PASSING $C^*$-CORRESPONDENCE RELATIONS TO THE CUNTZ-PIMSNER ALGEBRAS

M. ERYÜZLÜ

Abstract. We construct a functor that maps $C^*$-correspondences to their Cuntz-Pimsner algebras. Applications include a generalization of the well-known result of Muhly and Solel: Morita equivalent $C^*$-correspondences have Morita equivalent Cuntz-Pimsner algebras; as well as the result of Muhly, Pask, and Tomforde: regular strong shift equivalent $C^*$-correspondences have Morita equivalent Cuntz-Pimsner algebras.

1. Introduction

When the generalization of Cuntz-Pimsner algebras was fully completed by Katsura [Katsura, 2004], two natural questions arose.

• Can Morita theory be investigated in the $C^*$-correspondence setting?
• If there is a certain relation between two $C^*$-correspondences, what can be said about their Cuntz-Pimsner algebras?

In 1998, Muhly and Solel presented a ground breaking work on the investigation of Morita theory in the $C^*$-correspondence setting [Muhly and Solel, 2000]. They developed a notion of Morita equivalence for given $C^*$-correspondences $AX_A$ and $BY_B$, where there is an imprimitivity bimodule $AM_B$ so that

$$X \otimes_A M \cong M \otimes_B Y$$

as $A-B$ correspondences. They proved that if two injective $C^*$-correspondences are Morita equivalent then the corresponding Cuntz-Pimsner algebras are Morita equivalent in the sense of Rieffel. In [Elefrherakis et al., 2017], the authors presented an elegant proof of a generalized result where they drop the assumption of injectivity. In 2008, Muhly, Pask and Tomforde [Muhly et al., 2008] introduced a weaker relation between $C^*$-correspondences: strong shift equivalence. They proved that regular strong shift equivalent correspondences have Morita equivalent Cuntz-Pimsner algebras. Our motivation was to construct a method that significantly shortens the proofs of these results as well as allows us to easily determine the imprimitivity bimodule between the associated Cuntz-Pimsner algebras.

To study the relation between $C^*$-correspondences and their Cuntz-Pimser algebras, we define a functor from a category we call $ECCor$ to the enchilada category. We construct

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ECCor so that two objects being isomorphic is equivalent to them being Morita equivalent as C*-correspondences. The enchilada category has C*-algebras as objects, and isomorphism classes of C*-correspondences as morphisms. Our functor maps a given C*-correspondence $A_X$ to its Cuntz-Pimsner algebra $O_X$, and a morphism from $A_X$ to $O_Y$ is mapped to the isomorphism class of an $O_X - O_Y$ correspondence.

One of our initial hurdles was to assign an $O_X - O_Y$ correspondence to a given morphism $A_X 	o B_Y$ in ECCor. We overcome this issue by defining an injective covariant $(\pi, \Phi)$ representation for $A_X$ (Proposition 4.2). We prove that the representation $(\pi, \Phi)$ admits a gauge action (Proposition 4.11). Then the Gauge Invariant Uniqueness Theorem gives us one of our main results: $O_X$ is isomorphic to the C*-algebra generated by the representation $(\pi, \Phi)$.

In Section 6 we use our techniques to prove the results regarding Morita equivalent and strong shift equivalent correspondences as we aimed. There is in fact more accomplished in that section: we prove that the Morita equivalence between the Cuntz-Pimsner algebras associated to the regular strong shift equivalent correspondences is gauge equivariant (Theorem 6.7).

This paper serves as an updated version of (Eryüzlü, 2021). The previous definition of ECCor, unfortunately, yielded a subtle gap in the proof of (Eryüzlü, 2021, Theorem 5.1): the functor might not be well-defined. We conquer this problem by modifying the morphisms in ECCor, which does not affect any of our main results but one application. In fact, it is still an open problem whether (Eryüzlü, 2021, Theorem 6.7) holds.

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2. Preliminaries

A C*-correspondence $A_X$ is a right Hilbert $B$-module equipped with a left action given by a homomorphism $\varphi_X : A \to L(X)$, where $L(X)$ denotes the C*-algebra of adjointable operators on $X$. The correspondence $A_X$ is called non-degenerate if the set $A \cdot X = \{ \varphi_X(a)x : a \in A, x \in X \}$ is dense in $X$. Note here that by Cohen-Hewitt factorization theorem we have $A \cdot X = A \cdot X$. In this paper all our correspondences will be non-degenerate by standing hypothesis.

A C*-correspondence $A_X$ is called injective if the left action $\varphi_X : A \to L(X)$ is injective; it is called regular if the homomorphism $\varphi_X$ is injective and $\varphi_X(A)$ is contained in the C*-algebra $K(X)$ of compact operators on $X$.

A C*-correspondence homomorphism is a triple $(\Phi, \varphi_l, \varphi_r) : A_X \to B_Y$ consisting of a linear map $\Phi : X \to Y$, and homomorphisms $\varphi_l : A \to B$ and $\varphi_r : C \to D$ satisfying

1. $\Phi(a \cdot x) = \varphi_l(a) \cdot \Phi(x)$,
2. $\varphi_r(\langle x, z \rangle_C) = \langle \Phi(x), \Phi(z) \rangle_D$,

for all $a \in A$, and $x, z \in X$.

The triple $(\Phi, \varphi_l, \varphi_r)$ is called a C*-correspondence isomorphism if, in addition, $\Phi$ is bijective and $\varphi_l, \varphi_r$ are isomorphisms. In this paper we mostly deal with the situations
where $A = B$ and $C = D$. In those cases we just take $\varphi_l$ and $\varphi_r$ to be the identity maps on $A$ and $C$ respectively, and we simply denote the isomorphism $\Phi : A \to A$ by $\Phi$ instead of the triple $(\Phi, \text{id}_A, \text{id}_C)$. We denote by $\text{Ad} \Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ the associated $C^*$-algebra isomorphism.

Let $A_0$ and $B_0$ be dense $*$-algebras of $C^*$-algebras $A$ and $B$, respectively. An $A_0 - B_0$ bimodule $X_0$ is called a pre-correspondence if it has a $B_0$-valued semi-inner product satisfying

$$\langle x, y \cdot b \rangle = \langle x, y \rangle b, \quad \langle x, y \rangle^* = \langle y, x \rangle$$

and $\langle a \cdot x, a \cdot x \rangle \leq \|a\|^2 \langle x, x \rangle$ for all $a \in A_0, b \in B_0$ and $x, y \in X_0$. The Hausdorff completion $X$ of $X_0$ becomes an $A - B$ correspondence by taking the limits of the operations.

**Proposition 2.1.** ([Echterhoff et al., 2006, Lemma 1.23]) *Let $X_0$ be an $A_0 - B_0$ pre-correspondence, and let $Z$ be an $A - B$ correspondence. If there is a map $\Phi : X_0 \to Z$ satisfying

$$\Phi(a \cdot x) = \varphi_Z(a) \Phi(x) \quad \text{and} \quad \langle \Phi(x), \Phi(y) \rangle_B = \langle x, y \rangle_{B_0},$$

for all $a \in A_0$ and $x, y \in X_0$, then $\Phi$ extends uniquely to an injective $A - B$ correspondence homomorphism $\overline{\Phi} : X \to Z$.*

The balanced tensor product $X \otimes_B Y$ of an $A - B$ correspondence $X$ and a $B - C$ correspondence $Y$ is formed as follows: the algebraic tensor product $X \otimes Y$ is a pre-correspondence with the $A - C$ bimodule structure satisfying

$$a(x \otimes y)c = ax \otimes yc \quad \text{for } a \in A, x \in X, y \in Y, c \in C,$$

and the unique $C$-valued semi-inner product whose values on elementary tensors are given by

$$\langle x \otimes y, u \otimes v \rangle_C = \langle y, \langle x, u \rangle_B v \rangle_C \quad \text{for } x, u \in X, y, v \in Y.$$

This semi-inner product defines a $C$-valued inner product on the quotient of $X \otimes Y$ by the subspace generated by elements of form

$$x \cdot b \otimes y - x \otimes \varphi_Y(b)y \quad (x \in X, y \in Y, b \in B).$$

The completion, i.e., the Hausdorff completion of $X \otimes Y$, is an $A - C$ correspondence $X \otimes_B Y$, where the left action is given by

$$A \to \mathcal{L}(X \otimes_B Y), \quad a \mapsto \varphi_X(a) \otimes 1_Y,$$

for $a \in A$.

We denote the canonical image of $x \otimes y$ in $X \otimes_B Y$ by $x \otimes_B y$. The term balanced refers to the property

$$x \cdot b \otimes_B y = x \otimes_B b \cdot y \quad \text{for } x \in X, b \in B, y \in Y,$$

which is automatically satisfied.

**Lemma 2.2** ([Fowler et al., 2003]). *Let $X$ be a $C^*$-correspondence over $A$ and let $I$ be an ideal of $A$. Then we have the following.*
(1) There is an isometric embedding $\iota : K(XI) \to K(X)$ such that
\[ \theta_{\xi,\nu} \mapsto \theta_{\xi,\nu} \quad \text{for} \quad \xi, \nu \in XI. \]
Moreover, for $T \in K(XI)$, the operator $\iota(T)$ is the unique extension of $T$ to an operator in $L(X)$ whose range is contained in $XI$.

(2) Assume $Y$ is an $A - B$ correspondence. If the left action $\varphi_Y : A \to L(Y)$ is injective, the map $\iota : T \mapsto T \otimes 1_Y$ gives an isometric homomorphism of $L(X)$ into $L(X \otimes_A Y)$.
If, in addition, $\varphi_Y(A) \subseteq K(Y)$, then $\iota$ embeds $K(X)$ into $K(X \otimes_A Y)$.

A Hilbert bimodule $AX_B$ is a $C^*$-correspondence that is also equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle$, which satisfies
\[ A\langle ax, y \rangle = a\langle x, y \rangle \quad \text{and} \quad A\langle x, y \rangle^* = A\langle y, x \rangle \]
for all $a \in A$, $x, y \in X$, as well as the compatibility property
\[ A\langle x, y \rangle z = \langle y, z \rangle_B \quad \text{for} \quad x, y, z \in X. \]
A Hilbert bimodule $AX_B$ is left-full if the closed span of $A\langle X, X \rangle$ is all of $A$.

An imprimitivity bimodule $AX_B$ is an Hilbert bimodule that is full on both the left and the right. It’s dual $B\tilde{X}_A$ is formed as follows: write $\tilde{x}$ when a vector $x \in X$ is regarded as belonging to $\tilde{X}$, define $B - A$ bimodule structure by
\[ b\tilde{x}a = \widetilde{a^*xb^*} \]
and the inner product by
\[ B\langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle_B \quad \text{and} \quad \langle \tilde{x}, \tilde{y} \rangle_A = A\langle x, y \rangle \]
for $a \in A$, $b \in B$, and $x, y \in X$.

The identity correspondence on $A$ is the Hilbert bimodule $AA_A$ where bimodule structure given by multiplication, and the inner products are given by
\[ A\langle a, b \rangle = ab^*, \quad \langle a, b \rangle_A = a^*b, \quad \text{for} \quad a, b \in A. \]

**Lemma 2.3.** Let $AX_B$ be an imprimitivity bimodule and $B\tilde{X}_A$ be it’s dual. Then, the maps
\[ m_A : X \otimes_B \tilde{X} \to A, \quad x_1 \otimes_B \tilde{x}_2 \mapsto A\langle x_1, x_2 \rangle \]
\[ m_B : \tilde{X} \otimes_A X \to B, \quad \tilde{x}_1 \otimes_A x_2 \mapsto \langle x_1, x_2 \rangle_B \]
are $C^*$-correspondence isomorphisms satisfying the equality
\[ m_A(x \otimes_B \tilde{y}) \cdot z = x \cdot m_B(\tilde{y} \otimes_A x) \]
for all $x, y, z \in X$.

A representation $(\pi, t)$ of $AX_A$ on a $C^*$-algebra $B$ consists of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ such that
\[ \pi(a)t(x) = t(\varphi_X(a)(x)) \quad \text{and} \quad t(x)^*t(y) = \pi(\langle x, y \rangle_A), \]
for $a \in A$ and $x, y \in X$, where $\varphi_X$ is the left action homomorphism associated with $A_X A$.

An application of the $C^*$-identity shows that $t(x) \pi(a) = t(x \cdot a)$ is also valid. For each representation $(\pi, t)$ of $A_X A$ on $B$, there exist a homomorphism $\Psi_t : \mathcal{K}(X) \to B$ such that

$$\Psi_t(\theta_{x,y}) = t(x)t(y)^*$$

for $x, y \in X$. The representation $(\pi, t)$ is called injective if $\pi$ is injective, in which case $t$ is an isometry and $\Psi_t$ is injective. We denote the $C^*$-algebra generated by the images of $\pi$ and $t$ in $B$ by $C^*(\pi, t)$.

**Lemma 2.4.** [Katsura, 2004, Lemma 2.4] Let $(\pi, t)$ be a representation of a given $C^*$-correspondence $A_X A$. Then we have

(i) $\pi(a)\Psi_t(k) = \Psi_t(\varphi_X(a)k)$

(ii) $\Psi_t(k)t(x) = t(kx)$

for $a \in A$, $x \in X$, and $k \in \mathcal{K}(X)$.

Now, consider a $C^*$-correspondence $A_X A$. Let $X^{\otimes 0} = A$, $X^{\otimes 1} = X$, and for $n \geq 2$ let $X^{\otimes n} = X \otimes_A X^{\otimes (n-1)}$. Each $X^{\otimes n}$ is a $C^*$-correspondence over $A$ with

$$\varphi_n(a)(x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n) := \varphi_X(a)x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n$$

$$(x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n) \cdot a := x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A (x_n \cdot a).$$

The operator $\varphi_0(a) \in \mathcal{K}(X^{\otimes 0})$ is just the left multiplication operator on $A$. For any given representation $(\pi, t)$ of $A_X A$ on a $C^*$-algebra $B$, set $t^0 = \pi$ and $t^1 = t$. For $n \geq 2$, define a linear map

$$t^n : X^{\otimes n} \to B, \quad t^n(x \otimes_A y) = t(x)t^{n-1}(y),$$

where $x \in X$, $y \in X^{\otimes n-1}$. Then $(\pi, t^n)$ is a representation of $A_X^{\otimes n}$ on $B$. The associated homomorphism $\Psi_{t^n} : \mathcal{K}(X^{\otimes n}) \to B$ is given by

$$\Psi_{t^n}(\theta_{\xi,\mu}) = t^n(\xi)t^n(\mu)^*,$$

for $\xi, \mu \in X^{\otimes n}$. If $(\pi, t)$ is injective, then the linear map $t^n$ is isometric, and the homomorphism $\Psi_{t^n}$ is injective.

For a representation $(\pi, t)$ of $A_X A$ we have [Katsura, 2004, Proposition 2.7],

$$C^*(\pi, t) = \overline{\text{span}}\{t^n(x_n)t^m(y_m)^* : x_n \in X^{\otimes n}, y_m \in X^{\otimes m}, n, m \geq 0\}.$$ 

Consider a $C^*$-correspondence $A_X A$. The ideal $J_X$ is defined as

$$J_X = \varphi_X^{-1}(\mathcal{K}(X)) \cap (\text{Ker } \varphi_X)^\perp$$

$$= \{a \in A : \varphi_X(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \text{Ker } \varphi_X\},$$

and is called the *Katsura ideal*. The ideal $J_X$ is the largest ideal of $A$ such that the restriction map $J_X \to \mathcal{L}(X)$ is an injection into $\mathcal{K}(X)$. Notice here that if $A_X A$ is regular, i.e., the left action $\varphi_X : A \to \mathcal{L}(X)$ is injective and $\varphi_X(A) \subseteq \mathcal{K}(X)$, then $J_X = A$.

A representation $(\pi, t)$ of $A_X A$ is called *covariant* if $\pi(a) = \Psi_t(\varphi_X(a))$, for all $a \in J_X$.

The $C^*$-algebra generated by the universal representation of $A_X A$ is called the *Toeplitz algebra* $\mathcal{T}_X$ of $A_X A$. The $C^*$-algebra generated by the universal covariant representation of $A_X A$ is called the *Cuntz-Pimsner algebra* $\mathcal{O}_X$ of $A_X A$. 

3. Categories

As mentioned in the introduction, in the enchilada category our objects are $C^*$-algebras, and the morphisms from $A$ to $B$ are the isomorphism classes of $A - B$ correspondences. The composition of $[AX_B]: A \to B$ with $[BY_C]: B \to C$ is the isomorphism class of the balanced tensor product $A(X \otimes_B Y)_C$; the identity morphism on $A$ is the isomorphism class of the identity correspondence $AA$, and the zero morphism $A \to B$ is $[A0_B]$. It is a crucial fact for this work that a morphism $[AX_B]$ is an isomorphism in the enchilada category if and only if $A_XB$ is an imprimitivity bimodule (Echterhoff et al., 2006, Lemma 2.4). A detailed study of the enchilada category can be found in (Eryüzlü et al., 2020).

**Notation:** For any given $C^*$-correspondence $AX_B$ we denote the correspondence isomorphism $X \otimes_A A \to M, m \otimes a \mapsto m \cdot a$ by $i_{r,m}$, and the correspondence isomorphism $A \otimes_A M \to M, a \otimes m \mapsto a \cdot m$ by $i_{l,m}$.

**Definition 3.1.** Let $AX_A, BY_B$, and $AM_B$ be $C^*$-correspondences, and let $U_M : X \otimes_A M \to M \otimes_B Y$ be an $A - B$ correspondence isomorphism. Then the isomorphism class of the pair $(AM_B, U_M)$, denoted by $[AM_B, U_M]$, consists of pairs $(AN_B, U_N)$ such that

- there exists an isomorphism $\xi : AM_B \to AN_B$;
- $U_N : X \otimes_A N \to N \otimes_B Y$ is a $C^*$-correspondence isomorphism; and
- the diagram

$$
\begin{array}{ccc}
X \otimes_A M & \xrightarrow{\iota \otimes \xi} & X \otimes_A N \\
\downarrow U_M & & \downarrow U_N \\
M \otimes_B Y & \xrightarrow{\xi \otimes 1_N} & N \otimes_B Y
\end{array}
$$

commutes.

We now introduce our domain category:

**Theorem 3.2.** There exists a category $\text{ECCor}$ such that

- objects are $C^*$-correspondences;
- morphisms $AX_A \to BY_B$ are isomorphism classes of the pairs $(AM_B, U_M)$ where $U_M$ denotes an $A - B$ correspondence isomorphism $X \otimes_A M \to M \otimes_B Y$, and $AM_B$ is a regular correspondence satisfying $J_X \cdot M \subseteq M \cdot J_Y$;
- the composition $[BN_C, U_N] \circ [AM_B, U_M]$ is given by the isomorphism class $[A(M \otimes_B N)_C, U_{M \otimes_B N}]$ where $U_{M \otimes_B N}$ denotes the isomorphism $(1_M \otimes U_N)(U_M \otimes 1_N)$;
- the identity morphism on $AX_A$ is $[AA, U_A]$, where $U_A$ denotes the isomorphism $i_{l,x}^{-1} \circ i_{r,x} : X \otimes_A A \to A \otimes_A X$. 

Proof. Let \([AM_B, U_M] \in \text{Mor}(A X_A, B Y_B)\) and \([B N_C, U_N] \in \text{Mor}(B Y_B, C Z_C)\). Then, it is not difficult to verify that \([A(M \otimes_B N)_C, (1_N \otimes U_M)(U_M \otimes 1_N)]\) is in \(\text{Mor}(A X_A, C Z_C)\). Now, let \([C K_D, U_K] \in \text{Mor}(C Z_C, D R_D)\). The composition is associative:

\[
([C K_D, U_K] \circ [B N_C, U_N]) \circ [A M_B, U_M]
= [B(N \otimes_C K)_D, (1_N \otimes U_K)(U_N \otimes 1_K)] \circ [A M_B, U_M]
= [A(M \otimes_B (N \otimes_C K))_D, (1_M \otimes (1_N \otimes U_K)(U_N \otimes 1_K))(U_M \otimes 1_N \otimes_C K)]
= [A((M \otimes_B N) \otimes_C K)_D, (1_M \otimes (1_N \otimes U_K))(1_M \otimes U_N)(U_M \otimes 1_N) \otimes 1_K]
= [C K_D, U_K] \circ ([B N_C, U_N] \circ [A M_B, U_M]).
\]

It remains to prove that \([A A_A, U_A] \in \text{Mor}(A X_A, A X_A)\) is the identity morphism on \(A X_A\). Let \([A N_B, U_N] \in \text{Mor}(A X_A, B Y_B)\). We show that the following diagram commutes.

\[
\begin{array}{ccc}
X \otimes_A A \otimes_A N & \xrightarrow{1_X \otimes i_{1_N}} & X \otimes_A N \\
\downarrow (1_A \otimes U_N)(1_A \otimes U_N) & & \downarrow U_N \\
A \otimes_A N \otimes_B Y & \xrightarrow{U_N \otimes 1_Y} & N \otimes_B Y
\end{array}
\]

Let \(x \in X\), \(a \in A\), and \(n \in N\). On one hand we have

\[
(u_{t,n} \otimes 1_Y)(1_A \otimes U_N)(i_{t,x}^{-1} \otimes 1_N)(a \cdot x \otimes_A n) = (u_{t,n} \otimes 1_Y)(a \otimes_A U_N(x \otimes_A n))
= a \cdot U_N(x \otimes_A n)
= U_N(a \cdot x \otimes_A n).
\]

By linearity and density the equalities above hold for any element of \(X \otimes_A N\), i.e,

\[
(i_{t,n} \otimes 1_Y)(1_A \otimes U_N)(i_{t,x}^{-1} \otimes 1_N) = U_N. \tag{3.1}
\]

On the other hand,

\[
U_N(1_X \otimes u_{t,n})(x \otimes_A a \otimes_A n) = U_N(x \otimes_A a \cdot n) = U_N(x \cdot a \otimes_A n)
= U_N(i_{r,x} \otimes 1_N)(x \otimes_A a \otimes_A n).
\]

Again by linearity and density, we may conclude that

\[
U_N(1_X \otimes u_{t,n}) = U_N(i_{r,x} \otimes 1_N) \tag{3.2}
\]

Now, combining equations 3.1 and 3.2 we get

\[
U_N(1_X \otimes u_{t,n}) = U_N(i_{r,x} \otimes 1_N)
= (u_{t,n} \otimes 1_Y)(1_A \otimes U_N)(i_{t,x}^{-1} \otimes 1_N)(i_{r,x} \otimes 1_N)
= (u_{t,n} \otimes 1_Y)(1_A \otimes U_N)(U_A \otimes 1_N),
\]

which implies \([A N_B] \circ [A A_A] = [A(A \otimes_A N)_B, (1_A \otimes U_N)(U_A \otimes 1_N)] = [A N_B, U_N]. \tag*{□}
\]

**Proposition 3.3.** A morphism \([A M_B, U_M] : A X_A \rightarrow_B Y_B\) in **ECCor** is an isomorphism if and only if \(A M_B\) is an imprimitivity bimodule.

To prove Proposition 3.3 we first need the following lemma.
Lemma 3.4. Let $A M_B$ be an imprimitivity bimodule given with an $A - B$ correspondence isomorphism $U_M : X \otimes_A M \to M \otimes_B Y$. Let $B N_A$ be the dual of $A M_B$. Then, there exists a $B - A$ correspondence isomorphism $U_N : Y \otimes_B N \to N \otimes_A X$ such that

$$(i_{l,x} \otimes 1_M)(m_A \otimes 1_{X \otimes_A M})(1_{M \otimes_B N} \otimes U_M^{-1}) = U_M^{-1}(1_M \otimes i_{l,y})(1_M \otimes m_B \otimes 1_Y)$$

as operators on $M \otimes_B N \otimes_A M \otimes_B Y$, where $m_A : M \otimes_B N \to A$ and $m_B : N \otimes_A M \to B$ are the isomorphisms defined in Lemma 2.3.

**Proof.** Define a $B - A$ correspondence isomorphism $U_N : Y \otimes_B N \to N \otimes_A X$ as follows:

$$
\begin{array}{c}
Y \otimes_B N \xrightarrow{i_{l,y}^{-1} \otimes 1_N} B \otimes_B Y \otimes_B N \xrightarrow{1_N \otimes U_M^{-1} \otimes 1_N} N \otimes_A M \otimes_B Y \otimes_B N \\
N \otimes_A X \otimes_A M \otimes_B N \xrightarrow{1_N \otimes 1_X \otimes m_A} N \otimes_A X \otimes_A A \xrightarrow{1_N \otimes i_{r,x}} N \otimes_A X,
\end{array}
$$

i.e.,

$$U_N := (1_N \otimes i_{r,x})(1_{N \otimes A X} \otimes m_A)(1_N \otimes U_M^{-1} \otimes 1_N)(m_B^{-1} \otimes 1_Y \otimes_B N)(i_{l,y}^{-1} \otimes 1_N).$$

Now, notice that by linearity and density, it suffices to prove the required equality for the elements of form $\xi \otimes_B n \otimes_A \mu \otimes_B y$, where $\xi, \mu \in M, n \in N, y \in Y$:

$$(i_{l,x} \otimes 1_M)(m_A \otimes 1_{X \otimes_A M})(1_{M \otimes_A N} \otimes U_M^{-1}) (\xi \otimes_B n \otimes_A \mu \otimes_B y)
= (i_{l,x} \otimes 1_M) \left[m_A(\xi \otimes_B n) \otimes_A U_M^{-1}(\mu \otimes_B y)\right]
= m_A(\xi \otimes_B n) \cdot U_M^{-1}(\mu \otimes_B y)
= U_M^{-1}[m_A(\xi \otimes_B n) \cdot \mu \otimes_B y]
= U_M^{-1}[\xi \cdot m_B(n \otimes_A \mu) \otimes_B y]
= U_M^{-1}[\xi \otimes_B m_B(n \otimes_A \mu) \cdot y]
= U_M^{-1}(1_M \otimes i_{l,y})(1_M \otimes m_B \otimes 1_Y)(\xi \otimes_B n \otimes_A \mu \otimes_B y),$$

which completes the proof. \hfill \Box

**Proof of Proposition 3.3.** It is not difficult to see that $[A M_B, U_M]$ is an isomorphism in ECCor then $A M_B$ must be invertible. For the other direction, let $B N_A$ and $U_N$ be as in Lemma 3.4. We show that the diagram

$$
\begin{array}{c}
X \otimes_A M \otimes_B N \xrightarrow{1_X \otimes m_A} X \otimes_A A \\
\downarrow(1_M \otimes U_N)(U_M \otimes 1_N)
M \otimes_B N \otimes_A X \xrightarrow{m_A \otimes 1_X} A \otimes_A X \xrightarrow{U_A}
\end{array}
$$

commutes. This will allow us to conclude that $[M \otimes_B N, (1_M \otimes U_N)(U_M \otimes 1_N)] = [A A_A, U_A]$, i.e., $[B N_A, U_N]$ is a right inverse for $[A M_B, U_M]$. 
First, observe that the definition of $U_N$ and Lemma 3.4 together gives us the equality
\[(1_M \otimes U_N)(U_M \otimes 1_N) = (1_M \otimes 1_N \otimes i_{r,x})(m_A^{-1} \otimes 1_X \otimes m_A)(i_{l,x}^{-1} \otimes 1_M \otimes 1_N).\]
Now, let $x = a \cdot x' \in X$, where $a \in A$, $x' \in X$, and let $\xi \in M \otimes_B N$. Then we have
\[(m_A \otimes 1_X)(1_M \otimes 1_N \otimes i_{r,x})(m_A^{-1} \otimes 1_X \otimes m_A)(i_{l,x}^{-1} \otimes 1_M \otimes 1_N)(x \otimes_A \xi)\]
\[= (m_A \otimes 1_X)(1_M \otimes 1_N \otimes i_{r,x})(m_A^{-1} \otimes 1_X \otimes m_A)(a \otimes_A x' \otimes_A \xi)\]
\[= (m_A \otimes 1_X)(1_M \otimes 1_N \otimes i_{r,x})(m_A^{-1}(a) \otimes_A x' \otimes_A m_A(\xi))\]
\[= a \otimes_A x' \cdot m_A(\xi).\]
On the other hand, we have
\[U_A(1_X \otimes m_A)(x \otimes_A \xi) = i_{l,x}^{-1} \circ i_{r,x}(x \otimes_A m_A(\xi))\]
\[= i_{l,x}^{-1}(x \cdot m_A(\xi))\]
\[= i_{l,x}^{-1}(x) \cdot m_A(\xi)\]
\[= a \otimes_A x' \cdot m_A(\xi).\]
We have shown that $(m_A \otimes 1_X)(1_M \otimes U_N)(U_M \otimes 1_N) = U_A(1_X \otimes m_A)$, as desired. One can use the same technique to show that $[_{B}N_A, U_N]$ is also a left inverse for $[_{A}M_B, U_M]$. □

4. A Covariant Representation

In this section we define an injective covariant representation of a $C^*$-correspondence $A_{X_A}$ given with a morphism $[_{A}M_B, U_M]: A_{X_A} \to B_{Y_B}$ in $\text{ECCor}$. We prove that this representation in fact admits a gauge action. Since we use it frequently, we would like to remind the reader that any given Hilbert module isomorphism $U : X_A \to Y_A$ gives rise to an isomorphism $\text{Ad} U : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\text{Ad} U(T) = UTU^{-1}$ for any $T \in \mathcal{L}(X)$.

Let $_{A}X_A$, $_{A}M_B$ be given, where the latter is a regular $C^*$-correspondence. Consider the linear map
\[T : X \to \mathcal{L}(M, X \otimes_A M), \quad T(x)(m) := x \otimes_A m,\]
where $x \in X$, $m \in M$. Then we have
\[\langle T(x)m, y \otimes_A m' \rangle_B = \langle x \otimes_A m, y \otimes_A m' \rangle_B = \langle m, \varphi_M(\langle x, y \rangle_A)m' \rangle_B,\]
for $x, y \in X$ and $m, m' \in M$. This means the adjoint $T(x)^*$ satisfies
\[T(x)^*(y \otimes_A m) = T(x)^*T(y)m = \varphi_M(\langle x, y \rangle_A)m,\]
for any elementary tensor $y \otimes_A m \in (X \otimes_A M)$.

On the other hand, we know by regularity that the homomorphism $\varphi_M : A \to \mathcal{L}(M)$ is injective and $\varphi_M(a) \in \mathcal{K}(M)$ for any $a \in A$. This allows us to observe that
\[T(x) \in \mathcal{K}(M, X \otimes_A M) \iff T(x)^*T(x) \in \mathcal{K}(M) \iff \varphi_M(\langle x, x \rangle_A) \in \mathcal{K}(M),\]
which implies $T(x) \in \mathcal{K}(M, X \otimes_A M)$, for any $x \in X$.

Lemma 4.1. Let $(Y, t)$ be the universal covariant representation of $_{A}X_A$. Then we have the following.
(i) Consider the subspace
\[ \overline{(X)O_X} := \text{span}\{t(x)S : \ x \in X, S \in O_X\} \]
of \(O_X\). The map \(X \otimes_A O_X \rightarrow \overline{(X)O_X}\) determined on elementary tensors by
\[ x \otimes S \rightarrow t(x)S \]
is an \(A - O_X\) correspondence isomorphism.

(ii) \(J_X \cdot O_X \subseteq \overline{(X)O_X}\).

(iii) When \(A X_A\) is regular, the map defined in (i) gives an isomorphism
\[ A(X \otimes_A O_X)_{O_X} \rightarrow A(O_X)_{O_X}. \]

Proof. Let \(\Phi : X \otimes O_X \rightarrow \overline{(X)O_X}\) be the unique linear map such that \(x \otimes S \mapsto t(x)S\). It suffices to make our computations with elementary tensors. We first show that \(\Phi\) preserves the semi-inner product. Let \(x, y \in X\) and \(S, T \in O_X\). Then,
\[
\langle x \otimes S, y \otimes T \rangle_{O_X} = \langle S, \Upsilon((x, y)A)T \rangle_{O_X} = \langle S, t(x)^*t(y)T \rangle_{O_X} = \langle t(x)S, t(y)T \rangle_{O_X}.
\]
For \(a \in A\), the computation
\[
\Phi(a \cdot (x \otimes S)) = \Phi(ax \otimes S) = t(ax)S = \Upsilon(a)t(x)S = a \cdot \Phi(x \otimes S)
\]
shows that \(\Phi\) preserves the left action. It is clear that \(\Phi\) is surjective, and hence, \(\Phi\) extends to a unique \(C^*\)-correspondence isomorphism \(X \otimes_A O_X \rightarrow \overline{(X)O_X}\).

Part (ii) follows from the fact that for any \(a \in J_X\) we have
\[
\Upsilon(a) = \Psi_t(\varphi_X(a)) \in \Psi_t(K(X)) = \overline{\text{span}\{t(x)t(y)^* : x, y \in X\}}.
\]
For the last part, let \(A X_A\) be a regular \(C^*\)-correspondence. Then, since \(A = J_X\), we have
\[ O_X = A \cdot O_X = J_X \cdot O_X \subseteq \overline{(X)O_X}, \]
hence the map defined in (i) is an isomorphism onto \(O_X\). \(\square\)

**Proposition 4.2.** Let \([A M_B, U_M] : A X_A \rightarrow B Y_B\) be a morphism in \(\text{ECCor}\). Then \(A X_A\) has an injective covariant representation on \(\mathcal{K}(M \otimes_B O_Y)\).

**Proof.** Denote the universal covariant representation of \(B Y_B\) by \((\Upsilon, t)\). We use the \(C^*\)-correspondence isomorphisms
\[ U_M : X \otimes_A M \rightarrow M \otimes_B Y \quad \text{and} \quad V_Y : Y \otimes_B O_Y \rightarrow t(Y)O_Y \]
to construct a linear map \(\Phi : X \rightarrow \mathcal{K}(M \otimes_B O_Y)\).

For each \(x \in X\), define an operator \(T(x) : M \rightarrow M \otimes_B Y\) by
\[ T(x)m = U_M(x \otimes_A m). \quad (4.1) \]
Since \(A M_B\) is regular, as discussed above Lemma 4.1, we have
\[ T(x) \in \mathcal{K}(M, M \otimes_B Y). \]
In addition, since the $C^*$-correspondence $B(\mathcal{O}_Y)\mathcal{O}_Y$ is regular, we have
\[ T(x) \otimes 1_{\mathcal{O}_Y} \in \mathcal{K}(M \otimes_B \mathcal{O}_Y, M \otimes_B Y \otimes_B \mathcal{O}_Y) \quad \text{(Lemma 2.2)}. \]
And now, the operator
\[ \Phi(x) := (1_M \otimes V_Y)(T(x) \otimes 1_{\mathcal{O}_Y}) \in \mathcal{K}(M \otimes_B \mathcal{O}_Y, M \otimes_B \mathcal{O}_Y) \]
can be viewed as a compact operator on the Hilbert $\mathcal{O}_Y$-module $M \otimes_B \mathcal{O}_Y$ whose range is contained in the submodule $M \otimes_B \mathcal{O}_Y$.

Now, define a homomorphism $\pi : A \to \mathcal{L}(M \otimes_B \mathcal{O}_Y)$ by
\[ \pi(a) = \varphi_M(a) \otimes 1_{\mathcal{O}_Y}, \]
for $a \in A$. Note that $\pi$ is injective and $\pi(A) \subseteq \mathcal{K}(M \otimes_B \mathcal{O}_Y)$, by Lemma 2.2.

We claim that $(\pi, \Phi)$ is a representation of $A\mathcal{X}A$. Let $x, x' \in X$, and $a \in A$. Then we have
\[
\Phi(x)\Phi(x') = (T(x)^* \otimes 1_{\mathcal{O}_Y})(1_M \otimes V_Y)^*(1_M \otimes V_Y)(T(x') \otimes 1_{\mathcal{O}_Y})
\]
\[ = T(x)^*T(x') \otimes_B 1_{\mathcal{O}_Y} \quad (1_M \otimes V_Y \text{ is a unitary map})
\]
\[ = \varphi_M(\langle x, x' \rangle) \otimes_B 1_{\mathcal{O}_Y}
\]
\[ = \pi(\langle x, x' \rangle).
\]

It remains to show the equality $\pi(a)\Phi(x) = \Phi(\varphi_X(a)x)$. Observe that we have
\[ (\varphi_M(a) \otimes 1_{\mathcal{O}_Y})(1_M \otimes V_Y) = (1_M \otimes V_Y)(\varphi_M(a) \otimes 1_Y \otimes 1_{\mathcal{O}_Y}). \]
This allows us to make the following computation.
\[
\Phi(\varphi_X(a)x) = (1_M \otimes V_Y)(T(a \cdot x) \otimes 1_{\mathcal{O}_Y})
\]
\[ = (1_M \otimes V_Y)(\varphi_M(a) \otimes 1_Y \otimes 1_{\mathcal{O}_Y})(T(x) \otimes 1_{\mathcal{O}_Y})
\]
\[ = (\varphi_M(a) \otimes 1_{\mathcal{O}_Y})(1_M \otimes V_Y)(T(x) \otimes 1_{\mathcal{O}_Y})
\]
\[ = (\varphi_M(a) \otimes 1_{\mathcal{O}_Y})\Phi(x)
\]
\[ = \pi(\langle x, x' \rangle)\Phi(x).
\]

We now prove that the representation $(\pi, \Phi)$ is covariant. Let $\Psi_\phi : \mathcal{K}(X) \to \mathcal{K}(M \otimes_B \mathcal{O}_Y)$ be the injective homomorphism associated to the representation $(\pi, \Phi)$. Then, for $x, x' \in X$, we have the following equalities on $M \otimes_B \mathcal{O}_Y$.
\[ \Psi_\phi(\theta_{x,x'}) = \Phi(x)\Phi(x')^* 
\]
\[ = (1_M \otimes V_Y)(T(x) \otimes_B 1_{\mathcal{O}_Y})(T(x')^* \otimes 1_{\mathcal{O}_Y})^*(1_M \otimes V_Y)^*
\]
\[ = (1_M \otimes V_Y)(T(x)T(x')^* \otimes_B 1_{\mathcal{O}_Y})(1_M \otimes V_Y)^*
\]
\[ = (1_M \otimes V_Y)[\text{Ad}U_M(\theta_{x,x'} \otimes 1_M) \otimes 1_{\mathcal{O}_Y}](1_M \otimes V_Y)^*.
\]
This implies that, for any $k \in \mathcal{K}(X)$, we have
\[ \Psi_\phi(k)|_{M \otimes_B \mathcal{O}_Y} = (1_M \otimes V_Y)[\text{Ad}U_M(k \otimes 1_M) \otimes 1_{\mathcal{O}_Y}](1_M \otimes V_Y)^*.
\]
\[ ^1\text{as in the first item of Lemma 2.2.} \]
In particular, for \( a \in J_X \), we have
\[
\Psi_\Phi(\varphi_X(a)|_{M \otimes_B \overline{O_Y}}) = (1_M \otimes V_Y)[\text{Ad} U_M(\varphi_X(a) \otimes 1_M) \otimes 1_{O_Y}](1_M \otimes V_Y)^* \\
= (1_M \otimes V_Y)(\varphi_M(a) \otimes 1_Y \otimes 1_{O_Y})(1_M \otimes V_Y)^* \\
= \text{Ad}(1_M \otimes V_Y)(\varphi_M(a) \otimes 1_Y \otimes 1_{O_Y}) \\
= \varphi_M(a) \otimes 1_{O_Y}|_{M \otimes_B \overline{O_Y}} \\
= \pi(a)|_{M \otimes_B \overline{O_Y}}.
\]

On the other hand, for \( a \in J_X \), we know that the image of the operator \( \pi(a) \in \mathcal{K}(M \otimes_B \overline{O_Y}) \) is contained in
\[
J_X M \otimes_B \overline{O_Y} \subseteq MJ_Y \otimes_B \overline{O_Y} = M \otimes_B J_Y \overline{O_Y} \subseteq M \otimes_B \overline{t(Y)O_Y}.
\]
Coupling this with the fact that \( \Psi_\Phi(\varphi_X(a)) = \pi(a) \) on \( M \otimes_B \overline{t(Y)O_Y} \) we get
\[
\pi(a)^* \pi(a) = \Psi_\Phi(\varphi_X(a))^* \pi(a) \in \mathcal{K}(M \otimes_B \overline{O_Y}).
\]
In other words, we have
\[
\pi(a^* a) = \Psi_\Phi(\varphi_X(a))^* \pi(a) = (\pi(a^*) \Psi_\Phi(\varphi_X(a)))^* = \Psi_\Phi(\varphi_X(a^* a)).
\]
One can now show that
\[
\|\Psi_\Phi(\varphi_X(a)) - \pi(a)\|^2 = \|(\Psi_\Phi(\varphi_X(a)) - \pi(a))^*(\Psi_\Phi(\varphi_X(a)) - \pi(a))\| = 0,
\]
for any \( a \in J_X \), which completes the proof. \( \square \)

**Definition 4.3.** Let \([A_MB, U_M]: A X_A \rightarrow BY_B\) be a morphism in \( \text{ECCor} \). Then, the injective covariant representation \((\pi, \Phi)\) of \( A X_A \) defined as in the proof of Proposition 4.2 is called the \( C \)-covariant representation.

Now, the universality of the Cuntz-Pimsner algebra \( \mathcal{O}_X \) gives us the following result.

**Corollary 4.4.** Let \([A_MB, U_M]: A X_A \rightarrow BY_B\) be a morphism in \( \text{ECCor} \). Let \((\pi, \Phi)\) be the associated \( C \)-covariant representation of \( A X_A \). Then, there exists a unique homomorphism \( \sigma: \mathcal{O}_X \rightarrow \mathcal{K}(M \otimes_B \overline{O_Y}) \) such that
\[
\sigma(t_X(x)) = \Phi(x) \quad \text{and} \quad \sigma(t_X(a)) = \pi(a),
\]
for \( x \in X, a \in A \), where \((\mathcal{Y}_X, t_X)\) denotes the universal covariant representation of \( A X_A \). Thus, the regular \( C^* \)-correspondence \( A(M \otimes_B \overline{O_Y}) \overline{O_Y} \) can be viewed as an \( \mathcal{O}_X - \mathcal{O}_Y \) correspondence via the homomorphism \( \sigma \).

**Lemma 4.5.** Let \((\mathcal{Y}, t)\) be the universal covariant representation of \( A X_A \), and let \((\pi, \Phi)\) be the \( C \)-covariant representation of \( A X_A \) associated to the identity morphism \([A A_A, U_A]: A X_A \rightarrow A X_A\) in \( \text{ECCor} \). Then, the \( A \rightarrow \mathcal{O}_X \) correspondence isomorphism
\[
U: A \otimes_A \mathcal{O}_X \rightarrow \mathcal{O}_X, \quad a \otimes_A S \mapsto \mathcal{Y}(a)S
\]
preserves the left \( \mathcal{O}_X \) module structure, i.e.,
\[
\mathcal{O}_X(A \otimes_A \mathcal{O}_X) \mathcal{O}_X \cong \mathcal{O}_X \mathcal{O}_X \mathcal{O}_X.
\]
Proof. Let \( x \in X, \ S \in \mathcal{O}_X, \ a \in A. \) Note that \( x \cdot a = a' \cdot x' \) for some \( a' \in A, x' \in X, \) by Cohen-Hewitt factorization theorem. Now we have
\[
U[t(x) \cdot (a \otimes_A S)] = U[\Phi(x)(a \otimes_A S)] = U[a' \otimes_A t(x')S] = \Upsilon(a')t(x')S = t(x \cdot a)S = t(x)U(a \otimes_A S),
\]
which implies, by linearity and density, that \( U[t(x) \cdot m] = t(x)U(m) \) for any \( m \in A \otimes_A \mathcal{O}_X. \) Moreover, one can easily verify that \( U[\Upsilon(a) \cdot m] = \Upsilon(a) \cdot U(m), \) for any \( a \in A \) and \( m \in (A \otimes_A \mathcal{O}_X). \) This complete the proof as elements \( t(x) \) and \( \Upsilon(a) \) generate \( \mathcal{O}_X. \)

**Definition 4.6.** A \( C^* \)-correspondence \( A_X_A \) is said to be a nondegenerate subcorrespondence of \( bY_B \) if there exists an \( A - B \) correspondence homomorphism \( (\phi, \varphi): A_X_A \rightarrow bY_B \) such that

(i) the linear map \( \phi : X \rightarrow Y \) is injective;
(ii) the homomorphism \( \varphi : A \rightarrow B \) is injective and non-degenerate;
(iii) \( Y = \overline{\phi(X)B}. \)

Notice that any nondegenerate subcorrespondence of an injective correspondence is injective, by definition.

**Lemma 4.7.** Let \( A_X_A \) be a nondegenerate subcorrespondence of a \( C^* \)-correspondence \( bY_B. \) Then, in ECCor, there exists a morphism from \( A_X_A \) to \( bY_B \) if
\[
J_X \cdot B \subseteq J_Y. \tag{4.2}
\]
Condition (4.2) follows immediately when \( bY_B \) is injective.

**Proof.** Let \( (\phi, \varphi): A_X_A \rightarrow bY_B \) be as in Definition 4.6. Then, the homomorphism \( \varphi \) induces a regular correspondence \( A_{BB}. \) We first show that the unique linear map
\[
\xi : X \otimes B \rightarrow Y \quad x \otimes b \mapsto \phi(x)b,
\]
for \( x \in X, \ b \in B, \) extends to an \( A - B \) correspondence isomorphism \( X \otimes_A B \rightarrow Y. \) As usual, we make all our computations with elementary tensors, as it suffices. For \( x, x' \in X, \ a \in A, \) and \( b, b' \in B, \) we have
\[
\xi(a \cdot (x \otimes b)) = \xi(a \cdot x \otimes b) = \phi(a \cdot x)b = \varphi(a) \cdot \phi(x)b \quad \text{(by definition of} \ (\phi, \varphi))
\]
and
\[
\langle \xi(x \otimes b), \xi(x' \otimes b') \rangle_B = \langle \phi(x)b, \phi(x')b' \rangle_B = b^* \langle \phi(x), \phi(x') \rangle_B b' = b^* \varphi((x, x')_A)b' = \langle b, (x, x')_A \cdot b' \rangle_B = \langle x \otimes b, x' \otimes b' \rangle_B.
\]
Thus, by Proposition 2.1, the map $\xi$ extends to an injective $A - B$ correspondence homomorphism, which is clearly surjective.

On the other hand, we have the $A - B$ correspondence isomorphism

$$j: B \otimes_B Y \to Y \quad b \otimes_B y \mapsto b \cdot y.$$  

Then, the composition

$$U_B := j^{-1} \circ \xi: X \otimes_A B \to B \otimes_B Y \quad (4.3)$$

is an $A - B$ correspondence isomorphism. We may now conclude that the isomorphism class $[A_B, U_B]: A_X A \to B_Y B$ is a morphism in $\text{ECCor}$ if we are given the condition $J_X \cdot B \subseteq J_Y$.

We complete the proof by showing that if $B_Y B$ is injective then $J_X \cdot B \subseteq J_Y$: let $a \in J_X$. Since $A_B$ is regular we have $\varphi_X(a) \otimes 1_B \in K(X \otimes_A B)$; which implies

$$\text{Ad} \xi(\varphi_X(a) \otimes 1_B) = \varphi_Y(\varphi(a)) \in K(Y).$$

This means $\varphi(a) \in J_Y$, since $B_Y B$ is injective. Then for any $b \in B$, we have

$$a \cdot b = \varphi(a)b \in J_Y,$$

as desired. $\square$

**Proposition 4.8.** Let $A_X A$ and $B_Y B$ be injective correspondences, and let $(\phi, \varphi): A_X A \to B_Y B$ be as in Definition 4.6. Denote the $C^*$-covariant representation of $A_X A$ on $K(B \otimes_B O_Y)$ by $(\pi, \Phi)$, the universal covariant representation of $B_Y B$ by $(Y_Y, t_Y)$, and the natural $C^*$-algebra isomorphism $K(B \otimes_B O_Y) \to O_Y$ by $\iota$. Then,

$$\iota(\Phi(x)) = t_Y(\phi(x)) \quad \text{and} \quad \iota(\pi(a)) = Y_Y(\varphi(a))$$

for all $x \in X$, $a \in A$. In other words, the $C^*$-algebra $C^*(\pi, \Phi)$ is isomorphic to the $C^*$-algebra generated by $t_Y(\phi(X))$ and $Y_Y(\varphi(A))$.

**Proof.** Note first that the map $\iota$ is really the composition of the isomorphisms $s : K(O_Y) \to O_Y$ and $\text{Ad} \mu : K(B \otimes_B O_Y) \to K(O_Y)$, where $\mu$ denotes the $A - O_Y$ module isomorphism $B \otimes_B O_Y \to O_Y$ determined on elementary tensors by $\mu(b \otimes_B S) = Y_Y(b)S$, for $b \in B$, $S \in O_Y$. We will show that the the following diagram commutes.

![Diagram](image-url)
Let $U_B := j^{-1} \circ \xi : A(X \otimes_A B)_B \to A(B \otimes_B Y)_B$ be the isomorphism \[4.3\] defined in the proof of Lemma \[4.3\]. Let $x \in X, b \in B,$ and $S \in \mathcal{O}_Y.$ Then,

$$\Phi(x)(b \otimes_B S) = (1_B \otimes V_Y)(U_B(x \otimes_A b) \otimes_B S)$$

by construction, where $V_Y$ is the unitary map $Y \otimes_B \mathcal{O}_Y \to t_Y(Y)\mathcal{O}_Y.$ Note here that we have

$$U_B(x \otimes_A b) = j^{-1} \circ \xi(x \otimes_A b) = j^{-1}(\phi(x)b) = b' \otimes_B y,$$

for some $b' \in B, y \in Y$ satisfying $\phi(x)b = b' y,$ by Cohen-Hewitt factorization theorem. This gives us

$$\Phi(x)(b \otimes_B S) = (1_B \otimes V_Y)(U_B(x \otimes_A b) \otimes_B S) = b' \otimes_B t_Y(y)S,$$

which implies

$$\mu \circ \Phi(x)(b \otimes_B S) = \mu(b' \otimes_B t_Y(y)S) = \Upsilon_Y(b')t_Y(y)S
= t_Y(b' \cdot y)S
= t_Y(\phi(x)b)S
= t_Y(\phi(x))\Upsilon_Y(b)S
= t_Y(\phi(x))\mu(b \otimes_B S)
= s^{-1}(t_Y(\phi(x))[\mu(b \otimes_B S)]).$$

This computation allows one to conclude, by linearity and density, that

$$s^{-1}(t_Y(\phi(x))) = \mu \circ \Phi(x) \circ \mu^{-1}
= \text{Ad}(\mu(\Phi(x))),$$

which implies $\iota(\Phi(x)) = t_Y(\phi(x)),$ as desired. Lastly, for $a \in A,$ we have

$$\mu \circ \pi(a)(b \otimes_B S) = \mu(\varphi(a)b \otimes_B S) = \Upsilon_Y(\varphi(a)b)S = \Upsilon_Y(\varphi(a))\mu(b \otimes_B S),$$

which suffices to complete the proof. \[\square\]

Recall that a representation $(\pi, t)$ of $X$ admits a gauge action if for each $z \in \mathbb{T}$ there exists a homomorphism $\beta_z : C^*(\pi, t) \to C^*(\pi, t)$ such that

$$\beta_z(\pi(a)) = \pi(a) \quad \text{and} \quad \beta_z(t(x)) = zt(x)$$

for all $a \in A,$ and $x \in X.$ If it exists, the homomorphism $\beta_z$ is unique. The map

$$\beta : \mathbb{T} \to \text{Aut}(C^*(\pi, t)), \quad z \mapsto \beta_z$$

is called the gauge action. One can easily show that $\beta$ is a strongly continuous homomorphism.

**Theorem 4.9 (The Gauge Invariant Uniqueness Theorem).** Let the pair $(\Upsilon, t)$ be the universal covariant representation of $A \mathcal{X}_A.$ Assume $(\phi_X, t_X)$ is an injective covariant representation of $A \mathcal{X}_A$ on a $C^*$-algebra $B.$ If $(\phi_X, t_X)$ admits a gauge action, then the homomorphism $\rho : \mathcal{O}_X \to B$ is injective. In other words, the natural surjection $\rho : \mathcal{O}_X \to C^*(\phi_X, t_X)$ is an isomorphism.
A proof of the above theorem can be found in [Katsura, 2004].

**Remark 4.10.** Let $\gamma$ be the gauge action for the universal covariant representation $(\Upsilon, t)$ of $B Y_B$. Then, for any $z \in \mathbb{T}$, we have

$$
\gamma_z(t^n(y_n)) = z^n t^n(y_n) \quad \text{and} \quad \gamma_z(t^n(y_n)^*) = z^{-n} t^n(y_n)^*,
$$

where $y_n \in Y^\otimes n$. Now, for each $n \in \mathbb{Z}$ consider the subspace

$$
\mathcal{O}_Y^n := \{ T \in \mathcal{O}_Y : \gamma_z(T) = z^n(T), \text{ for all } z \in \mathbb{T} \}.
$$

We have

$$
\mathcal{O}_Y = \mathop{\text{span}} \{ t^n(y_t)t^m(y_m)^* : y_n \in Y^\otimes n, y_m \in Y^\otimes m, n, m \geq 0 \}
= \mathop{\text{span}} \{ T_s \in \mathcal{O}_Y^s : s \in \mathbb{Z} \},
$$

which implies that elements of form $m \otimes_B T_n$, where $m \in M$ and $T_n \in \mathcal{O}_Y^n$, densely span $M \otimes_B \mathcal{O}_Y$.

**Proposition 4.11.** Let $[A M_B, U M] : A X_A \to B Y_B$ be a morphism in $\text{ECCor}$. The associated $C^*$-covariant representation $(\pi, \Phi)$ of $A X_A$ admits a gauge action.

**Proof.** Let $\gamma$ be the gauge action for the universal covariant representation $(\Upsilon, t)$ of $B Y_B$, and let $z \in \mathbb{T}$. The linear map $1_M \otimes \gamma_z : M \otimes \mathcal{O}_Y \to M \otimes \mathcal{O}_Y$ satisfying

$$
(1_M \otimes \gamma_z)(m \otimes S) = m \otimes \gamma_z(S)
$$

for $m \in M$, $S \in \mathcal{O}_Y$ is bounded. Indeed, let $\sum_i m_i \otimes S_i \in M \otimes \mathcal{O}_Y$. We have

$$
\left\| (1_M \otimes \gamma_z)(\sum_i m_i \otimes S_i) \right\|_{\mathcal{O}_Y}^2 = \left\| \sum_{i,j} \langle \gamma_z(S_i), (m_i, m_j)_B \cdot \gamma_z(S_j) \rangle_{\mathcal{O}_Y} \right\|
= \left\| \sum_{i,j} \gamma_z(S_i)^* \gamma_z((m_i, m_j)_B \cdot S_j) \right\|
= \left\| \sum_{i,j} \gamma_z(S_i^*(m_i, m_j)_B \cdot S_j) \right\|
= \left\| \gamma_z(\sum_{i,j} S_i^*(m_i, m_j)_B \cdot S_j) \right\|
= \left\| \sum_{i,j} S_i^* (m_i, m_j)_B \cdot S_j \right\|
= \left\| \sum_i m_i \otimes S_i \right\|^2,
$$

where the symbol $\| \cdot \|$ represents the semi-norm on the pre-correspondence $M \otimes \mathcal{O}_Y$. Now by continuity we may conclude that $1_M \otimes \gamma_z$ extends to a well-defined bounded linear operator on $M \otimes_B \mathcal{O}_Y$. However, this operator is not adjointable. This can be easily seen with elementary tensors: let $m, n \in M$ and $T_i \in \mathcal{O}_Y^i$, $T_j \in \mathcal{O}_Y^j$. The computation

$$
\langle (1_M \otimes \gamma_z)(m \otimes_B T_i), n \otimes_B T_j \rangle_{\mathcal{O}_Y} = \langle m \otimes_B \gamma_z(T_i), n \otimes_B T_j \rangle_{\mathcal{O}_Y}
= \langle z^i T_i, (m, n)_B \cdot T_j \rangle_{\mathcal{O}_Y}
= \langle T_i, z^{-i} (m, n)_B \cdot T_j \rangle_{\mathcal{O}_Y},
$$

suffices.
We claim that the homomorphism $\beta_z : C^*(\pi, \Phi) \to C^*(\pi, \Phi)$ defined by

$$\beta_z(T) = (1_M \otimes \gamma_z) T (1_M \otimes \gamma_z)$$

is a gauge action for the representation $(\pi, \Phi)$. The key point here is that even though $1_M \otimes \gamma_z$ is not an adjointable operator on $M \otimes_B \mathcal{O}_Y$, the operator

$$(1_M \otimes \gamma_z)^k (1_M \otimes \gamma_z)$$

is adjointable for any $k \in \mathcal{K}(M \otimes_B \mathcal{O}_Y)$. It suffices to prove this for $k = \theta_{m_1 \otimes_B T_i, m_2 \otimes_B T_j}$, where $m_1, m_2 \in M$, $T_i \in \mathcal{O}_Y^i$, $T_j \in \mathcal{O}_Y^j$ by Remark 4.10. Let $m, n \in M$, $T_i \in \mathcal{O}_Y^i$ and $T_k \in \mathcal{O}_Y^k$. First observe that we have

$$
(1_M \otimes \gamma_z)^k (1_M \otimes \gamma_z) (m \otimes_B T_k) = m_1 \otimes \gamma_z (T_i \langle m \otimes_B T_j, z^{-k} T_k \rangle)_{\mathcal{O}_Y}
= m_1 \otimes \gamma_z (T_i T_j^* \langle m, m \rangle_B \cdot z^{-k} T_k)
= m_1 \otimes z^{-i+j} z^{-k} T_i T_j^* \langle m, m \rangle_B \cdot T_k.
$$

This allows us to make the following computation.

$$
\langle (1_M \otimes \gamma_z)^k (1_M \otimes \gamma_z) (m \otimes_B T_k), n \otimes_B T_l \rangle_{\mathcal{O}_Y}
= \langle m \otimes_B z^{-i+j} z^{-k} T_i T_j^* \langle m, m \rangle_B \cdot T_k, n \otimes_B T_l \rangle_{\mathcal{O}_Y}
= \langle z^{-i} T_i T_j^* \langle m, m \rangle_B \cdot T_k, n \otimes_B T_l \rangle_{\mathcal{O}_Y}
= \langle T_k, z^{-i} \langle m, m \rangle_B \cdot T_j T_i^* \langle m, n \rangle_B \cdot T_l \rangle_{\mathcal{O}_Y}
= \langle m \otimes_B T_k, (1_M \otimes \gamma_z)^k (1_M \otimes \gamma_z) (m \otimes_B T_l) \rangle_{\mathcal{O}_Y}.
$$

Now, let $z \in \mathbb{T}$. In order to complete the proof of our claim we need to show

$$
\beta_z(\Phi(x)) \xi = z \Phi(x) \xi \quad \text{and} \quad \beta_z(\pi(a)) \xi = \pi(a) \xi,
$$

for any $\xi \in (M \otimes_B \mathcal{O}_Y)$, $x \in X$, $a \in A$. We check the equalities for the elements of form $m \otimes_B T_n$ as it suffices. A crucial fact here is that for any $m \in M$, $T_n \in \mathcal{O}_Y^n$, we have $\Phi(x)(m \otimes_B T_n) \in M \otimes_B \mathcal{O}_Y^{n+1}$ by construction, and thus

$$
(1_M \otimes \gamma_z) \Phi(x)(m \otimes_B T_n) = z^{n+1} \Phi(x)(m \otimes_B T_n).
$$

(4.4)

This allows us to make the following computation.

$$
\beta_z(\Phi(x))(m \otimes_B T_n) = (1_M \otimes \gamma_z) \Phi(x)(m \otimes_B z^{-n} T_n)
= z^{-n} (1_M \otimes \gamma_z) \Phi(x)(m \otimes_B T_n),
= z^{-n} z^{n+1} \Phi(x)(m \otimes_B T_n)
= z \Phi(x)(m \otimes_B T_n).
$$

One can show very similarly that, for any $a \in A$, we have

$$
\beta_z(\pi(a))(m \otimes_B T_n) = \pi(a)(m \otimes_B T_n),
$$

(4.5)
since \( \pi(a)(m \otimes_B T_n) \in M \otimes_B \mathcal{O}^*_Y \), for any \( n \in \mathbb{Z} \).

The Gauge Invariant Uniqueness Theorem now gives us the following result.

**Theorem 4.12.** Let \([A M_B, U_M] : A X_A \to B X_B\) be a morphism in \(\text{EC Cor}\), and let the pair \((\pi, \Phi)\) be the \(C\)-covariant representation of \(A X_A\). Then, the associated homomorphism \(\sigma : \mathcal{O}_X \to \mathcal{K}(M \otimes_B \mathcal{O}_Y)\) is injective. Moreover, the Cuntz-Pimsner algebra \(\mathcal{O}_X\) is isomorphic to the \(C^*\)-algebra \(C^*(\pi, \Phi) \subseteq \mathcal{K}(M \otimes_B \mathcal{O}_Y)\).

**Corollary 4.13.** If \(A X_A\) is a nondegenerate subcorrespondence of \(B X_B\), then \(\mathcal{O}_X\) is isomorphic to a subalgebra of \(\mathcal{O}_Y\).

**Proof.** Follows from Proposition 4.14 and Theorem 4.12.

When \(A X_A\) is a regular \(C^*\)-correspondence we may view \([A X_A, 1_{X(n+1)}]_\otimes\) as a morphism from \(A X_A\) to \(A X_A\) in \(\text{EC Cor}\). Let \((\pi, \Phi)\) be the associated \(C\)-covariant representation, and \((\Upsilon, t)\) be the universal covariant representation of \(A X_A\). Then, the homomorphism \(\sigma : \mathcal{O}_X \to \mathcal{K}(X \otimes_A \mathcal{O}_X)\) defines a left action of \(\mathcal{O}_X\) on \((X \otimes_A \mathcal{O}_X)\). Now, let \(S \in \mathcal{O}_X\), and let \(y = (y_1 \otimes_A y_2 \otimes_A ... \otimes_A y_n) \in X \otimes^n\). By the construction of \((\pi, \Phi)\) we have

\[
\Phi(x)(y \otimes_A S) = (x \otimes_A y_1 \otimes_A ... \otimes_A y_n) \otimes_A t(y)S
\]

\[
\pi(a) = \varphi_n(a) \otimes 1_{X(n-1)}
\]

for any \(a \in A\), and \(x \in X\), where \(\varphi_n(a)\) denotes the operator \(\varphi_X(a) \otimes 1_{X(n-1)} \in \mathcal{L}(X \otimes^n)\). We now have the following Proposition.

**Proposition 4.14.** For a regular \(C^*\)-correspondence \(A X_A\), and for \(n > 0\), we have the isomorphism \(X \otimes_A \mathcal{O}_X \cong \mathcal{O}_X\) as \(\mathcal{O}_X - \mathcal{O}_X\) correspondences.

**Proof.** Let \(n > 0\). Let \((\Upsilon, t)\) denote the universal covariant representation of \(A X_A\), and let \((\pi, \Phi)\) be the \(C\)-covariant representation of \(A X_A\) on \(\mathcal{K}(X \otimes_A \mathcal{O}_X)\). Lemma 4.1(iii) implies that the map \(U : X \otimes_A \mathcal{O}_X \to \mathcal{O}_X\) determined on elementary tensors by \(z \otimes_A S \mapsto t^n(z)S\) is a Hilbert \(A - \mathcal{O}_X\) module isomorphism. Since \(t(x)\) and \(\Upsilon(a)\) generate \(\mathcal{O}_X\), it suffices to show

\[
U(t(x) \cdot \xi) = t(x)U(\xi), \quad \text{and} \quad U(\Upsilon(a) \cdot \xi) = \Upsilon(a)U(\xi),
\]

for any \(\xi \in (X \otimes_A \mathcal{O}_X)\). Let \(y_i = y_{i,1} \otimes_A y_{i,2} \otimes_A ... \otimes_A y_{i,n} \in X \otimes^n\), \(S_i \in \mathcal{O}_X\) for any \(i \in F \subseteq N\) finite. Then we have

\[
t(x) \cdot \sum_{i \in F} (y_i \otimes_A S_i) = \sigma_X(t(x)) \sum_{i \in F} (y_i \otimes_A S_i)
\]

\[
= \Phi(x) \sum_{i \in F} (y_i \otimes_A S_i)
\]

\[
= \sum_{i \in F} \Phi(x)(y_{i,1} \otimes_A y_{i,2} \otimes_A ... \otimes_A y_{i,n} \otimes_A S_i)
\]

\[
= \sum_{i \in F} x \otimes_A y_{i,1} \otimes_A y_{i,2} \otimes_A ... \otimes_A y_{i,n-1} \otimes_A t(y_{i,n}) S_i.
\]
This implies
\[
U \left( t(x) \cdot \sum_{i \in F} (y_i \otimes_A S_i) \right) = \sum_{i \in F} t^n(x \otimes_A y_{i,1} \otimes_A y_{i,2} \otimes_A \ldots \otimes_A y_{i,n-1}) t(y_i) S_i
\]
\[
= \sum_{i \in F} t(x) t^{n-1}(y_{i,1} \otimes_A y_{i,2} \otimes_A \ldots \otimes_A y_{i,n-1}) t(y_i) S_i
\]
\[
= t(x) \sum_{i \in F} t^n(y_{i,1} \otimes_A y_{i,2} \otimes_A \ldots \otimes_A y_{i,n} \otimes_A S_i)
\]
\[
= t(x) \cdot U \left( \sum_{i \in F} (y_i \otimes_A S_i) \right).
\]

Similarly, for \( a \in A \), we have
\[
\Upsilon(a) \cdot \sum_{i \in F} (y_i \otimes_A S_i) = \sigma_X(\Upsilon(a)) \sum_{i \in F} (y_i \otimes_A S_i)
\]
\[
= \pi(a) \sum_{i \in F} (y_i \otimes_A S_i)
\]
\[
= \sum_{i \in F} \varphi_n(a) y_i \otimes_A S_i.
\]

This implies that
\[
U \left( \Upsilon(a) \cdot \sum_{i \in F} (y_i \otimes_A S_i) \right) = U \left( \sum_{i \in F} \varphi_n(a) y_i \otimes_A S_i \right)
\]
\[
= \sum_{i \in F} t^n(\varphi_n(a) y_i) S_i
\]
\[
= \sum_{i \in F} \Upsilon(a) t^n(y_i) S_i
\]
\[
= \Upsilon(a) \cdot U \left( \sum_{i \in F} (y_i \otimes_A S_i) \right),
\]
which completes the proof.

\[\square\]

5. The Functor

**Theorem 5.1.** Let \([A M_B, U_M] : A X_A \to B Y_B\) be a morphism in ECCor. Then the assignments \( A X_A \mapsto O_X \) on objects and
\[
[A M_B, U_M] \mapsto [\bigcirc_X(M \otimes_B \bigcirc_Y)_{\bigcirc_Y}]
\]
on morphisms define a functor \( E \) from ECCor to the enchilada category.

**Proof.** Let \([A M_B, U_M] : A X_A \to B Y_B\) , and \([B N_C, U_N] : B Y_B \to C Z_C\) be morphisms in ECCor. We want to show
- \( E([A (M \otimes_B N)_C, U_{M\otimes_B N}] ) = E([A M_B, U_M]) \otimes_{\bigcirc_Y} E([B N_C, U_N]), \) and
- \( E([A A_A, U_A]) = [\bigcirc_X(O_X)_{\bigcirc_X}]. \)
We start with proving the isomorphism
\[ \varphi(X(M \otimes_B \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (N \otimes_C \mathcal{O}_Z))_{\mathcal{O}_Z} \cong \varphi(X(M \otimes_B N \otimes_C \mathcal{O}_Z))_{\mathcal{O}_Z}. \] (5.1)
Let \((\pi_1, \Phi_1)\), and \((\pi_2, \Phi_2)\) be the \(C\)-covariant representations of \(A\mathcal{X}_A\), as in Definition 4.2 on \(\mathcal{K}(M \otimes_B \mathcal{O}_Y)\) and \(\mathcal{K}(M \otimes_B N \otimes_C \mathcal{O}_Z)\), respectively. Let \((\pi, \Phi)\) be the \(C\)-covariant representation of \(B\mathcal{Y}_B\) on \(\mathcal{K}(N \otimes_C \mathcal{O}_Z)\). We already have the Hilbert \(A - \mathcal{O}_Z\) correspondence isomorphism
\[ U : (M \otimes_B \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (N \otimes_C \mathcal{O}_Z) \rightarrow (M \otimes_B N \otimes_C \mathcal{O}_Z), \]
which gives rise to the isomorphism
\[ AdU : \mathcal{L}(M \otimes_B \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (N \otimes_C \mathcal{O}_Z) \rightarrow \mathcal{L}(M \otimes_B N \otimes_C \mathcal{O}_Z). \]
Therefore, it suffices to show \(U\) preserves the left \(\mathcal{O}_X\)-module structure. Since \(U\) preserves the left action of \(A\), for any \(a \in A\) we observe
\[ AdU(\pi_1(a) \otimes 1_{N \otimes_C \mathcal{O}_Z}) = AdU((\varphi_M(a) \otimes 1_{\mathcal{O}_Y}) \otimes 1_{N \otimes_C \mathcal{O}_Z}) \]
\[ = \varphi_M(a) \otimes 1_N \otimes 1_{\mathcal{O}_Z} \]
\[ = \pi_2(a). \]
By following the construction of \(\Phi_1(x)\) and \(\Phi_2(x)\), we next show similarly that
\[ AdU(\Phi_1(x) \otimes 1_{N \otimes_C \mathcal{O}_Z}) = \Phi_2(x). \]
Let \(t_Y\) and \(t_Z\) be the linear maps associated to the universal covariant representations of \(B\mathcal{Y}_B\) and \(C\mathcal{Z}_C\), respectively. Consider the isomorphisms
\[ U_M : X \otimes_A M \rightarrow M \otimes_B Y, \quad U_N : Y \otimes_B N \rightarrow N \otimes_C Z, \]
\[ V_Y : Y \otimes_B \mathcal{O}_Y \rightarrow t_Y(Y)\mathcal{O}_Y, \quad V_Z : Z \otimes_C \mathcal{O}_Z \rightarrow t_Z(Z)\mathcal{O}_Z, \]
\[ U_{M \otimes_B N} : X \otimes_A M \otimes_B N \rightarrow M \otimes_B N \otimes_C Z. \]
For \(x \in X, y \in Y\), we have the linear maps
\[ T_1(x) : M \rightarrow M \otimes_B Y, \quad m \mapsto U_M(x \otimes_A m) \]
\[ T(y) : N \rightarrow N \otimes_C Z, \quad n \mapsto U_N(y \otimes_B n) \]
\[ T_2(x) : M \otimes_B N \rightarrow M \otimes_B N \otimes_C Z, \quad \nu \mapsto U_{M \otimes_B N}(x \otimes_A \nu). \]
Notice that for \(x \in X, m \in M\) and \(n \in N\) we have
\[ T_2(x)(m \otimes_B n) = U_{M \otimes_B N}(x \otimes_A m \otimes_B n) \]
\[ = (1_M \otimes U_N)[U_M(x \otimes_A m) \otimes_B n] \]
\[ = (1_M \otimes U_N)(T_1(x) \otimes 1_N)(m \otimes_B n), \]
which implies that
\[ \Phi_2(x) = (1_M \otimes V_Z)(T_2(x) \otimes 1_{\mathcal{O}_Z}) \]
\[ = (1_M \otimes V_Z)(1_M \otimes U_N \otimes 1_{\mathcal{O}_Z})(T_1(x) \otimes 1_N \otimes 1_{\mathcal{O}_Z}). \] (5.2)
On the other hand, recall that \(\Phi_1(x)\) and \(\Phi(y)\) are defined by
\[ \Phi_1(x) = (1_M \otimes V_Y)(T_1(x) \otimes 1_{\mathcal{O}_Y}), \quad \text{and} \quad \Phi(y) = (1_N \otimes V_Z)(T(y) \otimes 1_{\mathcal{O}_Z}). \] (5.3)
We aim to prove the equality
\[ \Phi_2(x)U = U(\Phi_1(x) \otimes 1_{N \otimes C O_Z}). \]  
(5.4)

Let \( \iota_{O_Y} \) be the isomorphism
\[ \iota_{O_Y} : B(O_Y) \otimes_{O_Y} (N \otimes_C O_Z)_{O_Z} \to B(N \otimes_C O_Z)_{O_Z} \]
determined by \( S \otimes_{O_Y} \nu \mapsto \sigma_Y(S)\nu \), for \( S \in O_Y \), \( \nu \in N \otimes_C O_Z \), where \( \sigma_Y \) denotes the left action of \( O_Y \) on the Hilbert module \( N \otimes_C O_Z \). We first claim
\[ (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(1_M \otimes 1_Y \otimes \iota_{O_Y}) = U(1_M \otimes V_Y \otimes 1_{N \otimes C O_Z}). \]  
(5.5)

It suffices to check equality (5.5) for the elements of form \((m \otimes_B y \otimes_B S) \otimes_{O_Y} \nu\), where \( m \in M, y \in Y, S \in O_Y, \nu \in (N \otimes_C O_Z)\): since \( V_Y(y \otimes_B S) = t_Y(y)S \) we have
\[
U(1_M \otimes V_Y \otimes 1_{N \otimes C O_Z})(m \otimes_B y \otimes_B S \otimes_{O_Y} \nu) \\
= U(m \otimes_B t_Y(y)S \otimes_{O_Y} \nu) \\
= m \otimes_B \sigma_Y(t_Y(y))\nu \\
= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes T(y) \otimes 1_{O_Z})(m \otimes_B \sigma_Y(S)\nu).
\]
(5.3)

Since by construction we have \((T(y) \otimes 1_{O_Z})(\xi) = (U_N \otimes 1_{O_Z})(y \otimes_C \xi)\), for any \( y \in Y, \xi \in N \otimes_C O_Z\), we may continue our computation as
\[
(1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(m \otimes_B y \otimes_B \sigma_Y(S)\nu) \\
= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(1_M \otimes 1_Y \otimes \iota_{O_Y})(m \otimes_B y \otimes_{O_Y} S \otimes_{O_Y} \nu),
\]
which completes the proof of our claim.

We are now ready to prove equality (5.4). Once again let \( m \in M, S \in O_Y, \nu \in (N \otimes_C O_Z)\). We have
\[
\Phi_2(x)U(m \otimes_B S \otimes_{O_Y} \nu) \\
\begin{align*}
&= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(T_1(x) \otimes 1_N \otimes 1_{O_Z})U(m \otimes_B S \otimes_{O_Y} \nu) \\
&= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(T_1(x) \otimes 1_N \otimes 1_{O_Z})[m \otimes_B \sigma_Y(S)\nu] \\
&= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})[U_M(x \otimes_A m) \otimes_B \sigma_Y(S)\nu] \\
&= (1_M \otimes 1_N \otimes V_Z)(1_M \otimes U_{N \otimes 1_{O_Z}})(1_M \otimes 1_Y \otimes \iota_{O_Y})(U_M(x \otimes_A m) \otimes_B S \otimes_{O_Y} \nu) \\
&= U(1_M \otimes V_Y \otimes 1_{N \otimes C O_Z})(U_M(x \otimes_A m) \otimes_B S \otimes_{O_Y} \nu) \\
&= U(1_M \otimes V_Y \otimes 1_{N \otimes C O_Z})(T_1(x) \otimes 1_{O_Y} \otimes 1_{N \otimes C O_Z})(m \otimes_B S \otimes_{O_Y} \nu) \\
&= U(\Phi_1(x) \otimes 1_{N \otimes C O_Z})(m \otimes_B S \otimes_{O_Y} \nu),
\end{align*}
\]
as desired. The equality (*) is followed by (5.2), and the equality (**) is followed by (5.5).

Now we have the following diagram.
This means, denoting by \((\Upsilon_X, t_X)\) the Cuntz-Pimsner representation of \(\mathcal{O}_X\), we have

\[
\begin{align*}
\text{• } \text{Ad} U(\sigma_X(\Upsilon_X(a)) \otimes 1_{N \otimes C \mathcal{O}_Z}) &= \text{Ad} U(\pi_1(a) \otimes 1_{N \otimes C \mathcal{O}_Z}) = \pi_2(a) \\
\text{• } \text{Ad} U(\sigma_X(t_X(x)) \otimes 1_{N \otimes C \mathcal{O}_Z}) &= \text{Ad} U(\Phi_1(x) \otimes 1_{N \otimes C \mathcal{O}_Z}) = \Phi_2(x)
\end{align*}
\]

for \(a \in A, x \in X\), which is enough to conclude that \(U\) preserves the left action of \(\mathcal{O}_X\), since the elements \(\Upsilon_X(a)\) and \(t_X(x)\) generate \(\mathcal{O}_X\).

It remains to show that \(\mathcal{E}\) maps the identity morphism \([A A_A, U_A]_A X_A A \to A X_A\) in \(\text{ECCor}\) to the identity morphism \([\mathcal{O}_X(\mathcal{O}_X)\mathcal{O}_X]\) in the enchilada category. This follows immediately from Lemma 4.5.

\[\square\]

**Remark 5.2.** Let \(X_A\) be a regular correspondence. Then, for any \(n>0\), we have

\[
\mathcal{E}([X^n, X^{n+1}]) = [\mathcal{O}_X(\mathcal{O}_X)\mathcal{O}_X],
\]

by Proposition 4.14.

6. **Applications**

6.1. **Muhly & Solel Theorem.** Muhly and Solel introduced the notion of Morita equivalence for \(C^*\)-correspondences (Muhly and Solel, 2000) as follows: \(A X_A\) and \(B Y_B\) are called Morita equivalent, denoted by \(A X_A \sim_{SME} B Y_B\), if there exists an imprimitivity bimodule \(A M_B\) such that

\[
A(X \otimes A M_B) \cong A(M \otimes B Y_B).
\]

They proved that Morita equivalent injective \(C^*\)-correspondences have Morita equivalent Cuntz-Pimsner algebras. In (Elefrherakis et al., 2017), the authors presented a proof for possibly non-injective \(C^*\)-correspondences. In this section, we discuss how our functor provides a very practical method to recover this result.

First, recall that in the enchilada category \([A M_B]\) is an isomorphism if and only if \(A M_B\) is an imprimitivity bimodule. On the other hand, by Proposition 3.3, we have that \([A M_B, U_M]: A X_A \to B Y_B\) is an isomorphism in \(\text{ECCor}\) if and only if \(A X_A\) and \(B Y_B\) are Morita equivalent.
Theorem 6.1. If two \( C^\ast \)-correspondences \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are Morita equivalent, then their Cuntz-Pimsner algebras \( \mathcal{O}_X \) and \( \mathcal{O}_Y \) are Morita equivalent (in the sense of Rieffel).

Proof. Since \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are Morita equivalent, there exists an imprimitivity bimodule with an isomorphism \( U_M : X \otimes_A M \to M \otimes_B Y \), which implies \([ \mathcal{A}_M, U_M ]\) is an isomorphism in ECCor. This means \( \mathcal{E}[\mathcal{A}_M, U_M] = [\mathcal{O}_X (M \otimes_B \mathcal{O}_Y) \mathcal{O}_Y] \) is an isomorphism in the enchilada category. Hence, the \( C^\ast \)-algebras \( \mathcal{O}_X \) and \( \mathcal{O}_Y \) are Morita equivalent. \( \square \)

6.2. Cuntz-Pimsner Algebras of Shift Equivalent \( C^\ast \)-correspondences. In 1973 Williams introduced "elementary strong shift equivalence" and "strong shift equivalence" for the class of matrices with non-negative integer entries, with the goal of characterizing the topological conjugacy of subshifts of finite type \([\text{Williams, 1973}]\). The relations were defined as follows: let \( X \) and \( Y \) be matrices as described.

- \( X \) and \( Y \) are elementary strong shift equivalent, denoted by \( X \sim^S Y \), if there exist matrices \( R \) and \( S \) with non-negative integer entries such that \( X = RS \) and \( Y = SR \).
- The transitive closure of the relation \( \sim^S \) is called strong shift equivalence.

Putting the result of Williams \([\text{Williams, 1973}]\) and the result of Cuntz and Krieger \([\text{Cuntz and Krieger, 1980}]\) together, one concludes that strong shift equivalent matrices (with non-negative integer entries) have Morita equivalent Cuntz-Krieger algebras. In \([\text{Muhly et al., 2008}]\), Muhly, Pask and Tomforde formulated this in the setting of \( C^\ast \)-correspondences as follows.

Definition 6.2. Two \( C^\ast \)-correspondences \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are called elementary strong shift equivalent, denoted by \( \mathcal{A}_X \sim^S \mathcal{A}_Y \), if there are \( C^\ast \)-correspondences \( \mathcal{A}_R \) and \( \mathcal{A}_S \) such that

\[
X \cong R \otimes_B S \quad \text{and} \quad Y \cong S \otimes_A R
\]

as \( C^\ast \)-correspondences.

\( C^\ast \)-correspondences \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are called strong shift equivalent, denoted by \( \mathcal{A}_X \sim^{\text{SSE}} \mathcal{A}_Y \), if there are \( C^\ast \)-correspondences \( \{ Z_i \}_{0 \leq i \leq n} \) such that \( Z_0 = X \), \( Z_n = Y \), and \( Z_i \sim^S Z_{i+1} \), for each \( i \).

Theorem 6.3 \([\text{Muhly et al., 2008}]\). If two regular \( C^\ast \)-correspondences \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are strong shift equivalent, then their Cuntz-Pimsner algebras \( \mathcal{O}_X \) and \( \mathcal{O}_Y \) are Morita equivalent.

Proof. Let \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) be elementary strong shift equivalent. Then, there exists correspondences \( \mathcal{A}_R \) and \( \mathcal{A}_S \) with the isomorphisms

\[
\phi_X : X \to R \otimes_B S \quad \text{and} \quad \phi_Y : Y \to S \otimes_A R.
\]

Define

\[
U_R = (1_R \otimes \phi_Y^{-1})(\phi_X \otimes 1_R) \quad \text{and} \quad U_S := (1_S \otimes \phi_X^{-1})(\phi_Y \otimes 1_S).
\]

Notice that we have

\[
(1_R \otimes U_S)(U_R \otimes 1_S) = (\phi_X \otimes \phi_Y^{-1}).
\]
This allows us to see that the diagram

\[
\begin{array}{ccc}
X \otimes_A (R \otimes_B S) & \xrightarrow{1_X \otimes \phi^{-1}} & X \otimes_A X \\
(R \otimes_B S) & \otimes_A & (R \otimes_B S) \\
\end{array}
\]

commutes, which implies the equality \([R \otimes_B S, U \otimes_B S] = [X, 1_X \otimes_B S]\).

We now show that \(E([S, U_S])\) and \(E([R, U_R])\) are inverses of each other:

\[
E([S, U_S]) \circ E([R, U_R]) = E([S, U_S] \circ [R, U_R]) = E([R \otimes_B S, U \otimes_B S]) = E([X, 1_X \otimes_B S]) = [\sigma_X \circ \sigma_Y \circ \sigma_X],
\]

where the last step follows by Remark 5.2. It can be seen similarly that

\[
E([R, U_R]) \circ E([S, U_S]) = [\sigma_X \circ \sigma_Y \circ \sigma_X].
\]

Hence, the correspondences \(\sigma_X(R \otimes_B \sigma_Y)\) and \(\sigma_X(S \otimes_A \sigma_X)\) are inverses of each other.

\[\Box\]

**Corollary 6.4.** If \(A X_A\) and \(B Y_B\) are regular elementary strong shift equivalent \(C^\ast\)-correspondences via \(A R_B\) and \(B S_A\), then the injective homomorphisms

\[
\sigma_Y : \sigma_Y \to \sigma_X(S \otimes_A \sigma_X) \quad \text{and} \quad \sigma_X : \sigma_X \to \sigma_X(R \otimes_B \sigma_Y)
\]

are surjective.

**Example 6.5.** Let \(A X_A\) be a regular \(C^\ast\)-correspondence. Then, the \(C^\ast\)-correspondence \(\chi(X)(X \otimes_A \chi(X))\) is regular, as well. Moreover, we have

\[
A X_A \cong \chi(X)(X \otimes_A \chi(X))\]

via \(\chi(X)X_A\) and \(\chi(X)X_A\). Denote \(X \otimes_A \chi(X)\) by \(\chi\). We know by Theorem 6.3 that \(\sigma_X\) and \(\sigma_X\) are Morita equivalent. However, Corollary 6.4 implies a stronger relation between these \(C^\ast\)-algebras:

\[
\sigma_X \cong \sigma_X(X \otimes_A \sigma_X) \cong \sigma_X,
\]

where the latter isomorphism is the natural \(C^\ast\)-algebra isomorphism as described in the proof of Proposition 4.8.

Let \(G\) be a locally compact group with \(\alpha : G \curvearrowright A\) and \(\beta : G \curvearrowright B\). An \(\alpha - \beta\) compatible action \(\gamma\) of \(G\) on \(A X_B\) is a homomorphism of \(G\) into the group of invertible linear maps on \(X\) such that

(i) \(\gamma_s(a \cdot x) = \alpha_s(a) \cdot \gamma_s(x)\)
(ii) \(\gamma_s(x \cdot b) = \gamma_s(x) \cdot \beta_s(b)\)
(iii) \(\langle \gamma_s(x), \gamma_s(y) \rangle_B = \beta_s(\langle x, y \rangle_B)\)
for each $s \in G$, $a \in A$, $x \in X$, and $b \in B$; and such that each map $s \mapsto \gamma_s(x)$ is continuous from $G$ into $X$.

**Definition 6.6.** Let $(\pi_1, t_1)$ be a representation of $AX_A$ that admits a gauge action $\alpha$, and let $(\pi_2, t_2)$ be a representation of $BY_B$ that admits a gauge action $\beta$. Let $M$ be a $C^*(\pi_1, t_1)$-$C^*(\pi_2, t_2)$ imprimitivity bimodule. The Morita equivalence between $C^*(\pi_1, t_1)$ and $C^*(\pi_2, t_2)$ is called *gauge equivariant* if there exists an $\alpha - \beta$ compatible action of $\mathbb{T}$ on $M$.

**Theorem 6.7.** The Morita equivalence in Theorem 6.3 is gauge equivariant.

**Proof.** Let $AX_A$ and $BY_B$ be regular elementary strong shift equivalent $C^*$-correspondences via $AR_B$ and $BS_A$. Denote the universal covariant representation of $AX_A$ by $(\Upsilon, t)$, and the $C$-covariant representation on $\mathcal{K}(R \otimes_B \mathcal{O}_Y)$ by $(\pi, \Phi)$. By Corollary 6.2 we have an isomorphism $\sigma : \mathcal{O}_X \to \mathcal{K}(R \otimes_B \mathcal{O}_Y)$ such that

$$\sigma(t(x)) = \Phi(x) \quad \text{and} \quad \sigma(\Upsilon(a)) = \pi(a)$$

for any $x \in X$, $a \in A$, which allows us to view $R \otimes_B \mathcal{O}_Y$ as an imprimitivity $\mathcal{O}_X - \mathcal{O}_Y$ bimodule. Now, denote by $\alpha$ the gauge action for $\mathcal{O}_X$ and by $\gamma$ the gauge action for $\mathcal{O}_Y$. We show that the homomorphism $z \mapsto 1_R \otimes \gamma_z$ is an $\alpha - \gamma$ compatible action of $\mathbb{T}$ on the imprimitivity bimodule $\mathcal{O}_X(R \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}$. To this end, we first prove the equality

$$(1_R \otimes \gamma_z)[T \cdot \xi] = \alpha_z(T) \cdot (1_R \otimes \gamma_z)(\xi)$$

(6.1)

for any $T \in \mathcal{O}_X$, $\xi \in R \otimes_B \mathcal{O}_Y$. Let $x \in X$ and $a \in A$. It suffices to let $T = t(x)$ and $T = \Upsilon(a)$ as such elements generate $\mathcal{O}_X$. For $r \in R$ and $S_n \in \mathcal{O}_Y$, we have

$$\alpha_z(t(x)) \cdot (1_R \otimes \gamma_z)(r \otimes_B S_n) = (zt(x)) \cdot [z^n(r \otimes_B S_n)]$$

$$= \sigma(zt(x))[z^n(r \otimes_B S_n)]$$

$$= z^{n+1} \Phi(x)(r \otimes_B S_n)$$

$$= (1_R \otimes \gamma_z)[t(x) \cdot (r \otimes_B S_n)],$$

where the last step follows from (4.4). One can verify (6.1) for $T = \Upsilon(a)$, very similarly.

Next, we show

$$\langle (1_R \otimes \gamma_z)\xi, (1_R \otimes \gamma_z)\nu \rangle_{\mathcal{O}_Y} = \gamma_z(\langle \xi, \nu \rangle_{\mathcal{O}_Y}),$$

for $\xi, \nu \in (R \otimes_B \mathcal{O}_Y)$. Let $r' \in R$, $S_m \in \mathcal{O}_Y^n$. We have

$$\langle (1_R \otimes \gamma_z)(r \otimes_B S_n), (1_R \otimes \gamma_z)(r' \otimes_B S_m) \rangle_{\mathcal{O}_Y} = z^{m-n}\langle r \otimes_B S_n, r' \otimes_B S_m \rangle_{\mathcal{O}_Y}$$

$$= z^{m-n} S_n^* \cdot \langle r, r' \rangle_B \cdot S_m$$

$$= \gamma_z(S_n^* \cdot (r, r')_B \cdot S_m)$$

$$= \gamma_z((r \otimes_B S_n, r' \otimes_B S_m)_{\mathcal{O}_Y}),$$

which completes the proof since elements of form $r \otimes_B S_n$ densely span $R \otimes_B \mathcal{O}_Y$. \qed
6.3. Pimsner Dilation. For an injective $C^*$-correspondence $A X_A$, one can construct a Hilbert bimodule that contains a copy of $X$ as a subspace. The Pimsner dilation $\tilde{X}$, which was first introduced by Pimsner (Pimsner, 2004), is the minimal Hilbert bimodule that contains $A X_A$ as a sub-correspondence ([Kakariadis and Katsoulis, 2014, Theorem 3.5]). To describe Pimsner dilations, we use Katsura’s so-called cores. The detailed information about these particular $C^*$-algebras can be found in (Katsura, 2004); here we give a quick review.

For each $n \in \mathbb{N}$ set $B_n = \Psi_n(\mathcal{K}(X^\otimes n)) \subseteq C^*(\pi, t)$. Note that $B_0 := \pi(A)$ and that $B_n \simeq \mathcal{K}(X^\otimes n)$ when $(\pi, t)$ is injective. For $m, n \in \mathbb{N}$ with $m \leq n$, define $B_{[m,n]} \subseteq C^*(\pi, t)$ by

$$B_{[m,n]} = B_m + B_{m+1} + \ldots + B_n.$$ 

We denote $B_{[m,n]}$ by $B_n$ for $n \in \mathbb{N}$. All $B_{[m,n]}$’s are $C^*$-subalgebras of $C^*(\pi, t)$. In addition, $B_{[m,n]}$ is an ideal of $B_{[m,n]}$ for $m, n \in \mathbb{N}$ with $m \leq k \leq n$. In particular, $B_n$ is an ideal of $B_{[0,n]}$ for each $n \in \mathbb{N}$. For $m \in \mathbb{N}$, define the $C^*$-subalgebra $B_{[m,\infty]}$ of $C^*(\pi, t)$ by

$$B_{[m,\infty]} = \bigcup_{n=m}^{\infty} B_{[n,\infty]}.$$ 

Notice that $B_{[m,\infty]}$ is an inductive limit of the increasing sequence of $C^*$-algebras $\{B_{[m,n]}\}_{n=m}^{\infty}$. The $C^*$-algebra $B_{[0,\infty]}$ is called the core of the $C^*$-algebra $C^*(\pi, t)$. The core $B_{[0,\infty]}$ naturally arises when $C^*(\pi, t)$ admits gauge action $\beta$, and it coincides with the fixed point algebra $C^*(\pi, t)^\beta$.

Now, the Pimsner dilation is defined as follows: let $(Y, t)$ be the universal covariant representation of an injective $C^*$-correspondence $A X_A$. Then

$$\tilde{X} := t(X) B_{[0,\infty]} = \text{span} \{ t(x) k : x \in X, k \in B_{[0,\infty]} \}$$

is a subspace of $O_X$. We may define right and left actions of $B_{[0,\infty]}$ on $\tilde{X}$ simply by multiplication. Notice that for any $\nu, \xi \in \tilde{X}$, we have $\langle \nu, \xi \rangle_{O_X} = \nu^* \xi \in B_{[0,\infty]}$. Moreover, we observe that

$$O_X \langle \tilde{X}, \tilde{X} \rangle = \tilde{X} \tilde{X}^* = t(X) B_{[0,\infty]} t(X)^* = B_{[1,\infty]},$$

and thus $\tilde{X}$ can be viewed as a $C^*$-correspondence over $B_{[0,\infty]}$ such that the left action homomorphism $\varphi_{\tilde{X}} : B_{[0,\infty]} \rightarrow \mathcal{L}(\tilde{X})$ is an isomorphism onto $\mathcal{K}(\tilde{X})$.

**Lemma 6.8.** Let $A X_A$ be an injective $C^*$-correspondence with the universal covariant representation $(Y, t)$. Then we have the following.

(i) The Hilbert $B_{[0,\infty]}$-modules $(X \otimes_A B_{[0,\infty]})$ and $\tilde{X}$ are isomorphic.

(ii) $J_{\tilde{X}} = B_{[1,\infty]}$.

(iii) The isomorphism class $[A B_{[0,\infty]} B_{[0,\infty]}]$ is a morphism $A X_A \rightarrow B_{[0,\infty]} \tilde{X} B_{[0,\infty]}$ in $\text{ECCor}$. 

**Proof.** It is straightforward to verify that the map $X \otimes B_{[0,\infty]} \rightarrow t(X) B_{[0,\infty]}$ determined on elementary tensors by $x \otimes S \mapsto t(x) S$ preserves the left-module structure and the semi-inner
product. Moreover, it is surjective. Hence, it extends to a Hilbert module isomorphism 
\((X \otimes_A \mathcal{B}_{[0,\infty)}) \rightarrow t(X)\). 

Item (ii) follows from the fact that \(\varphi_\Xi : \mathcal{B}_{[0,\infty)} \rightarrow \mathcal{L}(\bar{X})\) is an isomorphism onto \(\mathcal{K}(\bar{X})\). And, item (iii) follows from Lemma \[4.7\] since \(A_X\) is a nondegenerate subcorespondence of \(\mathcal{B}_{[0,\infty)}\).

Lemma \[6.8\] implies that any injective \(C^*\)-correspondence \(A_X\) has a \(C\)-covariant representation \((\pi, \Phi)\) on \(\mathcal{K}(\mathcal{B}_{[0,\infty)} \otimes \mathcal{B}_{[0,\infty)} \mathcal{O}_\bar{X})\). The \(C^*\)-algebras \(\mathcal{O}_X\) and \(\mathcal{O}_\bar{X}\) are isomorphic (Pimsner, 2001, Theorem 2.5), (Kakariadis and Katsoulis, 2012, Theorem 6.6). Then Corollary \[4.12\] tells us that \(\mathcal{O}_\bar{X}\) is nothing but the \(C^*\)-algebra generated by the \(C\)-covariant representation of \(A_X\). In the next theorem, we present an alternative proof for the isomorphism \(\mathcal{O}_X \cong \mathcal{O}_\bar{X}\) by using the \(C\)-covariant representation \((\pi, \Phi)\). The proof shows the exact relation between the generators of \(\mathcal{O}_X\), \(\mathcal{O}_\bar{X}\) and \(C^*(\pi, \Phi)\).

**Theorem 6.9.** Let \(A_X\) be an injective \(C^*\)-correspondence with the universal covariant representation \((\Upsilon, t)\), and let \(\mathcal{B}_{[0,\infty)} \bar{X} \mathcal{B}_{[0,\infty)}\) be as in \[6.2\]. Denote the associated \(C\)-covariant representation on \(\mathcal{K}(\mathcal{B}_{[0,\infty)} \otimes \mathcal{B}_{[0,\infty)} \mathcal{O}_\bar{X})\) by \((\pi, \Phi)\). Then, we have the isomorphisms \(\mathcal{O}_X \cong C^*(\pi, \Phi) \cong \mathcal{O}_\bar{X}\).

**Proof.** Let \((\Upsilon_\bar{X}, T)\) be the universal covariant representation of \(\bar{X}\). Let \(\iota\) denote the isomorphism 
\(\mathcal{K}(\mathcal{B}_{[0,\infty)} \otimes \mathcal{B}_{[0,\infty)} \mathcal{O}_\bar{X}) \rightarrow \mathcal{K}(\mathcal{O}_\bar{X}) \rightarrow \mathcal{O}_\bar{X}\).

Proposition \[4.8\] gives us
\[T(t(x)) = \iota(\Phi(x)) \quad \text{and} \quad \Upsilon_\bar{X}(\Upsilon(a)) = \iota(\pi(a))\]
for any \(x \in X\), \(a \in A\). We now have the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & \bar{X} \\
\downarrow{t} & & \downarrow{T} \\
\mathcal{O}_X & \xrightarrow{\mathcal{K}(\mathcal{B}_{[0,\infty)} \otimes \mathcal{B}_{[0,\infty)} \mathcal{O}_\bar{X})} & \mathcal{O}_\bar{X} \\
\downarrow{\Upsilon_\bar{X}} & & \downarrow{\Upsilon_\bar{X} \pi} \\
A & \xrightarrow{\mathcal{B}_{[0,\infty)}} & \mathcal{B}_{[0,\infty)}
\end{array}
\]

Since \(\Upsilon_\bar{X}(\mathcal{B}_{[0,\infty)})\) and \(T(\bar{X})\) generate \(\mathcal{O}_\bar{X}\), it suffices to show \(\Upsilon_\bar{X}(k) \in \iota(C^*(\pi, \Phi))\) and \(T(\xi) \in \iota(C^*(\pi, \Phi))\) for any \(k \in \mathcal{B}_{[0,\infty)}, \xi \in \bar{X}\). First, recall that \(\varphi_\bar{X} : \mathcal{B}_{[0,\infty)} \rightarrow \mathcal{L}(\bar{X})\) denotes the left action of \(\mathcal{B}_{[0,\infty)}\) on \(\bar{X}\), and \(J_\bar{X} = \mathcal{B}_{[1,\infty)}\). Now, for any \(\xi \in \bar{X}\) and \(x, y \in X\), we have
\[
\varphi_\bar{X}(\Psi_t(\theta_{x,y})) (\xi) = \Psi_t(\theta_{x,y}) \xi = t(x) t(y)^* \xi = t(x) (t(y), \xi) \mathcal{B}_{[0,\infty)}
\]
Since \((\Upsilon, T)\) is covariant, this allows us to observe that
\[
\Upsilon_X (\Psi_t(\theta_{x,y})) = \Psi_T(\theta_{t(x),t(y)})
= T(t(x))T(t(y))^* \\
= \iota(\Phi(x)\Phi(y))^* \\
= \iota(\Psi(\theta_{x,y})) \in \iota(C^*(\pi, \Phi)).
\]
Therefore, we obtain
\[
\Upsilon_X (k) = \Psi_T(\varphi_X(k)) = T(t(x))T(t(y))\Psi_t(\theta_{y_2,x_2})^* = T(t(x))\Upsilon_X (\Psi_t(\theta_{x_2,y_2}))T(t(y))^*.
\]
This computation allows one to conclude that \(\Upsilon_X (k) \in \iota(C^*(\pi, \Phi))\), for all \(k \in \mathcal{B}_{[1,\infty]}\).

On the other hand, if \(k \in \mathcal{B}_0\), we have \(k = \Upsilon(a)\) for some \(a \in A\). Thus, \(\Upsilon_X (k) = \Upsilon_X (\Upsilon(a)) = \iota(\pi(a)) \in \iota(C^*(\pi, \Phi))\).

To sum up, for any \(x \in X\) and \(k \in \mathcal{B}_{[0,\infty]}\), we have
\[
T(t(x))k = T(t(x))\Upsilon_X (k) \in \iota(C^*(\Phi, \pi)),
\]
which suffices to conclude that \(T(\xi) \in \iota(C^*(\pi, \Phi))\) for any \(\xi \in \tilde{X}\), since elements of form \(t(x)k\) densely span \(\tilde{X}\).

We have shown that \(C^*(\Phi, \pi) \cong \mathcal{O}_X\). By Corollary 4.12 we already have \(\mathcal{O}_X \cong C^*(\pi, \Phi)\), which completes the proof.

\[\square\]

7. Final Notes

There are more applications of the functor \(\mathcal{E}\) and of the \(\mathcal{C}\)-covariant representations. In an upcoming paper, we use the techniques presented in this paper to study the ideals and hereditary subalgebras of Cuntz-Pimsner algebras.

Meyer and Sehnem (Meyer and Sehnem, 2019) use a similar construction in the context of bicategories. We would like to note here that our development was completely independent; in fact, we were strongly motivated by the paper (Kaliszewski et al., 2013). However, the work of Meyer and Sehnem raises the question of what happens if we drop the injectivity condition on our morphisms in ECCor. In that case, we definitely would not have Corollary 4.11 as the representation \((\pi, \Phi)\) would not be injective. What we are not sure of is whether the injectivity condition is necessary for \(\mathcal{E}\) to be a functor.
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Department of Mathematics, University of Colorado Boulder, Boulder, CO 80309-0395

Email address: Menevse.Eryuzlu@colorado.edu