ON CLASSIFICATION OF DISCRETE, SCALAR-VALUED POISSON BRACKETS

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ABSTRACT. We address the problem of classifying discrete differential-geometric Poisson brackets (dDGPBs) of any fixed order on target space of dimension 1. It is proved that these Poisson brackets (PBs) are in one-to-one correspondence with the intersection points of certain projective hypersurfaces. In addition, they can be reduced to cubic PB of standard Volterra lattice by discrete Miura-type transformations. Finally, improving a consolidation lattice procedure, we obtain new families of non-degenerate, vector-valued and first order dDGPBs, which can be considered in the framework of admissible Lie-Poisson group theory.

Keywords: Discrete Poisson brackets, discrete Miura transformations, Lie-Poisson groups.

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1. INTRODUCTION

In this paper we deal with the following class of local Poisson brackets (PBs)

\[
\begin{align*}
\{u^i_n, u^j_{n+k}\}_M &= g^{ij}_k(u_n, \ldots, u_{n+k}), & 0 \leq k \leq M, \\
\{u^i_n, u^j_{n+k}\}_M &\equiv 0, & k > M,
\end{align*}
\]

defined on the phase space of infinite sequences

\[u : \mathbb{Z} \rightarrow \mathcal{M}^N, \quad n \mapsto u_n = (u^i_n)_{i=1,\ldots,N}\]

with values in the target manifold \(\mathcal{M}^N\) of dimension \(N\). The integer number \(M\), called the order of the PB, can be seen as the locality radius, i.e. the radius of the maximum local interaction between neighboring lattice variables. These PBs have been introduced by B. Dubrovin in [3] (see also A. Ya. Mal’tsev, [10]), as a
discretization of the differential geometric Poisson brackets (DGPBs), defined on the loop space $L(M) \doteq \{S^1 \to M\}$ by the formula

$$\{u^i(x), u^j(y)\}_M = \sum_{k=0}^{M} g^{ij}_k(u(x), u_x(x), \ldots, u^{(k)}(x)) \delta^{(M-k)}(x-y),$$

where $i, j = 1, \ldots, N$ and the functions $g^{ij}_k$ are graded-homogeneous polynomials. See for details the papers by S. Novikov and B. Dubrovin [4], [5].

PBs of type (1.1) are associated with lattice Hamiltonian equations of the following form

$$\dot{u}^i_n = \{u^i_n, H[u]\}_M = \sum_{m \in \mathbb{Z}} \sum_{p=1}^{N} \left\{u^i_n, u^p_m\right\}_M \frac{\delta H[u]}{\delta u^p_m},$$

where $H[u] = \sum_{m \in \mathbb{Z}} h(u_m, \ldots, u_{m+K})$ for some integer $K \geq 0$ and the function $h$ is defined on a finite interval of the lattice. In addition, we define the formal variational derivative as

$$\frac{\delta H[u]}{\delta u^p_n} = \frac{\partial}{\partial u^p_n} \left(1 + T^{-1} + \ldots + T^{-K}\right) h(u_n, \ldots, u_{n+K}),$$

where $T$ is the standard shift operator, satisfying

$$T^r h(u_n, \ldots, u_{n+K}) = h(u_{n+r}, \ldots, u_{n+r+K}),$$

for any integer $r$. The local Poisson structures of many fundamental integrable systems, such as the Volterra lattices, the Toda lattices, the Bogoyavlensky lattices (see Yu.B. Suris, [12]) belong to the class (1.1). However, the theory of such discrete PBs is much less developed than the corresponding of DGPBs (1.2) (see the survey of O. Mokhov [11] and references therein).

A classification of first order ($M = 1$) PBs (1.1) has been provided in [3], whereas it seems that the higher order PBs have not been studied yet. Moreover, to the best of our knowledge, in the literature there are no examples of PBs (1.1) of order $M > 2$. In [3], a correspondence between the following first order PBs

$$\{u^i_n, u^j_{n+1}\}_1 = g^{ij}_1(u_n, u_{n+1})$$
$$\{u^i_n, u^j_n\}_1 = g^{ij}_0(u_n),$$

and certain Lie-Poisson groups was discovered. More precisely, if the matrix $g^{ij}_1$ is non-singular (i.e. $\det g^{ij}_1(u_n, u_{n+1}) \neq 0$) the PBs (1.4) are induced by admissible Lie-Poisson group structures on the target manifold $M^N$ (see Definition 5.3 below).

Performing a consolidation lattice procedure, which is obtained by defining new variables of a larger target manifold by the formulas $v^i_{n+p} = u^i_{nM+p}$, $p = 0, \ldots, M - 1$, one can reduce any PB (1.1) to the form (1.4). However, this procedure leaves some unsolved questions:

(i) what are the relations between PBs (1.1) of order $M > 1$ and admissible Lie-Poisson groups associated to their consolidations?
(ii) how to produce examples of such admissible Lie-Poisson groups?

In the present paper, in order to give some partial answers, we classify scalar-valued \((N = 1)\) PBs \((1.1)\) of any positive order \(M\),

\[
\{u_n, u_{n+k}\}_M = g_k(u_n, \ldots, u_{n+k}), \quad 1 \leq k \leq M.
\]

First, we observe that PBs of type \((1.1)\) are invariant under local change of variable \((1.6)\)

\[
u_n^i \mapsto v_n^i = v^i(u_n), \quad i = 1, \ldots, N,
\]

where the coefficients \(g_k^{ij}\) transform according to the formula

\[
g_k^{ij}(u_n, \ldots, u_{n+k}) \mapsto \sum_{p,q=1}^N \frac{\partial v^i}{\partial u_n^p}(u_n) g_k^{pq}(u_n, \ldots, u_{n+k}) \frac{\partial v^j}{\partial u_n^{q+k}}(u_{n+k}).
\]

Two local PBs will be therefore considered equivalent if they can be related by a change of coordinates of type \((1.6)\).

Let us be more precise about the classification of scalar-valued PBs \((1.5)\). The coefficients \(g_k(u_n, \ldots, u_{n+k})\) satisfy the system of \(M^2\) bi-linear PDEs imposed by the Jacobi identity (see Section 2). It turns out that any PB \((1.5)\) is characterized by his leading order function \(g_M(u_n, \ldots, u_{n+M})\), according to the following results.

Lemma 1.1. For any PB of the form \((1.5)\), there exist a set of coordinates (canonical coordinates) and an integer \(\alpha > 0\), such that the leading order reduces to the form

\[
g_M(u_n, \ldots, u_{n+M}) = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}), \quad \xi = M - 2\alpha,
\]

where the function \(f^\xi\) is either constant or given by the formula

\[
f_n^\xi = f^\xi(u_n, \ldots, u_{n+\xi}) = \exp(z_n).
\]

Here,

\[
z_n = \sum_{i=0}^{\xi} \tau_i u_{n+i},
\]

and the parameters \(\tau_i, i = 0, \ldots, \xi\) satisfy a system of homogeneous polynomial equations (see below Theorem 2.7).

In the case when the leading coefficient \(g_M(u_n, \ldots, u_{n+M})\) is constant, it is not difficult to prove that all the other coefficients \(\{g_k\}_{k=1, \ldots, M-1}\) are also constant. The general, non-constant case is described by the following

Theorem 1.2. The coefficients \(g_k\) of a non-constant PB \((1.5)\) are given, in the canonical coordinates, by suitable linear combinations of the shifted generating function \(f^\xi\), according to the following formula

\[
g_{\alpha+\xi+p}(u_n, \ldots, u_{n+\alpha+\xi+p}) = \left(\min(\alpha+p, \alpha) \sum_{s=\max(0, p)}^{\min(\alpha+p, \alpha)} \lambda_p^s T^s\right) f^\xi(u_n, \ldots, u_{n+\xi}),
\]
where \( p = -\alpha, \ldots, \alpha \), and the constants \( \lambda \)'s can be expressed explicitly in terms of the parameters \( \tau \)'s (see equation (2.8), below).

This Theorem provides a complete classification of PBs of type (1.5). Note that the functional form of the coefficients \( g_k \) is fixed by the choice of a finite number of parameters \( \tau \).

In addition to the above results, we prove a Darboux-type theorem for PBs (1.5). This is done by considering the change of variable (1.7), that is a generalization of the local one (1.6). Notice that the formula (1.7) can be thought as a discrete analogue of Miura transformations, studied in the Hamiltonian PDEs theory (see [6]).

Splitting all the variables into \( \alpha + \xi \) families according to

\[
\left\{ v^{(p)}_n, v^{(p)}_{n+2}\right\} = \tau_0 \tau_\xi \exp(v^{(p)}_{n+1})
\]

\[
\left\{ v^{(p)}_n, v^{(p)}_{n+1}\right\} = \tau_0 \tau_\xi \left[\exp(v^{(p)}_n) + \exp(v^{(p)}_{n+1})\right],
\]

by direct computation, we obtain that any non-constant PB (1.5) in the \( z \)-coordinates can be reduced to the following simple form

\[
\{v_n^{(p)}, v_{n+2}^{(p)}\} = \tau_0 \tau_\xi \exp(v^{(p)}_{n+1})
\]

\[
\{v_n^{(p)}, v_{n+1}^{(p)}\} = \tau_0 \tau_\xi \left[\exp(v^{(p)}_n) + \exp(v^{(p)}_{n+1})\right],
\]

that are \( \alpha + \xi \) copies of cubic Volterra PB (see Theorem 3.2 below).

We consider next compatible pairs of PBs of type (1.5). Recall that a pair of PBs \( (P_1, P_2) \) is said to be compatible (or to form a pencil of PBs) if any linear combination with constant coefficients \( \mu P_1 + \nu P_2 \) is also a PB. This notion, first mentioned by F. Magri [9] and extended by I. Gel’fand and I. Dorfman [2] (see also [1]) provides a fundamental device for the integrability of Hamiltonian equations. In our setting, the study of compatible pair of PBs (1.5) might lead to the classification of bi-Hamiltonian lattice equations of type

\[
\dot{u}_n = F(u_{n-S}, \ldots, u_n, \ldots, u_{n+S}) = \{u_n, H_1[u]\}_{M_1} = \{u_n, H_2[u]\}_{M_2}
\]

for some integer \( S \geq 0 \) and local PBs of order \( M_1, M_2 \). It seems that higher order lattice equations have not been studied yet except for the Volterra type equations (i.e. \( S = 1 \)), analyzed by R.I. Yamilov and collaborators using the master symmetries approach (see review article [15] and references therein). The following result describes some necessary conditions for the classification of pencil of PBs (1.5). We expect these conditions to be also sufficient.

**Theorem 1.3.** Let us consider a pair \( (P, P') \) of non-constant PBs of type (1.5) and order \( M = 2\alpha + \xi \) and \( M' = 2\alpha' + \xi' \) respectively, with \( M \geq M' \). Then \( P \) and \( P' \) form a pencil of PBs only if \( \alpha = \alpha' \) and there exist coordinates on the manifold \( \mathcal{M}^N \) such that

\[
g_M(u_n, \ldots, u_{n+M}) = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}) = \sigma_M \exp \left( \sum_{i=0}^\xi \tau_i u_{n+i} \right)
\]

\[
g_{M'}(u_n, \ldots, u_{n+M'}) = f'^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi'}) = \sigma_{M'} \exp \left( \sum_{i=0}^{\xi'} \tau'_i u_{n+i} \right)
\]
where $\sigma_M, \sigma_{M'}$ are some constants and $\tau_p = \tau'_p = \tau_{-\xi' + p}$, for any $p = 0, \ldots, \xi'$.

The structure of the present paper is the following: in Section 2 we describe our classification procedure, proving formula (1.8) for the coefficients $g_k$. In Section 3, using the change of coordinates (1.7), suggested by the leading coefficient $g_M$, we prove that any PB of type (1.1) can be reduced to the cubic PB of Volterra lattice (Theorem 3.2). In Section 4, we present some necessary conditions for the compatibility of $(\alpha, \xi)$-brackets (Theorem 4.2 and Lemma 4.3). Finally, with Section 5, we introduce the concept of admissible Lie-Poisson groups and describe how to produce some new classes of non-degenerate vector-valued ($N > 1$) first order dDGBPs (Theorem 5.6).

2. Classification of $(\alpha, \xi)$-brackets

This Section is devoted to the classification of the scalar-valued PBs given by the formulas

\begin{equation}
\{u_n, u_{n+k}\}_M = g_k(u_n, \ldots, u_{n+k}), \quad 1 \leq k \leq M.
\end{equation}

The coefficients $g_k(u_n, \ldots, u_{n+k})$ are locally analytic functions (see R. Yamilov [15] for the precise definition), satisfying, for all values of the independent variables $u_n, n \in \mathbb{Z}$, the following bi-linear PDEs
given by Jacobi identity

\[ [p, q] \{u_n, u_{n+p}\}, u_{n+p+q} = \{u_n, \{u_{n+p}, u_{n+p+q}\}\} + \{\{u_n, u_{n+p}\}, u_{n+p}\}, \]

explicitly,

\[
\begin{align*}
\sum_{i=0}^{p} g_p(u_n, \ldots, u_{n+p}), u_{n+p-i}g_{q+i}(u_{n+p-i}, \ldots, u_{n+p+q}) + \\
\sum_{i=0}^{q} g_q(u_n, \ldots, u_{n+p+q}), u_{n+p+i}g_{p+i}(u_n, \ldots, u_{n+p+i}) \\
\end{align*}
\]

where $p, q = 1, \ldots, M$ and $g_0(\cdot) \equiv 0, g_k(\cdot) \equiv 0, k > M$.

Example 2.1. [Volterra lattice or discrete KdV equation]
The well-known Volterra lattice (VL) [14] is defined by the following equations of motion

\[ \dot{u}_n = u_n(u_{n+1} - u_{n-1}) \quad n \in \mathbb{Z}. \]

Performing the local change of variable $u_n \rightarrow \log u_n$, we obtain equations

\begin{equation}
\dot{u}_n = \exp(u_{n+1}) - \exp(u_{n-1}),
\end{equation}

that admit the following bi-hamiltonian representation (see [7])

\[ \dot{u}_n = \{u_n, H_1[u]\}_1 = \{u_n, H_2[u]\}_2, \]
where
\[
H_1(u) = \sum_k \exp(u_k) \begin{cases} u_n, u_{n+1} \end{cases}_1 = 1, \\
H_2(u) = \frac{1}{2} \sum_k u_k \begin{cases} u_n, u_{n+2} \end{cases}_2 = \exp(u_{n+1}) \begin{cases} u_n, u_{n+1} \end{cases}_2 = \exp(u_n) + \exp(u_{n+1}).
\]

These PBs belong to the class (2.1). Moreover, the coefficients of the cubic PB \( \{\cdot, \cdot\} \) are characterized by suitable linear combination of the leading order term, given by the exponential functions \( f_n^0 \equiv f^0(u_n) = \exp(u_n) \). This will be the typical behavior of non-constant PBs (2.1).

2.1. The leading-order coefficient. By the definition of locality radius, the leading-order term \( g_M \) might depend on variables \( u_n, \ldots, u_{n+M} \), i.e. \( g_M = g_M(u_n, \ldots, u_{n+M}) \).

**Lemma 2.2.** There exist canonical coordinates and an integer \( \alpha > 0 \) such that the leading term \( g_M \) reduce to the form
\[
g_M(u_n, \ldots, u_{n+M}) = f_n^{\xi} = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}), \quad \xi = M - 2\alpha,
\]
where
\[
(2.4) \quad \frac{\partial f_n^{\xi}}{\partial u_n} \frac{\partial f_n^{\xi}}{\partial u_{n+\xi}} \neq 0
\]
if \( f^\xi \) is a non-constant function.

**Proof.** Let us consider the bi-linear PDEs \([p, q]\), with \( p, q = 1, \ldots, M \). At first, we focus our attention on equation
\[
[M, M] \quad \{\{u_n, u_{n+M}\}, u_{n+2M}\} = \{u_n, \{u_{n+M}, u_{n+2M}\}\}
\]
that provides us
\[
\log g_M(u_n, \ldots, u_{n+M}), u_{n+M} = \log g_M(u_{n+M}, \ldots, u_{n+2M}), u_{n+M} = \hat{a}(u_{n+M})
\]
for some arbitrary function \( \hat{a}(u_{n+M}) \). Solving this logarithmic equation, we obtain the following factorization
\[
g_M(u_n, \ldots, u_{n+M}) = a(u_n) f^{M-\alpha-\beta}(u_{n+\alpha}, \ldots, u_{n+M-\beta}) a(u_{n+M})
\]
where \( \log a(u_n), u_n = \hat{a}(u_n) \) and \( f_{n+\alpha}^{M-\alpha-\beta} = f^{M-\alpha-\beta}(u_{n+\alpha}, \ldots, u_{n+M-\beta}) \) is a constant function or such that
\[
\frac{\partial f_{n+\alpha}^{M-\alpha-\beta}}{\partial u_{n+\alpha}} \frac{\partial f_{n+\alpha}^{M-\alpha-\beta}}{\partial u_{n+M-\beta}} \neq 0,
\]
for some integers \( \alpha, \beta \geq 1 \).

Performing a local change of the variables \( u_n \mapsto \tilde{u}_n = \varphi(u_n) \), we can reduce to
\[
g_M(\tilde{u}_n, \ldots, \tilde{u}_{n+M}) = f^{M-\alpha-\beta}(\tilde{u}_{n+\alpha}, \ldots, \tilde{u}_{n+M-\beta}).
\]

Finally, we prove \( \alpha = \beta \). Indeed, on the one hand, from equation
\[
[M, M-\beta] \quad \{\{u_n, u_{n+M}\}, u_{n+2M-\beta}\} = \{u_n, \{u_{n+M}, u_{n+2M-\beta}\}\}
\]
after some elementary computations, we have
\[
\log g_M(u_{n+\alpha}, \ldots, u_{n+M-\beta}), u_{n+M-\beta} =
\| g_{M-\beta}(u_{n+M}, \ldots, u_{n+2M-\beta}), u_{n+M} g_M(u_{n+M-\beta+\alpha}, \ldots, u_{n+2(M-\beta)})^{-1}
\]
where the variables appearing on the left-hand side do not intersect with those appearing on the right-hand side. Then, there exists a non-zero constant \(k\), such that
\[
g_{M-\beta}(u_{n+M}, \ldots, u_{n+2M-\beta}), u_{n+M} = k g_M(u_{n+M-\beta+\alpha}, \ldots, u_{n+2(M-\beta)})
\]
and this PDE makes sense only if \(\alpha \geq \beta\).

On the other hand, repeating the same argumentations for equation
\[ [M-\alpha, M] \{\{u_n, u_{n+M-\alpha}\}, u_{n+2M-\alpha}\} = \{u_n, \{u_{n+M-\alpha}, u_{n+2M-\alpha}\}\} \]
that is
\[
g_{M-\alpha}(u_n, \ldots, u_{n+M-\alpha}), u_{n+M-\alpha} g_M(u_{n+\alpha}, \ldots, u_{n+M-\beta})^{-1} =
\| \log g_M(u_{n+M}, \ldots, u_{n+2M-\beta-\alpha}), u_{n+M}
\]
we arrive at \(\alpha \leq \beta\), that implies \(\alpha = \beta\).

For any fixed order \(M\), the leading order functions, given by Lemma 2.2, define essentially different classes of PBs (2.1).

**Definition 2.3.** Let \((\alpha, \xi)\) be a pair of non-negative integers with \(\alpha \geq 1\), we call \((\alpha, \xi)\)-brackets the class of non-constant PBs (2.1) of order \(M = 2\alpha + \xi\) expressed in the canonical coordinates, i.e. \(g_M(u_n, \ldots, u_{n+M}) = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi})\).

For the constant case \(f \equiv \sigma_M\) one can immediately prove the following

**Proposition 2.4.** If there exists a set of coordinates reducing the leading order term to the constant form \(g_M(u_n, \ldots, u_{n+M}) = \sigma_M\), for some non-zero constant \(\sigma_M\), then all the coefficients \(g_k\), \(k = 1, \ldots, M\) are constant in such coordinates, i.e.
\[
\{u_n, u_{n+k}\}_M = g_k(u_n, \ldots, u_{n+k}) = \sigma_k, \quad k = 1, \ldots, M,
\]
where \(\sigma_k\) are complex constants.

**2.2. Classification theorem.**

**Theorem 2.5.** For any PB (2.1) of order \(M\), there exist a set of coordinates and an integer \(\alpha \geq 1\), such that the coefficients \(g_k(u_n, \ldots, u_{n+k})\), \(k = 1, \ldots, M\) are given by linear combination of the suitably shifted function \(f^\xi\) (see Lemma 2.2)
\[
f^\xi(u_n, \ldots, u_{n+\xi}) = \exp \left( \sum_{i=0}^{\xi} \tau_i u_{n+i} \right),
\]
where $\{\tau_i\}_{i=0,\ldots,\xi}$ are constrained complex parameters. Explicitly, we have

$$g_{\alpha+\xi+p}(u_n,\ldots,u_{n+\alpha+\xi+p}) = \left(\sum_{s=\max(0,p)}^{\min(\alpha+p,\alpha)} \lambda_s^\alpha \chi_s^\alpha\right) f^\xi(u_n,\ldots,u_{n+\xi}),$$

where $\xi \equiv M - 2\alpha, p = -\alpha,\ldots,\alpha, T$ is the shift operator (1.3) and $\lambda^s_{\alpha-r}$ are scalars such that

(i) they satisfy the multiplication rule

$$\lambda^s_{\alpha-r} = \lambda^\alpha_{\alpha-r} \ (\lambda_0^\alpha)^{-1} \lambda^\alpha_{\alpha-s} \ s, r = 0,\ldots,\alpha,$$

(ii) denoting $\theta \equiv \min(\alpha,\xi)$, they can be expressed by explicit formulas in terms of the $\theta + 1$ parameters $\tau_0, \tau_1,\ldots,\tau_{\theta-1}; \tau_\xi$

$$\lambda^\alpha_{\alpha-r} = \det A_r \{(-\theta^r \tau_0^{-1} \tau_s)_{s \geq 0}\} \ r = 0,\ldots,\alpha - 1$$

where $A_r$ is the band-Toeplitz matrix

$$A_r(a) = \begin{pmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_2 & a_1 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{r-1} & \ldots & \ldots & a_1 & a_0 \\ a_r & \ldots & \ldots & a_2 & a_1 \end{pmatrix}$$

associated to the sequence $a = (a_0, a_1,\ldots)$.

Here and below we adopt the notation $\tau_p \equiv 0$ if $p < 0$ or $p > \xi$.

**Remark 2.6.** The scalars appearing on formula (2.6) can be easily visualized looking at the rows of the following rhombus

$$g_{\alpha+\xi+\alpha} \rightarrow \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1}$$

$$g_{\alpha+\xi+\alpha-1} \rightarrow \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1}$$

$$\vdots$$

$$g_{\alpha+\xi} \rightarrow \lambda^\alpha_1 \lambda^\alpha_1 \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1}$$

$$g_{\alpha+\xi-1} \rightarrow \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-2} \lambda^\alpha_{\alpha-1}$$

$$\vdots$$

$$g_{\alpha+\xi-\alpha+1} \rightarrow \lambda^\alpha_{\alpha-1} \lambda^\alpha_{\alpha-1}$$

$$g_{\alpha+\xi-\alpha} \rightarrow \lambda^\alpha_{\alpha-1}$$

We complete the description above, specifying the non-trivial constraints that the Jacobi identity imposes on parameters $\tau$ in the following

**Theorem 2.7.** Let $(\alpha, \xi)$ be a pair of non-negative integers and define the sequence $\{\sigma_q\}_{q \geq 0} = \{(-)^q \lambda^\alpha_{\alpha-q}\}_{q \geq 0}$. Then, the $(\alpha, \xi)$-brackets are in one-to-one correspondence
with the intersection points of the $\theta$ projective hypersurfaces in $\mathbb{CP}^\theta$ defined by the homogeneous polynomial equations

(i) if $\xi \geq 2\alpha - 1$, for any $p = 0, \ldots, \alpha - 1$,

\begin{equation}
(\tau_0)^{p+1} \det \mathcal{A}_{\xi-p} (\{\sigma_q\}_{q \geq 0}) = (\tau_\xi)^{p+1} \det \mathcal{A}_p (\{\sigma_{a-q}\}_{q \geq 0})
\end{equation}

(ii) if $\xi < \alpha$, for any $p = 0, \ldots, \xi - 1$,

\begin{equation}
(\tau_0)^{\xi-p} \tau_p = (\tau_\xi)^{\xi-p+1} \det \mathcal{A}_{\xi-p} (\{\sigma_{a-q}\}_{q \geq 0})
\end{equation}

(iii) if $\alpha \leq \xi < 2\alpha - 1$, for any $p = 0, \ldots, \xi - \alpha$

\begin{equation}
(\tau_0)^{p+1} \det \mathcal{A}_{\xi-p} (\{\sigma_q\}_{q \geq 0}) = (\tau_\xi)^{p+1} \det \mathcal{A}_p (\{\sigma_{a-q}\}_{q \geq 0})
\end{equation}

and, for any $p = \xi - \alpha + 1, \ldots, \alpha - 1$

\begin{equation}
(\tau_0)^{\xi-p} \tau_p = (\tau_\xi)^{\xi-p+1} \det \mathcal{A}_{\xi-p} (\{\sigma_{a-q}\}_{q \geq 0}).
\end{equation}

The case $\theta = 0$ is described below in the Example 2.4.1.

2.3. Proof. In our computations we often encounter generalized Fibonacci sequences. Therefore, it might be useful to remind some elementary topics about them.

Remark 2.8. For any positive integer $k \geq 2$, the $k$-generalized Fibonacci sequence \( \{F_n\}_{n \geq 0} \) is defined by the order $k$ linear homogeneous recurrence relation

\begin{equation}
F_n = \sum_{i=1}^{k} (-1)^{i+1} a_i F_{n-i}, \quad \text{for } n \geq 1,
\end{equation}

for arbitrary coefficients $a_i$, $i = 0, \ldots, k$.

The generating function for $\{F_n\}_{n \geq 0}$ is given by

\[ \varphi(t) = \sum_{n \geq 0} F_n t^n = \frac{1}{1 - a_1 t + \cdots + (-)^k a_k t^k} \]

and, using the theory of lower triangular Toeplitz matrices (see for example [13] or [16]), one can express the $k$-generalized Fibonacci numbers in terms of determinants of matrices with entries given by the coefficients of their recurrence equation, that is

\begin{equation}
F_n = \det(a_{1-i+j})_{1 \leq i,j \leq n} = \det \mathcal{A}_n (a),
\end{equation}

where $a = (a_0 \equiv 1, a_1, \ldots, a_k, a_{k+1} \equiv 0, \ldots)$ and $\mathcal{A}$ the band Toeplitz matrix defined by (2.9).

Let us consider a PB of type (2.1). According to the Lemma 2.2 we can choose certain coordinates on the target manifold $\mathcal{M}^N$ such that the leading term $g_M$ reduces to the following form $f_{n+\alpha}^\xi = f_\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi})$. In our opinion, the simplest way to obtain the form of coefficients $g_k$, $k = 1, \ldots, M$ comes from considering equations

\[ [M - \alpha - p, M] \{\{u_n, u_{n+M-\alpha-p}\}, u_{n+2M-\alpha-p}\} = \{u_n, \{u_{n+M-\alpha-p}, u_{n+2M-\alpha-p}\}\} \]
\[ [M, M - \alpha - p] = \{u_n, u_{n+M}, u_{n+2M - \alpha - p}\} = \{u_n, \{u_{n+M}, u_{n+2M - \alpha - p}\}\} \]

for any \( p = 0, \ldots, \alpha + \xi - 1 \).

2.3.1. Formula (2.6). We start a detailed analysis of some initial cases (i.e. \( p = 0, 1 \)), that will suggest us how to organize the general computations. For the sake of simplicity, we suppose \( \xi > \alpha \).

\( p = 0 \). Equations \([M - \alpha, M]\) and \([M, M - \alpha]\) yield \((\log f_n^\xi)_{u_n} = \tau_0\) and \((\log f_n^\xi)_{u_{n+\xi}} = \tau_{\xi}\) for some non-zero constants \( \tau_0, \tau_{\xi} \) (see condition (2.4)) and

\[ g_{\alpha+\xi}(u_n, \ldots, u_{n+\alpha+\xi}) = \lambda_0^0 f_n^\xi + \ldots + \lambda_0^\alpha f_n^{\alpha+\xi}, \]

where dots stay for any arbitrary function depending at the most on the variables \( u_{n+1}, \ldots, u_{n+\alpha+\xi-1} \) and \( \lambda_0^0 = (\tau_0)^{-1} \tau_\xi, \lambda_0^\alpha = \tau_0 (\tau_\xi)^{-1} \).

\( p = 1 \). Analogously, equations \([M - \alpha - 1, M]\) and \([M, M - \alpha - 1]\) provide us \((\log f_n^\xi)_{u_{n+1}} = \tau_1\) and \((\log f_n^\xi)_{u_{n+\xi+1}} = \tau_{\xi-1}\) for some constants \( \tau_1, \tau_{\xi-1} \) and

\[ g_{\alpha+\xi-1}(u_n, \ldots, u_{n+\alpha+\xi-1}, u_{n+\alpha+\xi-1}) = \lambda_1^{\alpha} f_n^\xi (u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}) + \lambda_1^{\alpha-1} h_n^{\xi-1} (u_{n+\alpha}, \ldots, u_{n+\alpha+\xi-1}) + \lambda_1^{\alpha-1} h_n^{\xi-1} (u_{n+\alpha}, \ldots, u_{n+\alpha+\xi-1}), \]

\[ \tau_0 g_{M-1}(u_n, \ldots, u_{n+2\alpha+\xi-1}) + \tau_1 f_n^\xi (u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}) \]

\[ g_{\alpha+\xi-1}(u_n, \ldots, u_{n+\alpha+\xi-1}, u_n) \]

\[ \tau_\xi g_{M-1}(u_{n-\alpha}, \ldots, u_{n+\alpha+\xi-1}) + \tau_{\xi-1} f_n^\xi (u_{n-1}, \ldots, u_{n+\xi-1}). \]

It necessarily follows that the constants \( \tau_1, \tau_{\xi-1} \) are non-zero and

\[ g_{M-1} = g_{M-1}(u_{n+\alpha-1}, \ldots, u_{n+\alpha+\xi}) = \lambda_{\alpha-1}^{\alpha-1} h_n^{\xi-1} (u_{n+\alpha}, \ldots, u_{n+\alpha+\xi-1}) + \lambda_{\alpha-1}^{\alpha} f_n^\xi \]

with \( \lambda_{\alpha-1}^{\alpha-1} = (\tau_\xi)^{-1} \tau_{\xi-1}\) and \( \lambda_{\alpha-1}^{\alpha} = -(\tau_0)^{-1} \tau_1 \).

Remark 2.9. Notice that the function \( h_{n+\alpha}^{\xi-1} \) can be understood as the leading order function of a \((\alpha, \xi - 1)\)-bracket: \( h_{n+\alpha}^{\xi-1} = f_{n+\alpha}^{\xi-1} (u_{n+\alpha}, \ldots, u_{n+M-\alpha-1}) \).

We decide to forget about the contributions provided by the leading order functions \( f_{n+\alpha}^\xi \) of lower order PBs (i.e. \( M' < M \)), postponing to Section 4 the problem of classifying the compatible pairs of PBs (2.1).

According to Remark 2.9, we solve equations \([M - \alpha - 1, M]\) and \([M, M - \alpha - 1]\), finding

\[ g_{\alpha+\xi-1}(u_n, \ldots, u_{n+\alpha+\xi-1}) = \lambda_0^0 f_n^\xi + \ldots + \lambda_0^{\alpha-1} f_n^{\alpha+\xi}, \]

where \( \lambda_0^0 = (\lambda_0^\alpha)^{-1} \lambda_{\alpha-1}^{\alpha}, \lambda_0^{\alpha-1} = \lambda_0^\alpha \lambda_{\alpha-1}^{\alpha-1} \) and dots stay for any arbitrary function depending at the most on variables \( u_{n+1}, \ldots, u_{n+\alpha+\xi-2} \). Iterating this procedure, one can show that the coefficients \( \{g_k\}_{k=1,\ldots,M} \) depend on lattice variables according to formulas

\[ g_{\alpha+\xi+k} = g_{\alpha+\xi+k}(u_{n+k}, \ldots, u_{n+\alpha+\xi}), \quad k = 0, \ldots, \alpha \]

\[ g_{\alpha+\xi-k} = g_{\alpha+\xi-k}(u_n, \ldots, u_{n+\alpha+\xi-k}), \quad k = 1, \ldots, \alpha + \xi - 1 \]
and

\[ f^\xi(u_n, \ldots, u_{n+\xi}) = \lambda_0^\alpha \exp \left( \sum_{i=0}^{\xi} \tau_i u_{n+i} \right), \]

where

(i) \( \lambda_0^\alpha \) is a free multiplicative constant, that can be normalized (i.e. \( \lambda_0^\alpha \equiv 1 \)) choosing a suitable rescaling of the coordinates: \( u_n \to k u_n \), for some constant \( k \).

(ii) \( \{\tau_i\}_{i=0,\ldots,\xi} \) are non-zero complex parameters, thanks to condition (2.4).

Moreover, solving the equations \( [M - \alpha - p, M]_{p=0,\ldots,\alpha+\xi-1} \) (analogous results come from equations \( [M, M - \alpha - p] \)).

The constraints on constants \( \lambda \)'s can be encoded into the following linear system

\[
\begin{pmatrix}
\tau_0^\lambda \alpha_{-p}^0 \\
\vdots \\
0
\end{pmatrix} =
\begin{pmatrix}
\lambda_0^\alpha_{-p} & 0 & \ldots & \ldots & 0 \\
\lambda_0^\alpha_{-p+1} & \lambda_0^\alpha_{-p+1} & \ldots & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ldots & 0 \\
\lambda_0^\alpha_{-p} & \lambda_0^\alpha_{-p+1} & \ldots & \lambda_0^\alpha_{-1} & 0 \\
0 & \lambda_0^\alpha_{-p} & \lambda_0^\alpha_{-p+1} & \ldots & \lambda_0^\alpha_{-1} \lambda_0^\alpha
\end{pmatrix}
\begin{pmatrix}
\tau_0 \\
\vdots \\
\tau_{p-1} \\
\tau_p
\end{pmatrix}
\]

(2.17)

where \( \lambda_{s-r}^s \) is non-zero iff \( s, r = 0, \ldots, \alpha \), and recursively on \( p \) we have defined

(i) \( \lambda_0^\alpha_{-p} \equiv \lambda_0^\alpha_{-p} \)

(ii) \( \lambda_0^\alpha_{-p+s} \equiv -\tau_0^{-1} \left[ \sum_{i=1}^{s} \tau_i \lambda_0^\alpha_{-p+i} \right], \ s = 1, \ldots, p. \)

Combining (i) and (ii), by induction on \( p \) one can prove the following multiplication rule

\[
\lambda_{s-r}^s = \lambda_s^s \lambda_{a-r}^a, \quad s, r = 0, \ldots, \alpha, \]

that allows us to describe all constants \( \lambda \)'s in terms of the constants \( \{\lambda_{a-s}^a, \lambda_{a-s}^a\}_{s=0,\ldots,\alpha} \), appearing on the edges of rhombus (2.10).

2.3.3. Constants \( \lambda \)'s as function of parameters \( \tau \)'s. We start looking at the last row of the matrix appearing on (2.17). If \( 0 \leq p < \alpha \), we have the following order \( p \) recursive relations

\[
\lambda_0^\alpha_{-p} = -(\tau_0)^{-1} \sum_{i=1}^{p} \tau_i \lambda_0^\alpha_{-p+i},
\]

(2.19)

Denoting \( F_p \equiv \lambda_0^\alpha_{-p}, \ a_p \equiv (-)^p \tau_0^{-1} \tau_p \), we recognize the \( p \)-generalized Fibonacci sequence (see equation (2.15)). Then, according to Remark 2.8, we can express

\[
\lambda_0^\alpha_{-p}(\tau_0, \ldots, \tau_p) = \det A_p \left( \{(-)^s \tau_0^{-1} \tau_s\}_{s \geq 0} \right), \quad p = 0, \ldots, \alpha - 1.
\]
To write down the constants \( \{ \lambda_{\alpha-s}^s \}_{s=0,...,\alpha} \) as functions of the constants \( \{ \lambda_{\alpha-s}^s \}_{s=0,...,\alpha} \), we look at the equations
\[
[M - q, M - \alpha + q]_{q=0,...,\alpha} \{ \{ u_n, u_{n+M-q} \}, u_{n+2M-\alpha} \} = \{ u_n, \{ u_{n+M-q}, u_{n+2M-\alpha} \} \}.
\]
They split into
\[
\begin{align*}
(a) & \quad g(\alpha+\xi)+\alpha-q(u_n+\alpha-q, \ldots, u_n+\alpha+\xi, u_{n+\alpha+\xi}) = \lambda_{\alpha-q}^q f(\xi(u_n+\alpha, \ldots, u_n+\alpha+\xi), u_{n+\alpha+\xi}) \\
(b) & \quad g(\alpha+\xi+q)(u_n+q, \ldots, u_n+\alpha+\xi), u_{n+q}) = \lambda_{\alpha}^q f(\xi(u_n+q, \ldots, u_n+\alpha+\xi), u_{n+q})
\end{align*}
\]
that are compatible only if
\[
(2.20) \quad \lambda_{\alpha-q}^q = (\lambda_{\alpha}^0)^{-1} \lambda_{\alpha-q}^q.
\]
Substituting (2.20) in the formula (2.18), we hold the multiplication rule (2.7).

2.3.4. Constraints for parameters \( \tau \)'s. Supposing \( \xi > \alpha \), we can analyze when \( \alpha \leq p \leq \xi \). Analogously to (2.19), we have the following linear homogeneous recurrence relation of order \( \alpha \)
\[
(2.21) \quad \tau_p = -\sum_{q=1}^{\alpha} \lambda_{\alpha-q}^q \tau_{p-q}, \quad \text{for any } p \geq \alpha.
\]
For any \( q = 1, \ldots, \alpha \), denoting \( a_q \triangleq (-)^q \lambda_{\alpha-q}^q \), we arrive at \( \tau_p = \sum_{q=1}^{\alpha} (-)^q a_q \tau_{p-q} \) that is
\[
(2.22) \quad \tau_p(\tau_0, \ldots, \tau_{\alpha-1}; \tau_\xi) = \tau_0 \det \mathcal{A}_p \{ \{ (-)^s \lambda_{\alpha-s}^s \}_{s=0}^{\alpha} \}, \quad p = \alpha, \ldots, \xi.
\]
Finally, when \( \xi + 1 \leq p < \alpha + \xi \), we obtain analogue relations with respect to the case \( 1 \leq p < \alpha \): the equations \( [M - \alpha - p, M] \) give us
\[
\sum_{i=0}^{q} \lambda_{\alpha-q-i}^q \tau_{\xi-i} = 0, \quad q = 1, \ldots, \alpha - 1.
\]
Following again Remark 2.8, for any \( q = 1, \ldots, \alpha - 1 \), we obtain
\[
(2.23) \quad \tau_{\xi-q}(\tau_0, \ldots, \tau_{\alpha-1}; \tau_\xi) = \tau_0 \det \mathcal{A}_q \{ \{ (-)^s (\lambda_{\alpha-s}^q)^{-1} \lambda_{\alpha}^s \}_{s=0}^{\alpha} \}
\]
Observe that formulas (2.22) and (2.23) have to be consistent: this provides us the constraints on parameters \( \tau \)'s. According to the values of \( \alpha \) and \( \xi \), we immediately find out the homogeneous polynomial equations given by Theorem 2.7.

2.3.5. Remaining equations. Due to the relation (2.20), the symmetric set of equations \( [M, M - \alpha - p]_{p=0,...,\alpha+\xi-1} \) gives us the same constraints.
Moreover, replacing expression (2.6) on the other equations \( [p, q]_{p,q=1,...,M} \), with straightforward computations that generalize the previous ones, we can obtain that the relations for the parameters \( \tau_0, \ldots, \tau_{\theta-1}; \tau_\xi \) describe above are necessary and sufficient.

2.4. Examples. Looking at the number of the lattice variables which the leading order function can depend on, we study in detail the minimal and the maximal cases.
2.4.1. \((\alpha, 0)\)-brackets. The leading order coefficient can be chosen of the form \(g_{2\alpha} = f^0(u_{n+\alpha}) = \exp(\tau_0 u_{n+\alpha})\) and substituting in the formulas for the constants \(\lambda\)'s we obtain

\[
\begin{aligned}
\{u_n, u_{n+2\alpha}\}_M &= \exp (\tau_0 u_{n+\alpha}) \\
\{u_n, u_{n+\alpha}\}_M &= \exp (\tau_0 u_n) + \exp (\tau_0 u_{n+\alpha})
\end{aligned}
\]

where \(\tau_0\) is a free complex constant. Notice that the cubic PB of Volterra lattice \((2.3)\) belongs on the class of \((1, 0)\)-brackets.

2.4.2. \((1, \xi)\)-brackets. These PBs are given in the canonical coordinates by the formulas

\[
\begin{aligned}
\{u_n, u_{n+M}\} &= \exp (z_{n+1}) \\
\{u_n, u_{n+M-1}\} &= \tau^{-1}_\xi \exp(z_n) + \tau_0 \tau^{-1}_\xi \exp(z_{n+1}) \\
\{u_n, u_{n+M-2}\} &= \exp(z_n)
\end{aligned}
\]

where \(z_n = \sum_{i=0}^\xi (-\lambda)^i \tau_0 u_{n+i}\) and the pair \((\tau_0, \tau_\xi)\) belongs to the set of points of \(\mathbb{CP}^1\), described by the following equation (see equation \((2.11)\))

\[
\tau_0^{\alpha+\xi} + (-)^{\alpha+\xi} \tau_\xi^{\alpha+\xi} = 0 \quad \text{where} \quad [\tau_0 : \tau_\xi] \in \mathbb{CP}^1, \quad \alpha = 1.
\]

Analogously, the \((\alpha, 1)\)-brackets are parametrized by the same points \((2.24)\), with \(\xi = 1\).

2.4.3. A multi-parameters example: \((2, 2)\)-brackets. This is the first non trivial case in which the parameters \(\tau\)'s are described by the intersection points of certain hypersurfaces. From the Theorems \ref{thm:2.5} and \ref{thm:2.7}, it follows that the \((2, 2)\)-brackets are characterized by three parameters \(\tau_0, \tau_1, \tau_2\), constrained to satisfy the following systems

\[
\begin{aligned}
(\tau_0)^2 &= (\tau_2)^2 \\
(\tau_1)^2 &= 2\tau_0 \tau_2.
\end{aligned}
\]

3. A Darboux-type theorem

In Theorem \ref{thm:2.5} we have proven that, up to local point-wise change of coordinates, the leading order function \(g_M\) of any PB \((1.5)\) can be considered of the following form

\[
g_M = f_{n+\alpha}^\xi = \exp \left( \sum_{i=0}^\xi \tau_i u_{n+\alpha+i} \right), \quad \text{for some pair of non-negative integers } (\alpha, \xi).
\]

In the present Section, we deal with the problem of reduction of PBs \((2.1)\) to a canonical form by more general changes of variable than the local ones. Let us give the following

**Definition 3.1.** Let \((s_1, s_2)\) be a pair of non negative integers, we call a discrete Miura-type transformation any map of the form \(z_n = \varphi(u_{n-s_1}, \ldots, u_{n+s_2})\) that is a **canonical transformation**, i.e. a change of coordinates preserving the PBs.

Notice that these discrete Miura-type transformations, as the continuous ones, are **differential substitutions** (i.e. they depend on \(u_{n+1}, u_{n+2}, \ldots\)) and therefore, only
formally invertible. Defined new coordinates according to

\[ z_n = \sum_{i=0}^{\xi} \tau_i u_{n+i}, \]

we prove by direct computation the following

**Theorem 3.2.** Any \((\alpha, \xi)\)-bracket is mapped by the Miura-type transformation (3.1) into the following \((\alpha + \xi, 0)\)-bracket,

\[
\begin{align*}
\{z_n, z_{n+2(\alpha+\xi)}\} &= \tau_0 \tau_\xi \exp(z_n + \alpha + \xi) \\
\{z_n, z_{n+\alpha+\xi}\} &= \tau_0 \tau_\xi [\exp(z_n) + \exp(z_n + \alpha + \xi)].
\end{align*}
\]

**Remark 3.3.** Subdividing all the particles-variables into \(\alpha + \xi\) families, according to

\[
v_n^{(p)} \triangleq z_{(\alpha+\xi)(n-1)+p}, \quad p = 1, \ldots, \alpha + \xi,
\]

the PB (3.2) can be presented in the following simple form

\[
\begin{align*}
\{v_n^{(p)}, v_{n+2}\} &= \tau_0 \tau_\xi \exp(v_{n+1}^{(p)}) \\
\{v_n^{(p)}, v_{n+1}\} &= \tau_0 \tau_\xi [\exp(v_n^{(p)}) + \exp(v_{n+1}^{(p)})].
\end{align*}
\]

Therefore, it splits into \(\alpha + \xi\) copies of the cubic Volterra PB (2.3).

Furthermore, any constant PB of order \(M\), \(\{u_n, u_{n+k}\} = \sigma_k\), with \(k = 1, \ldots, M\), can be reduced by the lattice splitting \(v_n^{(p)} \triangleq u_{M(n-1)+p}\), to the following PB

\[
\begin{align*}
\{v_n^{(p)}, v_{n+2}\} &= \sigma_{M+q-p} \\
\{v_n^{(p)}, v_{n+1}\} &= \sigma_{q-p},
\end{align*}
\]

where \(\sigma_0 \equiv 0\) and \(p, q = 1, \ldots, M\). Notice that, if all constants \(\sigma_k\) are normalized (i.e. \(\sigma_k \equiv 1, k = 1, \ldots, M\)), the PB (3.3) becomes the quadratic PB for the Bogoyavlensky lattice (BL) of order \(M\) (see the definition of BL in Suris [12], Chapter 17).

**Proof.** We are interested on the expression of \((\alpha, \xi)\)-brackets in \(z\)-coordinates. Let be \(p = 0, 1, \ldots, \), we have to compute

\[
\{z_n, z_{n+p}\} = \{\sum_{i=0}^{\xi} \tau_i u_{n+i}, \sum_{j=0}^{\xi} \tau_j u_{n+p+j}\} = \sum_{i=0}^{\xi} \tau_i \left[ \sum_{j \in J_+^{\alpha}(i,p)} \tau_j \{u_{n+i}, u_{n+p+j}\} + \sum_{j \in J_-^{\alpha}(i,p)} \tau_j \{u_{n+p+j}, u_{n+i}\} \right]
\]

where we have defined the sets of admissible \(j\)'s,

\[
J_+^{\alpha}(i,p) \triangleq \{0, \ldots, \xi\} \cap \{i + \xi - p, \ldots, i + 2\alpha + \xi - p\}
\]

\[
J_-^{\alpha}(i,p) \triangleq \{0, \ldots, \xi\} \cap \{i - 2\alpha - \xi - p, \ldots, i - \xi - p\}.
\]

In the following, to avoid some technicalities, we suppose \(\xi > \alpha\). At first, we notice that when \(p > 2(\alpha + \xi)\), \(\{z_n, z_{n+p}\}\) vanishes.

Let us start considering the set \(J_-^{\alpha}(i,p)\). It is non-empty only if \(i = \xi\) and \(p = 0\).
When \( p = 0 \), \( J_a^+(i, p) \) is non-empty only if \( i = 0 \), then our summation vanishes, indeed
\[
\{z_n, z_n\} = \{\sum_{i=0}^{\xi} \tau_i u_{n+i}, \sum_{j=0}^{\xi} \tau_j u_{n+j}\} = \sum_{j>i} \tau_i \tau_j \{u_{n+i}, u_{n+j}\} - \sum_{i>j} \tau_i \tau_j \{u_{n+j}, u_{n+i}\} = 0.
\]

If \( p \geq 1 \), we can reduce to evaluate
\[
(3.4) \quad \{z_n, z_{n+p}\} = \sum_{i=0}^{\xi} \tau_i \left[ \sum_{j \in J_a^+(i, p)} \tau_j \{u_{n+i}, u_{n+p+j}\} \right].
\]

In the following, we give some details about the complete computation. Enforcing a recursive procedure on \( p \), for any \( i \) that runs from 0 to \( \xi \), we describe the set of admissible \( j: J_a^+(i, p) \).

**Step 1:** \( p = 2(\alpha + \xi) \). Then \( J_a^+(i, p) \neq \emptyset \) if and only if \( i = \xi \) and
\[
\{z_n, z_{n+2(\alpha+\xi)}\} = \tau_0 \tau_\xi \{u_{n+\xi}, u_{n+2(\alpha+\xi)}\} = \tau_0 \tau_\xi \exp(z_{n+\alpha+\xi}).
\]

**Step 2:** \( p = 2(\alpha + \xi) - q, q = 1, \ldots, \xi \). In the following table: fixed \( p \), we describe the non-empty sets \( J_a^+(i, p) \), as the index \( i \) changes.

| \( p \)          | \( i: J_a^+(i, p) \neq \emptyset \) | \( J_a^+(i, p) \) |
|------------------|----------------------------------|-------------------|
| 2(\( \alpha + \xi) \) - \( q \) | \( i = \xi \)                     | \{ \( q - 2\alpha \), ... , \( q - 1 \), \( q \) \} |
| \( \vdots \)                              |                                  | \{ \( \ldots \), \( \ldots \), \( \ldots \) \} |
| \( i = \xi - q + 2\alpha \)               | \{ 0, \( 1 \), \( \ldots \), \( 2\alpha - 1 \), \( 2\alpha \) \} |
| \( \vdots \)                              |                                  | \{ \( \ldots \), \( \ldots \), \( \ldots \) \} |
| \( i = \xi - q + \alpha \)                | \{ 0, \( \ldots \), \( \alpha - 1 \), \( \alpha \) \} |
| \( \vdots \)                              |                                  | \{ \( \ldots \), \( \ldots \) \} |
| \( i = \xi - q + 1 \)                     | \{ \( 0 \), \( 1 \) \}          |
| \( i = \xi - q \)                         | \{ \( \ldots \), \( \ldots \), \( \ldots \) \} |

Looking at sets \( J_a^+(i, p) \), we distinguish three cases depending on the number of elements \#\( J_a^+(i, p) \):

(i) \( \#J_a^+(i, p) = 2\alpha + 1 \), we have
\[
\sum_{i=0}^{\xi} \tau_i \sum_{j \in \{i+\xi-p, \ldots, i+2\alpha+\xi-p\}} \tau_j \{u_{n+i}, u_{n+p+j}\} = \sum_{i=0}^{\xi} \tau_i \sum_{l=0}^{2\alpha} \tau_{r-l} \{u_{n+i}, u_{n+i+M-l}\}
\]
where \( r = i + M - p \). Now,
\[
\sum_{l=0}^{2\alpha} \tau_{r-l} \{u_{n+i}, u_{n+i+M-l}\} = \sum_{s=0}^{\alpha} \sum_{l=0}^{s+\alpha} \tau_{r-l} \lambda_{\alpha-t}^s f^\xi(u_{n+\alpha-s}, \ldots, u_{n+\alpha+\xi-s})
\]
and
\[
\sum_{l=0}^{s+\alpha} \tau_{r-l} \lambda_{\alpha-t+s} = \sum_{q=0}^{\alpha} \tau_{r-s-q} \lambda_{\alpha-q} \equiv 0 \text{, according to the recurrence relations (2.21) for the parameters } \tau_{l}\text{'s.}
(ii) When \(1 \leq \# J^+_{a} (i, p) \leq \alpha + 1\), the set \(J^+_{a} (i, p)\) is given by \(J^+_{a} (i, p) = \{0, \ldots, r\}\), for some \(0 \leq r \leq \alpha\). The summation (3.4) can be written in the following way

\[
\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \left[ \sum_{j \in \{0, \ldots, r\}} \tau_j \{u_{n+i, n+p+j}\} \right] \quad r = i - \xi + q.
\]

Noticing that

\[
\sum_{j \in \{0, \ldots, r\}} \tau_j \{u_{n+i, n+p+j}\} = \sum_{s=0}^{r} \tau_{r-s} \{u_{n+i, n+i+M-s}\} = \tau_0 \lambda_{r}^{\alpha-r} f_{r_n+i+\alpha-r} f_{\xi}
\]

we obtain

\[
\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \left[ \sum_{j \in \{0, \ldots, r\}} \tau_j \{u_{n+i, n+p+j}\} \right] = \sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \lambda_{i-\xi+q}^\alpha f_{r_n+i+\alpha-r} f_{\xi}
\]

and \(\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \lambda_{i-\xi+q}^\alpha = \sum_{t=0}^{\alpha} \tau_{t-\xi+q} t \lambda_{t}^\alpha \equiv 0\), see the recurrence (2.21).

(iii) When \(\alpha + 1 < \# J^+_{a} (i, p) < 2\alpha + 1\), the set \(J^+_{a} (i, p)\) is \(J^+_{a} (i, p) = \{0, \ldots, r\}\), for some \(\alpha < r \leq 2\alpha\). With analogous calculations, one can directly prove that the summation \(\sum_{j \in \{0, \ldots, r\}} \tau_j \{u_{n+i, n+p+j}\}\) vanishes.

**Step 3:** \(p = 2\alpha + \xi - q, \quad q = 1, \ldots, \alpha + 1\). The sets \(J^+_{a} (i, p)\) are given by

| \(p\) | \(i : J^+_{a} (i, p) \neq \emptyset\) | \(J^+_{a} (i, p)\) |
|------|-------------------------------|------------------|
| \(2\alpha + \xi - q\) | \(i = \xi\) | \{\(\xi - 2\alpha + q\) \ldots \(\xi\)\} |
| \(\vdots\) | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} |
| \(i = \xi - q\) | \{\(\xi - 2\alpha\) \ldots \(\ldots\) \(\xi - 1\) \(\xi\)\} |
| \(\vdots\) | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} |
| \(i = 2\alpha - q\) | \{0 \ldots \(\ldots\) \(\ldots\) \(2\alpha\)\} |
| \(\vdots\) | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} | \{\(\ldots\) \ldots \(\ldots\) \ldots \(\ldots\)\} |
| \(i = 0\) | \{0 \ldots \(q\)\} | \{0 \ldots \(q\)\} |

We find out again the two situations

\[
\begin{align*}
(i) & \quad \# J^+_{a} (i, p) = 2\alpha + 1 \quad \checkmark \\
(ii) & \quad \alpha + 1 < \# J^+_{a} (i, p) < 2\alpha + 1 \quad \checkmark
\end{align*}
\]

that we have been already studied in the previous Step. Then the summation (3.4) vanishes.
**Step 4:** $p = \alpha + \xi$. In this case, we have

| $p$ | $J^+_{i,p}$ | $i : J^+_a \neq \emptyset$ | $J^+_a(i)$ |
|-----|-------------|-----------------------------|------------|
| $\alpha + \xi$ | $\{i - \alpha, \ldots, i + \alpha\}$ | $i = \xi$ | $\{\xi - \alpha, \ldots, \xi\}$ |
| | | $\vdots$ | $\vdots$ |
| | | $i = \xi - \alpha$ | $\{\xi - 2\alpha, \ldots, \xi\}$ |
| | | $\vdots$ | $\vdots$ |
| | | $i = \alpha$ | $\{0, \ldots, 2\alpha\}$ |
| | | $\vdots$ | $\vdots$ |
| | $i = 0$ | $\{\}$ | $0, \ldots, \alpha$ |

three different cases

1. $(i)$ \#$J^+_a(i, p) = 2\alpha + 1 \checkmark$
2. $(ii)$ $\alpha + 1 < \#J^+_a(i, p) < 2\alpha + 1 \checkmark$
3. $(iii)$ \#$J^+_a(i, p) = \alpha + 1 \ ?$

Only the last one provides some contributions. Indeed,

$$\sum_{j \in \{0, \ldots, \alpha\}} \tau_j \{u_{n+i}, u_{n+p+j}\} = \tau_0 \lambda_0^0 f^\xi(u_{n+i}, \ldots, u_{n+i+\xi})$$

and

$$\sum_{j \in \{\xi - \alpha, \ldots, \xi\}} \tau_j \{u_{n+i}, u_{n+p+j}\} = \tau_\xi \lambda_0^0 f^\xi(u_{n+\alpha+i}, \ldots, u_{n+i+\alpha+\xi}).$$

Our summation (3.4) becomes

$$\{z_n, z_{n+\alpha+\xi}\} = \tau_0 \tau_\xi f^\xi(z_n) + \tau_0 \tau_\xi f^\xi(z_{n+\alpha+\xi}) = \tau_0 \tau_\xi [\exp(z_n) + \exp(z_{n+\alpha+\xi})].$$

Finally, **Step 5:** $p = 0, \ldots, \xi + \alpha - 1$ can be analyzed similarly to Step 2 and Step 3.

\[\square\]

### 4. Compatible pairs

We are now in a position to study the compatible pairs $(P, P')$ of PBs (2.1). We provide some necessary conditions that we expect to be also sufficient. This might be the starting point for a future classification of the still little-understood bi-hamiltonian higher order scalar-valued difference equations.

Let $P$ and $P'$ two PBs of type (2.1). Their leading order functions are given respectively by

\begin{align}
(4.1) \quad g_M(u_n, \ldots, u_{n+M}) &= a_M(u_n) f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}) a_M(u_{n+M}) \\
(4.2) \quad g'_M(u_n, \ldots, u_{n+M'}) &= a'_M(u_n) f'^\xi(u_{n+\alpha'}, \ldots, u_{n+\alpha'+\xi'}) a'_M(u_{n+M'})
\end{align}
where \( f_{n+\alpha}^\xi = \sigma_M \exp \left( \sum_{p=0}^\xi \tau_p u_{n+\alpha+p} \right) \) and \( f_{n+\alpha'}^{\xi'} = \sigma_{M'} \exp \left( \sum_{p=0}^{\xi'} \tau'_p u_{n+\alpha'+p} \right) \)

for some non-zero constant \( \sigma_M \) and \( \sigma_{M'} \). It is not restrictive to suppose \( M \geq M' \).

**Lemma 4.1.** A pair of non-constant PBs (2.1) \((P, P')\) forms a pencil of PBs only if there exists a local change of variables, reducing the leading coefficients (4.2) to the formulas

\[
g_M(u_n, \ldots, u_{n+M}) = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi})
\]

and

\[
g_{M'}(u_n, \ldots, u_{n+M'}) = f^{\xi'}(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi'})
\]

for some functions \( f^\xi \) and \( f^{\xi'} \). Notice that \( \alpha = \alpha' \).

**Proof.** Let \( P \) and \( P' \) PBs defined respectively by leading function (4.1) and (4.2). From equation

\[
[M, M'] \quad \{u_n, u_{n+M'}\}, u_{n+M+M'} = \{u_n, \{u_{n+M'}, u_{n+M+M'}\}\}
\]

we find out \( \log a_M(u_n)_{u_n} = (\log a_{M'}(u_n))_{u_n} \), that implies \( a_M(u_n) = k a_{M'}(u_n) \) for some constant \( k \). A suitable change of coordinates leads us to (4.3) and (4.4), where the constant \( k \) has been absorbed in the multiplicative constant \( \sigma_{M'} \). We hold \( \alpha = \alpha' \) looking, for example, at equation

\[
[M, M' - \alpha] \quad \{u_n, u_{n+M}\}, u_{n+M+M'-\alpha} = \{u_n, \{u_{n+M}, u_{n+M+M'-\alpha}\}\}
\]

that makes sense if and only if \( M' - \alpha \leq M' - \alpha' \), i.e. \( \alpha \geq \alpha' \). In such case we have

\[
f_{n+\alpha, u_{n+\alpha+\xi}}^{\xi} f_{\alpha+\xi+\alpha', u_{n+\alpha+\xi+\alpha'}}^{\xi'} = g_{\alpha+\xi+\alpha', u_{n+M}}, u_{n+\alpha+\xi+\alpha'}= \tau_{\xi} f_{\alpha+\xi, u_{n+\alpha+\xi}}^{\xi'}(u_{n+M}, \ldots, u_{n+\alpha+\xi+\alpha'})
\]

that gives \( g_{\alpha+\xi+\alpha', u_{n+M}}, u_{n+\alpha+\xi+\alpha'}= \tau_{\xi} f_{\alpha+\xi, u_{n+\alpha+\xi}}^{\xi'}(u_{n+\alpha+\xi+\alpha'}, \ldots, u_{n+\alpha+\xi+\alpha'+\xi'+\alpha'}) \).

This equation makes sense if \( \alpha \leq \alpha' \). It follows \( \alpha = \alpha' \), so it is not restrictive to suppose \( \xi > \xi' \). \( \square \)

**Theorem 4.2.** A pair of non-constant PBs (2.1) \((P, P')\), defined by leading functions (4.3) and (4.4) for certain parameters \( \tau \) and \( \tau' \) satisfying algebraic constraints of Theorem 2.7, forms a pencil of PBs only if

\[
\tau_p = \tau'_p = \tau_{\xi-\xi'+p},
\]

for any \( p = 0, \ldots, \xi' \).

**Proof.** Let \( P \) and \( P' \) be a pair of PBs (2.1) respectively, of order \( M = 2\alpha + \xi \) and \( M' = 2\alpha + \xi' \) (i.e. \( M > M' \)), leading functions (4.3) and (4.4) and suppose that any linear combination \( \mu P + \nu P' \), for \( \mu, \nu \) arbitrary constants, is a PB of order \( M \) (i.e. it satisfies the bi-linear PDEs, coming from Jacobi identity).

Analogously to the procedure followed in the proof of Theorem 2.5, we find necessary
conditions looking recursively at equations

\[ [M', M - \alpha - p], \, [M - \alpha - p, M'] \quad p = 0, \ldots, \alpha + \xi' \]

When \( p = 0 \), equation \([M', M - \alpha]\) gives us

\[ \{\{u_n, u_{n+M}\}, u_{n+M+\alpha}\} = \{u_n, \{u_{n+M}, u_{n+M+M'-\alpha}\}\} \]

gives us

\[
\log f^{\xi'}(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi'}, u_{n+\alpha+\xi'}) = \lambda^0_{0} \log f^{\xi}(u_{n+M'}, \ldots, u_{n+M'+\xi}, u_{n+M})
\]

that enable us to identify \( \tau_{\xi'} \equiv \tau_\xi \). Analogously, equation \([M - \alpha, M']\) provides \( \tau_0 \equiv \tau_0 \). Iterating this procedure, when \( 1 \leq p \leq \alpha \), equations \([M - \alpha - p, M']\) restricted to the function \( f^{\xi'} \) originate some constraints that can be organized in the following matrix form

\[
\begin{pmatrix}
\tau_\xi \lambda^\alpha_{\alpha-p} & 0 & \ldots & \ldots & 0 \\
0 & \lambda^\alpha_{\alpha-p+1} & \lambda^\alpha_{\alpha-p+1} & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \lambda^\alpha_{\alpha-1} & \lambda^\alpha_{\alpha-1} & \ldots & \lambda^\alpha_{\alpha-1} \\
0 & \lambda^\alpha_{\alpha-p} & \lambda^\alpha_{\alpha-p+1} & \ldots & \lambda^\alpha_{\alpha-p}
\end{pmatrix}
\begin{pmatrix}
\tau_0 \\
\tau_1 \\
\vdots \\
\tau_{p-1} \\
\tau_p
\end{pmatrix}
\]

and recursively on \( p \), we obtain that \( \lambda^\alpha_{\alpha-p} \equiv \lambda^\alpha_{\alpha-p} \), for any \( p = 0, \ldots, \alpha \). Adding the contribution coming from equations \([M', M - \alpha - p]\), the following constraints on the parameters \( \tau \)'s hold

\[
\tau_p = \tau_{p}' \\
\tau_{\xi-p} = \tau_{\xi-p}' \quad p = 0, \ldots, \alpha - 1.
\]

The formula (4.6) immediately follows. \( \square \)

4.1. Compatibility with constant brackets. We devote this sub-section to study the compatibility of pairs \((P, P')\) of PBs (2.1), where \( P' \) can be reduced by local change of coordinates to the constant form.

**Lemma 4.3.** Any non-constant PB (2.1) of order \( M = 2\alpha + \xi \), with \( \xi > 0 \) is compatible with a constant bracket of order \( M' \) only if \( M' \leq \alpha \). Moreover, the constant coefficients \( \{\sigma_i\}_{i=1, \ldots, M'} \) have to satisfy the following recurrences

\[
\sigma^\alpha_{\alpha-k} = \left( \sum_{i=0}^{k} \lambda^\alpha_{\alpha-k} \right) \sigma_\alpha, \quad k = 0, \ldots, \alpha,
\]

\[
(4.7)
\]

Proof. Let us fix a pair of PBs (2.1) \((P, P')\) such that their leading functions are of the form

\[
\begin{align*}
g_M(u_n, \ldots, u_{n+M}) &= a_M(u_n) f^{\xi}(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}) a_M(u_{n+M}) \\
g'_M(u_n, \ldots, u_{n+M'}) &= a'_M(u_n) \sigma_{M'} a_{M'}(u_{n+M}), \quad \sigma_{M'} \neq 0.
\end{align*}
\]
As in the proof of Lemma 4.3, from equation \([M, M']\), we obtain \((\log a_M(u_n))_{u_n} = (\log a'_{M'}(u_n))_{u_n}\), that is \(a_M(u_n) = k a_{M'}(u_n)\), for some constant \(k\).

Then, after a change of variables, we can reduce to consider

\[
g_M(u_n, \ldots, u_{n+M}) = f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi})
\]

\[
g'_M(u_n, \ldots, u_{n+M'}) = \sigma_{M'}.
\]

Let us now suppose \(\alpha < M' \leq M\), then from the equation

\[
[M, M' - \alpha] \quad \{\{u_n, u_{n+M}\}, u_{n+M+M'-\alpha}\} = \{u_n, \{u_{n+M}, u_{n+M+M'-\alpha}\}\}
\]

we obtain \(f^\xi(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi})u_{n+\alpha+\xi} \sigma_{M'} = 0\) that implies \(\sigma_{M'} \equiv 0\).

Let us suppose \(\xi > 0\) and \(k = 1, \ldots, \alpha\), then \(M - k > \alpha\). We focus our attention on equations

\[
[k, M - k] \quad \{\{u_n, u_{n+k}\}, u_{n+M}\} = \{u_n, \{u_{n+k}, u_{n+M}\}\} + \{\{u_n, u_{n+M}\}, u_{n+k}\},
\]

obtaining the constraints system \(\lambda_{\alpha-k}^{-1} \sigma_{\alpha} = \tau_0^{-1} \sum_{i=0}^{k} \sigma_{\alpha+i-k} \tau_i\) that can be written in the form (4.7).

\[\square\]

**Remark 4.4.** When \(\xi = 0\), we immediately find that any \((\alpha, 0)\)-bracket is compatible with a constant PB of order \(\alpha\), given by the formulas

\[
\{u_n, u_{n+\alpha}\} = \sigma_{\alpha}
\]

\[
\{u_n, u_{n+\alpha-s}\} \equiv 0, \quad s = 1, \ldots, \alpha - 1.
\]

**4.2. Examples.** We complete this Section, adding the details for the relevant family of \((1, \xi)\)-brackets, where we are able to prove that our necessary conditions are also sufficient. Let \(P\) be a \((1, \xi)\)-bracket,

- if \(\tau_0 \neq \tau_\xi\), according to Theorem 4.2, there are not \((1, \xi')\)-brackets \(P'\), such that the pair \((P, P')\) forms a pencil of PBs. Looking for constant brackets, we have that \(P\) is compatible only with first order constant brackets if and only if \(\xi = 0, 1\).

- if \(\tau_0 = \tau_\xi\), then necessarily \(M = 2K\) for some positive integer \(K\). Any pair of \((1, 2q)\)-brackets, with \(q = 1, \ldots, K\), defined by building functions of the form

\[
f^{2q}(u_{n+1}, \ldots, u_{n+2q-1}) = \exp(z_{n+1}^{(2q)}),
\]

where \(z_{n+1}^{(2q)} \equiv \sum_{i=0}^{2q-1} (-\tau_0)^i u_{n+i}\), forms a pencil.

5. **On non-degenerate, vector-valued PBs**

In order to provide some examples of vector-valued PBs, we can define new lattice variables, according to the following
Proposition 5.1. Let us consider a pair of positive integers \((M, K)\), with \(K \leq M\). Any scalar-valued PB \((2.1)\) of order \(M\), according to the formula
\[ u_{n+1}^1 + p_n = u_{nK+p}^1, \quad p = 0, \ldots, K - 1, \]
is transformed into a non-degenerate PB (i.e. the leading order is given by a non-singular matrix) of order \(A\) and target space of dim. \(K\) iff \(M = AK\), for some positive integer \(A\).

We are interested on vector-valued PBs of first order (i.e. \(A = 1\)), then we put \(K = M\) in the above proposition. First, let us recall some preliminary definitions.

A Lie group \(G\), with a Poisson bracket \(\{ \cdot, \cdot \}_G\) is a Lie-Poisson group if the multiplication \(\mu : G \times G \rightarrow G\) is a mapping of Poisson manifolds, where on \(G \times G\) is defined the bracket \(\{ \phi, \psi \}_G\times_G(g, h) = \{\phi(\cdot , h), \psi(\cdot , h)\}_G(g) + \{\phi(g, \cdot), \psi(g, \cdot)\}_G(h)\), with \(h, g \in G\). Let \(c^k_{ij}\) be the structure constants of a Lie algebra \(g\). The couple \((g, \gamma)\) is a Lie bi-algebra if and only if

(i) \(\gamma\) is a 1-cocycle on \(g\) with values on \(g \otimes g\), where \(g\) acts on \(g \otimes g\) by the adjoint representation \(\text{ad}^2(\xi) = \text{ad}\xi \otimes 1 + 1 \otimes \text{ad}\xi\), that means: \(\delta_\text{ad}\gamma = 0\), i.e.
\[ \text{ad}^2(\gamma(\eta)) - \text{ad}^2(\gamma(\xi)) - \gamma([\xi, \eta]) = 0, \]
or, fixed a basis of \(g\),
\[ \gamma_{ij}^k \text{c}^k_{rs} = \text{c}^r_{ps} \gamma_{ij}^q + \text{c}^r_{pq} \gamma_{ij}^s - \text{c}^s_{pr} \gamma_{ij}^q - \text{c}^s_{ps} \gamma_{ij}^r. \]

(ii) \(\gamma : g^* \otimes g^* \rightarrow g^*\) defines a Lie bracket on \(g^*\): \([\xi, \eta]_{g^*} = \gamma(\xi \otimes \eta)\).

The correspondence between the Lie-Poisson groups and the Lie bi-algebras is clarified by the following

Theorem 5.2. Let \(G\) a Lie group, with tangent Lie algebra \(g\): locally a Lie-Poisson structure on \(G\) is uniquely (up to isomorphism) determined by a Lie algebra structure on the dual space \(g^*\), then \(g\) is a Lie bialgebra \((g, \gamma)\).

Proof. All the detail of the proof can be found in [8]. Here we recall some ideas about the direction from Lie bi-algebra to Lie-Poisson group. Fixed a basis of \((g, \gamma)\), the constants \(\gamma_{pq}^k\) define a Lie bracket on \(g^*\). Moreover, solving the differential equation
\[ \gamma_{pq}^k = \partial_k \pi_{G}^{pq}|_e, \]
we can define \(\{\phi, \psi\}_G \doteq \partial_\phi \pi_{G}^{pq} \partial_\psi, \psi,\) that is a PBs because the compatibility condition of the system
\[ (5.1) \quad \begin{cases} \partial_k \pi_{G}^{pq} = c^p_{sk} \pi_{G}^{sq} + c^q_{sk} \pi_{G}^{ps} + \gamma_{pq}^k \\ \pi_{G}^{pq}|_e = 0 \end{cases} \]
is guaranteed by \((i)\), i.e. \(\gamma\) is a 1–cocycle on \(g\). \(\square\)

We focus on the sub-class of Lie Poisson group, given by the following

Definition 5.3. A Lie-Poisson group \(\{G, \{, \}_G\}\) is called admissible, if there exist:
(i) A skew-symmetric matrix \( k \in \Lambda^2 g \), such that the cohomologous 1-cocycle \( \tilde{\gamma} \), defined by \( \tilde{\gamma} = \delta_{ad}k + \gamma \), i.e. \( \tilde{\gamma}^pq = \gamma^pq + \delta_{ad}k^q + k^pq c^q_{st} \), provides a Lie algebra structure on \( g^* \). Notice that \( k \) has to satisfy the Yang-Baxter equation

\[
k^q c^p_{st} k^r + k^p c^q_{st} k^r + k^p c^r_{sq} k^q = \gamma^pq k^r + \gamma^rq k^q + \gamma^qr k^p.
\]

(ii) A Lie algebra homomorphism \( r : (g^*, \gamma^pq_s) \longrightarrow (g, c^s_{pq}) \) such that \( r^*_s : r^{pq}_s = r^{qp} \) defines a Lie algebra homomorphism \( r^*_s : (g^*, \gamma^pq_s) \longrightarrow (g, c^s_{pq}) \).

We are now in a position to formulate the Dubrovin's theorem [3].

**Theorem 5.4.** An admissible Lie-Poisson group \( (G, \{, \}_G) \) together with corresponding matrices \( r, k \) defines a Poisson bracket of the form

\[
\{ u^i_n, u^j_n \}_1 = h^{ij}(u_n) \\
\{ u^i_n, u^j_{n+1} \}_1 = g^{ij}(u_n, u_{n+1})
\]

where \( u_n \in G \) for all \( n \), according to the following formulas

\[
\{ \varphi(u_n), \psi(u_{n+1}) \}_1 = \partial_\alpha \varphi(u_n) r^{\alpha\beta} \partial_\beta \psi(u_{n+1})
\]

where \( \partial_\alpha \) and \( \partial_\beta \) are left- and right-invariant vector fields on \( G \),

\[
\{ \varphi(u_n), \psi(u_n) \}_1 = h^{\alpha\beta}(u_n) \partial_\alpha \varphi(u_n) \partial_\beta \psi(u_n)
\]

where \( h^{\alpha\beta}(u_n) = \delta_{ad}^{\alpha\beta}(u_n) + \text{Ad}^{(2)}_{u_n} k^{\alpha\beta} \) and \( \pi^{\alpha\beta}_G(u_n) \) is determined by the system (5.1).

**Remark 5.5.** This theorem does not seem to have simple applications in the lattice systems, studied in the literature. For example, one can notice that the fundamental Toda lattice has three well-known local compatible PBs (see [12]), but only the quadratic one is non-degenerate and can be represent using the previous theorem on the algebra of \( \text{Aff}^0 \mathbb{R}^1 \), group of affine transformations of the straight line.

Our new vector-valued \( (N > 1) \) PBs (5.2) are provided by the following

**Theorem 5.6.** Any \( (\alpha, \xi) \)-bracket, after the consolidation lattice procedure, becomes a non-degenerate PB of the form (5.2), with associated Lie bi-algebra given by

\[
g_{(\alpha, \xi)} = \text{span} \{ L_s \}_{s=1,...,M} : [L_p, L_q] (v_n) = c^r_{pq} L_r (v_n) \\
g^*_{(\alpha, \xi)} = \text{span} \{ R_s \}_{s=1,...,M} : [R_p, R_q] (v_n) = \gamma^r_{pq} R_r (v_n),
\]

where the summation is over the repeated index \( r \) and \( c^r_{pq} \) and \( \gamma^r_{pq} \) are certain constants depending on the parameters \( \tau \)'s.

**Proof.** Choosing \( K = M \) in the Proposition 5.1, we have

\[
g^{ij}(v_n, v_{n+1}) = \begin{pmatrix}
\{ v^1_n, v^1_{n+1} \}_M \\
\{ v^2_n, v^2_{n+1} \}_M \\
\vdots \\
\{ v^M_n, v^M_{n+1} \}_M \\
\{ v^1_n, v^1_{n+1} \}_M \\
\vdots \\
\{ v^M_n, v^M_{n+1} \}_M
\end{pmatrix}
\]
and \( h^{ij}(v_n) = g^{ij}(v_n, v_n) - g^{ij}(v_n, v_n) \). Substituting the explicit formulas (2.6), we find out a decomposition of the leading order \( g^{ij}(v_n, v_{n+1}) = L^i_\mu(v_n) R^{\mu j}(v_{n+1}) \). The matrix \( L(v_n) \) is given by \( L(v_n) \equiv L \) \( \text{diag}(l_1, \ldots, l_M) \) \((v_n)\) where \( l_s(v_n^\alpha, \ldots, v_n^\xi) = \exp \left( \sum_{i=0}^\xi \tau_i v_n^{\alpha+s+i} \right) \) and

\[
\Lambda_L \equiv \left( \begin{array}{cccc}
\lambda_0^\alpha \\
\vdots \\
\lambda_0^0 \\
\vdots \\
\lambda_0^0 \\
\vdots \\
\lambda_0^0 \\
\lambda_0^0 \\
\lambda_0^\alpha
\end{array} \right)
\]

Analogously the matrix \( R(v_{n+1}) \) factorizes into \( R(v_{n+1}) \equiv R \) \( \text{diag}(r_1, \ldots, r_M) \) \( \Lambda_R \), where \( r_s(v_{n+1}^1, \ldots, v_{n+1}^\alpha) = \exp \left( \sum_{i=0}^{\alpha-s+1} \tau_i v_{n+1}^{\alpha-s+i} \right) \).

\[
\Lambda_R \equiv \left( \begin{array}{cccc}
\lambda_0^\alpha \\
\vdots \\
\lambda_0^0 \\
\vdots \\
\lambda_0^0 \\
\vdots \\
\lambda_0^0 \\
\lambda_0^0 \\
\lambda_0^\alpha
\end{array} \right)
\]

Notice that we are using the notation \( v_n^s \equiv 0 \) if \( s < 1 \) or \( s > M \). Moreover, to avoid some technicalities, we suppose \( \xi > \alpha \).

Now, let \( \{L_s\}_{s=1, \ldots, M} \) be the s-th column of matrix \( L \) and \( g_{(\alpha, \xi)} \equiv \text{span} \{L_s\} \) as vector space. By direct computation, we find that the following commutators

\[
[L_p, L_q]^k(v_n) = \left( L^s_q(v_n) L^k_p(v_n) - L^k_q(v_n) L^s_p(v_n) \right) = c_{pq}^r L_r(v_n)
\]

equip \( g_{(\alpha, \xi)} \) of a Lie algebra structure. At first, we observe that if \( \alpha + \xi < q \leq M \), \( L_q(v_n) \) are constant vector fields. Then

\[
[L_p, L_q]^k(v_n) = \sigma_{pq} L_p(v_n), \quad p = 1, \ldots, M,
\]

for some constants \( \sigma_{pq} \) that can be expressed in terms of parameters \( \tau \).

In particular, observe that \( [L_s, L_{\alpha+\xi+t}]^k(v_n) = \delta_{s,t} L_s \), where \( s, t = 1, \ldots, \alpha \) and \( \delta \) is the Kronecker symbol.

Let us now consider \( p, q = 1, \ldots, \alpha + \xi \). It is not restrictive to suppose \( q > p \). Denoting \( t = q - p \), the formulas (5.3) for the commutators reduce to

\[
[L_p, L_{p+t}]^k(v_n) = L^s_{p+t}(v_n) L^k_{p,s}(v_n).
\]

We distinguish two cases:
(i) When $t \leq \alpha$, we have $[L_p, L_{p+t}]^k (v_n) = \lambda_{\alpha-k+p}^{\alpha-k+p} \left( \lambda_{\alpha-t}^{\alpha-t} l_p (v_{n+t}) l_{p+t} (v_n) \right)$. Let us focus on the summation $\sum_{i=0}^{t} \lambda_{\alpha-t}^{\alpha-t} t_i$. According to formulas (2.7) and (2.21), we have

$$\sum_{i=0}^{t} \lambda_{\alpha-t}^{\alpha-t} t_i = \lambda_0^0 \sum_{i=0}^{t} \lambda_{\alpha-t}^{\alpha-t} t_i \equiv 0.$$

(ii) When $t > \alpha$, we obtain $[L_p, L_{p+t}]^k (v_n) = \lambda_{\alpha-k+p}^{\alpha-k+p} \left( \lambda_{\alpha-t}^{\alpha-t} l_p (v_{n+t}) l_{p+t} (v_n) \right)$, and we have $\sum_{i=0}^{\alpha} \lambda_{\alpha-t}^{\alpha-t} t_{\alpha-t}^{\alpha-t} t_i \equiv 0$, according to (2.21).

Finally, starting from the rows of matrix $R$, we can repeat the procedure above finding the Lie algebra structure on $\mathfrak{g}^*$.

□

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