Integrable crosscap states in $\mathfrak{gl}(N)$ spin chains

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Abstract: We study the integrable crosscap states of the integrable quantum spin chains and we classify them for the $\mathfrak{gl}(N)$ symmetric models. We also give a derivation for the exact overlaps between the integrable crosscap states and the Bethe states. The first part of the derivation is to calculate sum formula for the off-shell overlap. Using this formula we prove that the normalized overlaps of the multi-particle states are ratios of the Gaudin-like determinants. Furthermore we collect the integrable crosscap states which can be relevant in the AdS/CFT correspondence.
1 Introduction

In recent years there has been renewed interest for the integrable boundary states of 1+1 dimensional field theories \(^1\) and their spin chain versions \(^2\), \(^3\). These states appear in quite distinct parts of theoretical physics including statistical physics and the gauge/string duality. In statistical physics these quantities appear in the context of non-equilibrium dynamics of the integrable models. In a quantum quench (a parameter of the Hamiltonian is suddenly changed) the goal is to study the non-equilibrium dynamics, the emergence of steady states, and their properties \(^4\). The so-called Quench Action method \(^5\) is one of the main methods for the investigation of the steady states where the knowledge of the exact overlaps between the boundary states and energy eigenstates is an important input \(^6\), \(^7\). The exact overlap formulas also played a central role in the early studies of the Generalized Gibbs Ensemble (GGE) in interacting integrable models, see \(^8\), \(^9\).

The boundary states also play crucial role in the AdS/CFT correspondence. It turned out that the one-point functions in defect CFT \(^10\)–\(^17\) and three-point functions involving determinant operators \(^18\)–\(^20\), can be mapped to the overlaps between boundary states and energy eigenstates of integrable spin chains at weak coupling. Some of these results are also extended to any coupling at the asymptotic limit \(^18\), \(^21\)–\(^23\).

In recent years several exact overlaps were determined. Based on the early results on the overlaps of integrable boundary states, it was conjectured that the normalized overlap always can be written in the following form

\[
\frac{|\langle B|\mathbb{B}(\bar{u})\rangle|^2}{||\mathbb{B}(\bar{u})||^2} = \prod_{j,\nu} F_{\nu}(u_j^\nu) \times \frac{\det G^+}{\det G^-}, \tag{1.1}
\]

where \(\langle B\rangle\) is the integrable boundary state and \(\mathbb{B}(\bar{u})\) is a Bethe state\(^2\). This conjecture was proved in several ways \(^28\)–\(^32\) but these derivations could be used only for XXZ type spin chains (rank 1 symmetry). The first derivation for nested spin chains was presented in \(^33\). The derivation is based on the algebraic relation of the boundary state, the so-called the KT-relation. This KT-relation can be used to derive a recurrence equation for the off-shell overlap and applying the original Korepin’s idea this recursion is enough the prove the form (1.1). In \(^33\) the exact overlaps of \(\mathfrak{g}l(N)\) spin chain were proved for all of the \(\mathfrak{so}(N)\) and \(\mathfrak{gl}(\left\lfloor \frac{N}{2} \right\rfloor) \oplus \mathfrak{gl}(\left\lceil \frac{N}{2} \right\rceil)\) symmetric boundary states.

Recently, in \(^34\) the authors introduced an other type of initial states of spin chains which are the analogous versions of the crosscap states of 2d CFT \(^35\). Geometrically, these states correspond to non-orientable surfaces such as \(\mathbb{RP}^2\) and the Klein bottle. In \(^34\) the spin chain version of crosscap states was introduced for XXX spin chain. It was shown that this crosscap state satisfies the same integrability condition which was proposed for the boundary states in \(^2\). The authors of \(^34\) also proposed a formula for the overlap:

\[
\frac{|\langle C|\mathbb{B}(\bar{u})\rangle|^2}{||\mathbb{B}(\bar{u})||^2} = \frac{\det G^+}{\det G^-}, \tag{1.2}
\]

where \(\langle C\rangle\) is the integrable crosscap state. This overlap formula has not been proved, it was guessed and validated by numerical computations. We can see that the overlap of the crosscap state is similar to the boundary state but now the boundary dependent term is missing.

\(^1\)The boundary states have two types: two-site states and matrix product states (MPS). The form (1.1) corresponds to the two-site states. For MPS there is also a summation for the boundary dependent terms.

\(^2\)In this paper we use the notations of \(^24\)–\(^27\) where the symbol \(\mathbb{B}(\bar{u})\) without ket denotes the Bethe state.
Recently a holographic description of $\mathcal{N} = 4$ super Yang-Mills on the four-dimensional real projective space $\mathbb{RP}^4$ was proposed in [36]. The $\mathbb{RP}^4$ is the simplest unorientable four-manifold which is obtained by modding out the four dimensional sphere $S^4$ by the involution that identifies antipodal points. To formulate $\mathcal{N} = 4$ SYM on $\mathbb{RP}^4$ one has to specify how this involution acts on the elementary fields. In [36] the authors found two possibility which preserves half of the supersymmetry: the involution acts as identity or charge conjugation on the fields. They investigated the first version and showed that the one-point functions of scalar operators do not have selection rules we expect for an integrability preserving configuration. However the authors stated that for the second configuration there is a compelling guess for the holographic dual, as an orientifold projection of $AdS_5 \times S^5$, which in particular adds a crosscap on the worldsheet. In [34] it was shown that integrability survives in their presence. This suggests that the orientifold setup is in fact integrable. Based on these observations, the classification of the integrable crosscap states of $SO(6)$ spin chain can help to find the integrability preserving orientifold setups of the $\mathcal{N} = 4$ SYM.

The goals of this paper are the following: classifying the boundary crosscap states for every $\mathfrak{gl}(N)$ symmetric spin chain and deriving the proposed overlap formula (1.2) for as many states as possible. As mentioned earlier, for boundary states these results are consequences of the $KT$-relation therefore at first we generalize the algebraic framework of [33] for crosscap states, i.e., we introduce the $KT$-relation for crosscap states of the $\mathfrak{gl}(N)$ spin chains. Using this $KT$-relation we can classify the integrable crosscap states. It turns out that the possible residual symmetries are $\mathfrak{so}(N)$, $\mathfrak{sp}(N)$ and $\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$. We also discuss which of these states may be relevant in AdS/CFT. An other benefit of the $KT$-relation is that we can derive a sum formula for the off-shell overlap which can be used to derive the on-shell formula (1.2) for the $\mathfrak{so}(N)$ and $\mathfrak{gl}([N,2]) \oplus \mathfrak{gl}([N,2])$ symmetric crosscap states.

The paper is organized as follows. In section 2 we review the basic definitions of the $\mathfrak{gl}(N)$ spin chains. In section 3 we introduce the $KT$-relations and we also classify the solutions which give us the integrable crosscap states. In section 4 we discuss the potentially relevant states for the AdS/CFT correspondence. In section 5 we derive a sum formula for the off-shell overlaps. In section 6 we take the on-shell limit of the sum formula and show that the normalized overlap is equal to the ratio of Gaudin-like determinants.

2 Definitions

In this section we review the definitions of the $\mathfrak{gl}(N)$ spin chains. We use the notations of [25–27, 33, 37]. At first let us define the monodromy matrix $T(u) = \sum_{i,j=1}^N E_{i,j} \otimes T_{i,j}(u) \in \text{End}(\mathbb{C}^N) \otimes \text{End}(\mathcal{H})$ ($\mathcal{H}$ is the quantum space and $E_{i,j}$-s are the generators of $\mathfrak{gl}(N)$) by the usual $R T T$-relation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v),$$

(2.1)

where we used the $\mathfrak{gl}(N)$ R-matrix

$$R(u) = u1 + cP,$$

(2.2)

where $c$ is a constant and $I$ and $P$ are the identity and the permutation operators in the vector space $\mathbb{C}^N \otimes \mathbb{C}^N$. The monodromy matrix entries $T_{i,j}(u)$ generate the famous Yangian algebra $Y(N)$ [38]. We can define representations of the Yangian $Y(N)$ on the quantum space $\mathcal{H}$. A representation is highest weight if there exists a unique pseudo-vacuum $|0\rangle \in \mathcal{H}$ such that

$$T_{i,i} |0\rangle = \lambda_i(u) |0\rangle, \quad \text{for } i = 1, \ldots, N,$$

$$T_{j,i} |0\rangle = 0, \quad \text{for } 1 \leq i < j \leq N.$$

(2.3)
The \( \lambda_i(u) \)-s are the vacuum eigenvalues. The irreducible representations of \( \mathfrak{gl}(N) \) can be generalized for representations of \( Y(N) \). For the \( N \)-tuples \( \Lambda = (\Lambda_1, \ldots, \Lambda_N) \) we can define a representation \( \mathcal{Y}^\Lambda \). Let \( E_{i,j}^\Lambda \in \text{End}(\mathcal{Y}^\Lambda) \), \( |0^\Lambda\rangle \in \mathcal{Y}^\Lambda \) be the corresponding generators and highest weight state for which

\[
E_{i,j}^\Lambda |0^\Lambda\rangle = \Lambda_i |0^\Lambda\rangle, \quad \text{for } i = 1, \ldots, N,
\]
\[
E_{i,j}^\Lambda |0^\Lambda\rangle = 0, \quad \text{for } 1 \leq i < j \leq N.
\]

Using this representation we can define the matrices (Lax-operators)

\[
L^\Lambda(u) = 1 + \frac{c}{u} \sum_{i,j=1}^{N} E_{i,j} \otimes E_{j,i}^\Lambda \in \text{End}(\mathbb{C}^N) \otimes \text{End}(\mathcal{Y}^\Lambda),
\]

which are solutions of the RTT-relation. We use the following convention for the representation

\[
(E_{i,j}^\Lambda)^t = E_{j,i}^\Lambda,
\]

where \( t \) denotes the transposition. We can define the twisted Lax operator

\[
\tilde{L}^\Lambda_{1,2}(u) = V_1 \left(L^\Lambda_{1,2}(-u)\right)^{t_1} V_1 = 1 - \frac{c}{u} E_{N+1-j,N+1-i} \otimes E_{j,i}^\Lambda = 1 - \frac{c}{u} E_{i,j} \otimes E_{N+1-i,N+1-j}^\Lambda,
\]

where \([V]_{i,j} = \delta_{i,N+1-j} \). Defining the operator \( \mathcal{V}^\Lambda \in \text{GL}(\mathcal{Y}_\Lambda) \) as

\[
E_{N+1-i,N+1-j}^\Lambda = \mathcal{V}^\Lambda E_{i,j}^\Lambda \mathcal{V}^\Lambda,
\]

we obtain that

\[
\tilde{L}^\Lambda_{1,2}(u) = V_2^\Lambda \left(L^\Lambda_{1,2}(-u)\right)^{t_2} V_2^\Lambda.
\]

Let us also define the contra-gradient reps \( \tilde{\Lambda} = (-\Lambda_N, -\Lambda_{N-1}, \ldots, -\Lambda_1) \). Since the generators

\[
\tilde{E}_{i,j} := - \mathcal{V}^\Lambda E_{j,i}^\Lambda \mathcal{V}^\Lambda = - E_{N+1-j,N+1-i}^\Lambda
\]

have highest weights \((-\Lambda_N, -\Lambda_{N-1}, \ldots, -\Lambda_1)\), we can define the generators of rep \( \mathcal{Y}_{\tilde{\Lambda}} \) as

\[
E_{i,j}^{\tilde{\Lambda}} = \tilde{E}_{i,j}.
\]

We can also obtain the following identity:

\[
(L^\Lambda_{1,2}(u))^{t_1 t_2} = L^\Lambda_{1,2}(u).
\]

Let us consider the following tensor product quantum space \( \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \) and define monodromy matrices on each sub-spaces \( T^{(i)}(u) \in \text{End}(\mathbb{C}^N) \otimes \mathcal{H}^{(i)} \) for \( i = 1, 2 \). We can define a monodromy matrix (which satisfy the RTT-relation) on the tensor product space as

\[
T(u) = T^{(2)}(u)T^{(1)}(u).
\]

The consequence of this co-product property is that we can build more general monodromy matrices using the elementary ones (2.5):

\[
T_0(u) = L^{(1)}_{0,j} (u - \xi_j) \ldots L^{(1)}_{0,1} (u - \xi_1).
\]
We can also define the transfer matrix
\[ T(u) = \text{tr} T(u), \]
which gives commuting quantities
\[ [T(u), T(v)] = 0. \]  
(2.15)
For a given sets of complex numbers \( \bar{t}^\mu = \{ t_k^\mu \}_{k=1}^N, \) \( \mu = 1, \ldots, N-1, \) following \([25]\), one can define off-shell Bethe vectors
\[ \mathbb{B}(\bar{t}) = \mathbb{B}(\bar{t}^1, \ldots, \bar{t}^{N-1}). \]

The recursion for the definition of the off-shell Bethe vector can be found in the appendix A. We call the Bethe vector on-shell if the Bethe roots \( \bar{t}^\mu \) satisfies the Bethe Ansatz equations
\[ \alpha^\mu(t_k^\mu) := \frac{\lambda^\mu(t_k^\mu)}{\lambda_{\mu+1}(t_k^\mu)} = \frac{f(t_k^\mu, \bar{t}^\mu_k) f(t_k^{\mu+1}, \bar{t}^\mu_k)}{f(t_k^\mu, \bar{t}^\mu_k) f(t_k^\mu, \bar{t}^{\mu-1})}, \]
(2.17)
where
\[ f(u, v) = 1 + g(u, v) = \frac{u - v + c}{u - v}, \quad \bar{t}^\mu_k = \bar{t}^\mu \setminus t_k^\mu, \]
\[ f(u, \bar{t}^\mu) = \prod_{k=1}^{\nu} f(u, t_k^\mu), \quad f(\bar{t}^\mu, u) = \prod_{k=1}^{\nu} f(t_k^\mu, u), \quad f(\bar{t}^\mu, \bar{t}^\nu) = \prod_{k=1}^{\nu} f(t_k^\nu, \bar{t}^\mu). \]
(2.18)

The on-shell Bethe vectors are eigenvectors of the transfer matrix
\[ T(u) \mathbb{B}(\bar{t}) = \tau(u|\bar{t}) \mathbb{B}(\bar{t}), \]
with the eigenvalue
\[ \tau(u|\bar{t}) = \sum_{i=1}^{N} \lambda_i(u) f(\bar{t}^\mu, u) f(u, \bar{t}^{\mu-1}), \]
(2.20)
where \( r_0 = r_N = 0. \)

One can also define the left eigenvectors of the transfer matrix
\[ \mathbb{C}(\bar{t}) T(u) = \tau(u|\bar{t}) \mathbb{C}(\bar{t}), \]
(2.21)
and the square of the norm of the on-shell Bethe states satisfies the Gaudin hypothesis \([26]\)
\[ \mathbb{C}(\bar{t}) \mathbb{B}(\bar{t}) = \prod_{\nu=1}^{N-1} \prod_{k \neq \ell} f(t_k^\nu, t_k^\ell) \prod_{\nu=2}^{N-2} f(t_\nu+1, t_\nu) \det G, \]
(2.22)
where \( G \) is the Gaudin matrix given by
\[ G^{(\mu, \nu)}_{j, k} = -c \frac{\partial \log \Phi^{(\mu)}_j}{\partial t_k^\nu}, \]
(2.23)
where we defined the expressions
\[ \Phi^{(\mu)}_j \]
(2.24)
We can also define another monodromy matrix which satisfies the same RTT-algebra [37]. This transfer matrix can be obtained from one of the quantum minors as

\[
\hat{T}_{N+1-j,N+1-i}(u) = (-1)^{i+j} \hat{t}^{1,\ldots,j,1,\ldots,N}(u-c) \text{qdet}(T(u))^{-1},
\]

(2.25)

\[
\hat{t}^{a_1,a_2,\ldots,a_m}_{b_1,b_2,\ldots,b_m}(u) = \sum_{p} \text{sgn}(p) T_{a_1,b_{p(1)}}(u) T_{a_2,b_{p(2)}}(u-c) \cdots T_{a_m,b_{p(m)}}(u-(m-1)c),
\]

(2.26)

\[
\text{qdet}(T(u)) = \hat{t}_{1,2,\ldots,N}^{1,2,\ldots,N}(u).
\]

(2.27)

Here \(i\) and \(j\) mean that the corresponding indices are omitted. We call \(\hat{T}\) as twisted monodromy matrix. The twisted monodromy matrix \(\hat{T}\) is also a highest weight representation of \(Y(N)\) with

\[
\hat{T}_{i,i}(0) = \hat{\lambda}_i(u) |0\rangle, \quad \text{for } i = 1, \ldots, N,
\]

(2.28)

\[
\hat{T}_{j,i}(0) = 0, \quad \text{for } 1 \leq i < j \leq N,
\]

where

\[
\hat{\lambda}_i(u) = \frac{1}{\lambda_{N-i+1}(u-(N-i)c)} \prod_{k=1}^{N-i} \lambda_k(u-(k-1)c),
\]

(2.29)

therefore the ratios of the vacuum eigenvalues have the following form

\[
\hat{\alpha}_i(u) = \frac{\hat{\lambda}_i(u)}{\lambda_{i+1}(u)} = \alpha_{N-i}(u-(N-i)c).
\]

(2.30)

Let \(\hat{B}(\tilde{i})\) be the off-shell Bethe vector generated from \(\hat{T}_{i,j}\). In [37] the connection between the Bethe vectors \(B(\tilde{i})\) and \(\hat{B}(\tilde{i})\) was determined

\[
\hat{B}(\tilde{i}) = (-1)^{\#\tilde{i}} \left( \prod_{s=1}^{N-2} f(\tilde{i}^{s+1},\tilde{i}^{s}) \right)^{-1} B(\mu(\tilde{i})),
\]

(2.31)

where

\[
\mu(\tilde{i}) = \{\tilde{i}^{N-1} - c, \tilde{i}^{N-2} - 2c, \ldots, \tilde{i}^1 - (N-1)c\}.
\]

(2.32)

From this identity we can obtain the eigenvalue of the twisted transfer matrix

\[
\hat{\tau}(u) = \text{tr}\hat{T}(u) = \sum_{i=1}^{N} \hat{T}_{i,i}(u),
\]

(2.33)

\[
\hat{\tau}(u)\hat{B}(\tilde{i}) = \hat{\tau}(u|\tilde{i})\hat{B}(\tilde{i}).
\]

(2.34)

Using (2.31), (2.20) and the fact that \(\hat{T}\) satisfies the same RTT-relation we can obtain that

\[
\hat{\tau}(u|\tilde{i}) = \sum_{i=1}^{N} \hat{\lambda}_i(u)f(\tilde{i}^{N-i} + (N-i)c, u)f(\tilde{i}^{N-i+1} + (N-i+1)c).
\]

(2.35)

The twisted monodromy matrix \(\hat{T}\) is similar to the inverse of the original monodromy matrix \(T\):

\[
V\hat{T}(u)VT(u) = 1,
\]

(2.36)
where $V$ is an off-diagonal $N \times N$ matrix of the auxiliary space with the components $V_{i,j} = \delta_{i,N+1-j}$ and the superscript $t$ is the transposition in the auxiliary space, i.e. $\left[ \hat{T}^t(u) \right]_{i,j} = \hat{T}_{j,i}(u)$. Applying this equation to the $RTT$-relation we obtain the $R\hat{\hat{T}}T$-relation

\[
R_{1,2}(u - v)\hat{T}_1(u)\hat{T}_2(v) = T_2(v)\hat{T}_1(u)R_{1,2}(u - v),
\]

(2.37)

where we used the crossed $R$-matrix

\[
R_{1,2}(u) = V_2 R_{1,2}^2(-u)V_2.
\]

(2.38)

For a concrete form of the monodromy matrix (2.14) we can obtain the explicit form of twisted monodromy matrix using (2.36)

\[
\hat{T}_0(u) = \hat{L}_{0,j}^{(j)}(u - \xi_j) \cdots \hat{L}_{0,1}^{(1)}(u - \xi_1),
\]

(2.39)

where

\[
\hat{L}_{0,1}^{\lambda}(u) = V_0 \left( \left( \hat{L}_{0,1}^{\lambda}(u) \right)^{-1} \right) \text{to } V_0.
\]

(2.40)

We can see that twisted monodromy matrix has a same co-product property i.e.

\[
T(u) = T^{(2)}(u)T^{(1)}(u) \implies \hat{T}(u) = \hat{T}^{(2)}(u)\hat{T}^{(1)}(u).
\]

(2.41)

### 3 Integrable crosscap states

In this section we generalize the algebraic framework of the boundary states [33] for crosscap states.

#### 3.1 Untwisted case

At first we divide the quantum space as $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. Let us define the following KT-relation

\[
K(u)\langle C| T^{(1)}(u) = \langle C| T^{(2)}(-u)K(u),
\]

(3.1)

where $\langle C |$ is the crosscap state and $K(u) \in GL(N)$ is the $K$-matrix. We can derive a consistency condition for the $K$-matrix from the following to equations

\[
\langle C|T_{a}^{(1)}(u)T_{b}^{(1)}(v) = R_{a,b}^{-1}(u - v)\langle C|T_{b}^{(1)}(v)T_{a}^{(1)}(u)R_{a,b}(u - v) =
\]

\[
= R_{a,b}^{-1}(u - v)K_{b}^{-1}(v)\langle C|T_{b}^{(2)}(-v)T_{a}^{(1)}(u)K_{a}(v)R_{a,b}(u - v) =
\]

\[
= R_{a,b}^{-1}(u - v)K_{b}^{-1}(v)\langle C|T_{a}^{(1)}(u)T_{b}^{(2)}(-v)K_{a}(v)R_{a,b}(u - v) =
\]

\[
= R_{a,b}^{-1}(u - v)K_{b}^{-1}(v)\langle C|T_{a}^{(1)}(u)T_{b}^{(2)}(-v)K_{a}(v)R_{a,b}(u - v) =
\]

\[
= R_{a,b}^{-1}(u - v)K_{a}^{-1}(u)\langle C|T_{a}^{(2)}(-u)T_{b}^{(2)}(-v)K_{a}(u)K_{b}(v)R_{a,b}(u - v),
\]

(3.2)

and

\[
\langle C|T_{a}^{(1)}(u)T_{b}^{(1)}(v) = K_{a}^{-1}(u)\langle C|T_{a}^{(2)}(-u)T_{b}^{(1)}(v)K_{a}(u) =
\]

\[
= K_{a}^{-1}(u)\langle C|T_{b}^{(1)}(v)T_{a}^{(2)}(-u)K_{a}(u) =
\]

\[
= K_{a}^{-1}(u)K_{b}^{-1}(v)\langle C|T_{a}^{(2)}(-v)T_{b}^{(2)}(-u)K_{a}(v)K_{a}(u) =
\]

\[
= K_{a}^{-1}(u)K_{b}^{-1}(v)R_{a,b}^{-1}(u - v)\langle C|T_{a}^{(2)}(-u)T_{b}^{(2)}(-v)R_{a,b}(u - v)K_{b}(v)K_{a}(u).
\]

(3.3)
We can see that the consistency condition is
\[ K_a(u)K_b(v)R_{a,b}(u-v) = R_{a,b}(u-v)K_a(u)K_b(v). \] (3.4)
Substituting the explicit form of the \( R \)-matrix we obtain that
\[ K_a(u)K_b(v) = K_a(v)K_b(u), \] (3.5)
which has the following solution (up to a spectral parameter dependent normalization)
\[ K(u) = K, \] (3.6)
where \( K \) is a spectral parameter independent invertible \( N \times N \) matrix.

The crosscap states also have co-product property. Let us assume that crosscap states exist on the spaces \( \mathcal{H}^{(a)} = \mathcal{H}^{(1,a)} \otimes \mathcal{H}^{(2,a)} \) and \( \mathcal{H}^{(b)} = \mathcal{H}^{(1,b)} \otimes \mathcal{H}^{(2,b)} \) with the same \( K \)-matrix i.e.
\[ K|C^{(a)}|T^{(1,a)}(u)\rangle = |C^{(a)}|T^{(2,a)}(-u)K, \quad K|C^{(b)}|T^{(1,b)}(u)\rangle = |C^{(b)}|T^{(2,b)}(-u)K. \] (3.7)
On the product spaces \( \mathcal{H}^{(1)} = \mathcal{H}^{(1,a)} \otimes \mathcal{H}^{(1,b)} \) and \( \mathcal{H}^{(2)} = \mathcal{H}^{(2,a)} \otimes \mathcal{H}^{(2,b)} \) the monodromy matrices are
\[ T^{(1)}(u) = T^{(1,b)}(u)T^{(1,a)}(u), \quad T^{(2)}(u) = T^{(2,b)}(u)T^{(2,a)}(u). \] (3.8)
Let us consider the following expression
\[ K|C^{(a)}\rangle \langle C^{(b)}|T^{(1)}(u)\rangle = K|C^{(b)}\rangle \langle C^{(a)}|T^{(2,1)}(u)\rangle = \langle C^{(b)}|T^{(2,2)}(-u)K|C^{(a)}\rangle \langle T^{(1,1)}(u)\rangle =
\] \[ = \langle C^{(b)}|T^{(2,2)}(-u)|C^{(a)}\rangle \langle T^{(2,2)}(-u)K = \langle C^{(a)}|\langle C^{(b)}|T^{(2)}(-u)K, \] (3.9)
therefore on the product space \( \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} = \mathcal{H}^{(1,a)} \otimes \mathcal{H}^{(1,b)} \otimes \mathcal{H}^{(2,a)} \otimes \mathcal{H}^{(2,b)} \) the state \( \langle C^{(a)}|\langle C^{(b)}| \) is a crosscap state with \( K \)-matrix \( K \).

Using this co-product property we can define crosscap states for the general monodromy matrices (2.14) from the solution for the elementary representations \( L^A(u-\xi) \). Let us substitute \( T^{(1)}(u) = L^A(u-\xi_1) \) and \( T^{(2)}(u) = L^A(u-\xi_2) \) to the KT-relation (3.1)
\[ K_0(c|L^A_{0,1}(u-\xi_1)\rangle = \langle c|L^A_{0,2}(-u-\xi_2)K_0. \] (3.10)
Let us use the following parametrization
\[ K = \sum_{i,j} K_{i,j}E_{i,j}, \quad |c| = \sum_{a,b} \Psi_{b,a}e^A_{a} \otimes e^A_{b}, \quad L^A(u) = \sum_{i,j,a,b} L^A(u) \sum_{i,j,a,b} E_{i,j} \otimes E^A_{a,b}, \] (3.11)
where \( e^A_{i} \) are the canonical basis of the dual space of \( V^A \). After the substitution we obtain that
\[ K_{i,j}\Psi_{b,a}L^{A(1)}(u-\xi_1)^{k,b}(u-\xi_2)_{j,c} = L^{A(2)}(-u-\xi_2)_{j,c} \Psi_{b,c}K_{j,k}. \] (3.12)
This can be rewritten as
\[ K_1\Psi_2 L^{A(1)}_{1,2}(u-\xi_1)^{1,2} = L^{A(2)}_{1,2}(-u-\xi_2)\Psi_2 K_1. \] (3.13)
Using the explicit form of the Lax-operators
\[ \frac{c}{u-\xi_1} \sum_{i,j} K E_{i,j} \otimes \Psi \left( E^{A^{(1)}}_{j,i} \right)^T = -\frac{c}{u+\xi_2} \sum_{i,j} E_{i,j} K \otimes E^{A^{(2)}}_{j,i} \Psi, \] (3.14)
or equivalently

\[
\sum_{i,j} E_{i,j} \otimes E_{j,i}^{(2)} = \frac{u + \xi_2}{u - \xi_1} \sum_{i,j} KE_{i,j} K^{-1} \otimes \Psi \left(-E_{j,i}^{(1)}\right)^t \Psi^{-1}.
\] (3.15)

This equation requires that \( \xi_2 = -\xi_1 \) and the representation \( \Lambda^{(2)} \) is similar to the contra-gradient representation of \( \Lambda^{(1)} = \Lambda \), i.e. \( \Lambda^{(2)} = \Lambda \). Substituting back

\[
E_{i,j}^{(1)} = E_{i,j}^{\Lambda}, \quad E_{i,j}^{(2)} = E_{i,j}^{\Lambda} = -V^A E_{j,i}^{A} V^A = -E_{N+1-j,N+1-i}^{\Lambda},
\] (3.16)

therefore we just obtained that

\[
\sum_{i,j} E_{i,j} \otimes E_{j,i}^{\Lambda} = \sum_{i,j} K^{-1} E_{i,j} K \otimes \Psi^{-1} E_{N+1-j,N+1-i}^{\Lambda} = \sum_{i,j} (VK)^{-1} E_{i,j} VK \otimes \Psi^{-1} E_{j,i}^{\Lambda} \Psi.
\] (3.17)

It means that the matrix \( \Psi = \psi^A \) is the image of the matrix \( \psi = VK \) in the representation \( \Lambda \).

Using this solution, we just obtained a general solution of the \( KT \)-equation with the monodromy matrices

\[
T_0^{(1)}(u) = L_0^{(j/2)}(u - \xi_{j/2}) \ldots L_0^{(1)}(u - \xi_1),
\] (3.18)

\[
T_0^{(2)}(u) = L_0^{(j/2)}(u + \xi_{j/2}) \ldots L_0^{(1)}(u + \xi_1),
\] (3.19)

and crosscap states

\[
\langle C \rangle = \prod_{j=1}^{J/2} \langle c_j \rangle, \quad \langle c_j \rangle = \sum_{a,b} \psi_{b,a}^{(j)} \langle a | j \rangle \langle b | j + \phi/2 \rangle.
\] (3.20)

Now let us turn on the consequences for the transfer matrices. At first let us calculate that

\[
\langle C | T_a^{(2)}(u) T_b^{(1)}(u) \rangle = K_a \langle C | T_a^{(1)}(-u) T_b^{(1)}(u) K_a^{-1} = K_a R_{ab}(2u) \langle C | T_a^{(1)}(u) T_a^{(1)}(-u) R_{ab}^{-1}(2u) K_a^{-1} = K_a R_{ab}(2u) K_a^{-1} \langle C | T_b^{(2)}(-u) T_a^{(1)}(-u) K_a R_{ab}^{-1}(2u) K_a^{-1}.
\] (3.21)

Since

\[
T(u) = \text{tr}_{a,b} \left[ T_a^{(2)}(u) T_b^{(1)}(u) P_{a,b} \right],
\] (3.22)

we can obtain that

\[
\langle C | T(u) = \text{tr}_{a,b} \left[ K_a R_{ab}(2u) K_a^{-1} \langle C | T_b^{(2)}(-u) T_a^{(1)}(-u) K_b R_{ab}^{-1}(2u) K_a^{-1} P_{a,b} \right] = \text{tr}_{a,b} \left[ \langle C | T_b^{(2)}(-u) T_a^{(1)}(-u) K_b R_{ab}^{-1}(2u) K_a^{-1} K_a R_{ab}(2u) K_a^{-1} P_{a,b} \right] = \text{tr}_{a,b} \left[ \langle C | T_b^{(2)}(-u) T_a^{(1)}(-u) K_a^{-1} R_{ab}^{-1}(2u) K_b R_{ab}(2u) K_a^{-1} P_{a,b} \right],
\] (3.23)

where we used the \( GL(N) \) symmetry of the \( R \)-matrix

\[
K_a K_b R_{ab}(u) = R_{ab}(u) K_a K_b.
\] (3.24)

Assuming that

\[
K^2 = 1,
\] (3.25)

we obtain the untwisted integrability condition

\[
\langle C | T(u) = \langle C | T(-u).
\] (3.26)
The condition (3.25) means that the $K$-matrix has eigenvalues $\pm 1$ and let $\#(+1) = M$ and $\#(-1) = N - M$. For such $K$-matrices the residual symmetry is $\mathfrak{gl}(M) \otimes \mathfrak{gl}(N - M)$.

In [33] it was shown that the integrability condition (3.26) leads to achiral pair structure of Bethe roots of the on-shell Bethe states with non-vanishing overlaps i.e. $\bar{t}_t^\nu = -\bar{t}^{N-\nu}$ for $\nu < \frac{N}{2}$ and $\bar{t}_t^\nu = \bar{t}^{+} \cup \bar{t}^{-}$ where $\bar{t}^{+} = -\bar{t}^{-}$ for even $N$.

### 3.2 Twisted case

Let us define the following twisted KT-relations

$$K(u)\langle C| T^{(1)}(u) = \lambda_0(u)\langle C| \bar{T}^{(2)}(-u)K(u),$$

where $\langle C|$ is the twisted crosscap state and $\lambda_0(u)$ is spectral parameter dependent function for a proper normalization. Using (2.36) we have an equivalent form

$$\bar{K}(u)\langle C| T^{(2)}(u) = \lambda_0(-u)\langle C| \bar{T}^{(1)}(-u)\bar{K}(u),$$

where $\bar{K}(u) = V K^t(-u)V$.

We can repeat the previous analysis for this twisted equation. For the consistency condition we obtain that

$$K(u) = K,$$

where $K$ is a spectral parameter dependent invertible $N \times N$ matrix.

The twisted crosscap states also has a co-product property. Let us assume that twisted crosscap states exist on the spaces $\mathcal{H}^{(a)} = \mathcal{H}^{(1,a)} \otimes \mathcal{H}^{(2,a)}$ and $\mathcal{H}^{(b)} = \mathcal{H}^{(1,b)} \otimes \mathcal{H}^{(2,b)}$ with the same $K$-matrix i.e.

$$K\langle C^{(a)}| T^{(1,a)}(u) = \lambda_0^a(u)\langle C^{(a)}| \bar{T}^{(2,a)}(-u)K,$$

where $\langle C^{(a)}| \langle C^{(b)}|$ is a crosscap state with $K$-matrix $K$.

Using this co-product property we can define crosscap states for the general monodromy matrices (2.14) from the solutions for the elementary representations $L^{\Lambda}(u - \xi)$. Let us substitute to the KT-relation

$$K_0\langle c| L^{\Lambda^{(1)}}_{0,1} (u - \xi_1) = \lambda_0(u)\langle c| \bar{L}^{\Lambda^{(2)}}_{0,2}(-u - \xi_2)K_0.$$  

We know that the crossed Lax operator has the form (2.40)

$$\bar{L}^{\Lambda}_{0,1}(u) = V_0 \left( \left( L^{\Lambda}_{0,1}(u) \right)^{-1} \right)^{t_0} V_0.$$  

Let us concentrate on the representations with rectangular Young tableaux i.e let us assume that $\Lambda_i^{(2)} = s$ for $i \leq a$ and $\Lambda_i = 0$ for $i > a$. For these representations, the Lax operators have inversion property

$$\left( L^{\Lambda^{(2)}_{0,1}}(u) \right)^{-1} = \frac{u + sc - ac}{u - ac} L^{\Lambda^{(2)}_{0,1}}(-u - sc + ac).$$

After the substitution we just obtain that

$$V_0 K_0\langle c| L^{\Lambda^{(1)}_{0,1}}(u - \xi_1) = \langle c| \bar{L}^{\Lambda^{(2)}_{0,1}}(u + \xi_2 - sc + ac) V_0 K_0,$$

where we fixed the normalization as

$$\frac{1}{\lambda_0(u)} = \frac{-u - \xi_2}{-u - \xi_2 - ac}.\frac{-u - \xi_2 + sc - ac}{-u - \xi_2 + sc - ac}.$$
Let us use the following parametrization

\[ VK = \sum_{i,j} \psi^{-1}_{i,j} E_{i,j}, \quad \langle c | = \sum_{a,b} \Psi_{a,b} e^{(1)}_a \otimes e^{(2)}_b, \quad L^\Lambda (u) = \sum_{i,j,a,b} L(u)_{i,a}^{j,b} E_{i,j} \otimes E^\Lambda_{a,b}, \]  

(3.36)

where \( e^\Lambda_a \) are the canonical basis of the dual space of \( \mathcal{V}^\Lambda \). After the substitution we obtain that

\[ \psi^{-1}_{i,j} \Psi_{b,c} L^\Lambda (u) (u - \xi_1)_{j,a}^{k,b} = L^\Lambda (u + \xi_1 - sc + ac)_{j,c}^{i,b} \psi^{-1}_{i,b} \psi^{-1}_{j,k}. \]  

(3.37)

This can be rewritten as

\[ \psi^{-1}_{1} L^\Lambda_{1,2} (u - \xi_1) \Psi_2 = \Psi_2 L^\Lambda_{1,2} (u + \xi_2 - sc + ac) \psi^{-1}_1. \]  

(3.38)

This equation requires that \( \xi_2 = -\xi_1 + sc - ac \) and the representation \( \Lambda^{(2)} \) is similar to the representation \( \Lambda^{(1)} \) i.e. \( \Lambda^{(1)} = \Lambda^{(2)} = \Lambda \). The matrix \( \Psi = \psi^{\Lambda} \) is the image of the matrix \( \psi = (VK)^{-1} \) in the representation \( \Lambda \).

Using this solution, we just obtained the general solutions of the twisted KT-equation with the monodromy matrices

\[ T_0^{(1)}(u) = L^{\Lambda^{(j/2)}}_{0,j/2} (u - \xi_{j/2}) \ldots L^{\Lambda^{(1)}}_{0,1} (u - \xi_1), \]  

(3.39)

\[ T_0^{(2)}(u) = L^{\Lambda^{(j/2)}}_{0,j/2} (u + \xi_j - sc + ac) \ldots L^{\Lambda^{(1)}}_{0,j/2+1} (u + \xi_1 - sc + ac), \]  

(3.40)

and crosscap states

\[ \langle c | = \prod_{j=1}^{J/2} |c|_j, \quad |c|_j = \sum_{a,b} \psi^{(j)}_{a,b} (a|_j \langle b|_{j+\delta}, \]  

(3.41)

where \( \Lambda^{(j)}_i = s_j \) for \( i \leq a_j \) and \( \Lambda^{(j)}_i = 0 \) for \( i > a_j \).

Now let us turn on the consequences for the transfer matrices. At first let us calculate

\[ \langle c | T^{(2)}_a (u) T^{(1)}_b (u) = \lambda_0 (-u) K_a^{-1} \langle c | T^{(1)}_a (u) T^{(1)}_b (u) K_a = \lambda_0 (-u) K_a^{-1} R_{ab}^{-1} (-2u) K_b^{-1} \langle c | T^{(2)}_a (u) T^{(1)}_b (u) K_b R_{ab} (-2u) K_a = \lambda_0 (u) \lambda_0 (-u) K_a ^{-1} R_{ab} ^{-1} (-2u) K_b ^{-1} \langle c | T^{(2)}_a (u) T^{(1)}_b (u) K_b R_{ab} (-2u) K_a. \]  

(3.42)

Since

\[ T (u) = tr_{a,b} \left[ T^{(2)}_a (u) T^{(1)}_b (u) P_{a,b} \right], \]  

(3.43)

we can obtain that

\[ \langle c | T (u) = \lambda_0 (u) \lambda_0 (-u) tr_{a,b} \left[ K_a ^{-1} R_{ab} ^{-1} (-2u) K_b ^{-1} \langle c | T^{(2)}_a (u) T^{(1)}_b (u) K_b R_{ab} (-2u) K_a P_{a,b} \right] = \lambda_0 (u) \lambda_0 (-u) tr_{a,b} \left[ \langle c | T^{(2)}_b (u) T^{(1)}_a (u) K_b R_{ab} (-2u) K_a ^{-1} R_{ab} ^{-1} (-2u) K_b ^{-1} P_{a,b} \right] = \lambda_0 (u) \lambda_0 (-u) tr_{a,b} \left[ \langle c | T^{(2)}_b (u) T^{(1)}_a (u) K_b R_{ab} (-2u) K_a K_b ^{-1} R_{ab} ^{-1} (-2u) K_a ^{-1} P_{a,b} \right], \]  

(3.44)

where we used the GL(N) symmetry of the crossed R-matrix

\[ K_a R_{ab} (u) V_b K_b ^{i} V_b = V_b K_b ^{i} R_{ab} (u) K_a, \]  

(3.45)

and the fact that \( \bar{K} = VK^T V \). Assuming that

\[ K = \pm \bar{K}, \]  

(3.46)
we obtain the twisted integrability condition
\[
\langle C | T(u) = \lambda_0(u(0)\lambda_0(-u)\langle C | \tilde{T}(-u).
\]  \tag{3.47}

The condition (3.46) is equivalent to
\[
VK^dV = \pm K,
\]  \tag{3.48}
therefore we have two classes. For the positive and negative signs the residual symmetries are $\mathfrak{so}(N)$ and $\mathfrak{sp}(N)$, respectively.

In [33] it was shown that the integrability condition (3.47) leads to chiral pair structure of Bethe roots of the on-shell Bethe states with non-vanishing overlaps i.e. $\tilde{t}^\nu = \tilde{t}^+\nu \cup \tilde{t}^-\nu$ where \(\tilde{t}^\nu = -\tilde{t}^-\nu - \nu c\).

4 Application to AdS/CFT

In this section we collect the integrable crosscap states which can be relevant for the AdS/CFT correspondence. We would like to point out that this paper is not intended to provide an exact holographic description where these crosscap states appear. We only collect those crosscap states of the homogeneous $SO(6)$ and the alternating $SU(4)$ spin chains that preserve integrability. These states could be relevant for the \(\mathcal{N} = 4\) SYM and ABJM theories since these spin chains describe the scalar sectors of these field theories at weak coupling.

4.1 Scalar sector of the $\mathcal{N} = 4$ SYM

For the $SO(6)$ spin chain we have to use the result of the $\mathfrak{gl}(4)$ case. In this situation there are 4 classes of crosscap states.

4.1.1 Twisted case

We saw that there are two classes in the twisted case: the $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and the $\mathfrak{sp}(4) = \mathfrak{so}(5)$ case.

The $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ symmetric crosscap state in the real basis is built from the vector
\[
\langle c | j = + \langle 1 | j \langle 1 | j + \frac{\pi}{2} + \langle 2 | j \langle 2 | j + \frac{\pi}{2} + \langle 3 | j \langle 3 | j + \frac{\pi}{2}
\]
\[- \langle 4 | j \langle 4 | j + \frac{\pi}{2} + \langle 5 | j \langle 5 | j + \frac{\pi}{2} + \langle 6 | j \langle 6 | j + \frac{\pi}{2}.
\]  \tag{4.1}

The $\mathfrak{so}(5)$ symmetric crosscap state in the real basis is built from the vector
\[
\langle c | j = + \langle 1 | j \langle 1 | j + \frac{\pi}{2} - \langle 2 | j \langle 2 | j + \frac{\pi}{2} - \langle 3 | j \langle 3 | j + \frac{\pi}{2}
\]
\[- \langle 4 | j \langle 4 | j + \frac{\pi}{2} - \langle 5 | j \langle 5 | j + \frac{\pi}{2} - \langle 6 | j \langle 6 | j + \frac{\pi}{2}.
\]  \tag{4.2}

The global rotations are also integrable crosscap states. For these integrable states the pair structure is chiral.

4.1.2 Untwisted case

We saw that there are two classes in the untwisted case: the $\mathfrak{gl}(4)$, $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$ and the $\mathfrak{gl}(3)$ cases. These symmetries in the $SO(6)$ language are $\mathfrak{so}(6)$, $\mathfrak{so}(2) \oplus \mathfrak{so}(4)$ and the $\mathfrak{gl}(3)$, respectively.

The $\mathfrak{so}(6)$ symmetric crosscap state in the real basis is built from the vector
\[ \langle c \rangle_j = + \langle 1 \rangle_j \langle 1 \rangle_{j+\frac{d}{2}} + \langle 2 \rangle_j \langle 2 \rangle_{j+\frac{d}{2}} + \langle 3 \rangle_j \langle 3 \rangle_{j+\frac{d}{2}} + \langle 4 \rangle_j \langle 4 \rangle_{j+\frac{d}{2}} + \langle 5 \rangle_j \langle 5 \rangle_{j+\frac{d}{2}} + \langle 6 \rangle_j \langle 6 \rangle_{j+\frac{d}{2}}. \]  

The \( \mathfrak{so}(2) \oplus \mathfrak{so}(4) \) symmetric crosscap state in the real basis is built from the vector

\[ \langle c \rangle_j = + \langle 1 \rangle_j \langle 1 \rangle_{j+\frac{d}{2}} + \langle 2 \rangle_j \langle 2 \rangle_{j+\frac{d}{2}} - \langle 3 \rangle_j \langle 3 \rangle_{j+\frac{d}{2}} - \langle 4 \rangle_j \langle 4 \rangle_{j+\frac{d}{2}} - \langle 5 \rangle_j \langle 5 \rangle_{j+\frac{d}{2}} - \langle 6 \rangle_j \langle 6 \rangle_{j+\frac{d}{2}}. \]  

The \( \mathfrak{gl}(3) \) symmetric crosscap state in the real basis is built from the vector

\[ \langle c \rangle_j = + \langle 1 \rangle_j \langle 4 \rangle_{j+\frac{d}{2}} + \langle 2 \rangle_j \langle 5 \rangle_{j+\frac{d}{2}} + \langle 3 \rangle_j \langle 6 \rangle_{j+\frac{d}{2}} - \langle 4 \rangle_j \langle 1 \rangle_{j+\frac{d}{2}} - \langle 5 \rangle_j \langle 2 \rangle_{j+\frac{d}{2}} - \langle 6 \rangle_j \langle 3 \rangle_{j+\frac{d}{2}}. \]  

The global rotations are also integrable crosscap states. For these integrable states the pair structure is achiral.

### 4.2 Scalar sector of the ABJM

For the ABJM we have to use the result of the \( \mathfrak{gl}(4) \) case when representations are alternating: \( \Lambda^{(21)} = (1,0,0,0) \) and \( \Lambda^{(24)} = (0,0,0,1) \). In this convention the monodromy matrix has the form

\[ T_0(u) = \tilde{L}_{0,j}(u+c)L_{0,j-1}(u-c) \ldots \tilde{L}_{0,2}(u+c)L_{0,1}(u-c), \]

where we also used the conventions \( \xi_{2j-1} = c \), \( \xi_{2j} = -c \) for the inhomogenities and \( L(u) = L^{(1,0,0,0)}(u) \), \( \tilde{L}(u) = L^{(0,0,0,-1)}(u) \) see the definition (2.5). The vacuum eigenvalues are

\[ \lambda_1(u) = \left( \frac{u}{u-c} \right)^{\frac{d}{2}}, \quad \lambda_2(u) = \lambda_3(u) = 1, \quad \lambda_4(u) = \left( \frac{u}{u+c} \right)^{\frac{d}{2}}, \]

therefore the Bethe equations are

\[ \left( \frac{t^1_j}{t^2_j - c} \right)^L = -\prod_{k=1}^{r_1} \frac{t^1_j - t^1_k + c}{t^1_j - t^1_k - c} \prod_{k=1}^{r_2} \frac{t^2_j - t^2_k - c}{t^2_j - t^2_k + c}, \]

\[ 1 = -\prod_{k=1}^{r_1} \frac{t^3_j - t^3_k - c}{t^3_j - t^3_k + c} \prod_{k=1}^{r_2} \frac{t^4_j - t^4_k - c}{t^4_j - t^4_k + c}, \]

\[ \left( \frac{t^2_j + c}{t^3_j} \right)^L = -\prod_{k=1}^{r_1} \frac{t^3_j - t^3_k + c}{t^3_j - t^3_k - c} \prod_{k=1}^{r_2} \frac{t^4_j - t^4_k + c}{t^4_j - t^4_k - c}. \]

Re-defining the Bethe roots as \( t^1_j = u_j + c/2, t^2_j = w_j, t^3_j = v_j - c/2 \) \( (r_1 = K_u, r_2 = K_w, r_3 = K_v) \) and using the convention \( c = i \), the Bethe equations read as

\[ \left( \frac{u_j + i/2}{u_j - i/2} \right)^L = -\prod_{k=1}^{K_u} \frac{u_j - u_k + i/2}{u_j - u_k - i/2}; \]

\[ 1 = -\prod_{k=1}^{K_w} \frac{w_j - w_k + i/2}{w_j - w_k - i/2}; \]

\[ \left( \frac{v_j + i/2}{v_j - i/2} \right)^L = -\prod_{k=1}^{K_v} \frac{v_j - v_k + i/2}{v_j - v_k - i/2}. \]
which are the Bethe equations for the scalar sector of the ABJM theory [20].

We will see that the convention for the monodromy matrix (4.6) is compatible with the untwisted crosscap states but there is an alternative convention for the monodromy matrix (which will be compatible for the twisted case) since the \((0,0,0,-1)\) and \((1,1,1,0)\) are equivalent representations. Now let us define the monodromy matrix in the alternative way

\[
\tilde{T}_0(u) = \tilde{L}_{0,J}(u+c)L_{0,J-1}(u)\ldots\tilde{L}_{0,2}(u+c)L_{0,1}(u),
\]

(4.14)

where \(\tilde{L}(u) = L^{(1,1,1,0)}(u)\). Using the definition (2.5) we can easily show that

\[
\tilde{L}(u) = 1 + \frac{c}{u} \sum_{i,j=1}^{4} E_{i,j} \otimes E_{j,i}^{(1,1,1,0)} = 1 + \frac{c}{u} \sum_{i,j=1}^{4} E_{i,j} \otimes \left( E_{j,i}^{(0,0,0,-1)} + \delta_{i,j} \right)
\]

\[
= \left( \frac{u+c}{u} \right) 1 + \frac{c}{u} \sum_{i,j=1}^{4} E_{i,j} \otimes E_{j,i}^{(0,0,0,-1)} = \left( \frac{u+c}{u} \right) \tilde{L}(u+c),
\]

(4.15)

therefore the two conventions for the monodromy matrix are really equivalent as

\[
\tilde{T}_0(u) = \left( \frac{u+2c}{u+c} \right) \tilde{L}_{0,J}(u+2c)L_{0,J-1}(u)\ldots\tilde{L}_{0,2}(u+2c)L_{0,1}(u) = \left( \frac{u+2c}{u+c} \right) \tilde{T}_0(u+c).
\]

(4.16)

### 4.2.1 Twisted case

In the twisted case we used the following convention for the representations \(\Lambda_i^{(j)} = s_j\) for \(i \leq a_j\) and \(\Lambda_i^{(j)} = 0\) for \(i > a_j\) (see subsection (3.2)) therefore we have to use the second convention (4.14) for the monodromy matrix. We saw that in the integrability requires that \(\Lambda^{(j)} = \Lambda^{(j+\hat{T})}\) therefore \(J/2\) has to be even. We also saw that \(\xi_{j+\hat{T}} = -\xi_{j} + s_j c - a_j c\) (see (3.39),(3.40)). Since \(s_j = 1, a_j = 1\) and \(\xi_{j} = 0\) for the odd sites and \(s_j = 1, a_j = 3\) and \(\xi_{j} = -2c\) for the even sites the monodromy matrix (4.14) is compatible with the twisted crosscap states for even \(J/2\). We saw that there are two classes in the twisted case: the \(\mathfrak{so}(4)\) and the \(\mathfrak{sp}(4)\) case.

The \(\mathfrak{so}(4)\) symmetric crosscap state is built from the vector

\[
\langle c |_{2j-1} = \langle Y_1 |_{2j-1} Y_2 |_{2j-1} + Y_3 |_{2j-1} Y_4 |_{2j-1} + Y_2 |_{2j-1} Y_3 |_{2j-1} + Y_4 |_{2j-1} Y_1 |_{2j-1},
\]

\[
\langle c |_{2j} = \langle Y_1^\dagger |_{2j} Y_2^\dagger |_{2j} + Y_2^\dagger |_{2j} Y_3^\dagger |_{2j} + Y_3^\dagger |_{2j} Y_4^\dagger |_{2j} + Y_4^\dagger |_{2j} Y_1^\dagger |_{2j}.
\]

(4.17)

The \(\mathfrak{sp}(4)\) symmetric crosscap state is built from the vector

\[
\langle c |_{2j-1} = \langle Y_1 |_{2j-1} Y_2 |_{2j-1} + Y_3 |_{2j-1} Y_4 |_{2j-1} - Y_2 |_{2j-1} Y_1 |_{2j-1} - Y_4 |_{2j-1} Y_3 |_{2j-1},
\]

\[
\langle c |_{2j} = \langle Y_1^\dagger |_{2j} Y_2^\dagger |_{2j} + Y_2^\dagger |_{2j} Y_3^\dagger |_{2j} - Y_3^\dagger |_{2j} Y_2^\dagger |_{2j} - Y_1^\dagger |_{2j} Y_4^\dagger |_{2j}.
\]

(4.18)

The global rotations are also integrable crosscap states. For these integrable states the pair structure is chiral.

### 4.2.2 Untwisted case

For the untwisted case we saw that \(\Lambda^{(j)} = \Lambda^{(j+\hat{T})}\) (see (3.18),(3.19)). This requires that \(J/2\) has to be odd. We also saw that \(\xi_{j} = -\xi_{j+\hat{T}}\) (see (3.18),(3.19)) and it is compatible with the first convention
for the transfer matrix (4.6) where \(\xi_{2j-1} = c\) and \(\xi_{2j} = -c\). We saw that there are three classes in the untwisted case: the \(\mathfrak{gl}(4), \mathfrak{gl}(3)\) and the \(\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)\) symmetric states.

The \(\mathfrak{gl}(4)\) symmetric crosscap state is built from the vector

\[
\langle c |_{2j-1} = (Y_1 |_{2j-1} \langle Y_1^4 |_{2j-1} + \frac{1}{\xi} + Y_2 |_{2j-1} \langle Y_2^4 |_{2j-1} + \frac{1}{\xi} + Y_3 |_{2j-1} \langle Y_3^4 |_{2j-1} + \frac{1}{\xi} + Y_4 |_{2j-1} \langle Y_4^4 |_{2j-1} + \frac{1}{\xi},
\]

\[
\langle c |_{2j} = \langle Y_1 |_{2j} \langle Y_1^4 |_{2j} + \frac{1}{\xi} + Y_2 |_{2j} \langle Y_2^4 |_{2j} + \frac{1}{\xi} + Y_3 |_{2j} \langle Y_3^4 |_{2j} + \frac{1}{\xi} + Y_4 |_{2j} \langle Y_4^4 |_{2j} + \frac{1}{\xi}.
\] (4.19)

The \(\mathfrak{gl}(3)\) symmetric crosscap state is built from the vector

\[
\langle c |_{2j-1} = (Y_1 |_{2j-1} \langle Y_1^4 |_{2j-1} + \frac{1}{\xi} + Y_2 |_{2j-1} \langle Y_2^4 |_{2j-1} + \frac{1}{\xi} + Y_3 |_{2j-1} \langle Y_3^4 |_{2j-1} + \frac{1}{\xi} - Y_4 |_{2j-1} \langle Y_4^4 |_{2j-1} + \frac{1}{\xi},
\]

\[
\langle c |_{2j} = \langle Y_1 |_{2j} \langle Y_1^4 |_{2j} + \frac{1}{\xi} + Y_2 |_{2j} \langle Y_2^4 |_{2j} + \frac{1}{\xi} + Y_3 |_{2j} \langle Y_3^4 |_{2j} + \frac{1}{\xi} - Y_4 |_{2j} \langle Y_4^4 |_{2j} + \frac{1}{\xi}.
\] (4.20)

The \(\mathfrak{gl}(2) \otimes \mathfrak{gl}(2)\) symmetric crosscap state is built from the vector

\[
\langle c |_{2j-1} = (Y_1 |_{2j-1} \langle Y_1^4 |_{2j-1} + \frac{1}{\xi} + Y_2 |_{2j-1} \langle Y_2^4 |_{2j-1} + \frac{1}{\xi} + Y_3 |_{2j-1} \langle Y_3^4 |_{2j-1} + \frac{1}{\xi} + Y_4 |_{2j-1} \langle Y_4^4 |_{2j-1} + \frac{1}{\xi},
\]

\[
\langle c |_{2j} = \langle Y_1 |_{2j} \langle Y_1^4 |_{2j} + \frac{1}{\xi} + Y_2 |_{2j} \langle Y_2^4 |_{2j} + \frac{1}{\xi} + Y_3 |_{2j} \langle Y_3^4 |_{2j} + \frac{1}{\xi} + Y_4 |_{2j} \langle Y_4^4 |_{2j} + \frac{1}{\xi}.
\] (4.21)

The global rotations are also integrable crosscap states. For these integrable states the pair structure is achiral.

## 5 Sum formulas for the off-shell overlaps

In this section we derive a sum formula for the off-shell overlap

\[
\langle C | \overline{B}(\bar{w}).
\] (5.1)

The method uses the results of [25, 37] which are reviewed in the appendices A and B.

### 5.1 Untwisted case

In this section we focus on the crosscap states which correspond to the \(K\)-matrix \(K = V\). Using this \(K\)-matrix the \(KT\)-relation reads as

\[
\langle C | T^{(1)}_{ij}(u) = \langle C | T^{(2)}_{N+1-i,N+1-j}(u).
\] (5.2)

From this relation we can obtain that

\[
\lambda^{(1)}_i(u) = \lambda^{(2)}_{N+1-i}(-u), \Rightarrow \alpha^{(1)}_i(u) = \frac{1}{\alpha^{(2)}_{N+1-i}(-u)}.
\] (5.3)

Using the \(KT\)-relation and the recurrence relation for the off-shell Bethe vectors (A.5) we can obtain that

\[
\langle C | (\overline{B}^{(1)}(s) \otimes \overline{B}^{(2)}(\bar{t})) = \tilde{C}^{(1)}(\pi^a(s))\tilde{B}^{(2)}(\bar{t}) = \tilde{S}^{(2)}(\pi^a(s)|\bar{t}),
\] (5.4)
\[ \pi^a(\bar{s}) = (-\bar{s}^{N-1}, -\bar{s}^{N-2}, \ldots, -\bar{s}^1). \] (5.5)

The renormalized Bethe vector \( \tilde{\mathbf{B}}(\bar{\upsilon}) \) is defined in the appendix A. The derivation can be found in the Appendix C.1. Using the co-product formula (A.8) of the Bethe state we can obtain that

\[
\langle \mathcal{C} | \tilde{\mathbf{B}}(\bar{\upsilon}) \rangle = \sum_{\text{part}(\bar{\upsilon})} \prod_{\nu=1}^{N-1} \frac{\lambda^{(2)}_{\nu}(\bar{s}^\nu) \lambda^{(1)}_{\nu+1}(\bar{\upsilon}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{\upsilon}^{\nu+1}, \bar{s}^\nu)} \langle \mathcal{C} | \tilde{\mathbf{B}}^{(1)}(\bar{s}) \otimes \tilde{\mathbf{B}}^{(2)}(\bar{\upsilon}) \rangle = \sum_{\text{part}(\bar{\upsilon})} \prod_{\nu=1}^{N-1} \frac{\lambda^{(2)}_{\nu}(\bar{s}^\nu) \lambda^{(1)}_{\nu+1}(\bar{\upsilon}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{\upsilon}^{\nu+1}, \bar{s}^\nu)} \tilde{S}^{(2)}(\pi^a(\bar{s})|\bar{\upsilon}).
\] (5.6)

The sum goes over all the partitions of \( \bar{\upsilon}^\nu = \bar{s}^\nu \cup \bar{\upsilon}^\nu = \bar{t}^\nu \cup \bar{\upsilon}^\nu \) and \( \bar{s}^\nu = \bar{s}^\nu \cup \bar{s}^\nu \) where \#\( \tilde{\upsilon}^\nu = \#s_{N-\nu}^{\nu} \) and \#\( \tilde{\upsilon}_{\nu}^\nu = \#s_{N-\nu}^{\nu} \). Using the sum rule of the off-shell scalar product (B.2) we obtain that

\[
\langle \mathcal{C} | \tilde{\mathbf{B}}(\bar{\upsilon}) \rangle = \sum_{\text{part}(\bar{\upsilon})} \prod_{\nu=1}^{N-1} \frac{\lambda^{(2)}_{\nu}(\bar{s}^\nu) \lambda^{(1)}_{\nu+1}(\bar{\upsilon}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{\upsilon}^{\nu+1}, \bar{s}^\nu)} \times W(\pi^a(\bar{s}_i), \pi^a(\bar{s}_{\nu}^\nu)|\bar{t}_{\nu}, \bar{t}_{\nu}) \prod_{k=1}^{N-1} \lambda^{(2)}_k (-\bar{s}^{N-k}) \lambda^{(2)}_{k+1} (-\bar{s}^{N-k+1}) \lambda^{(2)}_{k+1} (\bar{t}_k^\nu) \lambda^{(2)}_k (\bar{t}_{\nu})^\nu.
\] (5.7)

After some simplification we obtain that

\[
\langle \mathcal{C} | \tilde{\mathbf{B}}(\bar{\upsilon}) \rangle = \sum_{\text{part}(\bar{\upsilon})} \prod_{\nu=1}^{N-1} \frac{\lambda^{(2)}_{\nu}(\bar{s}^\nu) \lambda^{(1)}_{\nu+1}(\bar{\upsilon}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{\upsilon}^{\nu+1}, \bar{s}^\nu)} \times W(\pi^a(\bar{s}_i), \pi^a(\bar{s}_{\nu}^\nu)|\bar{t}_{\nu}, \bar{t}_{\nu}) \prod_{k=1}^{N-1} \lambda^{(1)}_k (\bar{s}_k^\nu) \lambda^{(1)}_k (\bar{s}_k^\nu) \lambda^{(2)}_k (\bar{t}_k^\nu) \lambda^{(2)}_k (\bar{t}_{\nu}^\nu).
\] (5.8)

After a renormalization we have a sum formula for the off-shell overlap

\[
\langle \mathcal{C} | \mathbf{B}(\bar{\upsilon}) \rangle = \sum_{\text{part}(\bar{\upsilon})} \prod_{k=1}^{N-1} \frac{f(\bar{t}_k^\nu, \bar{s}_k^\nu) f(\bar{s}_k^\nu, \bar{s}_k^\nu) f(\bar{t}_{\nu}^\nu, \bar{t}_{\nu}^\nu)}{\prod_{k=1}^{N-2} f(\bar{t}_{\nu}^k, \bar{s}_k^\nu) f(\bar{s}_k^k, \bar{s}_k^\nu) f(\bar{t}_{\nu}^k, \bar{t}_{\nu}^k)} \times Z(\pi^a(\bar{s}_i)|\bar{t}_{\nu}) Z(\bar{t}_{\nu}|\pi^a(\bar{s}_{\nu})) \prod_{k=1}^{N-1} \alpha^{(2)}_k (\bar{s}_k^\nu) \alpha^{(2)}_k (\bar{s}_k^\nu) \alpha^{(2)}_k (\bar{t}_k^\nu),
\] (5.9)

where we used the identity (B.6).

### 5.2 Twisted case

In this section we focus on the crosscap states which correspond to the \( K \)-matrix \( K = V \). Using this \( K \)-matrix the twisted \( K \)-relations read as

\[
\langle \mathcal{C} | T_{i,j}^{(1)}(u) \rangle = \lambda_0(u) \langle \mathcal{C} | T_{N+1-i,N+1-j}^{(2)}(-u) \rangle, \quad \langle \mathcal{C} | T_{i,j}^{(2)}(u) \rangle = \lambda_0(-u) \langle \mathcal{C} | T_{N+1-i,N+1-j}^{(1)}(-u) \rangle.
\] (5.10)

From these relations we can obtain that

\[
\lambda^{(1)}_i(u) = \lambda_0(u) \tilde{\lambda}^{(2)}_{N+1-i}(-u), \quad \lambda^{(2)}_i(u) = \lambda_0(-u) \tilde{\lambda}^{(1)}_{N+1-i}(-u),
\] (5.11)
therefore
\[
\alpha_i^{(1)}(u) = \frac{1}{\alpha_i^{(2)}(-u - ic)}.
\]

Using the recursion formula of the off-shell Bethe state (A.5) and the KT-relation we can obtain that
\[
\langle C | \left( \bar{\mathcal{B}}^{(1)}(\bar{s}) \otimes \bar{\mathcal{B}}^{(2)}(\bar{f}) \right) \rangle = \frac{1}{\prod_{\nu=1}^{N-1} \lambda_0(\bar{s}^\nu)} \mathcal{C}^{(2)}(\pi^a(\bar{s}) \bar{B}^{(2)}(\bar{f})) = \frac{1}{\prod_{\nu=1}^{N-1} \lambda_0(\bar{s}^\nu)} \bar{S}^{(2)}(\pi^a(\bar{s}) | \bar{f}).
\]
(5.12)
The derivation can be found in the Appendix C.2. Using the co-product formula of the off-shell Bethe state (A.8) and the identity (2.31) we can obtain that
\[
\langle C | \bar{\mathcal{B}}(\bar{w}) \rangle = \sum_{\text{part}(\bar{w})} \frac{1}{\prod_{\nu=1}^{N-1} \lambda_{\nu+1}^{(2)}(\bar{s}^\nu) \lambda_{\nu+1}^{(1)}(\bar{f}) f(\bar{f}^\nu, \bar{s}^\nu)} \langle C | \bar{\mathcal{B}}^{(1)}(\bar{s}) \otimes \bar{\mathcal{B}}^{(2)}(\bar{f}) \rangle = \sum_{\text{part}(\bar{w})} \left( -1 \right)^{\# \bar{w}} \frac{1}{\prod_{\nu=1}^{N-1} \lambda_{\nu+1}^{(1)}(\bar{s}^\nu)} \times \frac{1}{\prod_{\nu=1}^{N-2} f(\bar{f}^\nu, \bar{s}^\nu)} \prod_{\nu=1}^{N-2} f(\bar{s}^\nu, \bar{s}^\nu+1+c) \bar{s}^{(2)}(\pi^c(\bar{s}) | \bar{f}),
\]
(5.13)
where
\[
\pi^c(\bar{s}) = \pi^a(\mu^{-1}(\bar{s})) = \{-\bar{s}^1 - c, -\bar{s}^2 - 2c, \ldots, -\bar{s}^{N-1} - (N - 1)c\}.
\]
(5.14)
The sum goes over all the partitions of \(\bar{w}^\nu = \bar{s}^\nu \cup \bar{f}^\nu\), \(\bar{f}^\nu = \bar{f}_{\bar{1}}^\nu \cup \bar{f}_{\bar{2}}^\nu\) and \(\bar{s}^\nu = \bar{s}_{\bar{1}}^\nu \cup \bar{s}_{\bar{2}}^\nu\) where \(#\bar{f}^\nu = #\bar{s}^\nu\) and \(#\bar{f}_{\bar{1}}^\nu = #\bar{s}_{\bar{1}}^\nu\). Using the sum rule of the scalar product (B.2) we obtain that
\[
\langle C | \bar{\mathcal{B}}(\bar{w}) \rangle = \sum_{\text{part}(\bar{w})} \left( -1 \right)^{\# \bar{w}} \frac{1}{\prod_{\nu=1}^{N-1} \lambda_{\nu+1}^{(1)}(\bar{s}^\nu)} \times \frac{1}{\prod_{\nu=1}^{N-2} f(\bar{f}^\nu, \bar{s}^\nu)} \prod_{\nu=1}^{N-2} f(\bar{s}^\nu, \bar{s}^\nu+1+c) \times \prod_{k=1}^{N-1} \lambda_k^{(2)}\left(-\bar{s}_{\bar{1}}^k - kc)\right) \lambda_{k+1}^{(2)}\left(-\bar{s}_{\bar{1}}^k - kc)\right) \lambda_{k+1}^{(2)}(\bar{f}_k^{\nu}) \lambda_{k}^{(2)}(\bar{f}_k^{\nu}) \lambda_{k}^{(2)}(\bar{f}_k^{\nu}),
\]
(5.15)
Using the identities (5.11) and (2.30) we obtain that
\[
\langle C | \bar{\mathcal{B}}(\bar{w}) \rangle = \sum_{\text{part}(\bar{w})} \prod_{\nu=1}^{N-1} \lambda_{\nu+1}^{(1)}(\bar{s}^\nu) \left( -1 \right)^{\# \bar{w}} \prod_{\nu=1}^{N-2} f(\bar{f}^\nu, \bar{s}^\nu) \prod_{\nu=1}^{N-2} f(\bar{s}^\nu, \bar{s}^\nu+1+c) \times \prod_{k=1}^{N-1} \lambda_k^{(1)}\left(\bar{s}_{\bar{1}}^k\right) \lambda_{k+1}^{(2)}(\bar{f}_k^{\nu}) \lambda_{k}^{(2)}(\bar{f}_k^{\nu}) \lambda_{k}^{(2)}(\bar{f}_k^{\nu}),
\]
(5.16)
After a renormalization we have a sum formula for the off-shell overlap
\[
\langle C | \bar{\mathcal{B}}(\bar{w}) \rangle = \sum_{\text{part}(\bar{w})} \left( -1 \right)^{\# \bar{w}} \prod_{\nu=1}^{N-1} f(\bar{f}^\nu, \bar{s}^\nu) f(\bar{s}_{\bar{1}}^\nu, \bar{s}^\nu) f(\bar{f}_{\bar{1}}^\nu, \bar{f}_{\bar{2}}^\nu) \prod_{\nu=1}^{N-2} f(\bar{s}^\nu, \bar{s}^\nu+1+c) \prod_{k=1}^{N-1} f(\bar{s}_{\bar{1}}^k, \bar{s}_{\bar{1}}^k + c) f(\bar{f}_{\bar{1}}^k, \bar{f}_{\bar{1}}^k) \times \prod_{k=1}^{N-1} Z(\pi^c(\bar{s}) | \bar{f}) Z(\bar{f}_{\bar{1}} | \pi^c(\bar{s}) \bar{f}_{\bar{1}}) \prod_{k=1}^{N-1} \alpha_k^{(1)}(\bar{s}_{\bar{1}}^k) \alpha_k^{(2)}(\bar{s}_{\bar{1}}^k) \alpha_k^{(2)}(\bar{f}_{\bar{1}}^k),
\]
(5.17)
where we used the identity (B.6).
6 On-shell limit

Let us continue with the on-shell limit of the overlaps. The calculation of this section is based on the derivation of the norm of the Bethe states [26] and the overlaps of the boundary states [33].

We saw that the non-vanishing overlaps require pair structures \( \tilde{w} = \tilde{w}^+ \cup \tilde{w}^- \). These new sets are defined for the untwisted case as

\[
\begin{align*}
\tilde{w}_k^+ & = w_k^+, \\
\tilde{w}_k^- & = w_k^{N-\nu}, & \nu < N/2, \\
\tilde{w}_k^{N/2} & = w_k^{N/2}, & k = 1, \ldots, \frac{rN/2}{2},
\end{align*}
\]

and

\[
\begin{align*}
\tilde{w}_k^+ & = w_k^+, \\
\tilde{w}_k^- & = w_k^{\nu}, & k = 1, \ldots, \frac{r\nu}{2},
\end{align*}
\]

for the twisted case. The pair structure limit is \( \tilde{w}^- \rightarrow -\tilde{w}^+ \) for untwisted case and \( \tilde{w}^- \nu \rightarrow -\tilde{w}^+ \nu - \nu c \) for the twisted case. Let us introduce a common notation for the pair structure limit \( \tilde{w}^- \rightarrow \pi(\tilde{w}^+) \) where \( \pi(\tilde{w}^+) = -\tilde{w}^+ \) or \( \pi(\tilde{w}^+ \nu) = -\tilde{w}^+ \nu - \nu c \) for the achiral or the chiral pair structures.

6.1 Gaudin-like determinants

We show that the on-shell overlaps are proportional to the Gaudin-like determinant \( \det G^+ \) where the Gaudin-like matrix is defined as

\[
G^{+,(\mu,\nu)}_{j,k} = -c \left( \frac{\partial}{\partial \tilde{w}_k^+} + \frac{\partial}{\partial \tilde{w}_k^-} \right) \log \Phi^{+,(\mu)}_j(\tilde{w}) \bigg|_{\tilde{w} = \pi(\tilde{w}^+)} ,
\]

where

\[
\Phi^{+,(\mu)}_k = \alpha_{\nu}(w_k^+,w_k^{\nu}) f(w_k^+,w_k^{\nu}) f(w_k^{\nu},w_k^+) .
\]

It is crucial that we first take the derivative in (6.3) and only after take the pair structure limit. The diagonal elements of this matrix contains derivatives of log \( \alpha \)-s for which we use the following notation

\[
X_j^{+,-\mu} = -c \frac{d}{du} \log \alpha_{\mu}(u) \bigg|_{u = w_j^{+,-\mu}} .
\]

We also use the derivatives of log \( \alpha^{(1)} \) and log \( \alpha^{(2)} \)

\[
X_j^{(1),-\mu} = -c \frac{d}{du} \log \alpha_{\mu}^{(1)}(u) \bigg|_{u = w_j^{(1),-\mu}} , \\
X_j^{(2),-\mu} = -c \frac{d}{du} \log \alpha_{\mu}^{(2)}(u) \bigg|_{u = w_j^{(2),-\mu}} .
\]

We can see that for a specific model the variables \( X_j^{+,\mu} \) are functions of the Bethe roots. Let us define a more general case where the variables \( X_j^{(1),-\mu} \) and \( w_j^{+,-\mu} \) are independent which gives us a more general version of the Gaudin determinant where we do not impose (6.5)

\[
F^{(r^+)}(\tilde{X}^{(1)},\tilde{X}^{(2)},\tilde{w}^+) = \det G^+ .
\]

This function depends on three sets of variables \( \tilde{X}^{(1)} = \cup \tilde{X}^{(1),\nu} \) and \( \tilde{w}^+ = \cup \tilde{w}^+ \nu \). The superscript denotes the number of Bethe roots as \( r^+ = \# \tilde{X}^{(1)} = \# \tilde{X}^{(2)} = \# \tilde{w}^+ \).

The function \( F^{(r^+)} \) obeys the following Korepin criteria.
Korepin criteria.

(i) The function \( \mathbf{F}^{(r+1)}(\bar{X}^{(1)}, \bar{X}^{(2)}, \bar{w}^+) \) is symmetric over the replacement of the pairs \((X_j^{(1)}, \mu, X_j^{(2)}, \mu, w_j^{+, \mu}) \leftrightarrow (X_k^{(1)}, \mu, X_k^{(2)}, \mu, w_k^{+, \mu})\).

(ii) It is linear function of each \( X_j^{(1)}, \mu \) and \( X_j^{(2)}, \mu \).

(iii) \( \mathbf{F}^{(1)}(X_1^{(1), \nu}, X_1^{(2), \nu}, w_1^{+, \nu}) = X_1^{(1), \nu} + X_1^{(2), \nu} \).

(iv) The coefficient of \( X_j^{(i), \mu} \) is given by the function \( \mathbf{F}^{(r+1)} \) with modified parameters \( X_k^{(i), \nu} \)

\[
\frac{\partial \mathbf{F}^{(r+1)}(\bar{X}^{(1)}, \bar{X}^{(2)}, \bar{w}^+)}{\partial X_j^{(i), \mu}} = \mathbf{F}^{(r+1)}(\bar{X}^{(1)}, \mu, \mu, X_j^{(1), \mu}, \mu, X_j^{(2), \mu}, \mu, w_1^{+, \mu}),
\]

where the original variables \( X_k^{(i), \nu} \) should be replaced by the modified expression \( X_k^{(i), \nu} \) which is defined in the Appendix D.

(v) \( \mathbf{F}^{(r+1)}(\bar{X}^+, \bar{w}^+) = 0 \), if all \( X_j^{(i), \mu} = 0 \).

It is easy to show that the functions \( \mathbf{F}^{(r+1)}(\bar{X}^{(1)}, \bar{X}^{(2)}, \bar{w}^+) \) satisfy the Korepin criteria. The reverse statement can be also proven easily and this proof is the same as the proof of Proposition 4.1. in [26].

### 6.2 On-shell formulas

It turns out that the results of the previous subsection are enough to derive the closed form of the on-shell overlaps.

Let us introduce a normalized overlap for the untwisted case

\[
N(\bar{w}) = \prod_{i=1}^{2N-1} f(\bar{w}^{+, \nu+1}, \bar{w}^{+, \nu}) \left[ f(\bar{w}^{+, \nu}, \bar{w}^{+, \nu-1}) \right]^2 \frac{1}{\prod_{\nu=1}^{N} \prod_{\nu \neq \mu} \alpha^{(2)}_{\nu} (\bar{w}^{+, \nu})} \langle C| \mathbb{B}(\bar{w}) \rangle,
\]

for even \( N \) and

\[
N(\bar{w}) = \prod_{i=1}^{2N-1} f(\bar{w}^{+, \nu+1}, \bar{w}^{+, \nu}) \left[ f(\bar{w}^{+, \nu}, \bar{w}^{+, \nu-1}) \right]^2 \frac{1}{\prod_{\nu=1}^{N-1} \prod_{\nu \neq \mu} \alpha^{(2)}_{\nu} (\bar{w}^{+, \nu})} \langle C| \mathbb{B}(\bar{w}) \rangle,
\]

for odd \( N \). For the twisted case the normalized overlap is

\[
N(\bar{w}) = \frac{(-1)^{\# \nu}}{\prod_{\nu=1}^{N-1} \prod_{i \neq \mu} f(w_i^{+, \nu}, w_i^{+, \nu}) f(\bar{w}^{+, \nu+1}, w_i^{+, \nu}) f(\bar{w}^{+, \nu}, w_i^{+, \nu}) \alpha^{(2)}_{\nu} (\bar{w}^{+, \nu})} \langle C| \mathbb{B}(\bar{w}) \rangle.
\]

In appendix F we show that a re-normalized version of the overlap in the on-shell limit satisfies the Korepin criteria therefore it is equal to the Gaudin-like determinant.

\[
N(\bar{w}) = \det G^+.
\]

Therefore the untwisted on-shell overlaps are

\[
\langle C| \mathbb{B}(\bar{w}) \rangle = \frac{\prod_{\nu=1}^{2N-1} \prod_{i \neq \mu} f(w_i^{+, \nu}, w_i^{+, \nu}) f(\bar{w}^{+, \nu}, \bar{w}^{+, \nu}) \left[ f(\bar{w}^{+, \nu}, \bar{w}^{+, \nu-1}) \right]^2 \prod_{\nu=1}^{N} \alpha^{(2)}_{\nu} (\bar{w}^{+, \nu}) \det G^+}{\prod_{\nu=1}^{2N-1} f(\bar{w}^{+, \nu+1}, \bar{w}^{+, \nu}) \left[ f(\bar{w}^{+, \nu}, \bar{w}^{+, \nu-1}) \right]^2 \prod_{\nu=1}^{2N-1} \alpha^{(2)}_{\nu} (\bar{w}^{+, \nu}) \det G^+},
\]
for even $N$

$$\langle C | B (\bar{w}) = \frac{\prod_{\nu=1}^{N-1} \prod_{i \neq j} f(w_i^{+ \nu}, w_j^{+ \nu})}{\prod_{\nu=1}^{N-1} f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu})} \prod_{\nu=1}^{N-1} \alpha_\nu^{(2)}(\bar{w}^{+ \nu}) \det G^+,$$

(6.13)

for odd $N$. For the twisted case the on-shell overlap is

$$\langle C | B (\bar{w}) = (-1)^{\frac{N-1}{2}} \prod_{\nu=1}^{N-1} \prod_{i \neq j} f(w_i^{+ \nu}, w_j^{+ \nu}) f(\bar{w}^{+ \nu}, \bar{w}^{+ \nu}) f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu}) \alpha_\nu^{(2)}(\bar{w}^{+ \nu}) \det G^+.$$

(6.14)

The scalar product for the achiral pair structure reads as

$$\langle \bar{w} | B (\bar{w}) = \frac{\prod_{\nu=1}^{N} \prod_{k \neq l} f(w_i^{+ \nu}, w_k^{+ \nu})^2 [f(\bar{w}^{+ \nu_1}, \bar{w}^{+ \nu_2}) f(\bar{w}^{+ \nu_1 + 1}, \bar{w}^{+ \nu_2})]^2}{\prod_{\nu=1}^{N} f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu})} \det G^+ \det G^-,$$

(6.15)

for even $N$

$$\langle \bar{w} | B (\bar{w}) = \prod_{\nu=1}^{N-1} \left[ \prod_{k \neq l} f(w_i^{+ \nu}, w_k^{+ \nu}) \right]^2 f(\bar{w}^{+ \nu}, \bar{w}^{+ \nu}) f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu}) \det G^+ \det G^-.$$

(6.16)

for odd $N$. For the chiral pair structure we have

$$\langle \bar{w} | B (\bar{w}) = \prod_{\nu=1}^{N-1} \left[ \prod_{k \neq l} f(w_i^{+ \nu}, w_k^{+ \nu}) \right]^2 f(\bar{w}^{+ \nu}, \bar{w}^{+ \nu}) f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu}) \det G^+ \det G^-.$$

(6.17)

We can also obtain the overlaps for the dual vectors (we only need to replace $\alpha_\nu^{(1)} \leftrightarrow \alpha_\nu^{(2)}$). Therefore, in the untwisted case, we have

$$\langle C | C \rangle = \frac{\prod_{\nu=1}^{N} \prod_{i \neq j} f(w_i^{+ \nu}, w_j^{+ \nu}) f(\bar{w}^{+ \nu_1}, \bar{w}^{+ \nu_2})}{\prod_{\nu=1}^{N} f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu})} \prod_{\nu=1}^{N} \alpha_\nu^{(1)}(\bar{w}^{+ \nu}) \det G^+,$$

(6.18)

for even $N$

$$\langle C | C \rangle = \prod_{\nu=1}^{N-1} \prod_{i \neq j} f(w_i^{+ \nu}, w_j^{+ \nu}) \prod_{\nu=1}^{N-1} \alpha_\nu^{(1)}(\bar{w}^{+ \nu}) \det G^+.$$

(6.19)

for odd $N$. For the twisted case we have

$$\langle C | C \rangle = (-1)^{\frac{N-1}{2}} \prod_{\nu=1}^{N-1} \prod_{i \neq j} f(w_i^{+ \nu}, w_j^{+ \nu}) f(\bar{w}^{+ \nu}, \bar{w}^{+ \nu}) f(\bar{w}^{+ \nu + 1}, \bar{w}^{+ \nu}) \alpha_\nu^{(1)}(\bar{w}^{+ \nu}) \det G^+.$$

(6.20)

For the normalized overlaps, in the untwisted case, we obtain that

$$\frac{\langle \bar{w} | C \rangle \langle C | \bar{w} \rangle}{\langle \bar{w} | B (\bar{w})} = \left[ \frac{1}{f(\bar{w}^{+ \nu_1}, \bar{w}^{+ \nu_2})} \right] \prod_{\nu=1}^{N} \alpha_\nu(\bar{w}^{+ \nu}) \det G^+,$$

(6.21)
for even $N$ in and
\[
\frac{\mathbb{C}(\vec{w})|\mathbb{C}\rangle\langle\mathbb{B}|(\vec{w})}{\mathbb{C}(\vec{w})\mathbb{B}(\vec{w})} = \frac{1}{f(\vec{w}_{\frac{N-1}{2}}, \vec{w}_{\frac{N+1}{2}})} \prod_{\nu=1}^{N-1} \alpha_{\nu}(\vec{w}_{\nu}, \vec{w}_{\nu+1}) \det G'^+ \det G'^-.
\] (6.22)
for odd $N$. In the twisted case we have
\[
\frac{\mathbb{C}(\vec{w})|\mathbb{C}\rangle\langle\mathbb{B}|(\vec{w})}{\mathbb{C}(\vec{w})\mathbb{B}(\vec{w})} = \prod_{\nu=1}^{N-1} \frac{f(\vec{w}_{\nu}, \vec{w}_{\nu+1})}{f(\vec{w}_{\nu}', \vec{w}_{\nu}'')} \frac{f(\vec{w}_{\nu}, \vec{w}_{\nu+1})}{f(\vec{w}_{\nu}', \vec{w}_{\nu}'')} \alpha_{\nu}(\vec{w}_{\nu}, \vec{w}_{\nu+1}) \det G'^+ \det G'^-.
\] (6.23)
Using the identity
\[
f(\vec{w}_{\nu+1}', \vec{w}_{\nu}) = f(-\vec{w}_{\nu+1}', -\vec{w}_{\nu}) = f(\vec{w}_{\nu}, \vec{w}_{\nu+1}) + c = \frac{1}{f(\vec{w}_{\nu+1}', \vec{w}_{\nu})},
\] (6.24)
we can obtain that
\[
\frac{\mathbb{C}(\vec{w})|\mathbb{C}\rangle\langle\mathbb{B}|(\vec{w})}{\mathbb{C}(\vec{w})\mathbb{B}(\vec{w})} = \prod_{\nu=1}^{N-1} \frac{f(\vec{w}_{\nu}, \vec{w}_{\nu+1})}{f(\vec{w}_{\nu}, \vec{w}_{\nu+1})} \alpha_{\nu}(\vec{w}_{\nu}, \vec{w}_{\nu+1}) \det G'^+ \det G'^-.
\] (6.25)

Using the Bethe Ansatz equations, the on-shell formulas (6.21), (6.22) and (6.25) can be written in the following universal form
\[
\frac{\mathbb{C}(\vec{w})|\mathbb{C}\rangle\langle\mathbb{B}|(\vec{w})}{\mathbb{C}(\vec{w})\mathbb{B}(\vec{w})} = \frac{\det G'^+}{\det G'^-}.
\] (6.26)

We just derived the overlap formulas which can be applied for a wide class of integrable crosscap states of the $\mathfrak{gl}(N)$ symmetric spin chains. For the AdS/CFT point of view it can be applied for the crosscap states (4.1) (chiral pair structure) and (4.4) (achiral pair structure) in the $\mathcal{N} = 4$ SYM. In the ABJM the formula (6.26) can be applied for the crosscap states (4.17) (chiral pair structure) and for the crosscap state (4.21) (achiral pair structure).

7 Conclusion

In this paper we generalized the algebraic method of [33] to the crosscap states of the $\mathfrak{gl}(N)$ symmetric spin chains. The main advantage of this approach is that it is independent from the quantum space. We classified the crosscap states and collected the states which can be relevant in the AdS/CFT correspondence. We also investigated the overlaps between the crosscap and Bethe states. Our main result is the equation (6.26) which give the exact overlaps for the $\mathfrak{so}(N)$ and $\mathfrak{gl}([\frac{N}{2}]) \otimes \mathfrak{gl}([\frac{N}{2}])$ symmetric crosscap states.

Finally we collect some possible future directions for research.

- Our proof is not complete even for the $\mathfrak{gl}(N)$ symmetric spin chains since it does not work for the $\mathfrak{sp}(N)$ and the $\mathfrak{gl}(M) \otimes \mathfrak{gl}(N-M)$ symmetric two-site states where $M < \lfloor \frac{N}{2} \rfloor$. It would be nice to find an extension of the method of this paper which contains these remaining cases.

- Our method is based on the $KT$-relation and the recurrence relations of the off-shell Bethe states. For the $U_q(\mathfrak{gl}(N))$ and $\mathfrak{so}(2N+1)$-invariant spin chains the recurrence relations are available [39–41]. Using these results it would be interesting to generalize our method to the $q$-deformed and orthogonal spin chains.
• For the applications in the AdS/CFT duality, it would be interesting to generalize our results to \( \mathfrak{gl}(N|M) \) symmetric spin chains. For the context of the \( \text{AdS}_5/CFT_4 \) duality the relevant case is the \( \mathfrak{gl}(4|4) \).

• It would be also interesting to combine the crosscap states and the SoV framework, like it was done for the boundary states in [32]. This method would give an alternative formula for the crosscap overlap which would contain Vandermonde determinants instead of Gaudin determinants.

• An other possible direction is to generalize the integrable crosscap states for long range spin chains which are relevant in \( \mathcal{N} = 4 \) SYM at higher loops. Combining this to the method of [42] we could investigate the wrapping corrections of the overlaps between crosscap and Bethe states.

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A Off-shell Bethe vectors

In this section we review the recurrence and action formulas of the off-shell Bethe vectors which are used in our derivations of the overlaps. The formulas of this section can be found in [25–27, 37].

The off-shell Bethe vectors can be calculated from the following sum formula [25]

\[
\mathcal{B}(\{ \bar{T}^i \}, \{ \bar{t}^k \}_{k=0}^{N-1}) = \sum_{j=2}^{N} \frac{T_{i,j}(z)}{\lambda_2(z)} \sum_{\text{part(} t)} \mathcal{B}(\{ \bar{t}^i \}, \{ \bar{t}^k \}_{k=2}^{j-1}) \frac{\prod_{\nu=1}^{j-1} \alpha_\nu(\bar{t}^i_\nu, \bar{t}^i_{\nu-1}) g(\bar{t}^i_\nu, \bar{t}^i_{\nu-1}) f(\bar{t}^i_\nu, \bar{t}^i_{\nu-1})}{\prod_{\nu=1}^{j-1} f(\bar{t}^{i+1}_\nu, \bar{t}^{i+1}_{\nu-1})},
\]

where the sum goes over all the possible partitions \( \bar{t}^\nu = \bar{t}_1^\nu \cup \bar{t}_2^\nu \) for \( \nu = 2, \ldots, j - 1 \) where \( \bar{t}_1^\nu, \bar{t}_2^\nu \) are disjoint subsets and \( \# \bar{t}_i^\nu = 1 \). We set by definition \( \bar{t}_1^1 = \{ z \} \) and \( \bar{t}_1^0 = \emptyset \).

There is another sum formula

\[
\mathcal{B}(\{ \bar{t}^k \}_{k=1}^{N-2}, \{ z, \bar{t}^{N-1} \}) = \sum_{j=1}^{N-1} \frac{T_{j,n}(z)}{\lambda_2(z)} \sum_{\text{part(} t)} \mathcal{B}(\{ \bar{t}^i \}, \{ \bar{t}^k \}_{k=1}^{j-1}) \frac{\prod_{\nu=1}^{j+1} g(\bar{t}_1^\nu, \bar{t}_2^\nu) f(\bar{t}_1^\nu, \bar{t}_2^\nu)}{\prod_{\nu=1}^{j+1} f(\bar{t}_1^{\nu+1}, \bar{t}_2^{\nu+1})},
\]

where the sum goes over all the possible partitions \( \bar{t}^\nu = \bar{t}_1^\nu \cup \bar{t}_2^\nu \) for \( \nu = j, \ldots, N - 2 \) where \( \bar{t}_1^\nu, \bar{t}_2^\nu \) are disjoint subsets and \( \# \bar{t}_i^\nu = 1 \). We set by definition \( \bar{t}_1^{N-1} = \{ z \} \) and \( \bar{t}_1^0 = \emptyset \).

We also use the following action formula for the off-shell Bethe vectors [27]

\[
T_{i,j}(z) B(\bar{t}) = \lambda_2(z) \sum_{\text{part(} \bar{w} \)} B(\bar{w}_1, \bar{w}_2) \frac{\prod_{s=j}^{i-1} f(\bar{w}_s, \bar{w}_{i-1})}{\prod_{s=j}^{i-2} f(\bar{w}_s, \bar{w}_{s+1})} \times \\
\prod_{s=1}^{i-1} \frac{f(\bar{w}_s, \bar{w}_s)}{h(\bar{w}_s, \bar{w}_{s+1})} \prod_{s=j}^{i-1} \frac{\alpha_\nu(\bar{w}_1^\nu, \bar{w}_2^\nu)}{h(\bar{w}_1^\nu, \bar{w}_2^\nu) f(\bar{w}_1^\nu, \bar{w}_2^\nu)},
\]

where \( \bar{w}^\nu = \{ z, \bar{t}^\nu \} \). The sum goes over all the partitions of \( \bar{w}^\nu = \bar{w}_1^\nu \cup \bar{w}_2^\nu \cup \bar{w}_3^\nu \) where \( \bar{w}_i^\nu, \bar{w}_j^\nu, \bar{w}_k^\nu \) are disjoint sets for a fixed \( \nu \) and \( \# \bar{w}_i^\nu = \Theta(i - 1 - \nu) \), \( \# \bar{w}_i^\nu = \Theta(\nu - j) \). We also set \( \bar{w}_i^0 = \bar{w}_i^N = \{ z \} \) and
\[ \bar{\omega}_i^0 = \bar{\omega}_i^0 = \bar{\omega}_i^N = \bar{\omega}_i^N = \emptyset. \] We also used the unit step function \( \Theta(k) \) which is defined as \( \Theta(k) = 1 \) for \( k \geq 0 \) and \( \Theta(k) = 0 \) for \( k < 0 \).

We can see that the diagonal elements \( T_{i,i}(u) \) do not change the quantum numbers \( r_j \). The creation operators i.e. \( T_{i,j}(u) \) where \( i < j \) increase the quantum numbers \( r_i, r_{i+1}, \ldots, r_{j-1} \) by one. The annihilation operators i.e. \( T_{j,i}(u) \) where \( i < j \) decrease the quantum numbers \( r_i, r_{i+1}, \ldots, r_{j-1} \) by one.

In some parts of the derivations it is more convenient to use the following re-normalized Bethe states

\[ \hat{\mathcal{B}}(\tilde{t}) = \prod_{\nu=1}^{N-1} \lambda_{\nu+1}(\tilde{t}_\nu^\nu) \mathcal{B}(\tilde{t}), \] (A.4)

for which the recurrence and action formulas for can be written as

\[ \hat{\mathcal{B}}(\{\bar{t}_i^k\}, \{\bar{t}_j^{N-1}\}) = \sum_{j=2}^{N} T_{i,j}(z) \sum_{\text{part}(\tilde{t})} \hat{\mathcal{B}}(\tilde{t}, \{\bar{t}_i^k\}_{k=2}^{j-1}, \{\bar{t}_j^{N-1}\}_{k=j}^{N-1}) \prod_{\nu=2}^{j-1} \lambda_{\nu}(\tilde{t}_\nu^\nu) g(\tilde{t}_\nu^\nu, \tilde{t}_{\nu-1}^\nu) f(\tilde{t}_\nu^\nu, \tilde{t}_\nu^{\nu-1}) \prod_{\nu=1}^{j-1} f(\tilde{t}_{\nu-1}^{\nu}, \tilde{t}_\nu^\nu), \] (A.5)

\[ \hat{\mathcal{B}}(\{\bar{t}_i^k\}_{k=1}^{N-2}, \{z, \bar{t}_i^{N-1}\}) = \sum_{j=1}^{N-1} T_{j,N}(z) \sum_{\text{part}(\tilde{t})} \hat{\mathcal{B}}(\{\bar{t}_i^k\}_{k=1}^{j-1}, \{\bar{t}_i^{N-2}\}_{k=j}^{N-1}, \{\bar{t}_i^{N-1}\}) \times \prod_{\nu=j}^{N-2} \lambda_{\nu+1}(\tilde{t}_\nu^\nu) g(\tilde{t}_\nu^\nu+1, \tilde{t}_\nu^{\nu-1}) f(\tilde{t}_\nu^\nu, \tilde{t}_\nu^{\nu-1}) \prod_{\nu=j}^{N-1} f(\tilde{t}_\nu^{\nu-1}, \tilde{t}_\nu^\nu), \] (A.6)

and

\[ T_{i,j}(z) \hat{\mathcal{B}}(\tilde{t}) = \sum_{\text{part}(\tilde{w})} \prod_{s=1}^{i-1} \lambda_{s+1}(\tilde{w}_s^s) \prod_{s=j}^{N-2} \lambda_{s}(\tilde{w}_s^s) \mathcal{B}(\tilde{w}_s^s) \times \] \[ \prod_{s=j}^{i-1} f(\tilde{w}_s^s, \tilde{w}_s^{s+1}) \prod_{s=1}^{i-1} f(\tilde{w}_s^s, \tilde{w}_s^{s-1}) \prod_{s=j}^{N-1} f(\tilde{w}_s^s, \tilde{w}_s^{s-1}) \prod_{s=j}^{i-2} f(\tilde{w}_s^{s+1}, \tilde{w}_s^{s+2}) h(\tilde{w}_s^{s+1}, \tilde{w}_s^{s+2}) f(\tilde{w}_s^{s+1}, \tilde{w}_s^{s+2}), \] (A.7)

We will also use the co-product formula of the off-shell Bethe vectors. Let \( \mathcal{H}^{(1)}, \mathcal{H}^{(2)} \) be two quantum spaces for which \( \mathcal{H} = \mathcal{H}^{(1)} \odot \mathcal{H}^{(2)} \) and the corresponding off-shell states are \( \hat{\mathcal{B}}^{(1)}(\tilde{t}), \hat{\mathcal{B}}^{(2)}(\tilde{t}) \). The co-product formula reads as [24]

\[ \hat{\mathcal{B}}(\tilde{t}) = \sum_{\text{part}(\tilde{w})} \prod_{\nu=1}^{N-1} \lambda_{\nu+1}(\tilde{t}_\nu^\nu) \prod_{\nu=1}^{N} \lambda_{\nu}(\tilde{w}_\nu^\nu) \mathcal{B}^{(1)}(\tilde{t}_\nu^\nu) \otimes \mathcal{B}^{(2)}(\tilde{w}_\nu^\nu), \] (A.8)

where the sum goes over all the possible partitions \( \tilde{t}_\nu^\nu = \tilde{t}_\nu^\nu \cup \tilde{w}_\nu^\nu \) where \( \tilde{t}_\nu^\nu, \tilde{w}_\nu^\nu \) are disjoint subsets.

We will also need the recursion formulas for the left off-shell vectors

\[ \hat{\mathcal{C}}(\{\bar{s}_i^k\}_{k=2}^{N-1}, \{z, \bar{s}_i^{N-1}\}) = \sum_{j=1}^{N-1} \sum_{\text{part}(\tilde{t})} \hat{\mathcal{C}}(\{\bar{s}_i^k\}_{k=1}^{j-1}, \{\bar{t}_i^{N-2}\}_{k=j}^{N-1}, \{\bar{t}_i^{N-1}\}) T_{i,j}(z) \times \] \[ \prod_{\nu=j}^{N-2} \lambda_{\nu+1}(\tilde{t}_\nu^\nu) g(\tilde{t}_\nu^\nu+1, \tilde{t}_\nu^{\nu-1}) f(\tilde{t}_\nu^\nu, \tilde{t}_\nu^{\nu-1}) \prod_{\nu=j}^{N-1} f(\tilde{t}_\nu^{\nu-1}, \tilde{t}_\nu^\nu), \] (A.9)
B Sum formula for the scalar product

In [25] it was shown that the off-shell scalar product has the following sum rule

\[
S(\bar{s}|\bar{t}) = C(\bar{s}) \mathbb{B}(\bar{t}) = \sum W(\bar{s}_i, \bar{s}_i|\bar{t}_i, \bar{t}_i) \prod_{k=1}^{N-1} \alpha_k(s^k_i)\alpha_k(t^k_i). \tag{B.1}
\]

The sum goes over all the partitions of \( \bar{t}^\nu = \bar{t}^\nu_\mu \cup \bar{t}^\nu_1 \) and \( \bar{s}^\nu = \bar{s}^\nu_\nu \cup \bar{s}^\nu_1 \) where \( \#\bar{t}^\nu_\mu = \#\bar{s}^\nu_\nu \) and \( \#\bar{t}^\nu_1 = \#\bar{s}^\nu_1 \). For the renormalized Bethe vector we obtain that

\[
\hat{S}(\bar{s}|\bar{t}) = \hat{C}(\bar{s}) \hat{\mathbb{B}}(\bar{t}) = \sum W(\bar{s}_i, \bar{s}_i|\bar{t}_i, \bar{t}_i) \prod_{k=1}^{N-1} \lambda_k(s^k_i)\lambda_{k+1}(s^k_{i+1})\lambda_k(t^k_i). \tag{B.2}
\]

One can introduce the highest coefficients (HC)

\[
W(\bar{s}, \emptyset|\bar{t}, \emptyset) = Z(\bar{s}|\bar{t}), \tag{B.3}
\]

\[
W(\emptyset, \bar{s}|\bar{t}, \emptyset) = \bar{Z}(\bar{s}|\bar{t}), \tag{B.4}
\]

for which we have the identity

\[
\bar{Z}(\bar{s}|\bar{t}) = Z(\bar{t}|\bar{s}). \tag{B.5}
\]

The coefficients \( W \) can be expressed with the HC-s as

\[
W(\bar{s}_i, \bar{s}_i|\bar{t}_i, \bar{t}_i) = Z(\bar{s}_i|\bar{t}_i)Z(\bar{t}_i|\bar{s}_i) \prod_{k=1}^{N-1} \frac{f(s^k_i, \bar{s}^k_i)f(t^k_i, \bar{t}^k_i)}{f(s^k_{i+1}, \bar{s}^k_i)f(t^k_{i+1}, \bar{t}^k_i)}. \tag{B.6}
\]

A recurrence relation for the HC-s can be found in [25] but in this paper we only need the pole structure of the HC-s [26]

\[
Z(\bar{s}|\bar{t}) \bigg|_{s_j^\mu \to t_j^\mu} = g(t^\mu_j, s^\mu_j) \frac{f(t^\mu_j, t^\mu_j)f(s^\mu_j, \bar{s}^\mu_j)}{f(t^\mu_{j+1}, t^\mu_j)f(s^\mu_j, \bar{s}^\mu_{j+1})} Z(\bar{s}\setminus\{s^\mu_j\}|\bar{t}\setminus\{t^\mu_j\}) + \text{reg}. \tag{B.7}
\]

where the "reg" contains the regular terms in the limit \( s^\mu_j \to t^\mu_j \).

C Relation between crosscap overlaps and scalar products

In this section we prove the identity

\[
\langle C| \hat{\mathbb{B}}^{(1)}(\bar{s}) \otimes \hat{\mathbb{B}}^{(2)}(\bar{t}) \rangle = \hat{C}^{(2)}(\pi^\nu(\bar{s}))\hat{\mathbb{B}}^{(2)}(\bar{t}), \tag{C.1}
\]

for the untwisted and the identity

\[
\langle C| \hat{\mathbb{B}}^{(1)}(\bar{s}) \otimes \hat{\mathbb{B}}^{(2)}(\bar{t}) \rangle = \frac{1}{\prod_{\nu=1}^{N-1} \lambda_\nu(\bar{s}^\nu)} \hat{C}^{(2)}(\pi^\nu(\bar{s}))\hat{\mathbb{B}}^{(2)}(\bar{t}), \tag{C.2}
\]

for the twisted case. We use induction in the number of roots \( r_1 = \#\bar{t}^1 = \#\bar{s}^1 \). Let assume that (C.1) and (C.2) is true up to \( r_1 \) Bethe roots. Let us increase the number of roots and using the recursion formula (A.9) we obtain a recursion for the scalar product

\[
\hat{C}^{(2)}(\{s^k\}_{k=2}^{N-1}, \{z, s^{N-1}\})\hat{\mathbb{B}}^{(2)}(\bar{t}) = \sum_{j=1}^{N-1} \sum_{\text{part}(\bar{t})} \tilde{C}^{(2)}(\{s^k\}_{k=2}^{j-1}, \{s^k\}_{k=j+1}^{N-2}, s^{N-1})T^{(2)}_{N,j}(z)\hat{\mathbb{B}}^{(2)}(\bar{t}) \frac{\prod_{\nu=1}^{N-2} \lambda^{(2)}(s^\nu_{\nu+1})(\bar{s}^\nu_{\nu+1}, \bar{s}^\nu_{\nu})f(s^\nu_{\nu+1}, s^\nu_{\nu})}{\prod_{\nu=1}^{N-1} f(s^\nu_{\nu+1}, \bar{s}^\nu_{\nu})}, \tag{C.3}
\]

where \( \#\bar{s}^1 = r_1 \) and \( \#\bar{t}^1 = r_1 + 1 \).
C.1 Untwisted case

Now we derive a recursion for the crosscap overlap

$$\langle C| \left( \hat{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \hat{B}^{(2)}(i) \right).$$

(C.4)

Using the recursion formula (A.5) we can obtain that

$$\langle C| \left( \hat{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \hat{B}^{(2)}(i) \right) =$$

$$\sum_{j=2}^{N} \langle C| \hat{T}^{(1)}_{1,j}(z) \sum_{\text{part}(i)} \hat{B}^{(1)}(s^1, \{ z^k \}_{k=2}^{j-1}, \{ z^k \}_{k=j}^{N-1}) \otimes \hat{B}^{(2)}(i) \prod_{\nu=2}^{j-1} \lambda^{(1)}_{\nu}(s^\nu) g(s^\nu, s^{\nu-1}) f(s^\nu, s^\nu) \prod_{\nu=1}^{j-1} f(s^\nu+1, s^\nu).$$

(C.5)

Using the KT-relation we obtain that

$$\langle C| \left( \hat{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \hat{B}^{(2)}(i) \right) =$$

$$\sum_{j=2}^{N} \sum_{\text{part}(i)} \langle C| \hat{B}^{(1)}(s^1, \{ z^k \}_{k=2}^{j-1}, \{ z^k \}_{k=j}^{N-1}) \otimes T^{(2)}_{N,N+1-j}(-z) \hat{B}^{(2)}(i) \prod_{\nu=2}^{j-1} \lambda^{(1)}_{\nu}(s^\nu) g(s^\nu, s^{\nu-1}) f(s^\nu, s^\nu) \prod_{\nu=1}^{j-1} f(s^\nu+1, s^\nu).$$

(C.6)

After some rearrangements we obtain that

$$\langle C| \left( \hat{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \hat{B}^{(2)}(i) \right) =$$

$$\sum_{j=1}^{N-1} \sum_{\text{part}(i)} \langle C| \hat{B}^{(1)}(s^1, \{ z^k \}_{k=2}^{N-j}, \{ z^k \}_{k=N+1-j}^{N-1}) \otimes T^{(2)}_{N,j}(-z) \hat{B}^{(2)}(i) \times$$

$$\prod_{\nu=j}^{N-2} \lambda^{(2)}_{\nu+1}(-s^\nu) g(-s^\nu, -s^{\nu+1}) f(-s^\nu, -s^\nu) \prod_{\nu=j}^{N-2} f(-s^\nu, -s^{\nu+1}).$$

(C.7)

We can see that this recursion is the same what we obtained for the scalar product (C.3) therefore we just derived (C.1).

C.2 Twisted case

Now we derive a recursion for the crosscap overlap

$$\langle C| \left( \tilde{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \tilde{B}^{(2)}(i) \right).$$

(C.8)

Using the recursion formula (A.5) we can obtain that

$$\langle C| \left( \tilde{B}^{(1)}(\{ z, z^1 \}, \{ z^k \}_{k=2}^{N-1}) \otimes \tilde{B}^{(2)}(i) \right) = \sum_{j=2}^{N} \langle C| \tilde{T}^{(1)}_{1,j}(z) \sum_{\text{part}(i)} \tilde{B}^{(1)}(s^1, \{ z^k \}_{k=2}^{j-1}, \{ z^k \}_{k=j}^{N-1}) \otimes \tilde{B}^{(2)}(i) \times$$

$$\prod_{\nu=2}^{j-1} \lambda^{(1)}_{\nu}(s^\nu) g(s^\nu, s^{\nu-1}) f(s^\nu, s^\nu) \prod_{\nu=1}^{j-1} f(s^\nu+1, s^\nu).$$

(C.9)
Using the twisted KT-relation we obtain that
\[
\langle C | \hat{\mathcal{B}}^{(1)}(\{z, \bar{z}\}, \{\bar{s}^k\}_{k=2}^{N-1}) \otimes \hat{\mathcal{B}}^{(2)}(\bar{t})) = \sum_{j=2}^{N} \sum_{\text{part}(\bar{t})} \langle C | \hat{\mathcal{B}}^{(1)}(\bar{s}, \{\bar{s}^k\}_{k=2}^{N-1}) \otimes T_{N,N+1-j}^{(2)}(-z) \hat{\mathcal{B}}^{(2)}(\bar{t}) \times \frac{1}{\lambda_0(z)} \prod_{\nu=2}^{j-1} \lambda_0^{(1)}(\bar{s}_\nu^\nu)g(\bar{s}_\nu^\nu, \bar{s}_\nu^{\nu-1})f(\bar{s}_\nu^\nu, \bar{s}_\nu^{\nu}) \rangle.
\] (C.10)

After some rearrangements we obtain that
\[
\langle C | \hat{\mathcal{B}}^{(1)}(\{z, \bar{z}\}, \{\bar{s}^k\}_{k=2}^{N-1}) \otimes \hat{\mathcal{B}}^{(2)}(\bar{t})) = \sum_{j=1}^{N-1} \sum_{\text{part}(\bar{t})} \langle C | \hat{\mathcal{B}}^{(1)}(\bar{s}, \{\bar{s}^k\}_{k=2}^{N-1}) \otimes T_{N,N+1-j}^{(2)}(-z) \hat{\mathcal{B}}^{(2)}(\bar{t}) \times \frac{1}{\lambda_0(\bar{s}_\nu^\nu)} \prod_{\nu=2}^{N-j} \lambda_0(\bar{s}_\nu^\nu)g(-\bar{s}_\nu^{\nu-1}, -\bar{s}_1^{\nu-1})f(-\bar{s}_1^{\nu-1}, -\bar{s}_1^{\nu-1}) \rangle.
\] (C.11)

We can see that this recursion is the same what we obtained for the scalar product (C.3) therefore we just derived (C.2).

D Definitions of the modified X-s

In the equation (6.7) the modification is trivial for $|\nu - \mu| > 1$ i.e. $X_k^{(b),\nu,m_a} = X_k^{(b),\nu}$ for $|\nu - \mu| > 1$. For the other cases we define the modified $X_k^{(b),\nu}$-s, separately. Let us define the following expressions

\[
F^{\pm,\mu} = -e \frac{d}{du} \log \frac{f(u^\pm, u)}{f(u, u^\pm)} \bigg|_{u=w_k^{\pm,-}} ,
\] (D.1)

\[
G^{\pm,\mu+1} = -e \frac{d}{du} \log f(u, u^\pm) \bigg|_{u=w_k^{\pm,-+1}} ,
\] (D.2)

\[
H^{\pm,\mu-1} = +e \frac{d}{du} \log f(u, u^\pm) \bigg|_{u=w_k^{\pm,--1}} .
\] (D.3)

For the twisted case the modified $X_k^{(b),\nu}$-s in the equation (6.7) are defined as

\[
X_k^{(1),\mu,m_1} = X_k^{(1),\mu} + F^{+,\mu} ,
\]

\[
X_k^{(1),\mu+1,m_1} = X_k^{(1),\mu+1} + G^{+,\mu+1} ,
\]

\[
X_k^{(1),\mu-1,m_1} = X_k^{(1),\mu-1} + H^{+,\mu-1} ,
\]

\[
X_k^{(2),\mu,m_2} = X_k^{(2),\mu} + F^{+,\mu} ,
\]

\[
X_k^{(2),\mu+1,m_2} = X_k^{(2),\mu+1} + G^{+,\mu+1} ,
\]

\[
X_k^{(2),\mu-1,m_2} = X_k^{(2),\mu-1} + H^{+,\mu-1} .
\] (D.4)
For the untwisted case the modified $X_{k}^{(b),\mu,s}$ in the equation (6.7) are defined as

$$
X_{k}^{(1),\mu,m_1} = X_{k}^{(1),\mu} + F^{+,\mu}, \quad X_{k}^{(2),\mu,m_2} = X_{k}^{(2),\mu} + F^{+,\mu},
$$

$$
X_{k}^{(1),\mu+1,m_1} = X_{k}^{(1),\mu+1} + G^{+,\mu+1}, \quad X_{k}^{(2),\mu+1,m_2} = X_{k}^{(2),\mu+1} + G^{+,\mu+1},
$$

$$
X_{k}^{(1),\mu-1,m_1} = X_{k}^{(1),\mu-1} + H^{+,\mu-1}, \quad X_{k}^{(2),\mu-1,m_2} = X_{k}^{(2),\mu-1} + H^{+,\mu-1},
$$

and $X_{k}^{(1),\mu,m_2} = X_{k}^{(1),\mu}, X_{k}^{(2),\mu,m_2} = X_{k}^{(2),\mu}$ for $\mu < \frac{N}{2} - 1$ and

$$
X_{k}^{(1),\mu-1,m_1} = X_{k}^{(1),\mu-1} + F^{+,\mu-1}, \quad X_{k}^{(2),\mu-1,m_2} = X_{k}^{(2),\mu-1} + F^{+,\mu-1},
$$

$$
X_{k}^{(1),\mu-2,m_1} = X_{k}^{(1),\mu-2} + H^{+,\mu-2}, \quad X_{k}^{(2),\mu-2,m_2} = X_{k}^{(2),\mu-2} + H^{+,\mu-2},
$$

for $\mu = \frac{N}{2} - 1$ and

$$
X_{k}^{(1),\mu,m_1} = X_{k}^{(1),\mu} + F^{+,\mu}, \quad X_{k}^{(2),\mu,m_2} = X_{k}^{(2),\mu} + F^{+,\mu},
$$

$$
X_{k}^{(1),\mu-1,m_1} = X_{k}^{(1),\mu-1} + H^{+,\mu-1}, \quad X_{k}^{(2),\mu-1,m_2} = X_{k}^{(2),\mu-1} + H^{+,\mu-1},
$$

for $\mu = \frac{N}{2}$ and

$$
X_{k}^{(1),\mu,m_1} = X_{k}^{(1),\mu} + F^{+,\mu}, \quad X_{k}^{(2),\mu,m_2} = X_{k}^{(2),\mu} + H^{+,\mu},
$$

$$
X_{k}^{(1),\mu-1,m_1} = X_{k}^{(1),\mu-1} + H^{+,\mu-1}, \quad X_{k}^{(2),\mu-1,m_2} = X_{k}^{(2),\mu-1} + H^{+,\mu-1},
$$

for $\mu = \frac{N-1}{2}$.

### E Pair structure limit

In this section we calculate the pair structure limits of the overlaps. In the untwisted case we have achiral pair structure for which we have to take the limit $w_{k}^{-,\nu} \rightarrow \pi^a (w_{k}^{+,\nu}) = -w_{k}^{-,\nu}$. In the twisted case we have chiral pair structure for which we have to take the limit $w_{k}^{-,\nu} \rightarrow \pi^c (w_{k}^{+,\nu}) = -w_{k}^{-,\nu} - \nu c$.

#### E.1 Twisted case

Let us start with the twisted case. Using the sum formula (5.17) and normalization (6.10), the normalized overlap reads as

$$
N(\bar{w}) = F(\bar{w}^{+}, \bar{w}^{-}) \sum_{\text{part}(w)} G(\tilde{s}_i, \tilde{s}_{ii}, \tilde{r}_i, \tilde{r}_{ii}) \prod_{\nu=1}^{N-1} \frac{\alpha_{\nu}^{(1)}(\tilde{s}_{\nu}) \alpha_{\nu}^{(2)}(\tilde{s}_{\nu}^{+}) \alpha_{\nu}^{(2)}(\tilde{s}_{\nu}^{-})}{\alpha_{\nu}^{(2)}(\tilde{w}^{+,\nu}) \alpha_{\nu}^{(2)}(\tilde{w}^{-,\nu})},
$$

(E.1)
where
\[ F(\bar{w}^+, \bar{w}^-) = \frac{1}{\prod_{\nu=1}^{N-1} \prod_{i \neq j} f(w_i^{+\nu}, w_j^{+\nu}) f(\bar{w}^{-\nu}, \bar{w}^{+\nu}) f(\bar{w}^{+\nu+1}, \bar{w}^{-\nu})} \tag{E.2} \]
and
\[ G(\bar{s}_i, \bar{s}_n|\bar{t}_i, \bar{t}_n) = \frac{\prod_{\nu=1}^{N-1} f(\bar{p}_i^\nu, \bar{s}_i^\nu) f(\bar{s}_i^\nu, \bar{p}_i^\nu) f(\bar{p}_n^\nu, \bar{t}_n^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{p}_n^\nu, \bar{s}_n^\nu) f(\bar{s}_n^\nu, \bar{p}_n^\nu)} \times \frac{\prod_{\nu=1}^{N-2} f(\bar{s}_n^\nu, \bar{s}_n^{\nu+1} + c) f(\bar{p}_n^\nu, \bar{p}_n^{\nu+1} + c) Z(\pi^c(\bar{s}_i)|\bar{t}_i) Z(\bar{t}_n|\pi^c(\bar{s}_n))} {Z(\pi^c(\bar{s}_i)|\bar{t}_i) Z(\bar{t}_n|\pi^c(\bar{s}_n))}. \tag{E.3} \]

Now we take the \( w_k^{+\nu} + w_k^{-\nu} + \nu \kappa \to 0 \) limit. Let us define the sets \( \bar{\omega} = \bar{w} \setminus \{w_k^{+\nu}, w_k^{-\nu}\}, \bar{\omega}^+ = \bar{w}^+ \setminus \{w_k^{+\nu}\} \) and \( \bar{\omega}^- = \bar{w}^- \setminus \{w_k^{-\nu}\} \). We can see that the formal poles appear only for the partitions (see equation (B.7)) for the partitions where
- \( w_k^{+\nu} \in \bar{p}_i^\nu, w_k^{-\nu} \in \bar{s}_i^\nu \) or
- \( w_k^{+\nu} \in \bar{p}_i^\nu, w_k^{-\nu} \in \bar{s}_i^\nu \) or
- \( w_k^{+\nu} \in \bar{s}_i^\nu, w_k^{-\nu} \in \bar{p}_i^\nu \) or
- \( w_k^{+\nu} \in \bar{s}_i^\nu, w_k^{-\nu} \in \bar{s}_i^\nu \).

Let us fix a partition as \( \bar{\omega} = \bar{\sigma}_i \cup \bar{\sigma}_n \cup \bar{\tau}_i \cup \bar{\tau}_n \). Let us get the G-term with \( \bar{t}_i = \{w_k^{-\nu}\} \cup \bar{\tau}_i, \bar{s}_i = \{w_k^{+\nu}\} \cup \bar{\sigma}_i; \)
\[ G(\bar{s}_i, \bar{s}_n|\bar{t}_i, \bar{t}_n) \mid_{w_k^{-\nu} \to \pi^c(w_k^{+\nu})} \] \[ \to G_1 = g(w_k^{-\nu}, -w_k^{+\nu} - \nu \kappa) f(w_k^{-\nu}, w_k^{+\nu}) \times \frac{f(\bar{\tau}_i^{\nu}, \bar{w}_k^{+\nu}) f(\bar{\sigma}_n^{\nu}, \bar{w}_k^{+\nu}) f(w_k^{+\nu}, \bar{\sigma}_n^{\nu})}{f(\bar{\tau}_i^{\nu+1}, \bar{w}_k^{+\nu+1}) f(\bar{\sigma}_n^{\nu+1}, \bar{w}_k^{+\nu+1}) f(w_k^{+\nu+1}, \bar{\sigma}_n^{\nu+1})} \times \frac{f(\bar{w}_k^{-\nu}, \bar{\sigma}_n^{\nu-1}) f(\bar{w}_k^{-\nu}, \bar{\tau}_i^{\nu}) f(w_k^{-\nu}, \bar{\tau}_i^{\nu})}{f(\bar{w}_k^{-\nu}, \bar{\tau}_i^{\nu+1}) f(\bar{w}_k^{-\nu}, \bar{\tau}_i^{\nu+1}) f(w_k^{-\nu}, \bar{\tau}_i^{\nu+1})} G(\bar{\sigma}_i, \bar{\sigma}_n|\bar{\tau}_i, \bar{\tau}_n) + \text{reg.} \tag{E.4} \]

where we used (B.7). The \( \alpha \) dependence for this partition is
\[ \prod_{\mu=1}^{N-1} \alpha_\mu^{(1)}(\bar{\sigma}_n^\nu) \alpha_\mu^{(2)}(\bar{\sigma}_n^\nu) \alpha_\mu^{(2)}(\bar{\tau}_i^\nu) \alpha_\mu^{(2)}(\bar{\omega}^+). \tag{E.5} \]

For the partition \( \bar{t}_i = \{w_k^{-\nu}\} \cup \bar{\tau}_i, \bar{s}_n = \{w_k^{+\nu}\} \cup \bar{\sigma}_n; \)
\[ G(\bar{s}_i, \bar{s}_n|\bar{t}_i, \bar{t}_n) \mid_{w_k^{-\nu} \to \pi^c(w_k^{+\nu})} \] \[ \to G_2 = g(-w_k^{+\nu} - \nu \kappa, w_k^{-\nu}) f(w_k^{-\nu}, w_k^{+\nu}) \times \frac{f(\bar{w}_k^{+\nu}, \bar{w}_k^{+\nu}) f(\bar{\tau}_i^{\nu}, \bar{w}_k^{+nu}) f(w_k^{+\nu}, \bar{\tau}_i^{\nu})}{f(\bar{w}_k^{+\nu+1}, \bar{w}_k^{+\nu+1}) f(\bar{\tau}_i^{\nu+1}, \bar{w}_k^{+\nu+1}) f(w_k^{+\nu+1}, \bar{\tau}_i^{\nu+1})} \times \frac{f(\bar{w}_k^{+\nu}, \bar{\tau}_i^{\nu-1}) f(\bar{w}_k^{+\nu}, \bar{\tau}_i^{\nu}) f(w_k^{+\nu}, \bar{\tau}_i^{\nu})}{f(\bar{w}_k^{+\nu}, \bar{\tau}_i^{\nu+1}) f(\bar{w}_k^{+\nu}, \bar{\tau}_i^{\nu+1}) f(w_k^{+\nu}, \bar{\tau}_i^{\nu+1})} G(\bar{\sigma}_i, \bar{\sigma}_n|\bar{\tau}_i, \bar{\tau}_n) + \text{reg.} \tag{E.6} \]

where we used (B.7). The \( \alpha \) dependence for this partition is
\[ \alpha_\nu^{(1)}(w_k^{+\nu}) \alpha_\nu^{(2)}(w_k^{-\nu}) \prod_{\mu=1}^{N-1} \alpha_\mu^{(1)}(\bar{\tau}_i^\nu) \alpha_\mu^{(2)}(\bar{\tau}_i^\nu) \alpha_\mu^{(2)}(\bar{\omega}^+). \tag{E.7} \]
Let us take the sum of the two terms corresponding the previous two partitions

$$G_1 \prod_{\mu=1}^{N-1} \frac{\alpha^{(1)}_\mu (\tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{\alpha^{(2)}_\mu (\tilde{\omega}^{+}_{\mu})} + G_2 \prod_{\mu=1}^{N-1} \frac{\alpha^{(1)}_\mu (w^{+}_{\mu} \nu) \alpha^{(2)}_\mu (w^{-}_{\mu} \nu)}{\alpha^{(2)}_\mu (\tilde{\omega}^{+}_{\mu})} = \left( G_1 + G_2 \right) \prod_{\mu=1}^{N-1} \frac{\alpha^{(1)}_\mu (\tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{\alpha^{(2)}_\mu (\tilde{\omega}^{+}_{\mu})}. \quad (E.8)$$

In this sum, the following expression appears

$$g(w^{+}_{\nu}, -w^{-}_{\nu} \nu c(1-\alpha^{(1)}_\mu (w^{+}_{\nu}) \alpha^{(2)}_\mu (w^{-}_{\nu})) \rightarrow -\frac{d}{d\nu} \log \alpha^{(1)}_\mu (u) \bigg|_{u=w^{+}_{\nu}} = X^{(1)}_{k} \nu, \quad (E.9)$$

therefore the previous sum simplifies as

$$\lim_{\nu \rightarrow \pi^{c}(w^{+}_{\nu})} f(w^{+}_{\nu} \mu, w^{-}_{\nu} \nu) \sum_{\mu} \frac{\alpha^{(1)}_\mu (\tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{\alpha^{(2)}_\mu (\tilde{\omega}^{+}_{\mu})} = X^{(1)}_{1} \nu \times \quad (E.10)$$

For the partition \( \tilde{r}_{\mu} = \{ w^{+}_{\nu} \mu, w^{-}_{\nu} \nu \} \cup \tilde{\sigma}, \tilde{s}_{\mu} = \{ w^{+}_{\nu} \mu, w^{-}_{\nu} \nu \} \cup \tilde{\sigma}:

$$G(\tilde{s}_{\mu}, \tilde{s}_{\mu} | \tilde{r}_{\mu}, \tilde{r}_{\mu}) \bigg|_{w^{+}_{\nu} \nu \rightarrow \pi^{c}(w^{+}_{\nu})} \rightarrow G_3 = g(w^{+}_{\nu} \mu, -w^{-}_{\nu} \nu c f(w^{+}_{\nu} \mu, w^{-}_{\nu} \nu) \times \quad (E.11)$$

$$\frac{f(w^{+}_{\nu} \mu, \tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{f(w^{+}_{\nu} \mu, \tilde{\tau}^{+}_{\nu} \mu) f(w^{+}_{\nu} \mu, \tilde{\tau}^{+}_{\nu} \mu)} = G(\tilde{s}_{\mu}, \tilde{s}_{\mu} | \tilde{r}_{\mu}, \tilde{r}_{\mu}) + \text{reg.}$$

where we used (B.7). The \( \alpha \) dependence for this partition is

$$\frac{\alpha^{(2)}_{\mu} (w^{-}_{\nu})}{\alpha^{(2)}_{\mu} (w^{+}_{\nu})} \prod_{\mu=1}^{N-1} \frac{\alpha^{(1)}_\mu (\tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{\alpha^{(2)}_\mu (\tilde{\omega}^{+}_{\mu})}. \quad (E.12)$$

For the partition \( \tilde{r}_{\mu} = \{ w^{+}_{\nu} \mu, \tilde{s}_{\mu} = \{ w^{+}_{\nu} \nu \} \cup \tilde{\sigma}:

$$G(\tilde{s}_{\mu}, \tilde{s}_{\mu} | \tilde{r}_{\mu}, \tilde{r}_{\mu}) \bigg|_{w^{+}_{\nu} \nu \rightarrow \pi^{c}(w^{+}_{\nu})} \rightarrow G_4 = g(-w^{-}_{\nu} \nu c, w^{+}_{\nu} \nu) f(w^{+}_{\nu} \mu, w^{-}_{\nu} \nu) \times \quad (E.13)$$

$$\frac{f(w^{+}_{\nu} \mu, \tilde{\sigma}_{\mu}) (\tilde{\rho}^{(2)}_\mu)}{f(w^{+}_{\nu} \mu, \tilde{\tau}^{+}_{\nu} \mu) f(w^{+}_{\nu} \mu, \tilde{\tau}^{+}_{\nu} \mu)} = G(\tilde{s}_{\mu}, \tilde{s}_{\mu} | \tilde{r}_{\mu}, \tilde{r}_{\mu}) + \text{reg.}$$
where we used (B.7). The α dependence for this partition is

$$\alpha_{\nu}^{(1)}(w_k^{-,\nu}) \alpha_{\nu}^{(2)}(w_k^{-,\nu}) \prod_{\mu=1}^{N-1} \alpha_{\mu}^{(1)}(\tilde{\sigma}_\mu) \frac{\alpha_{\mu}^{(2)}(\tilde{\sigma}_\mu) \alpha_{\mu}^{(2)}(\tilde{\tau}_\mu)}{\alpha_{\mu}^{(2)}(\tilde{\omega}+\nu)}.$$  \hfill (E.14)

An analogous way we can obtain that

$$\lim_{w_k^{-,\nu} \to \pi^-(w_k^{+,\nu})} = \left( G_{3} \frac{\alpha_{\nu}^{(2)}(w_k^{+,\nu})}{\alpha_{\nu}^{(2)}(w_k^{+,\nu})} + G_{4} \alpha_{\nu}^{(1)}(w_k^{-,\nu}) \alpha_{\nu}^{(2)}(w_k^{-,\nu}) \right) \prod_{\mu=1}^{N-1} \alpha_{\mu}^{(1)}(\tilde{\sigma}_\mu) \frac{\alpha_{\mu}^{(2)}(\tilde{\sigma}_\mu) \alpha_{\mu}^{(2)}(\tilde{\tau}_\mu)}{\alpha_{\mu}^{(2)}(\tilde{\omega}+\nu)} \times
$$

$$\frac{f(w_k^{+,\nu}, \tilde{\sigma}_\nu) f(\tilde{\sigma}_\mu, w_k^{+,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)}{f(w_k^{+,\nu}, \tilde{\sigma}_\mu) f(\tilde{\sigma}_\mu, w_k^{+,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)} \times$$

$$\frac{f(w_k^{-,\nu}, \tilde{\sigma}_\nu) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)}{f(w_k^{-,\nu}, \tilde{\sigma}_\mu) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)} \times
$$

$$G(\tilde{\sigma}, \tilde{\nu} \mid \tilde{\tau}, \tilde{\tau}_\mu) \prod_{\mu=1}^{N-1} \alpha_{\mu}^{(1)}(\tilde{\sigma}_\mu) \frac{\alpha_{\mu}^{(2)}(\tilde{\sigma}_\mu) \alpha_{\mu}^{(2)}(\tilde{\tau}_\mu)}{\alpha_{\mu}^{(2)}(\tilde{\omega}+\nu)} + \text{reg.}$$  \hfill (E.15)

Substituting back we obtain that

$$\hat{N}(\tilde{\omega}) \bigg|_{w_k^{-,\nu} = \pi^-(w_k^{+,\nu})} \to$$

$$X_k^{(1),\nu} f(w_k^{-,\nu}, w_k^{+,\nu}) F(\tilde{\omega}, \tilde{\omega}) \frac{f(\tilde{\omega}, w_k^{+,\nu}) f(\tilde{\omega}, w_k^{-,\nu})}{f(\tilde{\omega}+1, w_k^{+,\nu}) f(\tilde{\omega}+1, w_k^{-,\nu})} \times
$$

$$\sum_{\text{part}(\tilde{\omega})} \frac{f(w_k^{+,\nu}, \tilde{\sigma}_\nu) f(\tilde{\sigma}_\mu, w_k^{+,\nu}) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)}{f(\tilde{\sigma}_\mu, w_k^{+,\nu}) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)} \times
$$

$$G(\tilde{\sigma}, \tilde{\nu} \mid \tilde{\tau}, \tilde{\tau}_\mu) \prod_{\mu=1}^{N-1} \alpha_{\mu}^{(1)}(\tilde{\sigma}_\mu) \frac{\alpha_{\mu}^{(2)}(\tilde{\sigma}_\mu) \alpha_{\mu}^{(2)}(\tilde{\tau}_\mu)}{\alpha_{\mu}^{(2)}(\tilde{\omega}+\nu)} + \hat{N},$$  \hfill (E.16)

$$+ X_k^{(2),\nu} f(w_k^{-,\nu}, w_k^{+,\nu}) F(\tilde{\omega}, \tilde{\omega}) \frac{f(\tilde{\omega}, w_k^{-,\nu}) f(\tilde{\omega}, w_k^{-,\nu})}{f(\tilde{\omega}+1, w_k^{+,\nu}) f(\tilde{\omega}+1, w_k^{-,\nu})} \alpha_{\nu}(w_k^{-,\nu}) \times
$$

$$\sum_{\text{part}(\tilde{\omega})} \frac{f(w_k^{-,\nu}, \tilde{\sigma}_\nu) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)}{f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\sigma}_\mu, w_k^{-,\nu}) f(\tilde{\tau}_\mu, \tilde{\nu}_\mu)} \times
$$

$$G(\tilde{\sigma}, \tilde{\nu} \mid \tilde{\tau}, \tilde{\tau}_\mu) \prod_{\mu=1}^{N-1} \alpha_{\mu}^{(1)}(\tilde{\sigma}_\mu) \frac{\alpha_{\mu}^{(2)}(\tilde{\sigma}_\mu) \alpha_{\mu}^{(2)}(\tilde{\tau}_\mu)}{\alpha_{\mu}^{(2)}(\tilde{\omega}+\nu)} + \hat{N},$$

where \( \hat{N} \) is not depend on \( X_k^{(1),\nu} \) and \( X_k^{(2),\nu} \). Using the following form of the F-term (E.17)

$$F(\tilde{\omega}, \tilde{\omega}) = \frac{1}{f(w_k^{-,\nu}, w_k^{+,\nu}) f(w_k^{-,\nu}, w_k^{+,\nu})} \frac{f(\tilde{\omega}, w_k^{-,\nu}) f(\tilde{\omega}, w_k^{-,\nu})}{f(\tilde{\omega}+1, w_k^{+,\nu}) f(\tilde{\omega}+1, w_k^{-,\nu})} F(\tilde{\omega}, \tilde{\omega}),$$

and the Bethe equation

$$\alpha_{\nu}(w_k^{-,\nu}) = \frac{f(w_k^{-,\nu}, w_k^{+,\nu}) f(w_k^{-,\nu}, w_k^{+,\nu})}{f(w_k^{-,\nu}, w_k^{+,\nu}) f(w_k^{+,\nu}, w_k^{-,\nu})} \frac{f(\tilde{\omega}, w_k^{+,\nu}) f(\tilde{\omega}, w_k^{+,\nu})}{f(\tilde{\omega}, w_k^{+,\nu}) f(\tilde{\omega}, w_k^{+,\nu})} \frac{f(\tilde{\omega}+1, w_k^{-,\nu}) f(\tilde{\omega}+1, w_k^{-,\nu})}{f(\tilde{\omega}+1, w_k^{-,\nu}) f(\tilde{\omega}+1, w_k^{-,\nu})},$$  \hfill (E.18)
we obtain that

\[
\mathbb{N}(\bar{w}) \bigg|_{\bar{w}_k^{-\nu} = \pi^\nu(w_k^+)} \rightarrow \sum_{\text{part}(\bar{\omega})} f(\bar{w}_k^{-\nu}, \bar{w}_k^{+\nu}) f(w_k^{+\nu}, \bar{w}_k^{-\nu}) \times \\
\sum_{\text{part}(\bar{\omega})} \frac{f(w_k^{+\nu}, \bar{w}_k^{+\nu}) f(w_k^{-\nu}, \bar{w}_k^{-\nu})}{f(w_k^{+\nu}, \bar{w}_k^{+\nu}) f(w_k^{+\nu}, \bar{w}_k^{-\nu})} \times \\
\times G(\bar{\sigma}, \bar{\sigma}_i | \bar{\tau}_i, \bar{\tau}_i) \prod_{\nu=1}^{N-1} \frac{\alpha_\nu^{(1)}(\bar{\sigma}_i^{(1)}) \alpha_\nu^{(2)}(\bar{\sigma}_i^{(2)}) \alpha_\nu^{(1)}(\bar{\tau}_i^{(1)})}{\alpha_\nu^{(2)}(\bar{\omega}_i^{(2)})} + \tilde{N},
\]

where we also used the identities

\[
f(-u, -v) = f(v, u), \quad f(u, v + c) = f(u - c, v) = \frac{1}{f(v, u)}.
\]

This limit can be simplified as

\[
\mathbb{N}(\bar{w}) \bigg|_{\bar{w}_k^{-\nu} = \pi^\nu(w_k^+)} \rightarrow \sum_{\text{part}(\bar{\omega})} f(\bar{w}_k^{-\nu}, \bar{w}_k^{+\nu}) \times \\
\sum_{\text{part}(\bar{\omega})} \frac{f(w_k^{+\nu}, \bar{w}_k^{+\nu}) f(w_k^{-\nu}, \bar{w}_k^{-\nu})}{f(w_k^{+\nu}, \bar{w}_k^{+\nu}) f(w_k^{+\nu}, \bar{w}_k^{-\nu})} \times \\
\times G(\bar{\sigma}, \bar{\sigma}_i | \bar{\tau}_i, \bar{\tau}_i) \prod_{\nu=1}^{N-1} \frac{\alpha_\nu^{(1)}(\bar{\sigma}_i^{(1)}) \alpha_\nu^{(2)}(\bar{\sigma}_i^{(2)}) \alpha_\nu^{(1)}(\bar{\tau}_i^{(1)})}{\alpha_\nu^{(2)}(\bar{\omega}_i^{(2)})} + \tilde{N},
\]

where we introduced the modified \( \alpha \) variables as

\[
\begin{align*}
\alpha_\nu^{(1), \text{mod}_1}(u) &= \alpha_\nu^{(1)}(u) \frac{f(w_k^{+\nu}, u)}{f(u, w_k^{+\nu})}, \\
\alpha_{\nu+1}^{(1), \text{mod}_1}(u) &= \alpha_{\nu+1}^{(1)}(u) f(u, w_k^{+\nu}), \\
\alpha_{\nu-1}^{(1), \text{mod}_1}(u) &= \alpha_{\nu-1}^{(1)}(u) \frac{1}{f(w_k^{+\nu}, u)}, \\
\alpha_\nu^{(2), \text{mod}_1}(u) &= \alpha_\nu^{(2)}(u) \frac{f(w_k^{-\nu}, u)}{f(u, w_k^{+\nu})}, \\
\alpha_{\nu+1}^{(2), \text{mod}_1}(u) &= \alpha_{\nu+1}^{(2)}(u) f(u, w_k^{-\nu}), \\
\alpha_{\nu-1}^{(2), \text{mod}_1}(u) &= \alpha_{\nu-1}^{(2)}(u) \frac{1}{f(w_k^{-\nu}, u)},
\end{align*}
\]

(22)
and
\[ \alpha_{\nu}^{(2),mod_2}(u) = \alpha_{\nu}^{(2)}(u) \frac{f(w_k^+,\nu, u)}{f(u, w_k^{-}, \nu^*)}, \]
\[ \alpha_{\nu+1}^{(2),mod_2}(u) = \alpha_{\nu+1}^{(2)}(u) f(u, w_k^+, \nu), \]
\[ \alpha_{\nu-1}^{(2),mod_2}(u) = \alpha_{\nu-1}^{(2)}(u) \frac{1}{f(u, w_k^{-}, \nu)}, \]
\[ \alpha_{\nu}^{(1),mod_2}(u) = \alpha_{\nu}^{(1)}(u) \frac{f(w_k^-,\nu, u)}{f(u, w_k^{-}, \nu^*)}, \]
\[ \alpha_{\nu+1}^{(1),mod_2}(u) = \alpha_{\nu+1}^{(1)}(u) f(u, w_k^-, \nu), \]
\[ \alpha_{\nu-1}^{(1),mod_2}(u) = \alpha_{\nu-1}^{(1)}(u) \frac{1}{f(u, w_k^{-}, \nu)}, \]
\[ \text{and } \alpha_{\mu}^{(i),mod_4}(u) = \alpha_{\mu}^{(i),mod_2}(u) = \alpha_{\mu}(u) \text{ for } |\mu - \nu| > 1 \text{ and } i = 1, 2. \]

In the rhs of (E.21) we can recognize the sum rule of the normalized overlap (E.1) therefore the \( X_k^{(1),\nu} X_k^{(2),\nu} \) dependence simplifies as
\[ \mathcal{N}(\tilde{\omega}) \bigg|_{w_k^{-},\nu^* = \pi^*(w_k^{-},\nu)} \rightarrow X_k^{(1),\nu} \mathcal{N}(\tilde{\omega})^{mod_4} + X_k^{(2),\nu} \mathcal{N}(\tilde{\omega})^{mod_2} + \tilde{\mathcal{N}}. \] (E.24)

### E.2 Untwisted case

Let us continue with the untwisted case. Using the sum formula (5.17) and normalization (6.10), the normalized overlap reads as
\[ \mathcal{N}(\bar{\omega}) = F(\tilde{\omega}^+, \tilde{\omega}^-) \sum_{\text{part}(\bar{\omega})} G(\bar{\omega}, \bar{\nu}|\bar{f}, \bar{i}_n) \prod_{\nu=1}^{N-1} \alpha_{\nu}^{(1)}(\bar{\omega}) \alpha_{\nu}^{(2)}(\bar{\omega}), \] (E.25)

where
\[ F(\tilde{\omega}^+, \tilde{\omega}^-) = \begin{cases} \prod_{\nu=1}^{N-1} f(\tilde{\omega}^+, \nu) f(\tilde{\omega}^-, \nu)^2 & \text{for even } N, \\ \prod_{\nu=1}^{N-1} f(\tilde{\omega}^+, \nu) f(\tilde{\omega}^-, \nu) & \text{for odd } N, \end{cases} \] (E.26)

and
\[ G(\bar{s}, \bar{\nu}|\bar{f}, \bar{i}_n) = \prod_{k=1}^{N-1} f(\bar{\nu}^+, \bar{k}) f(\bar{\nu}^-, \bar{k}) f(\bar{i}_n^+, \bar{\nu}^+) f(\bar{i}_n^-, \bar{\nu}^-) Z(\pi^\alpha(\bar{s}), \bar{f}) Z(\bar{s}, \pi^\alpha(\bar{\nu})). \] (E.27)

We can repeat the calculation of the previous section for limit \( w_k^{+\nu} + w_k^{-\nu} \rightarrow 0 \) of the untwisted normalized overlap. The calculation is long but straightforward (there is nothing new in it, compared to the previous one). At the end we obtain the following formula
\[ \mathcal{N}(\bar{\omega}) \bigg|_{w_k^{-\nu} = \pi^\alpha(w_k^{-\nu})} \rightarrow X_k^{(1),\nu} \mathcal{N}(\tilde{\omega})^{mod_4} + X_k^{(2),\nu} \mathcal{N}(\tilde{\omega})^{mod_2} + \tilde{\mathcal{N}}, \] (E.28)

where \( \tilde{\mathcal{N}} \) does not depend on \( X_k^{(1),\nu}, X_k^{(2),\nu} \) and \( \mathcal{N}(\tilde{\omega})^{mod_4}, \mathcal{N}(\tilde{\omega})^{mod_2} \) contain the following modified \( \alpha \)-s:
\[ \alpha_{\nu}^{(1),mod_4}(u) = \alpha_{\nu}^{(1)}(u) \frac{f(w_k^{+\nu}, u)}{f(u, w_k^{-\nu})}, \]
\[ \alpha_{\nu+1}^{(1),mod_4}(u) = \alpha_{\nu+1}^{(1)}(u) f(u, w_k^{+\nu}), \]
\[ \alpha_{\nu-1}^{(1),mod_4}(u) = \alpha_{\nu-1}^{(1)}(u) \frac{1}{f(u, w_k^{-\nu})}, \]
\[ \alpha_{\nu}^{(2),mod_4}(u) = \alpha_{\nu}^{(2)}(u) \frac{f(w_k^{-\nu}, u)}{f(u, w_k^{-\nu})}, \]
\[ \alpha_{\nu+1}^{(2),mod_4}(u) = \alpha_{\nu+1}^{(2)}(u) f(u, w_k^{-\nu}), \]
\[ \alpha_{\nu-1}^{(2),mod_4}(u) = \alpha_{\nu-1}^{(2)}(u) \frac{1}{f(u, w_k^{-\nu})}, \] (E.29)
and

\[ \alpha_{N-\nu}^{(2),\text{mod}}(u) = \alpha_{N-\nu}^{(2)} \frac{f(w_{k}^{-\nu}, u)}{f(u, w_{k}^{-\nu})}, \quad \alpha_{N-\nu}^{(1),\text{mod}}(u) = \alpha_{N-\nu}^{(1)} \frac{f(w_{k}^{-\nu}, u)}{f(u, w_{k}^{-\nu})}, \]

\[ \alpha_{N-\nu+1}^{(2),\text{mod}}(u) = \alpha_{N-\nu+1}^{(2)}(u)f(u, w_{k}^{-\nu}), \quad \alpha_{N-\nu+1}^{(1),\text{mod}}(u) = \alpha_{N-\nu+1}^{(1)}(u)f(u, w_{k}^{-\nu}), \quad \alpha_{N-\nu+1}^{(1),\text{mod}}(u) = \alpha_{N-\nu+1}^{(1)}(u) \frac{1}{f(w_{k}^{-\nu}, u)}, \]

(6.10)

and \( \alpha_{\mu}^{(i),\text{mod}}(u) = \alpha_{\mu}^{(i),\text{mod}}(u) = \alpha_{\mu}(u) \) for \( |\mu - \nu| > 1 \) and \( |\mu + \nu - N| > 1 \).

F Normalized overlaps and the Korepin criteria

In this section we show that the normalized overlaps \( \mathbb{N}(\bar{w}) \) satisfy the Korepin criteria. Let us start with the twisted case. The criteria (i) is obviously true since the Bethe states and the normalization factors in (6.10) are symmetric over the replacement \( w_{j}^{+\mu} \leftrightarrow w_{j}^{-\mu} \). The properties (ii),(iv) follow from (E.24). The proof that the normalized crossgap overlaps satisfy the criteria (v) is the same as the proofs for the scalar products [26] and the boundary state overlaps [33]. The proof is based on the fact that on-shell overlaps are non-zero only for Bethe states with pair structure, i.e., the on-shell overlaps vanish for general Bethe states.

Only the criteria (iii) remains to prove. We can substitute to the sum rule (E.1). For the the magnon state \( \bar{w}^{\nu} = \{w_{1}^{\nu}, w_{2}^{\nu}\} \) and \( \bar{w}^{\mu} = \emptyset \) for \( \mu \neq \nu \), the sum formula simplifies as

\[ \mathbb{N}(\{w_{1}^{\nu}, w_{2}^{\nu}\}) = \frac{1}{f(w_{2}^{\nu}, w_{1}^{\nu})} G(\{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}, \emptyset) + \frac{1}{f(w_{2}^{\nu}, w_{1}^{\nu})} G(\emptyset, \{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}) \alpha_{\nu}^{(1)}(w_{1}^{\nu}) \alpha_{\nu}^{(2)}(w_{2}^{\nu}) + \frac{1}{f(w_{2}^{\nu}, w_{1}^{\nu})} G(\{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}, \emptyset) \frac{\alpha_{\nu}^{(2)}(w_{2}^{\nu})}{\alpha_{\nu}^{(2)}(w_{1}^{\nu})} + \frac{1}{f(w_{2}^{\nu}, w_{1}^{\nu})} G(\emptyset, \{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}) \alpha_{\nu}^{(1)}(w_{2}^{\nu}) \alpha_{\nu}^{(2)}(w_{2}^{\nu}), \]

(F.1)

and

\[ G(\{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}, \emptyset) = f(w_{2}^{\nu}, w_{1}^{\nu}) Z(-w_{1}^{\nu} - \nu c) w_{2}^{\nu}) Z(\emptyset, \emptyset) = f(w_{2}^{\nu}, w_{1}^{\nu}) g(w_{2}^{\nu}, -w_{1}^{\nu} - \nu c), \]

(F.2)

\[ G(\emptyset, \{w_{1}^{\nu}\}, \emptyset, \{w_{2}^{\nu}\}) = f(w_{2}^{\nu}, w_{1}^{\nu}) Z(\emptyset, \emptyset) Z(w_{2}^{\nu}) - w_{1}^{\nu} - \nu c) = f(w_{2}^{\nu}, w_{1}^{\nu}) g(-w_{1}^{\nu} - \nu c, w_{2}^{\nu}), \]

(F.3)

where we used that

\[ Z(\emptyset, \emptyset) = 1, \quad Z(\{s\}, \{t\}) = g(t, s). \]

(F.4)

Substituting back we obtain that

\[ \mathbb{N}(\{w_{1}^{\nu}, w_{2}^{\nu}\}) = g(w_{2}^{\nu}, -w_{1}^{\nu} - \nu c)(1 - \alpha_{\nu}^{(1)}(w_{1}^{\nu}) \alpha_{\nu}^{(2)}(w_{2}^{\nu}))+ \frac{f(w_{1}^{\nu}, w_{2}^{\nu}) \alpha_{\nu}^{(2)}(w_{2}^{\nu})}{f(w_{1}^{\nu}, w_{2}^{\nu}) \alpha_{\nu}^{(2)}(w_{1}^{\nu})} g(w_{1}^{\nu}, -w_{2}^{\nu} - \nu c)(1 - \alpha_{\nu}^{(2)}(w_{1}^{\nu}) \alpha_{\nu}^{(1)}(w_{2}^{\nu})). \]

(F.5)

Taking the limit \( w_{2}^{\nu} \to -w_{1}^{\nu} - \nu c \) we obtain that

\[ \mathbb{N}(\{w_{1}^{\nu}, w_{2}^{\nu}\}) = X_{k}^{(1),\nu} + \frac{f(w_{1}^{\nu}, w_{2}^{\nu})}{f(w_{1}^{\nu}, w_{2}^{\nu}) \alpha_{\nu}(w_{1}^{\nu})} X_{k}^{(2),\nu}, \]

(F.6)
where we used (E.9). Now we can take the on-shell limit where the following Bethe equation is satisfied

$$\alpha_\nu(w_1^\nu) = \frac{f(w_1^\nu, w_2^\nu)}{f(w_2^\nu, w_1^\nu)}.$$  

(F.7)

Substituting back we just obtained that

$$N(\{w_1^\nu, w_2^\nu\}) = X_k^{(1),\nu} + X_k^{(2),\nu},$$  

(F.8)

therefore we just proved the last criteria (iii). Since the normalized on-shell overlaps satisfy the Korepin criteria, they can be written as a Gaudin-like determinant:

$$N(\bar{w}) = \det G_+.$$  

(F.9)

The proof in the untwisted case is completely analogous.

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