

Research Article

Weighted Composition Operators from Derivative Hardy Spaces into \(n\)-th Weighted-Type Spaces

Nanhuí Hu\(^{1,2}\)

\(^1\)Department of Mathematics, Shantou University, Shantou 515063, Guangdong, China
\(^2\)Department of Mathematics, Jiaying University, Meizhou 514015, Guangdong, China

Correspondence should be addressed to Nanhuí Hu; hunhiju@163.com

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The boundedness, compactness, and the essential norm of weighted composition operators from derivative Hardy spaces into \(n\)-th weighted-type spaces are investigated in this paper.

1. Introduction

Let \(H(D)\) denote the space of analytic functions on the open unit disk \(D\). Let \(S(D)\) denote the set of all analytic self-maps of \(D\). Let \(\varphi \in S(D)\). The composition operator \(C_\varphi\) with the symbol \(\varphi\) is defined by

\[
(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(D).
\]

(1)

Let \(\varphi \in S(D)\) and \(\psi \in H(D)\). The weighted composition operator \(\psi C_\varphi\) is defined on \(H(D)\) by

\[
(\psi C_\varphi f)(z) = \psi(z)f(\varphi(z)), \quad f \in H(D).
\]

(2)

It is important to give function theoretic descriptions when \(\psi\) and \(\varphi\) induce a bounded or compact weighted composition operator on various function spaces. See references \([1, 2]\) for more information of this research field.

For \(0 < p < \infty\), the Hardy space, denoted by \(H^p\), consists of all functions \(f \in H(D)\) such that (see \([3]\))

\[
\|f\|_{H^p} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
\]

(3)

As usual, \(B^\infty\) denotes the space of bounded analytic functions in \(D\). If \(f' \in H^p\), we say that \(f\) belongs to the derivative Hardy space, denoted by \(\delta^p\). When \(p > 1\), the space \(\delta^p\) is a Banach space under the norm defined by

\[
\|f\|_{\delta^p} = |f(0)|^p + \|f'\|_{H^p}^p.
\]

(4)

When \(p = 2\), \(\delta^2\) is a Hilbert space. In \([4]\), Roan started the study of composition operators on the space \(\delta^p\). In \([5]\), MacCluer investigated composition operators on the space \(\delta^p\) in terms of Carleson measure. The boundedness and compactness of weighted composition operators on \(\delta^p\) were studied in \([6]\). See references \([4–7]\) and references therein for more study of composition operators and weighted composition operators on the space \(\delta^p\).

If \(\mu\) is a radial, positive, and continuous function on \(D\), then \(\mu\) is called a radial weight. Let \(\mu\) be a radial weight and \(n \in \mathbb{N}\), the set of all positive integers. Let \(W^n\) denote the \(n\)-th weighted space, which consists of all \(f \in H(D)\) such that

\[
\|f\|_{W^n} = \sum_{k=0}^{n-1} \left| f^{(k)}(0) \right| + \sup_{z \in D} |f(z)|^n < \infty.
\]

(5)

It is a Banach space with the norm \(\| \cdot \|_{W^n}\). When \(n = 1\) and \(n = 2\), \(W^n\) becomes the Bloch-type space \(B^\mu\), and the Zygmund-type space \(\mathcal{Z}^\mu\), respectively. Furthermore, when \(\mu(z) = (1 - |z|^2)^\alpha\), \(B^\mu = B\) is the Bloch space and \(\mathcal{Z}^\mu = \mathcal{Z}\) is the Zygmund space. For some results on the space \(W^n\), see references \([2, 8–13]\).

Let \(n, k \in \mathbb{N}_0\) with \(k \leq n\). Recall that the partial Bell polynomials are defined as follows:
where the sum taken over all sequences \( j_1, j_2, \ldots, j_{n-k+1} \) of nonnegative integers such that the following two conditions hold:

\[
j_1 + j_2 + \cdots + j_{n-k+1} = k
\]
\[
j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.
\]

See reference [14] for more information about Bell polynomials.

In [15], Stević studied the boundedness and compactness of the composition operator from \( A^p_\mu \) to \( \mathcal{W}^n_\mu \) on the unit disk. In [12], Stević studied the boundedness and compactness of weighted composition operators from derivative Hardy spaces \( \mathcal{H}^p_\mu \) to \( \mathcal{B} \). See references [8–10] for more characterizations for weighted composition operators from \( \mathcal{H}^p_\mu \) and the Bloch space to \( \mathcal{W}^n_\mu \). In [16], Zhu and Du studied the boundedness, compactness, and essential norm of weighted composition operators from weighted Bergman spaces with doubling weight \( A^p_\mu \) to \( \mathcal{W}^n_\mu \). Recall that the essential norm of a bounded linear operator \( T \colon X \rightarrow Y \) is its distance to the set of compact operators \( K \) mapping \( X \) into \( Y \), that is,

\[
\|T\|_{e,X \rightarrow Y} = \inf\{\|T - K\|_{X \rightarrow Y} : K \text{ is a compact operator}\}.
\]

In this section, the boundedness of weighted composition operators from \( \delta^p \) to \( \mathcal{W}^n_\mu \) is characterized.

\[2. \text{ Boundedness}\]

In this section, the boundedness of weighted composition operators from \( \delta^p \) to \( \mathcal{W}^n_\mu \) is characterized.

\[\text{Lemma 1.} \quad \text{Suppose} \ 1 < p < \infty \text{ and } k \in \mathbb{N}. \text{ Then, there exists a positive constant } C \text{ such that}
\]

\[
\|f\|_{\mathcal{B}} \leq C\|f\|_{\delta^p},
\]

\[
\left|f^{(k)}(z)\right| \leq \frac{C\|f\|_{\delta^p}}{(1 - |z|^2)^{k-1-\epsilon(1/p)}},
\]

for every \( f \in \delta^p \).

\[\text{Proof.} \quad \text{The first inequality follows from the fact that } \delta^p \text{ are contained in the disk algebra for } p > 1. \text{ In addition, it is well known that for every } f \in \mathcal{H}^p_\mu, \text{ there exists a positive constant } C \text{ such that}
\]

\[
\left|f^{(k)}(z)\right| \leq \frac{C\|f\|_{\mathcal{H}^p_\mu}}{(1 - |z|^2)^{k-\epsilon(1/p)}},
\]

which implies the second inequality.

For any \( a \in \mathbb{D}, \ 1 < p < \infty, \text{ and } j \in \{1, 2, \ldots, n+1\}, \text{ set}

\[
f_{j,a}(z) = \frac{(1 - |a|^2)^{j}}{(1 - az)^{j+1/(1/p)}}, \quad z \in \mathbb{D}.
\]

After a calculation, for each \( j \in \{1, 2, \ldots, n+1\} \),

\[
\sup_{\alpha \in \mathbb{D}} \|f_{j,a}\|_{\delta^p} < \infty.
\]

\[\text{Lemma 2.} \quad \text{Let} \ 1 < p < \infty \text{ and } 0 \neq a \in \mathbb{D}. \quad \text{For any} \ i \in \{1, \ldots, n\}, \text{ there exist constants } c^1_{i2}, \ldots, c^1_{i(n+1)} \text{ such that}
\]

\[
v_{i,a} = f_{i,a} + \sum_{j=2}^{i} c^j_{i} f_{j,a} \in \delta^p,
\]

\[
v_{i,a}(a) = 0,
\]

\[
v^{(k)}_{i,a}(a) = \begin{cases} 
\frac{\alpha}{(1 - |a|^2)^{j+1/(1/p)-1}}, & k = i, \\
0, & k \neq i.
\end{cases}
\]

Moreover, \( v_{i,a} \) converges to 0 uniformly in \( \mathbb{D} \).
Hence, we omit the details of the proof. For the simplicity of this paper, we define

$$I^n_t(z) = \frac{n}{\sum_{i=1}^n \left( \begin{array}{c} n \cr l \end{array} \right)} \psi(z)B_{l,i}(\psi'(z), \psi''(z), \ldots, \psi^{(l-1)}(z)).$$

(14)

Proof. From the definition, we see that, for example,

$$B_{0,0}(x_1) = 1, \quad B_{1,0}(x_1, x_2) = 0, \quad B_{1,1}(x_1) = x_1, \quad B_{2,0}(x_1, x_2, x_3) = 0, \quad B_{2,1}(x_1, x_2) = x_2, \quad B_{2,2}(x_1) = x_1^2.$$ (15)

Proof (sufficiency). Let \( f \in \mathcal{S}^p \). After a calculation, we have (see, e.g., [12])

$$\left( \psi C_{\mu} f \right)^{(n)}(z) = \sum_{i=0}^n f^{(i)}(\psi(z)) \sum_{l=0}^n \left( \begin{array}{c} n \cr l \end{array} \right) \psi^{(n-l)}(z)B_{l,i}(\psi'(z), \psi''(z), \ldots, \psi^{(l-1)}(z)).$$ (17)

From (18), for each \( j \in \{0, 1, \ldots, n - 1\}, \)

This fact will be used in the proof of the following theorem.

Now, we are in a position to state and prove the first result in this paper. □

**Theorem 1.** Let \( n \in \mathbb{N}, 1 < p < \infty, \varphi \in S(\mathbb{D}), \psi \in H(\mathbb{D}) \), and \( \mu \) be a weight. Then, the operator \( \psi C_{\mu} : \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n \) is bounded if and only if \( \psi \in \mathcal{W}_\mu^n \) and

$$\mu(z) \left| \sum_{i=1}^n \left( \begin{array}{c} n \cr l \end{array} \right) \psi^{(n-l)}(z)B_{l,i}(\psi'(z), \psi''(z), \ldots, \psi^{(l-1)}(z)) \right| < \infty.$$ (16)

$$\left( \psi C_{\mu} f \right)^{(n)}(z) \leq \mu(z)\left| \sum_{i=1}^n f^{(i)}(\psi(z))\left| I^n_t(z) \right| \right| = \mu(z) \left| \sum_{i=1}^n f^{(i)}(\psi(z)) \right| I^n_t(z),$$

$$\leq\| f \|_{\mathcal{S}^p} \| \psi \|_{\mathcal{W}_\mu^n} + \| f \|_{\mathcal{S}^p} \left( \sum_{l=1}^n \sup_{z \in \mathbb{D}} \frac{\mu(z)\left| I^n_t(z) \right|}{\left( 1 - |\psi(z)|^2 \right)^{1/(1/p) - 1}} \right) \left| I^n_0(z) \right|,$$

$$\left( \psi C_{\mu} f \right)^{(n)}(0) \leq |f(\psi(0))\psi^{(j)}(0)| + \sum_{i=1}^n |f^{(i)}(\psi(0))| I^n_t(0),$$

$$\leq\| f \|_{\mathcal{S}^p} \| \psi \|_{\mathcal{W}_\mu^n} + \| f \|_{\mathcal{S}^p} \left( \sum_{i=1}^n \left| I^n_t(0) \right| \right) \left( 1 - |\psi(0)|^2 \right)^{1/(1/p) - 1}$$

(20)
Therefore, from (19) and (20) and the fact that \( \psi \in \mathcal{H}^{n}_\mu \), we get that \( \psi C_{\psi} : \delta^p \rightarrow \mathcal{H}^n_\mu \) is bounded.

**Necessity.** Assume that \( \psi C_{\psi} : \delta^p \rightarrow \mathcal{H}^n_\mu \) is bounded. It is clear that \( \psi \in \mathcal{H}^n_\mu \), and for each \( j \in \{1, \ldots, n+1\} \) and \( a \in \mathbb{D} \),

\[
\sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_j a \|_{\mathcal{H}^n_\mu} < \infty, \quad (21)
\]

by the boundedness of \( \psi C_{\psi} : \delta^p \rightarrow \mathcal{H}^n_\mu \).

Next, we show that for \( i = 1, 2, \ldots, n \),

\[
\sup_{z \in \mathbb{D}} \mu(z) I^n_i(z) | < \infty. \quad (22)
\]

Applying the operator \( \psi C_{\psi} \) for \( h_1(z) = z \), by (17) and (14), we obtain

\[
\sup_{z \in \mathbb{D}} \mu(z) | I^n_0(z) \psi(z) + I^n_1(z) \rangle = \sup_{z \in \mathbb{D}} \mu(z) \left( (\psi C_{\psi} h_1)^{(n)}(z) \right) \leq \| \psi C_{\psi} h_1 \|_{\mathcal{H}^n_\mu} < \infty. \quad (23)
\]

Therefore, by the triangle inequality and the boundedness of \( \psi \), we obtain

\[
\sup_{z \in \mathbb{D}} \mu(z) | I^n_i(z) \rangle < \infty. \quad (24)
\]

Now, we assume that for \( 1 \leq i < j \leq (j \leq n) \), \( \sup_{z \in \mathbb{D}} \mu(z) | I^n_i(z) \rangle < \infty \). To get the desired result, we only need to show that

\[
\sup_{z \in \mathbb{D}} \mu(z) | I^n_j(z) \rangle < \infty. \quad (25)
\]

Applying the operator \( \psi C_{\psi} \) for \( h_j(z) = z^j \), we obtain

\[
\sup_{z \in \mathbb{D}} \mu(z) \left| \psi(z) I^n_0(z) + \sum_{k=1}^{j} j (j-1), \ldots, (j-k+1) \psi(z) I^n_k(z) \right| \leq \| \psi C_{\psi} h_j \|_{\mathcal{H}^n_\mu} < \infty. \quad (26)
\]

Hence, from the boundedness of \( \varphi \) and triangle inequality again, we get the desired result.

\[
\frac{\mu(a) | \psi(a) | I^n(a)}{(1 - | \psi(a) |^2)^{(1/p)-1}} \leq \sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_{j,a} \|_{\mathcal{H}^n_\mu} \leq \sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_{j,a} \|_{\mathcal{H}^n_\mu} \leq \sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_{j,a} \|_{\mathcal{H}^n_\mu} < \infty, \quad (27)
\]

where \( c_j \) are independent of the choice of \( a \). From the last inequality and (22), for any \( i \in \{1, \ldots, n\} \), we get

\[
\sup_{|\psi(a)\| \leq (1/2)} \frac{\mu(a) | I^n_i(a) |}{(1 - | \psi(a) |^2)^{(1/p)-1}} \leq \sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_{j,a} \|_{\mathcal{H}^n_\mu} + \sum_{j=2}^{n+1} | c_j | \sup_{z \in \mathbb{D}} \| \psi C_{\psi} f_{j,a} \|_{\mathcal{H}^n_\mu} < \infty, \quad (28)
\]

Therefore, for any \( i \in \{1, \ldots, n\} \) and \( \psi(a) \neq 0 \), by Lemma 2,

\[
\frac{\mu(a) | \psi(a) | I^n_i(a)}{(1 - | \psi(a) |^2)^{(1/p)-1}} < \infty, \quad (29)
\]

The proof is complete.

Let \( n = 1 \). We get the following result, which was first obtained in [17]. \square

**Corollary 1.** Let \( 1 < p < \infty \), \( \varphi \in S(\mathbb{D}) \), \( \psi \in H(\mathbb{D}) \), and \( \mu \) be a weight. Assume that \( \psi C_{\psi} : \delta^p \rightarrow \mathcal{B}_\mu \) is bounded if and only if \( \psi \in \mathcal{R}_\mu \) and

\[
\sup_{z \in \mathbb{D}} \frac{\mu(z) | \psi(z) \psi'(z) |}{(1 - | \psi(z) |^2)^{(1/p)} < \infty}. \quad (30)
\]
Let \( n = 2 \). We get the following result.

**Corollary 2.** Let \( 1 < p < \infty \), \( \varphi \in \mathcal{S}(\mathbb{D}) \), \( \psi \in H(\mathbb{D}) \), and \( \mu \) be a weight. Then, the operator \( \psi C_{\varphi} : \mathcal{D}^p \to \mathcal{X}_\mu \) is bounded if and only if \( \psi \in \mathcal{X}_\mu \).

\[
\sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi(z)|^2 |\psi(z)|^2 + 2 \varphi'(z) \psi'(z)}{(1 - |\varphi(z)|^2)^{(1/p)}} < \infty.
\]

(31)

### 3. Essential Norm

In this section, we obtain some estimates for the essential norm of the weighted composition operator \( \psi C_{\varphi} : \mathcal{D}^p \to \mathcal{W}_\mu \), and we need the following lemma.

**Lemma 3** (see [17]). Let \( X \) be a Banach space that is continuously contained in the disk algebra, and let \( Y \) be any Banach space of analytic functions on \( \mathbb{D} \). Suppose that

1. The point evaluation functionals on \( Y \) are continuous.
2. For every sequence \( \{f_n\} \) in the unit ball of \( X \), there exists \( f \in X \) and a subsequence \( \{f_{n_j}\} \) such that \( f_{n_j} \to f \) uniformly on \( \overline{\mathbb{D}} \).
3. The operator \( T : X \to Y \) is continuous if \( X \) has the topology of uniform convergence on compact sets

Then, \( T \) is a compact operator if and only if, given a bounded sequence \( \{f_n\} \) in \( X \) such that \( f_n \to 0 \) uniformly on \( \overline{\mathbb{D}} \), then \( \|Tf_n\|_Y \to 0 \) as \( n \to \infty \).

**Lemma 4** (see [17]). Let \( 1 < p < \infty \). Every sequence in \( \mathcal{D}^p \) bounded in norm has a subsequence which converges uniformly in \( \mathbb{D} \) to a function in \( \mathcal{D}^p \).

The following result is a direct consequence of Lemma 3 and Lemma 4.

**Lemma 5.** Let \( n \in \mathbb{N}, 1 < p < \infty \), \( \psi \in \mathcal{S}(\mathbb{D}) \), \( \varphi \in H(\mathbb{D}) \), and \( \mu \) be a weight such that \( \psi C_{\varphi} : \mathcal{D}^p \to \mathcal{W}_\mu \) is bounded. Then,

\[
\|\psi C_{\varphi}\|_{\mathcal{E}, \mathcal{D}^p} \to \mathcal{W}_\mu \approx \sum_{j=1}^{n} \limsup_{\varphi(z) \to 0} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{(1/p)-1}}.
\]

(32)

Proof. First, we prove that

\[
\sum_{i=1}^{n} \limsup_{\varphi(z) \to 0} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{(1/p)-1}} \leq \|\psi C_{\varphi}\|_{\mathcal{E}, \mathcal{D}^p} \to \mathcal{W}_\mu
\]

(33)

If \( \max_{z \in \mathbb{D}} |\varphi(z)| < 1 \), there is a number \( \delta \in (0, 1) \) such that \( \max_{z \in \mathbb{D}} |\varphi(z)| < \delta \). In this case, (32) is vacuously satisfied and the desired result follows.

Assume that \( \max_{z \in \mathbb{D}} |\varphi(z)| = 1 \). Let \( \{z_j\}_{j \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \). Since \( \psi C_{\varphi} : \mathcal{D}^p \to \mathcal{W}_\mu \) is bounded, for any compact operator \( K : \mathcal{D}^p \to \mathcal{W}_\mu \) and \( i \in \{1, \ldots, n\} \), by using Lemma 2 and Lemma 5, we obtain

\[
\|\psi C_{\varphi} - K\|_{\mathcal{D}^p} \to \mathcal{W}_\mu \geq \limsup_{j \to \infty} \|\psi C_{\varphi_{i,j}}(z_j)\|_{\mathcal{W}_\mu} - \limsup_{j \to \infty} \|K_{i,j}(z_j)\|_{\mathcal{W}_\mu}
\]

(34)

\[
\geq \limsup_{j \to \infty} \frac{\mu(z_j) |\varphi'(z_j)| |\varphi'(z_j)|}{(1 - |\varphi(z_j)|^2)^{(1/p)-1}}.
\]

Hence,

\[
\|\psi C_{\varphi}\|_{\mathcal{E}, \mathcal{D}^p} \to \mathcal{W}_\mu \geq \limsup_{j \to \infty} \frac{\mu(z_j) |\varphi'(z_j)| |\varphi'(z_j)|}{(1 - |\varphi(z_j)|^2)^{(1/p)-1}}.
\]

(35)

which implies that

\[
\|\psi C_{\varphi}\|_{\mathcal{E}, \mathcal{D}^p} \to \mathcal{W}_\mu \geq \limsup_{\varphi(z) \to 0} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{(1/p)-1}}.
\]

(36)

as desired.

Now, we show that
\[ \|\psi_{C_i}\|_{L^p(D)} - \|\psi_{C_n}\|_{L^p(D)} \leq \sum_{i=1}^{n} \limsup_{\|\varphi(z)\| \rightarrow \infty} \frac{\mu(z)|I_i^p(z)|}{(1 - |\varphi(z)|)^{1/(1/p) - 1}}. \]  

Let \( r \in [0, 1) \) and define \( K_r f(z) = f(rz) \). Then, \( K_r : \mathcal{D}^p \rightarrow \mathcal{D}^p \) is compact and \( \|K_r\|_{\mathcal{D}^p \rightarrow \mathcal{D}^p} \leq 1 \). It is clear that \( f_r \rightarrow f \) uniformly on compact subsets of \( \mathbb{D} \) as \( r \rightarrow 1 \). Let \( \{r_j\} \subset (0, 1) \) be a sequence such that \( r_j \rightarrow 1 \) as \( j \rightarrow \infty \). Then, for any positive integer \( j \), the operator \( \psi_{C_rK_{r_j}} : \mathcal{D}^p \rightarrow \mathcal{W}_p^n \) is compact. By the definition of the essential norm, we get

\[ \|\psi_{C_rK_{r_j}}\|_{\mathcal{D}^p \rightarrow \mathcal{W}_p^n} \leq \limsup_{j \rightarrow \infty} \|\psi_{C_r} - \psi_{C_rK_{r_j}}\|_{\mathcal{D}^p \rightarrow \mathcal{W}_p^n}. \]  

Hence, it is sufficient to show that

\[ \limsup_{j \rightarrow \infty} \|\psi_{C_r} - \psi_{C_rK_{r_j}}\|_{\mathcal{D}^p \rightarrow \mathcal{W}_p^n} \leq \sum_{i=1}^{n} \limsup_{\|\varphi(z)\| \rightarrow \infty} \frac{\mu(z)|I_i^p(z)|}{(1 - |\varphi(z)|)^{1/(1/p) - 1}}. \]  

For any \( f \in \mathcal{D}^p \) such that \( \|f\|_{\mathcal{D}^p} \leq 1 \),

\[ \left\| \left( \psi_{C_r} - \psi_{C_rK_{r_j}} \right) f \right\|_{\mathcal{W}_p^n} \]
\[ \leq \sum_{i=0}^{n-1} \left[ \left( f - f_{r_j} \right)^{(i)} (\varphi(0)) \right] L_i^p(0) + \sup_{\|\varphi(z)\| \rightarrow \infty} \mu(z) \sum_{i=1}^{n} \left[ \left( f - f_{r_j} \right)^{(i)} (\varphi(z)) \right] L_i^p(0) \]
\[ \leq \sum_{i=0}^{n-1} \left[ \left( f - f_{r_j} \right)^{(i)} (\varphi(0)) \right] L_i^p(0) + \left( \sup_{\|\varphi(z)\| \rightarrow \infty} \mu(z) \right) \left( f - f_{r_j} \right)(z) \left( L_i^p(z) \right) \]
\[ + \sup_{\|\varphi(z)\| \rightarrow \infty} \mu(z) \sum_{i=1}^{n} \left[ \left( f - f_{r_j} \right)^{(i)} (\varphi(z)) \right] L_i^p(z), \]

where \( N \in \mathbb{N} \) such that \( r_j \geq (2/3) \) for all \( j \geq N \). Since for any nonnegative integer \( s \), \( (f - f_{r_j})^{(i)} \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( j \rightarrow \infty \). It is clear that

\[ \limsup_{j \rightarrow \infty} \Omega_0 = 0 \]
\[ \limsup_{j \rightarrow \infty} \Omega_2 = 0. \]  

From Lemma 4,

\[ \lim_{j \rightarrow \infty} \Omega_1 \leq \|\psi\|_{\mathcal{W}_p^n} \lim_{j \rightarrow \infty} \sup_{\|\varphi(z)\| \rightarrow \infty} \left( f - f_{r_j} \right)(z) = 0, \]  

while

\[ \Omega_3 \leq \sum_{i=1}^{n} \sup_{\|\varphi(z)\| > r_0} \mu(z) \left( f^{(i)}(\varphi(z)) \right) L_i^p(z) p_j + \sum_{i=1}^{n} \sup_{\|\varphi(z)\| > r_0} \mu(z) \left( r_j^{(i)}(\varphi(z)) \right) L_i^p(z) Q_i. \]  

For any \( i \in \{1, \ldots, n\} \), by Lemma 1 and Lemma 2,
Let $n \in \mathbb{N}$, $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and $\mu$ be a weight. Assume that $\psi C_\varphi : \mathbb{D}^p \to \mathbb{H}^n_\mu$ is bounded, then the operator $\psi C_\varphi : \mathbb{D}^p \to \mathbb{H}^n_\mu$ is compact if and only if
\begin{equation}
\sum_{i=1}^{n} \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |f(z)|_{\mathbb{H}^n_\mu}}{(1 - |\varphi(z)|^2)^{n(1/p) - 1}} = 0.
\end{equation}

Let $n = 1$. We get the following result, which was first obtained in [17].

**Corollary 4.** Let $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and $\mu$ be a weight such that $\psi C_\varphi : \mathbb{D}^p \to \mathbb{H}^1_\mu$ is bounded. Then, the operator $\psi C_\varphi : \mathbb{D}^p \to \mathbb{H}^1_\mu$ is compact if and only if
\begin{equation}
\limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |\psi(z)|^2}{(1 - |\varphi(z)|^2)^{1/(1/p)}} = 0.
\end{equation}

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**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest.
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