Compactness in Spaces of Vector Valued Continuous Functions
and Asymptotic Almost Periodicity

Dedicated to Heron S. Collins on the occasion of his 65th birthday

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(Received December 12, 1985)

Summary. We characterize the precompact sets in spaces of vector valued continuous functions and use the resulting criteria to investigate asymptotic behaviour of such functions defined on a halfline. This problem arose in the context of a qualitative study of solutions to the abstract Cauchy problem. We give particular consideration to the relationship between vector valued asymptotically almost periodic functions on a subinterval \([a, \infty)\) of the real line and precompactness of the set of its translates. Our compactness criteria are also applied to a question concerning the approximation property for spaces of vector valued continuous functions with topologies induced by weighted analogues of the supremum norm, as well as to obtain nonlinear variants on factorization of compact operators through reflexive Banach spaces.

Introduction. Evolution equations can be used to model a wide range of time dependent physical processes, and it is frequently the case, at least in practice, that such a process can be represented by a (semi-) dynamical system \(\pi: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \times X \to X\) on an appropriate state space \(X\) — for definitions and general information concerning dynamical systems, evolution equations, and the relationship between them, we refer to [41]. In this setting, given an initial state \(x \in X\), questions about future-time behavior of the process then devolve to consideration of the motion \(f \in C(\mathbb{R}^+, X)\) defined for \(t \in \mathbb{R}^+\) by \(f(t) = \pi(t, x)\). For Banach state spaces, moreover, relative compactness of the (positive) orbit \(\gamma^+(x) = f(\mathbb{R}^+)\) has emerged as an important factor in describing asymptotic behavior of the motion \(f\) through \(x \in X\) (cf. [41, IV, Sections 4 and 5]). Of course, \(f(\mathbb{R}^+)\) is relatively compact in \(X\) exactly when the set \(H^+(f) = \{f_\omega: \omega \in \mathbb{R}^+\}\) of translates of \(f\) is precompact in \(C(\mathbb{R}^+, X)\) with respect to the topology of pointwise convergence. Stronger compactness conditions on \(H^+(f)\) lead to correspondingly stronger conclusions about the eventual behavior of the motion, and the question has been raised as to those conditions under which \(H^+(f)\) would actually be relatively compact in \(C_b(\mathbb{R}^+, X)\) with respect to the topology of uniform convergence on \(\mathbb{R}^+\). In the present paper, we develop characterizations of precompactness in spaces of vector valued continuous functions which we then use to resolve this problem.
Many concrete state spaces that arise in the theory of evolution equations are spaces of vector valued continuous functions with topologies induced by weighted analogues of the supremum norm (e.g., see [42])—a desirable situation in view of the computability of estimates, and we have chosen to place our results on compactness in a context which is broad enough to include such instances of weighted uniform convergence. As a consequence, these criteria will be applicable to the question of orbital precompactness of motions as well as to the problem immediately at hand.

Following a preliminary section, we begin in Section 2 by establishing our characterizations of precompactness (Theorems 2.1 and 2.2), and then relate these to the known results in this direction. Further, in the case of a FRÉCHET range space, we obtain nonlinear variants (Theorem 2.4) on factorization of compact operators through reflexive BANACH spaces. In Section 3, we focus on asymptotic behavior of vector valued continuous functions defined on a subinterval \([a, \infty)\) of the real line, using our general compactness criteria to extend and unify results connecting such behavior with various periodicity conditions that occur in this context. The central notion is that of a vector valued asymptotically almost periodic function—an idea that was introduced more than forty years ago by M. \textsc{Fréchet} [19], [20]. Of course, many of \textsc{Fréchet}'s original arguments for finite dimensional range spaces can be adapted to the case of functions taking values in arbitrary BANACH spaces. However, the extent to which these results carry over is apparently not in the realm of general knowledge, and we exploit our methods to provide a comprehensive treatment of this important concept. Another aspect of this third section is based upon interpretation of the nonlinear factorization result from Section 2 in the setting at hand. We then bring the paper to an end with a fourth section in which we apply our compactness criteria to resolve a question concerning the approximation property for weighted spaces of vector valued continuous functions.

\textbf{Acknowledgement.} Our work on the material presented here has been supported in part through dual funding from NSF-EPSCOR Grant ISO-8011447 and the State of Arkansas. Visits by the second author to the Universität Essen in the Summer of 1981 and the first named author's stay at the University of Arkansas in the Spring of 1982 have contributed significantly to carrying the project forward; we hereby express our gratitude to both institutions for their support in this regard.

We are particularly grateful to D. W. \textsc{Brewer} for the considerable insight and information that he has made available to us in connection with the theory of evolution equations.

1. \textbf{Preliminaries}

Topological notation and terminology will primarily follow the usage in [25], while that for locally convex duality theory derives from [34]. We next note some exceptions and supplementary conventions.
Throughout the remainder of the article, we let $T$ denote a completely regular Hausdorff space, and the phrase "locally convex space" will be taken to mean a Hausdorff locally convex topological vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$; there will be no loss of generality in tacitly assuming that $K = \mathbb{C}$. Further, the set of all continuous seminorms on a locally convex space $X$ will be denoted by $cs(X)$, while we write $C(T, X)$ to indicate the collection of all continuous functions from $T$ into $X$.

For a subset $A$ of a locally convex space $X$, its closed convex (respectively, absolutely convex) hull will be denoted by $\overline{co}A$ (respectively, $\overline{ac}A$); an absolutely convex subset of $X$ will be referred to simply as a disk. The topological dual $X'$ of $X$, when given the topology of uniform convergence on all compact disks in $X$, will be denoted by $X'_{\text{c}}$.

If $B$ is a closed and bounded disk in a locally convex space $X$, $X_B$ will denote the linear span of $B$ in $X$ under the norm defined by the gauge of $B$. In case $B$ is also compact, or even just sequentially complete, then certainly $X_B$ is complete.

1.1. Weighted spaces of vector valued continuous functions. A nonnegative upper semicontinuous function on $T$ will be called a weight (on $T$). If $V$ is a set of weights on $T$ such that, given any $t \in T$, there is some $v \in V$ for which $v(t) > 0$, we write $V > 0$. A set $V$ of weights on $T$ is said to be directed upward provided that, for every pair $v_1, v_2 \in V$ and each $\lambda > 0$, there exists $v \in V$ so that $\lambda v_i \leq v$ (pointwise on $T$) for $i = 1, 2$. Since there is no loss of generality, we hereafter assume that sets of weights are directed upward; a set $V$ of weights on $T$ which additionally satisfies $V > 0$ will be referred to as a system of weights on $T$.

Now, taking a system $V$ of weights on $T$ and a locally convex space $X$, we consider the following vector spaces (over $K$) of continuous functions associated with the triple $(T, V, X)$:

- $CV_0(T, X) = \{f \in C(T, X) : v_f \text{ vanishes at infinity on } T \text{ for all } v \in V\}$;
- $CV_p(T, X) = \{f \in C(T, X) : v_f(T) \text{ is precompact in } X \text{ for all } v \in V\}$;
- $CV_b(T, X) = \{f \in C(T, X) : v_f(T) \text{ is bounded in } X \text{ for all } v \in V\}$.

Obviously, $CV_p(T, X) \subseteq CV_b(T, X)$, while the upper semicontinuity of the weights yields that $CV_0(T, X) \subseteq CV_p(T, X)$. Thus, if for each $v \in V, q \in cs(X)$, and $f \in C(T, X)$, we put

$$p_{v, q}(f) = \sup \{q(f(t)) : t \in T\},$$

then $p_{v, q}$ can be regarded as a seminorm on either $CV_b(T, X), CV_p(T, X)$, or $CV_0(T, X)$; we assume that each of these three spaces is equipped with the Hausdorff locally convex topology induced by $\{p_{v, q} : v \in V, q \in cs(X)\}$.

In case $X = K$, we will omit $X$ from our notation and write, say, $CV_0(T)$ in place of $CV_0(T, \mathbb{K})$; we also then put $p_v = p_{v, q}$ for each $v \in V$, where $q(z) = |z|, z \in K$. Similarly, if $X = (X, q)$ is any normed space and $v \in V$, we write $p_v$ instead of $p_{v, q}$.

As a matter of further notational convenience, given $v \in V$ and $q \in cs(X)$, the closed unit ball corresponding to the seminorm $p_{v, q}$ in either $CV_0(T, X), CV_p(T, X)$,
or \( CV_0(T, X) \) will be denoted by \( B_{v,q} \), or simply \( B_v \) in case \( X=(X, q) \) is a normed space; this ambiguity should occasion no difficulty since the setting under consideration will always be clear from context.

The spaces \( CV_0(T) \) and \( CV_0(T) \) were first introduced by L. Nachbin (cf. [30]), and the corresponding vector valued analogues \( CV_0(T, X) \) and \( CV_0(T, X) \) were subsequently considered in detail by K.-D. Bierstedt [2], [3] and J. B. Prolla [33]. Aside from examples that arise in response to special considerations such as those inherent in the dynamics of age dependent populations (cf. [42]), many standard spaces of continuous functions can be realized in this general format, and we shall here list certain instances for future reference. To this end, given a completely regular Hausdorff space \( T \), and writing \( 1 \) to designate the characteristic function of a subset \( F \) of \( T \), we distinguish three systems of weights on \( T \): namely,

\[
K = K(T) = \{ \lambda 1_K : \lambda > 0, K \subseteq T, K \text{ compact} \}, \quad 1 = 1(T) = \{ \lambda 1_T : \lambda > 0 \},
\]

and the system \( U = U(T) \) consisting of all weights on \( T \) which vanish at infinity. Further, given a locally convex space \( X \), we put

\[
C_0(T, X) = \{ f \in C(T, X) : f \text{ vanishes at infinity on } T \},
\]

\[
C_p(T, X) = \{ f \in C(T, X) : f(T) \text{ is precompact in } X \},
\]

and

\[
C_b(T, X) = \{ f \in C(T, X) : f(T) \text{ is bounded in } X \}.
\]

1.1.1. Example. For any pair \( (T, X) \) consisting of a completely regular Hausdorff space \( T \) and a locally convex space \( X \),

(a) \( C_{K_0}(T, X) = C_{K_p}(T, X) = C_{K_b}(T, X) = \left( C(T, X), x \right) \),

where \( x \) denotes the compact-open topology;

(b) \( C_{U_0}(T, X) = C_{U_p}(T, X) = C_{U_b}(T, X) = \left( C_b(T, X), \beta_0 \right) \),

where \( \beta_0 \) denotes the substrict topology (cf. [17]);

(c) \( C_{1_0}(T, X) = \left( C_{0}(T, X), v \right) \), \( C_{1_p}(T, X) = \left( C_{p}(T, X), v \right) \),

and

\[
C_{1_b}(T, X) = \left( C_b(T, X), v \right),
\]

where we use \( v \) in each case to denote the topology of uniform convergence on \( T \).

1.2. Representation of weighted spaces as \( \varepsilon \)-products. The main tool in hand for treating the question of compactness in the vector case is the notion of \( \varepsilon \)-product (in the sense of L. Schwartz [38], [39]). Through the representations \( X \varepsilon CV_0(T) \) and \( X \varepsilon CV_0(T) \) for \( CV_0(T, X) \) and \( CV_0(T, X) \), respectively, recently developed results on compactness in arbitrary \( \varepsilon \)-products (cf. [35]) can be brought to bear. Moreover, this concept embodies a linearization principle for vector valued functions which allows the scalar and vector cases to be directly linked through functional analytic means.
1.2.1. Definition. The $\varepsilon$-product $XEY$ of two locally convex spaces $X$ and $Y$ is the locally convex space $L_+(X', Y)$ of all continuous linear operators from $X'$ into $Y$, equipped with the topology of uniform convergence on the equicontinuous subsets of $X'$.

For basic facts on $\varepsilon$-products and their relationship to the injective tensor product (of A. Grothendieck [23]) as well as a wealth of information concerning applications and the broad variety of concrete spaces that can thus be represented, we refer to [38] and [39]; we also refer to [10] in this regard. In our context, the question of $\varepsilon$-product representation has been considered by K.-D. Bierstedt [6]. Before stating Bierstedt's result in this direction, however, we first present one additional item of terminology.

A completely regular Hausdorff space $T$ is said to be a $V_\kappa$-space with respect to a given system $V$ of weights on $T$ if a function $f: T \to \mathbb{R}$ is necessarily continuous whenever, for each $v \in V$, the restriction of $f$ to $\{t \in T: v(t) \geq 1\}$ is continuous. This requirement on a pair $(T, V)$ can essentially be construed as a condition for the completeness of $CV_0(T)$ (cf. [22]), and we note in passing that it is a relatively modest restriction. Indeed, if $V = K(T)$, the only requirement is that $T$ be a $k_\kappa$-space (in the language of H. Buchwalter), which certainly holds, say, when $T$ is locally compact, while no restriction whatsoever is imposed on $T$ in case $V = 1(T)$.

1.2.2. Theorem (Bierstedt [6, p. 39]). Let $V$ be a system of weights on a completely regular Hausdorff space $T$, and assume that $X$ is a quasicomplete locally convex space. If $T$ is a $V_\kappa$-space, then

(a) $CV_0(T, X) \cong X_\varepsilon CV_0(T)$ and
(b) $CV_p(T, X) \cong X_\varepsilon CV_0(T)$.

The next result (cf. [6, p. 38]) is an immediate consequence of Theorem 1.2.2.

1.2.3. Corollary. Given a system $V$ of weights on a completely regular Hausdorff space $T$, if $T$ is a $V_\kappa$-space and $X$ is a quasicomplete (respectively, complete) locally convex space, then $CV_0(T, X)$ and $CV_p(T, X)$ are both quasicomplete (respectively, complete).

Finally, we mention a point which will be helpful for avoiding technicalities in the sequel. As has been observed by A. Goulet de Ruyog [22], if $V$ is any set of weights on a completely regular Hausdorff space $T$ such that $CV_0(T)$ is a complete Hausdorff locally convex space, then there is no essential loss of generality in assuming that $V$ is a system of weights on $T$ (i.e., $V > 0$) and that the following condition holds for the pair $(T, V)$:

1.2.4. For each $t \in T$, there exists $f_t \in CV_0(T)$ such that $f_t(t) \neq 0$.

2. The compactness criteria

In developing generalized Arzelà-Ascoli theorems for weighted spaces of type $CV_0(T, X)$ and type $CV_p(T, X)$, we work under a standing hypothesis based on the ideas discussed in Section 1 above — viz., $T$ is a completely regular Hausdorff
space, \( V \) is a system of weights on \( T \), and \( T \) is a \( V \)-space such that, for each \( t \in T \), there exists \( f_t \in CV_b(T) \) with \( f_t(t) = 0 \); these basic assumptions will be in force throughout the present section. The locally convex range space \( X \) will not be restricted, however, since certain of the applications to be treated in Section 3 require that \( X \) be considered under its weak topology, where presupposing quasicompleteness would then limit us to semireflexive range spaces (cf. [28, p. 299]). Our results of \( \text{ARZELÀ-ASCOLI} \) type are thus formulated as characterizations of the precompact subsets of \( CV_o(T, X) \) and \( CV_p(T, X) \), respectively. Of course, these automatically become criteria for relative compactness whenever \( X \) does happen to be quasicomplete.

2.1. Theorem. A subset \( H \) of \( CV_o(T, X) \) is precompact if, and only if, the following conditions are satisfied:

(i) \( H \) is equicontinuous;
(ii) \( H(t) = \{ h(t) : h \in H \} \) is precompact in \( X \) for each \( t \in T \); and
(iii) \( vH \) vanishes at infinity on \( T \) for every \( v \in V \) (i.e., given \( v \in V \), \( q \in \text{cs}(X) \), and \( \varepsilon > 0 \), there exists a compact set \( K \subseteq T \) such that, for all \( h \in H \) and \( t \in T \setminus K \), \( v(t) q(h(t)) < \varepsilon \))

Whereas Theorem 2.1 is a natural extension of the classical \( \text{ARZELÀ-ASCOLI} \) theorem, the case for \( CV_p(T, X) \) tends to reflect that of \( (C_b(T), v) \) when \( T \) is not assumed to be compact (e.g., see [15, IV. 6.5, p. 266]), and this is particularly true with regard to the covering conditions for \( T \) that occur. In formulating these, some additional notation will be helpful to us—viz., given \( H \subseteq CV_p(T, X) \), \( v \in V \), \( q \in \text{cs}(X) \), and \( \varepsilon > 0 \), if \( s \in T \), we put

\[
T_s(H, v, q, \varepsilon) = \{ t \in T : \sup \{ q(v(t) h(t) - v(s) h(s)) : h \in H \} \leq \varepsilon \}.
\]

2.2. Theorem. The following are equivalent for a subset \( H \) of \( CV_p(T, X) \):

1. \( H \) is precompact;
2. (i) \( H \) is equicontinuous,
   (ii) \( H(t) = \{ h(t) : h \in H \} \) is precompact in \( X \) for each \( t \in T \), and
   (iii) given \( v \in V \), \( q \in \text{cs}(X) \), and \( \varepsilon > 0 \), there exists a compact set \( K \subseteq T \) such that
       \( \{ T_s(H, v, q, \varepsilon) : s \in K \} \) covers \( T \);
3. (i) \( H(t) \) is precompact in \( X \) for each \( t \in T \), and
   (ii) given \( v \in V \), \( q \in \text{cs}(X) \), and \( \varepsilon > 0 \), there exists a finite set \( F \subseteq T \) such that
       \( \{ T_s(H, v, q, \varepsilon) : s \in F \} \) covers \( T \);
4. (i) \( vH(T) = \{ v(t) h(t) : t \in T, h \in H \} \) is precompact in \( X \) for each \( v \in V \), and
   (ii) given \( v \in V \), \( q \in \text{cs}(X) \), and \( \varepsilon > 0 \), there exists a finite set \( F \subseteq T \) such that
       \( \{ T_s(H, v, q, \varepsilon) : s \in F \} \) covers \( T \).

In the scalar case, it is known [6, p. 39] that precompact subsets of \( CV_b(T) \), and thus also of \( CV_o(T) \), are equicontinuous; we take this fact as our starting point in establishing 2.1 and 2.2. For completeness, however, as well as since the idea of proof provides additional insight into the conditions imposed under our
standing hypothesis, we briefly sketch the approach to the scalar case (in [6]) before proceeding to carry the result over to the present context.

To begin, since \( V > 0 \), the point evaluation \( \delta_t \) corresponding to any \( t \in T \) can be regarded as a continuous linear functional on \( CV_b(T) \). Hence, putting \( A(t) = \delta_t \) for each \( t \in T \), \( A \) is a function from \( T \) into \( CV_b(T') \); it is clear that \( A \) is continuous when \( CV_b(T') \) has its weak-star topology \( \sigma(CV_b(T'), CV_b(T)) \). For \( v \in V \) and \( F_v = \{ t \in T : v(t) \geq 1 \} \), \( A(F_v) \subseteq B_v^0 \), so that, in fact, \( A \mid F_v \) is a continuous map into \( CV_b(T) \). Since \( T \) is a \( V_n \)-space, this means (even in the case of vector valued functions (cf. [6, p. 381])) that \( A \in C(T, CV_b(T)') \), which is equivalent to the equicontinuity of the precompact subsets of \( CV_b(T) \) in view of Corollary 1.2.3.

2.3. Lemma. If \( H \) is a precompact subset of \( CV_p(T, X) \), then \( H \) is equicontinuous.

Proof. We first note that each \( f \in CV_p(T, X) \) canonically induces a linear operator \( f^* : X' \rightarrow CV_p(T) \) defined by \( f^*(x') = f' \circ f, x' \in X' \), and that, moreover, \( f^* \) is continuous when \( X' \) has the topology \( A = A(X) \) of uniform convergence on the precompact subsets of \( X \) since, given \( v \in V \), \( F = \{ t \in T : f(t) \in A(0, P) \} \subseteq B_v \). Now, fix \( q \in cs(X) \) and, following our convention for denoting the unit ball associated with \( \seminorm \), put \( B_q = \{ x' \in X : q(x') \leq 1 \} \); it will suffice to show that

\[
B_q^0 \circ H = H^*(B_q^0) = \{ h^*(x') : h \in H, x' \in B_q^0 \}
\]

is an equicontinuous subset of \( CV_b(T) \). According to the scalar result discussed above, the proof can thus be completed by showing that \( H^*(B_q^0) \) is precompact in \( CV_b(T) \). To this end, fix \( v \in V \) and take \( \varepsilon > 0 \). Since \( H \) is precompact in \( CV_p(T, X) \), there exists a finite set \( \{ h_i \}_{i=1}^n \subseteq H \) such that \( H \subseteq \bigcup_{i=1}^n (h_i + \varepsilon B_v) \), from which it follows that \( H^*(B_q^0) \subseteq \bigcup_{i=1}^n (h_i^*(B_q^0) + \varepsilon B_v) \). But \( B_q^0 \) is \( \lambda \)-compact whereby \( h_i^*(B_q^0) \) is compact in \( CV_b(T) \) for \( i = 1, \ldots, n \), and this brings us to the desired conclusion.

Proof of Theorem 2.1. Assume that \( H \) is a precompact subset of \( CV_b(T, X) \). Then Lemma 2.3 gives us that (i) holds, while (ii) follows from the fact that the weighted topology on \( CV_b(T, X) \) is finer than that of pointwise convergence. To establish (iii), given \( v \in V \), \( q \in cs(X) \), and \( \varepsilon > 0 \), choose a finite set \( \{ h_i \}_{i=1}^n \subseteq H \) so that \( H \subseteq \bigcup_{i=1}^n (h_i + \varepsilon/2 B_v) \), and put \( K = \bigcup_{i=1}^n \{ t \in T : v(t) q(h_i(t)) \leq \varepsilon/2 \} \). Then \( K \) is compact. Moreover, taking \( h \in H \) and \( t \in T \setminus K \), if we choose \( i \in \{ 1, \ldots, n \} \) such that \( h - h_i \in (\varepsilon/2) B_v \), then

\[
v(t) q(h(t)) \leq v(t) q(h(t) - h_i(t)) + v(t) q(h_i(t)) < \varepsilon \cdot
\]

For sufficiency, since \( H \) is an equicontinuous subset of \( C(T, X) \) which, in view of (ii), also happens to be precompact in the topology of pointwise convergence, it is straightforward to check (cf. [44, p. 289]) that, in fact, \( H \) is a precompact subset of \( (C(T, X), \varepsilon) \). Now, taking \( v \in V \), \( q \in cs(X) \), and \( \varepsilon > 0 \), we utilize (iii) to
select a compact set \( K \subseteq T \) such that, for all \( h \in H \) and \( t \in T \setminus K \),

\[
(v(t) g(h(t)) < \varepsilon/2 .
\]

Because \( v \) is upper semicontinuous, \( \|v\|_K = \sup \{v(t) : t \in K\} < \infty \), and hence we can choose a finite set \( \{h_i\}_{i=1}^n \subseteq H \) so that, given \( h \in H \), there exists \( i \in \{1, \ldots, n\} \) with

\[
\sup \{v(t) : t \in K\} = \varepsilon/(\|v\|_K + 1) .
\]

This observation, when combined with (*), gives us that \( H \subseteq \bigcup_{i=1}^n (h_i + \varepsilon B_{v,q}) \); i.e., \( H \) is precompact in \( CV_0(T, X) \).

**Proof of Theorem 2.2.** Assume, first of all, that \( H \) is a precompact subset of \( CV_0(T, X) \). Just as in the proof of 2.1, (i) and (ii) follow from Lemma 2.3 and the fact that \( V > 0 \), respectively. Next, let us fix \( v \in V \), \( q \in \mathcal{C}^0(X) \), and \( \varepsilon > 0 \). For any \( f \in CV_0(T, X) \), since \( v_f(T) \) is precompact in \( X \), it is obvious that, given \( \eta > 0 \), there is a finite set \( F \subseteq T \) such that \( \{T_s(f), v, q, \eta \} : s \in F \) covers \( T \), and a straightforward induction argument shows that this conclusion still holds when \( \{f\} \) is replaced by any finite subset of \( CV_0(T, X) \). With this in mind, we choose a finite set \( \{h_i\}_{i=1}^n \subseteq H \) so that \( H \subseteq \bigcup_{i=1}^n (h_i + \varepsilon/3 B_{v,q}) \) and a corresponding finite set \( K \subseteq T \) for which \( \{T_s(h_i)_{i=1}^n, v, q, \varepsilon/3 \} : s \in K \) covers \( T \). Now, fixing \( t \in T \), we take \( s \in K \) so that \( t \in T_s(h_i)_{i=1}^n, v, q, \varepsilon/3 \). Then, given \( h \in H \), if \( i \in \{1, \ldots, n\} \) is chosen so as to have \( h - h_i \in (\varepsilon/3) B_{v,q} \),

\[
g(v(t) h(t) - v(s) h(s)) \leq g(v(t) h(t) - v(t) h_i(t)) + g(v(s) h_i(t) - v(s) h(s)) \leq \varepsilon
\]

whereby \( t \in T_s(H, v, q, \varepsilon) \); i.e., (ii) must also hold.

Since \( V > 0 \), the implication \( 4. \Rightarrow 3. \) is clear, and so let us next show that 4. follows from condition 3. Assuming, therefore, that 2. of Theorem 2.2 is satisfied, we first note that 2. (i) and 2. (ii) combine to imply (again, just as in the proof of 2.1) that \( H \) is a precompact subset of \( (C(T, X), \mathcal{R}) \). Now, let us fix \( v \in V \), \( q \in \mathcal{C}^0(X) \), and \( \varepsilon > 0 \). By 2. (iii), there is a compact set \( K \subseteq T \) such that \( \{T_s(H, v, q, \varepsilon/4) : s \in K\} \) covers \( T \), and we select a corresponding finite set \( \{h_i\}_{i=1}^n \subseteq H \) so that, given any \( h \in H \), it is always possible to find \( i \in \{1, \ldots, n\} \) for which \( (h - h_i) (K) \subseteq \eta B_q \), where \( \eta = \varepsilon [4 (\|v\|_K + 1)]^{-1} \). Moreover, for each \( i \in \{1, \ldots, n\} \), since \( v h_i(K) \) is precompact in \( X \), we can choose a finite set \( \{t_{ij}\}_{j=1}^{n_i} \subseteq K \) such that \( v h_i(K) \subseteq \bigcup_{j=1}^{n_i} (v(t_{ij}) h_i(t_{ij}) + (+ (\varepsilon/4) B_q) \). Thus, for a fixed \( t \in T \) and any \( h \in H \), there exists \( s \in K \) with \( t \in T_s(H, v, q, \varepsilon/4) \), \( i \in \{1, \ldots, n\} \) with \( (h - h_i) (K) \subseteq \eta B_q \), and \( j \in \{1, \ldots, n_i\} \) so that \( g(v(s) h_i(s) - v(t_{ij}) h_i(t_{ij})) < \varepsilon/4 \); these estimates together yield

\[
(\#) \quad g(v(t) h(t) - v(t_{ij}) h_i(t_{ij})) < (3/4) \varepsilon,
\]

which certainly serves to establish that \( v H(T) \) is precompact in \( X \), while 4. (ii)
follows from (#) and the fact that
\[ g(\psi(t_1) h_1(t_1) - \psi(t_1) h(t_1)) = \varepsilon/4 , \]
where we take \( F = \bigcup_{j=1}^{n} \{ t_j : j = 1, \ldots, n \} \).

Only the implication \( 3. \Rightarrow 1. \) remains to be verified, and so we once again fix \( v \in V, \, q \in cs(X) \), and \( \varepsilon > 0 \). Assuming that \( 3. \) holds, we can find a finite set \( F \subseteq T \) such that \( T(H, v, q, \varepsilon/3) : s \in F \) covers \( T \). Further, since \( 3. \) implies that \( H \) is precompact in the topology of pointwise convergence, there is a finite set \( \{ h_i \}_{i=1}^{n} \subseteq H \) with the property that, given \( h \in H \), some \( i \in \{ 1, \ldots, n \} \) can be chosen for which
\[ (h - h_i) (F) \subseteq \eta B_q, \quad \text{where} \quad \eta = \varepsilon \left( \frac{1}{3} \right) \left( \| \psi \| + 1 \right)^{-1} ; \]
we claim that \( H \subseteq \bigcup_{i=1}^{n} (h_i + \varepsilon B_{v,q}) \).

To see this, fix \( h \in H \) and take \( i \in \{ 1, \ldots, n \} \) so that \( (h - h_i) (F) \subseteq \eta B_q \). Then, for any \( t \in T \), since \( t \in T_s(H, v, q, \varepsilon/3) \) for some \( s \in F \), we have that
\[ g(\psi(t) h(t) - \psi(t) h_i(t)) \]
\[ = g(\psi(t) h(t) - \psi(t) h_i(t)) + g(\psi(s) h_i(s) - \psi(s) h(s)) \]
\[ + g(\psi(s) h_i(s) - \psi(t) h_i(t)) < \varepsilon . \]

This, of course, establishes our claim, and the proof of Theorem 2.2 is thereby complete.

Remark. Under additional continuity conditions on the weights, the covering conditions occurring in Theorem 2.2 can be strengthened. In particular, for a precompact subset \( H \) of \( CV_p(T, X) \), if \( q \in cs(X) \), \( \varepsilon > 0 \), and \( v \in V \) is a continuous weight on \( T \), then the approach used above (in going from 1. to 2.) to obtain the (finite) cover \( \{ T_s(H, v, q, \varepsilon) : s \in K \} \) of condition 2. (iii) also shows that there is an open neighborhood \( O \) of each \( s \in K \) such that \( \{ O : s \in K \} \) covers \( T \) and, for each \( s \in K \) and any \( t \in O \), \( g(\psi(t) h(t) - \psi(t) h(s)) \) is for all \( h \in H \); if only the restriction of \( v \) to its nonzero set \( N(v) = \{ t \in T : \psi(t) > 0 \} \) is continuous, it is still possible to find a cover \( \{ O_s : s \in K \} \) of \( T \) where, for each \( s \in K, \, O_s \cap N(v) \) is relatively open in \( N(v) \), \( s \in O_s \), and the \( q \)-variation of \( vH \) on \( O_s \) is uniformly small.

In the presence of more structure, the general criteria of 2.1 and 2.2 can be augmented by nonlinear analogues of factorization results for compact (linear) operators.

2.4. Theorem. Let \( X \) be a Fréchet space, and assume that the weighted topology (or) on \( CV_p(T) \) is also metrizable. Then, fixing \( a \in \{ 0, p \} \), a subset \( H \) of \( CV_a(T, X) \) is relatively compact if, and only if, there is a compact disk \( K \) in \( X \) for which the following conditions are satisfied: (i) the Banach space \( X_K \) is reflexive; (ii) \( H(T) \subseteq X_K \); (iii) \( H \) is relatively compact as a subset of \( CV_a(T, X_K) \).

Before going to the proof, we note that Theorem 2.4 provides conditions under which relatively compact subsets of \( CV_a(T, X) \) can be uniformly factored through a reflexive Banach space while preserving relative compactness, whereby corresponding results for a single compact linear operator [24] and relatively compact
sets of compact linear operators [35] are extended to a nonlinear setting. As a compactness criterion, of course, primary interest lies in necessity; an application to vector valued almost periodic functions will be considered in Section 3.

Proof. We begin by fixing a decreasing sequence \( \{ u_n \} \) of closed disks in \( X \) which is a neighborhood base at zero. Similarly, we fix an increasing sequence \( \{ v_n \} \) of weights from \( V \) such that \( \{ B_{v_n} \} \) is a neighborhood base (at zero) in \( CV_b(T) \). Now, assuming that \( H \) is a relatively compact set in \( CV_a(T, X) \), we use Theorem 1.2.2 to identify \( H \) with the corresponding relatively compact subset \( H^* \) of \( X \cap CV_a(T) \). From [35, Theorem 1.5], we then have that

(a) \( H^*(u_n^2) \) is relatively compact in \( CV_a(T) \) for all \( n \in \mathbb{N} \), and

(b) \( H^* \) is an equicontinuous subset of \( L(X', CV_a(T)) \).

Since \( X' \) is a gDF space (cf. [36, p. 421]), it follows from (b) that \( W = \bigcap_{n=1}^{\infty} \left( u_n^2 + \frac{1}{n} \cdot (H^*)^{-1}(B_{v_n}) \right) \) is a zero neighborhood in \( X' \), while (a) allows us to conclude that \( H^*(W) \) is relatively compact in \( CV_a(T) \). (For additional details, refer to the proof of Theorem 2.2 in [36].) We thus have that \( K_1 = \overline{W}^0 \) is a compact disk in \( X \), and that

(c) \( H^*(\overline{W}^0) = K_1^0(H) \) is relatively compact in \( CV_a(T) \).

Applied to \( K_1^0(H) \), our general compactness criteria from Theorems 2.1 and 2.2 now yield that \( H \subseteq CV_a(T, X_{K_1}) \); even more, for each \( t \in T \), \( H(t) \) is a bounded subset of \( X_{K_1} \). Since \( X \) is a Fréchet space, a well known consequence of the Banach-Dieudonné theorem (cf. [36, pp. 422–423]) allows us to select compact disks \( K_2 \) and \( K_3 \) in \( X \) such that \( K_1 \) and \( K_2 \) are compact subsets of \( X_{K_2} \) and \( X_{K_3} \) respectively. Next, we use the fact that the compact injection of \( X_{K_2} \) into \( X_{K_3} \) can be factored through a reflexive Banach space. More directly, [13, Lemma 1] asserts that there exists a compact disk \( K \) in \( X \) such that \( X_K \) is reflexive and \( K \), whence also \( K_1 \), is a compact subset of \( X_K \). Finally, again applying our compactness criteria, both to the relatively compact subset \( K_3^0(H) \) of \( CV_a(T) \) and, in the other direction, to \( H \), it is now apparent that \( H \) is a relatively compact subset of \( CV_a(T, X_K) \). Since sufficiency is obvious, the proof is thus complete.

Remark. The compact disk \( K_1 \) constructed in the foregoing argument can be described explicitly in terms of \( H \), and this may be useful in certain special cases. Before giving these, however, let us also note that the relatively compact set \( H \) in \( CV_a(T, X) \), when viewed as a subset of \( CV_a(T, X_{K_1}) \), already satisfies all but possibly one of the conditions for relative compactness as set forth in either 2.1 if \( a = 0 \) or 2.2, say, of 2.2 when \( a = p \); the exception in both instances is the requirement that \( H(t) \) be relatively compact, and not just bounded, in \( X_{K_1} \) for all \( t \in T \). Now, proceeding to the promised descriptions, we continue under the hypotheses of 2.4, as well as make use of the notation established in the course of the proof.
For each \( n \in \mathbb{N} \), we put \( C_n = \overline{w}(r_n H(t)) \). (In view of 2.2, since \( H \) is assumed to be relatively compact in \( CV_n(T, X) \), each \( C_n \) is compact in \( X \).) Next, observe that

\[
(H^*)^{-1}(B_{x_n}) = \{ x' \in X' : x' \circ H \subseteq B_{x_n} \}
\]

\[
= \{ x' \in X' : (\langle r_n(t) h(t), x' \rangle) \leq 1 \text{ for all } h \in H \text{ and all } t \in T \}
\]

\[
= (r_n H(T))^0 = C_n, \quad n \in \mathbb{N}.
\]

whereby \( K_1 = W^0 = \left( \bigcap_{n=1}^{\infty} \left( n \mathbb{U}_n + \frac{1}{n} C_n^0 \right)^0 \right) \); i.e.,

\[
K_1 = \overline{w} \left( \bigcup_{n=1}^{\infty} \left( n \mathbb{U}_n + \frac{1}{n} C_n^0 \right)^0 \right).
\]

Moreover, according to [26, p. 148], if \( \alpha \in (0, 1) \) and \( A_n = n \mathbb{U}_n + \frac{1}{n} C_n^0, n \in \mathbb{N} \), then

\[
\alpha A_n^0 \subseteq \bigcup \left\{ \left( \frac{r}{n} \right) \mathbb{U}_n \cap (s n C_n) : 0 \leq r, s ; r+s \leq 1 \right\} \subseteq A_n^0,
\]

from which, equivalently,

\[
K_1 = \overline{w} \left( \bigcup_{n=1}^{\infty} \left( \left( \frac{r}{n} \right) \mathbb{U}_n \cap (s n C_n) : 0 \leq r, s ; r+s \leq 1 \right) \right).
\]

Having now established our general criteria, we proceed to illustrate these results in the context of several familiar function spaces. For notation and terminology relating to the special cases considered in the following sequence of corollaries to Theorems 2.1, 2.2, and 2.4, we refer back to Section 1 — particularly, Example 1.1.1.

2.5.1. Corollary. Let \( X \) be a quasicomplete locally convex space.

(a) If \( T \) is a \( k_h \)-space, then a subset \( H \) of

\[
(C(T, X), \chi) = C_{K_r}(T, X)
\]

is relatively compact if, and only if,

(i) \( H \) is equicontinuous, and

(ii) \( H(t) \) is relatively compact in \( X \) for every \( t \in T \).

(b) If \( T \) is locally compact and \( \alpha \)-compact, and if \( X \) is a Fréchet space, then the following set of conditions is necessary (and sufficient) for the relative compactness of a set \( H \subseteq (C(T, X), \chi) \):

There exists a compact disk \( K \) in \( X \) for which the Banach space \( X_K \) is reflexive and such that

(i) \( H \) is an equicontinuous subset of \( C(T, X_K) \);

(ii) \( H(t) \) is relatively compact in \( X_K \) for all \( t \in T \).

Comparing 2.5.1. (a) to the known Arzelà-Ascoli theorem for \( (C(T, X), \chi) \) in the case that \( T \) is a \( k \)-space (cf. [35, p. 234]), we note that equicontinuity of \( H \) on all of \( T \) is here unveiled as a necessary condition for relative compactness.
2.6.2. Corollary. Let $T$ be a locally compact Hausdorff space, and take $X$ to be a quasicomplete locally convex space.

(a) A subset $H$ of 
$$(C_0(T, X), v) = \mathcal{C} C_0(T, X)$$
is relatively compact if, and only if,

(i) $H$ is equicontinuous,
(ii) $H(t)$ is relatively compact in $X$ for each $t \in T$, and
(iii) $H$ uniformly vanishes at infinity on $T$; i.e., given $q \in cs(X)$ and $\varepsilon > 0$, there exists a compact set $C$ in $T$ such that $(qh(t)) < \varepsilon$ for all $t \in T \setminus C$ and all $h \in H$.

(b) In case $X$ is a Fréchet space, the following set of conditions is necessary (and sufficient) for a subset $H$ of $(C_0(T, X), v)$ to be relatively compact:

There exists a compact disk $K$ in $X$ for which the Banach space $X_K$ is reflexive and such that

(i) $H$ is an equicontinuous subset of $C_0(T, X_K)$;
(ii) $H(t)$ is relatively compact in $X_K$ for all $t \in T$;
(iii) given $\varepsilon > 0$, there is a compact set $C$ in $T$ so that $h(t) \in \varepsilon K$ for all $t \in T \setminus C$ and all $h \in H$.

The classical strict topology $\beta$ on $C_0(S)$, where $S$ here denotes a locally compact Hausdorff space, was introduced and first studied by R. C. Buck [9]. The following consequence of Theorem 2.1 extends the Arzelà-Ascoli theorem established for $(C_0(S), \beta)$ by J. B. Conway [11, Theorem 2.12], as well as subsequent version for the strict topology $\beta_0$ due to R. F. Wheeler (cf. [43, Theorem 13.6]). Since $(C_0(T), \beta_0)$ is metrizable only when $T$ is compact [40, p. 321], application of 2.4 in this setting is covered under (b) of either 2.5.1 or 2.5.2.

2.6.3. Corollary. Let $T$ be a completely regular Hausdorff $k_\sigma$-space, and assume that $X$ is a quasicomplete locally convex space. Then a subset $H$ of 
$$(C_0(T, X), \beta_0) = \mathcal{C} U_0(T, X)$$
is relatively compact if, and only if,

(i) $H$ is equicontinuous,
(ii) $H(t)$ is relatively compact in $X$ for all $t \in T$, and
(iii) $H$ is uniformly bounded (i.e., $H(T)$ is bounded in $X$).

We would note at this point that the conditions characterizing $\beta_0$ relative compactness formally agree with those set forth in the classical Arzelà-Ascoli theorem for $(C(T, X), v)$ in the case that $T$ is compact and $X$ is a Banach space. Thus, Corollary 2.6.3 adds further support to the thesis that the strict topology is the proper and natural generalization of the supremum norm topology in the setting of bounded continuous functions on completely regular spaces.

2.6.4. Corollary. Assume that $X$ is a quasicomplete locally convex space.

(a) A subset $H$ of 
$$(C_\beta(T, X), v) = \mathcal{C} \beta(T, X)$$
is relatively compact if, and only if,

(i) \( H(T) \) is relatively compact in \( X \), and

(ii) given \( g \in \text{cst}(X) \) and \( \varepsilon > 0 \), there is a finite open cover \( \{T_i\}_{i=1}^n \) of \( T \) such that, for any \( i \in \{1, \ldots, n\} \) and all \( s, t \in T_i \),

\[
\sup \{ g(\hat{h}(s) - \hat{h}(t)) : \hat{h} \in H \} \leq \varepsilon .
\]

(b) In case \( X \) is a Fréchet space, a subset \( H \) of \( (C_\varepsilon(T, X), v) \) will be relatively compact if, and only if, there exists a compact disk \( K \) in \( X \) for which the Banach space \( X_K \) is reflexive and such that

(i) \( H(T) \) is relatively compact in \( X_K \);

(ii) given \( \varepsilon > 0 \), there is a finite open cover \( \{T_i\}_{i=1}^n \) of \( T \) so that, for any \( i \in \{1, \ldots, n\} \) and all \( s, t \in T_i \), \( \hat{h}(s) - \hat{h}(t) \in \varepsilon K \) for every \( \hat{h} \in H \).

The classical compactness result for \( (C_\varepsilon(T), v) \) in case \( T \) is not compact (cf. [15, IV.6.5, p. 266]) is extended to a vector valued setting by Corollary 2.5.4. (a). As mentioned previously, interest in 2.5.4. (b) lies primarily in the necessity of the given condition; this formulation will be used to advantage in Section 3.

Clearly, 2.5.4 readily extends to more general situations (cf. [8]). In particular, let \( M \) be a collection of closed subsets of \( T \), put \( V = \{1_M : i > 0, M \in M\} \), and assume that \( T \) and \( M \) have been chosen so that the pair \( (T, V) \) meets the minimal requirements under which we are working. Then, given a quasicomplete locally convex space \( X \), \( C\varepsilon(T, X) \) consists of all \( X \)-valued continuous functions on \( T \) which transform each \( M \in M \) into a relatively compact subset of \( X \), and the topology on \( X \) that is that of uniform convergence on the members of \( M \); a subset \( H \) of \( C\varepsilon(T, X) \) is relatively compact if, and only if,

(i) \( H(M) \) is relatively compact in \( X \) for each \( M \in M \), and

(ii) given \( M \in M \), \( g \in \text{cst}(X) \), and \( \varepsilon > 0 \), there exists a finite open cover \( \{T_i\}_{i=1}^n \) of \( M \) such that, for any \( i \in \{1, \ldots, n\} \) and all \( s, t \in T_i \), \( \sup \{ g(\hat{h}(s) - \hat{h}(t)) : \hat{h} \in H \} \leq \varepsilon .
\]

For related results in this direction, compare [32, Chapter 5].

3. Compactness and asymptotic behavior

If \( A : \Theta(A) \subseteq X - X \) generates a \( C_0 \)-semigroup \( (S(t))_{t \geq 0} \) of bounded linear operators on a Banach space \( X \), then the motion \( x : R^+ \to X \) through \( x_0 \in \Theta(A) \) defined for \( t \geq 0 \) by \( x(t) = S(t) x_0 \) is the unique strong solution of the associated Cauchy problem

\[
(\text{CP}) \quad \begin{cases} \dot{x}(t) = A x(t), & t \in R^+ \\ x(0) = x_0, \end{cases}
\]

and an appropriately modified version of this assertion holds even when \( A \) is nonlinear or multivalued (cf. [41, Chapter III]). In a qualitative study of solutions to \((\text{CP})\), certainly one major concern is the behavior of \( x(t) = S(t) x_0 \) as \( t \to +\infty \).

With regard to describing asymptotic behavior of the motion \( S(\cdot) x_0 \), the (positive) orbit \( \gamma^+(x_0) = \{S(t) x_0 : t \in R^+ \} \) and corresponding \( \omega \)-limit set \( \Omega(x_0) \) have a basic part.

In particular, if \( \Omega(x_0) \) is nonvoid, then there is at least one sequence \( (t_n) \) in \( R^+ \) with
t_n \to +\infty$ and some $y \in X$ such that $S(t_n) x_0 \to y$, while an even stronger statement can clearly be made in this direction when $y^+(x_0)$ is relatively compact in $X$.

However, yet much more can be said about asymptotic behavior of the motion in case the set $H^+(S(\cdot) x_0) = \{S(\cdot + \omega) x_0 : \omega \in R^+\}$ of translates happens to be relatively compact in $C_{1b}(R^+, X)$. Indeed, assuming this to be the case, not only would $y^+(x_0)$ be relatively compact in $X$, if $(t_n)$ is any sequence in $R^+$ with $t_n \to +\infty$, there would then exist a subsequence $(t_{n_k})$ of $(t_n)$ and a function $y \in C_b(R^+, X)$ such that the sequence $(S(\cdot + t_{n_k}) x_0)_k$ converges uniformly to $y$ on $R^+$.

The foregoing considerations concerning asymptotic behavior of solutions to (CP) have prompted the following two problems:

1. Characterize those $f \in C_b(R^+, X)$ for which the set of translates $H^+(f) = \{f_\omega : \omega \in R^+\}$, where here $f_\omega(t) = f(t + \omega)$ for $t \in R^+$, is relatively compact in $C_{1b}(R^+, X)$;

2. Determine conditions on $A$ and $X$ under which the set of translates of the motion $x = S(\cdot) x_0$ is relatively compact in $C_{1b}(R^+, X)$.

As we announced in the Introduction, the compactness criteria developed in Section 2 can be used to derive a complete solution to Problem 1, and this shall be among our purposes in the present section. Problem 2 will be the subject of a future paper.

In the case of a Banach range space $X$, if $R^+$ is everywhere replaced by $R$ in the first problem posed above, then the solution is contained in a classical result due to S. Bochner [7, § 2]; namely, for $f \in C(R, X)$, $H(f) = \{f_\omega : \omega \in R\}$ is a relatively compact subset of $C_{1b}(R, X)$ if, and only if, $f$ is almost periodic. Some years later, M. Fréchet [19], [20] considered the question for subintervals of $R$ of the form $[a, \infty)$, $a \in R$, but with the range $X$ restricted to a finite dimensional space, obtaining an analogous result in terms of "fonctions asymptotiquement presque-périodiques." A scalar version of Fréchet's concept for functions defined on commutative topological semigroups has subsequently been noted by De Leeuw and Glicksberg [29, p. 112].

A modern survey of Fréchet's work on asymptotically almost periodic functions has been given by A. M. Fink [17, Chapter 9]. Since [17] seems to be a basic reference for experts in the field, however, we take this opportunity to point out that the proof given in [17] for one of the critical results—the implication (3) implies asymptotic almost periodicity in [17, Theorem 9.3]—contains a conceptual error. For a correct argument, we refer the reader to either Fréchet's original proof in [20] or the one of Theorem 3.4 below.

Starting from the precompactness criteria of Section 2, we were led quite naturally to the notion of asymptotic almost periodicity and a version of Fréchet's classical theorem in which any locally convex space could serve as range. Our methods apply as well in the situation covered by Bochner's result, and we now proceed to give a comprehensive development of these and related ideas in the general setting of an arbitrary locally convex range space.
To begin, for \( a \in \mathbb{R} \), let \( J_a = \{ t \in \mathbb{R} : t \equiv a \} \), and recall that a subset \( P \) of \( J_a \) is said to be \emph{relatively dense} in \( J_a \) whenever there exists \( l > 0 \) such that, for each \( t \in J_a \), the closed interval \([ t, t + l ]\) contains at least one member of \( P \).

**Definition.** (cf. [19], [20]). Let \( X \) denote a locally convex space, and fix \( a \in \mathbb{R} \). A continuous function \( f : J_a \to X \) will be termed \emph{asymptotically almost periodic (a.a.p.)} if, given any \( \varepsilon > 0 \) and any \( q \in \text{cs}(X) \), there exists \( r = r(\varepsilon, q) \equiv a \) and a relatively dense set \( P = P(\varepsilon, q) \) in \( J_a \) such that, for each \( t \in J_a \), with \( t + r \equiv r \), \( q \left( f(t + r) - f(t) \right) \equiv \varepsilon \).

The extension of Fréchet's theorem to which we alluded above can now be stated; this result serves to completely resolve Problem 1.

3.1. **Theorem.** Assume that \( X \) is a locally convex space, and take \( a \in \mathbb{R} \). For \( f \in C(\mathbb{R}, X) \), the set \( H^+(f) = \{ f_\omega : \omega \equiv 0 \} \) of translates is a precompact subset of \( C^1_b(\mathbb{R}, X) \) if, and only if, \( f \) is asymptotically almost periodic.

Since our argument to establish Theorem 3.1 can actually be used to show somewhat more, we shall proceed by stating and proving this stronger result (Theorem 3.3). First, however, additional preparation is required.

**Definition.** Let \( X \) be a locally convex space, and fix \( a \in \mathbb{R} \). A subset \( H \) of \( C(\mathbb{R}, X) \) will be called \emph{equi-asymptotically almost periodic} if, given \( \varepsilon > 0 \) and \( q \in \text{cs}(X) \), there exists \( r \equiv a \) and a relatively dense set \( P \) in \( J_a \) such that, for each \( t \in P \) and every \( t \in J_a \), with \( t + r \equiv r \), \( q \left( h(t + r) - h(t) \right) \equiv \varepsilon \) for all \( h \in H \).

Assuming that \( X \) is a Banach space, it is well known (cf. [7]) that an almost periodic function \( f \in C(\mathbb{R}, X) \) is uniformly continuous and has a relatively compact range. Indeed, this is just the assertion that the set \( \{ f_\omega : \omega \equiv R \} \) of translates is already relatively compact with respect to the compact-open topology \( \pi \) on \( C(\mathbb{R}, X) \), as can easily be seen from Corollary 2.5.1. If 3.1 is to hold, moreover, then similar reasoning would imply that an a.a.p. function must as well be uniformly continuous with precompact range. In the next result, we confirm and extend these implications. Our terminology and other usage pertaining to almost periodic (a.p.) functions will primarily follow that in [12].

3.2. **Lemma.** Let \( a \in \mathbb{R} \), put \( T = J_a \), and assume that \( H \) is a precompact subset of \( (C(T, X), \pi) \), where \( X \) is a locally convex space. If \( H \) is equi-asymptotically almost periodic (respectively, equi-almost periodic), then \( H \) is uniformly equicontinuous on \( T \) and \( H(T) \) is precompact in \( X \).

**Proof.** We shall only consider the case \( T = J_a \) with \( H \) equi-asymptotically almost periodic; except for obvious minor changes, the proof in the a.p. case follows in exactly the same way. Now, let \( q \in \text{cs}(X) \) and fix \( \varepsilon > 0 \). We then choose \( r \equiv a \). \( \varepsilon > 0 \) and a relatively dense set \( P \) in \( J_a \) such that, for each \( t \in P \) and every \( t \in J_a \), with \( t + r \equiv r \), \( q \left( h(t + r) - h(t) \right) \equiv \varepsilon / 3 \) for all \( h \in H \), while \( [ t, t + l ] \cap P = \emptyset \) for any \( t \in J_a \). Putting \( N = \max \{ r, l \} \), let us select \( t_k \in [ kN, (k + 1) N ] \cap P \), \( k = 1, 2, \ldots \). In view of Theorem 2.1, \( H \) is equicontinuous on \( J_a \), and thus \( H \) is uniformly equicontinuous on the closed interval \([ a, 5N ] \) whereby there exists \( \delta \in (0, N/2) \) such that \( q \left( h(t_1) - h(t_2) \right) \equiv \varepsilon / 3 \) whenever \( h \in H \) and \( t_1, t_2 \in [ a, 5N ] \) with \( | t_1 - t_2 | < \delta \).
Assume, on the other hand, that $t_1, t_2 > 4N$ with $|t_1 - t_2| < \delta$. Then, choosing $k \in \mathbb{N}$ so that $t_1, t_2 \in [kN, (k + 2)N]$, let us put $s_i = t_i - \tau_{k-2}$, $i = 1, 2$. Since $s_1, s_2 \in [N, 4N]$ and $|s_1 - s_2| < \delta$,

$$g \left( h(s_1) - h(s_2) \right) \leq g \left( h(s_1 + \tau_{k-2}) - h(s_2) \right) + g \left( h(s_1) - h(s_1 + \tau_{k-2}) \right) + g \left( h(s_2) - h(s_2 + \tau_{k-2}) \right) < \varepsilon$$

for any $h \in H$, which gives us that $H$ is indeed uniformly equicontinuous on all of $J_a$. To verify that $H(J_a)$ is precompact in $X$, we again start from the equi-continuity of $H$ to obtain a finite (open) cover $\{T_i\}_{i=1}^n$ of $[a, 3N]$ and $t \in T_i$, $i = 1, \ldots, n$, such that, for every $h \in H$, $g(t, h(t)) < \varepsilon/2$ in case $t \in T_i, i \in \{1, \ldots, n\}$. If $t > 3N$, let us choose $k \in \mathbb{N}$ so that $t \in [kN, (k + 1)N]$. Setting $s = t - \tau_{k-2}$, we then have that $s \in [N, 3N]$ whence $s \in T_i$ for some $i \in \{1, \ldots, n\}$, and therefore, given any $h \in H$,

$$g \left( h(t) - h(t_i) \right) \leq g \left( h(s + \tau_{k-2}) - h(s) \right) + g \left( h(s) - h(t_1) \right) < \varepsilon;$$

i.e., $H(J_a) \subseteq \bigcup_{i=1}^n \{H(t_i) + \varepsilon B_q\}$. For each $i \in \{1, \ldots, n\}$, however, $H(t_i)$ is precompact in $X$ by Theorem 2.1, which certainly suffices to conclude the argument.

We are now in a position to establish the promised extension of Theorem 3.1.

3.3. Theorem. Let $T = \mathbb{R}$ (respectively, $T = J_a$, where $a \in \mathbb{R}$), and assume that $X$ is a locally convex space. Then the following are equivalent for a subset $H$ of $C(T, X)$:

1. (i) $H$ is precompact in $(C(T, X), \|\cdot\|)$, and
    (ii) $H$ is equi-almost periodic (respectively, equi-asymptotically almost periodic);
2. the set $\check{H} = \{h_\omega : h \in H, \omega \in \mathbb{R}\}$ (respectively, $\check{H}^+ = \{h_\omega : h \in H, \omega \in \mathbb{R}^+\}$) of translates is a precompact subset of $C_0(T, X)$.

Proof. First of all, starting from 1. in the case $T = J_a$, assume that $H$ is a $\Gamma$-precompact and equi-asymptotically almost periodic subset of $C(J_a, X)$. From Lemma 3.2, we already know that $H(J_a)$ is precompact in $X$, which immediately places us in a context where Theorem 2.2 can be applied since $\check{H}^+$ must then be contained in $C_1(T, X)$. The precompactness of $H(J_a)$ also yields that $\check{H}^+(t)$ is precompact in $X$ for each $t \in J_a$, and we can therefore conclude that $\check{H}^+$ is precompact in $C_1(T, X)$ if we can as well show that the finite covering condition 3. (ii) of Theorem 2.2 holds for $\check{H}^+$ in this setting. To this end, given $\varepsilon \in \mathcal{O}(X)$ and $\varepsilon > 0$, we proceed (as in the proof of Lemma 3.2) to choose $\tau \equiv a$, $l > 0$, and a relatively dense set $P$ in $J_a$ such that, for each $t \in P$ and every $t \in J_a$, $g(h(t) - h(t_1)) < \varepsilon/2$ for all $h \in H$, while $[t, t + l] \cap P \neq \emptyset$ for any $t \in J_a$. Further, we put $N = \max \{r, l\}$, set $\tau_0 = 0$, and fix $\tau_0 \in [kN, (k + 1)N] \cap P$, $k = 1, 2, \ldots$. Since we have that $H$ is uniformly equicontinuous on $J_a$ from Lemma 3.2, $\check{H}^+$ is also clearly uniformly equicontinuous. In particular, this allows us to obtain a finite cover $\{S_i\}_{i=1}^n$ of $[N, 3N]$ by (relatively open) subsets of $J_a$ and $s_i \in S_i$, $i = 1, \ldots, n$, such that, for every $h \in H$ and all $\omega \in \mathbb{R}^+$, $g(h_\omega(s) - h_\omega(s_i)) < \varepsilon/2$ whenever $s \in S_i$.

Hence, we put $T_i = \bigcup_{k=0}^{n-1} (S_i + \tau_k)$, $i = 1, \ldots, n$. Now, taking $t \in T_i$ for $i \in \{1, \ldots, n\}$, choose $s \in S_i$ and $k \in \mathbb{N} \cup \{0\}$ so that $t = s + \tau_k$. According to the above estimates,
we then have
\[ q(h(t) - h(t_0)) \leq q(h(s + \omega + t) - h(s + \omega)) + q(h(s) - h(t_0)) \leq \varepsilon \]
for every \( h \in H \) and all \( \omega \in \mathbb{R}^+ \). Moreover, it is straightforward to check that
\[ J_N \subseteq \bigcup_{i=1}^n T_i \]
and so \( \{T_i \cap f^{-1}(H^+, v, q, \varepsilon) : i = 1, \ldots, n\} \) covers \( J_N \), where here, of course, \( v(t) = 1 \) for each \( t \in J_a \).

Since the equicontinuity of \( H^+ \) makes it possible to trivially cover \([a, N]\) by finitely many sets of this same prescribed form, we see that (ii) of Theorem 2.2. is indeed satisfied, whereby \( H \) is precompact in \( C(J, X) \).

In case \( T = \mathbb{R} \) and \( H \) is \( \varepsilon \)-precompact and equi-almost periodic subset of \( C(\mathbb{R}, X) \), the foregoing argument can readily be adapted to show that \( H = \{h_\omega : h \in H, \omega \in \mathbb{R}\} \) is a precompact subset of \( C_p(\mathbb{R}, X) \); the details are left to the reader.

In the converse direction, assume that (i) holds. Then 1. (i) is clearly satisfied. Moreover, keeping Theorem 2.1 in mind, \( H \) (respectively, \( H^+ \)) is a precompact subset of \( C_p(\mathbb{R}, X) \) (respectively, \( C_p(J, X) \)), whereby Theorem 2.2 can be brought to bear in showing that \( H \) is equi-almost periodic (respectively, equi-asymptotically almost periodic). Fixing \( q \in C(\mathbb{R}) \) and \( \varepsilon > 0 \), we use 3. (ii) of Theorem 2.2 to obtain a finite cover \( \{T_i\}_{i=1}^n \) of \( \mathbb{R} \) (respectively, \( J_b \), where \( b = \max \{a, 1\} \)) and \( T_i \subseteq J_b, i = 1, \ldots, n \) and \( t_i \in T_i, i = 1, \ldots, n \), such that, for every \( h \in H \) and all \( \omega \in \mathbb{R} \) (respectively, all \( \omega \in \mathbb{R}^+ \)), \( q(h(t) - h(t_i)) < \varepsilon \) whenever \( t \in T_i, i \in \{1, \ldots, n\} \).

At this point, let us turn our attention to the case \( T = \mathbb{J} \). Here, setting \( r = \max \{t_1, \ldots, t_n\} \), we note that \( r \geq a \) and \( r \geq 0 \). Next, put \( P = \bigcup_{i=1}^n (T_i - t_i) \). For any \( t \in J_b, t + l \equiv b \) so that \( t + l \in T_i \) for some \( i \in \{1, \ldots, n\} \), while \( t \leq (t + l) - t_i \equiv s + l \), and hence \( P \) is relatively dense in \( J_b \). Now, given \( t \in J_b \), \( \tau \in P \), and \( \varepsilon \in P \), let us choose \( i \in \{1, \ldots, n\} \) and \( s \in T_i \) such that \( \tau = s - t_i \). Since \( t - t_i \geq 0 \), we then have
\[ q(h(t + \tau) - h(t)) = q(h(t_i + s) - h(t_i + t_i)) < \varepsilon \]
for all \( h \in H \), which serves to establish that \( H \) is equi-a.a.p. Turning to the case \( T = \mathbb{R} \), we take \( l > 2 \max \{t_1, \ldots, t_n\} \), and put \( P = \bigcup_{i=1}^n (T_i - t_i) \). Then \( P \) is relatively dense in \( \mathbb{R} \); indeed, for any \( t \in \mathbb{R}, (t + l/2) - t_i \in [t, t + l] \cap P \) when \( i \in \{1, \ldots, n\} \) is chosen so that \( t + l/2 \in T_i \). Also, given \( t \in \mathbb{R} \) and any \( \tau \in P \), essentially the same argument as that used in the preceding case readily shows that \( q(h(t + \tau) - h(t)) < \varepsilon \)
for all \( h \in H \). We thus conclude that \( H \) is equi-a.a.p. in the present instance, and this then completes the proof.

Generalization aside, the arguments used to establish Theorem 3.3 offer a straightforward and relatively simple alternative to those given by Fréchet [20] in proving the classical version of Theorem 3.1. A further advantage of our methods, and one which we shall exploit later in this section, lies in the fact that special considerations are not needed in order to treat the case of a range space \( X \) with its associated weak topology, thereby allowing application to questions concerning a notion of weak almost periodicity. At this point, however, we turn to consider another view of asymptotically almost periodic functions.
Fixing \( a \in \mathbb{R} \), if \( g : \mathbb{R} \rightarrow X \) is a continuous almost periodic function and \( h \in C_0(J_a, X) \), then the function \( f = g \mid J_a + h \) is clearly asymptotically almost periodic on the interval \( J_a \). Fréchet \([20]\), moreover, has shown that every a.a.p. function in \( C(J_a, X) \) can be so represented in case \( X \) is finite dimensional. We shall use Theorem 3.3 to place this result in a broader context.

3.4. Theorem. Assume that \( X \) is a quasicomplete locally convex space, and fix \( a \in \mathbb{R} \). Then \( f \in C(J_a, X) \) is asymptotically almost periodic if, and only if, there is a unique almost periodic function \( g \in C(\mathbb{R}, X) \) and a unique function \( h \in C_0(J_a, X) \) such that

\[ f = g \mid J_a + h. \]

Proof. In view of Corollary 1.3.6, \( C_1(\mathbb{R}, X) \) and \( C_1(\mathbb{R}, X) \) are both quasicomplete. Therefore, from the outset, let us note that precompactness can be equated with relative compactness in either of these two spaces.

Assuming that the representation (\( * \)) does hold, we have yet to show that the functions \( g \) and \( h \) are necessarily unique. To this end, first observe that an almost periodic function \( \vartheta \in C(\mathbb{R}, X) \) must be identically zero on \( \mathbb{R} \) if \( \vartheta \mid J_a \in C_0(J_a, X) \). Thus, given almost periodic functions \( g \) and \( \varphi \) in \( C(\mathbb{R}, X) \), we need only verify that \( g + \varphi \) is also almost periodic. For this, choosing any net \( \{\omega_\lambda\}_\lambda \) in \( \mathbb{R} \), we apply Theorem 3.3 (twice) to obtain a subnet \( \{\omega_\lambda(\alpha)\}_\alpha \) such that the corresponding nets of translates \( \{g_\omega(\alpha)\}_\alpha \) and \( \{\varphi_\omega(\alpha)\}_\alpha \) both converge in \( C_1(\mathbb{R}, X) \). Another application of 3.3 now gives us that \( g + \varphi \) is indeed almost periodic.

We next consider an arbitrary a.a.p. function \( f \in C(J_a, X) \). Then, for each pair \((n, q) \in N \times cs(X)\), there exists \( r(n, q) = \max \{a, n\} \) and a relatively dense set \( P(n, q) \subset J_{r(n, q)} \) such that, for any \( \tau \in P(n, q) \) and every \( t \in J_{r(n, q)} \), \( q \left( f(t + \tau) - f(t) \right) < \varepsilon \). Let us equip \( A = N \times cs(X) \) with the usual product order; i.e., given \((m, p), (n, q) \in A \), \((m, p) \preceq (n, q)\) if, and only if, \( m \preceq n \) and \( p \preceq q \). Also, for each \( \alpha = (n, q) \in A \), we choose \( \tau_\alpha \in P(n, q) \). Again by Theorem 3.3, since \( \{\tau_\alpha\}_{\alpha \in A} \) is a net in \( R^+ \), there is a subnet \( \{\tau_\alpha(\beta)\}_{\beta \in \Lambda} \) of \( \{\tau_\alpha\}_{\alpha \in A} \) for which the net of translates \( \{f_\tau(\alpha)\}_{\alpha \in A} \) converges uniformly on \( J_a \) to some \( \varphi \in C_1(J_a, X) \), and we would show that \( \varphi \) has an almost periodic extension \( g \in C(\mathbb{R}, X) \). To begin, fixing \( b \in \mathbb{R} \backslash J_a \), there exists \( \lambda_b \in A \) so that \( b + \tau_\alpha(\lambda) \equiv a \) whenever \( \lambda \in A \) and \( \lambda \equiv \lambda_b \); put \( \Lambda_b = \{\lambda \in A : \lambda \equiv \lambda_b\} \). For each \( \lambda \in \Lambda_b \), let \( \tilde{f}_\tau(\lambda) : J_b \rightarrow X \) be the extension of \( f_\tau(\lambda) \) defined by \( \tilde{f}_\tau(\lambda)(s) = f(s + \tau_\alpha(\lambda)) \), \( s \in J_b \). Then \( \{\tilde{f}_\tau(\lambda)\}_{\lambda \in \Lambda_b} \) is clearly a bounded net in \( C_1(\mathbb{R}, X) \). Now, given \( \varepsilon > 0 \) and \( g \in cs(X) \), choose \( n \in N \) with \( n = \max \{3/\varepsilon, a - b\} \), let \( \tau \in P(n, g) \), and take \( \lambda_0 \in \Lambda_b \) for which the following conditions are satisfied: (i) if \( \lambda, \mu \in \Lambda_b \) and \( \lambda \equiv \lambda_0 \), then \( q \left( f_\tau(\lambda)(t) - f_\tau(\mu)(t) \right) < \varepsilon/3 \) for all \( t \in J_a \); (ii) in case \( \lambda \in \Lambda_b \) and \( \lambda \equiv \lambda_0 \), \( b + \tau_\alpha(\lambda) \equiv \tau(n, g) \).

For any \( s \in J_b \), if \( \lambda, \mu \in \Lambda_b \) and \( \lambda, \mu \equiv \lambda_0 \), we then have that

\[
q \left( f_\tau(\lambda)(s) - f_\tau(\mu)(s) \right) = q \left( f(s + \tau_\lambda(\lambda)) - f(s + \tau_\lambda(\mu) + \tau) \right) + q \left( f_\tau(\lambda)(s + \tau) - f_\tau(\mu)(s + \tau) \right) + q \left( f(s + \tau_\lambda(\mu) + \tau) - f(s + \tau_\lambda(\mu)) \right) \leq \varepsilon.\]
Since \( \{f_{\tau_a}\}_{\lambda \in \Lambda_b} \) is thus a Cauchy net in \( C_{1p}(J_b, X) \), as well as being bounded, it converges uniformly on \( J_b \) to a function \( \varphi_0 \in C_{1p}(J_b, X) \), and we also obviously have that \( \varphi_0 | J_a = \varphi \). If \( c \in \mathbb{R} \) with \( c < 0 \), and if \( \lambda \) is any element in \( \Lambda \) for which \( c + t_{\tau_a} \equiv a \) for all \( \lambda \in \Lambda \) such that \( \lambda \equiv \lambda_c \), then the corresponding net \( \{f_{t_{\tau_a}}\}_{\lambda \in \Lambda_c} \) of extensions from \( J_a \) to \( J_c \) will converge in \( C_{1p}(J_c, X) \) to a function \( \varphi_c \) which clearly must coincide with \( \varphi_0 \) on \( J_b \). Therefore, putting

\[
g(t) = \begin{cases} 
\varphi(t), & t \in J_a \\
\varphi_0(t), & t \in \mathbb{R} \setminus J_a,
\end{cases}
\]

\( g \) is certainly a well defined and continuous function from \( \mathbb{R} \) into \( X \), while \( g | J_a = \varphi \).

To see that \( g \) is also almost periodic, again consider \( s > 0 \) and \( \varphi \in \mathcal{S}(X) \). Taking \( n \in \mathbb{N} \) so that \( n \equiv 3/s \), the set \( P = P(n, \varphi) \cap \{t \in \mathbb{R} | t \not\equiv P(n, g)\} \) is necessarily relatively dense in \( \mathbb{R} \). Now, given \( t \in \mathbb{R} \) and \( s \in P \), let us choose \( b \in \mathbb{R} \setminus J_a \) with \( b \equiv \min \{t, t + r\} \), and then take \( \lambda \in \Lambda \) such that (i) \( b + t_{\tau_a} \equiv r(n, g) + |r| \) and (ii) \( q \left( \varphi_0(t) - f(s + t_{\tau_a}) \right) < \epsilon/3 \) for every \( s \in J_b \). Since

\[
q \left( (f(t + r) - g(t)) + q \left( \varphi_0 (t + r - f(t + t_{\tau_a})) \right) \right)
\]

we conclude that \( g \) is indeed almost periodic. The proof will thus be complete once we show that \( h = f - g | J_a \) vanishes at infinity on \( J_a \), and so, one more time, let us fix \( \epsilon > 0 \) and \( \varphi \in \mathcal{S}(X) \). For \( n \in \mathbb{N} \) with \( n \equiv 2/\epsilon \), there exists \( \lambda \in \Lambda \) such that \( \alpha(\lambda) \equiv (n, g) \) and \( q \left( (t_{\tau_a}) - \varphi(t) \right) < \epsilon/2 \) for all \( t \in J_a \); we may assume that \( \alpha(\lambda) \equiv (m, p) \), where \( m \in \mathbb{N} \) and \( p \in \mathcal{S}(X) \). Thus, if \( t \in J_{r(m, p)} \), then

\[
q(h(t)) = q(f(t) - \varphi(t)) \leq p \left( f(t) - f(t + t_{\tau_a}) \right) + q \left( f(t + t_{\tau_a}) - \varphi(t) \right) < \epsilon,
\]

as desired.

**Remark.** In the argument given above, we have seen that, starting from an a.a.p. function \( f \in C(J_a, X) \), a particular net \( \{t_{\tau_a}\}_{\lambda \in \Lambda} \) in \( \mathbb{R}^+ \) could be used to construct an almost periodic function \( g \in C(R, X) \) such that \( f - g | J_a \in C_0(J_a, X) \). The method for effecting this construction, however, does not entirely depend on the special choice of the net \( \{t_{\tau_a}\}_a \); indeed, given any net \( \{t_{\tau_a}\}_a \) in \( \mathbb{R}^+ \) which is eventually in each interval of the form \( J_a, n \in \mathbb{N} \), the same construction would show that every accumulation point in \( C_{1p}(J_a, X) \) of the corresponding net \( \{t_{\tau_a}\}_a \) of translates has an almost periodic extension in \( C(R, X) \)—assuming, of course, that the conditions imposed by Theorem 3.4 are still in place. We show in [37] that the notion of an a.a.p. function has some bearing on the question considered by V. V. Nemitskii and V. V. Stepanov [31] and L. G. Deyssach and G. R. Sell [14] as to when the \( \omega \)-limit set of a motion of a dynamical system is a minimal set of a.p. motions.

As has come to our attention after this paper had been accepted for publication, a version of Theorem 3.4 and one direction of Theorem 3.1 (namely, the observa-
tion that asymptotic almost periodicity of \( f \in C_b(\mathbb{R}^+, X) \) implies that \( H^+(f) \) is relatively compact in \( C_1(\mathbb{R}^+, X) \) have been given for the case of a Banach range space \( X \) via different methods in a recent book by S. Zaidman [45].

Taken together, Theorems 2.4 and 3.3 immediately yield a result on factoring Fréchet space valued a.p. and a.a.p. functions through a reflexive Banach space.

3.5. Theorem. Let \( T = \mathbb{R} \) (respectively, \( T = J_a \), where \( a \in \mathbb{R} \)), and assume that \( X \) is a Fréchet space. Then the following are equivalent for a subset \( H \) of \( C(T, X) \):

1. (i) \( H \) is relatively compact in \( (C(T, X), \pi) \), and
   (ii) \( H \) is equi-almost periodic (respectively, equi-asymptotically almost periodic);
2. there is a compact disk \( K \) in \( X \) such that
   (i) the Banach space \( X_K \) is reflexive,
   (ii) \( H(T) \subseteq X_K \), and
   (iii) the set \( \mathcal{H} = \{ h_\omega : h \in H, \omega \in \mathbb{R} \} \) (respectively, \( \mathcal{H}^+ = \{ h_\omega : h \in H, \omega \in \mathbb{R}^+ \} \) of translates is relatively compact as a subset of \( C_1_p(T, X_K) \);
3. there is a compact disk \( K \) in \( X \) such that
   (i) \( X_K \) is a reflexive Banach space,
   (ii) \( H(T) \subseteq X_K \),
   (iii) \( H \) is relatively compact as subset of \( (C(T, X_K), \pi) \), and
   (iv) \( H \) is equi-almost periodic (respectively, equi-asymptotically almost periodic) as a subset of \( C(T, X_K) \).

Thus, in the context of Theorem 3.5, a.p. and a.a.p. functions can be factored through a reflexive Banach space so as to maintain the respective periodicity properties. Moreover, equi-a.p. (respectively, equi-a.a.p.) sets of functions can even be uniformly factored in case such sets are also equicontinuous and pointwise relatively compact.

A locally convex space \( X \) under its associated weak topology \( \sigma(X, X') \) will hereafter be denoted by \( X_w \). For \( a \in \mathbb{R} \), a function \( f : J_a \rightarrow X \) will be called weakly asymptotically almost periodic (w.a.a.p.) if \( f : J_a \rightarrow X_w \) is asymptotically almost periodic.

In case \( X \) is a Banach space, a weakly almost periodic (w.a.p.) function \( f : \mathbb{R} \rightarrow X \) is known to be a.p. if, and only if, the range of \( f \) is relatively compact in \( X \) (cf. [1, p. 45]); this holds, as well, in our general setting.

3.6. Theorem. Let \( T = \mathbb{R} \) (respectively, \( T = J_a \), where \( a \in \mathbb{R} \)), and assume that \( X \) is a locally convex space. A function \( f : T \rightarrow X \) is almost periodic (respectively, asymptotically almost periodic) if, and only if, \( f \) is weakly almost periodic (respectively, weakly asymptotically almost periodic) and \( f(T) \) is precompact in \( X \).

Proof. Assume, first of all, that \( f : T \rightarrow X \) is w.a.p. (respectively, w.a.a.p.). Then, by Theorem 3.3, the set \( \mathcal{H}(f) = \{ f_\omega : \omega \in \mathbb{R} \} \) (respectively, \( \mathcal{H}^+(f) = \{ f_\omega : \omega \in \mathbb{R}^+ \} \) of translates is a precompact subset of \( C_1(p(T, X_w) \)). Assuming, furthermore, that \( f(T) \) is precompact in \( X \), since the topology induced by \( X \) coincides with the relative weak topology on \( f(T) \) (cf. [28, p. 385]), \( f \in C(T, X) \), and so we have that
$H(f) \subseteq C_{1p}(R, X)$ (respectively, $H^+(f) \subseteq C_{1p}(J_a, X)$). The topology of $X$ also coincides with $\sigma(X, X')$ on the precompact disk $K=\overline{w(f(T))}$, whereby the respective uniformities induced on $K$ are the same (cf. [28, p. 386]). From this, we readily see that $H(f)$ (respectively, $H^+(f)$) is actually precompact in $C_{1p}(T, X)$, and a second application of Theorem 3.3 now gives us that $f$ is a.p. (respectively, a.a.p.). Necessity being obvious, this then concludes the proof.

Here, of course, following [12] (or [1]), we are regarding a function $f: R \rightarrow X$ as being weakly almost periodic in case $f: R \rightarrow X_w$ is a.p., which is the same, according to Theorem 3.3, as requiring that the set $H(f) = \{f_\omega: \omega \in R\}$ of translates be precompact in $C_{1b}(R, X_w)$. With respect to conditions under which $H(f)$ will actually be relatively compact in $C_{1b}(R, X_w)$ when $f: R \rightarrow X$ is w.a.p., the usual quasicompleteness hypothesis becomes unduly restrictive; indeed, $X_w$ is quasicompact if, and only if, $X$ is semireflexive (cf. [28, p. 299]). As is plain, however, if $H(f)$ is to be a relatively compact subset of $C_{1b}(R, X_w)$, then $f: R \rightarrow X$ must necessarily be a w.a.p. function with weakly relatively compact range. In bringing this section to a close, we will note that, in fact, this condition is also sufficient.

3.7. Theorem. Let $T=R$ (respectively, $T=J_a$, where $a \in R$), and assume that $X$ is a locally convex space. For a function $f: T \rightarrow X$, the following are equivalent:

1. (i) $f$ is weakly almost periodic (respectively, weakly asymptotically almost periodic), and

(ii) $f(T)$ is weakly relatively periodic in $X$;

2. the set $H(f) = \{f_\omega: \omega \in R\}$ (respectively, $H^+(f) = \{f_\omega: \omega \in R^+\}$) of translates is relatively compact as a subset of $C_{1b}(T, X_w)$.

Proof. The implication 2.$\Rightarrow$1., of course, is an immediate consequence of Theorem 3.3 and the fact that, for each $t \in T$, the usual point evaluation $\delta_t$ is a continuous map from $C_{1b}(T, X_w)$ into $X_w$. For sufficiency, since the argument is basically the same in either instance, we shall only present the proof in the case where $T=R$ and $f: R \rightarrow X$ is w.a.p. In this setting, as we know from Theorem 3.3, $H(f)$ is precompact in $C_{1b}(R, X_w)$. Thus, given a net $\{f_{\omega_t}\}_t$ in $H(f)$, there is a subnet $\{f_{\omega_{t_k}}\}_k$, say, which is a Cauchy net in $C_{1b}(R, X_w)$. Going to the algebraic dual $X^*$ of $X$, we put $Y = (X^*, \sigma(X^*, X'))$. Since $Y$ is complete, therefore so also is $C_{1b}(R, Y)$, and $\{f_{\omega_{t_k}}\}_k$ consequently converges in $C_{1b}(R, Y)$ to some $g \in C_{1b}(R, Y)$. Now, from 1. (ii), the closure of $f(R)$ in $X_w$, call it $C$, is $\sigma(X, X^*)$-compact, whence $\sigma(X^*, X^*)$-compact, from which we see that $g(R) \subseteq C$. This then gives us that $g \in C_{1b}(R, X_w)$, and we thereby conclude the proof.

As an addendum to the sufficiency argument from the foregoing proof of 3.7, if $f$ were also to be uniformly continuous as a mapping from $R$ into $X$, we could further conclude that $g \in C_{1b}(R, X)$; i.e., aside from being compact in $C_{1b}(R, X_w)$, the closure of $H(f)$ in $C_{1b}(R, X_w)$ would even be contained in $C_{1b}(R, X)$. Under these circumstances, the question then arises as to whether $H(f)$ would, as well, be weakly relatively compact in $C_{1b}(R, X)$. If this should indeed be the case, then a definite link will have been established between the notion of weak almost
periodicity in the sense considered here and the alternative concept as introduced by W. F. Eberlein in [16].

4. The approximation property for weighted functions spaces

Under suitable restrictions on the triple \((T, V, X)\), it is known ([3, p. 205] and [27, p. 141]) that both \(CV_0(T, X)\) and \(CV_p(T, X)\) have the approximation property whenever \(X\) does. In each instance, however, the proof is based on abstract \(\varepsilon\)-product techniques, and the question has been raised by Bierstedt [5, pp. 18–19] as to whether one could argue directly through explicit construction of the requisite finite rank operators. This has been done for \(CV_0(T)\) in the special case that \(T\) is a locally compact subspace of \(\mathbb{R}^n\) and each weight \(v \in V\) is continuous [21, p. 180]; we next show that our characterization of the precompact sets in \(CV_0(T, X)\) serves to obviate the need for such limitations. Of course, the standing hypotheses imposed on the triple \((T, V, X)\) in Section 2 are also assumed to be in effect in the present section.

4.1. Theorem. [Bierstedt [3, Theorem 5.5]]. Assume that \(X\) is a quasicomplete locally convex space. If \(X\) has the approximation property, then so does \(CV_0(T, X)\).

Proof. Let \(H\) be a relatively compact subset of \(CV_0(T, X)\). Then, given \(v \in V\), \(q \in \mathcal{C}^0(X)\), and \(\varepsilon > 0\), it will suffice to produce a continuous finite rank operator \(L\) on \(CV_0(T, X)\) such that

\[
(*) \quad v(t) \left( Lh(t) - h(t) \right) \leq \varepsilon \quad \text{for all } h \in H \text{ and any } t \in T.
\]

To do this, we first apply 2.1 (iii) in order to obtain a compact set \(K\) in \(T\) for which

\[
(a) \quad v(t) g(h(t)) \leq \varepsilon / 3 \quad \text{whenever } h \in H \text{ and } t \in T \setminus K.
\]

Since \(v\) is upper semicontinuous, \(\|v\|_K = \sup \{v(t) : t \in K\} < \infty\) and \(U = \{t \in T : v(t) < -\|v\|_K + 1\}\) is an open set containing \(K\). Thus, using the equicontinuity of \(H\) (2.1(i)), we can find a finite cover \(\{U_i\}_{i=1}^n\) of \(K\) by open subsets of \(U\) so that, taking \(i \in \{1, \ldots, n\}\) and any \(s, t \in U_i\),

\[
(b) \quad g(h(s) - h(t)) \leq \varepsilon \left[ \frac{\|v\|_K + 1}{3} \right]^{-1} \quad \text{for all } h \in H.
\]

As our next step, because \(CV_0(T)\) is a module over \(C_b(T)\) (and Condition 1.2.4 holds), [30, Lemma 2, p. 69] applies to give us a set \(\{q_i\}_{i=1}^n\) in \(CV_0(T)\) with the following properties:

\[
(c) \quad 0 \leq q_i \leq 1 \quad \text{and} \quad q_i(T \setminus U_i) = 0 \quad \text{for } i \in \{1, \ldots, n\}, \quad \text{while}
\]

\[
\sum_{i=1}^n q_i(t) \leq 1 \quad \text{for all } t \in T \quad \text{and} \quad \sum_{i=1}^n q_i(t) = 1 \quad \text{when } t \in K.
\]

Finally, fixing \(s_i \in U_i, i = 1, \ldots, n\), 2.1(ii) asserts that each \(H(s_i)\) is a relatively compact subset of \(X\), and combining this with the fact that \(X\) has the approximation property allows us to select continuous finite rank operators \(\{L_i\}_{i=1}^n\) on \(X\).
such that
(d) \( q \left( L_i(x) - x \right) < \varepsilon \left( \|v\|_{\infty} + 1 \right)^{-1} \) for all \( x \in H(s_i), \quad i = 1, ..., n \).

Let us now consider the continuous finite rank operator \( L: CV_0(T, X) \to CV_0(T, X) \) defined by

\[
L(f) = \sum_{i=1}^{n} \varphi_i(t) L_d((s_i)) .
\]

Taking \( h \in H \) and \( t \in T \), if \( t \in K \), then we have from (b), (c), and (d) that

\[
v(t) q \left( Lh(t) - h(t) \right) = v(t) q \left( \sum_{i=1}^{n} \varphi_i(t) \left( L_d(h(s_i)) - h(t) \right) \right)
\]

\[
\leq v(t) \sum_{i=1}^{n} \varphi_i(t) \left[ q \left( L_d(h(s_i)) - h(s_i) \right) + q \left( h(s_i) - h(t) \right) \right]
\]

\[
\leq \sum_{i=1}^{n} \left( \varepsilon/3 \right) \varphi_i(t) + \sum_{i=1}^{n} \left( \varepsilon/3 \right) \varphi_i(t) = 2\varepsilon/3 ;
\]

if \( t \in T \setminus K \), then (a), (b), (c), and (d) together give us that

\[
v(t) q \left( Lh(t) - h(t) \right) = v(t) q \left( \sum_{i=1}^{n} \varphi_i(t) L_d(h(s_i)) - h(t) \right)
\]

\[
\leq v(t) \sum_{i=1}^{n} \varphi_i(t) \left[ q \left( L_d(h(s_i)) - h(s_i) \right) + q \left( h(s_i) - h(t) \right) \right]
\]

\[
+ v(t) \left( 1 - \sum_{i=1}^{n} \varphi_i(t) \right) q(h(t))
\]

\[
\leq 2 \sum_{i=1}^{n} \left( \varepsilon/3 \right) \varphi_i(t) + v(t) q(h(t)) < \varepsilon .
\]

Since \( L \) is thus such that (*) holds, the proof is complete.

Before considering the approximation property in \( CV_p(T, X) \), a moment of reflection on the foregoing proof may be appropriate. The key to this direct approach lies in condition (iii) of Theorem 2.1, which essentially allowed us to reduce the problem to one over a compact domain space. From this point, a standard partition of unity argument then led to the desired conclusion.

Unfortunately, Theorem 2.2 offers no such analogous reduction in the case of \( CV_p(T, X) \), nor is anything in this direction apparent from an examination of the classical examples, and we have not as yet been able to circumvent this difficulty in constructing a direct proof of G. KLEINSTÜCK's general result [27, Korollar 3.6]. Nonetheless, our methods do apply in the setting considered by BIERENTEDT [4, 5] in his study of the approximation property in \( CV_k(T) \); we illustrate this by establishing the following version of [4, Theorem 1.1] and [5, Theorem 21].

4.2. Theorem. In addition to the standing hypothesis, let \( (T, V, X) \) be such that \( X \) is quasicomplete and the following two conditions are satisfied:
(i) for each \( v \in V \), the restriction \( v \mid \text{spt}(v) \) (of \( v \) to its support) is continuous;
(ii) given \( f \in CV_p(T) \) and \( v \in V \), there exists \( g \in C_b(T) \) so that \( g \mid \text{spt}(v) = (fv) \mid \text{spt}(v) \).

Then \( CV_p(T, X) \) has the approximation property whenever \( X \) has the approximation property.

Remark. Conditions (i) and (ii) of Theorem 4.2 imply (the somewhat more technical) Bedingung (A) under which, in [4], Bierstedt established that \( CV_b(T) \) does have the approximation property. While not as general, (i) and (ii) of 4.2 were considered in [5] as being more convenient in terms of applications; these requirements are clearly satisfied, of course, in case each weight \( v \in V \) is continuous or, in the presence of (i), if either \( T \) is normal or each \( v \in V \) has compact support (cf. [4]).

Proof of Theorem 4.2. Let \( H \) be a relatively compact subset of \( CV_p(T, X) \), and fix \( v \in V, g \in C_b(X) \), and \( \varepsilon > 0 \). As before, we would obtain a continuous finite rank operator \( L \), this time on \( CV_p(T, X) \), such that
\[
(*) \quad \nu(t) q \left( L h(t) - h(t) \right) < \varepsilon \quad \text{for all } \ h \in H \text{ and any } t \in T .
\]
To this end, if \( f \in CV_p(T, X) \), then \( f_p = (vf) \mid \text{spt}(v) \) belongs to \( C_p(\text{spt}(v), X) \) in view of (i); we claim that \( f_p \) has a continuous extension to the closure of \( \text{spt}(v) \) in the Stone–Čech compactification of \( T \). Indeed, let \( S \) denote the closure of \( \text{spt}(v) \) in \( \beta T \), fix \( p \in S \setminus \text{spt}(v) \), and choose a net \( \{ t_\alpha \} \) in \( \text{spt}(v) \) that converges to \( p \). For \( x' \in X' \), \( x' \circ f \in CV_b(T) \), and so, by (ii), there exists \( g \in C_b(T) \) such that \( g \mid \text{spt}(v) = x' \circ f \) whence \( \{ x'(v(t_\alpha) f(t_\alpha)) \} \) converges to \( g(p) \), where \( g \) denotes the continuous extension of \( g \) to \( \beta T \). Since \( v(T) \) is relatively compact in \( X \), we can thus conclude that the net \( \{ v(t_\alpha) f(t_\alpha) \} \) converges (in the topology of \( X \)) to some \( x_p \in X \). Moreover, it is obvious that \( x_p \) does not depend on the choice of the net in \( \text{spt}(v) \) converging to \( p \), whereby putting \( f_p(p) = x_p \) for each \( p \in S \setminus \text{spt}(v) \) certainly yields the desired extension. Having established our claim, let us put \( l_p(f) = f_p \) for each \( f \in CV_p(T, X) \), in which case \( l_p \) is a continuous linear mapping from \( CV_p(T, X) \) into \( C_1_p(S, X) \). Since this means that \( l_p(H) \) is a relatively compact subset of \( C_1_p(S, X) \), we can apply Theorem 2.2 (and the Remark following its proof) to obtain a finite open cover \( \{ U_i \}_{i=1}^n \) of \( S \) such that, for \( i \in \{1, \ldots, n\} \) and any \( s, t \in U_i \),
\[
(a) \quad q(l_p(h)(s) - l_p(h)(t)) < \varepsilon/3 \quad \text{for all } \ h \in H .
\]
Without loss of generality, we may assume that there exists \( m \in N \) with \( m \leq n \) for which, given \( i \in \{1, \ldots, m\} \) and \( s \in U_i \), \( q(l_p(h)(s)) = 0 \) for some \( h \in H \), while there exists \( s \in U_i \) such that
\[
(b) \quad q(l_p(h)(s)) = 0 \quad \text{for all } \ h \in H
\]
in case \( m < n \) and \( i \in \{m+1, \ldots, n\} \); of course, \( \{ U_i \}_{i=1}^m \) is an open cover of the compact set \( K = S \setminus \bigcup_{i=m+1}^n U_i \). Next, if \( \varphi \in C(S) \), then \( \varphi \) has a continuous extension \( \varphi^* \) to \( \beta T \) whence \( \psi = \varphi^* | T \in C_b(T) \). Consequently, if \( f \in CV_p(T, X) \), then so also is \( \psi f \), and we have that \( l_p(\psi f) = \varphi^* l_p(f) \); i.e., \( E_\psi = \{ l_p(f) : f \in CV_p(T, X) \} \) is a module.
over $C(S)$. If we now put $M = \{x' \circ l_i(f) : x' \in X', f \in CV_p(T, X)\}$, then the linear span of $M$ is a vector subspace of $C(S)$ which likewise is a module over $C(S)$. Thus, applying [30, 23. Lemma 2], we can find finite sequences $(x'_k)_{k=1}^{n_i}$ in $X'$ and $(f_k)_{k=1}^{n_i}$ in $CV_p(T, X)$, $i = 1, ..., m$, so that, putting
\[ q_i = \sum_{k=1}^{n_i} x'_k \circ l_i(f_k) \text{ for } i = 1, ..., m, \]

(c) \[ 0 \leq q_i \leq 1 \text{ and } q_i(S \setminus \Omega_i) = 0 \text{ if } i \in \{1, ..., m\}, \]
while
\[ \sum_{i=1}^{m} q_i(s) \leq 1 \text{ for all } s \in S \text{ and } \sum_{i=1}^{m} q_i(s) = 1 \text{ when } s \in K. \]

Finally, fixing $s_i \in \Omega_i \cap \text{spt}(v)$, $i = 1, ..., m$, 2.2 also asserts that $l_i(H)(s_i)$ is relatively compact in $X$, which means that we can choose continuous finite rank operators $\{L_i\}_{i=1}^{m}$ on $X$ such that

(d) \[ q(L_i(x) - x) \leq \varepsilon/3 \text{ for all } x \in l_i(H)(s_i), \quad i = 1, ..., m. \]

Let us consider the continuous finite rank operator $L$ on $CV_p(T, X)$ defined by
\[ Lf(x) = \sum_{i=1}^{m} \psi(t_i) L_i(v(s_i) f(s_i)) \]
where $\psi_i = \sum_{k=1}^{n_i} x'_k \circ f_k$, $i = 1, ..., m$. Taking $h \in H$ and $t \in T$, if $t \in K$, then we have from (a), (c), and (d) that
\[ v(t) q(Lh(t) - h(t)) = v(t) q \left( \sum_{i=1}^{m} \psi(t) L_i(v(s_i) h(s_i)) - \sum_{i=1}^{m} q_i(t) h(t) \right) \]
\[ \leq v(t) \sum_{i=1}^{m} \psi(t) \left[ q(L_i(v(s_i) h(s_i)) - v(s_i) h(s_i)) \right. \]
\[ + q \left( v(s_i) h(s_i) - v(t) h(t) \right) \]
\[ \leq \sum_{i=1}^{m} (\varepsilon/3) q_i(t) + \sum_{i=1}^{m} (\varepsilon/3) q_i(t) = 2\varepsilon/3; \]

if $t \in \text{spt}(v) \setminus K$, then $t \in \Omega_f$ for some $f \in \{m + 1, ..., n\}$, whereby (a), (b), (c), and (d) combine to give us that
\[ v(t) q(Lh(t) - h(t)) = v(t) q \left( \sum_{i=1}^{m} \psi(t) L_i(v(s_i) h(s_i)) - h(t) \right) \]
\[ \leq v(t) \sum_{i=1}^{m} \psi(t) \left[ q(L_i(v(s_i) h(s_i)) - v(s_i) h(s_i)) \right. \]
\[ + q \left( v(s_i) h(s_i) - v(t) h(t) \right) \]
\[ + v(t) \left( 1 - \sum_{i=1}^{m} q_i(t) \right) q(h(t)) \]
\[ \leq 2 \sum_{i=1}^{m} (\varepsilon/3) q_i(t) + q(v(t) h(t) - l_0(h)(s_f)) - \varepsilon. \]
We have thus shown that $L$ is such that (*) is satisfied, and the proof is thereby complete.

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