THE MULTIPLICATIVE HILBERT MATRIX

OLE FREDRIK BREVIG, KARL-MIKAIL PERFEKT, KRISTIAN SEIP, ARISTOMENIS G. SISKAKIS,
AND DRAGAN VUKOTIĆ

ABSTRACT. It is observed that the infinite matrix with entries \( \left( \sqrt{m n \log(m n)} \right)^{-1} \) for \( m, n \geq 2 \) appears as the matrix of the integral operator \( Hf(s) := \int_{1/2}^{+\infty} f(w) \zeta(w + s) - 1)dw \) with respect to the basis \( (n^{-s})_{n \geq 2} \); here \( \zeta(s) \) is the Riemann zeta function and \( H \) is defined on the Hilbert space \( \mathcal{H}_0^2 \) of Dirichlet series vanishing at \( +\infty \) and with square-summable coefficients. This infinite matrix defines a multiplicative Hankel operator according to Helson's terminology or, alternatively, it can be viewed as a bona fide (small) Hankel operator on the infinite-dimensional torus \( \mathbb{T}^\infty \). By analogy with the standard integral representation of the classical Hilbert matrix, this matrix is referred to as the multiplicative Hilbert matrix. It is shown that its norm equals \( \pi \) and that it has a purely continuous spectrum which is the interval \( [0, \pi] \); these results are in agreement with known facts about the classical Hilbert matrix. It is shown that the matrix \( \left( \frac{1}{m + n + 1} \right)_{m,n \geq 0} \) has norm \( \pi/\sin(\pi/p) \) when acting on \( \ell^p \) for \( 1 < p < \infty \). However, the multiplicative Hilbert matrix fails to define a bounded operator on \( \mathcal{H}_0^p \) for \( p \neq 2 \), where \( \mathcal{H}_0^p \) are \( H^p \) spaces of Dirichlet series. It remains an interesting problem to decide whether the analytic symbol \( \sum_{n \geq 2} (\log n)^{-1} n^{-s - 1/2} \) of the multiplicative Hilbert matrix arises as the Riesz projection of a bounded function on the infinite-dimensional torus \( \mathbb{T}^\infty \).

1. INTRODUCTION

The classical Hilbert matrix
\[
A := \left( \frac{1}{m + n + 1} \right)_{m,n \geq 0}
\]
is the prime example of an infinite Hankel matrix, i.e., a matrix whose entries \( a_{m,n} \) only depend on the sum \( m + n \). The Hilbert matrix can be viewed as the matrix of the integral operator
\[
H_a f(z) := \int_0^1 f(t)(1 - zt)^{-1} dt
\]
with respect to the standard basis \( (z^n)_{n \geq 0} \) for the Hardy space \( H^2(\mathbb{D}) \). This representation was first used by Magnus [13] who found that the Hilbert matrix has no eigenvalues and that its continuous spectrum is \( [0, \pi] \).

The purpose of this paper is to identify and study a multiplicative analogue of \( A \). This means that we seek an infinite matrix with entries \( a_{m,n} \) that depend only on the product \( mn \)

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and with properties that parallel those of $A$. Our starting point is the multiplicative counterpart to (1) which we have found to be the integral operator

$$Hf(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w + s) - 1)dw$$

acting on Dirichlet series $f(s) = \sum_{n=2}^{\infty} a_n n^{-s}$. Here $\zeta(s)$ denotes the Riemann zeta function, and we assume that $f$ is in $H^2_0$, which means that

$$\|f\|^2_{H^2_0} := \sum_{n=2}^{\infty} |a_n|^2 < \infty.$$

By the Cauchy–Schwarz inequality, every $f$ in $H^2_0$ represents an analytic function in the half-plane $\sigma = \text{Re} \ s > 1/2$. As is readily seen, the reproducing kernel $K_w$ of $H^2_0$ is $K_w(s) = \zeta(s+w) - 1$. This implies that

$$\langle Hf, g \rangle_{H^2_0} = \int_{1/2}^{\infty} f(w)\overline{g(w)}dw$$

when $f$ and $g$ are Dirichlet polynomials. Since arc length measure on the half-line $(1/2, +\infty)$ is a Carleson measure for $H^2_0$ (see [15, Theorem 4]), we get that (3) in fact holds for arbitrary functions $f$ and $g$ in $H^2_0$, and hence $H$ is well defined and bounded on $H^2_0$. Taking into account that every $f$ in $H^2$ is analytic when $\sigma > 1/2$, we find that $\langle f, Hf \rangle_{H^2_0} = 0$ if and only if $f \equiv 0$. Hence (3) also implies that $H$ is a strictly positive operator.

Now an explicit computation of the integral on the right-hand side of (2) shows that the matrix of $H$ with respect to the orthonormal basis $(n^{-s})_{n \geq 2}$ is

$$M := \left( \frac{1}{\sqrt{mn \log(mn)}} \right)_{m,n \geq 2}.$$ 

We will refer to this matrix as the multiplicative Hilbert matrix. We will be interested in understanding $M$ as an operator on $\ell^2 = \ell^2(\mathbb{N} \setminus \{1\})$, which means that, equivalently, we will be concerned with the properties of the integral operator $H$.

Our main result reads as follows.

**Theorem 1.** The operator $H$ is a bounded and strictly positive operator on $H^2_0$ with $\|H\| = \pi$. It has no eigenvalues, and its continuous spectrum is $[0, \pi]$.

This theorem, which is in agreement with what is known about the classical Hilbert matrix, should be seen as an outgrowth of Helson’s last two papers [11, 12]. In these works, a study of multiplicative Hankel matrices was initiated, mainly focused on the question of to which extent Nehari’s theorem [14] extends to the multiplicative setting. We will return to this interesting question in the final section of this paper. At this point, we just wish to point out that the existence of a canonical operator like $H$, closely related to the Riemann zeta function, clearly demonstrates that multiplicative Hankel matrices may arise quite naturally.

The computation of the norm of $H$ is straightforward, by a simple adaption of the classical proof of [9, pp. 226–229]. In fact, this adaption leads us to consider an $\ell^p$ version of the
multiplicative Hilbert matrix $M$, namely

\[ M_p := \left( \frac{1}{m^{1/p} n^{(p-1)/p} \log(mn)} \right)_{m,n \geq 2}, \]

where $1 < p < \infty$. We will see that $M_p$ has norm $\pi / \sin(\pi / p)$, viewed as an operator on $\ell^p$, which is analogous to the classical fact that $A$ has norm $\pi / \sin(\pi / p)$ when it acts on $\ell^p$. We will explain this link in Section 2.

The identification of the spectrum is the hardest part of the proof of Theorem 1. Inspired by Magnus’s work [13], it is split into two main parts. First, in Section 3, we establish estimates near the singular point $s = 1/2$ for the anticipated solutions $f$ to equations of the form

\[(H - \lambda) f = c \cdot \psi,\]

where $c$ is a constant and $\psi$ is the analytic symbol of $H$. This means that $\psi$ is the primitive of $-(\zeta(s + 1/2) - 1)$ belonging to $\mathcal{H}^2_0$. The point of this estimation is to show that $f'(w)$ must be square integrable on $(1/2, \infty)$. Here we make use of the fact that $\zeta(s) - (s - 1)^{-1}$ is an entire function, which allows us to relate $H$ to a classical operator studied by Carleman. This analysis requires a fair amount of classical-type computations involving Mellin transforms. In Section 4 we may then finish the proof by resorting to the following commutation relation, obtained by integration by parts, between $H$ and the differentiation operator $D$:

\[ DHf(s) = -f(1/2)(\zeta(s + 1/2) - 1) - HDf(s). \]

After finishing the proof of Theorem 1 we turn to two questions related to Helson’s viewpoint, namely that multiplicative Hankel operators are bona fide (small) Hankel operators on the infinite-dimensional torus $T^\infty$. The first question is whether there is a counterpart to the result of [5, 6] saying that the norm of $H_a$ viewed as an operator on $H^p(\mathbb{D})$ is again $\pi / \sin(\pi / p)$. We will show in Section 5 that the analogy with $H_a$ breaks down at this point, or, more precisely, that $H$ does not extend to a bounded operator on the $H^p$ analogues of $\mathcal{H}^2_0$, which by Bayart’s work [1] can be associated with $H^p(T^\infty)$. This negative result is related to, though not a trivial consequence of, the fact that $H^p(T^\infty)$ is not complemented in $L^p(T^\infty)$ [7].

The final question to be discussed concerns the analytic symbol

\[ \psi(s) := \sum_{n=2}^{\infty} \frac{n^{-s}}{\sqrt{n \log n}} \]

of the multiplicative Hankel matrix. Since $-\psi$ is, up to a linear term, a primitive of the Riemann zeta function, it appears to be of interest to investigate it more closely. While it is known from [16] that Nehari’s theorem does not hold in the multiplicative setting, it could still be true that $\psi$ is the Riesz projection of a bounded function. In the final Section 6 we will explain the exact meaning of this statement and show how this question relates to a long-standing embedding problem for $H^p$ spaces of Dirichlet spaces.

A word on notation: Throughout this paper, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set in question, which may be a space of functions or a set of numbers. We write $U(z) \asymp V(z)$ to signify that both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$ hold.
2. THE NORM OF THE MATRIX $M_p$

In this section, $\|M_p\|_p$ will denote the norm of $M_p$ viewed as an operator on $\ell^p$. Our aim is to prove the following theorem, which in particular shows that $\|H\| = \pi$.

**Theorem 2.** We have $\|M_p\|_p = \pi / \sin(\pi / p)$ for $1 < p < \infty$.

**Proof.** The proof relies, as in [9, pp. 226–234], on the following homogeneity property of the kernel $(x + y)^{-1}$:

$$
\int_0^{\infty} x^{-1/p} \frac{1}{1 + x} dx = \int_0^{\infty} x^{-(p-1)/p} \frac{1}{1 + x} dx = \frac{\pi}{\sin(\pi / p)}.
$$

The exact computation of the integral can be found in [19, p. 254, Example 4].

We prove first that $\|M_p\|_p \leq \pi / \sin(\pi / p)$. We write $q = p / (p - 1)$ and assume that $(a_n)$ is in $\ell^p$ and $(b_n)$ is in $\ell^q$. By Hölder’s inequality, we find that

$$
\sum_{m,n \geq 2} |a_m| |b_n| m^{-1/p} n^{-1/q} (\log(mn))^{-1} \leq P \cdot Q,
$$

where

$$
P := \left( \sum_{m \geq 2} |b_m|^p \sum_{n \geq 2} n^{-1} \left( \frac{\log m}{\log n} \frac{1}{\log(mn)} \right)^{1/p} \right)^{1/q},
$$

and

$$
Q := \left( \sum_{n \geq 2} |a_n|^q \sum_{m \geq 2} m^{-1} \left( \frac{\log n}{\log m} \frac{1}{\log(mn)} \right)^{1/p} \right)^{1/q}.
$$

By a change of variables argument, each of the inner sums is dominated by the integral in (5), and hence we obtain the desired bound by duality.

To prove that the norm is bounded below by $\pi / \sin(\pi / p)$, we use the sequences

$$
a_n = n^{-1/q} (\log n)^{-1+\varepsilon/q} \quad \text{and} \quad b_n = n^{-1/p} (\log n)^{-1+\varepsilon/p}
$$

for which we have

$$
\|(a_n)\|_q = \frac{1}{\varepsilon} + O(1) \quad \text{and} \quad \|(b_n)\|_p = \frac{1}{\varepsilon} + O(1)
$$

when $\varepsilon \to 0^+$. We see that

$$
\sum_{m,n \geq 2} a_m b_n m^{-1/p} n^{-1/q} \frac{1}{\log(mn)} = \sum_{m,n \geq 2} (\log m)^{-1+\varepsilon/p} (\log n)^{-1+\varepsilon/q} m^{-1} n^{-1} \frac{1}{\log(mn)} \\
\geq \int_3^{\infty} \int_3^{\infty} x^{-(1+\varepsilon)/q} y^{-(1+\varepsilon)/p} \frac{1}{x+y} dx dy.
$$

This iterated integral can computed as the corresponding integral in [9] (9.5.2) p. 233) so that we get

$$
\sum_{m,n \geq 2} a_m b_n m^{-1/p} n^{-1/q} = \frac{1}{\varepsilon} \left( \frac{\pi}{\sin(\pi / p)} + o(1) \right)
$$

when $\varepsilon \to 0^+$. Combining this estimate with (5), we get the desired bound $\|M_p\|_p \geq \pi / \sin(\pi p)$.

$\square$
It is of interest to observe that when we replace the inner sum $s$ in (6) and (7) by the respective integrals in (5), we get a strict inequality. In particular, we get that
\[ \|Hf\|_{\mathcal{H}^2_0} < \pi \|f\|_{\mathcal{H}^2_0} \]
for every nontrivial function $f$ in $\mathcal{H}^2_0$. This means that we have already shown that $\pi$ is not an eigenvalue for $H$.

Another observation is that the matrix $M_p$ fails to be bounded on $\ell^{p'}$ when $p' \neq p$. This is most easily seen when $p' > p$ because we can find a sequence $a$ in $\ell^{p'}$ for which the entries in $M_p a$ become infinite. When $p' < p$, we can apply the same argument to the conjugate exponents $q$ and $q'$ and the matrix $M_q$.

3. Estimates for solutions of $(H - \lambda)f = c\psi$

In preparation for the characterization of the spectrum of $H$, we will in this section prove precise asymptotics as $s \to 1/2$ for solutions $f$ in $\mathcal{H}^2_0$ of the equation $(H - \lambda)f = c\psi$, where $c$ is a constant and $\psi$ is the analytic symbol of $H$ defined by (4). The considerations to come are in fact of a rather general nature, providing a spectral decomposition of $f$ in terms of generalized eigenvectors of the (shifted) Carleman operator \cite[p. 169]{3} defined by
\[ Cf(s) = \int_{1/2}^{\infty} \frac{f(w)}{s + w - 1} \, dw, \quad s > 1/2. \]

We choose to focus on $H$ for simplicity, but it will be clear from the proof of the next theorem that minor modifications yield similar results for other integral operators whose kernels are perturbations $K(s + w)$, $K$ analytic, of the Carleman kernel.

**Theorem 3.** Suppose that $0 < \lambda < \pi$, and let $\psi$ denote the analytic symbol of $H$, that is
\[ \psi(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n} n^{-s}, \quad \text{Re } s > 1/2. \]

If $f$ in $\mathcal{H}^2_0$ satisfies $(H - \lambda)f = c\psi$, then there exists a complex number $d$ and polynomially bounded sequences of complex numbers $(c_k)_{k=1}^{\infty}$ and $(d_k)_{k=1}^{\infty}$ such that $f$ has the series representation
\[ f(s) = cd + \sum_{k=1}^{\infty} \left( s - 1/2 \right)^{2k-1/2} \left( c_k (s - 1/2)^{-i\theta} + d_k (s - 1/2)^{i\theta} \right), \quad 1/2 < s < 3/2, \]
where $\theta$ is a real number dependent on $\lambda$, namely
\[ \theta = \frac{1}{\pi} \log \left( \frac{\pi}{\lambda} - \sqrt{\left( \frac{\pi}{\lambda} \right)^2 - 1} \right). \]

**Remark.** Note that for each $k$, the functions $s \mapsto (s - 1/2)^{2k-1/2} \pm i\theta$ are generalized eigenvectors of the Carleman operator $C$ belonging to the eigenvalue $\lambda$, $0 < \lambda < \pi$; see Lemma \[1\]. The constant function $s \mapsto cd$ is not such an eigenfunction, and its appearance in (10) will allow us to derive a contradiction in the case that $c \neq 0$.

It is also possible to treat the case $\lambda = \pi$ with the methods below, although we choose not to since we do not need it. Carrying out the details, one obtains for $\lambda = \pi$ a decomposition of
in terms of the eigenfunctions $s \mapsto (s - 1/2)^{2k-1/2}$ and $s \mapsto (s - 1/2)^{2k-1/2} \log(s - 1/2)$ of the Carleman operator.

To simplify the computations and to align our proof with the classical representation of the Carleman operator, we will in this section shift everything to $\mathbb{R}_+ = (0, \infty)$, and prove Theorem 3 on this ray. Shifting the representation back to $(1/2, +\infty)$ will then give (10). This means that we consider $\mathcal{H}_0^2$ the space of Dirichlet series

\[ f(s) = \sum_{n=2}^{\infty} \frac{a_n}{n^{s}}, \]

with coefficients $(a_n) \in \ell^2$, and the operator

\[ (11) \quad Hf(s) = \int_0^\infty f(w) (\zeta(s + w + 1) - 1)\, ds. \]

We let $\{x\}$ denote the fractional part of $x$, and use the well-known formula

\[ \zeta(s + w + 1) - 1 = \frac{1}{s} - (s + 1) \int_1^\infty (x) x^{-(s+1)} \frac{dx}{x} = \frac{1}{s} - (s + 1) \int_0^\infty (e^x) e^{-(s+1)x} \, dx =: 1/s - K(s). \]

The function $1/s$ is the kernel of Carleman's operator, defined on $L^2(\mathbb{R}_+)$ as

\[ Cf(s) = \int_0^\infty \frac{f(w)}{s + w} \, dw. \]

We will let $K$ denote the similarly defined integral operator with kernel $(s, w) \mapsto K(s + w)$, so that $H = C - K$. By the fact that the arc length measure on the ray $(0, +\infty)$ is a Carleman measure for $\mathcal{H}_0^2$, we have for $f \in \mathcal{H}_0^2$ that

\[ \int_0^\infty |f(s)|^2 \, ds \lesssim \sum_{n=2}^{\infty} |a_n|^2 = \|f\|_{\mathcal{H}_0^2}^2, \]

and hence $f$ is in $L^2(\mathbb{R}_+)$. For $0 < \lambda < \pi$ and $f \in \mathcal{H}_0^2$, we consider the equation $(H - \lambda)f = c\psi$, where $\psi$ denotes

\[ \psi(s) = \sum_{n=2}^{\infty} \frac{1}{n \log n} n^{-s}. \]

(Note that this function also differs by a $1/2$ shift from the actual symbol appearing in Theorem 3.) It is convenient to rewrite this equation in the form

\[ (13) \quad (C - \lambda)f = Kf + c\psi. \]

To analyze the equation (13), we will use the Mellin transform, which is defined by

\[ \mathcal{M} f(z) = \int_0^\infty s^z f(s) \frac{ds}{s}. \]

By the Cauchy–Schwarz inequality and (12), taking into account the rapid decay near infinity, we obtain that if $f$ is in $\mathcal{H}_0^2$, then the integral (14) converges absolutely when $\text{Re } z > 1/2$. This means that the function $\mathcal{M} f(z)$ is analytic in (at least) $\text{Re } z > 1/2$. Our first goals are thus to compute $\mathcal{M} Cf$ and $\mathcal{M} Kf$ for $f$ in $\mathcal{H}_0^2$, as well as the special transform $\mathcal{M} \psi$. 
**Lemma 1.** Suppose that $f$ is in $\mathcal{H}_0^2$. Then

$$\tag{15} (\mathcal{M}Cf)(z) = \frac{\pi}{\sin(\pi z)} (\mathcal{M}f)(z),$$

when $\text{Re } z > 1/2$ and $z \notin \mathbb{Z}$.

**Proof.** When $\text{Re } z < 1$, $z \notin \mathbb{Z}$ and $w > 0$, we have

$$\tag{16} \int_0^\infty \frac{s^{z-1}}{s+w} \, ds = \frac{\pi}{\sin(\pi z)} w^{z-1},$$

which is the same integral (5) which was used in the proof of Theorem 2. By this formula and Fubini’s theorem, we obtain (15) in the strip $1/2 < \text{Re } z < 1$. However, the right hand side of (15) has a meromorphic continuation to the domain $\text{Re } z > 1/2$. □

**Remark.** Note that the choice of $\theta$ is such that $\pi / \sin(\pi(i\theta + 1/2)) = \lambda$. This motivates the appearance of the functions $s \mapsto s^{2k-1/2+2i\theta}$ in (10) as generalized eigenfunctions to the Carleman operator. Compare with the remark following Theorem 3.

**Lemma 2.** Let $f$ be a function in $\mathcal{H}_0^2$. Then $(\mathcal{M}Kf)(z)$ has a meromorphic continuation to $\text{Re } z < 1$ with simple poles at the non-positive integers, and satisfies in the same half–plane when $|\text{Im } z| \geq \varepsilon$ for some positive $\varepsilon$, the estimate

$$\tag{17} (\mathcal{M}Kf)(z) \lesssim \|f\|_{\mathcal{H}_0^2} \, |z|^{3/2} e^{-\pi/2|\text{Im } z|}.$$

**Proof.** We begin by computing

$$\tag{18} Kf(s) = \int_0^\infty f(w)K(s+w) \, dw = \sum_{n=2}^\infty \frac{a_n}{\sqrt{n\log n}} \left( \alpha_n(s) + \beta_n(s) \right),$$

where

$$\alpha_n(s) = \int_0^\infty A_n(x) \, s e^{-sx} \, dx, \quad A_n(x) = \frac{1}{1+x/\log n} \frac{|e^x|}{x} \left( \frac{1}{x} e^{-x} \right),$$

$$\beta_n(s) = \int_0^\infty B_n(x) \, s e^{-sx} \, dx, \quad B_n(x) = \frac{1}{2} \left( \frac{1}{(1+x/\log n)^2} + \frac{1}{1+x/\log n} \right) \frac{|e^x|}{x} \left( \frac{1}{x} e^{-x} \right).$$

We will only need the estimates $A_n(x), B_n(x) \leq e^{-x}$, which imply that $Kf(s)$ is analytic in $\text{Re } s > -1$, since $(a_n/\sqrt{n\log n})$ is in $\ell^1$. We apply the Mellin transform of (18), initially with $0 < \text{Re } z < 1$, obtaining

$$\tag{19} (\mathcal{M}Kf)(z) = \sum_{n=2}^\infty \frac{a_n}{\sqrt{n\log n}} \left( \Gamma(1+z)\bar{\alpha}_n(z) + \Gamma(z)\bar{\beta}_n(z) \right),$$

where $\Gamma$ denotes the Gamma function and

$$\bar{\alpha}_n(z) = \int_0^\infty A_n(x) \, x^{1-z} \, \frac{dx}{x} \quad \text{and} \quad \bar{\beta}_n(z) = \int_0^\infty B_n(x) \, x^{2-z} \, \frac{dx}{x}.$$

When $\text{Re } z < 1$, we use the estimates $A_n(x), B_n(x) \leq e^{-x}$ along with the triangle inequality to obtain

$$|\bar{\alpha}_n(z)| \leq \Gamma(1-\text{Re } z) \quad \text{and} \quad |\bar{\beta}_n(z)| \leq 2\Gamma(2-\text{Re } z).$$
Hence $\mathcal{M}Kf$ has a meromorphic continuation to $\Re z < 1$, with simple poles at the poles of $\Gamma(z)$. Moreover, by the Cauchy–Schwarz inequality, we obtain that

$$|\langle \mathcal{M}Kf \rangle(z)| \lesssim \|f\|_{\mathcal{H}_0^2} \left( |\Gamma(1 + z)| \Gamma(1 - \Re z) + 2|\Gamma(z)| \Gamma(2 - \Re z) \right).$$

When $|\Im z| \geq \varepsilon$, we use the functional equation and reflection formula for the Gamma function, and estimate further that

$$|\langle \mathcal{M}Kf \rangle(z)| \lesssim \|f\|_{\mathcal{H}_0^2} \left( |\Gamma(1 + z)| |\Gamma(1 - \Re z)\right) = \|f\|_{\mathcal{H}_0^2} \frac{\pi}{|\sin(\pi z)| |\Gamma(-z)|}.$$ 

Using the assumption that $\Re z < 1$ along with the product formulas for $\Gamma(z)$ and $\sin(\pi z)$, we get

$$\frac{\Gamma(1 - \Re z)}{|\Gamma(-z)|} = |\Re z| \prod_{n=0}^{\infty} \left| \frac{n - z}{n - \Re z} \right| \leq |z| \prod_{n=2}^{\infty} \left| \frac{n - z}{n - \Re z} \right| = \frac{1 - z}{1 - \Re z} \prod_{n=0}^{\infty} \left| \frac{n - z}{n - \Re z} \right| = |z| \frac{1 - z}{1 - \Re z}$$

we therefore arrive at the desired bound

$$|\langle \mathcal{M}Kf \rangle(z)| \lesssim \|f\|_{\mathcal{H}_0^2} \frac{|z|^{3/2}}{|\sin(\pi z)|^{1/2}} \lesssim \|f\|_{\mathcal{H}_0^2} |z|^{3/2} e^{-\pi/2|\Im z|}. \qed$$

**Lemma 3.** For $\Re z > 0$, we have

$$\mathcal{M} \psi(z) = -\frac{1}{z^2} + \sum_{n=0}^{\infty} \frac{b_n}{z + n} + E_{\psi}(z),$$

where $|b_n|$ decays super-exponentially, and $E_{\psi}(z)$ is an entire function that, for every real number $R$, is bounded in the half-plane $\Re z < R$.

**Proof.** Set $h(s) = \psi(s) - \log s$. Since $h'(s) = \zeta(s + 1) - 1 - 1/s$, $h(s) = \sum_{n=0}^{\infty} b_n s^n$ is an entire function. Note now that for $\Re z > 0$ we have

$$\int_{0}^{1} s^{z-1} \log s \, ds = -\frac{1}{z^2},$$

while

$$\int_{0}^{1} s^{z-1} h(s) \, ds = \sum_{n=0}^{\infty} \frac{b_n}{z + n}.$$ 

We finish the proof by setting $E_{\psi}(z) := \int_{1}^{\infty} s^{z-1} \psi(s) \, ds$. \qed

**Proof of Theorem 3.** Suppose that $0 < \lambda < \pi$. Transforming the equation (13) by the Mellin transform and solving for $\mathcal{M}f$, we obtain

$$\mathcal{M}f(z) = \left( \frac{\pi}{\sin(\pi z)} - \lambda \right)^{-1} \left( \mathcal{M}Kf(z) + c\mathcal{M}\psi(z) \right), \quad \Re z > 1/2$$

Initially this formula is only valid for $1/2 < \Re z < 1$, but we note that we may continue the right-hand side meromorphically to $\Re z > 1/2$. 

The inverse Mellin transform is given by

\[ (M^{-1}h)(s) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} s^{-z} h(z) \, dz \]

for a suitable \( k \). Our expressions for \( Mf \) and \( M\psi \) show that the right-hand side of (20) is meromorphic in \( \text{Re} \, z < 1 \) with (possible) simple poles at the solutions of \( \sin(\pi z) = \pi / \lambda \) as well as at \( z = 0 \). Note here that the factor in front of \( Mf(z) + c M\psi(z) \) has simple zeroes at the integers. Note also that there actually are no poles in \( \text{Re} \, z > 1/2 \), since \( Mf(z) \) is analytic there.

Hence we are left with the pole \( z = 0 \) (if \( c \neq 0 \)) and those given by

\[ 1 - \frac{\lambda}{\pi} \sin(\pi z) = 0 \quad \text{Re} \, z \leq 1/2 \quad \iff \quad z = \pm i\theta + (2k + 1/2), \]

where \( k = 0, -1, -2, \ldots \)

We now compute (21) for \( h = Mf \) and \( k = 2/3 \) by the method of residues. Let \( J_n = [\theta] + n \) and form the rectangular contour \( J_n \) with corners in \( 2/3 \pm iJ_n \) and \( -(2J_n + 3/2) \pm iJ_n \), traversed counter-clockwise. Using (17) and (19), straightforward estimates show that for \( 0 < s < 1 \) we have

\[ \lim_{n \to \infty} \int_{J_n} s^{-z} Mf(z) \, dz = \frac{1}{2\pi i} \int_{2/3-i\infty}^{2/3+i\infty} s^{-z} Mf(z) \, dz. \]

Evaluating the left-hand side by residues, we obtain

\[ f(s) = cd + \sum_{k=0}^{\infty} s^{2k-1/2} \left( c_k s^{-i\theta} + d_k s^{i\theta} \right), \quad 0 < s < 1, \]

where \( cd, c_k \), and \( d_k \) are obtained as the residues of the right-hand side of (20) at \( z = 0 \), \( z = i\theta - 2k + 1/2 \) and \( z = -i\theta - 2k + 1/2 \), respectively. In fact, it is clear that \( c_0 \) and \( d_k \) grow at most polynomially in \( k \), as seen from the estimates of Lemma 2 and Lemma 3.

It remains to show that \( c_0 = d_0 = 0 \). However, either of them assuming a non-zero value contradicts the fact that \( f \) is in \( L^2(\mathbb{R}_+) \). Moving back to \( (1/2, +\infty) \), we obtain (10).

Note that in the excluded case \( \lambda = \pi \) one may use the same argument, but the representation of \( f \) is different because all poles of the right-hand side of (20) except \( z = 0 \) are double. We also note that a more careful analysis would show that the sequences \( (c_k) \) and \( (d_k) \) are in fact bounded, but since we don’t need this, we have not made an effort to optimize this part of the theorem.

4. The spectrum of the multiplicative Hilbert matrix

In this section we establish that \( H \) has the purely continuous spectrum \( [0, \pi] \) on \( H^2_0 \). Our argument is based on a commutation relation between \( H \) and the operator \( D \) of differentiation, \( Df(s) = f'(s) \). To establish this relation, we observe that

\[ DHf(s) = \int_{1/2}^{\infty} f(w) D(\zeta(w + s) - 1) \, dw, \quad s > 1/2. \]
Supposing that \( f' \) is integrable on the segment \((1/2, 1)\), we get that
\[
DHf(s) = -f(1/2)(\zeta(s + 1/2) - 1) - \int_{1/2}^{\infty} f'(w)(\zeta(w + s) - 1) \, dw
\]
\[
= -f(1/2)(\zeta(s + 1/2) - 1) - HDf(s), \quad s > 1/2,
\]
where we have defined \( f(1/2) = f(1) - \int_{1/2}^{1} f(w) \, dw \). Thus, \( D \) and \( H \) anti-commute up to an (unbounded) rank-one term. This observation is crucial for the characterization of the spectrum of \( H \).

To demonstrate that \( H \) has the purely continuous spectrum \([0, \pi]\), it suffices to show that \( H \) has no eigenvalues and that \( H - \lambda \) does not have full range for \( \lambda \) in \((0, \pi)\). Indeed, \( H \) is a positive operator with norm \( \pi \), and so it follows that its spectrum is \([0, \pi]\). Since any \( \lambda \) in the spectrum of a self-adjoint operator must either be an eigenvalue or part of the continuous spectrum, we can conclude that \( H \) has purely continuous spectrum. With this in mind we now finish the proof of Theorem 1.

**Theorem 4.** The operator \( H : \mathcal{H}_0^2 \to \mathcal{H}_0^2 \) has no point spectrum. Furthermore, if \( f \) in \( \mathcal{H}_0^2 \) solves the equation \((H - \lambda)f = c\psi\), where \( c \) is a complex number and
\[
\psi(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n} n^{-s},
\]
then \( f = c = 0 \). In particular, the spectrum of \( H \) is \([0, \pi]\) and purely continuous.

**Proof.** We have already proved that \( \lambda = 0, \pi \) are not eigenvalues, since we have shown in Section 2 that \( H \) is a strictly positive operator for which \( \|Hf\|_{\mathcal{H}_0^2} < \pi \|f\|_{\mathcal{H}_0^2}, \; f \neq 0 \). It is hence sufficient to verify the second part of Theorem 4 since it shows simultaneously that no \( \lambda \) in \((0, \pi)\) is an eigenvalue, and that \( H - \lambda \) does not have full range.

Accordingly, we suppose that \( f \) in \( \mathcal{H}_0^2 \) satisfies \((H - \lambda)f = c\psi\). By Theorem 3, we have the series representation \(\psi\). In particular \( f' \) is square-integrable on \((1/2, \infty)\) and \( f(1/2) = cd \). But noting that \( \psi'(s) = \zeta(s + 1/2) - 1 \) and using the commutation relation of \( H \) and \( D \), we then get that
\[
-(H + \lambda)f' - cd(\zeta(s + 1/2) - 1) = c(\zeta(s + 1/2) - 1).
\]
Since \( f' \) is \( L^2(1/2, \infty) \) while \( \zeta \) has a pole of order 1 in \( s = 1 \), it follows that \( d = -1 \). Hence, we have obtained that
\[
(H + \lambda)f' = 0.
\]
Note now that since \( f' \) is in \( L^2(1/2, \infty) \), we have that \( Hf' \) is in \( \mathcal{H}_0^2 \) (see 3), and thus we actually obtain that \( f' \) is \( \mathcal{H}_0^2 \) by (22). But since \( H \) is a positive operator, applying (22) again, we find that \( f'' = 0 \).

\[\square\]

5. Failure of boundedness of \( H \) on \( \mathcal{H}_0^p \) when \( p \neq 2 \)

We follow H and define \( \mathcal{H}_0^p \) as the completion of the set of Dirichlet polynomials \( P(s) = \sum_{n=1}^{N} a_n n^{-s} \) with respect to the norm
\[
\|P\|_{\mathcal{H}_0^p} := \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T |P(it)|^p \, dt \right)^{1/p}.
\]
The space $\mathcal{H}_0^P$ is the subspace of $\mathcal{H}^P$ consisting of Dirichlet series of the form $\sum_{n=2}^{\infty} a_n n^{-s}$, which means that series in $\mathcal{H}_0^P$ vanish at $+\infty$.

**Theorem 5.** $H$ does not act boundedly on $\mathcal{H}_0^P$ for $p \neq 2$.

The proof of this theorem requires us to associate $\mathcal{H}^P$ with $H^P(\mathbb{T}^\infty)$. This means that we need to invoke the so-called Bohr lift, which we now recall (see [10, 17] for further details). For every positive integer $n$, the fundamental theorem of arithmetic allows the prime factorization

$$n = \prod_{j=1}^{\pi(n)} p_j^{\kappa_j},$$

which associates $n$ to the finite non-negative multi-index $\kappa(n) = (\kappa_1, \kappa_2, \kappa_3, \ldots)$. The Bohr lift of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the power series

$$(23) \quad \mathcal{B} f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)},$$

where $z = (z_1, z_2, z_3, \ldots)$. Hence (23) is a power series in infinitely many variables, but each term contains only a finite number of these variables.

Under the Bohr lift, $\mathcal{H}^P$ corresponds to the Hardy space $H^P(\mathbb{T}^\infty)$, which we view as a subspace of $L^P(\mathbb{T}^\infty)$. This means that $\mathcal{B}$ is an isometric map from $\mathcal{H}^P$ onto $H^P(\mathbb{T}^\infty)$. We refer to [1, 10] for the details, mentioning only that the Haar measure of the compact abelian group $\mathbb{T}$ is simply the product of the normalized Lebesgue measures for each variable. In particular, $H^P(\mathbb{D}^d)$ is a natural subspace of $H^P(\mathbb{T}^\infty)$. We denote orthogonal projection (Riesz projection) from $L^2(\mathbb{T}^\infty)$ onto $H^2(\mathbb{T}^\infty)$ by $P_+$.

We require the following lemma which is established by direct computation. Here and in what follows, $L^P$ norm with respect to normalized Lebesgue measure on $\mathbb{T}$ (or $\mathbb{T}^\infty$) is denoted by $\| \cdot \|_p$.

**Lemma 4.** Let $\lambda$ be a real parameter and suppose that $0 < \epsilon (1 + |\lambda|) < 1$, $1 \leq p < \infty$. Then

$$\| 1 + \epsilon (z + \lambda \overline{z}) \|_p^P = 1 + \frac{p}{4} [(p-1)(1+\lambda)^2 + (1-\lambda)^2] \epsilon^2 + O(\epsilon^3).$$

The norm is minimal when $\lambda = (2-p)/p$:

$$\left\| 1 + \epsilon \left( z + \frac{(2-p)}{p} \overline{z} \right) \right\|_p^P = 1 + (p-1)\epsilon^2 + O(\epsilon^3).$$

**Proof.** We write $z = e^{i\theta}$ so that we have

$$|1 + \epsilon (z + \lambda \overline{z})|^P = (1 + 2\epsilon (1+\lambda) \cos \theta + \epsilon^2 (1+\lambda)^2 \cos^2 \theta + \epsilon^2 (1-\lambda)^2 \sin^2 \theta)^{P/2}$$

$$= 1 + p\epsilon (1+\lambda) \cos \theta$$

$$+ \frac{p}{2} \epsilon^2 \left[ 1 + 2 \left( \frac{p}{2} - 1 \right) \right] (1+\lambda)^2 \cos^2 \theta + \frac{p}{2} \epsilon^2 (1-\lambda)^2 \sin^2 \theta + O(\epsilon^3).$$

Integrating, we get

$$\| 1 + \epsilon (z + \lambda \overline{z}) \|_p^P = 1 + \frac{p}{4} [(p-1)(1+\lambda)^2 + (1-\lambda)^2] \epsilon^2 + O(\epsilon^3).$$

\qed
The point of the lemma is that $p^2/4 > p - 1$ whenever $p \neq 2$, so that (one-dimensional) Riesz projection acts expansively on the vector $1 + \epsilon(z + \lambda \overline{z})$.

Proof of Theorem 5. We will choose $G$ of norm 1 in $L^q(T^\infty)$ with $1/p + 1/q = 1$. Then

$$\|Hf\|_{\mathcal{H}^p} \geq \left| \int_{1/2}^\infty f(w) B^{-1} P_+ G(w) dw \right|.$$  

More specifically, we set

$$G(z) = \prod_{j=1}^\infty \left( 1 + \frac{2}{q} p_j^{-\alpha} \left( z + \frac{(2-q)}{q} z_j \right) \right) / c_j,$$

where $\alpha > 1/2$ and

$$c_j^q = \left\| 1 + \frac{2}{q} p_j^{-\alpha} \left( z + \frac{(2-q)}{q} z_j \right) \right\|_{L^q}^q = 1 + \frac{4(q-1)}{q^2} p_j^{-2\alpha} + O(p_j^{-3\alpha})$$

by Lemma 4. It follows that

$$P_+ G(w) = \prod_{j=1}^\infty \left( 1 + (2/q) p_j^{-\alpha} - w \right) / c_j \approx (2\alpha - 1)^{4/(pq)} / (\alpha + w - 1)^{2/q}.$$  

We now choose

$$f(s) = \prod_{j=1}^\infty \frac{1 + (2/p) p_j^{-\alpha} - s}{1 + (1/p) p_j^{-2\alpha}} - 1,$$

which has $\|f\|_{\mathcal{H}^p} \approx 1$. We see that

$$\|Hf\|_{\mathcal{H}^p} \gtrsim (2\alpha - 1)^{4/(pq) + 1/p - 1}.$$  

Here the exponent is negative if $p \neq 2$ since, in this case, $pq > 4$ so that $\|Hf\|_{\mathcal{H}^p} \to \infty$ when $\alpha \to 1/2$. 

\[ \square \]

6. Symbols of the multiplicative Hilbert matrix

To place our discussion in context, we begin with some general considerations concerning Hankel forms, i.e., the bilinear forms associated with (additive or multiplicative) Hankel matrices. We recall that any function $\psi$ in $H^2(D)$ defines a Hankel form $H_\psi$ by the relation

$$H_\psi(f, g) = \langle f, g \rangle_{L^2(D)} = \langle f, g \rangle_{L^2(\mathbb{T})},$$

which makes sense at least for polynomials $f$ and $g$. Nehari’s theorem [14] says that $H_\psi$ extends to a bounded form on $H^2(D) \times H^2(D)$ if and only if $\psi = P_+ \varphi$ for a bounded function $\varphi$ in $L^\infty(T)$. Moreover, $\|H_\psi\| = \|\varphi\|_{\infty}$ if we choose $\varphi$ to have minimal $L^\infty$ norm. Since

$$\sum_{n=1}^\infty n^{-1} e^{in\theta} = 2i P_+ \arg(1 - e^{i\theta}),$$

Nehari’s theorem gives an alternate proof of the fact that the Hilbert matrix has norm $\pi$. By the Hahn–Banach theorem, Nehari’s theorem is equivalent to the statement that

$$\left| \sum_{n=0}^\infty c_n \rho_n \right| \leq \|H_\psi\| \|f\|_1,$$
where \( f(z) = \sum_n c_n z^n \) and \( \psi(z) = \sum_n \rho_n z^n \). In the case of the Hilbert matrix, we have in fact the stronger inequality

\[
\sum_{n=0}^{\infty} |c_n|(n+1)^{-1} \leq \pi \|f\|_1,
\]

which was proved by Hardy and Littlewood [8].

We turn next to what is known about multiplicative Hankel forms. Every sequence \( \rho = (\varrho_1, \varrho_2, \varrho_3, \ldots) \) in \( \ell^2 \) defines in an obvious way a multiplicative Hankel matrix, and we associate with it the corresponding multiplicative Hankel form given by

\[
\rho(a, b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varrho_{mn} a_m b_n,
\]

which initially is defined at least for finitely supported sequences \( a \) and \( b \) in \( \ell^2 \). We will now explain, using the Bohr lift, that every multiplicative Hankel form can be uniquely associated with either a Hankel form on \( H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty) \) or equivalently a (small) Hankel operator acting on \( H^2(\mathbb{D}^\infty) \).

If \( f \), \( g \), and \( \varphi \) are Dirichlet series in \( \mathcal{H}^2 \) with coefficients \( a_n \), \( b_n \), and \( \overline{\varphi_n} \), respectively, a computation shows that

\[
\langle f g, \varphi \rangle_{\mathcal{H}^2} = \rho(a, b).
\]

A formal computation gives that

\[
\langle \mathcal{B} f \mathcal{B} g, \mathcal{B} \varphi \rangle_{L^2(\mathbb{T}^\infty)} = \langle f g, \varphi \rangle_{\mathcal{H}^2},
\]

allowing us to compute the multiplicative Hankel form \(25\) on \( \mathbb{T}^\infty \). This means that we may equivalently study Hankel forms

\[
H_\Phi(FG) = \langle FG, \Phi \rangle_{L^2(\mathbb{T}^\infty)}, \quad F, G \in H^2(\mathbb{T}^\infty).
\]

In our previous considerations we required that \( \Phi \) be in \( H^2(\mathbb{T}^\infty) \), but there is nothing to prevent us from considering arbitrary symbols \( \Phi \) from \( L^2(\mathbb{T}^\infty) \). Hence, each \( \Phi \) in \( L^2(\mathbb{T}^\infty) \) induces by \(25\) a (possibly unbounded) Hankel form \( H_\Phi \) on \( H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty) \). Of course, this is not a real generalization. Each form \( H_\Phi \) is also induced by a symbol \( \Psi \) in \( H^2(\mathbb{T}^\infty) \); setting \( \Psi = P_+ \Phi \) we have \( H_\Phi = H_\Psi \).

On the polydisc, the Hankel form \( H_\Phi \) is naturally realized as a (small) Hankel operator \( H_\Phi \), which when bounded acts as an operator from \( H^2(\mathbb{T}^\infty) \) to the anti-analytic space \( \overline{H^2(\mathbb{D}^\infty)} \). Letting \( P_+ \) denote the orthogonal projection of \( L^2(\mathbb{T}^\infty) \) onto \( \overline{H^2(\mathbb{D}^\infty)} \), we have at least for polynomials \( F \) in \( H^2(\mathbb{T}^\infty) \) that

\[
H_\Phi F = \overline{P_+(\Phi F)}.
\]

We now come to the question of to which extent Nehari’s theorem remains valid in the multiplicative setting. Note first that if \( \Psi \) is in \( L^\infty(\mathbb{T}^\infty) \), then the corresponding multiplicative Hankel form is bounded, since

\[
|H_\Psi(fg)| = |\langle fg, \Psi \rangle| \leq \|f\|_2 \|g\|_2 \|\Psi\|_{\infty}.
\]

We say that \( H_\Phi \) has a bounded symbol if there exists a \( \Psi \) in \( L^\infty(\mathbb{T}^\infty) \) such that \( H_\Phi = H_\Psi \). In [11], Helson proved that every Hankel form in the Hilbert–Schmidt class \( S_2 \) has a bounded
symbol, but it was shown in [16] that there exist bounded multiplicative Hankel forms without bounded symbols. This result was strengthened in [2] which showed that there are Hankel forms in Schatten classes $S_p$ without bounded symbols whenever $p > (1 - \log \pi / \log 4)^{-1} = 5.7388...$

In the opposite direction, we have the following positive result about Hankel forms with bounded symbols.

**Theorem 6.** If $\alpha(n)$ is a completely multiplicative function, then the symbol $B \sum_{n=1}^{\infty} \alpha(n)n^{-s}$ is the Riesz projection of a bounded function on $\mathbb{T}^\infty$ if and only if $\alpha(n)$ is in $\ell^2(\mathbb{N})$.

**Proof.** This result is just a reformulation of the fact that the set of bounded point evaluations for $H^1(D^\infty)$ is $D^\infty \cap \ell^2$ [4]. Following [4, p. 122], corresponding bounded functions can be found explicitly: For every point $z = (z_j)$ on $T^\infty$, we set

$$\Phi(z) = \prod_p \frac{1 - |\alpha(p)|^2}{1 - |\alpha(p)|^2} \prod_{j=1}^{\infty} \frac{1 - \alpha(p_j)z_j}{1 - \alpha(p_j)z_j}.$$

This is a bounded function on $T^\infty$ because the latter product is the ratio between two non-vanishing functions in $L^2(T^\infty)$. One may check that $B P \Phi(s) = \sum_{n=1}^{\infty} \alpha(n)n^{-s}$ by a direct computation or by checking that $\Phi$ represents the functional of point evaluation at $(\alpha(p_j))$. $\square$

The symbol (4) of the multiplicative Hilbert matrix does not have completely multiplicative coefficients, and we know from Theorem 1 that it is not compact. This means that the preceding discussion gives no answer to the following question.

**Question.** Does the multiplicative Hilbert matrix have a bounded symbol?

Equivalently, we may ask whether we have

$$|a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n \log n}}| \lesssim \|f\|_{H^1}$$

when $f(s) = \sum a_n n^{-s}$ is in $H^1$. We could even ask if the analogue of the Hardy–Littlewood inequality (24) is valid: Does (27) hold when we put absolute values on $a_n$?

We observe that the left-hand side of (27) can be written as an integral, so that another reformulation of the question is to ask if the linear functional defined by

$$Lf = \int_{1/2}^{\infty} f(w)dw$$

extends to a bounded linear functional on $H^1$. A Carleson measure argument (see [15, Theorem 4]) shows that if there exists an absolute constant $C$ such that

$$\int_0^1 |P(1/2 + it)| dt \leq C \|P\|_{H^1}$$

holds for all Dirichlet polynomials, then we also have

$$\int_{1/2}^{\infty} |f(w)| dw \leq C' \|f\|_{H^1}$$
for another absolute constant $C'$ and all $f$ in $\mathcal{H}_0^1$. Whether (28) holds has been an open problem for a number of years; we refer to [18] for a discussion of it. We conclude that a positive solution to this embedding problem would imply a positive answer to question (1), and conversely, a negative answer to question (1) would imply a negative solution to the embedding problem (28).

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OLE FREDRIK BREVIG, DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY
E-mail address: ole.brevig@math.ntnu.no

KARL-MIAKEL PERFETT, DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY
E-mail address: karl-mikael.perfekt@math.ntnu.no

KRISTIAN SEIP, DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY
E-mail address: seip@math.ntnu.no

ARISTOMENIS G. SISKAKIS, DEPARTMENT OF MATHEMATICS, ARISTOTLE UNIVERSITY OF THESSALONIKI, 54124, THESSALONIKI, GREECE
E-mail address: siskakis@math.auth.gr

DRAOAN VUKOTIĆ, DEPARTAMENTO DE MATEMÁTICAS, MÓDULO 17, FACULTAD DE CIENCIAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN
E-mail address: dragan.vukotic@uam.es