Finsleroids Reflect Future-Past Asymmetry of Space-Time

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Abstract

The Finslerian extension of the Euclidean metric is proposed and studied under rigorous conditions that the associated indicatrix is regular and convex. The relativistic pseudo-Euclidean metric is extended, too. The extensions show distinct violation of the $T$–parity, so that the future-past asymmetry of the physical world can stem from the $T$–asymmetry of the Finslerian indicatrix. Indeed the idea should compel much attention of physicists.
1. Introduction

Let us pose the question:

Is there a handy possibility to continue the Euclidean metric into the Finslerian domain such that the Euclidean sphere, \( S \), will go over into a regular and convex closed surface, an indicatrix \( \mathcal{I} \)?

On analyzing the query

\[
S \overset{\text{Finsler way}}{\leftrightarrow} \mathcal{I} \quad ?
\]  

we should note, first of all, that the surface \( \mathcal{I} \) cannot be “spherical-symmetric”, for the choice of a sphere for the indicatrix will immediately lead back to the precise-Euclidean metric.

**Whence the indicatrix \( \mathcal{I} \) should be asymmetric at least in a single direction.** In this respect, the simplest assumption is that indicatrix is a (hyper)surface of revolution around a preferred direction, to be called conventionally the \( T \)-direction.

In what follows, we would like to propose and study the corresponding Finslerian metric function obtained under a convenient condition that the indicatrix is a space of constant positive curvature. We call the respective indicatrices \( \mathcal{I} \) the Finsleroids.

For the spaces under study, the indicatrix equation is essentially non-linear, so that the generatrix equation

\[
T_{\text{Finslerian}} = T(g; |\mathbf{R}|),
\]

where \( g \) is the characteristic Finslerian parameter, is defined only implicitly. This is in contrast to the ordinary root dependence

\[
T_{\text{Euclidean}} = \sqrt{1 - |\mathbf{R}|^2}
\]  

(1.3)

or

\[
T_{\text{pseudo-Euclidean}} = \sqrt{1 + |\mathbf{R}|^2}.
\]

(1.4)

It occurs, however, that differentiations of the indicatrix equations yield sufficiently transparent equations which provide a convenient basis to study any peculiar feature of any given Finsleroid. We shall follow this method.

For fundamentals of Finsler Geometry, the reader is referred to [1-7]. In various respects, the present work continues our previous papers [8-12].

The organization of the paper is as follows.

On introducing the implied initial definitions in Section 2, we devote Section 3 to deriving the relations which are sufficient to prove the regularity and convexity of Finsleroids. In Section 4, we expose explicitly the relevant diffeomorphic spherical map

\[
S \overset{\tau}{\leftrightarrow} \mathcal{I}
\]

(1.5)

which justifies the above query (1.1) in positive. A direct way of finding the associated Hamiltonian function in an explicit way is proposed in Section 5, which enables us in Section 6 to understand that the corresponding Co-Finsleroid (the figuratrix) is \( gT \)-conjugate to the Finsleroid (to the indicatrix). After that, in Section 7, we formulate the relativistic counterpart of our theory, in which case \( T \) has the meaning of a physical time proper. In the last Section 8 we sum up some straightforward ways of physical applications of the extensions obtained.
2. Initial definitions

Suppose we are given an \((N - 1)\)-dimensional Euclidean space \(E_{N-1}\) with local Euclidean coordinates \(\{R^a\}\) and with the Euclidean metric tensor \(e = \{e_{ab} = \delta_{ab}\}\), where \(\delta\) stands for the Kronecker symbol. Let us form an \(N\)-dimensional topological-Euclidean space \(E\) to be the product

\[
E = E_{N-1} \times \mathbb{R},
\]

where \(\mathbb{R}\) is the real line, and use a canonical coordinate \(T\) in \(\mathbb{R}\) to decompose the vectors \(R \in E\):

\[
R = \{R^a\} = \{R^0 = T, R^a\}.
\]

The indices \((a, b, \ldots)\) and \((p, q, \ldots)\) will be specified over the ranges \((1, \ldots, N-1)\) and \((1, \ldots, N)\), respectively. In the vector notation, we have

\[
R = \{R^a\} = \{R^1, \ldots, R^{N-1}\}, \quad |R| = \sqrt{\delta_{ab}R^aR^b},
\]

and

\[
R = \{T, R\}.
\]

We put

\[
w^a = R^a/T, \quad w = |R|/T, \quad w_a = w^a.
\]

The variable \(w\) is compatible with the whole definition domain

\[
w \in (-\infty, \infty).
\]

Throughout this paper, vector indices are up, co-vector indices are down, and repeated up–down indices are automatically summed; \(N = 4\) in the proper space-time context.

Given a parameter \(g\) ranged over

\[
-2 < g < 2,
\]

let us introduce the convenient notation

\[
h = \sqrt{1 - \frac{1}{4}g^2}, \quad r = \frac{1}{h},
\]

\[
G = g/h,
\]

together with the characteristic quadratic form

\[
B(g; R) = |R|^2 + g|R||T + T^2
\]

whose discriminant is

\[
D = -4h^2 < 0.
\]

In terms of this notation, we propose the Finslerian metric function:

\[
K(g; R) = \sqrt{B(g; R)} \cdot j(g; R),
\]

where

\[
j(g; R) = \exp\left(\frac{1}{2}G\left(\frac{\pi}{2} - \arctg\frac{2|R| + gT}{2hT}\right)\right), \quad \text{if} \quad T \geq 0.
\]
\[ j(g; R) = \exp \left( -\frac{1}{2} G \left( \frac{\pi}{2} + \arctg \frac{2|\mathbf{R}| + gT}{2kT} \right) \right), \quad \text{if } T \leq 0, \]  
(2.14)

and

\[ j(g; R) = 1, \quad \text{if } T = 0. \]  
(2.15)

Under these conditions, we call the Minkowskian space \{\mathcal{E}, K\} the \( \mathcal{E}_{PD} \)-space:

\[ \mathcal{E}_{PD} = \{ \mathcal{E} = E_{N-1} \times \mathbb{R}; K(g; T, \mathbf{R}); g\}. \]  
(2.16)

We have

\[ K(g; -T, \mathbf{R}) \neq K(g; T, \mathbf{R}), \quad \text{unless } g = 0. \]  
(2.17)

Instead, the function \( K \) shows the property of \( gT \)-parity

\[ K(-g; -T, \mathbf{R}) = K(g; T, \mathbf{R}) \]  
(2.18)

and the property of \( \mathcal{P} \)-parity

\[ K(g; T, -\mathbf{R}) = K(g; T, \mathbf{R}). \]  
(2.19)

It is frequently convenient to rewrite the representation (2.12) in the form

\[ K(g; R) = |T|V(g; w) \]  
(2.20)

with the generating function

\[ V(g; w) = \sqrt{Q(g; w)} j(g; w), \]  
(2.21)

where \( Q(g; w) \) abbreviates \( B(g; R)/T^2 \), so that

\[ Q(g; w) = 1 + gw + w^2. \]  
(2.22)

We directly obtain

\[ V' = wV/Q, \quad V'' = V/Q^2, \quad j' = -\frac{1}{2} gj/Q, \]  
(2.23)

where the prime (\('\)) denotes the differentiation with respect to \( w \). Using Eqs. (2.19)–(2.23) in the Finslerian rule \( R_p = \frac{1}{2} \partial K^2 / \partial R^p \) yields

\[ R_a = w_a V K/Q, \quad R_0 = (1 + gw)V K/Q. \]  
(2.24)

3. Shape of Finsleroid

The metric function (2.12) defines an \((N - 1)\)-dimensional indicatrix hypersurface according to the equation

\[ K(g; T, |\mathbf{R}|) = 1. \]  
(3.1)

We call this particular hypersurface the Finsleroid, to be denoted as \( \mathcal{F}_{PD}^g \).

From (2.12)-(2.14) it follows directly that

\[ |\mathbf{R}|_{T=0} = 1. \]  
(3.2)
Also
\[ T|R|_0 = T_1(g), \quad \text{when} \quad T < 0; \quad T|R|_0 = T_2(g), \quad \text{when} \quad T > 0, \quad (3.3) \]
where
\[ T_1(g) = -e^{G\pi/4} \exp\left[ \frac{G}{2} \arctg \frac{G}{2} \right] \quad \text{and} \quad T_2(g) = e^{-G\pi/4} \exp\left[ \frac{G}{2} \arctg \frac{G}{2} \right]. \quad (3.4) \]
The equation (3.1) cannot be resolved for the function
\[ T = T(g; |R|) \quad (3.5) \]
in an explicit form, because of a complexity of the right-hand part of Eq. (2.12). Nevertheless, differentiating the identity
\[ K(g; T(|R|), |R|) = 1. \quad (3.6) \]
yields the simple result
\[ \frac{dT}{d|R|} = -\frac{|R|}{T + g|R|} \quad (3.7) \]
which just entails
\[ \frac{d^2T}{d|R|^2} = -\frac{B(g; R)}{(T + g|R|)^3} < 0. \quad (3.8) \]
We also get
\[ \left| \frac{dT}{d|R|} \right|_{|R|_0} = 0, \quad \frac{dT}{d|R|} \xrightarrow{T \to +0} -\frac{1}{g}. \quad (3.9) \]
Inversely, for the function
\[ |R| = |R|(T) \quad (3.10) \]
we obtain
\[ \frac{d|R|}{dT} = -\frac{T + g|R|}{|R|} \quad (3.11) \]
and
\[ \frac{d^2|R|}{dT^2} = -\frac{B(g; R)}{|R|^3} < 0. \quad (3.12) \]
We have
\[ \frac{d|R|}{dT} > 0, \quad \text{if} \quad T < -g|R|; \quad \text{and} \quad \frac{d|R|}{dT} < 0, \quad \text{if} \quad T > -g|R|. \quad (3.13) \]
Also,
\[ \frac{d|R|}{dT} = 0, \quad \text{if} \quad T = T^* \quad \text{with} \quad T^* = -g|R|. \quad (3.14) \]
Inserting this \( T^* \) in (3.1) yields
\[ T^* = f(g) \quad (3.15) \]
with
\[ f(g) = -g \exp \left( \frac{G}{2} \left( \frac{\pi}{2} - \arctg \frac{2 - g^2}{2gh} \right) \right). \quad (3.16) \]
and for the function
\[ k(g) = |R|(T^*) \] (3.17)
we obtain merely
\[ k(g) = \exp \left( \frac{G}{2} \frac{\pi^2}{2} - \arctg \frac{2 - g^2}{2gh} \right). \] (3.18)

The above formulae, particularly the negative sign of the second derivative (3.8), are useful to apply when verifying the following

**THEOREM 1.** The Finsleroid \( \mathcal{F}_{PD}^g \) is closed, regular, locally-convex and convex.

4. **Spherical map of Finsleroid**

Let us perform in the space \( \mathcal{E}_{PD} \) the nonlinear transformation given by the functions
\[ \tilde{R}^a = \tau^a(g; R) \] (4.1)
with
\[ \tau^0 = (R^0 + \frac{1}{2}g|R|)j(g; R^0, |R|)r(g), \quad \tau^a = R^a j(g; R^0, |R|), \] (4.2)
and call the result the \( \tau \)-transformation. Inserting these functions in an Euclidean metric function
\[ S(\tilde{R}) \overset{\text{def}}{=} \sqrt{(\tilde{R}^0)^2 + |\tilde{R}|^2} \] (4.3)
yields the remarkable identity
\[ K(g; R) = S(\tilde{R})/r(g), \] (4.4)
where \( K(g; R) \) is exactly the Finslerian metric function (2.12); \( h(g) \) and \( r(g) \) are the functions that were defined in Eq. (2.8). Therefore we have

**THEOREM 2.** The \( \tau \)-transformation turns over the Finsleroid \( \mathcal{F}_{PD}^g \) into the sphere \( S_{r(g)} \) of radius \( r(g) \).

An attentive consideration shows that the functions written out in Eq. (4.2) are smooth of at least class \( C^2 \) over all the definition range

\[ (R^0)^2 + |R|^2 > 0. \] (4.5)

Whence the \( \tau \)-transformation of the Finsleroid \( \mathcal{F}_{PD}^g \) into the sphere \( S_{r(g)} \) is a diffeomorphism.

By the help of (4.1) and (4.2) we find
\[ w = |\tilde{R}|/\tilde{I}, \] (4.6)
where
\[ \tilde{I} = \tilde{I}(g; \tilde{R}) = h(g)\tilde{R}^0 - \frac{1}{2}g|\tilde{R}|. \] (4.7)

Additional direct calculations lead to the relation
\[ Q(g; w) = \frac{(\tilde{R}^0)^2 + |\tilde{R}|^2}{r^2(g)I^2(g; \tilde{R})} \] (4.8)
which, when used together with the redefinition

\[ \tilde{j}(g; \tilde{R}) \overset{\text{def}}{=} j(g; R(g; \tilde{R})) = j\left(g; \frac{\tilde{R}}{\tilde{I}}\right), \] (4.9)

enables us to inverse the transformations (4.1)–(4.3):

\[ R^p = \lambda^p(g; \tilde{R}), \] (4.10)

where

\[ \lambda^0 = \tilde{I}(g; \tilde{R})/\tilde{j}(g; \tilde{R}), \quad \lambda^a = \tilde{R}^a/\tilde{j}(g; \tilde{R}). \] (4.11)

The functions (4.2) are homogeneous of degree 1 with respect to \( R \):

\[ \tau^q(g; b\tilde{R}) = b\tau^q(g; R), \quad b > 0, \] (4.12)

from which it follows that the identity

\[ \tau^q_p(g; R)R^p = \tilde{R}^q \] (4.13)

holds for the derivatives

\[ \tau^q_p(g; R) \overset{\text{def}}{=} \frac{\partial \tau^q(g; R)}{\partial R^p}. \] (4.14)

The simple representations

\[ \tau^0_0 = \left(1 - \frac{1}{2}gwEQ^{-1}\right)j/h, \] (4.15)

\[ \tau^0_a = \frac{1}{2}g(w_a/w)(E - Q)Q^{-1}j/h, \] (4.16)

\[ \tau^0_0 = -\frac{1}{2}gwQ^{-1}j, \quad \tau^a_0 = j\delta^a_0 + \frac{1}{2}g(w^aw_b/w)Q^{-1}j \] (4.17)

are obtained, where

\[ E = 1 + \frac{1}{2}gw. \] (4.18)

The determinant is equal to

\[ \det(\tau^q_p) = rj^N > 0. \] (4.19)

The relations

\[ \tau^a_bw^b = jw^a(E - w^2)Q^{-1}, \quad \tau^a_c\tau^b_d\tau^c_d = j^2\left[\delta^{ab} + g(w^aw^bwQ) + \frac{1}{4}g^2(w^aw^b/Q^2)\right] \]

are convenient to take into account in process of calculations involving coefficients \( \{\tau^q_p\} \).

Similarly to (4.12), we get

\[ \lambda^p(g; b\tilde{R}) = b\lambda^p(g; \tilde{R}), \quad b > 0, \] (4.20)

which entails the identity

\[ \lambda^p_q(g; \tilde{R})\tilde{R}^q = R^p \] (4.21)

for the derivatives

\[ \lambda^p_q(g; \tilde{R}) \overset{\text{def}}{=} \frac{\partial R^p(g; \tilde{R})}{\partial \tilde{R}^q}. \] (4.22)
Let us find the transform

\[ n^{pq}(g; \tilde{R}) \stackrel{\text{def}}{=} \tau^p_r(g; R) \tau^q_s(g; R) g^{rs}(g; R), \tag{4.23} \]

of the Finslerian metric tensor \( g^{rs}(g; R) \) (associated to the Finslerian metric function (2.12)) under the \( \tau \)-transformation. By the help of Eqs. (4.12)-(4.18), we get after rather lengthy calculations the following simple result:

\[ n^{pq} = \delta^{pq} + \frac{1}{4} G^2 l^p l^q, \quad n_{pq} = \delta_{pq} - \frac{1}{4} g^2 l_p l_q \tag{4.24} \]

\((n_{pr} n_{qr} = \delta_{pq}; G \text{ is given by Eq. (2.9))}, \) and

\[ \det(n_{pq}) = h^2, \tag{4.25} \]

where

\[ l^p = \tilde{R}^p / S(\tilde{R}) = l_p \tag{4.26} \]

are Euclidean unit vectors, obeying the rules

\[ l^p l_q = 1, \quad n_{pq} l^q = h^2 l_p, \quad n^{pq} l_q = l^p / h^2, \quad n_{pq} l^p l^q = h^2. \]

Inversing (4.23) reads

\[ g_{pq}(g; R) = n_{rs}(g; \tilde{R}) \tau^r_p(g; R) \tau^s_q(g; R). \tag{4.27} \]

We call the tensor \( \{n\} \) with components (4.24) the quasi-Euclidean metric tensor.

Thus we have proven

THEOREM 3. The \( \tau \)-transformations turns over the Finslerian metric tensor of the space \( \mathcal{E}_{PD} \) under study in the quasi-Euclidean metric tensor in accordance with (4.23) or (4.27).

The \( \tau \)-transformation defines obviously the \( \tau \)-map

\[ \mathcal{F}^P_D \xleftarrow{\tau} S_r(g) \tag{4.28} \]

which is a diffeomorphism. Vice versa, because the \( \tau \)-transformations are homogeneous, according to Eqs. (4.12) and (4.20), the knowledge of the \( \tau \)-map can be applied to restore totally the \( \tau \)-transformation.

5. Associated Hamiltonian function

To go over from the space (2.16) to its dual counterpart, \( \hat{\mathcal{E}} \), we ought to introduce the co-versions of Eqs. (2.2)-(2.6): \( \hat{R} \in \hat{\mathcal{E}} \) and

\[ \hat{R} = \{ R_p \} = \{ R_0 = \hat{T}, R_a \}, \tag{5.1} \]

\[ \hat{\mathbf{R}} = \{ R_1, \ldots, R_{N-1} \}, \quad |\hat{\mathbf{R}}| = \sqrt{\delta^{ab} R_a R_b}, \tag{5.2} \]

\[ \hat{R} = \{ \hat{T}, \hat{\mathbf{R}} \}, \tag{5.3} \]
\[ p_a = R_a / \hat{T}, \quad p = |\hat{R}| / \hat{T}, \quad p^a = p_a, \quad (5.4) \]

where \( p \in (-\infty, \infty) \), and consider the quadratic form

\[ \hat{B}(g; \hat{R}) = |\hat{R}|^2 - g \hat{T} |\hat{R}| + (\hat{T})^2 \quad (5.5) \]

conjugated to the basic form (2.10).

To find the Hamiltonian function \( H \) associated to the Finslerian metric function (2.12), we should resolve the equation set (2.24) with respect to the variables \( \{ R^a \} \) to construct

\[ H(g; \hat{R}) \overset{\text{def}}{=} K(g; R) \quad (5.6) \]

(see the respective general homogeneous Hamilton-Jacobi theory in \[13-14\]). This procedure yields

\[ H(g; \hat{R}) = \sqrt{\hat{B}(g; \hat{R}) \hat{j}(g; p)} \quad (5.7) \]

where

\[ \hat{j}(g; p) = \exp \left( \frac{1}{2} G(-\frac{\pi}{2} + \arctg \frac{2|\hat{R}| - g \hat{T}}{2h \hat{T}}) \right), \quad \text{if} \quad \hat{T} \geq 0, \quad (5.8) \]

\[ \hat{j}(g; p) = \exp \left( \frac{1}{2} G\left(\frac{\pi}{2} + \arctg \frac{2|\hat{R}| - g \hat{T}}{2h \hat{T}}\right) \right), \quad \text{if} \quad \hat{T} \leq 0. \quad (5.9) \]

and

\[ \hat{j}(g; p) = 1, \quad \text{if} \quad \hat{T} = 0. \quad (5.10) \]

We observe that

\[ H(g; -\hat{T}, \hat{R}) \neq H(g; \hat{T}, \hat{R}), \quad \text{unless} \quad g = 0. \quad (5.11) \]

At the same time, quite similarly to the properties (2.18) and (2.19), the function \( H \) shows the property of \( g \hat{T} \)-parity

\[ H(-g; -\hat{T}, \hat{R}) = H(g; \hat{T}, \hat{R}) \quad (5.12) \]

and the property of \( \hat{P} \)-parity

\[ H(g; \hat{T}, -\hat{R}) = H(g; \hat{T}, \hat{R}). \quad (5.13) \]

In an alternative way, we write

\[ H(g; \hat{R}) = |\hat{T}| W(g; p) \quad (5.14) \]

with

\[ W(g; p) = \sqrt{\hat{Q}(g; p) \hat{j}(g; p)}, \quad (5.15) \]

where

\[ \hat{Q}(g; p) = 1 - gp + p^2 \equiv \hat{B}(g; \hat{R}) / (\hat{T})^2. \quad (5.16) \]

Similarly to (2.23), we obtain

\[ W' = pW/\hat{Q}, \quad W'' = W/(\hat{Q})^2, \quad \hat{j}' = \frac{1}{2} g \hat{j}/\hat{Q}, \quad (5.17) \]
which entails for the components of the contravariant vector $R^a = \frac{1}{2} \partial H^2 / \partial R^p$, the following result:

$$R^a = p^a W H / \hat{Q}, \quad R^0 = (1 - gp) W H / \hat{Q}. \quad (5.18)$$

The identities

$$j(g; p) = 1/j(g; w), \quad (5.19)$$

$$Q(g; w) = \frac{\hat{Q}(g; p)}{(1 - gp)^2}, \quad (5.20)$$

and

$$V^2 W^2 = Q \hat{Q} \quad (5.21)$$

hold fine.

To verify that the representations (4.18) solve the set of equations (2.24), it is easy to note that Eqs. (5.4) and (2.24) entail the equality

$$p_a = w_a / (1 + gw), \quad (5.22)$$

which inverse is

$$w_a = p_a / (1 - gp). \quad (5.23)$$

In this way we find

$$p = \frac{w}{1 + gw}, \quad w = \frac{p}{1 - gp}, \quad 1 + gw = \frac{1}{1 - gp}. \quad (5.24)$$

When the last relations are used in the definition (2.21) for the function $Q$, the identities (5.19) and (5.20) are obtained, whereupon we take into account the second part of Eq. (2.24) to obtain the equality

$$\hat{T} = (1 + gw) K^2 / QT, \quad (5.25)$$

which entails

$$T = \frac{1 + gw}{Q(g; w)} \frac{K^2}{\hat{T}} = \frac{1 - gp}{\hat{Q}(g; p)} \frac{K^2}{\hat{T}}. \quad (5.26)$$

The conclusive step is to insert (5.26) in the right-hand part of (2.12).

6. Shape of Co-Finsleroid

The Hamiltonian function (5.7) gives rise an $(N - 1)$-dimensional figuratrix to be the hypersurface defined by the equation

$$H(g; \hat{T}, |\hat{\mathbf{R}}|) = 1. \quad (6.1)$$

We call this particular hypersurface the Co-Finsleroid, to be denoted as $\hat{F}^{PD}$. Evaluating the functions (5.7)-(5.9) at $\hat{T} = 0$ yields

$$|\hat{\mathbf{R}}|_{T=0} = 1. \quad (6.2)$$

Also

$$\hat{T} \big|_{|\mathbf{R}|=0} = -T_1(g), \quad \text{when} \ T > 0; \quad \hat{T} \big|_{|\mathbf{R}|=0} = -T_2(g), \quad \text{when} \ T < 0, \quad (6.3)$$
where $T_1(g)$ and $T_2(g)$ are exactly the functions given by Eq. (3.4).

Differentiating the identity

$$H(g; \hat{T}(|\hat{R}|), |\hat{R}|) = 1$$

leads to the simple result

$$\frac{d\hat{T}}{d|\hat{R}|} = -\frac{|\hat{R}|}{\hat{T} - g|\hat{R}|},$$

from which it follows that

$$\frac{d^2\hat{T}}{d|\hat{R}|^2} = -\frac{\hat{B}(g; R)}{(\hat{T} - g|\hat{R}|)^3} < 0.$$

We also get

$$\frac{d\hat{T}}{d|\hat{R}|}\bigg|_{|\hat{R}|=0} = 0, \quad \frac{dT}{d|\hat{R}|} \implies \frac{1}{\hat{T} \rightarrow 0} g.$$

Inversely, we obtain

$$\frac{d|\hat{R}|}{d\hat{T}} = -\frac{\hat{T} - g|\hat{R}|}{|\hat{R}|}$$

and

$$\frac{d^2|\hat{R}|}{d\hat{T}^2} = -\frac{\hat{B}(g; R)}{|\hat{R}|^3} < 0.$$

We have

$$\frac{d|\hat{R}|}{d\hat{T}} > 0, \quad \text{if} \quad \hat{T} < g|\hat{R}|; \quad \text{and} \quad \frac{d|\hat{R}|}{d\hat{T}} < 0, \quad \text{if} \quad \hat{T} > g|\hat{R}|. \quad (6.10)$$

Also,

$$\frac{d|\hat{R}|}{d\hat{T}} = 0, \quad \text{if} \quad \hat{T} = \hat{T}^* \quad \text{with} \quad \hat{T}^* = g|\hat{R}|. \quad (6.11)$$

Inserting this $\hat{T}^*$ in (6.1) yields

$$\hat{T}^* = \hat{f}(g) \quad (6.12)$$

with

$$\hat{f}(g) = g \exp\left(\frac{G}{2} (\pi + \arctg \frac{2 - g^2}{2gh})\right) \equiv f(-g) \quad (6.13)$$

(cf. Eq. (3.17)).

The above calculations enable us to obtain

**THEOREM 4.** The Co-Finsleroid $\mathfrak{F}^{PD}_g$ is closed, regular, locally-convex and convex. The $\mathfrak{F}^g_{PD}$-hyperboloid and the $\mathfrak{F}^{SR}_g$-co-hyperboloid mirror one another under the $g$-reflection:

$$\mathfrak{F}^{PD}_g \leftrightarrow g \mathfrak{F}^{PD}_{-g}. \quad (6.14)$$
7. Shape of $\mathcal{F}_{g}^{SR}$-hyperboloid

The special-relativistic Finslerian metric function

$$F_{SR}(g; R) = |T + g_-|R||^{G_+/2} |T + g_+|R||^{-G_-/2}$$  \hspace{1cm} (7.1)

can be adduced by the Hamiltonian function

$$H_{SR}(g; \dot{R}) = \left| \dot{T} - \frac{|\dot{R}|}{g^+} \right|^{G_+/2} \left| \dot{T} - \frac{|\dot{R}|}{g^-} \right|^{-G_-/2}$$  \hspace{1cm} (7.2)

(see [10-12]). In this case we ought to replace the definition (2.8) by

$$h \overset{\text{def}}{=} \sqrt{1 + \frac{1}{4}g^2}$$  \hspace{1cm} (7.3)

and use the notation

$$g_+ = -\frac{1}{2}g + h, \quad g_- = -\frac{1}{2}g - h,$$  \hspace{1cm} (7.4)

$$g^+ = 1/g_+ = -g_-, \quad g^- = 1/g_- = -g_+,$$  \hspace{1cm} (7.5)

$$g^+ = \frac{1}{2}g + h, \quad g^- = \frac{1}{2}g - h,$$  \hspace{1cm} (7.6)

$$G_+ = g_+/h, \quad G_- = g_-/h,$$  \hspace{1cm} (7.7)

$$G^+ = g^+/h, \quad G^- = g^-/h.$$  \hspace{1cm} (7.8)

The associated indicatrix equation

$$F_{SR}(g; T, |R|) = 1$$  \hspace{1cm} (7.9)

defines what we call the $\mathcal{F}_{g}^{SR}$-hyperboloid. From (7.1) we get

$$|R|_{T=0} = c(g),$$  \hspace{1cm} (7.10)

where

$$c(g) = (-g_-)^{-G_+/2}(g_+)^{G_-/2},$$  \hspace{1cm} (7.11)

and also

$$T|_{|R|=0} = 1, \quad \text{when} \quad T > 0; \quad T|_{|R|=0} = -1, \quad \text{when} \quad T < 0.$$  \hspace{1cm} (7.12)

Differentiating Eq. (7.9) yields the simple result

$$\frac{dT}{d|R|} = \frac{|R|}{T - g|R|}$$  \hspace{1cm} (7.13)
which entails
\[ \frac{d^2 T}{d|\mathbf{R}|^2} = \frac{B(g; R)}{(T - g|\mathbf{R}|)^3} > 0. \]  
(7.14)

We observe that
\[ \frac{dT}{d|\mathbf{R}|}|_{|\mathbf{R}|=0} = 0, \quad \frac{dT}{d|\mathbf{R}|} \xrightarrow{T \to 0} -\frac{1}{g}. \]  
(7.15)

Inversely,
\[ \frac{d|\mathbf{R}|}{dT} = \frac{T - g|\mathbf{R}|}{|\mathbf{R}|} \]  
(7.16)
leads to
\[ \frac{d^2|\mathbf{R}|}{dT^2} = \frac{B(g; R)}{|\mathbf{R}|^3} > 0. \]  
(7.17)

We have
\[ \frac{d|\mathbf{R}|}{dT} > 0, \text{ if } T > g|\mathbf{R}|; \quad \text{and } \frac{d|\mathbf{R}|}{dT} < 0, \text{ if } T < g|\mathbf{R}|. \]  
(7.18)

Also,
\[ \frac{d|\mathbf{R}|}{dT} = 0, \text{ if } T = T^* \text{ with } T^* = g|\mathbf{R}|. \]  
(7.19)

Inserting this \( T^* \) in (7.1) yields
\[ T^* = s(g) \]  
(7.20)
with
\[ s(g) = gz(g), \]  
(7.21)
where
\[ z(g) = (g_+)^{-G+/2}(-g_-)^{-G-}/2, \]  
(7.22)
and also
\[ R^* \overset{\text{def}}{=} |\mathbf{R}|(T^*) = z(g). \]  
(7.23)

Quite similar formulae are obtainable for the \( \hat{F}^{\text{SR}-\text{co-hyperboloid}} \) defined by the figuratrix equation
\[ H_{\text{SR}}(g; \hat{R}) = 1. \]  
(7.24)

From Eqs. (7.1)-(7.22) we derive the following counterpart of the \{PD\}-Theorems 1 and 2:

THEOREM 5. The \( F^{\text{SR}}_g \)-hyperboloid is everywhere regular and locally-convex. The same conclusion is applicable to the \( \hat{F}^{\text{SR}}_g \)-co-hyperboloid. These two hypersurfaces, the \( F^{\text{SR}}_g \)-hyperboloid and the \( \hat{F}^{\text{SR}}_g \)-co-hyperboloid, mirror one another under the \( g \)-reflection:
\[ F^{\text{SR}}_g \leftrightarrow g \hat{F}^{\text{SR}}_g. \]  
(7.25)

8. Conclusion: New Ways for Finslerian Physics?

There exists rather huge literature (see [15]) about possible violation of the Special Theory of Relativity (STR). In several instances, the authors proposed sensitive ways to test experimentally how well Lorentz invariance is obeyed in Nature (see [16-18]).
Nevertheless, one cannot say that the attempts made were conclusive. In fact, much more information that were obtained is needed actually to put reliable limits on Lorentz noninvariance.

To consider departures from the STR, the standard practice was to modify the Lorentz transformations while leaving the pseudo-Euclidean metric intact. However, we know (and teach students!) that the Lorentz transformations stem directly from the choice of the latter metric because they are playing actually the role of invariance transformations. Therefore, to investigate possible violation of Lorentz transformations in self-consistent way, one should modify the Lorentz transformations in conjunction with a due Finslerian extension of the metric. Accordingly, to rectify the practice, we propose to follow the concise Finslerian approach outlined in Sec.7 (and in the previous papers [8-12]), in which the characteristic parameter $g$ measures the degree of violation of the Finslerian metric function simultaneously with the degree of violation of the Lorentz transformations.

Our approach is everywhere compatible with the ordinary believe that "the laws of physics are invariant under spatial rotations". At the same time, the resultant Finslerian framework manifests the $T-$violation, so that we ought to conclude that the parameter $g$ measures also the degree of violation of the $T$-parity, and hence the CP-parity. Whence searches for the $CP-$violation (which are many; see, e.g., [19-20]) can, in principle, put interesting limits on the magnitude of the parameter $g$.

The theory of relativistic quantum fields adheres often to the so-called "Euclidean turn" which go over the theory in the Euclidean sector (see, e.g. [19-24]; such a sector is often used to consider the confinement of quarks). In the context of this, the positive-definite Finslerian metric function described in Secs. 2-6 may serve to give the base to continue "the Euclidean theory of quantum fields" in the Finslerian domain. Again, the continuation proves to be "$T-$asymmetric", for the Finsleroids are not symmetric under the $T-$reflection, - instead they are mirror-symmetric under the $gT$-type reflections (cf. Eq. (2.18)).

In general, the Finslerian approach outlined above seems to follow the proud thesis: "Sometimes a physical motivation precedes a mathematical theory, but it is not always so" [R.S.Ingarden, [25], p.213].

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