Variational study of the flux tube recombination in the two quarks and two quarks system i Lattice QCD

M. Cardoso, N. Cardoso, and P. Bicudo
CFTP, Departamento de Física, Instituto Superior Técnico (Universidade Técnica de Lisboa), Av. Rovisco Pais, 1049-001 Lisboa, Portugal

The color fields in a system composed by two static quarks and two static antiquarks are studied. In particular, we consider the four particles in the corners of a rectangle, and two possible alignment of the particles, one in which the quarks are at the same side of the rectangle, and the other where they are at opposite sides. We use a variational method, to probe not only the ground state but also the first excited state. This results permit us to observe and interpret the flux-tube recombination in the mesons to mesons and the tetraquark to mesons transitions, for both states. The results are compared with previous results for the static potential and the Casimir scaling predictions.

I. INTRODUCTION

Systems constituted by two quarks and two antiquarks are of extreme importance for strong interaction physics. Not only because they are a starting point for meson-meson scattering, but also because of the possible existence of bound-states — tetraquarks, initially predicted by Jaffe [1]. There are several observed resonances which are the candidates to tetraquarks [2–10]. The most recent are the Z_b± particles reported by the Belle collaboration [11, 12]. While it remains difficult to study tetraquarks in Lattice QCD with dynamical quarks [13, 14] the static limit as a Cornell potential:

\[ V_M = V_M(|r_1 - r_3|) + V_M(|r_2 - r_4|), \]

where the intrameson potential \( V_M \) is well fitted in the static limit as a Cornell potential: \( V_M(r) = C - \frac{\gamma}{r^2} + \frac{\sigma}{r} \). The potential \( V_T \), is the tetraquark potential and is given according to [10] and [15] by

\[ V_T = C + \alpha \sum \frac{\lambda_i}{2} \sum \frac{\lambda_j}{2} \frac{1}{r_{ij}} + \sigma L_{min} \]

where \( L_{min} \) is the minimal distance linking the four particles, as depicted in Fig. 1.

However, the value of the static potential alone, is not sufficient to understand the confinement in tetraquarks. Confinement can be researched both with the measurement of the color fields and with the study of the colour wave functions of the tetraquark. Recently we computed the color-electric and color-magnetic fields generated by a static tetraquark [16]. We confirmed that for quark and antiquark distances compatible with a tetraquark potential, tetraquark color flux tubes actually are created. Thus the mechanism of confinement in static tetraquarks is the localization of the color fields in fundamental flux tubes.

But it remains important to clarify what are the color wave functions of static tetraquarks, and to observe the flip-flop of both the flux tubes and the wave functions.
of static teraquarks. Here we apply the variational principle to observe whether the flip-flop transition occurs for the color flux tubes in a quark-quark-antiquark-antiquark \((QQ\bar{Q})\) system.

In Section II, we discuss the two possible color singlets of the four-particle system, while in section III, we review the Wilson Loop of the tetraquark system. Then, in section IV, we make the synthesis of the two previous sections and develop a variational method to describe the \(QQ\bar{Q}\bar{Q}\) system, either in a tetraquark color state or in a two meson state. In section V, the method we use to compute the color fields in the lattice is described. In section VI, this method is used to compute the color fields in the system, for different arrangements of the four particles. In section VII our results are discussed and in section VIII the conclusions are presented.

II. A TWO COMPONENT BASIS FOR THE TETRAQUARK COLOR WAVEFUNCTION

Notice a tetraquark must be a color singlet due to confinement and to gauge invariance, but there are two possible color wave functions for a color singlet tetraquark. For instance, if the system is on the domain where \(V_{FF} = V_I\) (domain I), we expect the color wavefunction of the ground state to be given by

\[
|I\rangle = \frac{1}{3} \sum_{ij} |Q_i Q_j \bar{Q}_i \bar{Q}_j\rangle .
\]

If the system is on domain II, we expect the ground state to be given by

\[
|II\rangle = \frac{1}{3} \sum_{ij} |Q_i Q_j \bar{Q}_j \bar{Q}_i\rangle .
\]

Note that these two systems are not orthogonal as \(\langle I|II\rangle = \frac{1}{3}\).

When \(V_{FF} = V_T\), we expect that the two quarks form an antitriplet and the two antiquarks a triplet. So, the wavefunction is given by,

\[
|A\rangle = N \epsilon_{ijk} |Q_i Q_j\rangle \epsilon_{klm} |Q_k Q_m\rangle ,
\]

where the \(A\) stand for antisymmetric since this wavefunction is antisymmetric for the exchange of two quarks or two antiquarks. Eq. (7) can be simplified by contracting the tensors, and by imposing the normalization \(\langle A|A\rangle = 1\),

\[
|A\rangle = \frac{1}{2\sqrt{3}} \left( |Q_i Q_j \bar{Q}_j \bar{Q}_i\rangle - |Q_i Q_j \bar{Q}_i \bar{Q}_j\rangle \right) = \frac{\sqrt{3}}{2} (|I\rangle - |II\rangle) .
\]

We can also construct a color symmetric state as

\[
|S\rangle = \frac{1}{2\sqrt{6}} \left( |Q_i Q_j \bar{Q}_j \bar{Q}_i\rangle + |Q_i Q_j \bar{Q}_i \bar{Q}_j\rangle \right) = \frac{\sqrt{3}}{2} (|I\rangle + |II\rangle) .
\]

Any color single state of two quarks and two antiquarks can be decomposed in two different sets of basis vectors, either \(|I\rangle\) and \(|II\rangle\) or \(|A\rangle\) and \(|S\rangle\). The second set of kets has the advantage of forming an orthonormal basis. \(|I\rangle\) and \(|II\rangle\) can be written as

\[
|I\rangle = \sqrt{\frac{7}{3}} |S\rangle + \frac{1}{\sqrt{3}} |A\rangle ,
\]

\[
|II\rangle = \sqrt{\frac{7}{3}} |S\rangle - \frac{1}{\sqrt{3}} |A\rangle .
\]

Therefore, we can describe our state with a two component vector and the potential itself as a two by two matrix. The lowest eigenvalue of the matrix is corresponds to the static potential measured on the lattice, with the corresponding eigenvector being \(|I\rangle\), \(|II\rangle\) or \(|A\rangle\) depending on the domain considered. Since the potential is hermitian, the eigenvector of the excited state of the potential is orthogonal to the one of the ground state. Therefore, these eigenvectors must be

\[
|\tilde{I}\rangle = -\frac{1}{\sqrt{3}} |S\rangle + \sqrt{\frac{2}{3}} |A\rangle ,
\]

\[
|\tilde{II}\rangle = -\frac{1}{\sqrt{3}} |S\rangle - \sqrt{\frac{2}{3}} |A\rangle .
\]
with flip-flop potential, the two-quark and two-antiquark system is a generalized singlet system where the two quarks form an antitriplet and the two antiquarks form a triplet. The two-meson potentials, given by the sum of two independent intra-meson potentials \( V_{\text{meson}} = V_{M_1} + V_{M_2} \), and \( V_T \) is the tetraquark potential which corresponds to the sector where the four particles are confined, linked by a single fundamental string.

For the special case where the four particles form a rectangle as in Fig. 2, \( L_{\text{min}} \) is given by \( L_{\text{min}} = \sqrt{3}r_1 + r_2 \) for \( r_2 > \frac{r_1}{\sqrt{3}} \). If we neglect the Coulomb part of \( V_{FF} \), the flip-flop potential with the linear potentials only produces a tetraquark domain for distances \( r_2 \gtrsim \sqrt{3}r_1 \).

\[ |II\rangle = \frac{1}{\sqrt{3}} |S\rangle + \sqrt{\frac{2}{3}} |A\rangle, \tag{13} \]

for the domain II. In the tetraquark domain the excited state eigenvector must be \(|S\rangle\), defined in Eq. (9).

### III. THE TETRAQUARK WILSON LOOP

The static potential for the tetraquark has been studied in the lattice by [10] and [9]. The Wilson loop operator for the tetraquark system, illustrated in Fig. 2, is given by

\[ W_{4Q} = \frac{1}{3} \text{Tr}[M_1 R_{12} M_2 L_{12}], \tag{14} \]

with

\[ R_{ij}^{kk'} = \frac{1}{2} \epsilon_{ijk} \epsilon_{i'j'k'} R_1^{kk'} R_2^{ij'}, \tag{15} \]

\[ L_{ij}^{kk'} = \frac{1}{2} \epsilon_{ijk} \epsilon_{i'j'k'}. \]

This Wilson Loop has the quantum numbers of color singlet system where the two quarks form an antitriplet and the two antiquarks form a triplet.

This lattice studies, indicate that the static potential of the two-quark and two-antiquark system is a generalized flip-flop potential,

\[ V_{FF} = \min(V_T, V_{M_1, M_2}, V_{M_3, M_4}), \tag{16} \]

where \( V_{M_1, M_2} \) and \( V_{M_3, M_4} \) are the two possible two-meson potentials, given by the sum of two independent intra-meson potentials \( V_{M_1, M_2} = V_{M_1} + V_{M_2} \), and \( V_T \) is the tetraquark potential which corresponds to the sector where the four particles are confined, linked by a single fundamental string.

For the special case where the four particles form as rectangle as in Fig. 2, \( L_{\text{min}} \) is given by \( L_{\text{min}} = \sqrt{3}r_1 + r_2 \) for \( r_2 > \frac{r_1}{\sqrt{3}} \). If we neglect the Coulomb part of \( V_{FF} \), the flip-flop potential with the linear potentials only produces a tetraquark domain for distances \( r_2 \gtrsim \sqrt{3}r_1 \).

### IV. VARIATIONAL METHOD FOR THE QQ\bar{Q}\bar{Q} SYSTEM

Now we extend the Wilson loop in order to obtain the true ground state of the tetraquark system and also to obtain the first excited state. To achieve this, we note that the Wilson Loop operator can be written as a correlation of a certain operator at different times \( W(t) = \langle \mathcal{O}(t) \mathcal{O}^\dagger(0) \rangle \). This can be generalized by considering instead a base of operators \( \mathcal{O}_i \). So, this way, our Wilson loop becomes a matrix \( W_{ij} = \langle \mathcal{O}_i \mathcal{O}_j^\dagger \rangle \). This can be used not only to improve the ground state overlap but also to obtain more energy levels of the system. The energy levels are the solution of the generalized eigensystem

\[ (W_{ij}(t))c_n^j(t) = w_n \langle W_{ij}(0) \rangle c_n^i(t). \tag{17} \]

We consider again the case where the four particles form a rectangles and also two different alignments of the \( QQ\bar{Q}\bar{Q} \) system. A parallel one, where the two quarks are on the same side of the rectangle and an antiparallel one, where the quarks are on opposite corners of the rectangle, see Fig. 3.

For both cases we use a basis of two operators, inspired in Ref. 28. In the parallel case the operators are the tetraquark operator \( \mathcal{O}_{4Q} \), and a two-meson operator. This gives a Wilson loop matrix where the diagonal elements are: the tetraquark Wilson Loop \( W_{4Q} = \langle \mathcal{O}_{4Q}(t) \mathcal{O}_{4Q}^\dagger(0) \rangle \) (Fig. 2), the correlation of two Wilson loops (one per meson composed of a quark and an antiquark) and the off-diagonal elements corresponding to the transition between the two states. The four matrix elements, each one corresponding to a different Wilson loop, are depicted in Fig. 4.

For the antiparallel alignment, the operators we utilize are the two meson-meson operators, giving the four matrix elements given on Fig. 5.

### V. CHROMO-FIELDS COMPUTATION

We compute the color electric and the color magnetic fields, utilizing the correlators of the plaquettes \( P_{\mu\nu} \) and the Wilson loops \( W_a \). We define the plaquettes as \( P_{\mu\nu} = 1 - \frac{1}{3} \text{Tr}[U_\mu(s)U_\nu(s+\mu)U_\mu^\dagger(s+\nu)U_\nu^\dagger(s)] \).

Figure 4: Elements of the Wilson loop matrix in the parallel alignment case.

Figure 5: Elements of the Wilson loop matrix in the antiparallel alignment case.
With this definition, the chromo-fields are given by

\[ \langle E_i^2 \rangle = \langle P_{0i} \rangle - \frac{\langle W_i P_{0i} \rangle}{\langle W \rangle} \]  

\[ \langle B_i^2 \rangle = \frac{\langle W P_{jk} \rangle}{\langle W_n \rangle} - \langle P_{jk} \rangle , \]  

with the indices \( j \) and \( k \) complementing index \( i \). The Lagrangian and Energy densities are given by \( \mathcal{L} = \frac{1}{2}(E^2 - B^2) \) and \( \mathcal{H} = \frac{1}{2}(E^2 + B^2) \). The Wilson loop operator for each state is given by \( W_n(t) = c_0^n(t) W_{ij} c_j^n(t) \), in conformity with Eq. (17).

The plaquettes are placed at \( T = T_{\text{min}} \), where \( T \) is the temporal extension of the Wilson Loops. We calculate the correlators for different values of \( T \) and fit the results from \( T = T_{\text{min}} \) to \( T = T_{\text{max}} \) to a constant, and calculate the \( \chi^2/d.o.f. \). We find that for \( T_{\text{min}} = 4 \) and \( T_{\text{max}} = 16 \), the values of the \( \chi^2/d.o.f. \) are generally acceptable, while still giving a clear signal. For this we consider the value of the fields as the result of the fit to a constant of the correlators from \( T = 4 \) to \( T = 16 \).

\section{VI. RESULTS FOR THE COLOR FIELDS}

In this work, we only consider the geometry of the four particles in a plane forming a rectangle with two distinct kinds of alignment, see Fig. 3. When the two quarks/antiquarks are on the same side of the rectangle, this corresponds to a parallel alignment; if they are in opposite sides, we have an antiparallel alignment. The parallel alignment is useful in the visualization of the transition between the tetraquark and the two meson states, while the antiparallel one is used to observe the transition between the two meson-meson states.

The results presented in this work are obtained using 1121 quenched lattice QCD configurations with dimension \( 24^3 \times 48 \) and \( \beta = 6.2 \) with lattice spacing, \( a, a = 0.07261(85) \) fm or \( a^{-1} = 2718(32) \) MeV The configurations were generated with GPUs using a combination of Cabbibo-Marinari, pseudo-heatbath and over-relaxation algorithms. We smearing and Hypercubic blocking are used to improve the signal to noise ratio.

In Figs. 6 and 7, we show the results of the Lagrangian density for the antiparallel alignment for the ground state, \( n = 0 \), and the first excited state, \( n = 1 \), respectively, with different values of \( r_1 \) and \( r_2 \). Note that, in this geometry the system is symmetric for the exchange of \( r_1 \) and \( r_2 \). When \( r_1 = r_2 \), the system forms a square-symmetric structure for both the ground and first excited states, thus the two states have to correspond to the symmetric and antisymmetric color wavefunctions. Inspecting the composition of the variational Wilson Loop for the two states, we are able to conclude that the ground state corresponds in this geometry to the color symmetric wavefunction, with the color antisymmetric corresponding to the first excited state. This is the reverse of what happens in the tetraquark sector, where the ground state is color antisymmetric. When \( r_1 < r_2 \), the groundstate results show a pure two-meson ground state. In what concerns the first excited state, it must have a color wavefunction orthogonal to the one of the ground state. We observe a first excited state different from the other two-meson state, where the confining string seems to be linking all the four particles together.

In Figs. 8 and 9, we depict the plot of the Lagrangian density of the parallel alignment, in the ground and first excited states respectively. These results are complemented with the Figs. 10 and 11 with cuts for \( x = 0 \) and \( y = 0 \), for \( r_1 = 6 \) and \( r_2 = 8 \).

We now analyse the ground states profiles. In upper graphics of both Fig. 10 and 11 we see the results for the cuts of the Lagrangian density (with \( r_1 = 6 \) and \( r_1 = 8 \) respectively), for \( x = 0 \) (on the left side) and \( y = 0 \) (on the right side). The first case corresponds to the center of the predictable diquark-diantiquark flux tube formed in the tetraquark domain, while the second corresponds to a transversal cut of the same flux tube and also of the two mesonic flux tubes, whose the centers should be in \(-r_1/2 \) and \( r_2/2 \). We observe the transition between the tetraquark and the two meson states.

More interesting and not so easily interpretable is the first excited state. We can with a fair amount of certainty claim that this first excited state is due to the recombination of the flux tubes and not due to the flux tube excitations as in. More easily interpretable (Figs. 7, 9, 10 and 11) contrarily to what we can naively be expect for the meson to meson transition. The first excited state clearly is not the other two meson state, but a different color state. Neither, is it a vibrational excitation of a flux tube.

Indeed the first excited state should be orthogonal in the color space to the ground state. Since both the ground state and the first excited state are eigenvectors of the static potential matrix since the phenomenon which is happening is the recombination of the flux tube, and since this matrix is hermitian, the two vectors have to be orthogonal to each other. Consequently, we expect that \( |I \rangle \) to be the color vector of the first excited state, when the ground state has the color of the meson system \( |J \rangle \). Similarly, we expect that the first excited state to be given by the symmetric color state \( |S \rangle \), when the ground state is the antisymmetric one \( |A \rangle \). This happens when we are in the tetraquark domain.
VII. DISCUSSION

As we discussed before, the results seem to agree with the previous ones, obtained for the static potential [9, 10], which support the generalized flip-flop picture for the ground state of two quarks and two antiquarks system. Namely, we observe the formation of the tetraquark string and of the two mesonic strings in the domains we would expect that to happen.

For the first excited states, the situation is not as clear, and so to better understand these states we calculate the Casimir scaling factors for the different color wavefunctions in Table I. This shall provide us with qualitative insight to what kind of interaction we expect in the system.
In Table I we see that Casimir Scalling predicts a repulsive quark-quark and antiquark-antiquark interaction and an attractive quark-antiquark interaction for the state \(|S\rangle\). As can be seen in Figs. 9, 10 and 11, the results qualitatively agree with this prediction, due to a suppression of the flux-tube between the two quarks and the two antiquarks. For the state \(|\bar{I}\rangle\), the prediction is the repulsion between the particles that form the two mesons in the ground states, while all other interactions are attractive. Again, the results do not qualitatively contradict this possibility, as shown in Fig. 9 for \((r_1, r_2) = (6,8)\) and \((r_1, r_2) = (8,10)\), where we again see a flux tube suppression between the particles that form mesons in the ground state.

So, the results for the first excited state seem to agree with the hypothesis that the first excited state is orthogonal to the ground state in the color space, with \(|\bar{I}\rangle\) and \(|II\rangle\) and \(|S\rangle\) being the excited states, where \(|I\rangle\), \(|II\rangle\) and \(|A\rangle\) respectively are the ground states.
Figure 10: Cuts of the Lagrangian density for $x = 0$ and for $y = 0$, with $r_1 = 6$, both for the ground state and for the first excited state.

VIII. CONCLUSION

In this work, we use a variational method to compute the chromo-fields of the system composed of two quarks and two antiquarks. With this method we can not only observe the region where the tetraquark state is the ground one, but also the region where the ground state is composed of two mesons, as well as the transitions between this two regimes and the one between the two possible meson-meson ground states. We are also able to observe, for the first time, the first excited state of this system.

For the ground state the results improve our previous work [11] where only the tetraquark operator was used. There we had difficulties with the measurement of the ground state outside of the tetraquark region due to the low overlap of the used operator with the true ground state. Here, we are able to overcome that difficulty by using a variational basis. The results are similar, in the tetraquark region, to those obtained there, while giving the expected transition to the meson-meson behavior outside that region.

The results for the first excited state are not, at the first view, as understandable as the ones for the ground state. We note however that since the ground state is well below the first gluonic excitation of the string, the only way by which we can explain this excitation is the flux tube recombination. This way we know that the first excited state has a color eigenfunction which is orthogonal to the ground state, be it a tetraquark or a two-meson state. By using, the Casimir factors, we compare the predictions of the aforementioned hypothesis with our results of the Lagrangian density for this state. We conclude that both results are compatible.

Note that this first excited state should be as important as the ground state of this system. Think for instance in the decay of a tetraquark into a two meson system. Since the color structure of the initial and final states are not the same, we can not consider the potential of the system to be a scalar in the color space. Instead it should be given by a two by two matrix as explained above. To reconstruct this matrix we need to know not only the potential of the ground state but also it’s color...
Figure 11: Cuts of the Lagrangian density for $x = 0$ and for $y = 0$, for $r_1 = 8$, both for the ground state and for the first excited state.

Table I: $C_{ij} = \langle \Psi | \frac{\lambda_i \cdot \lambda_j - 16}{3} | \Psi \rangle$

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This study should be complemented in the future by a detailed study of the potential and color composition of the first excited state in the lattice.

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composition as well as the potential of the first excited state.
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