Optimum transfer characteristics of the Tesla transformer on the first and second half-waves of output voltage

V Kladukhin¹ and S Khramtsov

Institute of Electrophysics, UB, RAS, 106 Amundsen Street, Ekaterinburg, 620016, Russia

¹E-mail: laepr@iep.uran.ru

Abstract. The elements of the theory of the Tesla transformer are stated, the exact solution of the equations of the dynamics of currents and voltages in the transformer circuits through the generalized parameters of the circuits (Q-factors of the primary and secondary circuits, the coupling coefficient of the circuits and mismatching factor of the natural resonance frequencies of the circuits) is given, under the assumption of their constancy. The optimal transfer characteristics of the processes of charging the capacitive storage of the secondary circuit of the transformer on the first and second half-waves are given, demonstrating the capabilities of the Tesla transformer.

1. Introduction

Tesla transformers are widely used in the design of sources of high-voltage pulses (microsecond and nanosecond duration) based on the charge of capacitive storage [1, 2]. Many publications are devoted to the analysis of the operating modes of the Tesla transformer, for example [3-6], however, the transfer characteristics of the transformer, in all publications known to the authors, were examined using various kinds of approximations. This paper presents the exact solution of the equations of the dynamics of currents and voltages in the transformer circuits under the assumption that the parameters of the circuits and their magnetic coupling are constant. Numerical values of the optimal transfer characteristics of the processes of charging the capacitive storage of the secondary circuit of the transformer at the first and second half-waves are given for the range of practically attainable parameter values. In order to generalize the results obtained, the dynamics of currents and voltages in the circuits, as well as the transfer characteristics of the transformer are presented through the generalized parameters of the transformer and the normalized values of the parameters of its primary circuit.

2. Model of Tesla transformer

Tesla transformer, equivalent circuit is shown in figure 1, are two inductively coupled oscillatory circuits used to transfer energy from the capacitive storage of the first circuit (C₁) to the capacitive storage of the second circuit (C₂). It is assumed that in the initial state (τ = 0) the switch S₁ is open, C₁ is charged (u₁ = u₀), and C₂ is discharged (u₂ = 0).

The process of energy transfer from C₁ to C₂ begins when the switch S₁ is closed and the current (i₁) appears in the primary and current (i₂) in the secondary circuits. The transfer of energy to the load
is started by closing the switch $S_2$ at the moment of the completion of the $C_2$ charge. It is assumed that the $C_2$ charge time is much longer than its discharge time to the load.

![Figure 1. Equivalent circuit of Tesla transformers.](image)

The inductance of the primary circuit ($L_1$), in addition to the inductance of the transformer coil, includes the inductance of the connecting conductors, capacitive storage and the switch. The inductance of the secondary circuit ($L_2$), in addition to the transformer coil, includes the inductance of the connecting wires and the capacitive storage. Resistance of active losses in the primary circuit ($R_1$) includes heating losses in the capacitive storage, switch, connecting wires, contacts, primary coil of the transformer and magnetic circuits.

The inductance of the secondary circuit ($L_2$), in addition to the transformer coil, includes the inductance of the connecting wires and the capacitive storage.

Resistance of active losses in the primary circuit ($R_1$) includes heating losses in the capacitive storage, switch, connecting wires, contacts, primary coil of the transformer and magnetic circuits.

The magnetic coupling of the circuits is determined by the mutual inductance ($M$).

The dynamics of the interaction of the Tesla transformer circuits are referred by the equations:

$$u_1(t) + R_1i_1(t) + L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} = 0,$$

$$u_2(t) + R_2i_2(t) + L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt} = 0,$$

$$u_1(t) = \frac{1}{C_1}\int_0^t i_1(\tau)d\tau,$$

$$u_2(t) = \frac{1}{C_2}\int_0^t i_2(\tau)d\tau.$$

Assuming the parameters of the circuits $R_1, L_1, C_1, R_2, L_2, C_2$ and their magnetic coupling $M$ as constants, the system of equations (1) can be reduced to a homogeneous differential equation of the fourth degree with respect to any of the variables $u_1, i_1, u_2, i_2$. To increase the generality, it is advisable to express the coefficients of the differential equation through the generalized parameters of the circuits: the quality factors ($Q_1 = \sqrt{L_1/C_1}/R_1, Q_2 = \sqrt{L_2/C_2}/R_2$), the coupling factor ($k = M/\sqrt{L_1L_2}$), the natural resonance frequencies ($\omega_{01} = 1/\sqrt{L_1C_1}, \omega_{02} = 1/\sqrt{L_2C_2}$), their mismatching factor ($\alpha = \omega_{02}/\omega_{01}$) and the transformation coefficient $n = \sqrt{L_2/L_1}$. In this case, the characteristic equation will take the form:

$$(1 - k^2)x^4 + \omega_{01}\left(\frac{1}{Q_1} + \frac{\alpha}{Q_2}\right)x^3 + \omega_{01}^2\left(1 + \alpha^2 + \frac{\alpha}{Q_1Q_2}\right)x^2 + \omega_{01}^3\left(\frac{\alpha^2}{Q_1} + \frac{\alpha}{Q_2}\right)x + \omega_{01}^4\alpha^2 = 0. \quad (2)$$

In order to obtain generalized results, it is advisable to investigate the dynamics of processes in the circuits on the normalized parameters of the primary circuit, assuming: $u_1 = 1, C_1 = 1, L_1 = 1$, then $\omega_{01} = 1, \omega_{02} = \alpha, C_2 = 1/(n^2\alpha^2)$.

In this case, the dynamics of currents and voltages in the circuits are referred by the equations:
$$u_1(\tau) = \sum_{j=1}^{4} A_i \exp(x_i \tau), \quad i_1(\tau) = \sum_{j=1}^{4} x_i A_i \exp(x_i \tau), \quad (3)$$
$$u_2(\tau) = \sum_{j=1}^{4} B_i \exp(x_i \tau), \quad i_2(\tau) = \frac{1}{n^2 \alpha^2} \sum_{j=1}^{4} x_i B_i \exp(x_i \tau), \quad (4)$$

where the constants $A_k, B_k, k = 1,4$ determined from the initial conditions have the form:

$$A_k = \prod_{j=1}^{4} x_j - 1 - k^2 \left( \sum_{j=1}^{4} x_j + \frac{1}{Q_1(1-k^2)} + \frac{k^2 \alpha^2}{Q_2(1-k^2)} \right) \prod_{j=1}^{4} (x_k - x_j)^{-1}, \quad (5)$$

$$B_k = \frac{k \alpha^2}{1-k^2} \left( \sum_{j=1}^{4} x_j + \frac{1}{Q_1(1-k^2)} + \frac{\alpha^2}{Q_2(1-k^2)} \right) \prod_{j=1}^{4} (x_k - x_j)^{-1}. \quad (6)$$

Since with the real roots of the characteristic equation, the voltage build-up on the secondary capacitive storage has an aperiodic form, at which the transfer coefficient tends to 25%, for energy transfer, as a rule, the coupling coefficients of the circuits are used that provide the complex roots of the characteristic equation. In this case, the transfer of energy from the capacitive storage of the primary circuit to the capacitive storage of the secondary circuit occurs in the form of oscillations and the completion of the process of charging the capacitive storage can be considered for the first, second, third, etc. extremum of voltage. It is advisable to calculate the roots of the characteristic equation $x_1, x_2, x_3, x_4$ by the Ferrari method.

For ease of display, it is advisable to scale the values of currents and voltages in the secondary circuit using the transformation ratio $i_s = n_i, u_s = u_i / n$.

The efficiency of the transformer is characterized by the transfer coefficient

$$\eta = C_2 \left[ u_2(T) \right]^2 / C_1 u_2^2 \left[ u_2(T) \right]^2 / \alpha^2, \quad \text{where} \quad T \quad \text{is the moment in time corresponding to the completion of the charging process.}$$

The potentially achievable efficiency of the transformer, with a known quality factor of the circuits, is given by the coefficient of optimal (maximum) energy transfer:

$$\eta_{opt} = Q_1, Q_2, T = \max_{\alpha, \beta} \eta(\alpha, \beta, Q_1, Q_2) = \eta(\alpha_{opt} (Q_1, Q_2), k_{opt} (Q_1, Q_2), Q_1, Q_2, T) \quad (7)$$

3. The one-half-wave charging process

The one-half-wave charging process continues until the first extremum of the voltage of the secondary circuit capacitance charge appears (d$u_2^*$ / dt = 0). The dynamics of normalized currents and voltages in the transformer circuits and the dynamics of energy localization in the elements of the Tesla transformer circuits with the parameters: $Q_1 = 10, Q_2 = 40, k = 0.94, \alpha = 1.05$, in a one-half-wave charging process, is illustrated in figure 2, where: $W_{C1}$ is the energy in a capacitive storage $C_1, W_{C2}$ is the energy in a capacitive storage $C_2, W_L$ is the energy in the inductances of the circuits, $W_R$ is the energy losses on the active resistances of the circuits.

The optimal coefficient of energy transfer to the capacitive storage of the secondary circuit at different $Q$-factors of the circuits is shown in figure 3 - on the left, on the right is shown the normalized energy remaining in the inductance of the primary circuit at the time of completion of the optimal charging process. Figure 4 shows the values of the coupling coefficients and the mismatching coefficients of the circuits that provide the optimal energy transfer coefficient. The optimal energy transfer coefficients are determined for the physically achievable range of $Q$-factors of the circuits: $Q_1 = 1 \div 100, Q_2 = 1 \div 100$. 

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Figure 2. Dynamics of currents, voltages in circuits and energies on circuit elements in a one-half-wave charging process, on the left - currents and voltages, on the right – energies.

Figure 3. Characteristics of the optimal energy transfer process in a one-half-wave charging process, on the left - the maximum energy transfer coefficient, on the right - the energy stored in the inductance at the time of completion of the charge.

Figure 4. Optimal coupling coefficients and frequency mismatching coefficients of the circuits in a one-half-wave charging process, on the left - the coupling coefficient, on the right - the mismatching coefficient.
4. The two-half-wave charging process

The two-half-wave charging process continues until the second voltage extremum appears on the capacitance $C_2$. The dynamics of normalized currents and voltages in the transformer circuits and the dynamics of energy localization in the elements of the Tesla transformer circuits with the parameters: $Q_1 = 10$, $Q_2 = 40$, $k = 0.59$, $\alpha = 1.02$, in a two-half-wave charging process, is illustrated in figure 5, where: $W_{C1}$ is the energy in a capacitive storage $C_1$, $W_{C2}$ is the energy in a capacitive storage $C_2$, $W_L$ is the energy in the inductances of the circuits, $W_R$ is the energy losses on the active resistances of the circuits.

![Figure 5](image5.png)

**Figure 5.** Dynamics of currents, voltages in circuits and energies on circuit elements in a two-half-wave charging process, on the left - currents and voltages, on the right – energies.

In figure 6 shows the optimal coefficient of energy transfer to the capacitive storage of the secondary circuit and the normalized energy remaining in the inductance of the primary circuit at the time of completion of the optimal charging process. Figure 7 shows the values of the coupling coefficients and the mismatching coefficients of the circuits that provide the optimal energy transfer coefficient.

![Figure 6](image6.png)

**Figure 6.** Characteristics of the optimal energy transfer process in a two-half-wave charging process, on the left - the maximum energy transfer coefficient, on the right - the energy stored in the inductance at the time of completion of the charge.
5. Conclusion

From the above graphs it can be seen that with the same $Q$-factor of the circuits, the optimal transfer coefficient is higher for a two-half-wave charging process, especially with a low $Q$-factor of one of the transformer circuits, which is due to the absence of residual energy in the inductance of the primary circuit at the end of the charging process. However, the two-half-wave charging process is characterized by a significantly longer duration of the charging process and the range of voltage oscillations on the secondary circuit capacitance $u_2(T_1) + u_2(T_2)$, where $T_1$, $T_2$ are the moments of time corresponding to the first and second voltage extremum), which at high charging voltages can impose excessively high requirements for insulation.

In addition, it should be remembered that the above analysis of the processes in the Tesla transformer was carried out under the assumption that the $R$, $L$, $C$ and $M$ parameters of the circuits are constant, i.e. the power of active losses ($P_A$) (in the switch, in connecting conductors, in contacts, in capacitors, in conductors of transformer coils and in magnetic systems), divided to the square of the current, remains constant during the entire charging process ($P_A/i^2 \equiv \text{const}$), and the inductances of the circuits and their magnetic coupling do not depend on the magnitude of the magnetic fluxes. Unfortunately, it is not possible to fully fulfill these conditions, however, the use of fast switches and magnetic systems based on air or open magnetic circuits makes it possible to approach the fulfillment of this condition.

6. Appendices

A variant of writing the roots of an algebraic equation of the fourth degree used by the authors when performing the work:

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$  \quad (A.1)

$$x_{1,2} = -\frac{1}{2}\left(\frac{a}{2} - \alpha\right) \pm \left[\frac{1}{4}\left(\frac{a}{2} - \alpha\right)^2 - \left(\frac{y_0}{2} - \beta\right)^2\right]^{1/2},$$ \quad (A.2)

$$x_{3,4} = -\frac{1}{2}\left(\frac{a}{2} + \alpha\right) \pm \left[\frac{1}{4}\left(\frac{a}{2} + \alpha\right)^2 - \left(\frac{y_0}{2} + \beta\right)^2\right]^{1/2},$$ \quad (A.3)

where
The sign \( \beta \) is chosen taking into account the fulfillment of the equation \( 2\alpha\beta = \frac{ay_0}{2} - c \), and the value of \( y_0 \) is calculated by the formula:

\[
y_0 = z + \frac{b}{3},
\]

where

\[
z = z_1 + z_2, \quad z_1 = \left( \frac{q}{2} + \left[ \frac{q^2}{2} + \left( \frac{p}{3} \right)^{3/2} \right]^{1/3} \right)^2, \quad z_2 = \frac{3}{z_1},
\]

\[
p = -\frac{b^2}{3} + ac - 4d, \quad q = -\frac{2b^3}{27} + \frac{b(ac - 4d)}{3} - \left[ d \left( a^2 - 4b \right) + c^2 \right].
\]