Algebra of potentials of the volume-preserving vector fields.

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Abstract

The algebra of volume-preserving vector fields is considered. The potentials for that fields are introduced, and induced algebra of potentials is considered. It is shown, that this algebra fails to satisfy the Jacoby identity. Analogy with hamiltonian mechanics is developed, as well as 3-cocycle interpretation of corresponding expressions.
1 Introduction.

The Lie algebras of vector fields on some n-dimensional manifold M, preserving the volume form $\omega$ on that manifold, have found some applications in physics recently [1, 2, 3, 4, 5, 6, 7]. They appear for $n = 2$ (i.e. for the case of area-preserving transformations) as an algebra of hidden symmetries of $d = 1$ closed string field theory [1, 2], and the same $d = 1$ string theory, at a certain radius of compactification, possess an $n = 3$ volume-preserving diffeomorphisms algebra. The algebras of vector fields, of which the algebra of volume-preserving fields is the subalgebra, prove their importance in physics and mathematics [8] in many important cases, and deserve further study.

The main aim of this letter is to construct an algebra of potentials for a divergenceless vector fields, i.e. the fields, preserving some volume form, and to show, that this algebra unavoidably fails to satisfy the Jacoby identity, so it is not a Lie algebra. The construction is carried out first in Sect.2 for three-dimensional case, when that potentials turn out to be one-forms on that manifolds, and is generalized to an arbitrary $n$ in Sect.5. This construction has a very close analogy with the construction of the algebra of hamiltonian vector fields, when one considers the subalgebra of vector fields, which maintain the symplectic structure, i.e. some nondegenerate closed two-form. The corresponding potentials are the usual Hamiltonians, which now are the scalar functions on the manifold. That analogy is considered in Sect.3.

The other point of view on the objects, appeared in Sec.2, is that we obtained a 3-cocycle of the algebra of volume-preserving vector fields, with values in the exact one forms. That approach will be discussed in Sect.4.

Some possible developments of these ideas are described in the Conclusion: the hamiltonian mechanics generalization, the generalization of the notion of the algebra of the symmetries of the theories, etc.
2 Construction of the algebra

Let's consider a three-dimensional manifold, which we shall take as $R^3$, (this is not a restriction, since our considerations will be local), with the volume 3-form, which can be brought to the simple form $\varepsilon_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda$. The vector fields $\xi^\mu$, which maintain that form, are characterized by the property

$$\partial_\mu \xi^\mu = 0$$

which means, that they can be parameterized by the vector-potentials $A_\mu$ as

$$\xi^\mu \equiv \xi^\mu (A) = \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

This parametrization is degenerate in a sense, that many potentials correspond to the same vector fields: the gauge transformed potential

$$A_\mu + \partial_\mu \varphi$$

leads to the same vector field $\xi^\mu$.

Evidently, divergenceless vector fields form the Lie algebra, since the commutator of such a fields gives again divergenceless field - from

$$[\xi_1, \xi_2] = \xi_3$$

follows that $\xi^\mu$ satisfies the eq. (1)

$$\partial_\mu \xi^\mu = 0$$

Let’s denote through $A, B$ and $C$ the potentials for the vector fields $\xi^\mu_i = 0, i = 1, 2, 3$, so $\xi^\mu_1 = \xi^\mu (A), \xi^\mu_2 = \xi^\mu (B), \xi^\mu_3 = \xi^\mu (C)$. $C$ is defined by $A$ and $B$ only up to the gauge transformation. The calculation shows, that $C$ is given by

$$C_\mu = \varepsilon_{\mu\nu\lambda} \xi^\nu_1 \xi^\lambda_2 + \partial_\mu \varphi$$

where $\varphi$ is an arbitrary function, which cannot be obtained from the commutation relation between vector fields. So, one can try to introduce an algebra of potentials, with the following binary operation (“Poisson bracket”, see next section)

$$[A, B]_\mu = C_\mu$$

where $C$ has to be given by (1) with some concrete choice for $\varphi$. $\varphi$ may depend on $A$ and $B$, and may satisfy some additional requirements. The natural requirements, aimed to keep the algebra of potentials as close as possible to the original algebra of vector fields, are the requirement of linearity over $A, B$, and the antisymmetry under the interchange of $A$ and $B$. The possible choice, which also has the property of being invariant with respect to the independent gauge transformations of $A$ and $B$, is $\varphi = 0$. Let’s accept this definition now, the more general choice will be discussed later.

The real problem arises when one considers the Jacobi identity for this bracket. Since the algebra of potentials simulate the algebra of vector fields up to a gauge transformations, nothing guarantee now that the rhs of Jacobi identity will be exactly zero, and not the pure gauge. The straightforward calculation shows
\[ [A, [B, C]]_\sigma + (cyc.\, perm.) = \partial_\sigma \epsilon_{\mu \nu \lambda} \xi^\mu(A) \xi^\nu(B) \xi^\lambda(C) \] (8)

where \( A, B, C \) are three independent vector-potentials. We observe the appearance of the violation of the Jacoby identity, by the pure gauge terms.

The non-trivial question is whether one can remove the rhs of the Jacoby identity by the appropriate choice of \( \phi \) in the commutation relation (8). It is easy to understand, that the only possible choice for \( \phi \), which gives in the rhs of Jacoby identity terms, similar to (8), is

\[ \phi = k \epsilon^{\mu \nu \lambda} \partial_\mu (A_\nu B_\lambda) \] (9)

where \( k \) is an arbitrary constant. This choice maintains the antisymmetry and linearity, but violates gauge invariance. With this new bracket the Jacoby identity looks like

\[ [A, [B, C]]_\sigma + (cyc.\, perm.) = \partial_\sigma \epsilon_{\mu \nu \lambda} \xi^\mu(A) \xi^\nu(B) \xi^\lambda(C) + (k^2 + k) \partial_\sigma \partial_\mu [\xi^\mu(A) \xi^\lambda(B) C_\lambda + (cyc.\, perm.)] \] (10)

The last term is gauge noninvariant (changes on \((k^2+k) \partial_\sigma \partial_\mu [\xi^\mu(A) \xi^\lambda(B) \partial_\lambda \sigma + (cyc.\, perm.)]\)) under gauge transformation of (e.g.) \( C : C \rightarrow C + \partial \sigma \), hence, cannot cancel first, gauge-invariant, term.

So, we conclude, that it is impossible to restore the Jacoby identity using the freedom in definition (8), by the appropriate choice of \( \phi \). The most natural choice is \( \phi = 0 \), since it has an additional property of gauge invariance. Thus, we end up with an algebra of the vector-potentials with the binary operation

\[ [A, B]_\mu = \epsilon_{\mu \nu \lambda} \xi^\nu(A) \xi^\lambda(B) \] (11)

which is antisymmetric, linear and gauge-invariant w.r.to the gauge transformations of \( A \) and \( B \), but does not satisfy the Jacoby identity.

### 3 Hamiltonian analogy

Let’s consider the usual hamiltonian formalism. One starts from some closed two-form \( \omega = \omega_{\mu \nu} dx^\mu \wedge dx^\nu \) instead of closed three-form of previous section, and consider the vector fields, which maintain that form - the hamiltonian vector fields. This is the requirement

\[ \partial_\mu \omega_{\lambda \nu} \xi^\nu - (\mu \leftrightarrow \nu) = 0 \] (12)

instead of (3), and so \( \xi^\mu \) can be parameterized as

\[ \xi^\mu \equiv \xi^\mu(H) = \omega^{\mu \nu} \partial_\nu H \] (13)

where \( H \) is the analog of \( A \), but now is a scalar function. The analog of gauge transformations (3) is the shift of \( H \) on some arbitrary constant

\[ H \rightarrow H + const \] (14)
so, \( H \) can be recovered from (13) only up to a constant. The commutator of two hamiltonian vector fields gives another hamiltonian vector field, and denoting the corresponding Hamiltonians as \( H_1, H_2, H_3 \), we obtain

\[
H_3 = \{H_1, H_2\} + c(H_1, H_2)
\]

where the bracket in the r.h.s. is the usual Poisson bracket, and \( c \) is a constant, which depends on \( H_1, H_2 \) and may satisfy some constraints. This expression is an analog of (\( \square \)). Imposing the natural conditions of antisymmetry and linearity, one can try also to fulfill the Jacoby identity, maintaining in that way the Lie algebra nature of hamiltonian vector fields. One easily finds, that this last condition leads to the equation on \( c(H_1, H_2) \) which is an equation on two-cocycle of the algebra of hamiltonian vector fields (more extensive discussion of cocycles equations see below):

\[
c(H_1, \{H_2, H_3\}) + (\text{cycl.perm.}) = 0
\]

The trivial solution \( c = 0 \) leads to a usual Poisson-Lie algebra structure on the space of Hamiltonians, solutions with \( c \neq 0 \) give a central extensions of that algebra.

4 The cocycle interpretation.

The expressions of preceding Section have an interpretation on the language of the cocycles of the Lie algebra of the volume-preserving vector fields. The notion of cocycles of the (gauge) Lie groups, was used intensively in the study of anomalies [9], where 2-cocycles appeared, and also possible appearance and physical meaning of 3-cocycles were discussed [10, 11]. In this section we shall interpret the rhs of Jacoby identity of Sect. 2 as a 3-cocycle of the group of a volume-preserving diffeomorphisms.

The cochains of the Lie algebra \( X \) are an antisymmetric polylinear functionals on that algebra with values in some \( X \)-modules. The coboundary operation on the space of cochains is given by the equation

\[
\delta c_n(x_1, \ldots, x_{n+1}) = \sum_{k<l} (-1)^{k+l} c_n([x_k, x_l], x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_l, \ldots, x_n) + \\
\sum_{k=1}^{n+1} x_k (-1)^k c_n(x_1, \ldots, \hat{x}_k, \ldots, x_{n+1})
\]

where hat on the argument means that it is absent.

For application to our case, we choose as basic Lie algebra the algebra of volume-preserving vector fields, and a module is a space of exact one-forms with trivial action of vector fields. Due to the construction (8), rhs of that equation gives the closed 3-cochain \( c_3(\xi_1, \xi_2, \xi_3) = \partial_\sigma \epsilon_{\mu\nu\lambda}(\xi_1^\mu, \xi_2^\nu, \xi_3^\lambda) \):

\[
\delta c_3(x_1, \ldots, x_4) = 0
\]

The nontriviality of that cocycle (i.e. whether it can be represented as the coboundary of some 2-cochain) requires more rigorous approach and will be discussed elsewhere.
5 Higher-dimensional generalization.

Higher-dimensional generalization of the algebra of potentials of volume-preserving transformations of three-dimensional space, introduced in Sect.2, is straightforward. The same constraint (1) on volume-preserving vector fields now has a solution parameterized in \( d \)-dimensional space by \( (d - 2) \)-th rank tensors \( A_{\mu_1 \ldots \mu_{d-2}} \):

\[
\xi^\mu \equiv \xi^\mu(A) = \epsilon^{\mu\mu_1 \ldots \mu_{d-1}} \partial_{\mu_1} A_{\mu_2 \ldots \mu_{d-1}}
\]

Induced algebra structure on the space of potentials is given by the evident generalization of (7):

\[
[A, B]_{\mu_1 \ldots \mu_{d-2}} = C_{\mu_1 \ldots \mu_{d-2}}
\]

where \((d-2)\)-form \( C \) is given by

\[
C_{\mu_1 \ldots \mu_{d-2}} = \epsilon_{\mu_1 \ldots \mu_d} \xi_{\mu_{d-1}} + \partial_{[\mu_1} \phi_{\mu_2 \ldots \mu_{d-2}]}\]

The main statement of Sect.2 of impossibility of fulfillness the Jacoby identity remains unchanged also in this case. Namely, the Jacoby identity now looks like

\[
[A, [B, C]]_{\mu_1 \ldots \mu_{d-2}} + (\text{cycl.perm.}) = \partial_{[\mu_1} \epsilon_{\mu_2 \ldots \mu_{d-2}]} \xi_{\mu_{d-1}} (A) \xi^\lambda (B) \xi^\sigma (C)
\]

Again, one cannot remove the rhs of Jacoby identity by changing the definition of bracket, using an arbitrariness in definition of \( C \) in (21).

6 Conclusion.

In previous sections we introduce an unconstrained potentials for a volume-preserving vector fields, and, in complete analogy with construction of Poisson bracket, construct a similar bracket on the space of potentials. The main difference is in that new bracket doesn’t satisfy the Jacoby identity. Construction is carried out in all dimensions.

The other interpretation of corresponding expressions is as a 3-cocycles of the Lie algebra of volume-preserving vector fields. This construction can be viewed as a tool for obtaining the cocycles of the algebra of volume-preserving vector fields, and one may try to generalize that to other subgroups of algebra of all vector fields.

The interesting direction of development follows from the analogy, discussed in Sect.3. Namely, it is possible to consider the “equations of motion” of one-forms, in the form, analogous to usual hamiltonian mechanics

\[
\frac{d}{dt} A = \{H, A\}
\]

where \( H \) is a given ”Hamiltonian” one-form, and ”the motion” means, that an arbitrary one-forms \( A \) change in time according to the equation (23). In usual hamiltonian mechanics the same equation may be interpreted as a change of function according to the
change of its arguments, so the whole mechanics consists from motion of point in the phase space. In our case such an interpretation is not straightforward, but nevertheless, more general point of view is possible, and one can consider different motions in the space of one-forms, induced by different Hamiltonian one-forms $H$. It is interesting to find and classify finite-dimensional orbits of such a motions. Actually, according to [12] in the three-dimensional case the present construction, in the special case of the 1-form $H$, given by the expression $H = F dG$, gives the Namby’s odd-dimensional mechanics [13]. Finally, we would like to mention another question, arising in connection with the algebras with Jacoby identity, violated by pure gauge terms. Since the observables in gauge theories are gauge invariant, one can try to study theories with non-Lie-algebraic type of symmetries, the deviation from the Lie algebras being purely gauge, and unessential on the level of observables. Another direction of investigation is the consideration of coadjoint action of the algebra of potentials.

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