ON THE RICCI-FLAT METRIC FOR THE
NAVIER-STOKES EQUATIONS

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Abstract

Examples of the Ricci-flat metrics associated with the equations of Navier-Stokes are constructed. Their properties are investigated.

1 Introduction

Properties of solutions of the Navier-Stokes equations to the incompressible fluid can be studied by geometric methods [1-3].

For this purpose we rewrite the NS- equations in equivalent form of conservation laws

\[\begin{align*}
U_t + (U^2 - \mu U_x + P)_x + (UV - \mu U_y)_y + (UW - \mu U_z)_z &= 0, \\
V_t + (UV - \mu V_x)_x + (V^2 - \mu V_y + P)_y + (VW - \mu V_z)_z &= 0, \\
W_t + (UW - \mu W_x)_x + (VW - \mu W_y)_y + (W^2 - \mu W_z + P)_z &= 0, \\
U_x + V_y + W_z &= 0,
\end{align*}\] where \(U, V, W\) and \(P\) are components of the velocity and the pressure of the fluid.

The system of equations (1) can be considered as conditions of equality to zero the Ricci tensor of 14-dimensional space \(D^{14}\) in local coordinates

\[X = (x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n) = (\tilde{x}, t, \eta, \rho, m, \Psi_l), \quad l = 1...7\]
endowed with the Riemann metric

\[\begin{align*}
14 ds^2 &= -2\Gamma^i_{jk}(\tilde{x}, t)\Psi_i dx^j dx^k + 2d\Psi_l dx^l.
\end{align*}\] (2)

The metric (2) is the metric of the Riemann extension [4] of seven-dimensional space \(D^7\) of affine connection in local coordinates \((x, y, z, t, \eta, \rho, m)\) with components of connection \(\Gamma^i_{jk}(\tilde{x}, t)\).

In explicit form it looks as follows

\[\begin{align*}
14 ds^2 &= 2 dx \, du + 2 dy \, dv + 2 dz \, dw + (-V(\tilde{x}, t)v - W(\tilde{x}, t)w - U(\tilde{x}, t)u) \, dt^2 +
\end{align*}\]

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\[ +\left( -u(U(\bar{x}, t))^2 + uP(\bar{x}, t) + u\mu U_x(\bar{x}, t) - vU(\bar{x}, t)V(\bar{x}, t) - U(\bar{x}, t)p \right) \, \, d\eta^2 \\
+ (v \mu U_y(\bar{x}, t) - wU(\bar{x}, t)W(\bar{x}, t) + w \mu U_z(\bar{x}, t)) \, \, d\eta^2 + 2 \, d^2 \eta \xi^+ \\
+ (-uU(\bar{x}, t)V(\bar{x}, t) + vP(\bar{x}, t) - V(\bar{x}, t)p + u\mu V_x(\bar{x}, t) - wV(\bar{x}, t)W(\bar{x}, t)) \, \, dp^2 + \\
+ \left( (v \mu V_y(\bar{x}, t) + w \mu V_z(\bar{x}, t) - v(V(\bar{x}, t))^2 \right) \, \, dp^2 + 2 \, d^2 \rho \chi^+ \\
+ \left( \omega P(\bar{x}, t) - W(\bar{x}, t)p - w(W(\bar{x}, t))^2 + w \mu W_z(\bar{x}, t) + v \mu W_y(\bar{x}, t) \right) \, \, dm^2 + \\
+ \left( -vV(\bar{x}, t)W(\bar{x}, t) + u \mu W_x(\bar{x}, t) - uU(\bar{x}, t)W(\bar{x}, t) \right) \, \, dm^2 + 2 \, dm \, dn + 2 \, dt \, dp. \tag{3} \]

The main property of the space with the metric (3) lies in the fact that it is a Ricci-flat if the functions \( U, V, W \) and \( P \) satisfy the NS-equations (1).

Despite the fact that all scalar invariants of the space \( D^{14} \) are equal to zero its geometric properties can be studied with the help of equations of geodesic and corresponding invariant differential operators.

### 2 Geodesic

The complete system of geodesics of the metric (2) consists of two parts

\[ \ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0, \quad \frac{\delta^2 \Psi_k}{ds^2} + R_{kj}^{i} \dot{x}^i \dot{x}^j \Psi_l = 0, \]

where

\[ \frac{\delta \Psi_k}{ds} = \ddot{\Psi}_k - \Gamma^l_{jk} \dot{\Psi}_l \dot{x}^j. \]

In considered case the first group of equations is

\[ \ddot{x} + 1/2 U(\bar{x}, t) \dot{t}^2 + 1/2 \dot{\eta} U(\bar{x}, t)^2 - 1/2 \dot{\eta}^2 \mu U_x(\bar{x}, t) - 1/2 \dot{\eta}^2 P(\bar{x}, t) + \\
+ 1/2 \dot{\rho}^2 U(\bar{x}, t)V(\bar{x}, t) - 1/2 \dot{\rho}^2 \mu V_x(\bar{x}, t) + 1/2 \ddot{m}^2 U(\bar{x}, t)W(\bar{x}, t) - \\
- 1/2 \ddot{m}^2 \mu W_x(\bar{x}, t) = 0, \]

\[ \ddot{y} + 1/2 V(\bar{x}, t) \dot{t}^2 + 1/2 \dot{\eta}(s) U(\bar{x}, t)V(\bar{x}, t) - 1/2 \dot{\eta}^2 \mu U_y(\bar{x}, t) + 1/2 \dot{\rho}^2 V(\bar{x}, t)^2 - \\
- 1/2 \dot{\rho}^2 \mu V_y(\bar{x}, t) - 1/2 \dot{\rho}(s) P(\bar{x}, t) + 1/2 \ddot{m}^2 V(\bar{x}, t)W(\bar{x}, t) - \\
- 1/2 \ddot{m}^2 \mu W_y(\bar{x}, t) = 0, \]

\[ \ddot{z} + 1/2 W(\bar{x}, t) \dot{t}^2 + 1/2 \dot{\eta}(s)^2 U(\bar{x}, t)W(\bar{x}, t) - 1/2 \dot{\eta}^2 \mu U_z(\bar{x}, t) + 1/2 \dot{\rho}^2 V(\bar{x}, t)W(\bar{x}, t) - \\
- 1/2 \dot{\rho}^2 \mu V_z(\bar{x}, t) + 1/2 \ddot{m}^2 W(\bar{x}, t)^2 - 1/2 \ddot{m}^2 \mu W_z(\bar{x}, t) - 1/2 \ddot{m}^2 P(\bar{x}, t) = 0, \]

\[ \ddot{t} + 1/2 U(x, y, z, t) \dot{t}^2 + 1/2 V(x, y, z, t) \dot{\rho}^2 + 1/2 W(x, y, z, t) \dot{m}^2 = 0, \]

\[ \ddot{\eta} = 0, \quad \ddot{\rho}(s) = 0, \quad \ddot{m}(s) = 0. \tag{4} \]

In particular case of 2D-potential flow

\[ U = \phi_y, \quad V = -\phi_x, \quad W = 0, \quad P = Q(x, y, t) \]

the system (4) takes the form

\[ 2 \ddot{x} + \alpha_1^2 \dot{\phi}_y^2 - \alpha_1^2 \mu \phi_{xy} - \alpha_1^2 Q - \alpha_2^2 \phi_y \phi_x + \alpha_2^2 \mu \phi_{xx} = 0, \]
\[2\ddot{y} - \phi_x t^2 - \alpha_1^2 \phi_y \phi_x - \alpha_1^2 \mu \phi_{yy} + \alpha_2^2 \phi_x^2 + \alpha_2^2 \mu \phi_{xy} - \alpha_2^2 Q = 0,\]
\[2 \ddot{z} - Q \alpha_3^2 = 0, \quad 2 \ddot{t} + \phi_y \alpha_1^2 - \phi_x \alpha_2^2 = 0,\]
\[\eta(s) = \alpha_1 s, \quad \rho(s) = \alpha_2 s, \quad m(s) = \alpha_3 s.\]

**Remark.** The coefficients of the system (4) are the components \(\Gamma_{ij}^k\) of affine connection of the seven-dimensional manifold in the local coordinates \((\vec{x}, t, \eta, \rho, m)\).

It is a Ricci-flat
\[R_{ij} = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{kl}^i \Gamma_{kj}^l - \Gamma_{im}^k \Gamma_{kj}^m = 0\]
on solutions of the NS-equations (1) and its properties can be studied independently of the enclosing 14-dimensional Riemann space with the metric (3).

Linear part of geodesic has the form of linear system of equations with variable coefficients
\[\ddot{\Psi}_i = A_i^k \dot{\Psi}_k + B_i^k \Psi_k,\]
where \(\Psi_k = [u, v, w, p, \xi, \chi, n]\) is vector-functions and \(A_i^k = A_i^k(\vec{x}, t)\) and \(B_i^k = B_i^k(\vec{x}, t)\) are the matrix-functions depending on coordinates \(X^a = (\vec{x}, t)\).

Properties of the seven-dimension of the space of affine connection can be studied with the help of solutions of a system of linear partial differential equations with respect to the components of the vector of motions \(\omega^k(\vec{x}, t, \eta, \rho, m)\) of the space
\[\partial^2_{bc} \omega^a + \omega^k \partial_k \Gamma_{bc}^a + \partial_b \omega^k \Gamma_{kc}^a + \partial_c \omega^k \Gamma_{bk}^a - \partial_k \omega^a \Gamma_{bc}^k = 0.\]

### 3 Parameters of Beltrami

The functions of coordinates \(\psi(x^k)\) which defined by the formulas
\[\Delta_2 \psi = g^{ij} \left( \frac{\partial^2 \psi}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \psi}{\partial x^k} \right),\]
and
\[\Delta_1 \psi = g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j}\]
are the invariants of the space.

Solutions of the equations
\[\Delta_2 \phi = 0, \quad \Delta_1 \phi = 0\]
are used to the study properties of solutions of the NS -equations.

As an example, consider the two-dimensional potential flow of fluid.

The metric of associated Riemann space in this case has the form
\[ds^2 = 2 dx du + 2 dy dv + 2 dz dw + (-\phi_y u + \phi_x v) dt^2 + 2 dt dp +\]
\[+ \left( -u \phi_y^2 + u \mu \phi_{xy} + u Q(x, y, t) + v \phi_y \phi_x + v \mu \phi_{yy} - \phi_y P \right) d\eta^2 +\]
\[+ 2 d\eta d\xi + \left( u \phi_y \phi_x - u \mu \phi_{xx} - v \phi_x^2 - v \mu \phi_{xy} + v Q(x, y, t) + \phi_x P \right) d\rho^2 +\]
\[+ 2 d\rho d\chi + w Q(x, y, t) dm^2 + 2 dm dn.\]
Components of the Ricci-tensor of the metric (7) are equal to zero \( R_{\eta\eta} = 0, R_{\rho\rho} = 0 \) on solutions of 2DNS-equations
\[ \phi_y \phi_{xy} - \mu \phi_{xxy} - Q_x - \phi_{yy} \phi_x - \mu \phi_{yy} + \phi_{yt} = 0, \]
\[ -\phi_y \phi_{xx} + \mu \phi_{xxx} + \mu \phi_{xyy} - Q_y - \phi_{xt} + \phi_{xy} \phi_x = 0, \]
where the function \( \phi(x, y, t) \) satisfies the condition of compatibility
\[ (\phi_{xx} + \phi_{yy})_t + \phi_y(\phi_{xx} + \phi_{yy})_x - \phi_x(\phi_{xx} + \phi_{yy})_y - \mu \Delta(\phi_{xx} + \phi_{yy}) = 0. \] (8)

Here is an example of the solution of equation
\[ g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} - 1 = 0. \] (9)

As is known [5] with the help of solutions equation (9) can be studied geodesics of the metric for arbitrary Riemann space.

For the metric (7) the equation (9) takes the form
\[ 2 \psi_x \psi_u + 2 \psi_y \psi_v + 2 \psi_z \psi_w + 2 \psi_t \psi_p + 2 \psi_t \psi_x + 2 \psi_z \psi_{\xi} + 2 \psi_m \psi_n + \psi^2 \phi_y u - \]
\[ -\psi^2 \phi_x v + \psi^2 \phi_y u - \psi^2 \mu \phi_{xy} - \psi^2 \mu \phi_{xy} - \psi^2 \mu \phi_{xy} + \psi^2 \phi_y p - \]
\[ -\psi x \phi_y \phi_x + \psi^2 \mu \phi_{xx} + \psi^2 \mu \phi_{xx} + \psi^2 \mu \phi_{xx} - \psi^2 \mu \phi_{xx} p - w Q \psi^2 - 1 = 0. \] (10)

After separation of variables in the equation (10) we find
\[ \psi(x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n) = c_3 z + c_6 \eta + c_6 \rho + c_7 m + c_1 \xi + \]
\[ + c_1 \chi + c_1 n + F(x, y, t, u, v, w, p) \]
where the function \( F(x, y, t, u, v, w, p) \) satisfies the equation
\[ 2 F_x F_u + 2 F_y F_v + 2 c_3 F_w + 2 F_t F_p + 2 c_5 c_12 + 2 c_6 c_13 + 2 c_7 c_14 + \]
\[ + F_p^2 \phi_y u - F_x^2 \phi_x v + c_12^2 u \phi_y^2 - c_12^2 u \phi_{xy} - c_12^2 u Q - c_12^2 v \phi_y \phi_x - \]
\[ -c_12^2 v \mu \phi_{yy} + c_12^2 \phi_y p - c_13^2 u \phi_y \phi_x + c_13^2 u \phi_{xx} + c_13^2 v \phi_y^2 + \]
\[ + c_13^2 v \mu \phi_{xy} - c_13^2 Q - c_13^2 \phi_x p - w Q c_14^2 - 1 = 0. \] (11)

Using presentation the function \( F(x, y, t, u, v, w, p) \) in the form
\[ F(x, y, t, u, v, w, p) = A(x, y, t) p + u B(x, y, t), c_14 = 0, c_13 = 0, c_12 = 1/2 c_5^{-1} \]
we get over determinant system of equations to the functions \( A(x, y, t), B(x, y, t) \)
\[ 8 B c_5^2 B_x + 8 A c_5^2 B_t + 4 A^2 \phi_y c_5^2 + \phi_y^2 - \mu \phi_{xy} - Q = 0, \]
\[ 8 B c_5^2 A_x + 8 A c_5^2 A_t + \phi_y = 0, -4 A^2 \phi_x c_5^2 - \phi_y \phi_x - \mu \phi_{yy} = 0. \] (12)

From conditions of compatibility we find that the system (12) has solutions if the functions \( \phi(x, y, t), Q(x, y, t) \) satisfy to the relation
\[ H(\phi, \phi_x, \phi_y, \phi_t, ... Q) = 0 \] (13)
containing 275 summands.

In particular, for the Euler system of equations, \( (\mu = 0) \) the relation (13) takes the form
\[ \phi_y t \phi_{xy} \phi_y - 4 \phi_y t \phi_y \phi_{xy} \phi_{xy} = 2 \phi_y \phi_{xxy} \phi_{yy} - 4 \phi_y t \phi_y \phi_{xy} - \]
\[ -3 \phi_y t \phi_{xy}^2 + 4 \phi_y t \phi_{xy} \phi_{xy} + 2 \phi_y t \phi_{xy} \phi_{yy} t - Q \phi_{xy}^2 = 0. \]
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