Free-energy formula for emittance-growth estimation in intense mismatched beams

Kazuya Osaki and Hiromi Okamoto

Graduate School of Advanced Sciences of Matter, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima 739-8530, Japan

E-mail: okamoto@sci.hiroshima-u.ac.jp

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We construct a theoretical model that allows a quick estimate of emittance growth in an intense charged-particle beam initially mismatched to an external linear focusing potential. The present theory is a natural generalization of Reiser’s free-energy model for coasting round beams in a uniform focusing channel. The free energy generated by a spatial mismatch, i.e. a discrepancy between the ideal beam size and an actual beam size, is calculated for an ellipsoidal bunch with an arbitrary aspect ratio. Following Reiser’s prescription, we assume that the excess free energy is converted into root-mean-squared emittance growth. Multi-particle simulations are performed for comparison with theoretical predictions, which indicates that an initially mismatched bunch eventually settles into a sort of thermally anisotropic state when the mismatch is large. It is shown that the free-energy formula can explain simulation results over a wide range of parameters if the degree of the temperature anisotropy in the final state is properly incorporated into the theory.

Subject Index G10, G11

1. Introduction

There are many potential sources of instabilities that seriously deteriorate the quality of a charged-particle beam. Even if the operating point of an accelerator is properly chosen on the tune diagram to avoid intrinsic resonance lines, the beam may still be unstable due to various extra factors including magnetic error fields, coupling-impedance sources, radio-frequency (rf) noises, etc. (see, e.g., Ref. [1]). At high beam density, the natural Coulomb potential also plays an important role, leading to significant emittance growth even without these external driving forces. A typical example is the instability caused by an initial mismatch between the ideal beam configuration in phase space and the actual beam shape. It is known that a low-density particle cloud called a beam halo is developed around the central core if the mismatch is large (Refs. [2–9], and R. A. Jameson, Los Alamos Report LA-UR-93-1209, 1993 (unpublished)).

The self-consistent treatment of such a collective effect is extremely difficult. Since the basic equations of motion are too complex to solve mathematically, we try numerical approaches in many cases. Particle-in-cell (PIC) codes are often employed for this purpose, but high-precision PIC simulations are quite time-consuming. In particular, extremely long CPU time is required to simulate the collective behavior of an intense long bunch containing a huge number of particles. It is thus useful in practice to have any mathematical formula that enables us to make a quick estimate of emittance growth expected in a space-charge-dominated beam under a certain non-ideal condition.
Consider an intense beam traveling through a uniform focusing channel. The total energy \( E \) per particle, which includes the kinetic energy and the potential energies of the external focusing field and Coulomb self-field, is conserved in this closed system. \( E \) should be minimum when the beam is in the perfect stationary state at the beginning. Any realistic beam is, however, more or less deviated from the stationary state, which implies that the system usually possesses an excess energy. Reiser assumed that this free energy is consumed to increase the beam emittance. He developed a simple analytic theory under this assumption to evaluate the possible root-mean-squared (rms) emittance growth in a non-stationary coasting round beam \[10\]. We here generalize his theory to treat an intense bunched beam focused by linear external forces in all three dimensions. An ellipsoidal bunch with rotational symmetry is assumed for the sake of simplicity because the horizontal and vertical betatron tunes are often close in ordinary beam transport channels. The excess free energy is calculated in the case where a stationary bunch in the thermal equilibrium is spatially distorted in both the transverse and longitudinal directions. Such a spatial mismatch is unavoidable in practice.

The paper is organized as follows. In Sect. 2, we first outline the Hamiltonian model employed for the present study and then derive formulas to estimate emittance growth rates in a non-stationary bunch. The free energy produced by a deformation of the bunch shape is calculated in Sect. 3. We also discuss temperature anisotropy developed during the relaxation process of the initially deformed (mismatched) bunch. On the basis of the free-energy and temperature-anisotropy equations given in Sect. 3, the possible emittance growth due to an initial mismatch is evaluated in Sect. 4 and compared with self-consistent multi-particle simulations. Concluding remarks are finally given in Sect. 5.

2. Model

2.1. Basic equations for a uniformly populated bunch

As is well known, the motion of a charged particle in a dense beam traveling through a linear focusing channel obeys the Hamiltonian

\[
H = \frac{p_x^2 + p_y^2 + p_z^2}{2} + V_{\text{ext}} + \frac{2\pi \varepsilon_0 K_p}{Nq} U_C, \tag{1}
\]

where \( V_{\text{ext}} \) is the external potential provided by beam-focusing magnets, \( U_C \) is the Coulomb self-field potential, \( q \) is the charge state of the particle, \( \varepsilon_0 \) is the vacuum permittivity, \( K_p \) is the generalized beam perveance, and the independent variable is the path length \( s \) along the design beam orbit. For a beam traveling at a speed \( \beta c \) with \( c \) being the speed of light, the perveance is defined by

\[
K_p = 2 N r_p / \beta^2 \gamma^3
\]

where \( N \) is the number of particles in a bunch, \( r_p \) is the classical particle radius, and \( \gamma \) is the Lorentz factor, i.e., \( \gamma = 1/(1 - \beta^2)^{1/2} \). Employing the smooth approximation, we can express \( V_{\text{ext}} \) as

\[
V_{\text{ext}} = \frac{1}{2} \left( k_x^2 x^2 + k_y^2 y^2 + k_z^2 z^2 \right), \tag{2}
\]

where \( (k_x, k_y) \) and \( k_z \) determine the beam-focusing strengths on the transverse \( x-y \) plane and in the longitudinal \( z \)-direction. These focusing parameters are proportional to the phase advances or, in other words, the tunes at zero beam intensity. The harmonic oscillator model as in Eq. (2) has been frequently used in past theoretical studies of intense beam dynamics \[11\]. For an ellipsoidal bunch with rotational symmetry, we can put \( k_x = k_y (\equiv k_\perp) \). At high beam intensity, the spatial particle distribution is homogenized due to the natural Debye screening effect. We thus assume that the charge density \( \rho \) is approximately uniform within the ellipsoidal boundary \( (x^2 + y^2) / a^2 + z^2 / b^2 = 1 \).
The corresponding Coulomb potential is given by

\[ U_C = -\frac{\rho a^2 b}{4\varepsilon_0} \int_0^\infty \frac{1}{(a^2 + \sigma)(b^2 + \sigma)^{1/2}} \left( \frac{x^2 + y^2}{a^2 + \sigma} + \frac{z^2}{b^2 + \sigma} \right) d\sigma. \]  

(3)

When the bunch contains \( N \) particles, \( \rho = Nq/(4\pi a^2 b/3) \). Substituting Eqs. (2) and (3) into Eq. (1), we have

\[ H = \frac{p_x^2 + p_y^2 + p_z^2}{2} + \frac{1}{2} k_\perp f_\perp(a, b) \left( x^2 + y^2 \right) + \frac{1}{2} k_\parallel f_\parallel(a, b) z^2, \]

(4)

where

\[ f_\perp(a, b) = 1 - \frac{3K_p}{4k_\perp^2} \int_0^\infty \frac{d\sigma}{(a^2 + \sigma)^2(b^2 + \sigma)^{1/2}} \]

and

\[ f_\parallel(a, b) = 1 - \frac{3K_p}{4k_\parallel^2} \int_0^\infty \frac{d\sigma}{(a^2 + \sigma)(b^2 + \sigma)^{3/2}}. \]

For later convenience, we introduce several useful equations of second moments. In the transverse \( x \)-direction the canonical equations of motion are derived from Hamiltonian (4) as \( dx/ds = p_x \) and \( dp_x/ds = -k_\perp f_\perp(a, b)x \). Provided that the particle distribution function obeys the Vlasov equation in phase space, the use of these canonical equations allows us to obtain \( da_{\text{rms}}/ds = \langle xp_x \rangle/a_{\text{rms}} \) and \( d\langle xp_x \rangle/ds = \langle p_x^2 \rangle - k_\perp f_\perp(a, b)a_{\text{rms}}^2 \), where the symbol \( \langle A \rangle \) stands for averaging the quantity \( A \) over all particles and \( a_{\text{rms}} \) is the rms beam size defined by \( a_{\text{rms}} = \langle x^2 \rangle^{1/2} \). \( a_{\text{rms}} \) satisfies the rms envelope equation

\[ \frac{d^2a_{\text{rms}}}{ds^2} + k_\perp^2 f_\perp(a, b)a_{\text{rms}} - \frac{\varepsilon_\perp^2}{a_{\text{rms}}^3} = 0, \]

(5)

where \( \varepsilon_\perp \) is the transverse rms emittance defined by \( \varepsilon_\perp = (a_{\text{rms}}^2 \langle p_x^2 \rangle - \langle xp_x \rangle^2)^{1/2} \). Similar second-moment equations hold for the other two directions. While a uniform particle density has been assumed here, the rms envelope equations are insensitive to the type of distribution function, as theoretically proven by Sacherer [12].

Ideally, an intense beam injected into an accelerator should be not only well matched to the machine lattice but also in thermal equilibrium (or, in other words, equipartitioned [13]). No emittance growth occurs in that case. The beam is perfectly stationary under the uniform restoring force generated by \( V_{\text{ext}} \), so we write \( a \equiv a_0(= \text{const.}) \) and \( b \equiv b_0(= \text{const.}) \). Since \( a_{\text{rms}} = a/\sqrt{5} \) and \( b_{\text{rms}} = b/\sqrt{5} \) for a uniformly populated bunch, the rms envelope equations lead to

\[ \varepsilon_\perp^{(0)} = \frac{k_\perp \eta_\perp a_0^2}{5} \quad \text{and} \quad \varepsilon_\parallel^{(0)} = \frac{k_\parallel \eta_\parallel b_0^2}{5}, \]

(6)

where \( \varepsilon_\perp^{(0)} \) and \( \varepsilon_\parallel^{(0)} \) represent the transverse and longitudinal rms emittances of the matched beam, and the so-called tune depressions have been introduced as \( \eta_\perp = [f_\perp(a_0, b_0)]^{1/2} \) and \( \eta_\parallel = [f_\parallel(a_0, b_0)]^{1/2} \). By definition, the tune depressions become unity at the low-beam-intensity limit, i.e. \( K_p \to 0 \). As the beam density increases, both parameters approach zero. Assuming a
matched beam initially equipartitioned, we obtain [13]

\[
\frac{\varepsilon_{\parallel}^{(0)}}{\varepsilon_{\perp}^{(0)}} = \frac{k_{\parallel} \eta_{\perp}}{k_{\parallel} \eta_{\parallel}} = R_0, \tag{7}
\]

where \( R_0 = b_0/a_0 \) is the aspect ratio of the matched ellipsoid. Under this condition, the tune depression factors can be written as

\[
\eta_{\perp}^2 = 1 - Q_0 I_{\perp}(R_0) \quad \text{and} \quad \eta_{\parallel}^2 = 1 - Q_0 \left( \frac{\eta_{\parallel}}{\eta_{\perp}} \right)^2 I_{\parallel}(R_0), \tag{8}
\]

where \( Q_0 = 3K_p/(4(k_{\perp}a_0)^2a_0) \),

\[
I_{\perp}(R_0) = \int_0^\infty \frac{d\sigma'}{(1 + \sigma')^2 \left( R_0^2 + \sigma' \right)^{3/2}} \quad \text{and} \quad I_{\parallel}(R_0) = \int_0^\infty \frac{d\sigma'}{(1 + \sigma') \left( R_0^2 + \sigma' \right)^{3/2}}.
\]

Equations (8) indicate that, given \( \eta_{\perp} \) and \( \eta_{\parallel} \), the parameter \( Q_0 \) and the aspect ratio \( R_0 \) are uniquely determined for the equipartitioned bunch.

### 2.2. Emittance growth due to excess free energy

If a bunch is perfectly matched to the external focusing potential, the total energy of the system takes the minimum value \( W_0 \). Unfortunately, it is impossible in any realistic cases to establish such a perfect stationary state at the beginning; the beam is more or less deviated from the ideal condition because of unavoidable artificial errors. The beam then possesses a greater energy \( W_i (> W_0) \) depending on the degree of the initial mismatch. The mismatched bunch cannot be stationary but starts to execute a complex collective motion. It is reasonable to expect that, after some relaxation period, the non-stationary beam will settle into a stationary state with final energy \( W_f \) [10]. The excess energy \( \Delta W = W_f - W_0 \) is, according to Reiser, consumed to increase the rms emittance. Since the energy conservation law requires \( W_f = W_i \), the emittance growth rate is directly linked to how much free energy is produced at the beginning by a certain mismatch.

The average total energy of the system per particle is the sum of the kinetic energy \( E_k \), the beam-focusing potential \( E_p \), and the Coulomb self-field energy \( E_C \). In the case of an initially matched beam, the second-moment equations yield the simple relations \( \langle p_{x}^2 \rangle = k_{\perp}^2 a_0^2 f_{\perp}(a_0, b_0)/5 \), etc. because everything is static; namely, all \( s \)-derivatives vanish. We then readily find

\[
E_k = \frac{1}{2} \left( \langle p_{x}^2 \rangle + \langle p_{y}^2 \rangle + \langle p_{z}^2 \rangle \right) = \frac{(k_{\perp} \eta_{\perp} a_0)^2}{5} + \frac{(k_{\parallel} \eta_{\parallel} b_0)^2}{10}. \tag{9}
\]

On the other hand, \( E_p \) is given by

\[
E_p = \frac{(k_{\perp} a_0)^2}{5} + \frac{(k_{\parallel} b_0)^2}{10}. \tag{10}
\]

Integrating the Coulomb potential over the whole bunch, we obtain the average self-field energy per particle:

\[
E_C = \frac{3K_p}{16} \int_0^\infty \frac{d\sigma}{(a_0^2 + \sigma) (b_0^2 + \sigma)^{1/2}} - \frac{(k_{\perp} a_0)^2}{10} \left( 1 - \eta_{\perp}^2 \right) - \frac{(k_{\parallel} b_0)^2}{20} \left( 1 - \eta_{\parallel}^2 \right). \tag{11}
\]

Hence, the minimum energy of the matched state can be calculated from

\[
W_0 = \frac{(k_{\perp} a_0)^2}{10} \left( 1 + 3\eta_{\perp}^2 \right) + \frac{(k_{\parallel} b_0)^2}{20} \left( 1 + 3\eta_{\parallel}^2 \right) + \frac{3K_p}{16} \int_0^\infty \frac{d\sigma}{(a_0^2 + \sigma) (b_0^2 + \sigma)^{1/2}}. \tag{12}
\]
In the final stationary state reached from a certain mismatched beam, the semi-axes of the bunch are no longer \( a_0 \) and \( b_0 \) but changed to, say, \( a_f \) and \( b_f \). The total energy \( W_f \) can be expressed similarly to Eq. (12) as long as the bunch density stays approximately uniform:

\[
W_f = \frac{(k_f a_f)^2}{10} \left[ 1 + 3 f_\perp (a_f, b_f) \right] + \frac{(k_f b_f)^2}{20} \left[ 1 + 3 f_\parallel (a_f, b_f) \right] \\
+ \frac{3K_p}{16} \int_0^\infty \frac{d\sigma}{(a_f^2 + \sigma)(b_f^2 + \sigma)^{1/2}}.
\] (13)

Although an initial mismatch often develops a low-density tail around the beam core, the number of these halo particles is typically a few percent of \( N \). We, therefore, assume a uniform density profile to be approximately valid in the final state, as Reiser did in his original work for a coasting beam [10]. By expanding \( W_f \) about the matched state and keeping only low-order terms, an approximate expression of the excess energy \( \Delta W \) takes the form

\[
\frac{\Delta W}{(k_\perp a_0)^2} \approx \frac{1}{20} \left( 11 \eta_\perp^2 - 3 \right) \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] + \frac{3Q_0}{20} \left[ 4I_1 (R_0) \left( \frac{a_f}{a_0} \right)^2 + I_2 (R_0) \left( \frac{b_f}{b_0} \right)^2 \right] \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] \\
+ \frac{3Q_0}{40} \left[ 2I_2 (R_0) \left( \frac{a_f}{a_0} \right)^2 + 3I_3 (R_0) \left( \frac{b_f}{b_0} \right)^2 \right] \left[ \left( \frac{b_f}{b_0} \right)^2 - 1 \right].
\] (14)

where we have used Eq. (7) and introduced the following integral functions of \( R_0 \):

\[
I_1 (R_0) = \int_0^\infty \frac{d\sigma'}{(1 + \sigma')^3 (R_0^2 + \sigma')^{1/2}}, \quad I_2 (R_0) = R_0^2 \int_0^\infty \frac{d\sigma'}{(1 + \sigma')^2 (R_0^2 + \sigma')^{3/2}}, \quad \text{and} \quad I_3 (R_0) = R_0^2 \int_0^\infty \frac{d\sigma'}{(1 + \sigma') (R_0^2 + \sigma')^{5/2}}.
\]

For a spherical beam where \( R_0 = 1 \), we have \( I_1 = I_2 = I_3 = 2/5 \).

After the relaxation process of an initially mismatched beam is completed, the transverse rms emittance has reached the approximate final value \( \varepsilon_\perp^{(f)} \approx k_\perp a_f^2 \sqrt{f_\perp (a_f, b_f)} \). The emittance growth rate can thus be estimated from

\[
\frac{\varepsilon_\perp^{(f)}}{\varepsilon_\perp^{(0)}} \approx \frac{1}{\eta_\perp} \left( \frac{a_f}{a_0} \right)^2 \sqrt{f_\perp (a_f, b_f)} \\
\approx \left( \frac{a_f}{a_0} \right)^2 \sqrt{1 + \frac{2Q_0 I_1 (R_0)}{\eta_\perp^2} \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] + \frac{Q_0 I_2 (R_0)}{2\eta_\perp^2} \left[ \left( \frac{b_f}{b_0} \right)^2 - 1 \right]}.
\] (15)

Similarly, we have the longitudinal emittance-growth formula:

\[
\frac{\varepsilon_\parallel^{(f)}}{\varepsilon_\parallel^{(0)}} \approx \frac{1}{\eta_\parallel} \left( \frac{b_f}{b_0} \right)^2 \sqrt{f_\parallel (a_f, b_f)} \\
\approx \left( \frac{b_f}{b_0} \right)^2 \sqrt{1 + \frac{Q_0 I_2 (R_0)}{\eta_\perp^2} \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] + \frac{3Q_0 I_3 (R_0)}{2\eta_\perp^2} \left[ \left( \frac{b_f}{b_0} \right)^2 - 1 \right]}.
\] (16)
As already mentioned, the values of \( Q_0 \) and \( R_0 \) are fixed once the tune depressions \( \eta_\perp \) and \( \eta_\parallel \) are chosen. Therefore, the additional pieces of information required to find the emittance growth rates are the ratios \( a_f/a_0 \) and \( b_f/b_0 \).

### 3. Determination of \( a_f/a_0 \) and \( b_f/b_0 \)

Unlike the previous study by Reiser [10], two independent conditions are necessary to determine the emittance growth rates because we have one more dimension. The first condition can be obtained from Eq. (14) by calculating \( \Delta W \) for a particular non-stationary initial state. We here focus our discussion on one of the most probable initial errors in general accelerators, i.e., a mismatch in the transverse and longitudinal beam sizes at injection. The corresponding free-energy formula can be given as a function of known parameters, which we employ to derive a relation of \( a_f/a_0 \) and \( b_f/b_0 \). Although we still need one more condition to determine these ratios, it seems difficult to deduce another useful relation from a simple physical hypothesis. In the latter part of this section, therefore, we try to introduce an empirical formula on the basis of information from self-consistent numerical simulations.

#### 3.1. Free-energy formula for a spatially mismatched bunch

Let us consider a bunch that initially has semi-axes of \( a_i \) (transverse) and \( b_i \) (longitudinal). Ideally, \( a_i \) and \( b_i \) must be adjusted precisely to the matched values \( a_0 \) and \( b_0 \), but, in reality, we can only achieve an approximate matching where \( a_i = \xi_\perp a_0 \) and \( b_i = \xi_\parallel b_0 \) with positive constants \( \xi_\perp \) and \( \xi_\parallel \) called mismatch factors. The total energy \( W_i \) of a mismatched bunch can be calculated in the same way as described in the last section:

\[
W_i = \frac{(k_\perp a_0)^2}{10} \left[ \xi_\perp^2 + 2 \left( \frac{\eta_\perp}{\xi_\perp} \right)^2 + \xi_\perp f_\perp (\xi_\perp a_0, \xi_\parallel b_0) \right] + \frac{(k_\parallel b_0)^2}{20} \left[ \xi_\parallel^2 + 2 \left( \frac{\eta_\parallel}{\xi_\parallel} \right)^2 + \xi_\parallel f_\parallel (\xi_\perp a_0, \xi_\parallel b_0) \right] + \frac{3K_p}{16} \int_0^\infty \frac{d\sigma}{(\xi_\perp^2 a_0^2 + \sigma)(\xi_\parallel^2 b_0^2 + \sigma)^{1/2}},
\]

where we have assumed that the mismatched beam initially has the same rms emittance as the matched beam. Provided that the mismatch is not too large, the free energy can be written as

\[
\frac{\Delta W}{(k_\perp a_0)^2} = \frac{W_i - W_0}{(k_\perp a_0)^2} \approx \frac{\xi_\perp^2 - 1}{20} \left[ \left( \frac{\eta_\perp}{\xi_\perp} \right)^2 (7\xi_\perp^2 - 4) - 3 + 4Q_0\xi_\perp^2I_1(R_0) + Q_0\xi_\perp^2I_2(R_0) \right] + \frac{\xi_\parallel^2 - 1}{40} \left[ \left( \frac{\eta_\parallel}{\xi_\parallel} \right)^2 (7\xi_\parallel^2 - 4) - 3 \left( \frac{\eta_\perp}{\eta_\parallel} \right)^2 + 2Q_0\xi_\parallel^2I_2(R_0) + 3Q_0\xi_\parallel^2I_3(R_0) \right].
\]

Given the design tune depressions \( \eta_\perp \) and \( \eta_\parallel \), we can readily evaluate \( \Delta W/(k_\perp a_0)^2 \) from Eq. (18) for a non-stationary bunch that has specific mismatch factors at injection. We then substitute the obtained value of \( \Delta W/(k_\perp a_0)^2 \) into the left-hand side of Eq. (14), which leads to a simple algebraic equation of \( a_f/a_0 \) and \( b_f/b_0 \).
3.2. Temperature anisotropy after relaxation

It may be rational to think that an initially mismatched bunch will relax into an equipartitioned state. If that is the case, the temperature ratio \( T_f \equiv k_\perp a_\perp^2 f_\perp (a_f, b_f)/k_\parallel b_\parallel^2 f_\parallel (a_f, b_f) \) should be close to unity after a final quasi-equilibrium is reached. We can then use the equation \( T_f \approx 1 \) as the second condition to determine \( a_f/a_0 \) and \( b_f/b_0 \). We have, however, found through numerical simulations that the final state is often quite anisotropic. Only when \( R_0 \approx 1 \) (a spherical bunch), the bunch stays near an equipartitioned state unless the mismatch is too large. In order to judge whether a mismatched beam has reached a quasi-equilibrium state, we paid attention to the time evolution of the transverse and longitudinal rms emittances. Figure 1 shows the typical emittance evolution calculated with the particle-in-cell code “Warp” [14]. When the mismatch is large, the rms emittances rapidly grow at the beginning in both the transverse and longitudinal directions and then come to a plateau. Since weak emittance oscillations do not completely vanish within a reasonable CPU time, we take an average over the last few tens of betatron periods to estimate the equilibrium emittance and temperature after relaxation\(^1\). The temperature ratio \( T_f \) in the final state is plotted in Fig. 2 at several different bunch densities and aspect ratios as a function of the transverse mismatch factor \( \xi_\perp \). The longitudinal mismatch factor \( \xi_\parallel \) is varied in the range \( 0.8 \leq \xi_\parallel \leq 1.2 \). The initial distribution of macro-particles is uniform in real space and Maxwellian in velocity space\(^2\). The figure suggests that the temperature anisotropy in the final state is enhanced as we increase the mismatch and/or bunch density. The observed dependence of the anisotropy on free parameters needs to be incorporated properly into our model. Among a number of possible choices, we here try the following function:

\[
T_f (\xi_\perp, \xi_\parallel; \eta_\perp, R_0) \approx 1 + \frac{R_0 - 1}{R_0} \left[ \left( A_1 e^{-A_2 R_0} + A_3 \right) \frac{\left( \xi_{\perp}^{\text{sgn}(\xi_{\perp} - 1)} - 1 \right)^2}{\eta_\perp} + \left( B_1 e^{-B_2 R_0} + B_3 \right) \frac{\left( \xi_{\parallel}^{\text{sgn}(\xi_{\parallel} - 1)} - 1 \right)^2}{\eta_\perp^3} \right], \tag{19}
\]

where the fitting constants are given by \( A_1 = 1.67, A_2 = 0.27, A_3 = 1.46, B_1 = -1.49, B_2 = 0.43, \) and \( B_3 = -1.71 \). As demonstrated in Fig. 2 (solid curves), this function fits the numerical results (colored dots) over a sufficiently wide range of parameters. Naturally, the temperature anisotropy in the final state is not symmetric about the matched line \( \xi_\perp = 1 \) because the bunch initially becomes denser with \( \xi_\perp < 1 \) and thinner with \( \xi_\perp > 1 \). From the equation \( T_f = k_\perp a_\perp^2 f_\perp (a_f, b_f) / k_\parallel b_\parallel^2 f_\parallel (a_f, b_f) \), we have

\[
T_f (\xi_\perp, \xi_\parallel; \eta_\perp, R_0) \cdot \left( \frac{b_f}{b_0} \right)^2 \left\{ 1 + \frac{Q_0 I_2 (R_0)}{\eta_\perp^2} \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] + \frac{3 Q_0 I_3 (R_0)}{2 \eta_\perp} \left[ \left( \frac{b_f}{b_0} \right)^2 - 1 \right] \right\} \approx \left( \frac{a_f}{a_0} \right)^2 \left\{ 1 + \frac{2 Q_0 I_1 (R_0)}{\eta_\perp^2} \left[ \left( \frac{a_f}{a_0} \right)^2 - 1 \right] + \frac{Q_0 I_2 (R_0)}{2 \eta_\perp^2} \left[ \left( \frac{b_f}{b_0} \right)^2 - 1 \right] \right\}, \tag{20}
\]

\(^1\) The number of numerical integration steps required for a good estimate of the equilibrium temperature tends to be larger for a longer bunch. We have so far followed the emittance evolution over at most a few thousand betatron oscillation periods in each Warp simulation.

\(^2\) We have confirmed that the bunch is stationary (no emittance growth) without the mismatch, i.e., under the condition \( \xi_\perp = \xi_\parallel = 1 \).
Fig. 1. Time evolution of the transverse and longitudinal rms emittances obtained from Warp simulations. Several different sizes of initial mismatches are applied to a bunch that has the aspect ratio $R_0 = 3$ and the transverse tune depression $\eta_\perp = 0.8$. The transverse mismatch is chosen as either $\xi_\perp = 0.8$, 1.0, or 1.2 while the longitudinal mismatch is fixed at $\xi_\parallel = 1.2$.

Fig. 2. Examples of Warp simulation results on the temperature anisotropy in the final quasi-equilibrium state. The transverse-to-longitudinal temperature ratio $T_f$ is plotted as a function of $\xi_\perp$. Three different values, i.e. 0.8, 1.0, and 1.2, are chosen for the longitudinal mismatch factor $\xi_\parallel$ in each panel. The aspect ratio is fixed at $R_0 = 3$ (short bunch) in the upper three panels while, in the lower three, $R_0 = 30$ (long bunch). The transverse tune depression $\eta_\perp$ is set at 0.98 (left), 0.90 (middle), and 0.80 (right). Colored dots represent multi-particle simulation data obtained with corresponding parameters. Solid lines are the fitting result based on Eq. (19).

which gives another explicit relation of $a_f/a_0$ and $b_f/b_0$ once the initial tune depressions and mismatch factors are chosen. Together with the free-energy equation, i.e. Eq. (14) equated to Eq. (18), we now have sufficient conditions to determine $a_f/a_0$ and $b_f/b_0$, the values of which are inserted into Eqs. (15) and (16) for emittance-growth evaluation.

4. Comparison with multi-particle simulations

Systematic Warp simulations were performed to verify how accurately the present theoretical model predicts the rms emittance growth of an initially mismatched bunch. We considered the fundamental parameters listed in Table 1. As explained above, we only need two given independent parameters
Table 1. Simulation parameters.

| Parameter                      | Values           |
|--------------------------------|------------------|
| Aspect ratio $R_0$             | 1, 2, 3, 6, 10, 30 |
| Transverse tune depression $\eta_\perp$ | 0.98, 0.90, 0.80 |
| Transverse mismatch factor $\xi_\perp$ | 0.7–1.5 |
| Longitudinal mismatch factor $\xi_\parallel$ | 0.8–1.2 |

Fig. 3. Rms emittance growth at $\eta_\perp = 0.98$. The growth rates in the transverse direction (upper panels) and longitudinal direction (lower panels) are plotted as a function of transverse mismatch factor $\xi_\perp$. Colored dots represent Warp simulation results based on the fundamental parameters in Table 1. Different colors stand for different longitudinal mismatch factors: $\xi_\parallel = 0.8$ (blue), 0.9 (red), 1.0 (black), 1.1 (green), and 1.2 (orange). Solid lines are the predictions from the free-energy model. The results obtained with other aspect ratios ($R_0 = 2, 6,$ and 10) are similar to those of $R_0 = 3$ and 30.

Let us first look at a relatively low-density case, setting $\eta_\perp = 0.98$. The longitudinal tune depression $\eta_\parallel$ ranges from 0.98 to 0.89 corresponding to the change of the aspect ratio $R_0$ from 1 to 30. Figure 3(a) shows the transverse rms emittance growth rate plotted as a function of $\xi_\perp$. Five different values of $\xi_\parallel$ are considered in each panel where the aspect ratio is fixed at either 1, 3, or 30. The five colored curves are theoretical predictions based on the emittance-growth formula in Eq. (15). Following the prescription described in previous sections, we first insert the given numbers of $\eta_\perp$ and $R_0$ into the free-energy formula (Eq. (14) together with Eq. (18)) and the temperature anisotropy formula (Eq. (20) together with Eq. (19)). These two conditions are then solved with specific mismatch factors to find the ratios $a_f/a_0$ and $b_f/b_0$ that are substituted in Eq. (15) to predict the emittance growth rate. Colored dots in each panel represent Warp simulation results. We see that the free-energy model can
explain the Warp data very well. Note that, at this tune depression, the transverse emittance growth rate is insensitive to the longitudinal mismatch factor $\xi_\parallel$. Similar results have been obtained for other aspect ratios, i.e. $R_0 = 2$, 6, and 10. Reasonable agreement between the theory and self-consistent simulations is also confirmed for the longitudinal degree of freedom, as demonstrated in Fig. 3(b). Interestingly, the longitudinal growth rate appears to be almost independent of $\xi_\perp$ and $R_0$, except for the spherical bunch ($R_0 = 1$).

We now increase the bunch density from $\eta_\perp = 0.98$ to higher levels where $\eta_\perp$ is set equal to 0.9 or 0.8 initially. Theoretical and simulation results in the case of $\eta_\perp = 0.9$ are summarized in Fig. 4. We again only show the data obtained with $R_0 = 1$, 3, and 30. The dependence of rms emittance growth on the mismatch factors is more or less similar to what we found in Fig. 3. In any case, the theoretical estimate from the free-energy model is in good agreement with the corresponding Warp results over the whole parameter ranges considered here. Figure 5 shows the results of $\eta_\perp = 0.8$. The transverse emittance growth is slightly enhanced compared to the lower-density cases in Figs. 3 and 4 while, in the longitudinal direction, no significant change is observed. We recognize that the accuracy of the theoretical prediction is somewhat worsened, especially in the longitudinal emittance-growth estimate. A possible reason for this is the deterioration of the fitting accuracy by Eq. (19). Another reason could be a deviation of the final bunch profile from the uniform distribution assumed in our model. In fact, a noticeable beam halo is inevitably formed around the core as we increase the bunch density and the degree of initial mismatches. Typical phase-space distributions of mismatched bunches after relaxation are shown in Fig. 6 for reference. Despite these limitations, it is evident from Figs. 3–5 that the present formulas are useful in estimating the rate of potential emittance growth.
Fig. 5. Rms emittance growth at $\eta_\perp = 0.8$. Theoretical predictions from the free-energy model are compared with Warp simulation results. Fundamental parameters considered here are the same as those in Fig. 3 except for the bunch density. The longitudinal tune depression $\eta_\parallel$ ranges from 0.8 to 0.47 corresponding to the change of $R_0$ from 1 to 30.

Fig. 6. Warp simulation results of initially mismatched bunches with $R_0 = 3$. Typical particle distributions after relaxation are plotted in the transverse and longitudinal phase spaces. The upper panels correspond to the case where $\eta_\perp = 0.98$ and $\xi_\perp = \xi_\parallel = 1.2$. The bunch density and initial mismatch are increased in the lower panel where $\eta_\perp = 0.8$ and $\xi_\perp = \xi_\parallel = 1.5$. 

5. Summary

We have developed a simple analytic theory to estimate the rms emittance growth of an intense beam when its spatial extent disagrees with the ideal matched value at injection. Reiser’s free-energy model for a coasting beam is generalized to treat an ellipsoidal bunch focused three-dimensionally by a linear external potential. Given the initial beam density, or, more correctly, two of the four free parameters $(\eta_\perp, \eta_\parallel, R_0, Q_0)$, the stationary matched state is uniquely defined. The transverse and longitudinal emittance growth rates can then be calculated from Eqs. (15) and (16) for a certain degree of initial bunch deformation characterized by the mismatch factors $\xi_\perp$ and $\xi_\parallel$. To find the ratios $a_f/a_0$ and $b_f/b_0$ in these equations, we employ the free-energy formula in Eq. (14) and the temperature-anisotropy formula in Eq. (20). The left-hand side of Eq. (14), i.e. the excess free energy of an initially mismatched bunch, is evaluated from Eq. (18) for specific mismatch factors, which yields an explicit relation of $a_f/a_0$ and $b_f/b_0$. Similarly, Eq. (20) gives another relation of $a_f/a_0$ and $b_f/b_0$ if the temperature anisotropy $T_f$ is known as a function of the mismatch factors. In the present study, we carried out systematic numerical simulations to clarify the parameter dependence of $T_f$. Equation (19) is a possible choice that fits the numerical data over a wide range of parameters at reasonable accuracy. Equation (14) with Eq. (18) and Eq. (20) with Eq. (19) make it possible to determine $a_f/a_0$ and $b_f/b_0$, which are substituted into Eqs. (15) and (16) to obtain the emittance growth rates. We compared theoretical predictions with time-consuming multi-particle simulations, confirming that the present model can explain the self-consistent numerical results fairly well. The simple algebraic equations derived in this paper thus enable us to estimate the degree of mismatch-induced emittance growth in an intense bunch easily and quickly.

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