On a conjecture of V. V. Shchigolev

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Abstract

V. V. Shchigolev has proven that over any infinite field $k$ of characteristic $p > 2$, the $T$-space generated by $G = \{x_1^p, x_2^p, \ldots\}$ is finitely based, which answered a question raised by A. V. Grishin. Shchigolev went on to conjecture that every infinite subset of $G$ generates a finitely based $T$-space. In this paper, we prove that Shchigolev’s conjecture was correct by showing that for any field of characteristic $p > 2$, the $T$-space generated by any subset $\{x_1^p x_2^p, x_3^p, \ldots\}$, $i_1 < i_2 < i_3 < \cdots$, of $G$ has a $T$-space basis of size at most $i_2 - i_1 + 1$.

1 Introduction

In [2] (and later in [3], the survey paper with V. V. Shchigolev), A. V. Grishin proved that in the free associative algebra with countably infinite generating set $\{x_1, x_2, \ldots\}$ over an infinite field of characteristic 2, the $T$-space generated by the set $\{x_1^2, x_2^2, \ldots\}$ is not finitely based, and he raised the question as to whether or not over a field of characteristic $p > 2$, the $T$-space generated by $\{x_1^p, x_2^p, \ldots\}$ is finitely based. This was resolved by V. V. Shchigolev in [4], wherein he proved that over an infinite field of characteristic $p > 2$, this $T$-space is finitely based. Shchigolev then raised the question in [4] as to whether every infinite subset of $\{x_1^p, x_2^p, \ldots\}$ generates a finitely based $T$-space. In this paper, we prove that over an arbitrary field of characteristic $p > 2$, every subset of $\{x_1^p, x_2^p, \ldots\}$ generates a $T$-space that can be generated as a $T$-space by finitely many elements, and we give an upper bound for the size of a minimal generating set.

Let $p$ be a prime (not necessarily greater than 2) and let $k$ denote an arbitrary field of characteristic $p$. Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set, and let $k_0(X)$ denote the free associative $k$-algebra over the set $X$.

Definition 1.1. For any positive integer $d$, let

$$S^{(d)} = S^{(d)}(x_1, x_2, \ldots, x_d) = \sum_{\sigma \in \Sigma_d} \prod_{i=1}^{d} x_{\sigma(i)},$$

where $\Sigma_d$ is the symmetric group on $d$ letters. Then define $S^{(d)}_1 = \{ S^{(d)} \}$, the $T$-space generated by $\{ S^{(d)} \}$, and for all $n \geq 1$, $S^{(d)}_{n+1} = (S^{(d)}_n S^{(d)}_1)^{S}$. 

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Let $I : i_1 < i_2 < \cdots$ be a sequence of positive integers (finite or infinite), and then for each $n \geq 1$, let $R_{n,I}^{(d)} = \sum_{j=1}^{n} S_{ij}^{(d)}$. When the sequence $I$ is understood, we shall usually write $R_{n,I}^{(d)}$ instead of $R_{n,I}^{(d)}$. Finally, let $R_{\infty,I}^{(d)}$ (even if the sequence is finite) denote the $T$-space generated by $\{ S_{ij}^{(d)} \mid i \in I \}$. We shall prove that $R_{\infty,I}^{(d)}$ has a $T$-space basis of size at most $i_2 - i_1 + 1$.

**Definition 1.2.** Let $H_1 = \{ x_1^p \}^S$, and for each $n \geq 1$, let $H_{n+1} = (H_n H_1)^S$.

Then for any positive integer $n$, let $L_{n,I} = \sum_{j=1}^{n} H_{ij}$, and let $L_{\infty,I}$ denote the $T$-space generated by $\{ h_i \mid i \in I \}$. We prove that $L_{\infty,I}$ is finitely generated as a $T$-space, with a $T$-space basis of size at most $i_2 - i_1 + 1$. In particular, this proves that Shchigolev’s conjecture is valid.

## 2 Preliminaries

In this section, $k$ denotes an arbitrary field of characteristic an arbitrary prime $p$, and $V_i$, $i \geq 1$, denotes a sequence of $T$-spaces of $k_0(X)$ satisfying the following two properties:

(i) $(V_i V_j)^S = V_{i+j}$;

(ii) for all $m \geq 1$, $V_{2m+1} \subseteq V_{m+1} + V_1$.

**Lemma 2.1.** For any integers $r$ and $s$ with $0 < r < s$, $V_{s+t(s-r)} \subseteq V_r + V_s$ for all $t \geq 0$.

**Proof.** The proof is by induction on $t$. There is nothing to show for $t = 0$. For $t = 1$, let $m = s - r$ in (ii) to obtain that $V_{2s-2r+1} \subseteq V_{s-r+1} + V_1$, then multiply by $V_{r-1}$ to obtain $V_{r-1} V_{2s-2r+1} \subseteq V_{r-1} V_{s-r+1} + V_{r-1} V_1 \subseteq (V_{r-1} V_{s-r+1})^S + (V_{r-1} V_1)^S = V_s + V_r$. But then $V_{2s-r} = (V_{r-1} V_{2s-2r+1})^S \subseteq V_s + V_r$, as required.

Suppose now that $t \geq 1$ is such that the result holds. Then $V_{s+t(s-r)} = (V_{s+t(s-r)} V_{s-r})^S \subseteq ((V_s + V_r) V_{s-r})^S = V_{2s-r} + V_s \subseteq V_r + V_s + V_s = V_r + V_s$. The result follows now by induction. □

For any increasing sequence $I : i_1 < i_2 < \cdots$ of positive integers, we shall refer to $i_2 - i_1$ as the initial gap of $I$.

**Proposition 2.1.** For any increasing sequence $I = \{ i_j \}_{j \geq 1}$ of positive integers, there exists a set $J$ of size at most $i_2 - i_1 + 1$ with entries positive integers such that the following hold:

(i) $1, 2 \in J$;

(ii) $\sum_{j=1}^{\infty} V_{ij} = \sum_{j \in J} V_{ij}$.

**Proof.** The proof of the proposition shall be by induction on the initial gap. By Lemma 2.1, for a sequence with initial gap 1, we may take $J = \{ i_1, i_2 \}$ . Suppose now that $l > 1$ is an integer for which the result holds for all increasing
sequences with initial gap less than \( l \), and let \( i_1 < i_2 < \cdots \) be a sequence with initial gap \( i_2 - i_1 = l \). If for all \( j \geq 3 \), \( V_{i_j} \subseteq V_{i_1} + V_{i_2} \), then \( J = \{ 1, 2 \} \) meets the requirements, so we may suppose that there exists \( j \geq 3 \) such that \( V_{i_j} \) is not contained in \( V_{i_1} + V_{i_2} \). By Lemma 2.1, this means that there exists \( j \geq 3 \) such that \( i_j \notin \{ i_2 + ql \mid q \geq 0 \} \). Let \( r \) be least such that \( i_r \notin \{ i_2 + ql \mid q \geq 0 \} \), so that there exists \( t \) such that \( i_2 + tl < i_r < i_2 + (t + 1)l \). Form a sequence \( I' \) from \( I \) by first removing all entries of \( I \) up to (but not including) \( i_r \), then prepend the integer \( i_2 + tl \). Thus \( i'_1 \), the first entry of \( I' \), is \( i_2 + tl \), while for all \( j \geq 2 \), \( i'_j = i_r + (i_2 + tl) \leq l - 1 \). By hypothesis, there exists a subset \( J' \) of size at most \( i'_2 - i'_1 + 1 \leq l = i_2 - i_1 \) that contains 1 and 2 and is such that \( \sum_{j=1}^{\infty} V_{i'} = \sum_{j \in J'} V_{j} \). Set

\[
J = \{ 1, 2 \} \cup \{ r + j - 2 \mid j \in J', \ j \geq 2 \}.
\]

Then \( |J| = |J'| + 1 \leq i_2 - i_1 + 1 \) and

\[
V_{i_2 + tl} + \sum_{j=r}^{\infty} V_{i_j} = \sum_{j=1}^{\infty} V_{j} = \sum_{j \in J'} V_{j} = V_{i_2 + tl} + \sum_{j \geq 2} V_{j}
\]

and by Lemma 2.1, \( V_{i_2 + tl} \subseteq V_{i_1} + V_{i_2} \), so

\[
V_{i_1} + V_{i_2} + \sum_{j=r}^{\infty} V_{i_j} = V_{i_1} + V_{i_2} + V_{i_2 + tl} + \sum_{j=r}^{\infty} V_{i_j} = V_{i_1} + V_{i_2} + V_{i_2 + tl} + \sum_{j \geq 2} V_{j}
\]

\[
= V_{i_1} + V_{i_2} + \sum_{j \geq 3} V_{i_j}.
\]

Finally, the choice of \( r \) implies that

\[
\sum_{j \in J} V_{i_j} = V_{i_1} + V_{i_2} + \sum_{j \in J} V_{i_j} = V_{i_1} + V_{i_2} + \sum_{j=r}^{\infty} V_{i_j} = \sum_{j=1}^{\infty} V_{i_j}.
\]

This completes the proof of the inductive step. \( \square \)

We remark that in Proposition 2.1, it is possible to improve the bound from \( i_2 - i_1 + 1 \) to \( 2(\log_2(2(i_2 - i_1))) \).

In the sections to come, we shall examine some important situations of the kind described above.

### 3 The \( R_n^{(d)} \) sequence

We shall have need of certain results that first appeared in [1]. For completeness, we include them with proofs where necessary. In this section, \( p \) denotes an arbitrary prime, \( k \) an arbitrary field of characteristic \( p \), and \( d \) an arbitrary positive integer.

The proof of the first result is immediate.
Lemma 3.1. Let $d$ be a positive integer. Then
\[
S^{(d+1)}(x_1, x_2, \ldots, x_{d+1}) = \sum_{i=1}^{d+1} S^{(d)}(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{d+1})x_i
\]  
(1)
\[
= S^{(d)}(x_1, x_2, \ldots, x_d)x_{d+1} + \sum_{i=1}^{d} S^{(d)}(x_1, x_2, \ldots, x_i, \ldots, x_{d}) \quad (2)
\]
\[
= x_{d+1}S^{(d)}(x_1, x_2, \ldots, x_d) + \sum_{i=1}^{d} S^{(d)}(x_1, x_2, \ldots, x_i, x_{d+1}, \ldots, x_d). \quad (3)
\]

Corollary 3.1. Let $d$ be any positive integer. Then modulo $S^{(d)}_1$,
\[
S^{(d+1)}(x_1, x_2, \ldots, x_{d+1}) \equiv S^{(d)}(x_1, \ldots, x_d)x_{d+1} \equiv x_{d+1}S^{(d)}(x_1, \ldots, x_d).
\]

Proof. This is immediate from (2) and (3) of Lemma 3.1. \qed

We remark that Corollary 3.1 implies that for every $u \in S^{(d)}_1$ and $v \in k_0(X)$, $[u, v] \in S^{(d)}_1$. While we shall not have need of this fact, we note that in \cite{1}, Shchigolev proves that if the field is infinite, then for any $T$-space $V$, if $v \in V$, then $[v, u] \in V$ for any $u \in k_0(X)$.

The next proposition is a strengthened version of Proposition 2.1 of \cite{1}.

Proposition 3.1. For any $u, v \in k_0(X)$,
\[(i) \ (S^{(d)}_1uv)^S \subseteq S^{(d)}_1 + (S^{(d)}_1u)^S + (S^{(d)}_1v)^S; \text{ and}\]
\[(ii) \ (uvS^{(d)}_1)^S \subseteq S^{(d)}_1 + (uS^{(d)}_1)^S + (vS^{(d)}_1)^S.\]

Proof. We shall prove the first statement; the proof of the second is similar and will be omitted. By (1) of Lemma 3.1
\[
\sum_{i=1}^{d} S^{(d)}(x_1, \ldots, \hat{x}_i, \ldots, x_{d+1})x_i = S^{(d+1)}(x_1, \ldots, x_{d+1}) - S^{(d)}(x_1, \ldots, x_d)x_{d+1}
\]
and by (2) of Lemma 3.1 $S^{(d+1)}(x_1, \ldots, x_{d+1}) - S^{(d)}(x_1, \ldots, x_d)x_{d+1} \in S^{(d)}_1$. Let $v \in k_0(X)$. Then
\[
S^{(d)}(x_2, \ldots, x_{d+1})v + \sum_{i=2}^{d} S^{(d)}(x_1, \ldots, \hat{x}_i, \ldots, x_{d+1})x_i v
\]
\[
= \sum_{i=1}^{d} S^{(d)}(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{d+1})x_i v \in (S^{(d)}_1v)^S.
\]

Now for each $i = 2, \ldots, d$, we use two applications of Corollary 3.1 to obtain
\[
S^{(d)}(x_1, \ldots, \hat{x}_i, \ldots, x_{d+1})x_i v \equiv S^{(d+1)}(x_1, \ldots, \hat{x}_i, \ldots, x_{d+1}, x_i v)
\]
\[
\equiv S^{(d)}(x_2, \ldots, \hat{x}_i, \ldots, x_{d+1}, x_i v)x_1 \mod S^{(d)}_1.
\]
Thus
\[ S^{(d)}(x_2, \ldots, x_{d+1})x_1v + \left(\sum_{i=2}^{d} S^{(d)}(x_2, \ldots, \hat{x}_i, \ldots, x_i)v\right)x_1 \in (S_1^{(d)}v)^S + S_1^{(d)}. \]

Thus for \( u \in k_0(X) \), we obtain \( S^{(d)}(x_2, \ldots, x_{d+1})uv \in (S_1^{(d)}u)^S + (S_1^{(d)}v)^S + S_1^{(d)}, \)
and so
\[ (S_1^{(d)}uv)^S \subseteq (S_1^{(d)}u)^S + (S_1^{(d)}v)^S + S_1^{(d)}, \]
as required. \( \square \)

**Corollary 3.2.** Let \( d \) be any positive integer. Then the sequence \( S^{(d)}_n \), \( n \geq 1 \), satisfies

(i) For all \( m, n \geq 1 \), \( (S_m^{(d)}S_n^{(d)})^S = S_{m+n+1}^{(d)} \).

(ii) For all \( m \geq 1 \), \( S_{2m+1}^{(d)} \subseteq S_m^{(d)} + S_1^{(d)} \).

**Proof.** The first statement follows immediately from Definition 3.1 by an elementary induction argument. For the second statement, let \( m \geq 1 \). Then by Proposition 3.1 for any \( u, v \in S_m^{(d)} \), \( (S_1^{(d)}uv)^S \subseteq S_1^{(d)} + (S_1^{(d)}u)^S + (S_1^{(d)}v)^S \), which implies that \( (S_1^{(d)}S_mS_n^{(d)})^S \subseteq S_1^{(d)} + (S_1^{(d)}S_m)^S \). By (i), this yields \( S_{2m+1}^{(d)} \subseteq S_1^{(d)} + S_m^{(d)} \), as required. \( \square \)

**Theorem 3.1.** Let \( I \) denote any increasing sequence of positive integers with initial gap \( g \). Then \( R^{(d)}_{\infty, I} \) is finitely based, with a \( T \)-space basis of size at most \( g + 1 \).

**Proof.** Denote the entries of \( I \) in increasing order by \( i_j, j \geq 1 \). By Corollary 3.2 and Proposition 2.1 there exists a set \( J \) of positive integers with \( |J| \leq i_2 - i_1 + 1 \) and \( R^{(d)}_{\infty, I} = R^{(d)}_{n, J} = \sum_{j \in J} S_j^{(d)} \). Since for each \( i \), the \( T \)-space \( S_i^{(d)} \) has a basis consisting of a single element, the result follows. \( \square \)

4 The \( L_n \) sequence

We shall make use of the following well known result. An element \( u \in k_0(X) \) is said to be essential if \( u \) is a linear combination of monomials with the property that each variable that appears in any monomial appears in every monomial.

**Lemma 4.1.** Let \( V \) be a \( T \)-space and let \( f \in V \). If \( f = \sum f_i \) denotes the decomposition of \( f \) into its essential components, then \( f_i \in V \) for every \( i \).

**Proof.** We induct on the number of essential components, with obvious base case. Suppose that \( n > 1 \) is an integer such that if \( f \in V \) has fewer than \( n \) essential components, then each belongs to \( V \), and let \( f \in V \) have \( n \) essential components. Since \( n > 1 \), there is a variable \( x \) that appears in some but not all essential components of \( f \). Let \( z_x \) and \( f_x \) denote the sum of the essential components of \( f \) in which \( x \) appears, respectively, does not appear. Then evaluate
at $x = 0$ to obtain that $f_x = f|_{x=0} \in V$, and thus $z_x = f - f_x \in V$ as well. By hypothesis, each essential component of $f_x$ and of $z_x$ belongs to $V$, and thus every essential component of $f$ belongs to $V$, as required. 

**Corollary 4.1.** $S_1^{(p)} \subseteq H_1$.

**Proof.** $S^{(p)}$ is one of the essential components of $(x_1 + x_2 + \cdots + x_p)^p$, and since $(x_1 + x_2 + \cdots + x_p)^p \in H_1$, it follows from Lemma 4.1 that $S^{(p)} \in H_1$. Thus $S_1^{(p)} \subseteq H_1$.

**Corollary 4.2.** For every $m \geq 1$, $S_m^{(p)} \subseteq H_m$.

**Proof.** The proof is an elementary induction, with Corollary 4.1 providing the base case.

**Corollary 4.3.** For any $u \in H_1$ and any $v \in k_0(X)$, $[u, v] \in H_1$.

**Proof.** It suffices to observe that

$$[x^p, v] = \sum_{i=0}^{p} x^i [x, v] x^{p-i} = \frac{1}{(p-1)!} S^{(p)}(x, \ldots, x, [x, v]),$$

which belongs to $H_1$ by virtue of Corollary 4.1.

We remark again that in [3], Shchigolev proves that if $k$ is infinite, then every $T$-space in $k_0(X)$ is closed under commutator in the sense of Corollary 4.3. Since we have not required that $k$ be infinite, we have provided this closure result (see also Lemma 4.1 below).

**Lemma 4.2.** For any $m, n \geq 1$, $(H_m H_n)^S = H_{m+n}$.

**Proof.** The proof is by an elementary induction on $n$, with Definition 1.2 providing the base case.

**Lemma 4.3.** For any $m \geq 1$, $(S_1^{(p)} H_{2m})^S \subseteq H_1 + H_{m+1}$ and $(H_{2m} S_1^{(p)})^S \subseteq H_1 + H_{m+1}$.

**Proof.** By Proposition 4.1 (i), for any $u, v \in H_m$, we have $S_1^{(p)} uv \subseteq S_1^{(p)} + (S_1^{(p)} u)^S + (S_1^{(p)} v)^S$. By Corollary 4.2, this gives $S_1^{(p)} H_m H_m \subseteq H_1 + (H_1 H_m)^S$, and then from Lemma 4.2, we obtain $S_1^{(p)} H_{2m} \subseteq H_1 + H_{m+1}$. The proof of the second part is similar.

**Lemma 4.4.** Let $m \geq 1$. For every $u \in H_m$ and $v \in k_0(X)$, $[u, v] \in H_m$.

**Proof.** The proof is by induction on $m$, with Corollary 4.3 providing the base case. Suppose that $m \geq 1$ is such that the result holds. It suffices to prove that for any $v \in k_0(X)$, $[x_1^p x_2^p \cdots x_m^p x_{m+1}^p, v] \in H_{m+1}$. We have

$$[x_1^p x_2^p \cdots x_m^p x_{m+1}^p, v] = [x_1^p x_2^p \cdots x_m^p, v] x_{m+1}^p + x_1^p x_2^p \cdots x_m^p [x_{m+1}^p, v].$$
By hypothesis, \([x_1^p x_2^p \cdots x_n^p, v] \in H_m\), while \(x_{m+1}^p \in H_1\) and thus by Corollary 4.3 \([x_{m+1}^p, v] \in H_1\) as well. Now by definition, \([x_1^p x_2^p \cdots x_n^p, v] x_{m+1}^p \in H_{m+1}\) and \(x_1^p x_2^p \cdots x_m^p x_{m+1}^p \in H_{m+1}\), which completes the proof of the inductive step. \(\square\)

**Lemma 4.5.** Let \(m \geq 1\). Then \(H_i S^{(p)} H_{2m-i} \subseteq H_1 + H_{m+1}\) for all \(i\) with \(1 \leq i \leq 2m - 1\).

**Proof.** Let \(m \geq 1\). We consider two cases: \(2m - i \geq m\) and \(2m - i < m\). Suppose that \(2m - i \geq m\), and let \(u \in H_i, w \in H_{m-1}\) and \(z \in H_{m-i+1}\). Then \(u S^{(p)} w z = ([u, S^{(p)} w] + S^{(p)} w u) z = [u, S^{(p)} w] z + S^{(p)} w u z\). Since \(u \in H_i\), it follows from Lemma 4.4 that \([u, S^{(p)} w] \in H_i\). But then by Lemma 4.2 \([u, S^{(p)} w] z \in H_{i+m-1} = H_{m+1}\). As well, by Corollary 4.1 and Lemma 4.2, \(S^{(p)} w u z \in S_1^{(p)} H_{m-1+i+m-i+1} = S_1^{(p)} H_{2m}\), and by Lemma 4.3 \(S^{(p)} H_{2m} \subseteq H_1 + H_{m+1}\). Thus \(u S^{(p)} w z \in H_1 + H_{m+1}\). This proves that \(H_i S^{(p)} H_{m-1} H_{m-i+1} \subseteq H_1 + H_{m+1}\), and so by Lemma 4.2 \(H_i S^{(p)} H_{2m-i} = H_i S^{(p)} (H_{m-1} H_{m-i+1})^S \subseteq H_1 + H_{m+1}\). The argument for the case when \(2m - i < m\) is similar and is therefore omitted. \(\square\)

**Proposition 4.1.** Let \(p > 2\). Then for every \(m \geq 1\), \(H_{2m+1} \subseteq H_1 + H_{m+1}\).

**Proof.** First, consider the expansion of \((x + y)^p\) for any \(x, y \in k_0(X)\). It will be convenient to introduce the following notation. Let \(J_p = \{1, 2, \ldots, p\}\). For any \(J \subseteq J_p\), let \(P_J = \prod_{i=1}^{p} z_i\), where for each \(i, z_i = x\) if \(i \in J\), otherwise \(z_i = y\). As well, for each \(i\) with \(1 \leq i \leq p - 1\), we shall let \(S^{(p)}(x, y; i) = S^{(p)}(x, x, \ldots, x, y, y, \ldots, y)\). Observe that \(S^{(p)}(x, y; i) = i!(p - i)! \sum_{|J| = i} P_J\).

We have
\[
(x + y)^p = \sum_{i=0}^{p} \sum_{J \subseteq J_p, |J| = i} P_J = y^p + x^p + \sum_{i=1}^{p-1} \frac{1}{i!(p - i)!} S^{(p)}(x, y; i).
\]

Let \(u = \sum_{i=1}^{p-1} \frac{1}{i!(p - i)!} S^{(p)}(x, y; i)\), so that \((x + y)^p = x^p + y^p + u\), and note that \(u \in S_1^{(p)}\). Then \((x + y)^{2p} = y^{2p} + x^{2p} + 2 x^p y^p + [y^p, x^p] + u^2 + (x^p + y^p) u + u(x^p + y^p)\).

Since \((x + y)^{2p}, x^{2p}, y^{2p}\), and, by Lemma 4.2 \([y^p, x^p]\) all belong to \(H_1\), it follows (making use of Corollary 4.2 where necessary) that \(2 x^p y^p \in H_1 + H_1 S_1^{(p)} + S_1^{(p)} H_1\).

Consequently, for any \(m \geq 1\),
\[
x_1^p \prod_{i=1}^{m} (2 x_i^p x_{2i+1}^p) \in H_1 (H_1 + H_1 S_1^{(p)} + S_1^{(p)} H_1)^m.
\]

By Corollary 4.1 Lemma 4.2 and Lemma 4.5 \(H_1 (H_1 + H_1 S_1^{(p)} + S_1^{(p)} H_1)^m \subseteq H_1 + H_{m+1}\), and since \(p > 2\), it follows that \(\prod_{i=1}^{2m+1} x_i^p \in H_1 + H_{m+1}\). Thus \(H_{2m+1} \subseteq H_1 + H_{m+1}\), as required. \(\square\)
Theorem 4.1 (Shchigolev’s conjecture). Let $p > 2$ be a prime and $k$ a field of characteristic $p$. For any increasing sequence $I = \{ i_j \}_{j \geq 1}$, $L_{\infty, I}$ is a finitely based $T$-space of $k_0\langle X \rangle$, with a $T$-space basis of size at most $i_2 - i_1 + 1$.

Proof. By Lemma 4.2 and Proposition 4.1, the sequence $H_n$ of $T$-spaces of $k_0\langle X \rangle$ meets the requirements of Section 2. Thus by Proposition 2.1, for any increasing sequence $I = \{ i_j \}_{j \geq 1}$ of positive integers, there exists a set $J$ of positive integers such that $|J| \leq i_2 - i_1 + 1$ and $L_{\infty, I} = \sum_{j=1}^{\infty} H_{i_j} = \sum_{j \in J} H_{i_j}$. Since for each $i$, $H_i$ has $T$-space basis $\{ x_1^{p_1} x_2^{p_2} \cdots x_i^{p_i} \}$, it follows that $L_{\infty, I}$ has a $T$-space basis of size $|J| \leq i_2 - i_1 + 1$.

Shchigolev’s original result was that for the sequence $I^+$ of all positive integers, $L_{\infty, I^+}$ is a finitely-based $T$-space, with a $T$-space basis of size at most $p$. It was then shown in [1], a precursor to this work, that $L_{\infty, I^+}$ has in fact a $T$-space basis of size at most 2 (the bound of Theorem 4.1 since $i_1 = 1$ and $i_2 = 2$).

It is also interesting to note that the results in this paper apply to finite sequences. Of course, if $I$ is a finite increasing sequence of positive integers, then $L_{\infty, I}$ has a finite $T$-space basis, but by the preceding work, we know that it has a $T$-space basis of size at most $i_2 - i_1 + 1$.

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