Dynamical fluctuations in the one particle density –
comparison of different approaches

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Abstract

Diffusion coefficients are obtained from linear response functions and from the quan-
tal fluctuation dissipation theorem. They are compared with the results of both the
theory of hydrodynamic fluctuations by Landau and Lifschitz as well as the Boltzmann-
Langevin theory. Sum rules related to conservation laws for total particle number,
momentum and energy are demonstrated to hold true for fluctuations and diffusion
coefficients in the quantum case.

1 Introduction

For a theoretical description of phenomena like nuclear multifragmentation mean field theories
have to be extended to include the dynamics of fluctuations. In the case of the semi-classical
Boltzmann-Ühling-Uhlenbeck (BUU) equation a first step was made by Bauer et al. [1]. Their
numerical realization of the collision term corresponds to adding to the BUU collision term a
further contribution. The average of the latter should vanish since BUU is meant to describe
the effect of the collisions on average. In this sense the additional term acts like a fluctuating
force. However, in [1] detailed properties of the former as its second moment (or correla-
tion function) were not investigated. Moreover, in a later work [2] the fluctuations that are
obtained within the method of [1] were judged unsatisfactory.
Possibilities to extend theories for average dynamics were proposed before by Landau and Lifschitz [3] for hydrodynamics, by Abrikosov and Khalatnikov [4] for Landau Fermi Liquid theory and by Bixon and Zwanzig [5] for the Boltzmann equation. These authors augment the equations for the averages with a fluctuating term and derive explicitly its correlation function, albeit in different ways.

The task to describe heavy ion collisions, especially multifragmentation, initiated the more recent approaches by Ayik and Grégoire [6], Randrup and Remaud [7] and Hofmann, the author and Tsekhmistrenko [8]. In [7] and [8] the equation of motion for the fluctuations (the second cumulants) of the distribution function (the Wigner transform of the one particle density matrix) is addressed, as first step. The inhomogeneity of this equation, called diffusion coefficient in analogy to Fokker Planck equations, can be related to the correlation function of the fluctuating force. The latter is derived in [6] by investigating the hierarchy of equations for the n-particle densities. It is found to coincide with the one obtained from the expression of [7] for the diffusion coefficient – for the case that the equation for the average distribution function is given by the BUU equation with a Markovian collision term.

In the approach of [8] the diffusion coefficient is determined to guarantee that stationary solutions of the equations of motion for the dynamical fluctuations are, for a stable system, equal to the equilibrium values. The latter can be determined by means of the fluctuation dissipation theorem [9] to include quantum effects beyond Pauli blocking. In a recent work [10] these quantum effects were investigated in the density range between one tenth of up to two times the saturation density. They were found to be sizable for temperatures below 15 MeV (This temperature range is of special interest as it is the one in which, in a certain density range below saturation density, homogeneous nuclear matter can become mechanically unstable against isothermal density variations, in other words, the temperature range of the isothermal spinodal region of nuclear matter.).

In the approaches of [6] and [7] – both now known under the name Boltzmann-Langevin theory – correlation function of the fluctuating force and diffusion coefficient, respectively, are determined as functionals of the distribution function. The latter is allowed to be the one of a non equilibrium state of the system. However, the numerical evaluation of these functionals turns out to be too time consuming as to presently allow for an application to a realistic (six dimensional) system. Therefore, recently [11] the expression for the diffusion coefficient was proposed to be evaluated at the equilibrium distribution, similar to the approach of [8].

It is the main aim of the present paper to work out the relation of the approach of [8] based on the (quantal) fluctuation dissipation theorem to those of Landau–Lifschitz and Bixon–Zwanzig as well as to the Boltzmann-Langevin theory. The connection to the latter will be established for an equilibrium state in the case that the classical form of the fluctuation dissipation theorem is applied.
In Boltzmann-Langevin theory sum rules reflecting conservation laws for total particle number, momentum and energy are fulfilled by the expressions obtained for the correlation function of the fluctuating force [6] and for the diffusion coefficients [12]. These sum rules will be demonstrated to hold true also for fluctuations and diffusion coefficient found in the approach of [8] – in the classical limit as well as for the quantum corrections. A proper interpretation will be given for previous results [8] which seem to violate these sum rules.

## 2 Review of general formalism

In this section the basic ideas of the approach of [8, 10] will be reviewed using a more generally applicable formulation. It will be explained in which way equations of motion for the fluctuations are obtained starting from the equations for the first moments or averages. The former turn out to be inhomogeneous. The knowledge of this inhomogeneity, the diffusion coefficient, can serve as the first step in the determination of the correlation function of the fluctuating force. Moreover, the equations obtained can be used to study the behaviour of the dynamical fluctuations, at least in the range of small amplitudes [13].

The starting point are the equations of motion for averages \( A_\nu \) of a set of dynamical variables \( \hat{A}_\nu \),

\[
A_\nu(t) = \langle \hat{A}_\nu \rangle_t ,
\]

the system is to be described with. These equations are assumed to have the form

\[
\frac{\partial}{\partial t} A_\nu + D_{\nu\nu'} A_{\nu'} + c_\nu = 0 .
\]

Here and in the following summation over repeated indices is understood to be performed. The linearity in \( A_\nu \) may have been obtained by linearization. The \( c_\nu \) are independent of \( A_\nu \) and symbolize terms containing either external fields or zero order contributions remaining from the linearization procedure or both.

To have a specific example in mind let us look at the case where the system is described by the distribution function (which is the Wigner transform of the one particle density matrix) \( n_\nu(p, r, t) \). In the semi classical limit the equation of motion of the latter is the BUU equation which is nonlinear in \( n_\nu(p, r, t) \). One, therefore, has to perform a linearization to obtain an equation that can be written in the form of (2). The averages \( A_\nu \) would then be the deviation

\[
A_\nu(t) \leftrightarrow \delta n_\nu(p, r, t) = n_\nu(p, r, t) - n_0^0(\nu, p) .
\]

The index \( \nu \) stands for position \( r \) and momentum \( p \) and eventually for a spin-isospin index, the latter being suppressed throughout the paper. So far, the reference distribution might be...
the one of a local equilibrium or some other state as well. One simply would obtain expressions for $D_{\nu\mu}$ and $c_\nu$ different from those found when linearizing around global equilibrium.

The fluctuations of the dynamical variables are defined as:

$$\sigma_{\nu\mu}(t) = \langle \frac{1}{2} \left[ \hat{A}_\nu, \hat{A}_\mu^\dagger \right] \rangle_t - A_\nu(t)A_\mu^*(t)$$  \hspace{1cm} (4)

From this form the property

$$\sigma_{\nu\mu} = \sigma_{\mu\nu}^*$$  \hspace{1cm} (5)

follows even for non hermitian $\hat{A}_\nu$, as one has to deal with considering the spatial Fourier transform of $\delta n_p(r, t)$.

Since the equations of motion (4) for the averages $A_\nu$ have been assumed linear, the corresponding ones for the $\hat{A}_\nu$ differ from the former only by a term, called the fluctuating force, whose average vanishes. Thus the equations for the fluctuations are found to read

$$\frac{\partial}{\partial t} \sigma_{\nu\mu} + D_{\nu\nu'}\sigma_{\nu'\mu} + (D_{\mu\mu'}\sigma_{\mu'\nu})^* = 2 d_{\nu\mu}$$  \hspace{1cm} \text{(6)}$$

where the diffusion coefficient $d_{\nu\mu}$ includes all terms arising from the fluctuating force. In general, $d_{\nu\mu}$ depends on the momentary state of the system: looking at the hierarchy of n-particle densities mentioned before it becomes clear that $d_{\nu\mu}$ stands for terms containing the correlated part of the three particle density. To calculate these terms exactly would be equivalent to solving the full hierarchy which is, of course, not possible. One, therefore, has to use approximations. The one made in the present approach is the following. For a stable system $d_{\nu\mu}$ is chosen in such a way that the stationary solution of (6) is equal to the equilibrium fluctuations; for an unstable system $d_{\nu\mu}$ is obtained \[10\] as the analytic continuation of the expressions found for the stable system. This is achieved by setting

$$2d_{\nu\mu} = D_{\nu\nu'}\sigma_{\nu'\mu}^{\text{st}} + (D_{\mu\mu'}\sigma_{\mu'\nu}^{\text{st}})^*$$  \hspace{1cm} \text{(7)}$$

where, for the stable system, $\sigma_{\nu\mu}^{\text{st}}$ stands for the equilibrium fluctuations, while it represents the analytical continuation of the latter in the unstable case.

The diffusion coefficient obtained in this way depends on both the equilibrium properties – via $\sigma_{\nu\mu}^{\text{st}}$ – and the dynamics of the system – via $D_{\nu\mu}$. Let us look again at the example previously discussed where the linearized BUU-equation is the one from which $D_{\nu\mu}$ is to be obtained. In this case $D_{\nu\mu}$ contains information not only about the drift terms (involving the equilibrium mean field) but also about the collision term and thus about the differential scattering cross section. Therefore, the diffusion coefficient depends on both the mean field and the residual interaction causing the two particle collisions.

Clearly, this choice for the diffusion coefficient is an approximation to the exact term appearing as inhomogeneity in (6). However, in the case that the system approaches equilibrium, the value of the latter approaches the one of the former, which thus can serve as an approximation valid for systems not too far from equilibrium.
Since $\mathcal{D}_{\nu\mu}$ is determined by the form (2) of the equation of motion for $A_{\nu}$, what is left in order to calculate the diffusion coefficient is the determination of the equilibrium fluctuations (and their analytical continuation to the unstable case). For this purpose the relation of the latter to dissipative processes in the average dynamics as expressed in the fluctuation dissipation theorem is exploited. The latter provides the connection between the equilibrium fluctuations and the corresponding response function which can be obtained from the equation of motion for the averages $A_{\nu}$ as follows.

Introducing an external field $U_{\text{ext}}$ in such a way that the change in the Hamiltonian due to $U_{\text{ext}}$ is given by

$$\delta H = \hat{A}_{\nu} U_{\nu}(t),$$

one defines the response function as the negative functional derivative of the Fourier transform $\langle \hat{A}_{\nu} \rangle (\omega)$ of the average $A_{\nu}(t)$ with respect to the external field:

$$\chi_{\nu\mu}(\omega) = -\frac{\delta \langle \hat{A}_{\nu} \rangle (\omega)}{\delta U_{\nu}(\omega)} \bigg|_{U_{\text{ext}}=0}$$

This derivative can be calculated from the equation of motion of the $A_{\nu}$. The dissipative processes are related to the quantity

$$\chi''_{\nu\mu}(\omega) = \frac{1}{2i} \left( \chi_{\nu\mu}(\omega) - \chi_{\mu\nu}(\omega^*)^* \right)$$

which, therefore, is usually called, for real arguments $\omega$, the dissipative part of the response function (9).

According to the fluctuation dissipation theorem (9) the $\sigma_{\nu\mu}^{st}$ are given by the sum of two contributions:

$$\sigma_{\nu\mu}^{st} = \tilde{\sigma}_{\nu\mu}^{st} + \Delta \sigma_{\nu\mu},$$

(11)

to be explained in the following.

The first contribution on the right hand side shall be given by

$$\tilde{\sigma}_{\nu\mu}^{st} = \hbar \int_C \frac{d\omega}{2\pi} \coth \left( \frac{\hbar \omega}{2T} \right) \chi''_{\nu\mu}(\omega),$$

(12)

where T stands for the temperature. The integral in (12) is to be taken along a contour $C$. For a stable system $C$ lies on the real axis. In this case the stationary solution $\sigma_{\nu\mu}^{st}$ (11) can be interpreted as the equilibrium values of the fluctuations and (11) with (12) is one version of the fluctuation dissipation theorem. For an unstable system, $C$ has to be deformed in such a way that it lies above all poles of the response function $\chi(\omega)$ and below all poles of the function $\chi(\omega^*)$ in the complex frequency plane, crossing the line Re $\omega = 0$ in the interval $-2\pi T/\hbar < \text{Im} \omega < 2\pi T/\hbar$. 5
Let us turn now to the second contribution to $\sigma^{st}$, i.e., to $\Delta\sigma_{\nu\mu}$. Its physical origin lies in the possibility that, due to conserved quantities, an initially equilibrated system does not relax to the new thermal equilibrium (grand-, micro- or canonic) after an adiabatic switch on of an external perturbation.

To shed some more light on this let us look at the case where the equilibrium distribution is assumed to be the one for the canonical ensemble. Then $\Delta\sigma_{\nu\mu}$ is given by the positive semi definite matrix

$$
\Delta\sigma_{\nu\mu} = T \left( \chi^T_{\nu\mu} - \chi_{\nu\mu}(\omega = 0) \right)
$$

(13)

where the isothermal susceptibility $\chi^T$ can be calculated from the behaviour of the equilibrium values of $A_\nu$ as functions of the external field:

$$
\chi^T_{\nu\mu} = - \left( \frac{\delta < \hat{A}_\nu >^\text{eq}}{\delta U^\text{ext}_\mu} \right) \bigg|_{U^\text{ext}=0}
$$

(14)

Here, the change in the equilibrium values of $A_\nu$ has to be taken under the subsidiary condition of a constant temperature. The difference between isothermal susceptibility and static response as appearing in (13) can then be expressed in the following way: using the isothermal susceptibility as scalar product in the space of dynamical variables one can determine an orthogonal basis $\{\mathcal{V}_c\}$ in the subspace of those variables representing conserved quantities. One then finds:

$$
\chi^T_{\nu\mu} - \chi_{\nu\mu}(\omega = 0) = \sum_c \chi^T_{\nu\mathcal{V}_c} \chi^T_{\mathcal{V}_c\mu} \chi^T_{\mathcal{V}_c\mathcal{V}_c}
$$

(15)

where the sum on the right hand side extends over all elements $\mathcal{V}_c$ of the basis. This formula clearly states the relation of $\Delta\sigma$ to the existence of conserved quantities.

These considerations are valid for a system in contact with a heat bath preserving the temperature. However, if the system to be investigated has no heat bath, one might conclude that, for such a case, $\Delta\sigma_{\nu\mu}$ has to be modified as compared to (13), like it was done in [10].

In the present paper the following point of view is taken: Quite generally a contribution $\Delta\sigma$ to $\sigma^{st}$ must be expected to exist; there will be no calculation of the former; for the comparison with the other approaches only $\tilde{\sigma}^{st}$ will be used. It will be found that the former are consistent with $\Delta\sigma = 0$. Actually, we would like to use this result as justification to identify $\tilde{\sigma}^{st}$ (12) with $\sigma^{st}$ (11). Nevertheless, $\tilde{\sigma}^{st}$ will be distinguished from $\sigma^{st}$ by keeping the tilde whenever the expression on the right hand side of (12) is to be referred to.

At this stage it is possible to briefly compare with the approach of Bixon and Zwanzig [6]. (For more details see the appendix A.) These authors address a stable system with the Boltzmann equation as the one governing the average dynamics of the distribution function. For comparison of the results of the two methods one, therefore, has to define $D_{\nu\mu}$ and $c_\nu$ by

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1 A matrix $M$ is called positive semi definite, if $\sum_{\nu,\mu} M_{\nu\mu} x_\nu x_\mu \geq 0$ for any real $x_\nu$. 

writing the linearized Boltzmann equation in the form (3) with (4). One finds the product of the expression on the right hand side of (7) with a $\delta-$function in time to equal the expression of [5] for the correlation function of the fluctuating force. This is consistent with the assumption of a vanishing memory for the fluctuating force made in [5]. The value for the equilibrium fluctuations used in [5] is the same as the one obtained from (12), provided the classical limit of the latter is taken and the response function is calculated from the Boltzmann equation (Please notice, that in the form of the latter used in [5] no particle interaction is taken into account.). This can be concluded from the results in [10].

3 Fluctuations for Hydrodynamics

In this section the method described in the previous one for the determination of the diffusion coefficients is applied to hydrodynamics. Agreement with the method of Landau and Lifschitz [3] will be established. For the case of hydrodynamics the index $\nu$ stands for a continuous and a discrete one. The former is given by the spatial position $r$ or – in the corresponding Fourier space – the wave vector $k$. The discrete one is meant to tell whether $A_{\nu}$ is the deviation $\delta \rho$ of the mass density from the equilibrium value $\rho_0$, the deviation $\delta q = T_0 \rho_0 \delta s$ of the density of heat or the momentum density $g$. (Here $s$ is the entropy per unit mass and $T_0$ is the equilibrium value of the temperature.)

The linearized hydrodynamic equations for the dynamical variables, i.e. those corresponding to $A_{\nu}$, read (see e.g. §132 of [3]):

$$\frac{\partial}{\partial t} \hat{\rho}(k, t) + i k \hat{g}(k, t) = 0$$

(16)

$$\frac{\partial}{\partial t} \hat{g}_i(k, t) + i k_j \Pi_{ij}(k, t) = i k_i s_{ij}(k, t)$$

(17)

$$\frac{\partial}{\partial t} \hat{q}(k, t) + i k_j j_T(k, t) = -i k G(k, t)$$

(18)

Here beside the usual stress tensor

$$\Pi_{ij}(k, t) = \delta_{ij} P(k, t) - i \left( D_l (k_j \hat{g}_i + k_i \hat{g}_j) + (D_l - 2 D_t) \delta_{ij} k \hat{g} \right)$$

(19)

and heat current

$$j_T(k, t) = -i k \kappa \delta T(k, t)$$

(20)

where $P$ is the pressure, $D_l$ and $D_t$ are the longitudinal and transversal diffusion constants, respectively, and $\delta T$ is the deviation of the temperature from $T_0$, the fluctuating forces $s_{ij}$ and $G$ appear. In the form suitable for later purposes their correlation functions are given by:

$$\left\langle \frac{1}{2} \left[ s_{ij}(k, \omega) , s_{mn}(-k', t) \right] \right\rangle = (2\pi)^3 \delta(k - k') 2E(\omega) e^{i\omega t} \rho_0 D_{ijmn}$$

(21)
\[
\left\langle \frac{1}{2} [G_i(\mathbf{k}, \omega), G_j(-\mathbf{k}', t)]_+ \right\rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') 2E(\omega) e^{i\omega t} \kappa T_0 \delta_{ij}
\]

where \(2E(\omega) = \hbar \omega \coth(\hbar \omega / 2T)\) and \(D_{ijmn} = D_t(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + (D_l - 2D_t) \delta_{ij} \delta_{mn}\). The mixed correlation vanishes: \(< G_{sij} >= 0\). In [3] the correlation functions are obtained using their relation (e.g. §121 of [16]) to coefficients appearing in the expression for the rate of change of the entropy (see §119 of [16]).

The appearance of the factor \(E(\omega)\) implies that in the quantum mechanical case, i.e. for \(\hbar \neq 0\), the correlation function exhibits, in general, a finite correlation time. Furthermore, \(E(\omega)\) reminds one that diffusion constants \(D_l, D_t\) as well as thermal conductivity \(\kappa\) are functions of frequency, the even parts of which decrease for large frequencies asymptotically, at least, as \(1/\omega\) (for the right hand sides of (21) and (22) to be proper Fourier transforms). This feature ensures convergence of frequency integrals which will have to be dealt with in the following.

The structure of the equations for the averages follow by putting the right hand side of (16)–(18) equal to zero. The equations of motion for the fluctuations of \(\delta \rho, g\) and \(\delta q\) can be obtained in different ways. One of them is to directly use the equations (16)–(18) for the dynamical variables exploiting the relations (21), (22) for the fluctuating forces. This one will be applied later. An other possibility is the one described in section 2. For the latter one needs to know the corresponding response functions, as the density density response.

If external fields are introduced in the equations of motion according to the rules

\[
\begin{align*}
P & \rightarrow P + \rho_0 U^{ce}_\rho \\
g & \rightarrow g + \rho_0 U^e \\
\delta T & \rightarrow \delta T + T_0 U^e_q
\end{align*}
\]

then the interaction energy takes the form [17]:

\[
H^{ext} = \int d\mathbf{r} \left( \delta \rho U^{ce}_\rho + g U^e + \delta q U^e_q \right)
\]

This is exactly of form (8). Therefore, the equations for the response functions can be obtained taking the functional derivatives of (16)–(18) with respect to the external fields introduced according to (23). For example, the set of equations containing the density density response \(\chi_{\rho\rho}(\mathbf{k}, \mathbf{k}', \omega) = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \chi_{\rho\rho}(\mathbf{k}, \omega)\) is found by calculating the derivative with respect to \(U^{ce}_\rho\). The solution reads:

\[
\chi_{\rho\rho}(\mathbf{k}, \omega) = \frac{-k^2}{\omega^2 + i\omega D_t k^2 - c^2 k^2 - i dk^4 c_s^2 \frac{1 - c_v/c_p}{\omega + idk}}
\]

if the following abbreviations are used for derivatives of thermodynamic functions:

\[
\begin{align*}
d & = \frac{\kappa}{\rho c_v} \\
c_x & = \frac{T}{V \rho} \left( \frac{\partial S}{\partial T} \right)_x \\
c^2 & = \left( \frac{\partial P}{\partial \rho} \right)_S \\
a & = \left( \frac{\partial T}{\partial \rho} \right)_S
\end{align*}
\]
The other response functions are found to be related to $\chi_{\rho\rho}$ in the following way:

$$\chi_{\rho_i\rho}(k, \omega) = \chi_{\rho\rho}(k, \omega) = k_i \frac{\omega}{k^2} \chi_{\rho\rho}(k, \omega)$$  \hspace{1cm} (27)$$

$$\chi_{q\rho}(k, \omega) = \chi_{\rho q}(k, \omega) = -i\kappa a k^2 \chi_{\rho\rho}(k, \omega)$$  \hspace{1cm} (28)$$

$$\chi_{\rho\rho}(k, \omega) = \chi_{\rho\rho}(k, \omega) = k_i \frac{\omega}{k^2} \chi_{\rho\rho}(k, \omega)$$  \hspace{1cm} (29)$$

$$\chi_{q\rho}(k, \omega) = \chi_{\rho q}(k, \omega) = -i\kappa a k^2 \chi_{\rho\rho}(k, \omega)$$  \hspace{1cm} (30)$$

$$\chi_{s_i\rho_j}(k, \omega) = \frac{k_i k_j}{k^2} \chi_l(k, \omega) + \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \chi_t(k, \omega)$$  \hspace{1cm} (31)$$

$$\chi_t(k, \omega) = \frac{\omega^2}{k^2} \chi_{\rho\rho}(k, \omega) + \rho_0$$  \hspace{1cm} (32)$$

$$\chi_t(k, \omega) = \rho_0 \frac{i D_i k^2}{\omega + i D_i k^2}$$  \hspace{1cm} (33)$$

From these functions the equilibrium values of the corresponding fluctuations can be calculated – at least the contributions $\tilde{\sigma}_{\rho\rho}$ – combinations of which form the diffusion coefficients.

According to (6), with the operator $D$ defined by (2) and (16)–(18), the equations of motion for the fluctuations read:

$$\frac{\partial}{\partial t} \sigma_{\rho\rho}(k, k') + i k_j \sigma_{\rho_j\rho} - i k'_j \sigma_{\rho\rho_j} = 0$$  \hspace{1cm} (34)$$

$$\frac{\partial}{\partial t} \sigma_{\rho\rho_i}(k, k') + i k_j \sigma_{\rho_j\rho_i} - i k'_j \sigma_{\rho\rho_i j} = 2 d_{\rho\rho_i}$$  \hspace{1cm} (35)$$

$$\frac{\partial}{\partial t} \sigma_{\rho\rho}(k, k') + i k_j \sigma_{\rho_j q} + \kappa k'^2 \sigma_{\rho T} = 2 d_{\rho q}$$  \hspace{1cm} (36)$$

$$\frac{\partial}{\partial t} \sigma_{s_i\rho_j}(k, k') + i k_l \sigma_{s_{l i}} - i k'_l \sigma_{s_{i j}} = 2 d_{s_i s_j}$$  \hspace{1cm} (37)$$

$$\frac{\partial}{\partial t} \sigma_{s_i q}(k, k') + i k_j \sigma_{s_{j i}} + \kappa k'^2 \sigma_{s_i T} = 2 d_{s_i q}$$  \hspace{1cm} (38)$$

$$\frac{\partial}{\partial t} \sigma_{q\rho}(k, k') + \kappa k^2 \sigma_{T q} + \kappa k'^2 \sigma_{q T} = 2 d_{q q}$$  \hspace{1cm} (39)$$

To simplify notation the arguments $k, k'$ are written only in those terms containing a derivative with respect to time. Please notice that in the present case the relation (3) translates to $\sigma_{AB}(k, k') = \sigma_{BA}(-k', -k)$, where $A, B = \rho, g, q$. To shorten the notation further the abbreviations

$$k_l \sigma_{s_{l i} A}(k, k') = k_i \sigma_{s_{i A}}(k, k') - i D_{m j n} k_m k_n \sigma_{s_{j A}}(k, k')$$  \hspace{1cm} (40)$$

9
\[
\sigma_{PA} = \frac{e^2}{\rho_0} a \sigma_{qA} + \frac{\rho_0}{T_0} a \sigma_{qA}
\]

and

\[
\sigma_{AT} = a \sigma_{Ap} + \frac{1}{\rho_0 c_v} \sigma_{AQ}
\]

are introduced for certain combinations of fluctuations.

The diffusion coefficients follow either from the general formula (7) together with (2), (16)–(18) or from (34)–(39):

\[
2 d_{pgi} (k, k') = i k_j \sigma_{gjqi}^{st} - i k_j' \sigma_{pjqi}^{st} = i k_j \sigma_{gjqi}^{st} - i k_j' \sigma_{jpqi}^{st}
\]

(43)

\[
2 d_{pq} (k, k') = i k_j \sigma_{gjqi}^{st} + \kappa \kappa' \sigma_{gqi}^{st} = \kappa \kappa' \sigma_{gqi}^{st}
\]

(44)

\[
2 d_{gij} (k, k') = i k_j \sigma_{gjqi}^{st} - i k_j' \sigma_{gjiq}^{st} = i k_j \sigma_{gjqi}^{st} - i k_j' \sigma_{gjiq}^{st}
\]

(45)

\[
2 d_{gq} (k, k') = i k_j \sigma_{gjqi}^{st} + \kappa \kappa' \sigma_{gqi}^{st} = \kappa \kappa' \sigma_{gqi}^{st}
\]

(46)

\[
2 d_{qq} (k, k') = \kappa \kappa' \sigma_{gqi}^{st}
\]

(47)

where \( \sigma_{AB}^{st} \equiv \sigma_{AB}^{st} (k, k') \). To obtain these results use was made of the behaviour of \( \rho, q, g \) under time reversal.

As mentioned earlier in this section, knowing the equations (16)–(18) for the dynamical variables one can determine the equations for the fluctuations and the diffusion coefficients in a second way. Realizing that the time dependence of \( \sigma_{\nu \mu} (t) \) can be interpreted as average of time dependent dynamical variables \( \dot{A}_\nu (t) \) one can calculate the time derivatives of the fluctuations by inserting the expressions for time derivatives of dynamical variables. The resulting equations for the fluctuations are identical to (34)–(39). For the diffusion coefficients one finds the following explicit formulas:

\[
2 d_{pgi}^{L} (k, k') = -i k_j \frac{1}{2} \langle \left[ \delta \dot{p} (k, t), s_{ij} (-k', t) \right] \rangle
\]

(48)

\[
2 d_{pq}^{L} (k, k') = i k_j \frac{1}{2} \langle \left[ \delta \dot{p} (k, t), G_j (-k', t) \right] \rangle
\]

(49)

\[
2 d_{gij}^{L} (k, k') = i k_j \frac{1}{2} \langle \left[ s_{ij} (k, t), \dot{g}_j (-k', t) \right] \rangle - i k_j' \frac{1}{2} \langle \left[ \dot{g}_i (k, t), s_{ij} (-k', t) \right] \rangle
\]

(50)

\[
2 d_{gq}^{L} (k, k') = i k_j \frac{1}{2} \langle \left[ s_{ij} (k, t), \dot{q} (-k', t) \right] \rangle + i k_j' \frac{1}{2} \langle \left[ \dot{g}_i (k, t), G_j (-k', t) \right] \rangle
\]

(51)

\[
2 d_{qq}^{L} (k, k') = -i k_j \frac{1}{2} \langle \left[ G_j (k, t), \dot{q} (-k', t) \right] \rangle + i k_j' \frac{1}{2} \langle \left[ \dot{q} (k, t), G_j (-k', t) \right] \rangle
\]

(52)

To evaluate them one has to express \( \delta \dot{p}, \dot{g}, \dot{q} \) as functions of the fluctuating forces with the help of (16)–(18). Using the response functions (23), (27)–(33) one can summarize the result by

\[
\dot{A}(k, \omega) = \chi_{Ap} (k, \omega) \frac{k_i k_j}{\rho_0 k^2} s_{ij} (k, \omega) + \chi_{Aq} (k, \omega) \frac{-i k_i}{\kappa T_0 k^2} G_i (k, \omega),
\]

(53)
where $A = \rho, k, g, q,$ and
\[
\left( g_i(k, \omega) \right)_i = \chi_i(k, \omega) \frac{i}{\rho_0 D_i k^2} \left( k_j s_{ij}(k, \omega) \right)_i.
\] (54)

Using (53), (54) as well as the correlation functions (21) and (22) one finds that the diffusion coefficients (48)–(52) are diagonal in $k, k'$:
\[
d^{\rho}_{i}(k, k') = (2\pi)^3 \delta(k - k') d_{\rho}(k)
\] (55)

where $A, B = \rho, g, q$. For the $d(k)$ one ends up with the following expressions:
\[
d_{\rho \rho}(k) = \int \frac{d\omega}{2\pi} E(\omega) (-iD_i) k_i \chi_{\rho \rho}(k, \omega)
\] (56)
\[
d_{\rho \rho}(k) = \int \frac{d\omega}{2\pi} E(\omega) \chi_{\rho \rho}(k, \omega)
\] (57)
\[
d_{g q}(k) = \int \frac{d\omega}{2\pi} E(\omega) \chi_{g q}(k, \omega)
\] (58)
\[
d_{q q}(k) = \int \frac{d\omega}{2\pi} E(\omega) \chi_{q q}(k, \omega)
\] (59)
\[
d_{i}(k) = \int \frac{d\omega}{2\pi} E(\omega) (-iD_i) k_i \chi_{g \rho}(k, \omega)
\] (60)
\[
d_{t}(k) = \int \frac{d\omega}{2\pi} E(\omega) \chi_{t}(k, \omega)
\] (61)

With the help of the expressions (25), (27)–(33) for the response functions one finds the same results for the diffusion coefficients (43)–(47), provided that for $\sigma_{\text{st}}$ only the contribution $\tilde{\sigma}_{\text{st}}$ is inserted, i.e.
\[
\tilde{d}_{AB}(k, k') = (2\pi)^3 \delta(k - k') d_{AB}(k) = d_{AB}(k, k'),
\] (62)

with $A, B = \rho, g, q$, where the tilde above $d$ reminds of the restriction on $\tilde{\sigma}_{\text{st}}$.

We thus see, that the theory of fluctuations of Landau and Lifschitz leads to the same expressions for the diffusion coefficients as our method, provided the additional contribution $\Delta \sigma$ to $\sigma_{\text{st}}$ (11) either vanishes exactly or is neglected.

### 4 Comparison with Boltzmann-Langevin theory

In this section connection will be established between the approach explained in section 2 and the Boltzmann-Langevin theory by comparing the expressions for the diffusion coefficients. To calculate the latter with the method of section 2, in general one has to know the behaviour of the response function as function of frequency. However, investigating (12) one finds that less information is needed to exploit the fluctuation dissipation theorem in the classical limit:
one only has to determine the static response, i.e. the value of the response function at \( \omega = 0 \). From (2) and (9) the equation for the latter follows as:

\[
D_{\nu\nu'} \chi_{\nu'}(\omega = 0) = \frac{\delta c_{\nu}}{\delta U^\text{ext}_\mu(\omega = 0)} \tag{63}
\]

The combination on the left hand side of (63) appears also in the expression (7) for the diffusion coefficient since the contribution \( \tilde{\sigma}^\text{st} \) (12) to the latter equals, in the classical limit, the product of static response and temperature. Therefore, it is not necessary to solve (63) for the static response – only the right hand side of (63) is needed.

Let us now look at the cases of BUU and Landau equation as the ones for the averages, i.e. the ones defining \( c_{\nu} \) via (2). Linearizing the equation for the distribution function in the deviation from the homogeneous equilibrium distribution one finds

\[
\frac{\partial}{\partial t} \delta n_P(r, t) + \left( v_p \nabla r - J_{P,r*} \right) \left( \delta n_P(r, t) - \frac{\partial n^0}{\partial \varepsilon_p} \delta \varepsilon_p(r, t) \right) = 0 . \tag{64}
\]

The linear operator \( J \) is defined by the collision term:

\[
I[n_p]_{\text{lin}} = J_{P,r} \ast \left( n_P(r, t) - n^0(\varepsilon_p) \right) \tag{65}
\]

The asterix reminds that, in general, \( J_{P,r} \) includes integrations over momentum and position. (In the non-Markov case, there will be an additional integration over time. An index \( t \) at \( J \), however, is omitted for the moment.)

Please notice that the energy appearing in the argument of the equilibrium distribution \( n^0 \) is taken to be the momentary one. The latter differs from \( \varepsilon^0_p \), which appears in (3) and which represents the energy in equilibrium without external field, by the amount

\[
\delta \varepsilon_p = \int \frac{dp}{h^3} f_{pp'} \delta n_{p'} + U^\text{ext}_{p} . \tag{66}
\]

The kernel \( f_{pp'} \) is given by \( f_{pp'} = \partial U/\partial \rho \) for the case of the Boltzmann equation with a momentum independent mean field \( U[\rho] \). In Landau Fermi Liquid theory \( f_{pp'} \) is called quasi particle interaction.

To calculate the diffusion coefficient one has to determine \( c_{\nu} \) as functional of the external field. Comparing (64) with the general form (2) of the equation for the averages one obtains

\[
c_{\nu} \leftrightarrow \left( v_p \nabla r - J_{P,r*} \right) \left( -\frac{\partial n^0}{\partial \varepsilon_p} U^\text{ext}_p(r, t) \right) . \tag{67}
\]

The right hand side of the equation (63) for the static response corresponds therefore to

\[
\frac{\delta c_{\nu}}{\delta U^\text{ext}_\mu(\omega = 0)} \leftrightarrow \left( v_{p_{\nu}} \nabla_{r_{\nu}} - J_{P_{\nu},r_{\nu}*} \right) \left( -\frac{\partial n^0}{\partial \varepsilon_{p_{\nu}}} h^3 \delta(p_{\nu} - p_{\mu}) \delta(r_{\nu} - r_{\mu}) \right) . \tag{68}
\]
Denoting by \( \tilde{d} \) the contribution from \( \tilde{\sigma}^{st} \) \(^{12} \) to the diffusion coefficient \(^{7} \), one finds from \(^{68} \)

\[
2\tilde{d}(p, r, p', r')_{cl} = -T \left( J_{p, r} \star \left( -\frac{\partial n_0^0}{\partial \varepsilon_p} h^3 \delta(p - p') \delta(r - r') \right) + (r, p \leftrightarrow r', p') \right)
\]

(69)
since the terms with the spatial derivative in \(^{68} \) cancel because of \((\nabla_r + \nabla_{r'}) \delta(r - r') = 0 \).

For a local, Markov collision term the linear operator \( J \) \(^{65} \) can be written as the sum of a diagonal and a non-diagonal contribution

\[
J_{p, r} \star h(p, r, t) = -\frac{1}{\tau_p} h(p, r, t) + \int \frac{dp'}{h^3} I_{pp'} h(p', r, t)
\]

(70)
where \( h(p, r, t) \) is some function of \( p, r, t \). The action of the diagonal part of \( J \) consists of a multiplication with a function of \( p \) only which, in \(^{71} \), is written as \( 1/\tau_p \) to indicate, that this factor has the dimension of an inverse time. It is uniquely determined by \( J \) under the subsidiary condition that the kernel \( I_{pp'} \) does not contain terms proportional to \( \delta(p - p') \).

For the collision term \(^{71} \) the diffusion coefficient follows as:

\[
\tilde{d}(p_1, r_1, p_2, r_2)_{cl} = \delta(r_1 - r_2) \left[ \frac{n_0^0 \pi_0^0}{\tau_p} h^3 \delta(p_1 - p_2) - \frac{1}{2} \left( I_{p_1, p_2, p_2} n_0^0 \pi_0^0 + I_{p_2, p_1, p_1} n_0^0 \pi_0^0 \right) \right],
\]

(71)
using the definition \( \pi_p = 1 - n_p \).

This result can be compared with the one obtained in the Boltzmann-Langevin theory for the (Markovian) BUU collision term. The comparison will be performed in two steps: In a first step results obtained in \(^{11} \) for the low temperature limit are used. In a second step the restriction to small temperatures will be removed.

In \(^{11} \) the Boltzmann-Langevin expressions for collision term and diffusion coefficient were evaluated in the low temperature limit and for small deviations from thermal equilibrium. The result is written in formula (21) of \(^{11} \). To perform the comparison one first has to determine \( \tau_p \) and \( I_{pp'} \) as defined by \(^{70} \). Comparing \(^{70} \) with the expression for the collision term in \(^{11} \) (there called drift coefficient) one finds that, in the approximation of \(^{11} \), \( 1/\tau_p \) is given by the momentum independent quantity \( W_0 \) of \(^{11} \) and the kernel \( I_{p_1, p_2} \) is given by \( f_1 f_2 C_{12} \) of \(^{11} \). With the help of these relations the expression \(^{71} \) turns into the lower one in formula (21) of \(^{11} \), i.e. into the Boltzmann-Langevin result for the diffusion coefficient in equilibrium and in the low temperature limit.

However, the restriction to low temperature can be released. To this end one starts from the general Boltzmann-Langevin expressions for drift- and diffusion coefficient \(^{12} \) \(^{8} \). One finds that the contribution to the collision term which is linear in \( \delta n_p \) can be written as (see appendix B)

\[
I[n_p]|_{lin} = -\frac{\alpha_0^2 \langle n^0_0 \rangle(p)}{2n_0^0 \pi_0^0} \delta n_p - \int \frac{dp_2}{h^3} \frac{\alpha_{cov} \langle n^0_0 \rangle(p, p_2)}{2n_0^0 \pi_0^0} \delta n_p
\]

(72)
where (in the notation of \cite{12}) \( \alpha[n](\mathbf{p}, \mathbf{r}, \mathbf{p}', \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \left( \alpha^2[n](\mathbf{p}) \hbar^3 \delta(\mathbf{p} - \mathbf{p}') + \alpha_{\text{cov}}[n](\mathbf{p}, \mathbf{p}') \right) \) appears as inhomogeneity in the equation for the fluctuations, i.e. is the equivalent to \( 2 \hat{d}(\mathbf{p}, \mathbf{p}') \) of the present paper. Comparing (72) with (70) one finds the identification:

\[
\begin{array}{c}
\text{Boltzmann—} \\
\text{Langevin} \\
\alpha_{\text{cov}}[n^0](\mathbf{p}_1, \mathbf{p}_2)
\end{array}
\leftrightarrow
\begin{cases}
2n_p^0 \pi_0^p / \tau_p & \text{present work} \\
-2 I_{\mathbf{p}, \mathbf{p}_2} n_{p_2}^0 \pi_{p_2}^0 & \text{work (73)}
\end{cases}
\]

Inserting (73) in the expression (71) results in

\[
2 \hat{d}(\mathbf{p}, \mathbf{r}, \mathbf{p}', \mathbf{r}')|_{\text{cl}} = \alpha[n^0](\mathbf{p}, \mathbf{r}, \mathbf{p}', \mathbf{r}')
\]

one only has to use the symmetry property \( \alpha_{\text{cov}}[n^0](\mathbf{p}, \mathbf{p}') = \alpha_{\text{cov}}[n^0](\mathbf{p}', \mathbf{p}) \). Thus, the equilibrium diffusion coefficient of the Boltzmann-Langevin theory turns out identical to the contribution \( \hat{d} \) to the one of the present approach provided the latter is calculated with the classical form of the fluctuation dissipation theorem, i.e. is identical to (71).

As mentioned before, in the Boltzmann-Langevin theory the correlation function for the fluctuating force, \( C(\mathbf{p}, \mathbf{r}, t, \mathbf{p}', \mathbf{r}', t') \), when calculated in equilibrium and for a Markovian collision term, is given \cite{3, 7} by

\[
C(\mathbf{p}, \mathbf{r}, t, \mathbf{p}', \mathbf{r}', t') = 2 \hat{d}(\mathbf{p}, \mathbf{r}, \mathbf{p}', \mathbf{r}')|_{\text{cl}} \delta(t - t') ,
\]

where the relation between the Boltzmann-Langevin diffusion coefficient and (71) was utilized. Using the method of Landau and Lifschitz, applied in the previous section, to calculate the correlation function of the fluctuating force for BUU or Landau equation (for the latter see e.g. \cite{3}) one obtains a result that agrees with (75), again for the classical fluctuation dissipation theorem and for a Markovian collision term.

In a recent work \cite{18} the Boltzmann-Langevin theory is extended to a non-Markovian collision term. In the framework of the theory of Landau and Lifschitz this generalization can be done without difficulty, since until the final step in the derivation one does not need to specify the form of the collision term. Indeed, one finds for \( C(\mathbf{p}, \mathbf{r}, t, \mathbf{p}', \mathbf{r}', t') \) an expression similar to the right hand side of (73). However, in the former an \( \delta \)-function \( \delta(t - t') \) appears besides those in the positions and in the momenta, and the operator \( J(\mathbf{p}, \mathbf{r}, t) \) is now non local in time:

\[
J_{\mathbf{p}, \mathbf{r}, t} \ast h(\mathbf{p}, \mathbf{r}, t) = \int \frac{d\mathbf{p}' d\mathbf{r}' dt'}{h^3} I(\mathbf{p}, \mathbf{r}, t; \mathbf{p}', \mathbf{r}', t') h(\mathbf{p}', \mathbf{r}', t')
\]

Following \cite{3} the quantum correlation function of the fluctuating force is obtained multiplying the Fourier transform (in time) of the classical one by \( E(\omega)/T \). Thus one finds the former to read:

\[
C(\mathbf{p}, \mathbf{r}, t, \mathbf{p}', \mathbf{r}', t') = \int \frac{d\omega}{2\pi} e^{-i\omega t} E(\omega) \int d\tau e^{i\omega \tau} J_{\mathbf{p}, \mathbf{r}, \tau} \ast \left( \frac{\partial n_0^0}{\partial \varepsilon_p} h^3 \delta(\mathbf{p} - \mathbf{p}') \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - t') \right) + \left( \mathbf{p}, \mathbf{r}, t \leftrightarrow \mathbf{p}', \mathbf{r}', t' \right)
\]

(77)
In the expression for $C$ the use of the kernel $I$ \((76)\) is omitted for the sake of an easier comparison with \((69)\) in the Markov limit.

It is straightforward to show that \((77)\) leads to the result found in \([18]\) for the correlation function of the Boltzmann-Langevin theory in equilibrium, provided the expression for the non Markovian collision term of \([18]\) is used to define $J$ according to \((65)\).

5 Sum rules – the case of zero wave vector

In finite, closed systems the total number of particles, the total momentum and the total energy are conserved quantities. Starting with an ensemble of systems all having the same particle number, momentum and energy, the dynamical variables corresponding to the latter should not exhibit any fluctuations at a later time. This feature constrains the possible forms for the equations of motion of the fluctuations. Especially, it implies sum rules for fluctuations $\sigma$ and diffusion coefficient $d$. Since the deviations of total particle number, momentum or energy from the corresponding equilibrium values are given by the values of the phase space integral of the deviation of the distribution function weighted with $1$, $p$ or $\varepsilon_p$, respectively, the sum rules read:

$$\int \frac{d^3\mathbf{r}d^3\mathbf{p}}{h^3} O_p b(p, r, p, r') = 0 \quad (78)$$

where $O_p = 1, p, \varepsilon_p$ and $b = \sigma, d$. In the following it will be demonstrated that the results for $\tilde{\sigma}$ and $\tilde{d}$ obtained with the method of section 2 together with the assumption $\Delta\sigma = 0$ are consistent with these sum rules.

At first, the classical limit shall be addressed. Here, the diffusion coefficient obtained in section 4 was found to agree with the result of the Boltzmann-Langevin theory in equilibrium. The former is, therefore, known \([6, 12]\) to fulfil the sum rules \((78)\).

For investigation of the fluctuations one might think of solving \((63)\) for the static response. However, there is some subtlety involved: response functions may be non analytic at $\omega = 0 = k$ and one has to choose carefully the correct sequence in performing limits. This fact is relevant for the present case since in the sum rules \((78)\) there occurs an integral over space. The value of the latter is thus equal to the one of the Fourier transform of the integrand taken at $k = 0$. Moreover, in the contribution $\tilde{\sigma}_{st}$ to the integrand there appears, in the classical limit, the response function at $\omega = 0$. The order in which the two limits -- $k, \omega \to 0$ -- have to be taken can be found as follows. As said before, the sum rules arise due to the conservation laws for particle number, momentum and energy. The deviations of the latter from their equilibrium values are the averages of dynamical variables given by

$$\hat{M}_{O_p}(k = 0) = \int \frac{d^3\mathbf{r}d^3\mathbf{p}}{h^3} O_p \delta\hat{n}_p(r) \quad , \quad (79)$$
where $O_p = 1, p, \varepsilon_p$ respectively and $< \delta n_p(r) >= \delta n_p(r)$. Due to the conservation laws $\dot{M}_{O_p}(k = 0)$ does not exhibit any fluctuations:

$$\sigma_{M_{O_p}(k=0), A} = 0$$  \hspace{1cm} (80)

where $\dot{A}$ is some dynamical variable. Inserting here (79) one recovers (78) for $b = \sigma$ when $A$ is taken as $n_p$. The fluctuations are related to the static response whose microscopic form is given by [17]

$$\chi_{M_{O_p}(k=0), A}(\omega = 0) = \int dt \frac{i}{\hbar} \langle [\dot{M}_{O_p}(k = 0), A^\dagger(-t)] \rangle \Theta(t) \hspace{1cm} (81)$$

The expression on the right hand side can be rewritten to give

$$\chi_{M_{O_p}(k=0), A}(\omega = 0) = \lim_{\omega \to 0} \int dt e^{i\omega t} \Theta(t) \hspace{1cm} (82)$$

If the limit $k \to 0$ can be commuted both with the averaging procedure and with the time integration, then one finds:

$$\chi_{M_{O_p}(k=0), A}(\omega = 0) = \lim_{\omega \to 0} \left( \lim_{k \to 0} \int \frac{dp}{h^3} O_p \chi_{pp^\dagger}(k, \omega) \right) \hspace{1cm} (83)$$

Therefore, the sum rules for the fluctuations ($b = \sigma^{st}$) read in the classical limit:

$$\Delta \sigma_{M_{O_p}(k=0), A} + T \lim_{\omega \to 0} \left( \lim_{k \to 0} \int \frac{dp}{h^3} O_p \chi_{pp^\dagger}(k, \omega) \right) = 0$$  \hspace{1cm} (84)

Thus, for calculating the left hand side of (78) one has first to perform the momentum integral of the response function at finite $k, \omega$ weighted with $O_p$. Then one has to take the limit $k \to 0$ (corresponding to the spatial integral) and, finally, $\omega \to 0$. Due to this sequence of limits the investigation about the fluctuations fulfilling the sum rules turns out to be not easier in the classical limit than in the general case.

There is a further, indirect argument that the sum rules are related to the order of limits stated, i.e. first $k \to 0$ and then $\omega \to 0$. It is well known [19], that taking first the limit for $\omega$ and then for $k$ leads to thermodynamic derivatives, e.g. to the compressibility in the case of the density density response function. However, a finite, closed system can have a finite compressibility, too. And indeed, solving (63) in the case of Landau Fermi Liquid theory one obtains [8] the known, finite value for the compressibility [20].

In the general (i.e. not classical) case one may proceed as follows in order to prove the sum rules to hold true for fluctuations and diffusion coefficient. Let again $O_p$ be either 1, $p$ or $\varepsilon_p$. Multiplying the equation of motion for the distribution function $n_p(r, t)$ by $O_p$ and integrating with respect to $r, p$ one finds [20]

$$\int \frac{dr dp}{h^3} O_p \frac{\partial}{\partial t} \delta n_p(r) = 0 \hspace{1cm} (85)$$
To obtain this result one only has to integrate by parts the terms containing the derivatives with respect to $r$ and $p$ and to use the feature of the collision term that number of particles, momentum and energy are preserved.

Taking the functional derivative of (85) with respect to the external field results in
\[ \frac{\partial}{\partial t} \int \frac{d^3r}{h^3} \frac{d^3p}{h^3} \chi_{pp}(r, t, r', t') = 0. \] (86)
Since the response function is proportional to the Heavyside-$\Theta$-function, $\chi_{pp}(r, t, r', t') \sim \Theta(t - t')$, reflecting causality, the solution of (86) reads
\[ \int \frac{d^3r}{h^3} \frac{d^3p}{h^3} \chi_{pp}(r, t, r', t') = 0. \] (87)
Due to (10) the dissipative part of $\chi$ has the same property. Therefore, it follows from (12) that
\[ \int \frac{d^3r}{h^3} \frac{d^3p}{h^3} \tilde{\sigma}_{st}^{pp}(r, r') = \int \frac{d^3p}{h^3} \tilde{\sigma}_{st}^{pp}(k = 0) = 0. \] (88)
We thus see that the contribution $\tilde{\sigma}_{st}$, when being calculated from the fluctuation dissipation theorem, fulfills the same sum rules as the fluctuations themselves. An approximation like $\Delta \sigma = 0$, in other words approximating $\sigma_{st}$ by $\tilde{\sigma}_{st}$, is, therefore, consistent with the sum rules (78) for the fluctuations.

Let us now turn to the diffusion coefficient. According to (7) the latter is given by
\[ d(r, p, r', p') = (v_p \nabla_r - J_p r \leftrightarrow p' r' \leftrightarrow p' r') \left( \sigma_{pp}^{st}(r, r') - \frac{\partial n_0}{\partial \varepsilon_p} f_{pp}(r, r'') \sigma_{pp}^{st}(r'', r') \right) + \left( p, r \leftrightarrow p', r' \right)^* \] (89)
where integration over double primed quantities is to be understood. The last term on the right hand side stands for the complex conjugate of the first term but with the unprimed variables exchanged with the single primed ones. Multiplying $d$ with $O_p$ and integrating the product with respect to position and momentum one finds that the contribution from the first term on the right hand side of (89) vanishes. This is due to the same reasons that lead to (85), i.e. directly due to the structure of the equation of motion for the distribution function. The other contribution vanishes due to (88) with (3), i.e. due to features of the fluctuations. Thus the sum rules (78) for $b = d$
\[ \int \frac{d^3r}{h^3} \frac{d^3p}{h^3} O_p d(p, r, p, r') = \int \frac{d^3p}{h^3} \frac{d^3p}{h^3} O_p d(p, p', k = 0) = 0. \] (90)
are proven.

The diffusion coefficient forms the inhomogeneity in the equation for the (non equilibrium) fluctuations. Due to (90) the equation of motion for the product of the fluctuations with $O_p$
integrated over $r, p$ is homogeneous. Therefore, the dynamic fluctuations obey the same sum rules as $\tilde{\sigma}^{st}$, i.e. \eqref{eq:sumRules}, for all times provided they did in the initial state.

It should be mentioned that these properties, especially the ones of the fluctuations, are restricted to $k = 0$. Indeed, it turns out \cite{ref}, that the limit $k \to 0$ of $\tilde{\sigma}_{pp}^{st}(k)$ differs from the value at $k = 0$. This means, that the fluctuations are not continuous functions at vanishing wave vector.

6 Summary

The problem to determine equations of motion of fluctuations and diffusion coefficients was addressed. A method \cite{ref1, ref2} for its solution using knowledge only of the average dynamics was reviewed using a more general formulation. The basic ingredient of the method is the fluctuation dissipation theorem and a suitable extension of the latter to instabilities. This extension consists in the prescription to determine the diffusion coefficients as the analytic continuation of the expression valid for the stable system.

The connection between the diffusion coefficient in the present approach and the correlation function of the fluctuating force in the one of Bixon and Zwanzig \cite{ref3} was explained. The method was then applied to hydrodynamics. Equation of motion of the fluctuations and diffusion coefficients were demonstrated to be the same as those obtained using the theory of fluctuations by Landau and Lifschitz \cite{ref4}. Subsequently, the case of BUU and Landau equation as equation for the average one (quasi) particle distribution was addressed. An expression for the diffusion coefficient in terms of the linearized, Markov collision operator was calculated using the classical form of the fluctuation dissipation theorem. This expression was found to coincide, for BUU, with the one obtained for the equilibrium distribution in the Boltzmann-Langevin theory \cite{ref5}. The connection with both the theory by Landau–Lifschitz and the Boltzmann-Langevin theory holds under the assumption that for the equilibrium fluctuations – and, consequently, their analytical continuation – only that contribution is taken into account which arises from the frequency integral of the dissipative part of the response function (weighted with $\hbar \coth(\hbar \omega / 2T) / \pi$).

In the framework of the theory of Landau and Lifschitz the generalization to a non-Markovian collision term in Boltzmann and Landau equation was presented. The correlation function of the fluctuating force obtained for the case of the Boltzmann equation coincides with the equilibrium one of the extension \cite{ref6} of the Boltzmann-Langevin theory provided the same form is used for the collision term.

To establish sum rules arising from conservation of particle number and momentum the behaviour of the response functions for vanishing wave vector and frequency was taken into account properly. Clarifying the sequence in which the limits $k \to 0$ and $\omega \to 0$ have to be
calculated, results of previous work could be interpreted correctly.

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Appendix:

A Detailed comparison with Bixon and Zwanzig

In this appendix the comparison of the present approach with the one of Bixon and Zwanzig summarized at the end of section 2 will be worked out in more detail.

Bixon and Zwanzig start with the classical Boltzmann equation without mean and/or external field as the equation for the average one particle distribution function. Augmenting the linearized Boltzmann equation by a fluctuating force, \( f_B F(t) \) in their notation, the authors write the result in the form

\[
\frac{\partial \phi}{\partial t} + L \phi = F(t) \tag{A.1}
\]

where \( \phi(t) \) is the relative deviation of the distribution from the Boltzmann distribution. The correlation function of the fluctuating force is written as

\[
\langle F(k_1, v_1 ; t_1)F(k_2, v_2 ; t_2) \rangle = 2B(k_1, v_1, k_2, v_2)\delta(t_1 - t_2) \tag{A.2}
\]

For \( B \) the authors find the expression

\[
2B(k_1, v_1, k_2, v_2) = (L_1 + L_2) \langle \phi_1 \phi_2 \rangle \tag{A.3}
\]

where the average on the right hand side has to be performed in equilibrium. For the calculation of the correlation function (A.2) the authors use the expression

\[
\langle \phi(k_1, v_1)\phi(k_2, v_2) \rangle = [f_B(v_2)]^{-1} \delta(v_1 - v_2)\delta(k_1 + k_2) \tag{A.4}
\]

After this repetition the comparison can be performed easily. This the more, as the equation for the average distribution following from (A.1) by averaging of both sides, is written directly in the form (2): one only has to identify \( \hat{A}_\nu \) with \( \phi(k_\nu, v_\nu) \). Clearly, one has

\[
\mathcal{D}_{\nu\mu} \hat{A}_\mu \leftrightarrow (L\phi)(k_\nu, v_\nu) \tag{A.5}
\]

From (7), (A.3) and (A.5) follows

\[
d_{\nu\mu} = B(k_\nu, v_\nu, -k_\mu, v_\mu) \tag{A.6}
\]

where the minus sign in front of \( k_\mu \) is due to the hermitian conjugation of the second dynamical variable in the definition (4) of the fluctuations. Thus the statement at the end of section 2 about the relation of the diffusion coefficient and the correlation function of the fluctuating force is proven.

In the rest of this appendix it will be demonstrated how the expression (A.4) for the equilibrium fluctuations in a classical ideal gas can be obtained from the related response
function. First it should be noted, that in (A.4) there appears the fluctuations at finite wave-vectors \( k_1, k_2 \). Next one realizes that in the classical limit only the static response is needed to obtain \( \tilde{\sigma} \). From (64) one finds after Fourier transformation of the time and putting \( \omega = 0 \):

\[
\left( v_r \nabla_r - J_{p,r} \right) \left( \delta n_p(r, \omega = 0) - \frac{\partial n^0}{\partial \varepsilon_p} \delta \varepsilon_p(r, \omega = 0) \right) = 0 \tag{A.7}
\]

The solution of this equation reads

\[
\delta n_p(r, \omega = 0) - \frac{\partial n^0}{\partial \varepsilon_p} \delta \varepsilon_p(r, \omega = 0) = c_p(r) \tag{A.8}
\]

where \( (v_r \nabla_r - J_{p,r}) c_p(r) = 0 \). This seems to be no progress at first site, but it is: \( J \) does not depend on the external field; consequently the same is valid for \( c_p(r) \). Therefore, taking the functional derivative of (A.8) with respect to \( U_{\text{ext}}(r', \omega = 0) \) one finds the static response of non interacting particles after Fourier transforming the spatial dependencies:

\[
\chi_{pp'}(k, k', \omega = 0) = -\frac{\partial n^0}{\partial \varepsilon_p} h^3 \delta(p - p')(2\pi)^3 \delta(k - k') \tag{A.9}
\]

Since the derivative of the Boltzmann distribution with respect to the energy equals the negative of the distribution divided by the temperature one ends up with the result for the equilibrium fluctuations:

\[
\tilde{\sigma}(p, k, p', k')_{\text{cl. id. gas}} = n^0(p) h^3 \delta(p - p')(2\pi)^3 \delta(k - k') \tag{A.10}
\]

where \( n^0(p) \) is the equilibrium distribution function for the momentum with the property \( \int d^3 p / h^3 n^0(p) = \int d^3 v f_B(v) \). Therefore, \( f_B(v) = (m/h)^3 n^0(mv) \) and from (A.10)

\[
\langle \delta f(v_1, k_1) \delta f(v_2, k_2) \rangle = f_B(v_1) \delta(v_1 - v_2)(2\pi)^3 \delta(k_1 + k_2) \tag{A.11}
\]

Dividing both sides by \( f_B(v_1)f_B(v_2) \) one finds (A.4), beside a factor \( (2\pi)^3 \) which is due to the different conventions of the Fourier transform.
B  Collision term in BL-theory

In this appendix the derivation of (72) will be presented. The notation used is the one of [12].

In a first step one observes that the linearized BUU collision term can be written in the form:

\[ I[n_p]_{\text{lin}} = \int (\prod_{i=2}^{4} dp_i) \frac{u_1234}{2} \left( \delta(p-p_3) - \delta(p-p_1) \right) \left[ (\delta n_1 n_2 + \delta n_2 n_1) \bar{n}_3 \bar{n}_4 - (\delta n_3 \bar{n}_4 + \delta n_4 \bar{n}_3) n_1 n_2 \right], \]

(B.12)

where as short hand notation a number \( i \) stands for \( p_i \) and, as before, \( \bar{n}_p = 1 - n_p \). Using the quantities \( W^x \) introduced in [12] one finds:

\[ I[n_p]_{\text{lin}} = -\left( W^+[n_p] + W^-[n_p] \right) \delta n_p \\
- \int dp_2 \left[ \overline{\pi}_p^0 \left( W_0^{++}(p, 2) - W_0^{+-}(p, 2) \right) + n_p^0 \left( W_0^{--}(p, 2) - W_0^{+-}(p, 2) \right) \right] \delta n_{p_2}, \]

(B.13)

where the index 0 indicates, that the corresponding quantity is to be taken at \( n^0 \). Making use of the properties of the Fermi function \( n^0 \) as well as of the energy conservation contained in the transition rate \( w \) one can rewrite the integrand in (B.13) as

\[ \overline{\pi}_p^0 (W_0^{++}(p, 2) - W_0^{+-}(p, 2)) + n_p^0 (W_0^{--}(p, 2) - W_0^{+-}(p, 2)) = \frac{n_p^0 Z_0^-(p, 2) + \overline{\pi}_p^0 Z_0^+(p, 2)}{2n_p^0 \bar{n}_p^0} \]

(B.14)

Again the definitions of the \( Z^\pm \) are to be found in [12].

Remembering that the non diagonal part to \( \alpha[n](p, p') \) is related to the \( Z^\pm \) via

\[ \alpha_{\text{cov}}[n^0](p, p') = n_p^0 Z_0^-(p, p') + \overline{\pi}_p^0 Z_0^+(p, p') \]

(B.15)

while the diagonal part is given by

\[ \alpha^2[n^0](p) = 2n_p^0 \overline{\pi}_p^0 \left( W^+[n_p] + W^-[n_p] \right) \]

(B.16)

one obtains (72) from (B.13).
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