Engineering interactions and anyon statistics by multicolor lattice-depth modulations

Lorenzo Cardarelli, Sebastian Greschner, and Luis Santos

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstr. 2, DE-30167 Hannover, Germany

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We show that a multicolor modulation of the depth of an optical lattice allows for a flexible independent control of correlated hopping, occupation-dependent gauge fields, effective on-site interactions without Feshbach resonances, and nearest-neighbor interactions. As a result, the lattice-depth modulation opens the possibility of engineering with minimal experimental complexity a broad class of lattice models in current experiments with ultra-cold atoms, including Hubbard models with correlated hopping, peculiar extended models, and two-component anyon-Hubbard models.

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Floquet engineering – the averaging of fast periodic modulations to obtain an effective time-independent system – is an ubiquitous tool for the manipulation and probing of various systems, ranging from NMR probes in solid state physics to atom-light interactions or Raman-dressed states [1]. In recent years, Floquet techniques have established themselves as a toolbox for the creation of novel Hamiltonians for ultra-cold atoms in optical lattices, including lattice shaking [2, 8], Raman-assisted hopping [9–12], and modulated interactions [13–18].

A major reason for the interest on Floquet techniques lies in the possibility of engineering gauge fields, i.e. complex hopping rates, for neutral atoms in optical lattices [2]. Most relevantly, synthetic magnetic fields have been created in the last years using Raman-assisted hopping [9–12]. Interestingly, various Floquet techniques have been recently proposed for the creation of occupation-dependent gauge fields (ODG) [10–22], in which the phase of the hopping depends on the site occupation. Under proper conditions, 1D models with ODG may be mapped into an anyon-Hubbard model (AHM) [19–22], in which the exchange statistics of the atoms may be externally modified. The 1D AHM presents a wealth of new physics, including statistically-induced transitions [22], novel superfluid phases [21], smooth fermionization [22], asymmetric momentum distributions [23, 24], and intriguing dynamics [22, 27]. The atomic back-action on the synthetic gauge field given by ODG could pave a way for the realization of dynamical gauge fields [22, 23], and leads to interesting physics, such as chiral solitons in Bose-Einstein condensates [30] or density-flux interplay in 2D lattices [31].

In this Letter we propose a novel method based on the multi-color modulation of the depth of a tilted optical lattice. As shown by Ma et al. [32] lattice-depth modulations may be employed to assist different occupation-dependent hoppings for sufficiently strong interactions. We show for the particular case of two-component fermions that a three-color modulation (3CM) of the lattice depth may be employed to achieve a separate flexible control of correlated hopping, ODG, effective on-site interactions without the need of Feshbach resonances, and nearest-neighbor (NN) interactions. As a result, 3CM allows with a minimal experimental complexity for engineering a broad class of lattice Hamiltonians using ultra-cold atoms, including Hubbard models with correlated hopping, peculiar extended models, and two-component AHM, whose basic properties we analyze as well.

Effective Hamiltonian.– We consider a balanced two-component (σ = ↑, ↓) Fermi gas in an optical lattice (equal for both components), whose depth is modulated in time, \( V(t) = V_0 + \delta V(t) \), with \( \delta V \ll V_0 \). We choose two-component fermions for simplicity, but similar ideas may be applied to bosons, and multi-component fermions. In the tight-binding regime, the hopping rate is \( \frac{\delta J(t)}{J_0} = \frac{s}{\sqrt{s^2 + \delta s(t)^2}} \exp(-2\sqrt{s}) \) [33], where \( s = V/E_{\text{rec}} = s_0 + \delta s(t) \), with \( E_{\text{rec}} \) the recoil energy associated to the laser that creates the lattice. Since \( \delta s \ll s_0 \), then \( J(t) = J_0 + \delta J(t) \), where \( J_0 = J(s_0) \), and \( \frac{\delta J(t)}{J_0} = \left( \frac{4}{3} - \sqrt{3} \right) \frac{\delta s(t)}{s_0} \), and hence the lattice modulation directly maps into a modulation of the hopping rate. We assume a tilted lattice, with an energy shift \( \Delta \) between neighboring sites (Fig. 1). The system is then described by the Fermi-Hubbard Hamiltonian:

\[
\mathcal{H}(t) = -J(t) \sum_{j, \sigma} \left[ c_{j+1, \sigma} c_{j, \sigma} + \text{H.c.} \right] + U \mathcal{H}_{\text{int}} + \Delta \mathcal{H}_{\text{tilt}},
\]

where \( c_{j, \sigma} \) is the annihilation operator of a fermion with spin \( \sigma \) at site \( j \), \( U \) characterizes the on-site interactions, \( \mathcal{H}_{\text{int}} = \sum_j n_{j, \uparrow} n_{j, \downarrow} \), and \( \mathcal{H}_{\text{tilt}} = \sum_{j, \sigma} J(t) n_{j, \sigma} \). Note that four different hoppings are possible (Fig. 1): (i) a single atom hops to an empty site to its right leading to an energy

\[
\delta J(t) = \frac{s}{\sqrt{s^2 + \delta s(t)^2}} \exp(-2\sqrt{s}) \]

and

\[
\Delta = \left( \frac{4}{3} - \sqrt{3} \right) s_0 \frac{\delta s(t)}{s_0}.
\]

FIG. 1: (Color online) Sketch of the lattice set-up and the relevant hoppings.
shift $\Delta E_1 = \Delta$; (ii) an atom with spin $\sigma$, initially alone at a given site, tunnels to the site at its right already occupied by a single atom with $\bar{\sigma} \neq \sigma$, resulting in a shift $\Delta E_2 = \Delta + U$; (iii) same as (ii) but the hopping is to the left – in this case $\Delta E_3 = U - \Delta$; and (iv) an atom of component $\sigma$ sharing a site with a $\bar{\sigma}$ atom, tunnels into the site at its right already occupied by a single atom with $\bar{\sigma}$ leading to $\Delta E_4 = \Delta$ (i.e. (iv) and (i) are resonant).

We assume that $J(t) \ll |\Delta|/|\Delta \pm U|$, and hence direct hopping is negligible. However, a periodic modulation of $\delta J(t)$ leads to assisted hopping if the modulation frequency matches the energy shift associated to the hopping process $\delta \phi$. Crucially, processes (i), (ii) and (iii) are characterized by different energy shifts (typically separated by several kHz, see below), and hence the different hoppings may be laser-assisted separately. The key point of our proposal is to address them separately but simultaneously using a 3CM of the laser intensity:

$$\delta V(t) = \sum_{s=1,2,3} \delta V_s (\omega_s t + \phi_s),$$

which, as mentioned above, translates into an equivalent modulation of the hopping,

$$\delta J(t) = \sum_s \delta J_s \cos(\omega_s t + \phi_s).$$

Each component of the modulation has an amplitude $\delta J_s$ and a phase $\delta \phi_s$, which may be independently controlled. The frequencies $\omega_1 = \Delta$, $\omega_2 = \Delta + U - \bar{U}$, and $\omega_3 = -\Delta + U - \bar{U}$, with $|U| \ll \bar{U}$ are chosen (quasi-)resonant to the hoppings (i) (and hence also (iv)), (ii), and (iii), respectively.

In interaction picture, $\mathcal{H} = \mathcal{U} \mathcal{H} \mathcal{U}^\dagger$, with $\mathcal{U} = \exp[-i(\Delta H_{\text{int}} + \bar{U} H_{\text{int}})]$:

$$\mathcal{H}(t) = J(t) \sum_{j,\sigma} \left[ c_{j+1,\sigma}^\dagger \delta J_1 (n_{j+1,\sigma} - n_{j,\sigma}) + c_{j,\sigma} + H.c. \right].$$

(2)

3CM introduces oscillating terms $e^{\pm i(\omega_s \pm \Delta E_s)t}$. For $|\Delta - U|, U \gg J_0$ the fast-oscillating terms average to zero (rotating wave approximation (RWA)), and only quasi-resonant terms remain $\delta \phi$. As a result, processes (i) and (iv), (ii), and (iii) present an effective hopping rate $\frac{b_1}{2} e^{i \phi_1}$, with $s = 1, 2, 3$, respectively. We consider below the particular case with $\delta J_{2,3} = \beta \delta J_1$, $\phi_1 = 0$, $\phi_{2,3} = \phi$. Undoing the interaction picture we obtain the effective time-independent Hamiltonian:

$$\mathcal{H}_{\text{eff}} = -\frac{\delta J_1}{2} \sum_{j,\sigma} c_{j+1,\sigma}^\dagger F[n_{j+1,\sigma} - n_{j,\sigma}] c_{j,\sigma} + \bar{U} H_{\text{int}},$$

(3)

where $F[0] = 1$, and $F[1] = \beta e^{i \phi}$. 3CM provides remarkable control possibilities. Both the amplitude and the phase of the hopping rate of the $\sigma$ component depend on the site occupation of the $\bar{\sigma}$ component. As shown below, this may be employed to realize ODG. Moreover, the detuning $\bar{U}$ results in an effective on-site interaction, allowing for controlling interactions even in those systems where Feshbach resonances are not available. This is in particular the case of alkaline-earth fermions in the lowest $^1S_0$ state $^{32}$. Since 3CM may be also used with multi-component fermions, this opens a novel way of controlling the properties of SU(N) fermions $^{33}$.

Although for $J(t) \ll |\Delta|/|\Delta \pm U|$ direct hoppings are energetically forbidden, virtual hoppings may induce effective interactions between NN sites $^{34}$ of the form

$$\mathcal{H}_{NN} = \sum_{i,j} \left[ \frac{2J_0^2}{\Delta + U} P_{ij}^0 P_{ij}^2 - \frac{2J_0^2}{\Delta - U} P_{ij}^2 P_{ij}^0 \right] + \frac{J_0^2}{\Delta} \left( (1 - n_i) P_{ij}^1 - P_{ij}^1 (1 - n_j) \right) + \frac{2U J_0^2}{\Delta^2 - U^2} \left( P_{ij}^1 P_{ij}^{1\dagger} + P_{ij}^{1\dagger} P_{ij}^1 - S_j^z S_i^z - S_i^z S_j^z \right),$$

(4)

where $S_i^+ = c_i^\dagger c_i$ and $S_i^- = c_i c_i^\dagger$ are spin operators, $n_i = n_{i,\downarrow} + n_{i,\uparrow}$, and we introduce the projector of two particles per site $P_{ij}^0 = n_{i,\downarrow} n_{i,\uparrow}$, zero particles $P_{ij}^0 = (1 - n_{i,\downarrow}) (1 - n_{i,\uparrow})$, and a single particle $P_{ij}^1 = (1 - n_{i,\downarrow}) n_{i,\uparrow}$, and a single particle $P_{ij}^{-1} = (1 - n_{i,\uparrow}) n_{i,\downarrow}$, and a single particle $P_{ij}^{1\dagger} + P_{ij}^{1\dagger}$. The peculiar NN interactions depend on $J_0^2/\Delta$ and $J_0^2/(U \pm \Delta)$, whereas the effective hopping is given by $\delta J_1$. Hence they may be separately controlled. For sufficiently small $J_0 \ll \Delta, |U \pm \Delta|$ we may neglect $\mathcal{H}_{NN}$. However, as shown below, $\mathcal{H}_{NN}$ opens additional interesting possibilities.

Non-equilibrium dynamics. Figure 2(a) depicts our results for the dynamics of the averaged probability of

![FIG. 2: (Color online) (a) Average double occupancy $\langle P_2 \rangle$ after a sudden-quench of $\delta V$ for a finite temperature $T = T_0$, $\Delta/J_0 = 20$, $U/J_0 = 10$, $\delta J_1/J_0 = 0.2$, $\beta = 1$, and different values of $U/\delta J_1$ and $\phi$ (exact diagonalization results using 6 particles in 6 sites); dashed (solid) curves depict the results of the effective (full) model; (b) $\langle P_2 \rangle$ for a quasi-adiabatic preparation (iTEBD results for $\rho = 1$). The system is initially prepared in a MI for $\delta V = 0$. $\delta V(t)$ is linearly increased to its final value for $0 < \delta t < 40$; we consider $U/J_0 = 5$, $\delta J_1/J_0 = 0.1, \bar{U}/\delta J_1 = 2$. $\beta = 1$. ($P_2(t)$ for the full (solid) and effective model (dashed) oscillates around the expected value (horizontal lines) for the ground state with the final $\delta V$.]

(\text{continued})
double occupancy, \( \langle P_2 \rangle \), based on exact diagonalization of small systems [36]. We initially prepare for \( \Delta V = 0 \) a Mott-insulator (MI) state at \( U \gg J_0 \), assuming an initial temperature \( T = J_0 \), and hence initially \( \langle P_2 \rangle \approx 0 \). At time \( t = 0 \) we abruptly turn on the modulation on the effective interactions \( \delta J(t) \).

The results show a very good agreement between the effective model \( \mathcal{H}_{eff} + \mathcal{H}_{NN} \), and the full model \( \mathcal{H} \). Figure 2(a) shows that non-equilibrium experiments should be able to reveal both the ODG, and the suppression of \( \langle P_2 \rangle \) resulting from the repulsive effective interactions \( U \).

The analysis of ground-state properties requires a (quasi-)adiabatic ramping of \( \delta V \). We present in Fig. 2(b) our results obtained using infinite time evolving block decimation (ITEBD) [37]. Starting again with \( \delta V = 0 \) from an initial MI state, we have studied the quasi-adiabatic preparation of different MI states. During the time \( 0 < t < t_{ramp} \) we linearly increase \( \delta V \) to its final value, monitoring \( \langle P_2 \rangle \). Again \( \mathcal{H}_{eff} + \mathcal{H}_{NN} \) reproduces very well the dynamics of the full model \( \mathcal{H} \). After the ramp, the heating induced by the quasi-adiabatic character of the finite ramping time results in oscillations of \( \langle P_2 \rangle \) around the value expected for the ground state of the effective model (see below).

**Phases of the effective Hamiltonian**—At this point we focus on the ground-state physics of \( \mathcal{H}_{eff} \), assuming that \( J_0 \ll \Delta, |\Delta \pm U| \), and hence that \( \mathcal{H}_{NN} \) may be neglected. For \( \beta \neq 1 \), \( \mathcal{H}_{eff} \) realizes a broad class of Hubbard models with correlated hopping extensively studied in the context of cuprate superconductors [38,42], and recently revisited for ultra-cold gases with modulated interactions [14,17]. For \( \phi \neq 0 \), the ODG gives rise to a particularly intriguing physics. For \( \beta = 1 \):

\[
\mathcal{H}_{eff} = -\frac{\delta J_1}{2} \sum_{\sigma,j} c_{j+1,\sigma}^\dagger e^{i\phi[n_{\sigma,j+1}-n_{\bar{\sigma},j}]} c_{j,\bar{\sigma}} + U\mathcal{H}_{int}. 
\]  

(5)

For a low lattice filling \( \rho \) for which processes (iv) may be neglected, a Jordan-Wigner like transformation [20], \( f_j = e^{i2\phi \sum_{i < j} n_i} e^{i\phi n_j} c_j \), maps \( \mathcal{H} \) into a two-component anyon-Hubbard model (2-AHM):

\[
\mathcal{H}_{AHM} = -\frac{\delta J_1}{2} \sum_{\sigma,j} (f_{j,\sigma} f_{j+1,\sigma} ^\dagger + \text{H.c.}) + U\mathcal{H}_{int}. 
\]  

(6)

where the operators \( f_{j,\sigma} \) and \( f_{j,\sigma} ^\dagger \) characterize anyon-like hardcore particles that fulfill a deformed exchange statistics (DES): \( f_{j,\sigma} f_{k,\sigma'} ^\dagger + Q_{j,k} ^\sigma \sigma' f_{k,\sigma'} ^\dagger f_{j,\sigma} = \delta_{jk} \delta_{\sigma,\sigma'} \) and \( f_{j,\sigma} f_{k,\sigma'} + Q_{j,k} ^\sigma \sigma' f_{k,\sigma'} f_{j,\sigma} = 0 \), with \( Q_{j,k} ^\sigma = e^{i2\phi} (j > k), 0 (j = k), e^{-i2\phi} (j < k) \). Specific cases of the 2-AHM have been studied in the context of exactly solvable models [13,44]. In contrast, the non-integrable DES discussed here does strongly modify the spectrum of the 2-AHM compared to the fermionic Hubbard model.

Figure 3(a) shows, as a function of \( \beta \) and the chemical potential \( \mu \), the ground-state phase diagram of \( \mathcal{H} \) for \( \phi = \pi/2 \) and \( \bar{U} = 0 \), obtained by means of density matrix renormalization group (DMRG) [43] simulations in finite-size open-boundary systems of up to 80 sites, keeping up to 600 matrix states [46]. For \( \beta = 0 \) doubly-occupied sites (doublons) and empty ones (holons) become mutually impenetrable, resulting at half filling in a non-conducting metal with a vanishing Drude weight (Kohn metal) [40]. For \( 0 < \beta < 1 \), in the absence of ODG, the system undergoes a smooth phase transition from a metal (M) with dominant spin-density wave (SDW) correlations, \( -1/2 \langle n_{\sigma,j} n_{\bar{\sigma},j-1} \rangle \), to a triplet superconductor [17]. On the contrary, for \( \phi = \pi/2 \), the M phase undergoes for \( \beta \leq 1.4 \) a commensurate-incommensurate (C-IC) phase transition, marked by a kink in the \( \mu(\rho) \) curve (Fig. 3(b)), to a peculiar gapless multi-component (MC) phase. The MC phase presents a central charge \( c \approx 3 \) [17,49]. In contrast, the metallic phase has \( c = 2 \). The MC phase smoothly connects to the Kohn-metal for \( \beta \rightarrow 0 \). For \( \beta \geq 1.4 \) and \( \rho 
eq 1 \), a spin gap \( \Delta_S \) opens and the kink in \( \mu(\rho) \) disappears marking the transition to a phase with dominant singlet-superconducting (SS) correlations, \( \langle Q_{j-k} \rangle \), with \( Q_{j-k} \equiv c_{j+1,\uparrow} c_{j,\downarrow} - c_{j+1,\downarrow} c_{j,\uparrow} \). Finally, at \( \rho = 1 \) we find a MI with dominant SDW correlations, and a totally gapped phase with bond-ordering.

FIG. 3: (Color online) (a) Phase diagram of \( \mathcal{H}_{eff} \) as function of \( \mu/\delta J_1 \) and \( \beta \) for \( \phi = \pi/2 \) and \( \bar{U} = 0 \) [46]. The dash-dotted lines mark the C-IC M-MC transition. The dashed (blue) lines denote the opening of \( \Delta_S \) that marks the MC-SS and MI-BOW transitions. Shaded regions depict the vacuum. (b) Equation of state \( \rho = \rho(\mu) \) for \( \beta = 0, 0.5, 1, \) and 1.5 for the parameters of Fig. (a). (c) Momentum distribution \( n_s(k) \) of Eq. (5) for \( \rho = 0.5 \), and \( \phi = \pi/2 \) \( (L = 60) \).
The nature of this phase is best understood for \( \phi = 0 \), \( U > 0 \) results in a MI (SDW) phase with a finite charge excitation gap \( \Delta_r > 0 \) but \( \Delta_s = 0 \), whereas for \( U < 0 \) \( \Delta_s > 0 \) and \( \Delta_r = 0 \). For sufficiently large \( J_0/\Delta \) the system is driven into a fully gapped density-wave (DW), characterized by a non-vanishing DW order \((-1)^j n_{\sigma n_j}\). For \( U > 0 \) we observe two MI phases with a suppressed doublon number, the above mentioned MI (SDW) and a region of phase separation of ferromagnetic domains (PS). The MI-DW transition is associated to the opening of \( \Delta_s \), characterized by the Luttinger-liquid parameter in the spin sector \( K_S = 1 \) (+-symbols in Fig. 4)\[51, 52\]. Since \( H_{NN} \) breaks the spatial reflection symmetry, we do not observe a separate BOW phase, as it is the case for Hubbard models with standard density-density NN interactions \[53\], but a non-zero BOW-order in the DW due to the preferred creation of excitations in a particular spatial direction.

**Outlook.**- A multicolor modulation of the lattice depth allows for a flexible separate manipulation of (a) correlated hopping, controlled by the modulation amplitudes \( \delta V_s \); (b) ODG, given by the dephasings \( \phi_s \); (c) effective on-site interactions, provided by the detuning \( U \); and (d) NN interactions, that depend on \( J_0/\Delta \) and \( J_0/|\Delta \pm U| \). 3CM thus provides an experimentally straightforward method for engineering a very broad class of lattice models, including Hubbard Hamiltonians with correlated hopping, peculiar extended models, and 2-AHM. In particular, the controllable quantum statistics of the 2-AHM results in a peculiar MC phase of coexisting superconducting and metallic components. The RWA requirements necessary for the 3CM are readily achievable experimentally. For example, for \( ^{172}\text{Yb} \) (scattering length of 199.4a_B and lattice spacing of 380nm \[11, 54\]) with \( s_0 = 6.9 \) \( (J_0/\hbar = 100\text{Hz}) \), one achieves \( U = 23J_0, \Delta = 16J_0, \Delta - |U| = 7J_0 \), well within the RWA requirements. For \( \delta J/J_0 = 0.2 \), the typical effective-tunneling time is \( \tau = \hbar/\delta J \approx 8 \text{ ms} \).

Multi-color modulation permits several further interesting extensions, including the control of three-body interactions \[57\]. In combination with a Raman-induced coupling of several spin components \[11, 12\] one may study density dependent magnetic fields \[31\]. Other scenarios could pave a realistic exploration path towards the simulation of dynamical gauge fields with cold atoms in optical lattices, exploring e.g. occupation-dependent non-Abelian fields and gauge fields in Fermi-Bose mixtures.

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Supplementary material for “Engineering interactions and anyon statistics by multicolor lattice-depth modulations”

Lorenzo Cardarelli, 1 Sebastian Greschner, 1 and Luis Santos 1

1 Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstr. 2, DE-30167 Hannover, Germany

In this Supplementary Material we discuss in more detail about the physics of the multi-component (MC) phase. We comment as well in more detail about the derivation of the effective Hamiltonian using a Magnus expansion.

I. MULTI-COMPONENT PHASE

A. Two-particle model

We assume for simplicity $\beta = 1$, and hence Model (5) of the main text. We are interested in the two-particle problem, with one ↑ particle and one ↓ particle. Let $|D(j)\rangle$ be a doubly occupied site ($j$ site), and $|S(j, j + l)\rangle$ a singlet state placed in sites $j$ and $j + l$. Then

$$H_{\text{eff}}|D(j)\rangle = -\frac{\delta J_1}{\sqrt{2}} \left[ e^{i\phi}|S(j, j + 1)\rangle + e^{-i\phi}|S(j - 1, j)\rangle \right] + \tilde{U}|D(j)\rangle,$$

$$H_{\text{eff}}|S(j, j + 1)\rangle = -\frac{\delta J_1}{\sqrt{2}} \left[ e^{i\phi}|D(j + 1)\rangle + e^{-i\phi}|D(j)\rangle \right] - \frac{\delta J_1}{2} \left[ |S(j, j + 1)\rangle + |S(j, j + 2)\rangle \right],$$

$$H_{\text{eff}}|S(j, j + l)\rangle \overset{\text{†}}{=} -\frac{\delta J_1}{2} \left[ |S(j - 1, j + l)\rangle + |S(j + 1, j + l)\rangle + |S(j, j + l - 1)\rangle + |S(j, j + l + 1)\rangle \right].$$

Let $|D(k)\rangle = \frac{1}{\sqrt{L}} \sum_l e^{ikl}|D(l)\rangle$ and $|S(j, k)\rangle = \frac{1}{\sqrt{L}} \sum_l e^{ik(l+j/2)}|S(l, l+j)\rangle$, with $k$ the center-of-mass momentum of the pair, and $L$ the number of sites. Then $H_{\text{eff}} = \sum_k H_{\text{eff}}(k)$, with $H_{\text{eff}}(k) = H_0(k) + H_1(k)$, where:

$$H_0(k) = \tilde{U}|D(k)\rangle\langle D(k)| - A(k) [|S(1, k)\rangle\langle D(k)| + h.c.|,$$

$$H_1(k) = -B(k) \sum_{j \geq 1} [|S(j, k)\rangle\langle S(j + 1, k)| + h.c.|,$$

with $A(k) = \sqrt{2}\delta J_1 \cos(k/2 - \phi)$ and $B(k) = \delta J_1 \cos(k/2)$. We may diagonalize $H_0$:

$$H_0(k) = E_+(k)|\tilde{P}(k)\rangle\langle \tilde{P}(k)| + E_-(k)|P(k)\rangle\langle P(k)|,$$

where the eigenenergies are $E_{\pm}(k) = \frac{\tilde{U}}{2} \pm \sqrt{\left( \frac{\tilde{U}}{2} \right)^2 + A(k)^2}$, and the corresponding eigenstates are $|\tilde{P}(k)\rangle = \cos \theta(k)|D(k)\rangle + \sin \theta(k)|S(1, k)\rangle$, and $|P(k)\rangle = -\sin \theta(k)|D(k)\rangle + \cos \theta(k)|S(1, k)\rangle$, with $\tan \theta(k) = \frac{\tilde{U}/2 - E_-(k)}{A(k)}$. The Hamiltonian $H_0$ characterizes deeply-bound pairs. We may then split $H_1(k) = H_c(k) + H_u(k)$, where

$$H_c(k) = -B(k) \sum_{j \geq 2} [|S(j, k)\rangle\langle S(j + 1, k)| + h.c.|$$

and

$$H_u(k) = -B(k) \left( \sin \theta(k)|P(k)\rangle + \cos \theta(k)|\tilde{P}(k)\rangle \right) [|S(2, k)\rangle + h.c.,$$

determines the physics of broken pairs, where the dynamics of relative coordinate $j$ is given by the hopping rate $B(k)$, and

characterizes the coupling between deeply-bound and unbound pairs. Note that such a coupling is also given by $B(k)$.

Let us consider $\phi = \frac{\pi}{2}$. In that case, $E_{\pm}(k) = \frac{\tilde{U}}{2} \pm \sqrt{\left( \frac{\tilde{U}}{2} \right)^2 + 2\delta J_1^2 \sin^2(k/2)}$. The minimal energy is clearly for $k = \pi$, $E_F \equiv E_-(\pi) = \frac{\tilde{U}}{2} \pm \sqrt{\left( \frac{\tilde{U}}{2} \right)^2 + 2\delta J_1^2}$. If existing, bound pairs will quasi-condense in $|P\rangle \equiv |P(\pi)\rangle$. Crucially, $B(\pi) = 0$, and hence $H_c = 0$. As a result, $|P\rangle$ remains a deeply-bound two-particle eigenstate, fully decoupled from the unbound pairs, irrespective of the value of $\tilde{U}/\delta J_1$. On the contrary for $\phi = 0$, i.e. without occupation-dependent gauge (ODG), the bound pairs
are fully connected with the rest and cannot be formed unless \( \tilde{U} < 0 \) dominates. For \( \phi \) in the vicinity of \( \pi/2 \) the coupling \( \mathcal{H}_c \) may be considered perturbative, and deeply-bound pairs due to the ODG still exist even if \( \phi \) is not exactly \( \pi/2 \).

The existence of these pairs that are deeply-bound by the ODG rather than by attractive interactions is crucial to understand the nature of the MC phase. The metallic (M) phase is stable if \( E_F < E_P/2 < E_F' \), where \( E_F \) the Fermi energy of the metal. However, for decreasing \( \tilde{U} > 0 \), \( E_F' < E_F/2 \), and hence it is energetically favorable to pair part of the Fermi sea into \( |P\rangle \) pairs, until reaching an equilibrium at a new Fermi energy \( E_{F}' = E_{P}/2 \). This partial pairing, and the corresponding coexistence of a two-component metal and a superconductor explains the MC phase, and its \( e = 3 \) central charge. For \( E_-(\pi) < -2\delta J_1 \) (which occurs at \( \tilde{U}/\delta J_1 \simeq -1 \)) the Fermi sea is completely depleted, and the system enters the fully-paired (SS) phase.

\[
\mathcal{H}^{(0)} = \frac{1}{T} \int_0^T dt_1 \mathcal{H}(t_1)
\]

provides Eq. (3) of the main text. The first order correction in \( \frac{U}{\delta J_1} \) may be expressed as \([1]\)

\[
\mathcal{H}^{(1)}_{ME} = \frac{-1}{2T} \int_0^T dt_2 \int_0^{t_2} dt_1 [\mathcal{H}(t_2), \mathcal{H}(t_1)].
\]

If the time periodic Hamiltonian is given by a Fourier series \( \mathcal{H}(t) = \mathcal{H}_0 + \sum V^{(k)} e^{ikwt} \), then

\[
\mathcal{H}^{(1)}_{ME} = \frac{1}{\omega} \sum_k \frac{1}{k} \left( [V^{(k)}, V^{(-k)}] - [V^{(k)}, \mathcal{H}_0] + [V^{(-k)}, \mathcal{H}_0] \right).
\]

In Eq. (2) of the main text we expand the exponential term \( e^{\pm i t U n \sigma} = 1 + (e^{\pm i t U} - 1)n_{J \sigma} \). Then

\[
\tilde{\mathcal{H}}(t) = (J_0 + \delta J(t)) \left( e^{i \Delta - U} \tilde{V}^{(1)} + e^{i \Delta U} \tilde{V}^{(2)} + e^{i |\Delta + U|} \tilde{V}^{(3)} + \text{H.c.} \right)
\]
with

\[ \bar{V}^{(1)} = \sum_{j,\sigma} d_{j,\sigma}^\dagger c_{j+1,\sigma} - d_{j,\sigma}^\dagger d_{j+1,\sigma}, \]

\[ \bar{V}^{(2)} = \sum_{j,\sigma} \left( d_{j,\sigma}^\dagger - c_{j,\sigma} \right) \left( d_{j+1,\sigma}^\dagger - c_{j+1,\sigma} \right), \]

\[ \bar{V}^{(3)} = \sum_{j,\sigma} c_{j,\sigma}^\dagger d_{j+1,\sigma} - d_{j,\sigma}^\dagger d_{j+1,\sigma}. \]

where we employ the correlated annihilation operator \( d_{j,\sigma} \equiv n_{j,\sigma} c_{j,\sigma} \). Neglecting terms of order \( J_0 \delta J \) and \( \delta J^2 \) we may write

\[ \mathcal{H}_{MB}^{(1)} = \frac{J_0^2}{\Delta - U} \left[ \bar{V}^{(1)}, \bar{V}^{(1)} \right] + \frac{J_0^2}{\Delta} \left[ \bar{V}^{(2)}, \bar{V}^{(2)} \right] + \frac{J_0^2}{\Delta + U} \left[ \bar{V}^{(3)}, \bar{V}^{(3)} \right] + \mathcal{O}(\delta J), \]

which after some algebra yields Eq. (4) of the main text.

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