P,T-invariant system of Chern-Simons fields:
Pseudoclassical model and hidden symmetries

Khazret S. Nirov\textsuperscript{a} and Mikhail S. Plyushchay\textsuperscript{b,c}

\textsuperscript{a}Institut für Theoretische Physik T30, Physik Department, Technische Universität München, D-85747 Garching, Germany
\textsuperscript{b}Departamento de Física — ICE, Universidade Federal de Juiz de Fora 36036-330 Juiz de Fora, MG Brazil
\textsuperscript{c}Institute for High Energy Physics, Protvino, Moscow region, 142284 Russia

Abstract

We investigate hidden symmetries of $P,T$-invariant system of topologically massive U(1) gauge fields. For this purpose, we propose a pseudoclassical model giving rise to this field system at the quantum level. The model contains a parameter, which displays a quantization property at the classical and the quantum levels and demonstrates a nontrivial relationship between continuous and discrete symmetries. Analyzing the integrals of motion of the pseudoclassical model, we identify U(1,1) symmetry and S(2,1) supersymmetry as hidden symmetries of the corresponding quantum system. Representing the hidden symmetries in a covariant form, we show that one-particle states realize an irreducible representation of a non-standard super-extension of the (2 + 1)-dimensional Poincaré group labelled by the zero eigenvalue of the superspin.

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\textsuperscript{1}Alexander von Humboldt fellow; on leave from the Institute for Nuclear Research, Moscow, Russia
\textsuperscript{2}E–mail: nirov@dirac.physik.uni-bonn.de
\textsuperscript{3}E–mail: plyushchay@mx.ihep.su
1 Introduction

Topologically massive gauge fields \[1\], originated from the \(\theta\)-vacuum of four-dimensional theories \[2\], turned out to be the basic tool for constructing models which possess quantum Hall effect \[3\] and high-temperature superconductivity \[4\]. Actually, considerable interest in 3d field theories is highly motivated by perspectives they give us for better understanding critical phenomena generic to 4d physics \[5\]. In this, a particular position of the high-temperature superconductivity is cogent: high-\(T_c\) superconducting materials have quasi-planar structures \[6\], so that they can effectively be described by three-dimensional models.

We deal here with the simplest \(P,T\)-invariant system of Chern-Simons vector U(1) gauge fields, given in terms of a self-dual free massive field theory \[7\]. The corresponding source-free equations are first order differential equations \(L^{\epsilon}_{\mu
u}F^\nu = 0\), where \(L^{\epsilon}_{\mu
u} \equiv (i\epsilon_{\mu\nu\lambda}P^\lambda + \epsilon m\eta_{\mu\nu})\), \(P^\mu = -i\partial^\mu\), \(\eta_{\mu\nu} = \text{diag}(-1, +1, +1)\), \(\epsilon = +\) or \(-\), and the totally antisymmetric tensor \(\epsilon^{\mu\nu\lambda}\) is normalized by \(\epsilon^{012} = 1\). Due to the basic equations, the field \(F^\mu\) satisfies also Klein-Gordon equation \((P^2 + m^2)F^\mu = 0\) and the transversality condition \(P^\mu F^\mu = 0\). It is clear from the definition above that \(F^\mu\) carries massive irreducible representation of spin \(-\epsilon\) of the 3d Poincaré group. As was shown in Ref. \[8\], this formulation of the theory is essentially equivalent to the original one \[1\].

Already in pioneering works \[1\] it was noted that topological mass terms are odd under the parity and time-reversal transformations, and the full set of discrete \(C\), \(P\) and \(T\) symmetries may be restored if one doubles the number of fields and introduces opposite sign mass terms. In the case under consideration, when taking the action

\[
A = \int d^3x \left( F^{\mu+}_{\mu\nu}L^{+\nu}_{\mu\nu} + F^{\mu-}_{\mu\nu}L^{-\nu}_{\mu\nu} \right), \tag{1.1}
\]

we get \(P,T\)-invariant system of topologically massive vector U(1) gauge fields \[1\]. This observation plays an essential role in constructing models of high-temperature superconductors. Actually, single spin state models predict observable parity and time-reversal violation in corresponding superconductors \[8\], for which experiments still give no evidence \[10\]. Besides, the problem of cancellation between single bare and radiatively generated Chern-Simons terms \[11\] arises in the conventional models \[12\]. For these fundamental reasons, it is desirable to have parity and time-reversal conserving system modelling high-\(T_c\) superconductors without, at least, these serious obstructions \[13\]. For the same reasons, we shall pay particular attention to the requirement of these discrete symmetries.

The relevance of 3d field theories to critical phenomena might be explained by some hidden symmetries. From this point of view, it is of interest to investigate properties of parity and time-reversal conserving systems which are considered to be relevant to high-temperature superconductivity. Such a program has been realized for planar fermions \[14\], where the authors elucidated a rich set of hidden symmetries. The results of Ref. \[14\] were obtained from analysis of the pseudoclassical model of a relativistic spinning particle proposed in Ref. \[15\]. Classical particle models are indeed useful for clarifying problems of more complicated quantum mechanical and field systems and for revealing hidden properties of corresponding quantum systems and understanding their nature.

In this paper we propose a pseudoclassical model by means of which we analyze \(P,T\)-invariant system of Chern-Simons fields. The model contains a \(c\)-number parameter at a mass...
term for spin variables. This one, which we call the model parameter, displays a quantization
property both at the classical and quantum levels. Although the variation of the parameter
does not affect the discrete space-time symmetries of the pseudoclassical model, its values
are crucial for continuous global symmetries. Actually, there are special discrete values of
the model parameter at which the system has a maximal number of integrals of motion.
The same values of the parameter turn out to be special quantum mechanically: they are
separated by the requirement of maximality of global symmetry of the physical state space
at the quantum level. Moreover, we shall see that only at these special values discrete
parity and time-reversal symmetries are conserved in the corresponding quantum theory.
This result does actually indicate a profound relationship between discrete and continuous
global symmetries. When considering algebras of the integrals of motion, we shall elucidate
hidden U(1,1) symmetry and S(2,1) supersymmetry of the P,T-invariant system of Chern-
Simons U(1) gauge fields. We shall also demonstrate that this system realizes an irreducible
representation of a non-standard super-extension of the (2 + 1)-dimensional Poincaré group,
namely ISO(2, 1|2, 1). The non-standard character of this supergroup means, in particular,
that, unlike the usual supersymmetries, the anticommutator of corresponding supercharges
results in an operator different from the Hamiltonian of the system.

It is interesting to observe that non-standard supersymmetries and quantization of pa-
rameters turn out to be generic to systems with nontrivial topology of the corresponding
configuration or phase spaces. For instance, when investigating space-time symmetries in
terms of motion of pseudoclassical spinning point particles [16]–[19], Gibbons, Rietdijk and
van Holten [20] elucidated the existence of a non-standard supersymmetry (see also Ref.
[21]). In this, the Poisson brackets of the odd Grassmann generators give rise to an even
integral of motion different from the Hamiltonian of the system. A non-standard supersym-
metry with the same feature appeared in studying hidden symmetries [22] of a 3d monopole
[23]. And the quantization of the dimensionless mass-coupling-constant ratio of the non-
Abelian 3d vector fields [1, 23], discovered by Deser, Jackiw and Templeton, is caused by
nontrivial homotopy properties of these fields.

The paper is organized as follows. In Section 2 the pseudoclassical model and its La-
grangian symmetries are described. Section 3 is devoted to Hamiltonian description of the
model. In this Section the solutions to the equations of motion and the set of the integrals of
motion are presented. In Section 4 the corresponding quantum theory is constructed. Pro-
vided that, the hidden (super)symmetries of the system under consideration are revealed.
Covariantization of the symmetry relations is performed in Section 5. After Concluding
remarks, some useful from a technical point of view formulas are gathered in the Appendix.

Everywhere in the text repeated indices imply the corresponding summation.

2 The pseudoclassical model and its symmetries

2.1 The action

The pseudoclassical model we are going to analyze here is given by the action

\[ A_q = \int_{\tau_i}^{\tau_f} L_q d\tau + \Gamma_\xi, \]  

(2.1)
where $L_q$ is the Lagrangian

$$L_q = \frac{1}{2e} \left( \dot{x}_\mu - i e_{\mu\nu\lambda} \xi^\nu_{\xi a} \xi^\lambda_{\xi a} \right)^2 - \frac{1}{2} \varepsilon^2 \dot{v}x_{\mu}^2 - \frac{1}{2} \varepsilon^2 \dot{v} \xi_{\mu}^a \xi_{\mu}^a + \frac{i}{2} \xi_{\mu}^a \dot{\xi}_{\mu}^a,$$

(2.2)

and $\Gamma_{\xi}$ means a boundary term, $\Gamma_{\xi} = \frac{i}{2} e_{\mu}(\tau) \xi_{\mu}(\tau)$. The configuration space of the system is described by the set of variables $x_{\mu}$, $\xi_{\mu}^a$, $a = 1, 2, e$ and $v$. In this, $x_{\mu}$, $\mu = 0, 1, 2$, denote space-time coordinates of the particle, $\xi_{\mu}^a$ are real Grassmann odd variables forming two Lorentz vectors, $e$ and $v$ are even Lagrange multipliers, and $q$ is a real $c$-number parameter.

The presence of the boundary term is caused by the form of the equations of motion for the Grassmann variables which are differential equations of the first order [24]. The action $A_q$ is extremal on the trajectories satisfying the boundary conditions $\delta \xi_{\mu}^a(\tau_i) + \delta \xi_{\mu}^a(\tau_f) = 0$.

### 2.2 Discrete symmetries

The system given by Eqs. (2.1)-(2.2) is invariant under the discrete parity and time-reversal transformations

$$P : X^\mu \rightarrow \bar{\varepsilon}(X^0, -X^1, X^2), \quad T : X^\mu \rightarrow \bar{\varepsilon}(-X^0, X^1, X^2),$$

(2.3)

where $X^\mu = x^\mu$, $\xi_{\mu}^1, \xi_{\mu}^2$, and

$$P : T : E \rightarrow \bar{\varepsilon}E,$$

(2.4)

where $E = e, v$. In these discrete symmetry transformation laws $\bar{\varepsilon} = +$ for the vector $x_{\mu}, \xi_{\mu}^1$ and scalar $e$ variables, and $\bar{\varepsilon} = -$ for $\xi_{\mu}^2$ and $v$ implying that $\xi_{\mu}^2$ is a pseudovector and $v$ is a pseudoscalar.

One of the most important features of our pseudoclassical model is that in the classical theory parity and time-reversal invariance take place for any value of the parameter $q$. Nevertheless, we shall see that the case of $|q| = 2$ is particular both at the classical and the quantum levels of the theory, and that the quantization of the model (2.1)-(2.2) results in the $P, T$-invariant system of topologically massive vector U(1) gauge fields.

### 2.3 Global symmetries

In addition to the Poincaré invariance, the action (2.1) is invariant against the following set of global transformations:

$$\delta_\lambda \xi_{\mu}^a = \lambda e_{ab} \xi_{\mu}^b,$$

(2.5)

where $e_{ab} = -e_{ba}$, $a, b = 1, 2$, $e_{12} = 1$,

$$\delta_\nu x_{\mu} = \nu e_{ab} \xi_{\mu}^a \xi_{\nu}^b P_{\lambda}^\lambda,$$

(2.6)

$$\delta_\nu \xi_{\mu}^a = -i \nu e_{ab} p_{\mu}^a P_{\lambda}^\lambda,$$

(2.7)

and

$$\delta_\theta x_{\mu} = i \theta e_{\mu\nu\lambda} \xi_{\nu}^a \xi_{\lambda}^a,$$

(2.8)

$$\delta_\theta \xi_{\mu}^a = -2i \theta e_{\mu\nu\lambda} \xi_{\nu}^a \xi_{\lambda}^a,$$

(2.9)

where we have introduced the notation $p_{\mu} = e^{-1} \left( \dot{x}_{\mu} - \frac{i}{2} e_{\mu\nu\lambda} \xi_{\nu}^a \xi_{\lambda}^a \right)$. In these transformations $\lambda, \nu$ and $\theta$ are even constant infinitesimal parameters. Further we shall find the corresponding generators of these global symmetries.
2.4 Local symmetries

The system has two local symmetries. One of them is a reparametrization invariance defined with respect to the transformations

$$\delta_\alpha E = \frac{d}{d\tau}(\alpha E), \quad \delta_\alpha X = \alpha \dot{X},$$

(2.10)

where $E = e, v$ and $X = x_\mu, \xi_{a\mu}$, changing the Lagrangian $L_q$ by $\delta_\alpha L_q = \frac{d}{d\tau}(\alpha L_q)$. As a consequence, the corresponding action is extremal if the boundary conditions $\alpha(\tau_i) = 0$ for the infinitesimal gauge parameter $\alpha$ are fulfilled.

Another local symmetry transformation is of the form

$$\delta_\beta x_\mu = \frac{i}{2} \beta \varepsilon_{\mu\nu\lambda} \xi_\nu^a \xi_\lambda^a,$$

(2.11)

$$\delta_\beta \xi_{a\mu} = -\beta \left( \varepsilon_{\mu\nu\lambda} p^\nu \xi_\lambda^a - q m \varepsilon_{ab} \xi_{b\mu} \right),$$

(2.12)

$$\delta_\beta v = \dot{\beta}.$$  

(2.13)

For the Lagrangian $L_q$ we obtain $\delta_\beta L_q = \frac{d}{d\tau} \left( \frac{i}{2} \beta \varepsilon_{\mu\nu\lambda} p^\mu \xi_\nu^a \xi_\lambda^a \right)$, so that the action is invariant provided that the boundary conditions $\beta(\tau_i) = \beta(\tau_f) = 0$ on the gauge parameter $\beta$ are imposed.

It is interesting to note here that if we formally relate global and local symmetry parameters as $\lambda = 2qm\theta = qm\beta$, we get the local $\beta$-symmetry to be a superposition of $\lambda$ and $\theta$ global symmetries,

$$\delta_\beta = \delta_\lambda + \delta_\theta.$$  

(2.14)

Having a Hamiltonian description of the system we shall obtain the generators of the local symmetry transformations and shall reveal the origin of the property (2.14).

3 Hamiltonian description of the model

3.1 Canonical structure and constraints

Let us construct the Hamiltonian description of the model. The nontrivial Poisson-Dirac brackets following from the Lagrangian (2.2) are

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu}, \quad \{\xi_\mu^a, \xi_\nu^b\} = -i \delta_{ab} \eta^{\mu\nu},$$

(3.1)

$$\{e, p_e\} = 1, \quad \{v, p_v\} = 1.$$  

(3.2)

The model possesses two sets of primary, $p_e \approx 0, p_v \approx 0$, and secondary,

$$\phi = \frac{1}{2}(p^2 + m^2) \approx 0, \quad \chi = \frac{i}{2} \left( \varepsilon_{\mu\nu\lambda} p^\mu \xi_\nu^a \xi_\lambda^a + q m \varepsilon_{ab} \xi_{a\mu} \xi_{b\nu} \right) \approx 0,$$

(3.3)

constraints forming the trivial algebra of the first class with respect to the above brackets. As a consequence of the reparametrization invariance, the Hamiltonian of our model is a linear combination of the constraints:

$$H = e\phi + v\chi + u_1 p_e + u_2 p_v,$$

(3.4)

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with the coefficients at the primary constraints being arbitrary functions of the evolution parameter $\tau$. It is easy to see that the reparametrization invariance is generated by the set of the constraints $p_v \approx 0$ and $\phi \approx 0$, while the generators of the local $\beta$-symmetry are the constraints $p_v \approx 0$ and $\chi \approx 0$ [24].

### 3.2 Equations of motion

Essential equations of motion of the system are

\[
\begin{align*}
\dot{p}_\mu &= 0, \\
\dot{x}_\mu &= e_{\mu} + \frac{i}{2} v \varepsilon_{\mu \lambda} \xi_\lambda^a \xi_a^b, \\
\dot{\xi}_{a\mu} &= -v (\varepsilon_{\mu \lambda} \chi_{b}^\lambda - q m e_{a b} \xi_{b \mu}).
\end{align*}
\]

From these equations we immediately find that the energy-momentum vector $p_\mu$ and the total angular momentum vector $J_\mu = -\varepsilon_{\mu \lambda} x^{\nu \lambda} + \frac{i}{2} \varepsilon_{\mu \lambda} \xi_\lambda^a \xi_a^b$ are integrals of motion. Since Eqs. (3.7) are generated by the nilpotent constraint only, $\dot{\xi}_{a\mu} = v \{\xi_{a\mu}, \chi\}$, it can be considered as a Hamiltonian of the spin variables.

The equations of motion for the Lagrange multipliers are not important for our analysis, therefore we shall not write them down.

To solve the equations for the spin variables $\xi^a_\mu$, it is convenient to use complex mutually conjugate odd variables $b^\pm_\mu = \frac{1}{\sqrt{2}} (\xi_{1\mu} \pm i \xi_{2\mu})$ with nontrivial brackets $[b^+_\mu, b^-_\mu] = -i \eta_{\mu \nu}$. The new odd variables satisfy the equations

\[
\dot{b}^\pm_\mu = -v (\varepsilon_{\mu \lambda} p^\nu \pm i q m n_{\mu \lambda}) b^{\pm \lambda}.
\]

Taking into account the mass-shell constraint, we introduce the general notation $f^{(\alpha)}_\mu \equiv f^{(\alpha)}_\mu e^{(0)}_\mu$ for the projection of any Lorentz vector $f^\mu$ onto the complete oriented triad $e^{(0)}_\mu(p)$, $\alpha = 0, 1, 2$, defined by the relations

\[
e^{(0)}_\mu(p) = \frac{p_\mu}{\sqrt{-p^2}}, \quad e^{(0)}_\mu \eta_{\alpha \beta} e^{(\beta)}_\nu = \eta_{\mu \nu}, \quad \varepsilon_{\mu \nu} \lambda e^{(0)}_\mu e^{(i)\nu} e^{(j)\lambda} = \varepsilon^{ij \lambda}.
\]

It is important to note here that $e^{(i)}_\mu(p)$ are not Lorentz vectors, and so, the projections $f^{(i)}$ of a Lorentz vector $f^\mu$ onto these triad components are not covariant quantities, while $f^{(0)}$ is a Lorentz scalar [25].

In terms of these, we find that the odd spin variables have the following evolution law:

\[
b^{(0)}\pm(\tau) = e^{\mp i \omega(\tau)} b^{(0)\pm}(\tau_i),
\]

\[
b^{(i)}\pm(\tau) = e^{\mp i \omega(\tau)} \left[ \cos \omega(\tau) b^{(i)\pm}(\tau_i) + \varepsilon^{0ij} \sin \omega(\tau) b^{(j)\pm}(\tau_i) \right],
\]

with $\omega(\tau) \equiv \omega(\tau; \tau_i) = m \int_{\tau_i}^{\tau} v(\tau') d\tau'$, so that $b^{(0)\pm}$ are harmonic-like variables, while the solution for $b^{(i)\pm}$ includes an additional SO(2) rotation.

In terms of the initial odd variables $\xi^a_\mu$ the solutions to the equations of motion can be written as follows:

\[
\xi_{a\mu}(\tau) = g_{a\nu}(\tau) \left( \xi^\nu_\lambda(\tau_i) \cos q \omega(\tau) + \epsilon_{a b} \xi^\nu_b(\tau_i) \sin q \omega(\tau) \right), \quad a = 1, 2,
\]

\[
x_\mu(\tau) = p_\mu \int_{\tau_i}^{\tau} e(\tau') d\tau' - \frac{1}{2m} e^{(0)}_\mu \pi_{\nu \lambda} \xi^\nu_\lambda(\tau_i) \xi^\lambda_\lambda(\tau) + \frac{i}{m} \xi_{a(0)} \xi_{a\mu}(\tau_i) + x_\mu(\tau_i),
\]

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where we have introduced the notations
\begin{equation}
g_{\mu\nu}(\tau) = -\varepsilon_\mu^{(0)} \varepsilon_\nu^{(0)} + \pi_{\mu\nu} \cos \omega(\tau) + \varepsilon_{\mu\nu\lambda} \varepsilon_\lambda^{(0)} \sin \omega(\tau), \quad \pi_{\mu\nu} = \eta_{\mu\nu} + \varepsilon_\mu^{(0)} \varepsilon_\nu^{(0)}. \quad (3.14)
\end{equation}

We see that for \( q \neq 0 \) the evolution mixes the initial data of the spin variables. The terms with the odd variables in the solution for the coordinates of the particle \((3.13)\) describe the pseudoclassical analog of the quantum Zitterbewegung \([26, 27, 17, 18, 25]\).

We have seen also that it is quite natural to use complex variables \( b_\mu^\pm \) instead of their real and imaginary parts \( \xi_\alpha^\mu \), and so, in what follows, we will do work in terms of these complex spin variables.

### 3.3 Integrals of motion

From the solutions to the equations of motion we obtain quadratic nilpotent integrals of motion
\begin{align*}
N_0 &= b^{(0)+}b^{(0)-}, \\
N_\perp &= b^{(i)+}b^{(i)-}, \\
S &= i\varepsilon^{0ij}b^{(i)+}b^{(j)-} \equiv J^{(0)}. \quad (3.15)
\end{align*}

The integral of motion \( N_0 \) is the generator of the global \( \nu \)-symmetry transformation \((2.6),(2.7)\). The global \( \lambda \)-symmetry transformation \((2.5)\) is generated by the combination \( N = -N_0 + N_\perp \). The global \( \text{SO}(2) \) rotations with the parameter \( \theta \) \((2.8),(2.9)\) are generated by the integral of motion \( S \sqrt{-p^2} \). We see that on the mass shell the constraint function \( \chi \) can be represented as a linear combination of the quadratic integrals of motion, \( \chi = m(S - qN) \), and so, regarding this nilpotent constraint as the generator of the gauge \( \beta \)-transformation and the integrals of motion as the generators of global symmetry transformations, we see the reason of the above mentioned relation \((2.14)\) of global and local symmetries of the model.

The case of \( q = 0 \) is dynamically degenerated with the variables \( b^{(0)+} \) being trivial integrals of motion, \( b^{(0)+}(\tau) = b^{(0)+}(\tau_i) \). In this case the constraint \( \chi \) generates only \( \text{SO}(2) \) rotations, which do not transform variables \( b^{(0)+} \). As we shall see, this special case is completely excluded on the quantum level.

Further, we have the nilpotent second order quantities
\begin{equation}
B_\pm^\pm = \left( b^{(2)+}b^{(2)-} - b^{(1)+}b^{(1)-} \right) \pm i \left( b^{(2)+}b^{(1)-} + b^{(1)+}b^{(2)-} \right) \quad (3.16)
\end{equation}
satisfying a simple evolution law \( \dot{B}_\pm^\pm = \pm 2i m v B_\pm^\pm \) with an obvious harmonic-like solution \( B_\pm^\pm(\tau) = e^{\pm 2i \omega(\tau)} B_\pm^\pm(\tau_i) \). Recalling the evolution law for the odd variables \( b^{(0)+} \) we obtain that if and only if \( |q| = 2 \), there are two additional third order nilpotent integrals of motion in the model, namely
\begin{align*}
B_\pm^+ &= B_\pm^\pm b^{(0)+}, \\
B_\pm^- &= (B_\pm^-)^* \quad \text{for } q = 2, \quad (3.17)
\end{align*}
or
\begin{align*}
B_\pm^+ &= B_\pm^\pm b^{(0)+}, \\
B_\pm^- &= (B_\pm^-)^* \quad \text{for } q = -2, \quad (3.18)
\end{align*}
which are \textit{local} in the evolution parameter \( \tau \) quantities.

The quadratic integrals of motion \( N_0, N_\perp \) and \( S \) form trivial algebra with respect to the canonical structure \( \{ b^{(\alpha)+}, b^{(\beta)-} \} = -i \eta^{\alpha\beta} \). Besides, we find that
\begin{equation}
N_0^2 = 0, \quad N_\perp S = 0, \quad N_\perp^2 = -S^2. \quad (3.19)
\end{equation}
We have also for \( q = 2 \) (the case of \( q = -2 \) can easily be reproduced by the change \( \xi^\mu \leftrightarrow \xi^\mu_0 \)):

\[
B^+_\tau B^-_\tau = 2N_0^2 N^2_\perp, \quad \{B^+_\tau, B^-_\tau\} = -2i\mathcal{H}_+, \quad \{B^+_\tau, \mathcal{H}_+\} = 0, \quad \{B^+_\tau, N_0\} = \mp iB^\perp_\tau, \quad \{B^+_\tau, S\} = \pm 2iB^\perp_\tau,
\]

where \( \mathcal{H}_+ = N^2_\perp + 2N_0 S \).

For arbitrary \( q > 0 \) or \( q < 0 \) one can construct nonlocal integrals of motion \( B^\pm_{\tau q} = B^\pm b^{(0)} e^{\pm i(q-2)\omega(\tau)} \) and \( B^\pm_q = B^\pm b^{(0)} e^{\mp i(q+2)\omega(\tau)} \), respectively. For the particular values of the model parameter, \( q = \pm 2 \), these quantities become local in \( \tau \), coinciding with the integrals of motion \( B^\pm_\tau \). These integrals generate global symmetry transformations, acting on the canonical variables \( X = x_\mu, b^\pm_\mu \) as \( \delta X = \gamma \{X, B^\pm_\tau\} \), where \( \gamma \) are corresponding odd constant infinitesimal transformation parameters.

Thus, here we have observed some phenomenon of classical quantization: there are two special values of the parameter \( q, q = \pm 2 \), when, and only when, the system has additional (local in \( \tau \)) nontrivial integrals of motion. These integrals are the generators of corresponding global symmetry transformations, and so, the system has maximal global symmetry at these two special values of the model parameter.

## 4 Quantization of the system

### 4.1 State space of the model

To describe the state space of the model, let us first remove from the theory the Lagrange multipliers \( e \) and \( v \) and their canonically conjugate momenta \( p_e \) and \( p_v \). To this end, we introduce gauge-fixing conditions \( e - e_0 \approx 0, v - v_0 \approx 0 \) for the primary constraints, where \( e_0 \) and \( v_0 \) are some constants. Using the notion of Dirac brackets, we can now define the quantum theory on the corresponding reduced phase space. Upon quantization, the odd variables \( b^\pm_\mu \) become the fermionic creation-annihilation operators \( \hat{b}^\pm_\mu \) having the only nonzero anticommutators \( \{\hat{b}^-_\mu, \hat{b}^+_\nu\}_+ = \eta_{\mu\nu} \). Then an arbitrary quantum state can be realized over the vacuum \( |0\rangle \), defined as \( \hat{b}^-_\mu |0\rangle = 0, \langle 0|0\rangle = 1 \):

\[
\Psi(x) = \left( f(x) + \mathcal{F}^\mu(x)\hat{b}^+_\mu + \frac{1}{2!}\varepsilon_{\mu\nu\lambda}\mathcal{F}^\mu(x)\hat{b}^+\nu\hat{b}^+\lambda + \frac{1}{3!}\tilde{f}(x)\varepsilon_{\mu\nu\lambda}\hat{b}^+\nu\hat{b}^+\lambda\right)|0\rangle. \quad (4.1)
\]

It is clear that Eq. (4.1) means an expansion of the general state vector into the complete set of eigenvectors of the fermion number operator \( \hat{N} \). The coefficients of this expansion are some square-integrable functions of the space-time coordinates. The quantum parity and time-reversal transformations are generated by the antiunitary operators

\[
U_P = V^0_+ V^1_- V^2_+, \quad U_T = V^0_- V^1_+ V^2_+,
\]

\[
U^\dagger_{P,T} = U_{P,T}, \quad U^2_{P,T} = -1,
\]

where \( V^\mu_\pm = \hat{b}^+\mu \pm \hat{b}^-\mu \), as follows:

\[
P, T : \Psi(x) \rightarrow \Psi'(x'_{P,T}) = U_{P,T} \Psi(x),
\]

\[
x'^\mu_P = (x^0, -x^1, x^2), \quad x'^\mu_T = (-x^0, x^1, x^2).
\]

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In correspondence with classical relations (2.3) we have
\[
U_p \hat{b}^{\pm}_{0,2} U_p^{-1} = \hat{b}^{\mp}_{0,2}, \quad U_p \hat{b}^{\pm}_{1} U_p^{-1} = -\hat{b}^{\mp}_{1},
\]
(4.6)

\[
U_T \hat{b}^{\pm}_{1,2} U_T^{-1} = \hat{b}^{\mp}_{1,2}, \quad U_T \hat{b}^{\pm}_{0} U_T^{-1} = -\hat{b}^{\mp}_{0}.
\]
(4.7)

We get that while acting on the general state \(\Psi(x)\) these operators induce mutual transformation of scalar, \(f(x) \leftrightarrow \tilde{f}(x)\), and vector, \(F^\mu(x) \leftrightarrow \tilde{F}^\mu(x)\), fields.

### 4.2 Physical subspace

The physical states should be singled out by the quantum analogs of the remaining first class constraints:

\[
(P^2 + m^2)\Psi = 0 \quad \hat{\chi}\Psi = 0,
\]
(4.8)

where we assume that \(P_\mu = -i\partial_\mu\). Note that the first class constraint corresponding to the even nilpotent function \(\chi\) admits no, even local, gauge condition, and so, the respective sector of the phase space can be quantized only by the Dirac method [28]. This peculiarity is caused by the homogeneous quadratic in Grassmann variables nature of \(\chi\), due to which there exists no gauge constraint \(\psi\) such that the bracket \(\{\psi, \chi\}\) would be invertible. From Eq. (4.8) we see also that the scalar and vector functions from the state vector belong actually to the so-called Schwartz space, which is a rigged Hilbert space [29].

Let us fix in the quantum operator \(\hat{\chi}\) the same ordering as in the corresponding classical constraint (3.3). This gives

\[
\hat{\chi} = i\varepsilon_{\mu\nu\lambda} P^\mu \hat{b}^{+\nu} \hat{b}^{-\lambda} - qm(\hat{b}^{+\mu} \hat{b}^{-\mu} - 3/2).
\]
(4.9)

As a consequence of the quantum constraints (4.8), (4.9), we find that

\[
f(x) = \tilde{f}(x) = 0,
\]
(4.10)

whereas the fields \(F^\mu(x)\) and \(\tilde{F}^\mu(x)\) satisfy the equations

\[
i\varepsilon_{\mu\nu\lambda} P^\nu F^\lambda - \frac{1}{2} qm F_\mu = 0, \quad i\varepsilon_{\mu\nu\lambda} P^\nu \tilde{F}^\lambda + \frac{1}{2} qm \tilde{F}_\mu = 0,
\]
(4.11)

and

\[
(P^2 + m^2) F_\mu = (P^2 + m^2) \tilde{F}_\mu = 0.
\]
(4.12)

Due to the linear equations (4.11) we have also

\[
P_\mu F_\mu = P_\mu \tilde{F}_\mu = 0
\]
(4.13)

and

\[
(P^2 + \frac{1}{4} q^2 m^2) F_\mu = (P^2 + \frac{1}{4} q^2 m^2) \tilde{F}_\mu = 0.
\]
(4.14)

Comparing Eqs. (4.12) and (4.14), we see that the quantum constraints are consistent, and so, have nontrivial solutions if and only if \(|q| = 2\). We have arrived at the same quantization condition which was obtained in the classical theory.
Note that we have obtained the transversality condition (4.13) for the vector fields while there was no corresponding constraint in the classical theory.

Putting \( q = \epsilon 2, \epsilon = + \) or \(-\), we finally see that the field \( F^\mu \) can be identified with the topologically massive vector \( U(1) \) gauge field \( F^\mu_\epsilon \), whereas the field \( \tilde{F}^\mu \) coincides with \( F^\mu_\epsilon \). This gives us the desirable \( P,T \)-invariant system [1].

Let us note here that the latter can be reformulated in terms of the gauge fields through the duality relation \( \varepsilon^{\mu\nu\lambda} F^\lambda = \tilde{F}^\epsilon_{\mu\nu} = \partial_\mu A^\nu_\epsilon - \partial_\nu A^\mu_\epsilon \). In this case the corresponding basic equations are of the second order, and can thus be compared with equations of motion for another \( P \) and \( T \) conserving system – gauge-non-invariant massive model. This one, the three-dimensional Proca theory, describes causally propagating massive field excitations of spin polarizations \(+1\) and \(-1\). So, the kinematical contents of the gauge-invariant and non-invariant cases are identical [1, 30]. However, our pseudoclassical model has led exactly to topologically massive gauge fields, but not to the Proca theory. The difference between these systems may appear dynamically, when the vector fields interact with matter fields. It is quite natural to expect that interactions will affect these free systems in different ways. In this respect, it would be worth investigating quantum symmetries of the three-dimensional gauge-non-invariant vector theory (see, for example, Ref. [31] where hidden parasupersymmetries of the four-dimensional Proca theory were analyzed) and conferring them with those we shall demonstrate in this paper. This could be done in the spirit of the present analysis with the help of a pseudoclassical model corresponding to the Proca theory. Probably, such a model could be found as a result of modification of pseudoclassical models considered in Ref. [32].

If we choose another ordering prescription for the quantum counterpart of the constraint function \( \chi \), we would have the same operator but with the constant term \(-3/2\) changed for \( \alpha - 3/2 \), where the constant \( \alpha \) specifies the ordering [15]. As a result, we would find that for \( \alpha \neq 0, +3/2, -3/2 \) under appropriate choice of the parameter \( q \) (note in this case \(|q| \neq 2\)) we have as a solution of the quantum constraints only one field \( F^\mu_\epsilon \) or \( F^\mu_\epsilon \) satisfying the corresponding linear differential equation. This would lead to the violation of the \( P \) and \( T \) symmetries at the quantum level. For \( \alpha = +3/2 \) (or \( q = 0 \)) or \( \alpha = -3/2 \) the physical states are respectively described by one scalar field \( f(x) \) or \( \tilde{f}(x) \), and for both these cases the discrete symmetries are broken.

We see that the same values of the parameter \( q, q = \pm 2 \), which we have separated classically, turn out to be also special quantum mechanically: for these the number of physical states is maximal, so that the maximal global symmetry group can be realized on the physical state space, and only at \( q = \pm 2 \) parity and time-reversal symmetries are conserved. This result indicates that discrete and continuous global symmetries are profoundly connected.

4.3 Scalar product and the field system

To deal with the field system obtained upon quantization of the pseudoclassical model (2.1)-(2.2), let us consider average value of the constraint operator \( \hat{\chi} \) over an arbitrary state. First, let us investigate the structure of the scalar product on the state space. We find

\[
\langle \Psi_2, \Psi_1 \rangle = \Psi_2^*(x) \Psi_1(x) = f_2^*(x)f_1(x) - \tilde{f}_2^*(x)\tilde{f}_1(x) + F_2^*(x)F_1^\mu(x) - \tilde{F}_2^*(x)\tilde{F}_1^\mu(x). \quad (4.15)
\]
From the last expression we see that the scalar product is indefinite in the doublets \( \varphi = \left( \frac{f}{\tilde{f}} \right) \) and \( \Phi = \left( \frac{f}{\tilde{f}} \right) \). Actually we have

\[
\langle \Psi, \Psi \rangle = \bar{\varphi} \varphi + \Phi \Phi,
\]

(4.16)

where \( \bar{\varphi} = \varphi^\dagger_3 \) and \( \bar{\Phi} = \Phi^\dagger_3 \). To have the norm of the state vectors defined from a positive-definite scalar product, we should modify the metrics in the doublets \( \varphi \) and \( \Phi \) as follows:

\[
\langle \Psi_2, \Psi_1 \rangle \rightarrow \langle \langle \Psi_2, \Psi_1 \rangle \rangle = \langle \Psi_2, \tilde{\eta} \Psi_1 \rangle = \bar{\varphi}_2^\dagger \varphi_1 + \Phi_2^\dagger \Phi_1,
\]

(4.17)

where the metric operator \( \tilde{\eta} = (-1)^{\frac{1}{2}N(\mathcal{N}-1)} \) is introduced. Remember that the discrete symmetry operators \( U_{P,T} \) are antiunitary with respect to the indefinite scalar product \( \langle \cdot \rangle \). Using the relation \( U_{P,T}\tilde{\eta} = -\tilde{\eta}U_{P,T} \), it is easy to verify that the operators \( U_{P,T} \) are unitary with respect to the modified scalar product \( \langle \langle \cdot \rangle \rangle \).

For the constraint operator \( \hat{\chi} \) we get the following average value:

\[
\langle \langle \hat{\chi} \rangle \rangle \equiv \langle \langle \Psi(x), \hat{\chi} \Psi(x) \rangle \rangle = -i\varepsilon^\alpha_{\mu\beta} \left( \mathcal{F}_\alpha^\dagger P^\mu \mathcal{F}^\beta + \tilde{\mathcal{F}}_\alpha^\dagger P^\mu \tilde{\mathcal{F}}^\beta \right) + \frac{1}{2}qm \left( \mathcal{F}_\gamma^\dagger \mathcal{F}^\gamma - \tilde{\mathcal{F}}_\gamma^\dagger \tilde{\mathcal{F}}^\gamma \right) + \frac{3}{2}qm \left( f^* f - \tilde{f}^* \tilde{f} \right).
\]

(4.18)

We see that unphysical scalar fields \( f(x) \) and \( \tilde{f}(x) \) are completely decoupled from the physical sector, and so, we can take into account the equations of motion (4.10) for these fields without changing physical contents of the theory, and put \( q = \epsilon 2 \). This gives

\[
\langle \langle \hat{\chi} \rangle \rangle = \Phi^\dagger \left( PJ \otimes 1 + \epsilon m \cdot 1 \otimes \sigma_3 \right) \Phi,
\]

(4.19)

where \( \Phi = (\mathcal{F}_c, \mathcal{F}_{-c}) \) (in transposed form) and we use the usual conveniences with Pauli matrices and the generators \( (J_\mu)^\alpha_{\beta} = -i\varepsilon^\alpha_{\mu\beta} \) in the vector representation of the 3d Lorentz group, \( [J_\mu, J_\nu] = -i\varepsilon_{\mu\nu\lambda}J^\lambda, J_\mu J^\mu = -2 \). The second factor in expressions with the direct product, as in Eq. (4.19), being either identity or Pauli matrices, acts in the two-dimensional space labelled by the index distinguishing spin \( \epsilon \) and \(-\epsilon \) components of the doublet \( \Phi \), whereas the first factor corresponds to its spin (vector) index. The modified scalar product \( \langle \langle \cdot \rangle \rangle \) allows us to give the components of the doublets \( \varphi \) and \( \Phi \), and consequently, the spin states \(+\epsilon\) and \(-\epsilon\) equal treatment. However, note that there is still an indefiniteness due to the metric tensor \( \eta_{\alpha\beta} \), \( \Phi_1^\dagger J_\mu \otimes 1 \Phi_1 = \mathcal{F}_2^\dagger \eta_{\alpha\beta} \mathcal{F}_{\beta}^\dagger + \tilde{\mathcal{F}}_2^\dagger \eta_{\alpha\beta} \tilde{\mathcal{F}}_{\beta}^\dagger \). The presence of \( \eta_{\alpha\beta} \) guarantees the spinor part of the total angular momentum operator of the system to be a self-adjoint operator, \( \Phi_1^\dagger J_\mu \otimes 1 \Phi_1^* = \Phi_1^\dagger J_\mu \otimes 1 \Phi_1^* \). But this indefiniteness only concerns pure gauge degrees of freedom present in the theory and does not actually play any role in our consideration.

Having incorporated the scalar fields into the theory, we provided the completeness of the basis vectors of the total state space, expressed by the expansion (4.1). The physical state space is its subspace, obtained by eliminating the scalar fields. In this sense, \( f(x) \) and \( \tilde{f}(x) \) have actually been used as auxiliary fields. We get that on the physical subspace the...
space-time integral of the average value of the constraint operator coincides with the action $A$ (1.1):

$$\int d^3x \langle \langle \hat{\chi} \rangle \rangle_\epsilon = A = \int d^3x \Phi^\dagger(x) (PJ \otimes 1 + \epsilon m \cdot 1 \otimes \sigma_3) \Phi(x).$$

Thus, the pseudoclassical model (2.1)-(2.2) leads to the $P,T$-invariant system of topologically massive vector $U(1)$ gauge fields in a natural way.

The corresponding procedure is reminiscent of that suggested in Ref. [34] for constructing a string field theory action and subsequently developed in Ref. [35]. There, a quantity $A = \int d\mu \langle \langle \Psi | \Omega | \Psi \rangle \rangle$, with a BRST operator $\Omega$ singling out physical states and $d\mu$ being an integration measure, was regarded as a string field theory action. In this, a scalar product $\langle \langle \rangle \rangle$ was proposed to provide hermiticity of the BRST operator. The underlying idea was originated from the observation that the functional $A$ is extremal on the physical subspace: the variational principle applied to the “action” $A$ results in “quantum equations of motion” encoded in $\Omega | \Psi \rangle = 0$, and besides, it keeps symmetries of the initial first-quantized theory. In our case, we have an analogous construction, with the constraint operator $\hat{\chi}$ instead of $\Omega$.

### 4.4 Quantum analogue of the Poisson-Dirac brackets

In what follows, we put for brevity $\epsilon = +$, that corresponds to $q = 2$. The case of $\epsilon = -$ ($q = -2$) can be achieved by obvious changes.

The quantum counterpart of the integrals of motion are operators acting in the state space with the arbitrary state vector (4.1). They form the following (super)algebra:

$$[\hat{B}_+^\pm, \hat{B}_-^\mp]_- = -2(\hat{S} - \hat{R}),$$

$$[\hat{B}_+^\pm, \hat{S}]_- = \mp2\hat{B}_+^\pm, \quad [\hat{B}_+^\pm, \hat{R}]_- = \pm2\hat{B}_+^\mp,$$

$$[\hat{B}_+^\pm, \hat{B}_-^\pm]_- = 2\hat{C}_+, \quad [\hat{B}_+^\pm, \hat{C}_+]_- = 0,$$

where we have introduced the notations

$$\hat{R} = (1 + 2\hat{N}_0)\hat{N}_\perp(2 - \hat{N}_\perp), \quad \hat{C}_+ = (\hat{S} - \hat{R})(1 + 2\hat{N}_0).$$

Note that $(1 + 2\hat{N}_0)^2 = 1$. Comparing Eqs. (3.20)-(3.22) and Eqs. (4.21)-(4.24), we see that the (super)algebras of integrals of motion in the classical and the quantum theories are essentially different. The reason of this modification occurred at the quantum level is that the corresponding operators are composite ones and, in particular, the integrals of motion $B_\pm^\pm$ are of the third order in (odd) spin variables (we discuss this point in detail in the concluding Section).

### 4.5 Quantum symmetry operators

The generators of continuous global symmetries in the one-particle sector of the $P,T$-invariant system (4.20) can be found by averaging the quantum counterpart of the third order nilpotent integrals of motion taking place at $q = \epsilon 2$. For $\epsilon = +$ we have

$$Q^\pm = -\frac{1}{2}(\langle \langle \hat{B}_+^\mp \rangle \rangle) = \Phi^\dagger(x)Q^\pm \Phi(x),$$

(4.25)
where the quantum mechanical nilpotent operators

\[ Q^\pm = \frac{1}{4i} J^2 \pm \sigma_\pm \]  

(4.26)

realize mutual transformation of the physical states of spins +1 and −1. Here we use the notation \( \sigma_\pm = \sigma_1 \pm i\sigma_2 \) and \( J_\pm = J^{(1)} \pm iJ^{(2)}, J^{(a)} = J^\mu e^{(a)}_\mu \). Commutation relation of these operators is

\[ [Q^+, Q^-]_\pm = \frac{1}{2}(S - \Pi), \]  

(4.27)

where \( S = J^{(0)} \otimes 1 \) and \( \Pi = J^{(0)}J^{(0)} \otimes \sigma_3 \). In this, \( S \) is the spin operator corresponding to the average value of the quantum counterpart of the integral \( S, \Phi^\dag S \Phi = \langle \hat{S} \rangle \), and \( \Pi \) is the operator associated with the projector onto the physical spin ±1 states, that is the quantum counterpart of the integral of motion \( R = (1 + 2N_0 )N_\perp (2 - N_\perp), \Phi^\dag \Pi \Phi = \langle \hat{R} \rangle \).

We have also

\[ [Q^\pm, S]_\pm = \pm 2Q^\pm, \quad [Q^\pm, \Pi]_\pm = \mp 2Q^\pm. \]  

(4.28)

Besides, we find the anticommutator of the physical operators

\[ [Q^+, Q^-]_+ = \frac{1}{2}(S\Pi - \Pi^2), \]  

(4.29)

where

\[ S\Pi = J^{(0)} \otimes \sigma_3, \quad \Pi^2 = J^{(0)}J^{(0)} \otimes 1. \]  

(4.30)

Taking into account that \( \Phi^\dag S\Pi \Phi = \langle \hat{S}(1+\hat{N}_o) \rangle \) and \( \Phi^\dag \Pi^2 \Phi = \langle \hat{N}_\perp (2-\hat{N}_\perp) \rangle \), we finally see that the operators \( Q^\pm \) reproduce exactly the (super)algebra of the quantum mechanical counterpart of the integrals \( B_\pm^\pm \) (4.21)-(4.24).

To obtain the above algebras of the quantum mechanical operators \( Q^\pm \), we have used the properties of the triad and Pauli matrices, as well as the relations of the generators \( J^{(\alpha)} \), \( \alpha = 0, 1, 2 \), listed in the Appendix.

As we have learned from Section 3, the nilpotent classical constraint function \( \chi \) of our model played the role of Hamiltonian for the spin variables. The operator

\[ D = PJ \otimes 1 + m \cdot 1 \otimes \sigma_3 \]  

(4.31)

is its quantum analog obtained on the physical subspace after removing auxiliary scalar fields. It means that we can treat it as a quantum one-particle Hamiltonian, specifying simultaneously the physical state space, \( D\Phi = 0 \). The operators \( Q^\pm \) generate symmetries of the operator \( D \). Actually, we have

\[ [Q^\pm, D]_\pm = \pm 2Q^\pm(\sqrt{-\Pi^2} - m) \approx 0, \]  

(4.32)

which means that \( Q^\pm \) are quantum symmetry operators of the Hamiltonian \( D \) [24].

When considering the linear combinations

\[ Q_0 = \frac{1}{4}(S - \Pi), \quad Q_1 = \frac{1}{2}(Q^+ + Q^-), \quad Q_2 = i\frac{1}{2}(Q^+ - Q^-), \]  

(4.33)
we obtain that the quantum physical operators $Q_\alpha$, $\alpha = 0, 1, 2$, form $su(1, 1)$ algebra:

$$[Q_\alpha, Q_\beta] = -i\varepsilon_{\alpha\beta\gamma}Q^\gamma.$$  

The generators $Q_\alpha$ and the Casimir operator

$$C = Q_\alpha\eta^{\alpha\beta}Q_\beta = \frac{3}{8}(S\Pi - \Pi^2)$$

of this algebra form also $s(2, 1)$ superalgebra [36],

$$[Q_\alpha, Q_\beta]_+ = \eta_{\alpha\beta}\frac{2}{3}C, \quad [Q_\alpha, C]_- = 0,$$

with even generator $C$ being different from the Hamiltonian $D$.

The Casimir operator $C$ is related to the average value of $\hat{C}_+$ from Eq. (4.24), $\Phi^\dagger C\Phi = \frac{3}{8}\langle \langle \hat{C}_+ \rangle \rangle$, and it takes the value $C = -3/4$ on the physical subspace given by two square-integrable transversal vector fields $F^\mu_1, F^\mu_2$ carrying spins $-1$ and $+1$.

We can also construct another combination of the physical operators $S$ and $\Pi$, namely

$$U = \frac{1}{2}(S + \Pi),$$

satisfying the relation

$$[Q_\alpha, U]_- = 0.$$ 

The operator $U$ is the generator of global $U(1)$ symmetry, taking zero value on the physical states. Hence, the set of the physical operators $Q_\alpha$ and $U$ form $U(1, 1) = SU(1, 1) \times U(1)$ group.

We have thus revealed hidden $U(1, 1)$ symmetry and $S(2, 1)$ supersymmetry of the $P,T$-invariant quantum system of topologically massive vector $U(1)$ gauge fields at the one-particle level.

5 Covariant form of the hidden (super)symmetries

5.1 Towards covariantization

The hidden dynamical symmetry we have revealed in the previous Section leads to a non-standard super-extension of the $(2+1)$-dimensional Poincaré group. To show this we have to construct a covariant form of the above algebra relations. Actually, the quantities $S$ and $\Pi$, as well as their combination $Q_0$,

$$Q_0 = \frac{1}{4}J^{(0)} \otimes 1 - \frac{1}{4}J^{(0)}J^{(0)} \otimes \sigma_3,$$  

are expressed as covariant (scalar) operators, whereas $Q_i$,

$$Q_1 = \frac{1}{4i} \left[ (J^{(1)})^2 - (J^{(2)})^2 \right] \otimes \sigma_1 - \frac{1}{4i} \left[ J^{(1)}J^{(2)} + J^{(2)}J^{(1)} \right] \otimes \sigma_2,$$

$$Q_2 = -\frac{1}{4i} \left[ J^{(1)}J^{(2)} + J^{(2)}J^{(1)} \right] \otimes \sigma_1 - \frac{1}{4i} \left[ (J^{(1)})^2 - (J^{(2)})^2 \right] \otimes \sigma_2,$$
are given in terms of non-covariant quantities $J^{(i)}$, $i = 1, 2$.

Taking into account that the hidden symmetry operators interchange spins $+1$ and $-1$, it is natural to consider as a candidate for their covariant form a rank-2 symmetric tensor operator
\[ X_{\mu\nu} = X^0_{\mu\nu} + X^\perp_{\mu\nu}, \]
where
\[ X^0_{\mu\nu} = e^\mu_\nu Q_0, \]
with $e^\mu_\nu$ defined below, and $X^\perp_{\mu\nu}$ is a symmetric, transversal and traceless tensor:
\[ X^\perp_{\mu\nu} = X^\perp_{\nu\mu}, \quad e^{(0)\mu} X^\perp_{\mu\nu} = 0, \quad \eta^{\mu\nu} X^\perp_{\mu\nu} = 0, \]
being a quantum physical operator:
\[ [D, X_{\mu\nu}]_\mp = [D, X^\perp_{\mu\nu}]_\mp \approx 0, \]
where the weak equality means the equality on the mass shell $P^2 + m^2 \approx 0$.

From the expressions of the quantum physical operators $Q_\alpha$ and $Q^\pm$ we see that it is convenient to introduce the quantities
\[ e^\pm_\mu_\nu = (e^{(1)\mu} \pm ie^{(2)\mu}) \cdot (e^{(1)\nu} \pm ie^{(2)\nu}) \]
and their linear combinations
\[ [1] e^\mu_\nu = \frac{1}{2\sqrt{2}} (e^\mu_\nu + e^\nu_\mu) = \frac{1}{\sqrt{2}} \left( e^{(1)\mu} e^{(1)\nu} - e^{(2)\mu} e^{(2)\nu} \right), \]
\[ [2] e^\mu_\nu = \frac{i}{2\sqrt{2}} (e^\mu_\nu - e^\nu_\mu) = -\frac{1}{\sqrt{2}} \left( e^{(1)\mu} e^{(2)\nu} + e^{(2)\mu} e^{(1)\nu} \right), \]
together with the obvious covariant construction of the form
\[ [0] e^\mu_\nu = ie^{(0)\mu} e^{(0)\nu} \]
providing projection of the tensor $X_{\mu\nu}$ onto the set of non-covariant quantum symmetry operators:
\[ [\alpha] e^{\mu\nu} X_{\mu\nu} = Q^\alpha = \eta^{\alpha\beta} Q_\beta, \quad \alpha, \beta = 0, 1, 2. \]

We find the solution to the equalities (5.6)-(5.7),(5.12) in the form
\[ X^\perp_{\mu\nu} = \frac{1}{4i} [1] A^\mu_\nu \otimes \sigma_1 + \frac{1}{4i} [2] A^\mu_\nu \otimes \sigma_2, \]
where the rank-2 tensors $[1] A^\mu_\nu$ and $[2] A^\mu_\nu$ are given by the expressions
\[ [1] A^\mu_\nu = \frac{1}{\sqrt{2}} (r^\mu r^\nu - s^\mu s^\nu), \quad [2] A^\mu_\nu = \frac{1}{\sqrt{2}} (r^\mu s^\nu + s^\mu r^\nu), \]
with
\[ r_\mu = \pi_{\mu\nu} J^\nu \equiv J_{\mu}^1 \equiv J_{\mu} + \epsilon^{(0)}_{\mu} J^{(0)}, \quad s_\mu = \epsilon_{\mu\alpha\beta} \epsilon^{(0)\alpha} J^\beta. \] (5.15)

Here \( \pi_{\mu\nu} \) is the quantum counterpart of the tensor introduced by Eq. (3.14).

Using the properties of \( r_\mu \) and \( s_\mu \), described in the Appendix, we obtain that the square of the tensor \( X_{\mu\nu} \) is the Casimir operator \( C \):
\[ X_{\mu\nu} \cdot X^{\mu\nu} = X_{\mu\nu}^0 \cdot X^{\mu\nu}_0 + X_{\mu\nu}^\perp \cdot X^{\mu\nu}_\perp = C. \] (5.16)

Let us introduce the notations
\[ \mathcal{G}^{\mu\nu|\rho\sigma} = e^{[a}_{\mu\nu} \eta_{\alpha\beta} e^{[b]}_{\rho\sigma}, \quad \mathcal{E}^{\mu\nu|\rho\sigma|\lambda\tau} = \epsilon_{\alpha\beta\gamma} e^{[a}_{\mu\nu} e^{[b]}_{\rho\sigma} e^{[c]}_{\lambda\tau}. \] (5.17)

The properties of the quantities \( e^{[a}_{\mu\nu} \) and of the tensors \( \mathcal{G}^{\mu\nu|\rho\sigma} \) and \( \mathcal{E}^{\mu\nu|\rho\sigma|\lambda\tau} \) are listed in the Appendix. Using these properties and the relation between the Casimir and \( Q_0 \) operators, we get that the tensor operator \( X_{\mu\nu}^\perp \) satisfies the equation
\[ X_{\mu\nu}^\perp X_{\rho\sigma}^\perp = \frac{1}{2} \left( \pi_{\mu\nu} \pi_{\rho\sigma} - \pi_{\mu\sigma} \pi_{\nu\rho} - \pi_{\mu\rho} \pi_{\nu\sigma} \right) Q_0^2 + \frac{1}{4} \left( \pi_{\mu\sigma} \epsilon_{\nu\rho\lambda} + \pi_{\nu\rho} \epsilon_{\mu\sigma\lambda} \right) e^{(0)\lambda} Q_0. \] (5.18)

The last equality leads finally to the symmetry algebra relation for the tensor operator \( X_{\mu\nu}: \)
\[ X_{\mu\nu} X_{\rho\sigma} = \mathcal{G}^{\mu\nu|\rho\sigma} \cdot \frac{1}{3} C - \frac{i}{2} \mathcal{E}^{\mu\nu|\rho\sigma|\lambda\tau} X^{\lambda\tau}. \] (5.19)

In addition to Eq. (5.19) we have the commutation relations
\[ [X_{\mu\nu}, C]_- = 0, \quad [X_{\mu\nu}, U]_- = 0, \] (5.20)
where the operator \( U \) is expressed by Eq. (4.37).

It is worthwhile seeing that, as required by tensor nature of the hidden symmetry generators, the covariantization is achieved by means of a kind of the bi-vector mapping, provided that the quantities \( e^{[a}_{\mu\nu} \) play the same role in the corresponding bi-vector space as the components of the complete oriented triad \( e^{(a)}_{\mu} \) do in the three-dimensional Minkowski space-time. This actually originates our notations of Eq. (5.17) and consequent properties, so that \( \mathcal{G}_{ab} \) and \( \mathcal{E}_{abc} \) turn out to be the metric and totally antisymmetric tensors of the bi-vector space.

Note that one can introduce a complex vector \( a_\mu \) related to the operators \( r_\mu \) and \( s_\mu \) as
\[ 2 a_\mu = r_\mu + i s_\mu, \quad a_\mu a^\mu = 0. \]
Then the tensor operator \( X_{\mu\nu}^\perp \) is represented in the form
\[ X_{\mu\nu}^\perp = \frac{1}{\sqrt{2}} \theta^\mu_{\sigma_1} \theta^\nu_{\sigma_2}, \]
where the 2 \( \times \) 2 block-matrix \( \theta_\mu \) is given by the expression
\[ \theta_\mu = \text{diag} \left( a_\mu, a^*_\mu \right). \]
Obviously, the same (super)algebras of the dynamical symmetry operators can be obtained in terms of this representation as well. Probably, for some particular problems the use of the complex-valued vector operator \( a_\mu \) would be more appropriate than of its real and imaginary parts \( r_\mu \) and \( s_\mu \), while the latter seem to be quite sufficient for the present analysis.
5.2 Non-standard super-extension of the Poincaré group

We have now the following set of covariant operators being the generators of the full dynamical symmetry algebra: \( X_{\mu\nu} \) — generators of the hidden symmetries just revealed, \( P_\mu \) — energy-momentum operator, \( M_\mu \) — total angular momentum operator explicitly given by the expression

\[
M_\mu = -\varepsilon_{\mu\nu\lambda} x^\nu P^\lambda \cdot 1 \otimes 1 + J_\mu \otimes 1.
\] (5.21)

One can easily get that the above operators are integrals of motion. Actually, the commutator of \( X_{\mu\nu} \) with the Hamiltonian \( D \) disappears in a weak sense on the surface defined by the constraint \( P^2 + m^2 \approx 0 \), while the generators \( P_\mu \) and \( M_\mu \) strongly commute with \( D \):

\[
[D, X_{\mu\nu}]_- \approx 0, \quad [D, M_\mu]_- = 0, \quad [D, P_\mu]_- = 0.
\] (5.22)

Nonzero (anti)commutation relations of these operators are of the form:

\[
[M_\mu, P_\nu]_- = -i\varepsilon_{\mu\nu\lambda} P^\lambda,
\] (5.23)

\[
[M_\mu, M_\nu]_- = -i\varepsilon_{\mu\nu\lambda} M^\lambda,
\] (5.24)

\[
[X_{\mu\nu}, X_{\rho\sigma}]_+ = \mathcal{G}_{\mu\nu|\rho\sigma} \cdot \frac{2}{3} C,
\] (5.25)

\[
[X_{\mu\nu}, X_{\rho\sigma}]_- = -i\varepsilon_{\mu\nu|\rho\sigma} X^{\lambda\tau},
\] (5.26)

\[
[M_\mu, X_{\rho\sigma}]_- = -i\varepsilon_{\mu\rho\lambda} X^\lambda_{\ |\sigma} - i\varepsilon_{\mu\sigma\lambda} X^\lambda_{\ |\rho}.
\] (5.27)

Hence, the physical operators \( P_\mu \), \( M_\mu \) and \( X_{\mu\nu} \) complete the set of generators of the super-extended Poincaré group ISO(2,1|2,1). The Casimir operators of this supergroup are \( P^2 \) and the superspin

\[
\Sigma = e^{(0)\mu} M_\mu + 2e^{0\mu\nu} X_{\mu\nu}.
\] (5.28)

From the explicit forms of the total angular momentum operator \( M_\mu \) and the hidden (super)symmetry generators \( X_{\mu\nu} \) we find

\[
\Sigma = \frac{1}{2}(S + \Pi) = U.
\] (5.29)

We get that the physical operator \( U \) has the sense of the superspin of the system. The eigenvalues of the superspin \( \Sigma \) are given by the set of numbers \((-1,0,0,0,0,1)\). As we noted in the preceding Section, the operator \( U \) takes zero value in the physical subspace. Therefore, we gain that the physical states are the eigenstates of the superspin operator with zero eigenvalue. The same result can be seen by expressing the operator \( C \) through the superspin as a quadratic function of the superspin: \( C = \frac{3}{4}(\Sigma^2 - J^{(0)\mu} J^{(0)\mu} \otimes 1) \). Consequently, the one-particle states of the quantum \( P,T \)-invariant system of topologically massive vector U(1) gauge fields realize an irreducible representation of the supergroup ISO(2,1|2,1) labelled by the zero eigenvalue of the superspin. Similar properties have been elucidated for the double fermion system \cite{14}, which is also considered to be relevant to high-temperature superconductivity \cite{13}. 16
6 Concluding remarks

In this paper, with the help of the proposed pseudoclassical model (2.1)-(2.2) we have uncovered a rich set of hidden symmetries of the $P,T$-invariant system of topologically massive vector U(1) gauge fields.

Let us stress once more on the difference between the (super)algebras formed by the integrals of motion at the classical and the quantum levels. In the classical theory we have the set of quadratic (in independent odd variables) integrals of motion and two additional third order quantities conserved at the special values of the model parameter. These two integrals of motion together with the “Hamiltonian” $\mathcal{H}_+\text{formed }N=2$ supersymmetry algebra (3.21) with respect to the Poisson-Dirac brackets, and the system reproduced this superalgebra at the quantum level: we have the relations (4.23) where the operator $\hat{C}_+$ plays the role of the quantum counterpart of the “Hamiltonian” $\mathcal{H}_+$ (the corresponding modification is actually due to the compositeness of the integrals of motion). At the quantum level we have also $su(1,1)$ algebra, given by Eqs. (4.21)-(4.22) with respect to commutators. This symmetry algebra can be reproduced at the classical level only partially: to the commutator of the quantum counterparts of the third order integrals of motion corresponds the relation (3.20) defined with respect to ordinary multiplication, so that we lose the usual correspondence between commutators and canonical brackets. To see the reason for such a breakdown of the standard quantization prescription [26], $\{,\} \rightarrow [\,,\,]/ih$, with respect to the hidden symmetry algebras, one has to reconstruct the spin variables in physical units, $\hat{b}_{+}^{\pm} \sim h^{1/2}$, so that $\hat{B}_{+}^{\pm} \sim h^{3/2}$. Now it is clear that (anti)commutators of the operators $\hat{B}_{+}^{\pm}$, divided by $ih$, vanish in the classical limit, $[\,,\,]/ih \rightarrow 0$ as $\hbar \rightarrow 0$. Actually, this observation just corresponds to the well-known fact that the terms of order $\hbar^{2+\kappa}$, $\kappa \geq 0$, have no classical analog in ordinary (without Grassmann variables) classical mechanics.

This situation is different from that one realized for the case of planar fermions [14]. In the latter, all the integrals of motion forming hidden dynamical symmetry group are quadratic in odd variables. Besides, there is an odd first order integral of motion in the double fermion system, which gives a possibility to change Grassmann parities of the integrals of motion simply multiplying them by this quantity. All this allows one to have one and the same (super)symmetry algebras in the corresponding classical and quantum theories. As we have seen above, the situation considered in this paper is essentially different, and so, quantum symmetries are reproduced at the classical level only in part. Nevertheless, exactly the quantum counterparts of the third order integrals of motion give us finally the set of revealed quantum symmetries.

It is interesting to compare the supersymmetry we have revealed in this paper with the BRST and anti-BRST type supersymmetry, obtained in Ref. [37] for the non-Abelian Chern-Simons theory and shown in Ref. [38] to be forming IOSp(3|2) supergroup. The latter has been proven only for Landau gauge, provided that namely in this particular gauge ghost and vector field sectors are respectively coupled. The hidden supersymmetry we have elucidated for $P,T$-invariant Abelian Chern-Simons theory is not related to any gauge-choice, it is a true supersymmetry leading to a nontrivial super-extension of the Poincaré group.

The pseudoclassical model we have proposed for $P,T$-invariant system of topologically massive vector U(1) gauge fields turned out itself to be very interesting. It has revealed the quantization of the parameter $q$ and nontrivial ‘superposition’ of the discrete ($P$ and $T$) and
continuous (hidden $U(1,1)$ and $S(2,1)$) (super)symmetries. These hidden continuous global symmetries are quite nontrivial since the corresponding generators act not only on spin $+1$ and $-1$ states as the whole, but they transform also components of the fields and, moreover, due to the dependence on $P_\mu$, the symmetry generators act nontrivially on the space-time coordinates.

A principal problem to be investigated developing these results is to obtain the quantum field analog of the hidden (super)symmetry generators. It is quite natural to expect that they should be generators of the corresponding field symmetry transformations. Having such an interpretation, one might further analyze systems with $P,T$-invariant matter coupling and study what happens with the revealed hidden (super)symmetries.

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A Appendix

A.1 Generators of the 3d Lorentz group

The projections of the generators of the 3d Lorentz group onto the triad, as introduced in Section 4.5, satisfy the relations

\[
\left[J^{(\alpha)}, J^{(\beta)}\right] = -i\varepsilon^{\alpha\beta\gamma} J^{(\gamma)},
\]

\[
\left[J^2, J^{(0)}\right] = \pm 2 J^2,
\]

\[
\left[J^2_+, J^2_-\right] = -4 J^{(0)},
\]

\[
\left[J^2_+, J^2_-\right]^+ = 4 J^{(0)} J^{(0)},
\]

and

\[
(J^{(0)})^{2k} = J^{(0)} J^{(0)}, \quad (J^{(0)})^{2k+1} = J^{(0)}
\]

for any positive integer $k$.

A.2 Vector operators $r_\mu$ and $s_\mu$ and their combinations

The vectors $r_\mu$ and $s_\mu$ have the properties

\[
e^{(i)\mu} r_\mu = J^{(i)}, \quad e^{(i)\mu} s_\mu = \delta^{i2} J^{(1)} - \delta^{i1} J^{(2)}, \quad e^{(0)\mu} r_\mu = e^{(0)\mu} s_\mu = 0,
\]

from which we get the transversality condition

\[
e^{(0)\mu} A_\mu^{[i]} = 0, \quad i = 1, 2.
\]
Besides, we obtain the equalities

\[ r_\mu r^\mu = s_\mu s^\mu = -2 + J^{(0)} J^{(0)}, \]  
\[ r_\mu s^\mu = -s_\mu r^\mu = -iJ^{(0)}, \]  
which provide tracelessness of the rank-2 tensors \( A_{\mu\nu} \):

\[ A_\mu^\mu = 0, \quad i = 1, 2. \]  

(A.10)

The following relations are useful to clarify properties of the tensors \( A_{\mu\nu} \):

\[ [r_\mu, r_\nu]_\pm = [s_\mu, s_\nu]_\pm = i\varepsilon_{\mu\nu\lambda} e^{(0)\lambda} J^{(0)}, \quad [r_\mu, s_\nu]_\pm = -i\pi_{\mu\nu} J^{(0)}, \]  
\[ [J^{(0)}, r_\mu]_\pm = is_\mu, \quad [J^{(0)}, s_\mu]_\pm = -ir_\mu, \]  
\[ \varepsilon_{\mu\alpha\beta} e^{(0)\alpha} r^{\beta} = s_\mu, \quad \varepsilon_{\mu\alpha\beta} e^{(0)\alpha} s^{\beta} = -r_\mu. \]  

(A.11)

(A.12)

(A.13)

It follows from Eqs. (A.12) that

\[ [J^{(0)}, A_{\mu\nu}]_\pm = 2i A_{\mu\nu}, \quad [J^{(0)}, A_{\mu\nu}]_\pm = -2i A_{\mu\nu}. \]  

(A.14)

The last equalities are necessary to prove that \( X_{\mu\nu} \) is a physical operator.

Using the properties of the operators \( r_\mu \) and \( s_\mu \), we obtain the relations

\[ A_{\mu\nu} \cdot A_{\mu\nu} = 2J^{(0)} J^{(0)}, \]  
\[ A_{\mu\nu} \cdot A_{\mu\nu} = 2i J^{(0)}, \]  
\[ A_{\mu\nu} \cdot A_{\mu\nu} = -A_{\mu\nu} \cdot A_{\mu\nu} = 2i J^{(0)}, \]  

(A.15)

(A.16)

which help us to relate the operator \( X_{\mu\nu} \) with the Casimir operator.

The operators \( A_{\mu\nu} \) fulfil also the equalities

\[ J^{(0)} J^{(0)} A_{\mu\nu} = A_{\mu\nu} = A_{\mu\nu} J^{(0)} J^{(0)}, \quad i = 1, 2, \]  
\[ i J^{(0)} A_{\mu\nu} = -A_{\mu\nu} = -i A_{\mu\nu} J^{(0)}, \]  
\[ i J^{(0)} A_{\mu\nu} = A_{\mu\nu} = -i A_{\mu\nu} J^{(0)}, \]  

(A.17)

(A.18)

(A.19)

which are necessary to obtain the (super)algebra of the operators \( X_{\mu\nu} \).
A.3 Structure functions of symmetry algebras

The list of useful properties of the quantities $\epsilon^{[\alpha}_\mu e_{\nu]}$ is

$\epsilon^{[\alpha}_\mu e_{\nu]} \cdot e^{[\beta}_\rho e_{\sigma]} = \eta^{\alpha\beta}$, (A.20)

$\varepsilon_{0ij} \epsilon^{[i}_\mu e^{j}_{\rho} e^{j}_{\sigma} e^{k}_{\lambda} \equiv \epsilon_{\mu \nu | \rho \sigma | \lambda \tau} = \frac{1}{4} \epsilon^{\lambda \tau}_{\alpha} (\pi^{\mu \rho} e_{\nu \sigma} + \pi^{\nu \sigma} e_{\mu \rho} + \pi^{\mu \sigma} e_{\nu \rho}) e^{(0)}_{\alpha} + \frac{1}{2} \epsilon^{\mu \nu}_{\alpha} e^{(0)}_{\alpha}$, (A.21)

$\epsilon^{[\alpha}_\mu e_{\nu]} e^{[\beta}_\rho e^{\nu}_{\lambda} e^{\rho}_{\sigma} e^{\lambda}_{\tau} = \frac{1}{4} \epsilon^{\lambda \tau}_{\alpha} (\pi^{\mu \rho} e_{\nu \sigma} + \pi^{\nu \sigma} e_{\mu \rho} + \pi^{\nu \sigma} e_{\mu \rho}) e^{(0)}_{\alpha}$, (A.22)

$\epsilon^{[\alpha}_\mu \eta^{\alpha \beta} e^{\beta}_{\rho} e^{\beta}_{\sigma} \equiv G^{\mu \nu | \rho \sigma} = -\epsilon^{[\mu \nu \rho]} e_{\sigma]} + \frac{1}{2} (\pi^{\mu \rho} e_{\nu \sigma} + \pi^{\nu \sigma} e_{\mu \rho} - \pi^{\mu \sigma} e_{\rho \nu})$, (A.23)

From the above relations we see that

$G^{\mu \nu | \rho \sigma} X_{\rho \sigma} = \epsilon^{\mu \nu} e^{\rho \sigma} Q_{0} + X^{\mu \nu} = X^{\mu \nu}$, (A.24)

$\epsilon^{\alpha \beta \gamma} e^{\mu \nu} e^{\rho \sigma} e_{\lambda \tau} = \frac{1}{6} (\pi^{\mu \rho} e_{\nu \sigma} + \pi^{\nu \sigma} e_{\mu \rho} - \pi^{\nu \sigma} e_{\mu \rho}) e^{(0)}_{\gamma} Q_{0}$

Then, taking into account the symmetry algebra relation

$Q^{\gamma} Q^{\beta} = \eta^{\alpha \beta} \frac{1}{3} C - i \frac{1}{2} \epsilon^{\alpha \beta \gamma} Q^{\gamma}$, (A.26)

we obtain a covariant equation

$X_{\mu \nu} = \frac{1}{6} (\pi^{\mu \rho} e^{\nu \sigma} + \pi^{\nu \sigma} e^{\mu \rho} - \pi^{\nu \sigma} e^{\mu \rho}) e^{(0)}_{\gamma} Q_{0}$.

Finally, the equality

$\epsilon^{[\mu \nu \lambda} e^{0}_{\rho \sigma]} e^{(0)}_{\tau} = \pi^{\mu \rho} e^{\nu \sigma} - \pi^{\nu \sigma} e^{\mu \rho}$

is implicated to write the structure functions of the hidden (super)symmetry algebra in a more appropriate form, given by Eqs. (A.20)-(A.23).

The tensor $E_{\alpha \beta \gamma}$ has useful properties

$\epsilon^{[\mu \nu} e_{\rho]} e^{[\nu \lambda} e^{\rho}_{\sigma]} = \epsilon^{[\alpha \beta} e_{\gamma]}^{\lambda \tau} \equiv \eta_{\gamma \delta} e^{\lambda \tau}$,

Besides, the tensor $G_{\mu \nu | \rho \sigma}$ is symmetric over the pair indices: $G_{ab} = G_{ba}$, where $a = (\mu \nu)$ $b = (\rho \sigma)$, while the tensor $E_{abc}$ with $a = (\mu \nu)$, $b = (\rho \sigma)$, $c = (\lambda \tau)$ is completely antisymmetric
over pair indices. It is remarkable that basic properties of the tensors $E$ and $G$ are the same as of the totally antisymmetric tensor $\varepsilon_{\mu\nu\lambda}$ and of the metric tensor $\eta_{\mu\nu}$ of the 3d Minkowski space-time. Actually, we have

$$E_{abc}E^{abc} = -d! = -6, \quad (A.30)$$
$$E_{acd}E_{b}^{cd} = -2G_{ab}, \quad (A.31)$$
$$E_{abf}E^{fcd} = G_{ad}G_{bc} - G_{ac}G_{bd}, \quad (A.32)$$

and

$$G_{ab} = G_{ba}, \quad G_{ac}G^{cb} = G_a^b, \quad G_{ab}G^{ab} = d = 3, \quad (A.33)$$

where $d = 3$ is the space-time dimension.

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