I. DERIVATION OF THE INTEGRATION EQUATIONS FOR $\sigma^2$

In this section we show how we derive the numerical expressions for the diffusion length (Eq. (8) in main text (MT)). Starting from the differential equation for the diffusion length we have:

$$\frac{d\sigma^2}{dt} - 2 \dot{\varepsilon}_z(t) \sigma^2 = 2D(t).$$

(1)

We consider the vertical strain rate due to densification:

$$\dot{\varepsilon}_z(t) = \frac{-\partial \rho}{\partial t} \frac{1}{\rho}$$

(2)

By substituting $t$ with $\rho$ and combining Eq. (1), (2) we get:

$$\frac{d\sigma^2}{d\rho} + \frac{2\sigma^2}{\rho} = 2 \left( \frac{d\rho}{dt} \right)^{-1} D(\rho)$$

(3)

Multiplication of both sides of Eq. (3) with the integrating factor $F(\rho) = e^{\int \frac{2}{\rho} d\rho} = \rho^2$,

$$\frac{d}{dt} (\rho^2 \sigma^2) = 2 \rho^2 \left( \frac{d\rho}{dt} \right)^{-1} D(\rho),$$

(4)

from which we get the result:

$$\sigma^2(\rho) = \frac{1}{\rho^2} \int_{\rho_0}^{\rho} 2 \rho^2 \left( \frac{d\rho}{dt} \right)^{-1} D(\rho) d\rho.$$}

(5)

In a similar way for the ice diffusion length (Eq. (12) in the MT) we have:

$$\frac{d\sigma^2}{dt} - 2 \dot{\varepsilon}_z(t) \sigma^2 = 2D(t),$$

(6)

where the total thinning is given by:

$$S(t') = e^{\int_0^{t'} -2\dot{\varepsilon}_z(t) dt}.$$}

(7)

We multiply both sides of Eq. (7) with the integrating factor $F(\rho) = e^{\int_0^{t'} -2\dot{\varepsilon}_z(t) dt}$,

$$\frac{d\sigma^2}{dt} - 2 \dot{\varepsilon}_z(t) \sigma^2 = 2D(t),$$

(8)

which results in

$$\frac{d}{dt} \left[ \sigma^2 e^{\int_0^{t'} -2\dot{\varepsilon}_z(t) dt} \right] = 2D(t)e^{\int_0^{t'} -2\dot{\varepsilon}_z(t) dt},$$

(9)

From this, we get the expression for the ice diffusion length

$$\sigma_{\text{ice}}^2(t') = S(t')^2 \int_0^{t'} 2D_{\text{ice}}(t) S(t)^{-2} dt.$$
• $\alpha$: Ice – Vapor fractionation factor. We use the formulations by Maiolatesi (1971) and Merlivat and Nier (1967) for $\alpha_{s/v}^2$ and $\alpha_{18}^{16}$ respectively.

$$\ln \alpha_{Ice/Vapor}^2(^2H/^1H) = 16288/T^2 - 9.34 \times 10^{-2}$$  \hspace{1cm} (15)

$$\ln \alpha_{Ice/Vapor}^{18O/16O} = 11.839/T - 28.224 \times 10^{-3}$$  \hspace{1cm} (16)

• $\tau$: The firm tortuosity. We use (Schwander et al., 1988; Johnsen et al., 2000):

$$\frac{1}{\tau} = \begin{cases} 1 - b_\tau \left(\frac{\rho}{\rho_{Ice}}\right)^2 & \text{for } \rho \leq \frac{\rho_{Ice}}{\sqrt{3}} \\ 0 & \text{for } \rho > \frac{\rho_{Ice}}{\sqrt{3}} \end{cases}$$  \hspace{1cm} (17)

where $b_\tau = 1.30$, implying a close-off density of $\rho_{co} = 804.3$ kgm$^{-3}$.

B. The ice diffusivity

Ice diffusion is believed to occur via a vacancy mechanism with transport of molecules within the ice lattice. Based on isotopic probe experiments, there is a strong consensus that the ice diffusivity coefficient is the same for $\text{H}_2^{18O}$, $\text{D}_2$O and $\text{T}_2$O (Ramseier, 1967; Blicks et al., 1966; Itagaki, 1967; Delibaltas et al., 1966). The dependence of the ice diffusivity parameter to temperature is described by an Arrhenius type equation

$$D = D_0 \exp(-Q/RT),$$  \hspace{1cm} (18)

where $Q$ is the activation energy and $D_0$ a pre-exponential factor. The results of the studies mentioned above agree well with each other. Here we plot the diffusivity parametrization coefficients suggested by those studies (Fig. 1). In this work we follow Ramseier (1967) and use $Q = 0.62$ eV and $D_0 = 9.2 \times 10^{-4}$ m$^2$s$^{-1}$. Note that the results of Ramseier (1967) are based on measurements of both artificially as well as naturally grown ice collected at Mendenhall glacier, Alaska.

Enhanced ice diffusion rates have been proposed to be the cause of the excess diffusion observed in the Holocene section of the GRIP ice core (Johnsen et al., 2000). For the early Holocene part of the GRIP core, the authors of that study observed higher diffusion rates than expected by the theory. In order to diminish the discrepancy between modeled and observed diffusion rates Johnsen et al. (2000) introduced the term “excess ice diffusion”, referring to a possibly higher diffusivity coefficient due to isotopic exchange in the liquid phase on thin water films and ice crystal veins. However, the thickness of the water films and the diameter of the ice crystal veins required, are unrealistically high.

Although the existence of an “excess ice diffusion” mechanism cannot be excluded based on the findings of our study, it should be mentioned that the diffusion model used in Johnsen et al. (2000) assumes an accumulation rate and temperature signal that is based on the $\delta^{18}$O record of the GRIP core. The GRIP $\delta^{18}$O signal is characterized by a rather flat curve throughout the Holocene showing no indication of an early Holocene optimum, a feature that is mostly due to ice sheet elevation effects “masking” the temperature change. A result, it is expected that the diffusion length calculations in Johnsen et al. (2000) would underestimate the diffusion signal throughout the Holocene. We conclude that the “excess ice diffusion” issue requires more work in the future. Considering that the ice diffusivity coefficient is very similar for all the isotopologues of water, a study focusing on the differential diffusion signal between $\delta^{18}$O and $\delta^D$ would provide a better insight in the problem. An effort will be undertaken as soon as an adequately long Holocene section of the NEEM ice core is analyzed for both $\delta^{18}$O and $\delta^D$ using dual isotope laser spectroscopy.

III. Examples of diffusion lengths for different ice core sites

In this section we present an ensemble of implementations of the diffusion–densification model for various combinations of surface forcings that represent typical modern day conditions for a number of ice core sites on Greenland and Antarctica. The contours in the plot are generated by integration of Eq. (6) and expressed in m ice eq. The forcing for each ice core site is given in Table 11 and the results are shown in Fig. 2.

IV. Estimation of $\sigma^2$ from the high resolution data set

In order to estimate the diffusion length value from high resolution water isotope data we minimize the 2-norm $\|P_s - \hat{P}_s\|$ where $P_s$ is an estimate of the power spectral density of a high resolution $\delta^{18}$O data section and $P_s$ is a model description of the power spectral density.

$\hat{P}_s$ is obtained by the use of the Burg’s spectral estimation method. The method fits an autoregressive model of order $\mu$ (AR-$\mu$) by minimizing the forward–backward prediction error filter (Hayes, 1996; Press et al., 2007; Andersen, 1974). For the theoretical model we have:

$$P_s = P_o + |\hat{\eta}(k)|^2,$$  \hspace{1cm} (19)

where $P_o = P_0 e^{-k^2 \sigma^2}$ is the effect of the firn diffusion process with squared diffusion length $\sigma^2$. Regarding the noise, we find red noise described by an AR-1 process with an autoregressive coefficient $q_1 = 0.15$ to provide a good description of the noise signal we observe. The spectrum of this signal is (Kay and Marple, 1981):

$$|\hat{\eta}(k)|^2 = \frac{\sigma^2_o \Delta}{|1 + q_1 \exp(-i k \Delta)|^2},$$  \hspace{1cm} (20)

where $\sigma^2_o$ is the variance of the noise. The angular frequency $k = 2\pi f$ is in the range $f \in [0, \frac{1}{2 \Delta}]$ defined by the Nyquist frequency and thus the sampling resolution $\Delta$. We vary the parameters $\sigma^2$, $P_0$, and $\sigma^2_o$ of the spectral model in order to minimize the misfit between $P_s$ and $\hat{P}_s$ in a least squares sense.

As it can be seen in Fig. 3 it is the characterization of the full spectrum that yields information on $\sigma^2$. It is thus not necessary to specifically study the relative attenuation of individual spectral peaks as for example the annual signal. This approach allows for a study of the diffusion signal even after the spectral signature of the annual signal diminishes.
V. AR ORDER SELECTION

An interesting feature of the Burg estimation method is that the order $\mu$ of the AR filter affects the spectral resolution of $P_x$ (Hayes, 1996; Press et al., 2007). Low $\mu$ values result in smoother spectra with inferior spectral resolution, while higher order spectra show better performance in resolving neighboring spectral peaks. This can be seen in the spectral estimates presented in Fig. 8 where we plot spectral estimates with $\mu = 30$ and $\mu = 40$. As described above, the goal of the $\sigma^2$ estimation is to characterize the overall shape of the spectrum. As a result, relatively low values of $\mu$ produce smooth spectra of relatively low spectral resolution and can be adequate for the purpose of our application.

We look into both the influence of the value of $\mu$ on the $\sigma^2$ estimate by performing 41 power spectrum estimates with $\mu \in [40, 80]$. Possible interferences of spectral features due to longer scale climate variability that could have an effect on the estimation of $\sigma^2$ are also investigated with this test. In Fig. 4 we show the mean value of the 41 spectral estimates before and after strain correction. The standard deviation of the 41 estimates suggests that a possible effect of spectral features due to low frequency climate variability is of second order. The same applies for the selection of the AR order $\mu$, in Burg’s spectral estimation.

VI. TEST WITH SYNTHETIC DATA

Additional to the MEM order selection sensitivity test shown in section V we investigate the precision and accuracy of the $\sigma^2$ estimation using synthetic data. We perform two tests which we describe below.

A. Synthetic data test 1

For the first test we investigate the influence of short or long memory of the $\delta^{18}$O time series due to climate variability. We generate high resolution synthetic $\delta^{18}$O data by assuming an AR–1 process with the AR–1 coefficient $\phi_1$ of the process being equal to 0.3 and 0.9995. The process is applied on Gaussian noise with a mean of zero and a standard deviation of $0.10 \sigma$. The time series generated have a spacing of $\Delta t = 10^{-3}$ m and a total length $L = 20$ m. A $\phi_1$ equal to 0.9995 with $\Delta t = 10^{-3}$ m corresponds to a time constant for the memory of the process equal to $\tau_{0,9995} = -\Delta t / \ln \phi_1 = 2 m$ (Percival, 1993). For the part of the record we study, this is equivalent to climate variability at decadal time scales. The high resolution AR–1 process is then convolved with the Gaussian filter of predefined variance, simulating the effect of firm diffusion. Measurement white noise is then added to the result of the convolution. We then sample the high resolution diffused time series at a resolution of $\Delta = 0.05$ m representing the width of a discrete ice core sample and perform the $\sigma^2$ estimation as described in section V.

The test is run using 3 different values for the diffusion length; 0.05, 0.035 and 0.025 m. We perform the procedure 100 times for every diffusion length generating a new AR–1 process for every repetition. This results in a total of $2 \times 3 \times 100 = 600$ experiments. We compare the estimated values for $\sigma^2$ with the target values and calculate the mean and RMS value for the estimation. In Fig. 5 we show one example for each of the three sets of different diffusion length values used presenting the raw time series for both values of $\phi_1$ and the respective power spectral densities. The results for the 6 sets of experiments are presented in Table 1.

B. Synthetic data test 2

For the second test we follow a similar procedure as in test 1 generating an AR–1 process that we then diffuse using a fixed value for the diffusion length. We choose $\sigma = 0.08$ m. The synthetic data sets are then sampled with 4 different resolutions with $\Delta = 0.05, 0.08, 0.10$ and $0.12$ m. The diffusion length is estimated and corrected for discrete sampling as described in section 4 of the MT. From the spectral estimation point of view, the effect of the coarser sampling resolution is equivalent to the ice flow thinning. The lower Nyquist frequency caused by the coarser sampling scheme results in an inferior estimation of the noise signal (Fig. 6). The diffusion length estimates that we present in Table 1 indicate that the estimation scheme is accurate and insensitive to the memory of the AR–1 process as well as the sampling scheme.

An RMS value of 0.5 cm based on the synthetic data tests is taken into account when calculating the confidence intervals in Fig. 7. The equivalent temperature uncertainty of ±0.5 cm is approximately 1 K for the NorthGRIP site.

VII. ICE FLOW THINNING EFFECTS

The layer thinning induced by the ice flow, impacts the diffusion length estimation mainly in two ways. First, due to the discrete sampling scheme as the diffusion length estimation moves towards the deeper parts of the core, a single sample averages more years of climate information. This effect is essentially taken care of by means of the discrete sampling correction described in section 4 of the MT. The term $\sigma_{\text{miss}}^2$ is constant with depth. However based on Eq. (20) of the MT one can see that the effective correction for discrete sampling scales with the total thinning function $S(z)$. In Fig. 8 we illustrate the effect of this correction with depth.

Second, the ice flow thinning will result in the diffusion length value decreasing with depth. With lower $\sigma^2$ values, a spectrum estimate up to the Nyquist frequency $1/2\Delta$ will contain a decreasing part of the noise signal $|\hat{\eta}(k)|^2$. After a certain depth, the sampling resolution is not high enough to resolve $|\hat{\eta}(k)|^2$. The result of this effect is that the estimation of the $P_x$ signal requires an assumption about $|\hat{\eta}(k)|^2$ and thus it can limit the extend to which the diffusion technique can be applied to the deeper parts of the core.

In Fig. 9 we plot the expected diffusion length value assuming a simple case of constant temperature and accumulation rate at the surface and a certain ice layer thinning.
ice layer thinning has reduced the diffusion length, a higher
Conclusively, for the deeper parts of the core where the
progressively smaller portion of the noise signal is resolved.
shape of the power spectral density. As depth increases, a
ice layer thinning as well as the sampling resolution on the
5 cm and \( \Delta^2 \). The plots illustrate the effect of the
ice layer thinning has reduced the diffusion length, a higher
sampling resolution (\( \Delta < 2.5 \text{ cm} \)) is preferable for an even
more accurate estimation of \( P_z \) and subsequently \( \sigma^2 \) to be
possible. For the NorthGRIP reconstruction we present here,
we are able resolve the noise signal down to the depth of
approximately 1450 m. For depths higher than 1450 m we
make the simplest possible assumption that the noise level
is equal to the average values we have observed in the Holocene
section.

The accuracy of the ice flow model in inferring the ice
thinning function has an influence on the uncertainty of our
temperature reconstruction. The value of the diffusion length
of a layer at depth \( z \), estimated from the spectral properties of a
set of \( \delta^{18}O \) data needs to be corrected for ice flow thinning. An
inaccurately estimated thinning function affects the inferred
values of \( \sigma^2_{iirm} \) in a linear way as we show in Eq. (13) and (20)
of the MT. As far as the inferred temperatures are concerned,
the ice thinning function impacts the slope of the signal, but
has no influence on its variability.

In Fig. 10 we performed the temperature calculation using
six different scenarios for the ice thinning function \( S(z) \). We
assume the simple scenario of a thinning function that varies
linearly with depth and a value of \( S(z = 2100 \text{ m}) \) equal to
0.22, 0.24, 0.26, 0.28, 0.30, 0.32. It is apparent that the change
in temperature due to thinning function uncertainties can be
significant. However this type of uncertainty only affects the
slope of the temperature signal. So, unrealistic thinning
function scenarios are relatively straightforward to rule out.
Estimates of temperature from other proxies for any point of
the record, can be useful in order to select a plausible scenario
for the thinning function. This allows for a more accurate
determination of slope of the temperature signal inferred by
means of the firn diffusion method.

A direct consequence of this is that the method can poten-
tially be useful in providing combined paleotemperature
and glaciological information. In this study the unrealistically
high temperature values we inferred for the Holocene climatic
optimum pointed to possible inaccuracies of the ice thinning
function used for the estimation. When fixing the temperature
gradient between the Holocene optimum and present condi-
tions to be approximately 3 K, we were able to propose a
more likely scenario for the ice thinning function and hence
the accumulation rate history. The temperature reconstruction
using the proposed ice thinning function for NorthGRIP is
presented in red color in Fig. 10.

VIII. FIRN DENSIFICATION UNCERTAINTIES

Uncertainties related to the densification model affect the
estimation of the diffusion length \( \sigma^2_{iirm} \). Hereby we examine
the influence of four parameters involved in the densification–
diffusion model. We run a set of sensitivity experiments where
the four firn densification parameters are perturbed in order
to create a family of 1000 implementations of the diffusion-
densification model for each experiment. For all the following
sensitivity tests we also consider the standard deviation of the
diffusion length spectral estimate as calculated in sections VII
and VI.

The first two parameters we consider are the surface and
close–off densities \( \rho_0 \) and \( \rho_{co} \). We perform two sensitivity
experiments where the values of \( \rho_0 \) and \( \rho_{co} \) are drawn from a
Gaussian distribution with a defined mean and standard
deviation (Table IV). For the close–off density a value of
804 \( \pm 20 \text{ kgm}^{-3} \) (1\( \sigma \)) is used (Schwander et al. 1988,
Jean-Baptiste et al. 1998, Johnsen et al. 2000). The range of
values we choose for \( \rho_{co} \) brackets within 2\( \sigma \) the more extreme
estimates of 775 and 840 kgm\(^{-3} \) shown in (Scher and Zallen
1970) and Stauffer et al. (1985) respectively. For the surface
density we vary a value of 320 \( \pm 40 \text{ kgm}^{-3} \) (1\( \sigma \)), based on
modern observations of the firm column density at
NorthGRIP. Previous high resolution density observations by
Albert and Shultz (2002) for Summit, Greenland indicate that
the surface density can vary within \( \pm 50 \text{ kgm}^{-3} \) of its mean
value. As a result, with a 1\( \sigma \) of 40 kgm\(^{-3} \) the Gaussian dis-
tribution of \( \rho_0 \) covers this range adequately in our sensitivity
experiments.

Additionally, we include two parameters that describe the
dependence of the densification rate to temperature. Based on
Herron and Langway (1980)

\[
\frac{dp(z)}{dt} = K(T) A^\rho \rho_{ice} - \rho(z) \rho_{ice}, \tag{21}
\]

where \( K(T) \) is a temperature dependent Arrhenius–type den-
sification rate coefficient described by:

\[
K(T) = 11 \exp \left( -\frac{10160}{RT} \right) \rho < 550 \text{ kgm}^{-3}, \tag{22}
\]

and

\[
K(T) = 575 \exp \left( -\frac{21400}{RT} \right) \rho \geq 550 \text{ kgm}^{-3}. \tag{23}
\]

In order to perturb the model we use the term \( K'(t) \) in Eq.
where \( K'(T) = fK(T) \) and \( f = 1 \pm 0.2 \) (1\( \sigma \)). This
results in a family of density profiles that are used for the
diffusion length calculation. In Fig. 11 \( \sigma \) and 2\( \sigma \) intervals
are illustrated together with firn density measurements from
NorthGRIP.

The results of these sensitivity experiments are illustrated in
Fig. 12. Based on these results we conclude that using a fixed
value for the surface and close–off densities is a plausible
approach. The combined uncertainty of the \( \rho_0 \) and \( \rho_{co} \) param-
eters is in the order of 1 K and thus the temperature history we
infer is consistent over a wide range of densification parameter
values. Combining all densification parameters the uncertainty
of the estimation is equal to \( \pm 2.5 \text{ K} \) (1\( \sigma \)). Combining this in a
Gaussian sense with the spectral estimation uncertainty from
section VII we get a total of \( \pm 2.7 \text{ K} \) (1\( \sigma \)).
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TABLE I: Summary of the synthetic data test 1. The mean and RMS value of the diffusion length estimation is given in cm for 3 different target values of the diffusion length. Statistics are based on 100 realizations for each experiment.

| Sampling Interval [cm] | 5   | 3.5 | 2.5 |
|------------------------|-----|-----|-----|
| \( \phi_1 = 0.2 \)    | 4.99 \pm 0.20 | 3.47 \pm 0.19 | 2.30 \pm 0.25 |
| \( \phi_1 = 0.9995 \)  | 5.06 \pm 0.15 | 3.50 \pm 0.13 | 2.56 \pm 0.39 |

TABLE II: Summary of the synthetic data test 2. The mean and RMS value of the diffusion length estimation given in cm, is based on 100 realizations of the experiment for each set of AR-1 coefficient \( \phi_1 \) and sampling interval \( \Delta x \).

| Site         | Location          | Accum. Rate [Myr\(^{-1}\)] | Temperature [C] | \( \sigma_T^2 \) [m] |
|--------------|-------------------|------------------------------|-----------------|---------------------|
| Dome C       | 75°06' S 123°21'E  | 0.027                        | -54.5           | 0.067               |
| GISP2        | 72°30'N 38°30'W    | 0.24                         | -31.4           | 0.079               |
| GRIP         | 72°35'N 57°38'W    | 0.23                         | -31.7           | 0.0795              |
| NEEM         | 77°45' S 51°00'W   | 0.27                         | -30             | 0.088               |
| NorthGRIP    | 75°10'N 42°32'W    | 0.207                        | -32             | 0.081               |
| SipleDome    | 81°40' S 148°46'W  | 0.087                        | -25             | 0.145               |
| South Pole   | 90°2 S 00°W        | 0.076                        | -51             | 0.054               |
| Vostoc       | 78°27' S 10°51'E   | 0.024                        | -55.5           | 0.067               |

TABLE III: Surface forcing used for the diffusion length calculations in Fig. 2.

| Experiment   | \( f_0 \) | \( f_1 \) | \( \rho_0 \) [kgm\(^{-3}\)] | \( \rho_\infty \) [kgm\(^{-3}\)] |
|--------------|-----------|-----------|-------------------------------|----------------------------------|
| Experiment 1 | 1         | 1         | 320 \pm 40 kgm\(^{-3}\)      | 804 \pm 20 kgm\(^{-3}\)          |
| Experiment 2 | 1         | 1         | 320 \pm 40 kgm\(^{-3}\)      | 804 kgm\(^{-3}\)                |
| Experiment 3 | 1 \pm 0.2 | 1 \pm 0.2 | 320 kgm\(^{-3}\)            | 804 kgm\(^{-3}\)                |
| Experiment 4 | 1 \pm 0.2 | 1 \pm 0.2 | 320 \pm 40 kgm\(^{-3}\)      | 804 \pm 20 kgm\(^{-3}\)          |

TABLE IV: Summary of the sensitivity experiments run

Fig. 1: Ice diffusivity parametrizations based on isotopic probe experiments for the temperature range 200 – 270 K
Fig. 2: Calculation of $\sigma_{18}^2$ for the close-off density of $\rho_{co} = 804.3 \text{ kg m}^{-3}$ in m of ice equivalent. $\rho_o = 330 \text{ kg m}^{-3}$ for all sites.
Fig. 3: Examples of raw δ¹⁸O data and estimated power spectral densities \( \hat{P}_s \) for four depth intervals using \( \mu = 30 \) (red spectra) and \( \mu = 40 \) (black spectra). The power spectral model is illustrated; \( P_\sigma \) in green, \( P_\sigma \) in cyan and \( |\hat{\eta}(k)|^2 \) in pink. The red vertical line in the top three plots indicates the approximate position of the frequency representing the annual layer thickness. For the bottom plot this frequency is \( \approx 40 \text{ m}^{-1} \) and omitted from the plot.
Fig. 4: Result of the AR order selection test (section V). The black curve is the mean value of the diffusion length calculated using a $\mu$ in the range [40, 80]. The blue curve is the diffusion length curve corrected for ice flow thinning (red curve) effects. The mean $P_0$ and $\sigma^2$ values are given in the two middle plots. In the top plot the standard deviation of the 41 estimates of the diffusion length for every depth in m ice eq. is shown.

Fig. 5: Synthetic data test 1. Time series after diffusion and discrete sampling for $\phi = 0.9995$ and 0.2. The power spectral densities and the estimated $\hat{P}_x$ are also presented. (a) $\sigma = 0.05$ m. (b) $\sigma = 0.035$ m. (c) $\sigma = 0.025$ m.
Fig. 6: Synthetic data test 2. $\sigma = 0.08 \text{ m}$ for all 4 sets of experiments. Power spectral densities (column 1) and time series after convolution with the diffusion filter and discrete sampling (columns 2 and 3) for both values of $\phi_1$ are presented. Here we show the 100th realization of the experiment for every set. (a) $\Delta = 0.05 \text{ m}$. (b) $\Delta = 0.08 \text{ m}$. (c) $\Delta = 0.10 \text{ m}$. (d) $\Delta = 0.12 \text{ m}$. 
Fig. 7: Effect of the ice layer thinning on the value of the diffusion length. For the calculation of $\sigma_{\text{firm}}$, the parameters we used for the H–L model were typical of Holocene conditions for the NorthGRIP site: $P = 0.7$ atm, $\rho_0 = 330$ kg m$^{-3}$, $\rho_{CO} = 804.3$ kg m$^{-3}$, $T = 242.15$ K, and $A = 0.2$ myr$^{-1}$ ice eq.

Fig. 8: Modeled power spectral densities for 6 different depths using diffusion length values from the calculation of Fig. 7 (Note that all plots represent identical accumulation rate and temperature forcing at the surface). We use the spectral model as in Eq. 19 with $q_1 = 0.15$, $P_{\text{ini}} = 0.2$. The blue highlighted area represents the expected spectra for the case of $\Delta = 5$ cm ($f_{Nyq} = 10$ m$^{-1}$) while the full range of the modeled spectra represents the case with $\Delta = 2.5$ cm ($f_{Nyq} = 20$ m$^{-1}$).
Fig. 9: Discrete sampling correction

Fig. 10: Ice thinning uncertainty. Temperature reconstructions for NorthGRIP using (from top to bottom) $S(2100m) = 0.22, 0.24, 0.26, 0.28, 0.30, 0.32$. Records filtered with a 500y low-pass filter.
Fig. 11: Firn density measurements from NorthGRIP (red) compared to implementations of the H-L densification model with varying values of $f$. Solid and dashed curves represent $1\sigma$ and $2\sigma$ respectively.
Fig. 12: Sensitivity tests results. The top panel illustrates the mean temperature history as calculated from Experiment 4 (all parameters varied) bracketed by the 95% confidence interval estimated with the sensitivity test. For the 95% confidence interval we have also taken into account the uncertainty of the spectral estimation based on the synthetic data tests and using an RMS value of $\pm 0.5 \text{ cm}$ that is equivalent to $\approx \pm 1 \text{ K}$ in temperature. In the bottom panel we present the value of the standard deviation ($\sigma$) for each sensitivity test.