Sphere-Packing Bound for Symmetric Classical-Quantum Channels

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Abstract. We provide a sphere-packing lower bound for the optimal error probability in finite blocklengths when coding over a symmetric classical-quantum channel. Our result shows that the pre-factor can be significantly improved from the order of the subexponential to the polynomial. This established pre-factor is essentially optimal because it matches the best known random coding upper bound in the classical case. Our approaches rely on a sharp concentration inequality in strong large deviation theory and crucial properties of the error-exponent function.

1. Introduction

The probability of decoding error is one of the fundamental criteria for evaluating the performance of a communication system. In Shannon’s seminal work \cite{Shannon1}, he pioneered the study of the noisy coding theorem, which states that the error probability can be made arbitrarily small as the coding blocklength grows when the coding rate $R$ is below the channel capacity $C$. Later, Shannon \cite{Shannon2} made a further step in exploring the exponential dependency of the optimal error probability $\epsilon^*(n, R)$ on the blocklength $n$ and rate $R$, and defined the reliability function as follows: given a fixed coding rate $R < C$, $E(R) := \limsup_{n \to +\infty} \frac{-1}{n} \log \epsilon^*(n, R)$. The quantity $E(R)$ then provides a measure of how rapidly the error probability approaches zero with an increase in blocklength. This asymptotic characterization of the optimal error probability under a fixed rate is hence called the error exponent analysis. For a classical channel, the upper bounds of the optimal error can be established using a random coding argument \cite{Gallager}. On the other hand, the lower bound was first developed by Shannon, Gallager, and Berlekamp \cite{Shannon3} and was called the sphere-packing bound. Alternative approaches by Haroutunian \cite{Haroutunian} and Blahut \cite{Blahut} were subsequently proposed.

In recent years, much attention has been paid to the finite blocklength regime \cite{Altug,Dalai}. Altuğ and Wagner employed strong large deviation techniques \cite{Altug2} to prove a sphere-packing bound with a finite blocklength $n$. Moreover, the pre-factor of the bound was significantly refined from the order of the subexponential $\exp\{O(\sqrt{n})\}$ \cite{Shannon3} to the polynomial \cite{Altug,Altug1}. This refinement is substantial especially at rates near capacity, where the error-exponent function is close to zero; hence, the pre-factor dominates the bound \cite{Altug2,Altug1}.

Error exponent analysis in classical-quantum (c-q) channels is much more difficult because of the noncommutative nature of quantum mechanics. Burnashev and Holevo \cite{Burnashev,Holevo} investigated reliability functions in c-q channels and proved the random coding upper bound for pure-state channels. Winter \cite{Winter} adopted Haroutunian’s method to derive a sphere-packing bound for c-q channels in the form of relative entropy functions \cite{Haroutunian}. Dalai \cite{Dalai} employed Shannon-Gallager-Berlekamp’s approach to establish a sphere-packing bound with Gallager’s expression \cite{Shannon3}. It was later pointed out that these two sphere-packing exponents are not equal for general c-q channels \cite{Winter1}. In this work, we initiate the study of the refined sphere-packing bound in the quantum scenario. In particular, we consider a “symmetric c-q channel” (see Section 2 for a detailed definition), which is an important class of covariant channels (e.g. \cite{Winter2}), and establish a sphere-packing bound with the pre-factor improved from the order of the subexponential...
in Dalai’s result [18] to the polynomial. Our result recovers Altu˘g and Wagner’s work [10] for classical symmetric channels including the binary symmetric channel and binary erasure channel. Furthermore, the proved pre-factor matches that of the best known random coding upper bound [21] in the classical case. Hence, our result yields the exact asymptotics for the sphere-packing bound in symmetric c-q channels. The main ingredients in our proof are a tight concentration inequality from Bahadur and Ranga Rao [9], [13] (see Appendix A) and the major properties of the sphere-packing exponent [22]. We remark that the result obtained in this paper might enable analysis in the medium error probability regime of a classical-quantum channel [12, 13, 14]. We leave the case for general c-q channels as future work [23].

This paper is organized as follows. We introduce the necessary notation and state our main result in Section 2. Section 3 includes the crucial properties of the error-exponent function. We provide the proof of the main result in Section 4. Section 5 concludes this paper.

2. NOTATION AND MAIN RESULT

2.1. Notation. Throughout this paper, we consider a finite-dimensional Hilbert space \( \mathcal{H} \). The set of density operators (i.e. positive semi-definite operators with unit trace) on \( \mathcal{H} \) are denoted as \( \mathcal{S}(\mathcal{H}) \). For \( \rho, \sigma \in \mathcal{S}(\mathcal{H}) \), we write \( \rho \ll \sigma \) if \( \text{supp}(\rho) \subset \text{supp}(\sigma) \), where \( \text{supp}(\rho) \) denotes the support of \( \rho \). The identity operator on \( \mathcal{H} \) is denoted by \( \mathbb{1}_\mathcal{H} \). When there is no possibility of confusion, we skip the subscript \( \mathcal{H} \) for the trace function. Let \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{R}_{>0} \) denote the set of integers, real numbers, and positive real numbers, respectively. Define \( [n] := \{1, 2, \ldots, n\} \) for \( n \in \mathbb{N} \). Given a pair of positive semi-definite operators \( \rho, \sigma \in \mathcal{S}(\mathcal{H}) \), we define the (quantum) relative entropy as \( D(\rho||\sigma) := \text{Tr} [\rho \log \rho - \log \sigma] \), where \( \rho \ll \sigma \), and \( +\infty \) otherwise. For every \( \alpha \in [0, 1] \), we define the (Petz) quantum Rényi divergences \( D_\alpha(\rho||\sigma) := \frac{1}{1-\alpha} \log \text{Tr} [\rho^\alpha \sigma^{1-\alpha}] \). For \( \alpha = 1 \), \( D_1(\rho||\sigma) := \lim_{\alpha \to 1} D_\alpha(\rho||\sigma) = D(\rho||\sigma) \). Let \( \mathcal{X} := \{1, 2, \ldots, |\mathcal{X}|\} \) be a finite alphabet, and let \( \mathcal{P}(\mathcal{X}) \) be the set of probability distributions on \( \mathcal{X} \). In particular, we denote by \( \mathcal{U}_\mathcal{X} \) the uniform distribution on \( \mathcal{X} \). A classical-quantum (c-q) channel \( \mathcal{W} \) maps elements of the finite set \( \mathcal{X} \) to the density operators in \( \mathcal{S}(\mathcal{H}) \), i.e., \( \mathcal{W} : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \). Let \( \mathcal{M} \) be a finite alphabetical set with size \( |\mathcal{M}| = |\mathcal{X}| \). An (n-block) encoder is a map \( f_n : \mathcal{M} \to \mathcal{X}^n \) that encodes each message \( m \in \mathcal{M} \) to a codeword \( x^n(m) := x_1(m) \cdots x_n(m) \in \mathcal{X}^n \). The codeword \( x^n(m) \) is then mapped to a state \( \mathcal{W}_{x^n(m)} = W_{x_1(m)} \otimes \cdots \otimes W_{x_n(m)} \in \mathcal{S}(\mathcal{H}^\otimes n) \). The decoder is described by a positive operator-valued measurement (POVM) \( \Pi_n = \{\Pi_{0,1}, \ldots, \Pi_{0,M}\} \) on \( \mathcal{H}^\otimes n \), where \( \Pi_{n,i} \geq 0 \) and \( \sum_{i=1}^M \Pi_{n,i} = \mathbb{1} \). The pair \( (f_n, \Pi_n) =: C_n \) is called a code with rate \( R = \frac{1}{n} \log |\mathcal{M}| \). The error probability of sending a message \( m \) with the code \( C_n \) is \( \epsilon_n(W; C_n) := 1 - \text{Tr} (\Pi_{n,m} \mathcal{W}_{x^n(m)}) \). We use \( \epsilon_{\max}(W; C_n) = \max_{m \in \mathcal{M}} \epsilon_n(W; C_n) \) and \( \bar{\epsilon}(W; C_n) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \epsilon_n(W; C_n) \) to denote the maximal error probability and the average error probability, respectively. Given a sequence \( x^n \in \mathcal{X}^n \), we denote by \( P_{x^n}(x) := \frac{1}{n} \sum_{i=1}^n 1 \{x = x_i\} \) the empirical distribution of \( x^n \).

Throughout this paper, we consider a symmetric c-q channel defined as

\[
W_x := V^{x-1} W_1 (V^\dagger)^{x-1}, \quad \forall x \in \mathcal{X},
\]

where \( W_1 \in \mathcal{S}(\mathcal{H}) \) is an arbitrary density operator, and \( V \) satisfies \( V^\dagger V = V V^\dagger = V^{\mathcal{X}} = \mathbb{1}_\mathcal{H} \). We define the following conditional entropic quantities for the channel \( W \) with \( P \in \mathcal{P}(\mathcal{X}) \):

\[
D_\alpha(W||\sigma |P) := \sum_{x \in \mathcal{X}} P(x) D_\alpha(W_x || P),
\]

the mutual information of the c-q channel \( W : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \) with prior distribution \( P \in \mathcal{P}(\mathcal{X}) \) is defined as \( I(P; W) := D(W|| \mathbb{1} P), \) where \( \mathbb{1} P := \sum_{x \in \mathcal{X}} P(x) W_x \). The (classical) capacity of the channel \( W : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \) is denoted by \( C := \max_{P \in \mathcal{P}(\mathcal{X})} I(P; W) \). Let

\[
E^{(1)}_{sp}(R, P) := \sup_{s \geq 0} \{ E_0(s, P) - s R \}
\]

\[
E^{(2)}_{sp}(R, P) := \sup_{0 < \alpha < 1} \min_{\sigma \in \mathcal{S}(\mathcal{H})} \frac{\alpha - 1}{\alpha} \left( R - D_\alpha(W||P \sigma |P) \right),
\]

where we denote by \( E_0(s, P) := -\log \text{Tr} \left((PW^{1/(1+s)})^{1+s}\right) \) an auxiliary function [16, 22]. The sphere-packing exponent is defined by

\[
E_{sp}(R) := \max_{P \in \mathcal{P}(\mathcal{X})} \frac{E^{(1)}_{sp}(R, P)}{E^{(2)}_{sp}(R, P)},
\]

\[
E^{(1)}_{sp}(R, P) = \max_{P \in \mathcal{P}(\mathcal{X})} E^{(2)}_{sp}(R, P),
\]

(2)
where the last equality follows from [24, Proposition IV.2]. Further, we define a rate [25, p. 152], [18]:

\[
R_{\infty} := \lim_{s \to +\infty} \max_{P \in \mathcal{P}(X)} \min_{\sigma \in S(H)} \sum_{x \in X} P(x) D_{\frac{1}{s}}(W_{x} \| \sigma)
\]

\[
= \max_{P \in \mathcal{P}(X)} \min_{\sigma \in S(H)} \sum_{x \in X} P(x) \text{Tr} \left[ W_{x}^{0} \sigma \right].
\]

(3)

It follows that \(E_{\text{sp}}(R) = +\infty\) for any \(R \leq R_{\infty}\) (see also [4, p. 69] and [3, Eq. (5.8.5)]).

Consider a binary hypothesis whose null and alternative hypotheses are \(\rho, \sigma \in S(H)\), respectively. The type-I error and type-II error of the hypothesis testing, for an operator \(0 \leq Q \leq 1\), are defined as \(\alpha(Q; \rho) := \text{Tr}[(I - Q) \rho]\), and \(\beta(Q; \sigma) := \text{Tr}[Q \sigma]\). There is a trade-off between these two errors. Thus, we can define the minimum type-I error, when the type-II error is below \(\mu \in (0,1)\), as

\[
\hat{\alpha}_{\mu}(\rho \| \sigma) := \min_{0 \leq Q \leq 1} \{ \alpha(Q; \rho) : \beta(Q; \sigma) \leq \mu \}.
\]

(4)

2.2. Main Result. Let us now consider any symmetric c-q channel with capacity \(C\).

**Theorem 1** (Exact Sphere-Packing Bound). For any rate \(R \in [0,C]\), there exists an \(N_{0} \in \mathbb{N}\) such that for all codes \(C_{n}\) of length \(n \geq N_{0}\), we have

\[
\epsilon_{\text{max}}(C_{n}) \geq \frac{1 - o(1)}{n^{2} |E_{\text{sp}}(R)|} \exp \left\{ -n E_{\text{sp}}(R) \right\},
\]

(5)

where \(E_{\text{sp}}^{(1)}(R) := \partial \max_{P \in \mathcal{P}(X)} E_{\text{sp}}(s, P) / \partial r \mid r = R\).

3. Properties of the Sphere-Packing Exponent

**Lemma 2** (Optimal Input Distribution). For any \(R > R_{\infty}\), the distribution \(U_{X}\) is a maximizer of \(E_{\text{sp}}^{(1)}(R, \cdot)\) and \(E_{\text{sp}}^{(2)}(R, \cdot)\).

**Proof.** We first prove that \(U_{X}\) attains \(\max_{P \in \mathcal{P}(X)} E_{0}(s, P)\). From Eq. (1), it is not hard to verify that \(U_{X}W^{\alpha} = VU_{X}W^{\alpha}V^{\dagger}\) for all \(\alpha \in (0,1]\). Hence, it follows that

\[
\text{Tr}[W_{x}^{\alpha}(U_{X}W^{\alpha})^{1-\alpha}] = \text{Tr}[V^{x-1}W_{1}^{\alpha}V^{\dagger} - 1(U_{X}W^{\alpha})^{1-\alpha}]
\]

(6)

\[
= \text{Tr}[W_{1}^{\alpha}(1) V^{x-1} - 1(U_{X}W^{\alpha})^{1-\alpha}]
\]

(7)

\[
= \text{Tr}[(U_{X}W^{\alpha})^{1}] - \text{Tr}[(U_{X}W^{\alpha})^{1-\alpha}]
\]

(8)

for all \(\alpha \in (0,1]\). The above equation shows that the distribution \(U_{X}\) that maximizes \(E_{0}(s, P), \forall s \geq 0\) [16, Eq. (38)]. Then we have

\[
E_{\text{sp}}^{(1)}(R, U_{X}) = \sup_{s \geq 0} \left\{ \max_{P \in \mathcal{P}(X)} E_{0}(s, P) - sR \right\} = E_{\text{sp}}(R).
\]

Further, Jensen’s inequality implies that \(E_{\text{sp}}^{(2)}(R, U_{X}) \geq E_{\text{sp}}^{(1)}(R, U_{X}) = E_{\text{sp}}(R)\), which completes the proof. \(\square\)

**Lemma 3** (Saddle-Point Property). Consider any \(R \in (R_{\infty}, C)\) and \(P \in \mathcal{P}(X)\). Let \(S_{P,W}(H) := \{ \sigma \in S(H) : \forall x \in \text{supp}(P), \text{supp}(W_{x}) \cap \text{supp}(\sigma) \neq \emptyset \}\). We define

\[
F_{R, P}(\alpha, \sigma) := \frac{\alpha - 1}{\alpha} (R - D_{\alpha}(W \| \sigma) P),
\]

(10)

on \((0,1] \times S_{P,W}(H)\), and let \(\mathcal{P}_{R}(X) := \{ P \in \mathcal{P}(X) : \min_{\sigma \in S(H)} \sup_{0 < \alpha < 1} F_{R, P}(\alpha, \sigma) \in \mathbb{R}_{>0} \}\). The following holds

(i) For any \(P \in \mathcal{P}(X)\), \(F_{R, P}(\cdot, \cdot)\) has a saddle-point with the saddle-value:

\[
\min_{\sigma \in S(H)} \sup_{0 < \alpha < 1} F_{R, P}(\alpha, \sigma) = \sup_{0 < \alpha < 1} \min_{\sigma \in S(H)} F_{R, P}(\alpha, \sigma) = E_{\text{sp}}(R, W, P).
\]

(11)
(ii) The saddle-point is unique for \( P \in \mathcal{P}_R(\mathcal{X}) \).

(iii) Let \( P \in \mathcal{P}_R(\mathcal{X}) \). The unique saddle-point \((\alpha, \sigma)\) of \( F_{R,P}(\cdot, \cdot) \) satisfies \( \alpha \in (0, 1) \) and

\[
\sigma = \frac{\left( \sum_{x \in \mathcal{X}} P(x) W_x^\alpha e^{(1-\alpha) D_{\alpha}(W_x || \sigma)} \right)^{1/\alpha}}{\text{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) W_x^\alpha e^{(1-\alpha) D_{\alpha}(W_x || \sigma)} \right)^{1/\alpha} \right]} \gg W_x, \quad \forall x \in \text{supp}(P). \tag{12}
\]

The proof is provided in Appendix B.1.

**Lemma 4 (Representation).** For any \( R \in (R_\infty, C) \), let \((\alpha^*_R, \sigma^*_R)\) be the saddle-point of \( F_{R,U}(\cdot, \cdot) \). It follows that

\[
(\alpha^*_R, \sigma^*_R) = \left( -E_{\text{sp}}'(R), \frac{U_X W_{\alpha}^* \left( 1/\alpha^*_R \right)}{\text{Tr} \left[ U_X W_{\alpha}^* \left( 1/\alpha^*_R \right) \right]} \right). \tag{13}
\]

**Proof.** Since Lemma 2 implies that \( U_X \) attains \( E_{\text{sp}}(R, \cdot) \), one observes from the definition of \( E_{\text{sp}}(R, \cdot) \) that all the quantities \( D_{\alpha^*_R}(W_x || \sigma^*_R) \), \( x \in \mathcal{X} \) are equal. By item (iii) of Lemma 3, we obtain a representation of \( \sigma^*_R \) in Eq. (13). The optimal \( \alpha^*_R = -\partial E_{\text{sp}}(r, U_X)/\partial r \mid_{r=R} \) follows from [22, Eq. (42)]. \( \square \)

**Lemma 5 (Invariance).** For any \( R \in (R_\infty, C) \), we have

\[
F_{R,P}(\alpha^*_R, \sigma^*_R) = E_{\text{sp}}(R) > 0, \quad \forall P \in \mathcal{P}(\mathcal{X}), \tag{14}
\]

where \( \alpha^*_R \) and \( \sigma^*_R \) are defined in Eq. (13).

**Proof.** Following the argument in Lemma 2 and recalling Eq. (13) in Lemma 4, one can verify that \( \sup_{s \geq 0} F_{R,P}(\alpha^*_R, \sigma^*_R) = \sup_{s \geq 0} \{ E(\bar{s}, U_X) - sR \} = E_{\text{sp}}(R) \) for all \( P \in \mathcal{P}(\mathcal{X}) \). Further, we obtain \( E_{\text{sp}}(R) > 0 \) for \( R \in (R_\infty, C) \) from the result in [22, Proposition 10]. \( \square \)

### 4. Proof of the Main Result

For rates in the range \( R \leq R_\infty \), we have \( E_{\text{sp}}(R) = +\infty \). The bound in Eq. (5) obviously holds. Hence, we consider the case of \( R \in (R_\infty, C) \) and fix the rate throughout the proof.

We first pose the channel coding problem into a binary hypothesis testing through Lemma 6, which originates from Blahut [6] for the classical case.

**Lemma 6 (Hypothesis Testing Reduction).** For any code \( C_n \) with message size \( e^{nr} \), there exists an \( x^n \in C_n \) such that

\[
\epsilon_{\max}(C_n) \geq \max_{\sigma \in \mathcal{S}(H)} \tilde{\alpha}_{\exp(-nr)} \left( W_{x^n}^\otimes || \sigma_n^\otimes \right). \tag{15}
\]

The proof is provided in Appendix B.2.

Let us now commence with the proof of Theorem 1. Fix arbitrary \( \gamma, \xi > 0 \). Let \( \gamma_n := \left( \frac{1}{2} + \gamma \right) \frac{\log n}{n} \) and \( R_n := R - \gamma_n \). The choice of the rate back-off term \( \gamma_n \) will become evident later. Choose \( N_1 \in \mathbb{N} \) such that \( R_n \geq R - \xi > R_\infty \). Let \( \sigma^*_R \) be defined in Eq. (13), and from Lemma 6, we have

\[
\epsilon_{\max}(C_n) \geq \tilde{\alpha}_{\exp(-nR_n)} \left( W_{x^n}^\otimes || \sigma^*_R \right). \tag{16}
\]

In the following, we provide a lower bound for the type-I error \( \tilde{\alpha}_{\exp(-nR_n)} \left( W_{x^n}^\otimes || \sigma^*_R \right) \). Let \( p^n := \bigotimes_{i=1}^n P_{x_i} \) and \( q^n := \bigotimes_{i=1}^n q_{x_i} \), where \( (p_i, q_i) \) are Nussbaum-Szkoła distributions [26] of \( (W_{x_i}, \sigma^*_R) \) for every \( i \in [n] \). Since \( D_{\alpha}(W_{x_i} || \sigma^*_R) = D_{\alpha}(p_{x_i} || q_{x_i}) \), for all \( \alpha \in (0, 1] \), we shorthand \( \phi_n(R_n) := \sup_{\alpha \in (0, 1]} F_{R_n, P_{x^n}}(\alpha, \sigma^*_R) \), where \( P_{x^n} \) is the empirical distribution of \( x^n \). Moreover, item (iii) in Lemma 3 implies that the state \( \sigma^*_R \) dominates all the channel outputs: \( \sigma^*_R \gg W_{x^n} \) for all \( x \in \text{supp}(P_{x^n}) \). Hence, we have \( p^n \ll q^n \). Subsequently, for every \( i \in [n] \), we let \( q_{x_i}(\omega) = 0 \), for all \( \omega \notin \text{supp}(p_{x_i}) \). We apply Nagaoka’s argument [27] by choosing \( \delta = \exp\{nR_n - n\phi_n(R_n)\} \) to yield, for any \( 0 \leq Q_n \leq 1 \),

\[
\alpha(Q_n; W_{x^n}^\otimes) + \beta(Q_n; \sigma_n^\otimes) \geq \frac{\alpha(U; p^n) + \beta(U; q^n)}{2}, \tag{17}
\]

where \( \alpha(U; p^n) := \sum_{\omega \in U} p^n(\omega) \), \( \beta(U; q^n) := \sum_{\omega \in U} q^n(\omega) \), and \( U := \{ \omega : p^n(\omega)e^{n\phi_n(R_n)} > q^n(\omega)e^{nR_n} \} \).
Next, we employ Bahadur-Ranga Rao’s concentration inequality, Theorem 9 in Appendix A, to further lower bound \( \alpha(\mathcal{U}; p^n) \) and \( \beta(\mathcal{U}; q^n) \). Before proceeding, we need to introduce some notation. We define the tilted distributions, for every \( i \in [n], \omega \in \text{supp}(p_{x_i}), \) and \( t \in [0,1] \) by

\[
\tilde{q}_{x_i,t}(\omega) := \frac{p_{x_i}(\omega)^{1-t} q_{x_i}(\omega)^t}{\sum_{\omega \in \text{supp}(p_{x_i})} p_{x_i}(\omega)^{1-t} q_{x_i}(\omega)^t}.
\]  

(18)

Let

\[
\Lambda_{0,x_i}(t) := \log \mathbb{E}_{p_{x_i}} \left[ e^{\frac{t \log \frac{q_{x_i}}{p_{x_i}}}{q_{x_i}}} \right];
\]

\[
\Lambda_{1,x_i}(t) := \log \mathbb{E}_{q_{x_i}} \left[ e^{\frac{t \log \frac{q_{x_i}}{p_{x_i}}}{q_{x_i}}} \right].
\]

(19)

Since \( p^n \) and \( q^n \) are mutually absolutely continuous, the maps \( t \mapsto \Lambda_{j,x_i}(t), \ j \in \{0,1\} \) are differentiable for all \( t \in [0,1] \). One can immediately verify the following partial derivatives with respect to \( t \):

\[
\Lambda'_{0,x_i}(t) = \mathbb{E}_{q_{x_i},t} \left[ \log \frac{q_{x_i}}{p_{x_i}} \right], \quad \Lambda''_{0,x_i}(t) = \mathbb{Var}_{q_{x_i},t} \left[ \log \frac{q_{x_i}}{p_{x_i}} \right],
\]

\[
\Lambda''_{0,x_i}(t) = \mathbb{Var}_{q_{x_i},t} \left[ \log \frac{q_{x_i}}{p_{x_i}} \right], \quad \Lambda'_{1,x_i}(t) = \mathbb{E}_{q_{x_i},1-t} \left[ \log \frac{p_{x_i}}{q_{x_i}} \right].
\]

(20)

With \( \Lambda_{j,x_i}(t) \) in Eq. (19), we can define

\[
\Lambda_{j,P_{X^n}}(t) := \sum_{x \in \mathcal{X}} P_{X^n}(x) \Lambda_{j,x}(t), \quad j \in \{0,1\};
\]

\[
\Lambda^*_{j,P_{X^n}}(z) := \sup_{t \in \mathbb{R}} \left\{ tz - \Lambda_{j,P_{X^n}}(t) \right\}, \quad j \in \{0,1\},
\]

(21)

(22)

where \( \Lambda^*_{j,P_{X^n}}(z) \) in Eq. (22) is the Fenchel-Legendre transform of \( \Lambda_{j,P_{X^n}}(t) \). The quantities \( \Lambda^*_{j,P_{X^n}}(z) \) would appear in the lower bounds of \( \alpha(\mathcal{U}; p^n) \) and \( \beta(\mathcal{U}; q^n) \) obtained by Bahadur-Randga Rao’s inequality as shown later.

In the following, we relate the Fenchel-Legendre transform \( \Lambda^*_{j,P_{X^n}}(z) \) to the desired error-exponent function \( \phi_n(R_n) \). Such a relationship is stated in Lemma 7; the proof is provided in Appendix B.3.

**Lemma 7.** Under the prevailing assumptions and for all \( R_n \in (R_{\infty}, C) \), the following holds:

(i) \( \Lambda^*_{0,P_{X^n}}(\phi_n(R_n) - R_n) = \phi_n(R_n); \)

(ii) \( \Lambda^*_{1,P_{X^n}}(R_n - \phi_n(R_n)) = R_n; \)

(iii) There exists a unique \( t^* = \frac{s^*}{1+s^*} \in (0,1) \), such that \( \Lambda^*_{0,P_{X^n}}(t^*) = \phi_n(R_n) - R_n \), where \( s^* := \frac{\partial \phi_n(t)}{\partial t} |_{t=R_n}. \)

Item (iii) in Lemma 7 shows that the optimizer \( t \) in Eq. (22) always lies in the compact set \([0,1]\). Further, Eqs. (19) and (20) ensure that \( \Lambda_{0,x_i}(t) = \Lambda_{1,x_i}(1-t), \Lambda'_{0,x_i}(t) = -\Lambda'_{1,x_i}(1-t), \Lambda''_{0,x_i}(t) = \Lambda''_{1,x_i}(1-t) \). We define the following quantities:

\[
V_{\max} := \max_{t \in [0,1], x \in \mathcal{X}} \Lambda'_{0,x}(t);
\]

\[
V_{\min} := \min_{t \in [0,1], x \in \mathcal{X}} \Lambda'_{0,x}(t);
\]

\[
T_{\max} := \max_{t \in [0,1], x \in \mathcal{X}} T_{0,x}(t);
\]

\[
T_{0,x}(t) := \mathbb{E}_{q_{x,t}} \left[ \log \frac{q_{x}}{p_{x}} - \Lambda_{0,x}(t) \right]^3 \];
\]

(23)

(24)

(25)

(26)

\( T_{1,x}(t) := T_{0,x}(1-t); \) and \( K_{\max} := 15\sqrt{2\pi T_{\max}/V_{\min}} \). Note that for every \( x \in \mathcal{X}, \Lambda''_{0,x}(-) \) and \( T_{0,x}(-) \) are continuous functions on \([0,1]\) from the definitions in Eqs. (20), (26) (see also [10, Lemma 9]). The maximization and minimization in the above definitions are well-defined and finite. Moreover, Lemma 8 guarantees that \( V_{\min} \) is bounded away from zero.
Lemma 8 (Positivity). For any \( R_n \in (R_\infty, C) \) and \( P_{\mathcal{X}^n} \in \mathcal{P}(\mathcal{X}) \), \( \Lambda''_{0,P_{\mathcal{X}}} (t) > 0 \), for all \( t \in [0,1] \).

Proof. Assume \( \Lambda''_{0,P_{\mathcal{X}}} (t) \) is zero for some \( t \in [0,1] \). This is equivalent to

\[
p_i(\omega) = q_i(\omega) \cdot e^{-\Lambda'(x_i,t)}, \quad \forall \omega \in p_i, \quad \forall i \in [n].
\]

Summing the right-hand side of Eq. (27) over \( \omega \in p_i \), gives \( 1 = \text{Tr} \left[ p_i^0 q_i \right] e^{-\Lambda'(x_i,t)}, \quad \forall i \in [n] \). Then, Eqs. (27) and the above equation imply that

\[
\phi_n(R_n) = \sup_{0 < \alpha \leq 1} \frac{\alpha - 1}{\alpha} \left( R_n + \sum_{x \in \mathcal{X}} P_{\mathcal{X}^n}(x) \log \text{Tr} \left[ p_i^0 q_i \right] \right)
\]

where we use the fact that \( R_n > R_\infty = -\sum_{x \in \mathcal{X}} P_{\mathcal{X}^n}(x) \log \text{Tr} \left[ p_i^0 q_i \right]; \text{ see Eq. (3)} \). However, Lemma 5 implies that \( \phi_n(R_n) = E_{\text{sp}}(R_n) > 0 \), which leads to a contradiction. \( \square \)

Now, we are ready to derive the lower bounds to \( \alpha(\mathcal{U}; p^n) \) and \( \beta(\mathcal{U}; q^n) \). Let \( N_2 \in \mathbb{N} \) be sufficiently large such that for all \( n \geq N_2 \),

\[
\sqrt{n} \geq \frac{1 + (1 + K_{\text{max}})^2}{\sqrt{V_{\text{min}}}}.
\]

Applying Bahadur-Randga Rao’s inequality (Theorem 9) to \( Z_i = \log q_i - \log p_i \) with the probability measure \( \lambda_i = p_i \), and \( z = R - \phi_n(R_n) \) gives

\[
\alpha(\mathcal{U}; p^n) = \text{Pr} \left\{ \frac{1}{n} \sum_{i=1}^{n} Z_i \geq R - \phi_n(R_n) \right\} \geq \frac{2A}{\sqrt{n}} \exp \left\{ -n \Lambda'_{0,p_{\mathcal{X}^n}} (\phi_n(R_n) - R_n) \right\}
\]

where \( A := \frac{e^{-K_{\text{max}}}}{\sqrt{4nV_{\text{max}}}} \).

Similarly, applying Theorem 9 to \( Z_i = \log p_i - \log q_i \) with the probability measure \( \lambda_i = q_i \), and \( z = \phi_n(R_n) - R_n \) yields

\[
\beta(\mathcal{U}; q^n) = \text{Pr} \left\{ \frac{1}{n} \sum_{i=1}^{n} Z_i \geq \phi_n(R_n) - R_n \right\} \geq \frac{2A}{\sqrt{n}} \exp \left\{ -n \Lambda'_{1,p_{\mathcal{X}^n}} (R_n - \phi_n(R_n)) \right\}.
\]

Continuing from Eq. (30) and item (i) in Lemma 7 gives

\[
\alpha(\mathcal{U}; p^n) \geq \frac{2A}{\sqrt{n}} \exp \left\{ -n \phi_n(R_n) \right\}.
\]

Eq. (32) together with item (iii) in Lemma 7 yields

\[
\beta(\mathcal{U}; q^n) \geq \frac{2A}{\sqrt{n}} \exp \left\{ -n R_n \right\} = 2An^\gamma \exp \left\{ -nR_n \right\}.
\]

Let \( N_3 \in \mathbb{N} \) such that \( An^\gamma > 1 \), for all \( n \geq N_3 \). Then Eq. (34) implies that \( \beta(\mathcal{U}; q^n) > 2 \exp \left\{ -nR_n \right\} \). Thus, we can bound the left-hand side of Eq. (17) from below by \( \frac{A}{\sqrt{n}} e^{-n\phi_n(R_n)} \). For any test \( 0 \leq Q_n \leq 1 \) such that \( \beta(Q_n; \sigma_{R}^{\otimes n}) \leq \exp \left\{ -nR_n \right\} \), we have

\[
\alpha_{\exp \left\{ -nR_n \right\}}(W_{\mathcal{X}^n} || \sigma_R^{\otimes n}) = \alpha(Q_n; p^n)
\]

\[
\geq \frac{A}{\sqrt{n}} \exp \left\{ -n \phi_n(R_n) \right\} = \frac{A}{\sqrt{n}} \exp \left\{ -nE_{\text{sp}}(R_n) \right\},
\]

where the last equality follows from Lemma 5.
Finally, it remains to remove the back-off term $R_n = R - \gamma_n$ in Eq. (35). By Taylor’s theorem, we have

$$E_{sp}(R - \gamma_n) = E_{sp}(R) - \gamma_n E'_{sp}(R) + \frac{\gamma_n^2}{2} E''_{sp}(R), \quad (36)$$

for some $\bar{R} \in (R - \xi, R)$ and $E''_{sp}(\bar{R}) := \left. \frac{\partial^2 E_{sp}(s, U_X)}{\partial s^2} \right|_{s = \bar{s}}$. Further, one can calculate that

$$E''_{sp}(\bar{R}) = -\left. \left( \frac{\partial^2 E_0(s, U_X)}{\partial s^2} \right) \right|_{s = \bar{s}}^{-1} \geq \left( \frac{(1 + \bar{s})^3}{\Lambda''_{0, U_X}(\bar{s})} \right) \leq \left( \frac{1}{V_{\min}} \right) =: \Upsilon, \quad (37)$$

where $\bar{s} = \frac{1 - \alpha^*}{\alpha_R}$. From item (iii) in Lemma 3, it follows that both $\bar{s}$ and $|E'_{sp}(\bar{R})| = s^*$ are both positive and finite for $\bar{R} \in (R_{\infty}, C)$ and $R \in (R_{\infty}, C)$. Together with the fact that $V_{\min} > 0$, we have $\Upsilon \in \mathbb{R}_{>0}$. We apply Taylor’s expansion on the function $n^{-1}$ again to yield

$$n^{-\frac{1}{2}(1 + |E_{sp}(R)|)} - \gamma_n \Upsilon = n^{-\frac{1}{2}(1 + |E_{sp}(R)|)} \cdot \left( 1 - \frac{\log n}{n^{1/2}} \right) \gamma_n \Upsilon 
= n^{-\frac{1}{2}(1 + |E_{sp}(R)|)} \cdot (1 - o(1)), \quad (39)$$

where the first equality holds for some $\bar{x} \in (0, \gamma_n)$, and the last line follows from the definition $\gamma_n = (\frac{1}{2} + \gamma) \frac{\log n}{n}$. Finally, by combining Eqs. (16), (36), and (39), we obtain the desired Eq. (5) for sufficiently large $n \geq N_0 := \max \{N_1, N_2, N_3\}$.

5. Discussion

In this work, we establish a sphere-packing bound with a refined polynomial pre-factor that coincides with the best classical results [10, Theorem 1] to date. As discussed by Altu˘g and Wagner [10, Sec. VII], the pre-factor is correct for binary symmetric channels but slightly worse for binary erasure channels (in the order of $1/\sqrt{n}$). On the other hand, our pre-factor matches the recent result of the random coding upper bound [21, Theorem 2], where the pre-factor has been shown to be exact. Hence, we conjecture that the established result is optimal for general symmetric c-q channels.

This work admits variety of potential extensions. First, the symmetric c-q channel studied in this paper is a covariant channel with a cyclic group:

$$W_{\mathcal{U}_n(x)\mathcal{U}_n(g)} = \mathcal{U}_n(g) W_x \mathcal{U}_n(g)^\dagger, \quad \forall g, x \in \mathcal{X}, \quad (40)$$

where $\mathcal{U}_n$ and $\mathcal{U}_out$ are the unitary representations on $\mathcal{X}$ and $\mathcal{S}(\mathcal{H})$ such that $\mathcal{U}_n(g) x \mathcal{U}_n(g)^\dagger = (x + g) \bmod |\mathcal{X}|$ and $\mathcal{U}_n(g) = V_g$. It would be interesting to investigate whether the refined sphere-packing bound can be extended to covariant quantum channels $\mathcal{N} : \mathcal{S}(\mathcal{H}_in) \rightarrow \mathcal{S}(\mathcal{H}_{out})$ with arbitrary compact groups. Second, the random coding bound in the quantum case has been proved only for pure-state channels [16]. It is promising to prove the bound for this class of c-q channels by employing the symmetry property. Finally, the refinement provides a new possibility for moderate deviation analysis in c-q channels [13], which is left as future work.

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Appendix A. A Tight Concentration Inequality

Let \((Z_i)_{i=1}^n\) be a sequence of independent, real-valued random variables whose probability measures are \(\lambda_i\). Let \(\Lambda_i(t) := \log \mathbb{E} [e^{tZ_i}]\) and define the Fenchel-Legendre transform of \(\frac{1}{n} \sum_{i=1}^n \Lambda_i(\cdot)\) to be: \(\Lambda^*_n(z) := \sup_{t \in \mathbb{R}} \left\{ zt - \frac{1}{n} \sum_{i=1}^n \Lambda_i(t) \right\}, \quad \forall z \in \mathbb{R}. \) Then there exists a real number \(t^* \in (0, 1]\) for every \(z \in \mathbb{R}\) such that \(z = \frac{1}{n} \sum_{i=1}^n \Lambda_i(t^*)\) and \(\Lambda^*_n(z) = z t^* - \frac{1}{n} \sum_{i=1}^n \Lambda_i(t^*).\) Define the probability measure \(\tilde{\lambda}_i\) via \(\frac{d\tilde{\lambda}_i}{d\lambda_i}(z_i) := e^{t^* z_i - \Lambda_i(t^*)}\), and let \(\tilde{Z}_i := Z_i - E_{\lambda_i} [Z_i]\). Furthermore, define \(m_{2,n} := \sum_{i=1}^n \text{Var}_{\lambda_i} [Z_i], m_{3,n} := \sum_{i=1}^n E_{\lambda_i} \left[ |\tilde{Z}_i|^3 \right]\), and \(K_n(t^*) := \frac{15\sqrt{2\pi}m_{3,n}}{m_{2,n}}.\) With these definitions, we can now state the following sharp concentration inequality for \(\frac{1}{n} \sum_{i=1}^n Z_i\):

**Theorem 9** (Bahadur-Ranga Rao’s Concentration Inequality [11, Proposition 5], [28]). Given

\[
\sqrt{m_{2,n}} \geq 1 + (1 + K_n(t^*))^2,
\]

it follows that

\[
\Pr \left\{ \frac{1}{n} \sum_{i=1}^n Z_i \geq z \right\} \geq e^{-\Lambda_n^*(z)} e^{-K_n(t^*)} \frac{1}{2\sqrt{2\pi}m_{2,n}}.
\]

Appendix B. Proofs of Miscellaneous Lemmas

B.1. **Proof of Lemma 3.** Let \(R > R_\infty\) and \(P \in \mathcal{P}(\mathcal{X})\) be arbitrary. It is convenient to reparameterize the function \(F_{R,P}\) by the substitution \(\alpha = \frac{1}{1+s}\):

\[
F_{R,P}(\alpha, \sigma) |_{\alpha=\frac{1}{1+s}} = -sR + sD_{\frac{1}{1+s}}(W||\sigma|P) =: K_{R,P}(s, \sigma).
\]

In the following, we prove the existence of a saddle-point of \(K_{R,P}(\cdot, \cdot)\) on \(\mathbb{R}_{\geq 0} \times S_{P,W}(\mathcal{H}),\) where \(\mathbb{R}_{\geq 0} := \{0, \infty\}.\) By Ref. [29, Lemma 36.2], \((s^*, \sigma^*)\) is a saddle point of \(K_{R,P}(\cdot, \cdot)\) if and only if the supremum in

\[
\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in S_{P,W}(\mathcal{H})} K_{R,P}(s, \sigma)
\]

is attained at \(s^*\), the infimum in

\[
\inf_{\sigma \in S_{P,W}(\mathcal{H})} \sup_{s \in \mathbb{R}_{\geq 0}} K_{R,P}(s, \sigma)
\]

is attained at \(\sigma^*\), and the two extrema in Eqs. (44), (45) are equal and finite. We first claim that

\[
\forall s \in \mathbb{R}_{\geq 0}, \quad \inf_{\sigma \in S_{P,W}(\mathcal{H})} K_{R,P}(s, \sigma) = \inf_{\sigma \in S(\mathcal{H})} K_{R,P}(s, \sigma).
\]

To see that, observe that for any \(s \in \mathbb{R}_{\geq 0},\) the definition of the \(\alpha\)-Rényi divergence yields

\[
\forall \sigma \in S(\mathcal{H}) \setminus S_{P,W}(\mathcal{H}), \quad D_{\frac{1}{1+s}}(W||\sigma|P) = +\infty,
\]

which, in turn, implies

\[
\forall \sigma \in S(\mathcal{H}) \setminus S_{P,W}(\mathcal{H}), \quad K_{R,P}(s, \sigma) = +\infty.
\]

Hence, Eq. (46) yields

\[
\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in S_{P,W}(\mathcal{H})} K_{R,P}(s, \sigma) = \sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in S(\mathcal{H})} K_{R,P}(s, \sigma) = \sup_{s \in \mathbb{R}_{\geq 0}} \min_{\sigma \in S(\mathcal{H})} K_{R,P}(s, \sigma),
\]

where the last equality in Eq. (49) follows from the lower semi-continuity of the map \(\sigma \mapsto D_{1/(1+s)}(W||\sigma|P)\) [24, Corollary III.25] and the compactness of \(S(\mathcal{H}).\) Further, by the fact \(R > R_\infty\) and the definition of \(E_{sp}^{(2)}\), we have

\[
E_{sp}^{(2)}(R, P) = \sup_{s \in \mathbb{R}_{\geq 0}} \min_{\sigma \in S(\mathcal{H})} K_{R,P}(s, \sigma) < +\infty,
\]

which guarantees the supremum in the right-hand side of Eq. (49) is attained at some \(s \in \mathbb{R}_{\geq 0},\) i.e.,

\[
\sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in S_{P,W}(\mathcal{H})} K_{R,P}(s, \sigma) = \max_{s \in \mathbb{R}_{\geq 0}} \min_{\sigma \in S(\mathcal{H})} K_{R,P}(s, \sigma) < +\infty.
\]
Thus, we complete our claim in Eq. (44). It remains to show that the infimum in Eq. (45) is attained at some \( \sigma^* \in S_{P,W}(\mathcal{H}) \) and the supremum and infimum are exchangeable. To achieve this, we will show that \((\mathbb{R}_{\geq 0}, S_{P,W}(\mathcal{H}), K_{R,P})\) is a closed saddle-element (see Definition 10 below) and apply Rockafellar’s saddle-point result, Theorem 11, to conclude our claim.

**Definition 10 (Closed Saddle-Element [29]).** The triple \((\mathcal{A}, \mathcal{B}, F)\) is called a closed saddle-element if for any \( x \in \text{ri} (\mathcal{A}) \) (resp. \( y \in \text{ri} (\mathcal{B}) \)):

(a) \( \mathcal{B} \) (resp. \( \mathcal{A} \)) is convex;

(b) \( F(x,.) \) (resp. \( F(.) , y \)) is convex (resp. concave) and lower (resp. upper) semi-continuous; and

(c) any accumulation point of \( \mathcal{B} \) (resp. \( \mathcal{A} \)) that does not belong to \( \mathcal{B} \) (resp. \( \mathcal{A} \)), say \( y_0 \) (resp. \( x_0 \)), satisfies \( \lim_{y \to y_0} F(x, y) = +\infty \) (resp. \( \lim_{x \to x_0} F(x, y) = -\infty \)).

**Theorem 11 (The Existence of Saddle-Points [29, Theorem 8], [30, Theorem 37.3]).** Let \((\mathcal{A}, \mathcal{B}, F)\) be any closed saddle-element on \( \mathbb{R}^m \times \mathbb{R}^n \).

(I) No non-zero \( x_0 \) has the property that, for all \( x \in \text{ri} (\mathcal{A}) \) and \( y \in \text{ri} (\mathcal{B}) \), the half-line \( \{ x + tx_0 : t \geq 0 \} \)

is contained in \( \mathcal{A} \) and \( F(x + tx_0 , y) \) is a non-zero and non-decreasing function for \( t \geq 0 \).

(II) No non-zero \( y_0 \) has the property that, for all \( x \in \text{ri} (\mathcal{A}) \) and \( y \in \text{ri} (\mathcal{B}) \), the half-line \( \{ y + ty_0 : t \geq 0 \} \)

is contained in \( \mathcal{B} \) and \( F(x, y + ty_0) \) is a non-increasing function for \( t \geq 0 \).

If condition (I) is satisfied, then

\[
\max_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} F(x, y) = \inf_{y \in \mathcal{B}} \sup_{x \in \mathcal{A}} F(x, y) < +\infty. \tag{52}
\]

If condition (II) is satisfied, then

\[
-\infty < \sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{B}} F(x, y) = \min_{y \in \mathcal{B}} \sup_{x \in \mathcal{A}} F(x, y). \tag{53}
\]

If (I) and (II) are both satisfied, then \( F \) has a saddle-point on \( \mathcal{A} \times \mathcal{B} \).

Fix an arbitrary \( s \in \text{ri} (\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0} \). We check that \((S_{P,W}(\mathcal{H}), K_{R,P}(s, .))\) fulfills the three items in Definition 10. (a) The set \( S_{P,W}(\mathcal{H}) \) is clearly convex. (b) Since the map \( \sigma \mapsto D_{1/(1+s)}(W \| \sigma | P) \)

is convex (owing to Lieb’s concavity theorem [31]) and lower semi-continuous on \( L(\mathcal{H})_+ \) [24, Corollary III.25], by Eq. (43), \( \sigma \mapsto K_{R,P}(\alpha , \sigma) \) is also convex and lower semi-continuous on \( S_{P,W}(\mathcal{H}) \). (c) Due to the compactness of \( S(\mathcal{H}) \), any accumulation point of \( S_{P,W}(\mathcal{H}) \) that does not belong to \( S_{P,W}(\mathcal{H}) \), say \( \sigma_o \), satisfies \( \sigma_o \in S(\mathcal{H}) \setminus S_{P,W}(\mathcal{H}) \). By Eqs. (47), (48), one finds \( K_{R,P}(\alpha , \sigma_o) = +\infty \).

Next, fix an arbitrary \( \sigma \in \text{ri} (S_{P,W}(\mathcal{H})) \). Owing to the convexity of \( S_{P,W}(\mathcal{H}) \), it follows that \( \text{ri} (S_{P,W}(\mathcal{H})) = \text{ri} \left(c1(S_{P,W}(\mathcal{H})) \right) \) (see e.g. [30, Theorem 6.3]). We first claim \( c1(S_{P,W}(\mathcal{H})) = S(\mathcal{H}) \). To see this, observe that \( S(\mathcal{H})_{++} \subset S_{P,W}(\mathcal{H}) \) since a full-rank density operator is not orthogonal with every \( W_x, x \in X \).

Hence,

\[
S(\mathcal{H}) = c1(S(\mathcal{H})_{++}) \subset c1(S_{P,W}(\mathcal{H})). \tag{54}
\]

On the other hand, the fact \( S_{P,W}(\mathcal{H}) \subset S(\mathcal{H}) \) leads to

\[
c1(S_{P,W}(\mathcal{H})) \subset c1(S(\mathcal{H})) = S(\mathcal{H}). \tag{55}
\]

By Eqs. (54) and (55), we deduce that

\[
\text{ri} (S_{P,W}(\mathcal{H})) = \text{ri} (c1(S_{P,W}(\mathcal{H}))) = \text{ri} (S(\mathcal{H})) = S(\mathcal{H})_{++}, \tag{56}
\]

where the last equality in Eq. (56) follows from [32, Proposition 2.9]. Hence, we obtain

\[
\forall \sigma \in \text{ri} (S_{P,W}(\mathcal{H})), \quad \forall x \in X, \quad \sigma \gg W_x. \tag{57}
\]

Now, we verify that \((\mathbb{R}_{\geq 0}, K_{R,P}(\cdot, \cdot))\) satisfies the three items in Definition 10. (a) The set \( \mathbb{R}_{\geq 0} \) is obviously convex. (b) From Eqs. (57) and the definition of the Rényi divergence, the map \( s \mapsto D_{1/(1+s)}(W \| \sigma | P) \)

is continuous on \( \mathbb{R}_{\geq 0} \). Further, \( s \mapsto sD_{1/(1+s)}(W \| \sigma | P) \) is concave on \( \mathbb{R}_{\geq 0} \) [24, Appendix B]. By Eq. (43), the map \( s \mapsto K_{R,P}(s, \sigma) \) is concave and continuous on \( \mathbb{R}_{\geq 0} \). (c) Since \( \mathbb{R}_{\geq 0} \) is closed, there is no accumulation point of \( \mathbb{R}_{\geq 0} \) that does not belong to \( \mathbb{R}_{\geq 0} \).

\footnote{We denote by \( \text{ri} \) and \( c1 \) the relative interior and the closure of a set, respectively.}
We are now in a position to prove item (i) of this Proposition. Since the set \( \mathcal{S}_{P,W}(\mathcal{H}) \) is bounded, condition (II) is satisfied. Equation (53) in Theorem 11 implies that

\[
-\infty < \sup_{s \in \mathbb{R}_{\geq 0}} \inf_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} K_{R,P}(s,\sigma) = \min_{\sigma \in \mathcal{S}_{P,W}(\mathcal{H})} \sup_{s \in \mathbb{R}_{\geq 0}} K_{R,P}(s,\sigma).
\]

Then Eqs. (51) and (58) lead to the existence of a saddle-point of \( K_{R,P}(\cdot,\cdot) \) on \( \mathbb{R}_{\geq 0} \times \mathcal{S}_{P,W}(\mathcal{H}) \). Note that \( K_{R,P}(s,\sigma) = F_{R,P}(1/(1+s),\sigma) \). We conclude the existence of a saddle-point of \( F_{R,P}(\cdot,\cdot) \) on \( (0,1] \times \mathcal{S}_{P,W}(\mathcal{H}) \). Hence, item (i) is proved.

We postpone the proof of the uniqueness of the optimizer to later and now show item (iii). Given any \( R \in (R_{\infty},C) \) and \( P \in \mathcal{P}(\mathcal{X}) \), one finds

\[
\min_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{0 < \alpha < 1} F_{R,P}(\alpha,\sigma) \in (0, +\infty).
\]

If \( \alpha^* = 1 \) and \( \sigma^* \) is a saddle point of \( F_{R,P}(\cdot,\cdot) \), by Eq. (10) we deduce that \( F_{R,P}(1,\sigma^*) = 0 \) for every possible \( \sigma^* \), which contradicts Eq. (59). Hence, \( \alpha^* = 1 \) is not a saddle point of \( F_{R,P}(\cdot,\cdot) \).

For any saddle-point \((\alpha^*,\sigma^*)\) of \( F_{R,P}(\cdot,\cdot) \), it holds that

\[
F_{R,P}(\alpha^*,\sigma^*) = \min_{\sigma \in \mathcal{S}(\mathcal{H})} F_{R,P}(\alpha^*,\sigma) = \frac{\alpha^* - 1}{\alpha^*} R + \frac{1 - \alpha^*}{\alpha^*} \min_{\sigma \in \mathcal{S}(\mathcal{H})} D_{\alpha^*}(W||\sigma|P).
\]

We claim the minimizer of Eq. (60) must satisfy

\[
\sigma^* = \frac{\left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha^*}}{\text{Tr}[W_x^{\alpha^*(1-\alpha^*)}]} \right)^{\frac{1}{\alpha^*}}}{\text{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha^*}}{\text{Tr}[W_x^{\alpha^*(1-\alpha^*)}]} \right)^{\frac{1}{\alpha^*}} \right]} = \frac{\left( \sum_{x \in \mathcal{X}} P(x) W_x^{\alpha^*} e(1-\alpha^*)D_{\alpha^*}(W_x||\sigma) \right)^{\frac{1}{\alpha^*}}}{\text{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) W_x^{\alpha^*} e(1-\alpha^*)D_{\alpha^*}(W_x||\sigma) \right)^{\frac{1}{\alpha^*}} \right]}
\]

for every \( \alpha^* \in (0,1) \). Our approach closely follows from Hayashi and Tomamichel [33, Lemma 5]. For two density operators \( \sigma, \omega \in \mathcal{S}(\mathcal{H}) \) and a map \( G : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})_{\text{sa}} \) (where \( \mathcal{L}(\mathcal{H})_{\text{sa}} \) denotes the self-adjoint operators on \( \mathcal{H} \)), define the Fréchet derivative (see e.g. [33, Appendix C], [34])

\[
\partial_\omega G(\sigma) := DG(\sigma)[\omega - \sigma].
\]

By letting

\[
g_\alpha(\sigma) := \sum_{x \in \mathcal{X}} P(x) \log \text{Tr} \left[ W_x^{\alpha} \sigma^{1-\alpha} \right],
\]

it follows that

\[
\sigma^* = \arg \min_{\sigma \in \mathcal{S}(\mathcal{H})} D_\alpha \left( W||\sigma|P \right) = \arg \max_{\sigma \in \mathcal{S}(\mathcal{H})} g_\alpha(\sigma), \quad \forall \alpha \in (0,1).
\]

Since the map \( \sigma \mapsto g_\alpha(\sigma) \) is strictly concave for every \( \alpha \in (0,1) \) [31], a sufficient and necessary condition for \( \sigma \) to be an optimizer of Eq. (64) is \( \partial_\omega g_\alpha(\sigma) = 0 \) for all \( \omega \in \mathcal{S}(\mathcal{H}) \). Direct calculation shows that

\[
\partial_\omega g_\alpha(\sigma) = \text{Tr} \left[ \sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\text{Tr}[W_x^{\alpha(1-\alpha)}]} \partial_\omega \sigma^{1-\alpha} \right].
\]

Next, we check that the fixed-points of the following map achieves the optimum:

\[
\sigma \mapsto \frac{\left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\text{Tr}[W_x^{\alpha(1-\alpha)}]} \right)^{\frac{1}{\alpha}}}{\text{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^{\alpha}}{\text{Tr}[W_x^{\alpha(1-\alpha)}]} \right)^{\frac{1}{\alpha}} \right]}.\]

\(^2\)We note that the Fréchet derivative of functions involving matrices has other applications in quantum information theory; see e.g. [35, 36, 37].
Let
\[
\chi_\alpha(\sigma) := \text{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^\alpha}{\text{Tr}[W_x^\alpha \sigma^{1-\alpha}]} \right) \frac{1}{\alpha} \right] > 0, \quad \forall \alpha \in (0, 1),
\]
and let \( \bar{\sigma} \) be a fix-point of the map in Eq. (66). Then, by Eqs. (66), (67), we have
\[
\chi_\alpha(\bar{\sigma}) \cdot \bar{\sigma} = \left( \sum_{x \in \mathcal{X}} P(x) \frac{W_x^\alpha}{\text{Tr}[W_x^\alpha \sigma^{1-\alpha}]} \right) \frac{1}{\alpha}.
\]
Substituting Eq. (68) into Eq. (65) yields
\[
\partial_\omega g_\alpha(\bar{\sigma}) = \text{Tr} \left[ \chi_\alpha(\bar{\sigma})^\alpha \sigma^{1-\alpha} \partial_\omega \bar{\sigma}^{1-\alpha} \right] = \text{Tr} \left[ \chi_\alpha(\bar{\sigma})^\alpha(1 - \alpha)\bar{\sigma}^{-\alpha}(\omega - \bar{\sigma}) \right] = (1 - \alpha)\chi_\alpha(\bar{\sigma})^\alpha \text{Tr}[\omega - \bar{\sigma}] = 0.
\]
By Brouwer’s fixed-point theorem, the map in Eq. (66) is indeed the optimizer for Eq. (64). Further, from Eq. (61), it is clear that
\[
\sigma^* \gg W_x, \quad \forall x \in \text{supp}(P),
\]
and thus item (iii) is proved.

Lastly, we show the uniqueness of the saddle-point. Since the map \( \sigma \mapsto D_\sigma(W\|\sigma|P) \) is strictly concave [31], the minimizer of Eq. (59) is unique for any \( \alpha \in (0, 1) \). Then, it remains to prove the uniqueness of the maximizer. Let \( \sigma^* \) attain the minimum in Eq. (59). By using the reparameterization again, we have
\[
K_{R,P}(s, \sigma^*) = -sR + sD_{\frac{1}{1+s}}(W\|\sigma^*|P)
\]
\[
= -sR + s \sum_{x \in \mathcal{X}} P(x)D_{\frac{1}{1+s}}(p_x\|q_x),
\]
where \( p_x, q_x \) are the Nussbaum-Szkola distributions of \( W_x \) and \( \sigma^* \). The second-order partial derivative can be calculated as
\[
\frac{\partial^2 K_{R,P}(s, \sigma^*)}{\partial s^2} = - \frac{1}{(1 + s)^3} \sum_{x \in \mathcal{X}} P(x) \text{Var}_{\frac{1}{1+s} x} \left[ \log \frac{q_x}{p_x} \right],
\]
where
\[
\hat{q}_{t,x}(\omega) := \frac{p_x(\omega)^{1-t} q_x(\omega)^t}{\sum_{\omega \in \text{supp}(p_x) \cap \text{supp}(q_x)} p_x(\omega)^{1-t} q_x(\omega)^t}, \quad \forall \omega \in \text{supp}(p_x) \cap \text{supp}(q_x), \quad t \in [0, 1].
\]
Now, we assume the right-hand side of Eq. (73) is zero, which is equivalent to
\[
p_x(\omega) = c_x \cdot q_x(\omega), \quad \forall \omega \in \text{supp}(p_x) \cap \text{supp}(q_x)
\]
for some constant \( c_x > 0 \) and \( x \in \text{supp}(P) \). From Eq. (70), one finds \( p_x \ll q_x \). Summing the right-hand side of Eq. (75) over \( \omega \in p_x^2 \) yields
\[
1 = c_x \cdot \text{Tr} \left[ p_x^2 q_x \right], \quad \forall x \in \text{supp}(P).
\]
By combining Eqs. (75) and (76), one can verify
\[
\sup_{s \in \mathbb{R}_{>0}} \left\{ -sR + s \sum_{x \in \mathcal{X}} P(x)D_{\frac{1}{1+s}}(p_x\|q_x) \right\} = \sup_{s \in \mathbb{R}_{>0}} \left\{ -sR - s \sum_{x \in \mathcal{X}} P(x) \log \text{Tr} \left[ p_x^0 q_x \right] \right\} = 0,
\]
where we rely the fact \( R > R_\infty(W) \geq -\sum_{x \in \mathcal{X}} P(x) \log \text{Tr} \left[ p_x^0 q_x \right] \) from Eq. (3). However, Eq. (77) contradicts the assumption \( P \in \mathcal{P}_R(\mathcal{X}) \), which in turn implies that the right-hand side of Eq. (73) is strictly negative. Therefore, the map \( s \mapsto K_{R,P}(s, \sigma^*) \) is strictly concave for all \( s \in \mathbb{R}_{>0} \) and thus the maximizer of Eq. (59) is unique. \( \square \)
B.2. Proof of Lemma 6. Let $x^n(m)$ be the codeword encoding the message $m \in \{1, \ldots, \exp\{nr\}\}$. We define a binary hypothesis testing problem as:

$$
\begin{align*}
H_0 : W_{x^n(m)}^n, \\
H_1 : \sigma^n := \bigotimes_{i=1}^n \sigma_i,
\end{align*}
$$

(78)

(79)

where $\sigma^n \in \mathcal{S}(\mathcal{H}^\otimes n)$ can be viewed as a dummy channel output. Since $\sum_{m=1}^M \beta(\Pi_{n,m}; \sigma^n) = 1$ for any POVM $\Pi_n = \{\Pi_{n,1}, \ldots, \Pi_{n,\exp\{nr\}}\}$, and $\beta(\Pi_{n,m}; \sigma^\otimes n) \geq 0$ for every $m \in \mathcal{M}$, there must exist a message $m \in \mathcal{M}$ for any code $C_n$ such that $\beta(\Pi_{n,m}; \sigma^n) \leq \exp\{-nr\}$. Let $x^n := x^n(m)$ be the codeword for that message $m$. Then

$$
\epsilon_m(C_n) \geq \epsilon_m(C_n) = \alpha(\Pi_{n,m}; W_{x^n}^\otimes n) \geq \tilde{\alpha}_{\exp\{-nr\}}(W_{x^n}^\otimes n) = \epsilon_n(C_n).
$$

(80)

Since the above inequality (80) holds for every $\sigma^n \in \mathcal{S}(\mathcal{H}^\otimes n)$, it follows that

$$
\epsilon_{\max}(C_n) \geq \max_{\sigma \in \mathcal{S}(\mathcal{H})} \epsilon_{\exp\{-nr\}}(W_{x^n}^\otimes n) = \epsilon_n(C_n).
$$

(81)

□

B.3. Proof of Lemma 7. This lemma closely follows from Altuğ and Wagner’s [11, Lemma 9]. However, the major difference is that we prove the claim using the expression $\phi_n$ as the error-exponent instead of the discrimination function: $\min \{D(\tilde{\tau}||\rho) : D(\tilde{\tau}||\sigma) \leq R_n\}$. This expression is crucial to obtaining the sphere-packing bound in Theorem 1 in the strong form of Gallager’s expression.

For convenience, we shorthand $r = R_n$. From Lemma 5, it can be verified that

$$
E_0(s) := -\frac{1+s}{n} \log \text{Tr} \left[ (q^n)^{1+s} (q^n)^{1-s} \right]
$$

(82)

$$
= -(1+s)\Lambda_{0,P_{x^n}} \left( \frac{s}{1+s} \right),
$$

(83)

where Eq. (83) follows from the definition of $\Lambda_{0,P_{x^n}}$ in Eq. (21). Then, we rewrite the error-exponent function $\phi_n(r)$ by the Legendre-Fenchel transform of $E_0(s)$, i.e.,

$$
\phi_n(r) = \sup_{s \geq 0} \left\{ \frac{\alpha - 1}{\alpha} \left( r - \sum_{x \in \mathcal{X}} P_{x^n}(x) D_\alpha(p_x||q_x) \right) \right\}
$$

(84)

$$
= \sup_{s \geq 0} \{-sr + E_0(s)\}.
$$

(85)

Direct calculation shows that

$$
\frac{\partial E_0(s)}{\partial s} = -\Lambda_{0,P_{x^n}} \left( \frac{s}{1+s} \right) - \frac{1}{1+s} \Lambda'_{0,P_{x^n}} \left( \frac{s}{1+s} \right),
$$

(86)

$$
\frac{\partial^2 E_0(s)}{\partial s^2} = -\frac{1}{(1+s)^2} \Lambda''_{0,P_{x^n}} \left( \frac{s}{1+s} \right).
$$

(87)

Now assume the second-order derivative $\Lambda''_{0,P_{x^n}}(t)$ in right-hand side of Eq. (87) is zero for some $t \in [0,1]$. This is equivalent to

$$
p_x(\omega) = q_x(\omega) \cdot e^{-\Lambda_{0,x}(t)}, \quad \forall \omega \in p_x, \quad \forall x \in \text{supp}(P_{x^n}).
$$

(88)

Summing the right-hand side of Eq. (88) over $\omega \in p_x$ gives

$$
1 = \text{Tr} \left[ p_x^0 q_x \right] e^{-\Lambda_{0,x}(t)}.
$$

(89)
Then, Eqs. (88) and (89) imply that

\[ \phi_n(r) = \sup_{0<\alpha\leq 1} \frac{\alpha - 1}{\alpha} \left( r - \sum_{x \in \mathcal{X}} P_{x^n}(x) \log \frac{P_{x^n}(x)}{p_0(x)} \right) \]

\[ = \sup_{0<\alpha\leq 1} \frac{\alpha - 1}{\alpha} \left( r + \sum_{x \in \mathcal{X}} P_{x^n}(x) \log \left[ D(p_0(x) || q_x) \right] \right) = 0, \]

where in Eq. (91) we use the fact that \( r > -\sum_{x \in \mathcal{X}} P_{x^n}(x) \log \left[ D(p_0(x) || q_x) \right] \); see Eq. (3). However, from Lemma 5 we know that \( \phi_n(r) = E_{s_p}(R) > 0 \), which leads to a contradiction. Hence, we obtain

\[ \Lambda''(t) > 0, \quad \forall t \in [0, 1], \]  

and prove item (i).

From Eqs. (87) and (92), the objective function \(-sr + E_0(s)\) in Eq. (85) is strictly concave in \( s \) for \( s \in \mathbb{R}_+ \). Further, by recalling that \( \phi_n(r) = E_{s_p}(R) > 0 \), \( s = 0 \) will not be an optimum in Eq. (85). We deduce that there exists a unique maximizer \( s^* \in \mathbb{R}_+ \) such that

\[ r = \frac{\partial E_0(s)}{\partial s} \bigg|_{s=s^*}, \]

\[ \phi_n(r) = E_0(s^*) - s^* \frac{\partial E_0(s)}{\partial s} \bigg|_{s=s^*}, \]

if \( r \) lies in the range:

\[ -\frac{1}{n} \log \text{Tr} \left[ (p^n) 0 q^n \right] = \lim_{s \to +\infty} \frac{\partial E_0(s)}{\partial s} \leq r \leq \frac{\partial E_0(s)}{\partial s} \bigg|_{s=0} = \frac{1}{n} D(p^n || q^n), \]

where the boundary values \(-\frac{1}{n} \log \text{Tr} \left[ (p^n) 0 q^n \right] \) and \( \frac{1}{n} D(p^n || q^n) \) can be obtained from Eqs. (86), (19) and (20). Substituting Eq. (93) into (86) gives

\[ r = -\Lambda_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) - \frac{1}{1 + s^*} \Lambda'_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right). \]

Further, Eqs. (85), (83), (96) imply that

\[ \phi_n(r) = -s^* r + E_0(s^*) \]

\[ = \frac{s^*}{1 + s^*} \Lambda'_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) - \Lambda_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right). \]

By comparing Eqs. (96) and (98), we obtain

\[ \Lambda'_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) = \phi_n(r) - r \]

which is exactly the optimum solution to the Fenchel-Legendre transform \( \Lambda_{0,P_{x^n}}(z) \) in Eq. (22) with

\[ t^* = \frac{s^*}{1 + s^*} \in (0, 1), \]

\[ z = \phi_n(r) - r. \]

From Eqs. (22), (99) and (98), we conclude the item (i) of Lemma 7:

\[ \Lambda_{0,P_{x^n}}(\phi_n(r) - r) = t^* z - \Lambda_{0,P_{x^n}}(t^*) \]

\[ = \frac{s^*}{1 + s^*} (\phi_n(r) - r) - \Lambda_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) \]

\[ = \frac{s^*}{1 + s^*} \Lambda'_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) - \Lambda_{0,P_{x^n}} \left( \frac{s^*}{1 + s^*} \right) \]

\[ = \phi_n(r). \]
Item (ii) follows from item (i), the symmetry $\Lambda_{0,x_1}(t) = \Lambda_{1,x_1}(1 - t)$ and $\Lambda'_{0,x_1}(t) = -\Lambda'_{1,x_1}(1 - t)$, and Eq. (22). $\Lambda^*_1(P^n_x) (r - \phi(r)) = r$.

For the item (iii), the positivity of $\Lambda''_{0,P^n_x}(t)$, for $t \in [0,1]$, implies that the objective function $tz - \Lambda_{0,P^n_x}(t)$ in Eq. (22) is strictly concave in $t$ for $t \in [0,1]$. Hence, by Eq. (100), the optimizer $t^* \in (0,1)$ exists uniquely. By recalling Eq. (99), we complete the claim in item (iii).

\[\square\]

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