Characterization of Multi-scale Invariant Fields

H. Ghasemi\textsuperscript{a}, S. Rezakhah\textsuperscript{a}, * and N. Modarresi\textsuperscript{b}

\textsuperscript{a} Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran.
\textsuperscript{b} Faculty of Mathematics and Computer Science, Allameh Tabataba’i University, Tehran, Iran.

Abstract

Applying certain flexible geometric sampling of a multi-scale invariant (MSI) field we provide a multi-dimensional multi-selfsimilar field which has a one to one correspondence with such sampled MSI field. This sampling enables us to characterize harmonic-like representation and spectral density function of the sampled MSI field. Imposing Markov property for the MSI field, we find that the covariance function and spectral density matrix of such sampled Markov MSI field are characterized by the covariance functions of samples of the first scale rectangle. We present an example of MSI field as two-dimensional simple fractional Brownian motion. We consider a real data example of the precipitation in some area of Brisbane in Australia for some special period. We show that precipitation on this area has MSI property and estimate time dependent scale and Hurst parameters of this MSI field in three dimension as latitude, longitude and time. Our method enables one to predict precipitation in time and place.

Mathematics Subject Classification MSC 2010: 60G18; 60G22; 62M15; 62H05.

Keywords: Multi-scale invariant; Quasi-Lamperti Transformation; Self-Similarity; Spectral Representation; Wide-Sense Markov Fields.

1 Introduction

Gaussian self-similar fields have been extensively studied and applied in various areas such as hydrology, biology, economics, finance and image processing \cite{28}. Multivariate random fields are mainly interested in spatial statistics as well as in many natural applications, say environmental, agricultural, and ecological sciences. Modeling of multivariate data that have been measured in space, such as temperature and pressure, are challenging tasks in environmental sciences. Genton et al. \cite{6} studied multi-selfsimilar (MSS) fields that have self-similar property and characterized their structure via their stationary counterpart by using some quasi

*Address correspondence to S. Rezakhah, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran; E-mails: rezakhah@aut.ac.ir, h.ghasemi@aut.ac.ir, n.modarresi@atu.ac.ir.
Lamperti transformation. They defined a random field \( \{ X(t); t \in \mathbb{R}^d \} \) to be MSS with index \( H = (H_1, \ldots, H_d)^T \in \mathbb{R}_+^d \) if for any \( \Lambda = (\lambda_1, \ldots, \lambda_d)^T \in \mathbb{R}_+^d \),

\[
\left\{ \left( \prod_{i=1}^d \lambda_i^{-H_i} \right) X(\Lambda t); t \in \mathbb{R}^d \right\} \overset{d}{=} \left\{ X(t); t \in \mathbb{R}^d \right\}
\]

where \( \Lambda t = (\lambda_1 t_1, \ldots, \lambda_d t_d)^T \) and \( \overset{d}{=} \) denotes equality of finite-dimensional distributions. If the above equality holds for some fixed \( \Lambda \) then the field is called multi-scale invariant (MSI) with index \( H \) and scale vector \( \Lambda \) and is denoted by \((H, \Lambda)\)-MSI. The family of distribution of such random fields are invariant up to certain dilation factors.

Applying some flexible geometric sampling of MSI field, adopted from Modarresi and Rezakhah [17, 18], we provide a corresponding multi-dimensional MSS field. Then applying some quasi Lamperti transform, the result of Gladyshev [10] enables us to characterize the spectral density and spectral representation of such sampled MSI field. Imposing Markov property for the MSI field, we show that the covariance function and spectral density of such sampled Markov MSI field are characterized by the covariance function of the first scale rectangles.

For the estimation of Hurst parameters of MSI fields we extend the method of Rezakhah et al. [23], [24] for estimation of the Hurst parameter of discrete scale invariant (DSI) processes. This paper is motivated by applications in environmental and climate phenomena. As an example of MSI field, real data of the precipitation in some part of Brisbane area of Australia for some special period of time are considered and the MSI behavior of this precipitation in three dimension as latitude, longitude and time are verified [2]. Also the corresponding time dependent scale and Hurst parameters of this MSI field are estimated. By estimating scale and Hurst parameters, it is possible to predict precipitation in time and place.

Section 2 is devoted to some preliminary definitions and also definitions of MSS and MSI fields. Our sampling scheme and a quasi-Lamperti transform are defined in this section. The definition of a two-dimensional sfBm as an example of MSI field is given in section 2. Section 3 is devoted to a harmonic-like representation and spectral density function of \( n \)-dimensional MSS fields. The characterization of covariance and spectral density functions of two dimensional scale invariant wide-sense Markov fields are presented in section 4. In section 5 we present some heuristic method for the estimation of Hurst parameters of MSI fields. Implying the rainfall data of Brisbane area which has been presented by Australian bureau of meteorology, their MSI property is verified and also the scale and Hurst parameters of this fields are estimated.

## 2 Theoretical Structure

A random field is simply a stochastic process, taking values in a Euclidean space, and defined over a parameter space of dimensionality at least one. In this section, we introduce multi-selfsimilar (MSS) and multi-scale invariant (MSI) fields. Then we present quasi-Lamperti transform which provides a one to one correspondence between MSS and stationary fields and also between MSI and periodic fields respectively.

First we present the definition of periodic field and we use them as the Lamperty counterpart of self-similar field to obtain the harmonic representation and spectral density of MSI filesd. According to Hurd et al. [9], we have the following definition.

**Definition 1.** Given \( \tau \in \mathbb{R}^d \), the shift operator \( S_\tau \) operates on random field \( \{X(t); t \in \mathbb{R}^d\} \),
where $A$ is any subset of distinct points of $T$ and scaling factor $\lambda$ as a renormalized dilation operator $\mathcal{D}_{\mathbf{H},\mathbf{A}}$ operates on random field $\{X(t); t \in \mathbb{R}^d\}$ as

$$
\mathcal{D}_{\mathbf{H},\mathbf{A}}X(t) := \left( \prod_{i=1}^{d} \lambda_i^{-H_i} \right) X(\mathbf{A}t),
$$

where $\mathbf{A}t = (\lambda_1 t_1, \ldots, \lambda_d t_d)^T$.

We present the following definition for MSS field from Genton et al. [6].

**Definition 3.** A random field $\{X(t); t \in \mathbb{R}^d\}$ is said to be MSS of index $\mathbf{H} = (H_1,\ldots,H_d)^T$ ($\mathbf{H}$-MSS), if for any $\mathbf{A} = (\lambda_1,\ldots,\lambda_d)^T$, $\lambda_i > 0$, $i = 1,\ldots,d$

$$
\{\mathcal{D}_{\mathbf{H},\mathbf{A}}X(t)\} \overset{d}{=} \{X(t)\},
$$

(2.1)

The random field is said to be MSI of index $\mathbf{H}$ and scaling factor $\mathbf{A}' = (\lambda_1',\ldots,\lambda_d')^T$ or $(\mathbf{H},\mathbf{A}')$-MSI, if (2.1) holds for some $\mathbf{A} = \mathbf{A}'$.

Extending definition of Modarresi et al. [17] for DSI process with some parameter space, we present the following definition.

**Definition 4.** A random field $\{X(k); k \in T\}$ is called $\mathbf{H}$-MSS with parameter space $T$, where $T$ is any subset of distinct points of $\mathbb{R}_+^d$, if for any $k_1 = (k_{11},k_{12},\ldots,k_{1d})^T$, $k_2 = (k_{21},k_{22},\ldots,k_{2d})^T \in T$

$$
\{X(k_2)\} \overset{d}{=} \left( \prod_{i=1}^{d} \left( \frac{k_{2i}}{k_{1i}} \right)^{H_i} \right) \{X(k_1)\}.
$$

(2.2)

A random field $X(.)$ is called $(\mathbf{H},\mathbf{L})$-MSI with parameter space $T$ and vector scale $\mathbf{L} = (l_1,\ldots,l_d)^T$ where $l_i > 0$ for $i = 1,\ldots,d$, if for any $k_1,k_2 \in T$ where $k_{2i} = l_i k_{1i}$, (2.2) holds.

**Remark 1.** If the random field $\{X(t); t \in \mathbb{R}_+^d\}$ is $(\mathbf{H},\mathbf{L})$-MSI with vector scale $\mathbf{L} = (l_1,\ldots,l_d)^T$ where $l_i = \alpha_i^{u_i}$ for fixed $\alpha_i > 1$, $u_i \in \mathbb{N}$, $i = 1,\ldots,d$, then by sampling of the field at points $\alpha^k, k \in \mathbb{W}^d$ where $\alpha^k = (\alpha_1^{k_1},\ldots,\alpha_d^{k_d})^T$, $\mathbb{W} = \{0,1,\ldots\}$, we have $X(.)$ as a $(\mathbf{H},\mathbf{L})$-MSI with parameter space $\bar{T} = \{\alpha^k; k \in \mathbb{W}^d\}$ and scale $\mathbf{L}$. If we consider sampling of $X(.)$ at points $\alpha^{nu+k} = (\alpha_1^{n_1 u_1+k_1},\ldots,\alpha_d^{n_d u_d+k_d})^T$, $n \in \mathbb{W}^d$, for $i = 1,\ldots,d$ and fixed $k_i = 0,1,\ldots,u_i - 1$, then $X(.)$ is $\mathbf{H}$-MSS field with parameter space $\bar{T} = \{\alpha^{nu+k}; n \in \mathbb{W}^d\}$. 

3
Following the definition of the wide-sense self-similar process presented by Nuzman et al. [21], we present the following definition.

**Definition 5.** A random field \( \{X(t); t \in \mathbb{R}^d\} \) is said to be wide-sense \( \mathbf{H} \)-MSS, if the following properties are satisfied for each \( a = (a_1, \ldots, a_d)^T \) where \( a_i > 0 \)
1. \( E[X^2(t)] < \infty \)
2. \( E[X(at)] = \left( \prod_{i=1}^{d} a_i^{H_i} \right) E[X(t)] \)
3. \( E[X(at_1)X(at_2)] = \left( \prod_{i=1}^{d} a_i^{2H_i} \right) E[X(t_1)X(t_2)] \)

This field is called wide-sense \((\mathbf{H}, \alpha')\)-MSI of index \( \mathbf{H} \) and scaling factor \( \alpha' = (a'_1, \ldots, a'_d)^T \) where \( a'_i > 0 \), if the above conditions hold for some \( a = \alpha' \).

In the rest of the paper we consider MSS and MSI in the wide-sense fields, so for simplicity we omit the term "in the wide sense" henceforth. Following Modarresi and Rezakhah [17], we present the following the quasi-Lamperti transformation.

**Definition 6.** The quasi-Lamperti transform with positive index \( \mathbf{H} \) and \( \alpha \), denoted by \( \mathcal{L}_{\mathbf{H}, \alpha} \) operates on a random field \( \{Y(t); t \in \mathbb{R}^d\} \) as

\[
\mathcal{L}_{\mathbf{H}, \alpha} Y(t) = \left( \prod_{i=1}^{d} t_i^{H_i} \right) Y(\log_{\alpha} t) \tag{2.3}
\]

where \( \log_{\alpha} t = (\log_{\alpha_1} t_1, \ldots, \log_{\alpha_d} t_d)^T \) and \( \alpha > 1 \) i.e. \( \alpha_i > 1 \) for \( i = 1, \ldots, d \). The corresponding inverse quasi-Lamperti transform \( \mathcal{L}_{\mathbf{H}, \alpha}^{-1} \) on field \( \{X(t); t \in \mathbb{R}^d\} \) acts as

\[
\mathcal{L}_{\mathbf{H}, \alpha}^{-1} X(t) = \prod_{i=1}^{d} \alpha_i^{-t_i^{H_i}} X(\alpha^t) \tag{2.4}
\]

where \( \alpha^t = (\alpha_1^{t_1}, \ldots, \alpha_d^{t_d})^T \).

It is easy to verify that \( \mathcal{L}_{\mathbf{H}, \alpha} \mathcal{L}_{\mathbf{H}, \alpha}^{-1} X(t) = X(t) \) and \( \mathcal{L}_{\mathbf{H}, \alpha}^{-1} \mathcal{L}_{\mathbf{H}, \alpha} Y(t) = Y(t) \). Note that in the above definition, if \( \alpha = (e, \ldots, e)^T \), we have the usual Lamperti transformation \( \mathcal{L}_{\mathbf{H}} \).

**Proposition 1.** The quasi-Lamperti transform assures an equivalence between the shift operator \( \mathcal{S}_{\log_{\alpha} k} \) and the renormalized dilation operator \( \mathcal{D}_{\mathbf{H}, k} \) in the sense that, for any \( k > 0 \)

\[
\mathcal{L}_{\mathbf{H}, \alpha}^{-1} \mathcal{D}_{\mathbf{H}, k} \mathcal{L}_{\mathbf{H}, \alpha} = \mathcal{S}_{\log_{\alpha} k}. \tag{2.5}
\]

**Proof.** By a similar method as in [17], the validation of (2.5) follows. \( \square \)

So we have the following corollaries.

**Corollary 1.** If \( \{Y(t); t \in \mathbb{R}^d\} \) is a stationary field, its quasi-Lamperti transformation \( \{\mathcal{L}_{\mathbf{H}, \alpha} Y(t); t \in \mathbb{R}^d\} \) is \( \mathbf{H} \)-MSS. Conversely if \( \{X(t); t \in \mathbb{R}^d\} \) is a \( \mathbf{H} \)-MSS, its inverse quasi-Lamperti transformation \( \{\mathcal{L}_{\mathbf{H}, \alpha}^{-1} X(t); t \in \mathbb{R}^d\} \) is stationary.

**Corollary 2.** If \( \{X(t); t \in \mathbb{R}^d\} \) is a \((\mathbf{H}, \alpha^U)\)-MSI then \( \mathcal{L}_{\mathbf{H}, \alpha}^{-1} X(t) = Y(t) \) is periodic field with period \( \mathbf{U} > 0 \). Conversely if \( \{Y(t); t \in \mathbb{R}^d\} \) is periodic field with period \( \mathbf{U} \) then \( \mathcal{L}_{\mathbf{H}, \alpha} Y(t) = X(t) \) is \((\mathbf{H}, \alpha^U)\)-MSI.
Remark 2. If $X(.)$ is a $\mathbf{H}$-MSS with parameter space $\mathcal{T} = \{\mathbf{L}; \mathbf{n} \in \mathbb{W} \}$ and Hurst vector $\mathbf{H} = (H_1, \ldots, H_d)^T$ and $\mathbf{L} = (\alpha_1^1, \ldots, \alpha_d^U)^T$, then it is easy to show that its stationary counterpart $Y(.)$ has parameter space $\mathcal{T} = \{\mathbf{nU}; \mathbf{n} \in \mathbb{W} \}$ where $\mathbf{nU} = (n_1U_1, \ldots, n_dU_d)^T$.

As an example of MSI field we introduce two-dimensional sfBm by the followings.

Example 1. A two dimensional sfBm $\{X(t); t \in \mathbb{R}_+^2 \}$ with Hurst and scale vectors $\mathbf{H} = (H_1, H_2)^T$ and $\mathbf{\lambda} = (\lambda_1, \lambda_2)^T$ is defined by

$$X(t_1, t_2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \lambda_1^{n_1(H_1-H')} \lambda_2^{n_2(H_2-H')} I_{[\lambda_1^{n_1-1}, \lambda_1^n]}(t_1) I_{[\lambda_2^{n_2-1}, \lambda_2^n]}(t_2) B(t_1, t_2),$$

where $B(.,.)$, $I(.)$ are the two-dimensional fractional Brownian motion indexed by $\mathbf{H}' = (H'_1, H'_2)^T$ and indicator function respectively and $H_1 > H'_1, H_2 > H'_2, \lambda_i > 1$ for $i = 1, 2$.

Let $A_{11} = [1, \lambda_1], A_{i2} = [\lambda_i, \lambda_i^2]$ and $A_{in} = [\lambda_i^{-1}, \lambda_i^n]$ for $i = 1, 2$ are disjoint sets. The field $X(.)$ is MSI. Let’s first we find the covariance function that is used to prove. For $(t_1, t_2) \in A_{1n_1} \times A_{2n_2}$ and $(s_1, s_2) \in A_{1n'_1} \times A_{2n'_2}$, $B(.,.)$ is the two-dimensional fractional Brownian motion with covariance

$$\text{Cov}(B(t_1, t_2), B(s_1, s_2)) = \frac{1}{2} \prod_{i=1}^{2} (t_i^{2H_i'} + s_i^{2H_i'} - |t_i - s_i|^{2H_i'}).$$

Hence the covariance function of the field is

$$\text{Cov}(X(t_1, t_2), X(s_1, s_2)) = \frac{1}{4} \prod_{i=1}^{2} \lambda_i^{(n_1+n'_1)(H_i-H')} (t_i^{2H_i'} + s_i^{2H_i'} - |t_i - s_i|^{2H_i'}).$$

If $(t_1, t_2) \in A_{1n} \times A_{2m}$, then $(\lambda_1 t_1, \lambda_2 t_2) \in A_{1(n+1)} \times A_{2(n+1)}$. Thus, for $(\lambda_1 t_1, \lambda_2 t_2) \in A_{1(n+1)} \times A_{2(n+1)}$ and $(\lambda_1 s_1, \lambda_2 s_2) \in A_{1(n'+1)} \times A_{2(n'+1)}$ we have

$$\text{Cov}(X(\lambda_1 t_1, \lambda_2 t_2), X(\lambda_1 s_1, \lambda_2 s_2)) = \frac{1}{4} \prod_{i=1}^{2} \lambda_i^{(n_1+n'_1+2)(H_i-H')} \text{Cov}(B(\lambda_1 t_1, \lambda_2 t_2), B(\lambda_1 s_1, \lambda_2 s_2))$$

$$= \frac{1}{4} \prod_{i=1}^{2} \lambda_i^{(n_1+n'_1+2)(H_i-H')} \lambda_i^{2H_i'} (t_i^{2H_i'} + s_i^{2H_i'} - |t_i - s_i|^{2H_i'})$$

$$= \frac{1}{4} \prod_{i=1}^{2} \lambda_i^{2H_i} \lambda_i^{(n_1+n'_1)(H_i-H')} (t_i^{2H_i'} + s_i^{2H_i'} - |t_i - s_i|^{2H_i'})$$

$$= \prod_{i=1}^{2} \lambda_i^{2H_i} \text{Cov}(X(t_1, t_2), X(s_1, s_2)).$$

So $X(t_1, t_2) \sim (\mathbf{H}, \mathbf{\lambda})$-MSI field. In the case $\mathbf{H}' = (\frac{1}{2}, \frac{1}{2})^T$, $B(.,.)$ is the two-dimensional Brownian motion.
3 Spectral Representation

In this section we characterize the harmonic-like representation and spectral density matrix of the MSI field. Following Hurd et al. \[9\] and Haghbin et al. \[8\], let \(\{Y(\mathbf{m});\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2\}\) be a periodic field with period \(U = (U_1, U_2)\) and define \(D_U = \{\mathbf{j} := (j_1, j_2), j_1 = 0, 1, \ldots, U_1 - 1, j_2 = 0, 1, \ldots, U_2 - 1\}\), then there exists a collection \(\{Z^{j_1,j_2}(\mathbf{m}), j_1 = 0, 1, \ldots, U_1 - 1, j_2 = 0, 1, \ldots, U_2 - 1, \mathbf{m} \in \mathbb{Z}^2\}\) of jointly stationary fields for which

\[
Y(\mathbf{m}) = \sum_{\mathbf{j} = (j_1, j_2) \in D_U} Z^{j_1,j_2}(\mathbf{m}) e^{-2\pi i \left(\frac{m_1 j_1}{U_1} + \frac{m_2 j_2}{U_2}\right)}
\]  

(3.6)

where \(Z^{j_1,j_2}(\mathbf{m})\) has the spectral representation

\[
Z^{j_1,j_2}(\mathbf{m}) = \int_{[0, \frac{2\pi}{U_1}] \times [0, \frac{2\pi}{U_2}]} e^{-i(m_1 j_1 + m_2 j_2) \psi_{\omega(j)}}(d\lambda)
\]  

(3.7)

where \(\psi_{\omega(j)}\) is a random spectral measure with orthogonal increments and the function \(\omega : D_U \to \{1, \ldots, U_1 U_2\}\) is defined by \(\omega(k_1, k_2) = k_2 U_1 + k_1 + 1\).

Also \(Y(\mathbf{m})\) can be represented by the integral

\[
Y(\mathbf{m}) = \int_{[0, 2\pi]^2} e^{-i(m_1 \lambda_1 + m_2 \lambda_2) \phi}(d\lambda)
\]  

(3.8)

where the spectral random measure \(\phi\) is given by \(\phi(d\lambda) = \psi_{\omega(\mathbf{n})}(d\lambda - 2\pi \frac{\mathbf{m}}{U})\), for \(\lambda \in \left[\frac{2\pi m_1}{U_1}, \frac{2\pi (m_1 + 1)}{U_1}\right] \times \left[\frac{2\pi m_2}{U_2}, \frac{2\pi (m_2 + 1)}{U_2}\right]\).

Furthermore, the spectral distribution \(F(., .)\), specified by \(F(d\lambda, d\lambda') = E[\phi(d\lambda)\phi(d\lambda')]\), is a measure on \([0, 2\pi]^2 \times [0, 2\pi]^2\) which is supported by \(S_n = \{(\lambda, \lambda') \in [0, 2\pi]^2 \times [0, 2\pi]^2, \lambda - \lambda' = 2\pi \frac{\mathbf{n}}{U}\}\) for \(n_1 = -U_1 + 1, \ldots, U_1 - 1\) and \(n_2 = -U_2 + 1, \ldots, U_2 - 1\). Considering \(F_n\) as the restriction of \(F\) on the set \(S_n\), then

\[
E[\psi_{\omega(\mathbf{m})}(d\mathbf{z}) \overline{\psi_{\omega(\mathbf{n})}(d\mathbf{z})}] = F_{\mathbf{m}-\mathbf{n}}(d\mathbf{z} + 2\pi \frac{\mathbf{m}}{U}), \quad \mathbf{z} \in \left[0, \frac{2\pi}{U_1}\right] \times \left[0, \frac{2\pi}{U_2}\right]
\]  

(3.9)

where the corresponding square matrix \(\mathcal{F}(d\mathbf{z}) = [\mathcal{F}_{\omega(\mathbf{m}), \omega(\mathbf{n})}(d\mathbf{z})]_{\mathbf{m}, \mathbf{n} \in D_U}\) is defined by

\[
\mathcal{F}_{\omega(\mathbf{m}), \omega(\mathbf{n})}(d\mathbf{z}) = F_{\mathbf{m}-\mathbf{n}}(d\mathbf{z} + 2\pi \frac{\mathbf{m}}{U}).
\]  

(3.10)

Let \(Q_{\mathbf{n}}(\mathbf{\tau}) = \text{Cov}(Y(\mathbf{m}), Y(\mathbf{m}+\mathbf{\tau}))\) be the covariance function which is periodic with respect to \(\mathbf{n}\) and has the representation

\[
Q_{\mathbf{n}}(\mathbf{\tau}) = \sum_{\mathbf{j} = (j_1, j_2) \in D_U} e^{-2\pi i \left(\frac{m_1 j_1}{U_1} + \frac{m_2 j_2}{U_2}\right) \mathcal{R}_j(\mathbf{\tau})}
\]  

(3.11)

where \(\mathcal{R}_j(\mathbf{\tau})\) is represented as

\[
\mathcal{R}_j(\mathbf{\tau}) = \int_{[0, 2\pi]^2} e^{-i(\tau_1 \lambda_1 + \tau_2 \lambda_2)} D_j(d\lambda)
\]  

(3.12)
Proposition 2. where

\[ \phi(u) \] where

\( \tau \) is defined by (3.12) for \( \tau \in \mathbb{R} \) and by (3.11) we have the Fourier series representation

\[ R_j(\tau) = \frac{1}{U_1 U_2} \sum_{(n_1,n_2) \in \mathcal{D}} e^{2\pi i (\frac{n_1}{U_1} + \frac{n_2}{U_2})} Q_n(\tau) \]  

and by (3.12) for \( A \subset A_k, k = 1, 2, 3, 4 \)

\[ D_j(A) = \frac{1}{2\pi} \sum_{(\tau_1,\tau_2) \in \mathcal{D}} e^{i(\tau_1 \delta_1 + \tau_2 \delta_2)} R_j(\tau) d\delta \]  

and the corresponding spectral density is denoted by \( d_j(\lambda) \) which is for \( A = (\lambda, \lambda + d\lambda) = (\lambda_1, \lambda_1 + d\lambda_1) \times (\lambda_2, \lambda_2 + d\lambda_2) \),

\[ d_j(\lambda) : = \frac{D_j(d\lambda)}{d\lambda} = \frac{1}{2\pi} \sum_{(\tau_1,\tau_2) \in \mathcal{D}} \left( \frac{1}{2\pi} \sum_{n=1}^{2} \left( \frac{1}{i\tau_n} \lim_{\delta \to 0} \frac{e^{i(\lambda+n\lambda)\tau_n} - e^{i\lambda_n\tau_n}}{d\lambda_n} \right) R_j(\tau) \right) \]

Proposition 2. If \( \{X(\alpha_{n_1}^{(1)}, \alpha_{n_2}^{(2)}); (n_1, n_2) \in \mathbb{W}^2\} \) is \( ((H_1, H_2), (\alpha U_1, \alpha U_2))\)-MSI field, then

\( (i) \) The harmonic-like representation of the field is

\[ X(\alpha_{n_1}^{(1)}, \alpha_{n_2}^{(2)}) = \alpha_n^{(1)} H_1 \alpha_n^{(2)} H_2 \int_{[0,2\pi]^2} e^{-i(n_1 \lambda_1 + n_2 \lambda_2)} \phi(d\lambda) \]  

where \( \phi \) is the corresponding spectral measure.

\( (ii) \) The spectral representation of the covariance function is

\[ Q_{\mathbf{m}}^{H}(\tau) = \alpha_n^{(1)} H_1 \alpha_n^{(2)} H_2 \sum_{(j_1,j_2) \in \mathcal{D}} e^{-2\pi i \left( \frac{m_1 j_1}{U_1} + \frac{m_2 j_2}{U_2} \right)} R_j(\tau) \]  

where \( R_j(\tau) \) is defined by (3.14).

\( (iii) \) The spectral density of \( X(\alpha_{\tau}^{(1)}, \alpha_{\tau}^{(2)}) \) for \( \lambda \in A_k, k = 1, 2, 3, 4 \), is

\[ d_j^{H}(\lambda) = \frac{1}{2\pi} \sum_{(\tau_1,\tau_2) \in \mathcal{D}} e^{i(\tau_1 \lambda_1 + \tau_2 \lambda_2)} R_j(\tau) \]  

and each \( D_j(\lambda) \) is defined by

\[ D_j(\lambda) = \begin{cases} F_j-U(\lambda), & \lambda \in A_1 = [0, \frac{2\pi}{U_1}] \times [0, \frac{2\pi}{U_2}] \\ F_j-U(\lambda), & \lambda \in A_2 = [0, \frac{2\pi}{U_1}] \times [\frac{2\pi}{U_2}, 2\pi] \\ F_j-U(\lambda), & \lambda \in A_3 = [\frac{2\pi}{U_1}, \frac{2\pi}{U_2}] \times [0, \frac{2\pi}{U_2}] \\ D_j(\lambda), & \lambda \in A_4 = [\frac{2\pi}{U_1}, \frac{2\pi}{U_2}] \times [\frac{2\pi}{U_2}, 2\pi] \end{cases} \]  

(3.13)
Proof. (i) According to corollary 2, the corresponding quasi-Lamperti transformation of $X(\alpha_1^{n_1}, \alpha_2^{n_2})$ is periodic field $Y(n_1, n_2)$ with period $U = (U_1, U_2)$. By (3.8), the spectral representation of field has the form
\[
X(\alpha_1^{n_1}, \alpha_2^{n_2}) = \mathcal{L}_{\mathbf{H}, \alpha} Y(\alpha_1^{n_1}, \alpha_2^{n_2}) = \alpha_1^{n_1} H_1 \alpha_2^{n_2} H_2 Y(n_1, n_2)
\]
\[
= \alpha_1^{n_1} H_1 \alpha_2^{n_2} H_2 \int_{[0,2\pi]^2} e^{-i(n_1 \lambda_1 + n_2 \lambda_2)} \phi(d\lambda)
\]
(ii) The covariance function is
\[
Q_{\mathbf{m}}^H(\tau) = \text{Cov}[X(\alpha_1^{m_1}, \alpha_2^{m_2}), X(\alpha_1^{m_1+\tau_1}, \alpha_2^{m_2+\tau_2})]
\]
\[
= \text{Cov}[\mathcal{L}_{\mathbf{H}, \alpha} Y(\alpha_1^{m_1}, \alpha_2^{m_2}), \mathcal{L}_{\mathbf{H}, \alpha} Y(\alpha_1^{m_1+\tau_1}, \alpha_2^{m_2+\tau_2})]
\]
\[
= \alpha_1^{(2m_1+\tau_1)H_1} \alpha_2^{(2m_2+\tau_2)H_2} \text{Cov}[Y(\mathbf{m}), Y(\mathbf{m} + \tau)]
\]
\[
= \alpha_1^{(2m_1+\tau_1)H_1} \alpha_2^{(2m_2+\tau_2)H_2} Q_{\mathbf{m}}(\tau)
\]
So by (3.11), the proof is completed.
(iii) In (3.17), let $\mathcal{R}_{\mathbf{j}}^H(\tau) = \alpha_1^{(2m_1+\tau_1)H_1} \alpha_2^{(2m_2+\tau_2)H_2} \mathcal{R}_{\mathbf{j}}(\tau)$. Hence as the spectral distribution function of $Q_\mathbf{j}(\tau)$ is $D_j(d\lambda)$, so using (3.11) and (3.12) and the relation of $Q_{\mathbf{m}}^H(\tau)$ and $Q_{\mathbf{m}}(\tau)$ as above, we find by (3.15) that the corresponding spectral density is
\[
d_j^H(\lambda) = \frac{1}{(2\pi)^2} \sum_{(\tau_1, \tau_2) \in D_U} e^{i(\lambda_1 \tau_1 + \lambda_2 \tau_2)} \mathcal{R}_{\mathbf{j}}(\tau)
\]
\[
= \frac{1}{(2\pi)^2} \sum_{(\tau_1, \tau_2) \in D_U} \alpha_1^{(2\tau_1 + \lambda_1)H_1} \alpha_2^{(2\tau_2 + \lambda_2)H_2} e^{i(\lambda_1 \tau_1 + \lambda_2 \tau_2)} \mathcal{R}_{\mathbf{j}}(\tau).
\]

4 Two dimensional Scale Invariant Markov Fields

The covariance function of a Markov random field \(X(s, t), (s, t) \in \mathbb{R}^2\) has separable property if it satisfies
\[
R(s_1, s_2; t_1, t_2) = R_1(s_1, t_1) R_2(s_2, t_2)
\]
where \(R(s_1, s_2; t_1, t_2) = \text{Cov}(X(s_1, s_2), X(t_1, t_2))\) and \(R_1(s_1, t_1), R_2(s_2, t_2)\) both have non-negative definite property and can be considered as the covariance function of some Markov process, see [7] and [25], p. 320. Also there exist some formal methods to test the separability of the covariance function of random fields, see [5]. By considering some rectangle grid over full area of consideration, if the covariance function \(R(s_1, s_2; t_1, t_2)\) has MSI property, then the covariance functions \(R_1(s_1, t_1), R_2(s_2, t_2)\) can be considered as the covariance functions of DSI Markov processes \(X_1(.)\) and \(X_2(.)\) corresponding to the vertical and horizontal strips respectively. So \(R_1(s_1, t_1) = \text{Cov}(X_1(s_1), X_1(t_1))\) and \(R_2(s_2, t_2) = \text{Cov}(X_2(s_2), X_2(t_2))\). In this section we show that the covariance function of wide sense Markov MSI (WM-MSI) field with separable property is characterized by the covariance functions of the field at points of the first scale rectangle.
Now we consider two dimensional scale invariant wide-sense Markov field \( \{X(\alpha^1_n, \alpha^2_n); (n_1, n_2) \in \mathbb{Z}^2\} \) which has separable covariance function with Hurst \( H = (H_1, H_2) \) and scale \( \alpha^T = (\alpha_{T_1}^1, \alpha_{T_2}^2) \). Let \( H^H_n(\tau) := R(n, u + \tau) = \text{Cov}[X(\alpha^1_{n_1+\tau_1}, \alpha^2_{n_2+\tau_2}), X(\alpha^1_n, \alpha^2_n)] \) for \( \tau \in \mathbb{Z}^2 \). Also assume that \( \{X_i(\alpha^k_i), k \in \mathbb{Z}\} \) be a DSI wide-sense Markov process with parameters \( (H_i, \alpha^T_i) \) and covariance function \( R^H_{i,n_i}(\tau_i) = \text{Cov}[X_i(\alpha^1_{n_1+\tau_1}), X_i(\alpha^1_{n_1})] \) for \( i = 1, 2 \). So we have

\[
R^H_n(\tau) = R^H_{1,n_1}(\tau_1)R^H_{2,n_2}(\tau_2).
\]

Thus by Theorem 3.2 in \cite{17}, we have that

\[
P^H_n(kT + \nu) = [\tilde{h}(\alpha^{T-1})]^k[\tilde{h}(\alpha^{\nu+n-1})]^{-1}P^H_n(0), \tag{4.19}
\]

and

\[
R^H_n(-kT + \nu) = \alpha^{-2kTH}P^H_{n+\nu}(kT - \nu)
\]

where

\[
\tilde{h}(\alpha^r) = \tilde{h}_1(\alpha^{r_1}_1)\tilde{h}_2(\alpha^{r_2}_2), \quad \alpha^{-2kTH} = \alpha_{-2kT1}^1H_1\alpha_{-2kT2}^2H_2
\]

and for \( i = 1, 2; k_i \in \{0, 1, 2, \ldots\}, \nu_i = 0, 1, \ldots, T_i - 1 \) and \( \tilde{h}_i(\alpha^{r_i}) = \prod_{j=0}^{r_i} h(\alpha^{j}_i) = \prod_{j=0}^{r_i} \frac{R^H_{i,n}(1)}{R^H_{i,n}(0)} \), \( \tilde{h}_i(\alpha^{n_i-1}) = 1 \).

**Proposition 3.** Let \( \{X(\alpha^1_n, \alpha^2_n); (n_1, n_2) \in \mathbb{Z}^2\} \) be a TS-MSI field with wide-sense Markov property which has Hurst parameter \( H = (H_1, H_2) \) and scale vector \( \alpha^T = (\alpha^T_1, \alpha^T_2) \). Then the covariance function can be characterized by the covariance functions in (4.19).

According to Remark 1 and corresponding to the TS-MSI field with wide-sense Markov property, \( \{X(\alpha^1_n, \alpha^2_n); (n_1, n_2) \in \mathbb{Z}^2\} \) with the scale \( L = (\alpha^T_1, \alpha^T_2) \), there exists a \( T_1T_2 \)-dimensional self-similar wide-sense Markov field \( Y(t_1, t_2) = (Y_{0,0}(t_1, t_2), Y_{0,1}(t_1, t_2), \ldots, Y_{(T_1-1),(T_2-1)}(t_1, t_2)) \) with parameter space \( T = \{L^n; n \in \mathbb{W}^2; L = (\alpha^T_1, \alpha^T_2)\} \), where

\[
Y_{k_1,k_2}(L^n) = Y_{k_1,k_2}(\alpha^{n_1T_1}_{k_1}, \alpha^{n_2T_2}_{k_2}) = X(\alpha^{n_1T_1+k_1}_1, \alpha^{n_2T_2+k_2}_2) \tag{4.20}
\]

and for \( i = 1, 2; k_i = 0, 1, \ldots, T_i - 1 \). Hence, the cross covariance function of \( Y_{j_1,j_2} \) and \( Y_{k_1,k_2} \) at points \( L^n \) and \( L^{n+\tau} \) can be written as

\[
Q^H_{j_1,j_2;k_1,k_2}(L^{n+\tau}, L^n) = \text{Cov}[Y_{j_1,j_2}(L^{n+\tau}), Y_{k_1,k_2}(L^n)]
\]

\[
\alpha^{2nT^1H_1}Cov[X(\alpha^{T_1+n_1j_1}_1, \alpha^{T_2+n_2j_2}_2), X(\alpha^{n_1}_1, \alpha^{n_2}_2)]
\]

\[
\alpha^{2nT^1H_1}\tilde{h}^{\alpha^T-1}(\tau \mathbf{T} + j - k)
\]

\[
\alpha^{2nT^1H_1}[\tilde{h}(\alpha^{T-1})]^\tau[\tilde{h}(\alpha^{j-1})][\tilde{h}(\alpha^{k-1})]^{-1}R^H_k(0)
\]

where \( \alpha^{2nT^1H_1} = \alpha^{2n_1T_1H_1}\alpha^{2n_2T_2H_2} \).

**Proposition 4.** Let \( \{X(\alpha^1_n, \alpha^2_n); (n_1, n_2) \in \mathbb{Z}^2\} \) be a TS-MSI field with wide-sense Markov property with the covariance function \( R^H_n(\cdot) \) and \( \{Y(L^n); n \in \mathbb{W}^2\} \) defined in (4.20), be its \( T_1T_2 \)-dimensional self-similar wide-sense Markov field with the cross covariance function \( Q^H(\cdot, \cdot) \). Then the cross covariance function of \( Y_{j_1,j_2} \) and \( Y_{k_1,k_2} \) at points \( L^n \) and \( L^{n+\tau} \) is

\[
Q^H_{j_1,j_2;k_1,k_2}(L^{n+\tau}, L^n) = \alpha^{2nT^1H_1}[\tilde{h}(\alpha^{T-1})]^\tau[\tilde{h}(\alpha^{j-1})][\tilde{h}(\alpha^{k-1})]^{-1}R^H_k(0) \tag{4.21}
\]
5 Data modeling

In this section we study the precipitation of rainfall data on a region of Brisbane area in Australia for two days (25 and 26 January 2013). This study follows by the evaluation of such precipitation on squares with side length 2km of grids over a 512 km $\times$ 512 km in this region. Thus the precipitation values are considered as a 256 $\times$ 256 matrix. The precipitation of rainfall in the area is depicted in Figure 1. The region was affected by extreme rainfall and subsequent flood. This rainfall data is provided by the Australian bureau of meteorology [2].

We consider MSI structure for precipitation in this field and estimate corresponding scales and Hurst parameters along three axes (horizontal, vertical and time) for certain specified area in this region. This circumscription is specified by yellow rectangle in Figure 1 and re-plotted in Figure 2. The selected part is a 120 km $\times$ 100 km area which correspone to 60 $\times$ 50 squares with side length 2km.

Let $X_{ij}$ be the value of precipitation on $ij$-th square with vertices in $(i,j)$, $(i,j-1)$, $(i-1,j)$, $(i-1,j-1)$ in this area in 25th and 26th January 2013, where $i = 1, \ldots, 60$ and $j = 1, \ldots, 50$. We consider sum of accumulated precipitation on these two days on horizontal and vertical strips with side length 2km in Figure 2. Let $X_i = \sum_{k=1}^{50} X_{ik}$ be sum of precipitation data on $i$-th vertical strip and $X_j = \sum_{l=1}^{60} X_{lj}$ is sum of precipitation data on $j$-th horizontal strip, where $i = 1, \ldots, 60$ and $j = 1, \ldots, 50$. Table 1 shows precipitation on vertical strips by their orders as cosecutive data on succesive rows of this table. Also the precipitation on horizontal strips are recorded on succesive rows of Table 2 as well. These precipitation on vertical and horizontal strips are plotted in Figures 3 and 4 respectivaly by blue. These accumulated precipitation on vertical and horizontal strips which are plotted by Figures 5 and 6 provide DSI processes. Following [1], [18] and [24] we fit some appropriate curve to these plots to detect the corresponding scale intervals, which are pointed out by vertical red lines. Thus we have three scale intervals for the accumulated precipitation on vertical strips in Figure 3 with end points $a_1 = 0$, $a_2 = 14$, $a_3 = 31$, $a_4 = 52$, and three scale intervals for precipitation on horizontal strips in Figure 4 with end points $b_1 = 0$, $b_2 = 10$, $b_3 = 21$, $b_4 = 39$. So we evaluate the corresponding time varying scale parameters by $\lambda_{1,n-1} = \frac{a_{n+1} - a_n}{a_n - a_{n-1}}$ and $\lambda_{2,n-1} = \frac{b_{n+1} - b_n}{b_n - b_{n-1}}$ for $n = 2, 3$. This leads to find $\lambda_{1,1} = 1.214$, $\lambda_{1,2} = 1.235$ or $\Lambda_1 := (\lambda_{1,1}, \lambda_{1,2}) = (1.214, 1.235)$, as the values of scale parameter for DSI process of accumulated precipitation on the vertical strips and $\lambda_{2,1} = 1.1, \lambda_{2,2} = 1.636$ or $\Lambda_2 := (\lambda_{2,1}, \lambda_{2,2}) = (1.1, 1.636)$, as the values of scale parameter for DSI process of accumulated precipitation on the horizontal strips.

For each scale interval we consider two equal subintervals which are indicated with green
Table 1: Sum of precipitation data on vertical strips as millimeters

|   | 11169 | 11448 | 11812 | 12174 | 12454 | 12620 | 12673 | 12673 | 12663 | 12636 |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|   | 12590 | 12545 | 12516 | 12499 | 12511 | 12557 | 12692 | 12855 | 13013 | 13201 |
|   | 13406 | 13561 | 13646 | 13694 | 13750 | 13802 | 13813 | 13754 | 13613 | 13460 |
|   | 13382 | 13409 | 13532 | 13696 | 13896 | 14138 | 14377 | 14560 | 14672 | 14730 |
|   | 14765 | 14788 | 14820 | 14876 | 14956 | 15051 | 15152 | 15235 | 15365 | 15433 |
|   | 15433 | 15496 | 15572 | 15687 | 15850 | 16044 | 16276 | 16536 | 16787 | 17009 |

Table 2: Sum of precipitation data on horizontal strips as millimeters

|   | 10593 | 10964 | 11379 | 11862 | 12443 | 13051 | 13578 | 13927 | 14105 | 14187 |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|   | 14171 | 14131 | 14184 | 14395 | 14835 | 15431 | 16064 | 16638 | 17039 | 17179 |
|   | 17161 | 17257 | 17546 | 17927 | 18188 | 18246 | 18303 | 18371 | 18387 | 18451 |
|   | 18728 | 19068 | 19215 | 19214 | 19152 | 19194 | 19419 | 19634 | 19668 | 19491 |
|   | 19246 | 19017 | 18882 | 18772 | 18575 | 18340 | 18184 | 18045 | 17809 | 17553 |

Figure 3: Fitted black curves for precipitation data on vertical strips and revealing the corresponding scale intervals by red lines.

Figure 4: Fitted black curves for precipitation data on horizontal strips and revealing the corresponding scale intervals by red lines.

dashed lines in Figures 5, 6. We consider 7 equally spaced samples in each subintervals in Figure 5; and 5 equally spaced samples in subintervals of Figure 6. These equally spaced samples provide some partitions for each subinterval. The accumulated precipitation on vertical and horizontal strips corresponding to these strips are evaluated as \( y_{(n,m)k} \) which denote the sum of precipitation on the \( k \)-th partition of the \( m \)-th subinterval in the \( n \)-th scale interval. Following [19, 24] we estimate the Hurst parameters of these DSI processes by calculation of quadratic variations of the \( m \)-th subinterval in the \( n \)-th scale interval as

\[
SS_{n,m} = \frac{1}{l_m} \sum_{k=1}^{l_m} y_{(n,m)k}^2
\]

(5.22)

where \( l_m \) is the number of partitions in the \( m \)-th subintervall for \( n = 1, 2, 3 \) and \( m = 1, 2 \). Then the corresponding Hurst parameters are estimated by

\[
\mathcal{H}_{(n,m)1} = \frac{\log(SS_{n+1,m}/SS_{n,m})}{2 \log \lambda_{1,m}}
\]

and

\[
\mathcal{H}_{(n,m)2} = \frac{\log(SS_{n+1,m}/SS_{n,m})}{2 \log \lambda_{2,m}}
\]
for precipitation data in subintervals along vertical and horizontal strips respectively. These method provide different estimation of Hurst parameters for the first subinterval \((m = 1)\) and second subintervals \((m = 2)\).

Hence,

\[ H_1 := (H_{(1,1)}, H_{(1,2)}, H_{(2,1)}, H_{(2,2)}) = (1.40, 1.42, 1.42, 1.49) \]

and

\[ H_2 := (H_{(1,1)}, H_{(1,2)}, H_{(2,1)}, H_{(2,2)}) = (3.42, 3.02, 1.46, 1.29) \]

are the Hurst parameters for precipitation in subintervals along vertical and horizontal strips respectively.

So for vertical strips we can evaluate the Hurst parameters of second subinterval with respect to first subinterval and for the third subinterval with respect to the second as

\[ H_{1,1} = \frac{H_{(1,1)} + H_{(1,2)}}{2} = 1.41, \quad H_{1,2} = \frac{H_{(2,1)} + H_{(2,2)}}{2} = 1.46, \]

Also for horizontal strips these Hurst parameters for the second subinterval with respect to the first and third subinterval with respect to the second are estimated as

\[ H_{2,1} = \frac{H_{(1,1)} + H_{(1,2)}}{2} = 3.22, \quad H_{2,2} = \frac{H_{(2,1)} + H_{(2,2)}}{2} = 1.375 \]

Now we are to consider the variation of precipitation with respect to time in the selected area shown in Figure 2 which shows another DSI behavior. For this we have considered the accumulated participation in this area for every 30 minutes on 25th and 26th January 2013 and depicted in Table 3 by their orders.

In Figure 7, Precipitation per 30 minutes at the two days are indicated by blue. The fitted black curves and red lines reveal the corresponding scale intervals which illuminates the DSI behavior of precipitation. The end points of these scale intervals are \(c_1 = 38, c_2 = 44, c_3 = 60, c_4 = 79\). Thus, the scale parameters can be evaluated by values \(\lambda_{n-1} = \frac{c_{n+1} - c_n}{c_n - c_{n-1}}\) for \(n = 2, 3\).

So, we have \(\lambda_{3,1} = 2.667, \lambda_{3,2} = 1.187\) or \(A_3 := (\lambda_{3,1}, \lambda_{3,2}) = (2.667, 1.187)\). According to prior method and dividing each scale interval to two equal subintervals which are indicated with green dashed lines in Figure 8 and the equally spaced red points in each subinterval, the corresponding Hurst parameters are obtained by

\[ H_{(n,m)} = \frac{\log(SS_{n+1,m}/SS_{n,m})}{2 \log \lambda_{3,m}} \]
Table 3: The precipitation data on the selected region in the Brisbane area for the two days (25 and 26 January 2013) per 30 minutes as millimeters.

```
|       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2311  | 2568  | 2702  | 2802  | 2351  | 2248  | 2496  | 2552  | 2061  |
| 1453  | 1823  | 2415  | 2596  | 2885  | 2947  | 2936  | 3113  | 2877  |
| 3511  | 3954  | 2339  | 1525  | 1527  | 2071  | 2907  | 2545  | 1965  |
| 3064  | 2455  | 1515  | 1516  | 1527  | 848   | 1837  | 2110  | 1064  |
| 1084  | 2163  | 4603  | 1671  | 17761 | 17274 | 15813 | 14470 | 13623 |
| 4130  | 5911  | 8518  | 8876  | 9693  | 11377 | 10841 | 12442 | 14649 |
| 15266 | 15964 | 15842 | 16711 | 17761 | 17274 | 15813 | 14470 | 13623 |
| 14326 | 14867 | 16331 | 16070 | 15880 | 18072 | 21312 | 22709 | 24026 |
| 21492 | 21567 | 16331 | 22416 | 23215 | 23309 | 21192 | 17992 | 17550 |
| 13795 | 11567 | 9122  | 7488  | 6425  | 4745  |
```

Thus,

\[ H_3 := (H_{1,1}^3, H_{1,2}^3, H_{2,1}^3, H_{2,2}^3) = (2.56, 1.94, 7.93, 3.48) \]

and

\[ H_{3,1} = \frac{H_{1,1}^3 + H_{1,2}^3}{2} = 2.25, \quad H_{3,2} = \frac{H_{2,1}^3 + H_{2,2}^3}{2} = 5.7 \]

**Acknowledgement**

We would like to thank Professor Alan Seed for providing the data used in this paper and the Australian Bureau of Meteorology as the source of the data.

**References**

[1] G. Balasis, C. Papadimitriou, I. A. Daglis, A. Anastasiadis, L. Athanasopoulou, K. Eftaxias. Signatures of discrete scale invariance in Dst time series. *Geophys. Res. Lett.* 38, L13103, doi:10.1029/2011GL048019.

[2] Commonwealth of Australia, Bureau of Meteorology (ABN 92 637 533 532), [http://www.bom.gov.au/other/copyright.shtml](http://www.bom.gov.au/other/copyright.shtml)

[3] C. R. Dietrich and G. N. Newsam. Fast and Exact Simulation of Stationary Gaussian Processes Through Circulant Embedding of the Covariance Matrix. *SIAM Journal on Scientific Computing.* 18(4):1088-1107, 1997.
[4] J. L. Doob. Stochastic Processes, Wiley, New York, 1953.

[5] M. Fuentes. Testing for separability of spatial-temporal covariance functions. *Journal of Statistical Planning and Inference*. 136, 447-466, 2006.

[6] M. G. Genton, O. Perrin and M. S. Taqqu. Self-Similarity and Lamperti Transformation for Random Fields. *Stochastic Models*. 23:397-411, 2007.

[7] M. G. Genton. Separable approximations of space-time covariance matrices. *Environmetrics*. 18:681-695, 2007.

[8] H. Haghbin and Z. Shishebor. On infinite dimensional periodically correlated random fields: Spectrum and evolutionary spectra. *Statistics and Probability Letters*. 110, 257-267, 2016.

[9] H. Hurd, G. Kallianpur and J. Farshidi. Correlation and spectral theory for periodically correlated random fields indexed on $\mathbb{Z}^2$. *Journal of Multivariate Analysis*. 90:359-383, 2004.

[10] E. G. Gladyshev. Periodically correlated random sequences. *Sov. Math*. 2:385-388, 1961.

[11] D. P. Kroese and Z. I. Botev. Spatial Process Generation. To appear in: V. Schmidt (Ed.). *Lectures on Stochastic Geometry, Spatial Statistics and Random Fields, Volume II: Analysis, Modeling and Simulation of Complex Structures*. Springer-Verlag, Berlin, 2014.

[12] N. Leonenko. Limit Theorems for Random Fields with Singular Spectrum. Springer Science+ Business Media Dordrecht, 1999.

[13] S. Lee and R.M. Rao. Self-Similar Random Field Models in Discrete Space. *IEEE Transactions On Image Processing*. 15(1), 160-168, 2006.

[14] Y. Li and Y. Xiao. Multivariate Operator-Self-Similar Random Fields. *Stochastic Processes and Their Applications*. 21(6), 1178-1200, 2011.

[15] A. Makagon. Stationary Sequences Associated with a Periodically Correlated Sequence. *Prob. Math. Stat*. 31(2), 263-283, 2011.

[16] M. Meerschaert, D. Wu and Y. Xiao. Local Times of Multifractional Brownian Sheets. *Bernoulli*. 14(3), 865-898, 2008.

[17] N. Modarresi and S. Rezakahah. Spectral Analysis of Multi-dimensional Self-Similar Markov Processes. *J. Phys. A, Math. Theor*. 43(12), 125004, 14 pp, 2010.

[18] N. Modarresi and S. Rezakahah. A New Structure for Analyzing Discrete Scale Invariant Processes: Covariance and Spectra. *J. Stat. Phys*. 153:162-176, 2013.

[19] N. Modarresi and S. Rezakahah. Characterization of discrete scale invariant Markov sequences. *Communications in Statistics: Theory and Methods*. 45(18), 5263-5278, 2016.

[20] S. Nemirovsky and M. Porat. On Texture and Image Interpolation Using Markov Models. *Signal Processing: Image Communication*. 24, 139-157, 2009.
[21] C. J. Nuzman and H. V. Poor. Linear estimation of self-similar processes via Lamperti transformation. *Journal of Applied Probability*. 37(2), 2000.

[22] H. Qian, G. M. Raymond and J. B. Bassingthwaighte. On Two-dimensional Fractional Brownian Motion and Fractional Brownian Random Field. *J. Phys. A: Math. Gen.* 31, L527-L535, 1998.

[23] S. Rezakhah, A. Philippe and N. Modarresi. Estimation of Scale and Hurst Parameters of Semi-Selfsimilar Processes. [arXiv:1207.2450v1 [math.ST]].

[24] S. Rezakhah and Y. Maleki. Discretization of Continuous Time Discrete Scale Invariant Processes: Estimation and Spectra. *J. Stat. Phys.* 164:438-448, 2016.

[25] A. Rosenfeld and A.C. Kak. Digital Picture Processing. Second ed, Vol. 1, Academic Press, New York, 1982.

[26] M. M. Rao. Random and Vector Measures, Series on Multivariate Analysis - Vol. 9, 2012.

[27] Y. A. Rozanov. Stationary Random Processes, Holden Day, San Francisco, 1967.

[28] G. Samorodnitsky, M.S. Taqqu. Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Stochastic Modeling, *Chapman and Hall, New York*, 1994.

[29] M. Shinozuka and G. Deodatis. Simulation of Stochastic Processes by Spectral Representation. *Appl Mech Rev*, 44(4):191-204, 1991.

[30] E. VanMarcke. Random Fields: Analysis and Synthesis. Revised and expanded new edition, *World scientific publishing*, 2011.

[31] D. Wu and Y. Xiao. On Local Times of Anisotropic Gaussian Random Fields. *Comm. Stoch. Anal.* 5, 15-39, 2011.