ANALYTICAL SIGNATURES AND PROPER ACTIONS

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Abstract. In this paper we compare Mishchenko’s definition of noncommutative signature for an oriented manifold with an orientation preserving proper action of a discrete, countable group $G$ with the (more analytical) counter part defined by Higson and Roe in the series of articles “Mapping Surgery to analysis”. A generalization of the bordism invariance of the coarse index is also addressed.

1. Introduction

There are different notions of non-commutative signatures that can be applied to oriented proper cocompact $G$-manifolds for a discrete group $G$. Higson and Roe studied the relation between a signature of $C^*$-algebras, an analytic signature and the coarse index of the signature operator, they also show that these signatures are bordism and homotopy invariants.

For these definitions, they consider two types of so-called Hilbert-Poincaré complexes: algebraic complexes of finitely generated projective modules over a $C^*$-algebra $C$ and analytically controlled complexes of Hilbert spaces. Both kind of complexes are required to satisfy suitable versions of Poincaré Duality. The algebraic signature has values in the $K$-theory $K_*(C)$ of the algebra $C$, and the analytic signature has values in the Mitchener $K$-theory of a suitable $C^*$-category.

All these signatures are defined for the case of a compact oriented smooth manifold $X$ and the authors showed that the analytic signature coincides with the $K$-theoretic index of the signature operator defined on the $L^2$-completion of the De Rham complex of $X$. In this case, it is proven that Mitchener $K$-theory coincides with the $K$-theory $K_*(C_r(G))$ of the reduced $C^*$-algebra of the group $G$.

Their $C^*$-algebra signature is defined for finitely generated projective Hilbert-Poincaré modules over the algebra $C_0(X)$ of continuous functions vanishing at infinity. In the case of an oriented smooth manifold $X$ with orientation preserving free action of a discrete group $G$ their definition makes no sense if the quotient $X = X/G$ is not compact, because the complexes considered are not finitely generated over this algebra and the representation of $C_0(X)$ on the given complex is not by chain maps. The analytic signature does make sense and the proof of its coincidence with the index of the signature operator generalizes to this context.

On the other hand, Mishchenko defined a signature for finitely generated projective algebraic Hilbert-Poincaré complexes over the reduced $C^*$-algebra $C^*_r(G)$ of the group $G$. This can be applied to a proper oriented co-compact smooth $G$-manifold $M$ with orientation preserving action of the group $G$. The analytic signature of Higson and Roe also makes sense in this context for the $L^2$-completion of the De Rham complex.

In this paper we show that, with slight modifications to the notion of algebraic Hilbert-Poincaré complex, the $C^*$-algebra signature defined by Higson and Roe coincides in even dimension with that of Mishchenko. A consequence of this is another proof of the homotopy and bordism invariance of the signature of Mishchenko. The
analytic version of the signature can be applied in this context to triangulated *bounded isotropy* proper oriented $G$-manifolds of even dimension. In this case, the coincidence of the analytic signature with the coarse index of the signature operator is a consequence of the results proven by Higson and Roe. Also, another version of bordism invariance is considered in this context. In the last section, we synthetize the relations between the signatures considered.

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### 3. Algebraic Hilbert-Poincaré complexes and their signature

In [9] a signature for a Hilbert-Poincaré complex was defined. This definition is as follows.

Let $C$ be a $C^*$-algebra. Recall that an $n$-dimensional Hilbert-Poincaré complex is a triple $(E, b, S)$ where $(E, b)$ is an $n$-dimensional chain complex

\[
E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_{n-1} \leftarrow E_n
\]

of finitely generated projective Hilbert modules over a $C^*$-algebra $C$, the $b_k, k = 1, \ldots, n$ are bounded adjointable maps, $b = \oplus_k b_k : E \to E$, $E = \oplus_k E_k$, and $S : E \to E$ is a self-adjoint operator such that

(i) $S_k : E_{n-k} \to E_k$, where $S_k = S|_{E_{n-k}}$,

(ii) $b_k S_k + S_{k-1} b_{n-k+1} = 0$ and

(iii) $S$ induces an isomorphism from the homology of the dual complex $(E, b^*)$ to the homology of the complex $(E, b)$.

The second condition means that $S : (E, -b^*) \to (E, b)$ is a chain map.

We recall the following definition.

**Definition 3.2.** (conf. [1 def.2.2,p.280]) The mapping cone of a chain map $A : (E', b') \to (E, b)$ is the complex

\[
E'_0 \leftarrow E'_1 \leftarrow \cdots \leftarrow E'_{n-1} \leftarrow E'_n
\]

where $E'_j = E'_{j-1} \oplus E_j$ and differential $b'' : E' \to E''$ defined by

\[
b''_j = \begin{pmatrix} -b'_{j-1} & 0 \\ A_{j-1} & b_j \end{pmatrix}
\]

Using the language and notations in [1], the definitions of the signature are as follows.

**Definition 3.5.** (Mishchenko, [9 sec.3]). Let $(E, b, S)$ be a Hilbert-Poincaré complex of Hilbert $C$-modules (with $S$ self-adjoint and $bS + Sb^* = 0$) and let $(E \oplus E, b)$ the mapping cone of $S$. Then, the signature of $(E, b, S)$ is the formal difference $[Q_+] - [Q_-]$ in $K_0(C)$ of the positive and negative projection of the restriction of
the map $B_S = b_S^* + b_S$ to the +1 eigenspace of the symmetry which exchanges the two copies of $E$ in $E \oplus E$.

**Remark 3.6.** In the previous definition we made use of the fact that the self-adjoint operator $B_S = b_S^* + b_S$ is invertible. Indeed, property (iii) in the definition of the Hilbert-Poincaré complex is equivalent to the acyclicity of the complex $E \oplus E$ by lemma 2.3 of [1]. This is equivalent to the invertibility of $B_S$ according to proposition 2.1 of [1].

**Remark 3.7.** In the construction of Mishchenko [9] the summands in the mapping cone are interchanged and this gives a different formula for the operator: $B_S = b_S + T b_S T$, where $T$ is the symmetry interchanging the two summands copies of $E$ in $E \oplus E$ (see [9] p.14) and notice the typo in the identity $H_k = T_k H_{n-k-2} T_{k-1}$.

In the notation used here, Mishchenko’s definition of the mapping cone would be $(Higson-Roe)$. Let $(E, b, S)$ be an even dimensional Hilbert-Poincaré complex of Hilbert $C^*$-modules, $(E, b, S)$ is the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$ respectively, where $B = b + b^*$. 

**Definition 3.10.** (Higson-Roe). Let $(E, b, S)$ be an even dimensional Hilbert-Poincaré complex of Hilbert $C$-modules. The signature of $(E, b, S)$ is the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$ respectively, where $B = b + b^*$.

**Proposition 3.11.** Definitions 3.6 and 3.10 coincide.

**Proof.** Using remark 3.7 one has that $P_+ = Q_+$. So, it is enough to prove that the positive projection of the operator $B - S$ is equivalent to the negative projection of $B + S = b + b^* + S$.

Consider the self-adjoint symmetry $\varphi : E \rightarrow E$ equal to the identity on $E_{2i}$, $i = 0, \ldots, 2l$ and minus the identity on $E_{2i+1}$, $i = 0, \ldots, 2l - 1$. This operator intertwines $B - S$ with $-B - S$ and, therefore, the positive projection of $B - S$ is equivalent to the negative projection of $b + b^* + S$. □

**Remark 3.12.** Higson-Roe definition of the index while more elaborated is more suitable for the aim of comparison with the index of the signature operator on the de Rham complex. Mishchenko’s definition is a more straightforward generalization of the signature of an algebraic Poincaré complex to the context of $C^*$-algebras.

4. **Bordism invariance of the algebraic signature**

Let $(E, b)$ be an $(n+1)$-dimensional complex of Hilbert $C$-modules, $(E_0, b_0) \subset (E, b)$ an $n$-dimensional subcomplex, more precisely $E = E_1 \oplus E_0$ for some $E_1$ and $b|_{E_0} = b_0$, and $S : E \rightarrow E$ is a self-adjoint operator such that

1. $S_k : E_{n+k} \rightarrow E_k$, where $S_k = S|_{E_{n+k}}$,
2. $(b_k S_k + S_{k-1} b^*_k) v$ for every $v \in E_0$ and
3. $S$ induces an isomorphism from the homology of the dual complex $(E, b^*)$ to the homology of the quotient complex $(E/E_0, b_1)$ where $b_1 : E/E_0 \rightarrow E/E_0$ is the induced boundary operator.

Such a complex is called an algebraic Hilbert-Poincaré complex with boundary.

Then one can show the analog to [10] lemma 1.1.p.503.
Lemma 4.1. The boundary complex $(E_0, \sqrt{-1}b_0, S_0)$, where $S_0 = bS + Sb^*|_{E_0}$, is an algebraic Hilbert-Poincaré complex (without boundary).

Proof. Consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & E_0 & \xrightarrow{i} & E & \xrightarrow{j} & E/E_0 & \rightarrow & 0 \\
\downarrow{s\star} & & \downarrow{s} & & \downarrow{j\star} & & \downarrow{\epsilon} & & \downarrow{0} \\
0 & \rightarrow & E/E_0 & \xrightarrow{j} & E & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

where the rows are exact. We will construct a map $b_0 : E_0 \rightarrow E_0$ such that $ib_0i^* = bS + Sd^*$.

On one hand the image $dS + Sd^*(E)$ is contained in the subcomplex $E_0$, i.e.

$bS + Sb^*(E) \subset E_0$,

so, it can be composed with the inverse $i^{-1} : \text{im } i \rightarrow E_0$, and, then $S_0$ can be defined as

$S_0 = i^{-1}[i^{-1}(bS + Sb^*)]^* = i^{-1}(bS + Sb^*)i^{-1*}$.

(this is just chasing de diagram).

On the other hand, the subcomplex $E_0$ is a direct summand of the complex $E$, that is, there is a subcomplex $E_1 \subset E$ such that $E = E_0 \oplus E_1$

so the homomorphism $S$ is represented by a matrix, i.e.

$S = \begin{pmatrix} S_2 & F \\ F^* & S_1 \end{pmatrix}$, $S_2^* = S_2$, $S_1^* = S_1$,

where $S_1 : E_1 \rightarrow E_1$, $S_2 : E_0 \rightarrow E_0$, $F : E_1 \rightarrow E_0$.

In this terms,

$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} \tilde{b}_0 & h \\ f & \bar{b}_1 \end{pmatrix}$,

where $ib_0 = bi$, with $b_0$ being the differential of the chain complex $E_0$, i.e.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}b_0 = \begin{pmatrix} \tilde{b}_0 & h \\ f & \bar{b}_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

i.e.

$\begin{pmatrix} b_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{b}_0 \\ f \end{pmatrix}$,

i.e. $f \equiv 0$, $\tilde{b}_0 = b_0$, and $jb = b_1j$ with $b_1$ the differential of the complex $E/E_0$, i.e.

$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} b_0 \\ 0 \end{pmatrix} = b_1\begin{pmatrix} 0 & 1 \end{pmatrix}$

which means $\tilde{b}_1 = b_1$.

Also,

$0 = b^2 = \begin{pmatrix} b_0 & h \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} b_0 & h \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} b_0^2 & b_0h + hb_1 \\ 0 & b_1^2 \end{pmatrix} = \begin{pmatrix} 0 & b_0h + hb_1 \\ 0 & 0 \end{pmatrix}$

i.e.

\[
b_0h + hb_1 = 0,
\]
where $h : E_1 \rightarrow E_0$.

Then,

$$bS + Sh^* =$$

$$= \left( \begin{array}{cc} b_0 & h \\ 0 & b_1 \end{array} \right) \left( \begin{array}{cc} S_2 & F \\ F^* & S_1 \end{array} \right) + \left( \begin{array}{cc} S_2 & F \\ F^* & S_1 \end{array} \right) \left( \begin{array}{cc} b_0^* & 0 \\ h^* & b_1^* \end{array} \right) =$$

$$= \left( \begin{array}{cc} b_0S_2 + hF^* & b_0F + hS_1 \\ b_1F^* & b_1S_1 \end{array} \right) + \left( \begin{array}{cc} S_2b_0^* + Fh^* & Fb_1^* \\ F^*b_0^* + S_1h^* & S_1b_1^* \end{array} \right) =$$

$$= \left( \begin{array}{cc} b_0S_2 + S_2b_0^* + hF^* + Fh^* & b_0F + hS_1 + Fb_1^* \\ b_1F^* + F^*b_0^* + S_1h^* & b_1S_1 + S_1b_1^* \end{array} \right).$$

From the equation $j(bS + Sh^*) = 0$ we obtain

(4.4) \[ b_1F^* + F^*b_0^* = -S_1h^*, \]

$$b_1S_1 + S_1b_1^* = 0. \]

In addition,

$$b_0F + hS_1 + Fb_1^* = (b_1F^* + F^*b_0^* + S_1h^*)^* = 0,$$

i.e.

(4.5) \[ b_0F + Fb_1^* = -hS_1, \]

$$bS + Sh^* =$$

$$= \left( \begin{array}{cc} b_0S_2 + S_2b_0^* + hF^* + Fh^* & 0 \\ 0 & 0 \end{array} \right)$$

i.e.

(4.6) \[ S_0 = b_0S_2 + S_2b_0^* + hF^* + Fh^*. \]

Let's now show that $S_0$ makes $E_0$ an algebraic Hilbert-Poincaré complex with the differential $\sqrt{-1}b_0$. In this case, we prove that $b_0S_0 - S_0b_0^* = 0$.

Indeed,

$$b_0S_0 - S_0b_0^* =$$

$$= b_0(b_0S_2 + S_2b_0^* + hF^* + Fh^*) - (b_0S_2 + S_2b_0^* + hF^* + Fh^*)b_0^* =$$

$$= b_0^2S_2 + b_0S_2b_0^* + b_0hF^* + b_0Fh^* - b_0S_2b_0^* - S_2b_0^* - hF^*b_0^* - Fh^*b_0^* =$$

$$= b_0hF^* + b_0Fh^* - Fh^*b_0^* - Fh^*b_0^*. \]

From (4.3), $b_0h = -hb_1$ and $h^*b_0^* = -b_1^*h^*$, substituting

$$b_0S_0 - S_0b_0^* = b_0hF^* + b_0Fh^* - hF^*b_0^* - Fh^*b_0^* =$$

$$= -hb_1F^* + b_0Fh^* - hF^*b_0^* + Fb_1^*h^* =$$

$$= -h(b_1F^* + F^*b_0^*) + (b_0F + Fb_1^*)h^* =$$

$$= hS_1h^* - hS_1h^* = 0. \]

Now we are able to prove algebraic bordism invariance.

**Theorem 4.7.** The signature of the boundary complex $(E_0, \sqrt{-1}b_0, S_0)$ is equal to zero.
Proof. By adding the rows in the diagram (4.2), we obtain the sequence
\[
0 \to E_0 \overset{I}{\to} (E \oplus E/E_0) \overset{J}{\to} (E/E_0 \oplus E) \overset{I^*}{\to} E_0^* \to 0
\]
where
\[
I = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} j & 0 \\ 0 & j^* \end{pmatrix}
\]
and the graduated module \( A = E \oplus E/E_0 \) is a chain complex with the differential
\[
H = \begin{pmatrix} b & Sj^* \\ 0 & b^* \end{pmatrix}
\]
, i.e.
\[
H^2 = \begin{pmatrix} b & Sj^* \\ 0 & b^* \end{pmatrix} \begin{pmatrix} b & Sj^* \\ 0 & b^* \end{pmatrix} = \begin{pmatrix} b^2 & bSj^* + Sj^*b^* \\ 0 & b^* \end{pmatrix},
\]
but
\[
bSj^* + Sb^*j^* = (bS + Sb^*)j^* = (j(bS + Sb^*))^* = 0.
\]
By construction, the sequence (4.8) is exact, i.e.
\[
\text{im } I = \text{im } i = \ker j = \ker J, \quad \text{im } J = \text{im } j \oplus \text{im } j^* = E/E_0 \oplus \ker i^* = \ker I^*.
\]
In terms of the decomposition \( E = E_0 \oplus E_1 \), and rearranging the terms, the sequence (4.8) takes the form
\[
0 \to E_0 \overset{I}{\to} E_0 \oplus E_1 \overset{J}{\to} E_0 \oplus E_1 \oplus E_1 \overset{I^*}{\to} E_0^* \to 0
\]
where
\[
I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} d_1 & h & F \\ 0 & d_2 & D_2 \\ 0 & 0 & d_2^* \end{pmatrix}
\]
Note that the complex \( E_1 \oplus E_1 \) is a chain complex with the differential
\[
\delta = \sqrt{-1} \begin{pmatrix} b_1 & S_1 \\ 0 & b_1^* \end{pmatrix}
\]
and an algebraic Hilbert-Poincaré complex with the isomorphism \( T : E_1 \oplus E_1 \to E_1 \oplus E_1 \) defined by the matrix
\[
T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
i.e. \( \delta T - T \delta^* = \)
\[
\begin{pmatrix} b_1 & S_1 \\ 0 & b_1^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1^* & 0 \\ S_1 & b_1 \end{pmatrix} = \begin{pmatrix} S_1 & b_1^* \\ 0 & S_1 & b_1 \end{pmatrix} = \begin{pmatrix} b_1^* & 0 \\ S_1 & b_1 \end{pmatrix} = 0
\]
Now consider the diagram
\[
\begin{array}{ccc}
E_1 \oplus E_1 & \xrightarrow{f} & E_0 \\
\uparrow T & & \uparrow s_0 \\
E_1 \oplus E_1 & \xrightarrow{f^*} & E_0^*
\end{array}
\]
where the homomorphism \( f : E_1 \oplus E_1 \to E_0 \) defined by the matrix \( f = \begin{pmatrix} h & F \end{pmatrix} \)
is a chain map, i.e.
\[
f \delta = \begin{pmatrix} h & F \end{pmatrix} \begin{pmatrix} b_1 & S_1 \\ 0 & b_1^* \end{pmatrix} = \begin{pmatrix} hb_1 & hS_1 + Fb_1^* \end{pmatrix} = \]
and $S_2$ defines a homotopy between these maps, i.e.

$$fTf^* = (h \ F) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h^* \\ F^* \end{pmatrix} = Fh^* + hF^*,$$

but by 4.6,

$$S_0 = fTf^* + b_0S_2 + S_2b_0^*.$$  

In particular, this equation means that

$$H(f)H(T)H(f^*) = H(fTf^*) = H(S_0)$$

is an isomorphism. That is, the complex $(E_0, \sqrt{-1}b_0, fTf^*)$ is an algebraic Hilbert-Poincaré complex and the complexes $(E_0, \sqrt{-1}b_0, fTf^*)$ and $(E_0, \sqrt{-1}b_0, S_0)$ are homotopy equivalent according to definition 4.1 of [1, p.285] with the homotopy given by the identity map $E_0 \to E_0$. They have the same signature by theorem 4.3 in the same paper.

Finally, it is obvious that the signature of the complex $(E_0, \sqrt{-1}b_0, fTf^*)$ is equal to zero. □

5. Analytically controlled Hilbert-Poincaré complexes over $C^*$-categories and their signature

Here we recall the definition of an analytically controlled Hilbert-Poincaré complex, its signature and other relevant constructions from [1].

Consider a triple $(H, b, S)$, where $(H, b)$ is an $n$-dimensional chain complex

$$H_0 \xrightarrow{b_1} H_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} H_{n-1} \xrightarrow{b_n} H_n$$

of Hilbert spaces, the operator $b = \oplus k b_k : H \to H$, where $H = \oplus k H$, is an unbounded, closed operator such that $b \cdot b$ is defined an equal to zero, i.e. $\text{Image}(b) \subset \text{Domain}(b)$, $b^2 = 0$. The map $S : H \to H$ is an everywhere defined self-adjoint operator such that

(i) $S_k : H_{n-k} \to H_k$, where $S_k = S|_{H_{n-k}}$;

(ii) $S : (H, -b^*) \to (H, b)$ is a chain map, i.e. $S(\text{Domain}(b^*)) \subset \text{Domain}(b)$

and $(bS + Sb^*)v = 0$ for every $v \in \text{Domain}(b^*)$;

(iii) $S$ induces an isomorphism from the homology of the dual complex $(H, b^*)$

to the homology of the complex $(H, b)$.

Such a triple is called an analytic Hilbert-Poincaré complex.

**Definition 5.2.** A $C^*$-category $\mathfrak{A}$ is an additive subcategory of all Hilbert spaces and bounded linear maps which is closed under taking adjoint of morphisms and such that the morphisms sets $\text{Hom}_\mathfrak{A}(H_1, H_2)$ are Banach subspaces of the set $\text{Hom}(H_1, H_2)$ of bounded linear operators from the Hilbert space $H_1$ to the Hilbert space $H_2$.

**Definition 5.3.** A $C^*$-category ideal $\mathfrak{J}$ of the $C^*$-category $\mathfrak{A}$ is a $C^*$-subcategory possibly without identity morphisms such that any composition of a morphism in $\mathfrak{A}$ with a morphism in $\mathfrak{J}$ is a morphism in $\mathfrak{J}$.

**Remark 5.4.** In the case of a $C^*$-category with a single object, this definition of ideal coincides with that of a (bilateral) ideal of a $C^*$-algebra of bounded operators on a fixed Hilbert space. In all of the following constructions, we will fix this Hilbert space.

**Definition 5.5.** An unbounded, self-adjoint Hilbert space operator $D : H \to H$ is said to be analytically controlled over the pair $(\mathfrak{A}, \mathfrak{J})$ if
(i) $H$ is an object of $\mathfrak{J}$,
(ii) the operators $(D \pm iI)^{-1}$ are morphisms of $\mathfrak{J}$, and
(iii) the operator $D(1 + D^2)^{\frac{1}{2}}$ is a morphism of $\mathfrak{A}$.

This definition means that $f(D)$ is a morphism of $\mathfrak{J}$ for every $f \in C_0(\mathbb{R})$ and $f(D)$ is a morphism of $\mathfrak{A}$ for every $f \in C_0[-\infty, \infty]$.

**Definition 5.6.** A complex $(H, b)$ of Hilbert spaces is said to be analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if the self-adjoint operator $B = b + b^*$ is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ according to definition [6].

**Definition 5.7.** An analytic Hilbert-Poincaré complex $(H, b, S)$ is said to be analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if the complex $(H, b)$ is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ in the sense of the previous definition, i.e. if $B = b + b^*$ is analytically controlled, and the duality operator $T$ is a morphism in $\mathfrak{A}$.

It is shown in [1] lemma 5.8 and the discussion on p.291 that for a Hilbert-Poincaré complex analytically controlled over $(\mathfrak{A}, \mathfrak{J})$, the difference $P_+ - P_-$ of the formal projections of the operators $B + S = b + b^* + S$ and $B - S = b + b^* - S$ (respectively) belongs to the ideal $J$ of $\mathfrak{A}$, where $A$ is the $C^*$-algebra of $\mathfrak{A}$-endomorphisms of the space $H$ and $J$ is the $C^*$-algebra of $\mathfrak{J}$-endomorphisms of the same space. This means that the formal difference $[P_+] - [P_-]$ is an element of the group $K_n(J)$. There is a natural map $K_n(J) \to K_n(\mathfrak{J})$ so there is a class in $K_n(\mathfrak{J})$ determined by the difference $[P_+] - [P_-]$.

**Definition 5.8.** Let $(H, b, S)$ be a Hilbert-Poincaré complex analytically controlled over $(\mathfrak{A}, \mathfrak{J})$. Its analytical signature is the class determined by the formal difference $[P_+] - [P_-]$ in $K_n(\mathfrak{J})$.

6. **Signatures of a $G$-manifold**

In this section we modify some the notions in [2] to extend the main results there to include proper, not necessarily free actions. Namely, we extend the definitions of the control categories. Also, we take into account the additional structure on the complex $C^*_G(M)$ of an oriented co-compact $G$-manifold $M$ (with orientation-preserving $G$-action) needed to make it an algebraic Hilbert-Poincaré complex over the reduced $C^*$-algebra $C^*_r(G)$. This complex also is interpreted by Higson and Roe as an analytically controlled Hilbert-Poincaré complex and, therefore, it has two signatures. The relation between this signatures is addressed.

6.1. **The algebraic signature of a triangulated smooth $G$-manifold.** In [3], [8] it is shown that a smooth manifold $M$ with proper action of a discrete group $G$ admits $G$-invariant triangulations. It is also shown the uniqueness of this piecewise linear structure up to barycentric subdivision. In this case one shall choose a triangulation such that every simplex is either fixed point-wise or permuted by the action.

Following [2] p.306-310 we denote by $C_*(M)$ the space of finitely supported simplicial chains on $M$ with complex coefficients. Then, for each $p$ the complex vector space $C_p(M)$ has a basis comprised of the $p$-simplices on $M$. Define an inner product on $C_p(M)$ such that this basis is orthonormal. The completion of this space is denoted by $C^q_p(M)$, in other words, this is the Hilbert space of square summable $p$-chains on $M$. The differentials $\partial_p : C_p(M) \to C_{p-1}(M)$ extend to operators $b_p : C^q_p(M) \to C^q_{p-1}(M)$.

The operators $b_p$ are bounded if the number of simplices in the triangulated space $M$ with a common boundary is bounded, and this assumption can in turn be
reduced to requiring that the number of simplices containing a point in the space $M$ is bounded. Such space $M$ is called of bounded geometry.

Also, the adjoint operators $b^*_p : C^*_{p-1}(M) \to C^*_p(M)$ identify with the extension of the co-boundary maps. This makes $(C^*_p(M), b)$ a complex of Hilbert spaces.

Denote by $C_0(M)$ the algebra of continuous functions vanishing at infinity. Define a representation of $C_0(M)$ on $C^*_p(M)$ as follows: for every $f \in C_0(M)$ and chain $c = \sum_\sigma c_\sigma [\sigma]$, 

$$f \cdot c = \sum_\sigma f(b_\sigma)c_\sigma [\sigma],$$

where $b_\sigma$ is the barycenter of the simplex $\sigma$. With this and the bounded geometry assumption, one might interpret $(C^*_p(M), b)$ as a complex of Hilbert $C_0(M)$-modules, but these spaces are not in general finitely generated over this algebra, and the representation of $C_0(M)$ on $C^*_p(M)$ is not by chain maps.

On the other hand, as the action of $M \times G \to M$ is simplicial, the complex $C_*(M)$ has a natural action of this group defined by the formula

$$c \cdot g = \sum_\sigma c_\sigma [\sigma] g = \sum_\sigma c_\sigma [\sigma g],$$

for $g \in G$. The action is simplicial, so it commutes with the boundary map. As the action either fixes simplices or permutes them, this action is by unitaries, and it extends to a representation of the reduced $C^*$-algebra $C^*_r(G)$ of the group $G$. This means that $(C^*_r(M), b)$ is a complex of $C^*_r(G)$-modules. If the quotient $M/G$ is compact, then the modules $C^*_p(M)$ are finitely generated: one may assume that there is a finite number of simplices in the triangulation of the compact quotient $X = M/G$ induced by the map $M \to M/G$, and this means that there is a finite number of $G$-orbits of simplices in $M$.

In order to analyze Poincaré duality in this context one shall first give some explicit expression of the action of $G$ on cochains. If $u : C_p(M) \to \mathbb{C}$ a $p$-cochain, this is defined by the rule

$$(u \cdot g)[\sigma] = u([\sigma g^{-1}]),$$

for a simplex $\sigma \in C^p(M)$.

The Poincaré duality homomorphism of an oriented, possibly non-compact manifold $M$ is given by the intersection $[M] \cap u$ of the fundamental class of the manifold with a finitely supported cochain $u$. More precisely, let $u : C^{n-p}(M) \to \mathbb{C}$ be a finitely supported $(n-p)$-cochain and $[M] = \sum_\sigma (-1)^{\epsilon(\sigma)}[\sigma]$ be the fundamental class, where $\epsilon(\sigma)$ denotes the orientation of the simplex $\sigma$ induced by the orientation of the manifold $M$, and the sum runs over all $n$-simplices in the triangulation of $M$. Then, the Poincaré duality homomorphism $T_p : C^{n-p}(M) \to C^p(M)$ is defined by the formula

$$T_p(u) = [M] \cap u = \sum_{\sigma = [v_0 \cdots v_n]} (-1)^{\epsilon(\sigma)} u([v_0 \cdots v_{n-p}]) [v_{n-p} \cdots v_n].$$

This map is $G$-equivariant, i.e. satisfies the identity

$$T_p(u \cdot g) = (T_p(u)) \cdot g.$$
Indeed, 
\[
(T_p(u)) \cdot g = \left( \sum_{\sigma = [v_0 \cdots v_n]} (-1)^{l(\sigma)} u([v_0 \cdots v_{n-p}]) [v_{n-p} \cdots v_n] \right) \cdot g = 
\]
\[
= \sum_{\sigma = [v_0 \cdots v_n]} (-1)^{l(\sigma)} u([v_0 \cdots v_{n-p}]) [v_{n-p} \cdots v_n] g = 
\]
\[
= \sum_{\sigma = [v_0 \cdots v_n]} (-1)^{l(\sigma)} u([v_0 \cdots v_{n-p}]) [v_{n-p} \cdots v_n] g = 
\]
\[
= \sum_{\sigma g = [v_0 \cdots v_n]} (-1)^{l(\sigma g)} (u \cdot g) [v_0 \cdots v_{n-p}][v_{n-p} \cdots v_n] g = 
\]
\[
= \sum_{\gamma = [w_0 \cdots w_n]} (-1)^{l(\gamma)} (u \cdot g) [w_0 \cdots w_{n-p}][w_{n-p} \cdots w_n] 
\]
\[
= T_p(u \cdot g).
\]

where \( \gamma = \sigma g \) and in the last step we have used the identity \( \epsilon(\gamma g^{-1}) = \epsilon(\gamma) \), that is, one must require that \( g \) preserves orientation. The equivariant map \( T : C^*(M) \to C_\epsilon(M) \) satisfies the classic Poincaré duality identities and extends to a \( G \)-linear map \( T : C^2_\epsilon(M) \to C^2_\epsilon(M) \). Then, if the dimension of \( M \) is even, the operator \( S : C^2_\epsilon(M) \to C^2_\epsilon(M) \) defined by the rule

\[ S_p = r^{p-1}T_p : C^2_{n-p}(M) \to C^2_p(M) \]

satisfies the properties:

(i) \( S \) is self-adjoint,
(ii) \( bS + Sb^* = 0 \) and
(iii) \( S \) induces an isomorphism from the homology of the dual complex \( (C^*(M), b^*) \) to the homology of the complex \( (C^*(M), b) \).

Therefore, \( (C^*(M), b, S) \) is an algebraic Hilbert-Poincaré complex over \( C^*_\epsilon(G) \) and has algebraic signature in \( K_0(C^*_\epsilon(G)) \) as in definitions 3.3 or 3.10.

With this structure, one obtains another proof of the following:

**Proposition 6.1.** The signature of Mishchenko is a homotopy invariant.

**Proof.** This is theorem 4.3 of [1] applied to the signature defined there and recalled here as [3,10] but using the algebraic Hilbert-Poincaré complex over the algebra \( C^*_\epsilon(G) \) that we have just constructed. These signatures coincide by proposition 3.11. \( \square \)

**Remark 6.2.** The construction of this Hilbert-Poincaré complex has been presented by Mishchenko in several conference talks before 2010, so the authors claim no originality. We refer to [3] sec.3] and check that this complex satisfies the definition given there.

6.2. The analytic signatures of a smooth \( G \)-manifold. Here we generalize the \( C^* \)-categories considered in [2] and reinterpret the complex \( (C^*(M), b, S) \) as an equivariant analytically controlled Hilbert-Poincaré complex. Then we show that the results about invariance of the analytic signature can be applied to bounded geometry spaces with bounded isotropy action, and that this is the case for proper spaces with bounded geometry quotient.
Definition 6.3. Let $M$ be a proper metric space. An $M$-module $H$ is a separable Hilbert space equipped with a non-degenerate representation of the $C^*$-algebra $C_0(M)$ of continuous, complex-valued functions on $M$ vanishing at infinity.

Definition 6.4. Let $G$ a finitely generated discrete group. A $G$-presented space $X$ is a proper geodesic metric space presented as the quotient $X = M/G$ of a proper geodesic metric space $M$ by an isometric proper action $\mu : G \times M \to M$ of the group $G$. The pair $(M, \mu)$ is called a $G$-presentation of $X$.

For fixed discrete group $G$ and space $X$, the presentations of $X$ together with equivariant maps form a category. We avoid the action in the notation and say that $M$ is a $G$-presentation of $X$. We shall assume in the following that all such presentations have an invariant non-empty open set where the action of the group $G$ is free.

Definition 6.5. An equivariant $G$-$X$-module is an $M$-module $H$, where $M$ is a $G$-presentation of $X$ equipped with a compatible (faithful) unitary representation of $G$.

In the case of an equivariant $G$-$X$-module $M$ we will require that the representation of the $C^*$-algebra $C_0(M)$ restricts to a non-degenerate representations of the subalgebra $C_0(U)$ for a $G$-invariant non-empty open set $U \subset M$.

Given a locally compact, separable and metrizable space, together with a non-degenerate representation on the Hilbert space $H$, that is, a nondegenerate continuous $*$-homomorphism

$$\rho : C_0(M) \to B(H),$$

we define the support of $\nu \in H$ to be the complement in $X$ of the union of all open subsets $U \subset X$ such that $\rho(f)(\nu) = 0$ for all $f \in C_0(U)$. An operator $T \in B(H)$ is locally compact on $X$ if $fT$ and $Tf$ are compact operators for all functions $f \in C_0(M)$.

Definition 6.6. The support of an operator $T \in B(H)$, denoted by $\text{Supp}(T)$, is the complement in $X \times X$ of the union of all open subsets $U \times V \subset X \times X$ such that $\rho(f)T\rho(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$. More generally, if $C_0(X)$ and $C_0(Y)$ are non-degenerate representations on Hilbert spaces $H_X$ and $H_Y$, then the support of a bounded operator $T : H_X \to H_Y$ is the complement in $X \times X$ of the union of all open subsets $U \times V \subset Y \times X$ such that $\rho_Y(f)T\rho_X(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$.

Definition 6.7. Let $X$ be a locally compact separable and metrizable space, proper in the sense of metric geometry, meaning that closed balls are compact. Let $\rho : C_0(X) \to B(H)$ be a nondegenerate representation on the Hilbert space $X$.

An operator $T \in B(H)$ is boundedly controlled if the support $\text{Supp}(T)$ is at bounded distance of the diagonal in $X \times X$, that means

$$\sup_{y \in \text{Supp}(T), x \in \Delta(X)} \{d_{X \times X}(y, x)\} < \infty.$$

An operator $T$ is locally compact on $X$ if $fT$ and $Tf$ are compact for all functions $f \in C_0(X)$.

Given an operator $T \in B(H)$, we define its propagation $\text{Prop}(T)$, to be the following extended real number:

$$\text{Prop}(T) = \sup\{d_{X \times X}(d(x, y) \mid x, y \in \text{Supp}(T))\},$$

and will say that an operator is of finite propagation if this number is finite.
Lemma 6.11. Let effective action with \( X \) geometrically controlled if \( f \) less than representation of the group \( G \) that a \( h \) and choose \( f \) elements. This space is non-trivial: take \( G \) a endomorphisms of a non-trivial object in \( C \) acting as the identity on \( v \) and, therefore, their \( K \)∞ \( N < \). We will follow two steps in the proof:

(i) Every non-trivial \( G \)-X-module \( H \) with an effective representation of \( G \) by unitaries contains a non-trivial subspace which can be endorsed with the structure of a Hilbert module over \( C_r(G) \) whose algebra of compact operators in the sense of Hilbert modules is isomorphic to the algebra of endomorphisms of \( H \) in the category \( \mathcal{E}(X, G, M) \).

(ii) There is an example of a Hilbert \( G \)-X-module \( H \) such that this algebra of compact operators is isomorphic to \( C_r(G) \).

Step (i). Let \( H \) be a non-trivial \( G \)-X-module for the presentation \( M \) (i.e. a \( C_0(M) \)-module) with a non-trivial representation of \( C_0(M) \) and compatible effective representation of the group \( G \) by unitary operators in the sense that \((f \cdot v) \cdot g = f^g \cdot (v \cdot g)\) for every \( f \in C_0(M), v \in H, g \in G \), where \( f^g(x) = f(gx), x \in M \). Recall that a \( v \in H \) is said to be compactly supported if there is a function \( f \in C_c(M) \) acting as the identity on \( v \), i.e. \( f \cdot v = v \). Denote by \( H_{C_c(M)} \) the vector space of such elements. This space is non-trivial: take \( f \in C_c(M) \) and \( v \in H \) such that \( f \cdot v \neq 0 \) and choose \( h \in C_c(M) \) such that \( hf = f \), then one has \( h \cdot (f \cdot v) = (hf) \cdot v = f \cdot v \).
Define a $\mathbb{C}[G]$-valued inner product on $H_{C_r(O)}$ by

$$\langle (v, w) \rangle = \sum_{g \in G} (v \cdot g, w)[g],$$

where $O \subset M$ is a non-empty $G$-invariant open subset such that the action $G \times O \to O$ is free. Note that, if $\text{supp}(f) \cap \text{supp}(h) = \emptyset$ then $fh = 0$. Assume that $\text{supp}(f) \subset U$ for some $U \subset O$ such that $gU \cap U = \emptyset$ for $g \neq 1$. This means that $\text{supp}(f) \cap \text{supp}(f^g) = \emptyset$, therefore

$$\langle (v, v) \rangle = \sum_{g \in G} (v \cdot g, v)[g] = \sum_{g \in G} \langle (f \cdot v) \cdot g, f \cdot v \rangle[g] = \sum_{g \in G} \langle f^g \cdot (v \cdot g), f \cdot v \rangle[g] = \sum_{g \in G} \langle (f f^g) \cdot (v \cdot g), v \rangle[g] = \langle (ff^g) \cdot v, v \rangle[1] = (f \cdot v, f \cdot v)[1] = \langle v, v \rangle[1] = [1],$$

if $\|v\| = 1$. Here we have used that the adjoint of the operator $f : H \to H$ is the conjugate $\bar{f}$ of this function, which has the same support.

Similarly

$$\langle (v, v \cdot g) \rangle = \langle (f f^g) \cdot (v \cdot g), v \cdot g \rangle[g] = \langle f^g \cdot (v \cdot g), f^g \cdot (v \cdot g) \rangle[g] = \langle v \cdot g, v \cdot g \rangle[g] = \langle v, v \rangle[g] = [g].$$

This computations show that every element in the group algebra $\mathbb{C}[G]$ can arise as the inner product of some elements in $H_{C_r(O)}$.

Then one checks positivity, completes simultaneously $\mathbb{C}[G]$ to $C_r(G)$ and $H_{C_r(O)}$ to a Hilbert $C_r(G)$-module $H_G$ and writes

$$\overline{H_{C_r(O)}} = H_G \otimes_{\chi} L^2(G)$$

where $\lambda : C_r(G) \to \mathcal{B}(L^2(G))$ is the left regular representation. Note that, as $C_r(O)$ is dense in $C_0(O)$ and $C_0(O)H$ is dense on $H$, one has that $C_r(O)H$ is also dense in $H$ and $\overline{H_{C_r(O)}} = H$.

Then one proves by analogy with lemma 2.2 and lemma 2.3 of [13] p. 243,244 that the algebra of compact operators of $H_G$ in the sense of Hilbert modules is isomorphic to the algebra of endomorphisms of $H$ in the category $\mathcal{C}(X, G, M)$.

Step (iii). The example is the Hilbert space is the completion $L^2(M)$ of $C_c(M)$ with respect to the norm defined by the complex-valued inner product

$$\langle f, g \rangle = \int f(x)h(x)d\mu(x)$$

where $\mu$ is a $G$-invariant measure finite on compact subsets. The proof that this is an example are precisely lemma 2.2 and lemma 2.3 of [13] p. 243,244.

Let $M$ be a simplicial complex, and let $G \times M \to M$ be a proper simplicial action of a discrete group $G$. Assume that the quotient $M/G$ is compact. Let $\mathfrak{M}_M$ the family of (finite) subgroups of $G$ having non empty fixed point set in $M$, i.e.

$$\mathfrak{M}_M = \{ H < G \mid MH \neq \emptyset \},$$

where

$$MH = \{ x \in M \mid hx = x, \text{ for every } h \in H \}.$$
Definition 6.12. The action $G \times M \rightarrow M$ is said to be of bounded isotropy if the order of the elements in $\mathfrak{g}_M$ is uniformly bounded, i.e. there is a constant $c_M$ such that $|H| < c_M$ for every $H \in \mathfrak{g}_M$.

Lemma 6.13. If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \rightarrow M$ is of bounded isotropy, then $M$ is of bounded geometry.

Proof. Take a point $x \in M$ and let $S(x)$ the set of simplices containing $x$. Denote by $p : M \rightarrow M/G$ the projection on the quotient. Then $p(S(x)) = S(p(x))$ and, therefore, $\#S(x) \leq \#S(p(x)) \cdot c_M \leq N \cdot c_M$, where $N$ is the bound on the number of simplices containing a point in $M/G$. □

Lemma 6.14. A proper space $M$ with proper, co-compact proper action $G \times M \rightarrow M$ of a discrete group $G$ is of bounded isotropy.

Proof. Choose finite family $(U_i, G_i), \ i = 1, \ldots, N'$ such that $U_i \subset M$ are open subsets and $G_i < G$ are finite subgroups such that, if a point $x \in gU_i$ for some $g \in G$, then one has that $G_x < gG_ig^{-1}$. Therefore $\beta = \max_i |G_i|$ is a bound on the orders of isotropy groups of points in the space $M$. □

We will not discuss the functorial properties of the $C^*$-algebras associated to coarse structures of a proper metric space, called in the literature "morphism covering a coarse map". However, we will need a restriction map for the inclusion of a boundary component into a bordism satisfying some additional assumptions, see the coments preceeding section 7.

Definition 6.15. Let $X$ be a proper space and $M$ a $G$-presentation of $X$. A Hilbert-Poincaré complex is equivariantly analytically controlled if it is analytically controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$, i.e. the modules in the complex are objects of these categories, the operator $B = b + b^*$ is controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$ and the duality operator $S$ is a morphism in the category $\mathfrak{A}(X, G, M)$.

In the following, by controlled in the case of a complex of Hilbert modules we mean equivariantly analytically controlled and in the case of an operator we mean controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$.

Theorem 6.16. If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \rightarrow M$ is of bounded isotropy and orientation preserving, then its Higson-Roe non-commutative signature is a homotopy and bordism invariant in the controlled category.

Proof. As $M$ is of bounded geometry, its simplicial chain and cochain complexes are geometrically controlled. The action either permutes or fixes simplices and is therefore unitary, and the fundamental cycle of such a triangulation is invariant. By theorem 3.14 in [2, p.309] geometric control implies analytic control. The comment before section 3.2 on [2, p.310] ensure that this is true also in the equivariant setting. This means that the $l^2$-chain complex $C^2(M)$ of $M$ is an example of an analytically controlled Hilbert-Poincare complex.

In the case of bordism invariance, one shall assume that one has a triangulated bordism such that the simplices in the boundary coincide with the given triangulation of $M$.

The result now follows as a corollary of theorems 5.12 and 7.9 of [1]. □

7. Bordism invariance of the coarse index

In this section, we review the approach to bordism invariance of the coarse index due to C. Wulff [14] and extend it to the context of manifolds with proper actions...
of a discrete group. This section’s results benefited in a fundamental way from
remarks of an anonymous referee. The authors thank her or him.

We recall that given a smooth manifold with a proper, smooth $G$-action $M$, the
existence of $G$-invariant Riemannian metrics due to Palais [12] implies the existence
of a $G$-invariant geodesic length metric on $M$. Recall that this geodesic metric is
proper in the sense of metric geometry, meaning that closed balls are compact. If the
original manifold is geodesically complete, then so is the one with the $G$-invariant
metric. In the case of a non co-compact manifold $M$, there might be many quasi-
isometry classes of $G$-invariant metrics on $M$. We will fix, however, a boundedly
controlled coarse structure coming from a particular $G$-invariant, complete geodesic
metric structure for the remain of this section.

In order to define adequately the (coarse index) boundary maps and the functori-
ality properties after $K$-theory, certain remarks on the bounded coarse structure
on a proper geodesic manifold are pertinent. References for the bounded coarse
bounded structure, and other ones defined on a geodesic metric space include [3],
chapter 6, although we specialize here to the Riemannian manifold case.

**Definition 7.1** (Coarse map in the bounded metric structure). Let $M$ and $N$ be
proper Riemannian $G$-manifolds equipped with $G$-invariant geodesic length metrics
$d_M$ and $d_N$. A map $f : N \to M$ is a coarse map if

1. The inverse image of every closed ball is compact.
2. For every $R > 0$, there exists $\delta > 0$, such that $d_N(x, x') < R$ implies
   $d_M(f(x), f(x')) < \delta$.

A coarse map induces a $C^*$-homomorphism of the algebras of locally compact and
finite propagation operators by lemma 6.3.12 in [3].

**Definition 7.2** (Referenced Manifolds). Let $M, N_1$ and $N_2$ be proper, oriented
$G$-manifolds of dimension $n$ furnished with an orientation preserving $G$-action.

Assume that $M, N_1$ and $N_2$ are furnished with the bounded coarse structure
associated to a $G$-invariant geodesic length metric. The manifolds $N_1$ and $N_2$ are
referenced manifolds with respect to $M$ if they are furnished with a $G$-equivariant
coarse map $i_1 : N_1 \to M$ and $i_2 : N_1 \to M$.

**Definition 7.3** (Referenced Bordism). Let $N_1$ and $N_2$ be referenced manifolds
with respect to $M$.

A referenced bordism from $(N_1, f_1)$ to $(N_2, f_2)$ is a referenced $G$-manifold $W$,
together with a coarse map $F : W \to M$, such that there exists a positive real number $K
with the property that the diagram depicting the inclusions of the boundary components $j_i : N_i \to \partial W$,

![Diagram](attachment://diagram.png)

commutes up to $K$, meaning that the inequalities

$$d_M(f_i(n), F \circ j \circ j_i(n)) < K$$

hold for $i = 1, 2$ and every $n \in N_i$. 
Definition 7.4. [Analytical referenced bordism] Let $M$ be a complete, proper metric $G$-space with an action of bounded isotropy. The analytical referenced bordism group $\Omega_{n,eq}(M)$ is the group with generators $(N, f, E, b)$, such that

- $N$ is an $n$-dimensional referenced manifold with respect to $M$, with bounded isotropy,
- $f : N \rightarrow M$ is an equivariant coarse map;
- $E$ is a $G$-$X$-Hilbert module with presentation $N$, i.e. an equivariant $N$-module with $X = N/G$,
- $b : E \rightarrow E$ is boundedly controlled operator.

Two of such generators $(N_1, f_1, E_1, b_1)$ and $(N_2, f_2, E_2, b_2)$ are said to be referenced-bordant with respect to $M$ if there exists a referenced bordism $(W, F, E, B)$ with respect to $M$, between $N_1$ and $N_2$, together with a coarse map $F : W \rightarrow M$, inclusions $j_i : N_i \rightarrow W$, which induce isometries of Hilbert spaces $E_i \rightarrow E$, and a controlled operator $B$, restricting to $b_i$.

If the space $M$ is a proper oriented manifold of bounded isotropy, then one defines the fundamental class in the group $\Omega_{n,eq}(M)$ by taking $f = \text{id}$ and, for example, $E = \Omega^*_{L^2}(M)$, the $L^2$-completion of the de Rham complex of $M$ and $b$ as the signature operator. Although this is an unbounded operator, the generalized conditions of analytical control meet (meaning that the Cayley transform is locally compact and of finite propagation and the resolvent has finite propagation).

One can also, take $E' = C^*_L(M) \oplus C^*_L(M)$ and $b = B_S$ as in [3], where $S$ is the Poincaré duality homomorphism completion. Both choices coincide in terms of index by theorems 5.5 and 5.12 in [2], using the version of analytic control defined in here.

Definition 7.5. [Coarse Fundamental Signature Class] Let $(N, f, E, b)$ be a referenced manifold with respect to $M$. The coarse fundamental class of $b$ is the class in $K_n-1(A(X, G, M)/C(X, G, M))$ of the boundedly controlled operator $b$ associated to the Hilbert-Poincaré complex $E$.

We interpret now the main result of [14] in an equivariant setting:

Theorem 7.6. The coarse fundamental class is a referenced bordism invariant.

Proof. The situation is completely analogous to [14], where the invariance is seen to be a consequence of the naturality of the assembly map. Consider the diagram of $G$-equivariant inclusions, which are assumed to give coarse maps.

\[
\begin{array}{ccc}
N_1 & \rightarrow & \partial W \leftarrow N_2 \\
\downarrow & & \downarrow \\
W & \rightarrow & M \\
\downarrow F & & \downarrow \\
& W & \end{array}
\]

The long exact sequence in $K$-theory of $C^*$-algebras gives:

\[
\begin{array}{ccc}
K_{p+1}(\mathfrak{A}(W/\partial W)/\mathfrak{C}(W/\partial W)) & \xrightarrow{\partial} & K_p(\mathfrak{A}(\partial W)/\mathfrak{C}(\partial W)) \\
\downarrow & & \downarrow \Lambda_W \\
K_p(\mathfrak{A}(W)/\mathfrak{C}(W)) & \rightarrow & K_p(\mathfrak{A}(W)/\mathfrak{C}(W)) \\
\downarrow i_* & & \downarrow i_* \\
K_p(C^*(\partial W)) & \rightarrow & K_p(C^*(W))
\end{array}
\]
Where the upper morphism $\partial$ is the connecting homomorphism, and the vertical morphisms are coarse assembly maps.

The functoriality of the index morphism, assembly map gives

$$F^*_i(A_W(i_2([b_2]))) = F^*_i(A_W(i_1([b_1]))) .$$

$\square$

**Corollary 7.7.** The coarse fundamental class gives a group homomorphism

$$C : \Omega^{an,eq}_n(M) \rightarrow K_{n-1}(\mathfrak{A}(X,G,M)/\mathfrak{E}(X,G,M)) .$$

**Definition 7.8 (Analytical signature).** The analytical signature of a referenced manifold $(M, f, E, b)$ is given as the composition of the coarse fundamental morphism $C$ together with the coarse assembly map. (Recall that the coarse assembly map for $X$ is the homomorphism $\mu : K^G_i(M) \cong K_{i+1}(D^*_G(M)/C^*_G(M)) \rightarrow K_i(C^*_G(M))$, where the first instance of $K$ denotes equivariant $k$-homology for spaces, and all the others $C^*$-algebra $K$-theory, the first isomorphism is given by Paschke duality, as written in [13, p.242], and the second is the boundary map in the long exact sequence of $K$-groups associated to the ideal $C^*_G(M)$ in $D^*_G(M).$)

In the following we shorten the notation $\mathfrak{A}(X,G,M)$, $\mathfrak{E}(X,G,M)$ by $\mathfrak{A}(M)$, $\mathfrak{E}(M)$ respectively.

Recall that a directed bordism, in the sense of 7.4, produces an algebraic Hilbert-Poincaré complex with boundary, as in the sense of 4.1. Hence, the algebraic signature constructed in 3.5 is well defined after passing to geometric bordism.

**Definition 7.9.** The algebraic signature $\Sigma$ is the group homomorphism

$$\Omega^{an,eq}_n(M) \rightarrow K_n(C^*_r(G))$$

described in 3.5. The fact that the algebraic signature descends to the referenced bordism groups follow from the fact that a referenced bordism gives an algebraic Hilbert-Poincaré complex with boundary, [13] proves that the algebraic signature is the same, and thus the map is well defined on referenced bordism classes.

### 8. Mapping surgery to analysis

In this section, we will state the main theorem of this paper:

**Theorem 8.1.** Let $M$ be a proper $G$ manifold with a bounded isotropy action. Assume that the quotient $M/G$ is compact. Then, we have the following homomorphism

$$\Omega^{an,eq}_n(M) \rightarrow K_n(C^*_r(G))$$

where the maps are defined as follows: the map $C$ is the coarse fundamental class, [13] the map $\omega_1$ is the isomorphism constructed in [13] (denoted by $\omega_4$ in page 242), the group homomorphism $\omega_2$ is induced by the up to $G$-equivariant homotopy unique map $M \rightarrow E_G$, and $\mu$ denotes the analytical Baum-Connes assembly map in $KK$-theory. We will call the composition

$$\mu \circ \omega_2 \circ \omega_1 \circ C : \Omega^{an,eq}_n(M) \rightarrow K_n(C^*_r(G))$$

the analytical signature.
Proof. The analytical Assembly map \( \mu : KK_G^n(C_0(\mathbb{E}G), \mathbb{C}) \to K_n(C^*_r(G)) \) is given by the composite of the descent homomorphism
\[
KK_G^n(C_0(\mathbb{E}G), \mathbb{C}) \to KK^n(C_0(\mathbb{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G)
\]
followed by composing with the map given by the Kasparov product with the Mishchenko-Fomenko line bundle for \( E_G \),
\[
KK^n(C_0(\mathbb{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G) \to KK^n(\mathbb{C}, C^*_r(G)).
\]
By \( KK \)-theoretical homotopy invariance, the composite map
\[
KK_G^n(C_0(M), \mathbb{C}) \xrightarrow{\omega_2} KK_G^n(C_0(\mathbb{E}G), \mathbb{C}) \xrightarrow{\mu} K_n(C^*_r(G))
\]
agrees with the composite
\[
KK_G^n(C_0(M), \mathbb{C}) \to KK^n(C_0(M) \rtimes_r G, \mathbb{C} \rtimes_r G) \to KK^n(\mathbb{C}, C^*_r(G)),
\]
which consists of the descent homomorphism followed by the Kasparov product with a Mishchenko-Fomenko element for \( C_0(M) \) (called \( w_5 \) and \( w_6 \) in [13], p. 242, respectively.)

By [13, 10], the bordism relations are compatible.

Finally, by commutativity of the diagram in page 242 of [13], the assembly maps commute. \( \square \)

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