Theoretical Consideration on Short- & Open-circuited Transmission Lines for Permeability & Permittivity Measurement

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The use of short- and open-circuited transmission lines is widely known to be a simple method of measuring the permeability and permittivity of magnetic materials. Lumped element approximation is one of the methods to analyze these transmission lines. However, this approximation generally involves rather large model errors. We describe here a way to reduce these errors. The distributed element expression of short- and open-circuited transmission lines is expanded by a Taylor series, and the first and the second terms are adopted in a limited form. These analyses have clarified that this approximation could hold within ±5% of model error if a phase shift βl is less than 90% of π/2 (1.4 radian), where β is the propagation constant and l is the sample length.

Key words: short-circuited, open-circuited, transmission line, permeability, permittivity, lumped element approximation, wideband measurement

1. Introduction

Along the recent development of cell phones, wireless LAN, RFID, and etc., the application area of magnetic materials at GHz band have been expanding. With this trend, the demand to measure permeability and permittivity of magnetic materials is increasing. For this purpose, there are two kinds of methods, namely resonant method and non resonant method.

Although the former has the advantage such as high sensitivity, it also has the problems in size and usefulness because if the wideband measurements are needed, it requires the fair number of cavities.

On the other hand, the latter, non resonant method, in which transmission line is used, has a possibility of wideband measurement. There exist two kinds of methods such as transmission method (2-port) and reflection method (1-port).

For the typical method of 2-port, the coaxial transmission line is used and the sample is loaded at the middle of transmission line. The S-parameters of \( S_1 \) and \( S_2 \) are measured by VNA (Vector Network Analyzer) for before and after sample loading, respectively. From them, permeability and permittivity are analytically derived simultaneously \(^1\). The coaxial transmission line method doesn’t require the demagnetization correction for permeability due to toroidal shape of sample and enable us wideband measurement. However, permittivity is measured as an effective permittivity including the contact situations between sample and outer and inner conductors.

In the case of permeability measurement by 1-port method, the sample is loaded at the short end of short-circuited transmission line and the reflection coefficients are measured for before and after sample loading. From them, permeability is derived. In the case of permittivity measurement, the sample is loaded at the open end of open-circuited transmission line. The reflection coefficients are measured for before and after sample loading. From them, permittivity is derived.

In order to analyze 1-port method mentioned above, the well known lumped element approximation is used. However, this approximation is so simple that it is only valid for lower frequency region \(^2, 3\). In this paper, considering the short- and open-circuited coaxial lines, we would like to evolve the lumped element approximation and, as the result, introduce the new approximation method. This paper also provides the theoretical bases of our previous papers \(^3, 5\) that were ambiguous then.

2. Short- and open-circuited transmission lines

The test fixtures are considered here to be the short-circuited transmission line for permeability and the open-circuited transmission line for permittivity. The transmission line types are possibly considered to be coaxial line, strip line, micro strip line, and parallel line. The theory introduced here is so general that it could be applicable for any types. However, in order to discuss the theory, the physical image should be established. Here, we would like to go forward the discussion by taking the examples of short-circuited transmission line as shown in Fig.1 and open-circuited transmission line as shown in Fig.2, where \( l \) is the length (thickness) of sample along the transmission direction. The characteristic impedance \( Z_0 \) is limited to be 50 ohm.

The intrinsic permeability and permittivity are defined for infinite medium. The measured permeability and permittivity at GHz band vary depending on the measurement method in the almost case. The reason is that the demagnetization effect and the depolarization effect can not be treated correctly.

For example, in the case of short-circuited line as shown in Fig.1, the toroidal shape of sample sheet is used and the demagnetization field is zero along
circumference but very large along the thickness \( l \) due to \( l < (b-a)/2 \) where \( a \) and \( b \) are inner- and outer-diameters of the sample. The measured result always reflects these effects.

In the case of open-circuited transmission line as shown in Fig.2, the outer and inner circumferences of toroidal shape sample are needed to tightly contact with the outer and inner conductors, respectively. However, this is a very difficult issue. The depolarization effect due to the gaps between the circumferences and the conductors is not negligible when the permittivity is larger.

Therefore, in this paper, we’d like to declare that the permeability and permittivity measured are not the intrinsic parameters as defined in the infinite medium but the effective parameters of them.

If the intrinsic permeability and permittivity were needed such as for electromagnetic simulation, they have to be expected from the measured ones by extracting demagnetization and depolarization influences.

3. Lumped element approximation and equivalent circuits

3.1 Short-circuited transmission line

In general, the input impedance of short-circuited transmission line containing a medium with relative permeability \( \mu \) and permittivity \( \varepsilon \) is expressed as follows:

\[
Z_{in} = jZ \tan \beta l \quad \text{------------------------ (1),}
\]

where \( l \) is the length of transmission line and

\[
Z = (\mu L/\varepsilon C)^{1/2} \quad \text{------------------------ (2),}
\]

\[
\beta = 2\pi \gamma/\omega = \omega (\mu \varepsilon C)_{\infty}^{1/2}, \quad \text{c: light speed (TEM) \textellipsis (3),}
\]

where \( L = Z_{L}/C \) and \( C = 1/(\varepsilon Z) \) are the inductance and capacitance per unit length of transmission line with air medium (\( Z \_o \) : the characteristic impedance). Here, the next equations are defined.

\[
L_o = L, \quad C_o = C \quad \text{------------------------ (4).}
\]

Equation (1) can be changed to the next equation of the input admittance.

\[
Y_{in} = 1/(jZ \beta \tan \beta \delta) \quad \text{------------------------ (5),}
\]

\[
= 1/[j(\mu L/\varepsilon C)^{1/2} \varepsilon \omega (\mu \varepsilon C)_{\infty}^{1/2} \beta \tan \beta \delta] = 1/[j(\mu L/\varepsilon C)^{1/2} \beta \tan \beta \delta]
\]

3.1.1 First order lumped element approximation

When \( \beta l \ll 1 \), the equation of \( \beta l / (\tan \beta \delta) = 1 \) can hold. Then the equation (5) can be changed as follows:

\[
Y_{in} = 1/(j\omega \mu L_o) \quad \text{------------------------ (6).}
\]

This means that the equivalent circuit is a simple inductance. So far, this approximation has been used and can be said as the first order lumped element approximation.

3.1.2 Second order lumped element approximation

Under the condition of \( \beta l \ll 1 \), the equation (5) of input admittance can be expanded in Taylor series and the first and second terms are limitedly written as the next equation.

\[
Y_{in} = 1/(j\omega \mu L_o) \left[ 1 - (1/3)(\beta l)^2 \right] = 1/(j\omega \mu L_o) + (1/3)j\omega \mu C_o \quad \text{------------------------ (7).}
\]

Equation (7) means that the equivalent circuit for short-circuited coaxial line is the parallel circuit of \( \mu L_o \) and \( (1/3)j\omega C_o \) as shown in Fig.1 (b).

When the sample is inserted into the short-circuited coaxial line, the circuit is influenced by permittivity as well as permeability of sample. Equation (7) can hold with full occupation and also with partial occupation by sample if the effective permeability \( \mu_e \) and effective permittivity \( \varepsilon_o \) are used in stead of \( \mu \) and \( \varepsilon \). This second order approximation makes much precise compared to the first order approximation due to the consideration of permittivity.

3.2 Open-circuited transmission line

In general, the input impedance of open-circuited transmission line with the length of \( l \) containing medium with permeability \( \mu \) and permittivity \( \varepsilon \) is expressed as follows:

\[
Z_{in} = -jZ \cot \beta \quad \text{------------------------ (8).}
\]

Equation (8) is changed to the next equation.

\[
Z_{in} = -jZ (\beta l/\beta l (\tan \beta \delta)) = -j(\mu L/\varepsilon C)^{1/2} \varepsilon \omega (\mu \varepsilon C)_{\infty}^{1/2} \beta l (\tan \beta \delta) = -j(\omega C) \beta l (\tan \beta \delta) \quad \text{------------------------ (9).}
\]

3.2.1 First order lumped element approximation

When \( \beta l \ll 1 \), the equation of \( \beta l / (\tan \beta \delta) = 1 \) can hold. Then, equation (9) becomes the next.

\[
Z_{in} = -j(\omega C) \quad \text{------------------------ (10).}
\]

This means the equivalent circuit is a simple capacitance. This is the first order lumped element approximation.
3.2.2 Second order lumped element approximation

Under the condition of $|\beta|<1$, the equation (9) of input impedance can be expanded in Taylor series and the first and second terms are limitedly written as the next equation.

$$Z_o = -j(\omega \varepsilon C_o)(1- (1/3)(\beta)^2)$$
$$= -j(\omega \varepsilon C_o)+(1/3)j\omega \mu L_o$$

Equation (11) means that the equivalent circuit for open-circuited coaxial line is the series circuit of $\varepsilon C_o$ and $(1/3) \mu L_o$ as shown in Fig.2 (b).

When the sample is inserted into the open·circuited coaxial line, the circuit is influenced by permeability as well as permittivity of sample. Equation (11) can hold with full occupation and also with partial occupation by sample if the effective permittivity $\varepsilon_e$ and effective permeability $\mu_e$ are used in stead of $\varepsilon$ and $\mu$. This second order approximation makes much precise compared to the first order approximation due to the consideration of permeability.

4. Approximation of error function

As shown in the equations of (5) and (9), both contain the function of $\beta/(\tan \beta)$ for short-circuited and open-circuited transmission lines. Here, we replace as $\beta = x$ and name $x \tan x$ “error function”. We'd like to investigate the properties of this error function.

4.1 First order lumped element approximation

In this case, the equation $x \tan x$ is assumed to be 1 when $x$ is small enough. The lumped element approximation method can be used only when small $x$ value, and the error of lumped element method can be defined as $1-(x \tan x)$. The relative error to the true value is defined as the next equation.

Error (%) = $100 \left\{ 1 - \frac{(x \tan x)}{(x \tan x)} \right\}$

Fig.3 shows the $x$ dependences of error function and Error(%) for the first order approximation. If Error(%) is required with in 5%, $x$ is needed less than 0.37 radian.

4.2 Second order lumped element approximation

In this case, the equation $x \tan x$ is approximated by the next equation.

$$x \tan x = 1-(1/3)x^2$$

Fig.4 shows the $x$ dependences of the equation $x \tan x$ and Error(%) for the second order approximation. Here, the relative error is defined as the next equation.

Error(%) = $100 \left\{ 1 - \frac{(x \tan x)}{(x \tan x)} \right\}$

Despite Taylor expansion under the condition of $x <1$, the permitted region of $x$ for Error(%) required less than 5% increased up to $x=1.04$. This value corresponds to about 3 times wider than the first order approximation. From this result, the second order approximation is obviously much precise.

Next, we’d like to introduce an arbitrary coefficient $A$ to expand the region of $\pm 5\%$. Then the equation $x \tan x$ was supposed as the next equation.

$$x \tan x = 1-(4/3)x^2$$

The influence of coefficient $A$ was investigated. The result is shown in Fig.5. With the increase of $A$, the region of $\pm 5\%$ becomes larger and has the maximum of $x=1.4$ at $A=1.15$. This value corresponds to about 4 times larger than the first order approximation.
5. How to use this approximation

In the above chapter, it is discussed that the second order approximation has the optimum value at $A=1.15$. So $A/3=0.38$, the equivalent circuits of short- and open-circuited transmission lines are expressed in Fig. 6. This approximation can hold up to $x=1.4$ within $\pm 5\%$ error.

The permitted region of $x$ is less than 0.37 for first order approximation and 1.4 for second order approximation. This means the applicable region of the second order approximation increased 4 times compared to the first order approximation.

This implies the upper limit of measurement frequency of permeability and permittivity using short- and open-circuited transmission lines become about 4 times higher. This analytical method enables us to measure permittivity including their frequency variation, how can we treat this issue?

The former example is the permeability measurement case for such as ferrite that is a good insulator. The permeability can be measured using the equivalent circuit of Fig. 6.

If permittivity is known and permeability is unknown, the permeability can be derived using the equivalent circuit of short-circuited line shown in Fig. 6(a). Similarly, the permittivity can be derived using the equivalent circuit of Fig. 6(b) based on the known permeability.

The former example is the permeability measurement case for such as ferrite that is a good insulator. The permeability can be measured using another way as a cavity method based on perturbation approximation. The frequency dependence of permittivity $\varepsilon$ is obtained to be almost constant.

The typical latter example is the permittivity measurement case for non magnetic dielectric materials in which $\mu=1$ always holds.

However, if we can’t know both of permeability and permittivity including their frequency variation, how can we treat this issue?

Here, we’d like to propose the inter-correction method and explain the points taking the example of coaxial lines as shown in Fig. 1 and Fig. 2. First, after loading a toroidal sample into the short-circuited line, the permeability is measured and derived based on $\varepsilon=1$. This is named $\mu_1$. Next, after loading one and the same sample into the open-circuited line, the permittivity is measured and derived based on $\mu=1$. This is named $\varepsilon_1$.

After then, using the data obtained at the first stage of short-circuited line, the permeability named $\mu_2$ is recalculated based on $\varepsilon=\varepsilon_1$. Similarly, the data obtained at the second stage of open-circuited line, the permittivity named $\varepsilon_2$ is recalculated based on $\mu=\mu_1$. By repeating this process, the accuracy will increase. Actually, by only one operation, namely $\mu_2$, $\varepsilon_2$, the considerable results were obtained.

6. Judgment of lumped element approximation

It is necessary to judge if the lumped element approximation proposed in this paper is reasonable or not. One of scaling is the range of $x=\beta l <1.4$ resulting in the above chapter 4.

The expression of $\beta l$ is as follows:

$$\beta l = (\varepsilon \mu)^{1/2}(2\pi/\lambda_o)$$  \hspace{1cm} (16),

where $\lambda_o$ is a free space wave length at measurement frequency $f$.

The permeability and permittivity obtained by this method are assumed to be

$$\mu = \mu' - j\mu''$$  \hspace{1cm} (17)

$$\varepsilon = \varepsilon' - j\varepsilon''$$  \hspace{1cm} (18)

The value of equation (16) is complex but $x$ is real. We can not compare them directly. So the absolute value of equation (16) is adopted to enable us to compare them.

$$|\beta| = (\varepsilon^2 + \mu^2)^{1/2}(2\pi/\lambda_o)$$ \hspace{1cm} (19)

If this value is less than 1.4, it is considered that the obtained result exists within lumped element approximation.

The lumped element approximation discussed here stands implicitly on the assumption that the electro-magnetic field is rather uniform in the medium and can reach the ends of transmission lines. The electro-magnetic field in the materials with high loss and high conductivity possibly can not reach the ends. In this case, before applying this lumped element approximation, it is demanded that the material characteristics such as skin depth and the optimum length (thickness) $l$ should be investigated.

7. Conclusion

We theoretically revealed that a short-circuited and an open-circuited transmission lines are modeled as a $LC$ parallel and $LC$ series circuits based on the lumped element approximation, respectively. We also verified that this approximation can hold within $\pm 5\%$ up to 1.4 radian phase shift by balancing the quantities of $L_o$ and $C_o$.

This approximation allows us up to 4 times phase shift larger compared to conventional simple lumped element approximation.

Using this method, the inter-correctional measurement method between permeability and permittivity are proposed.

There are two factors to determine the upper limit of measurement frequency in this method. The one is the sample length $l$, the another is the quantities of...
permeability and permittivity. The smaller these values are, the higher the upper limit measurement is. One of the upper measurement frequency expected is expressed as $f < \frac{c}{4\sqrt{\varepsilon\mu}}$ (c : light speed, $\mu$ and $\varepsilon$ : permeability and permittivity of air).

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