Higher Order Anomaly Consistency Conditions: 
Renormalization and Non-Locality

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Abstract

We study the Slavnov-Taylor Identities (STI) breaking terms, up to the second order in perturbation theory.
We investigate which requirements are needed for the first order Wess-Zumino consistency condition to hold true at the next order in perturbation theory. We find that:

a) If the cohomologically trivial contributions to the first order STI breaking terms are not removed by a suitable choice of the first order normalization conditions, the Wess-Zumino consistency condition is modified at the second order.

b) Moreover, if one fails to recover the cohomologically trivial part of the first order STI breaking local functional, the second order anomaly actually turns out to be a non-local functional of the fields and the external sources.

By using the Wess-Zumino consistency condition and the Quantum Action Principle, we show that the cohomological analysis of the first order STI breaking terms can actually be performed also in a model (the massive Abelian Higgs-Kibble one) where the BRST transformations are not nilpotent.

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1 Introduction

It is well known that classical symmetries, expressed in functional form by a set of Ward identities (WI) satisfied by the classical action $\Gamma^{(0)}$, may be broken at the quantum level [1]. The possible quantum breaking in perturbative QFT of the WI can have two origins. Either it is a truly physical obstruction to the restoration of the WI (and one is then faced with the problem of classifying all possible anomalous terms for the model under investigation) or it is an unwanted effect of the renormalization procedure, needed to handle UV divergences in the full quantum action $\Gamma$. The latter breaking can always be recovered by the choice of a more suited renormalization scheme, or of different renormalization conditions. Since changing the renormalization procedure does not alter the physical content of the theory, this is a spurious breaking of the WI.

In gauge theories, the BRST [2] transformations turn out to be a very powerful tool for proving the unitarity of the physical $S$ matrix [3]. Thus the restoration of BRST symmetry is an essential step in the perturbative construction of gauge theories. The crucial requirement in the proof of physical unitarity is nilpotency: nilpotency of the BRST transformations is a sufficient condition for a theory to be unitary, provided that the associated WI (known as Slavnov-Taylor identities - STI from now on) hold at the quantum level. The task is then to restore the STI, when this is physically possible (absence of anomalies), and to completely classify the form of the breaking terms in the anomalous case.

The nilpotency of the BRST transformations also allows for an effective cohomological analysis of the ST breaking terms. This is most easily seen in the framework of the field-antifield formalism, an extension of the original BRST formulation [4].

We consider a general gauge theory with fields $\phi_i$ and ghosts $c_k$, introduced by the covariant quantization of the model. The fields $\phi_i$ and the ghosts $c_k$ are collectively denoted by $\Phi^A$, $A = 1, \ldots, N$. In the field-antifield formalism, for each field $\Phi^A$ one introduces an antifield $\Phi^*_A$. The space $F$ of the functionals of $\Phi^A, \Phi^*_A$ is endowed with an odd symplectic structure $(\cdot, \cdot)$, the antibracket:

$$ (X, Y) = \sum_{A=1}^{N} \int d^4x \left( \frac{\delta r X}{\delta \Phi^A(x)} \frac{\delta l Y}{\delta \Phi^*_A(x)} - \frac{\delta r X}{\delta \Phi^*_A(x)} \frac{\delta l Y}{\delta \Phi^A(x)} \right). $$

(1)

The subscripts $r$ and $l$ denote right and left differentiation respectively. The classical action $\Gamma^{(0)}[\Phi, \Phi^*]$ is assumed to satisfy the classical master equation

$$ (\Gamma^{(0)}, \Gamma^{(0)}) = 0, $$

(2)

under the condition that $\Gamma^{(0)}[\Phi, \Phi^*]|_{\Phi^*=0}$ coincides with the classical gauge-fixed BRST invariant action. The quantization of the theory produces an effective action $\Gamma$

$$ \Gamma = \Gamma^{(0)} + \sum_{n \geq 1} \hbar^n \Gamma^{(n)}, $$

(3)

which satisfies the quantum extension of the classical master equation [2] [5]:

$$ \frac{1}{2}(\Gamma, \Gamma) = \hbar (A \cdot \Gamma), $$

(4)

where the insertion $(A \cdot \Gamma)$ represents the possible anomalous terms due to the quantum corrections. Using the graded Jacobi identity for the antibracket $(\cdot, \cdot)$

$$ ((X, Y), Z) + (-1)^{(r_x+1)(r_y+r_z)}((Y, Z), X) + (-1)^{(r_x+1)(r_x+r_y)}((Z, X), Y) = 0 $$

(5)

Though non-physical, these breaking terms nevertheless require a careful treatment. For example, in the Standard Model no regularization scheme is known to preserve all the symmetries of the theory, because of the presence of the $\gamma_5$ matrix and of the completely antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. Thus some procedures are needed to recover the Ward identities broken by the intermediate renormalization: suitably defined finite counter-terms must be added to the regularized quantum action in order to recover the relevant Ward identities.
(where $\epsilon_X = 0$ if $X$ obeys Bose statistics and $\epsilon_X = 1$ if $X$ obeys Fermi statistics), it is easy to deduce the following identity for $\Gamma$:

$$\langle\Gamma, (\Gamma, \Gamma)\rangle = 0$$  \hspace{1cm} (6)

or, taking into account eq.(4),

$$\hbar \langle\Gamma, (A \cdot \Gamma)\rangle = 0 .$$  \hspace{1cm} (7)

At the first order in perturbation theory the previous condition reduces to

$$\langle\Gamma^{(0)}, A_1\rangle = 0 ,$$  \hspace{1cm} (8)

where $A \cdot \Gamma = \sum_{j \geq 1} \hbar^{j-1} A_j$. Eq.(8) is the Wess-Zumino consistency condition written in the field-antifield formalism.

Since $\Gamma^{(0)}$ satisfies the classical master equation (2), the operator $\langle\Gamma^{(0)}, \cdot\rangle$ is nilpotent and eq.(8) gives rise to a cohomological problem. The most general solution of eq.(8) can be cast in the form

$$A_1 = \sum_{k \geq 1} \lambda_k C_k + \langle\Gamma^{(0)}, C_0\rangle ,$$  \hspace{1cm} (9)

where the sum is over the representatives $C_k$ of the independent non-trivial cohomology classes of the operator $\langle\Gamma^{(0)}, \cdot\rangle$ and $\langle\Gamma^{(0)}, C_0\rangle$ is an arbitrary element of the trivial cohomology class with FP-charge 1. The purely algebraic analysis which leads to eq.(9) puts no restrictions on the form of $C_k$ and $C_0$, apart from saying that they are functionals of $\Phi^A$, $\Phi^*_A$. In particular, $C_k$ and $C_0$ might very well be non-local functionals of $\Phi^A$, $\Phi^*_A$ and they might contain arbitrarily high powers of $\Phi^A$, $\Phi^*_A$, when expanded on a basis of the space $\mathcal{F}$.

However, if the theory is power-counting renormalizable and the quantization is performed by means of a renormalization procedure which satisfies the Quantum Action Principle (QAP) \[1, 7\], several strong restrictions are imposed on $C_k$ and $C_0$: they must be local functionals of $\Phi^A$, $\Phi^*_A$ and have dimensions that cannot exceed a finite upper limit, predicted by the QAP.

Taking into account these constraints on locality and power counting, one sees from eq.(8) that $\langle\Gamma^{(0)}, C_0\rangle$ can always be reabsorbed by adding finite counter-terms to $\Gamma^{(1)}$. Thus $\langle\Gamma^{(0)}, C_0\rangle$ is a spurious contribution to the anomaly.

If some of the coefficients $\lambda_k$ actually turn out to be non-zero (once their calculation has been performed in the intermediate renormalization scheme), the theory is truly anomalous: no matter how one changes the finite part of $\Gamma^{(1)}$, these terms cannot be reabsorbed. Moreover, it is believed that in every admissible renormalization scheme (compatible with Poincaré invariance and all other exact symmetries of the theory) the calculation of $\lambda^{(n)}$ will yield a non-zero result.

Some attempts have been made to push forward this kind of analysis of the anomalous terms to higher orders in perturbation theory \[8, 9\]. The strategy is to find out a suitable higher-order generalization of the Wess-Zumino consistency condition in eq.(8), relying on the consistency condition for the full quantum action $\Gamma$ in eq.(4) and the nilpotency of the operator $\langle\Gamma^{(0)}, \cdot\rangle$.

In this program, the properties of the renormalization procedure and the choice of the renormalization conditions turn out to be as crucial as the algebraic features of the cohomological analysis, dictated by the nilpotency of $\langle\Gamma^{(0)}, \cdot\rangle$.

\[4\]For the sake of definiteness we are supposing that the ST invariance is the only broken invariance of the model.
In this paper we discuss the second order generalization of the Wess-Zumino consistency condition, in the simple framework of the Abelian Higgs-Kibble model [10]. However, the conclusions one can draw from this example are general enough to be of interest for a wide class of gauge theories. Although the Abelian HK model exhibits spurious anomalous terms only (see [11, 12] for a detailed analysis), we prove that, if the cohomologically trivial contributions to $A_2$ are not recovered by suitably chosen finite counter-terms in $\Gamma^{(1)}$, at the next order the equation for the anomaly is no more the Wess-Zumino consistency condition

$$ (\Gamma^{(0)}, A_2) = 0 . $$

Moreover, we show that in this case $A_2$ must be non-local, in sharp contrast with the locality of the solutions of eq.(10). In our discussion we will relax the assumption of nilpotency of the BRST transformations, by adding to the classical action the following mass term

$$ \int d^4x \left( \frac{M^2}{2} A_\mu^2 + M^2 \bar{c} c - \frac{M^2}{2\alpha} (\phi_1^2 + \phi_2^2) \right). $$

Even though the price of this generalization is the loss of unitarity, in the massive framework it is simpler to appreciate the interplay between algebraic properties and the behavior of the quantum theory under the renormalization procedure. This also allows to discuss the conditions under which strict nilpotency of $(\Gamma^{(0)}, \cdot)$ is actually needed to carry out the construction of $A \cdot \Gamma$, to higher orders in perturbation theory.

## 2 Consistency conditions in the non-nilpotent case

In the Abelian Higgs-Kibble model the fields $A_\mu$ and $\bar{c}$ have linear BRST transformations. Thus one can actually avoid to introduce their antifields. Moreover, we work in the on-shell formalism, i.e. we have eliminated the auxiliary Nakanishi-Lautrup field $B$ [13] associated with the BRST variation of $\bar{c}$ in the off-shell formalism. This in turn allows for a simplification of the Feynman graphs involved in our analysis.

From now on we use the reduced antibracket

$$ (X, Y) = \int d^4x \left[ \frac{\delta X}{\delta J_1} \frac{\delta Y}{\delta \phi_1} + \frac{\delta X}{\delta J_2} \frac{\delta Y}{\delta \phi_2} - \frac{\delta X}{\delta \psi} \frac{\delta Y}{\delta \bar{\eta}} + \frac{\delta X}{\delta \bar{\psi}} \frac{\delta Y}{\delta \eta} \right]. $$

All functional derivatives are assumed to act from the left. $J_1, J_2, \eta, \bar{\eta}$ are the antifields of $\phi_1, \phi_2, \bar{\psi}, \psi$ respectively.

The ST identities for the Abelian Higgs-Kibble model then read

$$ S(\Gamma) = 0 , $$

where the ST operator is

$$ S(\Gamma) = \int d^4x \left[ \partial^\mu c \frac{\delta \Gamma}{\delta A^\mu} + \left( \partial A + \frac{ev}{\alpha} \phi_2 \right) \frac{\delta \Gamma}{\delta \bar{c}} \right] + (\Gamma, \Gamma) . $$

The complete antibracket $\frac{1}{2}(\Gamma, \Gamma)$ in eq. (4) becomes the ST operator in eq. (14).

In the on-shell formalism the ghost equation is:

$$ G = \alpha \Box c + M^2 c , $$

where the ghost operator $G$ is defined by:

$$ G = \frac{\delta (\cdot)}{\delta \bar{c}} - ev \frac{\delta (\cdot)}{\delta J_2} . $$
2.1 The massless case

In the massless (nilpotent) case, eq. (14) is translated into

\[ S_{\Gamma}(S(\Gamma)) = 0 \]  \hspace{1cm} (17)

where \( S_{\Gamma} \) denotes the linearization of the ST operator (14):

\[ S_{\Gamma}(\cdot) = \int d^4x \left[ \partial^\mu c \frac{\delta(\cdot)}{\delta A^\mu} + \left( \partial A + \frac{e_v}{\alpha} \phi_2 \right) \frac{\delta(\cdot)}{\delta c} \right] + (\Gamma, \cdot) + (\cdot, \Gamma). \]  \hspace{1cm} (18)

The identity (17) is valid for any \( \Gamma \) (even for a \( \Gamma \) which does not satisfy the STI \( S(\Gamma) = 0 \)).

We denote by \( S_0 \equiv S_{\Gamma(0)} \) the zero-th order ST linearization. Notice that

\[ \{ G, S_0 \} = 0. \]  \hspace{1cm} (19)

We perform a formal expansion for \( S(\Gamma) \) in powers of \( \hbar \):

\[ S(\Gamma) = \sum_{n \geq 0} \hbar^n S(\Gamma)^{(n)}. \]  \hspace{1cm} (20)

At the first order in perturbation theory eq.(17) becomes

\[ S_0(S(\Gamma)^{(1)}) = 0, \]  \hspace{1cm} (21)

which parallels eq.(14) and gives rise to the cohomological analysis of \( S(\Gamma)^{(1)} \). Thanks to the nilpotency of \( S_0 \) (guaranteed by the invariance of the classical action \( S(\Gamma^{(0)}) = 0 \)), it is possible to find the most general form of \( S(\Gamma)^{(1)} \) compatible with condition (21). \( S(\Gamma)^{(1)} \) can be cast in the form

\[ S(\Gamma)^{(1)} = Y^{(1)} + S_0(C_0), \]  \hspace{1cm} (22)

where \( Y^{(1)} \) is characterized as the most general local functional belonging to the kernel of \( S_0 \) and to the orthogonal complement of \( \text{Im} S_0 \). Written on a basis \( \{ C_k \}_{k \geq 1} \) of \( \ker S_0 \cap (\text{Im} S_0)^\perp \), one gets

\[ Y^{(1)} = \sum_{k \geq 1} \lambda_k C_k, \]  \hspace{1cm} (23)

for some coefficients \( \lambda_k \). This expansion separates truly anomalous terms (\( Y^{(1)} \)) from spurious ones (\( S_0(C_0) \)). The latter can be canceled by a suitable redefinition of the first-order counter-terms entering in the construction of \( \Gamma^{(1)} \).

2.2 The massive case

In the massive HK model eq. (14) is modified as follows

\[ S_{\Gamma} S(\Gamma) = \int d^4x \left( \Box c + \frac{e_v}{\alpha} \delta \Gamma \frac{\delta \Gamma}{\delta c} \right) \frac{\delta \Gamma}{\delta c}. \]  \hspace{1cm} (24)

This identity is valid for any functional \( \Gamma \), without restrictions as \( S(\Gamma) = 0 \) or the ghost equation (15).

Taking into account eq.(15) we get

\[ S_{\Gamma} S(\Gamma) = -\frac{M^2}{\alpha} \int d^4x \frac{\delta \Gamma}{\delta c}. \]  \hspace{1cm} (25)
The linearized ST operator $S_0$ is no more nilpotent; for any functional $F$ satisfying the ghost equation (15) we have now

$$S_0^2(F) = -\frac{M^2}{\alpha} \int d^4x \frac{\delta F}{\delta \bar{c}}.$$  \hfill (26)

At the classical level the STI are satisfied:

$$S(\Gamma^{(0)}) = 0.$$  \hfill (27)

At the first order in perturbation theory eq.(25) gives, taking into account eq.(27):

$$S_0(S(\Gamma)^{(1)}) = -\frac{M^2}{\alpha} \int d^4x \frac{\delta \Gamma^{(1)}}{\delta \bar{c}}.$$  \hfill (28)

Noticing that

$$S_0(S(\Gamma)^{(1)}) = S_2^0(\Gamma^{(1)}),$$  \hfill (29)

we conclude that eq.(26) with $F = \Gamma^{(1)}$ is embodied in eq.(25), under the assumption of the ST invariance of the classical action in eq.(27).

If we adopt a regularization consistent with locality, power-counting and all other unbroken symmetries of the theory (e.g. Lorentz invariance, C-parity, etc.), by using the Quantum Action Principle (QAP) we conclude that the R.H.S. of eq. (28) is zero. Indeed, from the QAP and the power counting theorem we know that $\int d^4x \frac{\partial \Gamma^{(1)}}{\delta \bar{c}}$ is a local $C$-even, Lorentz invariant functional and has dimension less or equal four and FP charge equal two. There are no terms with these properties, so we get the following equation:

$$S_2^0(\Gamma^{(1)}) = 0,$$  \hfill (30)

i.e. even in the massive (non-nilpotent) case the QAP and the power-counting imply that the linearized ST operator $S_0$ is nilpotent on the space of the first-order quantum corrections $\Gamma^{(1)}$.

Let us come back to the study of eq.(28). Since we know that its R.H.S. is zero, thanks to the QAP and the power counting theorem, we have the following equation for the breaking terms $A_1 = S(\Gamma)^{(1)}$:

$$S_0(A_1) = 0.$$  \hfill (31)

Since even in the massive case $S_0$ is nilpotent on the action-like functionals, one can apply the same decomposition of eq.(22). In the Abelian HK model there is just one cohomologically non trivial insertion, $\int d^4x \bar{c} \partial_\mu c A^\mu$. It has the right quantum numbers and the correct exact symmetries of the theory (it is Lorentz-invariant and C-even). However, it can be excluded thanks to the ghost equation (15), the QAP and the power counting. For $n \geq 1$ the ghost equation (15) can be written in the form

$$G \Gamma^{(n)} = 0.$$  \hfill (32)

Using eq.(32) and (19) we get:

$$G S_0(\Gamma^{(1)}) = -S_0(G \Gamma^{(1)}) = 0.$$  \hfill (33)

By power counting $S_0(\Gamma^{(1)})$ cannot contain the external source $J_2$ and from eq. (25) we conclude

$$\frac{\delta}{\delta \bar{c}} S_0(\Gamma^{(1)}) = 0.$$  \hfill (34)

Thus the non-trivial breaking term $\int d^4x \bar{c} \partial_\mu c A^\mu$ is not present and the HK model turns out to be non-anomalous; suitable counter-terms can be constructed at the first order in perturbation theory (actually at all orders), by which the STI can be restored (see [11], [12]).
It is worthwhile noticing that on purely algebraic grounds there are no reasons to exclude the anomalous insertion \( \int d^4 x \bar{c} c \partial_\mu c A^\mu \). To this extent, the properties of the renormalization procedure, dictated by the QAP and the power counting theorem, are essential.

Suppose now that the STI have been restored up to the \((n - 1)\)-th order in perturbation theory, i.e. we assume that suitable counter-terms have been added iteratively to \( \Gamma^{(j)} \), \( j = 1, \ldots, k \), in order to restore the STI till order \( n - 1 \):

\[
S(\Pi^{(k)}) = 0, \quad k = 1, 2, \ldots, n - 1
\]  

(35)

\( \Pi^{(k)} \) denotes the correct symmetric effective action at the \( k \)-th order in perturbation theory.

Then eq. (25) becomes

\[
S_0(S(\Gamma)^{(n)}) = - \frac{M^2}{\alpha} \int d^4 x c \delta \Gamma^{(n)} \frac{\delta c}{\delta \bar{c}}
\]  

or

\[
S_0 \left( S_0(\Gamma^{(n)}) + \sum_{j=1}^{n-1} (\Pi^{(n-j)}, \Pi^{(j)}) \right) = - \frac{M^2}{\alpha} \int d^4 x c \frac{\delta \Gamma^{(n)}}{\delta \bar{c}}.
\]  

(37)

Taking into account eq. (26) we arrive at the following consistency condition for the lower orders parts of the effective action:

\[
S_0 \left( \sum_{j=1}^{n-1} (\Pi^{(n-j)}, \Pi^{(j)}) \right) = 0.
\]  

(38)

This consistency condition is a consequence of the form of the linearized ST operator and of the lower order requirements (35). Again, it relies on the use of the QAP to ensure the fulfillment of STI at lower orders in perturbation theory.

### 3 Higher orders

We now consider eq. (25) at the second order in perturbation theory. We do not assume that the STI have been restored at the first order. Then eq. (25) can be written as

\[
S_0(S(\Gamma)^{(2)}) + S_{\Gamma^{(1)}}(S(\Gamma)^{(1)}) = - \frac{M^2}{\alpha} \int d^4 x c \delta \Gamma^{(2)} \frac{\delta c}{\delta \bar{c}}.
\]  

(39)

We show that, if we use a renormalization scheme where the QAP holds, the R.H.S. of eq. (39) is zero. We have to verify that

\[
- \frac{M^2}{\alpha} \int d^4 x c \delta \Gamma^{(n)} \frac{\delta c}{\delta \bar{c}} = 0,
\]  

(40)

for all \( n \). For \( n = 0 \) the classical action \( \Gamma^{(0)} \) (appendix \( A \)) satisfies eq. (40), as it can be checked by explicit computation. Suppose now that eq. (40) is verified till order \( n - 1 \):

\[
- \frac{M^2}{\alpha} \int d^4 x c \delta \Gamma^{(k)} \frac{\delta c}{\delta \bar{c}} = 0, \quad k = 1, \ldots, n - 1
\]  

(41)

\( ^5 \)Of course, this is possible only in the absence of anomalies, as it is in the Abelian HK model; in this case it can be proven that the restoration of the STI to the \( n \)-order doesn’t change the counter-terms needed to recover the STI up to the \((n - 1)\)-th order, and can be performed, if the STI are fulfilled till order \( n - 1 \), by a proper choice of counter-terms at the \( n \)-order only.
By using the QAP, at the next order in perturbation theory we get:

$$- \frac{M^2}{\alpha} \int d^4 x \frac{\delta \Gamma^{(n)}(x)}{\delta \epsilon} = \int d^4 x \Delta(x),$$

where \( \int d^4 x \Delta(x) \) is an integrated Lorentz invariant local polynomial \( \Delta(x) \) in the fields of the theory. \( \Delta(x) \) has dimension \( \leq 4 \), FP-charge +2 and it obeys all the exact symmetries of the model. Since there are no terms with these properties (\( \int d^4 x c c = 0 \), \( \int d^4 x c \square c = 0 \), \( \int d^4 x A_\mu c \partial \mu c \) is excluded by C-parity, and so on), we conclude that at the \( n \)-th order

$$- \frac{M^2}{\alpha} \int d^4 x c \frac{\delta \Gamma^{(n)}(x)}{\delta \epsilon} = 0.$$ 

(43)

This in turn implies that eq. (40) holds true for all \( n \).

This result can be demonstrated by a direct analysis of the Feynman graphs, arising in the perturbative expansion of \( \Gamma \).

Moreover, the QAP also implies that \( S(\Gamma)^{(1)} \) cannot depend on external sources. Thus eq. (39) simplifies to

$$S_0(\Gamma)^{(2)} = - \sum_i \int d^4 x \frac{\delta \Gamma^{(1)}(x)}{\delta J_i(x)} \frac{\delta (S(\Gamma)^{(1)}(x))}{\delta \phi_i(x)},$$

(44)

where the sum is extended to all the fields \( \phi_i \) whose BRST variation is non linear.

From eq. (44) we see that, if the STI have been restored at the first order (i.e. \( S(\Gamma)^{(1)} = 0 \)), the Wess-Zumino consistency condition holding for \( S(\Gamma)^{(1)} \) is true for \( S(\Gamma)^{(2)} \) too. In particular, \( S(\Gamma)^{(2)} \) is also local, like \( S(\Gamma)^{(1)} \).

On the contrary, if \( S(\Gamma)^{(1)} \neq 0 \), the Wess-Zumino consistency condition for \( S(\Gamma)^{(2)} \) is modified by the R.H.S. of eq. (44), which now is non-zero. Moreover, we show in eq. (45) that eq. (44) implies that \( S(\Gamma)^{(2)} \) receives non-local contributions, arising from the insertion of the local functional \( \int d^4 x \frac{\delta (S(\Gamma)^{(1)}(x))}{\delta \phi_i(x)} \) on the non-local quantities \( \frac{\delta \Gamma^{(1)}(x)}{\delta J_i(x)} \).

In order to simplify the notations, we define \( X \equiv S(\Gamma)^{(2)} \). We also define (in the momentum space) the fourth-order differential operator

$$\mathcal{P}(\cdot) \equiv \frac{\delta^4(\cdot)}{\delta \epsilon(q) \delta \epsilon(s) \epsilon(t) \epsilon(r)}$$

(45)

We apply \( \mathcal{P} \) on both sides of eq. (44) and then set the fields (including external sources) to zero. By using C-parity and the fact that the FP-charge of \( \Gamma \) is zero, we get

$$\mathcal{P} S_0(X)|_{\epsilon=0} = -i r_p X_{\epsilon(q) \epsilon(s) \epsilon(t) \epsilon(r)}|_{\epsilon=0} + ev X_{\epsilon(q) \epsilon(s) \epsilon(t) \epsilon(r)}|_{\epsilon=0} + e^2 v X_{\epsilon(s) \epsilon(t) \epsilon(r) \epsilon(-q-r)}|_{\epsilon=0}$$

$$+ \text{cycl. permutations of } (s, t, r)$$

(46)

and

$$\mathcal{P} \left[ - \sum_i \left( \int \frac{d^4 p}{(2\pi)^4} \Gamma^{(1)}_{\epsilon(q) \epsilon(s) \epsilon(t) \epsilon(r)} \right) \right]_{\epsilon=0} =$$

$$\int \frac{d^4 p}{(2\pi)^4} \left( \Gamma^{(1)}_{\epsilon(q) \epsilon(s) \epsilon(t) \epsilon(r)} \right)_{\epsilon=0} + \text{cycl. permutations of } (s, t, r)$$

(47)

Taking into account the conservation of momenta in 1PI Green functions, we see that the R.H.S. of eq. (47) is not zero for \( -s + q + r + t = 0 \), for \( -t + q + r + s = 0 \), or for \( -r + s + q + t = 0 \).

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\(^6\)We use a short hand for functional derivatives: \( Y_{\epsilon(p)} \) denotes \( \frac{\delta Y}{\delta \epsilon(p)} \), \( Y_{\epsilon(p) \phi(s)} \) denotes \( \frac{\delta^2 Y}{\delta \epsilon(p) \delta \phi(s)} \), and so on.
From eq.(50) we learn that some of the terms arising in the expansion of $S^\ast \Phi$ and the external sources $\Phi$ will show that the expansion of $S^\ast \Phi$ must also contain terms with an arbitrarily high number of fields $\phi$. It is worthwhile performing the construction of these insertions in a recursive way. It is possible to

By using eqs.(46), (47) and (49) we obtain the following equation:

$$\Gamma^{(1)}_{\bar{c}(q)c(r)c(t)J_2(p)}|_{\varphi=0} = \frac{i v^3}{\alpha^2} \int d^4 k \left( \frac{1}{k^2 + m_1^2} \frac{1}{(k+q)^2 + m_2^2} \frac{1}{(k+q+r)^2 + m_3^2} \frac{1}{(k-p)^2 + m_3^2} \right)$$

$$+ \frac{1}{k^2 + m_2^2} \frac{1}{(k+q)^2 + m_2^2} \frac{1}{(k+q+r)^2 + m_3^2} \frac{1}{(k-p)^2 + m_3^2}$$

$$- (r \leftrightarrow t) \cdot \delta^{(4)}(p+q+r+t)$$

(48)

All integrals are convergent (no subtraction required). For general momenta $p, q, r, t$ the R.H.S. of eq.(48) is non-zero. Moreover, it is non-polynomial in the independent external momenta: by applying Weinberg’s theorem [14] we conclude that for non-exceptional external momenta the amplitude $\Gamma^{(1)}_{\bar{c}(Qq)c(Qr)c(t)J_2(p)}|_{\varphi=0}$ behaves as $\sim Q^{-4}$ for $Q \to \infty$ and fixed $p, q, r, t$ (in the Euclidean region).

A direct calculation of $S(\Gamma^{(1)})$, obtained by applying the method described in [12], shows that

$$S(\Gamma^{(1)}_{c(s)\phi_2(p)})|_{\varphi=0} = (a + b s^2) \cdot \delta^{(4)}(s + p)$$

(49)

for some $c$-numbers $a, b$ (depending on the intermediate renormalization scheme used).

By using eqs.(46), (47) and (48) we obtain the following equation

$$-ir_{\mu} X_{\bar{c}(q)c(s)c(t)A_{\mu}}(r)|_{\varphi=0} + ev X_{\bar{c}(q)c(s)c(t)\phi_2(-r)}|_{\varphi=0} + c^2 v X_{c(s)c(t)J_1(-q-r)}|_{\varphi=0}$$

$$+ \text{cycl. permutations of } (s, t, r)$$

$$= (a + b s^2) \Gamma^{(1)}_{\bar{c}(q)c(r)c(t)J_2(-s)} + \text{cycl. permutations of } (s, t, r)$$

(50)

Eq.(48) implies that the R.H.S. of eq.(50) is non-polynomial in the variables $s, q, r, t$. Thus at least one of the amplitudes $X_{c\bar{c}cA_{\mu}}, X_{c\bar{c}c\phi_2}, X_{c\bar{c}J_1}$ is non-polynomial in the external momenta $s, q, r, t$. This in turn implies that $S(\Gamma^{(2)})$ is non-local.

From eq.(50) we learn that some of the terms arising in the expansion of $S(\Gamma^{(2)})$ on a basis of the fields $\Phi$ and the external sources $\Phi^*$ of the theory must contain an arbitrarily high number of derivatives. We will show that the expansion of $S(\Gamma^{(2)})$ on a basis of $\Phi, \Phi^*$ must also contain terms with an arbitrarily high number of fields $\phi_1$.

Consider the insertion of $k$ fields $\phi_1$ along the $\bar{c}c$-lines or the $\phi_1\bar{c}1$-lines in the graphs shown in figures [1] and [2]. It is worthwhile performing the construction of these insertions in a recursive way. It is possible to
insert a leg \( \phi_1(w_1) \) (carrying momentum \( w_1 \)) in the graph on the left of figure 3 by cutting a \( cc \)-propagator or by cutting a \( \phi_1 \phi_1 \)-line. The graph on the left of figure 3 thus generates four graphs contributing to the 1-PI amplitude \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p)\phi_1(w_1))_{\phi=0} \). They are shown in figure 3. The same construction can be applied to all other graphs appearing in figures 1 and 2, yielding a family \( \mathcal{E}^1 \) of graphs contributing to \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p)\phi_1(w_1))_{\phi=0} \). Moreover, the graphs in \( \mathcal{E}^1 \) exhaust all possible graphs contributing to \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p)\phi_1(w_1))_{\phi=0} \). Indeed, if \( \mathcal{T} \) is a graph contributing to \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p)\phi_1(w_1))_{\phi=0} \), the leg \( \phi_1(w_1) \) must be inserted either on a \( \phi_1 \phi_1 \)-line or on a \( cc \)-line. Thus removing the insertion of \( \phi_1(w_1) \) and gluing together the two propagators \( \phi_1 \phi_1 \) or \( cc-cc \) of the line with the \( \phi_1(w_1) \) insertion into a single \( \phi_1 \phi_1 \) or \( cc \) propagator respectively yields a graph contributing to \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p))_{\phi=0} \). The latter must be one of the graphs in figures 1 or 2. Hence \( \mathcal{T} \) belongs to \( \mathcal{E}^1 \).

From an analytic point of view, the construction of graphs in \( \mathcal{E}^1 \) amounts to the replacement

\[
\frac{1}{k^2 + m_2^2} \rightarrow \frac{1}{(k + w_1)^2 + m_2^2} \frac{1}{k^2 + m_2^2}
\]

(for a \( \phi_1(w_1) \)-insertion on a \( cc \)-line) or

\[
\frac{1}{k^2 + m_2^2} \rightarrow \frac{1}{(k + w_1)^2 + m_1^2} \frac{1}{k^2 + m_1^2}
\]

(for a \( \phi_1(w_1) \)-insertion on a \( \phi_1 \phi_1 \)-line).

No new cancellations arise from these replacements, as it can be seen from the explicit computation of the associated integrals, leading to a straightforward generalization of eq.(48). So one can conclude that \( \Gamma^{(1)}(\bar{c}(q)c(t)c(r)J_2(p)\phi_1(w_1))_{\phi=0} \) is non-zero.

This construction can be applied to the graphs of \( \mathcal{E}^1 \) to generate the family \( \mathcal{E}^2 \), whose elements are graphs contributing to \( \Gamma^{(1)}(J_2(p)\bar{c}(q)c(t)c(s)\phi_1(w_1)\phi_1(w_2))_{\phi=0} \). Again, the same remarks as before apply and one finds out that \( \Gamma^{(1)}(J_2(p)\bar{c}(q)c(t)c(s)\phi_1(w_1)\phi_1(w_2))_{\phi=0} \neq 0 \). The recursion can be iterated till the \( k \) insertions of \( \phi_1(w_j) \), \( j = 1, \ldots, k \) have been completed.

We now introduce an extension of the operator \( \mathcal{P} \) defined in eq.(43):

\[
\mathcal{P}_k(\cdot) = \frac{\delta^{k+1}(\cdot)}{\delta \phi_1(w_k)\delta \phi_0(w_{k-1}) \ldots \delta \phi_1(w_1)\delta \bar{c}(q)\delta c(s)\delta c(t)\delta c(r)}
\]

By applying \( \mathcal{P}_k \) to eq.(44) and setting next the fields to zero, one gets the following set of equations:

\[
\mathcal{P}_k \left[ S_0(S(\Gamma))^{(2)} \right]_{\phi=0} = - \sum_{l} \int d^4 x \mathcal{P}_k \left[ \frac{\delta \Gamma^{(1)}_n(S(\Gamma))^{(1)}}{\delta J_l(x)} \frac{\delta(S(\Gamma))^{(1)}}{\delta \phi_1(x)} \right]_{\phi=0}
\]
For $k = 0$ we recover eq. (50). The R.H.S. of eq. (54) contains functions of the form

$$ S(\Gamma^{(1)}_{c(p_1)c(p_2)c(p_3)c(p_4)}\prod_{a=1}^{k+1} \phi_1(q_a)) = Q(p_1, p_2, q_a)\delta^{(4)}(p_1 + p_2 + \sum_{a=1}^{k} q_a), $$

(55)

where $Q(p_1, p_2, q_a)$ is a polynomial of degree at most 2 in $p_1, p_2, q_a$. These functions are zero for $k > 4$, as it can be seen by a direct calculation applying the method in [12].

The R.H.S. of eq. (54) has the same feature as the R.H.S. of eq. (54): each configuration of external momenta, compatible with the delta functions contained in the R.H.S. of eq. (54), picks out one and only one of the terms which arise in the expansion of the R.H.S. of eq. (54) in terms of the amplitudes $\Gamma^{(1)}_{c(p_1)c(p_2)c(p_3)c(p_4)}\prod_{a=1}^{k+1} \phi_1(q_a)|_{\varphi=0}$, $j = 1, \ldots, k$. Thus, having shown that

$$ \Gamma^{(1)}_{c(p_1)c(p_2)c(p_3)c(p_4)}\prod_{a=1}^{k+1} \phi_1(q_a)|_{\varphi=0} \neq 0, $$

(56)

we conclude that the R.H.S. of eq. (54) is non-zero for every $k$. Eq. (54) implies that, in the expansion of $S(\Gamma^{(2)})$ on a basis of $\Phi, \Phi^*$, there are non-zero terms associated with monomials containing an arbitrary number of $\phi_1$ fields.

This has some interesting consequences. We have shown that, if improper finite counter-terms in $\Gamma^{(1)}$ are chosen, at the second order in perturbation theory the STI

$$ S(\Gamma^{(2)}) = A_2 $$

(57)

are broken by a non-local functional $A_2$. Eq. (57) gives rise upon differentiation with respect to a set of fields and external sources $\{\Phi^I, \Phi^*_I\}_{I \in \mathcal{I}}$ (with $I$ running in the set of indices $\mathcal{I}$) to a number of relations among 1-PI Green functions, once we set $\Phi^I = \Phi^*_I = 0$ after taking the relevant derivatives.

We can expand $A_2$ on a basis of monomials in $\Phi, \Phi^*$ and their derivatives. Since $A_2$ is non-local, an infinite number of monomials appears in this expansion. It may happen that there is a maximum finite number $O$ of $\Phi, \Phi^*$, appearing in every monomial of the expansion. In this case, the expansion is infinite because it contains monomials with arbitrarily high order derivatives. Thus we can differentiate eq. (57) with respect to a number of fields greater than $O$, yielding the same result as if $S(\Gamma^{(2)}) = 0$. Only a finite number of relations among 1-PI Green functions, valid in the invariant case $S(\Gamma^{(2)}) = 0$, is altered by this type of non-local breaking terms $A_2$. Notice that this is the same behavior one has when the breaking is local.

It may also happen that in the expansion of $A_2$ there appear monomials with an arbitrarily high number of $\Phi, \Phi^*$. Now an infinite set of relations among 1-PI Green functions, derived from eq. (57) upon differentiation, is changed with respect to the invariant case. In this sense, violation of locality by arbitrarily high number of $\Phi, \Phi^*$ is more severe than violation of locality by arbitrarily high number of derivatives only.

We briefly comment on the results of this section. Had we restored the STI at the first order in perturbation theory, eq. (21) would have read

$$ S_0(S(\Gamma^{(2)})) = 0. $$

(58)

If $S(\Gamma^{(1)}) = 0$, the same Wess-Zumino consistency condition holds true both for $S(\Gamma^{(1)})$ (see eq. (21)) and $S(\Gamma^{(2)})$. In particular, $S(\Gamma^{(2)})$ is local. Moreover, eq. (57) shows that if $S(\Gamma^{(1)}) \neq 0$, $S(\Gamma^{(2)})$ receives non-local contributions. Thus a necessary and sufficient condition for $S(\Gamma^{(2)})$ to be local is that $S(\Gamma^{(1)}) = 0$. Notice that one can impose in the Abelian HK model $S(\Gamma^{(1)}) = 0$ because the model does not possess physical anomalies.
This result admits a wider generalization. Suppose that the gauge theory under investigation is truly anomalous. Then at the second order in perturbation theory the consistency condition obeyed by $S(\Gamma)^{(2)}$ is eq. (44), where now $S(\Gamma)^{(1)}$ is non zero for any choice of the first order action-like counterterms. An argument similar to the one leading to eq. (46) thus entails that $S(\Gamma)^{(2)}$ must be non local, because of the contributions coming from $S_{\Gamma^{(1)}}(S(\Gamma)^{(1)})$.

4 Conclusions

In this paper we have shown that, if the action-like counter-terms entering in $\Gamma^{(1)}$ are not properly chosen, even a physically non-anomalous theory exhibits a non-local second order anomaly. This anomaly cannot be restored by local second order counterterms. Thus, an improper choice of the finite part of the first order counter-terms renders a first-order physically non-anomalous theory a second order truly anomalous one.

Moreover, we have argued that, if one starts with a truly anomalous theory, locality of the STI breaking terms is satisfied at the first order in perturbation theory only, no matter which renormalization scheme is adopted.

We conclude that locality of the STI breaking terms can be maintained to all orders if and only if there are no truly anomalous terms at the first order in perturbation theory.

Finally, we have shown that strict nilpotency of the BRST transformations (and consequently of the linearized ST operator) is not an essential requirement in order to perform the characterization of the STI breaking terms, independently on the order of the perturbative expansion.

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7 In [8] it was argued that, if one recovers the spurious contributions to the first order anomaly, $S(\Gamma)^{(2)}$ can be made local and chosen in such a way that it satisfies the same Wess-Zumino consistency condition as $S(\Gamma)^{(1)}$. To match these requirements, one needs to introduce the first order truly anomalous (local) terms as interaction vertices in the quantum effective action $\Gamma^{(1)}$, by coupling them to external sources of negative dimension.

In our opinion, this procedure generates a set of Feynman rules which spoil the validity of the QAP at the next order. In particular, at the next order $S_0(S(\Gamma)^{(2)})$ gets non-local contributions, which in our framework are embodied in $S_{\Gamma^{(1)}}(\Gamma^{(1)})$.

Notice that, if the first order physical anomalies had dimension $\leq 4$ (which is not forbidden by the QAP, saying only that they must have dimension less or equal to 5), no troubles would arise in including them as vertices in $\Gamma^{(1)}$. They could be coupled to external sources with non-negative dimension. Actually, truly anomalous terms have dimension 5 only (at least for a gauge group without Abelian factors) [15]. This in turn implies that they cannot just be thought as new interaction vertices, since these vertices must contain external sources with dimension $-1$. 

12
A  Classical action

The classical action for the HK model in the on-shell formalism is

\[ \Gamma^{(0)} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^2 + \frac{e^2 v^2}{2} A_\mu^2 ight. \\
\left. -\frac{\alpha}{2} \partial A^2 + \alpha \Box \bar{c} c + e^2 v^2 \bar{c} c + e^2 \bar{v} \bar{c} c \phi_1 \\
+\frac{1}{2}((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2) - \nu^2 \phi_1^2 - \frac{e^2 v^2}{2\alpha} \phi_2^2 \\
+e A_\mu (\phi_2 \partial^\mu \phi_1 - \partial^\mu \phi_2 \phi_1) + e^2 v \phi_1 A^2 + \frac{e^2}{2} (\phi_1^2 + \phi_2^2) A^2 \\
-\nu \phi_1 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2 \\
+\bar{\psi} i \gamma^\mu + G \bar{v} \gamma^\mu \psi + \frac{e}{2} \bar{v} \gamma^\mu \gamma_5 \psi A^\mu \\
+G \bar{v} \psi_1 - iG \bar{\psi} \gamma_5 \gamma_5 \psi_2 \\
+J_1 [-ee \phi_2] + J_2 ee (\phi_1 + v) + ie \bar{\psi} \gamma_5 \gamma_5 c + i e c \bar{\psi} \gamma_5 c \\
+\frac{M^2}{2} A_\mu^2 + M^2 \bar{c} c - \frac{M^2}{2\alpha} (\phi_1^2 + \phi_2^2) \right] \]

(59)

The explicit mass term is in evidence in the last line.

BRST transformations

Off-shell formalism

\[ sA_\mu = \partial_\mu c, \quad s\phi_1 = -ee \phi_2, \quad s\phi_2 = ee (\phi_1 + v) \]

\[ s\psi = -i e \gamma^5 \psi c, \quad s\bar{\psi} = i e \bar{c} \gamma_5 \psi, \quad s\bar{c} = B, \quad sB = -\frac{M^2}{\alpha} c, \quad sc = 0 \]

(60)

In the on-shell formalism the \( B \) field disappears and the BRST transformation of \( \bar{c} \) becomes

\[ s\bar{c} = \partial A + \frac{e \nu}{\alpha} \phi_2, \]

(61)

B  Feynman rules

We only recall the Feynman rules needed to evaluate the graphs in Figures 1 and 2.

Propagator for \( \phi_1 \phi_1 \)

\[ \Delta_{\phi_1 \phi_1} (p) = \frac{i}{p^2 - m_1^2 + i\epsilon}, \quad m_1^2 = 2\nu^2 + \frac{M^2}{\alpha} \]

(62)

The propagator for \( \phi_1 \phi_1 \) is denoted by a solid line.

Propagator for \( \bar{c}\bar{c} \)

\[ \Delta_{\bar{c}\bar{c}} (p) = \frac{-i}{\alpha (p^2 - m_2^2 + i\epsilon)}, \quad m_2^2 = \frac{e^2 v^2 + M^2}{\alpha} \]

(63)

The propagator for \( \bar{c}\bar{c} \) is denoted by a dashed line.
Vertices $eJ_2c\phi_1$ and $e^2vcc\phi_1$

Vertices

\[ ie(2\pi)^4\delta^{(4)}(\text{incoming momenta}) \]  

for the vertex $eJ_2c\phi_1$ and

\[ ie^2v(2\pi)^4\delta^{(4)}(\text{incoming momenta}) \]  

for the vertex $e^2vcc\phi_1$ (see Figure 4).

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