-U(y)}dy$. However, in some applications, explicitly computing $\nabla U$ is not an option and further numerical methods must be implemented which come with additional bias since only approximations of $\nabla U$ can be used in (1). In this paper, we precisely study this additional source of error. In particular, based on a chain defined by (1), we consider a second Markov chain $(\tilde{Y}_k)_{k \in \mathbb{N}}$ defined by the recursion

$$\tilde{Y}_{k+1} = \tilde{T}_\gamma(\tilde{Y}_k) + \sigma \sqrt{\gamma} \tilde{Z}_{k+1},$$  

where $\{\tilde{T}_\gamma : \gamma \in (0, \bar{\gamma}]\}$ is a family of functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that for any $\gamma$, $\tilde{T}_\gamma$ is an approximation of $T_\gamma$ in a sense specified below, and $(\tilde{Z}_k)_{k \geq 1}$ is a sequence of i.i.d. $d$-dimensional standard Gaussian random variables potentially correlated with $U$.

We will enforce below conditions that ensure that both $(Y_k)_{k \in \mathbb{N}}$ and $(\tilde{Y}_k)_{k \geq 1}$ are geometrically ergodic, and denote by $\pi_\gamma$ and $\tilde{\pi}_\gamma$ their invariant probability measures respectively. If for any $\gamma > 0$, $\tilde{T}_\gamma$ is close in some sense to $T_\gamma$, the overall process $(\tilde{Y}_k)_{k \in \mathbb{N}}$ can be seen as a perturbed version of $(Y_k)_{k \in \mathbb{N}}$, and $\tilde{\pi}_\gamma$ is expected to be close to $\pi_\gamma$. The main goal of this paper is to establish quantitative bounds on the Wasserstein and total variation distance between the finite-time laws of the two processes and between their equilibria. The study of perturbation of Markov processes has been the subject of many existing works; see e.g. [33, 28, 21, 32, 26] and the references therein. However, it turns out that these existing results do not apply as such. We pay particular attention to the dependency of these estimates on $\gamma$. Indeed, in the case of the Euler scheme of a continuous-time diffusion, $\pi_\gamma$ and the law of $Y_{\lfloor t/\gamma \rfloor}$ for some $t > 0$ converge to the invariant measure and law at time $t$ of the continuous-time process, and similarly for the perturbed chain. Hence, as $\gamma \to 0$, our estimates should not degenerate, but rather yield quantitative estimates for the continuous time process. More precisely, the present paper is the discrete-time counterpart of the study conducted by [18] in the continuous-time case, and as $\gamma$ vanishes we recover estimates that are consistent with those of [18].

As in [18], our results are based on the construction of a suitable coupling of the processes, i.e. a simultaneous construction of a pair $(Y_k, \tilde{Y}_k)_{k \in \mathbb{N}}$ of non-independent chains that marginally follow (1) and (2) respectively and are designed to get and stay close to each other. We use the maximal reflection coupling for Gaussian laws, namely at each step the two chains are coupled to merge with maximal probability and, otherwise, we use a reflection (see Section 2.2 below). Estimates on the laws of the chains then follow from the study of $(\|Y_k - \tilde{Y}_k\|)_{k \in \mathbb{N}}$, which is itself based on the analysis of a Markov chain $(W_k)_{k \in \mathbb{N}}$ on $[0, +\infty)$ that is such that, by design of the coupling, $\|Y_k - \tilde{Y}_k\| \leq W_k$ for all $k \in \mathbb{N}$. Thus, the question of establishing bounds between the laws of two $d$-dimensional Markov chains is reduced to the study of a single one-dimensional chain. Besides, together with the Markov property, the auxiliary chain has some nice features. At first, it is stochastically monotonous, i.e. if $(W'_k)_{k \in \mathbb{N}}$ is a Markov chain associated to the same Markov kernel as $(W_k)_{k \in \mathbb{N}}$ and such that $W_0 \leq W'_0$ then for any $k \in \mathbb{N}$, $W'_k$ is stochastically dominated by $W_k$, i.e. for any $t \geq 0$, $\mathbb{P}(W_k < t) \geq \mathbb{P}(W'_k < t)$. Secondly, $(W_k)_{k \in \mathbb{N}}$ has an atom at 0.

The main results and main steps of this study are the following. First, we prove that $(W_k)_{k \in \mathbb{N}}$ admits a unique invariant measure and that, independently of $\gamma$, the moments and mass on $(0, +\infty)$ of this equilibrium are small when the difference between $T_\gamma$ and $\tilde{T}_\gamma$ is
small. Secondly, we establish the geometric convergence of the chain towards its equilibrium, at an explicit rate (stable as \( \gamma \to 0 \)). Finally, we prove that, as \( \gamma \to 0 \), the chain \((W_k)_{k \in \mathbb{N}}\) converges in law to the continuous-time sticky diffusion that played the same role in [18]. This last part is not necessary to get estimates on the finite-time and equilibrium laws of (1) and (2) for a given \( \gamma > 0 \), but it sheds some new light on the limit sticky process which, in [18], is constructed as the limit of continuous-time diffusions with diffusion coefficients that vanish at zero, rather than discrete-time chains. In some sense, \((W_k)_{k \in \mathbb{N}}\) can be seen as a discretization scheme for the sticky process, see also [2] on this topic.

Besides the obvious continuous/discrete time difference between [18] and the present work, let us emphasize a few other distinctions. First, in [18], the one-dimensional sticky process has an explicit invariant measure. This is not the case in our framework, which makes the derivation of the bounds on the moments of the equilibrium a bit more involved. Secondly, in [18], although it is proven that the mass at zero and the first moment of the law of the sticky diffusion converge to their value at equilibrium (which is sufficient to get estimates on the laws of the two initial \(d\)-dimensional processes), the question of long-time convergence is not addressed for the sticky diffusion, whereas our long-time convergence results for \((W_k)_{k \geq 0}\) together with its convergence as \( \gamma \to +\infty \) furnish an explicit convergence rate for the sticky diffusion. The proof of the stability of the mass at zero and of the first moment in [18] relies on a concave modification of the distance (such as used e.g. in [16]), which is contracted by the chain before it hits zero. This method does not apply to, say, the second moment of the process. As a consequence, the results of [18] only concern the total variation and \(W_1\) Wasserstein distances, while we consider a broader class of distances.

Finally, our theoretical results are illustrated through numerical experiments. In particular, we study the influence of the discretization scheme generally needed to perform Bayesian inference for parameters of Ordinary Differential Equations (ODEs).

Notation and convention

We denote by \(\mathcal{B}(\mathbb{R}^d)\), the Borel \(\sigma\)-field of \(\mathbb{R}^d\) endowed with the Euclidean distance and by \(\varphi_{\sigma^2}\) the density of the one-dimensional Gaussian distribution with zero-mean and variance \(\sigma^2 > 0\). In the case \(\sigma = 1\), we simply denote this density by \(\varphi\). \(\Phi\) denotes the cumulative distribution of the one-dimensional Gaussian distribution with mean 0 and variance 1. \(\Delta_{\mathbb{R}^d}\) stands for the subset \(\{(x,x) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d\}\) of \(\mathbb{R}^d\) and for any \(A \subset \mathbb{R}^d\), \(A^c\) for its complement. Let \(\mu\) and \(\nu\) be two \(\sigma\)-finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\). If \(\nu\) is absolutely continuous with respect to \(\mu\), we write \(\nu \ll \mu\). We say that \(\nu\) and \(\mu\) are equivalent if and only if \(\nu \ll \mu\) and \(\mu \ll \nu\). We denote by \([\cdot]\) and \(\lceil \cdot \rceil\) the floor and ceiling function respectively. For \(d,n \in \mathbb{N}^*\), \(\mathcal{M}_{d,n}(\mathbb{R})\) stands for the set of \(d \times n\) real matrices. We denote by \(\mathcal{C}^k(U,A)\) the set of \(k\) times continuously differentiable functions from an open set \(U \subset \mathbb{R}^m\) to \(A \subset \mathbb{R}^p\). We use the convention \(\sum_{k=n}^p = 0\) and \(\prod_{k=n}^p = 1\) for \(p < n\), \(n,p \in \mathbb{N}\), and \(a/0 = +\infty\) for \(a > 0\).
2 Sticky reflection coupling

2.1 Main result

The Markov kernels $R_\gamma$ associated with $(Y_k)_{k \in \mathbb{N}}$ defined in (1) are given for any $y \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$R_\gamma(y,A) = (2\pi \sigma^2 \gamma)^{-d/2} \int_{\mathbb{R}^d} 1_A(y') \exp \left\{ - \|y' - T_\gamma(y)\|^2 / (2\sigma^2 \gamma) \right\} dy'.$$

Note that $\tilde{R}_\gamma$ associated with $(\tilde{Y}_k)_{k \in \mathbb{N}}$ is given by the same expression upon replacing $T_\gamma$ by $\tilde{T}_\gamma$. We consider the following assumption on the family $\{T_\gamma : \gamma \in (0, \bar{\gamma})\}$. This condition will ensure that $R_\gamma$ is geometrically ergodic (see Proposition 1) and it will be important to derive our main results regarding the distance of $R^k_\gamma$ and $\tilde{R}^k_\gamma$, for $k \in \mathbb{N}$.

**H 1.** Assume that $\sup_{\gamma \in (0, \bar{\gamma})} \gamma^{-1}\|T_\gamma(0)\| < +\infty$ and for any $\gamma \in (0, \bar{\gamma}]$, there exists a non-decreasing function $\tau_\gamma : [0, +\infty) \to [0, +\infty)$ satisfying

\begin{enumerate}[(i)]
\item $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq \tau_\gamma(\|x - \tilde{x}\|)$ for any $x, \tilde{x} \in \mathbb{R}^d$;
\item $\tau_\gamma(0) = 0$ and there exist $R_1, L \geq 0$ and $m > 0$ such that for any $\gamma \in (0, \bar{\gamma}]$,
\end{enumerate}

\begin{equation}
\sup_{r \in (0, +\infty)} \{\tau_\gamma(r) / r\} \leq 1 + \gamma L, \quad \sup_{r \in (R_1, +\infty)} \{\tau_\gamma(r) / r\} \leq 1 - \gamma m. \tag{4}
\end{equation}

Note that H 1 implies that $\bar{\gamma} \leq 1 / m$. Further, the condition that for any $\gamma \in (0, \bar{\gamma}]$, $\tau_\gamma$ is non-decreasing can be omitted upon replacing in our study $\tau_\gamma$ by the affine majorant

$$\bar{\tau}_\gamma : r \mapsto \begin{cases} (1 + L \gamma) r & \text{if } r \in [0, R_1], \\ (1 + L \gamma) R_1 + (1 - m \gamma)(r - R_1) & \text{otherwise}. \end{cases} \tag{5}$$

Indeed, by definition and (4), for any $r \in [0, +\infty)$, $\tau_\gamma(r) \leq \bar{\tau}_\gamma(r)$, therefore for any $x, \tilde{x} \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq \bar{\tau}_\gamma(\|x - \tilde{x}\|)$. In addition, an easy computation leads to setting $R_2 = 2R_1(L + m) / m$,

$$\sup_{r \in (0, +\infty)} \{\bar{\tau}_\gamma(r) / r\} \leq 1 + \gamma L, \quad \sup_{r \in (R_2, +\infty)} \{\bar{\tau}_\gamma(r) / r\} \leq 1 - \gamma m / 2.$$

Then, $\bar{\tau}_\gamma$ satisfies H 1 and is non-decreasing.

Note that H 1 implies that for any $r \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$, $\tau_\gamma(r) \leq (1 + \gamma L)r$, therefore $T_\gamma$ is $(1 + \gamma L)$-Lipschitz. The second condition in (4) ensures that for any $\gamma \in (0, \bar{\gamma}]$, $T_\gamma$ is a contraction at large distances, i.e. for any $x, \tilde{x} \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq (1 - \gamma m) \|x - \tilde{x}\|$, if $\|x - \tilde{x}\| \geq R_1$.

The assumption H 1 holds for the Euler scheme applied to diffusions with scalar covariance matrices, i.e. (1) with $T_\gamma(x) = x + \gamma b(x)$ and a drift function $b : \mathbb{R}^d \to \mathbb{R}^d$, if, for some $L_b, m_b, R_b > 0$, $b$ is $L_b$-Lipschitz continuous and satisfies

$$\langle x - y, b(x) - b(y) \rangle \leq -m_b \|x - y\|^2,$$

for all $x, y \in \mathbb{R}^d$ with $\|x - y\| \geq R_b$. Indeed, this implies that for any $x, y \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(y)\| \leq (1 + L_b \gamma) \|x - y\|$ and, provided $\gamma \in (0, m_b / L_b^2)$ and $\|x - y\| \geq R_b$,
\[ \| T_\gamma(x) - T_\gamma(y) \|^2 \leq (1 - m_b \gamma) \| x - y \|^2 . \]
Therefore, it suffices to consider \( \tau_\gamma \) defined by (5) with \( L = L_b, m = m_b/2 \) and \( R_1 = R_b \).

Our results will be stated in terms of Wasserstein distances and \( V \)-norms, whose definitions are the following. Consider a measurable cost function \( c : \mathbb{R}^{2d} \to [0, \infty) \). Then the associated Wasserstein distance \( \mathcal{W}_c \) is given for two probability measures \( \mu, \nu \) on \( \mathbb{R}^d \) by

\[ \mathcal{W}_c(\nu, \mu) = \inf_{\pi \in \Pi(\nu, \mu)} \int_{\mathbb{R}^{2d}} c(x, y) \pi(dx, dy), \]

where \( \Pi(\nu, \mu) \) is the set of transference plans or couplings between \( \nu \) and \( \mu \), namely the set of probability measures on \( \mathbb{R}^d \) whose first and second \( d \)-dimensional marginals are \( \nu \) and \( \mu \) respectively. In the particular case where \( c(x, y) = 1_{\Delta_2}(x, y) \), \( \mathcal{W}_c \) is simply the total variation distance \( \| \cdot \|_{TV} \). For \( V : \mathbb{R}^d \to [1, +\infty) \), the choice \( c(x, y) = 1_{\Delta_2}(x, y) \{ V(x) + V(y) \} \) yields the \( V \)-norm (see [11, Theorem 19.1.7]), i.e. \( \mathcal{W}_c(\nu, \mu) = \| \nu - \mu \|_V \). Finally, for \( c(x, y) = \| x - y \|^p \) with \( p \in [1, +\infty) \), \( \mathcal{W}_c \) is the \( p \)-th power of the usual Wasserstein distance of order \( p \).

We first show that \( \mathbf{H}1 \) implies that the Markov kernel \( R_\gamma \) is \( V_c \)-uniformly geometrically ergodic where for any \( c > 0 \) and \( x \in \mathbb{R}^d \), \( V_c(x) = \exp(c \| x \|^2) \), with a convergence rate that scales linearly with the step size \( \gamma \).

**Proposition 1.** Assume \( \mathbf{H}1 \). Then, for any \( \gamma \in (0, \bar{\gamma}) \), \( R_\gamma \) admits a unique stationary distribution \( \pi_\gamma \). In addition, there exist \( c > 0 \), \( \rho \in [0, 1) \) and \( C \geq 0 \) such that for any \( x \in \mathbb{R}^d \) and \( \gamma \in (0, \bar{\gamma}) \), \( \| \Delta_{\gamma} R_\gamma - \pi_\gamma \|_{V_c} \leq C \rho^\gamma V_c(x) \).

**Proof.** The proof of this result follows the same strategy as [9] but since we are not interested in sharp constants a more direct proof is postponed to Appendix A. \( \square \)

We now consider an assumption which quantifies the perturbation associated with \( \hat{T}_\gamma \), relatively to \( T_\gamma \), for \( \gamma \in (0, \bar{\gamma}) \).

\( \mathbf{H}2 \). There exists \( c_\infty > 0 \) such that \( \sup_{x \in \mathbb{R}^d} \| T_\gamma(x) - \hat{T}_\gamma(x) \| \leq \gamma c_\infty \) for all \( \gamma \in (0, \bar{\gamma}) \).

**Example 2.** The assumption \( \mathbf{H}2 \) holds for the Euler scheme applied to diffusions with scalar covariance matrices, i.e. (1) and (2) with

\[ T_\gamma(x) = x + \gamma b(x) \quad \text{and} \quad \hat{T}_\gamma(x) = x + \gamma \tilde{b}(x), \]

under the condition that \( \sup_{x \in \mathbb{R}^d} \| b(x) - \tilde{b}(x) \| \leq c_\infty \). This setting is exactly the one we introduced to motivate our study. In particular, in the case where \( b = -\nabla U \) for some potential \( U \), \( \tilde{b} \) may correspond to a numerical approximation of this gradient.

Note that compared to \( T_\gamma \), \( \gamma \in (0, \bar{\gamma}) \), we do not assume any smoothness condition on \( \hat{T}_\gamma \). More precisely, we do not assume that \( \hat{T}_\gamma \) satisfies \( \mathbf{H}1 \). Regarding the ergodicity properties of \( R_\gamma \), associated with \( \hat{T}_\gamma \), \( \gamma \in (0, \bar{\gamma}) \), we have the following result.

**Proposition 3.** Assume \( \mathbf{H}1 \) and \( \mathbf{H}2 \). Then, for any \( \gamma \in (0, \bar{\gamma}) \), \( \hat{R}_\gamma \) admits a unique stationary distribution \( \hat{\pi}_\gamma \). In addition, there exist \( c > 0 \) such that for any \( \gamma \in (0, \bar{\gamma}) \), there exist \( \rho_c \in [0, 1) \) and \( C_\gamma \geq 0 \) such that for any \( x \in \mathbb{R}^d \), \( \| \Delta_{\gamma} R_\gamma - \hat{\pi}_\gamma \|_{V_c} \leq C_\gamma \rho^\gamma V_c(x) \), where \( V_c(x) = \exp(c \| x \|^2) \).

\[ 5 \]
Proof. The proof is postponed to Appendix B.

Similarly to Proposition 1 with respect to \( R_\gamma \), Proposition 3 implies that \( \tilde{R}_\gamma \) is \( V_c \)-uniformly geometrically ergodic. However in contrast to Proposition 1, the dependency of the rate of convergence with respect to the step size \( \gamma \) is not explicit anymore since the results and the method employed in [9] or [17] cannot be applied anymore.

Note that Proposition 1 and Proposition 3 imply that \( \tilde{R}_\gamma \) and \( R_\gamma \) converge to \( \pi_\gamma \) and \( \tilde{\pi}_\gamma \) respectively in total variation and Wasserstein metric of any order \( p \in [1, +\infty) \).

Based on the two assumptions above, we can now state one of our main results. Our goal is to quantify the distance between the laws of the iterates of the two chains \( (Y_k)_{k \in \mathbb{N}} \) and \( (\tilde{Y}_k)_{k \in \mathbb{N}} \), in particular starting from the same initial point \( x \in \mathbb{R}^d \) or at equilibrium. Indeed, remark that, in view of Propositions 1 and 3, letting \( k \to +\infty \) in the next statement yields quantitative bounds on \( W_c(\pi_\gamma, \tilde{\pi}_\gamma) \) for any \( \gamma \in (0, \tilde{\gamma}] \).

**Theorem 4.** Assume \( H_1 \) and \( H_2 \) hold and let

\[
(\tilde{c}, \nu') \in \{(1, (0,+\infty), |.|+1), (|.|, |.|+1), (\exp(|.|)-1, \exp(|.|))\}.
\]

Then, there exist some explicit constants \( C, c \geq 0, \rho \in [0, 1) \) and \( \tilde{\gamma}_1 \in (0, \tilde{\gamma}] \) such that for any \( k \in \mathbb{N}, \gamma \in (0, \tilde{\gamma}_1) \) and \( x, \tilde{x} \in \mathbb{R}^d \),

\[
W_c(\delta_x \tilde{R}_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \leq C \rho^k \nu'(|x-\tilde{x}|) + cc_\infty,
\]

where \( c(x, \tilde{x}) = \tilde{c}(|x-\tilde{x}|) \).

**Remark 5.** It is also possible to treat the case of cost functions \( c \) of the form \( c(x,y) = \tilde{c}(|x-y|)(V(x)+V(y)) \) with \( \tilde{c} \) as in Theorem 4 and \( V \) a positive function, simply by using Hölder’s inequality. Indeed, for \( p, q > 1 \) with \( 1/p + 1/q = 1 \), we can bound

\[
W_c(\nu, \mu) \leq 2(W_{c_p}(\nu, \mu))^{1/p} \left((\nu(V^q))^{1/q} + (\mu(V^q))^{1/q}\right),
\]

with \( c_p(x,y) = \tilde{c}^p(|x-y|) \). Bounds on the \( W_{c_p} \) distance can then be established as in Theorem 4, while bounds on expected values of \( V^q \), independent of \( \gamma \), are classically obtained through Lyapunov arguments (see e.g. the proof of Proposition 1 in Appendix A).

The rest of this section is devoted to the proof of Theorem 4. In particular, we define in the following the main object of this paper.

### 2.2 The discrete sticky kernel

We define a Markovian coupling of the two chains \( (Y_k)_{k \in \mathbb{N}} \) and \( (\tilde{Y}_k)_{k \in \mathbb{N}} \) defined in (1) and (2) by using, at each step, the maximal reflection coupling of the two Gaussian proposals, which is optimal for the total variation distance (i.e that maximizes the probability of coalescence). Let \( (U_k)_{k \geq 1} \) be a sequence of i.i.d. uniform random variables on \([0,1] \) independent of \( (Z_k)_{k \geq 1} \) which we recall is a sequence of i.i.d. \( d \)-dimensional standard Gaussian random variables. We define the discrete sticky Markov coupling \( K_\gamma \) of \( R_\gamma \) and \( \tilde{R}_\gamma \) as the Markov kernel associated with the Markov chain on \( \mathbb{R}^{2d} \) given for \( k \in \mathbb{N} \) by

\[
\begin{align*}
X_{k+1} &= T_\gamma(X_k) + (\sigma^2 \gamma)^{1/2}Z_{k+1} \\
\tilde{X}_{k+1} &= X_{k+1}B_{k+1} + (1-B_{k+1})F_{\gamma}(X_k, \tilde{X}_k, Z_{k+1}),
\end{align*}
\]

(7)
where $B_{k+1} = \mathbf{1}_{[0, +\infty)}(p_{\gamma}(X_k, \tilde{X}_k, Z_{k+1}) - U_{k+1})$ and
\[ F_{\gamma}(x, \tilde{x}, z) = \tilde{T}_{\gamma}(\tilde{x}) + (\sigma^2{\gamma})^{1/2}\{\mathbf{1} - 2e(x, \tilde{x})e(x, \tilde{x})^T\}z, \]
\[ E(x, \tilde{x}) = \tilde{T}_{\gamma}^2(x) - T_{\gamma}(x), \quad e(x, \tilde{x}) = \begin{cases} \frac{E(x, \tilde{x})}{\|E(x, \tilde{x})\|} & \text{if } E(x, \tilde{x}) \neq 0 \\ e_0 & \text{otherwise} \end{cases} \]
\[ p_{\gamma}(x, \tilde{x}, z) = 1 \wedge \left[ \varphi_{\sigma^2{\gamma}} \left\{ \|E(x, \tilde{x})\| - (\sigma^2{\gamma})^{1/2}\langle e(x, \tilde{x}), z \rangle \right\} \right]. \]
where $e_0 \in \mathbb{R}^d$ is an arbitrary unit-vector, i.e. $\|e_0\| = 1$, and $\varphi_{\sigma^2{\gamma}}$ is the density of the one-dimensional Gaussian distribution with mean 0 and variance $\sigma^2{\gamma}$. In other words, $K_{\gamma}$ is given for any $\gamma \in (0, \hat{\gamma}]$, $(x, y) \in \mathbb{R}^{2d}$ and $A \in \mathcal{B}(\mathbb{R}^{2d})$ by
\[ K_{\gamma}((x, \tilde{x}), A) = \int_{\mathbb{R}^d} \mathbf{1}_A(T_{\gamma}(x) + (\sigma^2{\gamma})^{1/2}z, T_{\gamma}(x) + (\sigma^2{\gamma})^{1/2}z)p_{\gamma}(x, \tilde{x}, z) \frac{e^{-\|x\|/2}}{(2\pi)^{d/2}} \, dz \]
\[ + \int_{\mathbb{R}^d} \mathbf{1}_A \left( T_{\gamma}(x) + (\sigma^2{\gamma})^{1/2}z, F_{\gamma}(x, \tilde{x}, z) \right) (1 - p_{\gamma}(x, \tilde{x}, z)) \frac{e^{-\|x\|/2}}{(2\pi)^{d/2}} \, dz \]
In words, from the initial conditions $(x, \tilde{x})$, this coupling works as follows: first, a Gaussian variable $Z_{k+1}$ is drawn for the fluctuations of $X_{k+1}$ with probability $p(x, \tilde{x}, Z_{k+1})$ and, otherwise, the random variable $\tilde{Z}_{k+1}$ which determines the fluctuations of $\tilde{X}_{k+1}$ with respect to its average $\tilde{T}_{\gamma}(\tilde{x})$ is given by the orthogonal reflection of $Z_{k+1}$ in the direction $\tilde{T}_{\gamma}(\tilde{x}) - T_{\gamma}(x)$. It is well known that for any $(x, \tilde{x}) \in \mathbb{R}^d$, $K_{\gamma}((x, \tilde{x}), A \times \mathbb{R}^d) = R_{\gamma}(x, A)$ and $K_{\gamma}((x, \tilde{x}), \mathbb{R}^d \times A) = R_{\gamma}(x, A)$, see e.g. [4, Section 3.3], [13, Section 4.1], [17] or [9]. We coin the term sticky for the Markov coupling $K_{\gamma}$ after [18, 35] and since it aims at each stage $k + 1 \in \mathbb{N}$ to merge $X_k$ and $\tilde{X}_k$ with maximal probability under the constraint that these two processes must be Markov chains associated with $R_{\gamma}$ and $\tilde{R}_{\gamma}$ respectively, with prescribed initial conditions. However, unlike other works dealing with couplings, which also use this term sticky, $K_{\gamma}((x, x), \Delta_{\mathbb{R}^d}) < 1$ for any $x \in \mathbb{R}^d$ if $c_{\infty} > 0$ and therefore $\mathbb{P}(X_{k+1} \neq \tilde{X}_{k+1}, X_k = \tilde{X}_k||X_k, \tilde{X}_k) = \mathbf{1}_{\Delta_{\mathbb{R}^d}}(X_k, \tilde{X}_k)K_{\gamma}((X_k, \tilde{X}_k), \Delta_{\mathbb{R}^d}) > 0$.

The starting point of our analysis is the next result, which will enable to compare the coupling difference process $\|X_{k+1} - \tilde{X}_{k+1}\|$ with a Markov chain on $[0, +\infty)$. Define $(G_k)_{k \geq 1}$ for any $k \geq 1$ by
\[ G_k = (e(X_{k-1}, \tilde{X}_{k-1}), Z_k), \]
where $e$ is given by (8). For any $a \geq 0, g \in \mathbb{R}, u \in [0, 1]$ and $\gamma \in (0, \hat{\gamma}]$ define
\[ \mathcal{H}_{\gamma}(a, g, u) = \mathbf{1}_{[0, +\infty)}(u - p_{\sigma^2{\gamma}}(a, g))(a - 2(\sigma^2{\gamma})^{1/2}g), \]
where
\[ p_{\sigma^2{\gamma}}(a, g) = 1 \wedge \frac{\varphi_{\sigma^2{\gamma}}(a - (\sigma^2{\gamma})^{1/2}g)}{\varphi_{\sigma^2{\gamma}}((\sigma^2{\gamma})^{1/2}g)}. \]
Note that $\mathcal{H}_{\gamma}$ is non-negative function as proven in Lemma 24.
Proposition 6. Assume $H1$-(i) and $H2$ hold. Then for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$, almost surely, we have
\[
\|X_{k+1} - \tilde{X}_{k+1}\| \leq \mathcal{G}_\gamma(\|X_k - \tilde{X}_k\|, G_{k+1}, U_{k+1}) ,
\]
where $(X_k, \tilde{X}_k)_{k \in \mathbb{N}}$ are defined by (7), and for any $w \in [0, +\infty)$, $g \in \mathbb{R}$ and $u \in [0, 1]$, we have
\[
\mathcal{G}_\gamma(w, g, u) = \mathcal{K}_\gamma(\tau_\gamma(w) + \gamma c_\infty, g, u) .
\]
In addition, for any $g \in \mathbb{R}^d$ and $u \in [0, 1]$, $w \mapsto \mathcal{G}_\gamma(w, g, u)$ is non-decreasing.

Proof. The proof is postponed to Section 5.1.

Consider now the stochastic process $(W_k)_{k \in \mathbb{N}}$ starting from $\|X_0 - \tilde{X}_0\|$ and defined by induction on $k$ as follows,
\[
W_{k+1} = \mathcal{G}_\gamma(W_k, G_{k+1}, U_{k+1}) = \begin{cases} 
\tau_\gamma(W_k) + \gamma c_\infty - 2\sigma\sqrt{\gamma}G_k & \text{if } U_{k+1} \geq \bar{\mathcal{G}}_{\gamma}^2(\tau_\gamma(W_k) + \gamma c_\infty, G_k) \\
0 & \text{otherwise.}
\end{cases}
\]
(13)

By definition (9) and (8), an easy induction implies that $(G_k)_{k \geq 1}$ and $(U_k)_{k \geq 1}$ are independent, $(G_k)_{k \geq 1}$ are i.i.d. standard Gaussian random variables and $(U_k)_{k \geq 1}$ are i.i.d. uniform random variables on $[0, 1]$. Therefore, $(W_k)_{k \in \mathbb{N}}$ is a Markov chain with Markov kernel $Q_\gamma$ defined for $w \in [0, +\infty)$ and $A \in \mathcal{B}([0, +\infty))$ by
\[
Q_\gamma(w, A) = \delta_0(A) \int_{\mathbb{R}} \bar{\mathcal{G}}_{\gamma}^2(\tau_\gamma(w) + \gamma c_\infty, g)\varphi(g)dg 
+ \int_{\mathbb{R}} \mathbb{I}_A(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\gamma^{1/2}g)\{1 - \bar{\mathcal{G}}_{\gamma}^2(\tau_\gamma(w) + \gamma c_\infty, g)\}\varphi(g)dg ,
\]
where $\varphi$ is the density of the standard Gaussian distribution on $\mathbb{R}$.

By Proposition 6, we have almost surely for any $k \in \mathbb{N}$,
\[
\|X_k - \tilde{X}_k\| \leq W_k .
\]
(15)

Another consequence of Proposition 6 is that $Q_\gamma$ is stochastically monotonous (see e.g. [24] or [31]), more precisely if $(W_k)_{k \in \mathbb{N}}$ and $(\tilde{W}_k)_{k \in \mathbb{N}}$ are two chains given by (13) with the same variables $(G_k, U_k)_{k \in \mathbb{N}}$ with $W_0 = \tilde{W}_0$, then almost surely $W_k \leq \tilde{W}_k$ for all $k \in \mathbb{N}$. This nice property will be used several times in the analysis of this chain.

The main consequence of (15) is the following result.

Corollary 7. Assume $H1$-(i) and $H2$ hold. Let $c : \mathbb{R}^d \to [0, +\infty)$ of the form $c(x, y) = \bar{c}(\|x - y\|)$ for some non-decreasing function $\bar{c} : [0, +\infty) \to [0, +\infty)$, $\bar{c}(0) = 0$. For any $x, \tilde{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$,
\[
\mathcal{W}_c(\delta_\gamma R^k_{\gamma}, \delta_\gamma \tilde{R}^k_{\gamma}) \leq \int_{\mathbb{R}^d} c(y, \tilde{y})K^k_{\gamma}((x, \tilde{x}), d(y, \tilde{y})) \leq \int_0^{+\infty} \bar{c}(\tilde{w})Q^k_{\gamma}(\|x - \tilde{x}\|, d\tilde{w}) .
\]
Proof. Let $k \in \mathbb{N}$. By (15) and since $\bar{c}$ is non-decreasing, we get almost surely $\bar{c}(\|X_k - \tilde{X}_k\|) \leq \bar{c}(W_k)$. Taking the expectation concludes the proof.

From Corollary 7, the question to get bounds on $\mathcal{W}_c(\delta_\gamma R^k_{\gamma}, \delta_\gamma \tilde{R}^k_{\gamma})$ boils down to the study of the Markov kernel $Q_\gamma$ on $[0, +\infty)$, which is the main part of our work.
2.3 Analysis of the auxiliary Markov chain

We start with a Lyapunov/drift result.

**Proposition 8.** Assume $H1$-(ii). Then for any $w \geq 0$,

$$Q_\gamma \varphi^k(w) \leq (1 - \gamma m) \varphi^k(w) + (1 + \gamma L) \varphi^k(w) + \gamma c_\infty ,$$

where $Q_\gamma$ is defined by (14) and for any $w \in \mathbb{R}_+$, $\varphi^k(w) = w$.

**Proof.** The proof is postponed to Section 5.2. \qed

Proposition 8 implies in particular that for any $w \in \mathbb{R}_+$,

$$Q_\gamma \varphi^k(w) \leq (1 - \gamma m) \varphi^k(w) + \gamma [(L + m)R_1 + c_\infty] .$$

Then, a straightforward induction shows that for any $k \in \mathbb{N}$,

$$Q_\gamma \varphi^k(w) \leq (1 - \gamma m)^k \varphi^0(w) + [(L + m)R_1 + c_\infty]/m ,$$

and therefore by Corollary 7 taking $c(t) = t$,

$$\mathcal{W}_1(\delta_\gamma R^{k}_\gamma, \delta_\gamma \tilde{R}^{k}_\gamma) \leq (1 - \gamma m)^k \|x - \tilde{x}\| + [(L + m)R_1 + c_\infty]/m . \tag{16}$$

However, this result is not sharp as $k \to +\infty$. Indeed, in the case $c_\infty = 0, R_\gamma = \tilde{R}_\gamma$ and by Proposition 1, it holds that $\mathcal{W}_1(\delta_\gamma R^{k}_\gamma, \delta_\gamma \tilde{R}^{k}_\gamma) \to 0$ as $k \to +\infty$, while the right-hand side of (16) converges to $(L + m)R_1/m \neq 0$. In particular, that is why adapting existing results, such as the one established in [32], is not an option here. We need to refine our results in order to fill this gap. To this end, we need to analyze more precisely the long-time behavior of $Q_\gamma$. A first step to show that it is ergodic.

**Proposition 9.** Assume $H1$-(ii). For any $\gamma \in (0, \bar{\gamma}]$, $Q_\gamma$ admits a unique invariant probability measure $\mu_\gamma$ and is geometrically ergodic. In addition, $\mu_\gamma(\{0\}) > 0$ and $\mu_\gamma$ is absolutely continuous with respect to the measure $\delta_0 + \text{Leb}$ on $([0, +\infty), \mathcal{B}([0, +\infty)))$. Finally, in the case $c_\infty \neq 0$, $\mu_\gamma$ and $\delta_0 + \text{Leb}$ are equivalent.

**Proof.** The proof is postponed to Section 5.3. \qed

**Corollary 10.** Assume $H1$ and $H2$ hold. Let $c : \mathbb{R}^{2d} \to [0, +\infty)$ of the form $c(x, y) = \bar{c}(\|x - y\|)$ for some non-decreasing function $\bar{c} : [0, +\infty) \to [0, +\infty)$, $\bar{c}(0) = 0$. For any $x, \tilde{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$,

$$\mathcal{W}_c(\delta_\gamma R^{k}_\gamma, \delta_\gamma \tilde{R}^{k}_\gamma) \leq \int_0^{+\infty} \bar{c}(\tilde{w}) \{Q_\gamma^{k}(\|x - \tilde{x}\|, \cdot) - \mu_\gamma\}(d\tilde{w}) + \mu_\gamma(\bar{c}) . \tag{17}$$

where $\mu_\gamma$ is the stationary distribution of $Q_\gamma$ given by (14). In particular, if $x = \tilde{x}$, $\mathcal{W}_c(\delta_\gamma R^{k}_\gamma, \delta_\gamma \tilde{R}^{k}_\gamma) \leq \mu_\gamma(\bar{c})$.

**Proof.** The proof of (17) is a consequence of Proposition 9 and Corollary 7. The last statement follows from the fact that $Q_\gamma$ is stochastically monotonous. Indeed, by Proposition 6, for any $w, \tilde{w} \in [0, +\infty), w \leq \tilde{w},$ and $a \in [0, +\infty), Q_\gamma(w, [0, a]) \geq Q_\gamma(\tilde{w}, [0, a])$. Therefore, for any $a \in [0, +\infty)$, $w \mapsto Q_\gamma(w, [0, a])$ is non-increasing on $[0, +\infty)$ and for any non-increasing bounded function $f$, $Q_\gamma f(w) \geq Q_\gamma f(\tilde{w})$ for any $w, \tilde{w} \in [0, +\infty), w \leq \tilde{w}$. As a result, a straightforward induction shows that for any $k \in \mathbb{N}, w, \tilde{w} \in [0, +\infty), w \leq \tilde{w},$ and $a \in [0, +\infty)$, $Q_\gamma^k(w, [0, a]) \geq Q_\gamma^k(\tilde{w}, [0, a])$. Then, we obtain $\int_0^{+\infty} \mu_\gamma(\text{Leb}Q_\gamma^k(w, [0, a]) = \mu_\gamma([0, a])$. Since $\bar{c}$ is non-decreasing on $[0, +\infty)$, we get $Q_\gamma^k \bar{c}(0) \leq \mu_\gamma(\bar{c})$, which combined with (17) completes the proof. \qed
Corollary 10 then naturally brings us to derive moment bounds for the stationary distribution \( \mu_\gamma \), \( \gamma \in (0, \bar{\gamma}] \) and quantitative convergence bounds for \( Q_\gamma \) to \( \mu_\gamma \). Our next results address these two problems.

**Theorem 11.** Assume \( H1-(ii) \). For any \( \bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}] \) and \( \gamma \in (0, \bar{\gamma}] \),

\[
\int_{(0, +\infty)} w \mu_\gamma(dw) \leq c_\infty c_1 \quad \mu_\gamma((0, +\infty)) \leq c_\infty c_2 ,
\]

(18)

where \( \mu_\gamma \) is the stationary distribution of \( Q_\gamma \) given by (14), and, considering \( \zeta \) given below in (46),

\[
c_1 = \eta_1 R_1(1 + L/m) + 1/m ,
\]

\[
c_2 = e^{(\bar{\delta} + \gamma)L}(1 + \gamma L)/\bar{\delta}^{1/2}[\bar{\delta} + \gamma]^{1/2}/(\sqrt{2\pi}\sigma) + 2\zeta[\bar{\delta} + \gamma]^{1/2}e^{3(\bar{\delta} + \gamma)L}/\sigma^3 ,
\]

\[
\eta_1 = [\bar{\delta} + \gamma]^{1/2}/2\zeta e^{3(\bar{\delta} + \gamma)L}/\sigma^3 + e^{(\bar{\delta} + \gamma)L}/2\sqrt{2\pi}\sigma \). 
\]

Proof. The proof is postponed to Section 5.4.

**Theorem 12.** Assume \( H1-(ii) \). For any \( a > 0 \) and \( \gamma \in (0, \bar{\gamma}] \),

\[
\int_{\mathbb{R}_+} [\exp(aw) - 1]d\mu_\gamma(w) \leq c_\infty c_3 ,
\]

(19)

where \( \bar{\gamma}_1 \) and \( c_3 \) are explicitly given in the proof and \( \mu_\gamma \) is the stationary distribution of \( Q_\gamma \) given by (14).

Proof. The proof is postponed to Appendix C.

We now specify the convergence of \( Q_\gamma \) to \( \mu_\gamma \) for any \( \gamma \in (0, \bar{\gamma}] \).

**Theorem 13.** Assume \( H1 \) and \( H2 \) hold. There exist explicit constants \( \rho \in [0, 1), C \geq 0 \) and \( \bar{\gamma}_1 \in (0, \bar{\gamma}] \) such that for any \( \gamma \in (0, \bar{\gamma}_1] \), \( w \geq 0 \),

\[
||\delta \omega Q_\gamma^k - \mu_\gamma||_{\mathcal{V}} \leq C \rho^\gamma \mathcal{V}(w) ,
\]

where \( \mathcal{V}(w) = 1 + w \) or \( \mathcal{V}(w) = \exp(aw) \), for \( a > 0 \).

Proof. The proof is postponed to Section 5.5.

**Discussion on the bounds provided by Theorem 11.** In this paragraph, we discuss how the constants \( c_1, c_2 \) given in Theorem 11 behaves with respect to the parameters \( R_1, L, m \) in the limit \( c_\infty \to 0 \) and \( \bar{\gamma} \to 0 \). For ease of presentation, we also only consider the case \( \sigma = 1 \).

(1) First consider the case \( R_1 = 0 \). As \( m \to 0 \), \( c_1, c_2 \) are of order \( m^{-1} \) and \( 1/[m^{1/2}] + \bar{\delta}^{1/2} \) respectively for \( \bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}] \). Since \( L \) can be taken arbitrarily small (as \( R_1 = 0 \)), choosing \( \bar{\delta} = m^{-1} \), we obtain that \( c_2 \) is of order \( m^{-1/2} \). Note that the dependency of \( c_1, c_2 \) with respect to \( m \) is sharp; see Example 14 below.
(2) We now consider the case $R_1 \geq 1$, $L = 0$. Note that in this case $\tilde{\delta}$ can be chosen arbitrarily in $(0, 1)$. Then, for some universal constants $C_1, C_2, C_3, \eta_1 \geq C_1 \tilde{\delta}^{1/2}/\Phi (C_2 R_1 \tilde{\delta}^{1/2} + C_3 c_\infty \tilde{\delta}^{1/2})$. Therefore, taking $\tilde{\delta} = m^{-1} \vee R_1^2$, we get that for some universal constants $D_1, D_2, E \geq 0$, $c_1 \leq D_1 [(R_1 \vee m^{-1/2}) + m^{-1}], c_2 \leq E m^{-1/2} \vee R_1$. Note that the bound of $c_2$ with respect to $R_1$ and $m$ is consistent with the results obtained in [18] (see [18, Lemma 1]) for the stationary distributions of continuous sticky processes. Note that it is shown in [18, Example 2] that this bound is sharp with respect to $R_1$ and $m$.

(3) In the case $R_1 \wedge L \geq 1$, taking $\tilde{\delta} = L^{-1}$ since we are in the regime $c_\infty \to 0$, we get that up to logarithmic term and using $\tilde{\gamma} \leq L^{-1}$, $c_1, c_2$ are smaller than $C \exp[\epsilon T (R_1 L^{1/2} + c_\infty)^2]$ for some universal constant $C \geq 0$. The estimate for $c_2$ is also consistent with [18, Lemma 1] which holds for stationary distributions of continuous sticky processes.

Example 14. Consider the particular example of two auto-regressive processes for which

$$T_\gamma(y) = (1 - \varphi) y + \gamma a$$

and

$$T_{\tilde{\gamma}}(y) = (1 - \varphi) y + \gamma a$$

for $\gamma \in (0, \varphi^{-1})$ and $a, \varphi > 0$. Then, on the one hand, $H 1$ and $H 2$ are satisfied with $R_1 = 0$, $m = \varphi$ and $c_\infty = \gamma a$ which lead to $c_1 c_\infty \sim a$ and $c_2 c_\infty \sim \varphi a / \sqrt{2}$, as $\varphi \to 0$, for some universal constant $C \geq 0$. On the other hand, an easy computation (see e.g. [12]) shows that the stationary distributions $\pi_\gamma$ and $\tilde{\pi}_\gamma$, provided by Proposition 1 and Proposition 3 are $\mathcal{N}(0, \varphi^{-1} (2 - \gamma \varphi)^{-1})$ and $\mathcal{N}(a, \varphi^{-1} (2 - \gamma \varphi)^{-1})$ respectively. Therefore, we get $\mathcal{U}(\pi_\gamma, \tilde{\pi}_\gamma) = a$ and $\|\pi_\gamma - \tilde{\pi}_\gamma\|_{TV} \sim \varphi a / \sqrt{2}$ as $\varphi \to 0$.

3 Continuous-time limit

In the case where $T_\gamma$ and $\tilde{T}_\gamma$ are specified by (6), then under appropriate conditions on $b$ and $\tilde{b}$, it can be shown, see e.g. [9, Proposition 25], that for any $T \geq 0$ and $x \in \mathbb{R}^d$,

$$\lim_{m \to +\infty} \{ \| \delta_x R_T^m / m - \delta_x P_T \|_V + \| \delta_x \tilde{R}_T^m / m - \delta_x \tilde{P}_T \|_V \} = 0 ,$$

for some measurable function $V : \mathbb{R}^d \to [1, +\infty)$ and where $(P_t)_{t \geq 0}$ and $(\tilde{P}_t)_{t \geq 0}$ are the Markov semigroup corresponding to the diffusions $dX_t = b(X_t) + \sigma dB_t$ and $d\tilde{X}_t = \tilde{b}(\tilde{X}_t) + \sigma d\tilde{B}_t$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard $d$-dimensional Brownian motion. Then, this naturally implies convergence in total variation and also Wasserstein distance of order $p$ if $\inf_{x \in \mathbb{R}^d} \{ V(x) / \|x\|_p \} > 0$. As a consequence, results of Section 2 immediately transfer to the continuous-time processes. More precisely, let $c : \mathbb{R}^{2d} \to [0, +\infty)$ of the form $c(x, y) = \mathfrak{c}(\|x - y\|)$ for some non-decreasing function $\mathfrak{c} : [0, +\infty) \to [0, +\infty)$, $\mathfrak{c}(0) = 0$. If (20) holds and $\sup_{x, y \in \mathbb{R}^d} \{ c(x, y) / \{ V(x) + V(y) \} \} < +\infty$, we get if $\mathcal{U}_c$ satisfies the triangle inequality that for any $x, \tilde{x} \in \mathbb{R}^d, T > 0$, $\mathcal{U}_c(\delta_x P_T, \delta_x \tilde{P}_T) \leq \lim \inf_{m \to +\infty} \mathcal{U}_c(\delta_x R_T^m / m, \delta_x \tilde{R}_T^m / m)$. Then, results of Section 2 can be applied implying if $H 1$ and $H 2$ holds, that for any $x, \tilde{x} \in \mathbb{R}^d$ there exist $C_1, C_2 > 0$ such that for any $T > 0$, $\mathcal{U}_c(\delta_x P_T, \delta_x \tilde{P}_T) \leq C_1 \rho_T + C_2 c_\infty$. We therefore generalize the result provided in [18] which is specific to the total variation distance. We do not give a specific statement for this result which is mainly technical and is not the main subject of this paper. Instead, the goal of this section is to study the continuous-time limit of the coupling (7) (and not only of its marginals) toward some continuous-time sticky diffusion.

More precisely, let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of step sizes such that $\lim_{n \to +\infty} \gamma_n = 0, \gamma_n < \tilde{\gamma}$ and $w_0 \geq 0$. Then, consider the sequence of Markov chains $\{(W_k^{(n)})_{k \in \mathbb{N}} : n \in \mathbb{N}\}$ for any $n \in \mathbb{N}$, $(W_k^{(n)})_{k \in \mathbb{N}}$ is the Markov chain defined by (13) with $W_0^{(n)} = w_0, \gamma = \gamma_n$ and therefore
associated with the Markov kernel \(Q_{\gamma n}\). Let \(\{(W_t^{(n)})_{t \in (0, +\infty)} : n \in \mathbb{N}\}\) be the continuous linear interpolation of \(\{(W_k^{(n)})_{k \in \mathbb{N}} : n \in \mathbb{N}\}\), i.e. the sequence of continuous processes defined for any \(n \in \mathbb{N}, t \in (0, +\infty)\) by

\[
W_t^{(n)} = W_{\lfloor t/\gamma_n \rfloor} + \{W_{t/\gamma_n} - W_{\lfloor t/\gamma_n \rfloor}\}\{t/\gamma_n - \lfloor t/\gamma_n \rfloor\}.
\]

(21)

Note that for any \(k \in \mathbb{N}\) and \(h \in [0, \gamma_n]\), \(W_{k\gamma_n + h}^{(n)} = W_k + (h/\gamma_n)\{W_{k+1}^{(n)} - W_k^{(n)}\}\). We denote by \(W = C([0, +\infty), \mathbb{R})\) endowed with the uniform topology on compact sets, \(W\) its corresponding \(\sigma\)-field and \((W_t)_{t \geq 0}\) the canonical process defined for any \(t \in (0, +\infty)\) and \(\omega \in W\) by \(W_t(\omega) = \omega_t\). Denote by \((W_t)_{t \geq 0}\) the filtration associated with \((W_t)_{t \geq 0}\). Note that \(\{(W_t^{(n)})_{t \in (0, +\infty)} : n \in \mathbb{N}\}\) is a sequence of \(W\)-valued random variables. The main result of this section concerns the convergence in distribution of this sequence.

We consider the following assumption on the function \(\tau_\gamma\).

**A1.** There exists a function \(\kappa : [0, +\infty) \rightarrow \mathbb{R}\) such that for any \(\gamma \in (0, \bar{\gamma}]\), \(\tau_\gamma(w) = w + \gamma \kappa(w)\) and \(\kappa(0) = 0\). In addition, \(\kappa\) is \(L_\kappa\)-Lipschitz: for any \(w_1, w_2 \in (0, +\infty)\), \(|\kappa(w_1) - \kappa(w_2)| \leq L_\kappa |w_1 - w_2|\).

This is not a restrictive condition since, under H1, up to a possible modification of \(\tau_\gamma\), it is always possible to ensure A1.

Under A1, we consider a sticky process [34, 35, 18], which solves the stochastic differential equation

\[
dW_t = \{\kappa(W_t) + c_\infty\}dt + 2\sigma(0, +\infty)(W_t)dB_t,
\]

(22)

where \((B_t)_{t \geq 0}\) is a one-dimensional Brownian motion. Note that for any initial distribution \(\mu_0\) on \((\mathbb{R}, B(\mathbb{R}^d))\), (22) admits a unique weak solution by [18, Lemma 17, Theorem 22].

The main result of this section is the following.

**Theorem 15.** Assume A1. Then, the sequence \(\{(W_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}\) defined by (21) converges in distribution to the solution \((W_t)_{t \geq 0}\) of the SDE (22).

The proof of this theorem follows the usual strategy employed to show convergence of a sequence of continuous processes to a Markov process. A first step is to show that under A1, \(\{(W_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}\) is uniformly bounded in \(L^q\) for some \(q \geq 2\), on \([0, T]\) for any \(T > 0\).

**Proposition 16.** Assume A1. Then for any \(T > 0\), there exists \(C_T > 0\) such that

\[
\sup_{n \in \mathbb{N}} \mathbb{E}[\sup_{t \in [0, T]} \{|W_t^{(n)}|\}^q] \leq C_T
\]

where \((W_t^{(n)})_{t \geq 0}\) is defined by (21).

**Proof.** The proof is postponed to Section 5.6.

Then, we are able to obtain the tightness of the sequence of stochastic processes \(\{(W_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}\).

**Proposition 17.** Assume A1. Then, \(\{(W_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}\) is tight in \(W\).

**Proof.** The proof is postponed to Section 5.7.
\begin{align*}
\{(W_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\} \text{ is tight again and since (22) admits a unique weak solution, the proof of Theorem 15 will be completed. To establish this result, we use the characterization of solutions of SDEs through martingale problems. More precisely by [6, Theorem 1.27], the distribution } \mu \text{ on } \mathcal{W} \text{ of } (W_t)_{t \geq 0}, \text{ solution of (22), is the unique solution to the martingale problem associated with } \mu_0, \text{ the drift function } w \mapsto \kappa(w) + c_\infty \text{ and the variance function } 2\sigma 1_{(0, +\infty)}, \text{ i.e. it is the unique probability measure satisfying on the filtered probability space } (\mathcal{W}, \mathcal{W}, (W_t)_{t \geq 0}, \mu): \\
(a) \text{ the distribution of } W_0 \text{ is } \mu_0; \\
(b) \text{ the processes } (M_t)_{t \geq 0}, (N_t)_{t \geq 0} \text{ defined for any } t \geq 0 \text{ by}
\begin{align*}
M_t &= W_t - W_0 - \int_0^t \{c_\infty + \kappa(W_u)\}du, \\
N_t &= M_t^2 - 4\sigma^2 \int_0^t 1_{(0, +\infty)}(W_u)du,
\end{align*}
\text{are } (W_t)_{t \geq 0}-\text{local martingales.}
\end{align*}

In other words, it corresponds in showing that \((M_t)_{t \geq 0}\) is a \((W_t)_{t \geq 0}\)-local martingales and by [30, Theorem 1.8] identifying its quadratic variation \((\langle M_t \rangle_{t \geq 0})\) as the process \(4\sigma^2 \int_0^t 1_{(0, +\infty)}(W_u)du\). Therefore, Theorem 15 is a direct consequence of the following result.

\textbf{Theorem 18.} \textit{Assume A1. Let } \mu_\infty \text{ be a limit point of } (\mu_n)_{n \in \mathbb{N}}. \text{ Then, the two processes } (M_t)_{t \geq 0} \text{ and } (N_t)_{t \geq 0} \text{ defined by (23) are } (W_t)_{t \geq 0}-\text{martingales on } (\mathcal{W}, \mathcal{W}, (W_t)_{t \geq 0}, \mu_\infty).

\begin{proof}
\textit{The proof is postponed to Section 5.8.}
\end{proof}

\section{An application in Bayesian statistics: parameter estimation in an ODE}

\subsection{Setting and verifying the assumptions}

Consider an ordinary differential equation ODE on } \mathbb{R}^n \text{ of the form
\begin{equation}
\dot{x}_\theta(t) = f_\theta(x_\theta(t), t), \quad x_\theta(0) = x_0 \in \mathbb{R}^n,
\end{equation}
\text{where } \{f_\theta : \theta \in \mathbb{R}^d\} \text{ is a family of function from } \mathbb{R}^n \times [0, +\infty) \text{ to } \mathbb{R}^n \text{ parametrized by some parameter } \theta \in \mathbb{R}^d. \text{ In all this section } x_0 \in \mathbb{R}^n \text{ is assumed to be fixed and we consider the following assumption.}

\textbf{AO1.} \textit{For all } \theta \in \mathbb{R}^d \text{ there exists a unique solution of (24) defined for all positive times, which we denote by } (x_\theta(t))_{t \geq 0}. \text{ In addition, the functions } (\theta, x, t) \in \mathbb{R}^d \times \mathbb{R}^n \times [0, +\infty) \mapsto f_\theta(x, t) \text{ and } (\theta, t) \in \mathbb{R}^d \times [0, +\infty) \mapsto x_\theta(t) \text{ are continuously differentiable.}

\textbf{AO1.} \textit{In fact the continuous differentiability of } (\theta, t) \mapsto x_\theta(t) \text{ is a consequence of the one of } (\theta, x, t) \mapsto f_\theta(x, t), \text{ see e.g. [36, Theorem 4.D].}

\textbf{AO1.} \textit{To fix ideas, throughout this section, we will repeatedly discuss the following case of a logistic equation.
Example 19. For $r \in C^1(\mathbb{R}, \mathbb{R}_+)$, set $f_\theta(x) = x(1 - r(\theta)x)$ for any $\theta, x \in \mathbb{R}$, so that (24) reads

$$
\dot{x}_\theta(t) = x_\theta(t) (1 - r(\theta)x_\theta(t)) , \quad x_\theta(0) = x_0 ,
$$

with $x_0 \geq 0$. In this example, AO1 holds and, $r$ and $x_0$ being positive, for all $\theta \in \mathbb{R}$, the solution of (24) is such that $x_\theta(t) \in [0, e^t x_0]$ for all $t \geq 0$. Indeed, $x \mapsto x(1 - r(\theta)x)$ is locally Lipschitz continuous, which yields existence and uniqueness of a maximal solution. Since 0 is always an equilibrium, solutions stay positive, from which $x_\theta(t) \leq e^t x_0$ for all $t \geq 0$. This also implies non-explosion, hence the solution is defined on $[0, +\infty)$.

We consider the problem of estimating $\theta$ based on some observation of a trajectory of the ODE. More precisely, for $T > 0$, $N \in \mathbb{N}$, $(t_1, \ldots, t_N) \in \mathbb{R}^N$, $0 < t_1 < \cdots < t_N = T$, the statistical model corresponding to the observation $y = (y_i)_{i \in \{1, \ldots, N\}} \in (\mathbb{R}^n)^N$ is given by

$$
y_i = x_\theta(t_i) + \varepsilon_i ,
$$

for $\theta \in \mathbb{R}^d$ and where $(\varepsilon_i)_{i \in \{1, \ldots, N\}}$ are independent and identically random variables on $\mathbb{R}^n$ distributed according to some known positive density $\varphi_\varepsilon$ with respect to the Lebesgue measure. Given a prior distribution with positive density $\pi_0$ on $\mathbb{R}^d$, the a posteriori distribution for this model admits a positive density $\pi$ with respect to the Lebesgue measure which is characterized by the potential $U$ given by (up to an additive constant)

$$
- \log \pi(\theta) = U(\theta) = - \ln \pi_0(\theta) - \sum_{i=1}^N \ln \varphi_\varepsilon (y_i - x_\theta(t_i)) .
$$

We consider the following assumption on $\pi_0$ and $\varphi_\varepsilon$ setting $- \log(\pi_0) = U_0$. 

AO2. The functions $\pi_0$ and $\varphi_\varepsilon$ are twice continuously differentiable and there exist $m_U > 0$, $L_U, R_U > 0$ such that $\nabla U_0$ is $L_U$-Lipschitz continuous and for any $\theta, \tilde{\theta} \in \mathbb{R}^d$ with $\|\theta - \tilde{\theta}\| \geq R_U$,

$$
\langle \theta - \tilde{\theta}, \nabla U_0(\theta) - \nabla U_0(\tilde{\theta}) \rangle \geq m_U \|\theta - \tilde{\theta}\|^2 .
$$

In practice, expectations with respect to the posterior distribution can be approximated by ergodic means of the Unadjusted Langevin Algorithm (ULA), namely the Markov chain

$$
X_{k+1} = X_k - \gamma \nabla U(X_k) + \sqrt{2\gamma} Z_{k+1} ,
$$

where $\gamma > 0$ and $(Z_k)_{k \in \mathbb{N}}$ are independent and identically standard Gaussian variables. The long-time convergence of this algorithm and the numerical bias on the invariant measure due to the time discretization are well understood, see e.g. [7, 14, 15, 8, 10] and references therein. However, in the present case, it is not possible to sample this Markov chain, as the exact computation of

$$
\nabla U(\theta) = -\nabla_\theta \ln \pi_0(\theta) + \sum_{i=1}^N \nabla_\theta x_\theta(t_i) \nabla_x \ln \varphi_\varepsilon (y_i - x_\theta(t_i)) ,
$$

is not possible in most cases because of the term involving $x_\theta$ and $\nabla_\theta x_\theta$. Here $\nabla_\theta$ and $\nabla_x$ denote the gradient operator with respect to $\theta$ and $x$ respectively. Therefore, only approximations of these two functions can be used in place of $(x_\theta(t_i), \nabla_\theta x_\theta(t_i))_{i \in [0, N]}$, which leads to an
additional discretization bias. Our results based on the sticky coupling yields a quantitative bound on this error (with respect to the ideal ULA above). Let us detail this statement.

First, remark that $t \mapsto z_\theta(t) = (x_\theta(t), \nabla_\theta x_\theta(t))$ solves

$$
\dot{z}_\theta(t) = F_\theta(z_\theta(t), t) \quad z_\theta(0) = z_0 = (x_0, 0)
$$

on $\mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$ with for any $x \in \mathbb{R}^n$, $A \in \mathcal{M}_{d,n}(\mathbb{R})$, $\theta \in \mathbb{R}^d$, $t \geq 0$,

$$
F_\theta((x, A), t) = (f_\theta(x, t), \nabla_\theta f_\theta(x, t) + A \nabla_x f_\theta(x, t)) .
$$

Provided $f_\theta$, $\nabla_\theta f_\theta$ and $\nabla_x f_\theta$ are computable, in practice this ODE can be approximated by standard numerical schemes. For instance, a basic explicit Euler discretization with time-step $h > 0$ is given by

$$
z_\theta^h(0) = z_0, \quad z_\theta^h((k+1)h) = z_\theta^h(kh) + hF_\theta(z_\theta^h(kh), kh) \quad \forall k \in \mathbb{N}
$$

and

$$
z_\theta^h(t) = z_\theta^h(kh) + (t - kh)F_\theta(z_\theta^h(kh), kh) , \quad t \in [kh, (k+1)h) .
$$

To establish the consistency of this approximation (with some uniformity in $\theta$), we consider the following condition.

**AO 3.** There exist $L_F, L'_F, C_F, \delta > 0$ and a compact set $K \subset \mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$ such that the following holds. For all $t \in [0, T]$ and $\theta \in \mathbb{R}^d$, the ball centered at $z_\theta(t)$ and radius $\delta$ is included in $K$. Moreover, for all $z, \bar{z} \in K$, $t, s \in [0, T]$ and $\theta, \bar{\theta} \in \mathbb{R}^d$, $\|F_\theta(z, t)\| \leq C_F$ and

$$
\|F_\theta(z, t) - F_\theta(\bar{z}, s)\| \leq L_F \|\theta - \bar{\theta}\| + L'_F (\|z - \bar{z}\| + |t - s|) .
$$

**Proposition 20.** Assume AO 1 and AO 3. There exist $\bar{h}, C > 0$ such that for all $h \in (0, \bar{h}]$, $\theta \in \mathbb{R}^d$, and $t \in [0, T]$, $z_\theta^h(t) \in K$ and $\sum_{i=1}^{N} \left\|z_\theta(t_i) - \bar{z}_\theta^h(t_i)\right\| \leq Ch$, where $z_\theta$ solves (28) and $\bar{z}_\theta^h$ is given by (30).

**Proof.** The proof is postponed to Section 5.10. \hfill \Box

**Example 21** (Continuation of Example 19). Let us check for instance that AO 3 is satisfied for Example 19 provided that $r$ is twice continuously differentiable on $[0, +\infty)$ with, for some $L_r, L'_r, L''_r > 0$,

$$
r, r' \text{ and } r'' \text{ uniformly bounded respectively by } L_r, L'_r \text{ and } L''_r .
$$

We may consider for example $r : \theta \mapsto a_1 \theta^2/(\theta^2 + a_2)$ for $a_1, a_2 \in (0, +\infty)$. Recall that $f_\theta(x) = x(1 - r(\theta) x)$ and $x_\theta(t) \in [0, e^{t} x_0]$ for all $\theta \in \mathbb{R}$ and all $t \in [0, T]$, so that for any $t \geq 0$ and $\theta \in \mathbb{R}$,

$$
|\partial_\theta f_\theta(x_\theta(t))| = |r'(\theta) x_\theta^2(t)| \leq L'_r e^{2t} x_0^2 .
$$

Notice that $1/r(\theta)$ is an equilibrium of the equation, so that it cannot be crossed by other solutions. Hence, on the one hand, if $1 \leq r(\theta) x_0$ then $x_\theta$ is non-increasing (in particular $x_\theta(t) \leq x_0$ for all $t \geq 0$) while, on the other hand, if $1 \geq r(\theta) x_0$, then $1 \geq r(\theta) x_\theta(t)$ for all $t \geq 0$. In both cases, we get that for all $t \geq 0$,

$$
|\partial_x f_\theta(x_\theta(t))| = |1 - 2r(\theta) x_\theta(t)| \leq 1 + 2L_r x_0 .
$$
Combining the two previous bounds,
\[
|\partial_\theta x_\theta(t)| \leq \int_0^t \left( L e^{2t} x_\theta^2 + (1 + 2L_r x_\theta) |\partial_\theta x_\theta(s)| \right) ds,
\]
and thus by Grönwall’s inequality, \(|\partial_\theta x_\theta(t)| \leq M_t = L e^{t(3 + 2L_r)} \), for all \( t \in [0, T] \) and \( \theta \in \mathbb{R} \). Then, for any \( \delta > 0 \), \( AO4 \) is satisfied with \( K = [-\delta, e^T x_0 + \delta] \times [-M_T - \delta, M_T + \delta] \).

Since \( K \) is compact and as, by (29), for any \( x \in \mathbb{R}, A \in \mathcal{M}_{1, 1}(\mathbb{R}), \theta \in \mathbb{R}, t > 0, \)
\[
F_\theta((x, A), t) = \left( x(1 - r(\theta)x), -r'(\theta)x^2 + A(1 - 2r(\theta)x) \right),
\]
then (31) easily follows from the condition (32).

We aim to consider general discretization schemes with higher orders than the Euler discretization defined in (31). Therefore, in the following we consider a solver in practice to evaluate \( AO3 \) is satisfied with \( \bar{K} = [-\delta, e^T x_0 + \delta] \times [-M_T - \delta, M_T + \delta] \).

\[
\Phi_\theta(x, A) = \left( x(1 - r(\theta)x), -r'(\theta)x^2 + A(1 - 2r(\theta)x) \right),
\]

then (31) easily follows from the condition (32).

We aim to consider general discretization schemes with higher orders than the Euler discretization defined in (31). Therefore, in the following we consider a solver \( \Phi_\theta : \mathbb{R} \rightarrow (\mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R}))^N \) for \( h > 0 \) satisfying the condition:

AO4. There exist \( \bar{h}, C_{\Phi}, \alpha > 0 \) such that, for any \( \theta \in \mathbb{R}^d \) and \( h \in (0, \bar{h}] \),
\[
\sum_{i=1}^N ||z_\theta(t_i) - \Phi_\theta^h(\theta)|| \leq C_{\Phi}^h, \alpha,
\]
where \( z_\theta \) is a solution of (28) and \( \Phi_\theta^h : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R}) \) is the \( i \)-th component of \( \Phi_\theta^h \).

When AO3 and AO4 are both satisfied, without loss of generality, we assume furthermore that \( \bar{h} \) is sufficiently small so that \( C_{\Phi}^h \alpha \leq \delta \). This implies that \( \Phi_\theta^h(\theta) \in K \) for all \( \theta \in \mathbb{R}^d \), \( i \in \{1, \ldots, N\} \) and \( h \in (0, \bar{h}] \).

Writing \( \Phi_\theta^h(\theta) = (\bar{z}_\theta^h(t_i), \bar{C}_\theta^h(t_i)) \), we can consider for any \( \theta \in \mathbb{R}^d \),
\[
\bar{b}_\theta^h(\theta) = -\nabla_\theta \ln \pi_0(\theta) + \sum_{i=1}^N \bar{C}_\theta^h(t_i) \cdot \nabla_x \ln \varphi_\varepsilon(y_i - \bar{z}_\theta^h(t_i)),
\]
as an approximation of \( \nabla U \) (27) Remark that, now, in contrast to \( b(\theta) = \nabla U(\theta) \), it is possible in practice to evaluate \( \bar{b}_\theta^h(\theta) \) for \( \theta \in \mathbb{R}^d \), provided \( \nabla_x \ln \varphi_\varepsilon \) and \( \nabla_\theta \ln \pi_0 \) can be evaluated. We now assess the error due to the use of \( \bar{b}_\theta^h \) in place of the exact gradient in (26) by verifying that the assumption of Section 2 are satisfied. For \( \gamma, h > 0 \) and \( \theta \in \mathbb{R}^d \), denote
\[
T_\gamma(\theta) = \theta - \gamma \nabla U(\theta), \quad \tilde{T}_\gamma,h(\theta) = \theta - \gamma \tilde{b}_\theta^h(\theta).
\]
When AO2 and AO3 are both satisfied, there exist \( C_s, L_s > 0 \) such that for all \( i \in \{1, \ldots, N\} \), the function \( s_i \), given for any \( x \in \mathbb{R}^n \) and \( A \in \mathcal{M}_{d,n}(\mathbb{R}) \), by
\[
s_i(x, A) = A \nabla_x \ln \varphi_\varepsilon(y_i - x)
\]
is bounded by \( C_s \) and \( L_s \)-Lipschitz continuous on \( K \).

Proposition 22. Under AO1, AO2, AO3 and AO4, for any \( h \in (0, \bar{h}] \), the functions \( T_\gamma 
\] and \( \tilde{T}_{\gamma,h} \) given by (34) satisfy for any \( \bar{\gamma} \in (0, m/L^2) \), \( H1 \) and \( H2 \), with
\[
c_\infty = C_{\Phi} L_s h^\alpha, \quad R_1 = \frac{4NC_s}{mU} \vee R_U, \quad m = \frac{mU}{4}, \quad L = L_U + L_s L_{\varepsilon U} \sum_{i=1}^N t_i e^{L_{\varepsilon U} t_i}.
\]
Proof. The proof is postponed to Section 5.10.

Under the conditions of Proposition 22 and using the results of Section 2, we get that the Markov chains \((X_k)_{k \in \mathbb{N}}\) and \((\tilde{X}_k)_{k \in \mathbb{N}}\) associated to \(T_\gamma\) and \(\tilde{T}_{\gamma,h}\) given by (34) have unique invariant measure \(\pi_\gamma\) and \(\tilde{\pi}_{\gamma,h}\), and that there exist \(\tilde{\gamma}_i, \tilde{h}, C > 0\) such that for all \(\gamma \in (0, \gamma]\) and \(h \in (0, \bar{h})\),

\[
\|\pi_\gamma - \tilde{\pi}_{\gamma,h}\|_{TV} \leq C h . \quad (36)
\]

Example 23 (Continuation of Example 19). As a conclusion, consider the logistic case of Example 19 with the Euler scheme (30), assuming that \(r \in C^2(\mathbb{R}, [0, +\infty))\) satisfies (32). Then, by Example 19 and Example 21, \(AO1\), \(AO3\) and \(AO4\) hold. Assuming moreover that \(\pi_0\) and \(\varphi_*\) are Gaussian, then \(AO2\) also holds and we obtain (36).

4.2 Numerical results

We illustrate our findings on two particular ODEs. First, we consider the ODE associated with the Van der Pol oscillator corresponding to the second order ODE:

\[
\ddot{x}(t) - \theta(1 - x^2(t)) \dot{x}(t) + x(t) = 0 , \quad (37)
\]

where \(\theta \in \mathbb{R}\) is the parameter to infer. It corresponds to (24) with \(f_\theta(x_1, x_2) = (x_2, \theta(1 - x_1^2)x_2 - x_1)\). We generate synthetic data solving (37) using the 4th Runge-Kutta method for \(T = 10\) and \(\theta = 1\). We then select \((x_\theta(t_i))_{i=1}^{25}\) uniformly chosen in \([0, T]\). The observations \(y = (y_i)_{i=1}^{25}\) are obtained from \((x_\theta(t_i))_{i=1}^{25}\) adding i.i.d. zero-mean Gaussian noise with variance 0.5. We consider the corresponding statistical model (25) where \((\varepsilon_i)_{i=1}^{25}\) are i.i.d. zero-mean Gaussian random variables with variance 0.5. We consider as prior \(\pi_0\), the zero-mean Gaussian distribution with variance 0.5. We then use ULA with \(\gamma = 10^{-2}\), for which the gradient is estimated using the Euler method with the time steps \(h \in \{0.05, 0.01, 0.001\}\). Figure 1a represents the histograms corresponding to the different Markov chains after \(10^5\) iterations with a burn-in of \(10^4\) steps. Gaussian kernel density approximation of these histograms are estimated and used as proxy for the density of the invariant distributions \(\pi_\gamma, h\) of the Markov chain \((\tilde{X}_k)_{k \in \mathbb{N}}\) associated to \(\tilde{T}_{\gamma,h}\) given by (34). To obtain a proxy for the density of \(\pi_*\), the stationary distribution of \((X_k)_{k \in \mathbb{N}}\) associated to \(T_\gamma\), we use the same procedure but using the Euler method with \(h = 0.0001\). We then estimate the total variation between \(\pi_*\) and \(\pi_{\gamma,h}\) for \(h \in \{0.005, 0.004, 0.003, 0.0025, 0.0015, 0.001, 0.00075, 0.0005\}\) using numerical integration. The corresponding results over 10 replications are reported in Figure 1b. We can observe that the total variation distance linearly decreases with \(h\) which supports our findings.

For our second experiment, we consider the Lotka-Volterra model describing the evolution of the population of two interacting biological species denoted by \(t \mapsto x_\theta(t) = (u_\theta(t), v_\theta(t))\). The dynamics of these two populations are assumed to be governed by the system of equations given by:

\[
\begin{align*}
\dot{u}_\theta(t) &= (\alpha - \beta v_\theta(t))u_\theta(t) , \\
\dot{v}_\theta(t) &= (-\gamma + \delta u_\theta(t))v_\theta(t) ,
\end{align*}
\]

where \(\theta = (\alpha, \beta, \gamma, \delta)\) is the parameter to infer. We follow the same methodology presented in [25, Chapter 16]. For this experiment, we consider an other statistical model as previously and generate synthetic data \(y = (y_i)_{i=1}^{50}\) accordingly and associated with observation times \((t_i)_{i=1}^{50}\) uniformly spaced in \([0, T]\) for \(T = 10\) and the true parameter \(\theta_0 = (0.6, 0.025, 0.8, 0.025)\).

More precisely, for any \(i \in \{1, \ldots, 50\}\), \(y_i = (u_i^y, v_i^y)\) with \(u_i^y = u_\theta(t_i)e^{\epsilon u,i}, v_i^y = v_\theta(t_i)e^{\epsilon v,i}\) and
\((\varepsilon_{u,j}, \varepsilon_{v,j})_{j=1}^{50}\) are i.i.d. one dimensional zero-mean Gaussian random variables with covariance matrix \(I_2\). The prior \(\pi_0\) is set to be the Gaussian distribution on \(\mathbb{R}^4\) with means \((1, 0.05, 1, 0.05)\) and standard deviations \((0.5, 0.05, 0.5, 0.05)\). The posterior distribution is then given by \(\pi(\theta|y) \propto \exp(-U(\theta))\), where

\[
U(\theta) = -\log \pi_0(\theta) + \sum_{i=1}^{50} \left( \frac{\log u_i^y - \log u_0(t_i))^2 + (\log v_i^y - \log v_0(t_i))^2}{2\varsigma^2} \right).
\]

We then use ULA with \(\gamma = 5 \times 10^{-5}\), for which the gradient is estimated using the Euler method. We focus here on the second component of the chain. The results for the other components are similar. Figure 2a represents the histograms for the second component corresponding to the different Markov chains after \(10^7\) iterations with a burn-in of \(10^3\) steps. Gaussian kernel density approximation of these histograms are estimated and used as proxy for the marginal density of the invariant distributions \(\pi_{\gamma,h}\) of the Markov chain \((\tilde{X}_k)_{k \in \mathbb{N}}\) associated to \(\tilde{T}_{\gamma,h}\) given by (34). To obtain a proxy for the marginal density of \(\pi_{\gamma}\), the stationary distribution of \((X_k)_{k \in \mathbb{N}}\) associated to \(T_{\gamma}\), we use the same procedure but using the Euler method with \(h = 0.0001\). We then estimate the total variation between \(\pi_{\gamma}\) and \(\pi_{\gamma,h}\) for \(h \in \{k \times 10^{-2} \mid k \in \{1, \ldots, 10\}\}\) using numerical integration. The corresponding results over 10 replications are reported in Figure 2b. We can observe that the total variation distance still linearly decreases with \(h\).

5 Postponed proofs

5.1 Proof of Proposition 6

The proof is based on this technical lemma.

\textbf{Lemma 24.} For any \(g \in \mathbb{R}\), \(u \in [0, 1]\), \(\gamma \in (0, \bar{\gamma}]\) and \(c \in [0, +\infty)\),

\[
\mathcal{H}_\gamma(c, g, u) \geq 0,
\]

in addition, for any \(a, b \in [0, +\infty)\), \(a \leq b\),

\[
\mathcal{H}_\gamma(a, g, u) \leq \mathcal{H}_\gamma(b, g, u).
\]

(39)
(b) Numerical estimation of $\|\pi_\gamma - \pi_{\gamma,h}\|_{TV}$

Figure 2: Numerical illustrations for the Lotka-Volterra model (38)

Proof. Let $u \in [0, 1]$, $g \in \mathbb{R}$ and $a, b \in [0, +\infty)$, $a \leq b$. We first prove that for any $c \in \mathbb{R}_+$ such that $c - 2(\sigma^2\gamma)^{1/2}g < 0$,

$$\mathcal{H}_\gamma(c, g, u) = 0,$$  \hspace{1cm} (40)

which implies that for any $c \in \mathbb{R}_+$, $\mathcal{H}_\gamma(c, g, u) \geq 0$.

If $c - 2(\sigma^2\gamma)^{1/2}g < 0$ since $c \geq 0$, we have $0 \leq c < 2(\sigma^2\gamma)^{1/2}g$. This implies that $-(\sigma^2\gamma)^{1/2}g \leq c - (\sigma^2\gamma)^{1/2}g < (\sigma^2\gamma)^{1/2}g$ which gives $\varphi_{\sigma^2\gamma}(c - (\sigma^2\gamma)^{1/2}g) \geq \varphi_{\sigma^2\gamma}((\sigma^2\gamma)^{1/2}g)$. Finally, $\bar{p}_\gamma(c, g) = 1$ and therefore (40) holds.

We now show (39). It is straightforward by (40) if $0 > a - 2(\sigma^2\gamma)^{1/2}g$. If $0 \leq a - 2(\sigma^2\gamma)^{1/2}g$, By using $t \mapsto \varphi_{\sigma^2\gamma}(t)$ is decreasing on $[0, +\infty)$, we obtain

$$\bar{p}_{\sigma^2\gamma}(a, g) \leq \bar{p}_{\sigma^2\gamma}(b, g).$$  \hspace{1cm} (41)

Then, (39) follows from (10) and (41).

\[ \square \]

Proof of Proposition 6. Let $k \in \mathbb{N}$. By H1, H2 and the triangle inequality, for any $x, \tilde{x} \in \mathbb{R}^d$,

$$\|E(x, \tilde{x})\| \leq \tau_\gamma(\|x - \tilde{x}\|) + \gamma c_\infty.$$  \hspace{1cm} (42)

By using (7), and (10) we have,

$$\|X_{k+1} - \tilde{X}_{k+1}\| = (1 - B_{k+1})\| - E(X_k, \tilde{X}_k) + 2(\sigma^2\gamma)^{1/2}e(X_k, \tilde{X}_k)e(X_k, \tilde{X}_k)^T Z_{k+1}\|
\leq (1 - B_{k+1})\| - E(X_k, \tilde{X}_k)\| - 2(\sigma^2\gamma)^{1/2}G_{k+1}e(X_k, \tilde{X}_k)\|
\leq (1 - B_{k+1})\| - E(X_k, \tilde{X}_k)\| - 2(\sigma^2\gamma)^{1/2}G_{k+1}\| = \mathcal{H}_\gamma(\|E(X_k, \tilde{X}_k)\|, G_{k+1}, U_{k+1}).$$

This gives (12) when combined with (42). Finally, the last statement follows from Lemma 24 and H1 ensuring that $\tau_\gamma$ is non-decreasing on $[0, +\infty)$.  \[ \square \]
5.2 Proof of Proposition 8

The proof is an easy consequence of this technical lemma.

**Lemma 25.** For any \( w \in [0, +\infty) \) and \( \gamma \in (0, \bar{\gamma}) \), we have that
\[
Q_\gamma \psi_1^*(w) = \tau_\gamma(w) + \gamma c_\infty.
\]

**Proof.** For any \( w \in \mathbb{R} \), we have,
\[
Q_\gamma \psi_1^*(w) = \int_\mathbb{R} (1 - p_\sigma \cdot \gamma (\tau_\gamma(w) + \gamma c_\infty, g)) \left( \tau_\gamma(w) + \gamma c_\infty - 2(\sigma^2 \gamma)^{1/2} g \right) \varphi(g) dg
\]
\[
= \int_\mathbb{R} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left( \varphi(\sigma^2 \gamma)^{1/2}(g) - \varphi(\sigma^2 \gamma)^{1/2}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg
\]
\[
= \int_{-\infty}^{(\tau_\gamma(w) + \gamma c_\infty)/2} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left( \varphi(\sigma^2 \gamma)^{1/2}(g) - \varphi(\sigma^2 \gamma)^{1/2}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg
\]
\[
= \frac{1}{2} \int_\mathbb{R} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left( \varphi(\sigma^2 \gamma)^{1/2}(g) - \varphi(\sigma^2 \gamma)^{1/2}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg
\]
\[
= \tau_\gamma(w) + \gamma c_\infty.
\]

By using change of variable \( g \to \tau_\gamma(w) + \gamma c_\infty - g \) we have,
\[
\int_{-\infty}^{(\tau_\gamma(w) + \gamma c_\infty)/2} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left( \varphi(\sigma^2 \gamma)^{1/2}(g) - \varphi(\sigma^2 \gamma)^{1/2}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg
\]
\[
= \frac{1}{2} \int_\mathbb{R} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left( \varphi(\sigma^2 \gamma)^{1/2}(g) - \varphi(\sigma^2 \gamma)^{1/2}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg
\]
\[
= \tau_\gamma(w) + \gamma c_\infty.
\]

**Proof of Proposition 8.** By Lemma 25 and H1-(ii), for any \( w \in [0, +\infty) \),
\[
Q_\gamma \psi_1^*(w) = \tau_\gamma(w) + \gamma c_\infty
\]
\[
\leq (1 - \gamma m) \psi_1^*(w) 1_{[0, \infty)}(w) + (1 + \gamma L) \psi_1^*(w) 1_{[0, R]}(w) + \gamma c_\infty.
\]

This completes the proof.

5.3 Proof of Proposition 9

We first establish that \( Q_\gamma \) admits a unique invariant probability measure \( \mu_\gamma \) and is geometrically ergodic. To that end, we show that \( Q_\gamma \) is (a) irreducible and aperiodic, (b) any compact set of \([0, +\infty)\) is small and (c) there exists \( 1 > \lambda > 0 \) and \( b \geq 0 \) such that \( Q_\gamma \psi(w) \leq \lambda \psi(w) + b \) for any \( w \in [0, +\infty) \) with \( \psi(w) = w + 1 \). The proof then follows from [11, Corollary 14.1.6, Theorem 15.2.4].

(a) Let \( K \) be a compact set. Then for any \( w \in K \) we have
\[
Q_\gamma(w, \{0\}) \geq \int_{[-1,1]} \beta_{\sigma \gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \varphi(g) dg \geq \eta_K ,
\]
where using H1-(ii)
\[
\eta_K = \inf_{(r,g) \in K \times [-1,1]} \beta_{\sigma \gamma}(r + \gamma c_\infty, g) \int_{[-1,1]} \varphi(g) dg
\]
\[
\geq \inf_{(a,g) \in [0,M] \times [-1,1]} \beta_{\sigma \gamma}(a, g) \int_{[-1,1]} \varphi(g) dg ,
\]
and $M = (1 + \gamma L) \sup(K) + \gamma c_\infty$. Note that since $(a, g) \to \pi_{\sigma \gamma}(a, g)$ is a continuous positive function, and $[0, M] \times [-1, 1]$ is compact, $\gamma K > 0$. Therefore $\{0\}$ is an accessible $(1, \delta_0)$-small set and $Q_{\gamma}$ is irreducible. In addition, $Q_{\gamma}(0, \{0\}) > 0$ which implies that $Q_{\gamma}$ is strongly aperiodic.

(b) Let now $C$ be a compact set, we show that $C$ is small. By (43), for $A \in \mathcal{B}([0, +\infty))$ and $w \in [0, +\infty)$,

$$Q_{\gamma}^2(w, A) \geq \int_{\mathbb{R}} \mathds{1}_{[0]}(\tilde{w})Q_{\gamma}(\tilde{w}, A)Q_{\gamma}(w, d\tilde{w}) \geq \eta_C Q_{\gamma}(0, A).$$

Therefore $C$ is a $(2, Q_{\gamma}(0, \cdot))$-small set.

(c) In addition, by Proposition 8 we have, for any $w \in [0, +\infty)$,

$$Q_{\gamma} \nu'(w) \leq 1 + (1 - \gamma m) w \mathds{1}_{(R_1, +\infty)}(w) + (1 + \gamma L) w \mathds{1}_{(0, R]}(w) + \gamma c_\infty \leq (1 - \gamma m) \nu'(w) + \gamma R_1(m + L) + \gamma c_\infty + \gamma m.$$

The proof of the first part of the proposition is complete.

We now establish the second part. Let $A \in \mathcal{B}(\mathbb{R})$ such that $(\delta_0 + \text{Leb})(A) = 0$. Then $0 \not\in A$ and $\text{Leb}(A) = 0$ therefore for any $w \in \mathbb{R}$,

$$Q_{\gamma}(w, A) \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} A \left(\tau_{\gamma}(w) + \gamma c_\infty - 2\sigma \gamma^{1/2} g\right) dg = 0.$$

It follows that $\mu_\gamma(A) = \mu_\gamma Q_{\gamma}(A) = 0$ and $\mu_\gamma \ll (\delta_0 + \text{Leb})$.

Since for any $w \in [0, +\infty)$, $Q_{\gamma}(w, \{0\}) > 0$, $\delta_{\{0\}}$ is an irreducibility measure, and by [11, Theorem 9.2.15], $\mu_\gamma$ is a maximal irreducibility measure for $Q_{\gamma}$, $\delta_0 \ll \mu_\gamma$ implying that $\mu_\gamma(\{0\}) > 0$.

In the case $c_\infty \neq 0$, first remark that for any $w \in [0, +\infty)$ and $g \in \mathbb{R}$, if $\tau_{\gamma}(w) + \gamma c_\infty - 2\sigma \gamma^{1/2} g > 0$, then $|\tau_{\gamma}(w) + \gamma c_\infty - \sigma \gamma^{1/2} g| > \sigma \gamma^{1/2} g$ therefore $\pi_{\sigma \gamma}(\tau_{\gamma}(w) + \gamma c_\infty, g) < 1$ by (11). Then by (14), for any $A \in \mathcal{B}([0, +\infty))$, $\text{Leb}(A) > 0$ and $w \in [0, +\infty)$, $Q_{\gamma}(w, A) > Q_{\gamma}(0, A \setminus \{0\}) > 0$ and therefore $\text{Leb}$ is an irreducibility measure. Applying [11, Theorem 9.2.15] again, we get that $\delta_0 + \text{Leb} \ll \mu_\gamma$. This completes the proof since we have already shown that $\mu_\gamma \ll (\delta_0 + \text{Leb})$.

5.4 Proof of Theorem 11

All technical results are moved to Section 5.4.1. The main idea is to analyze how the mass of $(0, +\infty)$ evolves with the sequence of distributions defined by $Q^n(w, \cdot)$ for $w \in [0, +\infty)$. To do this, we first establish the following drift condition. Define $(\alpha_k)_{k \geq 1}, (\beta_k)_{k \geq 1}$ for any $k \geq 1$ by

$$\alpha_k = \gamma c_\infty \sum_{i=0}^{k-1} (1 + \gamma L)^{-i}, \quad \beta_k^2 = \gamma \sigma^2 \sum_{i=0}^{k-1} (1 + \gamma L)^{-2i}.$$

(44)

Lemma 26. Assume $H1-(ii)$. Let $\delta \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$, with the convention $1/0 = +\infty$. For any $\gamma \in (0, \bar{\gamma})$, $n \in \{0, \ldots, n_\gamma\}$, $n_\gamma = \lfloor \delta / \gamma \rfloor$, and $w \in [0, +\infty)$, it holds

$$\int_{[0, +\infty)} \mathds{1}_{[0, +\infty)}(\tilde{w})Q_{\gamma}^{n+1}(w, d\tilde{w}) \leq 1 - 2\Phi \left(-\frac{\tau_{\gamma}(w) + \alpha_{n+1}}{2\beta_{n+1}}\right) + \zeta \sum_{k=1}^{k_{n_\gamma}} \frac{\gamma \alpha_k}{\beta_k^2},$$

(45)
where \((\alpha_k)_{k \geq 1}, (\beta_k)_{k \geq 1}\) are defined in (44), where \(\Phi\) are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1 and

\[
\zeta = 2(1 + \gamma L)^2 \sigma^2(2\sqrt{2\pi})^{-1} \left[ \sup_{t \geq 0} \{t^2 \Phi(-t)\} \right] + 1/8 .
\] (46)

**Proof.** Let \(\gamma \in (0, \bar{\gamma}]\), \(w \in [0, +\infty)\). Note that for any \(n \in \{0, \ldots, n_\gamma\}\), by (44) and Lemma 34,

\[
\alpha_n/(2\beta_n) \leq 1 .
\] (47)

Then, by Lemma 28, (45) holds for \(n = 0\). Assume it holds for \(n \in \{0, \ldots, n_\gamma - 1\}\). Then, we get

\[
\int_{[0, +\infty)} \mathds{1}_{(0, +\infty)}(\bar{w})Q^{n+1}_\gamma(w, d\bar{w}) \leq \int_{(0, +\infty)} \left[ 1 - 2\Phi\left( -\frac{\tau_\gamma(w) + \alpha_n}{2\beta_n} \right) \right] Q_\gamma(w, \bar{w}) + \zeta \sum_{k=1}^{n-1} \frac{\gamma \alpha_k}{\beta_k^3} .
\]

The proof is then concluded by a straightforward induction using Lemma 33 and (47). \(\square\)

From Lemma 26, we can have the following bound on the mass \(\mu_\gamma\) at \((0,R)\) for \(R \geq 0\).

**Theorem 27.** Assume \(H 1-(ii)\). Let \(R \geq 0\). For any \(\gamma \in (0, \bar{\gamma}]\) and \(\bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]\), \(\mu_\gamma((0,R)) \leq \eta_{Rc_\infty}\), where

\[
\eta_R = [\bar{\delta} + \bar{\gamma}]^{1/2} \left[ \frac{2\zeta e^{3(\bar{\delta} + \bar{\gamma})L}}{\sigma^2} + \frac{e^{(\bar{\delta} + \bar{\gamma})L}}{2\sqrt{2\pi} \sigma} \right] / \Phi\left( \frac{(1 + \gamma L)R + (\bar{\delta} + \bar{\gamma})c_\infty}{2\bar{\delta}^{1/2} \sigma e^{-(\bar{\delta} + \bar{\gamma})L}} \right) .
\] (48)

**Proof.** Let \(\bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]\) and \(\gamma \in (0, \bar{\gamma}]\). Set \(n_\gamma = \lfloor \gamma/\bar{\gamma} \rfloor\). Note that \(\bar{\delta} \leq \gamma(n_\gamma + 1) \leq \bar{\delta} + \bar{\gamma}\). By Lemma 26, Proposition 9, integrating (45) with respect to \(\mu_\gamma\) and using that \(\tau_\gamma(0) = 0\), \(\Phi(-t) \leq 1/2\) for any \(t \geq 0\), gives

\[
\mu_\gamma((0, +\infty)) \leq \int_{\mathbb{R}} \left\{ 1 - 2\Phi\left( -\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) \right\} d\mu_\gamma(w) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3}
\] (49)

\[
\leq 1 - 2\Phi\left( -\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3}
+ 2 \int_{(0, +\infty)} \left\{ \Phi\left( -\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) - \Phi\left( -\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) \right\} d\mu_\gamma(w)
\leq 1 - 2\Phi\left( -\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \mu_\gamma((0, +\infty)) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3}
- 2 \int_{(0,R)} \Phi\left( -\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) d\mu_\gamma(w) .
\]

Rearranging the terms yields

\[
2 \int_{(0,R)} \Phi\left( -\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) d\mu_\gamma(w) \leq 1 - 2\Phi\left( -\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3} .
\] (50)
In addition, by $H1$-(ii) using Lemma 34 and $t \mapsto \Phi(-t)$ is decreasing on $\mathbb{R}$, we have
\[
\int_{(0,R)} \Phi \left( -\frac{\tau_{\gamma}(w) + \alpha_{n+1}}{2\beta_{n+1}} \right) \, d\mu_{\gamma}(w) \geq \Phi \left( -\frac{(1 + \gamma L)R + \alpha_{n+1}}{2\beta_{n+1}} \right) \mu_{\gamma}((0,R)) \geq \Phi \left( -\frac{(1 + \bar{\gamma} L)R + (\bar{\delta} + \bar{\gamma})c_{\infty}}{2\delta_{1/2}e^{-\delta_{1/2}}} \right) \mu_{\gamma}((0,R)), \tag{51}
\]
Using that $t \mapsto 1 - 2\Phi(t)$ is $\sqrt{2/\pi}$-Lipschitz and combining (50), (51) and Lemma 34 we get that
\[
\Phi \left( -\frac{(1 + \gamma L)R + (\bar{\delta} + \bar{\gamma})c_{\infty}}{2\delta_{1/2}e^{-\delta_{1/2}}} \right) \mu_{\gamma}((0,R)) \leq \alpha_{n+1}/(2\sqrt{2\pi}\beta_{n+1}) + \zeta \sum_{k=1}^{n_{\gamma}} \alpha_k / \beta_k^3 \\
\leq c_{\infty} [\bar{\delta} + \bar{\gamma}]^{1/2} \left[ 2\zeta e^{3(\bar{\delta} + \bar{\gamma})L/\sigma^3} + \frac{e^{(\bar{\delta} + \bar{\gamma})L}}{2\sqrt{2\pi}} \right],
\]
which implies that $\mu_{\gamma}((0,R)) \leq \eta R c_{\infty}$ and completes the proof.

Now we can easily complete the proof of Theorem 11.

Proof of Theorem 11. Let $\delta \in (0, \{L^{-1} \land (\sigma e^{-1}/c_{\infty})^2]\}$ and $\gamma \in (0, \bar{\gamma}]$. By Proposition 8 and using $\mu_{\gamma}$ is invariant for $Q_{\gamma}$, we obtain
\[
\int_{\mathbb{R}} w \, d\mu_{\gamma}(w) \leq (1 - \gamma m) \int_{[R_1, +\infty)} w \, d\mu_{\gamma}(w) + (1 + \gamma L) \int_{(0,R_1)} w \, d\mu_{\gamma}(w) + \gamma c_{\infty}.
\]
Then, rearranging the terms in this inequality yields
\[
\int_{[R_1, +\infty)} w \, d\mu_{\gamma}(w) \leq R_1 \mu_{\gamma}((0, R_1))L/m + c_{\infty}/m \\
\int_{(0, +\infty)} w \, d\mu_{\gamma}(w) \leq R_1 \mu_{\gamma}((0, R_1))(1 + L/m) + c_{\infty}/m,
\]
which, combined with Theorem 27 applied to $R \leftarrow R_1$, concludes the proof of the first inequality in (18). Finally, by (49), using that $t \mapsto 1 - 2\Phi(-t)$ is $\sqrt{2/\pi}$-Lipschitz, we have
\[
\mu_{\gamma}((0, +\infty)) \leq \left( \sqrt{2\pi} \beta_{n+1} \right)^{-1} \int_{[0, +\infty)} \{(1 + \gamma L)w + \alpha_{n+1}\}d\mu_{\gamma}(w) + \zeta \sum_{k=1}^{n_{\gamma}} \alpha_k / \beta_k^3 \\
\leq \left( c_{\infty}c_1 (1 + \gamma L) + \alpha_{n+1} \right) / (\sqrt{2\pi}\beta_{n+1}) + \zeta \sum_{k=1}^{n_{\gamma}} \alpha_k / \beta_k^3.
\]
This finishes the proof using $n_{\gamma} = \lfloor \delta / \gamma \rfloor$, $\delta \leq \gamma (n_{\gamma} + 1) \leq \bar{\delta} + \bar{\gamma}$ and Lemma 34.

5.4.1 Technical results

Lemma 28. For any $w \in \mathbb{R}$,
\[
Q_{\gamma}(w, \{0\}) = 2\Phi \left( -\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \sqrt{\gamma}} \right),
\]
where $Q_{\gamma}$ is defined by (14) and $\Phi$ is the cumulative distribution of the one-dimensional Gaussian distribution with mean 0 and variance 1.
Proof. Let \( w \in \mathbb{R} \). By (13) and the change of variable \( g \to \sigma \sqrt{\gamma} g \), we get
\[
Q_{\gamma}(w, \{0\}) = \int_{\mathbb{R}} \left( 1 - \varphi_{\sigma^2 \gamma} \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty} - \sigma \sqrt{\gamma} g}{\sigma \sqrt{\gamma}} \right) \right) \varphi(g) dg
\]
\[
= \int_{\mathbb{R}} \varphi_{\sigma^2 \gamma}(g) \wedge \varphi_{\sigma^2 \gamma} \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty} - g}{\sigma \sqrt{\gamma}} \right) \varphi_{\sigma^2 \gamma}(g) dg
\]
\[
= \int_{-\infty}^{(\tau_{\gamma}(w) + \gamma c_{\infty})/2} \varphi_{\sigma^2 \gamma}(g - \tau_{\gamma}(w) - \gamma c_{\infty}) dg + \int_{(\tau_{\gamma}(w) + \gamma c_{\infty})/2}^{+\infty} \varphi_{\sigma^2 \gamma}(g) dg
\]
\[
= \int_{-\infty}^{-(\tau_{\gamma}(w) + \gamma c_{\infty})/2} \varphi_{\sigma^2 \gamma}(g) dg + \int_{(\tau_{\gamma}(w) + \gamma c_{\infty})/2}^{+\infty} \varphi_{\sigma^2 \gamma}(g) dg = 2 \Phi \left( -\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \sqrt{\gamma}} \right),
\]
and the lemma follows. \( \square \)

Lemma 29. Let \( \sigma^2, \gamma > 0 \). For any \( t \geq 0 \) and \( a > 0 \), we have
\[
\int_{\mathbb{R}} \left( 1 - 2 \Phi \left( -\frac{t - 2\sigma \gamma^{1/2} g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(t, g) \varphi(g) dg = 0,
\]
where \( \varphi_{\sigma^2 \gamma} \) is defined by (11), \( \varphi \) and \( \Phi \) are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1 respectively.

Proof. Using the changes of variable \( g \to \sigma \gamma^{1/2} g \) and \( g \to t - g \), we obtain
\[
\int_{\mathbb{R}} \left( 1 - 2 \Phi \left( -\frac{t - 2\sigma \gamma^{1/2} g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(t, g) \varphi(g) dg
\]
\[
= \int_{\mathbb{R}} \left( 1 - 2 \Phi \left( -\frac{t - 2\sigma \gamma^{1/2} g}{2a} \right) \right) \left\{ 1 \wedge \frac{\varphi_{\sigma^2 \gamma} \left( t - \sigma \sqrt{\gamma} g \right)}{\varphi_{\sigma^2 \gamma} \left( \sigma \sqrt{\gamma} g \right)} \right\} \varphi(g) dg
\]
\[
= \int_{-\infty}^{t/2} \left( 1 - 2 \Phi \left( -\frac{t - 2g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(t - g) dg + \int_{t/2}^{+\infty} \left( 1 - 2 \Phi \left( -\frac{t - 2g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(g) dg
\]
\[
= \int_{t/2}^{+\infty} \left( 1 - 2 \Phi \left( -\frac{t - 2g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(g) dg + \int_{t/2}^{+\infty} \left( 1 - 2 \Phi \left( -\frac{t - 2g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(g) dg.
\]
Using that \( s \in \mathbb{R} \), \( 1 - 2 \Phi (s) = -[1 - 2 \Phi (-s)] \) completes the proof. \( \square \)

Lemma 30. Let \( \sigma^2, \gamma > 0 \). For any \( t, s \geq 0 \) and \( a > 0 \),
\[
\int_{\mathbb{R}} \left( 1 - 2 \Phi \left( -\frac{t + s - 2\sigma \sqrt{\gamma} g}{2a} \right) \right) \varphi_{\sigma^2 \gamma}(t, g) \varphi(g) dg
\]
\[
= 2 \Phi \left( \sigma \sqrt{\gamma} G \geq t/2, -s - t \leq 2a \tilde{G} - 2 \sigma \sqrt{\gamma} G \leq s - t \right),
\]
where \( G, \tilde{G} \) are two independent one-dimensional standard Gaussian random variables, \( \varphi_{\sigma^2 \gamma} \) is defined by (11), \( \varphi \) and \( \Phi \) are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1 respectively.
Proof. Using the changes of variable \( g \mapsto \sigma \gamma^{1/2} g, \ g \mapsto g - t \), we get
\[
\int_{\mathbb{R}} \left[ 1 - 2\Phi \left( \frac{-t + s - 2\sigma \sqrt{t} g}{2a} \right) \varphi_{\sigma^2 \gamma}(t, g) \varphi(g) \right] dg
= \int_{-\infty}^{-t/2} \left[ 1 - 2\Phi \left( \frac{-t + s - 2g}{2a} \right) \right] \varphi_{\sigma^2 \gamma}(g) \, dg
+ \int_{t/2}^{+\infty} \left[ 1 - 2\Phi \left( \frac{-t + s - 2g}{2a} \right) \right] \varphi_{\sigma^2 \gamma}(g) \, dg
= 2[\mathbb{P}(\sigma \gamma^{1/2} G \geq t/2) - A - B] \quad (52)
\]

In addition, we have since \((-G, G)\) has the same distribution than \((G, G)\),
\[
A = \mathbb{P} \left( \sigma \gamma^{1/2} G \leq -\frac{t}{2}, \ G \leq -\frac{t + s - 2\sigma \sqrt{t} G}{2a} \right)
= \mathbb{P} \left( \sigma \gamma^{1/2} G \geq \frac{t}{2}, \ G \geq -\frac{t + s - 2\sigma \sqrt{t} G}{2a} \right),
\]
and
\[
B = \mathbb{P} \left( \sigma \gamma^{1/2} G \geq \frac{t}{2}, \ G \leq -\frac{t + s - 2\sigma \sqrt{t} G}{2a} \right).
\]

Therefore, we obtain
\[
A + B
= \mathbb{P}(\sigma \gamma^{1/2} G \geq t/2) - \mathbb{P} \left( \sigma \gamma^{1/2} G \geq t/2, \ -\frac{t + s - 2\sigma \sqrt{t} G}{2a} \leq G \leq \frac{-t + s - 2\sigma \sqrt{t} G}{2a} \right)
\]
Plugging this expression in (52) concludes the proof.

Lemma 31. For any \( a \in \mathbb{R}, \ b \in [0, 1] \), it holds
\[
\Phi (a + b) - \Phi (a - b) \geq 1 - 2\Phi (-b) - a^2 b \exp(-b^2/2)/\sqrt{2\pi},
\]
where \( \Phi \) are the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1.

Proof. Define \( \psi : \mathbb{R} \times [0, 1] \to \mathbb{R} \) for any \( a \in \mathbb{R}, \ b \in [0, 1] \) by
\[
\psi(a, b) = \Phi (a + b) - \Phi (a - b) - 1 + 2\Phi (-b) + a^2 b \exp(-b^2/2)/\sqrt{2\pi}.
\]
We show that \( \psi(a, b) \geq 0 \) for any \( a \in \mathbb{R} \) and \( b \in [0, 1] \). Using that \( 1 - \Phi (-t) = \Phi (t) \) for any \( t \in \mathbb{R} \), we get that \( \psi(a, b) = \psi(-a, b) \) and therefore we only need to consider the case \( a \leq 0 \) and \( b \in [0, 1] \). In addition, for any \( b \in [0, 1] \), \( \psi(0, b) = 0 \) and thus, it is sufficient to establish that for any \( b \in [0, 1] \), \( a \mapsto \psi(a, b) \) is non-increasing on \( \mathbb{R}_- \).
For any $a \leq 0$ and $b \in (0, 1)$, we have using that \(\sinh(t) = \int_0^t \cosh(s) ds \leq t \cosh(t)\) and \(e^{-t^2/2} \cosh(t) \leq 1\) for any $t \in [0, +\infty)$,

\[
\sqrt{2\pi} \exp(t^2/2) \frac{\partial}{\partial a} \Phi(a, b) = 2 \exp(-a^2/2) \sinh(-ab) + 2ab < -2ab[\exp(-a^2b^2/2) \cosh(ab) - 1] \leq 0.
\]

By continuity, it also holds for $a \leq 0$ and $b \in [0, 1]$ which concludes the proof.

\[\square\]

**Lemma 32.** Assume $H1$-(ii). For any $w \in [0, +\infty)$ and $\alpha, \beta \in [0, +\infty)$, $\beta > 0$,

\[
\int_{(0, +\infty)} \left[1 - 2\Phi \left( -\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) \leq 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma L)}{2\sqrt{\sigma^2\gamma + \beta^2/(1 + \gamma L)^2}} \right) - \int_{\mathbb{R}} \left[1 - 2\Phi \left( -\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma g}}{2\beta/(1 + \gamma L)} \right) \right] \frac{p_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty, g)}{\varphi(g)} dg
\]

where $\Phi$ are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1.

**Proof.** Let $\alpha, \beta \geq 0$, $\beta > 0$. By (14), we have

\[
\int_{(0, +\infty)} \left[1 - 2\Phi \left( -\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) = \int_{\mathbb{R}} \left[1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma g} + \alpha}{2\beta} \right) \right] (1 - p_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg
\]

By $H1$-(ii), $t \mapsto 1 - 2\Phi(-t)$ is increasing, we have setting $\psi_\gamma(w) = \tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma L)$,

\[
1 - 2\Phi \left( -\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma g}}{2\beta/(1 + \gamma L)} \right) \leq 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma g} + \alpha}{2\beta/(1 + \gamma L)} \right).
\]

Using [15, Lemma 20], Lemma 29 and Lemma 30, we get

\[
\int_{\mathbb{R}} \left[1 - 2\Phi \left( -\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma g}}{2\beta/(1 + \gamma L)} \right) \right] (1 - p_{\sigma^2\gamma}(\psi_\gamma(w), g)) \varphi(g) dg
\]

which completes the proof.

\[\square\]
Lemma 33. Assume $H1$-(ii). For any $w \in [0, +\infty)$ and $\alpha, \beta \in [0, +\infty)$, $\beta > 0$, such that $\alpha/(2\beta) \leq 1$,

\[
\int_{[0, +\infty)} \left[ 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right) \right] Q_\gamma(w, dw) \leq 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right) + \frac{\zeta\alpha}{\beta^3},
\]

where $\zeta$ is defined in (46).

Proof. Let $\alpha, \beta \geq 0$ such that $\alpha/(2\beta) \leq 1$. By (14) and Lemma 32, we have

\[
\int_{[0, +\infty)} \left[ 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right) \right] Q_\gamma(w, dw) \leq 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right)
\]

\[
+ \left[ 1 - 2\Phi \left( -\frac{\tau_\gamma(0) + \alpha}{2\beta} \right) \right] 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right)
\]

\[
- \int_{\mathbb{R}} \left[ 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right) \right] \varphi(w, g) \phi(g) dg
\]

\[
\leq 1 - 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right) + \left[ 1 - 2\Phi \left( -\frac{\alpha}{2\beta} \right) \right] 2\Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right)
\]

\[
- 2\mathbb{P}\left( \{2\sigma\sqrt{G} \geq \tau_\gamma(w) + \alpha \} \cap A \right),
\]

where we used $\tau_\gamma(0) = 0$ and set

\[
A = \left\{ -\tau_\gamma(w) - \gamma c_\infty - \frac{\alpha}{1 + \gamma L} \leq \frac{2\beta\tilde{G}}{1 + \gamma L} - 2\sigma\sqrt{G} \leq -\tau_\gamma(w) - \gamma c_\infty + \frac{\alpha}{1 + \gamma L} \right\},
\]

and $G, \tilde{G}$ are two independent one-dimensional standard Gaussian random variables.

Define $\theta_\gamma : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ for any $w \in [0, +\infty)$ and $g \in \mathbb{R}$ by $\theta_\gamma(w, g) = (2\beta)^{-1}(1 + \gamma L)[-\tau_\gamma(w) - \gamma c_\infty + \gamma \sqrt{g}]$. Then, using that

\[
A = \left\{ \theta_\gamma(w, G) - \alpha/(2\beta) \leq \tilde{G} \leq \theta_\gamma(w, G) + \alpha/(2\beta) \right\},
\]

we have

\[
\mathbb{P}\left( \{2\sigma\sqrt{G} \geq \tau_\gamma(w) + \alpha \} \cap A \right)
\]

\[
= \int_{\mathbb{R}} \left[ 1_{[0, +\infty)} \left( g - \frac{\tau_\gamma(w) + \alpha}{2\beta} \right) \varphi \left( \theta_\gamma(w, g) + \frac{\alpha}{2\beta} \right) - \varphi \left( \theta_\gamma(w, g) - \frac{\alpha}{2\beta} \right) \right] \phi(g) dg.
\]

Since $\alpha/(2\beta) \leq 1$ by Lemma 31 we have, for any $a \in \mathbb{R}$,

\[
\Phi \left( a + \frac{\alpha}{2\beta} \right) - \Phi \left( a - \frac{\alpha}{2\beta} \right) \geq 1 - 2\Phi \left( -\frac{\alpha}{2\beta} \right) - \frac{a^2\alpha}{2\sqrt{2\pi\beta}} e^{-a^2/(8\beta^2)},
\]

which implies

\[
\mathbb{P}\left( \{2\sigma\sqrt{G} \geq \tau_\gamma(w) + \alpha \} \cap A \right) \geq \Phi \left( -\frac{\tau_\gamma(w) + \alpha}{2\beta} \right)
\]

\[
- \int_{\mathbb{R}} 1_{[0, +\infty)} \left( g - \frac{\tau_\gamma(w) + \alpha}{2\beta} \right) \frac{\theta_\gamma^2(g, w) \alpha}{2\sqrt{2\pi\beta}} e^{-a^2/(8\beta^2)} \phi(g) dg.
\]
Therefore, we obtain using that \( \mathbb{E}[\mathbb{1}_{[0, +\infty)}(G)G^2] = 1/2 \),
\[
\left(1 - 2\Phi\left(-\frac{\alpha}{2\beta}\right)\right)\Phi\left(-\frac{\tau_0(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}}\right) - \mathbb{P}\left(\{2\sigma\sqrt{\gamma}G \geq \tau_0(w) + \gamma c_\infty\} \cap A\right)
\leq \int_\mathbb{R} \mathbb{1}_{[0, +\infty)} \left( g - \frac{\tau_0(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}} \right) \frac{\phi_2^2(g, w)}{2\sqrt{2\pi}} e^{-\alpha^2/(8\beta^2)} \varphi(g) \, dg
\leq \frac{\alpha\gamma}{\beta^3} \left( e^{-\alpha^2/(8\beta^2)} \right) \left( \frac{\tau_0(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}} \right)^2 \Phi\left(-\frac{\tau_0(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}}\right)
- \frac{1}{\sqrt{2\pi}} \frac{1}{2\sigma\sqrt{\gamma}} \exp\left(-\left(\frac{\tau_0(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}}\right)^2/2\right) + 1/8.
\]

Plugging this inequality in (53) concludes the proof. \( \Box \)

**Lemma 34.** For any \( \gamma > 0, k \geq 1 \), we have
\[
k\gamma c_\infty e^{-k\gamma L} \leq \alpha_k \leq k\gamma c_\infty \quad , \quad (k\gamma)^{1/2} \sigma e^{-k\gamma L} \leq \beta_k \leq (k\gamma)^{1/2} \sigma ,
\]
\[
[(k\gamma)^{1/2} c_\infty/\sigma] e^{-k\gamma L} \leq \alpha_k/\beta_k \leq [(k\gamma)^{1/2} c_\infty/\sigma] e^{k\gamma L},
\]
\[
[e_\infty \gamma^{1/2}/(\sigma^2 k^{1/2})] e^{-k\gamma L} \leq \gamma \alpha_k/\beta_k \leq [e_\infty \gamma^{1/2}/(\sigma^2 k^{1/2})] e^{3k\gamma L}, \quad \gamma \sum_{i=1}^{k-1} \{\alpha_i/\beta_i^3\} \leq [2e_\infty (k\gamma)^{1/2}/(\sigma^2)] e^{3k\gamma L} \quad \text{(54)}
\]

where \( (\alpha_k)_{k \geq 1}, (\beta_k)_{k \geq 1} \) are defined in (44).

**Proof.** Let \( k \geq 1 \). Using for any \( i \in \mathbb{N} \), \( e^{-i\gamma L} \leq (1 + \gamma L)^{-i} \leq 1 \), we have
\[
k\gamma c_\infty e^{-k\gamma L} \leq \gamma c_\infty \sum_{i=0}^{k-1} (1 + \gamma L)^{-i} \leq k\gamma c_\infty \quad \text{(56)}
\]

In the same way, using for any \( i \in \mathbb{N} \), \( e^{-2i\gamma L} \leq (1 + \gamma L)^{-2i} \leq 1 \), we obtain
\[
k\gamma \sigma^2 e^{-2i\gamma L} \leq \gamma \sigma^2 \sum_{i=0}^{k-1} (1 + \gamma L)^{-2i} \leq k\gamma \sigma^2 \quad \text{(57)}
\]

Combining (56) and (57) completes the proof of the first four inequalities. Then, (55) is a simple consequence of (54) and a comparison test. \( \Box \)

### 5.5 Proof of Theorem 13

We show first that any compact sets of \( \mathbb{R}_+ \) are \( k+1 \)-small sets (also called \( \varepsilon_k, k \) Doeblin sets) with an explicit constant \( \varepsilon_k > 0 \) depending on \( k \in \mathbb{N} \). To this end, we need the following technical lemma.

We consider in what follows that \( (W_k)_{k \in \mathbb{N}} \) is the canonical process on \( ([0, +\infty)^{\mathbb{N}}, B([0, +\infty)^{\otimes \mathbb{N}})) \) and for any \( w \in [0, +\infty) \), \( \mathbb{P}_w \) and \( \mathbb{E}_w \) correspond to the probability and expectation respectively, associated with \( Q_w \) and the initial condition \( \delta_w \) on this space.
Lemma 35. Assume $H1-(ii)$. For any $k \in \mathbb{N}$ and $w \in (0, +\infty)$,
\[
\mathbb{P}_w\left(\min_{i \in \{0, \ldots, k+1\}} W_i > 0\right) \leq 1 - 2\Phi\left(-\frac{\tau_{\gamma}(w) + \alpha_{k+1}}{2\beta_{k+1}}\right),
\]
where $\alpha_{k+1}, \beta_{k+1}$ are given in (44).

Proof. The proof is by induction on $k \in \mathbb{N}$. The proof for $k = 0$ follows from Lemma 28. Assume that the result holds for $k-1 \in \mathbb{N}$ and for any $w \in (0, +\infty)$. Then, by the Markov property and the assumption hypothesis, for any $w \in (0, +\infty)$,
\[
\mathbb{P}_w\left(\min_{i \in \{0, \ldots, k\}} W_i > 0\right) = \mathbb{E}_w\left[\mathbb{1}_{(0, +\infty)}(W_1)\mathbb{P}_{\mathbb{W}_1}\left(\min_{i \in \{0, \ldots, k-1\}} W_i > 0\right)\right] \\
\leq \mathbb{E}_w\left[\mathbb{1}_{(0, +\infty)}(W_1)\left\{1 - 2\Phi\left(-\frac{\tau_{\gamma}(W_1) + \alpha_k}{2\beta_k}\right)\right\}\right].
\]
The proof is then completed upon using Lemma 32. \qed

We now ready to show that any compact set of $\mathbb{R}_+$ is $k+1$-small.

Lemma 36. Assume $H1-(ii)$. Then, for any $k \in \mathbb{N}$, $w, \tilde{w} \in [0, +\infty)$,
\[
\|\delta_wQ_{\gamma}^{k+1} - \delta_{\tilde{w}}Q_{\gamma}^{k+1}\|_{TV} \leq 1 - 2\Phi\left(-\frac{\tau_{\gamma}(w \vee \tilde{w}) + \alpha_{k+1}}{2\beta_{k+1}}\right),
\]
where $\alpha_{k+1}, \beta_{k+1}$ are given in (44).

Proof. We consider again $(G_k)_{k \geq 1}$ and $(U_k)_{k \geq 1}$ two independent sequences of i.i.d. standard Gaussian and $[0, 1]$-uniform random variables respectively. Define the Markov chains $(W_k)_{k \in \mathbb{N}}$ and $(\tilde{W}_k)_{k \in \mathbb{N}}$ starting from $w \in [0, +\infty)$ and $\tilde{w} \in [0, +\infty)$ respectively, for any $k \in \mathbb{N}$, $W_{k+1} = \mathcal{G}_{\gamma}(W_k, G_{k+1}, U_{k+1})$ and $\tilde{W}_{k+1} = \mathcal{G}_{\gamma}(\tilde{W}_k, G_{k+1}, U_{k+1})$. Note that the case $w = \tilde{w}$ is trivial so we only consider the converse and assume that $w < \tilde{w}, \tilde{w} > 0$. Then, we obtain by Proposition 6 that almost surely $W_k \leq \tilde{W}_k$ for any $k \in \mathbb{N}$, which implies that
\[
\|\delta_wQ_{\gamma}^{k+1} - \delta_{\tilde{w}}Q_{\gamma}^{k+1}\|_{TV} \leq \mathbb{P}\left(W_{k+1} \neq \tilde{W}_{k+1}\right) \leq \mathbb{P}\left(\min_{i \in \{0, \ldots, k\}} \tilde{W}_i > 0\right).
\]
Indeed, if $\min_{i \in \{0, \ldots, k\}} \tilde{W}_i = 0$, then there exists $i \in \{0, \ldots, k+1\}$, $\tilde{W}_i = 0$ which implies since $\tilde{W}_i \geq W_i \geq 0$ that $\tilde{W}_i = W_i$ and therefore $\tilde{W}_{k+1} = W_{k+1}$ by definition of the two processes. The proof is then completed by Lemma 35. \qed

We easily deduce then the following corollary.

Corollary 37. Assume $H1-(ii)$. Let $t_0 > 0$. Then, for any $w, \tilde{w} \in [0, +\infty)$,
\[
\|\delta_wQ_{\gamma}^{[t_0/\gamma]} - \delta_{\tilde{w}}Q_{\gamma}^{[t_0/\gamma]}\|_{TV} \leq 1 - 2\Phi\left(-\frac{\tau_{\gamma}(w \vee \tilde{w}) + t_0c_{\infty}}{2\sigma(1 - e^{-2\tau_{\gamma}/t_0})^{1/2}}\right).
\]

Proof. Note that by (44) for any $k \in \mathbb{N}$, $\alpha_{k+1} \leq k \gamma c_{\infty}$ and $\beta_{k+1}^2 = (\sigma^2(1 + \gamma L)/L)\{1 - (1 + \gamma L)^{-2k}\} \geq (\sigma^2(1 + \gamma L)/L)\{1 - e^{-2\tau_{\gamma}2k}\}$ using that $(1 + t) \leq e^t$ for any $t \in \mathbb{R}$. The proof is then completed using Lemma 36, $H1-(ii)$ and the previous bounds for $k \leftarrow \lfloor t_0/\gamma \rfloor$. \qed
Define $\mathcal{V}_1, \mathcal{W}_a : \mathbb{R}_+ \to [1, +\infty)$ for any $w \in \mathbb{R}_+$ by
\[
\mathcal{V}_1(w) = 1 + w, \quad \mathcal{W}_a(w) = \exp(aw).
\] 

Lemma 38. Assume H1-(ii). Let $a > 0$. Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \gamma_1)$,
\[
Q_{\gamma} \mathcal{V}_1(w) \leq (1 - \gamma m) \mathcal{V}_1(w) + \gamma R_1(m + L) + \gamma c_\infty + \gamma m, \quad Q_{\gamma} \mathcal{W}_a(w) \leq \lambda_1^w \mathcal{W}_a(w) + \gamma \alpha_a,
\]
where $Q_{\gamma}$ is defined by (14) and $\lambda_1, \gamma_1, R_a, B_a, D_a$ are defined by (99) and
\[
\alpha_a = \log(B_a/\lambda_a) \mathcal{W}_a(R_a) B_a^\gamma - \log(\lambda_a) + D_a.
\]

Proof. (59) is a simple consequence of Proposition 8. In addition, by Proposition 46, we have
\[
Q_{\gamma} \mathcal{W}_a(w) \leq \lambda_1^w \mathcal{W}_a(w) I_{[R_a, +\infty)}(w) + B_a^\gamma \mathcal{W}_a(w) I_{[0, R_a)}(w) + \gamma D_a I_{[0, R_a)}(w) + 1
\leq \lambda_1^w \mathcal{W}_a(w) + \lambda_1^w (B_a^\gamma \lambda_a^\gamma - 1) \mathcal{W}_a(R_a) + \gamma \alpha_a + 1 - \lambda_1^w.
\]
Using $e^t - 1 \leq te^t$ for any $t \geq 0$, completes the proof.

Proof of Theorem 13. Let $a > 0$ and $t_0 > 0$. By Lemma 38 and an easy induction, and using that $te^{-1} \leq 1 - e^{-t} \leq t$ for any $t \geq 0$, we have for any $k \in \mathbb{N}$,
\[
Q_{\gamma}^k \mathcal{V}_1(w) \leq \lambda_1^{\gamma^k} \mathcal{V}_1(w) + [R_1(m + L) + c_\infty + m]/m, \quad Q_{\gamma}^k \mathcal{W}_a(w) \leq \lambda_a^{k\gamma} \mathcal{W}_a(w) + k \gamma \alpha_a \lambda_a^{-\gamma},
\]
where $Q_{\gamma}$ is defined by (14), $\mathcal{V}_1, \mathcal{W}_a$ by (58), $\lambda_a$ by (99), $\alpha_a$ by (60) and
\[
\lambda_1 = e^{-a}, \quad \beta_1 = [R_1(m + L) + c_\infty + m]/m, \quad \beta_a = (t_0 + \gamma) \alpha_a \lambda_a^{-\gamma}.
\]

Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \gamma_1)$,
\[
Q_{\gamma}^{[t_0/\gamma]} \mathcal{V}_1(w) \leq \lambda_1^{t_0} \mathcal{V}_1(w) + \beta_1, \quad Q_{\gamma}^{[t_0/\gamma]} \mathcal{W}_a(w) \leq \lambda_a^{t_0} \mathcal{W}_a(w) + \beta_a.
\]

We now only complete the proof show for $\mathcal{V} = \mathcal{V}_1$. The result for $\mathcal{V} = \mathcal{W}_a$, $a > 0$ is similar upon replacing $\lambda_1$ and $\beta_1$ by $\lambda_a$ and $\beta_a$ given in (61) respectively.

Define $\delta_1 = 4 \beta_1/(1 - \lambda_1) - 1$ and $M_1 = \sup \{w \in [0, +\infty) : \mathcal{V}_1(w) \leq \delta_1\}$ which is well defined since $\lim_{w \to +\infty} \mathcal{V}_1(w) = +\infty$. Define in addition,
\[
\epsilon_1 = 2 \Phi \left[ -\frac{M_1 + t_0 c_\infty}{2 \sigma \{1 - e^{-2\epsilon_1 (t_0 + \gamma)}\}^{1/2}} \right] < 1.
\]

Then, $\{\mathcal{V}_1 \leq \delta_1\}$ is a $([t_0/\gamma], \epsilon_1)$-Doeblin set for $Q_{\gamma}$ and $\lambda_1 + 2 \beta_1/(1 + \delta_1) < 1$. Therefore, [11, Theorem 19.4.1] implies that for any $k \in \mathbb{N}$,
\[
\left\| \delta_u Q_{\gamma}^k - \mu_{\gamma} \right\|_{\mathcal{V}_1} \leq \tilde{C} \rho^k \{\mathcal{V}_1(w) + \mu_{\gamma}(\mathcal{V}_1)\},
\]
where a bound on $\mu_{\gamma}(\mathcal{V}_1)$ is provided by Theorem 11 and
\[
\log(\rho) = (t_0 + \gamma)^{-1} \log(1 - \epsilon_1) \log(\lambda_1)/\{\log(1 - \epsilon_1) + \log(\lambda_1) - \log(\beta_1)\}
\lambda_1 = \lambda_1^0 + 2 \beta_1/(1 + \delta_1), \quad \beta_1 = \lambda_1^0 \beta_1 + \delta_1
\tilde{C} = \rho^{-1}\{\lambda_1^0 + \beta_1\}/[1 + \beta_1/\{(1 - \epsilon_1)(1 - \lambda_1)\}].
\]

\footnote{There is a $b$ missing in Equation 19.4.2d in [11, Theorem 19.4.1] which has been confirmed by one of the authors of [11].}
5.6 Proof of Proposition 16

The proof is an easy consequence of Lemma 41 below and the definition of \((W^{(n)}_t)_{t \geq 0}\). Before stating and proving Lemma 41, we need the following technical results.

**Lemma 39.** Assume A1. Then, for any \(q \in [1, +\infty)\), we have for any \(\gamma \in (0, \gamma]\),

\[
\mathbb{E}[|W_1 - \tau_\gamma(W_0) - \gamma c_\infty|^q] \leq (4\sigma^2 \gamma)^{q/2} \{m_q + 2 \sup_{u \geq 0}[u^q \Phi(-u)]\},
\]

where \(W_1\) is defined by (13), \(m_q\) is the \(q\)-th moment of the standard Gaussian distribution and \(\Phi\) is its cumulative distribution function.

**Proof.** Let \(w_0 \in [0, +\infty)\) and \(\gamma \in (0, \gamma]\). By definition (14) and (11), we have setting \(\bar{\tau}_\gamma^\infty(w_0) = \{\tau_\gamma(w_0) + \gamma c_\infty\}/(2\sqrt{\sigma^2 \gamma})\),

\[
\int_{\mathbb{R}^+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) = (4\sigma^2 \gamma)^{q/2} \int_{-\infty}^{\bar{\tau}_\gamma^\infty(w_0)} |g|^{q} \varphi(g) dg
\]

\[-(4\sigma^2 \gamma)^{q/2} \int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g-2\bar{\tau}_\gamma^\infty(w_0)|^q \varphi(g) dg
\]

\[+ \int_{\mathbb{R}} |\tau_\gamma(w_0) + \gamma c_\infty|^q \varphi(2\bar{\tau}_\gamma^\infty(w_0) - g) \wedge \varphi(g) dg
\]

\[\leq (4\sigma^2 \gamma)^{q/2} m_q + 2^{q+1} \sigma^2 \gamma^{q/2}(\bar{\tau}_\gamma^\infty(w_0))^q \Phi\{-\bar{\tau}_\gamma^\infty(w_0)\},
\]

which completes the proof. \(\square\)

**Lemma 40.** Assume A1. Then, there exists \(C \geq 0\) such that for any \(k \in \mathbb{N}\) and \(\gamma \in (0, \gamma]\),

\[
\mathbb{E}[W_k^4] \leq e^{Ck\gamma}\{\mathbb{E}[W_0^4] + 1\},
\]

where \((W_k)_{k \in \mathbb{N}}\) is defined by (13).

**Proof.** By A1, (13) and (14) we have that for any \(w_0 \in [0, +\infty)\) and \(\gamma \in (0, \gamma]\), setting \(W_0 = w_0\) and \(\kappa_\infty(w_0) = \kappa(w_0) + c_\infty\),

\[
\int_{\mathbb{R}^+} w_1^4 Q_\gamma(w_0, dw_1) = \mathbb{E}[W_1^4] \leq \mathbb{E}[(w_0 + \gamma \kappa_\infty(w_0) - 2\sqrt{\sigma^2 \gamma} G_1)^4]
\]

\[= \{w_0 + \gamma \kappa_\infty(w_0)\}^4 + 48\sigma^2 \gamma \{w_0 + \gamma \kappa_\infty(w_0)\}^2 + 2^{4+1}\sigma^4 \gamma^2.
\]

By A1, for any \(\ell \in \{2, 4\}\), we have that for any \(w_0 \in [0, +\infty)\), \(\gamma \in (0, \gamma]\),

\[\{w_0 + \gamma \kappa_\infty(w_0)\}^\ell \leq w_0^\ell + 2^{\ell-1}(1 + \bar{\gamma})^\ell \bar{\gamma}(1 + L_n)^\ell \{w_0^\ell + c_\infty^\ell\}.
\]

Therefore, we obtain that there exists some constant \(C_1, C_2 \geq 0\), such that for any \(w_0 \in [0, +\infty)\), \(\gamma \in (0, \gamma]\),

\[
\int_{\mathbb{R}^+} w_1^4 Q_\gamma(w_0, dw_1) \leq w_0^4 + C_1 \gamma \{1 + w_0^2 + w_0^4\} \leq (1 + \gamma C_2) w_0^4 + \gamma C_2.
\]

By an easy induction, we get then that for any \(w_0 \in [0, +\infty)\), \(\gamma \in (0, \gamma]\) and \(k \in \mathbb{N}\),

\[
\int_{\mathbb{R}^+} w_1^4 Q_\gamma^k(w_0, dw_1) \leq (1 + C_2 \gamma)^k w_0^4 + C_2 \gamma \sum_{i=0}^{k-1} (1 + C_2 \gamma)^i \leq e^{k\gamma C_2} [w_0^4 + 1],
\]

which completes the proof by the Markov property. \(\square\)
Lemma 41. Assume A1. Then, there exists $C > 0$ such that for any $k \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}]$, $E[\max_{\ell \in \{0, \ldots, k\}} |W_{\ell} - W_0|^4] \leq C(k\gamma)^2 e^{Ck\gamma} \{E[W_0^4] + 1\}$, where $(W_k)_{k \in \mathbb{N}}$ is defined by (13).

Proof. Assume that $E[W_0^4] < +\infty$, otherwise the results holds. Denote by $(\mathcal{F}_k)_{k \in \mathbb{N}}$ the filtration associated with $(W_k)_{k \in \mathbb{N}}$. We consider the following decomposition for any $\ell \in \mathbb{N}$,

$$W_\ell - W_0 = A_\ell + B_\ell, \quad A_\ell = \sum_{i=0}^{\ell-1} \Delta M_i, \quad B_\ell = \sum_{i=0}^{\ell-1} H_i,$$

where using that $E[W_{\ell+1} | \mathcal{F}_\ell] = \tau_\gamma(W_\ell) + \gamma c_\infty$ by Lemma 25 and the Markov property,

$$\Delta M_i = W_{i+1} - E[W_{i+1} | \mathcal{F}_i] = W_{i+1} - \tau_\gamma(W_i) - \gamma c_\infty, \quad H_i = \tau_\gamma(W_i) + \gamma c_\infty - W_i. \quad (62)$$

Then, using Young’s inequality, we get for any $\gamma \in (0, \bar{\gamma}]$ and $k \in \mathbb{N},$

$$\max_{\ell \in \{0, \ldots, k\}} |W_{\ell} - W_0|^4 \leq 2^{\frac{3}{4}} \max_{\ell \in \{0, \ldots, k\}} A_\ell^4 + \max_{\ell \in \{0, \ldots, k\}} B_\ell^4. \quad (63)$$

We now bound the two last terms in the right hand side of this equation. First, by A1 and Young’s inequality, we get for any $\gamma \in (0, \bar{\gamma}]$ and $k \in \mathbb{N},$

$$E[\max_{\ell \in \{0, \ldots, k\}} B_\ell^4] \leq E[k^3 \sum_{i=0}^{k-1} H_i^4] \leq 2^3(k\gamma)^4 (1 + L_n)^4 \{\max_{i \in \{0, \ldots, k-1\}} E[W_i^4] + C_{\infty} \}. \quad (64)$$

In addition, by definition (62), $(\Delta M_i)_{i \in \mathbb{N}}$ are $(\mathcal{F}_i)_{i \in \mathbb{N}}$-martingale increments. It follows by Burkholder inequality [5, Theorem 3.2] and Young’s inequality that there exists $C_4 \geq 0$ satisfying for any $k \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}),$

$$E \left[ \max_{\ell \in \{0, \ldots, k\}} A_\ell^4 \right] \leq C_4 E[\sum_{i=0}^{k-1} (\Delta M_i)^4] \leq C_4 k \sum_{i=0}^{k-1} E[\Delta M_i^4].$$

Therefore by Lemma 39, we get that

$$E \left[ \max_{\ell \in \{0, \ldots, k\}} A_\ell^4 \right] \leq C_4 k \sum_{i=0}^{k-1} E[\Delta M_i^4] \leq C_4 (4\sigma^2 k\gamma)^2 \{m_4 + 2 \sup_{u \geq 0} [u^4 \Phi(-u)]\},$$

where $m_4$ is the fourth moment of the standard Gaussian distribution. Combining this result with (64) and using Lemma 40 in (63) concludes the proof.

5.7 Proof of Proposition 17

To show this result, we use the Komolgorov criteria [22, Corollary 14.9]: for any $T > 0$, there exist $C_T \geq 0$ such that for any $n \in \mathbb{N}$ and $s, t \in [0, T]$, $s \leq t,$

$$E \left[ |W_t^{(n)} - W_s^{(n)}|^4 \right] \leq C_T (t-s)^2.$$ 

Note that denoting $k_1^{(n)} = \lfloor s/\gamma_n \rfloor$ and $k_2^{(n)} = \lfloor t/\gamma_n \rfloor$, we have by (21)

$$E \left[ |W_t^{(n)} - W_s^{(n)}|^4 \right] \leq \begin{cases} (t-s)^4 \gamma_n^4 E[W_{k_2^{(n)+1}}^{(n)} - W_{k_1^{(n)}}^{(n)}]^4 & \text{if } k_2^{(n)} < k_1^{(n)} \\ 3^3 (t-s)^4 \gamma_n^4 E[(W_{k_2^{(n)+1}}^{(n)} - W_{k_2^{(n)}}^{(n)})^4 + (W_{k_2^{(n)}}^{(n)} - W_{k_1^{(n)-1}}^{(n)})^4] \\ + 3^3 E[(W_{k_2^{(n)}}^{(n)} - W_{k_1^{(n)}}^{(n)})^4] & \text{otherwise}. \end{cases}$$

Lemma 41, Lemma 40 and the Markov property complete the proof.
5.8 Proof of Theorem 18

Consider the differential operators $\mathcal{A}, \tilde{\mathcal{A}}$ defined for any $\psi \in C^2(\mathbb{R})$ by

$$
\mathcal{A}\psi(w) = \{\kappa(w) + c_\infty\}\psi'(w) + 2\mathbb{1}_{(0,\infty)}(w)\sigma^2\psi''(w) \quad (65)
$$

$$
\tilde{\mathcal{A}}\psi(w) = \{\kappa(w) + c_\infty\}\psi'(w) + 2\sigma^2\psi''(w),
$$

where $\kappa$ is arbitrary extended on $\mathbb{R}$. Note that $\mathcal{A}$ is the extended generator associated with (22).

A crucial step in the proof of Theorem 18 is the following.

**Proposition 42.** Let $\varphi \in C^3(\mathbb{R})$, satisfying

$$
\sup_{w \in \mathbb{R}} \{\varphi(w)/(1 + w^2) + |\varphi'(w)/(1 + |w|) + |\varphi''(w)| + |\varphi^{(3)}(w)|\} < +\infty. \quad (66)
$$

Then, for any $N \in \mathbb{N}$, $(t_1, \ldots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \cdots \leq t_N \leq s < t$, $\psi : [0, +\infty)^N \to \mathbb{R}$, nonnegative, continuous and bounded, it holds that

$$
\lim_{n \to +\infty} \frac{n}{w} \left[ \varphi(W^{(n)}_t) - \varphi(W^{(n)}_s) - \int_s^t \mathcal{A}\varphi(W^{(n)}_u)du \right] \psi(W^{(n)}_{t_1}, \ldots, W^{(n)}_{t_N}) = 0. \quad (67)
$$

If $\varphi''(w) \geq 0$ for any $w \in \mathbb{R}$, it holds that

$$
\lim_{n \to +\infty} \sup_{w \in \mathbb{R}} \left[ \varphi(W^{(n)}_t) - \varphi(W^{(n)}_s) - \int_s^t \tilde{\mathcal{A}}\varphi(W^{(n)}_u)du \right] \psi(W^{(n)}_{t_1}, \ldots, W^{(n)}_{t_N}) \leq 0. \quad (68)
$$

**Proof.** The proof is postponed to Section 5.9. \hfill \Box

Note that while Proposition 42-(67) is in general sufficient to conclude on the convergence of the sequence of processes $\{(W^{(n)}_t)_{t \geq 0} : n \in \mathbb{N}\}$ (see e.g. [19]), in our setting, it is not enough to complete the proof of Theorem 15 since the diffusion coefficient associated with $\mathcal{A}$ is discontinuous. To circumvent this issue, we adapt to our sequence the same strategy employed in [29, Proposition 6].

**Proposition 43.** Assume A 1. Let $\mu_\infty$ be a limit point of $(\mu_n)_{n \in \mathbb{N}}$. Then, $\mu_\infty$-almost everywhere, $\inf_{t \in [0, +\infty)} W_t \geq 0$.

**Proof.** Without loss of generality, we assume that $(\mu_n)_{n \in \mathbb{N}}$ converges to $\mu_\infty$. Since $\omega \mapsto \inf_{t \in [0, +\infty)} \omega_t$ is continuous, $F = \{\omega \in \mathbb{W} : \inf_{t \in [0, +\infty)} \omega_t \geq 0\}$ is closed. Therefore, by the Portmanteau theorem [23, Theorem 13.16], we obtain that $\mu_\infty(F) \geq \limsup_{n \to +\infty} \mu_n(F) = 1$. \hfill \Box

**Proof of Theorem 18.** Recall that we denote by $(\mu_n)_{n \in \mathbb{N}}$ the sequence of distribution on $\mathbb{W}$ associated with $\{(W^{(n)}_t)_{t \geq 0} : n \in \mathbb{N}\}$. Let $\mu_\infty$ be a limit point of this sequence for the convergence in distribution. Without loss of generality, we assume that $(\mu_n)_{n \in \mathbb{N}}$ converges in distribution to $\mu_\infty$. Note that by Proposition 16, for any continuous function $F : \mathbb{W} \to \mathbb{R}$ such that $|F| (\omega) \leq C_T \{1 + \sup_{t \in [0, T]} |\omega_t|^{\delta_c}\}$ for $\delta_c \in [0, 4]$, $T, C_T \geq 0$, then $F$ is uniformly integrable for $(\mu_n)_{n \in \mathbb{N}}$ and therefore (see e.g. [1, Lemma 5.1.7.])

$$
\lim_{n \to +\infty} \int_{\mathbb{W}} F d\mu_n = \int_{\mathbb{W}} F d\mu_\infty. \quad (69)
$$

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We divide then the proof into two parts. **First part:** we first show that under $\mu_\infty$, $(M_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-martingale. Considering

$$F_1 : \omega \mapsto \left( \varphi_1(\omega_t) - \varphi_1(\omega_s) - \int_s^t A\varphi_1(\omega_u)du \right) \psi(\omega_1, \ldots, \omega_t),$$

and applying Proposition 42-(67) to $\varphi_1(w) = w$ for any $w \in \mathbb{R}$, since $A\varphi_1$ is continuous under $A1$, for any $N \in \mathbb{N}$, $(t_1, \ldots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \cdots \leq t_N \leq s < t$, $\psi : \mathbb{R}_+^N \to \mathbb{R}_+$, continuous and bounded,

$$\mathbb{E}^{\mu_\infty} \left[ (M_t - M_s) \psi(W_{t_1}, \ldots, W_{t_N}) \right] = 0,$$

where $\mathbb{E}^{\mu_\infty} [\cdot]$ is the expectation under $\mu_\infty$ on $(\mathcal{W}, \mathcal{W})$. We obtain by the monotone class theorem and [30, Theorem 2.3, Chapter 0] that the first part of the result holds, i.e. $(M_t)_{t \geq 0}$ defined by (23) is a $(\mathcal{W}_t)_{t \geq 0}$-martingale on $(\mathcal{W}, \mathcal{W}, (\mathcal{W}_t)_{t \geq 0}, \mu_\infty)$.

**Second part:** It remains to show that under $\mu_\infty$, $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-martingale. We first establish setting $\varphi_2(w) = w^2$ for $w \in \mathbb{R}$, that

$$\hat{N}_t = \varphi_2(W_t) - \varphi_2(W_0) - \int_0^t A\varphi_2(W_u)du,$$

is a $(\mathcal{W}_t)_{t \geq 0}$-submartingale, which easily implies that $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-submartingale. Let $N \in \mathbb{N}$, $(t_1, \ldots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \cdots \leq t_N \leq s < t$, $\psi : \mathbb{R}_+^N \to \mathbb{R}_+$, continuous, nonnegative and bounded. Then, consider $F_2^+ = F_{2,1}^+ - F_{2,2}^+$ on $\mathcal{W}$ with :

$$F_{2,1}^+ : \omega \mapsto \left\{ \varphi_2(\omega_t) - \varphi_2(\omega_s) - 2 \int_s^t \omega_u(\kappa(\omega_u) + c_\infty)du \right\} \psi(\omega_1, \ldots, \omega_t),$$

$$F_{2,2}^+ : \omega \mapsto 4\sigma^2 \left\{ \int_s^t \mathbb{1}_{\mathbb{R}_+^+}(\omega_u)du \right\} \psi(\omega_1, \ldots, \omega_t).$$

Note that it is easy to check that $F_{2,1}^+$ is continuous and $F_{2,2}^+$ is bounded lower semi-continuous on $\mathcal{W}$, i.e. for any $(\omega^n)_{n \in \mathbb{N}}$ converging to $\omega_\infty$ in $\mathcal{W}$ endowed with the uniform convergence on compact set, $\liminf_{n \to +\infty} F_{2,2}^+(\omega^n) \geq F_{2,2}^+(\omega_\infty)$. Therefore, we obtain by the Portmanteau theorem [23, Theorem 13.16] and (69) that

$$\int_{\mathcal{W}} F_{2,1}^+ d\mu_\infty = \lim_{n \to +\infty} \int_{\mathcal{W}} F_{2,1}^+ d\mu_n, \quad \text{and} \quad \int_{\mathcal{W}} F_{2,2}^+ d\mu_\infty \leq \liminf_{n \to +\infty} \int_{\mathcal{W}} F_{2,2}^+ d\mu_n.$$

Therefore, Proposition 42-(67) applied with $\varphi \leftarrow \varphi_2$ implies that

$$0 = \limsup_{n \to +\infty} \int_{\mathcal{W}} F_{2}^+ d\mu_n \leq \int_{\mathcal{W}} F_{2}^+ d\mu_\infty.$$

Using the same arguments as before, we obtain that under $\mu_\infty$, $(\hat{N}_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-submartingale. Then, it is easy to verify that $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-submartingale. We complete then the proof by showing that $(N_t)_{t \geq 0}$ is also a $(\mathcal{W}_t)_{t \geq 0}$-supermartingale under $\mu_\infty$. To do so, we need the following lemma.

**Lemma 44.** Assume A1. Then, for any limit point $\mu_\infty$ of $(\mu_n)_{n \in \mathbb{N}}$, $\mu_\infty$-almost everywhere, $t \mapsto (M)_t - 4\sigma^2 t$ is nonincreasing, where $(M_t)_{t \geq 0}$ is the quadratic variation of $(M_t)_{t \geq 0}$.  

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Proof. Let $N \in \mathbb{N}$, $(t_1, \ldots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \cdots \leq t_N \leq s < t$, $\psi : \mathbb{R}_+^N \rightarrow \mathbb{R}$, continuous, nonnegative and bounded. Consider now the continuous map

$$F_2 : \omega \mapsto \left\{ \varphi_2(\omega_t) - \varphi_2(\omega_s) - \int_s^t \tilde{A}\varphi_2(\omega_u)\,du \right\} \psi(\omega_1, \ldots, \omega_{t_N}) .$$

Then, by (69) and Proposition 42-(68), we get $\lim_{n \rightarrow \infty} \int_\Omega F_2 \, d\mu_n \leq 0$. Using that under $\mu_\infty \ (M_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-martingale, we get that $(M_t^2 - 4\sigma^2t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-supermartingale. By the Doob-Meyer decomposition [22, Theorem 22.5], under $\mu_\infty$, there exists a unique nondecreasing, locally integrable and predictable process $(C_t)_{t \geq 0}$, such that $(M_t^2 - 4\sigma^2t + C_t)_{t \geq 0}$ is a local $(\mathcal{W}_t)_{t \geq 0}$-martingale. In addition, under $\mu_\infty$, by [30, Theorem 1.8, Chapter IV], the quadratic variation $(\langle M_t \rangle)_{t \geq 0}$ of $(M_t)_{t \geq 0}$ is a finite variation process satisfying $(M_t^2 - \langle M_t \rangle)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-martingale. Therefore, $(M_t - 4\sigma^2t - C_t)_{t \geq 0}$ is a local $(\mathcal{W}_t)_{t \geq 0}$-martingale and a finite variation process. By [30, Proposition 1.2, Chapter IV], $\mu_\infty$-almost everywhere, for any $t \in [0, +\infty)$, $(M_t - 4\sigma^2t + C_t = 0$, which completes the proof.

By Lemma 44, denoting by $(\langle M_t \rangle)_{t \geq 0}$, the quadratic variation of $(M_t)_{t \geq 0}$, see [30, Theorem 1.8, Chapter IV], $\mu_\infty$-almost everywhere, $t \mapsto \langle M_t \rangle - 4\sigma^2t$ is nonincreasing and therefore we get that for any $s, t \in [0, +\infty)$, $s \leq t$, $\mu_\infty$-almost everywhere,

$$\int_s^t \mathbb{1}_{\mathbb{R}_+^t}(W_u)d\langle M \rangle_u \leq 4\sigma^2\int_s^t \mathbb{1}_{\mathbb{R}_+^t}(W_u)\,du .$$

(70)

In addition, by the occupation times formula [30, Corollary 1.6, Chapter VI] applied twice and Proposition 43, $\mu_\infty$-almost everywhere,

$$\langle M \rangle_t = \int_0^t \langle M \rangle_u d\langle W \rangle_u = \int_0^t \mathbb{1}_{\mathbb{R}_+^t}(W_u)d\langle W \rangle_u = \int_{\mathbb{R}_+^t} L_u^q\,da = \int_{\mathbb{R}_+^t} L_u^q\,da = \int_0^t \mathbb{1}_{\mathbb{R}_+^t}(W_u)d\langle M \rangle_u .$$

Using this result and (70), we get that $\langle M \rangle_t - \langle M \rangle_s \leq 4\sigma^2\int_s^t \mathbb{1}_{\mathbb{R}_+^t}(W_u)\,du$, for any $s, t \in [0, +\infty)$, $s \leq t$. Therefore since $(M_t^2 - \langle M_t \rangle)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-martingale under $\mu_\infty$, we conclude that $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$-supermartingale which completes the proof.

5.9 Proof of Proposition 42

We preface the proof by the following technical lemma.

Lemma 45. Assume A1. Then, for any $q \in [1, +\infty)$, we have

(a) for any $\gamma \in (0, \bar{\gamma}]$ and $w_0 \in [0, +\infty)$,

$$-2(4\sigma^2 \gamma)^{q/2} \int_{\hat{\tau}_{\gamma}^\infty(w_0)}^{+\infty} |g|^q \varphi(g)\,dg \leq \int_{\mathbb{R}_+^t} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) - (4\sigma^2 \gamma)^{q/2}m_q \leq 0 ,$$

(71)

where $\hat{\tau}_{\gamma}^\infty(w_0) = \{\tau_\gamma(w_0) + \gamma c_\infty\}/(2\sqrt{\sigma^2 \gamma})$, $Q_\gamma$ is defined by (14), $m_q$ is the $q$-th moment of the standard Gaussian distribution and $\varphi$ is its probability density function;

(b) for any $\gamma \in (0, \bar{\gamma}]$,

$$\int_{\mathbb{R}_+^t} |w_1 - \gamma c_\infty|^q Q_\gamma(0, dw_1) \leq 3(\gamma c_\infty)^q + (2\gamma c_\infty)^q + q m_{q-1}(4\sigma^2 \gamma)^{q/2} \gamma^{1/2}c_\infty/\sigma .$$
Proof. (a) Let \( w_0 \in [0, +\infty) \) and \( \gamma \in (0, \bar{\gamma}] \). By definition (14) and (11), we have setting \( \bar{\tau}_\gamma(w_0) = \{ \tau_\gamma(w_0) + \gamma c_\infty \}/(2\sqrt{\sigma^2 \gamma}) \),

\[
\int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) = (4\sigma^2 \gamma)^{q/2} \int_{-\infty}^{\bar{\tau}_\gamma(w_0)} |g|^q \varphi(g)dg - (4\sigma^2 \gamma)^{q/2} \int_{\bar{\tau}_\gamma(w_0)}^{+\infty} |g - 2\bar{\tau}_\gamma(w_0)|^q \varphi(g)dg + \int_{\mathbb{R}} |\tau_\gamma(w_0) + \gamma c_\infty|^q \varphi(2\bar{\tau}_\gamma(w_0) - g) \wedge \varphi(g)dg .
\]

Therefore, we obtain that

\[
\int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) - (4\sigma^2 \gamma)^{q/2} m_q = -(4\sigma^2 \gamma)^{q/2} \int_{-\infty}^{\bar{\tau}_\gamma(w_0)} |g|^q \varphi(g)dg - (4\sigma^2 \gamma)^{q/2} \int_{\bar{\tau}_\gamma(w_0)}^{+\infty} |g - 2\bar{\tau}_\gamma(w_0)|^q \varphi(g)dg + 2(4\sigma^2 \gamma)^{q/2} \{ \bar{\tau}_\gamma(w_0) \} q \int_{\bar{\tau}_\gamma(w_0)}^{+\infty} \varphi(g)dg .
\]

Using that for \( g \in [\bar{\tau}_\gamma(w_0), +\infty) \), \( \{ \bar{\tau}_\gamma(w_0) \} q \leq 2^{-1} |g|^q + |2\bar{\tau}_\gamma(w_0) - g|^q \) and \( |2\bar{\tau}_\gamma(w_0) - g| \leq |g| \) completes the proof.

(b) By (72) and since \( \tau_\gamma(0) = 0 \) and \( \bar{\tau}_\gamma(0) = \gamma c_\infty/(2\sqrt{\sigma^2 \gamma}) \),

\[
\left| \int_{\mathbb{R}_+} |w_1 - \tau_\gamma(0) - \gamma c_\infty|^q Q_\gamma(0, dw_1) \right| \leq 2(4\sigma^2 \gamma)^{q/2} \int_{0}^{\gamma c_\infty/(2\sqrt{\sigma^2 \gamma})} |g|^q \varphi(g)dg + (4\sigma^2 \gamma)^{q/2} \int_{0}^{\gamma c_\infty/(2\sqrt{\sigma^2 \gamma})} |g|^q \varphi(g)dg + (4\sigma^2 \gamma)^{q/2} \int_{0}^{\gamma c_\infty/(2\sqrt{\sigma^2 \gamma})} |g|^q \varphi(g)dg + (4\sigma^2 \gamma)^{q/2} \int_{0}^{\gamma c_\infty/(2\sqrt{\sigma^2 \gamma})} \left[ (-g)^q - \left( -\frac{\gamma c_\infty}{\sqrt{\sigma^2 \gamma}} - g \right)^q \right] \varphi(g)dg + \gamma c_\infty)^q .
\]

Using that \( (4\sigma^2 \gamma)^{q/2} \int_{0}^{\gamma c_\infty/(2\sqrt{\sigma^2 \gamma})} |g|^q \varphi(g)dg \leq (\gamma c_\infty)^q \) and \( a^q - (a - h)^q \leq qa^{q-1}h \) for \( a, h \geq 0, a \geq h \), completes the proof.

\[ \square \]

Proof of Proposition 42. Proof of (67). Let \( \varphi \in C^\infty(\mathbb{R}^d) \) satisfying (66), \( (s, t) \in [0, +\infty)^2, s < t \). Note that we only need to show that

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \mathbb{E} \left[ \varphi(W^{(n)}_t) - \varphi(W^{(n)}_s) - \int_s^t A\varphi(W^{(n)}_u)du \right| \mathcal{G}^{(n)}_s \right] \right] = 0 ,
\]

setting for any \( n \in \mathbb{N} \), \( u \in [0, +\infty) \), \( \mathcal{G}^{(n)}_u = \mathcal{F}^{(n)}_{[u/\gamma_n]} \), where \( (\mathcal{F}^{(n)}_k)_{k \in \mathbb{N}} \) is the filtration corresponding to \( (W^{(n)}_k)_{k \in \mathbb{N}} \).
Define $k_1^{(n)} = [t/\gamma_n]$ and $k_2^{(n)} = [s/\gamma_n]$ and consider the following decomposition

$$
E \left[ \varphi(W_t^{(n)}) - \varphi(W_s^{(n)}) - \int_s^t A\varphi(W_u^{(n)}) du \right] G_s^{(n)} = E \left[ A_1^{(n)} + A_2^{(n)} + A_3^{(n)} \right] G_s^{(n)} \tag{73}
$$

$$
A_1^{(n)} = \varphi(W_t^{(n)}) - \varphi(W_s^{(n)}) - \{ \varphi(W_k^{(n)}) - \varphi(W_{k-1}^{(n)}) \}
$$

$$
A_2^{(n)} = - \int_s^t A\varphi(W_u^{(n)}) du + \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} A\varphi(W_k^{(n)})
$$

$$
A_3^{(n)} = \varphi(W_k^{(n)}) - \varphi(W_{k-1}^{(n)}) - \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} A\varphi(W_k^{(n)})
$$

We deal with these three terms separately.

First, since $\varphi$ satisfies (66), by the fundamental theorem of calculus, there exists $C \geq 0$ such that for any $w_0, w_1 \in \mathbb{R}$, $|\varphi(w_1) - \varphi(w_0)| \leq C (1+ \max (|w_0|, |w_1|)) |w_0 - w_1|$. By (21), Cauchy–Schwarz inequality and Lemma 41, we get that there exists $C \geq 0$ such that for any $n \in \mathbb{N}$,

$$
E[|A_1^{(n)}|^4] \leq E[|A_1^{(n)}|^2]^2 \leq C \gamma_n^2 \max_{i \in \{k_1^{(n)}-1, k_2^{(n)}-1, k_2^{(n)}\}} E[|W_i^{(n)}|^4] + 1 \right)^2.
$$

This implies by Lemma 40 that

$$
\lim_{n \to +\infty} E[|A_1^{(n)}|] = 0. \tag{74}
$$

Regarding $A_2^{(n)}$, we consider the decomposition,

$$
A_2^{(n)} = A_{2,1}^{(n)} + A_{2,2}^{(n)}
$$

$$
A_{2,1}^{(n)} = - \int_s^{k_2^{(n)}} A\varphi(W_u^{(n)}) du + \int_t^{k_1^{(n)}-1} A\varphi(W_u^{(n)}) du
$$

$$
A_{2,2}^{(n)} = \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \{ A\varphi(W_u^{(n)}) - A\varphi(W_k^{(n)}) \} du.
$$

Since $\varphi$ satisfies (66) and by Lemma 40, we get that $\lim_{n \to +\infty} E[|A_{2,1}^{(n)}|] = 0$. In addition, we have by definition of $A$ (65) that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\}$, $u \in (k\gamma_n, (k+1)\gamma_n)$,

$$
|A\varphi(W_u^{(n)}) - A\varphi(W_k^{(n)})| \leq B_{u,k}^{(n)} + 2A_k \sigma^2 \sup_{W_k^{(n)}} |\varphi''|
$$

$$
B_{u,k}^{(n)} = (|\kappa(W_k^{(n)})| + c_{\infty}) |\varphi(W_u^{(n)}) - \varphi(W_k^{(n)})|
$$

$$
+ |\varphi'(W_u^{(n)})| |\kappa(W_k^{(n)}) - \kappa(W_k^{(n)})| + 2A_k \sigma^2 |\varphi'(W_u^{(n)}) - \varphi'(W_k^{(n)})|,
$$

where $A_k = \{W_k^{(n)} = 0, W_{k+1}^{(n)} \neq 0\}$. Note that using that $\varphi'$ and $\varphi''$ are Lipschitz and $\sup_{u \in [0, + \infty)} |\varphi'(w)/(1+|w|) < +\infty$ by (66), A1, (21), Cauchy–Schwarz inequality, Lemma 40
and Lemma 41, we get that there exists $C > 0$ such that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\}$, $u \in (k\gamma_n, (k + 1)\gamma_n)$,

$\mathbb{E}[|B_{u,k}^{(n)}|^4] \leq \mathbb{E}[|B_{u,k}^{(n)}|^2] \leq C \gamma_n^2 \mathbb{E}[|W_0^{(n)}|^4] + 1^2$,

which implies that

$$\lim_{n \to +\infty} \sum_{k = k_2^{(n)}}^{k_1^{(n)} - 1} \int_{k\gamma_n}^{(k+1)\gamma_n} \mathbb{E}[B_{u,k}^{(n)}] \, du = 0.$$ 

To conclude that $\lim_{n \to +\infty} \mathbb{E}[|A_2^{(n)}|] = 0$, it remains to show that

$$\lim_{n \to +\infty} \gamma_n \sum_{k = k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{E}[\mathbb{I}_{A_k}] = 0. \quad (75)$$

Note that using that by definition, $(W_k^{(n)})_{k \in \mathbb{N}}$ is a Markov chain with Markov kernel $Q_{\gamma_n}$ (14), the Markov property and Lemma 28 implies that for any $n \in \mathbb{N}$ and $k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\}$,

$$\mathbb{E}[\mathbb{I}_{A_k}] = \mathbb{P}\left(W_k^{(n)} = 0, W_{k+1}^{(n)} \neq 0\right) \leq 1 - 2\Phi\left[-c\sqrt{\gamma_n}/(2\sigma)\right]. \quad (76)$$

Since $1 - 2\Phi(-u) \leq u$ for any $u \in [0, +\infty)$, we get that there exists $C > 0$ such that for any $n \in \mathbb{N}$, $\gamma_n \sum_{k = k_2^{(n)}}^{k_1^{(n)} - 1} 1 - 2\Phi\left[-c\sqrt{\gamma_n}/(2\sigma)\right] \leq C((t - s) + \gamma_n)\gamma_n^{1/2}$ and therefore

$$\lim_{n \to +\infty} \gamma_n \sum_{k = k_2^{(n)}}^{k_1^{(n)} - 1} \{1 - 2\Phi\left[-c\sqrt{\gamma_n}/(2\sigma)\right]\} = 0.$$ 

This result combined with (76) in (75) shows that

$$\lim_{n \to +\infty} \mathbb{E}[|A_2^{(n)}|] = 0. \quad (77)$$

Finally we deal with $A_3^{(n)}$ from the decomposition

$$A_3^{(n)} = \sum_{k = k_2^{(n)}}^{k_1^{(n)} - 1} \varphi(W_{k+1}^{(n)}) - \varphi(W_k^{(n)}) - \gamma_n A \varphi(W_k^{(n)}).$$

Set for any $k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\}$, $\Delta W_k^{(n)} = W_{k+1}^{(n)} - W_k^{(n)}$. Using that $\varphi$ is three times continuously differentiable, we get by Taylor’s theorem with Lagrange reminder, that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\}$, there exists $u_k \in [0, 1]$ satisfying

$$\varphi(W_{k+1}^{(n)}) - \varphi(W_k^{(n)}) = \varphi'(W_k^{(n)}) \Delta W_k^{(n)} + (\varphi''(W_k^{(n)})/2) \{\Delta W_k^{(n)}\}^2 + 6^{-1} \varphi^{(3)}(u_k W_k^{(n)} + (1 - u_k) W_k^{(n)}) \{\Delta W_k^{(n)}\}^3.$$
It follows from the definition (13), A1 and Young’s inequality, setting \( \kappa^\infty(w) = \kappa(w) + c_\infty \) that for any \( n \in \mathbb{N} \) and \( k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\} \),

\[
|\Delta W_{k+1}^{(n)}|^3 \leq 4\{\tau_n^3 \kappa^\infty(W_k^{(n)})|^3 + |W_k^{(n)} - (\tau_n(W_k^{(n)}) + \gamma_n c_\infty)|^3\}.
\]

It follows then using the definition of A (65), (66), Lemma 40 and Lemma 39 that

\[
\lim_{n \to +\infty} \mathbb{E}[|A_3^{(n)} - A_3^{(n)}|] = 0 , \quad \text{where } A_3^{(n)} = A_{3,1}^{(n)} + A_{3,2}^{(n)} , \quad (78)
\]

\[
A_{3,1}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} [\varphi'(W_k^{(n)})\Delta W_{k+1}^{(n)} - \gamma_n \varphi'(W_k^{(n)})\kappa^\infty(W_k^{(n)})] \]

\[
A_{3,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} [(\varphi''(W_k^{(n)})/2)\{\Delta W_{k+1}^{(n)}\}^2 - 2\gamma_n \sigma^2 \varphi''(W_k^{(n)})1_{\mathbb{R}_+'}(W_k^{(n)})] . \]

Note that by Lemma 25 and the Markov property, we have that for any \( n \in \mathbb{N} \) and \( k \in \{k_2^{(n)}, \ldots, k_1^{(n)} - 1\} \), \( \varphi'(W_k^{(n)})\mathbb{E}[\Delta W_{k+1}^{(n)} | \mathcal{F}_k^{(n)}] - \gamma_n \varphi'(W_k^{(n)})\kappa^\infty(W_k^{(n)}) = 0 \), which implies that

\[
\mathbb{E}[A_{4,1}^{(n)} | \mathcal{G}_s^{(n)}] = \mathbb{E}[A_{4,1}^{(n)} | \mathcal{F}_k^{(n)}] \]

\[
= \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E}[\varphi'(W_k^{(n)})\mathbb{E}[\Delta W_{k+1}^{(n)} | \mathcal{F}_k^{(n)}] - \gamma_n \varphi'(W_k^{(n)})\kappa^\infty(W_k^{(n)}) | \mathcal{F}_k^{(n)}] = 0 . \quad (79)
\]

We now show that \( \lim_{n \to +\infty} \mathbb{E}[|A_{4,2}^{(n)} | \mathcal{G}_s^{(n)}]| = 0 \) using the decomposition

\[
A_{4,2}^{(n)} = A_{4,2,1}^{(n)} + A_{4,2,2}^{(n)} + A_{4,2,3}^{(n)} , \quad (80)
\]

\[
A_{4,2,1}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} 1_{\mathbb{R}_+'}(W_k^{(n)})(\varphi''(W_k^{(n)})/2)[\varphi''(W_k^{(n)})/2 - 4\sigma^2 \gamma_n] \]

\[
A_{4,2,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} 1_{\{0\}}(W_k^{(n)})(\varphi''(0)/2)[\varphi''(W_k^{(n)})/2] \]

\[
A_{4,2,3}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} (\varphi''(W_k^{(n)})/2)[\{\Delta W_{k+1}^{(n)}\}^2 - \{\Delta W_{k+1}^{(n)}\}^2] ,
\]

where \( \tilde{\Delta} W_{k+1}^{(n)} = W_{k+1}^{(n)} - (\tau_n(W_k^{(n)}(\omega)) + \gamma_n c_\infty) \) and \( M \in \mathbb{R}_+^* \).

We first consider \( A_{4,2,2}^{(n)} \). By Lemma 45-(b) and the Markov property, we have

\[
|\mathbb{E}[A_{4,2,2}^{(n)} | \mathcal{G}_s^{(n)}]| = |\mathbb{E}[A_{4,2,2}^{(n)} | \mathcal{F}_k^{(n)}]| \leq (\varphi''(0)/2)[(t-s) + \gamma_n(7\gamma_n c_\infty^2 + 8\sigma^2 \gamma_n^2/2 c_\infty] ,
\]

showing that

\[
\lim_{n \to +\infty} \mathbb{E}[|A_{4,2,2}^{(n)} | \mathcal{G}_s^{(n)}]| = 0 . \quad (81)
\]
Then, by Lemma 45-(a), we have using the Markov property that
\[ \mathbb{E}[|A_{4,2,3}^{(n)}|] \leq \sup_{\mathbb{R}} \{|\varphi''|/2\} \mathbb{E} \left[ \sum_{k=k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{E} \left[ \{\Delta W_{k+1}^{(n)}\}^2 - \{\Delta W_{k+1}^{(n)}\}^2 \mid \mathcal{F}_k^{(n)} \right] \right]. \quad (82) \]

In addition by Lemma 25, the Markov property and A1, we have for any \( k \in \mathbb{N} \),
\[ \mathbb{E} \left[ \{\Delta W_{k+1}^{(n)}\}^2 - \{\Delta W_{k+1}^{(n)}\}^2 \mid \mathcal{F}_k^{(n)} \right] = \left| \gamma_n \kappa \infty \left( W_k^{(n)} \right) \mathbb{E} \left[ 2\Delta W_{k+1}^{(n)} - \gamma_n \kappa \infty \left( W_k^{(n)} \right) \mid \mathcal{F}_k^{(n)} \right] \right| \]
\[ = (\gamma_n \kappa \infty \left( W_k^{(n)} \right))^2 \]
\[ \leq 2\gamma_n^2 \left[ (1 + L \kappa)^2 \{W_k^{(n)}\}^2 + \epsilon_0^2 \right]. \]

Then by Lemma 40, there exists \( C \geq 0 \) such that for any \( k \leq k_1^{(n)} - 1 \) and \( n \in \mathbb{N}^* \),
\[ \mathbb{E} \left[ \mathbb{E} \left[ \{\Delta W_{k+1}^{(n)}\}^2 - \{\Delta W_{k+1}^{(n)}\}^2 \mid \mathcal{F}_k^{(n)} \right] \right] \leq 2\gamma_n^2 \left[ (1 + L \kappa)^2 c^{2}\sqrt{\mathbb{E}[W_0^4]} + 1 + \epsilon_0^2 \right]. \]

Combining this with (82) and using that \( \gamma_n(k_1^{(n)} - k_2^{(n)}) \leq (t-s+\gamma_n) \) we obtain
\[ \lim_{n \to +\infty} \mathbb{E}[|A_{4,2,3}^{(n)}|] = 0. \quad (83) \]

Regarding the first term in (80), we consider for \( M > 0 \), the following decomposition
\[ A_{4,2,1}^{(n)} = D_1^{(n)}(M) + D_2^{(n)}(M), \]
where
\[ D_1^{(n)}(M) = \sum_{k=k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{I}_{(0,M \sqrt{\gamma_n})}(W_k^{(n)}) (\varphi''(W_k^{(n)})/2) \{\Delta W_{k+1}^{(n)}\}^2 - 4\sigma^2 \gamma_n \]
\[ D_2^{(n)}(M) = \sum_{k=k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{I}_{(M \sqrt{\gamma_n},+\infty)}(W_k^{(n)}) (\varphi''(W_k^{(n)})/2) \{\Delta W_{k+1}^{(n)}\}^2 - 4\sigma^2 \gamma_n. \]

Then, by Lemma 45-(a), we have using the Markov property that
\[ \mathbb{E}[|D_1^{(n)}(M)|] \leq \sup_{\mathbb{R}} \{|\varphi''|/2\} \mathbb{E} \left[ \sum_{k=k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{I}_{(0,M \sqrt{\gamma_n})}(W_k^{(n)}) \mathbb{E} \left[ \{\Delta W_{k+1}^{(n)}\}^2 \mid \mathcal{F}_k^{(n)} \right] - 4\sigma^2 \gamma_n \right] \]
\[ \leq 4\gamma_n \sigma^2 \sup_{\mathbb{R}} \{|\varphi''|\} \sum_{k=k_2^{(n)}}^{k_1^{(n)} - 1} \mathbb{E} \left[ \mathbb{I}_{(0,M \sqrt{\gamma_n})}(W_k^{(n)}) \mathcal{G}(\gamma_n \infty (W_k^{(n)}(\omega))) \right], \quad (84) \]
where \( \Upsilon(u) = \int_u^{+\infty} |g|^2 \varphi(g)dg, \quad \bar{\tau}_r^\infty(W_k^{(n)}(\omega)) = [\tau_{n}(W_k^{(n)}(\omega)) + \gamma_n c_n] / \{2(\sigma^2 \gamma_n)^{1/2}\}. \) In addition by Lemma 28, A1 and using that for any \( x \in \mathbb{R}_+ \), \( 1 - 2\Phi(-x) \leq \sqrt{2/\pi}x \), we have for any \( w \in \mathbb{R}_+ \)

\[
Q_{\gamma_n} 1_{\mathbb{R}_+^\ast}(w) = 1 - 2\Phi \left( \frac{-\tau_{n}(w) + \gamma_n c_n}{2\sigma \sqrt{\gamma_n}} \right)
\leq 1_{\mathbb{R}_+^\ast}(w) - 2\Phi \left( \frac{(1 + L_n)M + \sqrt{\gamma_n} c_n}{2\sigma} \right) 1_{(0,M,\sqrt{\gamma})}(w) + (2\pi\sigma^2)^{-1/2} \sqrt{\gamma_n} c_n.
\]

Therefore for any \( l \in \mathbb{N}^* \),

\[
\sum_{k=0}^{l-1} Q_{\gamma_n} 1_{(0,M,\sqrt{\gamma})}(w) \leq 2^{-1}\Phi \left( \frac{(1 + L_n)M + \sqrt{\gamma_n} c_n}{2\sigma} \right) \left[ 1_{\mathbb{R}_+^\ast}(w) + l(2\pi\sigma^2)^{-1/2} \sqrt{\gamma_n} \right].
\]

Using that \( k_1^{(n)} - k_2^{(n)} \leq 1 + (t - s)/\gamma_n \), we have

\[
\gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E} \left[ 1_{(0,M,\sqrt{\gamma_n})}(W_k^{(n)}) \right] F_k^{(n)} = \gamma_n \sum_{k=0}^{k_1^{(n)}-k_2^{(n)}-1} Q_{\gamma_n} 1_{(0,M,\sqrt{\gamma})}(W_k^{(n)})
\leq 2^{-1}\Phi \left( \frac{(1 + L_n)M + \sqrt{\gamma_n} c_n}{2\sigma} \right) \left[ \gamma_n 1_{\mathbb{R}_+^\ast}(W_{k_2^{(n)}}^{(n)}) + (\gamma_n + t - s)(2\pi\sigma^2)^{-1/2} \sqrt{\gamma_n} \right].
\]

Combining (85) with (84) and using that \( \Upsilon(u) \leq \int_0^{+\infty} |g|^2 \varphi(g)dg \) for any \( u \in \mathbb{R}_+ \) we obtain that for any \( M \in \mathbb{R}_+^* \),

\[
\lim_{n \to +\infty} \mathbb{E}[|D_1^{(n)}(M)|] = 0.
\]

We now consider \( D_2^{(n)} \). Similarly to (84) we obtain

\[
\mathbb{E}[|D_2^{(n)}(M)|] \leq 4\gamma_n \sigma^2 \sup_{\mathbb{R}}|\varphi''| \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E} \left[ 1_{[M,\sqrt{\gamma_n},+\infty)}(W_k^{(n)}) \Upsilon(\bar{\tau}_r^\infty(W_k^{(n)}(\omega))) \right].
\]

In addition there exists \( N_1 \) such that for \( n \geq N_1 \), \( 1 - \gamma_n L_n > 1/2 \), which implies by A1 that for any \( n \geq N_1 \),

\[
1_{[M,\sqrt{\gamma_n},+\infty)}(W_k^{(n)}) \Upsilon(\bar{\tau}_r^\infty(W_k^{(n)}(\omega))) \leq 1_{[M,\sqrt{\gamma_n},+\infty)}(W_k^{(n)}) (\bar{\tau}_r^\infty(W_k^{(n)}(\omega)))^{-1} \sup_{t \in \mathbb{R}_+^\ast} t \Upsilon(t)
\leq 4\sigma \sup_{t \in \mathbb{R}_+^\ast} t \Upsilon(t)/M.
\]

Note that by using Cauchy–Schwarz inequality we have \( \Upsilon(t)^2 \leq \Phi(t)m_1 \), therefore \( \sup_{t \in \mathbb{R}_+^\ast} t \Upsilon(t) < +\infty \). Combining this with (87) and using that \( \gamma_n (k_1^{(n)} - k_2^{(n)}) \leq (t - s + \gamma_n) \) we obtain that for any \( M \in \mathbb{R}_+^* \),

\[
\limsup_{n \to +\infty} \mathbb{E}[|D_2^{(n)}(M)|] \leq 16\sigma^3 \sup_{\mathbb{R}}|\varphi''| \sup_{t \in \mathbb{R}_+^\ast} t \Upsilon(t)(t - s)/M.
\]
Then by (86) and (88),
\[
\limsup_{n \to +\infty} \mathbb{E}[A_{4,2,1}^{(n)}(M)] \leq \limsup_{n \to +\infty} \mathbb{E}[D_1^{(n)}(M)] + \mathbb{E}[D_2^{(n)}(M)] = 0.
\]  
(89)
Combining this result, (89)-(83)-(81)-(80)-(79)-(78), we get that
\[
\lim_{n \to +\infty} \mathbb{E}[\mathcal{A}_3^{(n)} | \mathcal{G}_s^{(n)}] = 0.
\]  
Plugging this result and (77)-(74) in (73) completes the proof.

**Proof of (68).** The proof follows exactly the same lines as (67) but we use that the only different and non negligible terms are \(A_{4,2,1}^{(n)}\) and \(A_{4,2,2}^{(n)}\) which becomes
\[
A_{4,2,1}^{(n)} + A_{4,2,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} (\varphi''(W_k^{(n)})/2)[\{\Delta W_{k+1}^{(n)}\}^2 - 4\gamma_n\sigma^2].
\]
Using (71) in Lemma 45, the assumption that \(\varphi''(w) \geq 0\) for any \(w \in \mathbb{R}\), and the Markov property, we get that for any \(n \in \mathbb{N}\), \(\mathbb{E}[A_{4,2,1}^{(n)} + A_{4,2,2}^{(n)} | \mathcal{G}_s] \leq 0\), which concludes the proof. \(\square\)

5.10 Postponed proofs of Section 4

**Proof of Proposition 20.** From Ao3 we know that \(z_\theta(t) \in K\) for all \(t \in [0,T]\). In particular, using that for all \(0 \leq s \leq t \leq T\), \(z_\theta(t) - z_\theta(s) = \int_s^t F_\theta(z_\theta(u),u)du\), we get that
\[
\|z_\theta(t) - z_\theta(s)\| \leq C_F(t-s).
\]  
(90)
Let \(k \in \mathbb{N}\) be such that \(\tilde{z}_\theta^h(kh) \in K\) (this is for instance the case of \(k = 0\)). Then, for \(t \in [kh,(k+1)h]\), using by (30) that
\[
z_\theta(t) - \tilde{z}_\theta^h(t) = z_\theta(kh) - \tilde{z}_\theta^h(kh) + \int_{kh}^t \left(F_\theta(z_\theta(s),s) - F_\theta(\tilde{z}_\theta^h(kh),kh)\right)ds,
\]
and setting \(f(t) = \|z_\theta(t) - \tilde{z}_\theta^h(t)\|\), we get by (90) and Ao3 for any \(h > 0\) and \(t \in [kh,(k+1)h]\),
\[
f(t) \leq (1 + L_F'h)f(kh) + L_F'(1 + C_F') \frac{h^2}{2}.
\]
Assuming that \(h \leq \tilde{h}\) where \(\tilde{h}\) is sufficiently small so that
\[
\frac{1}{2}L'_F(1 + C_F)hTE^{L'_F T} < \delta,
\]
we get by a direct induction that, for all \(k \in \{0, \ldots, \lfloor T/h \rfloor\}\), \(\tilde{z}_\theta^h(kh) \in K\) and for all \(t \in [0,T]\),
\[
f(t) \leq \frac{1}{2}L'_F(1 + C_F)hTE^{L'_F T}
\]
Conclusion follows with
\[
C = \frac{1}{2}L'_F(1 + C_F)NTe^{L'_F T}.
\]  
\(\square\)
Proof of Proposition 22. We have by \( AO2 \) and (35), \( T_\gamma(\theta) = \theta - \gamma \nabla U(\theta) \) with

\[
\nabla U(\theta) = \nabla U_0(\theta) + \sum_{i=1}^N s_i(z_\theta(t_i))
\]

(91)

where \( z_\theta \) solves (28) and \( s_i \) are defined in (35). For all \( \theta, \tilde{\theta} \in \mathbb{R}^d \) and \( t \in [0, T] \),

\[
z_\theta(t) - z_{\tilde{\theta}}(t) = \int_0^t (F_\theta(z_\theta(s), s) - F_{\tilde{\theta}}(z_{\tilde{\theta}}(s), s)) \, ds
\]

and thus, using (31), Grönwall’s inequality implies that

\[
\|z_\theta(t) - z_{\tilde{\theta}}(t)\| \leq L_F \|\theta - \tilde{\theta}\| t e^{L_F t} \quad \forall t \geq 0.
\]

In particular, by (91) and (35),

\[
\|\nabla U(\theta) - \nabla U(\tilde{\theta})\| \leq \left( L_U + L_N L_F \sum_{i=1}^N e^{L_U t_i} \right) \|\theta - \tilde{\theta}\|.
\]

Moreover, similarly, we get if \( \|\theta - \tilde{\theta}\| \geq R_U \),

\[
\langle \theta - \tilde{\theta}, \nabla U(\theta) - \nabla U(\tilde{\theta}) \rangle \geq m_U \|\theta - \tilde{\theta}\|^2 - 2 N C_s \|\theta - \tilde{\theta}\|.
\]

Combining the last two estimates yields H. Finally, H2 follows using \( AO4, (33) \) and (35) from

\[
\|\nabla U(\theta) - \tilde{b}_h(\theta)\| \leq L_N \sum_{i=1}^N \|z_\theta(t_i) - \Psi^h_i(\theta)\| \leq C_N L_N h^\alpha.
\]

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Recall that under $\textbf{H1}$, we have for any $\gamma \in (0, \bar{\gamma}]$,
\[
\sup_{x \in \mathbb{R}^d, x \neq 0} \{ \| T_\gamma(x) - T_\gamma(0) \| / \| x \| \} \leq (1 + \gamma L), \\
\sup_{x \in \mathbb{R}^d, \| x \| \geq R_1} \{ \| T_\gamma(x) - T_\gamma(0) \| / \| x \| \} \leq (1 - \gamma m). \tag{92}
\]

We show that there exist $\lambda \in (0, 1)$, $A \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathbb{R}^d$,
\[
R_\gamma V_c(x) \leq \lambda^3 V_c(x) + \gamma A. \tag{93}
\]
Denote $T_\infty = \sup_{\gamma \in (0, \gamma]} \gamma^{-1} \|T_\gamma(0)\|$, we show that it holds with

\[
\lambda = \exp(-cM^2/8), \quad M = R_1 \lor (16d\sigma^2/m)^{1/2} \lor \left[ 4T_\infty + 2\gamma T_\infty^2 \right] / m,
\]

\[
A = \exp(cM^2 + \bar{\gamma}\{cB_1M^2 + B_2 - \log(\lambda)\}\{cB_1M^2 + B_2 - \log(\lambda)\}),
\]

\[
B_1 = 4C_1 + 2\left(1 + 8c^2\gamma^2\right)\left(1 + \gamma L\right)T_\infty, \quad C_1 = C_2 \lor C_2 \lor \gamma, \quad C_2 = \left(2L + L^2\right) \lor (8c^2), \quad c = m/(32\sigma^2), \quad B_2 = 2dc^2 + 2\left(1 + 8c^2\bar{\gamma}^2\right)\left(1 + \gamma L\right)T_\infty + c\left(1 + 8c^2\bar{\gamma}\right)\gamma T_\infty^2.
\]

Define for any $x \in \mathbb{R}^d$, $T_\gamma(x) = T_\gamma(x) - T_\gamma(0)$. Note that for any $\gamma \in (0, \gamma]$, $2c\gamma^2 \gamma < 1$ by definition of $c$ (94) and using that $\gamma \ll 1/m$. Let $x \in \mathbb{R}^d$ and $\gamma \in (0, \gamma]$. Then, we obtain using that $\int_\mathbb{R} e^{ax + bx^2 - x^2/2}dz = (2\pi(1 - 2b)^{-1})^{1/2}e^{a^2/(2(1 - 2b))}$ for any $a \in \mathbb{R}$ and $b \in [0, 1/2)$,

\[
R_\gamma V_c(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(c\|T_\gamma(x) + (\sigma^2\gamma)^{1/2}z\|^2 - \|z\|^2/2)dz
= (1 - 2c^2\gamma)^{-d/2} \exp\left\{c\left(1 - 2c^2\gamma\right)^{-1}\|T_\gamma(x)\|^2\right\}
\leq (1 - 2c^2\gamma)^{-d/2} \exp\left\{c\left(1 - 2c^2\gamma\right)^{-1}\left[\|T_\gamma(x)\| + \|T_\gamma(0)\|^2\right]\right\}.
\]

We now distinguish the case $\|x\| \geq M$ and $\|x\| < M$. In the first case, we get by (92), $(1 - 2c^2\gamma)^{-1} \leq 1 + 8c^2\gamma$ and $(1 - m\gamma)^2 \leq 1 - m\gamma$ since $2c\gamma^2 \gamma \leq 1/2$ and $\gamma \leq 1/m$, by definition of $c$ (94) and using that $\gamma \ll 1/m$,

\[
R_\gamma V_c(x) \leq (1 - 2c^2\gamma)^{-d/2} \exp\left\{c\left(1 + 8c^2\gamma\right)\left[(1 - m\gamma)\|x\|^2 + 2\gamma\|x\| T_\infty + \gamma T_\infty^2\right]\right\}
\leq \exp\left\{c\left(1 - m\gamma/8\right)\|x\|^2 + 2dc^2\gamma - mc\gamma\|x\|^2/8\right\} \leq \lambda^7 V_c(x),
\]

where we used for the penultimate inequality that $-\log(1 - t) \leq 2t$ for $t \in [0, 1/2]$. For the case $\|x\| \leq M$, by (92) and $(1 - 2c^2\gamma)^{-1} \leq 1 + 8c^2\gamma$ since $2c\gamma^2 \gamma \leq 1/2$, $\gamma \leq 1/m$, and using $-\log(1 - t) \leq 2t$ for $t \in [0, 1/2]$ again,

\[
R_\gamma V_c(x) \leq (1 - 2c\gamma)^{-d/2} \exp\left\{c\left(1 + 8c^2\gamma\right)\left[(1 + L\gamma)\|x\|^2 + 2(1 + \gamma L)\|x\| T_\infty\right]\right\}
\times \exp\left\{c\left(1 + 8c^2\gamma\right)^{2}\gamma T_\infty^2\right\}
\leq \exp\left\{c\left(1 + \gamma B_1\right)\|x\|^2 + \gamma B_2\right\},
\]

where we used for the last line that for any $x \in \mathbb{R}^d$, $\|x\| \leq 1 + \|x\|^2$. Using that $e^t - 1 \leq te^t$ for $t \geq 0$, we obtain that

\[
R_\gamma V_c(x) \leq \lambda^7 V_c(x) + \lambda^7 V_c(x)\exp\left\{c\gamma B_1\|x\|^2 + \gamma B_2 - \gamma \log(\lambda)\right\} - 1
\leq \lambda^7 V_c(x) + V_c(x)\exp\left\{c\gamma B_1\|x\|^2 + B_2 - \log(\lambda)\right\} \exp\left\{\gamma B_1\|x\|^2 + B_2 - \log(\lambda)\right\},
\]

which combined with (95) completes the proof of (93).

In addition, by [15, Theorem 19] and using that for any $i \in \mathbb{N}$, $e^{-2i\gamma L} \leq (1 + \gamma L)^{-2i}$, we have, for any $x, y \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}^+$,

\[
\|\delta_x R_{t_0/\gamma}^i - \delta_y R_{t_0/\gamma}^j\|_{TV} \leq 1 - 2\Phi\left(-\frac{\|x - y\|}{2\sigma^2 t_0 e^{-2(t_0 + \gamma L)}}\right),
\]

where $\Phi$ is the cumulative distribution of the one-dimensional Gaussian distribution with mean 0 and variance 1.

The proof is completed by combining (93), (96) and [11, Theorem 19.4.1][2].

[2]There is a $b$ missing in Equation 19.4.2d in [11, Theorem 19.4.1]
B Proof of Proposition 3

First note that under $\textbf{H1}$ and $\textbf{H2}$, for all $x \in \mathbb{R}^d$, since

$$\|\tilde{T}_\gamma(x) - T_\gamma(0)\| \leq \gamma c_\infty + \|T_\gamma(x) - T_\gamma(0)\|$$

and from (92),

$$\sup_{x \in \mathbb{R}^d, \|x\| \geq R_1} \left[ \|\tilde{T}_\gamma(x) - T_\gamma(0)\|/\{(1 - \gamma m)\|x\| + \gamma c_\infty\} \right] \leq 1 . \tag{97}$$

In addition, by (3), for any compact set $K$, for any $y \in K$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\tilde{R}_\gamma(y, A) = (2\pi\sigma^2\gamma)^{-d/2} \int_{\mathbb{R}^d} \mathbb{1}_A(y') \exp \left\{-\frac{\|y' - \tilde{T}_\gamma(y)\|^2}{(2\sigma^2\gamma)}\right\} dy' \geq 2^{-d/2} \sup_{y \in K} \left( \frac{e^{-\frac{\|\tilde{T}_\gamma(y)\|^2}{(\sigma^2\gamma)}}}{(\pi\sigma^2\gamma)^{-d/2}} \int_{\mathbb{R}^d} \mathbb{1}_A(y') \exp \left\{-\frac{\|y'\|^2}{(\sigma^2\gamma)}\right\} dy' \right).$$

Note that by $\textbf{H1}$ and $\textbf{H2}$, for any compact set $K$, $\sup_{y \in K}(e^{-\|\tilde{T}_\gamma(y)\|^2/(\sigma^2\gamma)}) > 0$. As a result, by [11, Definition 9.3.5, Definition 9.2.2, Definition 9.1.1], $\tilde{R}_\gamma$ is strongly aperiodic, Lebesgue-irreducible and all compact sets are 1-small.

It is therefore sufficient to show by [27, Theorem 16.0.1] that there exists $c > 0$, such that for any $\gamma \in (0, \bar{\gamma})$, there exists $\lambda_\gamma \in [0, 1)$, a compact $K \subset \mathbb{R}^d$ and $A_\gamma \geq 0$, such that

$$\tilde{R}_\gamma V_c \leq \lambda_\gamma V_c + A_\gamma \mathbb{1}_K . \tag{98}$$

Let $c = m/(16\sigma^2)$ and define for any $x \in \mathbb{R}^d$, $\tilde{T}_\gamma(x) = \tilde{T}_\gamma(x) - T_\gamma(0)$. Note that for any $\gamma \in (0, \bar{\gamma})$, $2\sigma^2\gamma \leq 1$ by definition of $c$ and using that $\bar{\gamma} \leq 1/m$. Let $x \in \mathbb{R}^d$ and $\gamma \in (0, \bar{\gamma})$. Then, we obtain using that $\int_{\mathbb{R}} e^{ax + bx^2 - z^2/2}dz = (2\pi(1 - 2b)^{-1/2})e^{a^2/(2(1 - 2b))}$ for any $a \in \mathbb{R}$ and $b \in [0, 1/2)$,

$$\tilde{R}_\gamma V_c(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(c\|\tilde{T}_\gamma(x) + (\sigma^2\gamma)^{1/2}z\|^2 - \|z\|^2/2)dz \leq (1 - 2\sigma^2\gamma)^{-d/2} \exp\left\{ (1 - 2\sigma^2\gamma)^{-1}\left[\|\tilde{T}_\gamma(x)\|^2 + \|T_\gamma(0)\|^2\right] \right\}.$$

If $\|x\| \geq M$, we get by (97), $(1 - 2\sigma^2\gamma)^{-1} \leq 1 + 8\sigma^2\gamma$ and $(1 - m\gamma)^2 \leq 1 - m\gamma$ since $2\sigma^2\gamma \leq 1/2$ and $\gamma \leq 1/m$, by definition of $c$ and using that $\bar{\gamma} \leq 1/m$,

$$\tilde{R}_\gamma V_c(x) \leq (1 - 2\sigma^2\gamma)^{-d/2} \exp\left\{ \left[\left(1 + 8\sigma^2\gamma\right)\left(1 - m\gamma\right)\|x\| + \gamma c_\infty + \|T_\gamma(0)\|\right]^{2} \right\} \leq (1 - 2\sigma^2\gamma)^{-d/2} \exp\left\{ \left[\left(1 + m\gamma/2\right)\left(1 - m\gamma\right)\|x\| + \gamma c_\infty + \|T_\gamma(0)\|\right]^{2} \right\} \leq (1 - 2\sigma^2\gamma)^{-d/2} \exp\left\{ \left[\left(1 - m\gamma/2\right)\|x\|^2 + B_1\|x\| + B_2\right] \right\},$$

for some $B_{1,\gamma}, B_{2,\gamma} \in \mathbb{R}$.

Therefore, we get $\limsup_{\|x\| \to +\infty} [\tilde{R}_\gamma V_c(x)/V_c(x)] = 0$ which completes the proof of (98).
C  Proof of Theorem 12

For any $a > 0$, define $\mathcal{W}_a^a(w) : \mathbb{R}_+ \to [0, +\infty)$ for any $w \in \mathbb{R}_+$ by

$$\mathcal{W}_a^a(w) = \exp(aw) - 1 .$$

The proof of Theorem 12 is based on the following proposition which combined technical lemmas gathered in Appendix C.1.

**Proposition 46. Assume H1-(ii).** For any $w \in \mathbb{R}_+$ and $\gamma \in (0, \bar{\gamma})$,

$$Q_{\gamma} \mathcal{W}_a^a(w) \leq \lambda_{\alpha}^\gamma \mathcal{W}_a^a(w) \mathbb{1}_{[\bar{R}_a, +\infty]}(w) + B_a^\gamma \mathcal{W}_a^a(w) \mathbb{1}_{[0, \bar{R}_a]}(w) + \gamma D_a \mathbb{1}_{[0, \bar{R}_a]}(w) ,$$

where

$$\bar{R}_a = 1 \lor R_1 \lor ((4a^2 + 2c_\infty) / m) \lor 16a^2 / m, \quad \lambda_{\alpha} = \exp(-am\bar{R}_a / 8) ,$$

$$C_a = a(c_\infty + 2a^2 \gamma e^{a_\gamma(c_\infty + 2a^2 / m)} + 2a^2 (2\pi)^{-1/2} e^{(a + 2a^2 / m)^2 / 2} ,$$

$$R_a = \bar{R}_a \lor a^{-1} \log \left(1 + C_a / (-\log(\lambda_{\alpha})\lambda_{\alpha}^2)\right) .$$

$$\bar{\gamma}_1 = \bar{\gamma} \lor 1 / (-\log(\lambda_{\alpha})) \lor 1 / (4a^2) , \quad B_a = e^{a(c_\infty + 2a^2 + LR_a)} + 2a^2 e^{a(c_\infty + 2a^2 + LR_a)}$$

$$\quad + 2a^2 (2\pi)^{-1/2} e^{(a + 2a^2 / m)^2 / 2} + 2a^2 \lambda_{\alpha}^2 \mathcal{W}_a^a(R_a) .$$

**Proof.** Combining Lemma 47, Lemma 48 and Lemma 49 we have for any $w \in \mathbb{R}_+$ and $\gamma \in (0, \bar{\gamma})$,

$$Q_{\gamma} \mathcal{W}_a^a(w) \leq \left(e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\right) e^{2a^2 \gamma^2 \gamma} e^{2a^2 \gamma^2 (\pi)^{-1/2} e^{(a + 2a^2 / m)^2 / 2} / 2} + 2a^2 \gamma e^{2a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{2a(\gamma_1(w) + c_\infty)}.$$

In addition, for any $w \in \mathbb{R}_+$ and $\gamma \in (0, \bar{\gamma})$

$$\left(e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\right) e^{2a^2 \gamma^2 \gamma} e^{2a^2 \gamma^2 (\pi)^{-1/2} e^{(a + 2a^2 / m)^2 / 2} / 2} + 2a^2 \gamma e^{2a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{2a(\gamma_1(w) + c_\infty)}.$$

By H1-(ii),

$$e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\leq e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\leq e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\leq e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1} .$$

Then using that $R_a \geq \bar{R}_a$, for any $t \in \mathbb{R}_+$, $e^t - 1 \leq te^t$ and combining (101), (102) and (99)

$$\left(e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma - 1}\right) \leq \lambda_{\alpha}^\gamma \mathcal{W}_a^a(w) \mathbb{1}_{[\bar{R}_a, +\infty]}(w) + B_a^\gamma \mathcal{W}_a^a(w) \mathbb{1}_{[0, \bar{R}_a]}(w) + \gamma a(c_\infty + 2a^2 e^{a_\gamma(c_\infty + 2a^2 / m)} \mathbb{1}_{[\bar{R}_a, +\infty]}(w) + \gamma (ac_\infty + 2a^2 \gamma + aL\bar{R}_a) e^{a(\gamma_1(w) + c_\infty)} + 2a^2 \gamma e^{a_\gamma} .$$
In addition, using that for any $t \in \mathbb{R}_-$, $e^t - 1 \leq t + t^2/2$, $\gamma \leq 1/(-\log(\lambda_a))$ and by (99), for any $w \in [R_a, +\infty)$, and $\gamma \in (0, \gamma_1]$ we have
\[
\left(\lambda_a^{2\gamma} \mathcal{W}_a^\ast(w) + \gamma C_a\right) \leq \lambda_a^{2\gamma} \left(1 + \lambda_a^{2\gamma} - 1 + \gamma C_a/(\lambda_a^{2\gamma} \mathcal{W}_a^\ast(w))\right) \mathcal{W}_a^\ast(w) \\
\leq \lambda_a^{2\gamma} \left(1 + \gamma \log(\lambda_a^2) + \gamma^2 \log(\lambda_a^2)^2/2 + \gamma C_a/(\lambda_a^{2\gamma} \mathcal{W}_a^\ast(w))\right) \mathcal{W}_a^\ast(w) \\
\leq \lambda_a^{2\gamma} \left(1 + \gamma \log(\lambda_a^2)/2 + \gamma C_a/(\lambda_a^{2\gamma} \mathcal{W}_a^\ast(w))\right) \mathcal{W}_a^\ast(w) \leq \lambda_a^{2\gamma} \mathcal{W}_a^\ast(w)^2. 
\]

(104)

In addition, for any $w \in [R_a, +\infty)$ and $\gamma \in (0, \gamma_1]$
\[
\lambda_a^{2\gamma} \mathcal{W}_a^\ast(w) + 2\sigma^2 \gamma a^2 e^a(\gamma(w) + \gamma c) + 2\sigma^2 a^2 \leq \lambda_a^{2\gamma} \mathcal{W}_a^\ast(w) + 2\sigma^2 \gamma a^2 \lambda_a^{2\gamma} \mathcal{W}_a^\ast(w). 
\]

(105)

Using (99) we have
\[
\lambda_a^{2\gamma} \left(1 + \gamma 2\sigma^2 a^2\right) = e^{2\gamma \log(\lambda_a) + \log(1 + \gamma 2\sigma^2 a^2)} \leq e^{2\gamma \log(\lambda_a) + \gamma 2\sigma^2 a^2} \leq \lambda_a^\gamma, 
\]

which completes the proof when combined with (100), (104), (103) and (105).

Proof of Theorem 12. Let $\tilde{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$ and $\gamma \in (0, \gamma_1]$. We show that (19) holds with
\[
c_3 = (B_a/\lambda_a)^\gamma a (LR_a + 2\sigma^2 a + c_\infty + m\tilde{R}_a/8)\eta R_a \mathcal{W}_a^\ast(R_a)/|\log(\lambda_a)| \\
+ [D_a\eta R_a + A_a]/|\log(\lambda_a)|,
\]
where $\lambda_a, R_a, \tilde{R}_a, B_a, D_a$ are defined in (99), $A_a$ in Lemma 50, and $\eta R$ in (48).

By Lemma 50, Proposition 46 and since $\mathcal{W}(0) = 0$ and $\mu_\gamma$ is invariant for $Q_{\gamma}$, we have
\[
\int_{(0, +\infty)} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) = \int_{(0, +\infty)} Q_{\gamma} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) + Q_{\gamma} \mathcal{W}_a^\ast(0)\mu_\gamma(\{0\}) \\
\leq \lambda_a^\gamma \int_{(R_a, +\infty)} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) + B_a^\gamma \int_{(0, R_a)} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) \\
+ \gamma D_a\mu_\gamma((0, 1/\gamma)) + c_\infty A_a. 
\]

Rearranging terms yields
\[
\int_{(0, +\infty)} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) \leq \{B_a^\gamma - \lambda_a^\gamma\}/(1 - \lambda_a^\gamma) \int_{(0, R_a)} \mathcal{W}_a^\ast(w)\,d\mu_\gamma(w) \\
+ \gamma D_a\mu_\gamma((0, R_a))/(1 - \lambda_a^\gamma) + c_\infty A_a/(1 - \lambda_a^\gamma) \\
\leq \{B_a^\gamma \lambda_a^{-\gamma} - 1\}/(\lambda_a^{-\gamma} - 1) \mathcal{W}_a^\ast(R_a)c_\infty \eta R_a \\
+ c_\infty \gamma \lambda_a^{-\gamma} [D_a\eta R_a + A_a]/(\lambda_a^{-\gamma} - 1), 
\]
where we have used Theorem 27 applied to $R \leftarrow R_a$ in the last inequality. The proof is then completed upon using that for any $t \geq 0$, $t \leq e^t - 1 \leq te^t$. \qed
C.1 Technical results

Lemma 47. Let $a > 0$. Then, for any $\gamma > 0$, and $w \in \mathbb{R}_+$

$$Q_\gamma W^\prime_a (w) = e^{2a^2 \sigma^2} \left[ e^{a (\tau_\gamma (w) + \gamma c_\infty)} \left( \Phi \left( \frac{\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} \right) + 2 \sigma \gamma^{1/2} \right) 
- \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} \right) \right]$$

$$+ 2 \sinh \left( a (\tau_\gamma (w) + \gamma c_\infty) \right) \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} \right)$$

$$- 1 + 2 \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} \right),$$

where sinh is the hyperbolic sine.

Proof. By (14) and Lemma 28, for any $w \in \mathbb{R}_+, \gamma > 0$

$$Q_\gamma W^\prime_a (w) = \int_{\mathbb{R}} W^\prime_a (\tau_\gamma (w) + \gamma c_\infty - 2 \sigma \gamma^{1/2} g) \{1 - \boldsymbol{p}_{\sigma^2 \gamma} (\tau_\gamma (w) + \gamma c_\infty, g)\} \varphi(g) dg$$

$$= \int_{\mathbb{R}} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 \sigma \gamma^{1/2} g)} \{1 - \boldsymbol{p}_{\sigma^2 \gamma} (\tau_\gamma (w) + \gamma c_\infty, g)\} \varphi(g) dg$$

$$- \int_{\mathbb{R}} 2 \sinh \left( a (\tau_\gamma (w) + \gamma c_\infty) \right) \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} \right) \varphi(g) dg$$

In addition, by (11) and using changes of variable,

$$= \int_{\mathbb{R}} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 \sigma \gamma^{1/2} g)} \{1 - \boldsymbol{p}_{\sigma^2 \gamma} (\tau_\gamma (w) + \gamma c_\infty, g)\} \varphi(g) dg$$

$$= \int_{-\infty}^{(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \{1 - 1 \wedge \frac{\varphi_{\sigma^2 \gamma} (\tau_\gamma (w) + \gamma c_\infty - g)}{\varphi_{\sigma^2 \gamma} (g)}\} \varphi_{\sigma^2 \gamma} (g) dg$$

$$= \int_{-\infty}^{(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \varphi_{\sigma^2 \gamma} (g) dg$$

$$- \int_{-\infty}^{(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \varphi_{\sigma^2 \gamma} (\tau_\gamma (w) + \gamma c_\infty - g) dg$$

$$= \int_{-\infty}^{(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \varphi_{\sigma^2 \gamma} (g) dg$$

$$- \int_{-\infty}^{-(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \varphi_{\sigma^2 \gamma} (g) dg$$

$$= \int_{-(\tau_\gamma (w) + \gamma c_\infty)/2}^{(\tau_\gamma (w) + \gamma c_\infty)/2} e^{a (\tau_\gamma (w) + \gamma c_\infty - 2 g)} \varphi_{\sigma^2 \gamma} (g) dg$$

$$+ \int_{-(\tau_\gamma (w) + \gamma c_\infty)/2}^{-(\tau_\gamma (w) + \gamma c_\infty)/2} 2 \sinh \left( a (\tau_\gamma (w) + \gamma c_\infty) \right) e^{-2g} \varphi_{\sigma^2 \gamma} (g) dg$$

$$= e^{a (\tau_\gamma (w) + \gamma c_\infty) + 2a^2 \sigma^2} \left( \Phi \left( \frac{\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} a \right) - \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} a \right) \right)$$

$$+ 2 \sinh \left( a (\tau_\gamma (w) + \gamma c_\infty) \right) e^{2a^2 \sigma^2} \Phi \left( \frac{-\tau_\gamma (w) + \gamma c_\infty}{2 \sigma \gamma^{1/2}} + 2 \sigma \gamma^{1/2} a \right).$$
This concludes the proof. □

**Lemma 48.** Let \( a > 0 \). Then, for any \( \gamma > 0 \), and \( w \in \mathbb{R}_+ \)

\[
\Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) - \Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) \leq 1 - 2\Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right) + 2\sigma^2 \gamma a^2 .
\]

**Proof.** We consider the decomposition

\[
\Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) - \Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) = 1 - 2\Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right) + A_1 + A_2 ,
\]

where

\[
A_1 = \Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) - \Phi \left( \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right)
\]

\[
A_2 = \Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right) - \Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) .
\]

By Taylor’s theorem, we have for any \( \gamma > 0 \), and \( w \in \mathbb{R}_+ \)

\[
A_1 \leq 2\sigma \gamma^{1/2} a (2\pi)^{1/2} e^{-2\left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right)^2} .
\]

Similarly,

\[
\left| A_2 + 2\sigma \gamma^{1/2} a (2\pi)^{1/2} e^{-2\left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right)^2} \right| \leq \int_{\frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}}}^{\frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a} -t \phi(t) \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a - t \right) dt \leq \sup_{t \in \mathbb{R}} |t \phi(t)| \int_{0}^{2\sigma \gamma^{1/2} a} t dt .
\]

The proof is complete since \( \sup_{t \in \mathbb{R}} |t \phi(t)| \leq 1 \). □

**Lemma 49.** Let \( a > 0 \). Then, for any \( \gamma \in (0, 1/(4\sigma^2)] \), and \( w \in \mathbb{R}_+ \)

\[
-2\Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} \right) \left( e^{\alpha(t_{\gamma}(w) + \gamma c_{\infty}) + 2a^2 \sigma^2 \gamma} - 1 \right)
\]

\[
+ 2e^{2a^2 \sigma^2 \gamma} \sinh \left( \alpha(t_{\gamma}(w) + \gamma c_{\infty}) \right) \Phi \left( \frac{-\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2} a \right) \leq 4e^{2a^2 \sigma^2 \gamma^2} \gamma^2 a_2 \gamma (2\pi)^{-1/2} e^{a + 2\sigma \gamma^{1/2} a} .
\]

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Proof. Using that \( \sinh(t) - e^t + 1 \leq 0 \) for \( t \geq 0 \), we get

\[
-2\Phi \left( \frac{-\tau_\gamma(w) + \gamma c_\infty}{2\sigma \gamma^{1/2}} \right) (e^{a(\tau_\gamma(w) + \gamma c_\infty) + 2a^2\sigma^2\gamma} - 1)
\]

\[
+ 2e^{2a^2\sigma^2\gamma} \sinh (a(\tau_\gamma(w) + \gamma c_\infty)) \Phi \left( \frac{-\tau_\gamma(w) + \gamma c_\infty}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2}/a \right)
\]

\[
\leq 2\Phi \left( \frac{-\tau_\gamma(w) + \gamma c_\infty}{2\sigma \gamma^{1/2}} \right) e^{2a^2\sigma^2\gamma} \left[ \sinh (a(\tau_\gamma(w) + \gamma c_\infty)) - e^{a(\tau_\gamma(w) + \gamma c_\infty) + 1} \right]
\]

\[
+ 2e^{2a^2\sigma^2\gamma} \sinh (a(\tau_\gamma(w) + \gamma c_\infty)) A_1
\]

\[
\leq 2e^{2a^2\sigma^2\gamma} \sinh (a(\tau_\gamma(w) + \gamma c_\infty)) A_1 ,
\]

with \( A_1 = \left[ \Phi \left( \frac{-\tau_\gamma(w) + \gamma c_\infty}{2\sigma \gamma^{1/2}} + 2\sigma \gamma^{1/2}/a \right) - \Phi \left( \frac{-\tau_\gamma(w) + \gamma c_\infty}{2\sigma \gamma^{1/2}} \right) \right] .
\]

In addition for any \( t \in \mathbb{R}_+ \), since \( (e^u - 1)/u \leq e^u \) for any \( u \in \mathbb{R}_+ \), we have

\[
\Phi(-t + 2\sigma \gamma^{1/2}/a) - \Phi(-t) = \int_{-t}^{-t + 2\sigma \gamma^{1/2}/a} \varphi(x)dx \leq (2\pi)^{-1/2} e^{-t^2/2} \int_0^{2\sigma \gamma^{1/2}/a} e^{xt}dx \leq 2\gamma^{1/2} \sigma a(2\pi)^{-1/2} e^{-t^2/2 + 2\sigma \gamma^{1/2}/at} .
\]

Note that \( (2\sigma \gamma^{1/2})^{-1} \sinh (a(\tau_\gamma(w) + \gamma c_\infty)) \leq \sinh \left( a(\tau_\gamma(w) + \gamma c_\infty)(2\sigma \gamma^{1/2})^{-1} \right) \) using the condition \( (2\sigma \gamma^{1/2})^{-1} \geq 1 \). It yields

\[
\sinh (a(\tau_\gamma(w) + \gamma c_\infty)) A_1 \leq 4\sigma^2 \alpha (2\pi)^{-1/2} \sup_{t \in \mathbb{R}_+} \left( \sinh (at)e^{-t^2/2 + 2\sigma \gamma^{1/2}/at} \right) .
\]

The proof is complete since for any \( t \in \mathbb{R}_+ \), \( \sup_{t \in \mathbb{R}_+} (\sinh (at)e^{-t^2/2 + 2\sigma \gamma^{1/2}/at}) \leq 2^{-1} e^{(a + 2\sigma \gamma^{1/2}/a)^2/2} . \]

\( \square \)

Lemma 50. Assume \( \mathbf{H}1\)-(ii). Let \( a > 0 \). Then, for any \( \gamma \in (0, \bar{\gamma}] \),

\[
Q_\gamma W_a^\gamma (0) \leq \gamma c_\infty A_a ,
\]

where \( A_a = 4e^{\sigma\bar{\gamma}(c_\infty - 2a^2)} \left[ \bar{\gamma}(2a^2 + c_\infty/2)^2 + \sigma^2 \right]/(2\sigma^2) .
\]

Proof. By definition \( (14) \), we have

\[
Q_\gamma W_a^\gamma (0) = \int_{-\infty}^{\gamma c_\infty/2} (\exp(a\{\gamma c_- - 2\sigma \gamma^{1/2}g\}) - 1)\{1 - \varphi_{\sigma^2 \gamma}(\gamma c_- - g)\} \varphi(g)dg
\]

\[
= \int_{-\infty}^{\gamma c_\infty/2} (\exp(a\{\gamma c_- - 2g\}) - 1)\{\varphi_{\sigma^2 \gamma}(g) - \varphi_{\sigma^2 \gamma}(\gamma c_- - g)\} .
\]

In addition, by using that \( 1 - e^{-t} \leq t \) for any \( t \in \mathbb{R}_+ \), we have for any \( g \in [-\infty, \gamma c_-/2] \),

\[
\varphi_{\sigma^2 \gamma}(g) - \varphi_{\sigma^2 \gamma}(\gamma c_- - g) \leq \varphi_{\sigma^2 \gamma}(g) \left[ 1 - e^{-c_- (\gamma c_- - 2g)/(2\sigma^2)} \right] \leq \varphi_{\sigma^2 \gamma}(g) \left[ c_- (\gamma c_- - 2g)/(2\sigma^2) \right] .
\]

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Finally, using
\[
\int_{-\infty}^{\gamma c_{\infty}/2} (\gamma c_{\infty} - 2g)(\exp(a\{\gamma c_{\infty} - 2g\}) - 1)\varphi_{\sigma^2\gamma}(g)dg
\]
\[
\leq \int_{-\infty}^{\gamma c_{\infty}/2} (\gamma c_{\infty} - 2g)^2 \exp(a\{\gamma c_{\infty} - 2g\})\varphi_{\sigma^2\gamma}(g)dg
\]
\[
= 4 \int_{-\infty}^{0} g^2 \exp(-2ag)\varphi_{\sigma^2\gamma}(g + \gamma c_{\infty}/2)dg
\]
\[
= 4(\sqrt{2\pi}\sigma\gamma^{1/2})^{-1}e^{\sigma^2\gamma_{\infty} + 2a^2\sigma^2\gamma} \int_{-\infty}^{0} g^2 e^{-(g + \gamma c_{\infty}/2 + 2a\sigma^2\gamma)^2/(2\sigma^2\gamma)}dg
\]
\[
\leq 4e^{a\gamma c_{\infty} + 2a^2\sigma^2\gamma} \left[\gamma^2(2a\sigma^2 + c_{\infty}/2)^2 + \sigma^2\gamma\right] ,
\]
completes the proof. \qed