Non-universality of free fall in quantum theory

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Abstract
We show by embodying the Einstein equivalence principle and general covariance in quantum theory that wave-function spreading rules out universality of free fall, and vice versa. Assuming the former is more fundamental than the latter, we gain a quantitative estimate of the free-fall non-universality, which turns out to be empirically testable in atom interferometry.

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I. INTRODUCTION

According to Newton’s gravitational law, any body having a non-zero gravitational mass is a source of gravity. It is a consequence of numerous experiments that gravitational mass $M_g$ of a macroscopic body is equal with good accuracy to its inertial mass $M_i$. So, one might assume

$$\left(\frac{M_g}{M_i}\right)_{\text{classical}} = 1.$$  \hspace{1cm} (1)

In Newton’s theory, this observation implies that test bodies fall down equally fast, provided same initial position and velocity. The general theory of relativity (GR) promotes this result to the weak equivalence principle which is also known in the literature as the universality of free fall. This principle yields a key argument for modelling gravitational interaction through space-time geometry [1], where particles’ trajectories correspond to geodesic world lines.

In the framework of quantum theory, however, particles cannot be thought of as point-like objects which move along single world lines. Indeed, Heisenberg’s uncertainty principle forces to abandon the idea that position and momentum can be simultaneously defined with perfect precision for quantum particles [2]. This quantum fuzziness originates from the fact that wave functions have finite localisation in space, resulting in the probability of finding a particle at a given space-time point, which is always less than unity. This suggests that quantum particles might not obey the weak equivalence principle.

In this article, we study this conceptual conflict quantitatively. It becomes possible through the implementation of Einstein’s equivalence principle and general covariance in theoretical particle physics.

II. FREE FALL OF CLASSICAL PARTICLES

According to Einstein’s gravitational theory, matter is a source of a non-trivial spacetime curvature. The spacetime curvature is mathematically described by the Riemann tensor. This tensor has dimension of inverse length squared. In other words, we can characterise the space-time curvature by a length scale: The bigger this length scale, the weaker a gravitational field is. In particular, at the Earth’s surface, it reads

$$l_\oplus \equiv r_\oplus \left(\frac{r_\oplus}{r_{S,\oplus}}\right)^{\frac{1}{2}} \approx 1.71 \times 10^{11} \text{ m},$$  \hspace{1cm} (2)

where $r_\oplus \approx 6.37 \times 10^6 \text{ m}$ denotes the Earth’s radius, whereas $r_{S,\oplus} \approx 8.87 \times 10^{-3} \text{ m}$ stands for its Schwarzschild ($S$) radius. Thus, the Earth’s curvature plays a little role in the dynamics of microscopic objects in quantum processes taking place over time intervals much smaller than $l_\oplus/c \approx 9.52 \text{ min}$, where $c \approx 2.99 \times 10^8 \text{ m/s}$ is the speed of light in vacuum. For this reason, we shall neglect the Earth’s curvature in what follows until Sec. \[.] 

This approximation means that the metric tensor at the Earth’s surface can be replaced by the Minkowski metric $\eta \equiv \text{diag}(+1, -1, -1, -1)$ iff one considers local inertial coordinates.
To this end, let us introduce Riemann normal coordinates, $y^a$, defined at a given point at the Earth’s surface, which corresponds to $y^a = 0$. In the vicinity of this point, we have

$$ ds^2 = g_{ab}(y) \, dy^a dy^b \approx \eta_{ab} dy^a dy^b, \quad (3) $$

where the Latin indices lie in \{0, 1, 2, 3\}. We have neglected curvature-dependent terms on the right-hand side of (3), because of the weakness of the Earth’s gravitational field. These terms can be found in [3]. The very fact that the metric tensor can always be locally brought to the Minkowski-metric form is a result of Einstein’s equivalence principle – locally and at any non-singular point of the Universe, the special-relativity physics applies.

The general principle of relativity, saying that dynamical laws of nature are the same in all reference frames, ensures that physics does not depend on coordinates utilised. Nevertheless, the same physical process can look different in different coordinate frames. In particular, the local inertial coordinates $y$ and general coordinates $x \equiv (ct, x, y, z)$ are related as follows [3]:

$$ x^c \approx y^c - \frac{1}{2} \Gamma^c_{ab} y^a y^b, \quad (4) $$

where $\Gamma^c_{ab}$ are Christoffel symbols computed at the Earth’s surface and we have omitted terms which depend on higher-order derivatives of metric, in accord with the Minkowski-spacetime approximation (3). Taking into account that the Earth’s gravitational field is approximately described by the Schwarzschild geometry, we obtain

$$ \Gamma^0_{ab} y^a y^b \approx + \frac{2g_\oplus}{c^2} y^3 y^0, \quad (5a) $$

$$ \Gamma^1_{ab} y^a y^b \approx - \frac{2g_\oplus}{c^2} y^3 y^1, \quad (5b) $$

$$ \Gamma^2_{ab} y^a y^b \approx - \frac{2g_\oplus}{c^2} y^3 y^2, \quad (5c) $$

$$ \Gamma^3_{ab} y^a y^b \approx + \frac{g_\oplus}{c^2} \left( (y^0)^2 + (y^1)^2 + (y^2)^2 - (y^3)^2 \right), \quad (5d) $$

where the free-fall acceleration points down in the negative $z$-direction with the magnitude at the Earth’s surface reading

$$ g_\oplus \equiv \frac{c^2 r_S, \oplus}{2(r_\oplus)^2} \approx 9.81 \text{ m/s}^2. \quad (6) $$

Now, in the Riemann frame, all geodesics passing through $y^a = 0$ are straight world lines [3]. This is basically the condition which determines Riemann normal coordinates. Considering a classical (point-like) particle being initially at rest in the Riemann frame, we have

$$ y^a(\tau) = c \tau \delta^a_0, \quad (7) $$

where $\tau$ is the proper time and $\delta \equiv \text{diag}(+1, +1, +1, +1)$ is the Kronecker delta. It turns into

$$ x^a(\tau) \approx c \tau \delta^a_0 - \frac{1}{2} g_\oplus \tau^2 \delta^a_3 \quad (8) $$

in the non-inertial frame associated with the Earth’s surface, where we have substituted (7) into (4) and (5) to get (8). This is the well-known result of Newton’s gravitational theory, that explicitly demonstrates the universality of free fall in classical theory.
III. FREE FALL OF QUANTUM PARTICLES

Quantum field theory (QFT) is a mathematical formalism which enables us to successfully describe high-energy processes taking place between particles. This formalism is based on the unification of the underlying principles of quantum mechanics (QM) and the special theory of relativity (SR). The observable Universe cannot be described by Minkowski spacetime, which is a basic mathematical structure of SR. Hence, the application of QFT in theoretical particle physics relies on the Minkowski-spacetime approximation (3).

It is apparent that we need to go beyond this approximation in order to describe quantum particles in the presence of a gravitational field. We thereby wish to demand that the Einstein equivalence principle and the general principle of relativity be also implemented in quantum theory. The former principle implies then that quantum particles must locally be modelled by wave functions which, in local inertial frames, are given by plane-wave superpositions. In fact, it ensures that such quantum particles move along straight world lines in local inertial frames. The latter principle says in turn that wave functions must transform as tensors under general coordinate transformations. In particular, a spin-zero-particle wave function must correspond to a rank-zero tensor – scalar. This ensures that the semi-classical Einstein field equation is in accord with general covariance.

Thus, in the Riemann frame, Einstein’s equivalence principle tells us that the wave packet of a spin-zero particle of mass \( M > 0 \) is a superposition of plane waves, namely

\[
\psi(y) = \frac{1}{(2\pi \hbar)^3} \int d^3K \frac{F_P(K)}{2E_K} \exp \left( -\frac{iK \cdot y}{\hbar} \right),
\]

where \( \hbar \approx 1.05 \times 10^{-34} \text{ J} \cdot \text{s} \) is the reduced Planck constant and

\[
K \equiv (E_K/c, K) \equiv (\sqrt{(Mc)^2 + K^2}, K),
\]

\[
K \cdot y \equiv \eta_{ab}K^a y^b.
\]

The function \( F_P(K) \) must have a narrow peak at \( K \sim P \), where \( P \equiv (E_P/c, P) \) is an initial 4-momentum of the particle. This is in effect required for \( \psi(y) \) to be a localised-in-space packet. Furthermore, the general principle of relativity forces us to deal with \( F_P(K) = F(K \cdot P) \). For instance, a covariant Gaussian wave function \([4, 5]\) is characterised by

\[
F_P(K) \propto \exp \left( -\frac{K \cdot P}{2D^2} \right),
\]

where \( D > 0 \) stands for momentum variance. The covariance principle leads, thereby, to \( \psi(y) \) which is invariant under the (local) Lorentz transformations.

According to Born’s statistical interpretation, the wave function \( \psi(y) \) yields a probability amplitude of measuring the particle at a given place \([2]\). Thus, the probability to find it somewhere in space must be unity:

\[
\int d^3y \psi^* (y) \psi(y) = 1.
\]
This is a normalisation condition for the wave function $\psi(y)$ in QM. It is evident though that this normalisation condition is at odds with special covariance, since the integration measure in (11) is variant under the (local) Lorentz transformations. Therefore, it must be replaced in QFT by the Klein-Gordon product of $\psi(y)$ with itself:

$$i \int d^3y \left( \psi^*(y) \partial_0 \psi(y) - \psi(y) \partial_0 \psi^*(y) \right) = 1. \quad (12)$$

This equation is, in contrast, independent on (local) inertial frames, provided $\psi(y)$ is a scalar. It physically means that quantum particles are reference-frame-independent objects, i.e. their very existence does not depend on coordinates utilised [6, 7].

This observation suggests that the wave-function (centre-of-mass) position corresponds to

$$\langle y^a(\tau) \rangle \equiv i \int_\tau d^3y \ y^a \left( \psi^*(y) \partial_0 \psi(y) - \psi(y) \partial_0 \psi^*(y) \right), \quad (13)$$

which turns into the quantum-mechanics definition of position expectation value in the non-relativistic regime $|\mathbf{P}| \ll Mc$ [7]. Note that the position expectation value $\langle y^c(\tau) \rangle$ depends on the proper time $\tau$. This is a physical hypothesis, meaning that quantum particles measure $\tau$. This, however, can be justified by recalling that a lifetime of cosmic-ray (relativistic) muons is bigger than that of muons at rest. This discrepancy arises due to the time-dilation effect in SR [8]: The laboratory lifetime of the cosmic-ray muons is by the Lorentz factor bigger than their proper lifetime. This experimental result validates our hypothesis.

Consequently, we find from (9), (10), (12) and (13) for the spin-zero quantum particle being initially at rest ($|\mathbf{P}| = 0$) that

$$\langle y^a(\tau) \rangle = c\tau \delta^a_0 \quad (14)$$

in the Riemann or, in other words, local inertial frame, while, by bearing in mind (4) and (5),

$$\langle x^a(\tau) \rangle \approx c\tau \delta^a_0 - \frac{1}{2} g_{\oplus} \left( 1 + \frac{D^2}{(Mc)^2} \right) \tau^2 + \frac{\hbar^2}{4(Dc)^2} \delta_3^a \quad (15)$$

in the non-inertial frame associated with the Earth’s surface. The quantum result (15) differs from the classical one (8) by terms to depend on internal quantum-particle properties, which clearly show the non-universality of free fall in quantum theory [6, 7].

The origin of the free-fall non-universality in quantum theory is wave-function spreading. Indeed, this universal phenomenon follows from the circumstance that the wave function $\psi(y)$ obeys the Heisenberg uncertainty principle. This manifests itself through

$$\langle y^1 y^1 \rangle = \langle y^2 y^2 \rangle = \langle y^3 y^3 \rangle \approx \frac{\hbar^2}{4D^2} + \frac{D^2}{M^2} \tau^2, \quad (16)$$

meaning that $\psi(y)$ spreads in space. The combination of this quantum-mechanical result with (5d) explains the quantum corrections to (8) in (15).
Our result \((15)\) may be interpreted within Newton’s theory as \((1)\) cannot hold in quantum theory, namely we instead have
\[
\left(\frac{M_g}{M_i}\right)_{\text{quantum}} \approx 1 + \frac{D^2}{(Mc)^2},
\]
(17)
because it follows from \((15)\) that
\[
\frac{d^2}{d\tau^2} \langle z(\tau) \rangle \approx -g_{\oplus} \left(\frac{M_g}{M_i}\right)_{\text{quantum}}.
\]
(18)
We intend next to study whether this interpretation is at least approximately consistent with other observables.

IV. FOUR-MOMENTUM OF QUANTUM PARTICLES

With the stress-energy tensor of the quantum particle in the local Minkowski frame, i.e.
\[
T^{ab}(y) = \partial^a \psi^*(y) \partial^b \psi(y) + \partial^b \psi^*(y) \partial^a \psi(y) - \eta^{ab}(|\partial \psi(y)|^2 - (Mc/\hbar)^2 |\psi(y)|^2),
\]
(19)
we find for the particle being initial at rest that
\[
\langle p^a(\tau) \rangle \equiv \int d^3y \frac{\partial x^a}{\partial y^b} T^{b0}(y)
\approx Mc \left(1 + \frac{3}{2} \frac{D^2}{(Mc)^2}\right) \delta^a_0 - M g_{\oplus} \tau \left(1 + \frac{5}{2} \frac{D^2}{(Mc)^2}\right) \delta^a_3.
\]
(20)
This result can be immediately obtained from \(M_i \langle \dot{x}^a(\tau) \rangle\) with \((15)\), which, in classical theory, gives particle’s 4-momentum, where the inertial mass \(M_i\) has been defined via the Lagrangian mass \(M\) at the leading order of the approximation as follows:
\[
M_i \equiv M \left(1 + \frac{3}{2} \frac{D^2}{(Mc)^2}\right).
\]
(21)
These computations provide an independent support for the result \((17)\).

V. GEODESIC DEVIATION FOR QUANTUM PARTICLES

The free-fall acceleration is a non-inertial-frame effect which is, accordingly, absent in local inertial frames. In contrast, the spacetime curvature is non-vanishing in all reference frames. In particular, it shows itself as a relative acceleration between geodesics. This starts to play a role in satellite-borne experiments. In fact, if we consider an interferometer freely falling along the geodesic \((7)\) and the quantum particle \(\psi(y)\) being initially placed at \(y = (0, l_x, l_y, l_z)\), then its acceleration relative to the free-falling interferometer reads
\[
\frac{d^2}{d\tau^2} \langle y^c(\tau) \rangle \approx -\frac{1}{3(l_{\oplus}/c)^2} \left( l_x \delta_1^c + l_y \delta_2^c - 2l_z \delta_3^c \right) \left(1 + \frac{D^2}{(Mc)^2}\right).
\]
(22)
This result is in accord with the geodesic deviation equation up to the factor depending on the internal quantum-particle properties. This factor agrees with that in (18), suggesting that the gravitational mass of the quantum particle is by this factor bigger than its inertial mass. This is, apparently, in agreement with (17).

VI. QUANTITATIVE ESTIMATE

The result (18) implies quantum particles fall down faster than classical ones. This effect is negligibly small for macroscopic objects. In particular, one gram of iron has the size of about $6.24 \times 10^{-3}$ m, which can be equated to $\hbar/D$, according to Heisenberg’s uncertainty principle, giving $D/Mc \approx 5.63 \times 10^{-38}$. However, a rubidium atom, $^{85}$Rb, has the radius of $220 \times 10^{-12}$ m and, thus, we get the estimate $D/Mc \approx 5.63 \times 10^{-9}$.

A dimensionless parameter, which quantifies relative free-fall acceleration of a pair of test bodies of different composition, is known as the Eötvös parameter. We obtain from (18) that

$$\eta(A, B) \approx \frac{D_A^2}{(M_Ac)^2} - \frac{D_B^2}{(M_Bc)^2}.$$  

It approximately reads $3.16 \times 10^{-17}$ in case of $^{85}$Rb and a heavier atom. This is by five orders of magnitude smaller than the atom-interferometer sensitivity recently gained in [9] by quantum tests of the free-fall universality, where the heavier atom was the rubidium isotope $^{87}$Rb. Yet, the Eötvös parameter increases if lighter atoms are considered:

|                      | $(D/Mc)^2$ |
|----------------------|------------|
| One gram of iron     | $3.17 \times 10^{-75}$ |
| Rubidium atom ($^{85}$Rb) | $3.16 \times 10^{-17}$ |
| Potassium atom ($^{39}$K) | $1.78 \times 10^{-16}$ |
| Hydrogen atom (H)    | $1.14 \times 10^{-11}$ |

Satellite-borne experiments have much better sensitivity with respect to the Earth-based ones by quantum tests of the free-fall universality – at the $10^{-17}$ level or better, – where their main advantage consists in the fact that these tests can potentially be made over infinite free-fall times [10]. Their sensitivity will thus be sufficient to empirically discover if the Heisenberg uncertainty principle is more fundamental than the weak equivalence principle.

VII. CONCLUDING REMARKS

It is a result of lots of experiments that QFT over Minkowski space locally makes physical sense, although the observable Universe is actually curved. This observation implies that both Einstein’s equivalence principle and general covariance must be built into quantum theory for this to be in accordance with observations in particle colliders. This line of reasoning leads to
our model of quantum particles in the presence of a gravitational field. This model gives novel
results which, probably, will be empirically testable in the near future.

Yet, there are two possible outcomes of these tests. If the free-fall non-universality will be
experimentally discovered, then the weak equivalence principle – one of the underlying ideas
of GR – has to be re-thought in quantum theory. If otherwise, the wave-function description
of quantum particles has to be re-considered in GR. Remarkably, any of these two outcomes
will improve our insight of gravity.

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