Hawking radiation from the holographic screen

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Abstract

In this paper we generalize the Parikh-Wilczek scheme to a holographic screen in the framework of the ultraviolet self-complete quantum gravity. We calculate that the tunneling probabilities of the massless and massive particles depend on their energies of the particles and the mass of the holographic screen. The radiating temperature has not been the standard Hawking temperature. On the contrary, the quantum unitarity principle always remains.

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According to the no-hair theorem, a black hole is generated from a celestial body’s collapse can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum. In the classical Einstein gravitational theory, other information of the celestial body is confined inside the black hole. However, in 1974 Hawking\[1, 2] recovered that a black hole is not really black previously thought but radiates energy due to the quantum effects near the event horizon, and so that the black hole has a spectrum of a black body. The pure black body spectrum has no information and the total evaporation of the black hole give rise to the information loss paradox, which breaks the unitarity of quantum theory.

In 1999, Parikh and Wilczek \[3, 4]\ gave a semiclassical derivation of Hawking radiation as a tunneling process. They argued that with the continuous radiation of particles the energy of the black hole decreases and the contraction of radius of the horizon make the particles get across the classically forbidden trajectory, in other words, the potential barrier is created by the self-gravity of the system. They introduced a special coordinate system where the metric is not singular at the horizon, and then used the WKB method to compute the tunneling rate. In that article they worked out the emission rate of the massless and uncharged particles radiating from the Schwarzschild and R-N black holes, and calculated the Hawking temperature by comparing the exponential part of the emission rate with a Boltzmann factor when neglecting the quadratic term of the energy.

In recent years, the Parikh-Wilczek method has been generalized to the more complex cases. One case is that particles have mass or charge, or both, another case is that the spacetime is no longer so simple. For example, the noncommutativity idea \[5]\ originating from the ultraviolet divergence elimination in loop quantum theory\[6]\ has been introduced and a great many papers has been published\[7]. All the noncommutative spacetimes predict the existence of a minimal length of the order of the Planck scale. Unfortunately, many sorts of spacetimes constructed depend on external noncommutativity parameters.

In 2012, P. Nicolini and E. Spallucci\[8, 9]\ derived a static, neutral, non-rotating black hole metric whose extremal configuration radius is equal to the Planck length in order to avoid introducing an additional principle to justify the existence of a minimal length to provide a UV cut off. Below the sub-Planckian scale the interior of the black hole loses any physical meaning thus no singularity in the origin. The authors named the particular black hole as holographic screen. Moreover, they discussed the thermodynamics of the holographic screen and pointed out that the area law is corrected by a logarithmic term and a minimal holographic screen corresponding to the zero entropy existed.

Our aim is to generalize the Parikh-Wilczek method to the tunneling process of massless and uncharged particles from holographic screen. Our article is arranged as follows. In Sections 2, a brief introduction about holographic screen is given. In section 3, we work out the tunneling rate of the massless particles tunneling through the horizon of the holographic screen.
In addition, we derive the temperature of the holographic screen and study the extremal case when its mass are very large. In section 4, we discuss the massive particles tunneling through the horizon of the holographic screen. Finally in section 5, we present a conclusion.

1 A self-regular holographic screen

The mass density of a point particle with mass $M$ in spherical coordinates is proportional to the Delta function

$$\rho(r) = \frac{M}{4\pi r^2} \delta(r), \quad (1)$$

which can be generalized to a derivative of a smooth function $h(r)$ \[?\]

$$\rho(r) = \frac{M}{4\pi r^2} \frac{d}{dr} h(r) \equiv T^0_0 \quad (2)$$

to overcome the problem that at the sub-Planckian energy regime the Compton wave length of a particle is larger than a Schwarzschild black hole with the same mass. Taking the conservation equation $\nabla_{\mu} T^{\mu\nu} = 0$ into consideration the stress tensor takes the form

$$T^{\nu}_{\mu} = \text{diag} \left( -\rho, p_r, p_r + \frac{r}{2} \partial_r p_r, p_r + \frac{r}{2} \partial_r p_r \right) \quad (3)$$

with $p_r = -\rho$. By substituting eq.(3) into Einstein equation and assuming that the form of the left hand side of the Einstein equation remains unchanged we get the metric (gravitational constant $G = L_p^2$)

$$ds^2 = - \left( 1 - \frac{2L_p^2 m(r)}{r} \right) dt^2 + \left( 1 - \frac{2L_p^2 m(r)}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (4)$$

where the parameter $m(r)$ takes the form

$$m(r) = 4\pi \int_0^r dr' r'^2 \rho(r'). \quad (5)$$

The particular form of the function $h(r)$ must satisfy two rules\[?\]: i). Spacetime in the sub-Planckian regime has no physical meaning; ii). The characteristic scale of the system is provided by the spacetime itself of the scale of the Planck length, not imposed as a external parameter. The most natural and algebraically assumption can be written as

$$h(r) = \frac{r^2}{r^2 + L_p^2}, \quad (6)$$
where \( L_p \) is the Planckian length, and thus the smeared energy density \( \rho(r) \) is
\[
\rho(r) = \frac{ML_p^2}{2\pi r (r^2 + L_p^2)^2}.
\]

Now we have found the modified metric of holographic screen
\[
ds^2 = -\left(1 - \frac{2L_p^2m(r)}{r}\right) dt^2 + \left(1 - \frac{2L_p^2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2
\]
\[
= -\left(1 - \frac{2ML_p^2r}{r^2 + L_p^2}\right) dt^2 + \left(1 - \frac{2ML_p^2r}{r^2 + L_p^2}\right)^{-1} dr^2 + r^2 d\Omega^2.
\]

(8)

2 The Parikh-Wilczek Tunneling Mechanism and massless particles

Now we investigate the quantum tunneling through the holographic screen via Parikh-Wilczek Tunneling mechanism. To study the quantum tunneling through the holographic screen we consider the modified metric of the holographic screen
\[
ds^2 = -\left(1 - \frac{2ML_p^2r}{r^2 + L_p^2}\right) dt^2 + \left(1 - \frac{2ML_p^2r}{r^2 + L_p^2}\right)^{-1} dr^2 + r^2 d\Omega^2.
\]

(9)

The holographic screen admits two horizons provided \( M > L_p \)
\[
r_\pm = ML_p^2 \pm L_p \sqrt{M^2L_p^2 - 1}
\]

(10)
determined by
\[
1 - \frac{2ML_p^2r_\pm}{r^2 + L_p^2} = 0.
\]

(11)

At the beginning, it is necessary to choose the Painlevé coordinate that having no singularity at the horizon. The suitable choice can be written as [10]
\[
dt_s = dt \pm \sqrt{(r^2 + L_p^2)(r^2 + L_p^2 - 2ML_p^2r)} dr.
\]

(12)

where \( t \) is the Painlevé time. After the above transformation the Painlevé line element reads
\[
ds^2 = -\left(1 - \frac{2ML_p^2r}{r^2 + L_p^2}\right) dt^2 + 2\sqrt{\frac{2ML_p^2r}{r^2 + L_p^2}} dtdr + dr^2 + r^2 d\Omega^2.
\]

(13)
The radial null geodesics are calculated as

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2ML_p^2 r}{r^2 + L_p^2}};$$

(14)

where the plus (minus) sign corresponding to outgoing (ingoing) geodesics.

Now we consider a massless particle radiating from the holographic screen as a massless shell through its horizon. In the Parikh-Wilczek tunneling mechanism, the effect of self-gravitation the particles tunnel out of a holographic screen and its energy decreases because of the total energy conversation, which makes the mass of the holographic screen decline and the horizon of the black hole shrink smaller. Naturally, the metric of the holographic screen must be edited. Here we fix the total mass $M$ and denote the energy evaporating from the holographic screen as $\omega$ and use $M \rightarrow M - \omega$, the spacetime metric can be written as

$$ds^2 = -\left[1 - \frac{2(M - \omega)L_p^2 r}{r^2 + L_p^2}\right]dt^2 + 2\sqrt{\frac{2(M - \omega)L_p^2 r}{r^2 + L_p^2}}dtdr + dr^2 + r^2d\Omega^2;$$

(15)

and in the same way, the radial null geodesics are calculated as

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2(M - \omega)L_p^2 r}{r^2 + L_p^2}}.$$  

(16)

The characteristic length of the massless particle described as a spherically symmetric massless shell is infinitesimal near the horizon on account of the infinite blue-shift, hence the wavenumber inclines to infinity. In accordance with the WKB method, the imaginary part of the action that an s-wave massless particle traveling on the radial null geodesics tunnel across the outer horizon $r_+$ from the initial position $r_{in} = ML_p^2 + L_p\sqrt{M^2L_p^2 - 1}$ to the final position $r_{out} = (M - \omega)L_p^2 + L_p\sqrt{(M - \omega)^2L_p^2 - 1}$ can be exhibited as follows

$$\text{Im}\mathcal{I} = \text{Im} \int_{r_{in}}^{r_{out}} p_r dr = \text{Im} \int_{r_{in}}^{r_{out}} \int_{p_r}^{p_r^0} dp_r dr$$

(17)

with the Hamilton equation

$$dp_r' = \frac{dH}{r},$$

(18)

and $H = M - \omega'$. We can deform the contour of the $r$ integral around the pole at the horizon in order to ensure the positive energy solutions decay in time (choose the lower half $\omega'$ plane),

4
so the imaginary part of the action is worked out as

$$\text{Im}I = -\text{Im} \left[ \int_{0}^{\omega} d\omega' \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')L_p^2}{r^2+L_p^2}}} \right]$$

$$= \pi \int_{0}^{\omega} \frac{2(M-\omega)L_p^2 \left[ (M-\omega)L_p + \sqrt{(M-\omega)^2L_p^2-1} \right]}{\sqrt{(M-\omega)^2L_p^2-1}} d\omega'$$

$$= \pi \omega (2M-\omega)L_p^2 + \pi L_p M \sqrt{M^2L_p^2 - 1} - \pi L_p (M-\omega) \sqrt{(M-\omega)^2L_p^2 - 1}$$

$$+ \pi \ln \frac{ML_p + \sqrt{M^2L_p^2 - 1}}{(M-\omega)L_p + \sqrt{(M-\omega)^2L_p^2 - 1}} \tag{19}$$

The tunneling rate is

$$\Gamma \sim \exp(-2\text{Im}I). \tag{20}$$

Expanding \( \Gamma \) with respect \( \omega \)

$$\Gamma \approx e^{-4\pi \omega \left[ \frac{L_p^2 M^2}{(L_p^2 M^2 - 1)^{1/2}} \right] - 2\pi^2 \left[ -\frac{L_p^2}{(L_p^2 M^2 - 1)^{3/2}} + \frac{2L_p^2 M^2}{(L_p^2 M^2 - 1)^{1/2}} \right]}, \tag{21}$$

and comparing the first order of \( \omega \) with the Boltzmann factor \( e^{-\frac{\omega}{T}} \), we obtain the temperature of the holographic screen

$$T = \frac{\sqrt{M^2L_p^2 - 1}}{4\pi ML_p^2 \left( \sqrt{M^2L_p^2 - 1} + ML_p \right)}, \tag{22}$$

which demonstrates the temperature of the holographic screen is no longer equivalent to the Schwartzschild metric’s. We depict the the curves of the radiation temperature \( T \) versus the mass \( M \) in Figure 1 for comparisons. The entropy has a logarithmic correction,

$$S = \int_{L_p^{-1}}^{M} \frac{dM}{T} = 2\pi \left[ L_p^2 M^2 - 1 + L_p M \sqrt{L_p^2 M^2 - 1} + \ln \left( L_p M + \sqrt{L_p^2 M^2 - 1} \right) \right]. \tag{23}$$

The difference of the entropy after and before the radiation

$$\Delta S = S(M-\omega) - S(M)$$

$$= 2\pi \left[ (\omega - 2M)\omega L_p^2 + (M-\omega)L_p \sqrt{(M-\omega)^2L_p^2 - 1} - ML_p \sqrt{L_p^2 M^2 - 1} \right.$$\n
$$\left. + \ln \frac{(M-\omega)L_p + \sqrt{(M-\omega)^2L_p^2 - 1}}{ML_p + \sqrt{M^2L_p^2 - 1}} \right]. \tag{24}$$

We depict the the curves of the entropy \( S \) versus the mass \( M \) in Figure 2 for comparisons. \( M \gg L_p^{-1} \), the action can be simplified as
Figure 1: The radiation temperature $T$ versus the mass $M$. The black curve stands for the temperature of Schwartzschild black hole, and the red curve stands for the temperature of holographic screen. Here we use the Planck units.

Figure 2: The entropy $S$ versus the mass $M$. The black curve stands for the entropy of Schwartzschild black hole, and the red curve stands for the entropy of holographic screen. Here we use the Planck units.
\[
\text{Im}\mathcal{I} \approx 2\pi \omega (2M - \omega) L_p^2 + \pi \ln \frac{M}{M - \omega}.
\]

(25)

And the tunneling rate is

\[
\Gamma \sim e^{-8\pi ML_p^2\omega - 4\pi \omega^2 (2/M^2 - L_p^2)}.
\]

(26)

The temperature degenerates to the conventional results of the Schwarzschild black hole,

\[
T \approx \frac{1}{8\pi ML_p^2},
\]

(27)

while the entropy reduces to

\[
S \approx 4\pi L_p^2 M^2 + 2\pi \ln(2M_p M).
\]

(28)

3 Massive particles

In the section we discuss massive particles radiating from the holographic screen as a shell through its horizon. A massive particle because of the particles have which makes the mass of the holographic screen decline and the horizon of the black hole shrink smaller. A massive particle has no longer moved along a null geodesics but a time-like trajectory determined by the Lagrangian,

\[
\mathcal{L}(x^\mu, \tau) = \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau},
\]

(29)

where \(\tau\) stands for the proper time.

\[
d\tau^2 = - \left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right) dt^2 + 2\sqrt{\frac{2ML_p^2 r}{r^2 + L_p^2}} dtdr + dr^2 + r^2d\Omega^2.
\]

(30)

The fact that the Lagrangian does not contain canonical coordinate \(t\) implies in its corresponding Euler-Lagrange equation the corresponding canonical momentum of \(t\), also the particle’s energy is conserved \((\dot{t} = \frac{dt}{d\tau}, \dot{r} = \frac{dr}{d\tau})\),

\[
p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = m \left[- \left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right) \dot{t}_p + \sqrt{\frac{2ML_p^2 r}{r^2 + L_p^2}} \dot{r} \right] = -\omega,
\]

(31)
here the minus sign before $\omega$ results from the positivity of the energy of the tunneling particle. For a time-like trajectory, we also have

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1,$$

(32)

i.e.,

$$- \left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right)i^2 + 2\sqrt{\frac{2ML_p^2 r}{r^2 + L_p^2}}i\dot{r} + r^2 = -1,$$

(33)

and in consequence of Eq.(31) and (33) the particle’s trajectory along the radial direction is

$$\frac{dr}{dt} = \frac{-g_{00}\sqrt{g_{00}m^2 + \omega^2}}{\omega\sqrt{g_{01}^2 - g_{00} + g_{01}\sqrt{g_{00}m^2 + \omega^2}}}.$$

(34)

where

$$g_{00} = -\left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right), \quad g_{01} = \sqrt{\frac{2ML_p^2 r}{r^2 + L_p^2}}.$$

(35)

By repeating the same steps in Section 3, the imaginary part of the action that a massive particle along the radial direction tunnels across the outer horizon $r_+$ from $r_{in} = ML_p^2 + L_p\sqrt{M^2L_p^2 - 1}$ to $r_{out} = (M - \omega)L_p^2 + L_p\sqrt{(M - \omega)^2L_p^2 - 1}$ can be exhibited as follows

$$\text{Im}\mathcal{I} = \text{Im} \int_{r_{in}}^{r_{out}} \int_{m}^{\omega'} \frac{dH}{\dot{r}}dr$$

$$= -\text{Im} \int_{m}^{\omega} d\omega' \int_{r_{in}}^{\omega'} \left(\omega'\sqrt{g_{01}^2 - g_{00} + g_{01}\sqrt{g_{00}m^2 + \omega^2}}\right)dr$$

$$= -\text{Im} \int_{m}^{\omega} d\omega' \int_{r_{in}}^{\omega'} \frac{(r^2 + L_p^2)\left(\omega'\sqrt{g_{01}^2 - g_{00} + g_{01}\sqrt{g_{00}m^2 + \omega^2}}\right)}{(r - r_+)(r - r_-)\sqrt{g_{00}m^2 + \omega^2}}dr,$$

(36)

where the two horizons are

$$r_{\pm} = (M - \omega')L_p^2 \pm L_p\sqrt{(M - \omega')^2L_p^2 - 1}, \quad H = M - \omega'.$$

(37)

Clearly, there exists a pole at $r = (M - \omega')L_p^2 + L_p\sqrt{(M - \omega')^2L_p^2 - 1}$, and hence we can deform the contour of the $r$ integral around the pole at the horizon in order to ensure the positive energy solutions decay in time (choose the lower half $\omega'$ plane), so the imaginary part of the
The tunneling rate is

$$\Gamma \sim \exp(-2\text{Im}I).$$

Expanding $\Gamma$ with respect $\omega$

$$\Gamma \approx e^{-4\pi L^2 M \left( \frac{L^M}{\sqrt{L^2 M^2 - 1}} + 1 \right)} \omega$$

and comparing the first order of $\omega$ with the Boltzmann factor $e^{-\frac{\omega}{T}}$, we obtain the temperature of the holographic screen

$$T = \frac{\sqrt{M^2 L^2 - 1}}{4\pi M L^2 \left( \sqrt{M^2 L^2 - 1} + M L_p \right)};$$

which demonstrates the temperature of the holographic screen is no longer equivalent to the Schwartzschild metric’s.

## 4 Conclusion

In this paper, we have applied the the Parikh-Wilczek scheme on a holographic screen, analyzed the tunneling process of the massless and massive particles and derived their tunneling rates. We have pointed out that the radiation spectra are not purely thermal, and the temperatures of the holographic screen are not equal to the standard Hawking temperature of the Schwartzschild black hole. Moreover, we have also noticed that the changes of the entropies $\Delta S = -2\text{Im}I$ thus the unitarity principle remains in the tunneling process of the holographic screen model.

In our future work the approach in this article will be developed and massive charged particles emitting from the holographic screen will be studied. It is expected that several interesting results will be gained.
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