Market Making with Model Uncertainty\footnote{We thank the participants of the seminar held in 2015 at the Department of Management Science and Engineering, Stanford University for helpful comments.}

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Abstract

Pari-mutuel markets are trading platforms through which the common market maker simultaneously clears multiple contingent claims markets. This market has several distinctive properties that began attracting the attention of the financial industry in the 2000s. For example, the platform aggregates liquidity from the individual contingent claims market into the common pool while shielding the market maker from potential financial loss. The contribution of this paper is two-fold. First, we provide a new economic interpretation of the market-clearing strategy of a pari-mutuel market that is well known in the literature. The pari-mutuel auctioneer is shown to be equivalent to the market maker with extreme ambiguity aversion for the future contingent event. Second, based on this theoretical understanding, we present a new market-clearing algorithm called the Knightian Pari-mutuel Mechanism (KPM). The KPM retains many interesting properties of pari-mutuel markets while explicitly controlling for the market maker’s ambiguity aversion. In addition, the KPM is computationally efficient in that it is solvable in polynomial time.
1 Introduction

In this paper, we design a new platform for trading contingent claims, the Knightian Pari-mutuel Mechanism.

The term "pari-mutuel" originates from the automated horse race betting system invented in the 19th century. The pari-mutuel betting system automatically calculates payoff odds for each horse based on the amount of money bet on each horse. It also completely shields the market organizer from financial loss. To illustrate, suppose that people wager their money on the outcome of a race between two horses: A and B. People wager a total of $50 on Horse A and $100 on Horse B. The total premium of $150 is paid to those who correctly predicted the outcome. Thus, if Horse A wins, those who wagered money on Horse A receive $3 for each dollar they wagered. If Horse B wins, the winners make $1.5 for each dollar they wagered. Because the payment to the winners is financed exclusively by the fees collected from both winners and losers, the market maker does not need to worry about his/her loss. This is called the self-financing property of the market.

The rate of return from wagering on a particular horse conveys information on the collective perception of that horse’s chances of winning. Consider the example above. Wagering on Horse B yields a lower rate of return than wagering on Horse A because people wagered more money on Horse B than on Horse A. The more money people wager on a particular horse the lower the rate of return becomes. People bet money on a horse if they believe that horse is likely to win the race. Thus, the rate of return from wagering money on a horse is low if many people believe that the horse will win the race. The pari-mutuel system maps the popularity of horses to the rates of return from wagering money on those horses.

This simple pari-mutuel system subsequently evolved into more sophisticated prediction markets. For example, more recently developed markets (Peters et al., 2005; Peters et al., 2007) trade securities with fixed final payoffs. The prices of those securities fluctuate in a way that reflects their popularity in the market.

Despite considerable heterogeneity across various prediction markets, they typically exhibit three defining characteristics. First, the popularity of a particular security is mapped to a higher price of that security through an automated market-clearing algorithm. Second, the market maker’s maximum possible loss at maturity is bounded. Irrespective of the outcome at the time when contingent claims mature, the market maker is not expected to lose more than a certain pre-specified amount. In this paper, when we say that the market is completely pari-mutuel, we mean
that the market maker is not expected to lose money, regardless of the outcome.

Finally, the market aggregates liquidity across different markets into the common pool (Baron and Lange, 2007). For example, consider the horse race example above but with a slight modification. Suppose that people trade securities with fixed payoffs: Claim A and Claim B. Let "Claim A" refer to the contingent claim that pays $1 if and only if Horse A wins. Define the term "Claim B" similarly. The potential payout to the holders of Claim A is financed by the premiums collected from the holders of Claim B and vice versa. Therefore, it is as if the holders of Claim A and those of Claim B were transacting with one another via a common pari-mutuel auctioneer. Compare this case with an alternative situation in which potential buyers of Claim A (or Claim B) only trade with potential sellers of Claim A (or Claim B). The effective number of people trading with one another is larger in the former case. It is as if the common auctioneer pooled liquidity from individual markets - the market for Claim A and the market for Claim B - into the common pool. As a result, market participants can enjoy a more liquid market.

In the 2000s, researchers (Lange and Economide, 2005; Baron and Lange, 2007) noted that the pari-mutuel principle can be used to better organize a certain type of financial derivatives market. For example, note that the pari-mutuel auctioneer is well protected from financial loss at the time the claims mature. This property of the pari-mutuel market allows the auctioneer to be less concerned with fluctuations in the value of the inventory. Therefore, the pari-mutuel principle can be used to design a market if the market maker has difficulty hedging against inventory risk (Lange and Economide, 2005; Baron and Lange, 2007). For example, Lange and Economide (2005) designed the Pari-mutuel Digital Call Auction (PDCA) to trade options written on economic indices, for which delta hedging using the underlying asset is not feasible.

Longitude, a financial technology company, developed software to implement the PDCA. In collaboration with investment banks (e.g., Goldman Sachs) and financial exchanges (e.g., the International Securities Exchange (ISE)) new PDCA-based derivatives markets were launched. Due to lack of active market participation, the ISE shut down the auction in June 2007. However, the ISE has shown consistent interest in utilizing this technology in the near future (Burne, 2013).

Despite their use in the financial industry, many pari-mutuel auctions have design features that are different from the modeling assumptions that economists use. A potential reason for this is that pari-mutuel auctions have primarily been studied by scholars in operations research. For example, many pari-mutuel auctions optimally clear the market while placing a lower bound on the auctioneer’s maximum possible loss (Hanson, 2003; Pennock, 2004; Peters et al., 2005; Lange
and Economide, 2005; Chen and Pennock, 2007; Peters et al., 2007; Abernethy et al., 2013). The worst-case scenario can have a material impact on how the auctioneer clears the market even if such a scenario is very unlikely. In contrast, the market maker in the economists’ model is often exclusively concerned with maximizing his/her expected utility derived from the monetary payoff. When only the expected value of the future utility is concerned, extreme worst-case loss with a small probability of occurrence does not merit considerable attention.

The contribution of our paper is two-fold. First, we present a theoretical framework through which pari-mutuel auctions can be reconciled with standard economic models. Regarding the economics model, we focus on a market maker with extreme ambiguity aversion for the future contingent event on which claims are written. The decision maker with ambiguity aversion is uncertain of which probability distribution accurately describes the contingent event. For a pari-mutuel market, we consider the Convex Pari-mutuel Call Auction Mechanism (Peters et al., 2005), which is an improved version of the PDCA. We show that the market-clearing strategy of the market maker with extreme ambiguity aversion is asymptotically equivalent to that of the CPCAM auctioneer. By asymptotic equivalence, we imply making the CPCAM increasingly completely pari-mutuel.

Second, based on this unified theoretical framework, we design a new market called the Knightian Pari-mutuel Mechanism (KPM). The KPM has a solid microeconomic rationale behind its design. We derive the optimization problem of the KPM by modeling the market maker using the theory of decision making under ambiguity aversion. The market-clearing algorithm explicitly controls for the level of the market maker’s ambiguity aversion. In addition, we propose an algorithm that can compute an optimal solution to the optimization problem in polynomial time.

1.1 Literature Review

In the prediction market literature, the KPM is most similar to Chen and Pennock’s utility-based market maker (Chen and Pennock, 2007). The utility-based market maker prices contingent claims

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1 In the CPCAM, before the beginning of the regular trading session, the initial liquidity provider seeds the market with small initial orders. This initial order, which is typically called the starting order, is a unique design feature of the CPCAM. The starting order is introduced into the CPCAM only to ensure the existence of unique state prices, which are used to compute the market-clearing prices of contingent claims. However, the starting order exposes the market organizer to a financial loss at the time the claims mature. The larger the starting order, the greater the potential financial loss of the market maker. By asymptotic equivalence, I mean reducing the magnitude of the starting order toward zero. When the starting orders are infinitely small, the market-clearing strategy of the CPCAM auctioneer approaches that of the ambiguity-averse market maker.

In addition, I assume that the market organizer submits the same starting order for all possible states of the future. I describe what the starting order is in later sections.
in such a way that the transaction leaves the market maker’s expected utility over the future monetary payoff unaffected. Agrawal et al. (2011) suggests an improvement of Chen and Pennock’s (2007) market. The market maker may find it difficult to propose a unique probability distribution to describe the event for which the claims are written. The KPM addresses Agrawal et al. (2011)’s suggestion: It allows the market maker to indicate a set of multiple reasonable probability distributions instead of a single distribution. The KPM acknowledges that the market maker often cannot pin down a single subjective probability distribution.

Our paper is related to the recent literature that explains a wide variety of prediction markets from a unified theoretical perspective. The most notable work in this respect is Agrawal et al. (2011). They show that the four most well-known prediction markets in the literature can be unified under one theoretical framework. In a similar vein, we reconcile pari-mutuel mechanisms from the prediction market literature with the model from the economics literature.

My paper is related to a growing body of literature that focuses on the role of Knightian uncertainty in decision making. In the past decade, Knightian uncertainty has received a significant amount of attention in areas ranging from macroeconomic modeling (Hansen and Sargent, 2008) to market microstructure theory (Easley and O’Hara, 2009; Easley and O’Hara, 2010).

2 The Theory of Decision Making under Ambiguity

We present a brief overview of the theory of decision making under ambiguity. First, it is necessary to distinguish between risk and ambiguity. Risk applies to situations in which it is possible to attach a probability distribution to an unknown prospect. By contrast, ambiguity refers to situations in which it is impossible to do so. For example, consider a situation in which a person receives a dollar if and only if he/she draws a red ball from a box. The box contains both red balls and blue balls. If the person knows the fraction of balls that are red, he/she knows the probability of receiving a dollar. In this case, the person is said to be facing risk. On the other hand, suppose that the person does not know the fraction of balls in the box that are red. Then, the person cannot assign a number to the probability of winning a dollar. This person is said to be facing ambiguity. In the 1920s, Knight was the first to note the difference between these two concepts (Knight, 1936).

An ambiguous prospect requires a different analysis from that of a risky prospect. To this end, Theorem 1 reproduces the main finding of Gilboa and Schmeidler (1989) in the language of Ghirardato et al. (2004). Let $S$ denote the set of all possible states (e.g., the person chooses a red
ball, the person chooses a blue ball), and let $X$ denote the set of consequences (e.g., the person wins a dollar). Subsets of $S$ are called events. Let $\Sigma$ denote the algebra of subsets of the state space $X$. We are interested in the decision maker’s preference over different simple acts: A simple act is a $\Sigma$-measurable function $f : S \to X$ that is finite-valued (Ghirardato et al., 2004). Let $F$ denote the set of all simple acts. Suppose that the binary relations $\succeq$ and $\succ$ characterize the decision maker’s preference over different acts: $f \succeq (\succ) g$ if and only if the decision maker (strictly) prefers the simple act $f$ to the simple act $g$. Finally, let $u : X \to \mathbb{R}$ denote the decision maker’s utility function.

Theorem 1 (Decision Making under Ambiguity) (Gilboa and Schmeidler, 1989; Ghirardato et al., 2004) The decision maker’s preference relation $\succeq$ satisfies the set of six behavioral axioms\footnote{Please see Ghirardato et al. (2004) for the set of six behavioral axioms.} if and only if there exists a unique set $\Psi$ of probabilities on $(S, \Sigma)$ such that (1) holds for $\forall f, g \in F$. The set $\Psi$ is weakly compact, convex and nonempty\footnote{Ghirardato et al. (2004) presents three different versions of the theorem depending on the DM’s attitude toward ambiguity. However, we only work with the version that assumes aversion to ambiguity. Please refer to Ghirardato et al. (2004) for a more rigorous formal definition of aversion to ambiguity.}

$$f \succeq g \iff \min_{P \in \Psi} \int u(f) dP \geq \min_{P \in \Psi} \int u(g) dP$$  \hspace{1cm} (1)

Proof. See Gilboa and Schmeidler (1989) or Ghirardato et al. (2004). □

The set of probabilities $\Psi$ in Theorem 1 encapsulates the decision maker’s (hereafter called the DM) perception of ambiguity (Ghirardato et al., 2004). Recall that a DM facing ambiguity cannot attach a single probability distribution to the unknown prospect. Instead, the DM has a set of candidates $\Psi$ that he/she believes are fairly accurate predictions of the future (Ghirardato et al., 2004). In other words, the DM has a set of multiple priors (Gilboa and Schmeidler, 1989). The size of $\Psi$ represents the extent to which the DM feels ambiguous toward the unrealized future outcome (Ghirardato et al., 2004). A large size of $\Psi$ implies that the DM cannot easily narrow down the set of reasonable probability distributions because he/she is too ambiguous about the future outcome (Ghirardato et al., 2004).

Among the set of multiple priors, the DM is exclusively concerned with the worst possible scenario. First, for each $P \in \Psi$, the DM calculates the expected utility $\int u(f) dP$ from the unknown prospect assuming that $P$ is the true description of the future. Second, the DM finds the distribution that results in the lowest level of utility. Third, when comparing one act with another, the DM
uses the probability distribution associated with the worst scenario. The DM chooses the act whose worst-case scenario is better than the worst-case scenarios of the other acts. See the Appendix for a numerical example.

In practical modeling and implementation, the specification of the set $\Psi$ of the DM becomes another issue. Hansen and Sargent (2008) presents a useful solution in the context of modeling in macroeconomics. We first introduce Kullback’s cross-entropy function (Cover and Thomas, 2012) to quantify the extent to which two probability distributions differ from one another.

**Definition 1 (Kullback’s Cross-Entropy Function)** Suppose that there are two probability distributions $p$ and $q$ with the common support set $S$. Suppose that $q$ is the prior density over the set $S$. Then, the Kullback’s cross-entropy function is defined as (2) (Cover and Thomas, 2012). A large value of $S(p, q)$ implies that $p$ and $q$ are very different from one another.

$$S(p, q) = \int_S p(x) \ln \left[ \frac{p(x)}{q(x)} \right] \, dx$$

Hansen and Sargent (2008) use cross-entropy to restrict the set of probability distributions considered by the DM. Given the prior distribution $q$ and a parameter $\eta$, the DM’s set $\Pi$ includes all probability distributions $p$ for which $S(p, q) \leq \eta$. As long as the probability distributions are not too different from $p$, in which case $S(p, q) > \eta$, the DM considers those probability distributions to be equally acceptable.

A large value of the parameter $\eta$ quantifies the DM’s strong ambiguity aversion. With a larger value of $\eta$, the DM regards a larger set of probability distributions as candidates for accurate descriptions of the world. Hence, a large $\eta$ is equivalent to saying that the DM is more ambiguous about the real world.

3 The Microeconomic Analysis of the Convex Pari-mutuel Call Auction Mechanism (CPCAM)

3.1 The Market Setting

The CPCAM allows the common market maker to simultaneously handle different types of contingent claims as long as the claims are written on the same uncertain event (e.g., the outcome of the

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Illeditsch (2011) also uses the size of the set of possible models under the DM’s consideration as a proxy for the DM’s level of ambiguity aversion.
world cup, stock prices). Suppose that there are $N$ possible outcomes of the uncertain event, each of which is indexed by $i \in \{1, 2, ..., N\}$.

The CPCAM is a call auction. For simplicity, only buy orders are accepted. Suppose that the market participants as a whole submit $J$ orders to the market maker. Let the matrix $A \in \mathbb{R}^{N \times J}$ denote the payoff structure of $J$ orders. The $(i, j)$ element of $A$ denotes the per-share payoff of the $j$th order, where $j \in \{1, 2, ..., J\}$ if the $i$th outcome is realized. Define the vector $b \in \mathbb{R}^J$ such that the $j$th element of this vector is the limit price associated with the $j$th order. Define the vector $Q \in \mathbb{R}^J$ such that its $j$th element is the limit quantity for the $j$th order.

$\delta \in \mathbb{R}^N$ denotes the starting order. The starting order is a unique feature of pari-mutuel auctions (Lange and Economide, 2005; Peters et al., 2005). Before regular traders submit their orders, the market organizer seeds the market with the starting order $\delta$. For each $i$, the organizer purchases $\delta_i$ dollars’ worth of the Arrow-Debreu security that pays $1 per share if and only if the $i$th outcome is realized. Arrow-Debreu securities are introduced only for the starting order and thus are not traded in the regular trading session. Let “the $i$th Arrow-Debreu security” refer to the one that pays $1 per share if and only if the $i$th event is realized. At this point, the organizer does not know the number of shares of Arrow-Debreu securities he/she owns because those securities are not yet priced. The prices of Arrow-Debreu securities are determined only when the markets are cleared at the end of the regular trading session. The number of the $i$th Arrow-Debreu security the organizer holds is determined by dividing $\delta_i$ by the price of that security. Then, the auctioneer pays the market organizer just like any other trader. The starting orders are included in the model to ensure that the market clearing optimization problem yields a unique set of prices for contingent claims (Lange and Economide, 2005; Peters et al., 2005).

Equation (3) is the CPCAM. Let $\varepsilon \in \mathbb{R}^N$ denote the vector of state prices: The $i$th element of $\varepsilon$ is the state price for the $i$th outcome. $\varepsilon_i$ is the Lagrange multiplier associated with the constraint

$$\sum_{j=1}^{J} A_{i,j} x_j + s_i = M.$$  

The state prices are the building blocks on the basis of which all contingent claims traded on this market are priced. For example, the market-clearing price for the contingent claim with the payoff structure $A_{i,j}$ is $A_{i,j}^T \varepsilon$. $s \in \mathbb{R}^N$ and $M$ are dummy variables. $x \in \mathbb{R}^J$ is the vector of order fills. For example, the $j$th element of $x$ is the number of shares of the claim that the submitter of the $j$th order is allowed to purchase.
\[
\begin{align*}
\max_{\mathbf{x}, \mathbf{s}, M} \mathbf{b}^T \mathbf{x} - M + \sum_{i=1}^{N} \delta_i \log(s_i) \\
\text{such that} \\
(A) & \quad \sum_{j=1}^{J} A_{i,j} x_j + s_i = M \quad \text{for each } i \in \{1, 2, \ldots, N\} \\
(B) & \quad 0 \leq \mathbf{x} \leq \mathbf{Q} \\
(C) & \quad \mathbf{s} \geq \mathbf{0}
\end{align*}
\]

The Karush-Kuhn-Tucker (KKT) optimality condition for (3) implies the limit order logic (4) for each \(j\). The market maker can exercise his/her discretion if the bid price is exactly equal to the market-clearing price of the order.

\[
\begin{align*}
x_j &= 0 \quad \text{if} \quad \mathbf{A}_j^T \mathbf{v} > b_j \\
x_j &\in [0, Q_j] \quad \text{if} \quad \mathbf{A}_j^T \mathbf{v} = b_j \\
x_j &= Q_j \quad \text{if} \quad \mathbf{A}_j^T \mathbf{v} < b_j
\end{align*}
\]

The person who submitted the \(j\)th order pays the premium worth \(b_j x_j\) to the market maker. If the \(i\)th outcome is realized, the market maker pays the person \(A_{ij} x_j\).

The \(\sum_{i=1}^{N} \delta_i \log(s_i)\) term ensures the existence of a unique state price vector. However, the starting order subjects the market organizer to potential financial loss when the claims mature. To minimize organizer’s potential loss, Peters et al. (2005) suggest making the magnitude of \(\delta\) very small.

### 3.2 Equivalence with the Ambiguity-Averse Market Maker

Let \(u: \mathbb{R} \to \mathbb{R}\) denote the market maker’s utility function. Suppose that \(u\) is an increasing function. Unlike Peters et al. (2005), we suppose that the market maker uses the uniform starting order. That is, \(\delta_i\) is the same constant \(\delta\) for \(\forall i\). Let \(\mathbf{v}(\delta)\) denote the state price vector associated with (3) when \(\delta_i = \delta\) for \(\forall i\). Let \(\mathbf{x}(\delta)\) denote an optimal value of \(\mathbf{x}\) for (3).
Theorem 2 As $\delta \to 0$, $x(\delta)$ converges to an optimal solution for (5).

\[
\max_x \min_p \sum_{i=1}^N p_i u \left[ b^T x - \sum_{j=1}^J A_{i,j} x_j \right]
\]

such that

(A') $0 \leq x \leq Q$

(B') $p \geq 0, \sum_{i=1}^N p_i = 1$

(C') $\Psi = \{ p \in \mathbb{R}^{N \times 1} | p \geq 0, \sum_{i=1}^N p_i = 1, \sum_{i=1}^N p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega \}$

\[
(5)
\]

Proof. See the Appendix.

(5) is an optimization problem to which (6) converges as the value of $\Omega$ increases to infinity.

\[
\max_x \min_p \sum_{i=1}^N p_i u \left[ b^T x - \sum_{j=1}^J A_{i,j} x_j \right]
\]

such that

(A') $0 \leq x \leq Q$

(B') $p \geq 0, \sum_{i=1}^N p_i = 1$

(C') $\Psi = \{ p \in \mathbb{R}^{N \times 1} | p \geq 0, \sum_{i=1}^N p_i = 1, \sum_{i=1}^N p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega \}$

(6) is the optimization problem that the market maker should be solving if the market maker’s decision-making process obeys the theory of Gilboa and Schmeidler (1989) or Ghirardato et al. (2004). $b^T x$ is the total premium that the market maker collects from the traders. $\sum_{j=1}^J A_{i,j} x_j$ is what the market maker has to pay to traders if the $i$th outcome is realized. $\sum_{i=1}^N p_i u \left[ b^T x - \sum_{j=1}^J A_{i,j} x_j \right]$ is thus the expected utility for the market maker. The vector $q \in \mathbb{R}^N$ is the pivot prior probability distribution. The market maker considers any probability distribution $p$ reasonable as long as the Kullback-Leibler distance between $p$ and $q$ is not greater than $\Omega$.

Therefore, (5) is an optimization problem that the market maker solves if he/she is a DM with extreme Knightian ambiguity aversion. In Theorem 2, we show that the market-clearing order fill of the CPCAM is an optimal market-clearing strategy of a market maker with extreme ambiguity aversion.

Our result may be relevant to other pari-mutuel markets because the CPCAM is closely related to other pari-mutuel markets. First, the CPCAM is an improved version of the PDCA. Peters et al. (2005) developed the CPCAM to make the optimization problem convex. However, the CPCAM and the PDCA still yield the same equilibrium price.
Second, Agrawal et al. (2011) show that many important pari-mutuel markets in the literature (e.g., the Market Scoring Rule mechanism, cost-function based market makers, utility-based market makers, and the Sequential Convex Pari-mutuel Mechanism) can be understood under a common theoretical framework. The Sequential Convex Pari-mutuel Mechanism (SCPM) (Peters et al., 2007) is one of the pari-mutuel markets that Agrawal et al. (2011) analyze. In addition, the CPCAM and the SCPM are very closely related to one another. The only major difference is that the CPCAM is a call auction and the SCPM is a continuous market. Therefore, the CPCAM and other important pari-mutuel markets are closely related to one another. Given this close relationship between different market designs, our analysis of the CPCAM may also apply to other pari-mutuel markets. However, we leave that extension to future work.

4 The Knightian Pari-mutuel Mechanism (KPM)

In this section, we design a new market called the Knightian Pari-mutuel Mechanism (KPM).

4.1 The Market Setting

The limit order logic and the basic trading environment are similar to those of the PDCA (Lange and Economide, 2005). However, the algorithm through which the market maker clears the market is original. In particular, the probabilistic treatment of the market maker’s optimization problem is original.

Like the CPCAM, the KPM allows the common market maker to handle multiple types of contingent claims written on the same random event. There are $N$ possible states of the uncertain event, each of which is indexed by $i \in \{1, 2, ..., N\}$.

The KPM allows traders to submit both market orders and limit orders. Traders can submit market orders just as if they were submitting limit orders simply by making the limit price extremely high or low. Therefore, throughout the rest of the paper, we assume that people trade only limit orders. When submitting each limit order, the trader indicates the limit price, the limit quantity, and if the order is a buy or a sell.

For the sake of simplicity, we describe the setting in which the market is run as a call auction. However, the setting can be easily adjusted to accommodate continuous trading in the same manner as the CPCAM (Peters et al., 2005) is changed to the SCPM (Peters et al., 2007).

Suppose there is a total of $J$ limit orders outstanding in the limit order book. Let the matrix
$A \in \mathbb{R}^{N \times J}$ represent the payoff structure of those orders. The column matrix $A_j \in \mathbb{R}^{N \times 1}$ is the payoff structure of the contingent claim that the $j$th order attempts to transact. For example, suppose that the second order attempts to buy three shares of the contingent claim that pays $1$ per share if and only if state 1 is realized. In such a case, the column matrix $A_2$ is $\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}^T$.

When determining the market equilibrium price of each order, the market maker first determines the equilibrium price for each state. Let $\xi = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_N \end{bmatrix}^T$ denote the equilibrium state prices. Then, the market maker determines the market-clearing price of each contingent claim by taking the dot product between the payoff vector and $\xi$. For example, consider the contingent claim with payoff structure $\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$. Suppose that the equilibrium state price vector $\xi$ is $\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{bmatrix}^T$. Then, the market-clearing price of this contingent claim is $\xi_1 + \xi_3$. Therefore, to determine the equilibrium price of each order in the limit order book, the market maker only has to determine the value of $\xi$.

The binary variable $B_j$ is 1 if the $j$th limit order is a buy order and $-1$ if the $j$th limit order is a sell order. Let $b_j$ and $Q_j$ denote the limit price and the limit quantity associated with the $j$th order, respectively. Let $x_j$ denote the actual number of the shares of the claim that the submitter of the $j$th order is allowed to trade. We call $x_j$ the "order fill" for the $j$th order.

Once the equilibrium price of each order is determined, the market maker decides $x_j$, $\forall j$ according to the limit order logic. Consider a buy order. If the market-clearing price of an order is strictly higher than the limit price, $x_j$ is exactly equal to 0. If the market-clearing price is strictly lower than the limit price, $x_j$ is set to $Q$. In these two cases, the limit order logic automatically determines the order fill. In contrast, if the market-clearing price of an order is exactly equal to the limit price, the value of $x_j$ can be any number in the closed interval $[0, Q_j]$. The logic works similarly for a sell order. Define the vector $x \in \mathbb{R}^{J \times 1}$ such that the $j$th element of $x$ is $x_j$. Similarly, define $Q \in \mathbb{R}^{J \times 1}$ such that the $j$th element of $Q$ is $Q_j$. Define $b \in \mathbb{R}^{J \times 1}$ such that the $j$th element of $b$ is $b_j$.

Definition 2 (Limit Order Logic) $\sum_{i=1}^{N} A_{ij} \xi_i$ is the market-clearing price of the $j$th order.

$$
x_j = 0 \quad \text{if} \quad \sum_{i=1}^{N} A_{ij} \xi_i > B_j b_j
$$

$$
x_j \in [0, Q_j] \quad \text{if} \quad \sum_{i=1}^{N} A_{ij} \xi_i = B_j b_j
$$

$$
x_j = Q_j \quad \text{if} \quad \sum_{i=1}^{N} A_{ij} \xi_i < B_j b_j
$$
The market maker has two decision variables for his/her optimal clearing of the market: the equilibrium state prices $\xi$ and the order fill vector $x$.

Like other market makers in the financial markets, the market maker of the KPM also has an inventory of contingent claims. If the $i$th state is realized in the future, the inventory subjects the market maker to the monetary payoff of $w_i$. Let $\alpha$ denote the market maker’s risk aversion coefficient. Suppose that the constant absolute risk aversion (CARA) utility function $u(x) = -e^{-\alpha x}$ characterizes the market maker’s risk appetite.

The market maker has Knightian ambiguity toward the random future event on which the claims are written. Let the set $\Psi$ define the set of probability distributions that the market maker considers. $q \in \mathbb{R}^{N \times 1}$ is the market maker’s pivot probability distribution. Assume that every element in $q$ is strictly positive. Any probability distribution for which the Kullback-Leibler from $q$ is no greater than $\Omega$ is acceptable for the market maker. $\Omega$ quantifies the market maker’s level of ambiguity aversion. A large value of $\Omega$ implies that the market maker has strong ambiguity aversion. The $i$th elements of $p$ and $q$ describe the market maker’s probabilistic belief about the $i$th outcome.

$$\Psi = \left\{ p \in \mathbb{R}^{N \times 1} | p \geq 0, \sum_{i=1}^{N} p_i = 1, \sum_{i=1}^{N} p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega \right\}$$  \hspace{1cm} (7)

4.2 The Market-Clearing Optimization Problem

We assume that the market maker adheres to the standard decision-making theory under Knightian ambiguity aversion. The market maker’s optimization problem can be framed as (8). Unlike the CPCAM, the KPM asks the market participants to pay the market-clearing prices of the claims instead of the bid prices they submitted.

This optimization problem does not make any arbitrary assumptions. The problem is a corollary of the standard theory of decision making under ambiguity aversion. However, the constraints (E1) - (E3) and the objective function causes the problem to be non-convex. Finding a global optimal solution to a non-convex optimization problem is extremely difficult.
\[
\max_{\xi, x} \min_{p \in \Psi} \sum_{i=1}^{N} p_i \exp \left[ -\alpha w_i - \alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]
\]
such that
\[(A) \ \Psi = \{ p \in \mathbb{R}^{N \times 1} | p \geq 0, \sum_{i=1}^{N} p_i = 1, \sum_{i=1}^{N} p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega \}\n(B) \ \xi \geq 0 \n(C) \ \sum_{i=1}^{N} \xi_i = 1 \n(E1) \ \forall j \in \{1, 2, \ldots, J\}, \ x_j = 0 \quad \text{if} \quad (A^T \xi)_j > B_j b_j \n(E2) \ \forall j \in \{1, 2, \ldots, J\}, \ x_j \in [0, Q_j] \quad \text{if} \quad (A^T \xi)_j = B_j b_j \n(E3) \ \forall j \in \{1, 2, \ldots, J\}, \ x_j = Q_j \quad \text{if} \quad (A^T \xi)_j < B_j b_j \]

**Corollary 1** Suppose that the market maker holds zero inventory: \(w_i = 0\) for \(\forall i\). As the value of \(\Omega\) increases to infinity, the KPM becomes completely pari-mutuel. The market maker incurs no loss regardless of the outcome.

**Proof.** See the Appendix. □

The KPM may not be completely pari-mutuel in the sense that the market maker can lose money with positive probability. However, Corollary 1 shows that the KPM subsumes a completely pari-mutuel market. By adjusting the value of \(\Omega\), the market designer can fine-tune the extent to which the market is close to being completely pari-mutuel. The larger the value of \(\Omega\), the more completely pari-mutuel the market becomes.

For example, consider increasing the value of \(\Omega\). Problem (8) then models the auctioneer with a large level of ambiguity aversion. The ambiguity-averse DM is very sensitive to the worst-case scenario. Thus, the auctioneer clears the market such that he/she performs moderately even in the worst-case scenario. In other words, the auctioneer does not want to lose too much money even in the worst-case scenario. In the extreme case in which \(\Omega\) diverges to infinity, the auctioneer becomes so conservative that he/she does not want to lose any money under any circumstances. The market should become completely pari-mutuel.

### 4.3 The Market-Clearing Algorithm

Before further discussion, we introduce new notations: \(z_i = -e^{-\alpha w_i}\) and \(\theta_i = q_i e^\Omega\) for each \(i \in \{1, 2, \ldots, N\}\). In addition, let \(F\) be the set of pairs \((\xi, x)\) that satisfy the limit order logic constraints (E1), (E2), and (E3).

\footnote{The cost of this strategy is that the market maker may not be able to make a great deal of money on the upside.}
Lemma 1 \((\xi, x) = (\xi^*, x^*)\) is an optimal solution to (8) if and only if it is part of an optimal solution to (9).

\[
\min_{\xi, x, \mu, d, \zeta} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)
\]

such that

(A) \(-d_i = -z_i e^{\xi_i}\) for \(\forall i\)

(B) \(\zeta_i \geq \alpha \sum_{j=1}^{J} \left[ x_j A_{ij} - Q_j \left( A^T \xi \right)_j - B_j b_j (x_j - Q_j) \right]\) for \(\forall i\)

(C) \(\zeta_i \geq \alpha \sum_{j=1}^{J} \left[ x_j A_{ij} - B_j b_j x_j \right]\) for \(\forall i\)

(F) \(\mu \geq 0\)

(G) \(\zeta \geq 0\)

(H) \(\sum_{i=1}^{N} \xi_i = 1\)

(I) \((x, \xi) \in F\)

**Proof.** See the Appendix.

It is difficult to directly apply well-known optimization algorithms (e.g., the interior-point method) to solve (9) because the problem is non-convex. The problem is non-convex because \(F\) is not a convex set.

Suppose \(C\) is a convex set of pairs \((x, \xi)\). We define another optimization problem (10).

\[
\min_{\xi, x, \mu, d, \zeta} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)
\]

such that

(A) \(-d_i = -z_i e^{\xi_i}\) for \(\forall i\)

(B) \(\zeta_i \geq \alpha \sum_{j=1}^{J} \left[ x_j A_{ij} - Q_j \left( A^T \xi \right)_j - B_j b_j (x_j - Q_j) \right]\) for \(\forall i\)

(C) \(\zeta_i \geq \alpha \sum_{j=1}^{J} \left[ x_j A_{ij} - B_j b_j x_j \right]\) for \(\forall i\)

(F) \(\mu \geq 0\)

(G) \((x, \xi) \in C\)

Lemma 2 The optimization problem (10) is a convex optimization problem.

**Proof.** See the Appendix.

Our general strategy is as follows. First, we express the set of pairs \((x, \xi)\) that satisfies the constraints (G), (H) and (I) in (9) as a union of multiple convex sets \(C_1, C_2, ..., C_M\). Second, we
solve a convex optimization problem (10) with \( C \) replaced with each \( C_m, m \in \{1, \ldots, M\} \). Let \( L_m \) denote the optimal value of the objective function from solving the convex optimization problem (10) with \( C = C_m \). Let \((x_m, \xi_m)\) denote the optimal solutions to those problems. Third, we find \( m^* = \arg \max_m L_m \). \((x_{m^*}, \xi_{m^*})\) becomes the global optimal solution to the main optimization problem (9). By Lemma 1, \((x_{m^*}, \xi_{m^*})\) is the global optimal solution to (8).

### 4.3.1 Partitioning of the Feasible Set

Suppose that a total of \( K \) types of contingent claims are traded in the market. Let the vector \( P_k \in \mathbb{R}^N \) denote the payoff structure of the \( k \)th security \((1 \leq k \leq K)\). For example, if the \( i \)th outcome is realized, the person holding the claim receives \( P_{k,i} \) per share from the market maker.

Because there are \( J \) outstanding orders in the limit order book, there are \( J \) limit prices. Let \( n_k \) denote the number of distinct limit prices associated with the \( k \)th security. If there are multiple orders with the same limit price and the same security, only one is counted toward \( n_k \). Sort those bid prices in ascending order. Let \( B^l_k \) denote the \( l \)th smallest limit price associated with the \( k \)th security.

**Example 1** For the sake of simplicity, consider a market in which only Arrow-Debreu securities are traded. Suppose that \( N = 5 \). Suppose that there are \( K = 5 \) different Arrow-Debreu securities, one for each state of the world. The \( k \)th Arrow-Debreu security pays $1 per share to its holder if and only if the \( k \)th state is realized.

Five row vectors in (11) show the payoff structures of Arrow-Debreu securities. For example, the nonzero entry in the first element of \( P_1 \) implies that the first security pays $1 per share if the first outcome is realized.

\[
\begin{align*}
P_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
P_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
P_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\
P_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
P_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

(11)

Table 1 illustrates seven orders \((J = 7)\) outstanding in the limit order book. For example, the person who submitted the first order wants to buy the first Arrow-Debreu security. The entry 0.18
Table 1: An example of a limit order book

in the fourth column implies that he/she is willing to pay at most 0.18 dollars per share. The payoff matrix in the last five columns shows how the person will be paid by the market maker. For example, the market maker will pay $1 to the person who submitted the first order if and only if the first outcome is realized. The person who submitted the fourth order will receive $1 if and only if the fourth outcome is realized.

Based on this limit order book, we can determine the number of distinct limit prices associated with each Arrow-Debreu security. For example, for the first Arrow-Debreu security, there are three distinct limit prices: 0.18, 0.20 and 0.25. Thus, \( n_1 \) should be 3. Likewise, \( n_2 = 1 \), \( n_3 = 1 \), \( n_4 = 1 \), and \( n_5 = 0 \).

Next, we sort the limit prices in ascending order. For example, for the first Arrow-Debreu security, we have \( B^1_1 = 0.18 \), \( B^2_1 = 0.20 \), and \( B^3_1 = 0.25 \). In addition, \( B^1_2 = 0.18 \), \( B^3_2 = 0.18 \), and \( B^1_4 = 0.18 \). Because there is no limit order associated with the fifth Arrow-Debreu security, \( B^1_5 \) is undefined.

Suppose that there are \( n_k \) distinct limit prices. Let \( E \) denote the \( N \)-dimensional space defined as (12). We define \( n_k + 1 \) convex subsets of \( E \) such that if \( \xi \) is restricted to one of those subsets, (8) becomes a convex optimization problem. The intuition is as follows. The limit order logic constraints (E1) - (E3) are non-convex because we do not know which of the three conditions - \( (A^T \xi)_j > b_j \) or \( (A^T \xi)_j = b_j \) or \( (A^T \xi)_j < b_j \) - hold at an optimal solution. We define subsets to ensure that such ambiguity is resolved within each set. As a result, the limit order logic constraints can be replaced by \( x_j = 0 \) or \( x_j \in [0, Q_j] \) or \( x_j = Q_j \).

\[
E = \left\{ \xi \in \mathbb{R}^N | \sum_{i=1}^{N} \xi_i = 1, \xi \geq 0 \right\} 
\]  
(12)

16
Let us illustrate how we obtain subsets of $E$. Note that the market-clearing price of each Arrow-Debreu security is bounded below by 0 and above by 1. $n_k$ distinct bid prices associated with the $k$th Arrow-Debreu security define $2n_k + 1$ subsets of $[0,1]$: $[0,B^1_k], B^1_k, [B^1_k,B^2_k], B^2_k,...,B^{n_k}_k, [B^{n_k}_k,1]$.

These $2n_k + 1$ points or closed intervals can be used to define $2n_k + 1$ subsets of $E$: $E^1_k, E^2_k,..., E^{2n_k+1}_k$, as shown in (14).

$$E^1_k = \left\{ \xi \in \mathbb{R}^N \mid \sum_{i=1}^N \xi_i = 1, \xi \geq 0, P_k \xi \in [0,B^1_k] \right\}$$

$$E^2_k = \left\{ \xi \in \mathbb{R}^N \mid \sum_{i=1}^N \xi_i = 1, \xi \geq 0, P_k \xi = B^1_k \right\}$$

... (13)

$$E^{2n_k+1}_k = \left\{ \xi \in \mathbb{R}^N \mid \sum_{i=1}^N \xi_i = 1, \xi \geq 0, P_k \xi \in [B^{n_k}_k,1] \right\}$$

**Example 2** We continue with the earlier example. Let us begin with the first Arrow-Debreu security. Using the three distinct limit prices, we can define $2 \times 3 + 1 = 7$ subsets of $E = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0 \right\}$: $E_1^1, E_1^2,..., E_1^7$. Note that $P_1 \xi = \xi_1$.

- $E_1^1 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, 0 \leq \xi_1 \leq 0.18 \right\}$
- $E_1^2 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, \xi_1 = 0.18 \right\}$
- $E_1^3 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, 0.18 \leq \xi_1 \leq 0.2 \right\}$
- $E_1^4 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, \xi_1 = 0.2 \right\}$
- $E_1^5 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, 0.2 \leq \xi_1 \leq 0.25 \right\}$
- $E_1^6 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, \xi_1 = 0.25 \right\}$
- $E_1^7 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^5 \xi_i = 1, \xi \geq 0, 0.25 \leq \xi_1 \leq 1 \right\}$

There is only one distinct limit price for the second Arrow-Debreu security. Therefore, we can define three subsets of the set $E$ as (15). The third and the fourth Arrow-Debreu securities also

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6I assume that the bid prices are strictly larger than 0 and strictly smaller than 1.
have only one limit price. Therefore, the partitioning of $E$ should work in exactly the same way.

$$E_2^1 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, 0 \leq \xi_2 \leq 0.18 \right\}$$

$$E_2^2 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, \xi_2 = 0.18 \right\}$$

$$E_2^3 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, 0.18 \leq \xi_2 \leq 1 \right\}$$

Theorem. There is no outstanding order or limit price associated with the fifth Arrow-Debreu security. Thus, we can define only one subset of the set $E$: $E_5^1$.

$$E_5^1 = \left\{ \xi \in \mathbb{R}^5 \mid \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0 \right\}$$

We introduce new notation as (17). The idea is as follows. Having $n_1$ distinct limit prices associated with the first security yields $2n_1 + 1$ distinct subsets of $E$. Choose one subset out of these $2n_1 + 1$ subsets. Similarly, choose one of $2n_2 + 1$ subsets of $E$ that we generate from the limit prices associated with the second security. Repeat this process for the remaining Arrow-Debreu securities. Once we have one subset for each type of Arrow-Debreu security, we can obtain the intersection of those $K$ subsets, which is shown in (17). There are a total of $\prod_{k=1}^{K}(2n_k + 1)$ ways to choose a combination of subsets.

$$E(\ell_1, \ell_2, ..., \ell_K) = E_{\ell_1}^1 \cap E_{\ell_2}^2 \cap ... \cap E_{\ell_K}^K$$

where $1 \leq \ell_1 \leq 2n_1 + 1, ..., 1 \leq \ell_K \leq 2n_K + 1$ (17)

Now consider the optimization problem (9). Imagine replacing the constraint $\sum_{i=1}^{N} \xi_i = 1, \xi \geq 0$ with a more restrictive one (17). The part that causes problem (9) to be non-convex is (18). However, once the feasible set of the state price vector $\xi$ is restricted to a smaller set $E(\ell_1, \ell_2, ..., \ell_K)$, (18) can be replaced with $x_j = 0$ or $x_j \in [0, Q_j]$ or $x_j = Q_j$ for all $j$.

$$x_j = 0 \quad \text{if} \quad (A^T \xi)_j > B_j b_j$$

$$x_j \in [0, Q_j] \quad \text{if} \quad (A^T \xi)_j = B_j b_j \quad \text{for all } j \in \{1, ..., J\}$$

$$x_j = Q_j \quad \text{if} \quad (A^T \xi)_j < B_j b_j$$

(18)
Example 3 Again, we continue with the previous example. Because $n_1 = 3$, $n_2 = 1$, $n_3 = 1$, $n_4 = 1$, and $n_5 = 0$ there are in total $(3 \times 2 + 1) \times (1 \times 2 + 1) \times (1 \times 2 + 1) \times (1 \times 2 + 1) \times (0 \times 2 + 1) = 189$ different sets of the form $E(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$.

To illustrate, consider a particular case in which $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 2$, $\ell_4 = 2$, and $\ell_5 = 1$.

\[
E^{\ell_1} = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, 0 \leq \xi_0 \leq 0.18 \right\}
\]

\[
E^{\ell_2} = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, \xi_1 = 0.18 \right\}
\]

\[
E^{\ell_3} = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, \xi_2 = 0.18 \right\}
\]

\[
E^{\ell_4} = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, \xi_3 = 0.18 \right\}
\]

\[
E^{\ell_5} = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0 \right\}
\]

\[
E(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) = \left\{ \xi \in \mathbb{R}^5 \middle| \sum_{i=1}^{5} \xi_i = 1, \xi \geq 0, 0 \leq \xi_1 \leq 0.18, \xi_2 = \xi_3 = \xi_4 = 0.18 \right\}
\]

Suppose that we replace the usual constraint $\sum_{i=1}^{N} \xi_i = 1, \xi \geq 0$ with a more restrictive one in the main optimization problem. Then, the optimization should take the form of (21).

\[
\min_{\xi, x, \mu, \omega} \mu \ln \left( \sum_{i=1}^{5} \theta_i e^{-\frac{d_i}{\mu}} \right)
\]

such that

(A) $\omega_i \geq -z_ie^{\xi_i}$ for $\forall i$

(B) $\zeta_i \geq \alpha \sum_{j=1}^{7} \left[ x_j A_{ij} - Q_j (A^T \xi)_j - B_j b_j (x_j - Q_j) \right]$, $\forall i$

(C) $\zeta_i \geq \alpha \sum_{j=1}^{7} \left[ x_j A_{ij} - B_j b_j x_j \right]$, $\forall i$

(D) $\mu \geq 0$

(E) $\xi \in E(1, 2, 2, 2, 1)$

\[
x_j = 0 \text{ if } (A^T \xi)_j > B_j b_j
\]

(F) $x_j \in [0, Q_j] \text{ if } (A^T \xi)_j = B_j b_j \text{ for } \forall j$

\[
x_j = Q_j \text{ if } (A^T \xi)_j < B_j b_j
\]

As long as constraint (E) holds, constraint (F) can be replaced with those in the last column of
Table 2 An Example of How the Limit Order Logic Constraint Can be Simplified

| # | order | limit quantity | security # | bid price | market clearing price | relevant restriction on the price in $E(1,2,2,2,1)| restriction on the order fill | $x_j$ |
|---|------|----------------|------------|----------|-------------------|-----------------------------------------------|------------------------|---|
| 1 | 0.001 | 1              | 1          | 0.18     | $\xi_1$           | $\xi_1 \leq 0.18$                               | 0.001                  |
| 2 | 0.001 | 2              | 1          | 0.18     | $\xi_2$           | $\xi_2 = 0.18$                                  | 0.001                  |
| 3 | 0.001 | 3              | 1          | 0.18     | $\xi_3$           | $\xi_3 = 0.18$                                  | 0.001                  |
| 4 | 0.001 | 4              | 1          | 0.18     | $\xi_4$           | $\xi_4 = 0.18$                                  | 0.001                  |
| 5 | 0.002 | 1              | 1          | 0.20     | $\xi_1$           | $\xi_1 \leq 0.18$                               | 0.001                  |
| 6 | 0.001 | 1              | 1          | 0.25     | $\xi_1$           | $\xi_1 \leq 0.18$                               | 0.001                  |
| 7 | 0.001 | 1              | 1          | 0.20     | $\xi_1$           | $\xi_1 \leq 0.18$                               | 0.001                  |

Solving (21) is equivalent to solving (22).

\[
\min_{\xi, x, \mu, \zeta, \omega} \ell(\mu) = \mu \ln \left( \sum_{i=1}^{5} \theta_i e^{-\frac{d_i}{\mu}} \right),
\]

such that

(A) \(-d_i \geq -z_i e^{\xi_i} \) for \(\forall i\)

(B) \(\zeta_i \geq \alpha \sum_{j=1}^{7} [x_j A_{ij} - Q_j (A^T \xi)_j] - B_j b_j (x_j - Q_j)\), \(\forall i\)

(C) \(\zeta_i \geq \alpha \sum_{j=1}^{7} [x_j A_{ij} - B_j b_j x_j]\), \(\forall i\)

(D) \(\mu \geq 0\)

(E) \(\xi \in E(1,2,2,2,1)\)

(F) \(x_1 = x_5 = x_6 = x_7 = 0.001, 0 \leq x_2, x_3, x_4 \leq 0.001\)

For notational simplicity, we define new sets:

\[
X(\ell_1, \ell_2, ..., \ell_m) = \begin{cases} 
  x_j = 0 & \text{if } \max_{\xi \in E(\ell_1, \ell_2, ..., \ell_K)} (A^T \xi)_j > B_j b_j \text{ and} \\
  x_j \in [0, Q_j] & \text{if } \min_{\xi \in E(\ell_1, \ell_2, ..., \ell_K)} (A^T \xi)_j = \max_{\xi \in E(\ell_1, \ell_2, ..., \ell_K)} (A^T \xi)_j = B_j b_j, \forall j \\
  x_j = Q_j & \text{if } \min_{\xi \in E(\ell_1, \ell_2, ..., \ell_K)} (A^T \xi)_j < B_j b_j 
\end{cases}
\]
Example 4 We continue with the previous example. \(X(1, 2, 2, 2, 1)\) is defined as (24).

\[
X(1, 2, 2, 2, 1) = \{ x \in \mathbb{R}^7 \mid x_1 = x_5 = x_6 = x_7 = 0.001, 0 \leq x_2, x_3, x_4 \leq 0.001 \} \quad (24)
\]

4.3.2 The Pseudo-Code

If we apply an interactive method (e.g., the interior point method) to solve (10), \(\mu\) may converge toward zero along the path. However, the objective function is ill-defined when \(\mu\) is zero. Therefore, we define a new objective function as (25).

\[
L(\mu, \omega) = \begin{cases} 
\mu \ln \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right) & \text{if } \mu > 0 \\
\max_{1 \leq i \leq N} \omega_i & \text{if } \mu = 0
\end{cases}
\quad (25)
\]

Then, (9) can be reformulated as (26).

\[
\min_{\xi, x, \mu, \zeta, \omega} L(\mu, \omega)
\quad \text{such that}
\]

(A) \(\omega_i \geq -z_i e^{\xi_i} \) for \(\forall i\)

(B) \(\zeta_i \geq \alpha \sum_{j=1}^{J} \left[ x_j a_{ij} - Q_j (A^T \xi)_j - B_j b_j (x_j - Q_j) \right], \forall i\)

(C) \(\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j a_{ij} - B_j b_j x_j], \forall i\) \quad (26)

(D) \(\mu \geq 0\)

(E) \(\xi \geq 0\)

(F) \(\sum_{i=1}^{N} \xi_i = 1\)

(G) \((x, \xi) \in F\)
A global optimal solution to (26) can be obtained by executing the following pseudo-code.

$$\min_{\xi, x, \mu, \zeta, \omega} L(\mu, \omega)$$

such that

(A) $\omega_i \geq -z_i e^{\xi_i}$ for $\forall i$
(B) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - Q_j (A^T \xi)_j - B_j b_j (x_j - Q_j)], \forall i$
(C) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - B_j b_j x_j], \forall i$
(D) $\mu \geq 0$
(E) $x \in E(\ell_1, \ell_2, ..., \ell_K)$
(F) $\xi \in X(\ell_1, \ell_2, ..., \ell_K)$

for $\ell_1 = 1 : 1 : n_1$

for $\ell_2 = 1 : 1 : n_2$

... 

for $\ell_K = 1 : 1 : n_K$

if $E(\ell_1, \ell_2, ..., \ell_K) \neq \emptyset$

Solve (27) using the interior point method.

The optimal value of the objective function $\rightarrow L^*(\ell_1, \ell_2, ..., \ell_K)$

The optimizing value of $x \rightarrow x^*(\ell_1, \ell_2, ..., \ell_K)$

The optimizing value of $\xi \rightarrow \xi^*(\ell_1, \ell_2, ..., \ell_K)$

end

end

... 

for 

end

arg max_{\ell_1, \ell_2, ..., \ell_K, E(\ell_1, \ell_2, ..., \ell_K) \neq \emptyset} L^*(\ell_1, \ell_2, ..., \ell_K) \rightarrow \ell_1^*, \ell_2^*, ..., \ell_K^*$

$x^*(\ell_1, \ell_2, ..., \ell_K), \xi^*(\ell_1, \ell_2, ..., \ell_K) \rightarrow$ global optimal solution

4.3.3 The Computational Efficiency

In modern complexity analysis, the efficiency of an algorithm is assessed based on whether the number of iterations required is bounded above by a polynomial of the problem dimension (Luen-
berger and Ye, 2008). In our setting, the number of securities traded in the market typically does not grow in the order of thousands. Frequently, a growing number of outstanding orders in the limit order book demands significant computing power. Therefore, to prove that our algorithm is of practical value, we need to show that the algorithm is polynomial in the number of outstanding orders $J$. Theorem 3 does precisely this.

**Theorem 3** The number of iterations required to execute the pseudo-code (28) is bounded above by a polynomial function of the number of outstanding orders $J$.

**Proof.** See the Appendix.

5 Simulation

For simplicity, we simulate the market when only Arrow-Debreu securities are traded. The $i$th security pays $\$1$ per share to the holder if and only if state $i$ is realized at maturity, where $i \in \{1, 2, 3, 4, 5\}$.

Through this simulation exercise, we verify that our market-clearing algorithm gives the result that is consistent with economic intuition.

5.1 Simulation A: Market Maker’s Ambiguity Aversion

In this subsection, we present simulation results that show how the market maker’s level of ambiguity aversion affects how the market is cleared. We run simulations for five different parameters of ambiguity aversion: $\Omega = 0, 0.2, 0.4, 1, 2$. Table 3 shows the sample limit order book used for Simulation A. Table 4 summarizes the parameters used for each of the five iterations. The table reports the number of shares that traders as a whole hold. Thus, any positive number in the top-left part of the table implies that the market maker may have to incur additional loss at the time the securities mature.
Table 3 Sample Limit Order Book Used for Simulation A

| order | limit quantity | security | bid price per share | payoff matrix | buy or sell |
|-------|----------------|----------|---------------------|---------------|------------|
| #     | b              | #        | state 1 | state 2 | state 3 | state 4 | state 5 |               |
| 1     | 0.002          | 1        | 0.18    | 1       | 0       | 0       | 0       | 0               | buy         |
| 2     | 0.001          | 2        | 0.18    | 0       | 1       | 0       | 0       | 0               | buy         |
| 3     | 0.001          | 3        | 0.18    | 0       | 0       | 1       | 0       | 0               | buy         |
| 4     | 0.001          | 4        | 0.18    | 0       | 0       | 0       | 1       | 0               | buy         |
| 5     | 0.001          | 5        | 0.18    | 0       | 0       | 0       | 0       | 1               | buy         |

Table 4: The Set of Parameters Used for Each Iteration

In Table 3, we purposefully make the bid prices slightly lower than 0.2. For example, because of the limit order logic, the market-clearing price of the first state has to be equal to or smaller than 0.18 for the market maker to accept the first order. If he/she wants to accept all five outstanding orders, he/she has to make every state price lower than or equal to 0.18. However, because the state prices must sum to 1, it is impossible to do so. Therefore, the market maker has to strategically accept some orders while declining others. We investigate how the market maker’s ambiguity aversion affects this strategic decision making through this simulation.

When making a strategic choice over the five outstanding orders, there are two counteracting
forces. The first factor derives from the skewness in the pivot prior probability distribution. This factor causes the market maker to want to accept orders #1, #2, or #3. According to the market maker’s subjective probabilistic belief, he/she is very unlikely to be forced to pay money at the time the securities mature. However, the factor causes the market maker to not want to accept order #4 or #5.

This factor becomes weaker with an increasingly large value of $\Omega$. Suppose that the value of $\Omega$ becomes increasingly large. The set $\Psi$ of probability distributions that the market maker considers in his/her decision making becomes larger. As a result, the upper bound and the lower bound on the probability of a particular outcome becomes higher and lower, respectively. The widening gap between the upper and lower bounds causes the market maker’s probabilistic belief to be increasingly uninformative. For example, suppose that the value of $\Omega$ is extremely large. The probability of a particular outcome can be as high as 1 and as low as 0. In such a case, it is as if the market maker had no information about the event. In conclusion, a larger value of $\Omega$ causes the market maker to further disregard the pivot prior distribution in making the decision, thereby weakening the first factor.

The second factor derives from the market maker’s aversion to extreme downside risk. The price of any Arrow-Debreu security is between 0 and 1. Thus, the worst-case payoff of any Arrow-Debreu security is typically negative for the market maker. Fearing this worst-case scenario, the ambiguity-averse market maker will not want to purchase the security. This factor causes the market maker to not want to fill any outstanding order.

The second factor becomes stronger with an increasing value of $\Omega$. $\Omega$ is a parameter that captures the extent to which the market maker is ambiguity averse. The larger the value of $\Omega$, the more ambiguity averse the market maker. Ambiguity aversion causes the DM to become obsessed with the worst-case scenario. Therefore, a large value of $\Omega$ causes the second factor to become stronger.

Figure 1 shows the simulation result. The result can be easily interpreted using the two counteracting forces we just explained. First, consider the situation in which $\Omega$ is small. The first factor dominates the second factor. As a result, the market maker accepts orders #1, #2 and #3 while declining orders #4 and #5. Second, consider the case in which $\Omega$ is large. Here, the second factor dominates the first factor. The market maker does not accept any order when $\Omega$ is larger than 0.4.

\footnote{It is zero if and only if the price is 1.}
Figure 1: The graph shows the order fills for different values of $\Omega$, which parametrizes the market maker’s level of ambiguity aversion. For example, when $\Omega$ is 0.4, a 0.001 share of the second Arrow-Debreu security is filled. The second Arrow-Debreu security pays $1 to its holder if and only if the second state is realized at maturity.

5.2 Simulation B: Market Maker’s Pivot Probability Distribution

In this subsection, we show how the market maker’s pivot prior probabilistic belief affects the way our algorithm clears the market. Table 5 below shows the limit order book used for this subsection. Table 6 below shows the set of simulation parameters for both iterations.

Table 5 Sample Limit Order Book Used for Simulation B

| order # | limit quantity | security # | bid price per share | payoff matrix | buy or sell |
|---------|----------------|------------|---------------------|---------------|-------------|
| 1       | 0.001          | 1          | 0.18                | 0 0 0 0       | buy         |
| 2       | 0.001          | 2          | 0.18                | 0 1 0 0       | buy         |
| 3       | 0.001          | 3          | 0.18                | 0 0 1 0       | buy         |
| 4       | 0.001          | 4          | 0.18                | 0 0 0 1       | buy         |
| 5       | 0.001          | 5          | 0.18                | 0 0 0 0 1     | buy         |
Table 6: The Set of Parameters Used for Simulation B

Figure 2 shows the two prior distributions used in this simulation exercise. The market maker with the uniform prior has no information on what will happen in the future. With no valuable piece of evidence available to make an inference, the market maker simply assumes that each state is equally probable. In contrast, the market maker with the exponential prior is more assertive in deciding which state is more probable than the others. For example, he/she thinks that state 5 is at least sixty times more probable than state 1.

Figure 3 shows the outcome of the market-clearing algorithm for two prior distributions. We
Figure 3: The graph shows the market clearance result for different prior beliefs held by the market maker. The vertical axis shows the number of shares of each Arrow-Debreu security filled. For example, in the case of the exponential prior, 0.001 shares of the first Arrow-Debreu security are filled (the first Arrow-Debreu security pays $1 to its holder if and only if state 1 is realized at maturity).

interpret the result with the countervailing forces introduced in the previous section.

For the market maker with the completely uninformative prior, the second factor strongly dominates the first factor. The first factor is powerless because, with the uniform prior, the market maker does not face a larger risk of incurring a loss in one state than in the other states. Dominated by the second force, the market maker does not accept any order.

On the contrary, the first factor is much stronger for the market maker with the exponential prior. The first, the second, and the third states receive very small probability weights. Therefore, the market maker views selling the first security as an opportunity to make a riskless profit of 0.18 dollars per share. By a similar line of reasoning, the market maker has a strong disincentive against accepting the fifth order. The result is in accordance with this intuition. Figure 3 shows that the market maker with the exponential prior accepts only the first, second, and third orders.

6 The Strength of the KPM: Empirical Discussion

Based on a solid understanding of a pari-mutuel auctioneer from the perspective of market microstructure theory, we discuss why a market making firm may want to organize a derivative market based on the KPM.
6.1 Why Automate?

The KPM is an automated market maker. The main strength of an automated market maker relative to its human counterpart derives from its ability to update quotes for dozens of related securities almost instantaneously. This ability reduces adverse selection cost, thereby allowing market makers to provide more competitive quotes to customers.

In today’s increasingly electronic and automated trading environment, the market maker’s ability to quickly update his/her quotes is increasingly important. The various speeds with which market participants react to the arrival of new information represent a source of informational asymmetry (Foucault et al., 2003; Litzenberger, 2012). In particular, liquidity suppliers who are slow to react to new information can leave their stale quotes vulnerable to being adversely picked off by high-frequency traders (Hendershott and Riordan, 2013). The competition to respond to new information faster than anyone else has become so intense that trading firms want to place their computers the building where the exchange’s matching machine is: The time it takes for the light to travel from their computers to the matching machine matters (Litzenberger, 2012). Given this extraordinarily high-frequency trading environment, the automation of the quote-updating process is important for the liquidity supplier to survive.

This adverse selection cost becomes particularly important for a market maker involved in multiple related markets: Quotes need to be consistent with one another to ensure that there is no arbitrage opportunity. With more information to process, the comparative advantage of the automated market maker over the human counterpart can only become more significant (Gerig and Michayluk, 2013).

The KPM is an automated algorithm through which the liquidity supplier can quickly price multiple contingent claims while taking into account a variety of factors. The resulting prices reflect the market maker’s risk aversion and ambiguity aversion while ensuring that there is no arbitrage opportunity.

6.2 Other Well-Known Strengths of a Pari-mutuel Auction

First, the ISE is interested in the PDCA mainly because pari-mutuel markets can effectively mitigate counterparty risk (Burne, 2013). The pari-mutuel auctioneer can be thought of as the central clearing counterparty (CCP). In particular, the pari-mutuel auctioneer is the common CCP operating in multiple contingent claims markets. The fact that one auctioneer handles multiple markets
allows the pari-mutuel market to better mitigate counterparty risk.

Second, the auction performs better than other trading platforms, particularly in a low-liquidity environment. The auction aggregates liquidity dispersed in multiple individual markets into the common pool. Lastly, having the common market maker in multiple markets improves price efficiency (Lange and Economide, 2005). Please see the Appendix for further details.

6.3 Potential Areas of Application

The KPM is expected to be useful for options markets in which delta hedging the market maker’s inventory is not feasible. The KPM solves an optimization that is robust to worst-case scenarios. In particular, Corollary 1 shows that the KPM can become almost completely pari-mutuel when the value of $\Omega$ is very large. If the market is completely pari-mutuel, the market maker does not lose any money regardless of what happens at maturity. Therefore, the inability to delta hedge the inventory becomes less critical.

There are two specific options markets for which delta hedging may be particularly infeasible. The first example is options for which the underlying asset is not tradable (e.g., the market for economic derivatives written on U.S. non-farm payrolls) (Baron and Lange, 2007). The second example is options with extremely short time to maturity because the delta fluctuates too much (Baron and Lange, 2007).

7 Conclusion

In this paper, we first show that the market-clearing strategy of the Convex Pari-mutuel Call Auction Mechanism (CCPAM) is asymptotically equivalent to that of the market maker with extreme ambiguity aversion for the future contingent event. Because the CPCAM is closely related to other notable pari-mutuel auctions in the literature, we regard this conclusion as a basis for arguing that pari-mutuel auctions are closely related to ambiguity aversion.

With this understanding, we design a new market for trading contingent claims, the Knightian Pari-mutuel Mechanism (KPM). The main optimization problem of the KPM is what the market maker should solve if he/she adheres to the theory of decision making under ambiguity aversion. The algorithm clears the market while controlling for the market maker’s level of risk and ambiguity.

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8 Duffie and Zhu (2011) show that counterparty risk can be better managed if the same CCP is involved in more than one market.

9 Baron and Lange (2007) also argue that the PDCA is suitable for markets where delta hedging is difficult.
aversion. We present a polynomial-time algorithm to solve the optimization problem.

Our paper may contribute to facilitating the adoption of a pari-mutuel mechanism in the trading community. As Robert Shiller once noted, a pari-mutuel mechanism can be particularly useful in launching a wide variety of innovative derivatives markets, thereby enabling investors to hedge a new class of fundamental risks (Baron and Lange, 2007).

8 Appendix

8.1 Illustration of the Theory of Decision Making Under Uncertainty

Suppose that there is an urn that contains red, blue and green balls, of which there are 90 in total. While there are 30 red balls in the urn, the exact number of either blue balls or green balls is unknown to the DM. Suppose that five lotteries are available. Lottery R pays $1 if and only if the DM draws a red ball from the urn. Lotteries B and G pay $1 if and only if he/she draws a blue ball and a green ball, respectively. Similarly, lottery RB pays $1 if and only if either a red ball or a blue ball is drawn. Lottery BG pays $1 if and only if either a blue ball or a green ball is drawn. Empirical studies show that most people prefer lottery R to either lottery B or G. Moreover, most people prefer lottery BG to RB. It is well known that this empirical result contradicts Savage’s theory of utility maximization with subjective probability (Savage, 1954).

Let us reformulate the DM’s problem in the language of Theorem 1. The set of all possible states $S$ is \{red, blue, green\}. The set of consequences $X$ is \{0, 1\}, expressed in dollars. The DM is interested in five different acts: $f_R$, $f_B$, $f_G$, $f_{RB}$, and $f_{BG}$. The act $f_R: S \rightarrow X$ is a mapping such that $f_R(\text{red}) = 1$, $f_R(\text{blue}) = 0$ and $f_R(\text{green}) = 0$. We define the other four acts similarly.

Without knowing the exact number of either blue or green balls, the DM cannot attach a single probability distribution to $S$. Suppose that the DM’s set of candidates is $\Psi = \{(1/3, x, 2/3 - x) \in \mathbb{R}^3 | 0.1 \leq x \leq 0.4\}$, where $1/3, x,$ and $2/3 - x$ are the chances of drawing red, blue and green balls, respectively.

Let $u: X \rightarrow \mathbb{R}$ denote the DM’s utility function. Equations (29a) and (29b) should hold for the DM to prefer lottery R to the other two lotteries.

\[
\min_{0.1 \leq x \leq 0.4} \left[ \frac{1}{3}u(1) + \frac{2}{3}u(0) \right] \geq \min_{0.1 \leq x \leq 0.4} \left[ x \cdot u(1) + (1 - x) \cdot u(0) \right] \tag{29a}
\]
In addition, equation (30) must hold for the DM to prefer lottery BG to RB.

\[
\min_{0.1 \leq x \leq 0.4} \left[ \frac{2}{3} u(1) + \frac{1}{3} u(0) \right] \geq \min_{0.1 \leq x \leq 0.4} \left[ \left( \frac{1}{3} + x \right) \cdot u(1) + \left( \frac{2}{3} - x \right) \cdot u(0) \right]
\]

Equations (29a), (29b) and (30) hold as long as the utility function is non-decreasing. Theorem 1 successfully reconciles the theory with empirical observations. ■

8.2 The Proof of Theorem 2

If we assume that \( \delta_i = \delta \) for \( \forall i \), (3) is a barrier problem to (31).

\[
\max_{x,M} b^T x - M
\]

such that

(A) \( \sum_{j=1}^J A_{i,j} x_j \leq M \) for each \( i \in \{1,2,\ldots,N\} \)

(B) \( 0 \leq x \leq Q \)

Assume that the feasible set for (31) is not empty. Sending the value of the parameter \( \delta \) to zero is equivalent to reducing the duality gap along the primal-dual central path in the interior point method. Thus, as \( \delta \) approaches zero, \( x(\delta) \) should converge to an optimal solution to (31) (Luenberger and Ye, 2008), which we denote \( x^* \).

(31) is equivalent to (32).

\[
\max_x \left[ b^T x - \max_i \sum_{j=1}^J A_{i,j} x_j \right]
\]

such that

(A') \( 0 \leq x \leq Q \)

(32) is equivalent to (33).

\[
\max_x \min_i u \left[ b^T x - \sum_{j=1}^J A_{i,j} x_j \right]
\]

such that

(A') \( 0 \leq x \leq Q \)
is equivalent to (34).

\[
\max \min_x \sum_{i=1}^{N} p_i u \left[ b^T x - \sum_{j=1}^{J} A_{i,j} x_j \right]
\]

such that

(A') \quad 0 \leq x \leq Q \\
(B') \quad p \geq 0, \sum_{i=1}^{N} p_i = 1 \\

8.3 The Proof of Corollary 1

With the assumption that \( w_i = 0 \) for \( \forall i \), (8) converges to (35) as the value of \( \Omega \) increases to infinity.

\[
\max_{\xi, x} \min_p \sum_{i=1}^{N} p_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]
\]

such that

(A) \quad p \geq 0, \sum_{i=1}^{N} p_i = 1 \\
(B) \quad \xi \geq 0 \\
(C) \quad \sum_{i=1}^{N} \xi_i = 1 \\
(E1) \quad \forall j \in \{1, 2, ..., J\}, \quad x_j = 0 \quad \text{if} \quad (A^T \xi)_j > B_j b_j \\
(E2) \quad \forall j \in \{1, 2, ..., J\}, \quad x_j \in [0, Q_j] \quad \text{if} \quad (A^T \xi)_j = B_j b_j \\
(E3) \quad \forall j \in \{1, 2, ..., J\}, \quad x_j = Q_j \quad \text{if} \quad (A^T \xi)_j < B_j b_j \\

Let \( \xi^* \) and \( x^* \) denote the values of \( \xi \) and \( x \) that optimize (35), respectively. Define \( i^* \) as (36).

\( i^* \) may not be uniquely defined. In that case, we simply choose any of multiple \( i \)s that minimize

\[
- \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi^*)_j - A_{ij} \right) \right].
\]

\[
i^* = \arg \min_i - \exp \left[ -\alpha \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \right]
\]

Consider the inner minimization problem. To minimize the objective function, we need \( p_{i^*} = 1 \) and \( p_i = 0 \) for \( \forall i \neq i^* \). If we substitute \( p_{i^*} = 1 \) and \( p_i = 0 \) for \( \forall i \neq i^* \) into the objective function of (35), we obtain (37).

\[
- \exp \left[ -\alpha \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{i^*j} \right) \right] = \min_i - \exp \left[ -\alpha \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \right]
\]

If we substitute \( x = 0 \) into the objective function of (35), we obtain 1. Therefore, the optimal
value of the objective function of \( \text{(35)} \), which is \( \text{(37)} \), should be at least as large as 1.

\[
\min_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \right] \geq 1 \tag{38}
\]

\( \text{(38)} \) is equivalent to \( \text{(39)} \).

\[
\min_i \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \geq 0 \tag{39}
\]

\( \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \) is the monetary payoff for the market maker if the \( i \)th outcome is realized. Therefore, \( \min_i \sum_{j=1}^{J} x_j^* \left( (A^T \xi^*)_j - A_{ij} \right) \) is the worst possible monetary payoff that the market maker can ever receive. Inequality \( \text{(39)} \) shows that the market maker never loses money even in that worst-case scenario. The market is completely pari-mutuel.

8.4 The Proof of Lemma 1

8.4.1 The Dual Problem of the Inner Minimization Problem

\( \text{(40)} \) is the inner minimization problem isolated from \( \text{(8)} \). The optimal value of this inner optimization problem is an implicit function of \( \xi \) and \( x \). Because the objective function is linear in \( p \) and \( \Psi \) is a convex set, this problem is a convex optimization problem.

\[
\min_{p \in \Psi} \sum_{i=1}^{N} p_i z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]
\]

such that

\[
\begin{align*}
(A) & \quad \Psi = \left\{ p \in \mathbb{R}^{N \times 1} | p \geq 0, \sum_{i=1}^{N} p_i = 1, \sum_{i=1}^{N} p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega \right\}
\end{align*}
\]

\( x \) and \( \xi \) should be treated like constants when solving \( \text{(40)} \). To make notations simpler, we introduce new constants.

\[
d_i = z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right] \quad \text{for} \quad \forall i \in \{1, \ldots, N\} \tag{41}
\]

\[
d = \begin{pmatrix} d_1 & d_2 & \ldots & d_N \end{pmatrix}^T \tag{42}
\]
Then, minimization problem (40) reduces to (43).

$$\min_{\mathbf{p}} \mathbf{d}^T \mathbf{p}$$

such that

(A) $\mathbf{p} \geq 0$

(B) $\sum_{i=1}^{N} p_i = 1$

(C) $\sum_{i=1}^{N} p_i \ln \left( \frac{p_i}{q_i} \right) \leq \Omega$

The domain of the minimization problem is $D$ as defined in (44).

$$D = \{ \mathbf{p} \in \mathbb{R}^N | \mathbf{p} > 0 \}$$ (44)

The Lagrangian associated with problem (43) is (45). $\lambda_1, \lambda_2, ..., \lambda_N, \mu, \nu$ are Lagrange multipliers.

$$L(\mathbf{p}, \lambda, \mu, \nu) = \mathbf{d}^T \mathbf{p} + \sum_{i=1}^{N} \lambda_i (-p_i) + \mu \left( \sum_{i=1}^{N} p_i \ln \left( \frac{p_i}{q_i} \right) - \Omega \right) + \nu \left( \sum_{i=1}^{N} p_i - 1 \right)$$ (45)

The Lagrange dual function associated with problem (43) is (46).

$$g(\lambda, \mu, \nu) = \inf_{\mathbf{p} > 0} L(\mathbf{p}, \lambda, \mu, \nu)$$ (46)

$L(\mathbf{p}, \lambda, \mu, \nu)$ is a convex function of each $p_i$. The first order condition is

$$\frac{\partial L(\mathbf{p}, \lambda, \mu, \nu)}{\partial p_i} = (d_i - \lambda_i + \nu) + \mu \left\{ 1 + \ln \left( \frac{p_i}{q_i} \right) \right\} = 0$$ (47)

$$1 + \ln \left( \frac{p_i}{q_i} \right) = -\frac{d_i - \lambda_i + \nu}{\mu}$$ (48)

$$p_i = q_i \exp \left( -1 + \frac{\lambda_i - d_i - \nu}{\mu} \right) > 0$$ (49)
Because the function is convex, (49) is the global minimizer. We substitute (49) into (46).

\[
g(\lambda, \mu, \nu) = \sum_{i=1}^{N} (d_i - \lambda_i) q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} + \sum_{i=1}^{N} (-\mu + \lambda_i - d_i - \nu) q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} - \mu \Omega + \nu \sum_{i=1}^{N} q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} - \nu
\]

\[
= \sum_{i=1}^{N} (-\mu) q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} - \mu \Omega - \nu
\]

\[
= -\mu \sum_{i=1}^{N} q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} - \mu \Omega - \nu
\]

The Lagrange dual problem associated with the inner minimization problem is (51).

\[
\max_{\lambda, \mu, \nu} -\mu \sum_{i=1}^{N} q_i e^{-\frac{\lambda_i - d_i - \nu}{\mu}} - \mu \Omega - \nu
\]

s.t.
\[
\lambda_1, ..., \lambda_N \geq 0
\]
\[
\mu \geq 0
\]

Each \(q_i\) is assumed to be positive. \(\mu\) is implicitly assumed to be nonzero because it appears as a denominator in (51). Therefore, the objective function of (51) decreases with increasing \(\lambda_i\). The optimal value of each \(\lambda_i\) should thus be zero. (51) reduces to (52).

\[
\max_{\mu, \nu} -\mu \sum_{i=1}^{N} q_i e^{-\frac{d_i + \nu}{\mu}} - \mu \Omega - \nu
\]

s.t.
\[
\mu \geq 0
\]

8.4.2 Applying Strong Duality to the Inner Minimization Problem

We use the trick presented in Palomar (2009) to address the max-min problem. We replace the inner minimization problem with the dual maximization problem. This substitution is valid if and only if strong duality holds. Then, the overall structure of the problem is max-max instead of max-min. The double max structure can collapse to a more conventional problem with only one maximization operator.

We use the criteria in Boyd and Vandenberghe (2004) to determine whether strong duality holds. If the primal problem is convex and Slater’s condition holds, strong duality holds. Slater’s condition holds if there exists a strictly feasible \(p \in \text{relintD}\).
Slater’s condition holds in the context of our problem as long as $\Omega$ is a strictly positive number (i.e., $p$ such that $p_i = q_i$ for $\forall i$ is a strictly feasible solution). Therefore, strong duality holds for our inner minimization problem as long as $\Omega$ is strictly positive.

Using strong duality, our max-min problem can be transformed into (53).

\[
\max_{\xi, x, d} \mu \sum_{i=1}^{N} q_i e^{-1 - \frac{d_i + \nu}{\mu}} - \mu \Omega - \nu
\]

such that

(A) $d_i = z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]$ for $\forall i \in \{1, ..., N\}$

(B) $\mu \geq 0$

(C) $\xi \geq 0$

(D) $\sum_{i=1}^{N} \xi_i = 1$

(E) $(x, \xi) \in F$

Two maximization operators can be collapsed into a single operator.

\[
\max_{\xi, x, \mu, d} -\mu \sum_{i=1}^{N} q_i e^{-1 - \frac{d_i + \nu}{\mu}} - \mu \Omega - \nu
\]

such that

(A) $d_i = z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]$ for $\forall i \in \{1, ..., N\}$

(B) $\mu \geq 0$

(C) $\xi \geq 0$

(D) $\sum_{i=1}^{N} \xi_i = 1$

(E) $(x, \xi) \in F$

**8.4.3 Further Simplification through Algebraic Manipulation**

Because the objective function of (54) is a strictly concave function of $\nu$, we can find the global optimizing value of $\nu$ from the first-order condition.

\[
-\mu \sum_{i=1}^{N} \left( -\frac{1}{\mu} \right) q_i e^{-1 - \frac{d_i + \nu}{\mu}} - 1 = 0
\]

\[
\sum_{i=1}^{N} q_i e^{-1 - \frac{d_i + \nu}{\mu}} = 1
\]
\[ e^{-\frac{\mu}{\nu}} e^{-1} \sum_{i=1}^{N} q_ie^{-\frac{d_i}{\mu}} = 1 \]

\[-\frac{\nu}{\mu} - 1 + \ln \left( \sum_{i=1}^{N} q_ie^{-\frac{d_i}{\mu}} \right) = 0 \]

\[ \nu^* = -\mu + \mu \ln \left( \sum_{i=1}^{N} q_ie^{-\frac{d_i}{\mu}} \right) \] (55)

We substitute (55) into the objective function of (54).

\[-\mu \sum_{i=1}^{N} q_i e^{-1-\frac{d_i+\nu^*}{\mu}} - \mu \Omega - \nu^* \]

\[= -\mu \frac{e}{\sum_{i=1}^{N} q_i e^{-\frac{d_i}{\mu}}} \sum_{i=1}^{N} q_i e^{-1-\frac{d_i}{\mu}} - \mu \Omega + \mu \ln \left( \sum_{i=1}^{N} q_i e^{-\frac{d_i}{\mu}} \right) \]

\[= -\mu - \mu \Omega + \mu \ln \left( \sum_{i=1}^{N} q_i e^{-\frac{d_i}{\mu}} \right) \]

\[= -\mu \Omega - \mu \ln \left( \sum_{i=1}^{N} q_i e^{-\frac{d_i}{\mu}} \right) \] (56)

Substituting (56) into (54) further simplifies the problem.

\[\min_{\xi, x, \mu, d} \mu \Omega + \mu \ln \left( \sum_{i=1}^{N} q_i e^{-\frac{d_i}{\mu}} \right)\]

such that

(A) \( d_i = z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi) - A_{ij} \right) \right] \) for \( \forall i \in \{1, \ldots, N\} \)

(B) \( \mu \geq 0 \)

(C) \( \xi \geq 0 \)

(D) \( \sum_{i=1}^{N} \xi_i = 1 \)

(E) \( (x, \xi) \in F \)

We define new constants.

\[ \theta_i = e^{\Omega} q_i > 0 \] for \( \forall i \) (58)
Then, the objective function of (57) can be more succinctly represented as a function of $\mu$

$$\ell(\mu) = \mu \Omega + \mu \ln \left( \sum_{i=1}^{N} q_i e^{\frac{-d_i}{\mu}} \right) = \mu \ln \left( \sum_{i=1}^{N} q_i e^{\Omega - \frac{d_i}{\mu}} \right)$$

$$= \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)$$

(59)

The optimization problem then becomes:

$$\min_{\xi, x, \mu, d} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)$$

such that

(A) $d_i = z_i \exp \left[ -\alpha \sum_{j=1}^{J} x_j \left( (A^T \xi)_j - A_{ij} \right) \right]$ for $\forall i \in \{1, .., N\}$

(B) $\mu \geq 0$

(C) $\xi \geq 0$

(D) $\sum_{i=1}^{N} \xi_i = 1$

(E) $(x, \xi) \in F$

(60)

8.4.4 Linearization of Constraint (A)

Note that constraint (A) in (60) involves the quadratic terms $\sum_{j=1}^{J} x_j (A^T \xi)_j$. Therefore, in this section, we suggest a way to linearize this constraint.

Intuitively, $x_j (A^T \xi)_j$ is simply the market maker’s revenue from the $j$th order. Let $R_j$ denote the market maker’s revenue from the $j$th order. $x_j$ is the number of shares of the option traded. $(A^T \xi)_j$ is the market-clearing price of the $j$th order.

To begin the transformation, we first consider the feasible set of $x$ and $\xi$. We say that the pair $x$ and $\xi$ are feasible if and only if the pair satisfies (61).

(E1) $\forall j \in \{1, 2, .., J\}, \ x_j = 0 \ \text{if} \ \ (A^T \xi)_j > B_j b_j$

(E2) $\forall j \in \{1, 2, .., J\}, \ x_j \in [0, Q_j] \ \text{if} \ \ (A^T \xi)_j = B_j b_j$

(E3) $\forall j \in \{1, 2, .., J\}, \ x_j = Q_j \ \text{if} \ \ (A^T \xi)_j < B_j b_j$

(61)

We restrict attention to the $j$th order. Figure 4 shows the feasible set of pairs $x_j$ and $(A^T \xi)_j$. The figure is just a graphic illustration of (61).

Figure 5 is a three-dimensional graph. The graph shows the market maker’s revenue as a

\text{I can simply ignore cases for which $Q_j = 0$. I can also simply remove the $j$th order from my optimization problem.}
Figure 4: The feasible set of $x_j$ and $(A^T \xi)_j$.

function of the quantity filled ($x_j$) and the market-clearing price $(A^T \xi)_j$.

For illustrative purposes, Figure 6 dissects Figure 5 into three distinct regions. Region (1) corresponds to constraint (E1) in (61). Region (2) and (3) correspond to constraints (E2) and (E3), respectively.

If we assume that the market-clearing price $(A^T \xi)_j$ and the quantity traded $x_j$ can take any real values, the market maker’s revenue $R_j$ becomes a nonlinear term $(A^T \xi)_j x_j$. However, if we restrict attention to the feasible set in Figure 4, either the market-clearing price or the quantity filled is held constant in each of the three regions. The market maker’s revenue $R_j$ is a piece-wise linear function.

$$
R_j = \begin{cases} 
0 & \text{if } (A^T \xi)_j > B_j b_j \\
B_j b_j x_j & \text{if } (A^T \xi)_j = B_j b_j \\
Q_j (A^T \xi)_j & \text{if } (A^T \xi)_j < B_j b_j 
\end{cases} \tag{62}
$$

(62) is equivalent to (63) as long as $x_j$ and $(A^T \xi)_j$ belong to the feasible set that Figure 4 represents. $[(A^T \xi)_j - B_j b_j]^+$ is a short-hand notation for $\max \{0, (A^T \xi)_j - B_j b_j\}$.

$$
R_j = Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j) - Q_j [(A^T \xi)_j - B_j b_j]^+ \tag{63}
$$

For example, consider region (1) where $(A^T \xi)_j > B_j b_j$ and $x_j = 0$. Then, (63) reduces to (64).
Figure 5: The market maker’s revenue from the $j$th order as a function of the quantity filled $x_j$ and the market-clearing price of the $j$th order $(\mathbf{\bar{A}}^T \xi)_j$

Figure 6: The market maker’s revenue from the $j$th order as a function of the quantity filled $x_j$ and the market-clearing price of the $j$th order $(\mathbf{A}^T \xi)_j$
Note that (64) agrees with (62).

\[ R_j = Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j) - Q_j (A^T \xi)_j - B_j b_j \]

\[ = Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j) - Q_j (A^T \xi)_j - B_j b_j \]

\[ = B_j b_j x_j = B_j b_j \cdot 0 = 0 \quad (64) \]

We can simplify (63):

\[ R_j = Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j) - Q_j (A^T \xi)_j - B_j b_j \]

\[ = \min \left[ Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j), Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j) - Q_j \left( (A^T \xi)_j - B_j b_j \right) \right] \]

\[ = \min \left[ Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j), B_j b_j x_j \right] \quad (65) \]

Substitution of (65) into (60) yields (66). Constraint (E) in (66) ensures that the pair \((x, \xi)\) is within the feasible set shown in Figure 1. Replacement of the quadratic term with the piece-wise linear term is valid due to this restriction.

\[
\min_{\xi, x, \mu, d} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)
\]

such that

(A) \[-d_i = -z_i e^{\alpha \sum_{j=1}^{J} [x_j A_{ij} - \min\{Q_j (A^T \xi)_j + B_j b_j (x_j - Q_j), B_j b_j x_j\}]} \quad \text{for } \forall i \]

(B) \[\mu \geq 0\]

(C) \[\xi \geq 0\]

(D) \[\sum_{i=1}^{N} \xi_i = 1\]

(E) \[(x, \xi) \in \mathbf{F} \]

Because \(z_i\) is negative and \(\alpha\) is positive, constraint (A) of (66) is equivalent to (67).

\[-d_i = \max \left[ -z_i e^{\alpha \sum_{j=1}^{J} [x_j A_{ij} - (A^T \xi)_j - B_j b_j (x_j - Q_j)]}, -z_i e^{\alpha \sum_{j=1}^{J} [x_j A_{ij} - B_j b_j x_j]} \right] \quad (67) \]

To minimize the objective function of (66), \(-d_i\) must be minimized. Hence, the optimization
problem can be further reduced to (68). Note that constraints (B) to (F) are all linear.

$$\min_{\xi,x,\mu,\zeta} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)$$

such that

(A) $-d_i = -z_i e^{\zeta_i}$ for $\forall i$

(B) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - Q_j (A^T \xi)_j - B_j b_j (x_j - Q_j)]$ for $\forall i$

(C) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - B_j b_j x_j]$ for $\forall i$ (68)

(F) $\mu \geq 0$

(G) $\xi \geq 0$

(H) $\sum_{i=1}^{N} \xi_i = 1$

(I) $(x, \xi) \in F$

$\zeta \in \mathbb{R}^{N \times 1}$ is a dummy variable.

$\zeta = \begin{bmatrix} \zeta_1 & \cdots & \zeta_N \end{bmatrix}$

8.5 The Proof of Lemma 2

In optimization problem (69), to minimize the objective, $(-d_i)$ needs to be minimized. Define a new vector $\omega \in \mathbb{R}^{N \times 1}$ such that $\omega = [\omega_1, ..., \omega_N]$ Hence, (69) is equivalent to (69).

$$\min_{\xi,x,\mu,\zeta,\omega} \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{-\frac{d_i}{\mu}} \right)$$

such that

(A) $\omega_i \geq -z_i e^{\xi_i}$ for $\forall i$

(B) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - Q_j (A^T \xi)_j - B_j b_j (x_j - Q_j)]$, $\forall i$ (69)

(C) $\zeta_i \geq \alpha \sum_{j=1}^{J} [x_j A_{ij} - B_j b_j x_j]$, $\forall i$

(D) $\mu \geq 0$

(E) $(x, \xi) \in C$

My goal is to show that (69) is a convex optimization problem. A necessary preliminary step is to show that the set of pairs of $\omega_i$ and $\zeta_i$ that satisfy constraint (A) in (69) constitute a convex set.

Lemma 3 The set of pairs of $\omega_i$ and $\zeta_i$ that satisfy constraint (A) in (69) form a convex set.

Proof. Define a new function.

$$F(\omega_i, \zeta_i) = -\omega_i - z_i e^{\xi_i}$$ (70)
To prove the lemma, it suffices to show that function $F$ is convex. The Hessian is:

$$\nabla^2 F = \begin{bmatrix} \frac{\partial^2 F}{\partial \omega_i^2} & \frac{\partial^2 F}{\partial \omega_i \partial \omega_j} \\ \frac{\partial^2 F}{\partial \omega_j \partial \omega_i} & \frac{\partial^2 F}{\partial \omega_j^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -z_i e^{\omega_i} \end{bmatrix} \quad (71)$$

Because $z_i$ is negative, $\nabla^2 F$ is positive semidefinite. Therefore, $F$ is convex. □

The next step is to show that the objective function $\mu \ln \left( \sum_{i=1}^{N} \theta_i e^{\omega_i} \right)$ is convex. Define a new function.

$$G(\omega_1, \omega_2, ..., \omega_N, \mu) = \mu \ln \left( \sum_{i=1}^{N} \theta_i e^{\omega_i} \right) \quad (72)$$

Before proceeding with the proof, we present a very useful result from Boyd and Vandenberghe (2004).\(^{11}\)

**Lemma 4** Let the function $f(y)$ be defined as (73) where $a_k, y \in \mathbb{R}^n, b_k \in \mathbb{R}$.

$$f(y) = \ln \left( \sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \quad (73)$$

$f(y)$ is a convex function.

Therefore, the Hessian of $f(y)$ must be positive semidefinite.

**Lemma 5** Function $G$, which is defined as (72), is convex.

**Proof.** Note that function $G$ can be expressed as (74). With $\mu$ fixed, the structure of $G$ as a function of $\omega_1, \omega_2, ..., \omega_N$ is exactly analogous to (73).

$$G(\omega_1, \omega_2, ..., \omega_N, \mu) = \mu \ln \left( \sum_{i=1}^{N} \frac{1}{\mu} e^{\frac{1}{\mu} \omega_i + \ln \theta_i} \right) \quad (74)$$

With $\mu$ fixed, $G$ is a convex function of $\omega_1, \omega_2, ..., \omega_N$. Hence all the principal minors of the matrix in (73) are nonnegative.

$$\begin{bmatrix} \frac{\partial^2 G}{\partial \omega_1^2} & \frac{\partial^2 G}{\partial \omega_1 \partial \omega_2} & \cdots & \frac{\partial^2 G}{\partial \omega_1 \partial \omega_N} \\ \frac{\partial^2 G}{\partial \omega_2 \partial \omega_1} & \frac{\partial^2 G}{\partial \omega_2^2} & \cdots & \frac{\partial^2 G}{\partial \omega_2 \partial \omega_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 G}{\partial \omega_N \partial \omega_1} & \frac{\partial^2 G}{\partial \omega_N \partial \omega_2} & \cdots & \frac{\partial^2 G}{\partial \omega_N^2} \end{bmatrix} \quad (75)$$

\(^{11}\)See equation (4.44) on page 162
To prove the lemma, it suffices to show that the Hessian $\nabla^2 G$ is positive semidefinite. We need to show that all the principal minors of $\nabla^2 G$ in (76) are nonnegative.

$$\nabla^2 G = \begin{bmatrix} \omega_1 & \omega_2 & \mu \\ \omega_1 & \ddots & \ddots \\ \omega_N & \ddots & \ddots \\ \mu & \ddots & \ddots \end{bmatrix}$$

(76)

However, because we already know that all the principal minors of the matrix in (75) are nonnegative, it only remains to show that $\frac{\partial^2 G}{\partial \mu^2} \geq 0$ and $\det \nabla^2 G \geq 0$.

**First, we show that** $\frac{\partial^2 G}{\partial \mu^2} \geq 0$

Find the first derivative of $G$.

$$\frac{\partial G}{\partial \mu} = \ln \left( \sum_{i=1}^{N} \theta_i e^{\omega_i \mu} \right) + \mu \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}$$

$$= \ln \left( \sum_{i=1}^{N} \theta_i e^{\omega_i \mu} \right) - \frac{1}{\mu} \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}$$

(77)

Find the second derivative of $G$.

$$\frac{\partial^2 G}{\partial \mu^2} = \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}} + \frac{1}{\mu^2} \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}$$

$$- \frac{1}{\mu} \left[ \frac{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}} \right]^2 \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}$$

(78)

$$\frac{\partial^2 G}{\partial \mu^2} = \frac{1}{\mu} \left[ \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}} \right]^2 - \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\omega_i \mu}}{\mu^3} \left[ \frac{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}}{\sum_{i=1}^{N} \theta_i e^{\omega_i \mu}} \right]^2$$

(79)

The denominator of (79) is positive. Hence, it only remains to show that the numerator is nonneg-
Second, we show that

\[ \partial \frac{\partial^2 G}{\partial \omega_k \partial \mu} \] is always nonnegative.

\[
\sum_{i=1}^{N} \left( \sqrt{\theta_i e^{\frac{\omega_i}{\mu}}} \right)^2 \sum_{i=1}^{N} \left( \sqrt{\theta_i \omega_i^2 e^{\frac{\omega_i}{\mu}}} \right)^2 \geq \left[ \sum_{i=1}^{N} \sqrt{\theta_i e^{\frac{\omega_i}{\mu}}} \cdot \sqrt{\theta_i \omega_i^2 e^{\frac{\omega_i}{\mu}}} \right]^2
\]

(80)

\[
\therefore \partial^2 G \partial_{\mu^2} \] is always nonnegative.

**Second, we show that** \( \det \nabla^2 G \geq 0 \).

From [77], we calculate \( \partial^2 G / \partial \omega_k \partial \mu \) where \( k \in \{1, 2, \ldots, N\} \).

\[
\frac{\partial^2 G}{\partial \omega_k \partial \mu} = \frac{\partial}{\partial \omega_k} \left[ \ln \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right) - \frac{1}{\mu} \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right] - \frac{\partial}{\partial \mu} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right)
\]

\[
= \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right) - \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right)^2
\]

\[
= \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right) - \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right)^2
\]

\[
= \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu^2} \left[ \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right]^2 - \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu^2} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right)^2
\]

\[
= \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu^2} \left[ \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right]^2 - \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\mu^2} \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right)^2
\]

(81)

Similarly,

\[
\frac{\partial G}{\partial \omega_k} = \frac{\partial}{\partial \omega_k} \ln \left( \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right) = \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} = \frac{\theta_k e^{\frac{\omega_k}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}}
\]
Provided that \( j, k \in \{1, \ldots, N\} \) and \( j \neq k \),

\[
\frac{\partial^2 G}{\partial \omega_j \partial \omega_k} = \frac{\partial}{\partial \omega_j} \frac{\partial}{\partial \omega_k} \left[ \theta e^\mu \left( \sum_{i=1}^N \theta_i e^\mu \right) \right] = \sum_{i=1}^N \theta_i e^\mu \cdot \theta e^\mu - \theta_k e^\mu \cdot \theta e^\mu \mu \left( \sum_{i=1}^N \theta_i e^\mu \right)^2 = -\frac{1}{\mu} \theta_k \theta_j \frac{\sum_{i=1}^N \theta_i e^\mu}{\mu} \cdot \frac{\sum_{i=1}^N \theta_i e^\mu}{\mu} e^\mu \cdot e^\mu
\]

Thus, \( \nabla^2 G_{,k} \) in (82) shows the \( k \)th column of \( \nabla^2 G \). \( \nabla^2 G_{,1} \) denotes the first column, \( \nabla^2 G_{,2} \) denotes the second column, and so forth.
Consider the following linear combination of the columns.

\[
\sum_{k=1}^{N} \frac{\omega_k}{\mu} \nabla^2 G_{\cdot,k} = \sum_{k=1}^{N} \frac{\omega_k}{\mu} \begin{bmatrix}
\frac{\partial^2 G}{\partial \omega_1 \omega_k} \\
\vdots \\
\frac{\partial^2 G}{\partial \omega_N \omega_k} \\
\frac{\partial^2 G}{\partial \mu \omega_k}
\end{bmatrix}
\begin{bmatrix}
\theta_{1} e^{\omega_1 \mu} \\
\vdots \\
\theta_{N} e^{\omega_N \mu} \\
-\frac{1}{\mu} \theta_{1} e^{\omega_1 \mu} \sum_{i=1}^{N} \theta_{i} e^{\omega_i \mu} - \theta_{1} e^{\omega_1 \mu} \sum_{k=1}^{N} \theta_{k} \omega_k e^{\omega_k \mu}
\end{bmatrix}
\begin{bmatrix}
\theta_{1} e^{\omega_1 \mu} \sum_{i=1}^{N} \theta_{i} e^{\omega_i \mu} - \theta_{1} e^{\omega_1 \mu} \sum_{k=1}^{N} \theta_{k} \omega_k e^{\omega_k \mu} \\
\vdots \\
\theta_{N} e^{\omega_N \mu} \sum_{i=1}^{N} \theta_{i} e^{\omega_i \mu} - \theta_{k} e^{\omega_k \mu} \sum_{k=1}^{N} \theta_{k} \omega_k e^{\omega_k \mu} \\
-\frac{1}{\mu} \theta_{1} \omega_k e^{\omega_1 \mu} \sum_{i=1}^{N} \theta_{i} (\omega_i - \omega_k) e^{\omega_i \mu}
\end{bmatrix}
\begin{bmatrix}
\theta_{1} e^{\omega_1 \mu} \sum_{i=1}^{N} \theta_{i} e^{\omega_i \mu} - \theta_{1} e^{\omega_1 \mu} \sum_{k=1}^{N} \theta_{k} \omega_k e^{\omega_k \mu} \\
\vdots \\
\theta_{N} e^{\omega_N \mu} \sum_{i=1}^{N} \theta_{i} e^{\omega_i \mu} - \theta_{k} e^{\omega_k \mu} \sum_{k=1}^{N} \theta_{k} \omega_k e^{\omega_k \mu} \\
-\frac{1}{\mu} \theta_{1} \omega_k e^{\omega_1 \mu} \sum_{i=1}^{N} \theta_{i} (\omega_i - \omega_k) e^{\omega_i \mu}
\end{bmatrix}
\]

(83)
However, the last column of $\nabla^2 G$ is

$$\nabla^2 G_{N+1} = \begin{bmatrix}
\frac{\partial^2 G}{\partial \omega_1^2 \partial \mu} \\
\frac{\partial^2 G}{\partial \omega_2^2 \partial \mu} \\
\vdots \\
\frac{\partial^2 G}{\partial \omega_N^2 \partial \mu} \\
\frac{\partial^2 G}{\partial \mu^2}
\end{bmatrix}$$

$$= \frac{1}{\mu^2 \left[ \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \right]^2} \begin{bmatrix}
\theta_1 e^{\frac{\omega_1}{\mu}} \sum_{i=1}^{N} \theta_i (\omega_i - \omega_1) e^{\frac{\omega_i}{\mu}} \\
\theta_2 e^{\frac{\omega_2}{\mu}} \sum_{i=1}^{N} \theta_i (\omega_i - \omega_2) e^{\frac{\omega_i}{\mu}} \\
\vdots \\
\theta_N e^{\frac{\omega_N}{\mu}} \sum_{i=1}^{N} \theta_i (\omega_i - \omega_N) e^{\frac{\omega_i}{\mu}} \\
\frac{1}{\mu} \sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \omega_i^2 e^{\frac{\omega_i}{\mu}} - \frac{1}{\mu} \left[ \sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}} \right]^2
\end{bmatrix}$$

(84)

The combination of (83) and (84) yields (85).

$$\sum_{k=1}^{N} \frac{\omega_k}{\mu} \nabla^2 G_{.,k} + \nabla^2 G_{N+1} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix} $$

(85)

Therefore,

$$\det \nabla^2 G = \det \begin{bmatrix}
\nabla^2 G_{.,1} & \nabla^2 G_{.,2} & \ldots & \nabla^2 G_{.,N} & \nabla^2 G_{.,N+1}
\end{bmatrix}$$

$$= \det \begin{bmatrix}
\nabla^2 G_{.,1} & \nabla^2 G_{.,2} & \ldots & \nabla^2 G_{.,N} & \sum_{k=1}^{N} \frac{\omega_k}{\mu} \nabla^2 G_{.,k} + \nabla^2 G_{.,N+1}
\end{bmatrix}$$

$$= \det \begin{bmatrix}
\nabla^2 G_{.,1} & \nabla^2 G_{.,2} & \ldots & \nabla^2 G_{.,N} & 0
\end{bmatrix}$$

$$= 0$$

(86)

Because both $\frac{\partial G}{\partial \mu^2}$ and $\det \nabla^2 G$ are nonnegative, $G$ is a convex function. $\blacksquare$

(9) is a convex optimization problem because both the objective function and the feasible set are convex.
8.6 The Proof of Theorem 3

Note from our pseudo-code that the number of times we need to solve the problem (27) is \( \Pi^K_{k=1} n_k \)

\[
\Pi^K_{k=1} n_k \leq \Pi^K_{k=1} J = J^K
\]  

(87)

In addition, it is well known that, in principle, the interior-point method can solve any convex optimization in polynomial time of the problem dimension. Thus, (27) should also be solvable in polynomial time. Let \( \tau(\ell_1, \ell_2, ..., \ell_K) \) denote the time required to solve problem (27).

Let \( T \) denote the time required to execute the entire pseudo-code.

\[
T = \sum_{\ell_1=1}^{n_1} \sum_{\ell_2=1}^{n_2} ... \sum_{\ell_K=1}^{n_K} \tau(\ell_1, \ell_2, ... \ell_K) \leq J^K \max_{\ell_1, \ell_2, ..., \ell_K} \tau(\ell_1, \ell_2, ... \ell_K)
\]

\[
\max_{\ell_1, \ell_2, ..., \ell_K} \tau(\ell_1, \ell_2, ... \ell_K) \text{ is bounded above by a polynomial function of } J. \text{ Thus, } T \text{ is also bounded above by a polynomial of } J.
\]

8.7 Other Well-Known Strengths of the Pari-mutuel Auction

Please see Baron and Lange (2007) or Lange and Economide (2005) for a more thorough discussion. In this subsection, we briefly introduce some of the strengths of pari-mutuel auctions and our insights.

8.7.1 Liquidity Aggregation

The market maker can reduce his/her inventory holding cost by being involved in more than one market. This lower inventory holding cost allows the market maker to supply liquidity to each market at lower cost.

To illustrate, consider an exotic derivative market with the Consumer Price Index (CPI) as the underlying variable. Suppose that there are two types of options: a call option with the strike 0% and a put option with the same strike. For example, if the CPI is 1%, the call option pays $1, while the put option does not pay. Imagine that there is an overwhelming demand for both options.

First, consider the case in which two markets are fragmented. There is one dealer for each market. Overwhelming demand for each option forces the market maker to take a large short position. The inventory of each market maker becomes highly unbalanced, exposing him/her to
significant risk. This increased inventory cost leads to a larger bid-ask spread and reduced liquidity in each market (Stoll, 1978).

In contrast, consider having a common market maker serve both markets. Simultaneously taking large short positions in both the call option and put options is less risky than shorting only one option. As the underlying variable fluctuates, the price of the call and that of the put move in the opposite direction. Therefore, holding the call option can partly offset the risk of holding the put option and vice versa. A smaller inventory holding cost leads to a narrower bid-ask spread and enhanced liquidity in each market.

This effect is called "liquidity aggregation" because it is as if the common market maker is aggregating scarce liquidity from each market into the common pool (Lange and Economide, 2005; Baron and Lange, 2007).

The ability to aggregate liquidity is particularly important in introducing a new and innovative derivatives market (Shiller, 2008). One important reason is that there is a strong network externality effect when organizing a financial market (Stoll, 1992). People want to trade at a place where other people also tend to trade (Stoll, 1992; Shiller, 2008). Thus, it is difficult for the new market to gather a sufficient number of participants above a certain threshold to ensure smooth market operation (Shiller, 2008). In this respect, Robert Shiller notes that pari-mutuel auctions can serve as the springboard for new markets (Baron and Lange 2007). This approach can help new markets aggregate sufficient liquidity to compete with previously established markets (Baron and Lange, 2007).

8.7.2 Price Efficiency

Pari-mutuel mechanisms enhance price efficiency because information flows from one market to another through the common market maker (Baron and Lange, 2007). Prices of options with the same underlying asset or variable are closely related to one another. Hence, information in one market is relevant to the pricing of other options. Therefore, a common market maker is more efficient than market makers involved only in a single fragmented market. The common market maker can use information in multiple related markets when pricing each security.
9 Images and Tables

The files for images and tables used in this paper can be found at: https://sites.google.com/site/heesurohacademics/marketmaking
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