DYNAMIC EQUIVALENCE OF CONTROL SYSTEMS AND INFINITE PERMUTATION MATRICES

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Abstract. To each dynamic equivalence of two control systems is associated an infinite permutation matrix. We investigate how such matrices are related to the existence of dynamic equivalences.

1. Introduction

A control system is an underdetermined ODE system of the form
\[ \dot{x} = f(t, x, u), \]
where \( x = (x_i) \) are called the state variables, \( u = (u_α) \) the control variables. The meaning of “control” is clear: under suitable regularity conditions, specifying a control function \( u(t) \) and an initial value \( x(t_0) \) uniquely determines a local ‘trajectory’ \( x(t) \) that satisfies the ODE system and the initial value. When \( f \) does not explicitly depend on \( t \), the control system is said to be autonomous. In the current paper, all control systems considered are autonomous.

Let \( \dot{x} = f(x, u) \) and \( \dot{y} = g(y, v) \) be two control systems. Suppose that there exist mappings \( φ = φ(x, u, \hat{u}, \ddot{u}, ..., u^{(p)}) \) and \( ψ = ψ(y, v, \hat{v}, \ddot{v}, ..., v^{(q)}) \) such that,

- for any solution \( (x(t), u(t)) \) of the first system, the function \( (y, v) = φ(x, u, \hat{u}, \ddot{u}, ..., u^{(p)}) \) is a solution of the second;
- for any solution \( (y(t), v(t)) \) of the second system, the function \( (x, u) = ψ(y, v, \hat{v}, \ddot{v}, ..., v^{(q)}) \) is a solution of the first;
- moreover, applying \( φ \) and \( ψ \) successively to a solution \( (x(t), u(t)) \) of the first system yields the same solution \( (x(t), u(t)) \), and, similarly, applying \( ψ \) and \( φ \) successively to a solution \( (y(t), v(t)) \) of the second system yields the same solution \( (y(t), v(t)) \).

If all these conditions are satisfied, we say that the pair of maps \((φ, ψ)\) defines a dynamic equivalence between the two control systems. Intuitively, a dynamic equivalence provides a one-to-one correspondence between the spaces of solutions of the two control systems.

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Fixing a dynamic equivalence defined by a pair of maps \((\phi, \psi)\), one can always find the smallest \(p, q \geq 0\) so that \(\phi = \phi(x, u, \hat{u}, \ldots, u^{(p)})\) and \(\psi = \psi(y, v, \hat{v}, \ldots, v^{(q)})\). We call such a pair \((p, q)\) the *height* of the corresponding dynamic equivalence. A dynamic equivalence with height \((0, 0)\) is known as a *static (feedback) equivalence*, in which case \(\phi, \psi\) are inverses of each other as diffeomorphisms.

An immediate question is: *How much more general is the notion of dynamic equivalence than that of static equivalence.* Classical results (see [Pom95]) suggest that the answer depends on the number of control variables. In particular, a dynamic equivalence between two control systems with a single control variable is necessarily static. It is also well known that the number of control variables is invariant under a dynamic equivalence. However, in the cases of 2 or more controls, a precise answer to the question above remains largely unknown.

In [Sta13], the author considered all control-affine systems with 3 states and 2 controls, proving that three statically non-equivalent systems are pairwise dynamically equivalent at height \((1, 1)\). In addition, he introduced a new method of studying dynamic equivalences of two control systems. He proved that to each dynamic equivalence is associated an infinite permutation matrix. Intuitively, such a matrix tells us how the ‘generating 1-forms’ of certain prolongations of the two control systems (viewed as Pfaffian systems), when chosen appropriately, relate under a dynamic equivalence.

In the current work, we present further properties of dynamic equivalences that can be derived using the associated infinite permutation matrices. First we prove that there is a *rank matrix* (Definition 3.1) associated to a dynamic equivalence, which has a more ‘invariant’ nature than an associated infinite permutation matrix (Proposition 3.2). Then we prove several inequalities and equalities (Propositions 4.1 and 4.2) satisfied by the rank matrix. Using these results, we prove an inequality satisfied by the height \((p, q)\) of a dynamic equivalence (Theorem 4.1). In particular, this inequality implies the

**Theorem 4.3.** Two control systems with the same number of state variables and 2 control variables can only be dynamically equivalent at height \((p, q)\) with \(p = q\).

2. Control Systems and Dynamic Equivalence

2.1. Control Systems of Type \((n, s)\).

**Definition 2.1.** A control system of type \((n, s)\) \((s < n)\) is an underdetermined ODE system

\[
\dot{x} = f(x, u),
\]

where

\[\begin{align*}
x &= (x_i) \in \mathbb{R}^n, \\
u &= (u_\alpha) \in \mathbb{R}^s,
\end{align*}\]
and \( f = (f_i) : \mathbb{R}^{n+s} \to \mathbb{R}^n \) is a smooth function satisfying \( \text{rank} \left( \frac{\partial f_i}{\partial u_\alpha} \right) = s \) on some open domain\(^1\) in \( \mathbb{R}^{n+s} \). Here, \( x_i \) are called the state variables, \( u_\alpha \) the control variables.

For a control system, there is an equivalent geometric characterization. Let \( D \subset \mathbb{R}^n \) be the domain of the state variables \( x = (x_i) \). The admissible \( t \)-derivatives of \( x_i \), as imposed by the control system, are given by specifying a submanifold \( \Sigma \subset TD \) that submerses onto \( D \) with rank-\( s \) fibers. Each fiber is precisely parametrized by the control variables \( u = (u_\alpha) \). The submanifold \( \Sigma \) induces an embedding

\[
\iota : \mathbb{R} \times \Sigma \hookrightarrow \mathbb{R} \times TD,
\]

which is the identity in the \( \mathbb{R} \)-factor (with coordinate \( t \)) and \( t \)-independent in the \( \Sigma \)-factor. In coordinates, this embedding may be written as

\[
\iota(t, x, u) = (t, x, f(x, u))
\]

which satisfies

\[
\iota^* (dx_i - \dot{x}_i dt) = dx_i - f_i(x, u) dt.
\]

In other words, the system (1) corresponds to the Pfaffian system \((M, C_M)\), where \( M := \mathbb{R} \times \Sigma \), and \( C_M \) is the restriction to \( M \) of the standard contact system \( C = \langle dx_i - \dot{x}_i dt \rangle_{i=1}^n \) on the jet bundle \( J^1(\mathbb{R}, D) \cong \mathbb{R} \times TD \). Conversely, let \( \Sigma \subset TD \) be a submanifold that submerses onto a domain \( D \subset \mathbb{R}^n \) with rank-\( s \) fibers. The Pfaffian system \((M, C_M)\) corresponds to a control system of type \((n, s)\).

### 2.2. Prolongations of a Control System.

**Definition 2.2.** Let \((M, C_M)\) be a control system. For any \( p \in M \), an integral element at \( p \) is a 1-dimensional vector subspace \( L = \mathbb{R} u \subset T_p M \) satisfying \( \theta(u) = 0 \) for all 1-forms \( \theta \in C_M \) and \( dt(u) \neq 0 \).

In coordinates, at each point \( p \in M \), the 1-forms in \( C_M \) are linear combinations of \( dx_i - f_i(x, u) dt \) \( (i = 1, \ldots, n) \). It follows that, once we specify values of \( s \) auxiliary constants \( u_\alpha^{(1)} \), an integral element at \( p \) is uniquely determined by the vanishing of \( s \) extra 1-forms \( du_\alpha - u_\alpha^{(1)} dt \) \( (\alpha = 1, \ldots, s) \). In other words, the space of integral elements at \( p \in M \) is parametrized by the \( s \) parameters \( u_\alpha^{(1)} \). Let \( u^{(1)} := (u_\alpha^{(1)}) \).

**Definition 2.3.** The first total prolongation of a control system \((M, C_M)\) is the Pfaffian system \((M^{(1)}, C^{(1)})\), where \( M^{(1)} \) is the space of integral elements of \((M, C_M)\), with the standard coordinates \((t, x, u, u^{(1)})\); \( C^{(1)} \) is the Pfaffian system generated by

\[
\text{dx}_i - f_i(x, u) dt, \quad du_\alpha - u_\alpha^{(1)} dt, \quad (i = 1, \ldots, n; \ \alpha = 1, \ldots, s).
\]

\(^1\)Since our study is local, we henceforth assume that such a domain is the entire \( \mathbb{R}^{n+s} \).
Let $k$ be a positive integer. One can start from a control system $(M, C_M)$ of type $(n, s)$ and generate total prolongations successively for $k$ times. The result will be denoted as $(M^{(k)}, C^{(k)})$, where $M^{(k)}$ has the coordinates $(t, x, u, u^{(1)}, \ldots, u^{(k)})$ and $C^{(k)}$ is generated by the 1-forms
\[
dx_i - f_i(x, u) dt, \quad du_\alpha - u^{(1)}_\alpha dt, \quad du^{(\ell)}_\alpha - u^{(\ell+1)}_\alpha dt,
\]
\[
(i = 1, \ldots, n; \alpha = 1, \ldots, s; \ell = 1, \ldots, k - 1).
\]

When $k = 0$, we simply let $(M^{(0)}, C^{(0)})$ denote $(M, C_M)$.

It is clear that $(M^{(k)}, C^{(k)})$ is a control system of type $(n + ks, s)$.

### 2.3. Dynamic Equivalence.

Given two control systems, it is natural to regard them as equivalent if one can establish a one-to-one correspondence between their solutions.

Of course, two control systems $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ are equivalent in the sense above when they can be transformed into each other by a change of variables of the form $y = \phi(x), v = \psi(x, u)$ and $x = \phi^{-1}(y), u = \rho(y, v)$. This notion of equivalence is called static equivalence, which, in particular, requires that the two equivalent control systems are of the same type. However, it is possible for two systems of different types to have a one-to-one correspondence between their solutions, as is indicated by the following standard property of jet bundles.

**Proposition 2.1.** Let $(M, C_M)$ be a control system. Let $\pi : M^{(k)} \to M$ be the canonical projection from its $k$-th total prolongation. Any integral curve $\tau : \mathbb{R} \to M$ of $(M, C_M)$ has a unique lifting $\tau^{(k)} : \mathbb{R} \to M^{(k)}$ (i.e., satisfying $\pi \circ \tau^{(k)} = \tau$) to an integral curve of $(M^{(k)}, C^{(k)})$. In addition, for each integral curve $\sigma : \mathbb{R} \to M^{(k)}$ of $(M^{(k)}, C^{(k)})$, its projection $\pi \circ \sigma$ is an integral curve of $(M, C_M)$.

In other words, given two control systems, a one-to-one correspondence between their solutions may involve differentiation. This motivates the following notion of equivalence.

**Definition 2.4.** Two control systems $(M, C_M)$ and $(N, C_N)$ are said to be dynamically equivalent if there exist positive integers $p$ and $q$ and submersions $\Phi : M^{(p)} \to N$ and $\Psi : N^{(q)} \to M$ that satisfy

i. $\Phi, \Psi$ preserve the $t$-variable and are $t$-independent in the state and control components;

ii. $\Phi$ cannot factor through any $M^{(k)}$ for $k < p$; $\Psi$ cannot factor through any $N^{(\ell)}$ for $\ell < q$;

iii. for each integral curve $\tau : \mathbb{R} \to M$ of $(M, C_M)$, $\Phi \circ \tau^{(p)}$ is an integral curve of $(N, C_N)$; for each integral curve $\sigma : \mathbb{R} \to N$ of $(N, C_N)$, $\Psi \circ \sigma^{(q)}$ is an integral curve of $(M, C_M)$;

\[\text{A more careful definition would set the domains of } \Phi \text{ and } \Psi \text{ to be open subsets of } M^{(p)} \text{ and } N^{(q)}, \text{ respectively. For example, see[Pom94]. Since our results are local, for the economy of notations, we will be content with the definition presented here.}\]
iv. letting $\tau$ and $\sigma$ be as in iii, we have:

$$\tau = \Psi \circ (\Phi \circ \tau^{(p)})^{(q)}, \quad \sigma = \Phi \circ (\Psi \circ \tau^{(q)})^{(p)}.$$  

For the convenience of the reader, we present the commutative diagram:

\[ \begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tau} & M \\
\downarrow{p} & & \downarrow{p} \\
M^{(p)} & \xrightarrow{\pi} & N^{(q)} \\
\uparrow{q} & & \uparrow{q} \\
N & \xleftarrow{\sigma} & \mathbb{R}
\end{array} \]

**Remark.** (1) It is easy to verify that Definition 2.4 defines an equivalence relation. (2) By this definition, a control system $(M, C_M)$ is dynamically equivalent to each of its total prolongations $(M^{(k)}, C^{(k)})$. (3) A dynamic equivalence with $p = q = 0$ is a static equivalence. To see this, let $(t, x, u)$ and $(t, y, v)$ be coordinates on $M$ and $N$, respectively. Represent $\Phi$ in local coordinates as $(t, \phi(x, u), \psi(x, u))$. Since $\Phi$ maps integral curves of $(M, C_M)$ to integral curves of $(N, C_N)$, it is necessary that, for each $dy_i - g_i(y, v)dt \in C_N$, its pull-back

$$\Phi^*(dy_i - g_i(y, v)dt) = d(\phi_i(x, u)) - g_i(\phi(x, u), \psi(x, u))dt$$

is contained in $C_M$. It follows that $\phi(x, u)$ is independent of $u$. A similar argument applies to $\Psi$. Finally, Condition iv in Definition 2.4 implies that $\Phi$ and $\Psi$ are inverses of each other.

**Definition 2.5.** We call the pair of integers $(p, q)$ in Definition 2.4 the *height* of the corresponding dynamic equivalence.

Given two dynamically equivalent control systems $(M, C_M)$ and $(N, C_N)$, if needed, one could always apply a partial prolongation (for details, see [Sta13]) to one of them such that the resulting systems have the same number of states and are still dynamically equivalent. This perspective suggests understanding all dynamic equivalences between control systems with the same number of states.

**Proposition 2.2.** Let $(M, C_M)$ and $(N, C_N)$ be control systems with the same number of states. The height $(p, q)$ of a dynamic equivalence between them must satisfy either $p = q = 0$ or $p, q > 0$.

**Proof.** Suppose that the following commutative diagram represents a dynamic equivalence of height $(p, 0)$ $(p > 0)$ between $(M, C_M)$ and $(N, C_N)$:

\[ \begin{array}{ccc}
(M^{(p)}, C^{(p)}) & \xrightarrow{\pi} & (M, C_M) \\
\downarrow{\Phi} & & \downarrow{\Psi} \\
(N, C_N) & \xleftarrow{\sigma} & (N, C_N)
\end{array} \]

By the assumption, $\text{rank}(C_M) = \text{rank}(C_N)$. Since

$$\pi^* C_M = (\Psi \circ \Phi)^* C_M = \Phi^* (\Psi^* C_M) \subseteq \Phi^* C_N,$$
and since \( \pi, \Phi, \Psi \) are all submersions, it is necessary that \( \langle \pi^* C_M \rangle = \langle \Phi^* C_N \rangle \).

Now, \( \Phi \) is constant along the Cauchy characteristics of \( \Phi^* C_N \), since the Cartan system (see [BCG13]) of \( C_N \) generates the entire cotangent bundle of \( N \). On the other hand, the Cauchy characteristics of \( \pi^* C_M \) are precisely the fibres of \( \pi \). This proves that \( \Phi \) factors through \( M \), a contradiction to the choice of \( p \). The case when \( p = 0, q > 0 \) is similar. \( \square \)

3. Infinite Permutation Matrices Associated to a Dynamic Equivalence

In this section, we assume \( p, q > 0 \) unless otherwise noted.

Given a control system \( (M, C_M) \), one automatically obtains a system of projections

\[
\pi_{k,j} : M^{(k)} \to M^{(j)}, \quad (k \geq j).
\]

The inverse limit of this projective system is denoted as

\[
M^{(\infty)} := \lim_{k \to \infty} M^{(k)}.
\]

Let \( \pi_k : M^{(\infty)} \to M^{(k)} \) be the canonical projections. Since \( \pi_{k,j}^* C^{(j)} \subseteq C^{(k)} \) for all \( k \geq j \geq 0 \), one can define \( C^{(\infty)} \) to be the differential system generated by \( \bigcup_{k \geq 0} \pi_{k}^* C^{(k)} \).

In coordinates, if \( (M, C_M) \) corresponds to the system \( x = f(x, u) \), then \( M^{(\infty)} \) has the standard coordinates \( (t, x, u, u^{(1)}, \ldots) \); \( C^{(\infty)} \) is generated by the 1-forms

\[
dx_i - f_i(x, u)dt, \quad du_\alpha - u^{(1)}_\alpha dt, \quad du^{(\ell)}_\alpha - u^{(\ell+1)}_\alpha dt, \quad (i = 1, \ldots, n; \alpha = 1, \ldots, s; \ell \geq 1).
\]

The pair \( (M^{(\infty)}, C^{(\infty)}) \) is called the infinite prolongation of \( (M, C_M) \).

Now suppose that \( (M, C_M) \) \( (\dot{x} = f(x, u)) \) and \( (N, C_N) \) \( (\dot{y} = g(y, v)) \) are two control systems of types \( (n_1, s_1) \) and \( (n_2, s_2) \), respectively, between which a dynamic equivalence of height \( (p, q) \) is given by

\[
\Phi : M^{(p)} \to N, \quad \Psi : N^{(q)} \to M.
\]

It is shown in [Sta13] that \( \Phi \) and \( \Psi \) induce, respectively, maps

\[
\Phi^{(\infty)} : M^{(\infty)} \to N^{(\infty)}, \quad \Psi^{(\infty)} : N^{(\infty)} \to M^{(\infty)},
\]

which satisfy

\[
\Phi^{(\infty)} \circ \Psi^{(\infty)} = Id, \quad \Psi^{(\infty)} \circ \Phi^{(\infty)} = Id.
\]

Moreover, if we let

\[
\omega^0 = dx - f(x, u)dt, \quad \omega^k = du^{(k-1)} - u^{(k)} dt \quad (k \geq 1),
\]

\[
\eta^0 = dy - g(y, v)dt, \quad \eta^k = dv^{(k-1)} - v^{(k)} dt \quad (k \geq 1),
\]
then there exist matrices \( A^i_j, B^i_j (i, j \geq 0) \) satisfying

**P1.** for \( k \geq 0 \), we have

\[
\Psi^{(x)} \omega^k = A_0^k \eta^0 + A_1^k \eta^1 + \cdots + A_{p+k}^k \eta^{p+k},
\]

\[
\Phi^{(x)} \eta^k = B_0^k \omega^0 + B_1^k \omega^1 + \cdots + B_{q+k}^k \omega^{q+k},
\]

where \( A_{p+k}^k, B_{q+k}^k \neq 0 \) for all \( k \geq 0 \).

**P2.** All \( A_{p+k}^k \) are equal for \( k \geq 1 \); similarly for \( B_{q+k}^k \). Hence, we denote

\( A^8 := A_{p+k}^k, B^8 := B_{q+k}^k, p, q \geq 1 \).

**P3.** \( \text{rank}(A^0_0) = \text{rank}(A^\infty) > 0, \text{rank}(B^0_q) = \text{rank}(B^\infty) > 0. \)

For proofs of Properties P1 and P2, see [Sta13]. To see why P3 holds, note that there exists an \( n_1 \times s_1 \) matrix \( F \) such that

\[
d\omega^0 \equiv F \omega^1 \land dt \mod \omega^0
\]

\[
\equiv FA^1_{p+1} \eta^{p+1} \land dt \mod \eta^0, \eta^1, \ldots, \eta^p.
\]

On the other hand,

\[
d\omega^0 = d(A_0^0 \eta^0 + \cdots + A_p^0 \eta^p)
\]

\[
\equiv -A_p^0 \eta^{p+1} \land dt \mod \eta^0, \eta^1, \ldots, \eta^p.
\]

Consequently,

\[
FA^1_{p+1} = -A_p^0.
\]

By Definition 2.1, \( F \) has full rank. Therefore, \( \text{rank}(A^1_{p+1}) = \text{rank}(A_p^0) \). The case for \( B^i_j \) is similar. Property P3 follows.

### 3.1. Two Classical Theorems.

Concerning dynamic equivalences between two control systems, the following two classical theorems are of fundamental importance.

**Theorem 3.1.** ([Pom95]) Two dynamically equivalent control systems must have the same number of control variables.

**Theorem 3.2.** ([Car14]) A dynamic equivalence between two control systems of the same type \( (n, 1) \) can only have height \( (p, q) = (0, 0) \) (i.e., the equivalence is static).

It turns out that these two theorems are consequences of the fact that the following matrices are inverses of each other, after taking into account the maps \( \Phi^{(x)} \) and \( \Psi^{(x)} \):

\[
A = \begin{pmatrix}
A_0^0 & \cdots & A_0^p & 0 & \cdots \\
A_1^0 & \cdots & A_1^p & A_\infty & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
B = \begin{pmatrix}
B_0^0 & \cdots & B_0^q & 0 & \cdots \\
B_1^0 & \cdots & B_1^q & B_\infty & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

\[3\text{Here we have assumed } p, q > 0, \text{ otherwise, these properties may not hold. For example, consider the standard dynamic equivalence between a control system and its } k\text{-th total prolongation.}\]
In fact, suppose that \((M, C_M)\) and \((N, C_N)\) are of types \((n_1, s_1)\) and \((n_2, s_2)\), respectively. We can assume \(n_1 = n_2 = n\), because partial prolongations preserve the number of controls. Consequently, there are two possible cases: \(p = q = 0\) or \(p, q > 0\). In the former case, we have static equivalence. In the latter case, the form of matrix \(A\) implies that the \(n + rs_1\) linearly independent components of \(\omega^0, \ldots, \omega^r\) are all linear combinations of the \(n + (r + p)s_2\) components of \(\eta^0, \ldots, \eta^{r+p}\). When \(s_1 > s_2\), this is impossible because \(n + (r + p)s_2 < n + rs_1\) as long as

\[
r > \frac{ps_2}{s_1 - s_2}.
\]

Theorem 3.1 follows. Furthermore, when \(p, q > 0\), \(A\) and \(B\) being inverses of each other requires that either \(A_\infty B_\infty = 0\) or \(B_\infty A_\infty = 0\). This is impossible when \(s_1 = s_2 = 1\), in which case \(A_\infty\) and \(B_\infty\) are just nonvanishing functions. Theorem 3.2 then follows from Proposition 2.2.

More generally, we have the

**Lemma 3.3.** Suppose that \(E\) is a dynamic equivalence of height \((p, q)\) \((p, q > 0)\) between two control systems with \(s\) controls. It is necessary that the associated matrices \(A_\infty\) and \(B_\infty\) satisfy

\[
2 \leq \text{rank}(A_\infty) + \text{rank}(B_\infty) \leq s.
\]

*Proof.* This is because \(p, q > 0\) implies \(A_\infty B_\infty = 0\). \(\square\)

### 3.2. The Infinite Permutation Matrix \(\mathcal{S}\).

Let \((M, C_M), (N, C_N), \omega^i, \eta^j, A, B\) be as above. In [Sta13], it is proved that there exist transformations

\[
\begin{pmatrix}
\bar{\omega}^0 \\
\bar{\omega}^1 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
g^0_0 & 0 & \cdots \\
g^1_0 & g^1_1 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\omega^0 \\
\omega^1 \\
\vdots
\end{pmatrix},
\]

\[
\begin{pmatrix}
\bar{\eta}^0 \\
\bar{\eta}^1 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
h^0_0 & 0 & \cdots \\
h^1_0 & h^1_1 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\eta^0 \\
\eta^1 \\
\vdots
\end{pmatrix},
\]

where \(g^i_j = g^{i+1}_{i+1}, h^i_j = h^{i+1}_{i+1}\) for all \(i \geq 1\), such that, pointwise,

\begin{align*}
\text{C1.} \quad & \{ \text{span}\{\omega^0, \ldots, \omega^k\} = \text{span}\{\bar{\omega}^0, \ldots, \bar{\omega}^k\}, \quad (k \geq 0); \}
& \{ \text{span}\{\eta^0, \ldots, \eta^k\} = \text{span}\{\bar{\eta}^0, \ldots, \bar{\eta}^k\}, \quad (k \geq 0); \}
\text{C2.} \quad & \{ \frac{d\omega^\ell}{dt} = -\bar{\omega}^{\ell+1} \wedge dt \mod \bar{\omega}^0, \ldots, \bar{\omega}^r, \quad (\ell \geq 1); \}
& \{ \frac{d\eta^\ell}{dt} = -\bar{\eta}^{\ell+1} \wedge dt \mod \bar{\eta}^0, \ldots, \bar{\eta}^r, \quad (\ell \geq 1); \}
\text{C3.} \quad & \Phi_\infty^* \bar{\eta} = A \bar{\omega}, \quad \Psi_\infty^* \bar{\omega} = B \bar{\eta}, \quad \text{where } A, B \text{ take the same form as } (2) \text{ (in particular, all } A^{k}_{p+k} \text{ are equal for } k \geq 1, \text{ etc.}, \text{ and both } A \text{ and } B \text{ are infinite permutation matrices, that is, each row/column of } A \text{ and } B \text{ contains a single } 1 \text{ with the rest of the entries being all } 0. \}
\end{align*}

In addition, we have

**Proposition 3.1.** Assume that \(p, q > 0\). The matrices \(A\) and \(B\) satisfy \(A = B^T\). In particular, the infinite permutation matrix \(A\) must be of the
form

\[
S := \begin{pmatrix}
\bar{A}_0^0 & \bar{A}_1^0 & \cdots & \bar{A}_p^0 & 0 & \cdots \\
\bar{A}_0^1 & \bar{A}_1^1 & \cdots & \bar{A}_p^1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\bar{A}_0^q & \bar{A}_1^q & \cdots & \bar{A}_p^q & 0 & \cdots \\
0 & \bar{B}_{xq}^0 & \cdots & 0 & \cdots & \cdots
\end{pmatrix}
\]

in other words, \(\bar{A}_k^{q+k} = \bar{B}_{xk}^T (k \geq 1)\), and \(\bar{A}_k^{q+\ell} = 0\) for all \(\ell > k\).

**Proof.** The fact \(\bar{A} = B^T\) follows from \(\bar{A}B = BA = \text{diag}(1, 1, \ldots)\) and Property C3 above. Consequently, \(\bar{A}_k^{q+k} = (\bar{B}_{q+k}^k)^T = \bar{B}_{xk}^T\) for \(k \geq 1\). For a similar reason, \(\bar{A}_k^{q+\ell} = 0\) for all \(\ell > k\). \(\square\)

**Definition 3.1.** Let \(S\), taking the form of (5), be an infinite permutation matrix obtained from a dynamic equivalence with height \((p, q)\) \((p, q > 0)\) between two control systems. Let \(r_j^i = \text{rank}(\bar{A}_j^i)\). We define the rank matrix associated to \(S\) to be

\[\mathcal{R}(S) := (r_j^i)\]

Given a dynamic equivalence, an associated matrix \(S\) may depend on the choice of the transformations \(\bar{\omega} = G\bar{\omega}\) and \(\bar{\eta} = H\bar{\eta}\). However, we have

**Proposition 3.2.** If \(S_1\) and \(S_2\) are two infinite permutation matrices obtained from the same dynamic equivalence, then their rank matrices satisfy

\[\mathcal{R}(S_1) = \mathcal{R}(S_2)\]

**Proof.** Suppose that the underlying dynamic equivalence has height \((p, q)\) \((p, q > 0)\). One can write \(S_1\) and \(S_2\) in block forms:

\[
S_1 = \begin{pmatrix}
U_0^0 & \cdots & U_p^0 & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \ddots \\
U_0^q & \cdots & U_p^q & 0 & \cdots \\
0 & \cdots & 0 & 0 & \cdots
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
V_0^0 & \cdots & V_p^0 & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \ddots \\
V_0^q & \cdots & V_p^q & 0 & \cdots \\
0 & \cdots & 0 & 0 & \cdots
\end{pmatrix}
\]

Let \(u_j^i := \text{rank}(U_j^i)\) and \(v_j^i := \text{rank}(V_j^i)\). Since \(S_1, S_2\) arise from the same dynamic equivalence, there exist invertible block lower triangular matrices:

\[
K = \begin{pmatrix}
k_0^0 & 0 & \cdots \\
k_0^1 & k_1^1 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}, \quad L = \begin{pmatrix}
\ell_0^0 & 0 & \cdots \\
\ell_0^1 & \ell_1^1 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

\(^4\text{Respectively, the block sizes of } K \text{ and } L \text{ are the same as those of } G \text{ and } H.\)
where \( k_i^i = k_i^{i+1} \), \( \ell_i^i = \ell_i^{i+1} \) for all \( i \geq 1 \), such that

\[
S_1 K = LS_2 = \begin{pmatrix}
W_0^0 & \cdots & W_0^p & 0 & \cdots & \cdots & \cdots \\
W_0^1 & \cdots & W_1^p & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_0^q & \cdots & \cdots & W_p^q & 0 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

As results of the forms of \( K \) and \( L \), we have

(i) \( u_{p+i}^i = v_{p+i}^i \) for all \( i \geq 0 \). This is because \( W_{p+i}^i = U_{p+i}^i k_{p+i}^i = \ell_{p+i}^i V_{p+i}^i \), where both \( k_{p+i}^i \) and \( \ell_{p+i}^i \) are invertible.

(ii) \( u_i^0 = v_i^0 \) for all \( 0 \leq i < p \). To see why this is true, consider the submatrix \( (W_i^0 W_i^{i+1} \cdots W_i^p) \). For each \( i < p \), its row rank equals to \( v_i^0 + v_{i+1}^0 + \cdots + v_p^0 \), which must be equal to its column rank \( u_i^0 + u_{i+1}^0 + \cdots + u_p^0 \). To see why this is true, consider the submatrix \( (W_i^0 \cdots W_p^0) \). Its row rank \( (v_i^0 + \cdots + v_p^0) + (v_i^1 + \cdots + v_{p+1}^1) \) must be equal to its column rank \( (u_i^0 + \cdots + u_p^0) + (u_i^1 + \cdots + u_{p+1}^1) \). Let \( i \) decrease from \( p \) and use (ii). The desired result follows.

(iii) \( u_i^1 = v_i^1 \) for all \( 0 \leq i < p + 1 \). This can be verified by a similar comparison between the column and row ranks of the submatrices

\[
\begin{pmatrix}
W_i^0 & \cdots & W_i^p & 0 & \cdots & 0 \\
W_i^1 & \cdots & W_i^p W_{p+i}^1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_i^j & \cdots & W_i^j W_{p+i}^j & \cdots & W_{p+j}^j
\end{pmatrix}, \quad (i < p + j).
\]

This completes the proof. \( \square \)

As a consequence of Proposition 3.2, we have

**Corollary 3.4.** If \( S_1 \) and \( S_2 \) are two infinite permutation matrices obtained from a dynamic equivalence with height \((p, q)\) \( (p, q > 0) \), then there exist block diagonal matrices

\[
K = \text{diag}(k_i^i)_{i \geq 0}, \quad L = \text{diag}(\ell_i^i)_{i \geq 0},
\]

where \( k_i^i \) and \( \ell_i^i \) are usual permutation matrices of appropriate sizes\(^5\) such that

\[ S_1 = L S_2 K. \]

**Proof.** This is because \( S_1 \) and \( S_2 \) (i) are permutation matrices; and (ii) have the same rank in each pair of corresponding blocks. \( \square \)

\(^5\)That is, the sizes of \( k_i^i \) and \( \ell_i^i \) are consistent with the product of block matrices \( L S_2 K \).
Corollary 3.5. Let $S$ be an infinite permutation matrix obtained from a dynamic equivalence of height $(p, q)$ $(p, q > 0)$. Using the notations in (2), the associated rank matrix $R(S) = (r^k_j)$ satisfies

$$r^k_p = \text{rank}(A^\infty), \quad r^q_k = \text{rank}(B^\infty), \quad (k \geq 0).$$

Proof. To see why this is true, first notice that a transformation (4) preserves the ranks of $A^\infty$ and $B^\infty$; then use Property P3. □

4. The Height of a Dynamic Equivalence

Given two control systems $(M, C_M)$ (of type $(n_1, s)$) and $(N, C_N)$ (of type $(n_2, s)$) that are dynamically equivalent, it is interesting to ask: What are the possible heights of a dynamic equivalence? A particular instance is Theorem 3.2, which tells us that the height suggests how control systems with $s = 1$ and $s > 1$ are qualitatively different. The current section will present some new results in this direction.

4.1. Some Rank Equalities and Inequalities.

Throughout this section, let $(M, C_M)$ and $(N, C_N)$ be as above. Suppose that a dynamic equivalence between them has height $(p, q)$ with $p, q > 0$. Let $S$ be an associated infinite permutation matrix (Equation (5)), obtained from a choice of coframes $(\bar{\omega}^0, \bar{\omega}^1, ...)$ and $(\bar{\eta}^0, \bar{\eta}^1, ...)$. Let $R(S) = (r^k_j)$ be the corresponding rank matrix (Definition 3.1).

Proposition 4.1. $r^i_0$ satisfy the following equalities

$$\sum_{i \geq 0} r^i_0 = n_1, \quad \sum_{j \geq 0} r^0_j = n_2, \quad \sum_{i \geq 0} \sum_{j \geq 0} r^i_k = \sum_{j \geq 0} r^k_j = s, \quad (k = 1, 2, ...).$$

Proof. This is because the matrix $S$ is an infinite permutation matrix. □

Proposition 4.2. $r^i_j$ satisfy the following inequalities:

i. for $i, j \neq 0$, then

$$r^i_j \leq \min \left\{ \sum_{k=0}^{j+1} r^i_k + \sum_{k=0}^{i+1} r^k_j \right\};$$

ii. for $j \neq 0$,

$$r^0_j \leq \min \left\{ r^0_j + r^1_j + \sum_{k=0}^{j+1} r^1_k, \quad (n_2 - s) + \sum_{k=0}^{i+1} r^1_k \right\};$$

iii. for $i \neq 0$,

$$r^i_0 \leq \min \left\{ r^i_0 + r^i_1 + \sum_{k=0}^{i+1} r^k_1, \quad (n_1 - s) + \sum_{k=0}^{i+1} r^1_k \right\};$$

iv. $r^0_0 \leq \min \left\{ (n_1 - s) + r^0_0 + r^1_0 + (n_2 - s) + r^1_0 + r^1_1 \right\}$. 

Proof. To prove \( i \), the case when \( r_{j}^{i} = 0 \) is trivial. Otherwise, suppose that the submatrix \( A_{j}^{0} \) of \( S \) has 1’s precisely at positions \( (a_{k}, b_{k}) \), \( 1 \leq a_{k}, b_{k} \leq s \), \( 1 \leq k \leq r_{j}^{i} \). Dropping pullback symbols, we have

\[
\bar{n}_{a_{k}}^{i} = \bar{\omega}_{b_{k}}^{i}.
\]

Condition \( C2 \) demands
\[
\left(7\right) \quad d\bar{n}_{a_{k}}^{i} \equiv -\bar{n}_{a_{k}}^{i+1} \land dt \mod \eta^{0}, \ldots, \eta^{i}
\]
and
\[
\left(8\right) \quad d\bar{\omega}_{b_{k}}^{j} \equiv -\bar{\omega}_{b_{k}}^{j+1} \land dt \mod \omega^{0}, \ldots, \omega^{j}.
\]

At most \( \sum_{k=0}^{i} r_{j}^{i+1} \) congruences in \( \left(7\right) \) are reduced to the following form once the congruence is taken modulo \( \eta^{0}, \ldots, \eta^{i}, \omega^{0}, \ldots, \omega^{j} \):

\[
d\bar{n}_{a_{k}}^{i} \equiv 0 \mod \eta^{0}, \ldots, \eta^{i}, \omega^{0}, \ldots, \omega^{j};
\]
similarly, at most \( \sum_{k=0}^{i} r_{j}^{k+1} \) congruences in \( \left(8\right) \) are reduced to the following form once the congruence is taken modulo \( \omega^{0}, \ldots, \omega^{j}, \eta^{0}, \ldots, \eta^{i} \):

\[
d\bar{\omega}_{b_{k}}^{j} \equiv 0 \mod \omega^{0}, \ldots, \omega^{j}, \eta^{0}, \ldots, \eta^{i}.
\]

The remaining congruences in \( \left(7\right) \) and \( \left(8\right) \), reduced modulo \( \eta^{0}, \ldots, \eta^{i}, \omega^{0}, \ldots, \omega^{j} \), must match up as identical congruences; in particular, the corresponding \( \bar{n}_{a_{k}}^{i+1} \) and \( \bar{\omega}_{b_{k}}^{j+1} \) must be equal. Since the equalities \( \bar{n}_{a_{k}}^{i+1} = \bar{\omega}_{b_{k}}^{j+1} \) are at most \( r_{j}^{i+1} \) in number, we obtain the inequalities

\[
r_{j}^{i} \leq \sum_{k=0}^{j+1} r_{j}^{i+1}, \quad r_{j}^{i} \leq \sum_{k=0}^{i+1} r_{j}^{j+1},
\]

which justifies \( i \).

To prove \( iv \), suppose that \( r_{0}^{0} > 0 \) and that the submatrix \( A_{0}^{0} \) has 1’s precisely at positions \( (a_{k}, b_{k}) \), \( 1 \leq a_{k} \leq n_{2}, 1 \leq b_{k} \leq n_{1}, 1 \leq k \leq r_{0}^{0} \). We have

\[
\bar{n}_{a_{k}}^{0} = \bar{\omega}_{b_{k}}^{0}.
\]

There exist functions \( C_{k}^{0}, \ldots, C_{k}^{s}, D_{k}^{0}, \ldots, D_{k}^{s} \) such that

\[
\left(9\right) \quad d\bar{n}_{a_{k}}^{0} \equiv -(C_{k}^{0}\eta_{1}^{1} + \cdots C_{k}^{s}\eta_{1}^{s}) \land dt \mod \eta^{0},
\]

\[
\left(10\right) \quad d\bar{\omega}_{b_{k}}^{0} \equiv -(D_{k}^{0}\bar{\omega}_{1}^{1} + \cdots D_{k}^{s}\bar{\omega}_{1}^{s}) \land dt \mod \omega^{0}.
\]

Since the rank of \( d\eta^{0} \) (modulo \( \eta^{0} \)) is \( s \), we have

\[
\text{rank}(C_{k}^{0}) \geq s - (n_{2} - r_{0}^{0});
\]

similarly, we have

\[
\text{rank}(D_{k}^{0}) \geq s - (n_{1} - r_{0}^{0})).
\]

On the other hand, by taking the congruences \( \left(9\right) \) and \( \left(10\right) \) reducing modulo both \( \omega^{0} \) and \( \eta^{0} \), it is not hard to see that

\[
\text{rank}(C_{k}^{0}) \leq r_{0}^{1} + r_{1}^{1}, \quad \text{rank}(D_{k}^{0}) \leq r_{0}^{1} + r_{1}^{1}.
\]
Theorem 4.3. The height of the systems of the same type $p, n$.

Corollary 4.2. Let $p, q > 0$, then
\[
\min\{(p - 1)\delta + r_1p + n_1, \ (q - 1)\delta + r_2q + n_2\}
\geq \max\{r_1p + n_1, \ r_2q + n_2\},
\]
where $r_1 = \text{rank}(A_{\infty}), \ r_2 = \text{rank}(B_{\infty}), \ \delta = s - r_1 - r_2$.

Proof. The rank matrix associated to $E$ takes the form
\[
\mathcal{R} = \begin{pmatrix}
  r_0^0 & \cdots & r_0^p & 0 & \cdots \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  r_q^0 & \cdots & r_q^p \\
  0 & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots 
\end{pmatrix},
\]
where $r_{p+k}^{k} = \text{rank}(A_{\infty}) = r_1$, $r_{k+q}^{q+k} = \text{rank}(B_{\infty}) = r_2$ for all $k \geq 0$. Let
\[
C := \sum_{i=0}^{q-1} \sum_{j=p}^{q-i} r_j^i - \sum_{i=0}^{q-1} r_{p+i}^i, \quad D := \sum_{j=0}^{p-1} \sum_{i=q}^{p-1} r_j^i - \sum_{j=0}^{p-1} r_{q+j}^j.
\]
By Proposition 4.1, we have
\[
0 \leq C \leq (q - 1)\delta, \quad 0 \leq D \leq (p - 1)\delta.
\]
Furthermore, we have
\[
E := (q - 1)(s - r_1) - (n_1 - r_0^0 - r_2) - C
= (p - 1)(s - r_2) - (n_2 - r_0^0 - r_1) - D.
\]
Using (15), we obtain
\[
\max\{r_1p + n_1, \ r_2q + n_2\} \leq E - r_0^0 + n_1 + n_2 \\
\leq \min\{(p - 1)\delta + r_1p + n_1, \ (q - 1)\delta + r_2q + n_2\}.
\]
The conclusion follows.

Corollary 4.2. Let $E$ be a dynamic equivalence between two control systems of types $(n_1, s)$ and $(n_2, s)$, respectively. If $\text{rank}(A_{\infty}) + \text{rank}(B_{\infty}) = s$, then the height $(p, q)$ of $E$ must satisfy: when $p, q > 0$, we have
\[
n_1 + \text{rank}(A_{\infty}) \cdot p = n_2 + \text{rank}(B_{\infty}) \cdot q.
\]

Proof. This is an immediate consequence of setting $\delta = 0$ in (14).

Theorem 4.3. The height $(p, q)$ of a dynamic equivalence between two systems of the same type $(n, 2)$ must satisfy $p = q$. 

Proof. This is an immediate consequence of setting $\delta = 0$ in (14).
Proof. By Proposition 2.2, we have either $p = q = 0$ or $p, q > 0$. In the latter case, when $s = 2$, Lemma 3.3 implies that the only possibility is $\text{rank}(A_x) = \text{rank}(B_x) = 1$; then apply Corollary 4.2.

Remark. Theorem 4.3 suggests a qualitative distinction between control systems with $s = 2$ and those with $s > 2$. In fact, when $s > 2$, it may be, for a dynamic equivalence, that $\text{rank}(A_x) + \text{rank}(B_x) < s$, and (16) does not need to hold. For details, see Example 1 below.

4.3. Examples.

1. Let $(M, C_M)$ be a control system of type $(4, 3)$ where $M$ has coordinates $(t, x, u)$ and $C_M$ is generated by

\begin{equation}
\begin{align*}
\begin{array}{cccc}
\frac{dx_1}{dt} &= -u_1, & \frac{dx_2}{dt} &= -x_1, & \frac{dx_3}{dt} &= -u_2, & \frac{dx_4}{dt} &= -u_3.
\end{array}
\end{align*}
\end{equation}

A partial prolongation $(M, C_M)$ can be obtained by adjoining equations of the form $u_{a}^{(k)} = u_{a}^{(k+1)}$ to the original system. For example, consider $N_1 = M \times \mathbb{R}^3$ with the coordinates

$$(t, (x, u_1, u_2, u_2^{(1)}), (u_1^{(1)}, u_2^{(2)}, u_3)),$$

where $u_1^{(1)}$ and $u_2^{(2)}$ are coordinates on the $\mathbb{R}^3$ component; let $C_1$ be the Pfaffian system generated by (17) and the 1-forms

\begin{equation}
\begin{align*}
\begin{array}{cccc}
\frac{du_1}{dt} &= -u_1^{(1)}, & \frac{du_2}{dt} &= -u_2^{(1)}, & \frac{du_2^{(2)}}{dt} &= -u_2^{(2)}.
\end{array}
\end{align*}
\end{equation}

Alternatively, consider $N_2 = M \times \mathbb{R}^3$ with the coordinates

$$(t, (x, u_3, u_3^{(1)}, u_3^{(2)}), (u_1, u_2, u_3^{(3)})),$$

where $u_3^{(1)}$ and $u_3^{(2)}$ are coordinates on the $\mathbb{R}^3$ component; let $C_2$ be the Pfaffian system generated by (17) and the 1-forms

\begin{equation}
\begin{align*}
\begin{array}{cccc}
\frac{du_3}{dt} &= -u_3^{(1)}, & \frac{du_3^{(1)}}{dt} &= -u_3^{(2)}, & \frac{du_3^{(3)}}{dt} &= -u_3^{(3)}.
\end{array}
\end{align*}
\end{equation}

The standard submersions $\Phi : N_1^{(3)} \to N_2$, $\Psi : N_2^{(2)} \to N_1$ give rise to a dynamic equivalence between $(N_1, C_1)$ and $(N_1, C_2)$ (both of type $(7, 3)$) with height $(p, q) = (3, 2)$. The associated rank matrix is

\begin{equation}
\begin{align*}
\begin{array}{cccc}
4 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\vdots & & & \\
1 & 1 & 0 & 0 \\
\end{array}
\end{align*}
\end{equation}

In this example, (14) becomes an equality.
2. Let \((M, C_M) (\dot{x} = f(x, u))\) and \((N, C_N) (\dot{y} = f(y, v))\) be two copies of a same control system of type \((3, 2)\):
\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= f(x_2, x_3, u_2),
\end{align*}
\]
\[
\begin{align*}
\dot{y}_1 &= v_1, \\
\dot{y}_2 &= v_2, \\
\dot{y}_3 &= f(y_2, y_3, v_2).
\end{align*}
\]
For any \(p > 1\), the following pair of submersions \(\Phi : M^{(p)} \to N\) and \(\Psi : N^{(p)} \to M\) define a dynamic equivalence with height \((p, p)\) between \((M, C_M)\) and \((N, C_N)\):
\[
\begin{align*}
(y, v) &= \Phi(x, u, \ldots, u^{(p)}) = (u_2^{(p-1)} - x_1, x_2, x_3; u_2^{(p)} - u_1, u_2); \\
(x, u) &= \Psi(y, v, \ldots, v^{(p)}) = (v_2^{(p-1)} - y_1, y_2, y_3; v_2^{(p)} - v_1, v_2).
\end{align*}
\]

Of course, the dynamic equivalences given in Example 2 are somewhat artificial, since the underlying control systems are already statically equivalent. It is therefore more meaningful to ask: Fixing two control systems that are dynamically equivalent, is there any method of finding the minimum height \((p, q)\) of a dynamic equivalence between them? This question remains open.

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