A SEQUENTIAL LEAST SQUARES METHOD FOR ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

RUO LI AND FANYI YANG

Abstract. We develop a new least squares method for solving the second-order elliptic equations in non-divergence form. Two least-squares-type functionals are proposed for solving the equations in two steps. We first obtain a numerical approximation to the gradient in a piecewisely irrotational polynomial space. Then, together with the numerical gradient, we seek a numerical solution of the primitive variable in continuous finite element space. The variational setting naturally provides a posteriori error which could be used in an adaptive refinement algorithm. The error estimates in $L^2$ norm and energy norms for both two unknowns are derived. By a series of numerical experiments, we verify the convergence rates and show the efficiency of the adaptive algorithm.

keywords: non-divergence form, least squares method, piecewisely irrotational space, discontinuous Galerkin method.

1. Introduction

This work is concerned with the non-divergence form second-order elliptic equation, which is often encountered in many applications from areas such as probability and stochastic processes [18]. In addition, such problems also naturally arise as the linearization to fully nonlinear PDEs, as obtained by applying the Newton’s iterative method, see [7, 9]. Due to the non-divergence structure, it is invalid to derive a variational formulation by applying the integration by parts. Instead, the existence and uniqueness of the solutions to this problem are sought in the strong sense, we refer to [8, 18, 19, 2, 10] and the references therein for the well-posedness of the solutions to the non-divergence form second-order elliptic equation.

Recently several finite element methods have been proposed, though such a problem does not naturally fit within the standard Galerkin framework. Conforming finite element methods require $H^2$-regularity for approximating the strong solution, which naturally leads to a $C^1$ finite element space [6, 4]. But the $C^1$ finite elements are sometimes considered impractical. In [14], the authors introduced a mixed finite element method with $C^0$ finite element space via a finite element Hessian obtained in the same approximation space. In [8], the authors proposed and analyzed a finite element method with $C^0$ space by introducing an interior penalty term. But the coefficient matrix is assumed to be continuous. Gallistl introduced a conforming mixed finite element method based on a least squares functional, we refer to [10] for more details. In [17], the authors proposed a simple and convergent finite element method with $C^0$ finite element space. Based on discontinuous approximations, Smears and Süli proposed a discontinuous Galerkin method where the optimal convergence rate in $h$ with respect to broken $H^2$ norm is proven and the authors have extended this method to the Hamilton-Jacobi-Bellman equations [19, 11]. Besides, Wang et al proposed a weak Galerkin method and we refer to [20] for details.

In this paper, we propose a new least squares finite element method for solving the non-divergence elliptic problem. We rewrite the equation into an equivalent first-order system as a fundamental requirement in modern least squares method [5]. We employ two different approximation spaces to solve the gradient and the primitive variable sequentially, which is motivated from the idea in [15]. We first define a least squares functional to seek a numerical approximation.
to the gradient in a piecewisely irrotational polynomial space. Then we obtain the approximation to the primitive variable with the numerical gradient by solving another least squares problem in standard \( C^0 \) finite element space. Our method avoids solving a saddle-point problem of mixed formulation, and in contrast to \cite{18, 17, 11} our method only involves the first-order operator in each step. We prove the convergence rates for both variables in \( L^2 \) norm and energy norm. The least squares functional naturally serves as a \textit{a posteriori} error estimate and we introduce an adaptive algorithm for solving the problem of low regularity. By carrying out a series of numerical experiments, we verify the convergence orders in the error estimates and illustrate the efficiency of the adaptive algorithm.

The rest of this paper is organized as follows. Section 2 gives the notation that will be used throughout the paper and defines the considered problem. In Section 3, we introduce the piecewisely irrotational approximation space and give some properties of this space. In Section 4, we propose the least squares method for both two variables respectively and the error estimates are derived. In Section 5, a series of numerical experiments are presented for testing the accuracy of proposed scheme.

2. Preliminaries

Let \( \Omega \subset \mathbb{R}^d(d = 2, 3) \) be a bounded convex domain with smooth boundary. We denote by \( T_h \) a regular and shape-regular subdivision of \( \Omega \) into simplexes. Let \( \mathcal{E}_h^b \) be the set of all interior faces associated with the subdivision \( T_h \), \( \mathcal{E}_h \) the set of all faces lying on \( \partial \Omega \) and then \( \mathcal{E}_h = \mathcal{E}_h^b \cup \mathcal{E}_h^b \). We define \( h_K = \text{diam}(K), \ \forall K \in T_h, \ h_e = \text{diam}(e), \ \forall e \in \mathcal{E}_h \), and we set \( h = h_{\text{max}} = \max_{K \in T_h} h_K \).

We then introduce the trace operators commonly used in DG framework. Let \( K^+ \) and \( K^- \) be two adjacent elements sharing an interior face \( e = \partial K^+ \cap \partial K^- \in \mathcal{E}_h^b \) with the unit outward normal vectors \( \mathbf{n}^+ \) and \( \mathbf{n}^- \), respectively. Let \( v \) and \( \mathbf{v} \) be the scalar-valued and vector-valued functions that may be discontinuous across \( \mathcal{E}_h^b \). For \( v^+ := \left| v \right|_{e \subset \partial K^+}, \ v^- := \left| v \right|_{e \subset \partial K^-}, \ v^+ := \left| v \right|_{e \subset \partial K^+}, \ v^- := \left| v \right|_{e \subset \partial K^-}, \), we set the average operator \( \{ \cdot \} \) as
\[
\{ v \} := \frac{1}{2} (v^+ + v^-), \quad \{ \mathbf{v} \} := \frac{1}{2} (v^+ + v^-),
\]
and we set the jump operator \([ \cdot ]\) as
\[
[v] := v^+ n^+ + v^- n^-, \quad [\mathbf{v} \cdot \mathbf{n}] := v^+ \cdot n^+ + v^- \cdot n^-,
\]
\[
[v \times \mathbf{n}] := v^+ \times n^+ + v^- \times n^-, \quad [\mathbf{v} \otimes \mathbf{n}] := v^+ \otimes n^+ + v^- \otimes n^-.
\]
For \( e \in \mathcal{E}_h^b \), these definitions shall be modified as follows:
\[
\{ v \} := v, \quad \{ \mathbf{v} \} := \mathbf{v}, \quad [v] := v, \quad [v] := v, \quad [\mathbf{v} \cdot \mathbf{n}] := \mathbf{v} \cdot \mathbf{n}, \quad [\mathbf{v} \times \mathbf{n}] := \mathbf{v} \times \mathbf{n}, \quad [\mathbf{v} \otimes \mathbf{n}] := \mathbf{v} \otimes \mathbf{n}.
\]

Throughout this paper, let us note that \( C \) and \( C \) with a subscript are generic constants that may be different from line to line but are independent of \( h \). We would also use the standard notation and definition for the spaces \( L^r(D), L^r(D)^d, L^r(D)^d, H^r(D), H^r(D)^d, H^r(D)^d, H^r(D)^d \) with \( D \) a bounded domain and \( r \) a positive integer (may be \( \infty \)), and their associated inner products and norms. We define the Sobolev space of irrotational vector fields by
\[
\mathbf{\Gamma}(D) := \{ \mathbf{v} \in H^r(D)^d \mid \nabla \times \mathbf{v} = 0 \text{ in } \Omega \}.
\]
Further, for the partition \( T_h \) we would follow the standard definitions for the broken Sobolev spaces \( L^2(T_h), L^2(T_h)^d, L^2(T_h)^d, H^r(T_h), H^r(T_h)^d, H^r(T_h)^d \) and their corresponding broken norms \cite{1}.
The problem dealt with in this paper is to find numerical approximation to the strong solution for the elliptic problem in non-divergence form, which reads

\[ \mathcal{L}u := A : D^2u = f \quad \text{in } \Omega, \]
\[ \quad u = g \quad \text{on } \partial \Omega, \]

where \( : \) denotes the Frobenius inner product between two matrices. The coefficient matrix \( A(x) = \{a_{ij}(x)\} \in L^\infty(\Omega)^{d \times d} \) is assumed to be uniformly elliptic, i.e. there exist two positive constants \( \nu \) and \( \nu_s \) satisfying

\[ \nu |\xi|^2 \leq \xi^T A(x) \xi \leq \nu_s |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \text{a.e. in } \Omega. \]

We furthermore assume that the coefficient satisfies the Cordès condition: there exists a positive constant \( \varepsilon \leq 1 \) such that

\[ \frac{|A|^2}{(\text{tr } A)^2} \leq \frac{1}{d-1 + \varepsilon}, \quad \text{a.e. in } \Omega, \]

where \( |A| := \sqrt{A \cdot A} \) denotes the Frobenius norm. The uniform ellipticity of the coefficient cannot ensure the well-posedness of the problem (1), at least in three dimensions. If the condition (2) holds, there exists a unique strong solution \( u \in H^2(\Omega) \) to (1) with proper source term \( f \) and boundary condition \( g \), we refer to [18, 8, 19] for more regularity results of the problem (1). Particularly, the uniformly elliptic coefficient \( A \) directly implies the Cordès condition (2) for the planar case [19].

In this paper, we introduce the gradient variable \( p = \nabla u \) and the scalar elliptic problem in (1) may be rewritten into the first-order system:

\[ A : \nabla p = f \quad \text{in } \Omega, \]
\[ p - \nabla u = 0 \quad \text{in } \Omega, \]
\[ u = g \quad \text{on } \partial \Omega. \]

To transform the problem into first-order system is one of the fundamental ideas in modern least squares finite element method [5] and our proposed least squares method is based on the formulation (3).

3. The finite element space

In this section, we introduce the locally curl-free finite element space \( S^m_h \) with integer \( m \geq 0 \), which is defined as

\[ S^m_h = \{ v \in L^2(\Omega)^d \mid v|_K \in P_m(K)^d, \quad \nabla \times (v|_K) = 0, \quad \forall K \in \mathcal{T}_h \}. \]

We first give some basic properties of \( S^m_h \) which are very essential in convergence analysis. We set \( S^m(D) := P_m(D)^d \cap \Gamma^0(D) \) as the space of irrotational polynomials of degree at most \( m \) on the domain \( D \). Obviously, we have \( S^m_h = \Pi_{K \in \mathcal{T}_h} S^m(K) \).

**Lemma 1.** For \( q \in \Gamma^{m+1}(K) \) and an element \( K \in \mathcal{T}_h \), there exists a polynomial \( \tilde{q} \in S^m(K) \) such that

\[ ||q - \tilde{q}||_{H^{k+1}(K)} \leq Ch^{m+1-k}||q||_{H^{m+1}(K)}, \quad 0 \leq k \leq m + 1. \]

**Proof.** Based on the fact that \( \Gamma^{m+1}(K) = \nabla H^{m+2}(K) \) [13], we have that there exists a function \( v \in H^{m+2}(K) \) satisfying \( q = \nabla v \). We denote by \( \tilde{v} \in P_{m+1}(K) \) the standard nodal interpolation polynomial of \( v \). The estimate (4) is implied by the approximation property of \( \tilde{v} \) with \( q = \nabla \tilde{v} \in S^m(K) \). \( \square \)
Proof. We first proof for the planar case. Then the we could obtain the following local approximation property of \( \pi_K^{S,m} \) from Lemma 1.

**Lemma 2.** For any \( q \in \Gamma^{m+1}(\Omega) \) and any element \( K \in \mathcal{T}_h \), the following estimates hold:

\[
\| q - \pi_K^{S,m} q \|_{H^k(K)} \leq Ch_K^{m+1-k} \| q \|_{H^{m+1}(\Omega)}, \quad 0 \leq k \leq m + 1,
\]

\[
\| \partial^k (q - \pi_K^{S,m} q) \|_{L^2(\partial K)} \leq Ch_K^{m+1/2-k} \| q \|_{H^{m+1}(\Omega)}, \quad 0 \leq k \leq m.
\]

**Proof.** Obviously from (5) one has that

\[
\pi_K^{S,m} r = r, \quad \forall r \in S^m(K).
\]

Applying the inverse inequality could lead to

\[
\| q - \pi_K^{S,m} q \|_{H^k(K)} \leq \| q - \tilde{q} \|_{H^k(K)} + \| \pi_K^{S,m} (\tilde{q} - q) \|_{H^k(K)}
\]

\[
\leq \| q - \tilde{q} \|_{H^k(K)} + Ch_K^{-k} \| \pi_K^{S,m} (\tilde{q} - q) \|_{L^2(K)}
\]

\[
\leq \| q - \tilde{q} \|_{H^k(K)} + Ch_K^{-k} \| q - \pi_K^{S,m} q \|_{L^2(K)} + Ch_K^{-k} \| q - \pi_K^{S,m} q \|_{L^2(K)}
\]

\[
\leq Ch_K^{m+1-k} \| q \|_{H^{m+1}(\Omega)},
\]

where \( \tilde{q} \) is defined in Lemma 1. Similarly, by trace inequality it is trivial to obtain the trace estimate in (6), which completes the proof.

Furthermore, we define a global \( L^2 \)-projection \( \Pi_h^{S,m} \) in a piecewise manner: for any \( q \in \Gamma^{m+1}(\Omega) \), \( \Pi_h^{S,m} q \in S^m_h \) is denoted by

\[
\Pi_h^{S,m} q = \pi_K^{S,m} q, \quad \forall K \in \mathcal{T}_h.
\]

Clearly, the global \( L^2 \)-projection has the following approximation property:

**Lemma 3.** For any \( q \in \Gamma^{m+1}(\Omega) \) and any element \( K \in \mathcal{T}_h \), the following estimates hold:

\[
\| q - \Pi_h^{S,m} q \|_{H^k(K)} \leq Ch_K^{m+1-k} \| q \|_{H^{m+1}(\Omega)}, \quad 0 \leq k \leq m + 1,
\]

\[
\| \partial^k (q - \Pi_h^{S,m} q) \|_{L^2(\partial K)} \leq Ch_K^{m+1/2-k} \| q \|_{H^{m+1}(\Omega)}, \quad 0 \leq k \leq m.
\]

**Proof.** It is a direct extension of Lemma 2.

For the analysis of convergence, we may require the following estimate.

**Lemma 4.** Let \( S^m_h \) denote the \( m \)-th degree piecewisely irrotational finite element space with \( 1 \leq m \leq 2 \) if \( d = 3 \), and \( m \geq 1 \) if \( d = 2 \). Then for any \( p_h \in S^m_h \), the following inequality holds:

\[
\| \nabla p_h \|_{L^2(\mathcal{T}_h)} \leq \| \nabla \cdot p_h \|_{L^2(\mathcal{T}_h)} + C \left( \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \| [p_h \otimes n] \|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \| p_h \times n \|_{L^2(e)}^2 \right)^{1/2}.
\]

**Proof.** We first proof for the planar case \( d = 2 \). Since \( S^m_h = \Pi_{K \in \mathcal{T}_h} S^m(K) \) and the fact \( S^m(K) = \nabla P_{m+1}(K) \), there exists a function \( v_h \in V^m_h \) such that \( p_h = \nabla v_h \) in every element, where \( V^r_h \) denotes the \( r \)-th degree piecewisely polynomial space,

\[
V^r_h := \{ v_h \in L^2(\Omega) \mid v_h|_K \in P_r(K), \forall K \in \mathcal{T}_h \}.
\]
Let $\Pi_h^0$ be the $L^2$-projection from $L^2(\Omega)$ to the piecewise constant space $V_h^0$. More precisely,

$$(\Pi_h^0 v)|_K = \int_K v \, dx, \quad \forall K \in T_h, \quad \forall v \in L^2(\Omega).$$

We let $\tilde{v}_h := v_h - \Pi_h^0 v_h$ and it is clear that $p_h = \nabla \tilde{v}_h$ in any $K \in T_h$.

We then introduce an interpolation operator $E$ from $V_h^0$ to a $C^1$-conforming space consisting of macro-elements, which could be regarded as the high-order version of the Hsieh-Clough-Tocher macro-element [12]. The operator $E : V_h^r \to H^2(\Omega) \cap H^1_0(\Omega)$ with $r \geq 2$ satisfies the bound:

$$\sum_{K \in T_h} |\tilde{v}_h - E(\tilde{v}_h)|_{H^1(K)}^2 \leq C \left( \sum_{e \in E_h} h_e^{-2k} \|\tilde{v}_h\|_{L^2(e)}^2 + \sum_{e \in E_h} h_e^{3-2k} \|\nabla \tilde{v}_h \cdot n\|_{L^2(e)}^2 \right),$$

for $k = 0, 1, 2$. We refer to [12, 17] for more details about the operator $E$. Since $E(\tilde{v}_h) \in H^2(\Omega) \cap H^1_0(\Omega)$, by the Miranda-Talenti inequality $\|D^2 E(\tilde{v}_h)\|_{L^2(\Omega)} \leq \|\Delta E(\tilde{v}_h)\|_{L^2(\Omega)}$, we could conclude that

$$\|\nabla p_h\|_{L^2(T_h)} = \|D^2 v_h\|_{L^2(T_h)} \leq \|\nabla E(\tilde{v}_h)\|_{L^2(T_h)} \leq \|D^2 \tilde{v}_h\|_{L^2(T_h)} + \|D^2 E(\tilde{v}_h)\|_{L^2(T_h)}$$

and

$$\|\nabla \tilde{v}_h - E(\tilde{v}_h)\|_{L^2(T_h)} \leq \|\Delta \tilde{v}_h - E(\tilde{v}_h)\|_{L^2(T_h)} + \|\Delta E(\tilde{v}_h)\|_{L^2(T_h)} \leq e^{-2k} \|\nabla \tilde{v}_h \cdot n\|_{L^2(T_h)^2}.$$
The proof could be extended to the case \( d = 3 \) by using the similar interpolation operator. However, in three dimensions the degrees of freedom for higher-order Hsieh-Clough-Tocher spaces are not found in the literature [17]. We refer to [17] for the interpolation operator in low-order case. As a result, the estimate (4) is limited to \( 1 \leq m \leq 2 \) in three dimensions.

\[ \square \]

Remark 1. Lemma 4 is crucial in the convergence analysis, which requires \( m \leq 2 \) in three dimensions. In section 5, numerical results demonstrate the convergence of the numerical solution for \( m \geq 3 \). The theoretical verification for the case \( d = 3, m \geq 3 \) is considered as the future work.

To end this section, we outline a method for constructing bases for the space \( \mathbf{S}_m^h \). One could take the gradient of the natural basis polynomials
\[
1, x, y, x^2, xy, y^2, \ldots
\]
to get a basis for the finite elements of \( \mathbf{S}_m^h \). For an instance, in two dimensions if linear accuracy is considered, one could obtain the basis functions,
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1
\end{bmatrix},
\begin{bmatrix}
x \\
y
\end{bmatrix},
\begin{bmatrix}
y \\
x
\end{bmatrix}.
\]
Furthermore, there are also 4 second-order and 5 third-order basis functions:
\[
\begin{bmatrix}
x^2 \\
y^2
\end{bmatrix},
\begin{bmatrix}
x^2y \\
yx
\end{bmatrix},
\begin{bmatrix}
x^2y^2 \\
x^2y_z
\end{bmatrix},
\begin{bmatrix}
y^3 \\
wy
\end{bmatrix}.
\]
For the case \( d = 3 \), the basis functions could be constructed in a similar way: for \( m = 1 \), there are 9 basis functions which read
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1
\end{bmatrix},
\begin{bmatrix}
x \\
y
\end{bmatrix},
\begin{bmatrix}
y \\
x
\end{bmatrix},
\begin{bmatrix}
z \\
x
\end{bmatrix},
\begin{bmatrix}
y \\
z
\end{bmatrix},
\begin{bmatrix}
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
z
\end{bmatrix},
\begin{bmatrix}
0 \\
y
\end{bmatrix}.
\]
In our implementation, a normalization and a translation of the coordinates is applied to guarantee the numerical stability [16]. Taking 2D case as an example, we denote \((X,Y)\) in each element by
\[
X = \frac{x - x_c}{\sqrt{T}}, \quad Y = \frac{y - y_c}{\sqrt{T}},
\]
where \((x_c, y_c)\) is the barycenter of the triangular element and \( T \) is its area. Substituting \((X,Y)\) for \((x,y)\) in these basis functions could share a better numerical stability while the local irrotational property still holds.

4. Sequential Least Squares Method

In this section, we consider a least squares method based on the first-order system (3) to approximate \( p \) and \( u \) sequentially. Let us first define a least squares functional \( J_p^h(\cdot) \) by
\[
J_p^h(q) := \sum_{K \in \mathcal{T}_h} \| A : \nabla q - f \|_{L^2(K)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\mu}{h_e} \| [q \otimes n] \|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\mu}{h_e} \| q \times n - \nabla g \times n \|_{L^2(e)}^2,
\]
for seeking a numerical approximation of the variable \( p \). The functional \( J_p^h(\cdot) \) consists of the part related to the gradient \( p \) in (3) and the terms on the faces, and \( \mu \) is the penalty parameter which
will be specified later on. Minimizing the problem \((10)\) in the space \(S_h^m\) gives an approximation to the gradient \(p\), which reads
\[
\inf_{q_h \in S_h^m} J_h^P(q_h).
\]
Thus, the corresponding variational equation takes the form: find \(p_h \in S_h^m\) such that
\[
a_h^P(p_h, q_h) := \| p \otimes n \|_{L^2(e)}^2 + \frac{1}{h_e} \| p \times n \|_{L^2(e)}^2,
\]
where the bilinear form \(a_h^P(\cdot, \cdot)\) is
\[
a_h^P(p_h, q_h) = \sum_{K \in T_h} \int_K (A : \nabla p_h)(A : \nabla q_h)dx + \sum_{e \in E_h} \int_e \frac{\mu}{h_e} (p \otimes n)[q_h \otimes n]ds
\]
and the linear form \(l_h^P(\cdot)\) is
\[
l_h^P(q_h) = \sum_{K \in T_h} \int_K f(A : \nabla q_h)dx + \sum_{e \in E_h} \int_e \frac{\mu}{h_e} (p_h \times n) \cdot (q_h \times n)ds.
\]
We define a constant \(\gamma\) as
\[
\gamma = \frac{\text{tr}(A)}{|A|^2},
\]
and the Cordes condition \((2)\) provides the following inequality.

**Lemma 5.** Let \(\gamma\) be defined by \((14)\) and \(A(x) \in L^\infty(\Omega)^{d \times d}\) satisfy Cordes condition, then for any matrix \(B \in \mathbb{R}^{d \times d}\) we have that
\[
|\gamma A : B - \text{tr}(B)| \leq \sqrt{1 - \varepsilon} |B|,
\]
where \(\varepsilon\) is given in \((2)\).

**Proof.** By direct calculation, we obtain
\[
|\gamma A : B - \text{tr}(B)| = \left| \sum_{i,j=1}^d (\gamma a_{ij} - \delta_{ij}) b_{ij} \right| \leq \left( \sum_{i,j=1}^d |\gamma a_{ij} - \delta_{ij}| \right)^{1/2} |B|
\]
\[
\leq \sqrt{1 - \varepsilon} |B|,
\]
which completes the proof. \(\square\)

In particular, for any \(q_h \in S_h^m\) we set \(B = \nabla q_h\) in \((15)\) and one has the following estimate:
\[
|\gamma A : \nabla q_h - \nabla \cdot q_h| \leq \sqrt{1 - \varepsilon} |\nabla q_h|, \quad \text{a.e. in } \Omega,
\]
which is central in the convergence analysis.

Further we would focus on the continuity and coercivity of the bilinear form \(a_h^P(\cdot, \cdot)\). We begin by introducing an energy norm \(\| \cdot \|_P\):
\[
\|q\|_P := \left( \sum_{K \in T_h} \| \nabla q \|_{L^2(K)}^2 + \sum_{e \in E_h} \frac{1}{h_e} \| q \otimes n \|_{L^2(e)}^2 + \sum_{e \in E_h} \frac{1}{h_e} \| q \times n \|_{L^2(e)}^2 \right)^{1/2},
\]
for any \(q \in H^1(T_h)^d\). We present the following lemma to give a lower bound for the energy norm \(\| \cdot \|_P\).
Lemma 6. For any $q \in H^1(T_h)^d$, the following inequality holds:

$$\|q\|_{H^1(T_h)} \leq C\|q\|_p.$$  

Proof. It is sufficient to prove $\|q\|_{L^2(\Omega)} \leq C\|q\|_p$ for the estimate (17). To do so, we apply the Helmholtz decomposition of $L^2(\Omega)^d$. Here we proof for the planar case and it is trivial to extend the proof in three dimensions. Since $q \in L^2(\Omega)$, there exist functions $v \in H^1(\Omega)$ and $\phi \in H^1(\Omega)$ such that

$$q = \nabla v + \nabla \perp \times \phi := \begin{bmatrix} \partial_x v \\ \partial_y v \\ -\partial_z \phi \end{bmatrix},$$

and the following stability holds

$$\|v\|_{H^1(\Omega)} + \|\phi\|_{H^1(\Omega)} \leq C\|q\|_{L^2(\Omega)}.$$

We refer to [13, 3] for the detail of the decomposition. Then applying the integration by parts, together with the Helmholtz decomposition, we deduce that

$$\sum_{K \in T_h} \int_K \nabla \cdot q \, dx + \sum_{K \in T_h} \int_K (\nabla \perp \times \phi) \cdot q \, dx = - \sum_{K \in T_h} \int_K v \nabla \cdot q \, dx + \sum_{e \in E_h} \int_e [q \cdot n] \, ds - \sum_{K \in T_h} \int_K \nabla \cdot q \, dx + \sum_{e \in E_h} \int_e \phi \nabla \times q \, ds.$$

For the first term and third term, using the Cauchy-Schwarz inequality and the regularity estimate implies

$$\sum_{K \in T_h} \int_K v \nabla \cdot q \, dx + \sum_{K \in T_h} \int_K \nabla \cdot q \, dx \leq C(\|v\|_{L^2(\Omega)} \|\nabla \cdot q\|_{L^2(T_h)} + \|\phi\|_{L^2(\Omega)} \|\nabla \times q\|_{L^2(T_h)})$$

$$\leq C(\|v\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)})(\|\nabla \cdot q\|_{L^2(T_h)} + \|\nabla \times q\|_{L^2(T_h)})$$

$$\leq C\|q\|_{L^2(\Omega)}\|q\|_p.$$}

Moreover, we apply the trace inequality and Cauchy-Schwarz inequality to find

$$\sum_{e \in E_h} \int_e [q \cdot n] \, ds \leq \left( \sum_{e \in E_h} \int_e \frac{1}{h_e} |q \cdot n|^2 \, ds \right)^{1/2} \left( \sum_{e \in E_h} \int_e h_e |v|^2 \, ds \right)^{1/2},$$

and

$$h_e \|v\|^2_{L^2(e)} \leq C\|v\|^2_{H^1(K)}, \quad e \subset \partial K,$$

for any $K \in T_h$. Hence, we have

$$\sum_{e \in E_h} \int_e [q \cdot n] \, ds \leq C\|q\|_p \|v\|_{H^1(\Omega)} \leq C\|q\|_p \|q\|_{L^2(\Omega)},$$

and similarly we have the following estimate for the last term,

$$\sum_{e \in E_h} \int_e \phi |q \times n| \, ds \leq \left( \sum_{e \in E_h} \int_e \frac{1}{h_e} |q \times n|^2 \, ds \right)^{1/2} \left( \sum_{e \in E_h} \int_e h_e |\phi|^2 \, ds \right)^{1/2} \leq C\|q\|_p \|\phi\|_{H^1(\Omega)} \leq C\|q\|_p \|q\|_{L^2(\Omega)}.$$}

Combining all inequalities immediately gives the estimate $\|q\|^2_{L^2(\Omega)} \leq C\|q\|_p \|q\|_{L^2(\Omega)}$. By eliminating $\|q\|_{L^2(\Omega)}$ we reach the inequality (17), which completes the proof. \qed
Then we claim that the bilinear form $a_h^p(\cdot, \cdot)$ is bounded and coercive with respect to the energy norm $\| \cdot \|_p$ for any positive $\mu$.

**Theorem 1.** Let the coefficient $A(x) \in L^\infty(\Omega)^{d \times d}$ satisfy Cordès condition and let the bilinear form $a_h^p(\cdot, \cdot)$ be defined by (13), then $a_h^p(\cdot, \cdot)$ satisfies the properties of boundedness and coercivity with any positive $\mu$:

\begin{align}
|a_h^p(p, q)| &\leq C\|p\|_p\|q\|_p, \quad \forall p, q \in H^1(\mathcal{T}_h)^d, \tag{18} \\
a_h^p(p_h, p_h) &\geq C\|p_h\|_p^2, \quad \forall p_h \in S_m^h, \tag{19}
\end{align}

where $d$ and $m$ satisfy the condition in Lemma 4.

**Proof.** We first prove the boundedness property (18). Together with Cauchy-Schwarz inequality, one has that

\[
a_h^p(p, q) \leq \left( \sum_{K \in \mathcal{T}_h} \|A : \nabla p\|^2_{L^2(K)} + \sum_{e \in \mathcal{E}_h^K} \frac{\mu}{h_e} \|p \otimes \mathbf{n}\|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h^K} \frac{\mu}{h_e} \|p \times \mathbf{n}\|^2_{L^2(e)} \right)^{1/2}.
\]

Since $A \in L^\infty(\Omega)^{d \times d}$, we immediately get

\[
\|A : \nabla p\|_{L^2(\mathcal{T}_h)} \leq C\|p\|_{H^1(\mathcal{T}_h)}, \quad \|A : \nabla q\|_{L^2(\mathcal{T}_h)} \leq C\|q\|_{H^1(\mathcal{T}_h)},
\]

which implies the estimate (18).

Then we consider the term $a_h^p(p_h, p_h)$ and the definition of $\| \cdot \|_p$ indicates that it is sufficient to prove

\[
a_h^p(p_h, p_h) \geq C\|\nabla p_h\|^2_{L^2(\mathcal{T}_h)},
\]

for the coercivity of the bilinear form. Let $\gamma$ be defined by (14) and the triangle inequality shows that

\[
|\gamma A : \nabla p_h - \nabla \cdot p_h| + |\gamma A : \nabla p_h| \geq |\nabla \cdot p_h|, \quad \text{a.e. in } \Omega.
\]

Together with the inequality (16) and $\gamma \in L^\infty(\Omega)$, we obtain

\[
\sqrt{1 - \varepsilon}\|\nabla p_h\| + |\gamma|_{L^\infty(\Omega)}\|A : \nabla p_h\| \geq |\nabla \cdot p_h|, \quad \text{a.e. in } \Omega.
\]

By using the Cauchy-Schwarz inequality, we observe that

\[
(1 - \varepsilon)|\nabla p_h|^2 + |\gamma|_{L^\infty(\Omega)}|A : \nabla p_h|^2 + 2\sqrt{1 - \varepsilon}|\nabla p_h||\gamma|_{L^\infty(\Omega)}|A : \nabla p_h| \geq |\nabla \cdot p_h|^2
\]

\[
(1 - \varepsilon + C\sqrt{1 - \varepsilon})|\nabla p_h|^2 + \left( |\gamma|_{L^\infty(\Omega)}^2 + \frac{\|\gamma\|_{L^\infty(\Omega)}^2}{C} \right) |A : \nabla p_h|^2 \geq |\nabla \cdot p_h|^2 \quad \text{a.e. in } \Omega,
\]

for any $C > 0$. Since $1 - \varepsilon < 1$, we take a proper $C > 0$ such that there exist two constants $0 < C_1 < 1, C_2 > 0$ satisfying

\[
(1 - C_1)|\nabla p_h|^2 + C_2|A : \nabla p_h|^2 \geq |\nabla \cdot p_h|^2 \quad \text{a.e. in } \Omega.
\]

Integration over all elements gives us that

\[
(1 - C_1)\sum_{K \in \mathcal{T}_h} \|\nabla p_h\|^2_{L^2(K)} + C_2\sum_{K \in \mathcal{T}_h} |A : \nabla p_h|^2_{L^2(K)} \geq \sum_{K \in \mathcal{T}_h} |\nabla \cdot p_h|^2_{L^2(K)}.
\]
By the estimate (8), we first select a sufficiently large $\mu$ to derive
\[
(1 - C_1) \sum_{K \in \mathcal{T}_h} \| \nabla p_h \|^2_{L^2(K)} + C_2 \sum_{K \in \mathcal{T}_h} \| A : \nabla p_h \|^2_{L^2(K)} + \sum_{e \in \mathcal{E}_h^1} \frac{C_3 \mu}{h_e} \| q \otimes n \|^2_{L^2(e)}
+ \sum_{e \in \mathcal{E}_h^2} \frac{C_3 \mu}{h_e} \| q \times n \|^2_{L^2(e)} \geq \sum_{K \in \mathcal{T}_h} \| \nabla \cdot p_h \|^2_{L^2(K)} + \sum_{e \in \mathcal{E}_h^1} \frac{C_3}{h_e} \| q \otimes n \|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h^2} \frac{C_3}{h_e} \| q \times n \|^2_{L^2(e)} \geq \sum_{K \in \mathcal{T}_h} \| \nabla p_h \|^2_{L^2(K)},
\]
which actually yields
\[
a_h^p(p_h, p_h) \geq C \sum_{K \in \mathcal{T}_h} \| \nabla p_h \|^2_{L^2(K)}.
\]
With sufficiently large $\mu$, we have proven the coercivity (19). Note that by scaling arguments we conclude that for any positive $\mu$ the coercivity still holds, which completes the proof. \(\square\)

We have established the existence and uniqueness of the solution to the minimization problem (11) or equivalently to the problem (12). Then let us firstly give a priori error estimate of the method proposed for seeking an approximation to the gradient $p$ in (3).

**Theorem 2.** Let $p \in \mathbf{W}^{m+1}(\Omega)$ be the solution to (3) and let $p_h \in S_h^m$ be the solution to (12). Let the coefficient $A(x) \in L^\infty(\Omega)^{d \times d}$ satisfy the Cordès condition and let $d, m$ and $\mu$ satisfy the conditions in Theorem 1, then the following estimate holds:
\[
\| p - p_h \|_p \leq C h^m \| p \|_{H^{m+1}(\Omega)}.
\]

*Proof.* The orthogonal property directly follows from the definitions of the bilinear form $a_h^p(\cdot, \cdot)$ and linear form $l_h^p(\cdot)$: for any $q_h \in S_h^m$, one has that
\[
a_h^p(p_h, q_h) = 0.
\]
Then for any $q_h \in S_h^m$, together with the boundedness (18) and coercivity (19), there holds
\[
\| p_h - q_h \|_p^2 \leq C a_h^p(p_h - q_h, p_h - q_h) = C a_h^p(p - q_h, p_h - q_h) \leq C \| p - q_h \|_p \| p_h - q_h \|_p.
\]
By eliminating $\| p_h - q_h \|_p$, together with the triangle inequality, we observe that
\[
\| p_h - q_h \|_p \leq C \| p - q_h \|_p
\]
\[
\| p_h - q_h \|_p + \| p - q_h \|_p \leq C \| p - q_h \|_p
\]
\[
\| p - p_h \|_p \leq C \inf_{q_h \in S_h^m} \| p - q_h \|_p \leq C \| p - \Pi_h^{S,m} p \|_p.
\]
From Lemma 3, it is easy to deduce
\[
\| \nabla (p - \Pi_h^{S,m} p) \|_{L^2(K)} \leq C h^m \| p \|_{H^{m+1}(K)}, \quad \forall K \in \mathcal{T}_h,
\]
\[
h_e^{-1/2} \| (p - \Pi_h^{S,m} p) \times n \|_{L^2(e)} \leq C h^m \| p \|_{H^{m+1}(K)}, \quad \forall e \subset K, \quad \forall K \in \mathcal{T}_h.
\]
Hence, we conclude that
\[
\| p - p_h \|_p \leq C \| p - \Pi_h^{S,m} p \|_p \leq C h^m \| p \|_{H^{m+1}(\Omega)},
\]
which gives us the estimate (20) and completes the proof. \(\square\)
where the bilinear form \( a(u) \) and its corresponding variational problem takes the form: find \( u \) such that

\[
\langle a(u), v \rangle := \sum_{K \in T_h} \| \nabla v - p_h \|_{L^2(K)}^2 + \sum_{e \in \partial T_h} \frac{1}{h_e} \| v - g \|_{L^2(e)}^2,
\]

where \( p_h \) is the solution to (12) and \( g \) is the boundary condition in (3). We minimize the functional (21) on the standard \( C^0 \) finite element space \( \tilde{V}_h^m := V_h^m \cap H^2(\Omega) \) to get a numerical approximation to the variable \( u_h \). Precisely, the minimization problem reads

\[
\inf_{v_h \in \tilde{V}_h^m} J_h^m(v_h),
\]

and its corresponding variational problem takes the form: find \( u_h \in \tilde{V}_h^m \) such that

\[
a_h^m(u_h, v_h) = l_h^m(v_h), \quad \forall v_h \in \tilde{V}_h^m,
\]

where the bilinear form \( a_h^m(\cdot, \cdot) \) is defined as

\[
a_h^m(u_h, v_h) = \sum_{K \in T_h} \int_K \nabla u_h \cdot \nabla v_h \, dx + \sum_{e \in \partial T_h} \frac{1}{h_e} \int_{h_e} u_h v_h \, ds,
\]

and the linear form \( l_h^m(\cdot) \) is defined as

\[
l_h^m(v_h) = \sum_{K \in T_h} \int_K \nabla v_h \cdot p_h \, dx + \sum_{e \in \partial T_h} \frac{1}{h_e} \int_{h_e} v_h g \, ds.
\]

Let us define a natural energy norm \( \| \cdot \|_{u} \):

\[
\| u \|_{u}^2 := \sum_{K \in T_h} \| \nabla u \|_{L^2(K)}^2 + \sum_{e \in \partial T_h} \frac{1}{h_e} \| u \|_{L^2(e)}^2,
\]

for any \( v \in H^1(\Omega) \). Note that \( \| v \|_{u}^2 = a_h^m(v, v) \) for any \( v \in H^1(\Omega) \). Indeed we only need to prove that \( \| \cdot \|_{u} \) is actually a norm on the space \( H^1(\Omega) \) and the existence and uniqueness of the solution to (23) are then the direct consequences. In fact, we have the following inequality.

**Lemma 7.** For any \( v \in H^1(\Omega) \), the following estimate holds:

\[
\| v \|_{H^1(\Omega)} \leq C \| v \|_{u}.
\]

**Proof.** It is sufficient to prove \( \| v \|_{L^2(\Omega)} \leq C \| v \|_{u} \). Define \( \phi \in H^2(\Omega) \cap H_0^1(\Omega) \) by \( -\Delta \phi = v \) and the regularity estimate \( \| \phi \|_{H^2(\Omega)} \leq \| v \|_{L^2(\Omega)} \) holds. By integration by parts, we observe

\[
\| v \|_{L^2(\Omega)}^2 = \sum_{K \in T_h} \int_K -\Delta \phi \, v \, dx = \sum_{K \in T_h} \int_K \nabla \phi \cdot \nabla v \, dx + \sum_{e \in \partial T_h} \int_e v \nabla \phi \cdot n \, ds
\]

\[
\leq \left( \sum_{K \in T_h} \| \nabla v \|_{L^2(K)}^2 + \sum_{e \in \partial T_h} h_e^{-1} \| v \|_{L^2(e)}^2 \right) \left( \sum_{K \in T_h} \| \nabla \phi \|_{L^2(K)}^2 + \sum_{e \in \partial T_h} h_e \| \phi \|_{L^2(e)}^2 \right)
\]

\[
\leq \| v \|_{u} \| \phi \|_{H^2(\Omega)} \leq \| v \|_{u} \| v \|_{L^2(\Omega)}.
\]

The last inequality follows the trace inequality, which completes the proof. \( \square \)

With respect to the energy norm \( \| \cdot \|_{u} \), we have the following error estimate.
We end the proof by giving a bound for the term where \( p_h \) is the solution to (12).

\[
\|u - u_h\| \leq C \left( h^m \|u\|_{H^{m+1}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \right),
\]

where \( p_h \) is the solution to (12).

**Proof.** Let \( u_I \in \tilde{V}_h^m \) be the interpolant of \( u \) and we deduce that

\[
\|u - u_h\|_u = J^m_h(u_h) \leq J^m_h(u_I) \leq \sum_{K \in T_h} \|\nabla u_I - \nabla u + p - p_h\|_{L^2(K)}^2 + \sum_{e \in E_h^p} h_e^{-1} \|u_I - g\|_{L^2(e)}^2
\]

\[
\leq C \left( \|u - u_I\|_u^2 + \|p - p_h\|_{L^2(\Omega)}^2 \right).
\]

By trace inequality, it is trivial to obtain

\[
\|u - u_I\|_u \leq C h^m \|u\|_{H^{m+1}(\Omega)},
\]

which implies (25) and completes the proof.

Then we attain an error estimate with respect to \( L^2 \)-norm.

**Theorem 4.** Let \( u \in H^{m+1}(\Omega) \) be the solution to (3) and let \( u_h \in \tilde{V}_h^m \) be the solution to (23), then the following estimate holds:

\[
\|u - u_h\|_{L^2(\Omega)} \leq C \left( h^{m+1} \|u\|_{H^{m+1}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \right),
\]

where \( p_h \) is the solution to (12).

**Proof.** Let \( e_h = u - u_h \) and by direct calculation we could see

\[
0_h(e_h, v_h) = (p - p_h, \nabla v_h)_{L^2(\Omega)} + \sum_{e \in E_h^p} h_e^{-1} \|u_I - g\|_{L^2(e)}^2
\]

we let \( w \in H^2(\Omega) \cap H^1_0(\Omega) \) be the solution to the problem \(-\Delta w = \psi \) with \( \psi = e_h \). We denote \( w_I \in \tilde{V}_h^m \) as the linear interpolant of \( w \). One can observe that

\[
(e_h, \psi)_{L^2(\Omega)} = (\nabla e_h, \nabla w)_{L^2(\Omega)} - \left( e_h, \frac{\partial w}{\partial n} \right)_{L^2(\partial \Omega)}
\]

\[
= 0_h(e_h, w - w_I) + (p - p_h, \nabla w_I) - \left( e_h, \frac{\partial w}{\partial n} \right)_{L^2(\partial \Omega)}
\]

\[
\leq C \|e_h\|_{L^2(\Omega)} \|w - w_I\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \|\nabla w_I\|_{L^2(\Omega)} + \|e_h\|_{L^2(\Omega)} \|\nabla e_h\|_{L^2(\partial \Omega)}
\]

\[
\leq C \left( h^{m+1} \|u\|_{H^{m+1}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|e_h\|_{L^2(\partial \Omega)} \right) \|w\|_{H^2(\Omega)}.
\]

Together with the regularity inequality \( \|w\|_{H^2(\Omega)} \leq C \|\psi\|_{L^2(\Omega)} \), we immediately get

\[
\|e_h\|_{L^2(\Omega)} \leq C \left( h^{m+1} \|u\|_{H^{m+1}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|e_h\|_{L^2(\partial \Omega)} \right).
\]

We end the proof by giving a bound for the term \( \|e_h\|_{L^2(\partial \Omega)} \). We let \( \alpha \in H^1(\Omega) \) solve the problem

\[
-\Delta \alpha = 0, \quad \text{in } \Omega, \quad \alpha = \tau, \quad \text{on } \partial \Omega,
\]
with \( \tau = e_h \). We denote by \( \alpha_I \in \tilde{V}_h^{m} \) the interpolant of \( \alpha \), then we could obtain

\[
(e_h, e_h)_{L^2(\partial \Omega)} = (e_h, \alpha)_{L^2(\Omega)} = \sum_{e \in \mathcal{T}_h} \int_e e_h \alpha \, ds \leq h \sum_{e \in \mathcal{E}_h^b} h_{e}^{-1} \int_e e_h \alpha \, ds
\]

\[
= h \left( \sum_{e \in \mathcal{E}_h^b} h_{e}^{-1} \int_e e_h \alpha \, ds - a_h^{u}(e_h, \alpha_I) \right) + h(p - p_h, \nabla \alpha_I)_{L^2(\Omega)}
\]

\[
= h \left( \sum_{e \in \mathcal{E}_h^b} h_{e}^{-1} \int_e e_h \alpha \, ds - (\nabla e_h, \nabla \alpha_I)_{L^2(\Omega)} - \sum_{e \in \mathcal{E}_h^b} h_{e}^{-1} \int_e e_h \alpha_I \, ds \right) + h(p - p_h, \nabla \alpha_I)_{L^2(\Omega)}
\]

\[
\leq Ch \| e_h \|_u \left( \sum_{e \in \mathcal{E}_h^p} h_{e}^{-1} \| \alpha - \alpha_I \|_{L^2(e)} \right)^{1/2} \| \nabla \alpha_I \|_{L^2(\Omega)} + h(p - p_h, \nabla \alpha_I)_{L^2(\Omega)}
\]

\[
\leq Ch \| e_h \|_u \| \alpha \|_{H^1(\Omega)} + h(p - p_h)_{L^1(\Omega)} \| \alpha \|_{H^1(\Omega)}.
\]

The last inequality follows from the trace inequality and the approximation property of the interpolant \( \alpha_I \). Together with the regularity estimate \( \| \alpha \|_{H^1(\Omega)} \leq C \| \tau \|_{L^1(\partial \Omega)} \), we finally arrive at the bound

\[
\| e_h \|_{L^2(\partial \Omega)} \leq C (h^{m+1} \| u \|_{H^{m+1}(\Omega)} + h \| p - p_h \|_{L^1(\Omega)}),
\]

which yields the inequality (26) and completes the proof.

Remark 2. The Theorem 2 restricts \( 1 \leq m \leq 2 \) for the case \( d = 3 \), as required in Lemma 4. As we emphasize in Remark 1, the numerical results demonstrate that our method could also work for the case \( m \geq 3 \).

Remark 3. The optimal convergence order of \( \| u - u_h \|_{L^2(\Omega)} \) depends on the convergence order of the term \( \| p - p_h \|_{L^2(\Omega)} \). We can only prove a suboptimal \( L^2 \) convergence rate for the variable \( p \). However, the numerical results in next section demonstrate our proposed method produces an approximation for \( p \) with an optimal \( L^2 \) convergence rate. Actually, when one degree higher polynomials are employed to approximate \( p \), it is clear that the error \( \| u - u_h \|_{L^2(\Omega)} \) would converges optimally from Theorem 2 and Theorem 4.

Since the solution to problem (1) may be of low regularity, we note that the least squares functional can automatically serves as an a posteriori error estimator. Precisely, we define the element estimator \( \eta_K(p_h) \) as

\[
\eta_K^2(p_h) := \| A : \nabla p_h - f \|_{L^2(\Omega)}^2 + \| h_{e}^{-1/2}(p_h \otimes n) \|_{L^2(\partial K \cap \mathcal{E}_h^b)}^2 + \| h_{e}^{-1/2}(p_h \times n) \|_{L^2(\partial K \cap \mathcal{E}_h^b)}^2.
\]

As a direct consequences of former results, we have the following estimate:

Corollary 1. Let \( p \) be the solution to (3) and let \( p_h \in S_h^m \) and let \( \eta_K \) be piecewisely defined as (27), then the following inequality holds

\[
C_1 \| p - p_h \|_{L^2}^2 \leq \sum_{T \in \mathcal{T}_h} \eta_K^2(p_h) \leq C_2 \| p - p_h \|_{L^2}^2.
\]
We adopt the longest-edge bisection algorithm to avoid the hanging nodes. To close this section, we outline the following adaptive algorithm:

**Initialize**
Give the initial mesh $\mathcal{T}_0$ and a parameter $0 < \theta < 1$. Set $l = 0$.

**Solve**
Solve and obtain the numerical solution with respect to the mesh $\mathcal{T}_l$.

**Estimate**
Compute the error estimator $\eta_K$ on all elements in $\mathcal{T}_l$.

**Mark**
Construct the minimal subset $\mathcal{M} \subset \mathcal{T}_l$ such that $\theta \sum_{K \in \mathcal{T}_l} \eta_K^2 \leq \sum_{K \in \mathcal{M}} \eta_K^2$ and mark all elements in $\mathcal{M}$.

**Refine**
Refine all elements in $\mathcal{M}$ and generate a conforming mesh $\mathcal{T}_{l+1}$ from $\mathcal{T}_l$. Set $l = l + 1$ and repeat the loop.

5. **Numerical Results**

In this section, we carry out a series of numerical experiments to demonstrate the convergence rates predicted by theoretical analysis in section 4. In all cases, the parameter $\eta$ in the bilinear form $a_p^h(\cdot, \cdot)$ is taken as 10.

**Example 1.** In the first example, we consider a smooth problem in two dimensions. On the domain $\Omega = [0, 1]^2$, we select the exact solution $u(x, y)$ and the smooth coefficient $A(x, y)$ as

$$u(x, y) = xy \sin(2\pi x) \sin(3\pi y), \quad (x, y) \in \Omega,$$

and

$$A(x, y) = \begin{bmatrix} \sin(4(\pi(x - 0.5))) \cos(2\pi x) \\ \cos(4(\pi(x - 0.5))) \sin(2\pi y) \end{bmatrix}.$$

The source term and boundary condition are taken accordingly. We solve this problem on a sequence of triangular meshes with mesh size $h = 1/10, 1/20, \cdots, 1/160$, see Fig 1 for the coarsest mesh. We employ the finite element spaces $\mathbf{S}^m_h \times \mathbf{V}^m_h$ with $1 \leq m \leq 3$ to seek numerical solutions $(p_h, u_h)$ for approximating $(p, u)$ in (3). For the gradient, we plot the errors $\|p - p_h\|_P$ and $\|p - p_h\|_{L^2(\Omega)}$ in Fig 2. For fixed $m$, it is clear that the error $\|p - p_h\|_P$ converges to zero with the rate $O(h^m)$ and error $\|p - p_h\|_{L^2(\Omega)}$ converges to zero with the rate $O(h^{m+1})$ as the mesh size decreases to zero. All convergence rates are optimal and coincide with the Theorem 2. For $u$, we plot the numerical errors in both $L^2$ norm and energy norm in Fig 3. We also attain the optimal convergence rates $O(h^m)$ and $O(h^{m+1})$ for the errors $\|u - u_h\|_U$ and $\|u - u_h\|_{L^2(\Omega)}$, respectively. We note that all the convergence rates perfectly agree with the error estimates.
Example 2. In this example, we choose a discontinuous coefficients $A(x, y)$ which reads

$$A(x, y) = \begin{bmatrix} 2 & x & y \\ x & |x| & 0 \\ y & 0 & 2 \end{bmatrix}.$$
The exact solution and the triangular meshes and the approximating spaces are taken as the same as in Example 1. The numerically convergence rates are displayed in Fig 4 and Fig 5. Clearly, for both variables \( p \) and \( u \) the rates of convergence in \( L^2 \) norm and energy norm are \( m + 1 \) and \( m \), respectively, which again are in perfect agreement with theoretical results.

Example 3. This is a 3D example and we solve a problem in the unit cube \( \Omega = [0,1]^3 \). We partition the domain \( \Omega \) into a series of tetrahedral meshes with mesh size \( 1/4, 1/8, 1/16, 1/32 \), see Fig 1 for the tetrahedral mesh with \( h = 1/4 \). The analytical solution and the coefficient matrix are setup as

\[
 u(x,y,z) = \cos(2\pi x) \cos(2\pi y) \cos(2\pi z),
\]

and

\[
 A(x,y,z) = \begin{bmatrix}
 10 & xy & xz \\
 xy & |x| & yz \\
 xz & yz & |x| \\
 \end{bmatrix},
\]

and the boundary condition \( g \) and the source term \( f \) are taken suitably. We also use the finite element spaces \( S^m_h \times V^m_h \) with \( 1 \leq m \leq 3 \) to approximate \( p \) and \( u \), respectively. The numerical results are shown in Fig 6 and Fig 7. We note that for the case \( m = 3 \) our method shows the optimal convergence rates for all measurements although Lemma 4 has a restriction \( m \leq 2 \). Besides, all computed convergence orders agree with the theoretical results.

Example 4. In this example, we consider the problem on the domain \( [0,1]^2 \) and the exact solution is chosen to be

\[
 u(x,y) = |x|^\alpha,
\]

where \( \alpha \) is a positive constant. The discontinuous coefficient \( A(x,y) \) takes the form

\[
 A(x,y) = \begin{bmatrix}
 1 + \frac{x^2}{|x|^2} & \frac{xy}{|x|^2} \\
 \frac{xy}{|x|^2} & 1 + \frac{y^2}{|y|^2} \\
 \end{bmatrix},
\]
and the data function $f$ and $g$ are selected properly. Notice that $u$ belongs to the space $H^{\alpha+1-\delta}(\Omega)$ for arbitrary small $\delta$. In the following, we take $\alpha = 1.2$ to test the adaptive algorithm proposed in the previous section. The parameter $\theta$ is chosen $\theta = 0.4$ and we consider the linear accuracy $\mathbf{S}_h^1 \times \mathbf{V}_h^1$ in the approximation to the variables $p$ and $u$. The mesh size of initial triangular partition
Figure 7. Example 3. The convergence rates of $\|u - u_h\|_u$ (left) / $\|u - u_h\|_{L^2(\Omega)}$ (right).

is taken as $h = 0.1$, see left figure in Fig 1. The whole convergence history of uniform refinement and adaptive refinement is displayed in Fig 8. For uniform refinement, we observe the error $\|p - p_h\|_p$ decreases to zero at the speed $O(h^{0.2})$, which agrees with the convergence analysis. For the error $\|p - p_h\|_{L^2(\Omega)}$, the uniform refinement leads to a reduced convergence speed $O(h^1)$. The reason may be traced to the singularity of $u$ at the corner. Furthermore, for $u$ the errors $\|u - u_h\|_u$ and $\|u - u_h\|_{L^2(\Omega)}$ approach to zero at the rate $O(h^1)$, which matches with the theoretical analysis that the convergence rates in both norms for $u$ depend on the convergence rate of $\|p - p_h\|_{L^2(\Omega)}$. For the adaptive refinement, we note that all error measurements seem to be optimal. The triangular meshes after 6 adaptive refinement steps are shown in Fig 9. Clearly, the refinement is pronounced in the regions where the solution is of low regularity.

6. Conclusion

We proposed a sequential least squares finite element method for elliptic equations in non-divergence form. We employed a novel piecewisely curl-free approximate space to solve the gradient variable first and then we solve the primitive variable in the $C^0$ finite element space. We proved the convergence rates for both variables with respect to $L^2$ norm and energy norm. Optimal convergence orders for all measurements were detected in numerical experiments. We also tried an adaptive algorithm using $h$-adaptive method to improve numerical efficiency for a problem of low regularity.

Acknowledgements

This research is supported by the National Natural Science Foundation of China (Grant No. 91630310, 11421110001, and 11421101) and the Science Challenge Project, No. TZ2016002.
Figure 8. Example 4. The convergence history of $p$ (left) / $u$ (right).

Figure 9. Triangular mesh after 6 adaptive refinement steps (left) / elements in the red rectangle (right).

REFERENCES

1. D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. 39 (2001/02), no. 5, 1749–1779.
2. Ivo Babuška, Gabriel Caloz, and John E. Osborn, *Special finite element methods for a class of second order elliptic problems with rough coefficients*, SIAM J. Numer. Anal. 31 (1994), no. 4, 945–981.
3. Rickard E. Bensow and Mats G. Larson, *Discontinuous/continuous least-squares finite element methods for elliptic problems*, Math. Models Methods Appl. Sci. 15 (2005), no. 6, 825–842.
4. Bernard Bialecki, *Convergence analysis of orthogonal spline collocation for elliptic boundary value problems*, SIAM J. Numer. Anal. 35 (1998), no. 2, 617–631.
5. Pavel B. Bochev and Max D. Gunzburger, *Finite element methods of least-squares type*, SIAM Rev. **40** (1998), no. 4, 789–837.
6. Klaus Böhmer, *On finite element methods for fully nonlinear elliptic equations of second order*, SIAM J. Numer. Anal. **46** (2008), no. 3, 1212–1249.
7. Luis A. Caffarelli and Cristian E. Gutiérrez, *Properties of the solutions of the linearized Monge-Ampère equation*, Amer. J. Math. **119** (1997), no. 2, 423–465.
8. Xiaobing Feng, Lauren Hennings, and Michael Neilan, *Finite element methods for second order linear elliptic partial differential equations in non-divergence form*, Math. Comp. **86** (2017), no. 307, 2025–2051.
9. Xiaobing Feng and Michael Neilan, *Mixed finite element methods for the fully nonlinear Monge-Ampère equation based on the vanishing moment method*, SIAM J. Numer. Anal. **47** (2009), no. 2, 1226–1250.
10. Dietmar Gallistl, *Variational formulation and numerical analysis of linear elliptic equations in nondivergence form with Cordes coefficients*, SIAM J. Numer. Anal. **55** (2017), no. 2, 737–757.
11. Dietmar Gallistl and Endre Süli, *Mixed finite element approximation of the Hamilton-Jacobi-Bellman with Cordes coefficients*, SIAM J. Numer. Anal. **57** (2019), no. 2, 592–614.
12. Emmanuel H. Georgoulis, Paul Houston, and Juha Virtanen, *An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems*, IMA J. Numer. Anal. **31** (2011), no. 1, 281–298.
13. Vivette Girault and Pierre Arnaud Raviart, *Finite element methods for navier-stokes equations: Theory and algorithms*, Springer-Verlag, 1986.
14. Omar Lakkis and Tristan Pryer, *A finite element method for second order nonvariational elliptic problems*, SIAM J. Sci. Comput. **33** (2011), no. 2, 786–801.
15. R. Li and F. Y. Yang, *A sequential least squares method for Poisson equation using a patch reconstructed space*, arXiv:1901.06485 (2019).
16. Jiangguo Liu and Rachel Cali, *A note on the approximation properties of the locally divergence-free finite elements*, Int. J. Numer. Anal. Model. **5** (2008), no. 4, 693–703.
17. Michael Neilan and Mohan Wu, *Discrete Miranda-Talenti estimates and applications to linear and nonlinear PDEs*, J. Comput. Appl. Math. **356** (2019), 358–376.
18. Iain Smears and Endre Süli, *Discontinuous Galerkin finite element approximation of nondivergence form elliptic equations with Cordes coefficients*, SIAM J. Numer. Anal. **51** (2013), no. 4, 2088–2106.
19. ———, *Discontinuous Galerkin finite element approximation of Hamilton-Jacobi-Bellman equations with Cordes coefficients*, SIAM J. Numer. Anal. **52** (2014), no. 2, 993–1016.
20. Chunmei Wang and Junping Wang, *A primal-dual weak Galerkin finite element method for second order elliptic equations in non-divergence form*, Math. Comp. **87** (2018), no. 310, 515–545.

CAPT, LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China

E-mail address: rli@math.pku.edu.cn

School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China

E-mail address: yangfanyi@pku.edu.cn