The Minimality of the Georges–Kelmans Graph

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Abstract

In 1971, Tutte wrote in an article that it is tempting to conjecture that every
3-connected bipartite cubic graph is hamiltonian. Motivated by this remark, Horton
constructed a counterexample on 96 vertices. In a sequence of articles by different
authors several smaller counterexamples were presented. The smallest of these
graphs is a graph on 50 vertices which was discovered independently by Georges
and Kelmans. In this article we show that there is no smaller counterexample.

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As all non-hamiltonian 3-connected bipartite cubic graphs in the literature have cyclic 4-cuts—even if they have girth 6—it is natural to ask whether this is a necessary prerequisite. In this article we answer this question in the negative and give a construction of an infinite family of non-hamiltonian cyclically 5-connected bipartite cubic graphs.

In 1969 Barnette gave a weaker version of the conjecture stating that 3-connected planar bipartite cubic graphs are hamiltonian. We show that Barnette’s conjecture is true up to at least 90 vertices. We also report that a search of small non-hamiltonian 3-connected bipartite cubic graphs did not find any with genus less than 4.

1 Introduction

In this article all graphs will be simple and undirected unless explicitly stated otherwise. For any standard graph theoretical concepts not explicitly defined here, please refer to [9].

Tait conjectured in 1880 that every 3-connected planar cubic graph is hamiltonian. This conjecture was disproved in 1946 by Tutte [23], who constructed a counterexample on 46 vertices. If Tait’s conjecture had been true, it would have implied the Four Colour Theorem. In the years that followed, several researchers constructed smaller counterexamples. In 1988, Holton and McKay [15] completely settled the problem by showing that the smallest non-hamiltonian 3-connected planar cubic graphs have 38 vertices and that there are exactly 6 such graphs of that order (one of which is the famous Barnette–Bosák–Lederberg graph).

For any graph $H$, $V(G)$ is its vertex set and $E(G)$ is its edge set. A graph is cyclically $k$-edge-connected if the deletion of fewer than $k$ edges does not create two components both of which contain at least one cycle. Similarly, for cyclically $k$-vertex-connected. With the exception of a few graphs on 6 or fewer vertices, the cyclic edge-connectivity and cyclic vertex-connectivity of a connected cubic graph both equal the size of the smallest cut of independent edges [20]. (Edges are independent if they have no endpoints in common.) Consequently we will call this number just cyclic connectivity. With the same small exceptions, the complement of a shortest cycle has too many edges to be acyclic, so the cyclic connectivity is at most equal to the girth. Also note that cyclic 3-connectivity implies 3-connectivity.

In 1971, Tutte [24] wrote that it is tempting to conjecture that every 3-connected bipartite cubic graph is hamiltonian. This statement that is often cited as a conjecture was disproved in 1976 by Horton [3, p. 240], who constructed a counterexample on 96 vertices. In 1982 he found a smaller counterexample, on 92 vertices [16]. Later Ellingham [10] discovered an infinite family of non-hamiltonian 3-connected bipartite cubic graphs. The smallest member of his family has 78 vertices.

All of the examples mentioned so far have cyclic connectivity 3. In 1983, Ellingham and Horton [11] published a non-hamiltonian cyclically 4-connected bipartite cubic graph. Their 54-vertex graph can be seen in Figure 1(a).

Finally, Georges [12] and Kelmans [17, 18] independently discovered another infinite family of non-hamiltonian cyclically 4-connected bipartite cubic graphs, the smallest of which has 50 vertices. Their 1986 submission dates were only 15 days apart. This graph is shown in Figure 1(b) and is the smallest known.

In [17] it is written that Lomonosov and Kelmans proved without computer assistance that Tutte’s conjecture holds up to 30 vertices, but no reference is given. Recently,
Knauer and Valicov [19] verified Tutte’s conjecture up to 40 vertices using computational methods. Our aim in this paper is to prove that the Georges–Kelmans graph is in fact minimal.

**Theorem 1.1.** There is no smaller non-hamiltonian 3-connected bipartite cubic graph than the Georges–Kelmans graph (which has 50 vertices). Moreover if there is another example with 50 vertices, it has girth and cyclic connectivity exactly 6.

**Figure 1:** The Ellingham–Horton graph on 54 vertices (a) and the Georges–Kelmans graph on 50 vertices (b). (The asterisks are used in Section 3.)

Since we plan on liberal use of the computer, the reader may wonder why we do not just test all the 3-connected bipartite cubic graphs up to 48 vertices. The reason is that there are far too many of them. Based on exact counts for smaller sizes (Table 1 below; see [6] for some of the graphs) we estimate that there are about $5 \times 10^{17}$ cubic bipartite graphs with 48 vertices, most of them 3-connected. With our programs able to generate and test about 50,000 bipartite cubic graphs per second, this approach would take about 320,000 CPU years.

We will instead take an incremental approach that combines theory and computation, successively eliminating the possibilities that a non-hamiltonian 3-connected bipartite cubic graph smaller than the Georges–Kelmans graph is cyclically 3-connected, cyclically 4-connected, cyclically 5-connected, and finally show that it doesn’t exist. The required amount of computer time was large but available.

Furthermore, noting that all previous examples had cyclic connectivity at most 4, we construct an infinite family of non-hamiltonian cyclically 5-connected bipartite cubic graphs, of which the smallest has 300 vertices.

Another important strongly related conjecture is Barnette’s conjecture, which is a weakened combination of Tait and Tutte’s conjectures. Barnette [2] conjectured in 1969 that every 3-connected planar bipartite cubic graph is hamiltonian. Even after 50 years, this conjecture is still open. Holton, Manvel and McKay [14] showed that Barnette’s conjecture is true up to at least 64 vertices and this was later improved to 84 vertices by Aldred, Brinkmann and McKay [1]. We will show that Barnette’s conjecture is true up to at least 90 vertices. We also did a non-exhaustive search for non-hamiltonian 3-connected bipartite cubic graphs that are close to planar in the sense of having low genus. However we did not find any with genus less than 4.

The article is organised as follows. In Section 2 we give the theoretical and computational results necessary to prove Theorem 1.1. We verified the correctness of each of
Table 1: Counts of connected bipartite cubic graphs with girth at least 4, 6 or 8.

| Vertices | Girth at least 4 | Girth at least 6 | Girth at least 8 |
|----------|------------------|------------------|------------------|
| 6        | 1                |                  |                  |
| 8        | 1                |                  |                  |
| 10       | 2                |                  |                  |
| 12       | 5                |                  |                  |
| 14       | 13               | 1                |                  |
| 16       | 38               | 1                |                  |
| 18       | 149              | 3                |                  |
| 20       | 703              | 10               |                  |
| 22       | 4132             | 28               |                  |
| 24       | 29,579           | 162              |                  |
| 26       | 245,627          | 1,201            |                  |
| 28       | 2,291,589        | 11,415           |                  |
| 30       | 23,466,857       | 125,571          | 1                |
| 32       | 259,974,248      | 1,514,489        | 0                |
| 34       | 3,087,698,618    | 19,503,476       | 1                |
| 36       | 39,075,020,582   | 265,448,847      | 3                |
| 38       | 524,492,748,500  | 3,799,509,760    | 10               |
| 40       | 7,439,833,931,266| 57,039,155,060   | 101              |
| 42       | ?                | 896,293,917,129  | 2,510            |
| 44       | ?                | ?                | 79,605           |
| 46       | ?                | ?                | 2,607,595        |
| 48       | ?                | ?                | 81,716,416       |
| 50       | ?                | ?                | 2,472,710,752    |

our computational results by making an independent implementation of every program which was specifically developed for this project (often using an alternative algorithm) and using it to replicate each computational result as far as CPU time limits allowed. These correctness tests are described in Section 2 as well. In Section 3 we present an infinite family of non-hamiltonian cyclically 5-connected bipartite cubic graphs. In Section 4 we verify Barnette’s conjecture up to 90 vertices and give the results about the generation of the non-hamiltonian 3-connected bipartite cubic graphs constructed in this project. As this project required the discovery of more than $10^{14}$ hamiltonian cycles in cubic graphs of up to 50 vertices, we needed a very efficient practical algorithm for this. We describe this algorithm in Section 5.

2 Properties of a minimal non-hamiltonian bipartite cubic 3-connected graph

**Lemma 2.1.** In a bipartite graph with vertices of degree 2 and 3, the number of vertices of degree 2 of each colour (that is: of each bipartition class) are equal modulo 3.

**Proof.** Define $n_{d,c}$ to be the number of vertices of degree $d$ and colour $c$, for $d = 2, 3$ and $c = 0, 1$. Counting the edges in two different ways, we have $3n_{3,0} + 2n_{2,0} = 3n_{3,1} + 2n_{2,1}$, from which it follows that $n_{2,0}$ and $n_{2,1}$ are equal modulo 3. □
Lemma 2.2. A minimal non-hamiltonian 3-connected bipartite cubic graph is cyclically 4-connected.

Proof. Suppose that a non-hamiltonian 3-connected bipartite cubic graph $G$ has an independent 3-edge cut \{$e_1, e_2, e_3$\}. Divide $G$ into two parts at the cut. Due to Lemma 2.1 in each part the vertices of degree 2 have the same colour, so we can use one extra vertex for each part and connect it to the three vertices of degree 2 to form two bipartite cubic graphs $G_1, G_2$. It is an easy consequence of Menger’s theorem that they are both 3-connected. For $j = 1, 2$ we label the new edges \{$e_{1,j}, e_{2,j}, e_{3,j}$\} so that for $1 \leq i \leq 3$, $e_{i,j}$ is in $G_j$ and has one endpoint the same as $e_i$. As the cut was independent, $G_1$ and $G_2$ are smaller than $G$.

If one of $G_1, G_2$ was non-hamiltonian, $G$ would not be minimal, so we may assume that both $G_1$ and $G_2$ are hamiltonian.

Now assume that one smaller graph, w.l.o.g. $G_1$, has a hamiltonian cycle not containing $e_{i,1}$. If $G_2$ had a hamiltonian cycle not containing $e_{i,2}$, these cycles could be combined to form a hamiltonian cycle of $G$. If for one smaller graph $G_1$, each edge of \{$e_{1,j}, e_{2,j}, e_{3,j}$\} could be avoided by a hamiltonian cycle or for both smaller graphs there would be at least two edges that can be avoided, there would be a combination of hamiltonian cycles where the same edge would be avoided—so they could be combined to form a hamiltonian cycle of $G$. So in one smaller graph at most one edge can be avoided (which means that this forbidden edge is in no hamiltonian cycle) and in the other at most two edges can be avoided (which means that there is a forced edge that lies in each hamiltonian cycle).

A straightforward computation using the programs minibaum and cubhamg showed that forced edges first appear at 30 vertices and forbidden edges first appear at 34 vertices. (Minibaum \cite{4} is a generator for cubic graphs which can also generate bipartite cubic graphs efficiently and cubhamg is described in Section 5. The bipartite cubic graphs up to 30 vertices are available at the House of Graphs \cite{6}.) Using an independent implementation of a program to test if a graph contains forced or forbidden edges, we obtained exactly the same number of graphs with forced/forbidden edges up to 34 vertices. Therefore, this construction only yields non-hamiltonian graphs with at least 62 vertices—larger than the Georges–Kelmans graph.

Lemma 2.3. A minimal non-hamiltonian 3-connected bipartite cubic graph cannot have cyclic connectivity 4 unless it is the Georges–Kelmans graph. Moreover, a minimal non-hamiltonian 3-connected bipartite cubic graph has girth at least 6.

Proof. For a cubic graph $H$, define two types of edge $e$:

- Type 1: There is a hamiltonian cycle through at most 3 of the 4 paths of three edges whose central edge is $e$.
- Type 2: At least one of the auxiliary (non-bipartite) graphs $H', H''$ defined as in Figure 2 has no hamiltonian cycle.

Now let $G$ be a minimal non-hamiltonian bipartite cubic graph with cyclic connectivity 4. By Lemma 2.1 we can divide $G$ into two parts at a 4-edge cut and use new edges $e_1, e_2$ to complete the parts into bipartite cubic 3-connected graphs $G_1, G_2$ as shown in Figure 3. The 3-connectivity of $G_1$ and $G_2$ follows from the observation that a 2-cut in either of them would imply a cyclic 3-edge cut in $G$.

If $G_1$ has a hamiltonian cycle using $e_1$, then $e_2$ must have type 1, since otherwise $G$ would be hamiltonian. Similarly, if $G_1$ has a hamiltonian cycle avoiding $e_1$, then $e_2$ must have type 2, since otherwise $G$ would be hamiltonian.
Figure 2: Auxiliary graphs for type 2 edges used in the proof of Lemma 2.3.

Figure 3: Splitting a graph into two at a 4-edge cut.

A direct computation, again using minibaum and cubhamg, showed that type 1 or type 2 edges first appear at 18 vertices, so we have $|V(G_1)| \geq 18$, and similarly $|V(G_2)| \geq 18$. Consequently, it suffices to test combinations of graphs $G_1, G_2$ with $18 \leq |V(G_1)| \leq 26$ and $18 \leq |V(G_2)| \leq 54 - |V(G_1)|$ with $G_1$ restricted to graphs having an edge of type 1 or type 2. We could also restrict $G_2$ in the same way, but the number of possibilities for $G_1$ when $G_2$ is large is so small that simply testing every graph as $G_2$ is as fast as checking $G_2$ for edges of type 1 or type 2.

This computation yielded only the Georges–Kelmans graph. The counts are shown in Table 2 (which includes graphs that are connected but not 3-connected). Using an independent implementation, we obtained exactly the same number of graphs with type 1 or type 2 edges as in Table 2. Using another independent program we again tested all combinations of $G_1, G_2$ and this indeed only yielded the Georges–Kelmans graph.

For the second part of the lemma, it is only necessary to observe that the Georges–Kelmans graph has girth 6. 

$$\begin{array}{|c|c|c|c|c|c|}
\hline
|V(G_1)| & \text{type 1 count} & \text{type 2 count} & \text{G}_1 \text{ total} & |V(G_2)| & \text{G}_2 \text{ count} \\
\hline
18 & 1 & 1 & 0 & 18–36 & 6461410120 \\
20 & 1 & 1 & 2 & 18–34 & 3373711502 \\
22 & 5 & 3 & 8 & 18–32 & 286012884 \\
24 & 15 & 14 & 27 & 18–30 & 26038636 \\
26 & 71 & 56 & 121 & 18–28 & 2571779 \\
\hline
\end{array}$$

Table 2: The numbers of graphs that can act as $G_1, G_2$ in the proof of Lemma 2.3 with the candidates for $G_1$ also listed according to their type.

At this stage we could consider finishing the proof of Theorem 1.1 by computation alone. However, we estimate the number of bipartite cubic graphs with girth 6 up to 48 vertices to be around $4.5 \times 10^{15}$. With a lower generation rate of about 40,000 graphs per
second, this still amounts to 3,500 CPU years. By Lemma 2.3 we could also restrict our search to cyclic connectivity at least 5, but the counts are not much less and we don’t know of a fast generator.

**Lemma 2.4.** A minimal non-hamiltonian 3-connected bipartite cubic graph cannot have cyclic connectivity 5.

**Proof.** Define a 5-piece to be a connected bipartite graph of girth at least 6, cubic apart from 5 vertices of degree 2. By Lemma 2.1, a 5-piece has a vertex of degree 2 whose colour is different from the other vertices of degree 2; call that the *special vertex* of the 5-piece. Also, the number of vertices in the 5-piece with the same colour as the special vertex is one less than the number with the other colour.

Let \( G \) be a bipartite cubic graph with cyclic connectivity 5 and at most 50 vertices – so \( G \) is a candidate for a counterexample to the lemma. Separate \( G \) at a 5-cut into 5-pieces \( G_1 \) and \( G_2 \) as in Figure 4. We know that \( G_1 \) and \( G_2 \) are connected since otherwise \( G \) would have an edge cut of two independent edges. The special vertices of \( G_1 \) and \( G_2 \) are adjacent in \( G \); call that the *special edge* of \( G \).

The difference in the numbers of vertices in the two colour classes in 5-pieces imply:

(a) If a hamiltonian cycle in \( G \) uses 4 edges of the cut, then one of those edges is the special edge.

(b) If a hamiltonian cycle in \( G \) uses only 2 edges of the cut, then neither of those edges is the special edge.

Given a 5-piece \( H \), a “test graph” for \( H \) is formed by adjoining two vertices of degree 2, together adjacent to four distinct vertices of degree 2 in \( H \), one of which is the special vertex. There are 12 (non-bipartite) test graphs, three of which are shown in Figure 5. Classify 5-pieces as follows:
• Class 0: None of the test graphs is hamiltonian;
• Class 1: At least one of the test graphs is hamiltonian;
• Class 2: All of the test graphs are hamiltonian (a subset of Class 1).

Claim 1: If \( G_1 \) and \( G_2 \) are in Class 1, with one of them in Class 2, then \( G \) is hamiltonian.
Proof: Suppose that \( G_2 \) is in Class 2. Since \( G_1 \) is in Class 1, it can be covered by two paths with endpoints \( v_1, v_2 \) and \( w_1, w_2 \), where one of \( v_1, v_2, w_1, w_2 \) is the special vertex. Join these two paths into a hamiltonian cycle in \( G \) using the hamiltonian cycle in the test graph of \( G_2 \) where one vertex was connected to \( v_1, v_2 \) and the other to \( w_1, w_2 \).

Claim 2: If \( |V(G_1)| \leq 13 \), then \( G \) is hamiltonian.
Proof: As shown in Table 3 and checked twice by computer, there are no 5-pieces of order less than 11, one of order 11 and two of order 13. They are depicted in Figure 6. All of them are in Class 2. So, by Claim 1, the only possibility that \( G \) is non-hamiltonian is for \( G_2 \) to be in Class 0. A hamiltonian cycle in \( G \) that uses 4 edges of the cut would imply hamiltonicity of one of \( G_2 \)'s test graphs, so the only possibility is a hamiltonian cycle in \( G \) that uses 2 edges of the cut (none of them the special edge) and induces hamiltonian paths in \( G_1 \) and \( G_2 \).

Looking at the hamiltonian paths in \( G_1 \), we find that the non-special vertices of degree 2 can be labelled \( v_1, v_2, w_1, w_2 \) such that any pair of them can be joined by a hamiltonian path except \( v_1, v_2 \) and possibly \( w_1, w_2 \). (See Figure 6) Let \( v'_1, v'_2, w'_1, w'_2 \) be the vertices of \( G_2 \) adjacent in \( G \) to \( v_1, v_2, w_1, w_2 \), respectively. Now construct a bipartite cubic graph \( G^+_2 \) from \( G_2 \) by adjoining a path \( xyz \) with \( x \) adjacent to \( v'_1 \) and \( v'_2 \), \( y \) adjacent to the special vertex, and \( z \) adjacent to \( w'_1 \) and \( w'_2 \). Since \( |V(G^+_2)| < |V(G)| \), and it is 3-connected since otherwise \( G \) would have a 3-edge cut, \( G^+_2 \) is hamiltonian by the minimality of \( G \). Also, since \( G_2 \) is in Class 0, any hamiltonian cycle in \( G^+_2 \) cannot use exactly one of the edges \( xy \) and \( yz \) (else one of the test graphs of \( G_2 \) is hamiltonian), so it must use both \( xy \) and \( yz \). This provides a hamiltonian path in \( G_2 \) from a vertex in \( \{v'_1, v'_2\} \) to a vertex in \( \{w'_1, w'_2\} \). Any such path can be combined with a hamiltonian path in \( G_1 \) to make a hamiltonian cycle in \( G \). This completes the proof of Claim 2.

Now we can complete the proof of the lemma. We used the program multigraph to construct all 5-pieces and tested them with cubhamg. (Multigraph can generate all simple graphs or multigraphs with a given degree sequence. It implements the same ideas as minibaum [4] but for general degree sequences, and as it does not contain new ideas, it was never published. For a large number of degree sequences its results were tested and
confirmed by the program described in [13].) The computations showed that there are no 5-pieces of order at most 35 in Class 0. The number in Class 1 \ Class 2 are listed in Table 3. The total computation time was about 5 CPU years. We also independently determined all 5-pieces up to 29 vertices by using minibaum to generate all bipartite cubic graphs and using a separate program to remove one vertex and two of its neighbours in all possible ways of each input graph and retaining the connected graphs of girth at least 6 among the graphs resulting from this operation. We then determined the class of each 5-piece using another independent implementation and the results were in complete agreement with the counts reported in Table 3.

We joined all combinations of two of these 5-pieces in Class 1 \ Class 2 up to 50 vertices and found only the Georges–Kelmans graph among those that were 3-connected. (Also here we replicated this result using an independent implementation of the joining program.) The Georges-Kelmans graph has cyclic connectivity 4. Together with Claims 1 and 2, this completes the proof.

**Lemma 2.5.** All bipartite cubic graphs with girth at least 8 up to 50 vertices are hamiltonian.

**Proof.** This result was purely computational. There are only about $2.5 \times 10^9$ such graphs (see Table 1) but the much reduced generation time for girth 8 meant that it took about 22 CPU years.

After completing this long computation, we found that a modification of Meringer’s program genreg [22] made by the first author could generate the graphs in only 8 CPU months. This prompted us to perform a stronger test: all bipartite cubic graphs with girth at least 8 up to 50 vertices have the property that for each pair of distinct edges $e, e'$ there is a hamiltonian cycle using $e$ and not using $e'$.

**Proof of Theorem 1.1.** Let $G$ be a non-hamiltonian 3-connected bipartite cubic graph with at most 48 vertices. By Lemmas 2.2, 2.5 we know that $G$ is cyclically 6-connected and has girth 6.

Our approach is as follows: We define a reduction that transforms a bipartite cubic graph with girth 6 on $n$ vertices to a bipartite cubic graph with some 4-cycles on $n - 8$ vertices.

| $n$ | 5-pieces | Class $1 \setminus$ Class $2$ |
|-----|----------|-----------------------------|
| 11  | 1        | 0                           |
| 13  | 2        | 0                           |
| 15  | 12       | 2                           |
| 17  | 90       | 7                           |
| 19  | 754      | 14                          |
| 21  | 7,003    | 25                          |
| 23  | 70,639   | 68                          |
| 25  | 766,134  | 251                         |
| 27  | 8,862,333| 1,086                       |
| 29  | 108,917,294| 6,098                     |
| 31  | 1,417,268,482 | 44,842              |
| 33  | 19,471,253,036 | 393,423            |
| 35  | 281,715,327,672 | 3,887,896         |

Table 3: Computation for Lemma 2.4.
vertices. The set of reduced graphs will be much smaller than the set of original graphs, but there will also be irreducible graphs on \( n \) vertices. The irreducible graphs have to be generated and tested directly and on the reduced graphs some tests have to be performed in order to guarantee that they do not come from a non-hamiltonian graph, and reduced graphs that do not pass the test have to be extended and checked for hamiltonicity.

In order to be able to reduce the computation time to an amount available on a modern cluster, there has to be a balance between the two parts. On one hand the reduction should be so that there are not too many irreducible graphs and that it is possible to generate all irreducible graphs. On the other hand the tests necessary on the reduced graphs must not be too expensive, so that the reduced graphs can be tested in an affordable amount of time. The (admittedly very technical) reduction we used is the following:

Let \( \hat{G} \) be the bipartite graph with 16 vertices depicted on the left hand side of Figure 7. We will call a bipartite cubic graph \( G \) of girth 6 reducible if it contains \( \hat{G} \) in a way that has the following properties:

(i) Neither \( v'_8 \) and \( v'_3 \) nor \( v'_7 \) and \( v'_4 \) are adjacent.

(ii) At least one of the paths \( v'_1, v_1, v_2, v'_2 \) and \( v'_5, v_5, v_6, v'_6 \) lies on a 6-cycle.

(iii) If exactly one of the paths in (ii) lies on a 6-cycle, then in addition at least one of the paths \( v'_8, v_8, v_1, v_2, v'_3 \) and \( v'_4, v_4, v_5, v_6, v'_7 \) lies on an 8-cycle.

The reduced graph \( G_r \) is then obtained by deleting \( v_1, \ldots, v_8 \) and adding the edges \( e_1 = \{v'_1, v'_2\}, e_2 = \{v'_8, v'_2\}, e_3 = \{v'_5, v'_1\}, \) and \( e_4 = \{v'_6, v'_5\}, \) as depicted on the right hand side of Figure 7. It is obvious that \( G_r \) is bipartite and cubic. Moreover, since \( G \) has girth 6 and satisfies property (i), \( G_r \) is simple. Since \( G_r \) is cubic, each component of \( G_r \) contains a cycle. If cycles \( Z_1 \) and \( Z_2 \) lie in different components of \( G_r \), then with Menger’s theorem applied to the graph obtained by adding two new vertices and connecting them once to all vertices of \( Z_1 \) and once to all vertices of \( Z_2 \), the cyclic 5-connectivity of \( G \) means that there are at least 5 edge-disjoint paths in \( G \) from \( Z_1 \) to \( Z_2 \). However, at most 4 of these paths can contain vertices of the set \( \{v_1, \ldots, v_8\} \), so one of them connects \( Z_1 \) and \( Z_2 \) in \( G_r \), contradicting the assumption that \( Z_1 \) and \( Z_2 \) are in different components. Thus, \( G_r \) is connected.

\[
\begin{tikzpicture}

\node[fill=black, circle] (v1) at (0,0) {}; \node[draw=black, circle] (v2) at (0,1) {}; \node[draw=black, circle] (v3) at (1,0) {}; \node[draw=black, circle] (v4) at (1,1) {}; \node[draw=black, circle] (v5) at (2,0) {}; \node[draw=black, circle] (v6) at (2,1) {}; \node[draw=black, circle] (v7) at (3,0) {}; \node[draw=black, circle] (v8) at (3,1) {}; \node[draw=black, circle] (v9) at (4,0) {}; \node[draw=black, circle] (v10) at (4,1) {}; \node[draw=black, circle] (v11) at (5,0) {}; \node[draw=black, circle] (v12) at (5,1) {}; \node[draw=black, circle] (v13) at (6,0) {}; \node[draw=black, circle] (v14) at (6,1) {}; \node[draw=black, circle] (v15) at (7,0) {}; \node[draw=black, circle] (v16) at (7,1) {};

\foreach \a in {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16}
\draw (v\a) -- (v\a+1);
\draw (v16) -- (v1);
\end{tikzpicture}
\]

\[
\begin{tikzpicture}

\node[fill=black, circle] (v1) at (0,0) {}; \node[draw=black, circle] (v2) at (0,1) {}; \node[draw=black, circle] (v3) at (1,0) {}; \node[draw=black, circle] (v4) at (1,1) {}; \node[draw=black, circle] (v5) at (2,0) {}; \node[draw=black, circle] (v6) at (2,1) {}; \node[draw=black, circle] (v7) at (3,0) {}; \node[draw=black, circle] (v8) at (3,1) {}; \node[draw=black, circle] (v9) at (4,0) {}; \node[draw=black, circle] (v10) at (4,1) {}; \node[draw=black, circle] (v11) at (5,0) {}; \node[draw=black, circle] (v12) at (5,1) {}; \node[draw=black, circle] (v13) at (6,0) {}; \node[draw=black, circle] (v14) at (6,1) {}; \node[draw=black, circle] (v15) at (7,0) {}; \node[draw=black, circle] (v16) at (7,1) {};

\draw (v1) -- (v2) -- (v3) -- (v4) -- (v5) -- (v6) -- (v7) -- (v8) -- (v9) -- (v10) -- (v11) -- (v12) -- (v13) -- (v14) -- (v15) -- (v16);
\draw (v1) -- (v3) -- (v5) -- (v7) -- (v9) -- (v11) -- (v13) -- (v15) -- (v16);
\end{tikzpicture}
\]

\[\text{Figure 7: The reduction of an 8-cycle.}\]

If \( G_r \) is hamiltonian, this does not necessarily imply that \( G \) is also hamiltonian, but some hamiltonian cycles in \( G_r \) imply hamiltonicity in \( G \). We call a hamiltonian cycle \( H_r \) of \( G_r \) extendable for \( (e_1, \ldots, e_4) \) if \( E(H_r) \cap \{e_1, e_2, e_3, e_4\} \) is one of \( \{e_1\}, \{e_4\}, \{e_1, e_2\}, \ldots \).
\{e_2, e_3\} and \{e_3, e_4\}. In each of these cases it is easy to see that \(E(H_r) \setminus \{e_1, \ldots, e_4\}\) can be extended to a hamiltonian cycle in \(G\).

Our strategy now consists of two parts:

(a) Generate all irreducible bipartite cubic graphs of girth 6 on up to 48 vertices and test them for hamiltonicity.

(b) Generate all connected bipartite cubic graphs \(G_r\) with at most 40 vertices that may be the reduction of a bipartite cubic graph \(G\) of girth 6. For each 4-tuple \((e_1, \ldots, e_4)\) of edges of \(G_r\) that may be the new edges added in the reduction, determine whether \(G_r\) has a hamiltonian cycle extendable for \((e_1, \ldots, e_4)\). If not, reconstruct \(G\) and test it for hamiltonicity.

For part (a) we used the program \textit{minibaum}. The algorithm used in \textit{minibaum} constructs the graphs by recursively adding one edge at a time. The subgraph constructed at each step remains part of all descendants, so as soon as the subgraph on the left hand side of Figure 7 appears with conditions (i)–(iii) satisfied, we know that all cubic graphs descended from this step are reducible. Consequently, the generation tree can be pruned at this point.

The program was run on a cluster with various different processors. It needed about 7 CPU years and generated 136,941,076 irreducible graphs. Among the graphs were 4 non-hamiltonian ones, but they were not 3-connected. We also independently generated all irreducible graphs up to 40 vertices using the unmodified version of \textit{minibaum} to generate all bipartite cubic graphs of girth 6 and an independently implemented program to filter the irreducible graphs. The results were in complete agreement.

Part (b) of the proof was the most computationally expensive step in our whole project. We illustrate the magnitude of the task for \(G_r\) having 40 vertices. As indicated in Table 1 there are 7,439,833,931,266 connected bipartite cubic graphs with 40 vertices. In total (up to reversal), there were 129,922,879,860,637,000 possibilities for the 4-tuple \((e_1, e_2, e_3, e_4)\). In all but 417,626,620,084 of these (1 in 311,098) there was an extendable hamiltonian cycle. Since hamiltonian cycles can be extendable for many 4-tuples, the total number of hamiltonian cycles found was “only” 131,062,665,710,324. Of the 417,626,620,084 4-tuples for which there was no extendable hamiltonian cycle, the reconstructed graph on 48 vertices was hamiltonian except in 368 cases, none of them 3-connected. The total time was 5 CPU years for generation and 85 CPU years for hamiltonian cycle investigation. We also independently verified the computations for part (b) up to 34 vertices.

That completes the proof of the first part of Theorem 1.1. For the second part, recall that Lemmas 2.2–2.5 apply also to 50 vertices.

Remark 2.6. For each even girth \(g \geq 4\), there are infinitely many non-hamiltonian 3-connected bipartite cubic graphs of girth \(g\).

Proof. There exist 3-connected bipartite cubic graphs of arbitrary large even girth \(g\) (see e.g. the survey [25]). A non-hamiltonian 3-connected bipartite cubic graph of arbitrary even girth \(g\) can be obtained by replacing every vertex \(v\) of a non-hamiltonian 3-connected bipartite cubic graph \(G\) by a copy \(G'_v\) of a 3-connected bipartite cubic graph \(G'\) of girth \(g\) with one vertex removed. If \(\{v_1, v_2\}\) is an edge in \(G\), then a vertex of degree 2 in \(G'_{v_1}\) is connected to a vertex of degree 2 in \(G'_{v_2}\) in a way that a 3-regular graph is constructed.
3 Non-hamiltonian cyclically 5-connected bipartite cubic graphs

Cubic graphs of course cannot have connectivity greater than 3, but among all cubic 3-connected graphs, the cyclic connectivity provides a measure of how strong the connections between the parts of the graph are. Though the Georges–Kelmans graph has girth 6, which is a necessary prerequisite for being cyclically 6-connected, it is only cyclically 4-connected. To be exact: it has no cyclic edge-cuts of size 3, but 8 cyclic edge-cuts of size 4.

Extending a folklore technique (described to us by Carol Zamfirescu) for constructing graphs without hamiltonian paths from graphs without hamiltonian cycles, we will now describe how to construct non-hamiltonian cyclically 5-connected bipartite cubic graphs out of a suitable non-hamiltonian cyclically 5-connected bipartite cubic graph with lower cyclic connectivity, such as the Georges–Kelmans graph.

Recall that minimal cyclic edge cuts in cubic graphs are always independent edge cuts. We denote the distance between vertices \( v, w \) by \( d(v, w) \). Let \( G = (V, E) \) be a cyclically 4-connected cubic graph, so that there is a vertex \( v \in V \) with neighbours \( N(v) = \{w_1, w_2, w_3\} \) and an edge \( e = \{x, y\} \), so that \( d(v, x) \geq 3, d(v, y) \geq 3 \), and for each independent edge cut \( C \) of \( G \) with \(|C| = 4\) we have that neither \( v \) nor \( x \) are contained in an edge of \( C \) and that \( v \) and \( x \) are in different components of \( G \setminus C \). Such graphs have girth at least 5, as a 4-cycle containing \( v \) or \( x \) would imply a 4-cut containing \( v \) or \( x \), and a 4-cycle not containing \( v \) or \( x \) would imply a 4-cut with \( v \) and \( x \) in the same component.

In the figures showing the operations we use coloured vertices to illustrate that we can choose to maintain bipartiteness; nevertheless the construction is not restricted to bipartite graphs.

Take three copies \( G_1, G_2, G_3 \) of \( G \) with corresponding vertices \( v_i, w_{i,1}, w_{i,2}, w_{i,3}, x_i, y_i \) for \( 1 \leq i \leq 3 \) and add three new vertices \( z_1, z_2, z_3 \). Then remove \( v_1, v_2, v_3 \) and for \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 3 \) connect \( z_j \) with \( w_{i,j} \) and remove the edge \( \{x_i, y_i\} \). This operation is depicted on the left hand side of Figure 8. We call the result of this operation a \textit{brick}. Then we can take an arbitrary 3-edge connected cubic multigraph \( M \), replace each vertex by a triple and assign the three bricks to the three adjacent edges. Finally we connect vertices \( x_k \) and \( y_l \) from different bricks that are assigned to the same edge. On the right hand side of Figure 8 this is depicted for the graph \( M = K_4 \), but the smallest choice for \( M \) is the cubic multigraph with 2 vertices. We call this operation the \textit{brick-operation} \( \mathcal{T}(G, v, e, M) \).

\textbf{Theorem 3.1.} Let \( G = (V, E) \) be a cyclically 4-connected cubic non-hamiltonian graph and \( v \in V, e = \{x, y\} \in E \) so that \( d(v, x) \geq 3, d(v, y) \geq 3 \), and for each cyclic edge cut \( C \) of \( G \) with \(|C| = 4\) we have that neither \( v \) nor \( x \) are contained in an edge of \( C \) and that \( v \) and \( x \) are in different components of \( G \setminus C \). Furthermore let \( M \) be a 3-edge-connected cubic multigraph. Then \( \mathcal{T}(G, v, e, M) \) is a cyclically 5-connected non-hamiltonian graph.
Figure 8: Constructing a cyclically 5-connected graph from a cyclically 4-connected one. The colour of the vertices $z_i$ and the vertices $w_{i,j}$ can be the other way around. Between the vertices $x_i$ and $y_i$ there is no edge.

Proof. Let $C = \{e_1, \ldots, e_4\}$ be a set of four independent edges in $T(G, v, e, M)$. We have to show that $C$ is not a cut of $T(G, v, e, M)$.

Claim: If a brick $B$ contains at most 2 edges of $C$, then $B \setminus C$ is connected.

Proof: Assume that this is not the case and let $C' \subset C$ denote a set of minimal size so that $B \setminus C'$ is not connected. This means that $B \setminus C'$ has two components, say $P_1$ and $P_2$, each of which has at most two endpoints of $C'$. Since every vertex of $B$ has degree as least two, neither of $P_1$ and $P_2$ can be isolated vertices, and the only vertices that might have degree 1 in $B \setminus C'$ are $x, y, w_1, w_2, w_3$ (if they are also endpoints of edges in $C'$).

Since each component in $B \setminus C'$ is not an isolated vertex and has at most two vertices of degree 1, it has a cycle unless it is a non-trivial path within the vertices $x, y, w_1, w_2, w_3$. As any two vertices of $x, y, w_1, w_2, w_3$ have distance at least 2 from each other ($\{x, y\}$ is removed when forming $B$, $\{x, w_1\}$ or $\{x, w_i\}$ are not present as $d(v, x) \geq 3$, $d(v, y) \geq 3$ and an edge $\{w_i, w_j\}$ would be a triangle in $G$ contradicting that it is cyclically 4-connected), this cannot be the case. So both components contain a cycle.

One of $P_1, P_2$ would contain at most one of $w_1, w_2, w_3$; w.l.o.g. $P_1$ contains only $w_1$ or none of these vertices. If $P_1$ contains none of $x, y$ or both, then $C' \cup \{v, w_1\}$ (resp. $C'$) would be a cyclic edge-cut of $G$ with at most 3 edges, otherwise $C' \cup \{v, w_1\}, \{x, y\}$ (resp. $C' \cup \{x, y\}$) would be a cyclic edge-cut of $G$ with at most 4 edges containing the edge $\{x, y\}$. Both cases are in contradiction to the assumptions on $G$, so $B \setminus C$ is connected.

As only one brick can contain more than two edges of $C$, for at most one brick $B$ we have that $B \setminus C$ is not connected. Assume first that there is such a $B \setminus C$ in a triple $T$. Then there is at most one edge of $C$ not in $B$ and two bricks in the same triple (different from $T$) belong to the same component of $T(G, v, e, M) \setminus C$. Due to $M$ being 3-connected all triples different from $T$ belong to the same component. The two bricks in $T$ different from $B$ are connected to bricks in other triples, so they belong to the same component. If $B$ contains all 4 edges of $C$, then each component contains a vertex of $x, y, w_1, w_2, w_3$, as
otherwise $C$ would be a cut in $G$ not separating $v$ and $x$. As $x, y, w_1, w_2, w_3$ are connected to the other triples, in this case the graph is connected. If $B$ contains only 3 edges of $C$, each component contains at least two vertices of $x, y, w_1, w_2, w_3$ (otherwise we had a 4-cut in $G$ containing $x$ or $v$) and the result follows analogously.

In $B \setminus C$ each vertex is either in a component with all of $w_1, w_2, w_3$—at least two of which have a path to another brick in the triple—or with both of $x, y$ and at least one of them is connected to another triple. So in this case $T(G, v, e, M) \setminus C$ is connected.

Assume now that all bricks are connected after removing $C$. If no triple contains three or more edges of $C$, then all triples are connected and as $M$ is 3-edge connected (so with the edges doubled 6-edge connected), all triples are in the same component of $T(G, v, e, M) \setminus C$.

The last case is that a triple $T$ contains three or more edges of $C$ and is disconnected, which means that the edges are adjacent to the three new vertices, which are nevertheless still connected to at least two bricks in the triple. The other triples are still connected, and as each brick in $T$ is connected with at least one edge to a brick in another triple, the bricks of $T$ also belong to the same component. This completes the proof of cyclic 5-connectivity.

Next we prove the non-hamiltonicity. Assume that $T(G, v, e, M)$ has a hamiltonian cycle $H$. Each edge-cut must contain a positive even number of edges of $H$, so the 6-edge-cut isolating a triple from the rest of the graph contains 2, 4 or 6 edges of $H$. If it contains exactly 2 edges, one of the bricks has no edges going to another triple, so it must have 2 edges going to the vertices in the triple not belonging to a brick—but this implies a hamiltonian cycle in $G$.

So the edge-cut contains at least 4 edges of $H$ and there is a brick $B$ where both edges $e_1, e_2$ going to another triple belong to $H$. After $H$ enters $B$ at $e_1$, it must first leave at $e_2$; otherwise there is another segment of $H$ that enters $B$ at $e_2$ and these two segments together would allow us to construct a hamiltonian cycle in the non-hamiltonian graph $G$. The same argument applies to the other brick which $e_1$ is incident with, giving us a cycle made of segments of $H$ involving only two bricks, in contradiction to $H$ being a hamiltonian cycle.

**Corollary 3.2.** There are infinitely many non-hamiltonian cyclically 5-connected bipartite cubic graphs. The smallest one has at most 300 vertices.

**Proof.** This is a direct consequence of Theorem 3.1 and the fact that the construction can be carried out to preserve bipartiteness. The lower bound is obtained by applying the construction to the Georges–Kelmans graph and the cubic multigraph on 2 vertices.

Our construction raises the question of whether even higher cyclic connectivity can be achieved in a non-hamiltonian bipartite cubic graph. We are not able to answer this question and propose it as a research topic.

### 4 Barnette’s conjecture and the girth

A famous variation on the problem at hand is Barnette’s conjecture [2], which states that every 3-connected planar bipartite cubic graph is hamiltonian. Here our main result is the following.

**Theorem 4.1.** Let $G$ be a 3-connected planar bipartite cubic graph with $n$ vertices. Then
(a) $n \leq 90$ implies that $G$ is Hamiltonian;

(b) $n \leq 78$ implies that every edge of $G$ lies on a Hamiltonian cycle;

(c) $n \leq 66$ implies that for any two edges $e_1, e_2$ of $G$, there is a Hamiltonian cycle through $e_1$ but avoiding $e_2$.

Proof. Using the generator `plantri` and the program `cubhamg`, we established part (c) by direct computation. This required approximately 37 CPU years. We also found that the same property holds for those 3-connected planar bipartite cubic graphs on 68 or 70 vertices that do not have a 4-face adjacent to two other 4-faces. This required another 5 CPU years. Parts (a) and (b) now follow as in [14, Theorem 5].

Even assuming that Barnette’s conjecture is true, it is natural to ask how close to planarity non-Hamiltonian 3-connected bipartite cubic graphs can be; in particular, what is the minimum genus of a non-Hamiltonian 3-connected bipartite cubic graph?

Using the program `multi_genus` to determine the genus, we found that the Georges–Kelmans graph has genus 5. The Ellingham–Horton graphs on 54 and 78 vertices have genus 4 and 7, respectively. The graphs of Horton with 92 and 96 vertices have genus 8 or 9, and genus between 8 and 10, respectively.

During our project we compiled a collection of small non-Hamiltonian 3-connected bipartite cubic graphs by a mixture of unsystematic searches. For 50–64 vertices, the number of graphs in our collection is 1, 4, 30, 187, 1334, 3377, 29529, 204069, respectively. (The non-Hamiltonicity of these graphs was confirmed with a separate program.) The smallest genus that occurs in the collection is 4, and the smallest graph found with genus 4 has 52 vertices (see Figure 9).

![Figure 9: A non-Hamiltonian 3-connected bipartite cubic graph of genus 4 on 52 vertices. Join two vertices of degree 2 if they have the same colour and shape. The two edges for each colour can be drawn on one handle.](image)

Note that the condition of being 3-connected is essential for Barnette’s conjecture. Requiring only connectivity but not 3-connectivity, there is a non-Hamiltonian planar bipartite cubic graph already on 26 vertices. For genus 1 and 2, the smallest bipartite cubic non-Hamiltonian graphs have 24 and 20 vertices, respectively.
5 A fast practical algorithm for hamiltonian cycles

As mentioned, this project required the discovery of more than $10^{14}$ hamiltonian cycles, in cubic graphs of up to 50 vertices. A naive search that grows a path one edge at a time has no chance of achieving this feat in an acceptable amount of time. A much faster approach, available as the program cubhamg (which is part of the nauty package [21]), has been used in several investigations since [15] but never published. We now give a brief description here.

At each point of time, every edge has a label NO (not in the hamiltonian cycle), YES (in the hamiltonian cycle), or UNDEC (undecided). Every edge is initialized to UNDEC, except for edges we wish to force into or out of the hamiltonian cycle.

Given an edge labelling, a propagation process can be performed by applying the following rules until no further rules can be applied or a failure condition occurs.

(a) If a vertex has two incident NOs or three incident YESes, a failure condition occurs.
(b) If a vertex has one incident NO, then the other two incident edges will be labelled YES.
(c) If a vertex has two incident YESes, then the other incident edge will be labelled NO.
(d) Let $P$ be a maximal path of edges labelled YES.
   (i) Suppose $P$ is a hamiltonian path. If its ends are adjacent, then a hamiltonian cycle has been found; otherwise, a failure condition occurs.
   (ii) If $P$ is not a hamiltonian path and its ends are adjacent, then the edge between the ends will be labelled NO.
   (iii) If there are distinct vertices $x, y$ not in $P$, such that the neighbours of $x$ are $y$ and the two ends of $P$, then the edge ${x, y}$ will be labelled YES.

The overall structure is a backtrack search, with three branches according to which edges incident to an arbitrarily chosen vertex are labelled NO. If propagation ends with a hamiltonian cycle, we are done. If it finishes without either a hamiltonian cycle or a failure condition, there must be a vertex with one incident edge labelled YES and the other two labelled UNDEC (otherwise (a), (b) or (c) could be applied). Now the search branches into two cases depending on which of the two UNDEC edges is to be labelled NO. If propagation finishes with a failure condition, we backtrack to the nearest branching point where there is an unexplored branch.

The efficiency of this process is high because propagation usually gives new labels to many edges. Also, propagation can be carried out very quickly. Each vertex $v$ has an attribute $a_v$ whose meaning is “if this vertex is an end of a maximal path of edges labelled YES, then the other end is $a_v$”. With this simple data structure, each of the propagation operations can be carried out in constant time.

Note that the algorithm is not a heuristic for finding cycles, but a complete search that certifies the absence of hamiltonian cycles as well as their presence. When a hamiltonian cycle is found, checking it is trivial, but it is also worth pointing out that our proofs would still be valid if an error sometimes caused a hamiltonian graph to be misidentified as non-hamiltonian. As an example, if Table 2 contained a few graphs that didn’t belong, it would only mean that we have more pairs $G_1, G_2$ to join.
6 The graphs mentioned in this paper

For the reader’s convenience, in Table 4 we give the identification by which the graphs mentioned in this paper can be examined and downloaded at the House of Graphs [6].

| order | description of non-hamiltonian graph                             | HoG id |
|-------|----------------------------------------------------------------|--------|
| 20    | smallest connected bipartite cubic graph                        | 6923   |
| 24    | ditto, of genus 1                                               | 34282  |
| 26    | ditto, planar                                                   | 34286  |
| 38    | Barnette–Bosak–Lederberg graph                                  | 954    |
| 50    | Georges–Kelmans graph                                           | 1096   |
| 52    | 3-connected bipartite cubic graph of genus 4                   | 33805  |
| 54    | Ellingham–Horton graph                                          | 1059   |
| 78    | Ellingham–Horton graph                                          | 1061   |
| 92    | Horton graph                                                    | 1179   |
| 96    | Horton graph                                                    | 1181   |

Table 4: House of Graphs id numbers for graphs mentioned in this paper.

An archive of the software used for this project is available at [7].

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