BIVARIANT ALGEBRAIC $K$-THEORY

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Abstract. We show how methods from $K$-theory of operator algebras can be applied in a completely algebraic setting to define a bivariant, $\mathcal{M}_\infty$-stable, homotopy-invariant, excisive $K$-theory of algebras over a fixed unital ground ring $H$, $(A, B) \mapsto \mathcal{K}_*(A, B)$, which is universal in the sense that it maps uniquely to any other such theory. It turns out $\mathcal{K}_*$ is related to $C$. Weibel's homotopy algebraic $K$-theory, $KH$. We prove that, if $H$ is commutative and $A$ is central as an $H$-bimodule, then

$\mathcal{K}_*(H, A) = KH_*(A)$.

We show further that some calculations from operator algebra $KK$-theory, such as the exact sequence of Pimsner-Voiculescu, carry over to algebraic $\mathcal{K}_*$.

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2000 Mathematics Subject Classification. 19K35, 19D25, 18E30.

Key words and phrases. bivariant, excisive, homotopy invariant $K$-theory.

Cortiñas' research was partly supported by grants PICT03-12330, UBACyT-X294, VA091A05, and MTM00958.
Thom’s research was partly supported by the EU-Network Quantum Spaces and Noncommutative Geometry (Contract HPRN-CT-2002-00280) and the DFG (SFB 478 Münster, GK Metageometrie und analytische Topologie Münster, GK Gruppen und Geometrie Göttingen).
1. Introduction

We consider associative, not necessarily unital or central algebras over a fixed unital, not necessarily commutative ring $H$; we write $\text{Ass}_H$ for the category of such algebras. If $H$ is commutative, we consider also the category $\text{Ass}_{H}^c$ of central algebras. Let $\text{Alg}_H$ be either of $\text{Ass}_H$, $\text{Ass}_{H}^c$. Note that, by forgetting structure, we can embed $\text{Alg}_H$ faithfully into each of the categories of (central) $H$-bimodules, abelian groups and sets. Fix one of these underlying categories, call it $U$, and let $F : \text{Alg}_H \to U$ be the the forgetful functor. Let $E$ be the class of all exact sequences of $H$-algebras 
\[ (E) \quad 0 \to A \to B \to C \to 0 \]
such that $F(B) \to F(C)$ is a split surjection. We construct a triangulated category $kk$ with inverse suspension functor $\Omega$, a functor $j : \text{Alg}_H \to kk$, and a family of connecting maps 
\[ \{ \partial_E : \Omega(j(C)) \to j(A) \} \]
natural with respect to maps of exact sequences, such that the following conditions are satisfied.

i) For all $A \in \text{Alg}_H$, $j(A \to M_\infty A)$ and $j(A \to A[t])$ are isomorphisms.

ii) For every exact sequence $E \in E$, the following is a distinguished triangle in $kk$

\[ \Omega j(C) \xrightarrow{\partial_E} j(A) \xrightarrow{j} j(B) \xrightarrow{j} j(C). \]

In particular, the sequence of functors

\[ kk_n(A, B) := \text{hom}_{kk}(j(A), \Omega^n j(B)) \quad (n \in \mathbb{Z}) \]
forms a bivariant, homotopy invariant, $M_\infty$-stable homology theory which satisfies excision with respect to all sequences in $E$, that is, for any $E \in E$ as above, $n \in \mathbb{Z}$ and $D \in \text{Alg}_H$, we have long exact sequences

\[ \cdots \xrightarrow{(\partial_E)_*} kk_{n+1}(D, C) \xrightarrow{(\partial_E)^*} kk_n(D, A) \xrightarrow{kk_n(D, B)} kk_n(D, C) \xrightarrow{\cdots} \]

\[ \cdots \xrightarrow{(\partial_E)^*} kk_{n+1}(A, D) \xrightarrow{(\partial_E)^*} kk_n(C, D) \xrightarrow{kk_n(B, D)} kk_n(A, D) \xrightarrow{\cdots} \]

We call $kk_n$ bivariant algebraic $K$-theory. We show further that $j$ is universal among all functors $X : \text{Alg}_H \to T$ with values in a triangulated category $T$, and equipped with natural maps $\partial^X_E$ satifying the requirements above. Universality means that if $X$ is such a functor, then there is a
unique triangulated functor $\tilde{X} : \mathcal{K} \to \mathcal{T}$, compatible with connecting maps, such that $X = \tilde{X} \circ j$.

In particular $\tilde{X}$ induces, for each $n \in \mathbb{Z}$, a group homomorphism

$$kk_n(A, B) \to X_n(A, B) := \text{hom}_\mathcal{T}(X(A), \Omega^nX(B)) \quad (n \in \mathbb{Z})$$

compatible with all the structure. For example if $H$ is a field of characteristic zero, and $\text{Alg}_H = \text{Ass}_H$, then applying this universal property when $X$ is the Cuntz-Quillen pro-supercomplex, we obtain a product preserving Chern character to bivariant periodic cyclic cohomology

$$ch_* : kk_*(A, B) \to HP^*(A, B).$$

Bivariant periodic $K$-theory is related to C. Weibel’s homotopy algebraic $K$-theory, $KH_*$ ([31]). We prove that, for commutative $H$, $\text{Alg}_H = \text{Ass}_H$, and $B \in \text{Alg}_H$, we have

$$kk_*(H, B) = KH_*(B).$$

It follows from this and the universal property of $kk$, that if $X$ is any $M_\infty$-stable, homotopy invariant, excisive theory as above, then there is a map

$$KH_*(B) \to X_*(H, B).$$

For example, when $H$ is a field of characteristic zero, we obtain in this way the Chern character from $KH$ to periodic cyclic homology $HP_*(B) = HP^*(H, B)$.

The construction of $kk$ we present here is inspired in work of J. Cuntz on bivariant $K$-theory of topological algebras; we generalize and adapt his methods. In particular, no spectra are involved in the definition of $kk$. On the other hand, one can also use the machinery of symmetric spectra to construct a bivariant, homotopy invariant, $M_\infty$-stable excisive theory as follows. Recall (e.g. from [28] Appendix D) that $KH$ can be defined as the homotopy groups of a certain functorial symmetric spectrum $KH^* : \text{Alg}_H \to Sp^\Sigma$ with compatible external products $KH^*(A) \wedge KH^*(B) \to KH^*(A \otimes B)$. Thus for each $A \in \text{Alg}_H$, $KH^*(A)$ is a $KH^*(H)$-module spectrum. Let $[\cdot, \cdot]_{KH^*(H)}$ denote the homomorphisms in the homotopy category of $KH^*(H)$-module spectra. Put

$$KH_*(A, B) := [KH^*(A), \Omega^\infty KH^*(B)]_{KH^*(H)}.$$ 

By the universal property of $kk$, there is a natural map

$$kk_*(A, B) \to KH_*(A, B)$$

This map is an isomorphism in some cases, for example when $A = H$ and $\text{Alg}_H = \text{Ass}_H^0$; this yields [3]. Even for $H = \mathbb{Z}$ (in which case $\text{Ass}_H = \text{Ass}_H^0$) the problem of determining for which $A$ and $B$ the map [4] is an isomorphism is a difficult one. This problem is discussed in Subsection 8.3.

As said above, we adapt and generalize Cuntz’ methods to the algebraic setting; in particular continuous or differential homotopy has to be replaced by algebraic (i.e. polynomial) homotopy. In other words the interval $[0, 1]$ has to be replaced by the affine line $\mathbb{A}^1 = \mathbb{A}^1_\mathbb{Z} = \text{Spec}(\mathbb{Z}[t])$. This entails some technical difficulties. For example, in the topological setting, the basis for composing homotopies is that the interval $I = [0, 1]$ is homeomorphic to the amalgamated sum of two copies of it; $I = [0, 1/2] \cup [1/2, 1] \cong I \cup I$. This is no longer true in the algebraic setting; $\mathbb{A}^1 \cup \mathbb{A}^1 = \text{Spec}(\mathbb{Z}[t] \times_{\mathbb{Z}} \mathbb{Z}[t]) \neq \mathbb{A}^1$. Then there is also the fact that some of the homotopies used in the operator and topological algebra setting to prove some key results are not polynomial, so we need to come up with new algebraic homotopies to replace them. All of these problems are dealt with in Section 8. Modulo these technicisms, the construction of $kk$ is carried out pretty much as in the topological algebra setting (sections 4, 5, 6), and its universal property proved. Once the triangulated category $kk$ is constructed, some calculations performed by Cuntz in that setting carry over with essentially the same proof to the algebraic case. For example we show that if $A \to B$ and $A \to C$ are split algebra monomorphisms, then $j$ maps the coproduct $B \coprod_A C$ to the amalgamated sum $j(B) \oplus_{j(A)} j(C)$ in $kk$.

$$j(B \coprod_A C) \cong j(B) \oplus_{j(A)} j(C).$$
In particular we get
\[ kk_n(D, B \coprod_A C) = kk_n(D, B) \oplus kk_n(D, A) \quad (n \in \mathbb{Z}), \]
and similarly on the other variable. Moreover by the universal property of \( j : \text{Alg}_H \to kk \), it follows that if \( X : \text{Alg}_H \to \mathcal{T} \) is any functor satisfying conditions i) and ii) above, then
\[ X(B \coprod_A C) \cong X(B) \oplus_{X(A)} X(C). \]
In particular (6) is also valid for the bivariant groups \( X_* \) of (2). As another example consider an action \( \mathbb{Z} \to \text{Aut}_{\text{Alg}_H}(A) \), sending \( 1 \mapsto \alpha \), and write \( A \rtimes_{\alpha} \mathbb{Z} \) for the crossed product algebra. We show in 7.4 that there is a distinguished triangle in \( kk \)
\[ \Omega(j(A \rtimes_{\alpha} \mathbb{Z})) \longrightarrow j(A) \longrightarrow 1 - j(\alpha^{-1}) \]
Applying \( kk_* (D, ?) \) and \( kk_* (? , D) \) we get a version in our setting of the Pimsner-Voiculescu sequences of [24] (see also [8, 14.3]). Again by the universal property, the same is valid for any functor \( X \) satisfying i) and ii) above. Similarly, we prove that the Laurent polynomial ring is isomorphic in \( kk \) to the sum of \( A \) and its suspension
\[ j(A[t^\pm]) \cong j(A) \oplus \Sigma j(A). \]
For example if \( H \) is commutative and \( \text{Alg}_H = \text{Ass}_H \), then applying \( kk_*(H, ) \) and using (6), we obtain the fundamental theorem of homotopy \( K \)-theory (31)
\[ KH_n(A[t^\pm]) = KH_n(A) \oplus KH_{n-1}(A) \quad (n \in \mathbb{Z}). \]
Thus in light of the universal property of \( kk \), we can interpret (6) as saying that any \( M_{\infty} \)-stable, homotopy invariant, excisive theory satisfies the fundamental theorem.

The rest of this paper is organized as follows. In Section 2 we fix some notations used throughout the paper, and recall some basic facts about ind-objects. Section 3 deals with polynomial homotopy. In 3.1 we define the notion of homotopy between (ind)-algebra homomorphisms and introduce an ind-homomorphism from\( \text{Alg}_H \) to \( \mathcal{T} \), and recall some basic facts about ind-objects. Section 3 deals with polynomial homotopy.

\[ S^{op} \times \text{Alg}_H \to \text{Alg}_H, \quad (X, A) \mapsto A^X \]
such that
\[ \text{hom}_S(X, \text{hom}_{\text{Alg}_H}(A, B)) = \text{hom}_{\text{Alg}_H}(A, B^X) \]
For example \( B^{t^n} = B \otimes \mathbb{Z}[t_0, \ldots, t_n]/(\sum t_i - 1) \). There is also a pointed version \( B^{(X, *)} \). The functor
\[ \Omega B := B^{(S^1, *以外}} \]
provides the loopspace (i.e. the inverse suspension) in the triangulated category \( kk \) (see 6.5). In 7.2 we use 9 to translate the subdivision of simplicial sets to algebras, and obtain a fibrant version \( \text{HOM}_{\text{Alg}_H}(A, B) \) of the enrichment above (see Theorem 3.2.3). In 8.3 we define a path (or homotopy) between homomorphisms \( f_i : A \to B, \ i = 0, 1 \) as a map \( H : A \to B^{sd\Delta^1} \) for some \( n \geq 0 \), such that for the evaluations we have \( ev_i H = f_i \). Equivalently, a homotopy is an ind-homomorphism from \( A \) to the ind-algebra
\[ B^{sd\Delta^1} := \{ B^{sd\Delta^1} \}_{n \geq 0} \]
Similarly, a loop is an ind-homomorphism to \( B^{S^1} := B^{sd\Delta^1} \). We show in 9.3.2 that the set of homotopy classes of loops can be recovered as the fundamental group of the fibrant simplicial set \( \text{HOM}_{\text{Alg}_H}(A, B) \); we have
\[ [A, B^{S^1}] = \pi_1 \text{HOM}_{\text{Alg}_H}(A, B). \]
In Subsection 3.4 we observe that a rotational homotopy which appears in several proofs of topological algebra $K$-theory can be implemented in the polynomial homotopy setting. In Section 4 we introduce a number of (ind-) algebras and $F$-split extensions which appear in the definition of $kk$-theory and the $kk$-category. In 4.1 we introduce an ind-algebra which will play the role the compact (or rapid decay) operators play in topological setting. The following subsections define the notion of extension and introduce algebraic versions of some extensions, such as the path, universal, loop, and mapping path extensions. As in the topological algebra setting, the universal extension classifies all extensions up to homotopy (see 4.4.3). We also study (in 4.7) M. Karoubi’s cone extension (10)

\[ M_\infty A \to \Gamma A \to \Sigma A. \]

Interest in this extension comes from its use in delooping algebraic $K$-theory ([10], [30]). We show in Subsection 4.7 that (11) is split as a sequence of $A$-bimodules, whence $F$-split. In 4.8 we prove that $\Gamma A$ is an infinite sum ring in the sense of Wagoner [30]; this is used further below, in Section 6 to prove that $\Gamma A$ is equivalent to zero in $kk$ and thus that the functor $\Sigma$ is inverse to $\Omega$ in $kk$. Hence (11) is analogue to the operator algebra Calkin extension. The analogue of the operator algebra Toeplitz extension is considered in 4.10. Section 5 is devoted to split exact and $M_2$-stable functors defined on $\text{Alg}_{\mathbb{H}}$ with values in an abelian category. We show in 5.1 that $M_2$-stable functors are invariant under (generalized) inner automorphisms (see 5.1.2), and observe that this implies any such functor is Morita invariant in the usual unital sense (see 5.1.3). In particular this will apply to $kk$-theory. Then we recall the notion of quasi-homomorphism and the extended functoriality properties of split-exact $M_2$-invariant functors (see 5.2). Section 6 is devoted to the definition and basic properties of $kk$. The section starts with the definition of $kk(A, B)$; this makes use of the homotopy machinery developed in Section 3. The composition product $kk(A, B) \otimes kk(B, C) \to kk(A, C)$ and thus the category $kk$ are introduced in 6.2. In 6.3 we prove excision in both variables of $kk$ (see 6.3.6 and 6.3.7) and show that (11) induces an equivalence in $kk$ (see 6.3.8 and 6.3.9). In 6.4 we prove that the functor $\Sigma$ of (11) is an equivalence in $kk$, which is pseudo-inverse to $\Omega$. In 6.5 we show that $kk$ is triangulated. The universal property of $kk$ discussed above is proved in 6.6 where we also prove a second universal property of $kk$ (see 6.6.6). The latter says that any half exact, $M_\infty$-stable, homotopy invariant functor $G$ from $\text{Alg}_{\mathbb{H}}$ to an abelian category $\mathfrak{A}$ factors uniquely through a homological functor $\bar{G} : kk \to \mathfrak{A}$. In 6.7 we compare $kk$ with Kassel’s bivariant $K$ group (see [15]). The latter is a defined for pairs $(A, B)$ of unital algebras over commutative ground ring $\mathbb{H}$. By definition, $K(\mathbb{H}, B) = K_0(B)$ is the usual $K_0$; in particular, $K(\mathbb{R}, ?)$ is not homotopy invariant. However, it is equipped with a composition product and a Chern character $K(A, B) \to HC^0(A, B)$ with values in (nonperiodic) cyclic homology. We show that there is a map $K(A, B) \to kk(A, B)$ compatible with all the structure. Some computations, including (1), (2) and (3), are carried out in Section 7. We show further (in 7.2) that positively graded as well as nilpotent rings (satisfying a suitable $F$-splitting condition) go to zero in $kk$. In Section 8 we compare $kk$ and $KH$. In 8.1 we recall Weibel’s definition of $KH$; we also give an alternative definition which we prove is equivalent to the original one (see 8.1.1), and from which a new proof of excision and nilinvariance of $KH$ is easily deduced (see 8.1.2). This alternative definition is used in 8.2 to prove our main theorem (3). The problem of determining when (1) is an isomorphism is discussed in 8.3.

Acknowledgements. Many of the ideas that we are going to present go back to the work of J. Cuntz. He analyzed Kasparov’s $KK$-theory in algebraic terms; this pointed to new ideas which finally could be applied in many different settings. Our paper owes much to Cuntz’ pioneering work.

Part of the research for this article was carried out during visits of the first author to the universities of Göttingen and Münster. He is thankful to these institutions for their hospitality.
2. Conventions and preliminaries

Let $H$ be a unital ring. The tensor product $\otimes_H$ makes the category $H\text{-}\text{Bimod} = H \otimes \mathbb{Z} H^{op}$-$\text{Mod}$ of $H$-bimodules into a monoidal category. By an $H$-\textit{algebra} we understand a monoid in $(H\text{-}\text{Bimod}, \otimes_H)$, possibly without a neutral element; thus algebras will be nonunital in general. We denote by $\text{Ass}_H$ the category of $H$-algebras. If $H$ happens to be commutative, we consider also the full subcategory $\text{Ass}^c_H \subset \text{Ass}_H$ of central algebras. Let $\text{Alg}_H$ be either of $\text{Ass}_H$, $\text{Ass}^c_H$. Most of our results do not depend on which of these two choices we pick, so they will be formulated in terms of $\text{Alg}_H$. Whenever we need to restrict to one of the choices, this will be made clear. In what follows an $H$-algebra will be an object of $\text{Alg}_H$.

Throughout, we assume fixed an underlying category $\mathcal{U}$, which can be either the category of sets, of (central) bimodules or of abelian groups. In each case we have a faithful forgetful functor $F : \text{Alg}_H \to \mathcal{U}$ and a functor $\tilde{T} : \mathcal{U} \to \text{Alg}_H$, left adjoint to $F$.

If $\mathcal{C}$ is a category, we write $\text{ind}\neg\neg \mathcal{C}$ for the category of ind-objects of $\mathcal{C}$. It has as objects the directed diagrams in $\mathcal{C}$. An object in $\text{ind}\neg\neg \mathcal{C}$ is described by a filtering partially ordered set $(I, \leq)$ and a functor $X : I \to \mathcal{C}$. The set of homomorphisms from $(X, I)$ to $(Y, J)$ is

$$\text{colim}_{i \in I} \text{hom}_\mathcal{C}(X_i, Y_j).$$

(12)

There is a natural functor $\text{ind}\neg\neg (\text{ind}\neg\neg \mathcal{C}) \to \text{ind}\neg\neg \mathcal{C}$, mapping

$$((X_i, J_i), I) \to (X_{ij}, \coprod_{i \in I} J_i \times i).$$

(13)

We shall use this functor to collapse any ind-$\neg\neg$-object to an ind-$\neg\neg$-object. Any functor $F : \mathcal{C} \to \mathcal{D}$ extends to $\text{ind}\neg\neg \mathcal{C} \to \text{ind}\neg\neg \mathcal{D}$ by $F(X_i, i \in I) = (F(X_i), i \in I)$. In particular any functor $\mathcal{C} \to \text{ind}\neg\neg \mathcal{D}$ extends to $\text{ind}\neg\neg \mathcal{C} \to \text{ind}\neg\neg \mathcal{D}$, which after collapsing gives a functor $\text{ind}\neg\neg \mathcal{C} \to \text{ind}\neg\neg \mathcal{D}$.

These extensions and collapsing shall be implicit, so that whenever a functor is defined on a category we shall freely apply it to the ind-category. We shall identify objects of $\mathcal{C}$ with constant ind-objects, so that we shall view $\mathcal{C}$ as a subcategory of $\text{ind}\neg\neg \mathcal{C}$.

Throughout, the letters $A, B, C, \ldots$ denote (ind-)H-algebras. Thus $f : A \to B$ is a homomorphism of (ind-)H-algebras.

Let $L$ be a ring and $A$ be an H-algebra. For brevity, we write $LA$ for the H-algebra $L \otimes \mathbb{Z} A$. Similarly, $AL$ stands for $A \otimes \mathbb{Z} L$.

We write $S$ for the category of simplicial sets, and $\mathbb{N}$ for the set of natural numbers, which for us include $0$. Thus $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

We use the notation $:= $ to define the left side by the right side.

3. Homotopy

3.1. Algebraic homotopies. Let $A$ be an H-algebra. Put

$$A^A := A[t] = A \otimes \mathbb{Z} [t].$$

There are several natural morphisms relating $A$ and $A[t]$. We write $c_A : A \to A[t]$ for the inclusion of $A$ as constant polynomials in $A[t]$ and $\text{ev}_i : A[t] \to A$ for the evaluation of $t$ at $i$ ($i \in \{0, 1\}$). Note that $c_A$ is a section of $\text{ev}_i$.

Let $f_0, f_1 : A \to B$ be morphisms in $\text{Alg}_H$. We call $f_0$ and $f_1$ \textit{elementary homotopic} if there exists a morphism $h : A \to B[t]$ such that $\text{ev}_i h = f_i$ ($i \in \{0, 1\}$). Note that elementary homotopy is a reflexive and symmetric relation. In general, it is not transitive.

\textbf{Definition 3.1.1.} Let $f, g : A \to B$ be morphisms in $\text{Alg}_H$. We call $f$ and $g$ \textit{homotopic}, and write $f \sim g$, if they can be connected by a chain of elementary homotopies. We denote the set of
homotopy classes of morphisms from \( A \) to \( B \) by \([A, B] \). If now \( A = (A, I) \), \( B = (B, J) \in \text{Alg}_{\mathbb{H}}^{\text{ind}} \), we put
\[
[A, B] = \lim_j \text{colim}_i [A_j, B_i].
\]

Note that there is a natural map \( \hom_{\text{Alg}_{\mathbb{H}}^{\text{ind}}}(A, B) \rightarrow [A, B] \). Two homomorphisms \( f, g : A \rightarrow B \) in \( \text{Alg}_{\mathbb{H}}^{\text{ind}} \) are called homotopic if they have the same image in \([A, B] \).

In order to organize the notion of algebraic homotopy, we introduce a simplicial enrichment of the category of H-algebras. Note that the assignment
\[
Z^n : [n] \mapsto Z^n := \mathbb{Z} [t_0, \ldots, t_n]/\langle 1 - \sum t_i \rangle
\]
defines a simplicial unital ring. Let \( A \) be an H-algebra; put
\[
A^n : [n] \mapsto A \otimes \mathbb{Z} Z^n.
\]

Using this construction, we can enrich the category \( \text{Alg}_{\mathbb{H}} \) over simplicial sets, as follows. We have a mapping space functor \( \hom_{\text{Alg}_{\mathbb{H}}} : (\text{Alg}_{\mathbb{H}})^{\text{op}} \times \text{Alg}_{\mathbb{H}} \rightarrow \mathbb{S} \), given by
\[
(A, B) \mapsto ([n] \mapsto \hom_{\text{Alg}_{\mathbb{H}}} (A, B^n)).
\]

For \( A, B, C \in \text{Alg}_{\mathbb{H}} \), there is a simplicial map
\[
(14) \quad g \circ f := (1_C \otimes \mu)(g^n \circ f)
\]
Here \( g^n \) is the map the functor \((?)^n \) associates to \( g \). Furthermore, for every \( A \in \text{Alg}_{\mathbb{H}} \), the functor \( \hom_{\text{Alg}_{\mathbb{H}}} (?, A) : (\text{Alg}_{\mathbb{H}})^{\text{op}} \rightarrow \mathbb{S} \) has a left adjoint \( A^? : \mathbb{S} \rightarrow (\text{Alg}_{\mathbb{H}})^{\text{op}} \). If \( X \in \mathbb{S} \),
\[
A^X = \lim_{\Delta^n \rightarrow X} A^n = \int_n \prod_{x \in X} A^n.
\]

Here the first limit is taken over the category of simplices of \( X \) ([14], I.2) and the integral sign denotes an end [18, Ch IX, §5]. We have
\[
\text{hom}_{\text{Alg}_{\mathbb{H}}} (A, B^X) = \text{map}_\mathbb{S} (X, \hom_{\text{Alg}_{\mathbb{H}}} (A, B)).
\]

Applying this to \( A = \mathbb{H}[t] \) we obtain
\[
(15) \quad B^X = \text{map}_\mathbb{S} (X, B^\Delta).
\]

Let \( \mathbb{S}_* \) be the category of pointed simplicial sets. For \((K, \ast) \in \mathbb{S}_*\), put
\[
A^{(K, \ast)} := \text{map}_\mathbb{S}_* ((K, \ast), A^\Delta)
\]
(17)
\[
= \text{ker} (\text{map}_\mathbb{S} (K, A^\Delta) \rightarrow \text{map}_\mathbb{S} (\ast, A^\Delta))
\]
\[
= \text{ker} (A^K \rightarrow A).
\]

**Lemma 3.1.2.** Let \( j : K \rightarrow L \in \mathbb{S}, * \in K, \text{ and } A \in \text{Alg}_{\mathbb{H}} \). If \( j \) is a cofibration, then \( A^L \rightarrow A^K \) is surjective, and the sequence
\[
0 \rightarrow A^{(L/K, \ast)} \rightarrow A^{(L, \ast)} \rightarrow A^{(K, \ast)} \rightarrow 0
\]
is exact.
Proof. Under the identification (16), the map $A^L \to A^K$ is identified with $j^* : \text{map}_\mathbb{S}(L, A^\Delta) \to \text{map}_\mathbb{S}(K, A^\Delta)$

Because $A^\Delta$ is a weakly contractible, fibrant simplicial set, $j^*$ is surjective by the LLP of cofibrations ([27]). The remaining assertions of the lemma follow from the snake lemma applied to the following map of exact sequences

$$0 \to A^{(L/K, \ast)} \to A^L \to A^K \to 0$$

Because $A^\Delta$ is a weakly contractible, fibrant simplicial set, $j^*$ is surjective by the LLP of cofibrations ([27]). The remaining assertions of the lemma follow from the snake lemma applied to the following map of exact sequences

$$0 \to A^{(L/K, \ast)} \to A^L \to A^K \to 0.$$

□

Proposition 3.1.3. Let $K$ be a finite simplicial set, $\ast$ a vertex of $K$, and $A$ an $H$-algebra. Then $\mathbb{Z}^K$ and $\mathbb{Z}^{(K, \ast)}$ are free abelian groups, and there are natural isomorphisms

$$A \otimes \mathbb{Z}^K \sim \to A^K \quad A \otimes \mathbb{Z}^{(K, \ast)} \sim \to A^{(K, \ast)}$$

Proof. We consider the unpointed case first. The natural map

$$\eta : A \otimes \mathbb{Z}^K = A \otimes \lim_{\Delta^n \to K} \mathbb{Z}^{\Delta^n} \to \lim_{\Delta^n \to K} A^{\Delta^n}$$

is that induced by $A \otimes \mathbb{Z}^{\Delta^n} = A^{\Delta^n}$. We shall prove by induction on $\dim K$ that if $K$ is finite then $\mathbb{Z}^K$ is free and $\eta$ is an isomorphism for all $A$. If $\dim K = 0$ this is clear. Let $n \geq 0$ and assume both assertions true for all finite simplicial sets of dimension $n$. If $K$ is finite and $\dim K = n + 1$ we have a cocartesian square

$$\begin{array}{ccc}
\Pi_I \Delta^{n+1} & \to & K \\
\downarrow & & \downarrow \\
\Pi_I \partial^{n+1} & \to & \text{sk}^n K
\end{array}$$

where $I$ is a finite set. Applying the functor $A^?$ we get a cartesian square

$$\begin{array}{ccc}
\Pi_I A^{\Delta^{n+1}} & \to & A^K \\
\downarrow & & \downarrow \\
\Pi_I A^{\partial^{n+1}} & \to & A^{\text{sk}^n K}
\end{array}$$

Both vertical arrows are surjective by [3.1.2]. Moreover, because $I$ is finite, $\Pi_I = \bigoplus_I$ in Ab. Hence we have a short exact sequence of abelian groups

$$0 \to A^K \to A^{\text{sk}^n K} \oplus \bigoplus_I A^{\Delta^{n+1}} \to \bigoplus_I A^{\partial^{n+1}} \to 0. \quad (18)$$

Applying this to $A = \mathbb{Z}$, and taking into account that, by induction, $\bigoplus_I \mathbb{Z}^{\partial^{n+1}}$ is free, we get a split exact sequence. Thus $\mathbb{Z}^K$ is a direct summand of a sum of one copy of $\mathbb{Z}^{\text{sk}^n K}$, which is free by induction, and of a finite number of copies of polynomial rings, which are also free. Hence $\mathbb{Z}^K$ is free, and moreover, the sequence

$$0 \to A \otimes \mathbb{Z}^K \to A \otimes \mathbb{Z}^{\text{sk}^n K} \oplus A \otimes \bigoplus_I \mathbb{Z}^{\Delta^{n+1}} \to A \otimes \bigoplus_I \mathbb{Z}^{\partial^{n+1}} \to 0 \quad (19)$$

is exact. The natural map $\eta$ induces a map of exact sequences from (19) to (18). By induction this map is an isomorphism at both the middle and the right hand terms. It follows it is also an isomorphism at the left. This proves the unpointed part of the proposition. The part of the
proposition concerning the pointed case follows from the unpointed one and the fact that the sequence of abelian groups
\[ 0 \to \mathbb{Z}^{(K, \star)} \to \mathbb{Z}^K \to \mathbb{Z} \to 0 \]
is split.

Remark 3.1.4. The exponential law is not satisfied; in general
\[ A^{K \times L} \not\cong (A^K)^L \]
Thus \( A^K \) is not an \( A^K \)-object in the sense of \([27]\) II.1, Def. 3 (see \([27]\) II.1, Prop. 1), and therefore the axioms for a simplicial category in the sense of \([14]\) Def. 2.1 are not satisfied. The failure of the exponential law already occurs when \( K = \Delta^p \) and \( L = \Delta^q \). Indeed,
\[ (A^{\Delta^p})^{\Delta^q} = A^{\Delta^{p+q}}. \]
On the other hand \( \Delta^p \times \Delta^q \) is the amalgamated sum over \( \Delta^{p+q-1} \) of \( \binom{p+q}{q} \) copies of \( \Delta^{p+q} \). But since \( A^I \) has a right adjoint, it maps colimits in \( S \) to colimits in \((\text{Alg}_H)^{op}\), that is, to limits in \( \text{Alg}_H \). In particular, \( A^{\Delta^p \times \Delta^q} \) is the fiber product over \( A^{\Delta^{p+q-1}} \) of \( \binom{p+q}{q} \) copies of \( A^{\Delta^{p+q}} \). For example
\[ A^{\Delta^4 \times \Delta^2} = A^{\Delta^2 \times \Delta_1^3} = A^{\Delta^2} \times_{A^{\Delta^1}} A^{\Delta^2} \not\cong A^{\Delta^2}. \]
The reason for this is that \( A^{\Delta^p} \) is really the ring of functions on the algebro-geometric affine space \( R_{\Delta_p} \), and \( A^p_p \times A^q_q = A^{p+q}_{\Delta_p} \). Thus, with respect to products, affine spaces behave like cubes, not simplices. This seems to indicate that an approach using cubical sets instead of simplicial sets would be appropriate, but we do not follow this line.

3.2. Subdivision. Write \( \text{sd} : S \to S \) for the simplicial subdivision functor (see \([14]\) Ch. III.§4). It comes with a natural transformation \( h : \text{sd} \to 1_S \), which is usually called the last vertex map. We have an inverse system
\[ \text{sd}^n K : \text{sd}^0 K \to \text{sd}^1 K \to \cdots \]
We may regard \( \text{sd}^n K \) as a pro-simplicial set, that is, as an ind-object in \( S^{op} \). The ind-extension of the functor \( A^I : S_I^{op} \to \text{Alg}_H \) maps \( \text{sd}^n K \) to
\[ A^{\text{sd}^n K} = \{ A^{\text{sd}^n K} : n \in \mathbb{N} \}. \]
If we fix \( K \), we obtain a functor \( \text{sd}^* : \text{Alg}_H \to \text{Alg}_H^{\text{ind}} \), which extends to \( \text{sd}^* : \text{Alg}_H^{\text{ind}} \to \text{Alg}_H^{\text{ind}} \) in the usual manner explained in Section 2

Lemma 3.2.1. Let \( A \in \text{Alg}_H^{\text{ind}} \). The functor \( A^{\text{sd}^*} : S^{op} \to \text{Alg}_H^{\text{ind}} \) preserves finite limits.

Proof. Because \( \text{sd} \) has a right adjoint, it preserves all colimits. Similarly if \( B \) is an algebra then \( B^I \) maps colimits in \( S \) to limits in \( \text{Alg}_H \). Thus if \( A = (A, I) \in \text{Alg}_H^{\text{ind}} \), then
\[ A^{\text{sd}^* \text{colim}_I K} = \{ \lim_i A^{\text{sd}^n K} : (i, n) \in I \times \mathbb{N} \}. \]
To finish the proof it suffices to recall (e.g. from \([1]\) A.4) that finite limits in \( \text{Alg}_H^{\text{ind}} \) are computed levelwise.

Let \( A, B \in \text{Alg}_H^{\text{ind}} \). The mapping space of Section 3.1 extends to ind-algebras by
\[ \text{hom}^*_\text{Alg}_H(A, B) := ([n] \mapsto \text{hom}_{\text{Alg}_H^{\text{ind}}}(A, B^{\Delta^n})). \]
We also consider
\[ \text{HOM}^*_\text{Alg}_H(A, B) := ([n] \mapsto \text{hom}_{\text{Alg}_H^{\text{ind}}}(A, B^{\text{sd}^* \Delta^n})). \]
Proposition 3.2.2. Let $A$ and $B$ be ind–$H$-algebras and let $K$ be a finite simplicial set. There is a natural isomorphism
\[ \text{map}_S(K, \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)) = \text{hom}_{\text{Alg}^\text{ind}}(A, B^\text{sd} \ast K). \]

Proof. The adjointness relation is checked as follows.
\[
\text{map}_S(K, \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)) = \text{map}_S(\text{colim}_n \Delta^n \to K, \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)) = \lim_n \Delta^n \to K \text{ map}_S(\Delta^n, \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)) = \lim_n \Delta^n \to K \text{ hom}_{\text{Alg}^\text{ind}}(A, B^{\text{sd} \ast \Delta^n}) = \text{hom}_{\text{Alg}^\text{ind}}(A, \lim_n \Delta^n \to K B^{\text{sd} \ast \Delta^n}) = \text{hom}_{\text{Alg}^\text{ind}}(A, B^{\text{sd} \ast K}).
\]

\[ \square \]

Theorem 3.2.3. Let $A \in \text{Alg}_H$, $(B, J) \in \text{Alg}^\text{ind}_H$. Then
\[ \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B) = Ex^\infty \text{hom}^\bullet_{\text{Alg}^\text{ind}}(A, B). \]
In particular, $\text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)$ is fibrant.

Proof. The following chain of equalities is straightforward.
\[
\text{map}_S(\Delta^k, \text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)) = \text{colim}_{(j, n) \in J \times N} \text{hom}_{\text{Alg}^\text{ind}}(A, B^j, \Delta^n) = \text{colim}_{n \in N} \text{colim}_{j \in J} \text{map}_S(\Delta^n, \text{hom}^\bullet_{\text{Alg}^\text{ind}}(A, B_j)) = \text{colim}_{n \in N} \text{map}_S(\Delta^n, \text{hom}^\bullet_{\text{Alg}^\text{ind}}(A, B_j)) = \text{map}_S(\Delta^k, Ex^\infty \text{hom}_{\text{Alg}^\text{ind}}(A, B)).
\]

\[ \square \]

3.3 Paths and homotopies. Let $f_0, f_1 : A \to B$ be $H$-algebra homomorphisms. A path from $f_0$ to $f_1$ is an ind-homomorphism $h : A \to B^{\text{sd} \ast \Delta^1}$ such that $ev_i(h) = f_i$ ($i = 0, 1$). Thus a path from $f$ to $g$ is a 1-simplex $\text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)$. Note that any path can be represented by a homomorphism $A \to B^{\text{sd} \ast \Delta^1}$ for some $n$, and conversely each such map defines a path. We call a path from $f_0$ to $f_1$ elementary if it can be represented by a map $h : A \to B^{\Delta^1}$. We use the notation $P(f_0, f_1)$ for the set of paths from $f_0$ to $f_1$. The constant path of $P(f, f)$ is the degenerate path $s_0(f)$.

We consider two a priori distinct notions of homotopy between paths. Two paths $h_0, h_1$ from $f$ to $g$ are called cubically homotopic if there is an ind-map $\kappa : A \to B^{\text{sd} \ast \Delta^1} \hat{\otimes} \mathbb{Z}^{\text{sd} \ast \Delta^1}$ such that $(ev_i \otimes 1)(\kappa) = s_0(f_i)$, $(1 \otimes ev_i)(\kappa) = h_i$, ($i = 0, 1$). One checks that cubical homotopy of paths is an equivalence relation. We say that two paths $h_0, h_1$ as above are simplicially homotopic if they are homotopic as 1-simplices of $\text{HOM}^\bullet_{\text{Alg}^\text{ind}}(A, B)$ (Lemma 20.3.1). We shall show that these two notions are equivalent. Consider the free groupoid $\mathfrak{G}$ with set of objects $\text{hom}_{\text{Alg}^\text{ind}}(A, B)$ and one generating arrow $f \to g$ for each elementary path from $f$ to $g$, where we identify a degenerate path $s_0(f)$ with the identity map of the object $f$. Thus the following relation holds in $\mathfrak{G}$

i) $s_0(f) = 1_f$ ($f \in \text{hom}_{\text{Alg}^\text{ind}}(A, B)$).

Write $\mathfrak{G}$ for the quotient of $\mathfrak{G}$ by the following relation

ii) If $\kappa : A \to B[t, u] = B^{\Delta^2}$ then $\kappa(?, 1, t) \circ \kappa(?, t, 0) = \kappa(?, t, 1) \circ \kappa(?, 0, t)$.

Consider also the groupoid $\mathfrak{G}'$ which results from modding out $\mathfrak{G}$ by the following relation

ii') If $\kappa : A \to B[t, u]$ then $\kappa(?, t, 1 - t) \circ \kappa(?, 0, t) = \kappa(?, t, t)$.

Lemma 3.3.1. Let $A, B \in \text{Alg}_H$, $f, g \in \text{hom}_{\text{Alg}^\text{ind}}(A, B)$, and $h_1, h_2$ be paths from $f$ to $g$. Then $h_1$ and $h_2$ are cubically homotopic if and only if they are simplicially homotopic. Moreover there is a groupoid structure with set of objects $\text{hom}_{\text{Alg}^\text{ind}}(A, B)$, where the maps $f \to g$ are the homotopy classes of paths, and composition is given by concatenation. With this structure, $\text{hom}_{\text{Alg}^\text{ind}}(A, B)$ is isomorphic to both $\mathfrak{G}$ and $\mathfrak{G}'$. In particular the latter two groupoids are isomorphic to each other.
Proof. That concatenation is compatible with homotopy is well-known in the simplicial case and straightforward in the cubical case. Consider the map \( \theta : P(f,g) \to \mathfrak{F}(f,g) \), which sends \((h_0, \ldots, h_{2n-1}) : A \to B^{md^r \Delta} \) to the word \( h_{2n-1}^{-1} \circ h_{2n-2} \cdots \circ h_1^{-1} \circ h_0 \). It is clear that \( \theta \) is surjective, and that it sends concatenation to composition. Moreover we claim that \( \theta(h) = \theta(h') \) in \( \mathfrak{G} \) (respectively in \( \mathfrak{G}' \)) if and only if \( h \) and \( h' \) are cubically (simplicially) homotopic. We prove the cubical part of the claim; a similar argument proves the simplicial part. If \( h \) and \( h' \) are cubically homotopic paths then there is an \( n \geq 1 \) and a map \( \kappa : A \to B^{md^r \Delta} \otimes Z^{md^r \Delta} \) such that \((1 \otimes ev_0) \kappa \) represents \( h \), \((1 \otimes ev_1) \kappa \) represents \( h' \) and \((ev_0 \otimes 1) \kappa \) and \((ev_1 \otimes 1) \kappa \) represent respectively \( f := s_0(ev_0(h)) \) and \( g := s_0(ev_1(h')) \). If \( n = 1 \) then, by relations i) and ii) above, \( \kappa \) induces a commutative diagram in \( \mathfrak{G} \)

\[
\begin{array}{ccc}
  f & \theta h' & g \\
  \downarrow & & \downarrow \\
  f & \theta h & g
\end{array}
\]

Hence \( \theta(h) = \theta(h') \). For \( n \geq 2 \), \( \kappa \) induces a larger square of arrows in \( \mathfrak{G} \) made up of several commutative squares stuck together; for example if \( n = 2 \), we get

\[
\begin{array}{ccc}
  f & \theta(h_0) & g \\
  \downarrow & & \downarrow \\
  f & \theta(h_1) & g \\
  \downarrow & & \downarrow \\
  f & \theta(h_2) & g
\end{array}
\]

It follows that the top arrow equals that at the bottom; this proves that \( \theta \) maps cubically homotopic maps to equal maps in \( \mathfrak{G} \). Because \( \theta \) maps concatenation to composition, to prove the converse it suffices to show that if \((h_0, \ldots, h_{2n-1}) \) represents a loop \( h \) based at \( f \) and

\[
h_{2n-1}^{-1} \circ h_{2n-2} \cdots \circ h_1^{-1} \circ h_0 = 1_f
\]

in \( \mathfrak{G} \) then \( h \) is cubically homotopic to \( s_0(f) \). Now \( (21) \) means that the left hand side is equal in \( \mathfrak{G} \) to a composition of arrows of the form \( h_1 \circ h_0' \circ (h_1' \circ h_0)^{-1} \) where \( h_1 = (ev_1 \otimes 1) \kappa \) and \( h_1' = (1 \otimes ev_1) \kappa \) for some \( \kappa : A \to B^{t,u} \). Note strings of concatenable elementary paths which differ by the intercalation of constant (i.e. degenerate) elementary paths represent the same path. On the other hand any two strings without any configurations of the form

\[
\bullet \rightarrow h \rightleftharpoons \bullet
\]

which \( \theta \) maps to the same arrow in \( \mathfrak{G} \) represent the same path. Since \( \kappa(?,t,u) = h(?,t) \) is a homotopy from \( (22) \) to the constant path on \( h(?,0) \), it suffices to show that pruning a string from any configurations of the form

\[
\bullet \rightarrow h_0 \rightleftharpoons \bullet \rightarrow h_1 \rightleftharpoons \bullet \rightarrow h_0
\]

for \( h_1, h_1' \) as above, does not change the path homotopy type. This is straightforward. We show next that \( \mathfrak{G} = \mathfrak{G}' \). Let \( \kappa \) be as in condition ii) above, \( h_i(?,t) = \kappa(?,i,t) \), \( h_i'(?,t) = \kappa(?,t,i) \), \( u(?,t) = \kappa(?,t,1-t) \). Put \( \sigma : \mathbb{Z}[t] \to \mathbb{Z}[t], \sigma(f)(t) = f(1-t) \). Then for \( \tilde{\kappa} := (1 \otimes \sigma \otimes \sigma) \kappa \) we have \( \tilde{\kappa}(?,0,t) = \sigma(h_1')(?,t), \tilde{\kappa}(?,t,0) = \sigma(h_1)(?,t) \) and \( \tilde{\kappa}(?,t,1-t) = \sigma(u(?,t)) \). By ii'), we have the following identities in \( \mathfrak{G}' \):

\[
u \circ h_0 = h_0' \quad \sigma(u) \circ \sigma(h_1) = \sigma(h_1')
\]
On the other hand, if \( h \) is any path, then \( G(?, t, u) = h(?, u) \) satisfies \( G(?, 0, t) = h(?, t) \), \( G(?, t, 1) = h(?, t) \). Hence \( de \) is the inverse of \( h \) in \( \mathfrak{G}' \), by i) and ii). Thus it follows from (23) that the relation ii) holds in \( \mathfrak{G}' \). Conversely, \( \kappa(\?, t, u) := \kappa(\?, t, (1 \cdot t)u) \) satisfies \( \kappa(\?, 0, t) = h_0, \kappa(\?, t, 0) = h_u, \kappa(\?, 1, t) = s_0(\?, 1, 0) \) and \( \kappa(\?, t, 1) = u \). Hence relation ii) holds in \( \mathfrak{G} \). We have proved that \( \mathfrak{G} = \mathfrak{G}' \).

Let \( A \in \text{Alg}_{\text{H}} \). Define inductively

\[
A^{S^1} := A^{(\text{st} \ast S^1, \ast)}, \quad A^{S^{n+1}} := (A^{S^n})^{S^1}.
\]

**Theorem 3.3.2.** Let \( A \in \text{Alg}_{\text{H}}, B \in \text{Alg}_{\text{ind}}^{\text{H}} \). There is a natural isomorphism

\[
[A, B^{S^n}] = \pi_1 \text{HOM}_{\text{Alg}_{\text{H}}^{\text{ind}}}(A, B).
\]

**Proof.** The left hand side is the group of automorphisms of the zero map in the groupoid \( \text{hom}(A, B) \) of the previous Lemma, whose arrows are the cubical homotopy classes of paths. The right hand side is the set of loops based on the zero homomorphism modulo the simplicial homotopy relation. By the lemma, both sides are isomorphic.

**Remark 3.3.3.** It is apparent that the different group structures on \([A, B^{S^n}]\) distribute over each other and share a common neutral element. Therefore, by the Hilton-Eckmann argument, these group structures coincide and are abelian if \( n \geq 2 \).

### 3.4. Rotational homotopies

It was noted by several people in the field that most homotopies appearing in elementary proofs of properties of \( K \)-theoretic invariants in operator algebras are either algebraic or rotational. It is therefore not surprising that many ideas of proofs, which were discovered in computations of the topological \( K \)-theory of operator algebras, have applications to an algebraically homotopy invariant setting, once the rotational homotopies are under control.

For most of the purposes it is enough to construct an invertible matrix \( W \in M_2(\mathbb{Z}[t]) \), such that

\[
\text{ev}_0(W) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{ev}_1(W) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The matrix

\[
W = \begin{pmatrix} 1 - t^2 & t^3 - 2t \\ \frac{1}{t} & 1 - t^2 \end{pmatrix}
\]

is a concrete example.

**Remark 3.4.1.** Note that this matrix \( W \) is not orthogonal, in fact there is no orthogonal matrix with the required properties. Indeed, one easily checks that the existence of an orthogonal matrix with these properties is equivalent to finding a solution to the equation \( A^2 + B^2 = 1 \) in \( \mathbb{Z}[t] \) such that \( A_0 = 0, A_1 = 1, B_0 = 1 \) and \( B_1 = 0 \). Each solution of \( A^2 + B^2 = 1 \) in \( \mathbb{Z}[t] \) provides an invertible element \( A + Bi \in \mathbb{Z}[i, t] \). However it is known that the only units in \( \mathbb{Z}[i, t] \) are \( \pm 1 \) and \( \pm i \).

### 4. Extensions and Algebras

#### 4.1. Matrix algebras

Let \( n \) be a positive integer; write \( M_n \) for the algebra of \( n \times n \) matrices with integer coefficients. If \( n \leq m \), there is a natural inclusion \( t_{n,m} : M_n \rightarrow M_m \) of rings, sending \( M_n \) into the upper left corner of \( M_m \). Thus the sequence \( M_{\bullet} := (M_n)_n \) is an ind-ring; write \( M_\infty \) for its colimit. We also consider the tensor product of \( M_\bullet \) and \( M_\infty \); this is the ind-ring \( M_\infty = M_\bullet M_\infty \).

Let \( A \) and \( B \) be \( \text{ind-H} \)-algebras. Put

\[
[A, B] := [A, M_\infty B].
\]

Here \([\cdot, \cdot]\) denotes homotopy classes of ind-algebra homomorphisms, as defined in 3.2.1. If \( m \in \mathbb{Z}^{N \times N} \) and \( x \in M_\infty \) are matrices, then both \( m \cdot x \) and \( x \cdot m \) are well-defined elements of \( \mathbb{Z}^{N \times N} \). The set

\[
\Gamma^L := \{ m \in \mathbb{Z}^{N \times N} : m \cdot M_\infty \subset M_\infty \}
\]

is a concrete example.
consists of those matrices in $\mathbb{Z}^{N \times N}$ having finitely many nonzero elements in each row and column. Consider the free abelian group

$$\mathbb{Z}^{(N)} = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}.$$  

Note $\Gamma^t$ is a ring (called $\ell\mathbb{Z}$ in [30]) which is isomorphic to the subring of $\text{End}_{\text{Ab}}(\mathbb{Z}^{(N)})$ consisting of those endomorphisms $f$ whose transpose maps $(\mathbb{Z}^*)^{(N)}$ to itself. We remark that if $T \in \Gamma^t$, then for every $p$ there exists an $n$ such that $TM_p \subset M_n \supset M_pT$. Thus both left and right multiplication by $T$ define endomorphisms of the ind-abelian group $M_*$. In particular if $V, W \in \Gamma^t$ satisfy $VW = 1$, then the rule $a \mapsto WaV$ defines (ind-) ring endomorphisms $\psi_{V, W}$ of $M_\infty$.

**Lemma 4.1.1.** Let $V, W \in \Gamma^t$ be such that $VW = 1_{\mathbb{Z}^{[0]}}$, and $\psi = \psi_{V, W}$ and $\psi' = \psi_{V, W}$ as above. Then $\psi : M_\infty \to M_2M_\infty$ is homotopic to $\iota$, and $\psi' : M_* \to M_*$ and $1 \otimes \psi : M_\infty \to M_\infty$ are homotopic to the identity maps.

**Proof.** By [3, 4], $\iota$ is homotopic to the inclusion $\iota'$ in the lower right corner. Thus

$$\iota\psi_{V, W} = \psi_{V \oplus 1, W \oplus 1, \iota'} \sim \psi_{V \oplus 1, W \oplus 1, \iota'}. $$

Hence $\iota(1 \otimes \psi) : M_\infty \to M_2M_\infty$ is homotopic to $\iota$. Moreover the same calculation as above also shows that $\psi\iota'$ is homotopic to $\iota$. Hence it suffices to show that if $A$ and $B$ are ind-rings and $f, g : A \to M_*B$ are homomorphisms such that $f \sim g$, then $f \sim g$. This is immediate from the definition of the homotopy relation between ind-homomorphisms and the fact that $\iota_n, n_{2n}$ is the composite of $\iota \otimes 1_{M_n}$ and the isomorphism $M_2 \otimes M_n \cong M_{2n}$. $\square$

Next we recall two well-known operations in matrix rings. The first is the direct sum of matrices; those ind-homomorphisms $f$ induce one on $M_* \to \text{End}_{\text{Ab}}(\mathbb{Z}^{(N)})$.

**Theorem 4.1.2.** Let $A, B, C \in \text{Alg}_{\text{H}}^{\text{ind}}$. There is an associative composition law

$$\{B, C\} \times \{A, B\} \to \{A, C\}.$$

which makes $\text{Alg}_{\text{H}}^{\text{ind}}$ into a category, where the identity map of an object $A$ is the homotopy class of the inclusion $A \to M_\infty A$. Moreover, direct sum of matrix blocks defines an enrichment of $\text{Alg}_{\text{H}}^{\text{ind}}$ over abelian semigroups.

**Proof.** If $f : A \to M_\infty B$ and $g : B \to M_\infty C$ represent classes $[f] \in \{A, B\}$ and $[g] \in \{B, C\}$, define the composite $[g] [f]$ as the class of the following composite map

$$g \ast f : A \xrightarrow{f} M_\infty B \xrightarrow{1 \otimes g} M_\infty M_\infty C \xrightarrow{\ast 1} M_\infty C$$

Those ind-$H$-algebras which are isomorphic in $\text{Alg}_{\text{H}}^{\text{ind}}$ are called matrix-homotopy equivalent. The assignment $[A, B] \ni f \mapsto f \otimes \iota_\infty \in \{A, B\}$ extends to a functor from the category of ind-$H$-algebras with homotopy classes of morphisms to the category $\text{Alg}_{\text{H}}^{\text{ind}}$. Note that the induced map $[A, B] \to \{A, B\}$ is an isomorphism if $B$ is stable, i.e. isomorphic to $B \otimes M_\infty$. 


4.2. The notion of extension.

Definition 4.2.1. A sequence

\[(25) \quad A \xrightarrow{f} B \xrightarrow{g} C\]

of homomorphisms in \(\text{Alg}_{H}^{\text{ind}}\) is called an extension if \(f\) is a kernel of \(g\) and \(g\) a cokernel of \(f\). This implies that, up to isomorphism of arrows, \(f\) and \(g\) can be represented by level maps \(f_{\alpha}, g_{\alpha}\) such that \(f_{\alpha}\) is kernel of \(g_{\alpha}\) and \(g_{\alpha}\) is a cokernel of \(f_{\alpha}\) \([1.3\text{A.3}])

Definition 4.2.2. An extension of \(H\)-algebras \((25)\) is called \(F\)-split if there exists a section of \(F(g)\) in \(U\).

4.3. Path extension. Put \(\Omega = \mathbb{Z}(S^{1}, \ast)\)

Let \(A\) be an \(H\)-algebra. By \(3.1.3\) and \(3.1.2\), the inclusion \(\partial \Delta^{1} \subset \Delta^{1}\) induces an extension

\[(26) \quad \Omega A \xrightarrow{\Delta^{1}(\text{ev}_{0}, \text{ev}_{1})} A \oplus A\]

Note this extension is naturally \(F\)-split. An \(H\)-linear section of \((\text{ev}_{0}, \text{ev}_{1})\) is \((a_{0}, a_{1}) \mapsto (1-t)a_{0}+ta_{1}\).

4.4. Universal extension. Consider the composite functor \(T := \tilde{T} \circ F : \text{Alg}_{H}^{\text{ind}} \rightarrow \text{Alg}_{H}^{\text{ind}}\). Note that, for \(A \in \text{Alg}_{H}\), the counit map \(\eta_{A} : T(A) \rightarrow A\) is surjective (e.g. because \(F\) is faithful \([18\text{IV.3, Thm 1}]\)). Put \(J(A) := \ker \eta_{A}\). The following extension is called the universal extension of \(A\).

\[
\begin{array}{ccc}
J(A) & \xrightarrow{\iota_{A}} & T(A) \\
& \downarrow \cong & \downarrow \eta_{A} \\
& & A
\end{array}
\]

The name “universal” is justified by the following observation.

Proposition 4.4.1. Let \(A \rightarrow B \rightarrow C\) be an \(F\)-split extension. There exists a commutative diagram of extensions as follows.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \xi & & \downarrow 1_{C} \\
J(C) & \xrightarrow{\cong} & T(C) \\
& \downarrow \eta_{C} & \\
& & C
\end{array}
\]

Furthermore, \(\xi\) is unique up to elementary homotopy. Because of this, we shall abuse notation and refer to any such morphism \(\xi\) as the classifying map of the extension.

Proof. The existence of \(\xi\) is clear from the adjointness of \(F\) and \(T\); its uniqueness up to homotopy is straightforward from the fact that \((26)\) is \(F\)-split. \(\square\)

Note that if an extension is split in \(\text{Alg}_{H}\), then its classifying map can be chosen to be zero.

Proposition 4.4.2. Let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow g \\
A' & \xrightarrow{g} & C'
\end{array}
\]

be a commutative diagram of \(F\)-split extensions. Then there is a diagram

\[
\begin{array}{ccc}
J(C) & \longrightarrow & A \\
\downarrow J(g) & & \downarrow f \\
J(C') & \longrightarrow & A'
\end{array}
\]

of classifying maps, which is commutative up to elementary homotopy.
The following proposition clarifies the relation between classifying maps of extensions and tensor products.

**Proposition 4.4.3.** Let $L$ be a ring, $A \in \text{Alg}_H$ and $\mathcal{U}$ the underlying category. If $\mathcal{U}$ is additive, then the extension

$$J(A)L \to T(A)L \to AL$$

is $F$-split, and there is a choice for the classifying map $\phi_{A,L}: J(AL) \to J(A)L$ of this extension, which is natural in both variables. If $\mathcal{U} = \text{Sets}$ and $L$ is free as an abelian group, then (27) is an extension, and there is a choice of $\phi_{A,L}$ which is natural with respect to the first variable; in the second variable, it is natural with respect to basis-preserving ring homomorphisms.

**Proof.** If $\mathcal{U}$ is additive and $s: A \to T(A)$ the canonical splitting of the universal extension, then $s \otimes 1_L$ splits (27) and is natural in both variables. If $\mathcal{U} = \text{Sets}$ and $L = \bigoplus_i \mathbb{Z}$, then $\bigoplus_i s$ splits (27), and is natural in the first variable, and also in the second if we restrict to basis-preserving homomorphisms. The proposition follows from this. \(\square\)

The following corollary is easily implied by the preceding proposition and Proposition 3.1.3.

**Corollary 4.4.4.** Let $K$ be a finite pointed simplicial set. There is a homotopy class of maps $J(A^K) \to J(A)^K$, natural with respect to $K$, which is represented by a classifying map of the following the extension

$$J(A)^K \xrightarrow{\phi^K_{A,L}} T(A)^K \xrightarrow{\phi^K_{A,L}} A^K.$$

Naturality means that for any map of finite pointed simplicial sets $f: K \to L$, the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
J(A^K) & \xrightarrow{\phi^K_{A,L}} & T(A)^K \\
\downarrow & & \downarrow \\
J(A^K) & \xrightarrow{\phi^K_{A,L}} & A^K
\end{array}
$$

In particular, it follows from the corollary that if $f, g: A \to A'$ and $f \sim g$, then $J(f) \sim J(g)$. Moreover, Proposition 4.4.2 and Corollary 3.1.3 allow us to define the functor $J$ on the level of $\text{Alg}_H^\mathbb{M}$. Indeed, given a homomorphism $f: A \to B \otimes M_\infty$ representing a homotopy class in $\{A, B\}$, we define $J(f) \in \{J(A), J(B)\}$ as the homotopy class of the composite

$$J(A) \xrightarrow{J(f)} J(B \otimes M_\infty) \xrightarrow{\phi_{B,M_\infty}} J(B) \otimes M_\infty.$$

4.5. Loop extension. Write

$$P := \mathbb{Z}^{(\Delta^1, *)} \quad \text{and} \quad \mathcal{P} := \mathbb{Z}^{(\text{sd}^\bullet \Delta^1, *)}.$$

If $A$ is an $H$-algebra, then by 3.1.3 and 3.1.2, we have a diagram of extensions

$$
\begin{array}{ccc}
\Omega A & \xrightarrow{\text{ev}_1} & A \\
\downarrow & & \downarrow \\
\Omega A & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & A \oplus A
\end{array}
$$

$$
\begin{array}{ccc}
\Omega A & \xrightarrow{\text{ev}_0} & A \\
\downarrow & & \downarrow \pi_1 \\
A & \xrightarrow{1} & A
\end{array}
$$

Note that the rightmost column is split in $\text{Alg}_H$. Because (26) is naturally $F$-split, both rows and the middle column are. In particular

$$\Omega A \xrightarrow{\text{ev}_1} A$$

(28)
is naturally $F$-split; we call it the loop extension of $A$. Thus we can pick a natural choice for the classifying map, which we call

$$\rho_A : J(A) \to \Omega A.$$ 

A similar reasoning applies to the extension

$$A^{S^1} \quad \overset{\rho_A}{\longrightarrow} \quad \mathcal{P} A \quad \overset{\text{ev}_1}{\longrightarrow} \quad A$$

Since this is canonically split, it has a natural classifying map $J(A) \to A^{S^1}$, namely the composite $J(A) \overset{\rho_A}{\to} \Omega A \hookrightarrow A^{S^1}$. We abuse notation and call this map $\rho_A$ as well.

4.6. Mapping path extension. Let $f : A \to B$ be a morphism of $H$-algebras. The path extension of $f$ is the extension obtained from the loop extension of $B$ by pulling it back to $A$:

$$\begin{align*}
\Omega B & \quad \overset{1_B}{\longrightarrow} \quad PB \oplus_B A \quad \overset{\text{ev}_1}{\longrightarrow} \quad A \\
\Omega B & \quad \overset{\text{ev}_1}{\longrightarrow} \quad PB \quad \overset{f}{\longrightarrow} \quad B.
\end{align*}$$

It has a natural $H$-linear section because the loop extension does. Its natural classifying map is $\rho_B \circ J(f) : J(A) \to \Omega B$. We write $P_f := PB \oplus_B A$; $P_f$ is the path-algebra of $f$. It comes with a natural evaluation map $\pi_f : P_f \to A$ and a natural inclusion $\iota_f : \Omega B \to P_f$.

Again, similar reasoning applies in the subdivided setting, i.e. there is an extension

$$B^{S^1} \quad \overset{\rho_B}{\longrightarrow} \quad PB \oplus_B A \quad \overset{\pi_f}{\longrightarrow} \quad A .$$

We use the following notation

$$\mathcal{P}_f := \mathcal{P} B \oplus_B A .$$

4.7. Cone extension. Let $\Gamma$ be the ring of $\mathbb{N} \times \mathbb{N}$-matrices with integer coefficients which satisfy the following two properties.

(i) The set $\{a_{ij}, i, j \in \mathbb{N}\}$ is finite.

(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most $N$ nonzero entries.

It is easily checked that $\Gamma$ is indeed a ring with respect to entry-wise addition and the ordinary matrix multiplication law; hence it is a subring of Wagoner’s cone ring $\Gamma^\ell$ considered in (24). In particular $M_\infty \subset \Gamma$ is an ideal; we put

$$\Sigma = \Gamma / M_\infty.$$ 

We note that $\Gamma$, $\Sigma$ are the cone and suspension rings of $\mathbb{Z}$ considered by Karoubi and Villamayor in [16, p.269], where a different but equivalent definition is given. In keeping with our notational conventions, we shall write $\Gamma A$ and $\Sigma A$ for $\Gamma \otimes A$ and $\Sigma \otimes A$. As in [16], one may also consider the ring $\Gamma(A)$ of matrices with entries in $A$ satisfying i) and ii) above, and the quotient $\Sigma(A) = \Gamma(A)/M_\infty A$. There are natural ring homomorphisms

$$\begin{align*}
\Gamma A & \to \Gamma(A), \\
\Sigma A & \to \Sigma(A)
\end{align*}$$

Lemma 4.7.1. The maps (29) are isomorphisms.

Proof. It suffices to show that $\Gamma A \to \Gamma(A)$ is an isomorphism. Call this map $f$. For the purpose of this proof, $\mathbb{N} \times \mathbb{N}$-matrices will be regarded as maps on $\mathbb{N} \times \mathbb{N}$. Write $\pi_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ for each of the projection maps. The characteristic function $\chi_S$ of a subset $S \subset \mathbb{N} \times \mathbb{N}$ is in $\Gamma$ if and only if

$$S \in \Theta := \{T \subset \mathbb{N} \times \mathbb{N} : (\exists N)(\forall n \in \mathbb{N}), \#(\pi_i^{-1}(n) \cap T) \leq N, \quad i = 1, 2\}$$
If $a \in \Gamma A$, then

$$a = \sum_{\lambda \in \text{Im}(a)} \lambda \cdot \chi_{\{i,j):a_{ij} = \lambda\}}$$

$$= f\left( \sum_{\lambda \in \text{Im}(a)} \lambda \otimes \chi_{\{i,j):a_{ij} = \lambda\}} \right)$$

Hence $f$ is surjective. In particular $\Gamma$ is generated as an abelian group by the characteristic functions of the elements of $\Theta$. Thus every element $x$ of $\Gamma A$ can be written as a sum

$$x = \sum_{i=1}^{n} a_i \otimes \chi_{S_i}$$

for some $S_1, \ldots, S_n \in \Theta$ with $S_i \neq S_j$ if $i \neq j$. Put

$$S = \bigcup_{i=1}^{n} S_i.$$  

For each $F \subset \{1, \ldots, n\}$, write

$$S_F = \bigcap_{i \in F} S_i \cap \bigcap_{j \notin F} (S \setminus S_j)$$

Note the $S_F$ are disjoint from one another, and are elements of $\Theta$. Hence

$$\chi_{S_i} = \sum_{i \in F} \chi_{S_F}.$$  

Therefore the element (30) can be rewritten as

$$x = \sum_F \left( \sum_{i \in F} a_i \right) \otimes \chi_{S_F}.$$  

Thus if $f(x) = 0$, then $\sum_{i \in F} a_i = 0$ for all $F$, whence $x = 0$. □

An important property of the cone ring $\Gamma$ is that the sequence

$$0 \rightarrow M_{\infty} \rightarrow \Gamma \rightarrow \Sigma \rightarrow 0$$

is a split exact sequence of free abelian groups. This follows from the theory of Specker groups developed by G. Nöbeling in [23]. Almost by definition $M_{\infty} \subset \Gamma$ is an inclusion of Specker subgroups of the abelian group of bounded maps $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ (see [23]). By [23, Theorem 2], this implies that (31) is a split-exact sequence of free abelian groups. In particular, it follows that for every $H$-algebra $A$,

$$M_{\infty} A \longrightarrow \Gamma A \longrightarrow \Sigma A$$

is an extension, and is split as a sequence of $H$-bimodules.

The rings $\Gamma$ and $\Gamma^\ell$ share several formal properties. For example, Wagoner showed that $\Gamma^\ell$ is an infinite sum ring; the same is true of $\Gamma$, as we shall see in Lemma 4.8.2.

**Remark 4.7.2.** We remark that there is a formally similar definition of $\Gamma$ in [32, Sec. 10] which in fact does not define a ring. The definition in [32, Sec. 10] relaxes condition (ii) by requiring only that the number of nonzero elements in each row and column be finite. The resulting set of matrices is not closed under products. To see this, consider the matrix $A$ given by direct sum of matrix blocks $A_n$, $(n \in \mathbb{N}_{\geq 1})$, such that each block $A_n \in M_n \mathbb{Z}$ has all its entries equal to 1. The matrix $A$ satisfies (i) and has finitely many nonzero entries in each row and column; however $A^2$ does not satisfy (i).
4.8. Infinite sum rings. The notion of sum rings and infinite sum rings was introduced by Wag-
oner in [30]. In the context of operator algebras, similar algebras were introduced and sucessfully studied by Cuntz [6].

Definition 4.8.1.

- A sum $H$-algebra is an algebra $A \in \text{Alg}_H$ together with elements $\alpha_1, \alpha_2, \beta_1, \beta_2$, which satisfy the relations
  \[
  \alpha_1 \beta_1 = \alpha_2 \beta_2 = 1, \quad \beta_1 \alpha_1 + \beta_2 \alpha_2 = 1
  \]
  \[
  [\alpha_i, h] = [\beta_i, h] = 0, \quad (h \in H, \ i = 1, 2).
  \]
- Let $a$ and $b$ be elements in a sum $H$-algebra. We use the notation
  \[
a \oplus b = \beta_1 a \alpha_1 + \beta_2 b \alpha_2.
  \]
- Let $B$ be an $H$-algebra and let $\phi, \psi : B \to A$ be morphisms in $\text{Alg}_H$ into a sum $H$-algebra $A$. We write $\phi \oplus \psi$ for the $H$-algebra homomorphism which sends $B \ni b \mapsto \phi(b) \oplus \psi(b) \in A$.
- An infinite sum $H$-algebra is a sum $H$-algebra $A$ together with a $H$-algebra endomorphism $\phi^\infty : A \to A$ which satisfies
  \[
  1_A \oplus \phi^\infty = \phi^\infty.
  \]

Lemma 4.8.2. Let $A$ be a unital $H$-algebra. Then $\Gamma A$ is an infinite sum $H$-algebra.

Proof. First of all, $\Gamma A$ is a sum $H$-algebra, with
\[
\alpha_1 = \sum_{i=0}^\infty e_{i,2i}, \quad \beta_1 = \sum_{i=0}^\infty e_{2i, i}, \quad \alpha_2 = \sum_{i=0}^\infty e_{i,2i+1}, \text{ and } \beta_2 = \sum_{i=0}^\infty e_{2i+1,i}.
\]

Let $a \in \Gamma A$. Because the map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(k, i) \mapsto 2^{k+1}i + 2^k - 1$, is injective, the following assignment defines gives a well-defined, $\mathbb{N} \times \mathbb{N}$-matrix
\[
\phi^\infty(a) = \sum_{k=0}^\infty \beta_2^k \beta_1 \alpha_1 \alpha_2^k = \sum_{k, i, j} e_{2^{k+1}i+2^k-1,2^{k+1}j+2^k-1} \otimes a_{i,j}.
\]

One checks that in fact $\phi^\infty(a) \in \Gamma A$. Next note that $\alpha_2 \beta_1 = \alpha_1 \beta_2 = 0$. It follows that $\alpha_1 \alpha_2 \beta_2 \beta_1 = \delta_{ij}$ and hence that $\phi^\infty$ is a homomorphism of algebras. The conditions in the definition of infinite sum algebra are straightforward.

The following lemma is recalled from [30].

Lemma 4.8.3. (Wagoner, [30 page 355]) Let $A$ be a sum $H$-algebra. There is an invertible $3 \times 3$-matrix $Q \in M_3 A$ which satisfies
\[
Q \begin{pmatrix} a \oplus b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

for all $a, b \in A$.

4.9. Laurent extension. Write $\mathbb{Z}[t^{\pm 1}]$ for the Laurent polynomial ring. It comes with a natural evaluation map $ev_1 : \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ which sends $t$ to $1$. This surjection is split and defines a split extension as follows.

\[
\sigma A \longrightarrow A[t^{\pm 1}] \overset{ev_1}{\longrightarrow} A.
\]

Here, $\sigma = (t - 1)\mathbb{Z}[t^{\pm 1}] = \ker(ev_1)$. 

4.10. Toeplitz extensions. Let $\tau_H$ be the $H$-algebra which is unital and free on two generators $\alpha$ and $\beta$ satisfying the relations $\alpha \beta = 1$ and $[\alpha, h] = [\beta, h] = 0,$ for $h \in H.$ It is easy to verify that $\tau_H \cong \tau_2 \otimes_{\mathbb{Z}} H,$ as H-algebras. Where no confusion can arise, the subscript $H$ shall be omitted.

The natural map $\tau_2 \to \mathbb{Z}[t^{\pm 1}]$, mapping $\alpha$ to $t$ and $\beta$ to $t^{-1}$, is surjective and its kernel identifies with $M_\infty$ as the following lemma shows. It also follows from classical results of N. Jacobson in [10].

**Lemma 4.10.1.** The kernel of the natural map $\tau \to \mathbb{Z}[t^{\pm 1}]$ is isomorphic to $M_\infty$.

**Proof.** The elements $\hat{\alpha} = \sum_i e_{i,i+1}, \hat{\beta} = \sum_i e_{i+1,i}$ of $\Gamma$ satisfy $\hat{\alpha} \hat{\beta} = 1$. Write $\hat{}$ for the ring homomorphism $\tau \to \Gamma, \alpha \mapsto \hat{\alpha}, \beta \mapsto \hat{\beta}$. Since $1 - \hat{\beta} \hat{\alpha} = e_{11} \in M_\infty$, $\hat{}$ induces a map of extensions

$$\begin{array}{ccc}
\ker \pi & \to & \tau \\
\downarrow & & \downarrow \pi \\
M_\infty & \to & \mathbb{Z}[t^{\pm 1}]
\end{array}$$

As an abelian group, $\tau$ is generated by $\{\beta^p \alpha^q : p, q \geq 0\}$. Since the elements $\hat{\beta}^p \hat{\alpha}^q = \sum_j e_{j+p,j+q}$ are linearly independent, it follows that $\tau \to \Gamma$ is injective. In particular $\ker \pi \to M_\infty$ is injective. On the other hand

$$\hat{\beta}^p \hat{\alpha}^q - \beta^{p+1} \alpha^{q+1} = e_{p,q}$$

This proves that $\ker \pi \to M_\infty$ is surjective, whence an isomorphism. \hfill $\square$

Let $A$ be an $H$-algebra. From the lemma above, we get an extension

$$M_\infty A \longrightarrow \tau A \longrightarrow A[t^{\pm 1}],$$

and an analogous extension

$$M_\infty A \longrightarrow \tau_0 A \longrightarrow \sigma A.$$

In the last line, $\tau_0 = \tau \oplus_A t^{\pm 1} \sigma A$.

4.11. The algebras $q(A)$ and $Q(A)$. Let $A$ be an $H$-algebra; write $Q(A)$ for the coproduct of $A$ with itself in the category $\text{Alg}_H$. Note that there is a natural co-adjunction map $m : Q(A) \to A$. Write $q(A) := \ker m$. The algebra $q(A)$ was introduced by J. Cuntz in his study of $KK$-theory, see e.g. [7]. We have the following extension:

$$\begin{array}{ccc}
q(A) & \longrightarrow & Q(A) \longrightarrow A.
\end{array}$$

Note this extension is split in $\text{Alg}_H$ in two different ways, corresponding to each of the canonical inclusions $A \to Q(A)$. Moreover, one checks that it is universal among such bi-split extensions.

5. Split-exact and $M_2$-stable functors

5.1. Definitions. In this subsection we recall some basic facts about split-exact functors. Most of the results are well-known in the setting of topological algebras (see [7]) and adapt to the algebraic setting in a straightforward way.

A functor $E$ from $\text{Alg}_H$ to an abelian category is called split-exact, if it sends split-extensions in $\text{Alg}_H$ to split-extensions in the abelian category. A functor $E$ from $\text{Alg}_H$ to an abelian category is called $M_2$-stable, if for every $A \in \text{Alg}_H$, the map $E(A) \to E(M_2 A)$ induced by the natural inclusion $i : A \to M_2 A$ is an isomorphism.

**Proposition 5.1.1.** Let $E$ be a split-exact functor and $f, g : A \to B$ morphisms in $\text{Alg}_H$. If $f(a)g(b) = g(a)f(b) = 0$, for all $a, b \in A$, then $(f + g) : A \to B$ is a morphism in $\text{Alg}_H$ and $E(f) + E(g) = E(f + g)$.

The proof of the preceding proposition is an easy consequence of split-exactness.
Proposition 5.1.2. Let $E$ be an $M_2$-stable functor, $B$ an $H$-algebra, $A \subset B$ a subalgebra, and $V, W \in B$ elements such that

$$WA, VA \subset A, \quad aVWa' = aa' (a, a' \in A), \quad [V, H] = [W, H] = 0.$$  

Then

$$\phi^{V, W} : A \to A, \quad a \mapsto WaV$$

is an $H$-algebra homomorphism, and

$$E(\phi^{V, W}) = 1_{E(A)}.$$  

Proof. First of all, note that the inclusions

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$$

induce the same isomorphism $E(A) \to E(M_2A)$. This is true, since on $M_4A$, there are automorphisms of order 2 and 3, mapping the different inclusions onto each other; i.e. $M_2$-stability is used once more.

The argument is finished (as in the proof of Proposition 4.1.1) by noting that $\phi^{V, W}$ in the upper left corner and to $1_A$ in the lower right corner. \[\square\]

Remark 5.1.3. Note that $M_2$-stable functors are $M_n$-invariant for all $n$. Further, we shall presently use 5.1.2 to show that they are also Morita invariant. Let $A, B \subset C$ be subalgebras of an $H$-algebra $C$, $n \geq 1$ and $p, q \in C^n$ such that $[p, H] = [q, H] = 0$, $p_iA, Aq_i \subset B$ $(1 \leq i \leq n)$ and $a(\sum_i q_ip_i)a' = aa'$ $(a, a' \in A)$. Then

$$\xi_{p,q} : A \to M_nB, \quad a \mapsto e_{i,j} \otimes p_iq_j$$

is an $H$-algebra homomorphism. Thus if $E$ is an $M_2$-stable functor, we have a homomorphism

$$E(p, q) := E(\xi_{n})^{-1}E(\xi_{p,q}) : E(A) \to E(M_nB) \cong E(B).$$

In the particular case when $A = B$, for $V = \sum_i e_{i,1} \otimes q_i$ and $W = \sum_i e_{i,j} \otimes p_i$ we have $\xi_{p,q} = \phi^{V, W} \circ \xi_{n}$. Thus in this case $E(p, q)$ is the identity map, by 5.1.2. Now we apply this to show that if $A$ and $B$ are Morita equivalent unital algebras, then $E(A) \cong E(B)$. Suppose the equivalence is implemented by $H$-central bimodules $BPA, AQB$ and isomorphisms $f : P \otimes_A Q \to B$ and $g : Q \otimes_B P \to A$. Let

$$C = \begin{bmatrix} A & Q \\ P & B \end{bmatrix}.$$  

Choose $n, p, p' \in P^n$, $q, q' \in Q^n$ such that $f(\sum_i p_i \otimes q_i) = 1$, $g(\sum_i q_i' \otimes p_i') = 1$. Then by what we have just seen, $E(p, q)$ and $E(q', p')$ are inverse isomorphisms; in particular $E(A) \cong E(B)$. Hence $E$ is Morita invariant. Note that this gives an obvious way of introducing a notion of Morita invariance for nonunital $H$-algebras which will be automatically preserved by $M_2$-invariant functors (see [3] 87 for a similar notion in the topological algebra context).

5.2. Quasi-homomorphisms. The notion of quasi-homomorphism is a formalism which helps organize the extended functoriality properties of split-exact functors. We summarize some basic results from [11] and [7].

Definition 5.2.1. Let $A$ and $B$ be $H$-algebras. A quasi-homomorphism from $A$ to $B$ is a pair of $H$-algebra homomorphisms $\phi, \psi : A \to D$ into an $H$-algebra $D$, which contains $B$ as an ideal, and is such that

$$\phi(a) - \psi(a) \in B \quad (a \in A).$$

We use the notation

$$(\phi, \psi) : A \to D \triangleright B.$$
Example 5.2.2. The canonical inclusions $\iota_i : A \to q(A)$, $i = 1, 2$ define a quasi-homomorphism $(\iota_1, \iota_2) : A \to Q(A) \rhd qA$. This quasi-homomorphism is universal in the sense that for any quasi-homomorphism $(\phi, \psi) : A \to D \rhd B$ there is a unique homomorphism $\phi \ast \psi : Q(A) \to D$ making the obvious diagram commute. The induced map

$$\eta : q(A) \to B$$

is called the classifying map of $(\phi, \psi)$.

The following theorem summarizes the extended functoriality of split-exact functors under quasi-homomorphisms. For a proof, see [11, 3.2].

Theorem 5.2.3. Let $E$ be a split-exact functor from $\text{Alg}_H$ to an abelian category. Furthermore, let $(\phi, \psi) : A \to D \rhd B$ be a quasi-homomorphism. Then, there is an induced map

$$E(\phi, \psi) : E(A) \to E(B)$$

which satisfies the following naturality conditions:

- If $\psi = 0$, then $E(\phi, 0) = E(\phi)$.
- $E(\phi, \psi) = -E(\psi, \phi)$.
- If $\alpha : C \to A$ is an $H$-algebra homomorphism, then

$$E(\phi \alpha, \psi \alpha) = E(\phi, \psi)E(\alpha).$$

- If $\eta : D \to D'$ is an $H$-algebra homomorphism which maps $B$ into an ideal $B' \triangleleft D'$, then

$$E(\eta \phi, \eta \psi) = E(\eta)E(\phi, \psi).$$

The following corollary clarifies in what sense $(\iota_1, \iota_2) : A \to Q(A) \rhd q(A)$ is the universal quasi-homomorphism.

Corollary 5.2.4. Let $(\phi, \psi) : A \to D \rhd B$ be a quasi-homomorphism, $\eta : q(A) \to B$ its classifying map, and $E$ a split-exact functor. Then $E(\phi, \psi) = E(\eta) \circ E(\iota_1, \iota_2)$.

A proof of the following lemma can be found in [11].

Lemma 5.2.5. Let $E$ be a split exact functor. If $\phi, \psi : A \to B$ satisfy $\phi(a)\psi(b) = \psi(a)\phi(b) = 0$, for all $a, b \in A$, then $E(\phi + \psi, \phi) = E(\psi)$.

6. Algebraic $kk$-theory

6.1. Definition of $kk$-theory. Let $A$ and $B$ be ind $-H$-algebras. Let $f : A \to B$ be a morphism in $\text{Alg}_{H}^{\text{ind}}$. There is an associated map $Jf : B^{S^1} \to B$ which is given by $\rho_B Jf$. The map $\rho_B Jf$ is a classifying map for the mapping path extension

$$B^{S^1} \xrightarrow{\rho_B Jf} A.$$

Let us now assume that $A$ and $B$ are constant, i.e. just $H$-algebras. We set:

$$kk(A, B) = \text{colim}_{n \in \mathbb{N}} \{ J^n(A), B^{S^n} \} = \text{colim}_{n \in \mathbb{N}} \{ J^n(A), \mathcal{M}_\infty B^{S^n} \}.$$

Here, the connecting maps in the colimit are given by the assignment described above. Note that $\mathcal{M}_\infty B^{S^n}$ is an ind $-H$-algebra indexed over $\mathbb{N}^{n+1}$. A priori it is possible to write the definition of $kk$ without any mentioning of ind-algebras, but it would involve too many colimits to be understandable at an intuitive level.

Note that the sets appearing in the colimit carry abelian group structures and that the maps in the diagram of the colimit are homomorphisms of abelian groups. It is also easy to see that the semigroup product coming from direct sum of matrix blocks agrees with the group structure described above. This can be proved by giving an explicit rotational homotopy. However, we omit the proof, since it also follows immediately from Proposition 5.1.1 once the split-exactness (see Corollary 6.3.4) is proved. Thus, $kk(A, B)$ is an abelian group.
A priori, \( \mathrm{homomorphism} \) composition does not depend on either \( H \) or \( \mathcal{U} \). For another example, assume \( H \) is commutative, fix \( \mathcal{U} = H\text{-}\text{Bimod} \), and write \( kk^b \) and \( kk^{bc} \) for the a priori different \( kk \)-groups which result from the choices \( \mathrm{Alg}_H = \mathrm{Ass}_H \) and \( \mathrm{Alg}_H = \mathrm{Ass}^c_H \). If \( A \in \mathrm{Ass}^c_H \), then the value of \( T(A) \) is independent of whether we choose \( \mathrm{Alg}_H = \mathrm{Ass}^c_H \) or \( \mathrm{Alg}_H = \mathrm{Ass}_H \), and, moreover, \( T(A) \in \mathrm{Ass}_H \). It follows that for all \( n \), the algebra \( J^n(A) \) is central, and does not depend on the choice of \( \mathrm{Alg}_H \). On the other hand, the inclusion \( \mathrm{Ass}^c_H \subset \mathrm{Ass}_H \) has a right adjoint given by

\[
Z_H(B) = \{ b \in B : [H, b] = 0 \}
\]

Moreover, one checks that \( Z_H(B^{S^n}) = Z_H(B)^{S^n} \). It follows that

\[
(33) \quad kk^b(A, B) = kk^{bc}(A, Z_H B) \quad (A \in \mathrm{Ass}^c_H, \ B \in \mathrm{Ass}_H)
\]

Note that this can be further generalized to noncommutative \( H \), by considering the commutativization \( H_{ab} = H/[H, H] \); details are left to the reader.

### 6.2. Composition product.

We write \( \gamma_A : J(A^{S^1}) \to J(A)^{S^1} \) for the map discussed in Corollary 4.4.4.

**Lemma 6.2.1.** The map \( \rho \) of \([4.3]\) induces a natural transformation \( J(\cdot) \to (\cdot)^{S^1} \) of endofunctors of \( \mathrm{Alg}_H^{\text{ind}} \). That is, if \( f : A \to B \in \mathrm{Alg}_H^{\text{ind}} \), then

\[
f^{S^1} \rho_A = \rho_B J(f) \in \{ J(A), B^{S^1} \}.
\]

**Lemma 6.2.2.** Let \( A \) be an \( \text{ind }-H\text{-algebra} \). Then

\[
\gamma_A J(\rho_A) = \rho_{J(A)} : J^2(A) \to J(A)^{S^1}.
\]

**Theorem 6.2.3.** Let \( A, B \) and \( C \) be \( \text{ind }-H\text{-algebras} \). There is an associative composition product

\[
kk(B, C) \times kk(A, B) \to kk(A, C),
\]

which extends the composition of \( \mathrm{Alg}_H \) homomorphisms.

**Proof.** Let \([\alpha] \in kk(B, C)\) be represented by \( \alpha : J^n(B) \to C^{S^n} \) in \( \mathrm{Alg}_H^M \). Similarly, let \([\beta] \in kk(A, B)\) be represented by \( \beta : J^m(A) \to B^{S^m} \). We define the product to be represented by the composition

\[
J^{n+m}(A) \to J^n(B^{S^m}) \to J^n(B)^{S^m} \to C^{S^{n+m}}.
\]

The fact that this assignment is well-defined in the colimit and also the associativity of the product operation is a formal consequence of Lemma 6.2.1, Lemma 6.2.2 and the naturality of the map \( \gamma_A : J(A^{S^1}) \to J(A)^{S^1} \) discussed in Corollary 4.4.4. \( \square \)

**Definition 6.2.4.** We write \( kk \) for the category whose objects are those of \( \mathrm{Alg}_H \), and where the homomorphisms from \( A \to B \) are given by

\[
\hom_{kk}(A, B) = kk(A, B).
\]

Let us note that there is a sequence of maps

\[
\hom_{\mathrm{Alg}_H}(A, B) \to [A, B] \to \{A, B\} \to kk(A, B)
\]

which extend to functors between the different categories. In particular the composite defines a functor \( j : \mathrm{Alg}_H \to kk \). To alleviate notation, and since \( j \) is the identity on objects, we write \( A \) for \( j(A) \). Moreover, when no confusion can arise, we also write \( f \) for the image \( j(f) \in kk(A, B) \) of a homomorphism \( f \in \hom_{\mathrm{Alg}_H}(A, B) \).

A priori, \( kk \)-theory is only defined for \( H \)-algebras. However, if \( (A, J) \) is an \( \text{ind }-H\text{-algebra} \) for which all structure maps are \( kk \)-equivalences, we can equally well speak about its \( kk \)-groups. In some
Proof. Let \( kk \) be invertible in \( \mathbb{A} \). The statement that two ind-algebras \( B \) and \( C \) are \( kk \)-equivalent, has the rather strict meaning, that all structure maps of \( B \) and \( C \) are \( kk \)-equivalences and all morphisms that constitute the morphism of ind-algebras are \( kk \)-equivalences as well.

6.3. Excision.

Lemma 6.3.1. Let \( f : A \to B \) be a morphism in \( \mathbb{A} \) and let \( C \) be an H-algebra. The sequence

\[
kk(C, \mathcal{P}_f) \xrightarrow{j(f)} kk(C, A) \xrightarrow{j(f)_*} kk(C, B)
\]

is exact in the middle.

Proof. The statement is a rewriting of the definition of being null-homotopic. \( \Box \)

Let \( f : A \to B \) be an F-split surjection of H-algebras. The ind-algebra \( \mathcal{P}_f \) is indexed over \( \mathbb{N} \) and \( \mathcal{P}_{f,n} = B^{d^n} \oplus B A \). The natural inclusions \( \iota_n : \ker (f) \to \mathcal{P}_{f,n} \) assemble to a morphism of H-algebras \( \iota : \ker (f) \to \mathcal{P}_f \).

Lemma 6.3.2. Let \( f : A \to B \) be an F-split surjection. Then the inclusion \( \ker (f) \to \mathcal{P}_{f,n} \) is invertible in kk (\( n \geq 1 \)).

Proof. Let \( n \in \mathbb{N} \) be fixed. There is a commutative diagram of extensions as follows.

\[
\begin{array}{ccc}
\ker (f)^{sd^n} \to P_n & \xrightarrow{\iota} & \ker (f) \\
\downarrow & & \downarrow \\
\ker (f)^{sd^n} \to P_n A & \xrightarrow{\iota} & \mathcal{P}_{f,n} \\
\end{array}
\]

Here, an explicit homomorphism \( PA \to \mathcal{P}_f \) is needed. It is induced from a natural transformation \( (\delta)^{sd^n} \to (\delta \cap \delta)^{sd^n} \). The required natural transformation has a level presentation \( (\delta)^{sd^n} \to (\delta \cap \delta)^{sd^n} \) which we will describe. By Proposition 3.1.3 to give a natural map \( A^{sd^n} \to (A^{sd^n} \cap A^{sd^n}) \) it is sufficient to give \( n \) maps \( \eta_{k,l} : \mathbb{Z}^{sd^n} \to \mathbb{Z}[t_1, t_2] \) for \( 1 \leq k, l \leq n \) which paste together in the correct way. We map the \( i \)-th summand by

\[
\eta_{k,l} (s) = \begin{cases} 
  t_1 & l < i = k \\
  1 - t_1 - t_2 + t_1 t_2 & i = l = k \\
  t_2 & k < i = l.
\end{cases}
\]

One checks that these indeed paste together correctly.

By Proposition 4.3.2 the following diagrams of classifying maps commute up to homotopy.

\[
\begin{array}{ccc}
J(\ker (f)) & \xrightarrow{\rho_{\ker (f)}} & \ker (f)^{sd^n} \\
\downarrow J(\iota) & & \downarrow 1_{\ker (f)^{sd^n}} \\
J(\mathcal{P}_{f,n}) & \xrightarrow{\alpha} & \ker (f)^{sd^n} \\
\end{array}
\quad
\begin{array}{ccc}
J(\mathcal{P}_{f,n}) & \xrightarrow{\alpha} & \ker (f)^{sd^n} \\
\downarrow 1_{\mathcal{P}_{f,n}} & & \downarrow 1_{\mathcal{P}_{f,n}} \\
J(\mathcal{P}_{f,n}) & \xrightarrow{\rho_{\mathcal{P}_{f,n}}} & \mathcal{P}_{f,n}^{sd^n} \\
\end{array}
\]

This together implies that the class \( [\alpha] \in kk(\mathcal{P}_{f,n}, \ker (f)) \) is an inverse of \( j(\iota_n) \). \( \Box \)
It follows from the lemma above that all maps in the inductive system defining $P_f$ are kk-equivalences. Note also, that as far as kk-theory is concerned, there is no need to distinguish between $P_f$ and $P_f$, as long as $f$ is surjective. We will later see that the inclusion $P_f \hookrightarrow P_f$ is a kk-equivalence in general.

Applying the Lemma to the $F$-split surjection $P_n A \xrightarrow{ev^n} A$, we get that the inclusion $A^{sd^n S^1} \rightarrow P_{ev^n}$ is a kk-equivalence. Note that $P_{ev^n} = P_{ev_{1,n}}$, so that, again by the Lemma, we get that the inductive system

$$sd^* A : \Omega A \rightarrow A^{sd^1 S^1} \rightarrow A^{sd^2 S^1} \rightarrow A^{sd^3 S^1} \rightarrow \cdots$$

consists of kk-equivalences. In particular $\iota : \Omega A \rightarrow A^{S^1}$ is a kk-equivalence.

**Corollary 6.3.3.** Let $f : A \rightarrow B$ be a morphism of H-algebras, $\pi_f : P_f \rightarrow A$ the canonical map. The inclusion of ind-H-algebras $B^{S^1} \rightarrow P_{\pi_f}$ is invertible in kk.

**Proof.** The proof is an application of Lemma 6.3.1 in the case of the surjection $B \rightarrow 0$ and $ev : P_{f,n} \rightarrow A$. □

**Corollary 6.3.4.** Let $D$ be an H-algebra. The functor $kk(D, ?)$ is split-exact.

**Proof.** Let

$$A \xrightarrow{g} B \xrightarrow{f} C$$

be a split extension. By Lemma 6.3.1 we know that

$$0 \xrightarrow{kk(D, P_g)} kk(D, B) \xrightarrow{kk(D, C)} 0$$

is exact in the middle. Since the surjection $g : B \rightarrow C$ is split, it is also exact at $kk(D, C)$. By Lemma 6.3.2 the natural inclusion $\iota : A \rightarrow P_g$ is an equivalence in kk, so that $kk(D, P_g)$ identifies with $kk(D, A)$. It remains to show that

$$kk(D, P_g) \xrightarrow{j(\pi_f)} kk(D, B)$$

is injective. By Lemma 6.3.1

$$kk(D, P_{\pi_f}) \xrightarrow{kk(D, P_g)} kk(D, B)$$

is exact in the middle. Furthermore, by Corollary 6.3.3 the inclusion $\Omega C \rightarrow P_{\pi_g}$ is a kk-equivalence. Now, the composition $\Omega C \rightarrow P_g$ factors through $PC$, since $g$ is split. This shows that

$$kk(D, P_{\pi_g}) \xrightarrow{kk(D, P_g)}$$

is zero and thus, by exactness, $kk(D, P_g) \rightarrow kk(D, B)$ has to be injective. This finishes the proof. □

**Corollary 6.3.5.** Let $f : A \rightarrow B$ be an F-split surjection. Consider the natural inclusion $\iota_f : \Omega B \rightarrow P_f$. The natural map $\pi : P_{\iota_f} \rightarrow \Omega A$ is a kk-equivalence.

**Proof.** One shows that $\pi$ is a split-surjection and that the kernel is contractible. Then, the result follows from Corollary 6.3.4 □

**Theorem 6.3.6.** Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be an F-split extension. Then the following sequence is exact.

$$kk(D, \Omega B) \xrightarrow{j(\Omega g)} kk(D, \Omega C) \xrightarrow{\partial} kk(D, A) \xrightarrow{j(f)} kk(D, B) \xrightarrow{j(g)} kk(D, C)$$

Here, the map $\partial : kk(D, \Omega C) \rightarrow kk(D, A)$ is given by the composition of the first map in the diagram below followed by the inverse of the second

$$kk(D, \Omega C) \xrightarrow{kk(D, P_g)} kk(D, A).$$
Proof. This follows immediately from the iteration of Lemma 6.3.1 and the identifications which are possible due to Corollaries 6.3.3 and 6.3.5.

**Theorem 6.3.7.** Let
\[
A \xrightarrow{f} B \xrightarrow{g} C
\]
by an \(F\)-split extension. The following sequence is natural and exact.
\[
\text{kk}(C, D) \xrightarrow{j(g)} \text{kk}(B, D) \xrightarrow{j(f)} \text{kk}(A, D) \xrightarrow{\partial} \text{kk}(\Omega C, D) \xrightarrow{j(\Omega g)} \text{kk}(\Omega B, D)
\]
Here, the map \(\partial : \text{kk}(A, D) \rightarrow \text{kk}(\Omega C, D)\) is given by the composition of the inverse of the first map below followed by the second map
\[
\text{kk}(A, D) \xrightarrow{\sim} \text{kk}(P_\eta, D) \xrightarrow{\sim} \text{kk}(\Omega C, A).
\]

Proof. We start by showing that for any \(F\)-split surjection \(f : A \rightarrow B\), the sequence
\[
\text{kk}(B, D) \xrightarrow{j(f)} \text{kk}(A, D) \xrightarrow{j(\pi f)} \text{kk}(P_\eta, D)
\]
is exact in the middle. Having proved this, the result follows from Corollaries 6.3.3 and 6.3.5.

Let \([\alpha]\) be an element in \(\text{kk}(A, D)\) which vanishes after precomposing with \(j(\pi f)\). Let \([\alpha]\) be represented by \(\alpha : J^n(A) \rightarrow \mathcal{M}_\infty D^{S^n}\). We can choose \(n\) so large, that \(\alpha J^n(\pi f)\) is null-homotopic. This leads to the existence of a commutative diagram of extensions as follows. (Note here, that the functor \(J\) preserves \(F\)-split extensions.)

Consider the composite
\[
\beta : J^{n+1}(B) \rightarrow J^n(\Omega B) \rightarrow \ker (J^n(\pi f)) \rightarrow \mathcal{M}_\infty D^{S^{n+1}}
\]
Write \([\beta]\) for its class in \(\text{kk}(B, D)\). We claim that \(j(f)[\beta] = [\alpha]\). This follows by diagram chasing, using the uniqueness of the classifying map of an extension up to homotopy.

**Lemma 6.3.8.** Let \(A\) and \(B\) be \(H\)-algebras. There is a natural isomorphism
\[
\Omega : \text{kk}(A, B) \xrightarrow{\sim} \text{kk}(\Omega A, \Omega B).
\]

Proof. Let \([\alpha]\) be a class in \(\text{kk}(A, B)\) which is represented by \(\alpha : J^n(A) \rightarrow B^{S^n}\). To it we associate the class of the composition \(J^n(A^{S^1}) \rightarrow J^n(A)^{S^1} \xrightarrow{\alpha^{S^1}} B^{S^{n+1}}\) in \(\text{kk}(A^{S^1}, B^{S^1}) \cong \text{kk}(\Omega A, \Omega B)\). We call this assignment *looping*. Because the following diagram commutes in \(\text{Alg}_{\text{H}}^{\text{ind}}\), looping is well-defined at the level of \(kk\)-theory.

\[
\begin{array}{ccc}
J^{n+1}(A^{S^1}) & \xrightarrow{1} & J(J^n(A)^{S^1}) \xrightarrow{\gamma J^n(A)} J(B^{S^n+1}) \xrightarrow{\nu B^{S^n+1}} B^{S^{n+2}}. \\
J^{n+1}(A^{S^1}) & \xrightarrow{J^n(A)} J^{n+1}(A)^{S^1} & \xrightarrow{J^n(A)} J(B^{S^n})^{S^1} \xrightarrow{J(B^n)} B^{S^{n+2}}.
\end{array}
\]

Indeed, the lower row represents the effect of looping after having stabilized. The upper row represents the stabilization in \(kk\) after looping.
It remains to show that looping induces an isomorphism. Given a class in \( kk(A^{S^1}, B^{S^1}) \) which is represented by \( \beta : J^n(A^{S^1}) \to B^{S^{n+1}} \), we assign to it the class in \( kk(A, B) \) of the composition

\[
J^{n+1}(A) \xrightarrow{J^n(\rho_A)} J^n(A^{S^1}) \xrightarrow{\beta} B^{S^{n+1}}.
\]

Again, this is well defined at the level of \( kk \). We call this procedure delooping. The following diagram shows that delooping after having looped gives stabilization, and hence the identity in \( kk \).

\[
\begin{array}{ccc}
J^n(A^{S^1}) & \xrightarrow{\gamma} & J^n(A)^S^1 \\
\downarrow \rho_j(A) & & \downarrow \rho_j(S^1) \\
J^n(A) & \xrightarrow{1} & J^{n+1}(A) \xrightarrow{J(f)} J(B^{S^n})
\end{array}
\]

Indeed, starting from the lower left corner and going up and right gives the delooping of the loops, going right and up gives stabilization in \( kk \). The squares commute up to homotopy by Lemmas 6.2.2 and 6.2.1. It is similarly shown that delooping first and then looping also gives stabilization. This finishes the proof. \( \square \)

**Lemma 6.3.9.** Looping is compatible with composition. Thus it defines an endofunctor of the category \( kk \), which sends an algebra \( A \) to \( \Omega A \). We use the notation \( \Omega : kk \to kk \).

Lemma 6.3.8 shows that \( \Omega \) is fully faithful. We will see later that \( \Omega \) is also essentially surjective, i.e., an equivalence of categories.

**Lemma 6.3.10.** Let \( A \) be an \( H \)-algebra. The natural map \( \rho_A : J(A) \to \Omega A \) induces a \( kk \)-equivalence.

**Proof.** Note that \( \rho_A \) is part of a natural map between extensions, the universal extension and the path extension. In both extensions, the middle term is contractible and hence isomorphic to zero in \( kk \). The desired result follows from the naturality of the exact sequences associated to an \( F \)-split extension, which were established in Theorem 6.3.7. \( \square \)

The following lemma provides a useful description of an element in \( kk(A, B) \) which we will use in the subsequent sections.

**Lemma 6.3.11.** Let \( [f] \) be an element in \( kk(A, B) \), which is represented by \( f : J^n(A) \to B^{S^n} \). The composite of the first two maps in the following diagram, followed by the inverse of the third, induces \( \Omega^n[f] \) in \( kk \)-theory.

\[
\begin{array}{ccc}
\Omega^n A & \xrightarrow{(\sigma^n_{\Omega})^{-1}} & J^n(A) \\
& & \xrightarrow{f} B^{S^n} \\
& & \sim \Omega^n B
\end{array}
\]

6.4. **The pseudo-inverse to** \( \Omega \). We claimed that the functor \( \Omega \) is essentially surjective. Using the following theorem, we will provide a proof of this.

**Theorem 6.4.1.** Let \( (A, \phi^\infty) \) be a infinite sum \( H \)-algebra, and \( B \subset A \) an ideal such that \( \phi^\infty(B) \subset B \). Then \( B \) is \( kk \)-equivalent to zero.

**Proof.** From Lemma 4.8.3 and Proposition 5.1.2 we get that \( j(1_B \oplus \phi^\infty_B) = j(1_B) + j(\phi^\infty_B) \in kk(B, B) \). On the other hand, by definition of infinite sum \( H \)-algebra, \( j(1_B \oplus \phi^\infty_B) = j(\phi^\infty_B) \), so that \( j(1_B) \) has to vanish. This finishes the proof. \( \square \)

**Corollary 6.4.2.** Let \( A \) be an \( H \)-algebra. Then \( \Sigma A \) is a delooping of \( A \), i.e., there is a natural \( kk \)-equivalence

\[
\Omega \Sigma(A) \sim A.
\]

**Proof.** Immediate from 4.8.2 its proof, and the theorem above. \( \square \)
We use the stabilization isomorphism from Lemma 6.3.8 and Corollary 6.4.2 to define \( \mathbb{Z} \)-graded \( \mathbb{K} \)-groups as follows:

\[
kk_n(A, B) := \begin{cases} 
kk(A, \Omega^n B) & \text{for } n \geq 0 \\
kk(\Omega^{-n} A, B) \cong \kk(A, \Sigma^{-n} B) & \text{for } n < 0.
\end{cases}
\]

In order to organize the properties of \( \kk \), we state that \( \kk \) defines a bivariant homology theory in the sense of [28], i.e. it produces in each variable long exact sequences out of \( F \)-split extensions. In Subsection 6.5 below, we will show that \( \kk \) is indeed a triangulated category with a suitable class of distinguished triangles. This puts all the homological structure in an appropriate framework.

### 6.5. \( \mathbb{K} \) as a triangulated category

For the definition of triangulated category we refer to [17]. The original definition goes back to D. Puppe ([25]) and J.-L. Verdier ([29]), who studied the main motivating examples of triangulated categories, i.e. the stable homotopy category and derived categories. Over the last few years, several papers have appeared, in which triangulated categories of rings or algebras (in particular operator algebras) are considered (e.g. [26, 21, 28]).

Here, we will show that \( \kk \) is a triangulated category.

The definition of triangulated category we shall be using is that of [17]; axioms shall be recalled during the course of the proof of Theorem 6.5.2. This definition is a slight variant of that given for example in A. Neeman’s book [22]; instead of requiring that the endofunctor \( \Sigma \) be invertible, we just require that it be an equivalence of categories. It has been shown (see for example in the Appendix of [19]) that this yields no greater generality, i.e. each triangulated category in this weak sense is triangulated equivalent to an ordinary triangulated category.

The definition of triangulated category is self-dual, i.e. the opposite category of a triangulated category inherits a natural triangulated structure. It is therefore enough to check that \( \kk^{\text{op}} \) is triangulated, which is much more natural in our setting. In turn, checking this amounts to showing that \( \kk \) satisfies the axioms opposite to the usual axioms. These are most naturally stated in terms of a pseudo-inverse \( \Omega \) of \( \Sigma \).

#### Definition 6.5.1.

A diagram

\[
\Omega C \longrightarrow A \longrightarrow B \longrightarrow C
\]

of morphisms in \( \kk \) is called a distinguished triangle, if it isomorphic in \( \kk \) to the path sequence

\[
\Omega B' \xrightarrow{j(x)} P_{\ell} \xrightarrow{i_{(\pi_1)}} A' \xrightarrow{i_{(f)}} B'
\]

associated with a homomorphism \( f : A' \rightarrow B' \) of \( \mathbb{H} \)-algebras.

#### Theorem 6.5.2.

The category \( \kk \) is triangulated with respect to the endofunctor \( \Omega : \kk \rightarrow \kk \) and the class of distinguished triangles specified in Definition 6.5.1.

**Proof.** First of all, we need to show that \( \Omega \) is an equivalence. This follows from Lemma 6.3.8 and Corollary 6.4.2. Next we verify that all the requisite axioms for a triangulated category are satisfied.

**Axiom 6.5.3 (TR0).** Any distinguished triangle which is isomorphic to a distinguished triangle is a distinguished triangle. The triangle

\[
\Omega A \longrightarrow 0 \longrightarrow A \longrightarrow A
\]

is distinguished, for every object \( A \) in \( \kk \).

The first assertion is clear from Definition 6.5.1. To prove the second, consider the path sequence of \( 1_X \) and note that \( P_{1_X} \) is isomorphic to 0 in \( \kk \).
**Axiom 6.5.4 (TR1).** For any morphism $\alpha : A \to B$ in $kk$, there exists a distinguished triangle of the form
\[
\Omega B \to C \to A \xrightarrow{\alpha} B.
\]
Let $\alpha$ be represented by the ind–H-algebra homomorphism $f : J^n(A) \to BS^n$. By Lemma 6.3.11 the following diagram commutes
\[
\Omega^n A \xrightarrow{\Omega^n f} \Omega^n B
\]
\[
\rho^n _\alpha \downarrow \quad \quad \downarrow J^n(A) \xrightarrow{f} BS^n
\]
in $kk$-theory. This implies that $j(\Sigma^n(f))$ is isomorphic to $\alpha$ in the arrow category of $kk$. The path sequence of $\Sigma^n(f)$ provides a sequence which contains $\alpha$ up to isomorphism as desired.

**Axiom 6.5.5 (TR2).** Consider the two triangles
\[
\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C
\]
\[
\Omega B \xrightarrow{-\Omega(h)} \Omega C \xrightarrow{-f} A \xrightarrow{-g} B.
\]
If one is distinguished, then so is the other.

We call the lower triangle the rotation of the upper triangle. It is obvious that the threefold rotation of the path-sequence of $f$ is isomorphic to the path-extension of $\Omega f$. Also, the path-sequence of $\Omega \Sigma f$ is isomorphic to the path-extension of $f$. This implies that the threefold rotation of a triangle is distinguished if and only if the triangle is distinguished. Thus, it suffices to show that the rotation of a distinguished triangle is distinguished.

By Definition 6.5.1 we may assume that the first triangle is equal to
\[
\Omega B' \xrightarrow{j(\cdot)} \Omega P_f \xrightarrow{j(\pi_f)} A' \xrightarrow{j(f)} B'.
\]
The second triangle is then isomorphic to
\[
\Omega A' \xrightarrow{-j(\Omega f)} \Omega B' \xrightarrow{j(\cdot)} \Omega P_f \xrightarrow{j(\pi_f)} A'.
\]
By Lemma 6.3.3 the natural map of H-algebras $\Omega B' \to P_{\pi_f}$ is a $kk$-equivalence and makes the diagram
\[
\Omega A' \xrightarrow{-j(\Omega f)} \Omega B' \xrightarrow{j(\cdot)} \Omega P_f \xrightarrow{j(\pi_f)} A'
\]
commute. This finishes the proof of axiom TR2.

**Axiom 6.5.6 (TR3).** For any commutative diagram
\[
\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C
\]
\[
\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C',
\]
there exists a filler $j : A \to A'$ which makes the whole diagram commutative.
We may assume that the horizontal sequences are path sequences. Furthermore, choosing \( n \) sufficiently large, we may pick representatives of \( l \) and \( k \) of the form \( a : J^n(B) \to \Omega^n B' \) and \( b : J^n(C) \to \Omega^n C' \). Increasing \( n \) even more if necessary, we may assume that the diagram of H-algebras

\[
\begin{array}{ccc}
J^n(B) & \xrightarrow{J^n(h)} & J^n(C) \\
\downarrow{a} & & \downarrow{b} \\
\Omega^n B' & \xrightarrow{\Omega^n h'} & \Omega^n C'
\end{array}
\]

commutes up to homotopy. It follows from the properties of \( \Sigma \) and Lemma 6.3.11 that

\[
\begin{array}{ccc}
\Sigma^n J^n(B) & \xrightarrow{\Sigma^n J^n(h)} & \Sigma^n J^n(C) \\
\downarrow{\Sigma^n a} & & \downarrow{\Sigma^n b} \\
\Sigma^n \Omega^n B' & \xrightarrow{\Sigma^n \Omega^n h'} & \Sigma^n \Omega^n C'
\end{array}
\]

also commutes up to homotopy and is isomorphic in \( kk \) to the right-hand square in the diagram of triangles. It follows that the path sequences of \( \Sigma^n J^n(h) \) and \( \Sigma^n \Omega^n h' \) are isomorphic to those of \( h \) and \( h' \). To finish the proof, choose a null-homotopy for \( P_{\Sigma^n J^n(h)} \to \Sigma^n \Omega^n C' \) and use it to construct a map

\[
P_{\Sigma^n J^n(h)} \to P_{\Sigma^n \Omega^n h'} = P(\Sigma^n \Omega^n B') \oplus_{\Sigma^n \Omega^n C'} \Sigma^n \Omega^n B'.
\]

**Axiom 6.5.7 (TR4).** Let \( \alpha : A \to B \) and \( \beta : B \to C \) be morphisms in \( kk \). We set \( \gamma = \beta \alpha \). There exists a commuting diagram

\[
\begin{array}{cccc}
\Omega^2 C & \xrightarrow{\Omega^2} & \Omega D' & \xrightarrow{\Omega \beta} & \Omega C \\
\downarrow{0} & & \downarrow{1} & & \downarrow{0} \\
\Omega C & \xrightarrow{j} & D' & \xrightarrow{h} & C \\
\downarrow{1} & & \downarrow{1} & & \\
\Omega C & \xrightarrow{1} & D' & \xrightarrow{\alpha} & B \\
& & & \xrightarrow{\beta} & C
\end{array}
\]

in which each row and column is an exact triangle. Furthermore, the square

\[
\begin{array}{ccc}
\Omega B & \xrightarrow{\Omega \beta} & \Omega C \\
\downarrow{j} & & \downarrow{1} \\
D'' & \xrightarrow{h} & D''
\end{array}
\]

commutes in \( kk \).

Like in the proof of TR3, we replace the commuting square in the lower right corner by an isomorphic square which has a lift to a square in the category of H-algebras which commutes up to homotopy. Using a path-algebra, we may require that it strictly commutes.

The lower horizontal sequences are defined to be the path sequences of \( \beta \) and \( \gamma \), i.e. \( D' = P\beta \) and \( D'' = P\gamma \). Let \( \eta : P\gamma \to P\beta \) be the induced map. The square

\[
\begin{array}{ccc}
P\gamma & \xrightarrow{\eta} & A \\
\downarrow{\eta} & & \downarrow{\alpha} \\
P\beta & \xrightarrow{\alpha} & B
\end{array}
\]
commutes, so that there is a natural map between the path sequences of \( \eta \) and \( \alpha \). We put these into the middle vertical sequences. Note that the induced map \( P_\eta \to P_\alpha \) is a split surjection with contractible kernel, hence a \( \text{kk} \)-equivalence, so that we can put \( D'''' = P_\eta \). It is now clear that all rows and columns are exact triangles and all squares in the big diagram commute.

The commutativity of the small diagram follows from the commutativity of

\[
\begin{array}{ccc}
\Omega B & \xrightarrow{\Omega \beta} & \Omega C \\
\downarrow & & \downarrow \\
P_\alpha & \xrightarrow{P_\gamma} & P_\gamma
\end{array}
\]

and the identification \( P_\eta \to P_\alpha \). This finishes the proof of Axiom TR4. \( \square \)

6.6. Universal properties. Let \( \mathcal{E} \) be the class of all \( F \)-split extensions

\[(E) : A \to B \to C\]

in \( \text{Alg}_H \), and \( \mathcal{T} = (\mathcal{T}, \Omega) \) a triangulated category. An excisive homology theory for \( H \)-algebras with values in \( \mathcal{T} \) consists of a functor \( X : \text{Alg}_H \to \mathcal{T} \), together with a collection \( \{ \partial_E : E \in \mathcal{E} \} \) of maps \( \partial_E^X = \partial_E \in \text{hom}_\mathcal{T}(\Omega X(C), X(A)) \). The maps \( \partial_E \) are to satisfy the following requirements.

i) For all \( E \in \mathcal{E} \) as above,

\[
\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)
\]

is a distinguished triangle in \( \mathcal{T} \).

ii) If

\[
(E) : \\
(A) \xrightarrow{\alpha} (B) \xrightarrow{\beta} (C) \\
\downarrow \alpha \downarrow \beta \downarrow \gamma \\
(E') : \\
(A') \xrightarrow{\alpha'} (B') \xrightarrow{\beta'} (C')
\]

is a map of extensions, then the following diagram commutes

\[
\begin{array}{ccc}
\Omega X(C) & \xrightarrow{\partial_E} & X(A) \\
\downarrow \Omega X(\gamma) & & \downarrow X(\alpha) \\
\Omega X(C') & \xrightarrow{\partial_{E'}} & X(A)
\end{array}
\]

We say that the functor \( X : \text{Alg}_H \to \mathcal{T} \) is homotopy invariant if it maps homotopic homomorphisms to equal maps, or equivalently, if for every \( A \in \text{Alg}_H \), \( X \) maps the inclusion \( A \subset A[t] \) to an isomorphism. Call \( X \) \( M_\infty \)-stable if for every \( A \in \text{Alg}_H \), it maps the inclusion \( \iota_\infty : A \to M_\infty A \) to an isomorphism. Note that if \( X \) is \( M_\infty \)-stable, and \( n \geq 1 \), then \( X \) maps the inclusion \( \iota_n : A \to M_n A \) to an isomorphism.

Example 6.6.1. Let \( E \) be the \( F \)-split extension \( (34) \), and \( c_E \in \text{kk}(JC,A) \) the class of the classifying map. Define

\[(35) \quad \partial_E := c_E \circ \rho_A^{-1} \in \text{kk}(\Omega C, A).\]

Then the canonical functor \( j : \text{Alg}_H \to \text{kk} \), together with the collection \( \{ \partial_E \}_{E \in \mathcal{E}} \) is a homotopy invariant, \( M_\infty \)-stable, excisive homology theory in the sense defined above.
The homotopy invariant, $M_\infty$-stable, excisive homology theories form a category, where a homomorphism between the theories $X : \text{Alg}_M \rightarrow T$ and $Y : \text{Alg}_M \rightarrow \Omega$ is a triangulated functor $G : T \rightarrow \Omega$ such that

$$
\begin{array}{ccc}
\text{Alg}_M & \xrightarrow{X} & T \\
\downarrow{y} & & \downarrow{G} \\
\text{Alg}_M & \xrightarrow{Y} & \Omega
\end{array}
$$

commutes, and such that for every extension (35), the natural isomorphism $\phi : G(\Omega X(C)) \rightarrow \Omega Y(C)$ makes the following into a commutative diagram

$$
\begin{array}{ccc}
G(\Omega X(C)) & \xrightarrow{G(\partial^E_n)} & Y(A) \\
\downarrow{\phi} & & \downarrow{\partial^Y_n} \\
\Omega Y(C).
\end{array}
$$

**Theorem 6.6.2.** The canonical functor $j : \text{Alg}_M \rightarrow \text{kk}$, together with the maps (35), is an initial object in the category of all excisive, homotopy invariant and $M_\infty$-stable homology theories.

**Proof.** Let $X : \text{Alg}_M \rightarrow T$ be an excisive, homotopy invariant and $M_\infty$-stable theory. We have to show that there is a unique homomorphism of theories $\tilde{X} : j \rightarrow X$. By (30) we must have $\tilde{X}(A) = X(A)$. Because $X$ is homotopy invariant, the connecting map associated to any $F$-split extension (21) with $B$ contractible must be an isomorphism. It follows from this that $X$ sends each subdivision map $\Omega^n B \rightarrow B^\text{ad} S^n$ to an isomorphism. Further, by (37), $\phi = \partial^{-1}_n$, the inverse of the connecting map associated to the loop extension. Let $\alpha \in \text{kk}(A,B)$ be represented by a homomorphism $J^n A \rightarrow B^\text{ad} S^n \in \{ J^n A, B^\text{ad} S^n \}$. Then $\Omega^n \alpha \in \text{kk}(\Omega^n A, \Omega^n B)$ is the class of the composite of $\rho^{-n} \in \text{kk}(\Omega^n A, J^n A)$ followed by $f$, followed by the inverse $\mu$ of $\Omega^n B \rightarrow B^\text{ad} S^n$. Because $X$ is homotopy invariant and $M_\infty$-stable, it extends uniquely to a functor on the full subcategory of $\text{Alg}_M$ whose objects are the constant ind-algebras; we shall abuse notation and write $X$ for this extension. In particular $X$ is defined on $\mu^{-1} f \rho^{-n}$, and we must have $\tilde{X}(\Omega^n(\alpha)) = X(\mu^{-1} f \rho^{-n})$. Moreover, the condition that $\tilde{X}$ be triangulated determines that for the inverse $\Omega^{-1}$ of the bijection

$$
\Omega : \text{hom}_T(X(A),X(B)) \rightarrow \text{hom}_T(\Omega X(A),\Omega X(B))
$$

we must have

$$
\tilde{X}(\alpha) = \Omega^{-n}(\phi^n \tilde{X}(\Omega^n(\alpha))\phi^{-n}) = \Omega^{-n}(\partial^{-n}_n X(\mu)^{-1} X(f) X(\rho) \rho^{-n} \partial^{-n}_n)
$$

Essentially the same argument as that of the proof of 6.2.3 shows that (38) actually defines a functor $\tilde{X} : \text{kk} \rightarrow T$. It remains to show that the functor $\tilde{X}$ is triangulated. In view of 6.5.1 and of the rotation axiom, this boils down to proving that if $f \in \text{hom}_{\text{Alg}_M}(A,B)$, then $X$ maps

$$
\begin{array}{ccc}
\Omega A & \xrightarrow{\Omega f} & \Omega B \\
\downarrow{\Omega \iota} & & \downarrow{\Omega \pi} \\
P_f & \xrightarrow{\pi} & A
\end{array}
$$

to a distinguished triangle in $T$. Consider the extension $E$ formed by $\iota$ and $\pi$; we have a diagram

$$
\begin{array}{ccc}
\Omega X(B) & \xrightarrow{\Omega X(f)} & X(\Omega B) \\
\downarrow{\partial_1} & & \downarrow{X(\iota)} \\
\Omega X(A) & \xrightarrow{\partial_E} & X(\Omega B) \\
\downarrow{\partial_0} & & \downarrow{X(\pi)} \\
X(\Omega A) & \xrightarrow{X(\Omega f)} & X(A)
\end{array}
$$

In the diagram above the row is a distinguished triangle and the square on the left commutes, as do its upper and lower halves. It follows from this and the axioms of a triangulated category, that $X$ applied to $\mathfrak{U}$ is a distinguished triangle.

Next we give three examples which illustrate the applications of the theorem above.

**Example 6.6.3.** Algebraic $K$-theory is not excisive, nor is it $M_\infty$-stable or homotopy invariant. However there is a variant defined by C. Weibel [31], called homotopy algebraic $K$-theory, and denoted $KH$, which does have all these properties. The groups $KH_*(A)$ are defined as homotopy groups of a certain spectrum. There are several homotopy equivalent choices for this spectrum; one of them is recalled in [8,7]. There is also a variant which yields a functor $KH: \text{Ass} := \text{Ass}\mathbb{Z} \to \text{Sp}_\Sigma$ with values in the category of symmetric spectra (see [28]), and is equipped with suitably compatible products

$$KH^*(A) \land KH^*(B) \to KH^*(A \otimes B)$$

In particular $KH$ maps $\mathbb{H}$-algebras to $KH^*(\mathbb{H})$-modules. Composing with the canonical localization functor to the homotopy category $\mathcal{T}$ of $KH^*(\mathbb{H})$-modules, we get a functor $X : \text{Alg}_H \to \mathcal{T}$ which satisfies the hypothesis of Theorem 6.6.2. Thus we have a natural map

$$kk(A, B) \to KH(A, B) := \left[KH^*(A), KH^*(B)]_{KH^*(\mathbb{H})} := \text{hom}_\mathcal{T}(X(A), X(B)).$$

**Example 6.6.4.** Assume $H$ is a field of characteristic zero. Then bivariant periodic cyclic cohomology ([10]), $(A, B) \mapsto HP^*(A, B)$, is 2-periodic, excisive, homotopy invariant and $M_\infty$-stable in both variables. This bivariant theory is defined by means of a functor $X^\infty$ from $\text{Alg}_H$ to pro-supercomplexes. The latter form a closed model category (see [1] 4.3) whose homotopy category $\mathcal{T}$ is triangulated. One defines

$$HP^*(A, B) := \text{hom}_\mathcal{T}(X^\infty(A), \Omega^*X^\infty(B)).$$

The functor $X^\infty : \text{Alg}_H \to \mathcal{T}$ satisfies the hypothesis of Theorem 6.6.2 ([1]) so that we have a product-preserving Chern character

$$ch_* : kk_*(A, B) \to HP^*(A, B).$$

**Example 6.6.5.** Assume $H$ is commutative, put $\text{Alg}_H = \text{Ass}_H$, and let $\mathfrak{U}$ be the category of $H$-modules (i.e. central bimodules). Then $F$-split extensions remain $F$-split upon tensoring over $H$, so that for every algebra $B$ the functor $\iota \circ (? \otimes H B) : \text{Alg}_H \to kk$ is excisive. On the other hand, it is clear that $\iota \circ (? \otimes H B)$ is homotopy invariant and $M_\infty$-stable. Applying Theorem 6.6.2 we obtain a functor $\text{kk}_H B : kk \to kk$. It follows that there is an associative product

$$kk(A_1, B_1) \otimes kk(A_2, B_2) \to kk(A_1 \otimes_H A_2, B_1 \otimes_H B_2).$$

After dealing with functors into triangulated categories, we now concentrate on homological (i.e. half-exact) functors with values in some abelian category. Let $\mathfrak{A}$ be an abelian category. A functor $G : \text{Alg}_H \to \mathfrak{A}$ is called half exact if for every $F$-split extension [25] the sequence

$$G(A) \to G(B) \to G(C)$$

is exact.

**Theorem 6.6.6.** Let $\mathfrak{A}$ be an abelian category, and $G : \text{Alg}_H \to \mathfrak{A}$ a half exact, homotopy invariant, $M_\infty$-stable, additive functor. Then there exists a unique homological functor $\overline{G} : \text{kk} \to \mathfrak{A}$ such that the following diagram commutes.

$$\begin{array}{ccc}
\text{Alg}_H & \xrightarrow{k} & \text{kk} \\
\downarrow^G & & \downarrow^{\overline{G}} \\
\mathfrak{A} & \xrightarrow{\overline{G}} & \mathfrak{A}
\end{array}$$
Proof. Let \( f : B \to C \) be the surjective map in (33), and \( \iota : \Omega C \to P_f \) the inclusion. A standard argument shows (see for example [32 §21.4]) that \( G \) sends the natural map \( l : A \to P_f \) to an isomorphism, and that for \( \partial_E^G := G(l)^{-1} \circ G(\iota) \), the following sequence is exact

\[
G(\Omega B) \xrightarrow{G(\iota)} G(\Omega C) \xrightarrow{\partial_E^G} G(A) \xrightarrow{f} G(B) \xrightarrow{G(l)} G(C)
\]

Moreover one checks that the map \( \partial_E^G \) corresponding to the loop extension is the identity map \( G(\Omega C) \to G(\Omega C) \). On the other hand, (42) implies that \( \partial_E^G \) must be an isomorphism whenever \( B \) is contractible. In particular this applies to the connecting map \( \partial_E^G \) associated to the universal extension. It also follows from (42) that \( G(\rho) \) is an isomorphism. Hence \( \partial_E^G = G(\epsilon_E)G(\rho)^{-1} \) for every \( F \)-split extension (34). By (35), we must define \( \bar{G}(\partial_E) = \partial_E^G \). Now the argument of the proof of 6.6.2 shows that if \( \alpha \in \text{kk}(A,B) \) then we have a unique way of defining \( G(\Omega^n\alpha) \). Let \( \tau : \Sigma \Omega \to \Omega \Sigma \) be the natural isomorphism, \( \partial_c \in \{ \Omega \Sigma A, A \} \) the connecting map associated to the cone extension. The arguments of the proofs of 6.4.1 and 6.4.2 show that, under the current hypothesis on \( G \), \( G(\tau A) = 0 \). Hence \( \partial_c^G \) is an isomorphism. On the other hand, the class of \( \partial_c \tau \) in \( \text{kk}(\Sigma A, A) \) is a natural isomorphism \( \Omega \Sigma A \to A \). Thus we must have

\[
\bar{G}(\alpha) = \partial_E^G(\tau)\bar{G}(\Omega^n\alpha)\bar{G}(\tau^{-1})(\partial_E^G)^{-1}
\]

One checks that this rule does give a well-defined functor \( \text{kk} \to \mathfrak{A} \). To finish, we must show that \( \bar{G} \) is homological. This amounts to proving that \( G \) maps (39) to an exact sequence. By what we have already seen, the sequence

\[
G(\Omega A) \xrightarrow{G(\iota)} G(\Omega B) \xrightarrow{\partial_E^G} G(P_f) \xrightarrow{G(l)} G(A) \xrightarrow{G(f)} G(B)
\]

is exact everywhere except perhaps at \( G(A) \). Exactness at \( G(A) \) follows by comparing this sequence with the path sequence of the map \( \Sigma \Omega f \).

\[\square\]

**Corollary 6.6.7.** There is a natural action

\[\text{kk}(A, B) \otimes G(A) \to G(B), \quad \alpha \otimes \xi \mapsto \alpha \cdot \xi = \bar{G}(\alpha)(\xi)\]

**Example 6.6.8.** The functor \( KH_0 : \text{Alg}_H \to \text{Ab} \) satisfies the hypothesis of Theorem 6.6.6. Hence we have a natural map

\[
(43) \quad \text{kk}_0(H, A) \to \text{hom}_{ab}(KH_0(H), KH_0(A))
\]

As \( H \) is unital, there is a map \( \ell : \mathbb{Z} \to H \). Composing the map above with

\[
\text{hom}_{ab}(KH_0(H), KH_0(A)) \xrightarrow{\ell} \text{hom}_{ab}(KH_0(\mathbb{Z}), KH_0(A)) \cong KH_0(A)
\]

we obtain a homomorphism \( \text{kk}_0(H, A) \to KH_0(A) \). Applying this to \( \Omega^nA \) we obtain

\[
(44) \quad \text{kk}_n(H, A) \to KH_n(A)
\]

We will show in \( \mathfrak{K} \mathbb{Z} \) that (44) is an isomorphism.

### 6.7. Comparison with Kassel’s bivariant \( K \)-group

C. Kassel has constructed a bivariant abelian group \( K(A, B) \), defined for any pair of central unital algebras \( A, B \in \text{Ass}^c_H \) over a commutative unital ground ring \( H \), with unit-preserving structure maps. He defines \( K(A, B) \) as the Grothendieck group of the exact category \( \text{rep}_H(A, B) \) of those left \( A \otimes H \text{-modules} \) \( B^{op} \)-modules which are finitely generated projective as right \( B \)-modules. The exact structure is given by the short exact sequences of bimodules

\[
(45) \quad 0 \to P' \to P \to P'' \to 0.
\]

We point out that, as \( P'' \) is required to be projective, any exact sequence (45) is split in \( \text{B-Mod} \), and therefore in \( \text{H-Mod} \), so that the added requirement of [15 pp. 378] that (45) remains exact upon tensoring with any \( H \)-module \( V \) is automatic.

We wish to compare Kassel’s group with \( \text{kk} \), whenever both groups are defined. A map \( K(A, B) \to \text{kk}(A, B) \) is constructed as follows. If \( P \in \text{rep}(A, B) \), then there is an \( H \)-algebra homomorphism
Theorem 7.1.1. The map \( v : A \to \text{End}_B(P) \). Choose a finitely generated projective \( B \)-module \( Q \) such that \( P \oplus Q = B^n \) and obtain an inclusion \( \text{End}_B(P) \subset M_n(B) \). Composing with \( v \), we obtain a homomorphism
\[
(46) \quad u(P) : A \to M_n(B)
\]
which defines a class in \( kk(A, B) \). Because \( kk(A, B) \) is \( M_2 \)-stable and split-exact, the construction above preserves isomorphism classes. Moreover, a choice of a \( B \)-linear section of the map \( P \to P'' \) in (46) induces a block matrix decomposition of \( \text{End}_B(P) \) under which
\[
(47) \quad u(P)(A) \subset U := \left[ \begin{array}{cc} \text{End}_B(P') & \text{hom}_B(P', P'') \\ 0 & \text{End}_B(P'') \end{array} \right] \subset \text{End}_B(P).
\]
The projection \( U \to \text{End}_B(P') \oplus \text{End}_B(P'') \) is a surjection with square-zero kernel, and therefore a \( kk \)-equivalence. It follows that we have a group homomorphism
\[
(47) \quad u : K(A, B) \to kk(A, B).
\]
Kassel shows that the tensor product of bimodules defines a \( \mathbb{Z} \)-bilinear composition product
\[
K(A, B) \times K(B, C) \to K(A, C) \quad \text{via} \quad (P, Q) \mapsto P \otimes_B Q.
\]
It is straightforward to check that (47) maps this composition to that of \( kk \). Furthermore, the tensor product over \( H \) induces an external product (15 §3)
\[
(48) \quad K(A_1, B_1) \otimes K(A_2, B_2) \to K(A_1 \otimes_H A_2, B_1 \otimes_H B_2).
\]
In general there is no analogue of this product in \( kk \). However, if \( H \) is the category of \( H \)-modules, then we have the product (11). It is straightforward to show that (47) maps (48) to (11) whenever both products are defined. Finally, Kassel obtains a Chern character (15 §4)
\[
ch' : K(A, B) \to HC^0(A, B).
\]
He defines \( ch' \) as the composite of the trace map with the map (10) induces at the level of cyclic complexes. On the other hand, we have seen in Example 6.6.4 that if \( H \) is a field of characteristic zero we also have a map \( ch : kk(A, B) \to HP^0(A, B) \). It is immediate from the definitions that the canonical map \( HC^0(A, B) \to HP^0(A, B) \) makes the following diagram commute
\[
\begin{array}{ccc}
K(A, B) & \xrightarrow{u} & kk(A, B) \\
\downarrow{ch'} & & \downarrow{ch} \\
HC^0(A, B) & \xrightarrow{ch} & HP^0(A, B).
\end{array}
\]

7. Some computations

In this section we provide some computations in the triangulated category \( kk \). We first compute the \( kk \)-theory of an amalgamated product over a split sub-algebra. Then we deal with graded algebras and nilpotent extensions. The last two subsection contain the fundamental theorem in \( kk \)-theory and a computation for crossed products with \( \mathbb{Z} \).

7.1. Amalgamated free products. Let \( A, B \) be \( H \)-algebras containing a common subalgebra \( C \subset A, B \). Furthermore, the existence of retraction homomorphisms \( \alpha : A \to C \) and \( \beta : B \to C \) is assumed. In this situation, there is a natural map
\[
(49) \quad \vartheta : A \ast_C B \to A \oplus_C B,
\]
which is described by
\[
a \mapsto (a, \alpha(a)) \quad \text{and} \quad b \mapsto (\beta(b), b)
\]
This map was studied in [9], where it was shown to induce an isomorphism in topological algebra \( KK \)-theory. The same argument, which we repeat for sake of completeness, applies in our setting and shows that \( j(\vartheta) \) is invertible.

Theorem 7.1.1. The map \( \vartheta \) of (49) above is a \( kk \)-equivalence.
Proof. First of all, we note that the map \( \vartheta \) fits into a diagram of split extensions as follows.

\[
\begin{array}{ccc}
D_1 & \longrightarrow & A \ast C B \\
\downarrow{\vartheta'} & & \downarrow{\vartheta} \\
D_2 & \longrightarrow & A \oplus_C B \\
\end{array}
\]

Here, \( D_2 \) is equal to the subalgebra of those \((a, b)\), for which \( \alpha(a) = \beta(b) = 0 \). We are done if we can show that \( j(\vartheta') \) is invertible. This is implied by the following Lemma.

Lemma 7.1.2. The map \( \vartheta' : D_1 \rightarrow D_2 \) is a matrix-homotopy equivalence.

Proof. There is a homomorphism of \( H \)-algebras \( \eta : D_2 \rightarrow M_2D_1 \) which maps

\[(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2D_1\]

Note that the matrix \( a \oplus b \) is indeed in \( D_1 \), since \( \alpha(a) = \beta(b) = 0 \), by our assumption that \((a, b) \in D_2 \). The composite \( \eta \vartheta' : D_1 \rightarrow M_2(D_1) \) is induced by the map \( \varphi : A \ast C B \rightarrow M_2(A \ast C B) \) which sends

\[a \mapsto a \oplus \alpha(a) \quad \text{and} \quad b \mapsto \beta(b) \oplus b.\]

Using a rotational homotopy, one shows that \( \varphi \) is homotopic to \( 1_{A \ast C B} \oplus (\alpha \ast \beta) \). Since \( D_1 \) is a direct summand of \( A \ast C B \) in \( \text{kk} \), on which \( \alpha \ast \beta \) vanishes, we conclude that \( j(\eta \vartheta') \) is a \( \text{kk} \)-equivalence. The other composition \( M_2(\vartheta') \eta : D_2 \rightarrow M_2(D_2) \) is described by

\[(a, b) \mapsto (a, 0) \oplus (0, b).\]

It is clear, again by a rotational homotopy, that it is homotopic to the natural inclusion. \( \square \)

7.2. Graded algebras and nilpotent extensions.

Theorem 7.2.1. Let \( A \) be a \( \mathbb{N} \)-graded \( H \)-algebra. The inclusion \( A_0 \rightarrow A \) is a \( \text{kk} \)-equivalence.

Proof. This follows from the well-known fact that the inclusion \( A_0 \rightarrow A \) is an algebraic homotopy equivalence. The homotopy inverse is the projection \( A \rightarrow A_0 \). Indeed, the map \( A \rightarrow A[t] \) which sends an homogenous element \( a_n \in A_n \) to \( a_nt^n \), is a homotopy between the composite \( A \rightarrow A_0 \rightarrow A \) and the identity \( 1_A \). \( \square \)

Definition 7.2.2. An \( H \)-algebra is nilpotent if there exists a positive integer \( m \in \mathbb{N} \) such that \( A^m = 0 \), and \( F \)-nilpotent if it is nilpotent and, in addition, all quotient maps \( A^{2^n} \rightarrow A^{2^n}/A^{2^n+1} \) are \( F \)-split surjections.

Remark 7.2.3. In the particular case when \( \mathcal{U} = \text{Sets} \), and \( F \) the forgetful functor, then every extension is \( F \)-split, and thus an \( F \)-nilpotent algebra is the same as a nilpotent algebra.

Theorem 7.2.4. Let \( A \) be an \( F \)-nilpotent \( H \)-algebra. Then \( A \) is \( \text{kk} \)-equivalent to zero.

Proof. By induction the problem reduces to showing that square-zero algebras are \( \text{kk} \)-equivalent to zero. But any square-zero algebra \( A \) is graded, with \( A = A_1 \), \( A_n = 0 \) if \( n \neq 1 \). The theorem now follows from [7.2.1]. \( \square \)

7.3. The fundamental theorem. Let \( A \) be an \( H \)-algebra. The fundamental theorem provides a computation of the Laurent polynomial ring \( A[t^\pm] \) in the category \( \text{kk} \).

Theorem 7.3.1. Let \( A \) be an \( H \)-algebra. There is a \( \text{kk} \)-equivalence \( A[t^\pm] \cong A \oplus \Sigma A \).
Lemma 7.3.2. Let $\mathcal{A}$ be an abelian category, and $E : \text{Alg}_{\mathcal{A}} \to \mathcal{A}$ a split-exact, $M_\infty$-stable, homotopy invariant functor. Then $E(\tau_0) = 0$.

Proof. Again, the proof is a more or less straightforward modification of the corresponding proof in the topological setting, e.g. [8, Prop. 8.2]. The strategy of the proof is to construct a quasi-homomorphism from $\tau_0$ to $(M_\infty, \tau)^{\Delta_1}$, which is the natural inclusion if evaluated at $t = 0$ and which is zero at $t = 1$. The result then follows by Theorem 7.3.1.

The quasi-homomorphism is constructed as follows. First of all, we define several maps from $\tau_0$ to $\tau \otimes \tau$, using the universal properties of $\tau$.

$$
\psi(\alpha) = \beta \alpha^2 \otimes 1 \\
\phi_1(\alpha) = \beta \alpha^2 \otimes 1 + e \otimes \alpha \\
\phi_2(\alpha) = \alpha \otimes 1 \\
\phi_3(\alpha) = \beta \alpha^2 \otimes 1 + e \otimes 1
$$

Finally, we define invertible elements $u_1$ and $u_2$ in $(\tau \otimes \tau)[t]$ as follows:

$$
u_1 = \beta \alpha^2 \otimes 1 + \left( 1 - t^2 \beta \alpha \atop \frac{t \alpha}{t} \right) \left( t^3 - 2t \atop 1 - t^2 \right)
\nu_2 = \beta \alpha^2 \otimes 1 + \left( 1 - t^2 \beta \alpha \atop \frac{t \alpha}{t} \right) \left( t^3 - 2t \atop 1 - t^2 \right)
\nu_1^{-1} = \beta \alpha^2 \otimes 1 + \left( 1 - t^2 \beta \alpha \atop \frac{t \alpha}{t} \right) \left( t^3 - 2t \atop 1 - t^2 \right)
$$

Here, we are implicitly using the inclusion $M_2 \otimes \tau[t] \to \tau \otimes \tau[t]$. For $i = 1, 2$, we define homotopies $\Phi_i : \tau_0 \to (M_\infty, \tau)^{\Delta_1}$. This finishes the proof, using split-exactness once more.

Remark 7.3.3. In the topological algebraic setting, Theorem 7.3.1 provides a proof of Bott periodicity, since in this case $\Omega$ and $\sigma$ coincide, due to the existence of a logarithm. In the algebraic setting, this is no longer true.
7.4. **Crossed products by \( \mathbb{Z} \).** Let \( A \in \text{Alg}_{\mathbb{H}} \) and \( \sigma : A \to A \) be an automorphism of \( H \)-algebras. We can form the crossed product algebra \( A \rtimes_{\sigma} \mathbb{Z} \). It is defined to be a twisted Laurent polynomial ring as follows. As bimodule over \( H \), \( A \rtimes_{\sigma} \mathbb{Z} \) is isomorphic to \( A \otimes \mathbb{Z}[t, t^{-1}] \). Multiplication is determined by the rule

\[
\tau t^{-1} = \sigma(a).
\]

The analysis of the \( K \)-theory of a crossed product in the operator algebraic setting was first carried out by M. Pimsner and D. Voiculescu in [24]. They established a 6-term exact sequence, relating the operator \( K \)-theory of the crossed product \( A \rtimes_{\sigma} \mathbb{Z} \) to the \( K \)-theory of \( A \). Hereafter, this sequence was called Pimsner-Voiculescu exact sequence. Subsequently the argument of Pimsner and Voiculescu was extended to different settings of topological algebras, e.g. Cuntz has extended the argument to the setting of locally convex algebras in [8]. The proof of the existence of a Pimsner-Voiculescu sequence in our setting can be taken verbatim from Section 14 in [8], subject to suitable adjustments. We will recall the basic steps of the proof for convenience.

**Theorem 7.4.1.** Let \( A \in \text{Alg}_{\mathbb{H}} \) and \( \sigma : A \to A \) an automorphism. There is an distinguished triangle

\[
\Omega A \to A \to A \rtimes_{\sigma} \mathbb{Z}.
\]

**Proof.** Consider the subalgebra \( \tau_{\sigma} \) of \( \tau \otimes (A \rtimes_{\sigma} \mathbb{Z}) \) which is generated by \( 1 \otimes A \), \( \alpha \otimes t \) and \( \beta \otimes t^{-1} \). We get an extension

\[
0 \to M_{\infty} A \to \tau_{\sigma} \to A \rtimes_{\sigma} \mathbb{Z} \to 0.
\]

This \( F \)-split exact sequence induces the distinguished triangle above. The proof consists of two steps. First of all, one shows that the natural map \( \eta : A \to \tau_{\sigma} \) is an equivalence in \( kk \)-theory. Secondly, one identifies the induced maps so that the conclusion of the theorem holds.

The map \( \psi : \tau_{\sigma} \to \tau_{\sigma} \) which is defined by \( a \mapsto (\beta \otimes 1)a(\alpha \otimes 1) \) is part of a quasi-homomorphism \((1_{\tau_{\sigma}}, \psi) : \tau_{\sigma} \to \tau_{\sigma} \triangleright M_{\infty} A \). It can be shown that \((1_{\tau}, \psi) \) is a two-sided inverse of \( \eta : A \to \tau_{\sigma} \). Indeed, this is the content of [8, Prop. 14.1]. The proof can be taken from there, using the results from Lemma 7.3.2. The proof of [8, Prop. 14.2] gives the identification of the \( kk \)-elements as desired. \( \square \)

8. **Comparison with \( KH \)**

8.1. **Homotopy algebraic \( K \)-theory.** Let \( A \) be a unital ring. Put

\[
K(A) := BGl^+(A)
\]

for the connected \( K \)-theory space, and

\[
KV(A) := K(A^{\Delta})
\]

for the realization of the simplicial space \([n] \mapsto K(A^{\Delta^n})\). Set \( K_n(A) = \pi_n K(A) \), \( KV_n(A) = \pi_n KV(A) \) \((n \geq 1)\); these are respectively the Quillen and Karoubi-Villamayor \( K \)-theory groups. Both theories preserve products. In general if \( G \) is any product preserving functor from unital rings to abelian groups, and \( A \) is any ring, we consider its unitalization \( A = A \oplus \mathbb{Z} \) and set

\[
G(A) = \ker (G(A) \to G(\mathbb{Z})).
\]

The Karoubi-Villamayor groups can be equivalently defined as follows; for any ring \( A \) we have

\[
KV_n(A) = \pi_n(BGl(A^{\Delta})) = \pi_{n-1}(Gl(A^{\Delta})) \quad (n \geq 1).
\]

In particular

\[
KV_1(A) = \pi_0(Gl(A^{\Delta}))
\]

\[
KV_{n+1}(A) = KV_1(\Omega^n A) \quad (n \geq 1).
\]
Next we consider the nonconnective $K$-theory spectrum $K(A)$ of a unital ring $A$. To define it one uses the equivalence of connected spaces

$$K(A) \sim \Omega K(\Sigma A)_0$$

and defines $K(A)$ as the spectrum whose $n$th space is

$$nK(A) := \Omega K(\Sigma^{n+1} A)_0.$$

The equivalence (50) induces a map

$$\text{KV}(A) \to \Omega \text{KV}(\Sigma A).$$

Using this map one builds a nonconnective spectrum, the homotopy $K$-theory spectrum $KH(A)$, as follows. One defines $KH(A)$ as the spectrum whose $n$th space is

$$n_{KH}(A) := \Omega K((\Sigma^{n+1} A)^{\Delta}).$$

We put $K_n(A) = \pi_n(K(A))\ K_nH(A) = \pi_nKH(A) (n \in \mathbb{Z})$ for unital $A$, and extend these functors to the nonunital case as explained above. Note that, by definition of stable homotopy, if $A$ is unital we have

$$K_n(A) = \pi_n(K(A))\ K_nH(A) = \pi_nKH(A) (n \in \mathbb{Z})$$

In the next proposition we give an alternative formula for $KH_n(A)$. For this we need some natural maps defined as follows. Let $A$ be any ring. Because $K_1(\Gamma A) = KV_1(\Gamma A) = 0$, the surjection $K_1(\Sigma A) \to KV_1(\Sigma A)$ factors through $K_0(A)$, obtaining an epimorphism

$$K_n(A) \to KV_1(\Sigma A)$$

On the other hand the loop extension gives a map $\partial : K_1(A) \to K_0(\Omega A)$ whose kernel is $\text{Im}K_1(\Gamma A)$. A straightforward verification shows that $\text{Im}K_1(\Gamma A) = \ker(K_1(A) \to KV_1(A))$, so that $\partial(K_1(A)) \cong KV_1(A)$. In particular we have an injection

$$KV_1(A) \hookrightarrow K_0(\Omega A)$$

Applying (56) to $\Sigma A$ and composing with (55) we get a natural map $K_0(A) \to K_0(\Sigma A)$. Iterate this construction and put

$$KH'_n(A) := \text{colim}_r K_0(\Sigma^r \Omega^{n+r} A)$$

**Proposition 8.1.1.** Let $A$ be a ring. Then $KH_*(A) = KH'_*(A)$. 
Proof. It suffices to show that the composite of (56) and of the natural transformation (55) applied to $\Omega A$ coincides with the map used in the colimit (54). We have a commutative diagram

\[
\begin{array}{c}
K_1(A) \xrightarrow{\iota} \overline{KV_1(A)} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K_1(\overline{A}) \xrightarrow{\iota} \overline{KV_1(\overline{A})}
\end{array}
\]

Here $\iota$ is the inclusion $A \to \overline{A}$. The horizontal maps are induced by the comparison map $K \to KV$, and the oblique left to upper right maps are (induced by) the inverse of the edge map of the cone extension in $K$-theory. The vertical maps in the back square as well as $K_1(\overline{A}) \to K_0(\Omega A)$ are induced by the extension

$$\Omega A \to \overline{A} \to \overline{A}$$

and $KV_1(A) \to K_0(\Omega A)$ is as defined above. The homomorphism $KV_1(A) \to KV_1(\Omega A)$ of (55) is the composite shown in the diagram. It is straightforward to check that the composite map $K_0(\Omega A) \to KV_1(\Omega A)$ of the diagram is precisely the natural map (55) applied to $\Omega A$. The proposition follows by diagram chasing.

The following corollary is due to Weibel. We give a new proof using 8.1.1.

**Corollary 8.1.2.** (31) $KH$ satisfies excision and is invariant under nilpotent extensions.

**Proof.** Both assertions follow from the proposition above and properties of nonpositive $K$-theory. □

8.2. Main theorem.

**Theorem 8.2.1.** Let $H$ be a unital commutative ring. Assume $\text{Alg}_H = \text{Ass}_H$, and let $\mathfrak{U}$ be any of the underlying categories considered in Section 2. Write $kk$ for the bivariant $K$-theory constructed from these data. Then the map (43) induces an isomorphism

$$kk_*(H, A) \cong KH_*(A).$$

**Proof.** Let $A$ be an $H$-algebra, $\ell : Z \to H$ the structure map. Composition with $\ell$ induces a bijection

$$\text{hom}_{\text{Alg}_H}(H, A) \xrightarrow{\sim} \text{hom}_{\text{Ass}}(Z, A).$$

This implies the following relation between the coproduct functors $Q$ and $Q_Z$ computed respectively in $\text{Alg}_H$ and $\text{Ass}$:

$$Q(H) = H \otimes Q_Z(Z).$$

It follows that $q(H) = H \otimes q_Z(Z)$. Write $qq(Z, A)$ for the set of all quasi-homomorphisms

$$(e_0, e_1) : Z \to M_\infty A \rightrightarrows M_\infty A.$$

Define a map

$$qq(Z, A) \to K_0(\overline{A}), \quad (e_0, e_1) \mapsto [e_0] - [e_1].$$

The image of (57) is precisely the subgroup $K_0(A) = \ker (K_0(\overline{A}) \to K_0(Z))$. On the other hand, any element of $qq(Z, A)$ gives rise to a ring homomorphism $q_Z(Z) \to M_\infty A$, and thus to an $H$-algebra homomorphism $q(H) \to M_\infty A$. One checks that the map $qq(Z, A) \to kk_0(q(H), A)$ which sends a quasi-homomorphism to the composite of the class of the corresponding map $q(H) \to A$
factors through a group homomorphism \( K_0(A) \to kk_0(q(H), A) \). Composing the latter with the inverse of \( \delta : q(H) \to H \) we obtain
\[
(58) \quad \epsilon : K_0(A) \to kk_0(H, A).
\]
The map \((58)\) induces
\[
\alpha : KH_0(A) = \text{colim}_n K_0(\Sigma^n\Omega^n A) \to \text{colim}_n kk_0(\Omega^n A, \Sigma^n\Omega^n A) = kk_0(H, A).
\]
Conversely, if \( \partial_{ab}, \partial_e \) are the connecting maps corresponding respectively to the universal, loop (or any of its subdivided versions), and cone extensions, and \( e_0 \in K_0(\mathbb{Z}) \) the canonical generator, then the composite of canonical maps
\[
\beta : kk_0(H, A) \to \text{hom}_{ab}(KH_0(H), KH_0(A)) \xrightarrow{\ell'} \text{hom}_{ab}(KH_0(\mathbb{Z}), KH_0(A)) \cong KH_0(A)
\]
sends the class of \( \theta : J^n(H) \to M_\infty \Sigma^nJ^n(H) \) which represents the image of \( \ell(e_0) \) under
\[
\partial_e^{-n}\partial_{\Sigma^n}\theta : KH_0(\mathbb{Z}) \partial_e^{-n}\partial_{\Sigma^n} : KH_0(H) \to KH_0(A).
\]
In particular, if \( A \) is unital and \( E \in M_\infty A \) an idempotent, then the composite \( \beta\alpha \) sends the class of \( E \) to its image in \( KH_0(A) \). By construction of \( \alpha \), this is enough to prove that \( \beta\alpha \) is the identity. To finish the proof it suffices to show that \( \alpha \) is onto. It is clear that the class in \( kk_0(H, A) \) of any map \( H \to M_\infty A \) is in the image of \( \alpha \). Let \( n \geq 0 \) and \( \theta : J^n(H) \to M_\infty \Sigma^nJ^n(H) \) which represents the image of \( \ell(e_0) \) under \( \partial_e^{-n}\partial_{\Sigma^n} : KH_0(H) \to KH_0(\mathbb{Z}) \).

Note \( \alpha(\ell(e_0)) \) is the identity map in \( kk_0(H, H) \); hence we have the following equality in \( kk_0(H, \Sigma^nJ^n(H)) \)
\[
(59) \quad j(\kappa) \cdot j(\delta)^{-1} = \alpha((\partial_e^{-n}\partial_{\Sigma^n})(\ell(e_0))) = \partial_e^{-n}\partial_{\Sigma^n}.
\]
In other words the composite of \( \kappa \) with the inverse of \( \delta : q(H) \to H \) is the invertible excision element \( \in kk_0(H, \Sigma^nJ^n(H)) \).

Remark further that the following diagram commutes in \( kk \)
\[
\begin{array}{ccc}
\Sigma^n J^n(H) & \xrightarrow{\Sigma^n\theta} & \Sigma^n B^{sd\mathcal{P}\mathcal{S}^n} \\
\partial_e^{-n}\partial_{\Sigma^n} & \downarrow & \partial_e^{-n}\partial_{\Sigma^n} \\
H & \xrightarrow{\theta} & B
\end{array}
\]
Put \( \theta' := \Sigma^n\theta\partial_e^{-n}\partial_{\Sigma^n}(\ell(e_0)) \); by \((59)\), the composite of the upward arrow followed by top row is \( \epsilon(\theta') \). Let \( \nu : K_0(\Sigma^nB^{sd\mathcal{P}\mathcal{S}^n}) \to KH_0(B) \) be composite of \( K_0(\Sigma^nB^{sd\mathcal{P}\mathcal{S}^n}) \to KH_0(\mathbb{Z}) \to KH_0(\mathbb{Z}) \) followed by \( \partial_e^{-n}\partial_{\Sigma^n} \). Then
\[
\theta = \partial_e^{-n}\partial_{\Sigma^n}(\epsilon(\nu(\theta'))).
\]

\[\square\]

Remark 8.2.2. Let \( H \) be commutative, \( A \in \text{Ass}_H \). It follows from \((\ref{eq:53})\) and the theorem above that for the \( kk \)-groups \( kk^b \) obtained from the choices \( \text{Alg}_H = \text{Ass}_H \) and \( \mathfrak{A} = H \text{-Bimod} \), we have
\[
kk^b_0(H, A) = KH_*(Z_HA).
\]

Here, as in \((\ref{eq:53})\), \( Z_HA \subset A \) is the subring of those elements which commute with the action of \( H \).

8.3. Bootstrap Category. In this subsection we consider the natural question of whether the group \( kk_*(A, B) \) can be computed in terms of \( KH_*(A) \) and \( KH_*(B) \). For simplicity we restrict to the case \( H = \mathbb{Z} \). The construction of a Künneth spectral sequence can in principle be carried out in the category \( kk \); but the convergence of this spectral sequence is obstructed by a simple observation. Although the symmetric ring spectrum \( KH^*(\mathbb{Z}) \) is a generator (in the triangulated sense, i.e. its suspensions and de-suspensions form a generating set in the sense of \((\ref{eq:22})\) Definition 8.1.1)) of the triangulated category of \( KH^*-\text{module spectra} \), there is a priori no reason to expect that \( \mathbb{Z} \) is a generator for \( kk \). In the similar case of \( KK\)-theory for \( C^*-\text{algebras} \) it is not known
whether $\mathbb{C}$ is a generator for the corresponding triangulated category $KK$. Since the ingredients of the Künneth spectral sequence, i.e.

$$E_2^{p,q} = \text{Ext}_K^{p,q}(KH_*(A), KH_*(B)),$$

can only see $KH^*(\mathbb{Z})$-local information, the same is true for anything to which it could possibly converge. It is fairly easy to see that the passage from $kk$ for the triangulated category of $KH^*(\mathbb{Z})$-module spectra is precisely the colocalization at the subcategory of $\mathbb{Z}$-local objects, i.e. objects $A$ with $KH_*(A) = 0$.

The preceding remarks explain that a suitable Künneth spectral sequence should be constructed in the category of $KH^*(\mathbb{Z})$-module spectra and is in fact already constructed in [12] in a more general setting. The remaining interesting and difficult question is the following: Under what circumstances is the map $kk(A, B) \to [KH^*(A), KH^*(B)]_{KH^*(\mathbb{Z})}$ an isomorphism? The full subcategory formed by those $A$ for which the map above is an isomorphism for all $B$ is called the bootstrap category. It is clear that the map will be an isomorphism for all $B$, whenever $A$ is in the triangulated thick subcategory which is generated by $\mathbb{Z}$. Indeed, this follows just from an inductive construction of this subcategory and the 5-lemma. A further study of the size of the bootstrap category would require a better understanding of the behaviour of $kk_*$ under limits and colimits in the category of $\text{Alg}_{KH}$-algebras. However, the authors have no results in that direction.

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