From transversality condition to representation formalism of vector electromagnetic beams

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Abstract

A representation theory of finite electromagnetic beams in free space is formulated by introducing a mapping matrix that maps a 2-component Jones vector to a 3-component electric-field vector. It is shown that there exists one degree of freedom for the mapping matrix when the transversality condition is taken into account. This degree of freedom can be described by the azimuthal angle of a fixed unit vector with respect to the wave vector and is interpreted as mediating the interaction between the photon’s momentum and spin. For a finite beam, the interaction angle can be represented by the angle that a fixed unit vector makes with the beam’s propagation direction. The impact of the interaction angle on the properties of a beam is investigated in the first-order approximation under the paraxial condition. A transverse effect is found that a beam of elliptically-polarized angular spectrum is displaced from the center in the direction that is perpendicular to the plane formed by the fixed unit vector and the propagation direction. It is also shown with a beam of linearly-polarized angular spectrum that the transverse component is not uniformly polarized. The local polarization state is dependent on the interaction angle and is changing on propagation.

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I. INTRODUCTION

The vectorial feature or the polarization property of an electromagnetic beam, which concerns its angular momentum [1, 2, 3], intensity distribution [4, 5, 6], and diffraction characteristics [7, 8, 9], has become important in diverse areas of applications, including optical tweezers and spanner [10, 11, 12], optical data storage, optical trapping and manipulation [13, 14], and dark-field imaging [15].

In the last decade, more and more attention [16, 17, 18, 19, 20, 21, 22, 23, 24, 25] has been paid to the Imbert-Fedorov effect [26, 27], which means a transverse displacement of a reflected [18, 26, 27] or a transmitted [20, 25, 28] beam taking place at a dielectric interface. Recently, there was a controversy [21, 22] over the physical properties of the incident beams that were used to discuss the characteristics of the Imbert-Fedorov effect. Bliokh and Bliokh [23] made a detailed comparison of the incident beam in Ref. [21] with that in Ref. [22]. They found that the physical properties of the incident beam in Ref. [21] depend on the “incidence angle” and concluded that that kind of beam “is meaningless from the physical viewpoint”. Such a controversy concerns in fact the representation formalism of a finite beam.

The representation formalism and propagation characteristics of an electromagnetic beam in free space has drawn much attention [4, 5, 7, 29, 30, 31, 32, 33, 34, 35, 36, 37] after the advent of masers and lasers [38, 39, 40]. It was shown [29] that a linearly polarized solution of a beam is not compatible with the Maxwell equations, because it does not satisfy the transversality condition. It was also shown [4] that the polarization state is in general not a global property of a beam. Rather it is local and changes on propagation. It was argued by approaches of either vector potentials [5, 30] or Whittaker potentials [4, 31] that the representation of beams with a couple of fixed unit vectors, one in the transverse direction and the other in the longitudinal direction, would yield [31] a complete set of beam modes in terms of which an arbitrary electromagnetic beam might be expressed.

The purpose of this paper is to develop the representation formalism of a free-space electromagnetic beam from the transversality condition. In Section II we first introduce a mapping matrix (MM) that maps a 2-component Jones vector [41] to a 3-component electric-field vector associated with a plane wave. Then we show that the transversality condition allows us to have one degree of freedom to choose the MM. The degree of freedom
can be represented by the azimuthal angle $\Theta$ of a fixed unit vector $\mathbf{I}$ with respect to the wave vector. After proving a theorem on the property of the MM, we attempt to interpret the degree of freedom as the interaction angle between the photon’s momentum and spin. In Section III we put forward the general representation formalism for the field vector of a finite beam, letting the fixed unit vector $\mathbf{I}$ lie in the plane $zox$ and make an angle $\theta_I$ with the $z$-axis, the beam’s propagation direction. When the normalized Jones vector of the angular spectrum is independent of the wave vector, we obtain a MM for the field vector of a beam. The impact of the interaction angle $\theta_I$ on the properties of a beam is investigated in Section IV. After calculating the field vector in the first-order approximation under the paraxial condition and establishing the relation with the incident beams in Refs. [21] and [22], we find a transverse effect that a beam of elliptically polarized angular spectrum is displaced from the center in the direction that is perpendicular to the plane formed by the unit vector $\mathbf{I}$ and the propagation direction. The origin of this effect is also discussed. At last we show with a beam of linearly polarized angular spectrum that the local polarization state of the transverse component depends on the angle $\theta_I$ and changes on propagation. Conclusions and remarks are given in Section V.

II. THE DEGREE OF FREEDOM OF THE MM AND ITS PHYSICAL INTERPRETATION

A. Mapping matrix and the existence of one degree of freedom

In a source-free position space, the electric-field vector $\mathbf{F}(\mathbf{x})$ of a monochromatic finite electromagnetic beam satisfies the following vector Helmholtz equation,

$$\nabla^2 \mathbf{F}(\mathbf{x}) + k^2 \mathbf{F}(\mathbf{x}) = 0,$$

subject to the transversality condition,

$$\nabla \cdot \mathbf{F}(\mathbf{x}) = 0,$$

where $k$ is the wave number. The plane-wave angular-spectrum expression of the field vector, varying according to $\exp(-i\omega t)$ with the time, can be written as

$$\mathbf{F}(\mathbf{x}) = \frac{1}{2\pi} \int \int_{k_x^2 + k_y^2 \leq k^2} f(k_x, k_y) \exp(i\mathbf{k} \cdot \mathbf{x}) dk_x dk_y,$$
\[ k = k_x e_x + k_y e_y + k_z e_z = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \]

is the wave vector satisfying \[ k_x^2 + k_y^2 + k_z^2 = k^2, \]

\[ f(k_x, k_y) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \]

is the complex amplitude vector of the angular spectrum, \( f_j (j = x, y, z) \) is in general a complex function of \( k_x \) and \( k_y \), and \( e_j \) is the unit vector of the \( j \)-axis. Since \( f(k_x, k_y) \exp(ik_z z) = h(k_x, k_y; z) \) is the Fourier transformation of the field vector at a plane \( z = \text{constant} \),

\[
h(k_x, k_y; z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x) \exp[-i(k_x x + k_y y)] \, dx \, dy,
\]

we sometimes refer to \( f(k_x, k_y) \) as the field vector of the beam in wave-vector space, or in momentum space. Correspondingly, \( F(x) \) is referred to as the field vector in position space. Due to this transformation relation, we will be mainly concerned with \( f \) in this section.

Let us consider the field vector \( f \) of a plane wave associated with a particular wave vector \( k \). According to the transversality condition (2), \( f \) has only two mutually orthogonal polarization states, each of them being orthogonal to \( k \). Denoting respectively by \( p \) and \( s \) the two orthogonal linearly-polarized states, we can decompose \( f \) as

\[
f = f_p + f_s = f_p \tilde{p} + f_s \tilde{s},
\]

where \( f_p \) and \( f_s \) are respectively the \( p \)- and \( s \)-polarized complex amplitudes constituting the Jones vector

\[
\tilde{f} = \begin{pmatrix} f_p \\ f_s \end{pmatrix} = f_p \tilde{p} + f_s \tilde{s},
\]

\( \tilde{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \tilde{s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) form the basis of the Jones-vector space, \( p \) and \( s \) are respectively the \( p \)- and \( s \)-polarized unit vectors. To be clear, we assume in this paper that both \( p \) and \( s \) are real unit vectors. We can always do this as one may see from Eq. (5). They satisfy

\[
\begin{cases}
  \mathbf{p} \cdot \mathbf{k} = 0, \\
  \mathbf{s} \cdot \mathbf{k} = 0,
\end{cases}
\]
as well as
\[ \begin{align*}
\mathbf{p} \cdot \mathbf{p} &= 1, \\
\mathbf{s} \cdot \mathbf{s} &= 1, \\
\mathbf{p} \cdot \mathbf{s} &= 0.
\end{align*} \tag{8} \]

Eq. (5) means that the 3-component field vector is an element of a 2D space, rather than an element of a 3D space. Since the Jones-vector space is a 2D space, Eq. (5) defines in fact a mapping from the Jones-vector space to the 3-component 2D space. In order to describe this mapping, we change the form of Eq. (5) into
\[ \mathbf{f} = \mathbf{m} \tilde{\mathbf{f}}, \tag{9} \]
where
\[ \mathbf{m} = \begin{pmatrix} p_x & s_x \\ p_y & s_y \\ p_z & s_z \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{s} \end{pmatrix} \tag{10} \]
is the $3 \times 2$ MM, the column vectors of which are $\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$, respectively.

Any MM, the column vectors of which satisfy Eqs. (7) and (8), guarantees that the field vector $\mathbf{f}$ given by Eq. (9) satisfies the transversality condition whatever the Jones vector $\tilde{\mathbf{f}}$ may be. But there are only five equations for the six unknown elements of the MM. This shows that when the transversality condition is taken into account, we still have one degree of freedom to choose the MM. That is to say, the transversality condition itself is not sufficient to determine an electromagnetic wave from a given Jones vector.

**B. Description of the degree of freedom and its unique role**

It is well known \[4, 16, 17, 21, 31, 38\] that if the unit vectors $\mathbf{p}$ and $\mathbf{s}$ are defined from the wave vector $\mathbf{k}$ in terms of an arbitrary fixed real unit vector $\mathbf{I}$ as
\[ \begin{align*}
\mathbf{p} &= \mathbf{s} \times \frac{\mathbf{k}}{\lvert \mathbf{k} \rvert}, \\
\mathbf{s} &= \frac{\mathbf{k} \times \mathbf{I}}{\lvert \mathbf{k} \times \mathbf{I} \rvert},
\end{align*} \tag{11} \]
then they satisfy Eqs. (7) and (8). One might conclude that the degree of freedom is the unit vector $\mathbf{I}$. This is obviously not true, because we need two independent variables to
determine the orientation of a unit vector in a 3D space. But we can really use the real unit vector $\mathbf{I}$ to denote the degree of freedom somehow. To show this, let us look at a particular wave vector $\mathbf{k}$ and the unit vectors $\mathbf{p}$ and $\mathbf{s}$ that are defined by Eqs. (11) in terms of a fixed unit vector $\mathbf{I}$ as is schematically depicted in Fig. 1. It can be seen from this figure that the rotation of $\mathbf{I}$ around $\mathbf{k}$ by changing the azimuthal angle $\Theta$ of $\mathbf{I}$ with respect to $\mathbf{k}$ alters the orientation of $\mathbf{p}$ and $\mathbf{s}$. On the other hand, the rotation of $\mathbf{I}$ around $\mathbf{s}$ by changing the polar angle $\Phi$ of $\mathbf{I}$ with respect to $\mathbf{k}$ does not alter the orientation of $\mathbf{p}$ and $\mathbf{s}$. So it is the azimuthal angle $\Theta$ that plays the role of the degree of freedom and uniquely determines the MM. For a particular wave vector, different values of $\Theta$ represent different mapping matrices, and vice versa. The rotation of $\mathbf{I}$ around $\mathbf{s}$ forms a group $G(\Theta)$ that corresponds to one single value of the degree of freedom.

![FIG. 1: (Color online) The degree of freedom of the MM is denoted by the azimuthal angle $\Theta$ of $\mathbf{I}$ with respect to $\mathbf{k}$.](image)

Based on the above description of the MM’s degree of freedom, it is easy to prove the following theorem:

**Theorem:** A Jones vector can not be mapped to the same field vector by two different mapping matrices.

Consider an arbitrary Jones vector $\tilde{\mathbf{f}}$ that is mapped to a $3$-component field vector $\mathbf{f}$ by $m(\Theta)$ as is described by Eqs. (9)-(11). According to Eqs. (8), $m(\Theta)$ has the following property,

$$m^T(\Theta)m(\Theta) = 1,$$

(12)

where superscript $T$ stands for the transpose. Suppose that this field vector can also be mapped to from the same Jones vector by another MM, say, $m(\Theta')$. That is to say, suppose
that this field vector can also be given by
\[ f = m(\Theta') \tilde{f}, \tag{13} \]
where the two column vectors of \( m(\Theta') = (p(\Theta') \ s(\Theta')) \) are defined by
\[
\begin{align*}
p(\Theta') &= s(\Theta') \times \frac{k}{k}, \\
s(\Theta') &= \frac{k \times I(\Theta')}{|k \times I(\Theta')|}.
\end{align*}
\]
It follows from Eqs. (12), (13), and (9) that
\[ \tilde{f} = m^T(\Theta') f = m^T(\Theta') m(\Theta) \tilde{f}. \tag{14} \]
Because the definition of the MM is independent of the Jones vector, Eq. (14) requires
\[ m^T(\Theta') m(\Theta) = 1, \]
which is obviously impossible unless \( \Theta' = \Theta \). This proves the Theorem.

Now that one Jones vector can be mapped to an infinite number of field vectors due to
the MM’s degree of freedom, the whole 2D space of the 3-component field vector associated
with a particular wave vector is divided into an infinite number of subspaces. Each subspace
is mapped to from the Jones-vector space by a specific MM and is therefore isomorphic to
the Jones-vector space. The two column unit vectors of the MM are mapped to from the
basis of the Jones-vector space,
\[
\begin{align*}
p &= m\tilde{p}, \\
s &= m\tilde{s}.
\end{align*}
\tag{15}
\]
They act as the basis of the subspace. In view of this, \( f_p \) and \( f_s \) can be regarded as the
coordinates of the Jones vector \( (6) \) in the Jones-vector space and as the coordinates of the
field vector \( (5) \) in the field-vector subspace as well.

C. Interpretations

From the physical point of view, it is probable that the MM expresses somehow the
characteristic of the interaction between the photon’s momentum and spin. In order to
appreciate the role of \( I \) in such an interaction, let us look at the cross product \( k \times I \) in the
definition of \( s \). It is easily proven that
\[ k \times I = -\frac{i}{\hbar}(k \cdot \hat{S})I = \frac{i}{\hbar}(I \cdot \hat{S})k, \tag{16} \]
where $\hat{S} = \hat{S}_x e_x + \hat{S}_y e_y + \hat{S}_z e_z$, and

$$\hat{S}_x = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_y = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{S}_z = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  (17)

$\hat{S}_j$’s are Hermitian matrices satisfying the canonical commutation relations of the angular momentum,

$$[\hat{S}_i, \hat{S}_j] = \hat{S}_i \hat{S}_j - \hat{S}_j \hat{S}_i = i\hbar \epsilon_{ijk} \hat{S}_k,$$

and can be regarded as representing the three cartesian components of the photon’s spin operator $\hat{S}$ [42] acting on the 3-component field vector, where $\epsilon_{ijk}$ is the Levi-Civita symbol. Since the wave vector represents the photon’s momentum $\hbar k$, we may think that Eq. (16) expresses the interaction of the photon’s momentum with the spin, in which the unit vector $I$ plays a vital role. Due to the following property,

$$k \times (I + c_1 k) = k \times I,$$  (18)

where $c$ is an arbitrary constant, it is only the azimuthal angle $\Theta$ of $I$ with respect to $k$ that plays the role in the interaction. So we may conclude that the interaction of the photon’s momentum with the spin is mediated by the degree of freedom of the MM. In the following, we will refer to $\Theta$ as the interaction angle of the momentum with the spin.

III. REPRESENTATION FORMALISM FOR A FINITE BEAM

We have identified one degree of freedom of the MM and interpreted it as the interaction angle of the momentum of the photon with the spin. For a single plane wave, the impact of the interaction angle is not evident. So in the following we will investigate the impact of the interaction angle on the properties of finite beams.

A finite beam consists of an infinite number of plane waves. Each of the plane wave is different from one another in the direction of the wave vector. It seems impossible to realize physically the same interaction angle for all of the wave vectors. We assume that the fixed unit vector $I$ for a finite beam lies in the plane $zox$ and makes an angle $\theta_I$ with the $z$-axis, the propagation direction,

$$I(\theta_I) = e_z \cos \theta_I + e_x \sin \theta_I,$$  (19)
so that the MM takes the form of

\[
m = \frac{1}{k|k \times I|} \begin{pmatrix}
  (k_y^2 + k_z^2) \sin \theta_I - k_z k_x \cos \theta_I & kk_y \cos \theta_I \\
  -k_y (k_z \cos \theta_I + k_x \sin \theta_I) & k(k_z \sin \theta_I - k_x \cos \theta_I) \\
  (k_x^2 + k_y^2) \cos \theta_I - k_z k_x \sin \theta_I & -kk_y \sin \theta_I 
\end{pmatrix},
\]

(20)

where

\[
|k \times I| = \left| k^2 - (k_z \cos \theta_I + k_x \sin \theta_I)^2 \right|^{1/2}.
\]

(21)

In this case, the angle \( \theta_I \) plays the role of the interaction angle between the momentum and the spin. It should be noted that different values of \( \theta_I \) correspond to the same interaction angle for each of those wave vectors that also lie in the \( zox \) plane as we have shown before in Fig. [1]. Such a feature of \( \theta_I \) will produce a very interesting transverse effect as will be shown in Section IV.

The electric-field vector of a finite beam is given by Eq. (3). The amplitude vector \( f \) of the angular spectrum is factorized by Eq. (9) into the MM (20) and the Jones vector,

\[
\tilde{f} = \tilde{\alpha} f \equiv \begin{pmatrix} \alpha_p \\ \alpha_s \end{pmatrix} f,
\]

(22)

where \( \tilde{\alpha} = \begin{pmatrix} \alpha_p \\ \alpha_s \end{pmatrix} \) is the normalized Jones vector, \( \alpha_p \) and \( \alpha_s \) are complex numbers satisfying the normalization condition,

\[
|\alpha_p|^2 + |\alpha_s|^2 = 1,
\]

(23)

and \( f \) is referred to as the amplitude scalar of the angular spectrum. If the normalized Jones vector of the angular spectrum is independent of the wave vector, Eq. (3) can take a factorized form

\[
\mathbf{F}(x) = M(x) \tilde{\alpha},
\]

(24)

where

\[
M(x) = \frac{1}{2\pi} \int \int m f \exp(i k \cdot x) dk_x dk_y
\]

(25)

is the MM for the beam, and the integration limit \( k_x^2 + k_y^2 \leq k^2 \) is omitted for brevity.

In a circular cylindrical system with the \( z \)-axis being the symmetry axis, the integral (25) is changed into

\[
M(x) = \frac{1}{2\pi} \int_0^k \int_0^{2\pi} m f \exp(i k \cdot x) k \rho dk_\rho d\varphi,
\]

(26)
where \( \mathbf{x} = \mathbf{r} + \mathbf{z} \mathbf{e}_z, \mathbf{r} = r \mathbf{e}_r = \mathbf{e}_z r \cos \phi + \mathbf{e}_y r \sin \phi, \mathbf{e}_r \) and \( \mathbf{e}_\phi \) are respectively the unit vectors in the radial and azimuthal directions in position space; correspondingly, \( \mathbf{k} = \mathbf{k}_\rho + \mathbf{k}_z \mathbf{e}_z, \mathbf{k}_\rho = k_\rho \mathbf{e}_\rho = k_x \mathbf{e}_x + k_y \mathbf{e}_y, k_x = k_\rho \cos \phi, k_y = k_\rho \sin \phi, k_z = (k^2 - k_\rho^2)^{1/2} \), \( \mathbf{e}_\rho \) and \( \mathbf{e}_\phi \) are respectively the unit vectors in the radial and azimuthal directions in wave-vector space.

Since we are mainly concerned with the impact of the interaction angle on the properties of a beam, we will consider only, as an example, the following amplitude scalar of Gaussian distribution,

\[
f(k_\rho) = f_0 \exp \left(-\frac{w_0^2 k_\rho^2}{2}ight),
\]

which is \( \phi \)-independent, where \( w_0 \) denotes the transverse dimension of the beam at waist, and \( f_0 \) is a constant coefficient. For the sake of simplicity, we will consider only the paraxial case in which the divergence angle \( \Delta \theta = \frac{1}{kw_0} \) of the beam satisfies

\[
\Delta \theta \ll \frac{1}{2\pi}.
\]

**IV. IMPACT OF THE MOMENTUM-SPIN INTERACTION ON THE PROPERTIES OF A FINITE BEAM**

**A. Field vector distribution in the first-order approximation**

When \( \theta_I = 0 \), we arrive at the cylindrical vector beam \[37\]. Furthermore we have \( \mathbf{I}(\theta_I + \pi) = -\mathbf{I}(\theta_I) \). So for a reason that will be clear in the following, we will consider only the case in which \( \theta_I \) satisfies the condition

\[
\Delta \theta \ll |\theta_I| \leq \frac{\pi}{2}.
\]

Eq. \[27\] means that \( f(k_\rho) \) is appreciable only in a small region in which \( \frac{k_\rho}{k} \leq \Delta \theta \ll \frac{1}{2\pi} \). When integral \[26\] is taken into account, both \( \frac{k_z}{k} \) and \( \frac{k_\rho}{k} \) in the elements of the MM can be regarded as small numbers in comparison with unity. We rewrite Eq. \[21\] as

\[
|\mathbf{k} \times \mathbf{I}| = |k_z \sin \theta_I| \left( 1 - 2 \frac{k_x}{k_z} \cot \theta_I + \frac{k_x^2 \cos^2 \theta_I + k_y^2}{k_z^2 \sin^2 \theta_I} \right)^{1/2}.
\]

The condition \[29\] guarantees that the second and the third parts in the second factor are the first-order and the second-order terms in comparison with the first part. As a result, in
the zeroth-order approximation, we have for the MM,

\[ m \approx m_0 = \text{sgn}(\theta_I) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \]  

(30)

where sgn is the sign function. Substituting it into Eqs. (24) and (26) produces the field vector,

\[ \mathbf{F}(\mathbf{x}) \approx \mathbf{F}_T^{(0)}(\mathbf{x}) = m_0 \tilde{\alpha} \mathbf{F}_T^{(0)}, \]  

(31)

where

\[ F_T^{(0)} = \frac{1}{2\pi} \int_0^k f(k_\rho)k_\rho dk_\rho \int_0^{2\pi} d\varphi \exp(i\mathbf{k} \cdot \mathbf{x}) \]  

(32)

is the complex amplitude of the beam’s field vector. This represents a uniformly polarized beam, the polarization state being the same as that of the angular spectrum, \( \tilde{\alpha} \). Substituting Eq. (27) into Eq. (32) and making use of the following expansion,

\[ \exp(i\rho \cos \psi) = \sum_{m=-\infty}^{\infty} i^m J_m(\rho) \exp(im\psi), \]  

(33)

where \( J_m \)'s are the Bessel functions of the first kind, we find

\[ F_T^{(0)} = f_0 \int_0^k \exp \left( -\frac{w_0^2 k^2}{2k_\rho} \right) \exp(izk_z)J_0(rk_\rho)k_\rho dk_\rho. \]

Extending the upper integration limit to \( \infty \) and making the paraxial approximation \( k_z \approx k - \frac{k^2}{2k} \) in the exponential factor \( \exp(izk_z) \), we get

\[ F_T^{(0)} = \frac{f_0}{w^2} \exp \left( -\frac{r^2}{2w^2} \right) \exp(ikz) \equiv F_G(r, z), \]  

(34)

where

\[ w^2 = w_0^2 + iz/k. \]  

(35)

In the first-order approximation, we have

\[ m \approx m_0 + m_1, \]  

(36)

where

\[ m_1 = \text{sgn}(\theta_I) \begin{pmatrix} 0 & \frac{k_y}{k} \cot \theta_I \\ -\frac{k_y}{k} \cot \theta_I & 0 \\ -\frac{k_z}{k} & -\frac{k_y}{k} \end{pmatrix} \]  

(37)
is the first-order correction to the MM. $m_1$ provides not only with the longitudinal component but also with the first-order correction to the transverse component. Substituting Eqs. (36) and (27) into Eqs. (24) and (26), we find, following a similar procedure,

$$F(x) \approx F_T^{(0)}(x) + F_T^{(1)}(x) + F_L^{(1)}(x),$$

where

$$F_T^{(1)}(x) = i \text{sgn}(\theta_I)(\alpha_s e_x - \alpha_p e_y) \frac{y}{k w^2} F_G \cot \theta_I$$

is the first-order correction to the transverse component which is also uniformly polarized, and

$$F_L^{(1)}(x) = -i \text{sgn}(\theta_I) e_z \frac{\alpha_p x + \alpha_s y}{k w^2} F_G \equiv F_L^{(1)} e_z$$

is the longitudinal component. It is noticed that the polarization state of the first-order transverse term is different from that of the zeroth-order transverse term. As a matter of fact, they are orthogonal to each other. Combining Eqs. (31) and (39) together, we get the transverse component of the electric-field vector,

$$F_T(x) = \text{sgn}(\theta_I) \left[ (\alpha_p + i \frac{\alpha_s y}{k w^2} \cot \theta_I) e_x + (\alpha_s - i \frac{\alpha_p y}{k w^2} \cot \theta_I) e_y \right] F_G. \quad (41)$$

This shows that the transverse component of the beam is not in general uniformly polarized. The local polarization state is dependent on the value of $\theta_I$. It should be pointed out that when $|\theta_I| = \frac{\pi}{2}$, the first-order correction to the transverse component vanishes, $F_T^{(1)}(x) = 0$. In this case the zeroth-order term of the transverse component is the field vector of the well-known uniformly-polarized fundamental Gaussian beam. It satisfies, together with the first-order longitudinal component (40), the approximate transversality condition [29],

$$\nabla_T \cdot F_T^{(0)} + i k F_L^{(1)} = 0, \quad (42)$$

where $\nabla_T = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y}$. Defining $m_c = \frac{\alpha_s}{\alpha_p}$ and choosing $\alpha_p = -\frac{1}{(1 + |m_c|^2)^{1/2}}$ by use of the normalization condition (23), we find for the unit vector of the amplitude vector of the angular spectrum,

$$\frac{f}{|f|} = -\text{sgn}(\theta_I) \frac{(1 + m_c k y \cot \theta_I) e_x + (m_c - \frac{k_p}{k} \cot \theta_I) e_y - (\frac{k_p}{k} + m_c \frac{k_y}{k}) e_z}{(1 + |m_c|^2)^{1/2}}. \quad (43)$$
When $\theta_I = -\theta$ with $\theta > 0$, Eq. (43) turns into
\[
\frac{f}{|f|} = \frac{(1 - m_c \frac{k_y}{k} \cot \theta) e_x + (m_c + \frac{k_y}{k} \cot \theta) e_y - (\frac{k_y}{k} + m_c \frac{k_y}{k}) e_z}{(1 + |m_c|^2)^{1/2}},
\]
which is the same as the equation (20) of Ref. [23] if $\theta$ is interpreted as the incidence angle. This is exactly implied in Ref. [21] by the unit vector $n$ that plays the role of $I$ and is normal to the interface. In other words, the incident beam in Ref. [21] is a special case in which the interaction angle happens to be the minus incidence angle. This explains why the physical properties of the incident beam in Ref. [21] depend on the “incidence angle” [23].

Furthermore, when $\theta = \frac{\pi}{2}$, Eq. (44) reduces to
\[
\frac{f}{|f|} = \frac{e_x + m_c e_y - (\frac{k_y}{k} + m_c \frac{k_y}{k}) e_z}{(1 + |m_c|^2)^{1/2}}.
\]
Upon noticing that $m_c = \frac{k_y}{k}$ in this case, Eq. (45) is exactly the same as the equation (22) of Ref. [23]. This shows that the incident beam in Ref. [22] is nothing but the fundamental Gaussian beam, as long as the first-order longitudinal component is taken into account.

B. A transverse effect

Now we are ready to discuss the transverse effect. When $|\theta_I|$ is not equal to $\frac{\pi}{2}$, the first-order term of the transverse component does not vanish. Since this term is not axisymmetric as is clearly shown by Eq. (39), its interference with the zeroth-order term renders the intensity distribution deformed from the axisymmetry. To simplify discussions, we consider an angular spectrum of circular polarization,
\[
\tilde{\alpha}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}.
\]
Substituting it into Eq. (41) and noticing Eq. (35), we find
\[
F_T = \frac{\text{sgn}(\theta_I)}{\sqrt{2}} (e_x \pm ie_y) \frac{k w_0^2 y \cot \theta_I + iz}{kw^2} F_G;
\]
which shows that at a plane $z = \text{constant}$, the intensity of the transverse field is minimum on line
\[
y = y_n = \pm k w_0^2 \tan \theta_I.
\]
At the beam-waist plane $z = 0$, the minimum intensity is actually equal to zero. A direct consequence of the minimum intensity at a finite distance from the center is the following transverse effect: the barycenter of a beam of elliptically polarized angular spectrum is displaced from the center in the transverse $y$-direction; the displacement is dependent on the value of $\theta_I$ and on the polarization ellipticity of the angular spectrum.

To show this, let us define the $y$-position $y_b$ of the beam’s barycenter as the expectation of the $y$-coordinate of the beam,

$$y_b = \langle y \rangle = \frac{\int \int F^\dagger y F dx dy}{\int \int F^\dagger F dx dy},$$

(48)

where superscript $\dagger$ stands for the conjugate transpose. According to Eqs. (3) and (1), we have

$$\int \int F^\dagger y F dx dy = \int \int (i f^\dagger \frac{\partial f}{\partial k_y} + z \frac{k_y}{k_z} f^\dagger f) dk_x dk_y$$

and

$$\int \int F^\dagger F dx dy = \int \int f^\dagger f dk_x dk_y.$$  

Eqs. (9) and (22) tell us that

$$f^\dagger f = |f|^2$$

and

$$f^\dagger \frac{\partial f}{\partial k_y} = f^* \frac{\partial f}{\partial k_y} + \hat{\alpha}^T \frac{\partial m}{\partial k_y} \hat{\alpha} |f|^2.$$  

Substituting all these into Eq. (48) and noticing that $f$ is an even function of $k_y$, we obtain

$$y_b = \frac{i \int \int \hat{\alpha}^T \frac{\partial m}{\partial k_y} \hat{\alpha} |f|^2 dk_x dk_y}{\int \int |f|^2 dk_x dk_y},$$

(49)

which shows that the $y$-position of the beam’s barycenter is independent of $z$ as long as the amplitude scalar of the angular spectrum is an even function of $k_y$. Substituting Eq. (36) and noticing that $f$ is an even function of both $k_x$ and $k_y$, we get

$$y_b = \frac{i \frac{\alpha_p^* \alpha_s - \alpha_p \alpha_s^*}{k \tan \theta_I}}{k \tan \theta_I} = - \frac{\sigma_z}{k \tan \theta_I},$$

(50)

where $\sigma_z = -i(\alpha_p^* \alpha_s - \alpha_p \alpha_s^*)$ is the polarization ellipticity of the angular spectrum. This is the general expression for the transverse effect. It is inferred from this expression that

1. the beam is displaced a distance $y_b$ from the center when $|\theta_I|$ is not equal to $\frac{\pi}{2}$, because the $y$-position of the beam’s barycenter does not change on propagation;}
(2) the opposite ellipticity $\sigma_z$ corresponds to the opposite displacement for a given $\theta_I$;
(3) the opposite angle $\theta_I$ corresponds to the opposite displacement for a given $\sigma_z$.

It deserves mentioning that Schilling \[28\] obtained a similar result more than 40 years ago by assuming incidentally that the unit vector $\mathbf{I}$ is normal to the interface between two different media \[24\]. Recently, Bliokh and Bliokh \[23\] rediscovered Schilling’s result in a comparison of the incident beams in Refs. \[21\] and \[22\].

Substituting Eq. (46) into Eq. (50) yields

$$y_b = \mp \frac{1}{k \tan \theta_I} = \mp \frac{\lambda}{2\pi \tan \theta_I},$$

where $\lambda$ is the wavelength. This $y$-position of the beam’s barycenter corresponds to the minimum-intensity line (47) of the transverse field. The dependence of the displacement for $\tilde{\alpha}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ on the angle $\theta_I$ is schematically shown in Fig. 2 where we set $kw_0 = 1000$ so that $w_0 \approx 159\lambda$, $\Delta \theta = 10^{-3} \text{rad} \approx 0.0573^\circ$, the displacement is in units of wavelength, and the plot is drawn from $\theta_I = 20\Delta \theta \approx 1.15^\circ$ to $\theta_I = 90^\circ$.

FIG. 2: Dependence of the transverse displacement $y_b$ on the angle $\theta_I$ for beams of circularly-polarized angular spectrum, where $kw_0 = 1000$, and the displacement is in units of $\lambda$.

It is interesting to note that the position $y_n$ of the minimum-intensity point and the position $y_b$ of the barycenter satisfy a simple relation,

$$y_n y_b = -w_0^2.$$
C. The local polarization state depends on $\theta_I$ and changes on propagation

At last let us look at how the angle $\theta_I$ affects the polarization distribution of the transverse component. To this end, we set $\tilde{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Substituting into Eq. (41) gives

$$\mathbf{F}_T = \text{sgn}(\theta_I) F_G N \mathbf{v}, \quad (51)$$

where

$$\mathbf{v} = \frac{1}{N} \left( e_x - i e_y \frac{y}{k w^2} \cot \theta_I \right)$$

is the normalized vector of the transverse component, and

$$\frac{1}{N} = \left( 1 + \frac{y^2}{k^2 |w|^4} \cot^2 \theta_I \right)^{-1/2}$$

is the normalization factor. This shows that the beam is not uniformly polarized. The polarization ellipticity of this transverse component is

$$\Sigma_z = \frac{-2 u_0^2 y}{N^2 k |w|^4} \cot \theta_I. \quad (52)$$

It can be seen from Eqs. (51) and (52) that:

1. The local polarization state changes with $y$. The rotation direction of the polarization ellipse in half space $y > 0$ is opposite to that in the other half space $y < 0$. Only on line $y = 0$, is the polarization state linear.
2. The sign of the polarization ellipticity changes when the sign of $\theta_I$ is changed.
3. Since $w^2$ is dependent on $z$, Eq. (52) indicates that the local polarization state of the transverse component is changing on propagation.
4. The intensity distribution has the form of

$$|\mathbf{F}_T|^2 \propto 1 + \frac{y^2}{k^2 |w|^4} \cot^2 \theta_I,$$

which is not axisymmetric.

As an example, we show in Fig. 3 the dependence of $\Sigma_z$ on $y$ at the beam-waist plane $z = 0$ for $\theta_I = 20 \Delta \theta$, where all the parameters are the same as in Fig. 2.

V. CONCLUSIONS AND REMARKS

In conclusion, we have represented in Eq. (9) the amplitude vector $\mathbf{f}(k_x, k_y)$ of a beam’s angular spectrum in terms of the Jones vector $\tilde{\mathbf{f}}$ and the MM $m(k_x, k_y)$. It was shown that
the transversality condition allows us to have one degree of freedom to choose the MM. This
degree of freedom can be described by the azimuthal angle $\Theta$ of a fixed unit vector $\mathbf{I}$ with
respect to the wave vector $\mathbf{k}$ and was interpreted to mediate the interaction between the
photon’s momentum and spin. The cross product in the definition of the unit vector $\mathbf{s}$ is
expressed in terms of a set of $3 \times 3$ matrices (17). They satisfy the canonical commutation
relations of the angular momentum and can be regarded as the photon’s spin operators
acting on the 3-component field vector. It is remarked that the 3-component field vector $m_0\tilde{\alpha}_\pm$ that is mapped to by the MM (30) from the circular-polarized 2-component Jones
vector (46) is the eigen-state of the spin operator $\hat{\Sigma}_z = \frac{1}{\hbar}\hat{S}_z$ with eigenvalue $\pm 1$,
\[
\hat{\Sigma}_z(m_0\tilde{\alpha}_\pm) = \pm(m_0\tilde{\alpha}_\pm),
\]
in much the same way as the Jones vector (46) is the eigen-state of the Pauli matrix $\hat{\sigma}_z =
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}$ with eigenvalue $\pm 1$,
\[
\hat{\sigma}_z\tilde{\alpha}_\pm = \pm\tilde{\alpha}_\pm.
\]

The integral representation (3) for a beam’s electric-field vector in position space was
formulated by letting the unit vector $\mathbf{I}$ lie in the plane $zox$ and make an angle $\theta_I$ with the
$z$-axis. The integral representation (25) [or (26)] for the MM of a beam’s field vector was
obtained when the normalized Jones vector $\tilde{\alpha}$ is independent of the wave vector. When $\theta_I$
is equal to zero, we arrive at the unified description of the cylindrical vector beam (37). So

**FIG. 3** Dependence of $\Sigma_z$ on $y$-coordinate at the beam-waist plane $z = 0$ for a beam of linearly-
polarized angular spectrum, where $kw_0 = 1000$, $\Delta \theta = 10^{-3} \text{rad}$, $\theta_I = 20\Delta \theta$, and the $y$-coordinate
is in units of $w_0$. 
we investigated, at large values of angle $\theta_I$, the impact of the momentum-spin interaction on the properties of a beam in the first-order approximation under the paraxial condition \cite{28}. We found a transverse effect and showed that this effect originates from the beam’s deformation from the axisymmetry due to the momentum-spin interaction. The impact of this interaction turns out to be the interference between the zeroth-order and the first-order terms of the transverse component. We also showed that the local polarization state of a beam’s transverse component is dependent on the angle $\theta_I$ and is changing on propagation. It is clear that the representation of beams with only the transverse ($\theta_I = \pm \frac{\pi}{2}$) and the longitudinal ($\theta_I = 0$) unit vectors is far from complete \cite{31}. The properties of beams with small values of angle $\theta_I$ that are comparable with the divergence angle $\Delta \theta$ can not be discussed in the same way \cite{43} and will be presented elsewhere.

We showed that the incident beams in Ref. \cite{21} and Ref. \cite{22} are special cases of the beam discussed in this paper. At the same time we explained why the physical properties of the incident beam in Ref. \cite{21} is dependent on the “incidence angle”. Now that the cylindrical vector beam can be obtained experimentally \cite{44, 45} from the uniformly-polarized beam by changing the interaction angle from $\theta_I = \pm \frac{\pi}{2}$ into $\theta_I = 0$, it is expectable that the beam that shows the transverse effect can also be obtained in experiments from either the uniformly-polarized beam or the cylindrical vector beam by changing the interaction angle $\theta_I$ into a value that is neither equal to $\pm \frac{\pi}{2}$ nor to 0.

We have only discussed a special amplitude scalar \cite{27}. A physically allowed amplitude scalar $f(k_\rho, \varphi)$ in Eq. \cite{26} can be expanded as a Fourier series,

$$f(k_\rho, \varphi) = \sum_{l=-\infty}^{\infty} f_l(k_\rho) \exp(il\varphi).$$  \hfill (54)

One may consider the constituent term of the following form,

$$f(k_\rho, \varphi) = f_l(k_\rho) \exp(il\varphi),$$  \hfill (55)

and discuss the impact of the momentum-spin interaction on the resultant beam. It is expected that when $\theta_I = \pm \frac{\pi}{2}$, Eq. \hfill (55) together with Eqs. \hfill (24) and \hfill (26) will yield, according to Eq. \hfill (35), the eigen-state of the orbital angular momentum \cite{1} in the zeroth-order approximation.

According to the triad relation expressed by the first equation of \cite{11} and the principle
of duality in free space \[5\], we would like to point out that the following $3 \times 2$ matrix

$$
m_M = \begin{pmatrix}
    s_x & -p_x \\
    s_y & -p_y \\
    s_z & -p_z
\end{pmatrix}
$$

(56)

can be regarded as the MM for the magnetic-field vector, which maps a Jones vector to the 3-component magnetic-field vector for a particular wave vector.

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