LAD Regression and Nonparametric Methods for Detecting Outliers and Leverage Points

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Abstract

The detection of influential observations for the standard least squares regression model is a question that has been extensively studied. LAD regression diagnostics offers alternative approaches whose main feature is the robustness. In this paper a new approach for nonparametric detection of influential observations in LAD regression models is presented and compared with other classical methods of diagnostics.

Key words: Least Absolute Deviations Regression, Robustness, Outliers, Leverage Points.

1 Introduction

The robustness of LAD to low-leverage outliers, and its susceptibility to high-leverage outliers has been extensively studied in literature [2,3,4]. In this paper we propose a method for nonparametric detection of such influential observations by the use of a technique derived from LAD regression. Robust methods based on the \(L_1\)-norm have been proposed for example in [5,7]. The approach presented here considers suitable perturbations of a given data set and allows a detection of high-leverage observations and outliers from a new viewpoint. These methods answer to natural requirements for robustness, and give a new tool for the analysis of data.

Let \(S \subset \mathbb{R}^{p+1}\) be a finite discrete set of points. In statistics, such a set may represent observations in \(p + 1\) variables. Denote the elements of \(S\) as
(x_i, \ldots, x_{ip}, y_i), where the last variable is explained from the preceding ones by a linear regression model:

\[ y_i = \beta_0 + \sum_{j=1}^{p} \beta_j x_{ij} + \varepsilon_i \quad \text{for } i = 1, \ldots, n, \]

where \( p \) is the number of explanatory variables, \( \varepsilon_i \) are error terms, or deviations, and \( n \) is the number of observations. The LAD regression model is determined by minimizing the sum of the absolute deviations, i.e., the vector \((\beta_0, \beta_1, \ldots, \beta_p) \in \mathbb{R}^{p+1}\) is determined by minimizing on \( \beta_0, \beta_1, \ldots, \beta_p \) the function

\[
F(\beta_0, \beta_1, \ldots, \beta_p) := \sum_{i=1}^{n} \left| y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right|.
\]  

(1)

When a linear LAD regression model is fitted, the hyperplane always passes through at least \( p + 1 \) points [1], although the solution may be non-unique.

For our purposes, we assume that for every dataset and for each subset of it we deal with, the hyperplane which fits the linear LAD regression model is unique, as well as the observation with maximal absolute deviation. These assumptions are reasonable for datasets whose size is sufficiently large and/or the data contain sufficient significant digits. We suppose also that the dataset is such that every \( p + 2 \) points are not in the same hyperplane. With these assumptions the linear LAD regression model is unique and it passes through exactly \( p + 1 \) points. Furthermore, if \( n > p + 1 \), there is always a point which does not belong to the regression hyperplane, so having a positive absolute deviation.

Consider the \( n \) datasets composed by all possible subsets of \( S \) of size \( n - 1 \). Under the above assumptions for each dataset we have a unique solution. For each case, we assign the score 1 to each point through which the fitted model passes and 0 to the other points. We define the final score of each point as the sum of scores over all models fitting the \( n \) datasets. This score is produced by the repeated use of the same points, each time considering a different subset of the original data set, so in a certain sense by bootstrapping the linear LAD regression model.

The point \((x_{k1}, x_{k2}, \ldots, x_{kp}, y_k)\) will be also denoted by \( k \) and its score will be denoted by \( L(k) \).

Similarly, we may define another complementary score function, denoted by \( O(k) \), in the following way. Consider again the \( n \) datasets composed by all possible subsets of \( S \) of size \( n - 1 \). For each subset we consider the LAD regression line and we give the score 1 to the unique (according to the above
assumptions) point which maximizes the absolute distance from the LAD regression line.

We define the score \( O(k) \) as the sum (over all \( n \) possible subsets of \( S \) of size \( n - 1 \)) of scores arising from the LAD regression lines.

In Section 2 we discuss some elementary properties of the LAD regression model. These properties will justify our algorithms for the detection of outliers and leverage points presented in Section 3. In Section 4 we discuss some examples, and compare the results with those obtained using other classical methods.

2 Preliminary Considerations

Under the above assumptions, the sum of \( L \) scores of all points is \( n(p + 1) \), and the sum of \( O \) scores is \( n \), so, under the random variable viewpoint, \( E(L(k)) = p + 1 \) and \( E(O(k)) = 1 \). Now suppose that we have a set of observations, all concentrated in a region and an isolated observation horizontally very far from the others but such that the line of the LAD regression model will pass through it (a typical leverage point). It is likely that the \( L \) score of this observation will be quite high. So a large \( L \) score is synonymous of leverage point. On the other hand, suppose we have a dataset in which all points are roughly in a hyperplane, and a further point far above them (an outlier). The LAD regression model will be very near to this hyperplane, and the score \( L \) of the outlier will be probably zero, but the score \( O \) will be probably \( n - 1 \).

To justify these arguments we state the following theorems.

**Theorem 1** Let \( (x_{11}, \ldots, x_{1p}, y_{1}), \ldots, (x_{n1}, \ldots, x_{np}, y_{n}) \) be \( n \) points in \( \mathbb{R}^{p+1} \). Let \( (x_{n+11}, \ldots, x_{n+1p}, y_{n+1}) \) be an additional point, such that \( (x_{n+11}, \ldots, x_{n+1p}) \) belongs to the interior of the convex hull determined by the set \( \{(x_{11}, \ldots, x_{1p}), \ldots, (x_{n1}, \ldots, x_{np})\} \). If \( x_{n+11}, \ldots, x_{n+1p} \) are fixed and \( |y_{n+1}| \) is sufficiently large, a hyperplane relative to a linear LAD regression model does not passes through \( (x_{n+11}, \ldots, x_{n+1p}, y_{n+1}) \).

**Proof.** Suppose \( p = 1 \) and that for \( i = 1, \ldots, n, \ c < y_{i} < d \). The convex hull hypothesis reduces to

\[
a = \min_{i=1,\ldots,n} \{x_{i}\} < x_{n+1} < \max_{i=1,\ldots,n} \{x_{i}\} = b.
\]

Let \( y = \beta_{0} + \beta_{1}x \) be the line of the linear LAD regression model. Let \( \ell_{1} \) be the horizontal line \( y = (c + d)/2 \). The sum of the absolute deviations of \( (x_{11}, \ldots, x_{1p}, y_{1}), \ldots, (x_{n+11}, \ldots, x_{n+1p}, y_{n+1}) \) does not exceed \((d - c)(n + 1) + \)
\[ |y_{n+1}| + |d| + |c| \]. So
\[ F(\beta_0, \beta_1) < (d - c)(n + 1) + |d| + |c| + |y_{n+1}|. \]
On the other hand, if a line \( y = \beta_0^* + \beta_1^* x \) passes through \((x_{n+1}, y_{n+1})\), if it is a linear LAD regression model, it will pass also through another point \((x_i, y_i)\) for a suitable \( i \), so for sufficiently large \(|y_{n+1}|, |\beta_1^*| > |y_{n+1}|/(b - a)\). Hence there exists \( \alpha > 1 \) such that for sufficiently large \(|y_{n+1}|,\)
\[ F(\beta_0^*, \beta_1^*) > \alpha |y_{n+1}|. \]
For sufficiently large \(|y_{n+1}| \) we have
\[ F(\beta_0^*, \beta_1^*) > \alpha |y_{n+1}| > (d - c)(n + 1) + |d| + |c| + |y_{n+1}| > F(\beta_0, \beta_1), \]
and therefore, the line relative to the the LAD regression model cannot pass through \((x_{n+1}, y_{n+1})\).

For \( p > 1 \) the proof is similar. \( \square \)

**Theorem 2** Let \((x_{11}, \ldots, x_{1p}, y_1), \ldots, (x_{n1}, \ldots, x_{np}, y_n)\) be \( n \) points in \( \mathbb{R}^{p+1} \).
Let \((x_{n+11}, \ldots, x_{n+1p}, y_{n+1})\) be an additional point. If \( y_{n+1} \) is fixed and \( \sum_{i=1}^{p} |x_{n+1,i}| \)
is sufficiently large, a linear LAD regression model will pass through \((x_{n+11}, \ldots, x_{n+1p}, y_{n+1})\).

**Proof.** The proof is an exercise, and the approach is similar to the proof of Theorem 1. \( \square \)

**Theorem 3** Let \((x_{11}, \ldots, x_{1p}, y_1), \ldots, (x_{n1}, \ldots, x_{np}, y_n)\) be \( n \) points in \( \mathbb{R}^{p+1} \),
with \( n > p + 1 \). Let \( L(k) \) and \( O(k) \), for \( k = 1, \ldots, n \), defined as in Section 1.
Then \( L(k) + O(k) \leq n - 1 \).

**Proof.** This is a consequence of the fact that for each of the \( n \) subsets the scores are shared among distinct points, and, by the above assumptions, a point cannot collect a score for both \( L \) and \( O \), since, for \( L \) it must have zero residual, and for \( O \) a strictly positive absolute residual. And each point appears exactly \( n - 1 \) times in the \( n \) subsets. \( \square \)

### 3 The Algorithms

In this section we propose two algorithms based on the previous section. The aim of Algorithm 1 and Algorithm 2 is to detect leverage points and outliers respectively.
Algorithm 1.

(1) Consider a data set $S$ of size $n$. Let $A$ and $B$ be empty sets.
(2) Let $m$ be the size of $S$.
(3) Consider all $m$ subsets of $S$ of size $m - 1$ and fit the LAD-regression model for each subset.
(4) Compute $L(k)$ for each point of $S$ and select $k_1 \in S$ which maximizes $L(k)$.
(5) If $L(k_1) \geq \frac{8}{9}(m-1)$ and $L(k_1) \geq \frac{3}{4}(n-1)$ then move $k_1$ into $B$ and move the eventual points of $A$ into $S$; otherwise move $k_1$ into $A$.
(6) If the size of $S$ does not exceed $\frac{9}{10}n$ then the process stops and $B$ represents the leverage points; otherwise go to step 2.

In this algorithm, the elements of $S$ are transferred in a set $B$ of leverage points, or in a temporary set $A$ where points that did not reached a sufficient score to be classified as leverage points, are suitable to be reconsidered after that another point has been detected. This trick avoids the masking effect.

Discriminating values $\frac{8}{9}(m-1)$, $\frac{3}{4}(n-1)$ and $\frac{9}{10}n$ for the score function $L$, have been empirically determined, by testing on several data sets and several combinations of values. They have, however, a natural interpretation. When there is a unique leverage point, almost all $m$ regression models detect it, so its $L$ score is near to the maximum. When there are more leverage points, scores may be very different, and the masking effect can produce relatively small scores. Finally, we keep into account the size of $S$, to determine how many leverage points a data set may have. The process stops when the size of set $S$ does not exceed $\frac{9}{10}n$, so with our method a data set cannot have more than $\frac{1}{10}n$ leverage points.

Algorithm 2.

(1) Consider a data set $S$ of size $n$. Let $C$ and $D$ be empty sets.
(2) Initialize the last maximum score (LMS) by 0.
(3) Let $m$ be the size of $S$.
(4) Consider all $m$ subsets of $S$ of size $m - 1$ and fit the LAD-regression model for each subset.
(5) Compute $O(k)$ for each point of $S$ and select $k_1$ which maximizes $O(k)$.
(6) If $O(k_1) = m - 1$
   a) Then if $O(k_1) = \text{LMS} - 1$ or $\text{LMS} = 0$ then move $k_1$ into $D$, put $\text{LMS} = O(k_1)$ and move the eventual points of $C$ into $S$; otherwise the process stops and $D$ represents the outliers.
   b) Otherwise move $k_1$ into $C$.
(7) If the size of $S$ does not exceed $\frac{4}{5}n$ then the process stops and $D$ represents the outliers; otherwise go to step 3.
In this algorithm, the set $D$ contains the points classified as outliers and the set $C$ contains the points that did not reach the score to be classified as outliers, and that are suitable to be reconsidered in further steps until the algorithm stops.

We can note the two proposed algorithms have a similar structure. However, the main difference is a feature of Algorithm 2: the outliers have decreasing scores $O_1, O_1 - 1, O_1 - 2$, and so on. The algorithm stops when this sequence cannot be continued.

The process also stops when the size of set $S$ does not exceed $\frac{4}{5}n$, so here a data set cannot have more than $\frac{4}{5}n$ outliers.

Discriminating values for the score function $O$, have been also empirically determined.

4 Some Examples

In this section we illustrate the proposed algorithms and compare them with other two methods using several real and simulated data sets.

One of the method is the P-R plot proposed by Hadi [5] to aid in classifying observations as leverage points, outliers or combinations of both. Some authors suggested that points with $h_{ii} > \frac{2(p+1)}{n}$, where $h_{ii}$ is the $i$th diagonal element of matrix $H$, $p$ is the number of predictors and $n$ the number of observations, can be classified as leverage points and the points with $\frac{r_i}{\hat{\sigma}\sqrt{1-h_{ii}}} > 2$, where $r_i$ is the residual of the $i$th observation, $h_{ii}$ is the $i$th diagonal element of matrix $H$ and $\hat{\sigma}$ is an estimator of standard deviation of the errors, can be classified as outliers. In what follows the use of these suggested cut-off points to classify the observations will be intended as classical methods.

The first data set ‘Telephone’ relate the number of international telephone calls from Belgium (in tens of millions in minutes) to the variable year for 24 years and can be found in [7]. Cases 15-20 are unusually high and they are outliers. The second one ‘Hawkins’ consists of 75 observations in four dimensions, one response variable and three predictor variables, and can be found in [6]. It has been constructed for the study of special pathological phenomena in detection of outliers and leverage points and the cases 1-10 are outliers and leverage points. The data set ‘Scottish’ describes how the record times (in seconds) of 35 Scottish Hill races is related to two predictor variables, distance of race (in miles) and climb (in feet), and can be found in [5]. The data contain two clear outliers (observation 7 and 18). The last two data sets have been created by the authors. The data set ‘twovariables’ consists of 56
observations on one predictor variable and a response variable. The predictor was created as uniform \((0, 10)\) and the response variable to be consistent with the model \(Y = X_1 + 4 + \varepsilon \) with \(\varepsilon \sim N(0, 1)\). Three observations (51–53) have been conceived as leverage points and three others (54–56) as outliers. The other data set ‘threevariables’ is the three variable equivalent to the preceding one (two predictor variables).

Computation has been performed with a computer code in Splus, and the results are summarized in Table 1.

| Data       | Method          | Leverages | Outliers   |
|------------|-----------------|-----------|------------|
| Telephone  | Classical Method| -         | 20         |
|            | Hadi’s Method   | -         | 19, 20     |
|            | Our Results     | -         | 17-20      |
| Hawkins    | Classical Method| 12-14     | 7, 11-14   |
|            | Hadi’s Method   | 14        | 7, 11-14   |
|            | Our results     | 3-6, 9, 10, 13 | 11-14     |
| Scottish   | Classical Method| 7, 11, 33, 35 | 7, 18     |
|            | Hadi’s Method   | 7, 11    | 7, 18     |
|            | Our Results     | 11, 17, 35 | 7, 18, 33 |
| Twovariables| Classical Method| 51-53   | 54-56     |
|            | Hadi’s Method   | 51-53   | 54-56     |
|            | Our results     | 52      | 54-56     |
| Threevariables| Classical Method| 18, 51-53 | 54-56 |
|            | Hadi’s Method   | 51-53   | 54-56     |
|            | Our Results     | 51-53   | 9, 37, 54-56 |

Table 1. Detection of outliers and leverage points according to different methods.

As we can see in Table 1, our proposed method performed very well in detecting all outliers in the data set 'Telephone'. The other methods failed to identify all of them because the observation 19 and 20 mask all the others.

In the case of 'Hawkins' data, our proposed method as well as the other two methods failed to identify the outliers. The outliers are all also swamped in the good cases 11–14. On the other hand, our method detected almost all leverage points.

In the data set 'Scottish', Table 1 shows that all three methods identified
correctly the observation 7 and 18 as outliers. These observations mask the observation 33 detected by our proposed method. The observation 11 is suitably detected as leverage point by all three methods, but there are others observations identified as leverage points only in one or two methods.

For the simulated data sets, all methods performed very well in detecting all outliers. However, our proposed method failed to identify all leverage points.

5 Conclusion

The computation of the scores requires the determination of a certain number of LAD regression models, and this is computationally longer than usual methods. However it is important to note that the principle is very simple, and takes into account natural requirements for robustness in the detection of influential observations. Nowadays, the performances of common notebooks are largely sufficient to perform in a few seconds the computations for the above examples, so the new tools are suitable for applications in the statistical methodology.

A code has been implemented in Splus and is available at the web site of the second author [http://www.unine.ch/statistics/melfi/lad.html](http://www.unine.ch/statistics/melfi/lad.html). A variety of data sets, including simulated datasets used in Section 4, is also available on the same web site.

References

[1] Arthanari, T.S.; Dodge, Y., 1993. Mathematical Programming in Statistics. Classic edition, John Wiley and Sons, New York.

[2] Dodge, Y. (Ed.) 1987, Statistical data analysis based on the $L_1$-norm and related methods, Elsevier/North-Holland, New York; Amsterdam.

[3] Dodge, Y. (Ed.) 1997, $L_1$-statistical procedures and related topics, Institute of Mathematical Statistics, Hayward.

[4] Dodge, Y. (Ed.) 2002, Statistical Data Analysis Based on the $L_1$-Norm and Related Methods, Birkhäuser.

[5] Hadi, A., 1992. A new measure of overall potential influence in linear regression, Comp. Stat. and Data Analysis, 14, 1-27.

[6] Hawkins, D.M., Bradu, D. and Kass, G.V., 1984. Location of several outliers in multiple regression data using elemental sets, Technometrics, 26, 197-208.
[7] Rousseeuw, P.J. and Leroy, A.M., 1987. Robust Regression and Outlier Detection. John Wiley and sons, New York