SELF-APPROXIMATION OF DIRICHLET $L$-FUNCTIONS

RAMŪNAS GARUNKŠTIS

Abstract. Let $d$ be a real number, let $s$ be in a fixed compact set of the strip $1/2 < \sigma < 1$, and let $L(s, \chi)$ be the Dirichlet $L$-function. The hypothesis is that for any real number $d$ there exist 'many' real numbers $\tau$ such that the shifts $L(s + i\tau, \chi)$ and $L(s + id\tau, \chi)$ are 'near' each other. If $d$ is an algebraic irrational number then this was obtained by T. Nakamura. L. Pańkowski solved the case then $d$ is a transcendental number. We prove the case then $d \neq 0$ is a rational number. If $d = 0$ then by B. Bagchi we know that the above hypothesis is equivalent to the Riemann hypothesis for the given Dirichlet $L$-function. We also consider a more general version of the above problem.

1. Introduction

Let, as usual, $s = \sigma + it$ denote a complex variable. For $\sigma > 1$, the Dirichlet $L$-function is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s};$$

where $\chi(n)$ is a Dirichlet character mod $q$. For $q = 1$ we get $L(s, \chi) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function.

In [6] Bohr proved that if $\chi$ is a nonprincipal character, then the Riemann hypothesis for $L(s, \chi)$ is equivalent to the almost periodicity of $L(s, \chi)$ in the half plane $\sigma > 1/2$. A function $f(s)$ is almost periodic in a region $E \subset \mathbb{C}$ if for any positive $\varepsilon$ and any compact subset $K$ in $E$ there exists a sequence of real numbers $\cdots < \tau_1 < \tau_2 < \cdots$ such that

$$\lim \inf_{m \to \pm\infty} (\tau_{m+1} - \tau_m) > 0, \quad \lim \sup_{m \to \pm\infty} \frac{\tau_m}{m} < \infty$$

and

$$|f(s + i\tau_m) - f(s)| < \varepsilon$$

for all $s \in K$ and $m \in \mathbb{Z}$ hold. Bohr [6] also obtained that every Dirichlet series is almost-periodic in its half-plane of absolute convergence. Effective upper bounds for the almost periodicity of Dirichlet series with Euler products in the half-plane of absolute convergence were considered by Girondo and Steuding [8]. Note that every Dirichlet $L$-function is almost periodic in the sense of Besicovitch on any vertical line of the strip $1/2 < \sigma < 1$. For this and related results see Besicovitch [5] and Mauclaire [13], [14].

Supported by grant No MIP-94 from the Research Council of Lithuania.
Bagchi \[2\] proved that the Riemann hypothesis for \(L(s, \chi)\) (\(\chi\) is an arbitrary Dirichlet character) is true if and only if for any compact subset \(K\) of the strip \(1/2 < \sigma < 1\) and for any \(\varepsilon > 0\)

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in K} |L(s + i\tau, \chi) - L(s, \chi)| < \varepsilon \right\} > 0,
\]

where \(\text{meas} A\) stands for the Lebesgue measure of a measurable set \(A\). Bagchi says that the Dirichlet \(L\)-function \(L(s, \chi)\) is strongly recurrent on the strip \(\sigma_0 < \sigma < \sigma_1\) if (1) is valid for any compact \(K\) of the strip \(\sigma_0 < \sigma < \sigma_1\). The strong recurrence is connected with the universality property of Dirichlet series. More about the universality and the strong recurrence see Bagchi \[1\], \[2\], \[3\], and Steuding \[17\].

There are several unconditional results concerning the self-approximation of Dirichlet \(L\)-functions in the critical strip. Let \(K\) be a compact subset of the strip \(1/2 < \sigma < 1\) and let \(\lambda \in \mathbb{R}\) be such that \(K\) and \(K + i\lambda := \{s + i\lambda : s \in K\}\) are disjoint. From Kaczorowski, Laurinčikas and Steuding \[10\] it follows that for any character \(\chi\) and any \(\varepsilon > 0\)

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in K} |L(s + i\lambda + i\tau, \chi) - L(s + i\tau, \chi)| < \varepsilon \right\} > 0.
\]

Nakamura \[15\] considered the joint universality of shifted Dirichlet \(L\)-functions. His Theorem 1.1 leads to the following statement. If \(1 = d_1, d_2, \ldots, d_m\) are algebraic real numbers linearly independent over \(\mathbb{Q}\), then for any Dirichlet character \(\chi\) and any \(\varepsilon > 0\)

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j, k \leq m} \max_{s \in K} |L(s + id_j\tau, \chi) - L(s + id_k\tau, \chi)| < \varepsilon \right\} > 0.
\]

If \(m = 2\) then Pańkowski \[16\] using Six Exponentials Theorem showed that (2) holds for \(d_1, d_2\) are real numbers linearly independent over \(\mathbb{Q}\).

We prove the following theorem.

**Theorem 1.** Let \(1 = d_1, d_2, \ldots, d_m\) be nonzero algebraic real numbers and let \(K\) be a compact subset of the strip \(1/2 < \sigma < 1\). Then for any Dirichlet character \(\chi\) and any \(\varepsilon > 0\) the inequality (2) is valid.

Note that Theorem 1 remains true if \(d_1, d_2, \ldots, d_m\) are replaced by \(dd_1, dd_2, \ldots, dd_m\), where \(d \in \mathbb{R}\). The next theorem shows that ‘\(\liminf\)’ in the inequality (2) often can be replaced by ‘\(\lim\).

**Theorem 2.** Let \(d_1, d_2, \ldots, d_m\) be any real numbers, let \(\chi_1, \chi_2, \ldots, \chi_m\) be any Dirichlet characters, and let \(K\) be a compact subset of the strip \(1/2 < \sigma < 1\). Then for any \(\varepsilon > 0\), except an at most countable set of \(\varepsilon\), there exists a limit

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j, k \leq m} \max_{s \in K} |L(s + id_j\tau, \chi_j) - L(s + id_k\tau, \chi_k)| < \varepsilon \right\}.
\]
The mentioned results of Nakamura and Pańkowski together with Theorem 1 and Theorem 2 lead to the following corollary.

**Corollary 3.** Let \( d \) be a nonzero real number and let \( K \) be a compact subset of the strip \( 1/2 < \sigma < 1 \). Then for any Dirichlet character \( \chi \) and any \( \varepsilon > 0 \), except an at most countable set of \( \varepsilon \),

\[
\lim_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \max_{s \in K} |L(s + i\tau, \chi) - L(s + id\tau, \chi)| < \varepsilon \right\} > 0.
\]

From the proof of Theorem 2 we see that for any real numbers \( d_1, \ldots, d_m \) and for any Dirichlet characters \( \chi_1, \ldots, \chi_m \) the function

\[
g(\tau) = \max_{1 \leq j, k \leq m} \max_{s \in K} |L(s + id_j\tau, \chi_j) - L(s + id_k\tau, \chi_k)|
\]

is Besicovitch almost periodic function (for the definition see Section 3 above the proof of Theorem 2). Let \( \varepsilon > 0 \) be such that the limit (3) exists. For such \( \varepsilon \) we define a characteristic function \( I_{\varepsilon}(\tau) \), \( \tau \in \mathbb{R} \), by

\[
I_{\varepsilon}(\tau) = \begin{cases} 
1, & \text{if } g(\tau) < \varepsilon, \\
0, & \text{if } g(\tau) \geq \varepsilon.
\end{cases}
\]

It is known (Jessen and A. Wintner [7, Section 12]) that \( I_{\varepsilon}(\tau) \) is Besicovitch almost periodic function also. Thus we can say that self-approximations of Dirichlet \( L \)-functions, considered in this paper, usually appear in a regular way.

Theorem 1 and Theorem 2 are proved in Section 3. Next we state several lemmas.

2. **Lemmas**

We start from the following statement.

**Lemma 4.** Let \( K \) be a compact subset of the rectangle \( U \). Let

\[
d = \min_{z \in \partial U} \min_{s \in K} |s - z|.
\]

If \( f(s) \) is analytic on \( U \) and

\[
\int_U |f(s)|^2 \, d\sigma dt \leq \varepsilon,
\]

then

\[
\max_{s \in K} |f(s)| \leq \frac{\sqrt{\varepsilon/\pi}}{d}.
\]

**Proof.** The lemma can be found in Gonek [9] (Lemma 2.5).

**Lemma 5.** Let \( a_1, \ldots, a_N \) be real numbers linearly independent over the rational numbers. Let \( \gamma \) be a region of the \( N \)-dimensional unit cube with volume \( V \) (in
the Jordan sense). Let \( I_\gamma(T) \) be the sum of the intervals between \( t = 0 \) and \( t = T \) for which the point \((a_1t, \ldots, a_Nt)\) is \( \text{mod} \ 1 \) inside \( \gamma \). Then

\[
\lim_{T \to \infty} \frac{I_\gamma(T)}{T} = V.
\]

Proof. This is Theorem 1 in Appendix, Section 8, of Voronin and Karatsuba [1].

For a curve \( \omega(t) \) in \( \mathbb{R}^N \) we introduce the notation

\[
\{\omega(t)\} = (\omega_1(t) - \lceil \omega_1(t) \rceil, \ldots, \omega_N(t) - \lceil \omega_N(t) \rceil),
\]

where \( \lceil x \rceil \) denotes the integral part of \( x \in \mathbb{R} \).

Lemma 6. Suppose that the curve \( \omega(t) \) is uniformly distributed \( \text{mod} \ 1 \) in \( \mathbb{R}^N \). Let \( D \) be a closed and Jordan measurable subregion of the unit cube in \( \mathbb{R}^N \) and let \( \Omega \) be a family of complex-valued continuos functions defined on \( D \). If \( \Omega \) is uniformly bounded and equicontinuous, then

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\{\omega(t)\}) 1_D(t) dt = \int_D f(x_1, \ldots, x_N) dx_1 \ldots dx_N
\]

uniformly with respect to \( f \in \Omega \), where \( 1_D(t) \) is equal to 1 if \( \omega(t) \in D \) \( \text{mod} \ 1 \), and 0 otherwise.

Proof. The lemma is Theorem 3 in Appendix, Section 8, of Voronin and Karatsuba [1].

Lemma 7. Let \( p_n \) be the \( n \)th prime number and \( 1 = d_1, d_2, \ldots, d_l \) be algebraic real numbers which are linearly independent over \( \mathbb{Q} \). Then the set \( \{d_k \log p_n\}_{n \in \mathbb{N}}^{1 \leq k \leq l} \) is linearly independent over \( \mathbb{Q} \).

Proof. This is Proposition 2.2 in Nakamura [15]. The proof is based on Baker’s [14] Theorem 2.4] result.

3. PROOF OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. We define a truncated Dirichlet \( L \)-function

\[
L_v(s, \chi) = \prod_{p \leq v} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}.
\]

Roughly speaking, we first prove Theorem 1 for the truncated Dirichlet \( L \)-function and later we show that the tail is small.

Let \( \{d_1, d_2, \ldots, d_l\} \) be a maximal linearly independent (over \( \mathbb{Q} \)) subset of the set \( \{d_1, d_2, \ldots, d_m\} \). Then there are integers \( a \neq 0 \) and \( a_{k,1}, a_{k,2}, \ldots, a_{k,l} \) such that

\[
d_k = \frac{1}{a} (a_{k,1}d_1 + a_{k,2}d_2 + \cdots + a_{k,l}d_l) \quad \text{for} \quad l < k \leq m.
\]
Let
\[ A = \max_{l < k \leq m} \{ |a_{k,1}| + |a_{k,2}| + \cdots + |a_{k,l}| \}. \]

Denote by \( \| x \| \) the minimal distance of \( x \in \mathbb{R} \) to an integer. If
\[
\left\| \frac{\tau d_n \log p}{2\pi a} \right\| < \delta \quad \text{for} \quad p \leq v \quad \text{and} \quad 1 \leq n \leq l
\]
then
\[
\left\| \frac{\tau d_n \log p}{2\pi} \right\| < a\delta \quad \text{for} \quad p \leq v \quad \text{and} \quad 1 \leq n \leq l
\]
and, by the relation (5),
\[
\left\| \frac{\tau d_k \log p}{2\pi} \right\| < A\delta \quad \text{for} \quad p \leq v \quad \text{and} \quad l < k \leq m.
\]

By this and by the continuity of the function \( L_v(s, \chi) \) we have that for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for \( \tau \) satisfying (6)
\[
\max_{1 \leq k, n \leq m} \max_{s \in K} \left( \log L_v(s + id_k \tau, \chi) - \log L_v(s + id_n \tau, \chi) \right) < \varepsilon.
\]

For positive numbers \( \delta, v, \) and \( T \) we define the set
\[
S_T = S_T(\delta, v) = \left\{ \tau : \tau \in [0, T], \left\| \frac{\tau d_n \log p}{2\pi a} \right\| < \delta, \quad p \leq v, \quad 1 \leq n \leq l \right\}.
\]

Let \( U \) be an open bounded rectangle with vertices on the lines \( \sigma = \sigma_1 \) and \( \sigma = \sigma_2 \), where \( 1/2 < \sigma_1 < \sigma_2 < 1 \), such that the set \( K \) is in \( U \). Let \( y > v \). We have
\[
\frac{1}{T} \int_T \int_U \sum_{k=1}^m \left| \log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi) \right|^2 \, d\sigma dt d\tau
\]
\[
= \sum_{k=1}^m \frac{1}{T} \int_{S_T} \int_U \left| \log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi) \right|^2 \, d\tau d\sigma dt.
\]

For the inner integrals of the right-hand side of the last equality we will apply Lemma 6. Let \( p_n \) be the \( n \)th prime number. There are indexes \( M \) and \( N \) such that \( p_M \leq v < p_{M+1} \) and \( p_N \leq y < p_{N+1} \). By generalized Kronecker’s theorem (Lemma 5) and by Lemma 7 the curve
\[
\omega(\tau) = \left( \frac{\tau d_k \log p_n}{2\pi a} \right)_{1 \leq k \leq l, 1 \leq n \leq N}
\]
is uniformly distributed mod 1 in \( \mathbb{R}^{lN} \). Let \( R' \) be a subregion of the \( lN \)-dimensional unit cube defined by inequalities
\[
\| y_{k,n} \| \leq \delta \quad \text{for} \quad 1 \leq k \leq l \quad \text{and} \quad 1 \leq n \leq M
\]
and

\[ \left| y_{k,n} - \frac{1}{2} \right| \leq \frac{1}{2} \quad \text{for} \quad 1 \leq k \leq l \text{ and } M + 1 \leq n \leq N. \]

Let \( R \) be a subregion of the \( lM \)-dimensional unit cube defined by inequalities

\[ \| y_{k,n} \| \leq \delta \quad \text{for} \quad 1 \leq k \leq l \text{ and } 1 \leq n \leq M. \]

Clearly

\[ \text{meas } R' = \text{meas } R = (2\delta)^{lM}. \]

Note that

\[
\log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi) = \log \frac{L_y}{L_v}(s + id_k \tau, \chi)
\]

\[ = - \sum_{v<p \leq y} \log \left( 1 - \frac{\chi(p)}{p^{s+id_k \tau}} \right) = \sum_{v<p \leq y} \sum_{j=1}^{\infty} \frac{\chi^j(p)}{j p^{j(s+id_k \tau)}} \]

\[ = \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p_n)}{j p_n^{j(s+id_k \tau)}}. \]

Thus in view of the linear dependence (5) we get

\[
\lim_{T \to \infty} \frac{1}{T} \int_{s_T} \sum_{k=1}^{m} \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 d\tau
\]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{s_T} \left( \sum_{k=1}^{l} \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 + \sum_{k=l+1}^{m} \left| \log \frac{L_y}{L_v} \left( s + \frac{i}{a}(a_{k,1}d_1 + a_{k,2}d_2 + \cdots + a_{k,ld_l}) \tau, \chi \right) \right|^2 \right) d\tau. \]
By Lemma 6 and equality (9) we obtain that the last limit is equal to
\[
\int_{R} \left( \sum_{k=1}^{l} \left| \sum_{M<n \leq N} \sum_{j=1}^{\infty} \frac{\chi^{j}(p)e^{-2\pi ijy_{k,n}}}{jp^{s}_{n}} \right|^{2} \right) dy_{1,1} \ldots dy_{l,N}
+ \sum_{k=l+1}^{m} \left| \sum_{M<n \leq N} \sum_{j=1}^{\infty} \frac{\chi^{j}(p)e^{2\pi ij(a_{k,1}y_{1,n}+a_{k,2}y_{2,n}+\cdots +a_{k,l}y_{l,n})}}{jp^{s}_{n}} \right|^{2} \right) dy_{1,M+1} \ldots dy_{l,N}
= \text{meas } R \int_{0}^{1} \ldots \int_{0}^{1} \left( \sum_{k=1}^{l} \left| \sum_{M<n \leq N} \sum_{j=1}^{\infty} \frac{\chi^{j}(p)e^{-2\pi ijy_{k,n}}}{jp^{s}_{n}} \right|^{2} \right) dy_{1,1} \ldots dy_{l,N}
+ \sum_{k=l+1}^{m} \left| \sum_{M<n \leq N} \sum_{j=1}^{\infty} \frac{\chi^{j}(p)e^{2\pi ij(a_{k,1}y_{1,n}+a_{k,2}y_{2,n}+\cdots +a_{k,l}y_{l,n})}}{jp^{s}_{n}} \right|^{2} \right) dy_{1,M+1} \ldots dy_{l,N}
= m \text{meas } R \sum_{v<p \leq y} \sum_{j=1}^{\infty} \frac{1}{jp^{2\sigma}} \ll \text{meas } R \sum_{p>v} \frac{1}{p^{2\sigma}}.
\]

Consequently
\[
\frac{1}{T} \int_{S_{T}} \int_{U} \sum_{k=1}^{m} \left| \log L_{y}(s + id_{k}\tau, \chi) - \log L_{v}(s + id_{k}\tau, \chi) \right|^{2} d\sigma d\tau d\tau
\ll \text{meas } R \sum_{p>v} \frac{1}{p^{2\sigma}}.
\]

Again, by generalized Kronecker’s theorem (Lemma 5),
\[
\lim_{T \to \infty} \frac{1}{T} \text{meas } S_{T} = \text{meas } R.
\]

By (10) and (11), for large \(v\), as \(T \to \infty\), we have
\[
\text{meas } \left\{ \tau : \tau \in S_{T}, \int_{U} \sum_{k=1}^{m} \left| \log \left( \frac{L_{y}}{L_{v}}(s + id_{k}\tau, \chi) \right) \right|^{2} d\sigma dt < \sqrt{\sum_{p>v} \frac{1}{p^{2\sigma}}} \right\} > \frac{1}{2} T \text{meas } R.
\]

Then Lemma 4 gives
\[
\text{meas } \left\{ \tau : \tau \in S_{T}, \max_{s \in K} \sum_{k=1}^{m} \left| \log \left( \frac{L_{y}}{L_{v}}(s + id_{k}\tau, \chi) \right) \right|^{2} d\tau \leq \frac{1}{d \sqrt{\pi}} \left( \sum_{p>v} \frac{1}{p^{2\sigma}} \right)^{\frac{1}{4}} \right\}
> \frac{1}{2} T \text{meas } R,
\]
where $d = \min_{z \in \partial U} \min_{s \in K} |s - z|$. By the continuity of the logarithm we obtain that for any $\varepsilon > 0$ there is $v = v(\varepsilon)$ such that for any $y > v$

\[
(12) \quad \text{meas} \left\{ \tau : \tau \in S_T, \max_{s \in K} \sum_{k=1}^{m} |L_y(s + id_k \tau, \chi) - L_v(s + id_k \tau, \chi)|^2 d\tau < \varepsilon \right\} > \frac{1}{2} T \text{meas } R.
\]

Now we will prove that for any $\delta > 0$ there is $y = y(\delta)$ such that

\[
(13) \quad \text{meas} \left\{ \tau : \tau \in [0, T], \max_{s \in K} \sum_{k=1}^{m} |L(s + id_k \tau, \chi) - L_y(s + id_k \tau, \chi)|^2 d\tau < \delta \right\} > (1 - \delta)T.
\]

The last formula together with (7), (8) and (12) yields Theorem 1. We return to the proof of (13). By the mean value theorem of the Dirichlet $L$-function (Steuding [17], Corollary 6.11) and by Carlson’s Theorem (Titchmarsh [18], Chapter 9.51) we obtain

$$
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |L(s + ix \tau, \chi) - L_y(s + ix \tau, \chi)|^2 d\tau = \sum_{n>y} \frac{|\chi(n)|}{n^{2\sigma}},
$$

where $x$ is fixed. Thus (13) follows in view of

$$
\int_{0}^{T} \int_{U} \sum_{k=1}^{m} |L(s + id_k \tau, \chi) - L_y(s + id_k \tau, \chi)|^2 d\sigma dt d\tau \ll \sum_{n>y} \frac{|\chi(n)|}{n^{2\sigma}}.
$$

Theorem 1 is proved.

The proof of Theorem 2 is based on the ideas of Mauclaire [13], [14]. It uses the theory of Besicovitch almost periodic functions. We recall related definitions.

Let

$$
P(\tau) = \sum_{n \in F} a_n e^{i\lambda_n \tau},
$$

where $F$ is a finite set, $\lambda_n$ are any real numbers, and the coefficients $a_n$ are any complex numbers. For real $\tau$ we say that $P(\tau)$ is a trigonometric polynomial.

A function $f : \mathbb{R} \to \mathbb{C}$ is called uniformly almost periodic (U.A.P.) if given any $\varepsilon > 0$, there exists a trigonometric polynomial $P(\tau)$ such that

$$
\sup_{\tau \in \mathbb{R}} |f(\tau) - P(\tau)| \leq \varepsilon.
$$
A function \( f : \mathbb{R} \to \mathbb{C} \) is called \( B^q \) almost periodic (\( B^q.A.P. \)), \( q \geq 1 \), if given any \( \varepsilon > 0 \), there exists a trigonometric polynomial \( P(\tau) \) such that

\[
\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\tau) - P(\tau)|^q \, d\tau \leq \varepsilon.
\]  

(14)

If \( q = 1 \) then we write \( B.A.P. \) (Besikovitch almost periodic) instead of \( B^1.A.P. \).

For any \( q \geq 1 \) it is clear that every \( U.A.P. \) function is \( B^q.A.P. \) and that every \( B^q.A.P. \) function is \( B.A.P. \).

**Proof of Theorem 2.**

Let

\[
g(\tau) = \max_{1 \leq j, k \leq m} \max_{s \in \mathcal{K}} |L(s + id_j \tau, \chi_j) - L(s + id_k \tau, \chi_k)|
\]

and let

\[
F_T(x) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : g(\tau) < x \}
\]

be a distribution function of \( g(\tau) \). If \( g(\tau) \) is \( B.A.P. \) then it is known (see Jessen and Wintner [7, Theorem 27] or Laurinčikas [12, Theorem 6.3, Chapter 2]) that there is a distribution function \( F(x) \) such that \( F_T(x) \) converges weakly to \( F(x) \) for \( T \to \infty \). It means that if \( F(x) \) is continuous at \( x = \varepsilon \) then

\[
\lim_{T \to \infty} F_T(\varepsilon)
\]

exists. Thus to obtain Theorem 2 we need to show that \( g(\tau) \) is \( B.A.P. \).

We remark that if \( a(t) \) and \( b(t) \) are both non-negative \( B.A.P. \) functions of \( t \), then, \( t \mapsto \max(a(t), b(t)) \) is also \( B.A.P. \) since \( \max(a(t), b(t)) \) can be written as

\[
\max(a(t), b(t)) = \frac{1}{2} (|a(t) - b(t)| + (a(t) + b(t))),
\]

and the modulus of \( B.A.P. \) function is again \( B.A.P. \). By this we have that \( g(\tau) \) is \( B.A.P. \) if the function

\[
f(\tau) = \max_{s \in \mathcal{K}} |L(s + id_1 \tau, \chi_1) - L(s + id_2 \tau, \chi_2)|
\]

is \( B.A.P. \). In view of the note below the formula (14) the function \( f(\tau) \) is \( B.A.P. \) if there are \( U.A.P. \) functions \( f_N(\tau) \) such that

\[
\lim_{N \to +\infty} \left( \limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\tau) - f_N(\tau)|^2 \, d\tau \right) = 0.
\]  

(15)

Let

\[
L_N(s, \chi) = \sum_{n \leq N} \frac{\chi(n)}{n^s}
\]

be a partial sum of the Dirichlet series associated with \( L(s, \chi) \). Next we show that the equality (15) is true with

\[
f_N(\tau) = \max_{s \in \mathcal{K}} |L_N(s + id_1 \tau, \chi_1) - L_N(s + id_2 \tau, \chi_2)|.
\]
By repeating the proof of Proposition 12 of Mauclaire \cite{13} we get that \( f_N(\tau) \) is U.A.P. for any \( d_1, d_2 \in \mathbb{R} \). Note that the case when \( d_1 \) or \( d_2 \) is equal to zero is already included in Proposition 12 of Mauclaire \cite{13}.

Further we have that
\[
L(s + id_1\tau, \chi_1) - L(s + id_2\tau, \chi_2) = (L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1) + L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)) + (L_N(s + id_1\tau, \chi_1) - L_N(s + id_2\tau, \chi_2)),
\]
and as a consequence, we get that
\[
|f(\tau) - f_N(\tau)| \leq \sup_{s \in K} |L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1)| + \sup_{s \in K} |L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)|.
\]

Then, in view of the inequality \((a + b)^2 \leq 2a^2 + 2b^2\), we obtain that
\[
\frac{1}{2T} \int_{-T}^{T} |f(\tau) - f_N(\tau)|^2 \, dt \\
\leq \frac{1}{T} \int_{-T}^{T} \left( \sup_{s \in K} |L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1)| \right)^2 \, dt \\
+ \frac{1}{T} \int_{-T}^{T} \left( \sup_{s \in K} |L(s + id_2\tau, \chi_2) - L_N(s + id_2\tau, \chi_2)| \right)^2 \, dt.
\]

By Mauclaire \cite{14} Theorem 5.1 we have that, for any real \( d \),
\[
\lim_{N \to +\infty} \left( \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \left( \sup_{s \in K} |f(s + idt) - f_N(s + idt)| \right)^2 \, dt \right) = 0.
\]
This proves the equality \((15)\) and Theorem 2 \( \square \).

From the proof we see that Theorem 2 remains true with Dirichlet \( L \)-functions \( L(s, \chi_j), j = 1, \ldots, m \), replaced by any general Dirichlet series satisfying conditions of Theorem 5.1 of Mauclaire \cite{14}.

**Acknowledgment.** We thank Jean-Loup Mauclaire for suggesting Theorem 2 and for other useful comments which helped to improve the paper.

**Remark.** The ‘lim inf’ version of Corollary 3 is independently obtained by Takashi Nakamura in “The generalized strong recurrence for non-zero rational parameters”, arXiv:1006.1778v1 [math.NT].
References

[1] B. Bagchi, The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series, PhD Thesis, Calcutta: Indian Statistical Institute, 1981.

[2] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z., 181 (1982), 319-334.

[3] B. Bagchi, Recurrence in topological dynamics and the Riemann hypothesis, Acta Math. Hung., 50 (1987), 227-240.

[4] A. Baker, Transcendental number theory, London: Cambridge University Press. X, (1975).

[5] A.S. Besicovitch, Almost periodic functions, Dover, New York, (1954).

[6] H. Bohr, Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen L-Funktionen, Math. Ann., 85 (1922), 115-122.

[7] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Am. Math. Soc. 38 (1935), 48-88.

[8] E. Girondo and J. Steuding, Effective estimates for the distribution of values of Euler products, Monatsh. Math., 145, No. 2 (2005), 97-106.

[9] S.M. Gonek, Analytic properties of zeta and L-functions, Ph. D. Thesis, University of Michigan, 1979.

[10] J. Kaczorowski, A. Laurinčikas, and J. Steuding, On the value distribution of shifts of universal Dirichlet series, Monatsh. Math., 147 (2006), 309-317.

[11] A.A. Karatsuba and S.M. Voronin, The Riemann zeta-function, de Gruyter Expositions in Mathematics. 5. Berlin etc.: W. de Gruyter. xii, (1992).

[12] A. Laurinčikas, Limit theorems for the Riemann zeta-function, Mathematics and its Applications (Dordrecht). 352. Dordrecht: Kluwer Academic Publishers, (1995).

[13] J.-L. Mauclaire, Almost periodicity and Dirichlet series, Laurinčikas, A. (ed.) et al., Analytic and probabilistic methods in number theory. Proceedings of the 4th international conference in honour of J. Kubilius, Palanga, Lithuania, September 25–29, 2006. Vilnius: TEV, (2007),109-142.

[14] J.-L. Mauclaire, On some Dirichlet series, Proceedings of the conference New Directions in the Theory of Universal Zeta- and L-Functions, Würzburg, Germany, October 6-10, 2008. Shaker Verlag, (2009), 171-248.

[15] T. Nakamura, The joint universality and the generalized strong recurrence for Dirichlet L-functions, Acta Arith., 138 (2009), 357-362.

[16] L. Pańkowski, Some remarks on the generalized strong recurrence for L-functions, in: New Directions in Value Distribution Theory of zeta and L-Functions: proceedings of Würzburg Conference, October 6-10, 2008, Shaker Verlag, (2009), 305-315.

[17] J. Steuding, Value distribution of L-functions, Lecture Notes in Mathematics 1877, Springer, 2007.

[18] E.C. Titchmarsh, The theory of functions. 2nd ed., London: Oxford University Press. X, 1975.

Ramūnas Garunkštis, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania
E-mail address: ramunas.garunkstis@mif.vu.lt
URL: www.mif.vu.lt/~garunkstis