Linear Optics Computations

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Abstract
In this Chapter we briefly recall and summarize the main linear optics concepts of the accelerators beam dynamics theory. In doing so we put our emphasis on the related computational aspects: the reader will be provided with the basic elements to write a linear optics code. To this aim, we complement the text with few practical examples and code listings.

1 Introduction

The linear optics theory was developed more than 60 years ago [1] and its first success was to demonstrate the overall focusing effect of a sequence of alternating focusing and defocusing quadrupoles (the so-called alternating-gradient principle).

Despite based on simple linear algebraic concepts, the alternating-gradient principle was a breakthrough in the history of accelerators. Since then, the linear optics design of an accelerator is the very first step for understanding the particle motion and the foundation to study the non-linear behaviour of a lattice. Hence the importance to acquire a solid knowledge and familiarity with its concepts and the associated numerical methods. Presently the main challenges of the accelerator beam dynamics resides elsewhere (e.g., in the description of long-term behaviour of non-linear system): this Chapter has to be intended as an introduction to the problems and to the phenomena that will be presented in the following Chapters [2–4]. There is a rich bibliography covering the subject and hereby we indicated only a short and partial list of references [5–9]. The reader is assumed to be already familiar with the basic concept of linear optics.

The goal of the linear optics (and more in general of the beam dynamics) is to describe the motion of the particles traveling in the accelerator. The linear attribute refers to the assumptions or the approximation that the variation of the particle coordinates depends linearly on the coordinates themselves.

To introduce the linear optics theory three equivalent directions can be followed:

1. Integrating the equations of the motion. This is the historical approach and presents several limits when trying to generalize it to non-linear systems.
2. Using the Hamiltonian formalism to describe the particle motion. This approach is the natural one to generalize the solutions to non-linear dynamics problems [2–4].
3. using a computational approach, deriving the linear optics theory using linear algebra concepts. This is the approach behind standard linear optics codes and it is the one we will follow in this chapter.

1.1 The reference system

As presented in the previous paragraph, the aim of the beam dynamics is to describe the particle motion along the accelerator. For doing so we need to associate to each particle a set of coordinates with respect to a specific reference system and describe their evolution in time. The number of coordinates will depend on the degrees of freedom we are considering.

Several reference systems can in principle be chosen, e.g., a laboratory reference system in which we describe the phase-space \( [X, P_X, Y, P_Y, Z, P_Z] \) of the particle, where \( X, Y, Z \) are the three spatial coordinates of the particle and the \( P_X, P_Y \) and \( P_Z \) its momentum components. This reference system is not convenient to describe in an efficient way the particle motion: in fact we can simplify the formalism by expressing the motion as relative to a given particle, the reference particle. In other words we choose as reference system the one co-moving with the reference particle. The coordinates \( [x, p_x, y, p_y, z, p_z] \) have to be intended, therefore, as variations with respect to the reference particle. The reference particle defines the reference orbit of the circular machine (the orbit used to align the machine elements, i.e., the orbit defining the geometry of the machine. In single-passage machines, like in a LINAC, we will call it reference trajectory). All other dipolar contributions present in machine but not contributing in defining the circular machine geometry define the beam closed orbit (Fig. 1).

It is convenient to replace, as independent variable, the time \( t \), with the longitudinal position \( s \), along the reference orbit/trajectory. Under the conditions \( p_z \to p_0 \), where \( p_0 = 0 \) is the module of the particle momentum, the phase-space can be replaced with the trace-space

\[
\begin{align*}
[x, x'] &= \frac{dx}{ds}, \quad y, y' = \frac{dy}{ds}, \quad z, z' = \frac{dz}{ds} \\
\end{align*}
\]

If not explicitly stated, we will implicitly refer to the phase-space and not to the trace-space. In fact the phase-space, differently from the trace-space, features a conservation property that we will present in the following sections and will be central in linear optics computations.

1.2 Linear transformations

Our optics system is linear if and only if the evolution from the coordinates

\[
X_{s_1} = [x(s_1), p_x(s_1), y(s_1), p_y(s_1), z(s_1), p_z(s_1)]^T
\]
Figure 1: Reference systems. In the figure two reference systems are shown: the laboratory reference system \( \{X, Y, Z\} \) in blue and the co-moving reference system \( \{x(s), y(s), z(s)\} \) in black. It is worth noting that the latter depends on \( s \), the longitudinal abscissa on the reference orbit (black line). The reference orbit is described, by definition, by the fixed point \((0, 0, 0)\) with respect to the \( \{x(s), y(s), z(s)\} \) frame. The closed orbit (in red, see later) and the single turn oscillation (in orange, see later) are described with respect to the \( \{x(s), y(s), z(s)\} \) frame.

to \( X_{s_2} \) can be expressed as

\[
X_{s_2} = M X_{s_1}
\]

where \( M \) is a square matrix and does not depend on \( X_{s_1} \) or \( X_{s_2} \).

It is important to note that we are interested only in a special set of linear transformations: the so-called \textbf{symplectic linear transformations}, that is the ones associated to a \textbf{symplectic matrix}. In the following we will present the concept of symplectic matrix and its physical meaning. To do so we need to introduce the concept of bi-linear product.
1.3 The bi-linear product

The **bi-linear product** between the two vectors $V$ and $U$ associated to the square matrix $F$ is defined as the scalar

$$V^T F U.$$  \hspace{1cm} (6)

As an example, the dot product, is a bi-linear product with $F$ equal to the identity matrix.

As we will see in the following, it is interesting to study the properties of a linear transformations, $M$, that preserves the bi-linear product associated with $F$. Observing that

$$V^T F U = (MV)^T F MU \rightarrow F = M^T F M.$$  \hspace{1cm} (7)

we conclude that $M$ preserves the bi-linear product associated to $F$ if and only if

$$F = M^T F M.$$  \hspace{1cm} (8)

To be noted then, that if $M$ and $N$ are preserving the bi-linear product associated to $F$, then also $M \times N$ and $N \times M$ preserve it, therefore we can associate to a bi-linear product a group of linear transformations. In the following we will present two examples, the group of orthogonal and symplectic matrices.

1.3.1 **EXAMPLE. The orthogonal matrix**

Let us consider, for simplicity, the 1D case, that is, $U = (u_a, u_b)^T$ and $V = (v_a, v_b)^T$. Assuming

$$F = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (9)

the bi-linear transformation $I$ is the dot product between $V = (v_a, v_b)^T$ and $U = (u_a, u_b)^T$:

$$V^T I F U = v_a u_a + v_b u_b.$$  \hspace{1cm} (10)

A matrix $M$ preserves the bi-linear transformation $I$ (then the projections) if and only if

$$\overbrace{V^T M^T}^{(M V)^T} I M U = V^T I U \rightarrow M^T I M = I,$$  \hspace{1cm} (11)

then $M$ is called **orthogonal** matrix. The physical meaning of the orthogonal matrix lies in the fact that it preserves the projection between vectors (e.g., any rotation can be expressed as an orthogonal matrix).
1.3.2 **EXAMPLE. The symplectic matrix**

Assuming

$$F = \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(12)

the bi-linear transformation $\Omega$ is proportional to the amplitude of the cross product between $V = (v_a, v_b)^T$ and $U = (u_a, u_b)^T$:

$$V^T \begin{pmatrix} \Omega \\ F \end{pmatrix} U = v_a u_b - v_b u_a$$

(13)

that is **proportional to the area** defined by the vectors. A matrix $M$ preserves the bi-linear transformation $\Omega$ (related to the cross product) if and only if

$$V^T M^T \Omega M U = V^T \Omega U \rightarrow M^T \Omega M = \Omega,$$

(14)

then $M$ is called **symplectic matrix**. The physical meaning of the symplectic matrix lies on the fact that it preserves the area between two vectors.

In Fig. 2 we can see a graphical representation of an orthogonal and a symplectic linear transformation. Comparing two generic vectors between the identity transformation (upper plot) and the orthogonal transformation (middle plot) one can note that their dot product is conserved. In the symplectic transformation (lower plot) given a set on $n > 3$ vectors (defining a polygon) the surface of the polygon is preserved. For that reason a symplectic linear transformation preserves the phase-space areas. Behind this concept lies a much more general theorem of classical mechanics and the phase-space (**Liouville theorem**).

All the concepts we introduced so far can be generalized from the 1D to $n$D. In particular, $\Omega$ becomes a $2n \times 2n$ matrix:

$$\Omega = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ 0 & \ldots & -1 & 0 \end{pmatrix}.$$  

(15)

An example of symplectic matrix in 2D (not block-symplectic) is the following

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$  

(16)

It is worth recalling some important properties of the symplectic group:

- as already mentioned, if $M_1$ and $M_2$ are symplectic then $M = M_1 M_2$ is symplectic too.
Figure 2: Examples of identity, orthogonal and symplectic transformations.
– If $M$ is symplectic then $M^T$ is symplectic.
– Every symplectic matrix is invertible

$$M^{-1} = \Omega^{-1} M^T \Omega$$  \hspace{1cm} (17)

and $M^{-1}$ is symplectic. Therefore an inversion of a symplectic matrix can be very efficient in term of computational cost.

– A necessary (but not sufficient) condition for $M$ to be symplectic is that $\det(M) = +1$. For the 1D case, this condition is necessary and sufficient. An example of not symplectic matrix $M$ with $\det(M) = +1$ is the following

$$\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (18)

– There are symplectic matrices that are defective, that is they cannot be diagonalized, e.g.,

$$\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.$$  \hspace{1cm} (19)

1.4 Symplectic matrix and accelerators

From the concept of symplectic transformation we can define the basic building blocks that constitute all the linear transformation in an accelerator. In particular we can consider the following three matrices:

$$\begin{pmatrix}
G & 0 \\
0 & \frac{1}{G}
\end{pmatrix} \quad \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix}.$$  \hspace{1cm} (20)

Among the above matrices one can recognise the $L$-long drift and the thin quadrupole with focal length $f$. In addition there is also the thin telescope matrix. This matrix reduces to the identical transformation for $G = 1$, but for $G \neq 1$ introduces a discontinuity in the position coordinate, while the other thin matrix (the thin quadrupole) introduces a discontinuity on the momentum coordinate (thin kick). Conveniently combining drifts and thin quadrupole one find back also the well known matrices for the thick elements.

1.4.1 EXAMPLE. A thick quadrupole

One can derive the transfer matrix of a thick quadrupole of length $L$ and normalized gradient $K1$ by decomposing it in $N$ identical optics cells. Each cell is constituted by a $\frac{L}{N}$-long drift and a
thin quadrupole with focal length $\frac{N}{K_1 L}$. From this decomposition one can obtain the thick lens quadrupole matrix by solving the following limit

$$\lim_{n \to \infty} \left[ \begin{array}{cc} 1 & 0 \\ \frac{1}{K_1 L} & 1 \end{array} \right] \left[ \begin{array}{cc} \frac{L}{n} & 0 \\ 0 & 1 \end{array} \right]^n = \left( \begin{array}{cc} \cos (\sqrt{K_1 L}) & \frac{\sin (\sqrt{K_1 L})}{\sqrt{K_1}} \\ -\sqrt{K_1} \sin (\sqrt{K_1 L}) & \cos (\sqrt{K_1 L}) \end{array} \right). \tag{21}$$

To compute the above limit and, in general, for symbolic computations one can profit of the available symbolic computation tools (e.g., Mathematica™ or the Python package sympy). An example of symbolic calculation of the above limit is given in Listing 1.

We established a correspondence between elements along our machine (drift, bending, quadrupoles, solenoids,...) and symplectic matrices. For a rich list of matrix transformations in an accelerator refer to the Appendix in [7].

1.5 Tracking in a linear system

Given a sequence of elements $M_1, M_2, \ldots M_k$ (the lattice), the evolution of the coordinate, $X_n$, along the lattice for a given particle can be obtained as

$$X_n = M_n \ldots M_1 X_0 \text{ for } n \geq 1. \tag{23}$$

The transport of the particles along the lattice is called tracking. The tracking on a linear system is trivial and, as we will show in the following, unnecessary. In fact we will decompose the trajectory of the single particle in term of invariant of the motion and properties of the lattice. Via those properties we will describe not only the trajectory of the single particle but also the statistical evolution of an ensemble of particles (the beam). So, instead of tracking an ensemble of particles, we will concentrate to define and compute the properties of the lattice, on one hand, and of the beam, on the other hand.

2 Linear Lattices

2.1 Periodic lattice and stability

We study now the motion of the particles in a periodic lattice, that is lattice constituted by an indefinite repetition of the same basic $C$-long period. We represent with $M_{OTM}$, the so-called One-Turn-Map, that is the linear matrix of a single turn. Due to periodicity we have:

$$M_{OTM}(s_0) = M_{OTM}(s_0 + C). \tag{24}$$
From Eq. 23 we get
\[ X_m = M_{OTM}^m X_0, \]  
where we used the subscript \( m \) to refer to the turn number. In the following we study the property of \( M_{OTM} \) to have stable motion in the lattice, that is there is always a \( \tilde{X} \) such that
\[ |X_m| < |\tilde{X}| \text{ for all } X_0 \text{ and } m. \] (26)

In other words, to verify if the lattice is stable we need to verify that all the elements of the matrix \( M_{OTM}^m \) stay bounded for all \( m \). To solve this problem we use in the following three equivalent factorization forms:
- diagonal-factorization,
- R-factorization and,
- Twiss-factorization.

### 2.1.1 Diagonal-factorization
If \( M_{OTM} \) can be expressed as a diagonal-factorization (e.g., in diagonal form) \[ M_{OTM} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}, \] (27)
after \( m \)-turns, we have that
\[ M_{OTM}^m = D^m P \]  
Therefore the stability depends only on the eigenvalues of \( M_{OTM} \).

Note that if \( V \) is an eigenvector also \( kV, \ k \neq 0 \) is an eigenvector. Therefore \( P \) is not uniquely defined: we chose \( P \) such that \( \det(P) = -i \). There is not an physical meaning behind this convention, but as it will appear clearly later, it is convenient since is compatible with standard definitions of the accelerator dynamics. It is worth recalling the following properties:
- for a real matrix the eigenvalues, if complex, appear in complex conjugate pairs.
- For a symplectic matrix \( M_{OTM} \)
\[ \prod_{i}^{2n} \lambda_i = 1 \]  
(29)
where \( \lambda_i \) are the eigenvalues of \( M_{OTM} \).
Therefore for 2x2 symplectic matrix the eigenvalues can be written as \( \lambda_1 = e^{i\mu_{OTM}} \) and \( \lambda_2 = e^{-i\mu_{OTM}} \) (without loss of generality we consider \( \mu_{OTM} > 0 \)). This implies that

\[
D^m = D(m\mu_{OTM}).
\]  

(30)

Therefore a power of a matrix is reduced to a simple scalar multiplication.

**If \( \mu \) is real then the motion is stable** and we can define the fractional tune of the periodic lattice as \( \frac{\mu_{OTM}}{2\pi} \). We will describe in Section 2.3.1 how to compute the total phase advance of the machine and, therefore, the integer tune.

### 2.1.2 R-factorization

The diagonal-factorization we introduced is convenient to check the stability but not to visualize the turn-by-turn phase-space evolution of the particle. To do that it is worth considering the rotation-factorization

\[
M_{OTM} = \bar{P} \begin{pmatrix} \cos \mu_{OTM} & \sin \mu_{OTM} \\ -\sin \mu_{OTM} & \cos \mu_{OTM} \end{pmatrix} R(\mu_{OTM}) \bar{S}^{-1}.
\]

(31)

If \( M_{OTM} \) can be diagonalized then can be expressed also in a R-factorization. In fact, to go from diagonal-factorization to R-factorization we note that

\[
\begin{pmatrix} \cos \mu_{OTM} & \sin \mu_{OTM} \\ -\sin \mu_{OTM} & \cos \mu_{OTM} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\mu_{OTM}} & 0 \\ 0 & e^{-i\mu_{OTM}} \end{pmatrix} \begin{pmatrix} 1 & i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

(32)

where we introduced the matrix \( S \). Therefore

\[
R^m = R(m\mu_{OTM}).
\]

(33)

One can easily express \( \bar{P} \) as function of \( P \) and \( S \) observing that

\[
M_{OTM} = \frac{P}{\bar{P}} S^{-1} \underbrace{DS}_{R} S^{-1} \bar{P}^{-1}.
\]

(34)

i.e., \( \bar{P} = PS \). We note that by choosing \( \det P = -i \) we got \( \det \bar{P} = 1 \) that is we expressed \( M \) as the product of orthogonal and symplectic matrices. This result is very relevant, since it implies that the \( M_{OTM} \) is similar to a pure rotation. That is, with a convenient change of base (expressed by the matrix \( \bar{P}^{-1} \)), we can move from the physical phase-space to the normalized phase-space where the periodic motion is just a clockwise rotation of the angle \( \mu_{OTM} \).
2.1.3 Twiss-factorization of $M_{OTM}$

We note that

$$R(\mu_{OTM}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \mu_{OTM} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin \mu_{OTM},$$

yielding the, so called, Twiss-factorization

$$M_{OTM} = \bar{P} P^{-1} I \cos \mu_{OTM} + \bar{P} \Omega \bar{P}^{-1} \sin \mu_{OTM}.$$ (36)

It is worth observing that $J$ has three properties:

1. $\det(J) = 1$,
2. $J_{11} = -J_{22}$,
3. $J_{12} > 0$.

The last two expressions can be proved using the symbolic computation as show in Listing 2.

The following parametric expression of $J$ has been proposed [1]

$$J = \begin{pmatrix} \alpha & \beta > 0 \\ -1 + \alpha^2 & -\alpha \end{pmatrix} \begin{pmatrix} \beta > 0 \\ \sqrt{\beta} \end{pmatrix}$$

defining the Twiss parameters, $\alpha, \beta, \gamma$ of the lattice at the start of the sequence $M_{OTM}$. It is very important to not that they are not depending on the turn number $m$ since

$$M_{OTM}^m = I \cos(m\mu_{OTM}) + J \sin(m\mu_{OTM}).$$ (38)

In other words the Twiss parameters in a stable periodic lattice are periodic. From the definition of $J$, $J = P\Omega P^{-1}$, we can express $P, P^{-1}$ and $P$ as function of the Twiss parameters:

$$\bar{P} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ 0 & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha/\sqrt{\beta} & 1 \end{pmatrix},$$ (39)

$$\bar{P}^{-1} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix},$$ (40)
Figure 3: Similarity of two One-Turn-Map matrices referred to two different points $s_0$ and $s_1$.

and

$$P = \bar{P} S^{-1} = \begin{pmatrix} \sqrt{\beta} & \sqrt{\beta} \\ \frac{-\alpha + i}{\sqrt{2\beta}} & \frac{-\alpha - i}{\sqrt{2\beta}} \end{pmatrix}. \quad (41)$$

To summarise, if the matrix $M_{OTM}$ is diagonalizable and if $|\lambda_1|=1$, the lattice is stable and its fractional tune is $\frac{\text{phase}(\lambda_1)}{2\pi}$. From the eigenvector matrix $P$, conveniently normalized with $\det P = -i$, and from Eq. 41, one can find the Twiss parameters of the lattice at the $M_{OTM}$ starting point. In Listing 3, an example for computing the optical functions at the $M_{OTM}$ starting point is shown.

### 2.2 Twiss parameters along the machine

Given a C-long periodic lattice and two longitudinal positions $s_0$ and $s_1$ ($s_1 > s_0$), as depicted in Fig. 3, the transformation from $s_0$ to $s_1 + C$ can be expressed as

$$M_{OTM}(s_1) M = M M_{OTM}(s_0) \quad (42)$$

where $M$ is the transport matrix from $s_0$ to $s_1$. This implies

$$M_{OTM}(s_1) = M M_{OTM}(s_0) M^{-1} \quad (43)$$

that is the matrices $M_{OTM}(s_1)$ and $M_{OTM}(s_2)$ are similar and therefore they have the same eigenvalues. From this observation it yields that the $M_{OTM}$ is $s$-dependent but the tune is not.

#### 2.2.1 $\beta$ and $\alpha$ transport

We observe that $\beta$ and $\alpha$ are $s$-dependent functions. In fact we have:

$$M_{OTM}(s_1) = M M_{OTM}(s_0) M^{-1} = M (I \cos \mu_{OTM} + J(s_0) \sin \mu_{OTM}) M^{-1}, \quad (44)$$

therefore

$$\begin{pmatrix} \alpha(s_1) & \beta(s_1) \\ -\gamma(s_1) & -\alpha(s_1) \end{pmatrix} = M \begin{pmatrix} \alpha(s_0) & \beta(s_0) \\ -\gamma(s_0) & -\alpha(s_0) \end{pmatrix} M^{-1}. \quad (45)$$
From Eq. 17 (inverse of a symplectic matrix) we have
\[
\begin{pmatrix}
\alpha(s_1) & \beta(s_1) \\
-\gamma(s_1) & -\alpha(s_1)
\end{pmatrix} \Omega^{-1} = M \begin{pmatrix}
\alpha(s_0) & \beta(s_0) \\
-\gamma(s_0) & -\alpha(s_0)
\end{pmatrix} \Omega^{-1} M^T,
\]
that is
\[
\begin{pmatrix}
\beta(s_1) & -\alpha(s_1) \\
-\alpha(s_1) & \gamma(s_1)
\end{pmatrix} = M \begin{pmatrix}
\beta(s_0) & -\alpha(s_0) \\
-\alpha(s_0) & \gamma(s_0)
\end{pmatrix} M^T.
\]
Equation 47 allows us to propagate the initial condition of the optical functions \(\beta\) and \(\alpha\) along the lattice. It is worth noting that from Eq. 47 and remembering the definition of the \(\bar{P}^{-1}\) matrix (the one to transform the physical phase-space in the normalized phase-space), we get
\[
\bar{P}^{-1} \begin{pmatrix}
\beta(s_0) & -\alpha(s_0) \\
-\alpha(s_0) & \gamma(s_0)
\end{pmatrix} (\bar{P}^{-1})^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
hence, in the normalized phase-space, \(\beta\) and \(\alpha\) are 1 and 0, respectively.

### 2.2.2 EXAMPLE. The \(\beta\)-function in a drift

To compute the Twiss parameters in a drift starting from \(\beta_0\) and \(\alpha_0\), we can simply apply the previous equation
\[
\begin{pmatrix}
\beta(s) & -\alpha(s) \\
-\alpha(s) & \gamma(s)
\end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}
\]
yielding
\[
\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2
\]
and
\[
\alpha(s) = \alpha_0 - \gamma_0 s.
\]

### 2.2.3 The differential relation between \(\alpha\) and \(\beta\)

Up to now, we discussed how to compute the \(\alpha(s)\) and \(\beta(s)\) at the start of the lattice \((\alpha(s_0)\) and \(\beta(s_0))\) and how to propagate them along the lattice. We would like now to investigate if there is a differential relation between these two functions of the \(s\)-position. We consider the general \(\Delta M\) matrix for the infinitesimal quadrupole of length \(\Delta s\),
\[
\Delta M = \begin{pmatrix} 1 & \Delta s \\ -K(s)\Delta s & 1 \end{pmatrix}.
\]
Note that $\Delta M$ is just the product of a drift of length $\Delta s$ and a thin focusing quadrupole of gradient $K(s)\Delta s$ where we neglected the second order terms of $\Delta s$. $\Delta M$ is then symplectic only for $\Delta s \to 0$. From Eq. 47 we have that

$$
\begin{pmatrix}
\beta(s + \Delta s) & -\alpha(s + \Delta s) \\
-\alpha(s + \Delta s) & \gamma(s + \Delta s)
\end{pmatrix}
\frac{J(s+\Delta s)}{J(s)\Omega^{-1}}
\Delta M
\begin{pmatrix}
\beta(s) & -\alpha(s) \\
-\alpha(s) & \gamma(s)
\end{pmatrix}
\Delta M^T.
$$

(53)

Observing that

$$
\lim_{\Delta s \to 0} \frac{J(s + \Delta s) - J(s)}{\Delta s} \Omega^{-1} = \begin{pmatrix}
\beta'(s) & -\alpha'(s) \\
-\alpha'(s) & \gamma'(s)
\end{pmatrix}
$$

(54)

where we used standard notation $\frac{df}{ds} = f'$ and replacing Eq. 53 in eq:differentialJ, one obtains

$$
\beta'(s) = -2\alpha(s)
$$

(55)

$$
\alpha'(s) = -\gamma + K(s)\beta(s).
$$

(56)

Replacing $\alpha$ and $\gamma$ in the latter equation with functions of $\beta$, it yields the non-linear differential equation:

$$
\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K(s)\beta^2 = 1.
$$

(57)

It is important to note that, even if we are discussing linear optics, the differential equation between $\beta$ and $K$ is strongly non-linear. Therefore in order to avoid the linear tracking and to decompose the problem in properties of lattice and properties of the beam we introduced new functions of the $s$-positions ($\alpha, \beta, \gamma$) that are related by a non-linear differential equation to the lattice gradients.

2.2.4 EXAMPLE. From matrices to Hill's equation

Following the notation already introduced

$$
X(s + \Delta s) = \Delta M \ X(s)
$$

(58)

with $X(s) = (x(s), \frac{p_x(s)}{p_0})^T \approx (x(s), x'(s))^T$, therefore

$$
X'(s) = \begin{pmatrix}
x'(s) \\
x''(s)
\end{pmatrix} = \lim_{\Delta s \to 0} \frac{X(s + \Delta s) - X(s)}{\Delta s} = \begin{pmatrix}
x'(s) \\
-K(s)x(s)
\end{pmatrix}
$$

(59)

one can find back the Hill's equation

$$
x''(s) + K(s)x(s) = 0
$$

(60)

starting from a pure matrix approach, where we did not mentioned Lorentz force at all. This shows the full equivalence of the two formalisms.
2.3 Courant-Snyder invariant

Up to now we showed how to compute the optics functions of the lattice (that is functions independent on the particle initial conditions). In this Section we are going to investigate, given a particle with initial coordinate \( X \), if and how we can define a \( X \)-dependent quantity that is conserved during the motion of the particle in the machine. This invariant exists and is called **Courant-Snyder invariant** or **action** of the particle. It is defined as

\[
J_{CS} = \frac{1}{2} X^T \Omega^{-1} X.
\]  
(61)

We can show in fact from Eq. 47 that

\[
\frac{1}{2} X_1^T \Omega^{-1} X_1 = \frac{1}{2} X_0^T M^T (M \Omega^{-1} M^T)^{-1} M X_0 = \frac{1}{2} X_0^T \Omega^{-1} X_0.
\]  
(62)

That is \( J_{CS}(s_1) = J_{CS}(s_0) \). To be noted that in [1] the invariant of motion is defined as \( 2 J_{CS} \). In the normalized phase-space, remembering that \( X = \bar{P} \bar{X} \), we have

\[
\frac{1}{2} \bar{X}^T \bar{P}^T \bar{P} \Omega^{-1} \bar{P} \bar{X} = \frac{1}{2} \bar{X}^T \bar{X}
\]  
(63)

that is the \( J_{CS} \) is half of the square of the radius, \( \rho \), defined by the particle initial position in the normalized phase-space. The **angle** of the particle is defined as the the particle initial angle, \( \mu \), in the normalized phase-space and polar coordinates \( (\rho, \mu) \). Hence, the normalized phase-space is also called **action-angle** space. From Listing 4, going from the matrix form to the polynomial form, one finds back the definition of the invariant \( J_{CS} \) as function of the optics functions

\[
J_{CS} = \frac{\gamma}{2} x^2 + \alpha x p_x + \frac{\beta}{2} p_x^2.
\]  
(64)

It is worth noting that, under the assumptions of trace-space and phase-space equivalence (see Section 1.1), the invariant of motion can also be expressed in the trace-space variables as

\[
J_{CS}^{\text{trace-space}} = \frac{\gamma}{2} x^2 + \alpha x x' + \frac{\beta}{2} x'^2.
\]  
(65)

To be noted that Eq. 65 is not equivalent from a dimensional point of view to Eq. 64. Despite it, for the sake of simplicity and consistency with the existing conventions, we will use the same symbol \( J_{CS} \) for both invariants.

2.3.1 The betatron phase \( \mu(s) \)

In normalized space, we just observed that the transport from \( s \) to \( s + \Delta s \) does not change \( J_{CS} \) but the angle varies by \( \Delta \mu = \mu(s + \Delta s) - \mu(s) \).
What is the $\Delta \mu$ introduced by a linear matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$? To compute it we consider the normalized phase-space

$$X(s) = \bar{P}(s) \bar{X}(s) \quad \text{and} \quad X(s + \Delta s) = \bar{P}(s + \Delta s) \bar{X}(s) \quad (66)$$

and from

$$X(s + \Delta s) = M X(s), \quad (67)$$

it yields

$$\dot{X}(s + \Delta s) = \bar{P}(s + \Delta s)^{-1} M \bar{P}(s) \bar{X}(s) = \begin{pmatrix} \cos \Delta \mu & \sin \Delta \mu \\ -\sin \Delta \mu & \cos \Delta \mu \end{pmatrix} \bar{X}(s). \quad (68)$$

From the previous equation one gets

$$\tan \Delta \mu = \frac{\sin \Delta \mu}{\cos \Delta \mu} = \frac{m_{12}}{m_{11} \beta(s) - m_{12} \alpha(s)}, \quad (69)$$

that is the well known expression

$$\mu(s) = \int_{s_0}^s \frac{1}{\beta(\sigma)} d\sigma + \mu(s_0), \quad (73)$$

that is the phase advance from $s$ to $s + \Delta s$, that is the end of the $M$ transformation. The integer tune of the circular machine of length $C$ is defined as

$$\frac{1}{2\pi} \mu(C), \quad (70)$$

therefore it represents the number of betatron oscillations between $s = 0$ and $s = C$. It is worth noting that between the $\mu_{OTM}$ (e.g., Eq. 31) and $\mu(C)$ the following relation holds

$$\mu_{OTM} + 2\pi k = \mu(C) \quad (71)$$

where $k \in \mathbb{N}$ represents the integer number of betatron periods in the machine.

**2.3.2 EXAMPLE. The differential equation of $\mu(s)$**

From the previous equation, if $M = \begin{pmatrix} 1 & \Delta s \\ -K(s) \Delta s & 1 \end{pmatrix}$, one gets

$$\mu' = \lim_{\Delta s \to 0} \frac{\tan \Delta \mu}{\Delta s} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \frac{1}{\beta(s) - \alpha(s) \Delta s} = \frac{1}{\beta(s)}, \quad (72)$$

that is the well known expression
Figure 4: Betatron oscillations from $s_1$ to $s_2$ in the physical and normalized trace-spaces (referred as $X$ and $\bar{X}$, respectively). In this example we assumed a tune of $\mu_{OTM}/2\pi = 1/8$ and a phase advance between $s_1$ and $s_2$ of $\Delta \mu = \pi/2$. To be noted that all the transformations between these 4 trace-spaces are symplectic, therefore the $4 \times 8$ positions represented in the figure have the same $\mathcal{J}_{CS}$. The number of the markers indicates the turn number.
2.3.3 EXAMPLE. The betatron oscillations

We can describe a betatron oscillation from \( s_1 \) to \( s_2 \) in terms of the Twiss parameters and the particle initial conditions.

This can be easily done by transforming the vector \( X \) in the normalized phase-space in \( s_1 \), moving it from \( s_1 \) to \( s_2 \) in the normalized space (pure rotation of the phase \( \Delta \mu \)) and back transform it in the original phase-space, as shown in Fig. 4.

This is a very important result, since it implies that the motion of the particle is a pure rotation in the normalized phase-space also along the machine: this generalizes the result obtained from the R-factorization of the \( M_{OTM} \). As show in the Listing 6, one can express the \( M \) matrix as function of the optics function at \( s_1 \) and \( s_2 \), yielding

\[
M = \bar{P}(s_2) R(\Delta \mu) \bar{P}(s_1)^{-1} = \begin{pmatrix}
\sqrt{\frac{\beta_2}{\beta_1}} (\cos \Delta \mu + \alpha_1 \sin \Delta \mu) & \sqrt{\beta_1 \beta_2} \sin \Delta \mu \\
\frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \Delta \mu - \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \Delta \mu & 1 + \frac{\beta_1}{\beta_2} (\cos \Delta \mu - \alpha_2 \sin \Delta \mu)
\end{pmatrix}
\]

(74)

This is indeed the solution of the Hill’s equation given the particle initial conditions \( J_{CS} \) and \( \mu_0 \).

2.3.4 EXAMPLE. Solution of Hill’s equation

Remembering the important result that motion of the particle is a pure rotation in the normalized phase-space also along the machine, we can express the motion of the particle from its initial condition in the normalized phase-space (i.e., action and initial phase that is \( J_{CS} \) and \( \mu_0 \)). From the definition of \( J_{CS} \) it follows that the radial position of the particle in the normalized phase-space is \( \sqrt{2J_{CS}} \) and is angular position has a phase \( \mu(s) \) in addition to the initial phase \( \mu_0 \). Remembering that, for positive \( \mu \), the rotation is clockwise, one gets

\[
X(s) = \bar{P}(s) \begin{pmatrix}
\sqrt{J_{CS}} \cos(\mu(s) + \mu_0) \\
-\sqrt{J_{CS}} \sin(\mu(s) + \mu_0)
\end{pmatrix} = \begin{pmatrix}
\sqrt{J_{CS} \beta(s)} \cos(\mu(s) + \mu_0) \\
\sqrt{J_{CS} \beta(s)} [\alpha(s) \cos(\mu(s) + \mu_0) + \sin(\mu(s) + \mu_0)]
\end{pmatrix}
\]

(76)

This is indeed the solution of the Hill’s equation given the particle initial conditions \( J_{CS} \) and \( \mu_0 \).

2.4 EXAMPLE. From the CO matrix to the CO formula.

Up to now we implicitly assumed that the closed orbit (CO) corresponded to the reference orbit. This is not always true. In fact, during the machine operation one can switch on dipole correctors additional to the ones defining the alignment of the magnetic elements. Assuming a \( M_{OTM}(s_0) \) and a single thin kick \( \Theta \) at \( s_0 \) (independent from \( X_m \)) we can write

\[
X_{m+1}(s_0) = M_{OTM}(s_0) X_m(s_0) + \Theta.
\]

(78)
In the 1D case $\Theta$ can represent a kick of a dipole correction or misalignment of a quadrupole ($\Theta = (0, \theta)^T$). The closed orbit solution can be retrieved imposing $X_{m+1} = X_m$ (fixed point after 1-turn), yielding

$$X_n(s_0) = (I - M_{OTM}(s_0))^{-1}\Theta(s_0). \quad (79)$$

From the Eq. 79 we can find the fixed point in $s_0$. Please note that the CO is discontinuous in $s_0$ so the previous formula refers to the CO after the kick. Solving the Eq. 79 and transporting the fixed point from $s_0$ to $s$ using Eq. 75 as shown in Listing 7 we found back the known equation of the closed orbit

$$x_{CO}(s) = \sqrt{\beta(s)\beta(s_0)}\frac{\theta_{s_0}}{2}\cos(\mu(s) - \pi Q) \quad (80)$$

where $\mu(s)$ is the phase advance ($>0$) from $s_0$ to $s$. We can relax the last condition by replacing $\mu(s)$ with $|\mu(s) - \mu(s_0)|$. In presence of multiple $\theta(s_i)$ one can sum the single contributions along $s$.

### 2.5 Computing dispersion and chromaticity

Up to now we considered all the optics parameters for the on-momentum particle. To evaluate the off-momentum effect of the closed orbit and the tune we introduce the dispersion, $D_{x,y}(s, \frac{\Delta p}{p_0})$, and chromaticity, $\xi_{x,y}(\frac{\Delta p}{p_0})$, functions respectively, as

$$\Delta CO_{x,y}(s) = D_{x,y}\left(s, \frac{\Delta p}{p_0}\right) \times \frac{\Delta p}{p_0}, \quad D_{x,y}(s + C) = D(s) \quad (81)$$

and

$$\Delta Q_{x,y} = \xi_{x,y}\left(\frac{\Delta p}{p_0}\right) \times \frac{\Delta p}{p_0}. \quad (82)$$

In order to compute numerically the $D_{x,y}$ and $\xi_{x,y}$ we can compute the $CO_{x,y}$ and the $Q_{x,y}$ as function of $\frac{\Delta p}{p_0}$. To do that we have to compute $M_{OTM}(s, \frac{\Delta p}{p_0})$, that is evaluating the property of the element of the lattice as function of $\frac{\Delta p}{p_0}$.

- In a thin quadrupole the focal length linearly scales with the particle momentum:

  $$\begin{pmatrix}
  1 & 0 \\
  -\frac{1}{f\left(\frac{\Delta p}{p_0}\right)} & 1
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  1 & 0 \\
  -\frac{1}{f_0 \times \left(1 + \frac{\Delta p}{p_0}\right)} & 1
  \end{pmatrix}. \quad (83)$$
Figure 5: The trace-space of an ensemble of particles.

- A dipolar corrector $\theta$, scales with the inverse of the beam rigidity:

$$
\left( \theta \left( \frac{\Delta p}{p_0} \right) \right) \rightarrow \left( \frac{0}{\theta_0 \left( \frac{\Delta p}{p_0} \right)} \frac{\theta_0}{1 + \frac{\Delta p}{p_0}} \right).
$$

(84)

- For the dipolar magnet defining the reference orbit (e.g., the arc dipole of a synchrotron) it is important to consider only the differential kick due to the off-momentum:

$$
\left( \theta \left( \frac{\Delta p}{p_0} \right) - \theta_0 \right) \rightarrow \left( \frac{-\Delta p}{p_0} \frac{\theta_0}{1 + \frac{\Delta p}{p_0}} \frac{\theta_0}{1 + \frac{\Delta p}{p_0}} \right).
$$

(85)

3 Particle ensembles

3.1 The beam distribution

The beam can be considered as a set of $N$ particles (Fig. 5). To track $N$ particles we can use the same approach of the single particle tracking were $X$ becomes $X_B$, a $2n \times N$ matrix:

$$
X_B = (X_1, X_2, \ldots, X_n)
$$

(86)
We will restrict ourselves to the 1D case \((n = 1)\). We are looking for one or more statistical quantities that represents this ensemble and its evolution in the lattice.

A natural one is the average \( J_{CS} \) over the ensemble:

\[
\frac{1}{N} \sum_{i=1}^{N} J_{CS,i} = \langle J_{CS} \rangle
\]

From the definition it follows that the quantity is preserved during the beam evolution along the linear lattice.

### 3.1.1 The beam emittance

We will see that \( \langle J_{CS} \rangle \) converges, under specific assumptions (see later), to the **rms emittance** of the beam, \( \epsilon_{rms} \)

\[
\epsilon_{rms} = \sqrt{\det\left(\frac{1}{N} X_B X_B^T\right)}.
\]

where \( \frac{1}{N} X_B X_B^T \) represents the beam \( \sigma \) matrix.

One can see that the \( \epsilon_{rms} \) is preserved for the symplectic linear transformation \( M \) from \( s_0 \) to \( s_1 \) (see Cauchy-Binet theorem):

\[
\epsilon_{rms}^2(s_0) = \det\left(\frac{1}{N} X_B X_B^T\right)
\]

\[
\epsilon_{rms}^2(s_1) = \det(M \frac{1}{N} X_B X_B^T M^T) = \det(M \det(\frac{1}{N} X_B X_B^T) \det(M^T) \frac{1}{N} X_B X_B^T) = \det(M \det(\frac{1}{N} X_B X_B^T) \det(M^T) \frac{1}{N} X_B X_B^T)
\]

where \( X_B \) denotes \( X_B(s_0) \). Therefore we have that

\[
\sigma(s_1) = M \sigma(s_0) M^T
\]

and this transport equation is very similar to the one in Eq. 47

\[
\begin{pmatrix}
\beta(s_1) & -\alpha(s_1) \\
-\alpha(s_1) & \gamma(s_1)
\end{pmatrix}
= M
\begin{pmatrix}
\beta(s_0) & -\alpha(s_0) \\
-\alpha(s_0) & \gamma(s_0)
\end{pmatrix}
M^T.
\]

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3.1.2 The $\sigma$ matrix

By the $\sigma$-matrix definition (Eq. 88), it follows that (e.g., 1D trace-space)

$$
\sigma = \begin{pmatrix}
\frac{1}{N} \sum_{i=1}^{N} x_i x_i & \frac{1}{N} \sum_{i=1}^{N} x_ix'_i \\
\frac{1}{N} \sum_{i=1}^{N} x'_i x_i & \frac{1}{N} \sum_{i=1}^{N} x'_i x'_i
\end{pmatrix} = \begin{pmatrix}
x_{rms}^2 & \langle \bar{x}'^2 \rangle \\
\langle x x' \rangle & \langle x' x' \rangle
\end{pmatrix}
$$

(93)

and therefore we can write

$$
\epsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}.
$$

(94)

To summarize, from the transport equation of the $\sigma$ matrix (Eq. 91) and from the its statistical meaning (Eq. 93), we showed how to numerically transport the second-order moments of the beam distribution.

3.2 Matched beam distribution

A beam distribution is matched in $s_0$ to the specific optics functions $\alpha(s_0)$ and $\beta(s_0)$ if the corresponding normalized distribution $\bar{X}_B = P^{-1}X_B$ is statistically invariant by rotation. In other words $\bar{X}_B$ has an azimuthal symmetry therefore $\langle \bar{x} \bar{x}' \rangle = 0$ and $\langle \bar{x}'^2 \rangle = \langle \bar{x}^2 \rangle$. An example of matched and mismatched beams are presented in Fig. 6 and 7, respectively. It is worth noting that,
since $\bar{P}^{-1}$ is a symplectic matrix, $\epsilon_{\text{rms}} = \epsilon_{\text{rms}}$ and, for a matched beam, we have

$$\bar{\sigma} = \frac{1}{N} \bar{X}_B \bar{X}_B^T = \bar{P}^{-1} \bar{\sigma} \bar{P} = \begin{pmatrix} \bar{x}_{\text{rms}}^2 & 0 \\ \langle \bar{x}^2 \rangle & \langle \bar{x} \bar{x}' \rangle \\ \langle \bar{x} \bar{x}' \rangle & \bar{x}_{\text{rms}}'^2 \end{pmatrix} = \begin{pmatrix} \epsilon_{\text{rms}} & 0 \\ 0 & \epsilon_{\text{rms}} \end{pmatrix}. \quad (95)$$

We can conclude that the normalized $\sigma$ matrix, $\bar{\sigma}$, is diagonal.

For a beam distribution matched to the specific optics functions $\alpha(s_0)$ and $\beta(s_0)$ we have

$$\sigma = \bar{P} \begin{pmatrix} \epsilon_{\text{rms}} & 0 \\ 0 & \epsilon_{\text{rms}} \end{pmatrix} \bar{P}^{-1} = \epsilon_{\text{rms}} \begin{pmatrix} \beta(s_0) & -\alpha(s_0) \\ -\alpha(s_0) & \gamma(s_0) \end{pmatrix}. \quad (96)$$

where we found back the rms beam size and divergence formulas, $\sqrt{\beta\epsilon_{\text{rms}}}$ and $\sqrt{\gamma\epsilon_{\text{rms}}}$, respectively. From Eqs. 92 and 96, one can conclude that, if the beam is matched in the $s_0$-position, then is matched in all $s$. This implies that the second order statistical moments of a matched beam, in a periodic stable lattice and at given position $s$, are a turn-by-turn invariant.

Before concluding this chapter we demonstrate that, for matched beam, we have $\langle J_{CS} \rangle = \epsilon_{\text{rms}}$. 

Figure 7: A mismatched beam in the normalized phase-space.
This is straightforward in the normalized phase-space, in fact from Eq. 48 and 65

\[
J_{CS} = \frac{\bar{x}^2 + \bar{x}'^2}{2}.
\]  

(97)

Since the beam is matched then \( \langle \bar{x}^2 \rangle = \langle \bar{x}'^2 \rangle = \epsilon_{\text{rms}} \), yielding

\[
\langle J_{CS} \rangle = \frac{\langle \bar{x}^2 + \bar{x}'^2 \rangle}{2} = \frac{\langle \bar{x}^2 \rangle + \langle \bar{x}'^2 \rangle}{2} = \epsilon_{\text{rms}}.
\]

(98)

4 Conclusion

In this Chapter we recalled and summarized the main linear optics concepts of the accelerators beam dynamics theory with emphasis on the related computational aspects. Using a pure linear algebra approach and via symplectic matrices transformations, we introduced the concepts of lattice stability, optics functions, normalized phase-space and invariant of motions. In addition to the dynamics of the single particle, we studied the ensembles of particles presenting the statistical invariant of the ensemble and the concept of beam matching.

Appendix

A Code Listings

For the convenience of the reader an electronic version of the Mathematica™ and Python3 listings can be found in [11] and [12], respectively.

Listing 1: The Mathematica™ input to compute the thick quadrupole matrix as limit of thin quadrupoles and drifts.

```mathematica
(* INPUT to Mathematica *)
MD[ L_]={ {1, L}, {0, 1}}
MQ{ KL_}={ {1, 0}, {-KL, 1}}
FullSimplify[Limit[MatrixPower[MQ[ K1 L/n]. MD[L/n], n], n -> \[Infinity],
Assumptions -> { K1 > 0, L > 0}]]

(* OUTPUT *)
{{ Cos[Sqrt[K1] L], Sin[Sqrt[K1] L]/Sqrt[K1]}, {-Sqrt[K1] Sin[Sqrt[K1] L], Cos[Sqrt[K1] L]}}
```

Listing 1: The Mathematica™ input to compute the thick quadrupole matrix as limit of thin quadrupoles and drifts.

Listing 2: The Python3 input to compute the thick quadrupole matrix as limit of thin quadrupoles and drifts.

```python
# INPUT to python3
from sympy import *
import numpy as np
m11=Symbol('m11'); m12=Symbol('m12'); m21=Symbol('m21'); m22=Symbol('m22')
omega=Matrix([[0,1], [-1,0]])
pbar=Matrix([[m11,m12], [m21,m22]])
J=pbar @ omega @ pbar . inv ()
simplify(J.subs(m11*m22 - m12*m21,1))
```

Listing 2: The Python3 input to compute the thick quadrupole matrix as limit of thin quadrupoles and drifts.
Matrix ([
    [-m11 * m21 - m12 * m22, m11 **2 + m12 **2],
    [ -m21 **2 - m22 **2, m11 * m21 + m12 * m22]])

Listing 2: Symbolic expression for $J$ matrix.

```
# INPUT to python3
import numpy as np
la=np.linalg

# Drift
def DRIFT (L=1):
    '''
    This a matrix for a L-long drift.
    '''
    return np.array([[1, L],[0, 1]])

# Quadrupole
def QUAD (f=1):
    '''
    This a matrix for a this quadrupole of focal lenght f.
    '''
    return np.array([[1, 0],[ -1/f,1]])

# One turn maps of 100 m long FODO cell with 50 m focal length.
def M_OTM (f=50):
    M_OTM = DRIFT (50) @ QUAD (-f) @ DRIFT (50) @ QUAD (f)
    return M_OTM

eigenvalues,P= la.eig(M_OTM()) # P is >>before the normalization<<
P=P/((la.det(P)*1j)**(1/len(P))) # P is >>after the normalization<<
D=np.diag(eigenvalues)
beta=np.real(P[0,0]**2*2)
alpha=-np.real(P[1,0]*np.sqrt(2*beta))

print('The cell phase advance is ' + str(np.rad2deg(np.angle(D)[0][0]) ) + ' deg .')
print('The periodic beta at the start of the cell is ' + str(bbeta)+ ' m.')
print('The periodic alpha at the start of the cell is ' + str(alpha)+'.')
```

Listing 3: Basic linear optics code.

```
# INPUT to python3
from sympy import *
alpha=Symbol('alpha'); beta=Symbol('beta'); gamma=Symbol('gamma');
x=Symbol('x'); px=Symbol('px ')
omega=Matrix ([[0,1], [-1,0]])
J=Matrix([[alpha, beta], [-gamma,-alpha]]); X=Matrix([[x],[px]])
```

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Listing 4: From the matrix to the polynomial form of the $J_{CS}$.

```python
# INPUT to python3
import sympy as sy
beta0 = sy.Symbol('beta0'); alpha0 = sy.Symbol('alpha0');
beta1 = sy.Symbol('beta1'); alpha1 = sy.Symbol('alpha1');
m11 = sy.Symbol('m11'); m12 = sy.Symbol('m12'); m21 = sy.Symbol('m21'); m22 = sy.Symbol('m22');
Pbar0 = sy.Matrix([[sy.sqrt(beta0), 0], [-alpha0 /sy.sqrt(beta0), 1/sy.sqrt(beta0)]])
Pbar1 = sy.Matrix([[sy.sqrt(beta1), 0], [-alpha1 /sy.sqrt(beta1), 1/sy.sqrt(beta1)]])
M = sy.Matrix([[m11, m12], [m21, m22]])
pprint(sy.simplify(Pbar1.inv() @M@Pbar0))
```

Listing 5: The phase advance computation.

```python
# INPUT to python3
import sympy as sy
beta1 = sy.Symbol('beta1'); alpha1 = sy.Symbol('alpha1');
beta2 = sy.Symbol('beta2'); alpha2 = sy.Symbol('alpha2');
phi = sy.Symbol('phi'); Q = sy.Symbol('Q'); theta1 = sy.Symbol('theta1');
Pbar1 = sy.Matrix([[sy.sqrt(beta1), 0], [-alpha1 /sy.sqrt(beta1), 1/sy.sqrt(beta1)]])
Pbar2 = sy.Matrix([[sy.sqrt(beta2), 0], [-alpha2 /sy.sqrt(beta2), 1/sy.sqrt(beta2)]])
R = sy.Matrix([[sy.cos(phi), sy.sin(phi), [-sy.sin(phi), sy.cos(phi)]]])
pprint(sy.simplify(Pbar2@R@Pbar1.inv()))
```

Listing 6: Transport matrix as function of the optics parameter.

```python
# INPUT to python3
import sympy as sy
```

Listing 7: The polynomial form of the $J_{CS}$.

```python
expand(1/2*X.T@omega@J.inv()@X)[0,0].subs(alpha**2 - beta*gamma,-1)
```
\[ J = \text{sy. Matrix}([[\text{alpha1}, \text{beta1}], \left[-(1+\text{alpha1}^2)/\text{beta1}, -\text{alpha1}\right]]) \]

\[ I = \text{sy. Matrix}([[1, 0], [0,1]]) \]

\[ MCO = \text{sy. simplify}((I - (I*\text{sy. cos}(2*\text{sy. pi}*Q) + J*\text{sy. sin}(2*\text{sy. pi}*Q))).\text{inv()}) \]

\[ X_0 = \text{sy. simplify}((MCO@\text{sy. Matrix}([[0], [\text{theta1}]])) \]

\[ T = \text{sy. Matrix}([[(\text{sy. sqrt}(\text{beta2})*(\text{sy. cos}(\text{phi})+\text{alpha1})*\text{sy. sin}(\text{phi}))/\text{sy. sqrt}(\text{beta1})],\]
\[ -((\text{-alpha1+alpha2})*\text{sy. cos}(\text{phi})+\text{alpha1}*\text{alpha2})*\text{sy. sin}(\text{phi}))/\text{sy. sqrt}(\text{beta1})/\text{sy. sqrt}(\text{beta2}),\]
\[ \text{sy. sqrt}(\text{beta1})*\text{sy. sqrt}(\text{beta2})*\text{theta1}*(\text{sin}(\text{phi}) + \cos(\text{phi})/\tan(\text{pi}*Q))/2],\]
\[ -\text{sqrt}(\text{beta1})*\text{theta1}*(\text{alpha2})*\text{sin}(\text{phi}) + \text{alpha2})*\text{cos}(\text{phi})/\tan(\text{pi}*Q) + \text{sin}(\text{phi}))/\tan(\text{pi}*Q) - \cos(\text{phi}))/\text{(2*sqrt}(\text{beta2})))]) \]

Listing 7: Closed orbit computation.

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