An extension of cosmological dynamics with York time

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Abstract

It has been suggested that the York parameter $T$ (effectively the scalar extrinsic curvature of a spatial hypersurface) may play the role of a fundamental time parameter. In a flat, forever expanding cosmology the York parameter remains always negative, taking values $T = -\infty$ at the big bang and approaching some finite non-positive value as $t \to \infty$, $t$ being the usual cosmological time coordinate. Based on previous results concerning a simple, spatially flat cosmological model with a scalar field, we provide a temporal extension of this model to include ‘times’ $T > 0$, an epoch not covered by the cosmological time coordinate $t$, and discuss the dynamics of this ‘other side’ and its significance.

Introduction. Our goal in this brief note is to present an extension of a simple cosmological model beyond what is usually regarded as temporal infinity. The model we investigate will be a homogeneous isotropic universe with a scalar field in a flat space ($k = 0$), although we expect that analogous results may be obtained for any cosmology that does not undergo a phase of contraction in the finitely distant future.

This extension of the dynamics will be seen to follow naturally from the choice of a particular time parameter, called York time as a result of its first appearance in York’s work on the initial value problem of general relativity [4, 16, 17]. We recently developed classical and quantum cosmological dynamics in York time for
a scalar field using a reduced Hamiltonian formalism [11]. We gave some compelling reasons why the York parameter should, in fact, be considered a candidate for a physical time, based on the details of York’s solution as well as the recent development of ‘shape dynamics’ [6], which bears a close relationship to general relativity in the York-time gauge, and finally reasons based on quantum theory, in particular Hidden-Variable approaches [13–15].

**General remarks.** The York parameter $T$ is proportional to the trace of the extrinsic curvature, $K = g^{ij} K_{ij}$, where $g_{ij}$ is the metric on 3-space. In particular, $T = (12\pi G)^{-1} K$, where $G$ is the gravitational constant. Slices of constant $T$ are therefore slices of constant mean curvature (CMC). For a review of the notion of extrinsic curvature and associated concepts, we refer the reader to [11] and references therein, in particular Misner et al. [9]. In a homogeneous cosmological model the York parameter (as well as the trace of the extrinsic curvature) is furthermore proportional to the Hubble parameter, $T = -\frac{1}{4\pi G} H = -\frac{1}{4\pi G} \frac{\dot{a}}{a}$, where $a$ is the cosmological scale factor and the dot denotes differentiation with respect to cosmological time.

This property is central to the investigation in this paper.

In order to understand how the characteristics of York time enable (and indeed suggest) this extension, let us briefly recall some of them. In a homogeneous isotropic universe, in which the Friedmann-Lemaître equations are assumed to hold, one differentiates between cases of a closed ($k = 1$), flat ($k = 0$) and open ($k = -1$) universe. Let us focus on contrasting the closed and flat cases, leaving the open case for discussion at a later stage.

The general Friedmann-Lemaître equations are:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \rho$$  \hspace{1cm} (1)

$$\frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P),$$  \hspace{1cm} (2)

where $\rho$ is the energy density and $P$ the pressure. For a scalar field $\phi$ with a potential $V(\phi)$, these are $\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and $P = \frac{1}{2} \dot{\phi}^2 - V(\phi)$. The field $\phi$ is governed by the Klein-Gordon equation for a homogeneous field on expanding space:

$$\ddot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} + \frac{\partial V}{\partial \phi} = 0.$$  \hspace{1cm} (3)

In the closed case the universe expands for a finite amount of time before it begins a contraction phase. The Hubble parameter is infinite at the big bang,
reaches zero at the point of maximal expansion and then approaches minus infinity as the size of the universe returns to zero. Accordingly, York time $T$ takes values $-\infty$ at the big bang, zero at the point of maximal expansion and $+\infty$ at the final singularity. Both $H$ and $T$ evolve monotonically. Points on the York-time line $T \in (-\infty, \infty)$ are one-one with time slices constituting the history of the universe — another compelling reason to consider it a viable temporal parameter.

For the spatially flat universe the Hubble parameter is infinite at the big bang and asymptotically approaches zero in the $(t)$-infinite future. York time $T$ equals $-\infty$ at the big bang and then asymptotically approaches zero from below. Once again the evolution of both parameters is monotonic. Let the present cosmological time be denoted by $t_0$ and the present value of York time by $T_0$. It follows that an infinite ‘period’ of cosmological time, $(t_0, \infty)$, lies still ahead of us, but only a finite ‘period’ of York time, $(T_0, 0)$.

However, if we take the notion of $T$ as ‘time’ seriously then there is no a priori reason why time should end at zero or indeed any finite value. This is the essence of our extension. We consider the cosmological dynamics for $T \in (0, \infty)$ in the case of a spatially flat universe. This interval is not covered by the cosmological-time coordinate $t$, which approaches infinity before this era of positive $T$ is reached (see figure 1).

A reader bewildered by this may find it helpful to consider an analogy. In the case of a Schwarzschild black hole a new region of space-time is revealed by the use of Kruskal-Szekeres coordinates, extending to the ‘other side’ of the singularity (where the Kruskal-Szekeres variable usually denoted by $V$ takes negative values), a region not described by either Schwarzschild or Eddington-Finkelstein coordinates [9, Ch. 31].

Dynamics for the free field. Let us now investigate the dynamics on this ‘other side’, focusing on a free scalar field. If $T > 0$, then $H < 0$, and so (since the scale factor is positive) the universe is contracting. It is possible to perform the transformation $t \rightarrow T$ on equations 1, 2, 3. However the resulting equations are

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1In the case of a scalar field with a non-zero potential, this statement will have to be modified somewhat, as we will see.

2Arguably there is another, simpler example: the extension of dynamics beyond a black hole’s event horizon in the first place. The idea that spacetime extended beyond the horizon of a Schwarzschild black hole into the region $r < 2M$ took several decades to become apparent (5, 7, 12). However, a crucial difference here is that this extension concerns the removal of a mere coordinate singularity. No physical quantities become infinite (and geodesics crossing the surface $r = 2M$ are traversed in finite proper time). This example is therefore not truly analogous to the present extension.
complicated and rather unnatural to work with.

Instead, we perform a Hamiltonian reduction. Starting from the Einstein-Hilbert action for a homogeneous universe with variables $a$ and $\phi$, we bring it into ADM form [2] (introducing conjugate momenta $P_a$ and $p_\phi$). One obtains a Hamiltonian constraint. The momentum $P_T$ conjugate to $T$ is given by $-a^3$, the negative of the (normalised) volume of the universe. The Hamiltonian constraint is then transformed by the change of variables $(a, P_a, \phi, p_\phi) \rightarrow (T, P_T, \phi, p_\phi)$ and solved for $P_T$. The physical (non-vanishing) Hamiltonian is given by $H_{\text{phys}} = -P_T$.

This is the essence of the Hamiltonian reduction. Details are found in ref. [11]. The general consistency of this method was shown in Arnowitt et al. [1]. In [11] we showed that the physical Hamiltonian was given by

$$H_{\text{phys}} = -(12\pi G T^2 - 2V(\phi))^{-\frac{1}{2}} \cdot p_\phi. \quad (4)$$

Note that the pair $(a, P_a)$ has been eliminated as dynamical variables and ‘absorbed’ into the time parameter, which appears explicitly in the Hamiltonian. The only remaining dynamical variables are the scalar field and its conjugate momentum. The dynamics for the scale factor may be recovered by setting $a^3 = H_{\text{phys}}$ (since $a^3 = -P_T$ and we identified $-P_T = H_{\text{phys}}$), that is, the volume of the universe is given by the numerical value of the Hamiltonian.

Hamilton’s equation for $\phi$ in the case of a free field ($V(\phi) = 0$) is then the first-order equation

$$\frac{d\phi}{dT} = \frac{\partial H_{\text{phys}}}{\partial p_\phi} = -(12\pi G T^2)^{-\frac{1}{2}}. \quad (5)$$

The dynamics of $\phi$ is therefore decoupled from the dynamics of $p_\phi$, which needs not to be considered for our purposes. Its solution depends on whether $T > 0$ or $T < 0$ since different branches of the logarithm must be chosen for the two cases. For $T < 0$, ‘our’ side, we have

$$\phi(T)_{T<0} = (12\pi G)^{-\frac{1}{2}} \int \frac{d(-T)}{-T} = (12\pi G)^{-\frac{1}{2}} \ln(-T) + \text{const.}, \quad (6)$$

while for $T > 0$, the ‘other’ side, we see that

$$\phi(T)_{T>0} = (12\pi G)^{-\frac{1}{2}} \int \frac{dT}{-T} = -(12\pi G)^{-\frac{1}{2}} \ln T + \text{const.}. \quad (7)$$

The important feature here is the overall relative sign, which arose because in both cases we took the positive root, $\sqrt{T^2} > 0$, but opposite branches of the logarithm.
The behaviour at \( T = 0 \) is undefined but from the two limits we see that there is a discontinuity at the point of maximum expansion with \( \phi \to -\infty \) as \( T \to 0^- \) and \( \phi \to +\infty \) as \( T \to 0^+ \).

The discontinuity can be avoided by choosing the other root of \( T^2 \) for one of solutions 6 and 7 introducing a relative minus sign. Say this is done in eq. 7, then \( \phi \to \infty \) as \( T \to 0 \) from both above and below and there is no discontinuity. In the other case, taking the negative root for eq. 6 instead, the same dynamics is recovered except with the substitution \( \phi \to -\phi \). The symmetry between these two solutions is broken in the presence of a non-even potential function \( V(\phi) \).

The ‘other side’ \( (T > 0) \) is just the time-reversal of our side, as is clear from expressions 6 and 7 with \( |T| \) now increasing rather than decreasing. That the York-time dynamics matches those implied by the standard Friedmann-Lemaître equations (together with the Klein-Gordon equation for the field) has already been shown in [11], but it is worth noting explicitly that these equations have the same symmetries.

Equations 1, 2 and 3 are clearly invariant under the transformation \( t \to -t \). We can therefore view the dynamics during \( T > 0 \) in terms of these equations, but considering cosmological time to ‘run backwards’, possibly up to an overall sign in \( \phi \) depending on how the \( \phi \)-discontinuity at \( T = 0 \) was avoided. This symmetry implies that we can, in fact, describe the dynamics explicitly in terms of a cosmological-time-like parameter \( t' \), provided we take \( t' \to -\infty \) as \( T \to 0^+ \). As \( t' \) increases, the universe contracts at a rate determined by the Friedmann-Lemaître equations with the substitution \( t \to t' \).

\[ \text{Figure 1: Overview of corresponding values of cosmological time } t \text{ and York time } T \text{ in the case of a free scalar field. Only the left half of the time line is described by conventional cosmological time for a flat universe. The right hand side is however a natural extension if the York parameter is taken seriously as a fundamental time parameter. The finite ‘starting’ and ‘end’ values of } t \text{ and } t' \text{ respectively are up to convention (due to cosmological-time translation invariance).} \]

In heuristic terms, we saw that the infinite future history of the universe (as seen in terms of cosmological time) is ‘compressed’ into a finite York-time in-
terval. With \( d\phi/dT \sim T^{-1} \) we see that more and more ‘happens’ (\( \phi \) changes faster and faster) the closer the universe gets to \( T = 0 \). It is interesting that a similar observation was made about cosmological time regarding the very early history (close to \( t = 0 \)) in the 1960s by Misner [8], who suggested that \(- \ln a\) (or alternatively the logarithm of the homogeneous temperature, which turns out to roughly equivalent) might be a more appropriate choice of temporal parameter to describe the history of the universe. As he states, “[t]he universe is meaningfully infinitely old because infinitely many things have happened since the beginning” [8, p. 1331]. Another comparable choice is the parameter \( \ln t \) as had been advocated by Milne another twenty years earlier (cited in [8]).

**Dynamics with a cosmological constant.** Returning to our present purpose, we now wish to consider the effect of a non-zero cosmological constant \( \Lambda \). This corresponds to a constant non-zero potential \( V(\phi) = \Lambda/8\pi G \), or a cosmological constant arising from some other origin. Let us assume \( \Lambda > 0 \), leading to a universe that is forever expanding (‘forever’ referring to cosmological time), asymptotically approaching a constant non-zero fractional rate of expansion (a de Sitter universe) in the large-\( t \) limit. The cosmological constant will have interesting consequences in the York-time picture and similar qualitative features arise in the case of many non-constant potentials \( V(\phi) \).

In the cosmological-time picture the Hubble parameter approaches a final positive value \( H_\infty \) as \( t \to \infty \). The implication for York time is that \( T \to T_\infty = -\left(4\pi G\right)^{-1} H_\infty \) as \( t \to \infty \), not \( T \to 0 \). That is, from a cosmological-time perspective York time ‘ends’ not at zero, but a finite negative value \( T_\infty \) (figure 2). In the York-time reduced Hamiltonian formalism the equation governing \( \phi \) for non-zero \( \Lambda \) was shown to be \([11]\) (using \( V = \Lambda/8\pi G \))

\[
\frac{d\phi}{dT} = -\left(12\pi G T^2 - \frac{\Lambda}{4\pi G}\right)^{-\frac{1}{2}}. 
\]  

(8)

This equation is symmetric in \( T \). Similarly, the invariance of the Friedmann-Lemaître and Klein-Gordon equations (1, 2, 3) under the substitution \( t \to -t \) remains unbroken. From either of these facts we can infer that the dynamics on the ‘other side’—here referring to the epoch \( T > |T_\infty| \)—matches once again the ‘mirror image’ of those on our side. If a cosmological-time description is desired, we must once again define a new time parameter \( t' \) just as we did above.

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3This is further exemplified in our language to describe the early universe in terms of ‘epochs’ such as the ‘Planck’, ‘Grand Unifying’ and ‘Inflationary’ epochs, each of which is described by a vastly different order of magnitude of duration.
However, clearly there is a problem. What happens during the period \( T_{\infty} < T < -T_{\infty} \)?

The evolution of the Hubble parameter (and hence the universal scale factor) is easily deduced since it is proportional to \( T \). However, the one physical degree of freedom in this model, the scalar field, lacks well-defined dynamics. Once again this may be seen both in the cosmological-time as well as the York-time picture. In the former, we see that eq. [1] lacks real solutions for \( \dot{\phi} \) if \( H^2 \equiv \dot{a}^2/a^2 < \Lambda/3 \), while in the latter it follows from eq. [8] that \( d\phi/dT \) apparently becomes complex while \( 12\pi GT^2 < \Lambda/4\pi G \). The two perspectives on this problem are, of course, equivalent.

We are forced to conclude that at least one of the following is true:

1. York time is inadequate as a fundamental time parameter, or our notion of what a fundamental time parameter is must be revised (for example, time may end and restart, however one can make sense of this notion).

2. If York time is indeed to be considered physical time, then not all choices of possible matter fields provide a consistent dynamics. That is, a universe filled with a free real scalar field and a positive cosmological constant is manifestly inconsistent, as is any other set of matter fields (with or without a cosmological constant) such that its dynamical equations do not have solutions for every value of \( H \).

3. The universe may enter an epoch during which its matter dynamics is undefined. However, it may also emerge again from such an epoch.

Abandoning the idea of York time as fundamental is perhaps the simplest of these, although in that case it may, of course, be retained as an ‘effective’ choice of parameter along the lines of Bojowald et al. [3]. However, our purpose in this note is to investigate the consequences of taking York time seriously and this is therefore not an option for us.

The second choice is an interesting one. We know that the universe is not populated by a single real scalar field with a constant potential function and the problem that arises merely points to the fact that our matter model is inadequate. The real question is whether or not a similar problem arises if we take a more adequate matter content (such as the standard model). Expressed differently, York time as a fundamental choice of time parameter narrows down what the actual matter content of the universe could be. Thus taking York time as a true time parameter would make falsifiable predictions with regards to which possible mat-
ter fields are realised in nature. A priori we have no reason to accept or reject York time as such a parameter, but we must look at the observed matter and its dynamics for clues.

The last option, that the dynamics of the matter field is undefined during the interval $T_{\infty} < T < -T_{\infty}$, raises some curious philosophical issues. One could differentiate between two variants. The weaker conclusion might be that during this period the dynamics of $\phi$ cannot be described in terms of an action principle or equations of motion, but the value of $\phi$ during this period is nonetheless well defined. A stronger statement would be that $\phi$ ceases to take any meaningful value at all. What sort of picture of reality this implies is left to be debated by philosophers and lies outside the scope of this paper.

Figure 2: Overview of corresponding values of cosmological time $t$ and York time $T$ in the case of a scalar field with a positive cosmological constant. The left section of the line is described by conventional cosmological time $t$, the right one by a similar parameter $t'$. In the middle section, $T_{\infty} < T < -T_{\infty}$, the dynamics of the scalar field is undefined, raising philosophical questions.

Conclusion. Our purpose in this note was to consider the York parameter $T$ as a fundamental choice of time. By considering a cosmological model in which $T$ did not range over the entire real line in the standard description originally, we extended $T$ to do so after all and discussed the dynamics in the standardly inaccessible region. Depending on the details of our model, this may raise some serious questions.

Our discussion concerned only the simplest models, chosen to illustrate the principle of the York-time extension. For one, we restricted our consideration to a spatially flat, homogeneous universe. If homogeneity is broken (either through the introduction of perturbations or by considering the full non-linear theory) new features arise. For example, surfaces of constant York time do not match those of constant cosmological time and the boundary between ‘our’ and the ‘other

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4Recall however that our discussion is conditional on dealing with a flat universe.
side’ is space-time dependent. The presence of singularities introduces further complications.

Furthermore, the discussion of the dynamics was entirely on the classical level, although given that trajectories in the (de Broglie-Bohm formulation of the) quantum theory match those of the classical theory (as shown in [11]), similar issues will arise in the quantum theory.

Our proposed extension may turn out not have any observational consequences. Therefore whether or not we should give credence to predictions concerning the unobservable epoch $T > 0$ depends on the existence of other evidence for York time as a physically fundamental time parameter. For example, if dramatic progress is made in the development of some theory of quantum gravity that relies on York time, and observational evidence emerges that corroborates this theory, then this would be reason to take the extension seriously.

However, even in the absence of such evidence, at least when debating fundamental issues of time it is worth keeping the existence of such extensions in mind.

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[^5]: However, these are not as bad as one might think. The York time parameter is, in fact, remarkably well-behaved in the presence of singularities, as was illustrated by Qadir and Wheeler [10].
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