A second order expansion in the local limit theorem for a branching system of symmetric irreducible random walks

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Abstract

Consider a branching random walk, where the branching mechanism is governed by a Galton-Watson process, and the migration by a finite range symmetric irreducible random walk on the integer lattice $\mathbb{Z}^d$. Let $Z_n(z)$ be the number of the particles in the $n$-th generation at the point $z \in \mathbb{Z}^d$. Under the mild moment conditions for offspring distribution of the underlying Galton-Watson, we derive a second order expansion in the local limit theorem for $Z_n(z)$ for each given $z \in \mathbb{Z}^d$. That generalises the results for simple branching random walks obtained by Gao [2018, SPA].

Keywords. branching random walk, local limit theorem, second order expansion

1 Introduction and Main result

We consider a discrete-time branching random walk. That describes the evolution of a population of particles where spatial motion is present, hence generalizes the classical Galton-Watson branching processes. It has been a very active topic in probability, because of its own importance and close connection with many other random models, e.g. multiplicative cascades, infinite particle systems, random fractals and discrete Gaussian free field.

Since Harris [19, Chapter III §16] first proposed his conjecture on the question of central limit theorems for a branching random walk, the topic has been widely studied and in various forms. See e.g. [4, 7, 10, 13–15, 17, 24–27, 31, 32, 35, 39].

Révész [33] initiated the study of the convergence speed in local limit theorems for a simple branching random walk, where displacements of the particles are governed by a simple random walk on $\mathbb{Z}^d$. Specially, Révész gave a conjecture on the exact rate of the convergence speed for the simple branching random walk, which was proved by Chen [9]. Then Gao [11] improved and modified Chen’s result. As a further generalization, Gao [12] obtained the second order asymptotic expansion of the local limit theorem for the simple branching random walk. In this article, we continue the research line in [12] by extending the result therein to the case where the migration mechanism is governed by a finite range symmetric irreducible random walk on $\mathbb{Z}^d$.

We mention that some central limit theorems of different nature on the intrinsic martingale (Biggins martingale) in branching random walks have been established by Grübel and Kabluchko [16], Iksanov and...
Kabluchko [22], and some large deviations type results for branching random walks have been discussed by [21, 36]. For other aspects on branching random walks, see for example, [1, 18, 20, 23, 30, 34, 37, 38, 40] and references therein.

To introduce our main result, we need to make some basic settings.

In the $d$-dimensional lattice $\mathbb{Z}^d$, write $0 = (0,0,\cdots,0)$, $1 = (1,1,\ldots,1)$, and let

$$e_1 = (1,0,\ldots,0), \ldots, e_d = (0,0,\ldots,1)$$

be the standard basis of orthogonal unit vectors. By convention, denote by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^d$ and $\| \cdot \|$ the norm therein, i.e. for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$,

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d, \quad \| x \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$  

The multiplication of a vector $x \in \mathbb{R}^d$ by a real number $\lambda \in \mathbb{R}$ is defined by $\lambda x = (\lambda x_1, \lambda x_2, \cdots, \lambda x_d)$.

Let $\mathbb{N} = \{0, 1, 2, \cdots \}$. Suppose that $N$ is an integer-valued random variable with the probability law

$$\mathbb{P}(N = k) = p_k, \quad k \in \mathbb{N}, \quad \sum_{k=0}^{\infty} p_k = 1,$$

and that $L$ is a random variable with the probability law

$$\mathbb{P}(L = 0) = \zeta_0, \quad \mathbb{P}(L = re_s) = \mathbb{P}(L = -re_s) = \frac{1}{2} \zeta_{s,r}, \quad 1 \leq s \leq d, 1 \leq r \leq t_s, \quad (1.1)$$

where each $t_s$ ($1 \leq s \leq d$) is a positive integer, $\zeta_0 \in [0, 1)$, $\zeta_{s,r} \in [0, 1)$, and

$$\zeta_{s,t_s} > 0, \quad \zeta_0 + \sum_{s=1}^{d} \sum_{r=1}^{t_s} \zeta_{s,r} = 1.$$  

For $k \in \mathbb{N}$, set

$$\zeta_s(k) = \sum_{r=1}^{t_s} \zeta_{s,r} r^k, \quad 1 \leq s \leq d.$$  

Denote by $\Gamma_k = \text{diag}(\zeta_1(k), \cdots, \zeta_d(k))$ the diagonal matrix with diagonal entries $\zeta_1(k), \cdots, \zeta_d(k)$. It is easy to see that

$$\det \Gamma_k = \prod_{s=1}^{d} \zeta_s(k), \quad \text{tr}(\Gamma_k^{-1}) = \sum_{s=1}^{d} (\zeta_s(k))^{-1},$$

where $\det M$ is the determinant of a $d \times d$ matrix $M$, $M^{-1}$ is its inverse, and $\text{tr}(M)$ is its trace. By convention, for a vector $x$, the notion $Mx$ is viewed as the product of $M$ and the column matrix $x$.

Throughout the article, we shall assume that the law (1.1) of $L$ satisfies

$$\gcd\{r : \zeta_{s,r} > 0\} = 1, \quad s = 1, 2, \cdots, d, \quad (1.2)$$

where $\gcd$ denotes the greatest common divisor. This assumption implies that $V = \{re_s : \zeta_{s,r} > 0, 1 \leq r \leq t_s, s = 1, 2, \cdots, d\}$ is a generating set of $\mathbb{Z}^d$, which means that

$$\forall y \in \mathbb{Z}^d, \quad \exists \{k_{r,s}\} \subset \mathbb{Z}, \quad \text{s.t.} \quad y = \sum_{r \in V, \ s \in V} k_{r,s} re_s. \quad (1.3)$$

Moreover, the random walk $S_n$ with such increment distribution $L$ must be irreducible (meaning that each point in $\mathbb{Z}^d$ can be reached with positive probability [29]).

Under the hypothesis (1.2) for $L$, there are two possible cases:
(Ha) there exists one \( s \) such that either the set \( \{ r : \zeta_{s,r} > 0 \} \) contains at least one odd integer and one even, or the set \( \{ r : \zeta_{s,r} > 0 \} \) only contains odd integers and \( \zeta_0 > 0 \);

(Hb) \( \zeta_0 = 0 \) and for each \( 1 \leq s \leq d, \{ r : \zeta_{s,r} > 0 \} \) only contains odd integers.

Denote by \( A_d \) the set of all probability distributions \( L \) with the law (1.1) satisfying (1.2) and (Ha). When \( L \in A_d \), the random walk with increment distribution \( L \) is aperiodic, which means that each point on \( \mathbb{Z}^d \) can be reached after \( n \) steps with positive probability for all \( n \) sufficiently large.

Denote by \( B_d \) the set of all probability distributions \( L \) with the law (1.1) satisfying (1.2) and (Hb). For \( L \in B_d \), the associated random walk is bipartite, that means the random walks starting from a given point \( x_0 \) return to \( x_0 \) only after an even number of steps. In this case, \( \mathbb{Z}^d \) is divided into two disjoint sets \( \mathbb{Z}_o \) and \( \mathbb{Z}_e \), such that the walk starting from the origin reaches the states set \( \mathbb{Z}_o \) in an odd number of steps and reaches \( \mathbb{Z}_e \) in an even number of steps.

For example, if \( L \) with the law (1.1) satisfies

\[
\zeta_0 = \sigma \in [0, 1), \quad \zeta_{s,1} = (1 - \sigma)/d, \quad 1 \leq s \leq d,
\]

then when \( \sigma > 0 \), \( L \in A_d \) and the associated random walk is aperiodic; whereas when \( \sigma = 0 \), \( L \in B_d \) and the associated random walk is bipartite.

Consider a discrete-time branching random walk in the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \). Initially, an ancestor particle \( \emptyset \) is located at \( S_\emptyset = 0 \in \mathbb{Z}^d \). At time 1, it is replaced by \( N = N_\emptyset \) particles \((\emptyset 1, \ldots, \emptyset N)\), and each particle \( \emptyset i (1 \leq i \leq N) \) moves to \( S_{\emptyset i} = S_\emptyset + L_{\emptyset i} \). In general, at time \( n+1 \), each particle of generation \( n \), which is denoted by a sequence of positive integers of length \( n \), i.e. \( u = u_1 u_2 \cdots u_n \), is replaced by \( N_u \) new particles of generation \( n+1 \), \( \{ u_i : 1 \leq i \leq N_u \} \), with displacements \( L_{u1}, L_{u2}, \ldots, L_{un} \). That means for \( 1 \leq i \leq N_u \), each particle \( u_i \) moves to \( S_{ui} = S_u + L_{ui} \). Here, all \( N_u \) and \( L_{ui} \), indexed by finite sequences of integers \( u \), are independent random elements defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\): all \( N_u \) are independent copies of the integer-valued random variable \( N \); all \( L_{ui} \) are independent random vectors identically distributed (abbreviated as i.i.d) with \( L \) defined by the law (1.1).

The genealogy of all particles forms a Galton-Watson tree \( \mathbb{T} \) with \( \{N_u\} \) as defining elements. Let

\[
\mathbb{T}_n = \{ u \in \mathbb{T} : |u| = n \}
\]

be the set of particles of generation \( n \), where \(|u|\) denotes the length of the sequence \( u \) and represents the number of generation to which \( u \) belongs.

Let \( Z_n(\cdot) \) be the counting measure of particles in the \( n \)-th generation: for \( B \subset \mathbb{Z}^d \)

\[
Z_n(B) = \sum_{u \in \mathbb{T}_n} 1_B(S_u).
\]

In particular, we will frequently write \( Z_n(z) \) instead of \( Z_n(\{ z \}) \), which is the number of the \( n \)-th generation individuals located at \( z \in \mathbb{Z}^d \) by definition.

By the definitions, \( \{ Z_n(\mathbb{Z}^d) \}_{n} \) forms a Galton-Watson process. Throughout the paper, we assume

\[
m = \mathbb{E} N = \sum_{k=1}^{\infty} kp_k > 1, \quad \text{for } m = \mathbb{E} N = \sum_{k=1}^{\infty} kp_k > 1, \quad (1.4)
\]

and the process is called supercritical. In this case, the process survives with strictly positive probability. For ease of notion, we always assume that

\[
\mathbb{P}(N = 0) = 0, \quad (1.5)
\]
that implies the process survives with probability 1. Obviously this assumption could be removed, but then all results hold conditionally on non-extinction. Under this setting, it holds that \( P(Z_n(Z^d) \to \infty) = 1 \). Provided that \( \mathbb{E} N \log N < \infty \), the Kesten-Stigum theorem (\([3, 5]\)) asserts that the sequence \( W = Z_n(Z^d)/m^n \) converges almost surely (abbreviated as a.s.) to a finite positive random variable \( W \).

With the above notion, our main result can be stated as follows.

**Theorem 1.1.** Assume that the conditions (1.4) and (1.5) hold, \( \mathbb{E} N \log N^{1+\lambda} < \infty \) for some \( \lambda > 3(d+6) \) and \( L \) obeys the law (1.1). Then there exist some random variables \( V_1, V_2, V_3 \in \mathbb{R}^d \) and \( V_2^z, V_4 \in \mathbb{R} \), such that for each \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d \), as \( n \to \infty \),

(I) in the case \( L \in A_d \),

\[
\frac{1}{m^n} Z_n(z) = \left( \frac{2\pi n}{\sqrt{\det \Gamma_2}} \right)^{-d/2} \left[ W + \frac{1}{n} F_1(z) + \frac{1}{2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1) \quad \text{a.s.,} \tag{1.6}
\]

(II) in the case \( L \in B_d \), provided that \( n \equiv z_1 + z_2 + \cdots + z_d \mod 2 \),

\[
\frac{1}{m^n} Z_n(z) = 2\left( \frac{2\pi n}{\sqrt{\det \Gamma_2}} \right)^{-d/2} \left[ W + \frac{1}{n} F_1(z) + \frac{1}{2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1) \quad \text{a.s.,} \tag{1.7}
\]

where

\[
F_1(z) = \left( \tau_d - \frac{1}{2} \langle z, \Gamma_2^{-1}z \rangle \right) W + \langle V_1, \Gamma_2^{-1}z \rangle - \frac{1}{2} \langle V_2, \Gamma_2^{-1}1 \rangle, \tag{1.8}
\]

\[
\tau_d = \frac{8}{3} \text{tr}(\Gamma_2 G_2^{-2}) - \frac{1}{8} d(d+2), \tag{1.9}
\]

\[
F_2(z) = \left( \frac{1}{8} \langle \Gamma_2^{-1}z, z \rangle^2 - \langle \Lambda_d z, z \rangle + \chi_d \right) W + \left( \langle V_1, \left( 2\Lambda_d - \frac{1}{2} \langle z, \Gamma_2^{-1}z \rangle \Gamma_2^{-1} \right) z \rangle + \langle V_2, \left( \frac{1}{4} \langle z, \Gamma_2^{-1}z \rangle \Gamma_2^{-1} - \Lambda_d \right) 1 \rangle \right) + \frac{1}{2} V_2^2 - \frac{1}{2} \langle V_3, \Gamma_2^{-1} \rangle + \frac{1}{8} V_4 \tag{1.10}
\]

\[
\Lambda_d = \frac{1}{16} \left( \text{tr}(\Gamma_4 G_2^{-2}) - (d+2)(d+4) \right) \Gamma_2^{-1} + \frac{1}{4} \Gamma_4 \Gamma_2^{-3}, \tag{1.11}
\]

\[
\chi_d = -\frac{1}{64} (d+2)(d+4) \text{tr}(\Gamma_4 G_2^{-2}) + \frac{1}{12} \text{tr}(\Gamma_4 G_2^{-4}) + \frac{1}{128} \left( \text{tr}(\Gamma_4 G_2^{-2}) \right)^2 - \frac{1}{48} \text{tr}(\Gamma_6 G_2^{-3}) + \frac{1}{384} d(d+2)(d+4)(3d+2). \tag{1.12}
\]

Remark 1.2. In Section 2, we shall give the detailed accounts on the quantities \( V_1, V_2, V_3 \in \mathbb{R}^d \) and \( V_2^z, V_4 \in \mathbb{R} \). We prove that these random variables are limits of the following sequences:

\[
V_j \overset{a.s.}{\longrightarrow} \lim_{n \to \infty} N_{j,n}, \quad j = 1, 2, 3, 4; \quad V_2^z \overset{a.s.}{\longrightarrow} \lim_{n \to \infty} N_{2,n}^z,
\]

where

\[
N_{1,n} = \frac{1}{m^n} \sum_{u \in T_n} S_u,
\]

\[
N_{2,n} = \frac{1}{m^n} \sum_{u \in T_n} \left( \langle S_u, e_1 \rangle^2 - n c_1(2), \langle S_u, e_2 \rangle^2 - n c_2(2), \ldots, \langle S_u, e_d \rangle^2 - n c_d(2) \right),
\]

\[
N_{3,n} = \frac{1}{m^n} \sum_{u \in T_n} \left( \langle S_u, \Gamma_2^{-1}z \rangle^2 - n \langle \Gamma_2^{-1}z, z \rangle \right),
\]

\[
N_{4,n} = \frac{1}{m^n} \sum_{u \in T_n} \left( \langle S_u, \Gamma_2^{-1}S_u \rangle S_u - (d+2)n S_u \right),
\]

Moreover we also consider the convergence rate of the above limit procedure in the next section. Those play important roles in the proof of Theorem 1.1.
Comparing the bipartite case with the aperiodic case, a phase transition occurs. The result indicates that the periodicity of the random walk alternates the formulation of the limit laws with a factor 2. In the following Corollary 1.3, we discuss the example mentioned before when the underlying moving law is governed by a simple random walk or its lazy version (meaning stay there with positive probability).

**Corollary 1.3.** Suppose that the conditions (1.4) and (1.5) hold, \( E \ln N \leq \infty \) for some \( \lambda > 3(d+6) \), and the law of \( L \) in (1.1) satisfies \( \zeta_0 = \sigma, \zeta_{1,1} = \zeta_{2,1} = \cdots = \zeta_{d,1} = (1-\sigma)/d \). Then there exist some random variables \( V_1, V_2, V_3, V_4 \), such that for each \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d \), as \( n \to \infty \),

**I** when \( \sigma > 0 \),
\[
\frac{1}{m^n} Z_n(z) = \left( \frac{d}{2\pi n(1-\sigma)} \right)^{d/2} \left[ W + \frac{1}{n} \mathcal{H}_{0,1}(z) + \frac{1}{n^2} \mathcal{H}_{0,2}(z) \right] + \frac{1}{n^{2+d/2}} o(1) \text{ a.s.,}
\]

**II** when \( \sigma = 0 \), provided that \( n \equiv z_1 + z_2 + \cdots + z_d \pmod{2} \),
\[
\frac{1}{m^n} Z_n(z) = 2 \left( \frac{d}{2\pi n} \right)^{d/2} \left[ W + \frac{1}{n} \mathcal{H}_{0,1}(z) + \frac{1}{n^2} \mathcal{H}_{0,2}(z) \right] + \frac{1}{n^{2+d/2}} o(1) \text{ a.s.,}
\]

where for \( \sigma \in [0,1) \),
\[
\mathcal{H}_{0,1}(z) = \frac{d}{1-\sigma} \left( \frac{1}{8} \sigma(d+2) - \frac{1}{4} \frac{1}{2} \| z \|^2 \right) W + \langle z, V_1 \rangle - \frac{1}{2} \langle V_2, 1 \rangle,
\]
\[
\mathcal{H}_{0,2}(z) = \frac{d^2}{(1-\sigma)^2} \left( \frac{1}{8} \| z \|^4 + \mu_{\sigma,d} \| z \|^2 + \chi_{\sigma,d} \right) W + \left( 2\mu_{\sigma,d} - \frac{1}{2} \| z \|^2 \right) \langle V_2, 1 \rangle + \frac{2}{\sigma} \langle V_2, 1 \rangle - \frac{1}{2} \langle z, V_3 \rangle - \frac{1}{2} \langle z, V_4 \rangle,
\]

with
\[
\mu_{\sigma,d} = -\frac{1}{8} \left( 1 + \frac{4}{d} \right) + \sigma \left( \frac{d}{16} + \frac{3}{8} + \frac{1}{2d} \right),
\]
\[
\chi_{\sigma,d} = \frac{d}{48} - \frac{1}{32} + \frac{1}{24d} + \sigma \left( \frac{1}{64} + \frac{1}{3} \right) + \frac{1+\sigma}{3d}.
\]

**Remark 1.4.** In this corollary, the random variables \( V_1, V_2, V_3, V_4 \) are limits of the following sequences:
\[
V_j \overset{a.s.}{\longrightarrow} \lim_{n \to \infty} N_{j,n}, \quad j = 1, 2, 3, 4, \quad \bar{V}_j \overset{a.s.}{\longrightarrow} \lim_{n \to \infty} \bar{N}_{j,n},
\]

with
\[
N_{1,n} = \frac{1}{m^n} \sum_{u \in T_n} S_u,
\]
\[
N_{2,n} = \frac{1}{m^n} \sum_{u \in T_n} \left( \langle S_u, e_1 \rangle^2 - \frac{n(1-\sigma)}{d} \right),
\]
\[
N_{2,n}^\perp = \frac{1}{m^n} \sum_{u \in T_n} \left( \| S_u \|^2 S_u - \left( 1 + \frac{2}{d} \right) n(1-\sigma) S_u \right),
\]
\[
N_{3,n} = \frac{1}{m^n} \sum_{u \in T_n} \left( \| S_u \|^2 S_u - \left( 1 + \frac{2}{d} \right) n(1-\sigma) S_u \right),
\]
\[
N_{4,n}^\perp = \frac{1}{m^n} \sum_{u \in T_n} \left( \| S_u \|^4 - \left( 1 + \frac{2}{d} \right)^2 n(1-\sigma) \| S_u \|^2 + \left( 1 + \frac{2}{d} \right)^2 (1-\sigma)^2 n^2 - n \right).
\]

**Remark 1.5.** Corollary 1.3 generalises the main results of [11, 12]. Moreover, the expansions in Corollary 1.3 (II) was obtained in [12, Theorem 1.1] under the stronger moment condition \( \lambda > 6(d+5) \).

At the end of this section, we comment briefly on the strategy of proofs. Although the starting general ideas herein are close to those of [11, 12], the details of the proofs require more refined estimates and the
arguments are tricky. Specially what sets this work aside is the explicit expressions of the expansion terms. On the technical level, the proofs need much more complicated calculations.

The rest of the paper is organized as follows. In the next section, we discuss the convergence properties of the sequences \(\{N_{1,n}\}, \{N_{2,n}\}, \{N_{3,n}\}, \{N_{4,n}\}, \) and \(\{N_{2,n}^2\}\), as mentioned in Remark 1.2. In Section 3, we derive the second order expansion in the local limit theorem for a symmetric random walk on the \(d\)-dimensional integer lattice, which is used in the proof of Theorem 1.1. With the help of those preliminary results, the last section is devoted to the proof of Theorem 1.1.

2 Preliminary results

In this section, we shall describe the \(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathbb{R}^d\) and \(\mathcal{V}_2^*, \mathcal{V}_4 \in \mathbb{R}\) appeared in the main theorem. We show that they are limits of some martingale sequences mentioned in Remark 1.2.

Set
\[
\mathcal{D}_0 = \{\emptyset, \Omega\}, \quad \mathcal{D}_n = \sigma(N_u, L_u; : i \geq 1, |u| < n) \text{ for } n \geq 1. \tag{2.1}
\]

Denote respectively by \(P_{\mathcal{D}_n}\) and \(E_{\mathcal{D}_n}\) the conditional probability and conditional expectation with respect to \(\mathcal{D}_n\).

The results, which we shall need in the proof of Theorem 1.1, are stated as follows:

**Theorem 2.1.** The sequences \(\{N_{q,n}\}(q = 1, 2, 3, 4)\) and \(\{N_{2,n}^2\}\) are martingales w.r.t. the filtration \(\mathcal{D}_n\) \(n \geq 0\). Moreover, if (1.4) and (1.5) hold, \(\mathbb{E}N (\ln N)^{1+\lambda} < \infty\) for some \(\lambda > 4\) and \(L\) obeys the law (1.1), then \(\{N_{q,n}\}(q = 1, 2, 3, 4)\) converge to limit random variables (vectors) \(\mathcal{V}_q(q = 1, 2, 3, 4)\) and \(\mathcal{V}_2^*\) a.s. with rates \(o(n^{-(\lambda-3)})\), respectively.

**Remark 2.2.** For the simple branching random walk \((\zeta_0 = 0, \zeta_{1,1} = \cdots = \zeta_{d,1} = 1/d)\), the fact that \(\{N_{1,n}\}\) and \(\{\{N_{2,n}, 1\}\}\) are martingales was firstly observed by Chen [9], where their a.s. convergences are proved under the condition \(\mathbb{E}N^2 < \infty\).

Theorem 2.1 is an immediate consequence of the following general result:

**Theorem 2.3** ([12]). Let \(f\) be a real (or vector-valued) function of \((x, n) \in \mathbb{R}^d \times \mathbb{Z}\) satisfying
\[
\mathbb{E}f(x + (X - \mu), n + 1) = f(x, n) \text{ for } X \text{ a random vector with expectation } \mu.
\]

Consider a branching random walk with all \(N_u\) and \(L_u\) identically distributed with \(N\) and \(X\), respectively. Then the associated sequence \(\{\mathcal{M}_n\}_{n \geq 0}\) defined by
\[
\mathcal{M}_n = \frac{1}{m^n} \sum_{w \in T_n} f(S_u - n\mu, n) \tag{2.2}
\]
is a martingale with respect to the filtration \(\mathcal{D}_n\).

Assume that (1.4) and (1.5) hold. If given a constant \(Q \geq 0\), \(\mathbb{E}N (\ln N)^{1+M} < \infty\) for some \(M \geq \max\{2Q - 1, Q\}\) and \(\mathbb{E}\|X\|^{2Q} < \infty\), and the function \(f\) satisfies the inequality
\[
|f(x, n)| \leq C(\|x\|^{2Q} + n^Q) \quad \forall (x, n) \in \mathbb{R}^d \times \mathbb{N}, \ C \text{ a constant},
\]
then the martingale \(\{\mathcal{M}_n\}_{n \geq 0}\) converges to a random element \(\mathcal{V}\) a.s. and
\[
\mathcal{M}_n - \mathcal{V} = o(n^{-(M-\max\{2Q-1, Q\})}) \quad \text{a.s.} \tag{2.3}
\]
Remark 2.4. Throughout the paper, we use $C$ to denote various positive constants whose values are of no importance.

In the case that $f \equiv 1$ and $Q = 0$, this theorem implies Theorem 2 of [2]:

If $E N(\ln N)^{1+M} < \infty$, then a.s.

\[ W_n - W = o(n^{-M}). \]  

(2.4)

Now let us go to deduce Theorem 2.1.

Proof of Theorem 2.1. We first prove that $\mathcal{V}_4 = \lim_{n \to \infty} N_{4,n}$. Observe that for $u \in T_n, 1 \leq i \leq N_u$.

\[ \langle S_{ui}, \Gamma_2^{-1} S_{ui} \rangle^2 = \langle S_{ui}, \Gamma_2^{-1} S_u \rangle^2 + 4 \langle L_{ui}, \Gamma_2^{-1} S_u \rangle^2 + 2 \langle S_{ui}, \Gamma_2^{-1} S_u \rangle \langle L_{ui}, \Gamma_2^{-1} L_{ui} \rangle + 4 \langle S_{ui}, \Gamma_2^{-1} S_u \rangle \langle L_{ui}, \Gamma_2^{-1} S_u \rangle \]

\[ + 4 \langle L_{ui}, \Gamma_2^{-1} L_{ui} \rangle \langle L_{ui}, \Gamma_2^{-1} S_u \rangle, \]

\[ \langle S_{ui}, \Gamma_2^{-1} S_{ui} \rangle = \langle S_{ui}, \Gamma_2^{-1} S_u \rangle + \langle L_{ui}, \Gamma_2^{-1} L_{ui} \rangle + 2 \langle L_{ui}, \Gamma_2^{-1} S_u \rangle. \]

As all $L_{ui}$ obeys the law (1.1), we have

\[ \mathbb{E} \langle L_{ui}, \Gamma_2^{-1} L_{ui} \rangle^2 = \sum_{s=1}^{d} \sum_{r=1}^{t_{s,r}} \zeta_{s,r} (r_{e_{s}}, \Gamma_2^{-1} r_{e_{s}})^2 = \sum_{s=1}^{d} \sum_{r=1}^{t_{s,r}} \zeta_{s,r} (\zeta_{s}(2))^{-2} = \sum_{s=1}^{d} \zeta_{s}(4) (\zeta_{s}(2))^{-2} = \text{tr} \left( \Gamma_4 \Gamma_2^{-2} \right) \]

\[ \mathbb{E} \langle L_{ui}, \Gamma_2^{-1} L_{ui} \rangle = \sum_{s=1}^{d} \sum_{r=1}^{t_{s,r}} \zeta_{s,r} (r_{e_{s}}, \Gamma_2^{-1} r_{e_{s}}) = \sum_{s=1}^{d} \sum_{r=1}^{t_{s,r}} \zeta_{s,r} (\zeta_{s}(2))^{-1} = d, \quad \mathbb{E} L_{ui} = 0, \]

\[ \mathbb{E} \langle S_{ui}, \Gamma_2^{-1} S_{ui} \rangle = \langle S_{ui}, \Gamma_2^{-1} S_u \rangle + d, \]

\[ \mathbb{E} \langle S_{ui}, \Gamma_2^{-1} S_{ui} \rangle^2 = \langle S_{ui}, \Gamma_2^{-1} S_u \rangle^2 + (4 + 2d) \langle S_{ui}, \Gamma_2^{-1} S_u \rangle + \text{tr} \left( \Gamma_4 \Gamma_2^{-2} \right), \]

Then

\[ \mathbb{E} f(x, n) = \langle x, \Gamma_2^{-1} x \rangle^2 - (4 + 2d) \langle x, \Gamma_2^{-1} x \rangle + d(d + 2)n + d(d + 2)n - \text{tr} \left( \Gamma_4 \Gamma_2^{-2} \right) n, \]

the following relations hold:

\[ \mathbb{E}_n f(S_{ui}, n + 1) = f(S_{ui}, n), \quad |f(x, n)| \leq C(||x||^4 + n^2). \]

So the conditions of Theorem 2.3 is satisfied with $Q = 2$, therefore the sequences $(N_{4,n})_{n \geq 0}$ converges to some random variable $\mathcal{V}_4$ with rate $o(n^{-(\lambda-3)})$.

Similarly, we can prove that each sequence $(N_{q,n})_{n \geq 0}$ ($q = 1, 2, 3$) converges to some random variable $\mathcal{V}_q$ with rate $o(n^{-(\lambda+q+1)}) = o(n^{-(\lambda-3)})$, and $(N_{q,n})_{n \geq 0}$ to $\mathcal{V}_2$ with rate $o(n^{-(\lambda-3)})$. To avoid repetition, we omit the details.
We recall the following inversion formula (\cite{28}, Corollary 2.2.3):

$$\mathbb{P}(\tilde{S}_n = z) = (2\pi n)^{-d/2} \int_{[-\pi,\pi]^d} \cos(\varphi, z) \psi(\varphi)^n d\varphi.$$  \hfill (3.4)

By Taylor’s expansion about 0 of $\psi(\varphi)$,

$$\psi(\varphi) = 1 - \frac{1}{2} \sum_{s=1}^{d} \sum_{r=1}^{t_s} \zeta_s r^2 \varphi_s^2 + O(||\varphi||^4) = 1 - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \varphi_s^2 + O(||\varphi||^4).$$

Then we can find an $r \in (0, \pi/2)$ such that

$$|\psi(\varphi)| \leq 1 - \frac{1}{4} \zeta_0 ||\varphi||^2 \leq e^{-\zeta_0 ||\varphi||^2/4}, \text{ for } ||\varphi|| \leq r, \quad \zeta_0 := \min\{\zeta_1(2), \cdots, \zeta_d(2)\}.$$  \hfill (3.5)

(I) First we consider the case $L \in \mathcal{A}_d$.

By the assumption on $L$, $\tilde{S}_n$ is aperiodic, therefore from \cite{29}, Lemma 2.3.2, it follows that

$$\varrho := \sup\{||\varphi|| : \varphi \in [-\pi,\pi]^d, ||\varphi|| > r\} < 1.$$  \hfill (3.6)

3  Second order expansion in LLT for a finite range symmetric random walk

In this section, we shall derive second order expansions in the local limit theorem for a finite range symmetric random walk on the integer lattice. This is a useful complement of those results on local limit theorem summarized in \cite{29}, Chapter 2.

**Theorem 3.1.** Assume $\kappa \in (0, 1/6)$ and $L$ obeys the law (1.1). Let $\{\tilde{L}_n\}$ be independent copies of $L$ and set $\tilde{S}_n = \tilde{L}_1 + \cdots + \tilde{L}_n$. Then for $z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$, as $n \to \infty$,

(I) in the case $L \in \mathcal{A}_d$,

$$\mathbb{P}(\tilde{S}_n = z) = \frac{(2\pi n)^{-d/2}}{\det \Gamma_2} \left\{ 1 + \frac{1}{n} \left[ \frac{1}{4} \left( z, \Gamma_1^{-1} z \right) \right] \right\} + \frac{1}{n^{d/2 + 2}} \gamma_n(z), \quad (3.1)$$

(II) in the case $L \in \mathcal{B}_d$, provided that $n \equiv z_1 + \cdots + z_d \pmod{2}$,

$$\mathbb{P}(\tilde{S}_n = z) = \frac{2(2\pi n)^{-d/2}}{\det \Gamma_2} \left\{ 1 + \frac{1}{n} \left[ \frac{1}{4} \left( z, \Gamma_1^{-1} z \right) \right] \right\} + \frac{1}{n^{d/2 + 2}} \gamma_n(z), \quad (3.2)$$

where the quantities $\tau_n, \chi_n$ and $\Lambda_n$ are defined by (1.9) (1.12) and (1.11), and $\gamma_n(z)$ is infinitesimal satisfying

$$\sup_{||z|| \leq C n^\varrho} |\gamma_n(z)| \xrightarrow{n \to \infty} 0. \quad (3.3)$$

**Proof.** Consider the characteristic function $\psi$ of the random variable $L$, defined by the following

$$\psi(\varphi) = \mathbb{E}e^{i \langle \varphi, L \rangle} = \zeta_0 + \sum_{s=1}^{d} \sum_{r=1}^{t_s} \zeta_s r \cos(r \varphi_s), \quad \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_d) \in \mathbb{R}^d.$$  \hfill (3.7)

We recall the following inversion formula (\cite{28}, Corollary 2.2.3):

$$\mathbb{P}(\tilde{S}_n = z) = (2\pi n)^{-d} \int_{[-\pi,\pi]^d} \cos(\varphi, z) \psi(\varphi)^n d\varphi.$$  \hfill (3.4)
A second order expansion in LLT for BRW

Hence

\[ P(\tilde{S}_n = z) = I_n(z) + J_n(z), \quad (3.5) \]

where

\[ I_n(z) = (2\pi)^{-d} \int_{\|\varphi\| \leq r} \cos \langle \varphi, z \rangle \left( \psi(\varphi) \right)^n d\varphi, \quad J_n(z) \leq \varrho^n. \]

Hence we have

\[ J_n(z) = \frac{1}{n^{2 + d/2}} \alpha_n^e(z), \quad \text{with} \sup_{\|z\| \leq C n^\kappa} |\alpha_n^e(z)| \leq \varrho^{n^2 + d/2}. \]

Changing the variable \( \theta = \sqrt{n} \varphi \), we get

\[ I_n(z) = (2\pi)^{-d} n^{-d/2} \int_{\theta \leq \sqrt{n} r} \cos \left( \frac{1}{\sqrt{n}} \langle \theta, z \rangle \right) \left( \psi \left( \frac{\theta}{\sqrt{n}} \right) \right)^n d\theta. \]

Take a constant \( \delta = 1/12 - \kappa/2. \) We decompose \( I_n \) as follows:

\[ (2\pi \sqrt{n})^d I_n(z) = I_{1,n}(z) + I_{2,n}(z), \quad (3.6) \]

where

\[ I_{1,n}(z) = \int_{n^\delta \leq \|\theta\| \leq n^{1/2}} \cos \left( \frac{1}{\sqrt{n}} \langle \theta, z \rangle \right) \left( \psi \left( \frac{\theta}{\sqrt{n}} \right) \right)^n d\theta, \]

\[ I_{2,n}(z) = \int_{\|\theta\| < n^\delta} \cos \left( \frac{1}{\sqrt{n}} \langle \theta, z \rangle \right) \left( \psi \left( \frac{\theta}{\sqrt{n}} \right) \right)^n d\theta. \]

Observe that

\[ |I_{1,n}(z)| \leq \int_{n^\delta \leq \|\theta\| \leq n^{1/2}} e^{-\zeta_0 \|\theta\|^2/4} d\theta \leq (2r)^d n^{d/2} e^{-\zeta_0 n^{2\delta}/4}. \]

Then we can write that

\[ I_{1,n}(z) = \alpha_n^e(z) \frac{1}{n^2} \quad \text{with} \sup_{\|z\| \leq C n^\kappa} |\alpha_n^e(z)| \leq (2r)^d n^{2 + d/2} e^{-\zeta_0 n^{2\delta}/4}. \quad (3.7) \]

Next we pass to the analysis of the term \( I_{2,n}. \)

Note \( \delta + \kappa < 1/6. \) For \( \|\theta\| < n^\delta, \|z\| \leq C n^\kappa, \) we have

\[ |\langle \theta, z \rangle| \leq \|\theta\| \|z\| < C n^{\delta + \kappa} < n^{1/6}. \]

Then by Taylor’s expansion, we have that for \( \|\theta\| < n^\delta, \|z\| \leq C n^\kappa, \)

\[ \cos \left( \frac{1}{\sqrt{n}} \langle \theta, z \rangle \right) = 1 - \frac{1}{2n} \langle \theta, z \rangle^2 + \frac{1}{24n^2} \langle \theta, z \rangle^4 + \gamma_n(\theta, z) \frac{1}{n^2}, \quad (3.8) \]

where

\[ |\gamma_n(\theta, z)| \leq \frac{1}{720} \frac{\langle \theta, z \rangle^6}{n} \leq C n^{-1 + 6(\delta + \kappa)}. \]

By the definition of \( \psi \) and using Taylor’s expansion

\[ \cos x = 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \frac{r_x}{8!} x^8, \quad \text{with} |r_x| \leq 1, x \in \mathbb{R}, \]
we have

\[
\left( \psi\left( \frac{\theta}{\sqrt{n}} \right) \right)^n = \exp \left\{ n \ln \left[ \zeta_0 + \sum_{s=1}^{d} \sum_{r=1}^{t_s} \zeta_{s,r} \cos \left( \frac{\theta}{\sqrt{n}} \right) \right] \right\} \\
= \exp \left\{ n \ln \left[ \zeta_0 + \sum_{s=1}^{d} \sum_{r=1}^{t_s} \zeta_{s,r} \left( 1 - \frac{r^2 \theta^2}{2n} + \frac{r^4 \theta^4}{24n^2} - \frac{r^6 \theta^6}{6n^3} + \beta_{s,n}(\theta) \left( \frac{1}{n^3} \right) \right) \right] \right\} \\
= \exp \left\{ n \ln \left[ 1 - \frac{1}{2n} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s + \frac{1}{24n^2} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{6n^3} \sum_{s=1}^{d} \zeta_s(6) \theta^6_s + \beta_n(\theta) \left( \frac{1}{n^4} \right) \right] \right\}
\]

Here and below we use \( \beta_n(\cdot) \) as an infinitesimal as \( n \) tends to infinity, which may take different values even in the same line. Again using Taylor’s expansion

\[
\ln(1 + w) = w - \frac{1}{2} w^2 + \frac{1}{3} w^3 - \frac{r_w}{4} w^4, \quad |r_w| < 1, |w| < 1,
\]

we have

\[
\left( \psi\left( \frac{\theta}{\sqrt{n}} \right) \right)^n = \exp \left\{ n \left[ -\frac{1}{2n} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s + \frac{1}{24n^2} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{6n^3} \sum_{s=1}^{d} \zeta_s(6) \theta^6_s \\
- \frac{1}{2} \left( -\frac{1}{2n} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s + \frac{1}{24n^2} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s \right)^2 + \frac{1}{3} \left( -\frac{1}{2n} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^3 + \beta_n(\theta) \left( \frac{1}{n^4} \right) \right] \right\} \\
= \exp \left\{ -\frac{1}{2n} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right\} \left\{ \frac{1}{n} \left[ 1 + \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 \right] \\
+ \frac{1}{n^2} \left[ -\frac{1}{24} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^3 - \frac{1}{6} \left( \sum_{s=1}^{d} \zeta_s(6) \theta^6_s \right) + \frac{1}{48} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \sum_{s=1}^{d} \zeta_s(4) \theta^4_s \\
+ \frac{1}{2} \left( \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 \right)^2 \right] + \beta_n(\theta) \left( \frac{1}{n^4} \right) \right\}.
\]

Once again using Taylor’ expansion

\[
e^x = 1 + x + \frac{1}{2} x^2 + \frac{e^{rx}}{3} x^3, \quad |r_x| < |x|, x \in \mathbb{R},
\]

we get

\[
\left( \psi\left( \frac{\theta}{\sqrt{n}} \right) \right)^n = \exp \left\{ -\frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right\} \left\{ 1 + \frac{1}{n} \left[ \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 \right] \\
+ \frac{1}{n^2} \left[ -\frac{1}{24} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^3 - \frac{1}{6} \left( \sum_{s=1}^{d} \zeta_s(6) \theta^6_s \right) + \frac{1}{48} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \sum_{s=1}^{d} \zeta_s(4) \theta^4_s \\
+ \frac{1}{2} \left( \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 \right)^2 \right] + \beta_n(\theta) \left( \frac{1}{n^4} \right) \right\}, \quad (3.9)
\]

where all \( \beta_n(\theta) \) are infinitesimals satisfying \( \sup_{|\theta| \leq \theta_n} |\beta_n(\theta)| \leq C n^{-1+10\delta} \).

Combining (3.8) and (3.9), we get

\[
\cos \left( \theta \left( \frac{1}{\sqrt{n}} \right) \right) \left( \psi\left( \frac{\theta}{\sqrt{n}} \right) \right)^n \\
= \exp \left\{ -\frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right\} \left\{ 1 + \frac{1}{n} \left[ \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 - \frac{\langle \theta, z \rangle}{2} \right] \\
+ \frac{1}{n^2} \left[ \frac{1}{48} \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \sum_{s=1}^{d} \zeta_s(4) \theta^4_s + \frac{1}{2} \left( \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta^4_s - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta^2_s \right)^2 \right)^2 \right] - \frac{1}{6} \sum_{s=1}^{d} \zeta_s(6) \theta^6_s
\]
\[-\frac{1}{24} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^3 - \frac{(\theta, z)^2}{2} \left( \frac{1}{24} \sum_{s=1}^{d} \zeta_s(4) \theta_s^4 - \frac{1}{8} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^2 \right) + \frac{(\theta, z)^4}{24} \]

\[+ \tilde{\gamma}_n(\theta, z) \frac{1}{n^2}, \]

where the term $\tilde{\gamma}_n(\theta, z)$ satisfies

\[
\sup_{\{|z| \leq Cn^\kappa, |\theta| \leq n^\kappa\}} \left| \tilde{\gamma}_n(\theta, z) \right| \leq Cn^{-1 + \max\{6(\delta + \kappa), 105\}}.
\]

Then we see that

\[
\sup_{\{|z| \leq Cn^\kappa, |\theta| \leq n^\kappa\}} \left| \int_{|\theta| \leq n^\kappa} \tilde{\gamma}_n(\theta, z) e^{-\frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2} d\theta \right| \leq Cn^{-1 + \max\{6(\delta + \kappa), 105\}}.
\]

By elementary but tedious calculus, we obtain the following results:

1. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} (\theta, z)^2 d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} (z, \Gamma_2^{-1} z) + \alpha_n(\theta, z) \frac{1}{n^2},
\]

2. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} (\theta, z)^4 d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} 3 (z, \Gamma_2^{-1} z)^2 + \alpha_n(\theta, z) \frac{1}{n^2},
\]

3. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(4) \theta_s^4 \right)^2 (\theta, z)^2 d\theta
\]

\[= \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} (d + 2)(d + 4) (z, \Gamma_2^{-1} z) + \alpha_n(\theta, z) \frac{1}{n^2},
\]

4. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(4) \theta_s^4 \right) (\theta, z)^2 d\theta
\]

\[= \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} 3 (z, \Gamma_4^{-2}) (z, \Gamma_2^{-1} z) + \alpha_n(\theta, z) \frac{1}{n^2},
\]

5. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} + \varepsilon_n \frac{1}{n^2},
\]

6. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(4) \theta_s^4 \right) d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} 3 \text{tr}(\Gamma_4^{-2}) + \varepsilon_n \frac{1}{n^2},
\]

7. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^2 d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} \cdot d(d + 2) + \varepsilon_n \frac{1}{n^2},
\]

8. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^3 d\theta
\]

\[= \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} 3(d + 4) \text{tr}(\Gamma_4^{-2}) + \varepsilon_n \frac{1}{n^2},
\]

9. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(6) \theta_s^6 \right) d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} 15 \text{tr}(\Gamma_6^{-3}) + \varepsilon_n \frac{1}{n^2},
\]

10. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^3 d\theta = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} \cdot d(d + 2)(d + 4) + \varepsilon_n \frac{1}{n^2},
\]

11. \[
\int_{\|\theta\| < n^\kappa} \exp \left\{ - \frac{1}{2} \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right\} \left( \sum_{s=1}^{d} \zeta_s(2) \theta_s^2 \right)^4 d\theta
\]

\[= \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} \cdot d(d + 2)(d + 4) + \varepsilon_n \frac{1}{n^2}.
\]
Thus the integrand of \( (3.10) \) satisfies
\[
\sup_{\|z\| \leq Cn^e} |\alpha_n^e(z)| \leq Ce^{-n^{d/2}}, \quad \varepsilon_n^e \leq Ce^{-n^{d/2}}.
\]
Recall that
\[
I_{2,n}(z) = \int_{\|\theta\| \leq n^e} \cos \left( \frac{1}{n^e} (\theta, z) \right) \left( \psi \left( \frac{\theta}{n^e} \right) \right)^n d\theta.
\]
By using the above integrals and (3.10), we have
\[
I_{2,n}(z) = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} \left\{ 1 + \frac{1}{n} \left[ \frac{1}{8} (\Gamma_2 \Gamma_2^{-1}z - d(2) - d(2)) - \frac{1}{2} \langle \zeta, \Gamma_2^{-1}z \rangle \right] + \frac{1}{n^2} \left[ \frac{1}{8} \langle \zeta, \Gamma_2^{-1}z \rangle \right]^2 
- \frac{1}{16} \left[ \langle \Gamma_2 \Gamma_2^{-1}z - d(2)(d+4), \zeta \rangle - \frac{1}{4} \langle \Gamma_2 \Gamma_2^{-1}z, \zeta \rangle - \frac{1}{64} (d+2)(d+4) \langle \Gamma_2 \Gamma_2^{-1}z, \zeta \rangle \right]
+ \frac{1}{12} \left( \langle \Gamma_2 \Gamma_2^{-1}z \rangle + \frac{1}{128} (\Gamma_2 \Gamma_2^{-1}z)^2 \right) \right\}.
\]
where the term \( \alpha_n(z) \) satisfies
\[
\sup_{\|z\| \leq Cn^e} |\alpha_n(z)| \leq Cn^{-1+\max\{6(d+\kappa),10b\}}. \tag{3.11}
\]
With the notation \( \tau_d, \Lambda_d \) and \( \chi_d \) defined in (1.9), (1.11) and (1.12), we get
\[
I_{2,n}(z) = \frac{(2\pi)^{d/2}}{\sqrt{\det \Gamma_2}} \left\{ 1 + \frac{1}{n} \left[ \tau_d - \frac{1}{2} \langle \zeta, \Gamma_2^{-1}z \rangle \right] + \frac{1}{n^2} \left[ \frac{1}{8} \langle \zeta, \Gamma_2^{-1}z \rangle \right]^2 
- \langle \Lambda_d z, z \rangle + \frac{1}{n^2} \alpha_n(z) \right\}. \tag{3.12}
\]
where the term \( \alpha_n(z) \) satisfies (3.11).

Combining the equations (3.5) – (3.12), we obtain the desired result (3.1).

(II) Now we deal with the case \( L \in B_d \).

In this case, it holds that \( \zeta_0 = 0 \) and the set \( \{r : \zeta_s, r > 0, s = 1, 2, \cdots, d\} \) only contains odd numbers. Thus the integrand of (3.4) satisfies
\[
\cos \langle \varphi + \pi, z \rangle \psi(\varphi + \pi) \psi(\varphi)^n = (-1)^{n+z_1+\cdots+z_d} \cos \langle \varphi, z \rangle \psi(\varphi)^n = \cos \langle \varphi, z \rangle \psi(\varphi)^n.
\]
Therefore (3.4) turns out to be
\[
\mathbb{P}(\tilde{S}_n = z) = 2(2\pi)^{-d} \int_A \cos \langle \varphi, z \rangle \psi(\varphi)^n d\varphi, \tag{3.13}
\]
where \( A = [-\pi/2, \pi/2] \times [-\pi, \pi]^{d-1} \). Observe that
\[
\tilde{\varphi} := \sup \{ |\varphi| : \varphi \in A, \|\varphi\| > r \} < 1.
\]

Hence
\[
\mathbb{P}(\tilde{S}_n = z) = \tilde{I}_n + \tilde{J}_n := 2(2\pi)^{-d} \int_{\|\varphi\| \leq r} \cos \langle \varphi, z \rangle \left( |\varphi| \right)^n \, d\varphi + 2(2\pi)^{-d} \int_{A \setminus \{ |\varphi| \leq r \}} \cos \langle \varphi, z \rangle \left( |\varphi| \right)^n \, d\varphi,
\]
(3.14)
where \(|J_n(z)| \leq \tilde{\varphi}^n\). Then following the same procedures as in (I), we can obtain the desired (3.2). \( \square \)

4 Proof of Theorem 1.1 and Corollary 1.3

We first introduce some notation. As usual, we write \( \mathbb{N}^+ = \{1, 2, 3, \ldots\} \) and denote by
\[
U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n
\]
the set of all finite sequences, where \((\mathbb{N}^*)^0 = \{\emptyset\}\) contains the null sequence \(\emptyset\).

For all \( u \in U \), let \( T(u) \) be the shifted tree of \( T \) at \( u \) with defining elements \{\( N_{uv} \)\}: we have 1) \( \emptyset \in T(u) \), 2) \( v_i \in T(u) \Rightarrow v \in T(u) \) and 3) if \( v_i \in T(u) \), then \( v_i \in T(u) \) if and only if \( 1 \leq i \leq N_{uv} \).
Define \( T_n(u) = \{ v \in T(u) : |v| = n \} \). Recall that \( T = T(\emptyset) \) and \( T_n = T_n(\emptyset) \).

Let \( \kappa \) be a real number satisfying \( \frac{d+6}{2\lambda} < \kappa < \frac{1}{\mu} \) and set \( k_n = [n^\kappa] \), the largest integer not bigger than \( n^\kappa \).

From the additivity property of the branching process, it follows that
\[
Z_n(z) = \sum_{u \in T_{k_n}} \sum_{v \in T_{n-k_n}(u)} 1\{S_{uv} = z\},
\]
(4.1)
and
\[
\mathbb{P}_{\emptyset \varphi, k_n} \left( \sum_{v \in T_{n-k_n}(u)} 1\{S_{uv} = z\} \right) = m^{n-k_n} \mathbb{P}(\tilde{S}_{n-k_n} = z-y)\bigg|_{y=S_u}.
\]
Then we have the following decomposition:
\[
\frac{Z_n(z)}{m^n} = \frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \left( \frac{\sum_{v \in T_{n-k_n}(u)} 1\{S_{uv} = z\}}{m^{n-k_n}} - \mathbb{P}(\tilde{S}_{n-k_n} = z-y)\bigg|_{y=S_u} \right) + \frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \mathbb{P}(\tilde{S}_{n-k_n} = z-y)\bigg|_{y=S_u} =: D_{1,n} + D_{2,n}.
\]
(4.2)

So Theorem 1.1 will follow from the following two lemmas:

Lemma 4.1. Assume that the conditions (1.4) and (1.5) hold, \( \mathbb{E}N(\ln N)^{1+\lambda} < \infty \) for some \( \lambda > 3(d+6) \) and \( L \) has the law (1.1). Then we have
\[
n^{2+\frac{4}{d}} \mathbb{P}_{\emptyset \varphi} \xrightarrow{n \to \infty} 0 \quad \text{a.s.}
\]
(4.3)
Lemma 4.2. Assume that the conditions (1.4) and (1.5) hold, \( \mathbb{E}N(\ln N)^{1+\lambda} < \infty \) for some \( \lambda > 3(d + 6) \) and \( L \) has the law (1.1). Then as \( n \to \infty \), a.s.

(I) when \( L \in \mathcal{A}_d \),

\[
\mathbb{P}_{2,n} = \left( \frac{2(2n)^{-d/2}}{\sqrt{\det \Gamma_2}} \right) \left[ W + \frac{1}{n} F_1(z) + \frac{1}{n^2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1); \tag{4.4}
\]

(II) when \( L \in \mathcal{B}_d \) provided \( n \equiv z_1 + z_2 + \cdots + z_d \pmod{2} \),

\[
\mathbb{P}_{2,n} = \left( \frac{2(2n)^{-d/2}}{\sqrt{\det \Gamma_2}} \right) \left[ W + \frac{1}{n} F_1(z) + \frac{1}{n^2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1), \tag{4.5}
\]

where \( F_1(z) \) and \( F_2(z) \) are defined by (1.8) and (1.10) respectively.

Proof of Theorem 1.1. Combining Lemmas 4.1 with 4.2, together with (4.2), we obtain immediately Theorem 1.1.

Proof of Corollary 1.3. This corollary follows immediately from Theorem 1.1 by some elementary calculations.

Proof of Lemma 4.1. We start by introducing some notation. For \( u \in T_{k_n} \), set

\[
X_{n,u} = \frac{\sum_{v \in T_{n-k_n}(u)} 1(S_{n-v} = z)}{m^{n-k_n}} - \mathbb{P}(\tilde{S}_{n-k_n} = z - y) | y = S_u \}, \quad \mathbb{X}_{n,u} = X_{n,u} 1\{|X_{n,u}| \leq m^{k_n}\},
\]

\[
\mathbb{K}_n = \frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \mathbb{X}_{n,u}.
\]

It is easy to see the following fact:

\[
|X_{n,u}| \leq W_{n-k_n}(u) + 1, \quad \text{with } W_{n-k_n}(u) = m^{-(n-k_n)} \sum_{v \in T_{n-k_n}(u)} 1.
\tag{4.6}
\]

We remind that \( \{W_{n-k_n}(u) : u \in T_{k_n}\} \) are mutually independent and identically distributed as \( W_{n-k_n} \).

The lemma will be proved if we can show the following:

\[
\mathbb{P}(\mathbb{K}_n \neq \mathbb{A}_n \text{ i.o.}) = 0. \tag{4.7}
\]

\[
n^{d/2+2}(\mathbb{K}_n - \mathbb{E}_\mathbb{P}_{\mathbb{A}_n} \mathbb{K}_n) \xrightarrow{n \to \infty} 0 \quad \text{a.s.} \tag{4.8}
\]

\[
n^{d/2+2} \mathbb{E}_\mathbb{P}_{\mathbb{A}_n} \mathbb{K}_n \xrightarrow{n \to \infty} 0 \quad \text{a.s.} \tag{4.9}
\]

To this end, we shall need the following result.

Lemma 4.3. (8) Let \( W^* = \sup_n W_n \). Assume \( m > 1 \) and \( \mathbb{E}N(\log N)^{1+\lambda} < \infty \). Then

\[
\mathbb{E}(W^* + 1)(\log(W^* + 1))^\lambda < \infty. \tag{4.10}
\]

To prove (4.7), it suffices to show that

\[
\sum_{n=1}^{\infty} \mathbb{P}(\mathbb{K}_n \neq \mathbb{A}_n) < \infty. \tag{4.11}
\]

Observe that

\[
\mathbb{P}(\mathbb{A}_n \neq \mathbb{K}_n) \leq \mathbb{E} \sum_{u \in T_{k_n}} \mathbb{P}_{\mathbb{P}_{\mathbb{A}_n}} (X_{n,u} \neq \mathbb{X}_{n,u}) = \mathbb{E} \sum_{u \in T_{k_n}} \mathbb{P}_{\mathbb{P}_{\mathbb{A}_n}} (|X_{n,u}| \geq m^{k_n})
\]
Then (4.11) follows from the choice of $k_n$, the fact $\lambda \kappa > 1$ and (4.10).

Now we turn to the proof of (4.8). To this end, we will need the following inequality (by (5.3) in [6]):

for $1 < \alpha < 2,$

$$P_{\theta_k} \left[ \sum_{u \in \mathbb{T}_{k_n}} m^{-k_n} (X_{n,u} - E_{\theta_k} X_{n,u}) > \varepsilon n^{-d/2-2} \right] \leq K \frac{n^{(d/2+2)\alpha}}{\varepsilon \alpha} \left\{ m^{(1-\alpha)k_n} E(W^* + 1) \alpha I_{\{W^* + 1 \leq m^k\}} + m^{k_n} E_{1} \{W^* + 1 > m^k\}\right. \right\}.

Since $E_{Z_k} (Z^d) = m^k$, taking expectation in the both sides of the above gives the following

$$E_{P_{\theta_k}} \left[ \sum_{u \in \mathbb{T}_{k_n}} m^{-k_n} (X_{n,u} - E_{\theta_k} X_{n,u}) > \varepsilon n^{-d/2-2} \right] \leq K \frac{n^{(d/2+2)\alpha}}{\varepsilon \alpha} \left\{ m^{(1-\alpha)k_n} E(W^* + 1) \alpha I_{\{W^* + 1 \leq m^k\}} + m^{k_n} E_{1} \{W^* + 1 > m^k\}\right. \right\}.

Note that in the above formula and throughout the paper, $K$ denotes all constants, and thus its value may vary even in a single inequality. Thus by taking expected value of the above, we deduce that

$$\sum_{n=1}^{\infty} P(\bar{A}_n - E_{\theta_k} \bar{A}_n > \varepsilon n^{-(d/2+2)}) = \sum_{n=1}^{\infty} E_{P_{\theta_k}} \left[ \sum_{u \in \mathbb{T}_{k_n}} m^{-k_n} (X_{n,u} - E_{\theta_k} X_{n,u}) > \varepsilon n^{-d/2+2} \right] \leq \sum_{n=1}^{\infty} K \frac{n^{(d/2+2)\alpha}}{\varepsilon \alpha} \left\{ m^{(1-\alpha)k_n} E(W^* + 1) \alpha I_{\{W^* + 1 \leq m^k\}} + m^{k_n} E_{1} \{W^* + 1 > m^k\}\right. \right\} \leq K \varepsilon^{-\alpha} \left\{ (W^* + 1) \frac{(d+4)\alpha+2}{(2\alpha)-1} \right\}^{(d+4)\alpha+2}/(2\alpha)-1\right\},

which is finite, since $((d+4)\alpha+2)/(2\alpha)-1 < \lambda$ provided that $\alpha$ is sufficiently near one and $E(W^* + 1)(\log(W^* + 1))^{\lambda} < \infty$. Hence (4.8) follows by the Borel-Cantelli lemma.

It remains to prove (4.9). Since $E_{\theta_k} X_{n,u} = 0$, we see a.s.

$$|E_{\theta_k} \bar{A}_n| = \left| \frac{1}{m^k} \sum_{u \in \mathbb{T}_{k_n}} E_{\theta_k} X_{n,u} \right| \leq \frac{1}{m^k} \sum_{u \in \mathbb{T}_{k_n}} E_{\theta_k} (W_{n-k_n}(u) + 1) \alpha I_{\{W_{n-k_n}(u) + 1 \geq m^k\}} \leq W_{k_n} E(W_{n-k_n} + 1) \alpha I_{\{W_{n-k_n} + 1 \geq m^k\}} \leq W^* (W^* + 1) = k_n^{-\lambda} (\log m)^{-\lambda} W^* (W^* + 1) \log^\lambda (W^* + 1).$$
Combining this with the fact \( \lambda \kappa - d/2 - 2 > 1 \), we get

\[
\left| \sum_{n=1}^{\infty} n^{d/2 + 2} E_{\mathcal{B}^{\kappa}} \tau_n \right| \leq \sum_{n=1}^{\infty} n^{d/2 + 2} k_n^{-\lambda} W^* \mathbb{E}(W^* + 1) \log^\lambda(W^* + 1) < \infty \text{ a.s.}
\]

This implies the a.s. convergence of the series \( \sum_{n=1}^{\infty} n^{d/2 + 2} E_{\mathcal{B}^{\kappa}} \tau_n \), and accordingly (4.9) follows. The lemma has been proved.

**Proof of Lemma 4.2.** (I) First we consider the case \( L \in \mathcal{A}_d \).

Since \( ||S_u|| \leq Ck_n \) for \( u \in \mathbb{T}_k \), then by Theorem 3.1 (I), we have

\[
\mathbb{P}(\tilde{S}_{n-k_n} = z-y) = \frac{(2\pi)^{-d/2}}{\sqrt{\det \Gamma_2}} \left( \frac{1}{(n-k_n)^{d/2}} + \frac{1}{(n-k_n)^{d+2}} \left[ \tau_d - \frac{1}{2} \langle z - S_u, \Gamma_2 \rangle \right] \right)
\]

\[
+ \frac{1}{(n-k_n)^{2+d/2}} \left[ \frac{8}{9} \left( z - S_u, \Gamma_2 \right)^2 - \langle \Lambda_d(z - S_u), z - S_u \rangle + \chi_d \right]
\]

\[
+ \frac{\alpha_n(z, S_u)}{n^{d/2+2}}, \quad (4.12)
\]

where \( \tau_d, \chi_d \) and \( \Lambda_d \) are defined by (1.9) (1.12) and (1.11), and \( \alpha_n(z, S_u) \) are infinitesimals such that

\[
\sup_{u \in \mathbb{T}_n} |\alpha_n(z, S_u)| \xrightarrow{n \to \infty} 0.
\]

Observe that the following relations hold:

1. \( \langle z - S_u, \Gamma_2^{-1}(z - S_u) \rangle = \langle z, \Gamma_2^{-1}z \rangle - 2 \langle \Gamma_2^{-1}z, S_u \rangle + \langle S_u, \Gamma_2^{-1}S_u \rangle \),

2. \( \langle z - S_u, \Gamma_2^{-1}(z - S_u) \rangle^2 = \langle z, \Gamma_2^{-1}z \rangle^2 + 4 \langle \Gamma_2^{-1}z, S_u \rangle^2 + \langle S_u, \Gamma_2^{-1}S_u \rangle^2 + 2 \langle \Gamma_2^{-1}z, S_u \rangle \langle S_u, \Gamma_2^{-1}S_u \rangle - 4 \langle \Gamma_2^{-1}z, S_u \rangle \langle S_u, \Gamma_2^{-1}S_u \rangle \),

3. \( (n - k_n)^{-d/2} = \frac{1}{n^{d/2}} \left[ 1 + \frac{dk_n}{2n} + \frac{(d+2)k^2_n}{8n^2} + O \left( \frac{k^3_n}{n^{3+d/2}} \right) \right] \),

4. \( (n - k_n)^{-1-d/2} = \frac{1}{n^{d/2}} \left[ 1 + \frac{(d+2)k_n}{2n^2} + O \left( \frac{k^2_n}{n^3} \right) \right] \),

5. \( (n - k_n)^{-2} = \frac{1}{n^2} + O \left( \frac{k_n}{n^3} \right) \).

By the definitions of the quantities \( \tau_d, \Lambda_d \), we can get

\[
\text{tr}(\Gamma_2 \Lambda_d) = \frac{1}{16} d \left( \text{tr}(\Gamma_4 \Gamma_2^{-2}) - (d+2)(d+4) \right) + \frac{1}{4} \text{tr}(\Gamma_4 \Gamma_2^{-2}),
\]

hence

\[
\left( \frac{d}{2} + 1 \right) k_n \tau_d = k_n \text{tr}(\Gamma_2 \Lambda_d) + \frac{1}{8} \left( d(d+2) - \text{tr}(\Gamma_4 \Gamma_2^{-2}) \right) k_n.
\]

Substituting the above expressions into (4.12), we obtain that

\[
\mathbb{P}(\tilde{S}_{n-k_n} = z-y) \bigg|_{y=S_u} \quad \frac{(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}} \times \left[ 1 + \frac{dk_n}{2n} + \frac{(d+2)k^2_n}{8n^2} \right] + \frac{1}{n} + \frac{(d+2)k_n}{2n^2} \left[ \tau_d - \frac{1}{2} \langle z, \Gamma_2^{-1}z \rangle + \langle S_u, \Gamma_2^{-1}z \rangle - \frac{1}{2} \langle S_u, \Gamma_2^{-1}S_u \rangle \right].
\]
By the definitions of the martingales \( N_{2,n} \), we deduce

\[
\frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \left( \langle S_u, \Gamma_{2}^{-1} S_u \rangle - dk_u \right) = \langle N_{2,k_n}, \Gamma_{2}^{-1} 1 \rangle,
\]

\[
\frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \left( \langle S_u, \left( 2\Lambda_d - \frac{1}{4} \langle z, \Gamma_{2}^{-1} z \rangle \Gamma_{2}^{-1} \right) S_u \rangle \right) - k_u \left( \text{tr}(\Gamma_2 \Lambda_d) - \frac{1}{4} d \langle z, \Gamma_{2}^{-1} z \rangle \right) = \langle N_{2,k_n}, \left( \Lambda_d - \frac{1}{4} \langle z, \Gamma_{2}^{-1} z \rangle \Gamma_{2}^{-1} \right) 1 \rangle.
\]

By the linearity of inner product and the definitions of \( N_{1,n}, N_{3,n} \), we see

\[
\frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \langle S_u, \Gamma_{2}^{-1} z \rangle = \langle N_{1,k_n}, \Gamma_{2}^{-1} z \rangle,
\]

\[
\frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \left( \langle S_u, \left( 2\Lambda_d - \frac{1}{2} \langle z, \Gamma_{2}^{-1} z \rangle \Gamma_{2}^{-1} \right) S_u \rangle \right) = \langle N_{1,k_n}, \left( 2\Lambda_d - \frac{1}{2} \langle z, \Gamma_{2}^{-1} z \rangle \Gamma_{2}^{-1} \right) z \rangle,
\]

\[
\frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \langle \Gamma_{2}^{-1} z, \langle S_u, \Gamma_{2}^{-1} S_u \rangle S_u - (d + 2)k_u S_u \rangle = \langle \Gamma_{2}^{-1} z, N_{3,k_n} \rangle.
\]

Using the definitions of \( N_{2,n}^2, N_{4,n} \) and substituting all the above into (4.15), we derive that

\[
\mathbb{E}_{2,n} = \frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \mathbb{P} \left( S_{n-k_n} = z - y \right) \bigg| y = S_u
\]

\[
= \frac{(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}} \left[ W_{k_n} + \frac{1}{n} F_{1,k_n}(z) + \frac{1}{n^2} F_{2,k_n}(z) \right] + \frac{1}{m^{k_n}} \sum_{u \in T_{k_n}} \alpha_{n,u}. \quad (4.16)
\]

where

\[
F_{1,k_n}(z) = \left( \tau_d - \frac{1}{2} \langle z, \Gamma_{2}^{-1} z \rangle \right) W_{k_n} + \langle N_{1,k_n}, \Gamma_{2}^{-1} z \rangle - \frac{1}{2} \langle N_{2,k_n}, \Gamma_{2}^{-1} 1 \rangle,
\]

and \( \alpha_{n,u} (u \in T_{k_n}) \) denotes a family of infinitesimals dominated by an absolute infinitesimal \( \varepsilon_n \), i.e.

\[
\sup\{|\alpha_{n,u}| : u \in T_{k_n}\} \leq \varepsilon_n \rightarrow 0.
\]
\[ F_{2,k_n}(z) = \left( \frac{1}{8} \left( \langle z, \Gamma^{-1} z \rangle \right)^2 - \langle z, \Lambda_d z \rangle + \chi_d \right) W_{k_n} + \left\langle N_{1,k_n}, \left( 2\Lambda_d - \frac{1}{2} \langle z, \Gamma^{-1}_2 z \rangle \Gamma^{-1}_2 \right) z \right\rangle \\
- \left\langle N_{2,k_n}, \left( \Lambda_d - \frac{1}{4} \langle z, \Gamma^{-1}_2 z \rangle \Gamma^{-1}_2 \right) 1 \right\rangle + \frac{1}{2} N_{2,k_n}^2 - \frac{1}{2} \left( \langle z, \Lambda_d z \rangle + \chi_d \right) W_{k_n} + \frac{1}{8} N_{4,k_n}. \]

By the choice of \( \kappa \), we see that \( \lambda \kappa > 2 \) and \( (\lambda - 3) \kappa > 1 \). Hence by (2.4) and Theorem 2.1,

\[ W_{k_n} - W = o(1/n^2), \quad N_{q,k_n} - V_q = o(1/n), \quad q = 1, 2, \tag{4.17} \]
\[ N_{2,k_n}^2 - V_2^2 = o(1), \quad N_{q,k_n} - V_q = o(1), \quad q = 3, 4. \tag{4.18} \]

Therefore as \( n \) tends to infinity,

\[ F_{1,k_n}(z) - F_1(z) = o\left( \frac{1}{n} \right), \quad F_{2,k_n}(z) - F_2(z) = o(1), \quad \text{a.s.} \]

where \( F_1(z) \) and \( F_2(z) \) are defined by (1.8) and (1.10).

Observe that

\[ \left| \frac{1}{m_{k_n}} \sum_{w \in T_{k_n}} \alpha_{n,w} \right| \leq \varepsilon_n W_{k_n} \to 0. \]

Substituting these into (4.16), we have

\[ D_{2,n} = \frac{(2\pi n)^{-d/2}}{\sqrt{\det \frac{1}{2}}} \left[ W_{k_n} + \frac{1}{n} F_1(z) + \frac{1}{n^2} F_2(z) \right] + \frac{1}{n^{d+1/2}} o(1), \tag{4.19} \]

with \( F_1(z) \) and \( F_2(z) \) defined by (1.8) and (1.10). This is exactly what we want to prove.

(II) We consider the case \( L \in B_d \).

By using Theorem 3.1 (II) and following the arguments as in the first part (I), the formula (4.5) can be handled and we omit the details.

The lemma is proved.

\[ \square \]

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**References**

[1] S. Albeverio, L. V. Bogachev, S. A. Molchanov, and E. B. Yarovaya, *Annealed moment Lyapunov exponents for a branching random walk in a homogeneous random branching environment*, Markov Process. Related Fields 6 (2000), no. 4, 473–516.

[2] S. Asmussen, *Convergence rates for branching processes*, Ann. Probab. 4 (1976), no. 1, 139–146.

[3] S. Asmussen and H. Hering, *Branching processes*, Progress in Probability and Statistics, vol. 3, Birkhäuser Boston, Inc., Boston, MA, 1983.

[4] S. Asmussen and N. Kaplan, *Branching random walks. I*, Stochastic Process. Appl. 4 (1976), no. 1, 1–13.

[5] K. B. Athreya and P. E. Ney, *Branching processes*, Springer-Verlag, New York, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 196.
[6] J. D. Biggins, *Growth rates in the branching random walk*, Z. Wahrsch. verw. Geb. 48 (1979), no. 1, 17–34.

[7] J. D. Biggins, *The central limit theorem for the supercritical branching random walk, and related results*, Stochastic Process. Appl. 34 (1990), no. 2, 255–274.

[8] N. H. Bingham and R. A. Doney, *Asymptotic properties of supercritical branching processes. I. The Galton-Watson process*, Advances in Appl. Probab. 6 (1974), 711–731.

[9] X. Chen, *Exact convergence rates for the distribution of particles in branching random walks*, Ann. Appl. Probab. 11 (2001), no. 4, 1242–1262.

[10] X. Chen and H. He, *On large deviation probabilities for empirical distribution of supercritical branching random walks with unbounded displacements*, Probab. Theory Relat. Fields 175 (2019), 255–307.

[11] Z.-Q. Gao, *Exact convergence rate of the local limit theorem for branching random walks on the integer lattice*, Stoch. Process. Appl. 127 (2017), no. 4, 1282–1296.

[12] Z.-Q. Gao, *A second order asymptotic expansion in the local limit theorem for a simple branching random walk in \( \mathbb{Z}^d \)*, Stoch. Process. Appl. 128 (2018), no. 12, 4000–4017.

[13] Z.-Q. Gao and Q. Liu, *Exact convergence rate in the central limit theorem for a branching random walk with a random environment in time*, Stoch. Process. Appl. 126 (2016), no. 9, 2634–2664.

[14] Z.-Q. Gao and Q. Liu, *Second and third orders asymptotic expansions for the distribution of particles in a branching random walk with a random environment in time*, Bernoulli 24 (2018), no. 1, 772–800.

[15] Z.-Q. Gao, Q. Liu, and H. Wang, *Central limit theorems for a branching random walk with a random environment in time*, Acta Math. Sci. Ser. B Engl. Ed. 34 (2014), no. 2, 501–512.

[16] R. Grübel and Z. Kabluchko, *A functional central limit theorem for branching random walks, almost sure weak convergence and applications to random trees*, Ann. Appl. Probab. 26 (2016), no. 6, 3659–3698.

[17] R. Grübel and Z. Kabluchko, *Edgeworth expansions for profiles of lattice branching random walks*, Ann. Inst. H. Poincaré Probab. Statist. 53 (2017), no. 4, 2103–2134.

[18] O. Gün, W. König, and O. Sekulović, *Moment asymptotics for branching random walks in random environment*, Electron. J. Probab. 18 (2013), no. 63, 18.

[19] T. E. Harris, *The theory of branching processes*, Die Grundlehren der Mathematischen Wissenschaften, Bd. 119, Springer-Verlag, Berlin, 1963.

[20] C. Huang, X. Liang, and Q. Liu, *Branching random walks with random environments in time*, Frontiers of Mathematics in China 9 (2014), no. 4, 835–842.

[21] C. Huang, X. Wang, and X. Wang, *Large and moderate deviations for a \( \mathbb{R}^d \)-valued branching random walk with a random environment in time*, Stochastics 92 (2020), no. 6, 944–968.

[22] A. Iksanov and Z. Kabluchko, *A central limit theorem and a law of the iterated logarithm for the biggins martingale of the supercritical branching random walk*, J. Appl. Probab. 53 (2016), no. 4, 1178–1192.
[23] A. Iksanov, X. Liang, and Q. Liu, *On $L^p$-convergence of the Biggins martingale with complex parameter*, J. Math. Anal. Appl. **479** (2019), no. 2, 1653–1669.

[24] P. Jagers, *Galton-Watson processes in varying environments*, J. Appl. Probability **11** (1974), 174–178.

[25] A. Joffe and A. R. Moncayo, *Random variables, trees, and branching random walks*, Advances in Math. **10** (1973), 401–416.

[26] Z. Kabluchko, *Distribution of levels in high-dimensional random landscapes*, Ann. Appl. Probab. **22** (2012), no. 1, 337–362.

[27] N. Kaplan and S. Asmussen, *Branching random walks. II*, Stochastic Process. Appl. **4** (1976), no. 1, 15–31.

[28] G. F. Lawler, *Intersections of random walks*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2013, Reprint of the 1996 edition.

[29] G. F. Lawler and V. Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010.

[30] X. Liang and Q. Liu, *Regular variation of fixed points of the smoothing transform*, Stochastic Processes and their Applications **130** (2020), no. 7, 4104 – 4140.

[31] O. Louidor and W. Perkins, *Large deviations for the empirical distribution in the branching random walk*, Electron. J. Probab. **20** (2015), no. 18, 19.

[32] M. Nakashima, *Almost sure central limit theorem for branching random walks in random environment*, Ann. Appl. Probab. **21** (2011), no. 1, 351–373.

[33] P. Révész, *Random walks of infinitely many particles*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

[34] Z. Shi, *Branching random walks*, Lecture Notes in Mathematics, vol. 2151, Springer, Cham, 2015, Lecture notes from École d’Été de Probabilités de Saint-Flour XLII – 2012.

[35] A. J. Stam, *On a conjecture by Harris*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **5** (1966), 202–206.

[36] X. Wang and C. Huang, *Convergence of martingale and moderate deviations for a branching random walk with a random environment in time*, J. Theoret. Probab. **30** (2017), no. 3, 961–995.

[37] X. Wang and C. Huang, *Convergence of complex martingale for a branching random walk in a time random environment*, Electron. Commun. Probab. **24** (2019), Paper No. 41, 14.

[38] Y. Wang, Z. Liu, Q. Liu, and Y. Li, *Asymptotic Properties of a Branching Random Walk with a Random Environment in Time*, Acta Math. Sci. Ser. B (Engl. Ed.) **39** (2019), no. 5, 1345–1362.

[39] N. Yoshida, *Central limit theorem for branching random walks in random environment*, Ann. Appl. Probab. **18** (2008), no. 4, 1619–1635.

[40] O. Zeitouni, *Branching random walks and Gaussian fields*, Notes for Lectures, http:// www. wisdom.weizmann.ac.il/~zeitouni/pdf/ notesBRW.pdf, 2012.