Probabilities as Measures of Information

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Abstract

We analyze the notion that physical theories are quantitative and testable by observations in experiments. This leads us to propose a new, Bayesian, interpretation of probabilities in physics that unifies their current use in classical physical theories, experimental physics and quantum mechanics. Probabilities are the result of quantifying the domain of possibilities that results when we interpret observations within the framework of a physical theory. They could also be said to be measures of information used to make predictions based upon a physical theory.
I. INTRODUCTION

Even though the concept of probabilities has been used in physics for more than 200 years, Laplace and Gauss having made important contributions, there is no general agreement today concerning their interpretation, even when limited to their use in physics. There is not even a consensus among physicists as to whether they are ‘logical’ (‘subjective’, Bayesian) or ‘physical’ (‘objective’, statistical) when they are used in physics. Probabilities do not enter in all our physical theories. They are absent from classical mechanics, classical electromagnetic theory and the theories of relativity. However, they play an essential role in our most successful physical theory: quantum mechanics. Because there is complete agreement among physicists regarding how probabilities in physical theories should be tested via observations in experiments, the issue of why and/or how they arise in some physical theories and not in others is not thought to be fundamental to physics itself.

For some physicists, probabilities in physical theories are ‘physical’ in nature because there are some physical phenomena that are ‘stochastic’ or ‘random.’ For others, probabilities are ‘physical’ in a ‘statistical’ sense and they arise because of ‘incomplete’ descriptions, as could be considered the case for classical statistical mechanics. For others still, different physical mechanisms would be responsible for the probabilities of classical physics and quantum mechanics.

On the other side, during the last 50 years the notion that probabilities in physics were of a Bayesian nature has been getting more and more support. That this was the case for probabilities in classical physics is likely in large part due to Jeffreys[1] and Jaynes[2]. The notion that in experimental physics the concepts of Bayesian and statistical probabilities should be treated not only on an equal footing, but even numerically combined, has now received the imprimatur of the BIPM[3]. With the development of the field of quantum computation, during the last 10 years, the notion that the probabilities of quantum mechanics could also be Bayesian measures of information has been receiving some support (Caves et al[4], Mermin[5], Fuchs[6], Peres[7].) In our opinion, an essential obstacle to the general acceptance by the physics community that probabilities in physics are Bayesian in nature is the fact that they are said to be: “degrees of belief that an event will occur” as in the BIPM report[3], “degrees of truth of an assertion conditional upon the truth of some other assertion(s),” as advocated by Jaynes[2] following Cox[8], or similar concepts that are
thought by most physicists to lie outside the realm of physics. In section II, we show that starting from the notion that classical mechanics is quantitative and testable by observations in experiments we can generate the algebra of the probabilities used in classical physics. Since there are no probabilities in classical mechanics, we will obtain probabilities when we deal quantitatively with the ambiguities that must result when, given some observations in an experiment, we use classical mechanics to make predictions for future observations that could be considered as tests of classical mechanics. Consequently, the probabilities so generated will be Bayesian in nature and can be said to be measures of the information we have when making predictions for the values of the observable dynamical variables of classical mechanics. In section III, we show how, due to a very well understood difference between observing classical and quantum systems, classical mechanics must be modified for quantum systems in order to be quantitative and this leads to the probability amplitudes and to the principle of superposition of quantum mechanics. However, the probabilities in quantum mechanics will have precisely the same meaning as they have in our analysis of observations in experiments using classical mechanics. They will quantify the ambiguities concerning predictions that we can make for future observations on quantum systems after we have made on them unambiguous observations. Therefore they will also be measures of the information we have when making predictions for the values of the observable dynamical variables of quantum mechanics.

In our approach, probabilities in physics will be thoroughly defined entities in terms of well understood concepts of classical mechanics and now well established mathematical tools. What we propose is a simple explanation for why probabilities play a fundamental role in classical and quantum physics, explanation that we think is missing today. This explanation is not based upon any new physical mechanism. It is based upon an interpretation of a well established mathematical technique for dealing with a fundamental problem that experimentalists must face in every experiment. Consequently, it could be said that our argument is philosophical, rather than physical. While this does not present any problem concerning classical mechanics, since it is well understood. In the case of quantum mechanics, our interpretation of how and why its probabilities arise leaves us in the same conundrum as a plethora of other proposed interpretations that do not affect in any way the current use of quantum mechanics. However, what sets our argument apart from other interpretations of the nature and meaning of probabilities in physics, in particular in quantum mechanics, is
that it rests upon a simple statement that we think is non controversial and should be accept able to every physicist: ‘Physical theories are quantitative and testable by observations in experiments.’

II. CLASSICAL MECHANICS

We understand so well classical mechanics that there seems to be no need to discuss at length how classical mechanics is a quantitative theory: in its formulas the symbols stand for real numbers. These formulas will be numerically satisfied when we substitute into them the numbers that correspond to the values of the measurable dynamical variables in a given situation. Most frequently these formulas are used to make a quantitative prediction for an observation corresponding to one of the dynamical variable that was not measured in an experiment, on the basis of the numerical values of the other dynamical variables that were measured in that experiment. Quoting from a recent paper[9]: “The description of the system can be given, in classical mechanics, by a phase-space point. This point is the ‘true’ point — others are false — so the outcome of a measurement can be predicted with certainty.” We think that the overwhelming majority of physicists, if not all of them, would agree with this statement. The only trouble is that although we know that this statement is theoretically correct, for the last two hundred years we have not been able to do in practice, and will never be able to do, what we state we should theoretically be able to do!

It is frequently stated that in contradistinction with quantum mechanics, classical mechanics is deterministic, i.e. no probabilities enter in its formulas. However, it is a fact that probabilities have been used for a long time to convey the result of every measurement of the dynamical variables that enter classical mechanics: position, volume, momentum, length, etc... For some time now, no refereed physics journal will accept for publication an experimental paper that will not include a quantified discussion, using probabilities, of the measurements reported. Until rather recently these probabilities were said in textbooks to describe the ‘statistical errors’ in the measurements, but that there were also other experimental ‘errors’ that were said to be ‘systematic’ in nature and these were not ‘quantified’ by means of probabilities because there were no ‘statistics’ associated with them. Over the last thirty years or so, this point of view has gradually changed, one no longer speaks of experimental ‘errors,’ whether ‘statistical’ or ‘systematic,’ but instead of experimental
uncertainties. Most significantly, the BIPM[3] now recommends that all measurement uncertainties, including those that are not ‘statistical’ in nature, be quantified by means of probabilities. The probabilities used to quantify the systematic uncertainties are said to correspond “to a degree of belief that an event will occur.” Although the BIPM report reviews at length the concept of ‘statistical’ probabilities, only two references to the concept of ‘Bayesian statistics’ are given and how they should be evaluated in physics experiments is not discussed. Finally, the BIPM recommends that all the uncertainties, whether ‘statistical’ or ‘systematic’ in nature, be combined into a single probability statement.

‘Statistical’ uncertainties do not arise in many measurements of classical mechanics dynamical variables. We therefore postpone discussing them until we have dealt with the ‘systematic’ uncertainties that are impossible to avoid. Another reason we postpone discussing ‘statistical’ uncertainties in this paper is that the solution we propose for dealing with the ‘systematic’ uncertainties will give us the tool for dealing with the ‘statistical’ uncertainties. There is at least one reason why ‘systematic’ uncertainties are unavoidable, they must arise because in order to be quantitative we make measurements and in every measurement we will specify the outcome with only a finite number of digits. We are not at all concerned with the notion that some measurements may be accurate enough for some specific purpose and consequently can be considered exact for that purpose. What we analyze now is a fundamental problem that results from the fact that in practice we must report a measurement with a finite number of digits. How this arises can be made concrete via the example of measuring the length of an object using a set of ‘go, no-go’ gauges. With a set of ‘go, no-go’ gauges, if one repeats the measurement, one always (i.e. systematically) gets the same result: the value we seek is smaller than given by the ‘go’ gauge but larger than given by the ‘no-go’ gauge, whatever are the units used to label the set of gauges used. Clearly, these two different values are not the length of the object, since it must be a single real number. These two values are an attribute of the set of ‘go, no-go’ gauges we use to make the observation that we interpret as providing direct information regarding the length of the object. If we had used a finer set of ‘go, no-go’ gauges we would likely have obtained two different values, but we would still have obtained two numbers. In the situation we have just described, there are no distribution of values that could be dealt with using statistical concepts.

Many theoretical papers analyzing physical measurements in quantum mechanics view the
interaction of the observer with the classical measuring instrument as the reading of a ‘pointer on the dial of a gauge’. Due to the finite thickness of the tip of the pointer, and the finite size of the dial of a gauge, one can only state with certainty that it does not point below a certain value nor above another value, a result which is conceptually identical to using a set of ‘go, no-go’ gauges. In our digital age, we could replace the traditional ‘pointer on the dial of a gauge’ with a ‘digital counter’ that must necessarily have a finite number of digits. We therefore know that the ‘real number’ we seek is not less than the value shown on the counter but not more than the value obtained by increasing by 1 the least significant digit. From a mathematical point of view, as opposed to a practical one, the problem is the same whether we deal with real numbers that have a single digit or many digits: there is a continuum of real numbers between them. This source of ‘systematic’ uncertainty is present in every physics experiment. In many experiments there are other sources of ‘systematic’ uncertainties, but they are all of the same nature in the sense that they provide both a lower and an upper bound to the value sought. Of course, all sources of uncertainties in measurements must be combined to yield an overall uncertainty. In this paper we do not address this issue. In this section we analyze how one can deal quantitatively with the problem of ‘systematic’ uncertainties and we further limit ourselves to analyze this problem in the context of a classical mechanics experiment.

Is one testing classical mechanics when one reports the ‘measurement’ of the length, the position, the momentum, etc., of an object to which classical mechanics applies? In a very real sense yes because what is reported in experimental papers as having been measured in an experiment, is the result of a process that is often called the data analysis, or data reduction, where: from the observations made in the experiment, using the relations of classical mechanics, one makes a deduction concerning the value of the quantity reported as having been measured. Therefore, in this data analysis one is making a prediction on the basis of observations made in the experiment and the quantitative relations of classical mechanics in which the reported measured quantity enters and to which the observations apply. If subsequent observations contradicted the conclusions of the data reduction reported as the measured quantity one would suspect either that a mistake was made or, very unlikely, that classical mechanics was falsified in the experiment. Why should probabilities appear in this process of data analysis since there are no probabilities in the relations used in that data analysis? It is not because experimentalists are uncertain concerning what they have
observed in an experiment. Any observation that could be said to be uncertain should be categorically discarded in drawing predictions based on the relations of classical mechanics. Fundamentally, the reason why there are uncertainties in the predictions we make on the basis of observations in experiments is that these predictions are ambiguous. According to the relations of classical mechanics that relate what was observed and what we seek to predict: several different predictions, i.e. a range of values, would be consistent with the observations, but according to classical mechanics that prediction should be a unique real number. As discussed above, this must necessarily occur if only because we must use a finite number of digits.

Are the notions of “degrees of belief that an event will occur” or of “degree of truth of an assertion conditional upon the truth of some other assertion(s)” fully acceptable in physics in order to assign a probability measure to the possible values? We do not think so because, in our opinion, the concepts of ‘degrees of belief’ and ‘degree of truth’ do not belong in physics. In particular if, as we will show, what is accomplished by using them can be achieved using concepts that have long been considered essential in physics. Since physical theories are quantitative, when faced with the problem of ‘systematic’ uncertainties, what we can do as physicists is be quantitative about it and specify the number, or density, of different possible predictions for the dynamical variable reported as having been measured, given the unambiguous observations in the experiment and the relations of classical mechanics that were used in making these predictions. What we propose is to use a rationale and a mathematical technique that is fundamentally different from the one used today to quantify the ‘uncertainties’ in the predictions, i.e. measurements. Most of the time this new technique, as it should, will yield results, i.e. probabilities, that are close, if not identical, to those obtained today under a different justification. Therefore, what we propose is a new logical and mathematical foundation for the probabilities we use in classical physics.

The proposed new technique is based upon a very simple argument: since according to classical mechanics the ‘value’ (i.e. the real number) of the dynamical variable we seek to predict, the length, the position, the volume, etc., is postulated to be unique in the experiment in question, what we can do is formally transform this value, that we do not know but should be one of the possible values, from the value it has to all the values it has not: the other possible values. Because these entities in classical mechanics are real numbers the transformations in question will be additions or multiplications of real numbers and their
inverse. Since, given what was observed, every possible value could be the value it has in the experiment, the transformations in question must be applied using every possible value as the true value. Consequently, each of these transformations has an inverse and one of them must be an identity transformation. The transformations in question must therefore form a group. We refer to this group as the possibilities generating transformation group for what will be reported as the ‘measured’ dynamical variable in the experiment. What this does is immediately solve our problem of associating a quantitative measure with the set of possibilities. In the case of classical mechanics, the manifolds of the possibilities generating transformation groups are Euclidean spaces, because the phase-spaces of classical mechanics are Euclidean spaces. This is enough to establish that the algebra of a unique measure associated with the set of possibilities will have the algebra of probabilities of classical physics. This unique measure is often called the ‘volume element’ in the group manifold, the ‘Haar’ measure or the ‘weight function’ in group space.

We must emphasize that the probabilities so generated are fully objective in the sense that any one applying the same formulas of classical mechanics, given precisely what was observed in the experiment must come up with exactly the same probability measure for the set of possibilities. Such types of probabilities are often said to be ‘subjective’ or Bayesian. However, these probabilities are not ‘objective,’ in the sense in which this term is used to specify a concept of probabilities that would be ‘independent of any observer.’ We must emphasize that the concept of probabilities based upon the possibilities generating transformation groups we propose requires two human interventions: 1 - some observations must be made by someone, could be made by someone or a situation is postulated to be; 2 - a physical theory must be chosen to analyze the observations in question to quantify the number, or density, of possible values. When classical mechanics is the theory used in this analysis, there are no probabilities in that theory. We could say that the manifold of the possibilities generating transformation group for the value of a dynamical variable in an experiment represents our state of knowledge of that dynamical variable in the experiment, given that classical mechanics is used to analyze what was observed in the experiment. Another way to look at the probabilities we have obtained is that it is a measure associated with the set of possible values that would be obtained if we were to make another, but more accurate, measurement of that dynamical variable. To use a more modern terminology, we can say that the probabilities so obtained are a measure of the information we use to make
a prediction based upon a physical theory and some observations.

The notion that one could use group theory to formally deal with ambiguities is as old as group theory itself since at the time of his death Evariste Galois[10] stated that his main preoccupation for some time had been the application of the theory of ambiguity to transcendental analysis, according to the concluding remark in his celebrated letter addressed to Auguste Chevalier two days before his death. The idea that Galois’ theory of ambiguity was group theory and could be used to deal with uncertainties resulting from observations is not new and was pointed out by George Birkhoff[11] more than 60 years ago.

E. T. Jaynes[3] in the last 40 years has been a very eloquent champion of the point of view that probabilities were not physical, in particular in classical physics, including the use of transformation group methods. However, we must point out an essential and fundamental conceptual difference between what we have stated above and what Jaynes based his arguments upon. Jaynes defined probabilities as the “degree of truth of an assertion conditional upon the truth of some other assertion(s)”. Then, to justify his transformation group method, he introduced a “desideratum of consistency”[12]: “In two different problems where we have the same state of knowledge we should assign the same subjective probability.” Our objective is much more limited than the one stated by Jaynes, we stay strictly within the ambit of physics, doing what physicists have been doing for ages. Probabilities are not introduced at the outset, they are a consequence of testing the quantitative nature of our physical theories by observations in experiments. They are the result of enumerating all the possibilities according to a physical theory used in analyzing the observations made in an experiment. The transformation groups used are based upon the relations postulated in a specific physical theory and what is observed in the experiment. The fraction of all the possibilities that is associated with a subset of possibilities is what we call the probability associated with that subset of possible values. This is of course conditional upon having correctly identified what was observed in the physical theory used and that this theory would not be falsified in that experiment.

So far our argument has been completely abstract, we will now illustrate it with a few concrete examples. First, we will deal with two problems that have been associated with concepts of probabilities since antiquity. However, we will analyze them as if they were classical physics experiments in which some observations are made and classical mechanics is used to quantify the ambiguities of predictions based upon the observations. Next,
we will consider some whose solutions using probabilities are more recent. Finally, we will consider a well known 75 years old problem that, to our knowledge, has so far resisted any satisfactory solution. We have two further purposes in analyzing these classical mechanics problems. The first one is that although we will be using group theory, it is not because the problems we deal with have any ‘real’ symmetry associated with them, at best they could be said to present an ‘apparent’ symmetry, given what was observed and the theory used to analyze them. The second one is that these examples make very clear that the probabilities we generate are in no way ‘physical’ and the analysis of what was observed is based upon the use of a specific physical theory.

A. The Coin

This trivial example involves the information that a coin is at rest lying flat on one side, which side is facing up is not observed. This information, i.e. the data, could be the result of observing the coin on edge, or a ‘friend’ is telling us he observed the coin but does not tell us which way is facing up. We seek to reduce the data we have for which side is facing up. From classical mechanics, the possibilities generating transformation group for which side is facing up has two elements: a rotation of the coin of 180 degrees about a horizontal axis changes which side is facing up and a rotation of 360 degrees about the same axis is the identity transformation. This transformation group leaves invariant the data: the coin is lying flat on one side. Since this group is finite the same measure applies to each transformation and therefore equal probability must be assigned to each possibility. Obviously there is no symmetry in this situation. The coin need not be in any way symmetrical, nor need we be told that is was randomly placed, whatever this could mean in a classical mechanics context. We also did not have to consider “consistent betting behavior”, or any “Dutch-book argument”, for which side is facing up, as is sometimes done (Caves et al[4]) to justify the notion of Bayesian probabilities in physics. The instant our ‘friend’, who could be standing next to us, tells us which side of the coin is actually facing up, our possibilities generating transformation group, which was different from his, ‘collapses’ to its subgroup: the identity transformation, a rotation of 360 degrees.
B. The Playing Die

This second example is a little more interesting and deals with a conventional right handed playing die. Let us consider that the information, i.e. the data, we have concerning the die is that it is at rest on an horizontal surface with one of its vertical side facing North. We reduce this data for the possible orientations of the die. We are dealing again with a well known problem: the possibilities generating transformation group for the orientation of the die is the octahedral group (O) with 24 elements. We consequently obtain a probability of 1/24 for each possible orientation of the die. Again, there is no real symmetry involved, the playing die in question need not be symmetrical nor do we need to be told that it was randomly tossed. If we wanted to we could reduce the same data only for which side is facing up, ignoring the value on the side facing North. In this case the possibilities generating transformation group is the dihedral group ($D_3$) which has 6 elements. Each of the possibilities is assigned a probability of 1/6. In the conventional fashion we could call these probabilities ‘marginal’ since they ignore which side is facing North. Finally, we could reduce the data conditional upon a particular value on the side facing say North. The possibility generating transformation group for the side facing up is the cyclic group of order 4 ($C_4$). Consequently, we have in this case a ‘conditional’ probability of 1/4 for each of the possibilities for the side facing up. As an important aside, ‘Bayes theorem’ results from possibilities generating groups that are the direct product of two groups.

C. When and How Fast

We now show, by considering the instant of time and interval of time parameters of classical mechanics, how results that are often called the ‘Laplace prior’ and the ‘Jeffreys prior’ can be obtained. Let us say that in an experiment an event is observed to have occurred not before a time $t_1$ and not after a time $t_2$. Since the event must have occurred at a specific time $t$, we have the data: $t_1 < t < t_2$. In classical mechanics an instant of time is a continuous translation invariant parameter, the possibilities generating transformation group for the time $t$ is the one parameter Lie group $T_1$ where $t$ is transformed into $t'$ according to $t' = t + \alpha$ where $\alpha$ is the continuous parameter of the group. The volume measure in the group manifold of $T_1$ is constant. Therefore, the probability that the event occurred in an
interval of time $dt$ at any time $t$ between $t_1$ and $t_2$ is $dt/(t_2 - t_1)$. This result is often called a Laplace prior: the probability distribution is uniform in an interval.

Next we look at the scale parameter of the time-space of classical mechanics. Let us say that in this case the period $\tau$ of a cyclic phenomenon is observed to be not less than the time interval $\tau_1$ and no greater than the time interval $\tau_2$. The data to be reduced is: $\tau_1 < \tau < \tau_2$. Now the possibilities generating transformation group for the period $\tau$ is the scale transformation $\tau' = \tau \cdot \beta$ where $\beta$ is the continuous parameter of the group. The volume measure in the parameter space of this group is no longer constant but proportional to $1/\beta$. Therefore, the probability that the period of the cyclic phenomenon in question is in the interval $d\tau$ about $\tau$ in the range $\tau_1 < \tau < \tau_2$ is: $d\tau/(\tau \ln(\tau_2/\tau_1))$ and zero elsewhere. This result is often called a Jeffreys prior. In our proposed interpretation of the nature and meaning of probabilities in physics, the qualifier ‘prior’ to probabilities should never be used since two things are always required in order to assign probabilities: a physical theory and some observations.

D. von Mises’ Water and Wine Problem

One of the most interesting application by Jaynes[13] of his group theoretical method is his proposed solution to the ‘Bertrand Chord Paradox’[14]. Bertrand’s chord paradox is not a physics experiment but one of plane geometry. Since we know all the ‘quantitative relations’ between the elements of the geometrical figure involved, the groups of transformations to generate the possibilities are well defined, as Jaynes shows. Jaynes group theoretical method to solve ‘ambiguous’ problems has been widely criticized, see van Frassen[15], because of his inability to solve or explain why he could not solve von Mises’ water and wine problem. As we will show, we also cannot solve von Mises’s problem as he stated it, but we can explain why and we can solve it if we consider it a physics measurement problem.

Von Mises[16]’ original statement of his water and wine problem is: “We are given a glass containing a mixture of water and wine. All that is known about the concentrations of the liquids is that the ratio of water to wine is not less than 1 and not more than 2; this means that the mixture contains at least as much water as wine and at most, twice as much water as wine.” von Mises shows that if one applies Laplace’s prior first to the ratio of water to wine and then to the ratio of wine to water one gets a contradiction. von Mises water and
wine problem has long been used as a litmus test for theories of rational decisions in the face of uncertainty which, to our knowledge, none of them has passed.

Von Mises did not specify what attribute of water and wine was used to express the concentrations. If we are going to interpret this problem as one of data reduction in a physics experiment this must be done. It is only under such conditions that we could apply our theory of transformations because we need to invoke the relations of the physical theory used in which the given data enters. We will assume, for simplicity, that in the theory used to analyze von Mises’ data the volumes are the attributes in question and that in the mixture they are additive. Let us denote by $M$ the volume of the mixture in the glass, by $E$ the volume of water in the mixture and by $V$ the volume of wine in the mixture. Therefore, we consider that the physical theory used to analyze the data states that $M = E + V$. From which we obtain: $1 = f_e + f_v$ where $f_e$ is the fraction of the mixture which is water and $f_v$ the fraction which is wine. The given data is that $1 < (E/V) < 2$ from which we obtain:

$$1/2 < f_e < 2/3 \quad (2.1)$$

Given that we have: $1 = f_e + f_v$ the possibilities generating transformation group for $f_e$, the fraction of the mixture which is water, is the one parameter Lie group $T_1$ where $f_e$ is transformed into $f'_e$ according to $f'_e = f_e + \alpha$ and $\alpha$ is the continuous parameter of the group. The volume measure in the group manifold of $T_1$ is constant. Of course, because the theory we use states that: $1 = f_e + f_v$, to any transformation of $f_e$ must correspond a transformation of $f_v$. Therefore, the probability that the fraction of the mixture which is water is in the interval $df_e$ about any value of $f_e$ in the interval: $1/2$ to $2/3$ is given by: $6 \cdot df_e$. It is equally probable that $f_e$ is below or above $7/12$.

We must emphasize that we have not ‘solved’ von Mises water and wine problem as he stated it. We showed how to solve it if it was a physics experiment in which a specific physical theory ($M = E + V$) was used to analyze the observations. Because wine contains alcohol, we have used an incorrect ‘physical’ theory to ‘solve’ von Mises problem, we did so on purpose to point out that the same data analyzed with different theories could yield different probabilities. We do not know, and in fact as physicist we do not care, if von Mises water and wine problem can be solved as he stated it. As a referee for Physical Review we would have requested that von Mises specifies what was ‘measured’ before accepting his ‘experiment’ for publication.
We could go on and derive some well known results and distributions of classical physics[17]. As is the case for the Maxwellian and the normal distributions, it is a very simple matter to derive them using possibilities generating transformation groups. As a matter of fact, Maxwell[18] himself derived the Maxwellian distribution using an invariance argument. Most of Jaynes[2] derivations can be readily done using transformation group techniques, since from the context one can tell which physical ‘theory’ and which ‘observations’ are being used. At this stage it is important to investigate whether our proposed interpretation of the nature and meaning of probabilities in physics, which yields mathematical results that have been used for over a century in classical physics, is a step toward ‘understanding’ quantum mechanics.

III. NON RELATIVISTIC QUANTUM MECHANICS

The obstacles to understanding quantum mechanics can be appreciated when one realizes that it is 75 years old and even though many interpretations of it have been proposed, a definitive consensus has not yet been reached (Zeilinger[19].) This in spite of the fact that numerous international conferences have been devoted to the study of its foundation. Such conferences have been held at least yearly for more than 30 years without any apparent breakthrough. The situation concerning understanding quantum mechanics has barely changed in the 40 years since Feynman[20], who played a major role in one of its formulations, made his celebrated statement: “I think I can safely say that nobody today understands quantum mechanics.” Most, if not all, proposed interpretations of quantum mechanics focus upon understanding its probabilities. Could our proposed interpretation of the nature and meaning of probabilities in physics, based only upon the notion that physical theories are quantitative and testable by observations in experiments, provide a key to ‘understanding’ quantum mechanics? Only time will tell if what we propose is Rabi[21]’s ‘basic point’: “The problem is that the theory is too strong, too compelling. I feel we are missing a basic point. The next generation, as soon as they will have found that point, will knock on their heads and say: How could they have missed that?”

There are many different formulations of quantum mechanics. We do not think that at this stage it is essential to proceed via an analysis of an axiomatization of quantum mechanics. Because we claim to understand classical mechanics and how the fact that it is quantitative
and testable by observations in experiments leads to the probabilities of classical physics, we will focus upon a formulation of quantum mechanics that has a very close and direct relationship to classical mechanics. It is one of its earliest formulations: Dirac’s Hamiltonian formulation of non-relativistic quantum mechanics. Quoting Dirac[22] (page 3) “At this stage it is important to remember that science is concerned only with observable things and that we can observe an object only by letting it interact with some outside influence. An act of observation is thus necessarily accompanied by some disturbance of the object observed. We may define an object to be big when the disturbance accompanying our observation of it may be neglected, and small when the disturbance cannot be neglected.” Dirac then points out that one can give an absolute meaning to size, the emphasis is Dirac’s,: “...we have to assume that there is a limit to the finiteness of our powers of observation and the smallness of the accompanying disturbance — a limit which is inherent in the nature of things and can never be surpassed by improved technique or increased skill on the part of the observer. If the object under observation is such that the unavoidable limiting disturbance is negligible, then the object is big in the absolute sense and we may apply classical mechanics to it. If, on the other hand, the limiting disturbance is not negligible, then the object is small in the absolute sense and we require a new theory for dealing with it.” Systems for which the ‘limiting disturbance’ is not negligible we will call quantum systems. Those system for which the ‘limiting disturbance’ is negligible we will call classical systems.

Why can we not use classical mechanics for quantum systems? If classical mechanics was not modified and taken to apply to a quantum system, the state of the quantum system would be given by a point in an Euclidean phase space. As explained in section II, even though we could never experimentally prove it, our measurements should tell us that the state of the system could be a point within a domain of phase space that becomes smaller and smaller as we improve the accuracy of the measurements on a particular system. But because of the well understood manner in which a quantum system is disturbed when we measure canonically conjugate dynamical variables, the above argument breaks down completely when applied to quantum systems. The more accurately we measure one of the observable dynamical variable, the less accurately we can measure the conjugate dynamical variable. This means that the state of a quantum mechanical system cannot be given by a point in an Euclidean phase space. The concept of a phase space is based upon the fact that in the relations of classical mechanics we can take the symbols for conjugate dynamical
variables to stand for real numbers.

The testability, by observations in experiments, of the quantitative nature of a physical theory rests upon the fact that in order to be measurable the dynamical variables must be real numbers. This was fully understood by Dirac [22] (page 34) “When we make an observation we measure some dynamical variable. It is obvious physically that the result of such a measurement must always be a real number, so we expect that any dynamical variable that we can measure must be a real dynamical variable.” To solve this problem Dirac invented his transformation theory of non relativistic quantum mechanics in which the Hamiltonian formulation of classical mechanics is preserved but the symbols for the measurable dynamical variables are no longer directly real numbers, as they are for classical systems, but Hermitian operators in an Hilbert space. The eigenvalues of these Hermitian operators are the real numbers needed to make the theory quantitative and testable by observations in experiments.

Having obtained real numbers for the measurable dynamical variables of quantum systems, as eigenvalues of Hermitian operators, these can be measured in experiments. This is done by having the quantum systems interact with a classical system acting as the measuring instrument. We will have to face the same problem that we had with measuring dynamical variables of classical mechanics. That is to say, we will have systematic uncertainties associated with their experimental determination and these can be dealt with using possibilities generating transformation groups to quantify their possible values. An interesting aspect of the fact that some Hamiltonians will lead to a discrete set of eigenvalues for a dynamical variable is that one then does not need as high an accuracy in the classical measuring instrument to determine which state the quantum system is in, as is the case when we have a continuum of eigenvalues. A consequence of having obtained the real numbers for the measurable dynamical variables of quantum systems via Hermitian operators is that we have eigenfunctions corresponding to the eigenvalues. These eigenfunctions are rays in an Hilbert space and strictly speaking there is nothing in classical mechanics that corresponds to them.

We are going to show, with the simplest possible quantum system example, that these eigenfunctions can be interpreted as entities that are completely analogous to the manifolds of the possibilities generating transformation groups we introduced to deal quantitatively with ambiguities that occur when we test classical mechanics in experiments.

Let us consider having observed, via a Stern-Gerlach experiment, an electron traveling along
the $y$-axis having its spin up in the $z$-direction. Our observation of its spin up in the $z$-direction gives us, in an obvious basis, the eigenfunction:

$$|\Psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(3.1)

There is absolutely nothing ambiguous in that eigenfunction concerning what would happen if we were to repeat that measurement: the spin would be found to be up in the $z$-direction. However, what does it tell us about the outcome of a subsequent measurement of the spin in the $xz$ plane at an angle $\theta$ with the positive $z$-axis?

The operator $\hat{S}_\theta$ corresponding to that measurement can be expressed in terms of the projection of the $x$ and $z$ operators:

$$\hat{S}_\theta = \sin(\theta) \hat{S}_x + \cos(\theta) \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

(3.2)

The eigenvalues of $\hat{S}_\theta$ are $\hbar/2$ and $-\hbar/2$ and the corresponding eigenvectors are:

$$\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$$

and

$$\begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

(3.3)

We can therefore write the eigenfunction corresponding to our observation of the spin up in the $z$-direction as:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\theta/2) \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} - \sin(\theta/2) \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

(3.4)

What this results precisely means has been known for over 70 years. There are two possibilities: the spin could be found to be up or down and we have an ‘amplitude’ associated with each possibility. This is the simplest example of the ‘superposition principle’. Because, the eigenfunction is a ray in an Hilbert space, the ‘amplitude’ associated with each possibility is a complex number. In order to obtain a ‘real’ measure associated with each possibility we must use the ‘norm’ of these amplitudes. These norms are what we call the probabilities for the various possibilities.

Let us suppose we do perform the measurement of the spin observable at that angle $\theta$. The instant we observe which of the two possible eigenvalues prevails in this measurement, our state of knowledge concerning the value of the spin observable along that angle $\theta$ changes from ambiguous to certain. The eigenfunction which expressed the ambiguity concerning which eigenvalue would be observed collapses to the eigenfunction corresponding to the eigenvalue
that was observed. This new eigenfunction can be used to make quantitative predictions for the spin values that could be observed in subsequent experiments.

When we deal with classical systems the ambiguities concerning what could be observed in future experiments, given what was observed on this classical system, are expressed quantitatively via the Euclidean manifold of a possibilities generating transformation group. When we deal with a quantum system the same problem arises but now the quantification of the ambiguities concerning what could be observed in the future is given by the ray in Hilbert space which is the eigenfunction corresponding to the eigenvalue of the dynamical variable that was observed. Such observations upon which we base predictions for future observations are often said today to prepare a quantum system in a given state. Consequently the ‘algebra’ of the probabilities for quantum systems is different from the ‘algebra’ of the probabilities for classical systems. This provides a complete explanation of Bell’s inequalities which are applicable only to classical systems, and are indeed not applicable to quantum systems. From a logical point of view an even more fundamental difference between classical and quantum systems is their respective ‘principles of superposition’. In classical systems this principle refers to physical states of the systems. In quantum mechanics it refers to a state of ambiguity we have concerning predictions for future observations, when we have made some observations on, i.e. prepared, a quantum system.

IV. CONCLUSION

We have shown that based upon the requirement that physical theories be quantitative and testable via observations in experiments, we can interpret probabilities in both classical and quantum physics as being ‘logical’ rather than ‘physical’. Their Bayesian nature does not depend upon the introduction of concepts such as ‘degrees of belief’ or ‘degrees of truth of assertions,’ often associated today with the concept of Bayesian probabilities, even in physics. They are the result of quantifying the domain of possibilities that results when we interpret observations within the framework of a physical theory. Probabilities in physics could also be said to be measures of information interpreted within the framework of a physical theory.
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