On convolutions of slanted half-plane mappings

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ABSTRACT
The convolution of convex harmonic univalent functions in the unit disk, unlike analytic functions, may not be convex or even univalent. The main purpose of this work is to develop previous work involving the convolution of convex harmonic functions. Briefly, we obtain under which conditions, the convolution of a right half-plane harmonic mapping having a dilatation \(-z\) and a slanted half-plane harmonic mapping with \(\beta\) having a dilatation \(e^{\mu} - \frac{\mu + 1}{\mu + 2}\) \((|\mu| < 1, \mu \in \mathbb{R})\) is univalent and convex in the direction \(-\beta\). We also provide an example illustrating graphically with the help of Maple to illustrate the result.

1. Introduction
Consider the harmonic function \(k = v_1 + iv_2\) defined in the unit disc \(E = \{z \in \mathbb{C} : |z| < 1\}\) where \(v_1\) and \(v_2\) are real harmonic functions in \(E\). Such a function can be expressed as \(k = s + t\), where

\[
s(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad \text{and} \quad t(z) = \sum_{m=1}^{\infty} b_m z^m
\]

(1)

are analytic in \(E\). The dilatation of a harmonic function \(k\) defined by \(\varphi(z) = t'(z)/s'(z)\) is analytic and satisfies \(|\varphi(z)| < 1\) for all \(z \in E\) if and only if \(k\) is locally univalent (LU) and sense-preserving (SP) in \(E\). Denote by \(\mathcal{SH}_0\) the class of all harmonic, SP and univalent mappings \(k = s + t\) in \(E\), which are normalized by the conditions \(k(0) = k_0(0) = 0\). A simply connected subdomain of \(\mathbb{C}\) is said to be close-to-convex if its complement in \(\mathbb{C}\) can be written as the union of non-crossing half-lines. Further, let \(\mathcal{KH}_0\) and \(\mathcal{CH}_0\) be the subclass of \(\mathcal{SH}_0\) whose image domains are convex and close-to-convex domains, respectively. A domain \(M_\beta\) is said to be convex in the direction \(\beta\) if \(\beta \in \mathbb{R}\) and only if \(\beta = 0\). For every \(\beta \in \mathbb{C}\), the set \(M_\beta \cap \{z + e^{\mu} r : r \in \mathbb{R}\}\) is either connected or empty. In particular, the domain \(M_0\) is said to be convex in the direction of the real axis. Denote by \(\mathcal{CH}_0\), the class of functions \(k = s + t\) in \(\mathcal{SH}_0\) that map \(E\) onto the domain \(M_0\). It is clear that, \(\mathcal{CH}_0 \subset \mathcal{CH}_0\). The shearing method of Clunie and Sheil-Small \([1]\) enables us to construct new examples of harmonic univalent mappings with prescribed dilatations. In this method, they considered a harmonic function \(k = s + t\) which is LU in \(E\) (i.e. \(|\varphi(z)| < 1\) for all \(z \in E\)) and they proved that the function \(K = s - e^{2i/3} t\) is analytic univalent mapping of \(E\) onto the domain \(M_\beta\) if and only if \(k = s + t\) is a univalent mapping of \(E\) onto the domain \(M_\beta\).

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It is clear that, Theorem A implies for every \( k_\beta \in S H_\beta^0 \), \( K_0 \ast k_\beta \in C H_{\beta}^0 \) if \( K_0 \ast k_\beta \) is \( L_2 \) in \( E \). On the other hand, Dorff et al. [7] considered \( k \in S H_\beta^0 \) having a dilatation \( \sigma(z) = \frac{\rho + z}{1 + \rho z} \), with \( \rho \in (-1, 1) \) and they proved that \( K_0 \ast k \) is a member of the class \( C H_{\beta}^0 \). In [12], the authors collected some open problems on harmonic mappings. 

One of the open problems given in reference [12, Problem 3.26(a)] that will be addressed in this study was first raised by Dorff et al. [7]. The problem in question is as follows:

**Problem:** Let \( k = s + T \in S H_\beta^0 \) with the dilatation \( \sigma(z) = \frac{\rho + z}{1 + \rho z} \), \( |\rho| < 1 \). Determine other values of \( \rho \in E \) for the mapping \( K_0 \ast k \) to be a member of the class \( C H_{\beta}^0 \). Furthermore, Li and Ponnusamy [13] proved

**Theorem B ([13, Theorem 1.3]):** Let \( k_\beta = s_\beta + T_\beta \in S H_\beta^0 \) with \( \sigma(z) = \frac{\rho + z}{1 + \rho z} \), where \( \rho = |\rho| e^{i\lambda}, \lambda = \arg \rho, \) and \( |\rho| < 1, \lambda \in \mathbb{R} \). If 

\[
|\rho|^2 \left[ \cos^2 \left( \frac{\lambda - \frac{\beta}{2}}{2} \right) + 9 \sin^2 \left( \frac{\lambda - \frac{\beta}{2}}{2} \right) \right] \leq 1,
\]

but the conditions

\[
|\rho| \cos \left( \frac{\lambda - \frac{\beta}{2}}{2} \right) = - \cos \left( \frac{3\beta}{2} \right),
\]

\[
3 |\rho| \sin \left( \frac{\lambda - \frac{\beta}{2}}{2} \right) = - \sin \left( \frac{3\beta}{2} \right),
\]

then \( K_0 \ast k_\beta \in C H_{\beta}^0 \).

Recently, Jiang et al. [14] studied the convolution of \( K_0 \) with a mapping \( K_1 \in S H_\beta^0 \) and they proved

**Theorem C ([14, Theorem 1]):** Let \( K_1 = S_1 + T_1 \in S H_\beta^0 \) with \( \sigma(z) = \frac{\rho + z}{1 + \rho z} \), where \( \rho = |\rho| e^{i\lambda}, \lambda = \arg \rho, \) \( |\rho| < 1 \) and \( \lambda \in \mathbb{R} \). If 

\[
|\rho|^2 \left[ \cos^2 \left( \mu + \frac{\lambda}{2} \right) + 9 \sin^2 \left( \mu + \frac{\lambda}{2} \right) \right] \leq 1,
\]

and

\[
|\rho| \cos \left( \mu + \frac{\lambda}{2} \right) \neq - \cos \left( \frac{\lambda}{2} \right),
\]

then \( K_0 \ast K_1 \in C H_{\beta}^0 \).

In this paper, we deal mainly with the generalization of Theorem B such that considering \( \sigma \) of \( k_\beta \in S H_\beta^0 \) in a more general setting of the form

\[
\sigma(z) = e^{i\mu} \frac{z + \rho}{1 + \rho z} (\rho = |\rho| e^{i\lambda}, \lambda = \arg \rho, \ |\rho| < 1 \) and \( \lambda, \mu \in \mathbb{R} \),
\]

which is an automorphism of the unit disk. We investigate conditions which guarantee the mapping \( K_0 \ast k_\beta \) is a member of the class \( C H_{\beta}^0 \). Our main result is Theorem 3.1 in Section 3, which improves Theorem B, and also Theorem C. Furthermore, an example, illustrating graphically with the help of Maple, is given to illuminate the main result.

### 2. Preliminary results

**Lemma 2.1 ([13, Lemma 2.1]):** Let \( k_\beta = s_\beta + T_\beta \in S H_\beta^0 \) with \( \sigma(z) = \frac{\rho + z}{1 + \rho z} \). Then \( \sigma \) of \( K_0 \ast k_\beta \) is

\[
\hat{\sigma}(z) = -ze^{\beta i} \left( \frac{\sigma^2(z) + e^{2\beta i} \left[ \sigma(z) - \frac{1}{2} \sigma(\sigma(z)) + \frac{1}{2} e^{\beta i} \sigma(z) \right]}{1 + e^{-2\beta i} \left[ \sigma(z) - \frac{1}{2} \sigma(\sigma(z)) + \frac{1}{2} e^{\beta i} \sigma(z) \right]} \right).
\]

**Lemma 2.2:** Let \( k_\beta = s_\beta + T_\beta \in S H_\beta^0 \) with \( \sigma(z) = e^{i\mu} \frac{z + \rho}{1 + \rho z} \), where \( |\rho| < 1 \) and \( \mu \in \mathbb{R} \). Then \( \sigma \) of \( K_0 \ast k_\beta \) is

\[
\hat{\sigma}(z) = -ze^{-\beta i} \left( 2 + C(z) (2 + D) \right).
\]

where \( \varphi = \arg \left( e^{i\mu} \frac{z + \rho}{1 + \rho z} \right) \)

\[
(1 + Cz) (1 + Dz).
\]

Here \( -C, -D \) are the two zeros of \( \Psi(z) = 0 \), and \( C, D \) may be equal.

**Proof:** We have \( \sigma' = e^{i\mu} \frac{z + 1 - |\rho|^2}{1 + \rho z} \) for \( |\rho| < 1 \) and \( \mu \in \mathbb{R} \). Substituting \( \sigma \), \( \sigma' \) into the Equation (4) and simplifying, we obtain

\[
\hat{\sigma}(z) = -ze^{-\beta i} \Phi(z),
\]

where

\[
\Phi(z) = \frac{e^{i\mu} \frac{z + \rho}{1 + \rho z}^2 + e^{2\beta i} \left[ e^{i\mu} \frac{z + \rho}{1 + \rho z} - \frac{1}{2} e^{i\mu} \frac{z - |\rho|^2}{1 + \rho z} \right]}{1 + e^{-2\beta i} \left[ e^{i\mu} \frac{z + \rho}{1 + \rho z} - \frac{1}{2} e^{i\mu} \frac{z - |\rho|^2}{1 + \rho z} \right]}.
\]

Further, we expand the above equality and combine the coefficients in terms of \( z^2, z \) and \( e^{2\beta i} \) terms in both the numerator and denominator. Then we derive

\[
\Phi(z) = \left( e^{i\mu} + \frac{\rho}{e^{-i\mu} + \rho \ e^{-2\beta i}} \right) \times \left( \frac{z^2 + 4 e^{i\mu} + e^{3\beta i} e^{2\beta i} \ e^{2\beta i} + \ e^{2\beta i} \ e^{i\mu} \ e^{2\beta i} \ e^{2\beta i} \ e^{i\mu}}{2 (e^{i\mu} + e^{-i\mu})} \right) \times \left( \frac{z^2 + 2 e^{i\mu} + e^{2\beta i} + e^{2\beta i} \ e^{i\mu} \ e^{2\beta i} \ e^{2\beta i} \ e^{i\mu}}{2 (e^{i\mu} + e^{-i\mu})} \right) \times \left( \frac{z^2 + 2 e^{i\mu} + e^{2\beta i} + e^{2\beta i} \ e^{i\mu} \ e^{2\beta i} \ e^{2\beta i} \ e^{i\mu}}{2 (e^{i\mu} + e^{-i\mu})} \right).
\]
Here $\Psi(z)$ is given by (5) and
\[
\Psi^*(z) = 1 + \frac{4\rho e^{-i\mu} + e^{2i\beta} + 3 |\rho|^2 e^{-2i\beta}}{2 (e^{-i\mu} + \rho e^{-2i\beta})} z
+ \frac{2\rho^2 e^{-i\mu} + 2\rho e^{-2i\beta} + e^{-i\beta} (1 - |\rho|^2)}{2 (e^{-i\mu} + \rho e^{-2i\beta})} z^2.
\]
Suppose that $-C, -D$ are the two zeros of $\Psi(z) = 0$ ($C$ and $D$ may be equal). Then
\[
\Psi(z) = (z + C) (z + D)
\]
and
\[
\Psi^*(z) = z^2 \Psi \left( \frac{1}{z} \right) = (1 + C z) (1 + D z).
\]
As $|(e^{i\mu} + \rho e^{2i\beta})/(e^{-i\mu} + \rho e^{-2i\beta})| = 1$, the result follows.

**Lemma 2.3:** Let $\Psi(z)$ be defined by (5) so that $\Psi(z) = (z + C) (z + D)$. Also, let $\rho = |\rho| e^{i\lambda}$ where $\lambda = \arg \rho$ and $|\rho| < 1$. If
\[
|\rho|^2 \left[ \cos^2 \left( \frac{\lambda + \mu - \beta}{2} \right) + 9 \sin^2 \left( \frac{\lambda + \mu - \beta}{2} \right) \right] \leq 1,
\]
then $|CD| \leq 1$. Moreover, $|CD| = 1$ if and only if
\[
|\rho| \cos \left( \frac{\lambda + \mu - \beta}{2} \right) = - \cos \left( \frac{\mu - 3\beta}{2} \right)
\]
and
\[
3 |\rho| \sin \left( \frac{\lambda + \mu - \beta}{2} \right) = - \sin \left( \frac{\mu - 3\beta}{2} \right)
\]
for $\frac{1}{2} \leq |\rho| < 1$.

**Proof:** Since $\Psi(z) = (z + C) (z + D)$, it is clear that
\[
CD = \frac{2\rho^2 e^{i\mu} + 2\rho e^{2i\beta} + e^{i\beta} (1 - |\rho|^2)}{2 (e^{i\mu} + \rho e^{2i\beta})} = \frac{2\rho (e^{i\mu} + e^{i\beta}) + e^{i\beta} (1 - |\rho|^2)}{2 (e^{i\mu} + \rho e^{2i\beta})}.
\]
Further, we have
\[
|2\rho (e^{i\mu} + e^{i\beta}) + e^{i\beta} (1 - |\rho|^2)|^2 = 4 |\rho|^2 \Psi(\rho),
\]
where $\Psi(\rho)$ is real and
\[
\Psi(\rho) = 4 \Re \left( \rho^2 e^{i(\mu - \beta)} \right) + 4 \Re \left( \rho e^{i\beta} \right) - 8 \Re \left( \rho e^{i(-2\beta + \mu)} \right) - 3 - 5 |\rho|^2.
\]
Consideration of $\rho = |\rho| e^{i\lambda}$ gives
\[
\Psi(\rho) = 4 |\rho|^2 \cos (2\lambda + \mu - \beta) + 4 |\rho| \cos (\lambda + \mu + \beta)
- 8 |\rho| \cos (\lambda + \mu - 2\beta) - 3 - 5 |\rho|^2
\]
\[
= 4 |\rho|^2 \left[ 1 - 2 \sin^2 \left( \frac{\lambda + \mu - \beta}{2} \right) \right]
+ 4 |\rho| \cos \left( \lambda + \mu - \beta \right) \cos \left( \mu - 3\beta \right)
+ \sin \left( \lambda + \mu - \beta \right) \sin \left( \mu - 3\beta \right)
- 8 |\rho| \cos \left( \lambda + \mu - \beta \right) \cos \left( \mu - 3\beta \right)
- \sin \left( \lambda + \mu - \beta \right) \sin \left( \mu - 3\beta \right)
= 1 - \left[ |\rho| \cos \left( \lambda + \mu - \beta \right) + 2 \cos \left( \mu - 3\beta \right) \right] - 3 - 5 |\rho|^2
\]
\[
= \left[ 3 |\rho| \sin \left( \lambda + \mu - \beta \right) - 2 \sin \left( \mu - 3\beta \right) \right]^2.
\]
Using (6), we have a point
\[
H_1 \left( |\rho| \cos \left( \lambda + \mu - \beta \right), 3 |\rho| \sin \left( \lambda + \mu - \beta \right) \right)
\]
which lies in $E$, whereas the point
\[
H_2 \left( -2 \cos \left( \mu - 3\beta \right), 2 \sin \left( \mu - 3\beta \right) \right)
\]
lies on the circle $|z| = 2$. Thus, the distance between the points $H_1$ and $H_2$ must be at least 1. That is,
\[
\left[ \left| |\rho| \cos \left( \lambda + \mu - \beta \right) + 2 \cos \left( \mu - 3\beta \right) \right|^2
+ \left| 3 |\rho| \sin \left( \lambda + \mu - \beta \right) - 2 \sin \left( \mu - 3\beta \right) \right|^2 \right] ^{\frac{1}{2}} \geq 1
\]
which means that $\Psi(\rho) \leq 0$ i.e. $|CD| \leq 1$. In addition, in (10), equality holds if and only if the point $H_1$ lies on the unit circle. Note that $H_1(x_1, x_2)$ also lies on the ellipse $\left\{ z = x_1 + ix_2 : \frac{x_1^2}{m^2} + \frac{x_2^2}{\rho^2 m^2} = 1 \right\} \text{ for } 0 < |\rho| < 1$. Thus, we obtain conditions (7) and (8). In conclusion, (6) holds but not the (7) and (8), then $\Psi(\rho) < 0$ and hence, $|CD| < 1$ holds. If (7) and (8) hold, then $\Psi(\rho) = 0$ and hence, $|CD| = 1$. So, we are done.

An important tool to prove harmonic functions are LU and SP is Cohn’s Rule. This rule is given as follows

**Lemma 2.4 ([15]):** Given a polynomial
\[
r(z) = c_0 + c_1 z + \cdots + c_m z^m
\]
of degree $m$, let
\[
r^*(z) = z^{m-1}r \left( \frac{1}{z} \right) = \overline{c_m} + \overline{c_{m-1}} z + \cdots + \overline{c_0} z^m.
\]
Let $p$ and $q$ be the number of zeros of $r$ inside the unit circle and on it, respectively. If $|c_0| < |c_m|$, then
\[
r_1(z) = \frac{\overline{c_m} r(z) - c_0 r^*(z)}{z}
\]
is of degree $m-1$ with $p_1 = p - 1$ and $q_1 = q$ the number of zeros of $r_1$ inside the unit circle and on it, respectively.
3. Main result

**Theorem 3.1:** Suppose $k_0 = s_0 + \bar{c}_0 \in S\mathcal{H}_\beta^0$ with the dilatation $\varphi(z) = e^{i\beta}z^{\beta+2} + z^{\beta+2}$, where $\rho = |\rho|e^{i\mu}$, $\lambda = \arg \rho$, $|\rho| < 1$ and $\mu \in \mathbb{R}$. If

$$|\rho|^2 \left[ \cos^2 \left( \lambda + \frac{\mu - \beta}{2} \right) + 9 \sin^2 \left( \lambda + \frac{\mu - \beta}{2} \right) \right] \leq 1, \quad (11)$$

and

$$|\rho| \cos \left( \lambda + \frac{\mu - \beta}{2} \right) \neq - \cos \left( \frac{\mu - 3\beta}{2} \right), \quad (12)$$

$$3|\rho| \sin \left( \lambda + \frac{\mu - \beta}{2} \right) \neq - \sin \left( \frac{\mu - 3\beta}{2} \right), \quad (13)$$

then $K_0 \ast k_0 \in \mathcal{H}_\beta^0$.

**Proof:** Referring Lemma 2.3 and conditions (11)–(13), we obtain $|CD| < 1$. Then at least one of $-C, -D$ is in $E$. Assume, with no loss of generality, that $-C \in E$. Applying Lemma 2.4 to $\Psi(z)$ given by (5)

$$\Psi(z) = z^2 + c_1 z + c_0 = (z + C)(z + D),$$

then we obtain

$$\Psi_1(z) = \frac{\Psi(z) - c_0\Psi^*(z)}{z} = (1 - |c_0|^2) z + c_1 - c_0c_1$$

has a zero at

$$z_0 = \frac{c_0c_1 - c_1}{1 - |c_0|^2} = \frac{C(|D|^2 - 1) + D(|C|^2 - 1)}{1 - |CD|^2} = \frac{C(|D|^2 - 1) + D(|C|^2 - 1)}{1 - |CD|^2} = \frac{C(|D|^2 - 1) + D(|C|^2 - 1)}{1 - |CD|^2},$$

which, after simplification, is equivalent to

$$z_0 = e^{-i\beta} \left[ \frac{6\rho^2 e^{i(\mu - \beta)} + 8\rho e^{\beta} - 4\rho e^{i(2\beta - \mu)}}{4\rho e^{i(2\beta - \mu)} - 1 - 3|\rho|^2} \right].$$

Assume $z_0$ as $z_0 = e^{-i\beta} \frac{\varphi(\rho)}{\varphi(\rho)}$ where $\varphi(\rho)$ is given by (9) and

$$\varphi(\rho) = 6\rho^2 e^{i(\mu - \beta)} + 8\rho e^{\beta} - 4\rho e^{i(2\beta - \mu)} + 2 e^{i(3\beta - \mu)} - 1 - 3|\rho|^2.$$

Note that, by Lemma 2.3, if conditions (12) and (13) do not hold, then $\varphi(\rho) = 0$.

Now, we obtain that $|z_0| \leq 1$ if and only if $|\varphi(\rho)|^2 \leq |\varphi(\rho)|^2$. Using

$$|\varphi(\rho)|^2 = |\varphi(\rho)|^2 = \text{Im} \left( \varphi(\rho) \right)^2 + \left[ \Re \left( \varphi(\rho) \right) - \Re \left( \varphi(\rho) \right) \right]$$

and letting $\rho = |\rho|e^{i\beta}$ ($|\rho| < 1$, $\lambda \in \mathbb{R}$), we compute that

$$\Re \left( \varphi(\rho) \right) = 6|\rho|^2 \cos (2\lambda + \mu - \beta) + 8|\rho| \cos (\lambda + \beta) - 4|\rho| \cos (\lambda + \mu - 2\beta) + 2 \cos (\mu - 3\beta) - 1 - 3|\rho|^2, \quad (11)$$

and

$$\text{Im} \left( \varphi(\rho) \right) = 6|\rho|^2 \sin (2\lambda + \mu - \beta) + 8|\rho| \sin (\lambda + \beta) + 4|\rho| \sin (\lambda + \mu - 2\beta) + 2 \sin (3\beta - \mu)$$

$$+ 12|\rho|^2 \sin (\lambda + \mu - 2\beta) + 2 \sin (3\beta - \mu)$$

Also,

$$\Re \left( \varphi(\rho) \right) - \psi(\rho) = 2|\rho|^2 \left[ \cos (2\lambda + \mu - \beta) + 1 \right]$$

$$+ 4|\rho| \left[ \cos (\lambda + \mu - 2\beta) + \cos (\lambda + \beta) \right]$$

$$+ 2 \cos (\mu - 3\beta) + 1$$

$$+ 4|\rho|^2 \left[ \cos (\lambda + \mu - 2\beta) \right]$$

$$+ 8|\rho| \cos (\lambda + \mu - 2\beta) \cos (\mu - 3\beta)$$

$$+ 4 \cos^2 \left( \frac{\mu - 3\beta}{2} \right)$$

$$+ 4 \left[ |\rho| \cos (\lambda + \mu - 2\beta) + \cos \left( \frac{\mu - 3\beta}{2} \right) \right]^2$$

Note that, by Lemma 2.3, if conditions (12) and (13) do not hold, then $\psi(\rho) = 0$.

Now, we obtain that $|z_0| \leq 1$ if and only if $|\varphi(\rho)|^2 \leq |\varphi(\rho)|^2$. Using

$$|\varphi(\rho)|^2 = |\varphi(\rho)|^2 = \text{Im} \left( \varphi(\rho) \right)^2 + \left[ \Re \left( \varphi(\rho) \right) - \Re \left( \varphi(\rho) \right) \right]\times \left[ \Re \left( \varphi(\rho) \right) + \psi(\rho) \right]$$

and letting $\rho = |\rho|e^{i\beta}$ ($|\rho| < 1$, $\lambda \in \mathbb{R}$), we compute that

$$\Re \left( \varphi(\rho) \right) = 6|\rho|^2 \cos (2\lambda + \mu - \beta) + 8|\rho| \cos (\lambda + \beta)$$

$$- 4|\rho| \cos (\lambda + \mu - 2\beta) + 2 \cos (\mu - 3\beta) - 1 - 3|\rho|^2, \quad (11)$$

and

$$\text{Im} \left( \varphi(\rho) \right) = 6|\rho|^2 \sin (2\lambda + \mu - \beta) + 8|\rho| \sin (\lambda + \beta)$$

Further, we get

$$|\varphi(\rho)|^2 = |\varphi(\rho)|^2$$

$$= 8 \left[ |\rho| \cos \left( \frac{\lambda + \mu - 2\beta}{2} \right) \cos \left( \frac{\mu - 3\beta}{2} \right) \right]^2$$

$$\times \left[ \left( 3|\rho| \sin \left( \frac{\lambda + \mu - 2\beta}{2} \right) \right)^2$$

$$+ |\rho|^2 \left( 1 - 10 \sin^2 \left( \frac{\lambda + \mu - 2\beta}{2} \right) \right)$$

$$+ 12|\rho| \sin \left( \frac{\lambda + \mu - 2\beta}{2} \right) \sin \left( \frac{\mu - 3\beta}{2} \right)$$

$$- 2 \sin^2 \left( \frac{\mu - 3\beta}{2} \right) - 1 \right]$$


\[ = 8 \left| \rho \right| \cos \left( \lambda + \frac{\mu - \beta}{2} \right) + \cos \left( \frac{\mu - 3\beta}{2} \right) \right] ^2 \times \left\{ \left[ 9 \sin^2 \left( \lambda + \frac{\mu - \beta}{2} \right) + \cos^2 \left( \lambda + \frac{\mu - \beta}{2} \right) \right] \left| \rho \right|^2 - 1 \right\}.\]

Since \( |\rho| < 1 \), we observe that \( |z_0| < 1 \) if and only if

\[ |\rho|^2 \left[ 9 \sin^2 \left( \lambda + \frac{\mu - \beta}{2} \right) + \cos^2 \left( \lambda + \frac{\mu - \beta}{2} \right) \right] < 1.\]

Also, \( |z_0| = 1 \) if and only if

\[ |\rho|^2 \left[ 9 \sin^2 \left( \lambda + \frac{\mu - \beta}{2} \right) + \cos^2 \left( \lambda + \frac{\mu - \beta}{2} \right) \right] = 1.\]

Thus, if (11)–(13) hold, then \(-C \in E \) and \(-D = \bar{E} \). So, \( |\tilde{\delta}(z)| < 1 \) for each \( z \in E \). Referring Theorem A, we conclude that \( K_0 \ast k_\beta \in CH_0 \).\]

It can be remarked that this method can potentially be applicable to the estimation of eigenvalues. See the paper [16].

**Example 3.2:** Let \( k_\beta = s_\beta + t_\beta \in S \mathcal{H}_0^D \) with \( \beta = \frac{\pi}{2} \) and \( \sigma(\varphi) = \frac{\rho + z}{1 + \rho z} \). Thus, \( k_\beta \) is less than \( 1 \), then the hypothesis of Theorem B does not hold and Theorem B can not be used. Also, let \( K_1 = S_1 + t_1 \in S \mathcal{H}_0^D \) and \( \sigma(\varphi) = e^{i\beta} \frac{\rho + z}{1 + \rho z} \), then the hypothesis of Theorem C does not hold and Theorem C can not be used. However, if \( k_\beta = s_\beta + t_\beta \in S \mathcal{H}_0^D \) and \( \sigma(\varphi) = e^{i\beta} \frac{\rho + z}{1 + \rho z} \), then conditions (11), (12), and (13) hold. Thus, by Theorem 3.1, \( K_0 \ast k_\beta \in CH_0 \).\]

Now, we exemplify the above outcome by obtaining the images of \( E \) under the mappings \( k_\beta \) and \( K_0 \ast k_\beta \). Since

\[ s_\beta(z) - t_\beta(z) = \frac{z}{1 - iz} \] and \( d_\beta(z) = \frac{2iz - i}{2 - z} \rho \beta(z), \]

we have

\[ s_\beta(z) = \frac{1 - 2i}{2} \frac{z}{1 - iz} + \frac{3i}{4} \log \frac{(2 + i)(1 - iz)}{(-1 - 2i)z + 2 + i} \]

and

\[ t_\beta(z) = \frac{-1 - 2i}{2} \frac{z}{1 - iz} + \frac{3i}{4} \log \frac{(2 + i)(1 - iz)}{(-1 - 2i)z + 2 + i}. \]

Using (3), we yield

\[ S_0 \ast s_\beta = \frac{1 - 2i}{4} \frac{z}{1 - iz} + \frac{3i}{8} \log \frac{(2 + i)(1 - iz)}{(-1 - 2i)z + 2 + i} \]

\[ + \frac{(-z + 2)z}{2 (1 - iz)^2 ((-1 - 2i)z + 2 + i)} \]

and

\[ T_0 \ast t_\beta = \frac{-1 - 2i}{4} \frac{z}{1 - iz} + \frac{3i}{8} \log \frac{(2 + i)(1 - iz)}{(-1 - 2i)z + 2 + i} \]

\[ + \frac{(2iz - i)z}{2 (1 - iz)^2 ((-1 - 2i)z + 2 + i)}. \]

Applying Lemma 2.2 with \( \sigma(\varphi) = \frac{2iz - i}{2 - z} \), we obtain

\[ |\bar{\delta}(z)| = \left| \frac{(iz) \Psi(z)}{\Psi^*(z)} \right| = \left| \frac{z(z + C)(z + D)}{(1 + Cz)(1 + Dz)} \right| < 1, \]

where

\[ -C \approx 0.5929638 - 0.7762568i \]

and

\[ -D \approx 0.5570361 + 0.4762568i \]

are the zeros of the equation

\[ \Psi(z) = z^2 - \frac{1}{20} (23 - 3i) z + \frac{1}{20} (7 - 3i) = 0. \]

Figures 1 and 2 show the images of concentric circles inside \( E \) under \( k_\beta \) and \( K_0 \ast k_\beta \).
4. Conclusion

In this study, we considered a RHP mapping \( K_0 = H_0 + G_0 \) having a form \( H_0(z) + G_0(z) = \frac{z}{1-z} \) with a dilatation \(-z\) and a \( \beta - \) SHP mapping \( k_\beta = s_\beta + t_\beta \) having a form \( s_\beta(z) + e^{-\beta} t_\beta(z) = \frac{z}{1-e^{\beta} z} \) with a dilatation \( e^{i\mu} \rho + z \) (\( \rho \in E \) and \( \mu \in \mathbb{R} \)). We derived the convolution of these mappings is a member of the class \( C^CH^0_{-\beta} \) provided that the conditions of (11)–(13) hold. We would like to emphasize an important innovation of our study as follows: We generalized Theorem B by adding a rotation parameter \( \mu \) in the dilatation. In this context, as observed in Example 3.2, when the parameters are specially selected, Theorems B and C cannot be used, while our main result, Theorem 3.1 holds. In our further investigations we will consider some open problems related with the generalized RHP mapping defined by Muir [17, 18]

\[
L_\zeta (z) = \frac{1}{1 + \zeta} \left[ \frac{\zeta z}{(1-z)^2} + \frac{z}{1-z} \right]
+ \frac{1}{1 + \zeta} \left[ \frac{\zeta z}{(1-z)^2} - \frac{z}{1-z} \right]
\]

where \( \zeta > 0 \) and studied in recent papers such as [9, 10]. In this regard, considering the mapping \( L_\zeta \) instead of the mapping \( K_0 \), the conditions will be determined for which the results of Theorems B, C, and 3.1 hold.

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No potential conflict of interest was reported by the author(s).

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