Hydrodynamic model of BEC with anisotropic short range interaction and the bright solitons in the repulsive BEC

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The quantum hydrodynamic model is developed for the axial symmetric anisotropic short-range interaction. The quantum stress tensor presents the interaction. It is derived up to the third order by the interaction radius. The first order by the interaction radius contains the isotropic part only. It leads to the interaction in the Gross-Pitaevskii approximation. Terms existing in the third order by the interaction radius are caused by the isotropic and nonisotropic parts of the interaction. Each of them introduces the interaction constant. Therefore, three interaction constants are involved in the model. Atoms, except the alkali and alkali-earth atoms, can have anisotropic potential of interaction, particularly it is demonstrated for the lanthanides. The short-wavelength instability caused by the nonlocal terms appears in the Bogoliubov spectrum. Conditions for the stable and unstable behaviour are described. Bright solitons in the repulsive BEC are studied under influence of the anisotropic short-range interaction in the BEC of one species. Area of existence of this bright solitons corresponds to the area of the instability of the Bogoliubov spectrum. Approximate reduction of the nonlocal nonlinearity to the quintic nonlinearity at the description of the bright solitons is demonstrated either.

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I. INTRODUCTION

Solitons are famous nonlinear phenomena existing in various mediums. Being discovered on the water surface, solitons are found in plasmas [1], [2], [3], in lasers [4], in ultracold bosonic atoms [5], [6], in ultracold fermions [7], [8], in different setups of the condensed matter physics like spin waves [9], [10], [11], and charge density waves [12], [13], [14], [15]. Weak solitons observed over astronomical distances recirculating in an optical fibre loop [16]. Solitons show different features like attraction, repulsion, breaking up, merging, orbiting one another and the annihilation of two solitons [17], [18], [19], [20], [21]. Recently, it is demonstrated that solitons can absorb the resonant waves [22].

At the analysis of the nonlinear waves, many physical models of different physical systems lead to the nonlinear Schrodinger equation (NLSE) (such as the Gross-Pitasevskii (GP) equation) or the Korteweg-de Vries (KdV) equation with cubic nonlinearity, where the nonlinearity is mostly caused by the interparticle interaction. For instance, the attractive short-range interaction in Bose-Einstein condensate (BEC) of neutral particles leads to the bright soliton existence (the area of the concentration amplification) while the repulsive interaction leads to the dark soliton formation (the area of the concentration rarefication). It seems that the Fermi pressure is the cause of the dark solitons appearance in the ultracold fermions [8], where the Fermi pressure gives the effective repulsion, but the nonlinearity is fractional. However, more general models contain complex forms of nonlinearity, such as the quintic nonlinearity or the nonlocal nonlinearity. Generalised models predict existence of the bright soliton in mediums with repulsive meanfield interaction [23], [24].

There are two kinds of the nonlocal nonlinearities: the integral nonlinear term, like the dipolar BEC model [25], [26], [27], and the nonintegral nonlinearities containing spatial derivatives of the macroscopic wave function [28], [29], [30].

A model with nonlocal nonlinearity containing spatial derivatives is suggested in this paper at the analysis of the anisotropic short range interaction in the atomic BEC. Particularly, the paper is focused on study of the bright solitons in the repulsive BEC and the anisotropy of their properties.

Recent experiments with the BEC of rare-earth elements demonstrated phenomena requiring new physics for their explanation. It seems that it requires consideration of the quantum fluctuations in BEC in addition to the standard (meanfield) short-range interaction (SRI) and dipole-dipole interaction. To some extent, the "constructive" three-particle interaction [31], [32], [33], [34], [35], [36] described by the fifth-order nonlinearity in the Gross-Pitaevskii equation was applied to these phenomena [37]. However, the quantum fluctuations demonstrated better agreement with experimental data. All mentioned interaction and related to them quantum fluctuations are applicable to the alkali-earth (see Ref. [38] for $^{52}$Cr) atoms and rare-earth atoms (see Ref. [39], [40] for dysprosium $^{164}$Dy). However, the quantum Rosensweig instability is found for the rare-earth atoms. Hence, the following question can be formulated: is it quantitative or qualitative difference. The rare-earth atoms show a qualitative difference due to the...
anisotropy of the SRI. It shows that a model of the rare-earth atoms with the account of the anisotropic SRI is required. The analysis of the spectrum of Feshbach resonances in the rare-earth atoms demonstrates an extra feature: anisotropy of the quasi-potentials.

It is found that the quantum Rosensweig instability in BEC of lanthanides (which are atoms with relatively large dipole-dipole interaction) can be modeled by an effective NLSE with the local form of the quantum correlations

\[ i\hbar \partial_t \Psi = \left[ -\frac{\hbar^2 \nabla^2}{2m} + g |\Psi|^2 + \Phi(\mathbf{r}, t) + \gamma_{QF} |\Psi|^3 \right] \Psi, \quad (1) \]

where \( \Psi \) is the macroscopic (effective) wave function,

\[ \Phi(\mathbf{r}, t) = \mu^2 \int d\mathbf{r}' \frac{1 - 3 \cos^2 \theta'}{|\mathbf{r} - \mathbf{r}'|^3} |\Psi(\mathbf{r}', t)|^2 \quad (2) \]

is the potential of the magnetic dipole-dipole interaction

\[ \gamma_{QF} = \frac{32}{3} \frac{g^3}{\pi} \left( 1 + 3 \frac{e^d}{2} \right) \quad (3) \]

represents the coefficient at the fourth order nonlinear term caused by the quantum fluctuations, \( \hbar \) is the Planck constant, \( \theta \) is the angle in the spherical coordinates, \( g = 4\pi\hbar^2a_s/m \), \( e_d = a_{dd}/a_s \), \( a_{dd} = mm_s \mu^2/(2\pi\hbar^2) \) is the dipole length, \( a_s \) is the scattering length of the short range interaction, \( m \) is the mass of particle, \( \hbar \) is the Planck constant, \( \hbar \) is the time derivative. The dipolar part of the quantum fluctuations is found in Refs. [47, 48].

For the complete analysis of the Rosensweig instability in the rare-earth atoms, the many-particle quantum hydrodynamics method is applied to the derivation of the model with anisotropic SRI.

This paper is organized as follows. In Sec. II the formulation of basic ideas of the many-particle quantum hydrodynamics method is presented. In Sec III the quantum stress tensor is calculated in the first order by the interaction radius. In Sec. IV the quantum stress tensor is calculated in the third order by the interaction radius. In Sec. V generalization of the Bogoliubov spectrum is demonstrated. In Sec. VI the hydrodynamic equations are rewritten for the plane-like nonlinear objects propagating at the arbitrary angle to the preferable direction crated by the anisotropic interaction. In Sec. VII is on the possibility of the reduction of the nonlocal nonlinearity to a local one. In Sec. VIII models the bright solitons in the repulsive BEC appearing due to the nonlocal nonlinearities. In Sec. IX a brief summary of obtained results is presented.

II. ON THE DERIVATION OF HYDRODYNAMIC EQUATIONS

Derivation of the model is based on the many-particle Schrodinger equation containing the following Hamiltonian:

\[ \hat{H} = \sum_i \left( \frac{\hat{p}_i^2}{2m_i} + V_{ext}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i,j \neq i} U(\mathbf{r}_i - \mathbf{r}_j), \quad (4) \]

where \( m_i \) is the mass of \( i \)-th particle, \( \hat{p}_i = -i\hbar \nabla_i \) is the momentum of \( i \)-th particle, \( U_{ij} = U(\mathbf{r}_i - \mathbf{r}_j) \) is the potential of interparticle interaction, \( V_{ext}(\mathbf{r}_i, t) \) is the potential of external field acting on particles.

Derivation of the model starts with the definition of the simplest collective variable which is the quantum concentration of particles. It appears as the quantum mechanical average of corresponding operator:

\[ n(\mathbf{r}, t) = \int d\mathbf{r} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) |\psi^*(\mathbf{R}, t)\psi(\mathbf{R}, t)|, \quad (5) \]

where \( d\mathbf{R} = \prod_{i=1}^N d\mathbf{r}_i \) is the element of volume in \( 3N \) dimensional configurational space, with \( N \) is the number of particles. Its evolution calculated with the application of the many-particle Schrodinger equation and Hamiltonian \[ leads to the continuity equation: \[ \partial_t n + \nabla j = 0. \]

The last one gives the explicit form of the current:

\[ j(\mathbf{r}, t) = \int d\mathbf{r} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \frac{1}{2m_i} (\psi^*(\mathbf{R}, t)\hat{p}_i\psi(\mathbf{R}, t) + c.c.). \quad (6) \]

Next, the evolution of current leads to the Euler equation:

\[ m(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + \partial_\beta (p^{\alpha\beta} + \sigma^{\alpha\beta} + T^{\alpha\beta}) = -n \partial_\alpha V_{ext}, \quad (7) \]

where the current is represented via the velocity field \[ n(\mathbf{r}, t) = n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t), \quad p^{\alpha\beta} \] is the thermal pressure caused by the local distribution of particles on states with non-zero energy: the thermal effects or the quantum fluctuations, \( \sigma^{\alpha\beta} \) is the quantum stress tensor caused by the short range interaction (SRI), and \( T^{\alpha\beta} = -(\hbar^2/4m)[\partial^\alpha \partial^\beta n - (\partial_\alpha n)(\partial_\beta n)/n] \) is the quantum Bohm potential (this is a simplified form suitable for the BEC, while the general form is presented by equation (17) in Ref. \[ 28 \] and equation (10) in Ref. \[ 49 \]). Tensor \( \sigma^{\alpha\beta} \) appears at the calculation of the force field. It is assumed that the masses of all particles are equal to each other and written without subindex: \( m \). Originally, in the Euler equation, the force field appears as follows:

\[ \mathbf{F}(\mathbf{r}, t) = -\int d\mathbf{r} \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i)(\nabla U(\mathbf{r}_{ij}))|\psi^*(\mathbf{R}, t)|\psi(\mathbf{R}, t). \]

(8)

It can be symmetrized relatively pair of interacting particles

\[ \mathbf{F}(\mathbf{r}, t) = -\frac{1}{2} \int d\mathbf{r} \sum_{i,i,j \neq i} [\delta(\mathbf{r} - \mathbf{r}_i) - \delta(\mathbf{r} - \mathbf{r}_j)] \times \]
Next, introducing the coordinates of relative motion and center of mass for each pair of particles \( \mathbf{R}_{ij} = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j) \), \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \), represent coordinates of \( i \)-th and \( j \)-th particles via \( \mathbf{r}_j \) and \( \mathbf{R}_{ij} \) in equation (9).

The SRI is considered. Hence, the potential \( U_{ij} \) and its derivatives \( \partial_y U_{ij} \) go to zero for the large interparticle distances \( r_{ij} \). Therefore, equation (8) is nontrivial for the small values of \( r_{ij} \). Consequently, we can expand equation (8) in the Taylor series on parameter \( r_{ij} \). The \( \delta \) functions and the \( i \)-th and \( j \)-th arguments of the wave functions are involved in the expansion. The zeroth term and the even terms are equal to zero due to the symmetry of the equation (8) relatively \( i \)-th and \( j \)-th particles. Hence, we deal with odd terms. In the simplest case, we restrict our analysis with the first and third terms on the interaction radius.

This paper is a development of the many-particle quantum hydrodynamic method which gives a representation of the many-particle quantum system described by the time-dependent Schrodinger equation in terms of the collective variables. This method suggested in [29] for the quantum plasmas is applied to derivation of the Gross-Pitaevskii equation and its nonlocal generalization in [30]. It is done with accordance with the earlier proved possibility of such kind of derivation [51]. In this context, it is interesting to mention work [52], where the quantum dynamics of finite ultracold bosonic ensembles based on the Born-Bogoliubov-Green-Kirkwood-Yvon hierarchy for equations of motion for few-particle reduced density matrices with the necessary truncation. The multiconfiguration time-dependent Hartree method for bosons equations of motion of [33] for the specific wave function ansatz as their starting point.

III. QUANTUM STRESS TENSOR IN THE FIRST ORDER BY THE INTERACTION RADIUS

Considering the quantum stress tensor (QST) in the first order by the interaction radius find the following representation in terms of the many-particle wave function:

\[
\sigma_{1}^{\alpha\beta}(r, t) = -\frac{1}{2} \int dR' \sum_{i,j \neq i} \delta(r - R_{ij}) r_{ij}^{\beta} \times \frac{\partial U(r_{ij})}{\partial r_{ij}^\alpha} \psi^*(R', t) \psi(R', t),
\]

(10)

where \( R' = \{\mathbf{r}_1, ..., \mathbf{R}_{ij}, ..., \mathbf{R}_{ij}, ..., \mathbf{r}_N \} \) with \( \mathbf{R}_{ij} \) is located on the \( i \)-th and \( j \)-th places, but \( dR' = dR_{ij} dR_{ij} dR_{ij} \), where \( dR_{ij} \) does not contain the contribution of the \( i \)-th and \( j \)-th particles.

The particles under consideration have a preferable direction resulting in the nonisotropic interaction. Being in an arbitrary state the system consists of particles with the arbitrary orientation. The summation in the quantum stress tensor on all pair of particles leads to the cancelation or at least the decrease of the anisotropic part. However, if all particles oriented in the same direction the anisotropic part is not reduced. Moreover, the part of integral (10) and the QST in the higher orders containing \( r_{ij} \) can be extracted as the common multiplier. This assumption allows us to go to the simplified representation of the QST presented by the next equation. Let us mention here that the considered particles posses the magnetic moment. Therefore, orientation of the particles relatively the SRI is related to the creation of the spin polarization in the system. In the simplest case, the preferable direction of the SRI coincides with the spin direction. If it has an angle with spin the spin polarization create a partial orientation. Hence the perpendicular parts would cancel each other, but the part of the anisotropic SRI parallel to the spin is not zero and contribute in the collective interaction. This part enters the model developed in this paper.

The quantum stress tensor can be represented in a form containing two-particle concentration

\[
\sigma_{1}^{\alpha\beta}(r, t) = -\frac{1}{2} \text{Tr}(n_2(r, r', t)) \int r^{2l} \frac{\partial U(r)}{\partial r^\alpha} d\mathbf{r},
\]

(11)

where \( \text{Tr}n_2(r, r') = n_2(r, r) \) is the trace of the function of two arguments, and

\[
n_2(r, r', t) = \int dR \sum_{i,j \neq i} \delta(r - r_i) \delta(r' - r_j) \psi^*(R, t) \psi(R, t)
\]

(12)

is the explicit form of two-particle concentration.

Considering even and axial symmetric anisotropic potentials, include the expansion of the potential on the spherical functions

\[
U(r, \theta) = \sqrt{4\pi} \sum_{l=0}^{\infty} Y_{2l,0}(\theta) U_{2l}(r)
\]

(13)

find a generalization of the quantum stress tensor obtained in [29]

\[
\sigma_{1, \text{BEC}}^{\alpha\beta} = -\sqrt{\pi} \text{Tr}(n_2(r, r', t)) \sum_{l=0}^{\infty} \int d\mathbf{r} \left( Y_{2l,0}(\theta) \frac{r^{2l} \partial U_{2l}(r)}{r} + U_{2l}(r) \frac{\epsilon_{\phi}^{\alpha\beta} \partial Y_{2l,0}(\theta)}{\partial \theta} \right),
\]

(14)

where \( \epsilon_{\phi} \) is the unit vector in the spherical coordinate system \( r, \theta, \varphi \). Presence of \( \epsilon_{\phi}^{\alpha\beta} \) under the integral shows that two terms from expansion [13] give a contribution. These are terms proportional \( Y_{00}(\theta) \) and \( Y_{20}(\theta) \). The second of them brings anisotropy in the model. However, the calculation of the tensor structure of the quantum stress tensor in the first order by the
interaction radius shows cancelation of the term proportional to $Y_{20}(\theta)$. Hence, the anisotropy gives no contribution to the quantum stress tensor in the first order by the interaction radius even if anisotropic part of the potential is comparable with the isotropic one.

The two-particle concentration is calculated for the Bose-Einstein condensate in Ref. [28], where it is found that $Tr(n_2(\mathbf{r}, \mathbf{r}')) = n^2(\mathbf{r}, t)$.

After all, obtain the standard result for the quantum stress tensor in the first order by the interaction radius:

$$
\sigma_{\alpha\beta, \text{BEC}}(r, t) = \frac{1}{2} g \delta^{\alpha\beta} n^2,
$$

(15)

where $g = \int U_0(r) d\mathbf{r}$.

Substituting the found QST in the Euler equation, find the following result

$$
mn(\partial_t + \mathbf{v} \cdot \nabla)\psi^\alpha - \frac{\hbar^2}{2m} n \partial^\alpha \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} + gn\partial^\alpha n = -n \partial^\alpha V_{\text{ext}},
$$

(16)

where $p^{\alpha\beta} = 0$ since bosons at the zero temperature are considered.

The Euler equation (16) appears together with the continuity equation presented above. Here, we represent the continuity equation via the velocity field:

$$
\partial_t n + \nabla \cdot (n \mathbf{v}) = 0.
$$

(17)

Equations (15) and (17) correspond to the traditional Gross-Pitaevskii equation [54]:

$$
\hbar \partial_t \Psi = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + g \mid \Psi \mid^2 \right) \Psi,
$$

(18)

with

$$
\Psi(\mathbf{r}, t) = \sqrt{n} e^{im\phi/\hbar},
$$

(19)

where $\mathbf{v} = \nabla \phi$.

### IV. QUANTUM STRESS TENSOR IN THE THIRD ORDER ON THE INTERACTION RADIUS

The quantum hydrodynamic model of BEC of neutral atoms with the SRI considered up to the TOIR is developed in 2008 [28] for the isotropic potentials. Here it is considered for the anisotropic potentials.

An analog of equation (10) obtained in the TOIR is a large equation. Majority of the terms in this equation do not contribute in the BEC dynamics being related to the excited states (see Appendix A). For the BEC it simplifies to

$$
\sigma_{3, \text{BEC}}(\mathbf{r}, t) = -\frac{1}{48} \partial_\gamma \partial_\delta Tr(n_2(\mathbf{r}, \mathbf{r}'), t) \cdot \int r^{\beta} r^{\gamma} r^{\delta} \frac{\partial U(\mathbf{r})}{\partial r^{\alpha}} d\mathbf{r}.
$$

(20)

Calculation of the tensor structure of the QST in the TOIR gives the following result

$$
\int r^{\beta} r^{\gamma} r^{\delta} \frac{\partial U(\mathbf{r})}{\partial r^{\alpha}} d\mathbf{r} = \frac{1}{3} \bar{g}_{2,0} r^{\alpha} r^{\beta} \delta^{\gamma\delta} + \frac{1}{\sqrt{5}} \bar{g}_{2,2} r^{\alpha} r^{\beta},
$$

(21)

where $I_{0}^{\alpha\beta\gamma\delta} = \delta^{\alpha\beta} \delta^{\gamma\delta} + \delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma}$ is a symmetric tensor existing in the term describing isotropic part, and $I_{l}^{\alpha\beta\gamma\delta}$ is a nonsymmetric tensor, or more precisely, it is symmetric relatively permutations of the first three indexes, but it is not symmetric relatively permutations of the first index and other indexes. Tensor $I_{2}^{\alpha\beta\gamma\delta}$ has the following elements: $I_{2}^{xxxx} = I_{2}^{yyyy} = 1$, $I_{2}^{zzzz} = -2$, $I_{2}^{zzxx} = I_{2}^{yyzz} = -2/3$, $I_{2}^{xxyy} = I_{2}^{yxyy} = I_{2}^{zzxx} = I_{2}^{zzyy}$ is 1/3 and allowed permutations of indexes, other elements are equal to zero. Equation (21) contains two interaction constants $g_{2,1} = \int r^2 U_1(\mathbf{r}) d\mathbf{r}/24$, $g_{2,2} = \bar{g}_{2,1}/24$, where $l = 0, 2$.

Present the Euler equation with the SRI considered up to the TOIR

$$
mn(\partial_t + \mathbf{v} \cdot \nabla)\psi^\alpha + n \partial^\alpha V_{\text{ext}} - \frac{\hbar^2}{2m} n \partial^\alpha \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} + gn\partial^\alpha n + \frac{g_{2,0}}{2} n^2 \partial^\alpha \triangle \sqrt{n} - \frac{g_{2,2}}{2 \sqrt{5}} I_{2}^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta n^2 = 0.
$$

(22)

Obviously, the found generalized Euler equation corresponds to an approximation generalizing the Gross-Pitaevskii equation.

Interaction constants $g$ and $g_{2,0}$ are defined by $U_0$. Hence, their contributions $g_{2,0} n^2/2$ and $g_{2,2} \partial_\alpha \triangle n^2/2$ can be compared. Let us introduce the radius of interaction $r_0$ in a way that the interparticle interaction $U_2(\mathbf{r})$ or at least $U_0(\mathbf{r})$ is negligible for $r > r_0$. Consider dimensionally reduced parameters $\bar{g}$ and $\bar{g}_{2,0}$ introduced as follows $g = r_0^2 \bar{g}$ and $g_{2,0} = r_0^4 \bar{g}_{2,0}$, so $[\bar{g}] = [\bar{g}_{2,0}] = \epsilon r_0 g$. We have $\bar{g} = \int U_0(\mathbf{r}_0 \xi) d^3 \xi$ and $\bar{g}_{2,0} = \int \xi^2 U_0(\mathbf{r}_0 \xi) d^3 \xi$, where $\xi = r/r_0$ and $\xi < 1$ for the area of the nontrivial values of the potential $U_0$. Therefore, the presence of $\xi^2$ under the integral in the definition of $\bar{g}_{2,0}$ decreases its value in compare with $\bar{g}$. It shows $\bar{g}_{2,0} \ll \bar{g}$. Our comparison shows that the first order by the interaction radius dominates over higher orders. However, interaction constant $g_{2,2}$ is defined by another part of the anisotropic potential $U_2$. Therefore, constants $g_{2,0}$ and $g_{2,2}$ can have different relationships $g_{2,0} > g_{2,2}$ or $g_{2,0} < g_{2,2}$. If function $U_2$ is large in compare with $U_0$ (the large dipole anisotropy limit) the parameter $\bar{g}_{2,2} = \int \xi^2 U_2(\mathbf{r}_0 \xi) d^3 \xi$ can be large in compare with $\bar{g}_{2,0}$ and comparable with $\bar{g}$.

### V. COLLECTIVE EXCITATIONS

Spectrum of the collective excitations propagating as the plane wave in the infinite uniform BEC generalizing
The Bogoliubov spectrum is

$$\omega^2 = \frac{n_0}{m} \left( g k^2 + \frac{\hbar^2 k^4}{4 m n_0} - g_{2,0} k^4 - \frac{g_{2,2}}{\sqrt{2}} k^4 (3 \cos^2 \theta - 1) \right).$$  \hfill (23)

The first term on the right-hand side is the short-range interaction calculated in the first order by the interaction radius existing in the original Bogoliubov spectrum. The second term is the quantum Bohm potential contribution. The third and fourth terms are caused by the TOIR. The third term is related to the isotropic part of the TOIR. The last term in this equation is caused by the anisotropic part of the short range interaction. The third and fourth terms give a generalization of the Bogoliubov spectrum caused by the short range interaction in the third order by the interaction radius.

Equation (23) is found from the hydrodynamic equations (17) and (22). Derivation of equation (23) is made in the linear approximation on the small perturbations of the hydrodynamic functions relatively to the equilibrium state $n_0$, and $v_0 = 0$. Hence, the hydrodynamic variables are expressed as follows: $n = n_0 + \delta n$, and $v = 0 + \delta v$. The perturbations are presented in the following form: $\delta n = N \exp(-\omega t + ik x + ik z)$, and $\delta v = V \exp(-\omega t + ik x + ik z)$, where $N$ and $V$ are the amplitudes of the perturbations.

The terms proportional to $k^4$ dominate in the short-wavelength limit (the last three terms in equation (23)).

The second term is positive. The third and fourth terms can be negative and they can overcome the second term. Hence, the short-wavelength instability can occur in the BEC due to the interaction in the third order by the interaction radius.

Presented comparison of three terms demonstrates that the parameter $\hbar^2 / 4 m n_0$ is the natural unit for the interaction constants $g_{2,0}$ and $g_{2,2}$. Introduce corresponding dimensionless interaction constants $G_{2,0} = 4 m n_0 g_{2,0} \hbar^2$ and $G_{2,2} = 4 m n_0 g_{2,2} \hbar^2$.

Focus on the isotropic regime $g_{2,2} = 0$. The repulsive SRI $g > 0$ leads to the long-wavelength stability of the Bogoliubov spectrum. However, short-wavelength instability can exist if $G_{2,0}$ is larger then the quantum Bohm potential contribution 1, where $G_{2,0} > 0$ for the repulsive interaction since interaction constants $g$ and $G_{2,0}$ are the moments of the same partial potential $U_0$. If $1 > G_{2,0} > 0$ the spectrum (23) is stable, but the frequency $\omega$ decreases in the area of large $k$ in compare with the first order by the interaction radius approximation.

Interaction constant $g_{2,2}$ is the moment of the second partial potential $U_2$. Therefore, it does not have direct relation to $g_{2,0}$. Hence, it is possible to have positive or negative $g_{2,2}$ at the positive $g_{2,0}$. Anyway, angle dependence of the last term in equation (23) gives different signs of the last term depending on the wave propagation direction.

Negative contribution of the TOIR in the square of frequency is demonstrated in Fig. 1. Changing direction of wave propagation we change the relative contribution of the TOIR. If angle is small enough and the TOIR interaction constants are large enough the frequency get the zero value at some wave vector $\kappa > 1$. Spectrum becomes imaginary and condition for an instability occurs (see Fig. 2).

The instability can be caused by the isotropic part of the TOIR. However, in accordance with the described estimations, the anisotropic interaction constant can be larger then the isotropic TOIR constant. Therefore, the instability is easier detectable in the BEC of atoms with the anisotropic interaction.

VI. PROPAGATION OF THE NONLINEAR PERTURBATIONS IN AN ARBITRARY DIRECTION

Consider the wave propagation in the direction $h$ which has angle $\theta$ with the anisotropy axis (the $z$-axis). Change
of coordinate in \( h \) direction corresponds to the change of \( x \) and \( z \) (it is assumed that \( y = 0 \)) in accordance with the following relation \( h = x \sin \theta + z \cos \theta \). The velocity field perturbations in the direction \( h \) has corresponding form \( v_h = v_x \sin \theta + v_z \cos \theta \).

The continuity equation transforms in the following way for the perturbations propagating in direction \( h \)

\[
\partial_t n + \nabla \cdot (nv) = 0,
\]

\[
\partial_t n + \partial_x (nv_x) + \partial_z (nv_z) = 0,
\]

\[
\partial_t n + \sin \theta (nv_x)' + \cos \theta (nv_z)' = 0,
\]

and

\[
\partial_t n + (nv_h)' = 0,
\]

since \( \partial_x f = f' \sin \theta \), and \( \partial_z f = f' \cos \theta \), where \( f = f(h) \) an arbitrary hydrodynamic function \( n \), \( v_x \), \( v_z \) or their combination as a function of coordinate \( h \), and \( f' = df/dh \).

Corresponding modification is found for the \( x- \) and \( z- \)projections of the Euler equation

\[
m n (\partial_t v_x + v_x v_x') + n \sin \theta v_x' = -\frac{h^2}{2m} \sin \theta n \left[ \frac{\sqrt{n}''}{\sqrt{n}} \right]'
\]

\[+ gn \sin \theta n' + \frac{1}{2} g_{2,0} \sin \theta (n^2)''',\]

\[
- \frac{g_{2,2}}{2\sqrt{5}} \sin \theta \left( \sin^2 \theta - \frac{2}{3} \cos^2 \theta \right) (n^2)''' = 0,
\]

and

\[
m n (\partial_t v_z + v_z v_z') + n \cos \theta v_z' = -\frac{h^2}{2m} \cos \theta n \left[ \frac{\sqrt{n}''}{\sqrt{n}} \right]'
+ gn \cos \theta n' + \frac{1}{2} g_{2,0} \cos \theta (n^2)''',
\]

\[
- \frac{g_{2,2}}{2\sqrt{5}} \cos \theta \left( \frac{1}{3} \sin^2 \theta - 2 \cos^2 \theta \right) (n^2)''' = 0.
\]

The \( y- \) projection of the Euler equation is not involved.

Multiply equation \((28)\) on \( \sin \theta \) and equation \((29)\) on \( \cos \theta \) and combine obtained equations to get equation for \( \partial_t v_h \):

\[
m n (\partial_t v_h + v_h v_h') + n v_h' = -\frac{h^2}{2m} n \left[ \frac{\sqrt{n}''}{\sqrt{n}} \right]'
+ gn n' + \frac{g_{2,0}}{2} (n^2)'''
\]

FIG. 3: The figure shows 1. \( \Theta \equiv \sin^4 \theta - \frac{1}{3} \sin^2 \theta \cos^2 \theta - 2 \cos^4 \theta \) which is located in front of \( g_{2,2} \); 2. Parameter \( G_2 \equiv 4 m n_0 g_2 / h^2 \) at \( G_{2,0} = 1 \) and \( G_{2,2} = 0.1 \).

\[
- \frac{g_{2,2}}{2\sqrt{5}} \left[ \sin^4 \theta - \frac{1}{3} \sin^2 \theta \cos^2 \theta - 2 \cos^4 \theta \right] (n^2)''' = 0.
\]

Equation \((30)\) shows that the third order on the interaction radius allows to introduce a single effective constant combining isotropic and nonisotropic parts

\[
g_2 \equiv g_{2,0} - \frac{g_{2,2}}{2\sqrt{5}} \left[ \sin^4 \theta - \frac{1}{3} \sin^2 \theta \cos^2 \theta - 2 \cos^4 \theta \right].
\]

Hence, the last two terms in equation \((29)\) can be written in the following way \( \frac{g_{2,0}}{2} (n^2)''.\)

Dimensionless form of parameter \( g_2 \) is presented in Fig. \(3\) and \(4\) for chosen values of \( g_{2,0} \) and \( g_{2,2} \). It is found that parameter \( g_2 \) can change the sign. Hence, the repulsion existing at small angles and positive \( g_{2,2} \) can be changed by the attraction for the large enough \( g_{2,2} \) (see Fig. \(4\)).

VII. ON APPROXIMATE REDUCTION OF THE NONLOCAL NONLINEARITY TO THE QUINTIC NONLINEARITY AND THE BRIGHT SOLITON IN THE ATTRACTIVE BEC

Here we apply a method of nonperturbative analysis of the excitations in BEC. It allows to consider the large amplitude bright soliton in the one dimensional attractive
reduction is not a general property, but it approximately happens for the bright soliton structure.

Consider the one dimensional regime of the quantum hydrodynamic equations, for the arbitrary direction

\[ \rho n(\partial_t + v_h \partial_h) v_h = \frac{\hbar^2}{2m} n \partial_h \left( \frac{\partial^2 \sqrt{n}}{\sqrt{n}} \right) - g n \partial_h n - \frac{g_2}{2} \partial_h^2 n^2. \]

Divide equation (32) by \(mn\) and integrate it over \(n\) to find

\[ \partial_t \int v_h \sqrt{n} \Delta v + \frac{1}{2} \frac{\partial^2 v_h}{\partial \sqrt{n}} = \frac{\hbar^2}{2m^2} \frac{\partial^2 \sqrt{n}}{\sqrt{n}} - \frac{g}{m} n - \frac{g_2}{2m} \int \frac{\partial^3 n^2}{n} dh. \]

Next, multiply equation (33) by \(\sqrt{n} \partial_h \sqrt{n}\) and obtain the following

\[ \sqrt{n} \partial_h \sqrt{n} \cdot \partial_t \int v_h dh + \frac{1}{2} \frac{\partial^2 v_h}{\partial \sqrt{n}} \cdot \sqrt{n} = \frac{\hbar^2}{2m^2} \partial_h \sqrt{n} \cdot \partial^2 \sqrt{n} \]

\[ - \frac{g}{m} \sqrt{n} \partial_n \partial_h \sqrt{n} - \frac{g_2}{2m} \sqrt{n} \partial_h \sqrt{n} \cdot \int \frac{\partial^3 n^2}{n} dh. \]  

We continue with the representation of the one dimensional hydrodynamic equations following Ref. [56]. Next, we modify the continuity equation. First, the continuity equation is multiplied by the velocity and splitted in the following way \(n v_h \partial_t v_h = -v_h \partial_t n - \frac{\partial^2 v_h}{\partial n} \partial_h n\). Next, add the following term in both sides of the presented equations \(n \partial_t v_h\). The left-hand side of the obtained equation \(n(\partial_t v_h + v_h \partial_h v_h) = -v_h \partial_t n - \frac{\partial^2 v_h}{\partial n} \partial_h n + n \partial_t v_h\) coincides with the kinematic part of the Euler equation placed on the left-hand side of equation (32). Hence, the corresponding substitution is made in the Euler equation:

\[ m(n \partial_t v_h - v_h \partial_t n - \frac{\partial^2 v_h}{\partial n} \partial_h n) = \frac{\hbar^2}{2m} n \partial_h \left( \frac{\partial^2 \sqrt{n}}{\sqrt{n}} \right) \]

\[ - g n \partial_h n - \frac{g_2}{2} \partial_h^2 n^2. \]  

The quantum Bohm potential needs to be rewritten for further decompositions: \(\partial_h \frac{\partial^2 \sqrt{n}}{\sqrt{n}} = \frac{1}{n} \left( \partial^2 \sqrt{n}/2 - 4 \partial_h \sqrt{n} \cdot \partial^2 \sqrt{n}/3 \right)\).

Obtained equation

\[ m(n \partial_t v_h - v_h \partial_t n - \frac{\partial^2 v_h}{\partial n} \partial_h n) = \frac{\hbar^2}{4m} \partial_h^2 n \]

\[ - 2 \frac{\hbar^2}{m} \partial_h \sqrt{n} \cdot \partial^2 \sqrt{n}/2 - g n \partial_h n - \frac{g_2}{2} \partial_h^2 n^2. \]  

allows to represent \(-2 \hbar^2 \partial_\theta \sqrt{n} \cdot \partial^2 \sqrt{n}/m\) via other terms and substitute it in equation (34) to find a required intermediate result:

\[ -2 \partial_\theta n \cdot \partial_t \int v_h dh - n \partial_t v_h + v_h \partial_\theta n + \frac{\hbar^2}{4m^2} \partial_h^2 n \]

FIG. 4: The figure shows a regime opposite to Fig. (1), where \(g_{2,0} < g_{2,2}\). It demonstrates parameter \(G_2\) at \(G_{2,0} = 0.1\) and \(G_{2,2} = 1\).

FIG. 5: Modification of the bright soliton form under the influence of the interaction in the TOIR approximation. Thin continuous line shows bright soliton solution in the Gross-Pitaevskii approximation. The thick continuous line is made for \(\chi_1 = 0.1\). The dotted line is calculated for the larger values of \(\chi_2 = 0.4\). The dashed (upper) curve is obtained for the negative value of \(\chi\).

BEC. This method is a nonperturbative on the amplitude of the excitations. However, below, we explicitly use the condition of the weak interaction in the TOIR approximation in compare with the interaction in the Gross-Pitaevskii approximation (or the FOIR approximation).

A. General analysis of 1D regime

It has been shown that the nonlocal nonlinearity existing in the third order by the interaction radius in the isotropic regime proportional to \(\partial^3 \Delta n^2\) can be approximately reduced to the local form. It can be represented as a nonlinearity of fifth degree [55]. Hence, the Euler equation contain the following force field \(\beta \partial^3 \sqrt{n}\). This reduction is not a general property, but it approximately
Next, we consider a simplified regime which allows to describe the bright soliton in BEC and the reduction of the nonlocal nonlinearity existing in this regime.

The velocity field in BEC is the potential field \( v_\perp = \partial_t \Theta \). The nonzero value of the velocity is related to the solitons propagation moving with a constant velocity \( v_\perp = v_\parallel \). It leads to the following form of the velocity potential \( \Theta = v_0 \Theta - c_0 t \), where \( c_0 \) is a constant which is not depended on space \( h \) and time \( t \).

It is necessary to find a solution of hydrodynamic equations in the stationary wave form. Hence, the independent hydrodynamic function (the concentration) is assumed to be functions of \( \eta = h - v_0 t \).

Overall, we have that derivatives of the velocity in equation (37) go to zero and \( \partial_h \Theta \). The nonzero value of the velocity is related to the interaction constant for the effective fifth order nonlinearity found above. The decrease of width of the soliton is observed either. Our estimations show that the negative energy \( E < 0 \) and the following bound-

\[ \text{(38)} \]

Next, represent the last term in equation (41) via the concentration in accordance with equation (40). As the result, we find

\[ 2E_0 n + \frac{h^2}{4m^2} \partial_\eta^2 n - \frac{3g}{2m} n^2 \]

\[ - \frac{192g_2 n^4 | E_0 |^3}{h^2 g^2} \left[ \frac{\sin h - 4 \chi}{2} \right] \left[ \frac{\sin h - 4 \chi}{2} \right] = 0, (41) \]

where \( \chi = \sqrt{2 | E_0 |} \left[ m g g_2 / h^2 \right] \). The amplitude of the soliton is changed by the interaction in the TOIR approximation \( \chi \neq 0 \). For the isotropic case we can expect that interaction constants \( g \) and \( g_2 \) have same sign. Consequently, the ratio \( g_2 / g \) existing in \( \chi \) is positive. Hence, we have \( \chi > 0 \). It implies the increase of the soliton amplitude \( n_{\text{max}} = n_0 (1 + 3 \chi / 8) \).

Comparison of solutions (40) and (41) is presented in Fig. 6. Thick continuous and dashed curves show the increase of the soliton amplitude in accordance with equation found above. The decrease of width of the soliton is obtained either. Our estimations show that the negative value of \( \chi \) cannot be reached in the isotropic regime, but it can be obtained in the anisotropic case.

Let us assume that \( n_0 = 0 \). Hence, parameter \( | E | = v_\parallel / 2 \) is also simplified. Parameters \( \alpha \) and \( \chi \) transform into \( \alpha_0 = \alpha (c_0 = 0) = n_0 v_\parallel \theta / h \) and \( \chi_0 = \chi (c_0 = 0) = 16m^2v_\parallel^2g_2/h^2g^2 \). Parameter \( \chi_0 \) is proportional to higher degree of the soliton velocity \( v_\parallel / 2 \). Hence, the relative contribution of the TOIR effects (which are proportional to \( \chi_0 \)) grows at the increase of the soliton velocity. The amplitude \( n_{\text{max}} \) grows either.

\[ n(\eta) = \frac{n_0}{\sqrt{\chi_0^2 + 0.5 \chi_0 + 1} \cos \chi_0^2 (2\alpha_0) - 1 + \chi_0}, (44) \]

where \( \chi_0 = 32m^2 | E_0 | g_2 / h^2 g \). The amplitude of the soliton is changed by the interaction in the TOIR approximation \( \chi \neq 0 \). For the isotropic case we can expect that interaction constants \( g \) and \( g_2 \) have same sign. Consequently, the ratio \( g_2 / g \) existing in \( \chi \) is positive. Hence, we have \( \chi > 0 \). It implies the increase of the soliton amplitude \( n_{\text{max}} = n_0 (1 + 3 \chi / 8) \).

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\[ n(\eta) = \frac{n_0}{\sqrt{\chi_0^2 + 0.5 \chi_0 + 1} \cos \chi_0^2 (2\alpha_0) - 1 + \chi_0}, (44) \]
C. Anisotropic case

General structure of equations \((41), (40)\) conserves for the nonisotropic case, but it has a different meaning. Changing are coming via different and anisotropic behavior of parameter \(g_2 = g_2(\theta)\).

Parameter \(g_2\) enters equation \((40)\) via \(\chi\) only. Trace the charge of the soliton profile as a function of \(\theta\).

If \(G_{2,2}\) is relatively small, as in Fig. 3 parameter \(G_2\) has small change as the function of the angle \(\theta\), the Fig. 3 demonstrates the twofold decrease of the \(G_2\) at change of angle \(\theta\) from 0 to \(\pi/2\). Moreover, the sign of \(G_2\) does not change.

The regime of large \(G_{2,2}\) is demonstrated in Fig. 4. The parameter \(G_2\) changes from 0 to 4. Hence, it changes on several order on magnitude. Furthermore, the sign of \(G_0\) changes in this regime. At \(G_2 < 0\) the magnitude of \(|G_2|\) reaches \(|G_2| = 2\). It corresponds to the relatively large negative values of \(\chi\) (an example is presented by the dashed curve in Fig. 5).

\[ G_2 = \chi \frac{u}{v} \left( 1 - \frac{u^2}{2} \right) \]

\[ v = \frac{\varepsilon v_1 + \varepsilon^2 v_2}{2} + ... \]  

Presented in \((17)\) equilibrium concentration \(n_0\) is a constant. We put expansions \((16), (18)\) in equations \((27)\) and \((30)\). Consequently, the system of equation is divided into systems of equations in different orders on \(\varepsilon\).

Equations emerging in the first order by \(\varepsilon\) from the system of equations \((27)\) and \((30)\) have the following form

\[ u \partial_\xi n_1 - n_0 \partial_\xi v_1 = 0, \]

and

\[ m n_0 \partial_\xi v_1 = g n_0 \partial_\xi n_1, \]

and lead to the following expression for the phase velocity \(u\):

\[ u^2 = \frac{g n_0}{m}. \]

Square of phase velocity \(u^2\) should be positive. Consequently \(g\) is positive, i.e.

\[ g > 0. \]  

It corresponds to the repulsive SRI.

We also obtain from \((51)\) a relation between \(n_1\) and \(v_1\) and their derivatives

\[ \partial_\xi n_1 = \frac{n_0}{u} \partial_\xi v_1. \]  

Integrating this equation and using a boundary conditions

\[ n_1, v_1 \to 0 \text{ at } x \to \pm\infty \]

we have

\[ n_1 = \frac{n_0}{u} v_1. \]

In the second order by \(\varepsilon\), from equations \((27)\) and \((50)\), we derive

\[ -u \partial_\xi n_2 + u \partial_\xi n_1 + \partial_\xi (n_0 v_2 + n_1 v_1) = 0, \]

and

\[ -m u (n_0 \partial_\xi v_2 + n_1 \partial_\xi v_1) + m n_0 \partial_\xi v_1 + m n_0 v_1 \partial_\xi v_1 \]

\[ - \frac{h^2}{4m} \partial_{\xi}^2 n_1 = -g n_0 \partial_\xi n_2 - g n_0 \partial_\xi n_1 - g_2 n_0 \partial_{\xi}^2 n_1. \]  

We can express \(n_2\) via \(v_2\) and \(n_1, v_1\) using equation \((56)\) and put it in equation \((57)\). Using \((58)\), we exclude \(v_2\) from the obtained equation \((57)\). Thus, we obtain an equation which contains \(n_1\) and \(v_1\), only. Using \((55)\) and expressing \(v_1\) via \(n_1\) we get the Korteweg-de Vries equation for \(n_1\)

\[ \partial_\xi n_1 + p_{1D} n_1 \partial_\xi n_1 + q_{1D} \partial_{\xi}^2 n_1 = 0. \]
In this equation the coefficients $p_{1D}$ and $q_{1D}$ arise in the form
\[ p_{1D} = \frac{3}{2n_0}, \tag{59} \]
and
\[ q_{1D} = \frac{2mn_0g_2 - h^2}{4mn_0}. \tag{60} \]

From equation (58) we can find the solution in the form of a solitary wave using transformation $\kappa = \xi - V \tau$ and taking into account boundary condition $n_1 = 0$ and $\partial^2 n_1 = 0$ at $\kappa \to \pm \infty$. As the result we find the bright soliton perturbation:
\[ n_1 = \frac{2n_0V}{\cosh^2 \left( \frac{1}{2} \sqrt{\frac{V}{q_{1D}}} \kappa \right)}, \tag{61} \]
where $V$ is the velocity of soliton propagation to the right. From expression $p_{1D} = 3/2n_0$ and solution (61) we can find that a perturbation of concentration is positive. Consequently, obtained solution is a bright soliton (BS). A width of the soliton is given by equation $d = 2\sqrt{q_{1D}}/V$. The BS exists in the case $q_{1D}$ is positive. From conditions $q_{1D} > 0$ and (62) we have
\[ 2mn_0g_2 - h^2 > 0. \tag{62} \]
Relation (62) is fulfilled only in the case when $g_2$ is positive. In the absence of the second interaction constant $g_2$ (i.e. in the Gross–Pitaevskii approximation) the relation (62) does not fulfill and, consequently, the BS does not exist. From equation (62) we find that the second interaction constant $g_2$ should be positive and its module should be more than $h^2/2mn_0$:
\[ g_2 > \frac{h^2}{2mn_0}. \tag{63} \]

Relation (63) corresponds to two times larger value of $g_{2,0}$ than the critical value of $g_{2,0}$ leading to domination of the TOIR over the quantum Bohm potential in the Bogoliubov spectrum (23) (for the isotropic case).

Condition (63) can be considered explicitly for the anisotropic regime. Use parameters $G_{2,0}$ and $G_{2,2}$ introduced in the end of Sec. V. Then, the equation (63) reappears as follows
\[ G_{2,0} - \frac{1}{\sqrt{5}}G_{2,2} \left[ \sin^4 \theta - \frac{1}{3} \sin^2 \theta \cos^2 \theta - 2 \cos^4 \theta \right] > 2. \tag{64} \]

Figs. 1 and 2 show that this condition gives a strong restriction on the direction of the soliton propagation. Moreover, the existence of soliton itself can be restricted.

Fig. 2 shows that the propagation is possible in the small cone near the anisotropy axis. While Fig. 1 demonstrates that in the chosen parameter regime the soliton existence is prohibited. Wherein $G_2$ is positive in Fig. 3 for all angles, but it is too small to fulfill condition (64).

This is generalization of the result obtained earlier for the isotropic case (23). Moreover, the generalization of (23) for the boson-fermion mixtures is presented in Ref. 59. Expectedly the boson-fermion mixture has more complex properties. It demonstrates existence of the second kind solution found in Ref. 59 which has no analog in the single boson species.

**IX. CONCLUSION**

A hydrodynamic model for the BEC with anisotropic short range interaction has been developed.

Bright solitons in the repulsive BEC are described. They exist as a result of the SRI in the third order by the interaction radius. Influence of the anisotropic SRI has been considered for this phenomenon.

Weakly anisotropic behavior of the traditional bright solitons in the attractive BEC has also been demonstrated. The QHD method has been described in the paper allows to study any form of anisotropy of the SRI. Being restricted by the TOIR, the dipole anisotropy is the single contribution which has been explicitly modeled in this paper.

The TIOR approximation reveals in the nonlocal form of the force field. Hence, the force field is not just a gradient of function like for the GP approximation (the FOIR), but it is the gradient of the Laplassian of a function. Therefore, it contains higher derivatives of the concentration and it is more sensitive for the small scale perturbations. Found form of the force field does not allow to derive the Cauchy-Lagrangian integral even for the isotropic SRI. Consequently, an analog of the GP equation (a NLSE) does not exist either. It shows the complexity of the model. Anyway, the basic methods allow to study the fundamental properties of the system: Bogoliubov spectrum of bulk collective excitations and possible solitons.

The anisotropy of the SRI requires a complex form of the electron subshells of valence electrons. Hence, elements with incomplete $p$-, $d$-, $f$- subshells can show this property. The modeling of lanthanides has demonstrated the contribution of anisotropic parts of the SRI in measurable properties. Therefore, the presented hydrodynamic model is considered in context of these results.

The magnetic dipole-dipole interaction is cautiously neglected to stress our attention on the nonlocal and anisotropic parts of the meanfield SRI. Modern models of BEC of lanthanides includes the quantum fluctuations allowing qualitative and quantitative modeling of the quantum Rosenweig instability. The quantum fluctuations existing in the TOIR is the subject of future research which allows a comprehensive analysis of instability of the cloud and regimes of stabilization.

Described in the paper approximate relations between the TOIR nonlocal nonlinearity and the quintic nonlin-
earty addressed to the understanding of the bright solitons in the repulsive BEC found by different groups and by different methods and in different physical systems. Presented analysis provides a relation between whose models.

This relation gives a negative perspective for understanding of the quantum Rosensweig instability via the anisotropic SRI, since it is now well-known that the quintic nonlinearity does not give quantitative agreement with experiments while the quantum fluctuations provide the required results.

Nevertheless, approximate analysis of the force field gives the following comparison between the quantum fluctuations, the TOIR, and quintic nonlinearity: \( n^{5/2}, \Delta n^2, \ n^3 \sim n^{5/2}, n^{8/3}, n^3. \)

Therefore, the TOIR has an intermediate place in terms of the concentration dependence in compare with other two. Moreover, the TOIR shows a different treatment of the different scales since it contains derivatives in contrast with dependence on purely concentration.

Accordingly, the developed model opens a field of study which can accompany the dipolar BEC of lanthanides.

X. APPENDIX A: GENERAL EXPRESSIONS FOR THE QST UP TO THE TOIR APPROXIMATION

QST up to the third order on the interaction radius has the following form in terms of the microscopic many-particle wave function:

\[
\sigma^{\alpha\beta}(r, t) = -\frac{1}{2} \int dR \sum_{i,j,i \neq j} \delta(r - R_{ij}) \alpha_i \beta_j \frac{\partial U(r_{ij})}{\partial r_{ij}} \psi^*(R', t)\psi(R', t)
\]

\[
+ \frac{1}{48} \partial_{\alpha} \partial_{\beta} \int dR \sum_{i,j,i \neq j} \delta(r - R_{ij}) \frac{\partial U(r_{ij})}{\partial r_{ij}} \psi^*(R', t)\psi(R', t)
\]

\[
+ \frac{1}{8} \int dR \sum_{i,j,i \neq j} \delta(r - R_{ij}) \frac{\partial U(r_{ij})}{\partial r_{ij}} \psi^*(R', t)\psi(R', t)
\]

\[
\left[ \partial_{\gamma_1}\psi^*(R',t)\partial_{\delta_1}\psi(R',t) - [\partial_{\gamma_1}\psi^*(R',t)\partial_{\delta_2}\psi(R',t) - [\partial_{\gamma_2}\psi^*(R',t)\partial_{\delta_1}\psi(R',t)]\right]
\]

\[
- \partial_{\gamma_1}\partial_{\delta_1}\psi^*(R',t)\psi(R',t) + [\partial_{\gamma_2}\partial_{\delta_2}\psi^*(R',t)\psi(R',t) - [\partial_{\gamma_1}\partial_{\delta_2}\psi^*(R',t)\psi(R',t)]\right],
\]

where \( r_{ij} \) is the module \( r_{ij} \), \( \partial_{\alpha_1} \) and \( \partial_{\alpha_2} \) are the derivatives on \( R_{ij}^2 \) located in the \( i \)-th and \( j \)-th places correspondingly.

The terms in the second order by the interaction radius are not demonstrated since their contribution is equal to zero (in considering regime of the interaction anisotropy). This conclusion appears at the integration over \( r_{ij} \) (on the corresponding angle variables).

First two terms of equation (65) contribute in the BEC state. Other terms are related to the presence of particles in the excited states. Let us illustrate it for the isotropic regime:

\[
\sigma^{\alpha\beta}(r, t) = \frac{1}{2} g \delta^{\alpha\beta} \left[ 2n_{BEC} n_n + 2n_n^2 + \sum_{g} n_g (n_g - 1) |\varphi_g|^4 \right]
\]

\[
+ g_{2,0} \left[ \frac{1}{6} (\delta^{\alpha\beta} \Delta + 2 \partial^\alpha \partial^\beta) [2n_{BEC} n_n + 2n_n^2 + \sum_{g} n_g (n_g - 1) |\varphi_g|^4] \right]
\]

\[
- \frac{8}{h^2} \left[ m n_{BEC} (\delta^{\alpha\beta} \Pi_{\gamma}^\gamma + 2 \Pi_{n}^{\alpha\beta}) - \delta^{\alpha\beta} j_{BEC}^{\gamma} j_{BEC}^{\gamma} - 2 j_{BEC}^{\alpha} j_{BEC}^{\beta} + 4 n_{BEC} Tr[(\delta^{\alpha\beta} \varphi^\gamma \partial^\gamma + 2 \partial^\alpha \partial^\beta) \rho_n(r, r', t)] \right]
\]
\[-\frac{8}{\hbar^2}[mn_n(\delta^\alpha\beta \Pi_B^{\gamma\alpha\beta} + 2\Pi_B^{\gamma\alpha\beta}) - \delta^\alpha\beta j_B^{\gamma\alpha\beta} j_n^\gamma - 2j_n^{\alpha\beta j_B^{\gamma\alpha\beta}}] + 4n_T [\delta^\alpha\beta \rho^{\gamma\alpha\beta}_B] + 2\partial^\alpha \partial^\gamma \rho_B(r, r', t)]
\]

\[-\frac{8}{\hbar^2}[mn_n(\delta^\alpha\beta \Pi_B^{\gamma\alpha\beta} + 2\Pi_B^{\gamma\alpha\beta}) - \delta^\alpha\beta j_B^{\gamma\alpha\beta} j_n^\gamma - 2j_n^{\alpha\beta j_B^{\gamma\alpha\beta}}] + 4n_T [\delta^\alpha\beta \rho^{\gamma\alpha\beta}_B] + 2\partial^\alpha \partial^\gamma \rho_B(r, r', t)]
\]

\[+ (\delta^\alpha\beta \rho^{\gamma\alpha\beta} + \delta^\alpha\beta \rho^{\gamma\alpha\beta} + \delta^\alpha\beta \rho^{\gamma\alpha\beta}) \sum_g n_g (n_g - 1) [\varphi_g^* \varphi_g (\varphi_g \partial_r \varphi_g - \partial_r \varphi_g \varphi_g) + c.c.] , \quad (66)
\]

where \( \partial' = \partial/\partial r' \), and functions \( n_B^{\gamma\alpha\beta} \), \( n_n \), \( j_B^{\gamma\alpha\beta} \), \( \Pi_B^{\gamma\alpha\beta} \), \( \rho_B^{\gamma\alpha\beta} \), \( \varphi_g \) are functions of \( r \) and \( t \). Subindex \( \text{BEC} \) shows that the function describes the particles in the lower energy state. Subindex \( n \) shows functions related to the particles in the excited states. Summation index \( g \) is the full set of quantum numbers identifying the quantum states. Functions \( \varphi_g \) are the microscopic wave functions of the single particle state with quantum numbers \( g \).

The following representation of the hydrodynamic function is used in equation (66):

\[ n(r, t) = \sum_g n_g \varphi_g^*(r, t) \varphi_g(r, t) \]

(67)

\[ j^{\alpha\beta}(r, t) = \frac{1}{2m} \sum_g n_g [\varphi_g^*(r, t) \hat{p}^{\alpha\beta} \varphi_g(r, t) + \hat{p}^{\alpha\beta} \varphi_g^*(r, t) \varphi_g(r, t)] , \quad (68)
\]

\[ \Pi^{\alpha\beta}(r, t) = \frac{1}{4m} \sum_g n_g [\varphi_g^*(r, t) \hat{p}^{\alpha\beta} \varphi_g(r, t) + (\hat{p}^{\alpha\beta} \varphi_g^*(r, t)) \hat{p}^{\alpha\beta} \varphi_g(r, t) + c.c.] \]

(69)

Tensor \( \Pi^{\alpha\beta} \) is the momentum flux. Originally, \( \Pi^{\alpha\beta} \) appears in the model at the derivation of the Euler equation (see eq. (5) in Ref. 28 or eq. (8) in Ref. 43). Implicitly, tensor \( \Pi^{\alpha\beta} \) can be found in equation (4) as \( m n_v^\alpha v^\beta + p^\alpha v^\beta + T^{\alpha\beta} \), while term \( \partial_\beta (m n v^\beta) \) is modified via the application of the continuity equation.

Function \( 2n_B^{\text{BEC}} n + 2n_n^2 + \sum_g n_g (n_g - 1) |\varphi_g|^4 \) presented in the first term in equation (66) comes from the calculation of \( Tr(n_2(r, r', t)) \) in equation (14). Few intermediate steps of calculation are demonstrated in Sec. III of Ref. 28. The first term in equation (66) reproduces equation (30) of Ref. 28. It is mentioned in Ref. 28 that \( 2n_B^{\text{BEC}} n + 2n_n^2 \) comes from the terms describing two interaction particles located in two different quantum states. However, \( 2n_n^2 \) requires contribution of pairs of particles in the same quantum states (with no contribution of the ground state). Strictly, \( n_n^2 \) should be replaced by \( n_n^2 - \sum_{g, g' \neq g} n_g (n_g - 1) |\varphi_g|^4 \), where \( g_0 \) corresponds to the ground state. Hence, whole function transforms into \( 2n_B^{\text{BEC}} n + 2n_n^2 - \sum_{g, g' \neq g} n_g (n_g + 1) |\varphi_g|^4 + n_{g_0} (n_{g_0} - 1) |\varphi_{g_0}|^4 \). The last term plays major role for the BEC and it has the same form in both presentation.

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