A way to get a well-defined derivative expansion of real-time thermal effective actions

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We compute the quadratic part of the thermal effective action for real scalar fields which are initially in thermal equilibrium and vary slowly in time using a generalised real-time formalism proposed by Le Bellac and Mabilat. We derive both Real Time and Imaginary Time Formalisms and find that the result is analytic at the limits of zero external four-momenta when using our full time contour. We expand the fields in time up to the second derivative and discuss the initial time dependence of our result before and after the expansion in terms of equilibrium.

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I. INTRODUCTION

We consider a two real scalar field theory with fields \( \phi, \eta \) with Lagrangian \( L \) given by

\[
L[\phi, \eta] = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} g \phi \eta^2 + L_0
\]

where \( L_0 \) denotes the free Lagrangian for \( \phi \). If we integrate out the \( \eta \)-field fluctuations and use a one-loop approximation we find that the generating functional can be expressed as

\[
Z = \int C D\phi e^{iS_0[\phi] + iS_{\text{eff}}[\phi]} \]

where \( S_0 \) is the classical action and \( S_{\text{eff}} \) is given by

\[
S_{\text{eff}}[\phi] = \frac{i}{2} \text{Tr} \ln[1 - g D_c \phi] \]

In Eq. (3) \( D_c \) is the propagator for the \( \eta \) field and

\[
\text{Tr} = \int_C dt \int d^3x
\]

where the time path \( C \) starts at initial time \( t_i \) when the field \( \eta \) is in equilibrium and ends at time \( t_i - i \beta \) where \( \beta \) is the inverse temperature. The path is the one of

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We will expand our field in time up to the second order and we will compare our result with recent calculations by Evans and Berera et al. We will discuss the $t_i$-dependence of the “bubble” term before and after the expansion in terms of equilibrium and the periodicity of the fields.

II. THE METHOD

The propagators used are in their mixed representation $D_{c}(t, k)$ by Mills

$$D_{c}(t, k) = \int \frac{dk}{2\pi} e^{-ikc(t) + n(0)} \rho(k_0, k)$$

(6)

where $\theta_{c}(t)$ is a contour $\theta$ function, $n(0)$ is the Bose-Einstein distribution function given by

$$n(0) = \frac{1}{e^{\beta k_0} - 1}$$

(7)

and $\rho(k_0, k)$ is the (temperature independent) two-point spectral function given by

$$\rho(k_0, k) = 2\pi \varepsilon(k_0) \delta(k_0 - k_0)^2$$

(8)

where $\varepsilon(k_0)$ is the sign function and

$$\omega_k^2 = k^2 + m^2$$

(9)

The time contour $C$ is the one of Fig.1 where the horizontal path $C_H$ consists in the parts $[t_i, t_0]$, $[t_0, t_f]$, $[t_f, t_i]$ and $[t_0, t_1]$ and the vertical path $C_V$ is the $[t_i, t_i - i\beta]$ part. The ITF is recovered when $t_i = t_0$ where only the vertical path survives and the RTF when $t_i \to -\infty$ where only the horizontal path survives. In Eq. (6) we write the delta functions of $\rho(k_0, k)$ in their regularised form in order to be able to take the RTF limit and thus $\rho(k_0, k)$ is written as

$$\rho(k_0, k) = \frac{i}{2 \omega_k} \sum_{r, \xi = \pm 1} k_0 - \omega_k + ir \varepsilon$$

(10)

where $\varepsilon$ is the regulator which we keep finite until the $t_i \to -\infty$ limit has been performed. We Taylor expand the field $\phi(p, t_i)$ around $\phi(p, t_0)$ and write it as an exponential of the time derivative $\partial_t$ acting on $\phi(p, t)$

$$\phi(p, t_1) = \sum_{n=0}^{\infty} \frac{1}{n!} (t_1 - t_0)^n \left. \frac{\partial^n}{\partial t^n} \phi(p, t) \right|_{t = t_0}$$

$$= e^{(t_1 - t_0) \partial_t} \phi(p, t) \bigg|_{t = t_0}$$

(11)

Substituting in Eq. (6) the “bubble” term is written as

$$\Gamma^{(B)} = \int_C dt_1 \phi(p, t_0) D_{c}(t_0, t_1) e^{(t_1 - t_0) \partial_t} \phi(p, t) \bigg|_{t = t_0} D_{c}(t_1, t_0)$$

(12)

If we expand the exponential in powers of the time derivative $\partial_t$, $\Gamma^{(B)}$ is also written as

$$\Gamma^{(B)} = \int_{t_i}^{t_1 - i\beta} dt_1 \phi(p, t_0) \left[ \Gamma^{(0)} + \Gamma^{(1)} \partial_t + \Gamma^{(2)} \partial_t^2 + \ldots \right] \phi(p, t) \bigg|_{t = t_0}$$

(13)

In the following we will find the zeroth order term $\Gamma^{(0)}$ in the $\partial_t$ expansion which is related to the effective potential and the coefficients $\Gamma^{(n)}$ of the higher order derivatives up to the second order.

In Eq. (12) the exponential $e^{(t_1 - t_0) \partial_t}$ acts as an energy shift by $-i\partial_t = p^0$ in the energies of the propagators where $p^0$ is the external energy. The energies $\Omega, \omega$ in the dispersion relations of the propagators are related to the three momenta $k + p, k$ respectively where $p = -i \nabla$ is the external three momenta. Thus the zero four momenta limits correspond to taking the limits $\partial_t \to 0 (p^0 \to 0)$ and $\Omega \to \pm \omega (p \to 0)$.

III. THE BUBBLE TERM

We perform the $t_1$ integration in Eq. (12) along the whole time path for both horizontal and vertical parts and get

$$\Gamma^{(B)} = \sum_{\pm \omega, \Omega} \frac{im(n(\Omega))}{4\omega \Omega} \times \frac{\left(e^{\beta(\Omega + \omega)} - 1 - e^{-\beta A t} \left(e^{\beta(\omega + \Omega)} - 1\right)/A \right)}{A}$$

(14)

where

$$A = \omega + \Omega + i\beta \partial_t$$

and

$$\Delta t = t_1 - t_0$$

This result agrees with the one found by Evans using ITF. We notice that the $t_i$-dependence is included in the second term of Eq. (14) which is being multiplied by $(e^{-i\beta \partial_t} - 1)$. But

$$e^{-i\beta \partial_t} \phi(t) = \phi(t - i\beta)$$

and if the field $\phi(t)$ is periodic it is also

$$\phi(t - i\beta) = \phi(t)$$

This means that

$$(e^{-i\beta \partial_t} - 1) \phi(t) = 0$$

and the $t_i$-dependent term vanishes provided that we have included both horizontal and vertical paths of the contour. This agrees with Le Bellac and Mabilat who showed the $t_i$-independence for the case of the effective potential.
IV. ANALYTICITY

We perform the zero limits of the external four-momenta ($\Omega \rightarrow \pm \omega$, $\partial_t \rightarrow 0$) in both orders in the result of Eq. (13). In the case of $\Delta t$ finite, when both paths of the time contour contribute, the result is analytic and is the usual effective potential [2] given by

\[
\Gamma^{(0)} = i \frac{(2n(\omega) + 1)}{2\omega^3} + \frac{\beta n(\omega)(1+n(\omega))}{\omega^2} \tag{15}
\]

This is also the result for the ITF case of $\Delta t = 0$. For $\Delta t \rightarrow -\infty$ which is the RTF limit, we recover the result of Eq. (14), and after expanding in powers of the time-derivative $\partial_t \rightarrow 0$ first while in the opposite order of limits we get

\[
\lim_{\Omega \rightarrow \pm \omega, \partial_t \rightarrow 0} \Gamma^{(B)} = i \frac{(2n(\omega) + 1)}{2\omega^3}
\]

which is only the first term of the full result of Eq. (13), recovering thus the non-analyticity using RTF. It is therefore essential that we use both paths of the time contour to get a well-defined analytic result.

V. TIME-DERIVATIVE EXPANSION

For both $\Delta t$ cases we first take the limits $\Omega \rightarrow \pm \omega$ in Eq. (13) and then expand in powers of the time-derivative $\partial_t$ to get the coefficients $\Gamma^{(n)}$ up to the second order.

1. $\Delta t \rightarrow -\infty$ case

Performing the first limit ($\Omega \rightarrow \pm \omega$), Eq. (14) gives

\[
\lim_{\Omega \rightarrow \pm \omega} \Gamma^{(B)} = 2i \frac{(2n(\omega) + 1)}{\omega(4\omega^2 + \partial_t^2)}
\]

and after expanding in powers of the time-derivative $\partial_t$, we get a zeroth order in the time-derivative term which is the first part of the effective potential in agreement with before and a second order term of the form

\[
\Gamma^{(2)}(\Delta t \rightarrow -\infty) = -i \frac{(2n(\omega) + 1)}{8\omega^5}
\]

which agrees with a recent result by Berera et al [3]. In this case we did not find any coefficient $\Gamma^{(1)}$ of the term proportional to the single time-derivative of the field.

2. $\Delta t$ finite case

The expansion in powers of $\partial_t$ gives us a zeroth term which is the full effective potential of Eq. (13) as shown before, and a second order term which, in the limit of $\Delta t = 0$, is

\[
\Gamma^{(2)}(\Delta t = 0) = -\frac{i}{8\omega^5}[2(2n(\omega) + 1) - \beta \omega (2n(\omega) (n(\omega) + 1) + 1) + \beta^2 \omega^2 (2n(\omega) + 1) + \frac{4\beta^3 \omega^3 n(\omega)(n(\omega) + 1)}{3}]
\]

\[
\Gamma^{(2)}(\Delta t \rightarrow -\infty) + \text{other terms}
\]

This result is consistent with the one by Evans using ITF [4] but it is different to the one for $\Delta t$ infinite proving the $t_i$-dependence of the second order term $\Gamma^{(2)}$.

But in the finite $\Delta t$ case we also had a coefficient $\Gamma^{(1)}$ for the single time-derivative of the field unlike the zero temperature case where such a term vanishes. Evans [6] found a similar linear term in his imaginary time calculations. In the case of $\Delta t = 0$ (ITF) it is related to the effective potential $\Gamma^{(0)}$ by

\[
\Gamma^{(1)} = -\frac{i\beta}{2} \Gamma^{(0)}
\]

The existence of this term is due to the extra four-velocity with respect to the heat bath at finite temperature. The invariant quantity now is $U_\mu \partial^\mu$ where

\[
U_\mu = (1, 0, 0, 0)
\]

is the rest frame of the heat bath. Such a term did not exist in the expansion for the $\Delta t$ infinite case, where only zero and second order terms in the time-derivative survived. This makes sense since in the infinite time limit any interaction with the heat bath which gives rise to such linear terms will have been damped. Mathematically this term could arise due to the shape of the time contour, which in the finite $\Delta t$ case is non-symmetric. However this is not the case for the zero-temperature situation or the finite temperature one in the infinite $\Delta t$ limit where the symmetry of the contour will make any time integration of odd terms in the derivative expansion to vanish. This term is also $t_i$-dependent. We know that the effective action should be independent of the initial time for fields in thermal equilibrium. However in our case we have expanded time-dependent fields which have almost departed from equilibrium in order for such an expansion to be meaningful and therefore a dependence on the initial time is expected. Moreover a truncated expansion even of periodic fields is not necessarily periodic.

VI. CONCLUSIONS AND FUTURE WORK

We calculated the quadratic part of the thermal effective action for real scalar fields which vary slowly in time.
We found the result to be analytic in the limits of zero external four-momenta as long as we consider both horizontal and vertical paths of our time contour (\(\Delta t\) finite). We recovered the non-analyticity occurring in RTF as well as the ITF result in the appropriate limits of \(t_i\). We also found a simple way of computing higher derivative terms in the bubble and derived the complete bubble term. We expanded the bubble term up to the second derivative in the fields and our results were consistent with recent RTF and ITF calculations. The non zero, linear time-derivative term found in the finite \(\Delta t\) case is related to the heat bath frame at finite temperature. The physical meaning of such a term and in particular its sign and whether it is complex or real will determine whether it should be considered as a dissipative term or as a kind of “chemical potential”. We can solve the equations of motion for the \(\phi\) field in the truncated expansion and study the importance of this term for the dynamics of the field.

We also studied the \(t_i\)-dependence of our result before and after the expansion and discussed it in terms of equilibrium and the truncated expansion considered. We found that only for periodic \(\phi\) fields does the \(t_i\)-dependence cancel provided that we consider both horizontal and vertical paths of the time contour.

The extension of our calculation to higher derivative terms and to space-dependent fields will give the full effective action but the spatial contribution is quite trivial since the problems arising at finite temperature involve time. We have considered a two real scalar field theory, but we have also used our method in more physical models, such as a Yukawa theory, where the fermion field is the one integrated out and we have found a similar “linear term”. We can also consider scalar QED and gauge theories or even consider systems with time-dependent parameters. The possibility of evaluating quantum corrections can be more directly applied to phase transitions, where they may indicate us something about the order of the transition.

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