COMMUTATION RELATIONS ON THE COVARIANT DERIVATIVE

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Let \( g \) be a nonassociative algebra over a field \( k \). The operation in \( g \) will be denoted by the diamond sign \( \circ \), for example, \( (x \circ y) \circ z \in g \), \( x, y, z \in g \). Let \( T(g) = T_k(g) = \oplus_{n=0}^{\infty} g^\otimes n \) denotes the tensor algebra of \( g \) and \( L_u : T(g) \to T(g) \), \( u \in T(g) \), denotes the operator of left multiplication, \( L_u : v \mapsto u \otimes v \). For typographical reasons we shall write sometimes \( L(u) \) instead of \( L_u \). Denote by \( \tau_x \in \text{Der}_k(T(g)) \), \( x \in g \) the derivation of the tensor algebra defined by the condition \( \tau_x : y \mapsto x \circ y \), \( y \in g \). There exists a unique linear map \( K : T(g) \to T(g) \) such that \( K(1) = 1 \) and \( KL_x = L_x K - K \tau_x \), \( x \in g \). For example, \( K(x) = K(L_x 1) = L_x (K(1)) - K(\tau_x 1) = x \), \( K(x \circ y) = K(L_x y) = L_x (K(y)) - K(\tau_x y) = x \circ y - x \circ y \), etc. The author introduced this map in [1,2] (it is unlikely that such a map has never been considered before, however, I have not found an appropriate reference).

The problem we are interested in appears when \( g \) is a nonassociative algebra and a Lie algebra simultaneously (a "framed Lie algebra" in terms of [1]). In other words, it is an algebra with two operations, one of which is antisymmetric and satisfies the Jacobi identity. As \( g \) is a Lie algebra, there exists the exact sequence \( 0 \to I(g) \to T(g) \to U(g) \to 0 \), where \( U(g) \) is the universal enveloping algebra of \( g \) and \( I(g) \) is the two-sided ideal of \( T(g) \) generated by the elements of the form \( x \otimes y - y \otimes x - [x, y] \). Let \( \Sigma(g) \subset T(g) \) denotes the linear space of symmetric tensors (i.e. \( \Sigma_n(g) = \{ g \in T(g) : g \otimes (g \otimes \cdots) \in \Sigma_n \} \)). The map \( K \) has the property \( \text{deg}(K u - u) < \text{deg}(u) \), \( u \in T(g) \) hence it is invertible. Moreover, if char \( k = 0 \), then the restriction of the natural map to \( K(\Sigma(g)) \to U(g) \) is a linear isomorphism by the Poincare-Birkhoff-Witt theorem. Thus we have the decomposition \( T(g) = \Sigma(g) \oplus K^{-1}(I(g)) \).

The problem is to describe the corresponding projection \( \pi : T(g) \to \Sigma(g) \) explicitly.

The origin of the problem lies in differential geometry; it will be discussed below. Even in the case \( K = \text{id} \) (i.e. \( g \circ g = \{ 0 \} \)) it is not trivial [5]. In the general case it looks like a formidable task. This paper concerned with a much more simple related problem. Denote \( \Omega(g) = T(g) \otimes \wedge^2 g \), where \( \wedge^2 g \) is the exterior square of \( g \). Let \( \iota : \Omega(g) \otimes T(g) \to T(g) \) be the natural map, \( \iota : a \otimes x \wedge y \otimes b \mapsto a \otimes (x \wedge y - y \wedge x) \otimes b \).

The aim is to find in an explicit form a map \( R : \Omega(g) \otimes T(g) \to T(g) \) satisfying for any \( Q \in \Omega(g) \otimes T(g) \) the following two properties: \( R(Q) + \iota(Q) \in K^{-1}(I(g)) \) and \( \text{deg}(R(Q)) < \text{deg}(\iota(Q)) \). These properties do not determine the map \( R \) uniquely; however, the solution proposed below is probably the simplest one.

The projection \( \pi \) may be expressed via \( R \) as follows. Let \( s : T(g) \to S(g) \) be the natural algebra homomorphism and \( J(g) = \ker(s, T(g)) \). The restriction \( s : \Sigma(g) \to S(g) \) is a linear isomorphism, hence \( T(g) = \Sigma(g) \oplus J(g) \). Denote by \( \pi_0 : T(g) \to \Sigma(g) \) the corresponding projection. The map \( \iota : \Omega(g) \otimes T(g) \to J(g) \) is surjective, hence there exists a (non-unique) right inverse \( \iota^{-1} : J(g) \to \Omega(g) \otimes T(g) \).

Denote \( \rho = R \circ \iota^{-1} \circ (\text{id} - \pi_0) : T(g) \to T(g) \). One can choose an inverse map

\[ \text{Supported by part of the grant RFFI-08-01-92001.} \]
\[
\tau^{-1} \text{ satisfying the natural condition } \deg r^{-1}(u) < \deg u, \ u \in T(\mathfrak{g}). \text{ Then we have} \\
\deg \rho(u) < \deg u, \text{ hence the map } \text{id} + \rho \text{ is invertible. Let } \pi = \pi_0(\text{id} + \rho)^{-1}. \text{ By the definitions, } \rho : I(\mathfrak{g}) \to \{0\} \text{ and } \text{id} + \rho : J(\mathfrak{g}) \to K^{-1}(I(\mathfrak{g})). \text{ Hence } \pi \text{ is exactly the projection we need (though not in an explicit form). We shall see below that the map } R \text{ may be interpreted as a collection of commutation relations on the covariant derivative.}
\]

1. Relations

In this section, \( \mathfrak{g} \) is an algebra with two operations: the first one denoted by \( \odot \) and the antisymmetric second one denoted by the brackets \( [\cdot, \cdot] \). In the applications it is a Lie algebra, i.e. the second operation satisfies the Jacobi identity, but we shall not actually need this identity here. The base field \( k \) is an arbitrary one.

Let \( I(\mathfrak{g}) \subset T(\mathfrak{g}) \) be the two-sided ideal generated by the elements of the form \( x \odot y - y \odot x - [x, y], \ x, y \in \mathfrak{g} \). Note that if \( \mathfrak{g} \) is not a Lie algebra then \( T(\mathfrak{g})/I(\mathfrak{g}) \) is no more the universal enveloping algebra. Let \( \Omega(\mathfrak{g}) \) and \( \tau_x, \ x \in \mathfrak{g} \) are defined as above. The derivation \( \tau_x : T(\mathfrak{g}) \to T(\mathfrak{g}) \) can be naturally lifted to the map \( \tau_x : \Omega(\mathfrak{g}) \to \Omega(\mathfrak{g}) \) by the condition \( \tau_x \circ i = i \circ \tau_x \); namely \( \tau_x : a \odot y \wedge z \mapsto \tau_x(a) \odot y \wedge z + a \odot x \odot y \wedge z + a \odot y \wedge x \odot z \). The linear maps \( t : \Omega(\mathfrak{g}) \to \mathfrak{g}, r : \Omega(\mathfrak{g}) \to \text{Der}_k(T(\mathfrak{g})) \) and \( e : \Omega(\mathfrak{g}) \to I(\mathfrak{g}) \) are defined as follows. If \( x, y, z \in \mathfrak{g} \) and \( Q \in \Omega(\mathfrak{g}) \), then

\[
t(x \wedge y) = x \odot y - y \odot x - [x, y], \ t(x \odot Q + \tau_x Q) = x \odot t(Q),
\]

\[
r(x \wedge y) : z \mapsto x \odot (y \odot z) - y \odot (x \odot z) - [x, y] \odot z, \ r(x \odot Q + \tau_x Q) = [\tau_x, r(Q)],
\]

\[
e(x \wedge y) = x \odot y - y \odot x - [x, y], \ e(x \odot Q + \tau_x Q) = x \odot e(Q) - e(Q) \odot x.
\]

It is well known that tensor algebra is a bialgebra (actually it is a Hopf algebra but we make no use of antipode). The comultiplication \( \triangle : T(\mathfrak{g}) \to T(\mathfrak{g}) \odot T(\mathfrak{g}) \) is an algebra homomorphism defined by the condition \( \triangle : x \mapsto 1 \odot x + x \odot 1, \ x \in \mathfrak{g} \) (to avoid misunderstanding the "exterior" tensor product is denoted by the hatted sign, i.e. \( 1 \hat{\otimes} x \) and \( x \hat{\otimes} 1 \) are the elements of \( T(\mathfrak{g}) \odot T(\mathfrak{g}) \), not of \( T(\mathfrak{g}) \)). We shall use the common Sweedler notation, e.g. \( \triangle(u) = \sum_{(u)} u_{(1)} \hat{\otimes} u_{(2)} \).

**Theorem.** Let \( \mathfrak{g} \) be as above, \( u, v \in T(\mathfrak{g}) \) and \( \omega \in \wedge^2 \mathfrak{g} \). Then

\[
K(u \odot i(\omega) \odot v + \sum_{(u)} u_{(1)} \odot (t(u_{(2)} \odot \omega) \odot v + r(u_{(2)} \odot \omega) \odot v)) = \sum_{(u)} e(u_{(1)} \odot \omega) \odot K(u_{(2)} \odot \omega).
\]

As an easy consequence, the map

\[
R : \Omega(\mathfrak{g}) \odot T(\mathfrak{g}) \to T(\mathfrak{g}), \ R : u \odot \omega \odot v \mapsto \sum_{(u)} u_{(1)} \odot (t(u_{(2)} \odot \omega) \odot v + r(u_{(2)} \odot \omega) \odot v)
\]

has the required properties: if \( Q \in \Omega(\mathfrak{g}) \odot T(\mathfrak{g}) \), then \( R(Q) + i(Q) \in K^{-1}(I(\mathfrak{g})) \) and \( \deg R(Q) < \deg i(Q) \).

It is convenient to introduce the linear maps \( \lambda_x = L_x + \tau_x, \ x \in \mathfrak{g} \) and \( q(Q) = L_{\triangle(Q)} + r(Q) : T(\mathfrak{g}) \to T(\mathfrak{g}), \ Q \in \Omega(\mathfrak{g}) \).

Denote by \( Z(u, \omega) : T(\mathfrak{g}) \to T(\mathfrak{g}), \ u \in T(\mathfrak{g}), \omega \in \wedge^2 \mathfrak{g} \) the linear map defined by the equality

\[
Z(u, \omega) = KL(u)L(i(\omega)) + \sum_{(u)} KL(u_{(1)}) q(u_{(2)} \odot \omega) - L(e(u_{(1)} \odot \omega)) KL(u_{(2)}).
\]
The statement of the theorem may be written in the form $Z(u, \omega)v = 0$, so it remains to prove that $Z(u, \omega)$ is zero. By the definitions,

$$Z(1, x \wedge y) = K[L_x, L_y] + Kq(x \wedge y) - ([L_x, L_y] - L_{[x,y]} K).$$

Substituting

$$q(x \wedge y) = [\tau_x, L_y] - [\tau_y, L_x] + [\tau_x, \tau_y] - \lambda_{[x,y]}$$

and taking into account the equality $K\lambda_x = L_x K$, $x \in \mathfrak{g}$, we get

$$Z(1, x \wedge y) = K[\lambda_x, \lambda_y] - [L_x, L_y] K - K\lambda_{[x,y]} + L_{[x,y]} K = 0.$$

An easy computation shows that $L(\lambda_x u) = \lambda_x L(u) - L(u)\tau_x$, $q(\lambda_x Q) = [\tau_x, q(Q)]$

and $\Delta(\lambda_x u) = \sum \lambda_x u_{(1)} \otimes u_{(2)} + u_{(1)} \otimes \lambda_x u_{(2)}$ for any $x \in \mathfrak{g}$. Applying all these equalities we have

$$Z(\lambda_x u, \omega) + Z(u, \tau_x \omega) = KL(\lambda_x (u \otimes i(\omega))) + KL(\lambda_x (u_{(1)} \otimes \omega) +
+ KL(u_{(1)})(u_{(2)} \otimes \omega) - L(e(\lambda_x (u_{(1)} \otimes \omega)))[KL(u_{(2)}) - L(e(u_{(1)} \otimes \omega))]

= L_x KL(u_{(1)})(u_{(2)} \otimes \omega) - KL(u_{(1)}(u_{(2)} \otimes \omega) - KL(u_{(1)})(u_{(2)} \otimes \omega) +

+ KL(u_{(1)}(\tau_x, u_{(2)} \otimes \omega) - L(u \otimes e(u_{(1)} \otimes \omega) - e(u_{(1)} \otimes \omega) x) KL(u_{(2)}) -

- L(e(u_{(1)} \otimes \omega)) L_x KL(u_{(2)}) + L(e(u_{(1)} \otimes \omega)) KL(u_{(2)}) \tau_x = L_x Z(u, \omega) - Z(u, \omega) \tau_x.

One can write this as

$$Z(x \otimes u, \omega) = L_x Z(u, \omega) - Z(u, \omega) \tau_x - Z(\tau_x u, \omega) - Z(u, \tau_x \omega).$$

By the induction on the degree of $u$, we have $Z(u, \omega) = 0$.

2. Geometric interpretation

Let $\mathcal{M}$ be a smooth manifold, $\mathfrak{f}(\mathcal{M})$ be the algebra of smooth functions on $\mathcal{M}$ and $\mathcal{V}(\mathcal{M}) = \text{Der}_\mathbb{R}(\mathfrak{f}(\mathcal{M}))$ be the Lie algebra of smooth vector fields. Let us denote $\mathcal{V} = \mathcal{V}(\mathcal{M})$ and $T(\mathcal{V}) = T_\mathbb{R}(\mathcal{V})$.

Denote by $\Gamma(\mathcal{M}, T^n \mathcal{M})$ the space of smooth global sections of the rank $n$ tensor bundle $T^n \mathcal{M} = \bigotimes^n T \mathcal{M}$. For example, $\Gamma(\mathcal{M}, T^0 \mathcal{M}) = \mathfrak{f}(\mathcal{M})$ and $\Gamma(\mathcal{M}, T^1 \mathcal{M}) = \mathcal{V}(\mathcal{M})$. Denote $T(\mathcal{M}) = \bigoplus_{n=0}^{\infty} \Gamma(\mathcal{M}, T^n \mathcal{M})$. It is well known that $T(\mathcal{M}) = T_\mathbb{R}(\mathcal{M})(\mathcal{V})$ (e.g. [3, Ch. 1, Proposition 3.1]). Then there is a natural map $\iota : T(\mathcal{V}) \to T(\mathcal{M})$.

Denote by $\mathcal{D}(\mathcal{M})$ the algebra of scalar differential operators with smooth coefficients on $\mathcal{M}$. Any vector field is a first order differential operator. By the definition of $T(\mathcal{V})$, the natural inclusion map $\tau : \mathcal{V} \to \mathcal{D}(\mathcal{M})$ can be extended to the algebra homomorphism $\tau : T(\mathcal{V}) \to \mathcal{D}(\mathcal{M})$. By the definition of the Lie bracket, $\tau : x \otimes y - y \otimes x - [x, y] \to 0$, $x, y \in \mathcal{V}$, hence $\tau : I(\mathcal{V}) \to \{0\}$.

Let us suppose the manifold to be endowed with a smooth affine connection. Let $\mu : T(\mathcal{M}) \to \mathcal{D}(\mathcal{M})$ denotes the $\mathfrak{f}(\mathcal{M})$ – linear map defined by $\mu : v_1 \otimes \cdots \otimes v_n \mapsto \nabla^n_{v_1,\ldots,v_n}$, $v_1,\ldots,v_n \in \mathcal{V}$. Here $\nabla^n$ is the $n$-th order covariant derivative (in the notation of [3, Ch. III, §2] $\nabla^n_{v_1,\ldots,v_n} : f \mapsto f(;v_n;\ldots;v_1)$). For example, if $f, g \in \mathfrak{f}(\mathcal{M})$ and $v \in \mathcal{V}(\mathcal{M})$, then $\mu(f) : g \mapsto fg$ and $\mu(v) : g \mapsto v(g)$. The operator $\mu(v \otimes w) = \nabla^n_{v,w} = vw - (\nabla_v w)$ depends on the connection, as well as the images of the higher degree tensor fields.

Let $\Sigma(\mathcal{M}) = \{ \iota(\Sigma(V)) \subset T(\mathcal{M}) \}$ be the space of (formal sums of) symmetric tensor fields. By the methods of geometry it may be shown that there exists a unique map $\sigma : \mathcal{D}(\mathcal{M}) \to \Sigma(\mathcal{M})$, such that $\mu \circ \sigma = \text{id}_{\mathcal{D}(\mathcal{M})}$. This map is a surjective $\mathfrak{f}(\mathcal{M})$ – module homomorphism. The image $\sigma(A)$ is called a symbol of the differential operator $A$. In almost the same form the symbol map was introduced.
in [6,82] but actually it has been known long before (see the references in [6]). A proper investigation of the symbol map leads inevitably to the following natural question: what is the projection \( \Sigma = \sigma \circ \mu : T(\mathcal{M}) \to \Sigma(\mathcal{M}) \)? For example, this map is of importance when the symbol of a composition of two (pseudo)differential operators is considered. In some simple cases, Sharafutdinov has computed it in the unpublished supplements to [6].

The aforementioned projection \( \pi \) is closely related to \( \Sigma \). The space \( \mathcal{V} \) may be considered as an algebra with two operations: the Lie bracket and the covariant derivative (probably Nomizu was the first who take this view [4, Ch. III, §6]). Put \( v \circ w = \nabla_v w, \ v, w \in \mathcal{V} \). The corresponding map \( K : T(\mathcal{V}) \to T(\mathcal{V}) \) is then connected to \( \mu \) by the relation \( \mu \circ t = \tau \circ K \) [1, Prop. 1], [2, Lemma 2]. This relation is actually a simple consequence of the well known covariant derivation rules [3, Ch. III, Prop. 2.10]. Let \( \pi : T(\mathcal{V}) \to \Sigma(\mathcal{V}) \) be the projection defined above. By the definition, \( \id - \pi : T(\mathcal{V}) \to K^{-1}(I(\mathcal{V})) \), hence \( \mu \circ t = \mu \circ t \circ \pi \). If \( t^{-1} : T(\mathcal{M}) \to T(\mathcal{V}) \) is any right inverse of \( t^{-1} \), then \( \mu = \mu \circ t \circ \pi \circ t^{-1} \). The map \( \Sigma \) is determined uniquely by the property \( \mu = \mu \circ \Sigma \), hence \( \Sigma = \tau \circ \pi \circ t^{-1} \). Note that the equality does not depend on the choice of \( t^{-1} \), which means \( \pi : \ker(t(T(\mathcal{V}))) \to \ker(t(T(\mathcal{V}))) \). We shall call a linear map with this property a special one. In other words, the map \( T(\mathcal{V}) \to T(\mathcal{V}) \) is special if it may be lifted to a \( \mathfrak{F}(\mathcal{M}) \)- linear map \( T(\mathcal{M}) \to T(\mathcal{M}) \).

In the expression \( \pi = \pi_0(\id + \rho)^{-1} \) the map \( \pi_0 \) is special, so it is natural to ask for speciality of \( \rho \). Under natural assumptions on \( t^{-1} \) this is indeed the case, because the functions \( t \) and \( r \) are nothing but the (derivatives of) the torsion tensor and the curvature tensor respectively:

\[
t(v \wedge w) = T(v, w), \quad t(u \otimes v \wedge w) = (\nabla_u T)(v, w),
\]

\[
\rho(v \wedge w)h = R(v, w)h, \quad r(u \otimes v \wedge w)h = (\nabla_u R)(v, w)h,
\]

e tc., where \( u, v, w, h \in \mathcal{V}(\mathcal{M}) \) [4, Ch. III, §5], [1, §7]. The statement \( R(Q) + \iota(Q) \in K^{-1}(I(\mathcal{V})) \), \( Q \in \Omega(\mathcal{V}) \otimes T(\mathcal{V}) \) may then be considered as a series of commutation relations on the covariant derivative. For example, if \( Q = u \otimes v \wedge w \otimes h \), it takes the form \( u \otimes v \otimes w \otimes h - u \otimes w \otimes v \otimes h + u \otimes (v \wedge w) \otimes h + t(u \otimes v \wedge w) \otimes h + u \otimes r(v \wedge w)h + r(u \otimes v \wedge w)h \in K^{-1}(I(\mathcal{V})) \). Note that \( \mu \circ t = \tau \circ K : K^{-1}(I(\mathcal{V})) \to \{0\} \). Applying this map, we get

\[
\nabla^3_{u,v,w,h} - \nabla^3_{u,v,w,h} + \nabla^2_{u,T(v,w),h} + \nabla^2_{u,R(v,w)h} + \nabla_{u,R(v,w)h} = 0.
\]

Calculating relations of this kind manually is not an easy task even for relatively small degrees.

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