Nearly invariant subspaces for shift semigroups

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Abstract Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on an infinite-dimensional separable Hilbert space; a suitable definition of near \( \{T(t)\}_{t \geq 0} \) invariance of a subspace is presented in this paper. A series of prototypical examples for minimal nearly \( \{S(t)\}_{t \geq 0} \) invariant subspaces for the shift semigroup \( \{S(t)\}_{t \geq 0} \) on \( L^2(0, \infty) \) are demonstrated, which have close links with near \( T^*_\theta \) invariance on Hardy spaces of the unit disk for an inner function \( \theta \). Especially, the corresponding subspaces on Hardy spaces of the right half-plane and the unit disc are related to model spaces. This work further includes a discussion on the structure of the closure of certain subspaces related to model spaces in Hardy spaces.

Keywords nearly invariant subspace, \( C_0 \)-semigroup, shift semigroup, model space

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1 Introduction

The main aim of this paper is to investigate the near invariance problem for the shift semigroup \( \{S(t)\}_{t \geq 0} \) on \( L^2(0, \infty) \). In particular, we focus on characterizing a series of examples for the smallest nearly \( \{S(t)\}_{t \geq 0} \) invariant subspaces containing certain typical functions. The corresponding subspaces of these examples behave as model spaces in Hardy spaces of the right half-plane and the unit disc, which bring us a deeper understanding of near invariance and model spaces.

Let \( \mathcal{H} \) denote a separable infinite-dimensional Hilbert space and \( \mathcal{L}(\mathcal{H}) \) be the space of the bounded linear operator on \( \mathcal{H} \). If \( \{\mathcal{M}_i\}_{i \in I} \) is a family of subsets of the Hilbert space \( \mathcal{H} \), we denote by \( \bigvee_{i \in I} \mathcal{M}_i \) the closed linear span generated by \( \bigcup_{i \in I} \mathcal{M}_i \). Let the notation \( \bar{\mathcal{M}} \) denote the closure of \( \mathcal{M} \) for any subset \( \mathcal{M} \) of \( \mathcal{H} \). Here and throughout this paper, a subspace means a closed subspace.

A family \( \{T(t)\}_{t \geq 0} \) in \( \mathcal{L}(\mathcal{H}) \) is called a \( C_0 \)-semigroup if \( T(0) = I \), \( T(t+s) = T(t)T(s) \) for all \( s, t \geq 0 \) and \( \lim_{t \to 0^+} T(t)x = x \) for any \( x \in \mathcal{H} \). Given a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on a Hilbert space \( \mathcal{H} \), there exists a closed and densely defined linear operator \( A \) that determines the semigroup uniquely, called the generator of \( \{T(t)\}_{t \geq 0} \), defined as

\[
Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t},
\]
where the domain \( D(A) \) of \( A \) consists of all \( x \in \mathcal{H} \) for which this limit exists. If \( 1 \) is in the set
\[
\rho(A) := \{ \lambda \in \mathbb{C}, A - \lambda I : D(A) \subset \mathcal{H} \to \mathcal{H} \text{ is bijective} \},
\]
then \( (A - I)^{-1} \) is a bounded operator on \( \mathcal{H} \) by the closed graph theorem, and the Cayley transform of \( A \) defined by
\[
T := (A + I)(A - I)^{-1}
\]
is a bounded operator on \( \mathcal{H} \), since \( T - I = 2(A - I)^{-1} \). The operator \( T \) is the cogenerator and determines the semigroup uniquely, since the generator \( A \) does.

Let \( T \in \mathcal{L}(\mathcal{H}) \) be a left invertible isometric operator with finite multiplicity on \( \mathcal{H} \). We recall that a subspace \( \mathcal{M} \subset \mathcal{H} \) is nearly \( T^{-1} \) invariant if whenever \( g \in \mathcal{H} \) and \( Tg \in \mathcal{M} \), then \( g \in \mathcal{M} \). In [11], we have shown that the nearly \( T^{-1} \) invariant subspaces can be represented in terms of invariant subspaces under the backward shift. Especially, our result implies a characterization for the nearly \( T^n \) invariant subspaces in \( H^2(\mathbb{D}) \) when \( \theta \) is a finite Blaschke product. Here, \( H^2(\mathbb{D}) \) is the Hardy space defined on the unit disc with the form
\[
H^2(\mathbb{D}) = \bigg\{ f : \mathbb{D} \to \mathbb{C} \text{ is analytic, } f(z) = \sum_{k=0}^{\infty} a_k z^k, \| f \|^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty \bigg\}.
\]

However, there is no such simple description for the nearly \( T^n \) invariant subspaces in \( H^2(\mathbb{D}) \) for an infinite Blaschke product \( \theta \). In this paper, we explore some related investigations in this direction.

Before proceeding, we recall some preliminaries appearing in many books, including [12,13] for a detailed discussion. For an infinite Blaschke product \( \theta \), the Toeplitz operator \( T^n : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \) is universal (see, e.g., [4]) and it is similar to the backward shift \( S(1)^* \) on \( L^2(0, \infty) \), given by \( S(1)^* f(t) = f(t+1) \).

In general, the shift semigroup \( S(t) : L^2(0, \infty) \to L^2(0, \infty) \) with \( t \geq 0 \) is defined by
\[
(S(t)f)(\zeta) = \begin{cases} 0, & \zeta \leq t, \\ f(\zeta - t), & \zeta > t. \end{cases}
\]

It is obvious that \( S(1)^* \) is an element of the adjoint semigroup \( \{ S(t)^* \}_{t \geq 0} \) given as \( (S(t)^* f)(\zeta) = f(\zeta + t) \).

We recall \( H^2(\mathbb{C}_+) \) defined on the right half-plane \( \mathbb{C}_+ := \{ s = x + iy, x > 0 \} \) which contains all the analytic functions \( f : \mathbb{C}_+ \to \mathbb{C} \) such that
\[
\| f \|^2_{H^2(\mathbb{C}_+)} = \sup_{x > 0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy < \infty.
\]
The two-sided Laplace transform of \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) is given as
\[
(\mathcal{L}f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt.
\]

**Theorem 1.1** (See [16]). The Laplace transform gives a linear isomorphism from \( L^2(0, \infty) \) onto \( H^2(\mathbb{C}_+) \) such that
\[
\| \mathcal{L}(f) \|_{H^2(\mathbb{C}_+)} = \sqrt{2\pi} \| f \|_{L^2(0, \infty)} \text{ for } f \in L^2(0, \infty).
\]

It follows that \( S(1)^* \) is unitarily equivalent to the adjoint of the multiplication operator \( M_{e^{-s}} \) on the Hardy space \( H^2(\mathbb{C}_+) \). Regarding the Hardy spaces \( H^2(\mathbb{D}) \) and \( H^2(\mathbb{C}_+) \), there exists an isometric isomorphism \( V : H^2(\mathbb{D}) \to H^2(\mathbb{C}_+) \) given as
\[
(Vf)(s) = \frac{1}{\sqrt{\pi} (1 + s)} f(M(s)),
\]
where \( M : s \to \frac{1 - e^{-s}}{1 + e^{-s}} \) is a self-inverse bijection from \( \mathbb{C}_+ \) to \( \mathbb{D} \).

Meanwhile, the inverse map \( V^{-1} : H^2(\mathbb{C}_+) \to H^2(\mathbb{D}) \) is defined by
\[
(V^{-1}g)(z) = \frac{2\sqrt{\pi}}{1 + z} g(M(z)).
\]
Of great importance in operator-related function theory are the shift operators, ubiquitous in applications. It is well known that the inner functions arose from the representation of shift invariant subspaces in $H^2(D)$. Specifically, we say that $u$ is an inner function if it is a bounded analytic function on $D$ such that $|u(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. The celebrated theorem of Beurling says that the nontrivial invariant subspaces of $H^2(D)$ for the forward shift operator $Sf(z) = zf(z)$ are precisely $uH^2(D)$ with $u$ being an inner function. At the same time, the model space denoted by $K_u := (uH^2(D))^\perp = H^2(D) \ominus uH^2(D)$ is an invariant subspace of the backward shift operator $S^*f(z) = (f(z) - f(0))/z$; for a more detailed exposition on inner functions and model spaces, please see, e.g., [6,8].

Research on invariant subspaces leads to the concept of near invariance. The study of nearly invariant subspaces for the backward shift in $H^2(D)$ was first explored by Hayashi [9], Hitt [10] and then Sarason [14,15] in relation to kernels of Toeplitz operators. Afterwards, Câmara and Partington [1,2] continued the systematic investigations on near invariance and Toeplitz kernels. In particular, Hitt [10] proved the following most widely known characterization of nearly $S^*$ invariant subspaces in $H^2(D)$.

**Theorem 1.2** (See [10, Proposition 3]). The nearly $S^*$ invariant subspaces have the form $M = uK$ with $u \in M$ of the unit norm, $u(0) > 0$, and $u$ orthogonal to all the elements of $M$ vanishing at the origin. Here, $K$ is an $S^*$ invariant subspace, and the operator of multiplication by $u$ is isometric from $K$ into $H^2(D)$.

As a nontrivial extension of our recent work in [11], we study the nearly invariant subspaces for the shift semigroup on $L^2(0,\infty)$, which is related to near $T_\theta^*$ invariance on $H^2(D)$ for an infinite Blaschke product $\theta$. To the best of our knowledge, there have been no such investigations, even though there is a long history on invariant subspaces for a $C_0$-semigroup, which are defined as below.

**Definition 1.3.** Given a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ in $L(H)$, a subspace $M \subseteq H$ is said to be $\{T(t)\}_{t \geq 0}$ invariant if $T(t)M \subseteq M$ for all $t \geq 0$.

Based on Definition 1.3, it might be natural to call $N$ a nearly $\{T(t)^*\}_{t \geq 0}$ invariant subspace if whenever $T(t)x \in N$ for all $t > 0$, then $x \in N$. However, all the closed subspaces have this property since $x = \lim_{n \to \infty} T(t_n)x$ for any sequence $(t_n)$ tending to 0, so this definition is not useful. We provide a more suitable definition as follows.

**Definition 1.4.** Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup in $L(H)$ and $N \subseteq H$ be a subspace. If for every $x \in H$ whenever $T(t)x \in N$ for some $t > 0$, then $x \in N$, we call $N$ a nearly $\{T(t)^*\}_{t \geq 0}$ invariant subspace.

We say that $N$ is a trivial nearly $\{T(t)^*\}_{t \geq 0}$ invariant subspace if no element in $N$ satisfies the above condition in Definition 1.4. Then we study the starting question below.

**Question 1.** What is the structure of nontrivial nearly $\{S(t)^*\}_{t \geq 0}$ invariant subspaces of the shift semigroup $\{S(t)\}_{t \geq 0}$ on $L^2(0,\infty)$ given in (1.1)?

For the shift semigroup $\{S(t)\}_{t \geq 0}$ on $L^2(0,\infty)$, the maps in (1.4) and (1.2) imply the following commutative diagrams:

$\begin{align*}
L^2(0,\infty) &\xrightarrow{S(t)} L^2(0,\infty) \\
\downarrow \mathcal{L} &\quad \downarrow \mathcal{L} \\
H^2(C_+) &\xrightarrow{M(t)} H^2(C_+) \\
\downarrow V^{-1} &\quad \downarrow V^{-1} \\
H^2(D) &\xrightarrow{T(t)} H^2(D).
\end{align*}$

Here, the multiplication semigroup $\{M(t)\}_{t \geq 0}$ on $H^2(C_+)$ is defined by

$$(M(t)g)(s) = e^{-st}g(s), \quad s \in C_+$$
and \((M(t)^* g)(s) = P_{H^2(C_+)} e^{st} g(s)\). Moreover, \(\{T(t)\}_{t \geq 0} \) on \(H^2(D)\) is given as
\[
(T(t)h)(z) = \phi^t(z)h(z), \quad z \in D
\]
and \((T(t)^* h)(z) = P_{H^2(D)} \phi^{-t}(z)h(z), \quad z \in D \) with \(\phi^t(z) := \exp(-t \frac{1-z}{1+z})\), the power of a standard atomic inner function.

**Remark 1.5.** Since every Toeplitz kernel in \(H^2(C_+)\) or \(H^2(D)\) is nearly invariant under division by an inner function, it is also nearly \(\{M(t)^*\}_{t \geq 0}\) or \(\{T(t)^*\}_{t \geq 0}\) invariant in \(H^2(C_+)\) or \(H^2(D)\), respectively.

It is known that the cogenerator of a \(C_0\)-semigroup plays an important role in invariant subspaces, and the following theorem holds.

**Theorem 1.6** (See [7, Theorem 10-9]). Let \(\{T(t)\}_{t \geq 0}\) be a contractive semigroup and \(T\) be its infinitesimal cogenerator. A subspace \(M\) is invariant under \(\{T(t)\}_{t \geq 0}\) if and only if it is invariant under \(T\).  

**Remark 1.7.** However, the parallel conclusion in Theorem 1.6 does not hold for near invariance of a \(C_0\)-semigroup. For example, for \(\{M(t) = e^{-st}\}_{t \geq 0}\) on \(H^2(C_+)\), \(T := (A + I)(A - I)^{-1}\) is the cogenerator of \(\{M(t)\}_{t \geq 0}\) with \(Af := M_{e^{-t}} f\). Not every nearly invariant subspace in the usual sense (division by \((1-s)/(1+s)\)) is nearly \(\{M(t)^*\}_{t \geq 0}\) invariant, for example, \(e^{-s}H^2(C_+)\). Likewise \(\{(1-s)/(1+s)\}H^2(C_+)\) is nearly \(\{M(t)^*\}_{t \geq 0}\) invariant, but not nearly \(T^*\) invariant.

Hence, it is meaningful to construct nontrivial examples of near invariance for the above well-known \(C_0\)-semigroups. The rest of this paper is organized as follows. In Section 2, we explore a prototypical example of a smallest (cyclic) nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace in \(L^2(0, \infty)\) and deduce the corresponding results for multiplication \(C_0\)-semigroups on Hardy spaces. The second nontrivial example is also examined in Section 3, and this leads to a series of general examples presented using the Hardy space model. Especially, our results reveal that a wide class of nearly \(S\) invariant subspaces in the Hardy space \(H^2(D)\) are of finite codimension in model spaces. The relevant characterizations in the Hardy space \(H^2(C_+)\) are also addressed.

In the next two sections, \(N \subseteq L^2(0, \infty)\) is always supposed to be a nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace, and we denote the smallest (cyclic) nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace in \(N\) containing some nonzero vector \(f\) by \([f]_s\). There follow two possibilities.

(i) There is no function \(f \in N\), apart from the zero function, for which there exists some \(\delta > 0\) on \((0, \delta)\). In this case, \(N\) is a trivial nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace and \([f]_s = C \cdot f\) for all \(f \in N\).

(ii) There are \(\delta > 0\) and a function \(f \in N\) that vanishes almost everywhere on \((0, \delta)\) and not on \((0, \delta + \epsilon)\) for any \(\epsilon > 0\). Since \(S(\delta)S(\delta)^* f = f \in N\), the near \(\{S(t)^*\}_{t \geq 0}\) invariance implies \(g := S(\delta)^* f \in N\). Meanwhile, we have \(S(\lambda)g = S(\delta - \lambda)^* f \in N\) for all \(0 \leq \lambda \leq \delta\). So

\[
[f]_s = \{S(\lambda)g, \quad 0 \leq \lambda \leq \delta\}.
\]

This is the key to our work and we have not previously encountered subspaces defined in this way.

We begin with the simplest example, and explore the smallest (cyclic) nontrivial nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace in \(L^2(0, \infty)\) containing \(e_\delta(\zeta) := e^{-\zeta(\delta, \infty)}(\zeta)\) with \(\delta > 0\) such that \(e^{\delta L(e_\delta)(s)} = e^{-\delta s}(1 + s)^{-1}\). As an extension, we continue to take \(f_{\delta,n}(\zeta) := (\zeta - \delta)^n e_\delta(\zeta)/n!\) satisfying \(e^{\delta L(f_{\delta,n})(s)} = e^{-\delta s}(1 + s)^{-(n+1)}\) for integer \(n \geq 1\), and formulate the Laplace transform of the smallest nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspaces containing \(f_{\delta,n}\) in Hardy spaces. This offers a large class of important cases for Question 1.

## 2 Smallest nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspaces containing \(e_\delta\) for \(\delta > 0\)

In this section, we identify the smallest (cyclic) nearly \(\{S(t)^*\}_{t \geq 0}\) invariant subspace containing \(e_\delta(\zeta) := e^{-\zeta(\delta, \infty)}(\zeta)\) for some \(\delta > 0\) in \(L^2(0, \infty)\). After that we express such subspaces as model spaces in Hardy spaces of the right half-plane and the unit disk.
For \( \delta > 0 \), let \( f(\zeta) = e_\delta(\zeta) \in N \). For any \( 0 \leq \lambda \leq \delta \),
\[
(S(\delta - \lambda)^* e_\delta)(\zeta) = e_\delta(\zeta + \delta - \lambda) = e^{-(\delta - \lambda)}e_\lambda(\zeta),
\]
and then it holds that
\[
[e_\delta]_s := \vee\{e_\lambda, 0 \leq \lambda \leq \delta\} \subseteq N.
\]

We formulate a proposition for the smallest (cyclic) nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace \([e_\delta]_s\) in \( L^2(0, \infty) \).

**Proposition 2.1.** In \( L^2(0, \infty) \), the smallest nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace containing \( e_\delta \) with some \( \delta > 0 \) has the form
\[
[e_\delta]_s := \vee\{e_\lambda, 0 \leq \lambda \leq \delta\} = L^2(0, \delta) + C e^{-\zeta}.
\]

**Proof.** To simplify the writing, we define \( M := L^2(0, \delta) + C e^{-\zeta}. \) Since \( e_\lambda(\zeta) = -e^{-\zeta} \chi(0, \lambda) + e^{-\zeta} \in M \) for every \( 0 \leq \lambda \leq \delta \), we have \([e_\delta]_s \subseteq M\).

Conversely, for \( 0 \leq \lambda \leq \mu \leq \delta \), we have
\[
(e_\lambda - e_\mu)(\zeta) = e^{-\zeta} \chi(\lambda, \mu)(\zeta),
\]
and next we show that the closed linear span of \( e_\lambda - e_\mu \) is \( L^2(0, \delta) \).

Taking a function \( f \in C[0, \delta] \), we approximate \( f \) arbitrarily closely in \( L^\infty(0, \delta) \) by combinations of \( e_\lambda - e_\mu \). Since any function \( f \) can be written as \( u + iv \) with two real functions \( u \) and \( v \), we may suppose without loss of generality that \( f \) is real and \( \|f\|_{L^\infty(0, \delta)} \leq 1 \). Given \( \epsilon > 0 \), we use uniform continuity of \( f \) to partition \( [0, \delta) \) into \( N \) intervals \( I_k = [(k-1)\delta/N, k\delta/N) \) of length \( \delta/N \) such that
\[
\sup_{I_k} f - \inf_{I_k} f < \frac{\epsilon}{2} \quad \text{for each} \quad k = 1, \ldots, N.
\]

We also choose \( N \) large enough such that
\[
1 - e^{-\delta/N} < \frac{\epsilon}{2}. \tag{2.1}
\]

Then
\[
\left\| f - \sum_{k=1}^{N} a_k \chi_{I_k} \right\|_{L^\infty(0, \delta)} < \frac{\epsilon}{2} \tag{2.2}
\]
for some suitable \( a_k \in [-1, 1] \) and
\[
|a_k - a_k e^{-t+(k-1)\delta/N}| \leq |a_k| (1 - e^{-\delta/N}) \leq 1 - e^{-\delta/N} < \frac{\epsilon}{2} \quad \text{for} \quad t \in I_k \tag{2.3}
\]
due to \( -\delta/N < -t + (k-1)\delta/N \leq 0 \) and (2.1). Then (2.2) together with (2.3) gives
\[
\left\| f - \sum_{k=1}^{N} a_k e^{(k-1)\delta/N} \left(e_{(k-1)\delta/N} - e_{k\delta/N}\right) \right\|_{L^\infty(0, \delta)} \leq \frac{\epsilon}{2} + \max_{1 \leq k \leq N} |a_k - a_k e^{-t+(k-1)\delta/N}|
\]
\[
\leq \frac{\epsilon}{2} + \sum_{k=1}^{N} a_k \chi_{I_k} \left| \sum_{k=1}^{N} a_k \chi_{I_k} - \sum_{k=1}^{N} a_k e^{-t+(k-1)\delta/N} \chi_{I_k} \right|_{L^\infty(0, \delta)} \leq \frac{\epsilon}{2} \quad \text{for} \quad k = 1, \ldots, N.
\]

Since \( C[0, \delta] \) is dense in \( L^2(0, \delta) \) and \( \|f\|_{L^2(0, \delta)} \leq \sqrt{\delta}\|f\|_{L^\infty(0, \delta)} \), the desired result follows. \( \square \)
Using the transform $\mathcal{L} : L^2(0, \infty) \to H^2(C_+)$ in (1.2), we have
\[ e^\delta \mathcal{L}(e_\delta)(s) = e^\delta \int_0^\infty e^{-(s+1)t} dt = \frac{e^{-\delta s}}{1+s}. \tag{2.4} \]
Then the equation in Proposition 2.1 is mapped by $\mathcal{L}$ into
\[ \vee \left\{ \frac{e^{-\lambda s}}{1+s}, 0 \leq \lambda \leq \delta \right\} = K_{e^{-\delta s}} + C \frac{1}{1+s}. \tag{2.5} \]
By using the map $V^{-1} : H^2(C_+) \to H^2(D)$ in (1.4), we have
\[ V^{-1} : e^{-\delta s} \to \frac{2\sqrt{\pi}}{1+z} \phi^\delta(z) \quad \text{and} \quad V^{-1} : \frac{1-s}{1+s} e^{-\delta s} \to \frac{2\sqrt{\pi}}{1+z} z \phi^\delta(z). \tag{2.6} \]
Since $(1+z)^{-1}$ is an outer function in $H^2(D)$, (2.6) further implies the corresponding model spaces from $H^2(C_+)$ to $H^2(D)$:
\[ K_{e^{-\delta s}} \to K_{\phi^\delta} \quad \text{and} \quad K_{\frac{1-s}{1+s} e^{-\delta s}} \to K_{\sqrt{\pi} \phi^\delta}. \]
Next, we recall a lemma for model spaces from [8].

**Lemma 2.2** (See [8, Corollary 5.9]). If $\theta_1$ and $\theta_2$ are inner functions on $D$, then
\[ K_{\theta_1} \vee K_{\theta_2} = K_{\text{lcm}(\theta_1, \theta_2)}, \]
where lcm$(\theta_1, \theta_2)$ is the least common multiple of $\theta_1$ and $\theta_2$.

In Lemma 2.2, if one of the left-hand subspaces is finite-dimensional, then the closed linear span is the same as the sum. So it yields that
\[ K_{\sqrt{\pi} \phi^\delta} = K_z + K_{\phi^\delta} = C + K_{\phi^\delta}. \tag{2.7} \]
Switching (2.7) into $H^2(C_+)$ by the map (1.3), we deduce
\[ K_{\frac{1-s}{1+s} e^{-\delta s}} = C \frac{1}{1+s} + K_{e^{-\delta s}}. \tag{2.8} \]
Based on (2.5) and (2.8), we obtain a corollary in $H^2(C_+)$.  

**Corollary 2.3.** In $H^2(C_+)$, the Laplace transform of $[e_\delta]_s$ is
\[ \mathcal{L}([e_\delta]_s) = \vee \left\{ \frac{e^{-\lambda s}}{1+s}, 0 \leq \lambda \leq \delta \right\} = K_{e^{-\delta s}} + C \frac{1}{1+s} = K_{\frac{1-s}{1+s} e^{-\delta s}}, \]
where $K_{e^{-\delta s}}$ and $K_{\frac{1-s}{1+s} e^{-\delta s}}$ are model spaces in $H^2(C_+)$.  

Transferring Corollary 2.3 into $H^2(D)$ by
\[ V^{-1} : \frac{e^{-\lambda s}}{1+s} \to \sqrt{\pi} \phi^\lambda(z), \]
and using (2.7), we deduce a corollary in $H^2(D)$.

**Corollary 2.4.** In $H^2(D)$, it holds that
\[ V^{-1}(\mathcal{L}([e_\delta]_s)) = \vee \{ \phi^\lambda, 0 \leq \lambda \leq \delta \} = K_{\sqrt{\pi} \phi^\delta}. \]

The following corollary further shows that the closed linear span of the powers of the singular inner function $\exp((z-1)/(z+1))$ is $H^2(D)$.

**Corollary 2.5.** In $H^2(D)$, it holds that
\[ \vee \{ \phi^\lambda, 0 \leq \lambda < \infty \} = H^2(D). \tag{2.9} \]
Here, we use one-sided limits for $t$ implies that it is orthogonal to every

$$K_{z^\theta} = \bigvee \{ \phi^\lambda, 0 \leq \lambda < \infty \}.$$  

This means that $f$ is in the intersection of $z^\theta H^2(D)$ for every $\delta > 0$, and then $f = 0$ by the uniqueness of inner-outer factorization.

Using the isomorphism in (1.3), we have a corollary in $H^2(C_+)$, which can also be deduced from the fact that $1/(1+s)$ is an outer function.

**Corollary 2.6.** In $H^2(C_+)$, it holds that

$$\mathcal{C}_{1, 0}(1+\delta) \in \bigvee \{ e^{-\lambda s}, 0 \leq \lambda < \infty \} = H^2(C_+).$$

Next, we require a lemma in $H^2(C_+)$.

**Lemma 2.7.** If $g(s), sg(s) \in H^2(C_+$), then

$$sg(s)e^{-st} \in \mathcal{C}_{1, 0}(1+\delta)$$

for all $\epsilon > 0$.

**Proof.** Taking $s \in i\mathbb{R}$ and then differentiating with respect to the parameter $t$, we have

$$\lim_{\mu \to 0} \frac{g(s)(e^{-s(t+\mu)} - e^{-st})}{\mu} = -sg(s)e^{-st}.$$  

Moreover, by the mean value inequality, we deduce that

$$\left| \frac{g(s)(e^{-s(t+\mu)} - e^{-st})}{\mu} \right| \leq \sup_{|\lambda - t| < \mu} |sg(s)e^{-\lambda s}| = |sg(s)|$$

so that the above convergence not only is pointwise, but by the dominated convergence and the assumption that $|sg(s)| \in L^2(i\mathbb{R})$, takes place in the $H^2(C_+)$ norm. Thus the desired result follows.

**Remark 2.8.** (1) For the case $g(s) = 1/(1+s)^2$, we conclude that

$$\frac{se^{-st}}{(1+s)^2} \in B := \bigvee \left\{ \frac{e^{-\lambda s}}{(1+s)^2}, 0 \leq \lambda \leq \delta \right\}$$  

for all $t \in [0, \delta]$. Here, we use one-sided limits for $t = 0$ and $t = \delta$ in Lemma 2.7. By linearity it follows that $e^{-st}/(1+s) \in B$ which further implies the inclusion

$$\bigvee \left\{ \frac{e^{-\lambda s}}{1+s}, 0 \leq \lambda \leq \delta \right\} \subseteq \bigvee \left\{ \frac{e^{-\lambda s}}{(1+s)^2}, 0 \leq \lambda \leq \delta \right\}.$$  

By transferring to the unit disc by the map (1.4), we see that

$$\bigvee \{ \phi^\lambda, 0 \leq \lambda \leq \delta \} \subseteq \bigvee \{ (1+z)^n \phi^\lambda, 0 \leq \lambda \leq \delta \}. \quad (2.10)$$

(2) For the general $g_n(s) = 1/(1+s)^{n+1} \in H^2(C_+)$ with integer $n \geq 1$, it similarly holds on $H^2(C_+)$ and $H^2(D)$ as below:

$$\bigvee \left\{ \frac{e^{-\lambda s}}{(1+s)^n}, 0 \leq \lambda \leq \delta \right\} \subseteq \bigvee \left\{ \frac{e^{-\lambda s}}{(1+s)^{n+1}}, 0 \leq \lambda \leq \delta \right\},$$  

$$\bigvee \{ (1+z)^{n-1} \phi^\lambda, 0 \leq \lambda \leq \delta \} \subseteq \bigvee \{ (1+z)^n \phi^\lambda, 0 \leq \lambda \leq \delta \}. \quad (2.11)$$
3 Smallest nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspaces in more general situations

Recall that in Section 2, the function \( e_\delta \) satisfies (2.4). Next, it is natural to look at the smallest (cyclic) nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace in \( L^2(0, \infty) \) containing \( f = f_{\delta,1}(\zeta) := (\zeta - \delta)e_\delta(\zeta) \) such that \( e^\delta \mathcal{L}(f_{\delta,1})(s) = e^{-\delta s}(1 + s)^{-2} \) for some \( \delta > 0 \). In this case, we show that the mapped subspace in \( H^2(\mathbb{D}) \) is the closure of \( (1 + z)K_{z \phi^\delta} \) equaling the model space \( K_{z \phi^\delta} \). Afterwards, we describe the general formulas for the Laplace transform of the smallest (cyclic) nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace containing \( f_{\delta,n} \) in \( H^2(\mathbb{C}_+) \) and the corresponding subspaces in \( H^2(\mathbb{D}) \). This leads us to find some important characterizations for the closure of \( gK_{z \phi^\delta} \) with a more general \( g \in L^\infty(\mathbb{T}) \). Meanwhile, we also summarize the descriptions in \( H^2(\mathbb{C}_+) \).

3.1 Smallest (cyclic) nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspaces containing \( f_{\delta,1} \) for some \( \delta > 0 \)

For the vector \( f_{\delta,1}(\zeta) := (\zeta - \delta)e_\delta \), it holds that

\[ e^\delta \mathcal{L}(f_{\delta,1})(s) = \frac{e^{-\delta s}}{(1 + s)^2}. \]

Suppose that the nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace \( \mathcal{N} \) contains \( f_{\delta,1} \) with some \( \delta > 0 \). Since

\[ S(\delta - \lambda)^*f_{\delta,1} = e^{-\delta(\delta - \lambda)}(\zeta - \lambda)e_\lambda \in \mathcal{N} \text{ for all } 0 \leq \lambda \leq \delta, \]

the smallest (cyclic) nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace containing the vector \( f_{\delta,1} \) in \( \mathcal{N} \) is

\[ [f_{\delta,1}]_s = \mathbb{V} \{ (\zeta - \lambda)e_\lambda, 0 \leq \lambda \leq \delta \}. \]

In \( H^2(\mathbb{C}_+) \), the Laplace transform maps the subspace \( [f_{\delta,1}]_s \) onto

\[ \mathcal{L}([f_{\delta,1}]_s) = \mathbb{V} \left\{ \frac{e^{-\lambda s}}{(1 + s)^2}, 0 \leq \lambda \leq \delta \right\}. \]

Meanwhile, in \( H^2(\mathbb{D}) \), by the map \( V^{-1} \) in (1.4), we have

\[ V^{-1}(\mathcal{L}([f_{\delta,1}]_s)) = \mathbb{V} \{(1 + z)\phi^\lambda, 0 \leq \lambda \leq \delta \} = (1 + z)K_{z \phi^\delta} \subseteq K_{z \phi^\delta} + zK_{z \phi^\delta}. \] (3.1)

By the formula (2.10), it follows that

\[ \phi^\lambda, z\phi^\lambda \in (1 + z)K_{z \phi^\delta} \]

for \( 0 \leq \lambda \leq \delta \). This further implies

\[ K_{z \phi^\delta} + zK_{z \phi^\delta} \subseteq (1 + z)K_{z \phi^\delta}, \]

which together with (3.1) implies

\[ V^{-1}(\mathcal{L}([f_{\delta,1}]_s)) = K_{z \phi^\delta} + zK_{z \phi^\delta}. \] (3.2)

The next proposition gives the concrete form of (3.2).

**Proposition 3.1.** In \( H^2(\mathbb{D}) \), it holds that

\[ V^{-1}(\mathcal{L}([f_{\delta,1}]_s)) = \mathbb{V} \{(1 + z)\phi^\lambda, 0 \leq \lambda \leq \delta \} = K_{z \phi^\delta}. \] (3.3)

**Proof.** For any \( f \perp V^{-1}(\mathcal{L}([f_{\delta,1}]_s)) \), (3.2) implies \( f \perp K_{z \phi^\delta} \) and \( f \perp zK_{z \phi^\delta} \), which means \( f \in z\phi^\delta H^2(\mathbb{D}) \) and \( S^*f \in z\phi^\delta H^2(\mathbb{D}) \). So we can suppose \( f = z\phi^\delta h \) with some \( h \in H^2(\mathbb{D}) \), and then \( S^*f = \phi^\delta h \in z\phi^\delta H^2(\mathbb{D}) \), verifying that \( z \) divides \( h \). Hence, \( f \in z^2\phi^\delta H^2(\mathbb{D}) \), and this shows

\[ K_{z \phi^\delta} \subseteq V^{-1}(\mathcal{L}([f_{\delta,1}]_s)). \]

Furthermore, since \( K_{z \phi^\delta} \subseteq K_{z \phi^\delta} \) and \( zK_{z \phi^\delta} \subseteq K_{z \phi^\delta} \), combining (3.2), we obtain (3.3). \( \square \)
Proposition 3.2. The subspace \((1 + z)K_{z\phi^\delta} \) is not closed in \(H^2(\mathbb{D})\).

\(\text{Proof.} \quad \) Let

\[ k_v(z) = \frac{1 - \overline{v}\phi^\delta(v)\phi^\delta(z)}{1 - \overline{v}z} \]

denote the reproducing kernel for the model space \(K_{z\phi^\delta}\). Then

\[ ||k_v||^2 = k_v(v) = \frac{1 - |v\phi^\delta(v)|^2}{1 - |v|^2}. \]

For \(v \to -1\) nontangentially, it holds that \(\phi^\delta(v) \to 0\) and then

\[ ||k_v||^2 \to \infty. \quad (3.4) \]

On the other hand, it holds that \(||(1 + z)k_v||^2 = 2||k_v||^2 + 2\text{Re}(k_v, zk_v)\). Since \(k_v\) is orthogonal to \(\overline{v}\phi^\delta(v)z^2\phi^\delta(z)(1 - \overline{v}z)^{-1}\), we have

\[ (k_v, zk_v) = \left\langle \frac{1 - \overline{v}\phi^\delta(v)\phi^\delta(z)}{1 - \overline{v}z}, z \right\rangle \]
\[ = \left\langle \frac{1}{1 - \overline{v}z}, \frac{z}{1 - \overline{v}z} \right\rangle - \left\langle \overline{v}\phi^\delta(v)\phi^\delta(z), \frac{z}{1 - \overline{v}z} \right\rangle \]
\[ = \frac{\overline{v}(1 - |\phi^\delta(v)|^2)}{1 - |v|^2}, \]

where we used the fact that \((1 - \overline{v}z)^{-1}\) is the reproducing kernel for \(H^2(\mathbb{D})\). So we deduce that

\[ \|(1 + z)k_v\|^2 = 2\frac{1 - |v\phi^\delta(v)|^2}{1 - |v|^2} + 2\text{Re}\left(\frac{\overline{v}(1 - |\phi^\delta(v)|^2)}{1 - |v|^2}\right). \]

Let \(v = -r\) with \(0 < r < 1\), and then we get that

\[ \|(1 + z)k_v\|^2 = \frac{2 - 2r - 2r|\phi^\delta(-r)|^2}{1 - r^2} \]
\[ = \frac{2 + 2r}{1 - r^2} + \frac{2r|\phi^\delta(-r)|^2}{1 + r} \]
\[ \to 1 \quad \text{as} \quad r \to 1. \quad (3.5) \]

Then (3.4) together with (3.5) implies that the injective map \(f \mapsto (1 + z)f\) is not bounded below on \(K_{z\phi^\delta}\), so \((1 + z)K_{z\phi^\delta}\) is not closed in \(H^2(\mathbb{D})\).

In \(H^2(\mathbb{C}_+),\) Proposition 3.1 implies a characterization for the subspace \(\mathcal{L}([f_{\delta,1}])\) with some \(\delta > 0\).

Theorem 3.3. The Laplace transform of the smallest nearly \((S(t)^*)\)\textsuperscript{\(t\geq0\)} invariant subspace containing \(f_{\delta,1}\) with some \(\delta > 0\) has the form

\[ \mathcal{L}([f_{\delta,1}]) = \bigvee \left\{ \frac{e^{-2\pi s}}{(1 + s)^2}, 0 \leq \lambda \leq \delta \right\} = K_{(\frac{\lambda}{1 + s})z e^{-is}}, \]

where \(K_{(\frac{\lambda}{1 + s})z e^{-is}}\) is a model space in \(H^2(\mathbb{C}_+)\).
3.2 More general examples on the smallest (cyclic) nearly invariant subspaces

In this subsection, we suppose that the nearly \( \{S(t)^*\}_{t \geq 0} \) invariant subspace \( \mathcal{N} \subset L^2(0, \infty) \) contains the function \( f_{\delta,n} \) in Lemma 3.4, which can degenerate \( e_\delta \) and \( f_{\delta,1} \) with \( n = 0 \) and \( n = 1 \).

**Lemma 3.4.** Using the Laplace transform in (1.2), we have that

\[
e^\delta \mathcal{L}(f_{\delta,n})(s) = \frac{e^{-\delta s}}{(1 + s)^{n+1}}
\]

holds for the functions

\[
f_{\delta,n}(\zeta) = \frac{(\zeta - \delta)^n}{n!} e_\delta(\zeta)
\]

with \( e_\delta(\zeta) = e^{-\zeta \chi_{(\delta, \infty)}(\zeta)}, \delta > 0 \) and any nonnegative integer \( n \).

**Proof.** Define

\[
I_n(s) = \int_\delta^\infty e^{-(s+1)\zeta}(t - \delta)^n dt.
\]

It follows that

\[
I_n(s) = \frac{n}{1 + s} I_{n-1}(s).
\]

By iterations, we further have

\[
I_n(s) = \frac{n!}{(1 + s)^n} I_0(s) = \frac{n! e^{-(s+1)\delta}}{(1 + s)^{n+1}}
\]

by the display (2.4) for \( I_0(s) \). Then it turns out that

\[
\mathcal{L}(f_{\delta,n})(s) = \frac{I_n(s)}{n!} = \frac{e^{-(s+1)\delta}}{(1 + s)^{n+1}}.
\]

This means that the equation (3.6) is true. \( \square \)

Here, we first present the mapped subspaces of the smallest (cyclic) nearly invariant subspace \( [f_{\delta,n}]_s \) in Hardy spaces.

**Theorem 3.5.** For any nonnegative integer \( n \) and \( \delta > 0 \), the following statements are true:

1. In \( H^2(\mathbb{D}) \), it holds that the cyclic nearly \( \{T(t)^*\}_{t \geq 0} \) invariant subspace \( \bigvee \{(1 + z)^n \phi^\lambda, 0 \leq \lambda \leq \delta \} = K_{z^{n+1} \phi^\delta} \).

2. In \( H^2(\mathbb{C}_+) \), it holds that the cyclic nearly \( \{M(t)^*\}_{t \geq 0} \) invariant subspace \( \bigvee \{ e^{-\lambda \phi^\delta}/(1 + \phi^\delta)^n, 0 \leq \lambda \leq \delta \} = K_{(1 + e^{-\delta})^{n+1} \phi^\delta} \).

**Proof.** Since (2) can be deduced by using the map \( V \) in (1.4) and the result (1), we only need to prove (1). By mathematical induction, it holds for \( n = 0 \), and we suppose that it is true for \( n - 1 \), i.e.,

\[
\bigvee \{(1 + z)^{n-1} \phi^\lambda, 0 \leq \lambda \leq \delta \} = K_{z^n \phi^\delta}.
\]

Then it turns out that

\[
\bigvee \{(1 + z)^n \phi^\lambda, 0 \leq \lambda \leq \delta \} = (1 + z) \bigvee \{(1 + z)^{n-1} \phi^\lambda, 0 \leq \lambda \leq \delta \} = (1 + z)K_{z^n \phi^\delta}.
\]

By the formulae in (2.11), it follows that

\[
(1 + z)^{n-1} \phi^\lambda, z(1 + z)^{n-1} \phi^\lambda \subseteq (1 + z)K_{z^n \phi^\delta}
\]

for \( 0 \leq \lambda \leq \delta \) implying

\[
K_{z^n \phi^\delta} + zK_{z^n \phi^\delta} \subseteq (1 + z)K_{z^n \phi^\delta}.
\]
Since the above converse inclusion is obvious, it yields that
\[ \forall \{(1+z)^n \phi^\lambda, 0 \leq \lambda \leq \delta \} = K_{z^n \phi^\lambda} + z K_{z^n \phi^\lambda}. \]
By a similar proof of Proposition 3.1, we obtain
\[ K_{z^n \phi^\lambda} + z K_{z^n \phi^\lambda} = K_{z^{n+1} \phi^\lambda}. \]
This completes the proof.

Now we can formulate a corollary for the Laplace transform of \([f_{\delta, n}]_s\).

**Corollary 3.6.** The Laplace transform of the smallest nearly \(S(t)^*\) invariant subspace containing \(f_{\delta, n}\) in (3.7) has the form
\[ \mathcal{L}([f_{\delta, n}]_s) = K_{(1+\frac{t}{\pi})^{n+1} e^{-\delta t}} \]
for any nonnegative integer \(n\).

**Remark 3.7.** It is known that \(\{\sqrt{2\pi} p_n(t) e^{-t}\}_{n=0}^\infty\) forms an orthonormal basis for \(L^2(0, \infty)\) with \(p_n(t) = \pm L_n(2t)/\sqrt{\pi}\) (a real polynomial of degree \(n\)) and \(L_n\) denotes the Laguerre polynomial
\[ L_n(t) = e^t \frac{d^n}{dt^n} (t^n e^{-t}). \]
So when we consider the smallest (cyclic) nearly \(S(t)^*\) invariant subspace containing \(f_{\delta, n}\) in (3.7) for some \(\delta > 0\) and nonnegative integer \(n\), it covers many important cases for Question 1.

### 3.3 The characterization of \(gK_{z^n \phi^\lambda}\) for some general \(g\)

Inspired by the result (1) in Theorem 3.5, we continue to present the concrete formula of the subspace
\[ c(g) := gK_{z^n \phi^\lambda} = \forall \{g \phi^\lambda, 0 \leq \lambda \leq \delta\} \]
for a more general function \(g \in L^\infty(\mathbb{T})\). First of all, we demonstrate the subspace like \(gK_\theta\) is nearly \(S^*\) invariant for \(g \in H^\infty(\mathbb{D})\) with \(g(0) \neq 0\) (not necessarily an isometric multiplier) and a non-constant inner function \(\theta\).

**Theorem 3.8.** Let \(g \in H^\infty(\mathbb{D})\) with \(g(0) \neq 0\) and \(\theta\) be a non-constant inner function. Then \(gK_\theta\) is nearly \(S^*\) invariant, so by Hitt’s theorem it can be written as \(hK\), where \(K\) is \(S^*\) invariant (either a model space or \(H^2(\mathbb{D})\) itself) and \(h \in H^2(\mathbb{D})\) is a function such that multiplication by \(h\) is isometric on \(K\).

**Proof.** Since \(\theta\) is non-constant, there is an \(n_0 \geq 1\) such that \((S^{\*n_0} \theta)(0) \neq 0\). Without loss of generality, suppose \(g(0)(S^{\*n_0} \theta)(0) = 1\). Take \(f \in gK_\theta\) with \(f(0) = 0\), so there exist \(f_n \in gK_\theta\) with \(\|f_n - f\|_2 \to 0\) as \(n \to \infty\). Particularly, \(f_n(0) = f_n(0) - f(0) = (f_n - f, 1) \to 0\) as \(n \to \infty\).

Let \(F_n = f_n - f_n(0) g S^{*n_0} \theta \in gK_\theta\) (since \(S^{n_0} \theta \in K_\theta\)) with \(F_n(0) = 0\) and \(F_n \to f\) as \(n \to \infty\). Furthermore, it holds that \(F_n = gk_n\) with \(k_n \in K_\theta\) such that \(k_n(0) = 0\) and
\[ S^* F_n = F_n/z = gk_n/z = g S^* k_n \in gK_\theta \]
due to the fact that \(K_\theta\) is \(S^*\) invariant. Now we conclude that
\[ \|S^* F_n - S^* f\|_2 \leq \|S^*\| \cdot \|F_n - f\|_2 \to 0 \]
as \(n \to \infty\). This means that \(S^* f \in gK_\theta\), ending the proof.

It is known that every rational function \(p/q \in H^2(\mathbb{D})\) in its lowest terms has \(q\) invertible in \(H^\infty(\mathbb{D})\), and without loss of generality \(p\) is a polynomial with zeros on the unit circle \(T\), since zeros inside the open disc can be removed by using Blaschke factors. Next, we concentrate on finding \(c(\tilde{p})\) when \(\tilde{p}\) is a polynomial with zeros on \(T\).

Define \(\tilde{p}_N(z) := \prod_{j=1}^N (z + w_j), \; N \geq 1\) and \(w_j \in T\) for \(j = 1, \ldots, N\). Theorem 3.8 implies that \(c(\tilde{p}_N)\) is a nearly \(S^*\) invariant subspace in \(H^2(\mathbb{D})\). For a further description of \(c(\tilde{p}_N)\), we cite a result from [3]. We say that \(h \in H^2(\mathbb{D})\) is contained in a minimal Toeplitz kernel \(K_{\text{min}}(h)\), if every Toeplitz kernel \(K\) with \(h \in K\) contains \(K_{\text{min}}(h)\).
Lemma 3.9 (See [3, Theorem 3.3]). Let $h \in H^2 \setminus \{0\}$ and $h = IO$ be its inner-outer factorization. Then there exists a minimal Toeplitz kernel containing span$\{h\}$, denoted by $K_{\min}(h)$ written as

$$K_{\min}(h) = \ker T_{\overline{h}IO}.$$

Now Lemma 3.9 implies $c(\tilde{p}_N) \subseteq K_{\min}(\tilde{p}_N \phi^\delta)$, since being a Toeplitz kernel, it will contain $\tilde{p}_N \phi^\delta$ for all $0 \leq \delta \leq \lambda$. It yields that $c(\tilde{p}_N) \subseteq \ker T_{\overline{h}}$ with the function $d \in L^\infty(\mathbb{T})$ and

$$d(z) := \frac{z^{\phi^\delta}(z)\tilde{p}_N(z)}{\tilde{p}_N(z)} = z^{N+1} \lambda^\delta \left( \prod_{j=1}^{N} \overline{\mu_j} \right).$$

So we conclude that $c(\tilde{p}_N) \subseteq \ker T_{\overline{z^{N+1} \phi^\delta}} = K_{z^{N+1} \phi^\delta}$. Next, we explore the gap between $c(\tilde{p}_N)$ and $K_{z^{N+1} \phi^\delta}$.

**Proposition 3.10.** Let $\tilde{p}_N(z) := \prod_{j=1}^{N}(z + w_j)$ with $w_j \in \mathbb{T}$, $j = 1, \ldots, N$. It follows that

$$c(\tilde{p}_N) + \phi^\delta K_{\tilde{p}_N} = K_{z^{N+1} \phi^\delta}. \quad (3.8)$$

Hence, $c(\tilde{p}_N)$ has codimension at most $N$ in $K_{z^{N+1} \phi^\delta}$.

**Proof.** Since $\tilde{p}_N(z) := \prod_{j=1}^{N}(z + w_j)$ is an outer function in $H^2(\mathbb{D})$, by (2.9) we obtain

$$\forall \{\tilde{p}_N \phi^\lambda, 0 \leq \lambda < \infty\} = H^2(\mathbb{D}).$$

Define $B_N := \sqrt{\tilde{p}_N \phi^\lambda}$, $\delta \leq \lambda < \infty$ and use (2.9) again to obtain

$$B_N = \phi^\delta \lor \{\tilde{p}_N \phi^\lambda, 0 \leq \lambda < \infty\}$$

$$= \phi^\delta H^2(\mathbb{D})$$

$$= \phi^\delta(z^{N+1}H^2(\mathbb{D}) \oplus K_{z^{N+1}})$$

$$= z^{N+1} \phi^\delta H^2(\mathbb{D}) \oplus (\overline{\tilde{p}_N \phi^\delta} \lor \{z^k \phi^\delta, 0 \leq k \leq N - 1\})$$

$$= z^{N+1} \phi^\delta H^2(\mathbb{D}) \oplus (\overline{\tilde{p}_N \phi^\delta} + \phi^\delta K_{\tilde{p}_N}).$$

Since $c(\tilde{p}_N) + B_N = H^2(\mathbb{D})$ and $\tilde{p}_N \phi^\delta \in c(\tilde{p}_N)$, it always holds that

$$\overline{(c(\tilde{p}_N) + \phi^\delta K_{\tilde{p}_N})} + z^{N+1} \phi^\delta H^2(\mathbb{D}) = H^2(\mathbb{D}),$$

$$\overline{(c(\tilde{p}_N) + \phi^\delta K_{\tilde{p}_N})} \perp z^{N+1} \phi^\delta H^2(\mathbb{D}).$$

This means that

$$\overline{(c(\tilde{p}_N) + \phi^\delta K_{\tilde{p}_N})} \oplus z^{N+1} \phi^\delta H^2(\mathbb{D}) = H^2(\mathbb{D}).$$

which is equivalent to saying

$$(c(\tilde{p}_N) + \phi^\delta K_{\tilde{p}_N}) \oplus z^{N+1} \phi^\delta H^2(\mathbb{D}) = H^2(\mathbb{D}).$$

This further yields the desired result.

For a $g \in L^\infty(\mathbb{T})$, we deduce the following theorem on the closure of $gK_{z^N \phi^\delta}$.

**Theorem 3.11.** Suppose $g(z) = \tilde{p}_N(z) h(z)$ with $\tilde{p}_N(z) := \prod_{j=1}^{N}(z + w_j)$, where $w_j \in \mathbb{T}$, $j = 1, \ldots, N$, and $h$ is an invertible rational function in $L^\infty(\mathbb{T})$. Then $c(g)$ has codimension at most $N$ in $hK_{z^{N+1} \phi^\delta}$, i.e.,

$$c(g) + h \phi^\delta K_{\tilde{p}_N} = hK_{z^{N+1} \phi^\delta}. \quad (3.9)$$

**Proof.** By the fact that $h$ is invertible in $L^\infty(\mathbb{T})$, we can multiply the equation (3.8) by $h$ to deduce (3.9).
Remark 3.12. In Theorem 3.11, letting \( g(z) = (1 + z)^N h(z) \) with an invertible rational function \( h \in L^\infty(\mathbb{T}) \), we deduce
\[
\forall \{ (1 + z)^N h^\lambda \}, \ 0 \leq \lambda \leq \delta \} = hK_{z^{N+1}e^{\delta z}},
\]
which has codimension 0 in \( hK_{z^{N+1}e^{\delta z}} \). Particularly, for \( h(z) = 1 \), the result (1) of Theorem 3.5 implies
\[
\forall \{ (1 + z)^N \phi^\lambda \}, \ 0 \leq \lambda \leq \delta \} = K_{z^{N+1}e^{\delta z}},
\]
which has codimension 0 in \( K_{z^{N+1}e^{\delta z}} \).

In the sequel, we apply Theorem 3.11 to describe the corresponding case for rational functions in \( H^2(\mathbb{C}_+) \). Note that a rational function \( g \) in \( H^2(\mathbb{C}_+) \) can be factorized as \( g = g_l g_o \), where \( g_l \) is inner and hence invertible in \( L^\infty(\mathbb{R}) \) (which is a Blaschke product for the right half-plane), and \( g_o \) is outer (so all its zeros are in the closed left half-plane or at \( \infty \)). Then we can write \( g_o = g_o g_2 \), where \( g_1 \) is invertible in \( L^\infty(\mathbb{R}) \) and \( g_2 \) has zeros in \( \mathbb{R} \cup \{ \infty \} \). So every rational function \( g \in H^2(\mathbb{C}_+) \) can be represented by \( g = G_1 G_2 \), where \( G_1 \) is invertible in \( H^\infty(\mathbb{R}) \) and \( G_2 \) only has zeros in \( \mathbb{R} \cup \{ \infty \} \). Besides, we can always make the denominator of \( G_2 \) equal a power of \( s+1 \) and the power will be at least 1 as the function is in \( H^2(\mathbb{C}_+) \). Now suppose that the degrees of the numerator and the denominator of \( G_2 \) are \( m \) and \( n \), respectively. This means that \( m \) is the number of imaginary-axis zeros of \( g \) and \( g \) is asymptotic to \( s^{m-n} \) at \( \infty \), so \( n > m \). In particular, we write \( G_2(s) = \prod_{k=1}^m (s-y_k)/(s+1)^n \) with all \( y_k \in \mathbb{R} \). So it yields that
\[
V^{-1}(g) = 2\sqrt{\pi} G_1(M(z)) G_2(M(z))
\]
\[
= \sqrt{\pi} G_1 \left( \frac{1-z}{1+z} \right)^{m} \prod_{k=1}^m \left( \frac{1-z}{1+z} - y_k \right)^{-1} \left( \frac{1+z}{2} \right)^{n-1}
\]
\[
= 2^{1-n} \sqrt{\pi} G_1 \left( \frac{1-z}{1+z} \right)^{m} \prod_{k=1}^m \left( 1-y_k - z(1+y_k) \right)^{n-m-1}
\]
with \( G_1 \left( \frac{1-z}{1+z} \right) \) being rational and invertible in \( L^\infty(\mathbb{T}) \) and the polynomial
\[
\prod_{k=1}^m (1-y_k - z(1+y_k))(1+z)^{n-m-1}
\]
having \( n-1 \) zeros on \( \mathbb{T} \). Combining this with Theorem 3.11, we formulate
\[
e(V^{-1}g) + G_1 \left( \frac{1-z}{1+z} \right) K_{z^{n-1}} G_1 \left( \frac{1-z}{1+z} \right) K_{z^{n}e^{\delta z}}.
\]
Switching into \( H^2(\mathbb{C}_+) \) by the map \( V \) in (1.3), we obtain the following theorem in \( H^2(\mathbb{C}_+) \).

Theorem 3.13. Let \( g \in H^2(\mathbb{C}_+) \) be rational with \( m \) zeros on the imaginary axis and let \( n > m \) such that \( s^{n-m} g(s) \) tends to a finite nonzero limit at \( \infty \). Then \( g \) can be written as \( g = G_1 G_2 \), where \( G_1 \) is rational and invertible in \( L^\infty(\mathbb{R}) \) and \( G_2(s) = \prod_{k=1}^m (s-y_k)/(s+1)^n \) with all \( y_k \in \mathbb{R} \). Then it holds that
\[
\forall \{ g e^{-\lambda s}, \ 0 \leq \lambda \leq \delta \} + G_1 e^{-\delta s} K_{1^{n+1}} = G_1 K_{1^{n+1}} e^{-\delta s}.
\]

Remark 3.14. Let \( g(s) = G_1(s)/(1+s)^{n+1} \) in \( H^2(\mathbb{C}_+) \) with a rational and invertible \( G_1 \in L^\infty(\mathbb{R}) \). It holds that
\[
\forall \left\{ G_1 e^{-\lambda s}/(1+s)^{n+1}, \ 0 \leq \lambda \leq \delta \right\} + G_1 e^{-\delta s} K_{1^{n+1}} = G_1 K_{1^{n+1}} e^{-\delta s}.
\]

Particularly, for \( G_1(s) = 1 \), the result (2) of Theorem 3.5 implies that \( \forall \left\{ e^{-\lambda s}/(1+s)^{n+1}, \ 0 \leq \lambda \leq \delta \right\} \) has codimension 0 in \( K_{1^{n+1}} e^{-\delta s} \).
Remark 3.15. For \( \tilde{p}_N \) in Proposition 3.10, it also follows that
\[
(\tilde{c}(\tilde{p}_N)) + \phi^N K_{\phi^N} = K_{\phi^{N+1}}.
\]
(3.10)

Meanwhile, for \( g(s) = \prod_{k=1}^{m_0} (s - y_k) / (s + 1)^n \in \mathcal{H}^2(\mathbb{C}_+) \) with all \( y_k \in i\mathbb{R} \), the equation (3.10) implies that
\[
\sqrt{\mathcal{V}\{ge^{-\lambda s}, 0 \leq \lambda \leq \delta \} + K(\frac{1}{1+\gamma})e^{-\delta s}} = K(\frac{1}{1+\gamma})e^{-\delta s}.
\]

Proof. Taking the orthogonal complement of (3.8) in \( H^2(\mathbb{D}) \), we obtain
\[
(\tilde{c}(\tilde{p}_N)) \perp (\phi^N K_{\phi^N}) \perp (\phi^N \mathcal{H}^2 \oplus K_{\phi^N}) = z^{N+1} \phi^N \mathcal{H}^2.
\]

Here, we use [5, Lemma 2.3] to obtain \( (\phi^N K_{\phi^N}) \perp z^N \phi^N \mathcal{H}^2 \oplus K_{\phi^N} \). Since \( K_{\phi^N} \) is orthogonal to \( z^N \phi^N \mathcal{H}^2 \) and \( z^N \phi^N \mathcal{H}^2 \supseteq z^{N+1} \phi^N \mathcal{H}^2 \), the above formula implies
\[
(\tilde{c}(\tilde{p}_N)) \perp z^N \phi^N \mathcal{H}^2 = z^{N+1} \phi^N \mathcal{H}^2.
\]

Then the desired equation (3.10) can be deduced by taking the orthogonal complement of the above display in \( H^2(\mathbb{D}) \). This presents a link between the model spaces \( K_{\phi^{N+1}} \) and \( K_{\phi^N} \) in this context.

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