SUPERSYMMETRIC GAUGE THEORIES 
AND 
GRAVITATIONAL INSTANTONS

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Various string theory realizations of three-dimensional gauge theories relate them to gravitational instantons, Nahm equations and monopoles. We use this correspondence to model self-dual gravitational instantons of $D_k$-type as moduli spaces of singular monopoles, find their twistor spaces and metrics.

This work provides yet another example of how string theory unites seemingly distant physical problems. (See references for detailed results.) The central object considered here is supersymmetric gauge theories in three dimensions. In particular, we shall be interested in their vacuum structure. The other three problems that turn out to be closely related to these gauge theories are:

• Nonabelian monopoles of Prasad and Sommerfield, which are solutions of the Bogomolny equation 

$$*F = D\Phi$$

(where $F$ is the field-strength of a nonabelian connection $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ and $\Phi$ is a nonabelian Higgs field).

• An integrable system of equations named after Nahm

$$\frac{dT_i}{ds} = \frac{1}{2} \varepsilon_{ijk} [T_j, T_k],$$

for $T_i(s) \in u(n)$. These generalize Euler equations for a rotating top.

• Solutions of the Euclideanized vacuum Einstein equation called self-dual gravitational instantons, which are four-dimensional manifolds with self-dual curvature tensor

$$R_{\alpha \beta \gamma \delta} = \frac{1}{2} \varepsilon_{\alpha \beta \mu \nu} R_{\gamma \delta}^{\mu \nu}.$$
The latter provide compactifications of string theory and supergravity that preserve supersymmetry and are of importance in euclidean quantum gravity. The compact examples are delivered by a four-torus and K3. The noncompact ones are classified according to their asymptotic behavior and topology. Asymptotically Locally Euclidean (ALE) gravitational instantons asymptotically approach $\mathbb{R}^4/\Gamma$ ($\Gamma$ is a finite subgroup of $SU(2)$). These were classified by Kronheimer into two infinite ($A_k$ and $D_k$) series and three exceptional ($E_6$, $E_7$ and $E_8$) cases according to the intersection matrix of their two-cycles. Asymptotically Locally Flat (ALF) spaces approach the $(\mathbb{R}^3 \times S^1)/\Gamma$ metric. (To be more precise $S^3$ is Hopf fibered over the two-sphere at infinity of $\mathbb{R}^3$.) Sending the radius of the asymptotic $S^3$ to infinity we recover an ALE space of some type, which will determine the type of the initial ALF space. For example, the $A_k$ ALF is a $(k+1)$-centered multi-Taub-NUT space. Here we shall seek to describe the $D_k$ ALF space.

M theory on an $A_k$ ALF space is known to describe $(k+1)$ D6-branes of type IIA string theory. Probing this background with a D2-brane we obtain an $\mathcal{N} = 4$ $U(1)$ gauge theory with $(k+1)$ electron in the D2-brane worldvolume. As the D2-brane corresponds to an M2-brane in M theory, a vacuum of the above gauge theory corresponds to a position of the M2-brane on the $A_k$ ALF space we started with. Thus the moduli space of this gauge theory is the $A_k$ ALF. Next, considering M theory on a $D_k$ ALF one recovers $4k$ D6-branes parallel to an orientifold $O6^-$. On a D2-brane probe this time we find an $\mathcal{N} = 4$ $SU(2)$ gauge theory with $k$ matter multiplets. Its moduli space is the $D_k$ ALF. So far we have related gauge theories and gravitational instantons.

There is another way of realizing these gauge theories. Consider the Chalmers-Hanany-Witten configuration in type IIB string theory (Figure 1).
In the extreme infrared limit the theory in the internal D3-branes will appear to be three-dimensional with $\mathcal{N} = 4$ supersymmetry. This realizes the gauge theory we are interested in. A vacuum of this theory describes a particular position of the D3-branes. In the $U(2)$ theory on the NS5-branes the internal D3-branes appear as nonabelian monopoles, while every external semiinfinite D3-brane appears as a Dirac monopole in the $U(1)$ of the lower right corner of the $U(2)$. Thus the moduli space of such monopole configurations of non-abelian charge two and with k singularities is also a moduli space of the gauge theory in question.

Another way of describing the vacua of the three-dimensional theory on the D3-branes is by considering the reduction of the four-dimensional theory on the interval. For this reduction to respect enough supersymmetry the fields (namely the Higgs fields of the theory on the D3-branes) should depend on the reduced coordinate so that Nahm Equations (1) are satisfied. Thus the Coulomb branch of the three-dimensional gauge theory is described as a moduli space of solutions to Nahm Equations.

At this point we have two convenient descriptions of $D_k$ ALF space as a moduli space of solutions to Nahm equations and as a moduli space of singular monopoles. It is the latter description that we shall make use of here. Regular monopoles can be described by considering a scattering problem \((\vec{u} \cdot (\vec{\partial} + \vec{A}) - i\Phi)s = 0\) on every line $\gamma$ in the three-dimensional space directed along $\vec{u}$. The space of all lines is a tangent bundle to a sphere $T = TP^1$. Let $(\zeta, \eta)$ be standard coordinates on $T$, such that $\zeta$ is a coordinate on the sphere and $\eta$ on the tangent space. Then the set of lines on which the scattering problem has a bound state forms a curve $S \in T$. $S$ is called a spectral curve and it encodes the monopole data we started with.

In case of singular monopoles some of the lines $\gamma \in S$ will pass through the singular points. These lines define two sets of points $Q$ and $P$ in $S$, such that $Q$ and $P$ are conjugate to each other with respect to the change of orientation of the lines. Analysis of this situation shows, that in addition to the spectral curve, we have to consider two sections $\rho$ and $\xi$ of the line bundles over $S$ with transition functions $e^{i\eta/\zeta}$ and $e^{-i\eta/\zeta}$ correspondingly. Also $\rho$ vanishes at the points of $Q$ and $\xi$ at those of $P$.

Since we are interested in the case of two monopoles the spectral curve is given by $\eta^2 + \eta_2(\zeta) = 0$ where $\eta_2(\zeta) = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4$. $z,v$ and $w$ are the moduli. $z$ and $v$ are complex and $w$ is real. The sections $\rho$ and $\xi$ satisfy $\rho\xi = \prod_{i=1}^{k} (\eta - P_i(\zeta))$, where $P_i$ are quadratic in $\zeta$ with coefficients given by the coordinates of the singularities. The above equations provide the description of the twistor space of the singular monopole moduli space.
Knowing the twistor space one can use the generalized Legendre transform techniques to find the auxiliary function $F$ of the moduli

$$F(z, \bar{z}, v, \bar{v}, w) = \frac{1}{2\pi i} \oint_0 \frac{d\zeta}{\zeta^3} \left[ 2 \oint_{\omega_r} d\zeta \sqrt{-\eta_2} - \sum_a \frac{1}{2\pi i} \oint_{C_a} \frac{d\zeta}{\zeta^2} \left( \sqrt{-\eta_2} - z_a(\zeta) \right) \log(\sqrt{-\eta_2} - z_a(\zeta)) \right].$$

Imposing the consistency constraint $\partial F/\partial w = 0$ expresses $w$ as a function of $z$ and $v$. Then the Legendre transform of $F$ gives the Kähler potential for the $D_k$ ALF metric. This agrees with the conjecture of Chalmers.

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References

1. S. A. Cherkis and A. Kapustin, “Singular Monopoles and Gravitational Instantons,” hep-th/9711145 to appear in Nucl. Phys. B.
2. S. A. Cherkis and A. Kapustin, “$D_k$ Gravitational Instantons and Nahm Equations,” hep-th/9803112.
3. S. A. Cherkis and A. Kapustin, “Singular Monopoles and Supersymmetric Gauge Theories in Three Dimensions,” hep-th/9711143.
4. A. Sen, “A Note on Enhanced Gauge Symmetries in M- and String Theory,” JHEP 09, 1 (1997) hep-th/9707123.
5. M. Atiyah and N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton Univ. Press, Princeton (1988).
6. N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, “Hyperkähler Metrics and Supersymmetry,” Comm. Math. Phys. 108 535-589 (1987), Lindstrom, U. and Roček, M. “New HyperKähler metrics and New Supermultiplets,” Comm. Math. Phys 115, 21 (1988).
7. G. Chalmers, “The Implicit Metric on a Deformation of the Atiyah-Hitchin Manifold,” hep-th/9709084, “Multi-monopole Moduli Spaces for $SU(N)$ Gauge Group,” hep-th/9605182.