Choice and Attention Across Time

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Abstract

I study how past choices affect future choices in the framework of attention. Limited consideration causes a failure of “rationality”, where better options are not chosen because the DM has failed to consider them. I innovate and consider choice sequences, where past choices are necessarily considered in future choice problems. This provides a link between two kinds of rationality violations: those that occur in a cross-section of one-shot decisions and those that occur along a sequence of realized choices. In my setting, the former helps identify attention whereas the latter pins down true preferences. Both types of violations vanish over time and furnish a novel notion of rationality. A series of results shows that a DM may suffer from limited attention even though it cannot be detected when choices sequences are not considered. Moreover, attention across time is inherently related to models of attention at a given time, and a full characterization of compatible models is provided.

1 Introduction

Limited attention inspired a sizable choice theory literature that zoom in on each choice problem to study an attention mechanism, such as a search process that considers a subset of alternatives or a rule of thumb that eliminates alternatives from final decisions.

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In this paper, I zoom out and study how naturally evolving choices hint at the role of attention, where past choices possibly influence future ones as the DM becomes more aware of what is available. The two approaches offer complementing analysis, where the former studies one-shot decisions whereas the present studies how realized choices are interconnected.

When a decision maker fails to consider every alternative in every choice problem, even if the decision maker (DM) has a stable preference ranking, seemingly irrational choice behavior is observed. For example in Masatlioglu et al. (2012), a DM has a rational preference ranking over alternatives, but only considers a subset of each budget set. This leads to seemingly irrational choice patterns where in some choice sets the DM chooses a over b, but in other choice sets she fails to consider a and chooses b over a. In a different example, the DM in Manzini and Mariotti (2007) only considers shortlisted alternatives when making final decisions, and so a superior alternative that does not make the shortlist is forgone. Besides decision theoretic approaches, consideration set is also studied extensively in the marketing and finance literature.\(^1\) It provides straightforward explanations for violations of the weak axiom of revealed preferences (WARP) and presents itself as a form of bounded (consideration sets) rationality (utility maximization) (Aumann, 2005).

Yet, our ability to explain anomalies using limited consideration—a simple idea backed by compelling intuition and accumulating evidence—is not without cost. When attention cannot be directly observed and preferences not pinned down, our ability to explain burdens our ability to predict and analyze economic consequences, especially when it concerns welfare. This paper addresses this issue by exploiting the wealth of information contained in the natural evolution of choices. The DM is assumed to face choice sets one after another, and from each choice set she makes a choice. The key assumption is intuitive: realized choices should be automatically considered in the future, and so choice behavior becomes increasingly informative. Then, when a latter choice contradicts an earlier choice, the latter choice suggests true preference. It turns out that this simple assumption, backed by an axiomatic framework, allows us to confirm mistakes and fully identify true preferences.

The primitive of this paper is a dataset of choice sequences. It differs from the standard “one-shot” setting where a DM has a set of potential choice sets, faces one choice

\(^1\)See for example Wright and Barbour (1977); Hauser and Wernerfelt (1990); Roberts and Lattin (1991), and a more thorough review can be found in Masatlioglu et al. (2012).
set, and makes one choice. Choice sequences are inherently richer but nevertheless confined to observable choice. Then, when decisions from the same choice set depend on historical choice problems, they hint at a mechanism of evolving attention.

To illustrate, suppose you are unaware of Penang (an awesome island in Malaysia), a vacation destination within your budget set. Then, an analyst who observes your choice of Hawaii may falsely jump to the conclusion that you prefer Hawaii over Penang by the theory of revealed preference. However, if you were attending the Asian Meeting of the Econometric Society, you might discover and choose Penang for a drop-by vacation. This incident makes Penang then and forever an option you are aware of, and your future choices will more informatively convey your true preference between Penang and other destinations. The underlying intuition applies broadly. For instance, a DM who uses the iPad may or may not have considered a Surface Go, but a DM who converted to an iPad from a Surface Go probably prefers iPad to Surface Go. Likewise, a person who reads physical books may actually prefer e-readers, but one who left e-readers for physical books probably prefers physical books.

Building on this intuition, I propose an axiomatic framework that studies limited attention and revealed preference using the link between past and future choices. Three behavioral postulates underpin a DM with an Attention Across Time (AAT) representation: the DM has one utility function, considers a (weak) subset of alternatives to maximize utility, but always considers past choices. Specifically, the DM has a set-dependent default attention function that selects what she would normally consider from a choice set, \( \Gamma(A) \). As the DM experiences more decision problems, \( h \) (for “history”), from which \( c(h) \) is the set of previously chosen alternatives, the DM’s next decision solves

\[
\hat{c}(h)(A) = \max_{x \in \Gamma(A) \cup [c(h) \cap A]} u(x).
\]

The first two axioms introduce basic structure to the primitive of choice sequences. Axiom 2.1, Weak Stability, imposes restriction on behavior over time. In contrast to full compliance with WARP (which could be called full stability), Weak Stability allows for one-time switches between every pair of alternatives. This is accompanied by Axiom 2.2, Past Dependence, which limits the way past choices may affect future choices. When the experience of a recent choice problem changes the choice from the current

\footnote{Without further restriction, using choice sequences as primitive is richer in that choices may depend not only on the underlying choice sets but also on historical choice problems.}
choice set, the new choice is posited to be that recently chosen alternative. These axioms
do not at first appear related to limited attention, but it turns out that their, stronger,
standard counterparts will necessarily fail when full attention is ruled out. In words,
a non-trivial model of limited attention necessitates both WARP violations in realized
choices and future choices that are changing due to the experience of different histories
(Proposition 1).

The third and last axiom delivers the behavioral signature of attention. First, I de-
define revealed preference in this setting by means of two behavioral patterns. The first
indication of revealed preference is when $y$ is chosen over $x$ even though $x$ was al-
ready chosen in the past, which suggests the DM is aware of $x$ when $y$ was chosen.
The second indication is when $y$ is chosen over $x$ in the same choice set, even though
$x$ would have been chosen if there were no history, which also suggests that the DM
is aware of $x$ when $y$ was chosen. In both cases, I consider $y$ revealed preferred to $x$.

Axiom 2.3, Default Attention, posits that if $y$ is revealed preferred to $x$ then $x$ will never
be chosen from a choice set from which $y$ is chosen when there is no history—as in $y$
receives attention no matter what, and therefore a subjectively inferior alternative will
never be chosen.

I then turn to a series of analysis that investigates behavior under limited attention.
The first set of observations concerns counterfactual WARP violations. Unlike WARP vi-
olations within a sequence of (realized) choices, counterfactual violations are violations
that the DM prepares to make. This is the type of violations observed in a between-
subject design, where a population of subjects makes inconsistent decisions among dif-
ferent choice problems that they could be assigned to.$^3$ A series of examples asserts that
the lack of counterfactual WARP violations does not rule out limited attention. This is
because a DM who does not initially violate WARP may do so after some history. Also,
even if no counterfactual WARP violation is ever present, her choices could change from
one time to another due to limited attention. A sufficient and necessary condition for
counterfactual WARP violation is provided, which captures the intuition that they result
from a large enough discrepancy between initial behavior (choice without history) and
true preference (Proposition 2).

In AAT, preferences between alternatives are pinned down (Proposition 3). More-
over, by asking the right questions (choice sets) with the right timing (position in a

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$^3$Formally, a counterfactual WARP violation is a WARP violation in a cross-sectional choice function
$	ilde{c}(h): \mathcal{A} \rightarrow \mathbb{R}$.
sequence), preferences are fully revealed within just one sequence of realized choices (Proposition 4). This is because past experiences can help certain alternatives garner the DM’s consideration, even if they would normally be ignored. For instance, imagine a DM who does not by default consider $z$ when it appears in a choice set $S$, but she does consider $z$ when it appears in a different choice set $T$. Then, having experienced $T$ and choosing $z$, the model posits that she would consider $z$ when she faces $S$ next, and therefore her impending choice reveals her preference about $z$.\(^4\) Necessary revelation of preferences is both a testable prediction and a tool for preference elicitation. It can be shown that under AAT, WARP violations must be corrected in favor of the DM’s true preference when the “problematic” choice sets are presented in alternating order. Specifically, if a WARP violation occurs between the choice of $x$ from $\{x, y, z\}$ and the choice of $y$ from $\{x, y\}$, then the sequence of choice sets ($\{x, y\}, \{x, y, z\}$) would produce choice sequence $(y, x)$ if $u(x) > u(y)$, a revealed correction in favor of $x$. On the other hand if $u(y) > u(x)$, we would observe $(x, y)$ from ($\{x, y, z\}, \{x, y\}$), a revealed correction in favor of $y$. It turns out that convergence of this kind is necessary in AAT, ruling out sticky choice and past independence (Proposition 6 and Corollary 1).\(^5\)

Although preferences are pinned down, default attention $\Gamma$ is not. The DM can always pay or not pay attention to something she would never choose without changing her choice behavior.\(^6\) This observation helps pin down the exact set of admissible attention functions, which is characterized using a maximal set (Proposition 5).

The rest of the paper is organized as follows. Section 2 formalizes the primitive and provides the axiomatic foundation. This is followed by the main model representation theorem in Section 3, in which my model Attention Across Time (AAT) is formally introduced. In Section 4, I analyze how AAT is complementary to various one-shot decision making models, proposing a link between attention across time and attention at a given time.

\(^4\)In this example, $S$ could be a superset of $T$, or it can be a subset of $T$, or the two sets can be non-nested and at least intersect at $z$, the model does not place restrictions on $\Gamma(A)$.

\(^5\)Let $AxBy$ denote a choice sequence where $x$ is chosen from $A$ and then $y$ is chosen from $B$. Sticky choice would be $\{x, y, z\} x \{x, y\} x$ and $\{x, y\} y \{x, y, z\} y$—the first choice remains chosen when the second choice problem is presented. Past independence would be $\{x, y, z\} x \{x, y\} y$ and $\{x, y\} y \{x, y, z\} x$—choices that do not depend on the sequence. Convergence is the only other possibility, where $\{x, y, z\} x \{x, y\} a$ and $\{x, y\} y \{x, y, z\} a$ for some $a = x, y$.

\(^6\)The most straightforward example is the choice behavior given by a completely standard utility maximizer, who can either be paying attention to everything, or paying attention only to what is chosen, or any combination of the two, and none of these very different specifications would cause an inch of change in the DM’s behavior.
Related Literature

The challenge of trying to pin down or infer preferences within a framework of attention is not new. Caplin and Dean (2011) and Caplin et al. (2011) study the search process that took place as a DM works toward a final decision in a single choice problem. In an experiment, the DM must make tentative choices in response to an abrupt termination that would automatically confirm the currently chosen alternative. A dataset of tentative choices are informative about what the DM has considered, and preference is revealed whenever a tentative choice is switched to another. Masatlioglu and Nakajima (2013) also suggests a search process but does not assume that it can be directly observed, and therefore can be viewed an endogenization of search. A DM begins from a starting point and searches related alternatives, which leads to more related alternative that become searched, until the process ends. Kovach and Suleymanov (2021) pins down preference by focusing on the richness in a stochastic choice setting, where each reference point leads to a (potentially different) distribution of consideration set. Gossner et al. (2021) studies how behavior may react to an exogenous manipulation of attention, thereby using an intervention-based strategy to study attention.

This paper's strategy is to combine past and future choices of a same individual into one analysis. Although the underlying intuition is relatable to search in Caplin and Dean (2011); Caplin et al. (2011); Masatlioglu and Nakajima (2013), AAT does not observe or infer search behavior. Instead, ATT takes each choice problem to be completely standard (i.e., a choice set and a choice) and extracts information about attention through the evolution of actual choices. The underlying mechanisms are so different that AAT and Caplin and Dean (2011)’s alternative-based search and reservation-based search intersect only when no switches would occur.7

More broadly, whereas AAT studies attention across time, many models study attention at a given time. This paper refers to the latter as one-shot (decision making) models. For example in Masatlioglu et al. (2012)’s choice with limited attention (CLA), the removal of an not-considered alternatives does not alter consideration sets. Also in Lleras et al. (2017)’s overwhelming choice, a considered al-

7In Caplin and Dean (2011)’s models, the DM faces just one choice set, but her decision evolves as she searches through the choice set. A “switch” in this setting occurs when an option was temporarily chosen during this choice process but is not the final choice, as it was replaced by an option that is revealed better. In my model, ATT, a switch never occurs when the underlying choice set does not change, even if the same choice set is faced again and again. Instead, it is precisely because the underlying choice set has changed that other options have gained the DM’s attention.
ternative is always considered when the choice set shrinks. More fundamentally, heuristics that are seemingly unrelated to attention may give rise to consideration sets. This includes Manzini and Mariotti (2007)’s rational shortlist method (RSM), Masatlioglu et al. (2012)’s categorize-then-choose, and Cherepanov et al. (2013)’s rationalization, where potentially superior alternatives that do not survive an elimination process are excluded from final decisions.

The primitive that AAT considers is not just different, but strictly richer than those of one-shot models. AAT also does not place restriction on choices at a given time, which leaves room for a complementary relationship between AAT and one-shot models. Section 4 investigates the extent of this relationship. Proposition 8 demonstrates that choice behavior that satisfies CLA will result in future behavior that always satisfies CLA, so long that the evolution of choice are governed by AAT. Proposition 9 does the same for RSM. The resulting models are intuitive. Take RSM for instance, in which a DM first forms a shortlist and then makes a final decision. When attention at a given time is governed by RSM but attention across time is governed by AAT, the DM still uses a shortlist at every instance of decision making, but the shortlist is updated over time in the form of including past choices. This form of synergy led to a definition of full compatibility and a complete characterization of fully compatible one-shot models is provided (Theorem 3).

2 Primitive

2.1 Preliminaries

Let $X$ be a countable set of alternatives, and let $\mathcal{A}$ be the set of all finite subsets of $X$ with at least two elements. Let $\mathcal{A}^N$ be the set of all infinite sequences of choice sets and let $X^N$ be the set of all infinite sequences of alternatives. My primitive is a choice function that maps each infinite sequence of choice sets to an infinite sequence of choices, $c : \mathcal{A}^N \rightarrow X^N$, where for every sequence of choice sets $(A_n) \in \mathcal{A}^N$ and any natural number $k$, the corresponding alternative is an element of the corresponding choice set: $c ((A_n))_k \in (A_n)_k$. The first choice (when $k = 1$) can be treated as the first decision since the DM enters the analyst’s observation. When there is no confusion, I

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8Even though I work with infinitely long choice sequences, the main representation theorem only requires choice sequences to have at least length 4.
write $A_k$ for $(A_n)_k$.

Throughout, I limit the scope of this paper to choice situations in which the DM is not building a bundle over time. This restricts our attention to choice behavior where, for any two sequences of choice sets that are identical up to a certain point, the corresponding choices are too identical up to that point. Formally, for all $(A_n), (B_n) \in \mathcal{A}^N$, if $A_k = B_k$ for all $k \leq K$, then $c((A_n))_k = c((B_n))_k$ for all $k \leq K$. This property, henceforth future independence, rules out choices that are made in anticipation of future choice sets, and in so doing allows us to focus on how past choices affect future choices (instead of the opposite).

With future independence, each $c$ can be decomposed into a set of history-dependent choice functions that map choice sets to choices: Let $\mathcal{A}^{<N}$ be the set of all finite sequences of choice sets, including the empty sequence denoted by $\emptyset$. For each history $h \in \mathcal{A}^{<N}$, denote by $\tilde{c}(h) : \mathcal{A} \to X$ the cross-sectional choice function or one-shot choice function that gives the choice for each upcoming choice set (right after $h$), where $\tilde{c}(\emptyset) : \mathcal{A} \to X$ denotes the cross-sectional choice function without history.\footnote{The name “cross-sectional” comes from viewing $\tilde{c}(h)$ as a slice of $c$. Moreover, I call it “one-shot” because a DM who faces $A$ after history $h$ can never go back and face $B$ with history $h$, hence “one-shot”, even though this data can be extracted from a population of DMs.} Due to future independence, cross-sectional choice functions $\{\tilde{c}(h) : h \in \mathcal{A}^{<N}\}$ are fully and uniquely pinned down by $c$. Also, denote by $c(h)$ the accumulated set of chosen alternatives from history $h$; this too is pinned down by $c$.\footnote{When $h = (A_1, \ldots, A_K)$, $c(h) := \{\tilde{c}(\emptyset)(A_1)\} \cup \{\tilde{c}((A_1))(A_2)\} \cup \{\tilde{c}((A_1, \ldots, A_{i-1}))(A_i) : i = 3, \ldots, K\}$.} If a DM maximizes standard utility throughout, then $\tilde{c}(h) = \tilde{c}(h')$ for all $h, h' \in \mathcal{A}^{<N}$ (i.e., choices do not depend on history) and every collection of past choices is independent of the order of past choice sets.\footnote{Standard utility maximization means there exists $u : X \to \mathbb{R}$ such that $c((A_n))_k = \arg \max_{x \in A_k} u(x)$ for all $(A_n)$ and $k$.} In general, both are not true, even with future independence.

### 2.2 Weak Stability and Past Dependence

I begin with two axioms that provide basic structure to the primitive.

**Axiom 2.1** (Weak Stability). For any $(A_n) \in \mathcal{A}^N$ and $h < i < j$, if $c((A_n))_h = x$, $c((A_n))_i = y$, $x \in A_i$, and $y \in A_j$, then $c((A_n))_j \neq x$.

Section 2 imposes a version of the infamous weak axiom of revealed preferences (WARP) with two key differences. First, it is imposed within each sequence of choice
problems \((A_n) \in \mathcal{A}^N\). In particular, there is no restriction on how choices differ across sequences. Second, it does so without demanding full compliance with WARP, but limits the instances of WARP violations. A conforming DM may switch between \(x\) and \(y\), but she does not go back and fourth between them.

To illustrate, suppose a DM first chose \(x\) in the presence of \(y\), and then chooses \(y\) in the presence of \(x\). The latter choice violates WARP, and it may be due to the emerging consideration of \(y\). Axiom 2.1 does not exclude this behavior, but posits that, from here on, the choice between \(x\) and \(y\) has finalized. In other words, the DM may “flip”, but must not “flip-flop”.

Notice that a DM who never switches automatically satisfies this axiom—Axiom 2.1 without “\(c ((A_n))_n = x\)” equates WARP (within sequence).\(^{12}\) However, that assumption may deprive us the opportunity to separate utility from attention, since WARP is equally plausible to be the consequence of either standard utility maximization or conveniently paying attention to the right things at the right time. This is formalized by Proposition 1, which states that a DM who never switches always has a standard utility representation (assuming other axioms hold). On the other extreme, dropping Axiom 2.1 altogether allows for the DM’s behavior to go wild and identification of any form of utility maximization becomes impossible.

The next axiom allows past choices to affect future choices with limited scope.

**Axiom 2.2** (Past Dependence). *For any \((A_n) \in \mathcal{A}^N, B \in \mathcal{A}, \text{ and } K \in \mathbb{N},*

\[
\tilde{c} ((A_1, ..., A_K)) (B) \in \tilde{c} ((A_1, ..., A_{K-1})) (B) \cup \tilde{c} ((A_1, ..., A_{K-1})) (A_K).
\]

One way to understand this axiom is to contrast it with *past independence*, or \(\tilde{c} ((A_1, ..., A_K)) (B) = \tilde{c} ((A_1, ..., A_{K-1})) (B)\), which satisfies Axiom 2.2 trivially. In that case, the DM’s choice after history \((A_1, ..., A_K)\) does not depend on whether or not she had faced \(A_K\). Axiom 2.2 weakens past independence by allowing for one type of departure: the next choice is exactly the choice it succeeded. Then, after facing choice sets \((A_1, ..., A_K)\), what a DM chooses from \(B\) is either (i) what she would have chosen had she not faced \(A_K\) or (ii) exactly what she chose from \(A_K\).

The axiom is therefore a disciplined weakening of past independence. First, even though past choices may affect future choices, it must do so in a tractable manner, since

\(^{12}\)WARP stands for the Weak Axioms of Revealed Preferences. There are many (roughly) equivalent definitions. Here, since I consider choice functions (as opposed to correspondence), I use “if \(x\) is chosen in \(y\)’s presence, then \(y\) is never chosen in \(x\)’s presence”.

period $k - 2$’s decision can only affect period $k$’s decision via period $k - 1$’s decision. Moreover, said effect is limited to helping the chosen alternative gets chosen again, and all other forms of past dependence is prohibited.

### 2.3 Default Attention

Finally I introduce the key postulate that embodies the behavioral signature of (limited) attention. Let $c_0 (B)$ denote the choice from $B$ without history (that is, $c_0 (B) := \bar{c} (\emptyset) (B)$). Consider the following axiom.

**Definition 1.** Given $c$, define $x P y$ if at least one of the following is true for some $(A_n) \in \mathcal{A}^N$:

1. $c ((A_n))_j = x$, $y \in A_j$ and $c ((A_n))_i = y$ such that $i < j$.
2. $c ((A_n))_j = x$ such that $c_0 (A_j) = y$.

**Axiom 2.3 (Default Attention).** If $c_0 (A) P y$, then $y$ is never chosen from $A$.

Axiom 2.3 puts restrictions on how a DM can depart from her default choice. First, the relation $x P y$ captures the analyst’s inference that $x$ is better than $y$. This relationship is identified either when $x$ is chosen over $y$ when $y$ was chosen in the past (part 1 of Definition 1) or when $x$ is chosen from a choice set that $y$ is chosen by default (Condition 2 of Definition 1). Then, Axiom 2.3 requires that $y$ is never chosen from $A$ (no matter the sequence in which $A$ appears) if the default choice from $A$ is identified to be better than $y$, or $c_0 (A) P y$.

Taken together, Axiom 2.3 captures the idea that certain alternatives always receive attention when they appear in certain choice sets, regardless of what happened in the past. For example, these may be the most salient alternatives insofar as to attract attention: pizza is always in the consideration set for football night, even though the additional consideration of satay (Malaysian skewers) depends on whether the DM has learned of this dish.

Even though the axiom posits on the consistency of attention structure across periods, it stops short of limiting its form. A different interpretation of default attention is the contemporary attention structure when a DM first enters the analyst’s observation. Within the model, period 1 decisions are considered history-less, but they could be the consequence of past choices that occurred outside of our observation. In this
case, default attention simply captures the resulting attention structure formed in the (unobserved) past.

3 Model

3.1 Attention Across Time

We are ready for the main representation theorem!

Definition 2. \( c \) admits an Attention Across Time (AAT) representation if there exist

1. a utility function \( u : X \to \mathbb{R} \)

2. a default attention function \( \Gamma : \mathcal{A} \to 2^X \setminus \{\emptyset\}, \Gamma (A) \subseteq A \)

such that

\[
\tilde{c} (h) (A) = \arg \max_{x \in \tilde{\Gamma} (h) (A)} u (x)
\]

where \( \tilde{\Gamma} (h) (A) = \Gamma (A) \cup (c (h) \cap A) \).

Theorem 1. \( c \) satisfies Axioms 2.1, 2.2, and 2.3 if and only if it admits an Attention Across Time (AAT) representation.

AAT prescribes the following choice procedure: When DM faces choice set \( B \), she not only considers alternatives that she would always consider when she faces \( B \), but also the alternatives that she had chosen in the past. The former is history-independent and may capture what is salient (to her) in the underlying choice problems. The latter is history-dependent and receive her attention due to her past experiences. The intuition is straightforward—a DM may be unaware of certain alternatives that she had never chosen before, but she must be aware of the alternatives that she had chosen, which certainly depend on the choice sets she faced in the past.

Under AAT, an analyst knows definitively that DM prefers \( a \) to \( b \) if she chose \( b \) in the past and chooses \( a \) over \( b \) at the present, since both \( a \) and \( b \) are within the attention set of the latter choice problem. In fact, one way to elicit such preference is to first introduce a choice set under which the DM would choose \( b \), and then ask the DM to choose from a choice set from which she would normally choose \( a \).

A DM in this model may commit past-dependent behavior, where different past experiences result in different choices from the same future choice set, even if this would
result in WARP violations between past and future choices. It turns out that if behavior cannot be explained by standard utility maximization where the DM considers everything, then it must involve both WARP violations and past-dependent behavior; that is, these departures are unavoidable if full attention can be ruled out.

To formalize this observation, I introduce stronger versions of Axiom 2.2 and Axiom 2.3, and show that they each necessitates the special cases of AAT where choice behavior satisfies standard utility representation (i.e., there exists a utility function $u : X \to \mathbb{R}$ such that $\tilde{c}(h)(A) = \arg \max_{x \in A} u(x)$ for all $h,A$). Analogous to Weak Stability, consider the stronger postulate Full Stability where for any $(A_n) \in \mathcal{A}^N$, if $c((A_n))_i = x$, $\{x, y\} \subseteq (A_n)_i \cap (A_n)_j$, and $x \neq y$, then $c((A_n))_j \neq y$ (which is just WARP within sequence). Similarly, analogous to Past Dependence, consider the stronger postulate Past Independence where for any $(A_1, \ldots, A_K) \in \mathcal{A}^{<N}$ and $B \in \mathcal{A}$, $\tilde{c}((A_1, \ldots, A_K))(B) = \tilde{c}((A_1, \ldots, A_{K-1}))(B)$.

**Proposition 1.** Suppose $c$ admits an AAT representation. The following are equivalent:

1. $c$ satisfies Full Stability
2. $c$ satisfies Past Independence
3. $c$ admits a standard utility representation

I remark that Full Stability and Past Independence are, in general, non-nested conditions (i.e., choice behavior can satisfy one but not the other), and one may interpret them as different aspects of a “rational” behavior. However, Proposition 1 suggests that if non-standard behavior arises from limited attention, such as those captured by AAT, then these different aspects of irrationality may be inextricably linked in choice behavior.

### 3.2 Counterfactual WARP violations

Recall that for every history $h$, $\tilde{c}(h) : \mathcal{A} \to X$ is uniquely given by a future-independent $c$. I call $\tilde{c}$ a cross-sectional choice function, since it captures, at a given point in time, what the DM would choose for each possible upcoming choice set $A \in \mathcal{A}$. When $\tilde{c}(h)$ violates WARP (i.e., $\tilde{c}(h)(A) = x$, $\tilde{c}(h)(B) = y$ and $\{x, y\} \subseteq A \cap B$ for some $x, y, A, B$), I call it a *counterfactual* WARP violation, since these violations did not occur among realized choices.
This subsection inquires when, how, and why counterfactual WARP violations occur when a DM is AAT. I start off with the assertion full attention rules out counterfactual WARP violations (Example 1). Then, two examples send the clear message that the lack of counterfactual WARP violations may not rule out limited attention. When future choices are not considered, a DM who satisfies WARP at the beginning (c₀) may violate WARP in the future (Example 2). But even when future choices are also observed and studied, and the DM never commits counterfactual WARP violations, she could nevertheless be non-trivially influenced by limited attention (Example 3).

Example 1. Consider the AAT representation where Γ(A) = A for all A. Then, Γ(h)(A) = A for all h, which means the DM considers everything all the time, and therefore by utility maximization she does not commit counterfactual WARP violations.

Example 2. Suppose X = {x, y, z, z′} and c admits an AAT representation where Γ(A) = {z} if z ∈ A, Γ(A) = {x} if z ∉ A and x ∈ A, Γ({y, z′}) = {y}, and u(x) > u(y) > u(z) > u(z′). Notice that c₀ does not violate WARP, since it satisfies the maximization of preference ranking z ≻₀ y ≻₀ z′. Now consider the history h = ( {y, z′} ), then c(h)( {x, y, z, z′} ) = y but c(h)( {x, y} ) = x, a counterfactual WARP violation. In this example, even though the DM’s behavior at the beginning satisfies WARP, it is inconsistent with her true utility (>₀ is not represented by u), therefore WARP violation occurs at some history.

Example 3. Suppose X = {x, y, z} and c admits an AAT representation where Γ(A) = {x} if x ∈ A, Γ( {y, z} ) = {y}, and u(z) > u(y) > u(x). Notice that c₀ can be explained by the maximization of preference ranking x ≻₀ y ≻₀ z, thereby complies with WARP. Moreover, this DM will never commit counterfactual WARP violations, since a WARP violation between two alternatives requires the better alternative to receive attention only at certain choice sets, and this is not possible since default attention is only given to the worst alternative of each choice set. However, this DM is clearly affected by limited attention, since her choice from A = {x, y} without history is c₀(A) = x but after history h = ( {y, z} ) it changes to c(h)(A) = y.

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13Formally, for any h, let the set of previous chosen alternatives be Y = c(h) ⊆ X. A WARP violation at c(h) requires a, b ∈ X and A, B ∈ S such that {a, b} ⊆ A ∩ B, u(a) > u(b), a ∈ Γ(h)(A) and a ∉ Γ(h)(B). But this is not possible. If a ∈ Γ(A), then it is not possible that b ∈ A, since only the worst outcome is considered by default. If a ∉ Γ(A), then then so a ∈ c(h), which means a ∈ Γ(h)(B), a contradiction. This line of argument applies to any countable X where ≻₀ is represented by −u.
The next result outlines exactly when counterfactual WARP violations occur. Given $c$, let $\hat{X}$ denote the set of alternatives that are ever-chosen, i.e., $\hat{X} := \{x \in X : c((A_n))_i = x \text{ for some } (A_n) \text{ and } i\}$ (alternatives that are chosen from some choice set in some sequence).\textsuperscript{14}

**Proposition 2.** Suppose $c$ admits an AAT representation $(u, \Gamma)$ and suppose $c_0$ is explained by some complete and transitive $\succ_0$.\textsuperscript{15} There exists history $h$ such that $\tilde{c}(h)$ violates WARP if and only if there are $x, y, z \in \hat{X}$ such that $z \succ_0 x \succ_0 y$ but $u(x) > u(y) > u(z)$.

In AAT, counterfactual WARP violation occurs (assuming $c_0$ satisfies WARP, otherwise it is trivial) only if initial behavior reflects a preference that is sufficiently different to true preference. In Proposition 2, this is captured by alternative $z$’s relative ranking to alternatives $x$ and $y$. It can be shown that if the differences were limited to “one-steps”, such as $x \succ_0 z \succ_0 y$ with $u(x) > u(y) > u(z)$, then a counterfactual WARP violation will not occur.

The reverse is true to some extent. If initial behavior and true preference are sufficiently different in the aforementioned manner, then a counterfactual WARP violation will occur. However, Example 3 illustrated a case where initial behavior are the complete opposite of true preference, and yet a counterfactual WARP violation never occurs.

### 3.3 Uniqueness and Convergence

Next, a series of results asserts that standard economics problems that concern welfare and incentives, which rely on the identification of preferences, are possible to study even when the analyst cannot directly observe attention.

**Proposition 3.** If $c$ admits AAT representations $(u_1, \Gamma_1), (u_2, \Gamma_2)$ and $x, y \in \hat{X}$, then $u_1(x) > u_1(y)$ if and only if $u_2(x) > u_2(y)$.

In AAT, preferences are unique for ever-chosen alternatives. This is because the richness in a dataset of choice sequences resolves not some but all instances of preference misidentification even in an framework of (limited) attention. The intuition is

\textsuperscript{14}It can be shown that there is at most one never-chosen alternative, since the set of all binary choice sets leaves at most one alternative never chosen. Moreover, if an alternative is not chosen in period 1 from any choice set, then is never chosen ever (Lemma 3). This would arise when every choice set in which $x$ receives default attention contains a better alternative that also receives default attention.

\textsuperscript{15}Meaning $c_0(A) \succ_0 z$ for all $z \in A \setminus \{c_0(A)\}$.
straightforward: As long as two alternatives are sometimes chosen, a subsequent decision between the two will pin down preference. On the contrary, if we were to analyze past and future choices in isolation of one another, we often face an impossible to resolve dilemma trying to conclude whether choices are due to preferences or due to the lack of attention.

When the grand set of alternatives is finite, it takes just one sequence of choices sets to fully reveal preferences for ever-chosen alternatives. Proposition 4 captures the intuition that if we ask the right questions in the right order, we can in theory force one DM to pay attention to the universe of available alternatives and, as a consequence, allows us to pin down preferences using realized choices.

To formalize this statement, recall the argument that if $x$ is chosen over $y$ after $y$ was previously chosen, then this switch from $y$ to $x$ allows us to conclude that $x$ is better than $y$, since $y$ must have received attention in the latter choice problem. This notion of revealed preference is given in Definition 3 and will be used in the rest of the paper (“$S$” for “switch”).

**Definition 3.** Given $c$,

1. Let $x S_{(A_n)} y$ if $c((A_n))_i = y$ and $c((A_n))_j = x$ with $y \in A_j$ for some $i < j$.

2. Let $x S y$ if $x S_{(A_n)} y$ for some $(A_n) \in \mathcal{A}^N$.

**Proposition 4.** Suppose $X$ is finite.

1. For any $c$ that admits an AAT representation, there exists a sequence of choice sets $(A_n) \in \mathcal{A}^N$ such that $S_{(A_n)}$ is complete and transitive over $\hat{X}$.

2. There exists a sequence of choice sets $(A_n) \in \mathcal{A}^N$ such that for any $c$ that admits an AAT representation, there exists a subset of alternatives $\bar{X} \subseteq X$ such that $S_{(A_n)}$ is complete and transitive over $\bar{X}$ and $|X \setminus \bar{X}| \leq 1$.16

It remains to address whether default attention $\Gamma$ is also unique, and the answer is negative. Consider the extreme where choice behavior fully conforms with standard

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16 It is not possible to construct a $c$-independent sequence that fully identifies preferences. Counterexample: Let $X = \{x, y, z\}$. Consider $(u_1, \Gamma_1)$ where $u_1(x) > u_1(y)$, $\{A : y \in \Gamma_1(A)\} = \{\{x, y, z\}\}$, $\Gamma_1(\{x, y, z\}) = \{y\}$, $\Gamma_1(\{x, y\}) = \{x\}$, then $x, y \in \hat{X}$ and any sequence that reveals $x S_{(A_n)} y$ must present $\{x, y, z\}$ before $\{x, y\}$. Now consider $(u_2, \Gamma_2)$ where $u_2(x) > u_2(y)$, $\{A : y \in \Gamma_2(A)\} = \{\{x, y\}\}$, $\Gamma_2(\{x, y\}) = \{y\}$, $\Gamma_2(\{x, y, z\}) = \{x\}$, then $x, y \in \hat{X}$ and any sequence that reveals $x S_{(A_n)} y$ must present $\{x, y\}$ before $\{x, y, z\}$. 
utility maximization, then it makes no difference whether or not DM has considered not-chosen alternatives, since they are necessarily inferior. The same intuition applies even when behavior violates standard utility maximization sometimes, since we can alter $\Gamma$ by adding or removing inferior alternatives without changing the model’s predictions. This allows us to fully capture the extent of the generality of $\Gamma$.

**Proposition 5.** Let $\Gamma^+(A) := \{x \in A: c_0(A) \S x$ or $x = c_0(A) \text{ or } x \notin \hat{X}\}$. Suppose $c$ admits an AAT representation. Then, $c$ admits an AAT representation with default attention $\Gamma$ if and only if $c_0 \in \Gamma \subseteq \Gamma^+$.

In practice, this means an analyst who cannot directly observe a DM’s default attention has the freedom to a myriad of model specifications that would not alter choice predictions. This includes overestimating the size of default attention by combining possible candidates of $\Gamma$ or by taking a conservative approach using intersections. This is formalized in Lemma 1.

**Lemma 1.** Suppose $c$ admits AAT representations $(\Gamma, u)$ and $(\Gamma', u)$, then it also admits AAT representations $(\Gamma \cup \Gamma', u)$ and $(\Gamma \cap \Gamma', u)$.

The fact that preferences can be pinned down is linked to a testable prediction that choices must uniformly (independent of history) converge to a steady state. Recall that a DM may switch from choosing $y$ to choosing $x$ (over $y$) if she actually prefers $x$, which could appear as a WARP violation across time. Proposition 6 shows that these switches (and therefore, WARP violations) are not only unidirectional within sequences (Axiom 2.1) but also unidirectional across sequences. Then, observing any switches from any choice sequences allows us to definitively learn the DM’s preference.

**Proposition 6.** Suppose $c$ admits an AAT representation. If $x \S y$, then $u(x) > u(y)$ and not $y \S x$.

In practice, this observation allows us to build a test that compels the correction of “mistakes”, which immediately reveals preference. Specifically, suppose WARP is violated between two choice sets ($A$ and $B$). Presenting these “problematic” choice sets in alternating order (i.e., $AB$ or $BA$) will force the DM to reveal her true preference in her second choice. This is because both alternatives are considered at the second stage—if a DM first faces choice set $A$ and then faces choice set $B$, the choice from $A$ receives attention because it was chosen in the past and the choice from $B$ receives attention by default—so the choice would pin down true preference between the two alternatives.
Corollary 1. Suppose $c$ admits AAT representation. If $c_0(A) = x$, $c_0(B) = y$, $\{x, y\} \subseteq A \cap B$, and $x \neq y$ (a standard WARP violation), then $\tilde{c}((A))(B) = \tilde{c}((B))(A) = z$ for some $z \in \{x, y\}$. Moreover, $u(z) > u(z')$ where $z' \in \{x, y\} \setminus \{z\}$.

This observation rules out the remaining two possible choice patterns, namely sticky choice ($\tilde{c}((A))(B) = x$ and $\tilde{c}((B))(A) = y$) and past independence ($\tilde{c}((A))(B) = y$ and $\tilde{c}((B))(A) = x$), hence the necessity of convergence, which reveals preference.

4 Compatible models

AAT can be complemented by attention-based choice models in which past choices are not explicitly modeled. Although AAT introduces the propagation of attention from one period to the next, it says nothing about what, why, or how alternatives receive attention in the first place. On the other hand, AAT may complement attention-based choice models by exploiting the richness in choice sequence to verify inattentiveness and identify preferences. This section inquires the relationship between standard attention theories (attention at a given time) and AAT (attention across time).

Starting with an illustrative example, Subsection 4.1 presents a “vertical” merger between AAT and Masatlioglu et al. (2012)’s choice with limited attention (CLA)—an attention-based choice models that does not consider choice sequences. Subsection 4.2 does the same for Manzini and Mariotti (2007)’s rational shortlist methods (RSM). Then, Subsection 4.3 concludes with a complete characterization of compatible models.

Like before (Section 2), $X$ is assumed countable and $\mathcal{A}$ is the set of all finite subsets of $X$ with at least size 2.

4.1 Attention Filter in Sequences

This subsection demonstrates how AAT and Masatlioglu et al. (2012)’s Choice with Limited Attention (CLA) complement one another. Note that the former introduces a theory of attention across time and latter introduces a theory of attention across choice sets.

Recall that $\tilde{c}(\emptyset) : \mathcal{A} \to X$, also written as $c_0$ for convenience, describes choices without history. It is straightforward to see that AAT (Section 3) puts no restrictions on $c_0$.\footnote{That is, if $f : \mathcal{A} \to X$ is a choice function, there exists $c$ such that $c$ admits an AAT representation where $c_0(A) = f(A)$ for all $A \in \mathcal{A}$.} On the other hand, CLA does not consider choice sequences but puts restrictions
on \( c_0 \). I now reconcile both methods of identifying attention, demonstrate that they bridge the two setups, and argue that this finding is intuitive.

**Definition 4** (Masatlioglu et al. (2012)). A mapping \( \Gamma : \mathcal{A} \rightarrow \mathcal{A} \) is an attention filter if \( \Gamma(A) \subseteq A \) and \( y \notin \tilde{\Gamma}(A) \) implies \( \tilde{\Gamma}(A \setminus \{y\}) = \tilde{\Gamma}(A) \).

**Definition 5** (Masatlioglu et al. (2012)). A choice function \( \hat{c} : \mathcal{A} \rightarrow X \) is a choice with limited attention (CLA) if there exist \( \tilde{u} : X \rightarrow \mathbb{R} \) and an attention filter \( \tilde{\Gamma} : \mathcal{A} \rightarrow \mathcal{A} \) such that

\[
\hat{c}(A) = \arg \max_{x \in \tilde{\Gamma}(x)} \tilde{u}(x).
\]

In CLA, alternative \( x \) is inferred to be preferred to \( y \) if, in some choice set where \( x \) is chosen in the presence of \( y \), dropping \( y \) results in \( x \) no longer chosen. This refines the standard definition that simply requires \( x \) to be chosen in the presence of \( y \), since we do not know whether or not \( y \) is considered. The underlying intuition is that, if \( y \) were not initially considered, then dropping \( y \) should not result in a change in the consideration set, and hence \( x \) should remain chosen. If, instead, \( x \) is no longer chosen, it must be that \( y \) was considered, and hence \( x \) is inferred to be revealed preferred to \( y \). This line of argument builds on testable counterfactual choices, since each choice set is encountered without the experience of other choice sets.

The next axiom exploits a dataset of choice sequences to substantiate this very same argument using realized choices.

**Axiom 4.1.** If \( c_0(T) = x \) and \( c_0(T \setminus \{y\}) \neq x \), then there is no \( (A_n) \in \mathcal{A}^N \) such that \( y \notin S_{(A_n)}x \).

**Theorem 2.** \( c \) satisfies Axioms 2.1, 2.2, 2.3 and 4.1 if and only if it admits an Attention Across Time (AAT) representation \((u, \Gamma)\) where \( \Gamma \) is an attention filter.

Although attention is the foundation for both AAT and CLA, they infer attention and preference in seemingly different ways. In particular, CLA operates on the attention structure across choice sets whereas AAT operates across time. Theorem 2 asserts that a reconciliation is possible, in the sense that every choice function that is a CLA can be supported in AAT, and choice behavior in AAT where \( \Gamma \) is an attention filter is a CLA.

This reconciliation allows us to study how attention at a given time is related to attention across time. Revealed preference in CLA relies on the assumption that dropping alternatives that are not considered would not result in a change in the attention set.
Independently, revealed preference in AAT relies on the assumption that past choices are considered in the future. Axiom 4.1 bridges the two postulates so that both attention and preference can be identified in either way, making them complementary and inexplicably linked.

To see this, notice that if \( x \) is revealed preferred to \( y \) in the sense of CLA, then we will observe, in realized choices, evidence of \( x \) chosen over \( y \) after \( y \) was chosen in the past, i.e., \( x \preceq_{SA} y \).

**Proposition 7.** Suppose \( c \) admits an AAT representation \((u, \Gamma)\) where \( \Gamma \) is an attention filter. If \( c_0(T) = x \), \( c_0(T \setminus \{y\}) \neq x \), and \( y \in \mathcal{X} \), then \( x \preceq_{SA} y \).

In fact, at no extra cost, a DM who starts with an attention filter will satisfy CLA in the future, regardless of the history, where her evolving consideration set will always remain an (potentially different) attention filter. This is because AAT propels the system of attention captured by attention filters throughout a DM’s lifetime even though it is imposed only on starting behavior (that \( \Gamma \) is an attention filter). This observation further asserts that Axiom 4.1 manifests the behavioral signature of CLA in choices across time.

**Proposition 8.** If \( c \) admits an AAT representation \((u, \Gamma)\) where \( \Gamma \) is an attention filter, then for any history \( h \in \mathcal{X} \cup \{\emptyset\} \),

1. \( \tilde{\Gamma}(h) : \mathcal{X} \to \mathcal{X} \) is an attention filter,
2. \( \tilde{c}(h) : \mathcal{X} \to X \) is a CLA.

### 4.2 Shortlisting in Sequences

The compatibility between AAT and CLA is not exclusive. A second illustrative example is provided next with Manzini and Mariotti (2007)’s rational shortlist methods (RSM).

**Definition 6 (Manzini and Mariotti (2007)).** A choice function \( \hat{c} : \mathcal{X} \to X \) is a rational shortlist methods (RSM) if there exists an ordered pair \((P_1, P_2)\) such that

\[
\hat{c}(A) = \max \left( \max(A, P_1), P_2 \right).
\]

\(^{18}\)The reverse is not necessarily true, for instance if the DM has \( u(x) > u(y) \), always considers everything, and \( y \) is not the worst alternative, then she is AAT and will commit \( x \preceq_{SA} y \) but never violates WARP.
$P_1$ and $P_2$ are asymmetric binary relations on $X$ that are also called *rationales*, and $\max(S, P) := \{x \in S | \nexists y \in S : y P x\}$. The model describes a choice procedure that involves sequentially making a choice, where the DM first creates a shortlist using $P_1$, and then makes a final decision using $P_2$.

**Definition 7.** A mapping $\Gamma : \mathcal{A} \rightarrow \mathcal{A}$ is a *shortlist* if there exists an asymmetric binary relation $P \subseteq X \times X$ such that $\Gamma(A) = \max(A, P)$ for all $A$.

**Proposition 9.** If $c$ admits an AAT representation $(u, \Gamma)$ such that $\Gamma$ is a shortlist, then for any history $h \in \mathcal{A} \cup \{\emptyset\}$,

1. $\tilde{\Gamma}(h) : \mathcal{A} \rightarrow \mathcal{A}$ is a shortlist.
2. $\tilde{c}(h) : \mathcal{A} \rightarrow X$ is a RSM.

Combining RSM and AAT gives rise to a choice process where the DM shortlists alternatives before making final decisions, but has the opportunity to revise these shortlists. These revisions take place as the DM becomes increasingly experienced and includes past choices in future shortlists. After each history, the original rationale $P$ is revised so that nothing eliminates $y$ if $y$ was chosen in the past, guaranteeing the consideration of $y$ in the future. As a consequence, her decisions not only become increasingly informative of her true preferences, but also explains whether her past choices were in fact influenced by shortlisting. To see this, consider the observation that $x$ was at first chosen over $y$, and then $y$ became chosen in an incidental choice problem, and in the future $y$ is chosen over $x$; this behavioral pattern confirms that the choice of $x$ was driven by a rationale that eliminated $y$ instead of genuine preference.

### 4.3 General characterization of compatibility

I conclude this section with a full characterization of models compatible with AAT. Given $X$ and $\mathcal{A}$, let $\mathcal{C}_{\text{All}}$ be the collection of all choice functions $\hat{c} : \mathcal{A} \rightarrow X$ such that $\hat{c}(A) \in A$ for all $A \in \mathcal{A}$. A subset $\mathcal{C}$ of $\mathcal{C}_{\text{All}}$, which may include some choice functions and exclude others, characterizes a set of axioms. Let $\mathcal{C}_{\text{WARP}}$ characterize the set of all choice functions that satisfy WARP (i.e., $\hat{c} \in \mathcal{C}_{\text{WARP}}$ if and only if $\hat{c}(T) = \hat{c}(S)$ whenever $\hat{c}(S) \in T \subseteq S$). By convention, I write $f(\mathcal{A}) := \{f(A) : A \in \mathcal{A}\}$.

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19Formally, after history $h$ (which resulted in past choices $c(h)$), the rationale becomes $x P(h) y$ if and only if $x P y$ and $y \notin c(h)$. Then $\tilde{\Gamma}(h)(A) = \max(A, P(h))$. 20
Definition 8.

1. Let \( g \) be a \( \kappa \)-cousin of \( f \) if there exists \( T \subseteq f(\mathcal{A}) \) such that \( g(A) = \kappa(\{f(A)\} \cup [A \cap T]) \) if \(|\{f(A)\} \cup [A \cap T]| > 1 \) and \( g(A) = f(A) \) otherwise.

2. \( \mathcal{C} \) is WARP-convex if for all \( f \in \mathcal{C} \), there exists \( \kappa \in \mathcal{C}_{\text{WARP}} \) such that every \( \kappa \)-cousin of \( f \) is in \( \mathcal{C} \).

Definition 8 introduces a concept of inclusiveness using WARP as an anchor. Specifically, an axiom(s) (resp. model) is WARP-convex if, whenever a WARP-violating choice function satisfies the axiom, choice functions that are “in between” said choice function and some WARP-conforming choice function are also satisfy the axiom. This means the underlying axiom does not have “holes” in what it would include, in the sense that departure from WARP, whenever admissible by an underlying axiom, can find its way back to WARP-compliance without ever being excluded by the axiom. A handful of non-WARP models in the literature are WARP-convex, including Manzini and Mariotti (2007)’s Rational Shortlist Methods and Masatlioglu et al. (2012)’s Choice with Limited Attention. Moreover, if \( \mathcal{C} \) contains only WARP-conforming choice functions (even if \( \mathcal{C} \subsetneq \mathcal{C}_{\text{WARP}} \) ), then it is trivially WARP-convex, and therefore every model that implies WARP is WARP-convex, such as the expected utility model.\(^{20}\)

Theorem 3. The following are equivalent:

1. \( \mathcal{C} \) is WARP-convex

2. For any \( f \in \mathcal{C} \), there exists an AAT representation such that \( c_0 = f \) and \( \tilde{c}(h) \in \mathcal{C} \) for all \( h \in \mathcal{A}^<N \). In this case, we say \( \mathcal{C} \) is fully compatible with AAT.

Intuitively, Theorem 3 suggests that a choice model is compatible with AAT only if its WARP violations are correctable. This is linked to the fact that AAT compels the correction of WARP violations since as time progresses and more choices made, more

\(^{20}\)To elaborate, starting from a choice function \( f \), a \( \kappa \)-cousin of \( f \) is a modification on \( f \) in \( \kappa \)'s direction. Said modification can applied to an arbitrary subset of alternatives \( T \), giving \( f \) one \( \kappa \)-cousin for every variation of \( T \). This process generates a family of choice functions (cousins) related by \( \kappa \) (their bloodline). When \( T = \emptyset \), no modification is made, and therefore \( f \) is always a \( \kappa \)-cousin of itself (you are your own cousin). Now consider \( T \neq \emptyset \). For choice set \( A \) where \( A \cap T = \emptyset \) or \( \{f(A)\} = A \cap T \), no modification is made, and \( g(A) = f(A) \). But for \( A \) such that \(|\{f(A)\} \cup [A \cap T]| > 1 \), \( \kappa \) comes into play and sets \( g(A) := \kappa(\{f(A)\} \cup [A \cap T]) \), potentially resulting in \( g(A) \neq f(A) \). Every \( g \) generated from this process is called a \( \kappa \)-cousin of \( f \); \( g \) and \( f \) may not be identical, but their differences are always rooted in \( \kappa \). Lastly, when \( \kappa \) satisfies WARP, the modifications push \( f \) towards WARP-compliance.
alternatives would have been chosen at some point in time and therefore become considered in the future, effectively reducing WARP violations. But the result also says that a choice model is compatible whenever WARP violations are correctable, which highlights the fact that AAT does nothing more than fixing WARP violations in a specific way. In particular, if no WARP violations are present in the first place, then AAT would accommodate a choice behavior as is; and if there were WARP violations, then AAT would religiously correct them without introducing new or different types of WARP violations.

5 Conclusion

This paper treats past and future choices as related to one another, and introduces a framework that studies how past experience may lead to the consideration of previously chosen alternatives in future choice problems. A model, Attention Across Time, allows us to fully pin down preferences even if attention is not directly observed, paving the way to sharper welfare analysis in the presence of limited attention. This observation is supported by the testable prediction that WARP violations, if exist, must be resolved in favor of true preference when the “problematic” choice sets are presented in alternating order. Moreover, by asking the right questions at the right time, preference can be pinned down using just one sequence of realized choices.

The axiomatic framework has three components, but depends primarily on an axiom that essentially posits presence of default attention. Default attention is what the DM pays attention to in a particular choice set regardless of history. The axiom guarantees that for every choice set there is always a set of things that the DM pays attention to, ruling out the opposite behavior where she pays attention to a brand new set of things every time she faces the same choice set. Two other axioms allow the DM to commit limited WARP violations and past-dependent behavior. It turns out that these assumptions are indispensable in the sense that non-trivial limited attention necessitates WARP violations and past-dependent behavior.

Although the paper focuses on choices and attention across time, it shows that attention across time and attention at a given time may be inherently related. For example, Masatlioglu et al. (2012) introduces a model of attention at a given time, where predictive power comes from the assumption that considerations sets are attention filters. I show that when time is introduced, an attention filter will always be an attention
filter even through it is gradually changing to accommodate the consideration of past choices. An analogous demonstration is provided for Manzini and Mariotti (2007)’s introduction of shortlists, where I show that a shortlist will always be a (potentially different) shortlist even though it is no longer eliminating past choices from future considerations. A complete characterization of this relationship is provided using an axiom of axioms, satisfied by the above two models, which essentially requires that a model accommodates correction of WARP violations whenever it accommodates non-WARP behaviors.
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A  Proofs

A.1  Notations

First I introduce notations.

1. $A$: a choice set
2. $A$: a sequence of choice sets (of any length).
3. $[A]_{r=k}^l$: the subsequence of $A$ including only elements in positions $k$ through $l$.
4. $A \in A$: a choice set $A$ that is in the sequence of choice sets $A$.
5. $ABC$: the sequence of choice sets that starts with choice set $A$, followed by choice set $B$, and ends with choice set $C$.
6. $x$: an alternative
7. $Ax$: alternative $x$ is chosen from the choice set $A$, which is the only choice set in the sequence.
   (a) $A^y x$: alternative $x$ is chosen from the choice set $A$, which is the only choice set in the sequence, and alternative $y$ is in $A$.
   (b) $AxByC^z x$: the sequence of choice sets $A B C$ from which $x, y, z$ are chosen respectively, and alternative $x$ is also in $C$.
   (c) $AB y$: a sequence of choice sets $A$, followed by the choice set $B$ from which alternative $y$ is chosen.
   (d) $Ax z B y$: a sequence of choice sets $A$, from which alternative $x$ is chosen from some $A \in A$, and alternative $z$ is chosen from some $C \in A$ (in no particular order), followed by choice set $B$ from which alternative $y$ is chosen.

A.2  Basic Implications of Axioms

Condition A.1. If $A B x$, either $B x$ or $A x$.

Lemma 2. If $c$ satisfies Axiom 2.2, then it satisfies Condition A.1.
Proof. Take $ABx$, and suppose not $Bx$. Let $K$ be the length of $A$. By Axiom 2.2, either $[A]_{t=1}^{K-1} x$ or $[A]_{t=1}^{K-1} Bx$ (or both). If it is the former, we are done since $Ax$. Suppose it is the latter, then by Axiom 2.2 again we have either $[A]_{t=1}^{K-2} x$ or $[A]_{t=1}^{K-2} Bx$ (or both). Again, if it is the former, we are done, otherwise we keep moving backward until we find $1 \leq q < K$ such that $[A]_{t=1}^{q} [A]_{t=q+1}^{q+1} x$. If this process does not end when $q = 1$, then $[A]_{t=1}^{1} x$ by Axiom 2.2, so $Ax$. 

**Condition A.2.** If $Ax$, then $Bx$ for some $B$.

**Lemma 3.** If $c$ satisfies Axiom 2.2, then it satisfies Condition A.2.

Proof. This is due to Condition A.1. Say $Ax$, and in particular $x$ is chosen from the $K$-th element, or equivalently $[A]_{t=1}^{K-1} x$. By Condition A.1, either $[A]_{t=1}^{K} x$ or $[A]_{t=1}^{K-1} x$. If the former, let $B = [A]_{t=1}^{K}$ and we are done. If the latter, by Condition A.1 again, either $[A]_{t=1}^{K-1} x$ or $[A]_{t=1}^{K-2} x$. If the former, let $B = [A]_{t=1}^{K-1}$ and we are done. Otherwise we keep going backward until we find $1 \leq q < K$ such that $[A]_{t=1}^{q} x$. If this process does not end when $q = 2$, then it must be that $[A]_{t=1}^{1} x$, so $Bx$ where $B = [A]_{t=1}^{1}$.

**Condition A.3.**

1. If $AyB^\alpha y x$, then not $CxD^\alpha x y$.
2. If $Ax yBx$, then not $Cy yB y$.
3. If $Ax yB^\alpha y x$, then not $Ax yD^\alpha x y$.

**Lemma 4.** If $c$ satisfies Axiom 2.1 and Axiom 2.3, then it satisfies Condition A.3.

Proof. Properties 2 and 3 are immediate given property 1, which I now prove. Suppose for contradiction $AyB^\alpha y x$ and $CxD^\alpha x y$, and suppose without loss of generality $\{x, y\} x$. By Axiom 2.1, $CxD^\alpha x y \{x, y\} y$. But then $AyB^\alpha y x (xPy), \{x, y\} x (c_0(\{x, y\}) = x)$, and $CxD^\alpha x y \{x, y\} y$ (y is chosen over $c_0(\{x, y\})$) violate Axiom 2.3.

**Condition A.4.** Recall that $c(A)$ denotes the set of choices made from the finite sequence of choice sets $A$. If $c(A) = c(B)$, then $\tilde{c}(A)(D) = \tilde{c}(B)(D)$ for all $D$.

**Lemma 5.** If $c$ satisfies Axiom 2.1, Axiom 2.2, and Axiom 2.3, then it satisfies Condition A.4.
Proof. This proof uses Condition A.1 and Condition A.3. Say $Dz$. By Condition A.1, $	ilde{c}(A)(D), \tilde{c}(B)(D) \in c(A) \cup \{z\}$. Let $c_1 = \tilde{c}(A)(D)$ and $c_2 = \tilde{c}(B)(D)$, and suppose for contradiction $c_1 \neq c_2$. Say $c_1, c_2 \in c(A)$, then Condition A.3 is violated. Instead, suppose without loss of generality $c_1 \notin c(A)$, so $c_1 = z$ by Condition A.1, and since $c_2 \neq z$, $c_2 \in c(A)$ also by Condition A.1. We established $Ac_2Dz$. Now note under Axiom 2.3, $Dz$ and $BDc_2$ where $c_2 \neq z$ imply $\exists E$ such that $Ec_2Dz$, a contradiction. \qed

Condition A.4 states that the choice from $D$ depends on the history only insofar as it depends on the set of realized choices in the history, and not specifically on past choice sets, nor the order of said choices. That is, if two histories $A$ and $B$ resulted in identical sets of past choices $c(A) = c(B)$, then the choice from $D$ would be the same $(\tilde{c}(A)(D) = \tilde{c}(B)(D))$.

Condition A.5. If $Ax B x$, then not $B y A y$.

Lemma 6. If $c$ satisfies Axiom 2.3, then it satisfies Condition A.5.

Proof. Suppose for contradiction $Ax B x$ and $B y A y$. So $x, y \in A, B$. By future independence, $Ax$ and $B y$. Using the definition in Definition 1, $B y$ and $Ax B x$ together implies $x P y$. Since $Ax$, or $c_0(A) = x$, so $c_0(A) P y$. Then $B y A y$ violates Axiom 2.3’s restriction that $y$ must not be chosen from $A$ when $c_0(A) P y$. \qed

A.3 Proofs (chronological)

A.3.1 Proof of Theorem 1

Fix $c$. As is common, necessity of axioms (if) is straightforward, so I will focus on showing sufficiency of axioms (only if). The plan goes as follows. Stage 1, we construct $\succ$, the true underlying preference that the anticipated utility function represents. Stage 2, we show that the constructed $\succ$ has the desirable properties to be represented by a utility function. Stage 3, we construct $\Gamma$ and show that $(\succ, \Gamma)$ explains choices.

Stage 1, Construction of $\succ$ First we partition alternatives $x \in X$ into two parts: The group where an alternative is ever-chosen, $\hat{X} := \{x \in X : x = c((A_n))$, for some $(A_n)$ and $i\}$, and the group where an alternatives is never-chosen, $X \setminus \hat{X}$. By Condition A.2, $\hat{X}$ can be equivalently characterized by $\{x \in X : x = \tilde{c}(\emptyset)(A) \exists A \in \mathcal{A}\}$; that is $x \in \hat{X}$ if and only if $\exists A \in \mathcal{A}$ such that $Ax$.  


Moreover, $|X \setminus \hat{X}| \leq 1$: say $z \in X \setminus \hat{X}$, so $\{z, x\}$ for all $x \neq z$, which means $x \in \hat{X}$ for all $x \neq z$.

Consider any two alternatives $x, y \in \hat{X}$ where $x \neq y$ and suppose without loss of generality that $\{x, y\}$. Consider any $A$ such that $Ay$.

1. If $Ay \{x, y\}$, we set $x \succ_S y$. ($x$ is inferred better than $y$ due to a switch)

2. If $Ay \{x, y\}$, we set $y \succ_D x$. ($y$ is inferred better than $x$ because $y$ is chosen over $x$, which receives attention by default in $\{x, y\}$)

Note that whether $x \succ_S y$ or $y \succ_D x$ does not depend on $A$ as long as $Ay$. Suppose not, so for some $A$ we have $Ay \{x, y\}$ and for some $B$ we have $By \{x, y\}$, then this violates Condition A.4. Moreover, note that for every pair $x, y \in \hat{X}$, either $x \succ_S y$ or $y \succ_D x$ (and never both), meaning the joint relation $\succ_S \cup \succ_D$ is complete and antisymmetric on $\hat{X}$.

For every pair $x, y$ such that $x \in \hat{X}$ and $y \in X \setminus \hat{X}$, set $x \succ_P y$. Finally, for any $x \in X$, set $x \succ_R x$.

**Stage 2, Properties of $\succ$** We argued that $\succ_S \cup \succ_D$, a subset of the Cartesian product $\hat{X} \times \hat{X}$, is complete and antisymmetric. Note that $\succ_P$, a subset of $\hat{X} \times X \setminus \hat{X}$, is complete by construction and clearly antisymmetric. Hence the joint relation $\succ := \succ_S \cup \succ_D \cup \succ_P \cup \succ_R$, a subset of $X \times X$, is complete, antisymmetric, and reflexive.

We will soon argue that $\succ$ is transitive; but first we make the following observations.

**Claim 1.** If $\{x, y\}$ and $A \{x, y\}$ for some $A$, then $\{x, y\} xAy \{x, y\}$ (and hence there exists $AxBy^{xy}$).

**Proof.** Suppose $\{x, y\}$ and $A \{x, y\}$ for some $A$. By Condition A.1, $Ay$. By Axiom 2.2, either $\{x, y\} xAy$ or $\{x, y\} xAx$. The latter would violate Condition A.5. Hence, $\{x, y\} xAy$. Consider $\{x, y\} xAy \{x, y\} z$. Clearly, $z \in \{x, y\}$. Since $\{x, y\}$ $x$ and $A \{x, y\}$ for some $A$, Axiom 2.3 implies that $C y \{x, y\} x$. So $z = y$. To conclude, $\{x, y\} xAy \{x, y\}$, which means $AxBy^{xy}$. [Q.E.D.]

**Claim 2.** If either $x \succ_S y$ or $x \succ_D y$, then there exists $AxBy^{xy}$. Moreover, if $x \succ_i y$, $y \succ_j z$, $z \succ_k x$ where $i, j, k \in \{S, D\}$, then $Cxyz$.

**Proof.** First we prove the first statement. Say $x \succ_S y$, then by the construction of $\succ_S$ there exists $AxBy^{xy}$. Say $x \succ_D y$, then by the construction of $\succ_D$ the sufficient condition
of Claim 1 is satisfied, hence we have $AyB^\exists y x$. Next we prove the second statement. Say $x \succ_i y$, $y \succ_j z$, $z \succ_k x$ where $i, j, k \in \{S, D\}$ and suppose for contradiction there exists $Cxyz$. Now consider $Cxyz \{x, y, z\} w$. Clearly, $w \in \{x, y, z\}$. Suppose without loss of generality that $w = x$, then $Cxyz \{x, y, z\} x$ and $AxB^\exists z x$ (due to $z \succ_S x$ or $z \succ_D x$ and the first statement) violate Condition A.3.

Claim 3. $\succ$ is transitive.

Proof. Take any $x, y, z \in \hat{X}$.

Suppose some of $x, y, z$ are in $\hat{X}$. Since $|\hat{X}| \leq 1$, there is exactly one of $x, y, z$ in $\hat{X}$, say without loss of generality $x, y \in \hat{X}$ and $z \in \hat{X}$. Then either $x \succ y$ or $y \succ x$. Say without loss of generality $x \succ y$, then since $x \succ y, z$, no transitivity violation is possible.

Now suppose all of $x, y, z$ are in $\hat{X}$. Suppose for contradiction that transitivity is violated, that is, $x \succ y$, $y \succ z$, and $z \succ x$. Since $x, y, z \in \hat{X}$, each of these $\succ$'s is either $\succ_S$ or $\succ_D$. In the following cases of transitivity violations, we show the existence of $Cxyz$ for cases 1-3, which violates Axioms 1-3 as outlined by Claim 2.

1. Suppose none of the $\succ$'s are $\succ_D$, that is, $x \succ_S y$, $y \succ_S z$, and $z \succ_S x$. By the construction of $\succ_S$: $\{x, y\} x, \{y, z\} y, \{x, z\} z$. By Condition A.1, $\{x, y\} x \{y, z\} y$. By Axiom 2.3 and the construction of $\succ_S x$, $\{x, y\} x \{x, z\} z$. So by Axiom 2.2, $\{x, y\} x \{y, z\} y \{x, z\} z$. So there exists $Cxyz$, and by Claim 2 we have contradiction.

2. Suppose exactly one of the $\succ$'s is $\succ_D$. Without loss of generality, say $x \succ_S y$, $y \succ_S z$, and $z \succ_D x$. By the constructions of $\succ_S$ and $\succ_D$: $\{x, y\} x, \{y, z\} y, \{x, z\} x$. Since $z \succ_D x, Ax \{x, z\} z$ for some $A$. By Condition A.1, $Ax \{x, z\} z \{x, y\} x$. By Axiom 2.3 and the construction of $\succ_S x, Ax \{x, z\} z \{y, z\} y$. So by Axiom 2.2, $Ax \{x, z\} z \{x, y\} x \{y, z\} y$. So there exists $Cxyz$, and by Claim 2 we have contradiction.

3. Suppose exactly two of the $\succ$'s are $\succ_D$. Without loss of generality, say $x \succ_D y$, $y \succ_S z$, and $z \succ_D x$. By the construction of $\succ_D x$ and Claim 1, $\exists B x C z \exists y$. By the construction of $y \succ_S z, Az \{y, z\} y$ for some $A$ and $\{y, z\} y$. So by Axiom 2.3, $Bx C z \exists y \{y, z\} y$. So there exists $Cxyz$, and by Claim 2 we have contradiction.
4. Finally, suppose all three $\succ'$s are $\succ_D$. Suppose without loss of generality that $\{x, y, z\} x$.

From $z \succ_D x$ and Claim 1, $\{x, z\} x A \{x, y\} z$ for some $A$. Now consider $\{x, z\} x A \{x, y\} z \{x, y, z\} w$. If $w = y$, this violates Condition A.1. If $w = x$, this violates Axiom 2.1. Hence

$$\{x, z\} x A \{x, y\} z \{x, y, z\} z. \quad \text{(A.1)}$$

From $y \succ_D z$ and Claim 1, $\{y, z\} z B \{y, z\} y$ for some $B$. Now consider $\{y, z\} z B \{y, z\} y \{x, y, z\} w$. Since $\{x, y, z\} x$ and Equation A.1, $w = x$ would violate Axiom 2.3. If $w = z$, this violates Axiom 2.1. Hence

$$\{y, z\} z B \{y, z\} y \{x, y, z\} y. \quad \text{(A.2)}$$

From $x \succ_D y$ and Claim 1, $\{x, y\} y C \{x, y\} x$ for some $C$. Now consider $\{x, y\} y C \{x, y\} x \{x, y, z\} w$. If $w = z$, this violates Condition A.1. If $w = y$, this violates Axiom 2.1. Hence

$$\{x, y\} y C \{x, y\} x \{x, y, z\} x,$$

but this along with Equation A.2 and $\{x, y, z\} x$ violates Axiom 2.3, a contradiction.

$$\square$$

**Stage 3, Model Explains Choices**  
Now that we established completeness and transitivity of $\succ$ on $X$, and since $X$ is countable, let $u : X \to \mathbb{R}$ be real-valued function such that $u(x) > u(y)$ if and only if $x \succ y$. Moreover, define by $\Gamma(A) := \{\tilde{c}(\emptyset)(A)\}$, for all $A \in \mathcal{A}$, the default attention set $\Gamma : \mathcal{A} \to \mathcal{A}$.

Throughout, we use $c_{\text{model}}$ to label the choice function given by the model, and from it $\tilde{c}_{\text{model}}$ the cross-sectional choice functions.

**Claim 4.** The constructed $(u, \Gamma)$ explains $c$ when restricting attention to the first choice sets (i.e., without history). That is, for all $A \in \mathcal{A}$,

$$\tilde{c}(\emptyset)(A) = \arg \max_{x \in \Gamma(\emptyset)(A)} u(x).$$
Proof. This is a direct consequence of how $\Gamma$ was constructed, $\Gamma(A) := \{\hat{c}(\emptyset)(A)\}$ and the fact that $\hat{\Gamma}(\emptyset)(A) = \Gamma(A).$ \hfill \qed

We now show that $(u, \Gamma)$ explains the entire $c$. Take any sequence of choice sets $(A_n) \in \mathcal{A}^\mathbb{N}$, and suppose for contradiction that, for some $i$,

$$c ((A_n))_i \neq \arg \max_{x \in \Gamma((A_1, \ldots, A_{i-1}))(A_i)} u(x).$$

Let $i$ be the set of all such $i$'s; they point to the set of all choice sets in $(A_n)$ from which the actual choice is not the same as the model prediction. Denote the minimum element of $i$ by $i^* := \min i$, which is well-defined. $i^* \neq 1$ due to Claim 4.

Consider $i^* \geq 2$. For notational convenience, denote the realized choice and the model prediction by, respectively,

$$c^R = c ((A_n))_{i^*},$$

$$c^p = c_{\text{model}} ((A_n))_{i^*} = \arg \max_{x \in \Gamma((A_1, \ldots, A_{i^*-1}))(A_{i^*})} u(x).$$

Claim 5. $c^R, c^p \in \hat{X}$.

Proof. It is by definition that $c^R \in \hat{X}$. To establish $c^p \in \hat{X}$, we use the construction of $\Gamma$: Since $c^p \in \hat{\Gamma}((A_1, \ldots, A_{i^*-1}))(A_{i^*})$, denote by $A_j$ the earliest choice set in $(A_n)$ from which $c^p$ is chosen. Since $c^p$ is chosen from $A_{i^*}$, $A_j$ exists and is well-defined. Hence $c^p \in \hat{\Gamma}((A_1, \ldots, A_{j-1}))(A_j)$ and $c^p \notin c_{\text{model}} ((A_1, \ldots, A_{j-1})) := \{c_{\text{model}} ((A_n))_k : k < j\}$. By the definition of history-dependent attention set,

$$\hat{\Gamma}((A_1, \ldots, A_{j-1}))(A_j) = (\Gamma(A_j) \cup c_{\text{model}} ((A_1, \ldots, A_{j-1}))) \cap A_j,$$

it must be that $c^p \in \Gamma(A_j)$ (that is, $c^p$ is in the default attention set of $A_j$). Moreover, by the construction of $\Gamma$ (including only the alternatives that are chosen without history), the model predicts $\check{c}_{\text{model}} (\emptyset)(A_j) = c^p$. Finally by Claim 4, $\check{c}(\emptyset)(A_j) = \check{c}_{\text{model}} (\emptyset)(A_j) = c^p$, and hence $c^p \in \hat{X}$. \hfill \qed

Now that we know $c^R, c^p \in \hat{X}$, we leverage the way $u(c^R)$ and $u(c^p)$ were defined to complete the proof.

Since $c^p$ is the model’s prediction, we know that $c^p \in \hat{\Gamma}((A_1, \ldots, A_{i^*-1}))(A_{i^*})$. Also, Condition A.1 (which holds under Axiom 2.2 (Lemma 2)) implies $c^R \in$
\( \hat{\Gamma}((A_1, \ldots, A_{i-1}))(A_i) \). Formally, suppose for contradiction \( c^R \not\in \hat{\Gamma}((A_1, \ldots, A_{i-1}))(A_i) \).

By the definition of \( \hat{\Gamma} \), this means two things. For one, \( c^R \not\in \Gamma(A_i) \), which by Claim 4 implies (i) \( c^R \not\in \tilde{c}(\emptyset)(A_i) \). Also, \( c^R \not\in c_{\text{model}}((A_1, \ldots, A_{i-1})) := \{ c_{\text{model}}((A_n))_k : k < i^* \} \), which by the fact that \( i^* \) is the earliest choice set in \( (A_n) \) from which model prediction and actual choice disagree implies (ii) \( c^R \not\in c((A_n))_l \) for all \( l < i^* \). But then combining (i), (ii), and \( c^R = c((A_n))_{i^*} \) yields a direct contradiction of Condition A.1.

We continue with \( c^P, c^R \in \hat{\Gamma}((A_1, \ldots, A_{i-1}))(A_i) \). Since the model predicts \( c^P \) to be chosen over \( c^R \) even though \( c^R \) was paid attention to, we have \( u(c^P) > u(c^R) \). Given \( c^P, c^R \in \hat{X} \), this means either \( c^P >_S c^R \) or \( c^P >_D c^R \), both would imply there exists, by Claim 2, \( Ac^R B \triangleright_\mathcal{E} c^P \). Meanwhile, since \( c^P \in \hat{\Gamma}((A_1, \ldots, A_{i-1}))(A_i) \), either (i) \( c^P \in c_{\text{model}}((A_1, \ldots, A_{i-1})) := \{ c_{\text{model}}((A_n))_k : k < i^* \} \) or (ii) \( c^P = \tilde{c}(\emptyset)(A_i) \) or both. Under (i), since \( A_i \) is by definition the earliest choice set in \( (A_n) \) from which model prediction the actual choice disagree, this implies \( c((A_n))_k = c_{\text{model}}((A_n))_k = c^P \) for some \( k < i^* \). Along with \( c^R = c((A_n))_{i^*} \), where \( c^P \in A_{i^*} \), this gives \( (A_1, \ldots, A_{i-1}) c^P A_{i^*} c^R \), and so there exists \( B \) such that \( Bc^P A_{i^*} c^R \), which together with \( Ac^R B \triangleright_\mathcal{E} c^P \) would contradict Condition A.3.

If we instead have (ii), then by Claim 4 and \( c^R = c((A_n))_{i^*} \), there exists \( B \) such that \( A_{i^*} c^P \) and \( BA_{i^*} c^R \), which together with \( Ac^R B \triangleright_\mathcal{E} c^P \) would contradict Axiom 2.3.

We showed that if the model predictions and and actual choices mismatch for the first time in a sequence, that necessarily leads to a contradiction. But this means no mismatches can ever happen. Therefore, for all \( (A_n) \in \mathcal{A}^N \) and \( k \in \mathbb{N} \),

\[
    c((A_n))_k = c_{\text{model}}((A_n))_k = \arg\max_{x \in \hat{\Gamma}((A_1, \ldots, A_{k-1}))(A_k)} u(x).
\]

A.3.2 Proof of Proposition 1

It is straightforward that a standard utility representation implies the other two. The notation \( \hat{X} \), formally introduced in Subsection 3.3, is the set of ever-chosen alternatives, which has at least size \( |X| - 1 \) since at most one alternative is not chosen from binary choice sets.

**Full Stability \( \Rightarrow \) standard utility representation:** Suppose \( c \) admits an AAT representation \( (\Gamma', u) \). If \( X \setminus \hat{X} = \{ z^* \} \), we assume without loss that \( u(z) > u(z^*) \) for all \( z \in \hat{X} = X \setminus \{ z^* \} \). Fix any sequence of choice sets \( (A_n) \in \mathcal{A}^N \). Construct \( yPx \) if for some \( i, x \in (A_n)_i \) and \( y = c((A_n))_i \). Due to Full Stability, \( P \) is antisymmetric. I now
show $P$ is consistent with $u$ whenever defined. Suppose not, let $i$ be the earliest instance (smallest $i$) in $(A_n)$ such that for some $x, y \in X$, $u(x) > u(y)$ but $y = c((A_n))_i$ where $x \in (A_n)_i =: B$, a disagreement. By Lemma 3, there exist choice sets $D, E \in \mathcal{A}$ such that $Dx$ and $Ey$. By implications of AAT, there is a sequence of choice sets and choices that begins with $EyB^{xy}y Dx \{x, y\} x$ (where “$Dx$” is due to $x \in \Gamma(D)$ and $u(x) > u(z)$ for all $z \in \tilde{\Gamma}(EB)(D) \setminus \{x\}$, and “$\{x, y\} x$” is due to $x \in c(EBD)$, which implies $x \in \tilde{\Gamma}(EBD)(\{x, y\})$, and $u(x) > u(y)$). $EyB^{xy}y Dx \{x, y\} x$ violates Full Stability, and therefore a disagreement cannot occur. Since $P$ is consistent with $u$ whenever defined, and $P$ is itself defined without limited consideration, we conclude that $u$ explains the choices from $(A_n)$ when everything is always considered. Since this is true for all $(A_n) \in \mathcal{A}^N$, we arrive at a standard utility representation.

**Past Independence $\Rightarrow$ standard utility representation:** Under Past Independence, $\tilde{c}(h)(A) = \tilde{c}(\emptyset)(A)$ for all $h$ and $A$. Suppose $c$ admits an AAT representation $(\Gamma', u)$. If $X \setminus \hat{X} = \{z^*\}$, we assume without loss that $u(z) > u(z^*)$ for all $z \in \hat{X} = X \setminus \{z^*\}$. Now we show that $c$ also admits an AAT representation with specifications $(\Gamma, u)$ where $\Gamma(A) = A$ for all $A$, which would imply $\tilde{\Gamma}(h)(A) = A$ for all $A$, and after that standard utility representation is immediate. Suppose for contradiction that for some $h$ and $A$, the choice predicted by $(\Gamma, u)$, denoted by $x$, is not $\tilde{c}(h)(A)$. Since $x \in A = \Gamma(A) = \tilde{\Gamma}(h)(A)$, it must be that $u(x) \geq u(z)$ for all $z \in A$. If $\hat{X} = X$, then $x \in \hat{X}$; if $\hat{X} \neq X$, then by our selection of utility function $u$ we have $x \in \hat{X}$. Then, there is some $B$ such that $\tilde{c}(\emptyset)(B) = x$ (Lemma 3). Then, $\tilde{c}((B))(A) = x$ since $x \in \tilde{\Gamma'}((B))(A)$ and $u(x) \geq u(z)$ for all $z \in \tilde{\Gamma'}((B))(A) \subseteq A$. However, $x = \tilde{c}((B))(A) = \tilde{c}(\emptyset)(A) = \tilde{c}(h)(A) \neq x$, where the second and third equalities are due to Past Independence, hence a contradiction. To conclude, $c$ also admits an AAT representation with specifications $(\Gamma, u)$ where $\Gamma(A) = A$ for all $A$, and therefore it admits a standard utility representation.

**A.3.3 Proof of Proposition 2**

**If:** This is given by Example 2, with $c_0(\{y, z^\prime\}) = y$ replaced by $c_0(A) = y$ for some $A$, which is guaranteed by $y \in \hat{X}$ (otherwise, $y \neq c_0(A)$ for all $A$ and Lemma 3 gives $y \notin \hat{X}$, a contradiction).

**Only if:** Suppose $c_0$ satisfies WARP and $\tilde{c}(h)$ fails WARP for some $h \in \mathcal{A}^{<N}$. So there exist $x, y \in X$ and $A, B \in \mathcal{A}$ such that $\{x, y\} \subseteq A \cap B$, $\tilde{c}(h)(A) = x$, and $\tilde{c}(h)(B) =$
A.3.4 Proof of Proposition 3

If \( c \) admits an AAT representation, it satisfies Axiom 2.2. By Lemma 2 and Lemma 3, it also satisfies Condition A.1 and Condition A.2 respectively. Suppose for contradiction \( c \) admits AAT representations with specifications \((u_1, \Gamma_1)\) and \((u_2, \Gamma_2)\) but \( u_1(x) > u_1(y) \) and \( u_2(x) < u_2(y) \) for some \( x, y \in \hat{X} \). Since \( x, y \in \hat{X} \), there exist \( A, B \in \mathcal{A} \) such that \( Ax, By \) by Condition A.2. Now consider \( Ax Bz \). By Condition A.1, \( z \in \{x, y\} \). Under AAT, \( \{x, y\} \subseteq \hat{\Gamma}(\{A\})(B) \), so by \((u_1, \Gamma_1)\) we have \( z = x \) but by \((u_2, \Gamma_2)\) we have \( z = y \), hence they cannot both represent \( c \).

A.3.5 Proof of Proposition 4

(1) We prove by construction. Consider any \( c \) and an AAT representation \((u, \Gamma)\) that represents \( c \). Enumerate the alternatives in \( \hat{X} \) by \( \hat{u}() \) so that \( \{x_1, \ldots, x_n\} = \hat{X} \) and \( u(x_i) < u(x_{i+1}) \) for all \( i \). For each \( i \), find \( A_i \) such that \( c_0(A_i) = x_i \). Now construct the sequence that begins with \((A_1, \ldots, A_n)\), followed by the finite sequence of all possible binary choice problems \((B_1, \ldots, B_k)\) (in any order), with arbitrary completion of what happens next (since \((A_n)\) must be an infinite sequence). Note that \( c_0(A_1) = x_1 \) and for any \( j \in \{2, \ldots, n\} \), if \( \hat{\Gamma}([A_1, \ldots, A_{j-1}]) \in \Gamma(A_j) \cup \{x_1, \ldots, x_{j-1}\}, \) then \( \hat{c}([A_1, \ldots, A_{j-1}]) = \arg \max_{x \in \hat{\Gamma}([A_1, \ldots, A_{j-1}])} u(x) = x_j \). By induction, this gives \( c((A_1, \ldots, A_n)) = \hat{X} \). Then, in the \((B_1, \ldots, B_k)\) phase, either \( x S_{(A_n)} y \) or \( y S_{(A_n)} x \) for all \( x, y \in \hat{X} \). Since \( x S_{(A_n)} y \) if and only if \( u(x) > u(y) \), \( S_{(A_n)} \) is also transitive.

(2) We prove by construction. Consider any sequence \((A_n)\) that begins with the finite sequence of all possible binary choice problems \((B_1, \ldots, B_k)\) (in any order) and then repeats itself, with arbitrary completion of what happens next (since \((A_n)\) must be an
infinite sequence). Since $X$ is finite, the set of all binary choice problems is finite, so this sequence is possible. Now consider any $c$. Note that all but at most one alternative would have been chosen in the first iteration of $(B_1, ..., B_k)$ (if $z$ was not chosen, then all other alternatives have been chosen, so $z$ is the only alternative that has not been chosen), denote this set by $\hat{X}_c$. Then, during the repetition of $(B_1, ..., B_k)$, either $xS_{(u_n)}y$ or $yS_{(u_n)}x$ for all $x, y \in \hat{X}_c$. Moreover, since $c$ admits an AAT representation, due to Proposition 3 and $\bar{X}_c \subseteq \hat{X}$, we have $xS_{(u_n)}y$ only if $u(x) > u(y)$. Hence $S_{(u_n)}$ is also transitive.

A.3.6 Proof of Proposition 5

**Only if:** Fix $c$. Suppose $c$ admits an AAT representation $(\Gamma, u)$. Fix any $A \in \mathcal{A}$. Suppose $y \in \Gamma(A) \setminus \Gamma^+(A)$, so $yS_0(A)$ by definition of $\Gamma^+$, then by Proposition 6 we have $u(y) > u(c_0(A))$. In this case, $(\Gamma, u)$ gives $\arg\max_{x \in \Gamma(\emptyset)(A)} u(x) \neq c_0(A)$ a contradiction. So $\Gamma(A) \setminus \Gamma^+(A) = \emptyset$, or $\Gamma(A) \subseteq \Gamma^+(A)$. It is straightforward that $c_0(A) \in \Gamma(A)$, otherwise $\arg\max_{x \in \Gamma(\emptyset)(A)} u(x) \neq c_0(A)$, a contradiction.

**If:** Fix $c$. Suppose $c$ admits an AAT representation $(\Gamma', u')$. Consider $u$ constructed in the following way. Proposition 4 showed that $S$ is complete and transitive on $\hat{X}$ and note that $X \setminus \hat{X} \leq 1$ (the set of all binary choice sets leaves at most one alternative never chosen). If $X \setminus \hat{X} = \emptyset$, let $u$ represent $S$. If $X \setminus \hat{X} = \{z\}$, perform a completion of $S$ by letting $xS_x$ for all $x \in \hat{X}$ and then let $u$ represent $S$. It is straightforward that $(\Gamma', u)$ is an AAT representation for $c$. Now I show that $(\Gamma^+, u)$ is also AAT representation for $c$, and then the proof is complete by invoking Lemma 7. Consider $h = \emptyset$. For any $A$, $y \in \Gamma(\emptyset)(A) \equiv \Gamma^+(A)$ and $y \neq \emptyset(A)$ implies either $\emptyset(A)S_y$ or $y \notin \hat{X}$, and in both cases we have $u(\emptyset(A)) > u(y)$. Combined with the fact that $\emptyset(A) \in \Gamma^+(\emptyset)(A)$, we conclude $\emptyset(A)$ is uniquely chosen from $A$ when there is no history and $\Gamma^+$ is the default attention function. Then, the same applies to $h \neq \emptyset$ inductively, starting from history of length 1, then length 2, and so on. For any history $h$ and current choice set $A$, past choices $c(h)$ predicted by $(\Gamma^+, u)$ are correct (due to induction), so $y \in \Gamma^+(h)(A)$ implies either (i) $y \in \Gamma^+(A)$ or (ii) $y \in c(h)$ or both. In case (i), we have either $\emptyset(h)(A)S_y$ or $y \notin \hat{X}$, and so $u(\emptyset(h)(A)) > u(y)$ from the way $u$ is constructed. In case (ii), the fact that $\emptyset(h)(A)$ is actually chosen over $y$ after $y$ was chosen in the past means $\emptyset(h)(A)S_y$, and so $u(\emptyset(h)(A)) > u(y)$ from the way $u$ is constructed. We now have enough to conclude that $\emptyset(h)(A)$ is uniquely chosen from $A$ after history $h$ under $\Gamma^*$. 

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A.3.7 Proof of Lemma 1

The intuition is that a union (and similarly an intersection) always includes the choice and only includes alternatives that are inferior to the choice, at any history. Formally, let $\Gamma^* = \Gamma \cup \Gamma'$. We start from $h = \emptyset$. For any $A$, $y \in \hat{\Gamma}^*(\emptyset)(A)$ implies either $y \in \Gamma(A)$ or $y \in \Gamma'(A)$ or both, and they imply $u(\hat{c}(\emptyset)(A)) > u(y)$ whenever $\hat{c}(\emptyset)(A) \neq y$. Combined with the fact that $\hat{c}(\emptyset)(A) \in \Gamma(A)$, which implies $\hat{c}(\emptyset)(A) \in \hat{\Gamma}^*(\emptyset)(A)$, we conclude $\hat{c}(h)(A)$ is uniquely chosen from $A$ when there is no history and $\Gamma^*$ is the default attention function. Then, the same argument applies to $h \neq \emptyset$ inductively, starting from history of length 1, then length 2, and so on. For any history $h$ and current choice set $A$, past choices $c(h)$ predicted by $(\Gamma^*, u)$ are correct (due to induction), so $y \in \hat{\Gamma}^*(h)(A)$ implies at least one of $y \in c(h)$ or $y \in \Gamma(A)$ or $y \in \Gamma'(A)$, and they imply $u(\hat{c}(h)(A)) > u(y)$ whenever $\hat{c}(h)(A) \neq y$. So $\hat{c}(h)(A)$ is uniquely chosen from $A$ after history $h$ under $\Gamma^*$. This completes the proof for union operation. (This proof is related to but not implied by Proposition 5, the difference is that Proposition 5 has the freedom to $u$ whereas the present theorem does not).

Next I show the case for intersection operations using Lemma 7. Note that an intersection between $\Gamma$ and $\Gamma'$ guarantees that $\Gamma^* := \Gamma \cap \Gamma'$ satisfies $\Gamma^* \subseteq \Gamma$ and $c_0(A) \in \Gamma^*(A)$ for all $A$.

Lemma 7. If $c$ admits an AAT representation $(\Gamma, u)$ and $\Gamma^* \subseteq \Gamma$ such that $c_0(A) \in \Gamma^*(A)$ for all $A$, then $c$ also admit an AAT representation $(\Gamma^*, u)$.

Proof. Consider $h = \emptyset$. For any $A$, $y \in \hat{\Gamma}^*(\emptyset)(A)$ implies $y \in \Gamma(A)$, which implies $u(\hat{c}(\emptyset)(A)) > u(y)$ whenever $\hat{c}(\emptyset)(A) \neq y$. Combined with the fact that $\hat{c}(\emptyset)(A) \in \Gamma^*(A)$, which implies $\hat{c}(\emptyset)(A) \in \hat{\Gamma}^*(\emptyset)(A)$, we conclude $\hat{c}(\emptyset)(A)$ is uniquely chosen from $A$ when there is no history and $\Gamma^*$ is the default attention function. Then, the same applies to $h \neq \emptyset$ inductively, starting from history of length 1, then length 2, and so on. For any history $h$ and current choice set $A$, past choices $c(h)$ predicted by $(\Gamma^*, u)$ are correct (due to induction), so $y \in \hat{\Gamma}^*(h)(A)$ implies either $y \in c(h)$ or $y \in \Gamma(A)$ or $y \in \Gamma'(A)$ or both, and they all imply $u(\hat{c}(h)(A)) > u(y)$ whenever $\hat{c}(h)(A) \neq y$. So $\hat{c}(h)(A)$ is uniquely chosen from $A$ after history $h$ under $\Gamma^*$. □

A.3.8 Proof of Proposition 6

If $xS_{(A_n)}y$, which by definition means $c((A_n))_i = y$, $c((A_n))_j = x$, and $y \in A_j$ for some $i < j$, then in the AAT representation we have $\{x, y\} \subseteq \hat{\Gamma}((A_1, ..., A_{j-1}))(A_j)$, and so
\( c((A_n))_i = x \) implies \( u(x) > u(y) \). It is therefore not possible that \( y \in S_{(B_n)}x \) for some \((B_n)\), for we would otherwise deduce \( u(y) > u(x) \) which is a direct contradiction; so it is not the case that \( y \in S_{(B_n)}x \).

### A.3.9 Proof of Corollary 1

This is a direct consequence of the model. Note that \( \bar{\Gamma}((B))(A) = \Gamma(A) \cup \{y\} \) and \( \bar{\Gamma}((A))(B) = \Gamma(B) \cup \{x\} \). Also note that either \( u(x) > u(y) \) or \( u(y) > u(x) \) in the AAT representation that \( c \) admits. Suppose without loss of generality \( u(x) > u(y) \). Since \( c_0(A) = x \), so \( x \in \Gamma(A) \) and \( u(x) > u(z) \) for all \( z \in \Gamma(A) \setminus \{x\} \). Since \( c_0(B) = y \), so \( y \in \Gamma(B) \) and \( u(x) > u(y) \) for all \( z \in \Gamma(B) \setminus \{x\} \). Therefore \( u(x) > u(z) \) for all \( z \in (\Gamma(A) \cup \Gamma(B)) \setminus \{x\} \), so \( \bar{\Gamma}((A))(B) = \bar{\Gamma}((B))(A) = x \). Finally, since \( u(y) > u(x) \) would have implied, instead, convergence on \( y \), the fact that we have convergence on \( x \) implies \( u(x) > u(y) \).

### A.3.10 Proof of Theorem 2

We start with sufficiency of axioms. The key is to show that revealed preference in CLA is also revealed preference in AAT. The rest is completed with a proof technique similar to that for CLA in Masatlioglu et al. (2012), which allows for an arbitrary completion of preference, and other intrinsic compatibility between AAT and CLA.

Recall that \( \hat{X} := \{ x \in X : c((A_n))_i = x \text{ for some } (A_n) \text{ and } i \} \)

**Stage 1, AAT**  Since \( c \) satisfies Axiom 2.1, Axiom 2.2 and Axiom 2.3, by Theorem 1 it admits an AAT representation. We use the specification as constructed by the proof of Theorem 1, that is, \( \Gamma(A) = c(A) \) for all \( A \) and for all \( x \in \hat{X} \) and \( y \in X \setminus \hat{X} \) we have \( u(x) > u(y) \).

**Stage 2, xPy implies u(x) > u(y)**

**Lemma 8.** Suppose \( c \) admits an AAT representation. If \( x, y \in \hat{X}, \text{then there exists a sequence of two choice sets, } (A_1,A_2), \text{ such that } \{x, y\} \subseteq c((A_1,A_2)) \).

**Proof.** Say \( c \) admits an AAT representation with \( u \) and \( \Gamma \). Suppose without loss \( u(x) > u(y) \).
1. Suppose $\bar{c}(\emptyset)\{x, y\} = x$, and hence $x \in \Gamma\{x, y\}$. Since $y \in \hat{X}$, by Lemma 3 there exists $A \in \mathcal{A}$ such that $\bar{c}(\emptyset)(A) = y$, and hence $y \in \Gamma(A)$. By AAT representation, $\{x, y\} \subseteq \bar{\Gamma}(A)(\{x, y\})$, then since $u(x) > u(y)$, we have $\bar{c}(A)(\{x, y\}) = x$. So $\{x, y\} \subseteq c((A, \{x, y\}))$.

2. Suppose $\bar{c}(\emptyset)(\{x, y\}) = y$, and hence $y \in \Gamma\{x, y\}$. Since $x \in \hat{X}$, by Lemma 3 there exists $A \in \mathcal{A}$ such that $\bar{c}(\emptyset)(A) = x$, and hence $x \in \Gamma(A)$. By AAT representation, $\{x, y\} \subseteq \bar{\Gamma}(A)(\{x, y\})$, then since $u(x) > u(y)$, we have $\bar{c}(A)(\{x, y\}) = x$. So $\{x, y\} \subseteq c((A, \{x, y\}))$.

\[\square\]

**Definition 9.** Define $xP y$ if for some $T \in \mathcal{A}$, we have $\bar{c}(\emptyset)(T) = x$ and $\bar{c}(\emptyset)(T \backslash \{y\}) \neq x$.

**Lemma 9.** Suppose $c$ admits an AAT representation. Consider any $x, y \in \hat{X}$. Either $\exists (A_n)$ such that $c((A_n))_i = y$ and $c((A_n))_j = x$ where $y \in A_j$ and $i < j$ or $\exists (A_n)$ such that $c((A_n))_i = x$ and $c((A_n))_j = y$ where $x \in A_j$ and $i < j$ (and not both).

**Proof.** Since $x, y \in \hat{X}$, Lemma 8 guarantees the existence of $(A_1, A_2)$ such that $\{x, y\} \subseteq c((A_1, A_2))$. Suppose without loss of generality $\bar{c}((A_1, A_2))\{x, y\} = x$, then $(A_1, A_2, \{x, y\})$ is a sequence where $y$ was chosen in the past and $x$ is chosen over $y$ in the future. Furthermore, since $c$ admits an AAT representation, we must have $u(x) > u(y)$, which excludes the possibility of any sequence where $x$ was chosen in the past and $y$ is chosen over $x$ in the future. \[\square\]

**Lemma 10.** If $c$ (which is assumed to satisfy Axiom 2.1-Axiom 2.3) further satisfies Axiom 4.1, then $xP y$ implies $u(x) > u(y)$.

**Proof.** Suppose $xP y$.

1. Suppose $y \in \hat{X}$. Since $xP y$, $x \in \hat{X}$. Then by Axiom 4.1 and Lemma 9, $\exists (A_n)$ such that $c((A_n))_i = y$ and $c((A_n))_j = x$ where $y \in A_j$ and $i < j$, so $u(x) > u(y)$ since $(u, \Gamma)$ explains choices.

2. Suppose $y \notin \hat{X}$, then $u(x) > u(y)$ is by construction.

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Stage 3, Constructing the attention filter

Now we show that we can reconstruct the default attention $\Gamma$ into an attention filter $\Gamma^*$ while continue to explain $c$. First we restate a result provided in Masatlioglu et al. (2012) (with notational revisions).

**Lemma 11** (Masatlioglu et al. (2012)). Fix a choice function $c$. If $P$ is acyclic, for any arbitrary completion from $P$ to $\succ$ that is transitive and an attention filter constructed by

$$\Gamma^*(A) := \{ x \in A : \bar{c}(\emptyset)(A) \succ x \} \cup \{ \bar{c}(\emptyset)(A) \},$$

we have

$$\bar{c}(\emptyset)(A) = \{ x \in \Gamma^*(A) : x \succ y \text{ for all } y \in \Gamma^*(A) \}.$$ 

In Masatlioglu et al. (2012), an axiom (WARP(LA)) is used to guarantee the acyclicity of $P$. In our case, Lemma 10 guarantees that $P$ is a subset of $\succ_u$ (defined by $x \succ_u y$ if $u(x) > u(y)$), which is complete and transitive, so $P$ is acyclic. Moreover, $\succ_u$ is itself a transitive completion of $P$.

Stage 4, Shows that $u, \Gamma^*$ explains choice

We complete the proof by showing that $c$ admits an AAT representation with $(\Gamma^*, u)$, where

$$\Gamma^*(A) := \{ x \in A : u(\bar{c}(\emptyset)(A)) > u(x) \} \cup \{ \bar{c}(\emptyset)(A) \}.$$ 

Lemma 11 already states that $\Gamma^*$ constructed this way is an attention filter, and $\Gamma^*$ along with $u$ explains $\bar{c}(\emptyset) : \mathcal{A} \to X$. It remains to show that choices with history is also explained.

The intuition is straightforward: We added alternatives to $\Gamma$ to form $\Gamma^*$, but all those that are added are worse than what should be chosen, and hence does not affect the choice. Formally:

First note that $\Gamma \subseteq \Gamma^*$ (since $\Gamma(A) = \bar{c}(\emptyset)(A)$). Moreover, by construction of $\Gamma^*$, for all $(A_1, \ldots, A_n)$ and $A$,

$$\tilde{\Gamma}^*(A_1, \ldots, A_n)(A) \setminus \tilde{\Gamma}(A_1, \ldots, A_n)(A) \subseteq \{ x : u(\bar{c}(\emptyset)(A)) > u(x) \}.$$

Suppose

$$\bar{c}((A_1, \ldots, A_n))(A) = y,$$
then it must be that
\[ u(y) \geq \max \{ u(x) : x \in \Gamma (A_1, \ldots, A_n) (A) \} \geq \max \{ u(x) : x \in \Gamma (\emptyset ) (A) \} = u(\tilde{e} (\emptyset ) (A)). \]
by the AAT representation (the second weak inequality is due to \( \Gamma (A_1, \ldots, A_n) (A) \supseteq \Gamma (\emptyset ) (A) \)). So \( u(y) \geq u(\tilde{e} (\emptyset ) (A)) \), and hence
\[ u(y) \geq u(\tilde{e} (\emptyset ) (A)) \geq \max \{ u(x) : x \in \Gamma^* (A_1, \ldots, A_n) (A) \setminus \Gamma (A_1, \ldots, A_n) \}. \]
Since \( y \in \Gamma^* (A_1, \ldots, A_n) (A) \) and \( u(y) \geq u(x) \) for all \( x \in \Gamma^* (A_1, \ldots, A_n) (A) \), the model prediction with \( u \) and \( \Gamma^* \) conforms with choice (for any \( (A_1, \ldots, A_n) \) and \( A \)). And we are done!

**Necessity** Since \( c \) that admits an AAT representation, it satisfies Axiom 2.1-Axiom 2.3.

Suppose \( c \) admits an AAT representation where \( \Gamma \) is an attention filter. If \( \tilde{e} (\emptyset ) (T) = x \) and \( \tilde{e} (\emptyset ) (T \setminus \{ y \}) \neq x \), it must be that \( y \in \Gamma (T) \) since \( \Gamma \) is an attention filter and \( \Gamma (T) \neq \Gamma (T \setminus \{ y \}) \). Since \( \tilde{e} (\emptyset ) (T) = x \), it must be that \( u(x) > u(y) \). So for all \( (A_n) \), if \( c ((A_n))_i = x \), then \( c ((A_n))_j \neq y \) for all \( j > i \) such that \( x \in A_j \). Hence Axiom 4.1.

**A.3.11 Proof of Proposition 7**

This is a direct implication of Lemma 9 and the fact that \( u(x) > u(y) \), which is given by \( \tilde{e} (\emptyset ) (T) = x \) and \( \tilde{e} (\emptyset ) (T \setminus \{ y \}) \neq x \).

**A.3.12 Proof of Proposition 8**

Part 2 is immediate given Part 1. To show Part 1, that \( \tilde{e} (\emptyset ) : \mathcal{A} \rightarrow \mathcal{A} \) is an attention filter for all \( h \in \mathcal{A}^{<N} \), we begin with a lemma,

**Lemma 12.** For any \( T \subseteq X \), if \( \Gamma \) is an attention filter and \( \Gamma^* (A) := \Gamma (A) \cup [T \cap A] \) for all \( A \), then \( \Gamma^* \) is also an attention filter.

**Proof.** Fix any \( A \) and consider \( y \in A \setminus \Gamma^* (A) \), then \( y \in A \setminus \Gamma (A) \) by construction of \( \Gamma^* \), and then \( \Gamma (A \setminus \{ y \}) = \Gamma (A) \) since \( \Gamma \) is an attention filter. So \( \Gamma^* (A \setminus \{ y \}) = \Gamma (A \setminus \{ y \}) \cup [T \cap (A \setminus \{ y \})] = \Gamma (A) \cup [T \cap (A - \{ y \})] = \Gamma (A) \cup [T \cap A] = \Gamma^* (A) \), where the third inequality is due to \( y \notin T \) implied by \( y \in A \setminus \Gamma^* (A) \). Since this is true for any \( A \) and \( y \), \( \Gamma^* \) meets the properties of an attention filter. \( \Box \)
To wrap up, recall $\bar{\Gamma}(h)(A) = \Gamma(A) \cup [c(h) \cap A]$. Invoking Lemma 12 with $\Gamma^\ast = \bar{\Gamma}$ and $T = c(h)$ completes the proof for 2(a), which implies 2(b) by Masatlioglu et al. (2012).

A.3.13 Proof of Proposition 9

Part 2 is immediately given Part 1, where the corresponding $P_2$ (in RSM) is given by $u$ (in AAT) and $P_1$ (in RSM) is given by the rationale $P$ for shortlist $\bar{\Gamma}$ (in AAT). To show Part 1, that $\bar{\Gamma}(h) : \mathcal{A} \rightarrow \mathcal{A}$ is a shortlist for all $h \in \mathcal{A}^\ast < \mathbb{N}$, we prove by construction. Given $(u, \Gamma)$ where $\Gamma$ is a shortlist, let $P$ be the corresponding rationale. Consider any $h$. Define a new rationale $P^\ast \subseteq X \times X$ by $xP^\ast y$ if $xPy$ and $y \notin c(h)$. Then let $\Gamma^\ast$ be the shortlist defined $P^\ast$. Fix any $A \in \mathcal{A}$. If $y \in \bar{\Gamma}(h)(A)$, then either $y \in \Gamma(A)$ (so $\neg xPy$ for all $x \in X$) or $y \in c(h)$ (so $\neg xP^\ast y$ for all $x \in X$), so $y \notin \Gamma^\ast(A)$. If $y \in \Gamma^\ast(A)$, then $\neg xP^\ast y$ for all $x \in A$, which by definition of $P^\ast$ can be due to $\neg xPy$ for all $x \in A$ or $y \in c(h)$, so $y \in \bar{\Gamma}(h)(A)$. So $\bar{\Gamma}(h)(A) = \Gamma^\ast(A)$ for all $A \in \mathcal{A}$, and this holds for all $h \in \mathcal{A}^\ast < \mathbb{N}$.

A.3.14 Proof of Theorem 3

Proof for (1) $\Rightarrow$ (2) Take any $f \in \mathbb{C}$. Since $\mathbb{C}$ is WARP-convex, consider the $\kappa \in \mathbb{C}_{WARP}$ such that every $\kappa$-cousin of $f$ is in $\mathbb{C}$. Construct the AAT representation $(u, \Gamma)$ where $u$ represents $\kappa$ in standard utility maximization and $\Gamma(A) = \{f(A)\}$. It is clear that $c_0 = f$. Now consider any history $h \in \mathcal{A}^\ast < \mathbb{N}$, which results in past choices $c(h)$. The cross-sectional choice function is $\bar{c}(h)$ and attention function is $\bar{\Gamma}(h)$. Consider any $A \in \mathcal{A}$. By AAT, $\bar{\Gamma}(h)(A) = \Gamma(A) \cup [A \cap c(h)] = \{f(A)\} \cup [A \cap c(h)]$, so $\bar{c}(h)(A) = \arg\max_{x \in \{f(A)\} \cup [A \cap c(h)]} u(x) = \kappa \left(\{f(A)\} \cup [A \cap c(h)]\right)$, where the second equality is due to the fact that $u$ represents $\kappa$. By setting $T = c(h)$, we conclude that $\bar{c}(h)$ is a $\kappa$-cousin of $f$, and therefore $\bar{c}(h) \in \mathbb{C}$.

Proof for (2) $\Rightarrow$ (1) Suppose $\mathbb{C}$ is not WARP-convex, so for some $f \in \mathbb{C}$, there is no $\kappa \in \mathbb{C}_{WARP}$ such that every $\kappa$-cousin of $f$ is in $\mathbb{C}$. Consider this $f$, and suppose for contradiction the AAT representation $(u, \Gamma)$ is such that $c_0 = f$ and $\bar{c}(h) \in \mathbb{C}$ for all $h \in \mathcal{A}^\ast < \mathbb{N}$. Since $u$ represents some $\kappa \in \mathbb{C}_{WARP}$ in standard utility maximization, consider the $\kappa$-cousin of $f$ that is not in $\mathbb{C}$, and call it $g$, which is associated with some $T \subseteq f(\mathcal{A})$. 

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Now we construct the history \( h \in \mathcal{A}^{<N} \) such that \( c(h) = T \). Consider the alternatives \( x_1, \ldots, x_n \) such that \( \{x_1, \ldots, x_n\} = T \) and \( u(x_i) < u(x_{i+1}) \). For each \( i \), since \( x_i \in T \subseteq f(\mathcal{A}) \), there exists \( A_i \) such that \( c_0(A_i) = x_i \) (and therefore \( x_i = \arg\max_{x \in (A_i)} u(x) \)); consider the history \( h = (A_1, \ldots, A_n) \). By induction, \( c_0(A_1) = x_1 \) and at any \( j \), since \( \tilde{\Gamma}((A_1, \ldots, A_{j-\text{i}})) (A_j) = \Gamma(A_j) \cup \{x_1, \ldots, x_{j-\text{i}}\} \) and \( u(x_k) < u(x_j) \), we have \( \tilde{c}((A_1, \ldots, A_{j-\text{i}})) (A_j) = x_j. \) So \( c(h) = T \), and so \( \tilde{c}(h)(A) = \arg\max_{x \in (A_i) \cup (A \cap \Gamma)} u(x) = \arg\max_{x \in (f(A)) \cup (A \cap \Gamma)} u(x) = g(A) \) for all \( A \in \mathcal{A} \), so \( \tilde{c}(h) = g \) is not in \( \mathcal{C}. \)

B Extension to Choice Correspondences

As an extension, indifferences can too be accommodated in AAT, where both the choice and alternatives that are indifferent to it would be considered in future choice problems.

B.1 Primitive

I begin with the following changes to the primitive (\( X \) remains a countable set of alternatives and \( \mathcal{A} \) the set of all finite subsets of \( X \) with at least size 2). Let \( C : \mathcal{A}^{\mathbb{N}} \rightarrow (\mathcal{P}(\mathbb{X}) \setminus \{\emptyset\})^{\mathbb{N}} \), \( C((A_n))_i \subseteq (A_n)_i \) be a choice correspondence, therefore more than one choice could be chosen from any choice set in any sequence of choice sets. Let before, I assume future independence: for all \( (A_n), (B_n) \in \mathcal{A}^{\mathbb{N}} \) such that \( A_k = B_k \) for all \( k \leq K \in \mathbb{N} \), we have \( C((A_n))_k = C((B_n))_k \) for all \( k \leq K \). In other words, if two sequences of choice sets are identical up to a certain point, the corresponding choices up to that point are identical as well. Also similar to before, I define the cross-sectional choice correspondence \( \tilde{C} : \mathcal{A}^{<N} \times \mathcal{A} \rightarrow \mathcal{A} \), which maps a choice set \( A \in \mathcal{A} \) to a choice \( x \in A \) given a finite sequence of past choice sets \( (A_1, \ldots, A_K) \in \mathcal{A}^{<N}. \)

B.2 Axioms

I modify Axiom 2.1, Axiom 2.2, and Axiom 2.3 to accommodate indifferences. Additionally, I introduce a new axiom, Axiom B.4, which captures the basic intuition that if \( x \) is indifferent to \( a \), and \( y \) is indifferent to \( b \), then having observed a switch from \( y \) to \( x \) means we will not observe a switch from \( b \) to \( a \).

The first two modifications are straightforward adaptations from Axiom 2.1 and Axiom 2.2 to choice correspondences:
Axiom B.1 (Stability*). For any \((A_n) \in \mathcal{A}^N\) and \(k_1 < k_2 < k_3\),
\[
x \in C((A_n))_{k_1}, y \in C((A_n))_{k_2} \text{ with } x \in A_{k_2} \setminus C((A_n))_{k_2} \text{ and } y \in A_{k_3}
\]
implies \(x \notin C((A_n))_{k_3}\).

Axiom B.2 (Past Dependence*). For any \((A_n) \in \mathcal{A}^N\), \(B \in \mathcal{A}\), and \(K \in \mathbb{N}\),
\[
\tilde{C}((A_1, \ldots, A_K))(B) \subseteq \tilde{C}((A_1, \ldots, A_{K-1}))(B) \cup \tilde{C}((A_1, \ldots, A_{K-1}))(A_K).
\]

Next I modify Axiom 2.3, but in addition to simply rewriting it for choice correspondences, I add a property that extends the intuition of default attention to alternatives that are indifferent:

Definition 10. Define \(x W y\) if for some \((A_n) \in \mathcal{A}^N\) and \(i \in \mathbb{N}\), \(\{x, y\} \subseteq C((A_n))_i\).

Axiom B.3 (Default Attention*). If \(x \in C((A_n))_j\), \(y \in A_j \setminus C((A_n))_j\) and at least one of the following holds:

1. \(y \in C((A_n))_i\) where \(i < j\),
   
   (a) \(y \in \tilde{C}(\emptyset)(A_j)\).

Then for all \(x^* W x\) and \(y^* W y\), \(x^* \in \tilde{C}(\emptyset)(B_k) \cup C((B_n))_k\) implies \(y^* \notin C((B_n))_k\) for all \((B_n) \in \mathcal{A}^N\) and \(k \in \mathbb{N}\).

Finally I add an axiom that captures the crucial restriction that reflects a consistent preference ranking—if \(x\) is indifferent to \(a\), and \(y\) is indifferent to \(b\), then having observed a switch from \(y\) to \(x\) means I won’t observe a switch from \(b\) to \(a\).

Definition 11. Define \(a S x\) if for some \(a \in C((A_n))_i\), \(x \in C((A_n))_j\), and \(a \in A_j \setminus C((A_n))_j\) for some \(i < j\).

Axiom B.4. Suppose \(x W a\) and \(y W b\). If \(y S x\), then not \(a S b\).

B.3 Model

We are ready for the model and the representation theorem. As in Section 2 and Section 3, let \(h \in \mathcal{A}^{<\mathbb{N}}\) be an arbitrary history of past choice sets and let \(C(h)\) be the collection of choices from history \(h\).
**Definition 12.** $C$ admits an Attention Across Time (AAT) representation if there exist

1. a utility function $u : X \to \mathbb{R}$ and

2. a default attention function $\Gamma : \mathscr{A} \to 2^X \setminus \{\emptyset\}$, $\Gamma(A) \subseteq A$ such that

$$\tilde{C}(h)(A) = \arg\max_{x \in \tilde{\Gamma}(h)(A)} u(x)$$

where $\tilde{\Gamma}(h)(A) = \Gamma(A) \cup (C(h) \cap A)$.

**Theorem 4.** $C$ satisfies Axioms B.1-B.4 if and only if it admits an Attention Across Time (AAT) representation.

**B.4 Proofs**

**B.4.1 Proof of Theorem 4**

**Stage 1, Equivalent Classes** First, we show that under the axioms, if $x$ and $y$ are sometimes chosen simultaneously, and $y$ and $z$ are sometimes chosen simultaneously, then $x$ and $z$ are sometimes chosen simultaneously:

**Lemma 13.** Suppose $C$ satisfies Axiom B.3. For all $x \mathcal{W} y$, if $x \in C((A_n))_i$ and $y \in C((A_n))_j \cup \tilde{C}(\emptyset)(A_i)$ for some $j < i$, then $y \notin A_i \setminus C((A_n))_i$.

**Proof.** This is by direct application of Axiom B.3. Suppose $x \mathcal{W} y$. Suppose for contradiction $x \in C((A_n))_i$, $y \in B_i$, and $y \in C((A_n))_j \cup \tilde{C}(\emptyset)(A_i)$ for some $j < i$, but $y \notin C((A_n))_j$, then by Axiom B.3, $x \in C((B_n))_k$ implies $y \notin C((B_n))_k$ for all $(B_n) \in \mathscr{A}^N$ and $k \in \mathbb{N}$, but this contradicts $x \mathcal{W} y$. □

**Lemma 14.** If $x \mathcal{W} y$ and $y \mathcal{W} z$, then $x \mathcal{W} z$.

**Proof.** First we show that there exists a sequence of choice sets $(A_1, ..., A_K)$ such that $x, y, z$ were chosen at some point, $\{x, y, z\} \subseteq C((A_1, ..., A_K)) := \{C((A_n))_i : i \leq K\}$.

Since $y \mathcal{W} z$, there exists a sequence of choice sets $(B_n)$ such that $\{y, z\} \subseteq C((B_n))_i$ for some $i \in \mathbb{N}$.

1. Suppose $x \in \tilde{C}(\emptyset)(\{x, y\})$, then either $x \in \tilde{C}((B_1, ..., B_i))\{x, y\}$ or $y \in \tilde{C}((B_1, ..., B_i))\{x, y\}$, and the latter implies $x \in \tilde{C}((B_1, ..., B_i))\{x, y\}$ due to $x \in \tilde{C}(\emptyset)(\{x, y\})$ and Lemma 13. Hence $\{x, y, z\} \subseteq C((B_1, ..., B_i, \{x, y\}))$. 44
2. Suppose $x \notin \tilde{C}(\emptyset)(\{x, y\})$, so $\tilde{C}(\emptyset)(\{x, y\}) = \{y\}$. Since $xWy$, $x$ is ever-chosen $(x \in \tilde{X})$, so through an argument analogous to Lemma 3, Axiom B.2 implies $x \in \tilde{C}(\emptyset)(D)$ for some $D \in \mathcal{A}$. Due to Lemma 13 and $\tilde{C}(\emptyset)(\{x, y\}) = \{y\}$, $\tilde{C}((D))((\{x, y\}) = \{x, y\}$. Now consider $\tilde{C}((D, \{x, y\}))(B_i)$ where $B_i$ is defined above as the choice set from which both $y$ and $z$ were chosen in some sequence. If $\tilde{C}((D, \{x, y\}))(B_i) \cap \{x, y, z\} \neq \emptyset$, then due to $\{x, y\} \subseteq C((D, \{x, y\}))$, $xWy$ and $yWz$, once or twice application(s) of Lemma 13 guarantees $z \in \tilde{C}((D, \{x, y\}))(B_i)$, and we are done with $\{x, y, z\} \subseteq C((D, \{x, y\}, B_i))$. Suppose for contradiction $\tilde{C}((D, \{x, y\}))(B_i) \cap \{x, y, z\} = \emptyset$, let $a \in \tilde{C}((D, \{x, y\}))(B_i)$:

(a) Suppose $a \in \tilde{C}(\emptyset)(D)$, then $aWy$, and so $x \notin \tilde{C}((D, \{x, y\}))(B_i)$ is a contradiction of $x \in C((D, \{x, y\}))$ and Lemma 13.

(b) Suppose $a \notin \tilde{C}(\emptyset)(D)$, then through an argument analogous to Lemma 2, Axiom B.2 implies $a \in \tilde{C}(\emptyset)(B_i)$.

i. If $a \in C((B_n))$, then $aWy$, and so $y \notin \tilde{C}((D, \{x, y\}))(B_i)$ is a contradiction of $y \in C((D, \{x, y\}))$ and Lemma 13.

ii. If $a \notin C((B_n))$ (so $a \in B \setminus C((B_n))$), then $y \in C((B_n))$, $a \in \tilde{C}(\emptyset)(B_i) \setminus C((B_n))$, $y \in C((D, \{x, y\}))$ but $a \in \tilde{C}((D, \{x, y\}))(B_i)$ contradicts Lemma 13.

We have now established that there exists a sequence of choice sets $(A_1, \ldots, A_K)$ such that $\{x, y, z\} \subseteq C((A_1, \ldots, A_K))$. Now consider $\tilde{C}((A_1, \ldots, A_K))(\{x, y, z\})$. Since $xWy$ and $\{x, y\} \subseteq C((A_1, \ldots, A_K))$, Lemma 13 requires $\{x, y\} \subseteq \tilde{C}((A_1, \ldots, A_K))(\{x, y, z\})$ or $\{x, y\} \cap \tilde{C}((A_1, \ldots, A_K))(\{x, y, z\}) = \emptyset$. Similarly, since $yWz$ and $\{y, z\} \subseteq C((A_1, \ldots, A_K))$, Lemma 13 requires $\{y, z\} \subseteq \tilde{C}((A_1, \ldots, A_K))(\{x, y, z\})$ or $\{y, z\} \cap \tilde{C}((A_1, \ldots, A_K))(\{x, y, z\}) = \emptyset$. Therefore $\tilde{C}((A_1, \ldots, A_K))(\{x, y, z\}) = \{x, y, z\}$, and hence $xWy$.
Stage 2, Representation Without Indifferences  Suppose there are at least two non-intersecting indifferent classes (otherwise, the representation is simply $\Gamma (A) = \tilde{C} (\emptyset) (A)$ and $u(x) = u(y)$ for all $x, y \in X$).

Definition 13. We call $X^*$ an Indifference-Removed Set if $\bigcup_{x \in X^*} [x]_\sim = X$ and $[x]_\sim \cap [y]_\sim = \emptyset$ for all $x, y \in X^*$ and $x \neq y$.

That is, we pick an alternative $x$, remove all alternatives that are sometimes simultaneously chosen with $x$, remove them from $X$, and repeat until we are done. Consider the choice correspondence restricted to $X^*$ and the set of all finite choice sets of at least size two that $X^*$ generates: $\mathcal{A}_A^*$. This choice correspondence, $C^*: \mathcal{A}_A^* \rightarrow \mathcal{A}_A^* N$, is essentially a choice function, since we artificially removed all indifferences. Therefore, it admits an AAT representation using Theorem 1.

Stage 3, Representation for $C$  Pick any Indifference-Removed Set $X^* \subseteq X$. Since $C$ satisfies Axioms B.1-B.4 and $C^*: \mathcal{A}_A^* \rightarrow \mathcal{A}_A^* N$ is essentially a choice function, it admits an AAT representation with $(u^*, \Gamma^*)$, where $u^*: X^* \rightarrow \mathbb{R}$ and $\Gamma^*: \mathcal{A}_A^* \rightarrow \mathcal{A}_A^*$.

Now we construct $u: X \rightarrow \mathbb{R}$ from $u^*$ in the following way. For all $x \in \hat{X}^+$ (the set of chosen things in $X^*$ under $C^*$),

$$u(x) := u^*(x).$$

For $\bar{x} \in X^* \setminus \hat{X}^*$, we ask whether $\bar{x} \in \hat{X}$ or not. If not, then we set

$$u(x) := u^*(x).$$

If instead $\bar{x} \in \hat{X}$ (but $\bar{x} \in X^* \setminus \hat{X}^*$), this issue arises because there exists some set in $\mathcal{A}_A$—but no set in $\mathcal{A}_A^*$—from which $\bar{x}$ is chosen without history (recall that by an argument analogous to Lemma 3, if $\bar{x} \in \hat{X}$, $\bar{x} \in \tilde{C} (\emptyset) (A)$ for some $\mathcal{A}$). This means that the AAT representation $(u^*, \Gamma^*)$ did not correctly capture $\bar{x}$’s preference ranking in $C$. We now solve this issue.

Since $\bar{x} \in X^* \setminus \hat{X}^*$, $\tilde{C} (\emptyset) (\{\bar{x}, z\}) = \{z\}$ for all $z \in X^* \setminus \{\bar{x}\}$. Since $\bar{x} \in \hat{X}$, there exists $A \in \mathcal{A}_A$ such that $\bar{x} \in \tilde{C} (\emptyset) (A)$. Therefore we can partition $X^* \setminus \{\bar{x}\}$ into two parts: $\bar{x}^+$ and $\bar{x}^-$:

$$\bar{x}^+ := \{z \in X^* \setminus \{\bar{x}\} : \tilde{C} (A) (\{\bar{x}, z\}) = \{z\}\},$$

$$\bar{x}^- := \{z \in X^* \setminus \{\bar{x}\} : \tilde{C} (A) (\{\bar{x}, z\}) = \{\bar{x}\}\},$$

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the things that, in anticipation, are better than and worse than $\tilde{x}$ respectively. Note that $\tilde{C}(A)(\{\tilde{x}, z\}) = \{\tilde{x}, z\}$ is impossible by construction of $X^*$.

Since $|X^* \setminus \hat{X}^*| \leq 1$ (if $\tilde{C}(\emptyset)(\{\tilde{x}, z\}) = \{z\}$ for all $z \in X^* \setminus \{\tilde{x}\}$, then $z \in \hat{X}^*$ for all $z \in X^* \setminus \{\tilde{x}\}$), $\tilde{x}$'s preference ranking is well-defined, and we want to define $u(\tilde{x})$ to reflect that. One way to do that is:

$$u(\tilde{x}) := \sup_{z \in \tilde{x}^-} u(z) + \epsilon$$

for some $\epsilon > 0$ and add $2\epsilon$ to every $u(z)$ such that $z \in \tilde{x}^+$,

$$u_{\text{new}}(z) := u_{\text{old}}(z) + 2\epsilon \text{ for all } z \in \tilde{x}^+,$$

(The reason we do this is to go around the possible issue of $\sup_{z \in \tilde{x}^-} u(z) = \inf_{z \in \tilde{x}^+} u(z)$.) Consequently, for all $x, y \in X^* \cap \hat{X}$,

$$u(x) > u(y) \text{ if and only if } y S x. \quad (B.1)$$

Finally, for all $x \in X^*$,

$$u(y) := u^*(x) \text{ for all } y \in [x]_-. \quad (B.2)$$

Note, consequently, that for all $x, y \in \hat{X}$, there exists $x^*, y^* \in X^* \cap \hat{X}$ such that $x^* W x$ and $y^* W y$ such that

$$u(x) > u(y) \text{ if and only if } y^* S x^*, \quad (B.3)$$

as a consequence of Equation B.1 and Equation B.2. Moreover, for $y \in X \setminus \hat{X}$ (if it exists),

$$u(x) > u(y) \text{ for all other } x \in X, \quad (B.4)$$

since this means $y \in X^* \setminus \hat{X}^*$ and therefore $u(y) < u(x)$ for all $x \in X^*$ by construction of $u^*$ for $C^*$.

For $\Gamma$, we simply construct it by

$$\Gamma(A) := \tilde{C}(\emptyset)(A).$$
Stage 4, Model Explains Choices This proof is similar to that for Theorem 1, with two main differences. First, we need to deal with timely predictions of indifferences, which is primarily done using Lemma 14. Second, we will demonstrate contradictions of Axiom B.3 and Axiom B.4 when $x$ is predicted (but not chosen) but $y$ is chosen (but not predicted), where either $x$ or $y$ (or both) is not in $X^*.$

Lemma 15. The constructed $(u, \Gamma)$ explains $C$ when restricting attention to the first choice sets (i.e., without history). That is, for all $A \in \mathcal{A},$

$$\tilde{C}(\emptyset)(A) = \arg \max_{x \in \tilde{\Gamma}(\emptyset)(A)} u(x).$$

Proof. This is a direct consequence of how $\Gamma$ was constructed, $\Gamma(A) := \{\tilde{C}(\emptyset)(A)\}$ and the fact that $\tilde{\Gamma}(\emptyset)(A) = \Gamma(A).$ Moreover for all $x, y \in \tilde{C}(\emptyset)(A),$ it is by definition $x \preceq y$ and therefore $u(x) = u(y).$ Hence $\tilde{C}(\emptyset)(A) = \arg \max_{x \in \tilde{\Gamma}(\emptyset)(A)} u(x).$

We now show that $(u, \Gamma)$ explains the entire $C.$

Take any sequence of choice sets $(A_n),$ and suppose for contradiction that, for some $i,$

$$C((A_n))_i \neq \arg \max_{x \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_i)} u(x)$$

Let $i$ be the set of all such $i$’s; they point to the set of all choice sets in $(A_n)$ from which the actual choice(s) is not the same as the model prediction(s). Denote the minimum element of $i$ by $i^* := \min i,$ which is well-defined. $i^* \neq 1$ due to Claim 4.

Consider $i^* \geq 2.$ Denote by

$$C^R = C((A_n))_{i^*},$$

the realized choice and

$$C^P = C_{model}((A_n))_{i^*} = \arg \max_{x \in \tilde{\Gamma}((A_1, \ldots, A_{i^*-1}))(A_{i^*})} u(x)$$

the model prediction.

If $C^R \neq C^P,$ there are three cases: (i) $C^R \cap C^P \neq \emptyset$ where $C^R \setminus C^P \neq \emptyset,$ (ii) $C^R \cap C^P \neq \emptyset$ where $C^P \setminus C^R \neq \emptyset,$ and (iii) $C^R \cap C^P = \emptyset.$ We now show that each admits a contradiction. Note too that since $C^P$ is a well-defined maximization problem, $C^P \neq \emptyset.$ Clearly, $C^R \neq \emptyset$ by definition.

Claim 6. $C^R \cap C^P \neq \emptyset$ where $C^R \setminus C^P \neq \emptyset$ is impossible.
Proof. Suppose for contradiction there exists \( y \in C^R \setminus C^p \) and \( x \in C^R \cap C^p \). Since \( y \) and \( x \) are both in \( C^R \), \( y W x \), therefore \( u(y) < u(x) \) is not possible due to construction of \( u \) (Equation B.2). Therefore if \( y \notin C^p \), it must be that \( y \notin \Gamma((A_1, \ldots, A_{i-1}))(A_{i^*}) \), which means (by definition of \( \Gamma \)) \( y \notin \Gamma(A_{i^*}) \) and \( y \notin C_{model}((A_1, \ldots, A_{i-1})) \). By Lemma 15, \( y \notin \Gamma(A_{i^*}) \) implies

\[
y \notin \tilde{C}(\emptyset)(A_{i^*}).
\]

And by the fact that \( A_{i^*} \) is the first choice set in \( (A_n) \) from which the model and actual choice disagrees, \( C_{model}((A_1, \ldots, A_{i-1})) \) implies

\[
y \notin C((A_1, \ldots, A_{i-1})).
\]

Finally, since \( y \) is neither the default choice of \( A_{i^*} \) nor chosen in the past, \( y \in C^R \) contradicts Axiom B.2 by an argument analogous to Lemma 2. \( \square \)

Claim 7. \( C^R \cap C^p \neq \emptyset \) where \( C^R \setminus C^p \neq \emptyset \) is impossible.

Proof. Suppose for contradiction there exists \( y \in C^p \setminus C^R \) and \( x \in C^R \cap C^p \). Therefore \( x, y \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_{i^*}) \) and \( u(x) = u(y) \). By \( u(x) = u(y) \), we note that \( x W y \) (due to construction of \( u \), Equation B.2). By \( y \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_{i^*}) \), either \( y \in \Gamma(A_{i^*}) \) or \( y \in C_{model}((A_1, \ldots, A_{i-1})) \) (or both). If \( y \in \Gamma(A_{i^*}) \), by Lemma 15 we have

\[
y \in \tilde{C}(\emptyset)(A_{i^*}).
\]

If instead \( y \in C_{model}((A_1, \ldots, A_{i-1})) \), then since \( A_{i^*} \) is the first choice set in \( (A_n) \) from which the model and actual choice disagrees,

\[
y \in C((A_1, \ldots, A_{i-1})).
\]

Therefore \( y \) is either the default choice of \( A_{i^*} \) or chosen in the past, and combined with the fact that \( y W x \) and \( x \in C^R \), we have a contradiction of Axiom B.3 by Lemma 13. \( \square \)

Claim 8. \( C^R \cap C^p = \emptyset \) is impossible.

Proof. Suppose for contradiction \( C^R \cap C^p = \emptyset \). Take \( y \in C^R \) and \( x \in C^p \). Suppose \( y \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_{i^*}) \), so the fact that \( y \notin C^p \) and \( x \in C^p \) means \( u(x) > u(y) \). If \( y \notin \hat{X} \), an immediate contradiction to \( y \in C^R \) is established. If \( x \notin \hat{X} \), then by Equation B.4 it is impossible that \( u(x) > u(y) \). Therefore \( x, y \in \hat{X} \). By construction of
$u : X \rightarrow \mathbb{R}$, and specifically by Equation B.3, there exists $x^* W x$ and $y^* W y$ (it is possible that $x^* = x$ or $y^* = y$ or both) such that

$$y^* S x^*.$$  \hspace{1cm} (B.5)

Now since $x \in C^R$, it must be that either $x \in \Gamma (A_i^*)$ and therefore by Lemma 15,

$$x \in \tilde{C} (\emptyset) (A_i^*), \hspace{1cm} (B.6)$$

or $x \in C_{\text{model}}((A_1, ..., A_{i-1}))$ and therefore by the fact that $A_i^*$ is the first disagreement between the model and choice in $(A_n)$,

$$x \in C ((A_1, ..., A_{i-1})). \hspace{1cm} (B.7)$$

Both these cases result in contradictions:

| Equation       | Violates Axiom |
|----------------|----------------|
| B.6            | B.3            |
| B.7            | B.4            |

Ruling out all three cases means we are left with $C^R = C^P$, and therefore

$$C ((A_n))_i = \arg\max_{x \in \tilde{\Gamma}((A_1, ..., A_{i-1}))(A_i)} u (x)$$

for all $(A_n)$ and $i$. 

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