Persistent Edge Currents for Paired Quantum Hall States

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We study the behavior of the persistent edge current for paired quantum Hall states on the cylinder. We show that the currents are periodic with the unit flux \( \phi_0 = \hbar c/e \). At low temperatures, they exhibit anomalous oscillations in their flux dependence. The shape of the functions converges to the sawtooth function periodic with \( \phi_0/2 \).

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I. INTRODUCTION

One of the surprising aspects of the fractional quantum Hall effect is that the edge state forms a new kind of state of matter beyond Fermi liquid, called the chiral Tomonaga-Luttinger liquid [1]. Some experiments have already demonstrated the characteristic behavior of chiral Tomonaga-Luttinger liquid [2,3]. Recently the Aharonov-Bohm effect (AB effect) in such systems were studied by and Geller et al [4,5] and Chamon et al [6]. Especially, the latter authors predict new edge-current oscillations in the persistent edge current of the \( \nu = 1/q \) Laughlin state, which has no amplitude reduction from disorder and thus results in a universal non-Fermi-liquid temperature dependence. Also the persistent current for the annulus Laughlin state was recently investigated by Kettemann [7]. These studies show the current is periodic with a unit flux quantum \( \phi_0 = \hbar c/e \) in agreement with the theorem of Byers and Young [8]. The period would be \( q\phi_0 \) if the quasiparticle had a unit charge \( e \) instead of a fractional charge \( e/q \) [9].

Motivated by these recent studies, we investigated the persistent currents in paired fractional Hall states in Ref. [10]. We showed that the currents are periodic with the unit flux \( \phi_0 = \hbar c/e \). At low temperatures, they exhibit anomalous oscillations in their flux dependence. The shape of the functions converges to the sawtooth function periodic with \( \phi_0/2 \). In this paper, we extend these results to the states on the cylinder geometry and discuss some extremal limits.

The states we will consider are the 331 state [11], the Haldane-Rezayi state [12] and the Pfaffian state [13]. They are quantum Hall analogs of the BCS superconductor. The pairing symmetry is p-wave with \( S_z = 1, 0 \) for the Pfaffian and the 331 states respectively, and d-wave for the Haldane-Rezayi states. The Haldane-Rezayi state and the Pfaffian state are candidate states for \( \nu = 5/2 \) plateau [14]. They are supposed to exhibit some novel features beyond ordinary quantum Hall states, such as nonabelian statistics and specific degeneracy on a surface with nontrivial topology. On the other hand, the 331 state is a part of generalized hierarchy [15], but can be interpreted as a paired state. It is believed to be realized at \( \nu = 1/2 \) plateau observed in double layer systems [16].

The organization of this paper is as follows. In Sec.2, we recall the edge states of the paired quantum Hall states in Sec.3, we deduce the exact formulas of the persistent edge currents and discuss their analytic properties. We compare the persistent currents by numerical plots. In Sec.4, We consider some extremal limits and their effect on the persistent edge currents. Sec.6 summarizes our conclusions. The definitions of Jacobi \( \theta \) functions are given in Appendix.

II. EDGE STATES FOR PAIRED QUANTUM HALL STATES

Let us recall edge excitations on the circumference \( L \) of the \( \nu = 1/q \) paired quantum Hall states.

The charge sector is a chiral Luttinger liquid \( \varphi \). Its Hilbert space is generated by the \( U(1) \) Kac-Moody algebra formed by \( j = \sqrt{\nu} \partial \varphi \) and the zero modes corresponding to quasiparticles [10]. The Hamiltonian is given by \( H = \frac{1}{2} \sum_{n \in \mathbb{Z}} j_n j_{-n} - \frac{c}{\pi} \) where we include a Casimir factor with \( c = 1 \). The complete description of

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the edge excitations can be given by the grand partition function. For the sector with the charge \( r/q, \ r = 0, 1, \ldots, q - 1 \), they are given by

\[
\chi_{r/q}^{\text{even}}(\tau, \phi) = \frac{1}{\eta} \sum_{m \in \mathbb{Z}_{\text{even}}} e^{2\pi i \tau \left( \frac{mq + r}{q} \right)^2} e^{2\pi i \phi (m + \frac{r}{q})}
\]

(2.1)

\[
\chi_{r/q}^{\text{odd}}(\tau, \phi) = \frac{1}{\eta} \sum_{m \in \mathbb{Z}_{\text{odd}}} e^{2\pi i \tau \left( \frac{mq + r}{q} \right)^2} e^{2\pi i \phi (m + \frac{r}{q})}
\]

(2.2)

\[
\chi_{r/q} = \chi_{r/q}^{\text{even}} + \chi_{r/q}^{\text{odd}}.
\]

(2.3)

(2.4)

where even (odd) refers to the number of electrons and \( \eta \) is the Dedekind function:

\[
\eta(\tau) = x^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - x^n),
\]

(2.5)

\[
x = e^{2\pi i \tau}.
\]

(2.6)

The finite size \( L \) induces a temperature scale, which we will take as

\[
T_0 = \frac{\hbar v}{k_B L},
\]

(2.7)

where \( v \) is the Fermi velocity of the edge modes determined by the confining potential. For example, a Fermi velocity of \( 10^6 \) cm/s and circumference of 1 \( \mu \)m yields \( T_0 \sim 60 \) mK.

Let us suppose that the edge state is coupled to an Aharanov-Bohm flux \( \Phi \). The coupling of the edge state to an Aharanov-Bohm flux \( \Phi \) is achieved by putting \( \phi = \Phi/\phi_0 \) to \( \phi \) in (2.1) \( \Phi \) with \( \phi_0 = \hbar c/e \), the unit flux quantum.

Next we consider the extra internal degrees of freedom other than chiral Luttinger liquid. In the bulk conformal field theory description, pairing is due to the internal degrees of freedom given by some kind of fermion \( \psi \). The charge degrees of freedom is given by a chiral boson \( \varphi \). The operator for the electron is of the form \( \psi e^{i\sqrt{\eta}} \varphi \) where \( q \) is even. Thus the filling fraction \( \nu = 1/q \) have an even denominator.

The edge states of these states have a fermionic sector corresponding to \( \psi \). The fermionic and charge sectors are not decoupled in the following sense: the fermionic internal degree of freedom requires the global selection rules in the bulk and edge of the paired state. Thus the Hilbert spaces of the edge excitations of these states are not simply the direct sum of the sectors and the grand partition functions are not given by the direct product of the ones for these sectors.

The additional sectors of edge excitations for Pfaffian, Haldane-Rezayi, 331 states are given by Majorana-Weyl fermions (MW), symplectic fermions(Sf), Dirac fermions (D) respectively. These modes contribute to the partition function through the following functions for untwisted and twisted sectors with even or odd number of fermions :
\[ \chi_{1}^{\text{Sl}}(\tau, \phi) = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{1}^{\infty} (1 + x^n)^2 + \prod_{1}^{\infty} (1 - x^n)^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_2(\tau)}{2\eta(\tau)} + \eta(\tau) \right)^2, \]  
(2.14)
\[ \chi_{\psi}^{\text{Sl}}(\tau, \phi) = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{1}^{\infty} (1 + x^n)^2 - \prod_{1}^{\infty} (1 - x^n)^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_2(\tau)}{2\eta(\tau)} - \eta(\tau) \right)^2, \]  
(2.15)
\[ \chi_{\sigma}^{\text{Sl}}(\tau, \phi) = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{0}^{\infty} (1 + x^{n+\phi})^2 + \prod_{0}^{\infty} (1 - x^{n+\phi})^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_3(\tau)}{\eta(\tau)} + \theta_4(\tau) \right)^2, \]  
(2.16)
\[ \chi_{r}^{\text{Sl}}(\tau, \phi) = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{0}^{\infty} (1 + x^{n+\phi})^2 - \prod_{0}^{\infty} (1 - x^{n+\phi})^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_3(\tau)}{\eta(\tau)} - \theta_4(\tau) \right)^2, \]  
(2.17)
\[ \chi_{1}^{\text{D}} = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{0}^{\infty} (1 + x^{n+\phi})^2 + \prod_{0}^{\infty} (1 - x^{n+\phi})^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_3(\tau)}{\eta(\tau)} + \theta_4(\tau) \right)^2, \]  
(2.18)
\[ \chi_{\psi}^{\text{D}} = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{0}^{\infty} (1 + x^{n+\phi})^2 - \prod_{0}^{\infty} (1 - x^{n+\phi})^2 \right) \]
\[ = \frac{1}{2} \left( \frac{\theta_3(\tau)}{\eta(\tau)} - \theta_4(\tau) \right)^2, \]  
(2.19)
\[ \chi_{\sigma}^{\text{D}} = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{1}^{\infty} (1 + x^n)^2 + \prod_{1}^{\infty} (1 - x^n)^2 \right) \]
\[ = \frac{1}{2} \left( \theta_2(\tau) \frac{2\eta(\tau)}{\eta(\tau)} \right)^2, \]  
(2.20)
\[ \chi_{\sigma}^{\text{D}} = \frac{1}{2} x^{-\phi/\pi} \left( \prod_{1}^{\infty} (1 + x^n)^2 - \prod_{1}^{\infty} (1 - x^n)^2 \right) \]
\[ = \frac{1}{2} \left( \theta_2(\tau) \frac{2\eta(\tau)}{\eta(\tau)} \right)^2. \]  
(2.21)

### III. Persistent Edge Current for the States on the Cylinder

Let us derive the persistent edge current for the state on the cylinder. For the Laughlin state, the sum of the edge currents (for the annulus) was studied by Ketteman et al. [1]. In this section, we study the sum of the currents for Pfaffian, 331 and Haldane-Rezayi state.

The AB flux is through the center of the annulus. The low-energy excitations are edge excitations on the inner and our edges. On the bulk, it is necessary to consider the grand-canonical ensemble of the electrons, which leads to the constraints between the excitation on the two edges. The grand-canonical partition functions emerging here are the ones for the corresponding extended conformal field theories with the electron as the extending field. The grand-canonical partition functions with AB flux are [24,25]

**Pfaffian State**

\[ Z_{Pf}^{cyl}(\tau, \phi) = \sum_{r=0}^{q-1} \left[ |\lambda_1^{\text{MW}} x_{r/q}^{\text{even}} + \lambda_\psi^{\text{MW}} x_{r/q}^{\text{odd}}|^2 + |\lambda_\psi^{\text{MW}} x_{r/q}^{\text{even}} + \lambda_{1}^{\text{MW}} x_{r/q}^{\text{odd}}|^2 \right]. \]  
(3.30)
331 State

\[
Z_{331}^{cyl}(\tau, \phi) = \sum_{r=0}^{q-1} \left[ |\chi^0_{\text{D}, r/q} + \chi^0_{\text{D}, r+1/2, 2r/q}|^2 + 2|\chi^0_{s, \phi(r+1/2)/q}|^2 \right] \quad (3.31)
\]

Haldane-Rezayi State

\[
Z_{331}^{cyl}(\tau, \phi) = \sum_{r=0}^{q-1} \left[ |\chi^0_{\text{D}, r/q} + \chi^0_{\text{D}, r+1/2, 2r/q}|^2 + 2|\chi^0_{s, \phi(r+1/2)/q}|^2 \right]. \quad (3.32)
\]

By using the modular transformation, we end up with the following expressions in terms of theta functions \((t = -\frac{1}{\tau})\),

\[
Z_{331} = \frac{1}{2q\eta(q)} \sum_{r=0}^{q-1} \left[ \theta_4(0|t)\theta_3(\frac{\phi + r}{q}, \frac{t}{q})^2 + \theta_3(0|t)\theta_3(\frac{\phi + r + 1/2}{q}, \frac{t}{q})^2 + \theta_2(0|t)\theta_1(\frac{\phi + r}{q}, \frac{t}{q})^2 \right]. \quad (3.34)
\]

The formulae for the sum of the persistent currents on two edges are obtained by differentiation of the free energy \(-T\log Z\) in term of \(\phi\)

\[
I \equiv T \frac{\partial \ln Z(\tau, \phi)}{\partial \phi}. \quad (3.37)
\]

We plot in Fig. 3 the flux dependence of the persistent edge current for \(\nu = 1/2\) Haldane-Rezayi state at temperatures \(T/T_0 = 0.26, 0.23, 0.2, 0.17, 0.15, 0.135\). It shows a similar low-temperature behavior as in the case of disk. This is interpreted as an incorporation between the Byers-Yang and Mermin-Wagner theorems. The Pfaffian and 331 states also show a similar behavior but different. At low temperature limit, they converge as

\[
I \to -2\nu \frac{ev}{k_B L} (\phi - \frac{1}{2} r), \quad -\frac{1}{4} + \frac{1}{2} r < \phi < \frac{1}{4} + \frac{1}{2} r, \quad r \in \mathbb{Z}. \quad (3.38)
\]

Note that the total amount of the current is as twice as the one in the disk case.

Anomalous oscillations of the persistent composite fermions can be explained from the pairing of composite fermions in the bulk and edge state of paired states. Naively, the edge persistent currents which we have calculated should have a period \(\phi_0/2\) since the bulk states of paired states are in a BCS superconducting phase and the order parameter has a charge \(2e\). However, from the Mermin-Wigner theorem, the edge states cannot be a BCS-like condensate except at zero temperature. Then, from the behavior of the currents we have found, we see that as we lower the temperature, the edge states become closer to a BCS-like condensate, but the Mermin-Wigner theorem prevents the phase transition of the edge states. The effect of the pairing of composite fermions in the bulk changes continuously as a function of the temperature. The BCS pairing in the edge states becomes stronger at lower temperatures, but the condensation only occurs at zero temperature.

This phenomenon may be seen as an interesting bridge between superconductivity in 2+1 dimensions and superconductivity in 1+1 dimensions.

Thus we see that the predicted behavior of persistent edge currents can be used as a method to distinguish the bulk topological order. Especially the flux dependence can reveal the pairing of composite fermions in the bulk of fractional quantum Hall states.

Experimentally, the magnitude of the persistent currents is preferably measurable at low temperatures and small samples. As we have predicted, experiments at the even-denominator plateaus may detect the anomalous oscillations of the persistent current.
IV. FROM CYLINDER TO DISK

By considering different radii $R_1$, $R_2$ for two edges respectively in the formulas in the previous section, the persistent edge current for the annulus case can be treated. In this case, different temperature scales $T_1 = \frac{\hbar v}{2\pi k_B R_1}$ and $T_2 = \frac{\hbar v}{2\pi k_B R_2}$ arise. We will consider some extremal limits in this section. The extremal limits we will consider is the low temperature limit and the small radius limit. We will see that the disk grand partition function play an important role.

Let us consider the low temperature behavior of persistent current for a FQH on a cylinder. Generally the grand-partition function $Z(\tau, \phi)$ for a FQH on a cylinder has a form

$$Z(\tau, \phi) = \sum_{i,j} N_{ij} \bar{Z}_i Z_j$$

(4.39)

where $i$ and $j$ are indices for each sector in the radii $R_1$ and $R_2$ edge states respectively and $N_{ij}$ are integer coefficients.

As we change $T$ to 0, for a given value of $\phi$, the sector determined by $\phi$ gives a dominant contribution to $Z$. We put it as $i$-th sector, then the partition function at $T \sim 0$ becomes

$$Z(\tau, \phi) \sim \bar{Z}_i \sum_j N_{ij} Z_j.$$  

(4.40)

This decomposition is valid except around $\phi = \pm 1/4$. Thus the persistent edge current is divided into a sum

$$I \sim \bar{T}_1 + \bar{T}_2,$$  

(4.41)

where $\bar{T}_1, \bar{T}_2$ is calculated only from the one edge and actually same as the one calculated from the disk partition function $Z_{\text{disk}} = \sum_{i,j} N_{ij} Z_j$. Thus at sufficiently low temperature, the total persistent edge current is calculated from the one for the disk case except around $\phi = \pm 1/4$.

It is also interesting to consider an extremal limit $R_1 \to 0$ with the AB flux fixed at $\phi$. As we change $R_1$ to 0, $T_1$ goes to large. Thus for the small radius edge, we are effectively in the low temperature region. Thus the sector determined by $\phi$ gives a dominant contribution to $Z$. As above, the partition function at $R_1 \sim 0$ becomes

$$Z(\tau, \phi) \sim \bar{Z}_i \sum_j N_{ij} Z_j,$$  

(4.42)

except around $\phi = \pm 1/4$. Thus the persistent edge current is divided into a sum

$$I \sim \bar{T}_1 + \bar{T}_2,$$  

(4.43)

where $\bar{T}_2$ can be calculated from the disk partition function $Z_{\text{disk}}$ except around $\phi = \pm 1/4$.

Thus at small radius, the persistent edge current on the one edge can be calculated from the formula for the disk case. This observation can be extended to a FQH state on a region with arbitrary number of boundaries.

V. CONCLUSIONS

In this paper, we have deduced the following conclusions extending our previous result for the disk case:

1. The total amount of persistent edge currents in the paired quantum Hall states on the cylinder are flux periodic with period $\phi_0$ in agreement with the theorem of Byer and Young.

2. At low temperatures, the currents exhibit anomalous oscillations as functions of the Aharanov-Bohm flux. The shape of anomalous oscillations depends on the bulk topological order and converges to a sawtooth function with period $\phi_0/2$ at zero temperature.

3. This behaviors implies that, although the bulk states of paired states are in a BCS superconducting phase, the edge states are a BCS condensate only at zero temperature. The BCS pairing structure changes the flux dependence of the persistent current continuously and there is no phase transition at finite temperatures. This is in agreement with the Mermin-Wigner theorem.
4. These pairing effects, especially anomalous oscillations, may provide a means to observe paired quantum Hall states at the $\nu = 5/2$ plateau and the $\nu = 1/2$ plateau in double layers.

5. The phenomenon we predict may be seen as an interesting bridge between the superconductivity in 2+1 dimensions and in 1+1 dimensions.

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Appendix: Jacobi $\theta$ functions

The Jacobi $\theta$ functions $\theta_1, \theta_2, \theta_3, \theta_4$ are examples of modular forms. They are defined by

$$\theta_1(\zeta | \tau) = -i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r-1/2} y^r q^{r^2/2}, \quad (0.1)$$

$$\theta_2(\zeta | \tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} y^r q^{r^2/2}, \quad (0.2)$$

$$\theta_3(\zeta | \tau) = \sum_{n \in \mathbb{Z}} y^n q^{n^2/2}, \quad (0.3)$$

$$\theta_4(\zeta | \tau) = \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{n^2/2}, \quad (0.4)$$

with $q = \exp 2\pi i \tau$ and $y = \exp 2\pi i \zeta$. They also have the following infinite product forms:

$$\theta_1(\zeta | \tau) = -iy^{1/2} q^{1/8} \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=0}^{\infty} (1 - yq^{n+1})(1 - y^{-1}q^n), \quad (0.5)$$

$$\theta_2(\zeta | \tau) = y^{1/2} q^{1/8} \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=0}^{\infty} (1 + yq^{n+1})(1 + y^{-1}q^n), \quad (0.6)$$

$$\theta_3(\zeta | \tau) = \prod_{n=1}^{\infty} (1 - q^n) \prod_{r \in \mathbb{N} + \frac{1}{2}} (1 + yq^r)(1 + y^{-1}q^{r'}), \quad (0.7)$$

$$\theta_4(\zeta | \tau) = \prod_{n=1}^{\infty} (1 - q^n) \prod_{r \in \mathbb{N} + \frac{1}{2}} (1 - yq^r)(1 - y^{-1}q^{r'}). \quad (0.8)$$

The differential of $\theta(\zeta | \tau)$ with respect to $\zeta$ are denoted as $\theta'(\zeta | \tau)$.

The transformation property of Jacobi theta functions and the Dedekind $\eta$ function Eq.(2.5) under $S$ is

$$\theta_2(-1/\tau) = \sqrt{-i\tau} \theta_4(\tau), \quad (0.9)$$

$$\theta_3(-1/\tau) = \sqrt{-i\tau} \theta_3(\tau), \quad (0.10)$$

$$\theta_4(-1/\tau) = \sqrt{-i\tau} \theta_2(\tau), \quad (0.11)$$

$$\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (0.12)$$

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FIG. 1. The flux dependence of the total persistent current for the $\nu = 1/2$ Haldane-Rezayi state on the cylinder at temperatures $T/T_0 = 0.26, 0.23, 0.2, 0.17, 0.15, 0.135$. 