Direct Derivation of Scaling Relation of Prepotential in
$N = 2$ Supersymmetric $G_2$ Yang-Mills Theory

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Abstract

In contrast with the classical gauge group cases, any method to prove exactly the
scaling relation which relates moduli and prepotential is not known in the case of
exceptional gauge groups. This paper provides a direct method to establish this
relation by using Picard-Fuchs equations. In particular, it is shown that the scaling
relation found by Ito in $N = 2$ supersymmetric $G_2$ Yang-Mills theory actually holds
exactly.

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I. INTRODUCTION

It would be one of the greatest discoveries in 1990s that the holomorphic structure of the low energy effective prepotential of $N = 2$ supersymmetric Yang-Mills theory in four dimensions was actually related with the moduli space of a Riemann surface which possesses singularities when charged particles become massless. According to this mechanism found by Seiberg and Witten, the effective theory is parameterized by the vacuum expectation value of scalar components $\phi$ of $N = 1$ chiral multiplet, which can be identified with periods of a certain meromorphic differential on the Riemann surface, and accordingly the prepotential can be determined exactly also including instanton corrections. For example, for classical gauge group cases, the prepotential is known to be dictated by hyperelliptic curves, and instanton corrections to the prepotential obtained from these curves showed a good agreement with the so-called instanton calculus, a pure field theoretical method.

These hyperelliptic curves were also derived from a very different viewpoint, relation to integrable systems. In the language of integrable system, these hyperelliptic curves coincide with the spectral curves (the characteristic equation for Lax matrix) and the periods can be interpreted as the action integrals. This interpretation explains why the effective theory is solvable, and from this fact, it may be natural to expect that also for exceptional gauge group cases the relevant curves are given by hyperelliptic curves. In fact, several hyperelliptic curves for such cases were constructed but, unfortunately, in general, the spectral curves from integrable systems for exceptional gauge group cases can not be transformed into hyperelliptic form. For instance, the curve for $G_2$ gauge theory is related to the $(G_2^{(1)})^\vee$ Toda system and is given in the form

$$3 \left(z - \frac{\mu}{z}\right)^2 - x^8 + 2ux^6 - \left[u^2 + 6 \left(z + \frac{\mu}{z}\right)\right] x^4 + \left[v + 2u \left(z + \frac{\mu}{z}\right)\right] x^2 = 0,$$

where $x$ is the eigenvalue of the Lax operator matrix, $z$ is the spectral parameter, $u$ and $v$ are gauge invariant Casimirs called moduli (of the effective theory) and $\mu$ is a parameter which leads to the dynamical scale, but obviously this curve can not be transformed into hyperelliptic form. Of course, for these two formulations, which one is better must be decided by a comparison of instanton corrections to prepotential with that from instanton calculus. According to the result, there is a manifest difference between prepotentials from these two curves and only that from (1.1)
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can survive! Similar result follows also in E\textsubscript{6} gauge theory\textsuperscript{27} and therefore the substance of complex curves relevant to gauge theories is believed as a spectral curve of integrable system.

As a characteristic common problem concerning to these gauge theories with exceptional Lie groups, we can mention that the exact establishment of scaling relation\textsuperscript{8,12,28} of prepotential \( \mathcal{F} \) whose general form is typically represented in the form

\[
\sum_{i=1}^{g} a_i \frac{\partial \mathcal{F}}{\partial a_i} - 2\mathcal{F} = \beta \langle \text{Tr} \phi^2 \rangle,
\]

(1.2)

where \( g \) is the rank of the gauge group, \( a_i \) are the periods and \( \beta \) is the coefficient of one-loop beta function, is very hard. Although the formula (1.2) has been checked in the standpoint of instanton calculus in the SU(2) gauge theory\textsuperscript{24,25} as for a general proof of this formula for theories with any classical gauge groups, we know the method based on Whitham hierarchy\textsuperscript{35}. In such cases, the curves are hyperelliptic, and thus the verification of (1.2) can be done directly, but in contrast with these cases, for theories with exceptional gauge groups, similar discussions do not exist because of too complicated singularity structure of the spectral curves. For instance, in the case of the G\textsubscript{2} theory, we can explicitly see this from (1.1), which is actually an eight cover of \( z \)-plane.

On one hand, of course, there are strong supports for this formula (1.2) in exceptional gauge group cases and the validity of (1.2) was explicitly checked by first several instanton process levels with the help of explicit solutions to Picard-Fuchs equations\textsuperscript{26,27,36} but this does not mean that (1.2) holds exactly. Then, also in the theories with exceptional groups, does (1.2) hold exactly? To answer this question is the subject of this paper.

In this paper, we give a method to verify (1.2) exactly for G\textsubscript{2} gauge theory, although the method itself is applicable for all theories with any classical and exceptional gauge groups with or without massive hypermultiplets. Our starting point is to consider the differentiated versions of (1.2), defined by

\[
W = \sum_{i=1}^{2} (a_i \partial_u a_{D_i} - a_{D_i} \partial_u a_i), \quad w = \sum_{i=1}^{2} (a_i \partial_v a_{D_i} - a_{D_i} \partial_v a_i),
\]

(1.3)

and seek differential equations for (1.3). As a matter of fact, this can be proceeded by considering Picard-Fuchs equations, but since the Picard-Fuchs equations with multiple moduli are usually realized as a set of partial differential equations\textsuperscript{37,38} actually such equations do not have any advantages for a study of scaling relation of prepotential (in higher rank gauge groups), although the derivation
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using partial differential form of Picard-Fuchs equations was tried in the case of SU(3). However, if Picard-Fuchs equations represented by single kind of moduli derivatives can be found, we can easily construct an ordinary differential equation for respective quantities in (1.3). In addition, the basis of solutions to such ordinary differential equation can be uniquely fixed, so as the result, we can determine the right hand side of (1.2) by taking an appropriate “initial condition”.

II. PICARD-FUCHS EQUATIONS IN G\(_2\) THEORY

A. G\(_2\) Picard-Fuchs equations

To begin with, let us recall the Seiberg-Witten meromorphic differential on the G\(_2\) curve (1.1) given by

\[ \lambda = x \frac{dz}{z} \]  

(2.1)

and the definition of periods

\[ a_i = \oint_{\alpha_i} \lambda, \quad a_{Di} = \oint_{\beta_i} \lambda, \quad i = 1, 2, \]  

(2.2)

where \( \alpha_i \) and \( \beta_i \) are the canonical cycles on (1.1). Some of the properties of these period integrals were discussed by Masuda et al.\[40\] The classical relation among periods and moduli is given by\[24\]

\[ u = a_2^2 + (a_1 - a_2)^2 + (a_1 - 2a_2)^2, \quad v = a_2^2(a_1 - a_2)^2(a_1 - 2a_2)^2. \]  

(2.3)

These are invariant under the action of the Weyl group of G\(_2\)

\[ (a_1, a_2) \rightarrow (3a_2 - a_1, a_2), \quad (a_1, a_2) \rightarrow (a_1, a_1 - a_2). \]  

(2.4)

Then, the Picard-Fuchs equations can be given in the form\[24\]

\[ \left[ \frac{2(720u^2\mu + 2u^3v - 27v^2)}{-uv + 24\mu} \partial_u^2 + \frac{4(256u^4\mu - 3u^2v^2 - 720uv\mu + 13824\mu^2)}{-uv + 24\mu} \partial_u \partial_v \right. \]

\[ - \frac{6(-256u^3\mu + 96v\mu + 5uv^2)}{-uv + 24\mu} \partial_v - 1 \bigg] \lambda = 0, \]

\[ \left. \left[ \frac{1}{3}(8u^2v - 108v^2 + 2880u^2\mu)\partial_v^2 + \frac{1}{3}(8u^4 - 72uv + 6912\mu)\partial_u\partial_v + (4u^3 - 24v)\partial_v - 1 \right] \lambda = 0. \]  

(2.5)
The reader who wishes to know more details of Picard-Fuchs equations associated with non-hyperelliptic curves may consult the work of Isidro.

From these equations, we can construct differential equations satisfied by \((1.3)\), but such differential equations are not helpful for a direct proof of scaling relation because they are partial differential equations. For this reason, we seek a more convenient form of Picard-Fuchs equations. A candidate is an ordinary differential form because the right hand sides of respective quantities in \((1.3)\) are written in terms of only single variable derivative. Therefore, if Picard-Fuchs equations can take ordinary differential forms, it would be easy to obtain ordinary differential equations for \(W\) and \(w\). In addition to this, since they are ordinary differential equations, we can uniquely fix the basis of solution space and by this it becomes possible to verify \((1.2)\).

However, sadly, since the direct derivation of Picard-Fuchs equations in terms of single variable derivatives from the original period integral requires much labour, instead, let us try to derive such equations from \((2.5)\).

**B. Ordinary differential form of \(G_2\) Picard-Fuchs equations**

First, let us rewrite \((2.5)\) in the form

\[
\left[\partial_u^2 - c_1 \partial_u \partial_v - c_2 \partial_v - c_3\right] \lambda = 0, \quad \left[\partial_v^2 - d_1 \partial_u \partial_v - d_2 \partial_v - d_3\right] \lambda = 0. \tag{2.6}
\]

If there is any differential equation satisfied by \(\lambda\), it must be a linear combination of the two equations in \((2.5)\) and their differentiations. We would like to make an ordinary differential equation in terms of single moduli derivatives, e.g., \(\partial_u \lambda, \partial_u^2 \lambda\), etc, but in order to obtain such equation from \((2.6)\) by repeating differentiations, mixed derivatives and other moduli derivatives like \(\partial_u \partial_v \lambda\) or \(\partial_v \lambda\) must be eliminated. These irrelevant derivatives can be dropped by representing them in terms of \(\partial_u \lambda, \partial_u \partial_v \lambda\) and \(\partial_v \lambda\).

For example, regarding \(\partial_u^2 \partial_v = \partial_v (\partial_u^2)\), we get

\[
\partial_u^2 \partial_v \lambda = \left[\partial_v c_1 \partial_u \partial_v + \partial_v c_2 \partial_v + \partial_v c_3 + c_1 \partial_u (\partial_v^2) + c_2 \partial_v^2 + c_3 \partial_v\right] \lambda \tag{2.7}
\]

and further substituting \(\partial_v^2 \lambda\) from \((2.6)\) into this expression, we can obtain

\[
D\partial_u^2 \partial_v \lambda = \left[c_1 d_3 \partial_u + (c_3 + c_2 d_2 + \partial_v c_2 + c_1 \partial_u d_2) \partial_v + (c_2 d_1 + c_1 d_2 + \partial_v c_1 + c_1 \partial_u d_1) \partial_u \partial_v + (c_2 d_3 + \partial_v c_3 + c_1 \partial_u d_3)\right] \lambda, \tag{2.8}
\]
where

\[ D = 1 - c_1 d_1. \]  

(2.9)

In a similar manner, we can arrive at

\[
D \partial_u \partial_v^2 \lambda = \left[ d_3 \partial_u + (c_3 d_1 + c_2 d_1 d_2 + d_1 \partial_v c_2 + \partial_u d_2) \partial_v \\
+(c_2 d_1^2 + d_2 + \partial_u d_1 + d_1 \partial_v c_1) \partial_v \partial_v + (c_2 d_1 d_3 + \partial_u d_3 + d_1 \partial_v c_3) \right] \lambda. 
\]

(2.10)

With these in mind, eliminating \( \partial_u \partial_v \lambda, \partial_u \partial_v^2 \lambda \) and \( \partial_u \partial_v^2 \lambda \), we can obtain the fourth-order ordinary differential equation satisfied by \( \lambda \)

\[
\left[ \partial_u^4 - \frac{1}{\Delta} \left( \bar{c}_3 \partial_u^3 + \bar{c}_2 \partial_u^2 + \bar{c}_1 \partial_u + \bar{c}_0 \right) \right] \lambda = 0, 
\]

(2.11)

where we have denoted only the equation associated with \( u \)-derivatives and the coefficients are given by

\[
\begin{align*}
\Delta &= (-16u^6v^2 + 216u^3v^3 - 729v^4 + 1024u^8\mu - 14976u^5v\mu + 54432u^2v^2\mu + 421632u^4\mu^2 \\
&\quad -2985984u^3v^2\mu^2 + 47775744\mu^3)(648u^5v^5 - 8505u^2v^6 - 86016u^7v^3\mu + 1145664u^4v^4\mu \\
&\quad -326592u^9v\mu + 2097152u^6v^2\mu^2 - 24625152u^6v^2\mu^2 - 69672960u^3v^3\mu^2 - 24634368v^4\mu^2 \\
&\quad -163577856u^8\mu^3 + 2601123840u^5v^5\mu + 4824354816u^2v^2\mu^3 - 7270406536u^4\mu^4 \\
&\quad -88098471936uv\mu^4 + 1091580198912\mu^5), \\
\bar{c}_0 &= 4(-648u^7v^7 + 18711u^4v^8 + 120192u^9v^5\mu - 3530304u^6v^6\mu + 1191186u^3v^7\mu + 2480058v^8\mu \\
&\quad -11730944u^{11}v^3\mu^2 + 291824640u^8v^4\mu^2 - 158824800u^5v^5\mu^2 - 356682204u^2v^6\mu^2 \\
&\quad +100663296u^{13}v^\mu^3 - 2121007104u^{10}v^2\mu^3 - 42455384064u^7v^3\mu^3 + 51853167360u^4v^4\mu^3 \\
&\quad +53651973888u^5v^5\mu^3 - 13086228480u^{12}\mu^4 + 348024471552u^9v\mu^4 + 3595729895424u^6v^2\mu^4 \\
&\quad -4366362599424u^3v^3\mu^4 - 665208557568v^4\mu^4 - 13420581617664u^8\mu^5 \\
&\quad -115104252690432u^5v\mu^5 + 106830134820864u^2v^2\mu^5 + 1876392727805952u^4\mu^6 \\
&\quad -2191620144365568uv\mu^6 + 16936958366318592\mu^7), \\
\bar{c}_1 &= 4(-1944u^8v^7 + 44469u^5v^8 + 376320u^{10}v^5\mu - 8745408u^7v^6\mu + 3796632u^4v^7\mu - 2480058uv^8\mu \\
&\quad -25427968u^{12}v^3\mu^2 + 563576832u^9v^4\mu^2 - 206437248u^6v^5\mu^2 + 1138989600u^3v^5\mu^2 \\
&\quad -192735936v^7\mu^2 + 402653184u^{14}v\mu^3 - 7795113984u^{11}v^2\mu^3 - 41644523520u^8v^3\mu^3
\end{align*}
\]
\begin{align*}
-30988541952u^5v^4\mu^3 + 16587887616u^2v^5\mu^3 - 41875931136u^{13}\mu^4 + 102004634412u^{10}v\mu^4 \\
+2862297907200u^7v^2\mu^4 - 3324905127936u^4v^3\mu^4 - 69173305344uv^4\mu^4 \\
-3354986597248u^9\mu^5 - 38241489125376u^6v\mu^5 + 138188377423872u^3v^2\mu^5 \\
+232190115840v^3\mu^5 + 838333592764416u^5\mu^6 - 3213346090254336u^2v\mu^6 \\
+24521257588359168u\mu^7), \\
-\bar{c}_2 = 2(-47952u^9v^7 + 1093014u^6v^8 - 1594323u^3v^9 + 6200145v^{10} + 9277440u^{11}v^5\mu \\
-215229312u^8v^6\mu + 41115600u^5v^7\mu - 1285614828u^2v^8\mu - 637534208u^{13}v^3\mu^2 \\
+14181875712u^{10}v^4\mu^2 - 23978446848u^7v^5\mu^2 + 62058871872u^4v^6\mu^2 + 115596211968uv^7\mu^2 \\
+9663676416u^{15}v^3\mu^3 - 180703199232u^{12}v^2\mu^3 - 949095235584u^9v^3\mu^3 \\
+1489634758656u^6v^4\mu^3 \\
-7237021925376u^3v^5\mu^3 - 4614778552320v^6\mu^3 - 96314616128u^{14}\mu^4 \\
+23697146511360u^{11}v\mu^4 + 48477813866496u^8v^2\mu^4 - 7529438141024u^5v^3\mu^4 \\
+245288540749824u^2v^4\mu^4 - 763297561313280u^{10}v^5\mu^5 - 174881136181248u^7v\mu^5 \\
-4532971954765824u^4v^2\mu^5 - 2597743016017920uv^3\mu^5 + 14032107516985344u^6v\mu^6 \\
+301566403503194112u^2v\mu^6 + 88771668096319488v^2\mu^6 - 5754950530929524736u^2\mu^7), \\
-\bar{c}_3 = 2(-36288u^{10}v^7 + 750384u^7v^8 - 2493180u^4v^9 - 6200145uv^{10} + 7090176u^{12}v^5\mu \\
-149257728u^9v^6\mu + 548581248u^6v^7\mu + 1031074272u^3v^8\mu - 119042784v^9\mu \\
-446693376u^{14}v^3\mu^2 + 9473753088u^{11}v^4\mu^2 - 34651597824u^8v^5\mu^2 - 59910223104u^5v^6\mu^2 \\
-118816035840u^2v^7\mu^2 + 7516192768u^{16}v^3\mu^3 - 136549761024u^{13}v^2\mu^3 - 140014780416u^{10}v^3\mu^3 \\
+45325409173832u^7v^4\mu^3 + 9634653609984u^4v^5\mu^3 + 1729090768896uv^6\mu^3 \\
-670014898176u^{15}\mu^4 + 16657158242304u^{12}v\mu^4 - 2880983531200u^9v^2\mu^4 \\
-573860068982784u^6v^3\mu^4 - 146014044291072u^3v^4\mu^4 + 49247523569664v^5\mu^4 \\
-446693778653184u^{11}v\mu^5 + 2318736493117440u^8v\mu^5 + 27114473757081600u^5v^2\mu^5 \\
-1492703816712192u^2v^3\mu^5 - 21106281041362944u^7v\mu^6 - 739515240667938816u^4v\mu^6 \\
+1987695932645376uv^2\mu^6 + 8787962215324975104u^3\mu^7 \\
-1154955987665289216v\mu^7). \\
\end{align*}
Of course, a similar equation in terms of only \( v \)-derivatives can be found to follow by repeating the same algorithm (see Appendix). Below, we discuss only the case for \( u \)-derivatives.

### III. DIFFERENTIAL EQUATION FOR SCALING RELATION

#### A. Ordinary differential equation for \( W \)

We can now construct an ordinary differential equation satisfied by \( W \). To see this, let us define

\[
W_{ij} = \sum_{k=1}^{2}(\partial^i u a_k \partial^j a_{D_k} - \partial^j u a_k \partial^i a_{D_k}).
\]  

(3.1)

Similar quantities are often used for a calculation of Yukawa couplings in the context of mirror symmetry. Notice that \( W \) itself is given by \( W = W_{01} \). From (3.1), we can derive some relations among various \( W_{ij} \)

\[
\begin{align*}
W' &= W_{02}, & W'' &= W_{12} + W_{03}, & W'_{12} &= W_{13}, & W''' &= C_1 W + C_2 W' + C_3 W_{03} + 2W_{13}, \\
W_{03} - C_3 W_{03} &= C_1 W + C_2 W' + W_{13}, & W_{13} - C_3 W_{13} &= -C_0 W + C_2 W_{12} + W_{23}, \\
W'_{23} - C_3 W_{23} &= -C_0 W' - C_1 W_{12}, & C_i &= \frac{\tilde{c}_i}{\Delta},
\end{align*}
\]

(3.2)

where \( \partial/\partial u \).

From (3.2), we can construct a sixth-order ordinary differential equation satisfied by \( W \)

\[
\left[ \partial^5 u + \tilde{C}_4 \partial^4 u + \tilde{C}_3 \partial^3 u + \tilde{C}_2 \partial^2 u + \tilde{C}_1 \partial u + \tilde{C}_0 \right] \partial u W = 0,
\]

(3.3)

where \( \tilde{C}_i \) are very complicated and extremely lengthy rational functions in moduli and the scaling parameter \( \mu \).

#### B. Basis of solutions

In order to obtain the basis of solutions to (3.3), especially, to determine the indicial indices, by taking a Frobenius algorithm at the weak coupling region, it would be sufficient to consider (3.3) with \( \mu = 0 \). This is because the weak coupling solutions can be represented also in a series of \( \mu \) and the only the lowest order terms of this series are relevant in the determination of indicial indices. Then (3.3) turns to
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\[
\begin{align*}
&\left[u^5(4u^3 - 27v)^4(608u^6 + 32694u^3v - 192465v^2)\partial_u^5 \\
&+ 9u^4(4u^3 - 27v)^3(7296u^9 + 463280u^6v - 1645836u^3v^2 - 5196555v^3)\partial_u^4 \\
&+ 2u^3(4u^3 - 27v)^2(1162496u^{12} + 82291704u^9v - 292007736u^6v^2 - 238968387u^3v^3 \\
&- 2806139700u^4v)\partial_u^3 + 12u^2(4u^3 - 27v)(2687360u^{15} + 20558136u^{12}v \\
&- 1225216692u^9v^2 + 1404464913u^6v^3 + 6080288652u^3v^4 - 37882885950v^5)\partial_u^2 \\
&+ 8u(20422720u^{18} + 1666413804u^{15}v - 16557855750u^{12}v^2 \\
&+ 3377521721u^9v^3 + 1420502427u^6v^4 + 879666475221u^3v^5 - 3068513761950v^6)\partial_u \\
&+ 40(1337600u^{18} + 125474352u^{15}v - 812719908u^{12}v^2 + 133865410u^9v^3 + 7591437855u^6v^4 \\
&- 162446987646u^3v^5 + 613702752390v^6)\partial_u^W = 0,
\end{align*}
\]

(3.4)

and therefore we get the following set of indicial indices

\[\nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = (1, 2, 3, 5, 8)\]  

(3.5)

for

\[W' = u^\nu \tilde{W}.\]  

(3.6)

According to Frobenius algorithm, in general, there are two types of solutions due to the difference of two indices, one of which is a regular series and the other is logarithmic. In fact, the reader may be familiar to the following well-known fact:

- Suppose that \(\nu_1\) and \(\nu_2\) are the indicial indices to a second-order linear ordinary differential equation (unknown \(y\) and variable \(x\)) with regular singularities, say, at \(x = 0\). Then the general solution is given in the form

1. \(y = c_1 x^{\nu_1} \sum_{k=0}^{\infty} A_k x^k + c_2 x^{\nu_2} \sum_{k=0}^{\infty} B_k x^k\), \((\nu_1 - \nu_2 \neq \text{integer})\)

2. \(y = c_1 x^{\nu_1} \sum_{k=0}^{\infty} A_k x^k + c_2 \left[ c x^{\nu_1} \sum_{k=0}^{\infty} A_k x^k \ln x + x^{\nu_2} \sum_{k=1}^{\infty} B_k x^k \right]\), \((\nu_1 - \nu_2 = \text{integer})\),

where \(c_i\) are integration constants, \(c\) is a constant to be determined according to the difference of the indices (typically, \(c = 1\) when \(\nu_1 = \nu_2\), \(A_k\) and \(B_k\) are independent of \(x\). In both cases, when \(\nu_1 \neq \nu_2\), we can assume \(\nu_2 < \nu_1\) without loss of generality.
Summarizing this, we can say that in any case the solution can be factored by $x^{\nu_2}$ associated with the smaller index. Similar one to this fact also holds for a higher rank ordinary differential equation (of course, in this case, the solution may involve power of logarithm due to the difference of indicial indices). Therefore, for (3.3), $W'$ takes the form

$$W' = u^{\nu_1} \sum_{i=1}^{5} \rho_i u^{\nu_i - \nu_1} f_i(u, v, \mu),$$

where $\rho_i$ are some constants and $f_i$ are functions whose lowest order in expansion is logarithm or function which is independent of $u$. Integrating this gives the following function form

$$W = c(v) + uf,$$

where $c(v)$ is a function which may depend on $v$, by using a function $f$ whose lowest order in expansion is logarithm or function independent of $u$.

In order to make a contact with the weak coupling behavior, we must impose some “initial condition”. For this purpose, let us recall the definition of $W$ in (1.3). Substituting the solutions to Picard-Fuchs equations into (1.3), we will be able to compare it with the right hand side of (3.8). Of course, as a matter of fact, since the function form of $W$ is now uniquely determined as in (3.8), it is not necessary to know the solutions to Picard-Fuchs equations at all order in $\mu$ and is enough to know them only at the lowest order level. In fact, proceeding in this manner with the aid of the weak coupling behavior of periods from (2.3) (or explicitly solving (3.4) or using Ito’s result), we can see that the second term of (3.8) is suppressed and $c(v) = i/(4\pi)$. This indicates that

$$\sum_{i=1}^{2} (a_i \partial_u a_{D_i} - a_{D_i} \partial_u a_i) = \frac{i}{4\pi}$$

holds exactly! Therefore, the scaling relation found by Ito in the $G_2$ gauge theory based on the spectral curve (1.1) is actually an exact expression.

We can conclude that the scaling relation

$$\sum_{i=1}^{2} (a_i \partial_v a_{D_i} - a_{D_i} \partial_v a_i) = 0$$

holds exactly by repeating a similar discussion.

IV. SUMMARY

In this paper, we have proved the scaling relation of prepotential of $G_2$ Yang-Mills theory by using ordinary differential form of Picard-Fuchs equations. The direct verification of the scaling
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Relation (when multiple moduli are included) is very complicated, but it would be needless to say that our direct method presented here can be applied for any gauge group cases. That is, when one wish to establish the scaling relations like (3.9) and (3.10):

1. Represent the Picard-Fuchs equations in terms of single moduli derivatives.
2. Consider the ordinary differential equation satisfied by scaling relation using the above Picard-Fuchs equations.
3. Fix the basis of solutions to this equation.
4. Compare the result with that from solutions to Picard-Fuchs equation.

Note that a direct derivation of (1.2) by Picard-Fuchs equations as a system of partial differential equations is more involved than our presentation here.

In contrast with the classical gauge group cases, there are many problems concerning exceptional gauge group cases and in order to get more insight into these exceptional gauge theories application of Whitham hierarchy is necessary. To to this is very hard because of the complicated singularity structure of the spectral curves, but, also the problem of scaling relation in these cases should be understood in this framework.

APPENDIX: ANOTHER SCALING RELATION

In this appendix, we show that the Picard-Fuchs equation in terms of only $v$-derivatives and the ordinary differential equation for $w$.

The Picard-Fuchs equation can be represented as the fourth-order equation and takes the form

$$
\begin{aligned}
-9(37u^8 - 126u^5v + 81u^2v^2 - 81216u^4\mu + 171072uv\mu + 36578304\mu^2) \\
+24(68u^{11} - 1422u^8v + 12393u^5v^2 - 37179u^2v^3 + 61776u^7\mu - 2220048u^4v\mu + 10707552uv^2\mu \\
+204166656u^3\mu^2 - 1169012736v\mu^2)\partial_v - 24(24u^{14} - 1004u^{11}v + 15885u^8v^2 - 108540u^5v^3 \\
+273375u^2v^4 + 45648u^{10}\mu - 1823040u^7v\mu + 19801584u^4v^2\mu - 70123968uv^3\mu + 40061952u^6\mu^2 \\
-724847616u^3v\mu^2 + 4938071040v^2\mu^2 - 32356122624u^2\mu^3)\partial_v^2 \\
-32(u^4 - 9uv + 864\mu)(32u^{10}v - 792u^7v^2 + 6804u^4v^3 - 19683uv^4 + 5760u^9\mu - 81216u^6v\mu - 69984u^3v^2\mu)
\end{aligned}
$$
Derivation of scaling relation

\[ +2519424v^3\mu + 12130560u^5\mu^2 - 94058496u^2v\mu^2 + 2149908480u\mu^3 )\partial_v^3 \]
\[ +16(u^4 - 9uv + 864\mu)^2(-16u^6v^2 + 216u^3v^3 - 729v^4 + 1024u^8\mu - 14976u^5v\mu + 54432u^2v^2\mu + 421632u^4\mu^2 \]
\[ -2985984uv\mu^2 + 47775744\mu^3 )\partial_v^4 \lambda = 0. \]  
(A1)

Then the differential equation for \( w \) with \( \mu = 0 \) is given by

\[ \left[ -6(68u^{21} - 1458u^{18}v + 107406u^{15}v^2 - 2216889u^{12}v^3 + 21611934u^9v^4 - 121168548u^6v^5 ight. \\
\[ +382637520u^3v^6 - 516560652v^7) + 6(24u^{24} - 5656u^{21}v + 298890u^{18}v^2 - 7376508u^{15}v^3 \\
\[ +104622435u^{12}v^4 - 906638346u^9v^5 + 4757459832u^6v^6 - 13937571666u^3v^7 + 17563062168v^8 )\partial_v \\
\[ +27(4u^3 - 27v)(u^3 - 9v)v(32u^{18} - 2216u^{15}v + 61950u^{12}v^2 - 883845u^9v^3 + 6896583u^6v^4 \\
\[ -28199178u^3v^5 + 47475396v^6 )\partial_v^2 + (4u^3 - 27v)^2(u^3 - 9v)^2v^2(392u^{12} - 15108u^9v + 226503u^6v^2 \\
\[ -1515348u^3v^3 + 3831624v^4)\partial_v^3 + 5(4u^3 - 27v)^3(u^3 - 9v)^3v^3(8u^6 - 144u^3v + 729v^2)\partial_v^4 \\
\[ +(4u^3 - 27v)^4(u^3 - 9v)^4v^4\partial_v^5 \right] W = 0. \]  
(A2)

The set of indicial indices for (A2) is found to be

\[ \nu = (1, 1/2, 1/2, -1/2, -1/2), \quad w = v^n \bar{w}. \]  
(A3)
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