We propose a new interpretation of doubly special relativity based on the distinction between the momenta and the translation generators in its phase space realization. We also argue that the implementation of the theory does not necessarily require a deformation of the Lorentz symmetry, but only of the translation invariance.

Keywords: Doubly Special Relativity; Noncanonical phase space.

1. Introduction

In recent years, the idea that special relativity should be modified for energies close to the Planck scale $\kappa$, in such a way that $\kappa$ becomes an observer-independent parameter of the theory, like the speed of light, has been extensively debated. This hypothesis is motivated by the consideration that the Planck energy sets a limit above which quantum gravity effects become important, and its value should therefore not depend on the specific observer, as would be the case in special relativity. Of course, this postulate must be implemented in such a way that the principle of relativity, i.e. the equivalence of all inertial observers, still be valid. The theory based on these assumptions has been named doubly special relativity (DSR), and implies a deformation of the Poincaré invariance and consequently of the dispersion relations of elementary particles.

Although the most natural implementation of DSR seems to be the formalism of noncommutative spacetimes and quantum groups, an interpretation in terms of classical particle mechanics is useful in order to better understand its physical significance. Here we consider the realization of DSR in classical phase space and argue that the essence of the formalism relies on the distinction between the canonical momenta, interpreted as the physical variables, and the generators of translations.

A related observation is that, although DSR is usually associated with the deformation of the Lorentz symmetry, through a nonlinear realization of the Lorentz group on momentum space, its really distinguishing feature is the deformation of the translation symmetry. In fact, the main phenomenological consequences of DSR are a deformation of the addition law of momenta and of the dispersion law of the elementary particles, that clearly depend only on the nontrivial action of translations.

More specifically, the relation between the momentum spaces of special relativity and DSR can in general be obtained by defining the physical momenta $p_\mu$ in
terms of auxiliary variables $P_\mu$, with $p_\mu = U(P_\mu)$, that satisfy canonical transformation laws under the action of the Poincaré group. We show that the physical interpretation of the otherwise obscure auxiliary variables can be obtained through their identification with the generators of the deformed translations.

To illustrate these considerations, we discuss the Snyder model from a DSR point of view. This model was originally proposed in order to show the possibility of introducing a noncommutative spacetime without breaking the Lorentz symmetry, and later interpreted in terms of DSR.

### 2. The model

Let us start by considering the classical action of the Poincaré algebra on the phase space of special relativity. The Poincaré algebra is spanned by the Lorentz generators $J_{\mu\nu}$ and the translation generators $T_\mu$, obeying Poisson brackets

\[
\{J_{\mu\nu}, J_{\rho\sigma}\} = \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\mu\rho} J_{\nu\sigma},
\]

\[
\{J_{\mu\nu}, T_\lambda\} = \eta_{\mu\lambda} T_\nu - \eta_{\nu\lambda} T_\mu,
\]

\[
\{T_\mu, T_\nu\} = 0.
\]

Its realization in canonical phase space, with Poisson brackets

\[
\{X_\mu, X_\nu\} = \{P_\mu, P_\nu\} = 0, \quad \{X_\mu, P_\nu\} = \eta_{\mu\nu},
\]

is obtained through the identification

\[
J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \quad T_\mu = P_\mu,
\]

which yields the infinitesimal transformation laws for the phase space coordinates $X_\mu$ and $P_\mu$.

\[
\{J_{\mu\nu}, X_\lambda\} = \eta_{\mu\lambda} X_\nu - \eta_{\nu\lambda} X_\mu, \quad \{J_{\mu\nu}, P_\lambda\} = \eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu,
\]

\[
\{T_\mu, X_\nu\} = \eta_{\mu\nu}, \quad \{T_\mu, P_\nu\} = 0.
\]

As discussed previously, to derive the transformation laws of DSR under the action of the Poincaré group, one introduces the physical momenta $p_\mu$ as a nonlinear function of the variables $P_\mu$ that obey the standard transformation laws of special relativity. In the case of the Snyder model, this relation reads

\[
p_\mu = U(P_\mu) = \frac{P_\mu}{\sqrt{1 + \Omega P^2}},
\]

where $\Omega = 1/\kappa^2$ is the Planck area.

In a DSR interpretation, the Snyder model can then be characterized by the explicitly Lorentz invariant deformed dispersion relation $\frac{p^2}{(1 - \Omega p^2)} = m^2$, obtained by substituting (6) into the standard deformation relation $P^2 = m^2$. This can also be written $p^2 = m^2/(1 + \Omega m^2)$. In this form, the dispersion relation looks like a redefinition of the mass (notice that the dispersion relation for massless particles maintains its classical form); nevertheless, as we shall see, some nontrivial consequences follow. From the structure of (6) it is instead evident that the action of the Lorentz group on the momentum variables is not affected.
A full description of physics requires the definition of a spacetime structure compatible with the deformation of the Poincaré invariance. It is therefore natural to introduce position variables \( x_\mu \) that transform covariantly with respect to the momenta. These can be defined as\(^5,9\)

\[
x_\mu = \sqrt{1 + \Omega P^2} X_\mu.
\]  

With this definition, the Poisson brackets between the new phase space coordinates are no longer canonical, and the position space becomes noncommutative, realizing the proposal of Snyder,\(^7\)

\[
\{x_\mu, x_\nu\} = -\Omega (x_\mu p_\nu - x_\nu p_\mu), \quad \{p_\mu, p_\nu\} = 0, \quad \{x_\mu, p_\nu\} = \eta_{\mu\nu} - \Omega p_\mu p_\nu.
\]  

In terms of the physical coordinates \( x_\mu \) and \( p_\mu \), the generators of the Poincaré group read

\[
J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad T_\mu = P_\mu = \frac{p_\mu}{\sqrt{1 - \Omega p^2}},
\]

where we have identified the translation generators with the auxiliary variables \( P_\mu \). The transformation laws of \( x_\mu \) and \( p_\mu \) under the Lorentz subalgebra maintain the canonical form, while under translations become

\[
\{T_\mu, x_\nu\} = \frac{\eta_{\mu\nu}}{\sqrt{1 - \Omega p^2}}, \quad \{T_\mu, p_\nu\} = 0.
\]

Therefore, the effect of the translations on the position coordinates becomes momentum dependent and increases for near Planck-mass particles.

It can also be shown that the sum rule of momenta is modified. Moreover, it is possible to define a spacetime metric which is invariant under the transformations\(^10\). This is given by \( ds^2 = (1 - \Omega p^2) dx^2 \) and, as usual in DSR, depends explicitly on the momentum. More details on these topics can be found in Ref. 5.

References

1. G. Amelino-Camelia, *Phys. Lett. B* **510**, 255 (2001), *Int. J. Mod. Phys. D* **11**, 35 (2002), *Int. J. Mod. Phys. D* **11**, 1643 (2002).
2. J. Kowalski-Glikman, *Phys. Lett. A* **286**, 391 (2001).
3. J. Magueijo and L. Smolin, *Phys. Rev. Lett.* **88**, 190403 (2002); *Phys. Rev. D* **67**, 044017 (2003).
4. S. Majid and H. Ruegg, *Phys. Lett. B* **334**, 348 (1994); J. Lukierski, H. Ruegg and W.J. Zakrzewski, *Ann. Phys.* **243**, 90 (1995).
5. S. Mignemi, *Phys. Lett. B* **672**, 186 (2009).
6. S. Judes and M. Visser, *Phys. Rev. D* **68**, 045001 (2003).
7. H.S. Snyder, *Phys. Rev.* **71**, 38 (1947).
8. J. Kowalski-Glikman and S. Nowak, *Int. J. Mod. Phys. D* **13**, 299 (2003).
9. S. Mignemi, *Int. J. Mod. Phys. D* **15**, 925 (2006).