Slice Energy in Higher-Order Gravity Theories and Conformal Transformations

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Abstract

We study the generic transport of slice energy between the scalar field generated by the conformal transformation of higher-order gravity theories and the matter component. We give precise relations for this exchange in the cases of dust and perfect fluids. We show that, unless we are in a stationary spacetime where slice energy is always conserved, in non-stationary situations contributions to the total slice energy depend on whether or not test matter follows geodesics in both frame representations of the dynamics, that is on whether or not the two conformally related frames are physically indistinguishable.
1 Introduction

With recent advances in observational cosmology [1] $f(R)$ theories (and the closely related family of scalar-tensor ones) have in the last few years regained much attention both in cosmology [2] and in other contexts [3]. Perhaps the most economical and convenient way to study such modified gravity theories is through their well-known conformal relation to general relativity, a technique developed in the eighties by different groups [4].

In this conformal method the $f(R)$-vacuum equations on a spacetime $(\mathcal{V}, g)$ (the so-called Jordan frame) are transformed via a conformal transformation of the type $\tilde{g} = e^{\phi}g$, where $\phi$ is a function of $\partial f/\partial R$, to become on the conformally related spacetime $(\mathcal{V}, \tilde{g})$ (the so-called Einstein frame) Einstein equations for the metric $\tilde{g}$ with the scalar field $\phi$ having a self-interacting potential that depends on the function $f(R)$ and its first derivatives. This technique provides a refreshing way to view the $f(R)$-vacuum theory as a unified theory of gravitation (described in the Einstein frame by the metric $\tilde{g}$) and the scalar field $\phi$, that is as a theory uniting general relativity and the (lagrangian) theory of the scalar field. In the Jordan frame only one single geometric object appears, the ‘metric’ $g$, and the conformal transformation then serves as a tool to ‘fragment’ $g$ into its two pieces in the Einstein frame, namely the gravitational field $\tilde{g}$ and the scalar field $\phi$.

In a higher-order gravity theory with matter we have the following different pieces of information involved in the conformal transformation:

- gravity, the field $g$ or the conformally related field $\tilde{g}$
- the scalar field $\phi$.
- the various matter fields $\psi$ which couple non-minimally to $\tilde{g}$ and to $\phi$, or their conformal transform $\tilde{\psi}$ which couples minimally to $\tilde{g}$ but is not coupled to $\phi$.

The conformal transformation then relates the different pieces of matter and spacetime
geometry and describes the interaction between the components listed above in the context of \( f(R) \) theories.

We describe in this paper how interactions of this sort lead to an exchange of slice energy between the various fields and spacetime geometry. For more varied interactions and energy transfer models of interest to cosmological situations see [6, 7] and references therein.

The plan of this paper is as follows. The next Section is preliminary and includes two simple applications (Corollaries 2.1, 2.2) of the basic properties (Theorems 2.1, 2.2) of the slice energy. These basic properties and also their proofs are included here for easy reference and also to establish notation. Their applications deal with the simpler case of \( f(R) \) theories in vacuum. Section 3 is the heart of this paper. There, we find the general (slice) energy transport equation and study in detail its application to the case of a general \( f(R) \) theory coupled to matter, where by matter we mean a general perfect fluid-scalar field system. There are many properties analysed here but a particularly important one deals with conditions under which the choice of ‘physical’ metric influences the net contribution to slice energy of the system. We conclude with a discussion of these results in Section 4.

2 Energy on a slice

Consider a time-oriented spacetime \((\mathcal{V}, g)\) with \(\mathcal{V} = \mathcal{M} \times \mathbb{R}\), where \(\mathcal{M}\) is a smooth manifold of dimension \(n\), \(g\) a spacetime metric and the spatial slices \(\mathcal{M}_t (= \mathcal{M} \times \{t\})\) are spacelike submanifolds endowed with the time-dependent spatial metric \(g_t\). (In the following, Greek indices run from 0 to \(n\), while Latin indices run from 1 to \(n\). We also assume that the metric signature is \((+ - \cdots -)\).) On \((\mathcal{V}, g)\) we consider a family of matterfields denoted collectively as \(\psi\), assume that the field \(\psi\) arises from a lagrangian density which we denote by \(L\) and denote the stress tensor of the field \(\psi\) by \(T(\psi)\).

For \(X\) any causal vectorfield of \(\mathcal{V}\) we define the energy-momentum vector \(P\) of a stress
tensor $T$ relative to $X$ to be
\[ P^\beta = X^\alpha T^{\alpha \beta}. \] (2.1)
The energy on the slice $\mathcal{M}_t$ with respect to $X$ is defined by the integral (when it exists)
\[ E_t = \int_{\mathcal{M}_t} P^\alpha n_\alpha d\mu_t, \] (2.2)
where $n$ is the unit normal to $\mathcal{M}_t$ and $d\mu_t$ is the volume element with respect to the spatial metric $g_t$. We call $P^\alpha n_\alpha$ the energy density. Assume that $X$ and $T$ are smooth. Then we have
\[ \nabla_\alpha P^\alpha = \nabla_\alpha (X_\beta T^{\alpha \beta}) = \nabla_\alpha X_\beta T^{\alpha \beta} + X_\beta \nabla_\alpha T^{\alpha \beta} \]
or, equivalently,
\[ \nabla_\alpha P^\alpha = \frac{1}{2} T^{\alpha \beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + X_\beta \nabla_\alpha T^{\alpha \beta}. \] (2.3)
Thus, if $\mathcal{K} \subset \mathcal{V}$ is a compact domain with smooth boundary $\partial \mathcal{K}$, it follows from Stokes’ theorem that
\[ \int_{\mathcal{K}} \nabla_\alpha P^\alpha d\mu = \int_{\partial \mathcal{K}} P^\alpha n_\alpha d\sigma, \] (2.4)
where $d\mu$ is the volume element of $\mathcal{V}$ and $d\sigma$ that of $\partial \mathcal{K}$, and so we find
\[ \int_{\partial \mathcal{K}} P^\alpha n_\alpha d\sigma = \frac{1}{2} \int_{\mathcal{K}} T^{\alpha \beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) d\mu + \int_{\partial \mathcal{K}} X_\beta \nabla_\alpha T^{\alpha \beta} d\mu. \] (2.5)
Hence, when $\mathcal{M}$ is compact or the field falls off appropriately at infinity, on the spacetime slab $\mathcal{D} = \Sigma \times [t_0, t_1]$, $\Sigma \subset \mathcal{M}$, and with $T$ having support on $\mathcal{D}$ we have the following relation for the energies on the two end-slices
\[ \int_{\mathcal{M}_{t_1}} P^\alpha n_\alpha d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} P^\alpha n_\alpha d\mu_{t_0} = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha \beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) d\mu + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} X_\beta \nabla_\alpha T^{\alpha \beta} d\mu \] (2.6)
or
\[ E_{t_1} - E_{t_0} = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha \beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) d\mu + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} X_\beta \nabla_\alpha T^{\alpha \beta} d\mu. \] (2.7)
We therefore see that when $X$ is a Killing vectorfield the first term on the right-hand-side of Eq. (2.7) is zero and so we have

$$E_{t_1} - E_{t_0} = \int_{t_0}^{t_1} \int_{M_t} X_\beta \nabla_\alpha T^{\alpha \beta} d\mu.$$  (2.8)

Thus we have shown the following result (cf. [8], p. 87-88).

**Theorem 2.1** When $X$ is a Killing vectorfield and the field is conserved, i.e., $\nabla_\alpha T^{\alpha \beta} = 0$, we have

$$E_{t_1} = E_{t_0}. \quad (2.9)$$

This means that, when the energy-momentum tensor of a field is conserved, the same is true for its slice energy relative to a Killing vectorfield as a function of time.

In the next Section we pay particular attention to the case for which the field is a matter field $\psi$ interacting with a scalar field $\phi$ with potential $V(\phi)$. We take the scalar field lagrangian density to be

$$L = -\frac{1}{2} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi). \quad (2.10)$$

Then the energy-momentum tensor of $\phi$ is

$$T^{\alpha \beta}(\phi) = \partial^\alpha \phi \partial^\beta \phi - \frac{1}{2} g^{\alpha \beta} (\partial^\lambda \phi \partial_\lambda \phi - 2V(\phi)), \quad (2.11)$$

and we have the following result.

**Theorem 2.2** The energy density $P^\alpha n_\alpha$ of the scalar field $\phi$ with potential $V(\phi)$ is positive when $V(\phi) > 0$.

**Proof.** The proof, which we give here for a general scalar field potential $V(\phi)$, is a direct adaptation with slight modifications of that found in [8], p. 88, for a power-law potential. Using Eq. (2.11) we calculate

$$P^\alpha n_\alpha = -\frac{1}{2} X^\alpha n_\alpha \partial^\lambda \phi \partial_\lambda \phi + X^\alpha \partial_\alpha \phi n^\beta \partial_\beta \phi + X^\alpha n_\alpha V(\phi). \quad (2.12)$$
We define the quadratic form
\[ \gamma^{\lambda\mu} = -g^{\lambda\mu}X^\alpha n_\alpha + (X^\lambda n^\mu + X^\mu n^\lambda) \] (2.13)
and then we find that
\[ P^\alpha n_\alpha = -\frac{1}{2}X^\alpha n_\alpha g^{\lambda\mu}\partial_\lambda \phi \Phi_\mu \Phi + \frac{1}{2}(X^\lambda n^\mu + X^\mu n^\lambda)\partial_\lambda \phi \Phi_\mu \Phi + X^\alpha n_\alpha V(\phi). \] (2.14)
This means that
\[ P^\alpha n_\alpha = \frac{1}{2}\gamma^{\lambda\mu}\partial_\lambda \phi \Phi_\mu \Phi + X^\alpha n_\alpha V(\phi). \] (2.15)
Since \( \mathcal{M}_t \) is a \( t = \text{const.} \) hypersurface, we can choose coordinates such that \( X^0 = 1, X^i = 0, n_i = 0. \) Then \( n_0 = (g^{00})^{-1/2}, n^i = g^{i0}(g^{00})^{-1/2} \) and so it follows that the quadratic form \( \gamma \) is positive definite,
\[ \gamma^{00} = g^{00}n_0, \quad \gamma^{i0} = 0 \quad \gamma^{ij} = -g^{ij}n_0, \] (2.16)
(recall signature of \( g_{ij} \) is \((-\ldots-))\). We then find that
\[
P^\alpha n_\alpha &= \frac{1}{2}\left(\gamma^{00}\partial_0 \Phi_0 \Phi + 2\gamma^{0i}\partial_i \Phi_0 \Phi + \gamma^{ij}\partial_i \Phi \partial_j \Phi\right) + n_0 V(\phi) \\
&= \frac{1}{2}\left(n_0 g^{00}\dot{\phi}^2 - g^{ij}n_0 \partial_i \Phi \partial_j \Phi\right) + n_0 V(\phi)
\]
and therefore we conclude that the energy density \( P^\alpha n_\alpha \) is positive whenever \( V(\phi) > 0. \) This concludes the proof. □

To end this Section, for the following simple application of the preceding developments we restrict attention to \( n = 4 \) spacetime dimensions although everything we do becomes valid with minor modifications to arbitrary \( n. \) The following notation for conformally related quantities is used: Let \( g \) and \( \tilde{g} \) be two conformal metrics, \( \tilde{g} = \Omega^2 g, \) on the manifold \( V. \) This means that in two orthonormal moving frames, \( \theta^\alpha \) and \( \tilde{\theta}^\alpha, \) the two conformal metrics satisfy
\[ \tilde{g} = \eta_{\alpha\beta} \tilde{\theta}^\alpha \tilde{\theta}^\beta, \quad g = \eta_{\alpha\beta} \theta^\alpha \theta^\beta \quad \text{and} \quad \tilde{\theta}^\alpha = \Omega^{-1} \theta^\alpha, \] (2.17)
with $\eta_{\alpha\beta} = \text{diag}(+,-\cdots,-)$ being the flat metric. Setting $\Omega^2 = e^\phi$ we see that $\tilde{\theta}^\alpha = e^{-\phi/2}\theta^\alpha$ and obviously $\tilde{\theta}_\alpha = e^{\phi/2}\theta_\alpha$. The same rules are true for any 1-form or vectorfield on $\mathcal{V}$. Consider now the $f(R)$-vacuum equations,

$$L_{\alpha\beta} \equiv f' R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f' - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box g f' = 0,$$

(2.18)

where the left hand side satisfies the conservation identities (cf. [9], p. 140)

$$\nabla_\alpha L^{\alpha\beta} = 0.$$  

(2.19)

Then we conformally transform from $(\mathcal{V}, g)$ to the Einstein frame $(\mathcal{V}, \tilde{g})$, according to the prescription given in [4], that is, we set

$$\phi = \ln f',$$  

(2.20)

to obtain the Einstein equations with a scalar field ‘matter source’ of potential $V(\phi) = (1/2)(f')^{-2}(R f' - f)$ and energy-momentum tensor given by Eq. (2.11):

$$\tilde{G}_{\alpha\beta} = \tilde{T}_{\alpha\beta}(\phi).$$  

(2.21)

In this case we conclude that the field $\phi$ is conserved, i.e.,

$$\nabla_\alpha \tilde{T}^{\alpha\beta}(\phi) = 0,$$  

(2.22)

and, since

$$\nabla_\alpha \tilde{T}^{\alpha\beta}(\phi) = \partial^\beta \phi (\nabla_\alpha \partial^\alpha \phi + V'),$$

(2.23)

we find that the $\phi$-field is a scalar field satisfying the wave equation

$$\nabla_\alpha \partial^\alpha \phi + V' = 0.$$  

(2.24)

Further from Theorem 2.1 we have the following result.

**Corollary 2.1** The slice energy of the scalar field $\phi$ generated by the conformal transformation (2.20) to the Einstein frame of the $f(R)$-vacuum equations (2.18) relative to a Killing vectorfield of $\tilde{g}$ is conserved, i.e.,

$$\tilde{E}_t(\phi) = \int_{\mathcal{M}_t} \tilde{P}^\alpha \tilde{n}_\alpha \tilde{d}\tilde{\mu}_t = \text{const},$$  

(2.25)

with $d\tilde{\mu}_t$ being the volume element of $\tilde{g}_t$. 

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Secondly from Theorem 2.2 we have:

**Corollary 2.2** For all $f(R)$-vacuum theories \( (2.18) \) with a positive potential in the Einstein frame the energy density $\tilde{P}^\alpha \tilde{n}_\alpha$ of $\phi$ is positive.

Examples of theories in the last Corollary include, for instance, the choice $f(R) = R + \alpha R^2, \alpha > 0$.

### 3 $f(R)$-matter systems

Suppose now that we start by coupling a matter field $\psi$ to the geometry in $(\mathcal{V}, g)$ via the $f(R)$-matter field equations

$$f'R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box_g f' = T_{\alpha\beta}(\psi). \tag{3.1}$$

Because of the conservation identities \((2.19)\), the field $\psi$ satisfies the conservation laws

$$\nabla_\alpha T^{\alpha\beta}(\psi) = 0. \tag{3.2}$$

Then, if we conformally transform from $(\mathcal{V}, g)$ to the Einstein frame $(\mathcal{V}, \tilde{g})$ according to \((2.20)\), in place of equations \((2.21)\) we obtain

$$\tilde{G}_{\alpha\beta} = \tilde{T}_{\alpha\beta}(\phi) + \tilde{T}_{\alpha\beta}(\tilde{\psi}), \tag{3.3}$$

where now the whole tensor in the right-hand-side is conserved, namely

$$\tilde{\nabla}_\alpha \left( \tilde{T}^{\alpha\beta}(\phi) + \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right) = 0, \tag{3.4}$$

but the two components are not conserved separately, that is

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\phi) \neq 0 \tag{3.5}$$

and

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) \neq 0, \tag{3.6}$$
unless the conservation equations (3.2) for the field $\psi$ are conformally invariant (conditions for this are given in [10], p. 448). This result (already given in Ref. [11]) indicates that in higher order gravity theories there must be a generic, nontrivial $\phi - \tilde{\psi}$ interaction between the matter field $\tilde{\psi}$ and the $\phi$-field, and an associated exchange of energy between $\phi$ and $\tilde{\psi}$.

Writing Eq. (2.7) for the scalar field $\phi$ and substituting for the last term in the right-hand-side from Eq. (3.4) we find the general energy transport equation in the Einstein frame,

$$E_{t1}(\phi) - E_{t0}(\phi) = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{X}_\beta + \tilde{\nabla}_\beta \tilde{X}_\alpha) d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}, \quad (3.7)$$

with $d\tilde{\mu}$ being the volume element of $\tilde{g}$.

This result is only symbolic and has to be augmented by precise equations satisfied by the fields. Since the stress tensor of the $\phi$-field is not separately conserved, it follows that the $\phi$-field will not satisfy the usual scalar wave equation, $\tilde{\Box} \tilde{\psi} + V'(\phi) = 0$, but this equation will in general contain new terms. Similarly for the ‘ordinary matter’ $\tilde{\psi}$-field, whatever its form (scalar field, Maxwell, a fluid etc), its field equations have new terms indicating the $\phi - \tilde{\psi}$ interaction and associated energy exchange. For instance, if the $\tilde{\psi}$ field is another scalar field, then the equations satisfied by its conformal transform, $\tilde{\psi}$, have a general form of the type $\tilde{\Box} \tilde{\psi} + h(\phi) \partial_\alpha \phi \partial^\alpha \tilde{\psi} = 0$, where $h(\phi)$ is a smooth function of $\phi$ (often exponential).

To study this interaction and the associated energy exchange between $\phi$ and $\tilde{\psi}$ more closely we give some concrete examples. Suppose firstly that $\psi$ is a dust cloud on $(\mathcal{V},g)$ with 4-velocity $V^\alpha$ and stress tensor $T_{\alpha\beta}$, dust = $\rho V^\alpha V^\beta$, (3.8) satisfying the $f(R)$-dust equations in the Jordan frame, namely

$$f' R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box g' = \rho V^\alpha V^\beta. \quad (3.9)$$

Then

$$\nabla_\alpha (\rho V^\alpha V^\beta) = 0, \quad (3.10)$$
and it is obvious that here the dust streamlines are geodesics, that is \( V^\alpha \) is the tangent vectorfield to the geodesics. After the conformal transformation we find

\[
\tilde{G}_{\alpha\beta} = \tilde{T}_{\alpha\beta}(\phi) + \tilde{\rho} \tilde{V}_\alpha \tilde{V}_\beta, \tag{3.11}
\]

with \( \tilde{T}_{\alpha\beta}(\phi) \) given by (2.11) with tildes where appropriate, and with

\[
\tilde{V}_a = e^{-\phi/2} V_a, \quad \tilde{\rho} = e^{-2\phi}. \tag{3.12}
\]

(We have set \( \tilde{\rho} = \Omega^{-4} \rho \) and since \( \Omega^2 = e^\phi, \Omega^{-4} = e^{-2\phi} \).

What is the field equation satisfied by the scalar field \( \phi \)? From Eq. (3.11) the divergence of the stress tensor of \( \phi \) is minus that of the dust, but

\[
\tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}_\alpha \tilde{V}_\beta) = \nabla_a (\tilde{\rho} \tilde{V}^\alpha \tilde{V}^\beta) + A_{\alpha\gamma}^\beta \tilde{\rho} \tilde{V}^\gamma \tilde{V}^\beta + A_{\alpha\beta}^\gamma \tilde{\rho} \tilde{V}^\alpha \tilde{V}^\gamma, \tag{3.13}
\]

where

\[
A_{\beta\gamma}^\alpha = \frac{1}{2} \left( \delta^\alpha_{\beta} \partial_\gamma \phi + \delta^\alpha_{\gamma} \partial_\beta \phi - g_{\beta\gamma} g^{\alpha\delta} \partial_\delta \phi \right). \tag{3.14}
\]

From these equations and Eq. (3.10) we deduce the modified scalar field equation in the form

\[
\partial^\beta \phi (\tilde{\Box} \phi + V') + \frac{1}{2} \tilde{\rho} \tilde{V}^\alpha \tilde{V}^\beta \partial_\alpha \phi - \frac{1}{2} \tilde{\rho} \partial_\beta \phi = 0. \tag{3.15}
\]

Another way to derive the scalar field equation is as follows. Since

\[
\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{dust} = \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha \tilde{V}^\beta) = \tilde{V}^\beta \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha) + \tilde{\rho} (\tilde{\nabla} \tilde{V}^\beta) \tilde{V}^\alpha, \tag{3.16}
\]

and, since \( \tilde{V}^\beta \tilde{V}_\beta = 1 \), if we multiply Eq. (2.23) by \( \tilde{V}_\beta \) and use the fact that the divergence of the right hand side of Eq. (3.11) is zero to arrive at the following equation for the scalar field \( \phi \) in the Einstein frame, namely,

\[
\partial^\beta \phi (\tilde{\Box} \phi + V') + \tilde{V}^\beta \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha) = \tilde{\rho} \tilde{V}^\alpha \tilde{\nabla}_\alpha \tilde{V}^\beta = 0. \tag{3.17}
\]

Recalling that dust matter follows geodesics on the original Jordan frame, \( V^\alpha \nabla_\alpha V^\beta = 0 \), and taking it as a working hypothesis that the same is true in the conformally related
Einstein frame, we find that the last two terms in this equation are equal to the last two terms in Eq. (3.15) and so we conclude that Eq. (3.17) provides an equivalent form of Eq. (3.15).

We note that only in the very special case where we impose the constraint
\[ \bar{V}_\beta = \partial_\beta \phi, \quad (3.18) \]
which implies some sort of ‘alignment’ between the dust component and the scalar field, does the scalar field equation (3.15) becomes the standard one, namely,
\[ \bar{\Box} \phi + V' = 0. \quad (3.19) \]

We now study the behaviour of the total slice energy of the system comprised of \( \phi \) and the dust component. We choose \( V = X \) so that
\[ P^\alpha n_\alpha = X_\beta n_\alpha \rho V^\alpha V^\beta = \rho V^\alpha n_\alpha. \quad (3.20) \]

Hence, applying Stokes’ theorem we obtain
\[ \int_K \bar{\nabla}_\alpha (\bar{\rho} V^\alpha) d\bar{\mu} = \int_{\partial K} \bar{\rho} V^\alpha \bar{n}_\alpha d\bar{\sigma}. \quad (3.21) \]

Therefore Eq. (3.7) becomes
\[ E_{t_1}(\phi) - E_{t_0}(\phi) = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \bar{T}^{\alpha\beta}(\phi)(\bar{\nabla}_\alpha \bar{V}_\beta + \bar{\nabla}_\beta \bar{V}_\alpha) d\bar{\mu} \]
\[ - \left[ \int_{\mathcal{M}_{t_1}} \bar{\rho} V^\alpha \bar{n}_\alpha d\bar{\mu}_{t_1} - \int_{\mathcal{M}_{t_0}} \bar{\rho} V^\alpha \bar{n}_\alpha d\bar{\mu}_{t_0} \right] \]
\[ = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \bar{T}^{\alpha\beta}(\phi)(\bar{\nabla}_\alpha \bar{V}_\beta + \bar{\nabla}_\beta \bar{V}_\alpha) d\bar{\mu} + \int_{\mathcal{M}_{t_0}} \bar{\rho} V^\alpha \bar{n}_\alpha d\bar{\mu}_{t_0} \]
\[ - \int_{\mathcal{M}_{t_1}} \bar{\rho} V^\alpha \bar{n}_\alpha d\bar{\mu}_{t_1}, \quad (3.22) \]
or
\[ E_t(\phi) + E_{t}(\text{dust}) = E_{t_0}(\phi) + E_{t_0}(\text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \bar{T}^{\alpha\beta}(\phi)(\bar{\nabla}_\alpha \bar{V}_\beta + \bar{\nabla}_\beta \bar{V}_\alpha) d\bar{\mu}, \quad (3.23) \]
where by definition and Eq. (3.20), for any \( t \),
\[
E_t(\text{dust}) = \int_{\mathcal{M}_t} \rho V^\alpha n_\alpha d\mu_t. \tag{3.24}
\]

We see that the last term in Eq. (3.23) can be zero only when \( V \) is a Killing vectorfield.

We therefore arrive at the following result about the total slice energy with respect to the fluid itself.

**Theorem 3.1** The total slice energy with respect to the timelike vectorfield \( \tilde{V} \), tangent to the dust timelines, of the scalar field-dust system satisfying the field equations (3.9), satisfies
\[
E_t(\phi + \text{dust}) = E_0(\phi + \text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu}. \tag{3.25}
\]

In particular the slice energy of the scalar field-dust system is conserved when \( \tilde{V} \) is a Killing vectorfield of \( \tilde{g} \).

We also conclude that the property of the conservation of slice energy for dust is a conformal invariant. However, when \( V \) is not a Killing vectorfield, we see that there is a nontrivial contribution to the slice energy coming from the combined effect of the stress tensor of the scalar field generated by the conformal transformation coupled to the non-stationarity of the spacetime due to the lack of a Killing vector. Note that this contribution is also nonzero even in the special case that Eq. (3.18) is assumed for in that case the first term in \( \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) \) is zero because \( \tilde{V}^\alpha \tilde{V}^\beta(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) = 0 \), but the whole combination is still not zero as there are additional terms coming from the contributions of the other terms in Eq. (2.11) (unless the fluid satisfies an extra condition – see below).

We now proceed to see how this result changes when we assume that \((V, g)\) is filled with a perfect fluid. With our conventions the stress tensor of a perfect fluid with energy density \( \rho \) and pressure density \( p \) is \( T_{\alpha\beta} = (\rho + p)V_\alpha V_\beta - pg_{\alpha\beta} \) and the fluid satisfies the field equations
\[
f' R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box g f' = (\rho + p)V_\alpha V_\beta - pg_{\alpha\beta}. \tag{3.26}
\]
In this case, because the energy-momentum vector \( P^\alpha = \rho V^\alpha \), we have \( P^\alpha n_\alpha = \rho V^\alpha n_\alpha \) and so the slice energy with respect to the timelike vectorfield \( \tilde{V} \) in the Einstein frame is again

\[
E_t(\text{fluid}) = \int_{\mathcal{M}_t} \tilde{\rho} \tilde{\nabla}^\alpha \tilde{n}_\alpha d\tilde{\mu}_t.
\] (3.27)

Then

\[
\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{\text{fluid}} = \tilde{\nabla}_\alpha \left[ (\tilde{\rho} + \tilde{p}) \tilde{V}^\alpha \right] \tilde{V}^\beta + (\tilde{\rho} + \tilde{p}) \tilde{V}^\alpha \tilde{\nabla}_\beta - \tilde{\nabla}^\beta \tilde{p}.
\] (3.28)

Since \( \tilde{V}_\alpha \tilde{V}^\alpha = 1 \), we have \( \tilde{V}_\alpha \tilde{\nabla}_\beta \tilde{V}^\alpha = 0 \) and so on multiplication of Eq. (3.28) by \( \tilde{V}_\beta \) we find

\[
\tilde{V}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{\text{fluid}} = \tilde{\nabla}_\alpha \left[ (\tilde{\rho} + \tilde{p}) \tilde{V}^\alpha \right] - \tilde{V}^\beta \partial_\beta \tilde{p} = \tilde{p} \tilde{\nabla}_\alpha \tilde{V}^\alpha + \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha).
\] (3.29)

Integrating Eq. (3.29) on the spacetime slab \( \mathcal{D} \) and using Eq. (3.21) to re-express the \( \rho \)-term in (3.29) we have

\[
\int_{t_0}^{t_1} \int_{\mathcal{M}_t} -\tilde{V}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{\text{fluid}} d\tilde{\mu} = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} -\tilde{p} \tilde{\nabla}_\alpha \tilde{V}_\alpha d\tilde{\mu} - \left[ \int_{\mathcal{M}_{t_1}} \tilde{\rho} d\tilde{\mu}_{t_1} - \int_{\mathcal{M}_{t_0}} \tilde{\rho} d\tilde{\mu}_{t_0} \right]
\] (3.30)

and therefore we obtain from (3.27) the general energy transport equation in the form

\[
E_{t_1}(\phi) - E_{t_0}(\phi) = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi) (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu}
\]

\[
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{p} \tilde{\nabla}_\alpha \tilde{V}_\alpha d\tilde{\mu} + E_{t_0}(\text{fluid}) - E_{t_1}(\text{fluid}).
\] (3.31)

We thus arrive at the following result.

**Theorem 3.2** The total slice energy of the scalar field – perfect fluid system satisfying the field equations (3.20), depends upon the integrated pressure according to the formula

\[
E_{t_1}(\phi + \text{fluid}) = E_{t_0}(\phi + \text{fluid}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi) (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{p} \tilde{\nabla}_\alpha \tilde{V}_\alpha d\tilde{\mu}.
\] (3.32)

In particular, the slice energy is conserved when \( \tilde{V} \) is a Killing vectorfield for \( \tilde{g} \) (stationary spacetime).
When $V$ is not a Killing vector field, this slice energy is not generally conserved and this is true even in the special case of a fluid with zero expansion, $\tilde{\nabla}^a \tilde{V}_a = 0$, for which the last term in Eq. (3.32) is zero. In this case the term depending on the scalar field continues to have a nonzero contribution to the total slice energy. This term is given by

$$T^{\alpha\beta}(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) = \partial^\alpha \phi \partial^\beta \phi (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) - \tilde{\nabla}_\alpha \tilde{V}^\alpha (\partial^\lambda \phi \partial_\lambda \phi - 2V(\phi))$$ (3.33)

and so Eq. (3.32) becomes

$$E_{t_1}(\phi + \text{fluid}) = E_{t_0}(\phi + \text{fluid}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \partial^\alpha \phi \partial^\beta \phi (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\mu - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} (\frac{1}{2} \partial^\lambda \phi \partial_\lambda \phi + \tilde{p} - V(\phi)) \tilde{\nabla}^\alpha \tilde{V}_\alpha d\mu.$$ (3.34)

We conclude that the only other possible case for which we have slice energy conservation is when $V$ is not a Killing vector field for $\tilde{g}$, but the alignment condition (3.18) holds (making the middle term in Eq. (3.31) equal to zero) and the fluid has in addition zero expansion (last term in Eq. (3.34) is zero).

4 Discussion

The results of this paper allow us to make some comments concerning the problem of deciding which of the two frames (or metrics), Jordan or Einstein, is the physical one, meaning in which of the two representations of the dynamics test particles follow geodesics (assuming the validity of the principle of equivalence). In the case where test particles follow geodesics in both frames, one says that the two conformally related frames are physically equivalent. The main results of Section 3, in particular Eq. (3.25) (as well as its pressure extension - Eq. (3.32)), were proved under the implicit assumption that the vectorfield $V^a$ of the dust streamlines generates geodesics in both the Jordan frame and the conformally related Einstein frame.

But does $\tilde{V}^\alpha$ always generate a geodesic in the latter frame so that $\tilde{V}_\beta \tilde{\nabla}_\alpha T^{\alpha\beta}_{\text{dust}} = \tilde{V}_\alpha (\tilde{\rho} \tilde{V}^\alpha)$ (cf. Eq. (3.16))? In general, it will not do so and the two frames will not be
physically equivalent. In this case, we have
\[ \tilde{V}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha \beta}_{dust} = \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha) \tilde{V}^\beta + \tilde{\rho}(\tilde{\nabla}_\alpha \tilde{V}^\beta) \tilde{V}^\alpha \tilde{V}_\beta, \]  
(4.1)
so that the result of Theorem 3.1 becomes,
\[ E_t (\phi + \text{dust}) = E_0 (\phi + \text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha \beta} (\phi) (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu} \]
\[ - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{\rho}(\tilde{\nabla}_\alpha \tilde{V}^\beta) \tilde{V}^\alpha \tilde{V}_\beta d\tilde{\mu}, \]
\[ = E_0 (\phi + \text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} (\tilde{T}^{\alpha \beta} (\phi) - \tilde{\rho} \tilde{V}^\alpha \tilde{V}^\beta) (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu}. \]  
(4.2)

We may therefore conclude that the expressions for the total slice energy in the two situations considered here, namely, when test particles follow geodesics in both metrics \( g, \tilde{g} \) (cf. Eq. 3.25), or only in the original Jordan frame metric \( g \) (cf. Eq. 4.2), are different and in the latter case there is an extra term contributing to the total energy (i.e., the dust term in the integrand in the last term in Eq. 4.2). This additional contribution will appear as a measurable quantity which, if measured to be nonzero, will lead us to conclude that the two conformally related frames cannot be physically indistinguishable.

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