Effect of Container Walls on Vibrating Wire in $^3$He-$^4$He Mixtures in the Ballistic Limit

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Abstract. We study linearized kinetic equations describing mixtures of $^3$He and $^4$He at such low temperatures, that all of $^4$He is superfluid. The $^3$He part is described by Landau’s Fermi liquid theory, in the form derived by Khalatnikov in the case of mixtures. Here we apply the theory to the case of vibrating wire. The linearized Landau-Boltzmann kinetic equation with relaxation time approximation needs to be solved numerically at finite temperatures. In the ballistic limit, the collision term in the equation vanishes, enabling an analytic approach. There are many features affecting the system, including the coupling of $^3$He to the superfluid $^4$He, the effect of Fermi liquid interactions, and the dependence on the frequency $\omega$ of the vibrating wire. Also, the type of scattering of the quasiparticles from the surface of the wire can be either specular or diffusive. In this paper, we consider the effects of chamber walls near the wire. It has been speculated that the nearness of the container walls has affected the experimental results of Martikainen et al. We find that the presence of container walls near a slowly moving wire increases the dissipation experienced by the wire.

1. Introduction

In mixtures of $^3$He and $^4$He it is possible to study the properties of superfluids, but also the dynamics of degenerate Fermi liquids. At sufficiently low temperatures the $^4$He part is entirely superfluid, and its dynamical properties are independent of temperature. The $^3$He part is degenerate Fermi liquid, the behavior of which depends on the mean free path $\lambda$ of quasiparticles. In mixtures of $^3$He and $^4$He, it is possible to study the properties of normal Fermi liquids over the whole range of mean free paths, by varying the temperature. At high temperatures the $^3$He part can be described by hydrodynamic theory, but at lower temperatures, as $\lambda$ increases, kinetic theory is required. In the intermediate region we need to take into account the collision term in the kinetic equation, but in the ballistic limit the quasiparticle-quasiparticle collisions can be neglected, and analytic approach is possible [1, 2].

Experimentally, the dynamics of quantum liquids can be studied using vibrating objects such as wires. Consider a circular cylinder of radius $a$ moving with a slow velocity $u$ (perpendicular to the axis of the cylinder) through a normal Fermi liquid. We consider the force $F$ that the liquid exerts on the cylinder in the ballistic limit. It has been expressed as [3]

$$F = -Aan_3p_Fu,$$  \hspace{1cm} (1)

where $n_3$ is the number density of Fermi particles, $p_F$ the Fermi momentum. The prefactor $A$ has values $A = 3\pi/4$ for specular scattering of the quasiparticles from the wire, and $A = 43\pi/48$
for the diffusive case [2, 4]. These results take into account the Fermi liquid interactions, but are valid only in the limit $\omega = 0$ of the wire oscillation frequency. They also assume an unbounded liquid around the wire. In this paper we consider the case that the wire is in a finite container. The effect of container walls has been studied before in the hydrodynamic region [3], but not in the ballistic limit. The role of container walls close to an oscillator is interesting in the light of recent interest in the use of quartz forks as experimental probes of helium properties [5]. The industrially manufactured quartz forks come inside a cylindrical container, the radius of which is of the order of the oscillator size.

2. Fermi Liquid Theory
We will briefly describe the starting point of our calculations, which is Landau’s Fermi liquid theory, in the form derived by Khalatnikov in the case of mixtures [6]. See ref. [4] and references therein for more details. The quasiparticle energy near the Fermi surface can be written

$$\epsilon_p(r, t) = \epsilon_p^{(0)} + \delta \epsilon(p, r, t),$$

where $\epsilon_p^{(0)} = \mu_{30} + v_F(p - p_F)$, $\mu_{30}$ is the equilibrium chemical potential of $^3$He, and the Fermi velocity $v_F = p_F/m^*$ with effective mass $m^*$. Instead of the full distribution function $n(p, \mathbf{r}, t)$, it is sufficient to consider the deviation from the unperturbed Fermi distribution $n_0$, integrated over energy, $\phi(p, \mathbf{r}, t) = \int d\epsilon_p^{(0)} [n(p, \mathbf{r}, t) - n_0(p)]$. As a consequence $\phi$ depends only on the direction $\hat{p}$ of momentum in addition to $\mathbf{r}$ and $t$. The correction $\delta \epsilon$ of the quasiparticle energy is

$$\delta \epsilon(p, \mathbf{r}, t) = D p_F \hat{p} \cdot S(r, t) + \langle F(\hat{p}, \hat{p}') \phi(\hat{p}', \mathbf{r}, t) \rangle_{\hat{p}'} ,$$

where $D = 1 - m_3/m^*(1 + F_F^3/3)$, $F(\hat{p}, \hat{p}') = \sum_{l=-\infty}^{\infty} F_F^l P_l (\hat{p} \cdot \hat{p}')$, and $m_3$ is the bare fermion mass. The latter term in Eq. (3) is the standard Fermi-liquid interaction term, and the former term comes from the flow velocity $v_s$ of the superfluid $^4$He.

We define a new field $\psi(r, \hat{p}) = \phi(r, \hat{p}) + \delta \epsilon(r, \hat{p})$, which is useful in the following. The distribution function $\psi$ is determined from the kinetic equation and boundary conditions. The linearized Landau-Boltzmann kinetic equation for $\psi$ is

$$\frac{\partial \psi}{\partial t} + v_F \hat{p} \cdot \frac{\partial \psi}{\partial \mathbf{r}} = I ,$$

where $I$ is the collision term. The boundary conditions on the wire surface are derived in ref. [7]. For specular scattering the condition is

$$\psi(\hat{p}) = \psi(\hat{p} - 2\mathbf{n} \cdot \hat{p}) + 2p_F (\mathbf{n} \cdot \hat{p})(\mathbf{n} \cdot \hat{u}) ,$$

where $\mathbf{n}$ is surface normal pointing to the liquid. For diffusive scattering

$$\psi(\hat{p}_{\text{out}}) = -2(\mathbf{n} \cdot \hat{p}_{\text{in}} \psi(\hat{p}_{\text{in}}))_{\hat{p}_{\text{in}}} + p_F (\hat{p}_{\text{out}} + \frac{2}{3} \mathbf{n} \cdot \hat{u})$$

with an average over half of the unit sphere ($\mathbf{n} \cdot \hat{p}_{\text{in}} < 0$). The force exerted by $^3$He quasiparticles on a surface element $da$ is obtained from the formula

$$\frac{dF}{da} = -3n_3 (\mathbf{n} \cdot \hat{p}) \psi(\hat{p})_{\hat{p}} ,$$

where $n_3 = p_F^3/3\pi^2 h^3$ is the equilibrium fermion density.

We now consider a slowly moving wire in the ballistic limit, where the collisions play no role, $I = 0$. In the low frequency limit $\omega \to 0$, the time derivative $\partial \phi/\partial t$ can be neglected. Then it is
easy to solve the kinetic equation (4) for any incoming trajectory: \( \psi(\mathbf{p}_{in}) = 0 \). The distribution for outgoing trajectories is then obtained from the boundary condition (5) or (6). This gives

\[
\psi_{\text{spec}}(\mathbf{p}_{out}) = 2p_F (\mathbf{n} \cdot \mathbf{p}_{out})(\mathbf{n} \cdot \mathbf{u}), \quad \text{and} \quad \psi_{\text{diff}}(\mathbf{p}_{out}) = p_F (\mathbf{p}_{out} + \frac{2}{3} \mathbf{n}) \cdot \mathbf{u}. \tag{8}
\]

The force per unit length (1) is obtained from (7) by integrating over the wire surface, and it gives the aforementioned \( A_{\text{spec}} = 3\pi/4 \) and \( A_{\text{diff}} = 43\pi/48 \), or

\[
F_{\text{spec}} = -\frac{3\pi}{4} an_3 p_F \mathbf{u}, \quad \text{and} \quad F_{\text{diff}} = -\frac{43\pi}{48} an_3 p_F \mathbf{u}. \tag{9}
\]

3. Wire in Container

Now we extend the calculation by considering the effect of a container wall in the ballistic limit. For simplicity, we consider the case of a cylindrical container of radius \( R \gg a \), concentric with the wire. It should be noted that although the wire is moving, and the container is not, the wire is in the center of the cylinder. The situation can be understood as the \( \omega \to 0 \) limit of an oscillating wire, with time dependence \( \mathbf{u} = u_0 e^{-i\omega t} \). We require linearity in \( \mathbf{u} \), and thus \( \psi \propto e^{-i\omega t} \). In any case, the velocity of the wire (and the amplitude of the oscillations) is negligible compared to the Fermi velocity \( v_F \). In the \( \omega \to 0 \) limit \( \partial \psi / \partial t = 0 \), and the Landau-Boltzmann equation becomes

\[
\mathbf{p} \cdot \nabla \psi(\mathbf{p}, \mathbf{r}) = 0. \tag{10}
\]

The equation implies that \( \psi \) is constant along trajectories, and changes only when trajectory hits a wall. Here we will only address the case of a diffusive wire in a diffusive cylinder, other cases can be treated similarly. Now we can no longer assume \( \psi(\mathbf{p}_{in}) = 0 \). The boundary conditions on the wire (w) and the container (c) walls are

\[
\psi(\mathbf{p}_{w, out}) = g_w \mathbf{n}_w \cdot \mathbf{u} + p_F (\mathbf{p}_{w, out} + \frac{2}{3} \mathbf{n}_w) \cdot \mathbf{u}, \quad \text{and} \quad \psi(\mathbf{p}_{c, out}) = -g_c \mathbf{n}_c \cdot \mathbf{u}, \tag{11}
\]

where \( \mathbf{n}_w \) and \( \mathbf{n}_c \) are the surface normal vectors for wire and container, respectively. Here we have defined the averages \( g_w \) and \( g_c \) as

\[
g_w \mathbf{n}_w \cdot \mathbf{u} = -2(\mathbf{n}_w \cdot \mathbf{p}_{w, in} \psi(\mathbf{p}_{w, in}))_{\mathbf{p}_{w, in}}, \quad \text{and} \quad g_c \mathbf{n}_c \cdot \mathbf{u} = 2(\mathbf{n}_c \cdot \mathbf{p}_{c, in} \psi(\mathbf{p}_{c, in}))_{\mathbf{p}_{c, in}}. \tag{12}
\]

The simple dependence of these averages on the position through only \( \mathbf{n} \cdot \mathbf{u} \) is a necessary consequence of the symmetry of the system, and the requirement of linearity in \( \mathbf{u} \). The average on the outer boundary, \( g_c \), can be calculated in two parts: \( g_{cc} \) for the contribution of the other points of the container wall, and \( g_{cw} \) for the contribution of the trajectories coming from the wire. The former is easily obtained, to first order in \( a/R \), as

\[
g_{cc} \mathbf{n}_c \cdot \mathbf{u} = -2(\mathbf{n}_c \cdot \mathbf{p}_{c, in} g_c \mathbf{n}_{c2} \cdot \mathbf{u})_{\mathbf{p}_{c, in}} = -\frac{1}{3} g_c \mathbf{n}_c \cdot \mathbf{u} + \frac{a}{R} g_c \mathbf{n}_c \cdot \mathbf{u}, \tag{13}
\]

where the trajectories arrive from \( \mathbf{n}_{c2} = \mathbf{n}_c - 2\hat{\mathbf{p}}(\mathbf{n}_c \cdot \mathbf{p}) \), and \( \hat{\mathbf{p}} \) is the projection of \( \mathbf{p} \) into the \( xy \)-plane (perpendicular to the cylinder axis \( z \)). The latter term in Eq. (13) is due to the shadow of the wire, which is seen (in the \( xy \)-plane) in angle \( 2a/R \). For \( g_{cw} \) it is straightforward to calculate, to same order of accuracy, that

\[
g_{cw} \mathbf{n}_c \cdot \mathbf{u} = \left( \frac{\pi}{4} g_w + \frac{8}{3\pi} + \frac{\pi}{6} p_F \right) \frac{a}{R} (\mathbf{n}_c \cdot \mathbf{u}). \tag{14}
\]
The average on the wire surface, \( g_w \), is found in similar manner, by finding the point on the container wall where the incoming trajectories on the wire surface come from. The result is, to first order in \( a/R \),

\[
g_w = g_c \left( \frac{\pi}{4} + \frac{1}{3} \frac{a}{R} \right). \tag{15}
\]

Combining the results gives, to first order in \( a/R \),

\[
 g_c = \left( \frac{2}{\pi} + \frac{\pi}{8} \right) \frac{a}{R} pF, \quad \text{and} \quad g_w = \left( \frac{1}{2} + \frac{\pi^2}{32} \right) \frac{a}{R} pF. \tag{16}
\]

The force per surface element of the wire due to container walls is given by Eq. (7),

\[
 \frac{dF_g}{da} = -n_3 pF \left( \frac{2}{\pi} + \frac{\pi}{8} \right) \left[ \frac{16 + 3\pi^2}{24\pi} (\hat{n} \cdot \mathbf{u}) \hat{n} + \frac{2}{3\pi} \mathbf{u} \right] \frac{a}{R}. \tag{17}
\]

The total force on the wire per unit length is obtained by integrating over the wire surface, and is

\[
 F = -a n_3 pF \left( \frac{43\pi}{48} + \frac{256 + 32\pi^2 + \pi^4}{64\pi} \frac{a}{R} \right) \mathbf{u}, \tag{18}
\]

where the first term is due to diffusive boundary condition on the wire surface, Eq. (9), and the second term is the correction due to container walls, to first order in \( a/R \). The force is in phase with the wire velocity, and corresponds to damping of the wire. In particular, the presence of container walls increases the maximum dissipation. The closer the walls are, the higher is this extra dissipation. In the case of oscillating wire, the walls apparently have an effect on the inertia of the wire as well.

In the measurements of Martikainen et al. [8] they found slightly larger dissipation at the lowest temperatures than expected from the simple theory, Eq. (9), even in the diffusive case. Their highest value for the prefactor \( A = 2.84 \), is slightly higher than the theoretical value for diffusive wire, \( A = 2.81 \), in the unbounded case. The walls of the helium container in their experiment were close in the direction of the wire’s motion \( (a/R \sim 0.1) \), so the walls may have had an effect on the results. Taking into account the container walls according to eq. (18) gives, for \( a/R = 0.1 \), the value \( A = 3.14 \). The geometry in the experiment is more complicated than considered here, and the type of scattering is uncertain. In conclusion, the factor \( A \) depends on several features, and it is difficult to say what value it should attain in a particular experiment.

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