Grothendieck categories
and support conditions

We give examples of pairs \((G_1, G_2)\) where \(G_1\) is a Grothendieck category and \(G_2\) a full Grothendieck subcategory of \(G_1\), the inclusion \(G_2 \hookrightarrow G_1\) being denoted \(i\), for which \(R^+i : D^+G_2 \to D^+G_1\) (or even \(R_i : DG_2 \to DG_1\)) is a full embedding\(^1\). This yields generalizations of some results of Bernstein and Lunts, and of Cline, Parshall and Scott. To wit, Theorem 4 (resp. Theorem 6, resp. Theorem 7 and Corollary 8) below strengthen Theorem 17.1 in Bernstein and Lunts \([4]\) (resp. Example 3.3.c and Theorem 3.9.a of Cline, Parshall and Scott \([10]\), resp. Theorem 3.1 and Proposition 3.6 of Cline, Parshall and Scott \([9]\)).

We work in the axiomatic system defined by Bourbaki in \([6]\). We postulate in addition the existence of an uncountable universe \(U\) in the sense of Bourbaki \([7]\). All categories are \(U\)-categories.

By Alonso Tarrío, Jeremías López and Souto Salorio \([1]\), Theorem 5.4, or by Serpé \([19]\) Theorem 3.13, (or more simply by Spaltenstein \([20]\), proof of Theorem 4.5), the functor\(^2\) \(R\text{Hom}_{G_i}\) is defined on the whole of \(DG_i^{\text{op}} \times DG_i\). — Consider the following conditions.

\((R)\) : For all \(V, W \in DG_2\) the complexes \(R\text{Hom}_{G_1}(V, W)\) and \(R\text{Hom}_{G_2}(V, W)\) are canonically isomorphic in \(DZ\).

\((R+)\) : For all \(V, W \in D^+G_2\) the complexes \(R\text{Hom}_{G_1}(V, W)\) and \(R\text{Hom}_{G_2}(V, W)\) are canonically isomorphic in \(DZ\).

Let \(A\) be a commutative ring, let \(Y\) be a set of prime ideals of \(A\), let \(G_1\) (resp. \(G_2\)) be the category of \(A\)-modules (resp. of \(A\)-modules supported on \(Y\)). Do \((R)\) or \((R+)\) hold? (See Theorem 5 below for a partial answer.)

By the proof of Weibel \([22]\), Theorem A3, \((R)\) implies \((R+)\) \(^3\). Moreover, if \((R)\) (resp. \((R+)\)) holds, then \(R_i\) (resp. \(R^+_i\)) is a full embedding. Indeed we have \(\text{Hom}_{DG_i} = H^0 R\text{Hom}_{G_i}\) (resp. \(\text{Hom}_{D^+G_i} = H^0 R\text{Hom}_{G_i}\)) by Lipman \([18]\), I.2.4.2.

Let \(\text{Mod} A\) denote the category of left \(A\)-modules (whenever this makes sense), and let \(DA\) (resp. \(D^+A\), resp. \(KA\), resp. \(K^+A\)) be an abbreviation for

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\(^1\)The categories \(G_1\) and \(G_2\) will come under various names, but the inclusion will always be denoted by \(i\).

\(^2\)An example of category for which \(R\text{Hom}\) can be explicitly described is given in Appendix 1.

\(^3\)I know no cases where \((R+)\) holds but \((R)\) doesn’t.
D Mod $A$ (resp. $D^+ Mod A$, resp. $K Mod A$, resp. $K^+ Mod A$), where $K$ means “homotopy category”. (Even if $G_1$ or $G_2$ is not Grothendieck, it may still happen that $(R+)$ or $(R)$ makes sense and holds. In such a situation the phrase “$(R+)$ (resp. $(R)$) holds” shall mean “$(R+)$ (resp. $(R)$) makes sense and holds”.

Let $A$ be a sheaf of rings over a topological space $X$, let $Y$ be a locally closed subspace of $X$, let $B$ be the restriction of $A$ to $Y$, and identify, thanks to Section 3.5 of Grothendieck [12], $Mod B$ to the full subcategory of $A$-modules supported on $Y$.

**Theorem 1** The pair $(Mod A, Mod B)$ satisfies $(R)$.

**Proof.** Let $r : Mod A \rightarrow Mod B$ be the restriction functor.

**Case 1.** $Y$ is closed. — We have for $V \in KB$

\[ \mathbb{H}om^\bullet_B(V, ?) = \mathbb{H}om^\bullet_A(V, ?) \circ K : KB \rightarrow KZ. \]

Since $\iota$ is right adjoint to the exact functor $r$, it preserves $K$-injectivity in the sense of Spaltenstein [20]. By Lipman [13], Corollaries I.2.2.7 and I.2.3.2.3, we get

\[ \mathbb{R}hom_B(V, ?) \simeq \mathbb{R}hom_A(V, ?) \circ R \iota : DB \rightarrow DZ. \]

**Case 2.** $Y$ is open. — We have for $V \in KB$

\[ \mathbb{H}om^\bullet_A(V, ?) = \mathbb{H}om^\bullet_B(V, ?) \circ K : KA \rightarrow KZ. \]

As $r$ is right adjoint to the exact functor $\iota$, it preserves $K$-injectivity, and Lipman [13], Corollaries I.2.2.7 and I.2.3.2.3, yields $R r \circ R \iota = \text{Id}_{DB}$.

\[ \mathbb{R}hom_A(V, ?) = \mathbb{R}hom_B(V, ?) \circ R : DA \rightarrow DZ, \]

and thus

\[ \mathbb{R}hom_A(V, ?) \circ R \iota = \mathbb{R}hom_B(V, ?) : DB \rightarrow DZ. \]

**□**

**Proposition 2** Let $X$ and $A$ be as above, let $Y$ be a union of closed subspaces of $X$, and let $Mod(A, Y)$ be the category of $A$-modules supported on $Y$. Then the pair $(Mod A, Mod(A, Y))$ satisfies $(R+)$. 

**Proof.** See Grothendieck [12], Proposition 3.1.2, and Hartshorne [14], Proposition I.5.4. □
Let \((X, \mathcal{O}_X)\) be a noetherian scheme, \(A\) a sheaf of rings over \(X\) and \(\mathcal{O}_X \rightarrow A\) a morphism, assume \(A\) is \(\mathcal{O}_X\)-coherent, let \(Y\) be a subspace of \(X\), and denote by \(\text{QC}(\mathcal{A})\) (resp. \(\text{QC}(\mathcal{A}, Y)\)) the category of \(\mathcal{O}_X\)-quasi-coherent \(\mathcal{A}\)-modules (resp. \(\mathcal{O}_X\)-quasi-coherent \(\mathcal{A}\)-modules supported on \(Y\)).

**Theorem 3** The pair \((\text{QC}(\mathcal{A}), \text{QC}(\mathcal{A}, Y))\) satisfies \((R+)\). If in addition \(\text{Ext}^n_{\text{QC}(\mathcal{A})} = 0\) for \(n \gg 0\), then \((R)\) holds.\(^4\)

Let \(A\) be a left noetherian ring, let \(B\) be a ring, let \(A \rightarrow B\) be a morphism, let \(\mathcal{G}\) be a Grothendieck subcategory of \(\text{Mod} B\), let \((U_j)_{j \in J}\) be a family of generators of \(\mathcal{G}\) which are finitely generated over \(A\), and let \(I\) be an Artin-Rees left ideal of \(A\). For each \(V\) in \(\text{Mod} A\) set
\[
V_I := \{v \in V \mid I^n(v)v = 0 \text{ for some } n(v) \in \mathbb{N}\}.
\]
Assume that \(IV\) and \(V_I\) belong to \(\mathcal{G}\) whenever \(V\) does. Let \(\mathcal{G}_I\) be the full subcategory of \(\mathcal{G}\) whose objects satisfy \(V = V_I\).

**Example:** \(\mathcal{G}\) is the category of \((g, K)\)-modules defined in Section 1.1.2 of Bernstein and Lunts [4], \(A\) is \(Ug\), \(B\) is \(Ug \times \mathbb{C}K\), \(I\) is a left ideal of \(A\) generated by \(K\)-invariant central elements.

**Theorem 4** The pair \((\mathcal{G}, \mathcal{G}_I)\) satisfies \((R+)\). If in addition \(\text{Ext}^n_{\mathcal{G}} = 0\) for \(n \gg 0\), then \((R)\) holds. In particular if \((\mathcal{G}, \mathcal{G}_I)\) is as in the above Example and if \(K\) is reductive, then \((R)\) is fulfilled.

**Lemma 5** If \(E\) is an injective object of \(\mathcal{G}\), then so is \(E_I\).

**Lemma 5** implies **Theorem 4** By Theorem 1.10.1 of Grothendieck in [12], \(\mathcal{G}\) and \(\mathcal{G}_I\) have enough injectives. We have for \(V \in K^+\mathcal{G}_I\)
\[
\left[\text{Hom}^\bullet_{\mathcal{G}_I}(V, ?) = \text{Hom}^\bullet_{\mathcal{G}}(V, ?) \circ K^+I\right] : K^+\mathcal{G}_I \rightarrow K\mathbb{Z}
\]
and thus, by Lemma 5 and Hartshorne [14], Proposition I.5.4.b,
\[
\left[\text{RHom}_{\mathcal{G}_I}(V, ?) \Rightarrow \text{RHom}_{\mathcal{G}}(V, ?) \circ R^+I\right] : D^+\mathcal{G}_I \rightarrow D\mathbb{Z}.
\]
This proves the first sentence of the theorem. For the second one the argument is the same except for the fact we use Hartshorne [14], proof of Corollary I.5.3.γ.b.\(^\square\)

\(^4\)We regard \(\text{Ext}^n_{\mathcal{G}}\) as a functor defined on \(\mathcal{G}^{\text{op}} \times \mathcal{G}\) (and of course not on \(D\mathcal{G}^{\text{op}} \times D\mathcal{G}\)).
Proof of Lemma 5. Let \( W \subset V \) be objects of \( G \) and \( f : W \to E_I \) a morphism. We must extend \( f \) to \( g : V \to E_I \). We can assume, by the proof of Grothendieck [12] Section 1.10 Lemma 1, (or by Stenström [21], Proposition V.2.9), that \( V \) is finitely generated over \( A \). Since \( W \) is also finitely generated over \( A \), there is an \( n \) such that \( I^n f(W) = 0 \), and thus \( f(I^n W) = 0 \). Choose a \( k \) such that \( W \cap I^k V \subset I^n W \subset \ker f \) and set

\[
\nabla := \frac{V}{I^k V}, \quad \mathcal{W} := \frac{W}{W \cap I^k V}.
\]

Then \( f \) induces a morphism \( \nabla \to E_I \), which, by injectivity of \( E \), extends to a morphism \( \nabla \to E_I \), that in turn induces a morphism \( V \to \nabla \to E_I \), enabling us to define \( g \) as the obvious composition \( V \to \nabla \to E_I \). □

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, let \( \mathfrak{h} \subset \mathfrak{b} \) be respectively Cartan and Borel subalgebras of \( \mathfrak{g} \), put \( n := [\mathfrak{b}, \mathfrak{b}] \), say that the roots of \( \mathfrak{h} \) in \( n \) are positive, let \( \mathcal{W} \) be the Weyl group equipped with the Bruhat ordering, let \( O_0 \) be the category of those \( \text{BGG-modules} \) which have the generalized infinitesimal character of the trivial module. The simple objects of \( O_0 \) are parametrized by \( \mathcal{W} \). Say that \( Y \subset \mathcal{W} \) is an initial segment if \( x \leq y \) and \( y \in Y \) imply \( x \in Y \), and that \( w \in \mathcal{W} \) lies in the support of \( V \in O_0 \) if the simple object attached to \( w \) is a subquotient of \( V \). For such an initial segment \( Y \) let \( O_Y \) be the subcategory of \( O_0 \) consisting of objects supported on \( Y \subset \mathcal{W} \).

Theorem 6 The pair \((O_0, O_Y)\) satisfies (R).

Proof. In view of BGG [3] this will follow from Theorem 9. □

Let \( A \) be a ring, \( I \) an ideal, and \( B := A/I \) the quotient ring.

Theorem 7 Assume that \( \text{Ext}^n_A(B, B) \) vanishes for \( n > 0 \), and that there is a \( p \) such that \( \text{Ext}^n_A(B, W) = 0 \) for all \( n > p \) and all \( B \)-modules \( W \). Then the pair \((\text{Mod} A, \text{Mod} B)\) satisfies (R).

Proof. Step 1 : \( \text{Ext}^n_A(B, W) = 0 \) for all \( B \)-modules \( W \) and all \( n > 0 \). — By Theorem V.9.4 in Cartan-Eilenberg [3] we have \( \text{Ext}^n_A(B, F) = 0 \) for all free \( B \)-modules \( F \) and all \( n > 0 \). Suppose by contradiction there is an \( n > 0 \) such that \( \text{Ext}^n_A(B, W) \) does not vanish on all \( B \)-modules; let \( n \) be maximum for this property; choose a \( B \)-module \( V \) such that \( \text{Ext}^n_A(B, V) \neq 0 \); consider an exact sequence \( W \hookrightarrow F \twoheadrightarrow V \) with \( F \) free; and observe the contradiction \( 0 \neq \text{Ext}^n_A(B, V) \hookrightarrow \text{Ext}^{n+1}_A(B, W) = 0. \)
Step 2: Putting \( r := \text{Hom}_A(B, ?) \) we have \( Rr \circ Rr = \text{Id}_{DB} \). — The functor \( r \), being a right adjoint, commutes with products, and, having an exact left adjoint, preserves injectives. Let \( V \) be in \( DB \) and \( I \) a Cartan-Eilenberg injective resolution (CEIR) of \( V \) in \( \text{Mod} A \). By the previous step \( rI \) is a CEIR of \( rV = V \) in \( \text{Mod} B \). Weibel [22], Theorem A3, implies

(a) the complex \( \text{Tot}^I \in \text{DA} \), characterized by

\[
(\text{Tot}^I)^n = \prod_{p+q=n} P^{pq},
\]

is a K-injective resolution (see Spaltenstein [20]) of \( V \) in \( \text{Mod} A \),

(b) \( \text{Tot}^I rI = r\text{Tot}^I \) is a K-injective resolution of \( V = rV \) in \( \text{Mod} B \).

Statement (a) yields: (c) \( r\text{Tot}^I = RrV \). Then (b) and (c) imply that the natural morphism \( V \to RrV \) is a quasi-isomorphism.

Step 3: (R) holds. — See proof of Theorem 1, Case 2. □

Corollary 8 If there is a projective resolution \( P = (P_n \to \cdots \to P_1 \to P_0) \) of \( B \) by \( A \)-modules satisfying \( \text{Hom}_A(P_j, V) = 0 \) for all \( B \)-modules \( V \) and all \( j > 0 \), then pair \( (\text{Mod} A, \text{Mod} B) \) satisfies (R).

Let \( A \) be a ring, \( X \) a finite set and \( e_\bullet = (e_x)_{x \in X} \) a family of idempotents of \( A \) satisfying \( \sum_{x \in X} e_x = 1 \) and \( e_x e_y = \delta_{xy} e_x \) (Kronecker delta) for all \( x, y \in X \).

The support of an \( A \)-module \( V \) is the set \( \{x \in X \mid e_x V \neq 0\} \). Let \( \leq \) be a partial ordering on \( X \), and for any initial segment \( Y \) put

\[
A_Y := A \left/ \sum_{x \notin Y} Ae_x A \right.,
\]

so that \( \text{Mod} A_Y \) is the full subcategory of \( \text{Mod} A \) whose objects are supported on \( Y \). (Here and in the sequel, for any ring \( B \), we denote by \( BbB \) the ideal generated by \( b \in B \).) The image of \( e_y \) in \( A_Y \) will be still denoted by \( e_y \).

Assume that, for any pair \( (Y, y) \) where \( Y \) is an initial segment and \( y \) a maximal element of \( Y \), the module \( M_y := A_Y e_y \) does not depend on \( Y \), but only on \( y \). This is equivalent to the requirement that \( A_Y e_y \) be supported on \( \{x \in X \mid x \leq y\} \).
If \((V_\gamma)_{\gamma \in \Gamma}\) a family of \(A\)-modules, let \(\langle V_\gamma \rangle_{\gamma \in \Gamma}\) denote the class of those \(A\)-modules which admit a finite filtration whose associated graded object is isomorphic to a product of members of the family.

Assume that, for any \(x \in X\), the module \(Ae_x\) belongs to \(\langle M_z \rangle_{z < x}\).

**Theorem 9** The pair \((\text{Mod } A, \text{Mod } A_Y)\) satisfies \((R)\).

This statement applies to the categories satisfying Conditions (1) to (6) in Section 3.2 of Beilinson, Ginzburg and Soergel \cite{2}, like the categories of BGG modules \(O_\lambda\) and \(O^n\) defined in Section 1.1 of \cite{2}, or more generally the category \(\mathcal{P}(X, \mathcal{W})\) of perverse sheaves considered in Section 3.3 of \cite{2}. — Because of the projectivity of \(M_x = Ae_x\) we have

**Lemma 10** For any \(x, y \in X\) with \(x\) maximal there is a nonnegative integer \(n\) and an exact sequence \((Ae_x)^n \to Ae_y \to V\) such that \(V \in \langle M_z \rangle_{z < x}\). In particular \(e_x V = 0\). □

**Proof of Theorem** Assume \(Y = X \setminus \{x\}\) where \(x\) is maximal. Put \(e := e_x, I := AeA\) and \(B := A_Y = A/I\). By the previous Lemma there is a nonnegative integer \(n\) and an exact sequence \((Ae)^n \to A \to V\) with \(IV = 0\). Letting \(J \subset A\) be the image of \((Ae)^n \to A\), we have \(J = IA \subset J \subset I\), and thus \(I = J\). In particular \(I\) is \(A\)-projective and we have \(\text{Hom}_A(I, B) \simeq (eB)^n = 0\). Corollary \ref{cor:projectivity} applies, proving Theorem \ref{thm:projectivity} for the particular initial segment \(Y\). Lemma \ref{lem:exactness} shows that \((B, Y, (e_y)_{y \in Y})\) satisfies the assumptions of Theorem \ref{thm:projectivity} and an obvious induction completes the proof. □

For any complex Lie algebra \(g\) let \(I_g\) be the annihilator of the trivial module in the center of the enveloping algebra. Using the notation and definitions of Knapp and Vogan \cite{17}, let \((g, K)\) be a reductive pair, let \((g', K')\) be a reductive subpair attached to \(\theta\)-stable subalgebra, let \(R^S : C(g', K') \to C(g, K)\) be the cohomological induction functor defined in \cite{17}, (5.3.b), and let \(\mathcal{G}\) (resp. \(\mathcal{G}'\)) be the category of \((g, K)\)-modules on which \(I_g\) (resp. \(I_g')\) acts locally nilpotently. By \cite{17}, Theorem 11.225, the functor \(R^S\) maps \(\mathcal{G}'\) to \(\mathcal{G}\). Let \(F : \mathcal{G}' \to \mathcal{G}\) be the induced functor. By \cite{17}, Theorem 3.35.b, \(F\) is exact. It would be interesting to know if \(F\) satisfies Condition \((R)\).

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Proof of Theorem 3

Put $\mathcal{O} := \mathcal{O}_X$ and consider the following statements:

(a) Every object of $\text{QC}(\mathcal{O}, Y)$ is contained into an object of $\text{QC}(\mathcal{O}, Y)$ which is injective in $\text{QC}(\mathcal{O})$.

(b) Every object of $\text{QC}(\mathcal{A}, Y)$ is contained into an object of $\text{QC}(\mathcal{A}, Y)$ which is injective in $\text{QC}(\mathcal{A})$.

We claim (a) $\Rightarrow$ (b) $\Rightarrow$ Theorem 3.

(a) $\Rightarrow$ (b) : The functor $\text{Hom}_{\mathcal{O}}(\mathcal{A}, ?)$ preserves the following properties:

- quasi-coherence (by EGA I [13], Corollary 2.2.2.vi),
- the fact of being supported on $Y$ (by Grothendieck [12], Proposition 4.1.1),
- injectivity (by having an exact left adjoint). $\square$

(b) $\Rightarrow$ Theorem 3: See proof of Theorem 4 $\square$

Proof of (a). Let $M$ be in $\text{QC}(\mathcal{O}, Y)$ and let us show that $M$ is contained into an object of $\text{QC}(\mathcal{O}, Y)$ which is injective in $\text{QC}(\mathcal{O})$. We may, and will, assume that $Y$ is precisely the support of $M$.

Case 1. $M$ is coherent, $(X, \mathcal{O})$ is affine. — Write $A$ for $\Gamma \mathcal{O}$, where $\Gamma$ is the global section functor. Use the equivalence $\text{QC}(\mathcal{O}) \cong \text{Mod} A$ set up by $\Gamma$ to work in the latter category. Then $M$ “is” a finitely generated $A$-module, and $Y$ is closed by Proposition II.4.4.17 in Bourbaki [5]. Let $I \subset A$ be the ideal of those $f$ in $A$ which vanish on $Y$, and $\text{Mod}(A, Y)$ the full subcategory of $\text{Mod} A$ whose objects are the $A$-modules $V$ satisfying $V = V_I$ in the sense of Notation [11]. Corollary 2 to Proposition II.4.4.17 in Bourbaki [5] implies that $\Gamma$ induces a subequivalence $\text{QC}(\mathcal{O}, Y) \cong \text{Mod}(B, Y)$. The claim now follows from Theorem 4.

Case 2. $M$ is coherent. — Argue as in the proof of Corollary III.3.6 in Hartshorne [15], using Proposition 6.7.1 of EGA I [13].

Case 3. General case. — By Gabriel [11] Corollary 1 §II.4 (p. 358), Theorem 2 §II.6 (p. 362), and Theorem 1 §VI.2 (p. 443) we know that every object of $\text{QC} \mathcal{O}$ has an injective hull and that any colimit of injective objects of $\text{QC} \mathcal{O}$ is injective. The expression $M \preceq M'$, shall mean “$M'$ is an injective hull of $M$ and $M \subset M'$”. Let $M'$ be such a hull and $Z$ the set of pairs $(N, N')$ with $N \subset M$, $N' \subset M'$, $N \preceq N'$, $\text{Supp}(N') = \text{Supp}(N)$.
Then $Z$, equipped with its natural ordering, is inductive. Let $(N, N')$ is a maximal element of $Z$ and suppose by contradiction $N \neq M$. By Corollary 6.9.9 of EGA I [13] there is a $P$ such that $N \subset P \subset M$, $N \neq P$, and $C := P/N$ is coherent. Let $\pi : P \to C$ be the canonical projection and choose $P', C'$ such that $P \prec P'$, $C \prec C'$. By injectivity of $N'$ there is a map $f : P \to N'$ such that $[N \hookrightarrow P \xrightarrow{f} N'] = [N \hookrightarrow N']$ (obvious notation). Consider the commuting diagram

$$
\begin{array}{c}
N' \xrightarrow{N' \times C'} C' \\
\downarrow \quad \downarrow \pi \\
N \xrightarrow{P} C.
\end{array}
$$

We have $\text{Ker}(f \times \pi) = \text{Ker}(f) \cap \text{Ker}(\pi) = \text{Ker}(f) \cap N = 0$, i.e. $g := f \times \pi$ is monic. By injectivity of $N' \times C'$ there is a map $P' \to N' \times C'$ such that $[P \hookrightarrow P' \to N' \times C'] = [P \hookrightarrow g \to N' \times C']$, this map being monic by essentiality of $P \subset P'$; in particular

$$\text{Supp}(P') \subset \text{Supp}(N') \cup \text{Supp}(C').$$

A similar argument shows the existence of a monomorphism $P' \to M'$ such that $[P \hookrightarrow P' \to M'] = [P \to M \hookrightarrow M']$, meaning that we can assume $P' \subset M'$. Since $(P, P') \notin Z$, this implies $\text{Supp}(P') \neq \text{Supp}(P')$, and the equalities

$$\text{Supp}(N') = \text{Supp}(N) \quad (\text{because } (N, N') \in Z),$$

$$\text{Supp}(C') = \text{Supp}(C) \quad (\text{by Case 2}),$$

yield the contradiction

$$\text{Supp}(P') \subset \text{Supp}(N) \cup \text{Supp}(C) = \text{Supp}(P) \subset \text{Supp}(P'). \quad \square$$

Appendix 1

Let $k$ be a field and $\mathfrak{g}$ a Lie $k$-algebra. For $X, Y \in Dk$ put

$$\langle X, Y \rangle := \text{Hom}_k^\bullet(X, Y).$$

Let $C := U\mathfrak{g} \otimes \bigwedge \mathfrak{g}$ be the Koszul complex viewed as a differential graded coalgebra (here and in the sequel tensor products are taken over $k$).

In view of Weibel [22], Theorem A3, we can define $\text{RHom}_\mathfrak{g}$ by setting

$$\text{RHom}_\mathfrak{g}(X, Y) := \langle\langle C, X\rangle, \langle C, Y\rangle\rangle^\mathfrak{g}.$$
(As usual the superscript $g$ means “$g$-invariants”.) Recall that the Chevalley-Eilenberg complex, used to compute the cohomology of $g$ with values in $\langle X, Y \rangle$, is defined by $\text{CE}(X, Y) := \langle C, \langle X, Y \rangle \rangle^0$, and that there is a canonical isomorphism $F : \text{CE} \sim \text{RHom}_g$. Let

$$\text{ext}_{X,Y,Z} : \text{CE}(Y, Z) \otimes \text{CE}(X, Y) \to \text{CE}(X, Z)$$

be the exterior product and

$$\text{comp}_{X,Y,Z} : \text{RHom}_g(Y, Z) \otimes \text{RHom}_g(X, Y) \to \text{RHom}_g(X, Z)$$

the composition. Then the expected formula

$$\text{comp}_{X,Y,Z} \circ (F_{Y,Z} \otimes F_{X,Y}) = F_{X,Z} \circ \text{ext}_{X,Y,Z}$$

is easy to check.

Appendix 2

The following fact is used in various places (see for instance the proofs of Theorem I.3.3 in Cartan-Eilenberg [8], Theorem 1.10.1 in Grothendieck [12] and Lemma 4.3 in Spaltenstein [20]). We use the notation and definitions of Jech [16].

**Lemma 11** Let $P$ be a poset, $\alpha$ a cardinal $\geq |P|$, and $\beta$ the least cardinal $> \alpha$. Then every poset morphism $f : \beta \to P$ is stationary.

**Proof.** We can assume $P$ is infinite and $f$ is epic. The morphism $g : P \to \beta$ defined by $gp := \min f^{-1}p$ satisfies $fg = \text{Id}_p$. Put $\sigma := \sup gP$. For all $p \in P$ we have $|gp| \leq gp < \beta$, implying $|gp| \leq \alpha$ for all $p$, and $\sigma \leq \beta$. Statement (2.4) and Theorem 8 in Jech [16] entail respectively $\sigma = \bigcup_{p \in P} gp$ and $|P| \alpha = \alpha$, from which we conclude $|\sigma| \leq \alpha$; this forces $\sigma < \beta$, that is $\sigma \in \beta$. For any $\gamma \in \beta$, $\gamma > \sigma$ we have $f\gamma = fgf\gamma \leq f\sigma \leq f\gamma$. \[\Box\]

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