On the new identities of Dirichlet 
\textit{L}-functions

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Abstract

Let $q \geq 3$ be an integer, $\chi$ be a Dirichlet character modulo $q$, and $L(s, \chi)$ denote the Dirichlet $L$-functions corresponding to $\chi$. In this paper, we show some special function series, and give some new identities for the Dirichlet $L$-functions involving Gauss sums. Specially, we give specific identities for $L(2, \chi)$.

Keywords. Dirichlet $L$-functions; identity; Gauss sums; function series.

1. Introduction

Let $q \geq 3$ be an integer and $\chi$ be a Dirichlet character modulo $q$. Dirichlet $L$-functions $L(s, \chi)$ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $s$ is a complex number with $\text{Re}(s) > 1$. It can be extended to all $s$ using analytic continuation. Many scholars have studied the mean value of Dirichlet $L$-functions and have got some identities or asymptotic formulae \cite{1-10}. For example, Alkan \cite{1} has got the identical equations of Dirichlet $L$-functions when $s = 1, 2$ with the odd and even Dirichlet character $\chi$ respectively,

$$\sum_{\chi \mod q \chi(-1) = -1} |L(1, \chi)|^2 = \frac{\pi^2 \varphi(q)}{12} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) - \frac{\pi^2 \varphi^2(q)}{4q^2};$$

(1)
\[ \sum_{\chi \mod q} |L(2, \chi)|^2 = \frac{\pi^4 \varphi(q)}{180} \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + \frac{\pi^4 \varphi(q)}{18q^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right), \quad (2) \]

where \( \varphi(q) \) is the Euler’s totient function. In addition, in contrast with these results, he also has given approximate formulas which are relatively easy to be derived by Abel’s summation and the Polya-Vinogradov inequality, that is

\[ \sum_{\chi \mod q} |L(1, \chi)|^2 = \frac{\varphi(q)}{2} + O(\sqrt{q} \log q), \quad (3) \]

where \( \varphi(q) \) is defined the same as before and the signal \( O \) only depends on constants.

He also has remarked that if \( \chi \) and \( r \) have the same parity, then it is always possible to determine the average value of \( |L(r, \chi)|^2 \) when \( \chi \) ranges over all odd or even character modula \( q \).

To proceed, we shall introduce the Gauss sum \( G(z, \chi) [3] \) corresponding to \( \chi \mod q \) and Bernoulli numbers [3] which are defined as

\[ G(z, \chi) := \sum_{k=0}^{q-1} \chi(k) e^{\frac{2\pi ikz}{q}}; \]

\[ B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \]

where \( z \) is an integer, \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient and \( B_0 = 1 \).

In the proof of formulae (1) and (2), Alkan [2] has mainly constructed an identity between the Gauss sum and Dirichlet \( L \)-functions which is satisfied for the special conditions, that is,

\[ \frac{(-1)^{s+1}qs!}{r^s 2^s - \pi^s} L(s, \chi) = \sum_{j=1}^{q} G(j, \chi) \sum_{k=0}^{2[\frac{s}{2}]} \left(\begin{array}{c} s \\ k \end{array}\right) B_k \left(\frac{j}{q}\right)^{s-k}, \quad (4) \]

where \( s \geq 1 \) and \( \chi \) have the same parity.

Of course, Zhang [14] also has obtained formulae (1) and (2) with the help of the identity between the Dirichlet \( L \)-functions and the generalized Dedekind sum

\[ S(h, n, q) = \frac{(n!)^2}{4^{n-1} \pi^{2n} q^{2n-1}} \sum_{d|q} \varphi(d) \sum_{\chi \mod q} \chi(h)|L(n, \chi)|^2 \]

\[ -\frac{(n!)^2}{4^n \pi^{2n}} \left(\sum_{r=1}^{\infty} \frac{1 + (-1)^n}{r^n}\right)^2, \quad (5) \]

where the generalized Dedekind sum is defined by

\[ S(h, n, q) = \sum_{a=1}^{q} \bar{B}_n\left(\frac{a}{q}\right) \bar{B}_n\left(\frac{ah}{q}\right), \]
which \( \bar{B}_n(x) \) is defined as following

\[
\bar{B}_n(x) = \begin{cases} 
B_n(x - \lfloor x \rfloor), & \text{if } x \text{ is not an integer,} \\
0, & \text{if } x \text{ is an integer.}
\end{cases}
\] (6)

Although Alkan has obtained formulae (1) and (2) with the help of the connection in [2] between the Dirichlet L-function and Gauss sum, he can only give the connection for \( s \) and \( \chi \) having the same parity. In fact, when \( s \) and \( \chi \) had the different parity, Alkan has got the right of formula (4) is zero. Therefore following his method, he could not get the identities when \( s \) and \( \chi \) have the different parity. Based on that, we would like to study the connection between Dirichlet L-functions and Gauss sums under the condition that \( s \) and \( \chi \) have the different parity. By the method, we can also give a new connection between Dirichlet L-functions and Gauss sums under the condition \( s \) and \( \chi \) having the same parity. I think it’s very interesting and significant because we can know more about the connections between values of Dirichlet L-functions on positive integers and Gauss sum and do more research about Dirichlet L-functions.

In this paper, we will generate identities between Dirichlet L-functions and Gauss sums, and get some new identities of Dirichlet L-functions. That is, we will prove the following:

**Theorem 1** Let \( q \geq 3 \) be an integer and \( \chi \) be a Dirichlet character modulo \( q \). If \( s > 1 \) is an integer, \( s \) and \( \chi \) have the different parity, we have

\[
L(s, \chi) = \frac{(-i)^s \mod 2}{q} \sum_{j=1}^{q} G(j, \chi) \sum_{n=0}^{\infty} \frac{\sin((x+1) \mod 2) (\frac{2\pi j}{q}) \cos^{s \mod 2} (\frac{2\pi j}{q})}{n^s}, \] (7)

where \( G(j, \chi) \) is the Gauss sum corresponding the Dirichlet character \( \chi \) modulo \( q \).

**Theorem 2** Let \( q \geq 3 \) be an integer and \( \chi \) be a Dirichlet character modulo \( q \). If \( s \) is an integer, \( s \) and \( \chi \) have the same parity, we have

\[
L(s, \chi) = -\frac{(-i)^s}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s \pi x^{s-1}}{2} + \sum_{k=1}^{[s/2]} \zeta(2k)(x^{s})^{(2k)} \right]_{x = \frac{2\pi j}{q}}, \] (8)

where \( G(j, \chi) \) is the Gauss sum corresponding the Dirichlet character \( \chi \) modulo \( q \), \( \zeta(2k) \) is the Riemann zeta function at value \( 2k \), and \( \lfloor x \rfloor \) denotes the biggest integer less than \( x \).

Specially, if \( s = 2 \) and \( \chi \) is the even or odd Dirichlet character modulo \( q \) respectively, we have

**Corollary 1** Let \( q \geq 3 \) be an integer and \( \chi \) be an even and odd Dirichlet character modulo \( q \) respectively, we have

\[
L(2, \chi) = \frac{\pi^2}{q^2} \sum_{j=1}^{q} j^2 G(j, \chi) - \frac{\pi^2}{q^2} \sum_{j=1}^{q} j G(j, \chi) + \frac{\pi^2}{6q}, \chi(-1) = 1; \] (9)
\[ L(2, \chi) = \frac{1}{q} \sum_{j=1}^{q} G(j, \chi) \int_{0}^{2\pi/q} \log \left(2 \sin \frac{x}{2}\right) dx, \chi(-1) = -1, \quad (10) \]

where \(G(j, \chi)\) is the Gauss sum corresponding the Dirichlet character \(\chi\) modulo \(q\).

**Note.** Theoretically speaking, we can give all the identities for \(L(r, \chi)\) for every integer \(r \geq 1\) for the corresponding even and odd Dirichlet character \(\chi\) respectively except \(L(1, \chi)\) for the even character because of the condition of Theorem 2. How to get the identity of \(L(1, \chi)\) for an even character is still an open problem.

## 2. Some Lemmas

To prove the theorems, we give the following lemmas.

**Lemma 1** If a function sequence \(\{G_n(x)\}_{n=0,1,2,\ldots}\) satisfies the differential equation

\[
\frac{d}{dx} G_{n+1}(x) = (n + 1)G_n(x),
\]

with \(G_0(x)\) is a nonzero integer. Then the function sequence will be determined by a constant sequence \(\{a_n\}\) in the form of

\[
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k},
\]

where \(a_n\) is the constant term of polynomial \(G_n(x)\).

**Proof:** Obviously \(G_n(x)\) is a polynomial of degree \(n\), assume that

\[
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}.
\]

Multiplying both sides by \(n + 1\) and integrating, then we have

\[
\int (n + 1)G_n(x) dx = (n + 1) \sum_{k=0}^{n} \binom{n}{k} a_k \frac{x^{n-k+1}}{n-k+1} = \sum_{k=0}^{n} \frac{(n + 1)!}{k!(n - k + 1)!} a_k x^{n-k+1} + a_{n+1} \]

\[
= \sum_{k=0}^{n+1} \binom{n + 1}{k} a_k x^{n+1-k} = G_{n+1}(x).
\]

Taking the derivative of formula (11), then we can get

\[
(n + 1)G_n(x) = \frac{d}{dx} G_{n+1}(x).
\]

This has proved Lemma 1.
Lemma 2 If $x \in (0, 2\pi)$, we have
\[
\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}; \quad (12)
\]
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\log 2 \left( \sin \frac{x}{2} \right). \quad (13)
\]

Proof: As we know, for a complex variable $z$, when $|z| < 1$, we have the Taylor series expansion of $\log(1 - z)$ as
\[
-\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.
\]
We can also extend it to all $|z| \leq 1$ except $z = 1$ by using analytic continuation. Therefore, let $z = e^{ix}$, it is allowed for $x \in (0, 2\pi)$
\[
-\log(1 - e^{ix}) = \sum_{n=1}^{\infty} \frac{\cos nx}{n} + i \sum_{n=1}^{\infty} \frac{\sin nx}{n}. \quad (14)
\]
Substitute $1 - e^{ix} = \sqrt{2(1 - \cos x)e^{\frac{ix}{2}}}i$ into equation (14), we get
\[
-\log(1 - e^{ix}) = -\log \sqrt{2(1 - \cos x)} - \frac{x - \pi}{2}i
\]
\[
= -\log \left( 2 \sin \frac{x}{2} \right) - \frac{x - \pi}{2}i
\]
\[
= \sum_{n=1}^{\infty} \frac{\cos nx}{n} + i \sum_{n=1}^{\infty} \frac{\sin nx}{n}
\]
Consider real part and imaginary part respectively, we immediately have
\[
\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2};
\]
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\log 2 \left( \sin \frac{x}{2} \right).
\]
That proves Lemma 2.

2. Proofs of Theorems

With the help of these lemmas, it is feasible to prove theorems and the corollary. Firstly we will prove Theorem 1.

Proof of Theorem 1. In order to prove Theorem 1, we need to discuss two equations between Dirichlet $L$-functions and Gauss sums under the different condition $\chi(-1) = 1$ and $\chi(-1) = -1$ respectively.
When $\chi(-1) = 1$ and $s > 1$, we have

$$\sum_{j=1}^{q} G(j, \chi) \sum_{n=1}^{\infty} \frac{\cos \frac{2\pi nj}{q}}{n^s}$$

$$= \sum_{j=1}^{q} \sum_{m=1}^{q} \chi(m) e^{\frac{2\pi jm}{q}} \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi nj}{q}} + e^{-\frac{2\pi nj}{q}}}{2n^s}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{m=1}^{q} \chi(m) \sum_{j=1}^{q} \left( e^{\frac{2\pi (m+n)j}{q}} + e^{\frac{2\pi (m-n)j}{q}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{q}{2n^s} [\chi(-n) + \chi(n)]$$

$$= q \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$$= qL(s, \chi),$$

which is equivalent to

$$L(s, \chi) = \frac{1}{q} \sum_{j=1}^{q} G(j, \chi) \sum_{n=1}^{\infty} \frac{\sin \left( \frac{2\pi nj}{q} \right)}{n^s}. \quad (15)$$

When $\chi(-1) = -1$, we have

$$i \sum_{j=1}^{q} G(j, \chi) \sum_{n=1}^{\infty} \frac{\sin \left( \frac{2\pi nj}{q} \right)}{n^s}$$

$$= \sum_{j=1}^{q} \sum_{m=1}^{q} \chi(m) e^{\frac{2\pi jm}{q}} \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi nj}{q}} - e^{-\frac{2\pi nj}{q}}}{2n^s}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{m=1}^{q} \chi(m) \sum_{j=1}^{q} \left( e^{\frac{2\pi (m+n)j}{q}} - e^{\frac{2\pi (m-n)j}{q}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{q}{2n^s} [\chi(-n) - \chi(n)]$$

$$= -q \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$$= -qL(s, \chi),$$

which is equivalent to

$$L(s, \chi) = -\frac{i}{q} \sum_{j=1}^{q} G(j, \chi) \sum_{n=1}^{\infty} \frac{\cos \left( \frac{2\pi nj}{q} \right)}{n^s}. \quad (16)$$

According to the equations (15) and (16), when $s$ and $\chi$ have the same parity, we have

$$L(s, \chi) = \frac{(-i)^{s \mod 2}}{q} \sum_{j=1}^{q} G(j, \chi) \sum_{n=1}^{\infty} \frac{\sin^{s \mod 2 \left( \frac{2\pi nj}{q} \right)} \cos^{s \mod 2 \left( \frac{2\pi nj}{q} \right) + 1}}{n^s} \cos \left( \frac{2\pi nj}{q} \right). \quad (17)$$
When \( s \) and \( \chi \) have the different parity, \( s > 1 \), we have
\[
L(s, \chi) = \frac{(-i)^{s(\text{mod} \ 2)}}{q} q \sum_{j=1}^{q} G(j, \chi) \sum_{n=0}^{\infty} \frac{\sin^{(s+1)(\text{mod} \ 2)}(\frac{2\pi j}{q}) \cos^{(s)(\text{mod} \ 2)}(\frac{2\pi j}{q})}{n^s}.
\]

This proves Theorem 1. Next we will prove Theorem 2 from the formula (17).

**Proof of Theorem 2.** Consider the part of trigonometric function separately
\[
\sum_{n=1}^{\infty} \frac{\sin^{s(\text{mod} \ 2)}(\frac{2\pi n j}{q}) \cos^{(s+1)(\text{mod} \ 2)}(\frac{2\pi n j}{q})}{n^s},
\]
which is equivalent to the following function sequence we denote as \( \{F_s(x)\} \) at the point \( x = \frac{2\pi j}{q} \)
\[
\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, \sum_{n=1}^{\infty} \frac{\cos nx}{n^4}, \sum_{n=1}^{\infty} \frac{\sin nx}{n^5} \cdots (18)
\]

Therefore we can rewrite the formula (17) as following
\[
L(s, \chi) = \frac{(-i)^{s(\text{mod} \ 2)}}{q} q \sum_{j=1}^{q} G(j, \chi) F_s \left( \frac{2\pi j}{q} \right).
\]  
(19)

For formula (18), we have recursion obviously,
\[
\begin{align*}
F_1(x) &= \frac{x}{2} - \frac{\pi}{2}; \\
F_n(x) &= (-1)^n F_{n+1}(x), \\
\end{align*}
\]  
(20)

where the leading term is obtained by Lemma 2. If we give another function sequence \( \{G_n(x)\} \) which is defined by
\[
\begin{align*}
G_n(x) &= -n! F_n(x), \text{ when } n \equiv 0, 1(\text{mod} \ 4); \\
G_n(x) &= n! F_n(x), \text{ when } n \equiv 2, 3(\text{mod} \ 4). \\
\end{align*}
\]  
(21)

Then we have \( \{G_n(x)\} \) is a function sequence satifying the requirement of Lemma 1. Therefore, a sequence \( \{a_n\} \) exists to determine \( \{G_n(x)\} \). What means, if we find the general term formula of \( \{a_n\} \), we find that of \( \{F_n(x)\} \). When \( 0 < x < 2\pi \), through the formulae (18) and (20), we have
\[
\begin{align*}
G_0(x) &= 1/2; \\
G_1(x) &= \frac{x}{2} - \frac{\pi}{2}; \\
G_2(x) &= \frac{x^2}{2} - \pi x + \frac{\pi^2}{3}; \\
G_3(x) &= \frac{x^3}{2} - 3\pi x^2 + \pi^2 x + 0; \\
G_4(x) &= \frac{x^4}{2} - 2\pi x^3 + \pi^2 x^2 + 0x + \frac{4\pi^2}{15}; \\
G_5(x) &= \frac{x^5}{2} - 5\pi x^4 + 10\pi^2 x^3 - \frac{4\pi^2 x}{3} + 0; \\
\end{align*}
\]

\[\ldots\]\
Then list the constant term of the polynomial $G_n(x)$, we get the sequence $\{a_n\}$ as following

$$
a_0 = 1/2; \\
a_1 = -\pi/2; \\
a_{2k} = (2k)! \sum_{j=1}^{\infty} \frac{1}{n^{2k}}; \\
a_{2k+1} = 0, \\
(k = 1, 2, 3 \ldots)
$$

(22)

where $\sum_{j=1}^{\infty} \frac{1}{n^{2k}}$ is the value of Riemann $\zeta$ function on the even integers as

$$
\zeta(2k) = \frac{(-1)^{k+1} B_{2k}(2\pi)^{2k}}{2(2k)!}.
$$

Let $[t]$ denote the round down of $t$ and $(x^n)^{(t)}$ denote the $t$-order derivative of $x^n$, from Lemma 1 and formula (22), so we have

$$
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k} \\
= \frac{x^n}{2} - \frac{n\pi x^{n-1}}{2} + \sum_{k=1}^{\lfloor n/2 \rfloor} (2k)! \left( \binom{n}{2k} \zeta(2k) x^{n-2k} \right) \\
= \frac{x^n}{2} - \frac{n\pi x^{n-1}}{2} + \sum_{k=1}^{\lfloor n/2 \rfloor} \zeta(2k)(x^n)^{(2k)}.
$$

(23)

Plug the formula (23) into the formula (21), considering the formula (19) and we get

$$
L(s, \chi) = \begin{cases} \\
-\frac{1}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s\pi x^{s-1}}{2} + \sum_{k=1}^{\lfloor s/2 \rfloor} \zeta(2k)(x^s)^{(2k)} \right]_{x=\frac{2\pi j}{q}}, & s \equiv 0 \pmod{4}; \\
\frac{i}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s\pi x^{s-1}}{2} + \sum_{k=1}^{\lfloor s/2 \rfloor} \zeta(2k)(x^s)^{(2k)} \right]_{x=\frac{2\pi j}{q}}, & s \equiv 1 \pmod{4}; \\
\frac{1}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s\pi x^{s-1}}{2} + \sum_{k=1}^{\lfloor s/2 \rfloor} \zeta(2k)(x^s)^{(2k)} \right]_{x=\frac{2\pi j}{q}}, & s \equiv 2 \pmod{4}; \\
-\frac{i}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s\pi x^{s-1}}{2} + \sum_{k=1}^{\lfloor s/2 \rfloor} \zeta(2k)(x^s)^{(2k)} \right]_{x=\frac{2\pi j}{q}}, & s \equiv 3 \pmod{4}.
\end{cases}
$$

In conclusion

$$
L(s, \chi) = -\frac{(-i)^s}{qs!} \sum_{j=1}^{q} G(j, \chi) \left[ \frac{x^s}{2} - \frac{s\pi x^{s-1}}{2} + \sum_{k=1}^{\lfloor s/2 \rfloor} \zeta(2k)(x^s)^{(2k)} \right]_{x=\frac{2\pi j}{q}}.
$$

This proves Theorem 2.
Proof of Corollary 1 We can easily get formula (9) from Theorem 2 for $s = 2$. For formula (10), we remark that definitive integral appearing in the Corollary 1 is an improper integral. It converges because

$$\int_{0}^{\pi/2} \log(2 \sin \frac{x}{2}) \, dx = -4\pi \log 2.$$ 

Through the Lemma 2, when $x \in (0, 2\pi)$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = \int \sum_{n=1}^{\infty} \frac{\cos nx}{n} + C = \int_{0}^{x} - \log 2(\sin \frac{t}{2}) \, dt$$

So when $s = 2$ in Theorem 2, taking $x = \frac{2\pi j}{q}$, we have

$$L(2, \chi) = \frac{1}{q} \sum_{j=1}^{q} G(j, \chi) \int_{0}^{\frac{2\pi j}{q}} \log(2 \sin \frac{x}{2}) \, dx.$$ 

The proof of Corollary 1 has already been completed.

References

[1] Alkan E. On the mean square average of special values of L-functions. Journal of Number Theory, 2011, 131(8): 1470-1485.

[2] Alkan E. Values of Dirichlet L-functions, Gauss sums and trigonometric sums. The Ramanujan Journal, 2011, 26(3): 375-398.

[3] Apostol T M. Introduction to analytic number theory. Springer Science, Business Media, 1998.

[4] Balasubramanian R. A note on Dirichlet’s L-functions. Acta Arithmetica, 1980, 38: 273-283.

[5] Heath-Brown D R. An asymptotic series for the mean value of DirichletL-functions. Commentarii Mathematici Helvetici, 1981, 56(1): 148-161.

[6] Heath-Brown D R. Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression. Proceedings of the London Mathematical Society, 1992, 3(2): 265-338.

[7] Louboutin S. Quelques formules exactes pour des moyennes de fonctions L de Dirichlet. Canadian Mathematical Bulletin, 1993, 36(2): 190-196.
[8] Louboutin S. The mean value of $|L(k, \chi)|^2$ at positive rational integers $k \geq 1$, Colloquium Mathematicum. Instytut Matematyczny Polskiej Akademii Nauk, 2001, 90: 69-76.

[9] Ma R., Zhang Y., Grtzmann M. Some notes on identities for Dirichlet L-functions. Acta Mathematica Sinica, English Series, 2014, 30(5): 747-754.

[10] Ma R., Niu Y., Zhang Y., On asymptotic properties of the generalized Dirichlet L-functions. International Journal of Number Theory, 2019, 15(6): 1305-1321.

[11] Mordell L J. The reciprocity formula for Dedekind sums. American Journal of Mathematics, 1951, 73(3): 593-598.

[12] Walum H. An exact formula for an average of $L$-series. Illinois Journal of Mathematics, 1982, 26(1): 1-3.

[13] Zhang W. Lecture notes in contemporary mathematics, Beijing: Science Press China, 173-179, 1989.

[14] Zhang W. On the General Dedekind Sums and One Kind Identities of Dirichlet L-Functions. Acta Mathematica Sinica, 2001, 44(2): 269-272.