Classification of symmetric periodic trajectories in ellipsoidal billiards

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We classify nonsingular symmetric periodic trajectories (SPTs) of billiards inside ellipsoids of $\mathbb{R}^{n+1}$ without any symmetry of revolution. SPTs are defined as periodic trajectories passing through some symmetry set. We prove that there are exactly $2^{2n}(2^{n+1}-1)$ classes of such trajectories. We have implemented an algorithm to find minimal SPTs of each of the 12 classes in the 2D case ($\mathbb{R}^2$) and each of the 112 classes in the 3D case ($\mathbb{R}^3$). They have periods 3, 4 or 6 in the 2D case; and 4, 5, 6, 8 or 10 in the 3D case. We display a selection of 3D minimal SPTs. Some of them have properties that cannot take place in the 2D case.

I. INTRODUCTION

Smooth convex billiards are a paradigm of conservative dynamics, in which a particle collides with a fixed closed smooth convex hypersurface of $\mathbb{R}^{n+1}$. They provide examples of different dynamics: integrable, mostly regular, chaotic, etc. In this paper we tackle the integrable situation. Concretely, we find and classify symmetric periodic trajectories (SPTs) inside ellipsoids of $\mathbb{R}^{n+1}$. SPTs show different dynamics, which describe how they fold in $\mathbb{R}^{n+1}$. STs present symmetry with respect to a coordinate subspace of $\mathbb{R}^{n+1}$. Dynamics and symmetry are precisely the main aspects we consider in our classification of SPTs. We establish 112 classes of SPTs in the 3D case, and we find a representative of each class with the smallest possible period. Those minimal SPTs have periods 4, 5, 6, 8, or 10. We depict a selection of minimal 3D SPTs. Some of them have properties that cannot take place in the 2D case. SPTs are preserved under symmetric deformations of the ellipsoid. In a future paper we plan to study their bifurcations and the transition between stability and instability under such deformations.

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The authors settled both questions in Ref. 27.

We say that an ellipsoid of \( \mathbb{R}^{n+1} \) whose \( n + 1 \) axis have different lengths (i.e., an ellipsoid without any symmetry of revolution) is nondegenerate. Our current goal is to classify nonsingular symmetric periodic trajectories (SPTs) inside nondegenerate ellipsoids. We also look for minimal SPTs —representatives of each class with the smallest possible period.

The relevance of the SPTs relies in three facts. First, they persist under small enough symmetric perturbations of a nondegenerate ellipsoid. Second, each perturbed SPT can be computed by solving a nonlinear equation for which the corresponding unperturbed SPT is a good initial approximation. Third, this nonlinear equation is which the corresponding unperturbed SPT is a good initial approximation. Third, this nonlinear equation is nondegenerate.

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SPTs. Finally, by means of an argument involving wind-
ing numbers, we establish the existence of SPTs for all
those classes.

That concludes the analytical part of the work. We
have also implemented a numerical algorithm to find min-
imal SPTs of each of the 12 classes in the 2D case, and
each of the 112 classes in the 3D case. They have periods
3, 4 or 6 in the 2D case; and 4, 5, 6, 8 or 10 in the 3D
case. We depict all 2D minimal SPTs in Table V, and
just two in the 2D case. This has to do with the fact that, from a generic
point on the ellipsoid, we can trace four lines tangent to
a fixed couple of caustics. On the contrary, we can trace
just two in the 2D case.

Some of the ideas used in this paper were first in-
trived by Kook and Meiss,28 to classify symmet-
ric periodic orbits of some standard-like reversible 2n-
dimensional symplectic maps. The technical details of
our problem are harder —there are more symmetries,
caustic types play a role, and the classification does
not depend only on the evenness or oddness of some
integers—, but the main arguments do not change.

Some of our billiard SPTs could be considered as dis-
crete versions of the closed geodesics on triaxial ellipsoids
found in Ref. 29. For instance, it is interesting to com-
pare the SPT on the left side of Fig. 1 with the symmetric
closed geodesic shown in Fig. 4 of that paper. Of course,
they intersect.

We complete this introduction with a note on the or-
ganization of the article. We review the classical theory
of reversible maps in Sec. II following Ref. 30. Next,
we specialize that theory to billiards inside symmetric
hypersurfaces. In Sec. IV we review briefly some well-
known results about billiards inside ellipsoids, in order
to fix notations that will be used along the rest of the
paper. Billiards inside ellipses of \( \mathbb{R}^2 \) and inside triaxial
ellipsoids of \( \mathbb{R}^3 \), are thoroughly studied in Sec. V and
Sec. VI, respectively. Billiards inside nondegenerate el-
ipsoids of \( \mathbb{R}^{n+1} \) are revisited in Sec. VII. Perspectives
and conclusions are drawn in Sec. VIII.

II. REVERSIBLE MAPS AND SYMMETRIC ORBITS

In this section we state the concept of symmetric pe-
riodic orbit for a map. A map is reversible if each or-
bit is related to its time reverse orbit by a symmetry
transformation. Reversible maps can be characterized as
maps that factorize as the composition of two involutions.
Symmetric orbits must have points on certain symmetry
sets; namely, the fixed sets of the involutions (reversors)
which factorize the original map. A reversible map can
have different factorizations and, therefore, its symmetric
orbits can be classified according to the symmetry sets
they intersect.

Let \( f : M \to M \) be a diffeomorphism on a manifold
\( M \).

**Definition 1** The map \( f \) is symmetric when there exists
an involution \( s : M \to M \) (i.e., \( s \circ s = \text{Id} \)), such that
\( f \circ s = s \circ f \). Then, \( s \) is called a symmetry of the map \( f \).

**Definition 2** The map \( f \) is reversible when there exists
an involution \( r : M \to M \) such that \( f \circ r = r \circ f^{-1} \). Then,
\( r \) is called a reversor of the map \( f \) (i.e., \( f \) is \( r \)-reversible).

**Remark 1** The composition of a symmetry and a revers-
or (provided both commute) is another reversor.

**Definition 3** If \( r \) is a reversor, we denote by \( \text{Fix}(r) = \{ m \in M : r(m) = m \} \) its set of fixed points, which is
called the symmetry set of the reversor.

**Definition 4** An orbit of the map \( f \) is a sequence \( O = \{ m_j \}_{j \in \mathbb{Z}} \) such that \( m_j = f(m_{j-1}) = f^j(m_0) \).

**Definition 5** An orbit \( O \) of a \( r \)-reversible map \( f \) is called
\( r \)-symmetric when \( f(O) = O \).

The following characterization of reversible maps goes
back to G. D. Birkhoff.

**Lemma 1** A map \( f \) is reversible if and only if it can be
factorized as the composition of two involutions, in which
case both of them are reversors of \( f \).

**Proof** Let us assume that \( f \) is \( \hat{r} \)-reversible. Then \( \hat{r} = f \circ \hat{r} = \hat{r} \circ f^{-1} \) is another reversor, because:
\[
\bullet \quad f \circ \hat{r} = f \circ f^{-1} = \hat{r} \\
\bullet \quad \hat{r}^2 = \hat{r} \circ \hat{r} = f \circ f^{-1} = \text{Id}.
\]

Therefore, the map \( f = f \circ \hat{r}^2 = \hat{r} \circ \hat{r} \) is the composition of
two involutions.

On the other hand, if \( f = \hat{r} \circ \hat{r} \) and \( \hat{r}^2 = \text{Id} \), then:
\[
\bullet \quad f \circ \hat{r} \circ f = \hat{r} \circ \hat{r} \circ \hat{r} = \hat{r}^2 \circ \hat{r} = \hat{r} \\
\bullet \quad f \circ \hat{r} \circ f = \hat{r} \circ \hat{r} \circ \hat{r} \circ \hat{r} = \hat{r} \circ \hat{r} = \hat{r}.
\]
sos both involutions \( \hat{r} \) and \( \hat{r} \) are reversors of \( f \).

The factorization of a reversible symmetric map as a
composition of two involutions is not unique. From any
given factorization \( f = \hat{r} \circ \hat{r} \), we can construct infinitely
many more, namely, \( f = \hat{r}_k \circ \hat{r}_k \), where \( \hat{r}_k = f^{k+1} \circ \hat{r} = f^k \circ \hat{r} = \hat{r} \circ f^{-k} \) and \( \hat{r}_k = f^k \circ \hat{r} \) for any \( k \in \mathbb{Z} \). Neverthe-
less, these new factorizations do not provide new sym-
matic orbits, because \( \hat{r}(O) = \hat{r}(O) = \hat{r}_k(O) = \hat{r}_k(O) \),
i.e., an orbit is invariant under \( \hat{r} \) (or \( \hat{r} \)) if and only if it
is invariant under all the reversors \( \hat{r}_k \) and \( \hat{r}_k \). On the
contrary, the existence of a symmetry \( s \) commuting with
the reversors $\tilde{r}$ and $\hat{r}$ is more promising, because then $f = (s \circ \tilde{r}) \circ (s \circ \hat{r})$ is a new factorization, which could give rise to new kinds of symmetric orbits. The more such symmetries exist, the more kinds of symmetric orbits we can try to find.

Given a factorization $f = \tilde{r} \circ \hat{r}$, we say that $\hat{r}$ and $\tilde{r}$ are associated reversors, and $\text{Fix}(\hat{r})$ and $\text{Fix}(\tilde{r})$ are associated symmetry sets. The importance of these concepts is clarified in the following characterization of symmetric orbits, which can be found in Ref. 30.

**Theorem 1** Let $f = \tilde{r} \circ \hat{r}$ be any factorization of a reversible map as the composing of two involutions. Then an orbit of this map is:

a) $\tilde{r}$-symmetric if and only if it is $\hat{r}$-symmetric.

b) $\tilde{r}$-symmetric if and only if it has at least one point on $\text{Fix}(\hat{r}) \cup \text{Fix}(\tilde{r})$, in which case it has no more than two points on $\text{Fix}(\hat{r}) \cup \text{Fix}(\tilde{r})$.

c) $\tilde{r}$-symmetric and periodic if and only if it has exactly two points on $\text{Fix}(\hat{r}) \cup \text{Fix}(\tilde{r})$, in which case it has a point on each symmetry set if and only if it has odd period. In particular,

i. An orbit is $\tilde{r}$-symmetric with period 2k if and only if it has a point $m_k$ such that either $m_k \in \text{Fix}(\hat{r}) \cap f^k \text{Fix}(\tilde{r})$ or $m_k \in \text{Fix}(\tilde{r}) \cap f^k \text{Fix}(\hat{r})$, in which case $f^k(m_k)$ is on the same symmetry set as $m_k$.

ii. An orbit is $\tilde{r}$-symmetric orbit with period 2k + 1 if and only if it has a point $m_k$ such that $m_k \in \text{Fix}(\hat{r}) \cap f^k \text{Fix}(\tilde{r})$, in which case $f^{k+1}(m_k) \in \text{Fix}(\tilde{r})$.

**Proof** This is an old result, so we skip most of the details. We focus in those aspects that are more relevant for our goals.

a) It is immediate as $\hat{r}(O) = \tilde{r}(O)$.

b) To begin with, let us assume that an orbit $O = (m_j)_{j \in \mathbb{Z}}$ is $\tilde{r}$-symmetric, so there exists some $l \in \mathbb{Z}$ such that $m_l = \hat{r}(m_0)$. Then $m_{l+1} = f(m_l) = \hat{r}(m_0)$, because $\hat{r} = f \circ \tilde{r}$. We distinguish two cases:

- If $l = 2k$ is even then $m_k \in \text{Fix}(\tilde{r})$ as:
  
  $$m_k = f^{-k}(m_{2k}) = f^{-k}(\hat{r}(m_0)) = \hat{r}(m_k).$$

- If $l = 2k + 1$ is odd then $m_{k+1} \in \text{Fix}(\hat{r})$ as:
  
  $$m_{k+1} = f^{-(k+1)}(m_{2k+2}) = f^{-(k+1)}(\tilde{r}(m_0)) = \tilde{r}(m_{k+1}).$$

On the other hand, let us assume that $O$ has a point, say $m_k$, in one of those fixed sets. Then there exist two possibilities. In the case $m_k \in \text{Fix}(\tilde{r})$, $\forall j \in \mathbb{Z}$:

$$m_{k+j} = f^j(m_k) = f^j(\tilde{r}(m_k)) = \hat{r}(f^{-j}(m_k)) = \hat{r}(m_{k-j}).$$

In the case $m_k \in \text{Fix}(\hat{r})$, from $\hat{r} = f^{-1} \circ \tilde{r}$, $\forall j \in \mathbb{Z}$:

$$m_{k+j-1} = f^{j-1}(m_k) = f^{j-1}(\hat{r}(m_k)) = f^j(\tilde{r}(m_k)) = \hat{r}(f^{-j}(m_k)) = \tilde{r}(m_{k-j}).$$

Hence, $O$ turns out to be $\tilde{r}$-symmetric in both cases.

c) Next, we study the symmetric periodic orbits. First, if $O$ has two points, say $m_0$ and $m_k$, on $\text{Fix}(\hat{r})$, then $O$ is 2k-periodic, because:

$$m_{2k} = f^k(m_k) = f^k(\hat{r}(m_k)) = \hat{r}(f^{-k}(m_k)) = \hat{r}(m_0) = m_0.$$ 

Second, if $O$ has one point on each symmetry set, say $m_0 \in \text{Fix}(\tilde{r})$ and $m_{k+1} \in \text{Fix}(\tilde{r})$, then $O$ is (2k + 1)-periodic, because:

$$m_{2k+1} = f^k(m_{k+1}) = f^k(\tilde{r}(m_{k+1})) = f^{k+1}(\tilde{r}(m_{k+1})) = \tilde{r}(f^{-(k+1)}(m_{k+1})) = \tilde{r}(m_0) = m_0.$$ 

Therefore, the computation of symmetric orbits with period 2k can be reduced to the computation of points on symmetry sets, which are mapped onto the same symmetry set after $k$ iterations of the map. On the other hand, the computation of symmetric orbits with period 2k + 1 can be reduced to the computation of points on symmetry sets, which are mapped onto their associated symmetry sets after $k + 1$ iterations of the map.

**III. BILLIARDS INSIDE SYMMETRIC HYPERSURFACES OF $\mathbb{R}^{n+1}$**

Let $Q$ be a (strictly) convex smooth hypersurface of $\mathbb{R}^{n+1}$. The billiard motion inside $Q$ can be modelled by means of a diffeomorphism $f$ defined on the phase space $M = \{(q,p) \in Q \times \mathbb{R}^n : p$ is directed outward $Q$ at $q\}$.

We define the **billiard map** $f : M \to M$, $f(q,p) = (q',p')$, as follows. The new velocity $p'$ is the reflection of $p$, with respect to the tangent hyperplane $T_qQ$. That is, $p' = p_t - p_n = p - 2p_n$, where $p_t$ and $p_n$ are the tangent and normal components at $q$ of the old velocity:

$$p = p_t + p_n, \quad p_t \in T_qQ, \quad p_n \in N_qQ.$$ (1)

Let $q + \langle p' \rangle$ be the line through $q$ with direction $p'$. Then the new impact point $q'$ is determined by the condition

$$Q \cap (q + \langle p' \rangle) = \{q, q'\}.$$ 

That is, $q'$ is the intersection of the ray $\{q + \mu p' : \mu > 0\}$ with the hypersurface $Q$. This intersection is unique and transverse by convexity.
Definition 6 A billiard orbit is a sequence of points \( \{m_j\}_{j \in \mathbb{Z}} \subset M \) such that \( m_j = f(m_{j-1}) = f^j(m_0) \). If the points \( m_j = (q_j, p_j) \) form a billiard orbit, then the sequence of impact points \( \{q_j\}_{j \in \mathbb{Z}} \subset Q \) is a billiard (or impact) configuration, whose joining by polygonal lines form a billiard trajectory; the sequence of outward velocities \( \{p_j\}_{j \in \mathbb{Z}} \subset \mathbb{S}^n \) is a velocity configuration.

The distinction between orbits and trajectories is clear. We refer to orbits when we are working in the phase space \( M \), whereas we refer to trajectories when we are drawing in \( \mathbb{R}^{n+1} \). There exists an one-to-one correspondence between billiard orbits and billiard configurations. The velocities of the billiard orbit that correspond to the billiard configuration \( \{q_j\}_{j \in \mathbb{Z}} \) are:

\[
p_j = \frac{q_j - q_{j-1}}{\|q_j - q_{j-1}\|}.
\]

From now on we shall assume that the hypersurface \( Q \) is symmetric with respect to all coordinate hyperplanes of \( \mathbb{R}^{n+1} \). Then the billiard map admits \( 2^{n+1} \) factorizations as a composition of involutions. We need the following notations in order to describe them.

Let \( \Sigma \) be the set made up of the reflections—that is, involutive linear transformations—with respect to the \( 2^{n+1} \) coordinate subspaces of \( \mathbb{R}^{n+1} \). We represent its elements as diagonal matrices whose diagonal entries are equal to 1 or -1, so

\[
\Sigma = \{ \sigma = \text{diag}(\sigma_1, \ldots, \sigma_{n+1}) : \sigma_j \in \{-1, 1\} \text{ for all } j \}. 
\]

Given \( (q, p) \in M \), there exists a unique point \( \hat{q} \) such that

\[
Q \cap (q + \langle p \rangle) = \{q, \hat{q}\}.
\]

Thus, \( \hat{q} \) denotes the previous impact point. Finally, let \( \hat{p} \) be the reflection of the velocity \( p \) with respect to the normal line \( N_q Q \). That is,

\[
\hat{p} = -p' = p_n - p_t = p - 2p_t,
\]

where \( p_t \) and \( p_n \) were defined in the decomposition (1).

In what follows, symbols \( q', p', \hat{q}, \) and \( \hat{p} \) have the meaning given in the previous paragraphs. We emphasize that they make sense only after both impact point \( q \), and unitary outer velocity \( p \), are given. Next we describe the \( 2^{n+1} \) factorizations of the billiard map.

Proposition 1 Let \( f : M \to M \), \( f(q, p) = (q', p') \), be the billiard map inside a closed convex symmetric hypersurface \( Q \subset \mathbb{R}^{n+1} \). Let \( \tilde{r}, \hat{r} : M \to M \) be the maps

\[
\tilde{r}(q, p) = (q, \hat{p}), \quad \hat{r}(q, p) = (\hat{q}, -p).
\]

Let \( s_\sigma, \tilde{r}_\sigma, \hat{r}_\sigma : M \to M \) be the maps

\[
s_\sigma(q, p) = (\sigma q, \sigma p), \quad \tilde{r}_\sigma = s_\sigma \circ \tilde{r}, \quad \hat{r}_\sigma = s_\sigma \circ \hat{r}
\]

for any reflection \( \sigma \in \Sigma \). Then:

\( a) f = \hat{r} \circ \tilde{r}, \) and \( \hat{r}^2 = \tilde{r}^2 = \text{Id}. \)

\( b) f \circ s_\sigma = s_\sigma \circ f, \) \( s_\sigma^2 = \text{Id}, \) \( \tilde{r} \circ s_\sigma = s_\sigma \circ \tilde{r}, \) and \( \hat{r} \circ s_\sigma = s_\sigma \circ \hat{r}. \)

\( c) f = \tilde{r}_\sigma \circ \hat{r}_\sigma, \) and \( \tilde{r}_\sigma^2 = \hat{r}_\sigma^2 = \text{Id}. \)

Proof

\( a) \tilde{r}(\tilde{r}(q, p)) = \tilde{r}((q, \hat{p})) = \tilde{r}(q, -p') = (q', p'). \) Besides, \( \tilde{r}^2(q, p) = \tilde{r}(q, -p) = (q, p) \), as \( Q \cap (q + (-p)) = \{q, \hat{q}\} \). Finally, \( \hat{r}^2(q, p) = \hat{r}(q, \hat{p}) = (q, p) \).

\( b) \) Clearly \( s_\sigma \) is an involution. Since \( Q \) is symmetric with respect to the coordinate hyperplanes, it turns out that if \( p_t \) and \( p_n \) are the tangent and normal components of a velocity \( p \) at an impact point \( q \), then \( \sigma p_t \) and \( \sigma p_n \) are the tangent and normal components of \( \sigma p \) at the impact point \( \sigma q \). On the other hand, if we write \( q' = q + \mu (q, p') \), then \( \mu(\sigma q, \sigma p') = \mu(q, p') \), again by the symmetry of \( Q \). Hence, \( f \circ s_\sigma = s_\sigma \circ f \).

The proof of \( \tilde{r} \circ s_\sigma = s_\sigma \circ \tilde{r} \) and \( \hat{r} \circ s_\sigma = s_\sigma \circ \hat{r} \) follows the same lines.

\( c) \) It is a direct consequence of Remark 1. \( \Box \)

No symmetry has been required to obtain the factorization \( f = \tilde{r} \circ \hat{r} \). Therefore, all convex hypersurfaces \( Q \subset \mathbb{R}^{n+1} \) give rise to reversible billiard maps, although symmetric hypersurfaces have much more factorizations.

We introduce the acronyms SO, ST, SPO, and SPT for symmetric orbit, symmetric trajectory, symmetric periodic orbit, and symmetric periodic trajectory, respectively. If \( r \) is any reverser of the billiard map, we can deal with r-SOS, r-STs, r-SPOs, and r-SPTs.

Once these \( 2^{n+1} \) factorizations \( f = \tilde{r}_\sigma \circ \hat{r}_\sigma \) have been found, we describe their symmetry sets. In the next proposition we prove that only two symmetry sets are empty. We also provide an explicit geometric description of the \( 2^{n+2} - 2 \) nonempty symmetry sets where one can look for SOs. As a consequence, we shall see that these symmetry sets are mutually exclusive—that is, they do not intersect— but at some very specific points which are described in detail. As before, some notations are required.

We recall that given a reflection defined on an Euclidean space, we can decompose the space as the orthogonal sum of the eigenspaces of eigenvalues \(-1\) and \(1\). This is called the spectral decomposition of the reflection. Let \( \mathbb{R}^{n+1} = V^- \oplus V^+ \) be the spectral decomposition associated to any reflection \( \sigma \in \Sigma \). That is,

\[
V^\pm = \{p \in \mathbb{R}^{n+1} : \sigma p = \pm p\}.
\]

For instance, if \((x_1, x_2, x_3)\) are the Cartesian coordinates in \( \mathbb{R}^3 \) and \( \sigma = \text{diag}(-1, 1, 1) \), then \( V^+ \) is the \( x_2x_3 \) plane, and \( V^- \) is the \( x_1 \)-axis. We also introduce the sections

\[
Q_\sigma = Q \cap V^+_\sigma = \{q \in Q : \sigma q = q\}, \\
P_\sigma = \mathbb{S}^n \cap V^-_\sigma = \{p \in \mathbb{S}^n : \sigma p = -p\}.
\]
Given two linear varieties $V_1, V_2 \subset \mathbb{R}^{n+1}$, the symbol $V_1 \perp V_2$ means that they have a orthogonal intersection. Finally, we recall that a line in $\mathbb{R}^{n+1}$ is a chord of $Q$ when it intersects orthogonally $Q$ at two different points.

**Proposition 2** The symmetry sets of the reversors $\tilde{r}_\sigma$ and $\hat{r}_\sigma$, $\sigma \in \Sigma$, are:

\[
\text{Fix}(\tilde{r}_\sigma) = \{ (q,p) \in M : q \in Q_\sigma \text{ and } p \in N_q Q_\sigma \}, \\
\text{Fix}(\hat{r}_\sigma) = \{ (q,p) \in M : \sigma q + (p) \text{ and } p \in P_\sigma \} = \{ (q,p) \in M : (q + (p)) \perp V_\sigma^+ \}.
\]

Only the reversors $\tilde{r}_{-\text{Id}}$ and $\hat{r}_{\text{Id}}$ have empty symmetry sets. Moreover, if a point $(q,p) \in M$ belongs simultaneously to two different symmetry sets, then the line $q + (p)$:

- Is contained in some coordinate hyperplane of $\mathbb{R}^{n+1}$, in which case so is its whole billiard trajectory; or
- Is a chord of the hypersurface $Q$ through the origin, in which case its billiard trajectory is 2-periodic.

**Proof** To begin with, we deduce from the definitions of reversors $\hat{r}_\sigma$ and $\tilde{r}_\sigma$, and symmetries $s_\sigma$ that

\[
(q,p) \in \text{Fix}(\tilde{r}_\sigma) \iff s_\sigma(q,p) = \tilde{r}(q,p) \\
(q,p) \in \text{Fix}(\hat{r}_\sigma) \iff s_\sigma(q,p) = \hat{r}(q,p).
\]

To understand the condition $\sigma p = \tilde{p}$, we compare the spectral decomposition $\mathbb{R}^{n+1} = V_\sigma^+ \perp V_\sigma^-$ with the spectral decomposition $\mathbb{R}^{n+1} \simeq T_q \mathbb{R}^{n+1} = \hat{V}_q^+ \perp \hat{V}_q^-$ associated to the reflection $p \mapsto \tilde{p}$. We note that

\[
\hat{V}_q^- = \{ p \in T_q \mathbb{R}^{n+1} : \tilde{p} = -p \} = T_q Q, \\
\hat{V}_q^+ = \{ p \in T_q \mathbb{R}^{n+1} : p = \tilde{p} \} = N_q Q.
\]

If $q \in Q_\sigma$, then

\[
N_q Q_\sigma = N_q Q \perp V_\sigma^- = \hat{V}_q^+ \perp V_\sigma^-, \\
T_q Q_\sigma = T_q Q \cap V_\sigma^+ = \hat{V}_q^- \cap V_\sigma^+, \\
V_\sigma^- \subset \hat{V}_q^-.
\]

The equality on $N_q Q_\sigma$ is due to $(A \cap B)^\perp = A^\perp \perp B^\perp$. The inclusions follow from $V_\sigma^+$ being a symmetry subspace of $Q$. Those relations imply that

\[
T_q \mathbb{R}^{n+1} = N_q Q_\sigma \perp T_q Q_\sigma = \hat{V}_q^+ \perp V_\sigma^- \perp (\hat{V}_q^- \cap V_\sigma^+) = (V_\sigma^+ \cap \hat{V}_q^+) \perp (V_\sigma^- \cap \hat{V}_q^-) \perp (\hat{V}_q^- \cap V_\sigma^+).
\]

Therefore, if we fix any $q \in Q_\sigma$, then $\sigma p = \tilde{p} \iff p \in (V_\sigma^+ \cap \hat{V}_q^+) \perp (V_\sigma^- \cap \hat{V}_q^-) = \hat{V}_q^+ \perp V_\sigma^- = N_q Q_\sigma$, so

\[
\text{Fix}(\tilde{r}_\sigma) = \{ (q,p) \in M : q \in Q_\sigma, p \in N_q Q_\sigma \}.
\]

Next, we study the other symmetry sets for $\sigma \neq \text{Id}$. If $p \in P_\sigma$, then $p \perp V_\sigma^+$ and $p \parallel V_\sigma^-$, which implies that $\sigma q + (p) \iff q + (p) \perp V_\sigma^+$. Therefore,

\[
\text{Fix}(\hat{r}_\sigma) = \{ (q,p) \in M : q + (p) \perp V_\sigma^+ \}.
\]

Clearly, $Q_\sigma$ is empty if and only if $\sigma = -\text{Id}$, and $P_\sigma$ is empty if and only if $\sigma = \text{Id}$.

Finally, let us check that if the point $(q,p) \in M$ belongs to two different symmetry sets, then $q + (p)$ is a chord of $Q$ or is contained in some coordinate hyperplane like

\[
H_j = \{ q = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_j = 0 \}.
\]

First, if $(q,p) \in \text{Fix}(\tilde{r}_\sigma) \cap \text{Fix}(\tilde{r}_\gamma)$, then using the relations written at the beginning of this proof, we get

\[
\sigma q = q = \tau q, \\
\sigma p = \tilde{p} = \tau p.
\]

If $\sigma \neq \tau$, then $\sigma_j \neq \tau_j$ for some $j$, so that $q \in H_j$, $p \parallel H_j$, and $q + (p) \subset H_j$.

Second, if $(q,p) \in \text{Fix}(\tilde{r}_\sigma) \cap \text{Fix}(\hat{r}_\tau)$, then using again the same relations, we get that

\[
\sigma q = q = \hat{q}, \\
\sigma p = \tilde{p} = -p', \\
\tau q = \hat{q}, \\
\tau p = -p.
\]

If $\sigma \neq \text{Id}$, then $\sigma_j = -1$ for some $j$, so that $q, \hat{q} \in H_j$, $p \parallel H_j$, and $q + (p) \subset H_j$. If $\sigma = \text{Id}$ but $\tau \neq -\text{Id}$, then $\tau_j = 1$ for some $j$, so that $p' = -p \in N_q Q$, $p \parallel H_j$, and $p' \parallel H_j$. Thus, $p = p_0 \parallel H_j$ and $N_q Q \subset H_j$, which implies that $q \in H_j$, because $Q$ is symmetric, smooth, and (strictly) convex. Therefore, $q + (p) \subset H_j$. If $\sigma = \text{Id}$ and $\tau = -\text{Id}$, then $p' = -p \in N_q Q$ and $q = -q$, so that the line $q + (p)$ intersects orthogonally $Q$ at $q$ and $-q$.

Third, if $(q,p) \in \text{Fix}(\tilde{r}_\sigma) \cap \text{Fix}(\hat{r}_\tau)$, then

\[
\sigma q = \hat{q} = \tau q, \\
\sigma p = -p = \tau p,
\]

and we apply the same reasoning as in the first case. □

Hence, only very specific billiard orbits —the ones contained in a coordinate hyperplane or 2-periodic— can have a point in the intersection of two symmetry sets. For instance, it turns out that any 2-periodic billiard orbit inside a nondegenerate ellipsoid is contained in more than half of the $2^{n+2} - 2$ nonempty symmetry sets.

Nevertheless, there exists other billiard orbits that intersect different symmetry sets at different points. Indeed, all SPOs with odd period have points on two associated symmetry sets, whereas all SPOs with even period have two points on some symmetry set; see item c) of Theorem 1. However, a special situation arises when $Q$ is an ellipsoid: most of its SPOs with even period have exactly four —instead of the expected two— points in the symmetry sets described in Proposition 2, in which case they must intersect two different symmetry sets (see item b) of Theorem 1).

That motivates the following definition.

**Definition 7** An SPO inside a symmetric hypersurface $Q \subset \mathbb{R}^{n+1}$, is a doubly SPO when it intersects two different symmetry sets.

**Remark 2** Let $O$ be a $\tilde{r}_\sigma$-SO or, equivalently, a $\hat{r}_\sigma$-SO (see Theorem 1). By definition, $O$, as a subset of the phase space $M$, is invariant under the reversors

\[
\tilde{r}_\sigma(q,p) = (\sigma q, \sigma p), \\
\hat{r}_\sigma(q,p) = (\sigma q, -\sigma p).
\]
In particular, the billiard configuration associated to $O$, viewed as a subset of the configuration space $Q$, is invariant under the map $\sigma_{Q} : q \mapsto \sigma q$, whereas its velocity configuration, viewed as a subset of the velocity space $\mathbb{S}^{n}$, is invariant under the map $-\sigma_{Q} : p \mapsto -\sigma p$. This motivates the following definitions for the $2D$ and $3D$ cases. A billiard configuration inside a symmetric curve/surface is called central, axial or specular when it is symmetric with respect to the origin, some axis of coordinates or some plane of coordinates, respectively. Similar definitions apply to velocity configurations.

IV. BILLIARDS INSIDE ELLIPSOIDS OF $\mathbb{R}^{n+1}$

The billiard dynamics inside ellipsoids has several important properties. For instance, it is completely integrable in the sense of Liouville. We present some results about such billiards. First, we list the classical ones, which can be found in the monographs Refs. 31-33. Next, we describe a dual property discovered in Refs. 14 and 16. Finally, we detail the behaviour of elliptic coordinates of billiard trajectories inside ellipsoids given in Ref. 22.

A. Caustics and elliptic coordinates

The following results go back to Jacobi, Chasles, Poncelet, and Darboux. We consider a billiard inside the ellipsoid

$$Q = \{ q \in \mathbb{R}^{n+1} : \langle Dq, q \rangle = 1 \}, \quad (4)$$

where $D^{-1} = \text{diag}(a_{1}, \ldots, a_{n+1})$ is a diagonal matrix such that $0 < a_{1} < \cdots < a_{n+1}$. The degenerate cases in which the ellipsoid has some symmetry of revolution are not considered here. This ellipsoid is an element of the family of confocal quadrics given by

$$Q_{\mu} = \{ q \in \mathbb{R}^{n+1} : \langle D_{\mu}q, q \rangle = 1 \}, \quad \mu \in \mathbb{R},$$

where $D_{\mu} = (D^{-1} - \mu \text{Id})^{-1}$. The meaning of $Q_{\mu}$ in the singular cases $\mu \in \{ a_{1}, \ldots, a_{n+1} \}$ must be clarified. If $\mu = a_{j}$, we define it as the $n$-dimensional hyperplane $(3)$.

Theorem 2 Let $Q$ be the nondegenerate ellipsoid $(4)$.

a) Any generic point $q \in \mathbb{R}^{n+1}$ belongs to exactly $n+1$ distinct nonsingular quadrics $Q_{\mu_{1}}, \ldots, Q_{\mu_{n}}$ such that $\mu_{0} < a_{1} < \mu_{1} < a_{2} < \cdots < a_{n} < \mu_{n} < a_{n+1}$. Besides, those $n+1$ quadrics are mutually orthogonal at $q$.

b) Any generic line $\ell \subset \mathbb{R}^{n+1}$ is tangent to exactly $n$ distinct nonsingular confocal quadrics $Q_{\lambda_{1}}, \ldots, Q_{\lambda_{n}}$ such that $\lambda_{1} < \cdots < \lambda_{n}$, $\lambda_{i} \in (a_{i-1}, a_{i}) \cup \{ a_{i} \}$, and $\lambda_{i} \in (a_{i-1}, a_{i}) \cup \{ a_{i}, a_{i+1} \}$, for $i = 2, \ldots, n$.

Set $a_{0} = 0$. If a generic point $q$ is in the interior of the ellipsoid $Q$, then $\mu_{1} > 0$, so $a_{0} < \mu_{0} < a_{1}$. In the same way, if a generic line $\ell$ has a transverse intersection with the ellipsoid $Q$, then $\lambda_{1} > 0$, so $a_{1} \in (a_{0}, a_{1}) \cup (a_{1}, a_{2})$. The values $\mu_{0} = 0$ and $\lambda_{0} = 0$ are attained just when $q \in Q$ and $\ell$ is tangent to $Q$, respectively.

We denote by $q = (\mu_{0}, \ldots, \mu_{n}) \in \mathbb{R}^{n+1}$, the Jacobi elliptic coordinates of the point $q = (x_{1}, \ldots, x_{n+1})$. Cartesian and elliptic coordinates are linked by relations

$$x_{j}^{2} = \prod_{i=0}^{n} (a_{j} - \mu_{i}) \prod_{i \neq j} (a_{j} - a_{i}), \quad j = 1, \ldots, n + 1. \quad (5)$$

Elliptic coordinates define a coordinate system on each of the $2^{n+1}$ open orthants of the Euclidean space $\mathbb{R}^{n+1}$, but they become singular at the $n+1$ coordinate hyperplanes, because the map $q \mapsto q$ is not one-to-one in any neighborhood of these hyperplanes.

A point is generic, in the sense of Theorem 2, if and only if it is outside all coordinate hyperplanes. From Eq. (5), we deduce that when the point $q$ tends to the hyperplane $H_{j}$, some elliptic coordinate $\mu_{i}$ tends to $a_{j}$. In fact, $i = j$ or $i = j - 1$, because of the inequalities $a_{i} < \mu_{i} < a_{i+1}$.

A line is generic, in the sense of Theorem 2, if and only if it is neither tangent to a singular confocal quadric nor contained in a nonsingular confocal quadric.

If two lines obey the reflection law at a point $q \in Q$, then both lines are tangent to the same confocal quadrics. This shows a tight relation between elliptic billiards and confocal quadrics: all lines of a billiard trajectory inside the ellipsoid $Q$ are tangent to exactly $n$ confocal quadrics $Q_{\lambda_{1}}, \ldots, Q_{\lambda_{n}}$, which are called caustics of the trajectory. We will say that $\lambda = (\lambda_{1}, \ldots, \lambda_{n}) \in \mathbb{R}^{n}$ is the caustic parameter of the trajectory.

Definition 8 A billiard trajectory inside a nondegenerate ellipsoid $Q$ is nonsingular when it has $n$ distinct nonsingular caustics; that is, when its caustic parameter belongs to the nonsingular caustic space

$$\Lambda = \{ (\lambda_{1}, \ldots, \lambda_{n}) \in \mathbb{R}^{n} : 0 < \lambda_{1} < \cdots < \lambda_{n}, \lambda_{i} \in (a_{i-1}, a_{i}) \cup (a_{i}, a_{i+1}) \}. \quad (6)$$

We will only deal with nonsingular billiard trajectories along this paper.

Remark 3 The set $\Lambda$ has $2^{n}$ connected components, being each one associated to a different type of caustics. For instance, in the 2D case ($n = 1$) the two connected components correspond to ellipses and hyperbolas.

Theorem 3 If a nonsingular billiard trajectory closes after $l$ bounces, all trajectories sharing the same caustics also close after $l$ bounces.

Poncelet and Darboux proved this theorem for ellipses and triaxial ellipsoids of $\mathbb{R}^{3}$, respectively. Later on, it was generalized to any dimension in Ref. 14.
B. A dual transformation

Let us present a map $g : M \to M$ that takes place only for billiards inside ellipsoids. It exchanges the role of positions and velocities —so, it could be seen as a dual transformation—, and was introduced in Refs. 14 and 16. The following explicit formulæ are required to define $g$.

**Proposition 3** The billiard map $f : M \to M$ inside the ellipsoid $\{ q \}$ is expressed by $f(q,p) = (q',p')$, where

\[
\begin{aligned}
q' &= q + \mu(q,p')p', \quad \mu(q,p') = -2\langle Dq, p' \rangle / \langle Dp', p' \rangle; \\
p' &= p + \nu(q,p)Dq, \quad \nu(q,p) = -2\langle Dq, Dq_p \rangle.
\end{aligned}
\]

Besides, the reversors $\hat{r}, \tilde{r} : M \to M$ are given by:

\[
\begin{aligned}
\hat{r}(q,p) &= (\bar{q}, -p), \quad \bar{q} = q + \mu(q,p)p, \\
\tilde{r}(q,p) &= (q, \bar{p}), \quad \bar{p} = -p - \nu(q,p)Dq.
\end{aligned}
\]

**Proof** The formulæ for the billiard map are well known. See, for instance, Ref. 16. Next, to obtain the formula for $\hat{r}$ we recall that $\hat{q} = q$ is the previous impact point, so

\[
\hat{q} = q + \mu(q,-p)(-p) = q + \mu(q,p)p.
\]

Finally, the formula for $\tilde{r}$ follows from $\tilde{p} = -p'$. \hfill \Box

We are ready to introduce the dual transformation $g$. We check that, in certain sense, $g$ is the square root of $f$. In particular, $g$ has the same symmetries and reversors as $f$. From our point of view, its most useful property is that it interchanges some symmetry sets.

**Proposition 4** Let $g : M \to M$, $g(q,p) = (\bar{q}, \bar{p})$, where:

\[
\begin{aligned}
\bar{q} &= C p' = C p + \nu(q,p)C^{-1}q, \\
\bar{p} &= -C^{-1}q,
\end{aligned}
\]

with $C = D^{-1/2}$. Then, the following relations hold:

a) $g^2 = -f$, and $f \circ g = g \circ f$.

b) $g \circ s_\sigma = s_\sigma \circ g$ for all $\sigma \in \Sigma$.

c) $\hat{r} \circ g = -g \circ \hat{r}$, $g \circ \tilde{r} \circ g = \tilde{r}$, and $g \circ \tilde{r} \circ g = \hat{r}$.

d) $g(\text{Fix}(\hat{r}_\sigma)) = \text{Fix}(\tilde{r}_-\sigma) = \text{Fix}(f \circ \tilde{r}_-\sigma)$ for all $\sigma \in \Sigma$.

**Proof**

a) We observe two relations:

\[
\begin{aligned}
\nu(q,p) &= -2\langle D\bar{q}, \bar{p} \rangle / \langle D\bar{q}, D\bar{q} \rangle \\
&= -\mu(q,p') = \mu(q,\bar{p}), \\
\mu(q,p) &= -2\langle D\bar{q}, \bar{p} \rangle / \langle D\bar{p}, \bar{p} \rangle \\
&= -\nu(q,p') = \nu(q,\bar{p}).
\end{aligned}
\]

Thus $\nu \circ g = \mu \circ \hat{r}$ and $\mu \circ g = \nu \circ \tilde{r}$, and then $g^2 = -f$:

\[
g^2(q,p) = g(\bar{q}, \bar{p}) = (C\bar{p} + \nu(q,\bar{p})C^{-1}\bar{q}, -C^{-1}\bar{q})
\]

\[
= (-q - \mu(q,p')p', -p') = (-q', -p').
\]

It is immediate that $g$ is odd. Therefore,

\[
f \circ g = (-g^2) \circ g = -g^3 = g \circ (-g^2) = g \circ f.
\]

b) It is obvious.

c) By using a) and definition of $g$:

\[
g(q,\bar{p}) = (C\bar{p} + \nu(q,\bar{p})C^{-1}q, -C^{-1}q)
\]

\[
= (-\bar{q} - \mu(q,\bar{p})\bar{p}, \bar{p}) = -\hat{r}(\bar{q}, \bar{p}).
\]

As a consequence $\hat{r} \circ g = -g \circ \hat{r}$ and besides:

\[
\hat{r} = f \circ \hat{r} = -g^2 \circ \hat{r} = g \circ \hat{r} \circ g,
\]

\[
\tilde{r} = f \circ \tilde{r} \circ f = \hat{r} \circ f = \hat{r} \circ (-g^2)
\]

\[
= -\hat{r} \circ g^2 = g \circ \tilde{r} \circ g.
\]

d) Because $g$ is a diffeomorphism we have that:

\[
m \in \text{Fix}(\hat{r}_\sigma) \iff g(m) = g(\hat{r}_\sigma(m)) = -\hat{r}_\sigma(g(m)) \iff g(m) \in \text{Fix}(\hat{r}_-\sigma).
\]

We have used that $-\hat{r}_\sigma = \hat{r}_{-\sigma}$. Consequently,

\[
g(\text{Fix}(\hat{r}_\sigma)) = \text{Fix}(\hat{r}_{-\sigma}) = \text{Fix}(f \circ \tilde{r}_{-\sigma}).
\]

This proposition has a practical corollary.

**Corollary 1** The dual transformation $g : M \to M$ gives an explicit one-to-one correspondence between $\hat{r}_\sigma$-SPOs and $(f \circ \tilde{r}_{-\sigma})$-SPOs of the same period.

**Proof** Let $m \in \text{Fix}(\hat{r}_\sigma)$ be a point of the phase space such that its orbit $O$, is $l$-periodic. Thus, $O$ is a $\hat{r}_\sigma$-SPO. Let us consider the orbit $O$ by the point $\bar{m} = g(m)$. First,

\[
f^l(\bar{m}) = f^l(g(m)) = g(f^l(m)) = g(m) = \bar{m}.
\]

Thus, $O$ is $l$-periodic. Second, $\bar{m} = g(m) \in \text{Fix}(\hat{r}_{-\sigma})$, so $O$ is a $\hat{r}_{-\sigma}$-SPO. Therefore, we have explicitly constructed the correspondence $g : O \mapsto \hat{O}$, which is one-to-one since $g^2 = -f$. \hfill \Box

In subsequent sections we carry out some computations on SPOs only for reversors $\{\hat{r}_\sigma : \sigma \in \Sigma\}$, since from this corollary, we deduce the analogous results for reversors $\{\hat{r}_\sigma = f \circ \tilde{r}_\sigma : \sigma \in \Sigma\}$.
C. Elliptic coordinates of SPTs

The behaviour of elliptic coordinates along nonsingular billiard trajectories inside a nondegenerate ellipsoid, can be summarized as follows. If a particle obeys the billiard dynamics, describing a polygonal curve of \(\mathbb{R}^{n+1}\) whose vertices are on the ellipsoid, then its \(i\)-th elliptic coordinate \(\mu_i\) oscillates inside some interval \(I_i = [c_{2i}, c_{2i+1}]\) in such a way that its only critical points are attained when \(\mu_i \in \partial I_i\). In other words, the oscillation has amplitude \(c_{2i+1} - c_{2i}\). This is a classical result that can be found, for instance, in Ref. 22. Let us state it as a formal theorem.

In order to describe the intervals \(I_i\), we set

\[
\{c_1, \ldots, c_{2n+1}\} = \{a_1, \ldots, a_{n+1}\} \cup \{\lambda_1, \ldots, \lambda_n\},
\]

once ellipsoid parameters \(a_1, \ldots, a_{n+1}\), and caustic parameters \(\lambda_1, \ldots, \lambda_n\), are fixed.

If \(\lambda \in \Lambda\), then \(c_1, \ldots, c_{2n+1}\) are pairwise distinct and positive, so we can assume that

\[
c_0 := 0 < c_1 < \cdots < c_{2n+1}.
\]

Then \(I_i = [c_{2i}, c_{2i+1}], 0 \leq i \leq n\), are the intervals that we were looking for.

**Theorem 4** Let \(q(t)\) be an arc-length parameterization of a nonsingular billiard trajectory inside the ellipsoid (4), sharing caustics \(Q_{\lambda_1}, \ldots, Q_{\lambda_n}\). Let \(q(t) = (\mu_0(t), \ldots, \mu_n(t))\) be the corresponding parameterization in elliptic coordinates. The following properties hold:

a) \(c_{2i} \leq \mu_i(t) \leq c_{2i+1}\) for all \(t \in \mathbb{R}\).

b) Functions \(\mu_i(t)\) are smooth everywhere, except \(\mu_0(t)\) which is nonsmooth at impact points— that is, when \(q(t) \in Q^-\), in which case \(\mu_0'(t^+) = -\mu_0'(t^-) \neq 0\).

c) If \(\mu_i(t)\) is smooth at \(t = t^*\), then

\[
\mu_i'(t^*) = 0 \iff \mu_i(t^*) \in \{c_{2i}, c_{2i+1}\}.
\]

d) If the trajectory is periodic with length \(L_0\), then \(q(t)\) is \(L_0\)-periodic and there exist some positive integers \(m_0, \ldots, m_n\), called winding numbers, such that \(\mu_i(t)\) makes exactly \(m_i\) complete oscillations (round trips) inside the interval \(I_i = [c_{2i}, c_{2i+1}]\) along one period \(0 \leq t \leq L_0\). Besides,

i. \(m_0\) is the period of the billiard trajectory.

ii. \(m_i\) is even when \(\{c_{2i}, c_{2i+1}\} \cap \{a_1, \ldots, a_{n+1}\} \neq \emptyset\).

iii. \(q(t)\) has period \(L_0/2\) if and only if all winding numbers are even, but it never has period \(L_0/4\).

iv. Not all winding numbers can be multiples of four.

Therefore, all billiard trajectories sharing the caustics \(Q_{\lambda_1}, \ldots, Q_{\lambda_n}\) are contained in the \((n+1)\)-dimensional cuboid

\[
\mathcal{C}_\lambda := I_0 \times \cdots \times I_n = [0, c_1] \times \cdots \times [c_{2n}, c_{2n+1}] \subset \mathbb{R}^{n+1},
\]

when they are expressed in elliptic coordinates. Besides, a billiard trajectory drawn in elliptic coordinates, has elastic reflections with the \(n\)-dimensional face \(\{\mu_0 = 0\}\), of the cuboid, but inner tangent contacts with the other \(2n+1\) faces. This behaviour can be observed in Table V, where several SPTs are drawn in Cartesian and elliptic coordinates. The cuboid is just a rectangle in those cases.

Based on extensive numerical experiments, it has been conjectured\(^{27}\) that

\[
2 \leq m_n < \cdots < m_1 < m_0,
\]

but we are not aware of any proof.

V. 2D CASE

A. Caustics and elliptic coordinates

We adapt the previous setting of billiards inside ellipsoids of \(\mathbb{R}^{n+1}\) to the 2D case; that is, \(n = 1\). Then the configuration space is an ellipse \(Q\), which, in order to simplify the exposition, we write as

\[
Q = \left\{ q = (x, y) \in \mathbb{R}^2 : \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\}, \quad a > b > 0.
\]

As we said in Subsec. IV A, any nonsingular billiard trajectory inside \(Q\) is tangent to one confocal caustic

\[
Q_\lambda = \left\{ q = (x, y) \in \mathbb{R}^2 : \frac{x^2}{a - \lambda} + \frac{y^2}{b - \lambda} = 1 \right\},
\]

where the caustic parameter \(\lambda\) belongs to the nonsingular caustic space

\[
\Lambda = E \cup H, \quad E = (0, b), \quad H = (b, a).
\]

The names of the connected components of \(\Lambda\) come from the fact that \(Q_\lambda\) is an confocal ellipse for \(\lambda \in E\) and a confocal hyperbola for \(\lambda \in H\). The singular cases \(\lambda = b\) and \(\lambda = a\) correspond to the \(x\)-axis and \(y\)-axis, respectively. We say that the caustic type of a billiard trajectory is \(E\) or \(H\), when its caustic is an ellipse or a hyperbola. We also distinguish between E-caustics and H-caustics.

We denote by \(q = (e, h)\) the Jacobi elliptic coordinates of the point \(q = (x, y)\). That is, \(q\) belongs to the orthogonal intersection of the confocal ellipse \(Q_e\) and the confocal hyperbola \(Q_h\). We recall from Eq. (5) that

\[
x^2 = \frac{(a - e)(a - h)}{a - b}, \quad y^2 = \frac{(b - e)(h - b)}{a - b}.
\]

Besides, \(0 < e < b < h < a\) if the point \(q\) is contained in the interior of \(Q\), and \(e = 0\) at impact points \(q \in Q\). The \(y\)-axis is \(\{h = a\}\), the \(x\)-axis is \(\{e = b\} \cup \{h = b\}\), and the foci of the ellipse verify \(e = h = b\), in elliptic coordinates. Likewise, the points \((\pm x, \pm y)\) have associated the same elliptic coordinates \((e, h)\), so this system of coordinates do not distinguish among the four quadrants in \(\mathbb{R}^2\).
We also know that all billiard trajectories sharing the caustic \( Q_\lambda \) are contained in the rectangle

\[
\mathcal{C}_\lambda = [0, \min (b, \lambda)] \times [\max (b, \lambda), a] \subset \mathbb{R}^2
\]

when they are expressed in terms of \((e, h)\). Moreover, the coordinates \( e \) and \( h \) have a monotone behaviour except at the endpoints of the intervals that enclose them. The geometric meaning of the sides and vertexes of the rectangle \( \mathcal{C}_\lambda \) is described in Fig. 2. In the figure, black, red and gray sides means impact with the ellipse \( Q_\lambda \), tangency with the caustic \( Q_\lambda \), and crossing some coordinate axis, respectively. This color code is repeated in Table V.

**B. Rotation number and winding numbers**

We recall some concepts related to periodic trajectories of billiards inside ellipses. These results can be found, for instance, in Refs. 14 and 27. To begin with, we introduce the function \( \rho : \Lambda \rightarrow \mathbb{R} \) given by the quotient of elliptic integrals

\[
\rho (\lambda) = \frac{\int_0^{\min (b, \lambda)} ds}{2 \int_{\max (b, \lambda)}^{\min (b, \lambda)} ds} \frac{\sqrt{(\lambda - s)(b - s)(a - s)}}{\sqrt{\lambda (\lambda - s)(b - s)(a - s)}}.
\]

It is called the rotation number and characterizes the caustic parameters that give rise to periodic trajectories. To be precise, the billiard trajectories sharing the caustic \( Q_\lambda \) are periodic if and only if

\[
\rho (\lambda) = m_1 / 2m_0 \in \mathbb{Q}
\]

for some integers \( 2 \leq m_1 < m_0 \), which are the winding numbers. It turns out that \( m_0 \) is the period, \( m_1 / 2 \) is the number of turns around the ellipse \( Q_\lambda \) for E-caustics, and \( m_1 \) is the number of crossings of the \(-\)y-axis for H-caustics. Thus, \( m_1 \) is always even. Besides, all periodic trajectories with H-caustics have even period.

**C. Reversors and their symmetry sets**

We also change the notations for reversors. We denote by \( R \) the reversor previously written as \( \bar{r} \). Besides, \( R_x, R_y, \) and \( R_{xy} \) are the reversors obtained by composing \( R \) with the symmetries defined on the phase space and associated to the reflections \((x, y) \mapsto (-x, y)\), \((x, y) \mapsto (x, -y)\), and \((x, y) \mapsto (-x, -y)\), respectively. Finally, \( f \circ R \), \( f \circ R_x \), \( f \circ R_y \), and \( f \circ R_{xy} \) denote the four reversors of the form \( \hat{r}_\sigma = f \circ \bar{r}_\sigma : \sigma \in \Sigma \). We proved in Proposition 2 that the symmetry sets of \( R_{xy} \) and \( f \circ R \) are empty sets, and following the same proposition we get:

- \( \text{Fix}(R) = \{ (q,p) \in M : p \in N_q Q \} \)
- \( \text{Fix}(R_x) = \{ (q,p) \in M : q = (0, \pm \sqrt{b}) \} \)
- \( \text{Fix}(R_y) = \{ (q,p) \in M : q = (\pm \sqrt{a}, 0) \} \)
- \( \text{Fix}(f \circ R_x) = \{ (q,p) \in M : p = (0, \pm 1) \} \)
- \( \text{Fix}(f \circ R_y) = \{ (q,p) \in M : p = (0, \pm 1) \} \)
- \( \text{Fix}(f \circ R_{xy}) = \{ (q,p) \in M : q + p \} \)

Here, \( M = \{ (q,p) \in Q \times \mathbb{S} : p \in q + \langle p \rangle \} \) is the phase space. We write \( q = (x, y) \) and \( p = (u,v) \).

**D. Characterization of STs**

In order to find caustics \( Q_\lambda \), whose tangent trajectories are periodic, we solve equation \( \rho (\lambda) = m_1 / 2m_0 \), for some winding numbers \( 2 \leq m_1 < m_0 \). Hence, once we fix a caustic parameter \( \lambda \) such that \( \rho (\lambda) = m_1 / 2m_0 \in \mathbb{Q} \), and a reversor \( r : M \rightarrow M \), it is natural to look for points \((q,p) \in \text{Fix}(r)\) such that \( q + p \) is tangent to \( Q_\lambda \). If we can find those points, billiard trajectories associated to them are \( r \)-SPTs of period \( m_0 \).
In the 2D case there are only 6 (\(\#2^3-2\)) nonempty symmetry sets where one can look for STs; see previous subsection. We write the six formulae for \((\text{symmetry sets where one can look for STs}; \text{see previous subsection})\). We look for a velocity \(p \in M\) such that \(p \in N_qQ\) and \(q + \langle p \rangle\) is tangent to \(Q\). If \(p \in N_qQ\), then there exists \(\mu > 0\) such that \(p = \mu(x/a, y/b)\). If, in addition, \(q + \langle p \rangle\) is tangent to \(Q\), then \(Q\) is a hyperbola and \(q \in Q \cap Q\). Here, we have used that confocal ellipses and confocal hyperbolae are mutually orthogonal. Thus, the elliptic coordinates of \(q = (x, y)\) are \((e, h) = (0, \lambda)\), and so \(x^2 = a(\lambda - a)/b - a)\) and \(y^2 = b(\lambda - b)/(a - b)\). Finally, we recall that \(p\) is a unit velocity:

\[
1 = u^2 + v^2 = \mu^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{\lambda \mu^2}{ab} \Rightarrow \mu^2 = \frac{ab}{\lambda}.
\]

\(R_y\). We look for a velocity \(p = (u, v) \in S\) such that \(q + \langle p \rangle\) has exactly one point on \(Q\), for \(q = (0, \pm \sqrt{b})\). This condition turns into the uniqueness of solutions in \(t\) of

\[
\alpha t^2 + 2\beta t + \gamma = 0,
\]

where

\[
\alpha = \frac{u^2}{a - \lambda} + \frac{v^2}{b - \lambda}, \quad \beta = \frac{\pm \sqrt{b}v}{b - \lambda}, \quad \gamma = \frac{\lambda}{b - \lambda},
\]

or equivalently \(\beta^2 = \alpha \gamma\). This last equality together with \(u^2 + v^2 = 1\) gives \(v^2 = \lambda/a\) and \(u^2 = (a - \lambda)/a\).

\(R_{x}\). It is obtained directly from the previous case by exchanging the role of coordinates \(x\) and \(y\).

Next, we characterize nonsingular STs inside noncircular ellipses, as the trajectories passing, in elliptic coordinates, through some vertex of the rectangle \(C\).

We only have to check that any vertex \(q_\ast \in C\), corresponds to some point \(q_\ast \in q + \langle p \rangle\), such that \(q + \langle p \rangle\) is tangent to \(Q\) and \((q, p) \in \text{Fix}(r)\) for some reversor \(r\).

We list such correspondences in Table II. The point \(q_\ast\) is not unique, because elliptic coordinates do not distinguish among the four quadrants in \(\mathbb{R}^2\). In its last column we describe the feasible reversors for each caustic type; see Subsec. V E for more detailed information. For instance, let us focus on the vertex \(q_\ast = (0, \lambda)\). It is a vertex of \(C\) if and only if \(b < \lambda < a\), so the type of caustic is H. We note that \(q_\ast \in Q\) since \(e = 0\), and \(q_\ast \in Q\) since \(h = \lambda\). Let \(p \in S\) be the unique outward velocity such that \(q_\ast + \langle p \rangle\) is tangent to \(Q\). Then \(p \in N_qQ\), since the confocal hyperbola \(Q\) has an orthogonal intersection at \(q_\ast\) with the ellipse \(Q\). Hence, \((q_\ast, p) \in \text{Fix}(R)\). The other cases are similar. We only observe that \(e \neq 0\) in the last three cases, so \(q_\ast \notin Q\); but \(q_\ast\) is the middle point of two consecutive impact points.

### Table II. Points in STs that correspond to vertexes of \(C\).

| Vertex \((e, h)\) | \(q_\ast\) belongs to | Reversor | Type |
|------------------|------------------------|----------|------|
| \((0, \lambda)\) | \(Q \cap Q\) | \(R\) | H |
| \((0, a)\)       | \(Q \cap \{y\text{-axis}\}\) | \(R_a\) | any |
| \((0, b)\)       | \(Q \cap \{x\text{-axis}\}\) | \(R_b\) | E |
| \((\lambda, a)\) | \(Q \cap \{y\text{-axis}\}\) | \(R \in R_x\) | E |
| \((\lambda, b)\) | \(Q \cap \{x\text{-axis}\}\) | \(R \in R_y\) | any |
| \((b, a)\)       | \{(0, 0)\} | \(f \in R_{xy}\) | H |

### Table III. Feasible and forbidden reversors for each caustic type.

| Type | Feasible | Forbidden |
|------|----------|-----------|
| E    | \(R_x, R_y, f \circ R_x, f \circ R_y\) | \(R, f \circ R_{xy}\) |
| H    | \(R_x, R_y, f \circ R_x, f \circ R_y\) | \(R_y, f \circ R_{xy}\) |

## E. Forbidden reversors for each type of caustic

Although there are 6 nonempty symmetry sets, once the caustic type (E or H) is fixed, only 4 of them give rise to STs of that type. The correspondences between caustic types and reversors whose symmetry sets are feasible or forbidden are listed in Table III. To check that all couples caustic type/feasible reversor take really place, we show some examples in Table V.

We can prove Table III in three different ways. Firstly, by means of analytical arguments based on the formulae listed in Table I. Secondly, we could deduce it by using elliptic coordinates, as shown in Table II. Finally, we could write a geometric proof. The first way is the simplest one. If \((q, p) \in \text{Fix}(R)\), then \(v^2 = b(\lambda - b)/(a - b)\), so \(\lambda > b\) and \(Q\) is a hyperbola. If \((q, p) \in \text{Fix}(f \circ R_{xy})\), then \(v^2 = (\lambda - b)/(a - b)\), so \(\lambda > b\) and \(Q\) is a hyperbola. If \((q, p) \in \text{Fix}(R_{xy})\), then \(v^2 = (b - \lambda)/b\), so \(\lambda < b\) and \(Q\) is an ellipse. If \((q, p) \in \text{Fix}(f \circ R_x)\), then \(v^2 = b - \lambda\), so \(\lambda < b\) and \(Q\) is an ellipse. This ends the proof.

## F. Characterization and classification of STs

Next, we give a complete classification of nonsingular STs inside noncircular ellipses. We classify them by caustic type (E or H) and symmetry sets they intersect. We also characterize them as trajectories connecting, in elliptic coordinates, two vertexes of the rectangle \(C\).

Along this subsection, \(O\) denotes a \(r\)-SO through a point \((q, p) \in \text{Fix}(r)\), for some reversor \(r : M \to M\). We
set \((q_j, p_j) = f^j(q, p), q_j = (x_j, y_j),\) and \(p_j = (u_j, v_j).\) The properties listed in Table IV are easily deduced using geometric arguments from the symmetry of the ellipse. For instance, let us explain the case of reversor \(R_y\). If \((q, p) \in \text{Fix}(R_y)\), then \(x_0 = \pm \sqrt{a}\) and \(y_0 = 0\). Let us assume that \(q_j = (x_j, y_j)\) and the billiard configuration is symmetric with respect to the \(x\)-axis; see Fig. 3.

We note that all properties concerning velocities \(p_j\) can be directly deduced from the properties of impact points \(q_j\), by using identity (2).

Below, we suppose \(O\) is periodic as well, with winding numbers \(2 \leq m_1 < m_0, m_0\) being its period. We need a technical lemma on SPOs of even period.

**Lemma 2** Let us assume that the period is even: \(m_0 = 2l\), and the reversor is \(r = R_x\), so that \(x_0 = 0\).

a) If the caustic is an ellipse, \(q_1 = -q_0\) and \(p_1 = -p_0\).
Thus, \(q_{j+1} = -q_j\) and \(p_{j+1} = -p_j\), for all \(j \in \mathbb{Z}\).

b) If the caustic is a hyperbola, then \(q_1 = (-1)^j q_0\) and
\[
p_1 = \begin{cases} (u_0, v_0), & \text{if } l \text{ odd and } m_1/2 \text{ even}, \\ (-u_0, v_0), & \text{if } l \text{ even}, \\ (-u_0, -v_0), & \text{if } l \text{ odd and } m_1/2 \text{ odd}. \end{cases}
\]

**Proof** The hypothesis \(r = R_x\) in a), implies \(x_{-j} = -x_j\); see Table IV. Then, as \(m_0 = 2l\), we have \(x_l = x_{-l} = -x_l\), so \(x_l = 0\) and \(q_l \in \{q_0, -q_0\}\). If \(q_l = q_0\), then \(p_l = p_0\), since the line \(q_l + p_l\) must be tangent to the ellipse \(Q_\lambda\) and the billiard trajectory turns around \(Q_\lambda\) in a constant—clockwise or anticlockwise—direction. But the period cannot be smaller than \(m_0\) by hypothesis. Hence, \(q_l = -q_0\). Finally, the identity \(p_l = -p_0\) is necessary to keep the rotation direction around \(Q_\lambda\). The formulae for a general integer \(j\) are obvious.

We skip the proof for H-caustics in b): it is similar. □

Following we prove that all SPOs inside a noncircular ellipse are doubly SPOs and that there are 12 classes of SPOs, as claimed in the introduction. We recall that SPOs are classified by caustic type and intersected symmetry sets.

**Theorem 5** All SPOs inside a noncircular ellipse are doubly SPOs. SPTs inside noncircular ellipses are characterized as trajectories connecting, in elliptic coordinates, two different vertices of their rectangle \(C_\lambda\). There are 12 classes of SPTs, listed in Table V.

**Proof** Let \(O\) be a \(r\)-SPO through a point \((q, p) \in \text{Fix}(r)\) with winding numbers \(2 \leq m_1 < m_0\), being \(m_0\) its period. We must prove that there exists another reversor \(\tilde{r} \neq r\) and some index \(j \in \mathbb{Z}\) such that \((q_j, p_j) \in \text{Fix}(\tilde{r})\).

First, let us consider the caustic type \(E\). Then, according to Table III, we have only four possibilities:

\[
r \in \{R_x, f \circ R_x, R_y, f \circ R_y\}. \tag{8}
\]

The easiest situation is when \(m_0 = 2k + 1\), because we know from the last item in Theorem 1 that

\[
(q, p) \in \text{Fix}(R_x) \iff (q_{k+1}, p_{k+1}) \in \text{Fix}(f \circ R_x),
\]

and the same occurs with the couple of associated reversors \(R_y\) and \(f \circ R_y\). This proves the first row of Table V. The formulae in the last column are deduced from the symmetry sets in Subsec. V C.

Next, we assume that \(m_0 = 4k\). That is, we deal with the second row of Table V. We study each one of the four feasible reversors listed in Eq. (8).

- If \(r = R_x\), then \(q_0 \in \{y\text{-axis}\}\), so \(x_0 = 0\). Thus, since \(m_0 = 4k\), we have \(y_{3k} = y_{-k} = y_k = -y_{3k}\). The last equality follows from Lemma 2a) for \(l = 2k\) and \(j = k\). The above equalities imply \(y_{3k} = y_k = 0\) and thus we can take \(\tilde{r} = R_y\) and \(j \in \{k, 3k\}\).

- The case \(r = R_y\) is similar. Suffice it to exchange variables \(x\) and \(y\).

- The cases \(r \in \{f \circ R_x, f \circ R_y\}\) follow directly from the two previous ones by using Corollary 1.

Finally, we assume that \(m_0 = 4k + 2\), which corresponds to the third row of Table V. As before, the four feasible reversors listed in Eq. (8) are studied separately.

- If \(r = R_x\), then \(q_0 \in \{y\text{-axis}\}\), so \(x_0 = 0\). Thus, since \(m_0 = 4k + 2\), we have \(u_{3k+2} = u_{-k} = u_{k+1} = -u_{3k+2}\). The last equality follows from Lemma 2a)
for \( l = 2k + 1 \) and \( j = k + 1 \). These equalities imply 
\[ u_{3k+2} = u_{k+1} = 0 \] 
and thus we can take \( \mathbf{r} = f \circ R_y \) 
and \( j \in \{ k + 1, 3k + 2 \} \).

- **The case** \( r = f \circ R_y \) **follows directly from** the case 
\( r = R_x \) **by using Corollary 1.**

- **The cases** \( r \in \{ R_y, f \circ R_x \} \) **are similar to the previous** 
one. Sufficient it to exchange variables \( x \) and \( y \).

The proofs for the three rows about H-caustics follow the same lines, but using Lemma 2b). We skip the details.

Once we know that all SPOs are doubly SPOs, we deduce that, any doubly SPT connects two different vertexes, namely, the ones corresponding to the couple of reversors of the doubly SPT. Indeed, we simply take into account the 1-to-1 correspondence between reversors and vertexes of rectangles \( C_\lambda \), as listed in Table II. Besides, there are \( 12 = 2 \times 6 \) classes of SPTs, since there are two types of caustic (E and H) and any rectangle has four vertexes, and so, six couples of vertexes.

Finally, we check that the 12 classes are realizable. That is, we find one SPT of each class. This can be achieved by properly choosing the winding numbers. See Table V.

\[ \square \]

G. Minimal SPTs

We end the study of SPTs inside ellipses by showing an SPT of each one of the 12 classes with the smallest period \( m_0 \). We call minimal such SPTs. Therefore, \( m_0 \in \{ 3, 4, 6 \} \) for E-caustics, and \( m_0 \in \{ 4, 6 \} \) for H-caustics. Period \( m_0 = 2 \) is discarded, since two-periodic billiard trajectories are singular. The 12 minimal SPTs are drawn in Table V, both in Cartesian and elliptic coordinates. All of them connect two different vertexes of the rectangle \( C_\lambda \), as claimed in Theorem 5.

We realize that any minimal SPT of type E has a “twin” of type H when both are depicted in elliptic coordinates. This has to do with the fact that both trajectories have the same rotation number and connect the same vertexes.

Any SPT, minimal or not, can be computed as follows. First, we fix a caustic type (E or H), a reversor \( r \), a period \( m_0 \) (which must be even for H-caustics), and an even winding number \( m_1 \), such that \( 2 \leq m_1 < m_0 \). We have listed in Table V all possible combinations. Then we find the caustic parameter through numerical inversion of the relation \( \rho (\lambda) = m_1/2m_0 \). Of course, we solve that equation for \( \lambda \in E \) or \( \lambda \in H \), depending on the caustic type we are looking for. Finally, we take from Table I the starting point \( (q, p) \in \text{Fix}(r) \) such that \( q + (p) \) is tangent to \( Q_\lambda \), and iterating the billiard map \( f : M \rightarrow M \), we get the whole SPO in the phase space \( M \) and the whole SPT in \( \mathbb{R}^2 \).

Any \( R \)-SPT is travelled twice in opposite directions, since it hits orthogonally the ellipse at points \( q \in Q \cap Q_\lambda \). It can be observed in the last three rows of Table V. Therefore, there exist SPTs of even period \( m_0 = 2l \geq 4 \), with only \( l + 1 \) distinct impact points on the ellipse. We have found similar SPTs inside ellipsoids of \( \mathbb{R}^3 \). See Fig. 1.

Finally, we stress that any SPT has a dual version of the same period and caustic type, given by the dual transformation \( g \); see Corollary 1. For instance, this duality is evident for the two SPTs with caustic type E that have common period \( m_0 = 4k \). Suffice it to compare the two groups of associated formulae in the second row of Table V, and to recall that \( g \) exchanges the role of positions and velocities. In particular, the rectangular trajectory of that row is the dual version of the diamond-shaped one. The same applies to rows 1, 4 and 5. Instead, trajectories in rows 3 and 6 are dual of themselves.

VI. 3D CASE

The previous section sets the basis of this one. Roughly speaking, we apply similar ideas to find analogous results.

A. Caustics and elliptic coordinates

Let us recall some classical facts about caustics and elliptic coordinates. All the notations introduced in this subsection will be used later on in this section.

We write the ellipsoid as
\[
Q = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \right\},
\]
where \( 0 < a_1 < a_2 < a_3 \). Any nonsingular billiard trajectory inside \( Q \) is tangent to two distinct confocal caustics \( Q_{\lambda_1} \) and \( Q_{\lambda_2} \) (with \( \lambda_1 < \lambda_2 \)) of the family
\[
Q_\mu = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{j=1}^{3} \frac{x_j^2}{a_j - \mu} = 1 \right\},
\]
where \( \mu \notin \{ a_1, a_2, a_3 \} \). If we set
\[
E = (0, a_1), \quad H_1 = (a_1, a_2), \quad H_2 = (a_2, a_3),
\]
then \( Q_\mu \) is an ellipsoid for \( \mu \in E \), an one-sheet hyperboloid if \( \mu \in H_1 \), and a two-sheet hyperboloid when \( \mu \in H_2 \).

Not all combinations of caustics exist. For instance, both caustics cannot be ellipsoids. The four possible combinations are denoted by EH1, H1H1, EH2, and H1H2. Hence, the caustic parameter \( \lambda = (\lambda_1, \lambda_2) \) belongs to the nonsingular caustic space
\[
\Lambda = (E \times H_1) \cup (H_1 \otimes H_1) \cup (E \times H_2) \cup (H_1 \times H_2),
\]
for \( H_1 \otimes H_1 = \{ (\lambda_1, \lambda_2) \in H_1 \times H_1 : \lambda_1 < \lambda_2 \} \), in order to avoid the singular case \( \lambda_1 = \lambda_2 \).
Let us write the three coordinates planes of $\mathbb{R}^3$ as

$$\Pi_l = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_l = 0\}, \quad l \in \{1, 2, 3\}.$$  

We know (see Theorem 2) that given any $q \not\in \Pi_1 \cup \Pi_2 \cup \Pi_3$, there exist an ellipsoid $Q_e$, a one-sheet hyperboloid $Q_{h_1}$, and a two-sheet hyperboloid $Q_{h_2}$ such that

$$q \in Q_e \cap Q_{h_1} \cap Q_{h_2},$$  

(10)

those being quadrics mutually orthogonal at $q$. Therefore, $e < a_1 < h_1 < a_2 < h_2 < a_3$. Let $q = (e, h_1, h_2)$ be the elliptic coordinates of $q = (x_1, x_2, x_3)$. The relation between Cartesian and elliptic coordinates (5) becomes

$$x_i^2 = \frac{(a_i - e)(a_i - h_1)(a_i - h_2)}{(a_i - a_j)(a_i - a_k)},$$  

(11)

for any $\{i, j, k\} = \{1, 2, 3\}$. To understand what happens at nongeneric points, we define the three singular caustics as

$$Q_{al} = \Pi_l, \quad l \in \{1, 2, 3\}.$$  

### Table V: Classification and symmetry properties of SPTs together to the minimal SPTs in Cartesian and elliptic coordinates.

| Type | Period | Minimal SPTs in Cartesian coordinates | Minimal SPTs in elliptic coordinates | Couples of reversors and symmetry properties |
|------|--------|--------------------------------------|--------------------------------------|-----------------------------------------------|
| 2k + 1 | $R_x, f \circ R_y$ | $q_0 = 2q_{2k+1} \in \{y\text{-axis}\}$ | $p_{k+1} \parallel \{x\text{-axis}\}$ | |
| 4k | $R_x, R_y$ | $q_0 = q_{4k} = -q_{2k} \in \{y\text{-axis}\}$ | $q_{3k} = -q_k \in \{x\text{-axis}\}$ | |
| 4k + 2 | $R_y, f \circ R_y$ | $q_0 = q_{4k+2} = -q_{2k+1} \in \{x\text{-axis}\}$ | $p_{k+1} = -p_{k+1} \parallel \{y\text{-axis}\}$ | |
| $m_1/2$ even | $R_x, f \circ R_y$ | $q_0 = (x_0, y_0) = q_{2k+2} \in Q \cap Q_{\lambda}$ | $q_{2k+1} = (x_0, -y_0) \in Q \cap Q_{\lambda}$ | |
| H | 4k | $R_x, R_y$ | $q_0 = q_{4k} = q_{2k} \in \{y\text{-axis}\}$ | $q_{3k} = (x, y) \in Q \cap Q_{\lambda}$ | |
| 4k + 2 | $R_y, f \circ R_y$ | $q_0 = q_{4k+2} \in Q \cap Q_{\lambda}$ | $q_{3k+2} = q_{k} = -q_{k+1} \parallel \{y\text{-axis}\}$ | |
| $m_1/2$ odd | $R_x, f \circ R_y$ | $q_0 = q_{4k+2} = -q_{2k+1} \in \{y\text{-axis}\}$ | $p_{k+1} = -p_{k+1} \parallel \{y\text{-axis}\}$ | |
We also recall that the focal ellipse and focal hyperbola shared by the family of confocal quadrics \( \{ Q_\mu : \mu \in \mathbb{R} \} \) are
\[
\mathcal{E} = \left\{ (0, x_2, x_3) \in \mathbb{R}^3 : \frac{x_2^2}{a_2 - a_1} + \frac{x_3^2}{a_3 - a_1} = 1 \right\} \subset \Pi_1,
\]
\[
\mathcal{H} = \left\{ (x_1, 0, x_3) \in \mathbb{R}^3 : \frac{x_2^2}{a_2 - a_1} - \frac{x_3^2}{a_3 - a_1} = 1 \right\} \subset \Pi_2.
\]

These notations are useful because of the following lemma, which is a known extension of the first item in Theorem 2.

**Lemma 3** If \( q \notin \mathcal{E} \cup \mathcal{H} \), there exists \( q = (e, h_1, h_2) \) such that \( e \leq a_1 \leq h_1 \leq a_2 \leq h_2 \leq a_3 \), \( e \neq h_1, h_1 \neq h_2 \), and property (10) holds, including its orthogonal character.

Thus, we can express in elliptic coordinates all points along nonsingular billiard trajectories. The trajectory is contained inside \( Q \), so \( e \geq 0 \). Impact points \( q \in Q \) correspond to \( e = 0 \). Tangency points \( q \in \mathcal{Q}_\lambda \) correspond to \( e = \lambda_1 \in E \), \( h_1 = \lambda_j \in H_1 \) or \( h_2 = \lambda_j \in H_2 \) depending on the type of the caustic \( Q_{\lambda_j} \). Crossing points \( q \in \Pi_1 \) have the following elliptic coordinates: \( \Pi_1 \) is \( \{ e = a_1 \} \cup \{ h_1 = a_1 \} \), \( \Pi_2 \) is \( \{ h_1 = a_2 \} \cup \{ h_2 = a_2 \} \), and \( \Pi_3 \) is \( \{ h_2 = a_3 \} \).

Indeed, one could agree that \( \mathcal{E} = \{ e = a_1 = h_1 \} \) and \( \mathcal{H} = \{ h_2 = a_2 = h_2 \} \), although we do not need it, because all nonsingular billiard trajectories sharing the caustics \( Q_{\lambda_1} \) and \( Q_{\lambda_2} \) are contained, in elliptic coordinates, inside the cuboid
\[
\mathcal{C}_\lambda = \left\{ 0, \lambda_1 \right\} \times \left\{ a_1, \lambda_2 \right\} \times \left\{ a_2, a_3 \right\}, \text{ for } \text{EH1},
\left\{ 0, \lambda_1 \right\} \times \left\{ a_1, \lambda_2 \right\} \times \left\{ a_2, a_3 \right\}, \text{ for } \text{HH1},
\left\{ 0, \lambda_1 \right\} \times \left\{ a_1, \lambda_2 \right\} \times \left\{ a_2, a_3 \right\}, \text{ for } \text{EH2},
\left\{ 0, a_1 \right\} \times \left\{ \lambda_1, a_2 \right\} \times \left\{ a_2, a_3 \right\}, \text{ for } \text{HII2}.
\]

We note that \( e \neq h_1 \neq h_2 \) for all \( q = (e, h_1, h_2) \in \mathcal{C}_\lambda \), since \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \). Hence, all points belonging to \( \mathcal{C}_\lambda \) verify Eqs. (10) and (11).

### B. Frequency map and winding numbers

The following results related to periodic trajectories of billiards inside ellipsoids can be found in Ref. 27.

There exists a map \( \omega : \Lambda \to \mathbb{R}^2 \), called frequency map and defined by means of six hyperelliptic integrals, that characterizes all caustic parameters \( \lambda = (\lambda_1, \lambda_2) \) that give rise to periodic trajectories. (Its explicit formula is given in Ref. 27.) To be precise, the billiard trajectories sharing the caustics \( Q_{\lambda_1} \) and \( Q_{\lambda_2} \) are periodic if and only if
\[
\omega (\lambda_1, \lambda_2) = (m_1, m_2) / 2m_0 \in \mathbb{Q}^2
\]
for some positive integers \( m_0 \), \( m_1 \), and \( m_2 \). These are the winding numbers introduced in item d) of Theorem 4. They describe how trajectories fold in \( \mathbb{R}^3 \). First, \( m_0 \) is the period. Second, \( m_1 \) is the number of \( \Pi_1 \)-crossings and \( m_2 \) is twice the number of turns around the \( x_1 \)-axis for

### C. Reversors and their symmetry sets

Let \( \sigma_1, \sigma_{mn}, \sigma_{123} : \mathbb{R}^3 \to \mathbb{R}^3 \) be the \( x_m x_n \)-specular reflection, the \( x_1 \)-axial reflection, and the central reflection, respectively. Here, \( \{ l, m, n \} = \{ 1, 2, 3 \} \). Two samples are \( \sigma_3 (x_1, x_2, x_3) = (x_1, x_2, -x_3) \) and \( \sigma_{13} (x_1, x_2, x_3) = (-x_1, x_3, x_3) \). Let \( s_l, s_{mn}, s_{123} : M \to M \) be the symmetries given by \( s_l (q, p) = (\sigma_l q, \sigma_l p) \), \( s_{mn} (q, p) = (\sigma_{mn} q, \sigma_{mn} p) \), and \( s_{123} (q, p) = (\sigma_{123} q, \sigma_{123} p) \). We denote by \( R \) the reversor written as \( \bar{r} \) in Sec. III. Likewise, \( R_l = s_l \circ R, R_{mn} = s_{mn} \circ R \), and \( R_{123} = s_{123} \circ R \).

**Remark 4** Henceforth, we shall adopt the following index notation for the sake of brevity. If \( l, m, \) and \( n \) appear without an explicit explanation of their ranges, it must be understood that they represent any choice such that \( \{ l, m, n \} = \{ 1, 2, 3 \} \). This convention is used, for instance, in Tables VI and VII. Moreover, in the second to fifth rows of Table VII, index \( l \) has a distinguished meaning from indexes \( m \) and \( n \); and the formulæ contained in those rows are invariant under permutation of \( m \) and \( n \).

This has to do with the fact that \( \sigma_{mn} = \sigma_{nm} \).

The three planar sections of the ellipsoid are
\[
S_l = Q \cap \Pi_l, \quad l \in \{ 1, 2, 3 \}.
\]

The symmetry sets of the 16 = 2^4 reversors are listed in Table VI, which follows directly from Proposition 2. We note that only the symmetry sets of \( R_{123} \) and \( f \circ R \) are empty. We usually write coordinates \( q = (x_1, x_2, x_3) \) in the configuration space \( Q \), and \( p = (u_1, u_2, u_3) \) in the velocity space \( S^2 \).

### D. Characterization of STs

We want to characterize nonsingular STs inside nondegenerate ellipsoids, as trajectories passing, in elliptic

| \( r \) | \( \text{Fix}(r) \) | \( \text{Fix}(f \circ r) \) |
|---|---|---|
| \( R \) | \( p \in N_s Q \) | \( \emptyset \) |
| \( R_l \) | \( q \in S_l \) and \( p \in N_s S_l \) | \( u_m = u_n = 0, u_1^2 = 1 \) |
| \( R_{mn} \) | \( x_m = x_n = 0, x_1^2 = a_1 \) | \( (q + p) \perp \{ x_1 \text{-axis} \} \) |
| \( R_{123} \) | \( \emptyset \) | \( 0 \in q + (p) \) |

**TABLE VI.** The symmetry sets of the 16 reversors of the billiard inside a triaxial ellipsoid \( Q \subset \mathbb{R}^3 \).
coordinates, through some vertex of the cuboid $C_{\lambda}$. In order to accomplish this goal, we must prove both implications. In Proposition 5 we check that any vertex of $C_{\lambda}$ is visited by some (indeed, eight) STs. In Proposition 6 we see that any ST passes through some vertex of $C_{\lambda}$.

**Proposition 5** Let $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ be fixed. Given any vertex $q_{\ast} = (e, h_1, h_2)$ of $C_{\lambda}$, let $q_e \in Q_e \cap Q_{h_1} \cap Q_{h_2}$. Then there exists some reversor $r$, and some point $(q, p) \in \text{Fix}(r)$ such that $q_{\ast} \in q + (p)$ and the line $q + (p)$ is tangent to $Q_{\lambda_1}$ and $Q_{\lambda_2}$. If $e = 0$, then $q = q_{\ast}$; otherwise $q_{\ast}$ is the middle point of the consecutive impact points $q$ and $q$.

We clarify some aspects of this result before its proof:

- We follow the index notation explained in Remark 4.
- We know that two symmetry sets only intersect at points $(q, p) \in M$ such that the line $q + (p)$ is contained in some coordinate plane; see Proposition 2. But we are dealing with nonsingular trajectories. Therefore, we can associate just one reversor to each vertex.
- Correspondence $q_{\ast} \mapsto (q, p)$ is 1-to-8; see Table VII. This has to do with two facts. First, elliptic coordinates do not distinguish among the eight octants in $\mathbb{R}^3$. Moreover, unit velocities $p$ directed inward $Q$ at impact points $q$ are excluded from Table VII; otherwise, $(q, p) \notin M$. The billiard trajectory associated to any of those eight points $(q, p) \in M$ is $\pi$-symmetric and passes, in elliptic coordinates, through vertex $q_{\ast}$.

**Proof** To begin with, we know from Lemma 3 that $Q_e$, $Q_{h_1}$, and $Q_{h_2}$ are mutually orthogonal at $q_{\ast}$. There are several kinds of vertexes. The first distinction is $e = 0$ or $e \neq 0$; that is, $q_{\ast} \in Q$ or $q_{\ast} \notin Q$. The second distinction is the number of coordinate planes through $q_{\ast}$. It turns out that there are six kinds of vertexes. We study each kind separately.

a) If the point $q_{\ast}$ belongs to $Q$ but is outside all three coordinate planes, then $q_{\ast} = (0, \lambda_1, \lambda_2)$. From Eq. (12), the type is H1H2. Thus, $Q_{\lambda_1}$ is a one-sheet hyperboloid, $Q_{\lambda_2}$ is a two-sheet hyperboloid, and $Q$, $Q_{\lambda_1}$ and $Q_{\lambda_2}$ are pairwise orthogonal at $q_{\ast}$. Thus, if $p$ is the outward unit normal velocity to $Q$ at $q$, then $(q, p) \in \text{Fix}(R)$ and the line $q + (p)$ is tangent to $Q_{\lambda_1}$ and $Q_{\lambda_2}$. Finally, let us check the formulae for $q$ and $p$ given in Table VII.

The formula for $q$ follows directly from Eq. (11). Thus, there are exactly eight choices for $q$. If $p \in N_qS_1$ and $p$ points outward $Q$ at $q$, then there exists $\mu > 0$ such that $u_l = \mu x_l/a_l$. And using that $p \in S^2$ we get that

$$1 = \mu^2 \left( \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \right) \implies \mu = \sqrt{\frac{a_1 a_2 a_3}{\lambda_1 \lambda_2}}.$$  

In summary, only $R$-STs with H1H2-caustics can take place for this vertex.

b) If the point $q_{\ast}$ belongs to $Q$ and is contained in only one coordinate plane, say $\Pi_1$, then $q_{\ast} = (0, h_1, h_2)$ with $\{h_1, h_2\} = \{\alpha_l, \lambda_1\}$ for some $\alpha \in [1, 2]$. In that case, $q_{\ast} = Q \cap \Pi_1 \cap Q_{\lambda_1}$. We do not get any restriction on the type of the caustics, but only some restrictions on the indexes; see Eq. (12). Namely,

- $l \in \{2, 3\}$ and $j = 2$ for type EH1;
- $l \in \{2, 3\}$ for type H1H1;
- $l \in \{1, 2\}$ and $j = 2$ for type EH2; and
- $l = j = 2$ or $(l, j) = (3, 1)$ for type H1H2.

We observe that any $(q, p) \in M$ such that $q = q_{\ast}$ and $p \in N_qS_1$ is contained in $\text{Fix}(R_l)$. Thus, the formula for $q$ follows again from relations (11), and we find four choices for $q$. Next, we look for an outward velocity $p = (u_1, u_2, u_3) \in S^2$ such that $p \perp T_qS_1$ and $q + (p)$ is tangent to both $Q_{\lambda_1}$ and $Q_{\lambda_2}$ with $(j, k) = (1, 2)$.

Condition $p \perp T_qS_1$ reads as $x_m u_m/a_n = x_m u_n/a_m$. In other words, we look for some $\nu \in \mathbb{R}$ such that

$$\left( u_m, u_n \right) = \nu \left( a_n x_m, a_m x_n \right).$$

Then, tangency with $Q_{\lambda_1}$ is immediate, since $Q_{\lambda_1}$ is orthogonal to the planar section $S = Q \cap \Pi_1$ at $q$. It only remains to impose the tangency with $Q_{\lambda_2}$. Thus, the quadratic equation

$$\frac{(x_1 + tu_l)^2}{a_l - \lambda_k} + \frac{(x_m + tu_m)^2}{a_m - \lambda_k} + \frac{(x_n + tu_n)^2}{a_n - \lambda_k} = 1$$

must have a unique solution in $t$, which is equivalent to the vanishing of its discriminant: $\beta^2 = \alpha \gamma$, where

$$\alpha = \frac{u_l^2}{a_l - \lambda_k} + \frac{u_m^2}{a_m - \lambda_k} + \frac{u_n^2}{a_n - \lambda_k} = \frac{1}{a_l - \lambda_k} \left( 1 + \left( \frac{a_l - a_m}{a_l - \lambda_k} x_m + \frac{a_l - a_n}{a_l - \lambda_k} x_n \right) \nu^2 \right),$$

$$\beta = \frac{x_m u_m}{a_m - \lambda_k} + \frac{x_n u_n}{a_n - \lambda_k} = \frac{a_n u_n (\lambda - \beta) \nu}{(a_m - \lambda_k)(a_n - \lambda_k)},$$

$$\gamma = \frac{x_m^2}{a_m - \lambda_k} + \frac{x_n^2}{a_n - \lambda_k} - 1 = \frac{\lambda_k (\lambda - \lambda_k)}{(a_m - \lambda_k)(a_n - \lambda_k)}.$$  

Here, we have used that $u_l^2 = 1 - u_m^2 - u_n^2$. After some calculations we find $\nu^2 = \lambda_k/\lambda_m a_m$, and from here we get the desired formulae in Table VII. We must impose in turn, that $p$ is an outward velocity, so

$$0 < \langle Dq, p \rangle = \nu \left( a_n x_m/a_m + a_m x_n/a_n \right),$$

and $\nu > 0$. Thus, once we fix $q$, there are two choices for $u_l$, and exactly eight choices for $(q, p)$. We finish this item by stressing the consequences of the restrictions on the index $l$. They mean that the reversor $R_1$ cannot take place for types EH1, H1H1, and H1H2, whereas $R_3$ is forbidden for type EH2. This information is displayed in Table IX.
TABLE VII. Relations among vertexes \(q_*\) of the cuboid \(C_\lambda\), types of caustics, reversors \(r\), and points \(q_* \in q + (p)\) such that \(q + (p)\) is tangent to \(Q_\lambda\) and \(Q_{\lambda_2}\), with \((q,p) \in \text{Fix}(r)\). Notation for indexes \((i,m,n) = \{1,2,3\}\) and \((j,k) = \{1,2\}\) is described in the text.

| Vertex \(q_* = (e,h_1,h_2)\) | Type \(q_*\) belongs to | \(q = (x_1,x_2,x_3)\), \(p = (u_1,u_2,u_3)\) |
|-----------------------------|--------------------------|------------------------------------------------|
| \((0,\lambda_1,\lambda_2)\) | H1H2 \(Q \cap \bigcap \{Q_{\lambda_1},Q_{\lambda_2}\}\) | \(R\) |
| \(e = 0, \{h_1,h_2\} = \{a_i,\lambda_j\}\) | all \(Q \cap \Pi_i \cap \Pi_{\lambda_j}\) | \(R_i\) |
| \(e = 0, \{h_1,h_2\} = \{a_m,a_n\}\) not \(\text{H1H1}\) \(Q \cap \Pi_m \cap \Pi_n\) | \(R_{mn}\) |
| \(e,h_1,h_2\) = \{ai,\lambda_1,\lambda_2\} not \(\text{H1H1}\) \(\Pi_i \cap Q_{\lambda_1} \cap Q_{\lambda_2}\) | \(f \circ R_i\) |
| \(e,h_1,h_2\) = \{am,an,\lambda_j\} all \(\Pi_m \cap \Pi_n \cap \Pi_{\lambda_j}\) | \(f \circ R_{mn}\) |
| \((a_1,a_2,a_3)\) | H1H2 \(\Pi_1 \cap \Pi_2 \cap \Pi_3\) | \(f \circ R_{123}\) |

\(c)\) If \(q_*\) belongs to \(Q\) and is contained in two coordinate planes, say \(\Pi_m\) and \(\Pi_n\), then \(q_* = (0,h_1,h_2)\) with \(\{h_1,h_2\} = \{a_m,a_n\}\). This prevents \(\text{H1H1}\)-caustics; see Eq. (12). Furthermore, we get again some restrictions on the indexes. Namely,

- \(1 \in \{m,n\}\) for type \(\text{EH1}\);
- \(3 \in \{m,n\}\) for type \(\text{EH2}\); and
- \(1 \notin \{m,n\}\) for type \(\text{H1H2}\).

Besides, \(q_*\) is one of the two vertexes of the ellipsoid on the \(x_l\)-axis. Hence, \((q_*,p) \in \text{Fix}(R_{mn})\) for any \((q_*,p) \in M\). Finally, let us check that the formulæ given for \(p = (u_1,u_2,u_3)\) in Table VII are correct.

We look for an outward velocity \(p \in S^2\) such that the line \(q + (p)\) is tangent to \(Q_{\lambda_k}\), for \(k = 1,2\). Thus, the quadratic equation

\[
\frac{(x_l+t_u)^2}{a_l-\lambda_k} + \frac{(x_m+tu_m)^2}{a_m-\lambda_k} + \frac{(x_n+tu_n)^2}{a_n-\lambda_k} = 1
\]

must have a unique solution in \(t\), which is equivalent to the vanishing of its discriminant: \(\beta^2 = \alpha \gamma\), where

\[
\alpha = \frac{u_l^2}{a_l-\lambda_k} + \frac{u_m^2}{a_m-\lambda_k} + \frac{u_n^2}{a_n-\lambda_k}
= \frac{1}{a_n-\lambda_k} \left( \frac{a_m-a_l}{a_l-\lambda_k} u_l^2 + \frac{a_n-a_m}{a_m-\lambda_k} u_m^2 \right),
\]

\[
\beta = \frac{x_ltu_l}{a_l-\lambda_k},
\]

\[
\gamma = \frac{x_l^2}{a_l-\lambda_k} - 1 = \frac{\lambda_k}{a_l-\lambda_k}.
\]

After some calculations we obtain that

\[
a_n(a_m-\lambda_k)u_l^2 + (a_m-a_n)\lambda_ku_m^2 = \lambda_k(a_m-\lambda_k).
\]

for \(k = 1,2\). These two equations along with condition \(u_l^2 + u_m^2 + u_n^2 = 1\) form a system of three linear equations in the unknowns \(u_l^2, u_m^2,\) and \(u_n^2\). A simple computation shows that the formulæ given in Table VII are the unique solution of this system.

Finally, we impose that \(p\) is an outward velocity, so

\[0 < \langle Dq,p \rangle = x_lu_l/a_l.\]

and \(x_lu_l > 0\). Thus, once fixed the vertex \(q_*\), there are two choices for \(u_m\) and two others for \(u_n\). Therefore, there are exactly eight choices for \((q,p)\).

We finish again by stressing the consequences of the restrictions on the indexes \(m\) and \(n\). They mean that the reversor \(R_{123}\) cannot take place for type \(\text{EH1}\), \(R_{12}\) is forbidden for type \(\text{EH2}\), both \(R_{13}\) and \(R_{12}\) are not allowed for type \(\text{H1H2}\), and all reversors of the form \(R_{mn}\) cannot be found for type \(\text{H1H1}\). This information is also displayed in Table IX.

\(d)\) If \(q_*\) does not belong to \(Q\) but it is in all coordinate planes, then \(q_* = (0,0,0) = 0\) and \(q_* = (a_1,a_2,a_3)\). In that case the type is \(\text{H1H2}\); see Eq. (12). We look for lines \(q_* + (p) = (p)\) that are tangent to the one-sheet hyperboloid \(Q_{\lambda_1}\) and to the two-sheet hyperboloid \(Q_{\lambda_2}\). Since \(q_*\) is the center of both quadrics, \(p = (u_1,u_2,u_3)\) must be an asymptotic direction of
them, so
\[ \frac{u_1^2}{a_1 - \lambda_k} + \frac{u_2^2}{a_2 - \lambda_k} + \frac{u_3^2}{a_3 - \lambda_k} = 0, \quad k \in \{1, 2\}. \]

These equations along with \( u_1^2 + u_2^2 + u_3^2 = 1 \) form a system of three linear equations in the unknowns \( u_1', u_2', \) and \( u_3' \). A simple computation shows that the formulae given in Table VII are the unique solution of this system. Thus, there are eight choices for \( p \).

The line \( q_* + (p) = 0 + (p) \) intersects the ellipsoid \( Q \) at some points \( q = \mu p \) and \( \hat{q} = -\mu p \) where \( \mu > 0 \). So, \( x_l = \mu u_l \). And using that \( q \in Q \) we get that
\[ 1 = \mu^2 \left( \frac{u_1^2}{a_1^2} + \frac{u_2^2}{a_2^2} + \frac{u_3^2}{a_3^2} \right) \implies \mu = \sqrt{\frac{a_1 a_2 a_3}{\lambda_1 \lambda_2}}. \]

Finally, we realize that \( (q, p) \in \text{Fix}(R_{123}) \) and \( q_* \) is the middle point of \( \hat{q} \) and \( q \).

In summary, only \( (f \circ R_{123}) \)-STs with H1H2-caustics can take place for this vertex.

e) If \( q_* \) does not belong to \( Q \) but it is in two coordinate planes, say \( \Pi_m \) and \( \Pi_n \), then \( q_* = (e, h_1, h_2) \) with \( \{e, h_1, h_2\} = \{m, a_n, \lambda_3\} \). This case is the “dual” of case b), so it is associated to the reversors \( f \circ R_{mn} \). The computations do not involve new ideas. We skip them. We just stress that there are no restriction on the type of the caustics, but only some restrictions on the indexes; see Eq. (12). Namely,
\[ \bullet \ 1 \in \{m, n\} \text{ for types EH1, H1H1, and H1H2}; \text{ and} \\
\bullet \ 3 \in \{m, n\} \text{ for types EH2}. \]

They mean that the reversor \( f \circ R_{23} \) cannot take place for types EH1, H1H1, and H1H2, whereas \( f \circ R_{12} \) is forbidden for type EH2.

f) If \( q_* \) does not belong to \( Q \) and is contained in only one coordinate plane, say \( \Pi_l \), then \( q_* = (e, h_1, h_2) \) with \( \{h_1, h_2\} = \{a_l, \lambda_1, \lambda_2\} \). In that case, \( q_* \in \Pi_l \cap Q_{\lambda_1} \cap Q_{\lambda_2} \). This case is the “dual” of case c), so it is associated to the reversors \( f \circ R_l \). We also skip the computations.

We only remark that no H1H1-caustics can be associated to this case. Besides, we get again some restrictions on the indexes; see Eq. (12). Namely, \( l \neq 1 \) for type EH1: \( l \neq 3 \) for type EH2; and \( l = 1 \) for type H1H2. Consequently, \( f \circ R_l \) cannot take place for type EH1, \( f \circ R_3 \) is forbidden for type EH2, both \( f \circ R_2 \) and \( f \circ R_3 \) are not allowed for type H1H2, and all reversors of the form \( f \circ R_l \) cannot be found for type H1H1. \( \Box \)

Remark 5 Opposed vertexes of a given cuboid \( C_\lambda \) provide points on dual symmetry sets. At the same time, the formulae for \( (q, p) \) of reversors \( f \circ R_l \), \( f \circ R_{mn} \), and \( f \circ R_{123} \) follow from the formulae of their dual reversors \( R_{mn} \), \( R_l \), and \( R \), respectively, by the change \( u_i^2 \leftrightarrow x_i^2 / a_i \). This does not come as a surprise; see Corollary 1. Suffice it to realize that the dual transformation \( q : M \to M, g(q, p) = (\bar{q}, \bar{p}) \), verifies that \( \bar{p} = -C^{-1}q, C^2 = \text{diag}(a_1, a_2, a_3) \).

All implications in the proof of Proposition 5 can be reversed. Hence, we can move along Table VII in both directions: from left to right, and from right to left.

Nevertheless, we prefer to prove Proposition 6 following a reasoning whose generalization to an arbitrary dimension is straightforward.

**Proposition 6** Let \( O \) be a nonsingular \( r \)-SO through a point \((q, p) \in \text{Fix}(r) \) for some reversor \( r : M \to M \). Let \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) be its caustic parameter. Then there exists a vertex \( q_* = (e, h_1, h_2) \) of the cuboid \( C_\lambda \) and a point \( q_* \in Q_e \cap Q_{h_1} \cap Q_{h_2} \) such that \( q_* \in q + (p) \).

**Proof** First, we consider the reversors of the form \( R, R_l \) or \( R_{mn} \). Let \( \hat{q} : \mathbb{R} \to \mathbb{R}^3, \hat{q}(t) = (\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t)) \), be any of the arc-length parameterizations of the billiard trajectory through the impact point \( q := q \) with unit velocity \( p \), determined by the conditions \( \hat{q}(0) = q_* \) and \( \hat{q}(0) = p \). Clearly, it is smooth except at impact points. Besides, it has the symmetry properties listed in Table VIII, which can be deduced from comments after Remark 2 and elementary geometric arguments.

Let \( \tilde{q} : \mathbb{R} \to \mathcal{C}_\lambda, \tilde{q}(t) = (\tilde{e}(t), \tilde{h}_1(t), \tilde{h}_2(t)) \), be the corresponding parameterization in elliptic coordinates. We know that components \( \tilde{e}(t), \tilde{h}_1(t), \) and \( \tilde{h}_2(t) \), oscillate in some intervals in such a way that their only critical points are attained at the extremes of these intervals.

The cuboid \( C_\lambda \) is the product of these three intervals. Besides, the functions \( h_1(t) \) and \( h_2(t) \) are smooth everywhere.

The above symmetries of \( \tilde{q}(t) \) imply that \( \tilde{q}(-t) = \tilde{q}(t) \), since elliptic coordinates do not distinguish among the eight octants in \( \mathbb{R}^3 \). Therefore,
\[ \tilde{h}_1'(0) = 0, \quad \tilde{h}_2'(0) = 0. \]

In addition, \( \tilde{e}(0) = 0 \), since we have taken \( q_* = q \in Q \). Hence, \( q_* = (e, h_1, h_2) := q(0) = (0, \tilde{h}_1(0), \tilde{h}_1(0)) \) is a vertex of \( C_\lambda \); see item c) of Theorem 4.

It remains to consider reversors of the form \( f \circ R_l f \circ R_{mn} f \circ R_{123} \). Let \( \tilde{q} : \mathbb{R} \to \mathbb{R}^3 \) be the arc-length parameterization of the billiard trajectory that begins at the middle point \( q_* := (q + \hat{q}) / 2 \) with unit velocity \( p \).
TABLE IX. Forbidden reversors for each type of caustics.

| Type  | Forbidden reversors |
|-------|---------------------|
| EH1   | \( R, f \circ R_{123}, R_1, f \circ R_{23}, R_{23}, f \circ R_1 \) |
| EH2   | \( R, f \circ R_{123}, R_3, f \circ R_{12}, R_{12}, f \circ R_3 \) |
| H1H1  | \( R, R_1, R_{12}, R_{23}, f \circ R_{123} \) |
| H1H2  | \( f \circ R_1, f \circ R_2, f \circ R_3, f \circ R_{23} \) |

We recall that \( \tilde{q} \) is the previous impact point, that is, \( Q \cap (q + (p)) = \{q, \tilde{q}\} \).

The symmetry properties of these parameterizations are also listed in Table VIII. For instance, if \( r = f \circ R_{123} \), then \( q_s = (0, 0, 0) \), which implies \( \tilde{q}(-t) = -\tilde{q}(t) \). Then, we apply exactly the same argument as before, with just one difference. The function \( \tilde{e}(t) \) is smooth at \( t = 0 \), because \( \tilde{q}(0) = q_s \notin Q \) and \( \tilde{e}(0) \neq 0 \). Hence, we also get that \( \tilde{e}'(0) = 0 \), so \( q_s = (e, h_1, h_2) := \tilde{q}(0) = (\tilde{e}(0), h_1(0), h_1(0)) \) is a vertex of \( C_\lambda \).

Finally, the intersection property \( q_s \in Q_r \cap Q_{h_1} \cap Q_{h_2} \) follows from Lemma 3.

Corollary 2 Nonsingular STs inside triaxial ellipsoids of \( \mathbb{R}^3 \) are characterized as trajectories passing, in elliptic coordinates, through some vertex of their cuboid.

E. Forbidden reversors for each type of caustics

Next, we emphasize that, although there are 14 non-empty symmetry sets, once fixed the caustic type, some of them cannot take place. To be more precise, there are ten forbidden symmetry sets for H1H1-caustics, but only six otherwise. These results have been obtained along the proof of Proposition 5, and we organize them in Table IX. Forbidden reversors appear in dual couples—couples whose symmetry sets are interchanged by the dual transformation \( g \).

A glance at cuboid (12) and Table VII shows that, two different vertexes of the same cuboid cannot be associated to the same reversor, but for H1H1-caustics. This agrees with the previous discussion, where we found \( 8 = 14 - 6 \) feasible symmetry sets for caustics of type EH1, EH2, and H1H2, but only \( 4 = 14 - 10 \) for H1H1-caustics.

F. Characterization and classification of SPTs

Let us characterize and classify SPTs inside triaxial ellipsoids of \( \mathbb{R}^3 \). We classify them by the caustic type and the couple of vertexes of the cuboid they connect.

Theorem 6 Nonsingular SPTs inside triaxial ellipsoids of \( \mathbb{R}^3 \) are characterized as trajectories connecting, in elliptic coordinates, two different vertexes of their cuboid. All nonsingular SPOs are doubly SPOs, but a few with

H1H1-caustics. Besides, there are exactly 112 classes of nonsingular SPTs, listed in Tables X–XIII.

Proof Let \( O \) be a nonsingular \( r\)-SPO through a point \( (q, p) \in \text{Fix}(r) \), for some reversor \( r : M \to M \), with winding numbers \( m_0, m_1, m_2 \). Let \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) be its caustic parameter. Let \( L_0 \) be its length.

Let \( \tilde{q} : \mathbb{R} \to \mathbb{R}^3 \) be the arc-length parameterization of the billiard trajectory that begins at the distinguished point \( q \), introduced in Proposition 6.

Let \( \tilde{q} : \mathbb{R} \to C_\lambda \) be the corresponding parameterization in elliptic coordinates. Then \( \tilde{q}(t) \) is even —see the proof of Proposition 6— and \( L_0 \)-periodic —see Theorem 4.

The key trick is as simple as to realize that

\[
\tilde{q}(L_0/2 - t) = \tilde{q}(t - L_0/2) = \tilde{q}(t + L_0/2), \quad \forall t \in \mathbb{R}.
\]

Hence, \( \tilde{q}(t) \) is even with respect to \( L_0/2 \), and so

\[
q_s = (e^*, h_1^*, h_2^*); \quad q_s = (e^*, h_1^*, h_2^*); \quad \tilde{q}(L_0/2)
\]

are vertexes of the cuboid \( C_\lambda \). The proof for \( q_s \) was already explained in Proposition 6; the proof for \( q_s \) is equal.

Now, taking into account the interpretation of the winding numbers as, the number of complete oscillations of each elliptic coordinate when the arc-length parameter \( t \) moves from 0 to \( L_0 \)—see item d) of Theorem 4—, we consider two cases:

- If some winding number is odd, then \( q_s \neq q_s \), because

\[
e^* = e^* \Leftrightarrow m_0 \in 2Z, \quad h_1^* = h_1^* \Leftrightarrow m_i \in 2Z.
\]

- If all winding numbers are even, then \( \tilde{q}(t) \) has period \( L_0/2 \) and \( q_s = q_s \) —see item d) of Theorem 4—, in which case we repeat the same strategy to find a second vertex \( q_o = (e^*, h_1^*, h_2^*) \) such that

\[
e^* = e^* \Leftrightarrow m_0 \in 4Z, \quad h_1^* = h_2^* \Leftrightarrow m_i \in 4Z.
\]

Then, \( q_o \neq q_s \) —see item d) of Theorem 4.

Therefore, the trajectory connects two different vertexes in both cases. Next, we prove that any trajectory \( \tilde{q}(t) \) connecting, in elliptic coordinates, two vertexes is an SPT. Thus, we assume that \( q_s = \tilde{q}(t_* ) \) and \( q_s = \tilde{q}(t_* ) \) are two different vertexes with \( t_* < t_* \). From Proposition 5 we know that this trajectory is \( r_\gamma \)-symmetric and \( r_\gamma \)-symmetric for some reversors \( r_\gamma \) and \( r_\gamma \). The case \( r_\gamma = r_\gamma \) is not excluded. Repeating the reasoning in Proposition 6, we deduce that \( \tilde{q}(t) \) is symmetric with respect to \( t = t_* \) and \( t = t_* \). In particular,

\[
\tilde{q}(2t_* + t) = \tilde{q}(-t) \quad \text{and} \quad \tilde{q}(2t_* + t) = \tilde{q}(-t)
\]

for all \( t \in \mathbb{R} \). Hence, \( \tilde{q}(t) \) is \( T \)-periodic, and so, \( \tilde{q}(t) \) is periodic with period \( T \) or \( 2T \); see item d) of Theorem 4. This proves the characterization of SPTs.
TABLE X. Classification of SPTs for caustic type EH1. Notation for kinds “e”,”o”, “f”, and “v” is described in the text. For each kind of winding numbers we list its minimal representative, and its four couples of reversors, whose order inside the couple is irrelevant.

| $(m_0, m_1, m_2)$ | Couples of reversors |
|-------------------|----------------------|
| (t, t, t)         | $\{R_5, f \circ R_{12}\}, \{R_2, f \circ R_{13}\}$ |
| (10, 6, 2)        | $\{R_{13}, f \circ R_2\}, \{R_{12}, f \circ R_3\}$ |
| (10, 6, 4)        | $\{R_{13}, f \circ R_3\}, \{R_{12}, f \circ R_{12}\}$ |
| (6, 4, 2)         | $\{R_{13}, f \circ R_2\}, \{R_2, f \circ R_3\}$ |
| (6, 4, 2)         | $\{R_{13}, f \circ R_{12}\}, \{R_{12}, f \circ R_{13}\}$ |
| (o, e, e)         | $\{R_5, f \circ R_3\}, \{R_2, f \circ R_2\}$ |
| (5, 4, 2)         | $\{R_{13}, f \circ R_{13}\}, \{R_{12}, f \circ R_{12}\}$ |
| (8, 6, 2)         | $\{R_5, R_{13}\}, \{f \circ R_3, f \circ R_{13}\}$ |
| (8, 6, 2)         | $\{R_2, R_{13}\}, \{f \circ R_2, f \circ R_{13}\}$ |
| (f, f, t)         | $\{R_5, R_{12}\}, \{f \circ R_3, f \circ R_{13}\}$ |
| (8, 4, 2)         | $\{R_2, R_{12}\}, \{f \circ R_2, f \circ R_{12}\}$ |

TABLE XI. Analogous of Table X for caustic type EH2.

| $(m_0, m_1, m_2)$ | Couples of reversors |
|-------------------|----------------------|
| (t, t, t)         | $\{R_1, f \circ R_{23}\}, \{R_5, f \circ R_{13}\}$ |
| (10, 6, 2)        | $\{R_{13}, f \circ R_2\}, \{R_{23}, f \circ R_3\}$ |
| (10, 6, 4)        | $\{R_{13}, f \circ R_3\}, \{R_{23}, f \circ R_{12}\}$ |
| (6, 4, 2)         | $\{R_{13}, f \circ R_1\}, \{R_{23}, f \circ R_2\}$ |
| (f, t, t)         | $\{R_1, R_{23}\}, \{f \circ R_3, f \circ R_{23}\}$ |
| (8, 6, 2)         | $\{R_2, R_{23}\}, \{f \circ R_2, f \circ R_{23}\}$ |
| (f, f, t)         | $\{R_1, R_{23}\}, \{f \circ R_3, f \circ R_{23}\}$ |
| (8, 4, 2)         | $\{R_2, R_{23}\}, \{f \circ R_2, f \circ R_{23}\}$ |

TABLE XII. Analogous of Table X for caustic type H1H2.

| $(m_0, m_1, m_2)$ | Couples of reversors |
|-------------------|----------------------|
| (t, t, t)         | $\{R, f \circ R_{12}\}, \{R_3, f \circ R_3\}$ |
| (10, 6, 2)        | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (10, 6, 4)        | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (6, 4, 2)         | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (f, t, t)         | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (8, 6, 2)         | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (f, f, t)         | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |
| (8, 4, 2)         | $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}$ |

Once we have established that any SPT connects two different vertexes of its cuboid, it is easy to deduce that, all SPOs inside triaxial ellipsoids of $\mathbb{R}^3$ whose type of caustics are EH1, EH2 or H1H2, must be doubly SPOs. Suffice it to recall that each vertex of the cuboid is associated to a different reversor for those three types.

The number 112 comes from $4 \times 28$, since there are 4 types of caustics and any 3-dimensional cuboid has eight vertexes, and so $28 = (8 \times 7) / 2$ couples of vertexes.

We must check that none of the 112 classes of SPTs is fictitious. To do it, we present an algorithm that provides a minimal SPT of each class. Minimal means that it has the smallest possible period. The key step of the algorithm is to properly choose the winding numbers $(m_0, m_1, m_2)$. In accordance with the previous discussion, we distinguish four kinds of winding number. Namely, even: “e”, odd: “o”, multiple of four: “f”, and even but not multiple of four: “v”. For instance, all winding numbers of kind (t, t, t) connect opposite vertexes of their cuboids. The kind (f, f, f) never takes place —see item d) of Theorem 4.

Let us explain the algorithm by using an example. We want to connect opposite vertexes $q_1 = (e, h_1^1, h_2^1) = (0, \lambda_1, a_3)$ and $q_2 = (e, h_1^2, h_2^2) = (\lambda_1, a_1, a_2)$ of the cuboid

$C_\lambda = [0, \lambda_1] \times [a_1, \lambda_2] \times [a_2, a_3],$

which corresponds to caustic type EH1.

This problem about vertexes is equivalent to find an SPT of type EH1 that is simultaneously $R_3$-symmetric and $(f \circ R_{12})$-symmetric, because the reversors associated to $q_1$ and $q_2$ are $R_3$ and $f \circ R_{12}$, respectively (cf. Table VII). As we said above, winding numbers must be of kind $(t, t, t)$. Then, taking for granted conjecture (6), the minimal choice is $(m_0, m_1, m_2) = (10, 6, 2)$. Finally, if we solve Eq. (13) for $\lambda = (\lambda_1, \lambda_2) \in E \times H_1$, with $(m_0, m_1, m_2) = (10, 6, 2)$, and draw the billiard trajectory through the point $(q, p)$, given by the formulae of the second row of Table VII for $l = 3$, then we obtain a trajectory of period 10, type EH1, $R_3$-symmetric, and $(f \circ R_{12})$-symmetric.

All winding numbers of kind (t, t, t) connect opposite vertexes of their cuboids. Hence, their SPTs of type EH1 can display the following double symmetries: $\{R_3, f \circ R_{12}\}, \{R_2, f \circ R_3\}, \{R_{23}, f \circ R_3\}, \{R_{12}, f \circ R_3\}$. They are obtained by looking at Table VII the relations between vertexes and reversors.

Other kinds of winding numbers and other caustic types can be studied in a completely analogous way. And so, we complete Tables X–XIII.

The caustic type H1H1 has the following peculiarity: both caustics $Q_{\lambda_1}$ and $Q_{\lambda_2}$ are 1-sheet hyperboloids. We will denote $Q_{\lambda_1}$ as the outer hyperboloid and $Q_{\lambda_2}$ as the inner one, since $\lambda_1 < \lambda_2$. Note that two vertexes of the form $(e, \lambda_1, h_2)$ and $(e, \lambda_2, h_2)$ are associated to the same
reversor; see Table VII. In particular, SPTs connecting vertexes of this form are not doubly symmetric. Likewise, different connections can give rise to the same couple of reversors. We still classify those connections as different classes, because they have different geometries. Symbols \((R_2, R_1)\), \((R_3, R_2)\), \((R_2, R_3)\), and \((R_2, R_3)\) denote four classes with caustic type H1H1 and the same couple of reversors: \(\{R_2, R_3\}\). They are depicted in Table XVI. But the points \(q_\ast\), associated to each reversor (Table VII) are on the outer (respectively, inner) hyperboloid when the reversor is written to the left (respectively, right) of symbol \(|\). This notation is used in Table XIII.

We have seen that, once fixed the caustic type, there exists a tight relation between the symmetry sets associated to an SPT and the kind of winding numbers. A similar result for standard-like maps was obtained by Kook and Meiss.\(^{26}\) Compare Tables X–XIII with their Table I, where they list the 36 classes of SPOs for the 4D symplectic Froeschl´e map. Our 4D symplectic billiard map has more classes because a new factor enters in the classification: the caustic type.

### G. Gallery of minimal SPTs

We have 112 classes of SPTs listed in Tables X–XIII, so we tackle the task of finding a minimal representative in each class. They have periods \(m_0 \in \{4, 5, 6, 8, 10\}\).

The algorithm for the caustic type EH1 is: 1) Choose one of the seven minimal winding numbers \((m_0, m_1, m_2)\) in Table X; 2) Find caustic parameters \(\lambda_1 \in E\) and \(\lambda_2 \in H_1\) such that Eq. (13) holds; 3) Choose one of the four couples of reversors \((r, \tilde{r})\) in the corresponding row of Table X; 4) Get a point \((q, p)\) from Table VII using the reversor \(r\) (or \(\tilde{r}\)); and 5) Draw the doubly SPT through \(q\) with velocity \(p\).

Only step 2) is problematic, because it requires the inversion of the frequency map. The main obstacle is that Eq. (13) might not have solution in \(E \times H_1\) for some of the minimal winding numbers at hand. Nevertheless, it is known\(^{27}\) that \(\omega|_{E \times H_1}\) is a diffeomorphism and

\[
\lim_{a_1 \to 0^+} \omega(E \times H_1) = \{ (\omega_1, \omega_2) \in \mathbb{R}^2 : 0 < \omega_2 < \omega_1 < \frac{1}{2} \}.
\]

Consequently, Eq. (13) has a unique solution in \(E \times H_1\) provided that ellipsoid (9) is flat enough.

The other three caustic types can be dealt with the same algorithm. There is just one difference among them. Namely, we must consider different shapes of ellipsoids to ensure that Eq. (13) has a solution. More precisely, we must consider almost flat ellipsoids \((a_1)\) for caustic types EH1 and H1H1, and almost “segments” (both \(a_1\) and \(a_2\) small) for caustic types EH2 and H1H2. These results can be found in Ref. 27. This explains why some ellipsoids in Tables XIV–XVII are so flat. We have chosen not too extreme ellipsoids whenever it has been possible.

We have found a minimal SPT of each class. Some of them are displayed in Tables XIV–XVII. Arrows in their column headings point in the direction of increasing \(x_1\), \(x_2\) or \(x_3\). Each line of Tables XIV, XV and XVII shows four different perspectives of an SPT. The 3D image corresponds to an isometric view of the first octant. The three projected planes are viewed from the positive missing axis. We only display 3D views in Table XVI. Billiard trajectories are depicted in red. Green and yellow lines represent intersections of \(Q\) with H1-caustics and H2-caustics, respectively. The E-caustic is also shown in cases EH1 and EH2.

Period four takes only place for four classes of SPTs of type H1H1. Period five takes only place for eight classes of SPTs, half of type EH1 and half of type EH2. We display half of these minimal SPTs in Table XIV. Their announced symmetries can be verified by observing their projections. We can check that trajectories of caustic type H1H1 give one turn around the \(x_1\)-axis and have three tangential touches with the outer (respectively, inner) 1-sheet hyperboloid, trajectories of type EH1 give one turn around the \(x_1\)-axis and cross four times the plane \(\Pi_1\), whereas trajectories of type EH2 give two turns around the \(x_3\)-axis and cross twice the plane \(\Pi_2\). This is consistent with the geometric interpretation given in Subsec. VIB of winding numbers.

We show in Table XV the minimal representatives of the classes listed in the third row of Table XII. Since all of them have \((6, 4, 2)\) as winding numbers, they are 6-periodic, cross four times the plane \(\Pi_2\), and cross twice the plane \(\Pi_1\).

We draw several SPTs of type H1H1 in Table XVI. They show the difference among the six classes \((R_2, R_2)\), \((R_3, R_3)\), \((R_2, R_3)\), \((R_3, R_2)\), and \((R_2, R_3)\). We can locate those classes in the last rows of Table XIII. One can observe, for instance, that the SPT of class \((R_2, R_3)\) has two impacts on the intersection \(Q \cap \Pi_2 \cap Q_{\lambda_1}\) and two more on \(Q \cap \Pi_3 \cap Q_{\lambda_1}\). On the contrary, the
TABLE XIV. Two minimal SPTs of period 4 and four minimal SPTs of period 5. The “Data” column includes consecutively: caustic type, winding numbers \((m_0, m_1, m_2)\), ellipsoid parameters \(a_2\) and \(a_1\)—we have set \(a_3 = 1\)—, caustic parameters \(\lambda_2\) and \(\lambda_1\), and all reversors whose symmetry sets intersect the corresponding SPOs.

| 3D \((x_1 :↑, x_2 :\downarrow, x_3 :\nearrow)\) | Plane \(\Pi_1\) \((x_2 :↑, x_3 :→)\) | Plane \(\Pi_2\) \((x_1 :↑, x_2 :←)\) | Plane \(\Pi_3\) \((x_1 :↑, x_2 :→)\) | Data |
|---|---|---|---|---|
| \(H1H1\) \((4, 3, 2)\) | 0.8 | 0.13 | 0.648376 | 0.130077 \(R_2\) |
| | | | | | \(f \circ R_{13}\) |
| \(EH2\) \((5, 4, 2)\) | 0.3969 | 0.2 | 0.762965 | 0.199523 \(R_1\) |
| | | | | | \(f \circ R_1\) |
| \(EH1\) \((5, 4, 2)\) | 0.49 | 0.25 | 0.260266 | 0.231635 \(R_{12}\) |
| | | | | | \(f \circ R_{12}\) |
| \(EH1\) \((5, 4, 2)\) | 0.49 | 0.25 | 0.260266 | 0.231635 \(R_3\) |
| | | | | | \(f \circ R_3\) |
TABLE XV. The four classes of SPTs with minimal winding numbers \((m_0, m_1, m_1) = (6, 4, 2)\) and H1H2-caustics. “Data” as in Table XIV.

| Data                  | Plane \(\Pi_1\) \((x_2 : \uparrow, x_3 : \rightarrow)\) | Plane \(\Pi_2\) \((x_1 : \uparrow, x_3 : \rightarrow)\) | Plane \(\Pi_3\) \((x_1 : \uparrow, x_2 : \rightarrow)\) |
|-----------------------|-----------------------------------------------------|-----------------------------------------------------|-----------------------------------------------------|
| H1H2 \((6, 4, 2)\)    |                                                     |                                                     |                                                     |
| 0.45                  |                                                     |                                                     |                                                     |
| 0.13                  |                                                     |                                                     |                                                     |
| 0.967756              |                                                     |                                                     |                                                     |
| 0.133273              |                                                     |                                                     |                                                     |
| \(R \circ R_{13}\)   |                                                     |                                                     |                                                     |
| H1H2 \((6, 4, 2)\)    |                                                     |                                                     |                                                     |
| 0.45                  |                                                     |                                                     |                                                     |
| 0.13                  |                                                     |                                                     |                                                     |
| 0.967756              |                                                     |                                                     |                                                     |
| 0.133273              |                                                     |                                                     |                                                     |
| \(R \circ R_{123}\)  |                                                     |                                                     |                                                     |
| H1H2 \((6, 4, 2)\)    |                                                     |                                                     |                                                     |
| 0.45                  |                                                     |                                                     |                                                     |
| 0.13                  |                                                     |                                                     |                                                     |
| 0.967756              |                                                     |                                                     |                                                     |
| 0.133273              |                                                     |                                                     |                                                     |
| \(R \circ R_{1}\)    |                                                     |                                                     |                                                     |
| H1H2 \((6, 4, 2)\)    |                                                     |                                                     |                                                     |
| 0.45                  |                                                     |                                                     |                                                     |
| 0.13                  |                                                     |                                                     |                                                     |
| 0.967756              |                                                     |                                                     |                                                     |
| 0.133273              |                                                     |                                                     |                                                     |
| \(R \circ R_{23}\)   |                                                     |                                                     |                                                     |

SPT of class \((|R_2, R_3|)\) has two impacts on the intersection \(Q \cap \Pi_2 \cap Q_{\lambda_1}\) and two more on \(Q \cap \Pi_1 \cap Q_{\lambda_2}\). Since \(\lambda_1 < \lambda_2\), \(Q_{\lambda_1}\) and \(Q_{\lambda_2}\) are the outer and inner one-sheet hyperboloids, respectively.

Finally, in Table XVII we present half of the 6-periodic minimal SPTs with an ellipsoidal caustic. They correspond to the third row of Tables X–XI.

Some comments about these trajectories are in order:

- The 4-periodic trajectories in Table XIV are the simplest examples of nonplanar periodic trajectories. Besides, they are one of the few simply SPTs. Both trajectories have two points on the same symmetry set. However, each point is associated to a different caustic: inner or outer. That property is easier to see in the first one. Hence, they are of class \((R_2|R_2)\) and \((f \circ R_{13}|f \circ R_{13})\) in the notation used in Table XIII.

- The 5-periodic trajectories in Table XIV are the simplest examples of nonplanar periodic trajectories with odd period. They have one point on a symmetry set and another on the associated symmetry set. We recall that only SPTs with odd period have this property; see item c) of Theorem 1.

- Many projections onto the horizontal plane \(\Pi_1\) look like 2D SPTs (cf. Table V).

- Any \(R\)-SPT is travelled twice in opposite directions, since it hits orthogonally the ellipsoid at some point \(q \in Q \cap Q_{\lambda_1} \cap Q_{\lambda_2}\). Therefore, there exist SPTs of even period \(m_0 = 2l \geq 6\) with only \(l + 1\) distinct impact points on the ellipsoid, although all of them are of caustic type H1H2. We show a 6-periodic sample (the simplest one) in the first row of Table XV.

VII. THE GENERAL CASE

We have characterized STs and SPTs in terms of the vertexes of some rectangles (for the 2D case) and cuboids...
TABLE XVI. Six SPTs (3D view) for H1H1-caustics. They show different classes of SPTs corresponding to reversors $R_2$ and/or $R_3$. “Data” as in Table XIV.

| 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$) | Data | 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$) | Data |
|--------------------------------------------------|------|--------------------------------------------------|------|
| ![Image](image1.png) | ![Table Data](data1) | ![Image](image2.png) | ![Table Data](data2) |
| ![Image](image3.png) | ![Table Data](data3) | ![Image](image4.png) | ![Table Data](data4) |

TABLE XVII. Four minimal SPTs with winding numbers $(m_0, m_1, m_2) = (6, 4, 2)$ and an ellipsoidal caustic. “Data” as in Table XIV.

| 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$) | Plane $\Pi_1$ ($x_2 : \uparrow, x_3 : \rightarrow$) | Plane $\Pi_2$ ($x_1 : \uparrow, x_3 : \leftarrow$) | Plane $\Pi_3$ ($x_1 : \uparrow, x_2 : \rightarrow$) | Orbit data |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|------------|
| ![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png) | ![Table Data](data5) |
| ![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) | ![Table Data](data6) |
| ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png) | ![Table Data](data7) |
| ![Image](image17.png) | ![Image](image18.png) | ![Image](image19.png) | ![Image](image20.png) | ![Table Data](data8) |

Data: 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$)

Plane $\Pi_1$: $x_2 : \uparrow, x_3 : \rightarrow$

Plane $\Pi_2$: $x_1 : \uparrow, x_3 : \leftarrow$

Plane $\Pi_3$: $x_1 : \uparrow, x_2 : \rightarrow$

Orbit data:

| EH2 | (6, 4, 2) | 0.45 |
| --- | --- | --- |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |

Data: 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$)

Plane $\Pi_1$: $x_2 : \uparrow, x_3 : \rightarrow$

Plane $\Pi_2$: $x_1 : \uparrow, x_3 : \leftarrow$

Plane $\Pi_3$: $x_1 : \uparrow, x_2 : \rightarrow$

Orbit data:

| EH2 | (6, 4, 2) | 0.45 |
| --- | --- | --- |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |

Data: 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$)

Plane $\Pi_1$: $x_2 : \uparrow, x_3 : \rightarrow$

Plane $\Pi_2$: $x_1 : \uparrow, x_3 : \leftarrow$

Plane $\Pi_3$: $x_1 : \uparrow, x_2 : \rightarrow$

Orbit data:

| EH2 | (6, 4, 2) | 0.45 |
| --- | --- | --- |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |

Data: 3D ($x_1 : \uparrow, x_2 : \searrow, x_3 : \swarrow$)

Plane $\Pi_1$: $x_2 : \uparrow, x_3 : \rightarrow$

Plane $\Pi_2$: $x_1 : \uparrow, x_3 : \leftarrow$

Plane $\Pi_3$: $x_1 : \uparrow, x_2 : \rightarrow$

Orbit data:

| EH2 | (6, 4, 2) | 0.45 |
| --- | --- | --- |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
| EH1 | (6, 4, 2) | 0.8 |
(for the 3D case). We have proved that there are exactly 12 and 112 classes of SPTs in these low-dimensional cases. See Theorems 5 and 6.

Next, we generalize both characterizations and the final classification to nondegenerate ellipsoids of $\mathbb{R}^{n+1}$.

We say that two SPTs are of the same class when they have the same type of caustics and they connect the same couple of vertexes of their respective cuboids. All SPTs inside the same class are associated to the same reversors. For instance, looking at Table II, we see that all SPTs whose type of caustic is $E$ and that connect the two left (respectively, right) vertexes of the rectangle (7) are associated to the reversors $R_x$ and $R_y$ (respectively, $f \circ R_x$ and $f \circ R_y$).

We have the following result.

**Theorem 7** Nonsingular STs (respectively, nonsingular SPTs) inside nondegenerate ellipsoids of $\mathbb{R}^{n+1}$ are characterized as trajectories passing, in elliptic coordinates, through some vertex (respectively, two different vertexes) of their cuboid. All nonsingular SPOs are doubly SPOs, but a few with caustics of repeated types. There are $2^n(2^{n+1} - 1)$ classes of nonsingular SPTs.

This theorem is obtained by means of small refinements of the techniques used in this article, although some checks become rather cumbersome.

The number $2^n(2^{n+1} - 1)$ has a simple explanation. There are $2^n$ types of caustics — see Remark 3 —, and any $(n + 1)$-dimensional cuboid has $2^{n+1}$ vertexes, and so $2^n(2^{n+1} - 1)$ couples of vertexes.

Once fixed the ellipsoid and the caustic type, all $2^n(2^{n+1} - 1)$ couples of vertexes are realizable. That is, each couple is connected by some SPT. Of course, one should choose suitable winding numbers for each couple. That choice is guided by the following observation.

Let $\mathcal{C}_\lambda$ be a cuboid such that its billiard trajectories are periodic with winding numbers $m_1, \ldots, m_n$. Let $\hat{\mathbf{q}} = (\mu_1, \ldots, \mu_n)$ and $\hat{\mathbf{q}}' = (\mu_2, \ldots, \mu_n)$ be any couple of different vertexes of $\mathcal{C}_\lambda$ connected by an SPT. If some winding number is odd, then $\mu_i = \mu_i' \Leftrightarrow m_i \in 2\mathbb{Z}$. If all winding numbers are even, then $\mu_i = \mu_i' \Leftrightarrow m_i \in 4\mathbb{Z}$. Here, $i$ is any integer index such that $0 \leq i \leq n$.

**VIII. CONCLUSIONS AND FUTURE WORK**

We have shown that the billiard map associated to a convex symmetric hypersurface $Q \subset \mathbb{R}^{n+1}$ is reversible. Even more, it admits $2^{n+1}$ factorizations as a composition of two involutions. Therefore, its SPOs can be classified by the symmetry sets they intersect. We have carried out this classification for nondegenerate ellipsoids of $\mathbb{R}^{n+1}$. The characterization of STs and SPTs in terms of their elliptic coordinates has been the key tool.

The existence of SPOs is useful for several reasons. We indicate just two.

First, it reduces the numerical computations needed to find POs. Concretely, we can halve the dimension of the problem, by restricting the search of periodic points on the $n$-dimensional symmetry sets. (This issue will be greatly appreciated when dealing with perturbed ellipsoids, for which no frequency map exists, since the integrability is lost.) Second, an SPO of a completely integrable reversible map, persists under symmetric perturbations, when the symmetry sets are transverse to the Liouville invariant tori of the map at some point of the SPO.

These two facts are crucial to study billiards inside symmetrically perturbed ellipsoids, which is our next goal. To be more precise, we plan to generalize the results about the break-up of resonant invariant curves for billiards inside perturbed circles and perturbed ellipses.

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1. G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc. Colloq. Publ., Vol. IX (AMS, Providence, RI, 1966).
2. T. Gramchev and G. Popov, *Ann. Inst. Fourier (Grenoble)* 45, 859–895 (1995).
3. I. K. Babenko, *Math. USSR-Sb.* 71, 1–13 (1992).
4. M. Farber, *Duke Math. J.* 115, 559–585 (2002).
5. M. Farber, *Duke Math. J.* 115, 587–621 (2002).
6. M. Farber and S. Tabachnikov, *Manuscripta Math.*, 108, 431–437 (2002).
7. G. Popov and P. Topalov, *Ergodic Theory Dynam. Systems* 23, 225–248 (2003).
8. G. Popov and P. Topalov, *Ergodic Theory Dynam. Systems* 28, 1657–1684 (2008).
9. G. Popov and P. Topalov, *Comm. Math. Phys.* 303, 721–759 (2011).
10. M. Berger, *Bull. Soc. Math. France* 123, 107–116 (1995).
11. M. Gruber, *Math. Ann.* 303, 185–194 (1995).
12. A. V. Poncelet, *Traité des Propriétés Projectives des Figures* (Bachelier, Paris, 1822).
13. A. Cayley, *Philos. Mag.* 7, 339–345 (1854).
14. J. Chang and R. Friedberg, *J. Math. Phys.* 29, 1537–1550 (1988).
15. V. Dragović and M. Radnović, *J. Math. Phys.* 39, 5866–5869 (1998).
16. A. P. Veselov, *Funct. Anal. Appl.* 22, 1–13 (1988).
17. J. Moser and A. P. Veselov, *Commun. Math. Phys.* 139, 217–243 (1991).
18. J. Chang and B. Crespi and K. J. Shi, *J. Math. Phys.* 34, 2242–2256 (1993).
19. J. Chang and B. Crespi and K. J. Shi, *J. Math. Phys.* 34, 2257–2289 (1993).
20. V. Dragović and M. Radnović, *J. Math. Phys.* 39, 355–362 (1998).
21. Yu. Fedorov, *Acta Appl. Math.* 55, 251–301 (1999).
22. V. Dragović and M. Radnović, *J. Mathématiques Pures Appliquées* 85, 758–790 (2006).
23. A. Abenda and Yu. Fedorov, *Lett. Math. Phys.* 76, 111–134 (2006).
24. H. Waalkens, J. Wiersig, and H. R. Dullin, *Ann. Phys. (NY)* 276, 64–110 (1999).
25. H. Waalkens and H. R. Dullin, *Ann. Phys. (NY)* 295, 81–111 (2002).
26 V. Dragović and M. Radnović, *Regul. Chaotic Dyn.* **14**, 479–494 (2009).

27 P. S. Casas and R. Ramírez-Ros, *SIAM J. Appl. Dyn. Syst* **10**, 278–324 (2011).

28 H. Kook and J. D. Meiss, *Phys. D* **35**, 65–86 (1989).

29 Yu. Fedorov, *Regul. Chaotic Dyn.* **10**, 463–485 (2005).

30 J. S. W. Lamb and J. A. G. Roberts, *Phys. D* **112**, 1–39 (1998).

31 V. V. Kozlov and D. V. Treshchev, *Billiards, a Genetic Introduction to the Dynamics of Systems with Impacts*, Trans. Math. Monog., Vol. 89 (AMS, 1991).

32 S. Tabachnikov, *Billiards*, Panor. Synth., Vol. 1 (SMF, Paris, 1995).

33 S. Tabachnikov, *Geometry and Billiards*, Stud. Math. Libr., Vol. 30 (AMS, Providence, RI, 2005).

34 R. Ramírez-Ros, *Phys. D* **214**, 278-87 (2006).

35 S. Pinto-de-Carvalho and R. Ramírez-Ros, “Nonpersistence of resonant caustics in perturbed elliptic billiards,” e-print arXiv:1108.5582, to appear in *Ergodic Theory Dynam. Systems*. 