HEIGH T PAIRINGS FOR ALGEBRAIC CYCLES ON
THE PRODUCT OF A CURVE AND A SURFACE

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Abstract. For the product \( X = C \times S \) of a curve and a surface over a
number field, we prove Beilinson–Bloch’s conjecture about the existence
of a height pairing \( \langle \cdot, \cdot \rangle \) between homologically trivial cycles. Then,
for an embedding \( f : C \to S \), we construct an arithmetic diagonal cycle
modified from the graph of \( f \) and study its height. This work extends the
previous work of Gross and Schoen \([9]\) to \( X \) the product of three curves,
and makes Gan–Gross–Prasad conjecture unconditional for \( \text{O}(1, 2) \times \text{O}(2, 2) \) and \( \text{U}(1, 1) \times \text{U}(2, 1) \).

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1. Introduction

Let \( K \) be a number field or the function field of a curve \( B \) over a field \( k \),
and let \( X = C \times S \) be the product of a curve and a surface, both smooth
and projective over \( K \). This paper aims to extend the main results of Gross
and Schoen \([9]\) in the case \( S \) is the product of two curves. There are two
goals.

Our first goal is to prove Beilinson–Bloch’s conjecture about the existence
of the height pairing:

\[
\langle \cdot, \cdot \rangle_B : \text{Ch}^2(X)^0 \otimes \text{Ch}^2(X)^0 \to \mathbb{R},
\]

where \( \text{Ch}^2(X)^0 \) denote the Chow group of homologically trivial cycles of
codimension 2. We recover the result of Gross and Schoen for triple products
of curves with a new proof. Gross and Schoen used Tate’s conjectures, which
are known for triple products of curves. Furthermore, our proof extends
their result to the case of products of curves and surfaces, for which Tate’s conjecture is open but for which Grothendieck’s standard conjectures can be shown to be held and imply the existence of Beilinson–Bloch height pairings by our recent work [18].

Our second goal is to construct homologically trivial cycles $\gamma \in \text{Ch}^2(X)^0$ modified from the graph of finite morphisms $f : C \rightarrow S$. As an attempt to generalize our results in [16] from triple products of curves to products of curves and surfaces, we will provide a formula for the height $\langle \gamma, \gamma \rangle_B$ in the case that $K$ is a function field $K = k(B)$ and where both $C$ and $S$ have smooth models over $B$. When $S$ is a K3 surface in the function field case with good reduction, our formula shows that Beilinson’s Hodge index conjecture [1] gives an inequality between the $\omega_C^2$ and the canonical height of $S$. We expect a precise formula to hold in general for function field and number field situations and have interesting applications to Diophantine problems.

This work arose from an attempt to understand a conjecture proposed by Gan, Gross, and Prasad (GGP) [7, 19, 17], which relates the derivatives of $L$-series to the conjectured height pairings of arithmetic diagonal cycles for embeddings of Shimura varieties. Our work provides such height pairings and thus makes the GGP conjecture unconditional for $O(1,2) \times O(2,2)$ and $U(1,1) \times U(2,1)$. More precisely, when $f$ embeds a Shimura curve into a Shimura surface, then for each Hecke operator $t$ of $X$, we have a cycle $t_\ast \gamma \in \text{Ch}^2(X)^0$. Moreover, when $t$ is an idempotent for some new form in some automorphic representation $\pi$, then $\langle t_\ast \gamma, t_\ast \gamma \rangle_B$ is essentially equal to the central derivative of the specific $L$-series $L(s, \pi)$.

The plan of this paper is as follows. In §2-3 for any given polarizations $\eta$ of $C$ and $\xi$ of $S$, we will first give some canonical decompositions for the groups $A^\ast(X)$ of algebraic cohomology cycles and $\text{Ch}^\ast(X)^0$ of homologically trivial cycles. There are two consequences of these decompositions: one is Grothendieck’s standard conjecture for $A^\ast(\tilde{X})$ for any $\tilde{X}$ obtained from $X$ by successive blow-ups along smooth curves and points, another is a decomposition $\text{Ch}^\ast(X) = \text{Ch}^\ast(X)^0 \oplus A^\ast(X)$. Then we define certain quotients $J^\ast(X)$ of $\text{Ch}^\ast(X)^0$, which we call the intermediate Jacobians and the bi-primitive subgroup $J^2(X)_{00}$ annihilated by both $\xi$ and $\eta$. Finally, we provide a canonical way to modify a cycle in $\text{Ch}^\ast(X)$ to become a cycle in $J^2(X)_{00}$.

In §4-5, when both $C$ and $S$ have strictly semistable models $\mathcal{C}$ and $\mathcal{S}$ over a discrete valuation ring $R$, we will first construct a strictly semistable model $\mathcal{X}$ for $X$ by successive blowing up along the strict transforms of the irreducible components in the special fiber of $\mathcal{C} \times_R \mathcal{S}$. The results in section §2-3 will imply that the resulting irreducible components in the special fiber of $\mathcal{X}$ satisfy the Grothendieck standard conjectures. Then we apply our recent work in [18] to define the Beilinson–Bloch height pairings $\langle \cdot, \cdot \rangle_B$ for $\text{Ch}^\ast(X)^0$ when $X$ is defined over number fields or function fields. Finally, we extend the height pairing to general $C$ and $S$ using de Jong’s alteration [5].
In §6-7, first, we define the arithmetic diagonal \( \gamma \) as mentioned above. Then, we restrict our study in the case \( H^1(S) = 0 \) and \( \deg f^*f_*C \neq 0 \). In this case, we have a simpler formula describing the cycle \( \gamma \). Finally, we give a formula for \( \langle \gamma, \gamma \rangle_B \) in the function field case \( K = k(B) \) when both \( C \) and \( S \) have smooth models over \( B \).

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2. Algebraic cohomology cycles

Let \( X \) be a smooth and projective variety over an algebraically closed field \( K \) of dimension \( n \). For an integer \( i \), let \( \text{Ch}^i(X) \) denote the group of Chow cycles of codimension \( i \) with rational coefficients modulo rational equivalence, and consider the cohomological class map:

\[
\text{cl}^i : \text{Ch}^i(X) \to H^{2i}(X)(i)
\]

in a fixed Weil cohomology theory. Let \( A^i(X) \) denote the image of \( \text{cl}^i \) and let \( \text{Ch}^i(X)^0 \) denote the kernel of \( \text{cl}^i \).

When \( i = 1 \) or \( i = n \), these groups are related to abelian varieties by the identity \( \text{Ch}^1(X)^0 = \text{Pic}^0(X)_\mathbb{Q} \) and the Abel–Jacobi map

\[
\text{AJ} : \text{Ch}^n(X)^0 \to \text{Alb}(X)(K) \otimes \mathbb{Z} \mathbb{Q}.
\]

Let \( L \) be an ample line bundle on \( X \). Then taking the intersection with \( c_1(L) \) defines a Lefschetz operator \( L : A^i(X) \to A^{i+1}(X) \). Here is the Grothendieck standard conjecture.

**Conjecture 2.1** (Grothendieck). Let \( i \leq n/2 \).

1. **Hard Lefschetz theorem:** The following induced map is bijective:

\[
L^{n-2i} : A^i(X) \to A^{n-i}(X).
\]

2. **Hodge index theorem:** For a nonzero \( \alpha \in A^i(X) \) such that \( L^{n+1-2i}\alpha = 0 \),

\[
(-1)^i\alpha \cdot L^{n-2i}\alpha > 0.
\]

**Remark** 2.2. For \( i = 0 \), the conjecture is trivially true. For \( i = 1 \), the Hodge index theorem is true; see [2, Exposé XIII, Corollary 7.4]. This implies that \( L^{n-2} : A^1(X) \to A^{n-1}(X) \) is always injective and that \( \dim A^1(X) < \infty \). Thus, the conjecture is true for \( n = 1, 2 \), and is equivalent to the inequality \( \dim A^1(X) \geq \dim A^2(X) \) for \( n = 3 \).

In the rest of this section, we will assume that \( X \) is of the form \( C \times S \) with \( C \) a curve and \( S \) a surface. Let \( \pi_C \) and \( \pi_S \) denote two projections. We will
consider $\text{Ch}^*(X)$ as the group correspondences between $\text{Ch}^*(C)$ and $\text{Ch}^*(S)$ as usual: for any $\alpha \in \text{Ch}^*(X)$ and $\beta \in \text{Ch}^*(C)$ and $\gamma \in \text{Ch}^*(S)$, we define
\[
\alpha_*(\beta) = \pi_{S*}(\alpha \cdot \pi_C^* \beta) \in \text{Ch}^*(S), \quad \alpha^*(\gamma) = \pi_{C*}(\alpha \cdot \pi_S^* \gamma) \in \text{Ch}^*(C).
\]
For classes $\alpha \in \text{Ch}^i(C)$ and $\beta \in \text{Ch}^j(S)$, we denote
\[
\alpha \boxtimes \beta = \pi_C^* \alpha \cdot \pi_S^* \beta \in \text{Ch}^{i+j}(X).
\]
We will use the same notation for the induced operation in the Künneth decomposition:
\[
\boxtimes : A^*(C) \otimes A^*(S) \rightarrow A^*(X).
\]
We will fix ample classes $\xi \in \text{Ch}^1(S), \eta \in \text{Ch}^1(C)$ such that $\deg \eta = 1$. We consider them as cycles on $X$ via pull-backs. Let $A^i(X)_{\text{rig}}$ denote the subgroup of $A^i(X)$ of elements $\alpha$ with rigidifications:
\[
\alpha_*(\eta) = 0, \quad \alpha_*(|C|) = 0.
\]
It can be checked that $A^i(X)_{\text{rig}}$ is the intersection of $A^i(X)$ and $H^1(C) \otimes H^{2i-1}(S)$ in the Künneth decomposition:
\[
H^{2i}(X)(i) = \bigoplus_{j=0}^{2} H^i(C) \boxtimes H^{2i-j}(S)(i).
\]
Our starting point is a description of the group $A^*(X)$ in terms of $\xi, \eta, A^*(C), A^*(S)$, and $A^*(X)_{\text{rig}}$.

**Proposition 2.3.** Let $X = C \times S$ and $\xi$ and $\eta$ be as above. Then

1. $A^0(X) = \mathbb{Q}[X], A^3(X) = \mathbb{Q}(\eta \boxtimes \xi^2)$.
2. For $i = 1, 2$, $A^i(X) = A^i(X)_{\text{rig}} \oplus [C] \boxtimes A^i(S) \oplus \eta \boxtimes A^{i-1}(S)$ with respect to the decomposition
   \[
   H^{2i}(X) = H^i(C) \boxtimes H^{2i-1}(S) \oplus [C] \boxtimes H^i(S) \oplus \eta \boxtimes H^{2i-2}(S).
   \]
   The projection to the last two components is given by
   \[
   (\text{2.1}) \quad \alpha \mapsto [C] \boxtimes \alpha_*(\eta) + \eta \boxtimes \alpha_*([C]).
   \]
3. The intersection with $\xi$ gives a bijection $A^1(X)_{\text{rig}} \sim A^2(X)_{\text{rig}}$ with respect to the isomorphism given by intersection with $\pi_S^* \xi$:
   \[
   H^1(C) \boxtimes H^1(S) \sim H^1(C) \boxtimes H^3(S).
   \]

**Proof.** The first part is clear. For the second part, we notice that the map 2.1 takes cycles in $A^i(X)$ to cycles in $A^i(X)$. Thus it induces a decomposition on $A^i(X)$. For the third part, we only need to prove the surjectivity. Let $\alpha \in A^2(X)_{\text{rig}}$. We lift $\alpha$ to a class $\tilde{\alpha} \in \text{Ch}^2(X)$ and consider the morphism of abelian varieties:
\[
\tilde{\alpha}^* : \text{Pic}^0(S) \rightarrow \text{Pic}^0(C).
\]
This map does not depend on the choice of lifting $\tilde{\alpha}$ because $\alpha^*$ induces the following map on cohomology:
\[
H^1(\text{Pic}^0(C)) = H^1(C) \xrightarrow{\alpha^*} H^3(S)(1) = H^1(\text{Pic}^0(S)).
\]
Now we combine the map $\tilde{\alpha}^*$ with the following polarization map defined by $\xi$:

$$\varphi : \text{Pic}^0(S) \to \text{Alb}(S) : \mathcal{M} \mapsto A\mathcal{J}(\mathcal{M} : \xi),$$

and obtain an $f := \tilde{\alpha}^* \cdot \varphi^{-1} \in \text{Hom}(\text{Alb}(S), \text{Pic}^0(C))_\mathbb{Q}$. This $f$ is represented by a unique class $\mu \in \text{Ch}^1(X)$ such that $\mu_*(\eta) = 0$ and $\mu^*(\xi^2) = 0$. From construction, it is clear that

$$\alpha = c_1(\mu) \cdot \xi.$$

$\square$

**Remark 2.4 (Functoriality and correspondences).** Let

$$f = f_{C'} \times f_{S'} : X' := C' \times S' \to C \times S$$

be a finite morphism with $C'$ and $S'$ smooth and projective. Then, we have the induced homomorphisms of groups:

$$f_* : A^*(X') \to A^*(X), \quad f^* : A^*(X) \to A^*(X').$$

Let $\xi', \eta'$ be polarizations of $S'$ and $C'$ with $\xi' \in \mathbb{Q} f^* \xi$ and $\eta' \in \mathbb{Q} f^* \eta$. These two maps respect the decomposition in Proposition 2.3. More generally, we have two actions by correspondence:

$$A^3(X \times X) \to \text{End}(A^*(X)) : \alpha \mapsto \alpha_*, \quad \alpha \mapsto \alpha^*.$$

One is a homomorphism of algebras over $\mathbb{Q}$, and another is an anti-homomorphism:

$$(\alpha_1 \circ \alpha_2)^* = \alpha_2^* \circ \alpha_1^*, \quad (\alpha_1 \circ \alpha_2)_* = \alpha_1_* \circ \alpha_2_*.$$

Let $A^1(C \times C)_{\text{rig}}$ (resp. $A^2(S \times S)_{\text{rig}}$) be the subgroup of $A^1(C \times C)$ (resp. $A^2(S \times S)$) of elements $\alpha$ such that both $\alpha_*$ and $\alpha^*$ fix the line $\mathbb{Q} \eta$ (resp. $\mathbb{Q} \xi$). Define

$$A^3(X \times X)_{\text{rig}} := A^1(C \times C)_{\text{rig}} \boxtimes A^2(S \times S)_{\text{rig}}.$$

Then, the above actions restricted to $A^3(X \times X)_{\text{rig}}$ will respect the decomposition in Proposition 2.3.

By Proposition 2.3, $A^2(X)$ and $A^1(X)$ have the same dimension. Thus, we have the following Corollary.

**Corollary 2.5.** Let $X = C \times S$ be the product of a curve and a surface. Then Grothendieck’s standard conjecture holds.

We also need the standard conjectures for blow-ups of $X$ to define Beilinson–Bloch height pairings in §5:

**Proposition 2.6.** Let $Y$ be a smooth and projective threefold. Let $\pi : \tilde{Y} \to Y$ be a blow-up of $Y$ along a smooth subvariety $Z$. Then, the standard conjectures for $Y$ and $\tilde{Y}$ are equivalent.
Proof. Let $\iota : E \to \bar{Y}$ be the exceptional divisor. Then $A^*(\bar{Y})$ is generated by $\pi^*A^*(Y)$ and $\iota_* A^*(E)$. Notice that $E$ is a projective bundle over $Z$. So $A^*(E)$ is generated over $A^*(Z)$ by the first Chern class of $\mathcal{O}_E(1)$. Thus we have $A^1(\bar{Y}) = \pi^*A^1(Y) + \mathbb{Q}[E]$, and $A^2(\bar{Y}) = \pi^*A^2(Y) + \mathbb{Q}E^2$. This shows that $\dim A^1(Y) = \dim A^2(Y)$ is equivalent to that $\dim A^1(\bar{Y}) = \dim A^2(\bar{Y})$. See [6, §6.7] for more details.

Corollary 2.7. Let $\bar{X}$ be a smooth and projective three-fold obtained from the product $X = C \times S$ of curve and surface by successively blowing up along smooth curves. Then, the standard conjecture holds for $X$.

3. Homologically trivial cycles

In this section, let $K$ be an algebraically closed field and $X = C \times S$ be the product of a curve and a surface, both smooth and projective over $K$. We will fix classes $\xi \in \text{Ch}^1(S)$, $\eta \in \text{Ch}^1(C)$ such that $\xi$ is ample and $\deg \eta = 1$. Then we can define analogously the subgroup $\text{Ch}^1(X)_{\text{rig}}$ of $\text{Ch}^1(X)$ of elements with following rigidifications:

$$\alpha_\ast(\eta) = 0, \quad \alpha^\ast(\xi^2) = 0.$$

Then the map $\text{Ch}^1(X) \to A^1(X)$ defines a bijection $\text{Ch}^1(X)_{\text{rig}} \to A^1(X)_{\text{rig}}$. Thus Proposition 2.3 defines maps for $i = 1, 2$: $\alpha \mapsto \alpha_\mu : \text{Ch}^i(X) \to \text{Ch}^i(X)_{\text{rig}}$.

The combination of parts 2 and 3 of Proposition 2.3 gives the following

Proposition 3.1. For $i = 1, 2$, we have a decomposition:

$$\text{Ch}^i(X) = \text{Ch}^i(X)^0 \oplus [C] \boxtimes \text{Ch}^i(S) \oplus \eta \boxtimes \text{Ch}^{i-1}(S) \oplus \text{Ch}^i(X)_{\text{rig}}\xi^{i-1},$$

such that the projection to the last three components is given by

$$\alpha \mapsto [C] \boxtimes \alpha_\ast(\eta) + \eta \boxtimes \alpha_\ast[C] + \alpha_\mu \xi^{i-1}.$$

Remark 3.2 (Functoriality and correspondences). Let

$$f = f_{C'} \times f_{S'} : X' := C' \times S' \to C \times S$$

be a generically finite morphism with $C'$ and $S'$ smooth and projective. Then we have group homomorphisms

$$f_* : \text{Ch}^*(X') \to \text{Ch}^*(X), \quad f^* : \text{Ch}^*(X) \to \text{Ch}^*(X').$$

Let $\xi', \eta'$ be polarizations of $S'$ and $C'$ with $\xi' \in \mathbb{Q}f^*\xi$ and $\eta' = \mathbb{Q}f^*\eta$. Then, the above homomorphisms respect the decomposition in Proposition 3.1. Moreover, we have two actions by the correspondence $\alpha$:

$$\text{Ch}^3(X \times X) \to \text{End}(\text{Ch}^*(X)) : \alpha \mapsto \alpha_\ast, \quad \alpha \mapsto \alpha^\ast.$$

One is a $\mathbb{Q}$-algebra homomorphism; the other is a $\mathbb{Q}$-algebra anti-homomorphism. Let $\text{Ch}^1(C \times C)_{\text{rig}}$ (resp. $\text{Ch}^2(S \times S)_{\text{rig}}$) be the subgroup of $\text{Ch}^1(C \times C)$ (resp. $\text{Ch}^2(S \times S)$) of elements $\alpha$ such that both $\alpha_\ast$ and $\alpha^\ast$ fix the line $\mathbb{Q}\eta$ (resp. $\mathbb{Q}\xi$). Let $\text{Ch}^3(X \times X)_{\text{rig}}$ denote the product $\text{Ch}^1(C \times C)_{\text{rig}} \boxtimes \text{Ch}^2(S \times S)_{\text{rig}}$. Then, the above actions restricted to $\text{Ch}^3(X \times X)_{\text{rig}}$ will respect the decomposition in Proposition 3.1.
We want to define some “intermediate Abel–Jacobi maps” $\text{AJ}^i : \text{Ch}^i(X)^0 \to J^i(X)$ with actions by $A^*(X)$. For $i = 1$, we take $J^1(X) = \text{Pic}^0(X)/(K)_Q$: for $i = 3$, we take $J^3(X) := \text{Alb}(X)(K)_Q$; for $i = 2$, we define the modified group by

$$J^2(X) = \text{Ch}^2(X)^0 / \sum_f f_c(\text{Ch}^1(X)^0) \cdot \text{Ch}^1(X^0).$$

where $f$ runs generically finite morphisms

(3.1) $f = f_{c'} \times f_{s'} : X' = C' \times S' \to X = C \times S$

with $C'$ and $S'$ smooth and projective curves and surfaces, respectively.

Notice that for an embedding $K \subset C$, when $i = 2$, there is a Griffiths’ intermediate Jacobian defined by Hodge theory:

$$J^i_G(X_C) := F^iH^{2i-1}(X(C), C) / H^{2i-1}(X(C), C)/H^{2i-1}(X(C), \mathbb{Z}).$$

There will be a map

$$J^i(X_C) \to J^i_G(X(C)) \otimes \mathbb{Q}.$$

By definition, this is an isomorphism when $i = 1$ or 3. When $i = 2$ and $K = \mathbb{Q}$, by a conjecture of Beilinson [1, p.18], the restriction on $J^2(X)$ of this map is injective.

**Proposition 3.3.** The intersection pairing on $\text{Ch}^*(X)$ induces an action:

$$A^i(X) \otimes J^j(X) \to J^{i+j}(X).$$

**Proof.** This follows immediately from the following Lemma. \hfill \Box

**Lemma 3.4.** Let $\alpha \in \text{Ch}^1(X)^0$ and $\beta \in \text{Ch}^2(X)^0$. Then $\text{AJ}(\alpha \beta) = 0$.

**Proof.** For any $\beta \in \text{Ch}^2(X)$, the map $\alpha \mapsto \text{AJ}(\alpha \beta)$ defines a morphism of abelian groups $\varphi_\beta : \text{Pic}^0(X)(K) \to \text{Alb}(X)(K)$. We claim that a morphism of abelian varieties induces this. Using duality, $\text{Alb}(X) = \text{Pic}^0(\text{Pic}^0(X))$, we need to construct a line bundle $\mathcal{M}$ on $\text{Pic}^0(X) \times \text{Pic}^0(X)$ so that $\varphi_\beta(\alpha)$ is given the restriction of $\mathcal{M}$ on $\{\alpha\} \times \text{Pic}^0(X)$.

Fixed base point $x \in X$, then we have a universal bundle $\mathcal{P}$ on $X \times \text{Pic}^0(X)$ with trivializations on $X \times \{0\}$ and $\{x\} \times \text{Pic}^0(X)$ so that for any $\alpha \in \text{Pic}^0(X)$, the restriction of $\mathcal{P}$ on $X \times \{\alpha\}$ is a line bundle representing $\alpha$. Then consider variety $X \times \text{Pic}^0(X) \times \text{Pic}^0(X)$ and cycles $p^*_{12}c_1(\mathcal{P})$, $p^*_{13}c_1(\mathcal{P})$ and $p^*_1\beta$ defined by projections to the products $X \times \text{Pic}^0(X)$ and $X$ respectively. So we get a class

$$c_1(\mathcal{M}) := p_{23*}(p^*_{12}c_1(\mathcal{P}) \cdot p^*_{13}c_1(\mathcal{P}) \cdot p^*_1\beta) \in \text{Ch}^1(\text{Pic}^0(X) \times \text{Pic}^0(X)).$$

This is the class inducing $\varphi_\beta$.

This class of $\mathcal{M}$ certainly has a trivial restriction on $\{0\} \times \text{Pic}^0(X)$ and $\text{Pic}^0(X) \otimes \{0\}$. Thus, it is determined by its cohomological class in $H^2(\text{Pic}^0(X) \times \text{Pic}^0(X))$ which can be computed by the same formula

$$c^1(\mathcal{M}) = p_{23*}(p^*_{12}c^1(\mathcal{P}) \cdot c^1p^*_{13}c_1(\mathcal{P}) \cdot p^*_1c^2(\beta))$$. 

Thus $\mathcal{M}$ is trivial if $c^2\beta = 0$. Thus $\beta \in \text{Ch}^2(X)^0$ implies that $\mathcal{M} = 0$ and that $\varphi_B = 0$. \hfill \Box

**Remark 3.5.** Let $f : X' \to X$ be as in (3.1). Then we have morphisms

\[ f_* : J^\ast(X') \to J^\ast(X), \quad f^* : J^\ast(X) \to J^\ast(X'). \]

More generally, we have two actions by correspondence:

\[ \text{Ch}^3(X \times X) \to \text{End}(J^\ast(X)) : \alpha \mapsto \alpha_s, \quad \alpha \mapsto \alpha^\ast. \]

We don’t know if these actions factor through the action by $A^*(X \times X)$ in general. However, if $K = \mathbb{Q}$, then by Beilinson’s conjecture, $J^3(X)$ can embed into the Griffith intermediate Jacobian $J_G(X) \otimes \mathbb{Q}$. Thus, the action of $\text{Ch}^3(X \times X)$ conjecturally factors through action by $A^3(X \times X)$ in this case.

Write $\mathcal{L} = \xi \boxtimes \eta$ as an ample line bundle on $X$. The intersection with $c_1(\mathcal{L})$ then defines an operator

\[ L : J^i(X) \to J^{i+1}(X). \]

Notice that $L^2 : J^1(X) \to J^3(X)$ is a polarization. For $J^2(X)$, let $J^2(X)_0$ denote the kernel of $L$ called the primitive part of $J^2(X)$. Then, we have a decomposition:

\[ J^2(X) = L^2 J^1(X) \oplus J^2(X)_0. \]

For simplicity, we will work with the subgroup $J^2(X)_{00}$ of $J^2(X)_0$.

Let $\pi_C, \pi_S$ and $\pi^\ast$ be the primitive part of $\pi_C, \pi_S, \pi^\ast$. We call this the group of the bi-primitive cycles. These conditions are easy to describe by noticing that

\[ J^3(X) \to \text{Alb}(C) \times \text{Alb}(S) : \quad \alpha \mapsto (\pi_C \alpha, \pi_S \alpha, \alpha^\ast). \]

Thus an element $\alpha \in J^2(X)$ is in $J^2(X)_{00}$ if and only if

\[ \alpha_s(\eta) = 0, \quad \eta \cdot \alpha^\ast S = 0, \quad \alpha^\ast(\xi) = 0, \quad \alpha_s(C) \cdot \xi = 0. \]

Since $\alpha^\ast S = 0$ always true, and $\alpha_s(C) \xi = 0$ is equivalent to $\alpha_s(C) = 0$, we need only check the following three conditions:

\[ \alpha^\ast(\xi) = 0, \quad \alpha_s(\eta) = 0, \quad \alpha_s(C) = 0. \]

Thus, we have proved the following:

**Proposition 3.6.** Let $\xi^\vee = \xi / (\deg \xi^2)$. Then, we have a decomposition:

\[ J^2(X) = J^2(X)_{00} \oplus [C] \boxtimes \text{Alb}(S) \oplus \eta \boxtimes \text{Pic}^0(S) \oplus \text{Pic}^0(C) \boxtimes \xi^\vee, \]

such that the projection to the last three components is given by

\[ \alpha \mapsto [C] \boxtimes \alpha_s(\eta) + \eta \boxtimes \alpha_s([C]) + \alpha^\ast(\xi) \boxtimes \xi^\vee. \]

**Remark 3.7.** Let $f : X' \to X$ and $\xi', \eta'$ be as in Remark 3.2. Then the maps

\[ f_* : J^\ast(X') \to J^\ast(X), \quad f^* : J^\ast(X) \to J^\ast(X') \]

respect to the above decomposition. So do the actions by correspondences in $\text{Ch}^3(X \times X)_{\text{rig}}$ defined in Remark 3.2.
Combining with Proposition 2.3, we obtain the following:

**Corollary 3.8.** The inclusion
\[ J^2(X)_{00} \rightarrow \text{Ch}^2(X) / \sum f_*(\text{Ch}^1(X')^0) \cdot \text{Ch}^1(X)^0 \]
has a retraction given by
\[ \alpha \mapsto \alpha - [C] \otimes \alpha_*([\eta]) - \eta \otimes \alpha_*([C]) - \mu_\alpha \xi - (\alpha^*(\xi) - \deg \alpha^* \xi \cdot \eta) \otimes \xi^\bigvee. \]

Let NS(S) denote the orthogonal complement of \( \xi \) in NS(S) under the intersection pairing \( \langle \cdot, \cdot \rangle_{\text{NS}} \). The \( J^2(X)_{00} \) can be decomposed into the subspace \( J^2(X)_{000} \) annihilated by all elements in \( \pi^*_S \text{NS}(S) \). Then, we have a further decomposition.

**Proposition 3.9.** There is a decomposition
\[ J^2(X)_{00} = J^2(X)_{000} \oplus \text{Pic}^0(C) \otimes \text{NS}(S)_0 \]
so that the projection to the second component is given as follows:
\[ \alpha \mapsto \sum \alpha^*(h_i) \otimes h_i^\bigvee \]
where \( \{h_1, \ldots, h_t\} \) is a base of \( \text{NS}(S)_0 \), and \( \{h_1^\bigvee, \ldots, h_t^\bigvee\} \) its dual base in the sense that \( \langle h_i, h_j^\bigvee \rangle_{\text{NS}} = \delta_{ij} \).

4. **Integral models**

In this section, we consider the local situation. Let \( K \) be a discrete valuation field with the valuation ring \( R = O_K \), a uniformizer \( \pi \), and the residue field \( k = R/\pi \). For an \( R \)-variety \( \mathcal{V} \) and a closed point \( P \in \mathcal{V}_k \), we say that \( \mathcal{V} \) is strictly semistable at \( P \), if the the formal completion of \( \mathcal{V} \) at \( P \) admits a finite \( \acute{e} \text{tale} \) map to the following formal scheme:
\[ \text{Spf} R[[x_1, \ldots, x_n]]/(\prod_{i=1}^t x_i - \pi) \]
for some \( t \leq n \). We say that \( \mathcal{V} \) is strictly semistable if it is strictly semistable at each close point. The property of strict semistability does not change if we replace \( R \) with an unramified extension. So, in the following, we assume that \( R \) is complete and \( k \) is algebraically closed. In this case, the above \( \acute{e} \text{tale} \) map is an isomorphism.

Let \( X = C \times S \) be the product of a smooth proper curve and a proper smooth surface over \( K \). Assume that \( C \) (resp. \( S \)) has a strictly semistable model \( \mathcal{C} \) (resp. \( \mathcal{S} \)) over \( R \). Following Gross and Schoen [9] (see also Hartl [11]), we have a semistable model \( \mathcal{X} \) for \( X \) by blowing-up \( \mathcal{X}_1 := \mathcal{C} \times_R \mathcal{S} \). More precisely, take any order of the irreducible components of \( \mathcal{X}_{1,k} \):
\[ C_1^1, \ldots, C_1^t. \]
Then we define models \( \mathcal{X}_i \) for \( i = 1, \ldots, t \) as follows: \( \mathcal{X}_2 \) is the blow-up of \( \mathcal{X}_1 \) over \( C_1^i \). Then the special fiber of \( \mathcal{X}_2 \) is the union of the preimage of \( C_2^i \) of \( C_1^i \). Take \( \mathcal{X}_3 \) be the blow-up of \( \mathcal{X}_2 \) over \( C_2^2 \), etc.
Our main result in this section (Proposition 4.2) is a description of $C^i_t$ in terms of blow-ups from $C^j$ without reference to $\mathcal{X}$. We need to review the construction of $\mathcal{X}$ to prove this result.

**Proposition 4.1** (Hartl [11]). The scheme $\mathcal{X}_i$ is strictly semistable.

*Proof.* This is a very special case of a general result by Hartl [11, 9, Proposition 2.1]. For our application, the proof of the next Proposition, we repeat the proof in our case. Let $P$ be a closed point of $\mathcal{X}_i$. We need to show that the completion of $\mathcal{X}_i$ at $P$ has the form

$$\text{Spf} R[[t_1, \ldots, t_4]]/\left(\prod_{i=1}^q t_i - \pi\right)$$

for some integer $q$ between 1 and 4. Let $P_i$ be the image of $P$ on $\mathcal{X}_i$, and let $\widehat{\mathcal{X}}_i = \text{Spf} \mathcal{O}_i$ be the completion of $\mathcal{X}_i$ at $P_i$.

Write $P_i = (p, q)$ with $p \in C_k$ and $q \in \mathcal{I}_k$. If $p$ or $q$ is a smooth point, then $\mathcal{X}_1$ is strictly semistable at $P_1$. Thus near $P_1$, all irreducible components of $\mathcal{I}_k$ are Cartier. So, a blow-up along such a component changes the local completion at $P_1$. In this case, all $\mathcal{O}_i$ are isomorphic to each other. So, we may assume that both $p$ and $q$ are singular in the following discussion. In this case, $p$ is the intersection of two components $A_1 \cdot A_2$, and $q$ is the intersection of two or three components $B_j$. Then, the complete local ring of $\mathcal{O}_i$ is given as one of the following two cases:

$$\mathcal{O}_i = \begin{cases} R[[x_1, x_2, y_1, y_2, y_3]]/(y_1 y_2 - \pi, x_1 x_2 - y_1 y_2), & \text{or} \\ R[[x_1, x_2, y_1, y_2, y_3]]/(y_1 y_2 y_3 - \pi, x_1 x_2 - y_1 y_2 y_3) \end{cases}$$

Modulo the uniformizer $\pi$, $\mathcal{I}_{ik}$ has 4 or 6 irreducible components $C_{i,j}$ defined by ideals $(x_i, y_j)$ for $i = 1, 2$, and $j = 1, 2$, or 3. They are formal completions at $P_i$ of irreducible components $C_{i,j}^a$ for $1 \leq i_1 < \cdots < i_s \leq t$, where $s = 4$ or 6. It follows that $\mathcal{O}_i = \mathcal{O}_1$ for $i \leq i_1$. Without loss of generality, we assume that $C_{1,1} = C_{1,1}^a$. Then $P_{i+1}$ is on the blowing up $\text{Spf} \mathcal{O}_{i+1}$ by the ideal $(x_1, y_1)$. The formal completion of two affines covers this formal scheme with formal rings in two cases

$$\begin{cases} R[x_1', x_2, y_1, y_3]/(x_1' x_2 y_1 - \pi), & R[x_1, y_1', y_2, y_3]/(x_1 y_1' y_2 - \pi), & \text{or} \\ R[x_1', x_2, y_1, y_2, y_3]/(y_1 y_2 y_3 - \pi, x_1' x_2 - y_2 y_3), & R[x_1, y_1', y_2, y_3]/(x_1 y_1' y_2 y_3 - \pi) \end{cases}$$

These are $\mathcal{O}_1$-algebra generated by $x'_1 = x_1/y_1, y'_1 = y_1/x_1$. The point $P_{i+1}$ is defined by the ideal of $P_i$ and an element $z$ equal to $x'_1$ or $y'_1 - a$ for some $a \in R$. The above rings are strictly semistable at $P_i$ except one case:

$$R[x_1', x_2, y_1, y_2, y_3]/(y_1 y_2 y_3 - \pi, x_1' x_2 - y_2 y_3).$$

In this case, we have

$$\mathcal{O}_{i+1} = R[[x_1', x_2, y_1, y_2, y_3]]/(y_1 y_2 y_3 - \pi, x_1' x_2 - y_2 y_3), \quad z = x'_1.$$ 

One more blow-up will make $P_i$ a strictly semistable point for $\mathcal{X}_i$. $\square$

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Next, we want to show that $C^i_j$ can be obtained from $C^j_i$ by several blow-ups along smooth curves. Let $Z^{i,j} = C^i_1 \cap C^j_1$ for $i \neq j$. Then $Z^{i,j}$ has the form $(A_1 \cap A_2) \times (B_1 \cap B_2)$. It is either a smooth curve (when $A_1 \neq A_2$, $B_1 \neq B_2$), or a cartier divisor (when $A_1 = A_2$ or $B_1 = B_2$).

**Proposition 4.2.** For each $i$ between 1 and $t$, the projection $C^i_j \to C^i_1$ can be obtained by blowing up successively along the strict transforms of the following smooth curves:

\[ Z^1,i, \ldots, Z^{i-1,i}, Z^{i+1,i}, \ldots, Z^t,i \]

**Proof.** For each $j$ between 1 and $t$ and different than $i$, let $D^j_i$ denote the variety by successively blowing-up from $C^j_i$ along the strict transforms of $Z^{i,j-1}$'s if $j \neq i + 1$. We let $D^j_{i+1} = D^j_i$. The construction of $C^j_i$ is the same except for one extra blowing up on $\mathcal{X}_i$ along $C^i_i$. Thus we have natural morphisms $f_j : C^j_i \to D^j_i$. The $f_j$ is an isomorphism when $j \leq i$. From the proof of Proposition 4.1, the $C^i_i$ is not Cartier only when locally $\mathcal{X}_i$ looks one of the following three situations

\[
\begin{align*}
R[[x_1, x_2, y_1, y_2, y_3]]/(y_1 y_2 - \pi, x_1 x_2 - y_1 y_2), \\
R[[x_1, x_2, y_1, y_2, y_3]]/(y_1 y_2 y_3 - \pi, x_1 x_2 - y_1 y_2 y_3), \\
R[[x_1, x_2, y_1, y_2, y_3]]/(y_1 y_2 y_3 - \pi, x_1 x_2 - y_1 y_2 y_3).
\end{align*}
\]

Assume that $C^i_i$ is defined by $(x_1, y_1)$. Then $C^i_i$ has complete local ring $k[[x_2, y_2, y_3]]$. Blowing up $(x_1, y_1)$ means gluing two affines by adding rational functions $x_1/y_1$ and $y_1/x_1$, respectively. In the first and third cases, $x_1/y_1 = y_2/x_2$. Thus $f_{i+1} : C^i_{i+1} \to D^i_{i+1}$ is a blow-up along the ideal $(x_2, y_2)$. In other words, $C^i_{i+1} = D^i_k$ for some $k > i + 1$. This implies that $f_i$ is an isomorphism.

In the second case, $x_1/y_1 = y_2 y_3/x_2$. Thus $f_{i+1} : C^i_{i+1} \to D^i_{i+1}$ is a blow-up along the ideal $(x_2, y_2 y_3)$. For constructing $D^i_k$ for some $k > i + 1$, we need to blow-up ideals $(x_2, y_2)$ and $(x_2, y_3)$ in some order. After these blow-ups, the ideal sheaf $(x_2, y_2 y_3)$ will be invertible. By universal property of blowing-up, there is a map $g_{i+1} : D^i_l \to C^i_{i+1}$ such that $f_{i+1} : g_{i+1}$ is the projection $D^i_l \to D^i_{i+1}$. This implies that $f$ is an isomorphism. \qed

5. Height pairings

In this section, we let $K$ be a number field and $X$ be a smooth, geometrically connected, and projective variety over $K$ of dimension $n$. Let $\mathcal{X}$ be a regular, flat, and projective model over $\mathcal{O}_K$. But note that all results work for the function field $K$ of a smooth and projective curve $B$ over a field with the following minor modification:

1. the integral model $\mathcal{X}$ over $\mathcal{O}_K$ is replaced by the integral model $\mathcal{X}$ over $B$, and
2. the Chow group $\widehat{\text{Ch}}^*(\mathcal{X})$ is replaced by $\text{Ch}^*(\mathcal{X})$. 

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Recall that for each integer $i$, we have a cycle class map
\[ \text{cl}^i : \text{Ch}^i(X) \to H^{2i}(X)(i). \]
Let $A^*(X)$ and $\text{Ch}^i(X)^0$ denote the image and kernel, respectively. Then Beilinson [1] and Bloch [3] constructed a height pairing
\[ \langle \cdot, \cdot \rangle_B : \text{Ch}^i(X)^0 \times \text{Ch}^{n+1-i}(X)^0 \to \mathbb{R} \]
under the conjecture that Beilinson–Bloch’s condition always holds: $X$ has a regular integral model $\mathcal{X}$ over $\mathcal{O}_K$ such for each integer $i$ and each cycle $\alpha \in \text{Ch}^i(X)$ there is an integral cycle $\bar{\alpha} \in \text{Ch}^i(\mathcal{X})$ with the following properties:

1. the restriction of $\bar{\alpha}$ on the generic fiber $X$ is $\bar{\alpha}_K = \alpha$;
2. for any vertical cycle $\beta \in \text{Ker}(\text{Ch}^{n+1-i}(X) \to \text{Ch}^{n+1-i}(X))$,
   \[ \bar{\alpha} \cdot \beta = 0. \]

Such an integral cycle can be extended into an arithmetic cycle $\alpha^b = (\bar{\alpha}, g_\alpha)$ by adding a green current $g_\alpha$ by the equation $\partial g_\alpha = \delta_\alpha$. The Beilinson–Bloch height pairing is then defined as the intersection number
\[ \langle z, w \rangle_B = z^b \cdot w^b. \]
This pairing does not depend on the choice of integral models and base change and has the following functorial properties:

1. Let $K'$ be a finite extension of $K$, and $X'$ and $X$ are smooth and projective varieties of dimension $n$ over $K'$ and $K$, respectively, such that both $X'$ and $X$ satisfy Beilinson–Bloch condition. Let $f : X' \to X$ be a morphism over $K$. Then for any $\alpha \in \text{Ch}^i(X')^0$ and $\beta \in \text{Ch}^{n+1-i}(X)^0$, we have the following projection formula:
   \[ \langle \alpha, f^* \beta \rangle_B = \langle f_* \alpha, \beta \rangle_B. \]
   See for example [12, Lemma 4.1 and Formula (18) on page 787] for more general statements.
2. The Beilinson–Bloch pairing is an extension of the Néron–Tate height pairing $\langle \cdot, \cdot \rangle_{NT}$ on $\text{Pic}^0(X) \times \text{Alb}(X)$ in the following sense: for any $\alpha \in \text{Ch}^i(X)^0$, $\beta \in \text{Ch}^n(X)^0$, we have
   \[ \langle \alpha, \beta \rangle_B = -\langle \alpha, \text{AJ}(\beta) \rangle_{NT}. \]
3. For any $\alpha \in \text{Ch}^i(X)^0$, $\beta \in \text{Ch}^j(X)^0$, $\gamma \in \text{Ch}^k(X)$ such that $i + j + k = n + 1$. Then
   \[ \langle \alpha \cdot \gamma, \beta \rangle_B = \langle \alpha, \gamma \cdot \beta \rangle_B. \]

Indeed, for any lifting $\tilde{\gamma} \in \tilde{\text{Ch}}^k(X)$, we have
\[ (\alpha \cdot \gamma)^b = \alpha^b \cdot \tilde{\gamma}, \quad (\gamma \cdot \beta)^b = \tilde{\gamma} \cdot \beta^b. \]
It follows that
\[ \langle \alpha \cdot \gamma, \beta \rangle_B = \tilde{\gamma} \cdot \alpha^b \cdot \beta^b = \langle \alpha, \gamma \cdot \beta \rangle_B. \]
In the rest of this section, we assume that \( X = C \times S \) is a product of a curve and surface, both smooth and projective. The key result of this paper is as follows:

**Theorem 5.1.** Assume that \( C \) and \( S \) have strict semistable model on \( \text{Spec} \mathcal{O}_K \). Then \( X \) satisfies Beilinson-Bloch’s condition. In particular, the Beilinson–Bloch height pairing is well-defined on \( X \).

**Proof.** By assumption, \( C \) and \( S \) have strictly semistable models \( C' \) and \( S' \) over \( \mathcal{O}_K \). By Proposition 4.1, the blow-up \( C' \times_{\mathcal{O}_K} S' \) along its irreducible components in the special fiber gives a strictly semistable model \( X' \) for \( X \). By Corollary 2.7, and Proposition 4.2, we know that each irreducible component of the special fiber of \( X' \) satisfies the standard conjecture. Then, we apply Theorem 1.5.1 and Proposition 1.6.3 in [18] to get that \( X' \) satisfies the Beilinson–Bloch condition. \( \square \)

**Remark 5.2.** The idea of using standard conjecture to define height pairing was first used by K"unnemann [13] based on early work of Bloch, Gillet, and Soulé [4]. K"unnemann’s work [13] covers the case of abelian varieties over local fields with total degeneration and the case of varieties uniformized by the Drinfeld upper-half spaces. Furthermore, in [14], K"unnemann extended his work to general abelian varieties by constructing appropriate regular integral models.

Now, we want to extend the definition of the Beilinson–Bloch height pairing to the case where \( C \) or \( S \) does not have strictly semistable models over \( \mathcal{O}_K \). Using de Jong’s alteration [5], there is a finite extension \( K' \) of \( K \) and generically finite morphisms \( f_C : C' \longrightarrow C_{K'} \) and \( f_S : S' \longrightarrow S_{K'} \) from smooth and projective curve and surface respectively such that \( C' \) and \( S' \) both have strictly semistable reduction over \( \mathcal{O}_{K'} \). Let \( X' = C' \times S' \) and \( f : X' \longrightarrow X \) the induced morphism. Then we define a Beilinson–Bloch height pairing on \( \text{Ch}^i(X)^0 \) by the formula:

\[
\langle \alpha, \beta \rangle_{B} := \frac{1}{\deg f}(f^* \alpha, f^* \beta)_B, \quad \alpha \in \text{Ch}^i(X)^0, \beta \in \text{Ch}^{4-i}(X)^0.
\]

This definition does not depend on the choice of covering \( X' \longrightarrow X \) because of the following two facts:

1. A third one will dominate any two such coverings.
2. If \( g : X'' \longrightarrow X' \) is a generically finite morphism from another variety \( C'' \times S'' \) over a finite extension \( K'' \) of \( K \) with strictly semistable models over \( \mathcal{O}_{K''} \), then for any \( \alpha \in \text{Ch}^i(X')^0, \beta \in \text{Ch}^{4-i}(X')^0, \)

\[
\langle g^* \alpha, g^* \beta \rangle_B = \langle g_*, g_* \alpha, \beta \rangle_B = \deg g \langle \alpha, \beta \rangle_B.
\]

In the last step, we used the projection formula from intersection theory [6, Example 8.1.7] to conclude that for a generically finite proper map between regular varieties \( f_*, f^* \alpha = (\deg f) \alpha \).

We want to translate the height pairing to the intermediate Jacobians \( J^i(X) \) defined in §3.
Proposition 5.3. The Beilinson–Bloch pairing is induced by a pairing on
\[ J^i(X) \times J^{4-i}(X) \rightarrow \mathbb{R}. \]

Proof. We need only check the case \( i = 2 \): for all \( \alpha, \beta \in \text{Ch}^1(X)^0, \gamma \in \text{Ch}^2(X)^0 \), we have identity \( \langle \alpha \cdot \beta, \gamma \rangle_B = 0 \). This identity follows from the adjoint formula and relation with the Néron–Tate formula and Lemma 3.4:
\[
\langle \alpha \cdot \beta, \gamma \rangle_B = \langle \alpha, \beta \gamma \rangle_B = -\langle \alpha, \text{AJ}(\beta \gamma) \rangle_{\text{NT}} = 0.
\]
\[\square\]

Remark 5.4. Let \( f : X' \rightarrow X \) and \( \xi', \eta' \) be as in Remark 3.2. Then for any \( \alpha \in J^i(X), \beta \in J^{4-i}(X') \), we have a projection formula:
\[
\langle f^* \alpha, \beta \rangle_B = \langle \alpha, f_* \beta \rangle_B.
\]

More generally for any \( \gamma \in \text{Ch}^3(X \times X) \), we have
\[
(5.1) \quad \langle \gamma^* \alpha, \beta \rangle_B = \langle \alpha, \gamma_* \beta \rangle_B.
\]

This can be proved using operational arithmetic Chow groups of Gillet-Soule. See the proof of [12, Formula (18), page 787]. More precisely, let \( \tilde{\gamma} \in \widetilde{\text{Ch}}_3(X' \times X') \) be a lifting of \( \gamma \) as an element in the homological Chow group. Then, in terms of
\[
(\gamma^* \alpha)^{\flat} = p_1^*(\alpha^b \cdot p_2 \tilde{\gamma}), \quad (\gamma_* \beta)^{\flat} = p_2^*(\beta^b \cdot p_1 \tilde{\gamma}).
\]
Thus, the equation \(5.1\) becomes
\[
\tilde{\deg}(\beta^b \cdot p_1 (\alpha^b \cdot p_2 \tilde{\gamma})) = \tilde{\deg}(\alpha^b \cdot p_1 (\beta^b \cdot p_2 \tilde{\gamma})).
\]

Let \( \mathcal{L} \in \text{Ch}^1(X) \) be ample cycle. The intersection with \( c_1(\mathcal{L}) \) then defines an operator
\[
L : J^i(X) \rightarrow J^{i+1}(X).
\]
Notice that \( L^2 : J^1(X) \rightarrow J^3(X) \) is a polarization. Thus \( L^2 \) is an isomorphism, and for any \( \alpha \in J^1(X), \)
\[
-\langle \alpha, L^2 \alpha \rangle_B > 0.
\]

For \( J^2(X) \), let \( J^2(X)_0 \) denote the kernel of \( L \) called the primitive part of \( J^2(X) \). Then, we have a decomposition:
\[
J^2(X) = L J^1(X) \oplus J^2(X)_0.
\]

Here is an arithmetic Hodge index conjecture:

Conjecture 5.5 (Beilinson). The Beilinson–Bloch height pairing on \( J^2(X)_0 \) is positive definite.
6. Arithmetic diagonals

This section assumes that $K$ is a number field or a function field and that $f : C \to S = C \times C$ is the diagonal embedding to its self-product. Then $X = C \times C \times C$. Let $\xi$ be an ample class in $\text{Ch}^1(S)$ such that $\eta := f^* \xi$ has degree 1. Then by Corollary 3.8, we obtain a modified class in $J^2(X)_{00}$ given by

$$\gamma := \Gamma - [C] \otimes f_* \eta - \eta \otimes f_* [C] - \mu_f \cdot \xi$$

where $\mu_f \in \text{Ch}^1(X)_{00}$ is characterized by the following identities in $\text{Alb}(C)$ and $\text{Alb}(S)$:

$$\mu_f^*(\xi^2) = 0, \quad f_* (p) - f_* \eta = \mu_f(p) \cdot \xi, \quad \forall p \in C.$$

We call $\gamma$ an arithmetic diagonal.

**Example 6.1.** Assume that $f : C \to S = C \times C$ is the diagonal embedding to its self-product. Then $X = C \times C \times C$. Let $\xi = \frac{1}{2} e \times C + \frac{1}{2} C \times e$ be an ample class with $e \in \text{Ch}^1(C)$ of degree 1. Then $\eta = e$. In this way, $\gamma$ is the modified diagonal defined by Gross and Schoen [9]. They showed that $\gamma = 0$ if $C$ is rational, elliptic, or hyperelliptic. When $g(C) \geq 2$, the height of this modified diagonal has been computed in our previous paper [16], Theorem 1.3.1:

$$\langle \gamma, \gamma \rangle = \frac{2g + 1}{2g - 2} \omega^2 + \langle x_e, x_e \rangle_{\text{NT}} + \sum_v \varphi(X_v) \log N(v).$$

where $\omega$ is the admissible relative dualizing sheaf of $C$, $x_e = e - K_e/(2g - 2) \in \text{Ch}^1(C)^0$, and $v$ runs through the set of places of $K$, and $\varphi(X_v)$ are local invariants of $v$-adic curve $X_v$. The $\text{NS}(S)_0$ is generated by $e \times C - C \times e$ and the subgroup $\text{NS}(S)_{00}$ of classes of line bundles whose restriction on $e \times C$ and $C \times e$ are both trivial. For each $h \in \text{NS}(X)_0$, $f^* h$ is the divisor of fixed points of $h$.

By Proposition 3.9, we can also compute the projection of $\gamma$ to the subspace $\text{Pic}^0(C) \otimes \text{NS}(S)$ with respect to an orthonormal base $h_i$ with coefficient in $\mathbb{R}$ in the sense $\langle h_i, h_j \rangle_{\text{NS}} = -\delta_{ij}$:

$$\sum_i \gamma^* h_i \otimes h_i = \sum_i f^*_\eta h_i \otimes h_i, \quad f^*_\eta h_i := (f^* h_i - \deg f^* h_i) \otimes h_i.$$

Thus, Beilinson’s Hodge index conjecture implies the following inequality:

$$\langle \gamma, \gamma \rangle_{\mathbb{R}} \geq \sum_i \langle f^*_\eta h_i, f^*_\eta h_i \rangle_{\text{NT}}.$$

**Remark 6.2** (Correspondences and arithmetic GGP). We can define a family of cycles $t_* \gamma$ indexed by elements by rigid $t \in \text{Ch}^3(X \times X)_{\text{rig}}$ as defined in Remark 3.2. When $f : C \to S$ is an embedding of a Shimura curve into a Shimura surface, and $\xi, \eta$ are Hodge classes, our construction of height pairings $\langle t_* \gamma, t_* t \gamma \rangle_{\mathbb{R}}$ gives an unconditional formulation of the arithmetic Gan–Gross–Prasad conjecture [7, 19, 17] which relates the heights of these cycles.
to the derivative of the $L$-series. In the triple product case, $f$ is the diagonal embedding $\Delta : C \to S = C \times C$, already studied in Gross–Kudla [10], and in our recent work [15]. In the general case, we take Hecke eigenforms $\alpha \in \Gamma(C, \Omega^1_C)_{\mathbb{C}}$ and $\beta \in \Gamma(S, \Omega^2_S)_{\mathbb{C}}$ for some embedding $K \hookrightarrow \mathbb{C}$, and Hecke operators $t$ with complex coefficients whose action on $\Gamma(X, \Omega^2_X)_{\mathbb{C}}$ is an idempotent with image $\mathbb{C} \alpha \beta$. Then, by GGP conjecture, $(t^* \gamma, t^* \gamma)_B$ will be related to the derivative of certain Rankin–Selberg $L$-series $L(\pi_\alpha \times \pi_\beta, s)$ at its center of symmetry. Here is a simple example: take $B$, an indefinite quaternion algebra over $\mathbb{Q}$ (could be $M_2(\mathbb{Q})$), and $F$, a real quadratic field embedded in $B$. Let $\mathcal{O}_B$ be a maximal order of $B$ containing $\mathcal{O}_F$. Then we have the Hilbert moduli surface $S$ parameterizing polarized abelian surfaces $A$ with action by $\mathcal{O}_F$ with some level structure, and the Shimura curve $C$ inside $S$ parameterizing polarized abelian surfaces with action by $\mathcal{O}_B$.

In the rest of this paper, we assume that $H^1(S) = 0$. Then, we can write
$$\gamma = \Gamma - [C] \boxtimes f_* \eta - \eta \boxtimes f_* C.$$More generally, for any cycle $e \in \text{Ch}^1(C)$ of degree 1, and define
$$\gamma_e = \Gamma - [C] \boxtimes f_* e - e \boxtimes f_* C \in J^2(X).$$This class is killed by any polarization $\eta \in \text{Ch}^1(C)$ but not necessarily by polarization $\xi \in \text{Ch}^1(S)$. It is killed by $\xi$ if and only if $e = f^* \xi$. For two different $e_1, e_2 \in \text{Ch}^1(C)$ of degree 1, the difference is given by
$$\gamma_{e_2} = \gamma_{e_1} + (e_1 - e_2) \boxtimes f_* C.$$Let $\varphi = f^* f_* C \in \text{Ch}^1(C)$ and $d = \deg \varphi$. Then, we can compute the difference between their heights as follows:
$$\langle \gamma_{e_2}, \gamma_{e_2} \rangle_B = \langle \gamma_{e_1}, \gamma_{e_1} \rangle_B - 2\langle \gamma_{e_1} f_* C, e_1 - e_2 \rangle_{\text{NT}} - \langle e_2 - e_1, e_2 - e_1 \rangle_{\text{NT}} d$$
$$= \langle \gamma_{e_1}, \gamma_{e_1} \rangle_B - 2\langle \varphi - de_1, e_1 - e_2 \rangle_{\text{NT}} - \langle e_1 - e_2, e_1 - e_2 \rangle_{\text{NT}} d$$

If $d = 0$, we have
$$\langle \gamma_{e_2}, \gamma_{e_2} \rangle_B = \langle \gamma_{e_1}, \gamma_{e_1} \rangle_B - 2\langle \varphi, e_1 - e_2 \rangle_{\text{NT}}$$If $\varphi = 0$, then the height of $\gamma_e$ does not depend on the choice of $e$.

If $d \neq 0$, we take $e_0 = \varphi/d$, to obtain for any $e$,
$$\langle \gamma_e, \gamma_e \rangle_B = \langle \gamma_{e_0}, \gamma_{e_0} \rangle_B - d\langle e - e_0, e - e_0 \rangle_{\text{NT}}$$Thus $\langle \gamma_e, \gamma_e \rangle_B$ has its maximal value at $e_0$ if $d > 0$ and minimal value at $e_0$ if $d < 0$.

**Example 6.3.** Take $S = \mathbb{P}^2_B$ with a hyperplane $H$, and and $e_S = H^2 \in \text{Ch}^2(S)$. Let $f : C \to S$ be an embedding from a smooth curve. Then, as a Chow cycle
$$\Delta_S = e_S \boxtimes S \boxplus e_S \boxplus H \boxplus H, \quad \Gamma = C \boxtimes e_S + f^* H \boxplus H$$It follows that $\gamma = 0$ for any polarization $\xi$ on $S$. 

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Example 6.4. Take $S = \mathbb{P}^1 \times \mathbb{P}^1$ with two hyperplanes $H_1 = 0 \times \mathbb{P}^1, H_2 = \mathbb{P}^1 \times \infty$. Then $\text{NS}(S) = \mathbb{Q}H_1 + \mathbb{Q}H_2$. Let $f : C \to S$ be an embedding from a smooth curve linearly equivalent to $aH_1 + bH_2$ with both $a, b > 0$. Then we can define an orthogonal basis for $\text{NS}(S) = \text{Ch}^1(S)$:

$$h_1 = [\nu C] = aH_1 + bH_2, \quad h_2 = aH_1 - bH_2.$$  

Thus, as Chow cycles, we have the following identity:

$$\Delta_S = e_S \boxtimes S + S \boxtimes e_S + h_1 \boxtimes h_1/(h_1, h_1) + h_2 \boxtimes h_2/(h_2, h_2)$$

where $e_S = 0 \times \infty \in \text{Ch}^2(S)$. Take $\xi = [C]/[C]^2$ and $\eta = f^*\xi$. Then

$$\Gamma = C \boxtimes e_S + \eta \boxtimes f(C) + \nu h_2 \boxtimes h_2/(h_2, h_2).$$

It follows that

$$\gamma = f^*h_2 \boxtimes h_2/(h_2, h_2).$$

It is clear that $\alpha := f^*(h_2)$ has degree 0. In fact, let $\pi_1, \pi_2 : C \to \mathbb{P}^1$ be two projections induced by $f$, then $\alpha = a\pi_1^*\infty - b\pi_2^*\infty$. Thus

$$\langle \gamma, \gamma \rangle_B = -\langle \alpha, \alpha \rangle_{NT}/(h_2, h_2) = \frac{1}{2ab}(\alpha, \alpha)_{NT}.$$  

Moreover, the projection of $\gamma$ on $J^2(X)_{00} = \text{Alb}(C) \boxtimes \text{NS}(S)_{0}$ is given by

$$\alpha \boxtimes h_2/(h_2, h_2).$$

7. Unramified calculation

Now we assume that $K = k(B)$ is the function field of a smooth and projective curve $B$ over an algebraically closed field $k$, and $f : C \to S$ an embedding from a smooth and proper curve to a smooth and proper surface over $K$. We assume that $d := \deg f^*f_*C \neq 0$. As in §6, let $e = f^*f_*C/d \in \text{Ch}^1(C)$ of degree 1, and define a homologically trivial cycle

$$\gamma := \Gamma(f) - C \boxtimes f_*(e) - e \boxtimes f_*C \in J^2(X).$$

We want to compute its heights when $f$ extends to a morphism $\tilde{f} : \mathcal{C} \to \mathcal{S}$ of a smooth and proper curve over $B$ to a smooth and proper surface over $B$. Denote

$$\omega_\mathcal{C} := \Omega^1_{\mathcal{C}/B}, \quad \omega_\mathcal{S} = \kappa^2\Omega^1_{\mathcal{S}/B}.$$  

Theorem 7.1.

$$\langle \gamma, \gamma \rangle_B = -\frac{1}{d}((d + 1)\omega_\mathcal{C} - \tilde{f}^*\omega_\mathcal{S}) \cdot (\omega_\mathcal{C} - \tilde{f}^*\omega_\mathcal{S}).$$  

Proof. First we want to extend $\gamma$ to a cycle on the four-fold $\mathcal{X} = \mathcal{C} \times_B \mathcal{S}$

$$\tilde{\gamma} = \Gamma(\tilde{f}) - \tilde{e} \boxtimes \tilde{f}_*(\mathcal{C}) - \mathcal{C} \boxtimes \tilde{f}_*\tilde{e} \in \text{Ch}^2(\mathcal{X}).$$

where

$$\tilde{e} := \tilde{f}^*\tilde{f}_*\mathcal{C}/d + \kappa F \in \text{Ch}^1(\mathcal{C}).$$

where $F$ is the fiber class of $\mathcal{C}$ over $B$ and $\kappa = -\frac{1}{2}(f^*f_*\mathcal{C}/d)^2$ so that $\tilde{e}^2 = 0$ on $\mathcal{C}$.
Let \( i : \mathcal{Y} := \mathcal{C} \times_B \mathcal{C} \rightarrow \mathcal{X} \) denote the embedding induced by \( f \) and let \( \delta \) denote the cycle \( \Delta - e \times \mathcal{C} - \mathcal{C} \times e \). Then we have \( \bar{\gamma} = i_* \delta \). Thus

\[
\langle \gamma, \gamma \rangle_B = i_* \delta \cdot i_* \delta = \delta \cdot \delta \cdot i_* i_* \mathcal{Y} = \delta \cdot \delta \cdot i_* \pi_\mathcal{S}^* f_\mathcal{S}^* \mathcal{C}
\]

\[
= \delta \cdot \delta \cdot p_{2*} f^* \bar{f}_* \mathcal{C} = p_{2*}(\delta \cdot \delta) \cdot \bar{f}^* f_\mathcal{S}^* \mathcal{C}
\]

We need to compute \( p_{2*}(\delta \cdot \delta) \):

\[
p_{2*}(\delta \cdot \delta) = p_{2*}(\Delta \cdot (\Delta - 2\bar{e} \boxtimes \mathcal{C} - 2\mathcal{C} \boxtimes \bar{e})) + p_{2*}((\bar{e} \boxtimes \mathcal{C} + C \boxtimes \bar{e})^2).
\]

For the first term, we use the projection formula for the diagonal embedding:

\[
p_{2*}(\Delta \cdot (\Delta - 2\bar{e} \boxtimes \mathcal{C} - 2\mathcal{C} \boxtimes \bar{e})) = p_{2*}\Delta_* \Delta^*(\Delta - 2\bar{e} \boxtimes \mathcal{C} - 2\mathcal{C} \boxtimes \bar{e}))
\]

\[
= \Delta^*(\Delta - 2\bar{e} \boxtimes \mathcal{C} - 2\mathcal{C} \boxtimes \bar{e})) = -\omega_{\bar{e}} - 4\bar{e}.
\]

For the second term, we compute it directly,

\[
p_{2*}((\bar{e} \boxtimes \mathcal{C} + \mathcal{C} \boxtimes \bar{e})^2) = p_{2*}(\bar{e}^2 \boxtimes \mathcal{C} + 2\bar{e} \boxtimes \bar{e} + \mathcal{C} \boxtimes \bar{e}^2) = 2\bar{e}.
\]

Thus we have

\[
p_{2*}(\delta \cdot \delta) = -\omega_{\bar{e}} - 2\bar{e}.
\]

Put these two terms together, we get

\[
\langle \gamma, \gamma \rangle_B = -(\omega_{\bar{e}} + 2\bar{e}) \cdot \bar{f}^* \bar{f}_* \mathcal{C} = -\omega_{\bar{e}} \cdot \bar{f}^* \bar{f}_* \mathcal{C} + 2d\kappa = -\omega_{\bar{e}} \cdot \bar{f}^* \bar{f}_* \mathcal{C} - \frac{1}{d}(\bar{f}^* \bar{f}_* \mathcal{C})^2.
\]

The final formula in Theorem 7.1 follows from the adjunction formula:

\[
f^* f_* \mathcal{C} = \omega_{\bar{e}} - f^* \omega_{\mathcal{S}}.
\]

\( \Box \)

**Example 7.2.** We assume that \( \mathcal{S} / B \) is a smooth family of K3 surfaces. Then \( \omega_{\mathcal{S}} \) is a vertical class \( h(S)^F \), for the canonical height \( h(S) \in \mathbb{R} \) of \( \mathcal{S} \). Then \( f^* f_* \mathcal{C} = \omega_{\bar{e}} - h(S)^F \) with degree \( d = 2g - 2 \). It follows that

\[
\langle \gamma, \gamma \rangle_B = -\frac{2g - 1}{2g - 2} \omega_{\bar{e}}^2 + 2gh(S).
\]

If \( g \geq 2 \), then by Beilinson’s index conjecture, \( \langle \gamma, \gamma \rangle_B \geq 0 \). Thus, we have the following conjectured inequality:

\[
\omega_{\bar{e}}^2 \leq \frac{4g(g-1)}{2g-1} h(S).
\]

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