Asymptotic charges in 3d gravity with torsion

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Abstract. We discuss some new developments in three-dimensional gravity with torsion, based on Riemann-Cartan geometry. Using the canonical approach, we study the structure of asymptotic symmetry, clarify its fundamental role in defining the gravitational conserved charges, and explore the influence of the asymptotic structure on the black hole entropy.

1. Introduction
Although general relativity (GR) successfully describes all the known observational data, such fundamental issues as the nature of classical singularities and the problem of quantization remain without answer. Faced with such difficulties, one is naturally led to consider technically simpler models that share the same conceptual features with GR. A particularly useful model of this type is three-dimensional (3d) gravity [1, 2, 3, 4, 5].

Following a widely spread belief that GR is the most reliable approach to describe the gravitational phenomena, 3d gravity has been studied mainly in the realm of Riemannian geometry. However, there is a more general conception of gravity, based on Riemann-Cartan geometry [6], in which both the curvature and the torsion are used to describe the gravitational dynamics. Here, we focus our attention on some new developments in 3d gravity, in the realm of Riemann-Cartan geometry [7, 8, 9, 10]. We show that the symmetry of anti-de Sitter asymptotic conditions is described by two independent Virasoro algebras with different central charges, in contrast to GR. We also derive the expressions for the related conserved charges, energy and angular momentum, and discuss the new form of the black hole entropy.

2. Basic dynamical features
Theory of gravity with torsion can be formulated as Poincaré gauge theory (PGT), with an underlying geometric structure described by Riemann-Cartan space [6].

Basic gravitational variables in PGT are the triad field $b^i$ and the Lorentz connection $A^i_j = -A^j_i$ (1-forms). The corresponding field strengths are the torsion and the curvature:

\[ T^i = db^i + A^i_m \wedge b^m, \quad R^i_j = dA^i_j + A^i_m \wedge A^m_j \] (2-forms). Gauge symmetries of the theory are local translations and local Lorentz rotations, parametrized by $\xi^\mu$ and $\epsilon^{ij}$.

In 3D, we simplify the notation by introducing $A^{ij} = -\epsilon^{ijk}\omega_k$, $R^{ij} = -\epsilon^{ijk}R_k$, $\epsilon^{ij} = -\epsilon^{ijk}\theta_k$. In local coordinates $x^\mu$, we have $b^i = b^i_\mu dx^\mu$, $\omega^i = \omega^i_\mu dx^\mu$. The field strengths take the form

\[ T^i = db^i + \epsilon^{ijk}\omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \epsilon^{ijk}\omega^j \wedge \omega^k, \] (2.1)
and gauge transformations are

\[ \delta_0 b^i_\mu = -\varepsilon^i_{jk} b^j_\mu \theta^k - (\partial_\mu \xi^e) b^i_\rho - \xi^\rho \partial_\rho b^i_\mu, \quad \delta_0 \omega^i_\mu = -\nabla_\mu \theta^i - (\partial_\mu \xi^e) \omega^i_\rho - \xi^\rho \partial_\rho \omega^i_\mu, \]  

(2.2)

where \( \nabla_\mu \theta^i = \partial_\mu \theta^i + \varepsilon^i_{jk} \omega^j_\mu \theta^k \) is the covariant derivative of \( \theta^i \).

To clarify the geometric meaning of PGT, we introduce the metric tensor as a bilinear combination of the triad fields: \( g = \eta_{ij} b^i \otimes b^j \equiv g_{\mu \nu} dx^\mu \otimes dx^\nu \), where \( \eta_{ij} = (+, -, -) \). Although metric and connection are in general independent geometric objects, in PGT they are related to each other by the metricity condition: \( \nabla g = 0 \). Consequently, the geometric structure of PGT is described by Riemann-Cartan geometry.

General gravitational dynamics is defined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, Mielke and Baekler proposed a topological model for 3D gravity [7], defined by the action

\[ I = a I_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_M, \]

\[ I_1 \equiv 2 \int b^i \wedge R_i, \quad I_2 \equiv -\frac{1}{3} \int \varepsilon_{ijk} b^i \wedge b^j \wedge b^k, \]

\[ I_3 \equiv \int \left( \omega^i \wedge d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^j \wedge \omega^k \wedge \omega^k \right), \quad I_4 \equiv \int b^i \wedge T_i, \]

(2.4)

where \( I_M \) is a matter contribution. The first term, with \( a = 1/16\pi G \), is the usual Einstein-Cartan action, the second term is a cosmological term, \( I_3 \) is the Chern-Simons action for the Lorentz connection, and \( I_4 \) is a torsion counterpart of \( I_1 \). The Mielke-Baekler model is a natural generalization of Riemannian GR with a cosmological constant (GR\(_\Lambda\)).

For isolated gravitational systems, gravitational sources can be practically ignored in the asymptotic region. Hence, the asymptotic structure of the theory is determined by the vacuum field equations. In the sector \( \alpha_3 \alpha_4 - a^2 \neq 0 \), these equations take the simple form

\[ 2T^i = p \varepsilon^i_{jk} b^j \wedge b^k, \quad p \equiv \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \]

\[ 2R^i = q \varepsilon^i_{jk} b^j \wedge b^k, \quad q \equiv -\frac{(\alpha_4)^2 + a \Lambda}{\alpha_3 \alpha_4 - a^2}. \]

(2.5)

Thus, the vacuum configuration is characterized by constant torsion and constant curvature. For \( p = 0 \) or \( q = 0 \), the vacuum geometry is Riemannian \( (T^i = 0) \) or teleparallel \( (R^i = 0) \).

In Riemann-Cartan spacetime, one can use the identity (2.3) to express the curvature \( R^i(\omega) \) in terms of its Riemannian piece \( \tilde{R}^i = R^i(\tilde{\omega}) \) and the contortion: \( R^i(\omega) = \tilde{R}^i + \nabla K^i - \frac{1}{2} \varepsilon^{imn} K_m \wedge K_n \). This result, combined with the field equations (2.5), leads to

\[ 2\tilde{R}^i = \Lambda_{\text{eff}} \varepsilon^i_{jk} b^j \wedge b^k, \quad \Lambda_{\text{eff}} \equiv q - \frac{1}{4} p^2, \]

(2.6)

where \( \Lambda_{\text{eff}} \) is the effective cosmological constant. Thus, our spacetime is maximally symmetric: for \( \Lambda_{\text{eff}} < 0 \) (\( \Lambda_{\text{eff}} \geq 0 \)), the spacetime manifold is anti-de Sitter (de Sitter, Minkowski). In what follows, our attention will be focused on the anti-de Sitter sector: \( \Lambda_{\text{eff}} = -1/\ell^2 < 0 \).
3. The black hole with torsion

For $\Lambda_{\text{eff}} < 0$, equation (2.6) has a well known solution for the metric — the BTZ black hole. In the static coordinates $x^\mu = (t, r, \phi)$ with $0 \leq \phi < 2\pi$, the black hole metric is given by

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\phi + N_\phi dt)^2,$$

$$N^2 = \left(-8Gm + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right), \quad N_\phi = \frac{4GJ}{r^2}, \quad (3.1)$$

Using local Lorentz invariance, we can choose $b^i$ to have the simple, “diagonal” form:

$$b^0 = N dt, \quad b^1 = N^{-1} dr, \quad b^2 = r (d\phi + N_\phi dt). \quad (3.2a)$$

To find the connection, we combine the relation $K^i = (p/2) b^i$, which follows from the first field equation in (2.5), with the identity (2.3). This yields

$$\omega^i = \tilde{\omega}^i + \frac{p}{2} b^i,$$

$$\tilde{\omega}^0 = - N d\phi, \quad \tilde{\omega}^1 = N^{-1} N_\phi dr, \quad \tilde{\omega}^2 = - \frac{r}{\ell^2} dt - r N_\phi d\phi. \quad (3.2b)$$

where $\tilde{\omega}^i$ is Riemannian connection, defined by $d\tilde{\omega}^i + \varepsilon^{ijk} \tilde{\omega}^j b^k = 0$. Equations (3.2) define the analogue of the BTZ black hole in Riemann–Cartan spacetime [8, 9].

As a constant curvature spacetime, the black hole is locally isometric to the AdS solution (AdS$_3$), obtained formally from (3.2) by the replacement $J = 0$, $8Gm = -1$.

4. Asymptotic conditions

For isolated gravitational systems, matter is absent from the asymptotic region, but it can influence global properties of spacetime through the asymptotic conditions. The symmetries of these conditions are closely related to the gravitational conserved charges [10].

For $\Lambda_{\text{eff}} < 0$, maximally symmetric AdS solution has the role analogous to the role of Minkowski space in the $\Lambda_{\text{eff}} = 0$ case. Following the analogy, we could choose that all the fields approach the single AdS$_3$ configuration at large distances, leading to the global AdS symmetry. However, this choice would exclude the important black hole solution. This motivates us to introduce the asymptotic AdS configurations, determined by the following requirements:

(a) the asymptotic conditions include the black hole configuration,
(b) they are invariant under the action of the AdS group $SO(2, 2)$, and
(c) the asymptotic symmetries have well defined canonical generators.

The asymptotics of the triad field $b^i_\mu$ that satisfies (a) and (b) reads:

$$b^i_\mu = \begin{pmatrix} \frac{r}{\ell} + O_1 & O_4 & O_1 \\ O_2 & \frac{\ell}{r} + O_3 & O_2 \\ O_1 & O_4 & r + O_1 \end{pmatrix}. \quad (4.1a)$$

Here, for any $O_n = c/r^n$, we assume that $c$ is not a constant, but a function of $t$ and $\varphi$, $c = c(t, \phi)$, which is the simplest way to ensure the global $SO(2, 2)$ invariance.

The asymptotic form of $\omega^i_\mu$ is defined in accordance with (3.2b):

$$\omega^i_\mu = \begin{pmatrix} \frac{pr}{2\ell} + O_1 & O_4 & - \frac{r}{\ell} + O_1 \\ O_2 & \frac{p\ell}{2r} + O_3 & O_2 \\ - \frac{r}{\ell^2} + O_1 & O_4 & \frac{pr}{2} + O_1 \end{pmatrix}. \quad (4.1b)$$
A verification of the third condition (c) is left for the next section.

Having chosen the asymptotic conditions, we now wish to find the subset of gauge transformations (2.2) that respect these conditions. They are defined by restricting the original gauge parameters in accordance with (4.1), which yields

\[
\begin{align*}
\xi^0 &= \ell \left[ T + \frac{1}{2} \left( \frac{\partial^2 T}{\partial t^2} \right) \frac{\ell^4}{r^2} \right] + O_4, \\
\xi^1 &= -\ell \left( \frac{\partial T}{\partial t} \right) r + O_1, \\
\theta^0 &= -\frac{\ell^2}{r} \partial_0 \partial_2 T + O_3, \\
\theta^1 &= \partial_2 T + O_2, \\
\theta^2 &= \frac{\ell^3}{r} \partial_0^2 T + O_3.
\end{align*}
\]

The functions \( T \) and \( S \) are such that \( \partial_\pm (T \mp S) = 0 \), with \( x^\pm \equiv x^0/\ell \pm x^2 \), which implies \( T + S = g(x^+) \), \( T - S = h(x^-) \), where \( g \) and \( h \) are two arbitrary, periodic functions.

The commutator algebra of the Poincaré gauge transformations (2.2) is closed: \([\delta^0, \delta^0] = \delta^0\), where \( \delta^0 = \delta_0(\xi', \theta') \) and so on. Using the related composition law with the restricted parameters (4.2), and keeping only the lowest order terms, one finds the relation

\[
\begin{align*}
T'' &= T'\partial_2 S'' + S'\partial_2 T'' - T''\partial_2 S' - S''\partial_2 T', \\
S'' &= S'\partial_2 S'' + T'\partial_2 T'' - S''\partial_2 T' - T''\partial_2 ST'.
\end{align*}
\]

Let us separate the parameters (4.2) into two pieces: the leading terms containing \( (T, S) \) transformation, while the rest defines the residual (pure gauge) transformation. The PGT commutator algebra implies that the commutator of two \( (T, S) \) transformations produces not only a \( (T, S) \) transformations, but also an additional pure gauge transformation. This result motivates us to introduce an improved definition of the asymptotic symmetry: it is the symmetry defined by the parameters (4.2), modulo pure gauge transformations. As we shall see in the next section, this symmetry coincides with the conformal symmetry.

5. Canonical generators and conserved charges

We continue our study of the asymptotic symmetries and conservation laws in the canonical formalism [10]. Introducing the canonical momenta \((\pi^I, \Pi^I)\), corresponding to the Lagrangian variables \((b_i^\mu, \omega^i_\mu)\), we find that the primary constraints of the theory (2.4) are of the form:

\[
\begin{align*}
\phi_i^0 &\equiv \pi^0_i \approx 0, \\
\phi_i^\alpha &\equiv \pi^\alpha_i - \alpha \varepsilon^{0\alpha\beta} b_{i\beta} \approx 0, \\
\Phi_i^0 &\equiv \Pi^0_i \approx 0, \\
\Phi_i^\alpha &\equiv \Pi^\alpha_i - \varepsilon^{0\alpha\beta}(2ab_{i\beta} + \alpha_3\omega_{i\beta}) \approx 0.
\end{align*}
\]

Up to an irrelevant divergence, the total Hamiltonian reads

\[
\begin{align*}
H_T &= b^i_0 \mathcal{H}_i + \omega^i_0 K_i + w^0_0 \pi^0_i + v^0_0 \Pi^0_i, \\
\mathcal{H}_i &= -\varepsilon^{0\alpha\beta} \left( aR_{i\alpha\beta} + \alpha_1 T_{i\alpha\beta} - \Lambda \varepsilon_{ijk} b^k_\alpha b^\beta_\beta \right) - \nabla_\beta \phi_i^\beta + \varepsilon_{imn} b^m_\beta \left( p^\alpha n^\beta + q^\Phi n^\beta \right), \\
K_i &= -\varepsilon^{0\alpha\beta} \left( aT_{i\alpha\beta} + \alpha_3 R_{i\alpha\beta} + \alpha_4 \varepsilon_{imn} b^m_\alpha b^n_\beta \right) - \nabla_\beta \Phi_i^\beta - \varepsilon_{imn} b^m_\beta \Phi^n_\beta.
\end{align*}
\]

The constraints \((\pi^0_i, \Pi^0_i, \mathcal{H}_i, K_i)\) are first class, \((\phi_i^\alpha, \Phi_i^\alpha)\) are second class.

Applying the general Castellani’s algorithm [6], we find the canonical gauge generator:

\[
\begin{align*}
G &= -G_1 - G_2, \\
G_1 &= \xi^0 \left( b^i_0 \mathcal{H}_i + \omega^i_0 \Pi^0_i \right) + \xi^0 \left[ b^i_0 \mathcal{H}_i + \omega^i_0 K_i + (\partial_\rho b^i_\rho) \pi^0_i + (\partial_\rho \omega^i_\rho) \Pi^0_i \right], \\
G_2 &= \theta^i \Pi^0_i + \theta^i \left[ K_i - \varepsilon_{ijk} \left( b^j_0 \pi^{k0} + \omega^{j0} \Pi^{k0} \right) \right].
\end{align*}
\]
Here, the time derivatives $\dot{b}_\mu$ and $\dot{\omega}_\mu$ are shorts for $u^i_\mu$ and $\psi_\mu$, respectively, and the integration symbol $\int d^2x$ is omitted in order to simplify the notation. The transformation law of the fields, $\delta_0 \phi \equiv \{ \phi, G \}$, is in complete agreement with the gauge transformations (2.2) on shell.

The adopted asymptotic conditions guarantee differentiability and finiteness of $\tilde{\phi}$ is also conserved. The charge is completely determined by the boundary term $\Gamma$. Note that $\Gamma$ depends on $\xi$ not on pure gauge parameters. For $\xi = 0$ we obtain the spatial rotation generator. The corresponding surface terms, calculated for $\xi = 1$ and $\xi = 2$, respectively, have the meaning of energy and angular momentum:

$$E = \int_0^{2\pi} d\varphi \, \mathcal{E}^1, \quad M = \int_0^{2\pi} d\varphi \, \mathcal{M}^1.$$  

Energy and angular momentum are conserved gravitational charges. Using these results, one can calculate the conserved charges for the black hole [8, 9]:

$$E = m + \frac{\alpha_3}{a} \left( \frac{p \mu}{2} - \frac{J}{\ell^2} \right), \quad M = J + \frac{\alpha_3}{a} \left( \frac{p J}{2} - m \right).$$  

They differ from the corresponding expressions in Riemannian GR$_A$, where $\alpha_3 = 0$.

6. Canonical algebra

The structure of the asymptotic symmetry is encoded in the Poisson bracket algebra of the improved generators. In the notation $G' \equiv G[T', S']$, $G'' \equiv G[T'', S'']$, and so on, the Poisson bracket algebra is found to have the form $\{ \tilde{G}'', \tilde{G}'' \} = \tilde{G}''' + C'''$, where the parameters $T'''$, $S'''$ are determined by the composition rules (4.3), and $C'''$ is the central term of the algebra. Expressed in terms of the Fourier modes, this algebra takes a more familiar form—the form of two independent Virasoro algebras with classical central charges:

$$\{ \mathcal{L}_n, \mathcal{L}_m \} = -i(n - m)\mathcal{L}_{n+m} - \frac{c}{12} \delta_{n,-m},$$

$$\{ \bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m \} = -i(n - m)\bar{\mathcal{L}}_{n+m} - \frac{\bar{c}}{12} \delta_{n,-m},$$

and $\{ \mathcal{L}_n, \bar{\mathcal{L}}_m \} = 0$. The central charges have the form:

$$c = \frac{3\ell}{2G} + 24\pi \alpha_3 \left( \frac{p \ell}{2} + 1 \right), \quad \bar{c} = \frac{3\ell}{2G} + 24\pi \alpha_3 \left( \frac{p \ell}{2} - 1 \right).$$

Asymptotically, the gravitational dynamics is characterized by the conformal symmetry with two different central charges, in contrast to Riemannian GR$_A$, where $c = \bar{c} = 3\ell/2G$. The improved canonical generator $\tilde{G}$ reads:

$$\tilde{G} = G + \Gamma, \quad \Gamma = -\int_0^{2\pi} d\varphi \left( \xi^0 \mathcal{E}^1 + \xi^2 \mathcal{M}^1 \right),$$  

$$\xi^\alpha \equiv 2e^{\beta\alpha\beta} \left[ \left( a + \frac{\alpha_3}{2} b \right) \omega^\alpha - b^\alpha \right],$$

$$\mathcal{E}^1 = \left[ \left( a + \frac{\alpha_3}{2} b \right) \omega_\beta - b \right] b^\beta,$$  

$$\mathcal{M}^1 = \left[ \left( a + \frac{\alpha_3}{2} b \right) \omega_\beta - b \right] b_\beta.$$  

The adopted asymptotic conditions guarantee differentiability and finiteness of $\tilde{G}$. Moreover, $\tilde{G}$ is also conserved.

The value of the improved generator $\tilde{G}$ defines the gravitational charge. Since $\tilde{G} \approx \Gamma$, the charge is completely determined by the boundary term $\Gamma$. Note that $\Gamma$ depends on $T$ and $S$, but not on pure gauge parameters. For $\xi^2 = 0$, $G$ reduces to the time translation generator, while for $\xi^0 = 0$ we obtain the spatial rotation generator. The corresponding surface terms, calculated for $\xi^0 = 1$ and $\xi^2 = 1$, respectively, have the meaning of energy and angular momentum:
7. The black hole entropy

A particularly interesting consequence of the theory defined by the action (2.4) is that the entropy of the black hole with torsion differs from the corresponding Riemannian result. Indeed, the semi-classical calculation based on the Hamiltonian form of the action leads to

$$S = \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( p r_+ - \frac{r_+}{\ell} \right),$$  \tag{7.1}

where $r_\pm$ are the zeros of $N^2$. For GR$\Lambda$ with Riemannian Chern-Simons term ($p = 0$, $\alpha_4 = 0$, but $\alpha_3 \neq 0$), our formula for $S$ yields Solodukhin’s result [11]. Using the new expressions (5.4) for energy and angular momentum, one can easily verify that the first law of black hole thermodynamics takes the form

$$dE = TdS + \Omega dM,$$  \tag{7.2}

Thus, the existence of torsion is in complete agreement with the first law of thermodynamics.\(^1\)

8. Concluding remarks

- 3d gravity with torsion, defined by the action (2.4), is based on an underlying Riemann-Cartan geometry of spacetime.
- The theory possesses the black hole solution (3.2), a generalization of the Riemannian BTZ black hole. Energy and angular momentum of the black hole differ from the corresponding Riemannian expressions in GR$\Lambda$.
- The AdS asymptotic conditions (4.1) imply the conformal symmetry in the asymptotic region, which is described by two independent Virasoro algebras with different central charges.
- The existence of different central charges ($\alpha_3 \neq 0$) modifies the black hole entropy, but remains in agreement with the first law of thermodynamics.

Acknowledgments

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\(^1\) Thermodynamics of black holes is properly formulated in the Euclidean sector.