USING THE SWING LEMMA AND $C_1$-DIAGRAMS FOR CONGRUENCES OF PLANAR SEMIMODULAR LATTICES

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Abstract. A planar semimodular lattice $K$ is slim if $M_3$ is not a sublattice of $K$. In a recent paper, G. Czédli found four new properties of congruence lattices of slim, planar, semimodular lattices, including the No Child Property. Let $\mathcal{P}$ be the ordered set of join-irreducible congruences of $K$. Let $x, y, z \in \mathcal{P}$ and let $z$ be a maximal element of $\mathcal{P}$. If $x \neq y$ and $x, y \prec z$ in $\mathcal{P}$, then there is no element $u$ of $\mathcal{P}$ such that $u \prec x, y$ in $\mathcal{P}$.

We are applying my Swing Lemma, 2015, and a type of standardized diagrams of Czédli’s, to verify his four properties.

1. Introduction

Let $K$ be a planar semimodular lattice. We call the lattice $K$ slim if $M_3$ is not a sublattice of $K$. In the paper [17, Theorem 1.5], I found a property of congruences of slim, planar, semimodular lattices. In the same paper (see also Problem 24.1 in G. Grätzer [16]), I proposed the following:

Problem. Characterize the congruence lattices of slim planar semimodular lattices.

G. Czédli [4, Corollaries 3.4, 3.5, Theorem 4.3] found four new properties of congruence lattices of slim, planar, semimodular lattices.

Theorem. Let $K$ be a slim, planar, semimodular lattice with at least three elements and let $\mathcal{P}$ be the ordered set of join-irreducible congruences of $K$.

(i) Partition Property: The set of maximal elements of $\mathcal{P}$ can be divided into the disjoint union of two nonempty subsets such that no two distinct elements in the same subset have a common lower cover.

(ii) Maximal Cover Property: If $v \in \mathcal{P}$ is covered by a maximal element $u$ of $\mathcal{P}$, then $u$ is not the only cover of $v$.

(iii) No Child Property: Let $x \neq y \in \mathcal{P}$ and let $u$ be a maximal element of $\mathcal{P}$. If $x, y \prec u$ in $\mathcal{P}$, then there is no element $z \in \mathcal{P}$ such that $z \prec x, y$ in $\mathcal{P}$.

(iv) Four-Crown Two-pendant Property: There is no cover-preserving embedding of the ordered set $\mathcal{R}$ in Figure 2 into $\mathcal{P}$ satisfying the property: any maximal element of $\mathcal{R}$ is a maximal element of $\mathcal{P}$.

In this paper, we will provide a short and direct proof of this theorem using only the Swing Lemma and $C_1$-diagrams, see Section 3.

Date: June 6, 2021.

2000 Mathematics Subject Classification. 06C10.

Key words and phrases. Rectangular lattice, slim planar semimodular lattice, congruence lattice.
Figure 1. The Four-crown Two-pendant ordered set $R$ with notation; the covering $S_7$ sublattice with edge and element notation.

Outline. Section 2 provides the motivation for Czédli's Theorem. Section 3 provides the tools we need: the Swing Lemma, $C_1$-diagrams, and forks. Section 4 proves the Partition Property, Section 5 does the Maximal Cover Property, while Section 6 verifies the No Child Property. Finally, The Four-Crown Two-pendant Property is proved in Section 7.

2. Motivation

In my paper [27] with H. Lakser and E. T. Schmidt, we proved that every finite distributive lattice $D$ can be represented as the congruence lattice of a semimodular lattice $L$. To our surprise, the semimodular lattice $K$ we constructed was planar.

G. Grätzer and E. Knapp [22]–[26] started the study of planar semimodular lattices. I continued it with my “Notes on planar semimodular lattices” series (started with Knapp): [13], [28] (with T. Wares), [6] (with G. Czédli), [19], [20]. See also G. Czédli and E. T. Schmidt [10] and G. Czédli [1]–[5].

A major subchapter of the theory of planar semimodular lattices started with the observation that in the construction of the lattice $K$, as in the first paragraph of this section, $M_3$ sublattices play a crucial role. It was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in G. Grätzer [17]). In [17], I found the first necessary condition and G. Czédli [2] proved that this condition is not sufficient (see also my related papers [15] and [19]).

A number of papers developed tools to tackle this problem: the Swing Lemma (G. Grätzer [14]), trajectory coloring (G. Czédli [1]), special diagrams (G. Czédli [3]), lamps (G. Czédli [4]). Some of these results require long proofs. The proof of the trajectory coloring theorem is just shy of 20 pages, while the basic theory of lamps and its application to Theorem 1 is 23 pages.

There are a number of surveys of this field, see the book chapters G. Czédli and G. Grätzer [7] and G. Grätzer [11] in G. Grätzer and F. Wehrung, eds. [29]. My presentation [21] provides a gentle review for the background of this topic.

3. The tools we need

Most basic concepts and notation not defined in this paper are available in Part I of the book [16], see

https://www.researchgate.net/publication/299594715

arXiv:2104.06539
It is available to the reader. We will reference it, for instance, as [CFL2, page 52]. In particular, we use the notation $C \sim D$, $C \uparrow D$, and $C \downarrow D$ for perspectivity, up-perspectivity, and down-perspectivity, respectively. As usual, for planar lattices, a prime interval (or covering interval) is called an edge. For a finite lattice $K$ and a finite ordered set $R$, a cover-preserving embedding $\varepsilon : R \to K$ is an embedding mapping edges of $R$ to edges of $K$. We define a cover-preserving sublattice similarly.

For the lattice $S_7$ of Figure 1, we need a variant: an $S_7$ sublattice (isomorphic to $S_7$) is a peak sublattice if the three top edges ($L$, $M$, and $R$ in Figure 1) are edges in $K$.

By G. Grätzer and E. Knapp [25], every slim, planar, semimodular lattice $K$ has a congruence-preserving extension (see [CFL2, page 43]) $\hat{K}$ to a slim rectangular lattice. Any of the properties (i)–(iv) holds for $K$ iff it holds for $\hat{K}$. Therefore, in the rest of this paper, we can assume that $K$ is a slim rectangular lattice, simplifying the discussion.

3.1. Swing Lemma. For an edge $E$ of an SPS lattice $K$, let $E = [0_E, 1_E]$ and define $\text{col}(E)$, the color of $E$, as $\text{con}(E)$, the (join-irreducible) congruence generated by collapsing $E$ (see [CFL2, Section 3.2]). We write $P$ for $J(\text{Con} K)$, the ordered set of join-irreducible congruences of $K$.

As in my paper [14], for the edges $U, V$ of an SPS lattice $K$, we define a binary relation: $U \text{ swings to } V$, in formula, $U \swungto V$, if $1_U = 1_V$, the element $1_U = 1_V$ of $K$ covers at least three elements, and $0_V$ is neither the left-most nor the right-most element covered by $1_U = 1_V$; if also $0_U$ is such, then the swing is interior, otherwise, it is exterior, denoted by $U \swungto V$ and $U \exswungto V$, respectively.

Swing Lemma [G. Grätzer [14]]. Let $K$ be an SPS lattice and let $U$ and $V$ be edges in $K$. Then $\text{col}(V) \leq \text{col}(U)$ iff there exists an edge $R$ such that $U$ is up-perspective to $R$ and there exists a sequence of edges and a sequence of binary relations

$$R = R_0 \varrho_1 R_1 \varrho_2 \cdots \varrho_n R_n = V,$$

where each relation $\varrho_i$ is $\downarrow$ (down-perspective) or $\uparrow$ (swing). In addition, this sequence also satisfies

$$1_{R_0} \geq 1_{R_1} \geq \cdots \geq 1_{R_n}.$$

The following statements are immediate consequences of the Swing Lemma, see my papers [14] and [18].

Corollary 1. We use the assumptions of the Swing Lemma.

(i) The equality $\text{col}(U) = \text{col}(V)$ holds in $P$ iff there exist edges $S$ and $T$ in $K$, such that

$$U \uparrow S, \ S \in T, \ T \downarrow V.$$

(ii) Let us further assume that the element $0_U$ is meet-irreducible. Then the equality $\text{col}(U) = \text{col}(V)$ holds in $P$ iff there exists an edge $T$ such that $U \in T \downarrow V$.

(iii) If the lattice $K$ is rectangular and $U$ is on the upper boundary of $K$, then the equality $\text{col}(U) = \text{col}(V)$ holds in $P$ iff $U \downarrow V$.

Note that in (i) the edges $S, T, U, V$ need not be distinct, so we have as special cases $U = V$, $U \sim V$, $S = T$, and others.
Corollary 2. We use the assumptions of the Swing Lemma.

(i) The covering \( \text{col}(V) \prec \text{col}(U) \) holds in \( P \) iff there exist edges \( R_1, \ldots, R_4 \) in \( K \), such that

\[
U \uparrow R_1, \ R_1 \downarrow R_2, \ R_2 \uparrow R_3, \ R_3 \downarrow R_4, \ R_4 \downarrow V.
\]

(ii) If the element \( 0_U \) is meet-irreducible, then the covering \( \text{col}(V) \prec \text{col}(U) \) holds in \( P \) iff there exist edges \( S, T \) in \( K \), so that

\[
U \downarrow S \uparrow T \downarrow V.
\]

Corollary 3. Let \( K \) be a slim rectangular lattice, let \( U \) and \( V \) be edges in \( K \), and let \( U \) be in the upper-left boundary of \( K \).

(i) The covering \( \text{col}(V) \prec \text{col}(U) \) holds in \( P \) iff there exist edges \( S, T \) in \( K \), such that

\[
(1) \quad U \downarrow S \uparrow T \downarrow V.
\]

(ii) Define the element \( t = 1_S = 1_T \in K \) and let \( S = E_1, E_2, \ldots, E_n = W \) enumerate, from left to right, all the edges \( E \) of \( K \) with \( 1_E = t \). Then

\[
(2) \quad \text{col}(S) \neq \text{col}(W),
\]

\[
(3) \quad \text{col}(E_2) = \cdots = \text{col}(E_{n-1}) = \text{col}(T),
\]

\[
(4) \quad \text{col}(T) \prec \text{col}(S), \text{col}(W).
\]

Corollary 4. Let the edge \( U \) be on the upper edge of the rectangular lattice \( K \). Then \( \text{col}(U) \) is a maximal element of \( P \).

The converse of this statement is stated in Corollary 8.

3.2. \( \mathcal{C}_1 \)-diagrams. In the diagram of a planar lattice \( K \), a normal edge (line) has a slope of 45° or 135°. If it is the first, we call it a normal-up edge (line), otherwise, a normal-down edge (line). Any edge of slope strictly between 45° and 135° is steep.

Definition 5. A diagram of an rectangular lattice \( K \) is a \( \mathcal{C}_1 \)-diagram if the middle edge of any covering \( S_7 \) is steep and all other edges are normal.

This concept was introduced in G. Czédli [3, Definition 5.3(B)], see also G. Czédli [4, Definition 2.1] and G. Czédli and G. Grätzer [8, Definition 3.1]. The following is the existence theorem of \( \mathcal{C}_1 \)-diagrams in G. Czédli [3, Theorem 5.5].

Theorem 6. Every rectangular lattice lattice \( K \) has a \( \mathcal{C}_1 \)-diagram.

See the illustrations in this paper for examples of \( \mathcal{C}_1 \)-diagrams. For a short and direct proof for the existence of \( \mathcal{C}_1 \)-diagrams, see my paper [20].

In this paper, \( K \) denotes a slim rectangular lattice with a fixed \( \mathcal{C}_1 \)-diagram and \( P \) is the ordered set of join-irreducible congruences of \( K \).

Let \( C \) and \( D \) be maximal chains in an interval \([a, b]\) of \( K \) such that \( C \cap D = \{a, b\} \). If there is no element of \( K \) between \( C \) and \( D \), then we call \( C \cup D \) a cell. A four-element cell is a 4-cell. Opposite edges of a 4-cell are called adjacent. Planar semi-modular lattices are 4-cell lattices, that is, all of its cells are 4-cells, see G. Grätzer and E. Knapp [22, Lemmas 4, 5] and [CFL2, Section 4.1] for more detail.

The following statement illustrates the use of \( \mathcal{C}_1 \)-diagrams.
**Lemma 7.** Let $K$ be a slim rectangular lattice $K$ with a fixed $C_1$-diagram and let $X$ be a normal-up edge of $K$. Then $X$ is up-perspective either to an edge in the upper-left boundary of $K$ or to a steep edge.

**Proof.** If $X$ is not steep nor it is in the upper-left boundary of $K$, then there is a 4-cell $C$ whose lower-right edge is $X$. If the upper-left edge is steep or it is in the upper-left boundary, then we are done. Otherwise, we proceed the same way until we reach a steep edge or an edge the upper-left boundary. □

**Corollary 8.** Let the edge $U$ be on the upper edge of $K$. Then $\text{col}(U)$ is a maximal element of $P$. Conversely, if $u$ is a maximal element of $P$, then there is an edge $U$ on the upper edge of $K$ so that $\text{col}(U) = u$.

### 3.3. Trajectories

G. Czédli and E. T. Schmidt [9] introduced a trajectory in $K$ as a maximal sequence of consecutive edges, see also [CFL2, Section 4.1]. The top edge $T$ of a trajectory is either in the upper boundary of $K$ or it is steep by Lemma 7. For such an edge $T$, we denote by $\text{traj}(T)$ the trajectory with top edge $T$.

By G. Grätzer and E. Knapp [22, Lemma 8], an element $a$ in an SPS lattice $K$ has at most two covers. Therefore, a trajectory has at most one top edge and at most one steep edge. So we conclude the following statement.

**Lemma 9.** Let $K$ be a slim rectangular lattice $K$ with a fixed $C_1$-diagram. Let $X$ and $Y$ be distinct steep edges of $K$. Then $\text{traj}(X)$ and $\text{traj}(Y)$ are disjoint.

### 4. The Partition Property

First, we verify the Partition Property for the slim rectangular lattice $K$ and with a fixed $C_1$-diagram. We start with a lemma.

**Lemma 10.** Let $X$ and $Y$ be distinct edges on the upper-left boundary of $K$. Then there is no edge $Z$ of $K$ such that $\text{col}(Z) \prec \text{col}(X), \text{col}(Y)$.

**Proof.** By way of contradiction, let $Z$ be an edge such that $\text{col}(Z) \prec \text{col}(X), \text{col}(Y)$. Since $X$ and $Y$ are on the upper-left boundary, Corollary 3(i) applies. Therefore, there exist normal-up edges $S_X, S_Y$ and steep edges $T_X, T_Y$ such that

$$X \sim S_X \text{ ex } T_X, \quad Y \sim S_Y \text{ ex } T_Y, \quad Z \in \text{traj}(T_X) \cap \text{traj}(T_Y).$$

By Lemma 9 the third formula implies that $T_X = T_Y$ and $xo X = Y$, contrary to the assumption. □

By Corollary 8 the set of maximal elements of $P$ is the same as the set of colors of edges in the upper boundaries. We can partition the set of edges in the upper boundaries into the set of edges $\mathcal{L}$ in the upper-left boundary and the set of edges $\mathcal{R}$ in the upper-right boundary. If $X$ and $Y$ are distinct edges in $\mathcal{L}$, then there is no edge $Z$ of $K$ such that $\text{col}(Z) \prec \text{col}(X), \text{col}(Y)$ by Lemma 10. By symmetry, this verifies the Partition Property.

### 5. The Maximal Cover Property

Next, we verify the Maximal Cover Property for the slim rectangular lattice $K$ and with a fixed $C_1$-diagram.

Let $x \in P$ be covered by a maximal element $u$ of $P$ in $K$. By Corollary 8, we can choose an edge $U$ of color $u$ on the upper boundary of $K$, by symmetry, on the upper-left boundary of $K$. By Corollary 3(ii), we can choose the edges $S, T$ in $K$
so that $U \cong S \cup T$, $\text{col}(S) = u$, and $\text{col}(T) = x$. By Corollary 3(ii), specifically, by equations (2) and (4), we have $x \prec u, \text{col}(W)$ and $u \neq \text{col}(W)$, verifying the Maximal Cover Property.

6. The No Child Property

In this section, we verify the No Child Property for the slim rectangular lattice $K$ and with a fixed $\mathcal{C}_1$-diagram.

Let $x, y, z, u \in P$ with $x \neq y \in P$, let $u$ be a maximal element of $P$, and let $x, y \prec u$ in $P$. By way of contradiction, let us assume that there is an element $z \in P$ such that $z \prec x, y$ in $P$.

By Corollary 8, the element $u$ colors an edge $U$ on the upper boundary of $K$, say, in the upper-left boundary. By Corollary 2(i), for $z \prec x \in P$, we get a peak sublattice $S_7$ in which the middle edge $Z$ is colored by $z$ and upper-left edge $X$ is colored by $x$, or symmetrically. The upper-right edge $Y$ must have color $y$.

Now we apply Corollary 3(ii) to the edge $U$ and middle edge $Z$ of the peak sublattice $S_7$, obtaining that $U \cong Y \cup Z$, in particular, $U \cong Y$. This is a contradiction, since $U$ is normal-up and $Y$ is normal-down.

7. The Four-Crown Two-pendant Property

Finally, we verify the Four-Crown Two-pendant Property for the slim rectangular lattice $K$ and with a fixed $\mathcal{C}_1$-diagram.

By way of contradiction, assume that the ordered set $\mathcal{R}$ of Figure 1 is a cover-preserving ordered subset of $P$, where $a, b, c, d$ are maximal elements of $P$. By Corollary 8, there are edges $A, B, C, D$ on the upper boundary of $K$, so that $\text{col}(A) = a$, $\text{col}(B) = b$, $\text{col}(C) = c$, $\text{col}(D) = d$. By left-right symmetry, we can assume that the edge $A$ is on the upper-left boundary of $K$. Since $p \prec a, b$ in $P$, it follows from Lemma 10 that the edge $B$ is on the upper-right boundary of $K$, and so is $D$. Similarly, $C$ is on the upper-left boundary of $K$.

There are four cases, (i) $C$ is below $A$ and $B$ is below $D$; (ii) $C$ is below $A$ and $D$ is below $B$; and so on. The first two are illustrated in Figure 2.

Figure 2. Illustrating the proof of The Four-Crown Two-pendant Property
We consider the first case. By Corollary 2(ii), there is a peak sublattice $S_7$ with middle edge $P$ (as in the first diagram of Figure 2) so that $A$ and $B$ are down-perspective to the upper-left edge and the upper-right edge of this peak sublattice, respectively. We define, similarly, the edge $Q$ for $C$ and $B$, the edge $S$ for $A$ and $D$, the edge $R$ for $C$ and $D$, and the edge $U$ for $R$ and $P$.

The ordered set $R$ is a cover-preserving subset of $P$, so we get, similarly, the peak sublattice $S_7$ with middle edge $U$. Finally, $v \prec q, s$ in $R$, therefore, there is a peak sublattice $S_7$ with middle edge $V$ with upper-left edge $V_l$ and the upper-right edge $V_r$ so that $S \cong V_l$ and $S \cong V_r$, or symmetrically.

This concludes the proof of the Four-Crown Two-pendant Property and of Czédli’s Theorem.

Of course, the diagrams in Figure 2 are only illustrations. The grid could be much larger, the edges $A, C$ and $B, D$ may not be adjacent, and there may be lots of other elements in $K$. However, our argument does not utilize the special circumstances in the diagrams.

The second case is similar, except that we get the edge $V$ and cannot get the edge $U$. The third and fourth cases follow the same way.

Appendix A. Two more illustrations for Section 7

![Figure 3. Two more illustrations for Section 7](image)

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