Symplectic four-manifolds and conformal blocks

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Abstract

We apply ideas from conformal field theory to study symplectic four-manifolds, by using modular functors to “linearise” Lefschetz fibrations. In Chern-Simons theory this leads to the study of parabolic vector bundles of conformal blocks. Motivated by the Hard Lefschetz theorem, we show the bundles of SU(2) conformal blocks associated to Kähler surfaces are Brill-Noether special, although the associated flat connections may be irreducible if the surface is simply connected and not spin.

1 Introduction

This note is intended to publicise the following juxtaposition, the potential of which is surely not fully realised. (i) A symplectic four-manifold $X$ can be described via Lefschetz pencils $f : X \longrightarrow \mathbb{P}^1$, which are algebraically encoded as representations $\rho_{X,f}$ of free groups in mapping class groups. These representations are not canonical, but become so (asymptotically) under a stabilisation procedure which involves a sewing operation on the underlying fibres of the pencil. (ii) Chern-Simons theory gives rise to (projective) linear representations $\rho_{G,k}$ of mapping class groups, once a compact Lie group $G$ and level $k$ are fixed. These representations are not independent, but behave coherently under sewing operations of underlying families of surfaces.

Although the similarity above motivates our study, we are unable to take real advantage of the coherence, and accordingly achieve only modest results. A flat connection in a vector bundle over a punctured curve determines a parabolic bundle; the moduli space of parabolic bundles is stratification by the “Brill-Noether” loci of bundles which admit more holomorphic sections than predicted by Riemann-Roch. By studying restriction maps from holomorphic bundles on a Kähler surface $X$ to bundles on embedded curves we will prove:

(1.1) Theorem:

(A) If $X$ is simply connected and not spin then $\rho_{SU(2),1} \circ \rho_{X,f}$ is irreducible.

(B) For $k \gg 0$ $\rho_{SU(2),k} \circ \rho_{X,f}$ is Brill-Noether special.

Result (A) represents the failure of a “non-abelian” Hard Lefschetz theorem, and is a genericity result for the SU(2) Chern-Simons representations. (The result apparently generalises

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from level \(k = 1\) to the case \(k + 2\) prime, but we will not prove that here; the \(k = 1\) case is sufficient to answer in the negative a question of Tyurin, as we discuss.) In contrast, result (B) shows that the representations arising from Kähler Lefschetz pencils are special from at least one point of view, and in principle provides a new obstruction to integrability for symplectic four-manifolds. (Other known obstructions that go beyond topology come from gauge theory; we contrast Result (B) with related ideas from Donaldson theory at the end of the paper.)

The next section briefly recalls background material; the third discusses the Hard Lefschetz theorem and the first result above, and the final section discusses parabolic bundles. This paper fits into a general programme which replaces fibres of Lefschetz fibrations by moduli spaces of objects on those fibres. Surprisingly, when the substituted moduli space is linear, the resulting object seems less tractable than e.g. when one replaces fibres by their symmetric products [9]. Probably the right setting for these ideas has not yet been found; we hope, despite its preliminary flavour, the paper may encourage other people to think along these lines.

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2 Background

A Lefschetz pencil (or complex Morse function) on a smooth oriented four-manifold \(X\) is a map \(f : X \setminus \{b_1, \ldots, b_n\} \to \mathbb{S}^2\) defined on the complement of a finite set, submersive away from a disjoint finite set \(\{p_1, \ldots, p_{n+1}\}\), and conforming to local models \((z_1, z_2) \mapsto z_1/z_2\) near \(b_j\) and \((z_1, z_2) \mapsto z_1 z_2\) near \(p_i\), where the \(z_i\) are oriented local almost complex co-ordinates. Donaldson [7] has proved that all symplectic manifolds admit this structure. Topologically, \(X\) is swept out by surfaces, finitely many of which have complex ordinary double point singularities (at the \(p_i\)) and all of which meet at the \(b_j\) and are otherwise disjoint. There are other helpful viewpoints:

(i) A Lefschetz pencil \(f : X \to \mathbb{S}^2\) induces a representation \(\rho_{X,f} : \pi_1(\mathbb{S}^2 \setminus \{\text{Crit}\}) \to \Gamma_g\), which is well-defined up to global conjugation and the action of the braid group by automorphisms of the domain. A loop encircling one critical value maps to a positive Dehn twist about the vanishing cycle [16].

(ii) A metric on \(X\) gives a map \(\mathbb{S}^2 \setminus \{b_j\} \to M_g\) which extends to a map \(\phi_f : \mathbb{S}^2 \to \overline{M}_g\) into the Deligne-Mumford moduli space of stable curves. The homology class of the image is characterised by the fact that \(\sigma(X) = 4\langle \lambda_g, [\phi_f(\mathbb{S}^2)] \rangle - \delta\) where \(\lambda_g\) denotes the Hodge class and \(\delta\) the total number of singular fibres of the pencil. As a consequence, one can show that \(\langle \lambda, [\phi_f(\mathbb{S}^2)] \rangle > 0\) always [36].

Donaldson’s existence theorem is canonical in a certain asymptotic limit: the closures of fibres of the Lefschetz pencil are symplectic submanifolds Poincaré dual to \(\kappa[\omega]/2\pi\), and if the degree \(\kappa\) is large enough – depending on the particular \(X\) – then the representation of
the free group above is canonically associated to $X$ (up to global conjugation and the action of the braid group $B_n$). There is an explicit procedure \[2\], \[37\] which relates pencils of one degree $\kappa$ to pencils of a larger degree $2\kappa$; the degree $2\kappa$ pencil is obtained by perturbation of a degenerate family of hyperplane sections $\{s^2 + \lambda st\}_{\lambda \in \mathbb{P}^1}$ for $s, t$ degree $\kappa$ sections. One feature of this stabilisation is that the generic fibre (i.e. far away from $\lambda = 0$) at degree $2\kappa$ is obtained by connect summing two fibres of the degree $\kappa$ pencil at all their intersection points (the base-points $\{b_j\}$ above). This is where a family surgery enters, in the vein of the opening remarks of the Introduction, justifying the first half of the “juxtaposition”.

Now we briefly review Chern-Simons theory, as relevant for our needs. Proofs and details can be found in \[33\], \[11\]. Fix a Riemann surface $\Sigma$ and a gauge group $G$, for us always $SU(2), SO(3)$ or $U(1)$. Let $M_{2,L}(\Sigma)$ denote the moduli space of rank two stable bundles on $\Sigma$ with fixed determinant equal to $L$; $G = SU(2)$ corresponds to $L \cong O$. The moduli space is a smooth complex variety, closed if $\text{deg}(L)$ is odd, and with compactification the moduli space of semistable torsion-free coherent sheaves (with fixed Hilbert polynomial) when $\text{deg}(L)$ is even. The compactification locus is a copy of the Kummer variety of the curve, which arises as the singular locus in the moduli space if $g > 2$. The resulting projective varieties have Picard group $\mathbb{Z}$, generated by a determinant line bundle $L_{\text{det}}$ (described in more detail in the proof of \[33\]).

Let $V_k(\Sigma)$ denote the space of holomorphic global sections of $L_{\text{det}} \to M_g$, see \[41\]. (In fact, $V_k$ has a natural parabolic structure over $M_g \setminus M_g$, since the flat connexion on $V_k$ provided by the theorem above has simple poles along the divisor of nodal curves, but we shall not use this.)

\begin{equation}
V_k(g) = \text{rank}(V_k(\Sigma_g)) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\frac{1}{\sin(j\pi/(k+2))}\right)^{2g-2}. (2.1)
\end{equation}

(first conjectured by Verlinde \[47\]). We suppress the group $G$, equivalently the Chern classes of the relevant semistable sheaves, from the notation for simplicity. The Verlinde bundle $V_k \to M_g$ is the holomorphic vector bundle over the moduli space of curves whose fibre at $\Sigma$ is $V_k(\Sigma)$; that these spaces fit together to give a vector bundle follows from elliptic regularity.

(2.2) Theorem: \[15\], \[32\], \[22\] Fix a gauge group $G$ and level $k$. The holomorphic vector bundle $V_k \to M_g$ carries a projectively flat connexion, defining a representation $\rho_{G,k} : \Gamma_g \to \mathbb{P} \text{End}(V_k)$.

We will need several other properties of these bundles:

(i) The bundle $V_k \to M_g$ has a distinguished extension to a holomorphic vector bundle $V_k \to \overline{M}_g$, see \[41\]. (In fact, $V_k$ has a natural parabolic structure over $\overline{M}_g \setminus M_g$, since the flat connexion on $V_k$ provided by the theorem above has simple poles along the divisor of nodal curves, but we shall not use this.)

(ii) The first chern class $c_1(V_k \to \overline{M}_g) = \frac{3kV_k(g)}{k+1} \lambda \in H^2(\overline{M}_g)$. This doesn’t seem to be well-known: we give a (loop-group inspired) proof in the last section.

(iii) If $G = U(1)$ the conformal blocks are theta-functions (by lifting holomorphic sections of a line bundle over the Jacobian to periodic functions on the universal cover), cf. \[54\]. The representation $\rho_{U(1),k}$ factors through the automorphism group of a finite Heisenberg group $0 \to (\mathbb{Z}/k\mathbb{Z})^{2g} \to \text{Aut} \to S_{\mathbb{P}^{2g}(\mathbb{Z}/k\mathbb{Z})} \to 0$, \[23\], hence is irreducible.
(iv) The representations $\rho_{SU(2),1}$ and $\rho_{U(1),2}$ are equal; this is part of "rank-level duality" \[\text{[11], see also [22, 46]}.\]

For surfaces with marked points or boundary, there are also spaces of conformal blocks defined using moduli spaces of parabolic bundles with fixed conjugacy [22]. The extended theory is a "modular functor" [33], i.e. if $\Sigma = \Sigma_1 \cup C \Sigma_2$ where $C \subset \Sigma_i$ is a closed one-manifold, then $V_k(\Sigma) = V_k(\Sigma_1) \otimes V_k(\Sigma_2)$. Modularity implies that the Verlinde bundles are compatible under sewing in the following sense: given two families of Riemann surfaces over a base $B$ and identification diffeomorphisms of (some subsets of) the boundaries over the base, the projectively flat connexion in the vector bundle associated to the glued surfaces splits in the tensor product decomposition of the vector bundle. This justifies the second half of the "juxtaposition" from the Introduction.

3 Hard Lefschetz

Let $f: X \to S^2$ be a Lefschetz fibration; then $H^1(X)$ is the subgroup of monodromy invariants of $H^1(F)$, and if $X$ is Kähler the Hard Lefschetz theorem asserts that $H^1(X) \subset H^1(F)$ is a symplectic subspace, equipped with the non-degenerate skew-form $(a,b) = \int_X \omega \wedge a \wedge b$. (For general symplectic pencils the monodromy representation will not be completely reducible and the invariant subgroup of $H^1(F)$ and the subgroup generated by Poincaré duals of vanishing cycles will not be orthogonal with respect to cup-product.)

(3.1) Proposition: If $X$ is Kähler and $b_1(X) > 0$ then the representations $\rho_{U(1),k} \circ \rho_{X,f}$ are reducible for every $k$.

Proof: The abelian Verlinde representations $\rho_{U(1),k}$ factor through the metaplectic representation of the double cover $Mp_{2g}(\mathbb{Z})$ of the symplectic group. The vector space for a genus $g$ surface at level $k$ is a $k^g$ dimensional space of theta-functions:

$$H^0(\text{Jac}(\Sigma_g) ; \mathcal{L}^k) = \langle \vartheta_i^j \mid 1 \leq i \leq k, 1 \leq j \leq g \rangle$$

$$\vartheta_i^j(\tau, \omega) = \sum_{\mathbf{u} \in \mathbb{Z}^g + e_{i}^j} \exp\left(\frac{i\pi}{k}(\mathbf{u} \cdot \tau \cdot \mathbf{u}) + 2i\pi(\mathbf{u} \cdot \omega)\right)$$

where $\tau \in \mathfrak{h}_g$, the Siegel upper half-space, $\omega \in \mathbb{C}^g$ and $e_{i}^j$ has value $i/k$ in the $j$-th position and zeroes elsewhere. This set of generators for the space of conformal blocks leads to a "factorisation" property.

Suppose a symplectic vector space $W$ is written as a product of symplectic subspaces $W = U \oplus U'$ of dimensions $2h, 2g - 2h$ respectively. We have a natural inclusion of symplectic groups $Sp_{2h}(\mathbb{Z}) \times Sp_{2g-2h}(\mathbb{Z}) \hookrightarrow Sp_{2g}(\mathbb{Z})$. On the image of this inclusion, the metaplectic representations factorise as a tensor product:

$$\rho_{U(1),k}[g](A \oplus B) = \rho_{U(1),k}[h](A) \otimes \rho_{U(1),k}[g-h](B)$$
in an obvious notation. This is proved by the following calculation with exponentials. Fix a vector \( l = l_1 \oplus l_2 \) which indexes a particular choice of \( \vartheta \)-characteristic \([23]\) which above is given by the choice of labels \( e_j \quad i \). Then we have

\[
\sum_{m \in \mathbb{Z}^{h+g-h}} \exp \left\{ \frac{i\pi}{k} \left( (m + l)^t \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} (m + l) \right) \right\} = \\
\left( \sum_{\alpha \in \mathbb{Z}^{h}} \exp \left\{ \frac{i\pi}{k} \left( (\alpha + l_1)^t A(\alpha + l_1) \right) \right\} \right) \left( \sum_{\beta \in \mathbb{Z}^{g-h}} \exp \left\{ \frac{i\pi}{k} \left( (\beta + l_2)^t B(\beta + l_2) \right) \right\} \right).
\]

For a Kähler Lefschetz fibration there is a symplectic splitting \( H^1(F) = H^1(X) \oplus \text{Ann}(H^1(X)) \), and the homological monodromy is trivial on the first factor. It follows that the Verlinde representation is of the form \((\text{id} \otimes \phi)\) which is clearly reducible.

To generalise, we think of the reducibility arising from the presence of line bundles on the surface \((b_1(X) > 0)\); Gieseker and O’Grady \([28]\) have shown that every projective surface has a positive-dimensional moduli space of stable bundles with trivial determinant and fixed \( c_2 = r \) once \( r \) is sufficiently large. However, there is no analogous reducibility property for the representations in general. To see this we begin with a statement about the monodromy groups of Lefschetz pencils, obtained jointly with Denis Auroux.

\[\textbf{3.2) Proposition:} \quad \text{Let } X \text{ be a symplectic manifold. If } X \text{ is spin, the homological monodromy representation of any Lefschetz pencil is not onto } Sp_{2g}(\mathbb{Z}). \text{ If } X \text{ is not spin and } H_1(X; \mathbb{Z}/2\mathbb{Z}) = 0, \text{ the homological monodromy representation is surjective for any pencil given by stabilisation (degree doubling).}\]

\[\textbf{Proof:} \quad \text{The key ingredient is a result of Janssen } [20] \text{ which in turn relies on work of Gabrielov and Chmutov } [5]; \text{ we give their results translated into our language. Take a pencil of curves satisfying the constraints that (i) two vanishing cycles have homological intersection number one and (ii) all the (homology classes of the) vanishing cycles are conjugate under the monodromy group of the pencil. There are pencils that violate these conditions, for instance the genus two pencil on } \mathbb{T}^2 \times S^2. \text{ However, the first condition always holds after stabilisation, by a quick look at the pictures of Auroux and Katzarkov } [2], \text{ whilst the second condition holds for large enough degree by a result of Amoros, Munoz and Presas } [1]. \text{ (In the algebraic setting, the second condition follows from the irreducibility of the dual variety for a projective embedding of the surface, together with the Lefschetz hyperplane theorem; these imply that the fundamental group of the complement of the dual variety is normally generated by one element.) Chmutov proves that for any such pencil of curves, the homological monodromy contains the kernel of the natural map } Sp_{2g}(\mathbb{Z}) \rightarrow Sp_{2g}(\mathbb{Z}/2\mathbb{Z}) \text{ (this is generated by the squares of Dehn twists). Janssen builds on this to deduce that either the monodromy of a} \]
pencil is contained in the hyperelliptic mapping class group (which is an easy exceptional case, not preserved by doubling), or is full, or maps onto the subgroup of $Sp_2(Z/2Z)$ which preserves a quadratic form.

Now we remark that a four-manifold with $H_1(X;Z/2Z) = 0$ is spin iff the intersection form is even. Using this, it is easy to see that $X$ is spin iff the associated Lefschetz fibration over a disc (blow up the base points and remove a smooth fibre) is spin. However, as Stipsicz points out in [38], this Lefschetz fibration has a distinguished handle decomposition, in which handles are added to $\Sigma \times D$ along the vanishing cycles with framing $-1$. A spin structure $q: H_1(\Sigma;Z/2Z) \to Z/2Z = \{0,1\}$ on $\Sigma \times D$ extends across a handle whenever the associated vanishing cycle evaluates to $+1$.

There are two quadratic forms $q$ on $H_1(\Sigma_g;Z/2Z)$, to isomorphism, determined by their Arf invariant (in suitable symplectic bases they correspond to $(x,y) \mapsto \sum x_iy_i$ or to $(x,y) \mapsto \sum x_iy_i + x_1^2 + x_2^2$). If either of these forms is preserved by the monodromy of a pencil, $X$ is spin; fixing a form fixes either an even or odd spin structure on a fibre of the pencil, hence a spin structure on $D \times \Sigma$ along the vanishing cycles. (Conversely, a spin structure on $X$ induces one on the codimension zero subset $D \times (\Sigma_g \setminus \{b_j\})$ and hence one on $\Sigma$, the parity of which is reflected in the monodromy group of the pencil.)

It is possible that some symplectic manifolds admit Lefschetz pencils of arbitrarily high degree whose monodromy group is the entire mapping class group; the amazing computations of Auroux and Katzarkov [2] give an obvious route to attack such a question. Roberts [31] has shown that the $SU(2)$-theory mapping class group representations on conformal blocks remain irreducible whenever $k + 2$ is prime. His combinatorial method of proof, together with the pictures of [2], strongly suggests the composite representations $\rho_{SU(2),k} \circ \rho_{X,f}$ are irreducible for all such $k$. The previous Proposition, together with Remarks (iii) and (iv) after (2.2), prove Theorem (A) from the Introduction.

The Proposition is also relevant to a question of Tyurin; to see this we need to gather some facts about the behaviour of rank two stable bundles under restriction. We will use a strong form of the restriction theorem that can be garnered from [43], [12] and [19]. Let $X$ be a projective surface with hyperplane class $H$. Let $C$ denote an arbitrary smooth element of the linear system $|NH|$.

- Let $V$ be an $H$-stable $SU(2)$ or $SO(3)$ bundle on $X$. If $N > -p_1(adV)$ then $V$ restricts to a stable bundle on $C$.
- For $N \geq N(p_1) \gg 0$ restriction defines a holomorphic embedding $M_r(X) \to M_{2,\mathcal{O}}(C)$ for each moduli space with $0 \leq r \leq -p_1$.

For non-hyperelliptic curves the determinant line is very ample, i.e. $M_{2,\mathcal{O}}(C) \hookrightarrow \mathbb{P}(V_k(C)^*)$ for all $k \geq 1$, and we can compose the restriction maps with these embeddings.

**Question:** (Tyurin, [43], [42]) With notation as above, at level $k = 1$, does the canonical connexion preserve the image of the fibrewise restriction map as we vary $C$ amongst smooth curves (all or none of which are hyperelliptic) in its linear system?
Tyurin conjectured this was true as we vary $C$ locally and asked when it was true globally; in which case we’ll say that the stable bundles on the surface $X$ are globally invariant. Not surprisingly, the irreducibility property for $\rho_{SU(2),1} \circ \rho_X$, described above implies that global invariance is exceptional. We’ll call a polarisation $H$ of an algebraic surface $X$ “appropriate” if (i) $H$ is even and $K_X$ is even or (ii) $H = 2nH + K_X$ for some arbitrary polarisation $H$ and $n > 0$, when $K_X$ is odd. These obviously exist, so we can consider “appropriate” pencils $\mathbb{P}^1 \subset |\kappa H|$.

(3.4) Proposition: Let $X$ be a non-spin surface of general type with $H_1(X; \mathbb{Z}/2\mathbb{Z}) = 0$; fix an appropriate polarisation $H$ on $X$ and an appropriate pencil of degree $\kappa$. The stable bundles on $X$ of Chern class $c_2$ are not globally invariant for $\kappa$ sufficiently large.

Proof: Complex surfaces have non-empty moduli spaces $M_r(X)$ of $SU(2)$ bundles (indeed the moduli space contains smooth points as soon as $c_2 = r > b_4(X) + 2$ [12]), with well-understood compactifications. We will take the second Chern class large enough for the moduli space to be “generically smooth” [9], meaning it is of the expected dimension and the singular locus has codimension greater than one (explicit bounds on the required $c_2$ are known). The singularities are normal and algebraic sections of line bundles over the moduli space contains smooth points as soon as $c$.

Fix a smooth curve $C \in |\kappa H|$ in $X$, $\kappa \gg r$; by restriction we can suppose $M_r(X) \subset M_{2,0}(C)$. Let $L_\kappa = |\det_{im(r)}| \to M_{stab}(X)$ be the line bundle on $M_r(X)$ given by restricting the determinant line from $M_{2,0}(C)$. We claim that $h^0(L_\kappa)$ grows at most polynomially with $\kappa$. This is not completely trivial, since $\mbox{Pic}(M_r(X))$ is complicated and the dependence of $L_\kappa$ on $\kappa$ is not linear [19]; but it reduces to an argument of Donaldson from [6]. The determinant line bundle on $M_{stab}(C)$ at a bundle $E \to C$ is defined as having fibre

$$\left(\mathcal{L}_{det}\right)_E = \Lambda^{max} H^0(E \otimes K_C^{1/2}) \otimes \Lambda^{max} H^1(E^* \otimes K_C^{-1/2})^{-1};$$

by Serre duality (and recalling that $E^* \cong E$ for $SU(2)$ bundles), we can simplify this to

$$\left(\mathcal{L}_{det}\right)_E = \Lambda^{max} H^0(E \otimes K_C^{1/2})^2.$$  

The choice of spin structure on $C$ plays no serious role, as explained in [8], p.382. The divisor sequence on $X$

$$0 \to F \to F(D) \to F(D)|_D \to 0$$

with $F = E \otimes \mathcal{O}(-NH + K_X)^{1/2}$ and $D = C \subset |NH|$, and the adjunction formula $K_C = (K_X + \mathcal{O}(C)|_C)$, gives the long exact sequence in cohomology

$$0 \to H^0(E \otimes \mathcal{O}(-NH + K_X)^{1/2}) \to H^0(E \otimes (NH + K_X)^{1/2}) \to \cdots$$

$$\cdots H^0(E|_C \otimes K_C^{1/2}) \to H^1(E \otimes \mathcal{O}(-NH + K_X)^{1/2}) \to \cdots$$

For $H$ ample and $N$ large enough the bundles $F \otimes \mathcal{O}(-NH + K_X)^{1/2}$ will have no cohomology except in the top dimension. Taking determinants, we have that

$$\left(\mathcal{L}_{det}\right)_{|C} = \Lambda^{max} H^0(E \otimes (NH + K_X)^{1/2})^2$$
where the relevant bundles and cohomology group all live on $X$. Although the dependence of this bundle on $N$ is not linear as we vary the choice of linear system, over the open smooth locus of $M_r(X)$ where a universal bundle on $X \times M_{\text{stab}}(X)$ exists, all of the bundles $L_\kappa$ are pushforwards $\text{det}(R^0\pi_*E \otimes L^\kappa)$, where $E$ is the universal bundle, $L$ is the bundle on $X$ with first Chern class $H$ pulled back to $X \times M_{\text{stab}}(X)$, and $\pi$ denotes the projection to $M_r(X)$. Moreover, once $N$ is large, Li has shown the bundles $L_\kappa$ are ample [25], [19]. It follows that $h^0(L_\kappa)$ can be computed as an Euler characteristic by Grothendieck-Riemann-Roch, hence is a polynomial in the characteristic classes of $E \otimes L^\kappa$ and the Pontrjagin classes of $M_{\text{stab}}(X)$, yielding the claim.

With this established, suppose stable bundles on $X$ are globally invariant for a Lefschetz pencil $f$ of high degree. There is a monodromy representation $\rho_{SU(2),1} \circ \rho_{X,f}: \mathbb{F}_n \to \mathbb{P}GL(N_\kappa)$. Each matrix in the image of this representation preserves the subvariety $M_{\text{stab}}(X) \subseteq \mathbb{P}^N_{\kappa}$ and hence in particular preserves the locus of hyperplanes

$$\{h \in (\mathbb{P}^N_{\kappa})^* \mid M_{\text{stab}}(X) \subset h\},$$

that is the set of conformal blocks vanishing completely on the subset of restriction. These hyperplanes are exactly the rays in the kernel of the natural map from $H^0(V_1(C)) \to H^0(L_\kappa)$. The rank of the first group grows exponentially with the degree $\kappa$ of the pencil on $X$, whereas the rank of the second group grows polynomially by the above. Hence, once $\kappa$ is large enough, the kernel is non-empty. It is obviously not full; the representation therefore admits a non-trivial invariant subspace, contradicting Theorem (A).

Global restrictions on monodromy shed no light on the “local” version of Tyurin’s question, which in any case can apparently not be sensibly formulated for symplectic as opposed to Kähler Lefschetz pencils. However, there are properties which make sense in general and which appear special in the Kähler context, which we address next.

4 Brill-Noether

For every fibre genus $g$, level $k$ and number of critical fibres $r$, fix once and for all a model of the symplectic representation space

$$\text{Hom}_+(g,k,r;c_1) = \text{Hom}_+(\pi_1(S^2 \setminus \{p_1, \ldots, p_r\}), \mathbb{P}U_{\kappa}(g))/\langle \text{Conj} \rangle.$$

Here the subscript $+$ indicates that we are fixing the holonomy data at each puncture to conform to the matrix $\rho_k(\tau)$ given by a positive Dehn twist in a non-separating curve, and $c_1$ denotes the topological degree of the bundles. This space is a connected symplectic orbifold, whose dimension is given by

$$\dim(\text{Hom}_+(g,k,r)) = r \dim(C_\tau) - 2 \dim(\mathbb{P}U_{\kappa}(g))$$

where $C_\tau$ denotes the conjugacy class in $\mathbb{P}U_{\kappa}(g)$ of the matrix $\rho_k(\tau)$. If we fix a projective unitary representation of the mapping class group, a Lefschetz fibration gives rise to a point of the quotient of $\text{Hom}_+$ by the braid group action of Hurwitz moves. The braid group
acts ergodically \[14\], so in the absence of invariant open sets one can look for invariant stratifications.

Any choice of complex structure \( J \in \mathcal{M}_{0,r} \) on \( \mathbb{S}^2 \setminus \{ p_i \} \) defines a projective moduli space of parabolic bundles \( \mathcal{M}_{\text{par}}(J) \), with the flags and monodromy at each puncture fixed to be the same local model, and a homeomorphism \( \psi_J : \mathcal{M}_{\text{par}}(J) \xrightarrow{\sim} \text{Hom}_+ \). The space of parabolic bundles \( E \) carries a natural stratification, given by the upper semicontinuous function \( E \mapsto h^0(E^*) \) in the case where \( c_1 > \text{rank}(E) \); the union of all the lower strata \( \{ E \mid h^0(E^*) > 0 \} \) is the Brill-Noether locus, a complex subscheme which is carried by \( \psi_J \) to a closed real subvariety of \( \text{Hom}_+(g,k,r;c_1) \). Taking the union over the images of these strata as we vary \( J \in \mathcal{M}_{0,r} \) defines a sequence of braid-group invariant subsets of \( \text{Hom}_+ \).

Each of these is nowhere dense for large \( k \): the Brill-Noether loci may have excess dimension, but their actual codimension grows with the virtual codimension, hence with \( k \). By contrast, the space \( \mathcal{M}_{0,r} \) is a smooth complex manifold of dimension \( r - 3 \) independent of \( k \). The dense open subset

\[ \mathcal{U} = \{ \rho \in \text{Hom}_+ \mid h^0(\rho^*; j) = 0 \text{ for every } j \in \mathcal{M}_{0,n} \}. \]

comprises the parabolic bundles which are Brill-Noether general for every complex structure on the base (there are moduli here since there are marked points). Note that the condition \( c_1(E) > \text{rk}(E) \) holds for conformal block bundles over \( \mathbb{P}^1 \) by Remark (ii) after \([22]\), proved below.

The Arakelov-Parsin theorem shows that in fact only finitely many conjugacy classes of representation \( \rho \) are realised by Kähler pencils, but it seems very hard to characterise or identify properties of this distinguished finite set of braid-group orbits, which lends the following some interest. Fix \( c_2 = r \) large enough for moduli spaces of bundles on \( X \) to be singular only in high codimension and fix a Lefschetz pencil of degree \( \kappa \gg r \) so that restriction maps are well-defined embeddings.

**4.1 Theorem:** If \( X \) is Kähler and \( f \) is a pencil of degree \( \kappa \), then \( \rho_{SU(2),k} \circ \rho_{X,f} \in \text{Hom}_+ \setminus \mathcal{U} \) for all sufficiently large levels \( k \).

**Proof:** If \( C \in |\kappa H| \), the restriction kernel is the subspace of conformal blocks \( \{ s \in V_k(C) \mid s|_{\mathcal{M}_r(X)} = 0 \} \) which vanish identically on the image of the restriction map on \( \mathcal{M}_r(X) \). If \( k \) is sufficiently large, as we vary \( C \) in its linear system we claim there is a short exact sequence

\[ 0 \to RK_k \to V \to \mathbb{C}^n \to 0 \]

of vector bundles over the base \( \mathbb{P}^1 \). The restriction of the determinant line bundle \( \mathcal{L}_{\text{det}} \) from \( \mathcal{M}_{2,O}(C) \) to \( \mathcal{M}_r(X) \) gives an ample line bundle \( \mathcal{L}_{\text{rest}} \) on the latter space which Tyurin proves does not depend on the choice of (smooth or nodal) curve \( C \) within its linear system \([13]\), and we need to see that all sections of this line bundle extend to \( \mathcal{M}_{2,O}(C) \). The cokernel of the restriction map is given by \( H^1(\mathcal{I}_{\mathcal{M}_r(X)} \otimes \mathcal{L}_{\text{rest}}^k) \), where \( \mathcal{I} \) is the ideal sheaf. Since \( \mathcal{L}_{\text{rest}} \) is ample, this higher cohomology group will eventually vanish. This gives the claim.

For a Lefschetz fibration \( f : X \to \mathbb{S}^2 \) the index of the d-bar operator on the dual of the Verlinde bundle is negative (cf. the start of this section). Using Grothendieck’s theorem that all vector bundles on the line split, together with \([12]\), the most stable splitting type for the
bundle $V_k^*$ will be $\mathcal{O}(1 - 3\lambda) \oplus \cdots \oplus \mathcal{O}(1 - 3\lambda) \oplus \mathcal{O}(-3\lambda) \oplus \cdots \oplus \mathcal{O}(-3\lambda)$, where the ratio of the number of factors of the first sort to the total rank goes to zero as $k$ increases. (In other words, the index is more negative than the rank, and the ratio of $c_1$ to rank as $k \to \infty$ approaches $-3\lambda$ from above.) Since $\langle \lambda, [S^2] \rangle > 0$ we see that generically (in the space of holomorphic structures) $V_k^*$ has no sections; but restriction kernels give rise to sections of $V_k^*$.

We still owe the computation of $c_1(V_k)$. This requires a background remark on the conformal field theory connexions. Fix a level $k$, and consider the family of connexions in bundles over moduli spaces $M_g^n$ (of genus $g$ curves with $n > 0$ parametrised boundary components) that arise in $SU(2)$ Chern-Simons theory. Each connexion has scalar curvature $c$, and the scalar – normalised with respect to a natural Kähler form – is independent of $g$ and $n$ (determined only by the level $k$, in fact $c = 3k/(k+2)$). This is proved in Segal’s loop group framework [33], by using the modularity of the Verlinde bundles to “localise” the curvature computation to the case where the surface is an annulus. There is a determinant line bundle over the moduli space $\overline{M}_g$, with fibre given by the determinant line of the underlying $\overline{\mathcal{J}}$-operator on the surface: $L_{\det}(C) = \Lambda^g H^0(K_C)^*$, and with $c_1(L_{\det}) = -\lambda$. Segal observed in [33, Appendix B] that determinant lines give modular functors (in particular $L_{\det}(\Sigma) = L_{\det}(\Sigma_0)$ canonically, when $\Sigma_0 = \Sigma \setminus D$), and that the scalar curvature of the associated theory is $c = -2$.

Recall that $\overline{M}_g$ is a homology manifold, and by [48] there is an integral basis for $H^2(\overline{M}_g)$ comprising the classes $\lambda$ and the Poincaré duals of the subvarieties defined by curves which are separated by a node into components of genus $i$ and $g - i$, with $0 \leq i \leq [g/2]$.

\textbf{(4.2) Proposition:} The first Chern class $c_1(V_k) = \frac{3k}{4\pi^2} \text{rk}(V_k) \lambda \in H^2(\overline{M}_g)$.

\textbf{Proof:} The tensor product of modular functors is also a modular functor with the central charge behaving additively, and it follows that if we take the association

$$\Sigma \mapsto V_k(\Sigma)^{(k+2)} \otimes \text{Det}(\Sigma)^{3k}$$

we define a flat (not just projectively flat) vector bundle over the moduli space $M_g$. Hence this bundle has trivial first Chern class, from which we quickly deduce that the given formula holds in $H^2(M_g)$. Strictly speaking, this argument only applies as it stands to spaces $M_g^n$ with $n > 0$, because the “loop group definition” of the connexions is only valid for surfaces with boundary (for a closed surface $\Sigma$ Segal defines $V_k(\Sigma) = V_k(\Sigma \setminus D)$ and shows this is independent of the disc $D$ which is removed). However, as will become clear below, $H_2(M_g)$ is generated by a 2-cycle which admits a lift to $M_g^1$ (arising from a fibred four-manifold $X$ which has a section $D$). Removing a neighbourhood of $D \subset X$ gives a coherent family of decompositions of the fibres into open surfaces union discs, and enables one to reduce the computations for closed surfaces to surfaces with non-empty boundary. This shows the required formula does hold in $H^2(M_g)$; to lift this to $H^2(\overline{M}_g)$ we proceed as follows.

Fix $g > 2$ and construct a surface bundle with fibre genus $g - 1$, with two disjoint sections of non-zero square $-i$, and with total space having non-zero signature $a$. (This can be easily done by modifying appropriate Lefschetz fibrations, as in [?].) Gluing the sections together...
and picking a fibrewise metric gives a family of nodal curves of genus $g$, and the base defines a curve $C \subset \overline{M}_g$, the image of a curve $C \subset M^2_{g-1}$. $C$ lies entirely in the divisor $\Delta$ of nodal stable curves; the normal bundle $\nu_{\Delta/\overline{M}_g}$ is canonically identified along $C$ with the tensor product of the tangent spaces to the two exceptional sections. It follows that $C \cdot \Delta = -2l$. Also fix a smooth surface bundle of genus $g$, parametrised by a curve $\Sigma_{sm} \subset M_g$, with total space having non-zero signature: $\Sigma_{sm} \cdot \lambda = N \neq 0$. We can assume $\Sigma_{sm}$ is the image of a surface $\Sigma_{sm} \subset M^2_g$ by insisting the surface bundle has a section.

Given $\Sigma \subset \overline{M}_g$, a curve arising from a fibration of curves with no reducible members, the homology class of $\Sigma$ is completely determined by the numbers $\Sigma \cdot \lambda = t, \Sigma \cdot \lambda = t'$, since $H^2(\overline{M}_g, \mathbb{Z}) = \langle \lambda, PD[\Delta_i] \rangle$, with $\Delta_i$ the components of the divisor of stable curves. Trivial algebra yields that

$$[\Sigma] = \frac{1}{N(t + t'a/2l)}[\Sigma_{sm}] - \frac{t'}{2l}[C]$$

in $H^2(\overline{M}_g; \mathbb{Q})$. Linearity and modularity now reduce the computation of $\langle c_1(V_k), [\Sigma] \rangle$ to the analogous pairings with $\Sigma_{sm}$ and $\tilde{C}$, which can be computed in terms of pairings with the relevant determinant lines by our initial reduction. From [2], the Hodge class $\lambda_g$ pulls back under the obvious map $\overline{M}_{g-1} \to \overline{M}_{g-1,2} \to \Delta$ to the Hodge class $\lambda_{g-1}$ on $\overline{M}_{g-1}$ (lifted via the forgetful map to the moduli space of curves with two marked points) so $C \cdot \lambda_g = a$.

The result now follows from [15].

Since $V_k \to M_g$ carries a projectively flat connexion, the higher Chern classes are incidentally given by $c_1 = \left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right)(\frac{a}{n})^1$, where $n = v_k(g)$ is the rank. This completes our treatment of Theorem (B); one can go a little further, and compare it with the kind of information obtained using Donaldson invariants. We will just outline the connection. Let $(X, \omega)$ be an integral symplectic manifold, and fix a level $k$ and a Lefschetz pencil $f$. The $k$-depth of $(X, f)$ is defined as $\max_{j \in M_0,} \{ h^0(V_k(f_N)^*, j) \}$; since we have factored out the choice of $j$, this is a symplectic invariant of the Lefschetz pencil. It refines Theorem (B) in the sense that it measures the depth of the braid group orbit of $\rho_{SU(2), k} \circ \rho_{X, f}$ in the Brill-Noether stratification.

There is a homeomorphism [10] between (i) the space of instantons on an $SU(2)$ bundle $E \to X$ with $c_2(E) = r$, where $X$ is equipped with its Kähler metric, and (ii) the space of stable holomorphic bundles topologically equivalent to $E$. Donaldson invariants are suitable intersection pairings on $M_r(X)$, while the depth invariants are related to $h^0(L^{k}_{\text{rest}})$, which for large $k$ is also an intersection pairing by Riemann-Roch. This pairing gives a bound on the size of trivial quotients of $V_k$ for one holomorphic structure on the base of the Lefschetz fibration, hence an estimate on the supremum over all complex structures, and hence a lower bound on the depth.

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