Exact Solutions for the (2+1)-Dimensional Boiti-Leon-Pempielli System

Y.H. Hu\textsuperscript{1} and C.L. Zheng\textsuperscript{1, 2}

\textsuperscript{1}College of Mathematics and Physics, Zhejiang Lishui University, Lishui 323000, China

\textsuperscript{2}Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China

Email: hyh323000@163.com

Abstract. The object reduction approach is applied to the (2+1)-dimensional Boiti-Leon-Pempielli system using a special conditional similarity reduction. Abundant exact solutions of this system, including the hyperboloid function solutions, the trigonometric function solutions and a rational function solution, are obtained.

1. Introduction
Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts which are not simplify understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose which has been quite untouched. As is known, to search for exact solutions of nonlinear partial differential equations, we can exploit different approaches, such as the Exp-function method [1, 2], the Lie group method of infinitesimal transformations [3], the nonclassical Lie group method [4, 5], the Clarkson and Kruskal (CK) direct method [6, 7] and the conditional similarity reduction method, etc [8, 9].

The basic idea of the CK direct method is that: for a given n-dimensional partial differential equation (PDE) with independent variables $x \equiv (x_0 = t, x_1, x_2, \cdots, x_n)$ and dependent variable $u$

\begin{equation}
F(u, u_t, u_{x_1}, u_{x_2}, \cdots) = 0,
\end{equation}

where $F$ is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. For many real physical systems, one may use its ansatz in the following form

\footnote{To whom any correspondence should be addressed.}
\[ u = \alpha(x_1, x_2, \cdots, x_n) + \beta(x_1, x_2, \cdots, x_n) P(\xi(x_1, x_2, \cdots, x_n)). \]  

Instead of the more general one
\[ u = W(x_1, x_2, \cdots, x_n, P(\xi(x_1, x_2, \cdots, x_n))). \]

Using the ansatz (2) to reduce the above original PDE (1), one can derive the m-dimensional PDE
\[ G(x_1, x_2, \cdots, x_m, P, P_{\xi}, P_{\xi\xi}, \cdots) \equiv G(P) = 0, \]

Where \( m < n \) and the function \( P \) satisfies only one reduction equation. Recently, Tang and Lou [10] proposed a conditional similarity reduction approach. By conditional similarity reduction, the function \( P \) is reduced to satisfy more than one reduction equation, and have got several similarity reduction equations. The main difference between the two kinds of methods is that, in the CK direct method, the function \( P \) satisfies only one equation, while Lou’s method do not require the function \( P \) to be a solution of one PDE only, thus it is necessary to require that all the ratios of different derivatives and powers of \( P \) in the former but not in the latter are functions of \( \xi_i \). Actually, the final reduction equations obtained by these approaches are often unexpected in advance and complicated. Therefore, solving these equations is still a difficult task for researchers. Fortunately, with the help of the mapping approach, we have derived abundant exact excitations of several (2+1)-dimensional PDEs, such as the (2+1)-dimensional dispersive long water-wave equation, the (2+1)-dimensional generalized Broer-Kaup-Kupershmidt system and the (2+1)-dimensional generalized Broer-Kaup system [11-14]. Usually, the mapping approach is to construct the solutions of some unknown equations using certain known equations such as Riccati equation or ellipse functional equation, called as mapping equations or projective equations. However, the projective equations are factitiously taken in the ansatz of the concerned physical model. Here, an important and interesting issue is that can we naturally derive these projective equations (we call the projective equation as an objective equation) based on the fundamental ideas of similarity reduction? In the following part, we apply a special conditional similarity reduction approach, or in other name for brevity, an object reduction approach [18].

The main steps of this method are as follows:

Step 1 For a given partial differential system
\[ U(u, u_t, u_x, u_{xx}, \cdots) = 0, \]

we suppose its ansatz in the following form:
\[ u = \alpha(x_1, x_2, \cdots, x_n) + \beta(x_1, x_2, \cdots, x_n) P(\xi(x_1, x_2, \cdots, x_n)). \]

Step 2 Substituting the ansatz (6) into Eq. (5), we have
\[ \sum_i R_i F_i(p, p', p'', \cdots) = 0, \]

where \( R_i \equiv R_i(\alpha, \beta, \xi, \cdots) \) are \( p \)-independent functions of \( \alpha, \beta, \xi \) and their derivatives,
$F_i = \mathcal{F}_i(p, p', p'', \cdots)$ are some polynomials of $p$ and its derivatives (here $'$ denotes the derivative of the function $p$ with respect to $\xi$).

Step 3 In order to obtain objective equations which we expect, we may separate some $R_i$ in Eq. (7) into several parts through setting one coefficient of a special $R_i$ as the normalizing coefficient and making $p$ in every parts of Eq. (7) satisfy the same objective equation.

Step 4 Based on the solutions of the objective equation, we can obtain exact reduction solutions of Eq. (5). As an illustration applied in a nonlinear physical model, we take the (2+1)-dimensional Boiti-Leon-Pempielli (BLP) system

$$u_{y} = (u^2 - u_x)_{xy} + 2v_{xx}, \quad (8)$$

$$v_{y} = v_{xx} + 2uv_x, \quad (9)$$

as a concrete example in the following section. The (2+1)-dimensional BLP system is related to the sine-Gordon equation or the sinh-Gordon equation by certain transformations [15]. The soliton-like, the multisoliton-like and the period form solutions of the system were obtained by using an extended tanh method [16].

2. Object reduction approach of the (2+1)-dimensional BLP system

In this section, we choose the Riccati equation as the reduction equation for Eqs. (8) and (9). We rewrite the concrete type of ansatz (6) in the following form:

$$u = A(x, y, t) + B(x, y, t)p(\xi(x,y,t)) , \quad (10)$$

$$v = E(x, y, t) + F(x, y, t)p(\xi(x,y,t)) , \quad (11)$$

where $p(\xi(x,y,t))$ is a function of $\xi$ to be determined, $A = A(x, y, t)$, $B = B(x, y, t)$, $E = E(x, y, t)$, $F = F(x, y, t)$ and $\xi = \xi(x,y,t)$ are functions of $(x, y, t)$ to be determined. When supposing the $\xi_x \cdot \xi_y \neq 0$, substituting ansatzs (10) and (11) into Eqs. (8), (9), respectively, we have

$$\sum_i R_i F_i(p, p', p'', \cdots) = R_i p''' + R_j p'' + R_i p''' + R_i p'' + R_i p' + R_j p + R_j p^2 + R_j = 0, \quad (12)$$

$$\sum_j r_j f_j(p, p', p'', \cdots) = r_j p''' + r_j p'' + r_j p' + r_j p + r_j p^2 + r_j = 0, \quad (13)$$

where $R_i = R_i(x, y, t)$, $r_j = r_j(x, y, t)$ are $p$-independent functions, $F_i = F_i(p, p', p'', \cdots)$, $f_j = f_j(p, p', p'', \cdots)$ are some polynomials of $p$ and its derivatives, and

$$R_i = -B\xi_x \xi_x^2 + 2F\xi_x^3,$$

$$R_2 = 2B^2 \xi_x \xi_x,$$

$$R_3 = 2B^2 \xi_x \xi_x,$$
In the usual CK direction, the function $\rho$ only satisfies one reduction equation. In order to obtain an objective reduction equation shown by the Riccati equation for Eqs. (8), (9), we separate some $R_i$ and $r_i$ into two parts, for example,

\begin{align}
R_4 &= R_{41} + R_{42}, \\
R_6 &= R_{61} + R_{62}, \\
r_5 &= r_{51} + r_{52},
\end{align}

where

\begin{align}
R_{41} &= B_\xi_3 \xi_x + B_\xi_y \xi_{yx} + 2AB_\xi_3 \xi_y, \\
R_{42} &= -2B_\xi_3 \xi_x - 2B_\xi_y \xi_{yx} - B_\xi_3^2 + 6F_\xi_3 \xi_{xx} + 6F_\xi_y \xi_{yx} - 2B_\xi_y \xi_x, \\
R_{61} &= 4AB_\xi_3 \xi_y + 2AB_\xi_y \xi_{yx} + 4B_\xi_y \xi_x + 2B_\xi_y \xi_x + 2B_\xi_3 \xi_y + \frac{2B_\xi_3 \xi_{xy} + B_\xi_y \xi_{xy} + 2B_\xi_3 \xi_y + B_\xi_y \xi_y}{\xi_x}, \\
R_{62} &= 2A_y B_\xi_x + B_\xi_y + B_\xi_y - 3B_\xi_3 \xi_y + 2A_y B_\xi_y - B_\xi_x \xi_y + 6F_{\xi_x \xi_y} + 6F_\xi_3 \xi_{xx} - 2AB_\xi_y - 2B_\xi_y \xi_x - 2AB_\xi_y \xi_{xx} - 3B_\xi_y \xi_{xx} - B_\xi_y \xi_{xx} - 2B_\xi_y \xi_{xx} - \frac{2B_\xi_3 \xi_{xy} + B_\xi_y \xi_{xy} + 2B_\xi_3 \xi_y + B_\xi_y \xi_y}{\xi_x}, \\
r_{51} &= F_\xi_3 + 2AF_\xi_3 + F_\xi_y, \\
r_{52} &= 2F_\xi_3 \xi_x.
\end{align}

Then, we can rewrite Eqs. (12), (13) as

\begin{align}
R_4 p'' + R_2 pp'' + R_{41} p' + R_{42} p' + R_3 p + R_{41} p + R_{61} p' + R_{62} p' + R_4 + R_5 p^2 + R_6 = 0, \quad (17)
\end{align}
And require that the ratios of different derivatives and powers of \( p \) in each part are functions of \( \xi \). In other words, we may take \( R_1 \) as the normalizing coefficient for the first part, \( R_{42} \) as the normalizing coefficient for the second part, \( R_{62} \) as the normalizing coefficient for the third part, \( r_1 \) as the normalizing coefficient for the fourth part and \( r_{32} \) as the normalizing coefficient for the fifth part, respectively, namely

\[
R_2 = \lambda_1 R_1, \quad R_3 = \lambda_2 R_1, \quad R_{41} = \lambda_4 R_1, \\
R_5 = \lambda_4 R_{42}, \quad R_{61} = \lambda_3 R_{42}, \\
R_7 = \lambda_0 R_{62}, \quad R_8 = \lambda_7 R_{62}, \quad R_9 = \lambda_6 R_{62}, \\
r_2 = \lambda_1 r_1, \quad r_3 = \lambda_1 r_1, \\
r_4 = \lambda_1 r_{32}, \quad r_5 = \lambda_3 r_{32}, \quad r_6 = \lambda_1 r_{32},
\]

where \( \lambda_i \) \((i = 1, 2, \cdots, 13)\) are some functions of \( \xi \) to be determined. To determine \( A, B, E, F \) and \( \xi \) as the usual CK direct method, we may use some rules to simplify the calculations.

- **Rule 1** If \( A \) (or \( E \)) has the form \( A = \alpha_0(x, y, t) + \alpha_1(x, y, t)\Omega(\xi) \), we can take \( \Omega(\xi) \equiv 0 \);
- **Rule 2** If \( B \) (or \( F \)) has the form \( B = \beta_0(x, y, t)\Omega(\xi) \), we can take \( \Omega(\xi) \equiv C = \text{constant} \);
- **Rule 3** If \( \xi(x, y, t) \) is determined by an equation of the form \( \Omega(\xi) = \xi_0(x, y, t) \) (where \( \Omega \) is an invertible function), we can take \( \Omega(\xi) = \xi \).

Applying Rule 2 to Eq. (19), we obtain

\[
A = -\xi_{yy} + \xi_x + b \xi_x^2, \\
B = -c \xi_x, \\
F = -c \xi_y, \\
E = \phi(y),
\]

where \( b \) and \( c \) are arbitrary constants, and \( \phi = \phi(y) \) is arbitrary function of variable \( y \). At the same time, we obtain

\[
\lambda_1 = \lambda_2 = \lambda_4 = \lambda_9 = -2c, \quad \lambda_3 = \lambda_5 = \lambda_{10} = -b
\]

and the other \( \lambda_i \) \((i = 6, 7, 8, 11, 12, 13)\) are arbitrary functions.

After some calculations, we have

\[
\xi_{xxt} \xi_x + \xi_{x} \xi_{xx} = 0,
\]
\[-2\varepsilon_t\varepsilon_x^2\varepsilon_{yt}\varepsilon_{xx} - b^2\varepsilon_t^4\varepsilon_x^2 - 4\varepsilon_t\varepsilon_x\varepsilon_y\varepsilon_{xt} + 4\varepsilon_t\varepsilon_x\varepsilon_{yy}\varepsilon_{xx} + \varepsilon_t^3\varepsilon_{xxx} - \varepsilon_t^3\varepsilon_{yy} + b^2\varepsilon_t^5\varepsilon_{xxx} - \varepsilon_t^5\varepsilon_{yy}^2 + 3\varepsilon_t\varepsilon_{xx}\varepsilon_y^2 + 3\varepsilon_t\varepsilon_{xx}\varepsilon_{yy}^2 + 2b\varepsilon_t^4\varepsilon_{xx} - \varepsilon_t^2\varepsilon_{xx}\varepsilon_{xxx} - 2\varepsilon_t\varepsilon_{yy}\varepsilon_{xxx}\varepsilon_{xx} + 2\varepsilon_t\varepsilon_{yy}\varepsilon_t\varepsilon_x + \varepsilon_t^2\varepsilon_{yy}\varepsilon_{yyt}\varepsilon_t - 2\varepsilon_t^2\varepsilon_{yy}\varepsilon_{yyt}\varepsilon_{yy} = 0,\]

\[\varepsilon_t\varepsilon_{yy} = 0,\]

\[\varepsilon_t\varepsilon_{yy} + \varepsilon_t\varepsilon_{yy} - \varepsilon_t\varepsilon_{yy} - b\varepsilon_t^2\varepsilon_x = 0,\]

i.e. $\xi$ should satisfy the above equations. Substituting Eqs. (20), (21) and the solutions of (22) into Eqs. (17), (18), we can obtain the similarity reduction equations of Eqs. (8), (9), i.e.

\[\left[\varepsilon_t^2\varepsilon_x^2\varepsilon_{xx} + \varepsilon_t^3\varepsilon_x^2(3\varepsilon_x\varepsilon_{xx} + 2\varepsilon_y\varepsilon_{yy})\varepsilon_x + 2(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xy}\varepsilon_{yy})\right](p' - a - bp - cp^2) = 0,\]

\[(\varepsilon_{yy}\varepsilon_x^2\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_x^2\varepsilon_{yy})(p' - a - bp - cp^2) = 0,
\]

where $a$ is an arbitrary constant. Obviously, when we take

\[p' = a + bp + cp^2,\]

Eqs. (23), (24) will be satisfied automatically, which also means that the final reduction equation (25) is a Riccati equation.

3. Abundant solutions of the (2+1)-dimensional BLP system

In this section, we derive abundant exact solutions for the (2+1)-dimensional Boiti-Leon-Pempelli system. To observe Eq. (22), one special solution can be expressed as follows

\[\xi = \chi(x,t) + \eta(y)\]

where $\chi \equiv \chi(x,t)$ and $\eta \equiv \eta(y)$ are two arbitrary functions of $(x,t)$ and $y$, respectively. Based on Eqs. (10), (11), (20) and (26) and the solution of Eq. (25), we can derive new exact solution of Eqs. (8) and (9),

\[u = -\frac{\chi_{xx} + \chi_x^2 + b\chi_x^2}{2\chi_x} + c\chi_x p(\xi),\]

\[v = \phi(y) + c\eta(y) p(\xi),\]

where $p(\xi)$ is a solution of the Riccati equation (25), $b$ and $c$ are two arbitrary constants.

As well-known, Eq. (25) possesses the following solutions [17]:

1. When $a = 1$, $b = 0$, $c = -1$, $p(\xi) = \tanh \xi$, $\coth \xi$,

2. When $a = 1$, $b = 0$, $c = 1$, $p(\xi) = \tan \xi$,

3. When $a = -1$, $b = 0$, $c = -1$, $p(\xi) = \cot \xi$. 


4. When $a = 1/2$, $b = 0$, $c = 1/2$, $p(\xi) = \tan \xi \pm \sec \xi$, $\tan \xi / (1 \pm \sec \xi)$,
5. When $a = -1/2$, $b = 0$, $c = -1/2$, $p(\xi) = \cot \xi \pm \csc \xi$, $\cot \xi / (1 \pm \csc \xi)$,
6. When $a = 1/2$, $b = 0$, $c = -1/2$,
   
   \[ p(\xi) = \coth \xi \pm \csc \phi \xi, \quad \tanh \xi \pm \sec \phi \xi, \quad \tan \xi / (1 \pm \sec \phi \xi), \quad \coth \xi / (1 \pm \csc \phi \xi), \]
   
   where $i^2 = -1$,
7. When $a = 1$, $b = 0$, $c = -4$, $p(\xi) = \tanh \xi / (1 + \tanh^2 \xi)$,
8. When $a = 1$, $b = 0$, $c = 4$, $p(\xi) = \tan \xi / (1 - \tan^2 \xi)$,
9. When $a = -1$, $b = 0$, $c = -4$, $p(\xi) = \cot \xi / (1 - \cot^2 \xi)$,
10. When $a = b = 0$, $c \neq 0$, $p(\xi) = -1 / (c \xi + c_0)$, where $c$ and $c_0$ are arbitrary constants.

As a result, we can obtain abundant solutions for Eqs. (8) and (9). These solutions include the hyperboloid function solutions, the trigonometric function solutions and a rational function solution. In the same time, as $\chi \equiv \chi(x,t)$ and $\eta \equiv \eta(y)$ are two arbitrary functions of $(x,t)$ and $y$, respectively, the coherent structures of these solutions of Eqs. (8), (9) are rather abundant.

4. Summary
In summary, we have presented the objective reduction approach (ORA) in this paper. Applying this method to the (2+1)-dimensional Boiti-Leon-Pempilie system, we have successfully obtained abundant exact solutions for this system, including the hyperboloid function solutions, the trigonometric function solutions and a rational function solution. We should point out that though the special conditional similarity reduction equations obtained by this method are not as many as those by other similarity reduction approaches, however, by this method one can obtain many explicit and exact solutions of the given partial differential equations. We are sure that the ORA may be an interesting and practical technique for researchers.

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